Dictionary Learning with Low-rank Coding Coefficients for Tensor Completion

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Abstract—In this paper, we propose a novel tensor learning and coding model for third-order data completion. Our model is to learn a data-adaptive dictionary from the given observations, and determine the coding coefficients of third-order tensor tubes. In the completion process, we minimize the low-rankness of each tensor slice containing the coding coefficients. By comparison with the traditional pre-defined transform basis, the advantages of the proposed model are that (i) the dictionary can be learned based on the given data observations so that the basis can be more adaptively and accurately constructed, and (ii) the low-rankness of the coding coefficients can allow the linear combination of dictionary features more effectively. Also we develop a multi-block proximal alternating minimization algorithm for solving such tensor learning and coding model, and show that the sequence generated by the algorithm can globally converge to a critical point. Extensive experimental results for real data sets such as videos, hyperspectral images, and traffic data are reported to demonstrate these advantages and show the performance of the proposed tensor learning and coding method is significantly better than the other tensor completion methods in terms of several evaluation metrics.

Index Terms—Tensor completion, dictionary learning, tensor singular value decomposition (t-SVD), low-rank coding.

I. INTRODUCTION

TENSOR completion is a problem of filling the missing or unobserved entries of the complete observed data, playing an important role in a wide range of real-world applications, such as color image inpainting [1],[4], high-speed compressive video [5], magnetic resonance imaging (MRI) data recovery [6], and hyperspectral data inpainting [7]. Generally, many real-world tensors are inner correlated, the spectral redundancy [8] of the hyperspectral images (HSIs) for example. Therefore, it is effective to utilize the global low-dimensional structure to characterize the relationship between the missing entries and observed ones.

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Fig. 1. An illustration of the TNN based LRTC and the DTNN based LRTC. SR denotes the sampling rate.

Generally, like the matrix case, the low-rank tensor completion (LRTC) can be formulated as

\[
\min \text{rank}(\mathcal{X}) \quad \text{s.t.} \quad \mathcal{X}_\Omega = \mathcal{O}_\Omega, \quad (1)
\]

where \(\mathcal{X}\) is the underlying tensor, \(\mathcal{O}\) is the observed incomplete tensor as shown in the top-left of Fig. 1. \(\Omega\) is the index set corresponding to the observed entries, and \(\mathcal{X}_\Omega = \mathcal{O}_\Omega\) enforces the entries of \(\mathcal{X}\) in \(\Omega\) equal to the observation \(\mathcal{O}\). However, unlike the matrix cases, the definition of the tensor rank is still not unique and has received considerable attentions in recent researches. This work fixes attentions on a novel notion of the tensor rank, i.e., the tubal-rank, which is derived from the tensor singular value decomposition (t-SVD) framework [9, 10].

For a third-order tensor \(\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}\), its t-SVD is given as \(\mathcal{X} = \mathcal{U} \ast \mathcal{S} \ast \mathcal{Y}^\Lambda\), where \(\mathcal{U}\) and \(\mathcal{V}\) are orthogonal tensors (see Def. 3) of size \(n_1 \times n_1 \times n_3\) and \(n_2 \times n_2 \times n_3\) respectively, and \(\ast\) denotes the t-prod (see Def. 3). The tubal-rank of \(\mathcal{X}\) is defined as the number of non-zero singular tubes of \(\mathcal{S}\). Since that the LRTC problem associated with the tubal-rank is NP-hard, Zhang et al. [11] turn to minimize the tensor nuclear norm (TNN), which is a convex surrogate of the tubal-rank, and give some theoretical guarantees. The TNN based LRTC
model is given as
\[
\min \|\lambda\|_{\text{TNN}} \quad \text{s.t.} \quad \Omega = 0.
\] (2)

The t-prod is based on a convolution-like operation and can be implemented using the discrete Fourier transform (DFT) \[12\]. Consequently, the TNN can be computed by using the DFT matrix. More specifically, for \( \lambda \in \mathbb{R}^{n_1 \times n_2 \times n_3} \), after respectively denoting the DFT matrix and inverse DFT matrix as \( F_{n_3} \) and \( F_{n_3}^{-1} \), its Fourier transformed (along the third mode) tensor \( Z \) can be obtained by \( Z = \lambda \times_3 F_{n_3} \), where \( \times_3 \) is the tensor-matrix product (See Sec. II-B), and we have \( \lambda = Z \times_3 F_{n_3}^{-1} \). Then, \( \|\lambda\|_{\text{TNN}} = \|\text{bdiag}(Z)\|_\ast = \sum_{k=1}^{n_3} \|Z(:,:,k)\|_\ast \), where \( \text{bdiag}(\cdot) \) is the block diagonal operation (see Def. 5). Thus, (2) is equivalent to
\[
\min_{Z} \|\text{bdiag}(Z)\|_\ast \quad \text{s.t.} \quad (Z \times_3 F_{n_3}^{-1})_{\Omega} = 0. \quad (3)
\]

As shown in the top-right of Fig. 1, the TNN based LRTC model can be comprehended as finding the coding coefficients tensor \( Z \), which is slice-wisely low-rank, of \( \lambda \) with a predefined dictionary \( F_{n_3}^{-1} \).

From the transform perspective, Kernfeld et al. \[12\] note that the t-SVD framework can be defined via any invertible transform. The TNN in (2) can be alternatively implemented by using other transforms, e.g., the discrete cosine transform (DCT) adopted by Lu et al. \[13\] and Xu et al. \[14\], and the Haar wavelet transform exploited in \[15\]. Furthermore, Jiang et al. \[16\] introduce the framelet transform, which is semi-invertible, and break through the restriction of invertibility. Nonetheless, all these methods can be rewritten in the form of (3) and interpreted as to find a slice-wisely low-rank coding \( Z \) of the original tensor \( \lambda \) with some predefined dictionaries, e.g., the inverse DCT transform matrix and the inverse framelet transform matrix. Within these transform based TNN methods, the typical pipeline is applying one selected transform along the third dimension, and minimizing the low-rankness of slices of the transformed data (coefficients) for the completion. Once the tubes of the original tensor are highly correlated, the frontal slices of the transformed data would be low-rank \[15\] \[16\].

An unavoidable issue is that the correlations along the third mode are different for various types of data. For example, the redundancy of HSIs along the third mode are much higher than videos with changing scenes. Thus, predefined dictionaries, viz the inverse transform matrices, usually lack flexibility and could not be suitable for all kinds of data. Therefore, to address this issue, we construct a dictionary, which can be adaptively inferred form the data, instead of inverse transform matrices mentioned above. We formulate the LRTC model as
\[
\min_{Z, D} \|\text{bdiag}(Z)\|_\ast \quad \text{s.t.} \quad (Z \times_3 D)_{\Omega} = 0, \quad (4)
\]
where \( D \in \mathbb{R}^{n_3 \times d} \) (generally \( d > n_3 \)). To avoid the situation where the elements in \( D \) reach arbitrarily high values allowing for arbitrarily low (but non-zero) values of \( Z \), we additionally constrain \( D \)'s columns as \( \|D(:,i)\|_2 = 1 \) for \( i = 1, 2, \cdots, d \).

The bottom-right part in Fig. 1 shows the coefficients and the dictionaries obtained by our method. We can see that the dictionary learned for the HSI completion is smoother than that for the MRI data. The front slices of the original data \( \lambda \) are linear combinations of front slices of the coefficients tensor \( Z \). Meanwhile, as the number of atoms in \( D \) is much bigger than the length of the tubes, the low-rank representation of tensor tubes is expected to be more efficiently. With the adaptively learned dictionary and corresponding low-rank coding, the performance of our method is significant better than the TNN based LRTC method (see the peak signal-to-noise ratio (PSNR) values exhibited in Fig. 1).

As mentioned by the authors of \[13\] \[15\], it is interesting to learn the transform for implementing the t-SVD from the data in different tasks. Our approach can be viewed as learning the inverse transform from this perspective and indeed enriches the research on this topic. The methods, which utilize DFT, DCT, and Framelet, can be viewed as specific instances of our method with fixed dictionaries, i.e., the inverse discrete transformation matrices. From the view of dictionary learning, our method can also be interpreted as learning a dictionary with low-rank coding. We enforce the low-rankness of the coding coefficients in a tensor manner, and this allows the linear combination of features, namely, the atoms of the dictionary.

The main contributions of this paper mainly consist of three aspects:

- We propose novel tensor learning and coding model, which is to adaptively learn a dictionary from the observations and determine the low rank coding coefficients, for the third-order tensor completion.
- A multi-block proximal alternating minimization algorithm is designed to solve the proposed non-convex model. We theoretically prove its global convergence to a critical point.
- Extensive experiments are conducted on various types of real-world third-order tensor data. The results illustrate that our method outperforms compared LRTC methods.

This paper is organized as follows. Sec. II introduces related works and the basic preliminaries. Our method is given in Sec. III. We report the experimental results in Sec. IV. Finally, Sec. V draws some conclusions.

II. RELATED WORK AND PRELIMINARIES

A. Related Work

The t-SVD framework is proposed in \[9\] \[10\] and it induces the definition of the tubal-rank. Zhang et al. \[11\] utilize the TNN, which is the convex relaxation of the tubal-rank, for LRTC problems and give theoretical bounds from limited sampling for third-order tensor recovery in their journal extension \[17\]. Jiang et al. \[18\] and Wang et al. \[19\] tackle the robust tensor completion task, in which the incomplete observations are corrupted by sparse outliers, via minimizing TNN. In \[13\] \[15\] \[16\], other transformations are selected as substitutes of the DFT to formulate the transform based TNN and the corresponding LRTC models.

As mentioned before, different kinds of tensor rank notions exist and have also been widely investigated. For instance, the CANDECOMP/PARAFAC (CP)-rank, based on the CP decomposition, is defined as the minimal rank-one tensors to
express the original data \[20\]. Although determining the CP-rank of a given tensor is NP-hard \[21\]. CP decomposition has been successfully applied for tensor recovery problem \[22\]. The Tucker-rank, corresponding to the Tucker decomposition \[23\], is defined as a vector constituted of the ranks of the unfolding matrices. Liu et al. \[4\] propose a convex surrogate of the Tucker-rank and minimize it for the LRTC problem while Zhang et al. \[25\] resort to use a family of nonconvex functions onto the singular values. Another newly emerged one is the tensor train (TT)-rank derived from the TT decomposition \[27\]. In this framework, the tensor is decomposed in a chain manner with nodes being third-order tensors. Bengua et al. \[28\] minimize a nuclear norm based on the TT-rank for the color image and video recovery. The TT-rank has also been applied for the HSI super-resolution \[29\] and the tensor-on-tensor regression \[30\]. When factors cyclically connected, it becomes the tensor ring (TR) decomposition \[31\]. Yuan et al. \[32\] exploit the low-rank structure of the TR latent space and regularize the latent TR factors with the nuclear norm. Yu et al. \[33\] introduce the tensor circular unfolding for the TR decomposition and perform parallel low-rank matrix factorizations to all circularly unfolded matrices for tensor completion. For a comprehensive overview of the LRTC problem, please refer to \[34\]–\[35\].

B. Preliminaries

Throughout this paper, lowercase letters, e.g., \(x\), boldface lowercase letters, e.g., \(x\), boldface upper-case letters, e.g., \(X\), and boldface calligraphic letters, e.g., \(X\), are used to denote scalars, vectors, matrices, and tensors, respectively. Given a third-order tensor \(X \in \mathbb{R}^{n_1 \times n_2 \times n_3}\), we use \(X_{ijk}\) to denote its \((i,j,k)\)-th element. The \(k\)-th frontal slice of \(X\) is denoted as \(X^{(k)}\) (or \(X^{(i,:,:,k)}\), \(X^{(k,:,:)}\)), and the mode-3 unfolding matrix of \(X\) is denoted as \(X_{(3)} \in \mathbb{R}^{n_1 \times n_2 \times n_3}\). We use \(\text{fold}_3\) and \(\text{unfold}_3\) to denote the folding and unfolding operations along the third dimension, respectively, and we have \(X = \text{fold}_3(\text{unfold}_3(X)) = \text{fold}_3(X_{(3)})\). The mode-3 tensor-matrix product is denoted as \(\times_3\), and we have \(X \times_3 A \leftrightarrow A\text{unfold}_3(X)\). The tensor Frobenius norm of a third-order tensor \(X\) is defined as \(\|X\|_F := \sqrt{\langle X, X \rangle} = \sqrt{\sum_{ijk} X_{ijk}^2}\).

**Definition 1** (tensor conjugate transpose \[36\]). The conjugate transpose of a tensor \(A \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) is tensor \(A^H \in \mathbb{C}^{n_1 \times n_2 \times n_3}\) obtained by conjugate transposing each of the frontal slice and then reversing the order of transposed frontal slices 2 through 3, i.e., \((A^H)^{(1)} = (A^{(1)})^\dagger\) and \((A^H)^{(i)} = (A^{(n_3+i-2)}\ldots A^{(2)} A^{(1)})^\dagger\) for \(i = 2, \ldots, n_3\).

**Definition 2** (t-prod \[36\]). The tensor-tensor-product (t-prod) \(C = A \times B\) of \(A \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) and \(B \in \mathbb{R}^{n'_2 \times n'_3}\) is a tensor of size \(n_1 \times n_4 \times n_3\), where the \((i,j)\)-th tube \(c_{ij}\) is given by
\[
c_{ij} := C(i,j,:) = \sum_{k=1}^{n_2} A(i,k,:) \times B(k,j,:)
\]
where \(\times\) denotes the circular convolution between two tubes of same size.

**Definition 3** (special tensors \[36\]). The identity tensor \(I \in \mathbb{R}^{n_1 \times n_1 \times n_3}\) is the tensor whose first frontal slice is the \(n_1 \times n_1\) identity matrix, and whose other frontal slices are all zeros. A tensor \(Q \in \mathbb{C}^{n_1 \times n_1 \times n_3}\) is orthogonal if it satisfies
\[
Q^H \ast Q = Q \ast Q^H = I.
\]
A tensor \(A\) is called f-diagonal if each frontal slice \(A^{(i)}\) is a diagonal matrix.

**Theorem 1** (t-SVD \[10\], \[36\]). For \(A \in \mathbb{R}^{n_1 \times n_2 \times n_3}\), the t-SVD of \(A\) is given by
\[
A = U \ast S \ast V^H
\]
where \(U \in \mathbb{R}^{n_1 \times n_1 \times n_2 \times n_3}\) and \(V \in \mathbb{R}^{n_2 \times n_2 \times n_3}\) are orthogonal tensors, and \(S \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) is an f-diagonal tensor.

**Definition 4** (tensor tubal-rank \[11\]). The tubal-rank of a tensor \(A \in \mathbb{R}^{n_1 \times n_2 \times n_3}\), denoted as \(\text{rank}_t(A)\), is defined to be the number of non-zero singular tubes of \(S\), where \(S\) comes from the t-SVD of \(A\): \(A = U \ast S \ast V^T\). That is \(\text{rank}_t(A) = \#\{i : S(i,:) \neq 0\}\).

**Definition 5** (block diagonal operation \[11\]). The block diagonal operation of \(A \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) is given by
\[
\text{bdia}(A) \triangleq \begin{bmatrix} A^{(1)} & A^{(2)} & \ldots & A^{(n_3)} \end{bmatrix}
\]
where, and \(\text{bdia}(A) \in \mathbb{C}^{n_1 n_2 \times n_3}\).

**Definition 6** (tensor-nuclear-norm (TNN) \[11\]). The tensor nuclear norm of a tensor \(A \in \mathbb{R}^{n_1 \times n_2 \times n_3}\), denoted as \(\|A\|_{\text{TNN}}\), is defined as
\[
\|A\|_{\text{TNN}} \triangleq \|\text{bdia}(Z)\|_*,
\]
where \(Z \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) is the Fourier transformed (along the third mode) tensor of \(A\). The TNN can be computed via the summation of the matrix nuclear norm of \(Z\)’s frontal slices. That is \(\|A\|_{\text{TNN}} = \sum_{i=1}^{n_3} \|Z(i,:,,:)\|_\star\).

III. MAIN RESULTS

A. Proposed Model

The proposed tensor learning and coding model is formulated as
\[
\min_{Z, D} \|\text{bdia}(Z)\|_*,
\]
s.t. \((Z \times_3 D)_{i,:} = \Omega_{i,:}\)
\[
\|D(:,i)\|_2 = 1 \text{ for } i = 1, 2, \ldots, d,
\]
where \(D \in \mathbb{R}^{n_3 \times d}\) and \(Z \in \mathbb{R}^{n_1 \times n_2 \times d}\) are respective the dictionary and the low-rank coding coefficients. Since that our LRTC model is very similar to the TNN based model in \[3\], we term it as dictionary based TNN (DTNN)\footnote{We remark here that, although the objective function of \[10\] is in the same form of TNN in \[3\], it could not be derived to a normative definition of a norm as Def \[6\]. Given a tensor \(X\) and a certain dictionary \(D\), the coefficients in \(Z\) here could not be directly obtained and satisfy \((Z \times_3 D) = X\). It is needed to optimize \[10\] simultaneously with respect to the dictionary and coefficients (with \(\Omega\) indexing all the entries).}.
While resembling in form, our model in is distinct from the TNN based LRTC model. The main reference is that our model is more flexible for different kinds of data because of the data-adaptive dictionary term. The dictionary used in can be viewed as the inverse transform. This is also different from previous works tailoring the linear or unitary transform.

Traditional dictionary learning techniques utilize overcomplete dictionaries, the amount of whose atoms is always more than the dimension of the signal, and find the sparse representations. In , although is much bigger than, is still not big enough to overcompletely represent , which is of big volume, with sparse coefficients. Therefore, we need the specific low-rank structure of the coefficients, which allows the linear combination of features, together with the learned dictionary, to accurately complete . Thus, our method is distinct from previous tensor dictionary learning methods, which enforce the sparsity of coefficients, e.g., . Please see Sec. IV-E1 for detailed comparisons of sparsity and low-rankness.

B. Proposed Algorithm

To optimize the specific structured problem in the proposed model, we tailored a multi-block proximal alternating minimization algorithm. Let

\[ \Phi(X) = \begin{cases} 0, & X_\Omega = 0, \\ \infty, & \text{otherwise}, \end{cases} \]

and

\[ \Psi(D) = \begin{cases} 0, & \|D(:,i)\|_2 = 1 \text{ for } i = 1, 2, \cdots, d, \text{ otherwise}. \end{cases} \]

Thus, the problem in can be rewritten as the following unconstrained problem

\[
\min_{Z,D} \Phi(Z \times_3 D) + \sum_{k=1}^{d} \|Z^{(k)}\|_* + \Psi(D) \tag{11}
\]

is difficult to be directly optimized. Therefore, we resort to the half quadratic splitting (HQS) technique and turn to solve the following problem

\[
\min_{Z,D,X} \frac{\beta}{2} \|X - Z \times_3 D\|_F^2 + \Phi(X) + \sum_{k=1}^{d} \|Z^{(k)}\|_* + \Psi(D) \tag{12}
\]

We denote the objective function in as . Then, we update each variable alternatively:

\[
Z_{k+1} = \arg \min_Z \left\{ L(Z, D_k, X_k) + \frac{\rho^Z_k}{2} \|Z - Z_k\|_F^2 \right\},
\]

\[
D_{k+1} = \arg \min_D \left\{ L(Z_{k+1}, D, X_k) + \frac{\rho^D_k}{2} \|D - D_k\|_F^2 \right\},
\]

\[
X_{k+1} = \arg \min_X \left\{ L(Z_{k+1}, D_k, X) + \frac{\rho^X_k}{2} \|X - X_k\|_F^2 \right\},
\]

where \((\rho^Z_k)_{k \in \mathbb{N}^*}, (\rho^D_k)_{k \in \mathbb{N}^*}\) and \((\rho^X_k)_{k \in \mathbb{N}}\) are three positive sequences.

1) Updating and : Following the updating strategy in , the coefficient (or equivalently denoted as ) and the dictionary ) can be respectively decomposed as followings:

\[
Z_{(3)} = \begin{bmatrix} z_1^T \\ \vdots \\ z_d^T \end{bmatrix} = \begin{bmatrix} \text{vec}(Z')^T \\ \vdots \\ \text{vec}(Z'')^T \end{bmatrix} = \begin{bmatrix} \text{vec}(Z(:, 1))^T \\ \vdots \\ \text{vec}(Z(:, d))^T \end{bmatrix} \tag{14}
\]

and

\[
D_k = [d_{k,1}^1, \cdots, d_{k,1}^d, \cdots, d_{k,d}^1, \cdots, d_{k,d}^d], \tag{15}
\]

where \(Z' = (\cdot, i)\) indicates the \(i\)-th frontal slice of the tensor \(Z\), \(Z_i = \text{vec}(Z')\), and \(d_i = D(:, i)\) is the \(i\)-th atom of \(D\). Thus, the \(Z\) subproblem and the \(D\) subproblem can be respectively split into \(d\) problems. Then, we update the pair of \(Z'\) and \(d_i\) from \(i = 1\) to \(d\). This updating scheme is the same as the well-known KSVD technique.

Firstly, we define

\[
\begin{align*}
Z' &= \left[ z_{k+1,1}^T, \cdots, z_{k+1,1}^T, z_{k+1,1}^T, \cdots, z_{k+1,1}^T \right]^T, \\
D' &= \left[ d_{k+1,1}^T, \cdots, d_{k+1,1}^T, d_{k+1,1}^T, \cdots, d_{k+1,1}^T \right], \\
R' &= \text{unfold}_3(X_k) - D' z_{k,1}^T, \\
M' &= \rho^Z_k Z_{k,1}^T + \beta \text{vec}^{-1}\left( (R' )^T d_{k,1}^T \right), \\
p' &= \frac{\beta}{\beta + \rho^Z_k} \text{vec}^{-1}\left( (R' )^T d_{k,1}^T \right).
\end{align*}
\]

The \(Z'\) and \(d_i\) \((i = 1, \cdots, d)\) subproblems are respectively as following:

\[
\min_{Z'} \frac{\beta}{2} \|R' - d_i z_{k+1,1}^T\|_F^2 + \|Z'\|_* + \frac{\rho^Z_k}{2} \|Z' - Z_{k,1}^T\|_F^2, \tag{17}
\]

and

\[
\min_{d_i} \frac{\beta}{2} \|R' - d_i z_{k+1,1}^T\|_F^2 + \|D'\|_* + \frac{\rho^D_k}{2} \|d_i - d_{k,1}^T\|_F^2. \tag{18}
\]

Note that two quadratic terms in can be combined as

\[
\frac{\beta}{2} \|R' - d_i z_{k+1,1}^T\|_F^2 + \frac{\rho^Z_k}{2} \|Z' - Z_{k,1}^T\|_F^2
\]

\[
= \frac{\beta}{2} \left( \langle R', z_{k+1,1}^T \rangle - 2 \langle R', d_{k+1,1}^T z_{k+1,1}^T \rangle + \langle d_{k+1,1}^T z_{k+1,1}^T \rangle \right) + \frac{\rho^Z_k}{2} \left( \langle Z', Z_{k+1,1}^T \rangle - 2 \langle Z', Z_{k,1}^T \rangle + \langle Z_{k,1}^T, Z_{k,1}^T \rangle \right)
\]

\[
= \frac{\beta}{2} \left( \langle R', z_{k+1,1}^T \rangle - 2 \text{vec}^{-1}\left( (R' )^T d_{k,1}^T \right) + \langle Z', Z' \rangle \right) + \frac{\rho^Z_k}{2} \left( \langle Z_{k+1,1}^T, Z_{k+1,1}^T \rangle - 2 \langle Z_{k,1}^T, Z_{k+1,1}^T \rangle + \langle Z_{k,1}^T, Z_{k,1}^T \rangle \right)
\]

\[
= \frac{\beta}{2} \|Z_{k+1,1}^T - \frac{\rho^Z_k}{\beta + \rho^Z_k} Z_{k,1}^T + \beta \text{vec}^{-1}\left( (R' )^T d_{k,1}^T \right) \|_F^2 + \frac{\rho^Z_k}{2} \|Z_{k+1,1}^T - \frac{\rho^Z_k}{\beta + \rho^Z_k} Z_{k,1}^T \|_F^2
\]

\[
- \frac{1}{2} \left( \frac{\beta}{\beta + \rho^Z_k} \|Z_{k+1,1}^T - \frac{\rho^Z_k}{\beta + \rho^Z_k} Z_{k,1}^T \|_F^2 + \frac{\rho^Z_k}{2} \|Z_{k+1,1}^T - \frac{\rho^Z_k}{\beta + \rho^Z_k} Z_{k,1}^T \|_F^2 \right),
\]

where vec\(^{-1}\) denotes the inverse reshaping operation of vec in . Therefore, leaving terms independent of \(Z'\) and adding
the nuclear norm term, the minimization problem in (17) is equivalent to:
\[
Z_{k+1} = \arg\min_{Z^i} \|Z^i\|_F + \frac{\beta}{2} \|X - M^k\|_F^2,
\]
where \(M^{k,i} = \rho_k^2 Z_{k+1} + (\mathbb{R}^{k,i})^\ast \mathbf{d}_i^k\). Then, we can directly derive the closed form solution of (19) with the singular value thresholding (SVT) operator \(\mathcal{S}\) as
\[
Z_{k+1} = \mathcal{S} T_k \left( \frac{1}{\lambda_{\max}(M^{k,i})} \right) U \left( S - \frac{1}{\beta + \rho_k^2} \right) V^T,
\]
where \((U, S, V)\) comes from the SVD of \(M^{k,i}\), \(S\) is a diagonal matrix with \(M^{k,i}\)'s singular values, and \((\cdot)_+\) means keeping the positive values and setting the negative values as 0.

Similarly, we can obtain the closed form solution of (18) as following:
\[
d_i^{k+1} = (\|p^{k,i}\|_2)^{-1} p^{k,i}.
\]

Afterwards, we obtain the \(Z_{k+1}\) with its \(i\)-th frontal slice equal to \(Z_{k+1}^i\) and \(D_{k+1} = [d_1^{k+1}, \ldots, d_{k+1}^{k+1}, \ldots, d_{n_3}^{k+1}]\).

2) Updating \(X\): We update \(X\) via solving the following minimization problem:
\[
\min_{X} \frac{\beta}{2} \|X - Z_{k+1}\|_F^2 + \Phi(X) + \frac{\rho_k^2}{2} \|X - X_k\|_F^2.
\]
\(X_{k+1}\) is updated via the following steps:
\[
\begin{align*}
X_{k+1} & = \frac{\beta \mathcal{F}(D_{k+1}, Z_{k+1}(3)) + \rho_k^2 X_k}{\beta + \rho_k^2}, \\
X_{k+1} & = (X_{k+1})_{\Omega^c} + X_{\Omega}.
\end{align*}
\]
Finally, the pseudocode to solve (12) is summarized in Algorithm 1. The computational complexity of our algorithm at each iteration is \(O(dn_1n_2(dn_3 + \min(n_1, n_2) + n_3))\), given an input with size \(n_1 \times n_2 \times n_3\).

Algorithm 1 Proximal alternating minimization algorithm for solving (12)

Input: The observed tensor \(\mathcal{O} \in \mathbb{R}^{n_1 \times n_2 \times n_3}\); the set of observed entries \(\Omega\).

Initialization: \(X^{(0)}\), \(D^0\), and \(Z^0\);

1: while not converged do
2: for \(i = 1 \) to \(d\) do
3: Update \(Z^{k+1}(; ; ; i)\) via Eq. (19);
4: Update \(D^{k+1}(; ; ; i)\) via (21);
5: end for
6: Update \(X^k\) (22);
7: end while

Output: The reconstructed tensor \(X\).

C. Convergence analysis

In this part, we are really to establish the theoretical guarantee of convergence on our algorithm. We first define K-L functions and semi-algebraic functions, which provide the basic ingredients for the convergence analysis.

Definition 7. (Kurdyka-Lojasiewicz property [32]) A proper lower semi-continuous function \(f : \mathbb{R}^n \rightarrow \mathbb{R} \cup +\infty\) is said to have the K-L property at \(\bar{x} \in \text{dom}(f)\) if there exist \(\eta \in (0, +\infty]\), a neighborhood \(U\) of \(\bar{x}\) and a continuous concave function \(\phi : [0, \eta) \rightarrow [0, +\infty]\), which satisfies \(\phi(0) = 0, \phi\) is \(C^1\) on \((0, \eta)\), and \(\phi(s) > 0, \forall s \in (0, \eta)\) such that for each \(x \in U \cap [f(\bar{x}) < f < f(\bar{x}) + \eta]\) the K-L inequality holds:
\[
\phi(f(x) - f(\bar{x})) \text{dist}(0, \partial f(x)) \geq 1.
\]

If \(f\) satisfies the K-L property at each point of \(\text{dom} f\) then \(f\) is called a K-L function.

Definition 8. (Semi-algebraic sets and functions [32]) A subset \(S \subseteq \mathbb{R}\) is called the semi-algebraic set if there exists a finite number of real polynomial functions \(g_{ij}, h_{ij}\) such that \(S = \bigcap_i \bigcup_j \{x \in \mathbb{R}^n : g_{ij}(x) = 0, h_{ij}(x) < 0\}\). A function \(f\) is called the semi-algebraic function if its graph \(\{(x, t) \in \mathbb{R}^n \times \mathbb{R} : f(x) = t\}\) is a semi-algebraic set.

Lemma 1. (32) A semi-algebraic real valued function \(f\) is a K-L function, i.e., \(f\) satisfies K-L property at each \(x \in \text{dom} f\).

For convenience, we define the following formularies,
\[
\begin{align*}
\Phi(X) & = \Phi(X), \\
\Phi(D) & = \Psi(D), \\
Q(Z, D, X) & = \frac{\beta}{2} \|X - Z\|_F^2, \\
L(Z, D, X) & = F(Z) + \delta_X(X) + \delta_D(D) + Q(Z, D, X).
\end{align*}
\]

and
\[
\begin{align*}
Z_{k+1} \in \arg\min_{Z} \{M_1(Z|Z^k) := F(Z) \\
& + Q(Z, D_k, X_k) + \frac{\rho_k^2}{2} \|Z - Z_k\|_F^2\}, \\
D_{k+1} \in \arg\min_{D} \{M_2(D|D^k) := \delta_D(D) \\
& + Q(Z_{k+1}, D, X_k) + \frac{\rho_k^2}{2} \|D - D_k\|_F^2\}, \\
X_{k+1} \in \arg\min_{X} \{M_3(X|X^k) := \delta_X(X) \\
& + Q(Z_{k+1}, D_{k+1}, X) + \frac{\rho_k^2}{2} \|X - X_k\|_F^2\}.
\end{align*}
\]

We first begin to show a descent Lemma for \(L(Z, D_k, X_k)\).

Lemma 2 (Descent Lemma). Assume that \(L(Z, D_k, X_k)\) is a \(C^1\) function with locally Lipschitz continuous gradient and \(\rho_k^2, \rho_k^2, \rho_k^2 > 0\). Let \(\{Z_k, D_k, X_k\}_{k \in \mathbb{N}}\) is generated by (24). Then
\[
\begin{align*}
F(Z_{k+1}) + Q(Z_{k+1}, D_k, X_k) + \frac{\rho_k^2}{2} \|Z_{k+1} - Z_k\|_F^2 & \leq F(Z_k) + Q(Z_k, D_k, X_k), \\
\delta_D(D_{k+1}) + Q(Z_{k+1}, D_{k+1}, X_k) + \frac{\rho_k^2}{2} \|D_{k+1} - D_k\|_F^2 & \leq \delta_D(D_k) + Q(Z_{k+1}, D_k, X_k), \\
\delta_X(X_{k+1}) + Q(Z_{k+1}, D_{k+1}, X_{k+1}) + \frac{\rho_k^2}{2} \|X_{k+1} - X_k\|_F^2 & \leq \delta_X(X_k) + Q(Z_{k+1}, D_{k+1}, X_k).
\end{align*}
\]
Proof. When $D_{k+1}$ and $X_{k+1}$ are optimal solutions of $M_2$ and $M_3$, $\delta_D = 0$ and $\delta_X = 0$. By the definitions of $M_1$, $M_2$, and $M_3$, we clearly have that

$$F(Z_{k+1}) + Q(Z_{k+1}, D_k, X_k) + \frac{\rho_k^d}{2} \| Z_{k+1} - Z_k \|_F^2 = M_1(Z_{k+1} | Z_k) = F(Z_k) + Q(Z_k, D_k, X_k),$$

$$\delta_D(D_{k+1}) + Q(Z_{k+1}, D_{k+1}, X_k) + \frac{\rho_k^d}{2} \| D_{k+1} - D_k \|_F^2 = M_2(D_{k+1}|D_k),$$

$$\delta_X(X_{k+1}) + Q(Z_{k+1}, D_{k+1}, X_{k+1}) + \frac{\rho_k^e}{2} \| X_{k+1} - X_k \|_F^2 = M_3(X_{k+1}|X_k) = \delta_X(X_k) + Q(Z_{k+1}, D_{k+1}, X_k).$$

The descent lemma has been proved.

Then, we show the relative error Lemma.

**Lemma 3** (Relative error Lemma). \{Z_k, D_k, X_k\}$_{k\in N}$ is generated by (24) and $\rho_k^d, \rho_k^d, \rho_k^e > 0$. Then there exists $V_{1,k+1}, V_{2,k+1}, V_{3,k+1}$, which satisfy the following formulas.

$$\| V_{1,k+1} + \nabla_Z Q(Z_{k+1}, D_k, X_k) \|_F^2 \leq \frac{\rho_k^d}{2} \| Z_{k+1} - Z_k \|_F^2,$$

$$\| V_{2,k+1} + \nabla_D Q(Z_{k+1}, D_k, X_k) \|_F^2 \leq \frac{\rho_k^d}{2} \| D_{k+1} - D_k \|_F^2,$$

$$\| V_{3,k+1} + \nabla_X Q(Z_{k+1}, D_k, X_k) \|_F^2 \leq \frac{\rho_k^e}{2} \| X_{k+1} - X_k \|_F^2,$$

where $V_{1,k+1} \in \partial F(Z_k), V_{2,k+1} \in \partial \delta_D(D_k)$, and $V_{3,k+1} \in \partial \delta_X(X_k)$.\[\]

**Proof.** $F(Z)$ has the subgradient by Theorem 3.7 of [44]. Thus, we have

$$\mathcal{R}(\text{U} \text{diag}(\sigma_j(\text{bdiag}(Z))) \text{V}^T) \in \partial F(Z),$$

where $U$ and $V$ are the singular value decomposition matrices of $\text{bdiag}(Z)$ and $\text{diag}(\sigma_j(\text{bdiag}(Z)))$ denotes a diagonal matrix with diagonal entries $\sigma_j(\text{bdiag}(Z)), \ldots, \sigma_{\min(m_1,n_2)}(\text{bdiag}(Z))$. By the definition of $M_1$, we have that

$$0 \in \partial F(Z_{k+1}) + \nabla_Z Q(Z_{k+1}, D_k, X_k) + \rho_k^e(Z_{k+1} - Z_k).$$

Thus,

$$0 = \mathcal{R}(\text{U} \text{diag}(\sigma_j(\text{bdiag}(Z_{k+1}))) \text{V}^T) - \beta D_k^T (\text{unfold}_3(X_k) - D_k Z_{(3)(k+1)}) + \rho_k^e(Z_{k+1} - Z_k).$$

Let

$$V_{1,k+1} := \beta D_k^T (\text{unfold}_3(X_k) - D_k Z_{(3)(k+1)}) - \rho_k^e(Z_{k+1} - Z_k) \in \partial F(X_{k+1}).$$

It is clear that

$$\| V_{1,k+1} + \nabla_Z Q(Z_{k+1}, D_k, X_k) \|_F^2 = \rho_k^e \| (Z_{k+1} - Z_k) \|_F^2.$$

Consequently, there exists

$$0 \in \partial \delta_D(D_{k+1}) - \beta (\text{unfold}_3(X_k) - D_k Z_{(3)(k+1)}) Z_{(3)(k+1)}^T + \rho_k^d(D_{k+1} - D_k),$$

$$0 \in \partial \delta_X(X_{k+1}) + \beta (X_{k+1} - Z_{k+1} \times_3 D_{k+1}) + \rho_k^e(X_{k+1} - X_k).$$

Then, we define that

$$V_{2,k+1} := \beta (\text{unfold}_3(X_k) - D_k Z_{(3)(k+1)}) Z_{(3)(k+1)}^T - \rho_k^d(D_{k+1} - D_k) \in \partial \delta_D(D_{k+1}),$$

$$V_{3,k+1} := - \beta (X_{k+1} - Z_{k+1} \times_3 D_{k+1}) - \rho_k^e(X_{k+1} - X_k) \in \partial \delta_X(X_{k+1}).$$

It can be seen that

$$\| V_{2,k+1} + \nabla_D Q(Z_{k+1}, D_k, X_k) \|_F^2 \leq \rho_k^d \| (D_{k+1} - D_k) \|_F^2,$$

$$\| V_{3,k+1} + \nabla_X Q(Z_{k+1}, D_k, X_k) \|_F^2 \leq \rho_k^e \| (X_{k+1} - X_k) \|_F^2.$$}

The proof of relative error Lemma has been finished.

Now, we begin to establish our proof of the global convergence of the sequence generated by (24).

**Theorem 2.** The sequence generated by (24) is bounded, and it converges to a critical point of $L(Z, D, X)$.\[\]

**Proof.** It is easy to verify that $Q$ is $C^1$ function with locally Lipschitz continuous gradient and $F$, $\delta_D$, and $\delta_X$ are proper and lower semi-continuous. Thus, $L$ is a proper lower semi-continuous function. The nuclear norm and Frobenius norm are semialgebraic [45]. Additionally, the function with semialgebraic sets is semialgebraic [45]. Thus the function $L$ is a semi-algebraic function.

From Lemma 2, we have that the objective function value monotonically decreases. Firstly, we can see that the indicator function $\delta_D(D) = \Phi(D)$ should be 0 from its definition. Thus,

$$\| D \|_F^2 = \sum \| D(:,i) \|_2^2 = d,$$

which means \{D_k\}$_{k\in N}$ is bounded. Meanwhile, from the monotonic decreasing, the nonnegative terms $F(Z) = \sum_{k=1}^{d} \| Z^{(k)} \|_*, \text{ and } Q(Z, D, X) = \frac{d}{2} \| X - Z \times_3 D \|_F^2$ are bounded. Then,

$$\| Z \|_F^2 \leq \sum_{k=1}^{d} \| Z^{(k)} \|_F^2 \leq \sum_{k=1}^{d} \| (Z^{(k)}) \|_2^2.$$\[\]

That is, \{Z_k\}$_{k\in N}$ is bounded. Next, from the triangle inequality, we have

$$\| X \|_F - \| Z \|_F \| D \|_F \leq \| X \|_F - \| Z \times_3 D \|_F \leq \| X - Z \times_3 D \|_F.$$\[\]

This is equivalent to

$$\| X \|_F \leq \| X \|_F + \| Z \|_F + \| D \|_F.$$\[\]

Therefore, \{X_k\}$_{k\in N}$ is bounded.

By Lemma 1, the sequence \{Z_k, D_k, X_k\}$_{k\in N}$ is a bounded sequence with the $K-L$ property at each point.

Combining Lemma 2 and Lemma 3 with the above property of $L$, the process of updating in (24) is actually a special instance of the algorithm 4 described in [42]. Lemma 2 and Lemma 3 correspond to the (64)-(65)-(66) in [42]. Under these
Algorithm 1 is a direct multi-block generalization of (24). The proof of its convergence accords with the proof of Theorem 2 here. Therefore, we establish the proof of Theorem 2 here.

IV. NUMERICAL EXPERIMENTS

In this section, we compare our method with other state-of-the-art methods. Compared methods consist of: one baseline Tucker-rank based method HaLRTC [1], a Bayesian CP-factorization based method (BCPF) [2], a tensor ring decomposition based method (TRLRF) [3], a t-SVD based method (TNN) [4], a DCT induced TNN minimization method (DCTNN) [5], and a framelet represented TNN minimization method (FTNN) [6]. We select four types of tensor data, including videos, HSIs, traffic data, and MRI data, to show that our method is adaptive to different types of data.

Since the algorithm of our method is a non-convex approach, the initialization of our algorithm is important. We use a simple linear interpolation strategy, which is employed in [46] and convenient to implement with low cost, to fill the missing pixels and obtaining \( \lambda_0 \) for our method. The normalized tubes of \( \lambda_0 \) are used to initialize the dictionary \( \mathbf{D} \).

Then, we fix \( \mathcal{Y} = \lambda_0 \) and run 10 iterations of our method to initialize the \( \mathcal{Z}_0 \) with random inputs.

A. Video Data

In this subsection, we test our method for the video data completion and select four videos named “foreman” “carphone” “highway” and “container” of the size \( 144 \times 176 \times 50 \) (height×width×frame) to conduct the comparisons. The sampling rate (SR) varies from 10% to 50%. We compute the peak signal-to-noise ratio (PSNR), the structural similarity index (SSIM) [47], and the universal image quality index (UIQI) [48] of the results by different methods. Higher values of these three quality metrics indicate better completion performances.

In Tab. 1, we report the quantitative metrics of the results obtained by different methods and the average running time on the video data. From Tab. 1, it can be found that the results by TRLRF are promising when the sampling rate is low. The performance of FTNN is better than TNN and DCTNN for the video “foreman”, while DCTNN exceeds FTNN and TNN for the video “container”. This reveals the predefined transformations lack flexibility. Meanwhile, with minor exceptions, our DTNN achieves the best performance for different sampling rates, illustrating the superior of the data adaptive dictionary.

Fig. 2 exhibits one frame of the results by different methods on the video data. From the enlarged area, it can be found that our DTNN well restores edges in “foreman” and “highway”.

### TABLE I

| Video  | SR        | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | Time (s) |
|-------|-----------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|--------|
|       | 10%       | 20%  | 30%  | 40%  | 50%  | 10%  | 20%  | 30%  | 40%  | 50%  | 10%  | 20%  | 30%  | 40%  | 50%  |        |
|       | Method    |       |      |      |      |      |      |      |      |      |      |      |      |      |      |        |
| Foreman | Observed  | 39.04 | 0.909 | 0.715 | 35.55 | 0.963 | 0.853 | 38.81 | 0.979 | 0.905 | 41.55 | 0.986 | 0.934 | 44.01 | 0.991 | 0.954 | 29    |
|         | HaLRTC    | 31.96 | 0.855 | 0.693 | 33.43 | 0.927 | 0.855 | 35.03 | 0.963 | 0.884 | 36.61 | 0.973 | 0.906 | 118   |      |        |
|         | BCPF      | 29.36 | 0.839 | 0.657 | 31.58 | 0.880 | 0.740 | 33.43 | 0.949 | 0.855 | 35.03 | 0.963 | 0.884 | 118   |      |        |
|         | TNN       | 31.86 | 0.900 | 0.763 | 32.89 | 0.943 | 0.838 | 35.49 | 0.964 | 0.879 | 38.71 | 0.987 | 0.922 | 275   |      |        |
|         | DCTNN     | 31.61 | 0.930 | 0.762 | 32.87 | 0.977 | 0.802 | 42.69 | 0.989 | 0.940 | 45.91 | 0.993 | 0.960 | 48.43 | 0.996 | 0.972 | 19    |
|         | FTNN      | 32.43 | 0.948 | 0.809 | 37.38 | 0.978 | 0.904 | 41.41 | 0.988 | 0.946 | 44.42 | 0.992 | 0.958 | 47.16 | 0.994 | 0.971 | 146   |
|         | DTNN      | 32.50 | 0.896 | 0.763 | 37.98 | 0.945 | 0.829 | 43.72 | 0.992 | 0.959 | 47.28 | 0.995 | 0.972 | 49.82 | 0.997 | 0.980 | 355   |

***Implementation details***

- HaLRTC: [source code](https://github.com/jamiezeminzhang/Tensor_Completion_BPCA)
- BCPF: [source code](https://github.com/jamiezeminzhang/Tensor_Completion_BPCA)
- TRLRF: [source code](https://github.com/vuanlonghieu/TRLRF)
- DCTNN: [source code](https://github.com/vuanlonghieu/TRLRF)
- FTNN: [source code](https://github.com/vuanlonghieu/TRLRF)
- DTNN: [source code](https://github.com/vuanlonghieu/TRLRF)

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the hair in “carphone”, and the ship’s outline in “container”. The homogeneous areas are also protected by our method. We can conclude that the visual effect of our method is the best.

B. Hyperspectral Images

In this subsection, 2 HSIs, i.e. a subimage of Pavia City Center dataset\(^\text{[10]}\) of the size 200 \(\times\) 200 \(\times\) 80 (height\(\times\)width\(\times\)band), and a subimage of Washington DC Mall dataset\(^\text{[10]}\) of the size 256 \(\times\) 256 \(\times\) 191 are adopted as the testing data. Since the redundancy between HSIs’ slices is so high that all the methods perform very well with SR=40\%, we add the case with SR=5\%. Thus, the sampling rates vary from 5\% to 40\%. Three numerical metrics, consisting of PSNR, SSIM, and the mean Spectral Angle Mapper (SAM)\(^\text{[49]}\) are selected to quantitatively measure the reconstructed results. Lower values of SAM indicate better reconstructions.

In Tab.\(^\text{[11]}\) we show the quantitative comparisons of different methods on HSIs. FTNN and TRLRF perform well for the low sampling rate. We can also see that DCTNN and FTNN alternatively achieves the second best place in many cases, showing that DCT and framelet transformation fit the HSI data better than DFT. For different metrics, our DTNN obtains the best values in all cases. As sampling rates arise, the superior of our method over compared methods is more evident. For example, when dealing with Pavia City Center, the margins of our method over compared methods is more evident. For different metrics, our DTNN obtains the best values in all cases. As sampling rates arise, the superior of our method over compared methods is more evident. For example, when dealing with Pavia City Center, the margins of our method over compared methods is more evident. For different metrics, our DTNN obtains the best values in all cases. As sampling rates arise, the superior of our method over compared methods is more evident. For example, when dealing with Pavia City Center, the margins of our method over compared methods is more evident. For different metrics, our DTNN obtains the best values in all cases. As sampling rates arise, the superior of our method over compared methods is more evident. For example, when dealing with Pavia City Center, the margins of our method over compared methods is more evident.

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C. Traffic Data

In this subsection, we test all the methods on the traffic data\(^\text{[11]}\) which is provided by Grenoble Traffic Lab (GTL). A set of traffic speed data of 207 days (April 1, 2015 to

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\(^{[10]}\) http://www.ehu.eus/ccwintco/index.php?title=Hyperspectral_Remote_Sensing_Scenes

\(^{[11]}\) https://gtl.inrialpes.fr/data_download
Observed HaLRTC BCPF TRLRF TNN DCTNN FTNN DTNN Original

Fig. 3. The pseudo-color images and the corresponding enlarged areas of the results by different methods. Top: Pavia City Center (R-4 G-12 B-68) with SR = 5%. Bottom: Washington DC Mall (R-1 G-113 B-116) with SR = 10%.

### TABLE III

| Method   | SR 5%        | RMSE | MAPE       | RMSE | MAPE       | RMSE | MAPE       | RMSE | MAPE       | RMSE | MAPE       | RMSE | MAPE       | Time (s) |
|----------|--------------|------|------------|------|------------|------|------------|------|------------|------|------------|------|------------|----------|
| Observed | 0.9148       | 95.71% | 0.8942     | 91.44% | 0.8731     | 87.15% | 0.8514     | 82.87% | 0.8291     | 78.56% | 0.8061     | 74.26% | 0         |          |
| HaLRTC   | 0.3592       | 17.51% | 0.3581     | 16.72% | 0.3575     | 16.26% | 0.3571     | 15.94% | 0.3569     | 15.68% | 0.3566     | 15.47% | 13        |          |
| BCPF     | 0.3594       | 17.25% | 0.3576     | 16.45% | 0.3571     | 16.13% | 0.3568     | 15.97% | 0.3567     | 15.84% | 0.3566     | 15.76% | 218       |          |
| TRLRF    | 0.1781       | 9.09%  | 0.1753     | 8.62%  | 0.1749     | 8.42%  | 0.1747     | 8.29%  | 0.1745     | 8.20%  | 0.1744     | 8.08%  | 107       |          |
| TNN      | 0.0509       | 3.63%  | 0.0387     | 2.75%  | 0.0336     | 2.33%  | 0.0304     | 2.03%  | 0.0283     | 1.82%  | 0.0267     | 1.65%  | 27        |          |
| DCTNN    | 0.0480       | 3.38%  | 0.0387     | 2.71%  | 0.0338     | 2.31%  | 0.0316     | 1.96%  | 0.0287     | 1.69%  | 0.0265     | 1.49%  | 18        |          |
| FTNN     | 0.0452       | 3.33%  | 0.0358     | 2.31%  | 0.0316     | 1.96%  | 0.0287     | 1.69%  | 0.0265     | 1.49%  | 0.0247     | 1.32%  | 129       |          |
| DTNN     | 0.0428       | 2.76%  | 0.0354     | 2.19%  | 0.0305     | 1.84%  | 0.0278     | 1.60%  | 0.0256     | 1.42%  | 0.0237     | 1.26%  | 264       |          |

Fig. 4. The 88-th lateral slice of the reconstructions by different methods on the traffic data (SR = 30%).

October 24, 2015), 1440 time windows\textsuperscript{12} and 21 detection points, is downloaded and constitutes a third-order tensor of the size $1440 \times 207 \times 21$. A subset of the data with the size $400 \times 200 \times 21$ is clipped as the ground truth complete testing data. We select the root mean square error (RMSE\textsuperscript{13}) and the mean absolute percentage error (MAPE\textsuperscript{14}) to quantitatively measure the quality of the results. Higher values of RMSE and MAPE indicate better reconstructions. After random sampling the elements with SR $\in \{5\%, 10\%, 15\%, \cdots, 30\%\}$, 3 adjacent frontal slices in a random location are set as unobserved. This is to simulate the situations in which some detectors are broken. The 200-th lateral slice of the observation is shown in the top-left of Fig. 4, the missing slices corresponding to the blue columns.

Tab. III gives the quantitative metrics of the results by different methods with different sampling rates. We can find that the capabilities of HaLRTC, BCPF, and TRLRF is limited and this phenomenon accord with the visual results shown in Fig. 4. The effectiveness of these three methods is severely affected due to the missing frontal slices. TNN and DCTNN get better metrics while their performance is also not well...
TABLE IV
PSNR, SSIM, AND UIQI OF RESULTS BY DIFFERENT METHODS WITH DIFFERENT SAMPLING RATES ON THE MRI DATA. THE BEST AND THE SECOND BEST VALUES ARE RESPECTIVELY HIGHLIGHTED BY BOLDFACE AND UNDERLINE.

| SR    | 10% | 20% | 30% | 40% | 50% | Time (s) |
|-------|-----|-----|-----|-----|-----|----------|
| Method | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI | PSNR | SSIM | UIQI |
| Observed | 8.09 | 0.043 | 0.020 | 8.60 | 0.070 | 0.050 | 9.18 | 0.099 | 0.086 | 9.85 | 0.132 | 0.127 | 10.64 | 0.167 | 0.173 |       |       |       |       |       |       |       |
| HaLRTC | 23.89 | 0.637 | 0.606 | 25.48 | 0.720 | 0.682 | 26.36 | 0.760 | 0.722 | 27.17 | 0.794 | 0.756 | 28.06 | 0.825 | 0.786 | 921   |       |       |       |       |       |       |
| BCPF   | 22.63 | 0.577 | 0.518 | 24.70 | 0.678 | 0.631 | 25.37 | 0.710 | 0.663 | 25.55 | 0.720 | 0.671 | 25.71 | 0.727 | 0.687 |       |       |       |       |       |       |       |
| TNN    | 22.41 | 0.577 | 0.550 | 27.12 | 0.789 | 0.757 | 30.01 | 0.874 | 0.833 | 32.55 | 0.922 | 0.876 | 35.01 | 0.953 | 0.905 | 60    |       |       |       |       |       |       |
| DCTNN  | 23.79 | 0.644 | 0.617 | 27.63 | 0.808 | 0.773 | 30.56 | 0.888 | 0.844 | 33.15 | 0.952 | 0.885 | 35.62 | 0.960 | 0.911 | 45    |       |       |       |       |       |       |
| FTNN   | 25.15 | 0.743 | 0.695 | 29.02 | 0.872 | 0.825 | 31.96 | 0.928 | 0.883 | 34.49 | 0.958 | 0.916 | 36.89 | 0.975 | 0.937 | 253   |       |       |       |       |       |       |
| DTNN   | 27.19 | 0.835 | 0.790 | 31.03 | 0.917 | 0.870 | 33.65 | 0.949 | 0.902 | 35.83 | 0.966 | 0.920 | 37.91 | 0.978 | 0.934 | 568   |       |       |       |       |       |       |

Fig. 5. The 61-th frontal slice of the results on the MRI data by different methods (SR = 30%).

considering the location of missing slices. FTNN and DTNN recover the rough structure of the missing slices and the metrics of their results also achieve the best and the second best places. The reconstruction of our DTNN in the area of missing slices is closer to the original data than FTNN.

D. MRI Data

In this section, all the methods are conducted on the MRI data of the size 142 × 178 × 121. The sampling rates are set from 10% to 50%. Similar to the video data, we compute the mean values of PSNR, SSIM, and UIQI of each frontal slices and report them in Tab. IV. From Tab. IV, we can find that DTNN outperforms compared methods while DCTNN and FTNN alternatively obtain the second best values. Fig. 5 presents the 61-th frontal slice of the results by different methods. For the enlarged white manner area, which is smooth, the results by our DTNN is the cleanest compared with the results by other methods.

E. Discussions

1) Sparsity VS low-rankness: In this part, we discuss the situation where the objective function $\|\text{bdiag}(Z)\|_*$ in our model is replaced by a simple sparsity term such as $\|Z\|_1$. After this, it is similar to the traditional dictionary learning methods with sparse constrained coefficients. We also use the multi-block proximal alternating minimization algorithm to solve this model. The algorithm is the similar to Algorithm 1, except for solving the $Z_i^k (i = 1, 2, \cdots, d)$ subproblems. We update $Z_i^{k+1}$ with the soft thresholding as

$$Z_i^{k+1} = \text{sign}(M^{k,i}) \odot \left( |M^{k,i}| - \frac{1}{\beta + \rho_k} \right)_+,$$

where $\odot$ denotes the element-wise product and $|\cdot|$ means the absolute value. We use the same initializations as our DTNN. We use “DTNN (sparsity)” to denote the sparsity based method. The parameters of DTNN (Sparsity) are manually tuned for the best performances. Meanwhile, we enlarge $\rho$ at the 15-th, 20-th, and 25-th iterations by multiplying the factor 1.5 and enlarge $\rho$ by multiplying the factor 1.2 at each iteration from the 30-th iteration until satisfying the condition of convergence. We test DTNN (sparsity) on the HSI data Pavia City Center. We list the PSNR values of the results by our DTNN and DTNN (sparsity) in Tab. V. From Tab. V, it can be found that the performance of DTNN (sparsity) is inferior to DTNN. Especially, when the sampling rate is low, the PSNR values obtained by DTNN (sparsity) are poor.

TABLE V
PSNR OF RESULTS BY DTNN AND DTNN(S) WITH DIFFERENT SAMPLING RATES ON THE HSI DATA PAVIA CITY CENTER.

| SR    | 5%   | 10%  | 20%  | 30%  | 40%  |
|-------|------|------|------|------|------|
| DTNN  | 34.26| 40.86| 53.20| 65.00| 67.15|
| DTNN (sparsity) | 17.48| 23.00| 34.78| 46.48| 61.14|

15https://brainweb.bic.mni.mcgill.ca/brainweb/selection
11

Fig. 6. The PSNR and SSIM values of our method with different $\rho^z$, $\rho^d$, and $\rho^x$ on the video “foreman” (SR = 50%).

Fig. 7. The learned dictionaries (left) and the tubes of the original data (right). Top: the video “foreman”. Bottom: the HSI Pavia City Center.

This also supports our statement at the end of Sec. III-A that we need both the learned dictionary and the specific low-rank structure of the coefficients for the accurate completion of the data.

2) Parameters: Throughout all the experiments in this paper, the selected parameters of the proposed method are set as: $d = 5n_3$, $\beta = 10$, $\rho^z = 20$, $\rho^d = 1$, and $\rho^x = 1$. In the framework of the HQS algorithm, the penalty parameter $\rho$ is required to reach infinite when iteration goes on. Therefore, we enlarge $\rho$ at the 15-th, 20-th, and 25-th iterations by multiplying the factor 1.5 and enlarge $\rho$ by multiplying the factor 1.2 at each iteration from the 30-th iteration until satisfying the condition of convergence.

Next, to test the effects of $\rho^z$, $\rho^d$, and $\rho^x$, we take the video “foreman” as an example and set the sampling rate as 50%. Then, we alternately change each of them, keeping others the same as our default setting. We illustrate the PSNR and SSIM values with respect to different parameters in Fig. 6. From Fig. 6 we can see that the performance of our method is more sensitive to $\rho^z$. Our method could obtain satisfactory results with a wide range of $\rho^d$ and $\rho^x$.

3) Learned dictionaries: In Fig. 7 we exhibit the first 100 columns of the learned dictionaries together with the plotting of three tubes of the original data. From the red boxes with dashed line, we can see that when the tubes, i.e., the vectors along the third dimension, of the original data fluctuate, the corresponding areas of the dictionaries’ atoms (columns) tend not to be smooth. This reflects that the dictionaries learned by our method is flexible and adaptive to different types of data.

4) Convergency behaviors: When the largest relative change of the variables, i.e., $\max\left\{\frac{\|Z_k - Z_{k-1}\|}{\|Z_k\|}, \frac{\|D_k - D_{k-1}\|}{\|D_k\|}, \frac{\|X_k - X_{k-1}\|}{\|X_k\|}\right\}$, is smaller than $10^{-3}$, we consider that our algorithm converges and stop the iterations. In Fig. 8 we present the relative changes with respect to the iterations in our experiments on the HSI data Pavia City Center and the video data “foreman”. Three obvious fluctuations in each curve are in accord with our parameter setting of enlarging $\rho$ at the 15-th, 20-th, and 25-th iterations. The overall downward trend of the curves in Fig. 8 illustrates that our method converges quickly.

V. CONCLUSIONS

In this paper, we have introduced the data-adaptive dictionary and low-rank coding for third-order tensor completion. In the completion model, we have proposed to minimize the low-rankness of each tensor slice containing the coding coefficients. To optimize this model, we design a multi-block proximal alternating minimization algorithm, the sequence generated by which would globally converge to a critical point. Numerical experiments conducted on various types of real-world data show the superior of the proposed method.

As a future research work, we will consider how to use the proposed model and idea to analyze and study a tensor-based representation learning method for multi-view clustering. Here multi-view data as a third-order tensor expresses each tensorial
data point as a low rank representation of the learned learned
dictionary basis.

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