Renormalons in static QCD potential: review and some updates

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Abstract We give a brief review of the current understanding of renormalons of the static QCD potential in coordinate and momentum spaces. We also reconsider estimate of the normalization constant of the $u = 3/2$ renormalon and propose a new way to improve the estimate.

1 Introduction

The static QCD potential is an essential quantity for understanding the QCD dynamics, and at the same time it is suitable to understand renormalon of perturbative QCD. This is due to the following reasons. First, it is practically possible to observe renormalon in the perturbative series of the static QCD potential since it exhibits renormalon divergence at quite early stage, say at NLO. This is caused by the $u = 1/2$ renormalon, which is a very close singularity to the origin of the Borel $u$-plane. Second, the perturbative series is known up to $O(\alpha_s^2)$ [1–9]. This is the highest order that has been reached so far for physical observables. The explicit large-order coefficients are helpful to examine if the perturbative coefficients indeed follow the theoretically expected asymptotic form. Actually theoretical arguments already revealed detailed asymptotic behaviors of the perturbative coefficients caused by the renormalon at $u = 1/2$ and also that at $u = 3/2$.

In this paper, we first give a review of the current theoretical understanding of the renormalons in the static QCD potential. We discuss it both in coordinate space and momentum space, where totally different features are found. In particular, we explain a simple formula, presented recently, to analyze renormalons in momentum space. Second, we move on to discussion on estimation of normalization constants of renormalons. Normalization constants are the only parameter which cannot be determined by the current theoretical argument. One needs to know it to subtract renormalons in some methods [10–13]. In this paper, we perform a detailed test on methods to extract normalization constants. This aims at reconsidering the conclusions in Refs. [14,15]. Ref. [14] concluded that the normalization constant of the $u = 3/2$ renormalon cannot be estimated reliably with the NNNLO perturbative series while Ref. [15] stated that it is possible and estimated the normalization constant from the same series. Since this difference mainly stems from the difference in analysis method, we examine validity of different methods. After this examination, we propose a new way to improve the estimate; we propose to use the scale consistent with the scaling behavior of asymptotic form of perturbative coefficients, instead of minimal sensitivity scale. This is a new proposal in this paper. Finally, we give conclusions and supplementary discussion. In Appendix, we summarize the notation used here and basic relations to discuss renormalons.

2 Renormalons in coordinate space

The first IR renormalon of the static QCD potential is located at $t = 1/(2b_0)$ (or $u_*=1/2$), which is called the $u = 1/2$ renormalon [16]. Here $b_0 = (11 - 2n_f/3)/(4\pi)$ is the first coefficient of the beta function, where $n_f$ is the number of quark flavors. See Appendix for our notation, where the meaning of parameters $t$ and $u_*$ is explained. This induces the $O(r^0)$ renormalon uncertainty to $V_{\text{QCD}}(r)$. The important feature of this renormalon is that it is cancelled in the total energy of the heavy quark and anti-quark system [17–19],

$$E = V_{\text{QCD}}(r) + 2m_{\text{pole}},$$

(1)

once the heavy quark pole mass $m_{\text{pole}}$ is expanded perturbatively in terms of a short distance mass. Considering analogy to the multipole expansion in classical electrodynamics, one can understand this cancellation as a consequence of the fact that the $O(r^0)$ term couples to the total charge of the system [14,20]. Since the system is color neutral, there should not be the $O(r^0)$ term and such an uncertainty. Once we recognize that the cancellation takes place in the total energy, we can...
conclude that the $u = 1/2$ renormalon uncertainty of $V_{\text{QCD}}(r)$ (that of $m_{\text{pole}}$) is independent of $r$ ($m_{\text{pole}}$). Otherwise, the cancellation does not hold. Hence, the renormalon uncertainty is exactly proportional to the QCD dynamical scale:

$$\text{Im} V_{\text{QCD}}(r)_{\pm} \big|_{u_\gamma = 1/2} = \pm K_{1/2} A_{\text{MS}}^3 \ . \quad (2)$$

See Appendix for the definition of a renormalon uncertainty. The constant $K_{1/2}$ is the undetermined parameter in this argument.

The second IR renormalon is considered to be located at $u_\gamma = 3/2$ from the study in the large-$\beta_0$ approximation and from the structure of the multipole expansion in pNRQCD [16, 21]. The uncertainty is roughly given by $\sim A_{\text{MS}}^3 r^2$. Recently the detailed structure of the second IR renormalon has been investigated [14, 15] within the multipole expansion, which gives the static potential as

$$V_{\text{QCD}}(r) = V_S(r) + \delta E_{\text{US}}(r) + \cdots . \quad (3)$$

Here $V_S(r)$ is a Wilson coefficient in pNRQCD and identified as the perturbative computation of the static potential. Hence, $V_S(r)$ contains the $u = 3/2$ renormalon. $\delta E_{\text{US}}(r)$ is the first non-trivial correction in the $r$ expansion, given by

$$\delta E_{\text{US}}(r) = -\frac{V_A^2(r; \mu)}{6} \int_{0}^{\infty} dt \ e^{-i \Delta V(r) t} \times \langle \mathbf{r} \cdot \mathbf{E}(0, 0) \rangle,$$

whose $r$ dependence is roughly given by $\mathcal{O}(r^2)$. Here $V_A(r)$ is a Wilson coefficient in pNRQCD and $\Delta V(r) \equiv V_O(r) - V_S(r)$ denotes the difference between the potentials of the octet and singlet states. Since the $u = 3/2$ renormalon uncertainty in $V_S(r)$ is considered to be canceled against that of $\delta E_{\text{US}}(r)$ [21], the uncertainty should be the same $r$-dependence as $\delta E_{\text{US}}(r)$. Hence, we reveal the detailed $r$-dependence of $\delta E_{\text{US}}(r)$ to understand the detailed form of the $u = 3/2$ renormalon. In Eq. (4), $r$-dependent quantities are $V_A(r; \mu)$ and $\Delta V(r)$ besides the power term, $r^2$. However, after the IR renormalon in $V_S(r)$ is canceled against the UV contribution ($t \sim 0$) of $\delta E_{\text{US}}(r)$, we can approximate $e^{-i \Delta V(r) t} \sim 1$ in our present analysis and $\Delta V(r)$ is not relevant here. Therefore the $u = 3/2$ renormalon uncertainty is given by

$$\text{Im} V_S(r)_{\pm} \big|_{u_\gamma = 3/2} = \pm K_{3/2} \exp \left[ -2 \int_{0}^{a_{\alpha_s}(r^{-1})} dx \frac{\gamma(x)}{\beta(x)} \right] \times V_A^2(r; \mu) \ .$$

Here we have solved the RG equation,

$$V_A(r; \mu_0) = \exp \left[ - \int_{a_{\alpha_s}(\mu_0)}^{a_{\alpha_s}(\mu)} dx \frac{\gamma(x)}{\beta(x)} \right] V_A(r; \mu_0)$$

where

$$\frac{\mu^2 dV_A(r; \mu)}{d\mu^2} = \gamma(a_{\alpha_s}) V_A(r; \mu)$$

and taken $\mu = r^{-1}$ to show the uncertainty in terms of $\alpha_s(r^{-1})$. In the last line of Eq. (5), we used $\gamma_1 = 0$ and $V_A(r) = 1 + \mathcal{O}(\alpha_s^2(r^{-1}))$. Again $K_{3/2}$ is the undetermined constant. Although the renormalon uncertainty can be different from $A_{\text{MS}}^3 r^2$, the correction factor, $1 + \mathcal{O}(\alpha_s^2(r^{-1}))$, turns out to be small.

### 3 Renormalons in momentum space

Even though the perturbative series of the static potential in coordinate space suffers from seriously divergent behavior, that in momentum space has a good convergence property. Recently a simple formula to quantify the renormalon uncertainties of the momentum-space potential has been proposed [14]. In this formula, one considers Fourier transform of a coordinate-space renormalon uncertainty. Since renormalon uncertainties in coordinate space can be revealed systematically within the multipole expansion as seen above, it provides us with a clear way to study momentum-space renormalon uncertainties.

The momentum-space potential $\alpha_V(q)$ is defined by

$$-4 \pi C_F \frac{\alpha_V(q)}{q^2} = \int d^3 r e^{-inr} V_S(r).$$

Let us first consider a renormalon uncertainty of simple form in coordinate space:

$$\text{Im} v(r)_{\pm} = \pm K_{u_\gamma} (A_{\text{MS}}^2 r^2)^{u_\gamma}, \quad (8)$$

where $v(r) := r V_S(r)$ is the dimensionless potential. The $u = 1/2$ renormalon uncertainty indeed takes this form. We calculate the corresponding renormalon uncertainty in $\alpha_V(q)$ by considering Fourier transform of the above renormalon uncertainty. In other words,
we replace \( V_S(r) \) in Eq. (7) with \( \text{Im} \left[ v(r)/r \right] \) to obtain \( \text{Im} \alpha_V(q)_\pm \). We obtain

\[
\text{Im} \alpha_V(q)_\pm = \pm \frac{K_{u_*}}{C_F} \left( \frac{A_{\text{MS}}^2}{q^2} \right)^{u_*} \Gamma(2u_* + 1) \cos(\pi u_*).
\]

(9)

If \( u_* \) is a positive half-integer, this uncertainty completely vanishes since \( \cos(\pi u_*) = 0 \) and \( \Gamma(2u_* + 1) \) is finite. Hence, we conclude that the \( u = 1/2 \) renormalon is absent in the momentum-space potential. This is a revisit of the old conclusion obtained in Ref. [19]. Our argument does not rely on diagrammatic analysis.

We can easily extend this argument to study renormalon structure in momentum space beyond the \( u = 1/2 \) renormalon. Since a general renormalon uncertainty may include logarithms \( \log(\mu^2 r^2) \) when we rewrite \( \alpha_s(r^{-1}) \) in terms of \( \alpha_s(\mu) \) [for instance, see Eq. (5)], we assume that a renormalon uncertainty is given by

\[
\text{Im} \left[ v(r) \right]_{\pm} = \pm K_{u_*} \left( \frac{A_{\text{MS}}^2}{q^2} \right)^{u_*} u \sum_{n \geq 0} a_n \frac{\partial^n}{\partial u^n} \left( \mu^2 r^2 \right)^u \bigg|_{u = 0},
\]

(10)

where \( a_n \) is a function of \( \alpha_s(\mu) \). (In the case of the \( u = 1/2 \) renormalon studied above, \( a_n = 0 \) for \( n \geq 1 \), because its uncertainty is exactly proportional to \( A_{\text{MS}} \) and does not have \( \mathcal{O}(\alpha_s^{-1}) \) correction.) Repeating a similar calculation, we obtain the renormalon uncertainty in momentum space induced by the above coordinate-space renormalon uncertainty as

\[
\text{Im} \alpha_V(q)_\pm = \pm \frac{K_{u_*}}{C_F} \left( \frac{A_{\text{MS}}^2}{q^2} \right)^{u_*} \sum_n a_n \frac{\partial^n}{\partial u^n} \left( \mu^2 q^2 \right)^u \Gamma(2u_* + 1) \cos(\pi u_* + u) \bigg|_{u = 0}.
\]

(11)

Using this formula one can generally study the detailed renormalon structure in momentum space. In the case of the \( u = 3/2 \) renormalon the uncertainty is given by

\[
\text{Im} v(r)_{|u_*=3/2} = \pm K_{3/2}(\mu) V_A(r; \mu) r^{3/2} A_{\text{MS}}^3
\]

\[
= \pm K_{3/2}(\mu) \left( \frac{A_{\text{MS}}^2}{q^2} \right)^{3/2} \{1 + 2e_2 a_*^2(\mu) + 2(e_3 + (2b_0 e_2 + \gamma_2) \log(\mu^2 r^2)) a_*^3(\mu) + \cdots \},
\]

(12)

where we denote the perturbative series of \( V_A(r) \) as \( V_A(r) = 1 + e_2 a_*^2(\mu) + (e_3 + (2b_0 e_2 + \gamma_2) \log(\mu^2 r^2)) a_*^3(\mu) + \cdots \), \( e_2 \) and \( e_3 \) are log-independent constants. Here we used \( e_1 = \gamma_0 = \gamma_1 = 0 \). Then we obtain the \( u = 3/2 \) renormalon uncertainty in momentum space as

\[
\text{Im} \alpha_V(q)_{|u_*=3/2} = \pm \frac{K_{3/2}(\mu)}{C_F} \left( \frac{A_{\text{MS}}^2}{q^2} \right)^{3/2} \times [12\pi(2b_0 e_2 + \gamma_2) a_*^3(\mu) + \cdots].
\]

(13)

We can see that the momentum-space renormalon uncertainty is suppressed by \( \mathcal{O}(a_*^3) \).

We saw that in momentum space the \( u = 1/2 \) renormalon is absent and the \( u = 3/2 \) renormalon is fairly suppressed. It means that the renormalons in coordinate space are caused by the \( q \sim 0 \) region in the Fourier integral

\[
V_S(r) = -4\pi C_F \int \frac{d^3q}{(2\pi)^3} e^{iqr} \frac{\alpha_V(q)}{q^2}.
\]

(14)

This is exactly the case for the \( u = 1/2 \) renormalon and this is the case to a large extent also for the \( u = 3/2 \) renormalon. Hence, if we introduce an IR cutoff to the Fourier integral,

\[
V_S(r; \mu_f) = -4\pi C_F \int_{q^2<\mu_f^2} \frac{d^3q}{(2\pi)^3} e^{iqr} \frac{\alpha_V(q)}{q^2},
\]

(15)

it is almost free from the IR renormalons at \( u = 1/2 \) and \( u = 3/2 \). We note that the absence of the \( u = 1/2 \) renormalon in this quantity was revealed in Ref. [19] and this gave a motivation to define the potential subtracted (PS) mass.

### 4 Estimates of normalization constant

The normalization constants of renormalons, \( K_\mu \), cannot be determined by the above theoretical arguments. However, it can be estimated from fixed order perturbative coefficients since the size of normalization constants is related to the asymptotic form of perturbative coefficients. The following two methods have been adopted in the literature to extract the normalization constant of a leading IR renormalon. Hereafter we study \( N \) instead of \( K \); see Eq. (33) for their relation.

#### 4.1 Method A [22]

From Eq. (29), one considers the function,

\[
(1 - b_0 t/u_*)^{1+r_*} B_{\rho}(t)/(\mu^2 r^2)^{u_*}.
\]

(16)

Expanding this function in \( t \) and then substituting \( t \to u_*/b_0 \), one obtains the normalization constant \( N_{u_*} \). Note that the convergence radius of the series expansion of the above function is \( \rho = u_*/b_0 \) and on the convergence radius it gives us the correct value.
4.2 Method B [15,23,24]

Since the asymptotic behavior of perturbative coefficients can be predicted except for the overall constant, one can determine the normalization constant by

\[
N_u = \lim_{n \to \infty} \frac{d_n^{\mu_r(\text{asy})}}{d_n^{\mu_r(\text{asy})}/N_u}.
\]

\(d_n^{\mu_r(\text{asy})}\) is given by Eq. (30).

Both methods should give an accurate answer if an all-order perturbative series is known.

It is stated in Refs. [15,23,24] that Method B practically shows faster convergence than Method A and gives more stable estimate against scale variation. Related to this, different conclusions were obtained in Refs. [14,15] about the estimate of the \( u = 3/2 \) renormalon normalization constant, \( N_{3/2} \). In Ref. [14], using Method A we concluded that we cannot reasonably estimate the normalization constant because a large uncertainty remains with the NNLO perturbative series. In Ref. [15], on the other hand, the authors presented an estimation of the normalization constant using Method B from the same order perturbative series.

In this section, we reconsider the conclusion obtained in Ref. [14] by examining whether Method B actually gives more accurate results or not. We perform a validity test by considering a model-like all order perturbative series where exact normalization constants are known. In fact, since there are little cases where the nontrivial series where exact normalization constants are known. In the following analysis, we use the N^3LL model series. (We regularize the IR divergence in the three-loop coefficients in Scheme A defined in Ref. [14].)

We demonstrate how we can obtain model series in the simplest case, i.e. the LL model series. In momentum space, we have the LL result,

\[
\alpha_V(q)|_{\text{LL}} = \alpha_s(q) = \alpha_s(\mu) \sum_{n=0}^{\infty} \left[ b_0 \alpha_s(\mu) \log(\mu^2/q^2) \right]^n.
\]

(19)

This is an all-order series in terms of \( \alpha_s(\mu) \) in momentum space. The all-order perturbative series in coordinate space can be obtained by

\[
V_S(r)|_{\text{LL}} = -4\pi C_F \int \frac{d^3 q}{(2\pi)^3} e^{i q \cdot r} \frac{\alpha_V(q)|_{\text{LL}}}{q^2}
\]

\[
= -4\pi C_F \sum_{n=0}^{\infty} \alpha_s(\mu) (b_0 \alpha_s(\mu))^n \times \int \frac{d^3 q}{(2\pi)^3} e^{i q \cdot r} \left[ \log \left( \frac{\mu^2}{q^2} \right) \right]^n.
\]

(20)

From the integration of logarithms in a small-\( q \) region, \( V_S(r)|_{\text{LL}} \) has renormalons at \( u = 1/2, 3/2, \ldots \). The resummed result is given by the right-hand side of the first equality. (We deform the integration path to avoid the singularity of \( \alpha_V(q) = \alpha_s(q) \).) In this expression, the renormalon uncertainties stem from the simple pole of \( \alpha_s(q) \) at \( q = \Lambda_{\text{MS}}^2 \). The normalization constants of the renormalons can be calculated by the contour integral surrounding the pole. This idea can be generalized for the N^kLL case [14]. We note that, by construction, if one considers the N^kLL model series, the perturbative coefficients up to the \( O(\alpha^{k+1}_s) \) order are exactly obtained, while the higher order coefficients are estimated based on the expectation that the logarithmic terms in \( \alpha_V(q) \) dominantly determine the perturbative coefficients in \( V_S(r) \).

Since we are now interested in the \( u = 3/2 \) renormalon, we consider the QCD force, \( f(r) = r^2 dV_S(r)/dr \) to eliminate the \( u = 1/2 \) renormalon. We denote its perturbative coefficient by \( d_n^{\ell} \). Once we obtain the normalization constant of the force \( N_{3/2}^{\ell} \) we can readily obtain the normalization constant of the potential by the relation \( N_{3/2}^{\ell} = 2N_{3/2}^\nu \). Hereafter \( N_{3/2}^\nu \) means \( N_{3/2}^\nu \).

We assume the number of flavors to be \( n_f = 3 \) throughout this analysis.

We explain how we estimate a central value and its error from the nth order truncated series using Method A or B. We adopt a parallel estimation method to Ref. [15]. The central value at the nth order (which is estimated from the N^kLO perturbative series) is determined at the minimal sensitivity scale \( \mu_r \) of a nor-
To estimate the error we vary \( \mu r \) around the minimal sensitivity scale by the factor \( \sqrt{2} \) or \( 1/\sqrt{2} \). In addition we obtain the \((n-1)\)th order result at the minimal sensitivity scale of the \(n\)th order result and examine the difference. The procedure so far is common to Methods A and B. In Method B, we also examine the difference caused by including \( \frac{1}{n} \) correction [i.e. \( k = 1 \) term in Eq. (30)] in \( d_n^{μ_r}(\text{asym}) \) or not. Finally combining the two (three) errors in Method A (Method B) in quadrature,\(^4\) we obtain the \(n\)th order result with the total error.

We show the results in Fig. 1.

We can see that Method B gives smaller error than Method A and shows faster convergence. This indeed agrees with the statement in Refs. \([ 15,23,24]\), and we consider Method B superior.

However, it is worth noting that in both methods the error size does not show healthy convergence at small \( n \) as seen from Fig. 1; the estimated error does not always get smaller as \( n \) is raised in the region \( n \lesssim 10 \) in Method B and such a tendency is worse in Method A.

To improve the estimate, we propose to use, instead of the minimal sensitivity scale, a reasonable scale from the viewpoint of the asymptotic behavior. As shown in Eq. (30), \( d_n^{\mu_r}(\text{asym}) \) should behave as \( d_n^f \propto (μ^2 r^2)^{3/2} \) for scale variation\(^5\) if the \( u = 3/2 \) renormalon dominates the \( n\)th order perturbative coefficient. In this case, \( d\log d_n^f/dL \) should be (or very close to) \( 3/2 \), where \( L = \log(μ^2 r^2) \).

From this figure, we consider that the dominance of the \( u = 3/2 \) renormalon sets in around \( n \gtrsim 5 \). In the estimate of the normalization constant, we propose to use the scale where \( d\log d_n^f/dL \) is close to \( 3/2 \). Quantitatively we choose the optimal scale \( μ_0 \) such that the integral

\[
\int_{\log(μ_0/\sqrt{2})^2 r^2}^{\log((\sqrt{2}μ_0)^2 r^2)} \frac{dL}{dL} \left| \frac{d\log d_n^f}{dL} - \frac{3}{2} \right|
\]

is minimized. Then we determine a central value at \( μ = μ_0 \) using Method B. We call this estimation method Method B'. In this method, as seen from Fig. 2, larger scale \( L \sim 2 \) is favored, although in the previous analysis the minimal sensitivity scale appeared \( L \sim 0 \). The way to estimate the error is parallel to the previous case: we examine the difference caused by the scale variation by the factor \( \sqrt{2} \) or \( 1/\sqrt{2} \) and examine the difference of the \((n-1)\)th and \(n\)th order results at \( μ = μ_0 \). We also examine the impact of the \( 1/n \) correction.

We show the result in Method B' in Fig. 3.

The convergence is faster than Method B, and remarkably, the error gets smaller almost monotonically as \( n \) is raised, especially at \( n \gtrsim 5 \). Hence, Method B' is optimal as far as we have tested. Although the estimate at \( n = 3, N_{3/2} = 0.64 \pm 0.29 \), deviates from the exact value 0.143, this is not surprising because the renormalon would not be relevant enough at this order as suggested from Fig. 2.

Although we have used the \( N^3LL \) model series so far, we did a parallel analysis using the \( N^2LL \) model series. The situation was almost parallel. We found that (i) Method B shows faster convergence than Method A, and (ii) Method B' makes the central value and its error converge faster than Method B.

### 4.4 NNNLO estimate of \( N_{3/2} \)

We give the NNNLO estimate using Method B’. So far we have regularized the IR divergence in the three-loop coefficient [\( 21,26–28 \)] in Scheme A which is defined in [\( 14\)].\(^6\) (In this case our estimate at NNNLO reads \( N_{3/2} = 0.64 \pm 0.29 \)) as shown above, although the scaling behavior is far from the expected one from the \( u = 3/2 \) renormalon.) In this analysis, we adopt the same regularization as Ref. [\( 15\)] to make comparison easy. We show \( d\log d_n^f/dL \) in Fig. 4.

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\(^3\) If the minimal sensitivity scale is not found in the range \( 1/2 < μ r < 5 \), we treat \( μ r = 1 \) as the minimal sensitivity scale.

\(^4\) The final error is estimated in this way in Ref. [\( 15\)] and we follow it.

\(^5\) Rigorously speaking, \( d_n^{\mu_r}(\text{asym}) \propto (μ^2 r^2)^{3/2} \) does not exactly hold in general cases because \( c_k u_r \) is a polynomial of \( \log(μ^2 r^2) \). When a renormalon uncertainty is exactly proportional to \( A_{\text{ren}}^{\mu_r} \), \( c_k u_r \) does not have \( \log(μ^2 r^2) \) dependence and \( d_n^{\mu_r}(\text{asym}) \propto (μ^2 r^2)^{3/2} \) is exact.

\(^6\) In Scheme A, we assume dimensional regularization in calculating the three-loop coefficient. Then we drop the divergent term \( 1/\epsilon \) (associated with the IR divergence) and set the renormalization scale to \( 1/r \). (Both of the soft and ultra-soft renormalization scales are set to \( 1/r \).)
Fig. 2 $d \log(d^n_f)/dL$ for various $n$. The expected value $3/2$ is shown by the black line in each figure.

Fig. 3 Estimate of the normalization constant $N_{3/2}$ using Method B', where the scale consistent with the $u = 3/2$ renormalon is used instead of the minimal sensitivity scale. The black line shows the exact answer. The range of the vertical axis is taken the same as Fig. 1.

The difference from the upper left figure in Fig. 2 for $n = 3$ comes from the difference in regularization of the IR divergence. In this case, the behavior of $d \log(d^n_f)/dL$ is closer to $3/2$ than that of Fig. 2. We obtain

$$N_{3/2} = 0.17 \pm 0.05_{\text{scale}} \pm 0.02_{\text{NNLO}} \pm 0.02_{1/n} \pm 0.004_{\text{us}} = 0.17(5),$$

or

$$N_{3/2}^f = 0.35 \pm 0.1_{\text{scale}} \pm 0.04_{\text{NNLO}} \pm 0.05_{1/n} \pm 0.008_{\text{us}}$$

where the latter one is the result of the normalization for the force and can be compared with Eq. (4.4) in Ref. [15], which reads $0.37(17)$. The error analysis is also parallel to Ref. [15] (we assume symmetric errors in the first place though) and the last error shown by "us" shows the error associated with the ultrasoft contribution. In our analysis using Method B', the central value is extracted at $\mu r = 1.82$ [or $L = \log(\mu^2 r^2) = 1.20$ (see Fig. 4)] while in the analysis in Ref. [15] the central value is extracted at the minimal sensitivity scale $\mu r = 1.52$. 

Eq. (23)
5 Conclusions and discussion

In this paper, we gave a brief review of the current understanding of the renormalons at \( u = 1/2 \) and \( u = 3/2 \) of the static QCD potential in coordinate and momentum spaces. We also reconsidered estimation of the normalization constant of the \( u = 3/2 \) renormalon \([14,15]\). We examined the efficiency of different estimation methods based on a model-like all order series. Our study agrees with the statement in \([15,23,24]\) that Method B is superior to Method A. To improve the estimate further, we proposed to use the consistent scale with an asymptotic behavior of perturbative coefficients, instead of the minimal sensitivity scale. We call it Method B'. As far as we tested, the proposed method gives a stable result and is most efficient, in particular in the sense that it basically makes the error smaller monotonically as the order of perturbation theory is raised.

We did not mention the complexity caused by IR divergences in perturbative coefficients \([21,26–28]\) in this paper. However, related to this, it was pointed out in Ref. \([14]\) that an unfamiliar renormalon may arise at \( u = 1/2 \), whose uncertainty is specified as \( \sim A^2_{\text{ren}} \Delta V^2 \). Also ways to renormalize these IR divergences consistently with the renormalon uncertainties are discussed therein. These issues need to be further investigated for more precise understanding of renormalons in the static QCD potential.

Finally, we briefly mention renormalon subtraction methods. Although we did not mention how one can cope with renormalon uncertainties in this paper, methods to subtract renormalon uncertainties are being developed \([10–13,15]\). Recently, a new method has been proposed \([29]\), which uses the mechanism of renormalon suppression in momentum space. We argued in Sect. 3 that renormalons vanish or are fairly suppressed in momentum space. Using this mechanism one can largely suppress renormalons of a general physical observable by considering Fourier transform to fictional “momentum space” \([29]\). Higher order computation combined with renormalon subtraction will be an important direction to give more accurate QCD predictions.

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Appendix A: Notation and basic relations

In this appendix we summarize basic knowledge on renormalon and clarify the notation used in this paper. The beta function is given by

\[
\mu^2 \frac{d\alpha_s(\mu)}{d\mu^2} = \beta(\alpha_s) = -b_0\alpha_s^2 - b_1\alpha_s^3 - \cdots .
\]  

The QCD dynamical scale in the \(\overline{\text{MS}}\) scheme is defined by

\[
A^2_{\overline{\text{MS}}} / \mu^2 = \exp \left[ - \left( \frac{1}{b_0\alpha_s^2(\mu)} + \frac{b_1}{b_0^2} \log(b_0\alpha_s(\mu)) \right) + \int_0^{\alpha_s(\mu)} dx \left( \frac{1}{\beta(x)} + \frac{1}{b_0x^2} - \frac{b_1}{b_0^2x^3} \right) \right] .
\]  

(25)

We denote the dimensionless static QCD potential by \(v(r)\),

\[
v(r) = rV_S(r) = \sum_{n=0}^{\infty} d^n(\mu r)\alpha_s^{n+1}(\mu),
\]  

(26)

and the dimensionless QCD force by \(f(r)\),

\[
f(r) = r^2 \frac{dV_S}{dr} = 2 \frac{dv}{dL} - v = \sum_{n=0}^{\infty} d^n(\mu r)\alpha_s^{n+1}(\mu),
\]  

(27)

where \(L = \log(\mu^2r^2)\). We define the Borel transform of such a perturbative series by

\[
B_X(t) := \sum_{n=0}^{\infty} d^n(\mu r) t^n,
\]  

(28)

where \(X\) is \(v(r)\) or \(f(r)\) (or momentum-space potential \(n(1)\)). Around the singularity at \( t = u_*/b_0 > 0 \), it behaves as

\[
B_X(t) = (\mu r)^{u_*} \frac{N_{u_*}}{(1 - b_0/tu_*)^{1 + v_{u_*}}} \times \sum_{k=0}^{\infty} c_k(u_*) (1 - \frac{b_0t}{u_*})^k + \cdots , \quad (c_0 = 1),
\]  

(29)

where \(N_{u_*}, v_{u_*}\), and \(c_k(u_*)\) are parameters, and \(\cdots\) denotes a regular function at \( t = u_*/b_0 \). The asymptotic behavior of the perturbative coefficient due to the first IR renormalon \( t = u_*/b_0 \) follows from the above singular Borel transform as

\[
d_n^{\text{asym}} = N_{u_*} (\mu^2r^2)^{u_*} \frac{\Gamma(n + 1 + v_{u_*})}{\Gamma(1 + v_{u_*})} \left( \frac{b_0}{u_*} \right)^n \times \sum_{k=0}^{\infty} c_k(u_*) (\nu_{u_*}(\nu_{u_*} - 1) \cdots (\nu_{u_*} - k + 1) \cdots (n + v_{u_*} - k + 1).}
\]  

(30)

The renormalon uncertainty of \( X \) is defined by the imaginary part of a regularized Borel integral:

\[
\text{Im} X_{\pm} = \text{Im} \int_{0^{\pm}}^{\infty} dt B_X(t) e^{-t/\alpha_s(\mu)}
\]  

\[
= \pm \frac{\pi}{b_0} \frac{\mu^2r^2(\nu_{u_*} + 1 + v_{u_*}) e^{-\frac{v_{u_*}}{b_0}\log(b_0\alpha_s(\mu))} - \nu_{u_*}}{\Gamma(1 + v_{u_*})} \times \sum_{k=0}^{\infty} \nu_{u_*}(\nu_{u_*} - 1) \cdots (\nu_{u_*} - k + 1)
\]  

\[
\times (b_0/u_*)^k c_k(u_*) (\mu r)^{\nu_{u_*}(\nu_{u_*} + 1)}.
\]  

(31)

This is renormalization scale independent. Writing the renormalon uncertainty as

\[
\text{Im} X_{\pm} = \pm K_{u_*} e^{-u_*/(b_0\alpha_s(\mu))} (b_0\alpha_s(\mu))^{v_{u_*}}.
\]
\[
\sum_{k=0}^{\infty} s_{k,u_0} a_0^k (r^{-1})
\]

with \( s_0 = 1 \), we have the following relations,
\[
K_{u_0} = \frac{\pi}{b_0} \frac{N_{u_0}}{F(1 + \nu)} u_0^{1 + \nu u_0},
\]

and
\[
s_{k,u_0} = \nu u_0 (\nu u_0 - 1) \cdots (\nu u_0 - k + 1) \\
\times (b_0 / u_0)^k c_k(\mu r = 1) \quad \text{for} \quad k \geq 1.
\]

As discussed in Sect. 2, since the renormalon uncertainties in coordinate space are given by
\[
K_{u_0}(A_{\text{MIN}}^2 r^2)^{\nu u_0} [1 + \mathcal{O}(\alpha_s^2 (r^{-1}))],
\]

(where \( \mathcal{O}(\alpha_s^2) \) can be zero) one can see that \( \nu u_0 \) in Eq. (32) is given by
\[
\nu u_0 = u_0 b_1 / b_0^2
\]

for \( u_0 = 1/2 \) or \( 3/2 \) (where Eq. (25) is used). One can also calculate \( s_{k,u_0} \) and thus \( c_k, r_{u_0} \) by expanding Eq. (2) or (5) in \( \alpha_s \).

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