Quiver Schur algebras for linear quivers

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Abstract
We define a graded quasi-hereditary covering of the cyclotomic quiver Hecke algebras \( R^\Lambda_n \) of type \( A \) when \( e = 0 \) (the linear quiver) or \( e > n \). We prove that these algebras are quasi-hereditary graded cellular algebras by giving explicit homogeneous bases for them. When \( e = 0 \), we show that the Khovanov–Lauda–Rouquier grading on the quiver Hecke algebras is compatible with the Koszul grading on the blocks of parabolic category \( O^\Lambda \) given by Backelin, building on the work of Beilinson, Ginzburg and Soergel. As a consequence, \( e = 0 \) our cyclotomic quiver Schur algebras are Koszul over fields of characteristic zero. Finally, we give an Lascoux–Leclerc–Thibon-like algorithm for computing the graded decomposition numbers of the cyclotomic quiver Schur algebras in characteristic zero.

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1. Introduction
Khovanov and Lauda [37, 38] and Rouquier [53] have introduced a remarkable family of \( \mathbb{Z} \)-graded algebras that are now known to categorify the canonical bases of Kac–Moody algebras [15, 20, 56]. Brundan and Kleshchev [14] initiated the study of ‘cyclotomic’ quotients of these algebras by showing that they are isomorphic to the degenerate and non-degenerate cyclotomic Hecke algebras of type \( G(\ell, 1; n) \); see also [53].

This paper defines and studies certain graded quasi-hereditary covers \( S^\Lambda_n \) of the cyclotomic quiver Hecke algebras of the linear quiver. These algebras are graded analogues of the cyclotomic Schur algebras \( S_{\Lambda, n}^{DJM} \) of type \( G(\ell, 1; n) \) [13, 24]. This paper studies the cyclotomic quiver Schur algebras for the linear quiver and ‘large’ cyclic quiver. Let \( \mathcal{P}^\Lambda_n \) be the poset of multipartitions of \( n \) ordered by dominance. The first main result of this paper is the following.

**Theorem A.** Suppose that \( e = 0 \) or \( e > n \) and let \( Z = K \) be an arbitrary field. The algebra \( S^\Lambda_n \) is a quasi-hereditary graded cellular algebra with graded standard modules \( \{ \Delta^\mu \mid \mu \in \mathcal{P}^\Lambda_n \} \) and irreducible modules \( \{ L^\mu \mid \mu \in \mathcal{P}^\Lambda_n \} \). Moreover, there is an equivalence of (ungraded)
highest weight categories

\[ F^\Lambda_n : S^\Lambda_n \text{-Mod} \longrightarrow S^{DJM}_n \text{-Mod} \]

that preserves the labelling of the standard modules and simple modules.

In fact, the quiver Schur algebra \( S^\Lambda_n \) is defined over an arbitrary integral domain.

Like the cyclotomic Schur algebras of \([24]\), the quiver Schur algebra \( S^\Lambda_n \) is defined to be the endomorphism algebra of a direct sum of ‘graded permutation modules’; see Definition 4.15. Incorporating the grading into this picture is surprisingly difficult, not least because many of the structural results in this paper fail when \( 1 < e \leq n \). After we have defined the quiver Schur algebras, and shown that they are quasi-hereditary (Theorem 4.24), most of the work in the first six sections of the paper is geared towards proving the non-trivial result that the cyclotomic quiver Schur algebras are Morita equivalent, as ungraded algebras, to the cyclotomic \( q \)-Schur algebras (Theorem 6.13). Constructing a graded analogue of the Schur functor (Proposition 4.30) is also not completely straightforward.

If \( e = 0 \) and we work over the field of complex numbers, then Brundan and Kleshchev have shown that the degenerate cyclotomic Schur algebras are Morita equivalent to a sum of certain integral blocks \( O^\Lambda_\beta \) of parabolic category \( O^\Lambda \) for the Lie algebra of the general linear group \([13]\). By results of Backelin \([8]\), and Beilinson, Ginzburg and Soergel \([9]\), the blocks of parabolic category \( O \) can be endowed with a Koszul grading. By \([13]\), the endomorphism algebra of a prinjective generator of the sum of these blocks is Morita equivalent to the degenerate cyclotomic Hecke algebra \( H^\Lambda_n \) of type \( A \). Therefore, the Koszul grading on the blocks \( O^\Lambda_\beta \) induce a grading on the module category of \( H^\Lambda_n \). This gives two ostensibly different gradings on the degenerate cyclotomic Hecke algebra \( H^\Lambda_n \): one coming from parabolic category \( O^\Lambda \) and the Khovanov–Lauda–Rouquier (KLR) grading given by the Brundan–Kleshchev isomorphism \( H^\Lambda_n \cong R^\Lambda_n \) when \( e = 0 \).

**Theorem B.** Suppose that \( e = 0 \) and \( Z = \mathbb{C} \) is the field of complex numbers. Then category \( O^\Lambda \) and the quiver Hecke algebra \( R^\Lambda_n \) induce graded Morita equivalent gradings on \( H^\Lambda_n \text{-Mod} \).

To prove Theorem B, we first show that the cyclotomic quiver Schur algebra \( S^\Lambda_\beta \) is graded Morita equivalent to the (non-isomorphic) quiver Schur algebras recently constructed by Stroppel and Webster \([55]\). This allows us to show that the prinjective modules of \( S^\Lambda_\beta \) are rigidly graded and using this we can construct an explicit isomorphism between the basic algebras of \( O^\Lambda \) and \( S^\Lambda_\beta \).

Building on Theorem B, in §7.3 we prove a graded analogue of \([13, \text{Theorem C}]\), thus lifting Brundan and Kleshchev’s ‘higher Schur–Weyl duality’ to the graded setting.

**Theorem C.** Suppose that \( e = 0 \), \( \beta \in \mathbb{Q}^+ \) and \( Z = \mathbb{C} \). Then there are graded Schur functors \( F^O_\beta : O^\Lambda_\beta \text{-Mod} \longrightarrow R^\Lambda_\beta \text{-Mod} \) and \( F^S_\beta : S^\Lambda_\beta \text{-Mod} \longrightarrow R^\Lambda_\beta \text{-Mod} \) and a graded equivalence \( E^O_\beta : O^\Lambda_\beta \longrightarrow S^\Lambda_\beta \text{-Mod} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
O^\Lambda_\beta & \xrightarrow{E^O_\beta} & S^\Lambda_\beta \text{-Mod} \\
\downarrow F^O_\beta & & \downarrow F^S_\beta \\
R^\Lambda_\beta \text{-Mod} & & \\
\end{array}
\]

In particular, \( S^\Lambda_\beta \text{-Mod} \) is Koszul.
This result can be interpreted as saying that the KLR grading on \( \mathcal{H}_n^A \) induces the Koszul grading on parabolic category \( O^A \). As a consequence, via our graded Schur algebra, we obtain a very explicit and new combinatorial description of the grading on parabolic category \( O^A \).

Our description of the grading on parabolic category \( O^A \) gives new information. For example, we use it in §7.5 to give a fast algorithm for computing the graded decomposition numbers of \( S^\Lambda_n \) and parabolic category \( O^A \), which are certain parabolic Kazhdan–Lusztig polynomials, that is similar in spirit to the Lascoux–Leclerc–Thibon (LLT) algorithm for the Hecke algebras of type A [40]. What is really interesting about our ‘LLT algorithm’ is that it computes the graded decomposition numbers of the quiver Schur algebras when \( e = 0 \). In contrast, the extension of the LLT algorithm to the \( q \)-Schur algebras [42] is non-trivial because it requires first computing the action of the bar involution on the Fock space.

If \( \Lambda \) is a dominant weight of level 2, then we show in Corollary B.6 that \( S^\Lambda_n \) is isomorphic, as a graded algebra, to the corresponding quasi-hereditary cover of the Khovanov’s diagram algebra as introduced by Brundan and Stroppel [19]. This is quite surprising because the definition of these two algebras is very different. The key is to show that if \( \Lambda \) is a weight of level 2, then \( S^\Lambda_n \) is a positively graded basic algebra (Theorem B.3). This positivity result, and the uniqueness of Koszul gradings and Theorem C, provides the bridge to Brundan and Stroppel’s algebra.

### Index of notation

- \( A,M \) A graded algebra or module
- \( A_n \) An ungraded algebra or module
- \( D^\mu \) Simple \( R^\Lambda_n \)-module
- \( d(t), d'(t) \) Permutations: \( t = t^\mu d(t) = t^\mu d'(t) \)
- \( d^\mu(q) \) \([\Delta^\mu : L^\nu]_q \) for \( S^\Lambda_n \)
- \( d^\mu(q) \) \([\Delta^\mu : L^\nu]_q \) for \( S^\Omega_n \)
- \( \deg t \) Tableau degree
- \( \text{cog}d \) Tableau codegree
- \( \text{def } \beta \) Defect of \( \beta \in Q^+ \)
- \( \text{dim}_M \) Graded dimension of \( M \)
- \( \Delta^\Lambda, \nabla^\Lambda \) Weyl and costandard modules
- \( \Delta^\Lambda, \nabla^\Lambda \) Sign dual (co)standard modules
- \( e^\mu, e_\mu \) KLR idempotents \( e(i^\mu), e(1^\mu) \)
- \( e_\mu(q) \) Inverse decomposition number
- \( E^\mu \) Graded exterior powers
- \( F^\Lambda_n, F^\Omega_n \) Graded Schur functors
- \( G^\mu, G_\mu \) Graded permutation modules
- \( G^\Lambda_n \) \( \bigoplus_{\mu \in P^+} G^\mu \)
- \( \text{End}_A \) Degree preserving endomorphisms
- \( \text{End}_A \) All \( A \)-module endomorphisms
- \( \mathcal{F}^\Lambda \) Combinatorial Fock space
- \( \mathcal{H}_n^A, \mathcal{H}_n^\Lambda \) Cyclotomic Hecke algebras
- \( \text{Hom}_A \) Degree preserving maps in \( A \)-Mod
- \( \text{Hom}_A \) All \( A \)-module homomorphisms
- \( i^\beta, i_\beta \) \( \{ i \in I^\Lambda \mid \sum_{r=1}^t \alpha_{i_r} = \beta \} \)
- \( \kappa \) Multicharge determining \( \Lambda = \Lambda(\kappa) \)
- \( \kappa^\Lambda \) Restricted multipartitions for \( R^\Lambda_n \)
- \( \Lambda \) Dominant weight determined by \( \kappa \)
- \( \Lambda^\mu \) Simple \( S^\Lambda_n \)-module
- \( \mu^\lambda \) Conjugate multipartition
- \( O^\beta \) Parabolic category \( O \)
- \( P^+ \) Positive weight lattice
- \( P^\mu \) Projective cover of \( L^\nu \)
- \( P^\Lambda_n \) Kazhdan–Lusztig polynomial
- \( P^\Lambda_n \) Multipartitions of \( n \)
- \( P^\Lambda_n \) \( \{ \mu \in P^\Lambda_n \mid i^\mu \in I^\beta \} \)
- \( p_{\mu \rho}(q) \) Inverse graded decomposition numbers
- \( p_\rho^\theta(q) \) Inverse graded decomposition numbers
- \( \psi_{\mu \lambda}^{\nu \kappa} \) Basis elements of \( S^\Lambda_n \)
- \( \psi_{\mu \lambda}^{\nu \kappa} \) Basis elements of \( R^\Lambda_n \)
- \( \psi_{\mu \lambda}^{\nu \kappa} \) Identity map on \( G^\mu \)
- \( Q^+ \) Positive root lattice
- \( Q_n^+ \) \( \{ \beta \in Q^+ \mid P^\beta_n \neq 0 \} \)
- \( \text{res} \) Residue sequence of tableau
- \( \mathcal{R}_n^\Lambda, \mathcal{R}_n^\Omega \) Cyclotomic quiver Hecke algebra
- \( \mathcal{R}_n^\Lambda, \mathcal{R}_n^\Omega \) A block of \( \mathcal{R}_n^\Lambda \)
- \( \text{sgn} \) Sign isomorphism
- \( S^\Lambda_n, S^\Omega_n \) Cyclotomic quiver Schur algebra
- \( S^\Lambda_n, S^\Omega_n \) Schur algebra of \( O^\Lambda_n \)
- \( S^\Lambda_n, S^\Omega_n \) A block of \( S^\Lambda_n \)
- \( S^\Lambda_n, S^\Omega_n \) Sign-dual quiver Schur algebra
- \( S_n^\Lambda, S_n^\Omega \) Graded (dual) Specht modules
- \( \text{Std}(P^\Lambda_n) \) Standard tableaux
- \( \text{Std}(P^\Lambda_n) \) \( \{ t \mid t \geq t^\mu \text{ and } \text{res}(t) = i^\mu \} \)
- \( \text{Std}(P^\Lambda_n) \) \( \{ t \mid t \geq t^\mu \text{ and } \text{res}(t) = i^\mu \} \)
- \( T^\Lambda \) \( \{ (\mu, s) \mid s \in \text{Std}^\mu(\lambda) \} \)
- \( T^\Lambda \) \( \{ (\mu, s) \mid s \in \text{Std}^\mu(\lambda) \} \)
- \( T^\mu \) Tilting module
- \( \tau_3 \) Trace form on \( \mathcal{R}_n^\Lambda \)
- \( t^\mu, t^\mu_\mu \) Initial and final \( \mu \)-tableaux
- \( \omega(T^\Lambda) \) The weight of \( T^\Lambda \)
- \( y^\mu, y^\mu_\mu \) \( \psi_{\mu \lambda}^{\nu \kappa} = e^\mu y^\mu, \psi_{\mu \lambda}^{\nu \kappa} = e^\mu y^\mu_\mu \)
- \( Y^\mu, Y^\mu_\mu \) Young modules
- \( Z \) Commutative ring
- \( Z^\mu \) Graded symmetric power
- \( \triangleright, \blacktriangleright \) Domination orderings
- \( [M:L^\nu]_q \) Graded decomposition number
- \( [N:D^\mu]_q \) Graded decomposition number
- \( \text{Hom}_A(?, Z) \)-dual
- \( \text{Hom}_A(?, A) \)-dual
2. Graded representation theory

In this section, we set our notation and give the reader some quick reminders about graded modules and graded algebras, by which we will always mean \( \mathbb{Z} \)-graded modules and \( \mathbb{Z} \)-graded algebras. Expert readers may wish to skip this section.

2.1. Graded modules and algebras

Throughout this paper, \( \mathcal{Z} \) will be an integral domain. Unless otherwise stated, all modules and algebras will be free and of finite rank over their base ring. We also assume that if \( A \) is a \( K \)-algebra, then \( A \) is split over \( K \). As we work mainly with (graded) cellular algebras, which are always split, there is no loss of generality in assuming this.

In this paper, a graded \( \mathcal{Z} \)-module is a \( \mathcal{Z} \)-graded \( \mathcal{Z} \)-module \( M \). That is, as \( \mathcal{Z} \)-module, \( M \) has a direct sum decomposition

\[
M = \bigoplus_{d \in \mathcal{Z}} M_d.
\]

By assumption, \( M \) is \( \mathcal{Z} \)-free and of finite rank so \( M_d \neq 0 \) for only finitely many \( d \). If \( M \) is a graded \( \mathcal{Z} \)-module, let \( M \) be the ungraded \( \mathcal{Z} \)-module obtained by forgetting the grading on \( M \).

If \( m \in M_d \), for \( d \in \mathcal{Z} \), then \( m \) is homogeneous of degree \( d \) and we set \( \deg m = d \). If \( M \) is a graded \( \mathcal{Z} \)-module and \( s \in \mathcal{Z} \), let \( M(s) \) be the graded \( \mathcal{Z} \)-module obtained by shifting the grading on \( M \) up by \( s \). That is, \( M(s)_d = M_{d-s} \), for \( d \in \mathcal{Z} \). Let \( q \) be an indeterminate. If \( \mathcal{Z} = K \) is a field, then graded dimension of \( M \) is the Laurent polynomial

\[
\dim_q M = \sum_{d \in \mathcal{Z}} (\dim_K M_d) q^d \in \mathbb{N}[q,q^{-1}].
\]  

(2.1)

In particular, \( \dim_K M = (\dim_q M)|_{q=1} \).

If \( M \) is a graded module, and if \( f(q) = \sum_{d \in \mathcal{Z}} f_d q^d \in \mathbb{N}[q,q^{-1}] \) is a Laurent polynomial with non-negative coefficients \( \{f_d\}_{d \in \mathcal{Z}} \), then define

\[
f(q)M = \bigoplus_{d \in \mathcal{Z}} M(d)^{\oplus f_d}.
\]

Then \( f(q)M \) is again free and of finite rank and \( \dim_q(f(q)M) = f(q) \dim_q M \).

A graded \( \mathcal{Z} \)-algebra is a unital associative \( \mathcal{Z} \)-algebra \( A = \bigoplus_{d \in \mathcal{Z}} A_d \) that is a graded \( \mathcal{Z} \)-module such that \( A_d A_e \subseteq A_{d+e} \), for all \( d, e \in \mathcal{Z} \). It follows that \( 1 \in A_0 \) and that \( A_0 \) is a graded subalgebra of \( A \). A graded (right) \( A \)-module is a graded \( \mathcal{Z} \)-module \( M \) such that \( M_d \) is an \( A \)-module and \( M_d A_e \subseteq M_{d+e} \), for all \( d, e \in \mathcal{Z} \). Graded submodules, graded left \( A \)-modules and so on are all defined in the obvious way.

Let \( A \)-\text{Mod} be the category of finitely generated graded \( A \)-modules with degree preserving maps. Then

\[
\text{Hom}_A(M,N) = \{ f \in \text{Hom}_A(M,N) \mid f(M_d) \subseteq N_d \text{ for all } d \in \mathcal{Z} \},
\]

for all \( M, N \in A \)-\text{Mod}. The elements of \( \text{Hom}_A(M,N) \) are homogeneous maps of degree 0. More generally, for each \( d \in \mathcal{Z} \) set

\[
\text{Hom}_A(M,N)_d = \text{Hom}_A(M,N(-d)) \cong \text{Hom}_A(M(d),N).
\]

Thus, \( \text{Hom}_A(M,N) = \text{Hom}_A(M,N)_0 \). If \( f \in \text{Hom}_A(M,N)_d \), then \( f \) is homogeneous of degree \( d \) and we set \( \deg f = d \). Define

\[
\text{Hom}_A(M,N) = \bigoplus_{d \in \mathcal{Z}} \text{Hom}_A(M,N)_d = \bigoplus_{d \in \mathcal{Z}} \text{Hom}_A(M,N(-d)).
\]

Any map is a sum of its homogeneous components so \( \text{Hom}_A(M,N) \cong \text{Hom}_A(M,N) \) as a \( \mathcal{Z} \)-module. Set \( \text{End}_A(M) = \text{Hom}_A(M,M) \) and \( \text{End}_A(M) = \text{Hom}_A(M,M) \).
If \( r \geq 0 \) and \( M \) and \( N \) are graded \( A \)-modules, let \( \text{Ext}_A^r(M,N) \) be the space of \( r \)-fold extensions of \( M \) by \( N \) in the category \( A\text{-Mod} \) of (graded) \( A \)-modules. We emphasize that \( \text{Hom}_A \) and \( \text{Ext}_A \) are the spaces of homomorphisms and extensions in the category \( A\text{-Mod} \) of finitely generated (graded) \( A \)-modules. These should not be confused with \( \text{Hom}_A \) and \( \text{Ext}_A \) in the (ungraded) category \( A\text{-Mod} \).

Now suppose that \( A \) comes equipped with a homogeneous anti-isomorphism \( * \). Then the **graded dual** of the graded \( A \)-module \( M \) is the graded \( A \)-module

\[
M^\oplus = \bigoplus_{d \in \mathbb{Z}} \text{Hom}_Z(M(d), Z),
\]

where \( Z \) is concentrated in degree zero and where the action of \( A \) on \( M^\oplus \) is given by \((fa)(m) = f(am^*)\) for all \( f \in \mathcal{M}^\oplus \), \( a \in A \) and \( m \in M \). The module \( M \) is **self-dual** if \( M \cong M^\oplus \) as graded \( A \)-modules. If \( \mathcal{Z} = K \) is a field, then, as a vector space, \( M_d^\oplus = \text{Hom}_Z(M_{-d}, K) \), so that \( \dim_q M^\oplus = \dim_q M \), where the bar involution \(-: \mathbb{Z}[q, q^{-1}] \rightarrow \mathbb{Z}[q, q^{-1}]\) is the \( \mathbb{Z} \)-linear map determined by \( q^k \mapsto q^{-k} \), for all \( k \in \mathbb{Z} \).

If \( m \) is an \( A \)-module, then a **graded lift** of \( m \) is an \( A \)-module \( M \) such that \( M \cong \mathcal{M} \) as \( A \)-modules. In general, there is no guarantee that an \( A \)-module will have a graded lift but if \( A \) is a graded Artin algebra (for example, a finite-dimensional algebra over a field), it is easy to see that if a finitely generated indecomposable \( A \)-module has a graded lift, then this lift is unique up to isomorphism and grading shift; see, for example, [9, Lemma 2.5.3]. In this case, the irreducible and projective indecomposable \( A \)-modules always have graded lifts; see [28].

Suppose that \( M \) is a graded \( A \)-module and that \( X = \{ X^\mu \mid \mu \in \mathcal{P} \} \) is a collection of \( A \)-modules such that \( \{ X^\mu \mid \mu \in \mathcal{P} \} \) are pairwise non-isomorphic \( A \)-modules. Then \( M \) has a \( X \)-module filtration if there exists a filtration

\[
M = M_0 \supset M_1 \supset \cdots \supset M_s = 0
\]

such that there exist \( \mu_r \in \mathcal{P} \) and \( d_r \in \mathbb{Z} \) with \( M_r/M_{r+1} \cong X^{\mu_r}(d_r), \) for \( 0 \leq r < s \). The **graded multiplicity** of \( X^\mu \) in \( M \) is the Laurent polynomial

\[
(M : X^\mu)_q = \sum_{0 \leq r < s, \mu_r = \mu} q^{d_r} \in \mathbb{N}[q, q^{-1}].
\]

In general, this multiplicity will depend upon the choice of filtration but for many modules, such as irreducible modules and Weyl modules, the Laurent polynomial \( (M : X^\mu)_q \) will be independent of this choice.

2.2. **Cellular algebras**

Many of the algebras considered in this paper are (graded) cellular algebras, so we quickly recall the definition and some of the important properties of these algebras. Cellular algebras were defined by Graham and Lehrer [30] with their natural extension to the graded setting given in [32].

**Definition 2.4** (Graded cellular algebra [30, 32]). Suppose that \( A \) is a \( \mathbb{Z} \)-graded \( \mathcal{Z} \)-algebra that is free of finite rank over \( \mathcal{Z} \). A **graded cell datum** for \( A \) is an ordered quadruple \((\mathcal{P}, T, B, \text{deg})\), where \((\mathcal{P}, \triangleright)\) is the weight poset, \( T(\lambda) \) is a finite set for \( \lambda \in \mathcal{P} \), and

\[
B : \prod_{\lambda \in \mathcal{P}} T(\lambda) \times T(\lambda) \longrightarrow A; (s, t) \mapsto b_{st}, \quad \text{and} \quad \text{deg} : \prod_{\lambda \in \mathcal{P}} T(\lambda) \longrightarrow \mathbb{Z}
\]

are two functions such that \( B \) is injective and

\begin{align*}
\text{(GC}_1\text{)} & \quad \text{If } \lambda \in \mathcal{P} \text{ and } s, t \in T(\lambda), \text{ then } b_{st} \text{ is homogeneous of degree } \text{deg } b_{st} = \text{deg } s + \text{deg } t. \\
\text{(GC}_2\text{)} & \quad \{ b_{st} \mid s, t \in T(\lambda) \text{ for } \lambda \in \mathcal{P} \} \text{ is a } \mathcal{Z} \text{-basis of } A.
\end{align*}
(GCₐ) If $s, t \in T(\lambda)$, for some $\lambda \in \mathcal{P}$, and $a \in A$, then there exist scalars $r_{tv}(a)$, which do not depend on $s$, such that

$$b_{st}a = \sum_{v \in T(\lambda)} r_{tv}(a)b_{sv} \pmod{A^{\lambda}}$$

where $A^{\lambda}$ is the $Z$-submodule of $A$ spanned by $\{ b_{ab}^\mu \mid \mu \triangleright \lambda$ and $a, b \in T(\mu) \}$.

(GC₃) The $Z$-linear map $* : A \rightarrow A$ determined by $(b_{st})^* = b_{ts}$, for all $\lambda \in \mathcal{P}$ and all $s, t \in T(\lambda)$, is a homogeneous anti-isomorphism of $A$.

A graded cellular algebra is a graded algebra that has a graded cell datum. The basis $\{b_{st}\}$ is a graded cellular basis of $A$.

If we omit the degree assumption (GCₐ), then we recover Graham and Lehrer’s [30] definition of an (ungraded) cellular algebra.

Fix a graded cellular algebra $A$ with graded cellular basis $\{b_{st}\}$. If $\lambda \in \mathcal{P}$, then the graded cell module is the $Z$-module $\Delta^{\lambda}$ with basis $\{ b_t \mid t \in T(\lambda) \}$ and with $A$-action

$$b_t a = \sum_{v \in T(\lambda)} r_{tv}(a)b_v,$$

where the scalars $r_{tv}(a) \in Z$ are the same scalars appearing in (GCₐ). One of the key properties of the graded cell modules is that by [32, Lemma 2.7] they come equipped with a homogeneous bilinear form $\langle \cdot , \cdot \rangle$ of degree zero that is determined by the equation

$$\langle b_t, b_u \rangle b_v = b_t b_{uv},$$

for $s, t, u, v \in T(\lambda)$. The radical of this form,

$$\text{rad} \Delta^{\lambda} = \{ x \in \Delta^{\lambda} \mid \langle x, y \rangle = 0 \text{ for all } y \in \Delta^{\lambda} \},$$

is a graded $A$-submodule of $\Delta^{\lambda}$ so that $L^{\lambda} = \Delta^{\lambda}/\text{rad} \Delta^{\lambda}$ is a graded $A$-module.

As in §2.1, if $M$ is an $A$-module, let $M^\oplus$ be the (graded) dual of $M$.

**Theorem 2.5** [32, Theorem 2.10]. Suppose that $Z$ is a field and that $A$ is a graded cellular algebra. Then the following properties hold

(a) If $L^{\lambda} \neq 0$, for $\lambda \in \mathcal{P}$, then $L^{\lambda}$ is an absolutely irreducible graded $A$-module and $(L^{\lambda})^\oplus \cong L^{\lambda}$.

(b) $\{ L^{\lambda}(k) \mid \lambda \in \mathcal{P}, L^{\lambda} \neq 0 \text{ and } k \in \mathbb{Z} \}$ is a complete set of pairwise non-isomorphic irreducible (graded) $A$-modules.

Suppose that $Z = K$ is a field and let $M$ be a (graded) $A$-module and $L^\mu$ be a graded simple $A$-module, for $\mu \in \mathcal{P}$. Define

$$[M : L^\mu]_q = \sum_{d \in \mathbb{Z}} [M : L^\mu(d)]q^d$$

(2.6)

to be the graded multiplicity of $L^\mu$ in $M$. By the Jordan–Hölder theorem, $[M : L^\mu]_q$ depends only on $M$ and $L^\mu$ and not on the choice of composition series for $M$. Moreover, $[M : L^\mu]_q \in \mathbb{N}[q, q^{-1}]$ and $[M : L^\mu]_{q=1} = [M : L^\mu]$ is the usual decomposition multiplicity of $L^\mu$ in $M$.

**Corollary 2.7** [32, Lemma 2.13]. Suppose that $Z$ is a field and that $\lambda, \mu \in \Lambda$ with $L^\mu \neq 0$. Then $[\Delta^\mu : L^\mu]_q = 1$ and $[\Delta^\lambda : L^\mu]_q \neq 0$ only if $\lambda \triangleright \mu$.

Let $\mathcal{P}_0 = \{ \mu \in \mathcal{P} \mid L^\mu \neq 0 \}$. Then $D_A(q) = ([\Delta^\lambda : L^\mu]_q)_{\lambda, \mu \in \mathcal{P}_0}$ is the decomposition matrix of $A$. For each $\mu \in \mathcal{P}_0$, let $P^\mu$ be the projective cover of $L^\mu$ in $A$-Mod. Then $C_A(q) = ([P^\lambda : L^\mu]_q)_{\lambda, \mu \in \mathcal{P}_0}$ is the Cartan matrix of $A$. 

If $m = (m_{ij})$ is a matrix let $m^{tr} = (m_{ji})$ be its transpose. We will need the following fact.

**Corollary 2.8** (Brauer–Humphreys reciprocity [32, Theorem 2.17]). Suppose that $Z = K$ is a field, $\lambda \in \mathcal{P}$ and $\mu \in \mathcal{P}_0$. Then $P^\mu$ has a cell filtration in which $\Delta^\lambda$ appears with graded multiplicity $(P^\mu : \Delta^\lambda)_q = [\Delta^\lambda : L^\rho]_q$. Consequently, $C_A(q) = D_A(q)^qD_A(q)$ is a symmetric matrix.

Finally, we note the following criterion for a cellular algebra to be quasi-hereditary. In particular, this implies that $A$-Mod is a highest weight category. The definitions of these objects can be found, for example, in [25, Appendix]. Alternatively, the reader can take the following result to be the definition of a (graded split) quasi-hereditary algebra (with a graded duality).

**Corollary 2.9** [30, Remark 3.10]. Suppose that $A$ is a graded cellular algebra. Then $A$ is a split quasi-hereditary algebra, with standard modules $\{ \Delta^\mu \mid \mu \in \mathcal{P} \}$, if and only if $L^\mu \neq 0$ for all $\mu \in \mathcal{P}$.

### 2.3. Basic algebras and graded Morita equivalences

Let $Z = K$ be a field. Recall that a finite-dimensional ungraded split $K$-algebra $B_0$ is a basic algebra if every irreducible $B_0$-module is one-dimensional. It is well known that, up to isomorphism, every finite-dimensional (ungraded) split $K$-algebra $B$ is Morita equivalent to a unique basic algebra $B_0$. In fact, if $\{P_1, \ldots, P_r\}$ is a complete set of pairwise non-isomorphic projective indecomposable $B$-modules, then the basic algebra of $B$ is isomorphic to $\text{End}_B(P_1 \oplus \cdots \oplus P_r)^{op}$. These facts can be found, for example, in [10, §2.2].

We need analogues of these results for graded categories.

A graded category is any category whose objects are finite-dimensional $Z$-graded $K$-modules and whose morphisms are homogeneous maps of degree zero, where $K$ is a field. If $C$ is a graded category, let $\sigma_C$ be the shift functor that sends a module $M \in C$ to $M(1)$. If $D$ is another graded category then a graded functor $F : C \to D$ is any functor such that $F \circ \sigma_C = \sigma_D \circ F$. Similarly, a graded equivalence is an equivalence given by a graded functor.

Let $A$ and $B$ be two finite-dimensional $Z$-graded $K$-algebras. Following [28, §5], the $K$-algebras $A$ and $B$ are graded Morita equivalent if there is a graded equivalence of graded module categories $A$-Mod $\cong$ $B$-Mod. Equivalently, by the results of [28, §5], $A$ and $B$ are graded Morita equivalent if and only if there is an (ungraded) Morita equivalence $E : A$-Mod $\cong$ $B$-Mod and a graded functor $G : A$-Mod $\to$ $B$-Mod such that the following diagram commutes:

$$
\begin{array}{ccc}
A \text{-Mod} & \xrightarrow{G} & B \text{-Mod} \\
\text{Forget} \downarrow & & \text{Forget} \downarrow \\
A \text{-Mod} & \xrightarrow{E} & B \text{-Mod}
\end{array}
$$

where the vertical functors are the natural forgetful functors. Let $\{P_1, \ldots, P_r\}$ be a complete set of pairwise non-isomorphic graded projective indecomposable $A$-modules such that $P_i \not\cong P_j(k)$ for any $i \neq j$ and $k \in Z$. The graded basic algebra of $A$-Mod is the endomorphism algebra

$$A_b = \text{End}_A(P_1 \oplus \cdots \oplus P_r)^{op}.$$  

By construction, $A_b$ is naturally $Z$-graded and, on forgetting the grading, $A_b$ is the basic algebra of $A$. Unlike the ungraded case, two graded Morita equivalent graded basic algebras need not be isomorphic as graded algebras because, for example, we can change the degree shifts on $P_1, \ldots, P_r$. This is discussed in more detail after [28, Corollary 5.10].
2.4. Schur functors

Several places in this paper rely on Auslander’s theory of ‘Schur functors’, or quotient functors. We briefly recall how this works in the graded setting following [12, §3.1].

Let $A$ be a finite-dimensional graded algebra (with 1) that is split over a field $K$ and let $A\text{-Mod}$ be the category of finite-dimensional graded right $A$-modules. Suppose that $e \in A$ is a non-zero homogeneous idempotent and consider the subalgebra $eAe$ of $A$. Then $eAe$ is a graded algebra with identity element $e$. (In all of our applications, $A$ will be a quasi-hereditary graded cellular algebra.)

Define functors $F: A\text{-Mod} \to eAe\text{-Mod}$ and $G: eAe\text{-Mod} \to A\text{-Mod}$ by

$$F(M) = Me \cong \text{Hom}_A(eA, M) \quad \text{and} \quad G(N) = N \otimes_{eAe} eA,$$

(2.10)

for $M \in A\text{-Mod}$ and $N \in eAe\text{-Mod}$, together with the obvious action on morphisms. Both of these functors are graded because they respect the gradings on both categories. In general, however, these functors will not be equivalences between the (graded) module categories of $A$ and $eAe$.

To define a graded equivalence between $eAe\text{-Mod}$ and a subcategory of $A\text{-Mod}$, we need to work a little harder. Suppose that $M$ is a graded $A$-module and define $O_e(M)$ to be the largest graded submodule $M'$ of $M$ such that $F(M') = 0$ and define $O^e(M)$ to be the smallest graded submodule $M''$ of $M$ such that $F(M/M'') = 0$. Any $A$-module (degree preserving) homomorphism $M \to N$ sends $O_e(M)$ to $O_e(N)$ and $O^e(M)$ to $O^e(N)$, so $O_e$ and $O^e$ define graded functors on the category of $A$-modules.

Let $\mathcal{A}_e$ be the full subcategory of $A\text{-Mod}$ with objects all graded $A$-modules $M$ such that $O_e(M) = 0$ and $O^e(M) = M$. It is easy to check that any $A$-module homomorphism $M \to N$ induces a well-defined map $M/O_e(M) \to N/O_e(N)$ so that there is an exact graded functor

$$H: A\text{-Mod} \to \mathcal{A}_e; M \mapsto M/O_e(M).$$

By [12, Lemma 3.1a], the functors $H \circ G \circ F$ and $F \circ H \circ G$ are isomorphic to the identity functors on $\mathcal{A}_e$ and on $eAe\text{-Mod}$, respectively. This implies the following.

**Theorem 2.11** [12, Theorem 3.1d]. The restrictions of the functors $F$ and $H \circ G$ induce mutually inverse graded equivalences of categories between $\mathcal{A}_e$ and $eAe\text{-Mod}$.

In [12], this result is proved only for ungraded algebras, however, the proof there generalizes without change to graded module categories.

2.5. Koszul algebras

In this section, we recall the definition of Koszul algebras and the properties of these algebras that we will need. Throughout this section, we work over a field $K$.

Let $A = \bigoplus_{d \in \mathbb{Z}} A_d$ be a finite-dimensional graded $K$-algebra. Then $A$ is **positively graded** if $A_d = 0$ whenever $d < 0$. That is, all of the homogeneous elements of $A$ have **non-negative** degree.

Suppose that $M$ is a (graded) $A$-module. A **linear projective resolution** of $M$ is a (graded) projective resolution of $M$ of the form

$$\cdots \to P^2 \to P^1 \to P^0 \to M \to 0$$

such that $P^d = P^d_A$ is generated by its elements of degree $d$, for $d \geq 0$. Dually, a **linear injective coresolution** of $M$ is a (graded) injective coresolution with the $d$th term cogenerated in degree $-d$, for $d \geq 0$. 
DEFINITION 2.12 [9, Definition 1.2.1]. A Koszul algebra is a positively graded algebra $A = \bigoplus_{d \geq 0} A_d$ such that $A_0$ is semisimple and $A_0$ has a linear projective resolution.

More generally, if $A$ is a graded algebra, then $A$-Mod is Koszul if it is graded Morita equivalent to the module category of a Koszul algebra. By definition, if the category $A$-Mod is Koszul, then $A$ is not necessarily a Koszul algebra. For example, by Theorem C, in characteristic zero the category $\mathcal{S}_n$-Mod is Koszul when $e = 0$, however, the graded Schur algebra $\mathcal{S}_n^\ell$ is not usually positively graded so it is not a Koszul algebra in general.

Koszul algebras play a key role in Section 7 where we prove Theorem C. For us, one of the most important properties of a Koszul algebra is that their grading determines the radical and socle filtrations of certain modules. To make this statement precise, let $A$ be a Koszul algebra and suppose that $M$ is a finite-dimensional $A$-module. By definition, if the category $A$-Mod is Koszul, then $S_0$ is Koszul, then the category $A$-Mod is Koszul when $e = 0$, however, the graded Schur algebra $\mathcal{S}_n^\ell$ is not usually positively graded so it is not a Koszul algebra in general.

Definition 2.12 of a Koszul algebra is that $\text{rad} M$ and $\text{soc} M$ are irreducible, then the radical, socle and grading filtrations of $M$ coincide up to a constant shift. That is, there exists $d \in \mathbb{Z}$ such that $\text{rad}^k M = \mathcal{G}_{r}^k M$, for all $k$.

(a) If $M/\text{rad} M$ is irreducible, then the radical filtration of $M$, up to constant shift. That is, there exists $d \in \mathbb{Z}$ such that $\text{rad}^k M = \mathcal{G}_{r}^k M$, for all $k$.

(b) If $\text{soc} M$ and $M/\text{rad} M$ are irreducible and $M/\text{rad} M$ concentrated in degree 0, then $M$ is rigidly graded.

Proof. By [9, Corollary 2.3.3], any Koszul algebra $A$ is a quadratic algebra. That is, $A$ is generated by $A_0$ and $A_1$ subject only to relations in degree 2. Therefore, part (a) is a special case of [9, Proposition 2.4.1]. In turn, part (a) implies that if both $M/\text{rad} M$ and $\text{soc} M$ are irreducible, then the radical, socle and grading filtrations of $M$ coincide up to a constant shift. In particular, $M$ is rigid. Finally, if $M/\text{rad} M$ is concentrated in degree zero, then the radical and grading filtrations of $M$ coincide exactly, so $M$ is rigidly graded. □

If $A$ is a Koszul algebra, then the Koszul dual of $A$ is the algebra

$$E(A) := \text{Ext}^\bullet_A(A_0, A_0),$$

(2.14)

which we consider as a positively graded algebra under Yoneda product. By [9, Theorem 1.2.5], if $A$ is Koszul, then $E(A)$ is a Koszul and, moreover, $E(E(A)) \cong A$, whenever $A_d$ is a finitely generated as a left $A_0$-module, for $d \geq 0$. We use Koszul duality implicitly in Theorem 7.6 to define the Koszul algebras that we use to prove Theorem C.
All of the Koszul algebras that we consider in this paper will be quasi-hereditary, where there is a strengthening of these ideas. Let $A$ be a positively graded (split) quasi-hereditary basic algebra such that $A_0$ is semisimple and all of its standard modules have linear projective resolutions and all of its costandard modules have linear injective coresolutions. Then $A$ is standard Koszul in the sense of [1]. By [1, Theorem 1], any standard Koszul algebra is Koszul. (The paper [1] defines standard Koszul algebras in terms of top projective resolutions, however, under our assumptions top projective resolutions coincide with linear projective resolutions by the remarks in [1, §3].) Moreover, by [1, Theorem 3] and the remarks above, if $A$ is standard Koszul, then so is $E(A)$. Further, if the simple $A$-modules are indexed by the poset $(X, \geq)$ then the simple $E(A)$-modules are indexed by the opposite poset $(X, \leq)$.

Suppose that $A$ is a (finite-dimensional) standard Koszul quasi-hereditary basic $K$-algebra with graded simple modules $\{L^\mu \mid \mu \in X\}$, concentrated in degree zero, and graded standard modules $\{\Delta^\lambda \mid \lambda \in X\}$ such that the natural map $\Delta^\lambda \rightarrow L^\lambda$ is homogeneous of degree zero, for $\lambda \in X$. Let $P^\mu$ be the projective cover of $L^\mu$ and let $t_\mu$ be the degree zero homogeneous idempotent such that $P^\mu = t_\mu A$, for $\mu \in X$. Define two matrices $D_A(q) = ([\Delta^\lambda : L^\mu]_{\lambda, \mu \in X}$ and $C_A(q) = (\dim_{q t_\lambda A t_\mu})_{\lambda, \mu \in X}$, where the rows and columns of these matrices are ordered in a way that is compatible with the partial order on $X$. Then $D_A(q)$ is the decomposition matrix of $A$ and $C_A(q)$ its Cartan matrix.

Let $\{L^\mu \mid \mu \in X\}$ and $\{\Delta^\lambda \mid \lambda \in X\}$ be the graded simple and standard modules of $E(A)$. Then we have matrices $D_{E(A)}(q)$ and $C_{E(A)}(q)$ as above.

We have not found the next result in the literature, even though we think it is well known.

**Lemma 2.15.** Suppose that $A$ is a standard Koszul quasi-hereditary basic algebra over a field $K$. Then $D_A(q)^{-1} = D_{E(A)}(-q)^{tr}$.

**Proof.** Without loss of generality, we may assume that $A$ and $E(A)$ are both basic algebras. By standard arguments, $C_A(q) = D_A(q)^{tr}D_A(q)$ and $C_{E(A)}(q) = D_{E(A)}(q)^{tr}D_{E(A)}(q)$; compare with Corollary 2.8. On the other hand, since $A$ and $E(A)$ are basic algebras, the matrices $C_A(q)$ and $C_{E(A)}(q)$ coincide with the Hilbert polynomials of $A$ and $E(A)$, respectively, as defined in [9, §2.11]. By the numerical condition for Koszulity given in [9, Lemma 2.11.1], $C_A(q)C_{E(A)}(-q)^{tr} = I_X$, where $I_X$ is the $|X| \times |X|$ identity matrix. Expanding,

$$D_A(q)^{tr}D_A(q)D_{E(A)}(-q)^{tr}D_{E(A)}(-q) = I_X.$$

Hence, $D_A(q)D_{E(A)}(-q)^{tr} = (D_{E(A)}(-q)D_A(q)^{tr})^{-1}$. As noted above, the quasi-hereditary structures on $A$ and $E(A)$ are governed by opposite posets, so the matrix $D_A(q)D_{E(A)}(-q)^{tr}$ is upper unitriangular, whereas the matrix $(D_{E(A)}(-q)D_A(q)^{tr})^{-1}$ is lower unitriangular. Therefore,

$$D_A(q)D_{E(A)}(-q)^{tr} = I_X = (D_{E(A)}(-q)D_A(q)^{tr})^{-1}.$$

Hence, $D_{E(A)}(-q)^{tr} = D_A(q)^{-1}$ as claimed. □

3. **Cyclotomic Quiver Hecke algebras and combinatorics**

In this section, we recall the facts about the cyclotomic quiver Hecke algebras of type $A$ and the cyclotomic Hecke algebras of type $G(\ell, 1, n)$ that are needed in this paper.

3.1. **Cyclotomic quiver Hecke algebras**

Khovanov and Lauda [37, 38] and Rouquier [53] introduced (cyclotomic) quiver Hecke algebras for arbitrary oriented quivers. In this paper, we consider mainly the linear quiver of type $A_\infty$. 
Fix a non-negative integer $n$ and an integer $e \in \{0, 2, 3, 4 \ldots \}$. Let $\Gamma_e$ be the quiver with vertex set $I = \mathbb{Z}/e\mathbb{Z}$ and edges $i \rightarrow i + 1$, for all $i \in I$. Following [35, Chapter 1], attach to $\Gamma_e$ the standard Lie theoretic data of a Cartan matrix $(a_{ij})_{i,j \in I}$, simple roots $\{ \alpha_i \mid i \in I \}$, fundamental weights $\{ \Lambda_i \mid i \in I \}$, the positive weight lattice $P^+ = \bigoplus_{i \in I} \mathbb{N} \Lambda_i$ and the positive root lattice $Q^+ = \bigoplus_{i \in I} \mathbb{N} \alpha_i$. Let $(\cdot, \cdot)$ be the usual invariant form, normalized so that

\[(\alpha_i, \alpha_j) = a_{ij} \quad \text{and} \quad (\Lambda_i, \alpha_j) = \delta_{ij}, \quad \text{for} \quad i, j \in I.\]

All of the bases for the modules and algebras in this paper depend implicitly on the choice of $\kappa$ even though the algebras themselves depend only on $\Lambda$.

Let $\mathfrak{S}_n$ be the symmetric group on $n$ letters and let $s_r = (r, r + 1)$, for $1 \leq r < n$. Then $\{s_1, s_2, \ldots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathfrak{S}_n$. The group $\mathfrak{S}_n$ acts from the left on $I^n$ by place permutations. More explicitly, if $1 \leq r < n$ and $i = (i_1, \ldots, i_n) \in I^n$, then $s_r \cdot i = (i_1, \ldots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, \ldots, i_n) \in I^n$.

Fix $\beta \in Q^+$ with $\sum_{i \in I}(\Lambda_i, \beta) = n$ and let

\[I^\beta = \{ i \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \beta \}.\]

Then $I^\beta$ is a $\mathfrak{S}_n$-orbit of $I^n$ and every $\mathfrak{S}_n$-orbit can be written uniquely in this way for some $\beta \in Q^+$.

**Definition 3.2.** Suppose that $n \geq 0$, $e \in \{0, 2, 3, 4 \ldots \}$ and $\beta \in Q^+_n$. Define $\mathcal{R}_\beta$ to be the unital associative $\mathbb{Z}$-algebra with generators

\[
\{\psi_1, \ldots, \psi_{n-1}\} \cup \{y_1, \ldots, y_n\} \cup \{e(i) \mid i \in I^\beta\}
\]

and relations

\[
\begin{align*}
y_1(e(i)) &= 0, & e(i)e(j) &= \delta_{ij}e(i), & \sum_{i \in I^\beta} e(i) &= 1, \\
y_r e(i) &= e(i)y_r, & \psi_r e(i) &= e(s_r \cdot i)e_r, & y_r y_s &= y_s y_r, \\
y_r y_{r+1} e(i) &= (y_r \psi_r + \delta_{r,s} \psi_i) e(i), & y_{r+1} \psi_r e(i) &= (\psi_r y_r + \delta_{r,s} \psi_i) e(i), \\
\psi_r y_s &= y_s \psi_r & \text{if } s \neq r, r + 1, \\
\psi_r \psi_s &= y_s \psi_r & \text{if } |r - s| > 1,
\end{align*}
\]

for $i, j \in I^\beta$ and all admissible $r$ and $s$. 

\[
\begin{align*}
\psi_r \psi_{r+1} \psi_r e(i) &= \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\
(y_r + 1 - y_r)e(i) & \text{if } i_r \rightarrow i_{r+1}, \\
(y_r - y_{r+1})e(i) & \text{if } i_r \leftarrow i_{r+1}, \\
(y_r - y_r)(y_r - y_{r+1})e(i) & \text{if } i_r \Rightarrow i_{r+1} \\
e(i) & \text{otherwise}, \\
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\psi_r \psi_{r+1} \psi_r e(i) &= \begin{cases} \psi_r + 1 + 1)e(i) & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\
\psi_r + 1 + 1)e(i) & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\
\psi_r + 1 + 1)e(i) & \text{if } i_r = i_{r+2} \Rightarrow i_{r+1} \\
\psi_r + 1 + 1)e(i) & \text{otherwise}, \\
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\psi_r \psi_{r+1} \psi_r e(i) &= \begin{cases} \psi_r + 1 + 1)e(i) & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\
\psi_r + 1 + 1)e(i) & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\
\psi_r + 1 + 1)e(i) & \text{if } i_r = i_{r+2} \Rightarrow i_{r+1} \\
\psi_r + 1 + 1)e(i) & \text{otherwise}. \\
\end{cases}
\end{align*}
\]

for all admissible $r$ and $s$. 

\[
\]
The cyclotomic quiver Hecke algebra, or cyclotomic Khovanov–Lauda–Rouquier algebra, of weight $\Lambda$ and type $\Gamma_e$ is the algebra $\mathcal{R}_n^\Lambda = \bigoplus_{\beta \in Q^+} \mathcal{R}_\beta^\Lambda$. The algebras $\mathcal{R}_n^\Lambda = \mathcal{R}_n^\Lambda (\mathcal{Z})$, and $\mathcal{R}_\beta^\Lambda = \mathcal{R}_\beta^\Lambda (\mathcal{Z})$ for $\beta \in Q^+$ are $\mathbb{Z}$-graded with degree function determined by

$$\deg_e(i) = 0, \quad \deg_y = 2 \quad \text{and} \quad \deg_{\psi_e} i = -a_{i, i+1},$$

for $1 \leq r \leq n$, $1 \leq s < n$ and $i \in I^n$.

Inspecting the relations in Definition 3.2, there is a unique anti-isomorphism $\ast$ of $\mathcal{R}_n^\Lambda$ that fixes each of the generators of $\mathcal{R}_n^\Lambda$. Thus $\ast$ is homogeneous of order 2. Hence, by twisting with $\ast$ we can define the graded dual $M^\ast$ of an $\mathcal{R}_n^\Lambda$-module $M^\oplus = \text{Hom}_\mathcal{Z} (M, \mathcal{Z})$ as in (2.2).

In this paper, we will mainly be concerned with the special cases when either $e = 0$ or $e > n$. The presentation of $\mathcal{R}_n^\Lambda$ depends on the orientation of $\Gamma_e$, however, different orientations of $\Gamma_e$ yield isomorphic algebras; see, for example, [53, Proposition 3.12].

### 3.2. Cyclotomic Hecke algebras

Recall that $\Lambda \in P^+$ and that we have fixed an integer $e \in \{0, 2, 3, 4, \ldots \}$. We now define the ‘integral’ cyclotomic Hecke algebras $\mathcal{H}_n^\Lambda$ of type $G(\ell, 1, n)$, where $\ell = \sum_{i \in I} (\Lambda, \alpha_i)$ is the level of $\Lambda$.

Fix an integral domain $\mathcal{Z}$ that contains an element $\xi = \xi (e)$ such that one of the following holds:

(i) $e > 0$ and $\xi$ is a primitive $e$th root of unity in $\mathcal{Z}$;
(ii) $e = 0$ and $\xi$ is not a root of unity;
(iii) $\xi = 1$ and $e$ is the characteristic of $\mathcal{Z}$.

Define $\delta_{\xi 1} = 1$ if $\xi = 1$ and $\delta_{\xi 1} = 0$ otherwise. For $k \in \mathbb{Z}$ set

$$\xi^{(k)} = \begin{cases} \xi^k & \text{if } \xi \neq 1, \\ k & \text{if } \xi = 1. \end{cases} \quad (3.5)$$

The definition of $\xi = \xi (e)$ above ensures that $\xi^{(i)} = \xi^{(i+e)}$. Hence, $\xi^{(i)}$ is well-defined for all $i \in I = \mathbb{Z}/e\mathbb{Z}$.

**Definition 3.6.** The (integral) cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda (\mathcal{Z}, \xi)$ of type $G(\ell, 1, n)$ is the unital associative $\mathcal{Z}$-algebra with generators $L_1, \ldots, L_n, T_1, \ldots, T_{n-1}$ and relations

$$\prod_{i \in I} (L_1 - \xi^{(i)} (\Lambda, \alpha_i) ) = 0, \quad L_r L_t = L_t L_r,$$

$$(T_r + 1) (T_r - \xi) = 0, \quad T_r L_t + \delta_{\xi 1} = L_{r+1} (T_r - \xi + 1),$$

$$T_r T_{s+1} T_s = T_{s+1} T_r T_{s+1},$$

$$T_r L_t = L_t T_r \quad \text{if } t \neq r, r + 1,$$

$$T_r T_s = T_s T_r \quad \text{if } |r - s| > 1,$$

where $1 \leq r < n$, $1 \leq s < n - 1$ and $1 \leq t \leq n$.

It is well known that $\mathcal{H}_n^\Lambda$ decomposes into a direct sum of simultaneous generalized eigenspaces for the elements $L_1, \ldots, L_n$ (cf. [31]). Moreover, the possible eigenvalues for $L_1, \ldots, L_n$ belong to the set $\{ \xi^{(i)} \mid i \in I \}$. Hence, the generalized eigenspaces for these elements are indexed by $I^n$. For each $i \in I^n$, let $e (i)$ be the corresponding idempotent in $\mathcal{H}_n^\Lambda$ (or zero if the corresponding eigenspace is zero).
THEOREM 3.7 (Brundan–Kleshchev [14, Theorem 1.1]). Suppose that \( Z = K \) is a field, \( \xi \in K \) as above, and that \( \Lambda = \Lambda(\kappa) \). Then there is an isomorphism of algebras \( R_n^\Lambda \cong H_n^\Lambda \) that sends \( e(i) \mapsto e(i) \), for all \( i \in I^n \) and

\[
y_r \mapsto \begin{cases} 
\sum_{i \in I^n} (1 - \xi^{-1} L_r) e(i) & \text{if } \xi \neq 1, \\
\sum_{i \in I^n} (L_r - i_r) e(i) & \text{if } \xi = 1.
\end{cases}
\]

\[
\psi_s \mapsto \sum_{i \in I^n} (T_s + P_s(i)) Q_s(i)^{-1} e(i),
\]

where \( P_s(i), Q_s(i) \in \mathbb{Z}[y_s, y_{s+1}] \), for \( 1 \leq r \leq n \) and \( 1 \leq s < n \).

By [14, Theorem 1.1], the inverse isomorphism \( H_n^\Lambda \cong R_n^\Lambda \) is determined by

\[
L_r \mapsto \begin{cases} 
\sum_{i \in I^n} \xi^{i_r} (1 - y_r) e(i) & \text{if } \xi \neq 1, \\
\sum_{i \in I^n} (i_r + y_r) e(i) & \text{if } \xi = 1,
\end{cases}
\]

\[
T_s \mapsto \sum_{i \in I^n} (\psi_s Q_s(i) - P_s(i)) e(i),
\]

for \( 1 \leq r \leq n \) and \( 1 \leq s < n \).

Henceforth, we identify the algebras \( R_n^\Lambda \) and \( H_n^\Lambda \) under this isomorphism. In particular, we will not distinguish between the homogeneous generators of \( R_n^\Lambda \) and their images in \( H_n^\Lambda \) under the isomorphism of Theorem 3.7.

Even though we will not distinguish between \( R_n^\Lambda \) and \( H_n^\Lambda \) we will usually write \( R_n^\Lambda \) when we are working with graded representations and \( H_n^\Lambda \) for ungraded representations.

3.3. Tableau combinatorics

This section sets up the tableau combinatorics that will be used throughout this paper. Recall that a partition of \( m \) is a weakly decreasing sequence \( \mu = (\mu_1 \geq \mu_2 \geq \cdots) \) of non-negative integers that sum to \( m \). Set \( |\mu| = m \).

A multipartition of \( n \) is an \( \ell \)-tuple \( \mu = (|\mu^{(1)}| \cdots |\mu^{(\ell)}|) \) of partitions such that \( |\mu^{(1)}| + \cdots + |\mu^{(\ell)}| = n \). We identify a multipartition with its diagram

\[
\mu = \{ (r, c, l) \mid 1 \leq c \leq \mu^{(l)}_r \text{ for } r \geq 1 \text{ and } 1 \leq l \leq \ell \},
\]

which we think of as an \( \ell \)-tuple of boxes in the plane. For example,

\[
(3, 2|2, 1^2|3, 1) = \left( \begin{array}{ccc}
\square & \square & \square \\
\square & \square & \square \\
\square & \square & \square
\end{array} \right).
\]

The partitions \( \mu^{(1)}, \ldots, \mu^{(\ell)} \) are the components of \( \mu \) and we identify \( \mu^{(l)} \) with the subdiagram

\[
\{ (r, c, l) \mid 1 \leq c \leq \mu^{(l)}_r \text{ for } r \geq 1 \}
\]

of \( \mu \). A node is any triple \( A = (r, c, l) \in \mathbb{N}^2 \times \{1, 2, \ldots, \ell\} \). In particular, the elements of (the diagram of) \( \mu \) are nodes.
Let $P_n^\Lambda$ be the set of multipartitions of $n$. Then $P_n^\Lambda$ is a poset under the dominance order $\succeq$, where $\lambda \succeq \mu$, for multipartitions $\lambda$ and $\mu$ of $n$, if
\[
\sum_{k=1}^{l-1} |\lambda^{(k)}| + \sum_{j=1}^{i} \lambda_j^{(l)} \geq \sum_{k=1}^{l-1} |\mu^{(k)}| + \sum_{j=1}^{i} \mu_j^{(l)},
\]
for $1 \leq l \leq \ell$ and $i \geq 1$. If $\lambda \succeq \mu$ and $\lambda \neq \mu$, then we write $\lambda \rhd \mu$.

Suppose that $\mu \in P_n^\Lambda$ is a multipartition of $n$. A $\mu$-tableau is a bijection $t: \mu \rightarrow \{1, 2, \ldots, n\}$. We think of a $\mu$-tableau $t = (t^{(1)}, \ldots, t^{(\ell)})$ as a labelling of (the diagram of) $\mu$, where $t^{(r)}$ is the restriction of $t$ to $\mu^{(r)}$. In this way, we talk of the rows, columns and components of a tableau $t$. For example,

\[
\begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 & 12 \\
13 & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
5 & 8 \\
6 & 7 \\
1 & 3 & 4 \\
10 & 12 \\
\end{array}
\]

are two $(3, 2|2, 1^2|3, 1)$-tableaux. If $t = (t^{(1)}, \ldots, t^{(\ell)})$ is a $\mu$-tableau, then define $\text{Shape}(t) = \mu$, so that $\text{Shape}(t^{(r)}) = \mu^{(r)}$, for $1 \leq r \leq \ell$. If $t^{-1}(k) = (r, c, l)$, then we set $\text{comp}_l(k) = l$.

A $\mu$-tableau $t$ is standard if its entries increase along the rows and down the columns of each component. For example, the two tableaux above are standard. If $t$ is a standard tableau, let $t_{\downarrow k}$ be the subtableau of $t$ that contains $1, 2, \ldots, k$. Then a tableau $t$ is standard if and only if $\text{Shape}(t_{\downarrow k})$ is a multipartition for $1 \leq k \leq n$. The dominance order induces a partial order on the set of tableaux where $s \succeq t$ if
\[
\text{Shape}(s_{\downarrow k}) \succeq \text{Shape}(t_{\downarrow k}) \quad \text{for } 1 \leq k \leq n,
\]
for $s \in \text{Std}(\lambda)$ and $t \in \text{Std}(\mu)$, where $\lambda, \mu \in P_n^\Lambda$. Again we write $s \succeq t$ if $s \succeq t$ and $s \neq t$. Let $\text{Std}(\mu)$ be the poset of standard $\mu$-tableau and set $\text{Std}^2(\mu) = \text{Std}(\mu) \times \text{Std}(\mu)$, $\text{Std}(P_n^\Lambda) = \bigcup_{\mu \in P_n^\Lambda} \text{Std}(\mu)$ and $\text{Std}^2(P_n^\Lambda) = \bigcup_{\mu \in P_n^\Lambda} \text{Std}^2(\mu)$.

We extend the dominance order to $\text{Std}^2(P_n^\Lambda)$ by declaring that $(s, t) \rhd (u, v)$ if $s \succeq u$ and $t \succeq v$. We write $(s, t) \rightarrow (u, v)$ if $(s, t) \rhd (u, v)$ and $(s, t) \neq (u, v)$.

If $\mu \in P_n^\Lambda$, let $\mu' = (\mu^{(1)}', \ldots, \mu^{(n)}') \in P_n^\Lambda$ be the conjugate multipartition, which is obtained from $\mu$ by reversing the order of its components and then swapping the rows and columns in each component. Similarly, the conjugate of the $\mu$-tableau $t$ is the $\mu'$-tableau $t'$ that is obtained from $t$ by reversing the order of its components and then swapping its rows and columns in each component. The reader is invited to check that $\lambda \succeq \mu$ if and only if $\mu' \succeq \lambda'$ and that $s \succeq t$ if and only if $t' \succeq s'$, for $\lambda, \mu \in P_n^\Lambda$ and for $s, t \in \text{Std}(P_n^\Lambda)$.

Fix a multipartition $\mu \in P_n^\Lambda$. Define $t^\mu$ to be the unique standard $\mu$-tableau such that $t^\mu \succeq t$, for all $t \in \text{Std}(\mu)$. More explicitly, $t^\mu$ is the $\mu$-tableau that has the numbers $1, 2, \ldots, n$ entered in order, from left to right, and then top to bottom, along the rows of the components $\mu^{(1)}, \ldots, \mu^{(\ell)}$ of $\mu$. Define $t^t_t = (t^\mu)'$. By the last paragraph $t^t$ is the unique $\mu$-tableau such that $t \succeq t^t$, for all $t \in \text{Std}(\mu)$. The numbers $1, 2, \ldots, n$ are entered in order down the columns of the components $\mu^{(1)}, \ldots, \mu^{(\ell)}$ of $\mu$. The two tableaux displayed above are $t^\mu$ and $t^t$, respectively, for $\mu = (3, 2|2, 1^2|3, 1)$.

Recall from §3.1 that we have fixed a multicharge $\kappa \in \mathbb{Z}^\ell$. The residue of the node $A = (r, c, l)$ is $\text{res}(A) = k_i + c - r + e\mathbb{Z}$. Thus, $\text{res}(A) \in I$. A node $A$ is an $i$-node if $\text{res}(A) = i$. If $t$ is a $\mu$-tableau and $1 \leq k \leq n$, then the residue of $k$ in $t$ is $\text{res}_t(k) = \text{res}(A)$, where $A \in \mu$ is the unique node such that $t(A) = k$. The residue sequence of $t$ is
\[
\text{res}(t) = (\text{res}_t(1), \text{res}_t(2), \ldots, \text{res}_t(n)) \in I^n.
\]
As two important special cases we set $i^\mu = \text{res}(t^\mu)$ and $i^t = \text{res}(t^t)$, for $\mu \in P_n^\Lambda$. 


Following Brundan, Kleshchev and Wang [17, Definition 3.5], we now define the degree and codegree of a standard tableau. Suppose that $\mu \in \mathcal{P}_n^\Lambda$. A node $A$ is an addable node of $\mu$ if $A \notin \mu$ and $\mu \cup \{A\}$ is (the diagram of) a multipartition of $n+1$. Similarly, a node $B$ is a removable node of $\mu$ if $B \in \mu$ and $\mu \setminus \{B\}$ is a multipartition of $n-1$. Given any two nodes $A = (r, c, \ell), B = (r', c', \ell')$, say that $B$ is strictly below $A$, or $A$ is strictly above $B$, if either $l' > l$ or $l' = l$ and $r' > r$. Suppose that $A$ is an $i$-node and define integers

$$d_A(\mu) = \# \{ \text{addable } i\text{-nodes of } \mu \ \text{strictly below } A \} - \# \{ \text{removable } i\text{-nodes of } \mu \ \text{strictly below } A \},$$

and

$$d^A(\mu) = \# \{ \text{addable } i\text{-nodes of } \mu \ \text{strictly above } A \} - \# \{ \text{removable } i\text{-nodes of } \mu \ \text{strictly above } A \}.$$ 

If $t$ is a standard $\mu$-tableau, then its degree and codegree are defined inductively by setting $\deg t = 0 = \codeg t$ when $n = 0$. If $n > 0$, then define

$$\deg t = \deg t_{(n-1)} + d_A(\mu) \quad \text{and} \quad \codeg t = \codeg t_{(n-1)} + d^A(\mu),$$

where $A = t^{-1}(n)$ is the node containing $n$.

The definitions of the residue, degree and codegree of a tableau all depend on the choice of the choice of multicharge $\kappa$. We write $\res^\kappa$, we want to emphasize this choice.

Fix $\beta \in Q^+$ and set $\mathcal{P}^\beta = \{ \lambda \in \mathcal{P}_n^\Lambda \mid \lambda \in P^\beta \}$. The defect of $\beta$ is the integer

$$\def_A \beta = (A, \beta) - \frac{1}{2}(\beta, \beta).$$

When $A$ is clear we write $\def \beta = \def_A \beta$. The defect of $\beta \in Q^+$ is closely related to the degree and codegree of the corresponding tableaux.

**Lemma 3.10** [17, Lemma 3.12]. Suppose that $\beta \in Q^+$ and $s \in \Std(\mu)$, for $\mu \in \mathcal{P}_n^\beta$. Then $\deg s + \codeg s = \def \beta$.

### 3.4. Standard homogeneous bases

We are now ready to define some bases for the cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$. Recall from the last section that $\mathcal{S}_n$ is the symmetric group on $n$ letters and that $\{s_1, s_2, \ldots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathcal{S}_n$. If $w \in \mathcal{S}_n$, then the length of $w$ is the integer

$$\ell(w) = \min \{ k \mid w = s_{r_1} \cdots s_{r_k} \text{ for some } 1 \leq r_1, \ldots, r_k < n \}.$$ 

A reduced expression for $w$ is a word $w = s_{r_1} \cdots s_{r_k}$ such that $k = \ell(w)$. It is a general fact from the theory of Coxeter groups that any reduced expression for $w$ can be transformed into any other reduced expression using just the braid relations $s_r s_t = s_t s_r$, if $|r - t| > 1$, and $s_r s_{r+1} s_r = s_{r+1} s_r s_{r+1}$, for $1 \leq r < n - 1$.

Hereafter, unless otherwise stated, we fix a reduced expression $w = s_{r_1} \cdots s_{r_k}$ for each element $w \in \mathcal{S}_n$, with $1 \leq r_1, \ldots, r_k < n$. We define $\psi_w = \psi_{r_1} \cdots \psi_{r_k}$. By **Definition 3.2**, the generators $\psi_r$, for $1 \leq r < n$, do not satisfy the braid relations. Therefore, the element $\psi_w \in \mathcal{R}_n^\Lambda$ depends upon our choice of reduced expression for $w$.

The symmetric group $\mathcal{S}_n$ acts from the right on the set of tableaux by composition of maps. If $t \in \Std(\mu)$ define two permutations $d(t)$ and $d'(t)$ in $\mathcal{S}_n$ by $t = t(d(t))$ and $t = t(d'(t))$. Conjugating either of the last two equations shows that $d'(t) = d(t')$. Let $w_{\mu} = d(t_{\mu})$. Then it is easy to check that $w_{\mu} = d(t)d'(t)^{-1}$ and $w_{\mu} = d(t)d'(t)^{-1}$ and $\ell(w_{\mu}) = \ell(d(t)) + \ell(d'(t))$, for all $t \in \Std(\mu)$. 
Recall from §3.3 that $i^\mu = \text{res}(t^\mu)$ and that $i_\mu = \text{res}(t_\mu)$.

**Definition 3.11** [32, Definitions 4.9, 5.1 and 6.9]. Suppose that $\mu \in \mathcal{P}_n^\Lambda$. Define non-negative integers $d_1^\mu, \ldots, d_n^\mu$ and $d_1^\mu, \ldots, d_n^\mu$ recursively by requiring that

$$d_1^\mu + \cdots + d_k^\mu = \text{deg}(t_{1:k}^\mu) \quad \text{and} \quad d_1^\mu + \cdots + d_k^\mu = \text{codeg}(t_{1:k}^\mu),$$

for $1 \leq k \leq n$. Now set $e^\mu = e(i^\mu)$, $e_\mu = e(i_\mu)$,

$$y^\mu = y_1^\mu \cdots y_n^\mu \quad \text{and} \quad y_\mu = y_1^\mu \cdots y_n^\mu.$$

For a pair of tableaux $(s, t) \in \text{Std}^2(\mu)$ define

$$\psi_{st} = \psi_{d(s)}^* e^\mu y^\mu \psi_{d(t)} \quad \text{and} \quad \psi'_{st} = \psi_{d'(s)}^* e_\mu y_\mu \psi_{d'(t)}.$$

**Remark 3.12.** We warn the reader that the element $\psi'_{st}$ is equal to the element $\psi'_{s't'}$ in the notation of [32, 33], so care should be taken when comparing the results in this paper with those in [32, 33]. We have changed notation because Definition 3.11 makes several subsequent definitions and results more intuitive. For example, see Corollary 3.19 and Proposition 3.27.

In general, the elements $\psi_{st}$ and $\psi'_{st}$ depend upon the choice of reduced expression that we fixed in Definition 3.11 because $\psi_1, \ldots, \psi_{n-1}$ do not satisfy the braid relations. Similarly, $\psi_{d(s)-1}$ and $\psi_{d'(s)}$ will generally be different elements of $\mathcal{R}_n^\Lambda$.

It follows from Definition 3.11 and the relations that if $(s, t) \in \text{Std}^2(\mathcal{P}_n^\Lambda)$, then

$$e(i)\psi_{st} e(j) = \delta_{i,\text{res}(s)} \delta_{j,\text{res}(t)} \psi_{st} \quad \text{and} \quad e(i)\psi'_{st} e(j) = \delta_{i,\text{res}(s)} \delta_{j,\text{res}(t)} \psi'_{st},$$

(3.13)

for all $i, j \in I^n$. More importantly, we have the following.

**Theorem 3.14** (Hu–Mathas [32, Theorems 5.8 and 6.11, Li [43])]. Let $\mathcal{Z}$ be an integral domain. Then:

(a) $\{ \psi_{st} \mid (s, t) \in \text{Std}^2(\mathcal{P}_n^\Lambda) \}$ is a graded cellular basis of $\mathcal{H}_n^\Lambda$ with weight poset $(\mathcal{P}_n^\Lambda, \geq)$ and degree function $\deg \psi_{st} = \deg s + \deg t$;

(b) $\{ \psi'_{st} \mid (s, t) \in \text{Std}^2(\mathcal{P}_n^\Lambda) \}$ is a graded cellular basis of $\mathcal{H}_n^\Lambda$ with weight poset $(\mathcal{P}_n^\Lambda, \leq)$ and degree function $\deg \psi'_{st} = \deg s + \text{codeg } t$.

This result was proved under some assumptions on $e$ and $\mathcal{Z}$ in [32]. Li [43] has proved that $\mathcal{R}_n^\Lambda(\mathcal{Z})$ is free as a $\mathcal{Z}$-module with basis $\{ \psi_{st} \}$. This implies that the results of [32] extend to an arbitrary integral domain because all of the arguments in [32] can now be carried out over $\mathcal{Z}$ using the embedding $\mathcal{R}_n^\Lambda \hookrightarrow \mathcal{R}_n^\Lambda(\mathcal{Z})$. The results involving the $\psi'$-basis also hold over $\mathcal{Z}$ in view of Proposition 3.27.

The $\psi$-basis and the $\psi'$-basis are dual to each other in the following sense.

**Lemma 3.15** [33, Corollary 3.10]. Suppose that $(s, t), (u, v) \in \text{Std}^2(\mathcal{P}_n^\Lambda)$. Then:

(a) $\psi_{st} \psi'_{uv} \neq 0$ only if $\text{res}(t) = \text{res}(u)$ and $u \succeq t$;

(b) $\psi'_{uv} \psi_{st} \neq 0$ only if $\text{res}(s) = \text{res}(v)$ and $v \succeq s$.

We need the following dominance results. Recall from §3.3 that $(s, t) \succeq (u, v)$ if $s \succeq u$ and $t \succeq v$. 
We now show how Theorem 3.14 restricts to give a basis for the blocks, or the indecomposable 3.5.
The blocks of two-sided ideals, of $R$, $\triangleright$
in homogeneous) standard basis of $R$ that $\Psi$ is a graded cellular algebra.

3.6. Trace forms and graded duality
Recall that a trace form on $A$ is a map $\tau : A \rightarrow \mathbb{Z}$ such that $\tau(ab) = \tau(ba)$, for all $a, b \in A$. The trace form $\tau$ is non-degenerate if whenever $a \in A$ is non-zero, then $\tau(ab) \neq 0$ for some $b \in A$. An algebra $A$ is a symmetric algebra if it has a non-degenerate trace form.

\[ \psi_{st} y_r = \sum_{(u,v) \in P_n^\lambda} a_{uv} \psi_{uv} \] and $\psi_{st} y_r = \sum_{(u,v) \in P_n^\lambda} b_{uv} \psi_{uv}^\prime,$
for some scalars $a_{uv}, b_{uv} \in \mathbb{Z}$.

The next result strengthens [32, Lemma 5.7].

Lemma 3.17. Suppose that $\psi_{st}, \hat{\psi}_{st}$ and $\psi'_{st}, \hat{\psi}'_{st}$ are defined using possibly different reduced expressions for $d(s), d(t)$ and $d'(s), d'(t)$, where $s, t \in \text{Std}(\lambda)$ for some $\lambda \in P_n^\lambda$. Then $\psi_{st} - \hat{\psi}_{st} = \sum_{(u,v) \in P_n^\lambda} s_{uv} \psi_{uv}$ and $\psi'_{st} - \hat{\psi}'_{st} = \sum_{(s,t) \in P_n^\lambda} t_{uv} \psi_{uv}^\prime,$
where $s_{uv} \neq 0$ only if $\text{res}(u) = \text{res}(s), \text{res}(v) = \text{res}(t)$ and $\deg u + \deg v = \deg s + \deg t$ and $t_{uv} \neq 0$ only if $\text{res}(u) = \text{res}(s), \text{res}(v) = \text{res}(t)$ and $\text{codeg} u + \text{codeg} v = \text{codeg} s + \text{codeg} t$.

Proof. By [33, Theorem 3.9], the transition matrices between the $\psi$-basis and the (non-homogeneous) standard basis of $H_n^\lambda$ from [24] is triangular with respect to strong dominance partial order $\triangleright$. The same remark applies to the $\hat{\psi}$-basis, which is defined using possibly different choices of reduced expressions. Moreover, the transition matrices for the $\psi$-basis and the $\hat{\psi}$-basis have the same elements on the diagonal. Applying this result twice to rewrite $\psi_{st}$ in terms of the $\psi$-basis, via the standard basis, proves the first statement. The second statement can be proved similarly. □

3.5. The blocks of $R_n^\lambda$
We now show how Theorem 3.14 restricts to give a basis for the blocks, or the indecomposable two-sided ideals, of $R_n^\lambda$. By Definition 3.2, if $\beta \in Q^+$, then $R_n^\lambda = e_\beta R_n^\lambda e_\beta$, where $e_\beta = \sum_{i \in J^\beta} e(i)$. Set $Q_n^+ = \{ \beta \in Q^+ \mid e_\beta \neq 0 \in R_n^\lambda \}$. By [45, Theorem 2.11] and [11, Theorem 1], if $Z = K$ is a field and $\beta \in Q_n^+$, then $R_n^\lambda$ is a (non-zero) block of $R_n^\lambda$ and $e_\beta$ is a central primitive idempotent. That is,

\[ R_n^\lambda = \bigoplus_{\beta \in Q_n^+} R_n^\lambda \] (3.18)
is the decomposition of $R_n^\lambda$ into blocks. Theorem 3.7 implies that $R_n^\lambda \cong H_n^\lambda$, where $H_n^\lambda = e_\beta H_n^\lambda e_\beta$.

Recall that $P_n^\lambda = \{ \lambda \in P_n^\lambda \mid 1^\lambda \in I^\beta \}$. Combining Theorem 3.14, (3.13) and (3.18) we obtain the following.

Corollary 3.19 [32]. Suppose that $Z = K$ is a field and that $\beta \in Q_n^+$. Then $\{ \psi_{st} \mid s, t \in \text{Std}(\lambda) \text{ for } \lambda \in P_n^\lambda \}$ and $\{ \psi'_{st} \mid s, t \in \text{Std}(\lambda) \text{ for } \lambda \in P_n^\lambda \}$ are graded cellular bases of $R_n^\lambda$. In particular, $R_n^\lambda$ is a graded cellular algebra.
THEOREM 3.20 [32, Theorem 6.17]. Suppose that \( \beta \in \mathbb{Q}_n^+ \) and that \( Z = K \) is a field. Then there is a non-degenerate homogeneous trace form \( \tau_\beta : \mathcal{R}^\Lambda_n \rightarrow K \) of degree \(-2\) def \( \beta \) such that 
\[
\tau_\beta(\psi_{st}\psi_{vu}) \neq 0 \text{ only if } (u, v) \models (s, t), \text{ for } (s, t), (u, v) \in \text{Std}^2(\mathcal{P}^\Lambda_n) .
\]
Moreover, \( \tau_\beta(\psi_{st}\psi_{ta}) \neq 0 \), for all \( (s, t) \in \text{Std}^2(\mathcal{P}^\Lambda_n) \). Consequently, \( \mathcal{R}^\Lambda_\beta \) is a graded symmetric algebra.

By the results in §2.2, the two cellular bases \( \{ \psi_{st} \} \) and \( \{ \psi_{un} \} \) both determine cell modules for \( \mathcal{R}^\Lambda_n \). Suppose that \( \mu \in \mathcal{P}^\Lambda_n \). The Specht module \( S^\mu \) is the cell module of \( \mathcal{R}^\Lambda_n \) indexed by \( \mu \) determined by the \( \psi \)-basis and the dual Specht module \( S_\mu \) is the cell module indexed by \( \mu \) determined by the \( \psi \)-basis. In more detail, where we use the notation of §2.2, as a \( Z \)-module \( S^\mu \) has homogeneous basis \( \{ \psi_t \mid t \in \text{Std}(\mu) \} \), with \( \deg \psi_t = \deg t \), and the \( \mathcal{R}^\Lambda_n \)-module structure on \( S^\mu \) is determined by requiring that for any \( s \in \text{Std}(\mu) \) the map
\[
S^\mu(\deg s) \rightarrow \mathcal{R}^\Lambda_n/(\mathcal{R}^\Lambda_n)^{\Delta \mu} : \psi_t \mapsto \psi_{st} + (\mathcal{R}^\Lambda_n)^{\Delta \mu},
\]
is an \( \mathcal{R}^\Lambda_n \)-module isomorphism. Similarly, \( S_\mu \) has homogeneous basis \( \{ \psi_t' \mid t \in \text{Std}(\mu) \} \), with \( \deg \psi_t' = \text{codeg} t \), and where the \( \mathcal{R}^\Lambda_n \)-action is determined in the exactly same way except that we use the \( \psi \)-basis of \( \mathcal{R}^\Lambda_n \).

The modules \( S^\mu \) and \( S_\mu \) are dual to each other in the following sense.

PROPOSITION 3.21 [32, Proposition 6.19]. Suppose that \( \mu \in \mathcal{P}^\Lambda_\beta \), where \( \beta \in \mathbb{Q}_n^+ \). Then \( S^\mu \cong S_\mu(\deg \beta) \) as graded \( \mathcal{R}^\Lambda_n \)-modules.

We warn the reader that the module \( S_\mu \) is denoted \( S_{\mu'} \) in [32, §6]. This change in notation is a consequence of Remark 3.12. The notation for Specht modules and dual Specht modules in this paper is compatible with [39].

As in §2.2, define \( D^\mu = S^\mu/\text{rad } S^\mu \). Let \( \mathcal{K}^\Lambda_n = \{ \mu \in \mathcal{P}^\Lambda_n \mid D^\mu \neq 0 \} \) be the set of Kleshchev multipartitions. By [4, 6, 15], there is a recursive description of the Kleshchev multipartitions. Observe that \( \mathcal{K}^\Lambda_n \) depends on the choice of the \( \ell \)-tuple \( (\overline{r}_1, \ldots, \overline{r}_\ell) \) and not just on \( \Lambda \).

In the graded setting the irreducible \( \mathcal{R}^\Lambda_n \)-modules were first constructed by Brundan and Kleshchev [15, Theorem 4.11, Theorem 5.10]. By [32, Corollary 5.1], Brundan and Kleshchev’s irreducible modules coincide exactly with the irreducible \( \mathcal{R}^\Lambda_n \)-modules constructed using the \( \psi \)-basis and the cellular algebra framework of Theorem 2.5.

PROPOSITION 3.22 [15, 32]. Suppose that \( Z = K \) is a field. Then
\[
\{ D^\mu(d) \mid \mu \in \mathcal{K}^\Lambda_n \text{ and } d \in Z \}
\]
is a complete set of pairwise non-isomorphic irreducible graded \( \mathcal{R}^\Lambda_n \)-modules.

When \( e = 0 \) it is well known that \( \mathcal{K}^\Lambda_n \) is in bijection with the set of FLOTW multipartitions from [27], which have a particularly simple description. In fact, these two sets of multipartitions coincide. As with Proposition 3.22, the main point is to match the labelling of the simple modules with the cellular algebra structure of Theorem 2.5.

COROLLARY 3.23. Suppose that \( e = 0 \) or \( e > n, \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_\ell \) and \( \mu \in \mathcal{P}^\Lambda_n \). Then \( \mu = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \mathcal{K}^\Lambda_n \) if and only if \( \mu^{(l)}_{r + \kappa_l - \kappa_{l+1}} \leq \mu^{(l+1)}_r \), for \( 1 \leq l < \ell \) and \( r \geq 1 \).

Proof. This result is part of the folklore for these algebras when \( e = 0 \). As far as we are aware, however, it is not stated explicitly in the literature. This said, when \( e = 0 \) the result does follow from [15, Remark 3.4(2) and Theorem 5.3] and it is implicit in the papers [5, 13, 16, 57].

To prove the result directly, it is enough to observe that if \( \mu \in \mathcal{P}^\Lambda_n \), then \( \mu \) has at most one addable or removable \( i \)-node in each component, for any \( i \in I \), because \( e = 0 \) or \( e > n \).
Therefore, in view of [27, (11)], up to height $n$ the crystal graphs determining the sets of FLOTW and Kleshchev multipartitions coincide exactly, which gives the result.

3.7. The sign isomorphism

Following [39, §3.2], we now introduce an analogue of the sign involution of the symmetric groups for the quiver Hecke algebras. Unlike the case of the symmetric groups, this map is generally not an automorphism of $\mathcal{R}_n^\Lambda$. 

In §3.1, we fixed the multicharge $\kappa = (\kappa_1, \ldots, \kappa_\ell) \in \mathbb{Z}^\ell$ that determines $\Lambda = \Lambda(\kappa)$. Define $\kappa' = (-\kappa_1, \ldots, -\kappa_\ell) \in \mathbb{Z}^\ell$ and let $\Lambda' = \Lambda(\kappa')$. Then $\Lambda' \in P^+$. More precisely, if $\Lambda = \sum_{i=1}^\ell l_i \Lambda_i$, for $l_i \in \mathbb{N}$, then $\Lambda' = \sum_{i=1}^\ell l_i \Lambda_{-i}$. Similarly, if $\beta = \sum_{i \in I} b_i \alpha_i$, for some $b_i \in \mathbb{N}$, then define $\beta' = \sum_{i \in I} b_i \alpha_{-i}$. Then $\beta' \in Q^+$ and $\text{def} \Lambda' \beta' = \text{def} \Lambda \beta$.

As noted in [39, §3.2], the relations of $\mathcal{R}_3^\Lambda$ given in Definition 3.2 imply that there is a unique degree preserving isomorphism of graded algebras $\text{sgn} : \mathcal{R}_3^\Lambda \rightarrow \mathcal{R}_3^{\Lambda'}$ such that

$$e(i) \mapsto e(-i), \quad y_r \mapsto -y_r \quad \text{and} \quad \psi_s \mapsto -\psi_s,$$

for $i \in I^\beta$, $1 \leq r \leq n$, and $1 \leq s < n$. The map $\text{sgn}$ induces a graded equivalence

$$\mathcal{R}_3^\Lambda \text{-Mod} \xrightarrow{\sim} \mathcal{R}_3^{\Lambda'} \text{-Mod}$$

in the sense of §2.3. This equivalence sends an $\mathcal{R}_3^{\Lambda'}$-module $M$ to the $\mathcal{R}_3^\Lambda$-module $M^{\text{sgn}}$, where $M^{\text{sgn}}$ is equal to $M$ as a graded vector space and where the $\mathcal{R}_3^\Lambda$-action on $M^{\text{sgn}}$ is given by $m \cdot a = m \text{sgn}(a)$, for $a \in \mathcal{R}_3^\Lambda$ and $m \in M^{\text{sgn}}$.

In §3.3, we defined residues, degrees and codegrees for tableaux, all as functions of the multicharge $\kappa$. The same definitions, but with respect to the multicharge $\kappa'$, give analogous statistics for $\mathcal{R}_3^{\Lambda'}$. To distinguish these definitions from the previous ones set $\text{res}' = \text{res}^\kappa$, $\text{deg}' = \text{deg}^\kappa$, and $\text{codeg}' = \text{codeg}^\kappa$. In particular, if $A = (r, c, l)$ is a node, then $\text{res}'(A) = \kappa'_l + c - r + c\mathbb{Z}$. Residue sequences, degrees and codegrees are now defined exactly as before using $\text{res}'$. In this way, the set of multipartitions $\mathcal{P}_{\beta'} = \{ \mu \in \mathcal{P}_{n}^\Lambda \mid \text{res}'(\mu) \in I^\beta \}$ is attached to $\mathcal{R}_3^{\Lambda'}$ in the same way that $\mathcal{P}_{\beta'}$ is attached to $\mathcal{R}_3^\Lambda$.

Recall that $d'(t) \in \mathcal{S}_n$ is the permutation determined by $t = t_\mu d'(t)$ and that $t'$ is the tableau that is conjugate to $t$.

**Lemma 3.26.** Suppose that $\beta \in Q_n^+$. Then $\mathcal{P}_{\beta'} = \{ \mu' \mid \mu \in \mathcal{P}_{\beta} \}$. Moreover, if $t \in \text{Std}(\mathcal{P}_{\beta})$, then $t' \in \text{Std}(\mathcal{P}_{\beta'})$ and $\text{res}'(t') = -\text{res}(t)$, $\text{deg}' t' = \text{deg} t$, $\text{codeg}' t' = \text{codeg} t$ and $d'(t') = d(t')$.

**Proof.** By definition, if $t \in \text{Std}(\mu)$, then $t = t_\mu d(t)$. Conjugating shows that $t' = t_\mu d'(t)$ and hence that $d(t') = d'(t)$. By definition, $\kappa'_l = -\kappa_{\ell-l+1}$ for $1 \leq l \leq \ell$. Therefore, if the node $A = (r, c, l) \in \mu$ has residue $i = \text{res}(A)$, then the ‘conjugate node’ $A' = (c, r, \ell - l + 1) \in \mu'$ has residue $\text{res}'(A') = -i$. Consequently, $\mu \in \mathcal{P}_{\beta}$ if and only if $\mu' \in \mathcal{P}_{\beta'}$ and if $t \in \text{Std}(\mathcal{P}_{\beta})$, then $\text{res}'(t') = -\text{res}(t)$. As $\text{res}'(t') = -\text{res}(t)$, it is now easy to see that conjugation interchanges the definitions of degrees and codegrees and hence that $\text{deg}' t' = \text{codeg} t$ and $\text{codeg}' t' = \text{deg} t$.

Using the residue and degree functions defined using the multicharge $\kappa'$, Definition 3.11 gives $\psi$ and $\psi'$-bases for $\mathcal{R}_3^{\Lambda'}$. We emphasize that here and below the bases for $\mathcal{R}_3^{\Lambda'}$ will always be defined using $\kappa'$. The bases of $\mathcal{R}_3^\Lambda$ and $\mathcal{R}_3^{\Lambda'}$ are interchanged by the $\text{sgn}$ automorphism. More precisely, we have the following.
Proposition 3.27. Suppose that $\beta \in Q^+_n$, $\mu \in \mathcal{P}_\beta$, that $\text{sgn}: \mathcal{R}_\beta^\Lambda \to \mathcal{R}_\beta^\Lambda'$ is the sign isomorphism and $(s,t) \in \text{Std}^2(\mu)$. Then
\[
\text{sgn}(\psi_{st}) = \varepsilon_{st} \psi'_{s't'} \quad \text{and} \quad \text{sgn}(\psi'_{st}) = \varepsilon'_{st} \psi_{s't'},
\]
where $\varepsilon_{st} = (-1)^{\text{deg}t + \ell(d(s)) + \ell(d(t))}$ and $\varepsilon'_{st} = (-1)^{\text{codeg}t + \ell(d'(s)) + \ell(d'(t))}$.

Proof. By Lemma 3.26, $\text{deg}t' = \text{codeg}t_{\nu}$ for all $\nu \in \mathcal{P}_\beta^\Lambda$. This implies that $\text{deg}(t'_{1\mu}) = \text{codeg}(t_{\nu})$, for $1 \leq k \leq n$. Therefore, $\text{sgn}(y_{\mu}) = (-1)^{\text{deg}t'_{\nu}} y_{\mu'}$ by Definition 3.11. Hence, $\psi_{s't'} = (-1)^{\text{deg}t'_{\nu}} \psi'_{s't'}$, since $\text{res}(t'_{\nu}) = -\text{res}(t_{\nu})$ by Lemma 3.26. It follows that $\text{sgn}(\psi_{st}) = \varepsilon_{st} \psi'_{s't'}$ and $\text{deg}(s') = \text{degree}(s)$ and $\text{deg}(t') = \text{degree}(t)$ by Lemma 3.26. That $\text{sgn}(\psi'_{st}) = \varepsilon'_{st} \psi_{s't'}$ is proved similarly.

The isomorphism $\text{sgn}$ sends the $\psi$-basis of $\mathcal{R}_\beta^\Lambda$ to the $\psi'$-basis of $\mathcal{R}_\beta^\Lambda'$, up to sign. Therefore, $\text{sgn}$ induces an isomorphism between the Specht modules of $\mathcal{R}_\beta^\Lambda$, which are constructed using the $\psi$-basis, and the dual Specht modules of $\mathcal{R}_\beta^\Lambda'$, which are constructed using the $\psi'$-basis. That is, we have the following.

Corollary 3.28 [39, Theorem 8.5]. Suppose that $\mathcal{Z}$ is an integral domain. Then $S^\mu \cong (S_{\mu'})^{\text{sgn}}$ and $S_{\mu} \cong (S^\mu')^{\text{sgn}}$.

Note that no degree shifts are required in Corollary 3.28 because $\text{sgn}$ is homogeneous. Alternatively, Lemma 3.26 shows that the combinatorics giving the degrees on both sides of these isomorphisms agree.

4. Graded Schur algebras

In this section, we introduce the cyclotomic quiver Schur algebras. We will show that they are quasi-hereditary graded cellular algebras. Unless otherwise stated, the following assumption will be in force for the rest of this paper.

Assumption 4.1. We assume that $e = 0$ or $e > n$.

Unless otherwise stated, we work over an arbitrary integral domain $\mathcal{Z}$.

4.1. Permutation modules

Following the recipe in [24], we define the graded cyclotomic Schur algebra to be the algebra of graded $\mathcal{R}_n^\Lambda$-endomorphisms of a particular $\mathcal{R}_n^\Lambda$-module $G_n^\Lambda$. In this section, we introduce and investigate the summands of ‘graded tensor space’ $G_n^\Lambda$, as defined below.

Definition 4.2. Suppose that $\mu \in \mathcal{P}_n^\Lambda$. Define $G^\mu$ and $G_n^\mu$ to be the $\mathcal{R}_n^\Lambda$-modules:
\[
G^\mu = \psi_{\mu} \mathcal{R}_n^\Lambda \langle -\text{deg}t' \rangle \quad \text{and} \quad G_n^\mu = \psi_{\mu} \mathcal{R}_n^\Lambda \langle -\text{codeg}t \rangle.
\]
Set $G_n^\Lambda = \bigoplus \mu G^\mu$ and $G_n^\Lambda = \bigoplus \mu G_n^\mu$.

The degree shifts appear in Definition 4.2 because we want $G^\mu$ to have a graded Specht filtration in which $S^\mu$ has graded multiplicity one and we want $G_n^\mu$ to have a graded dual Specht filtration in which $S^\mu$ appears with graded multiplicity one.

The modules $G^\mu$ and $G_n^\mu$ are closely related. To explain this recall the isomorphism $\text{sgn}: \mathcal{R}_n^\Lambda \to \mathcal{R}_n^\Lambda'$ from (3.24).
LEMMA 4.3. Suppose that \( \mu \in \mathcal{P}_\beta^A \), for \( \beta \in \mathbb{Q}_n^+ \). Then

\[
G^\mu \cong (G^\mu)^{sgn} \quad \text{and} \quad G_\mu \cong (G_\mu)^{sgn}
\]
as graded \( \mathcal{R}_n^A \)-modules.

Proof. This result is a consequence of Definition 4.2 and Proposition 3.27. We give the details because applying the \( sgn \) involution is an important theme throughout this paper. For the proof, let \( \psi_{u,v} \in \mathcal{R}_n^A \) and \( \psi'_{u,v} \in \mathcal{R}_n^A \) be the corresponding basis elements of \( \mathcal{R}_n^A \) and \( \mathcal{R}_n^A \), respectively. We remind the reader that we have to use the definitions of \( \mathcal{S} \) and \( \mathcal{Z} \) respectively. We remind the reader that we have to use the definitions of \( \mathcal{S} \) and \( \mathcal{Z} \) respectively. We remind the reader that we have to use the definitions of \( \mathcal{S} \) and \( \mathcal{Z} \) respectively. We remind the reader that we have to use the definitions of \( \mathcal{S} \) and \( \mathcal{Z} \) respectively. We remind the reader that we have to use the definitions of \( \mathcal{S} \) and \( \mathcal{Z} \) respectively. We remind the reader that we have to use the definitions of \( \mathcal{S} \) and \( \mathcal{Z} \) respectively.
Proof. We argue by downwards induction on \( r \). If \( r = n \), then \( \psi_{n,r} = 1 \) and there is nothing to prove since, by assumption, \( d_r \geq d_{n+1} \). Suppose then that \( r < n \). We divide the proof into two cases.

First suppose that \( i_r \neq i_{r+1} \). Then, using (3.3), we have that

\[
\psi_{n,r}(y^d_{r} y^d_{r+1} \cdots y^d_{n}) = \psi_{n,r+1}(y^d_{r} y^d_{r+1} \cdots y^d_{n}) (i)
\]

\[
= \psi_{n,r+1}(y^d_{r} y^d_{r+1} \cdots y^d_{n}) y^d_{r+1} \cdots y^d_{n} e(i)
\]

\[
= \psi_{n,r+1}(y^d_{r} y^d_{r+1} y^d_{r+2} \cdots y^d_{n} e(i))
\]

\[
= y^d_{r+1} \psi_{n,r+1}(y^d_{r} y^d_{r+1} y^d_{r+2} \cdots y^d_{n} e(s_i)) \psi_r.
\]

Therefore, \( \psi_{n,r}(y^d_{r} y^d_{r+1} \cdots y^d_{n}) e(i) \in y^d_{r+1} y^d_{r+2} \cdots y^d_{n} e(s_i) \mathcal{R}^A_n \) by induction because the sequence \( s_i \) and the non-negative integers \( d_r, d_{r+2}, \ldots, d_{n+1} \) satisfy the assumptions of the lemma.

Now consider the remaining case when \( i_r = i_{r+1} \). A quick calculation using (3.3) shows that \( \psi_r \) commutes with any symmetric polynomial in \( y_r \) and \( y_{r+1} \), so \( \psi_{r}(y_r y_{r+1}) = (y_r y_{r+1}) \psi_r \). By assumption, \( d_r \geq d_{r+1} \geq d_{n+1} \), so

\[
\psi_{n,r}(y^d_{r} \cdots y^d_{n}) e(i) = \psi_{n,r+1}(y^d_{r} y^d_{r+1} \cdots y^d_{n}) y^d_{r+1} \cdots y^d_{n} e(i)
\]

\[
= \psi_{n,r+1}(y^d_{r} y^d_{r+1} \cdots y^d_{n}) (s_i) \psi_r y^d_{r+1} \cdots y^d_{n}
\]

\[
= y^d_{r+1} \psi_{n,r+1}(y^d_{r} y^d_{r+1} \cdots y^d_{n}) e(s_i) \psi_r y^d_{r+1} \cdots y^d_{n}.
\]

Since \( i_r = i_{r+1} \) the sequence \( s_i \) and the integers \( d_{r+1}, \ldots, d_n \) again satisfy the assumptions of the Lemma. Hence, the result follows by induction. \( \square \)

Before we can give bases for \( G^\mu \) and \( G^\nu \) we need to introduce a special choice of reduced expression. Recall the definition of \( \psi_{r,n} \) and \( \psi_{r,r} \) from above Lemma 4.5, for \( 1 \leq r \leq n \). We remind the reader that, by convention, the symmetric group \( \mathcal{G}_n \) acts from the right on \( \{1, 2, \ldots, n\} \). It is well known and easy to prove that

\[
\mathcal{G}_n = \bigsqcup_{r=1}^{n} s_{r,n} \mathcal{G}_{n-1} \quad \text{(disjoint union),}
\]

and that \( \ell(s_{r,n} w) = \ell(s_{r,n}) + \ell(w) = \ell(w) + n - r \), for all \( w \in \mathcal{G}_{n-1} \). Hence, we have the following.

**Lemma 4.6.** Suppose that \( w \in \mathcal{G}_n \). Then there exist unique integers \( r_2, \ldots, r_n \), with \( 1 \leq r_k \leq k \), such that \( w = s_{r_2,n} \cdots s_{r_2,2} \) and \( \ell(w) = \ell(s_{r_2,n}) + \cdots + \ell(s_{r_2,2}) \).

The factorization \( w = s_{r_2,n} \cdots s_{r_2,2} \) in Lemma 4.6 gives a reduced expression for \( w \). As a temporary notation, define \( \hat{\psi}_w = \psi_{r_2,n} \cdots \psi_{r_2,2} \in \mathcal{R}^A_n \) and if \( (s, t) \in \text{Std}^2(\lambda) \) let \( \hat{\psi}_{st} = \hat{\psi}_t \hat{\psi}_s \lambda^{y_{\lambda}} \hat{\psi}_{d(t)} \) and let \( \hat{\psi}_t' = \hat{\psi}_{d'(t)} \lambda^{y_{\lambda}} \hat{\psi}_{d'(t)} \).

Observe that the choice of reduced expression used to define \( \hat{\psi}_{st} \) is compatible with the natural embeddings \( \mathcal{G}_m \hookrightarrow \mathcal{G}_n \), for \( 1 \leq m \leq n \). More precisely, if \( n \) appears in \( t \) in the same position as \( r \) appears in \( s \), then \( \ell(d(t)) = s_{r,n} d(\Pi_{1}^{n-1}) \) and \( \ell(d(t)) = n - r + \ell(d(\Pi_{1}^{n-1})) \). Consequently, \( \hat{\psi}_{d(t)} = \psi_{r,n} \psi_{1}^{n-1} \). Similarly, if \( n \) appears in \( t \) in the same position as \( r \) appears in \( t \), then \( \ell(d'(t)) = s_{r,n} d'(\Pi_{1}^{n-1}) \) and \( \ell(d'(t)) = n - r + \ell(d'(\Pi_{1}^{n-1})) \) so that \( \hat{\psi}_{d'(t)} = \psi_{r,n} \psi_{1}^{n-1} \).

The next lemma makes heavy use of Assumption 4.1.

**Lemma 4.7.** Suppose that \( e = 0 \) or \( e > n \) and \( s \in \text{Std}^\mu(\lambda) \) and \( u \in \text{Std}_\mu(\lambda) \), for some \( \lambda \in \mathcal{P}_n^A \). Then \( \hat{\psi}_{st} \in G^\mu \) and \( \hat{\psi}_{st} \in G^\mu \).
Proof. By Proposition 3.27 and Lemma 4.3, both statements are equivalent so we prove only that $\psi_{st^1} \in G^\mu$.

We argue by induction on $n$. If $n = 1$, then $s = \lambda^1$ so that $\hat{\psi}_{st^1} = G^\mu = e^\mu y_1^{\deg s}$. Similarly, $\hat{\psi}_{t^1} = e^\mu y_1^{\deg t^1}$. Now $\deg s \geq \deg t^1$, since by assumption $s \geq t^1$ and $n = 1$, so $\hat{\psi}_{st^1} = y_1^{\deg s - \deg t^1} \hat{\psi}_{t^1} \in G^\mu$ as claimed.

Now assume that $n > 1$. Let $s_1 = s_{1(n-1)}$, $\lambda^{s_1} = \text{Shape}(s_1)$ and $\mu_1 = \text{Shape}(t^1_{(n-1)})$. Then $s_1 \in \text{Std}^\mu(\lambda^{s_1})$. Let $r$ be the integer such that $r$ appears in the same position in $(\lambda^s, n)$ as $s$ does in $s_1$. Let $\nu = \text{Shape}(s_{1(r-1)})$. By definition, $\hat{\psi}_{s_1} = \psi_{r,s_1} \hat{\psi}_{s_1}$, so, recalling the definition of the integers $d^\lambda_1, \ldots, d^\lambda_n$ from Definition 3.11, we have

$$\hat{\psi}_{st^1} = \hat{\psi}_{s_1}^{\nu} e^{\lambda_1} = \hat{\psi}_{s_1}^{\nu} y^\nu r^{d^\lambda_1} \cdots y^n e^\lambda,$$

where the last equality follows because if $r \leq j < n$, then $\psi_j$ commutes with $y^\nu$ by (3.3). We want to apply Lemma 4.5 to the sequence $d_1 = d^\lambda_1, \ldots, d_n = d^\lambda_n, d_{n+1} = d^\mu_n$, so we have to check that $d^\lambda_i \geq d^\lambda_i \geq d^\lambda_i \geq d^\mu_i$ whenever there exist $s$ and $t$ such that $r \leq s \leq t < n$ and $i^\lambda = i^\lambda = i^\lambda$. Suppose then, if possible, that $r \leq s \leq t < n$ and $i^\lambda = i^\lambda = i^\lambda$. If $s \in \mathcal{P}^\Lambda_n$, then, because $e = 0$ or $e > n$, each component of $\sigma$ contains at most one addable or removable $i$-node, for all $i \in I$. Therefore,

$$d^\mu_m = \# \{ 1 \leq l \leq \ell \mid l > \text{comp}_t(m) \text{ and } \kappa_l \equiv i^r_n \mod (\text{mod} e) \},$$

for $1 \leq m \leq n$. (Here, as usual, $i^r = \text{res}(i^r) \in I^\Lambda_n$.) By assumption, $r \leq s \leq t < n$ so $\text{comp}_t(r) \leq \text{comp}_t(s) \leq \text{comp}_t(t)$ and consequently $d^\mu_i \geq d^\mu_i \geq d^\mu_i$ since $i^\lambda = i^\lambda = i^\lambda$. Moreover, $\text{comp}_t(t) \leq \text{comp}_t(n)$ since $s \geq t^1$ and $i^\lambda = i^\lambda = i^\lambda$, so that $d^\lambda_i \geq d^\mu_i$. Therefore, $d^\lambda_i \geq d^\lambda_i \geq d^\mu_i$ whenever $r \leq s \leq t < n$ and $i^\lambda = i^\lambda = i^\lambda$. Consequently, Lemma 4.5 applies and we deduce that

$$\hat{\psi}_{st^1} = \hat{\psi}_{s_1}^{*} y^\nu r^{d^\lambda_1} \cdots y^{n-1} e(s_{n-1}, i^\lambda)h,$$

for some $h \in \mathcal{R}_n^\Lambda$. Now $y^\lambda e^\lambda = y^\nu r^{d^\lambda_1} \cdots y^{n-1} e(s_{n-1}, i^\lambda)h$, by definition, and $y_n$ commutes with $\hat{\psi}_{s_1}$ by (3.4). Therefore, by induction, there exists $h' \in \mathcal{R}_n^\Lambda$ such that

$$\hat{\psi}_{s_1} = y_n^{e^\mu} y_1^{e^\mu} \cdots y_{n-1}^{e^\mu} h' \in e^\mu y^\mu \mathcal{R}_n^\Lambda.$$

This completes the proof of the lemma.

If we drop Assumption 4.1, then it is easy to construct examples where Corollary 4.8 fails when $0 < e \leq n$.

COROLLARY 4.8. Suppose that $e = 0$ or $e > n$ and $s \in \text{Std}^\mu(\lambda)$ and $u \in \text{Std}_u(\lambda)$, for $\lambda \in \mathcal{P}^\Lambda_n$. Then for any $t \in \text{Std}(\lambda)$, $\psi_{st} \in G^\mu$ and $\psi_{ut} \in G^\mu$.

Proof. We show only that $\psi_{st} \in G^\mu$. If $\psi_{st} = \hat{\psi}_{st}$, then the result follows by Lemma 4.7. Otherwise, by Lemma 3.17, there exist $s_{uv} \in \mathbb{Z}$ such that

$$\psi_{st} = \psi_{st^1} + \sum_{(u,v) \in \text{Std}^2(\mathcal{P}^\Lambda_n) \atop (u,v) \vDash (s,t)} s_{uv} \psi_{uv},$$

where $s_{uv} \neq 0$ only if $\text{res}(u) = \text{res}(s)$. Consequently, if $s_{uv} \neq 0$, then $u \geq s \geq t^1$ and $u \geq t$ so that $u \in \text{Std}^\mu(\nu)$, for some $\nu \geq \lambda$. By induction on dominance, $\psi_{uv} = \psi_{uv} \psi_{uv}$ belongs to $G^\mu$ whenever $(u,u) \vDash (s,t)$. Moreover, $\psi_{st} \in G^\mu$ by Lemma 4.7. Hence, $\psi_{st} \in G^\mu$ as we wanted to show.

We can now give bases for $G^\mu$ and $G^\mu$. Almost everything in this paper relies on the next result.
THEOREM 4.9. Suppose that $\mu \in \mathcal{P}_n^\Lambda$. Then

(a) The set $\{ \psi_{st} | s \in \text{Std}^\mu(\nu) \text{ and } t \in \text{Std}(\nu) \}$ is a basis of $G^\mu$;

(b) The set $\{ \psi_{uv} | u \in \text{Std}_\mu(\nu) \text{ and } v \in \text{Std}(\nu) \}$ is a basis of $G_{\mu}$.

Proof. Parts (a) and (b) are equivalent by Lemma 4.3 and Proposition 3.27, so it is enough to prove (a). Suppose first that $\mathbb{Z} = K$ is a field. By Corollary 4.8, $\psi_{st} \in G^\mu$ whenever $s \in \text{Std}^\mu(\nu)$ and $t \in \text{Std}(\nu)$, for some multipartition $\nu \in \mathcal{P}^\Lambda_n$. Therefore, by Theorem 3.14,

$$\dim_K G^\mu \geq \sum_{u \in \text{Std}^\mu(\nu)} \# \text{Std}(\nu).$$

On the other hand, by Lemma 4.4 the dimension of $G^\mu$ is at most the number on the right-hand side. Hence, the set in the statement of the theorem is a basis of $G^\mu$, so that the lemma holds over any field $K$.

To prove the proposition when $\mathbb{Z}$ is not a field it suffices to consider the case where $\mathbb{Z} = \mathbb{Z}$. Let $G$ be the free $\mathbb{Z}$-submodule of $G^\mu$ spanned by the basis elements $\{ \psi_{st} | s \in \text{Std}^\mu(\nu), t \in \text{Std}(\nu), \nu \in \mathcal{P}^\Lambda_n \}$. Then $G$ is a pure $\mathbb{Z}$-submodule of $\mathcal{R}^\Lambda_n$, by Theorem 3.14, and hence a pure $\mathbb{Z}$-submodule of $G^\mu$. As a result, $G$ is a direct summand of $G^\mu$ as a $\mathbb{Z}$-module. Therefore, there is a short exact sequence of $\mathbb{Z}$-modules

$$0 \rightarrow G \rightarrow G^\mu \rightarrow G^\mu / G \rightarrow 0,$$

that splits as sequence of $\mathbb{Z}$-modules. Therefore, for every field $K$ there is an exact sequence

$$0 \rightarrow G \otimes K \rightarrow G^\mu \otimes K \rightarrow G^\mu / G \otimes K \rightarrow 0.$$

By Lemma 4.4, $\dim G^\mu \otimes K \leq \dim G \otimes K$. Hence the first homomorphism in the last exact sequence must be an isomorphism. It follows that $G^\mu / G \otimes K = 0$ for any field $K$ that is an $\mathbb{Z}$-algebra. Applying Nakayama’s lemma (see, for example, [7, Proposition 3.8]), $G^\mu / G = 0$. That is, $G^\mu = G$. Hence, elements in the statement of the theorem are a basis for $G^\mu$ as required.

Theorem 4.9 has several useful corollaries. We first note that it gives explicit formulæ for the graded dimensions of these two modules:

\[
\dim_q G^\mu = q^{-\text{deg } t^{\mu}} \sum_{s \in \text{Std}^\mu(\nu), t \in \text{Std}(\nu)} q^{\text{deg } s + \text{deg } t},
\]

\[
\dim_q G_{\mu} = q^{-\text{codeg } t_{\mu}} \sum_{u \in \text{Std}_{\mu}(\nu), v \in \text{Std}(\nu)} q^{\text{deg } u + \text{codeg } v}.
\]

COROLLARY 4.10. Suppose that $\mu, \lambda \in \mathcal{P}_n^\Lambda$. Then

$\{ \psi_{st} | s \in \text{Std}^\mu(\nu) \text{ and } t \in \text{Std}^\lambda(\nu), \nu \in \mathcal{P}^\Lambda_n \}$

is a basis of $G^\mu \cap (G^\lambda)^*$ and

$\{ \psi_{uv} | u \in \text{Std}_\mu(\nu) \text{ and } v \in \text{Std}_\nu(\lambda), \nu \in \mathcal{P}^\Lambda_n \}$

is a basis of $G_{\mu} \cap (G_{\lambda})^*$.

Proof. Suppose that $a \in G^\mu \cap (G^\lambda)^*$ and write $a = \sum_{(s,t) \in \mathcal{P}^\Lambda_n} r_{st} \psi_{st}$, for $r_{st} \in \mathbb{Z}$ and $(s,t) \in \text{Std}(\mathcal{P}^\Lambda_n)$. Then $r_{st} \neq 0$ only if $s \in \text{Std}^\mu(\mathcal{P}^\Lambda_n)$ by Theorem 4.9. Similarly, since $a^* \in G^\lambda$ we see that $r_{st} \neq 0$ only if $t \in \text{Std}^\lambda(\mathcal{P}^\Lambda_n)$. Moreover, if $s \in \text{Std}^\mu(\nu)$ and $t \in \text{Std}^\lambda(\nu)$, then $\psi_{st} \in G^\mu \cap (G^\lambda)^*$ by two more applications of Theorem 4.9. This proves the first claim. The second statement follows similarly.
Corollary 4.11. Suppose that \( \mu \in \mathcal{P}_n^\Lambda \).

(a) Write \( \text{Std}^\mu(\mathcal{P}_n^\Lambda) = \{S_1, \ldots, S_m\} \), ordered so that \( i \leq j \) whenever \( S_i \geq S_j \) and set \( \nu^i = \text{Shape}(S_i) \), for \( 1 \leq i, j \leq m \). Then \( G^\mu \) has a (graded) Specht filtration

\[
G^\mu = G_m \geq G_{m-1} \geq \cdots \geq G_1 \geq G_0 = 0
\]
such that \( G_i/G_i^{i-1} \cong S^\nu(\deg S_i - \deg \nu^i) \), for \( 1 \leq i \leq m \).

(b) Write \( \text{Std}_\mu(\mathcal{P}_n^\Lambda) = \{U_1, \ldots, U_l\} \), ordered so that \( i \geq j \) whenever \( U_i \geq U_j \) and set \( \nu_i = \text{Shape}(U_i) \), for \( 1 \leq i, j \leq l \). Then \( G_\mu \) has a (graded) dual Specht filtration

\[
G_\mu = G_1 \geq G_{l-1} \geq \cdots \geq G_l \geq G_0 = 0
\]
such that \( G_i/G_i-1 \cong S_{\nu_i}(\text{codeg } U_i - \text{codeg } t_{\nu_i}) \), for \( 1 \leq i \leq l \).

Proof. Suppose that \( 1 \leq i \leq m \). Define \( G^i \) to be the \( Z \)-submodule of \( G^\mu \) spanned by

\[
\{ \psi_{s,j,t} \mid 1 \leq j \leq i \text{ and } t \in \text{Std}(\nu^j) \}.
\]

Then \( G^i \) is a submodule of \( G^\mu \) by Theorem 4.9 and (GC2) of Definition 2.4. Finally, \( G^i/G^i-1 \cong S^\nu(\deg S_i - \deg \nu^i) \) by the construction of the cell modules given in §2.2. More precisely, recalling that the Specht module \( S^\nu \) has basis \( \{ \psi_t \mid t \in \text{Std}(\nu^i) \} \), the isomorphism is given by \( \psi_t \mapsto \psi_{s,j,t} + G^i-1 \), for all \( t \in \text{Std}(\nu^i) \). This map has degree \( \deg S_i - \deg \nu^i \) because \( \psi_{s,j,t} \) has degree \( \deg S_i + \deg t - \deg \nu^i \) when considered as an element of \( G^\mu = \psi_{t,\nu, \mathcal{R}_n^\Lambda \langle -\deg \nu^i \rangle}. \)

The proof of (b) is almost identical.

In particular, note that \( S^\mu \) is a quotient of \( G^\mu \) and that \( S_\mu \) is a quotient of \( G_\mu \).

Corollary 4.12. Suppose that \( \mu \in \mathcal{P}_n^\Lambda \). Then:

(a) The set \( \{ \psi_{t,\nu, \psi_{u,j}^{\nu} \mid u \in \text{Std}^\mu(\nu) \text{ and } t \in \text{Std}(\nu) \text{, for } \nu \in \mathcal{P}_n^\Lambda \} \) is a basis of \( G^\mu \);

(b) The set \( \{ \psi_{t,\nu, \psi_{s,j}^{\nu} \mid s \in \text{Std}^\mu(\nu) \text{ and } t \in \text{Std}(\nu) \text{, for } \nu \in \mathcal{P}_n^\Lambda \} \) is a basis of \( G_\mu \).

Proof. By Lemma 4.4 and Theorem 3.14(b), the elements in (a) span \( G^\mu \), so it remains to show that they are linearly independent. This is a direct consequence of Theorem 4.9. The proof of (b) is similar.

Corollary 4.13. Suppose that \( \mu \in \mathcal{P}_n^\Lambda \). Using the notation of Corollary 4.11:

(a) The module \( G^\mu \) has a dual Specht filtration \( G^\mu = H_0 \geq H_1 \geq \cdots \geq H_{m-1} \geq H_m = 0 \) such that \( H_i/H_{i+1} \cong S^\nu_{i+1}(\deg \nu^i + \text{codeg } \nu_{i+1}) \), for \( 0 \leq i < m \);

(b) The module \( G_\mu \) has a Specht filtration \( G_\mu = H_0 \geq H_1 \geq \cdots \geq H_{l-1} \geq H_l = 0 \) such that \( H^i/H^{i+1} \cong S^\nu_{i+1}(\text{codeg } t_{\nu_i} + \text{deg } u_{i+1}) \), for \( 0 \leq i < l \).

Proof. We prove only (b). Part (a) can be proved in a similar way. Mirroring the proof of Corollary 4.11, define \( H^i \) to be the \( Z \)-submodule of \( G_\mu \) spanned by the elements

\[
\{ \psi_{t,\nu, \psi_{u,j}^{\nu} \mid t \in \text{Std}(\nu^j) \text{ and } 1 + i \leq j \leq l \}.
\]

This is an \( \mathcal{R}_n^\Lambda \)-submodule of \( G_\mu \) by Theorem 3.14 and (GC2) of Definition 2.4. As in the proof of Corollary 4.11, it is easy to verify that \( H^i/H^{i+1} \cong S^\nu_{i+1}(\text{codeg } t_{\nu_i} + \text{deg } u_{i+1}) \); compare with [33, Corollaries 3.11, 3.12]. The degree shift is just the difference of the degrees of the basis elements of \( S^\nu_{i+1} \) and the degrees of the elements \( \psi_{t,\nu, \psi_{u,i+1}^{\nu}} \).
Recall from (3.18) that $R^n_\Lambda = \bigoplus_\beta R^n_\beta$ and that $R^n_\beta$ carries a non-degenerate homogeneous trace form $\tau_\beta$ of degree $-2\text{def} \beta$ by Theorem 3.20. The following argument is lifted from [47, Proposition 5.13].

**Theorem 4.14.** Suppose that $Z = K$ is a field and that $\mu \in \mathcal{P}^+_\beta$, for $\beta \in Q^+_n$. Then, as $\mathcal{R}^n_\Lambda$-modules,

$$G^\mu \cong (G^\mu)^\oplus (2 \text{def} \beta) \quad \text{and} \quad G_\mu \cong (G^\mu)^\oplus (2 \text{def} \beta).$$

**Proof.** Both isomorphisms can be proved similarly, so we consider only the first one. Using Theorem 4.9 and Corollary 4.12, define a pairing $G^\mu \times G^\mu \rightarrow Z$ by

$$\langle \psi_{st}, \psi_{tu} \psi_{uv}^* \rangle_\mu = \tau_\beta(\psi_{st} \psi_{uv}^*),$$

for all $s \in \text{Std}^\mu(\lambda)$, $t \in \text{Std}(\lambda)$, $u \in \text{Std}^\mu(\nu)$, $v \in \text{Std}(\nu)$, for some $\lambda, \nu \in \mathcal{P}^+_\beta$. By Theorem 3.20, $\tau_\beta(\psi_{st} \psi_{uv}^*) \neq 0$ and $\tau_\beta(\psi_{st} \psi_{uv}^*) \neq 0$ only if $(u, v) \triangleright (s, t)$. Therefore, the Gram matrix of $\langle \cdot, \cdot \rangle_\mu$ is upper triangular with non-zero elements on the diagonal, so $\langle \cdot, \cdot \rangle_\mu$ is non-degenerate. Recalling the degree shift in the definition of $G^\mu$ from Definition 4.2, it is easy to check that $\langle \cdot, \cdot \rangle_\mu$ is a homogeneous bilinear map of degree $-2\text{def} \beta$.

We claim that $\langle \cdot, \cdot \rangle_\mu$ is associative in the sense that

$$\langle \psi_{st} h, \psi_{tu} \psi_{uv}^* \rangle_\mu = \langle \psi_{st}, \psi_{tu} \psi_{uv}^* h^* \rangle_\mu,$$

for all $h \in \mathcal{R}^n_\Lambda$ and all $(s, t)$ and $(u, v)$ as above. Write $\psi_{uv}^* h^* = \sum r_{ab} \psi_{ab}$, where in the sum $(a, b) \in \text{Std}^2(\mathcal{P}^+_\beta)$ and $r_{ab} \in Z$. Then the left-hand side is equal to

$$\langle \psi_{st} h, \psi_{tu} \psi_{uv}^* \rangle_\mu = \tau_\beta(\psi_{st} \psi_{uv}^*) = \sum_{(a, b) \in \text{Std}^2(\mathcal{P}^+_\beta)} r_{ab} \tau_\beta(\psi_{st} \psi_{ba}^*).$$

Now $\tau_\beta$ is a trace form, so $\tau_\beta(\psi_{st} \psi_{ba}^*) = \tau_\beta(\psi_{ba} \psi_{st})$ is non-zero only if $a \triangleright s$ and res($a$) = res($s$) by Lemma 3.15, so that $a \in \text{Std}^\mu(\mathcal{P}^+_\beta)$. Consequently,

$$\langle \psi_{st} h, \psi_{tu} \psi_{uv}^* \rangle_\mu = \sum_{a \in \text{Std}^\mu(\nu), b \in \text{Std}(\nu)} r_{ab} \tau_\beta(\psi_{st} \psi_{ba}^*),$$

where the last equality follows using Lemma 3.15 and Corollary 4.12. Hence, the form $\langle \cdot, \cdot \rangle_\mu$ is associative. Taking duals reverses the grading. Therefore, the map $x \mapsto \langle x, ? \rangle_\mu$, for $x \in G^\mu$, gives the required isomorphism. \qed

### 4.2. Quiver Schur algebras

We are now ready to define the quiver Schur algebras of type $\Gamma$, which are the main objects of study in this paper. Recall that $Z$ is an arbitrary integral domain.

**Definition 4.15.** Suppose that $\Lambda \in P^+$ and let $G^\Lambda_n = \bigoplus_{\mu \in \mathcal{P}^+_n} G^\mu$. The quiver Schur algebra of type $(\Gamma, \Lambda)$ is the endomorphism algebra

$$S^\Lambda_n = S_n^\Lambda(\Gamma) = \mathcal{E}nd_{\mathcal{R}^\Lambda_n}(G^\Lambda_n).$$
By definition $S_n^\Lambda$ is a graded $\mathcal{Z}$-algebra. As a $\mathcal{Z}$-module, $S_n^\Lambda$ admits a decomposition
\[ S_n^\Lambda = \bigoplus_{\nu, \mu \in \mathcal{P}_n^\Lambda} \text{Hom}_{\mathcal{R}_n^\Lambda}(G^\nu, G^\mu). \]

By Theorem 3.20, $\mathcal{R}_n^\Lambda$ is a graded symmetric algebra, so by [22, 61.2]
\[ \text{Hom}_{\mathcal{R}_n^\Lambda}(G^\nu, G^\mu) \cong G^\mu \cap (G^\nu)^* \]
as graded $\mathcal{Z}$-modules, where an isomorphism is given by $\Psi(\nu \rightarrow \nu^\nu$). By Corollary 4.10, if $s \in \text{Std}^\mu(\lambda)$ and $t \in \text{Std}^\nu(\lambda)$, for $\lambda \in \mathcal{P}_n^\Lambda$, then $\psi_{st} \in G^\mu \cap (G^\nu)^*$ so we can define a homomorphism $\Psi_{st}^\mu \in \text{Hom}_{\mathcal{R}_n^\Lambda}(G^\nu, G^\mu)$ by
\[ \Psi_{st}^\mu(e^\nu y^\nu h) = \psi_{st} h \quad \text{for all } h \in \mathcal{R}_n^\Lambda. \]

We think of $\Psi_{st}^\mu$ as an element of $S_n^\Lambda$ in the obvious way. By definition, $\Psi_{st}^\mu$ is homogeneous of degree $(\text{deg } s - \text{deg } t^\nu) + (\text{deg } t - \text{deg } \nu^\nu$ since $\psi_{st}$ has degree $\text{deg } s + \text{deg } t - \text{deg } \nu^\nu$ when considered as an element of $G^\nu$.

**Example 4.18.** It is necessary to include $\mu$ and $\nu$ in the notation $\Psi_{st}^\mu$ because a given tableau can belong to $\text{Std}^\nu(\nu)$ for many different $\mu$. The simplest example of this phenomenon occurs when $t = (1, 1)$ and $\kappa = (0, 0)$, so that $\Lambda = 2\Lambda_0$. Let $\mu = (1|e)$ and $\nu = (1|1)$. Then $t \in \text{Std}^\nu(\mu)$ and $\psi_{tt} = e^\nu y^\nu \in G^\mu \cap G^\nu \cap (G^\nu)^* \cap (G^\nu)^*$ by Corollary 4.10. Therefore, the tableau $t$ determines four different maps in $S_n^\Lambda$:
\[ \Psi_{tt}^\mu : G^\mu \rightarrow G^\mu; e^\nu y^\nu h \rightarrow \psi_{tt} h, \quad \Psi_{tt}^\nu : G^\nu \rightarrow G^\nu; e^\nu y^\nu h \rightarrow \psi_{tt} h, \quad \Psi_{tt}^\nu : G^\nu \rightarrow G^\nu; e^\nu y^\nu h \rightarrow \psi_{tt} h, \quad \Psi_{tt}^\nu : G^\nu \rightarrow G^\nu; e^\nu y^\nu h \rightarrow \psi_{tt} h. \]

We have $\text{deg } \Psi_{tt}^\mu = 0$, $\text{deg } \Psi_{tt}^\mu = 1 = \text{deg } \Psi_{tt}^\mu \text{ deg } \Psi_{tt}^\mu = 1 = \text{deg } \Psi_{tt}^\mu \text{ and } \text{deg } \Psi_{tt}^\mu = 2$. For $\lambda \in \mathcal{P}_n^\Lambda$, let $T^\Lambda = \{ (\mu, s) \mid s \in \text{Std}^\mu(\lambda) \text{ for } \mu \in \mathcal{P}_n^\Lambda \}$.

**Theorem 4.19.** Suppose that $e = 0$ or $e > n$ and that $\mathcal{Z}$ is an integral domain. Then $S_n^\Lambda$ is a graded cellular algebra with cellular basis $\{ \Psi_{st}^\mu \mid (\mu, s), (\nu, t) \in T^\Lambda \text{ and } \lambda \in \mathcal{P}_n^\Lambda \}$, weight poset $(\mathcal{P}_n^\Lambda, \triangleleft)$ and degree function $\text{deg } \Psi_{st}^\mu = \text{deg } s - \text{deg } t^\nu + \text{deg } t - \text{deg } \nu^\nu$.

**Proof.** By Corollary 4.10 and (4.16), the maps in the statement of the theorem are a basis of $S_n^\Lambda$. As in [24, §6], it is now a purely formal argument to show that this basis is a cellular basis of $S_n^\Lambda$. We have already verified axioms (GC$_d$) and (GC$_1$) from §2.2. Axiom (GC$_3$) is a straightforward calculation using the fact that $\psi_{st}^\mu = \psi_{tu}$ by Theorem 3.14; see [24, Proposition 6.9]. It remains to check (GC$_2$) but this follows by repeating the argument from [24, Theorem 6.6(ii)], essentially without change, using Corollary 4.10 and Theorem 3.14.

**Remark 4.20.** In [24, Theorem 6.6], the cellular basis of the cyclotomic $q$-Schur algebras is labelled by semistandard tableaux of type $\nu$. The tableaux in $T^\Lambda$ are, in fact, closely related to semistandard tableaux. Using the notation of [24, Definition 4.2], if $(\nu, t) \in T^\Lambda$, then $\nu(t)$ is a semistandard $\lambda$-tableau of type $\nu$. In fact, Assumption 4.1 implies that the entries in each row of $\mu$ have distinct residues. Consequently, if $s, t \in \text{Std}^\nu(\lambda)$, then $s = t$ if and only if $\nu(s) = \nu(t)$.

**Example 4.21.** If $\ell = 2$, then $S_2^\Lambda$ is positively graded by Theorem B.3, proved in the appendix. If $\ell > 2$, then $S_2^\Lambda$ is, in general, not positively graded. For example, suppose that $\Lambda = 3\Lambda_0$, $\mu = (1|2), 1|2^2$ and
\[ t = \left( \begin{array}{c|c|c|c} 1 & 6 & 2 & 3 \\
7 & 4 & 8 & 5 \end{array} \right). \]
Then it is easy to check that $t \in \text{Std}^{\mu}(2,1|2^2|1)$ and that $\deg t = 2 < \deg t^\mu = 3$. So, $\deg \Psi_t = -2$. □

Now we know that $S_n^\lambda$ is a graded cellular algebra we can use the general theory from §2.2 to construct cell modules and irreducible $S_n^\lambda$-modules.

Suppose that $\lambda \in \mathcal{P}_n^\lambda$. The graded Weyl module $\Delta^\lambda$ is the cell module for $S_n^\lambda$ corresponding to $\lambda$. More explicitly, $\Delta^\lambda$ is the $S_n^\lambda$-module with basis

$$\{ \Psi_{t}^\lambda \mid (\nu, t) \in T^\lambda \} \quad (4.22)$$

such that $(\Phi^\lambda_{\lambda_1} S_n^\lambda + (S_n^\lambda)^{>\lambda})/(S_n^\lambda)^{\geq \lambda} \cong \Delta^\lambda$ under the map that sends $\Psi_{t}^{\lambda'} + (S_n^\lambda)^{>\lambda}$ to $\Psi_{t}^\lambda$, for $(\nu, t) \in T^\lambda$. (Note that $\deg \Psi_{t}^\lambda = 0$, for all $\lambda \in \mathcal{P}_n^\lambda$.)

As in §2.2, the graded Weyl module $\Delta^\lambda$ comes equipped with a homogeneous bilinear form $\langle , \rangle$ of degree zero such that

$$\langle \Psi_{s}^\mu, \Psi_{t}^\nu \rangle \Psi_{t}^\lambda = \Psi_{s}^{\mu} \Psi_{t}^{\nu \lambda} \quad \text{for } (\mu, s), (\nu, t) \in T^\lambda. \quad (4.23)$$

Define $L^\lambda = \Delta^\lambda / \text{rad} \Delta^\lambda$, where $\text{rad} \Delta^\lambda$ is the radical of this form. Set $\nabla^\lambda = (\Delta^\lambda)^\oplus$.

**Theorem 4.24.** Suppose that $e = 0$ or $e > n$ and that $\mathcal{Z} = K$ is a field. Then $S_n^\lambda$ is a quasi-hereditary graded cellular algebra with:

(i) weight poset $(\mathcal{P}_n^\lambda, \geq)$;

(ii) graded standard modules $\{ \Delta^\lambda \mid \lambda \in \mathcal{P}_n^\lambda \}$;

(iii) graded costandard modules $\{ \nabla^\lambda \mid \lambda \in \mathcal{P}_n^\lambda \}$;

(iv) graded simple modules $\{ L^\lambda \langle k \rangle \mid \lambda \in \mathcal{P}_n^\lambda \text{ and } k \in \mathbb{Z} \}$.

Moreover, $L^\lambda \cong (L^\lambda)^\oplus$ for all $\lambda \in \mathcal{P}_n^\lambda$.

**Proof.** By definition, $\Psi_{t}^{\lambda_1} S_n^\lambda$ is the identity map on $G^\lambda$, so $\langle \Psi_{t}^{\lambda_1}, \Psi_{t}^\lambda \rangle = 1$ by (4.23). Consequently, $L^\lambda \neq 0$ for all $\lambda \in \mathcal{P}_n^\lambda$. Therefore, $L^\lambda \cong (L^\lambda)^\oplus$, for $\lambda \in \mathcal{P}_n^\lambda$, and

$$\{ L^\lambda \langle k \rangle \mid \lambda \in \mathcal{P}_n^\lambda \text{ and } k \in \mathbb{Z} \}$$

is a complete set of pairwise non-isomorphic irreducible $S_n^\lambda$-modules by Theorem 2.5. In turn, this implies that $S_n^\lambda$ is a quasi-hereditary algebra by Corollary 2.9, with standard and costandard modules as stated. □

For each $\lambda \in \mathcal{P}_n^\lambda$ set $\Psi^\lambda = \Psi_{t}^{\lambda_1} S_n^\lambda$. Then $\Psi^\lambda$ (restricts to) the identity map on $G^\lambda$ and $\sum_{\lambda} \Psi^\lambda$ is the identity element of $S_n^\lambda$. As a $\mathcal{Z}$-module, every $S_n^\lambda$-module $M$ has a weight space decomposition

$$M = \bigoplus_{\lambda \in \mathcal{P}_n^\lambda} M_\lambda, \quad \text{where } M_\lambda = M \Psi^\lambda. \quad (4.25)$$

In particular, if $\lambda, \nu \in \mathcal{P}_n^\lambda$, then $\{ \Psi_{t}^\nu \mid (\nu, t) \in T^\lambda \}$ is a basis of $\Delta^\lambda$ by (4.22).

**Remark 4.26.** Although we will not need this, the reader can check that if $(\nu, t) \in T^\lambda$, then we can identify $\Psi_{t}^{\nu}$ with the homomorphism $G^\nu \to S^\lambda$ that sends $\psi^{\nu \nu \cdot h}$ to $\psi_{t \lambda} h$, for $h \in R_n^\lambda$. In this way, $\Delta^\lambda$ can be identified with a $S_n^\lambda$-submodule of $\text{Hom}_{\mathcal{R}_n^\lambda}(G^\nu, S^\lambda)$. By Corollary 4.11, there is a projection map $\pi^\lambda : G^\lambda \to S^\lambda$ such that $\pi^\lambda(\psi_{t \lambda} h) = \psi_{t \lambda} h$, for all $h \in R_n^\lambda$. By Theorem 4.19, and the remarks after (4.22), the weight space $\Delta^\lambda$ of the Weyl module $\Delta^\lambda$ can be identified with the set of maps in $\text{Hom}_{\mathcal{R}_n^\lambda}(G^\nu, S^\lambda)$ that factor through $\pi^\lambda$ so that the
4.3. Graded Schur functors

We now define an exact functor from the category of graded $S_n^\lambda$-modules to the category of graded $R_n^\lambda$-modules and use this to relate the graded decomposition numbers of the two algebras. To do this, we introduce a slightly bigger version of the quiver Schur algebra $S_n^\lambda$. The idea is to enlarge $S_n^\lambda$ so that it contains a copy of $\text{End}_{R_n^\lambda}(R_n^\lambda)$ and then use this to construct a graded Schur functor via (2.10).

Let $\tilde{S}_n^\lambda = S_n^\lambda \cup \{\omega\}$, where $\omega$ is a dummy symbol, and set $G^\omega = R_n^\lambda$ and $G_n^\lambda = G_n^\lambda \oplus G^\omega$. The extended quiver Schur algebra is the algebra

$$\tilde{S}_n^\lambda = \text{End}_{R_n^\lambda}(G_n^\lambda).$$

Suppose that $\nu \in \tilde{S}_n^\lambda$. For convenience of notation, set $\text{Std}^\nu(\nu) = \text{Std}(\nu)$ and define $e^\omega = 1 = y^\omega \in R_n^\lambda$ so that $G^\omega = e^\omega y^\omega R_n^\lambda$. Let $t^\omega = 1$ and set $\psi_{t^\omega t^\omega} = e^\omega y^\omega = 1$ and define $\deg t^\omega = 0$. We consider $S_n^\lambda$ is a graded subalgebra of $\tilde{S}_n^\lambda$ in the obvious way.

Extending (4.17), if $\nu, \mu \in \tilde{S}_n^\lambda$ and $s \in \text{Std}^\mu(\nu)$ and $t \in \text{Std}^\nu(\nu)$, then define

$$\Psi_s^\mu t^t (e^\nu y^\nu h) = \psi_{sht}$$

for all $h \in R_n^\lambda$.

Then $\Psi_s^\mu t^t \in \tilde{S}_n^\lambda$ and $\deg \Psi_s^\mu t^t = \deg s - \deg t^\mu + \deg t - \deg t^t$. For each multipartition $\lambda \in \tilde{S}_n^\lambda$ set $T^\lambda = \{(\nu, t) \mid t \in \text{Std}^\nu(\lambda) \text{ for } \nu \in \tilde{S}_n^\lambda\} = T^\lambda \cup \{\omega\} \times \text{Std}(\lambda)$.

**Proposition 4.27.** The algebra $\tilde{S}_n^\lambda$ is a graded cellular algebra with basis

$$\{ \Psi_s^\mu t^t \mid (\mu, s), (\nu, t) \in T^\lambda \text{ for } \lambda \in \tilde{S}_n^\lambda \},$$

weight poset $(\tilde{S}_n^\lambda, \geq)$ and degree function

$$\deg \Psi_s^\mu t^t = \deg s - \deg t^\mu + \deg t - \deg t^t.$$

Moreover, if $Z = K$ is a field, then $\tilde{S}_n^\lambda$ is a quasi-hereditary algebra with standard modules $\{ \Delta^\lambda \mid \lambda \in \tilde{S}_n^\lambda \}$ and simple modules $\{ \tilde{L}^\lambda(\langle k \rangle) \mid \lambda \in \tilde{S}_n^\lambda \text{ and } k \in Z \}$.

**Proof.** By definition, $S_n^\lambda$ is a subalgebra of $\tilde{S}_n^\lambda$ and, as a $Z$-module,

$$\tilde{S}_n^\lambda = S_n^\lambda \oplus \text{Hom}_{R_n^\lambda}(G^\omega, G_n^\lambda) \oplus \text{Hom}_{R_n^\lambda}(G_n^\lambda, G^\omega) \oplus \text{End}_{R_n^\lambda}(G^\omega).$$

For $\mu \in \tilde{S}_n^\lambda$, there are isomorphisms of graded $Z$-modules $G^\mu \cong \text{Hom}_{R_n^\lambda}(G^\omega, G^\mu)$ given by $\psi_{sht}$ for $s \in \text{Std}^\mu(\nu)$ and $t \in \text{Std}^\nu(\nu)$ and $\nu \in \tilde{S}_n^\lambda$. Therefore, the elements in the statement of the proposition give a basis of $\tilde{S}_n^\lambda$ by Theorems 4.19 and 4.9.

Now suppose that $Z = K$ is a field. Repeating the arguments from Theorems 4.19 and 4.24 shows that $\tilde{S}_n^\lambda$ is a quasi-hereditary graded cellular algebra.

By Proposition 4.27, there exist Weyl modules $\tilde{\Delta}^\lambda$ and simple modules $\tilde{L}^\lambda = \tilde{\Delta}^\lambda / \text{rad} \tilde{\Delta}^\lambda$ for $\tilde{S}_n^\lambda$, for each $\lambda \in \tilde{S}_n^\lambda$. As in (4.22), let $\{ \Psi_s^\mu t^t \mid (\mu, t) \in T^\lambda \}$ be the basis of $\tilde{\Delta}^\lambda$.

Set $\Psi_n^\lambda = \sum_{\mu \in \tilde{S}_n^\lambda} \Psi_s^\mu$ and let $\Psi^\omega$ be the identity map on $G^\omega = R_n^\lambda$. Then $\Psi_n^\lambda$ is the identity element of $S_n^\lambda$ and $\Psi_n^\lambda + \Psi^\omega$ is the identity element of $\tilde{S}_n^\lambda$. By definition, $\Psi_n^\lambda$ and $\Psi^\omega$ are both
idempotents in $\mathcal{S}_n^\Lambda$ and $\Psi_n^\Lambda \mathcal{S}_n^\Lambda \Psi_n^\Lambda \cong \mathcal{S}_n^\Lambda$. Therefore, by (2.10), there are exact graded functors

\[ \hat{F}_n^\omega : \mathcal{S}_n^\Lambda \text{-Mod} \rightarrow \mathcal{S}_n^\Lambda \text{-Mod}\] and \[ \hat{G}_n^\omega : \mathcal{S}_n^\Lambda \text{-Mod} \rightarrow \mathcal{S}_n^\Lambda \text{-Mod}\]
given by $\hat{F}_n^\omega(M) = M \Psi_n^\Lambda$ and $\hat{G}_n^\omega(N) = N \otimes_{\mathcal{S}_n^\Lambda} \Lambda_n^\Lambda \Psi_n^\Lambda$. By §2.4, there are graded functors

\[ H_{n,\omega} := H \Psi_n^\Lambda, \quad O_{n,\omega} := O \Psi_n^\Lambda, \quad O^{n,\omega} := O \Psi_n^\Lambda \]
from $\mathcal{S}_n^\Lambda$-Mod to $\mathcal{S}_n^\Lambda$-Mod such that $H_{n,\omega}(M) = M/O_{n,\omega}(M)$.

**Lemma 4.28.** Suppose that $Z = K$ is a field. Then the functors $\hat{F}_n^\omega$ and $\hat{G}_n^\omega$ induce mutually inverse graded equivalences of categories between $\mathcal{S}_n^\Lambda$-Mod and $\mathcal{S}_n^\Lambda$-Mod. Moreover,

\[ \hat{F}_n^\omega(\Delta^\lambda) \cong \Delta^\lambda \text{ and } \hat{F}_n^\omega(\Lambda^\lambda) \cong \Lambda^\lambda, \]
for all $\lambda \in \mathcal{P}_n^\Lambda$.

**Proof.** Let $M$ be an $\mathcal{S}_n^\Lambda$-module. Then, extending (4.25), $M$ has a weight space decomposition

\[ M = \bigoplus_{\mu \in \mathcal{P}_n^\Lambda} M_\mu, \quad \text{where } M_\mu = M \Psi_\mu. \]

Then, essentially by definition, $\hat{F}_n^\omega(M) = \bigoplus_{\lambda \in \mathcal{P}_n^\Lambda} M_\lambda$. That is, $\hat{F}_n^\omega$ removes the $\omega$-weight space of $M$. In particular, $\hat{F}_n^\omega(\Delta^\mu) = \Delta^\mu$ and $\hat{F}_n^\omega(\Lambda^\mu) = \Lambda^\mu$, for all $\mu \in \mathcal{P}_n^\Lambda$. The fact that $\hat{F}_n^\omega(\mu) = \mu$ for all $\mu \in \mathcal{P}_n^\Lambda$ implies that $O^{n,\omega}(M) = M, O_{n,\omega}(M) = 0$, for all $M \in \mathcal{S}_n^\Lambda$-Mod. Therefore, $H_{n,\omega}$ is the identity functor and $\hat{G}_n^\omega \cong H_{n,\omega} \circ \hat{G}_n^\omega$. Hence, the lemma is an application of the theory of quotient functors given in Theorem 2.11. □

The identity map $\Psi^\omega$ on $\mathcal{R}_n^\Lambda = G^\omega$ is idempotent in $\mathcal{S}_n^\Lambda$ and there is a graded isomorphism of $Z$-algebras $\Psi^\omega \mathcal{S}_n^\Lambda \Psi^\omega \cong \mathcal{R}_n^\Lambda$. Therefore, by (2.10), there are functors

\[ \hat{F}_n^\Lambda : \mathcal{S}_n^\Lambda \text{-Mod} \rightarrow \mathcal{R}_n^\Lambda \text{-Mod}\] and \[ \hat{G}_n^\Lambda : \mathcal{R}_n^\Lambda \text{-Mod} \rightarrow \mathcal{S}_n^\Lambda \text{-Mod}\]
given by $\hat{F}_n^\Lambda(M) = M \Psi^\omega = M_\omega$ and $\hat{G}_n^\Lambda(N) = N \otimes_{\mathcal{R}_n^\Lambda} \Psi^\omega \mathcal{S}_n^\Lambda$.

**Proposition 4.30.** Suppose that $Z = K$ is a field and $\beta \in Q^+_\Lambda$. Then there is an exact graded functor $\hat{F}_n^\Lambda : \mathcal{S}_n^\Lambda$-Mod $\rightarrow \mathcal{R}_n^\Lambda$-Mod given by $\hat{F}_n^\Lambda(M) = (M \otimes_{\mathcal{S}_n^\Lambda} \Psi_n^\Lambda \mathcal{S}_n^\Lambda) \Psi^\omega$, for $M \in \mathcal{S}_n^\Lambda$-Mod, such that if $\lambda, \mu \in \mathcal{P}_n^\Lambda$, then $\hat{F}_n^\Lambda(\Delta^\lambda) \cong \Delta^\lambda, \hat{F}_n^\Lambda(\nabla^\lambda) \cong \nabla^\lambda, \hat{F}_n^\Lambda(\mu) \cong \mu$, for all $\mu \in \mathcal{P}_n^\Lambda$.

\[ \hat{F}_n^\Lambda(\mu) \cong \begin{cases} D^\mu & \text{if } \mu \in \mathcal{K}_n^\Lambda, \\ 0 & \text{if } \mu \notin \mathcal{K}_n^\Lambda. \end{cases} \]

**Proof.** By definition, $\hat{F}_n^\Lambda = \hat{F}_n^\omega \circ \hat{G}_n^\Lambda$, so $\hat{F}_n^\Lambda$ is an exact graded functor from $\mathcal{S}_n^\Lambda$-Mod to $\mathcal{R}_n^\Lambda$-Mod. The functor $\hat{F}_n^\Lambda$ is nothing more than projection onto the $\omega$-weight space. Hence, if $\lambda \in \mathcal{P}_n^\Lambda$, then $\hat{F}_n^\Lambda(\Delta^\lambda)$ is spanned by the maps $\{ \Psi^\omega_t \mid t \in \text{Std}(\lambda) \}$, since $\text{Std}(\lambda) = \text{Std}(\Lambda)$. The map $\Psi^\omega_t \mapsto \psi_t$, for $t \in \text{Std}(\lambda)$, defines an isomorphism $\hat{F}_n^\Lambda(\Delta^\lambda) \cong \mathcal{S}_n^\Lambda$ of $\mathcal{R}_n^\Lambda$-modules. Therefore, $\hat{F}_n^\Lambda(\Delta^\lambda) \cong \mathcal{S}_n^\Lambda$ by Lemma 4.28. The functor $\hat{F}_n^\Lambda$ is easily seen to commute with duality (in $\mathcal{S}_n^\Lambda$ and $\mathcal{R}_n^\Lambda$), so $\hat{F}_n^\Lambda(\nabla^\lambda) \cong \hat{F}_n^\Lambda(\Delta^\lambda)^{\oplus} \cong \mathcal{S}_n^\Lambda(\text{def } \beta)$ by Proposition 3.21.

By Theorem 2.11, $\hat{F}_n^\Lambda(\mu)$ is an irreducible $\mathcal{R}_n^\Lambda$-module whenever it is non-zero. A straightforward argument by induction on the dominance ordering using $\hat{F}_n^\Lambda(\Delta^\lambda) \cong \mathcal{S}_n^\Lambda$, Corollary 2.7 and Proposition 3.22 now shows that $\hat{F}_n^\Lambda(\mu) \cong D^\mu$ if $\mu \in \mathcal{K}_n^\Lambda$ and that $\hat{F}_n^\Lambda(\mu) = 0$ otherwise.

Since $\hat{F}_n^\Lambda$ is exact and graded, we obtain the promised relationship between the graded decomposition numbers of $\mathcal{S}_n^\Lambda$ and $\mathcal{R}_n^\Lambda$. □
Corollary 4.31. Suppose that $Z = K$ is a field and that $\lambda \in \mathcal{P}_n^\Lambda$ and $\mu \in \mathcal{K}_n^\Lambda$. Then 
$[S^\lambda : D^\mu]_q = [\Delta^\lambda : L^\mu]_q$.

The graded decomposition multiplicities $[\Delta^\lambda : L^\mu]_q$ are one of the main objects of interest in this paper so we give them a special name.

Definition 4.32. Suppose that $\lambda, \mu \in \mathcal{P}_n^\Lambda$. Set

$$d_{\lambda\mu}(q) = [\Delta^\lambda : L^\mu]_q = \sum_{d \in \mathbb{Z}} [\Delta^\lambda : L^\mu(d)]_q^d.$$

Let $D_{S^\lambda_n}(q) = (d_{\lambda\mu}(q))_{\lambda, \mu \in \mathcal{P}_n^\Lambda}$ and $D_{R^\lambda_n}(q) = (d_{\lambda\mu}(q))_{\lambda \in \mathcal{P}_n^\Lambda, \mu \in \mathcal{K}_n^\Lambda}$ be the graded decomposition matrices of $S^\lambda_n$ and $R^\lambda_n$, respectively.

By Corollary 4.31, $D_{R^\lambda_n}(q)$ can be considered as a submatrix of $D_{S^\lambda_n}(q)$. For future use, we note the following important property of these Laurent polynomials. This is the general property of (graded) cellular algebras given in Corollary 2.7.

Corollary 4.33. Suppose that $\lambda, \mu \in \mathcal{P}_n^\Lambda$. Then $d_{\mu\mu}(q) = 1$ and $d_{\lambda\mu}(q) \neq 0$ only if $\lambda \geq \mu$ and $\lambda, \mu \in \mathcal{P}_n^\beta$ for some $\beta \in Q_n^+$. 

4.4. Blocks of quiver Schur algebras

We now give the block decomposition of the graded Schur algebra $S^\lambda_n$. The key observation is the following double centralizer result.

Recall from §4.3 that $G^\Lambda_n = G^\Lambda_n \oplus R^\Lambda_n$ and $\hat{S}^\Lambda_n = \text{End}_{R^\Lambda_n}(\hat{G}^\Lambda_n)$.

Lemma 4.34 (A double centralizer property). There are canonical isomorphisms of graded algebras such that $\hat{S}^\Lambda_n \cong \text{End}_{R^\Lambda_n}(\hat{G}^\Lambda_n)$ and $R^\Lambda_n \cong \text{End}_{S^\Lambda_n}(\hat{G}^\Lambda_n)$. In particular, the functor $\psi^\Lambda_n$ is fully faithful on projectives.

Proof. The first isomorphism is the definition of $\hat{S}^\Lambda_n$, whereas the second follows directly from the definition of $\hat{S}^\Lambda_n$, because

$$R^\Lambda_n \cong \text{Hom}_{R^\Lambda_n}(R^\Lambda_n, R^\Lambda_n) \cong \psi^\Lambda_n \hat{S}^\Lambda_n \psi^\Lambda_n \cong \text{End}_{R^\Lambda_n}(\psi^\Lambda_n \hat{S}^\Lambda_n),$$

and $\psi^\Lambda_n \hat{S}^\Lambda_n \cong \hat{G}^\Lambda_n$ as a right $\hat{S}^\Lambda_n$-module.

In order to describe the block decomposition of $S^\Lambda_n$ we set $G^\Lambda_\beta = \bigoplus_{\mu \in \mathcal{P}_n^\beta} G^{\mu}$ and define $S^\Lambda_\beta = \text{End}_{R^\Lambda_n}(G^\Lambda_\beta)$ if $\beta \in Q_n^+$. Equivalently, $S^\Lambda_\beta = \psi^\beta \hat{S}^\Lambda_n \Psi^\beta$, where $\Psi^\beta = \sum_{\mu \in \mathcal{P}_n^\beta} \Psi^\mu$.

The subalgebras $S^\Lambda_\beta$ of $S^\Lambda_n$ are the blocks of $S^\Lambda_n$. More precisely, we have the following.

Theorem 4.35. Suppose that $Z = K$ is a field and $\Lambda \in P^+$. Then

$$S^\Lambda_n = \bigoplus_{\beta \in Q_n^+} S^\Lambda_\beta,$$

is the block decomposition of $S^\Lambda_n$ into a direct sum of indecomposable two-sided ideals. Moreover, if $\beta \in Q_n^+$, then the cellular basis of $S^\Lambda_n$ in Theorem 4.19 restricts to give a graded cellular basis of $S^\Lambda_\beta$. Consequently, $S^\Lambda_\beta$ is a quasi-hereditary graded cellular algebra.
Proof. Let \( \alpha, \beta \in \mathcal{Q}^+ \) and \( \mu \in \mathcal{P}_n^\Lambda \). By Theorem 4.9, if \( g \in \mathcal{G}^\mu \), then \( g e_\alpha = \delta_{\alpha \beta} g \). Therefore, if \( \alpha \neq \beta \) and \( \lambda \in \mathcal{P}_n^\Lambda \), then \( \text{Hom}_{\mathcal{R}_n^\Lambda}(G^\lambda, G^\mu) = 0 \). Hence, as \( \mathcal{Z} \)-modules,

\[
S^\Lambda_n = \text{End}_{\mathcal{R}_n^\Lambda}(G^\Lambda_n) = \bigoplus_{\alpha, \beta \in Q^+_n} \text{Hom}_{\mathcal{R}_n^\Lambda}(G^\Lambda_\alpha, G^\Lambda_\beta)
\]

\[
= \bigoplus_{\beta \in Q^+_n} \text{End}_{\mathcal{R}_n^\Lambda}(G^\Lambda_\beta) = \bigoplus_{\beta \in Q^+_n} S^\Lambda_\beta.
\]

It follows that the cellular basis of Theorem 4.19 restricts to give cellular bases for the algebras \( S^\Lambda_\beta \), for \( \beta \in Q^+_n \). Therefore, \( S^\Lambda_\beta \) is a quasi-hereditary graded cellular algebra for each \( \beta \in Q^+_n \).

It remains to show that each of the algebras \( S^\Lambda_\beta \) is indecomposable. By the double centralizer property (Lemma 4.34), the algebras \( \mathcal{R}_n^\Lambda \) and \( S^\Lambda_n \) have the same number of blocks and \( S^\Lambda_n \) and \( S^\Lambda_\beta \) have the same number of indecomposable two-sided ideals by Lemma 4.28. By (3.18), the blocks of \( \mathcal{R}_n^\Lambda \) are indexed by \( Q^+_n \). As the elements of \( Q^+_n \) also index the subalgebras \( S^\Lambda_\beta \), the non-zero algebras \( S^\Lambda_\beta \) must be indecomposable giving the result. \( \square \)

For each \( \beta \in Q^+_n \) define \( \mathcal{F}_n^\Lambda(\beta) = \mathcal{F}_n^\Lambda(\mathcal{M}(\Psi^\beta)) \), for an \( S^\Lambda_\beta \)-module \( M \). Then \( \mathcal{F}_n^\Lambda(\beta) \) is the subfunctor of \( \mathcal{F}_n^\Lambda \) obtained by first projecting onto the block \( S^\Lambda_\beta \). Hence, we have the following refinement of Proposition 4.30.

**Corollary 4.36.** The functor \( \mathcal{F}_n^\Lambda(\beta) \) is fully faithful on projective \( S^\Lambda_\beta \)-modules. Moreover, there is a decomposition of functors \( \mathcal{F}_n^\Lambda \cong \bigoplus_{\beta \in Q^+_n} \mathcal{F}_n^\Lambda(\beta) \), where \( \mathcal{F}_n^\Lambda : S^\Lambda_\beta \text{-Mod} \to \mathcal{R}_n^\Lambda \text{-Mod} \) is the restriction of \( \mathcal{F}_n^\Lambda \) to \( S^\Lambda_\beta \text{-Mod} \) for \( \beta \in Q^+_n \).

**Proof.** By definition, \( \mathcal{F}_n^\Lambda \cong \bigoplus_{\beta \in Q^+_n} \mathcal{F}_n^\Lambda(\beta) \) so we only need to check that \( \mathcal{F}_n^\Lambda(\beta) \) is fully faithful on projectives. This follows because \( \mathcal{F}_n^\Lambda = \mathcal{F}_n^\Lambda \circ \hat{\mathcal{G}}^\Lambda_n \) and the functor \( \hat{\mathcal{F}}^\Lambda_n \) is fully faithful on projectives by Lemma 4.34 (and \( \hat{\mathcal{G}}^\Lambda_n \) is an equivalence of categories). \( \square \)

**Corollary 4.37.** Suppose that \( \beta \in Q^+_n \). Then \( S^\Lambda_\beta \) is a quasi-hereditary cover of \( \mathcal{R}_n^\Lambda \) in the sense of Rouquier [54, Definition 4.34].

**Proof.** Recall that \( \hat{\mathcal{G}}^\Lambda_n \cong \Psi^\omega \hat{\mathcal{S}}^\Lambda_n \) is a projective \( \hat{\mathcal{S}}^\Lambda_n \)-module. Using the graded Morita equivalence between \( S^\Lambda_n \) and \( \hat{\mathcal{S}}^\Lambda_n \), we see that \( G^\Lambda_\beta \) is a projective \( S^\Lambda_\beta \)-module. By Corollary 4.36, the functor \( \hat{\mathcal{F}}^\Lambda_n \) is fully faithful on projectives, and so is \( \mathcal{F}_n^\Lambda(\beta) \) because \( \mathcal{F}_n^\Lambda(\beta) \) is the composition of \( \hat{\mathcal{F}}^\Lambda_n \) with an equivalence of categories. This implies that \( S^\Lambda_\beta \) is a quasi-hereditary cover of \( \mathcal{R}_n^\Lambda \) in the sense of Rouquier [54, Definition 4.34]. \( \square \)

4.5. Sign-dual quiver Schur algebras

In this section, we construct a twisted version of the quiver Schur algebras by considering endomorphisms of the graded permutation modules \( G_\mu \), for \( \mu \in \mathcal{P}_n^\Lambda \). The twisted quiver Schur algebras really come from twisting by the sign automorphism \( \text{sgn} \) of \( \mathcal{R}_n^\Lambda \) defined in § 4.5. The twisted Schur algebras turn out to be Ringel dual to the algebras \( S^\Lambda_\beta \). We need the twisted Schur algebras in order to understand the \( \Delta \)-filtration multiplicities of tilting modules in § 7.4.

Suppose that \( \beta \in Q^+_n \) and recall the sign isomorphism \( \text{sgn} : \mathcal{R}_n^\Lambda \to \mathcal{R}_n^{\Lambda'} \) from (3.24). Consider the \( \mathcal{R}_n^{\Lambda'} \)-module

\[
G^\Lambda_{\alpha'} = \bigoplus_{\mu \in \mathcal{P}_n^{\Lambda'}} G_\mu.
\]
The sign-dual quiver Schur algebra of type \( (\Gamma, \Lambda')_{\beta'} \) is the algebra
\[
S_{\beta'}^{\beta} = S_{\beta}^{\beta}(\Gamma_{\epsilon}) = \mathcal{E}nd_{R_{\beta}^{\beta'}}(G_{\Lambda}^{\beta'}).
\]
By (3.24) and Lemma 4.3, we have
\[
S_{\beta'}^{\beta} = \mathcal{E}nd_{R_{\beta}^{\beta'}} \left( \bigoplus_{\mu \in \mathcal{P}_{\beta}^{\Lambda}} G_{\mu} \right) \cong \mathcal{E}nd_{R_{\beta}^{\beta'}} \left( \bigoplus_{\mu \in \mathcal{P}_{\beta}^{\Lambda}} G_{\mu} \right) = S_{\beta}^{\Lambda}.
\]
(4.38)
That is, \( S_{\beta'}^{\beta} \cong S_{\beta}^{\Lambda} \) as graded algebras. For \( \lambda \in \mathcal{P}_{\beta}^{\Lambda} \), let
\[
T_{\lambda} = \{ (\nu', t') \mid t \in \text{Std}_{\nu}(\lambda') \text{ for } \nu \in \mathcal{P}_{\beta}^{\Lambda} \} = \{ (\nu', t') \mid (\nu, t) \in T_{\lambda} \}.
\]
As noted in (4.16), \( \mathcal{H}om_{R_{\beta}^{\beta'}}(G_{\mu'}, G_{\nu'}) \cong G_{\mu'} \cap G_{\mu'}^{\nu'} \) as graded vector spaces. Therefore, by Corollary 4.10, the algebra \( S_{\lambda}^{\beta'} \) is free as a \( \mathcal{Z} \)-module with basis
\[
\{ \Psi_{\mu'}^{\nu'}(s) = \delta_{\mu', \nu'} \} \text{ for } (\mu', \nu'), (\nu', t') \in T_{\lambda} \text{ for } \lambda \in \mathcal{P}_{\beta}^{\Lambda'}, \text{ (4.39)}
\]
where \( \Psi_{\mu'}^{\nu'}(s) \) is the \( \mathcal{R}_{\beta'}^{\beta} \)-endomorphism of \( G_{\mu'}^{\nu'} \) given by
\[
\Psi_{\mu'}^{\nu'}(t) = e_{s} \psi_{t}^{\nu'} h,
\]
for \( (\mu', \nu'), (\nu', t') \in T_{\lambda} \) as above and \( \sigma \in \mathcal{P}_{\beta}^{\Lambda} \). By Proposition 3.27 and Lemma 4.3, the isomorphism \( S_{\beta'}^{\beta} \cong S_{\beta}^{\Lambda} \) above sends the basis element \( \Psi_{\mu'}^{\nu'}(s) \) of \( S_{\beta}^{\Lambda} \) to \( \text{sgn}(\sigma) \Psi_{\mu'}^{\nu'}(s) \). Therefore, by Theorems 4.19 and 4.24, this basis makes \( S_{\beta'}^{\beta} \) into a quasi-hereditary graded cellular algebra with weight poset \( (\mathcal{P}_{\beta}^{\Lambda'}, \leq) \).

If \( \lambda \in \mathcal{P}_{\beta}^{\Lambda} \), let \( \Delta_{\lambda} \) be the corresponding Weyl module of \( S_{\beta'}^{\beta} \) determined by the cellular basis \( \{ \Psi_{\mu'}^{\nu'}(s) \} \) given above and let \( L_{\lambda} = \Delta_{\lambda} / \text{rad} \Delta_{\lambda} \) be its simple head.

**Theorem 4.40.** Suppose that \( \Lambda \in P^{+} \text{ and } \beta \in Q^{+} \). The sign isomorphism \( \text{sgn}: \mathcal{R}_{\beta}^{\Lambda} \rightarrow \mathcal{R}_{\beta'}^{\Lambda'} \) induces a canonical degree persevering, poset reversing, isomorphism of quasi-hereditary graded cellular algebras \( \text{sgn}: S_{\beta}^{\Lambda} \rightarrow S_{\beta'}^{\beta} \). Moreover, when \( \mathcal{Z} = K \) is a field there are isomorphisms
\[
\Delta_{\mu} \cong \Delta_{\mu'}^{\text{sgn}} \text{ and } L_{\mu} \cong L_{\mu'}^{\text{sgn}}
\]
of \( S_{\beta}^{\Lambda} \)-modules, for \( \mu \in \mathcal{P}_{\beta}^{\Lambda} \). Consequently, \( [\Delta_{\lambda} : L_{\mu}]_{\mathcal{Z}} = [\Delta_{\lambda} : L_{\mu'}]_{\mathcal{Z}} \), for all \( \lambda, \mu \in \mathcal{P}_{\beta}^{\Lambda} \).

**Proof.** By (4.38), the automorphism \( \text{sgn}: \mathcal{R}_{\beta}^{\Lambda} \rightarrow \mathcal{R}_{\beta'}^{\Lambda'} \) induces an isomorphism \( \text{sgn}: S_{\beta}^{\Lambda} \rightarrow S_{\beta'}^{\beta} \) of graded algebras. This isomorphism sends the basis element \( \Psi_{\mu'}^{\nu'}(s) \) of \( S_{\beta}^{\Lambda} \) to \( \text{sgn}(s) \Psi_{\mu'}^{\nu'}(s) \) of \( S_{\beta'}^{\beta} \). Hence, the sign isomorphism identifies the cell module \( \Delta_{\mu} \) of \( S_{\beta}^{\Lambda} \) with the cell module \( \Delta_{\mu'} \) of \( S_{\beta'}^{\beta} \) (compare with Proposition 3.27 and Corollary 3.28). All of the remaining claims now follow. 

To each pair of weights \( (\Lambda, \beta) \in P^{+} \times Q^{+} \), we have now attached four different quiver Schur algebras \( S_{\beta}^{\Lambda}, S_{\beta}^{\beta}, S_{\beta'}^{\Lambda}, S_{\beta'}^{\beta} \). Each of these algebras is a quasi-hereditary graded cellular algebra. To avoid confusion, we clarify the relationships between these four algebras. By definition,
\[
S_{\beta}^{\Lambda} = \mathcal{E}nd_{R_{\beta}^{\Lambda}} \left( \bigoplus_{\mu \in \mathcal{P}_{\beta}^{\Lambda}} G_{\mu} \right) \text{ and } S_{\beta}^{\beta} = \mathcal{E}nd_{R_{\beta}^{\beta}} \left( \bigoplus_{\mu \in \mathcal{P}_{\beta}^{\Lambda}} G_{\mu} \right).
\]
Both of these algebras are defined using the cyclotomic quiver Hecke algebra $R_{13}^\Lambda$ and the multicharge $\kappa$. The cellular basis of $S_\beta^\Lambda$ is obtained by lifting the $\psi$-bases of the graded permutation modules $\{G^\mu\}$ and the graded cellular basis of $S_\Lambda^\beta$ is defined by lifting the $\psi'$-bases of $\{G_{\mu}\}$. Using the $\sgn$ isomorphism we can define isomorphic versions of both of these algebras using $R_{13}^\Lambda$ and the ‘conjugate’ multicharge $\kappa'$. That is, define

$$S_\beta^\Lambda = \text{End}_{R_{13}^\Lambda} \left( \bigoplus_{\mu \in P_{\beta}} G_{\mu} \right) \quad \text{and} \quad S_{\beta'}^\Lambda = \text{End}_{R_{13}^\Lambda} \left( \bigoplus_{\mu \in P_{\beta'}} G_{\mu'} \right).$$

Then the isomorphism $\sgn: R_{\beta'}^\Lambda \to R_{\beta}^\Lambda$ induces algebra isomorphisms $S_{\beta'}^\Lambda \cong S_{\beta}^\Lambda$ and $S_{\beta'}^\Lambda \cong S_{\beta}^\Lambda$. Therefore, via the algebra $S_{\beta'}^\Lambda$, we can transfer all of the theory that we have developed for $S_{\beta'}^\Lambda$ to $S_{\beta}^\Lambda$. The algebras $S_{\beta'}^\Lambda$ and $S_{\beta}^\Lambda$ are not isomorphic. Having both algebras is useful because the two algebras give rise to different graded Schur functors

$$F_{\beta}^\Lambda: S_{\beta'}^\Lambda \text{-Mod} \to R_{\beta'}^\Lambda \text{-Mod} \quad \text{and} \quad F_{\beta}^\Lambda: S_{\beta}^\Lambda \text{-Mod} \to R_{\beta}^\Lambda \text{-Mod},$$

where $F_{\beta}^\Lambda = \sgn \circ F_{\beta}^\Lambda \circ \sgn^{-1}$. We are abusing notation in the definition of $F_{\beta}^\Lambda$ because the left-hand $\sgn$ is the equivalence $\sgn: R_{\beta'}^\Lambda \text{-Mod} \to R_{\beta}^\Lambda \text{-Mod}$ of (3.25), whereas $\sgn^{-1}: S_{\beta'}^\Lambda \text{-Mod} \to S_{\beta}^\Lambda \text{-Mod}$ is the inverse of the equivalence given by Theorem 4.40.

We will show in Theorem 5.20 below that $(S_{\beta'}^\Lambda, S_{\beta}^\Lambda)$ and $(S_{\beta'}^\Lambda, S_{\beta'}^\Lambda)$ are both pairs of Ringel dual algebras. Thus, ultimately, the $\sgn$ isomorphism of the cyclotomic quiver Hecke algebras induces Ringel duality at the level of the cyclotomic quiver Schur algebras.

## 5. Tilting modules

In this section, we introduce the tilting modules for $S_\mu^\Lambda$, and the closely related Young modules for $R_{13}^\Lambda$. In Section 6, we use the Young modules to prove that the cyclotomic quiver Schur algebra $S_\mu^\Lambda$ is Morita equivalent to the (ungraded) cyclotomic Schur algebras of [13, 24], whereas the tilting modules give one of the Kazhdan–Lusztig bases of the Fock space in §7.4. Throughout this section, we continue working over a field and we maintain our standing Assumption 4.1 that $e = 0$ or $e > n$.

### 5.1. Young modules

In this section, we show that there exists a family of indecomposable $R_{13}^\Lambda$-modules indexed by $P_\Lambda^\beta$ and that $G^\mu$ is a direct sum of these modules, for each $\mu \in P_\Lambda^\beta$.

Fix $\beta \in Q_n^+$ and recall from (4.25) that every $S_\beta^\Lambda$-module has a weight space decomposition. Analogously, as a right $S_\beta^\Lambda$-module, the regular representation of $S_\beta^\Lambda$ has a decomposition into a direct sum of left weight spaces:

$$S_\beta^\Lambda = \bigoplus_{\mu \in P_\beta^\Lambda} Z^\mu, \quad \text{where} \quad Z^\mu = \Psi^\mu S_\beta^\Lambda \text{ for } \mu \in P_\beta^\Lambda. \quad (5.1)$$

Since $\Psi^\mu$ is an idempotent in $S_\beta^\Lambda$, $Z^\mu$ is a projective $S_\beta^\Lambda$-module. By Theorem 4.19, the module $Z^\mu$ has basis $\{\Psi^\mu_{s_{\lambda}} | (\mu, s), (\nu, t) \in \mathcal{T}_\Lambda \text{ and } \lambda \in P_\beta^\Lambda\}$.

Let $P^\mu$ be the projective cover of $L^\mu$ (in the category of graded $S_\beta^\Lambda$-modules). By Corollary 2.8, $P^\mu$ has a filtration by Weyl modules in which $\Delta^\lambda$ appears with graded multiplicity $(P^\mu : \Delta^\lambda)_q = |\Delta^\lambda : L^\mu|_q$. We now describe an analogous $\Delta$-filtration of $Z^\mu$.

Fix a total ordering $\text{Std}^\nu(\mathcal{P}_\beta) = \{s_1, \ldots, s_x\}$ such that $a > b$ whenever $\lambda_a > \lambda_b$, where $\lambda_c = \text{Shape}(s_c)$. In particular, $s_1 = e^\mu$. If $a \geq 1$, let $M_a$ be the submodule of $Z^\mu$ spanned by

$$\{\Psi^\mu_{s_{\lambda}} | t \in \text{Std}^\nu(\lambda_b) \text{ for } \nu \in P_\beta^\Lambda \text{ and } b \geq a\}.$$
By Theorem 4.19 and (GC$_2$) of Definition 2.4, the Weyl filtration of $S^\lambda_\beta$ restricts to $Z^\mu$ showing that

$$Z^\mu = M_1 \supset M_2 \supset \cdots \supset M_z \supset 0$$

(5.2)

is an $S^\lambda_\beta$-module filtration of $Z^\mu$ with $M_a/M_{a+1} \cong \Delta^\lambda_{\mu_a} (\deg s_a - \deg t^\mu)$, for $1 \leq a \leq z$. Thus, in the notation of §4.3, $Z^\mu$ has a $\Delta$-filtration in which $\Delta^\lambda$ appears with graded multiplicity

$$\langle Z^\mu : \Delta^\lambda \rangle_q := \sum_{s \in \text{Std}^\mu(\lambda)} q^{\deg s - \deg t^\mu}.$$  

(5.3)

Since $S^\lambda_\beta$ is quasi-hereditary, $(Z^\mu : \Delta^\lambda)_q$ is independent of the choice of $\Delta$-filtration.

By the last paragraph $(Z^\mu : \Delta^\mu)_q = 1$ and there is a surjection $Z^\mu \twoheadrightarrow \Delta^\mu$. Moreover, $\Delta^\lambda$ appears in $Z^\mu$ only if $\lambda \geq \mu$. Therefore, since $Z^\mu$ is projective, it follows that

$$Z^\mu = P^\mu \oplus \bigoplus_{\lambda \geq \mu} z_{\lambda\mu}(q)P^\lambda,$$

(5.4)

for some Laurent polynomials $z_{\lambda\mu}(q) \in \mathbb{N}[q, q^{-1}]$. This observation will be used later to compute the graded decomposition numbers of $S^\lambda_\beta$ in characteristic zero.

Analogously, let $Z_\mu = \Psi_\mu S^\beta_\Lambda$, where $\Psi_\mu = \Psi_{\mu\mu'} \in S^\beta_\Lambda$ is the identity map on $G_\mu$. Then, as before, $Z_\mu$ is a projective $S^\beta_\Lambda$-module so there exist Laurent polynomials $z_{\lambda\mu}(q) \in \mathbb{N}[q, q^{-1}]$ such that $Z_\mu = P_\mu \oplus \bigoplus_{\lambda \geq \mu} z_{\lambda\mu}(q)P_\lambda$, where $P_\lambda$ is the projective cover of $L_\lambda$ in $S^\lambda_\Lambda$-Mod. As in §4.3, there is a graded Schur functor $F^\beta_\Lambda : S^\beta_\Lambda$-Mod $\rightarrow \mathcal{R}^\beta_\Lambda$-Mod for $S^\beta_\Lambda$. Equivalently, $F^\beta_\Lambda = \sgn \circ F^\beta_{\mu'} \circ \sgn^{-1}$ by Theorem 4.40.

**Definition 5.5.** Suppose that $\mu \in \mathcal{P}^\Lambda_\beta$. The graded Young modules are the $\mathcal{R}^\Lambda_\beta$-modules

$$Y^\mu = F^\Lambda_\beta(P^\mu) \quad \text{and} \quad Y_\mu = F^\beta_\Lambda(P_\mu).$$

The next result gives some justification for this terminology. In Lemma 6.11, we will show that the graded Young modules are graded lifts of the Young modules for $\mathcal{H}^n$ introduced in [47].

**Proposition 5.6.** Suppose that $\beta \in Q^+_n$ and that $\lambda, \mu \in \mathcal{P}^\Lambda_\beta$. Then the following hold.

(a) The Young modules $Y^\mu$ and $Y_\mu$ are indecomposable $\mathcal{R}^\Lambda_\beta$-modules. Moreover, $Y^\mu \cong Y^{\sgn\mu}$.

(b) If $d \in \mathbb{Z}$, then $Y^\mu \cong Y^\nu(d)$ if and only if $\lambda = \mu$ and $d = 0$. Similarly, $Y_\mu \cong Y_\lambda(d)$ if and only if $\lambda = \mu$ and $d = 0$.

(c) We have $G^\mu \cong Y^\mu \oplus \bigoplus_{\lambda \geq \mu} z_{\lambda\mu}(q)Y^\lambda$ and $G_\mu \cong Y_\mu \oplus \bigoplus_{\lambda \geq \mu} z_{\lambda\mu}(q)Y_\lambda$.

(d) The Young module $Y^\mu$ has a graded Specht filtration in which $S^\lambda$ appears with graded multiplicity

$$\langle Y^\mu : S^\lambda \rangle_q = [\Delta^\lambda : L^\mu]_q$$

and $Y_\mu$ has a graded dual Specht filtration in which $S^\lambda$ appears with graded multiplicity

$$\langle Y_\mu : S^\lambda \rangle_q = [\Delta^\lambda : L_\mu]_q.$$  

Proof. By Corollary 4.36, the functor $F^\Lambda_\beta$ is fully faithful on projective modules, so $\mathcal{E}nd_{\mathcal{R}^\Lambda_\beta}(Y^\mu) \cong \mathcal{E}nd_{S^\beta_\Lambda}(P^\mu)$ is a local ring since $P^\mu$ is indecomposable. Similarly, $\mathcal{E}nd_{\mathcal{R}^\Lambda_\beta}(Y_\mu)$ is a local ring. Hence, $Y^\mu$ and $Y_\mu$ are indecomposable $\mathcal{R}^\Lambda_\beta$-modules. Moreover, the fact that $F^\beta_\Lambda$ is fully faithful on projectives also implies (b) since the $P^\mu(d)$ are pairwise non-isomorphic.

Applying the Schur functor from Proposition 4.30,

$$F^\beta_\Lambda(Z^\mu) = \Psi^\mu S^\Lambda_\beta \Psi_{\nu'} \cong \text{Hom}_{\mathcal{R}^\Lambda_\beta}(R^\Lambda_\beta, G^\mu) \cong G^\mu.$$
Hence, the first formula in part (c) follows from (5.4). Similarly, $F^\beta_\lambda(Z_\mu) \cong \text{Hom}_{R^\beta}(R^\beta_\lambda, G_\mu) \cong G_\mu$, which implies the second formula in part (c). By Lemma 4.3, $G_\mu \cong (G_\mu)^{\text{sgn}}$ so it follows by induction on dominance that $Y_\mu \cong Y_\mu^{\text{sgn}}$ for all $\mu \in \mathcal{P}_\beta$, completing the proof of (a).

Now consider (d). If $\mu \in \mathcal{P}_\beta$, then $L^\mu \neq 0$ by Theorem 4.24. Therefore,

$$(P^\mu : \Delta^\lambda)_q = [\Delta^\lambda : L^\mu]_q$$

by Corollary 2.8, for $\lambda \in \mathcal{P}_\beta$. Therefore, $(Y^\mu : S^\lambda)_q = (P^\mu : \Delta^\lambda)_q = [\Delta^\lambda : L^\mu]_q$ by the exactness of $F^\beta_\lambda$ and Proposition 4.30. By the same argument, $(Y_\mu : S^\lambda)_q = (P_\mu : \Delta^\lambda)_q = [\Delta^\lambda : L_\mu]_q$, so (d) holds. (Note that we are not claiming that the graded Specht filtration multiplicities for $Y^\mu$ and $Y_\mu$ are independent of the choice of filtration.)

**Remark 5.7.** The Laurent polynomials $z^\beta_\lambda(q)$ in part (c) of Proposition 5.6 should be computed using the analogue of (7.36) for the algebra $S^\lambda_\beta \cong \tilde{S}^\beta_\lambda$, whereas the graded decomposition number $[\Delta^\lambda : L_\mu]_q$ in part (d) is for the algebra $S^\lambda_\beta \cong \tilde{S}^\beta_\lambda$.

**Corollary 5.8.** Suppose that $\mu \in \mathcal{P}_\beta$, for $\beta \in Q^+_n$. Then, as $R^\beta_\lambda$-modules

$$Y^\mu \cong (Y^\mu)^{\otimes/(2 \text{def } \beta)}$$

and

$$Y_\mu \cong (Y_\mu)^{\otimes/(2 \text{def } \beta)}.$$ 

**Proof.** As both isomorphisms can be proved similarly we consider only the first one. If $\mu$ is maximal in $\mathcal{P}_\beta$, then $Y^\mu = G^\mu$ by Proposition 5.6(c), so in this case the result is a special case of Theorem 4.14. Now, $G^\mu \cong (G^\mu)^{\otimes/(2 \text{def } \beta)}$ by Theorem 4.14. So if $\mu$ is not maximal the result now follows by induction on dominance using parts (b) and (c) of Proposition 5.6.

We want to identify the projective Young modules. Recall that $P^\mu$ is the projective cover of $L^\mu$ and $F^\beta_\lambda(L^\mu) = D^\mu$ if $\mu \in K^\lambda_n$ by Proposition 4.30.

**Proposition 5.9.** Suppose that $\mu \in K^\lambda_n$, for $\beta \in Q^+_n$. Then $Y^\mu$ is the projective cover of $D^\mu$.

**Proof.** As $F^\lambda_n$ is exact there is a surjective map $Y^\mu \twoheadrightarrow D^\mu$. Therefore, it suffices to show that $Y^\mu$ is projective since $Y^\mu$ is indecomposable by Proposition 5.6(a).

Recall from §4.3, that $S^\lambda_n = \text{End}_{\mathcal{R}^\lambda_n}(G^\lambda_n)$, where $G^\lambda_n = G^\lambda_n \oplus \mathcal{R}^\lambda_n$. By (4.29), there is a graded Schur functor $F^\lambda_n$ from $S^\lambda_n$-Mod to $\mathcal{R}^\lambda_n$-Mod given by $F^\lambda_n(M) = M_\mu$. In particular, $F^\lambda_n(G^\lambda_n) \cong \mathcal{R}^\lambda_n$ as graded $\mathcal{R}^\lambda_n$-modules.

As an $S^\lambda_n$-module, $G^\lambda_n \cong \Psi^\omega S^\lambda_n$. In particular, $G^\lambda_n$ is a projective $S^\lambda_n$-module. If $\lambda \in \mathcal{P}_\lambda$, let $P^\lambda$ be the projective cover of the irreducible $S^\lambda_n$-module $L^\lambda$. The graded multiplicity of $P^\lambda$ as a summand of $G^\lambda_n$ is equal to

$$\dim_q \text{Hom}_{S^\lambda_n}(G^\lambda_n, L^\lambda) = \dim_q \text{Hom}_{\mathcal{R}^\lambda_n}(\Psi^\omega S^\lambda_n, L^\lambda) = \dim_q L^\lambda \Psi^\omega = \dim_q D^\lambda,$$

where the first equality follows because $\Psi^\omega$ is an idempotent and the second comes from Proposition 4.30. Consequently, $\tilde{G}^\lambda_n \cong \bigoplus_{\lambda \in K^\lambda_n} (\dim_q D^\lambda) P^\lambda$ as an $S^\lambda_n$-module. By definition, $Y^\lambda = F^\lambda_n(P^\lambda) = F^\lambda_n(\tilde{P}^\lambda)$, for all $\lambda \in \mathcal{P}_\lambda$. Therefore

$$\mathcal{R}^\lambda_n = F^\lambda_n(\tilde{G}^\lambda_n) \cong \bigoplus_{\lambda \in K^\lambda_n} (\dim_q D^\lambda) Y^\lambda$$

as a right $\mathcal{R}^\lambda_n$-module. The result follows.
A prinjective module for an algebra is a module that is both projective and injective.

**Corollary 5.10.** Suppose that \( \mu \in \mathcal{K}_{\beta}^{\Lambda} \) and \( \beta \in Q_{\Lambda}^{+} \). Then \( P_{\mu} \cong (P_{\mu})^{\circ}(2 \text{def } \beta) \).

Consequently, \( P_{\mu} \) is a prinjective \( \mathcal{S}_{\Lambda}^{\mu} \)-module.

**Proof.** By the proof of Proposition 5.9, \( \hat{G}_{n}^{\Lambda} \) is a projective \( \mathcal{S}_{\Lambda}^{\mu} \)-module. Moreover, \( \hat{G}_{n}^{\Lambda} \cong (\hat{G}_{n}^{\Lambda})^{\circ}(2 \text{def } \beta) \) as an \( \mathcal{R}_{\beta}^{\Lambda} \)-module by Theorems 4.14 and 3.20. By a standard argument, see [23, (1.5), (1.6)], this implies that \( \hat{G}_{n}^{\Lambda} \cong (\hat{G}_{n}^{\Lambda})^{\circ}(2 \text{def } \beta) \) as an \( \mathcal{S}_{\Lambda}^{\mu} \)-module. The proof of Proposition 5.9 shows that \( \hat{G}_{n}^{\Lambda} \) is a filtration by the graded duals of shifted Weyl modules. Consequently, up to shift, \( P_{\mu}^{\circ} \) and \( (P_{\mu})^{\circ} \) are both summands of \( \hat{G}_{n}^{\Lambda} \). Therefore, there exists \( \nu \in \mathcal{K}_{\Lambda}^{\mu} \) and \( d \in \mathbb{Z} \) such that \( (P_{\mu})^{\circ}(2 \text{def } \beta) \cong P_{\nu}(d) \). Applying the graded Schur functor and Corollary 5.8, we deduce that \( Y_{\nu} \cong (Y_{\mu})^{\circ}(2 \text{def } \beta) \cong Y_{\nu}(d) \). Hence, \( d = 0 \) and \( \nu = \mu \) by Proposition 5.6(b), completing the proof. \( \square \)

### 5.2. Tilting modules

By Theorem 4.24, \( \mathcal{S}_{\Lambda}^{\mu} \) is a quasi-hereditary algebra. An \( \mathcal{S}_{\Lambda}^{\mu} \)-module \( T \) is a (graded) tilting module if it has both a filtration by shifted Weyl modules \( \Delta^{\lambda}(k) \), for \( \lambda \in \mathcal{P}_{\Lambda}^{\beta} \) and \( k \in \mathbb{Z} \), and a filtration by the graded duals of shifted Weyl modules.

On forgetting the grading, Theorem 4.24 says that the ungraded algebra \( \mathcal{S}_{\Lambda}^{\mu} \) is quasi-hereditary. Therefore, by a famous theorem of Ringel [52], for each \( \mu \in \mathcal{P}_{\beta}^{\Lambda} \) there exists a unique \( \mathcal{S}_{\Lambda}^{\mu} \)-module \( T_{\mu} \) such that

- (a) \( T_{\mu} \) is indecomposable;
- (b) \( T_{\mu} \) has both a \( \Delta \)-filtration and a \( \nabla \)-filtration;
- (c) \( (\Delta_{\mu} : \Delta^{\mu}) = 1 \) and \( (T_{\mu} : \Delta^{\lambda}) \neq 0 \) only if \( \mu \trianglerighteq \lambda \).

Ringel’s construction extends to the graded case to show that every tilting module for \( \mathcal{S}_{\Lambda}^{\mu} \) has a graded lift; see [50, 58]. Since \( T_{\mu} \) is indecomposable it follows that there is a unique graded lift \( T_{\mu}^{n} \) of \( T_{\mu}^{n} \) such that \( (T_{\mu}^{n} : \Delta^{\mu})_{q} = 1 \). The aim of this section is to show that \( T_{\mu}^{n} \cong (T_{\mu}^{n})^{\circ} \) is graded self-dual. To prove this, we need another description of the graded tilting modules.

Fix \( \mu \in \mathcal{P}_{\beta}^{\Lambda} \) and let \( \theta_{\mu} \in \text{Hom}_{\mathcal{R}_{\beta}^{\Lambda}}(\mathcal{R}_{\beta}^{\Lambda}, G_{\mu}) \) be the map in \( \mathcal{S}_{\Lambda}^{\mu} \) given by

\[
\theta_{\mu}(h) = \psi_{t_{\mu}}^{\prime} h \quad \text{for all } h \in \mathcal{R}_{\beta}^{\Lambda}.
\]

We define analogues of the exterior powers for \( \mathcal{S}_{\beta}^{\Lambda} \) using the functor \( \hat{F}_{\mu}^{\varphi} \) from (4.29).

**Definition 5.11.** Suppose that \( \mu \in \mathcal{P}_{\beta}^{\Lambda} \). Define \( E_{\mu}^{n} = \hat{F}_{\mu}^{\varphi}(\theta_{\mu}^{\Lambda} \mathcal{S}_{\Lambda}^{\mu})(- \text{def } \beta) \).

Observe that \( E_{\mu}^{n} \) is a right \( \mathcal{S}_{\Lambda}^{\mu} \)-module under composition of maps because, by definition, \( E_{\mu}^{n} \) is the set of maps from \( G_{\mu}^{\Lambda} \) to \( G_{\mu}^{\Lambda}(- \text{def } \beta) \) which factor through \( \theta_{\mu}^{\Lambda} \):

\[
G_{\mu}^{\Lambda} \xrightarrow{\exists \theta^{\prime}} \mathcal{R}_{\beta}^{\Lambda} \xrightarrow{\theta} G_{\mu}^{\Lambda}(- \text{def } \beta)
\]

This is similar to the description of the Weyl module \( \Delta^{\mu} \) given in Remark 4.26.

Our first aim is to give a basis for \( E_{\mu}^{n} \). Note that if \( \lambda \in \mathcal{P}_{\beta}^{\Lambda} \), \( s \in \text{Std}_{\mu}(\lambda) \) and \( t \in \text{Std}^{\circ}(\lambda) \), then \( \psi_{t_{\mu}, \psi_{s_{\mu}}}^{\prime} \in G_{\mu}^{\Lambda} \cap (G_{\mu}^{\circ})^{\ast} \) by Corollary 4.12. Therefore, we can define
\( \theta_{st}^{\mu\nu} \in \text{Hom}_{R_\beta} (G^\nu, G_\mu (\text{def } \beta)) \) by

\[
\theta_{st}^{\mu\nu} (e^\nu y^\nu h) = \psi'_{t_u t_v} \psi_{st} h,
\]
for all \( h \in R_\beta^\lambda \). Recall that \( T^\lambda = \{ (\nu, t) \mid t \in \text{Std}^\nu (\lambda) \text{ for } \nu \in \mathcal{P}_\beta^\lambda \} \) and that \( \hat{T}^\lambda \) is defined in the same way except that \( \nu \in \hat{\mathcal{P}}_\beta^\lambda = \mathcal{P}_\beta^\lambda \cup \{ \omega \} \).

**Theorem 5.12.** Suppose that \( \mu \in \mathcal{P}_\beta^\lambda \), for \( \beta \in Q_+^n \). Then

\[
\{ \theta_{st}^{\mu\nu} \mid s \in \text{Std}_\mu (\lambda) \text{ and } (\nu, t) \in T^\lambda \text{ for some } \lambda \in \mathcal{P}_\beta^\lambda \}
\]
is a basis of \( E^\mu \). Moreover, \( \text{deg } \theta_{st}^{\mu\nu} = \text{deg } s - \text{deg } t_\mu + \text{deg } t - \text{deg } t^\nu - \text{deg } \beta \).

**Proof.** Let \( \hat{E}^\mu = \theta_\mu \mathcal{S}_n^\Lambda \). Then \( \hat{E}^\mu \) is a right \( \mathcal{S}_n^\Lambda \)-module and \( E^\mu = \hat{E}^\mu (\text{def } \beta) \). By Proposition 4.27, \( \hat{E}^\mu \) is spanned by the maps \( \theta_\mu \Psi_{st}^{\nu \sigma} \), for \( \sigma, \nu \in \mathcal{P}_n^\Lambda \), \( (\sigma, s), (\nu, t) \in T^\lambda \), and \( \lambda \in \mathcal{P}_\beta^\lambda \). By definition,

\[
\theta_\mu \Psi_{st}^{\nu \sigma} (e^\nu y^\nu h) = \delta_{s, \sigma} \psi'_{t_u t_v} \psi_{st} h.
\]
Hence, applying Lemma 3.15, \( \theta_\mu \Psi_{st}^{\nu \sigma} \) is non-zero only if \( \sigma = \omega \), \( \text{res}(s) = \text{res}(t_\mu) \) and \( t_\mu \geq s \), so that \( s \in \text{Std}_\mu (\mathcal{P}_\beta^\lambda) \). Consequently, in this case, \( \theta_{st}^{\mu\nu} = \theta_{st}^{\mu\nu} \). Therefore, the elements

\[
\{ \theta_{st}^{\mu\nu} \mid s \in \text{Std}_\mu (\lambda) \text{ and } (t, \nu) \in \hat{T}^\lambda \text{ for some } \lambda \in \mathcal{P}_\beta^\lambda \}
\]
span \( \hat{E}^\mu \). On the other hand, these elements are linearly independent because \( \{ \theta_{st}^{\mu\nu} (e^\nu y^\nu) \} = \{ \psi'_{t_u t_v} \psi_{st} \} \) is a linearly independent subset of \( G_\mu \) by Corollary 4.12(a). Hence, we have found a basis for \( E^\mu \). Applying the functor \( \hat{F}^\nu \) kills the \( \omega \)-weight space of \( \hat{E}^\mu \). So \( \hat{F}^\nu \) maps the basis \( \{ \theta_{st}^{\mu\nu} \} \) of \( E^\mu \) to the elements in the statement of the theorem, or to zero if \( \nu = \omega \). Hence, \( \{ \theta_{st}^{\mu\nu} \} \) is a basis of \( E^\mu \).

To complete the proof it remains to compute \( \text{deg } \theta_{st}^{\mu\nu} \), for \( s \in \text{Std}_\mu (\lambda) \), \( (\nu, t) \in T^\lambda \), and \( \lambda \in \mathcal{P}_\beta^\lambda \). Recalling the degree shifts in the definition of the three modules\( G^\nu, G_\mu \) and \( E^\mu \),

\[
\text{deg } \theta_{st}^{\mu\nu} = \text{deg } t_\mu + \text{deg } s + \text{deg } t - \text{deg } t^\nu - \text{deg } \beta.
\]
By Lemma 3.10, \( \text{deg } t_\mu - \text{deg } \beta = -\text{deg } t_\mu \), so \( \text{deg } \theta_{st}^{\mu\nu} = \text{deg } s - \text{deg } t_\mu + \text{deg } t - \text{deg } t^\nu \) as required.

By definition, \( s \in \text{Std}_\mu (\mathcal{P}_n^\Lambda) \) only if \( t_\mu \geq s \) which implies that \( \mu \geq \text{Shape}(s) \). Order \( \text{Std}_\mu (\mathcal{P}_\beta^\lambda) = \{ s_1, \ldots, s_y \} \) so that \( a > b \) whenever \( \lambda_a > \lambda_b \), where \( \lambda_c = \text{Shape}(s_c) \), for \( 1 \leq c \leq y \). (In particular, \( s_y = t_\mu \).) The proof of Theorem 5.12 shows that \( \theta_{st}^{\mu\nu} = \theta_{st}^{\mu\nu} \) so, as in (5.2), the cell filtration of \( \mathcal{S}_\beta^\Lambda \) gives the following.

**Corollary 5.13.** Suppose that \( \mu \in \mathcal{P}_\beta^\lambda \). Then \( E^\mu \) has a \( \Delta \)-filtration

\[
E^\mu = E_1 > E_2 > \cdots > E_y > 0,
\]
such that \( E_r / E_{r+1} \cong \Delta^{\lambda_r} (\text{deg } s_r - \text{deg } t_\mu) \), for \( 1 \leq r \leq y \). In particular, \( \Delta^\mu \) is a submodule of \( E^\mu \), \( (E^\mu : \Delta^\mu)_q = 1 \) and \( (E^\mu : \Delta^\lambda)_q \neq 0 \) only if \( \mu \geq \lambda \).

We now give a second basis of \( E^\mu \) and use it to show that \( E^\mu \) is a tilting module. Suppose that \( \nu \in \text{Std}_\mu (\nu) \) and \( \psi \in \text{Std}_\nu (\psi) \), for \( \lambda, \nu \in \mathcal{P}_\beta^\Lambda \). Then \( \psi'_{\nu \psi} \psi_{\nu \psi} \in G_\mu (\text{def } \beta) \) by Corollary 4.11. Therefore, we can define \( \theta_{\mu\nu}^{\lambda \lambda} \in \text{Hom}_{R_\beta^\Lambda} (G^\lambda, G_\mu (\text{def } \beta)) \) by

\[
\theta_{\mu\nu}^{\lambda \lambda} (e^\lambda y^\lambda h) = \psi'_{\nu \psi} \psi_{\nu \psi} h,
\]
for \( h \in R_\beta^\Lambda \).
Lemma 5.14. Suppose that \( Z = K \) is a field and that \( \mu \in \mathcal{P}_\beta^\lambda \). Then
\[
\{ \theta_{\mu \nu} \mid (\mu, u) \in \mathcal{T}_\lambda, (\nu, v) \in \mathcal{T}_\lambda^\mu \text{ for some } \lambda \in \mathcal{P}_\beta^\lambda \}
\]
is a basis of \( E^\mu \). Moreover, \( \deg \theta_{\mu \nu} = \operatorname{codeg} u - \operatorname{codeg} t_\mu + \operatorname{codeg} v - \operatorname{codeg} t'_\nu \).

Proof. We first show that \( \theta_{\mu \nu} \in E^\mu \) whenever \( u \in \operatorname{Std}_\mu(\lambda) \) and \( v \in \operatorname{Std}_v(\lambda) \), for some \( \lambda \in \mathcal{P}_\beta^\lambda \). By Theorem 4.9, \( \psi_{uv} = \psi_{x,t_\mu}t_\mu, x \), for some \( x \in \mathcal{R}_\beta^\lambda \). Therefore,
\[
\theta_{\mu \nu}(\nu') h = \psi_{uv}\psi_{t_\mu}t_\mu h = \psi_{x,t_\mu} x\psi_{t_\mu}t_\mu h = \theta_{\mu}(x\psi_{t_\mu}t_\mu h).
\]
That is, \( \theta_{\mu \nu} \) factors through \( \theta_\mu \) so that \( \theta_{\mu \nu} \in E^\mu \) as claimed. The elements in the statement of the theorem are linearly independent because \( \{ \theta_{\mu \nu}(\nu') \} \) is a linearly independent subset of \( G_\mu \) by applying \( \ast \) to Corollary 4.12(a). Therefore, as we are working over a field, these elements give a basis of \( E^\mu \) by counting dimensions using Theorem 5.12.

Finally, as in the last paragraph of the proof of Theorem 5.12, the formula for the degree of \( \theta_{\mu \nu} \) follows using Lemma 3.10.

Note that, unlike Theorem 5.12, the basis of Lemma 5.14 does not obviously yield a \( \Delta \)-filtration of \( E^\mu \) because it is not clear how to write the basis elements \( \theta_{\mu \nu} \) in terms of the cellular basis of \( S^\lambda_\beta \). By appealing to Theorem 4.40, it is possible to construct a \( \nabla \)-filtration of \( E^\mu \) using the basis of Lemma 5.14. The existence of a \( \nabla \)-filtration is also implied by the next result.

Recall the homogeneous trace form \( \tau_\beta \) from Theorem 3.20.

Theorem 5.15. Suppose that \( Z = K \) is a field and that \( \mu \in \mathcal{P}_\beta^\lambda \), for \( \beta \in Q^+_\mu \). Then \( E^\mu \cong (E^\mu)^\otimes \).

Proof. Using the two bases of \( E^\mu \) given by Theorem 5.12 and Lemma 5.14, define \( <, >_\mu : E^\mu \times E^\mu \rightarrow K \) to be the unique bilinear map such that
\[
<\theta_{st}^{\mu \lambda}, \theta_{vr}^{\mu \tau}>_\mu = \tau_\beta(\psi_{st}\psi_{vu}'),
\]
for \( (\mu, s) \in \mathcal{T}_\lambda, (\nu, t) \in \mathcal{T}_\lambda^\mu, (\mu, u) \in \mathcal{T}_\nu \) and \( (\tau, v) \in \mathcal{T}_\nu^\tau \) for some \( \lambda, \sigma \in \mathcal{P}_\beta^\lambda \). By Theorem 3.20, \( <\theta_{st}^{\mu \lambda}, \theta_{vr}^{\mu \tau}>_\mu = 0 \) if and only if \( (\mu, u) \equiv (s, t) \) and \( \deg(\psi_{st}\psi_{vu}') = 2 \). Therefore, \( <, >_\mu \) is a non-degenerate bilinear form.

We claim that the bilinear form \( <, >_\mu \) is homogeneous. To see this, suppose that \( <\theta_{st}^{\mu \lambda}, \theta_{vr}^{\mu \tau}>_\mu = 0 \), for basis elements as above. Then \( \deg(\psi_{st}\psi_{vu}') = 2 \) since \( \tau_\beta \) is homogeneous of degree \(-2\)\( \deg(\beta) \). Using the degree formulae in Theorem 5.12, together with Lemma 5.10,
\[
\deg \theta_{st}^{\mu \lambda} + \deg \theta_{vr}^{\mu \tau} = \deg s - \deg t_\mu + \deg v - \deg t'_\nu \cdot \operatorname{codeg} u - \operatorname{codeg} t_\mu + \operatorname{codeg} v - \operatorname{codeg} t'_\nu = \deg(\psi_{st}\psi_{vu}') - 2 \deg(\beta) = 0.
\]
Hence, \( <, >_\mu \) is a homogeneous bilinear form of degree zero.

To complete the proof it is enough to show that the form \( <, >_\mu \) is associative because then the map that sends \( \theta_{st}^{\mu \lambda} \) to the function \( x \mapsto <\theta_{st}^{\mu \lambda}, x>_\mu \) is an \( S^\lambda_\beta \)-module homomorphism. This can be proved by repeating the argument from the proof of Theorem 4.14. We leave the details for the reader.

Corollary 5.16. Suppose that \( Z = K \) is a field and that \( \mu \in \mathcal{P}_\beta^\lambda \). Then \( E^\mu \) is a tilting module. Moreover,
\[
E^\mu = T^\mu \oplus \bigoplus_{\mu > \lambda} t_{\lambda \mu}(q)T^\lambda,
\]
for some Laurent polynomials $t_{\lambda}(q) \in \mathbb{N}[q, q^{-1}]$ such that $t_{\lambda}(q) = t_{\lambda}(q^{-1})$.

**Proof.** By Corollary 5.13, $E^{\mu}$ has a $\Delta$-filtration. Therefore, since $E^{\mu} \cong (E^{\mu})^\circ$ it also as a $\nabla$-filtration. Hence, $E^{\mu}$ is a tilting module so that $E^{\mu}$ can be written uniquely as a direct sum of indecomposable tilting modules. By Corollary 5.13, $(E^{\mu} : \Delta^{\mu})_q = 1$ and $(E^{\mu} : T^{\mu})_q \neq 0$ only if $\mu \succeq \lambda$. Therefore, if $d \in \mathbb{Z}$, then $T^{\lambda}(d)$ is a summand of $E^{\lambda}$ only if $\mu \succeq \lambda$ and $T^{\mu}$ is a summand appearing with multiplicity 1 = $(E^{\mu} : \Delta^{\mu})_q$. Hence, $E^{\mu} = T^{\mu} \oplus \bigoplus_{\mu \triangleright \lambda} t_{\lambda}(q)T^{\lambda}$ for some polynomials $t_{\lambda}(q) \in \mathbb{N}[q, q^{-1}]$. Finally, $t_{\lambda}(q) = t_{\lambda}(q^{-1})$ because $E^{\mu}$ is graded self-dual and because $T^{\lambda} \cong T^{\mu}(d)$ only if $\lambda = \mu$ and $d = 0$.

Arguing by induction on dominance we obtain the main result of this section.

**Corollary 5.17.** Suppose that $\mu \in \mathcal{P}_{\beta}^{\lambda}$. Then $(T^{\mu})^\circ \cong T^{\mu}$.

**Proof.** To start the induction note that if $\mu$ is a minimal element of $\mathcal{P}_{\beta}^{\lambda}$, with respect to dominance, then $E^{\mu} = T^{\mu}$ is self-dual by Theorem 5.15 and Corollary 5.16. The general case now follows by induction using Corollary 5.16.

### 5.3. Graded Ringel duality

This section introduces the Ringel duality in the graded setting. The main aim, however, is to compute the $\Delta$-filtration multiplicities in the tilting modules. This will allow us to identify the tilting modules with one of the canonical bases of the Fock space in §7.4.

A full tilting module $E^{\lambda}_{\beta}$ for $S^{\lambda}_{\beta}$ is a tilting module that contains every indecomposable tilting module, up to shift, as a direct summand. Hence,

$$E^{\lambda}_{\beta} = \bigoplus_{\mu \in \mathcal{P}_{\beta}^{\lambda}} E^{\mu}$$

is a full tilting module for $S^{\lambda}_{\beta}$. Define the *Ringel dual* of $S^{\lambda}_{\beta}$ to be the graded algebra $\text{End}_{S^{\lambda}_{\beta}}(E^{\lambda}_{\beta})$. (Strictly speaking, this is a Ringel dual of $S^{\lambda}_{\beta}$.)

Recall the graded Schur functor $F^{\lambda}_{\beta} : S^{\lambda}_{\beta}-\text{Mod} \longrightarrow \mathcal{R}_{\beta}^{\lambda}-\text{Mod}$ from Corollary 4.36.

**Lemma 5.18.** Suppose that $\mu \in \mathcal{P}_{\beta}^{\lambda}$. Then $F^{\lambda}_{\beta}(E^{\mu}) \cong G_{\mu}(\dashv \beta)$ as an $\mathcal{R}_{\beta}^{\lambda}$-module.

**Proof.** By Proposition 4.30 and Lemma 4.28, and Definition 5.11,

$$F^{\lambda}_{\beta}(E^{\mu}) = F^{\lambda}_{\beta}(\theta_{\mu}(\nabla^{\mu}(\dashv \beta))) = \theta_{\mu}(\nabla^{\mu}(\dashv \beta)) = G_{\mu}(\dashv \beta),$$

as required.

**Corollary 5.19.** Suppose that $\mu \in \mathcal{P}_{\beta}^{\lambda}$. Then $F^{\lambda}_{\beta}(T^{\mu}) \cong Y_{\mu}(\dashv \beta)$ as an $\mathcal{R}_{\beta}^{\lambda}$-module.

**Proof.** If $\mu$ is minimal with respect to dominance in $\mathcal{P}_{\beta}^{\lambda}$, then $E^{\mu} = T^{\mu}$ by Corollary 5.16. In fact, $E^{\mu} = \Delta^{\mu} = \nabla^{\mu}$ by Corollary 5.13 and the fact that $T^{\mu}$ is self-dual. Therefore, $F^{\lambda}_{\beta}(T^{\lambda}) = F^{\lambda}_{\beta}(\nabla^{\mu}) = \theta_{\mu}(\dashv \beta)$ by Proposition 4.30 and Lemma 5.18. On the other hand, $G_{\mu} = S_{\mu}$ by Corollary 4.11, so $F^{\lambda}_{\beta}(T^{\mu}) \cong Y_{\mu}(\dashv \beta)$ as claimed. If $\mu$ is not minimal in $\mathcal{P}_{\beta}^{\lambda}$ the result follows by induction on the dominance order using Lemma 5.18 since $E^{\mu} = T^{\mu} \oplus \bigoplus_{\mu \triangleright \lambda} t_{\lambda}(q)T^{\lambda}$ by Corollary 5.16 and $G_{\mu} = Y_{\mu} \oplus \bigoplus_{\mu \triangleright \lambda} z_{\lambda}(q)Y_{\lambda}$ by Proposition 5.6(c). (Moreover, $t_{\lambda}(q) = q^{-\def \beta} z_{\lambda}(q)$.)
Theorem 5.20. Suppose that $\beta \in Q^+_n$. Then the Ringel dual of $S^3_\beta$ is isomorphic to $S^3_\Lambda$. In particular, $\mathcal{End}_{S^3_\beta}(E^\Lambda_\beta)$ is a quasi-hereditary graded cellular algebra.

Proof. There is a natural map $\mathcal{Hom}_{\mathcal{R}^\Lambda}(G, G) \rightarrow \mathcal{Hom}_{S^3_\beta}(E^\mu, E^\nu)$ given by composition of maps, for $\mu, \nu \in \mathcal{P}^\Lambda_\beta$. By Lemma 5.18, this map is injective. On the other hand, since $E^\mu$ and $E^\nu$ are tilting modules for the quasi-hereditary algebra $S^3_\beta$, it is well known (compare [25, Proposition A2.2 and Proposition A3.7]) that

$$\dim \text{Hom}_{S^3_\beta}(E^\mu, E^\nu) = \dim \text{Hom}_{S^3_\beta}(E^\mu, E^\nu) = \sum_{\sigma \in \mathcal{P}^\Lambda_\beta} (E^\mu : \Delta^\sigma)(E^\mu : \Delta^\sigma) = \dim \text{Hom}_{\mathcal{R}^\Lambda}(G, G),$$

where the last equality comes from (4.39) and Corollary 5.13. Therefore, comparing dimensions, the Ringel dual of $S^3_\beta$ is isomorphic to $\mathcal{End}_{\mathcal{R}^\Lambda}(G^\Lambda_\beta) = S^3_\Lambda$, as a graded algebra. \qed

By Theorem 4.40, $S^3_\Lambda \cong S^3_\Lambda^\prime$ as graded algebras. Note, however, that this is not an isomorphism of quasi-hereditary algebras because the isomorphism reverses the partial ordering on the standard modules of the two algebras.

We now identify $S^3_\beta$ and the Ringel dual of $S^3_\beta$. The Ringel duality functor $\mathcal{Hom}_{S^3_\beta}(E^\Lambda_\beta, ?)$, combined with Theorem 5.20, defines a duality $\text{Rin}^\Lambda_\beta : S^3_\beta \text{-Mod} \rightarrow S^3_\beta \text{-Mod}$ that sends an $S^3_\beta$-module $M$ to $\mathcal{Hom}_{\mathcal{R}^\Lambda}(G^\Lambda_\beta F^\Lambda_\beta(M))$, where $F^\Lambda_\beta$ is the graded Schur functor for $S^3_\beta$. It is a standard fact that Ringel duality sends tilting modules to projective modules and costandard modules to standard modules; for example, see [25, §A4]. That $\text{Rin}^\Lambda_\beta(T^\Lambda) \cong P^\Lambda_{(-\beta)}$ is immediate from the definitions, whereas using the graded Schur functor $F^\Lambda_\beta : S^3_\beta \text{-Mod} \rightarrow \mathcal{R}^\Lambda_\beta \text{-Mod}$ and Remark 4.26, shows that $\text{Rin}^\Lambda_\beta(\nabla^\mu) \cong \Delta^\mu_{(-\beta)}$.

The next result is the graded analogue of [25, Lemma A4.6]. This is the result that we need in §7.4 to identify the tilting modules with one of the canonical bases of the Fock space.

Corollary 5.21. Suppose that $\lambda, \mu \in \mathcal{P}^\Lambda_\beta$ for $\beta \in Q^+_n$. Then

$$(T^\Lambda : \Delta^\mu)_q = \frac{[\Delta^\mu : L^\Lambda_\lambda]_q}{[\mu : \lambda]_q},$$

where $[\Delta^\mu : L^\Lambda_\lambda]_q$ is a graded decomposition number for the sign-dual quiver Schur algebra $S^3_\Lambda$.

Proof. Using Corollary 2.8 and the remarks in the last paragraph,

$$[\Delta^\mu : L^\Lambda_\lambda]_q = (P^\Lambda : \Delta^\mu)_q = (\text{Rin}^\Lambda_\beta(T^\Lambda) : \text{Rin}^\Lambda_\beta(\nabla^\mu))_q = (T^\Lambda : \Delta^\mu)_q.$$ 

Therefore, $(T^\Lambda : \Delta^\mu)_q = (T^\Lambda : \Delta^\mu)_q = (T^\Lambda : \nabla^\mu)_q = [\Delta^\mu : L^\Lambda_\lambda]_q$ as required. \qed

6. Cyclotomic Schur algebras

We are now ready to connect the quiver Schur algebras with the (ungraded) cyclotomic Hecke algebras introduced in [13, Theorem C; 24].

6.1. Cyclotomic permutation modules

Throughout this section, we work with the ungraded Hecke algebra $H^\Lambda_\mu$. Consequently, as in Theorem 3.7, we assume that $\mathcal{Z} = K$ is a suitable field. If $w \in \mathcal{S}_n$, then define $T_w = T_{i_1} \cdots T_{i_k}$,
where \( w = s_{i_1} \cdots s_{i_k} \) is a reduced expression for \( w \). Unlike the element \( \psi_w \in \mathcal{R}_n^\Lambda \), \( T_w \) is independent of the choice of reduced expression for \( w \).

Suppose that \( \mu \in \mathcal{P}_n^\Lambda \). Recall that if \( 1 \leq k \leq n \) and \( t = (t^{(1)}, \ldots, t^{(\ell)}) \) is a tableau, then \( \text{comp}_t(k) = s \) if \( k \) appears in \( t^{(s)} \). Define \( m^\mu = u^\mu x^\mu \), where

\[
u^\mu = \prod_{k=1}^n \prod_{s=\text{comp}_t(k)+1}^\ell (L_k - \xi^{(k)}) \quad \text{and} \quad x^\mu = \sum_{w \in \mathcal{I}_\mu} T_w,
\]

where \( \xi^{(k)} \) is as defined in (3.5). These definitions reduce to [24, Definition 3.5] when \( \xi \neq 1 \) and to [13, (6.12)–(6.13)] when \( \xi = 1 \).

**Definition 6.1** [13, 24]. Suppose that \( \mu \in \mathcal{P}_n^\Lambda \) and define \( M^\mu = m^\mu H^\Lambda_n \).

We write \( M^\mu \) rather than \( M^\mu \) to emphasize that \( M^\mu \) is not (naturally) \( \mathbb{Z} \)-graded. We will not define a graded lift of \( M^\mu \). Instead, the aim of this section is to show that \( G^\mu \) is a direct summand of \( M^\mu \).

We remind the reader of our standing assumption that \( e = 0 \) or \( e > n \) from Assumption 4.1. This is crucial for the next result, and consequently for all of the results in this section.

**Lemma 6.2.** Suppose that \( \lambda \in \mathcal{P}_n^\Lambda \) and \( 1 \neq w \in \mathfrak{S}_\lambda \). Then \( e^\lambda \psi_w e^\lambda = 0 \).

**Proof.** By Definition 3.2, \( \psi_w e^\lambda = e(j) \psi_w \), where \( j = w \cdot i^\lambda \). Now, the assumption that \( e = 0 \) or \( e > n \) implies that all of the nodes in row \( a \) of \( \lambda^{(l)} \) have pairwise distinct residues whenever \( \lambda_n^{(l)} \neq 0 \), for \( a \geq 0 \) and \( 1 \leq l \leq \ell \). Consequently, \( j \neq i^\lambda \) since \( 1 \neq w \in \mathfrak{S}_\lambda \). Therefore, \( e^\lambda \psi_w e^\lambda = e^\lambda e(j) \psi_w = 0 \).

**Lemma 6.3.** Suppose that \( \lambda \in \mathcal{P}_n^\Lambda \). Then \( e^\lambda u^\lambda = g^\lambda(y) e^\lambda y^\lambda \), where \( g^\lambda(y) \) is an invertible element of \( K[y_1, \ldots, y_n] \).

**Proof.** We prove the lemma only when \( \xi \neq 1 \) and leave the case when \( \xi = 1 \), which is similar, to the reader. Write \( i^\lambda = (i_1, \ldots, i_n) \) and let \( d^\lambda_1, \ldots, d^\lambda_n \) be as defined in Definition 3.11, so that \( d^\lambda_r = \{ \text{comp}_t(r) < t \leq \ell \mid i_r = \kappa_l \pmod{e} \} \), for \( 1 \leq r \leq n \). Then, using (3.8),

\[
e^\lambda u^\lambda = \prod_{r=1}^n \prod_{t=\text{comp}_t(r)+1}^\ell e^\lambda(L_r - \xi^{(r)})
\]

\[
= \prod_{r=1}^n \prod_{t=\text{comp}_t(r)+1}^\ell e^\lambda(\xi^{(r)} - \xi^{(r)} - \xi^{(r)} y_r)
\]

\[
= \prod_{r=1}^\ell \prod_{c = \text{comp}_t(r) < t \leq \ell} \prod_{i_r \equiv \kappa_l \pmod{e}} e^\lambda(\xi^{(r)} - \xi^{(r)} - \xi^{(r)} y_r)
\]

\[
= e^\lambda y^\lambda \cdot \prod_{r=1}^n (-\xi^{(r)}) d^\lambda_r \prod_{c = \text{comp}_t(r) < t \leq \ell} \prod_{i_r \equiv \kappa_l \pmod{e}} (\xi^{(r)} - \xi^{(r)} - \xi^{(r)} y_r).
\]

The factor to the right of \( e^\lambda y^\lambda \) in the last equation is a polynomial in \( K[y_1, \ldots, y_n] \) with non-zero constant term. Since each \( y_r \) is nilpotent (it has positive degree), it follows that \( g(y) \) is invertible. All of the terms in the last equation commute, so the lemma follows. \( \square \)
Theorem 6.4. Suppose that $e = 0$ or $e > n$ and let $\lambda \in \mathcal{P}_n^\lambda$. Then there exists an invertible element $f^\lambda(y) \in K[y_1, \ldots, y_n]$ such that

$$e^\lambda m^\lambda e^\lambda = f^\lambda(y)e^\lambda y^\lambda.$$

Proof. By Lemma 6.3, there exists an invertible element $g^\lambda(y) \in K[y_1, \ldots, y_n]$ such that

$$e^\lambda m^\lambda e^\lambda = e^\lambda u^\lambda x^\lambda e^\lambda = g^\lambda(y)y^\lambda \sum_{w \in \mathcal{G}_\lambda} e^\lambda T_w e^\lambda.$$

By (3.9), if $w \in \mathcal{G}_n$ and $j \in I^n$, then $T_w e(j) = (\psi T_r(j) - P_r(j))e(j)$ so the last equation can be rewritten as

$$e^\lambda m^\lambda e^\lambda = g^\lambda(y)y^\lambda \sum_{w \in \mathcal{G}_\lambda} r_w(y) e^\lambda \psi_w e^\lambda,$$

for some $r_w(y) \in K[y_1, \ldots, y_n]$. Applying Lemma 6.2, this sum collapses to give

$$e^\lambda m^\lambda e^\lambda = g^\lambda(y)y^\lambda r_1(y) = f^\lambda(y)e^\lambda y^\lambda,$$

for some polynomial $f^\lambda(y) \in K[y_1, \ldots, y_n]$. It remains to show that $f^\lambda(y)$ is invertible or, equivalently, that it has non-zero constant term. By [33, Corollary 3.11], if $1 \leq r \leq n$ and $(s, t) \in \text{Std}^2(\mathcal{P}_n^\lambda)$, then $y_r \psi_{st}$ is a linear combination of terms $\psi_{uv}$, where $(u, v) \triangleright (s, t)$. Therefore, since $e^\lambda y^\lambda = \psi_{11\lambda}$, there exist scalars $b_{uv} \in K$ such that

$$f^\lambda(y)e^\lambda y^\lambda = b_{11\lambda} \psi_{11\lambda} + \sum_{(u, v) \triangleright (1, 1)} b_{uv} \psi_{uv},$$

where $b_{11\lambda} = f^\lambda(0)$ is the constant term of $f^\lambda(y)$. On the other hand, by [33, Theorem 3.9] there exist scalars $c_{uv} \in K$ such that $c_{11\lambda} \neq 0$ and

$$e^\lambda m^\lambda e^\lambda = e^\lambda \left( \sum_{u, v \triangleright 1\lambda} c_{uv} \psi_{uv} \right) e^\lambda = c_{11\lambda} \psi_{11\lambda} + \sum_{u, v \in \text{Std}^2(\mathcal{P}_n^\lambda)} c_{uv} \psi_{uv},$$

where the second equality follows from (3.13). Hence, $f^\lambda(0) = c_{11\lambda} \neq 0$ by Theorem 3.14, and the proof is complete.

Remark 6.5. Using Theorem 6.4, it is possible to show that $e^\lambda m^\lambda = f^\lambda(y)e^\lambda y^\lambda + \epsilon$, where $\epsilon$ is a linear combination of homogeneous terms of degree strictly greater than $2 \deg \lambda = \deg(e^\lambda y^\lambda)$. To see this, first show that $e^\lambda m^\lambda$ is a linear combination of terms of the form $e^\lambda m_j e(j)$, where $j \in I^\lambda = \{i \in I^n \mid i = \sigma \cdot 1^\lambda \text{ for some } \sigma \in \mathcal{G}_\lambda \}$. The key observation is then that $\deg \psi_w e(j) > 0$ whenever $1 \neq w \in \mathcal{G}_\lambda$ and $j \in I^\lambda$, which can be proved by adapting the argument of Lemma 6.2. Consequently, $e^\lambda y^\lambda$ is the homogeneous component of $e^\lambda m^\lambda$ of minimal degree. Examples show that this does not always happen if we drop the assumption that $e = 0$ or $e > n$.

Recall from Definition 6.1 that $M^\lambda = m^\lambda \mathcal{H}_n^\lambda$.

Corollary 6.6. Suppose that $\lambda \in \mathcal{P}_n^\lambda$. Then

$$e^\lambda M^\lambda = e^\lambda m^\lambda \mathcal{H}_n^\lambda = e^\lambda m^\lambda e^\lambda \mathcal{H}_n^\lambda = e^\lambda y^\lambda \mathcal{H}_n^\lambda = G^\lambda.$$

Proof. By definition, $e^\lambda M^\lambda = e^\lambda m^\lambda \mathcal{H}_n^\lambda$ and $G^\lambda = e^\lambda y^\lambda \mathcal{H}_n^\lambda$ so we only need to check the two middle equalities. By Theorem 6.4, there exists an invertible element $f^\lambda(y)$ such that $e^\lambda m^\lambda e^\lambda =$
depends implicitly on the dominant weight $\Lambda$. To complete the proof, it is enough to show that $e^\lambda m^\lambda e^\lambda H_n^\Lambda = e^\lambda y^\lambda H_n^\Lambda$. This is immediate because $e^\lambda m^\lambda = e^\lambda u^\lambda x^\lambda \in e^\lambda y^\lambda H_n^\Lambda$ by Lemma 6.3. \qed

**Definition 6.7.** Suppose that $\lambda \in \mathcal{P}_n^\Lambda$. Let $\pi^\lambda : M^\lambda \rightarrow e^\lambda M^\lambda = G^\lambda$ be the surjective $H_n^\Lambda$-module homomorphism given by $\pi^\lambda(h) = e^\lambda h$, for $h \in M^\lambda$.

**Proposition 6.8.** Suppose that $\lambda \in \mathcal{P}_n^\Lambda$. Then the epimorphism $\pi^\lambda$ splits. That is, $\pi^\lambda$ has a right inverse $\phi^\lambda$ and $M^\lambda \cong e^\lambda M^\lambda \oplus \ker(\pi^\lambda)$.\footnote{This follows directly from the proof of Corollary 6.6.}

**Proof.** By Theorem 6.4, $e^\lambda m^\lambda e^\lambda = f^\lambda(y)e^\lambda y^\lambda$, where $f^\lambda(y)$ is an invertible element of $H_n^\Lambda$. Define $\phi^\lambda$ to be the map $\phi^\lambda : e^\lambda M^\lambda \rightarrow M^\lambda; e^\lambda y^\lambda h \mapsto m^\lambda e^\lambda f^\lambda(y)^{-1} h,$

for $h \in H_n^\Lambda$. To prove that $\phi^\lambda$ is well-defined suppose that $e^\lambda y^\lambda h = 0$ for some $h \in H_n^\Lambda$. By Corollary 6.6, there exists $h^\lambda \in H_n^\Lambda$ such that $e^\lambda m^\lambda = e^\lambda y^\lambda h^\lambda$. Let $*$ be the non-homogeneous anti-isomorphism of $H_n^\Lambda$ that fixes each of the non-homogeneous generators $T_r$ and $L_s$, for $1 \leq r < n$ and $1 \leq s \leq n$. Then $(e^\lambda y^\lambda h^\lambda)^* = (h^\lambda)^* e^\lambda y^\lambda$ because $e^\lambda$ and $y^\lambda$ are polynomials in $L_1, \ldots, L_n$ by [32, Proposition 4.8] and Theorem 3.7, respectively. Therefore,

$$ m^\lambda e^\lambda f^\lambda(y)^{-1} h = (e^\lambda y^\lambda h^\lambda)^* f^\lambda(y)^{-1} h = (h^\lambda)^* f^\lambda(y)^{-1} e^\lambda y^\lambda h = 0. $$

That is, $\phi^\lambda(e^\lambda y^\lambda h) = 0$. Hence, $\phi^\lambda$ is a well-defined $H_n^\Lambda$-module homomorphism. Moreover, if $h \in H_n^\Lambda$, then

$$(\pi^\lambda \circ \phi^\lambda)(e^\lambda y^\lambda h) = e^\lambda m^\lambda e^\lambda f^\lambda(y)^{-1} h = e^\lambda y^\lambda f^\lambda(y) f^\lambda(y)^{-1} h = e^\lambda y^\lambda h.$$}

That is, $\pi^\lambda \circ \phi^\lambda$ is the identity map on $e^\lambda M^\lambda$. Hence, $\pi^\lambda$ splits as claimed. \qed

**Corollary 6.9.** Suppose that $\lambda \in \mathcal{P}_n^\Lambda$. Then $\phi^\lambda$ induces an $H_n^\Lambda$-module isomorphism $e^\lambda m^\lambda H_n^\Lambda \cong m^\lambda e^\lambda H_n^\Lambda$.

**Proof.** This follows directly from the proof of Proposition 6.8. In fact, we have that $\phi^\lambda(e^\lambda m^\lambda H_n^\Lambda) = m^\lambda e^\lambda H_n^\Lambda$. \qed

6.2. **Cyclotomic Schur algebras**

We are now ready to show that $S_n^\Lambda$ is Morita equivalent to the corresponding cyclotomic Schur algebras introduced in [13, 24].

**Definition 6.10**[13, 24]. The **cyclotomic Schur algebra** is the algebra

$$ S_n^{\text{DJM}} = \text{End}_{H_n^\Lambda} \left( \bigoplus_{\mu \in \mathcal{P}_n^\Lambda} M^\mu \right). $$

Again, we write $S_n^{\text{DJM}}$ to emphasize that $S_n^\Lambda$ is not $\mathbb{Z}$-graded. Note that the algebra $S_n^{\text{DJM}}$ depends implicitly on the dominant weight $\Lambda$.

By [24, Corollary 6.18], $S_n^{\text{DJM}}$ is a quasi-hereditary cellular algebra with Weyl modules $\Delta_n^{\text{DJM}}$ and irreducible modules $L_n^{\mu, \text{DJM}}$ for $\lambda, \mu \in \mathcal{P}_n^\Lambda$. By [11, 45, Theorem 2.11], the blocks of $S_n^{\text{DJM}}$ are again labelled by $Q^\Lambda_+$; however, the direct summands of $M^\mu$ do not necessarily belong to the same block so it is difficult to describe the blocks of $S_n^{\text{DJM}}$ explicitly; however, see [49, Theorem 4.5].
Recall the graded Young modules $Y^\mu$, for $\mu \in P_n^\Lambda$, from Definition 5.5.

**Lemma 6.11.** Suppose that $\mu \in P_n^\Lambda$. Then $M^\mu \cong Y^\mu \oplus \bigoplus_{\lambda \triangleright \mu} (Y^\lambda)^{m_{\lambda\mu}}$ for some integers $m_{\lambda\mu} \in \mathbb{N}$.

**Proof.** By [47, (3.5)], there is a family of pairwise non-isomorphic (ungraded) indecomposable $\mathcal{H}_n^\Lambda$-modules \{\(y^\mu \mid \mu \in P_n^\Lambda\}\} that are uniquely determined, up to isomorphism, by the property that
\[
M^\mu \cong y^\mu \oplus \bigoplus_{\lambda \triangleright \mu} (y^\lambda)^{m_{\lambda\mu}}
\]
for some (in general, unknown) integers $m_{\lambda\mu} \in \mathbb{N}$. We show by induction on the dominance ordering that $Y^\nu \cong y^\nu$, for all $\nu \in P_n^\Lambda$.

First suppose that $\mu \in P_n^\Lambda$ is maximal in the dominance ordering. Then $M^\mu = y^\mu$ by (6.12) and $G^\mu = Y^\mu$ by Proposition 5.6(c). Therefore, $Y^\mu \cong y^\mu$ since $G^\mu$ is a summand of $M^\mu$ by Proposition 6.8.

Now suppose that $\mu$ is not maximal in the dominance ordering. Then $Y^\mu$ is isomorphic to an indecomposable direct summand of $G^\mu$ by Proposition 5.6(c). Therefore, there exists a multipartition $\lambda \triangleright \mu$ such that $Y^\lambda \cong y^\lambda$ by Proposition 6.8 and (6.12). By induction, if $\nu \triangleright \mu$, then $y^\nu \cong Y^\nu$, so this forces $\lambda = \mu$ by Proposition 5.6(b). That is, $Y^\mu \cong y^\mu$ as claimed. This completes the proof.

**Theorem 6.13.** Suppose that $\mathbb{Z}$ is a field and that $e = 0$ or $e > n$. Then there is an equivalence of highest weight categories
\[
E_{\text{DJM}}: S_n^\Lambda \text{-Mod} \sim \rightarrow S_n^{\text{DJM}} \text{-Mod}
\]
such that $E_{\text{DJM}}(\Delta^\lambda) \cong \Delta_n^{\text{DJM}}$ and $E_{\text{DJM}}(L^\mu) \cong L_n^{\text{DJM}}$, for all $\lambda, \mu \in P_n^\Lambda$.

**Proof.** By Lemma 6.11, the algebra
\[
\text{End}_{\mathcal{H}_n^\Lambda} \left( \bigoplus_{\mu \in P_n^\Lambda} Y^\mu \right)
\]
is the basic algebra of $S_n^\Lambda$ and it is also the basic algebra of $S_n^{\text{DJM}}$. Hence the result follows because (ungraded) basic algebras are unique up to isomorphism, as discussed in §2.3.

Using the combinatorics of the cellular bases of the algebras $S_n^{\text{DJM}}$ and $S_n^\Lambda$, it is easy to see that if $\mathbb{Z}$ is a field, then $\dim S_n^\Lambda \leq \dim S_n^{\text{DJM}}$. Moreover, this inequality is strict except for small $n$; compare with Remark 4.20. In particular, the algebras $S_n^\Lambda$ and $S_n^{\text{DJM}}$ are not isomorphic in general.

**Corollary 6.14.** Suppose that $\mathbb{Z}$ is a field and that $e = 0$ or $e > n$. Then, up to Morita equivalence, $S_n^{\text{DJM}}$ depends only on $e$, $\Lambda$ and the characteristic of $\mathbb{Z}$.

In particular, if $e = 0$ or $e > n$, then the decomposition numbers of the degenerate and non-degenerate cyclotomic Schur algebras depend only $e$, $\Lambda$ and the characteristic of the field. This generalizes [14, Corollary 6.3], which is the analogous result for the cyclotomic Hecke algebras (without any restriction on $e$).

Using Lemma 6.11, it is not hard to show that the degenerate and non-degenerate cyclotomic Schur algebras are isomorphic over any field when $e = 0$ or $e > n$. Gordon and Losev [29,
Proposition 6.6] have constructed an explicit isomorphism between these algebras over \( \mathbb{C} \) when \( e = 0 \), extending Brundan and Kleshchev’s isomorphism Theorem 3.7.

6.3. Signed permutation modules

The arguments in the last two sections apply equally well to the signed, or twisted, permutation modules defined in [47, § 4]. Mirroring the definitions in § 6.1, for \( \mu \in P_n^\Lambda \) define

\[
 u_\mu = \prod_{k=1}^n \prod_{s=1}^{\text{comp}_\mu(k) - 1} (L_k - \xi^{(\kappa_s)}) \quad \text{and} \quad x_\mu = \sum_{w \in \mathfrak{S}_\mu^\prime} (-\xi^{t(w)}) T_w.
\]

Note that \( \mathfrak{S}_\mu^\prime \) is the column stabilizer of \( t_\mu \). Let \( n_\mu = u_\mu x_\mu \) and define \( N_\mu = n_\mu H_n^\Lambda \). (For similar reason as in Remark 3.12, this module is denoted \( N^\mu \) in [47, § 4].) By [47, Proposition 4.3], there is an isomorphism of ungraded algebras \( S_n^{\text{DJM}} \cong \text{End}_{H_n^\Lambda}(\bigoplus N_\mu) \). In fact, Theorem 4.40 should be considered as a graded analogue of this result. Let \( \{ \Delta^\Lambda_\lambda \mid \lambda \in P_n^\Lambda \} \) and \( \{ L^\mu_n \mid \mu \in P_n^\Lambda \} \) be the sets of standard modules and simple modules, respectively, of the quasi-hereditary algebra \( \text{End}_{H_n^\Lambda}(\bigoplus \mu N_\mu) \), which we consider as \( S_n^{\text{DJM}} \)-modules.

By adapting the arguments leading to Lemma 6.11, we obtain the following.

**Lemma 6.15.** Suppose that \( \mu \in P_n^\Lambda \). Then \( N_\mu \cong Y_\mu \oplus \bigoplus_{\lambda>\lambda}(Y_\lambda)^{n_\lambda\mu} \) for some integers \( n_{\lambda\mu} \in \mathbb{N} \).

6.4. Positivity

In this section, we show that, in characteristic zero, the graded decomposition numbers of \( S_n^\Lambda \) are polynomials, rather than Laurent polynomials, by showing that \( S_n^\Lambda \) is graded Morita equivalent to one of the cyclotomic Quiver Schur algebras introduced by Stroppel and Webster [55]. Stroppel and Webster’s results can be summarized as follows.

**Theorem 6.16 (Stroppel and Webster [55]).** Suppose that \( Z = \mathbb{C} \) and \( \Lambda \in P^+ \). Then there exists a graded cellular \( Z \)-algebra \( S_n^{SW} \) such that:

(a) as ungraded algebras \( S_n^{SW} \cong S_n^{\text{DJM}} \);

(b) the algebra \( S_n^{SW} \) is a quasi-hereditary graded cellular algebra with weight poset \((P_n^\Lambda, \leq)\), standard modules \( \{ \Delta_n^{SW} \mid \lambda \in P_n^\Lambda \} \) and simple modules \( \{ L_n^\mu \mid \mu \in P_n^\Lambda \} \) such that \( \Delta_n^{SW} \cong \Delta_n^{\text{DJM}} \) and \( L_n^\mu \cong L_n^{\text{DJM}} \) as \( S_n^{\text{DJM}} \)-modules;

(c) if \( \lambda, \mu \in P_n^\Lambda \), then \( [\Delta^\Lambda_\lambda : L^\mu_n]_q = \delta_{\lambda\mu} + q[n][q] \).

By Theorem 6.13, there is an ungraded equivalence between the module categories of \( S_n^\Lambda \) and \( S_n^{SW} \). The next result says that this lifts to a graded equivalence.

**Theorem 6.17.** Suppose that \( Z \) is a field and that \( e = 0 \) or \( e > n \). Then there is a graded equivalence of highest weight categories

\[
 E_n^{SW} : S_n^\Lambda \text{-Mod} \simto S_n^{SW} \text{-Mod}
\]

such that \( E_n^{SW}(\Delta_\lambda) \cong \Delta_\lambda^{SW} \) and \( E_n^{SW}(L_\mu) \cong L_\mu^{SW} \), for all \( \lambda, \mu \in P_n^\Lambda \).

**Proof.** Recall from § 6.3 that \( N_\mu \) is a signed permutation module and that \( S_n^{\text{DJM}} \cong \text{End}_{H_n^\Lambda}(\bigoplus \mu N_\mu) \). By [55, Theorem 6.3], the module \( N_\mu \) has a graded lift \( N_\mu \), which is a graded \( R_n^\Lambda \)-module. The Stroppel–Webster cyclotomic quiver Hecke algebra is (isomorphic to) the algebra \( S_n^{SW} = \text{End}_{R_n^\Lambda}(\bigoplus \lambda N_\lambda) \). (Stroppel and Webster first define their algebra geometrically and then show that it is isomorphic to this algebra.) As the module \( N_\mu \) is graded, the
Theorem 6.13 shows that the Koszul grading on category $S$ gives the result. By Lemma 6.15, up to shift, every indecomposable summand of $N_\mu$ is isomorphic to $Y_\mu$, for some $\mu \in \mathcal{P}_n$. Arguing by induction on the dominance order it follows that there exist integers $a_\mu \in \mathbb{Z}$ such that $Y^{{SW}}_\mu \cong Y_\mu(a_\mu)$, for all $\mu \in \mathcal{P}_n$.

Let $c^{{SW}}_\mu(q) = \dim_q \text{Hom}_{S_n^{{SW}}}(P^{{SW}}_\mu, P^{{SW}}_\lambda)$ be a graded Cartan number of $S_n^{{SW}}$, for $\lambda, \mu \in \mathcal{P}_n$. Since Schur functors are fully faithful on projective modules,

$$c_\mu(q) = \dim_q \text{Hom}_{S_n^{{SW}}}(P^{{SW}}_\mu, P_\lambda) = \dim_q \text{Hom}_{\Lambda_n^{{SW}}}(Y_\mu, Y_\lambda) = \dim_q \text{Hom}_{\Lambda_n^{{SW}}}(Y^{{SW}}_\mu(-a_\mu), Y^{{SW}}_\lambda(-a_\lambda)) = q^{a_\mu - a_\lambda} \dim_q \text{Hom}_{\Lambda_n^{{SW}}}(Y^{{SW}}_\mu, Y^{{SW}}_\lambda) = q^{a_\mu - a_\lambda} c^{{SW}}_\mu(q).$$

By Theorem 6.16(b), $S_n^{{SW}}$ is a graded cellular algebra. Applying Corollary 2.8, we deduce that $c_\mu(q) = 6^{a_\mu - a_\lambda} c^{{SW}}_\mu(q) = q^{2(a_\mu - a_\lambda)} c_\mu(q)$. On the other hand, $S^\Lambda_n$ is a graded cellular algebra, so Corollary 2.8 now forces $a_\lambda = a_\mu$ whenever $c_\mu(q) \neq 0$. For $\beta \in Q^+_n$, the algebra $S^\Lambda_n$ is an indecomposable block of $S^\Lambda_n$ by Theorem 4.40. So, there exist integers $a_\beta \in \mathbb{Z}$ such that $a_\lambda = a_\beta$ whenever $\beta \in \mathcal{P}_n$. Therefore, $S_n^{{SW}}$ is graded Morita equivalent to the algebra

$$\bigoplus_{\beta \in Q^+_n} \text{End}_{\Lambda_n^{{SW}}} \left( \bigoplus_{\mu \in \mathcal{P}_n} Y^{{SW}}_\mu \right) \cong \bigoplus_{\beta \in Q^+_n} \text{End}_{\Lambda_n^{{SW}}} \left( \bigoplus_{\mu \in \mathcal{P}_n} Y_\mu(a_\beta) \right) \cong \bigoplus_{\beta \in Q^+_n} \text{End}_{\Lambda_n^{{SW}}} \left( \bigoplus_{\mu \in \mathcal{P}_n} Y_\mu \right).$$

The last algebra in the displayed equation is a graded basic algebra for $S^\Lambda_n$. Hence, there is a graded equivalence $S_n^{{SW}} \rightarrow S^\Lambda_n$. The remaining claims follow using Theorem 6.13.

**Corollary 6.18.** Suppose that $Z = \mathbb{C}$ and that $\lambda, \mu \in \mathcal{P}_n$. Then $d_\lambda(q) = d_\mu(q) = \delta_\lambda \mu + qN[q]$.

**Proof.** By Theorem 6.17 with Theorem 6.16(c), $[\Delta_\lambda : L_\mu]_q = [\Delta^{{SW}}_\lambda : L^{{SW}}_\mu]_q = \delta_\lambda \mu + qN[q]$. Applying Theorem 4.40 gives the result.

**7. Parabolic category $O$ and the Fock space**

Theorem 6.13 shows that $S^\Lambda_n$ induces a grading on the category of finite-dimensional modules for the cyclotomic Schur algebras $S^\Lambda_{DJM}_n$. On the other hand, for the degenerate case ($e = 0$) in characteristic zero Brundan and Kleshchev [13] have constructed an equivalence of categories $O^\Lambda_n \cong S^\Lambda_{DJM}_n$. Deep results of Beilinson, Ginzburg and Soergel [9, Theorem 1.1.3] and Backelin [8, Theorem 1.1] show that $O^\Lambda_n$ admits a Koszul grading, so it follows that $S^\Lambda_{DJM}_n$ can be endowed with a Koszul grading as well. The aim of this chapter is to show that the Koszul grading on category $O$ coincides with the grading on $O^\Lambda_n$. More precisely, we prove Theorem C and show that Brundan and Kleshchev’s equivalence can be lifted to a graded equivalence $O^\Lambda_n \cong S^\Lambda_n$. We now give a brief outline of the main arguments in this chapter.

Fix $\beta \in Q^+_n$ and let $S^O_\beta$ be the basic algebra of category $O^\Lambda_\beta$ and $S^O_\beta$ be the basic algebra of $S^\Lambda_\beta$. (We define the category $O^\Lambda_\beta$ in §7.1.) We define these algebras below to be the graded endomorphism algebras of minimal projective generators in their respective categories. The algebras $S^O_\beta$ and $S^O_\beta$ are both graded and, as remarked above, $S^O_\beta$ is a Koszul algebra and $S^O_\beta \cong S^O_\beta$ as ungraded algebras. Unfortunately, we are not able to compare the algebras $S^O_\beta$ and $S^O_\beta$
directly. Instead, the idea is to compare the endomorphism algebras of their minimal \textit{prinjective generators}. Let \( \{ P_{\mu}^\lambda \mid \mu \in \mathcal{P}_\lambda \} \) and \( \{ P^\mu \mid \mu \in \mathcal{P}_\lambda \} \) be complete sets, up to shift, of the pairwise non-isomorphic projective indecomposable \( S^\lambda_{\beta} \) modules and \( S^\lambda_{\beta} \) modules, respectively. We choose the labelling of these modules so that \( \bigoplus_{\mu \in \mathcal{K}_{\beta}^\lambda} P_{\mu}^\lambda \) is a minimal \textit{prinjective generator} for \( S^\lambda_{\beta} \) and that \( \bigoplus_{\mu \in \mathcal{K}_{\beta}^\lambda} P^\mu \) is a minimal \textit{prinjective generator} for \( S^\lambda_{\beta} \). Define

\[
\mathcal{R}^\lambda_{\beta} = \mathcal{E}nd_{S^\lambda_{\beta}} \left( \bigoplus_{\mu \in \mathcal{K}_{\beta}^\lambda} P_{\mu}^\lambda \right)_{\text{op}} \quad \text{and} \quad \mathcal{R}^\mu_{\beta} = \mathcal{E}nd_{S^\mu_{\beta}} \left( \bigoplus_{\mu \in \mathcal{K}_{\beta}^\lambda} P^\mu \right)_{\text{op}}.
\]

Then \( \mathcal{R}^\lambda_{\beta} \) and \( \mathcal{R}^\mu_{\beta} \) are both graded basic algebras. Moreover, on forgetting the gradings, \( \mathcal{R}^\lambda_{\beta} \cong \mathcal{R}^\mu_{\beta} \) is the basic algebra of \( \mathcal{H}^\lambda_{\beta} \). Using \cite{15}, we can determine the graded decomposition numbers of \( \mathcal{R}^\lambda_{\beta} \) and the graded decomposition numbers of \( \mathcal{O}_{\beta}^\lambda \) can be computed using the results of \cite{8, 9}. In fact, it turns out that the decomposition matrices of \( \mathcal{R}^\lambda_{\beta} \) and \( \mathcal{R}^\mu_{\beta} \) are equal, which implies that

\[
dim_q \mathcal{R}^\mu_{\beta} = \sum_{\lambda, \mu \in \mathcal{K}_{\beta}^\lambda} c_{\lambda \mu}(q) = \dim_q \mathcal{R}^\lambda_{\beta}.
\]

The next step is to explicitly construct a homogeneous basis of \( \mathcal{R}^\mu_{\beta} \). The key point is that because \( S^\lambda_{\beta} \) is Koszul the \textit{prinjective modules} \( P_{\mu}^\lambda \), for \( \mu \in \mathcal{K}_{\beta}^\lambda \), are \textit{rigidly graded} by Proposition 2.13. Consequently, the socle, radical and grading filtrations of \( P_{\mu}^\lambda \) coincide. As \( P_{\mu}^\lambda \) is rigid, the \( S^\lambda_{\beta} \)-module \( P^\mu \) is also rigid, so that the radical and socle filtrations of \( P^\mu \) coincide. Using Corollary 6.18, we show that the corresponding projective indecomposable module \( P^\mu \) for the basic algebra \( S^\lambda_{\beta} \) is also rigidly graded. This observation allows us to use the radical filtrations of \( P_{\mu}^\lambda \) and \( P^\mu \) to construct explicit bases of \( \mathcal{R}^\lambda_{\beta} \) and \( \mathcal{R}^\mu_{\beta} \). As a consequence, it follows that \( \mathcal{R}^\lambda_{\beta} \cong \mathcal{R}^\mu_{\beta} \) as graded algebras.

The argument so far is based entirely on the \textit{prinjective modules} and it says nothing about the projective modules \( P_{\mu}^\lambda \) and \( P^\mu \), for \( \mu \in \mathcal{P}_\lambda \setminus \mathcal{K}_{\beta}^\lambda \). Nonetheless, we have essentially completed the proof. There is a graded Schur functor \( F_{\beta}^\lambda : S^\lambda_{\beta} - \text{Mod} \longrightarrow R^\lambda_{\beta} - \text{Mod} \) and the modules \( Y_{\mu}^\lambda = F_{\beta}^\lambda(P_{\mu}^\lambda) \), for \( \mu \in \mathcal{P}_\lambda \), are graded lifts of the Young modules considered in §6.2. Therefore, \( Y_{\mu}^\lambda \cong Y^\mu(\kappa_\mu) \), for some integers \( k_\mu \in \mathbb{Z} \), because graded lifts of indecomposable modules are unique up to grading shift, when they exist. Using BGG reciprocity we show that these shifts are constant on \( \mathcal{K}_{\beta}^\lambda \) (in fact, \( k_\mu = 0 \) for all \( \mu \)). As Schur functors are fully faithful on projectives, we therefore obtain graded algebra isomorphisms

\[
S^\lambda_{\beta} \cong \mathcal{End}_{S^\lambda_{\beta}} \left( \bigoplus_{\mu \in \mathcal{P}_\lambda} P_{\mu}^\lambda \right)_{\text{op}} \cong \mathcal{End}_{R^\lambda_{\beta}} \left( \bigoplus_{\mu \in \mathcal{P}_\lambda} Y_{\mu}^\lambda \right)_{\text{op}} \cong \mathcal{End}_{R^\lambda_{\beta}} \left( \bigoplus_{\mu \in \mathcal{P}_\lambda} Y_{\mu}^\lambda \right)_{\text{op}} \cong S^\lambda_{\beta}.
\]

Hence, \( S^\lambda_{\beta} \) is Koszul, and \( S^\lambda_{\beta} - \text{Mod} \) is a Koszul category, as claimed.

To make this argument work we need closely related irreducible and projective indecomposable modules in several categories. For the readers convenience, we summarize this notation now.

| Algebra | Irreducible | PIM | Index set |
|---------|-------------|-----|-----------|
| \( S^\lambda_{\beta} \) | \( L_{\mu}^\lambda \) | \( P_{\mu}^\lambda \) | \( \mu \in \mathcal{P}_\lambda \) |
| \( S^\lambda_{\beta} \) | \( L^\mu \) | \( P^\mu \) | \( \mu \in \mathcal{P}_\lambda \) |
| \( S^\lambda_{\beta} \) | \( L_{\mu}^\beta \) | \( P_{\mu}^\beta \) | \( \mu \in \mathcal{P}_\beta \) |

We use similar notation for the functors between the module categories of these algebras, for the corresponding ungraded modules and for the cell modules of these algebras. Even though
a large amount of notation is needed, we hope that this consistent pattern for labelling these modules will help the reader to understand our results.

7.1. Parabolic category \( \mathcal{O} \)

Following [13, 16], the first step in the programme outlined above is to use Brundan and Kleshchev’s higher Schur–Weyl duality for the degenerate cyclotomic Hecke algebras to connect the representation theory of the cyclotomic quiver Schur algebras with the blocks of parabolic category \( \mathcal{O} \) for general linear Lie algebras; see, for example, [34, Chapter 9]. Our focus is somewhat different to that of [13, 16] because we are interested in the blocks of category \( \mathcal{O} \) that correspond to \( S^\Lambda_n \) for a particular \( n \). This difference of perspective makes it difficult to extract the information that we need from [13, 16], so we are generous with the details.

Fix \( \beta \in \mathbb{Q}^n_+ \) and \( \Lambda \in P^+ \). Recall from §3.1 that the dominant weight \( \Lambda = \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell} \) is determined by our fixed choice of multicharge \( \kappa = (\kappa_1, \ldots, \kappa_\ell) \). For the rest of this section, we assume that

\[
\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_\ell.
\]

There is no loss of generality in making this assumption because we can permute the numbers in the multicharge without changing the isomorphism type of \( \mathcal{H}_n^\Lambda \) or the graded isomorphism type of \( \mathcal{R}_n^\Lambda \).

We describe a special case of the definitions and results in [13, 16] that is sufficient to capture all of the blocks of \( S^\Lambda_n \), and hence of \( \mathcal{R}_n^\Lambda \), when \( e = 0 \). Set

\[
J = \{ \kappa_1 + n - 1, \kappa_1 + n - 2, \ldots, \kappa_\ell + 2 - n, \kappa_\ell + 1 - n \}
\]

and let \( J_+ = J \cup (J + 1) \) (note that \( J_+ = I_+ \) in the notation of [16]). The motivation for this definition is that if \( t \in \text{Std}(\mathcal{B}_n^\Lambda) \), then \( \text{res}_k(t) \in J_+ \) for \( 1 \leq k \leq n \). Consequently, if \( e(i) \neq 0 \) in \( \mathcal{R}_n^\Lambda \), then \( i \in J_+ \) by Theorem 7.14. This is necessary for Lemma 7.3.

To help explain what Brundan and Kleshchev do, let \( \mathfrak{g}l_{J_+}(\mathbb{C}) \) be the general linear group of \( J_+ \times J_+ \) matrices, where we index the rows and columns of the matrices in \( \mathfrak{g}l_{J_+}(\mathbb{C}) \) by \( J_+ \). We label the fundamental and simple roots of \( U(\mathfrak{g}l_{J_+}(\mathbb{C})) \) by \( J_+ \). In this way, we identify \( \Lambda \) and \( \beta \) with weights for \( \mathfrak{g}l_{J_+}(\mathbb{C}) \).

Let \( \pi = (\pi_1, \ldots, \pi_\ell) \) be the partition defined by \( \pi_c = n + \kappa_c - \kappa_\ell \), for \( 1 \leq c \leq \ell \), and let \( N = \pi_1 + \cdots + \pi_\ell \). Note that \( \pi_c = \kappa_c + 1 - \inf(J) \), so this agrees with the definitions in [16, §3.1]. (In the notation of [13, §1], \( \pi \) corresponds to the partition \( (q_1 \geq \cdots \geq q_\ell) \).) Consider the Lie algebra \( \mathfrak{g}l_N(\mathbb{C}) \) of all \( N \times N \) matrices over \( \mathbb{C} \). Let \( \mathfrak{h} \) be the standard Cartan subalgebra of diagonal matrices in \( \mathfrak{g}l_N(\mathbb{C}) \) and let \( \mathfrak{b} \supseteq \mathfrak{h} \) the Borel subalgebra of upper triangular matrices. Define \( \mathfrak{p} \) to be the standard parabolic subalgebra of \( \mathfrak{g}l_N(\mathbb{C}) \) with Levi subalgebra \( \mathfrak{g}l_\pi(\mathbb{C}) = \mathfrak{g}l_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{g}l_{n_\ell}(\mathbb{C}) \), so that \( \mathfrak{p} = (\mathfrak{b}, \mathfrak{g}l_\pi(\mathbb{C})) \). By definition, \( \mathfrak{p} \) depends only on \( \pi \) and hence on \( \kappa \) (or \( \Lambda \)), and \( n \).

Let \( \mathcal{O}^\Lambda = \mathcal{O}^p \) be the category of all finitely generated \( \mathfrak{g}l_N(\mathbb{C}) \)-modules that are locally finite-dimensional over \( \mathfrak{p} \) and semisimple over \( \mathfrak{h} \). This is the usual parabolic analogue of the BGG category \( \mathcal{O} \) except that we are only allowing modules with integral weights or, equivalently, integral central characters. The irreducible modules in category \( \mathcal{O}^\Lambda \) are naturally parametrized by highest weights, however, following [13, 16] we will use a different labelling of the irreducible \( \mathcal{O}^\Lambda \)-modules that comes from the categorification of the \( \mathfrak{g}l_{J_+}(\mathbb{C}) \)-module \( \Lambda^\pi V = \bigwedge_{i=1}^{n_1} V \otimes \cdots \otimes \bigwedge_{i=1}^{n_\ell} V \) by the blocks of \( \mathcal{O}^\Lambda \). Here \( V \) is the defining representation of \( \mathfrak{g}l_{J_+}(\mathbb{C}) \).

Following Brundan and Kleshchev [16, §2], define the \( \Lambda \)-diagram, of column shape \( \pi \), to be the array of boxes with rows indexed by \( \{\kappa_1, \kappa_1 - 1, \ldots, \kappa_\ell + 1 - n\} \), in decreasing order from top to bottom, and columns indexed by \( \{1, \ldots, \ell\} \), in increasing order from left to right, with the rows left justified and the columns bottom justified. In particular, column \( c \) of the \( \Lambda \)-diagram has a node in row \( j \) if and only if \( \kappa_c \geq j \geq \kappa_\ell + 1 - n \). (The \( \Lambda \)-diagrams should...
not be confused with the Young diagrams defined in §3.3.) A \( \Lambda \)-tableau is any filling of the \( \Lambda \)-diagram by numbers in \( J_+ \). The ground state \( \Lambda \)-tableau is the \( \Lambda \)-tableau with a \( j \) in all of the boxes in row \( j \), whenever \( \kappa_1 \geq j \in \kappa_\ell + 1 - n \).

**Example 7.1.** Let \( n = 3 \), \( \ell = 4 \), \( \kappa = (1, 0, 0, -2) \). Then \( \Lambda = \Lambda_1 + 2\Lambda_0 + \Lambda_{-2} \) and \( \pi = (6, 5, 5, 3) \). The rows of the \( \Lambda \)-tableaux are indexed by \( \{1, 0, -1, \ldots, -4\} \), so the ground state \( \Lambda \)-tableau is

\[
\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 \\
-2 & -2 & -2 & -2 \\
-3 & -3 & -3 & -3 \\
-4 & -4 & -4 & -4 \\
\end{array}
\]

Let \( \text{Col}^\Lambda \) be the set of column-strict \( \Lambda \)-tableaux, which are those \( \Lambda \)-tableaux with strictly decreasing entries, from top to bottom, in each column. A \( \Lambda \)-tableau is standard if it is column strict and its entries are weakly increasing from left to right in each row. Brundan and Kleshchev [16, (2.2) and (2.3)] define the weight of a \( \Lambda \)-tableau and they let \( \text{Col}_\beta^\Lambda \) and \( \text{Std}_\beta^\Lambda \) be the sets of column-strict and standard tableau of weight \( \Lambda - \beta \), respectively. Alternatively, one can use Lemma 7.3 to define the weight of a \( \Lambda \)-tableau.

Following [16, (2.50); 11, Lemma 5.4], if \( \lambda \in \mathcal{P}_\beta^\Lambda \) define the \( \Lambda \)-tableau of \( \lambda \) to be the \( \Lambda \)-tableau \( T^\lambda \) that has \( \lambda_{c,j} \) in row \( j \) and column \( c \in \{1, 2, \ldots, \ell\} \), where \( \kappa_c \geq j \geq \kappa_\ell + 1 - n \). That is, column \( c \) of \( T^\lambda \) is obtained by adding the parts of \( \lambda(c) \) to the ground state \( \Lambda \)-tableau. In particular, the \( \Lambda \)-tableau of the empty multipartition \( (0) \cdots (0) \) is the ground state \( \Lambda \)-tableau. The point of these definitions is that the \( \Lambda \)-tableaux naturally index a basis of \( \Lambda^\pi V \).

**Example 7.2.** Continuing Example 7.1, some \( \Lambda \)-tableaux are:

\[
\begin{array}{c|c|c|c}
\lambda & (2, 1)|0|0) & (1|1)|0|1) & (1)|0|1|1) & (0)|0|2|1) \\
T^\lambda & \begin{array}{cccc}
1 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 \\
-2 & -2 & -2 & -2 \\
-3 & -3 & -3 & -3 \\
-4 & -4 & -4 & -4 \\
\end{array} & \begin{array}{cccc}
0 & 1 & 0 & 0 \\
-1 & -1 & -1 & -1 \\
-2 & -2 & -2 & -2 \\
-3 & -3 & -3 & -3 \\
-4 & -4 & -4 & -4 \\
\end{array} & \begin{array}{cccc}
0 & 0 & 1 & 0 \\
-1 & -1 & -1 & -1 \\
-2 & -2 & -2 & -2 \\
-3 & -3 & -3 & -3 \\
-4 & -4 & -4 & -4 \\
\end{array} & \begin{array}{cccc}
0 & 0 & 1 & 0 \\
-1 & -1 & -1 & -1 \\
-2 & -2 & -2 & -2 \\
-3 & -3 & -3 & -3 \\
-4 & -4 & -4 & -4 \\
\end{array} \\
\end{array}
\]

Only the last two of these \( \Lambda \)-tableaux are standard and, using Corollary 3.23, it is easy to see that the standard \( \Lambda \)-tableaux correspond to the Kleshchev multipartitions in this list.

If \( T \) is a \( \Lambda \)-tableau let \( \text{col}(T) = (t_1, \ldots, t_N) \) be the column reading of \( T \), that is, the sequence obtained by reading the entries of \( T \) in order from top to bottom down the columns and reading the columns in order from left to right. The symmetric group acts from the right on such sequences by place permutations.

**Lemma 7.3** (Brundan and Kleshchev [16, (2.50) and (2.52)]). Suppose that \( \lambda \in P^+ \) and \( \beta \in Q_n^+ \), for \( n \geq 0 \). Then the map \( \lambda \mapsto T^\lambda \) defines a bijection \( \mathcal{P}_\beta^\Lambda \sim \text{Col}_\beta^\Lambda \) that restricts to a bijection \( \mathcal{K}_\beta^\Lambda \sim \text{Std}_\beta^\Lambda \). Moreover, if \( \lambda, \mu \in \mathcal{P}_\beta^\Lambda \), then \( \text{col}(T^\lambda) = \text{col}(T^\mu)w \), for some \( w \in \mathcal{S}_N \).
Proof. By the remarks above, res_k(t) ∈ J, for all t ∈ Std(𝔥^β_λ) and 1 ≤ k ≤ n. Therefore, the map λ ↦ T^λ defines a bijection P^λ_β ↠ Col^λ_β in view of [16, (2.50)]. Furthermore, by Corollary 3.23 and [16, (2.52)], this map restricts to a bijection K^λ_β ↠ Std^λ_β. For the final claim, if λ, µ ∈ P^λ_β, then the column readings of T^λ and T^µ belong to the same $Γ_N$-orbit in view of [16, (2.3)]. Alternatively, all of the statements in the lemma follow easily from standard facts about abacuses once one realizes we can identify T^λ with an $ℓ$-tuple of abacuses corresponding to the multipartition λ as in [45, § 3.1]. □

Brundan and Kleshchev index the irreducible representations of $𝒪^Λ$ by the $Λ$-tableaux. To this end, let $ε_1, ..., ε_N ∈ h^*$ be the standard coordinate functions on $h$ so that if $t = (t_{ij}) ∈ h$, then $ε_i(t) = t_{ii}$ picks out the $i$th diagonal entry of $t$. If $λ ∈ P^λ_β$, let $L^λ_β$ be the irreducible highest weight $gl_N(ℂ)$-module of highest weight

$$\omega(λ) = t_1ε_1 + (t_2 + 1)ε_2 + \cdots + (t_N + N - 1)ε_N,$$

(7.4)

where col(T^λ) = (t_1, ..., t_N) is the column reading of T^λ. (In the notation of [16], $L^λ_β$ is the module $L_{T^λ}$.) By construction, $L^λ_β$ belongs to $𝒪^Λ_β$.

For $β ∈ Q^+_n$, let $𝒪^Λ_β$ be the Serre subcategory of $𝒪^Λ$ generated by the irreducible $gl_N(ℂ)$-modules $\{ L^μ_β \mid μ ∈ P^μ_β \}$. Then $𝒪^Λ_β$ is the full subcategory of $𝒪^Λ$ consisting of the modules that have all of their composition factors in $\{ L^μ_β \mid μ ∈ P^μ_β \}$. Brundan [11, Theorem 2] shows that $𝒪^Λ_β$ is an indecomposable block of parabolic category $𝒪^Λ$. All of the blocks of $𝒪^Λ$ can be described this way, however, we are only interested in the blocks that correspond to $S^λ_n$. Accordingly, set

$$𝒪^Λ_n = \bigoplus_{β ∈ Q^+_n} 𝒪^Λ_β.$$  (7.5)

All of these categories have enough projectives. Let $P^λ_β$ be the projective cover of $L^λ_β$ in $𝒪^Λ_β$, for $λ ∈ P^λ_β$.

Following Backelin [8], we now introduce a grading on $𝒪^Λ_β$. For any module $M$, let $i_M$ be the identity map on $M$. The reader might like to recall the definition of Koszul categories and Koszul duality from § 2.5.

**Theorem 7.6 (Backelin [8, Theorem 1.1]).** Suppose that $Λ ∈ P^+$ and $β ∈ Q^+_n$. Then the category $𝒪^Λ_β$ is Koszul. Moreover, there exists a Koszul dual category $𝒪^Λ_α$ with simple modules $\{ L^n_ν \mid ν ∈ P^λ_β \}$ such that

$$\text{End}_{𝒪^β}(P_β^{α, β})^{op} ≃ \text{Ext}^{∗}_{𝒪^Λ}(L^n_β, L^n_β),$$

where $P^n_β = \bigoplus_{λ ∈ P^λ_β} P^n_β$ and $L^n_β = \bigoplus_{λ ∈ P^λ_β} L^n_β$,

and where the algebra on the right-hand side is a positively graded Koszul algebra under the Yoneda product. Moreover, this isomorphism can be chosen so that it sends $i_{P^n_β}$ to $i_{L^n_β}$, for $ν ∈ P^λ_β$.

Proof. The existence of $𝒪^Λ_β$ and such an isomorphism is proved by Backelin [8, Theorem 1.1]. The isomorphism can be chosen so that it identifies $i_{P^n_β}$ and $i_{L^n_β}$ by [8, Remark 3.8]. □

Define $S^n_β = \text{Ext}^{∗}_{𝒪^Λ}(L^n_β, L^n_β)$. By Theorem 7.6, $S^n_β$ is a finite-dimensional Koszul algebra and there is an equivalence of categories $E^n_β : 𝒪^Λ_β ↠ S^n_β$-Mod.

Backelin proves Theorem 7.6 more generally for the blocks of parabolic category $𝒪$ for an arbitrary semisimple complex Lie algebra. The proof that $𝒪^Λ_β$ is Koszul follows easily from fundamental work of Beilinson, Ginzburg and Soergel [9, Theorem 1.1.3], which considers
the Borel-parabolic and singular-regular case, whereas Backelin considers the more general parabolic-parabolic and singular–singular case. Given [9], the deepest statement in Theorem 7.6 is the explicit description of Koszul duality for the categories $\mathcal{O}^\Lambda_\beta$ and $\mathcal{O}^\Lambda_\Lambda$. In Appendix A, we give Backelin’s description of $\mathcal{O}^\Lambda_\beta$ and use it to compute the graded decomposition numbers of $\mathcal{O}^\Lambda_\beta$. Mazorchuk [50, Theorem 7.2] has given an algebraic proof of Theorem 7.6 starting from the Kazhdan–Lusztig conjecture, which is known for $\mathcal{O}^\Lambda_\beta$. In fact, Mazorchuk shows that the algebra $S^\beta_0$ is standard Koszul.

By Theorem 7.6, $S^\beta_0 = \bigoplus_{d \geq 0} S^\beta_{0,d}$ is Koszul so its degree zero component $S^\beta_{0,0}$ is semisimple by Definition 2.12. By Theorem 7.6, the irreducible $S^\beta_0$-modules are labelled by $\mathcal{P}^\Lambda_\beta$ and are concentrated in degree zero. For $\nu \in \mathcal{P}^\Lambda_\beta$, let $L^\nu_\mathcal{O} = \iota_{L^\nu_\mathcal{O}} S^\beta_{0,0}$ be the unique irreducible $S^\beta_0$-module that is concentrated in degree zero such that $L^\nu_\mathcal{O}$ is isomorphic to $E^\nu_\beta(L^\nu_\mathcal{O})$ when we forget the grading. We are abusing notation here because $L^\nu_\mathcal{O}$ is one-dimensional so that $L^\nu_\mathcal{O}$ is not the module obtained from $L^\nu_\mathcal{O}$ by forgetting the grading. Let $S^\beta_{0,+} = \bigoplus_{d > 0} S^\beta_{d,d}$. Then, by Theorem 7.6,

$$S^\beta_0/S^\beta_{0,+} \cong \bigoplus_{\nu \in \mathcal{P}^\Lambda_\beta} L^\nu_\mathcal{O}. \quad (7.7)$$

Let $P^\nu_\mathcal{O} = \iota_{P^\nu_\mathcal{O}} S^\beta_0$ be the projective cover of $L^\nu_\mathcal{O}$ in $S^\beta_0$-Mod, so that $P^\nu_\mathcal{O}$ is isomorphic to $E^\nu_\beta(P^\nu_\mathcal{O})$ when we forget the grading. As with $L^\nu_\mathcal{O}$, the module $P^\nu_\mathcal{O}$ is not obtained from $P^0_\mathcal{O}$ by just forgetting the grading.

The ungraded category $\mathcal{O}$ has a duality \( \varnothing \) that is induced by the anti-isomorphism of $\mathfrak{gl}_n(\mathbb{C})$ that maps a matrix to its transpose. The duality \( \varnothing \) restricts to a duality, also denoted \( \varnothing \), on the (ungraded) subcategories $\mathcal{O}^\Lambda_\Lambda$ and $\mathcal{O}^\Lambda_\beta$. Since $(L^\nu_\mathcal{O})^\varnothing \cong L^\nu_\mathcal{O}$ taking duals induces natural isomorphisms

$$\text{Ext}^k_{\mathcal{O}^\Lambda_\beta}(L^\nu_\mathcal{O}, L^\nu_\mathcal{O}) \cong \text{Ext}^k_{\mathcal{O}^\Lambda_\beta}(L^\nu_\mathcal{O}, L^\nu_\mathcal{O}), \quad (7.8)$$

for $\lambda, \nu \in \mathcal{P}^\Lambda_\beta$ and $k \in \mathbb{Z}_{>0}$. Therefore, \( \varnothing \) induces a homogeneous anti-isomorphism $\theta$ on $S^\beta_0$. If $M$ is a graded $S^\beta_0$-module, let $M^\circ = \text{Hom}_{\mathcal{O}}(M, \mathcal{O})$ be the graded dual of $M$ where the $S^\beta_0$-action is given by twisting by $\theta$. Note that $\dim_q M^\circ = \dim_{q^{-1}} M$.

By Theorem 7.6, $\{ L^\nu_\mathcal{O} | \nu \in \mathcal{P}^\Lambda_\beta \}$ is a complete set of pairwise orthogonal homogeneous idempotents that sum to 1 in $S^\beta_0$. For any finite-dimensional $S^\beta_0$-module, define the graded character of $M$ to be

$$\text{ch}_q M := \sum_{\nu \in \mathcal{P}^\Lambda_\beta} \dim_q (M L^\nu_\mathcal{O}) \nu \in \mathbb{Z}[q, q^{-1}][[\mathcal{P}^\Lambda_\beta]].$$

In (ungraded) category $\mathcal{O}^\Lambda_\beta$, $(L^\nu_\mathcal{O})^\varnothing \cong L^\nu_\mathcal{O}$. Therefore, $L^\nu_\mathcal{O}$ is fixed by $\theta$, for $\nu \in \mathcal{P}^\Lambda_\beta$, and so $\text{ch}_q M^\circ = \text{ch}_q M$.

**Lemma 7.9.** Suppose that $\mu \in \mathcal{P}^\Lambda_\beta$. Then $(L^\mu_\mathcal{O})^\varnothing \cong L^\mu_\mathcal{O}$ as graded $S^\beta_0$-modules.

**Proof.** As $\theta$ is homogeneous, and $L^\mu_\mathcal{O}$ is concentrated in degree 0, $(L^\mu_\mathcal{O})^\varnothing \cong L^\mu_\mathcal{O}$ for some $\nu \in \mathcal{P}^\Lambda_\beta$. By (7.7), $\text{ch}_q L^\nu_\mathcal{O} = \lambda$, for all $\lambda \in \mathcal{P}^\Lambda_\beta$. Therefore, by the last paragraph, $\nu = \text{ch}_q (L^\mu_\mathcal{O})^\varnothing = \text{ch}_q M^\circ = \text{ch}_q M$, as required.

In Corollary 5.10, we proved that $P^\mu_\mathcal{O}$ is a prinjective module for all $\mu \in \mathcal{K}^\Lambda_\beta$. We now prove the corresponding result in category $\mathcal{O}$. Recall from §2.5 that a module is rigidly graded if its radical, socle and grading filtrations coincide. The following rigidity result is crucial for the proof of Theorem C.
Proposition 7.10. Suppose that $\beta \in Q_n^+$ and $\mu \in K_\beta$. Then $P_\mu^\alpha$ is a prinjective $S_\beta^O$-module. Consequently, $P_\mu^\alpha$ is rigidly graded.

Proof. By Lemma 7.9, $(P_\mu^\alpha)^\circ$ is an injective hull of $L_\mu^\alpha$. If $\nu \in K_\beta$, then $(P_\nu^\alpha)^\circ \cong P_\nu^\alpha$ by [16, (2.52), Lemma 3.2]. Therefore, $P_\nu^\alpha$ is a prinjective $S_\beta^O$-module. Consequently, $(P_\mu^\alpha)^\circ \cong P_\nu^\alpha(k_\mu)$ for some integer $k_\mu \in \mathbb{Z}$, because graded lifts of indecomposable modules are unique up to shift. Hence, $P_\nu^\alpha$ is a prinjective $S_\beta^O$-module. It follows that so $P_\nu^\alpha \cong L_\nu^\alpha(-k_\mu)$ and $P_\mu^\alpha/\text{rad} P_\mu^\alpha \cong L_\mu^\alpha$ are both irreducible $S_\beta^O$-modules. By Theorem 7.6, $S_\beta^O$ is Koszul, so $P_\mu^\alpha$ is rigidly graded by Proposition 2.13(b).

Let $\Delta_\lambda^O$ be the parabolic Verma module of highest weight $\omega(\lambda)$ in $O_\lambda^\beta$. Then there is an indecomposable graded $S_\beta^O$-module $\Delta_\lambda^O$ such that $\Delta_\lambda^O \cong E_\alpha^O(\Delta_\mu^O)$ by [9, Proposition 3.5.7] and the proof of [8, Proposition 3.2]. As remarked above, Mazorchuk [50, Theorem 5.1] has shown that $S_\beta^O$ is a standard Koszul algebra. In particular, $S_\beta^O$ is graded quasi-hereditary, so using this framework we can take $\{\Delta_\lambda^O \mid \lambda \in \mathcal{P}_\beta\}$ to be the graded standard modules of $S_\beta^O$. Since $\Delta_\lambda^O$ is indecomposable, we fix the grading on $\Delta_\lambda^O$ by requiring that the surjection $\Delta_\lambda^O \twoheadrightarrow L_\lambda^O$ is a homogeneous map of degree zero in $S_\beta^O$-Mod.

Since $S_\beta^O$ is a standard Koszul algebra with a duality $\Diamond$ that fixes the simple $S_\beta^O$-modules, the category $O_\beta^\lambda$ is a graded highest weight category with duality in the sense of [21, §1.2]. As a consequence, a graded analogue of BGG reciprocity holds in $O_\beta^\lambda$. The following result is essentially [21, Proposition 1.2.4], however, we sketch a proof for completeness.

Corollary 7.11 (Graded BGG reciprocity). Suppose that $\lambda, \mu \in \mathcal{P}_\beta$, for $\beta \in Q_n^+$. Then

$$(P_\lambda^\mu \colon \Delta_\lambda^O)_q = [\Delta_\lambda^O : L_\mu^\alpha]_q.$$  

Consequently, $[P_\mu^\alpha : L_\lambda^\beta]_q = \sum_q (P_\lambda^\mu \colon \Delta_\lambda^O)_q [\Delta_\lambda^O : L_\mu^\alpha]_q$.

Proof. The costandard modules of $S_\beta^O$ are the $\nabla_\lambda^O = (\Delta_\lambda^O)^\circ$, for $\lambda \in \mathcal{P}_\beta$. As $O_\beta^\lambda$ is a highest weight category, well-known arguments show that $\text{Ext}^1_{S_\beta^O}(\nabla_\lambda^O, \nabla_\mu^O(k)) = 0$, for all $\lambda, \mu \in \mathcal{P}_\beta$ and $k \in \mathbb{Z}$. Therefore, the functor $\text{Hom}_{S_\beta^O}(?, \nabla_\lambda^O)$ is exact on the subcategory of $\Delta$-filtered $S_\beta^O$-modules. The projective indecomposable module $P_\mu^\alpha$ has a $\Delta$-filtration, so if $k \in \mathbb{Z}$, then

$$(P_\mu^\alpha \colon \nabla_\lambda^O(k)_q) = \text{dim}_q \text{Hom}_{S_\beta^O}(P_\mu^\alpha, \nabla_\lambda^O(k)_q) = [\nabla_\lambda^O(k)_q : L_\mu^\alpha]_q = [\Delta_\lambda^O : L_\mu^\alpha(k)_q],$$

where the last equality follows by applying $\Diamond$. Hence, $[P_\mu^\alpha \colon \Delta_\lambda^O]_q = [\Delta_\lambda^O : L_\mu^\alpha]_q$. As $P_\mu^\alpha$ has a $\Delta$-filtration the multiplicity formula for $[P_\mu^\alpha : L_\lambda^\beta]_q$ now follows easily.

We are now ready to make the link between parabolic category $O$ and the quiver Schur algebras. The following result is a reformulation of some of Brundan and Kleshchev’s main results from [13, 16]. Our Theorem C from the introduction is a graded analogue of this result.

Theoreme 7.12 (Brundan and Kleshchev). Suppose that $e = 0$ and $\mathbb{C}$. Then there is an equivalence of categories $E_\lambda^O : O_n^\lambda \rightarrow S_\lambda^O$-Mod. Moreover, $E_\lambda^O(\Delta_\lambda^O) \cong \Delta_\lambda^O$ and $E_\lambda^O(L_\mu^\alpha) \cong L_\mu^\alpha$, for all $\lambda, \mu \in \mathcal{P}_n^\beta$.

Proof. By [13, Theorem C], there is an equivalence of categories from $O_\lambda^\beta$ to $S_\lambda^{\mathcal{D}, \mathcal{J}, \mathcal{M}}$-Mod, which sends $\Delta_\lambda^O$ to $\Delta_{\lambda^{\mathcal{D}, \mathcal{J}, \mathcal{M}}}^O$ and $L_\mu^\alpha$ to $L_{\mu^{\mathcal{D}, \mathcal{J}, \mathcal{M}}}^\alpha$, for $\lambda, \mu \in \mathcal{P}_n^\beta$. Hence, by Theorem 6.13 and the remarks above, there is an equivalence of categories $E_\lambda^O : O_n^\lambda \rightarrow S_\lambda^O$-Mod such that $E_\lambda^O(\Delta_\lambda^O) \cong \Delta_\lambda^O$ and $E_\lambda^O(L_\mu^\alpha) \cong L_\mu^\alpha$. 

By projecting onto the blocks, there are equivalences $\mathcal{O}_\beta^\lambda \xrightarrow{\sim} \mathcal{S}_\beta^\lambda$-Mod, for each $\beta \in Q_n^+$. Ultimately, we want to compare the grading on the cyclotomic quiver Schur algebra $\mathcal{S}_\beta^\lambda$ with the Koszul grading on $\mathcal{S}_\beta^\varnothing$ coming from Theorem 7.6. Recall that $d_{\lambda \mu}(q) = [\Delta^\lambda : L^\mu]_q$, for $\lambda, \mu \in \mathcal{P}_\beta^\lambda$. Similarly, set $d_{\lambda \mu}^\varnothing(q) = [\Delta^\lambda : L^\mu]_q$. Recalling Lemma 2.15, define ‘Koszul dual’ polynomials $p_{\lambda \mu}(q)$ and $p_{\lambda \mu}^\varnothing(q)$ by the matrix equations $(p_{\lambda \mu}(q))^{tr} = (d_{\lambda \mu}(-q)^{-1} - 1)$ and $(p_{\lambda \mu}^\varnothing(q))^{tr} = (d_{\lambda \mu}^\varnothing(-q))^{-1}$. Then $d_{\lambda \mu}(q) = d_{\lambda \mu}^\varnothing(q)$ for all $\lambda \in \mathcal{P}_\beta^\lambda$ and $\mu \in \mathcal{K}_\beta^\lambda$ if and only if $p_{\lambda \mu}(q) = p_{\lambda \mu}^\varnothing(q)$ for all $\lambda \in \mathcal{P}_\beta^\lambda$ and $\mu \in \mathcal{K}_\beta^\lambda$. To describe these polynomial explicitly, if $x, y \in \mathcal{S}_N$, let $P_{x,y}(t) \in \mathbb{Z}[t]$ be the corresponding Kazhdan–Lusztig polynomial introduced in [36, Theorem 1.1].

If $\lambda \in \mathcal{P}_\beta^\lambda$ recall that $\text{col}(T^\lambda) = (t_1, \ldots, t_N)$ is the column reading of the $\Lambda$-tableau $T^\lambda$, as defined before Lemma 7.3. As in [16, §2.3], define $w_\lambda \in \mathcal{S}_n$ to be the unique permutation of minimal length such that $\text{col}(T^\lambda)w_\lambda$ is weakly increasing. Let $Z_\lambda$ be the stabilizer of $\text{col}(T^\lambda)w_\lambda$. By Lemma 7.3, $Z_\lambda$ depends only on $\Lambda$ and $\beta$. Further, $w_\lambda$ is a minimal length left coset representative of $Z_\lambda$ in $\mathcal{S}_N$ (see Appendix A for more details).

We prove the following result in Appendix A. This proposition may be well known to experts, however, we include a proof because we have not found it in the literature.

**Proposition 7.13.** Suppose that $\lambda, \mu \in \mathcal{P}_\beta^\lambda$, for $\beta \in Q_n^+$. Then

$$p_{\lambda \mu}(q) = q^{\ell(w_\mu) - \ell(w_\lambda)} \sum_{z \in Z_\lambda} (-1)^{\ell(z)} P_{w_\lambda z, w_\mu}(q^{-2}) \in \mathbb{N}[q, q^{-1}].$$

By Lemma 2.15, $p_{\lambda \mu}^\varnothing(q)$ is a graded decomposition number for the Koszul dual category $\mathcal{O}_\beta^\lambda$. As we describe in Appendix A, $\mathcal{O}_\beta^\lambda$ is also a block of parabolic category $\mathcal{O}$, although it is associated to another parabolic subalgebra $\mathfrak{g}_N(\mathbb{C})$ when $q = b$ this result is equivalent to [9, Theorem 3.11.4(i)(iv)], which a consequence of the Kazhdan–Lusztig conjecture for $\mathfrak{g}_N(\mathbb{C})$.

Given Proposition 7.13, the next result is essentially a reformulation of results of Brundan and Kleshchev [15, 16], building on Ariki’s categorification theorem [3]. This result is the key to comparing the gradings on $\mathcal{R}_\beta^\varnothing$ and $\mathcal{R}_\beta^\lambda$.

**Theorem 7.14.** Suppose that $e = 0$, $\beta \in Q_n^+$ and $\mathcal{Z} = \mathbb{C}$. Then $d_{\lambda \mu}(q) = d_{\lambda \mu}^\varnothing(q)$ is a parabolic Kazhdan–Lusztig polynomial, for $\lambda \in \mathcal{P}_\beta^\lambda$ and $\mu \in \mathcal{K}_\beta^\lambda$. In particular, $d_{\lambda \mu}(q) \in \mathcal{R}_\beta^\varnothing$.

**Proof.** By [15, Theorem 5.14 and Corollary 5.15], the Grothendieck group of the finitely generated projective $\mathcal{R}_n^\lambda$-modules categorifies the integral highest weight module $L(\Lambda)$ for $U_q(\hat{\mathfrak{sl}}_n)$, with the projective indecomposable $\mathcal{R}_n^\lambda$-modules corresponding to the Lusztig–Kashiwara canonical basis of $L(\Lambda)$. Thus, the graded decomposition numbers of $\mathcal{R}_n^\lambda$ give the transition matrix between the standard basis and the canonical basis of $L(\Lambda)$.

As we are assuming that $e = 0$, the algebra $\mathcal{R}_n^\lambda \cong \mathcal{H}_n^\lambda$ is a degenerate cyclotomic Hecke algebra. Brundan and Kleshchev [16, Theorem 3.1 and (2.18)] proved the analogue of Ariki’s categorification theorem in the degenerate case. In particular, they showed that the transition matrix between the standard basis and the canonical basis of the Fock space is given by certain polynomials $\{d_{T^\lambda T^\mu}(q)\}$, where $T^\lambda \in \text{Col}_n^\lambda$ and $T^\mu \in \text{Std}_n^\lambda$ are the column-strict and standard $\Lambda$-tableaux defined before Lemma 7.3. The two papers [15, 16] use the same bar involution by [16, §2.5] and the remarks after [15, (3.60)]. (As explained in [16, §2.2], the infinite-dimensional space considered in [15] agrees with the finite-dimensional space considered in [16] after truncation.) Therefore, it follows by induction on dominance that $d_{\lambda \mu}(q) = d_{T^\lambda T^\mu}(q)$, for all $\lambda \in \mathcal{P}_\beta^\lambda$ and $\mu \in \mathcal{K}_\beta^\lambda$. 

For $\lambda, \mu \in K^A_\beta$ define polynomials $p_{\gamma^\lambda \gamma^\mu}(q)$ by the matrix equation
\[
(p_{\gamma^\lambda \gamma^\mu}(q))^{tr} = (d_{\gamma^\lambda \gamma^\mu}(q))^{-1}_{\lambda, \mu \in K^A_\beta}.
\]
These polynomials coincide with the polynomials defined by Brundan and Kleshchev [16, (2.17)], by [16, (2.39)]. Brundan and Kleshchev explicitly describe the polynomials $p_{\gamma^\lambda \gamma^\mu}(q)$ as parabolic Kazhdan–Lusztig polynomials in [16, (2.18)]. In fact, their formula for these polynomials coincides exactly with Proposition 7.13. Therefore, $d_{\gamma^\lambda \gamma^\mu}(q) = d_{\lambda \mu}(q)$, for all $\lambda, \mu \in K^A_\beta$. Therefore, $d_{\gamma^\lambda \gamma^\mu}(q) = d_{\lambda \mu}(q)$, for all $\gamma \in O^A_\beta$ and $\mu \in K^A_\beta$, since the polynomials indexed by multipartitions in $K^A_\beta$ uniquely determine the remaining polynomials under categorification by [15, Corollary 5.15].

Finally, observe that $d_{\gamma^\lambda}(q) = d_{\lambda \mu}(q) \in \delta_{\lambda \mu} + q\mathbb{N}[q]$ because $S^O_{\beta}$ is Koszul.

**Corollary 7.15.** Suppose that $e = 0$, $\beta \in Q_n^+$ and $Z = \mathbb{C}$. Then $c_{\lambda \mu}(q) = c_{\lambda \mu}(q)$ for all $\lambda, \mu \in K^A_\beta$.

**Proof.** Applying in turn Corollary 7.11, Theorem 7.14 and Corollary 2.8, shows that
\[
c_{\lambda \mu}(q) = \sum_{\nu \in O^A_\beta} d_{\lambda \nu}(q)d_{\nu \mu}(q) = \sum_{\nu \in O^A_\beta} d_{\nu \lambda}(q)d_{\nu \mu}(q) = c_{\lambda \mu}(q),
\]
as required.

**7.2. Comparing the KLR and category $O$ gradings**

We want to lift Theorem 7.12 to the graded setting. As a first step, we show that the KLR and category $O^A_\beta$ gradings induce the same grading on $H^A_\beta$. To do this, we need a graded analogue of the Hecke algebra $H^A_\beta$ with a grading that comes from category $O^A_\beta$.

We start by describing $S^O_\beta$ as an endomorphism algebra. Let $P^O_\beta = \bigoplus_{\mu \in O^A_\beta} P^\mu_\beta$.

**Lemma 7.16.** Suppose that $\beta \in Q_n^+$. Then there is an isomorphism of graded rings
\[
S^O_\beta \cong \mathcal{End}_{S^O_\beta}(P^O_\beta)^{op}.
\]
In particular, $\mathcal{End}_{S^O_\beta}(P^O_\beta)$ is Koszul.

**Proof.** By definition, $P^\mu_\beta$ is the projective cover of $L^\mu_\beta$ so, by (7.7), $P^O_\beta$ is a direct summand of $S^O_\beta$ by the universal property of projective modules. Therefore, $S^O_\beta \cong P^O_\beta$ as a right $S^O_\beta$-module and, consequently, $S^O_\beta \cong \mathcal{End}_{S^O_\beta}(P^O_\beta)^{op}$ as graded algebras. In fact, by the argument of [9, Corollary 2.5.2], the isomorphism $S^O_\beta \cong \mathcal{End}_{S^O_\beta}(P^O_\beta)$ is unique.

Henceforth, we identify $S^O_\beta$ and $\mathcal{End}_{S^O_\beta}(P^O_\beta)$ via Lemma 7.16. Motivated by Theorem 7.12, define $e^O_\beta = \sum_{\mu \in K^A_\beta} P^\mu_\beta$. Then $e^O_\beta \in S^O_\beta$ is a homogeneous idempotent of degree zero. Now define
\[
R^O_\beta = e^O_\beta S^O_\beta e^O_\beta \cong \mathcal{End}_{S^O_\beta} \left( \bigoplus_{\mu \in K^A_\beta} P^\mu_\beta \right)^{op}.
\]
By (2.10), there is an exact graded functor
\[
F^O_\beta : S^O_\beta \text{-Mod} \rightarrow R^O_\beta \text{-Mod}; M \mapsto Me^O_\beta \quad \text{for} \ M \in S^O_\beta \text{-Mod}.
\]
The anti-isomorphism $\theta$ of (7.8) fixes $S^O_{\beta,0}$, the degree zero component of $S^O_{\beta}$, so $\theta(e^O_{\beta}) = e^O_{\beta}$. Therefore, $\theta$ restricts to a homogeneous anti-isomorphism of $R^A_{\beta}$. Abusing notation, let $\circ$ be the corresponding duality on $R^A_{\beta}\text{-Mod}$. By construction, there is an isomorphism of functors $\circ \circ F^O_{\beta} \cong F^O_{\beta} \circ \circ$.

Let $R^O_{\beta} = \bigoplus_{d \in \mathbb{Z}} R^O_{\beta,d}$ be the decomposition of $R^O_{\beta}$ into its homogeneous components.

**Proposition 7.18.** Suppose that $\beta \in Q^+_n$. Then:

(a) The algebra $R^O_{\beta}$ is a positively graded basic algebra;

(b) The degree zero subalgebra $R^O_{\beta,0}$ is semisimple;

(c) the ungraded algebras $R^O_{\beta}$, $R^A_{\beta}$ and $H^A_{\beta}$ are Morita equivalent.

**Proof.** By Theorem 7.6, $S^O_{\beta}$ is a Koszul algebra so it is positively graded and its degree zero component is semisimple by Definition 2.12. Hence, parts (a) and (b) follow because $R^O_{\beta} = e^O_{\beta} S^O_{\beta} e^O_{\beta}$. Finally, by Theorem 7.12, the algebra $R^O_{\beta}$ is Morita equivalent to the Hecke algebra $H^A_{\beta}$. Hence, part (c) follows because $R^A_{\beta} \cong H^A_{\beta}$ by Theorem 3.7 and the remarks after (3.18). \(\square\)

Using Theorem 7.14, we now to replicate the results that we have just proved for the algebra $R^O_{\beta}$ for the KLR algebra $R^A_{\beta}$. To do this, we need to work with a basic algebra for $R^A_{\beta}$. We start by defining $S^\flat_{\beta} = \text{End}_{S^A_{\beta}} \left( \bigoplus_{\mu \in \mathcal{P}^A_{\beta}} P^\mu \right)^{\text{op}}$, which a basic algebra for $S^A_{\beta}$. Write $S^\flat_{\beta} = \bigoplus_d S^\flat_{\beta,d}$ for the decomposition of $S^\flat_{\beta}$ into its homogeneous components.

**Lemma 7.19.** The algebra $S^\flat_{\beta}$ is a positively graded algebra that is graded Morita equivalent to $S^A_{\beta}$. Moreover, $S^\flat_{\beta,0}$ is semisimple.

**Proof.** This is immediate from Corollary 6.18. \(\square\)

Eventually, we will show that $S^O_{\beta} \cong S^\flat_{\beta}$ as graded algebras. Mirroring the definition of $R^O_{\beta}$, define $R^\flat_{\beta} = \text{End}_{S^A_{\beta}} \left( \bigoplus_{\mu \in \mathcal{K}^A_{\beta}} Y^\mu \right)^{\text{op}}$, so that $R^\flat_{\beta} = e^A_{\beta} S^A_{\beta} e^A_{\beta}$, where $e^A_{\beta} = \sum_{\beta \in \mathcal{K}^A_{\beta}} \iota_{P^\mu}$. Note that $R^\flat_{\beta}$ is positively graded by Lemma 7.19 because it is a subalgebra of $S^\flat_{\beta}$.

**Lemma 7.20.** There is a graded equivalence of categories $R^A_{\beta}\text{-Mod} \cong R^\flat_{\beta}\text{-Mod}$.

**Proof.** By Lemma 4.34, the Schur functor $F^A_{\beta}$ is fully faithful on projective modules so, by Definition 5.5,

$$\text{Hom}_{S^\flat_{\beta}}(P^\lambda, P^\mu) \cong \text{Hom}_{R^\flat_{\beta}}(Y^\lambda, Y^\mu)$$

for all $\lambda, \mu \in \mathcal{K}^A_{\beta}$. Therefore, $R^\flat_{\beta} \cong \text{End}_{R^\flat_{\beta}}(\bigoplus_{\mu \in \mathcal{K}^A_{\beta}} Y^\mu)^{\text{op}}$. By Proposition 5.9, $\bigoplus_{\mu \in \mathcal{K}^A_{\beta}} Y^\mu$ is a minimal progenerator for $R^A_{\beta}$, so $R^\flat_{\beta}$ is graded Morita equivalent to $R^A_{\beta}$ as claimed. \(\square\)
By definition, $c_{\lambda \mu}(q) = \dim_q \mathcal{H}om_{R^A}(Y^\mu, Y^\lambda)$. Therefore, the proof of Lemma 7.20, together with Corollary 7.15, implies that $\dim_q \mathcal{R}_\beta^A = \sum_{\lambda, \mu \in K^A} c_{\lambda \mu}(q) = \dim_q \mathcal{R}_\beta^O$.

**Lemma 7.21.** Suppose that $\lambda, \mu \in K^A$.

Then $0 \leq \deg c_{\lambda \mu}(q) \leq 2 \deg \beta$ with equalities if and only if $\lambda = \mu$, so there homogeneous map. Moreover, $P^\mu_O$ and $P^\mu$ both have Loewy length 2 def $\beta$.

**Proof.** First observe that $c_{\lambda \mu}(q) \in \delta_{\lambda \mu} + q\mathbb{N}[q]$ by Corollary 6.18. Moreover, $(Y^\mu)^{\otimes (2 \deg \beta)} \cong Y^\mu$ by Corollary 5.8 if $\mu \in K^A$. Therefore, $0 \leq \deg c_{\lambda \mu}(q) \leq 2 \deg \beta$ with equalities if and only if $\lambda = \mu$.

To prove the remaining claims statement first recall that $P^\mu_O$ is rigidly graded by Proposition 7.10. Therefore, $P^\mu_O$ has Loewy length $\deg c_{\mu \mu}(q) = 2 \deg \beta$ because $\mathcal{R}_\beta^O$ by Corollary 7.15. On the other hand, $E^A_{\beta}(P^\mu_O) \cong P^\mu$ by Theorem 7.12. In particular, $P^\mu_O$ and $P^\mu$ both have the same Loewy length, completing the proof.

We would like to say that $P^\mu$ is rigidly graded when $\mu \in K^A$. We cannot say this, however, because we have only defined grading filtrations, and rigidly graded modules, for positively graded algebras. To remedy this, let $E^A_{\beta}: \mathcal{S}^A_{\beta} \rightarrow \mathcal{S}^A_{\beta}$ be the graded equivalence given by

$$M \mapsto \mathcal{H}om_{S^A_{\beta}} \left( \bigoplus_{\nu \in \mathcal{P}^A_{\beta}} P^\nu, M \right).$$

Set $P^\mu_y = E^A_{\beta}(P^\mu)$, for $\mu \in \mathcal{P}^A_{\beta}$. Then $P^\mu_y$ is an indecomposable projective $S^A_{\beta}$-module.

**Proposition 7.22.** Suppose that $\mu \in K^A$. Then $P^\mu_y$ is rigidly graded.

**Proof.** As $E^A_{\beta}$ is an equivalence, $P^\mu$ and $P^\mu_y$ both have Loewy length 2 def $\beta$ by Lemma 7.21. The grading filtration $\{ \mathcal{G}r_d P^\mu_y \}$ of $P^\mu_y$ also has length 2 def $\beta$ because $\deg c_{\mu \mu}(q) = 2 \deg \beta$ by Lemma 7.21. Moreover, for $0 \leq d \leq 2 \deg \beta$ the module $\mathcal{G}r_d P^\mu / \mathcal{G}r_{d+1} P^\mu_y$ is semisimple because it is an $S^A_{\beta,0}$-module and $S^A_{\beta,0}$ is semisimple by Lemma 7.19. Therefore, the grading filtration of $P^\mu_y$ is a grading with semisimple quotients and length equal to the Loewy length of $P^\mu_y$. As $P^\mu_y$ is rigid by Proposition 7.10 and $E^A_{\beta}(P^\mu_y) \cong P^\mu$ by Theorem 7.12, it follows that $P^\mu_y$ is rigid. Hence, the grading, radical and socle filtrations of $P^\mu_y$ all coincide, so $P^\mu_y$ is rigidly graded as claimed.

We now use the radical filtration of $P^\mu_y$ to construct a basis of $R^A_{\beta}$.

The key point is that if $\mu \in K^A$, then $P^\mu_y$ is rigidly graded by Proposition 7.10. In particular, the radical filtration of $P^\mu_y$ is equal to its grading filtration. Therefore, writing $c_{\lambda \mu}(q) = \sum_{d \geq 0} c_{\lambda \mu}^{(d)} q^d$, for $c_{\lambda \mu}^{(d)} \in \mathbb{N}$,

$$\text{rad}^d P^\mu_y / \text{rad}^{d+1} P^\mu_y \cong \bigoplus_{\lambda \in \mathcal{P}^A_{\beta}} (L^A_{\beta}(d))^{\oplus c_{\lambda \mu}^{(d)}},$$

for $0 \leq d \leq z$. Fix $\lambda \in \mathcal{P}^A_{\beta}$ and $d \geq 0$ with $c_{\lambda \mu}^{(d)} \neq 0$. Since $P^A_{\beta}(d)$ is the projective cover of $L^A_{\beta}(d)$ there exist homogeneous maps $\theta_{\lambda \mu}^{(d)} \in \mathcal{H}om_{S^A_{\beta}}(P^\lambda, \text{rad}^d P^\mu_y)$ such that
By Proposition 7.22, we can write $\text{Hom}_1(\mu)$ and the set $\lambda \mu$ is a homogeneous basis of $E_k$ of degree $\leq 1$ for $1 \leq s \leq c_{\lambda \mu}^{(d)}$, where $c_{\lambda \mu}^{(d)}$ is the identity map followed by shifting all degrees by $d$. By embedding $\text{rad}^d P_\mu^\lambda$ into $P_\mu^\lambda$, we consider $\theta_{\lambda \mu}^{(d,s)}$ as a homogeneous element of degree $d$ in $S_\beta^O$.

**Lemma 7.24.** Suppose that $\beta \in Q_+^\circ$. Then

$$\Theta_\beta^O = \{ \theta_{\lambda \mu}^{(d,s)} \mid 1 \leq s \leq c_{\lambda \mu}^{(d)}, 0 \leq d \leq 2 \text{ def } \beta \text{ and } \lambda, \mu \in P_\beta^\Lambda \}$$

is a homogeneous basis of $R_\beta^O$. Moreover, $\deg \theta_{\lambda \mu}^{(d,s)} = d$ for all $\theta_{\lambda \mu}^{(d,s)} \in \Theta_\beta^O$.

**Proof.** Suppose that $\lambda, \mu \in P_\beta^\Lambda$. By construction, $\theta_{\lambda \mu}^{(d,s)}$ is a homogeneous map of degree $d$ and the set $\{ \theta_{\lambda \mu}^{(d,s)} \mid 1 \leq s \leq c_{\lambda \mu}^{(d)} \}$ is linearly independent. Counting graded dimensions, this set is a homogeneous basis of $\text{Hom}_1^{S_\beta^O}(P_\lambda^\Lambda, P_\mu^\Lambda)$. The lemma follows. \(\square\)

We are ready to prove the main result of this section.

**Theorem 7.25.** Suppose that $e = 0$, $Z = C$ and $\beta \in Q_+^\circ$. Then there is a homogeneous algebra isomorphism $\Xi: R_\beta^O \sim \to R_\beta^\circ$ of degree zero.

**Proof.** By Theorem 7.12, there is an equivalence of categories $E_\lambda^\circ: S_\beta^O \to \text{Mod} \to P_\beta^\Lambda \to \text{Mod}$ such that $E_\lambda^\circ(P_\lambda^\Lambda) \cong P_\lambda^{\circ \lambda}$, for all $\lambda \in P_\beta^\Lambda$. By Lemma 7.24, $\Theta_\beta^O = \{ \theta_{\lambda \mu}^{(d,s)} \}$ is a basis of $\text{Hom}_1^{S_\beta^O}(P_\lambda^\Lambda, P_\mu^\Lambda)$. For each $\theta_{\lambda \mu}^{(d,s)} \in \Theta_\beta^O$, set $\vartheta_{\lambda \mu}^{(d,s)} = E_\lambda^\circ(\theta_{\lambda \mu}^{(d,s)})$. Forgetting the gradings for the moment, $\vartheta_{\lambda \mu}^{(d,s)} \in \text{Hom}_1^{S_\beta^O}(P_\lambda^\Lambda, P_\mu^\Lambda)$ and $\vartheta_{\lambda \mu}^{(d,s)}$ is a basis of $\text{Hom}_1^{S_\beta^O}(P_\lambda^\Lambda, P_\mu^\Lambda)$. Moreover, because $E_\lambda^\circ$ preserves radical filtrations, if $\lambda, \mu \in K_\beta^\Lambda$, then (7.23) shows that there is a commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & \text{rad}^{d+1}(P_\mu) & \longrightarrow & \text{rad}^{d}(P_\mu) & \longrightarrow & L_\lambda^\Lambda(d) & \longrightarrow & 0 \\
& & \downarrow \vartheta_{\lambda \mu}^{(d,s)} & & \downarrow & & \downarrow & & \\
& & P_\lambda^\Lambda & & P_\mu^\Lambda(d) & & & & \\
\end{array}
$$

(7.23)

By Proposition 7.22, we can write $\vartheta_{\lambda \mu}^{(d,s)} = \sum_{k \geq d} \vartheta_{\lambda \mu,k}^{(d,s)}$, where $\vartheta_{\lambda \mu,k}^{(d,s)}$ is homogeneous of degree $k$ and $\vartheta_{\lambda \mu,k}^{(d,s)} \neq 0$. By replacing $\vartheta_{\lambda \mu}^{(d,s)}$ with $\vartheta_{\lambda \mu,d}^{(d,s)} \neq 0$, if necessary, we may assume that $\vartheta_{\lambda \mu}^{(d,s)}$ is homogeneous of degree $d$ whenever $\lambda, \mu \in K_\beta^\Lambda$. The map $\Xi: R_\beta^O \to R_\beta^\circ$ given by $\Xi(\theta_{\lambda \mu}^{(d,s)}) = \vartheta_{\lambda \mu}^{(d,s)}$, for $\theta_{\lambda \mu}^{(d,s)} \in \Theta_\beta^O$ with $\lambda, \mu \in K_\beta^\Lambda$, defines an isomorphism of graded vector spaces $R_\beta^O \sim \to R_\beta^\circ$. As $E_\lambda^\circ$ respects composition of maps, and as multiplication in $R_\beta^O$ and
in $R_\beta^b$ is homogeneous, it follows $\Xi$ is an algebra homomorphism. Hence, $R_\beta^0 \cong R_\beta^b$ as graded algebras as claimed.

The proof of Theorem 7.25 is quite subtle because we have to work with the indecomposable prinjective modules $P^\mu$ for the quiver Schur algebra $S_\beta^\Lambda$ and use the fact that these modules are rigid. In many ways, it would be more natural to prove this result using the graded Young modules $Y^\mu$ but as these modules are not known to be rigid we cannot argue this way. It would be nice to know when the Young modules are rigid. Examples show that the algebras $R_\beta^0$ are not always quadratic, so the ideas underpinning Proposition 2.13 are not sufficient to answer this question.

7.3. Graded decomposition numbers when $e = 0$

Bellinson, Ginzburg and Soergel showed that a Koszul algebra has a unique positive grading [9, Corollary 2.5.2]. In the same spirit, we now show that the gradings on the Schur algebras $S_\beta^0$ and $S_\beta^\Lambda$ coincide.

Recalling the functor $F^\beta_\Lambda$ from (7.17), define analogues of the Young modules for $R_\beta^0$ by setting $Y^\mu_\beta = F^\beta_\Lambda(P^\mu_\beta)$, for $\mu \in \mathcal{P}_\beta^\Lambda$. Similarly, recalling that $R_\beta^b = \beta_\beta S_\beta^\Lambda e_\beta$, let $Y^\mu_\beta = P^\mu e_\beta$ be a Young module for $R_\beta^b$. Using the isomorphism $\Xi : R_\beta^0 \cong R_\beta^b$ from Theorem 7.25 we can consider $Y^\mu_\beta$ as an $R_\beta^b$-module.

**Lemma 7.26.** Suppose that $\mu \in \mathcal{P}_\beta^\Lambda$. Then $Y^\mu_\beta \cong Y^\mu_\beta(a_\mu)$ as $R_\beta^b$-modules, for some $a_\mu \in \mathbb{Z}$.

**Proof.** Recall the (ungraded) Young modules from [47, (3.5)] that were used in the proof of Lemma 6.11. By [13, Lemma 6.11] and Lemma 6.11), $E^\beta_\Lambda(Y^\mu_\beta)$ and $Y^\mu_\beta$ are both graded lifts of $y^\mu$. As $y^\mu$ is indecomposable, and graded lifts of indecomposable modules are unique up to shift, it follows that $Y^\mu_\beta \cong Y^\mu_\beta(a_\mu)$, for some $a_\mu \in \mathbb{Z}$. \qed

**Theorem 7.27.** Suppose that $e = 0$, $\mathbb{Z} = \mathbb{C}$ and $\beta \in Q^+_n$. Then $S_\beta^b \cong S_\beta^0$ as graded algebras. In particular, $S_\beta^b$ is Koszul.

**Proof.** For any $\lambda, \mu \in \mathcal{P}_\beta^\Lambda$, let $c_{\lambda\mu}(q)$ and $c^\beta_{\lambda\mu}(q)$ be the graded Cartan numbers of $S_\beta^b$ and $S_\beta^\Lambda$, respectively. By construction, $\text{dim}_q S_\beta^b = \sum_{\lambda, \mu \in \mathcal{P}_\beta^\Lambda} c_{\lambda\mu}(q)$. By definition, for any $\lambda, \mu \in \mathcal{P}_\beta^\Lambda$,

$$c_{\lambda\mu}(q) = \text{dim}_q \text{Hom}_{S_\beta^b}(P^\mu, P^\lambda) = \text{dim}_q \text{Hom}_{S_\beta^\Lambda}(Y^\mu, Y^\lambda), \quad \text{by Corollary 4.36},$$

$$= \text{dim}_q \text{Hom}_{R_\beta^b}(Y^\mu_\beta, Y^\lambda_\beta) = \text{dim}_q \text{Hom}_{R_\beta^\Lambda}(Y^\mu_\beta(a_\mu), Y^\lambda_\beta(a_\lambda)), \quad \text{by Lemma 7.26},$$

$$= q^{a_\lambda - a_\mu} \text{dim}_q \text{Hom}_{R_\beta^b}(Y^\mu_\beta, Y^\lambda_\beta) = q^{a_\lambda - a_\mu} c^\beta_{\lambda\mu}(q).$$

By graded BGG reciprocity (Corollary 7.11), $c^\beta_{\lambda\mu}(q) = c_{\mu\lambda}(q)$, so that $c_{\lambda\mu}(q) = q^{2(a_\lambda - a_\mu)} c_{\lambda\mu}(q)$. However, the Cartan matrix of $S_\beta^b$ is symmetric by Corollary 2.8. Arguing as in the second last paragraph in the proof of Theorem 6.17 it follows that $a_\lambda = a_\mu$ for all $\lambda, \mu \in \mathcal{P}_\beta^\Lambda$ since $S_\beta^\Lambda$ is indecomposable by Theorem 4.35. On the other hand, $a_\mu = 0$ for all $\mu \in K_\beta^\Lambda$ because if $\mu \in K_\beta^\Lambda$, then $\text{dim}_q Y^\mu_\beta = \text{dim}_q Y^\mu_\beta$ by Theorem 7.25. Therefore, $c_{\lambda\mu}(q) = c^\beta_{\lambda\mu}(q) \in \mathbb{N}[q]$ and, consequently, $\text{dim}_q S_\beta^b = \text{dim}_q S_\beta^\Lambda$.\hfill \qed
Moreover, \( S_\beta^O \cong \text{End}_{S_\beta^O} \left( \bigoplus_{\mu \in \mathcal{P}_\beta^A} P_\mu^\mu \right)^{\text{op}} \cong \text{End}_{R_\beta^A} \left( \bigoplus_{\mu \in \mathcal{P}_\beta^A} Y_\mu^\mu \right)^{\text{op}} \cong \text{End}_{S_\beta^A} \left( \bigoplus_{\mu \in \mathcal{P}_\beta^A} P_\mu^{\mu} \right)^{\text{op}} \cong S_\beta^O \), where the third isomorphism follows from Lemma 7.26 using the fact that \( a_\mu = 0 \) for all \( \mu \in \mathcal{P}_\beta^A \). Hence, \( S_\beta^O \cong S_\beta^O \) is Koszul by Theorem 7.6.

Define non-negative integers \( d^{(s)}_{\lambda\mu} \) by \( d_{\lambda\mu}(q) = \sum_{s \geq 0} d^{(s)}_{\lambda\mu} q^s \), for \( \lambda, \mu \in \mathcal{P}_\beta^A \).

**Corollary 7.28.** Suppose that \( e = 0 \), \( Z = \mathbb{C} \) and \( \beta \in Q_\beta^+ \) and let \( \lambda, \mu \in \mathcal{P}_\beta^A \). Then \( d_{\lambda\mu}(q) \in \delta_{\lambda\mu} + q\mathbb{N}[q] \) and \( c_{\lambda\mu}(q) \in \delta_{\lambda\mu} + q\mathbb{N}[q] \) and if \( s \geq 0 \), then

\[
[\text{rad}^s \Delta^\lambda \div \text{rad}^{s+1} \Delta^\lambda : L^\mu(s)]_q = d^{(s)}_{\lambda\mu}.
\]

**Proof.** Since \( S_\beta^O \) is Koszul, \( d_{\lambda\mu}(q) \in \delta_{\lambda\mu} + q\mathbb{N}[q] \). Consequently, \( c_{\lambda\mu}(q) \in \delta_{\lambda\mu} + q\mathbb{N}[q] \) by Corollary 2.8. Finally, since \( \Delta^\mu \div \text{rad} \Delta^\mu \cong L^\mu \) is irreducible the last statement follows from Proposition 7.10(a).

Combining the results in this section, we obtain a more precise version of Theorem C from the introduction.

**Theorem 7.29.** Suppose that \( \beta \in Q_\beta^+ \), \( e = 0 \) and \( Z = \mathbb{C} \). Then there is a graded equivalence of categories \( E_\beta^O : O_\beta^A \rightarrow S_\beta^A \)-Mod such that the following diagram commutes:

\[
\begin{array}{ccc}
E_\beta^O & \rightarrow & S_\beta^A \text{-Mod} \\
\downarrow & & \downarrow \\
F_\beta^A & \rightarrow & R_\beta^A \text{-Mod}
\end{array}
\]

Moreover, \( E_\beta^O(\Delta^\lambda_\beta) \cong \Delta^\lambda \) and \( E_\beta^O(L^\mu_\beta) \cong L^\mu \), for all \( \lambda, \mu \in \mathcal{P}_\beta^A \).

### 7.4. The Fock space

The aim of this subsection is to realize the projective indecomposable and irreducible modules for \( S_n^A \) as the canonical and dual canonical bases of the higher level Fock space. Throughout this subsection, we work over \( \mathbb{C} \) and assume that either \( e = 0 \) or \( e > n \).

Let \( \text{Rep}(S_n^A) \) be the Grothendieck group of finitely generated \( S_n^A \)-modules. If \( M \) is an \( S_n^A \)-module, let \( [M] \) be its image in \( \text{Rep}(S_n^A) \). Observe that \( \text{Rep}(S_n^A) \) is naturally a \( \mathbb{Z}[q, q^{-1}] \)-module, where \( q \) acts by grading shift: \( q[M] = [M(1)] \), for \( M \in S_n^A \)-Mod. Similarly, let \( \text{Proj}(S_n^A) \) be the Grothendieck group of the category of finitely generated projective \( S_n^A \)-modules. The Cartan pairing is the sesquilinear map (anti-linear in the first argument, linear in the second)

\[
(\ , \ ) : \text{Proj}(S_n^A) \times \text{Rep}(S_n^A) \rightarrow \mathbb{Z}[q, q^{-1}], \quad ([P], [M]) = \text{dim}_q \text{Hom}_{S_n^A}(P, M),
\]

for \( [P] \in \text{Proj}(S_n^A) \) and \( [M] \in \text{Rep}(S_n^A) \). There is a natural embedding \( \text{Proj}(S_n^A) \hookrightarrow \text{Rep}(S_n^A) \).
Define the *combinatorial Fock space* of weight $\Lambda$ to be
$$\mathfrak{F}^\Lambda = \bigoplus_{n \geq 0} \text{Rep}(S_n^\Lambda).$$

Thus, $\mathfrak{F}^\Lambda$ is a free $\mathbb{Z}[q,q^{-1}]$-module of infinite rank. Let $\mathcal{P}^\Lambda = \bigcup_{n \geq 0} P_n^\Lambda$. The Fock space $\mathfrak{F}^\Lambda$ is equipped with the following distinguished bases:

(i) The irreducible modules: $\{ [L^\mu] \mid \mu \in \mathcal{P}^\Lambda \}$.

(ii) The standard modules $\{ [\Delta^\mu] \mid \mu \in \mathcal{P}^\Lambda \}$.

(iii) The projective indecomposable modules $\{ [P^\mu] \mid \mu \in \mathcal{P}^\Lambda \}$.

(iv) The tilting modules $\{ [T^\mu] \mid \mu \in \mathcal{P}^\Lambda \}$.

These four sets are all bases for $\mathfrak{F}^\Lambda$ as a $\mathbb{Z}[q,q^{-1}]$-module because the graded decomposition matrix of $S_n^\Lambda$ is invertible over $\mathbb{Z}[q,q^{-1}]$ by Corollary 6.18.

The aim of this section is to clarify the relationships between these bases and to give an algorithm for computing the graded decomposition numbers of $S_n^\Lambda$.

There is a natural duality on $\text{Rep}(S_n^\Lambda)$ that induces an involution on $\mathfrak{F}^\Lambda$. Let $M$ be an $S_n^\Lambda$-module. Recall that $M^\oplus = \text{Hom}_C(M, \mathbb{C})$ is the graded dual of $M$. Similarly, define $M^\# = \text{Hom}_{S_n^\Lambda}(M, S_n^\Lambda)$, where $S_n^\Lambda$ acts on $M^\#$ by $(f \cdot s)(x) = f(xs)$, for $f \in M^\#$ and $x \in M$, $s \in S_n^\Lambda$. Then # restricts to a duality on $\text{Proj}(S_n^\Lambda)$.

**Lemma 7.30.** Suppose that $M$ is an $S_n^\Lambda$-module. Then
$$([P^\#], [M]) = ([P], [M^\#]),$$
for all $[P] \in \text{Proj}(S_n^\Lambda)$ and $[M] \in \text{Rep}(S_n^\Lambda)$. Moreover, if $\mu \in \mathcal{P}_n^\Lambda$, then $(L^\mu)^\oplus \cong L^\mu$, $(T^\mu)^\oplus \cong T^\mu$ and $(P^\mu)^\# \cong P^\mu$.

**Proof.** The first statement is well known; see, for example, [15, Lemma 2.5]. This implies that $(P^\mu)^\# \cong P^\mu$ since $(L^\mu)^\oplus \cong L^\mu$ by Theorem 2.5. Finally, $(T^\mu)^\oplus \cong T^\mu$ by Corollary 5.17.

A map $f : M \to N$ of $\mathbb{Z}[q,q^{-1}]$-modules is *semilinear* if it is $\mathbb{Z}$-linear and $f(q^k m) = q^{-k} f(m)$, for all $m \in M$ and $k \in \mathbb{Z}$.

**Lemma 7.31.** The maps $\oplus$ and $\#$ induce semilinear involutions on $\mathfrak{F}^\Lambda$ such that
$$(M(d))^\oplus \cong M^\oplus (-d) \text{ and } (N(d))^\# \cong N^\# (-d),$$
for all $[M] \in \text{Rep}(S_n^\Lambda)$, $[N] \in \text{Proj}(S_n^\Lambda)$ and $d \in \mathbb{Z}$.

**Proof.** It follows easily from the definitions that $\oplus$ is a duality on $\text{Rep}(S_n^\Lambda)$ and that $\#$ is a duality on $\text{Proj}(S_n^\Lambda)$. This immediately implies that $\oplus$ induces an involution on $\mathfrak{F}^\Lambda$ with the required properties. Moreover, $\#$ extends to an automorphism of $\mathfrak{F}^\Lambda$ because $\{ [P^\mu] \mid \mu \in \mathcal{P}_n^\Lambda \}$ is a $\mathbb{Z}[q,q^{-1}]$-basis of $\mathfrak{F}^\Lambda$. The map induced by $\#$ is an involution because $(P^\mu)^\# \cong P^\mu$ by Lemma 7.30, for $\mu \in \mathcal{P}_n^\Lambda$.

We emphasize that both of these maps are semilinear, that is, $\mathbb{Z}$-linear but not $\mathbb{Z}[q,q^{-1}]$-linear. This is implicit in the displayed equation of Lemma 7.31 because, for example, $(q[M])^\# = [M(1)]^\# = [M^\oplus (-1)] = q^{-1} [M^\oplus]$.

Recall from § 2.1 that the *bar involution* on $\mathbb{Z}[q,q^{-1}]$ is the $\mathbb{Z}$-linear automorphism of $\mathbb{Z}[q,q^{-1}]$ determined by $\overline{q} = q^{-1}$. A Laurent polynomial $f(q)$ in $\mathbb{Z}[q,q^{-1}]$ is *bar invariant* if $f(q) = \overline{f(q)}$. 
Lemma 7.32. Suppose that $\lambda \in \mathcal{P}^A_n$. Then

$$[\Delta^\lambda]_\oplus = [\Delta^\lambda] + \sum_{\mu \in \mathcal{P}^A_n, \lambda \trianglerighteq \mu} f_{\mu}(q)[\Delta^\mu] \quad \text{and} \quad [\Delta^\lambda]_\# = [\Delta^\lambda] + \sum_{\mu \in \mathcal{P}^A_n, \mu \trianglerighteq \lambda} g_{\mu}(q)[\Delta^\mu],$$

for some Laurent polynomials $f_{\mu}(q), g_{\mu}(q) \in \mathbb{Z}[q, q^{-1}]$.

Proof. Recall that $(d_{\lambda\mu}(q))_{\lambda, \mu \in \mathcal{P}^A_n}$ is the graded decomposition matrix of $S^A_n$. Let $(e_{\lambda\mu}(q))_{\lambda, \mu \in \mathcal{P}^A_n}$ be the inverse graded decomposition matrix. Using Lemma 7.32, we compute

$$[\Delta^\lambda]_\oplus = \left( \sum_{\mu \in \mathcal{P}^A_n, \lambda \trianglerighteq \mu} d_{\lambda\mu}(q)[L^\mu] \right)_\oplus = \sum_{\mu \in \mathcal{P}^A_n} d_{\lambda\mu}(q^{-1})[L^\mu]$$

since $(L^\mu)_\oplus \cong L^\mu$ by Theorem 4.24. Therefore,

$$[\Delta^\lambda]_\oplus = \sum_{\mu \in \mathcal{P}^A_n, \lambda \trianglerighteq \mu} d_{\lambda\mu}(q^{-1}) \sum_{\nu \in \mathcal{P}^A_n, \mu \trianglerighteq \nu} e_{\nu\mu}(q)[\Delta^\nu]$$

$$= [\Delta^\lambda] + \sum_{\nu \in \mathcal{P}^A_n, \lambda \trianglerighteq \nu} \left( \sum_{\mu \in \mathcal{P}^A_n, \lambda \trianglerighteq \mu} e_{\nu\mu}(q)d_{\lambda\mu}(q^{-1}) \right)[\Delta^\nu],$$

where the last line follows because both the graded decomposition matrix and its inverse are triangular with respect to dominance by Corollary 4.33. The formula for $[\Delta^\lambda]_\#$ is proved in exactly the same way by first writing $[\Delta^\lambda] = \sum_{\mu \geq \lambda} e_{\mu\lambda}(q)[P^\mu]$.

By a well-known result of Lusztig [44, Lemma 24.2.1], Lemma 7.32 implies that $\mathfrak{g}^A$ has several uniquely determined ‘canonical bases’ that are invariant under $\oplus$ and $\#$. Using Corollary 6.18, we can describe these bases explicitly. Let $\mathfrak{S}_q^A(>\mu)$ (respectively, $\mathfrak{S}_q^A(<\mu)$) be the $\mathbb{Z}[q]$-sublattice of $\mathfrak{g}^A$ with basis the images of the standard modules $\{[\Delta^\lambda] \mid \mu \triangleleft \lambda \in \mathcal{P}^A \}$ (respectively, $\{[\Delta^\lambda] \mid \mu \triangleright \lambda \in \mathcal{P}^A \}$) in $\mathfrak{g}^A$. Let $\mathfrak{S}_q^{A^{-1}}(<\mu)$ be the $\mathbb{Z}[q^{-1}]$-sublattice of $\mathfrak{g}^A$ spanned by the images of the standard modules $\{[\Delta^\lambda] \mid \mu \triangleright \lambda \in \mathcal{P}^A \}$.

Theorem 7.33. Suppose that $e = 0$ and $Z = C$. Then the three bases

$$\{[P^\mu] \mid \mu \in \mathcal{P}^A \}, \quad \{[L^\mu] \mid \mu \in \mathcal{P}^A \} \quad \text{and} \quad \{[T^\mu] \mid \mu \in \mathcal{P}^A \}$$

are ‘canonical bases’ of $\mathfrak{g}^A$ that, for $\mu \in \mathcal{P}^A$, are uniquely determined by:

(a) $[P^\mu]_\# = [P^\mu]$ and $[P^\mu]_\cong [\Delta^\mu] (\text{mod } q\mathfrak{S}_q^A(>\mu))$;
(b) $[L^\mu]_\oplus = [L^\mu]$ and $[L^\mu]_\oplus = [\Delta^\mu] (\text{mod } q\mathfrak{S}_q^A(<\mu))$;
(c) $[T^\mu]_\oplus = [T^\mu]$ and $[T^\mu]_\oplus = [\Delta^\mu] (\text{mod } q^{-1}\mathfrak{S}_q^{A^{-1}}(<\mu))$.

Proof. The existence and uniqueness of bases of $\mathfrak{g}^A$ with these properties follows from what is by now a standard argument (see [44, Lemma 24.2.1]), using the triangularity of the involutions $\oplus$ and $\#$ from Lemma 7.32. If $\mu \in \mathcal{P}^A_n$, then $[P^\mu]_\# = [P^\mu]$, $(L^\mu)_\oplus \cong L^\mu$ and $(T^\mu)_\oplus \cong T^\mu$ by Lemma 7.30. Furthermore, $[P^\mu]_\oplus = \sum_{\lambda} d_{\lambda\mu}(q)[\Delta^\lambda]$ and $[L^\mu]_\# = \sum_{\lambda} e_{\lambda\mu}(q)[\Delta^\lambda]$, where $d_{\lambda\mu}(q)$ and $e_{\lambda\mu}(q)$ are polynomials in $\mathbb{Z}[q]$ with constant term $d_{\lambda\mu}(0) = \delta_{\lambda\mu} = e_{\lambda\mu}(0)$ by Corollary 6.18. Therefore, if $\mu \in \mathcal{P}^A_n$, then $[P^\mu]$ and $[L^\mu]$ belong to $\mathfrak{S}_q^A(\geq \mu)$ and, moreover,

$$[P^\mu] \equiv [\Delta^\mu] (\text{mod } q\mathfrak{S}_q^A(>\mu)) \quad \text{and} \quad [\Delta^\mu] \equiv [L^\mu] (\text{mod } q\mathfrak{S}_q^A(<\mu)).$$
Hence, parts (a) and (b) follow. Finally, by Corollaries 5.17 and 5.21, \((T^\mu)^* \cong T^\mu\) and \([T^\mu] \equiv [\Delta^\mu] \pmod{q^{-1}\mathbb{F}_{q-1}(<\mu)}\). This completes the proof. \(\square\)

We call \(\{[T^\mu] \mid \mu \in \mathcal{P}^\Lambda\}\) the canonical basis of \(\mathfrak{S}^\Lambda\) and \(\{[L^\lambda] \mid \mu \in \mathcal{P}^\Lambda\}\) the dual canonical basis. By Theorem 5.20, Ringel duality induces an automorphism of \(\mathfrak{S}^\Lambda\) that interchanges, setwise, the canonical basis \(\{[P^\mu]\}\) and the basis \(\{[T^\lambda]\}\) of tilting modules. We remark that Theorem 7.33 should lift to a categorification of the canonical bases of \(\mathfrak{S}^\Lambda\) as a \(U_q(\mathfrak{gl}_\infty)\)-module.

Remark 7.34. Abusing notation slightly, let \(\#\) be the automorphism of \(\text{Rep}(\mathcal{R}_n^\Lambda)\) defined by \(M^\# = \text{Hom}_{\mathcal{R}_n^\Lambda}(M, \mathcal{R}_n^\Lambda)\). Then, as noted in [15, Remark 4.7], it follows from Theorem 3.20 and [53, Theorem 3.1] that there is an isomorphism of functors \(\# \cong \langle 2 \text{ def } \beta \rangle \circ \otimes\). Therefore, \(\{q^{\text{def } \beta}[D^\mu] \mid \mu \in K^\Lambda_\beta \text{ for } \beta \in Q^+\}\) is a \(\#\)-invariant basis of \(\bigoplus_n \text{Rep}(\mathcal{R}_n^\Lambda)\) that has similar uniqueness properties to the tilting module basis of \(\mathfrak{S}^\Lambda\). Similarly, \(\{q^{-\text{def } \beta}[Y^\mu] \mid \mu \in K^\Lambda_\beta\}\) is a ‘canonical’ \(\otimes\)-invariant basis of \(\bigoplus_{n \geq 0} \text{Rep}(\mathcal{R}_n^\Lambda)\).

7.5. An LLT algorithm for \(\mathcal{S}_n^\Lambda\)

Theorem 7.33 is not surprising because it is a well-established mantra that the indecomposable tilting modules should correspond to Lusztig’s canonical basis, the simple modules to the dual canonical basis and so on. The main reason for our introducing the Fock space is that we can now use the tableau combinatorics for \(\mathcal{S}_n^\Lambda\) to explicitly compute the graded decomposition numbers for \(\mathcal{S}_n^\Lambda\) and hence for parabolic category \(\mathcal{O}^\Lambda\). Thus, not only does this combinatorics explicitly describe the grading on parabolic category \(\mathcal{O}^\Lambda\) but it also gives an effective way of computing graded decomposition multiplicities of \(\mathcal{S}_n^\Lambda\), which are certain parabolic Kazhdan–Lusztig polynomials.

If \(\Lambda = \Lambda_0\), then \(\mathcal{H}_n^\Lambda \cong \mathcal{R}_n^\Lambda\) is isomorphic to the Iwahori–Hecke algebra of the symmetric group. In this case, Lascoux, Leclerc and Thibon [40] have given an efficient algorithm for computing the canonical basis of the irreducible \(U_q(\mathfrak{sl})\)-module \(L(\Lambda_0)\). By Ariki’s theorem [2, 15], the LLT algorithm computes the (graded) decomposition matrices of the Iwahori–Hecke algebra of the symmetric group.

In this section, we give an LLT-like algorithm for computing the canonical basis of \(\mathfrak{S}^\Lambda\). By Theorem 7.33, this gives an algorithm for computing the graded decomposition numbers for \(\mathcal{R}_n^\Lambda\) and \(\mathcal{S}_n^\Lambda\). To this end, if \(f(q) = \sum_{d \in \mathbb{Z}} f_d q^d\) is a non-zero Laurent polynomial in \(\mathbb{Z}[q, q^{-1}]\), let \(\text{mindeg } f(q) = \min \{d \in \mathbb{Z} \mid f_d \neq 0\}\).

Suppose that \(\mu \in \mathcal{P}^\Lambda_\beta\), where \(\beta \in Q^+\). Recall from (5.1) that \(Z^\mu = \Psi^\mu \mathcal{S}_n^\Lambda\). Similarly, recall from §2.1 that if \(M\) is an \(\mathcal{S}_n^\Lambda\)-module and \(f(q) = \sum_k f_k q^k \in \mathbb{N}[q, q^{-1}]\), then \(f(q)M = \bigoplus_k M(k)^{\oplus f_k}\).

Lemma 7.35. Suppose that \(\mu \in \mathcal{P}_\beta^\Lambda\). Then \((Z^\mu)^* \cong Z^\mu\) and 

\[Z^\mu = P^\mu \oplus \bigoplus_{\lambda > \mu} z_{\lambda\mu}(q)P^\lambda,\]

for some bar invariant polynomials \(z_{\lambda\mu}(q) \in \mathbb{N}[q, q^{-1}]\).

Proof. By definition, \(Z^\mu\) is a direct summand of \(\mathcal{S}_n^\Lambda\) and \((\Psi^\mu)^* = \Psi^\mu\), so \((Z^\mu)^* \cong Z^\mu\). We already noted in (5.4) that \(Z^\mu = P^\mu \oplus \bigoplus_{\lambda \neq \mu} z_{\lambda\mu}(q)P^\lambda\), for some Laurent polynomials \(z_{\lambda\mu}(q) \in \mathbb{N}[q, q^{-1}]\), because \(Z^\mu\) is projective. In view of Lemma 7.30, these polynomials are bar invariant. \(\square\)
Next observe that (5.3) implies that in $\mathfrak{F}_\Lambda$

$$[Z^\mu] = [\Delta^\mu] + \sum_{\nu \in \mathfrak{P}_\beta} q^{\deg s - \deg t^\nu} [\Delta^\nu].$$  

(7.36)

We now show how to use Lemma 7.35 and (7.36) to inductively compute $[P^\mu]$, for $\mu \in \mathfrak{P}_\beta$, as a linear combination of standard modules in $\mathfrak{F}_\Lambda$. Since $[P^\mu] = \sum_{\nu} d_{\lambda \mu}(q)[\Delta^\mu]$ this will give an algorithm for computing the graded decomposition numbers of $S^\beta_{\lambda \mu}$.

If $\mu$ is maximal in $\mathfrak{P}_\beta$, with respect to dominance, then $Z^\mu = P^\mu = \Delta^\mu$ by Lemma 7.35. So $[P^\mu] = [\Delta^\mu]$ in this case and there is nothing to do.

Now suppose that $\mu$ is not maximal in $\mathfrak{P}_\beta$ and that $[P^\lambda]$ is known whenever $\lambda \in \mathfrak{P}_\beta$ and $\lambda \triangleright \mu$. By (7.36), we can write

$$[Z^\mu] = [\Delta^\mu] + \sum_{\nu \in \mathfrak{P}_\beta} y_{\nu \mu}(q)[\Delta^\nu]$$

for some Laurent polynomials $y_{\nu \mu}(q) \in \mathbb{N}[q, q^{-1}]$ that are not all zero since $\mu$ is not maximal in $\mathfrak{P}_\beta$. Let $\lambda \triangleright \mu$ be any multipartition such that $y_{\lambda \mu}(q) \neq 0$ and

$$\operatorname{mindeg} y_{\nu \mu}(q) \leq \operatorname{mindeg} y_{\nu \mu}(q),$$

for all $\nu \in \mathfrak{P}_\beta$. Let $d = \operatorname{mindeg} y_{\lambda \mu}(q)$.

If $d > 0$, then $[Z^\mu] \equiv [\Delta^\nu] \pmod{q \delta_{\lambda \mu}(\mu)}$ by (7.36). Now $[Z^\mu] \# = [Z^\mu]$, by Lemma 7.35. This forces $[Z^\mu] = [P^\mu]$ because $[Z^\mu]$ satisfies the two properties that uniquely determine $[P^\mu]$ in Theorem 7.33(a).

Now suppose that $d \leq 0$. Let $y_{\lambda \mu}^{(d)}$ be the coefficient of $q^d$ in $y_{\lambda \mu}(q)$ and set

$$z_{\lambda \mu}^{(d)} = \begin{cases} y_{\lambda \mu}^{(d)}(q^d + q^{-d}) & \text{if } d < 0, \\ y_{\lambda \mu}^{(d)} & \text{if } d = 0. \end{cases}$$

Since $[P^\mu] \equiv [\Delta^\nu] \pmod{q \delta_{\lambda \mu}(\mu)}$, for all $\nu \in \mathfrak{P}_\beta$, the minimally of $d$ together with Lemma 7.35 implies that $z_{\lambda \mu}^{(d)} P^\lambda$ is a direct summand of $Z^\mu$. Since $[P^\lambda]$ is known by induction we can now replace $[Z^\mu]$ with $[Z^\mu] - \sum_{\nu \in \mathfrak{P}_\beta} z_{\lambda \mu}^{(d)} P^\lambda$, which is still $\#$-invariant. By repeating this process of stripping off the bar invariant minimal degree terms, we can rewrite $[Z^\mu]$ as a linear combination of canonical bases elements as in Lemma 7.35. This recursively computes $[P^\mu]$ and so determines the graded decomposition numbers $d_{\lambda \mu}(q)$.

Note that the Laurent polynomials $z_{\lambda \mu}(q)$ in Lemma 7.35 are given by $z_{\lambda \mu}(q) = \sum_{d \leq 0} z_{\lambda \mu}^{(d)}$. Hence, this algorithm also decomposes $Z^\mu$ into a direct sum of projective modules.

**Remark 7.37.** Note that $(E^\mu) \circ E^\mu$ by Theorem 5.15. An equivalent version of this algorithm computes $[T^\mu]$ by applying the same ‘straightening algorithm’ to the element $[E^\mu] = [E^\mu] \circ E^\mu$, where we use Corollary 5.13 in place of (7.36) and Corollary 5.13 in place of Lemma 7.35.

**Example 7.38.** Suppose that $e = 0$, $\Lambda = 3\lambda_0$ and that $\beta = \alpha_1 - 3\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Then $S^\beta_{\lambda_0}$ is a block of defect 4. The maximal multipartition in $\mathfrak{P}_\beta$ is $(4, 2|1|0)$ so $P^{(4, 2|1|0)} = \Delta^{(4, 2|1|0)}$. Taking $\mu = (4, 1|1|1)$ the tableaux in $\text{Std}^\mu(\mathfrak{P}_\beta)$ are

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
5 & & & & \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
6 & 7 & & & \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
5 & 6 & & & \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
5 & 6 & & & \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
6 & & & - & \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
5 & 6 & & & \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
5 & 6 & & & \\
\hline
\end{array} \quad - \quad - \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
6 & & & - & \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
5 & 6 & & & \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
5 & 6 & & & \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & \\
\hline
6 & & & - & \\
\hline
\end{array} \quad - \quad -$$
Therefore, \([Z^\mu] = [\Delta^{(4,1,1)(1)}] + q[\Delta^{(4,2,0)(1)}] + (q^2 + 1)[\Delta^{(4,2,1)(0)}]\). Applying our algorithm, \([Z^\mu] = [P^\mu] + [P^{(4,2,1)(0)}]\). Using our LLT algorithm, the full graded decomposition matrix of \(S^A_\beta\) in characteristic zero is:

\[
\begin{array}{c|cc}
0 & 1 & 4, 2 \\
0 & 4, 2 & 1 \\
1 & 0 & 4, 2 \\
1 & 1 & 4, 1 \\
1 & 1, 2 & 4 \\
1 & 4, 1 & 1 \\
1 & 4, 2 & 0 \\
1 & 2 & 1 \\
1 & 2 & 4 \\
4 & 1 & 1 \\
4 & 1, 2 & 1 \\
4 & 1 & 1 \\
4 & 2 & 0 \\
4 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccccccccccc}
 & q^2 & q & q^2 & q & q^2 & q & q \\
(0, 1) & 1 & 4, 2 & 1 & 1 & q & q & 1 & 1 & q & q & 1 & 1 \\
(0, 2) & q & q & q & q & q & q & q & q & q & q & q & q \\
(1, 1, 0) & q^2 & q & q^2 & q & q^2 & q & q & q & q & q & q & q \\
(1, 4) & q^2 & q & q^2 & q & q^2 & q & q & q & q & q & q & q \\
(1, 2) & q & q & q^2 & q & q^2 & q & q & q & q & q & q & q \\
(4, 1) & q^2 & q & q^2 & q & q^2 & q & q & q & q & q & q & q \\
(4, 2) & q^2 & q & q^2 & q & q^2 & q & q & q & q & q & q & q \\
(4, 2) & q^2 & q & q^2 & q & q^2 & q & q & q & q & q & q & q \\
\end{array}
\]

The Kleshchev multipartitions in this block are \((0, 1, 4, 2)\) and \((1, 1, 4, 1)\). If \(\Lambda\) is any weight of level \(\ell = 2\), then the graded decomposition numbers of \(S^A_\beta\) are monomials in \(q\) by Theorem B.3. The algebra \(S^A_\beta\) is one of the smallest examples of a block that has a graded decomposition number that is not a monomial.

Theorem B.3 in the appendix shows that all of the results in this section are valid over an arbitrary field when \(\Lambda\) is a dominant weight of level 2 and \(e = 0\) or \(e > n\).

Appendix A. Graded decomposition numbers for \(O^A_\beta\)

This appendix proves Proposition 7.13, which explicitly describes the polynomials \(p^O_{\mu}(q)\) as inverse parabolic Kazhdan–Lusztig polynomials. Throughout this section, we fix \(\Lambda \in \mathcal{P}^{\dagger}\) and \(\beta \in Q^+_{\nu}\) as in §7.1.

To prove Proposition 7.13, we need to interpolate between Brundan and Kleshchev’s work from [13, 16], as described in Section 7, and the more standard Lie theoretic notation used in [8, 9]. Backelin considers blocks \(O^\tau_{\nu}\) of parabolic category \(O\) (of arbitrary type), indexed by dominant integral weights \(\tau, \sigma \in \mathfrak{h}^*\). He defines \(R^\tau_{\nu}\) to be the endomorphism algebra of a minimal projective generator of \(O^\tau_{\nu}\). Set \(O_\tau = O^\mu_{\nu}\), a block of category \(O = O^0\) for \(\mathfrak{g}\mathfrak{l}_N(\mathbb{C})\). Recall that \(\mathcal{G}_N\) acts on \(\mathfrak{h}^*\) via the dot-action and let \(\mathcal{G}(\tau) = \{ w \in \mathcal{G}_N \mid w \cdot \tau = \tau \}\) be the stabilizer of \(\tau\) in \(\mathcal{G}_N\). Recall that the simple modules in \(O^\mu_{\nu}\) are in bijection with the set \(X^+_\tau := \{ \sigma \in \mathfrak{h}^* | (\sigma, \epsilon_i - \epsilon_{i+1}) \geq 0 \}\), whenever \((i, i + 1) \in \mathcal{G}(\tau)\). We set

\[
\begin{align*}
D_\tau &= \{ x \in \mathcal{G}_N \mid \ell(wx) \geq \ell(x) \text{ for all } w \in \mathcal{G}(\tau) \}, \\
D_\tau^+ &= \{ x \in \mathcal{G}_N \mid \ell(wx) \leq \ell(x) \text{ for all } w \in \mathcal{G}(\tau) \}.
\end{align*}
\]

Then \(D_\tau\) is the set of minimal length right coset representatives of \(\mathcal{G}(\tau)\) in \(\mathcal{G}_N\), while \(D_\tau^+\) is the set of maximal length right coset representatives of \(\mathcal{G}(\tau)\) in \(\mathcal{G}_N\). It is well known that \(w \cdot 0 \in X^+_\tau\) if and only if \(w \in D_\tau\). The reader can check that the map \(x \mapsto xw_0\) defines a bijection \(D_\tau \rightarrow D_\tau^+\) using the fact that \(\ell(wx_0) = \ell(w_0) - \ell(w)\), for all \(w \in \mathcal{G}_N\). As a result, the simple modules in \(O_\tau\) are the modules \(\{ \mathcal{L}(x^{-1} \cdot \tau) \mid x \in D_\tau^+ \}\), and \(\{ \mathcal{L}(xw_0 \cdot 0) \mid x \in D_\tau^+ \}\) is a complete set of simple modules in \(O^\mu_{\nu}\), where \(w_0\) is the (unique) element of longest length in \(\mathcal{G}_N\).
We need to describe the block $O^\Lambda_\beta$ using the more standard Lie theoretic notation introduced above following [8, 9]. Fix multipartitions $\lambda, \mu \in \mathcal{P}^\Lambda_\beta$ and recall from Lemma 7.3 that $T^\Lambda$ is the $\Lambda$-tableau corresponding to $\lambda$ and that $\text{col}(T^\Lambda)$ is the column reading of $T^\Lambda$. In §7.1, following [16], we defined $w_\lambda$ to be the unique minimal length permutation such that the sequence $\text{col}(T^\Lambda)w_\lambda$ is weakly increasing and that $Z_\lambda$ is the stabilizer of $\text{col}(T^\Lambda)w_\lambda$. Let $D_\lambda$ be the set of minimal length right coset representatives of $Z_\lambda$ in $\mathfrak{S}_N$. Similarly, let $v_\lambda \in \mathfrak{S}_N$ be the unique minimal length permutation such that $(s_1, \ldots, s_N) = \text{col}(T^\Lambda)v_\lambda$ is weakly decreasing. Fix the dominant integral weight $\phi = s_1 \varepsilon_1 + (s_2 + 1) \varepsilon_2 + \cdots + (s_N + N - 1) \varepsilon_N \in \mathfrak{h}^*$ for the rest of this appendix. By Lemma 7.3, $\phi$ depends only on $\Lambda$ and $\beta$. As above, let $\mathfrak{S}(\phi)$ be the stabilizer of $\phi$ under the dot action. Then $v_\lambda^{-1} \in D_\phi$ and $v_\lambda^{-1}w_0 \in D_\phi^+$.

By definition the subgroups $Z_\lambda$ and $\mathfrak{S}(\phi)$ are conjugate in $\mathfrak{S}_N$. Recall that $P_{x,y}$ is the Kazhdan–Lusztig polynomial indexed by $x, y \in \mathfrak{S}_N$.

**Lemma A.1.** Suppose that $\lambda \in \mathcal{P}^\Lambda_\beta$ and let $\phi \in \mathfrak{h}^*$ be as above. Then $Z_\lambda = w_0\mathfrak{S}(\phi)w_0$. Moreover, if $x, y \in D_\phi$ and $z \in \mathfrak{S}(\phi)$, then $P_{x',y'} = P_{xx,y}$, where $x' = w_0xw_0$, $y' = w_0yw_0$ and $z' = w_0zw_0$.

**Proof.** The definitions above ensure that $Z_\lambda = w_0\mathfrak{S}(\phi)w_0$. Finally, $P_{xx,y} = P_{w_0x,w_0yw_0,y'}w_0 = P_{x',y'}$ since $P_{u,v} = P_{w_0u,w_0v,w_0v_0}$, for any $u, v \in \mathfrak{S}_N$. \hfill \Box

The definitions above show that $\text{col}(T^\Lambda)v_\lambda w_\phi^0 w_0 = \text{col}(T^\Lambda)w_\lambda$, where $w_\phi^0$ is the element of longest length in $\mathfrak{S}(\phi)$. Therefore, by the minimality of $w_\lambda$, $w_\lambda = v_\lambda w_\phi^0 w_0$, by Lemma A.1, and consequently $\omega(\lambda) = v_\lambda \cdot \phi$, where $\omega(\lambda)$ is defined in (7.4). Using Theorem 7.12, it follows that $E^\Lambda_\beta(L(v_\lambda \cdot \phi)) \cong L^\Lambda$. Hence, $O^\beta_\phi$ is a subcategory of $O_\phi$.

Recall from §7.1 that the partition $\pi$ determines the parabolic subalgebra $p$. Let $\psi \in \mathfrak{h}^*$ be any dominant integral weight such that $E(\psi) = \mathfrak{S}_\pi$, where $\mathfrak{S}_\pi$ is the Young subgroup determined by $\pi$. Using Backelin’s notation, $O^\Lambda = O^p = O^\psi$ so that $O^\beta_\phi = O^\psi_\phi$ and $O^\beta_\phi = R^\psi_\phi$.

Consequently, by [8, Theorem 1.1], a precise statement of Theorem 7.6 is that $R^\psi_\phi$ is Koszul and $E(R^\xi_\phi) = R^\xi_\phi$, where $\xi = -w_0\psi$ (no dot action!). Consequently, we identify the categories $O^\beta_\Lambda$ and $R^\phi_\xi$-Mod.

Using Lemma A.1 and the fact that $w_\lambda = v_\lambda w_\phi^0 w_0$, it is easy to see that the next result will turn out to equivalent to Proposition 7.13. Let $q = q(\phi)$ be the parabolic subalgebra of $gl\Lambda(\mathbb{C})$ with Weyl group $\mathfrak{S}(\phi)$. When $q = b$, Proposition A.2 is a restatement of [9, Theorem 3.11.4(ii)] and (iv)]. (When comparing Proposition A.2 with [9] note that, in their formulas, $x$ and $y$ are maximal length left coset representatives, whereas $v_\lambda$ and $v_\mu$ are minimal length left coset representatives of $\mathfrak{S}(\phi)$ in $\mathfrak{S}_N$.)

**Proposition A.2.** Suppose that $\lambda, \mu \in \mathcal{P}^\Lambda_\beta$. Then

$$P^O_{\lambda\mu}(q) = q^\ell(v_\mu) - \ell(v_\lambda) \sum_{z \in \mathfrak{S}(\phi)} (-1)^{\ell(z)} P_{v_\lambda w_\phi^0 z \cdot w_0, v_\mu w_\phi^0 w_0} (q^{-2}).$$

**Proof.** As we have already noted, by Theorem 7.6 and [50, Theorem 5.1], the algebras $R^\psi_\phi$ and $R^\xi_\phi$ are standard Koszul algebras. By the remarks above, $\{ L(v_\mu \cdot \phi) \mid \mu \in \mathcal{P}^\Lambda_\beta \}$ is a complete set of simple modules in the category $O^\psi_\phi$. It follows that the Koszul dual category $O^\xi_\phi$ has (ungraded) simple modules $\{ L(w_\phi^0 v_\mu^{-1} w_0 \cdot \xi) \mid \mu \in \mathcal{P}^\Lambda_\beta \}$ and that the standard modules for $O^\phi_\psi$ correspond to the parabolic Verma modules $\{ M_q(w_\phi^0 v_\mu^{-1} w_0 \cdot \xi) \mid \mu \in \mathcal{P}^\Lambda_\beta \}$. Let $\{ L(w_\phi^0 v_\mu^{-1} w_0 \cdot \xi) \}$ and $\{ M_q(w_\phi^0 v_\mu^{-1} w_0 \cdot \xi) \}$ be the corresponding graded simple and standard
modules for $R^\phi_\xi$. The graded lifts of these modules exist because $R^\phi_\xi$ is a standard Koszul algebra by [50, Theorem 5.1].

By Lemma 2.15, $p^\phi_{\mu}(q) = [M_q(w^\phi_{\lambda}v_{\lambda}^{-1}w_0 \cdot \xi) : L(w^\phi_{\mu}v_{\mu}^{-1}w_0 \cdot \xi)]_q$. The proposition is equivalent to the identity:

$$[M_q(w^\phi_{\lambda}v_{\lambda}^{-1}w_0 \cdot \xi) : L(w^\phi_{\mu}v_{\mu}^{-1}w_0 \cdot \xi)]_q = q^{l(x)} - l(x) \sum_{z \in \mathcal{S}(\phi)} (-1)^{l(z)} P_{v_{\lambda}w^\phi_{\lambda}z w_0 v_{\mu}w^\phi_{\mu}w_0}(q^{-2}),$$

(A.3)

for all $\lambda, \mu \in \mathcal{B}_\beta^\Lambda$. Let $I_\psi$ be the Levi subalgebra of $\mathfrak{q}$ and set $\nu = x \cdot \xi$ for $x \in D_\phi^+$, so that $\nu$ is a dominant weight for $I_\psi$. By definition, $M_\psi(\nu) = U(\mathfrak{g}) \otimes U(\mathfrak{q}) L_\psi(\nu)$, where $L_\psi(\nu)$ is the (ungraded) irreducible highest weight $I_\psi$-module of highest weight $\nu$. The irreducible module $L_\psi(\nu)$ lives in the BGG category $O(I_\psi)$, for the Lie algebra $I_\psi$, so it has a BGG resolution of the form

$$0 \longrightarrow C^\psi_m \longrightarrow \cdots \longrightarrow C^\psi_0 \longrightarrow L_\psi(\nu) \longrightarrow 0,$$

where $C^\psi_k = \bigoplus_{z \in \mathcal{S}(\phi)} M_\psi(z \cdot \nu)$ and $M_\psi(z \cdot \nu)$ is an ungraded Verma module in $O(I_\psi)$. In particular, $C^\psi_0 = M_\psi(\nu)$ and $m = \ell(w^\phi_0)$. Applying the parabolic inflation–induction functor $U(\mathfrak{g}) \otimes U(\mathfrak{q})$ gives an exact sequence

$$0 \longrightarrow C_m \longrightarrow \cdots \longrightarrow C_0 = M(\nu) \longrightarrow M_\psi(\nu) \longrightarrow 0,$$

and $M(\nu)$ is an ungraded Verma module. By construction, all of the terms of this resolution live in the block $O^\psi$ of (ungraded) category $O$. Let $M(\nu)$ be the graded Verma module for $R^\phi_\xi$ constructed in [9, §3.11]. We claim that this resolution of the ungraded parabolic Verma module $M_\psi(\nu)$ lifts to a exact sequence of graded $R^\phi_\xi$-modules

$$0 \longrightarrow C_m \longrightarrow \cdots \longrightarrow C_0 = M(\nu) \longrightarrow M_\psi(\nu) \longrightarrow 0,$$

(A.4)

and all of the maps are homogeneous of degree 1 except for the map $C_0 = M(\nu) \rightarrow M_\psi(\nu)$, which is homogeneous of degree 0. Up to a scalar, there is a unique surjective map $M(\nu) \rightarrow M_\psi(\nu)$, namely the canonical quotient map which is homogeneous of degree zero. On the other hand, there is a non-zero homogeneous map $M(\sigma) \rightarrow M(\tau)$ of degree $k$ only if $[M(\tau) : L(\sigma)(k)] \neq 0$. By [9, Theorem 3.11.4(ii)(iv)] (and Proposition 2.13), if $x, y \in (D_\xi^+)^{-1}$, then

$$[M(x \cdot \xi) : L(y \cdot \xi)]_q = q^{l(y) - l(x)} P_{x,y}(q^{-2}).$$

(A.5)

Note that $v_{\lambda} \in D_\phi^{-1} \cap D_\psi$, because $v_{\lambda} \cdot \phi \in X^+_\psi$, and $D_\xi^+ = w_0 D_\psi^+ w_0$. If $z \in \mathcal{S}(\phi)$, then $v_{\lambda}w_0^\phi z^{-1} \in D_\psi$ because $v_{\lambda}w_0^\phi z^{-1} \cdot \phi = v_{\lambda} \cdot \phi \in X^+_\psi$. Hence, $v_{\lambda}w_0^\phi z^{-1}w_0 \in D_\psi^+$. It follows that

$$z w_0^\phi v_{\lambda}^{-1}w_0 \in (D_\xi^+)^{-1} \quad \text{and} \quad w_0^\phi v_{\mu}^{-1}w_0 \in (D_\xi^+)^{-1}.$$  

(A.6)

Recall from [36] that $P_{x,y} \neq 0$ only if $x \leq y$ (where $\leq$ is the Bruhat order) and $\deg_{x,y} P_{x,y}(q) \leq \frac{1}{2}(l(y) - l(x) - 1)$. Therefore, if $l(y) = l(x) + 1$, then $[M(x \cdot \xi) : L(y \cdot \xi)]_q \neq 0$ if and only if $y = sx$, for some simple reflection $s \in \mathcal{S}_N$, in which case $[M(x \cdot \xi) : L(y \cdot \xi)]_q = g$. Therefore, up to a scalar, if $y = sx$, then there is at most one map $M(y \cdot \xi) \rightarrow M(x \cdot \xi)$, which must be homogeneous of degree one, and if $y \neq sx$, then $\Hom_{\mathcal{O}^\psi}(M(y \cdot \xi), M(x \cdot \xi)) = 0$. Arguing by induction on $k$ to determine the degree shifts on the Verma modules, it follows that the ungraded resolution of $M_\psi(\nu)$ lifts to a graded ‘BGG resolution’ of $M_\psi(\nu)$ as in (A.4).

We now complete the proof of the theorem. Fix $\lambda, \mu \in \mathcal{B}_\beta^\Lambda$ and set $x = w_0^\phi v_{\lambda}^{-1}w_0$ and $y = w_0^\phi v_{\mu}^{-1}w_0$ in the formulas above. Using (A.4) for the first equality, and (A.5) and (A.6) for the second.
the second,

\[ M_q(w_0^\phi v_\lambda^{-1} w_0 \cdot \xi) : L(w_0^\phi v_\mu^{-1} w_0 \cdot \xi) \]

\[ = \sum_{z \in \mathcal{O}(\phi)} (-q)^{\ell(z)} M(z w_0^\phi v_\lambda^{-1} w_0 \cdot \xi) : L(w_0^\phi v_\mu^{-1} w_0 \cdot \xi) \]

\[ = \sum_{z \in \mathcal{O}(\phi)} (-q)^{\ell(z)} q^{\ell(v_\mu^{-1}) - \ell(v_\lambda^{-1})} P_{w_0^\phi v_\lambda^{-1} w_0, w_0^\phi v_\mu^{-1} w_0}(q^{-2}) \]

\[ = q^{\ell(v_\mu) - \ell(v_\lambda)} \sum_{z \in \mathcal{O}(\phi)} (-1)^{\ell(z)} P_{w_0 v_\lambda z, w_0 v_\mu w_0^\phi}(q^{-2}) \]

\[ = q^{\ell(v_\mu) - \ell(v_\lambda)} \sum_{z \in \mathcal{O}(\phi)} (-1)^{\ell(z)} P_{w_\lambda w_0^\phi z, w_\mu w_0^\phi w_0}(q^{-2}), \]

where the last two equalities use Lemma A.1 and the well-known facts that \( P_{u,v} = P_{u^{-1}, v^{-1}} \) and \( P_{u,v} = P_{w_0 u w_0, w_0 v w_0} \), for all \( u, v \in \mathcal{O}_N \). This completes the proof of our claim (A.3) and hence the proof of the proposition.

Combining Lemma A.1 and Proposition A.2, we obtain Proposition 7.13, which is what this appendix set out to prove. As a corollary of Proposition A.2, using Lemma 2.15 together with [26, Theorem 4.6] or [46, Proposition 3.17], we can identify the graded decomposition numbers of \( S_\beta^\lambda \) with certain inverse parabolic Kazhdan–Lusztig polynomials. By Theorem 7.27, these are also the graded decomposition numbers of \( S_\beta^\lambda \).

Appendix B. Quiver Schur algebras of level two

In this appendix, we assume that either \( e = 0 \) or \( e > n \) and we fix a dominant weight \( \Lambda \) of level \( \ell = 2 \). We will show that \( S_\beta^\Lambda \) is a positively graded basic algebra and, as a consequence, we will give a beautiful closed formula for the graded decomposition numbers for these algebras. Our formula for these graded decomposition numbers is new, however, by Theorem C when \( e = 0 \) these graded decomposition numbers can be computed in parabolic category \( \mathcal{O} \) where different formulations of this result are already known, all going back to the work of Lascoux and Schützenberger [41]. When \( e = 0 \) all of the results in this section have been obtained by Brundan and Stroppel [18, 20] using different methods. The extension of these results to the case \( e > n \) is new.

Suppose that \( t = (t^{(1)}, t^{(2)}) \) is a standard tableau. Let \( t^{(c)} = \{ 1 \leq k \leq n \mid \text{comp}_t(k) = c \} \) be the integers in component \( c \) of \( t \), for \( c = 1, 2 \). By assumption, \( e = 0 \) or \( e > n \) so the nodes of constant residue in \( t^{(c)} \) all appear on the same diagonal \( \{(a + d, b + d, c) \in \mu \mid d \in \mathbb{Z}\} \) in \( t^{(c)} \). Therefore, the tableau \( t \) is uniquely determined by its residue sequence \( i = \text{res}(t) \in \mathcal{P}_n \) and the sets \( t^{(1)} \) and \( t^{(2)} \).

Although we will not need this, the last paragraph implies that if \( i_{r+2} = i_r = i_{r+1} \pm 1 \), where \( 1 \leq r \leq n - 2 \) and \( i = \text{res}(t) \) for some tableau \( t \in \text{Std}(\mathcal{P}_n^\Lambda) \), then the permutation \( d(t) \) fixes \( r, r + 1 \) and \( r + 2 \). Consequently, \( s_r s_{r+1} s_r \) cannot appear in any reduced expression for \( d(t) \) so that \( \psi_{d(t)} \) depends only on \( t \) and not on a choice of reduced expression for \( d(t) \). Therefore, in level two the basis elements \( \psi_{st} \) depend only on \( s \) and \( t \), and not on the choices of reduced expressions. We warn the reader that this does not imply that the \( \psi_r \) satisfy the braid relations in \( R_\Lambda \).

Following [48], define a tableau \( t \) to be regular if its entries increase along the diagonals in each component. It is easy to see that all standard tableaux are regular and that there exist regular tableaux that are not standard. By the last paragraph, given a sequence \( i \in \mathcal{P}_n \) and disjoint sets \( A_1 \) and \( A_2 \) such that \( A_1 \cup A_2 = \{1, 2, \ldots, n\} \) there exists a unique regular tableau \( t \) such that \( \text{res}(t) = i \) and \( t^{(c)} = A_c \), for \( c = 1, 2 \). Note that \( t \) is not necessarily standard and, in general, that the shape of \( t \) need not be a bipartition.
Two nodes \((e, c, l)\) and \((r', c', l')\) are adjacent if \(l = l'\) and either \(r = r'\) and \(c = c' \pm 1\), or \(c = c'\) and \(r = r' \pm 1\). A set \(X\) of nodes is connected if for any \(x, y \in X\) there is a sequence \(x = x_1, \ldots, x_z = y\) of nodes in \(X\) such that \(x_i\) and \(x_{i+1}\) are adjacent, for \(1 \leq i < z\).

**Lemma B.1.** Suppose that \(t \in \text{Std}^\mu(\lambda)\). Then \(\deg t \geq \deg t^\mu\) with equality if and only if \(t = t^\mu\).

**Proof.** By definition, \(t \supsetneq t^\mu\) and \(\text{res}(t) = \text{res}(t^\mu)\). We argue by induction on dominance. If \(t = t^\mu\), then there is nothing to prove, so suppose that \(t \supsetneq t^\mu\). Let \(a\) be the smallest number in \(t^{(1)} \cap t^{(2)}\). As remarked above, the tableau \(t\) is uniquely determined by its residue sequence \(\text{res}(t)\) and the sets \(t^{(c)}\), for \(c = 1, 2\). Therefore, if \(1 \leq b \leq a\), then \(b\) appears in exactly the same position in \(t\) and in \(t^\mu\). In particular, \(a\) is larger than all of the numbers in \(t^{(1)}\). Moreover, \(a\) is uniquely determined by \(\lambda\) and \(\mu\).

Let \(X\) be the set of nodes in \(\lambda^{(1)} \setminus \mu^{(1)}\) that are connected to \(t^{-1}(a)\) such that they are either adjacent to a node in \(\mu^{(1)}\) or they are in the first row or in the first column of \(\lambda^{(1)}\). Since \(a\) is uniquely determined by \(\lambda\) and \(\mu\), it follows that \(X\) also depends only on \(\lambda\) and \(\mu\).

Let \(A = \{t(x) \mid x \in X\} \subseteq t^{(1)}\). Define \(t_A = (t_A^{(1)}, t_A^{(2)})\) to be the unique regular tableau with residue sequence \(\text{res}(t) = \text{res}(t^\mu)\) such that \(t_A^{(1)} = t^{(1)} \setminus A\) and \(t_A^{(2)} = t^{(2)} \cup A\). That is, \(t_A\) is the regular tableau obtained by moving the numbers in \(A\) from the first component of \(t\) to the second component.

For example, suppose that \(e = 0\) and \(\kappa = (0, 1)\), so that \(A = \Lambda_0 + \Lambda_1\). Let \(\mu = (2, 1^2|4^2, 2)\) and \(\lambda = (5^2, 2|1^2)\). Then \(t \in \text{Std}^\mu(\lambda)\), \(a = 6\) and

\[
\begin{bmatrix}
1 & 2 & 6 & 7 & 8 \\
3 & 9 & 10 & 11 & 12 \\
4 & 5 & 14 \\
\end{bmatrix}
\sim t_A =
\begin{bmatrix}
1 & 2 & 11 & 12 \\
3 & 4 \\
9 & 10 & 13 \\
\end{bmatrix}
\]

The shaded nodes in \(t\) mark the elements of \(A = \{6, 7, 8, 9, 10, 13\}\).

By the remarks in the first paragraph, the elements of \(A\) occupy the same positions in \(t_A\) as they do in \(t^\mu\). By definition, the elements of \(A\) have distinct residues. Moreover, we can order \(A = \{a_0, \ldots, a_z\}\) so that \(\text{res}_1(a_i) = \text{res}_1(a_0) + i\), for \(0 \leq i \leq z\). The definition of \(X\) implies that if \(b \in A\), then \(b\) is the smallest element of \(t^{(2)} \cap t^{(1)}\) with residue \(\text{res}_1(b)\). Therefore, two elements of \(A\) are in the same row of \(t^{(1)}\) if and only if they are in the same row of \(t_A^{(1)}\). Hence, the elements of \(A\) that are in the same row of \(t\) or of \(t^\mu\), are consecutive. It follows that the set \(A\) is also determined by \(a\) (and \(\text{res}(t^\mu)\)), and hence that \(A\) is uniquely determined by \(\lambda\) and \(\mu\).

By definition, \(t_A\) is obtained by moving the numbers in \(A\) from the first component to the second component, without changing their ‘shape’, then ‘sliding’ numbers down the diagonals in the first component to fill the gaps where the elements of \(A\) used to be, and then sliding numbers up the diagonal in the second component to make way for the elements of \(A\). Since \(t\) and \(t^\mu\) are both standard it follows that \(t_A\) is also standard and that \(t \supsetneq t_A \supsetneq t^\mu\). In particular, \(t_A \in \text{Std}^\mu(\mathcal{P}_n^A)\).

As remarked earlier, the elements in \(A\) occurring in a given row are consecutive. By definition, \(a\) is the smallest element of \(A\) and \(t_{(a-1)} = t_{A \setminus (a-1)}\). It is easy to see that \(\deg t_{1a} = \deg t_{A \setminus a} + 1\).

Adding the elements of \(A\) row by row to \(t_{1(a-1)}\) and to \(t_{A \setminus (a-1)}\) it is easy to see that the only difference in the degrees of the tableaux \(t\) and \(t_A\) occurs when adding \(a\), which appears in the ‘first row’ of \(A\), and that the subsequent rows in \(A\) do not change the degrees of \(t\) or of \(t_A\). Hence, \(\deg t_{1z} = \deg t_{A \setminus z} + 1\), where \(z\) is the largest element of \(A\). In view of the sliding construction of \(t_A\), this implies that \(\deg t = \deg t_A + 1\). Therefore, by induction, \(\deg t > \deg t_A \geq \deg t^\mu\) as required.

☐
The set $A$ in the proof of Lemma B.1 is uniquely determined by the bipartitions by the bipartitions $\lambda$ and $\mu$. Moreover, $t$ is uniquely determined by the bipartitions $\lambda$ and $\mu$. Hence as byproduct of the proof, we have the following.

**Corollary B.2.** Suppose that $\lambda, \mu \in \mathcal{P}_n^\Lambda$. Then $\# \operatorname{Std}^\mu(\lambda) \leq 1$.

If $\operatorname{Std}^\mu(\lambda) \neq \emptyset$ let $t^\mu_\lambda$ be the unique $\lambda$-tableau in $\operatorname{Std}^\mu(\lambda)$.

**Theorem B.3.** Suppose that $Z$ is a field, $e = 0$ or $e > n$ and that $\Lambda \in P^+$ is a weight of level 2. Then $S_n^\Lambda$ is a positively graded basic algebra. Moreover,

$$[\Delta^\lambda : L^\mu]_q = \begin{cases} q^{\deg t^\lambda_\mu - \deg t^\mu} & \text{if } \operatorname{Std}^\mu(\lambda) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for $\lambda, \mu \in \mathcal{P}_n^\Lambda$.

**Proof.** By Theorem 4.19, $S_n^\Lambda$ is a quasi-hereditary cellular algebra with cellular basis

$$\{ \Psi_{st}^{\mu \nu} \mid (\mu, s), (\nu, t) \in T^\Lambda \text{ and } \lambda \in \mathcal{P}_n^\Lambda \}. $$

Moreover, if $\Psi_{st}^{\mu \nu}$ is one of these basis elements, then $\deg \Psi_{st}^{\mu \nu} = (\deg s - \deg t^\mu) + (\deg t - \deg t^\nu) \geq 0$ by Lemma B.1. Therefore, the quiver Schur algebra $S_n^\Lambda$ is positively graded.

Now suppose that $\lambda \in \mathcal{P}_n^\Lambda$. Then $\{ \Psi_{st}^{\mu \nu} \mid (\mu, s), (\nu, t) \in T^\Lambda \}$ is a basis of $\Delta^\lambda$ by (4.22). Moreover, by Lemma B.1, $\deg \Psi_{st}^{\mu \nu} \geq 0$ with equality if and only if $(t, \nu) = (t^\lambda, \lambda)$. It follows that the simple module $L^\lambda$ is one-dimensional with basis vector $\Psi_{t^\lambda, \lambda}^{\mu \nu}$ since $\dim_q L^\lambda = \dim_q (L^\lambda)^\otimes = \dim_q L^\lambda$ by Theorem 2.5. Thus, $\dim_q L^\lambda = 1$, for all $\lambda \in \mathcal{P}_n^\Lambda$, and $S_n^\Lambda$ is a basic algebra.

Finally, since $S_n^\Lambda$ is positively graded, $Z^\mu = P^\mu$ for all $\mu \in \mathcal{P}_n^\Lambda$, for example by applying our LLT algorithm from §7.5. Therefore, by (7.36),

$$[\Delta^\lambda : L^\mu]_q = \sum_{t \in \operatorname{Std}^\mu(\lambda)} q^{\deg t^\lambda_\mu - \deg t^\mu} = \begin{cases} q^{\deg t^\lambda_\mu - \deg t^\mu} & \text{if } \operatorname{Std}^\mu(\lambda) \neq \emptyset, \\ 0 & \text{if } \operatorname{Std}^\mu(\lambda) = \emptyset, \end{cases}$$

where the last equality comes from Corollary B.2. $\square$

**Corollary B.4.** Suppose that $Z$ is a field, $e = 0$ or $e > n$ and that $\Lambda \in P^+$ is a weight of level 2. Then $G^\mu = Y^\mu$ is an indecomposable graded Young module, for all $\mu \in \mathcal{P}_n^\Lambda$.

**Corollary B.5.** Suppose that $Z$ is a field, $e = 0$ or $e > n$ and that $\Lambda \in P^+$ is a weight of level 2. Then the graded decomposition numbers of $S_n^\Lambda$ and $H_n^\Lambda$, and the graded dimensions of their graded simple modules, are independent of the characteristic of $Z$.

By Theorem C, [20, Corollary 8.20] and the uniqueness of Koszul gradings [9, Corollary 2.5.2], we obtain the link with Brundan and Stroppel’s work.

**Corollary B.6.** Suppose that $e = 0$. Then $S_n^\Lambda$ is isomorphic, as a graded algebra, to the quasi-hereditary algebra $K_n^\Lambda$ defined by Brundan and Stroppel [18].

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