APPROXIMATE CONSTRUCTION OF RATIONAL
APPROXIMATIONS AND THE EFFECT OF
ERROR AUTOCORRECTION. APPLICATIONS.

G. L. Litvinov

Abstract. Several construction methods for rational approximations to functions
of one real variable are described in the present paper; the computational results that
characterize the comparative accuracy of these methods are presented; an effect of
error autocorrection is considered. This effect occurs in efficient methods of ratio-
nal approximation (e.g., Padé approximations, linear and nonlinear Padé–Chebyshev
approximations) where very significant errors in the coefficients do not affect the
accuracy of the approximation. The matter of import is that the errors in the nu-
merator and the denominator of a fractional rational approximant compensate each
other. This effect is related to the fact that the errors in the coefficients of a rational
approximant are not distributed in an arbitrary way but form the coefficients of a new
approximant to the approximated function. Understanding of the error autocal-
correction mechanism allows to decrease this error by varying the approximation procedure
depending on the form of the approximant. Some applications are described in the
paper. In particular, a method of implementation of basic calculations on decimal
computers that uses the technique of rational approximations is described in the
Appendix.

To a considerable extent the paper is a survey and the exposition is as elementary
as possible.

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Whenever he has some money to spare, he goes to a shop and buys some kind of useful book. Once he bought a book that was entitled “Inverse trigonometrical functions and Chebyshev polynomials”.

N.N. Nosov “Happy family”. Moscow, 1975, p.91

§1. Introduction

The author came across the phenomenon of error autocorrection at the end of seventies while developing nonstandard algorithms for computing elementary functions on small computers. It was required to construct rational approximants of the form

\[ R(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m} \]

to certain functions of one variable \( x \) defined on finite segments of the real line. For this purpose a simple method (described in [1] and below) was used: the method allows to determine the family of coefficients \( a_i, b_j \) of the approximant (1) as the solution of a certain system of linear algebraic equations. These systems turned out to be ill conditioned, i.e., the problem of determining the coefficients of the approximant is, generally speaking, ill-posed in the sense of [2]. Nevertheless, the method ensures a paradoxically high quality of the obtained approximants whose errors are close to the best possible [1].
For example, for the function \( \cos x \) the approximant of the form (1) on the segment \([-\pi/4, \pi/4]\) obtained by the method mentioned above for \( m = 4, n = 6 \) has the relative error equal to \( 0.55 \cdot 10^{-13} \), and the best possible relative error is \( 0.46 \cdot 10^{-13} \) [3]. The corresponding system of linear algebraic equations has the condition number of order \( 10^9 \). Thus we risk losing 9 accurate decimal digits in the solution of calculation errors. Computer experiments show that this is a serious risk. The method mentioned above was implemented as a Fortran program. The calculations were carried out with double precision (16 decimal positions) by means of ICL–4–50 and ES–1045 computers. These computers are very similar in their architecture, but when passing from one computer to another the system of linear equations and the computational process are perturbed because of calculation errors, including round-off errors. As a result, the coefficients of the approximant mentioned above to the function \( \cos x \) experience a perturbation already at the sixth–ninth decimal digits. But the error in the rational approximant itself remains invariant and is \( 0.4 \cdot 10^{-13} \) for the absolute error and \( 0.55 \cdot 10^{-13} \) for the relative error. The same thing happens for approximants of the form (1) to the function \( \arctg x \) on the segment \([-1,1]\) obtained by the method mentioned above for \( m = 8, n = 9 \) the relative error is \( 0.5 \cdot 10^{-11} \) and does not change while passing from ICL–4–50 to ES–1045 although the corresponding system of linear equations has the condition number of order \( 10^{11} \), and the coefficients of the approximant experience a perturbation with relative error of order \( 10^{-4} \).

Thus the errors in the numerator and the denominator of a rational approximant compensate each other. The effect of error autocorrection is connected with the fact that the errors in the coefficients of a rational approximant are not distributed in an arbitrary way, but form the coefficients of a new approximant to the approximated function. It can be easily understood that the standard methods of interval arithmetic (see, for example, [54]) do not allow to take into account this effect and, as a result, to estimate the error in the rational approximant accurately.

Note that the application of standard procedures known in the theory of ill-posed problems results in this case in losses in accuracy. For example, if one applies the regularization method, two thirds of the accurate figures are lost [4]; in addition, the amount of calculations increases rapidly. The matter of import is that the exact solution of the system of equations in the present case is not the ultimate goal; the aim is to construct an approximant which is precise enough. This approach allows to “rehabilitate” (i.e., to justify) and to simplify a number of algorithms intended for the construction of the approximant, to obtain (without additional transforms) approximants in a form which is convenient for applications.

Professor Yudell L. Luke kindly drew the author’s attention to his papers [5, 6] where the effect of error autocorrection for the classical Padé approximants was revealed and was explained at a heuristic level. The method mentioned above leads to the linear Padé–Chebyshev approximants if the calculation errors are ignored.

In the present paper, using heuristic arguments and the results of computer experiments, the error autocorrection mechanism is considered for quite a general situation (linear methods for the construction of rational approximants, nonlinear generalized Padé approximations). The efficiency of the construction algorithms used for rational approximants seems to be due to the error autocorrection effect (at least in the case when the number of coefficients is large enough).

Our new understanding of the error autocorrection mechanism allows us, to some extent, to control calculation errors by changing the construction procedure.
depending on the form of the approximant.

In the paper the construction algorithm for linear Padé–Chebyshev approximants is considered and the corresponding program is briefly described (see [7]). It is shown that the appearance of a control parameter allowing to take into account the error autocorrection mechanism ensures the decrease of the calculation errors in some cases. Results of computer calculations that characterize the possibilities of the program and the quality of the approximants obtained as compared to the best ones are presented. Some other (linear and nonlinear) construction methods for rational approximants are described. Construction methods for linear and nonlinear Padé–Chebyshev approximants involving the computer algebra system REDUCE (see [8]) are also briefly described. Computation results characterizing the comparative precision of these methods are given. With regard to the error autocorrection phenomenon the effect described in [9] and connected with the fact that a small variation of an approximated function can lead to a sharp decrease in accuracy of the Padé–Chebyshev approximants is analyzed. Some applications are indicated. In particular, a method of implementation of basic calculations on decimal computers that uses the technique of rational approximations is described in the Appendix.

To a considerable extent the paper is a survey and the exposition is as elementary as possible. In the survey part of the paper we tried to present the required information clearly and consistently, to make it self-contained. But this part does not claim to be complete: the number of papers concerning rational approximations theory and its applications in numerical analysis (including computer calculation of functions, numerical solving of equations, acceleration of convergence of series, and quadratures), in theoretical and experimental physics (including quantum field theory, scattering theory, nuclear and neutron physics), in the theory and practice of experimental data processing, in mechanics, in control theory, and other branches is much too vast; see, in particular, the reviews and reference handbooks [3, 10–16].

The author is grateful to Yudell L. Luke for stimulating conversations and valuable instructions. The author also wishes to express his thanks to I. A. Andreeva, A. Ya. Rodionov and V. N. Fridman who participated in the programming and organization of computer experiments. This paper would not have been written without their help. A preliminary version of the paper was published in [56].

§2. Best approximants

We shall need some information and results pertaining to ideas of P. L. Chebyshev, see [17]. Let \([A,B]\) be a real line segment (i.e., \([A,B]\) is the set of all real numbers \(x\) such that \(A \leq x \leq B\)) and \(f(x)\) be a continuous function defined on this segment. Consider the absolute error function of the approximant of the form (1) to the function \(f(x)\), i.e., the quantity

\[
\Delta(x) = f(x) - R(x),
\]

and the absolute error of this approximant, i.e., the quantity

\[
\Delta = \max_{A \leq x \leq B} |\Delta(x)| = \max_{A \leq x \leq B} |f(x) - R(x)|.
\]

A classical problem of approximation theory is to determine, for fixed degrees \(m\) and \(n\) in (1), the coefficients in the numerator and the denominator of expression (1)
so that (3) is the smallest possible. The corresponding approximant is called best (with respect to the absolute error). An important role is played by the following result.

**Generalized de la Vallée–Poussin theorem** [17]. *If the polynomials*

\[
P(x) = \tilde{a}_0 + \tilde{a}_1 x + \cdots + \tilde{a}_{n-\nu} x^{n-\nu},
\]

\[
Q(x) = \tilde{b}_0 + \tilde{b}_1 x + \cdots + \tilde{b}_{m-\mu} x^{m-\mu},
\]

*where \(0 \leq \mu \leq m, 0 \leq \nu \leq n, b_{m-\mu} \neq 0,\) have no common divisor (i.e., the fraction \(\tilde{R}(x) = \tilde{P}(x)/\tilde{Q}(x)\) is irreducible), the expression \(\tilde{R}(x) = \tilde{P}(x)/\tilde{Q}(x)\) is finite on the segment \([A, B]\), and at successive points \(x_1 < x_2 < \cdots < x_N\) of the segment \([A, B]\) the error function \(\tilde{\Delta}(x) = f(x) - \tilde{R}(x)\) of the approximant \(\tilde{R}\) takes nonzero values \(\lambda_1, -\lambda_2, \ldots, (-1)^N \lambda_N\) with alternating signs (so that the numbers \(\lambda_i\) are either all positive or all negative), \(N = m + n + 2 - d\), where \(d\) is the smallest of the numbers \(\mu, \nu\), then the error \(\Delta\) of any approximant of the form (1) satisfies the inequality*

\[
\Delta \geq \lambda = \min \{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_N|\}.
\]

**Proof.** Suppose that there exists an approximant \(R(x)\) of the form (1) for which the inequality (5) is not satisfied. Consider the difference

\[
\varepsilon(x) = R(x) - \tilde{R}(x) = [f(x) - \tilde{R}(x)] - [f(x) - R(x)] = \tilde{\Delta}(x) - \Delta(x).
\]

From our assumption it follows that the numbers \(\varepsilon(x_1), \varepsilon(x_2), \ldots, \varepsilon(x_N)\) differ from zero and have alternate signs. And this, in its turn, by virtue of the continuity of the function \(\varepsilon(x)\) on the segment \([A, B]\) implies that the function \(\varepsilon(x)\) has at least \(N - 1 = m + n + 1 - d\) zeros inside the segment \([A, B]\). On the other hand, the definition of the function \(\varepsilon(x)\) implies the equality \(\varepsilon(x) = U(x)/V(x)\), where \(U(x)\) and \(V(x)\) are polynomials and the degree of \(U(x)\) does not exceed \(m + n - d\). So the function \(\varepsilon(x)\) cannot have more than \(N - 2 = m + n - d\) zeros. This contradiction proves the theorem.

The quantity \(d\) which is mentioned in the theorem is called the *defect of the approximant \(\tilde{R}(x)\);* in practice usually \(d = \mu = \nu = 0\). The generalized de la Vallée–Poussin theorem gives us a sufficient condition for the approximant \(\tilde{R}(x) = \tilde{P}(x)/\tilde{Q}(x)\), where \(\tilde{P}(x)\) and \(\tilde{Q}(x)\) are polynomials of the form (4), to be best. The points \(x_1 < x_2 < \cdots < x_N\) of the segment \([A, B]\) are called *Chebyshev alternation points* for the approximant \(\tilde{R}(x)\) if the error function \(\tilde{\Delta}(x) = f(x) - \tilde{R}(x)\) at these points has values which coincide with the absolute error \(\tilde{\Delta}\) of the approximant \(\tilde{R}\) in absolute value and are alternate in sign. In other words, at the points \(x_1, \ldots, x_N\) the error function \(\tilde{\Delta}(x)\) has extrema with alternating signs which coincide with each other in absolute value. From the generalized de la Vallée–Poussin theorem, it follows that the presence of Chebyshev alternation points is sufficient for the approximant \(\tilde{R}(x)\) to be best.

**Chebyshev theorem.** *The presence of Chebyshev alternation points is a necessary and sufficient condition under which the approximant is best. Such an approximant exists and is unique if two fractions that coincide after cancellation are not regarded as different.*

A comparatively simple proof is given in [17]. Note that P. L. Chebyshev and Vallée–Poussin considered the case of polynomial approximants. The general case
was first considered by N. I. Akhiezer, the results mentioned being valid also in
the case when the expression $\Delta_\rho(x) = (f(x) - R(x))/\rho(x)$, where the weight $\rho$ is
nonzero, is taken for the error function; if the weight satisfies certain additional
conditions, then the segment $[A, B]$ need not be assumed finite [17]. Note, that for
$\rho(x) \equiv 1$ we obtain the absolute error (2); and if $f(x)$ has no zeros on the segment
$[A, B]$, then for $\rho(x) = f(x)$ we shall obtain the relative error function

\[
\delta(x) = \frac{\Delta(x)}{f(x)} = \frac{f(x) - R(x)}{f(x)} = 1 - \frac{R(x)}{f(x)}.
\]

Correspondingly, the quantity

\[
\delta = \max_{A \leq x \leq B} |\delta(x)|
\]

is the relative error, and one can speak of the best approximants with respect to the
relative error.

Suppose that the segment $[A, B]$ is symmetric with respect to zero, i.e., $A = -B$. If
the function $f(x)$ is even, then it is not difficult to verify that all its best rational
approximants on this segment (in the sense of the absolute error or of the relative
one) are also even functions, so that one can immediately look for them in the form
$R(x^2) = P(x^2)/Q(x^2)$, where $P$ and $Q$ are polynomials. If the function $f(x)$ is odd,
then its best approximants are also odd functions and one can immediately look for
them in the form $xR(x^2) = xP(x^2)/Q(x^2)$, where $P$ and $Q$ are polynomials. One
can speak of the best approximants with respect to the relative error, if an odd
function $f(x)$ is zero only for $x = 0$, is continuously differentiable, and $f'(0) \neq 0$.
In this case $f(x)$ can be represented in the form $x\varphi(x)$, where $\varphi(x)$ is a continuous
even function that never equals zero. Then describing rational approximants to the
function $f(x)$ with best relative error reduces to solving the same problem for the
even function $\varphi(x)$; indeed,

\[
\frac{f(x) - xR(x^2)}{f(x)} = \frac{x\varphi(x) - xR(x^2)}{x\varphi(x)} = \frac{\varphi(x) - R(x^2)}{\varphi(x)}.
\]

§3. Construction methods for best approximants

Suppose that a rational approximant of the form (1) is the best approximant to
a continuous function $f(x)$ on the segment $[A, B]$. For simplicity, further we shall
assume that the defect is zero. Let $x_1, x_2, \ldots, x_{m+n+2}$ be the Chebyshev alternation
points. Then the error function $\Delta_\rho(x)$ corresponding to the weight $\rho$ (see above)
satisfies the following system of equalities:

\[
\Delta_\rho(x_k) = (-1)^k \lambda,
\]

where $|\lambda| = \Delta_\rho = \max_{A \leq x \leq B} |\Delta_\rho(x)|; k = 1, \ldots, m + n + 2$. For fixed values
of $x_1, \ldots, x_{m+n+2}$, relations (8) can be regarded as a system of $m + n + 2$ equations
with respect to the unknowns $a_i, b_j, \lambda$, where $i = 0, \ldots, n, j = 0, \ldots, m$. Since one
can multiply the numerator and the denominator of the fraction $R(x)$ by the same
number, we see that one more condition, for example, $b_0 = 1$, can be added to sys-
tem (8), so that the number of equations coincides with the number of unknowns.
The iteration method of computation of coefficients in the approximant $R(x)$ (suggested by A. Ya. Remez see [18]) for polynomial approximants and generalized to the general case) is based on this idea. Different versions of the generalized Remez method were considered in many papers; see, for example [3, 12, 20–27].

The approximant is constructed as follows. On the first step, the initial approximations $x_1 < x_2 < \cdots < x_{m+n+2}$ to the Chebyshev alternation points are chosen on the segment $[A, B]$ and the system of equations (8) is solved. As a result we obtain some rational approximant $R_1(x)$ with error function $\Delta_1(x) = (f(x) - R_1(x))/\rho(x)$. For this function the extremum points are found, and the information obtained is used to modify the set $\{x_1, \ldots, x_{m+n+1}\}$. Then the procedure is repeated anew, a new approximant $R_2(x)$ is obtained, and so on.

Taking into account the fact that $\Delta_\rho(x) = (f(x) - R(x))/\rho(x)$ and $R(x)$ has the form (1), system (8) can be rewritten in the form

$$f(x) - \sum_{i=0}^{n} a_i(x_k)^i = (-1)^k \rho(x) \cdot \lambda,$$

whence, as the result of elementary transformations, we get the system of equations

$$\sum_{i=0}^{n} a_i(x_k)^i + \mu_k(\lambda) \sum_{j=0}^{m} b_j(x_k)^j = 0,$$

where $\mu_k(\lambda) = (-1)^k \rho(x_k) \lambda - f(x_k)$, $k = 1, 2, \ldots, m + n + 2$.

Note that for a fixed value of $\lambda$ (as well as for the alternation points $x_1, \ldots, x_{m+n+2}$) the coefficients $a_i$, $b_j$ of the approximant satisfy the system of linear homogeneous algebraic equations (9). But $\lambda$ must also be determined; this transforms (9) into a nonlinear system of equations which is rather difficult to solve. The case when it is necessary to find the polynomial approximant, i.e., the case $m = 0$, is an exception to what was just noted. In this case the system (9) becomes linear.

The solution of nonlinear system of equations (9) is usually reduced to the iterated solution of systems of linear equations. The following method is comparatively popular (see, for example, [3, 12, 21, 22, 25, 28]) and was used to compile the well-known tables of rational approximants to elementary and special functions [3]. Let $b_0 = 1$ (normalization); then (9) takes the following form

$$\sum_{i=0}^{n} a_i(x_k)^i + \mu_k(\lambda) \sum_{j=1}^{m} b_j(x_k)^j + (-1)^k \rho(x_k) \lambda = f(x_k).$$

Substituting a fixed number $\lambda_0$ for $\lambda$ in the nonlinear terms of system (9'), we get the linear system

$$\sum_{i=0}^{n} a_i(x_k)^i + \mu_k(\lambda_0) \sum_{j=1}^{m} b_j(x_k)^j + (-1)^k \rho(x_k) \lambda = f(x_k).$$

The iteration process is applied to the initial collection of values of the critical points $\{x_k\}$, i.e., of initial approximations to the Chebyshev alternation points, and to the given value $\lambda_0$. First, from (10) one determines the new value of $\lambda$
and substitutes it for $\lambda$ in the nonlinear terms of equation (10); then the system of
equations (10) is solved again, and the next value of $\lambda$ is determined, and so on.
As a result a new value of $\lambda$ and the collection of the coefficients $a_i, b_j$ are defined.
The next step is to determine a new collection of critical points $\{x_k\}$ as extremum
points of the error function for the approximant obtained on the previous step.
Both steps form one cycle of an iteration process. The calculation is finished when
the value of $\lambda$ with precision given in advance coincides in absolute value with the
maximal value of the error function. A complete text of the corresponding Algol
program is given in [22].

Unfortunately the iteration process described above can be nonconvergent even
in the case when the initial approximation differs from the solution of the problem
infinitesimally; see [28]. For some versions of the Remez method it is proved that the
iteration process converges if the initial approximation is sufficiently good, see [20,
23–25, 29, 12]. Nevertheless, in each particular case it is often difficult to indicate
a priori (i.e., before the start of calculations) the initial approximant that ensures
the convergence of the iteration process, and for a given initial approximation it is
difficult to verify whether the conditions which ensure the convergence are satisfied.
One of the methods which is applied in practice is to construct, at first, the best
polynomial approximant of degree $m + n$ (in this case no difficulties arise); next,
using the Chebyshev alternation points of this approximant as the initial collection
of critical points for the iteration process one constructs the best approximant
having the form of a polynomial of degree $m + n − 1$ divided by a linear function.
Finally, in the same manner, the degree of the numerator is successively reduced
and the degree of the denominator is successively raised till an approximant of the
required form (1) is obtained, see [3, 12].

Together with iteration methods for constructing the best rational approximants,
methods of linear and convex programming are used, see [18, 30]. Iteration methods,
as a rule, are more efficient [27], but cannot be generalized directly to the case of
functions of several variables.

§4. THE ROLE OF APPROXIMATE METHODS AND AN
ESTIMATE OF THE QUALITY OF APPROXIMATION

The construction algorithms for the best rational approximants are compara-
tively complicated, so simpler methods that give an approximate solution of the
problem are used on a large scale, see, for example, [1, 5, 7–9, 11–15, 24, 25, 31–
37]. Below we describe methods which are easily implemented, use comparatively
little computation time and yield approximants that are close to best. Such an ap-
proximant can be used as an initial approximant for an iteration algorithm which
gives the exact result. The approximant that is best in the sense of the absolute
error is not necessary best in the sense of the relative error. It is usually important
in practice for both the absolute error and the relative one to be small. So rather
than the best approximants, the approximants constructed by means of an approx-
imate method and having appropriate absolute and relative errors are often more
convenient. Finally, one can also apply methods giving an approximate solution
of the rational approximation problem to those cases when the information about
an approximated function is incomplete (for example, there are known values of
a function only for a finite number of the argument values, or there are known
only the first terms of the function expansion in a series, or the initial information
contains an error, and so on).

The generalized de la Vallée–Poussin theorem (see §2 above) allows to estimate the proximity of an approximate solution of the approximation problem to the best approximant even in the case when this best approximant is unknown.

For example, suppose we want to estimate the proximity of a given approximant of the form (1) to an approximant of the same form with best absolute error to a given function \( f(x) \). Suppose for simplicity that the defect of the best approximant is zero (in practice this condition usually holds). Then, by virtue of Chebyshev’s alternation theorem, in the case when the given approximant \( R(x) \) is sufficiently close to the best one, at the successive points \( x_1 < \cdots < x_{m+n+2} \) belonging to the interval where the argument \( x \) varies the absolute error function \( \Delta(x) \) takes the nonzero values \( \lambda_1, -\lambda_2, \ldots, (-1)^{m+n+1}\lambda_{m+n+2} \) having alternating signs. In this case we shall say that alternation appears. If \( |\lambda_1| = |\lambda_2| = \cdots = |\lambda_{m+n+2}| \), then this alternation is Chebyshev’s. Denote by \( \Delta_{\text{min}} \) the best possible absolute error of approximants of the form (1) to the function \( f(x) \) (the numbers \( m \) and \( n \) are fixed). Suppose \( \lambda = \min\{|\lambda_1|, \ldots, |\lambda_{m+n+2}|\} \). Then, due to the generalized de la Vallée–Poussin theorem, the inequality \( \Delta_{\text{min}} \geq \lambda \) is valid; thus

\[
(11) \quad \Delta \geq \Delta_{\text{min}} \geq \lambda.
\]

It is clear that \( \Delta \) coincides with the greatest (in absolute value) extremum of the function \( f(x) \), and one can take the least (in absolute value) extremum of this function for \( \lambda \) (up to a sign). The quantity

\[
(12) \quad q = \frac{\lambda}{\Delta}
\]

characterizes the proximity of the error of the given approximant to the error of the best approximant. It is clear that \( 0 < q \leq 1 \) and \( q = 1 \) if the given approximant is best. The closer the quantity \( q \) to 1 the higher the approximant quality. From (11) and (12) it follows that

\[
(13) \quad \Delta_{\text{min}} \geq q \cdot \Delta.
\]

Usually, the estimate (13) is rather rough. The appearance of the alternation itself indicates to the closeness of the error of the given approximant to the best one, and the quantity \( \Delta_{\text{min}}/\Delta \) is, in general, much greater than the value of \( q \).

Similarly, the quality of an approximant with respect to the best relative error is evaluated.

If we can calculate the values of the approximated function for all the points of the segment \([A, B]\) (or for a sufficiently “dense” set of such points), and if the coefficients of the rational approximant \( R(x) \) are already known, then it is not hard to determine the points of local extremum of the error function and to calculate the quantities \( \lambda_1, \lambda_2, \ldots, \lambda_{m+n+2} \), and also the quantities \( \lambda \) and \( q \) by means of a special standard subroutine. The same subroutine is also necessary for the construction of the best approximants by means of an iteration method. A good program package for the construction of rational approximants must contain a subroutine of this sort as well as a good subroutine for solving systems of linear algebraic equations and must have, as a component part, routines which implement both the algorithms for approximate solving the approximation problem and the construction algorithms for best approximants.
§5. CHEBYSHEV POLYNOMIALS AND POLYNOMIAL APPROXIMATIONS

Chebyshev polynomials play an important role in approximation theory and in computational practice (see, for example, [12, 13, 17, 18, 24, 33, 38]). We shall consider Chebyshev polynomials of the first kind.

These polynomials were defined by P. L. Chebyshev in the form

\[ T_n(x) = \cos(n \arccos x), \]

where \( n = 0, 1, \ldots \). Assume that \( \varphi = \arccos x \); representing \( \cos n\varphi \) via \( \sin \varphi \) and \( \cos \varphi \), it is not difficult to verify that the right-hand side of formula (14) coincides indeed with a certain polynomial. In particular,

- \( T_0(x) = \cos 0 = 1 \),
- \( T_1(x) = \cos \varphi = \cos(\arccos x) = x \),
- \( T_2(x) = \cos 2\varphi = \cos^2\varphi - \sin^2\varphi = T_1^2(x) - (1 - T_1^2(x)) = 2x^2 - 1 \)

and so on. For an actual computation of \( T_n(x) \) the recurrence relation

\[ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \]

is usually used. Sometimes it is more convenient to consider the polynomials \( \bar{T}_n(x) = 2^{-n+1}T_n(x) \) since the coefficient at \( x^n \) of the polynomial \( \bar{T}_n(x) \) is equal to 1. The polynomials mentioned above satisfy the recurrence relation

\[ \bar{T}_n(x) = x\bar{T}_{n-1}(x) - \frac{1}{4}\bar{T}_{n-2}(x). \]

Consider a particular case of the problem of the best approximation, the approximant to the function \( f(x) = x^n \) on the segment \([-1, 1]\) being looked for in the form of a polynomial \( P(x) \) of degree \( n - 1 \). From the de la Vallée-Poussin theorem it follows that the approximant in question has the form \( P(x) = x^n - \bar{T}_n(x) \). In this case the error function \( \Delta(x) \) coincides with \( \bar{T}_n(x) \) and one can explicitly obtain the Chebyshev alternation points: \( x_k = -\cos \frac{k\pi}{n} \), where \( k = 0, 1, \ldots, n \). Indeed,

\[ \bar{T}_n(x_k) = 2^{-n+1}\cos n\left(\pi - \frac{k\pi}{n}\right) = 2^{-n+1}\cos(n - k)\pi = \frac{(-1)^{n-k}}{2^{n-1}}, \]

i.e., \( \bar{T}_n(x) \) takes its maximum value \( 1/2^{n-1} \) with alternate signs at the points indicated above. This implies an important consequence: the best polynomial approximant of degree \( n - 1 \) to the polynomial \( a_0 + a_1x + \cdots + a_nx^n \) on the segment \([-1, 1]\) has the form \( a_0 + a_1x + \cdots + a_nx^n - a_nT_n(x) \). This result allows to reduce the degree of a polynomial (for example, of some polynomial approximant) with a minimum loss of accuracy. The reduction of the polynomial degree by means of successively applying the method indicated above is called economization. The economization method is due to C. Lanczos, see [38].
The monomials \( x^0, x^1, \ldots, x^m \) can be expressed via the Chebyshev polynomials \( T_0, T_1, \ldots, T_m \). For \( m > 0 \) the following formula is valid:

\[
(17) \quad x_m = 2^{1-m} \sum_{k=0}^{[m/2]} a_k \binom{m}{k} T_{m-2k}(x),
\]

where \( \binom{m}{k} \) are the binomial coefficients, \( [m/2] \) is the integer part of the number \( m/2, \) \( a_k = 1/2 \) for \( k = m/2 \) and \( a_k = 1 \) for \( k \neq m/2 \). The expansion of the polynomial \( T_m \) in powers of \( x \) for \( m > 0 \) is given by the formula

\[
(18) \quad T_m(x) = \frac{1}{2} m \sum_{k=0}^{[m/2]} \frac{(-1)^k (m-k-1)!}{(m-2k)!} (2x)^{m-2k}.
\]

Finally, \( x^0 = T_0 = 1 \). It is clear that the set of polynomials of the form \( \sum_{i=0}^{n} c_i T_i \), where \( c_i \) are numerical coefficients, coincides with the set of all polynomials of degree \( n \).

The economization procedure mentioned above can also be described in the following way. The initial polynomial \( \sum_{i=0}^{n} a_i x^i \) can be represented by means of formula (17) in the form \( \sum_{i=0}^{n} c_i T_i \). The polynomial of degree \( k \) obtained as the result of economization coincides with \( \sum_{i=0}^{k} c_i T_i \). For functions represented in the form of power series \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) it is not difficult to obtain, by means of the economization method, polynomial approximants on the segment \([-1,1]\) close to the best ones. For this purpose it is necessary to replace \( f(x) \) by its truncated Taylor series at the point \( x = 0 \), i.e., by the polynomial \( \sum_{i=0}^{n} a_i x^i \) approximating this function with a high degree of accuracy, and then to obtain, by means of the economization of this polynomial, the polynomial \( \sum_{i=0}^{k} c_i T_i \) of the given degree \( k \).

As \( n \to \infty \), the quantity \( \sum_{i=0}^{k} c_i T_i \) tends to the sum of the first \( k + 1 \) terms of the expansion of \( f(x) \) into Fourier series with respect to Chebyshev polynomials.

Denote by \( L_w^2 \) the Hilbert space of square integrable (with respect to the measure \( w(x) \, dx \)) functions on the segment \([-1,1]\). Suppose \( w(x) = \sqrt{1-x^2} \). It is not hard to verify that the Chebyshev polynomials \( T_n \) form an orthogonal (but not orthonormal) basis in \( L_w^2 \). The expansion

\[
(19) \quad f(x) = \sum_{i=0}^{\infty} c_i T_i
\]

of a function \( f(x) \) into the series in Chebyshev polynomials (the Fourier–Chebyshev series) is easily reduced to the expansion of the function \( f(\cos x) \) into the standard Fourier series in cosines. Among the polynomials of degree \( n \), the polynomial \( P_n(x) = \sum_{i=0}^{n} c_i T_i \) gives the best approximation to the function \( f(x) \) in \( L_w^2 \). The following result shows that this approximant on the segment \([-1,1]\) is close to the best one in the sense of the absolute error.

**Cheney theorem.** Let \( \varphi(x) \) be a function integrable on the segment \([-1,1]\). If for \( i = 0, 1, 2, \ldots, k \) the equality

\[
(20) \quad \int_{-1}^{1} \varphi(x) T_i(x) w(x) \, dx = 0
\]
is valid, then \( \varphi(x) \) either changes its sign in \([-1, 1]\) at least \( k + 1 \) times or vanishes almost everywhere.

**Proof.** Assume that \( \varphi(x) \) has exactly \( m \) sign changes at the points \( x_1, \ldots, x_m \), where \( 0 \leq m \leq k \). Suppose \( P(x) = \prod_{i=1}^m (x - x_i) \); since \( P(x) \) is a polynomial of degree \( m \), we see that it can be represented in the form of a linear combination of the polynomials \( T_0, T_1, \ldots, T_m \). Thus (20) implies \( \int_{-1}^1 \varphi(x) P(x) w(x) \, dx = 0 \). It is clear that the function \( \varphi(x) P(x) \) has no sign changes; thus the equality just obtained means that \( \varphi(x) \) vanishes almost everywhere. The latter proves the theorem.

In a more general case, this result is proved in [39, p.110]. The proof given above allows a generalization to the case of systems of orthogonal polynomials of a sufficiently general form and arbitrary segments of integration (including infinite ones), see [40].

Now let us return to the function (19) and to its approximant \( P_n(x) = \sum_{i=0}^n c_i T_i \). The absolute error function

\[
\Delta(x) = f(x) - P_n(x) = \sum_{i=n+1}^{\infty} c_i T_i
\]

is orthogonal to the polynomials \( T_0, T_1, \ldots, T_n \), i.e.,

\[
\int_{-1}^1 \Delta(x) T_i(x) w(x) \, dx = 0
\]

for all \( i = 0, 1, \ldots, n \). If the function \( f(x) \) is continuous, then the error function \( \Delta(x) \) is also continuous. In this case from the Cheney theorem it follows that either \( \Delta(x) \) is identically zero or has \( n + 1 \) sign changes. This means that alternation is present, i.e., the approximant \( P_n(x) \) is close to the best one, and their proximity can be evaluated by means of relations (11)—(13).

While the truncated Taylor series \( \sum_{i=0}^n \frac{f^{(n)}(0)}{n!} x^n \) gives the best approximant only in a neighborhood of the origin, the truncated Fourier–Chebyshev series \( \sum_{i=0}^n c_i T_i \) for the function \( f(x) \) with the same number of terms gives an approximant which is close to best one on the entire segment \([-1, 1]\).

The change of the variable \( x \rightarrow \frac{1}{2}[(B - A)x + A + B] \) reduces the problem of approximation on an arbitrary finite segment \([A, B]\) to the case of the segment \([-1, 1]\). Further, as a rule, we shall consider the latter case.

**§6. Ill-conditioned problems and rational approximations**

Let \( \{\varphi_0, \varphi_1, \ldots, \varphi_n\} \) and \( \{\psi_0, \psi_1, \ldots, \psi_m\} \) be collections consisting of linearly independent functions of the argument \( x \) belonging some (possibly multidimensional) set \( X \). Consider the problem of constructing an approximant of the form

\[
R(x) = a_0 \varphi_0 + a_1 \varphi_1 + \cdots + a_n \varphi_n / b_0 \psi_0 + b_1 \psi_1 + \cdots + b_m \psi_m
\]

to a given function \( f(x) \) defined on \( X \). If \( X \) coincides with a real line segment \([A, B]\), \( \varphi_k = x^k \) and \( \psi_k = x^k \) for all \( k \), then the expression (21) turns out to be a rational function of the form (1) (see the Introduction). It is clear that expression (21) also
gives a rational function in the case when we take Chebyshev polynomials $T_k$ or, for example, Legendre, Laguerre, Hermite, etc. polynomials as $\varphi_k$ and $\psi_k$.

Fix an abstract construction method for an approximant of the form (21) and consider the problem of computing the coefficients $a_i$, $b_j$. Quite often this problem is ill-conditioned, i.e., small perturbations of the approximated function $f(x)$ or a calculation errors lead to considerable errors in the values of coefficients. For example, the problem of computing coefficients for best rational approximants (including polynomial approximants) for high degrees of the numerator or the denominator is ill-conditioned.

The instability with respect to the calculation error can be related both to the abstract construction method of approximation (i.e., with the formulation of the problem) and to the particular algorithm implementing the method. The fact that the problem of computing coefficients for the best approximant is ill-conditioned is related to the formulation of this problem. This is also valid for other construction methods for rational approximants with a sufficiently large number of coefficients. But an unfortunate choice of the algorithm implementing a certain method can aggravate troubles connected with ill-conditioning.

Several construction methods for approximants of the form (21) are connected with solving systems of linear algebraic equations. This procedure can lead to a large error if the corresponding matrix is ill-conditioned. Consider an arbitrary system of linear algebraic equations

\begin{equation}
Ay = h,
\end{equation}

where $A$ is a given square matrix of order $N$ with components $a_{ij}$ ($i,j = 1, \ldots, N$), $h$ is a given vector column with components $h_i$, and $y$ is an unknown vector column with components $y_i$. Define the vector norm by the equality

\begin{equation}
\|y\| = \sum_{i=1}^{N} |x_i|
\end{equation}

(this norm is more convenient for calculations than $\sqrt{x_1^2 + \cdots + x_N^2}$). Then the matrix norm is determined by the equality

\begin{equation}
\|A\| = \max_{\|y\|=1} \|Ay\| = \max_{1 \leq j \leq N} \sum_{i=1}^{N} |a_{ij}|.
\end{equation}

If a matrix $A$ is nonsingular, then the quantity

\begin{equation}
\text{cond}(A) = \|A\| \cdot \|A^{-1}\|
\end{equation}

is called the condition number of the matrix $A$ (see, for example, [41]). Since $y = A^{-1}h$, we see that the absolute error $\Delta y$ of the vector $y$ is connected with the absolute error of the vector $h$ by the relation $\Delta y = A^{-1} \Delta h$, whence

\begin{equation}
\|\Delta y\| / \|y\| \leq \|A^{-1}\| \cdot \|\Delta h\|
\end{equation}

and
\begin{equation}
\|\Delta y\| / \|h\| \leq \|A^{-1}\| \cdot (\|h\| / \|y\|)(\|\Delta h\| / \|h\|)).
\end{equation}
Taking into account the fact that $\|h\| \leq \|A\| \cdot \|y\|$, we finally obtain

\begin{equation}
\|\Delta y\|/\|y\| \leq \|A\| \cdot \|A^{-1}\| \cdot \|\Delta h\|/\|h\|,
\end{equation}

i.e., the relative error of the solution $y$ is estimated via the relative error of the vector $h$ by means of the condition number. It is clear that (26) can turn into an equality. Thus, if the condition number is of order $10^k$, then, because of round-off errors in $h$, we can lose $k$ decimal digits of $y$.

Similarly, the contribution of the error of the matrix $A$ is evaluated. Finally, the dependence of $\text{cond}(A)$ on the choice of a norm is weak. A method of rapid estimation of the condition number is described in [41, §3.2]. The analysis of the cases when the condition number gives a much too pessimistic error estimate is given in [42].

As an example, we note that the coefficients of the polynomial $P_n(x)$ which give the best approximant to the function $f(x)$ in the metric of the Hilbert space $L^2_w$ (see §5 above) can be determined from the system of equations

\begin{equation}
\int_{-1}^{1} (f(x) - P_n(x))x^k w(x) \, dx = 0,
\end{equation}

where $k = 0, 1, \ldots, n$. With respect to coefficients of the polynomial $P_n(x)$ (in powers of $x$ or in Chebyshev polynomials) these equations are linear and algebraic. But due to the fact that the monomials $x^k$ are “almost linearly dependent”, system (27) is very ill-conditioned. The equivalent system

\begin{equation}(27')\int_{-1}^{1} (f(x) - P_n(x))T_k(x) w(x) \, dx = 0
\end{equation}

is better conditioned, but in this case it is also preferable to use the economization procedure or to determine the coefficients $c_i$ in (19) by formulas

\begin{equation}
c_0 = \frac{1}{\pi} \int_{-1}^{1} f(x) w(x) \, dx,
\end{equation}

\begin{equation}
c_i = \frac{2}{\pi} \int_{-1}^{1} f(x) T_i(x) w(x) \, dx, \quad i \geq 1.
\end{equation}

We recall that here $w(x) = 1/\sqrt{1-x^2}$.

§7. THE EFFECT OF ERROR AUTOCORRECTION

Fix an abstract construction method (problem) for an approximant of the form (21) to the function $f(x)$. Let the coefficients $a_i, b_j$ give an exact or an approximate solution of this problem, and let the $\tilde{a}_i, \tilde{b}_j$ give another approximate solution obtained in the same way. Denote by $\Delta a_1, \Delta b_j$ the absolute errors of the coefficients,
i.e., $\Delta a_i = \hat{a}_i - a_i$, $\Delta b_j = \hat{b}_j - b_j$; these errors arise due to perturbations of the approximated function $f(x)$ or due to calculation errors. Set

$$P(x) = \sum_{i=0}^{n} a_i \varphi_i, \quad Q(x) = \sum_{j=0}^{m} b_j \psi_j,$$

$$\Delta P(x) = \sum_{i=0}^{n} \Delta a_i \varphi_i, \quad \Delta Q(x) = \sum_{j=0}^{m} \Delta b_j \psi_j,$$

$$\tilde{P}(x) = P + \Delta P, \quad \tilde{Q}(x) = Q + \Delta Q.$$

It is easy to verify that the following exact equality is valid:

$$\frac{P + \Delta P}{Q + \Delta Q} - \frac{P}{Q} = \frac{\Delta Q}{Q} \left( \frac{\Delta P}{\Delta Q} - \frac{P}{Q} \right). \tag{29}$$

As mentioned in the Introduction, the fact that the problem of calculating coefficients is ill-conditioned can nevertheless be accompanied by high accuracy of the approximants obtained. This means that the approximants $P/Q$ and $\tilde{P}/\tilde{Q}$ are close to the approximated function and, therefore, are close to each other, although the coefficients of these approximants differ greatly. In this case the relation $\Delta Q/\tilde{Q} = \Delta Q/(Q + \Delta Q)$ of the denominator considerably exceeds in absolute value the left-hand side of equality (29). This is possible only in the case when the difference $\Delta P/\Delta Q - P/Q$ is small, i.e., the function $\Delta P/\Delta Q$ is close to $P/Q$, and, hence, to the approximated function. Thus the function $\Delta P/\Delta Q$ will be called the error approximant. For a special case, this concept was actually introduced in [5]. In the sequel, we shall see that in many cases the error approximant provides indeed a good approximation for the approximated function, and, thus, $P/Q$ and $\tilde{P}/\tilde{Q}$ differ from each other by a product of small quantities in the right-hand side of (29). The thing is that the errors $\Delta a_i$, $\Delta b_j$ are not arbitrary, but are connected by certain relations.

Let an abstract construction method for the approximant of the form (21) be linear in the sense that the coefficients of the approximant can be determined from a homogeneous system of linear algebraic equations. The homogeneity condition is connected with the fact that, when multiplying the numerator and the denominator of fraction (21) by the same nonzero number, the approximant (21) does not change. Denote by $y$ the vector whose components are the coefficients $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$. Assume that the coefficients can be obtained from the homogeneous system of equations

$$H y = 0, \tag{30}$$

where $H$ is a matrix of dimension $(m + n + 2) \times (m + n + 1)$.

The vector $\hat{y}$ is an approximate solution of system (30) if the quantity $\|Hy\|$ is small. If $y$ and $\hat{y}$ are approximate solutions of system (30), then the vector $\Delta y = \hat{y} - y$ is also an approximate solution of this system since $\|H \Delta y\| = \|H \hat{y} - H y\| \leq \|H \hat{y}\| + \|H y\|$. Thus it is natural to assume that the function $\Delta P/\Delta Q$ corresponding to the solution $\Delta y$ is an approximant to $f(x)$. It is clear that the order of the residual of the approximate solution $\Delta y$ of system (30), i.e., of the
quantity \( \| H \Delta y \| \), coincides with the order of the largest of the residuals of the approximate solutions \( y \) and \( \tilde{y} \). For a fixed order of the residual the increase of the error \( \Delta y \) is compensated by the fact that \( \Delta y \) satisfies the system of equations (30) with greater “relative” accuracy, and the latter, generally speaking, leads to the increase of accuracy of the error approximant.

To obtain a certain solution of system (30), one usually adds to this system a normalization condition of the form

\[
(31) \quad \sum_{i=0}^{n} \lambda_i a_i + \sum_{j=0}^{m} \mu_j b_j = 1,
\]

where \( \lambda_i, \mu_j \) are numerical coefficients. As a rule, the equality \( b_0 = 1 \) is taken as the normalization condition (but this is not always successful with respect to minimizing the calculation errors).

Adding equation (31) to system (30), we obtain a nonhomogeneous system of \( m+n+2 \) linear algebraic equations of type (22). If the approximate solutions \( y \) and \( \tilde{y} \) of system (30) satisfy condition (31), then the vector \( \Delta y \) satisfies the condition

\[
(31') \quad \sum_{i=0}^{n} \lambda_i \Delta a_i + \sum_{j=0}^{m} \mu_j \Delta b_j = 0,
\]

It is clear that the above reasoning is not rigorous; for each specific construction method for approximations it is necessary to carry out some additional analysis. More accurate reasoning is given below, in §8, for the classical Padé approximants, and in §14, for the linear and nonlinear Padé–Chebyshev approximants. The presence of the error autocorrection mechanism described above is also verified by a numerical experiment (see below).

The effect of error autocorrection reveals itself for certain nonlinear construction methods for rational approximations as well. One of these methods is considered below, in §12–14 (nonlinear Padé–Chebyshev approximation).

It must be emphasized that (as noted in §3) the coefficients of the best Chebyshev approximant satisfy the system of linear algebraic equations (9) and are computed as approximate solutions of this system on the last step of the iteration process in algorithms of Remez’s type. Thus, the construction methods for the best rational approximants can be regarded as linear. At least for some functions (say, for \( \cos \pi/4x, -1 \leq x \leq 1 \)) the linear and the nonlinear Padé–Chebyshev approximants are very close to the best ones in the sense of the relative and the absolute errors, respectively. The results that arise when applying calculation algorithms for Padé–Chebyshev approximants can be regarded as approximate solutions of system (9) which determines the best approximants. Thus the presence of the effect of error autocorrection for Padé–Chebyshev approximants gives an additional argument in favor of the conjecture that this effect also takes place for the best approximants.

Finally, note that the basic relation (29) becomes meaningless if one seeks an approximant in the form \( a_0 \varphi_0 + a_1 \varphi_1 + \cdots + a_n \varphi_n \), i.e., the denominator in (21) is reduced to 1. However, in this case the effect of error autocorrection (although much weakened) is also possible; this is connected with the fact that the errors \( \Delta a_i \) approximately satisfy certain relations. Such a situation can arise when using the least squares method.
§8. Padé approximations

Let the expansion of a function \( f(x) \) into a power series (the Taylor series at zero) be given, i.e.,

\[
f(x) = \sum_{i=0}^{\infty} c_i x^i.
\]

The classical Padé approximant for \( f(x) \) is a rational function of the form

\[
R(x) = P_n(x)/Q_m(x),
\]

where \( P_n(x) \) and \( Q_m(x) \) are polynomials of degree \( n \) and \( m \), respectively, satisfying the relation

\[
Q_m(x)f(x) - P_n(x) = O(x^{m+n+1}).
\]

Let

\[
P_n(x) = a_0 + a_1 x + \cdots + a_n x^n,
\]
\[
Q_m(x) = b_0 + b_1 x + \cdots + b_m x^m.
\]

If \( b_0 \neq 0 \), then (34) means that

\[
f(x) - P_n(x)/Q_m(x) = O(x^{m+n+1}),
\]

i.e., the first \( m + n + 1 \) terms of the Taylor expansion in powers of \( x \) (to \( x^{m+n} \) inclusive) of \( f(x) \) and \( R(x) \) are the same. The Padé approximation gives the best approximant in a small neighborhood of zero; it is a natural generalization of the expansion of functions into Taylor series and is closely connected with the expansion of functions into continued fractions. Numerous papers are devoted to the Padé approximation; see, for example, [11–16, 5, 6].

One can evaluate the coefficients \( b_j \) in the denominator of fraction (33) by solving the homogeneous system of linear equations

\[
\sum_{j=1}^{m} c_{n+k+j} b_j = -b_0 c_{n+k},
\]

where \( k = 1, \ldots, m \) and \( c_l = 0 \) for \( l < 0 \). One can take any nonzero constant as \( b_0 \).

The coefficients \( a_i \) are given by the formulas

\[
a_i = \sum_{k=0}^{i} b_k c_{i-k} = \sum_{k=0}^{i} b_{i-k} c_k.
\]

The text of the corresponding Fortran program is given in [11].

For large \( m \) the system (36) is ill-conditioned. Moreover, the problem of computation for coefficients of Padé approximants is also ill-conditioned independent of a particular solving algorithm for this problem, see [6, 43, 44]. In Y. L. Luke’s
paper [5] the following reasoning is given. Let $\Delta a_i, \Delta b_j$ be the errors in the coefficients $a_i, b_j$ which arise when numerically solving system (36). We shall ignore the errors of the quantities $c_i$ and $x$ and we shall consider that, according to (37), the errors in the coefficients $a_i$ have the form

$$(37') \quad \Delta a_i = \sum_{k=0}^{i} \Delta b_{i-k} c_k.$$  

From (37') it follows that

$$f \Delta Q - \Delta P = \sum_{j=0}^{m} \Delta b_j x^j \sum_{i=0}^{\infty} c_i x^i - \sum_{i=0}^{n} \Delta a_i x^i$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{\infty} \Delta b_j c_i x^{i+j} - \sum_{i=0}^{n} \sum_{k=0}^{i} \Delta b_{i-k} c_k x^i,$$

the latter, after the change of indices, yields the relation similar to (34):

$$\Delta Q(x) f(x) - \Delta P(x) = O(x^{n+1}).$$

Thus, there are reasons to expect that the error approximant approximates indeed the function $f(x)$ and the effect of error autocorrection takes place. In [5] the corresponding experimental data for the function $e^{-x}$ for $x = 2, m = n = 6, 7, \ldots, 14$, and for $x = 5$ are given and the experiments with the functions $x^{-1} \ln(1+x)$, $(1+x)^{\pm 1/2}$, $xe^x \int_{x}^{\infty} t^{-1} e^{-t} dt$ are briefly described; see also [6].

A natural generalization of the classical Padé approximant is the multipoint Padé approximant (or Padé approximant of the second kind), i.e., a rational function of the form (33) whose values coincide with values of the approximated function $f(x)$ at some points $x_i$ ($i = 1, 2, \ldots, m+n+1$). This definition is extended to the case of multiple points, and for $x_i = 0$ for all $i$ it leads to the classical Padé approximations see [11, 14, 15]. The calculation of coefficients in the multipoint Padé approximant can be reduced to solving a system of linear equations, and there are reasons to suppose that in this case the effect of error autocorrection takes place as well.

§9. LINEAR PADÉ–CHEBYSHEV APPROXIMATIONS AND THE PADE PROGRAM

Consider the approximant of the form (33) to the function $f(x)$ on the segment $[-1, 1]$. The absolute error function of this approximant has the following form:

$$\Delta(x) = \Phi(x)/Q_m(x),$$

where

$$(38) \quad \Phi(x) = f(x) Q_m(x) - P_n(x).$$

The function $R_{m,n}(x) = P_n(x)/Q_m(x)$ is called the linear Padé–Chebyshev approximant to the function $f(x)$ if

$$(39) \quad \int_{-1}^{1} \Phi(x) T_k(x) w(x) \, dx = 0, \quad k = 0, 1, \ldots, m + n,$$
where $T_k(x)$ are the Chebyshev polynomials, $w(x) = 1/\sqrt{1-x^2}$. This concept (in a different form) was introduced in [45] and allows a generalization to the case of other orthogonal polynomials (see [11, 33, 34, 39, 40]). Approximants of this kind always exist [39]. Reasoning in the same way as in §5 and applying Cheney’s theorem, we can find out why the linear Padé–Chebyshev approximants are close to the best ones.

Let $P_n(x)$ and $Q_m(y)$ be represented in the form (35). Then the system of equations (39) is equivalent to the following system of linear algebraic equations with respect to the coefficients $a_i, b_j$:

$$\sum_{j=0}^{m} b_j \int_{-1}^{1} x^j T_k(x) f(x) \frac{dx}{\sqrt{1-x^2}} - \sum_{i=0}^{n} a_i \int_{-1}^{1} x^i T_k(x) \frac{dx}{\sqrt{1-x^2}} = 0.$$  \hspace{1cm} (40)

The homogeneous system (40) can be transformed into a nonhomogeneous one by adding a normalization condition; in particular, any of the following equalities can be taken as this condition:

$$b_0 = 1, \hspace{1cm} (41)$$
$$b_m = 1, \hspace{1cm} (42)$$
$$a_m = 1. \hspace{1cm} (43)$$

In [1, 9] the program PADE (in Fortran, with double precision) which allows to construct rational approximants by solving the system of equations of type (40) is briefly described. The complete text of a certain version of this program and its detailed description can be found in the Collection of algorithms and programs of the Research Computer Center of the Russian Acad. Sci [7]. For even functions the approximant is looked for in the form

$$R(x) = \frac{a_0 + a_1 x^2 + \cdots + a_n (x^2)^n}{b_0 + b_1 x^2 + \cdots + b_m (x^2)^m}, \hspace{1cm} (44)$$

and for odd functions it is looked for in the form

$$R(x) = \frac{x a_0 + a_1 x^2 + \cdots + a_n (x^2)^n}{x b_0 + b_1 x^2 + \cdots + b_m (x^2)^m}, \hspace{1cm} (45)$$

respectively. The program computes the values of coefficients of the approximant, the absolute and the relative errors, and gives the information which allows to estimate the quality of the approximation (see §4 above). In particular, a version of the PADE program is implemented by means of minicomputer of SM–4 class constructs the error curve, determines the presence of alternation, and produces the estimate of the quality of the approximation by means of quantity (12). Using a subroutine, the user introduces the function defined by means of any algorithm on an arbitrary segment $[A, B]$, introduces the boundary points of this segment, the numbers $m$ and $n$, and the number of control parameters. In particular, one can choose the normalization condition of type (41)–(43), look for an approximant in the form (44) or (45) and so on. The change of the variable reduces the approximation
on any segment \([A, B]\) to the approximation on the segment \([-1, 1]\). Therefore, we shall consider the case when \(A = -1, B = 1\) in the sequel unless otherwise stated.

For the calculation of integrals, the Gauss–Hermite–Chebyshev quadrature formula is used:

\[
\int_{-1}^{1} \frac{\varphi(x)}{\sqrt{1 - x^2}} \, dx = \frac{\pi}{s} \sum_{i=1}^{s} \varphi \left( \cos \frac{2i - 1}{2s} \pi \right),
\]

where \(s\) is the number of interpolation points; for polynomials of degree \(2s - 1\) this formula is exact, so that the precision of formula (46) increases rapidly as the parameter \(s\) increases and depends on the quality of the approximation of the function \(\varphi(s)\) by polynomials. To calculate the values of Chebyshev polynomials, recurrence relation (15) is applied.

If the function \(f(x)\) is even and an approximant is looked for the form (44), then system (40) is transformed into the following system of equations:

\[
\sum_{i=0}^{n} a_i \int_{-1}^{1} \frac{x^{2i}T_{2k}(x)}{\sqrt{1 - x^2}} \, dx - \sum_{j=0}^{m} b_j \int_{-1}^{1} \frac{x^{2j}T_{2k}(x)f(x)}{\sqrt{1 - x^2}} \, dx = 0,
\]

where \(k = 0, 1, \ldots, m + n\). If \(f(x)\) is an odd function and an approximant is looked for in the form (45), then, first, by means of the solution of system (47) complemented by one of the normalization conditions, one determines an approximant of the form (44) to the even function \(f(x)/x\), and then the obtained approximant is multiplied by \(x\). This procedure allows to avoid a large relative error for \(x = 0\).

The possibilities of the PADE program are demonstrated in Table 1. This table contains errors of certain approximants obtained by means of this program. For every approximant, the absolute error \(\Delta\), the relative error \(\delta\), and (for comparison) the best possible relative error \(\delta_{\text{min}}\) taken from [3] are indicated. The function \(\sqrt{x}\) is approximated on the segment \([1/2, 1]\) by the expression of the form (1), the function \(\cos \frac{x}{2}x\) is approximated on the segment \([-1, 1]\) by the expression of the form (44), and all the others are approximated on the same segment by the expression of the form (45).

Table 1

| Function              | \([1/2, 1]\) | \([-1, 1]\) |
|-----------------------|--------------|-------------|
| \(\sqrt{x}\)         | \(\Delta\)   | \(\delta\)  |
| \(\cos \frac{x}{2}x\)| \(\Delta_{\text{min}}\) | \(\delta_{\text{min}}\) |
The PADE program is comparatively simple and compact; it includes the standard subroutine DGELG for solving systems of linear algebraic equations (this subroutine is taken from [46]) and a subroutine of numerical integration, and also a number of service, test and auxiliary modules. No additional software is used. The program needs minimum hardware requirements and can be implemented by means of any computer having a Fortran compiler, random-access memory of sufficient volume, a printer or a display.

The version of the PADE program described in [7] is implemented by means of computers of IBM 360/370 class and requires 60 K bytes of main memory; the volume of this program in Fortran (including comments) is 581 lines (cards). The program execution time depends on the type of the computer, on the approximated function, and on the values of control parameters. For example, the CPU time for determining, by means of the PADE program, an approximant of the form (1) to the function $\sqrt{x}$ on the segment $[1/2, x]$ for $m = n = 2$ is 4.4s. In this case the normalization (43) is applied, and the number of checkpoints used while estimating the error is 1200; the compilation time is not taken into account\(^1\).

One of the versions of the program gives the estimate of the quality of the approximant obtained according to formula (12) (see §4 above). For example, for

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Function & $m$ & $n$ & $\Delta$ & $\delta$ & $\delta_{\text{min}}$ \\
\hline
$\sqrt{x}$ & 2 & 2 & $8 \cdot 10^{-6}$ & $1.13 \cdot 10^{-6}$ & $6 \cdot 10^{-6}$ \\
$\sqrt{x}$ & 3 & 3 & $1.9 \cdot 10^{-9}$ & $2.7 \cdot 10^{-9}$ & $1.12 \cdot 10^{-9}$ \\
$\cos \frac{x}{x}$ & 0 & 3 & $0.28 \cdot 10^{-7}$ & $0.39 \cdot 10^{-7}$ & $0.32 \cdot 10^{-7}$ \\
$\cos \frac{x}{x}$ & 1 & 2 & $0.24 \cdot 10^{-7}$ & $0.34 \cdot 10^{-7}$ & $0.29 \cdot 10^{-7}$ \\
$\cos \frac{x}{x}$ & 2 & 2 & $0.69 \cdot 10^{-10}$ & $0.94 \cdot 10^{-10}$ & $0.79 \cdot 10^{-10}$ \\
$\cos \frac{x}{x}$ & 0 & 5 & $0.57 \cdot 10^{-13}$ & $0.79 \cdot 10^{-13}$ & $0.66 \cdot 10^{-13}$ \\
$\cos \frac{x}{x}$ & 2 & 3 & $0.4 \cdot 10^{-13}$ & $0.55 \cdot 10^{-13}$ & $0.46 \cdot 10^{-13}$ \\
$\sin \frac{x}{x}$ & 0 & 4 & $0.34 \cdot 10^{-11}$ & $0.48 \cdot 10^{-11}$ & $0.47 \cdot 10^{-11}$ \\
$\sin \frac{x}{x}$ & 2 & 2 & $0.32 \cdot 10^{-11}$ & $0.45 \cdot 10^{-11}$ & $0.44 \cdot 10^{-11}$ \\
$\sin \frac{x}{x}$ & 0 & 5 & $0.36 \cdot 10^{-14}$ & $0.55 \cdot 10^{-14}$ & $0.45 \cdot 10^{-14}$ \\
$\sin \frac{x}{x}$ & 1 & 1 & $0.14 \cdot 10^{-3}$ & $0.14 \cdot 10^{-3}$ & $0.12 \cdot 10^{-3}$ \\
$\sin \frac{x}{x}$ & 0 & 4 & $0.67 \cdot 10^{-8}$ & $0.67 \cdot 10^{-8}$ & $0.54 \cdot 10^{-8}$ \\
$\sin \frac{x}{x}$ & 2 & 2 & $0.63 \cdot 10^{-8}$ & $0.63 \cdot 10^{-8}$ & $0.53 \cdot 10^{-8}$ \\
$\sin \frac{x}{x}$ & 3 & 3 & $0.63 \cdot 10^{-13}$ & $0.63 \cdot 10^{-13}$ & $0.5 \cdot 10^{-13}$ \\
$\tan \frac{x}{x}$ & 1 & 1 & $0.64 \cdot 10^{-5}$ & $0.64 \cdot 10^{-5}$ & $0.57 \cdot 10^{-5}$ \\
$\tan \frac{x}{x}$ & 2 & 1 & $0.16 \cdot 10^{-7}$ & $0.16 \cdot 10^{-7}$ & $0.14 \cdot 10^{-7}$ \\
$\tan \frac{x}{x}$ & 2 & 2 & $0.25 \cdot 10^{-10}$ & $0.25 \cdot 10^{-10}$ & $0.22 \cdot 10^{-10}$ \\
$\arctg x$ & 0 & 7 & $0.75 \cdot 10^{-7}$ & $10^{-7}$ & $10^{-7}$ \\
$\arctg x$ & 2 & 3 & $0.16 \cdot 10^{-7}$ & $0.51 \cdot 10^{-7}$ & $0.27 \cdot 10^{-7}$ \\
$\arctg x$ & 0 & 9 & $0.15 \cdot 10^{-8}$ & $0.28 \cdot 10^{-8}$ & $0.23 \cdot 10^{-8}$ \\
$\arctg x$ & 3 & 3 & $0.54 \cdot 10^{-9}$ & $1.9 \cdot 10^{-9}$ & $0.87 \cdot 10^{-9}$ \\
$\arctg x$ & 4 & 4 & $0.12 \cdot 10^{-11}$ & $0.48 \cdot 10^{-11}$ & $0.17 \cdot 10^{-11}$ \\
$\arctg x$ & 5 & 4 & $0.75 \cdot 10^{-13}$ & $3.7 \cdot 10^{-13}$ & $0.71 \cdot 10^{-13}$ \\
\hline
\end{tabular}
\end{center}

\(^1\)One can sufficiently decrease the number of checkpoint without considerable loss of accuracy of error estimation (in the present case, for example, to 200 points).
the function \( \sin \frac{\pi}{2}x \) for \( m = n = 2 \), and for the relative error we have \( q = 0.0025 \).

whence it follows that \( \delta_{\text{min}} \geq q\delta \approx 0.4 \cdot 10^{-9} \).

This estimate is rough and in fact, as is shown in Table 1, \( \delta_{\text{min}}/\delta \approx 0.84 \).

For the absolute error the program gives in this case \( q = 0.71 \). The latter indicates to the closeness of this error to the best possible.

The version of the program mentioned above allows to carry out the calculations in interactive mode varying the degrees \( m \) and \( n \), the boundary points of the segment \([A, B]\), the branches of the algorithm, the number of checkpoints when the errors are calculated, the number of interpolation points in the quadrature formula (46), and to estimate rapidly the quality of the approximation according to the error curve.

**Remark.** The program of constructing classical Padé approximants given in [11] is also called PADE, but, of course, here and in [11] different programs are discussed.

§10. **THE PADE PROGRAM. ANALYSIS OF THE ALGORITHM**

The quality of an approximant obtained by means of the PADE program mainly depends on the behavior of the denominator of this approximant and on the calculation errors. The fact that the corresponding systems of algebraic equations are ill-conditioned is the most unpleasant the source of errors of the method under consideration. Seemingly, the methods of this kind are not widely used due to this reason.

The condition numbers of systems of equations that arise while calculating, by means of the PADE program, the approximants considered above are also very large, for example, while calculating the approximant of the form (5) on the segment \([-1, 1]\) to \( \sin \frac{\pi}{2}x \) for \( m = n = 3 \), the corresponding condition number is of order \( 10^{13} \).

As a result, the coefficients of the approximant are determined with a large error.

In particular, a small perturbation of the system of linear equations arising when passing from computer ICL 4–50 to ES–1045 (because of the calculation errors) gives rise to large perturbations in the coefficients of the approximant. Fortunately, the effect of error autocorrection (see §7 above) improves the situation, and the errors of the approximant have no substantial changes under this perturbation. This fact is described in the Introduction, where concrete examples are also given.

Consider some more examples connected with passing from ICL 4–50 to ES–1045. The branch of the algorithm which corresponds to the normalization condition (41) (i.e., to \( b_0 = 1 \)) is considered. For \( \arctg x \) the calculation of an approximant of the form (45) on the segment \([-1, 1]\) for \( m = n = 5 \) by means of ICL–4–50 computer gives an approximation with the absolute error \( \Delta = 0.35 \cdot 10^{-12} \) and the relative error \( \delta = 0.16 \cdot 10^{-11} \).

The corresponding system of linear algebraic equations has the condition number of order \( 10^{30} \! \). Passing to ES–1045 we obtain the following: \( \Delta = 0.5 \cdot 10^{-14} \), \( \delta = 0.16 \cdot 10^{-12} \), the condition number is of order \( 10^{14} \), and the errors \( \Delta a_1 \) and \( \Delta b_1 \) in the coefficients \( a_1 \) and \( b_1 \) in (45) are greater in absolute value than 1! This example shows that the problem of computing condition number of an ill-conditioned system is, in its turn, ill-conditioned. Indeed, the condition number is, roughly speaking, determined by values of coefficients of the inverse matrix (see §6 above, eqs (24) and (25)), every column of the inverse matrix being the solution of the system of equations with the initial matrix of coefficients, i.e., of an ill-conditioned system.

Consider in more detail the effect of error autocorrection for the approximant of the form (44) on the segment \([-1, 1]\) to the function \( \cos \frac{\pi}{4}x \) for \( m = 2, n = 3 \).
Constructing this approximant both on the ICL–4–50 and the ES–1045 computer results in the approximation with the absolute error $\Delta = 0.4 \cdot 10^{-13}$ and the relative error $\delta = 0.55 \cdot 10^{-13}$ which are close to the best possible. In both the cases the condition number is of order $10^9$. The coefficients of the approximants obtained by means of the computers mentioned above and the coefficients of the error approximant (see §7 above) are as follows:

\[ \tilde{a}_0 = 0.999999999999600, \quad a_0 = 0.999999999999610, \quad \Delta a_0 = -10^{-15}, \]
\[ \tilde{a}_1 = -0.2925310453579570, \quad a_1 = -0.2925311264716216, \quad \Delta a_1 = 10^{-7} \cdot 0.811136646, \]
\[ \tilde{a}_2 = 10^{-1} \cdot 0.1105254254716866, \quad a_2 = 10^{-1} \cdot 0.110525658556549, \quad \Delta a_2 = -10^{-7} \cdot 0.2330839683, \]
\[ \tilde{a}_3 = 10^{-3} \cdot 0.1049474500904401, \quad a_3 = 10^{-3} \cdot 0.1049482094850086, \quad \Delta a_3 = 10^{-9} \cdot 0.7593947685, \]
\[ \tilde{b}_0 = 1, \quad b_0 = 1, \quad \Delta b_0 = 0, \]
\[ \tilde{b}_1 = 10^{-1} \cdot 0.1589409217324021, \quad b_1 = 10^{-1} \cdot 0.1589401105960337, \quad \Delta b_1 = 10^{-7} \cdot 0.8111363684, \]
\[ \tilde{b}_2 = 10^{-3} \cdot 0.1003359011092697, \quad b_2 = 10^{-3} \cdot 0.1003341918083529, \quad \Delta b_2 = 10^{-8} \cdot 0.17093009168. \]

Thus, the error approximant has the form

\[ (48) \quad \frac{\Delta P}{\Delta Q} = \frac{\Delta a_0 + \Delta a_1 x^2 + \Delta a_2 x^4 + \Delta a_3 x^6}{\Delta b_1 x^2 + \Delta b_2 x^4}. \]

If the relatively small quantity $\Delta a_0 = -10^{-15}$ in (48) is omitted, then, as testing by means of a computer shows (2000 checkpoints), this expression is an approximant to the function $\cos \frac{\pi}{4} x$ on the segment $\([-1, 1]\]$ with the absolute and the relative errors $\Delta = \delta = 0.2 \cdot 10^{-6}$.

But the polynomial $\Delta Q$ is zero at $x = 0$, and the polynomial $\Delta P$ takes a small, but nonzero value at $x = 0$. Fortunately, equality (29) can be rewritten in the following way:

\[ (49) \quad \frac{\tilde{P}}{\tilde{Q}} - P = \frac{\Delta P}{\Delta Q} \cdot \frac{\Delta Q}{Q} = -P. \]

Thus, as $\Delta Q \to 0$, the effect of error autocorrection arises because the quantity $\Delta P$ is close to zero, and the error of the approximant $P/Q$ is determined by the error of the coefficient $a_0$. The same situation also take place when the polynomial $\Delta Q$ vanishes at an arbitrary point $x_0$ belonging to the segment $[A, B]$ where the function is approximated. It is clear that if one chooses the standard normalization ($b_0 = 1$), then the error approximant has actually two coefficients less than the initial one. Relations (38) and (39) show that in the general case the normalization conditions
$a_n = 1$ or $b_m = 1$ result in the following: the coefficients of the error approximant form an approximate solution of the homogeneous system of linear algebraic equations whose exact solution determines the Padé–Chebyshev approximant having one coefficient less than the initial one. The effect of error autocorrection improves again the accuracy of this error approximant; thus, “the snake bites its own tail”. A situation also arises in the case when the approximant of the form (44) to an even function is constructed by solving the system of equations (47).

Sometimes it is possible to decrease the error of the approximant by means of the fortunate choice of the normalization condition. As an example, consider the approximation of the function $e^x$ on the segment $[-1, 1]$ by rational functions of the form (1) for $m = 15$, $n = 0$. For the traditionally accepted normalization $b_0 = 1$, the PADE program yields an approximant with the absolute error $\Delta = 1.4 \cdot 10^{-14}$ and the relative error $\delta = 0.53 \cdot 10^{-14}$. After passing to the normalization condition $b_{15} = 1$, the errors are reduced nearly one half: $\Delta = 0.73 \cdot 10^{-14}$, $\delta = 0.27 \cdot 10^{-14}$. Note that the condition number increases: in the first case it is $2 \cdot 10^6$, and in the second case it is $0 \cdot 10^{16}$. Thus the error decreases notwithstanding the fact that the system of equations becomes drastically ill-conditioned. This example shows that the increase of accuracy of the error approximant can be accompanied by the increase of the condition number, and, as experiments show, by the increase of errors of the numerator and the denominator of the approximant. The fortunate choice of the normalization condition depends on the particular situation.

A specific situation arises when the degree of the numerator (or of the denominator) of the approximant is equal to zero. In this case the unfortunate choice of the normalization condition results in the following: the error approximant becomes zero or is not well-defined. For $n = 0$ it is expedient to choose condition (42), as it was done in the example given above. For $m = 0$ (the case of the polynomial approximation) it is usually expedient to choose condition (43). Otherwise the situation will be reduced to solving the system of equations (27') in the case described in §6 above.

Since the double precision regime of ES–1045 corresponds to 16 decimal digits of mantissa in the computer representation of numbers, while running computers of this type it makes sense to vary the normalization condition only in case the condition number exceeds $\delta \cdot 10^{16}$, where $\delta$ is the relative error of the obtained approximant. The value of the condition number of the corresponding system of linear algebraic equations is given by the PADE program simultaneously with other computation results.

The theoretical error of the method is determined, to a considerable extent, by the behavior of the approximant’s denominator. It is convenient for the analysis, by dividing the numerator and the denominator of the fraction by $b_0$ to equate $b_0$ to 1. If the coefficients $b_1, b_2, \ldots, b_m$ are small in comparison with $b_0 = 1$, which often happens in computation practice, then the absolute error $\Delta(x)$ and its numerator $\Phi(x) = f(x)Q(x) - P(x)$ are of the same order, so that the minimization of $\Phi(x)$ leads to the minimization of the error $\Delta(x)$, see §9 above. Note that the coefficients of approximant (45) to the function arctg $x$ on the segment $[-1, 1]$ are not small in comparison with $b_0$. For example, for $m = n = 3$ the coefficient $b_1$ is almost one and half times greater than the coefficient $b_0$. Thus, as shown in Table 1, the errors of the approximant to arctg $x$ obtained by means of the PADE program are several times greater than the errors of the best approximants.

Note that sometimes it is possible to improve the denominator of the approxi-
mant or to reduce the condition number of the corresponding system of equations
by extending the segment \([A, B]\) where the function is approximated. Such an ef-
fect is observed, for example, when approximants to some hyperbolic functions are
calculated.

Note that the replacement of the standard subroutine DGELG for solving system
of linear algebraic equations by another subroutine of the same kind (for example,
by the DECOMP program from [41]) does not essentially affect the quality of
approximants obtained by means of the PADE program.

One could seek the numerator and the denominator of the approximant in the
form

\[
P = \sum_{i=0}^{n} a_i T_i, \\
Q = \sum_{j=0}^{m} b_j T_j,
\]

where \(T_i\) are the Chebyshev polynomials. In this case the system of linear equations
determining the coefficients would be better conditioned. But the calculation of
the polynomials of the form (50) by, for example, the Chenshaw method, results in
lengthening the computation time, although it has a favorable effect upon the error
of calculations, see [47, Chapter IV, §9]. The transformation of the polynomials
\(P\) and \(Q\) from the form (50) into the standard form (35) also requires additional
efforts.

In practice it is more convenient to use approximants represented in the form (1),
(44), or (45), and calculate the fraction’s numerator and denominator according the
Horner scheme. In this case the normalization \(a_n = 1\) or \(b_m = 1\) allows to reduce
the number of multiplications. Thus the PADE program gives coefficients of the
approximant in the two forms: with the condition \(b_0 = 1\) and with one of the
conditions \(a_n = 1\) or \(b_m = 1\) no matter which one of the conditions (41)–(43) is
actually used while solving the system of equations of type (39) or (40).

The PADE program (and the corresponding algorithm) can be easily modified,
for example, to take into account the case when some coefficients are fixed before-
hand. One can vary the systems of equations under consideration by changing the
weight \(w(x)\), the interval where the functions are approximated, and the system
of orthogonal polynomials. By a certain increase in complexity of the system of
equations (40) it is possible to minimize the norm of the numerator \(\Phi(x)\) of the
error function \(\Delta(x)\) in the Hilbert space \(L^2_w\) (see §5 above).

The use of the PADE program does not require that the approximated function
be expanded into a series or a continued fraction beforehand. Equations (39) or (40)
and the quadrature formula (46) show that the PADE program uses only the values
of the approximated function \(f(x)\) at the interpolation points of the quadrature
formula (which are zeros of some Chebyshev polynomial).

On the segment \([-1, 1]\] the linear Padé–Chebyshev approximants give a consider-
ably smaller error than the classical Padé approximants. For example, the Padé
approximant of the form (1) to the function \(e^x\) for \(m = n = 2\) has the absolute error
\(\Delta(1) = 4 \cdot 10^{-3}\) at the point \(x = 1\), but the PADE program gives an approximant
of the same form with the absolute error \(\Delta = 1.9 \cdot 10^{-4}\) (on the entire the segment),
i.e., the latter is 20 times smaller than the previous one. The absolute error of the
best approximant is \(0.87 \cdot 10^{-4}\).
§11. THE “CROSS–MULTIPLIED” LINEAR
PÄDE–CHEBYSHEV APPROXIMATION SCHEME

As a rule, linear Padé–Chebyshev approximants are constructed according to
the following scheme [45, 3, 11, 12]. Let the approximated function be decomposed
into the series in Chebyshev polynomials

\[ f(x) = \sum_{i=0}^{\infty} c_i T_i(x) = \frac{1}{2} c_0 + c_1 T_1(x) + c_2 T_2(x) + \ldots, \]

where the notation \( \sum_{i=0}^{m} u_i \) means that the first term \( u_0 \) in the sum is replaced by \( u_0/2 \). The rational approximant is looked for in the form

\[ R(x) = \frac{\sum_{i=0}^{n} a_i T_i(x)}{\sum_{j=0}^{m} b_j T_j(x)}; \]

the coefficients \( b_j \) are determined by means of the system of linear algebraic equations

\[ \sum_{j=0}^{m} b_j (c_{i+j} + c_{i-j}) = 0, \quad i = n + 1, \ldots, n + m, \]

and the coefficients \( a_i \) are determined by the equalities

\[ a_i = \frac{1}{2} \sum_{j=0}^{m} b_j (c_{i+j} + c_{i-j}) = 0, \quad i = 0, 1, \ldots, n. \]

It is not difficult to verify that this algorithm must lead to the same results as the
algorithm described in §9 if the calculation errors are not taken into account.

The coefficients \( c_k \) for \( k = 0, 1, \ldots, n + 2m \), are present in (53) and (54), i.e.,
it is necessary to have the first \( n + 2m + 1 \) terms of series (51). The coefficients \( c_k \) are known, as a rule, only approximately. To determine them one can take the
truncated expansion of \( f(x) \) into the series in powers of \( x \) (the Taylor series) and
by means of the economization procedure transform it into the form

\[ \sum_{i=0}^{n+2m} \tilde{c}_i T_i(x). \]

§12. NONLINEAR PÄDE–CHEBYSHEV APPROXIMATIONS

A rational function \( R(x) \) of the form (1) or (52) is called a nonlinear Padé–
Chebyshev approximant to the function \( f(x) \) on the segment \([-1, 1] \), if

\[ \int_{-1}^{1} (f(x) - R(x)) T_k(x) w(x) \, dx = 0, \quad k = 0, 1, \ldots, m + n, \]
where $T_k(x)$ are the Chebyshev polynomials, $w(x) = 1/\sqrt{1 - x^2}$. Cheney’s theorem (see §5 above) shows that the absolute error function $\Delta(x) = f(x) - R(x)$ has alternation. Thus, there are reasons to assume that the nonlinear Padé–Chebyshev approximants are close to the best ones in the sense of the absolute error.

In the paper [32] the following algorithm of computing the coefficients of the approximant indicated above is given. Let the approximated function $f(x)$ be expanded into series (51) in Chebyshev polynomials. Determine the auxiliary quantities $\gamma_i$ from the system of linear algebraic equations

\begin{equation}
\sum_{j=0}^{m} \gamma_j c_{|k-j|} = 0, \quad k = n + 1, n + 2, \ldots, n + m,
\end{equation}

assuming that $\gamma_0 = 1$. The coefficients of the denominator in expression (52) are determined by the equalities

$$b_j = \mu \sum_{i=0}^{m-j} \gamma_i \gamma_i+j,$$

where $\mu^{-1} = 1/2 \sum_{i=1}^{n} \gamma_i^2$; this implies $b_0 = 2$. Finally, the coefficients of the numerator are determined by formula (54). It is possible to solve system (57) explicitly and to indicate the formulas for computing the quantities $\gamma_i$. One can also estimate explicitly the absolute error of the approximant. This algorithm is described in detail in the book [33]; see also [11].

In contrast to the linear Padé–Chebyshev approximants, the nonlinear approximants of this type do not always exist, but it is possible to indicate explicitly verifiable conditions guaranteeing the existence of such approximants [33]. The nonlinear Padé–Chebyshev approximants (in comparison with the linear ones) have, as a rule, a somewhat smaller absolute errors, but can have larger relative errors. Consider, as an example, the approximant of the form (1) or (52) to the function $e^x$ on the segment $[-1, 1]$ for $m = n = 3$. In this case the absolute error for a nonlinear Padé–Chebyshev approximant is $\Delta = 0.258 \cdot 10^{-6}$, and the relative error, $\delta = 0.252 \cdot 10^{-6}$; for the linear Padé–Chebyshev approximant $\Delta = 0.33 \cdot 10^{-6}$ and $\delta = 0.20 \cdot 10^{-6}$.

§13. APPLICATIONS OF THE COMPUTER ALGEBRA SYSTEM REDUCE TO THE CONSTRUCTION OF RATIONAL APPROXIMANTS

The computer algebra system REDUCE [48, 49] allows to handle formulas at symbolic level and is a convenient tool for the implementation of algorithms of computing rational approximants. The use of this system allows to bypass the procedure of working out the algorithm of computing the approximated function if this function is presented in analytical form or when either the Taylor series coefficients are known or are determined analytically from a differential equation. The round-off errors can be eliminated by using the exact arithmetic of rational numbers represented in the form of ratios of integers.

Within the framework of the REDUCE system, the program package for enhanced precision computations and construction of rational approximants is implemented; see, for example [8]. In particular, the algorithms from §11 and §12 (which are similar to each other in structure) are implemented, the approximated function
being first expanded into the power (Taylor) series, \( f = \sum_{k=0}^{\infty} f^{(k)}(0)x^k/k! \), and then the truncated series

\[
\sum_{k=0}^{N} f^{(k)}(0)\frac{x^k}{k!},
\]

consisting of the first \( N + 1 \) terms of the Taylor series (the value \( N \) is determined by the user) being transformed into a polynomial of the form (55) by means of the economization procedure.

The algorithms implemented by means of the REDUCE system allow to obtain approximants in the form (1) or (52), estimates of the absolute and the relative error, and the error curves. The output includes the Fortran program of computing the corresponding approximant, the constants of rational arithmetic being transformed into the standard floating point form. When computing the values of the obtained approximant, this approximant can be transformed into the form most convenient for the user. For example, one can calculate values of the numerator and the denominator of the fraction of the form (1) according to the Horner scheme, and for the fraction of the form (52), according to Clenshaw scheme, and transform the rational expression into a continued fraction or a Jacobi fraction as well.

The ALGOL-like input language of the REDUCE system and convenient tools for solving problems of linear algebra guarantee simplicity and compactness of the programs. For example, the length of the program for computing linear Padé–Chebyshev approximants is sixty two lines.

§14. THE EFFECT OF ERROR AUTOCORRECTION FOR NONLINEAR PADÉ–CHEBYSHEV APPROXIMATIONS

Relations (56) can be regarded as a system of equations for the coefficients of the approximant. Let the approximants \( R(x) = P(x)/Q(x) \) and \( \tilde{R}(x) = \tilde{P}(x)/\tilde{Q}(x) \), where \( P(x), \tilde{P}(x) \) are polynomials of degree \( n \) and \( Q(x), \tilde{Q}(x) \) are polynomials of degree \( m \), be obtained by approximate solving the indicated system of equations. Consider the error approximant \( \Delta P(x)/\Delta Q(x) \), where \( \Delta P(x) = \tilde{P}(x) - P(x) \), \( \Delta Q(x) = \tilde{Q}(x) - Q(x) \). Substituting \( R(x) \) and \( \tilde{R}(x) \) in (56) and subtracting one of the obtained expressions from the other, we see that the following approximate equality holds:

\[
\int_{-1}^{1} \left( \frac{\tilde{P}(x)}{\tilde{Q}(x)} - \frac{P(x)}{Q(x)} \right) T_k(x)w(x) \, dx \approx 0, \quad k = 0, 1, \ldots, m + n.
\]

This and equality (29) imply the approximate equality

\[
\int_{-1}^{1} \left( \frac{\Delta P(x)}{\Delta Q(x)} - \frac{P(x)}{Q(x)} \right) \frac{\Delta Q}{Q} T_k(x)w(x) \, dx \approx 1,
\]

where \( k = 0, 1, \ldots, m + n, w(x) = 1/\sqrt{1-x^2} \). If the quantity \( \Delta Q \) is relatively not small (this is connected with the fact that the system of equations (57) is ill-conditioned), then, as follows from equality (59), we can naturally expect that
the error approximant is close to $P/Q$ and, consequently, to the approximated function $f(x)$. Due to the fact that the arithmetic system of rational numbers is used, the software described in §13 allows to eliminate the round-off errors and to estimate the “pure” influence of errors in the approximated function on the coefficients of the nonlinear Padé–Chebyshev approximant. In this case the effect of error autocorrection can be substantiated by a more accurate reasoning which is valid both for nonlinear Padé–Chebyshev approximants and for linear ones, and even for the linear generalized Padé approximants connected with different systems of orthogonal polynomials. This reasoning is analogous to Y. L. Luke’s considerations [5] given in §8 above.

Assume that the function $f(x)$ is expanded into series (51) and that the rational approximant $R(x) = P(x)/Q(x)$ is looked for in the form (52).

Let $\Delta b_j$ be the errors in coefficients of the approximant’s denominator $Q$. In the linear case these errors arise when solving the system of equations (53), and in the nonlinear case, when solving the system of equations (54). In both the cases the coefficients in the approximant’s numerator are determined by equations (54), whence we have

$$\Delta a_i = \frac{1}{2} \sum_{j=0}^{m'} \Delta b_j (c_{i+j} + c_{|i-j|}), \quad i = 0, 1, \ldots, n.$$  

This implies the following fact: the error approximant $\Delta P/\Delta Q$ satisfies the relations

$$\int_{-1}^{1} \left( f(x) \Delta Q(x) - \Delta P(x) \right) T_i(x) w(x) dx = 0, \quad i = 0, 1, \ldots, n,$$

which are analogous to relations (39) defining the linear Padé–Chebyshev approximants. Indeed, let us use the well-known multiplication formula for Chebyshev polynomials:

$$T_i(x)T_j(x) = \frac{1}{2} \left[ T_{i+j}(x) + T_{|i-j|}(x) \right],$$

where $i, j$ are arbitrary indices; see, for example [11–13, 33]. Taking (62) into account, the quantity $f\Delta Q - \Delta P$ can be rewritten in the following way:

$$f\Delta Q - \Delta P = \left( \sum_{j=0}^{m'} \Delta b_j T_j \right) \left( \sum_{i=0}^{\infty} c_i T_i \right) - \sum_{i=0}^{n'} \Delta a_i T_i$$

$$= \frac{1}{2} \sum_{i=0}^{\infty} \left[ \sum_{j=0}^{m'} \Delta b_j (c_{i+j} + c_{|i-j|}) \right] T_i - \sum_{i=0}^{n'} \Delta a_i T_i.$$  

This formula and (60) imply that

$$f\Delta Q - \Delta P = O(T_{n+1}),$$
i.e., in the expansion of the function \( f \Delta Q - \Delta P \) into the series in Chebyshev polynomials, the first \( n + 1 \) terms are absent, and the latter is equivalent to relations (61) by virtue of the fact that the Chebyshev polynomials form an orthogonal system. When carrying out actual computations, the coefficients \( c_i \) are known only approximately, and thus the equalities (60), (61) are also satisfied approximately.

Consider the results of computer experiments\(^2\) that were performed by means of the software implemented within the framework of the REDUCE system and briefly described in §13 above. We begin with the example considered in §10 above, where the linear Padé–Chebyshev approximant of the form (44) to the function \( \cos \frac{\pi}{4} x \) was constructed on the segment \([-1, 1]\) for \( m = 2, n = 3 \). To construct the corresponding nonlinear Padé–Chebyshev approximant, it is necessary to specify the value of the parameter \( N \) determining the number of terms in the truncated Taylor series (58) of the approximated function. In this case the calculation error is determined, in fact, by the parameter \( N \).

The coefficients in approximants of the form (44) which are obtained for \( N = 15 \) and \( N = 20 \) (the nonlinear case) and the coefficients in the error approximant are as follows\(^3\):

\[
\begin{align*}
\tilde{a}_0 &= 0.4960471034987563, & \Delta a_0 &= 10^{-8} \cdot 0.07230059, \\
{a}_0 &= 0.4960471027757504, & \\
\tilde{a}_1 &= -0.1451091945278387, & \Delta a_1 &= 10^{-8} \cdot 0.16522651, \\
{a}_1 &= -0.1451091928755736, & \\
\tilde{a}_2 &= 10^{-2} \cdot 0.548258654334515, & \Delta a_2 &= 10^{-9} \cdot 0.42224856, \\
{a}_2 &= 10^{-2} \cdot 0.548258121085953, & \\
\tilde{a}_3 &= 10^{-4} \cdot 0.5205903601778259, & \Delta a_3 &= 10^{-10} \cdot 0.13635919, \\
{a}_3 &= 10^{-4} \cdot 0.5205902238186334, & \\
\tilde{b}_0 &= 0.4960471034987759, & \Delta b_0 &= 10^{-8} \cdot 0.07230061, \\
{b}_0 &= 0.4960471027757698, & \\
\tilde{b}_1 &= 10^{-2} \cdot 0.7884201590727615, & \Delta b_1 &= 10^{-10} \cdot 0.1429272, \\
{b}_1 &= 10^{-2} \cdot 0.788420301999351, & \\
\tilde{b}_2 &= 10^{-4} \cdot 0.4977097973870693, & \Delta b_2 &= 10^{-10} \cdot 0.300388, \\
{b}_2 &= 10^{-4} \cdot 0.4977100977750249, & 
\end{align*}
\]

Both the approximants have absolute errors \( \Delta \) equal to \( 0.4 \cdot 10^{-13} \) and the relative errors \( \delta \) equal to \( 0.6 \cdot 10^{-13} \), these values being close to the best possible. The condition number of the system of equations (57) in both the cases is \( 0.4 \cdot 10^8 \). The denominator \( \Delta Q \) of the error approximant is zero for \( x = x_0 \approx 0.70752 \ldots \); the point \( x_0 \) is also close to the root of the numerator \( \Delta P \) which for \( x = x_0 \) is of order \( 10^{-8} \). Such a situation was considered in §10 above. Outside a small neighborhood of the point \( x_0 \) the absolute and the relative errors have the same order as in the “linear case” considered in §10.

\(^2\)At the author’s request, computer calculations were carried out by A. Ya. Rodionov.

\(^3\)Here we have in mind the coefficients of the expansions of the approximant’s numerator and denominator in powers of \( x \).
Now consider the nonlinear Padé–Chebyshev approximant of the form (44) on the segment $[-1, 1]$ to the function $\tan \frac{\pi}{4}x$ for $m = n = 3$. In this case the Taylor series converges very slowly, and, as the parameter $N$ increases, the values of coefficients of the rational approximant undergo substantial (even in the first decimal digits) and intricate changes. The situation is illustrated in Table 2, where the following values are given: the absolute errors $\Delta$, the absolute errors $\Delta_0$ of error approximants$^4$ (there the approximants are compared for $N = 15$ and $N = 20$, for $N = 25$ and $N = 35$, for $N = 40$ and $N = 50$), and also the values of the condition number $\text{cond}$ of the system of linear algebraic equations (57). In this case the relative errors coincide with the absolute ones. The best possible error is $\Delta_{\text{min}} = 0.83 \cdot 10^{-17}$.

Table 2

| $N$ | $15$ | $20$ | $25$ | $35$ | $40$ | $50$ |
|-----|------|------|------|------|------|------|
| $\text{cond}$ | $0.76 \cdot 10^7$ | $0.95 \cdot 10^8$ | $0.36 \cdot 10^{10}$ | $0.12 \cdot 10^{12}$ | $0.11 \cdot 10^{12}$ | $0.11 \cdot 10^{12}$ |
| $\Delta$ | $0.13 \cdot 10^{-4}$ | $0.81 \cdot 10^{-6}$ | $0.13 \cdot 10^{-7}$ | $0.12 \cdot 10^{-10}$ | $0.75 \cdot 10^{-12}$ | $0.73 \cdot 10^{-15}$ |
| $\Delta_0$ | $0.7 \cdot 10^{-4}$ | $0.7 \cdot 10^{-8}$ | $0.2 \cdot 10^{-9}$ | $0.2 \cdot 10^{-9}$ | $0.2 \cdot 10^{-9}$ | $0.2 \cdot 10^{-9}$ |

$^4$A small neighborhood of the root of the polynomial $\Delta Q$ is eliminated as before.
§15. Small deformations of approximated functions
and acceleration of convergence of series

Let a function \( f(x) \) be expanded into the series in Chebyshev polynomials,
\[
f(x) = \sum_{i=0}^{\infty} c_i T_i
\]
consider a partial sum
\[
\hat{f}_N(x) = \sum_{i=0}^{N} c_i T_i
\]
of this series. Using formula (62), it is easy to verify that the linear Padé-Chebyshev
approximant of the form (1) or (52) to the function \( f(x) \) coincides with the linear
Padé-Chebyshev approximant to polynomial (63) for \( N = n + 2m \), i.e., it depends
only on the first \( n + 2m + 1 \) terms of the Fourier-Chebyshev series of the function
\( f(x) \): a similar result is valid for the approximant of the form (44) or (45) to even
or odd functions, respectively. Note that for \( N = n + 2m \) the polynomial \( \hat{f}_N \)
is the result of application of the algorithm of linear (or nonlinear) Padé-Chebyshev
approximation to \( f(x) \), the exponents \( m \) and \( n \) being replaced by 0 and \( 2m + n \).

The interesting effect mentioned in [9] consists in the fact that the error of the
polynomial approximant \( \hat{f}_{n+2m} \) depending on \( n + 2m + 1 \) parameters can exceed
the error of the corresponding Padé-Chebyshev approximant of the form (1) which
depends only on \( n + m + 1 \) parameters. For example, consider an approximant
of the form (45) to the function \( \tan \frac{\pi}{4} x \) on the segment \([-1, 1]\]. For \( m = n = 3 \)
the linear Padé-Chebyshev approximant to \( \tan \frac{\pi}{4} x \) has the error of order \( 10^{-17} \), and
the corresponding polynomial approximant of the form (63) has the error of order
\( 10^{-11} \). This polynomial of degree \( 15 \) can be regarded as a result of deformation
of the approximated function \( \tan \frac{\pi}{4} x \). This deformation does not affect the first
twenty terms in the expansion of this function in Chebyshev polynomials and,
consequently, does not affect the coefficients in the corresponding rational Padé-
Chebyshev approximant, but leads to a several orders increase of its error. Thus,
a small deformation of the approximated function can result in a sharp change in
the order of error of a rational approximant.

Moreover the effect just mentioned means that the algorithm extracts from
polynomial (63) additional information concerning the next components of the
Fourier-Chebyshev series. In other words, in this case the transition from Fourier-
Chebyshev series to Padé-Chebyshev approximant accelerates convergence of se-
ries. A similar effect of acceleration of convergence of power series by passing to
the classical Padé approximant is known (see [11, 14, 15]).

It is easy to see that the nonlinear Padé-Chebyshev approximant of the form (1)
to the function \( f(x) \) depends only on the first \( m + n + 1 \) terms of the Fourier-
Chebyshev series for \( f(x) \), so that for such approximants a more pronounced effect
of the type indicated above takes place.

Since one can change the “tail” of the Fourier-Chebyshev series in a quite arbitrary
way without affecting the rational Padé-Chebyshev approximant, the effect
of acceleration of convergence can take place only for the series with an especially
regular behavior (and for the corresponding “nice” functions).

Note that the effect of error autocorrection indicates to the fact that the variation
of an approximated function under deformations of a more general type may have

\footnote{Odd functions are in question, and hence \( m = n = 3 \) in (45) corresponds to \( m = 6, n = 7 \)
in (1).}
little effect on the rational approximant considered as a function (whereas the coefficients of the approximant can have substantial changes). Accordingly, while deforming the functions for which good rational approximation is possible, the approximant’s error can rapidly increase.

There are interesting results distinguishing the classes of functions for which an efficient rational approximation is possible, for example, the classes of functions which are approximated by rational fractions considerably better (with a higher rate of convergence), then by polynomials; see, in particular, [10, 50–52]. The reasoning given above indicate that of a special interest are “individual” properties of functions which guarantee their effective rational approximation. There are reasons to suppose that solutions of certain functional and differential equations possess properties of this kind. Note that in papers [16, 37], starting from the fact that elementary functions satisfy simple differential equations, it is shown that these functions are better approximated by rational fractions than by polynomial ones (we have in mind the best approximation); because of complicated calculations only the following cases were considered: the denominator of a rational approximant is a linear function or (for even and odd functions) is a polynomial of degree 2.

§16. APPLICATIONS TO COMPUTER CALCULATION

Ti computer calculation of function values is reduced in fact to carrying out a finite set of arithmetic operations with the argument and constants, i.e. to computing the value of a certain rational function. Now we list some typical applications of methods for constructing rational approximants. Often it happens that a function \( f(x) \) is to be computed many times (for example, when solving numerically a differential equation) and with a given accuracy. In this case the construction of a rational approximant to this function (with a given accuracy) often produces the most economic algorithm for computation of values of \( f(x) \). For example, if \( f(x) \) is a complicated aggregate of elementary and special functions every one which can be calculated using the corresponding standard programs, then values of the function \( f(x) \) can, of course, be computed by means of these programs. But such an algorithm is often too slow and produces an unnecessary extra precision.

Standard computer programs for elementary and special functions, in their turn, are based, as a rule, on rational approximants. Note that although the accuracy of rational and polynomial approximants to a given function is the same, the computation of the rational approximant usually requires a lesser number of operations, i.e., it is more speedy; see, for example [1, 3, 12, 13, 24, 25, 31].

The coefficients of rational approximants to basic elementary and special functions can be found in reference handbooks; we note especially the fundamental book [3], see, also, for example, [12, 13]. But a computer can have certain specific properties requiring algorithms and approximants (for effective standard programs of computing functions) which are absent in reference handbooks. In that case the construction programs for rational approximants, including the PADE program described in §9 above (see also [1, 7, 9]), can be useful.

For example, decimal computers (including calculators) are widely used at present. The reason is that the use of decimal arithmetic system (instead of the standard binary one) enables the user to avoid a considerable loss of computing time needed for the transformation of numbers from the decimal representation to the binary one and vice versa. This is especially important if the amount of
the input/output operations is relatively large; the latter situation is characteristic for calculations in the interactive mode. A method of computing elementary functions on decimal computers which uses the technique of rational approximants is described in the Appendix below. The main idea of this method consists in the fact that the computation of values of various elementary functions, by means of simple algorithms, is reduced to the computation of a rational function of a fixed form. Roughly speaking, all basic elementary functions are calculated according to the same formula. Only the coefficients of the rational expression depend on a calculated function.

§17. **Nonlinear models and rational approximants**

One of the main problems of mathematical modeling is to construct analytic formulas (models) that approximately describe the functional dependence between different quantities according to given “experimental” data concerning the values of these quantities. In particular, let the set of real numbers \( x_1, \ldots, x_\nu \) which are values of the “independent” variable \( x \) be given, and for every value \( x_i \) of this variable the value \( y_i \) of the “dependent” variable \( y \) be given. The problem is to construct a function \( y = F(x) \) such that the functional dependence can be represented by an analytic formula of a certain form, and the approximate equality \( y_i \approx F(x_i) \) be valid for all \( i = 1, 2, \ldots, \nu \), where the function \( F(x) \) should take “reasonable values” at points \( x \) lying between the given points \( x_i \). In practice the values \( y_i \) are usually given with errors.

As it was noted above, computer calculation of functions is finally reduced to computation of some rational functions. Thus in many cases it is natural to construct an analytic model in the form of the rational function (1), where the degrees of the numerator and the denominator and also the values of the coefficients are determined in the process of modeling, see [14]. Of course, in this case we have in mind only the one-factor models. One can construct multi-factor models by using rational functions of several variables.

If we have a simple program of constructing rational approximants to continuous functions defined on finite segments of the real line, then we can reduce the construction of a model to constructing rational approximants to a continuous function (although in numerical analysis, as a rule, the goal is to reduce continuous problems to discrete ones). The construction of a model is carried out in two steps. On the first step a continuous function \( f(x) \) such that \( f(x_i) = y_i \) is constructed. A linear or a cubic spline (depending on the user’s choice) is used as \( f(x) \). The function whose graph coincides with the polygonal line consisting of segments of straight lines that connect the points \( (x_i, y_i) \) with the coordinates \( x_i, y_i \) is the linear spline; the cubic spline is described, for example, in [41]. On the second step the model is constructed by means of the PADE program. This approach guarantees the regular behavior of the model on the entire range of the argument.

If there are reasons to assume that the initial data lie on a sufficiently smooth and regular curve, then it is expedient to use a cubic spline. And if there are reasons to assume that the initial data contain considerable errors or deviations from theoretically admissible data, then it is expedient to use a linear spline: the behavior of a cubic spline at intermediate points in this case will be irregular.

The method for constructing models described above was implemented (together with I. A. Andreeva) as the SPLINE–PADE program. This program prints out the
graphs of splines and rational approximants (together with the initial data), and this facilitates the analysis of models. Of course, while choosing and analyzing models, it is necessary to take into account the theoretical requirements on the model which are connected with specific features of a particular problem.

Example. Let the points \( x_1, \ldots, x_\nu \) be uniformly distributed on the segment \([-\frac{\pi}{4}, \frac{\pi}{4}]\), \( x_1 = -\frac{\pi}{4}, \nu = 32, x_\nu = \frac{\pi}{4}, y_i = \cos x_i \). The rational approximant of the form (45) to the linear spline for \( m = n = 2 \) gives an approximant to \( \cos x \) on \([-\frac{\pi}{4}, \frac{\pi}{4}]\) with the absolute error \( \Delta = 10^{-3} \). If a cubic spline is applied, then the absolute error \( \Delta \) is \( 0.35 \cdot 10^{-6} \) in this case.

Other approaches to the construction of models in the form of rational functions can be found, for example, in [14].

The above results connected with the effect of error autocorrection show that similar models can have quite different coefficients. Thus the coefficients of models of this kind are, generally speaking, unstable; and one should be very careful when trying to give a substantial interpretation for these coefficients.

APPENDIX

A METHOD OF IMPLEMENTATION OF BASIC CALCULATIONS ON DECIMAL COMPUTERS

1. Introduction. A large relative amount of input/output operations is a characteristic feature of modern interactive computer systems. This results in a waste of computing time of systems with binary number representation: numbers are transformed from the decimal representation to the binary one and vice versa. Therefore, certain computers use decimal arithmetic system. As a rule, the use of decimal arithmetic system leads to a decrease in the rate of calculations and to additional memory requirements connected with specific coding of decimal numbers. The decrease in the rate of calculations is due to the fact that the implementation of decimal operations, as compared to that of binary ones, is more complicated; moreover, the binary representation is more convenient for implementing algorithms for calculating certain functions than the decimal one. Since the performance rate of floating point arithmetic operations and the rate of calculating elementary functions determine, to a considerable extent, the rate of mathematical data performing, the quality of the corresponding algorithms is, especially for cheap personal systems, of great importance.

Here we consider methods of implementation of the floating point arithmetic system and of organizing computations for elementary functions. These methods are convenient to use on decimal computers (this pertains both to the software and hardware implementation). They guarantee a sufficient economy of memory simultaneously with a relatively high performance rate of calculations. Examples of effective software implementation of these methods are given in [1, 53]. The hardware implementation is described in the patent [55]. The methods under consideration are also of interest for octal and hexadecimal computers.

2. Floating point arithmetic system. When carrying out arithmetic operations with floating point numbers, the exponents of these numbers undergo only the operations of addition, subtraction, and comparison. Almost all computers have means for these operations since they are necessary for the command and the address codes operations. This fact provides an opportunity to use the binary
representation for the exponents when implementing the floating point arithmetic system. Since exponents are integers lying in certain bounds, the transformation of exponents from binary to decimal representations does not encounter serious obstacles. The choice of an appropriate algorithm depends on the structure of a computer and the method of coding of decimal numbers. For the standard coding 8421, when each decimal digit corresponds to a binary tetrad, it is possible to use the fact that in this case the numbers from 0 to 9 have the same coding in the binary and the decimal representations. Therefore the binary representation \( x_2 \) of a number \( x \) can be converted into the decimal representation \( x_{10} \) by successively subtracting (in the binary arithmetic system) the numbers from 0 to 9 from \( x_2 \) and forming the number \( x_{10} \) from the sums of these numbers (in the decimal arithmetic system). Similarly, a decimal integer can be converted into a binary one.

Binary representation of exponents enables one to save memory, and the combination of decimal operations with more rapid binary operations of addition type enhances the performance rate. As a rule, the software implementation of the floating point arithmetic system leads to the fact that floating point operations take two orders as much time when compared with fixed point operations. The implementation described in [1] is much more efficient: for seven decimal digit numbers, the transition from the fixed point to the floating point regimes results in double computing time for multiplication and division, and to reduction of the rate of addition and subtraction by one decimal order.

3. The design of computation for elementary functions. The calculation of values of each of the basic elementary functions (at the reduction stage) is reduced to calculation of values of an odd function on a symmetric (with respect to the origin) interval. This odd function is approximated by a rational fraction of the form

\[
R(y) = y \frac{a + by^2 + cy^4}{\alpha + \beta y^2 + y^4},
\]

where \( y \) is the reduced argument, and the coefficients \( a, b, c, \alpha, \beta \) depend on the approximated function. Thus all algorithms of computation for basic elementary functions have the common block (1), and this fact guarantees an economy of memory. This block can be implemented both as a carefully devised part of software or as a part of hardware; this can enhance the performance rate. For the reduction algorithms described below, the approximant of the form (1) can guarantee 8–9 accurate decimal digits. Because of specific features of a particular computer and the way the common block is implemented, it can be required that expression (1) be transformed into a certain form, for example, into the form

\[
R(y) = y \frac{a + y^2(b + cy^2)}{\alpha + y^2(\beta + y^2)},
\]

or into a Jacobi fraction of the form

\[
R(y) = y \left( c + \frac{\mu}{y^2 + \nu + \frac{\mu}{y^2 + \lambda}} \right).
\]

The calculation of elementary functions with enhanced precision is organized according to a similar scheme. The approximant of the form (1) is replaced by the
expression

\[ R(y) = \frac{a + by^2 + cy^4 + dy^6}{\alpha + \beta y^2 + \gamma y^4 + y^6} \]

which can be transformed into the form similar to \((1')\) or \((1'')\), i.e.,

\[ R(y) = \frac{a + y^2(b + y^2(c + dy^2))}{\alpha + y^2(\beta + y^2(\gamma + y^2))} \]

\[ R(y) = y\left(d + \frac{\xi}{y^2 + \eta + \frac{\mu}{y^4 + \lambda}}\right). \]

The coefficients \(a, b, c, d, \alpha, \beta, \gamma, \xi, \eta, \mu, \nu, \lambda\) in formulas \((1'), (1''), (2), (2')\), \((2'')\) are constants that depend on the approximated function. The approximants of the form \((2), (2')\) or \((2'')\) guarantee 12–13 accurate decimal digits\(^6\).

The reduction algorithms are uniform; in particular, for calculations with ordinary and enhanced precision the same reduction algorithms are used. These algorithms are described in section 4 below. The errors of approximants and values of the coefficients in expressions \((1), (2)\) and in their modifications are given below. These coefficients are either taken from \([3]\), or calculated by means of the PADE program described in §9 above.

4. Algorithms. The relative, mean relative, absolute, and mean absolute errors are denoted by \(\delta, \bar{\delta}, \Delta, \bar{\Delta}\), respectively.

4.1. Calculation of logarithms. Let the argument \(x > 0\) have the form \(x = x_0 \cdot 10^p\), where \(0.1 \leq x < 1\), \(p\) is an integer. Suppose

\[ y = \frac{x_0 - \sqrt{10}}{x_0 + \sqrt{10}} \]

then we have

\[ x_0 = 10^{-\frac{1}{2}} \cdot \frac{1 + y}{1 - y} \]

whence

\[ \lg x = p - \frac{1}{2} + \lg \frac{1 + y}{1 - y}. \]

Substituting the approximant of the form \(R(y)\) with the best possible absolute error for the odd function

\[ \log \frac{1 + y}{1 - y} = \frac{2 \text{Arcth} y}{\ln 10}, \]

we finally obtain \(\lg(x) \approx p - 1/2 + R(y)\) for

\[ \frac{1 - \sqrt{10}}{1 + \sqrt{10}} \leq y < \frac{\sqrt{10} - 1}{\sqrt{10} + 1}. \]

\(^6\)Of course, the values of the argument for which the loss of precision is inevitable are an exception. For example, if \(x = 1 + \Delta x\), then \(\ln x \approx \Delta x\), and the number of significant digits of \(\ln x\) is smaller than the number of significant digits of the argument \(x\) by the number of zeros after the decimal point in the number \(\Delta x\).
For $0.1 \leq x \leq 1$ and ordinary precision, $\Delta = 0.23 \cdot 10^{-8}$, $\bar{\Delta} = 0.14 \cdot 10^{-8}$. For enhanced precision, $\Delta = 0.85 \cdot 10^{-12}$, $\bar{\Delta} = 0.53 \cdot 10^{-12}$. It is impossible to minimize the relative error on the given interval since this error is inevitable in a neighborhood of the point $x = 1$.

The calculation of the natural logarithm is reduced to the case of the decimal logarithm by means of the relation $\ln x = (\ln 10)(\lg x)$.

4.2. Calculation of exponentials. Consider a nonstandard (at the first sight) algorithm of reduction of the function $10^x$, which, nevertheless, is dual to the algorithm of reduction of $\lg x$ described above. Represent the argument $x$ in the form $x = y + p$, where $-1 < y < 1$, $p$ is an integer (for example, $-12.85 = -12 + (-0.85)$). Then

$$10^x \approx 10^{y} rac{1 + R(y)}{1 - R(y)},$$

where $R(y)$ is the approximant of the form (1) or (2) on the interval $[-1, 1]$ to the function $\ln(10 \cdot \lg x)$ for $-1 \leq x \leq 1$ and ordinary precision, $\delta = 0.23 \cdot 10^{-8}$, $\bar{\delta} = 0.6 \cdot 10^{-9}$, $\Delta = 0.23 \cdot 10^{-7}$, $\bar{\Delta} = 0.17 \cdot 10^{-8}$. For enhanced precision, $\delta = 0.17 \cdot 10^{-13}$, $\bar{\delta} = 0.3 \cdot 10^{-14}$, $\Delta = 0.16 \cdot 10^{-12}$, $\bar{\Delta} = 0.85 \cdot 10^{-14}$.

For calculation of the functions $e^x$ and $x^y$, the relations $e^x = 10^{\lg e}$ and $x^y = 10^{\lg x}$ are used.

4.3. Calculation of $\sin x$ and $\cos x$. Denote by $R(x)$ the approximant of the form (1) or (2) with the best possible relative error to the function $\sin x$ on the segment $[-\pi/2, \pi/2]$. Since the function $\sin x$ is odd, it is sufficient to consider the case $x > 0$. Denote by $\{a\}$ the fractional part of a positive number $a$; for example, $\{12.08\} = 0.08$. Set $y = \{x/2\} \cdot 2\pi$, then $0 \leq y < 2\pi$. If $y \leq \pi/2$, then we set $\sin x \approx R(y)$; if $\pi/2 < y \leq 3\pi/2$, then $\sin x \approx R(z)$, where $z = \pi - y$, and if $3\pi/2 < y < 2\pi$, then $\sin x \approx R(z)$, where $z = y - 2\pi$. For $-\pi/2 \leq x \leq \pi/2$ and ordinary precision, $\delta = \Delta = 0.53 \cdot 10^{-8}$, $\bar{\delta} = 0.34 \cdot 10^{-8}$, $\bar{\Delta} = 0.21 \cdot 10^{-8}$. For enhanced precision, $\delta = \Delta = 0.56 \cdot 10^{-13}$, $\bar{\delta} = 0.32 \cdot 10^{-13}$, $\bar{\Delta} = 0.2 \cdot 10^{-13}$. The calculation of $\cos x$ is reduced to the calculation of $\sin x$ by means of the relation $\cos x = \sin(\pi/2 - x)$.

4.4. Calculation of $\tan x$. Let $R(x)$ be the approximant of the form (1) or (2) with the best possible relative error to the function $\tan x$ on the segment $[-\pi/4, \pi/4]$, the approximant (2) satisfying the additional condition $d = 0$. The algorithm of reduction is quite similar to the algorithm for $\sin x$ given above. For $x > 0$ set $y = \{x/\pi\} \pi$; in this case $0 \leq y < \pi$. Hence $\tan x \approx R(y)$ for $y \leq \pi/4$; for $\pi/4 < y \leq 3\pi/4$ we have $\tan x \approx 1/R(z)$, where $z = \pi/2 - y$; finally, for $3\pi/4 < y < \pi$ we have $\tan x \approx R(z)$, where $z = \pi - y$. For $x < 0$ we use the relation $\tan(-x) = -\tan x$. For $-\pi/4 \leq x \leq \pi/4$ and ordinary precision $\Delta = \delta = 0.22 \cdot 10^{-10}$, $\bar{\Delta} = 0.63 \cdot 10^{-11}$, $\bar{\delta} = 0.13 \cdot 10^{-10}$. For enhanced precision, $\Delta = \delta = 0.26 \cdot 10^{-13}$, $\bar{\delta} = 0.15 \cdot 10^{-13}$, $\bar{\Delta} = 0.67 \cdot 10^{-14}$. The algorithm for calculating $\tan x$ described above has essential advantages in accuracy and speed as compared with the algorithm using the relation $\tan x = \sin x/\cos x$ and the algorithms for calculating $\sin x$ and $\cos x$.

4.5. Calculation of $\arctan x$. Let $R(x)$ be the approximant of the form (1) or (2) with the best possible relative error to the function $\arctan x$ for $|x| \leq \pi/8 = \sqrt{2} - 1$. The reduction is standard: if $0 \leq x < \sqrt{2} - 1$, then $\arctan x \approx R(x)$; if $\sqrt{2} - 1 \leq x < 1$, then $\arctan x \approx \pi/4 - R(y)$, where $y = (1 - x)/(1 + x)$; if $x > 1$, then $\arctan x \approx \pi/2 - R(1/x)$; for $x < 0$ the relation $\arctan(-x) = -\arctan x$
is used. For \( |x| \leq \sqrt{2} - 1 \) and ordinary precision, \( \delta = 0.29 \cdot 10^{-9} \), \( \bar{\delta} = 0.18 \cdot 10^{-9} \), \( \Delta = 0.11 \cdot 10^{-9} \), \( \bar{\Delta} = 0.37 \cdot 10^{-10} \). For enhanced precision, \( \Delta = 0.11 \cdot 10^{-13} \), \( \bar{\Delta} = 0.36 \cdot 10^{-14} \).

4.6. Calculation of \( \arcsin x \). Let \( R(x) \) be the approximant of the form (1) or (2) with the best possible relative error to the function \( \arcsin x \) on the interval \([-1/2, 1/2]\). Since the function is odd, it is sufficient to consider the case \( x > 0 \). If \( 0 < x \leq 1/2 \), then \( \arcsin x \approx R(x) \); if \( 1/2 < x \leq 1 \), then \( \arcsin x \approx \pi/2 - 2R(y) \), where \( y = \sqrt{(1-x)/2} \). For \(-1/2 \leq x \leq 1/2 \) and ordinary precision, \( \delta = 0.25 \cdot 10^{-8} \), \( \bar{\delta} = 0.16 \cdot 10^{-8} \), \( \Delta = 0.13 \cdot 10^{-8} \), \( \bar{\Delta} = 0.41 \cdot 10^{-9} \); for enhanced precision, \( \delta = 0.82 \cdot 10^{-12} \), \( \bar{\delta} = 0.52 \cdot 10^{-12} \), \( \Delta = 0.43 \cdot 10^{-12} \), \( \bar{\Delta} = 0.13 \cdot 10^{-12} \).

4.7. The reduction algorithms for \( \lg x \), \( \arcsin x \), and \( \text{arctg} x \) described above are taken from [3]. The reduction algorithms for \( \sin x \) and \( \tan x \) were proposed by the author and R. M. Borisyuk [53]. Of course, in particular cases the general scheme is supplemented by special ruses. For example, \( \sin x \), \( \arcsin x \), and \( \text{arctg} x \) are replaced by \( x \) for small values of the argument, and so on.

5. Coefficients. For every function the coefficients of approximants that are used while computing values of this function are indicated below (see Table 3 and Table 4). For every function the coefficients of approximants (1), (1′) and (1″) are listed according to the following order: \( a, b, c, \alpha, \beta, \gamma, d, \xi, \eta, \mu, \lambda, \nu, \alpha' \); the coefficients of approximants (2), (2′) and (2″) are listed according to the following order: \( a, b, c, d, \alpha, \beta, \gamma, \xi, \eta, \mu, \lambda, \nu, \alpha'' \); mantissas (significands) are separated from exponents by the letter \( D \). The accuracy of the coefficients (16 decimal digits of the mantissa) is, of course, excessive.

6. Analysis of the algorithms. It is easy to see that the algorithms of calculating trigonometric and inverse trigonometrical functions do not depend on on the arithmetic system of the computer. On the contrary, while implementing the computing algorithms for exponentials, logarithms and functions that are expressed through them (hyperbolic and inverse hyperbolic functions\(^7\), \( x^y \)) the binary arithmetic system has an essential advantage over the decimal one. For example, for binary arithmetic system the computation of the logarithm is reduced to finding an approximant on the segment \([1/2, 1]\) (and not on the segment \([1/10, 1]\)); since \(1/2\) is much closer to zero than \(1/10\), this implies that the approximation rate increases considerably. While computing \( \ln x \) according to the scheme described above on a binary computer, the approximant of the form (1) which depending on five parameters can be replaced by a more exact approximant (on a smaller segment) which depending only on three parameters. A similar situation arises while calculating an exponential. But the use of the decimal arithmetic system leads to a certain equilibrium between the difficulty of computing logarithmic and exponential functions, on one hand, and trigonometric functions, on the other. Thus in this case the use of a separate common block of the form (1) or (2) is justified.

7. Implementation of algorithms for calculating elementary functions. For the software implementation it is expedient to use representations (1′′) and (2′′) for rational approximants in the form of Jacobi fractions; this allows to minimize the number of arithmetic operations. The rate of computation of functions can be

\(^7\)Note that it is also convenient to use the common block of type (1) or (2) while calculating hyperbolic and inverse hyperbolic functions.
increased by implementing the calculation of Jacobi fractions mentioned above by means of the fixed point arithmetic system as described in [1].

A method of hardware implementation for algorithms under consideration is described in the patent [55]. In this case it is expedient to use representations (1') and (2') for rational approximants and to carry out computations of the fraction numerator and denominator in parallel. For example, when computing expression (2'), the value $y^2$ being computed beforehand, it is possible to use the summatord to compute $\gamma + y^2$ and the multiplier to compute $d \cdot y^2$ simultaneously. Then $y^2$ is multiplied by $(\gamma + y^2)$ and simultaneously the quantity $c$ is added to $d \cdot y^2$, and so on. Under such an implementation, additional hardware requirements are minimal since almost all computers have a summatord and a multiplier.
Table 3. Ordinary precision

| lg x       | 10^x               |
|------------|--------------------|
| 0.3187822082024000D | 0.4184196402707361D 02 |
| -0.2655807946600000D | 0.6132751585841820D 01 |
| 0.2668632700470000D | 0.7526525036394230D -01 |
| 0.3670115625115000D | 0.3634436682941857D 02 |
| -0.4280973292830000D | 0.2138428615920360D 02 |
| ** **   | ** **              |
| 0.2668632700470000D | 0.7526525036394230D -01 |
| -0.1513374262751513D | 0.4523257934215174D 01 |
| -0.1459257686092834D | 0.8645663007168292D 01 |
| -0.2821715606737166D | 0.1273862315203531D 02 |
| -0.447495619843138D | -0.7379037474539062D 02 |
| sin x    | tg x               |
| 0.2051458702138878D | 0.6260411195474330D 02 |
| -0.2731535822018325D | -0.6971684006294421D 01 |
| 0.6635009992122553D | 0.6730991258759150D -01 |
| 0.2051458712958973D | 0.6260411195336056D 02 |
| 0.687599504020228D | -0.2783972122004270D 02 |
| ** **   | ** **              |
| 0.6635009992122553D | 0.6730991258759150D -01 |
| -0.7293840555054680D | -0.5097817546843699D 01 |
| 0.1585035693573942D | -0.1145397751592885D 02 |
| 0.529065810446287D | -0.1638574370411366D 02 |
| 0.1212885465090180D | -0.1250778280153322D 03 |
| arctg x  | arcsin x           |
| 0.4482500977985320D | 0.5603629044813127D 01 |
| 0.3372473371827000D | -0.461453094664500D 01 |
| 0.2742666270116000D | 0.4955994747873100D 00 |
| 0.4482500979270910D | 0.5603629030606043D 01 |
| 0.4866639968788300D | -0.5548466599346680D 01 |
| ** **   | ** **              |
| 0.2742666270116000D | 0.4955994747873100D 00 |
| 0.2037716459063295D | -0.1864713814153353D 01 |
| 0.1596444173439070D | -0.1515767952614660D 01 |
| 0.3270195795349230D | -0.4032986646705020D 01 |
| -0.7381840442193144D | -0.5099063407308178D 00 |
Table 4. Enhanced precision

| $\lg x$ | $10^x$ |
|---------|--------|
| -0.8625170319686105D | 0.2416631060448244D |
| 0.117003513942458D | 0.4101315802439533D |
| -0.3932918863942010D | 0.1179794185292238D |
| 0.196631750250090D | 0.4067164397089984D |
| -0.9930994301066197D | 0.209959068979363D |
| 0.16786051701465279D | 0.1283652206816926D |
| -0.8155425915212830D | 0.856902248348670D |

| $\sin x$ | $\arctg x$ |
|---------|-----------|
| 0.5896178692831105D | 0.196631750250090D |
| -0.8390788000005352D | -0.2413382332334080D |
| 0.2686819888613831D | 0.5896178692831105D |
| -0.2413382332334080D | 0.1436176428157609D |
| 0.1669650188299142D | -0.2078359665454338D |

| $\tg x$ | $\arcsin x$ |
|--------|-----------|
| 0.5896178692831105D | 0.1669650188299142D |
| -0.8390788000005352D | -0.2413382332334080D |
| 0.2686819888613831D | 0.1436176428157609D |
| -0.2413382332334080D | 0.1669650188299142D |

| $\arctg x$ | $\arcsin x$ |
|---------|-----------|
| 0.3113858735833039D | 0.2413382332334080D |
| 0.5018932379116295D | 0.1436176428157609D |
| 0.2013116125542811D | 0.1669650188299142D |
| 0.1360993213361806D | 0.1669650188299142D |
| 0.2149368337314323D | 0.1669650188299142D |
| 0.9463307253423236D | 0.1669650188299142D |

| $\arctg x$ | $\arcsin x$ |
|---------|-----------|
| 0.3113858735833039D | 0.2413382332334080D |
| 0.5018932379116295D | 0.1436176428157609D |
| 0.2013116125542811D | 0.1669650188299142D |
| 0.1360993213361806D | 0.1669650188299142D |
| 0.2149368337314323D | 0.1669650188299142D |
| 0.9463307253423236D | 0.1669650188299142D |

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1. Litvinov, G.L. e.a., *Mathematical algorithms and programs for small computers.*, “Finansy i statistika”, Moscow, 1981. (in Russian)
2. Tikhonov, A.N., Arsenin, V.Ya., *Methods for the solution of ill-posed problems.*, “Nauka”, Moscow, 1979. (in Russian)
3. Hart, J.F. e.a., *Computer approximations*, Wiley, New York a.o., 1968.
4. Voevodin, V.V., *Numerical principles of linear algebra.*, “Nauka”, Moscow, 1977. (in Russian)
5. Luke, Y.L., *Computations of coefficients in the polynomials of Padé approximations by solving systems of linear equations.*, J. Comp. and Appl. Math. 6 (1980), no. 3, 213–218.
6. Luke, Y.L., *A note on evaluation of coefficients in the polynomials of Padé approximants by solving systems of linear equations.*, J. Comp. and Appl. Math. 8 (1982), no. 2, 93–99.
7. Andreeva, I.A., Litvinov, G.L., Rodionov, A.Ya., Fridman V.N., *The PADE-program for the calculation of rational approximants. The program specification and its code*, Fond algorithmov i programm NIVTs AN SSSR, Puschino, 1985, pp. 32. (in Russian)
8. Kryukov, A.P., Litvinov, G.L., Rodionov, A.Ya., *Construction of rational approximation by means of REDUCE.*, Proceeding of the ACM–SIGSAM Symposium on Symbolic and Algebraic Computation (SYMSAC’86), Univ. of Waterloo, Canada, 1986, pp. 31–33.
9. Litvinov, G.L., Fridman, V.N., *Approximate construction of rational approximants*, C. R. Acad. Bulgare Sci. 36 (1983), no. 1, 49–52. (in Russian)
10. Tikhomirov, V.M., *Approximation theory*, Sovremennye problemy mat., Fundamental’nye napravleniya, v.14, Itogi nauki i tekhniki, VINITI, Moscow, pp. 103–260, 272. (in Russian)
11. Baker, G.A., Graves-Morris, P., *Padé approximants. Part I: Basic theory. Part II: Extensions and applications*, Encyclopedia of Mathematics and its Applications, vol. 13, 14, Addison-Wesley Publishing Co., Reading, Mass., 1981.
12. Popov, B.A., Tesler, G.S., *Computer calculation of functions*, “Naukova dumka”, Kiev, 1984. (in Russian)
13. Luke, Y.L., *Mathematical functions and their approximations*, Academic Press Inc., New York–San Francisco–London, 1975.
14. Vinogradov, V.N., Gai, E.V., Rabotnov, N.S., *Analytic approximation of data in nuclear and neutron physics*, Energoatomizdat, Moscow, 1987. (in Russian)
15. Zinn-Justin, J., *Strong interactions dynamics with Padé approximants*, Physics Reports (Section C of Phys.Lett.) 1 (1971), no. 3, 55–102.
16. Dzyadyk, V.K., *Approximation methods for solving differential and integral equations; their applications and development*, Akad. Nauk Ukrain. SSR, Inst. Mat. Preprint 86.31, 1986. (in Russian)
17. Akhiezer, N.I., *Lectures on approximation theory*, “Nauka”, Moscow, 1965. (in Russian)
18. Remez, E.Ya., *Fundamentals of numerical methods for Chebyshev approximations*, “Naukova dumka”, Kiev, 1969. (in Russian)
19. Remez, E.Ya., Gavril’yuõ, V.T., *Computer development of certain approaches to the approximate construction of solutions of Chebyshev problems nonlinearly depending on parameters*, Ukr. Mat. Zh. 12 (1960), 324–338. (in Russian)
20. Gavril’yuõ, V.T., *Generalization of the first polynomial algorithm of E.Ya.Remez for the problem of constructing rational-fractional Chebyshev approximations*, Ukr. Mat. Zh. 16 (1961), 575–585. (in Russian)
21. Fraser, W., Hart, J.F., *On the computation of rational approximations to continuous functions*, Comm. of the ACM 5 (1962), 401–403, 414.
22. Cody, W.J., Fraser, W., Hart, J.F., *Rational Chebyshev approximation using linear equations*, Numer.Math. 12 (1968), 242–251.
23. Gravalu, T., *Rational Chebyshev approximation by Remez’ algorithms*, Numer.Math. 7 (1965), no. 4, 322–330.
24. A. Gravalu, T., *Rational Chebyshev approximation*, Mathematical Methods for Digital Computers v. 2 (Ralston A., Wilf H., eds.), Wiley, New York, 1967, pp. 264–284.
25. Cody, W.J., *A survey of practical rational and polynomial approximation of functions*, SIAM Review 12 (1970), no. 3, 400–423.
26. Kalenchuk-Porkhanova, A.A., *Algorithms and error analysis of the best Chebyshev approximations for functions of one variable*, Theory of approximation of functions (Proc. Internat. Conf., Kaluga, 1975), “Nauka”, Moscow, 1977, pp. 213–218. (in Russian)
27. Ivanov, V.V., Kalenchuk, A.A., Efficiency of algorithms for polynomial and rational Chebyshev approximations, Constructive function theory' 81 (Varna, 1981), Bulgar. Acad. Sci., Sofia, 1983, pp. 72–77. (in Russian)

28. Dunham, Ch.B., Convergence of the Fraser-Hart algorithm for rational Chebyshev approximation, Math. Comp. 29 (1975), no. 132, 1078–1082.

29. Barrar, R.B., Loeb, H.J., On the Remez algorithm for non-linear families, Numer.Math. 15 (1970), 382–391.

30. Collatz, L., Krabs, W., Approximations Theorie. Tschebyscheffsche Approximation mit Anwendungen, B.G. Teubner, Stuttgart, 1978.

31. Spielberg, K., Representation of power series in terms of polynomials, rational approximations and continuous fractions, Journal of the ACM 8 (1961), 613–627.

32. Clenshaw, C.K., Lord, K., Rational approximations from Chebyshev series, Studies in Numerical Analysis (B.K. P. Scaife, ed.), Academic Press, London and New York, 1974, pp. 95–113.

33. Paszkowski, S., Zastosowania numeryczne wielomianów i szeregów Czebyszewa, Państwowe Wydawnictwo Naukowe, Warszawa, 1975 (in Polish); Russian translation: Numerical applications of Chebyshev polynomials and series, “Nauka”, Moscow, 1983.

34. Juhász, K., Németh, G., Padé approximation and its generalizations, ATOMKI Közlémenyek 22 (1980), no. 4, 281–300.

35. Németh, G., Zimányi, M., Polynomial type Padé approximants, Math.Comp. 38 (1982), no. 158, 553–565.

36. Dzyadyk, V.K., The A-method and rational approximation, Ukr. Math. Zh. 37 (1985), no. 2, 250–252. (in Russian)

37. Kravchuk, V.R., Effective approximation of elementary functions by rational polynomials of order (n.1), Ukr. Mat. Zh. 37 (1985), no. 2, 175–180. (in Russian)

38. Lanczos, C., Applied analysis, Englewood Cliffs., N.J., Prentice-Hall, Inc., 1956.

39. Cheney, E.W., Introduction to approximation theory, McGraw-Hill, New York, 1966.

40. Lubinski, D.S., Sidi, A., Convergence of linear and nonlinear Padé approximants from series of orthogonal polynomials, Trans. Amer. Math. Soc. 278 (1983), no. 1, 333–345.

41. Forsythe, G.E., Malcolm, M., Moler, C., Computer methods for mathematical computations, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1977.

42. Fletcher, R., Expected conditioning, IMA J. Numer. Anal 5 (1985), no. 3, 247–273.

43. Hopkins, T.R., On the sensitivity of the coefficients of Padé approximants with respect to their defining power series coefficients, J. Comp. Appl. Math. 8 (1982), no. 2, 105–109.

44. Wuytack, L., On the conditioning of the Padé approximation problem, Lect. Notes Math. 888 (1983), no. 1, 78–89.

45. Maehly, H.J., Rational approximations for transcendental functions, Proceedings of the International Conference on Information Processing, UNESCO, Butterworths, London, 1960, pp. 57–62.

46. System/360 Scientific Subroutine Package (360 A-CM-03X). Version III. Programmer’s Manual, vol. 1, 2, Fourth Edition, IBM, Technical Publication Department, New York, 1970, 1971.

47. Bachvalov, N.S., Numerical methods (analysis, algebra, ordinary differential equations), “Nauka”, Moscow, 1973. (in Russian)

48. Hearne, A.C., REDUCE User’s Manual, Rand Publ., 1982.

49. Edneral, V.F., Kryukov, A.P., Rodionov, A.Ya., Analytic computations language REDUCE, Part I, Part II, Moscow State University, Moscow, 1983, 1986. (in Russian)

50. Popov, V.A., Petrushev, P.P., The exact order of the best uniform approximation of convex functions by means of rational functions, Mat. Sb. 7(103) (1979), no. 2, 285–291. (in Russian)

51. Gonchar, A.A., The rate of rational approximation of analytic functions, Trudy Mat. Inst. Steklov 166 (1984), 52–60. (in Russian)

52. Gonchar, A.A., The rate of approximation of functions by rational fractions and properties of the functions., Proc. Internat. Congr. Math.(Moscow, 1966), “Mir”, Moscow, 1968, pp. 329–356. (in Russian)

53. Borisyuk, R.M., Litvinov, G.L., A software support of algorithms for computing rational functions by means of “Mir”-computers, Mathematical algorithms and programmes for small computers, “Finansy i statistika”, Moscow, 1981, pp. 103–109.

54. Alefeld, G., Herzberger, J., Introduction to interval computations, Academic Press, Inc., New York–London, 1983.
55. Fet, Ya.I., Litvinov, G.L., *Device for approximation of functions. Patent no. 1488838 of August 7, 1986, "Otkrytiya. Izobreteniya". Ofitsialnyi byulleten' Gosudarstvennogo komiteta po izobreteniyam i otkrytiyam pri Gosudarstvennom komite po nauke i tekhnike SSSR* (1989), no. 23, 234. (in Russian)

56. Litvinov, G.L., *Approximate construction of rational approximations and an effect of error autocorrection*, Mathematics and modeling, NIVTs AN SSSR, Puschino, 1990, pp. 99–141. (in Russian)

E-mail: litvinov@islc.msk.su
glitvinov@mail.ru