$K$–structure of $\mathcal{U}(\mathfrak{g})$ for $\mathfrak{su}(n, 1)$ and $\mathfrak{so}(n, 1)$

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Abstract. Let $G$ be the adjoint group of a real simple Lie algebra $\mathfrak{g}_0$ equal either $\mathfrak{su}(n, 1)$ or $\mathfrak{so}(n, 1)$, $K$ its maximal compact subgroup, $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ and $\mathcal{U}(\mathfrak{g})^K$ its subalgebra of $K$–invariant elements. By a result of F. Knopp [3] $\mathcal{U}(\mathfrak{g})$ is free as a $\mathcal{U}(\mathfrak{g})^K$–module, so there exists a $K$–submodule $E$ of $\mathcal{U}(\mathfrak{g})$ such that the multiplication defines an isomorphism of $K$–modules $\mathcal{U}(\mathfrak{g})^K \otimes E \rightarrow \mathcal{U}(\mathfrak{g})$. We prove that $E$ is equivalent to the regular representation of $K$, i.e. that the multiplicity of every $\delta \in \hat{K}$ in $E$ equals its dimension. As a consequence we get that for any finitedimensional complex $K$–module $V$ the space $(\mathcal{U}(\mathfrak{g}) \otimes V)^K$ of $K$–invariants is free $\mathcal{U}(\mathfrak{g})^K$–module of rank $\dim V$.

1 Introduction

Let $\mathfrak{g}_0$ be a real simple Lie algebra of noncompact type. Denote by $G$ its adjoint group and choose its maximal compact subgroup $K$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ be the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{p}_0$, respectively. Denote by $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{k}) \subseteq \mathcal{U}(\mathfrak{g})$ the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{k}$. Furthermore, denote by $S(\mathfrak{g})$ and $S(\mathfrak{k})$ the symmetric algebras over $\mathfrak{g}$ and $\mathfrak{k}$ and by $\mathcal{P}(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{k})$ the polynomial algebras over $\mathfrak{g}$ and $\mathfrak{k}$. Then $\mathcal{P}(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{k})$ can be identified with the symmetric algebras $S(\mathfrak{g}^*)$ and $S(\mathfrak{k}^*)$ over dual spaces $\mathfrak{g}^*$ and $\mathfrak{k}^*$ of $\mathfrak{g}$ and $\mathfrak{k}$. The Killing form $B$ on $\mathfrak{g}$ allows us to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{k}$ with $\mathfrak{k}^*$. Thus the algebras $\mathcal{P}(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{k})$ are identified with $S(\mathfrak{g})$ and $S(\mathfrak{k})$. Considering polynomials as complex functions on $\mathfrak{g}$ and $\mathfrak{k}$, the inclusion $\mathcal{P}(\mathfrak{k}) \subseteq \mathcal{P}(\mathfrak{g})$ is obtained via the projection $pr : \mathfrak{g} \rightarrow \mathfrak{k}$ along $\mathfrak{p}$.

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The adjoint action of the group $G$ on $\mathfrak{g}$ extends uniquely to the action by automorphisms on the algebras $\mathcal{U}(\mathfrak{g})$, $S(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{g})$, and the subgroup $K$ acts also by automorphisms on the algebras $\mathcal{U}(\mathfrak{t})$, $S(\mathfrak{t})$ and $\mathcal{P}(\mathfrak{t})$. Denote by superscript $G$ (resp. $K$) the subalgebras of $G$–invariants (resp. $K$–invariants). Then, of course, $\mathcal{U}(\mathfrak{g})^G$ is the center $Z(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{t})^K$ is the center $Z(\mathfrak{t})$ of $\mathcal{U}(\mathfrak{t})$. Obviously, the multiplication defines algebra homomorphisms

$$Z(\mathfrak{g}) \otimes Z(\mathfrak{k}) \rightarrow \mathcal{U}(\mathfrak{g})^K, \quad S(\mathfrak{g})^G \otimes S(\mathfrak{t})^K \rightarrow S(\mathfrak{g})^K, \quad \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{t})^K \rightarrow \mathcal{P}(\mathfrak{g})^K.$$ 

In [3] F. Knopp has proved the following highly nontrivial results:

**Theorem 1.**

(a) $Z(\mathfrak{g}) \otimes Z(\mathfrak{t}) \rightarrow \mathcal{U}(\mathfrak{g})^K$ is an isomorphism onto the center of the algebra $\mathcal{U}(\mathfrak{g})^K$.

(b) The algebra $\mathcal{U}(\mathfrak{g})^K$ is commutative (i.e. $\mathcal{U}(\mathfrak{g})^K = Z(\mathfrak{g})Z(\mathfrak{t})$) if and only if $\mathfrak{g}$ is either $\mathfrak{su}(n,1)$ or $\mathfrak{so}(n,1)$. In these cases $\mathcal{U}(\mathfrak{g})$ is free as a $\mathcal{U}(\mathfrak{g})^K$–module.

The symmetrization $\mathcal{U}(\mathfrak{g}) \rightarrow S(\mathfrak{g}) \simeq \mathcal{P}(\mathfrak{g})$ is an isomorphism of vector spaces and of $G$–modules and (a) implies that the homomorphism

$$\mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{t})^K \rightarrow \mathcal{P}(\mathfrak{g})^K$$

is always injective and by (b) in the cases $\mathfrak{g} = \mathfrak{su}(n,1)$ and $\mathfrak{g} = \mathfrak{so}(n,1)$ this is an isomorphism; furthermore, the last sentence in (b) implies that in these two cases $\mathcal{P}(\mathfrak{g})$ is free as a $\mathcal{P}(\mathfrak{g})^K$–module.

**2  \(K\)–harmonic polynomials and the structure of the \(\mathcal{P}(\mathfrak{g})^K\)–module \(\mathcal{P}(\mathfrak{g})\)**

Consider for a while a more general situation. Let $V$ be a complex finite-dimensional vector space and let $L$ be a closed subgroup of $\text{GL}(V)$ acting fully reducibly on $V$. Denote by $S(V)$ and $\mathcal{P}(V)$ the symmetric and the polynomial algebra over $V$. For $x \in V$, let $\partial(x) : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ be the derivation in the direction $x$. The map $\partial : V \rightarrow \text{End}(\mathcal{P}(V))$ extends uniquely to an isomorphism $\partial$ of the symmetric algebra $S(V)$ onto the algebra $\mathcal{D}(V)$ of
linear differential operators on $\mathcal{P}(V)$ with constant coefficients. Now, one defines the bilinear form $\langle \cdot, \cdot \rangle$ on $S(V) \times \mathcal{P}(V)$ by
\[
\langle u, f \rangle = [\partial(u)f](0), \quad u \in S(V), \ f \in \mathcal{P}(V).
\]
This is a pairing, i.e. nondegenerate in each variable. Now, consider the sub-
algebras of $L-$invariants $S(V)^L$ and $\mathcal{P}(V)^L$ and their maximal ideals (of codimension 1)
\[
S^+(V)^L = \bigoplus_{k>0} S^k(V)^L, \quad \mathcal{P}^+(V)^L = \bigoplus_{k>0} \mathcal{P}^k(V)^L = \{ f \in \mathcal{P}(V)^L; \ f(0) = 0 \}.
\]
Define the (graded) space of so called $L-$harmonic polynomials on $V$ :
\[
\mathcal{H}_L(V) = \{ f \in \mathcal{P}(V); \ \partial(u)f = 0 \ \forall u \in S^+(V)^L \}.
\]
As noticed in [4] and [5] the obvious equality
\[
\langle uv, f \rangle = \langle u, \partial(v)f \rangle, \quad u, v \in S(V), \ f \in \mathcal{P}(V),
\]
implies easily that
\[
\mathcal{H}_L(V) = \{ f \in \mathcal{P}(V); \ \langle u, f \rangle = 0 \ \forall u \in S(V)S^+(V)^L \}.
\]
Part of the Helgason’s results in [2] (see also Propositions 3 and 4 in [4]) can be stated as follows:

**Proposition 1.** Suppose that the group $L$ is connected and that there exists
an $L-$invariant symmetric bilinear form $B : V \times V \longrightarrow \mathbb{C}$ and a real form $V_0$ of $V$ such that the restriction of $B$ to $V_0 \times V_0$ is a scalar product and that
the group $L$ is the complexification of its subgroup $L_0 = \{ g \in L; \ gV_0 = V_0 \}$. Then
\[
\mathcal{P}(V) = \mathcal{P}(V)\mathcal{P}^+(V)^L \oplus \mathcal{H}_L(V).
\]

Note that the conditions on the pair $(L, V)$ in Proposition 1 are obviously satisfied for the action of the complexification $K^C$ of the group $K$ on $\mathfrak{g}$, especially in the cases $\mathfrak{g}_0 = \mathfrak{su}(n, 1)$ and $\mathfrak{g}_0 = \mathfrak{so}(n, 1)$.

Consider any subgroup $L \subseteq \text{GL}(V)$ acting fully reducibly on a finitedime-
ensional complex vector space $V$. If $N$ is any graded subspace of $\mathcal{P}(V)$ such that
\[
\mathcal{P}(V) = \mathcal{P}(V)\mathcal{P}^+(V)^L \oplus N \quad (1)
\]
then it is easy to see (Proposition 1 in [4]) that the multiplication defines a surjective map

$\mathcal{P}(V)^L \otimes N \longrightarrow \mathcal{P}(V)$.

(2)

Kostant’s Lemma 1 in [4] can be stated as follows:

**Proposition 2.** The following properties are mutually equivalent:

(a) For every $N$, such that (1) holds true, the map (2) is also injective, i.e. an isomorphism.

(b) For some $N$, such that (1) holds true, the map (2) is injective.

(c) $\mathcal{P}(V)$ is free as a $\mathcal{P}(V)^L$–module.

Thus, by the last sentence in (b) of Theorem 1 we get from Propositions 1 and 2:

**Theorem 2.** For $g = \mathfrak{su}(n, 1)$ and for $g = \mathfrak{so}(n, 1)$ we have:

(a) $\mathcal{P}(g) = \mathcal{P}(g)\mathcal{P}_+(g)^K \oplus \mathcal{H}_K(g)$.

(b) The multiplication defines an isomorphism $\mathcal{P}(g)^K \otimes \mathcal{H}_K(g) \simeq \mathcal{P}(g)$.

3 The $K$–module of $K$–harmonic polynomials

Let $\mathcal{N}$ be the zero set in $g$ of the ideal $\mathcal{P}(g)\mathcal{P}_+(g)^K$ generated by $\mathcal{P}_+(g)^K$ in $\mathcal{P}(g)$:

$\mathcal{N} = \{ x \in g; f(x) = 0 \ \forall f \in \mathcal{P}(g)\mathcal{P}_+(g)^K \} = \{ x \in g; f(x) = 0 \ \forall f \in \mathcal{P}_+(g)^K \}$.

By Proposition 16 in [4] the zero set

$\mathcal{N}_G = \{ x \in g; f(x) = 0 \ \forall f \in \mathcal{P}_+(g)^G \}$

is exactly the set of all nilpotent elements in the Lie algebra $g$. Analogously

$\mathcal{N}_K = \{ x \in \mathfrak{k}; f(x) = 0 \ \forall f \in \mathcal{P}_+(\mathfrak{k})^K \}$

is the set of all nilpotent elements in the reductive Lie algebra $\mathfrak{k}$. Now, $\mathcal{P}(g)^K = \mathcal{P}(g)^G \otimes \mathcal{P}^K(\mathfrak{k})$ by the Knopp’s theorem, so we get
Proposition 3. $N$ is the set of all nilpotent elements in $g$ whose projection to $k$ along $p$ is nilpotent in the reductive Lie algebra $k$:

$$N = \{ x \in g; x \in N_G, \text{pr } x \in N_K \}.$$ 

We call the elements of $N$ $K$-nilpotent elements in $g$.

By the Harish-Chandra isomorphism and by the Chevalley’s theorem on Weyl group invariants we know that the algebra $P(g)^G$ is generated by $\ell = \text{rank } g$ homogeneous algebraically independent $G$-invariant polynomials $f_1, \ldots, f_\ell$ and the algebra $P(k)^K$ is generated by $k = \text{rank } k$ homogeneous algebraically independent $K$-invariant polynomials $\varphi_1, \ldots, \varphi_k$. Since in the cases $g_0 = \mathfrak{su}(n,1)$ and $g_0 = \mathfrak{so}(n,1)$

$$P(g)^K = P(g)^G P(k)^K \simeq P(g)^G \otimes P(k)^K,$$

the algebra $P(g)^K$ is generated by $\ell + k$ homogeneous algebraically independent polynomials $f_1, \ldots, f_\ell, \varphi_1, \ldots, \varphi_k$. Thus,

$$N = \{ x \in g; f_1(x) = \cdots = f_\ell(x) = \varphi_1(x) = \cdots = \varphi_k(x) = 0 \},$$

so the set $N$ is a Zariski closed subset of $g$ of dimension

$$\dim N = \dim g - \ell - k.$$ 

More generally, for any $(\xi, \eta) = (\xi_1, \ldots, \xi_\ell, \eta_1, \ldots, \eta_k) \in \mathbb{C}^{\ell + k}$ we define a $K^C$-stable Zariski closed subset $N(\xi, \eta)$ of $g$:

$$N(\xi, \eta) = \{ x \in g; f_j(x) = \xi_j, j = 1, \ldots, \ell, \varphi_i(x) = \eta_i, i = 1, \ldots, k \}.$$ 

Obviously,

$$\dim N(\xi, \eta) = \dim g - \ell - k, \quad (\xi, \eta) \in \mathbb{C}^{\ell + k}.$$ 

As in [4] and [5] we conclude from Theorem 2(a):

Proposition 4. The restriction of polynomials in $P(g)$ to the set $N(\xi, \eta)$ induces an isomorphism of $K$-modules

$$\mathcal{H}_K(g) \simeq P(N(\xi, \eta)) = \mathcal{R}(N(\xi, \eta)), \quad (\xi, \eta) \in \mathbb{C}^{k+\ell}.$$
Here for any subset \( S \subseteq \mathfrak{g} \) we set
\[
\mathcal{P}(S) = \{ f|S; \ f \in \mathcal{P}(\mathfrak{g}) \}
\]
and for any algebraic variety \( S \mathcal{R}(S) \) denotes the algebra of regular functions on \( S \).

The dimensions and the ranks \( \ell = \text{rank} \mathfrak{g} \) and \( k = \text{rank} \mathfrak{k} \) in our cases are the following:

| \( \mathfrak{g} \)       | \( \text{dim} \mathfrak{g} \) | \( \text{dim} \mathfrak{k} \) | \( \ell \) | \( k \) |
|-------------------------|-----------------|-----------------|------|------|
| \( \mathfrak{su}(n,1) \) | \( n^2 + 2n \) | \( n^2 \)        | \( n \) | \( n \) |
| \( \mathfrak{so}(2n,1) \) | \( 2n^2 + n \)  | \( 2n^2 - n \)  | \( n \) | \( n \) |
| \( \mathfrak{so}(2n+1,1) \) | \( 2n^2 + 3n + 1 \) | \( 2n^2 + n \)  | \( n + 1 \) | \( n \) |

So we see that in each case
\[
\dim \mathcal{N}(\xi, \eta) = \dim \mathfrak{k} = \dim K^C, \quad (\xi, \eta) \in \mathbb{C}^{\ell+k}, \quad (3)
\]

**Remark:** By the exercise 13) in §13 in [1] (p. 268) we can choose the following generators \( f_i, \varphi_j \) of \( \mathcal{P}(\mathfrak{g})^K \):

(a) For \( \mathfrak{g}_0 = \mathfrak{su}(n,1) \)
\[
f_i(x) = \text{Tr} \ x^{i+1}, \quad 1 \leq i \leq n, \quad \varphi_j(x) = \text{Tr} (pr \ x)^j, \quad 1 \leq j \leq n.
\]

(b) For \( \mathfrak{g}_0 = \mathfrak{so}(2n,1) \)
\[
f_i(x) = \text{Tr} \ x^{2i}, \quad 1 \leq i \leq n, \quad \varphi_j(x) = \text{Tr} (pr \ x)^{2j}, \quad 1 \leq j \leq n - 1,
\]
\[
\varphi_n(x)^2 = (-1)^n \det (pr \ x).
\]

(c) For \( \mathfrak{g}_0 = \mathfrak{so}(2n+1,1) \)
\[
f_i(x) = \text{Tr} \ x^{2i}, \quad 1 \leq i \leq n, \quad f_{n+1}(x)^2 = (-1)^{n+1} \det x,
\]
\[
\varphi_j(x) = \text{Tr} (pr \ x)^{2j}, \quad 1 \leq j \leq n.
\]

Consider the action of the complex group \( K^C \) on \( \mathfrak{g} \). For \( x \in \mathfrak{g} \) denote by \( \mathcal{O}_x \) its \( K^C \)-orbit. Then of course
\[
\dim \mathcal{O}_x = \dim K^C / K_x^C = \dim K^C - \dim K_x^C, \quad (4)
\]
where \( K_x^C \) denotes the stabilizer of the point \( x \) in the group \( K^C \). So, if \( K_x^C \) is trivial
\[
\dim \mathcal{O}_x = \dim K^C = \dim \mathcal{N}(\xi, \eta). \quad (5)
\]
Lemma 1. There exists \( x \in \mathfrak{g} \) such that the stabilizer \( K^C_x \) is trivial. In this case let \( (\xi, \eta) = (f_1(x), \ldots, f_\ell(x), \varphi_1(x), \ldots, \varphi_k(x)) \). The orbit \( O_x \) is open in \( \mathcal{N}(\xi, \eta) \).

We prove this Lemma in Section 4.

Let \( x \in \mathfrak{g} \) be as in Lemma 1, i.e. such that its stabilizer in \( K^C \) is trivial. Set
\[
(\xi, \eta) = (f_1(x), \ldots, f_\ell(x), \varphi_1(x), \ldots, \varphi_k(x)) \in \mathbb{C}^{\ell+k}.
\]
We know that \( \dim \mathcal{N}(\xi, \eta) \leq \dim \mathcal{N}(\xi, \eta) - 2 \),
\[
(\xi, \eta) = (f_1(x), \ldots, f_\ell(x), \varphi_1(x), \ldots, \varphi_k(x)) \in \mathbb{C}^{\ell+k}.
\]
We know that \( \dim O_x = \dim \mathcal{N}(\xi, \eta) \), so the \( K^C \)-orbit \( O_x \) is open in \( \mathcal{N}(\xi, \eta) \). Thus, the restriction to \( O_x \) is an isomorphism of \( \mathcal{P}(\mathcal{N}(\xi, \eta)) = \mathcal{R}(\mathcal{N}(\xi, \eta)) \) onto \( \mathcal{P}(O_x) \). Now, if the algebraic variety \( \mathcal{N}(\xi, \eta) \) would be irreducible and if we would have
\[
\dim \mathcal{N}(\xi, \eta) \setminus O_x \leq \dim \mathcal{N}(\xi, \eta) - 2,
\]
(this holds true in the settings of [4] and [5] since the dimensions of all the orbits have the same parity) we could conclude by a theorem from algebraic geometry that \( \mathcal{P}(O_x) = \mathcal{R}(O_x) \simeq \mathcal{R}(K^C) \) as \( K^C \)-modules and by the Frobenius reciprocity we could get that the multiplicity \( m(\delta) \) of any irreducible finitedimensional representation \( \delta \) of \( K^C \) in the \( K^C \)-module \( \mathcal{H}_K(\mathfrak{g}) \simeq \mathcal{R}(O_x) \) equals its dimension \( d(\delta) \). Unfortunately, (6) is not true. In fact, in the case \( \mathfrak{g} = \mathfrak{su}(n, 1) \) the algebraic set \( \mathcal{N} = \mathcal{N}(0, 0) \) is even not irreducible -- there exist two open orbits in \( \mathcal{N} \), and in the complement of these two orbits there exist orbits of dimension \( \dim \mathcal{N} - 1 \). In the case \( \mathfrak{g} = \mathfrak{so}(n, 1) \), \( n \geq 3 \), there also exist \( K^C \)-orbits in \( \mathcal{N}(\xi, \eta) \) of dimension \( \dim \mathcal{N}(\xi, \eta) - 1 \).

So, we get only the inclusion of \( K \)-modules \( \mathcal{H}_K(\mathfrak{g}) \hookrightarrow \mathcal{R}(K^C) \) and we may conclude only that
\[
m(\delta) \leq d(\delta)
\]
for every irreducible finitedimensional representation \( \delta \) of \( K \). In fact, the equality holds true although we do not know a priori that \( \mathcal{P}(O_x) = \mathcal{R}(O_x) \); it comes out a posteriori:

Theorem 3. The multiplicity of every irreducible finitedimensional representation \( \delta \) of the compact group \( K \) in the \( K \)-module \( \mathcal{H}_K(\mathfrak{g}) \) of \( K \)-harmonic polynomials on \( \mathfrak{g} \) is equal to its dimension \( d(\delta) \).

To prove Theorem 3 we use the compact form \( K \) of the complex group \( K^C \). Denote by \( \mathcal{P}(Kx) \) the restriction of the polynomial algebra \( \mathcal{P}(\mathfrak{g}) \) to the
$K$–orbit $Kx$. Note that the fact that $K^C$ is the complexification of $K$ easily implies that the restriction $O_x \to Kx$ induces an isomorphism of $K$–modules $\mathcal{P}(O_x)$ onto $\mathcal{P}(Kx)$. Thus, as a $K$–module we have

$$\mathcal{P}(Kx) = \bigoplus_{\delta \in \hat{K}} m(\delta)\delta.$$  \hfill (8)

The subalgebra $\mathcal{P}(Kx)$ of the algebra $C(Kx)$ of all complex continuous functions on the compact space $Kx$ evidently distinguishes the points of $Kx$. Furthermore, this subalgebra is closed under complex conjugation. This is implied by the fact that the set $Kx$ is contained in a real form of the complex vector space $g$. This follows from the fact that the compact group $K$ is contained in a maximal compact subgroup $U$ of the complex group $G^C = \text{Int}(g)$ and the Lie algebra $u$ of $U$ is a real form of $g$. Finally, the algebra $\mathcal{P}(Kx)$ obviously contains constants. Thus, by the Stone–Weierstrass theorem the subalgebra $\mathcal{P}(Kx)$ is uniformly dense in $C(Kx)$. Now, the Peter–Weyl theorem implies that $m(\delta) = d(\delta)$ for all $\delta \in \hat{K}$. This proves Theorem 3.

The symmetrization $U(g) \to S(g) \simeq \mathcal{P}(g)$ is an $K$–module isomorphism. Let $H_K$ be the inverse image of $H_K(g)$ in $U(g)$. The immediate consequence of Theorems 2 and 3 is

**Theorem 4.** The multiplication induces an isomorphism of $K$–modules $U(g)^K \otimes H_K \simeq U(g)$. The multiplicity of every $\delta \in \hat{K}$ in the $K$–module $H_K$ is equal to its dimension $d(\delta)$.

**Corollary 1.** Let $V$ be a finitedimensional $K$–module. Then the space of $K$–invariants $(U(g) \otimes V)^K$ is a free $U(g)^K$–module of finite rank $\dim V$.

By Theorem 4 we have

$$(U(g) \otimes V)^K \simeq (U(g)^K \otimes H_K \otimes V)^K = U(g)^K \otimes (H_K \otimes V)^K.$$ 

Thus, $U(g)$ is a free $U(g)^K$–module of rank $\dim (H_K \otimes V)^K$. Now, let $n(\varepsilon)$ be the multiplicity of $\varepsilon \in \hat{K}$ in $V$. Then

$$(H_K \otimes V)^K \simeq \left( \left( \bigoplus_{\delta \in \hat{K}} d(\delta)\delta \right) \otimes (\bigoplus_{\varepsilon \in \hat{K}} n(\varepsilon)\varepsilon) \right)^K = \bigoplus_{\delta, \varepsilon \in \hat{K}} d(\delta)n(\varepsilon)(\delta \otimes \varepsilon)^K,$$

so

$$\dim (H_K \otimes V)^K = \sum_{\delta, \varepsilon \in \hat{K}} d(\delta)n(\varepsilon) \dim (\delta \otimes \varepsilon)^K.$$
By the Schur’s lemma \( \dim (\delta \otimes \varepsilon)^K \) is 1 if \( \delta \) and \( \varepsilon \) are contragredient to each other and 0 otherwise. Since the dimensions of contragredient representations are equal, we get

\[
\dim (H_K \otimes V)^K = \sum_{\delta \in \hat{K}} n(\delta) d(\delta) = \dim V.
\]

4 Proof of Lemma 1

(1) \( g_0 = \mathfrak{su}(n, 1) \). We realize this Lie algebra as

\[
g_0 = \{ A \in \mathfrak{sl}(n + 1, \mathbb{C}); \ A^* = -\Gamma A \Gamma \},
\]

where \( \Gamma = \text{diag}(1, \ldots, 1, -1) \). Then \( g = \mathfrak{sl}(n + 1, \mathbb{C}) \) and \( K^C = \tilde{K}^C / Z \), where

\[
Z = \{ \text{diag}(\alpha, \ldots, \alpha); \alpha^{n+1} = 1 \} \text{ is the center of } \text{SL}(n + 1, \mathbb{C}) \text{ and }
\]

\[
\tilde{K}^C = \left\{ \begin{bmatrix} B & 0 \\ 0 & (\text{det } B)^{-1} \end{bmatrix} : B \in \text{GL}(n, \mathbb{C}) \right\}.
\]

Now, we can take for \( x \) the elementary \((n+1) \times (n+1)\) Jordan block:

\[
x = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]

The centralizer \( M_x \) of \( x \) in the algebra of all \((n+1) \times (n+1)\) matrices consists of all polynomials in \( x \), i.e.

\[
M_x = \left\{ \begin{bmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\
0 & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\
0 & 0 & \alpha_0 & \cdots & \alpha_{n-3} & \alpha_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \alpha_0 & \alpha_1 \\
0 & 0 & 0 & \cdots & 0 & \alpha_0
\end{bmatrix} : \alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{C} \right\}.
\]

So, we conclude that the centralizer of \( x \) in \( \tilde{K}^C \) is precisely the center \( Z \) of \( \text{SL}(n + 1, \mathbb{C}) \), thus the stabilizer of \( x \) in \( K^C \) is trivial.

(2) \( g_0 = \mathfrak{so}(2n + 1, 1) \). We choose the following realizations:

\[
g = \mathfrak{so}(2n + 2, \mathbb{C}) = \{ A \in \mathfrak{gl}(2n + 2, \mathbb{C}); \ A^t = -\Gamma A \Gamma \},
\]
Here the superscript $t$ denotes the matrix transpose and

$$\Gamma_0 = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Gamma_0 & 0 \\ 0 & 1 \end{bmatrix},$$

$I_n$ being the $n$ by $n$ identity matrix. Denoting as usual the space of all $n \times m$ complex matrices by $M_{n,m}(\mathbb{C})$ and $M_n(\mathbb{C}) = M_{n,n}(\mathbb{C})$, we have

$$k = \begin{cases} \begin{bmatrix} A & B & a & 0 \\ C & -A^t & b & 0 \\ -b^t & -a^t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; & A, B, C \in M_n(\mathbb{C}), \quad B^t = -B, \quad C^t = -C, \quad a, b \in M_{n,1}(\mathbb{C}) \end{cases}$$

and

$$g = \begin{cases} \begin{bmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & \alpha \\ -d^t & -c^t & -\alpha & 0 \end{bmatrix}; & X \in k, \quad c, d \in M_{n,1}(\mathbb{C}), \quad \alpha \in \mathbb{C} \end{cases}.$$
(3) \( \mathfrak{g} = \mathfrak{so}(2n,1) \). We choose the following realizations

\[
\mathfrak{g} = \mathfrak{so}(2n + 1, \mathbb{C}) = \{ A \in \mathfrak{gl}(2n + 1, \mathbb{C}); \ A^t = -\Gamma A \Gamma \} ,
\]

\[
\mathfrak{h} = \left\{ \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} : B \in \mathfrak{gl}(2n, \mathbb{C}), \ B^t = -\Gamma_0 B \Gamma_0 \right\} ,
\]

\[
\Gamma_0 = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} , \quad \Gamma = \begin{bmatrix} \Gamma_0 & 0 \\ 0 & 1 \end{bmatrix} ,
\]

Then

\[
\mathfrak{h} = \left\{ \begin{bmatrix} A & B & 0 \\ C & -A^t & 0 \\ 0 & 0 & 0 \end{bmatrix} : A, B, C \in M_n(\mathbb{C}), \ B^t = -B, \ C^t = -C \right\}
\]

and

\[
\mathfrak{g} = \left\{ X + \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -b^t & -a^t & 0 \end{bmatrix} : X \in \mathfrak{h}, \ a, b \in M_{n,1}(\mathbb{C}) \right\} .
\]

As in (2) let \( J \) denote the elementary \( n \) by \( n \) Jordan block and let

\[
\Delta = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in M_n(\mathbb{C}).
\]

The matrix

\[
x_0 = \begin{bmatrix} J & \Delta & 0 \\ 0 & -J^t & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

is a representative of the \( K^\mathbb{C} \)-orbit of all principal nilpotent elements of \( \mathfrak{h} \). By the Kostant’s results in [3] the stabilizer \( K_{x_0}^\mathbb{C} \) of \( x_0 \) in \( K^\mathbb{C} \) is an \( n \)-dimensional connected simply connected unipotent subgroup whose Lie algebra is the centralizer \( \mathfrak{h}_{x_0} \) of \( x_0 \) in \( \mathfrak{h} \).

(3a) Suppose first that \( n \) is odd, \( n = 2k + 1 \). By solving a system of linear equations one finds that \( \mathfrak{h}_{x_0} \) consists of all matrices of the form

\[
\begin{bmatrix} A & B & 0 \\ 0 & -A^t & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad (9)
\]
where $B$ is $n$ by $n$ antisymmetric matrix such that for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ its first row is
\[
\begin{bmatrix}
0 & \alpha_1 & 0 & \alpha_2 & 0 & \cdots & 0 & \alpha_k & \alpha_{k+1}
\end{bmatrix},
\]
its last column is
\[
\begin{bmatrix}
\alpha_{k+1} & \alpha_{k+2} & 0 & \alpha_{k+3} & 0 & \cdots & 0 & \alpha_{2k+1} & 0
\end{bmatrix}^t,
\]
the inner entries of $B$ are either 0, or $\pm \alpha_j$, $2 \leq j \leq k$, or $\pm 2\alpha_j$, $k+2 \leq j \leq 2k$, and $A$ is a strictly upper triangular $n$ by $n$ matrix whose first row is
\[
\begin{bmatrix}
0 & \alpha_{2k+1} & 0 & \alpha_{2k} & 0 & \cdots & 0 & \alpha_{k+2} & -\alpha_{k+1}
\end{bmatrix},
\]
and every parallel with the main diagonal is constant (i.e. $A$ is a polynomial in $J$). E.g. for $n = 7$ ($k = 3$)
\[
A = \begin{bmatrix}
0 & \alpha_7 & 0 & \alpha_6 & 0 & \alpha_5 & -\alpha_4 \\
0 & 0 & \alpha_7 & 0 & \alpha_6 & 0 & \alpha_5 \\
0 & 0 & 0 & \alpha_7 & 0 & \alpha_6 & 0 \\
0 & 0 & 0 & 0 & \alpha_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 & \alpha_4 \\
-\alpha_1 & 0 & -\alpha_2 & 0 & -\alpha_3 & 0 & \alpha_5 \\
0 & \alpha_2 & 0 & \alpha_3 & 0 & -2\alpha_5 & 0 \\
-\alpha_2 & 0 & -\alpha_3 & 0 & 2\alpha_5 & 0 & \alpha_6 \\
0 & \alpha_3 & 0 & -2\alpha_5 & 0 & -2\alpha_6 & 0 \\
-\alpha_3 & 0 & 2\alpha_5 & 0 & 2\alpha_6 & 0 & \alpha_7 \\
-\alpha_4 & -\alpha_5 & 0 & -\alpha_6 & 0 & -\alpha_7 & 0
\end{bmatrix}.
\]

(3b) Consider now the case of $n$ even, $n = 2k$. As in (3a) one finds that $\mathfrak{L}_{x_0}$ consists of all matrices of the form (9) where $B$ is $n$ by $n$ antisymmetric matrix whose first row is
\[
\begin{bmatrix}
0 & \alpha_1 & 0 & \alpha_2 & 0 & \cdots & 0 & \alpha_k
\end{bmatrix},
\]
its last column is
\[
\begin{bmatrix}
\alpha_k & 0 & \alpha_{k+2} & 0 & \alpha_{k+3} & 0 & \cdots & 0 & \alpha_{2k} & 0
\end{bmatrix}^t,
\]
the inner entries of its antidiagonal are $\pm \alpha_{k+1}$, all the other inner entries are either 0, or $\pm \alpha_j$, $2 \leq j \leq k - 1$, or $\pm 2\alpha_j$, $k + 2 \leq j \leq 2k - 1$, and $A$ is the strictly upper triangular $n$ by $n$ matrix whose first row is

$$
\begin{bmatrix}
0 & \alpha_{2k} & 0 & \alpha_{2k-1} & 0 & \cdots & 0 & \alpha_{k+2} & 0 & \alpha_{k+1} - \alpha_k
\end{bmatrix}
$$

and every parallel with the main diagonal is constant. E.g. for $n = 6$ ($k = 3$)

$$
A = 
\begin{bmatrix}
0 & \alpha_6 & 0 & \alpha_5 & 0 & \alpha_4 - \alpha_3 \\
0 & 0 & \alpha_6 & 0 & \alpha_5 & 0 \\
0 & 0 & 0 & \alpha_6 & 0 & \alpha_5 \\
0 & 0 & 0 & 0 & \alpha_6 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_6 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
B = 
\begin{bmatrix}
0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 \\
-\alpha_1 & 0 & -\alpha_2 & 0 & -\alpha_4 & 0 \\
0 & \alpha_2 & 0 & \alpha_4 & 0 & \alpha_5 \\
-\alpha_2 & 0 & -\alpha_4 & 0 & 2\alpha_5 & 0 \\
0 & \alpha_4 & 0 & 2\alpha_5 & 0 & \alpha_6 \\
-\alpha_3 & 0 & -\alpha_5 & 0 & -\alpha_6 & 0
\end{bmatrix}.
$$

Now, since $\mathfrak{p}$ is $K^C$-stable, for any $y \in \mathfrak{p}$ the stabilizer (resp. the centralizer) of $x = x_0 + y$ in $K^C$ (resp. $\mathfrak{k}$) is the stabilizer (resp. the centralizer) of $y$ in $K^C_{x_0}$ (resp. $\mathfrak{k}_{x_0}$). Let us compute the centralizer of

$$
y = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & e_1 \\
e_1^t & 0 & 0
\end{bmatrix} \in \mathfrak{p}^C
$$

in $\mathfrak{k}_{x_0}$. An element (9) of $\mathfrak{k}_{x_0}$ centralizes $y$ if and only if

$$
Be_1 = 0 \quad \text{and} \quad A^t e_1 = 0.
$$

Now, in the case (3a) we have

$$
Be_1 = \begin{bmatrix}
0 & -\alpha_1 & 0 & -\alpha_2 & 0 & \cdots & -\alpha_k & -\alpha_{k+1}
\end{bmatrix}^t,
$$

$$
A^t e_1 = \begin{bmatrix}
0 & \alpha_{2k+1} & 0 & \alpha_{2k} & 0 & \cdots & 0 & \alpha_{k+2} & -\alpha_{k+1}
\end{bmatrix}^t.
$$
and in the case (3b)

\[ Be_1 = \begin{bmatrix} 0 & -\alpha_1 & 0 & -\alpha_2 & 0 & \cdots & 0 & -\alpha_k \end{bmatrix}^t, \]

\[ A^t e_1 = \begin{bmatrix} 0 & \alpha_2 & 0 & \alpha_{2k-1} & 0 & : & 0 & \alpha_{k+2} & 0 & \alpha_{k+1} - \alpha_k \end{bmatrix}^t. \]

In both cases we conclude that (5) is in the centralizer of \( y \) in \( kx_0 \) if and only if \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \), i.e. if and only if \( A = B = 0 \). Thus,

\[ x = x_0 + y = \begin{bmatrix} J & \Delta & 0 \\ 0 & -J^t & e_1 \\ -e_1^t & 0 & 0 \end{bmatrix} \]

is an element of \( g \) whose stabilizer in \( K^c \) is trivial. This completes the proof of Lemma 1.

References

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