A New Expanded Mixed Finite Element Method for Parabolic Integro-Differential Equations with Nonlinear Memory

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Abstract: A new extended mixed finite element method (MFEM) is suggested for parabolic integrodifferential equations (PIDEs) with nonlinear memory. On the contrary, of the extended mixed scheme, the modern extended mixed element system refers to an asymmetric positive well known and the two gradient equation as well as the flux equation can be isolated from the scalar undefined equation. The presence and uniqueness of the semi-discrete system can be confirmed and error estimates can be achieved for semidiscrete. Fully discrete can be said to be discretization.

Keywords: Expanded mixed finite element method, semi discrete, error estimates, parabolic integrodifferential equations with nonlinear memory.

1. Introduction

Consider the following PIDEs [1]

\[
\begin{align*}
    u_t - \Delta u + \int_0^t k(t-s) \left\{ -\nabla \cdot (\alpha(x,u)\nabla u + \beta(x,u) + \gamma(x,u) \cdot \nabla u + g(x,u)) \right\} ds &= f(x,t), \quad (x,t) \in \Omega \times J, \\
    u(x,t) &= 0, \quad (x,t) \in \partial \Omega \times J, \\
    u(x,0) &= u_0(x), \quad x \in \Omega, 
\end{align*}
\]

with nonlinear memory, where \(J = (0,T]\) is the period interval, \(\Omega\) can be known as a bounded polygonal scope in \(\mathbb{R}^d (d \geq 1)\), besides boundary \(\partial \Omega\), in addition \(f\) is a put function. \(K\) be a memory kernel as well as supposed to become smooth or non-smooth, \(u\) be scalar function. \(\alpha(x,u)\) is a tensor one, \(\beta(x,u), \gamma(x,u)\) can be regarded vector functions and \(g(u)\) be scalar function.

Equation (1.1) is got through several physical operations in that it becomes necessary to consider the impacts of memory according to the deficiency of habitual diffusion equations [2]. More than that, practical problems like the stream of fluids within fissured rock heat [3] transfer dilemmas with memory [4, 5], the nonlocal reactive moves in porous media [6, 7]. There are also many research works on other IDEs [8, 9, 10, 11].

Lately, several researchers have searched the numerical ways for PIDEs problems like the methods of finite difference [12,13] as well as the finite element methods (FEMs) [14,15,16], MFEs [17,18], finite volume element methods (FVEMs) [19].
On the contrary to traditional MFEM, the extended mixed way is an improved meaning in which three variables become intelligibility approximated, particularly, the scalar undefined, gradient, and flux (the tensor coefficient times the gradient). Arbogast et al. [20] studied and utilized a connectedness among the extended mixed-method and a certain cell-centered finite difference method. Chen [21] proposed a new mixed method, which is known as an extended MFEs and confirmed a few mathematical theories of the linear equation of second-order. Besides, Chen affirms several mathematical theories concerning the second-order type of equations of quasilinear elliptic [22] in addition to fourth-order elliptic difficulties [23]. The improvement of the extended MFE. It has been done to several evolution equations. In this regard, Woodward and Dawson [24] investigated the extended of MFE to the equations of nonlinear parabolic. Many error estimates for the kind of extended mixed element of the parabolic equation are put forward in [25]. For Wu and Chen et al. [26–30], the two-grid ways for extended MFE resolution of the kind of semilinear reaction-diffusion equations. In the same vein, Guo and Chen [31] improved and studied an extended property-MFE for that of a convection-dominated transport difficulty. Song and Yuan [32] suggested the extended upwind mixed multistep way for the problem of the miscible displacement within those three dimensions. expanded mixed way of a nonlinear parabolic type of equations in porous medium flow, as well as Liu and Li [33] investigated the $H^1$-Galerkin expanded mixed method for the pseudo hyperbolic equation. Liu [34], studied the $H^1$-Galerkin expanded mixed method for the RLW Burgers equation and argued semidiscrete in addition to completely discrete optimal error estimates. Jiang and Li [35] searched the extended mixed semidiscrete scheme that is designated to a problem of merely longitudinal movement of the homogeneous bar. In [36, 37], the extended mixed covolume method has been investigated for that linear IDE of a parabolic kind and elliptic difficulties, consecutively. In [38], the following error evaluator for extended mixed hybrid ways has been studied Yang [39] suggested a modern MFEM known the sharp affirmative definite MFE rule of tackling the tension equation of parabolic kind during a nonlinear parabolic order showing a method to compressible stream movement during a porous instrument. On the contrary, the formal mixed ways that have numerical replies that are very hard resulting in the decrease of positive definite characteristics, the suggested one can not result in several saddle point difficulties. From then on, the way is carried out on the hyperbolic equations [40] as well as those of pseudo-hyperbolic [41].

In this study, the objective is suggesting and investigating a new extended mixed-method depending on the positive-definite system [42] concerning PIDEs. On the contrary, the extended mixed methods, the suggested mixed element order is symmetric positive-definite and keep off most saddle point difficulties. More than this, both gradient equations, as well as flux equations, can be got away from the scalar undefined equation. The existence, as well as the uniqueness of the semidiscrete scheme, can be affirmed and error estimates can be obtained for the fully discrete system.

By the paper, $C$ refers to a broad positive constant that cannot rely upon the space mesh parameter $h$ as well as the temporary discretization parameter $\Delta t$. Meanwhile, almost definitions, notations, and standards of the Sobolev spaces like [43, 44] are utilized. The study indicates that the natural interior yield in $Y = L^2(\Omega)$ or $[L^2(\Omega)]^2$ by $(\cdot , \cdot)$ with the equivalent standard $\| \cdot \|$, and present the function space $V = H(div, \Omega) = \{ v \in [L^2(\Omega)]^2; \nabla \cdot v \in L^2(\Omega) \}$.

2. Mixed weak Formulation

Re-write a equation (1.1) as following:

$$u_t - \nabla \cdot \left( \nabla u - \int_0^t k(t-s) \left( a(u) \nabla u + \beta(u) \right) ds \right) + \int_0^t k(t-s) \left( \gamma(u) \cdot \nabla u + g(u) \right) ds = f .$$

(2.1)

Presenting the following auxiliary variables:

$$p = \nabla u,$$

$$\lambda = \nabla u - \int_0^t k(t-s) \left( a(u) \nabla u + \beta(u) \right) ds = p - \int_0^t k(t-s) \left( a(u)p + \beta(u) \right) ds,$$

then, for the problem (1.1) have the equivalent system:
(a) \( u_t - \nabla \cdot \lambda + \int_{0}^{t} k(t-s)(\gamma(u) \cdot p + g(u))\, ds = f, \)

(b) \( p - \nabla u = 0, \) \hspace{1cm} (2.2)

(c) \( \lambda - p + \int_{0}^{t} k(t-s)(\alpha(u)p + \beta(u))\, ds = 0, \)

and the initial values \( p(x,0) = \nabla u_0(x), \lambda(x,0) = \nabla \lambda_0(x), \) with \( u(x,0) = u_0(x), \lambda(x,t) \in \Omega \times J. \)

Then, the extended mixed variational formulation of (2.2) may get by the means:

(a) \( (u_t, v) - (\nabla \cdot \lambda, v) + \left( \int_{0}^{t} k(t-s)(\gamma(u) \cdot p + g(u)), v \right)\, ds = (f, v), \quad \forall v \in Y, \)

(b) \( (p, w) + (u_t, \nabla \cdot w) = 0, \quad \forall w \in V, \) \hspace{1cm} (2.3)

(c) \( (\lambda, z) - (p, z) + \left( \int_{0}^{t} k(t-s)(\alpha(u)p + \beta(u)), z \right)\, ds = 0, \quad \forall z \in V. \)

We get (2.3)(b) with regard to \( t' \)

\( (p_t, w) + (u_t, \nabla \cdot w) = 0 \) \hspace{1cm} (2.4)

Taking \( v = \nabla \cdot w \) in (2.3)(a) for \( w \in V, \) to have

\( (u_t, \nabla \cdot w) = (\nabla \cdot \lambda, \nabla \cdot w) - \left( \int_{0}^{t} k(t-s)(\gamma(u) \cdot p + g(u)), \nabla \cdot w \right)\, ds + (f, \nabla \cdot w), \)

and then replacing the aforementioned equation into (2.4), we have.

\( (p_t, w) + (\nabla \cdot \lambda, \nabla \cdot w) - \left( \int_{0}^{t} k(t-s)(\gamma(u) \cdot p + g(u)), \nabla \cdot w \right)\, ds = -(f, \nabla \cdot w). \)

Now new expanded mixed weak formulation of the system (2.3): is to find \( \{u, p, \lambda\} : [0,T] \to Y \times V \times V \) such that

(a) \( (u_t, v) - (\nabla \cdot \lambda, v) + \left( \int_{0}^{t} k(t-s)(\gamma(u) \cdot p + g(u)), v \right)\, ds = (f, v), \quad \forall v \in Y, \)

(b) \( (p, w) + (\nabla \cdot \lambda, \nabla \cdot w) - \left( \int_{0}^{t} k(t-s)(\gamma(u) \cdot p + g(u)), \nabla \cdot w \right)\, ds = -(f, \nabla \cdot w), \quad \forall w \in V, \) \hspace{1cm} (2.5)

(c) \( (\lambda, z) - (p, z) + \left( \int_{0}^{t} k(t-s)(\alpha(u)p + \beta(u)), z \right)\, ds = 0, \quad \forall z \in V. \)
Then, the semidiscrete MFE scheme for (2.5) is to find: \( \{u_h, p_h, \lambda_h\} : [0, T] \rightarrow Y_h \times V_h \times V_h \) like

\[
\begin{align*}
(a) \quad & (u_{ht}, v_h) - \left( \nabla \cdot \lambda_h, v_h \right) + \int_0^t k(t-s) \left( \gamma(u_h) \cdot p_h + g(u_h) \right) ds = (f, v_h), \quad \forall v_h \in Y_h, \\
(b) \quad & (p_{ht}, w_h) + \left( \nabla \cdot \lambda_h, \nabla \cdot w_h \right) - \int_0^t k(t-s) \left( \gamma(u_h) \cdot p_h + g(u_h) \right) \nabla \cdot w_h ds = -(f, \nabla \cdot w_h), \quad \forall w_h \in V_h, \\
(c) \quad & (\lambda_h, z_h) - \left( p_h, z_h \right) + \int_0^t k(t-s) \left( \alpha(u_h) p_h + \beta(u_h) \right) z_h ds = 0, \quad \forall z_h \in V_h. \quad (2.6)
\end{align*}
\]

and taken an initial approximation \( (u_h^0, p_h^0, \lambda_h^0) \in Y_h \times V_h \times V_h \).

Where \( Y_h \subset H \) and \( V_h \subset V \) is finite element spaces clarified according to the partitions \( T_h \) and \( \tau_h \), consecutively, such that \( T_h \) and \( \tau_h \) are the families of quasi-regular partitions to the domain \( \Omega \), that can be the identical one or different, so the factors in the partitions get the diameters bounded by \( h_u \) and \( h_\lambda \), respectively.

**Theorem 2.1.** There exists a unique discrete solution to the system (2.6).

**Proof.** Let \( \{\phi_i\}_{i=1}^{N_1} \) and \( \{\psi_i(x)\}_{i=1}^{N_2} \) be bases of \( V_h \) and \( Y_h \) respectively. Let

\[
\begin{align*}
    u_h &= \sum_{i=1}^{N_1} u_i(t) \phi_i(x), \quad p_h = \sum_{j=1}^{N_2} p_j(t) \psi_j(x), \quad \lambda_h = \sum_{k=1}^{N_3} \lambda_k(t) \psi_k(x), \quad (2.7)
\end{align*}
\]

substituting the terms into (2.6) and selecting, \( v_h = \phi_m \), \( w_h = \psi_l \) to have the equations below.

\[
\begin{align*}
(a) \quad & \sum_{i=1}^{N_1} (\phi_i, \phi_m) u_{ti} - \sum_{k=1}^{N_2} (\nabla \cdot \psi_k, \phi_m) \lambda_k + \sum_{j=1}^{N_2} \left( \int_0^t \left( \sum_{i=1}^{N_1} u_i(t) \phi_i(x) \right) \psi_j ds, \phi_m \right) p_j \\
& \quad = (f, \phi_m) + \left( \int_0^t \left( \sum_{i=1}^{N_1} u_i(t) \phi_i(x) \right) ds, \phi_m \right), \\
(b) \quad & \sum_{j=1}^{N_2} (\psi_j, \psi_l) p_{tj} + \sum_{k=1}^{N_2} (\nabla \cdot \psi_k, \nabla \cdot \psi_l) \lambda_k - \sum_{j=1}^{N_2} \left( \int_0^t \left( \sum_{i=1}^{N_1} u_i(t) \phi_i(x) \right) \psi_j ds, \nabla \cdot \psi_l \right) p_j \\
& \quad = -(f, \nabla \cdot \psi_l) + \left( \int_0^t \left( \sum_{i=1}^{N_1} u_i(t) \phi_i(x) \right) ds, \nabla \cdot \psi_l \right), \quad (2.8)
\end{align*}
\]

\[
\begin{align*}
(c) \quad & \sum_{k=1}^{N_3} (\psi_k, \psi_l) \lambda_k = \sum_{j=1}^{N_2} (\psi_j, \psi_l) p_j + \sum_{j=1}^{N_2} \left( \int_0^t \left( \sum_{i=1}^{N_1} u_i(t) \phi_i(x) \right) \psi_j ds, \psi_l \right) p_j \\
& \quad + \left( \int_0^t k(t-s) \beta \left( \sum_{i=1}^{N_1} u_i(t) \phi_i(x) \right) ds, \psi_l \right) = 0,
\end{align*}
\]

then the following the equations can be reached:
\[ AU'(t) - BA(t) + \int_0^t k(t-s) CP(s) ds = F + \int_0^t k(t-s) G ds, \]

\[ DP'(t) + EA(t) - \int_0^t k(t-s) HP(s) ds = -I + \int_0^t k(t-s) M ds, \quad (2.9) \]

\[ DA(t) - DP(t) + \int_0^t k(t-s) NP(s) ds + \int_0^t k(t-s) R ds = 0, \]

where

\[ A = (\varphi_i, \varphi_m)^{N_1 \times N_1}, \quad B = (\nabla \cdot \psi_j, \varphi_m)^{N_2 \times N_2}, \quad C = (\alpha(U) \cdot \psi_j, \varphi_m)^{N_2 \times N_1}, \]

\[ F = (f, \varphi_m)^{1 \times N_1}, \quad G = (g(U), \varphi_m)^{1 \times N_1}, \quad D = (\psi_j, \psi_i)^{N_2 \times N_2}, \]

\[ E = (\nabla \cdot \psi_j, \nabla \cdot \psi_i)^{N_2 \times N_2}, \quad H = (\alpha(U) \cdot \psi_j, \nabla \cdot \psi_i)^{N_2 \times N_2}, \quad I = (f, \nabla \cdot \psi_i)^{1 \times N_2}, \]

\[ M = (g(U), \nabla \cdot \psi_l)^{1 \times N_2}, \quad N = (\alpha(U) \psi_j, \psi_l)^{N_2 \times N_2}, \quad R = (\beta(U), \psi_l)^{1 \times N_2}, \]

\[ U(t) = (u_1(t), u_2(t), \ldots, u_{N_1}(t))^T, \quad P(t) = (p_1(t), p_2(t), \ldots, p_{N_2}(t))^T, \]

\[ \Lambda(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_{N_2}(t))^T, \]

Simply the two \( A, D \) and \( E \) seem symmetric positive definite. From (2.9), the problems may be recorded as follows:

\[ (a) \quad U'(t) = A^{-1}BA(t) - A^{-1}C \int_0^t k(t-s) P(s) ds + A^{-1}F + A^{-1}G \int_0^t k(t-s) ds, \]

\[ (b) \quad P'(t) = D^{-1}H \int_0^t k(t-s) P(s) ds - D^{-1}E \Lambda(t) - D^{-1} + D^{-1}M \int_0^t k(t-s) ds, \]

\[ (c) \quad \Lambda(t) = P(t) - D^{-1}N \int_0^t k(t-s) P(s) ds - D^{-1}R \int_0^t k(t-s) ds. \]

So, via the theory of differential equations \([45, 46]\), (2.11) gets an exceptional solution, and Equally (2.6) has a distinctive solution.

### 3. New expanded mixed projection and some Lemmas

Let, \( Y_h \) and \( V_h \) be finite-dimensional subspaces of, \( L^2(\Omega) \) and \( W \) respectively, with the inverse property \([47]\) and the following approximation properties \([48-52]\): for \( 0 \leq p \leq +\infty \) and \( m, m^*, r \) positive integers

\[ \inf_{v_h \in H_h} \| v - v_h \|_{L^p(\Omega)} \leq C h^{r+1} \| v \|_{W^{r+1,p}(\Omega)}, \quad \forall v \in L^2(\Omega) \cap W^{r+1,p}(\Omega), \]

\[ \inf_{w_h \in V_h} \| w - w_h \|_{L^p(\Omega)} \leq C h^{m+1} \| v \|_{W^{m+1,p}(\Omega)}, \quad \forall w \in H(div, \Omega) \cap [W^{m+1,p}(\Omega)]^d, \]

\[ \inf_{w_h \in V_h} \| \nabla \cdot (w - w_h) \|_{L^p(\Omega)} \leq C h^{m^*} \| v \|_{W^{m^*,p}(\Omega)}, \quad \forall w \in H(div, \Omega) \cap [W^{r+1,p}(\Omega)]^d, \quad (3.1) \]
where \( m^* = m + 1 \) for the Brezzi-Douglas-Fortin-Marini spaces [52] and the Raviart-Thomas spaces [51] and \( m^* = m \) for the Brezzi-Douglas-Marini spaces [48,51].

For the sake of the following error analysis, the study introduces two operators according to the following lemmas.

**Lemma 3.1.**[53,54] There has an operator of a projection \( \Pi_h: \left( L^2(\Omega) \right)^2 \rightarrow W_h \) such that

\[
(\nabla \cdot (\lambda - \Pi_h \lambda), \phi_h) = 0, \quad \forall \phi_h \in \nabla \cdot V_h = \{ \phi_h = \nabla \cdot w_h, w_h \in V_h \},
\]

and

\[
\| \lambda - \Pi_h \lambda \|_{L^2(\Omega)} \leq C h^{m+1} \| \lambda \|_{W^{m+1,p}(\Omega)},
\]

\[
\| \nabla \cdot (\lambda - \Pi_h \lambda) \|_{L^2(\Omega)} \leq C h^{m} \| \nabla \cdot \lambda \|_{W^{m,p}(\Omega)}.
\]

**Lemma 3.2.**[55] There exists a projection operator \( P_h: L^2(\Omega) \rightarrow V_h \) such that

\[
(v - P_h v, v_h) = 0, \quad \forall v \in L^2(\Omega), \ v_h \in Y_h;
\]

and

\[
\| v - P_h v \|_{L^p(\Omega)} \leq C h^{r+1} \| v \|_{H^{r+1}(\Omega)}, \quad \forall v \in H^{r+1}(\Omega).
\]

Employing the explanations of the operators \( \Pi_h \) and \( P_h \) we can easily obtain the following lemma.

**Lemma 3.3.** Suppose that the solution of system (2.5) has the regular properties that \( u_t \in L^2(H^{r+1}(\Omega)), \ p_t, p_{tt}, \lambda_t \in L^2(H^{m+1}(\Omega)) \), then one has the following estimates:

\[
\| (p - \Pi_h p)_t \|_{L^p(\Omega)} \leq C h^{m+1} \| p_t \|_{W^{m+1,p}(\Omega)},
\]

\[
\| (\lambda - \Pi_h \lambda)_t \|_{L^p(\Omega)} \leq C h^{m+1} \| \lambda_t \|_{W^{m+1,p}(\Omega)},
\]

\[
\| (u - P_h u)_t \|_{L^p(\Omega)} \leq C h^{r+1} \| u_t \|_{H^{r+1}(\Omega)}.
\]

### 4. Semi-Discrete Error Estimates

In this section, we discuss a priori error estimates as.

Subtracting (2.6) from (2.5), and auxiliary projections (3.2) and (3.5) we can get the error equations

\[
(a) (\xi_t, v_h) - (\nabla \cdot \theta, v_h) = - \left( \int_0^t k(t-s)(\gamma(u) - \gamma(u_h)), p, v_h \right) ds - \left( \int_0^t k(t-s)(\gamma(u_h) \cdot \xi, v_h) \right) ds
\]

\[
- \left( \int_0^t k(t-s)(g(u) - g(u_h)), v_h \right) ds \quad \forall \ v_h \in Y_h,
\]

\[
(b) (\xi_t, w_h) + (\nabla \cdot \theta, \nabla \cdot w_h) = -(\rho_t, w_h) - \left( \int_0^t k(t-s)(\gamma(u) - \gamma(u_h)), p, \nabla \cdot w_h \right) ds
\]

\[
+ \left( \int_0^t k(t-s)(\gamma(u_h) \cdot \xi, \nabla \cdot w_h) \right) ds + \left( \int_0^t k(t-s)\rho(u_h), \rho, \nabla \cdot w_h \right) ds
\]
\[
+ \left( \int_0^t k(t-s)(g(u) - g(u_h)), \nabla \cdot w_h \right) ds, \qquad \forall w_h \in V_h.
\]

\[
(\theta, z_h) - (\xi, z_h) = - (\delta, z_h) + (\rho, z_h) - \left( \int_0^t k(t-s)(\alpha(u) - \alpha(u_h))p, z_h \right) ds
\]

\[
+ \left( \int_0^t k(t-s)\alpha(u_h)\rho, z_h \right) ds - \left( \int_0^t k(t-s)(\gamma(u) - \gamma(u_h)).p, \nabla \cdot \xi \right) ds
\]

\[
+ \left( \int_0^t k(t-s)(\alpha(u) - \alpha(u_h))p, \xi \right) ds + \left( \int_0^t k(t-s)(\alpha(u) + \beta(u_h)), \xi \right) ds, \quad \forall \xi_h \in V_h, \quad (4.1)
\]

where

\[
u - u_h = u - P_n u + P_n \eta = \eta + \zeta,
\]

\[
p - p_h = p - \Pi_n p + \Pi_n \rho = \rho + \xi, \quad (4.2)
\]

\[\lambda - \lambda_h = \lambda - \lambda_h = \delta + \theta.\]

**Theorem 3.1.** Assume that the approximate properties (3.1) hold, and the solution of system (2.5) has regular properties that

\[u_t \in L^2(H^{r+1}(\Omega)), p_t, \lambda_t \in L^2(H^{m+1}(\Omega)),\]

Then one has the error estimates

\[
\|p - p_h\|_{L^\infty(L^2(\Omega))} + \|\lambda - \lambda_h\|_{L^\infty(L^2(\Omega))} \leq C h^{\min(m+1, r+1)},
\]

\[
\|\nabla \cdot (\lambda - \lambda_h)\|_{L^\infty(L^2(\Omega))} \leq C \left( h^{m+1} + h^{\min(m+1, r+1)} \right), \quad (4.3)
\]

\[
\|(u - u_h)\|_{L^\infty(L^2(\Omega))} \leq C h^{\min(m+1, r+1)},
\]

**Proof.** Choose \(w_h = \theta \) in (4.1)(b) and \(z_h = \xi_t \) in (4.1)(c), then add the result two equations to obtain

\[
(\xi, \xi_t) + (\nabla \cdot \theta, \nabla \cdot \theta) = (\delta, \xi_t) - (\rho, \xi_t) - (\rho_t, \theta) - \left( \int_0^t k(t-s)(\gamma(u) - \gamma(u_h)), p, \nabla \cdot \theta \right) ds
\]

\[
+ \left( \int_0^t k(t-s)\gamma(u_h) \cdot \xi, \nabla \cdot \theta \right) ds + \left( \int_0^t k(t-s)\gamma(u_h), \rho, \nabla \cdot \theta \right) ds
\]

\[
+ \left( \int_0^t k(t-s)(g(u) - g(u_h)), \nabla \cdot \theta \right) ds + \left( \int_0^t k(t-s)(\alpha(u_h)\rho, \xi_t \right) ds
\]

\[
+ \left( \int_0^t k(t-s)(\alpha(u) - \alpha(u_h))p, \xi_t \right) ds + \left( \int_0^t k(t-s)(\alpha(u) + \beta(u_h)), \xi_t \right) ds
\]

\[
+ \left( \int_0^t k(t-s)(\alpha(u) + \beta(u_h)), \xi_t \right) ds, \quad (4.4)
\]

Which can be written as
\[
\frac{1}{2} \frac{d}{dt} \| \xi \|^2_{L^2(\Omega)} + \| \nabla \cdot \theta \|^2_{L^2(\Omega)} = \frac{d}{dt} (\delta, \xi) - (\delta_t, \xi) - \frac{d}{dt} (\rho, \xi) + (\rho_t, \theta) \\
- \left( \int_0^t k(t-s) (\gamma'(u) - \gamma'(u_h)), p, \nabla \cdot \theta \right) ds + \left( \int_0^t k(t-s) \gamma(u_h) \cdot \nabla \cdot \theta \right) ds \\
+ \left( \int_0^t k(t-s) \gamma(u_h), \rho, \nabla \cdot \theta \right) ds + \left( \int_0^t k(t-s) (g(u) - g(u_h)), \nabla \cdot \theta \right) ds \\
+ \frac{d}{dt} \left( \int_0^t k(t-s) \alpha(u_h) \rho, \xi \right) ds - \left( \int_0^t k(t-s) \alpha(u_h) p, \xi \right) ds \\
- \left( \int_0^t k(t-s) (\alpha(u) - \alpha(u_h)) p, \xi \right) ds - (k(t-s) \alpha(u_h) p, \xi) \\
- \left( \int_0^t k(t-s) (\alpha(u) u_t - \alpha(u_h) u_{th}) p, \xi \right) ds + \frac{d}{dt} \left( \int_0^t k(t-s) \alpha(u_h) \xi, \xi \right) ds \\
- \left( \int_0^t k(t-s) \alpha(u_h) \xi, \xi \right) ds - (k(t-s) \alpha(u_h) \xi, \xi) \\
+ \frac{d}{dt} \left( \int_0^t k(t-s) (\alpha(u) + \beta(u_h)), \xi \right) ds - (k(t-s) (\alpha(u) + \beta(u_h)), \xi) \\
- \left( \int_0^t k(t-s) (\alpha(u) u_t + \beta(u_h) u_{th}), \xi \right) ds. \quad (4.5)
\]

By integrating from 0 to \( t \) and applying the Cauchy-Schwarz's as well as the Young's inequalities with \( \xi(0) = 0 \), then

\[
\int_0^t \int_0^\tau |\phi(s)|^2 ds d\tau \leq c \int_0^t |\phi(s)|^2 ds. \quad (4.6)
\]

where \( \phi \) is an integrable function in \([0, t]\), \( \tau \in [0, T] \).

\[
T_1 = \frac{1}{2} \frac{d}{dt} \| \xi \|^2_{L^2(\Omega)} + \frac{1}{2} \int_0^t \frac{d}{dt} \| \xi \|^2_{L^2(\Omega)} = \frac{1}{2} \| \xi \|^2_{L^2(\Omega)} \\
T_2 = \| \nabla \cdot \theta \|^2_{L^2(\Omega)} = \int_0^t \| \nabla \cdot \theta \|^2_{L^2(\Omega)} d\tau,
\]

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\begin{align*}
T_3 &= \frac{d}{dt} (\delta, \xi) = \int_0^t \frac{d}{dt} (\delta, \xi) d\tau = (\delta, \xi) = |(\delta, \xi)| \leq \|\delta\|_{L^2(\Omega)} + \varepsilon \|\xi\|_{L^2(\Omega)}, \\
T_4 &= -(\delta_t, \xi) = - \int_0^t (\delta_t, \xi) d\tau = - \int_0^t (\delta_t, \xi) d\tau \leq c \int_0^t \|\delta_t\|_{L^2(\Omega)} d\tau + c \int_0^t \|\xi\|_{L^2(\Omega)} d\tau, \\
T_5 &= \frac{d}{dt} (\rho, \xi) = \int_0^t \frac{d}{dt} (\rho, \xi) d\tau = (\rho, \xi) = |(\rho, \xi)| \leq \|\rho\|_{L^2(\Omega)} + \varepsilon \|\xi\|_{L^2(\Omega)}, \\
T_6 &= (\rho_t, \xi) = \int_0^t (\rho_t, \xi) d\tau = \int_0^t (\rho_t, \xi) d\tau \leq c \int_0^t \|\rho_t\|_{L^2(\Omega)} d\tau + c \int_0^t \|\xi\|_{L^2(\Omega)} d\tau, \\
T_7 &= -(\rho_t, \theta) = \int_0^t -(\rho_t, \theta) d\tau \leq c \int_0^t \|\rho_t\|_{L^2(\Omega)} d\tau + c \int_0^t \|\theta\|_{L^2(\Omega)} d\tau, \\
T_9 &= - \left( \int k(t-s)(\gamma(u) - \gamma(u_h)) \cdot \rho, \nabla \cdot \theta \right) ds = \left| \int \left( \int k(t-s)(\gamma(u) - \gamma(u_h)) \cdot \rho ds, \nabla \cdot \theta \right) d\tau \right| \\
&= c_1 c_2 \int_0^t \left( \|\eta\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\Omega)} \right) d\tau + c_1 c_2 \int_0^t \|\nabla \cdot \theta\|_{L^2(\Omega)} d\tau, \\
T_{10} &= \left( \int_0^t k(t-s)\gamma(u_h) \cdot \xi, \nabla \cdot \theta \right) ds = \int_0^t \left( \int_0^s k(t-s)\gamma(u_h) \cdot \xi ds, \nabla \cdot \theta \right) d\tau \\
&\leq c_1 c_3 \int_0^t \|\xi\|_{L^2(\Omega)} d\tau + c_1 c_3 \int_0^t \|\nabla \cdot \theta\|_{L^2(\Omega)} d\tau, \\
T_{11} &= \left( \int_0^t k(t-s)\gamma(u_h) \cdot \rho, \nabla \cdot \theta \right) ds = \int_0^t \left( \int_0^s k(t-s)\gamma(u_h) \cdot \rho ds, \nabla \cdot \theta \right) d\tau \\
&\leq c_1 c_3 \int_0^t \|\rho\|_{L^2(\Omega)} d\tau + c_1 c_3 \int_0^t \|\nabla \cdot \theta\|_{L^2(\Omega)} d\tau, \\
T_{12} &= \left( \int_0^t k(t-s)(g(u) - g(u_h)) \cdot \nabla \cdot \theta \right) ds = \left| \int_0^t \left( \int_0^s k(t-s)(g(u) - g(u_h)) ds, \nabla \cdot \theta \right) d\tau \right| \\
&\leq c_1 \int_0^t \left( \|\eta\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\Omega)} \right) d\tau + c_1 \int_0^t \|\nabla \cdot \theta\|_{L^2(\Omega)} d\tau, \\
T_{13} &= \frac{d}{dt} \left( \int_0^t k(t-s)\alpha(u_h) \cdot \rho d\xi \right) ds = \int_0^t \left( \frac{d}{dt} \int_0^s k(t-s)\alpha(u_h) \cdot \rho ds, \xi \right) d\tau = \left| \int_0^t k(t-s)\alpha(u_h) \cdot \rho ds, \xi \right| \\
&\leq \|\alpha\|_{L^2(\Omega)} + \varepsilon \|\xi\|_{L^2(\Omega)},
\end{align*}
\[ \leq c_1 c_4 \int_0^t ||\rho||^2_{L^2(\Omega)} dt + c_1 c_4 \int_0^t ||\xi||^2_{L^2(\Omega)} dt, \]

where \( c_4 \) relies on a bound of \( \alpha(u_h) \).

\[ T_{14} = \int_0^t \int_0^s (k(t-s)\alpha(u_h)\rho, \xi) ds dt = \int_0^t \int_0^s \left| \left( \int_0^s (k(t-s)\alpha(u_h)\rho ds, \xi) \right) dt \right| \]

\[ \leq c_1 c_4 \int_0^t ||\rho||^2_{L^2(\Omega)} dt + c_1 c_4 \int_0^t ||\xi||^2_{L^2(\Omega)} dt, \]

\[ T_{15} = -(k(t-s)\alpha(u_h)\rho, \xi) = \int_0^t -(k(t-s)\alpha(u_h)\rho, \xi) dt \leq c_1 c_4 \int_0^t ||\rho||^2_{L^2(\Omega)} dt + c_1 c_4 \int_0^t ||\xi||^2_{L^2(\Omega)} dt, \]

\[ T_{16} = -\int_0^t k(t-s)\alpha(u_h)u_{th}\rho, \xi) ds = \int_0^t \left| \left( \int_0^s (k(t-s)\alpha(u_h)u_{th}\rho ds, \xi) \right) dt \right| \]

\[ \leq c_1 \int_0^t \left( \left| \int_0^s (\alpha(u_h)(u_{th} - u_t)\rho ds, \xi) \right| + \int_0^s (\alpha(u_h)u_t\rho ds, \xi) \right) dt \]

\[ \leq c_1 c_4 c_7 \int_0^t \left( ||\eta_t||^2_{L^2(\Omega)} + ||\xi_t||^2_{L^2(\Omega)} + ||\xi||^2_{L^2(\Omega)} \right) dt + c_4 c_5 \int_0^t \left( ||\rho||^2_{L^2(\Omega)} + ||\xi||^2_{L^2(\Omega)} \right) dt, \]

\( c_5 \) relies on \( ||u||_{W_b^L(L^\infty)} \) and \( c_7 \) depends on \( ||u||_{W_b^L(L^\infty)} \).

\[ T_{17} = \int_0^t \int_0^s k(t-s)(\alpha(u) - \alpha(u_h))p, \xi) ds dt = \int_0^t \frac{d}{dt} \left( \int_0^s k(t-s)(\alpha(u) - \alpha(u_h))p, \xi) ds \right) \]

\[ = \left( \int_0^t k(t-s)(\alpha(u) - \alpha(u_h))p, \xi) ds \right) \]

\[ \leq c_1 c_2 \int_0^t \left( ||\eta||^2_{L^2(\Omega)} + ||\xi||^2_{L^2(\Omega)} \right) dt + c_1 \int_0^t \left( ||\xi||^2_{L^2(\Omega)} \right) dt, \]

\[ T_{18} = -\int_0^t k(t-s)(\alpha(u) - \alpha(u_h))p, \xi) ds = \int_0^t \left( \int_0^s (k(t-s)(\alpha(u) - \alpha(u_h))p ds, \xi) \right) dt \]

\[ \leq c_2 c_1 \int_0^t \left( ||\eta||^2_{L^2(\Omega)} + ||\xi||^2_{L^2(\Omega)} \right) dt + c_1 \int_0^t \left( ||\xi||^2_{L^2(\Omega)} \right) dt, \]

\[ T_{19} = -(k(t-s)(\alpha(u) - \alpha(u_h))p, \xi) = \int_0^t -(k(t-s)(\alpha(u) - \alpha(u_h))p, \xi) dt \]

\[ \leq c_4 c_2 \int_0^t \left( ||\eta||^2_{L^2(\Omega)} + ||\xi||^2_{L^2(\Omega)} \right) dt + c_1 \int_0^t \left( ||\xi||^2_{L^2(\Omega)} \right) dt, \]

\[ T_{20} = -\int_0^t k(t-s)(\alpha(u)u_t - \alpha(u_h)u_{th})p, \xi) ds = \int_0^t \left( \int_0^s (k(t-s)(\alpha(u)u_t - \alpha(u_h)u_{th})p ds, \xi) \right) dt \]
\[
\int_0^t \left| \int_0^z k(t-s)(\alpha(u_i) pds, \xi) \right| dt \geq \int_0^t \left| \int_0^z k(t-s)(\alpha(u_{ih}) pds, \xi) \right| dt \leq c_1 c_5 c_6 \int \|\xi\|^2_{L^2(\Omega)} dt + c_1 c_4 c_5 \int (\|\eta\|^2_{L^2(\Omega)} + \|\zeta\|^2_{L^2(\Omega)}) dt + c_4 c_5 \int (\|\xi\|^2_{L^2(\Omega)}) dt.
\]

c_6 depends on \(\alpha(u)\),

\[
T_{21} = \frac{d}{dt} \left( \int_0^t k(t-s)\alpha(u_{ih}) \xi, \xi \right) ds = \int_0^t \frac{d}{dt} \left( \int_0^z k(t-s)\alpha(u_{ih}) \xi, \xi \right) ds, \xi \right) \right| dt \leq c_4 c_5 \int (\|\xi\|^2_{L^2(\Omega)}) dt,
\]

\[
T_{22} = \int_0^t \left| \int_0^z k(t-s)\alpha(u_{ih}) \xi, \xi \right| ds \leq c_1 c_4 \int (\|\xi\|^2_{L^2(\Omega)}) dt,
\]

\[
T_{23} = -(k(t-s)\alpha(u_{ih}), \xi) = \int_0^t \left| -(k(t-s)\alpha(u_{ih}), \xi) \right| dt \leq c_1 c_4 \int (\|\xi\|^2_{L^2(\Omega)}) dt,
\]

\[
T_{24} = \int_0^t \left| \int_0^z k(t-s)\alpha(u_{ih}) u_{ih} \xi, \xi \right| ds \leq c_1 c_4 \int (\|\eta\|^2_{L^2(\Omega)} + \|\zeta\|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}) dt + c_1 c_4 \int (\|\xi\|^2_{L^2(\Omega)}) dt,
\]

\[
T_{25} = \frac{d}{dt} \left( \int_0^t k(t-s)(\alpha(u_i) + \beta(u_{ih})), \xi \right) ds = \int_0^t \frac{d}{dt} \left( \int_0^z k(t-s)(\alpha(u_i) + \beta(u_{ih})), \xi \right) ds \leq c_1 \int (\|\eta\|^2_{L^2(\Omega)} + \|\zeta\|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}) dt + c_1 \int (\|\xi\|^2_{L^2(\Omega)}) dt.
\]

\[
T_{26} = -(k(t-s)(\alpha(u_i) + \beta(u_{ih})), \xi) = \int_0^t \left| -(k(t-s)(\alpha(u_i) + \beta(u_{ih})), \xi) \right| dt \leq c_1 \int (\|\eta\|^2_{L^2(\Omega)} + \|\zeta\|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}) dt + c_1 \int (\|\xi\|^2_{L^2(\Omega)}) dt.
\]

\[
T_{27} = \int_0^t \left| \int_0^z k(t-s)(\alpha(u_{ih}) u_{ih} + \beta(u_{ih}) u_{ih}), \xi \right| ds \leq \int_0^t \left| \int_0^z k(t-s)(\alpha(u_{ih}) u_{ih} + \beta(u_{ih}) u_{ih}), \xi \right| ds \leq c_1 \int (\|\eta\|^2_{L^2(\Omega)} + \|\zeta\|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}) dt + c_1 \int (\|\xi\|^2_{L^2(\Omega)}) dt.
\]

\[
T_{28} = \int_0^t \left| \int_0^z k(t-s)(\alpha(u_i) u_{ih}), \xi \right| ds \leq \int_0^t \left| \int_0^z k(t-s)(\alpha(u_i) u_{ih}), \xi \right| ds \leq c_1 \int (\|\eta\|^2_{L^2(\Omega)} + \|\zeta\|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}) dt + c_1 \int (\|\xi\|^2_{L^2(\Omega)}) dt.
\]
\[ \leq cc_1c_5 \int_0^t \| \xi \|_{L^2(\Omega)}^2 \, dt + cc_1c_4 \int_0^t (\| \eta \|_{L^2(\Omega)}^2 + \| \zeta \|_{L^2(\Omega)}^2) \, dt + c_1c_4 c_5 \int_0^t (\| \epsilon \|_{L^2(\Omega)}^2) \, dt, \]

Thus, by combining the above inequalities from \( (T_1) \) to \( (T_{27}) \), we get

\[ \| \xi \|_{L^2(\Omega)}^2 + 2 \int_0^t \| \nabla \cdot \theta \|_{L^2(\Omega)}^2 \, dt \leq \| \delta \|_{L^2(\Omega)}^2 + \| \rho \|_{L^2(\Omega)}^2 + \epsilon \| \xi \|_{L^2(\Omega)}^2. \]  

(4.7)

\[ + C_1 \int_0^t \left( \| \rho \|_{L^2(\Omega)}^2 + \| \rho \|_{L^2(\Omega)}^2 \right) \, dt + \int_0^t \left( \| \eta \|_{L^2(\Omega)}^2 + \| \zeta \|_{L^2(\Omega)}^2 \right) \, dt, \]

Here, \( C_1 = C_1(c_1, c_2, c_3, c_4, c_5, c_6, c_7). \)

Select \( z_h = \theta \) in (4.1)(c) to get

\[ (\theta, \theta) = (\xi, \theta) - (\delta, \theta) + (\rho, \theta) - \left( \int_0^t (k-t-s)(\alpha(u) - \alpha(u_h))p, \theta \right) ds + \left( \int_0^t (k-t-s)\alpha(u_h)p, \theta \right) ds \]

\[ - \left( \int_0^t (k-t-s)\alpha(u_h)\xi, \theta \right) ds - \left( \int_0^t (k-t-s)(\alpha(u) + \beta(u_h), \theta) \right) ds, \]  

(4.8)

We apply Young's inequalities to the bound the right aspect expression by the expression

\[ (\theta, \theta) \leq \| \theta \|_{L^2(\Omega)}, \]

\[ (\xi, \theta) \leq \| \xi \|_{L^2(\Omega)} + \epsilon \| \theta \|_{L^2(\Omega)}, \]

\[ (\delta, \theta) \leq \| \delta \|_{L^2(\Omega)} + \epsilon \| \theta \|_{L^2(\Omega)}, \]

\[ (\rho, \theta) \leq \| \rho \|_{L^2(\Omega)} + \epsilon \| \theta \|_{L^2(\Omega)}, \]

\[ \left( \int_0^t (k-t-s)(\alpha(u) - \alpha(u_h))p, \theta \right) ds \leq c_2c_1 \int_0^t \left( \| \eta \|_{L^2(\Omega)}^2 + \| \zeta \|_{L^2(\Omega)}^2 \right) \, dt + \epsilon \| \theta \|_{L^2(\Omega)}, \]

\[ \left( \int_0^t (k-t-s)\alpha(u_h)p, \theta \right) ds \leq c_1c_4 \int_0^t \| \rho \|_{L^2(\Omega)}^2 \, dt + \epsilon \| \theta \|_{L^2(\Omega)}, \]

\[ \left( \int_0^t (k-t-s)\alpha(u_h)\xi, \theta \right) ds \leq c_1c_4 \int_0^t \| \xi \|_{L^2(\Omega)}^2 \, dt + \epsilon \| \theta \|_{L^2(\Omega)}, \]

\[ \left( \int_0^t (k-t-s)(\alpha(u) + \beta(u_h), \theta) \right) ds \leq c_1 \int_0^t \left( \| \eta \|_{L^2(\Omega)}^2 + \| \zeta \|_{L^2(\Omega)}^2 \right) \, dt + \epsilon \| \theta \|_{L^2(\Omega)}, \]

thus (4.8) turns to

\[ \| \theta \|_{L^2(\Omega)} \leq C_2 \left( \| \xi \|_{L^2(\Omega)} + \| \delta \|_{L^2(\Omega)} + \| \rho \|_{L^2(\Omega)} \right) + \int_0^t \left( \| \eta \|_{L^2(\Omega)} + \| \zeta \|_{L^2(\Omega)} \right) \, dt + \epsilon \| \theta \|_{L^2(\Omega)}. \]  

(4.9)

Here, \( C_2 = C_1(c_1, c_2, c_4). \)
Combining (4.7) as well as (4.9), we have

\[ \| \xi \|_{L^2(\Omega)}^2 + 2 \int_0^t \| \nabla \cdot \theta \|_{L^2(\Omega)}^2 \, dt + \| \theta \|_{L^2(\Omega)}^2 \leq \delta \| \xi \|_{L^2(\Omega)}^2 + \| \rho \|_{L^2(\Omega)}^2 + \varepsilon \| \xi \|_{L^2(\Omega)}^2 \]  

(4.10)

\[ + \int_0^t (\| \rho \|_{L^2(\Omega)}^2 + \| \rho \pi \|_{L^2(\Omega)}^2 + \| \delta \|_{L^2(\Omega)}^2 + \| \theta \|_{L^2(\Omega)}^2 + \| \eta \|_{L^2(\Omega)}^2 + \| \eta \|_{L^2(\Omega)}^2 + \| \varepsilon \|_{L^2(\Omega)}^2 + \| \varepsilon \|_{L^2(\Omega)}^2) \, dt. \]

Select \( v_h = \eta_t \) in (4.1)(a) to have

\[ (\eta_t, \eta_t) + (\nabla \cdot \theta, \eta_t) = -\left( \int_0^t k(t-s)(y(u) - y(u_h)) \cdot p, \eta_t \right) ds - \left( \int_0^t k(t-s)y(u_h) \cdot \xi, \eta_t \right) ds \]

(4.11)

Applying Young inequalities with suitable small \( \varepsilon \), we have

\[ (\eta_t, \eta_t) = |(\eta_t, \eta_t)| \leq \| \eta_t \|_{L^2(\Omega)}^2, \]

\[ (\nabla \cdot \theta, \eta_t) = |(\nabla \cdot \theta, \eta_t)| \leq c_8 \| \nabla \cdot \theta \|_{L^2(\Omega)}^2 + \varepsilon \| \eta_t \|_{L^2(\Omega)}^2, \]

\[ -\left( \int_0^t k(t-s)(y(u) - y(u_h)) \cdot p, \eta_t \right) ds = \left| \left( \int_0^t k(t-s)(y(u) - y(u_h)) \cdot p, \eta_t \right) ds \right| \leq c_4 \frac{t}{\varepsilon} \int_0^t (\| \eta \|_{L^2(\Omega)}^2 + \| \xi \|_{L^2(\Omega)}^2) ds \]

\[ -\left( \int_0^t k(t-s)y(u_h) \cdot \xi, \eta_t \right) ds = \left| \left( \int_0^t k(t-s)y(u_h) \cdot \xi, \eta_t \right) ds \right| \leq c_4 \frac{t}{\varepsilon} \int_0^t (\| \eta \|_{L^2(\Omega)}^2 + \| \xi \|_{L^2(\Omega)}^2) ds + \varepsilon \| \xi_t \|_{L^2(\Omega)}^2, \]

\[ -\left( \int_0^t k(t-s)(g(u) - g(u_h)), \xi_t \right) ds = \left| \left( \int_0^t k(t-s)(g(u) - g(u_h)), \xi_t \right) ds \right| \leq c_4 \frac{t}{\varepsilon} \int_0^t (\| \eta \|_{L^2(\Omega)}^2 + \| \xi \|_{L^2(\Omega)}^2) ds \]

thus (4.8) turns to

\[ \| \xi_t \|_{L^2(\Omega)}^2 + \| \nabla \cdot \theta \|_{L^2(\Omega)}^2 \leq C \left( \int_0^t (\| \eta \|_{L^2(\Omega)}^2 + \| \xi \|_{L^2(\Omega)}^2 + \| \xi \|_{L^2(\Omega)}^2) ds \right), \]

(4.12)

Since \( \| \xi \|_{L^2(\Omega)}^2 \leq \| \xi \|_{L^2(\Omega)}^2 \), \( \forall \ \xi \in L^p(\Omega) \), then

\[ \| \xi \|_{L^2(\Omega)}^2 \leq C \left( \int_0^t (\| \eta \|_{L^2(\Omega)}^2 + \| \xi \|_{L^2(\Omega)}^2 + \| \xi \|_{L^2(\Omega)}^2) ds \right), \]

(4.13)

we apply Gronwall inequalities, to get
\[ \parallel \xi \parallel_{L^2(\Omega)}^2 \leq C \left( \int_0^t \left( \parallel \eta \parallel_{L^2(\Omega)}^2 + \parallel \xi \parallel_{L^2(\Omega)}^2 \right) ds \right), \quad (4.14) \]

Now substitute (4.13) with (4.14) into (4.10), we get

\[ \parallel \xi \parallel_{L^2(\Omega)}^2 + 2 \int_0^t \parallel \nabla \cdot \theta \parallel_{L^2(\Omega)}^2 d\tau + \parallel \theta \parallel_{L^2(\Omega)}^2 \leq \parallel \delta \parallel_{L^2(\Omega)}^2 + \parallel \rho \parallel_{L^2(\Omega)}^2 + \epsilon \parallel \xi \parallel_{L^2(\Omega)}^2 \]

\[ + \int_0^t \left( \parallel \rho \parallel_{L^2(\Omega)}^2 + \parallel \rho_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \delta_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \eta \parallel_{L^2(\Omega)}^2 + \parallel \eta_{\tau} \parallel_{L^2(\Omega)}^2 \right) d\tau. \quad (4.15) \]

In addition, utilizing Gronwall inequalities, to have

\[ \parallel \xi \parallel_{L^2(\Omega)}^2 + 2 \int_0^t \parallel \nabla \cdot \theta \parallel_{L^2(\Omega)}^2 d\tau + \parallel \theta \parallel_{L^2(\Omega)}^2 \leq \parallel \delta \parallel_{L^2(\Omega)}^2 + \parallel \rho \parallel_{L^2(\Omega)}^2 + \int_0^t \left( \parallel \rho \parallel_{L^2(\Omega)}^2 + \parallel \rho_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \delta_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \eta \parallel_{L^2(\Omega)}^2 + \parallel \eta_{\tau} \parallel_{L^2(\Omega)}^2 \right) d\tau, \quad (4.16) \]

Then, according to the equation up, we obtain

\[ \parallel \xi \parallel_{L^2(\Omega)}^2 \leq \parallel \delta \parallel_{L^2(\Omega)}^2 + \parallel \rho \parallel_{L^2(\Omega)}^2 + \int_0^t \left( \parallel \rho \parallel_{L^2(\Omega)}^2 + \parallel \rho_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \delta_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \eta \parallel_{L^2(\Omega)}^2 + \parallel \eta_{\tau} \parallel_{L^2(\Omega)}^2 \right) d\tau, \quad (4.17) \]

or,

\[ \parallel \theta \parallel_{L^2(\Omega)}^2 \leq \parallel \delta \parallel_{L^2(\Omega)}^2 + \parallel \rho \parallel_{L^2(\Omega)}^2 + \int_0^t \left( \parallel \rho \parallel_{L^2(\Omega)}^2 + \parallel \rho_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \delta_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \eta \parallel_{L^2(\Omega)}^2 + \parallel \eta_{\tau} \parallel_{L^2(\Omega)}^2 \right) d\tau, \quad (4.18) \]

Substitute (4.17) into (4.14) we have

\[ \parallel \xi \parallel_{L^2(\Omega)}^2 \leq C \int_0^t \left( \parallel \eta \parallel_{L^2(\Omega)}^2 + \parallel \delta \parallel_{L^2(\Omega)}^2 + \parallel \rho \parallel_{L^2(\Omega)}^2 \right) ds \]

\[ + \int_0^t \left( \parallel \rho \parallel_{L^2(\Omega)}^2 + \parallel \rho_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \delta_{\tau} \parallel_{L^2(\Omega)}^2 + \parallel \eta \parallel_{L^2(\Omega)}^2 + \parallel \eta_{\tau} \parallel_{L^2(\Omega)}^2 \right) d\tau, \quad (4.19) \]

using (4.17),(4.18),(4.19),(3.3) and (3.4) as well as Lemma 3.3, we apply the triangle inequality to Perfect the proof of (4.3)

\[ \Box \]

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