Binary Icosahedral Flavor Symmetry
for Four Generations of Quarks and Leptons

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Abstract

To include the quark sector, the $A_5 \equiv I$ (icosahedron) four generation lepton model is extended to a binary icosahedral symmetry $I'$ flavor model. We find the masses of fermions, including the heavy sectors, can be accommodated. At leading order the CKM matrix is the identity and the PMNS matrix, resulting from same set of vacua, corresponds to tribimaximal mixings.
I. INTRODUCTION

The current version of the standard model (SM) consists of three generations of quarks and leptons. However, a fourth generation is not excluded by the electroweak precision data \[1\], so the existence of a fourth generation is still a viable phenomenological possibility and can provide an explanation of the observed anomaly of CP asymmetries in the B meson system \[2\] and the baryon asymmetry of the universe \[3\] with additional mixings and CP phases. Recently we proposed \[4\] a four generation lepton model based on the non-abelian discrete symmetry \(A_5 \equiv I\) (icosahedron), in which the best features of the three family \(A_4 \equiv T\) (tetrahedral) model survive. Besides the new heavy degrees of freedom in the \(A_5\) model, which satisfy the experimental constraints, we retain tribimaximal neutrino mixings, three light neutrino masses, and three SM charged lepton masses in the three light generation sector.

In this paper, we explore a generalization of our \(A_5\) model to include four generations of both quarks and leptons. First we recall the three family scenarios in which the binary tetrahedral group \(T' \equiv SL_2(3)\) is capable of providing a model of both the quark and lepton with tribimaximal mixings and a calculable Cabibbo angle \[5\]. The \(T'\) group is the double covering group of \(A_4\). It has four irreducible representations (irreps) with identical multiplication rules to those of \(A_4\), one triplet \(3\) and three singlets \(1, 1', 1''\), plus three additional doublet irreps \(2, 2', 2''\). The additional doublets allow the implementation of the \(2 \oplus 1\) structure to the quark sector \[6-10\], thus the third family of quarks are treated differently and are assigned to a singlet. Hence they can acquire heavy masses \[11, 12\]. One should note that \(A_4\) is not a subgroup of \(T'\), therefore, the inclusion of quarks into the model is not strictly an extension of \(A_4\), but instead replaces it \[13\]. Based on the same philosophy, we study the model of four families of quarks and leptons by using the binary icosahedral group \(I' \equiv SL_2(5)\). The relation between \(I'\) and \(A_5\) is similar to that for \(T'\) and \(A_4\). The icosahedral group \(A_5 \subset SO(3)\) has double-valued representations that are single-valued representations of the double icosahedral group \(I' \subset SU(2)\). Hence, besides the odd dimensional irreps that are coincident with those of \(A_5\), there are four additional spinor-like irreps \(2_s, 2'_s, 4_s, 6_s\) of \(I'\). We shall be able to assign quarks to the spinor-like representations, but to discuss model building using \(I'\), we must first review our lepton model based on \(A_5\), which will remain essentially unchanged when generalized to \(I'\). Some
useful group theory details have been relegated to the Appendix.

II. THE LEPTONIC $A_5$ MODEL

The irreps of $A_5$ are one singlet $1$, two triplets $3$ and $3'$, one quartet $4$, and one quintet $5$. The model is required to be invariant under the flavor symmetry of $A_5 \times Z_2 \times Z_3$ and the particle content is given by Table I.

| Field $L_i$ $l_{R5}$ $l_{R3}^c$ $l_{R1}^{(1),(2)}$ $N_{R5}$ $N_{R5}^{(1)R}$ $S_4$ $H_4$ $H_4'$ $\Phi_3$ |
|-------------|-----|-----|----------------|-----|----------------|-----|-----|-----|-----|
| $SU(2)_L$ 2 1 1 1 1 1 1 1 1 1 2 2 2 |
| $A_5$ 4 5 3 1 5 1 4 4 4 3 |
| $Z_2$ 1 -1 -1 -1 1 1 1 1 -1 1 |
| $Z_3$ $\omega$ 1 1 1 1 1 1 1 1 1 $\omega^2$ $\omega^2$ $\omega^2$ |

**TABLE I.** Particle content of the lepton $A_5$ model.

Here $L_i = (\nu_i, l_i)^T$ is the left-handed $SU(2)_L$ doublet with generation index $i$, $l_R$’s and $N_R$’s are right-handed charged leptons and neutrinos respectively. $H_4$, $H_4'$ and $\Phi_3$ are $SU(2)_L$ doublet scalar fields, while $S_4$ is a singlet scalar. The representations of $SU(2)_L$ gauge symmetry should not be confused with the representations of the non-abelian discrete symmetry $A_5$. The most general Yukawa interactions invariant under the symmetries can be expressed as

$$L_{Y(\text{lepton})} = \frac{1}{2} M_1 N_R^{(1)R} N_R^{(1)} + \frac{1}{2} M_5 N_{R5} N_{R5}$$
$$+ Y_{S1}(S_4 N_{R5} N_{R5}) + Y_{S2}(S_4(l_{R3}^c)^c l_{R5})$$
$$+ Y_1(L_{L4} N_R^{(1)} H_4) + Y_2(L_{L4} N_{R5} H_4)$$
$$+ Y_3(L_{L4} N_{R5} \Phi_3) + Y_4(L_{L4} l_{R5} H_4')$$
$$+ Y_5(L_{L4} l_{R1} H_4') + Y_6(L_{L4} l_{R2} H_4') + \text{H.c.}$$

(1)

If the scalar $S_4$ develops the vacuum expectation value (VEV) $\langle S_4 \rangle = (V_S, 0, 0, 0)$, then $A_5$ will break to $A_4$ causing the $A_5$ irreps to decompose as $1 \rightarrow 1$, $3 \rightarrow 3$, $3' \rightarrow 3$, $4 \rightarrow 1 \oplus 3$, and $5 \rightarrow 1' \oplus 1'' \oplus 3$. There is one vector-like $SU(2)_L$ singlet charged lepton
field with the mass given by $\langle S_4 \rangle$. The masses of the four generations of charged leptons are generated by scalar field $H'_4$ with an interesting result that the electron mass is predicted to be zero at tree level and induced through quantum corrections. We argued in [4] that there is enough freedom to fit the observed charged lepton masses. The canonical seesaw mechanism is responsible for the left-handed neutrino masses in the $A_4$ model\(^1\), while the Dirac mass terms are provided by the scalars fields $H_4$ and $\Phi_3$. We are able to obtain one heavy neutrino and three SM light neutrino masses through arranging the VEVs of the two scalar fields without severe fine-tuning. Therefore, the mass structure of the three families under $A_4$ symmetry is retained. We refer the reader to Ref. [4] for the details of the model. Now we discuss the inclusion of the quark sector by extending $A_5$ to its double covering $I'$.

III. $I'$ SYMMETRY AND THE QUARK SECTOR

The irreps of $I'$ are one singlet 1, two triplets 3 and $3'$, one quartet 4, and one quintet 5, which are also the irreps of $A_5$, plus $I'$ has four spinor-like irreps $2_s, 2'_s, 4_s,$ and $6_s$. The characters and the multiplication rules of $I'$ symmetry can be found in Table II and Table III of the Appendix.

In this work we confine ourselves to minimally extending the $A_5$ model, i.e., to include the four generations of quarks while minimizing the introduction of other new degrees of freedom. The assignment of the quark sector under $I' \times Z_2 \times Z_3$ is given as follows:

$$
\begin{pmatrix}
  u \\
  d \\
  c \\
  s
\end{pmatrix}_L \quad \text{and} \quad \begin{pmatrix}
  t \\
  b \\
  t' \\
  b'
\end{pmatrix}_L
$$

(2)

for the left-handed doublets, and

$$
\begin{pmatrix}
  d_R, s_R \\
  u_R, c_R
\end{pmatrix} \quad , \quad \begin{pmatrix}
  b_R, b'_R \\
  t_R, t'_R
\end{pmatrix}
$$

(3)

for the right-handed singlets. Here the fields $(t', b')_L, b'_R$ and $t'_R$ denote the chiral fields of the fourth generation quarks and the Higgs sector is the same as the $A_5$ model. Thus, we

\(^1\) Also see Ref. [14] for a general discussion of neutrino masses with four generations of fermions.
can write the most general Yukawa interactions between quarks and scalar fields as

\[ L_{Y(\text{quark})} = f_1(U_{1L} \otimes D_{sR}) \otimes \Phi_3 + f_2(U_{2L} \otimes D_{bR}) \otimes H'_4 + f_3(U_{2L} \otimes D_{tR}) \otimes H_4 + \text{H.c.} \]  

(4)

Recall that in the $A_5$ model the first step of symmetry breaking $A_5 \rightarrow A_4$ was caused by the vacuum expectation values (VEVs) of the $SU(2)_L$ singlet scalars $\langle S_4 \rangle = (V_S, 0, 0, 0)$. Here the $S_4$ VEV breaks $I'$ to $T'$ symmetry. The decomposition of the irreps at this stage of symmetry breaking $I' \rightarrow T'$ is given in Table IV in the Appendix. Therefore, the quark fields decompose as

\[
U_{1L}(2_s, +1, \omega) \rightarrow U_{1L}(2, +1, \omega), \\
U_{2L}(2_s, +1, \omega^2) \rightarrow U_{2L}(2, +1, \omega^2), \\
D_{sR}(4_s, +1, +1) \rightarrow S_R(2', +1, +1) + C_R(2'', +1, +1), \\
D_{bR}(2'_s, -1, \omega^2) \rightarrow D_{bR}(2, -1, \omega^2), \\
D_{tR}(2'_s, +1, \omega^2) \rightarrow D_{tR}(2, +1, \omega^2); 
\]  

(5)

while for the scalars we have

\[
S_4(4, +1, +1) \rightarrow S_1(1, +1, +1) + S_3(3, +1, +1), \\
H_4(4, +1, \omega^2) \rightarrow H_1(1, +1, \omega^2) + H_3(3, +1, \omega^2), \\
H'_4(4, -1, \omega^2) \rightarrow H'_1(1, -1, \omega^2) + H'_3(3, -1, \omega^2), \\
\Phi_3(3, +1, \omega^2) \rightarrow \Phi_3(3, +1, \omega^2). 
\]  

(6)

The Yukawa interactions of Eq. [4] now reads

\[
 f_1[U_{1L}(2) \otimes (S_R(2') \oplus C_R(2''))] \otimes \Phi_3 \\
+ f_2[U_{2L}(2) \otimes D_{bR}(2)] \otimes (H'_1 \oplus H'_3) \\
+ f_3[U_{2L}(2) \otimes D_{tR}(2)] \otimes (H_1 \oplus H_3) + \text{H.c.} 
\]

(7)

The Clebsch-Gordan coefficients of $T'$ can be found in the review [15] and were already calculated in [7, 10, 16]. We adopt the coefficients shown in Ref. [15] and ignore the phases.
for simplicity. The Yukawa couplings are divided into terms involving the up-type quarks
\[
f_1 \left[ -\begin{pmatrix} u & c \\ d & s \end{pmatrix} L \Phi_{3_1} + \begin{pmatrix} c \\ s \end{pmatrix} R \Phi_{3_2} + \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ d \end{pmatrix} L \Phi_{3_3} \right] \\
+ f_3 \sqrt{2} \begin{pmatrix} t \\ b \end{pmatrix} L t_R (H_1 + H_{3_1}) - \begin{pmatrix} t' \\ b' \end{pmatrix} L t_R (H_1 - H_{3_1}) \right] \\
+ f_3 \begin{pmatrix} c \\ s \end{pmatrix} R L \Phi_{3_1} + \begin{pmatrix} c \\ s \end{pmatrix} R L \Phi_{3_2} + \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ d \end{pmatrix} L \Phi_{3_3} \right] \\
+ f_3 \sqrt{2} \begin{pmatrix} t \\ b \end{pmatrix} L t_R (H_1 + H_{3_1}) - \begin{pmatrix} t' \\ b' \end{pmatrix} L t_R (H_1 - H_{3_1}) \right]
\]
and the down-type quarks
\[
f_1 \left[ \begin{pmatrix} c \\ s \end{pmatrix} R L \Phi_{3_1} + \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ d \end{pmatrix} L d_R \Phi_{3_2} - \begin{pmatrix} u \\ d \end{pmatrix} L d_R \Phi_{3_3} \right] \\
+ f_2 \sqrt{2} \begin{pmatrix} t \\ b \end{pmatrix} L b_R (H'_1 + H'_{3_1}) - \begin{pmatrix} t' \\ b' \end{pmatrix} L b_R (H'_1 - H'_{3_1}) \right] \\
+ f_2 \begin{pmatrix} c \\ s \end{pmatrix} R L \Phi_{3_1} + \begin{pmatrix} c \\ s \end{pmatrix} R L \Phi_{3_2} + \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ d \end{pmatrix} L \Phi_{3_3} \right] \\
+ f_2 \sqrt{2} \begin{pmatrix} t \\ b \end{pmatrix} L b_R (H'_1 + H'_{3_1}) - \begin{pmatrix} t' \\ b' \end{pmatrix} L b_R (H'_1 - H'_{3_1}) \right]
\]
respectively. Here we express the components of scalar fields as \( \Phi_3 = (\Phi_{3_1}, \Phi_{3_2}, \Phi_{3_3}) \), \( H_4 = H_1 + H_3 \) where \( H_3 = (H_{3_1}, H_{3_2}, H_{3_3}) \), and \( H'_4 = H'_1 + H'_3 \) where \( H'_3 = (H'_{3_1}, H'_{3_2}, H'_{3_3}) \) according to the breaking \( I' \to T' \) shown in Eq. (6). The masses of up-type fermions are generated by the \( \Phi_3 \) and \( H_4 \) VEVs, while down-type fermion masses are generated by the \( \Phi_3 \) and \( H'_4 \) VEVs.

IV. Masses and Mixings

We notice that each up- and down-type quark mass matrix is divided into two 2 \times 2 block matrices and can be expressed as
\[
M_U = \begin{pmatrix} M_{uu} & 0 \\ 0 & M_{tt'} \end{pmatrix} \quad \text{and} \quad M_D = \begin{pmatrix} M_{uu} & 0 \\ 0 & M_{bb'} \end{pmatrix}.
\]
This indicates the first two generations mix only with the third and fourth generations through higher order corrections. The 2 \times 2 block mass matrices are given by
\[
M_{uu} = f_1 \left( -\frac{\langle \Phi_{3_1} \rangle}{\sqrt{2}} \frac{\langle \Phi_{3_3} \rangle}{\langle \Phi_{3_2} \rangle} \right), \quad M_{ds} = f_1 \left( -\frac{\langle \Phi_{3_3} \rangle}{\sqrt{2}} \frac{\langle \Phi_{3_2} \rangle}{\langle \Phi_{3_1} \rangle} \right),
\]
\[
\]
\[ M_{t'w'} = f_3 \begin{pmatrix}
\frac{1}{\sqrt{2}}(-\langle H_1 \rangle + \langle H_{31} \rangle) & \frac{1}{\sqrt{2}}(\langle H_1 \rangle + \langle H_{31} \rangle) \\
\frac{1}{\sqrt{2}}(-\langle H_3 \rangle + \langle H_{31} \rangle) & \langle H_{31} \rangle
\end{pmatrix}, \]

and

\[ M_{\bar{w}'w'} = f_2 \begin{pmatrix}
\frac{1}{\sqrt{2}}(-\langle H_1' \rangle + \langle H_{31}' \rangle) & \frac{1}{\sqrt{2}}(\langle H_1' \rangle + \langle H_{31}' \rangle) \\
\frac{1}{\sqrt{2}}(-\langle H_3' \rangle + \langle H_{31}' \rangle) & \langle H_{31}' \rangle
\end{pmatrix}. \]

With the \( \langle \Phi_3 \rangle = v(1, 1, 1) \) VEV, which is enforced by the requirement of tribimaximal mixings in neutrinos at leading order \cite{4}, we find that the masses of \( u, d, s, c \) are degenerate \( m_u = m_d = m_s = m_c = \sqrt{\frac{3}{2}}f_1 v \). Also, the mass matrices of both \( M_{uc} \) and \( M_{ds}M_{ds}^\dagger \) take the same form. Therefore, the matrices \( M_{uc}M_{uc}^\dagger \) and \( M_{ds}M_{ds}^\dagger \) are diagonalized by using the same unitary matrix. Thus we conclude that, at first order, the CKM matrix is forced to be the identity \cite{2}, which is an acceptable first approximation to \( V_{CKM} \). It is interesting that the light quark masses \( (u, d, s, c) \) and three SM light neutrino masses \cite{3} have the same origin; both come from the VEVs of \( \Phi_3 \) field, and the unit CKM matrix and the tribimaximal PMNS mixings arise from the single subgroup \( Z_3 \), which is the remnant symmetry left in the vacuum \( \langle \Phi_3 \rangle = v(1, 1, 1) \). To correct the CKM mixings by the high-order effects, the relevant dimension-five and -six effective operators are \( U_{2L}D_{sR}\Phi_3^2, U_{2L}D_{sR}\Phi_3 H_4, U_{2L}D_{sR}H_4^2, \)

\( U_{2L}D_{sR}H_4^2 \) and \( U_{1L}D_{bR}H_4^2H_4', U_{1L}D_{tR}H_4^3, U_{1L}D_{tR}H_4^2H_4 \) respectively. Also, as mentioned in Ref. \cite{4}, perturbations of the \( \Phi_3 \) VEVs are needed to accommodate realistic neutrino masses. These high-order corrections will link together the derivations of the Cabibbo angle \( \theta_C \) and \( \theta_{13} \) in the quark and lepton mixing matrices respectively. A nonzero value of \( \theta_{13} \) has recently been indicated by the T2K experiment \cite{17} and by global analyses \cite{18}. If we consider perturbations of the VEVs by taking \( \langle \Phi_3 \rangle = (v + \Delta_1, v + \Delta_2, v) \), the light quark masses are calculated to be

\[ m_{u,c}^2 = \frac{f_1^2}{2} \left[ 3v^2 + 2v(\Delta_2 - \Delta_1) + (\Delta_1^2 + \Delta_2^2) \right. \right.

\[ \left. \left. \pm \sqrt{(\Delta_1^2 + \Delta_2^2)^2(6v^2 - 4v(\Delta_1 + \Delta_2))} \right] \]

and

\[ m_{d,s}^2 = \frac{f_1^2}{2} \left[ 3v^2 + 2v(\Delta_1 + \Delta_2) + (\Delta_1^2 + \Delta_2^2) \right. \right.

\[ \left. \left. \pm \Delta_1 \sqrt{(2v + \Delta_1)^2 + 2(v + \Delta_2)^2} \right] . \]

\(^2\) The third and fourth generations can mix largely in principle.

\(^3\) Neutrino masses are generated through the seesaw mechanism, and therefore they are further suppressed by the lepton number breaking scale.
This indicates how the degeneracy of the light quark masses can be lifted.

For the masses of the heavy sectors, the current 95% CL mass limits on 4th generation quarks from the PDG [19] are $m_{t'} > 256 \text{ GeV}$ and $m_{b'} > 128 \text{ GeV}$ (CC decay) or 199 GeV (for 100% NC decay). Recall that in the lepton $A_5$ model, $H_4$ is responsible for the Dirac masses of neutrinos, and we require the condition $\langle H_1 \rangle \equiv V_1 \gg \langle H_{31,2,3} \rangle \equiv V_{31,2,3}$ in order to decouple the 4th generation neutrino from the three light SM neutrinos. Therefore, from $M_{t,t'}$, we obtain the masses of $t$ and $t'$ as $m_{t,t'}^2 \approx \left[ V_{1}^2 + (V_{31}^2 + V_{32}^2 + V_{33}^2) \mp V_1 \sqrt{4V_{31}^2 + 2(V_{32} + V_{33})^2} \right]/2$. For $M_{b,b'}$, we also follow the lepton $A_5$ model by taking $\langle H'_4 \rangle = (V'_1, V', V', V')$ since this gives masses to charged leptons too. $m_{b,b'}^2$ are then given by $[V_1'^2 + 3V'^2 \mp 2\sqrt{3V'_1V'}]/2$. In general, we have enough parameters to accommodate the heavy quark mass spectrum.

V. CONCLUSION

In summary, we construct a model of four fermion generations based on the binary icosahedral symmetry group $I'$. Many properties of the SM with three families are accommodated such as the mass spectrum, tribimaximal mixings in the neutrino sector, and an identity CKM matrix at leading order[5]. In addition, quarks and leptons relations are intimately connected as their masses are provided from the same set of scalars. We believe this makes the model both interesting and challenging. For example, one has to strike a balance between the result of tribimaximal mixings in the neutrino sector and derivation a realistic Cabibbo angle from perturbations in the quark sector. It is still not clear to us whether the higher order corrections will lead to a realistic Cabibbo angle or if we need extra degrees of freedom to realize it. We will leave it and other phenomenological aspects of this model to future work.

VI. APPENDIX

$A_5$ is the only simple finite discrete subgroup of $SO(3)$. Its 60 elements can be generated by products of the two generators $a$ and $b$, which satisfy

$$a^2 = b^3 = (ab)^5 = e,$$

\[10\]

The four component VEVs, in general, can be different.

Both $A_5$ and $I'$ have also recently been used in the context of three generation models of fermion masses and mixings [22].
where $e$ is the identity element.

$I'$ is the double covering of $A_5$, therefore it has 120 elements. The representation in terms of generators is similar to that of $A_5$, namely

$$a^2 = b^3 = (ab)^5.$$  \hspace{1cm} (11)

We note that $a^2$ is no longer the identity, but the negative of the identity, i.e., $a^4 = b^6 = e$.

Any finite subgroup of $SU(2)$ must have (at least) one spinor doublet $2_s$. By using the multiplication rules, the irreducible representations of the group $[20, 21]$ can be visualized as an extended Dynkin diagrams depicted in Fig. 1.

Recalling that the exact sequence relation between $SU(2)$ and $SO(3)$ is

$$1 \rightarrow Z_2 \rightarrow SU(2) \rightarrow SO(3) \rightarrow 1,$$  \hspace{1cm} (12)

we can restrict to the discrete cases

$$1 \rightarrow Z_2 \rightarrow T' \rightarrow T \rightarrow 1,$$  \hspace{1cm} (13)

and

$$1 \rightarrow Z_2 \rightarrow I' \rightarrow I \rightarrow 1,$$  \hspace{1cm} (14)

to demonstrate the double coverings. As our interest is in $I'$ we first reproduce its character table in Table II, from which we can easily calculate the multiplication table for $I'$ irreps presented in Table III. The symmetry breaking to $T'$ is also easily obtained, as given in Table IV.
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
 & $C_1(1)$ & $C_2(1)$ & $C_3(12)$ & $C_4(12)$ & $C_5(12)$ & $C_6(12)$ & $C_7(30)$ & $C_8(20)$ & $C_9(20)$ \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & $1 - \phi$ & $1 - \phi$ & $\phi$ & $\phi$ & $-1$ & 0 & 0 \\
3' & 3 & 3 & $\phi$ & $\phi$ & $1 - \phi$ & $1 - \phi$ & $-1$ & 0 & 0 \\
4 & 4 & 4 & $-1$ & $-1$ & $-1$ & $-1$ & 0 & 1 & 1 \\
5 & 5 & 5 & 0 & 0 & 0 & 0 & 1 & $-1$ & $-1$ \\
\hline
2_s & 2 & $-2$ & $\phi - 1$ & $1 - \phi$ & $-\phi$ & $\phi$ & 0 & $-1$ & 1 \\
2_s' & 2 & $-2$ & $-\phi$ & $\phi$ & $\phi - 1$ & $1 - \phi$ & 0 & $-1$ & 1 \\
4_s & 4 & $-4$ & $-1$ & $1$ & $-1$ & 1 & 0 & 1 & $-1$ \\
6_s & 6 & $-6$ & 1 & $-1$ & 1 & $-1$ & 0 & 0 & 0 \\
\hline
\end{tabular}
\caption{Character table of $I'$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.}
\end{table}

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TABLE III. Multiplication rules for the binary icosahedral group $I'$.

| ⊗ | 1  | 3  | 3'  | 4  | 5  | 2_s | 2'_s | 4_s | 6_s |
|---|----|----|-----|----|----|-----|------|-----|-----|
| 1 | 1  | 3  | 3'  | 4  | 5  | 2_s | 2'_s | 4_s | 6_s |
| 3 | 3  | 1+3+5 | 4+5 | 3'+4+5 | 3'+3'+4+5 | 2_s+4_s | 6_s | 2_s+4_s+6_s | 2_s+4_s+6_s |
| 3' | 3  | 1+3'+5 | 3+4+5 | 3'+3'+4+5 | 5 | 6_s | 2'_s+4_s | 2'_s+4_s+6_s | 2_s+2'_s+6_s |
| 4 | 4  | 3'+4+5 | 3+4+5 | 1+3+3'+4+5 | 3'+3'+4+5 | 5 | 2'_s+6_s | 2_s+6_s | 4_s+6_s+6_s | 2_s+2'_s+6_s |
| 5 | 5  | 3+3'+4+5 | 3+3'+4+5 | 3+3'+4+5 | 5 | 6_s | 4_s+6_s | 2_s+2'_s+6_s | 2_s+2'_s+6_s |
| 2_s | 2_s | 2_s+4_s | 6_s | 2'_s+6_s | 4_s+6_s | 1+3 | 4 | 3+5 | 3'+4+5 |
| 2'_s | 2'_s | 6_s | 2'_s+4_s | 4 | 1+3' | 3'+5 | 3+4+5 |
| 4_s | 4_s | 2_s+4_s+6_s | 4_s+6_s+6_s | 2_s+2'_s | 3+5 | 3'+5 | 3+4+5 | 3+3'+4+5 | 4+5+5 |
| 6_s | 6_s | 2'_s+4_s+6_s | 4_s+4_s+6_s | 6_s+6_s+6_s | 3'+4+5 | 3+4+5 | 3+3'+4+5 | 4+5+5 | 4+5+5+5 |

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TABLE IV. $I' \rightarrow T'$ symmetry breaking.

| $I'$ → $T'$ | $I' \rightarrow T'$ |
|------------|-----------------|
| 1 1       | $2_s$ 2         |
| 3 3       | $2'_s$ 2        |
| 3' 3      | $4_s$ $2' + 2''$ |
| 4 1 + 3   | $6_s$ $2 + 2' + 2''$ |
| 5 1' + 1'' + 3 |                     |

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