AN AMBARZUMIAN-TYPE THEOREM ON GRAPHS WITH ODD CYCLES

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We consider an inverse problem for Schrödinger operators on a connected equilateral graph $G$ with standard matching conditions. The graph $G$ consists of at least two odd cycles glued together at a common vertex. We prove an Ambarzumian-type result, i.e., if a specific part of the spectrum is the same as in the case of zero potential, then the potential must be equal to zero.

1. Introduction

The addressed problem originates from a work by Ambarzumian [2] on reconstruction of a differential operator according to its eigenvalues. As another source of the problem, we can mention the so-called quantum graphs, i.e., differential operators on graphs [3, 21, 26]. For the classical theory of Sturm–Liouville equations, we refer the reader to [7, 17]; for special Ambarzumian-type inverse problems, see [10, 18]. Some previous results for graphs can be found in [6, 8, 12, 20, 22, 25]. Both in forward and in inverse problems on graphs, a usual ingredient is the evaluation of spectral determinants (or, alternatively, functional determinants or characteristic functions) [1, 5, 9, 11, 13–16, 19, 23, 24, 27]. For a more detailed discussion of these results, see the introduction in [20].

2. Results and Discussion

Let $r \geq 2$. Consider $r$ cycle graphs $C_1, C_2, \ldots, C_r$ with odd cycle lengths $n_1, n_2, \ldots, n_r$ ($n_j = 1$ is also possible). Let the vertices of $C_j$ be $v_{j0}, v_{jn_j} = v_{j0}$. We form the graph $G$ as the union of $C_j$’s by identifying the vertex $v_{j0}$ for all $j$. We say that $G$ is a graph consisting of $r \geq 2$ odd cycles glued together at a common vertex. The edge of $G$ between $v_{jk}$ and $v_{jk}$ is sometimes denoted by $e_{jk}$. However, if the specific location of the edges is not important, we denote them as follows: $e_1, e_2, \ldots, e_{|E|}$.

Choosing an arbitrary orientation, we parametrize each edge with $x \in [0, 1]$ and consider a Schrödinger operator with potential $q_j(x) \in L^1(0, 1)$ on the edge $e_j$ and with Neumann (or Kirchhoff) boundary conditions (sometimes called standard matching conditions), i.e., the solutions are required to be continuous at the vertices and, for the local coordinate pointing outward, the sum of derivatives is equal to zero. More formally, we consider the eigenvalue problem

$$-y'' + q_j(x)y = \lambda y$$

(2.1)
on $e_j$ for all $j$ with the following conditions:

$$y_j(\kappa_j) = y_k(\kappa_k)$$

(2.2)

if $e_j$ and $e_k$ are incident edges attached to a vertex $v$, where $\kappa = 0$ for the outgoing edges, $\kappa = 1$ for the incoming
edges (and can be both 0 or 1 for loops), and, at every vertex \( v \), we get
\[
\sum_{e_j \text{ leaves } v} y'_j(0) = \sum_{e_j \text{ enters } v} y'_j(1)
\] (2.3)
(the loops are counted on both sides).

**Theorem 2.1.** Consider the eigenvalue problem (2.1)–(2.3). Let \( G \) be a graph formed by \( r \geq 2 \) odd cycles glued together at a common vertex. If \( \lambda = 0 \) is the smallest eigenvalue and, for infinitely many \( k \in \mathbb{Z}^+ \), there are \( r - 1 \) eigenvalues (counting multiplicities) such that \( \lambda = (2k + 1)^2\pi^2 + o(1) \), then \( q = 0 \) a.e. on \( G \).

If all lengths of the odd cycles are equal to 1, i.e., all cycles are loops, then the statement of the theorem reduces to the statement of Theorem 2.1 in [28], which states the following:

Suppose that \( G \) is a flower-like graph, i.e., a single vertex with attached \( r \) loops of length 1. For \( k = 1, 2, \ldots \), let \( m_k \) be a sequence of integers with \( \lim m_k = +\infty \). If the eigenvalues are nonnegative and \( \lambda_k = (2m_k + 1)^2\pi^2 \) are eigenvalues with multiplicities \( r - 1 \), where \( m_k \) is a strictly ascending infinite sequence of positive integers, then \( q_j(x) = 0 \) a.e. on \([0, 1]\) for each \( j = 1, 2, \ldots, r \). It is necessary to require that \( r \geq 2 \) for the consequence to hold.

3. **Evaluation of the Spectral Determinant**

Denote by \( c_j(x, \lambda) \) the solution of (2.1) satisfying the conditions
\[
c_j(0, \lambda) - 1 = c'_j(0, \lambda) = 0.
\]
In addition, by \( s_j(x, \lambda) \) we denote the solution of (2.1) satisfying the conditions
\[
s_j(0, \lambda) = s'_j(0, \lambda) - 1 = 0.
\]
Each \( y_j(x, \lambda) \) can be represented in the form of a linear combination
\[
y_j(x, \lambda) = A_j(\lambda) c_j(x, \lambda) + \sqrt{\lambda} B_j(\lambda) s_j(x, \lambda).
\]
Thus, \( y_j(0, \lambda) = A_j(\lambda) \) are identical on each outgoing edge. Hence, as in [20], we index the functions \( A(\lambda) \) by vertices. This enables us to write
\[
y_j(x, \lambda) = A_v(x, \lambda) c_j(x, \lambda) + \sqrt{\lambda} B_j(\lambda) s_j(x, \lambda),
\]
if \( e_j \) starts from \( v \). If the eigenfunctions are normalized, i.e.,
\[
\sum_j \| y_j(x, \lambda) \|_2^2 = 1,
\]
then \( A_v(\lambda) = B_j(\lambda) = O(1) \) [8, 20, 28]. The coefficients \( A_v \) and \( B_j \) form a \( (|V| + |E|) \)-dimensional vector satisfying \( |V| \) Kirchhoff conditions at the vertices and \( |E| \) continuity conditions at the incoming ends of the edges,
namely, for all \( v \in V(G) \),

\[
\begin{align*}
\sum_{e_j : \ldots \rightarrow v} \frac{1}{\sqrt{\lambda}} A_{v_j}(\lambda)c_j'(1, \lambda) + B_j(\lambda)s_j'(1, \lambda) - \sum_{e_j : v \rightarrow \ldots} \frac{1}{\sqrt{\lambda}} B_j(\lambda) &= 0, \\
\sum_{e_j : \ldots \rightarrow v} \frac{1}{\sqrt{\lambda}} v_j'(1, \lambda) - \sum_{e_j : v \rightarrow \ldots} \frac{1}{\sqrt{\lambda}} v_j'(0, \lambda) &= 0,
\end{align*}
\]

where, in the first sum, \( v_j \) denotes the starting point of \( e_j \). Moreover, for all \( e_j \in E(G) \),

\[
A_u(\lambda)c_j(1, \lambda) + \sqrt{\lambda}B_j(\lambda)s_j(1, \lambda) - A_v(\lambda) = 0,
\]

if \( e_j \) points from \( u \) to \( v \) (see equations (2.3) and (2.4) in [20]).

The matrix \( M \) of this homogeneous linear system of equations has a special structure. For the sake of convenience of the reader, we repeat its description from [20]. Namely,

\[
M = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix},
\]

where

\[
A \text{ is a } |V| \times |V| \text{ matrix, } a_{u,v} = \frac{1}{\sqrt{\lambda}} \sum_j c_j'(1, \lambda); \text{ the sum is taken over the edges pointing from } u \text{ to } v; \text{ in the case of zero potential, } A \text{ is the } - \sin \sqrt{\lambda} \text{ times (transpose of the) directed adjacency matrix of } G;
\]

\( B \) and \( C \) are similar incidence matrices;

\[
b_{v,j} = \begin{cases} 
  s_j'(1, \lambda), & \text{if } e_j \text{ ends at } v, \\
  -1 & \text{if } e_j \text{ starts from } v, \\
  s_j'(1, \lambda) - 1, & \text{if } e_j \text{ is a loop at } v, \\
  0, & \text{otherwise},
\end{cases}
\]

and

\[
c_{j,v} = \begin{cases} 
  -1, & \text{if } e_j \text{ ends at } v, \\
  c_j(1, \lambda), & \text{if } e_j \text{ starts from } v, \\
  -1 + c_j(1, \lambda), & \text{if } e_j \text{ is a loop at } v, \\
  0, & \text{otherwise};
\end{cases}
\]

\( D \) is an \( |E| \times |E| \) diagonal matrix and \( d_{j,j} = \sqrt{\lambda}s_j(1, \lambda) \).

The determinant of the matrix \( M \) is the so-called spectral determinant of problem (2.1)–(2.3).
Example. Consider a flower-like graph, i.e., a single vertex with \( r \) loops. Then

\[
M = M_1 = \begin{bmatrix}
\frac{1}{\sqrt{\lambda}} \sum_{k=1}^{r} c_k'(1, \lambda) & s_1'(1, \lambda) - 1 & \ldots & s_r'(1, \lambda) - 1 \\
-1 + c_1(1, \lambda) & \sqrt{\lambda} s_1(1, \lambda) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-1 + c_r(1, \lambda) & 0 & \ldots & \sqrt{\lambda} s_r(1, \lambda)
\end{bmatrix}
\]

with the determinant

\[
\det M_1 = \lambda^{r-1} \left( \sum_{k=1}^{r} c_k'(1, \lambda) \prod_{j=1}^{r} s_j(1, \lambda) - \sum_{k=1}^{r} (s_k'(1, \lambda) - 1)(-1 + c_k(1, \lambda)) \prod_{j \neq k} s_j(1, \lambda) \right)
\]

corresponding to relation (2.9) in [28].

For \( \lambda = (2k + 1)^2 \pi^2 + d + o(1) \), the elements of \( M \) satisfies the following asymptotic relations independent of \( k \) (see [8], equation (2.3), or [22], Lemma 3.1):

\[
\frac{1}{\sqrt{\lambda}} c_j'(1, \lambda) = \frac{1}{2\sqrt{\lambda}} \left( d - \int_0^1 q_j \right) + o \left( \frac{1}{\sqrt{\lambda}} \right), \quad (3.1)
\]

\[
s_j'(1, \lambda) = -1 + o \left( \frac{1}{\sqrt{\lambda}} \right), \quad (3.2)
\]

\[
c_j(1, \lambda) = -1 + o \left( \frac{1}{\sqrt{\lambda}} \right), \quad (3.3)
\]

\[
\sqrt{\lambda} s_j(1, \lambda) = \frac{1}{2\sqrt{\lambda}} \left( \int_0^1 q_j - d \right) + o \left( \frac{1}{\sqrt{\lambda}} \right). \quad (3.4)
\]

Remark. By using these asymptotic relations, we get

\[
\det M_1 = -4 \sum_{k=1}^{r} \prod_{j \neq k} \sqrt{\lambda} s_j(1, \lambda) + o(\lambda^{-\frac{5}{2}}).
\]

This is a special case of relations (4.1) and (4.2) presented in what follows.

4. Proofs of the Main Result

Lemma 4.1. The total multiplicity of the eigenvalues \( \lambda = (2k + 1)^2 \pi^2 + O(1) \) is exactly equal to \( |E| - |V| \).

Proof. If \( q = 0 \), then \( \det M \) is a polynomial of \( \cos \sqrt{\lambda} \) and \( \sin \sqrt{\lambda} \). Hence, its zeros are \( 2\pi \)-periodic in \( \sqrt{\lambda} \); \( \lambda = (2k + 1)^2 \pi^2 + O(1) \) with periodicity implies that \( \sqrt{\lambda} = (2k + 1)\pi \) exactly. For \( \lambda = (2k + 1)^2 \pi^2 \),
\(A\) and \(D\) are null matrices. Thus, the rank of \(M\) is \(2|V|\) and its null space is exactly \((|E| - |V|)\)-dimensional. We now write, for the moment, \(\lambda = \lambda(q)\) to denote the dependence of eigenvalues on potential. If we arrange the eigenvalues in an increasing sequence \(\lambda_1, \ldots, \lambda_n, \ldots\), then there is a constant \(c\) depending only on the graph and the \(L^1\)-norm of the potential, which does not depend on \(n\), i.e., the particular index of the eigenvalue such that \(|\lambda_n(q) - \lambda_n(0)| \leq c\) for all \(n\) [20]. Hence, the total multiplicity of eigenvalues \(\lambda = (2k + 1)^2\pi^2 + O(1)\) is the same for all \(q \in L^1\).

**Lemma 4.2.** The determinant of \(M\) for \(\lambda = (2k + 1)^2\pi^2 + O(1)\) is \(O(\lambda^{- \frac{1}{2}(|E| - |V|)})\).

**Proof.** Each term in the Leibniz formula for the determinant must contain at least \((|E| - |V|)\) factors from \(A\) and \(D\) with a magnitude of \(O\left(\frac{1}{\sqrt{\lambda}}\right)\).

**Lemma 4.3.** Assume that a graph has the same number of edges as the vertices. Then the determinant of its (unoriented) incidence matrix is equal to zero, except the case where there is no even cycles in the graph and every component contains exactly one (odd) cycle. In this case, the determinant is equal to \(\pm 2^\kappa\), where \(\kappa\) denotes the number of components.

**Proof.** If the graph contains an even cycle, then the corresponding rows are dependent. If a component contains no cycles, then the corresponding columns are dependent. The number of cycles (including loops) is equal to the number of components. Hence, if a component contains more than one cycle or one loop, then there must be another component without cycles. In all indicated cases, the determinant of the incidence matrix is zero. Otherwise, it suffices to prove the statement for connected graphs, as the incidence matrix is a direct sum of that of \(G\), which have \(|E|\) vertices and \(|V|\) edges. Then the required statement follows from the previous lemma.

**Lemma 4.4.** If \(\lambda = (2k + 1)^2\pi^2 + O(1)\), then the determinant of a \(|V| \times |V|\) submatrix of \(C\) (and of \(B\)) is \(\pm 2^\kappa + o\left(\frac{1}{\sqrt{\lambda}}\right)\) if the indices of the rows in \(C\) (the columns in \(B\)) correspond to the edges of a subgraph, which has no even cycles and every component contains exactly one (odd) cycle. Otherwise, the determinant is \(o\left(\frac{1}{\sqrt{\lambda}}\right)\).

**Proof.** Omitting the \(o\left(\frac{1}{\sqrt{\lambda}}\right)\) terms in the submatrix, we make only an \(o\left(\frac{1}{\sqrt{\lambda}}\right)\) error in its determinant. What we get is the negative of an incidence matrix of a subgraph with \(|V|\) vertices and \(|V|\) edges. Then the required statement follows from the previous lemma.

**Theorem 4.1.** If \(\lambda = (2k + 1)^2\pi^2 + O(1)\), then

\[
\det M = (-1)^{|V|} \sum_\tau 4^\kappa(\tau) \prod_{e_j \notin \tau} \sqrt{\lambda}s_j(1, \lambda) + O(\lambda^{- \frac{|E| - |V| + 1}{2}}),
\]

where the sum is taken over the subgraphs \(\tau\) of \(G\), which have \(|V|\) vertices, and their incidence matrix is nonsingular (i.e., \(\tau\) has no even cycles but has \(\kappa\) components each of which contains exactly one (odd) cycle).

**Proof.** The main terms in the Leibniz formula for the determinant are the terms containing exactly \((|E| - |V|)\) elements from \(D\). The product of a fixed set of \((|E| - |V|)\) elements in \(D\) is weighted by the determinant of the respective minor, with all other elements of \(D\) substituted by zeros. The remaining rows in \(C\) and columns in \(B\)
look like a disordered incidence matrix of the graph \( \tau \) spanned by the remaining \(|V|\) edges. Thus, the determinant of the minor is \((-1)^{|V|}\) times the squared determinant of the incidence matrix of \( \tau \).

**Corollary 4.1.** If \( \lambda = (2k + 1)^2 \pi^2 + O(1) \) and the graph \( G \) consists of \( r \) odd cycles of length \( n_1, \ldots, n_r \), glued together at a common vertex, then

\[
\det M = -4 \sum_{i=1}^{r} \prod_{j \neq i} n_j \sqrt{\lambda} s(1, \lambda, q_{ji}) + O(\lambda^{-\frac{r}{2}}),
\]

where \( q_{ji} \) is the potential on the \( l \)th edge of the \( j \)th cycle.

**Proof.** The incidence matrix of a subgraph of \( G \) is nonsingular if and only if we omit one edge in all but one cycles. Note also that \(|V|\) is odd and \(|E| - |V| = r - 1\). Thus, the statement follows from (4.1).

Substituting asymptotics (3.1)–(3.4), we get the following corollary:

**Corollary 4.2.** If \( \lambda = (2k + 1)^2 \pi^2 + d + o(1) \) and the graph \( G \) consists of \( r \) odd cycles of length \( n_1, \ldots, n_r \), glued together at a common vertex, then

\[
\det M = -4 \left( \frac{-1}{2 \sqrt{\lambda}} \right)^{r-1} p(d) + o(\lambda^{-\frac{r-1}{2}}),
\]

where

\[
p(d) = \sum_{i=1}^{r} \prod_{j \neq i} \left( n_j d - \sum_{l=1}^{n_j} \int_{0}^{1} q_{jl} \right).
\]

**Lemma 4.5.** Under the assumptions of Theorem 2.1,

\[
p(d) = \sum_{i=1}^{r} \prod_{j \neq i} n_j d^{r-1}.
\]

**Proof.** \( \lambda \) is an eigenvalue of the eigenvalue problem (2.1)–(2.3) if and only if \( \det M(\lambda) = 0 \). We denote the distinct roots of \( p(d) \) by \( d_1, \ldots, d_l \). By the previous corollary, for \( \lambda = (2k + 1)^2 \pi^2 + O(1) \), the distinct roots of \( \det M(\lambda) \) are exactly of the form \( \lambda = (2k + 1)^2 \pi^2 + d_j + o(1), 1 \leq j \leq l \). By Lemma 4.1, the total multiplicity of these eigenvalues is equal to \(|E| - |V|\). In Theorem 2.1, it is assumed that there is the same number of eigenvalues such that \( \lambda = (2k + 1)^2 \pi^2 + o(1) \). Hence, \( d_j = 0 \), for all \( j \) and, thus, \( p(d) \) is a constant multiple of \( d^{r-1} \). The principal coefficient is given by (4.3).

**Proof of Theorem 2.1.** We introduce

\[
Q_j = \sum_{l=1}^{n_j} \int_{0}^{1} q_{jl}.
\]

For fixed \( m \), substituting \( d = \frac{Q_m}{n_m} \) in (4.3), we get

\[
\sum_{i=1}^{r} \frac{1}{n_i} \left( \frac{Q_m}{n_m} \right)^{r-1} = \frac{1}{n_m} p\left( \frac{Q_m}{n_m} \right) = \frac{1}{n_m} \prod_{j=1}^{r} \left( \frac{Q_m}{n_m} - \frac{Q_j}{n_j} \right).
\]
Introducing \( h_j = \frac{Q_j}{n_j} \), we find

\[
\sum_{i=1}^{r} \frac{1}{n_i} h_m^{r-1} = \frac{1}{n_m} \prod_{j \neq m} (h_m - h_j), \quad m = 1, 2, \ldots, r.
\]

We can assume that \( h_1 \geq h_2 \geq \ldots \geq h_r \). Thus, for \( m = 2 \), the left-hand side is nonpositive. Therefore, \( h_2 \leq 0 \).

Similarly, \( h_{r-1} \geq 0 \).

Hence, for \( m = 1 \), we get

\[
\frac{1}{n_1} (h_1 - h_r) = \sum_{i=1}^{r} \frac{1}{n_i} h_i^{r-1}.
\]

If \( h_r = 0 \), then \( h_1 = 0 \) as \( n_j \)'s are positive. Similarly, \( h_1 = 0 \) implies that \( h_r = 0 \). If neither of these is zero, then we get

\[
\frac{1}{n_1} (h_1 - h_r) = \sum_{i=2}^{r} \frac{1}{n_i} (h_1 - h_r) = 0.
\]

As \( n_j \)'s are positive, if \( r > 2 \), then \( h_1 = h_2 = \ldots = h_r = 0 \), while for \( r = 2 \), we can write \( h_1 n_1 + h_2 n_2 = 0 \). In both cases,

\[
\sum_j Q_j = \sum_{e_{jl} \in G} \int_0^1 q_{jl} = \int_G q = 0.
\]

We denote the operator of the eigenvalue problem (2.1)–(2.3) by \( L \). Thus,

\[
\langle \varphi, L \varphi \rangle \geq \lambda_0 = 0
\]

and the equality holds if and only if \( \varphi \) is an eigenfunction of \( L \). It is clear that the constant 1 must be an eigenfunction corresponding to the eigenvalue 0. Substituting this in (2.1), we find \( q(x) = 0 \).

Theorem 2.1 is proved.

5. Glossary

A walk \( W \) in a graph is an alternating sequence of vertices and edges, say, \( X_0, e_1, \ldots, e_l, X_l \) where \( e_i = X_{i-1}X_i, \; 0 < i < l \). The length of \( W \) is \( l \). This walk \( W \) is called a trail if all its edges are distinct. A path is a walk with distinct vertices. A trail whose end vertices coincide (closed trail) is called a circuit. To be precise, a circuit is a closed trail without distinguished end vertices and direction such that, e.g., two triangles sharing a single vertex give rise to exactly two circuits with six edges. If a walk \( W = X_0, e_1, \ldots, e_l, X_l \) is such that \( l \geq 3 \), \( X_0 = X_l \), and the vertices \( X_i, 0 < i < l \), differ from each other and from \( X_0 \), then \( W \) is said to be a cycle [4, p. 5].
The incidence matrix of a graph has a row for each vertex and a column for each edge. It is defined as follows:

\[ R = (r_{ij}), \quad r_{ij} = \begin{cases} 
0, & \text{if } e_j \text{ is not incident to } v_i, \\
1, & \text{if } e_j \text{ is not a loop and incident to } v_i, \\
2, & \text{if } e_j \text{ is a loop at } v_i.
\]

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