MARKOV PROCESSES ON PARTITIONS

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ABSTRACT. We introduce and study a family of Markov processes on partitions. The processes preserve the so-called z-measures on partitions previously studied in connection with harmonic analysis on the infinite symmetric group. We show that the dynamical correlation functions of these processes have determinantal structure and we explicitly compute their correlation kernels. We also compute the scaling limits of the kernels in two different regimes. The limit kernels describe the asymptotic behavior of large rows and columns of the corresponding random Young diagrams, and the behavior of the Young diagrams near the diagonal.

Our results show that recently discovered analogy between random partitions arising in representation theory and spectra of random matrices extends to the associated time-dependent models.

INTRODUCTION

In a series of papers (see [BO1], [Ol2], references therein, and also [BO5]) we have been studying a remarkable family of probability distributions on partitions (equivalently, Young diagrams) called z-measures. These objects have a representation theoretic origin, they arise in harmonic analysis on the infinite symmetric group, see [KOV1], [KOV2]. Surprisingly enough, the z-measures turned out to be related to a number of probabilistic models of random matrix theory, stochastic growth, random tilings, percolation theory, etc. In this paper, we introduce and study a family of Markov processes on partitions which preserve the z-measures. Our main result is the computation of the dynamical correlation functions for these Markov processes. We also compute the scaling limits of the correlation functions corresponding to two different limit regimes as the size of partitions tends to infinity. In the first regime we look at the largest rows and columns of the random Young diagram while in the second one we focus on the boundary of the random Young diagram near the diagonal.

Examples of dynamical models of random matrix type are well known. The sources of dynamics may be very different: in the Gaussian random matrix ensembles one allows the matrix elements to evolve according to the stationary Ornstein–Uhlenbeck process (Dyson’s Brownian motion [Dy]), in tiling models one reads the two–dimensional picture section by section [Jo2], [Jo5], [Jo6], [OkR], in growth models the time parameter is present from the very beginning [PS].

In our setting the construction of dynamics is different; it is based on representation theory. We heavily rely on the fact that the z-measures define characters of the infinite symmetric group and thus possess a special coherence property. It reflects the consistency of restrictions of a character of the infinite symmetric group

1This limit regime has a representation theoretic meaning, see [BO2], [Ol2].
to various finite subgroups. The resulting Markov processes are analogous to those arising in other models, and in degenerations they even coincide with some of those, see [BO7]. It is rather surprising that the similarity of the $z$-measures to measures of different origin extends to dynamics associated with those models.

One of the elements of our construction is a special family of birth and death processes associated with Meixner orthogonal polynomials. Such birth and death processes, among many others, were extensively studied by Karlin–McGregor [KMG1], [KMG2]. Certain degenerations of our Markov processes admit a natural description in the language of Karlin–McGregor, see §7.5 below.

Let us now describe our results in more detail.

Let $Y$ denote the set of all Young diagrams. We consider a family $M_{z, z', \xi}$ of probability measures on $Y$ which depend on two complex parameters $z$ and $z'$ and a real parameter $\xi \in (0, 1)$. The weight of a Young diagram $\lambda$ with respect to $M_{z, z', \xi}$ is given by

$$M_{z, z', \xi}(\lambda) = (1 - \xi)^{zz'} \xi^{|\lambda|} (z)_{\lambda} (z')_{\lambda} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2.$$

Here

$$(z)_{\lambda} = \prod_{(i, j) \in \lambda} (z + j - i)$$

(product over the boxes of $\lambda$) is the generalized Pochhammer symbol, and $\dim \lambda$ is the dimension of the irreducible representation of the symmetric group of degree $|\lambda|$ associated to $\lambda$. In order for $M_{z, z', \xi}(\lambda)$ to be nonnegative for all $\lambda \in Y$, we need to impose certain restrictions on $z$ and $z'$, for instance, $z' = \bar{z}$. All possibilities for $(z, z')$ are given before Proposition 1.2 below.

Let $Y_n$ denote the set of all Young diagrams with $n$ boxes. Restricting $M_{z, z', \xi}$ to $Y_n \subset Y$ and renormalizing it, we obtain a probability measure $M_{z, z', \xi}^{(n)}$ on $Y_n$, which does not depend on $\xi$. The measure $M_{z, z', \xi}$ may be viewed as a mixture of the finite level measures $M_{z, z'}^{(n)}$.

The Markov processes that we are about to construct, are jump processes with countable state space $Y$ and continuous time $t \in \mathbb{R}$. The jumps are of two types: one either adds a box to the random Young diagram, or one removes a box from the diagram.

The event of adding or removing a box is governed by a birth and death process $N_{c, \xi}(t) := |\lambda(t)|$ on $\mathbb{Z}_+$. This process depends on $\xi$ and the product $c = zz'$, and its jump rates are given by

$$\text{Prob}\{N_{c, \xi}(t + dt) = n + 1 \mid N_{c, \xi}(t) = n\} = \frac{\xi(c + n)dt}{1 - \xi},$$

$$\text{Prob}\{N_{c, \xi}(t + dt) = n - 1 \mid N_{c, \xi}(t) = n\} = \frac{n dt}{1 - \xi}.$$ 

This is special case of the birth and death processes considered in [KMG2]. Its invariant distribution, the so-called negative binomial distribution, is the weight function for the Meixner orthogonal polynomials.

Conditioned on the jump $n \to n + 1$, the choice of the box $(i, j)$ to be added to $\lambda$ is made according to the transition probabilities

$$p^\uparrow(n, \lambda; n + 1, \nu) = \frac{(z + j - i)(z' + j - i) \dim \nu}{(zz' + n)(n + 1) \dim \lambda},$$
and conditioned on the jump $n \to n - 1$, the choice of the box $(i, j)$ to be removed from $\lambda$ is made according to the cotransition probabilities

$$p^\downarrow(n, \lambda; n - 1, \mu) = \frac{\dim \mu}{\dim \lambda}.$$  

The transition and cotransition probabilities are naturally associated with finite level measures $M_{z, z'}^{(n)}$. These probabilities were introduced in [VK] in the context of general characters of the infinite symmetric group (see also [Ke2]).

The jump rates $\lambda \not\rightarrow \nu$ and $\lambda \not\leftarrow \mu$ correctly define a stationary Markov process $\Lambda_{z, z', \xi}(t)$ on $Y$. The measure $M_{z, z', \xi}$ is the invariant measure for this process. Moreover, $\Lambda_{z, z', \xi}$ is reversible. In the degenerate case of $z$ or $z'$ being an integer, $\Lambda_{z, z', \xi}$ can be interpreted in terms of finitely many independent birth and death processes subject to a nonintersection condition, see §7.5 below.

One can also construct Markov chains which preserve the finite level measures $M_{z, z'}^{(n)}$. The key idea is that finite level measures are preserved by transition and cotransition probabilities. Thus, adding a random box and removing a random box afterwards leaves $M_{z, z'}^{(n)}$ invariant. Alternatively, one can first remove a box and then add a box. These two procedures yield two different Markov chains. They were suggested by Kerov a long time ago (unpublished). The same idea was independently exploited by Fulman [Fu]. It should be noted that our methods based on determinantal point processes are not directly applicable to such Markov chains. The idea of mixing all finite level measures together is essential for us, it allows us to obtain explicit formulas for dynamical correlation functions, as we explain below.

It is well known that Young diagrams can be viewed as infinite subsets (point configurations) in a one-dimensional lattice. This parametrization of Young diagrams turns out to be very useful.

Let $Z'$ be the lattice of (proper) half–integers

$$Z' = \mathbb{Z} + \frac{1}{2} = \{ \ldots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \}.$$  

For any $\lambda \in Y$ we set

$$\overline{X}(\lambda) = \{ \lambda_i - i + \frac{1}{2} \mid i = 1, 2, \ldots \} \subset Z'.$$  

For instance, for the empty diagram $\lambda = \varnothing$, $\overline{X}(\varnothing) = \{ \ldots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2} \}$. Using the correspondence $\lambda \mapsto \overline{X}(\lambda)$ we interpret the measure $M_{z, z', \xi}$ on $Y$ as a probability measure on $2^{Z'}$. This makes it possible to speak about the dynamical correlation functions of $\Lambda_{z, z', \xi}$ which uniquely determine the process. They are defined by

$$\rho_n(t_1, x_1; t_2, x_2; \ldots; t_n, x_n) = \text{Prob} \{ \overline{X}(\lambda) \text{ at time } t_i \text{ contains } x_i \text{ for } 1 \leq i \leq n \}.$$  

Here $n = 1, 2, \ldots$, and the $n$th correlation function $\rho_n$ is a function of $n$ pairwise distinct arguments $(t_1, x_1), \ldots, (t_n, x_n) \in \mathbb{R} \times Z'$.

The notion of the dynamical correlation functions is a hybrid of the finite-dimensional distributions of a stochastic process and standard correlation functions of probability measures on point configurations.

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2which may be viewed as a passage to the grand canonical ensemble, cf. [Ve]
The reason why we are interested in dynamical correlation functions is the same as in the “static” (fixed time) case: As we take scaling limits of our processes, the notion of weight of a point configuration ceases to make any sense because the space of relevant point configurations becomes uncountable. On the other hand, the scaling limits of the correlation functions do exist, and they carry complete information about the asymptotic behavior of our processes.

**Theorem A (Part 1).** The dynamical correlation functions of \( \Lambda_{z,z',\xi} \) have the determinantal form \((n = 1, 2, \ldots)\)

\[
\rho_n(t_1, x_1; \ldots; x_n, t_n) = \det[K_{z,z',\xi}(t_i, x_i; t_j, x_j)]_{i,j=1}^n,
\]

where the correlation kernel \( K_{z,z',\xi}(s, x; t, y) \) is a function on \((\mathbb{R} \times \mathbb{Z}')^2\) which can be explicitly computed.

One way of writing the kernel is by a double contour integral

\[
K_{z,z',\xi}(s, x; t, y) = e^{\frac{i}{2}(s-t)} \frac{\Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2})(-1)^{x+y+1}}{(\Gamma(-z - x + \frac{1}{2}) \Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2}) \Gamma(-z' - y + \frac{1}{2}))^2} \times \frac{1 - \xi}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} \left(1 - \sqrt{\xi} \omega_1\right)^{-z'} \left(1 - \sqrt{\xi} \omega_1^{-1}\right)^{z} \left(1 - \sqrt{\xi} \omega_2\right)^{z'} \left(1 - \sqrt{\xi} \omega_2^{-1}\right)^{z'} \frac{\omega_1^{-x - \frac{1}{2}} \omega_2^{-y - \frac{1}{2}} d\omega_1 d\omega_2}{e^{s-t} (\omega_1 - \sqrt{\xi}) (\omega_2 - \sqrt{\xi}) - (1 - \sqrt{\xi} \omega_1) (1 - \sqrt{\xi} \omega_2)}
\]

with the contours \(\{\omega_1\}\) and \(\{\omega_2\}\) of \(\omega_1\) and \(\omega_2\) satisfying the following conditions:

- \(\{\omega_1\}\) and \(\{\omega_2\}\) go around 0 in positive direction and pass between \(\sqrt{\xi}\) and \(1/\sqrt{\xi}\);
- The contours are chosen so that the denominator in the formula above does not vanish. There are two possibilities of doing that: one of them is used for \(s \geq t\), and the other one is used for \(s < t\), see Theorem 7.1 below for details.

This integral representation is convenient for computing the scaling limits of the correlation functions. However, it does not reveal important structural features of the kernel. Let us now present another way of writing the correlation kernel.

Consider a second order difference operator \(D\) on \(\mathbb{Z}'\), depending on parameters \((z, z', \xi)\) and acting on functions \(f(\cdot) \in \ell^2(\mathbb{Z}')\) as follows

\[
(Df)(x) = \sqrt{\xi(z + x + \frac{1}{2})} (z' + x + \frac{1}{2}) f(x + 1) + \sqrt{\xi(z + x - \frac{1}{2})} (z' + x - \frac{1}{2}) f(x - 1) - (x + \xi (z + z' + x)) f(x).
\]

This is a self-adjoint operator with discrete simple spectrum \(\text{Sp} \ D = \{(1 - \xi) \mathbb{Z}'\}\).

Its eigenfunctions \(\psi_\alpha\),

\[
D \psi_\alpha = (1 - \xi) \alpha \cdot \psi_\alpha,
\]

are explicitly written through the Gauss hypergeometric function, see (2.1) below. We normalize them by the condition \(\|\psi_\alpha\| = 1\).
Theorem A (Part 2). The correlation kernel for the dynamical correlation functions of the Markov process \( \Lambda_{z,z',\xi} \) can also be written as

\[
K_{z,z',\xi}(s,x,y) = \pm \sum_{a=\pm \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots} e^{-a|s-t|} \psi_{\pm a}(x) \psi_{\pm a}(y)
\]

with “+” taken for \( s \geq t \) and “−” taken for \( s < t \).

The functions \( \{ \psi_a \} \) form an orthonormal basis in \( \ell^2(Z') \). Thus, for \( s = t \) the kernel \( K_{z,z',\xi} \) defines a projection operator whose range is the span of the eigenfunctions of \( D \) corresponding to the positive part of the spectrum of \( D \). In this case (see Comments at the end of §3) the kernel can be written in a simpler, so-called integrable form:

\[
K_{z,z',\xi}(x,y) = \frac{P(x)Q(y) - Q(x)P(y)}{x - y},
\]

where \( P \) and \( Q \) are expressed through the Gauss hypergeometric function.

The formula of Theorem A (Part 2) shows that our Markov process is determined by the following data: a state space \( X \), a Hilbert space \( H \) of functions on \( X \), a self-adjoint operator \( D \) in \( H \), and two complementary spectral projection operators \( P_{\pm} \) for \( D \). In our case, \( X = Z' \), \( H = \ell^2(Z') \), \( D \) is the difference operator given above, and \( P_{\pm} \) are projections on the positive and negative parts of the spectrum of \( D \).

It seems that generating Markov processes with determinantal correlation functions by data \((X, H, D, P_{\pm})\) of this type is a rather general phenomenon. Similar structures have appeared earlier in the dynamics arising in polynuclear growth models [PS], [Jo3], in tiling models [Jo5], [Jo6], and in random matrix theory [NF], [Jo4], [TW]. Following the terminology of those papers, we call the kernel of Theorem A the extended hypergeometric kernel.

The reader might notice that in our Theorem A as well as in all the papers cited above, the values of the extended (dynamical) kernels are always given by somewhat different expressions depending on the relative order of the time variables. This dichotomy is unavoidable because of a certain discontinuity of the dynamical correlation functions. For example, we must have

\[
s \approx t \Rightarrow \rho_2(s, x; t, y) \approx \begin{cases} \rho_2(s, x; s, y), & x \neq y, \\ \rho_1(s, x), & x = y. \end{cases}
\]

If we assume the determinantal structure of the dynamical correlation functions with a kernel \( K(s, x; t, y) \) then

\[
\rho_1(s, x) = K(s, x; s, s), \quad \rho_2(s, x; t, y) = \begin{vmatrix} K(s, x; s, s) & K(s, x; t, y) \\ K(t, y; s, s) & K(t, y; t, y) \end{vmatrix}.
\]

If we further assume that the kernel is continuous in \( s, t \) subject to the condition \( s \geq t \), then the above relations imply

\[
K(s, x; s, y) = \lim_{\epsilon \to 0^+} K(s, x; s - \epsilon, y) = \lim_{\epsilon \to 0^+} K(s, x; s + \epsilon, y) + \delta_{xy}.
\]

3We could have used \( s \leq t \) equally well, this is a question of convention. For instance, transposition of the kernel does not affect the correlation functions, and this operation turns \( s \geq t \) into \( s \leq t \).
The validity of the last relation for the extended hypergeometric kernel can be immediately observed from Part 2 of Theorem A using the fact that \{ψ₁\} form an orthonormal basis.

Let us now describe our results on scaling limits of the dynamical correlation functions. In our previous works we considered three asymptotic regimes for random Young diagrams without dynamics: one for largest rows and columns, one for rows and columns of intermediate growth, and one for the behavior of the boundary of the Young diagrams near the diagonal, see [BO5] and references therein. In all three limit regimes the parameter ξ tends to 1, which makes the expected number of boxes in the random Young diagram go to infinity.

In this paper we concentrate on the first and the third limit regime, but with the presence of dynamics. Let us start with the behavior of largest rows and columns.

In order to catch the largest rows and columns in the limit \(\xi \to 1\), we need to scale them by \((1 - \xi)\). This leads to scaling of the state space \(Z'\) by the same factor. That is, \(Z'\) is replaced by \((1 - \xi)Z'\) which in the limit turns into \(\mathbb{R}^*\).

The parametrization of Young diagrams by point configurations \(X(\lambda)\) is not suitable for this limit transition. Or, rather to say, the positive part of \(X(\lambda)\) indeed reflects the behavior of largest rows, while the behavior of the largest columns is captured by the complement of the negative part of \(X(\lambda)\) in \(\{\ldots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}\}\). Thus, instead of encoding λ by \(X(\lambda)\) we use the map

\[
\lambda \mapsto X(\lambda) = \left(\frac{X(\lambda)}{2} \cap \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}\right) \cup \left(\{\ldots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}\}\setminus X(\lambda)\right).
\]

We refer the reader to [BO2], [Ol2] for representation theoretic interpretation of this map and for further details.

**Theorem B.** The scaling limits, as \(\xi \to 1\), of the dynamical correlation functions of \(X_{s,z',\xi}\) corresponding to the map \(\lambda \mapsto X(\lambda)\), under the rescaling of \(Z'\) by \((1 - \xi)\), have determinantal form with the correlation kernel \(K_\infty(s,u; t,v)\) on \((\mathbb{R} \times \mathbb{R}^*)^2\).

This kernel has four blocks according to the choices of signs of \(u\) and \(v\).

The block with \(u, v > 0\) has an integral representation

\[
K_\infty^W(s,u; t,v) = e^{\pi i (z_1 + z_2)} (u/v)\frac{dz_1 dz_2}{2\pi i} e^{\frac{1}{2}(s-t)}
\]

\[
\times \int_{+\infty}^{0-} \int_{+\infty}^{0-} \int_{+\infty}^{0-} \int_{+\infty}^{0-} \zeta_1^z (1 + \zeta_1) \zeta_2^z (1 + \zeta_2) \frac{e^{-u(\zeta_1 + \frac{1}{2}) - v(\zeta_2 + \frac{1}{2})} d\zeta_1 d\zeta_2}{e^{s-t}(1 + \zeta_1)(1 + \zeta_2) - \zeta_1 \zeta_2}
\]

with different choices of contours for \(s \geq t\) and \(s < t\), see Theorem 9.4 below.

The same block has a series representation

\[
K_\infty^W(s,u; t,v) = \pm \sum_{a=\frac{1}{2}, \frac{1}{2}, \ldots} e^{-a|s-t|} w_{\pm a}(u) w_{\pm a}(v),
\]

where “+” is taken for \(s \geq t\), “−” is taken for \(s < t\), and

\[
w_a(u) = \lim_{\xi \to 1} (1 - \xi)^{\frac{1}{2}} \psi_a([(1 - \xi)^{-1}u])
\]

are eigenfunctions of a second order differential operator on \(\mathbb{R}_+:\)

\[
w''_a(u) + w'_a(u) + \left(-\frac{u}{4} + \frac{z+z'}{2} - \frac{(z-z')^2}{4a}\right) w_a(u) = aw_a(u),
\]
which are explicitly written through the Whittaker functions, see (9.1) below.

Similar expressions are available for three other blocks of \( K_{z,z'}^W(s,u,t,v) \), see Theorems 9.2 and 9.4 below.

We call \( K_{z,z'}^W(s,u,t,v) \) the extended Whittaker kernel.

In the “static” case \( s = t \) the kernel admits a simpler “integrable” form, see [BO1], [BO2], [B1], [O12], and (9.4) below.

Let us now proceed to the other limit regime which describes the behavior of the Young diagrams near the diagonal. This just means that we stay on the lattice \( \mathbb{Z}' \).

For this asymptotic regime it does not really matter whether we use Young diagrams near the diagonal. This just means that we stay on the lattice \( \mathbb{Z}' \).

The symmetry property

\[
K_{z,z'}^\text{gamma}(\sigma,x;\tau,y) = K_{z',z}^\text{gamma}(-1)_{\tau,x;\sigma,y},
\]

where \( \sigma < \tau \) and as a single integral

\[
K_{z,z'}^\text{gamma}(\sigma,x;\tau,y) = \int_0^{+\infty} e^{-u(\sigma-\tau)} w_x(u;-z,-z') w_y(u;-z,-z') du.
\]

The values of the kernel for \( \sigma < \tau \) are obtained from the above formulas using the symmetry property

\[
K_{z,z'}^\text{gamma}(\sigma,x;\tau,y) = (-1)^{x+y} K_{z,-z'}^\text{gamma}(\tau,-x;\sigma,-y), \quad \sigma \neq \tau.
\]

For \( \sigma = \tau \) the kernel admits a simpler expression of “integrable” type

\[
P(x)Q(y) - Q(x)P(y)
\]

\[
\frac{1}{x - y}
\]

where \( P \) and \( Q \) are are expressed through gamma functions only, see [BO5] and (10.3) below. That kernel was called the gamma kernel, and for this reason we call \( K_{z,z'}^\text{gamma}(\sigma,x;\tau,y) \) the extended gamma kernel.

Note that the extended gamma kernel fits into the same abstract scheme as the extended hypergeometric kernel: one takes \( \mathbf{X} = \mathbb{Z}' \), \( H = \ell^2(\mathbb{Z}') \), \( D \) is the special case of the difference operator given above corresponding to the limit value
\[ \xi = 1. \] The spectrum of this operator fills the whole real axis, the eigenfunctions are \( x \mapsto w_x(u; -z, -z') \), and the spectral projections \( P_{\pm} \) again correspond to the positive and negative parts of the spectrum.

The functions \( \psi_a(x) = \psi_a(x; z, z', \xi) \) used in the discussion of the extended hypergeometric kernel have the following symmetry:

\[ \psi_a(x; z, z', \xi) = \psi_x(a; -z, -z', \xi). \]

This means, in particular, that \( \psi_a(x) \) satisfies second order difference equations both in \( a \) and \( x \) (the bispectrality property, see [Gr]). The two limit transitions considered above (Theorems B and C) correspond to taking continuous limits in \( x \) and \( a \), respectively. This explains why we end up with the same functions \( w_a(u) \) in Theorems B and C.

Let us make some remarks about our proof of Theorem A. As a matter of fact, we prove the theorem in a greater generality. We introduce certain time inhomogeneous Markov processes on partitions. Their fixed time distributions are also the measures \( M_{z, z', \xi} \), but now \( \xi = \xi(t) \) varies with time \( t \). The construction of these processes is similar to the stationary ones except that the birth and death process on \( \mathbb{Z}_+ \) becomes time inhomogeneous. In particular, we consider pure birth and pure death processes for which the Young diagrams either always gain new boxes or always lose their boxes. These “pure” processes are simpler, their transition probabilities can be evaluated explicitly. They can also be viewed as building blocks of general processes, more exactly, the transition matrix \( P(s, t) \) for a general process can be represented as a product \( P(s, t) = P^k(s, u)P^l(u, t) \) of transition matrices of “pure” processes for a suitable intermediate time moment \( u \in (s, t) \).

This product representation of the transition matrix plays an important role in the proof of Theorem A. We first prove the theorem for a degenerate case, when one of the parameters \( z, z' \) is an integer, and the process is “finite-dimensional”, that is, it lives on the Young diagrams with bounded number of rows or columns. Then the needed formulas are derived from a version of Eynard-Mehta theorem on spectral correlations of coupled random matrices [EM].\(^4\) The passage from the degenerate case to the general one is based on analytic continuation in the parameters \( z \) and \( z' \). This passage is not trivial since we need to extrapolate from the integer points to a complex domain. The needed analytic properties of the dynamical correlation functions are derived from the product formula for the transition matrix \( P(s, t) \) mentioned above. Let us also emphasize that in our approach, the introduction of time inhomogeneous processes is necessary for handling the stationary case.

There is one more subtle issue that we would like to mention here. Generally speaking, even for birth and death processes, jump rates do not determine the transition matrix uniquely, see e.g., [Fe2, ch. XVII, §10]. Since we want to define our processes by their jump rates, we need to ensure the uniqueness. We were unable to find suitable results in the literature and, therefore, we were forced to invent a special sufficiency condition which was suitable for our purposes, see §4.

Let us point out that there exists another way of obtaining the dynamical correlation functions of Theorem A, based on the formalism of infinite wedge Fock space. In [Ok2] Okounkov gave an elegant derivation of static \( (s = t) \) correlation functions (initially computed in [BO2]) using a representation of \( SL(2) \) by the so-called Kerov

\(^4\)Other proofs of this theorem can be found in [NF], [Jo3], [TW], [BR].
operators. We can extend Okounkov’s approach to derive the formula of Theorem A. This alternative path bears some similarity to the formalism of Schur processes of [OkR], [Ok3]. However, the Schur processes seem to be not applicable in our situation. Note also that despite the beauty of Okounkov’s idea, a rigorous realization of this approach would have to overcome certain nontrivial technical difficulties.

One more important subject that we do not touch upon in this paper, is a family of Markov processes on partitions related to Plancherel measures. In the limit $z, z' \to \infty$, $\xi \to 0$, $zz'\xi \to \theta > 0$, the measures $M_{z,z',\xi}$ tend to the so-called poissonized Plancherel measure on $\mathcal{Y}$ with Poisson parameter $\theta$. This connection was used in [BOO] to study the asymptotics of the Plancherel measures. Using the general scheme presented in this paper, one constructs Markov processes on $\mathcal{Y}$ which preserve the poissonized Plancherel measures. These processes may be viewed as degenerations of the processes considered in this paper. They are equivalent to the droplet model of polynuclear growth. Our results on this other family of Markov processes and their scaling limits are presented in [BO7]. Let us note that the analog of Theorem A for those processes can be obtained either by limit transition from Theorem A or by using the Schur process of [OkR].

The present paper is organized as follows. In Section 1 we introduce the z-measures, the associated transition and cotransition probabilities, and other notions related to the Young graph. In Section 2 we study the eigenfunctions $\psi_a$ of the second order difference operator $D$ on $\mathbb{Z}'$. In Section 3 we prove the static variant of Theorem A using the method of analytic continuation and reduction to the degenerate case of integral parameters. In Section 4 we introduce time homogeneous and inhomogeneous Markov processes on $\mathcal{Y}$, prove their existence and uniqueness, and compute the transition probabilities for “pure” ascending and descending processes. In Section 5 we evaluate the transition matrices for integral values of parameters. In Section 6 we study the analytic nature of the dependence of the dynamical correlation functions on the parameters. In Section 7 we prove Theorem A first in the degenerate case using Eynard–Mehta theorem and Meixner polynomials, and then in the general case using analytic continuation. In Section 8 we derive the dynamical correlation functions of $\Lambda_{z,z',\xi}$ corresponding to the map $\lambda \mapsto X(\lambda)$ (as opposed to the map $\lambda \mapsto X^0(\lambda)$ used in Theorem A). In Section 9 we prove Theorem B, and in Section 10 we prove Theorem C.

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1. Z-measures

As in Macdonald [Ma] we identify partitions and Young diagrams. By $\mathcal{Y}_n$ we denote the set of partitions of a natural number $n$, or equivalently, the set of Young diagrams with $n$ boxes. By $\mathcal{Y}$ we denote the set of all Young diagrams, that is, the disjoint union of the finite sets $\mathcal{Y}_n$, where $n = 0, 1, 2, \ldots$ (by convention, $\mathcal{Y}_0$ consists of a single element, the empty diagram $\varnothing$). Given $\lambda \in \mathcal{Y}$, let $|\lambda|$ denote the number of boxes of $\lambda$ (so that $\lambda \in \mathcal{Y}_{|\lambda|}$), let $\ell(\lambda)$ be the number of nonzero rows in $\lambda$ (the length of the partition), and let $\lambda'$ denote the transposed diagram.

For two Young diagrams $\lambda$ and $\mu$ we write $\mu \not\supset \lambda$ (equivalently, $\lambda \not\subset \mu$) if $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$, or, in other words, $\mu$ is obtained from $\lambda$ by removing one box.
The Young graph is the graph whose vertices are the elements of \( Y \) and the edges join all pairs \((\mu, \lambda)\) such that \( \mu \not< \lambda \). The Young graph will also be denoted by \( Y \).

Clearly, \( \mu \not< \lambda \) implies \( \mu' \not< \lambda' \), so that the transposition operation \( \lambda \mapsto \lambda' \) induces an involutive automorphism of the Young graph.

For any \( \lambda \in Y_n \), standard Young tableaux of shape \( \lambda \) can be viewed as paths

\[
\varnothing \xrightarrow{(1)} \lambda^{(1)} \xrightarrow{(2)} \cdots \xrightarrow{(n)} \lambda^{(n)} = \lambda
\]

in \( Y \). Let \( \dim \lambda \) be the number of all such paths. A convenient explicit formula for \( \dim \lambda \) is

\[
\dim \lambda = \frac{n!}{\prod_{i=1}^{N} (\lambda_i + N - i)!} \prod_{1 \leq i < j \leq N} (\lambda_i - i - \lambda_j + j), \quad \lambda \in Y_n,
\]

where \( N \) is an arbitrary integer \( \geq \ell(\lambda) \) (the above expression is stable in \( N \)).

For \( \lambda \in Y_n, \mu \in Y_{n-1} \) set

\[
p^{\downarrow}(n, \lambda; n-1, \mu) = \begin{cases} \frac{\dim \mu}{\dim \lambda}, & \mu \not< \lambda, \\ 0, & \text{otherwise} \end{cases} \tag{1.1}
\]

and note that

\[
\sum_{\mu \in Y_{n-1}} p^{\downarrow}(n, \lambda; n-1, \mu) = 1.
\]

The numbers \( p^{\downarrow}(n, \lambda; n-1, \mu) \) are called the cotransition probabilities of the Young graph.

A family \( \{M^{(n)}\} \) of probability measures \( M^{(n)} \) on \( Y_n, n = 0, 1, 2, \ldots \), is called a coherent system on \( Y \) if the measures are consistent with the cotransition probabilities in the following sense:

\[
M^{(n-1)}(\mu) = \sum_{\lambda \in Y_n} M^{(n)}(\lambda)p^{\downarrow}(n, \lambda; n-1, \mu), \quad \mu \in Y_{n-1}, \quad n = 1, 2, \ldots \tag{1.2}
\]

This concept has an important representation theoretic meaning. Namely, there is a 1–1 correspondence between coherent systems on \( Y \) and normalized positive definite class functions on the infinite symmetric group, see [VK], [Ke2], [Ol2].

Note that the cotransition probabilities are invariant under the involution \( \lambda \mapsto \lambda' \) of the Young graph. Consequently, the push–forward of a coherent system under this involution is again a coherent system.

Example 1.1. The Plancherel measures defined by

\[
M^{(n)}_{\text{Plancherel}}(\lambda) = \frac{(\dim \lambda)^2}{n!}
\]

form a coherent family of probability measures, see [VK].
This is a compact topological space (a closed subset of the product space $\prod_{n \geq 0} \mathbb{Y}_n$).

A probability measure $\mathcal{M}$ on $\mathcal{T}$ is called central if for any $n = 1, 2, \ldots$ and any $\lambda \in \mathbb{Y}_n$, the mass of each cylinder set consisting of all paths with fixed $\lambda^{(1)}, \ldots, \lambda^{(n)} = \lambda$ depends on $\lambda$ only (and does not depend on $\lambda^{(1)}, \ldots, \lambda^{(n-1)}$).

Any coherent system $\{M^{(n)}\}$ generates a central measure $\mathcal{M}$ on $\mathcal{T}$. By definition, the mass of the cylinder set mentioned above equals $M^{(n)}(\lambda)/\dim \lambda$. The relation (1.2) ensures that $\mathcal{M}$ is correctly defined. This defines a one-to-one correspondence between coherent systems $\{M^{(n)}\}$ on $\mathcal{Y}$ and central measure $\mathcal{M}$ on $\mathcal{T}$, see [VK], [Ke2], [Ol2].

For any central measure $\mathcal{M}$,

$$p^\uparrow(n, \lambda; n - 1, \mu) = \text{Prob}\{\lambda^{(n-1)} = \mu \mid \lambda^{(n)} = \lambda\},$$

which is a justification of the term “cotransition probability”.

Assuming $M^{(1)}(\lambda) > 0$ for all $\lambda \in \mathcal{Y}$, set

$$p^\uparrow(n, \lambda; n + 1, \nu) = \text{Prob}\{\lambda^{(n+1)} = \nu \mid \lambda^{(n)} = \lambda\}, \quad n = |\lambda|,$$

In contrast to $p^\downarrow(n, \lambda; n - 1, \mu)$, these numbers depend on $\mathcal{M}$. We call them the transition probabilities of the central measure $\mathcal{M}$ (or of the corresponding coherent system $\{M^{(n)}\}$). The transition probabilities define $\mathcal{M}$ and $\{M^{(n)}\}$ uniquely.

Note an important relation between the transition and cotransition probabilities:

$$M^{(n)}(\lambda)p^\uparrow(n, \lambda; n + 1, \nu) = p^\downarrow(n + 1, \nu; n, \lambda)M^{(n+1)}(\nu). \quad (1.3)$$

It implies, in particular, that

$$p^\uparrow(n, \lambda; n + 1, \nu) = \begin{cases} \frac{M^{(n+1)}(\nu) \dim \lambda}{M^{(n)}(\lambda) \dim \nu}, & \lambda \not\subseteq \nu, \\ 0, & \text{otherwise}. \end{cases} \quad (1.4)$$

If $M^{(1)}(\lambda)$ vanishes for some $\lambda \in \mathcal{Y}$ then the definition has to be slightly modified. Namely, let $\text{supp} \mathcal{M}$ be the set of those $\lambda \in \mathcal{Y}$ for which $M^{(1)}(\lambda) > 0$. Equivalently, $\lambda \in \text{supp} \mathcal{M}$ if the set of paths passing through $\lambda$ has positive mass with respect to $\mathcal{M}$. Note that $\lambda \in \text{supp} \mathcal{M}$ implies $\mu \in \text{supp} \mathcal{M}$ for all $\mu \not\subseteq \lambda$.

The set $\text{supp} \mathcal{M}$ spans a subgraph of $\mathcal{Y}$ (which may be called the support of $\mathcal{M}$), and the transition probabilities are correctly defined on this subgraph by the same formula (1.3). Again, the initial central measure $\mathcal{M}$ is uniquely determined by its support and the transition probabilities.

Note two useful equations

$$M^{(n-1)}(\mu) = \sum_{\lambda} M^{(n)}(\lambda)p^\downarrow(n, \lambda; n - 1, \mu), \quad (1.5)$$

$$M^{(n+1)}(\nu) = \sum_{\lambda} M^{(n)}(\lambda)p^\uparrow(n, \lambda; n + 1, \nu). \quad (1.6)$$

We shall need the generalized Pochhammer symbol $(z)_\lambda$:

$$(z)_\lambda = \prod_{i=1}^{\ell(\lambda)} (z - i + 1)_{\lambda_i}, \quad z \in \mathbb{C}, \quad \lambda \in \mathcal{Y},$$
where
\[(x)_k = x(x+1)\ldots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}\]
is the conventional Pochhammer symbol. Note that
\[(z)_\lambda = \prod_{(i,j) \in \lambda} (z+j-i)\]
(product over the boxes of \(\lambda\)), which implies at once the symmetry relation
\[(z)_\lambda = (-1)^{|\lambda|}(-z)_{\lambda'}.

For two complex parameters \(z, z'\) set
\[M^{(n)}_{z,z'}(\lambda) = \frac{(z)_\lambda(z')^{\lambda}(\dim \lambda)^2}{(zz')_n n!}, \quad n = 0, 1, \ldots, \quad \lambda \in \mathbb{Y}_n,\]
where \(\dim \lambda\) was defined in the beginning of the section. The expression (1.7) makes sense if \((zz')_n\) does not vanish, i.e., if \(zz' \notin \{0, -1, -2, \ldots\}\). Obviously, (1.7) is symmetric in \(z, z'\).

Note that (see Example 1.1)
\[\lim_{z, z' \to \infty} M^{(n)}_{z,z'}(\lambda) = M^{(n)}_{\text{Plancherel}}(\lambda).\]

Let us say that two nonzero complex numbers \(z, z'\) form an admissible pair of parameters if one of the following three conditions holds:

- The numbers \(z, z'\) are not real and are conjugate to each other.
- Both \(z, z'\) are real and are contained in the same open interval of the form \((m, m+1)\), where \(m \in \mathbb{Z}\).
- One of the numbers \(z, z'\) (say, \(z\)) is a nonzero integer while \(z'\) has the same sign and, moreover, \(|z'| > |z| - 1\).

**Proposition 1.2.** If \((z, z')\) is an admissible pair of parameters then \(\{M^{(n)}_{z,z'}\}\) is a coherent family of probability measures.

**Proof.** It is readily checked that if (and only if) one of the conditions above holds then \((z)_\lambda(z')_\lambda \geq 0\) for all \(\lambda\), see [BO5, Proposition 1.8]. Moreover, \((zz')_n > 0\) for all \(n\). Hence (1.7) is nonnegative. The fact that each \(M^{(n)}_{z,z'}\) is a probability measure and the coherency property can be proved in several ways. See, e.g., [Ol1], [BO3]. □

We call the measures \(M^{(n)}_{z,z'}\) the \(z\)-measures on the floors \(\mathbb{Y}_n\) of the Young graph. Depending on which of the three conditions of Proposition 1.1 holds we will speak about the principal, complementary or degenerate series of \(z\)-measures, respectively.

By virtue of (1.8), the \(z\)-measures may be viewed as a deformation of the Plancherel measure (for any fixed \(n\)). The principal series of \(z\)-measures first appeared in [KOV1], see also [KOV2]. For more information about the \(z\)-measures and their generalizations, see [BO2], [BO3], [BO4], [BO5], [BO6], [Ke1].

Note that the involution \(\lambda \mapsto \lambda'\) of the Young graph takes \(M^{(n)}_{z,z'}\) to \(M^{(n)}_{-z,-z'}\).
Let $M_{z,z'}$ be the central measure corresponding to the coherent family $\{M_{z,z'}^{(n)}\}_{n=0,1,\ldots}$. In the case of the principal or complementary series the support of $M_{z,z'}$ is the whole $\mathbb{Y}$. For the degenerate series it is a proper subset of $\mathbb{Y}$: if $z = k = 1, 2, \ldots$ and $z' > k - 1$ then $\text{supp} M_{z,z'}$ consists of diagrams with at most $k$ rows, and if $z = -k = -1, -2, \ldots$ and $z' < -(k - 1)$ then $\text{supp} M_{z,z'}$ consists of diagrams with at most $k$ columns.

The transition probabilities of the $z$–measures are given by

$$p_{z,z'}^+(n, \lambda; n + 1, \nu) = \frac{(z + c(\nu/\lambda))(z' + c(\nu/\lambda)) \dim \nu}{(zz' + n)(n + 1) \dim \lambda}, \quad \lambda \not\supset \nu,$$

where $c(\nu/\lambda)$ denotes the content of the box $(i, j) = \nu/\lambda$, that is, $c = j - i$. Indeed, (1.9) follows immediately from (1.4) and (1.7). Note that if $\lambda$ is in $\text{supp} M_{z,z'}$ while $\nu$ is not (which may happen for the degenerate series) then (1.9) vanishes due to vanishing of one of the factors $z + c(\nu/\lambda)$, $z' + c(\nu/\lambda)$.

For the Plancherel measure, the transition probabilities are

$$p_{\text{Plancherel}}^+(n, \lambda; n + 1, \nu) = \frac{\dim \nu}{(n + 1) \dim \lambda}, \quad \lambda \not\supset \nu,$$

see [VK].

Consider a special case of the negative binomial distribution on $\mathbb{Z}_+$ depending on two parameters $a > 0$ and $\xi \in (0, 1)$:

$$\pi_{a,\xi}(n) = (1 - \xi)^a \frac{(a)n^\xi}{n!}, \quad n = 0, 1, 2, \ldots$$

The next formula defines a probability measure on $\mathbb{Y}$ which is the mixture of all $z$–measures $M_{z,z'}^{(n)}$ with given fixed parameters $z, z'$ and varying $n$ by means of the distribution (1.10) on $n$’s, with parameters $a = zz'$ and $\xi$:

$$M_{z,z',\xi}(\lambda) = M_{z,z'}^{(|\lambda|)}(\lambda) \pi_{z,z',\xi}(|\lambda|) = (1 - \xi)^{zz'} \xi^{|\lambda|} (z)_\lambda (z')_{\lambda} \left(\frac{\dim \lambda}{|\lambda|!}\right)^2. \quad (1.11)$$

We call (1.11) the mixed $z$–measure. An interpretation of formula (1.11) is given in [BO5, Definition 1.4].

Likewise, consider a mixture of the Plancherel measures, depending on a parameter $\theta > 0$:

$$M_{\text{Plancherel}, \theta}(\lambda) = M_{\text{Plancherel}}^{(\lambda)}(\lambda) e^{-\theta |\lambda|} = e^{-\theta |\lambda|} \left(\frac{\dim \lambda}{|\lambda|!}\right)^2. \quad (1.12)$$

We call (1.12) the poissonized Plancherel measure. Note that it can be obtained as a limit case of the mixed $z$–measures:

$$\lim_{\substack{z, z' \to \infty \\
\xi \to 0}} M_{z,z',\xi}(\lambda) = M_{\text{Plancherel}, \theta}(\lambda).$$

The main objects of this paper are the $z$–measures and related Markov processes. One can also develop a parallel theory associated with the Plancherel measure. We do not pursue this goal in the present paper. An interested reader can found the statements of the main results related to the Plancherel measure in our paper [BO7].
2. **A basis in the $\ell^2$ space on the lattice and the Meixner polynomials**

In this section we examine a nice orthonormal basis in the $\ell^2$ space on the 1–dimensional lattice. The elements of this basis are eigenfunctions of a second order difference operator. They can be obtained from the classical Meixner polynomials via analytic continuation with respect to parameters.

Throughout the section we will assume (unless otherwise stated) that $(z, z')$ is in the principal series or in the complementary series but not in the degenerate series. In particular, $z, z'$ are not integers.

Consider the lattice of (proper) half–integers

$$Z' = \mathbb{Z} + \frac{1}{2} = \{ \ldots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \}.$$  

We introduce a family of functions on $Z'$ depending on a parameter $a \in Z'$ and also on our parameters $z, z', \xi$:

$$\psi_a(x; z, z', \xi) = \frac{\Gamma(x + z + \frac{1}{2}) \Gamma(x + z' + \frac{1}{2})}{\Gamma(z' - a + \frac{1}{2})} \xi^{\frac{1}{2}(x + a)} (1 - \xi)^{\frac{1}{2}(z + z') - a} \times \frac{F(-z + a + \frac{1}{2}, -z' + a + \frac{1}{2}; x + a + 1; \frac{\xi}{\xi - 1})}{\Gamma(x + a + 1)},$$  

where $F(A, B; C; w)$ is the Gauss hypergeometric function.

Let us explain why this expression makes sense. Since, by convention, parameters $z, z'$ do not take integral values, $\Gamma(x + z + \frac{1}{2})$ and $\Gamma(x + z' + \frac{1}{2})$ have no singularities for $x \in Z'$. Moreover, the admissibility assumptions on $(z, z')$ (see §1) imply that

$$\Gamma(x + z + \frac{1}{2}) \Gamma(x + z' + \frac{1}{2}) > 0, \quad \Gamma(z' - a + \frac{1}{2}) \Gamma(z' - a + \frac{1}{2}) > 0,$$

so that we can take the positive value of the square root in (2.1). Next, since $\xi \in (0, 1)$, we have $\xi/(\xi - 1) < 0$, and as is well known, the function $w \to F(A, B; C, w)$ is well defined on the negative semi–axis $w < 0$. Finally, although $F(A, B; C, w)$ is not defined at $C = 0, -1, -2, \ldots$, the ratio $F(A, B; C, w)/\Gamma(C)$ is well defined for all $C \in \mathbb{C}$.

Note also that the functions $\psi_a(x; z, z', \xi)$ are real–valued. Their origin will be explained below.

Further, we introduce a second order difference operator $D(z, z', \xi)$ on the lattice $Z'$, depending on parameters $z, z', \xi$ and acting on functions $f(x)$ (where $x$ ranges over $Z'$) as follows

$$D(z, z', \xi)f(x) = \sqrt{\xi(z + x + \frac{1}{2})(z' + x + \frac{1}{2})} f(x + 1)$$

$$+ \sqrt{\xi(z + x - \frac{1}{2})(z' + x - \frac{1}{2})} f(x - 1) - (x + \xi(z + z' + x)) f(x).$$

Note that $D(z, z', \xi)$ is a symmetric operator in $\ell^2(Z')$.

**Proposition 2.1.** The functions $\psi_a(x; z, z', \xi)$, where $a$ ranges over $Z'$, are eigenfunctions of the operator $D(z, z', \xi)$,

$$D(z, z', \xi)\psi_a(x; z, z', \xi) = a(1 - \xi)\psi_a(x; z, z', \xi).$$  

(2.2)
Proof. This equation can be verified using the relation
\[ w(C - A)(C - B)F(A, B; C + 1; w) - (1 - w)C(C - 1)F(A, B; C - 1; w) + C(C - 1 - (2C - A - B - 1)w)F(A, B; C; w) = 0 \]
for the Gauss hypergeometric function, see, e.g., [Er, 2.8 (45)]. □

The next lemma provides us a convenient integral representation for functions \( \psi_a \).

**Lemma 2.2.** For any \( A, B \in \mathbb{C}, M \in \mathbb{Z}, \) and \( \xi \in (0, 1) \) we have
\[
F(A, B; M + 1; \frac{\xi}{\xi - 1}) = \frac{\Gamma(-A + 1)\xi^{-M/2}(1 - \xi)^B}{\Gamma(-A + M + 1)} \times \frac{1}{2\pi i} \int_{\{\omega\}} (1 - \sqrt{\xi} \omega)^{A - 1} \left(1 - \frac{\sqrt{\xi}}{\omega}\right)^{-B} \omega^{-M} \frac{d\omega}{\omega}.
\]
(2.3)

Here \( \xi \in (0, 1) \) and \( \{\omega\} \) is an arbitrary simple contour which goes around the points 0 and \( \sqrt{\xi} \) in the positive direction leaving \( 1/\sqrt{\xi} \) outside.

**Comments.**

1. The branch of the function \( (1 - \sqrt{\xi} \omega)^{A - 1} \) is specified by the convention that the argument of \( 1 - \sqrt{\xi} \omega \) equals 0 for real negative values of \( \omega \), and the same convention is used for the function \( (1 - \sqrt{\xi} \omega)^{-B} \).

2. Like the Euler integral formula, formula (2.3) does not make evident the symmetry \( A \leftrightarrow B \).

3. The right–hand side of formula (2.3) makes sense for \( A = 1, 2, \ldots \), when \( \Gamma(-A + 1) \) has a singularity. Then the whole expression can be understood, e.g., as the limit value as \( A \) approaches one of the points 1, 2, \ldots.

**Proof.** Since both sides of (2.3) are real–analytic functions of \( \xi \) we may assume that \( \xi \) is small enough. Then we may apply the binomial formula which gives
\[
\xi^{-M/2}(1 - \sqrt{\xi} \omega)^{A - 1} \left(1 - \frac{\sqrt{\xi}}{\omega}\right)^{-B} \omega^{-M} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-A + 1)_k (B)_l}{k! l!} \xi^{(k+l-M)/2} \omega^{k-l-M}.
\]
After integration only the terms with \( k = l + M \) survive. It follows that the right–hand side of (2.3) is equal to
\[
\frac{(1 - \xi)^B}{\Gamma(-A + M + 1)} \sum_{l \geq \max(0, -M)} \frac{\Gamma(-A + M + 1 + l) (B)_l}{\Gamma(l + M + 1)!} \xi^l.
\]
We may replace the inequality \( l \geq \max(0, -M) \) simply by \( l \geq 0 \) because for negative integral values of \( M \) (when we have to start summation from \( l = -M \)), the terms with \( l = 0, \ldots, -M - 1 \) automatically vanish due to the factor \( \Gamma(l + M + 1) \) in the denominator. Consequently, our expression is equal to
\[
\frac{(1 - \xi)^B F(-A + 1 + M, B; M + 1; \xi)}{\Gamma(M + 1)} = \frac{F(A, B; M + 1; \xi \frac{\xi}{\xi - 1})}{\Gamma(M + 1)},
\]
where we used [Er, 2.9 (4)]. □
Proposition 2.3. We have the following integral representations
\[
\psi_a(x; z, z', \xi) = \frac{1}{2\pi i} \oint_{\{\omega\}} \left(1 - \sqrt{\xi \omega}\right)^{-z+a-\frac{1}{2}} \left(1 - \sqrt{\xi \omega}^{-1}\right)^{z-a-\frac{1}{2}} \omega^{-x-a} d\omega
\]
and
\[
\psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi) = \varphi_{z, z'}(x, y)
\]
where
\[
\varphi_{z, z'}(x, y) = \sqrt{\frac{\Gamma(x + z + \frac{1}{2}) \Gamma(x + z' + \frac{1}{2}) \Gamma(y + z + \frac{1}{2}) \Gamma(y + z' + \frac{1}{2})}{\Gamma(x + z' + \frac{1}{2}) \Gamma(y + z + \frac{1}{2})}}
\]
Here each contour is an arbitrary simple loop, oriented in positive direction, surrounding the points 0 and \(\sqrt{\xi}\), and leaving \(1/\sqrt{\xi}\) outside. We also use the convention about the choice of argument as in Comment 1 to Lemma 2.2.

Proof. Indeed, (2.4) immediately follows from (2.1) and (2.3). To prove (2.5) we multiply out the integral representation (2.4) for the first function and the same representation for the second function, but with \(z\) and \(z'\) interchanged. The transposition \(z \leftrightarrow z'\) in (2.4) is justified by the fact the initial formula (2.1) is symmetric with respect to \(z \leftrightarrow z'\). As a result of this trick the gamma prefactors involving \(a\) are completely cancelled out, and we obtain (2.5) \(\square\)

Proposition 2.4. The functions \(\psi_a = \psi_a(x; z, z', \xi)\), where \(a\) ranges over \(\mathbb{Z}'\), form an orthonormal basis in the Hilbert space \(l^2(\mathbb{Z}')\).

Proof. From (2.4) it is not difficult to see that the function \(\psi_a(x; z, z', \xi)\) has exponential decay as \(x \to \pm\infty\). In particular, it is square integrable. Since \(\psi_a\) is an eigenfunction of a symmetric difference operator whose coefficients have linear growth at \(\pm\infty\), and since to different indices \(a\) correspond different eigenvalues, we conclude that these functions are pairwise orthogonal in \(l^2(\mathbb{Z}')\).

Let us show that \(|\psi_a|^2 = 1\). Write (2.5), where we set \(x = y\). Then the whole expression simplifies because (2.6) becomes equal to 1. Next, in the double contour integral, we replace the variable \(\omega_2\) by its inverse. We obtain
\[
(\psi_a(x; z, z', \xi))^2 = \frac{1}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} \left(1 - \sqrt{\xi \omega_1}\right)^{-z+a-\frac{1}{2}} \left(1 - \sqrt{\xi \omega_1^{-1}}\right)^{z-a-\frac{1}{2}} \omega_1^{-x-a} d\omega_1 d\omega_2 \times \left(1 - \sqrt{\xi \omega_2^{-1}}\right)^{-z+a-\frac{1}{2}} \left(1 - \sqrt{\xi \omega_2}\right)^{z-a-\frac{1}{2}} \frac{\omega_1}{\omega_2}^{-x-a} \frac{\omega_1}{\omega_2} \frac{d\omega_1}{d\omega_2} \frac{d\omega_2}{d\omega_1}
\]
To evaluate the squared norm we have to sum this expression over \( x \in Z' \). We split the sum into two parts according to the splitting \( Z' = Z'_{-} \cup Z'_{+} \). We take as the contours concentric circles such that \( |\omega_{1}| < |\omega_{2}| \) in the sum over \( Z'_{-} \), and \( |\omega_{1}| > |\omega_{2}| \) in the sum over \( Z'_{+} \). This gives us

\[
\sum_{x \in Z'} (\psi_{a}(x; z, z', \xi))^{2} = \oint_{|\omega_{1}|<|\omega_{2}|} \frac{F(\omega_{1}, \omega_{2})}{\omega_{2} - \omega_{1}} \frac{d\omega_{1}}{\omega_{1}} \frac{d\omega_{2}}{\omega_{2}} + \oint_{|\omega_{1}|>|\omega_{2}|} \frac{F(\omega_{1}, \omega_{2})}{\omega_{1} - \omega_{2}} \frac{d\omega_{1}}{\omega_{1}} \frac{d\omega_{2}}{\omega_{2}}
\]

with

\[
F(\omega_{1}, \omega_{2}) = \frac{1 - \xi}{(2\pi i)^{2}} \left(1 - \sqrt[\xi]{\omega_{1}}\right)^{-1 - a - \frac{1}{2} \xi} \left(1 - \sqrt[1 - \xi]{\omega_{1}^{-1}}\right)^{-1 - a - \frac{1}{2} (1 - \xi)}
\]

\[
\times \left(1 - \sqrt[\xi]{\omega_{2}^{-1}}\right)^{-1 - a - \frac{1}{2} \xi} \left(1 - \sqrt[1 - \xi]{\omega_{2}}\right)^{-1 - a - \frac{1}{2} (1 - \xi)} \omega_{1}^{\frac{1}{2} - a} \omega_{2}^{\frac{1}{2} + a}
\]

Recall that both contours go in positive direction.

Let us transform the second double–contour integral: keeping the second contour fixed we move the first contour inside the second contour. Then we obtain a double–contour integral arising from the residue of the function \( \omega_{1} \to (\omega_{1} - \omega_{2})^{-1} \):

\[
\frac{1 - \xi}{2\pi i} \oint F(\omega, \omega) \frac{d\omega}{\omega^{2}} = \frac{1 - \xi}{2\pi i} \oint (1 - \sqrt[\xi]{\omega})(\omega - \sqrt[1 - \xi]{\omega}) = 1
\]

Thus, we have shown that the functions \( \psi_{a} \) form an orthonormal family in \( f^{2}(Z') \), and it remains to prove that this family is complete. For \( x \in Z' \), let \( \delta_{x} \) stand for the delta function at \( x \). Since the functions \( \delta_{x} \) form an orthonormal basis, it suffices to check that

\[
\sum_{a \in Z'} (\delta_{x}, \psi_{a})_{f^{2}(Z')} = \sum_{a \in Z'} (\psi_{a}(x; z, z', \xi))^{2} = 1, \quad \forall x \in Z'.
\]

But this follows from the previous claim and the symmetry \( a \leftrightarrow x \) established in the next proposition. □

**Proposition 2.5.** The following symmetry relation holds

\[
\psi_{a}(x; z, z', \xi) = \psi_{a}(a; -z, -z', \xi).
\]

**Proof.** Using the classical formula

\[
\Gamma(A + \frac{1}{2})\Gamma(A - \frac{1}{2}) = \frac{\pi}{\cos(\pi A)}
\]

and the fact that both \( x + \frac{1}{2} \) and \( a + \frac{1}{2} \) are integers we check that

\[
\frac{\Gamma(z + x + \frac{1}{2})\Gamma(z' + x + \frac{1}{2})}{\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2})} = \frac{\Gamma(-z + a + \frac{1}{2})\Gamma(-z' + a + \frac{1}{2})}{\Gamma(-z - x + \frac{1}{2})\Gamma(-z' - x + \frac{1}{2})}.
\]

Applying this to (2.1) and using another classical formula,

\[
F(A, B; C; w) = (1 - w)^{C - A - B} F(C - A, C - B; C; w),
\]

we get the required relation.

Another way is to make a change of the variable in integral (2.4):

\[
\omega \mapsto \omega' = \frac{\omega - \sqrt{\xi}}{\sqrt{\xi\omega - 1}}.
\]

This is an involutive transformation such that \( 0 \leftrightarrow \sqrt{\xi} \) and \( \infty \leftrightarrow 1/\sqrt{\xi} \). As is readily verified, it leads to transformation \( (a, x, z, z') \to (x, a, -z, -z') \). □
**Corollary 2.6.** The functions $\psi_a = \psi_a(x; z, z', \xi)$ satisfy the following three-term relation
\[
(1 - \xi)x\psi_a = \sqrt{\xi(z - a + \frac{1}{2})(z' - a + \frac{1}{2})}\psi_{a-1} \\
+ \sqrt{\xi(z - a - \frac{1}{2})(z' - a - \frac{1}{2})}\psi_{a+1} + (-a + \xi(z + z' - a))\psi_a.
\]

**Proof.** Under symmetry $x \leftrightarrow a$ (Proposition 2.4), this turns into the formula stated in Proposition 2.1. Of course, a direct verification is also possible. □

The formulas of Proposition 2.1 and Corollary 2.6 show that the functions $\psi_a(x; z, z', \xi)$ possess the bispectrality property in the sense of [Gr].

**Proposition 2.7.** One more symmetry relation holds:
\[
\psi_a(x; z, z', \xi) = (-1)^x a \psi_{-a}(-x; -z, -z', \xi), \quad x, a \in \mathbb{Z}.
\]

**Proof.** This follows from the relation
\[
\frac{F(A, B; C; w)}{\Gamma(C)} = w^{1-C} \frac{\Gamma(A-C+1)\Gamma(B-C+1)}{\Gamma(A)\Gamma(B)} \\
\times \frac{F(A-C+1, B-C+1; 2-C; w)}{\Gamma(2-C)}, \quad C \in \mathbb{Z},
\]
see [Er, 2.8 (19)]. Another way is to make a change of the variable, $\omega \rightarrow 1/\omega$, in integral (2.4). □

In the remaining part of the section we will explain how the functions $\psi_a$ are related to the Meixner polynomials.

Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Elements of $\mathbb{Z}_+$ will be denoted by symbols $\tilde{x}, \tilde{y}$ (we reserve the notation $x$ and $y$ for points of the lattice $\mathbb{Z}'$). Recall that the Meixner polynomials are the orthogonal polynomials with respect to the weight function
\[
W_{\alpha, \xi}(\tilde{x}) = \frac{(\alpha + 1)\tilde{x}^\xi}{\tilde{x}!} = \frac{\Gamma(\alpha + 1 + \tilde{x})\xi^\tilde{x}}{\Gamma(\alpha + 1)\tilde{x}!}, \quad \tilde{x} \in \mathbb{Z}_+ \quad (2.7)
\]
on $\mathbb{Z}_+$, where $\alpha > -1$ and, as before, $\xi \in (0, 1)$. Our notation for these polynomials is $\mathfrak{M}_n(\tilde{x}; \alpha; \xi)$. We use the same normalization of the polynomials as in the handbook [KS] (there is only a minor difference in notation: our parameter $\alpha$ corresponds to parameter $\beta = \alpha + 1$ in [KS], while our $\xi$ is precisely parameter $c$ in [KS]).

Set
\[
\mathfrak{M}_n(\tilde{x}; \alpha, \xi) = (-1)^n \frac{\mathfrak{M}_n(\tilde{x}; \alpha, \xi)}{\|\mathfrak{M}_n(\cdot; \alpha, \xi)\|} \sqrt{W_{\alpha, \xi}(\tilde{x})}, \quad \tilde{x} \in \mathbb{Z}_+, \quad (2.8)
\]
where
\[
\|\mathfrak{M}_n(\cdot; \alpha, \xi)\|^2 = \sum_{\tilde{x}=0}^{\infty} \mathfrak{M}_n^2(\tilde{x}; \alpha, \xi)W_{\alpha, \xi}(\tilde{x}).
\]
The factor $(-1)^n$ is introduced for convenience: it will compensate the same factor in formula (2.10) below.
Proposition 2.8. Drop the assumption that \((z, z')\) is not in the degenerate series, and assume, just on the contrary, that \(z = N\) and \(z' = N + \alpha\), where \(N = 1, 2, \ldots\) and \(\alpha > -1\). Then expression (2.1) for the functions \(\psi_a(x; z, z', \xi)\) still makes sense provided that

\[
\tilde{x} := x + N - \frac{1}{2}, \quad n := N - a - \frac{1}{2}
\]

(2.9)

are in \(\mathbb{Z}_+\), and in this notation we have

\[
\psi_a(x; z, z', \xi) = \tilde{M}_n(\tilde{x}; \alpha, \xi).
\]

Proof. As is well known, the Meixner polynomials can be expressed through the Gauss hypergeometric function in two different ways:

\[
\mathcal{M}_n(\tilde{x}; \alpha, \xi) = F(-n, -\tilde{x}; \alpha + 1; \frac{\xi}{1 - \xi} - 1)
\]

and

\[
\mathcal{M}_n(\tilde{x}; \alpha, \xi) = \frac{\Gamma(\tilde{x} + 1)\Gamma(-\alpha - n)}{\Gamma(-\alpha)} \left(\frac{1 - \xi}{\xi}\right)^n \frac{\Gamma(-n, -\alpha - n; \tilde{x} + 1 - n; \frac{\xi}{1 - \xi})}{\Gamma(\tilde{x} + 1 - n)}
\]

(2.10)

see [KS]. Although the first expression looks simpler, it turns out that only the second expression is suitable for our purposes. Note that

\[
\|\mathcal{M}_n(\cdot; \alpha, \xi)\|^2 = \frac{\xi^n(1 - \xi)^{\alpha + 1} \Gamma(\alpha + n + 1)}{\Gamma(\alpha + 1)\Gamma(n + 1)}.
\]

From the last two formulas and the definition of \(\tilde{M}_n\), we obtain

\[
\tilde{M}_n(\tilde{x}; \alpha, \xi) = \sqrt{\frac{\Gamma(\tilde{x} + 1)\Gamma(\tilde{x} + \alpha + 1)}{\Gamma(n + 1)\Gamma(n + \alpha + 1)}} \xi^{(\tilde{x} - n)/2}(1 - \xi)^{(\alpha + 2n)/2}
\]

\[
\times \frac{F(-n, -\alpha - n; \tilde{x} + 1 - n; \frac{\xi}{1 - \xi})}{\Gamma(\tilde{x} + 1 - n)}.
\]

Comparing this with (2.1) and taking into account (2.9) we get the required equality. □

Thus, our functions \(\psi_a\) can be obtained from the Meixner polynomials by the following procedure:

- We replace the initial polynomials \(\mathcal{M}_n\) by the functions \(\tilde{M}_n\). This step is quite clear: as a result we get functions which form an orthonormal basis in the \(\ell^2\) space on \(\mathbb{Z}_+\) with respect to the weight function 1.
- Next, we make a change of the argument. Namely, we introduce an additional parameter \(N = 1, 2, \ldots\) and we set \(x = \tilde{x} - N + \frac{1}{2}\). Then we get orthogonal functions on the subset

\[
\{-N + \frac{1}{2}, -N + \frac{3}{2}, -N + \frac{5}{2}, \ldots\} \subset \mathbb{Z}'
\]
which exhausts the whole \( Z' \) in the limit as \( N \) goes to infinity.

- Then we also need a change of the index. Namely, instead of \( n \) we have to take \( a = N - n - \frac{1}{2} \). We cannot give a conceptual explanation of this transformation, it is dictated by the formulas. Again, the range of the possible values for \( a \) becomes larger together with \( N \), and in the limit as \( N \to +\infty \) we get the whole lattice \( Z' \).

- Finally, we make a (formal) analytic continuation in parameters \( N \) and \( \alpha \), using an appropriate analytic expression for the Meixner polynomials. Note that the difference equation of Proposition 2.1 and the three–term relation of Corollary 2.6 precisely correspond to similar relations for the Meixner polynomials.

We hope that this detailed explanation will help the reader to perceive the analytic continuation arguments in Sections 3 and 7.

Of course, instead of the lattice \( Z' \) we could equally well deal with the lattice \( \mathbb{Z} \), and then numerous “1/2” would disappear. However, dealing with the lattice \( Z' \) makes main formulas more symmetric.

## 3. The Discrete Hypergeometric Kernel

Let \( \mathfrak{X} \) be a countable set. By a point configuration in \( \mathfrak{X} \) we mean any subset \( X \subseteq \mathfrak{X} \). Let \( \text{Conf}(\mathfrak{X}) \) be the set of all point configurations; this is a compact space. Assume we are given a probability measure on \( \text{Conf}(\mathfrak{X}) \) so that we can speak about the random point configuration in \( \mathfrak{X} \). The \( n \)th correlation function of our probability measure (where \( n = 1, 2, \ldots \)) is defined by

\[
\rho_n(x_1, \ldots, x_n) = \text{Prob}\{\text{the random configuration contains} \ x_1, \ldots, x_n\},
\]

where \( x_1, \ldots, x_n \) are pairwise distinct points in \( \mathfrak{X} \). The collection of all correlation functions determines the initial probability measure uniquely.

We say that our probability measure is determinantal if there exists a function \( K(x, y) \) on \( \mathfrak{X} \times \mathfrak{X} \) such that

\[
\rho_n(x_1, \ldots, x_n) = \det [K(x_i, x_j)]_{i,j=1}^n, \quad n = 1, 2, \ldots \tag{3.1}
\]

It is worth noting that if such a function \( K(x, y) \) exists, then it is not unique. Indeed, any “gauge transformation” of the form

\[
K(x, y) \rightarrow \frac{f(x)}{f(y)} K(x, y), \tag{3.2}
\]

where \( f \) is a nonvanishing function on \( \mathfrak{X} \), does not affect the determinants in the right–hand side of (3.1).

Any function \( K(x, y) \) satisfying (3.1) will be called a correlation kernel of the initial determinantal measure. Two kernels giving the same system of correlation functions will be called equivalent.

As in §2, we are dealing with the lattice \( Z' \) of (proper) half–integers. We split it into two parts, \( Z' = Z'_- \cup Z'_+ \), where \( Z'_- \) consists of all negative half–integers and \( Z'_+ \) consists of all positive half–integers. For an arbitrary \( \lambda \in \mathbb{Y} \) we set

\[
\mathfrak{X}(\lambda) = \{ \lambda_i - i + \frac{1}{2} \mid i = 1, 2, \ldots \} \subset Z'.
\]
For instance, $X(\emptyset) = Z'$. The correspondence $\lambda \mapsto X(\lambda)$ is a bijection between the Young diagrams $\lambda$ and those (infinite) subsets $X \subset Z'$ for which the symmetric difference $X \triangle Z'$ is a finite set with equally many points in $Z'_+$ and $Z'$. Note that

$$X(\lambda') = -(Z' \setminus X(\lambda)).$$

Using the correspondence $\lambda \mapsto X(\lambda)$ we can interpret any probability measure $M$ on $\mathcal{Y}$ as a probability measure on $\text{Conf}(Z')$. This makes it possible to speak about the correlation functions of $M$. Our goal is to compute them explicitly for the $z$–measures.

Now we can state the main results of the section.

**Theorem 3.1.** For any admissible pair of parameters $(z, z')$, see §1, the corresponding mixed $z$–measure $M_{z, z', \xi}$ is a determinantal measure.

**Theorem 3.2.** If $(z, z')$ is not in the degenerate series (so that $z$ and $z'$ are not integers) then the correlation kernel of $M_{z, z', \xi}$ can be written in the form

$$K_{z, z', \xi}(x, y) = \sum_{a \in Z'_+} \psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi), \quad x, y \in Z',$$

where the functions $\psi_a$ are defined in (2.1).

Note that the series in the right–hand side is absolutely convergent. Indeed, since $\{\psi_a\}$ is an orthonormal basis in $\ell^2(Z')$ (Proposition 2.4), this follows from the fact that the series can be written as

$$\sum_{a \in Z'_+} (\delta_x, \psi_a)(\psi_a, \delta_y),$$

where $\delta_x$ stands for the delta–function at point $x$ on the lattice $Z'$, and $(\cdot, \cdot)$ denotes the inner product in $\ell^2(Z')$.

Formula (3.3) simply means that $K_{z, z', \xi}(x, y)$ is the matrix of the orthogonal projection operator in $\ell^2(Z')$ whose range is the subspace spanned by the basis vectors $\psi_a$ with index $a \in Z'_+ \subset Z'$.

**Theorem 3.3.** The correlation kernel (3.3) can also be written in the form

$$K_{z, z', \xi}(x, y) = \varphi_{z, z'}(x, y) \tilde{K}_{z, z', \xi}(x, y)$$

where, as in (2.6),

$$\varphi_{z, z'}(x, y) = \sqrt{\frac{\Gamma(x + z + \frac{1}{2}) \Gamma(x + z' + \frac{1}{2}) \Gamma(y + z + \frac{1}{2}) \Gamma(y + z' + \frac{1}{2})}{\Gamma(x + z' + \frac{1}{2}) \Gamma(y + z + \frac{1}{2})}}$$

These results are essentially not new, see [BO2], [BO5], [Ok2], [BOk, Example 3], and the comments at the end of the section. However, the method of proof is new. The same method, with suitable modifications, is applied in §7 for the computation of the dynamical correlation functions.
and

$$\hat{K}_{z,z',\xi}(x,y) = 1 - \xi \int \frac{1}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} \frac{1 - \sqrt{\xi} \omega_1}{\omega_1} \frac{1 - \sqrt{\xi} \omega_2}{\omega_2} \frac{1 - \sqrt{\xi}}{\omega_1 \omega_2 - 1} \times \omega_1^{-x-1/2} \omega_2^{-y-1/2} d\omega_1 d\omega_2$$

(3.6)

where \(\{\omega_1\}\) and \(\{\omega_2\}\) are arbitrary simple contours satisfying the following three conditions:

- both contours go around 0 in positive direction;
- the point \(\xi^{-1/2}\) is in the interior of each of the contours while the point \(\xi^{-1/2}\) lies outside them;
- the contour \(\{\omega_1^{-1}\}\) is contained in the interior of the contour \(\{\omega_2\}\) (equivalently, \(\{\omega_2^{-1}\}\) is contained in the interior of \(\{\omega_1\}\)).

The kernels \(K_{z,z',\xi}(x,y)\) and \(\hat{K}_{z,z',\xi}(x,y)\) are equivalent. Namely, they are related by a “gauge transformation”:

$$\hat{K}_{z,z',\xi}(x,y) = \frac{f_{z,z'}(x)}{f_{z,z'}(y)} K_{z,z',\xi}(x,y), \quad x, y \in \mathbb{Z}'.$$

(3.7)

The kernel \(\hat{K}_{z,z',\xi}(x,y)\) can serve as a correlation kernel for all admissible values of parameters \((z, z')\), including the degenerate series.

Proof of Theorems 3.1–3.3. We prove these three theorems simultaneously. Let \(\rho_n^{(z,z',\xi)}(x_1, \ldots, x_n)\) denote the \(n\)-point correlation function of \(M_{z,z',\xi}\). The proof splits into two parts.

In the first part, we compute \(\rho_n^{(z,z',\xi)}\) for special values of the parameters (the degenerate series): \(z = N, 1, 2, \ldots\) and \(z' = N + \alpha\), where \(\alpha > -1\). Here we use the fact that for such \((z, z')\), the mixed \(z\)-measure can be interpreted as the so-called \(N\)-particle Meixner ensemble. We show that formula

$$\rho_n^{(z,z',\xi)}(x_1, \ldots, x_n) = \det[K_{z,z',\xi}(x_i, x_j)]_{i,j=1}^n$$

is valid (in particular, the values of the kernel in the right–hand size are well defined) when \(z = N, z' = z + \alpha\), provided that \(N\) is so large that the numbers \(x_i + N - 1/2\) are nonnegative. Then we check that in that formula, the kernel \(K_{z,z',\xi}\) can be replaced by the kernel \(\hat{K}_{z,z',\xi}\) :

$$\rho_n^{(z,z',\xi)}(x_1, \ldots, x_n) = \det[\hat{K}_{z,z',\xi}(x_i, x_j)]_{i,j=1}^n$$

In the second part, we extend the latter formula to arbitrary admissible \((z, z')\). To do this we show that both sides are analytic functions in parameters \((z, z', \xi)\).
Moreover, these functions are of such a kind that they are uniquely defined by their values at points \((z = N, z' = N + \alpha, \xi)\).

We proceed to the detailed proof.

Let, as in §2, \(N\) be a natural number and \(\alpha > -1\). Consider the Meixner weight function \(W_{\alpha, \xi}\) on \(\mathbb{Z}_+\), see (2.7). The \(N\)-point Meixner ensemble is formed by random \(N\)-point configurations \(\tilde{X} = (\tilde{x}_1 > \cdots > \tilde{x}_N)\) in \(X = \mathbb{Z}_+\), where

\[
\text{Prob}(\tilde{X}) = \text{const} \cdot \prod_{1 \leq i < j \leq N} (\tilde{x}_i - \tilde{x}_j)^2 \prod_{i=1}^N W_{\alpha, \xi}(\tilde{x}_i).
\]

By the \(N\)th Meixner measure we mean the corresponding probability measure on \(\text{Conf}(\mathbb{Z}_+)\).

**Lemma 3.4.** The \(N\)th Meixner measure is a determinantal measure. As its correlation kernel on \(\mathbb{Z}_+ \times \mathbb{Z}_+\) one can take the “Meixner kernel”

\[
K_{N,\alpha, \xi}^\text{Meixner}(\tilde{x}, \tilde{y}) = \sum_{m=0}^{N-1} \tilde{M}_m(\tilde{x}; \alpha, \xi) \tilde{M}_m(\tilde{y}; \alpha, \xi), \\
\tilde{x}, \tilde{y} \in \mathbb{Z}_+,
\]

where the functions \(\tilde{M}_m(\tilde{x}; \alpha, \xi)\) are defined in (2.8).

**Proof.** This is a special case of a well–known general claim about orthogonal polynomial ensembles, see, e.g., [De2]. □

Let \(\mathcal{Y}(N) \subset \mathcal{Y}\) denote the set of Young diagrams \(\lambda\) with \(\ell(\lambda) \leq N\). Recall that the mixed \(z\)-measure with parameters \(z = N, z' > N - 1\) is concentrated on \(\mathcal{Y}(N)\). We define a bijection between Young diagrams \(\lambda \in \mathcal{Y}(N)\) and \(N\)-point configurations \(\tilde{X} \subset \mathbb{Z}_+\) as follows

\[
\lambda \mapsto \tilde{X}(\lambda) = (\tilde{x}_1, \ldots, \tilde{x}_N), \\
\tilde{x}_i = \lambda_i - i + N, \quad i = 1, \ldots, N.
\]

**Lemma 3.5** ([BO2, Proposition 4.1]). The correspondence \(\lambda \mapsto \tilde{X}(\lambda)\) takes the \(z\)-measure \(M_{z, z', \xi}\) with parameters \(z = N, z' = N + \alpha\) to the \(N\)th Meixner measure with parameters \(\alpha, \xi\).

**Proof.** Direct verification. □

Recall that we identify \(M_{z, z', \xi}\) with its push–forward under the correspondence \(\lambda \mapsto X(\lambda)\).

**Corollary 3.6.** Let \(z = N = 1, 2, \ldots\) and \(z' = z + \alpha\) with \(\alpha > -1\). Assume that \(x_1, \ldots, x_n\) lie in the subset \(\mathbb{Z}_+ - N + 1 \times 1 \subset \mathbb{Z}'\), so that the points \(\tilde{x}_i := x_i + N - \frac{1}{2}\) are in \(\mathbb{Z}_+\). Then

\[
\rho_n^{(z, z', \xi)}(x_1, \ldots, x_n) = \det \left[ K_{N,\alpha, \xi}^\text{Meixner}(\tilde{x}_i, \tilde{x}_j) \right]_{i,j=1}^n
\]

**Proof.** Let \(\lambda \in \mathcal{Y}(N)\). Comparing the definition of the infinite configuration \(\underline{X}(\lambda) \subset \mathbb{Z}'\) with that of the \(N\)-point configuration \(\tilde{X}(\lambda)\) we see that

\[
\tilde{X}(\lambda) = (\underline{X}(\lambda) + N - \frac{1}{2}) \cap \mathbb{Z}_+.
\]

Then the claim follows from Lemmas 3.4 and 3.5. □

We take (3.3) as the definition of the kernel \(K_{z, z', \xi}(x, y)\).
Lemma 3.7. Let \( z = N = 1, 2, \ldots \) and \( z' = z + \alpha \) with \( \alpha > -1 \). Assume that \( x \) and \( y \) lie in the subset \( \mathbb{Z}_+ - N + \frac{1}{2} \subset \mathbb{Z}' \), so that \( \tilde{x} := x + N - \frac{1}{2} \) and \( \tilde{y} := y + N - \frac{1}{2} \) are in \( \mathbb{Z}_+ \).

Then expression (3.3) for the kernel \( K_{z, z', \xi}(x, y) \) is well defined and we have

\[
K_{z, z', \xi}(x, y) = K_{N, \alpha, \xi}^{\text{Meixner}}(\tilde{x}, \tilde{y}).
\]

Proof. We have to prove that

\[
\sum_{a \in \mathbb{Z}_+} \psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi) = \sum_{m=0}^{N-1} \mathcal{M}_m(\tilde{x}; \alpha, \xi) \mathcal{M}_m(\tilde{y}; \alpha, \xi) \quad (3.8)
\]

We recall that the functions \( \psi_a(x; z, z', \xi) \) were defined under the assumption that both \( z, z' \) are not integers. However, as it can be seen from (2.1), each summand in the left–hand side of (3.8) makes sense under the hypotheses of the lemma.

Set

\[
a(m) = N - m - \frac{1}{2}, \quad m = 0, 1, \ldots, N - 1
\]

By Proposition 2.8,

\[
\psi_{a(m)}(x; z, z', \xi) = \mathcal{M}_m(\tilde{x}; \alpha, \xi), \quad \psi_{a(m)}(y; z, z', \xi) = \mathcal{M}_m(\tilde{y}; \alpha, \xi),
\]

which implies that

\[
\sum_{a = \frac{1}{2}}^{N-\frac{1}{2}} \psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi) = \sum_{m=0}^{N-1} \mathcal{M}_m(\tilde{x}; \alpha, \xi) \mathcal{M}_m(\tilde{y}; \alpha, \xi) \quad (3.9)
\]

Finally, observe that

\[
\left. \frac{1}{\Gamma(z - a + \frac{1}{2})} \right|_{a=N+\frac{1}{2}, N+\frac{3}{2}, \ldots} = \left. \frac{1}{\Gamma(N - a + \frac{1}{2})} \right|_{a=N+\frac{1}{2}, N+\frac{3}{2}, \ldots} = 0
\]

We conclude that the infinite sum in the left–hand side of (3.8) actually coincides with the finite sum in (3.9). \( \square \)

Together with Corollary 3.6 this implies

Corollary 3.8. Let \( z = N = 1, 2, \ldots \) and \( z' = z + \alpha \) with \( \alpha > -1 \). Assume that \( x_1, \ldots, x_n \) lie in the subset \( \mathbb{Z}_+ - N + \frac{1}{2} \subset \mathbb{Z}' \), so that the points \( \tilde{x}_i := x_i + N - \frac{1}{2} \) are in \( \mathbb{Z}_+ \).

Then

\[
\rho_n^{(z, z', \xi)}(x_1, \ldots, x_n) = \det [K_{z, z', \xi}(x_i, x_j)]_{i,j=1}^n
\]
Lemma 3.9. Assume that
- either \((z, z')\) is not in the degenerate series and \(x, y \in \mathbb{Z}\) are arbitrary
- or \(z = N = 1, 2, \ldots, z' > N - 1\), and both \(x, y\) are in \(\mathbb{Z} - N + \frac{1}{2}\).

Then the kernel \(\hat{K}(z, z'; x, y)\) of Theorem 3.3 is related to the kernel \(K(z, z'; x, y)\) by equality (3.4). Equivalently, the kernels are related by the “gauge transformation” (3.2),
\[
\hat{K}(z, z'; x, y) = \frac{f(x) f(y)}{f(z) f(z')} K(z, z'; x, y),
\]
where \(f\) is defined in (3.7).

Proof. Let us start with expression (3.3) of the kernel \(K(z, z'; x, y)\) and let us replace each summand by its integral representation (2.5). It is convenient to set \(t = \frac{1}{z} = k\) so that as \(k\) ranges over \(\mathbb{Z}_+\), \(k\) ranges over \(\{0, 1, 2, \ldots\}\). Then we obtain
\[
K(z, z'; x, y) = \varphi(z; x) \frac{1}{(2\pi)^{2}} \sum_{k=0}^{\infty} \int_{\{\omega_1\}} \int_{\{\omega_2\}} (1 - \sqrt{k}(\omega_1) - \sqrt{k}(\omega_2))^{z-1} (1 - \sqrt{k}(\omega_1))^{-z} \left(1 - \sqrt{k}(\omega_2)\right)^{z'-1} \times \omega_1^{-y} \omega_2^{-y} \left(\frac{1 - \sqrt{k}(\omega_1)(1 - \sqrt{k}(\omega_2))}{(\omega_1 - \sqrt{k})(\omega_2 - \sqrt{k})}\right)^k \frac{d\omega_1 d\omega_2}{\omega_1 \omega_2}.
\]

We can choose the contours \(\{\omega_1\}\) and \(\{\omega_2\}\) so that they are contained in the domain \(|\omega| > 1\). Since the fractional–linear transformation \(\omega \mapsto \frac{1 - \sqrt{k}(\omega)}{\omega - \sqrt{k}}\) preserves the unit circle \(|\omega| = 1\) and maps its exterior \(|\omega| > 1\) into its interior \(|\omega| < 1\), we have on the product of the contours a bound of the form
\[
\frac{|(1 - \sqrt{k}(\omega_1))(1 - \sqrt{k}(\omega_2))|}{(\omega_1 - \sqrt{k})(\omega_2 - \sqrt{k})} \leq q < 1.
\]

Therefore, we can interchange summation and integration and then sum the arising geometric progression in the integrand:
\[
\sum_{k=0}^{\infty} \left(\frac{(1 - \sqrt{k}(\omega_1))(1 - \sqrt{k}(\omega_2))}{(\omega_1 - \sqrt{k})(\omega_2 - \sqrt{k})}\right)^k \frac{1 - \sqrt{\xi}}{\omega_1 \omega_2} = \frac{(1 - \sqrt{k}(\omega_1))(1 - \sqrt{k}(\omega_2))}{(1 - k)(\omega_1 \omega_2 - 1)}
\]

Then we obtain equality (3.4) with integral (3.6), as desired. Finally, we can relax the assumption on the contour: it suffices to assume that \(\{\omega_1^{-1}\}\) is strictly contained inside \(\{\omega_2\}\), as in the formulation of Theorem 3.3.

It remains to show that (3.4) is equivalent to (3.10). According to (3.5) consider the expression
\[
\frac{1}{\varphi(z, z'; x, y)} = \frac{\Gamma(x + z' + \frac{1}{2}) \Gamma(y + z + \frac{1}{2})}{\sqrt{\Gamma(x + z + \frac{1}{2}) \Gamma(x + z' + \frac{1}{2}) \Gamma(y + z + \frac{1}{2}) \Gamma(y + z' + \frac{1}{2})}}
\]
Let us show that
\[ \frac{1}{\varphi_{z,z'}(x,y)} = \frac{f_{z,z'}(x)}{f_{z,z'}(y)} \]
Indeed, \(1/\varphi_{z,z'}\) has the form
\[ a(x) b(y) \sqrt{a(x)b(x)a(y)b(y)} \]
and our hypotheses imply that \(a(x)b(x)\) and \(a(y)b(y)\) are real and strictly positive.
We also have
\[ f_{z,z'}(x) = \frac{a(x)}{\sqrt{a(x)b(x)}}. \]
Therefore, we get
\[ \frac{f_{z,z'}(x)}{f_{z,z'}(y)} = \frac{a(x)\sqrt{a(y)b(y)}}{\sqrt{a(x)b(x)a(y)b(y)} \ a(y)} = \frac{a(x)a(y)b(y)}{a(x)b(x)a(y)b(y)} \ a(y) = 1 \]
\[ \varphi_{z,z'}(x,y) \]
\[ \Box \]

**Corollary 3.10.** Let \(z = N = 1, 2, \ldots\) and \(z' > N - 1\). Then
\[ \rho_{n}(z,z',\xi)(x_1,\ldots,x_n) = \det \left[ \hat{K}_{z,z',\xi}(x_i, x_j) \right]_{i,j=1}^{n} \]  
(3.11)
provided that all the points \(x_1,\ldots,x_n \in \mathbb{Z}' \) lie in the subset \(\mathbb{Z}_{+} - N + \frac{1}{2} \subset \mathbb{Z}'\).

**Proof.** Indeed, this follows from Lemma 3.9 and Corollary 3.8. \(\Box\)

This completes the first part of the proof. Now we proceed to the second part.

**Lemma 3.11.** (i) Fix an arbitrary set of Young diagrams \(D \subset \mathbb{Y}\). For any fixed admissible pair of parameters \((z,z')\), the function
\[ \xi \mapsto \sum_{\lambda \in D} M_{z,z',\xi}(\lambda), \]
which is initially defined on the interval \((0,1)\), can be extended to a holomorphic function in the unit disk \(|\xi| < 1\).
(ii) Consider the Taylor expansion of this function at \(\xi = 0\),
\[ \sum_{\lambda \in D} M_{z,z',\xi}(\lambda) = \sum_{k=0}^{\infty} G_{k,D}(z,z')\xi^k. \]
Then the coefficients \(G_{k,D}(z,z')\) are polynomial functions in \(z,z'\). That is, they are restrictions of polynomial functions to the set of admissible values \((z,z')\).
Proof. (i) Set $D_n = D \cap Y_n$. By the definition of $M_{z,z',\xi}$,
\[
\sum_{\lambda \in D} M_{z,z',\xi}(\lambda) = \sum_{n=0}^{\infty} \left( \sum_{\lambda \in D_n} M_{z,z',\xi}^{(n)}(\lambda) \right) \pi_{z,z',\xi}(n)
= (1 - \xi)^{zz'} \sum_{n=0}^{\infty} \left( \sum_{\lambda \in D_n} M_{z,z',\xi}^{(n)}(\lambda) \right) \frac{(zz')_n \xi^n}{n!}.
\]
Each interior sum is nonnegative and does not exceed 1. On the other hand,
\[
\sum_{n=0}^{\infty} |\pi_{z,z',\xi}(n)| = |1 - \xi|^{zz'} \sum_{n=0}^{\infty} \frac{(zz')_n |\xi|^n}{n!} < \infty, \quad \xi \in \mathbb{C}, \quad |\xi| < 1.
\]
This proves the first claim.

(ii) By (1.11),
\[
\sum_{\lambda \in D} M_{z,z',\xi}(\lambda) = (1 - \xi)^{zz'} \sum_{n=0}^{\infty} \sum_{\lambda \in D_n} (z)(z')_\lambda \xi^n \left( \frac{\dim \lambda}{n!} \right)^2.
\]
It follows that
\[
G_{k,D}(z,z') = \sum_{n=0}^{k} \frac{(-zz')_{k-n}}{(k-n)!} \sum_{\lambda \in D_n} (z)(z')_\lambda \left( \frac{\dim \lambda}{n!} \right)^2.
\]
Since each $D_n$ is a finite set, this expression is a polynomial in $z, z'$. □

Now we can complete the proof of the theorems. Fix $n$ and an arbitrary $n$–point subset $X = \{x_1, \ldots, x_n\} \subset \mathbb{Z'}$, and regard $\rho_n^{(z,z',\xi)}(x_1, \ldots, x_n)$ as a function of parameters $z, z', \xi$. We want to show that equality (3.11) holds for any admissible $(z, z')$. Apply Lemma 3.11 to the set $D$ of those diagrams $\lambda$ for which $X(\lambda)$ contains $X$, and observe that
\[
\rho_n^{(z,z',\xi)}(x_1, \ldots, x_n) = \sum_{\lambda \in D} M_{z,z',\xi}(\lambda).
\]
It follows that $\rho_n^{(z,z',\xi)}(x_1, \ldots, x_n)$ is a real–analytic function of $\xi \in (0,1)$ which admits a holomorphic extension to the open unit disk $|\xi| < 1$. Moreover, the Taylor coefficients of this function depend on $z, z'$ polynomially.

On the other hand, from the expression (3.6) for the kernel $\tilde{K}_{z,z',\xi}(x,y)$ it follows that this kernel (and hence the right–hand side of (3.11)) has the same property, with $\xi$ replaced by $\sqrt{\xi}$.

Thus, both sides of (3.11) can be viewed as (restrictions of) holomorphic functions in $\sqrt{\xi}$ with polynomial Taylor coefficients. Since the set
\[
\{(z,z') \mid z \text{ is a large natural number } N \text{ and } z' > N - 1\}
\]
is a set of uniqueness for polynomials in two variables, we conclude that equality (3.11) is true for any admissible $(z, z')$. 

This proves Theorem 3.1 and Theorem 3.3. Now, Theorem 3.2 follows from Theorem 3.3 and Lemma 3.9.

Comments. 1. The correlation functions of the $z$–measures $M_{z,z',\xi}$ were first computed in [BO2] in a different form: in that paper we dealt with another embedding of partitions into the set of lattice point configurations (in the notation of §8, we used the map $\lambda \mapsto X(\lambda)$, instead of $\lambda \mapsto \chi(\lambda)$). The kernel $K_{z,z',\xi}(x,y)$ coincides with one of the “blocks” of the kernel considered in [BO2]. The relation between both kernels is discussed in detail in [BO5] (see also §8 below). The proofs in [BO2] and [BO5] are very different from the arguments of the present section.

2. Two other derivations of the kernel $K_{z,z',\xi}(x,y)$ are given in Okounkov’s papers [Ok2] and [Ok1]. In both these papers, the correlation functions are expressed through the vacuum state expectations of certain operators in the infinite wedge Fock space. A (substantial) difference between the methods of [Ok2] and [Ok1] consists in the concrete choice of operators. The general formalism of Schur measures presented in [Ok1] is complemented by explicit computations in [BOK, §4].

3. As shown in the papers listed above, the kernel $K_{z,z',\xi}(x,y)$ can be written in the form

$$K_{z,z',\xi}(x,y) = \frac{P(x)Q(y) - Q(x)P(y)}{x - y},$$

(3.12)

where $P$ and $Q$ are certain functions on $Z'$ depending on parameters $z, z', \xi$. Since $P$ and $Q$ are expressed through the Gauss hypergeometric function, we called $K_{z,z',\xi}(x,y)$ the discrete hypergeometric kernel. In general, kernels admitting such an expression are called integrable kernels, in accordance with the terminology of [HIKS], [De1], [B2].

4. The integrable form (3.12) can be readily derived from (3.3) using the three–term relation for functions $\psi_a(x) = \psi_a(x; z, z', \xi)$ given in Corollary 2.6. Specifically, we obtain

$$K_{z,z',\xi}(x,y) = \sqrt{z\xi} \left\{ \psi_{-\frac{1}{2}}(x)\psi_{\frac{1}{2}}(y) - \psi_{\frac{1}{2}}(x)\psi_{-\frac{1}{2}}(y) \right\} \frac{1}{x - y},$$

(3.13)

This derivation of (3.13) from (3.3) is quite similar to the standard derivation of the Christoffel–Darboux formula for an arbitrary system of orthogonal polynomials. Since, as explained in §2, the functions $\psi_a$ are closely related to the Meixner polynomials, this analogy is not surprising.

5. Once we know that the functions $\psi_a$ form an orthonormal basis (Proposition 2.4), the series expression (3.3) for the kernel $K_{z,z',\xi}(x,y)$ immediately implies that it is a projection kernel. This fact was first proved in [BO5, §5] in a different way.

6. The series representation (3.3) is equivalent to formula (3.16) in [Ok2]. A double contour integral representation of various correlation kernels related to Schur measures appeared earlier in [BOK].

4. Construction of Markov processes

The goal of this section is to explain the construction of the continuous time Markov processes on partitions which will be studied in the rest of the paper. Their fixed time distributions are the $z$-measures considered in the previous sections.

It is fairly easy to give the jump rates for these processes. However, it is not a priori clear why these rates define the process uniquely. Since we were unable to
find suitable uniqueness theorems in the literature, we will actually prove that the
rates define the process uniquely and compute the transition probabilities for an
underlying birth-death process.

4.1. Preliminaries on Markov processes. Let us recall some basic facts about
continuous time Markov processes and introduce the notation.

The time parameter \( t \) always ranges over an open interval \((t_{\text{min}}, t_{\text{max}})\) where \( t_{\text{min}} \in \mathbb{R} \cup \{-\infty\} \) and \( t_{\text{max}} \in \mathbb{R} \cup \{+\infty\} \). Let us denote the state space by \( \mathbb{A} \), it is assumed to be either finite or countable.

We also denote by \( P(s,t), \ s \leq t \), the matrix of transition probabilities of a
Markov process. This is a matrix with rows and columns marked by elements of \( \mathbb{A} \), its elements will be denoted by \( P_{ab}(s,t), \ a, b \in \mathbb{A} \). By definition, \( P_{ab}(s,t) \)
the probability that the process will be in the state \( b \) at the time moment \( t \)
conditioned that it is in the state \( a \) at time \( s \). Thus, all matrix elements of \( P(s,t) \)
are nonnegative, and its sum is equal to one along any row. Such matrices are called stochastic. The transition matrices \( P(s,t) \) also satisfy the Chapman-Kolmogorov equation

\[
P(s,t)P(t,u) = P(s,u), \quad s \leq t \leq u. \tag{4.1}
\]

We assume that there exist \( \mathbb{A} \times \mathbb{A} \) matrices \( Q(t) \) with continuously depending
on \( t \) entries, such that

\[
P_{ab}(s,t) = \delta_{ab} + Q_{ab}(t)(t-s) + o(|t-s|), \quad |t-s| \to 0.
\]

This relation implies that \( Q_{ab}(t) \geq 0 \) for \( a \neq b \) and \( Q_{aa}(t) \leq 0 \). Further, we assume that

\[
\sum_{b \neq a} Q_{ab}(t) = -Q_{aa}(t), \quad \text{for any } a \in \mathbb{A}. \tag{4.2}
\]

This is the infinitesimal analog of the condition \( \sum_{b \in \mathbb{A}} P_{ab}(s,t) = 1 \).

It is well known that (4.1) then implies that \( P(s,t) \) then satisfies Kolmogorov’s
backward equation

\[
-\frac{\partial}{\partial s} P(s,t) = Q(s)P(s,t), \quad s \leq t, \tag{4.3}
\]

with the initial condition

\[
P(t,t) \equiv \text{Id}.
\]

Under certain additional conditions, \( P(s,t) \) will also satisfy Kolmogorov’s forward equation

\[
\frac{\partial}{\partial t} P(s,t) = P(s,t)Q(t). \tag{4.5}
\]

In our concrete situation we would like to define a Markov process by specifying the transition rates \( Q(t) \). However, it may happen that this does not specify the process uniquely (then the backward equation has many solutions \( P(s,t) \)). Uniqueness always holds if \( \mathbb{A} \) is finite or, more generally, if \( \mathbb{A} \) is infinite but the functions \( |Q_{aa}(t)| \) are bounded on any closed time interval (see, e.g. [Fe1]). However, these conditions are not satisfied in our case. There exist other, more involved uniqueness conditions for time homogeneous (stationary) Markov processes. However, in our approach, even if we restrict our attention to stationary processes, we still need to handle some non stationary processes as auxiliary objects. For these reasons we had to find some more special uniqueness condition.
Let us write \( Q(t) \) in the form
\[
Q(t) = -R(t) + \tilde{Q}(t),
\]
where \(-R(t)\) is the diagonal part of \( Q(t) \) and \( \tilde{Q}(t) \) is the off-diagonal part of \( Q(t) \). In other words,
\[
R_{ab}(t) = -\delta_{ab}Q_{aa}(t), \quad \tilde{Q}_{ab}(t) = \begin{cases} Q_{ab}(t), & a \neq b, \\ 0, & a = b. \end{cases}
\]

For \( s \leq t \) set
\[
F(s, t) = \exp \left( -\int_s^t R(\tau)d\tau \right), \quad G(s, t) = F(s, t)\tilde{Q}(s, t).
\]

Define \( P^{[n]}(s, t) \) recursively by
\[
P^{[0]}(s, t) = F(s, t), \quad P^{[n]}(s, t) = \int_s^t G(s, \tau)P^{[n-1]}(\tau, t)d\tau, \quad n \geq 1,
\]
and set
\[
\overline{P}(s, t) = \sum_{n=0}^{\infty} P^{[n]}(s, t), \quad s \leq t.
\]

**Theorem 4.1** [Fe1].

(i) The matrix \( \overline{P}(s, t) \) is substochastic (i.e., its elements are nonnegative and \( \sum_b P_{ab}(s, t) \leq 1 \)). Its elements are absolutely continuous and almost everywhere differentiable with respect to both \( s \) and \( t \), and it provides a solution of Kolmogorov’s backward and forward equations \((4.3), (4.5)\) with the initial condition \((4.4)\).

(ii) \( \overline{P}(s, t) \) also satisfies the Chapman-Kolmogorov equation \((4.1)\).

(iii) \( \overline{P}(s, t) \) is the minimal solution of \((4.3)\) (or \((4.5)\)) in the sense that for any other solution \( P(s, t) \) of \((4.3)\) (or \((4.5)\)) with the initial condition \((4.4)\) in the class of substochastic matrices, one has \( P_{ab}(s, t) \geq \overline{P}_{ab}(s, t) \) for any \( a, b \in \mathbb{A} \).

**Corollary 4.2.** If the minimal solution \( \overline{P}(s, t) \) is stochastic (the sums of matrix elements along the rows are all equal to 1) then it is the unique solution of \((4.3)\) (or \((4.5)\)) with the initial condition \((4.4)\) in the class of substochastic matrices.

Let us note that the construction of \( \overline{P}(s, t) \) is very natural: the summands \( P^{[n]}(s, t) \) are the probabilities to go from \( a \) to \( b \) in \( n \) jumps. The condition of \( \overline{P}(s, t) \) being stochastic exactly means that we cannot make infinitely many jumps in a finite amount of time.

Our next goal is to provide a convenient sufficient condition for \( \overline{P}(s, t) \) to be stochastic.

Fix \( s \in (t_{\min}, t_{\max}) \) and \( a \in \mathbb{A} \). For any finite \( X, X \subset \mathbb{A}, a \in X \), we denote by \( T_{s, a, X} \) the time of the first exit from \( X \) under the condition that the process is in \( a \) at time \( s \). Formally, we can modify \( \mathbb{A} \) and \( Q(t) \) by contracting all the states \( b \in \mathbb{A} \setminus X \) into one absorbing state \( \tilde{b} \) with \( Q_{b,c} \equiv 0 \) for any \( c \in X \cup \{\tilde{b}\} \). We obtain a process with a finite number of states for which the solution \( \tilde{P}(s, t) \) of the backward equation is unique. Then \( T_{s, a, X} \) is a random variable with values in \((s, +\infty]\) defined by
\[
\text{Prob}\{T_{s, a, X} \leq t\} = \tilde{P}_{ab}(s, t).
\]
Proposition 4.3. Assume that for any \( a \in \mathbb{A} \) and any \( s < t, \varepsilon > 0 \), there exists a finite set \( X(\varepsilon) \subset \mathbb{A} \) such that

\[
\text{Prob}(T_{s,a,X(\varepsilon)} \leq t) \leq \varepsilon.
\]

Then the minimal solution \( \overline{P}(s,t) \) provided by Theorem 4.1 is stochastic.

Proof. Consider the modified process on the finite state space \( X(\varepsilon) \cup \{ b \} \) described above. Since its transition matrix \( \overline{P}(s,t) \) is stochastic,

\[
\sum_{b \in X(\varepsilon)} \overline{P}_{ab}(s,t) = 1 - \overline{P}_{ab}(s,t) \geq 1 - \varepsilon.
\]

The construction of the minimal solution as the sum of \( P^{|n|} \)'s, see above, immediately implies that \( P_{ab}(s,t) \geq \overline{P}_{ab}(s,t) \). Thus, \( \sum_b P_{ab}(s,t) \geq 1 - \varepsilon \) for any \( \varepsilon > 0 \). \( \square \)

4.2. An application to birth-death processes. A birth-death process is a continuous time Markov process on \( \mathbb{A} = \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \) such that the rates \( Q_{mn}(t) \) vanish if \( |n - m| > 1 \). In other words, the process can make jumps only of size 1. Our assumption (4.2) means that

\[
-Q_{nn}(t) = R_{nn}(t) = Q_{n,n+1}(t) + Q_{n,n-1}(t).
\]

Proposition 4.4. Assume that for any closed segment \([t', t''] \subset (t_{\text{min}}, t_{\text{max}})\) there exists a sequence \( \{\gamma_n\}_{n=0}^{\infty} \) of positive real numbers such that \( R_{nn}(t) \leq \gamma_n \) for any \( t \in [t', t''] \), and \( \sum_n \gamma_n^{-1} = \infty \). The the minimal solution \( \overline{P}(s,t) \) is stochastic.

Proof. We will apply Proposition 4.3. Let us fix \( a \in \mathbb{A} = \mathbb{Z}_+ \). As \( X = X(\varepsilon) \) we will take a set of the form \( \{0, 1, \ldots, n-1\} \) for a suitable \( n \). Then \( T_{s,a,X} \) is the moment of the first arrival at \( n \) given that we start at \( a \) at the time moment \( s \). To simplify the notation, set \( T_n = T_{s,a,\{0,1,\ldots,n-1\}} \).

In order to estimate \( T_n \) we will compare our inhomogeneous birth-death process to the pure birth homogeneous process with transition rates \( \hat{Q}_{n,n+1} = \gamma_n, \hat{Q}_{n+1,n} = 0 \), for all \( n \in \mathbb{Z}_+ \). Let \( \hat{T}_n \) denote the time of reaching \( n \) given that we start at \( a \) at time \( s \). Note that since this is a pure birth process, once the process leaves \( \{0,1,\ldots,n-1\} \) it never comes back.

It is known that for the pure birth process with rates \( \gamma_n \) the minimal solution is stochastic if and only if \( \sum_n \gamma_n^{-1} = \infty \), see [Fe1], [Fe3, ch. XIV, §8]. By our hypothesis, \( \sum_n \gamma_n^{-1} = \infty \), and we may denote by \( \hat{P}(s,t) \) the unique stochastic solution of the backward and forward equations. Clearly,

\[
\text{Prob}(\hat{T}_n \leq t) = \sum_{b=n}^{\infty} \hat{P}_{ab}(s,t),
\]

which tends to zero as \( n \to \infty \). Thus, the statement of this proposition will follow from Proposition 4.3 if we show that \( \text{Prob}(T_n \leq t) \leq \text{Prob}(\hat{T}_n \leq t) \) for any \( t > s \).

For any \( n = a + 1, a + 2, \ldots \), set

\[
F_n(t) = \text{Prob}(T_n \leq t), \quad \hat{F}_n(t) = \text{Prob}(\hat{T}_n \leq t).
\]
We will prove that $F_n(t) \leq \hat{F}_n(t)$ for all $n > a$ using induction on $n$.

Let us start with $n = a + 1$. Clearly,

$$\hat{F}_{a+1}(t) = 1 - e^{-\gamma_a(t-s)}$$

because the time of the jump $a \to a + 1$ is exponentially distributed with parameter $\gamma_a$. On the other hand, the probability that our birth-death process will jump from $a$ to either $a - 1$ or $a + 1$ before time $t$ equals

$$1 - e^{-\int_s^t (Q_{a,a+1}(t) + Q_{a,a-1}(t)) dt} = 1 - e^{-\int_s^t R_{aa}(t) dt}.$$ 

Thus, $F_{a+1}(t) \leq 1 - \exp(-\int_s^t R_{aa}(t) dt)$, and since $R_{aa}(t) \leq \gamma_a$ by hypothesis, the estimate follows.

In order to prove the induction step, note that for $n \geq a + 2$ we have

$$F_n(t) = \int_s^t dF_{n-1}(\tau) \text{Prob}\{T_n - T_{n-1} \leq t - \tau \mid T_{n-1} = \tau\},$$

$$\hat{F}_n(t) = \int_s^t d\hat{F}_{n-1}(\tau) \text{Prob}\{\hat{T}_n - \hat{T}_{n-1} \leq t - \tau \mid \hat{T}_{n-1} = \tau\}.$$ 

Arguing exactly as in the case $n = a + 1$ above, we see that

$$\text{Prob}\{\hat{T}_n - \hat{T}_{n-1} \leq t - \tau \mid \hat{T}_{n-1} = \tau\} = 1 - e^{-\gamma_{n-1}(t-\tau)},$$

$$\text{Prob}\{T_n - T_{n-1} \leq t - \tau \mid T_{n-1} = \tau\} \leq 1 - e^{-\gamma_{n-1}(t-\tau)}.$$ 

Hence, it suffices to verify that

$$\int_s^t dF_{n-1}(\tau)(1 - e^{-\gamma_{n-1}(t-\tau)}) \leq \int_s^t d\hat{F}_{n-1}(\tau)(1 - e^{-\gamma_{n-1}(t-\tau)}).$$ 

When we integrate by parts both sides of this inequality, we notice that the non-integral terms vanish (because $F_{n-1}(s) = \hat{F}_{n-1}(s) = 0$). Thus, we obtain the equivalent inequality

$$\gamma_{n-1} \int_s^t F_{n-1}(\tau)e^{-\gamma_{n-1}(t-\tau)} dt \leq \gamma_{n-1} \int_s^t \hat{F}_{n-1}(\tau)e^{-\gamma_{n-1}(t-\tau)} dt$$

which immediately follows from $F_{n-1}(\tau) \leq \hat{F}_{n-1}(\tau)$. □

**Remark 4.5.** It is very plausible that Proposition 4.4 holds under the weaker assumption $Q_{n,n+1}(t) \leq \gamma_n$. However, the proof of such a statement would require additional considerations.

**4.3. Birth-death process associated with Meixner polynomials.** From now on we restrict our attention to birth-death processes with

$$Q_{n,n+1} = \alpha(t)(c + n), \quad Q_{n,n-1} = \beta(t)n,$$ 

(4.6)

where $\alpha(t) \geq 0$, $\beta(t) \geq 0$ are continuous functions on $(t_{\min}, t_{\max})$ and $c > 0$ is a constant. Proposition 4.4 implies that for any process of this kind there exists a unique stochastic solution $P(s,t)$ of Kolmogorov’s backward and forward equations. By breaking the time interval into finitely many subintervals, we may as well assume that $\alpha(t)$ and $\beta(t)$ are piecewise continuous functions with finitely many points of discontinuity at which they have finite left and right limits.

The *negative binomial distribution* $\pi_{c,\xi}$ on $\mathbb{Z}_+$ with parameters $c > 0$ and $\xi \in (0,1)$ is defined by

$$\pi_{c,\xi}(n) = (1 - \xi)^c \frac{(c)^n}{n!} \xi^n, \quad n = 0, 1, 2, \ldots .$$

It will be convenient to interpret $\pi_{c,\xi}$ as an infinite row-vector.
Proposition 4.5. Let $\xi(t)$ be a continuous, piecewise continuously differentiable function in $t$ with values in $(0, 1)$. Assume that $\xi(t)$ solves the differential equation

$$\frac{\dot{\xi}(t)}{\xi(t)(1 - \xi(t))} = \frac{\alpha(t) - \beta(t)}{\xi(t)}, \quad t \in (t_{\min}, t_{\max}).$$  \hspace{1cm} \text{(4.7)}$$

Then the row vector $\pi_{c,\xi(t)}$ solves $\pi_{c,\xi(s)}P(s, t) = \pi_{c,\xi(t)}$ for any $s \leq t$.

Proof. Let us differentiate $\pi_{c,\xi(s)}P(s, t)$ with respect to $s$ and use Kolmogorov’s backward equation. Collecting the coefficients of $P_{xy}(s, t)$ in the $y$th coordinate gives

$$\pi_{c,\xi(s)}(x)P_{xy}(s, t) \left(-\frac{c\xi}{1 - \xi} + \frac{c\xi}{\xi} + (\alpha(x) + \beta x)
- \alpha(x + 1)\pi_{c,\xi(s)}(x + 1) - \beta(x + 1)\pi_{c,\xi(s)}(x + 1)\right).$$

Simplifications show that this expression is zero for all $m, n$ if (4.7) holds. The initial condition $\pi_{c,\xi(s)}P(s, t)|_{s=t}=\pi_{c,\xi(t)}$ is obviously satisfied. \qed

Once we have a family of distributions $\pi_{c,\xi(t)}$ satisfying $\pi_{c,\xi(s)}P(s, t) = \pi_{c,\xi(t)}$, we can define a birth-death process by the matrix of transition probabilities $P(s, t)$ (which is uniquely determined by the jump rates) and one-dimensional distributions $\pi_{c,\xi(t)}$.

It is not a priori clear what is a convenient way to parametrize these processes. In particular, multiplying both $\alpha(t)$ and $\beta(t)$ by the same function of $t$ leads only to a reparametrization of time in our process. In order to eliminate this freedom, we will always use one specific choice of time in our processes which we call interior or canonical time of the corresponding process. The convenience of this choice will soon become clear.

The interior time is uniquely determined by the condition that $\alpha(t)$ and $\beta(t)$ are expressed through $\xi(t)$ by

$$\alpha(t) = \left(1 + \frac{\xi(t)}{2\xi(t)}\right)\frac{\xi(t)}{1 - \xi(t)}, \quad \beta(t) = \left(1 - \frac{\xi(t)}{2\xi(t)}\right)\frac{1}{1 - \xi(t)}.$$  \hspace{1cm} \text{(4.8)}$$

Evidently, these formulas imply (4.7). Moreover, for any $(\alpha(t), \beta(t), \xi(t))$ satisfying (4.7), if $\alpha(t)$ and $\beta(t)$ do not vanish simultaneously, we can choose a new time variable $\tau(t)$ with

$$\dot{\tau} = \frac{1}{2} \left(\frac{\alpha}{\xi} + \beta\right) (1 - \xi)$$

so that $(\dot{\tau}^{-1}\alpha(t(\tau)), \dot{\tau}^{-1}\beta(t(\tau)), \xi(t(\tau)))$ satisfy both (4.7) and (4.8) as functions in $\tau$.

Thus, from now on we will parametrize our processes by continuous, piecewise continuously differentiable functions $\xi(t)$ taking values in $(0, 1)$ such that $|\xi(t)/\xi(t)| \leq 2$ (this condition is necessary to guarantee the nonnegativity of $\alpha$ and $\beta$). Such curves $\xi(t)$ will be called admissible. Then the corresponding birth-death process is determined by jump rates given by (4.6), (4.8) and one-dimensional distributions $\pi_{c,\xi(t)}$. We will denote this process by $N_{c,\xi(t)}$. 
In other words, if we set \( A(t) = -\frac{1}{2} \ln \xi(t) \) then \( A(t) \) has to satisfy three conditions: \( A(t) \geq 0 \), for all \( t \); \( |A(t)| \leq 1 \) for all \( t \); and \( A(t) \) is continuous and piecewise continuously differentiable.

In terms of \( A(t) \) it is convenient to single out important special cases: \( A(t) \equiv \text{const} \) corresponds to the homogeneous birth–death process; \( A(t) = t + \text{const} \) corresponds to pure death processes; and \( A(t) = -t + \text{const} \) corresponds to pure birth processes.

Note that in case of a pure birth process \( A(t) \) will hit zero in finite time which means that in terms of the canonical time parametrization, the process reaches infinity in a finite amount of time.

The connection of the processes \( N_{c,\xi} \) with Meixner polynomials discussed in the previous section is already obvious from the fact that the distributions \( \pi_{c,\xi} \) coincide, up to a constant factor, with the weight functions \( W_{c,\xi} \), see (2.7). Our next goal is to express \( P(s,t) \) in terms of the Meixner polynomials. We will use the notation (2.8).

**Proposition 4.6.** The matrix \( P(s,t) \) of transition probabilities for the birth–death process \( N_{c,\xi} \) has the form

\[
P_{xy}(s,t) = \left( \frac{\pi_{c,\xi(t)}(y)}{\pi_{c,\xi(s)}(x)} \right)^{\frac{s}{t}} \sum_{n=0}^{\infty} e^{n(s-t)} \overline{M}_n(x; c, \xi(s)) \overline{M}_n(y; c, \xi(t)),
\]

(4.9)

where \( s \leq t \) and \( x, y \in \mathbb{Z}_+ \).

**Comments.**
1. In the stationary case \( \xi(t) \equiv \text{const} \) this formula was derived by Karlin and McGregor [KMG2] as a part of a much more general formalism, see also [KMG1].
2. The formula implies that \( P(s,t) \) depends on the initial value \( \xi(s) \), final value \( \xi(t) \) and the length \( t - s \) of the time interval. However, \( P(s,t) \) does not depend on the behavior of the curve \( \xi(t) \) inside this time interval, as one might expect.
3. The simplicity of the factor \( e^{n(s-t)} \) is a consequence of our choice of the interior time of the process.
4. Since \( \overline{M}_0(x; c, \xi) = (\pi_{c,\xi}(x))^{\frac{t}{s}} \), the prefactor may be rewritten as

\[
\left( \frac{\pi_{c,\xi(t)}(y)}{\pi_{c,\xi(s)}(x)} \right)^{\frac{s}{t}} = \frac{\overline{M}_0(y; c, \xi(t))}{\overline{M}_0(x; c, \xi(s))}.
\]

5. The formula implies that the kernel on \( \mathbb{Z}_+ \times \mathbb{Z}_+ \)

\[
(x, y) \mapsto \sum_{n=0}^{\infty} q^n \overline{M}_n(x; c, \zeta) \overline{M}_n(y; c, \eta)
\]

(4.10)

takes nonnegative values for \( \zeta, \eta \in (0, 1) \) and \( 0 < q \leq \min \left\{ \sqrt{\frac{\zeta}{\eta}}, \sqrt{\frac{\eta}{\zeta}} \right\} \). The bound on \( q \) follows from the inequality \( |\zeta/\xi| \leq 2 \).

Our proof of Proposition 4.6 consists of few steps. Let us denote the right–hand side of (4.9) by \( \hat{P}_{xy}(s,t) \).
Lemma 4.7. The following relations hold

\[ \sqrt{\xi(x+1)(x+c)} \mathcal{M}_n(x+1; c, \xi) + \sqrt{\xi(x+c-1)} \mathcal{M}_n(x-1; c, \xi) \]

\[ -(x+1+c)\mathcal{M}_n(x; c, \xi) = -n(1-\xi)\mathcal{M}_n(x; c, \xi), \]  
(4.11)

\[ 2\xi(1-\xi)\frac{\partial}{\partial \xi} \mathcal{M}_n(x; c, \xi) + \sqrt{\xi(x+1)(x+c)} \mathcal{M}_n(x+1; c, \xi) \]

\[ - \sqrt{\xi(x+c-1)} \mathcal{M}_n(x-1; c, \xi) = 0. \]  
(4.12)

Proof. Straightforward computation using

\[ \xi^2 \frac{\partial}{\partial \xi} \mathcal{M}_n(x; c, \xi) = \frac{nx}{c} \mathcal{M}_{n-1}(x-1, c+1, \xi) \]

and [KS, 1.9.5, 1.9.6, 1.9.8]. \(\square\)

Proof of Proposition 4.6. First of all, we need to verify that \(\hat{P}(s, t)\) (the right-hand side of (4.9)) satisfies the backward equation. This is the equality

\[ -\frac{\partial}{\partial s} \hat{P}_{xy}(s, t) = Q_{xx}(s)\hat{P}_{xy}(s, t) + Q_{x,x+1}(s)\hat{P}_{x+1,y}(s, t) + Q_{x,x-1}(s)\hat{P}_{x-1,y}(s, t) \]

(4.13)

with \(Q_{xx}(s) = -Q_{x,x+1}(s) - Q_{x,x-1}(s)\) and

\[ Q_{x,x+1}(s) = (c+x) \left(1 + \frac{\xi(s)}{2\xi(s)}\right) \frac{\xi(s)}{1-\xi(s)}, \quad Q_{x,x-1}(s) = x \left(1 + \frac{\xi(s)}{2\xi(s)}\right) \frac{1}{1-\xi(s)}. \]

The computation proceeds as follows. One substitutes the sum in the right-hand side of (4.9) into the needed equality (4.13) and collects the coefficients of \(\mathcal{M}_n(y; c, \xi(t))\) using the relation \(\partial/\partial s = \xi(s) \partial/\partial(\xi(s))\). Each such coefficient has two parts: one of them does not involve \(\xi(s)\) while the other one is equal to \(\xi(s)\) times an expression not involving \(\xi(s)\). It turns out that each of these parts vanishes, for the first part the needed relation is (4.11), and for the second part one uses (4.12). The details are tedious but straightforward, and we omit them.

As was mentioned before, it remains to prove that \(\hat{P}_{xy}(s, t)\) is always nonnegative.

Let us use the notation \(P^+(s, t), P^-(s, t)\) for \(P(s, t)\) when we consider a pure birth or a pure death process (that is, \(\xi(\tau) = e^{const+2\tau}\) or \(\xi(\tau) = e^{const-2\tau}\), respectively).
Lemma 4.8. $P^\dagger(s, t)$ is the unique solutions of the backward equation for the pure death process with the initial condition $P(t, t) \equiv 1$, and $P^\ddagger(s, t)$ is the unique solution of the forward equation for the pure birth process with the same initial condition. Furthermore, with the notation $\zeta = \xi(s)$, $\eta = \xi(t)$, we have

$$P^\dagger_{xy}(s, t) = \begin{cases} \frac{(1-\zeta)\eta^x}{(1-\eta)\zeta} \frac{\zeta-\eta}{(1-\zeta)\eta} \frac{x!}{(x-y)!}, & x \geq y, \\ 0, & x < y, \end{cases} \quad (4.14)$$

$$P^\ddagger_{xy}(s, t) = \begin{cases} \frac{1-\eta}{1-\zeta} \frac{c+x}{1-\zeta} \frac{y-x}{(y-x)!}, & x \leq y, \\ 0, & x > y. \end{cases} \quad (4.15)$$

Proof. Consider $P^\dagger(s, t)$ first. Since $Q_{x, x+1} \equiv 0$, Kolmogorov’s backward equation takes the form

$$-\frac{\partial}{\partial s} P^\dagger_{xy}(s, t) = -Q_{x, x-1} P^\dagger_{xy}(s, t) + Q_{x, x-1} P^\dagger_{x-1, y}(s, t) = \frac{2x}{1-\zeta(s)} (-P^\dagger_{xy}(s, t) + P^\dagger_{x-1, y}(s, t)). \quad (4.16)$$

If we fix $y$ then these differential equations can be solved recursively: we subsequently find $P^\dagger_{0, y}$, $P^\dagger_{1, y}$, $P^\dagger_{2, y}$, \ldots, using the initial conditions $P^\dagger_{j, y}(t, t) = \delta_{xy}$. This shows that the backward equation for the pure death process has a unique solution. A straightforward calculation shows that the expression in the right–hand side of (4.14) satisfies this equation (with $\xi(s) = e^{\text{const} - 2s}$).

The case of the pure birth process is completely analogous. $\square$

Since for the pure death process the backward equation has a unique solution, we have just shown that $\tilde{P}(s, t) = P^\dagger(s, t)$ is the corresponding transition matrix.

In order to make a similar conclusion for the pure birth process, we need to know that $\tilde{P}(s, t)$ satisfies the forward equation. This fact can be proved directly using Lemma 4.7. It can also be reduced to the case of the backward equation as follows.

Note that $\pi_{c, \tilde{\xi}(s)}(x) \tilde{P}_{xy}(s, t)$ remains invariant under the changes

$$s \mapsto -t, \quad t \mapsto -s, \quad \tilde{\xi}(\tau) \mapsto \tilde{\xi}(\tau) := \xi(-\tau), \quad x \leftrightarrow y. \quad (4.17)$$

Thus, instead of computing $\frac{\partial}{\partial u} \tilde{P}_{xy}(s, t)$ we may compute

$$-\frac{\partial}{\partial u} \left( \pi_{c, \tilde{\xi}(u)}(y) \tilde{P}_{yx}(u, v) (\pi_{c, \tilde{\xi}(v)}(x))^{-1} \right) \bigg|_{u=-t, v=-s}$$

with the new $\tilde{\xi}(\tau)$ obtained from $\xi(\tau)$ by the time inversion. Since we already know that $\tilde{P}_{xy}(s, t)$ solves the backward equation, for $\tilde{\xi}(\tau) = e^{\text{const} + 2\tau}$ we obtain,\footnote{The argument goes through for any admissible curve $\xi(\cdot)$, it just becomes more tedious.} cf. (4.16),

$$-\frac{\partial}{\partial u} \frac{\pi_{c, \tilde{\xi}(u)}(y) \tilde{P}_{yx}(u, v)}{\pi_{c, \tilde{\xi}(v)}(x)} = \left( -\frac{2c \tilde{\xi}(u)}{1-\tilde{\xi}(u)} + 2y \right) \frac{\pi_{c, \tilde{\xi}(u)}(y) \tilde{P}_{yx}(u, v)}{\pi_{c, \tilde{\xi}(v)}(x)}$$

$$+ \frac{2y}{1-\tilde{\xi}(u)} \frac{\pi_{c, \tilde{\xi}(u)}(y)}{\pi_{c, \tilde{\xi}(v)}(x)} \left( -\tilde{P}_{yx}(u, v) + \tilde{P}_{y-1, x}(u, v) \right).$$
Using \( \pi_{c,\xi}(y) = \pi_{c,\xi}(y-1) \cdot (c+y-1)\xi(u)/y \) and substituting \( u = -t, v = -s \), we obtain the needed forward equation

\[
\frac{\partial}{\partial t} \hat{P}_{xy}(s,t) = -\frac{2(c+y)\xi(t)}{1-\xi(t)} \hat{P}_{xy}(s,t) + \frac{2(c+y-1)\xi(t)}{1-\xi(t)} \hat{P}_{x,y-1}(s,t).
\]

The conclusion is that in case of the pure birth process, \( \hat{P}(s,t) \) satisfies the forward equation, and by Lemma 4.8 we have \( \hat{P}(s,t) = P^\tau(s,t) \).

The nonnegativity of \( \hat{P}(s,t) \) for arbitrary admissible curves \( \xi(\cdot) \) follows from

Lemma 4.9. Let \( \xi(\cdot) \) be an admissible curve and \( N_{c,\xi(\cdot)} \) be the corresponding birth–death process. Then for any \( s < t \), \( \hat{P}(s,t) \) is a product of \( P^1(s,u) \) with \( \xi(\tau) = e^{-2(\tau-s)+\ln \xi(s)} \) and \( P^1(u,t) \) with \( \xi(t) = e^{2(\tau-t)+\ln \xi(t)} \) for a certain choice of \( u \). Specifically, \( u \) is determined from the continuity condition:

\[
e^{-2(u-s)+\ln \xi(s)} = e^{2(u-t)+\ln \xi(t)} \iff u = \frac{s + t}{2} + \frac{\ln \xi(s) - \ln \xi(t)}{4}.
\]

Proof. This statement follows from the Chapman–Kolmogorov equation (4.1), which \( \hat{P}(s,t) \) obviously satisfies due to the orthogonality of Meixner polynomials, and from the fact that \( \hat{P}(s,t) \) does not depend on the specific form of the curve \( \xi(\cdot) \), see Comment 2 after the statement of Proposition 4.6. Thus, we may just replace \( \xi(\tau) \) by a continuous combination of \( e^{-2\tau+\text{const}} \) and \( e^{2\tau+\text{const}} \) and preserve \( \xi(s) \), \( \xi(t) \), and \( t-s \). Note that the fact that \( u \) given by the formula above is between \( s \) and \( t \) follows from the inequality \( |\xi/\xi| < 2 \). □

Lemma 4.9 implies that \( \hat{P}_{xy}(s,t) \) is always nonnegative, and this completes the proof of Proposition 4.6. □

Corollary 4.10. The process obtained from \( N_{c,\xi(\cdot)} \) by the time reversal is also of the form \( N_{c,\xi(\cdot)} \) with \( \tilde{\xi}(\tau) = \xi(-\tau) \).

Proof. \( N_{c,\xi(\cdot)} \) is characterized by the fact that it is a Markov process with two-dimensional distributions

\[
\text{Prob}\{N_{c,\xi(s)} = x, N_{c,\xi(t)} = y\} = \pi_{c,\xi(s)}(x)P_{xy}(s,t).
\]

As was already mentioned above, the right-hand side of (4.9) multiplied by \( \pi_{c,\xi(s)}(x) \) is invariant with respect to (4.17). This implies the statement. □

Note that, in particular, time inversion turns our pure birth process into the pure death process and vice versa (essentially, we gave a proof of this fact before Lemma 4.9), and the stationary process \( N_{c,\xi} \) with \( \xi \equiv \text{const} \) is reversible. This is well known; any stationary birth–death process with an invariant measure is reversible with respect to this measure.

4.4. Markov processes on partitions. Our next goal is to extend birth–death processes \( N_{c,\xi(\cdot)} \) to partitions in the following sense. We construct continuous time Markov processes on the state space \( Y \) (the set of all Young diagrams, see §1) parametrized by admissible pairs \( (z, z') \), see §1, and admissible curves \( \xi(\cdot) \). The
projection of such a process on $\mathbb{Z}_+$ obtained by looking at the number of boxes of the random Young diagrams, coincides with $N_{zz',\xi(\cdot)}$.

Let us fix a pair $(z, z')$ of admissible parameters and set $c = zz' > 0$. Given an admissible curve $\xi(\cdot)$, we define the matrix $Q$ of jump rates of our future Markov process $\Lambda_{z,z',\xi}$ on $\mathbb{Y}$ by (set $n = |\lambda|$)

$$Q_{\lambda\mu}(s) = \begin{cases} (c + n) \left(1 + \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{\xi(s)}{1 - \xi(s)} \cdot p^+_{zz'}(n, \lambda; n + 1, \mu), & \lambda \nearrow \mu, \\ a \left(1 - \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{1}{1 - \xi(s)} \cdot p^i(n, \lambda; n - 1, \mu), & \lambda \searrow \mu, \\ -(c + n) \left(1 + \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{\xi(s)}{1 - \xi(s)} - n \left(1 - \frac{\dot{\xi}(s)}{2\xi(s)}\right) \frac{1}{1 - \xi(s)}, & \mu = \lambda, \\ \end{cases}$$

and $Q_{\lambda\lambda} = 0$ in all other cases. Here $p^+_{zz'}$ and $p^i$ are transition and cotransition probabilities from $\lambda$, see (1.9) and (1.1), and the expressions involving $\xi$ come from (4.8). Note that under the projection $\mathbb{Y} \to \mathbb{Z}_+$, $\lambda \mapsto |\lambda|$, this matrix $Q$ turns into the matrix of jump rates for $N_{c,\xi(\cdot)}$.

**Proposition 4.11.** The minimal solution $P(s, t)$ of Kolmogorov’s backward equation with the matrix $Q$ defined above is stochastic.

**Proof.** We apply Propositions 4.3, 4.4. In the proof of Proposition 4.4 it was shown that for any $a \in \mathbb{Z}_+$ there exists a set of the form $X = \{0, 1, \ldots, n - 1\}$ such that the probability of exiting $X$ during the time period from $s$ to $t$ with the initial state $a$ is smaller than any given positive number $\varepsilon$. This means that if we start at time $s$ from $\lambda \in \mathbb{Y}$ with $|\lambda| = a$ then the probability of exiting $\mathbb{Y}_0 \cup \mathbb{Y}_1 \cup \cdots \cup \mathbb{Y}_{n-1}$ before time $t$ is just the same as for the birth-death process and, hence, is less than $\varepsilon$. Proposition 4.3 concludes the proof. $\square$

**Proposition 4.12 (cf. Proposition 4.5).** For any $s < t$

$$M_{z,z',\xi(\cdot)}(s) P(s, t) = M_{z,z',\xi(\cdot)}(t),$$

where $P(s, t)$ is the transition matrix of Proposition 4.11, and $M_{z,z',\xi}$ is the mixed $z$-measure (1.11) viewed as a row-vector with coordinates marked by elements of $\mathbb{Y}$.

**Proof.** Since the formula obviously holds for $s = t$, it suffices to show that the derivative with respect to $s$ of the left–hand side of (4.20) vanishes. Thus, it suffices to show that

$$-\frac{\partial}{\partial s} M_{z,z',\xi(\cdot)}(\mu) + \sum_{\lambda \in \mathbb{Y}} M_{z,z',\xi(\cdot)}(\lambda) Q_{\lambda\mu} = 0$$

for any $\mu \in \mathbb{Y}$. Recall that

$$M_{z,z',\xi}(\lambda) = M_{z,z'}^{(n)}(\lambda) \pi_{c,\xi(n)} \quad \text{with} \quad c = zz', \quad n = |\lambda|.$$  

Substituting this relation into (4.21) we notice that we can perform the summation over $\lambda$ using (1.5) and (1.6). Factoring out $M_{z,z'}^{(\mu)}(\mu)$ leads to the formula which states that the derivative of $\pi_{c,\xi(\cdot)} P(s, t)$ with $P(s, t)$ being the transition matrix
for the birth-death process $N_{e,\xi}$, with respect to $s$ vanishes. But this has already been proved in Proposition 4.5. □

We conclude that given an admissible pair $(z, z')$ and an admissible curve $\xi(\tau)$, there exists a unique continuous time Markov process on $\mathbb{Y}$ with jump rates $Q$ defined above and with one-dimensional distributions $M_{z,z':\xi(\tau)}$. This Markov process will be denoted by $\Lambda_{z,z':\xi(\cdot)}$.

As for the birth–death processes, we single out three important special cases: the stationary process $\xi \equiv \text{const}$, the ascending process $\xi(\tau) = e^{2\tau + \text{const}}$ and the descending process $\xi(\tau) = e^{-2\tau + \text{const}}$. The projections of these processes on $\mathbb{Z_+}$ are the stationary birth–death process, the pure birth and the pure death processes, respectively.

As in §1, for $\lambda \in \mathbb{Y}$ we denote by $\dim \lambda$ the number of ascending paths in the Young graph leading from $\emptyset$ to $\lambda$. More generally, we denote by $\dim(\mu, \lambda)$ the number of ascending paths in $\mathbb{Y}$ leading from $\mu$ to $\lambda$; if there are no such paths we set $\dim(\mu, \lambda) = 0$. Also, for $\mu, \lambda \in \mathbb{Y}$ such that $\mu \subset \lambda$ we set

$$ (x)_{\lambda \setminus \mu} = \prod_{(i,j) \in \lambda \setminus \mu} (x + j - i), \quad x \in \mathbb{C}, $$

where the product is taken over all boxes in $\lambda \setminus \mu$.

**Proposition 4.13** (cf. Lemma 4.8). The transition matrix of the descending process $\Lambda_{z,z':\xi(\cdot)}$ has the form

$$ P_{\Lambda_{\mu}}(s,t) = \left( \frac{(1-\zeta)\eta}{(1-\eta)\zeta} \right)^x \left( \frac{\zeta - \eta}{1 - \zeta} \right)^{x-y} \frac{x!}{(x-y)!y!} \frac{\dim \mu \dim (\mu, \lambda)}{\dim \lambda} \tag{4.22} $$

and the transition matrix of the ascending process $\Lambda_{z,z':\xi(\cdot)}$ has the form

$$ P_{\Lambda_{\mu}}(s,t) = \left( \frac{1-\eta}{1-\zeta} \right)^{zz' + x} \left( \frac{\eta - \zeta}{1 - \zeta} \right)^{y-x} \frac{x!}{(y-x)!y!} \frac{\dim \mu \dim (\lambda, \mu)}{\dim \lambda} \cdot (z)_{\mu \setminus \lambda} \cdot (z')_{\mu \setminus \lambda} \tag{4.23} $$

where $\zeta = \xi(s)$, $\eta = \xi(t)$, $x = |\lambda|$, $y = |\mu|$.

**Proof.** Let us consider the descending process first. It is immediate to check that the matrix $P_{\Lambda_{\mu}}(s,t)$ obtained from the transition matrix $P_{\Lambda_{\mathbb{2}}}^\downarrow(s,t)$ of the pure death process by

$$ P_{\Lambda_{\mu}}(s,t) = P_{\Lambda_{\mathbb{2}}}^\downarrow(s,t) \times \sum p_{\downarrow}(x, \lambda; x-1, \mu^{(x-y-1)})p_{\downarrow}(x-1, \mu^{(x-y-1)}; x-2, \mu^{(x-y-2)}) \cdots p_{\downarrow}(y+1, \lambda^{(1)}; y, \mu) $$

where the sum is taken over all paths $\mu = \mu^{(0)} \nearrow \mu^{(1)} \nearrow \cdots \nearrow \mu^{(x-y)} = \lambda$ from $\mu$ to $\lambda$, satisfies the backward equation. All terms in the above sum are equal to $\dim \mu / \dim \lambda$, and the number of terms is equal to $\dim(\mu, \lambda)$. Together with (4.14) this implies (4.22).

Similarly, for the ascending process one has

$$ P_{\Lambda_{\mu}}(s,t) = P_{\Lambda_{\mathbb{2}}}^\uparrow(s,t) \times \sum p_{\mathbb{2}}^\uparrow(x, \lambda; x+1, \lambda^{(1)})p_{\mathbb{2}}^\uparrow(x+1, \lambda^{(1)}; x+2, \lambda^{(2)}) \cdots p_{\mathbb{2}}^\uparrow(y-1, \lambda^{(y-x-1)}; y, \mu) $$
where the sum is taken over all paths $\lambda = \lambda^{(0)} \triangleright \lambda^{(1)} \triangleright \cdots \triangleright \lambda^{(e-y)} = \mu$ from $\lambda$ to $\mu$. Again, the product of transition probabilities does not depend on the path and it is equal to
\[
\frac{1}{(c + x)^{y-x} y! \dim \lambda} \cdot (z)_{\mu|\lambda}(z')_{\mu|\lambda}
\]
while the number of paths is equal to $\dim(\lambda, \mu)$. Together with (4.15) this gives (4.23). \qed

5. Transition matrix for integral values of $z$.

Our main goal in this section is to obtain a formula for the transition matrix of the process $\Lambda_{z', \xi(\cdot)}$ in the case when $z$ is a nonnegative integer. For $z = 1$, the process $\Lambda_{z', \xi(\cdot)}$ coincides with the birth–death process $N_{z', \varepsilon}$ (because it lives on the Young diagrams with only one row), and our formula is reduced to (4.9).

Fix $z = N \in \{1, 2, \ldots\}$. In order for $(z, z')$ to be an admissible pair, we must have $z' \in \mathbb{R}$ and $z' > N - 1$. We will use the notation $z' = N + \alpha$, $\alpha > -1$. As before, we set $c = zz' = N(N + \alpha)$.

As was mentioned in §1, the support of $M_{N, N+\alpha, \xi}$ consists of the Young diagrams with no more than $N$ rows. It is convenient to parameterize such diagrams $\lambda$ by sequences of $N$ strictly decreasing nonnegative integers $(x_1, \ldots, x_N)$,
\[
x_i = \lambda_i + N - i, \quad i = 1, \ldots, N.
\]
Given an admissible curve $\xi(\cdot)$, set
\[
v_{s,t}(x, y) = \sum_{k=0}^{\infty} e^{k(s-t)} \tilde{M}_k(x; \alpha, \xi(s)) \tilde{M}_k(y; \alpha, \xi(t)), \quad x, y \in \mathbb{Z}_+.
\]

**Theorem 5.1.** Let $\lambda$, $\mu$ be Young diagrams with no more than $N$ rows, and let $(x_1, \ldots, x_N)$, $(y_1, \ldots, y_N)$ be the corresponding sets of decreasing nonnegative integers. For any admissible curve $\xi(\cdot)$ the transition matrix of the Markov process $\Lambda_{N, N+\alpha, \xi(\cdot)}$ has the form
\[
P_{\lambda\mu}(s, t) = e^{(z - \alpha)(N - N - 1)/2} \left( \frac{M_{N, N+\alpha, \xi(t)}(\mu)}{M_{N, N+\alpha, \xi(s)}(\lambda)} \right)^{1/2} \det [v_{s,t}(x_i, y_j)]_{i,j=1}^N.
\]

We will use the term Karlin–McGregor representation for this formula.

**Proof.** The arguments follow the same pattern as in the proof of Proposition 4.6 (which is a special case of this theorem). The first step is to show that the right-hand side of (5.2) satisfies Kolmogorov’s backward equation. After that we prove that this solution is stochastic.

We will use the notation
\[
x = (x_1, \ldots, x_N), \quad y = (y_1, \ldots, y_N), \quad \varepsilon_r = (0, \ldots, 0, 1, 0, \ldots, 0), \quad 1 \leq r \leq N,
\]
\[
\zeta = \xi(s), \quad \eta = \xi(t), \quad n = |\lambda| = \sum_{i=1}^{N} x_i - \frac{N(N - 1)}{2},
\]
\[
f_k(\cdot) = \tilde{M}_k(\cdot; \alpha, \xi(s)), \quad g_k(\cdot) = \tilde{M}_k(\cdot; \alpha, \xi(t)).
\]
Also, denote the right–hand side of (5.2) by \( \hat{P}_{xy}(s, t) \).

The formulas of §1 imply

\[
\left( \frac{M_{N,N+\alpha,\xi(t)}(\mu)}{M_{N,N+\alpha,\xi(s)}(\lambda)} \right)^{\frac{1}{2}} = \frac{\eta}{\xi} \left( \prod_{j=1}^{N} \Gamma(x_j + \alpha + 1) \Gamma(y_j + 1) \right)^{\frac{1}{2}} \frac{V(y)}{V(x)}
\]

where \( V(u) = \prod_{1 \leq i < j \leq N} (u_i - u_j) \) stands for the Vandermonde determinant.

Note that adding to \( \lambda \) one box or removing from \( \lambda \) one box is equivalent to adding \( \varepsilon_r \) to \( x \) or subtracting \( \varepsilon_r \) from \( x \), where \( r \) is the row number of the box. We have, cf. (1.9), (1.1),

\[
p_{N,N+\alpha}(n, x; n+1, x + \varepsilon_r) = \frac{x_r + \alpha + 1}{c + n} \frac{V(x + \varepsilon_r)}{V(x)},
\]

\[
p^{\dagger}(n, x; n-1, x - \varepsilon_r) = \frac{x_r}{n} \frac{V(x - \varepsilon_r)}{V(x)}.
\]

The right-hand sides of these relations conveniently vanish exactly when \( x \pm \varepsilon_r \) does not represent a Young diagram (two of the coordinates of \( x \pm \varepsilon_r \) are equal). Using (4.19) we can now write down the needed backward equation:

\[
-\frac{\partial}{\partial s} \hat{P}_{xy}(s, t) = - \left( 1 + \frac{\hat{\xi}(s)}{2 \hat{\xi}(s)} \right) \frac{\xi(s)(c + n)}{1 - \xi(s)} + \left( 1 - \frac{\hat{\xi}(s)}{2 \hat{\xi}(s)} \right) \frac{n}{1 - \xi(s)} \hat{P}_{xy}(s, t)
\]

\[
+ \left( 1 + \frac{\hat{\xi}(s)}{2 \hat{\xi}(s)} \right) \frac{\xi(s)}{1 - \xi(s)} \sum_{r=1}^{N} \frac{(x_r + \alpha + 1)V(x + \varepsilon_r)}{V(x)} \hat{P}_{x+\varepsilon_r,y}(s, t)
\]

\[
+ \left( 1 - \frac{\hat{\xi}(s)}{2 \hat{\xi}(s)} \right) \frac{1}{1 - \xi(s)} \sum_{r=1}^{N} \frac{x_r V(x - \varepsilon_r)}{V(x)} \hat{P}_{x-\varepsilon_r,y}(s, t).
\]

It is time to use the definition of \( \hat{P}_{xy}(s, t) \). The Cauchy–Binet identity (see, e.g., [Ga, ch. I, §2]) implies (in all determinants below the indices run from 1 to \( N \))

\[
det[v_{s,t}(x_i, y_j)] = \det \left[ \sum_{k=0}^{\infty} e^{(s-t)k} f_k(x_i) g_k(y_j) \right]
\]

\[
= \sum_{k_1 > k_2 > \ldots > k_N \geq 0} e^{(s-t)(k_1 + \ldots + k_N)} \det[f_{k_i}(x_j)] \det[g_{k_j}(y_j)].
\]

Let us use this relation and (5.2) for \( \hat{P}_{xy}(s, t), \hat{P}_{x+\varepsilon_r,y}(s, t) \) in the backward equation and collect the coefficients of \( \det[g_{k_j}(y_j)] \). Factoring out

\[
\left( \frac{M_{N,N+\alpha,\xi(t)}(\mu)}{M_{N,N+\alpha,\xi(s)}(\lambda)} \right)^{\frac{1}{2}} e^{(s-t)(k_1 + \ldots + k_N - \frac{N(N-1)}{2})}
\]
we obtain (Vandermonde determinants cancel out quite conveniently)

\[
\left( \frac{N(N-1)}{2} - (k_1 + \cdots + k_N) + \frac{\xi(s)}{2} \left( n - \frac{c}{\zeta - 1 - \zeta} \right) \right) \det[f_k(x_j)]
\]

\[-\frac{\partial}{\partial s} \det[f_k(x_j)] = -\left( 1 + \frac{\dot{\xi}(s)}{2\zeta} \right) \frac{\zeta(c+n)}{1-\zeta} + \left( 1 - \frac{\dot{\xi}(s)}{2\zeta} \right) \frac{n}{1-\zeta} \det[f_k(x_j)]
\]

\[+ \left( 1 + \frac{\dot{\xi}(s)}{2\zeta} \right) \frac{1}{1-\zeta} \sum_{r=1}^{N} \sqrt{(x_r + 1)(x_r + \alpha + 1)} \zeta \det[f_k((x + \epsilon_r)_j)]
\]

\[+ \left( 1 - \frac{\dot{\xi}(s)}{2\zeta} \right) \frac{1}{1-\zeta} \sum_{r=1}^{N} \sqrt{x_r(x_r + \alpha)} \zeta \det[f_k((x - \epsilon_r)_j)].\]

We claim that the part the left–hand side of this relation that does not involve derivatives \(\xi(s)\) and \(-\frac{\partial}{\partial s}\) equals the part of the right–hand side without \(\dot{\xi}(s)\), and the part of the left–hand side with derivatives equals that of the right–hand side with \(\dot{\xi}(s)\).

The part without derivatives gives

\[
\left( \zeta(c+n) + n + \frac{N(N-1)}{2} (1-\zeta) - (k_1 + \cdots + k_N)(1-\zeta) \right) \det[f_k(x_j)]
\]

\[= \sum_{r=1}^{N} \left( \sqrt{(x_r + 1)(x_r + \alpha + 1)} \det[f_k((x + \epsilon_r)_j)]
\]

\[+ \sqrt{x_r(x_r + \alpha)} \zeta \det[f_k((x - \epsilon_r)_j)] \right). \quad (5.6)
\]

The right–hand side of this equality can be rewritten as

\[
\sum_{\sigma \in S_N} \prod_{r=1}^{N} \text{sgn} \sigma f_{\sigma(1)}(x_1) \cdots f_{\sigma(N)}(x_N)
\]

\[\times \frac{\sqrt{(x_r + 1)(x_r + \alpha + 1)} f_{\sigma(r)}(x_r + 1) + \sqrt{x_r(x_r + \alpha)} f_{\sigma(r)}(x_r - 1)}{f_{\sigma(r)}(x_r)}.
\]

By (4.11) the last ratio equals \(x_r(1+\zeta) + (\alpha + 1)\zeta - k_{\sigma(r)}(1-\zeta)\), and the whole expression equals

\[
((x_1 + \cdots + x_N)(1+\zeta) + N(\alpha + 1)\zeta - (k_1 + \cdots + k_N)(1-\zeta)) \det[f_k(x_j)],
\]

which is exactly the left–hand side of (5.6) with \(c = N(N + \alpha)\) and \(n = \sum x_i - N(N - 1)/2\).

The part of (5.5) with derivatives gives

\[-\frac{2\zeta(1-\zeta)}{\xi(s)} \frac{\partial}{\partial s} \det[f_k(x_j)] = \sum_{r=1}^{N} \left( \sqrt{(x_r + 1)(x_r + \alpha + 1)} \zeta \det[f_k((x + \epsilon_r)_j)]
\]

\[-\sqrt{x_r(x_r + \alpha)} \zeta \det[f_k((x - \epsilon_r)_j)] \right).\]
The same operation with determinants and (4.12) show that this equality holds.

This concludes the proof of the fact that \( \hat{P}_{\lambda \mu}(s, t) \) (the right-hand side of (5.2)) satisfies the backward equation for \( \Lambda_{z, z', \xi(\cdot)} \). It remains to verify that the \( \mathbb{Y} \times \mathbb{Y} \) matrix \( \hat{P}_{\lambda \mu}(s, t) \) is stochastic.

Let us check that \( \sum_{\mu} \hat{P}_{\lambda \mu}(s, t) = 1 \) for any \( \lambda \in \mathbb{Y} \) and \( s < t \).

Using elementary row operations on the Vandermonde matrix, we obtain

\[
V(u) = \prod_{1 \leq i < j \leq N} (u_i - u_j) = \det[u_j^{N-i}] = (-1)^{N(N-1)/2} \det[\mathfrak{M}_{N-i}(u_j; \alpha, \xi)]
\] (5.7)

where the sign \((-1)^{N(N-1)/2}\) appears because the highest coefficient of \( \mathfrak{M}_k(u; \alpha, \xi) \) is \((-1)^k\). Then (5.2) and (5.3) imply

\[
\hat{P}_{\lambda \mu}(s, t) = e^{(\tau-t)/2} \frac{\det[g_{N-i}(y_j)]}{\det[f_{N-i}(x_j)]} \det[v_{s,t}(x_i, y_j)].
\]

Apply (5.4) and sum the result over all \( y_1 > \cdots > y_N \geq 0 \). The Cauchy–Binet identity gives

\[
\sum_{y_1 > \cdots > y_N \geq 0} \det[g_{k_i}(y_j)] \det[g_{N-i}(y_j)] = \det \left[ \sum_{u=0}^{\infty} g_{k_i}(u) g_{N-i}(u) \right].
\] (5.8)

Orthogonality of Meixner polynomials means that \( \sum_{u \geq 0} g_{k_i}(u) g_{l}(u) = \delta_{kl} \). Hence, the last determinant equals 1 if \( k_i = N - i \) for all \( i = 1, \ldots, N \) and vanishes otherwise. This gives the desired result.

Finally, the nonnegativity of \( \hat{P}_{\lambda \mu}(s, t) \) follows from Proposition 4.13 and

**Lemma 5.2 (cf. Lemma 4.9).** For any admissible curve \( \xi(\cdot) \), the matrix \( \hat{P}(s, t) \) given by the right-hand side of (5.2) is the product of the transition matrix \( P^\downarrow(s, u) \) for the descending process with \( \xi(\tau) = e^{-2(\tau-s) + \ln \xi(s)} \) and the transition matrix \( P^\uparrow(u, t) \) for the ascending process with \( \xi(\tau) = e^{2(\tau-t) + \ln \xi(t)} \) with \( u \) given by (4.18):

\[
u = \frac{s + t}{2} + \frac{\ln \xi(s) - \ln \xi(t)}{4}.
\]

The proof of this lemma is very similar to that of Lemma 4.9. The Chapman–Kolmogorov equation for \( \hat{P}(s, t) \) is easily verified by means of (5.4), the orthogonality of Meixner polynomials, and the trick with the Cauchy–Binet formula used above.

Since the transition matrices of the descending and ascending processes have nonnegative matrix elements (4.22), (4.23), the matrix elements of \( \hat{P}(s, t) \) are also nonnegative, and the proof of Theorem 5.1 is complete. □

**Corollary 5.3.** The kernel (4.10) is totally positive.

Indeed, Theorem 5.1 shows that the minors of this kernel are, up to positive factors, transition probabilities.
§6. Analytic continuation

The goal of this section is to employ the analytic continuation in the parameters \(z, z'\) in two directions. First, we show that the factorization of the transition matrix for \(\Lambda_{z,z';\xi(\cdot)}\) into the product of transition matrices for descending and ascending processes is carried over from the case of integral \(z\) (Lemma 5.2 above) to the general case. Second, we prove an analog of Lemma 3.11 for finite-dimensional distributions of \(\Lambda_{z,z';\xi(\cdot)}\). This result will be used later to compute the dynamical correlation functions of \(\Lambda_{z,z';\xi(\cdot)}\).

**Proposition 6.1.** The statement of Lemma 5.2 holds for arbitrary admissible pair of parameters \((z, z')\). That is, the transition matrix \(P(s, t)\) for \(\Lambda_{z,z';\xi(\cdot)}\) is equal to the product \(P_{\downarrow}(s, u)P_{\uparrow}(u, t)\) with the descending and ascending processes and the time moment \(u\) specified as in Lemma 5.2.

**Comments.**

1. This statement together with Proposition 4.13 allows to write \(P(s, t)\) down rather explicitly. Namely, we obtain

\[
P_{\lambda\nu}(s, t) = \frac{(\zeta - \theta)^{l}(\eta - \theta)^{m}(1 - \eta)^{zz'}(1 - \theta)^{l} \dim \nu}{l! \dim \lambda} \times \sum_{\mu \in Y} \left(1 - \eta\right)^{m} \frac{(1 - \theta)^{m} \eta^{m}}{(l - m)! (n - m)!} \frac{\dim(\mu, \lambda) \dim(\mu, \nu)}{(z_{\mu} z'_{\mu})_{\mu}}
\]

with the notation

\[
\zeta = \xi(s), \quad \eta = \xi(t), \quad \theta = e^{s-t} \sqrt{\zeta \eta}, \quad l = |\lambda|, \quad m = |\mu|, \quad n = |\nu|.
\]

The sum above is actually finite; \(\mu\) ranges over all Young diagrams which are smaller than both \(\lambda\) and \(\nu\).

2. The formula for \(P_{\lambda\mu}(s, t)\) above after the multiplication by

\[
M_{z,z';\xi}(\lambda) = (1 - \zeta)^{zz'} \zeta^{l}(z_{\lambda} z'_{\lambda})_{\lambda} \frac{\dim^{2} \lambda}{l!}
\]

becomes symmetric with respect to \((\lambda, \zeta) \leftrightarrow (\nu, \eta)\). This shows that the time reversion of \(\Lambda_{z,z';\xi(\cdot)}\) is again a process of this form with new \(\tilde{\xi}(\tau) = \xi(-\tau)\), cf. Corollary 4.10.

**Proof.** Set \(\widehat{P}(s, t) = P_{\downarrow}(s, u)P_{\uparrow}(u, t)\). Clearly, this is a stochastic matrix that satisfies the initial condition \(\widehat{P}(t, t) = \Id\). Thus, it suffices to verify that it satisfies Kolmogorov’s backward equation

\[-\frac{\partial}{\partial s} \widehat{P}_{\lambda\mu}(s, t) = Q_{\lambda\lambda}(s) \widehat{P}_{\lambda\lambda}(s, t) + \sum_{\mu: \lambda \supset \mu} Q_{\lambda\mu}(s) \widehat{P}_{\mu\mu}(s, t) + \sum_{\mu: \lambda \sqsubset \mu} Q_{\lambda\mu}(s) \widehat{P}_{\mu\mu}(s, t)\]

with \(Q(s)\) given by (4.19). As we substitute the expression in the right–hand side of (6.1) for \(\widehat{P}(s, t)\), we see that both sides of the equality above, as function in \(z\) and \(z'\), have the form

\[
\left(1 - \eta\right)^{zz'} \times \{\text{a polynomial in } z, z'\}.
\]
Since the equality has already been established for \((z, z') = (N, N + \alpha)\) with \(N = 1, 2, \ldots, \alpha > -1\), it must hold for arbitrary \((z, z')\).

The statement that we prove next will be used in the derivation of the dynamical correlation functions later in the next section.

Take an admissible pair of parameters \((z, z')\) and an admissible curve \(\xi(\cdot)\), and consider the Markov process \(\Lambda_{z, z', \xi(\cdot)}\). Let \(t_1 < t_2 < \cdots < t_n\) be arbitrary time moments. Set

\[
\xi_i = \xi(t_i), \quad \eta_{i,i+1} = e^{t_i-t_{i+1}} \sqrt{\xi_i \xi_{i+1}}.
\]

Proposition 4.13 shows that this is in fact a function in \(z, z^\prime\). Thus, in order to compute these finite-dimensional distributions we may replace our process by a sequence of alternating descending and ascending processes:

We start off at the time moment \(t_1\) and go up till time

\[
u_{12} = \frac{t_1 + t_2}{2} + \frac{\ln \xi_1 - \ln \xi_2}{4},
\]

the value of \(\xi\) at this moment is exactly \(\eta_{12}\). Then we go up till \(\xi_2\) and then again down, etc. The time moments when we change directions are

\[
t_1 < \nu_{12} < t_2 < \nu_{23} < \cdots < \nu_{n,n+1} < t_n
\]

with \(\nu_{i,i+1} = (t_i + t_{i+1})/2 + (\ln \xi_i - \ln \xi_{i+1})/4\), and the values of \(\xi\) at these points are \(\eta_{i,i+1}\). At \(t_i\)'s we switch from going up to going down, and at \(\nu_{i,i+1}\)'s we switch from going down to going up.

Fix arbitrary subsets \(D_1, D_2, \ldots, D_n; D_{12}, D_{23}, \ldots, D_{n-1,n}\) of \(Y\). Let us compute the probability that our new descending-ascending process hits all \(D_i, D_{j,j+1}\) at the time moments \(t_i, \nu_{j,j+1}\), respectively. It is equal to the sum

\[
\sum_{\lambda(i) \in D_{i, i=1,...,n}} \sum_{\mu(j,j+1) \in D_{j,j+1}, j=1,...,n-1} M_{z,z', \xi}(\lambda(1)) P_{\lambda(1), \mu(1,2)}^\downarrow(t_1, \nu_{12}) P_{\mu(1,2), \lambda(2)}^\uparrow(\nu_{12}, t_2) \cdots P_{\lambda(2), \mu(2,3)}^\downarrow(t_2, \nu_{23}) P_{\mu(2,3), \lambda(3)}^\uparrow(\nu_{23}, t_3) \cdots \\
\cdots P_{\lambda(n-1), \mu(n-1,n)}^\downarrow(t_{n-1}, \nu_{n-1,n}) P_{\mu(n-1,n), \lambda(n)}^\uparrow(\nu_{n-1,n}, t_n).
\]

Proposition 4.13 shows that this is in fact a function in \(\xi_i, \eta_{i,i+1}\) which we will denote by \(F(\xi, \eta)\) with \(\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_{n-1,n})\). The parameters \(\xi_i\) and \(\eta_{i,i+1}\) may take arbitrary values between 0 and 1 subject to the inequalities

\[
\xi_1 \geq \eta_{12} \leq \xi_2 \geq \eta_{23} \leq \xi_3 \geq \cdots \leq \xi_{n-1} \geq \eta_{n,n-1} \leq \xi_n.
\]
Proposition 6.2 (cf. Lemma 3.11). (i) The function \( F(\xi, \eta) \) is a real-analytic function in \( \xi \) and \( \eta \).

(ii) The function \( \varepsilon \mapsto F(\varepsilon \xi; \varepsilon \eta) \) which has been defined so far for \( \varepsilon > 0 \), can be analytically continued to a nonempty disc of the form \( |\varepsilon| < \text{const} \). The coefficients of the Taylor decomposition of this function at \( \varepsilon = 0 \) are polynomial functions in \( z, z' \).

Comment. This statement implies that the function \( F(\xi, \eta) \) viewed also as a function in \( z, z' \), is uniquely determined by its values on the arguments \( (\xi, \eta, z, z') \) with arbitrary \( \xi, \eta, 0 < \xi, \eta_{j+1} < 1 \), satisfying (6.3) and \( (z, z') = (N, N + \alpha) \) with \( N = 1, 2, \ldots \) and \( \alpha > -1 \). This uniqueness will be used in the next section to extend certain formulas derived in the case of integral \( z \) to the general case.

Proof. In order to prove (i), by Weierstrass’ uniform convergence theorem it suffices to check that the series (6.2) with \( M_{z', z}(\lambda(1)) \) and all \( P^{\lambda}, P^{\tau} \) replaced by their expressions given by (1.11), (4.22), (4.23), converges absolutely and uniformly in \( M \) to check that the series (6.2) with \( \xi, \eta \) arbitrary expressions given by (1.11), (4.22), (4.23), converges absolutely and uniformly in \( \xi, \eta \) varying in small discs around their values. It is more convenient to work with \( \xi, \eta \) varying in small discs around their values. It is more convenient to work with the case when all \( D_i \) and \( D_{i,i+1} \) coincide with \( Y \); clearly, the needed convergence of the restricted sum follows from that of the unrestricted sum.

Set \( l_i = |\lambda(i)|, m_{i,i+1} = |\mu(i, i+1)| \). As seen from the proof of Proposition 4.13, the matrix elements \( P^{\lambda}_{\lambda, \mu} \) and \( P^{\tau}_{\mu, \lambda} \) split into products of transition probabilities for pure death and pure birth processes of Lemma 4.8 and (co)transition probabilities on the Young graph. By (1.11), \( M_{z', z}(\xi(\lambda)) \) is also a product of the negative binomial distribution \( \pi_{\xi, \sigma}(n) \) on \( \mathbb{Z}_+ \) and the probability distributions \( M_{z', z}(\lambda) \) on \( \mathbb{Y}_n \)’s. The probabilities related to the Young graph do not depend on \( \xi \) and \( \eta \). Thus, we can split the sum in (6.2) (remember that all \( D_i \) and \( D_{i,i+1} \)’s are equal to \( Y \)) into two: the outer sum is taken over all nonnegative integers \( l_1, \ldots, l_n \) and \( m_{12}, \ldots, m_{n-1,n} \) satisfying

\[
  l_1 \geq m_{12} \leq l_2 \geq m_{23} \leq \cdots \geq m_{n-1,n} \leq l_n,
\]

and the inner sum is taken over all Young diagrams \( \lambda(i) \in \mathbb{Y}_{l_i}, \mu(i, i+1) \in \mathbb{Y}_{m_{i,i+1}} \).

The relations (1.5), (1.6) applied to the \( z \)-measures imply that the inner sum is equal to

\[
  \pi_{z', z}(l_1) P^{\lambda}_{l_1, m_{12}}(t_1, u_{12}) P^{\tau}_{m_{12}, l_2}(u_{12}, t_2) \cdots P^{\lambda}_{l_{n-1}, m_{n-1,n}}(t_{n-1}, u_{n-1,n}) P^{\tau}_{m_{n-1,n}, l_n}(u_{n-1,n}, t_n)
\]

with \( P^{\lambda} \) and \( P^{\tau} \) given by Lemma 4.8. Thus, we need to verify the uniform convergence of the sum of such products taken over nonnegative integers satisfying (6.4).

Denote

\[
  u_i = \frac{\xi_i - \eta_{i-1,i}}{1 - \eta_{i-1,i}}, \quad i = 1, \ldots, n, \quad \text{with} \quad \eta_{0,1} := 0,
\]

\[
  v_{i,i+1} = \frac{(1 - \xi_i) \eta_{i,i+1}}{(1 - \eta_{i,i+1}) \xi_i}, \quad i = 1, \ldots, n - 1.
\]

The inequalities (6.3) imply that \( u_i \in (0, 1) \) and \( v_{i,i+1} \in (0, 1] \) for all \( i \). Clearly, small (complex) variations of \( \xi \) and \( \eta \) lead to small (complex) variations of \( u_i \)’s and \( v_{i,i+1} \)’s. What is important for us here is that if the variations are small enough than \( u_i \)’s are bounded away from 1 and \( v_{i,i+1} \)’s are bounded away from 0.
Take the absolute value of (6.5) and sum it over \( l_n \). We have
\[
\sum_{l_n \geq m_{n-1,n}} \left| P_{m_{n-1,n},t_n}^+ (u_{n-1,n}, t_n) \right| = |1 - u_n|^{1 - m_{n-1,n}} \times \sum_{l_n \geq m_{n-1,n}} |u_n|^{l_n - m_{n-1,n}} \frac{(c + m_{n-1,n})l_n - m_{n-1,n}}{(l_n - m_{n-1,n})!} = \left( \frac{|1 - u_n|}{1 - |u_n|} \right)^{c + m_{n-1,n}} \tag{6.6}
\]
We conclude that this expression can be estimated, as a function of \( m_{n-1,n} \), by a constant times a geometric progression of the form \( r^{m_{n-1,n}} \) with a suitable \( r > 0 \). If the variation of \( u_n \) is small enough, it is close to the real axis. Hence, by decreasing the variations we can take \( r \) arbitrarily close to 1.

Let us substitute this estimate into (6.5) and sum over \( m_{n-1,n} \). We obtain
\[
\text{const} \sum_{m_{n-1,n} \leq l_{n-1}} \left| P_{l_{n-1}, m_{n-1,n}} (t_{n-1}, u_{n-1,n}) \right| = \text{const} |v_{n-1,n}|^{l_{n-1}} - 1 \times \sum_{m_{n-1,n} \leq l_{n-1}} \frac{1}{v_{n-1,n}^{l_{n-1} - m_{n-1,n}}} r^{m_{n-1,n}} \frac{l_{n-1}!}{(l_{n-1} - m_{n-1,n})!} |m_{n-1,n}|! = \text{const} \left( \frac{1}{v_{n-1,n}} - 1 \right) + r |v_{n-1,n}|^{l_{n-1}}.
\]
Again, this is bounded by a constant times a geometric progression \( \bar{r}^{l_{n-1}} \) where \( \bar{r} \) can be made arbitrarily close to 1 by considering small enough variations of \( u_n \) and \( v_{n-1,n} \).

The next step, summation over \( l_{n-1} \), is performed similarly to (6.6). The only difference is in the presence of the additional geometric progression \( \tilde{r}^{l_{n-1}} \). The summation yields
\[
\text{const} \left( \frac{|1 - u_{n-1}|}{1 - \tilde{r} |u_{n-1}|} \right)^{a + m_{n-2,n-1}}.
\]
Once again, for small variations of \( u_{n-1}, v_{n-1,n}, u_n, v_{n-1,n} \), this is bounded by a constant times a geometric progression with exponent \( m_{n-2,n-1} \) and a ratio that is close to 1. Induction on \( n \) and the presence of \( \xi_i \) in \( \pi_{z',\xi} (l_1) \) complete the proof of the uniform convergence of the series.

Let us prove (ii). The first step is the same: the sum (6.2) is split into the outer sum over nonnegative integers \( l_1, \ldots, l_n \) and \( m_{12}, \ldots, m_{n-1,n} \) satisfying (6.4), and the inner sum over Young diagrams \( \lambda (i), \mu (i, i + 1) \) with \( |\lambda (i)| = l_i, |\mu (i, i + 1)| = m_{i,i+1} \) restricted by \( \lambda (i) \in \mathcal{D}_i \) and \( \mu (i, i + 1) \in \mathcal{D}_{i,i+1} \). As was mentioned above, if all \( \mathcal{D}'s \) are equal to \( \mathcal{Y} \), the inner sum yields (6.5). Since all the summands are nonnegative, with arbitrary \( \mathcal{D}'s \) the inner sum yields (6.5) multiplied by a constant which depends on \( l's \) and \( m's \), does not depend on \( \xi \) and \( \eta \), and is between 0 and 1. It is also worth noting that these constants are polynomials in \( z \) and \( z' \) because the inner sums are always finite.

As we replace \( \xi \) and \( \eta \) by \( \varepsilon \xi \) and \( \varepsilon \eta \) in the expression obtained by using the formulas of Lemma 4.8 in (6.5), we use the estimates
\[
\left| \left( \frac{1 - \varepsilon \xi}{1 - \varepsilon \eta} \right) \left( \frac{\varepsilon \xi - \varepsilon \eta}{1 - \varepsilon \xi \varepsilon \eta} \right)^{l_{n-1} - m} \frac{l_{n-1}!}{(l_{n-1} - m)!} \right| \leq \text{const}^l, \quad \left| \left( \frac{1 - \varepsilon \xi}{1 - \varepsilon \eta} \right)^{z' + m} \left( \frac{\varepsilon \xi - \varepsilon \eta}{1 - \varepsilon \xi} \right)^{l_{n-1} - m} \frac{(c + x)(l_{n-1} - m)}{(l_{n-1} - m)!} \right| \leq |\varepsilon|^l \cdot \text{const}^{l - m},
\]
where the bounds are uniform in \( |\varepsilon| \) varying in the unit disc \( |\varepsilon| \leq 1 \). These estimates together with \(|\pi_{c,\varepsilon}(l)| \leq |\varepsilon|^l \) \( \text{const}^l \) imply that (6.5) is bounded by

\[
|\varepsilon|^{l_1 + \cdots + l_n - m_{12} - \cdots - m_{n-1,n}} \cdot \text{const}^{l_1 + \cdots + l_n}.
\]

Let us show that the radius of convergence of the power series in \( |\varepsilon| \) obtained by adding expressions (6.7) with all nonnegative \((l_1, \ldots, l_n, m_{12}, \ldots, m_{n-1,n})\) satisfying (6.4), is positive. Indeed, it is not hard to see that for any numbers satisfying the system of inequalities (6.4), one has

\[
l_1 + \cdots + l_n - m_{12} - \cdots - m_{n-1,n} \geq \max\{l_1, \ldots, l_n\}.
\]

We can split the power series into parts according to which of \( l_i \)'s is the largest one. It suffices to check the convergence of each part. Let us take one of such subseries, say, assume that \( l_j = \max\{l_1, \ldots, l_n\} \) for some \( j \). Then for \( |\xi| \leq 1 \)

\[
|\varepsilon|^{l_1 + \cdots + l_n - m_{12} - \cdots - m_{n-1,n}} \leq |\varepsilon|^l \cdot \text{const}^{l_1 + \cdots + l_n} \leq \max(1, \text{const}^n l)
\]

Finally, the number of sets of \( 2(n-1) \) nonnegative numbers \( \{l_i\} \) (not including \( l_j = l \)) and \( \{m_{i,i+1}\} \) bounded by \( l \) from above is \( (l+1)^{2(n-1)} \). Thus, our series is majorized by

\[
\sum_{l=0}^{\infty} (l+1)^{2(n-1)} \text{const}^n l |\varepsilon|^l,
\]

and this series converges for small enough \( |\varepsilon| \).

Thus, we have verified that the series of expressions (6.5) absolutely and uniformly converges when \( |\varepsilon| \) is small enough. Clearly, the multiplication of the terms of this series by constants between 0 and 1 which we mentioned at the beginning of the proof of (ii) (these are the inner sums over Young diagrams) does not affect the convergence. This proves the first statement of (ii).

The second statement of (ii) is easy: the terms of (6.5) with \( \xi, \eta \) replaced by \( \varepsilon \xi, \varepsilon \eta \), have a zero of order at least \( l_1 + \cdots + l_n - m_{12} - \cdots - m_{n-1,n} \) at \( \varepsilon = 0 \); it comes from the factors \( (\varepsilon \xi_i - \varepsilon \eta_{i-1})^{l_i-m_{i-1,i}} \) in \( P^\uparrow \)'s and from \( \xi_1^l \) in \( \pi_{c,\varepsilon}(l) \). Thus, only finitely many terms contribute to a fixed Taylor coefficient. Each of these terms involve polynomial expressions in \( z, z' \) and expressions of the form \( (1 - \varepsilon \xi_i)^{z z'}, (1 - \varepsilon \eta_{i+1})^{-z z'} \), and their Taylor coefficients at \( \varepsilon = 0 \) are also polynomials in \( z, z' \). □

7. Dynamical correlation functions

7.1. Definitions. Consider a continuous time stochastic process \( \Lambda(t) \) with the state space \( \mathcal{Y} \) (all Young diagrams). As in §3 we view the Young diagrams as point configurations (=subsets) of \( \mathbb{Z}' = \mathbb{Z} + \frac{1}{2} \) via

\[
\lambda \in \mathcal{Y} \mapsto \mathbf{X}(\lambda) = (x_1, x_2, \ldots) \subset \mathbb{Z}', \quad x_i = \lambda_i - i + \frac{1}{2}, \quad i = 1, 2, \ldots.
\]

Then \( \Lambda(t) \) is equivalent to the corresponding process with values in point configurations in \( \mathbb{Z}' \); let us denote this process by \( \mathbf{X}(t) \).
For any \( n = 1, 2, \ldots \) define the \( n \)th dynamical correlation function of \( n \) pairwise distinct arguments \((t_1, x_1), \ldots, (t_n, x_n) \in (t_{\text{min}}, t_{\text{max}}) \times \mathbb{Z}'\) by
\[
\rho_n(t_1, x_1; t_2, x_2; \ldots; t_n, x_n) = \text{Prob}\{X(t_i) \text{ contains } x_i \text{ for all } i = 1, \ldots, n\}.
\]

In other words, the dynamical correlation functions describe probabilities of events of the following type: for given time moments \( s_1 < \cdots < s_m \) and given finite sets \( Y_1, \ldots, Y_m \), the random point configurations \( \mathbf{X}(s_1), \ldots, \mathbf{X}(s_m) \) contain \( Y_1, \ldots, Y_m \), respectively. Thus, the notion of the dynamical correlation functions is a hybrid of the finite-dimensional distributions of a stochastic process and standard correlation functions of probability measures on point configurations.

Clearly, the dynamical correlation functions uniquely determine the finite-dimensional distributions of the process and, thus, the process itself. The reason why we are interested in these quantities is the same as in the “static” (fixed time) case: As we take scaling limits of our processes, for the limiting object the notion of the weight of a point configuration does not make sense anymore. Thus, the scaling limits of the correlation functions are well-defined and, moreover, carry a lot of useful information about the limit process.

We say that the process \( \mathbf{X}(t) \) is determinantal (cf. §3) if the exists a kernel
\[
K : ((t_{\text{min}}, t_{\text{max}}) \times \mathbb{Z}') \times ((t_{\text{min}}, t_{\text{max}}) \times \mathbb{Z}') \to \mathbb{C}
\]
such that for any \( n = 1, 2, \ldots \)
\[
\rho_n(t_1, x_1; \ldots; x_n, t_n) = \text{det}[K(t_i, x_i; t_j, x_j)]_{i,j=1}^n.
\]
As in the “static” case, if such a kernel exists then it is not unique. In particular, transformations of the form
\[
K(s, x; t, y) \longrightarrow \frac{f(s, x)}{f(t, y)} K(s, x; t, y)
\]
do not change the correlation functions.

### 7.2. Main results.

**Theorem 7.1.** Let \((z, z')\) be a pair of admissible parameters and \(\xi(\cdot)\) be an admissible curve. Consider the Markov process \(\Lambda_{z, z', \xi(\cdot)}\) defined in §4, and denote by \(\mathbf{X}_{z, z', \xi(\cdot)}\) the corresponding process with values in the space of point configurations in \(Z'\). Then the process \(\mathbf{X}_{z, z', \xi(\cdot)}\) is determinantal.

Recall that in (2.1) we introduced the functions \(\psi_a(x; z, z', \xi)\) which form, for any \(\xi \in (0, 1)\), an orthonormal basis in \(l^2(Z')\). These functions were defined under the condition that \((z, z')\) belong to either principal or complementary series.

**Theorem 7.2.** Assume that \((z, z')\) is either in principal or complementary series. Then the kernel
\[
\mathbf{K}_{z, z', \xi(\cdot)}(s, x; t, y) = \pm \sum_{a \in \mathbb{Z}_{\pm}'} e^{\pm a(t-s)} \psi_{\pm a}(x; z, z', \xi(s)) \psi_{\pm a}(y; z, z', \xi(t))
\]
with “+” taken for \(s \geq t\) and “−” taken for \(s < t\), is a correlation kernel of the process \(\mathbf{X}_{z, z', \xi(\cdot)}\).
Theorem 7.3. The correlation kernel (7.2) can also be written in the form

$$K_{z, z'; \xi(\cdot)}(s, x; t, y) = e^{\frac{1}{2}(s-t)} \varphi_{z, z'}(x, y) \tilde{K}_{z, z'; \xi(\cdot)}(s, x; t, y)$$  \hfill (7.3)$$

where, as in (2.6),

$$\varphi_{z, z'}(x, y) = \frac{\sqrt{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})\Gamma(y+z'+\frac{1}{2})}}{\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})}$$

and the kernel $\tilde{K}_{z, z'; \xi(\cdot)}(s, x; t, y)$ can be written as a double contour integral (set $\zeta = \sqrt{\xi(s)}$, $\eta = \sqrt{\xi(t)}$)

$$\tilde{K}_{z, z'; \xi(\cdot)}(s, x; t, y) = \frac{\sqrt{(1-\zeta)(1-\eta)}}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} \frac{1}{(1-\zeta \omega_1)^{-z'} (1-\zeta \omega_1^{-1})^z (1-\eta \omega_2)^{-z} (1-\eta \omega_2^{-1})^{z'}}$$

$$\times \frac{\omega_1^{-\frac{1}{2}} \omega_2^{-\frac{1}{2}}}{e^{s-t}(\omega_1-\zeta)(\omega_2-\eta) -(1-\zeta \omega_1)(1-\eta \omega_2)} d\omega_1 d\omega_2$$  \hfill (7.4)$$

with the contours $\{\omega_1\}$ and $\{\omega_2\}$ of $\omega_1$ and $\omega_2$ satisfying the following conditions:
- $\{\omega_1\}$ goes around $0$ in positive direction and passes between $\zeta$ and $\zeta^{-1}$;
- $\{\omega_2\}$ goes around $0$ in positive direction and passes between $\eta$ and $\eta^{-1}$;
- if $s \geq t$ then the image of $\{\omega_1\}$ under the fractional-linear map

$$\omega \mapsto \omega \left( e^{s-t}\eta - \zeta \right) + 1 - e^{s-t}\zeta \eta$$

$$\omega \left( e^{s-t} - \zeta \eta \right) + \eta - e^{s-t}\zeta$$  \hfill (7.5)$$

is contained inside $\{\omega_2\}$;
- if $s < t$ then the domain bounded by $\{\omega_2\}$ does not intersect the image of $\{\omega_1\}$ under the map above.

The kernels $K_{z, z'; \xi(\cdot)}(s, x; t, y)$ and $\tilde{K}_{z, z'; \xi(\cdot)}(s, x; t, y)$ are equivalent. Namely, they are related by a “gauge transformation” (7.1),

$$\tilde{K}_{z, z'; \xi(\cdot)}(s, x; t, y) = \frac{f_{z, z'}(s, x)}{f_{z, z'}(t, y)} K_{z, z'; \xi(\cdot)}(x, y), \quad x, y \in \mathbb{Z}'$$

where

$$f_{z, z'}(s, x) = e^{-\frac{s}{2} \Gamma(x+z'+\frac{1}{2})} \sqrt{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})}$$  \hfill (7.6)$$

The kernel $\tilde{K}_{z, z; \xi(\cdot)}(s, x; t, y)$ can serve as a correlation kernel for all admissible values of parameters $(z, z')$, including the degenerate series.

Comments. 1. The fractional-linear transformation (7.5) arises from the condition that the denominator in the integral representation (7.4) has to be nonzero. Solving the equation denominator=0 with respect to $\omega_1$ yields the right-hand side of (7.5) with $\omega = \omega_2$. 

2. It is not a priori clear why the needed contours \{\omega_1\} and \{\omega_2\} exist. Let us show that it is indeed so. Set \( q = e^{s-t} \). Note that (7.5) maps \( \zeta \mapsto \eta^{-1} \) and \( \zeta^{-1} \mapsto \eta \).

Consider the case \( q \geq 1 \) (i.e. \( s \geq t \)) first. Let us take a circle with center at the origin and radius \( r \) slightly smaller than \( \zeta^{-1} \) as \( \{\omega_1\} \). Then its image is again a circle which is symmetric with respect to the real axis and which passes through the images of \( r \) and \( -r \). The image of \( r \) is close to the image of \( \zeta^{-1} \) which is \( \eta \), and the image of \( -r \) is close to the image of \( -\zeta^{-1} \) which is equal to

\[
\frac{-\eta(\zeta + \zeta^{-1}) + 2}{-\eta(\zeta + \zeta^{-1}) + 2\eta}.
\]

Since \( \zeta \) and \( \eta \) are strictly between 0 and 1, we immediately see that the denominator is negative, and the whole expression is \( < \eta \). Thus, the image of \( \{\omega_1\} \) is a finite circle that lies to the left of \( \eta \) plus a small number. Clearly, there exists \( \{\omega_2\} \) that passes between \( \eta \) and \( \eta^{-1} \) and encircles both 0 and the image of \( \{\omega_1\} \).

Let us consider the case \( q < 1 \) now. As \( \{\omega_1\} \) we again take a circle with center at the origin but with radius slightly greater than \( \zeta \). Then its image is a circle which is symmetric with respect to the real axis and which passes through images of points that are close to the image of \( \zeta \) which is \( \eta^{-1} \), and to the image of \( -\zeta \) which is \( \frac{\zeta + \zeta^{-1} - 2\eta q}{\eta(\zeta + \zeta^{-1}) - 2\eta} \).

If the denominator is negative then the whole expression is negative, and there exists a contour \( \{\omega_2\} \) inside this circle that passes between \( \eta \) and \( \eta^{-1} \) and goes around the origin. If the denominator is positive then the whole expression is \( > \eta^{-1} \), and \( \{\omega_2\} \) can be a circle of radius between \( \eta \) and \( \eta^{-1} \) with center at the origin. □

Theorems 7.1, 7.2, and 7.3 are generalizations of Theorems 3.1, 3.2, and 3.3, respectively.

### 7.3. Proof of Theorems 7.1, 7.2, and 7.3.

The ideas used in the proof are similar to those of §3. The first step is to consider the case \( z = N \in \{1, 2, \ldots\} \), \( z' = N + \alpha, \alpha > -1 \). Then the state space of our Markov process \( \Lambda_{z, z', \zeta} \) is smaller than the whole set \( \mathcal{Y} \). Namely, since \( M_{N, N+\alpha, \zeta}(\lambda) \) vanishes if \( \ell(\lambda) \) (the number of nonzero rows of \( \lambda \)) is greater than \( N \), our process lives on the set \( \mathcal{Y}(N) \) of Young diagrams with no more than \( N \) rows. Consider an embedding of \( \mathcal{Y}(N) \) into the set of \( N \)-point subsets of \( \mathbb{Z}_+ \) given by

\[
\lambda \in \mathcal{Y}(N) \mapsto \tilde{X}(\lambda) = (\tilde{x}_1, \ldots, \tilde{x}_N), \quad \tilde{x}_i = \lambda_i + N - i, \quad i = 1, \ldots, N,
\]

and denote by \( \tilde{X}_{N, N+\alpha, \zeta} \) the corresponding stochastic process on the space of \( N \)-point configurations in \( \mathbb{Z}_+ \). Obviously, the processes \( \tilde{X}_{N, N+\alpha, \zeta} \) and \( \tilde{X}_{N, N+\alpha, \zeta} \) are equivalent: if \( \lambda \in \mathcal{Y}(N) \) then \( \tilde{X}(\lambda) = (x_1, x_2, \ldots) \) and \( \tilde{X}(\lambda) = (\tilde{x}_1, \ldots, \tilde{x}_N) \) are related by

\[
x_i = \begin{cases} 
\tilde{x}_i - N + \frac{1}{2}, & i = 1, \ldots, N, \\
-\tilde{i} + \frac{1}{2}, & i \geq N + 1.
\end{cases}
\]

This implies that the dynamical correlation functions \( \rho_t \) of \( \tilde{X}_{N, N+\alpha, \zeta} \) and \( \tilde{\rho}_t \) of \( \tilde{X}_{N, N+\alpha, \zeta} \) are related by

\[
\rho_t(\tau_1, x_1; \ldots; \tau_t, x_t) = \tilde{\rho}_t(\tau_1, \tilde{x}_1; \ldots; \tau_t, \tilde{x}_t)
\]
with \( \tilde{x}_i = x_i + N - \frac{1}{2} \in \mathbb{Z}_+ \) for \( i = 1, \ldots, l \).

Define the extended Meixner kernel \( K_{N,\alpha,\xi}^{\text{Meixner}}(s, \tilde{x}; t, \tilde{y}) \) by, cf. Lemma 3.4,

\[
K_{N,\alpha,\xi}^{\text{Meixner}}(s, \tilde{x}; t, \tilde{y}) = \begin{cases} 
K_{N,\alpha,\xi}^{\text{Meixner},+}(s, \tilde{x}; t, \tilde{y}), & s \geq t, \\
K_{N,\alpha,\xi}^{\text{Meixner},-}(s, \tilde{x}; t, \tilde{y}), & s < t,
\end{cases}
\]

with \( \tilde{x}, \tilde{y} \in \mathbb{Z}_+ \) and

\[
K_{N,\alpha,\xi}^{\text{Meixner},+}(s, \tilde{x}; t, \tilde{y}) = \sum_{m=0}^{N-1} e^{m(s-t)} \tilde{m}_m(\tilde{x}; \alpha, \xi(s)) \tilde{m}_m(\tilde{y}; \alpha, \xi(t)),
\]

\[
K_{N,\alpha,\xi}^{\text{Meixner},-}(s, \tilde{x}; t, \tilde{y}) = - \sum_{m=N}^{\infty} e^{m(s-t)} \tilde{m}_m(\tilde{x}; \alpha, \xi(s)) \tilde{m}_m(\tilde{y}; \alpha, \xi(t)).
\]

**Lemma 7.4.** The process \( \tilde{X}_{N,\alpha,\xi}(s) \) is determinantal. Its correlation functions have the form

\[
\rho_l(\tau_1, \tilde{x}_1; \ldots; \tau_l, \tilde{x}_l) = \det [K_{N,\alpha,\xi}^{\text{Meixner}}(\tau_i, \tilde{x}_i; \tau_j, \tilde{x}_j)]_{i,j=1}^l, \quad l = 1, 2, \ldots
\]

We postpone the proof of this lemma till §7.4.

The next step is to connect the extended Meixner kernel \( K_{N,\alpha,\xi}^{\text{Meixner}} \) and the kernel \( K \) of Theorem 7.2 (cf. Lemma 3.7).

**Lemma 7.5.** We have

\[
K_{N,\alpha,\xi}^{\text{Meixner}}(s, \tilde{x}; t, \tilde{y}) = e^{(N-\frac{1}{2})(s-t)} K_{N,\alpha,\xi}(s, x; t, y)
\]

with \( \tilde{x} = x + N - \frac{1}{2} \in \mathbb{Z}_+ \), \( \tilde{y} = y + N - \frac{1}{2} \in \mathbb{Z}_+ \).

**Proof.** We argue as in the proof of Lemma 3.7. For \( s \geq t \) take (7.7) and change the summation index \( m = N - a - \frac{1}{2} \). Then, see Proposition 2.8,

\[
e^{m(s-t)} = e^{(N-\frac{1}{2})(s-t)} \cdot e^{(t-s)}, \quad \tilde{m}_m(\tilde{x}; \alpha, \xi) = \psi_a(x; N, N + \alpha, \xi)
\]

with \( \tilde{x} = x + N - \frac{1}{2} \). Furthermore, \( \psi_a(x; N, N + \alpha, \xi) \equiv 0 \) if \( a = N + \frac{1}{2}, N + \frac{3}{2}, \ldots \) because of the factor \( \Gamma(\varepsilon + a + \frac{1}{2}) \) in (2.1). This yields (7.9). For \( s < t \) the argument is similar; it uses (7.8), the summation index change \( m = N - a - \frac{1}{2} \) and

\[
e^{m(s-t)} = e^{(N-\frac{1}{2})(s-t)} \cdot e^{-a(s-t)}, \quad \tilde{m}_m(\tilde{x}; \alpha, \xi) = \psi_{-a}(x; N, N + \alpha, \xi).
\]

\( \square \)

Lemma 7.4 and Lemma 7.5 imply that the correlation functions \( \rho_l \) of \( X_{N,\alpha,\xi}(s) \) can be written as

\[
\rho_l(\tau_1, x_1; \ldots; \tau_l, x_1) = \det \left[ K_{N,\alpha,\xi}(\tau_i, x_i; \tau_j, x_j) \right]_{i,j=1}^l, \quad l = 1, 2, \ldots
\]

if \( x_i \geq -N + \frac{1}{2} \) for all \( i = 1, \ldots, l \) (indeed, the factor \( e^{(N-\frac{1}{2})(s-t)} \) in (7.9) does not affect the correlation functions).

The next claim is a counterpart of Lemma 3.9.
Lemma 7.6. Assume that

- either \((z, z')\) is not in the degenerate series and \(x, y \in \mathbb{Z}'\) are arbitrary
- or \(z = N = 1, 2, \ldots, z' > N - 1\), and both \(x, y\) are in \(\mathbb{Z}_+ - N + \frac{1}{2}\).

Then the kernel \(\hat{K}_{z, z', \xi(t)}(s, x; t, y)\) of Theorem 7.3 is related to the kernel \(\hat{K}_{z, z', \xi(t)}(s, x; t, y)\) of Theorem 7.2 by equality (7.3). Or, that is the same, by the “gauge transformation”

\[
\hat{K}_{z, z', \xi(t)}(s, x; t, y) = \frac{f_{z, z'}(s, x)}{f_{z, z'}(t, y)} \hat{K}_{z, z', \xi(t)}(s, x; t, y),
\]

where \(f_{z, z'}\) is defined in (7.6).

Proof. Applying formula (2.4) of Proposition 2.3 to \(\psi_a\) and setting \(a = k + \frac{1}{2}\) we obtain

\[
\psi_{k + \frac{1}{2}}(x; z, z', \xi(s)) = \left(\frac{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})}{\Gamma(z - k)\Gamma(z' - k)}\right)^{\frac{1}{2}} \frac{\Gamma(z' - k)(1 - \xi(s))^{\frac{z' - k + 1}{2}}}{\Gamma(x + z' + \frac{1}{2})} \times \oint \frac{(1 - \sqrt{\xi(s)}\omega_1)^{z' + k}}{(1 - \sqrt{\xi(s)}\omega_1^{-1})^z} \frac{\omega_1^{-x - k - \frac{1}{2}}}{\omega_1} d\omega_1,
\]

\[
\psi_{k + \frac{1}{2}}(y; z, z', \xi(t)) = \left(\frac{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})}{\Gamma(z - k)\Gamma(z' - k)}\right)^{\frac{1}{2}} \frac{\Gamma(z' - k)(1 - \xi(t))^{\frac{z' - k + 1}{2}}}{\Gamma(x + z' + \frac{1}{2})} \times \oint \frac{(1 - \sqrt{\xi(t)}\omega_2)^{z' + k}}{(1 - \sqrt{\xi(t)}\omega_2^{-1})^z} \frac{\omega_2^{-y - k - \frac{1}{2}}}{\omega_2} d\omega_2.
\]

(As in the proof of Proposition 2.3, we used the symmetry \(\psi_a(x; z, z', \xi) = \psi_a(x; z', z, \xi)\) to get the second relation.)

Note that both sides of (7.3) are real-analytic functions of \(q = e^{s-t}\) on \(q \geq 1\) and \(0 < q < 1\) with all other parameters \((z, z', \xi(s), \xi(t), x, y)\) being fixed. Thus, it suffices to prove (7.3) for \(q\) large enough in the case of \(q \geq 1\), and for \(q\) small enough in the case \(q < 1\).

Let us consider the case \(q \geq 1\). Substituting the integral representations above in (7.2), we observe that the summation index \(k = a - \frac{1}{2} \in \mathbb{Z}_+\) enters the resulting expression only as the exponent in

\[
\left(\frac{1 - \sqrt{\xi(s)}\omega_1}{q(\omega_1 - \sqrt{\xi(s)})\omega_2 - \sqrt{\xi(t)}}\right)^k.
\]

For large enough \(q\) the geometric progression \(\sum_{k \geq 0}\) with this ratio converges uniformly on any fixed contours \(\{\omega_1\}, \{\omega_2\}\). Computing the sum yields (7.3), (7.4).

Similarly, in the case \(q < 1\) the computation reduces to summing the geometric progression with the ratio

\[
\frac{q(\omega_1 - \sqrt{\xi(s)})\omega_2 - \sqrt{\xi(t)}}{(1 - \sqrt{\xi(s)}\omega_1)(1 - \sqrt{\xi(t)}\omega_2)}
\]

which always makes sense for small enough \(q\).

For large \(q\) the image of any finite contour under (7.5) is concentrated near \(\eta = \sqrt{\xi(t)}\), and for small \(q\) such image is concentrated near \(\eta^{-1}\). These points
are always inside/outside of any contour \( \{ \omega_2 \} \) provided by Proposition 2.3. The conditions on contours in the statement of Theorem 7.3 reflect the deformations of the contours of Proposition 2.3. Note that the denominator of (7.4) must stay away from zero while deforming the contours, which means that the image of \( \{ \omega_1 \} \) under (7.5) is not allowed to intersect \( \omega_2 \). \( \square \)

The relations (7.10) and (7.3) yield the following formula for the correlation functions of \( X_{N,N+\alpha},\xi \):

\[
\rho_l(\tau_1, x_1; \ldots; \tau_l, x_l) = \det \left[ \tilde{K}_{z,z',\xi(\cdot)}(\tau_i, x_i; \tau_j, x_j) \right]_{i,j=1}^l, \quad l = 1, 2, \ldots, (7.11)
\]

where \( (z, z') = (N, N + \alpha) \) and \( x_i + N - \frac{1}{2} \in \mathbb{Z}_+ \) for all \( i = 1, \ldots, l \). In order to prove Theorem 7.3, we need to extend this formula to arbitrary admissible \( (z, z') \) and arbitrary \( x_i \in \mathbb{Z}' \). We will do that by means of Proposition 6.2.

Up till now the time moments \( \tau_1, \ldots, \tau_l \) were arbitrary, they were not ordered and some of them were allowed to coincide. Let us denote by \( t_1, \ldots, t_n, n \leq l \), the same numbers but ordered and without repetitions. Thus, each \( \tau_i \) is equal to one and only one \( \tau_j \). As in §6, we set

\[
\xi_i = \xi(t_i), \quad 1 \leq i \leq n, \quad \eta_{i,i+1} = e^{t_i-t_{i+1}} \sqrt{\xi_i} \sqrt{\xi_{i+1}}.
\]

Then in the notation of §6, see (6.2) and below, \( \rho_l(\tau_1, x_1; \ldots; \tau_l, x_l) \) is equal to \( F(\xi, \eta) \) with a suitable choice of the sets \( D_1, \ldots, D_n \) and \( D_{12}, \ldots, D_{n-1,n} \). Namely, we take all \( D_{i,i+1} \) equal to \( \mathcal{Y} \), and the set \( D_i \) is determined according to the following recipe: take all numbers \( j \in \{ 1, \ldots, l \} \) such that \( \tau_j = t_i \), then the corresponding points \( x_j \in \mathbb{Z}' \) must be pairwise distinct. Then

\[
D_i = \{ \lambda \in \mathcal{Y} \mid X_\lambda(\lambda) \text{ contains all } x_j \text{ such that } \tau_j = t_i \}.
\]

Proposition 6.2 says that \( \rho_l(\tau_1, x_1; \ldots; \tau_l, x_l) \) is a real analytic function in \( \xi, \eta \), and after the substitution \( (\xi, \eta) \to (\varepsilon \xi, \varepsilon \eta) \) the corresponding function in \( \varepsilon \) can be analytically continued in a neighborhood of \( \varepsilon = 0 \). Moreover, its Taylor coefficients at this point are polynomials in \( z \) and \( z' \).

Now let us look at the right–hand side of (7.11) with \( N, N + \alpha \) replaced by \( z, z' \). The values \( \tilde{K}_{z,z',\xi(\cdot)}(\tau_i, x_i; \tau_j, x_j) \) of the kernel are given by (7.4). This formula involves \( \sqrt{\xi(\tau_i)}, \sqrt{\xi(\tau_j)}, \) and \( e^{\tau_i-\tau_j} \), which is expressible through \( \sqrt{\xi_i}'s \) and \( \eta_{i,i+1}'s \) in a polynomial fashion. Moreover, \( e^{\tau_i-\tau_j} \) do not change if we scale \( \xi \) and \( \eta \) by \( \varepsilon \).

The integral representation (7.4) implies that \( \tilde{K}_{z,z',\xi(\cdot)}(\tau_i, x_i; \tau_j, x_j) \) are real analytic functions in \( \sqrt{\xi_i}'s \) and \( \eta_{i,i+1}'s \). Further, if we scale \( \xi \) and \( \eta \) by \( \varepsilon \), then (7.4) viewed as a function in \( \delta = \varepsilon \frac{1}{\varepsilon} \), extends to an analytic function in a small enough disc \( \{ \delta : |\delta| < \text{const} \} \). Moreover, its Taylor coefficients at \( \delta = 0 \) are polynomials in \( z \) and \( z' \) because the Taylor coefficients of \( (1-u)^{\kappa} \) at \( u = 0 \) are polynomials in \( \kappa \).

We conclude that both sides of (7.11) are uniquely determined by their values for \( (z, z') = (N, N + \alpha) \) with any large enough \( N \) and any \( \alpha > -1 \). This completes the proof of Theorem 7.3.

Theorem 7.2 is a direct corollary of Theorem 7.3 and Lemma 7.6. Theorem 7.1 follows from Theorem 7.3. \( \square \)

\footnote{It is worth noting that for small \( \varepsilon \) we can choose the contours of integration in (7.4) which would be independent of \( \varepsilon \); it suffices to consider suitable circles centered at the origin.}
7.4. Eynard–Mehta theorem and the proof of Lemma 7.4. Here we state the Eynard–Mehta theorem [EM] in a form which is convenient for our purposes and show that Lemma 7.4 is a corollary of this theorem.

Let $m$ be a fixed natural number and let the index $k$ range over $\{1, \ldots, m\}$. Consider the Hilbert space $\ell^2(\mathbb{Z}_+)$ taken with respect to the counting measure on the set $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Assume that for each $k$ we are given an orthonormal basis $\{\phi_{k,n}\}_{n=0,1,\ldots}$ of real–valued functions in $\ell^2(\mathbb{Z}_+)$. Next, assume that for each $k = 1, \ldots, m-1$ and each $n = 0, 1, \ldots$ we are given a number $c_{k,k+1,n} > 0$. As $n \to \infty$, these numbers have to decay fast enough to make convergent certain infinite sums specified below. Finally, we will impose on these data certain positivity conditions, see below.

We aim to construct a probability measure on collections $(X_1, \ldots, X_m)$, where each $X_k$ is an arbitrary $N$–point subset in $\mathbb{Z}_+$ and $N$ is a fixed natural number. This measure can be regarded as a Markov process with “time” $k = 1, \ldots, m$, the state space being the set of $N$–point subsets in $\mathbb{Z}_+$.

The construction goes as follows. For an arbitrary $N$–point subset $X = (x_1 < \cdots < x_N) \subset \mathbb{Z}_+$ we introduce the $N \times N$ matrix $\phi_k(X)$ with the entries $\phi_{k,i}(x_j)$, where the row index $i$ takes values in $\{0, \ldots, N-1\}$, and the column index $j$ takes values in $\{1, \ldots, N\}$.

As $X$ ranges over all $N$–point subsets of $\mathbb{Z}_+$, one has

$$\sum_X (\det \phi_k(X))^2 = 1.$$ 

The proof follows from the Cauchy-Binet identity and orthonormality of $\phi_{k,n}$’s, cf. (5.8).

Thus, for each $k = 1, \ldots, m$ we have a probability measure $\sigma_k$ on $N$–point subsets in $\mathbb{Z}_+$ which assigns to a subset $X$ its weight $(\det \phi_k(X))^2$. The measures $\sigma_k$ are the 1–dimensional distributions for our future Markov process.

For each $k = 1, \ldots, m-1$ we set

$$v_{k,k+1}(x,y) = \sum_{n=0}^{\infty} c_{k,k+1,n} \phi_{k,n}(x) \phi_{k+1,n}(y), \quad x,y \in \mathbb{Z}_+,$$

where the sum is assumed to be convergent. Since $\{\phi_{k,n}\}$ is an orthonormal basis for each fixed $k$, we have

$$\sum_{x \in \mathbb{Z}_+} \phi_{k,n}(x) v_{k,k+1}(x,y) = c_{k,k+1,n} \phi_{k,n}(y),$$

$$\sum_{y \in \mathbb{Z}_+} v_{k,k+1}(x,y) \phi_{k+1,n}(y) = c_{k,k+1,n} \phi_{k,n}(x). \quad (7.12)$$

For arbitrary subsets $X = (x_1 < \cdots < x_N)$ and $Y = (y_1 < \cdots < y_N)$ we form an $N \times N$ matrix $v_{k,k+1}(X,Y)$ with entries $v_{k,k+1}(x_i,y_j)$, and we set

$$\sigma_{k,k+1}(X,Y) = \frac{\det \phi_k(X) \det v_{k,k+1}(X,Y) \det \phi_{k+1}(Y)}{\prod_{n=0}^{N-1} c_{k,k+1,n}}.$$
Once again, using the Cauchy-Binet identity and (7.12), it is not hard to show that for any \( k = 1, \ldots, m - 1 \) one has

\[
\sum_Y \sigma_{k,k+1}(X,Y) = \sigma_k(X), \quad \sum_X \sigma_{k,k+1}(X,Y) = \sigma_{k+1}(Y).
\]

Assume that \( \sigma_{k,k+1}(X,Y) \geq 0 \) for all \( X \) and \( Y \). Then we may regard \( \sigma_{k,k+1}(X,Y) \) as a probability measure on couples \((X,Y)\) with marginal measures \( \sigma_k \) and \( \sigma_{k+1} \). This is the “one step” 2–dimensional distribution of our Markov process. We also assume that \( \det \phi_k(X) \) does not vanish. Then we define the “one step” transition probability function as follows

\[
P_{X,Y}(k, k+1) = \frac{\sigma_{k,k+1}(X,Y)}{\sigma_k(X)} = \frac{\det \psi_{k,k+1}(X,Y) \det \phi_{k+1}(Y)}{\det \phi_k(X) \prod_{n=0}^{N-1} c_{k,k+1;n}}.
\] (7.13)

We regard this as a matrix \( P(k, k+1) \) whose rows and columns are labelled by \( N \)–point subsets.

Finally, we define a Markov process \( X(k) \), where the “time” \( k \) takes values from 1 to \( m \) and \( X(k) \) is an \( N \)–point subset of \( \mathbb{Z}_+ \), using the initial distribution \( \sigma_1 \) and the “one step” transition probabilities (7.13):

\[
\operatorname{Prob}(X(1), \ldots, X(m)) = \sigma_1(X_1) P_{X(1),X(2)}(1, 2) \cdots P_{X(m-1),X(m)}(m-1, m).
\]

For arbitrary indices \( k, l \) such that \( 1 \leq k < l \leq m \) we set

\[
k, l \vdash c_{k,l;n} = c_{k,k+1;n} c_{k+1,k+2;n} \cdots c_{l-1,l;n}, \quad n = 0, 1, \ldots,
\]

\[
v_{k,l}(x, y) = \sum_{n=0}^{\infty} c_{k,l;n} \phi_k(x) \phi_l(y), \quad x, y \in \mathbb{Z}_+.
\]

**Theorem 7.7 (Eynard–Mehta [EM]).** Under the above assumptions, let us regard the Markov process \( X(k) \) as a probability measure on \( mN \)–point configurations \( X = (X(1), \ldots, X(m)) \) in the space \( \{1, \ldots, m\} \times \mathbb{Z}_+ \). Then this measure is determinantal, and its correlation kernel has the form

\[
K(k, x; l, y) = \sum_{i=0}^{N-1} \frac{1}{c_{l,k;i}} \phi_{k,i}(x) \phi_{l,i}(y), \quad k \geq l,
\]

(where we agree that \( c_{k,k;i} = 1 \)) and

\[
K(k, x; l, y) = \sum_{i=0}^{N-1} c_{k,l;i} \phi_{k,i}(x) \phi_{l,i}(y) - v_{k,l}(x, y)
\]

\[
= - \sum_{i=N}^{\infty} c_{k,l;i} \phi_{k,i}(x) \phi_{l,i}(y), \quad k < l.
\]

In other words, for any \( n = 1, 2, \ldots \) we have

\[
\rho_n(k_1, x_1; \ldots; k_n, x_n) = \operatorname{Prob}(X(k_i) \ni x_i \text{ for each } i = 1, \ldots, n) = \det [K(k_i, x_i; k_j, x_j)]_{i,j=1}^{n},
\]
where \((k_i, x_i) \in \{1, \ldots, m\} \times \mathbb{Z}\) and \((k_i, x_j) \neq (k_j, x_j)\) for \(i \neq j\).

**Proof.** See [EM], [NF], [Jo3], [TW], [BR]. □

**Proof of Lemma 7.4.** The process \(\Lambda_{N,N+\alpha,\xi}\) restricted to any finite sequence of time moments \(t_1 < \cdots < t_m\) fits into this formalism perfectly. Indeed, we take

\[
\phi_{k,n}(x) = \mathbb{M}_n(x; \alpha, \xi(t_k)), \quad c_{k,k+1;n} = e^{n(t_k-t_{k+1})},
\]

Then (5.7) or Lemma 3.4 imply that \(M_{N,N+\alpha,\xi(t_k)}\) is exactly \(\sigma_k\), and the transition matrix (5.2) coincides with (7.13).\(^8\) Lemma 7.4 is thus a direct corollary of Theorem 7.7. □

### 7.5. An interpretation via nonintersecting paths

In this section we interpret the stationary process \(\Lambda_{N,N+\alpha,\xi}\) in terms of \(N\) nonintersecting trajectories of independent birth–death processes. This is done using formulas of Karlin–McGregor [KMG3] and an idea of Johansson [Jo2].

Instead of dealing with \(\Lambda_{N,N+\alpha,\xi}\) we will use the associated process \(\tilde{X}_{N,N+\alpha,\xi}\) introduced in the beginning of §7.3. Recall that its state space consists of \(N\)-point configurations in \(\mathbb{Z}_+\).

In the special case \(N = 1\) the process \(\tilde{X}_{1,1+\alpha,\xi}\) is just the birth–death process \(N_{1+\alpha,\xi}\). We aim to construct \(\tilde{X}_{N,N+\alpha,\xi}\) directly in terms of \(N_{1+\alpha,\xi}\).

Let us take a large \(T > 0\) and consider a new process \(Y_{N,\alpha,\xi,T}\) introduced as follows. This process is defined on the time interval \([-T,T]\). Let us take \(N\) independent birth–death processes which start at the moment \(-T\) at the points \(a_1 < \cdots < a_N\) and end up at the moment \(T\) at the points \(b_1 < \cdots < b_N\) conditioned on the event that the trajectories \(x_i(t)\) do not intersect on \([-T,T]\):

\[x_1(t) < x_2(t) < \cdots < x_N(t), \quad -T \leq t \leq T.\]

The boundary conditions \(\{a_i\}\) and \(\{b_i\}\) are arbitrary but fixed while the parameter \(T\) will vary.

**Theorem 7.8.** In the above notation, as \(T \to \infty\) the processes \(Y_{N,\alpha,\xi,T}\) converge to \(\tilde{X}_{N,N+\alpha,\xi}\) in the sense of convergence of the finite dimensional distributions.

**Proof.** Let us fix arbitrary time moments \(t_1 < \cdots < t_k\) inside \((-T,T)\). Then by [KMG3] and Theorem 5.1, the corresponding finite-dimensional distribution of \(Y_{N,\alpha,\xi,T}\) has the form

\[
\Pr\{Y_{N,\alpha,\xi,T}(t_i) = (y_{1}^{(i)}) < \cdots < (y_{N}^{(i)})\} \quad \text{for all } i = 1, \ldots, k = \frac{\det[v_{-T,T}(a_i, y_{j}^{(1)})] \det[v_{t_1,t_2}(y_{i}^{(1)}, y_{j}^{(2)})] \cdots \det[v_{t_{k-1},t_k}(y_{i}^{(k-1)}, y_{j}^{(k)})] \det[v_{T,0}(y_{i}^{(k)}, b_j)]}{\det[v_{-T,T}(a_i, b_j)]},
\]

where \(v_{s,t}(x,y)\) is given by (5.1) with \(\xi(\cdot) \equiv \xi\).

On the other hand, the finite-dimensional distribution of \(\tilde{X}_{N,N+\alpha,\xi}\) is given by

\[
\Pr\{\tilde{X}_{N,N+\alpha,\xi}(t_i) = (y_{1}^{(i)}) < \cdots < (y_{N}^{(i)})\} = \frac{e^{i(z_{k-1})\sum_{i=1}^{N-1}}}{\det[\phi_{i,j}(y_{1}^{(1)})]} \frac{\det[v_{t_1,t_2}(y_{i}^{(1)}, y_{j}^{(2)})] \cdots \det[v_{t_{k-1},t_k}(y_{i}^{(k-1)}, y_{j}^{(k)})]}{\det[\phi_{i,j}(y_{j}^{(k)})]}.\]

\(^8\)Recall that the Young diagrams with no more than \(N\) rows are identified with \(N\)-point subsets of \(\mathbb{Z}_+\) via \(\lambda \mapsto (\lambda_1 + N - 1, \lambda_2 + N - 2, \ldots, \lambda_N)\).
with \( \phi_n(x) = \hat{\mathcal{M}}_n(x; \alpha, \xi) \), see Theorem 5.1.

Note that we have the following asymptotic relation: for arbitrary \( x_1', x_2', \ldots, x'_N, x'_N \in \mathbb{Z}_+ \)
\[
\det \left[ \sum_{n=0}^{\infty} \epsilon^n \phi_n(x'_i) \phi_n(x'_j) \right] = \epsilon^{N(N-1)/2} \det[\phi_{i-1}(x'_j)] \det[\phi_{i-1}(x'_j)] + O \left( \epsilon^{N(N-1)/2 + 1} \right)
\]
as \( \epsilon \to 0 \), cf. (5.4). Applying this asymptotic relation to \( v_{-T, t_1}, v_{t_2, T}, v_{-T, T} \), we obtain
\[
\det[v_{-T, t_1}(a_i, y^{(1)}_{j})] \sim e^{-\frac{2(t_1 + T)N(N-1)}{2}} \det[\phi_{i-1}(a_j)] \det[\phi_{i-1}(y^{(1)}_{j})],
\]
\[
\det[v_{t_2, T}(y^{(k)}_{j}, b_j)] \sim e^{-\frac{2(t_2 - T)N(N-1)}{2}} \det[\phi_{i-1}(y^{(k)}_{j})] \det[\phi_{i-1}(b_j)],
\]
\[
\det[v_{-T, T}(a_i, b_j)] \sim e^{-T^N(N-1)} \det[\phi_{i-1}(a_j)] \det[\phi_{i-1}(b_j)],
\]
as \( T \to +\infty \). This completes the proof. \( \square \)

8. Particle–hole involution

For any set \( X \) and its subset \( \mathcal{Y} \) one can define an involution on point configurations \( X \subset X \leftrightarrow X \triangle \mathcal{Y} \). This map leaves intact the “particles” of \( X \) outside of \( \mathcal{Y} \), and inside \( \mathcal{Y} \) it picks the “holes” (points of \( \mathcal{Y} \) free of particles). This involution is called the particle-hole involution on \( \mathcal{Y} \).

The goal of this section is to give a different description of the z-measures using a new identification of Young diagrams and point configurations on \( \mathbb{Z}' \). Instead of using the configurations
\[
\overline{X}(\lambda) = \{ \lambda_i - i + \frac{1}{2} | i = 1, 2, \ldots \}
\]
we will use the configurations
\[
X(\lambda) = (\overline{X}(\lambda) \cap \mathbb{Z}'_+) \cup (\mathbb{Z}'_+ \setminus \overline{X}(\lambda)) = \overline{X}(\lambda) \triangle \mathbb{Z}'_-
\]
which are obtained from \( \overline{X}(\lambda) \) by applying the particle-hole involution on \( \mathbb{Z}'_+ \).

The parametrization of Young diagrams \( \lambda \) by configurations \( X(\lambda) \) corresponds to considering the Frobenius coordinates of \( \lambda \), see [BOO, §1.2] for details. The reason for passing to \( X(\lambda) \) is very simple: in the continuous limit \( \xi \not\rightarrow 1 \) which will be considered below in §9, the point process generated by \( \overline{X}(\lambda) \) does not survive, while the process corresponding to \( X(\lambda) \) has a well defined limit.

Observe that \( X(\lambda') = -X(\lambda) \) for any \( \lambda \in \mathcal{Y} \).

Given an arbitrary kernel \( K(x, y) \) on \( \mathcal{X} \times \mathcal{X} \), and a subset \( \mathcal{Y} \) of \( \mathcal{X} \), we assign to it another kernel,
\[
K^{\circ}(x, y) = \begin{cases} K(x, y), & x \notin \mathcal{Y}, \\ \delta_{xy} - K(x, y), & x \in \mathcal{Y}, \end{cases}
\]
where \( \delta_{xy} \) is the Kronecker symbol. Slightly more generally, given an arbitrary map \( \varepsilon : \mathcal{X} \rightarrow \mathbb{R}^* \), we set
\[
K^{\circ, \varepsilon}(x, y) = \varepsilon(x) K^{\circ}(x, y) \varepsilon(y)^{-1}.
\]
Proposition 8.1. Let \( P \) be a probability measure in point configurations on a discrete space \( \mathcal{X} \) and let \( P^\circ \) be the image of \( P \) under the particle-hole involution on \( \{\emptyset\} \subset \mathcal{X} \). Assume that the correlation functions of \( P \) have determinantal form with a certain kernel \( K(x, y) \),

\[
\rho_m(x_1, \ldots, x_m \mid P) = \det_{1 \leq i, j \leq m} [K(x_i, x_j)], \quad m = 1, 2, \ldots.
\]

Then the correlation functions of the measure \( P^\circ \) also have a similar determinantal form, with the kernel \( K^\circ(x, y) \) as defined above or, equally well, with the kernel \( K^{\circ, \varepsilon}(x, y) \), where the map \( \varepsilon : \mathcal{X} \to \mathbb{R}^* \) may be chosen arbitrarily,

\[
\rho_m(x_1, \ldots, x_m \mid P^\circ) = \det_{1 \leq i, j \leq m} [K^\circ(x_i, x_j)] = \det_{1 \leq i, j \leq m} [K^{\circ, \varepsilon}(x_i, x_j)],
\]

\[ m = 1, 2, \ldots. \]

Proof. The factor \( \varepsilon(\cdot) \) does not affect the values of determinants in right–hand side of the above formula, so that we may take \( \varepsilon(\cdot) \equiv 1 \). Then the result is obtained by applying the inclusion/exclusion principle, see Proposition A.8 in [BOO]. □

Later on we choose the function \( \varepsilon(\cdot) \) in a specific way (see (8.3) below) which is appropriate for the limit transition of \( \xi \).

The main result of this section is a determinantal formula for the dynamical correlation functions of \( \Lambda_{z,z',\xi} \) computed in terms of \( X(\lambda) \). For any \( n = 1, 2, \ldots \), define the \( n \)th dynamical correlation function of \( n \) pairwise distinct arguments \((t_1, x_1), \ldots (t_n, x_n) \in (t_{\min}, t_{\max}) \times \mathcal{Y} \) by

\[
\rho_n(t_1, x_1; t_2, x_2; \ldots; t_n, x_n) = \text{Prob}\{X(\lambda) \text{ at the moment } t_i \text{ contains } x_i \text{ for all } i = 1, \ldots, n\}.
\]

Here and in what follows we denote by \( \Lambda_{z,z',\xi} \) the stationary Markov process corresponding to the constant curve \( \xi(t) \equiv \xi \), where \( \xi \in (0, 1) \) is fixed.

Theorem 8.2. Let \((z, z')\) be a pair of admissible parameters not from the degenerate series. Consider the Markov process \( \Lambda_{z,z',\xi} \), and denote by \( X_{z,z',\xi} \) the process with values in the space of point configurations in \( \mathcal{Z}' \), obtained from \( \Lambda_{z,z',\xi} \) via \( \lambda \mapsto X(\lambda) \).

Then the process \( X_{z,z',\xi} \) is determinantal and its correlation kernel has the form (in all the formulas below \( x \) and \( y \) are positive, in \( \pm \) and \( \mp \) the upper sign corresponds to the case \( s \geq t \), and the lower sign corresponds to \( s < t \))

\[
K_{z,z',\xi}(s, x; t, y) = \pm \sum_{a \in \mathbb{Z}'_+} e^{-a|s-t|} \psi_{\pm a}(x; z, z', \xi) \psi_{\pm a}(y; z, z', \xi),
\]

\[
K_{z,z',\xi}(s, x; t, -y) = \pm \sum_{a \in \mathbb{Z}_+} (-1)^{a-\frac{1}{2}} e^{-a|s-t|} \psi_{\pm a}(x; z, z', \xi) \psi_{-a}(y; -z, -z', \xi),
\]

\[
K_{z,z',\xi}(s, -x; t, y) = \mp \sum_{a \in \mathbb{Z}_+} (-1)^{a-\frac{1}{2}} e^{-a|s-t|} \psi_{-a}(x; -z, -z', \xi) \psi_{\pm a}(y; z, z', \xi),
\]

\[
K_{z,z',\xi}(s, -x; t, -y) = \mp \sum_{a \in \mathbb{Z}'_+} e^{-a|s-t|} \psi_{-a}(x; -z, -z', \xi) \psi_{\pm a}(y; -z, -z', \xi),
\]

(8.1)
where the fourth formula is valid for $s \neq t$, and for $s = t$ we have

$$K_{z, z', \xi}(s, -x; s, -y) = \sum_{a \in \mathbb{Z}_+'} \psi_a(x; -z, -z', \xi)\psi_a(y; -z, -z', \xi). \quad (8.2)$$

Comments. 1. For $s = t$ this kernel coincides with the hypergeometric kernel derived in [BO2], see also [BO4], [BO5]. In those papers the kernel was written in another, so-called “integrable form”, see Remark 8.3 below.

2. The kernel $K_{z, z', \xi}$ has the following symmetries $(x, y \in \mathbb{Z}')$:

$$K_{z, z', \xi}(s, x; t, y) = (-1)^{\text{sgn } x \cdot \text{sgn } y} K_{z, z', \xi}(s, y; t, x),$$

$$K_{z, z', \xi}(s, x; t, y) = \begin{cases} K_{-z, -z', \xi}(t, -x; s, -y), & s \neq t, \\ (-1)^{\text{sgn } x \cdot \text{sgn } y} K_{-z, -z', \xi}(t, -x; s, -y), & s = t. \end{cases}$$

Proof. We use Proposition 8.1. As the initial kernel we take the expression for $K_{z, z', \xi}(x, y)$ given in Theorem 7.2, the set $\mathcal{X}$ is the union of finitely many copies of $\mathbb{Z}'$ which correspond to times at which we evaluate the dynamical correlation function, and $\mathfrak{Z}$ is the union of the same number of copies of $\mathbb{Z}'_-$. On each copy of $\mathbb{Z}'$ the function $\varepsilon(\cdot)$ is chosen in the following way:

$$\varepsilon(x) = \begin{cases} 1, & x > 0, \\ (-1)^{1-x^{1/2}}, & x < 0. \end{cases} \quad (8.3)$$

The statement then follows from Proposition 2.7. The last formula (for $s = t$) arises from the relation

$$\delta_{x, y} - \sum_{a \in \mathbb{Z}_+'} \psi_{-a}(x; -z, -z', \xi)\psi_{-a}(y; -z, -z', \xi)$$

$$= \sum_{a \in \mathbb{Z}_+'} \psi_a(x; -z, -z', \xi)\psi_a(y; -z, -z', \xi),$$

which follows from the fact that $\psi_a$ form an orthonormal basis, see Proposition 2.4. \qed

Remark 8.3. Denote by $K_{z, z', \xi}(x, y)$ the kernel that is obtained from the kernel $K_{z, z', \xi}(x, y)$ of $\mathfrak{Z}$ by the procedure described above. That is, $K_{z, z', \xi} = (K_{z, z', \xi})^{\varepsilon(\cdot)}$ with $\varepsilon$ given by (8.3). This is a correlation kernel for the $z$–measure $\zeta_{z, \xi}$, corresponding to the map $\lambda \mapsto X(\lambda)$. Clearly, $K_{z, z', \xi}(x, y)$ coincides with the specialization of the kernel of Theorem 8.2 at $s = t$. Let us abbreviate

$$\psi_{\pm \frac{1}{2}}(x) = \psi_{\pm \frac{1}{2}}(x; z, z', \xi), \quad \psi_{\pm \frac{1}{2}}(x) = \psi_{\pm \frac{1}{2}}(x; -z, -z', \xi).$$

We have for $x, y \in \mathbb{Z}_+'$ (cf. (3.13))

$$K_{z, z', \xi}(x, y) = \sqrt{\frac{z' \xi}{1 - \xi}} \frac{\psi_{\pm \frac{1}{2}}(x)\psi_{\pm \frac{1}{2}}(y) - \psi_{\pm \frac{1}{2}}(x)\psi_{\pm \frac{1}{2}}(y)}{x - y}$$

$$K_{z, z', \xi}(x, -y) = \sqrt{\frac{z' \xi}{1 - \xi}} \frac{\psi_{\pm \frac{1}{2}}(x)\psi_{\pm \frac{1}{2}}(y) + \psi_{\pm \frac{1}{2}}(x)\psi_{\pm \frac{1}{2}}(y)}{x + y}$$

$$K_{z, z', \xi}(-x, y) = -\sqrt{\frac{z' \xi}{1 - \xi}} \frac{\psi_{\pm \frac{1}{2}}(x)\psi_{\pm \frac{1}{2}}(y) + \psi_{\pm \frac{1}{2}}(x)\psi_{\pm \frac{1}{2}}(y)}{x + y}$$

$$K_{z, z', \xi}(-x, -y) = \sqrt{\frac{z' \xi}{1 - \xi}} \frac{\psi_{\pm \frac{1}{2}}(x)\psi_{\pm \frac{1}{2}}(y) - \psi_{\pm \frac{1}{2}}(x)\psi_{\pm \frac{1}{2}}(y)}{x - y}. \quad (8.4)$$
Indeed, the first three formulas are easily obtained from (3.13), and for the last formula we use the symmetry relation

\[ K_{z,z',\xi}(-x,-y) = K_{z,-z',\xi}(x,y), \quad x,y \in \mathbb{Z}_+^\prime. \]

These four formulas coincide with the expressions obtained in [BO2, Theorem 3.3].

9. Limit transition to the Whittaker kernel

In this section we compute the scaling limit of the kernel \( K_{z,z',\xi} \) of §8 as \( \xi \to 1 \) and the arguments \( x \) and \( y \) are scaled by \( (1 - \xi) \). In this limit the lattice \( \mathbb{Z}^\prime \) turns into the punctured real line \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \).

Let us introduce the continuous analogs of the functions \( w_a \). These new functions \( w_a(u; z, z') \) are indexed by \( a \in \mathbb{Z}^\prime \) and the argument \( u \) varies in \( \mathbb{R}_{>0} \). They are expressed through the classical Whittaker functions \( W_{\kappa,\mu}(u) \), see [Er, ch. 6] for the definition, as follows:

\[
 w_a(u; z, z') = (\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2}))^{-\frac{1}{2}} u^{-\frac{1}{2}} W_{\frac{1}{2} + a, a - \frac{1}{2}}(u). \tag{9.1}
\]

Since \( W_{\kappa,\mu}(u) = W_{\kappa,-\mu}(u) \), this expression is symmetric with respect to \( z \leftrightarrow z' \).

It will be convenient for us to use the following integral representation of \( w_a \):

\[
 w_a(u; z, z') = \frac{\Gamma(z' - a + \frac{1}{2})e^{\pi i(z'-a)u^{-\frac{1}{2}+\frac{1}{2}}}}{2\pi (\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2}))^{\frac{1}{2}}} \times \int_0^\infty \zeta^{-z'+a-\frac{1}{2}}(1 + \zeta)^{z-a-\frac{1}{2}} e^{-u(\zeta+\frac{1}{2})} d\zeta. \tag{9.2}
\]

The (standard) notation for the contour of integration means that we start at \(+\infty\), go along the real axis, then around the origin in the clockwise direction, and back to \(+\infty\) along the real axis. On the last part of the contour we choose the principal branch of \( \zeta^{-z'+a-\frac{1}{2}} \), which uniquely determines the values of this function on the whole contour.

This formula is easily seen to be equivalent to one of the classical integral representations for the confluent hypergeometric function \( \Psi \), see [Er, 6.11.2(9)].

Proposition 9.1. If \( \xi \to 1 \) and \( x \in \mathbb{Z}_+^\prime \) goes to \(+\infty\) so that \( (1 - \xi)x \to u > 0 \), then

\[
 \psi_a(x; z, z', \xi) \sim (1 - \xi)^{\frac{1}{2}} w_a(u; z, z').
\]

Proof. This statement can be proved in a number of ways, see e.g. [Er, 6.8(1)]. We will give an argument which uses the integral representations of \( \psi_a \) and \( w_a \). A similar argument will also be employed in the proof of Theorem 9.2 below.

We start with the integral representation (2.4) for \( \psi_a \). Let us choose as \( \{\omega\} \) the following contour \( C(R, r, \xi) \), where \( r > 0 \) is small enough (smaller than the distance between \( \sqrt{\xi} \) and \( 1/\sqrt{\xi} \)) and \( R > 0 \) is big enough (greater than \( 1/\sqrt{\xi} + r \)):

The contour starts at the point \( \omega = R \), goes along the full circle \( |\omega| = R \) in the positive direction, then along the real line until the point \( \omega = 1/\sqrt{\xi} + r \), further along the full circle \( |\omega - 1/\sqrt{\xi}| = r \) in the negative direction, and back along the
real line to \( \omega = R \). Thus, \( C(R,r,\xi) \) consists of a “big circle” of radius \( R \), a “small circle” of radius \( r \), and a “bridge” between them.

We now fix \( R \), pick \( r \) of order \( (1-\xi) \), and take the limit \( \xi \nearrow 1 \) of the integral. The integration over the “big circle” \( |\omega| = R \) converges to zero exponentially in \( (1-\xi)^{-1} \) thanks to the factor \( \omega^{-\alpha-a} \). To take care of the rest of the integral, we make the change of the integration variable

\[
\omega = 1/\sqrt{\xi} + (1-\xi)\zeta.
\]

Then we have

\[
1 - \sqrt{\xi}\omega = -(1-\xi)\zeta \sqrt{\xi}, \quad 1 - \sqrt{\xi}/\omega = (1-\xi)(1+\zeta) \cdot \frac{(1/\sqrt{\xi} - \sqrt{\xi})(1-\xi)^{-1} + \zeta}{(1+\zeta)(1/\sqrt{\xi} + (1-\xi)\zeta)}.
\]

Note that the second factors in the formulas above are asymptotically equal to 1 for \( \xi \) close to 1 and \( \zeta \) bounded, and are uniformly bounded away from 0 and \( \infty \) for \( \xi \) close to 1 and \( \zeta \) corresponding to arbitrary \( \omega \) on the contour. Hence, the rest of the integral is asymptotically equal to the absolutely convergent integral

\[
\frac{(1-\xi)^{z-z'}}{2\pi i} \int_{+\infty}^{0-} (-\zeta)^{-z'+a-\frac{1}{2}}(1+\zeta)^{z-a-\frac{1}{2}} e^{-u(\zeta+\frac{i}{2})} d\zeta.
\]

Taking into account the convention stated in Comment 1 to Lemma 2.2 one can check that \( \arg(-\zeta) = -\pi i \) on the last part of the contour. Therefore, changing \( (-\zeta)^{-z'+a-\frac{1}{2}} \) to \( (-\zeta)^{-z'+a-\frac{1}{2}} \) produces the factor

\[
e^{\pi i(z'-a+\frac{1}{2})} = i \ e^{\pi i(z'-a)}.
\]

Finally, the prefactor in (2.4) asymptotically equals

\[
(1-\xi)^{z'-z+\frac{1}{2}} \frac{\Gamma(z'-a+\frac{1}{2})u^{z'-2}}{(\Gamma(z-a+\frac{1}{2})\Gamma(z'-a+\frac{1}{2}))^{\frac{1}{2}}}.
\]

Thus, (2.4) asymptotically equals

\[
(1-\xi)^{\frac{1}{2}} \frac{\Gamma(z'-a+\frac{1}{2})e^{\pi i(z'-a)u^{z'-2}}}{2\pi (\Gamma(z-a+\frac{1}{2})\Gamma(z'-a+\frac{1}{2}))^{\frac{1}{2}}} \times \int_{+\infty}^{0-} \zeta^{-z'+a-\frac{1}{2}}(1+\zeta)^{z-a-\frac{1}{2}} e^{-u(\zeta+\frac{i}{2})} d\zeta.
\]

\[\square\]

**Theorem 9.2.** Consider the extended hypergeometric kernel \( K_{z,z',\xi}(s,x;t,y) \) as described in Theorem 8.2. Let \( \xi \nearrow 1 \) and assume that \( x,y \to \infty \) inside \( z' \) so that \((1-\xi)x \to u, (1-\xi)y \to v \), where \( u,v \in \mathbb{R}^* \).

Then there exists a limit kernel \( K^{W}_{z,z'}(s,u;t,v) \) on \( \mathbb{R}^* \times \mathbb{R}^* \):

\[
\lim_{\xi \nearrow 1} (1-\xi)^{-1} K_{z,z',\xi}(s,x;t,y) = K^{W}_{z,z'}(s,u;t,v).
\]  

(9.3)
For \( s \neq t \) the formulas for the limit kernel are obtained from formulas (8.1) for the kernel \( K_{z,z'} \) by replacing \( \psi_a \)’s with \( w_a \)’s and setting \( \xi = 1 \): for \( u, v > 0 \)

\[
K_{z,z'}^W(u,v) = \sum_{a \in Z'_+} e^{-a|s-t|} w_{\pm a}(u; z, z') w_{\mp a}(v; z, z'),
\]

\[
K_{z,z'}^W(s, u; t, v) = \sum_{a \in Z'_+} (-1)^a \frac{e^{-a|s-t|}}{2} w_{\pm a}(u; z, z') w_{\mp a}(v; -z, -z'),
\]

\[
K_{z,z'}^W(s, -u; t, v) = \sum_{a \in Z'_+} (-1)^a \frac{e^{-a|s-t|}}{2} w_{\mp a}(u; -z, -z') w_{\pm a}(v; z, z'),
\]

\[
K_{z,z'}^W(s, -u; t, v) = \sum_{a \in Z'_+} e^{-a|s-t|} w_{\mp a}(u; -z, -z') w_{\pm a}(v; -z, -z').
\]

Comments. 1. The prefactor \((1-\xi)^{-1}\) in (9.3) is due to rescaling of the state space \( Z' \) by \((1-\xi)\).

2. The reason of the restriction \( s \neq t \) in above formulas is the divergence of the series for \( K_{z,z'}^W(s, u; s, v) \) and \( K_{z,z'}^W(s, -u; s, v) \). The series for \( K_{z,z'}^W(s, u; s, v) \) and \( K_{z,z'}^W(s, -u; s, v) \) do converge and give the correct answer. For \( s = t \) there exist analogs of formulas (8.4):

\[
K_{z,z'}^W(u, v) = \sqrt{zz'} \frac{w_{-\frac{1}{2}}(u)w_{\frac{1}{2}}(v)}{u-v}
\]

\[
K_{z,z'}^W(u, -v) = \sqrt{zz'} \frac{w_{-\frac{1}{2}}(u)w_{\frac{1}{2}}(v) + w_{\frac{1}{2}}(u)w_{-\frac{1}{2}}(v)}{u+v}
\]

\[
K_{z,z'}^W(-u, v) = -\sqrt{zz'} \frac{\bar{w}_{\frac{1}{2}}(u)w_{\frac{1}{2}}(v) + \bar{w}_{\frac{1}{2}}(u)w_{-\frac{1}{2}}(v)}{u+v}
\]

\[
K_{z,z'}^W(-u, -v) = \sqrt{zz'} \frac{\bar{w}_{-\frac{1}{2}}(u)\bar{w}_{\frac{1}{2}}(v) - \bar{w}_{\frac{1}{2}}(u)\bar{w}_{-\frac{1}{2}}(v)}{u-v}
\]

Here we abbreviate

\[
w_{\pm \frac{1}{2}}(u) = w_{\pm \frac{1}{2}}(u; z, z'), \quad \bar{w}_{\pm \frac{1}{2}}(u) = w_{\pm \frac{1}{2}}(u; -z, -z').
\]

Formulas (9.4) can be derived from (8.4) using Proposition 9.1. They were previously obtained in [B1], [BO2, §5].

3. In accordance with the terminology of these papers (where the kernel (9.4) was called the Whittaker kernel) we call the limit kernel \( K_{z,z'}^W(s, u; t, v) \) the extended Whittaker kernel.

Proof. We use Proposition 9.1. In order to prove the theorem, we need to justify the interchange of the summation and the limit transition in (8.1). To do this it suffices to show that the series converge uniformly in \( \xi \).

We will prove that each of the four expressions

\[
|\psi_{\pm a}(x; z, z', \xi)\psi_{\mp a}(y; z, z', \xi)|, \quad |\psi_{\mp a}(x; -z, -z', \xi)\psi_{\pm a}(y; -z, -z', \xi)|, \tag{9.5}
\]

\[
|\psi_{\mp a}(x; z, z', \xi)\psi_{\mp a}(y; -z, -z', \xi)|, \quad |\psi_{\mp a}(x; -z, -z', \xi)\psi_{\pm a}(y; z, z', \xi)|, \tag{9.6}
\]

For
is estimated from above by \(\text{const}(u, v)(1 - \xi) \cdot q^{|a|}\), where \(q > 1\) can be chosen arbitrarily close to 1, and \(\text{const}(u, v)\) does not depend on \(a\) and \(\xi\). Together with the factors \(e^{-\alpha|a-t|}\) in (8.1) this ensures the needed uniform convergence.

Both expressions (9.5) are estimated in the same way, let us handle the first one. We apply formula (2.5) of Proposition 2.3 and we get a double contour integral, in which we single out the terms involving \(a\); we observe that all together they can be written in the form \((F(\omega_1, \omega_2; \xi))^k\), where \(k := a - \frac{1}{2}\) ranges over \(\mathbb{Z}_+\), and \(\omega_1\) and \(\omega_2\) are the variables of integration. Let us write down precisely the whole expression separately for the upper and the lower choice of sign in the subscript \(\pm a\):

\[
\psi_a(x; z, z', \xi)\psi_a(y; z, z', \xi) = \frac{(\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})\Gamma(y + z + \frac{1}{2})\Gamma(y + z' + \frac{1}{2}))^{\frac{1}{2}}}{(1 - \xi)}
\]

\[
\times \frac{1}{(2\pi i)^2} \oint_{(\omega_1)} \oint_{(\omega_2)} (F_{++}(\omega_1, \omega_2; \xi))^k \left(1 - \sqrt{\xi}\omega_1\right)^{-z'} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{z - 1} (1 - \sqrt{\xi}\omega_2)^{-z} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z' - 1} \omega_1^{-x - \frac{1}{2}} \omega_2^{-y + \frac{1}{2}} d\omega_1 \ d\omega_2
\]

(9.7)

where

\[
F_{++}(\omega_1, \omega_2; \xi) = F_+(\omega_1; \xi)F_+(\omega_2, \xi),
\]

For \(\psi_{-a}(x; z, z', \xi)\psi_{-a}(y; z, z', \xi)\) we obtain a similar expression:

\[
\psi_{-a}(x; z, z', \xi)\psi_{-a}(y; z, z', \xi) = \frac{(\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})\Gamma(y + z + \frac{1}{2})\Gamma(y + z' + \frac{1}{2}))^{\frac{1}{2}}}{(1 - \xi)}
\]

\[
\times \frac{1}{(2\pi i)^2} \oint_{(\omega_1)} \oint_{(\omega_2)} (F_{--}(\omega_1, \omega_2; \xi))^k \left(1 - \sqrt{\xi}\omega_1\right)^{-z'} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{z - 1} (1 - \sqrt{\xi}\omega_2)^{-z} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z' - 1} \omega_1^{-x + \frac{1}{2}} \omega_2^{-y + \frac{1}{2}} d\omega_1 \ d\omega_2
\]

(9.9)

where

\[
F_{--}(\omega_1, \omega_2; \xi) = F_-(\omega_1; \xi)F_-(\omega_2, \xi),
\]

For any \(q > 1\) there exists a contour \(C_q(\xi, q)\) which is of the same kind as in the proof of Proposition 9.1, and such that

\[
|F_\pm(\omega, \xi)| \leq q \quad \forall \omega \in C_q(\xi, q).
\]

(9.11)
Proof of Lemma 9.3. Recall that in the proof of Proposition 9.1 we used a specific family \(\{C(R, r, \xi)\}\) of contours. We will show that it is possible to take \(C_\pm(\xi, q) = C(R, r, \xi)\) with an appropriate choice of parameters \(R\) and \(r\) (the radii of the "big circle" and the "small circle" in \(C(R, r, \xi)\)).

Consider first the case of \(F_+\). Fix \(q > 1\) and let \(\bar{\omega}\) be related to \(\omega\) by the equivalent relations
\[
\bar{\omega} = F_+(\omega, \xi)/q = \frac{1 - \sqrt{\xi} \omega}{q \omega - q \sqrt{\xi}}, \quad \omega = \frac{1 + q \sqrt{\xi} \bar{\omega}}{q \bar{\omega} + \sqrt{\xi}}.
\]

To fulfill inequality (9.11) the contour \(C_+(\xi, q)\) must be contained in the image of the unit disk \(|\bar{\omega}| \leq 1\) under the conformal map \(\bar{\omega} \mapsto \omega\). Let \(S_+(\xi, q)\) denote the image of the circle \(|\bar{\omega}| = 1\); \(S_+(\xi, q)\) is the circle that is symmetric with respect to the real axis and passes through the real points
\[
\frac{q \sqrt{\xi} - 1}{q - \sqrt{\xi}}, \quad \frac{q \sqrt{\xi} + 1}{q + \sqrt{\xi}}.
\]
(these are the images of \(-1\) and \(1\), respectively). Since we are interested in the limit transition as \(\xi \nearrow 1\) we may assume that \(\xi\) is so close to \(1\) that \(q > 1/\sqrt{\xi}\). Then we have
\[
\frac{q \sqrt{\xi} - 1}{q - \sqrt{\xi}} < \sqrt{\xi} < \frac{q \sqrt{\xi} + 1}{q + \sqrt{\xi}} < \frac{1}{\sqrt{\xi}}.
\]  
(9.12)

Observe that the image of the disk \(|\bar{\omega}| \leq 1\) is the exterior of \(S_+(\xi, q)\) (for instance, this follows from the fact that the image of 0 is the point \(1/\sqrt{\xi}\) which is outside \(S_+(\xi, q)\)). Now we take \(C_+(\xi, q) = C(R, r, \xi)\), where \(R\) and \(r\) are chosen so that both the "big circle" and the "small circle" in \(C(R, r, \xi)\) are in the exterior of the circle \(S_+(\xi, q)\): the "big circle" surrounds \(S_+(\xi, q)\) while the "small circle" lies to the right of \(S_+(\xi, q)\). This is possible due to inequalities (9.12) and the fact that the distance between the points \(\frac{q \sqrt{\xi} - 1}{q - \sqrt{\xi}}\) and \(\frac{1}{\sqrt{\xi}}\) is of order \(1 - \xi\).

The case of \(F_-\) is handled analogously. We define \(\bar{\omega}\) by the equivalent relations
\[
\bar{\omega} = F_-(\omega, \xi)/q = \frac{\omega - \sqrt{\xi}}{q - q \sqrt{\xi} \omega}, \quad \omega = \frac{q \bar{\omega} + \sqrt{\xi}}{q \sqrt{\xi} \omega + 1}.
\]

Instead of the circle \(S_+(\xi, q)\) we have another circle, denoted by \(S_-(\xi, q)\), which is symmetric with respect to the real axis and passes through the points
\[
\frac{q + \sqrt{\xi}}{q \sqrt{\xi} + 1}, \quad \frac{q - \sqrt{\xi}}{q \sqrt{\xi} - 1}.
\]

We note that
\[
\sqrt{\xi} < \frac{q + \sqrt{\xi}}{q \sqrt{\xi} + 1} < \frac{1}{\sqrt{\xi}} < \frac{q - \sqrt{\xi}}{q \sqrt{\xi} - 1}.
\]

The contour \(C_-(\xi, q)\) must lie in the exterior of \(S_-(\xi, q)\). We again can take \(C_-(\xi, q) = C(R, r, \xi)\) with appropriate \(R\) and \(r\). But, in contrast to the case of \(F_+\), now the "small circle" in \(C(R, r, \xi)\) must surround \(S_-(\xi, q)\) (because \(1/\sqrt{\xi}\) is
inside $S_-(\xi, q)$, see the inequalities above). This requirement can be satisfied because the diameter of $S_-(\xi, q)$ (the distance between the points $\frac{\sqrt{\xi}}{q^2} + 1$ and $\frac{\sqrt{\xi}}{q^2} - 1$) is of order $(1 - \xi)$.

This completes the proof of Lemma 9.3. □

We return to the proof of Theorem 9.2. Let us estimate (9.7). The product of the prefactors is asymptotically

$$(1 - \xi) u^\frac{1}{2}(z - z') v^\frac{1}{2}(z' - z).$$

To estimate the integral we take as contours $\{\omega_1\}$ and $\{\omega_2\}$ the contour $C_+ (\xi, q^\frac{1}{2})$ as described in Lemma 9.3. According to Lemma 9.3, on the product of these contours, $|F_{+,+}(\omega_1, \omega_2; \xi)| \leq q$, whence we get

$$
\oint_{\{\omega_1 \in C_+ (\xi, q^\frac{1}{2})\}} \int_{\{\omega_2 \in C_+ (\xi, q^\frac{1}{2})\}} \left| F_{+,+}(\omega_1, \omega_2; \xi)^k \left( 1 - \sqrt{\xi} \omega_1 \right)^{-z} \left( 1 - \sqrt{\xi} \omega_1 \right)^{-1} \right|
\times \left( 1 - \sqrt{\xi} \omega_2 \right)^{-z} \left( 1 - \sqrt{\xi} \omega_2 \right)^{-1} \omega_1^{-x - \frac{1}{2}} \omega_2^{-y - \frac{1}{2}} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2}
\leq q^k \oint_{\{\omega_1 \in C_+ (\xi, q^\frac{1}{2})\}} \int_{\{\omega_2 \in C_+ (\xi, q^\frac{1}{2})\}} \left| \left( 1 - \sqrt{\xi} \omega_1 \right)^{-z'} \left( 1 - \sqrt{\xi} \omega_1 \right)^{-1} \right|
\times \left( 1 - \sqrt{\xi} \omega_2 \right)^{-z'} \left( 1 - \sqrt{\xi} \omega_2 \right)^{-1} \omega_1^{-x - \frac{1}{2}} \omega_2^{-y - \frac{1}{2}} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2}
$$

Arguing as in the proof of Proposition 9.1 we check that the last integral is uniformly bounded as $\xi \nearrow 1$. This yields for (9.7) the required estimate of the form $\text{const}(1 - \xi) q^k$ with arbitrary $q > 1$. The estimate for (9.9) is obtained in exactly the same way, by using the contours $\{\omega_1\} = \{\omega_2\} = C_- (\xi, q^{\frac{1}{2}})$.

The quantities in (9.6) are estimated similarly. We leave the details to the reader, and only point out two minor differences. First, while writing the double integral representation, we do not need to switch parameters $z$ and $z'$ (as in the proof of Proposition 2.3) to get rid of the gamma factors containing $a$. Second, we have to use as $\{\omega_1\}$ and $\{\omega_2\}$ two distinct contours: either $(C_+(\xi, q^{\frac{1}{2}})$ and $C_-(\xi, q^{\frac{1}{2}}))$ or $(C_-(\xi, q^{\frac{1}{2}})$ and $C_+(\xi, q^{\frac{1}{2}}))$. □

Our next goal is to give an integral representation for the extended Whittaker kernel computed in Theorem 9.2. Let us introduce contours $C_{\pm}(q)$ for $q > 1$. They can be viewed as limits of the images of the contours $C_{\pm}(\xi, q)$ as $\xi \nearrow 1$ in the new variable $\xi$ where

$$\omega = 1/\sqrt{\xi} + (1 - \xi)\xi.$$

The contour $C_+(q)$ starts at $+\infty$, goes along the real axis, circles around 0 in the negative direction, and returns to $+\infty$ along the real axis. It has to leave on its left the point $-1$ together with the circle which is symmetric with respect to the real axis and passes through the points $-q/(q - 1)$ and $-q/(q + 1)$ (this circle contains $-1$).
The contour $C_-(q)$ also starts at $+\infty$, goes along the real axis, circles around 0 in the negative direction, and returns to $+\infty$ along the real axis. It has to leave on its left the point $-1$, and it has to leave on its right the circle which is symmetric with respect to the real axis and passes through the points $-1/(q+1)$ and $1/(q-1)$ (this circle contains 0).

Note that if $\zeta \in C_+(q)$ then $|\zeta/(1+\zeta)| < q$, and if $\zeta \in C_-(q)$ then $|(1+\zeta)/\zeta| < q$.

**Theorem 9.4.** The extended Whittaker kernel $K_{z,z'}^W(s,u;t,v)$ of Theorem 9.2 for $s \neq t$ has the following integral representation ($u > 0$, $v > 0$):

$$K_{z,z'}^W(s,u;t,v) = e^{\pi i (s+z')}(u/v) \frac{k_{z'} e^{\frac{1}{2}(s-t)}}{k_z}$$

$$\times \frac{1}{(2\pi i)^2} \oint_{C_{\pm}} \oint_{C_{\pm}} \frac{\zeta \zeta' (1+\zeta)(1+\zeta')}{e^{s-t}(1+\zeta)(1+\zeta') \zeta \zeta'} \frac{e^{-u(\zeta+\frac{1}{2})-v(\zeta'+\frac{1}{2})} d\zeta d\zeta'}{e^{s-t}(1+\zeta)(1+\zeta') \zeta \zeta'}$$

where for $s > t$ both contours $\{\zeta_1\}$ and $\{\zeta_2\}$ are of the form $C_+(e^{\frac{1}{2}(s-t)})$, and for $s < t$ both contours are of the form $C_-(e^{\frac{1}{2}(t-s)})$;

$$K_{z,z'}^W(s,u;t,-v) = \frac{\sin(\pi z) \sin(\pi z')}{\sin(\pi z')} \frac{k_{z'} e^{\frac{1}{2}(s-t)}}{k_z}$$

$$\times \frac{1}{(2\pi i)^2} \oint_{C_{\pm}} \oint_{C_{\pm}} \frac{\zeta \zeta' (1+\zeta)(1+\zeta')}{e^{s-t}(1+\zeta)(1+\zeta') \zeta \zeta'} \frac{e^{-u(\zeta+\frac{1}{2})-v(\zeta'+\frac{1}{2})} d\zeta d\zeta'}{e^{s-t}(1+\zeta)(1+\zeta') \zeta \zeta'}$$

where for $s > t$, the contour $\{\zeta_1\}$ is of the form $C_+(e^{\frac{1}{2}(s-t)})$ and the contour $\{\zeta_2\}$ is of the form $C_-(e^{\frac{1}{2}(t-s)})$, while for $s < t$ the contour $\{\zeta_1\}$ is of the form $C_-(e^{\frac{1}{2}(t-s)})$ and the contour $\{\zeta_2\}$ is of the form $C_+(e^{\frac{1}{2}(s-t)})$;

$$K_{z,z'}^W(s,-u;t,v) = -K_{z,z'}^W(s,v;t,-u);$$

$$K_{z,z'}^W(s,-u;t,-v) = K_{z,-z'}^W(t,u;s,v).$$

The contours may be chosen differently by deforming the contours above so that the denominators of the integrands do not vanish.

**Proof.** We take the integral representation (9.2) for the functions $w_n$ and plug it in into the series of Theorem 9.2. Computing the sum of geometric progression under the integral yields the formulas above. The contours are chosen in such a way that the absolute values of the ratios of geometric progressions involved are less than one.

As in the proof of Proposition 2.3, in the derivation of the first formula we switch $z$ and $z'$ in the integral representation of the second factor, which cancels the gamma factors involving the summation index. In the derivation of the second formula we do not need to do that, the gamma factors disappear thanks to the relation $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$. □

### 10. Limit Transition to the Gamma Kernel

In this section we compute the limit of the extended hypergeometric kernel $K_{z,z';\xi}(s,x;t,y)$ as $\xi \nearrow 1$ and scaling of time $s = (1 - \xi)\sigma$, $t = (1 - \xi)\tau$ with finite $\sigma, \tau \in \mathbb{R}$.
**Theorem 10.1.** There exists a limit of the extended hypergeometric kernel

\[ K_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) = \lim_{\xi \to 1} K_{z,z';\xi}((1 - \xi)\sigma, x; (1 - \xi)\tau, y) \]

where \( x, y \in \mathbb{Z}' \), \( \sigma, \tau \in \mathbb{R} \).

For \( \sigma \geq \tau \), the correlation kernel can be written in two different ways: as a double contour integral

\[
K_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y)
= \frac{\Gamma(-z' - x + \frac{3}{2})\Gamma(-z - y + \frac{1}{2})e^{-\pi i(z + z')(1 - \tau - y + \frac{1}{2})}}{(\Gamma(-z - x + \frac{3}{2})\Gamma(-z' - x + \frac{3}{2})\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2}))^{\frac{1}{2}}}
\times \frac{1}{(2\pi i)^2} \int_{0}^{0-} \int_{0-}^{0} \frac{e^{\frac{1}{2}(1 + \zeta_1) - z - x - \frac{1}{2}\zeta_2 + y - \frac{1}{2}}(1 + \zeta_2)}{1 + (\sigma - \tau) + \zeta_1 + \zeta_2} \, d\zeta_1 \, d\zeta_2
\tag{10.1}
\]

and as a single integral

\[
K_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) = \int_{0}^{+\infty} e^{-u(\sigma - \tau)} w_x(u; -z, -z') w_y(u; -z, -z') \, du. \tag{10.2}
\]

The values of the kernel for \( \sigma < \tau \) are obtained from the above formulas using the symmetry property

\[
K_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) = (-1)^{x+y+1} K_{z,z'}^{\text{gamma}}(\tau, -x; \sigma, -y), \quad \sigma \neq \tau.
\]

**Comment.** For \( \sigma = \tau \) the kernel \( K_{z,z'}^{\text{gamma}} \) has a simpler “integrable” expression

\[
K_{z,z'}^{\text{gamma}}(x; y) = \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi(z - z'))}
\times \left\{ \Gamma(z + x + \frac{1}{2}) \Gamma(z' + x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) \right\}^{-1/2}
\times \frac{\Gamma(z + x + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) - \Gamma(z' + x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2})}{x - y}
\]

see [BO5, Theorem 2.3]. We called this kernel the **gamma kernel**. The more general kernel \( K_{z,z'}^{\text{gamma}}(\sigma, x; \tau, y) \) of Theorem 10.1 will be called the **extended gamma kernel**.

**Proof.** We start with the series representation of the extended hypergeometric kernel, Theorem 7.2. Using the symmetry relations of Propositions 2.5 and 2.7, we rewrite the kernel in the following form

\[
K_{z,z';\xi}(s, x; t, y)
= \sum_{a \in \mathbb{Z}_+} e^{-a|s-t|} \psi_x(a; -z, -z', \xi) \psi_y(a; -z, -z', \xi), \quad s \geq t,
\]

\[
= \left\{ (-1)^{s+y+1} \sum_{a \in \mathbb{Z}_+} e^{-a|s-t|} \psi_{-x}(a; z, z', \xi) \psi_{-y}(a; z, z', \xi), \quad s < t. \right. \]
This formula implies that if we prove the statement for \( \sigma \geq \tau \), then the case \( \sigma < \tau \) will follow by symmetry. Thus, we continue with the assumption \( s \geq t \) and therefore \( \sigma \geq \tau \).

Formula (10.2) is the limit variant of the formula for \( K_{z,z',\xi}(s,x;t,y) \) above. Indeed, by virtue of Proposition 9.1,

\[
\psi_x(a; -z, -z', \xi) \psi_y(a; -z, -z', \xi) \sim (1 - \xi) w_x(u; -z, -z') w_y(u; -z, -z')
\]

provided that \( a \sim (1 - \xi)^{-1} u \). The factor \( (1 - \xi) \) is responsible for turning the sum into an integral over \( u \); this sum is just an approximation to the integral.

This empirical argument needs a rigorous justification. It is simpler to turn the series representation for \( K_{z,z',\xi}(s,x;t,y) \) into a double contour integral and then pass to the limit in the integral. The limit integral will be identified with the right-hand side of (10.2).

Using formula (2.5) with appropriately changed parameters, we obtain

\[
K_{z,z',\xi}(s,x;t,y) = \frac{\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})}{(\Gamma(-z - x + \frac{1}{2})\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2}))^{\frac{1}{2}}}
\]

\[
\times \frac{1 - \xi}{(2\pi)^2} \oint \oint \frac{e^{-a(s-t)}}{\omega_1} \left(1 - \sqrt{\xi \omega_1}\right)^{z' + x - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{-z - x - \frac{1}{2}}
\]

\[
\times \left(1 - \sqrt{\xi \omega_2}\right)^{z + y - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{-z' - y - \frac{1}{2}} \frac{\omega_1^{-a} \omega_2^{-a} d\omega_1 d\omega_2}{\omega_1 \omega_2}
\]

with contours \( \{\omega_1\} \) and \( \{\omega_2\} \) chosen as in Proposition 2.3. Now we want to sum the geometric progression inside the integrals. The ratio of the geometric progression is \( e^{1 - \xi} \omega_1^{-1} \omega_2^{-1} \). In order to justify the interchange of summation and integration we need to ensure that the absolute value of this ratio, as a function in \( \omega_1, \omega_2 \), is bounded from above by a constant strictly less than one. This is easy to arrange by requiring, for example, that both contours contain the unit circle.

Performing the summation, we obtain

\[
K_{z,z',\xi}(s,x;t,y) = \frac{e^{\frac{1}{2}(s-t)}\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})}{(\Gamma(-z - x + \frac{1}{2})\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2}))^{\frac{1}{2}}}
\]

\[
\times \frac{1 - \xi}{(2\pi)^2} \oint \oint \left(1 - \sqrt{\xi \omega_1}\right)^{z' + x - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{-z - x - \frac{1}{2}}
\]

\[
\times \left(1 - \sqrt{\xi \omega_2}\right)^{z + y - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{-z' - y - \frac{1}{2}} \frac{\omega_1^{-a} \omega_2^{-a} d\omega_1 d\omega_2}{\omega_1 \omega_2}
\]

Recall the notation \( C(R,r,\xi) \) for certain type of contours introduced in the proof of Proposition 9.1. We assume that both integration variables range over such a contour with \( R \) being a fixed number greater than 1, and \( r \) being of order \( 1 - \xi \), and such that \( 1/\sqrt{\xi} - r > 1 \).
Let us split each of the contours into two parts: the first one is the “big” circle \(|\omega| = R\), and the second part is the rest. If both \(\omega_1\) and \(\omega_2\) range over their big circles then the integrand is uniformly bounded, and the prefactor \(1 - \xi\) sends the whole expression to zero as \(\xi \to 1\).

If one of the variables, say, \(\omega_2\) ranges over its big circle and \(\omega_1\) ranges over the second part of its contour, we observe that all the factors of the integrand involving \(\omega_2\) are uniformly bounded. The absolute value of the remaining part of the integrand

\[
(1 - \sqrt{\xi} \omega_1)^{z' + x - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{-z - x - \frac{1}{2}}
\]

is uniformly bounded by

\[
\text{const} \cdot \left| (1 - \xi)^{z'-z-1} \zeta_1^{z'+x-\frac{1}{2}} (1 + \zeta_1)^{-z-x-\frac{1}{2}} \right|
\]

with \(\omega_1 = 1/\sqrt{\xi} + (1 - \xi)\zeta_1\), where we used the same argument as in the proof of Proposition 9.1. Thus, our double integral is bounded in absolute value by the following one-dimensional integral in \(\zeta_1\):

\[
\text{const} \cdot (1 - \xi)^{\Re(z' - z)} \int_R^R \left| \zeta_1^{z'+x-\frac{1}{2}} (1 + \zeta_1)^{-z-x-\frac{1}{2}} d\zeta_1 \right|
\]

Hence, our expression is bounded by

\[
\text{const} \cdot (1 - \xi)^{\Re(z' - z)} \int_1^1 \zeta_1^{\Re(z' - z) - 1} d\zeta_1
\]

which is either bounded by a constant (if \(\Re(z - z') \neq 0\)) or by \(|\ln(1 - \xi)|\) (if \(\Re(z - z') = 0\)). In both cases, the prefactor \(1 - \xi\) in the integral representation for \(K_{z, z'}(s, x; t, y)\) sends the whole expression to zero.

The only asymptotically significant part of the integral comes from the case when both \(\omega_1\) and \(\omega_2\) vary over the second parts of their contours. Making the change of variables

\[
\omega_1 = 1/\sqrt{\xi} + (1 - \xi)\zeta_1, \quad \omega_2 = 1/\sqrt{\xi} + (1 - \xi)\zeta_2,
\]

and arguing as in the proof of Proposition 9.1, we conclude, using the asymptotic relation

\[
\frac{1 - \xi}{e^{x-1} \omega_1 \omega_2 - 1} \sim \frac{1}{1 + (\sigma - \tau) + \zeta_1 + \zeta_2},
\]

that the limit value of the kernel is given by the right-hand side of (10.1). Note that the integral in (10.1) is absolutely convergent. To see this we use the estimate

\[
|1 + (\sigma - \tau) + \zeta_1 + \zeta_2| \geq \text{const} \cdot |\zeta_1|^{\nu} |\zeta_2|^{1 - \nu}
\]

which holds for any \(\zeta_1, \zeta_2\) on our contours and any \(\nu \in (0, 1)\). We apply this inequality with \(\nu = \frac{1}{2} + \Re(z' - z)/2\). The fact that \(\nu \in (0, 1)\) follows from our basic assumptions on \(z, z'\), see §1.

Thus, we have proved the integral representation (10.1). To see the equivalence of (10.1) and (10.2) we substitute the integral representation (9.2) into (10.2) and integrate explicitly over \(u\). \(\square\)
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