State-injection schemes of quantum computation in Spekkens’ toy theory

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Spekkens’ toy theory is a non-contextual hidden variable model with an epistemic restriction, a constraint on what the observer can know about the reality. It has been shown in [3] that for qudits of odd dimensions it is operationally equivalent to stabiliser quantum mechanics by making use of Gross’ theory of discrete Wigner functions. This result does not hold in the case of qubits, because of the unavoidable negativity of any Wigner function representation of qubit stabiliser quantum mechanics. In this work we define and characterise the subtheories of Spekkens’ theory that are operationally equivalent to subtheories of stabiliser quantum mechanics. We use these Spekkens’ subtheories as a unifying framework for the known examples of state-injection schemes where contextuality is an injected resource to reach universal quantum computation. In addition, we prove that, in the case of qubits, stabiliser quantum mechanics can be reduced to a Spekkens’ subtheory in the sense that all its objects that do not belong to the Spekkens’ subtheory, namely non-covariant Clifford gates, can be injected. This shows that within Spekkens’ subtheories we possess the toolbox to perform state-injection of every object outside of them and it suggests that there is no need to use bigger subtheories to reach universal quantum computation via state-injection. We conclude with a novel scheme of computation suggested by our approach which is based on the injection of CCZ states and we also relate different proofs of contextuality to different state injections of non-covariant gates.

Spekkens’ toy theory (ST) is a non-contextual hidden variable model made to advocate the epistemic view of quantum mechanics, where quantum states are seen as states of incomplete knowledge about a deeper underlying reality [1, 2]. The idea of the model is to reproduce quantum theory through a phase-space inspired theory with the addition of a constraint on what an observer can know about the ontic state (identified with a phase space point) describing the reality of a system. In the case of odd dimensional systems ST has been proven to be operationally equivalent to qudit stabiliser quantum mechanics (SQM) [3], where SQM is a subtheory of quantum mechanics that only allows eigenstates of tensors of Pauli operators, Clifford unitaries and Pauli measurement observables [4]. “Operationally equivalent” means that the two theories provide the same statistics of outcomes, given certain states, transformations and measurements. This equivalence between ST and qudit SQM is proven by representing qudit SQM through Gross’ Wigner functions [17]. These turn out to be exactly equivalent to Spekkens’ epistemic states and measurements. The measurement update rules are consistent and positivity-preserving. Clifford gates are mapped into consistent symplectic affine transformations [3].

In the case of qubits the above equivalence does not hold. ST is non-contextual by construction, while qubit SQM shows state-independent contextuality, as witnessed by the Peres-Mermin square argument [9–11]. This is reflected into the impossibility of finding a non-negative Wigner function that maps non-negatively qubit SQM into ST [18, 19]. Note that the non-negativity is needed in order to interpret the epistemic states and measurements as well-defined probabilities and thus state the operational equivalence. Even if ST and qubit SQM are not operationally equivalent, some restricted versions of qubit SQM and ST do show the same statistics. Our aim is to identify the subtheories of ST that are operationally equivalent to subtheories of qubit SQM. We define subtheories of Spekkens’ toy theory compatible with subtheories of stabiliser quantum mechanics (SS) as closed subtheories of quantum mechanics whose states and measurements are non-negatively represented by covariant Wigner functions.

In this work we use Spekkens’ subtheories to provide an application in the field of quantum computa-
More precisely, we relate SS with state injection schemes of quantum computation (QCSI schemes). The latter constitute nowadays the leading model to implement fault tolerant universal quantum computation (UQC). These schemes are composed by a “cheap” part which consists of quantum circuits that are efficiently simulatable by a classical computer (usually stabiliser circuits), and by magic resources (that are usually distilled from many copies of noisy states through magic state distillation [7]) that boost the computation to universal [5]. In 2014 Howard et al. [15] proved that in a QCSI scheme of qudits (odd prime dimensions), with the cheap part composed by stabiliser circuits, the contextuality possessed solely by the magic resource is a necessary resource for UQC. In terms of systems of dimensions 2 a similar result due to Delfosse et al. [16] holds for rebits, where the classical non-contextual cheap part is composed by CSS circuits. However, as already pointed out, an analogue version of Howard’s result for qubit cannot be found, since qubit SQM is already contextual. Nevertheless it has been proven [12] that in any QCSI scheme of qubits where we get rid of the state-independent contextuality (e.g. Peres-Mermin square), the contextuality possessed solely by the magic resource is necessary for UQC. A more complete version of this result is also treated in [13], where a general framework for qubit QCSI schemes with contextuality as a resource is provided.

More precisely, in [13], Raussendorf et al. develop a framework for building non-negative and non-contextual subtheories of qubit SQM from the choice of the phase function $\gamma$ defining the Weyl operators and consequently the Wigner functions (see equation (1)). They require tomographically completeness, i.e. any state can be fully measured by the observables allowed in the cheap part of the scheme, and they allow also gates that introduce negativity in the Wigner functions, i.e. non-covariant gates, the reason being that the gates can always be absorbed in the measurements without altering the outcome distribution of the computation. In addition to this, in [14], Wallman and Bartlett address the issue of finding the subtheories of qubit QM that are non-negative in some quasiprobability representation (and so are classically simulatable and correspond to non-contextual ontological models). They construct the so-called 8–state model, which can be seen as a generalisation of ST with an enlarged ontic space. The non-negativity for states and measurements is guaranteed by considering all the possible Wigner representations.

It is important to point out that in the mentioned frameworks of [12] and [13], the definition of QCSI is quite broad and it also includes schemes based on measurement-based quantum computation with cluster states [6]. We here consider only standard QCSI schemes instead (see equation (3) for the precise definition), like the ones in [15] and [16], and we show that Spekkens’ subtheories are an intuitive and effective tool to treat this cases. We first use SS to represent the non-contextual cheap part of the known examples of QCSI schemes, both for qubits and qudits, where contextuality arises as a resource [15, 16]. These can be unified in the following framework (figure 1): $SS + Magic\ state(s) \rightarrow UQC$. Secondly, we prove in theorem 1 that qubit SQM can be reduced to a SS since all its objects that do not belong to the Spekkens’ subtheory, namely non-covariant Clifford gates, can be injected, where the circuit needed for the injection is always made of objects belonging to a SS. This means that SS contain all the tools for performing state-injection computation. There is no need to consider bigger non-covariant subtheories in the cheap part.

The proof of theorem 1 suggests a novel state-injection scheme where contextuality is a resource based on injection of $CCZ$ states. State injection schemes with the related Toffoli (CCNOT) gates are already known [20–26], but our scheme differs from them as our non-contextual cheap part of the computation (a strict subset of the CSS rebit subtheory considered in [16]) is such that it is not possible to remove any object from it without denying the possibility of obtaining UQC via QCSI. The price to pay for this minimality is the injection of the control-Z state, $CZ\{++\}$, too (which also provides the Hadamard gate). By analysing this example we can associate different proofs of contextuality to different state injections of non-covariant gates. More precisely we show how the Clifford non-covariant $CZ$ gate (as well as the phase gate $S$) can provide proofs of the Peres-Mermin square contextuality and the GHZ paradox, while the injection of the $T\{\}+\}$ magic state, where $T$ is the popular $\frac{\pi}{2}$ non-Clifford gate, allows, in addition to the previous proofs of contextuality (as $T^2 = S$), also to obtain the maximum quantum violation of the CHSH inequality.

In the reminder of the paper we start by providing the definition of a Spekkens’ subtheory in section 1. In section 2 we describe Howard’s and Delfosse’s cases for qudits [15] and rebits [16] respectively and we prove that they fit in our framework where the cheap parts of the computation, qudit SQM and CSS rebits respectively, are SS. We then set the instructions to construct a SS from the choice of a non-negative Wigner function in section 3. We do so in line with [13] and we find that the main difference from Raussendorf et al’s
formulation (and also from the 8-state model) consists of demanding for the covariance of the Wigner function with respect to the allowed gates. Moreover we do not demand for tomographical completeness. By exploiting this comparison we then prove theorem 1. It basically shows that any standard QCSI scheme can be reduced to our framework, since any object not present in SS can be injected by using an injection scheme made of objects in SS. In section 4 we provide a novel example of state injection for qubits (rebits) based on CCZ magic states. We analyse the presence of different proofs of contextuality in correspondence of different state-injected gates in section 5 and we recap all the results and the future directions in the conclusion section.

1 Characterisation of Spekkens’ subtheories

We focus here on subtheories of quantum mechanics that we call Spekkens’ subtheories. Before providing the definition of SS we recall some basic notions characterising Spekkens’ theory and Wigner functions.

We denote the phase space as \( \Omega = \mathbb{Z}_N^d \). Spekkens’ epistemic states are probability distributions over the phase space, \( P_{V,\mathbf{w}}(\lambda) = \frac{1}{N} \delta_{V+\mathbf{w}}(\lambda) \), where \( V \) is the isotropic subspace of known variables, \( \mathbf{w} \) is the evaluation shift vector and \( \lambda \in \Omega \). The function \( \delta_{V+\mathbf{w}}(\lambda) \) takes value 1 when \( \lambda \) belongs to the set \( V+\mathbf{w} \) and 0 otherwise. The transformations \( G \) are symplectic affine transformations in the phase space (in general a subset of the permutations), i.e. \( G(\lambda) = S\lambda + \mathbf{a} \), where \( S \) is a symplectic matrix and \( \mathbf{a} \) a shift vector. The elements of a sharp measurement \( \Pi \) have an epistemic representation analogue to the states according to the dual representation of states and observables, where we denote with \( V_\Pi \) and \( \mathbf{r}_k \) the subspace of known variables and the evaluation shift vector associated with the \( k \) element respectively.

The Wigner function of a quantum state \( \rho \) is defined by the function \( \gamma \),

\[
W_\rho^\gamma(\lambda) = Tr(A^\gamma(\lambda)\rho),
\]

where the phase-point operator is

\[
A^\gamma(\lambda) = \frac{1}{N_\Omega} \sum_{\lambda' \in \Omega} \chi(\lambda,\lambda') T^\gamma(\lambda'),
\]

and the Weyl operator is defined by

\[
T^\gamma(\lambda) = w^{\gamma(\lambda)} Z(\lambda_Z) X(\lambda_X),
\]

where the phase-space point is \( \lambda = (\lambda_Z,\lambda_X) \in \Omega \). We will omit the superscript \( \gamma \) in the future in order to soften the notation. The normalisation \( N_\Omega \) is such that \( Tr(A(\lambda)) = 1 \) and the square brackets indicate the symplectic inner product. The operators \( Z(\lambda_Z), X(\lambda_X) \) represent the (generalised) Pauli operators. The functions \( \chi \) and \( w \) will be appropriately characterised in the case of qubits and qudits in the next section, as well as \( \gamma \).

A Spekkens’ subtheory is defined as a set \( (\mathcal{S},\mathcal{T},\mathcal{M}) \) of quantum states, transformations and measurements which satisfies the following conditions.

1. Subtheory. The set must be closed, which means that any allowed gate cannot bring from one allowed state to a non-allowed one.

\[
\forall U \in \mathcal{T}, U\rho U^\dagger \in \mathcal{S} \forall \rho \in \mathcal{S}. \tag{4}
\]

2. Spekkens representability. There must be an operational equivalence between the subtheory of QM \( (\mathcal{S},\mathcal{T},\mathcal{M}) \) and a subtheory of Spekkens’ toy theory \( (\mathcal{S}_s,\mathcal{T}_s,\mathcal{M}_s) \).

The operational equivalence means that the statistics of the two subtheories \( (\mathcal{S},\mathcal{T},\mathcal{M}) \) and \( (\mathcal{S}_s,\mathcal{T}_s,\mathcal{M}_s) \) are the same. We state this equivalence by finding a non-negative Wigner function that maps the states \( \rho \) and the measurement elements \( \Pi_k \) in \( (\mathcal{S},\mathcal{M}) \) to epistemic states and measurement elements in \( (\mathcal{S}_s,\mathcal{M}_s) \), i.e.

\[
W_\rho(\lambda) = \frac{1}{N} Tr(\rho A(\lambda)) = \frac{1}{N} \delta_{V+\mathbf{r}_k}(\lambda); \tag{5}
\]

\[
W_{\Pi_k}(k/\lambda) = \frac{1}{N'} Tr(\Pi_k A(\lambda)) = \frac{1}{N'} \delta_{V_m+\mathbf{r}_k}(\lambda). \tag{6}
\]

The \( N, N' \) are the normalisation factors so that \( \sum_{\lambda \in \Omega} W(\lambda) = 1 \) for all the above Wigner functions and \( k \) denotes the outcome associated with

\[
\text{Figure 1: Computational scheme. Schematic representation of the computational scheme of the paper.}
\]
the measurement element \( \Pi_k \). Notice at this point that the measurement update rules in [3] guarantee that also a state after a measurement is non-negatively represented if the measurement and the original state have non-negative Wigner functions, as they involve only sums and products of Wigner functions.

The operational equivalence in terms of transformations is implied if the Wigner function (5) satisfies the property of covariance for the allowed unitaries \( U \in T \), which means that (in accordance with the definition used by David Gross in [17, theorem 7]), for all the allowed states \( \rho \in S \),

\[
W_{U,\rho U^\dagger}(\lambda) = W_\rho(S\lambda + a), \quad (7)
\]

i.e. the transformation in quantum mechanics corresponds to a symplectic affine transformation in Spekkens’ theory. Notice that the property of covariance is defined in terms of Wigner functions and not directly in terms of the phase point operators. Therefore we do not necessarily need to demand for the standard Wigner function of the transformation, \( W_U(\lambda/\lambda') = \frac{1}{N} Tr(A(\lambda)U(\lambda') U^\dagger) \), to be non-negative in all the elements, once the previous requirements, non-negativity and covariance of \( W_\rho \), are satisfied. The transition matrix corresponding to the allowed permutation of the phase points can be always found, as shown by the following lemma.

**Lemma 1.** Given a non-negative Wigner function representation, \( W_\rho, W_{\rho'} \), of any two allowed states \( \rho, \rho' \in S \) such that \( \rho' = U\rho U^\dagger \), \( U \in T \), and covariance holds, i.e. \( W_{\rho'}(\lambda) = W_\rho(S\lambda + a) \), there always exists a (non-negative) transition matrix \( P_U : \Omega \times \Omega \rightarrow [0, 1] \) representing the transformation \( U \in T \),

\[
P_U(\lambda/\lambda') = \frac{1}{N''} \delta_{\lambda, S\lambda'+a}, \quad (8)
\]

where \( N'' \) is the normalisation factor, such that

\[
W_{\rho'}(\lambda) = \sum_{\lambda' \in \Omega} P_U(\lambda/\lambda') W_\rho(\lambda'). \quad (9)
\]

**Proof.** A matrix made of non-negative elements \( P_U(\lambda/\lambda') \) proportional to Kronecker deltas always exists because it corresponds to the transition matrix representing the permutation that brings \( W_\rho \) to \( W_{\rho'} \). More precisely, non-negative solutions \( P_U(\lambda/\lambda') \) to the equations (9) for every \( \lambda \), given the non-negative \( W_{\rho'}(\lambda') \), \( W_\rho(\lambda) \) defined in (1), always exist. For every fixed \( \lambda \), the \( P_U(\lambda/\lambda') \) are vectors with all zero components apart from one, i.e. they are proportional to Kronecker deltas.

The covariance property (7) guarantees that this permutation corresponds to a symplectic affine transformation on the phase space points (independent on the state \( \rho \) that \( U \) is acting on).

The non-negative functions (5), (6) and (8) can be interpreted as probability distributions and guarantee that the theories \( (S, T, M) \) and \( (S_s, T_s, M_s) \) are operationally equivalent, i.e. they provide the same statistics:

\[
p(k) = \frac{Tr(\Pi_k \rho U U^\dagger)}{\sum_{\lambda \in \Omega} W_\Omega(k/\lambda)} \sum_{\lambda' \in \Omega} P_U(\lambda/\lambda') W_\rho(\lambda'). \quad (10)
\]

To sum up, a SS is a (closed) subtheory of quantum mechanics whose states (and measurements) are represented by non-negative and covariant Wigner functions. We say that a SS is *maximal* if the set \( (S, T, M) \) is such that by adding either another state, gate or observable to the set of allowed states, transformations and observables contradicts at least one of the conditions above, i.e. it is no longer a subtheory or Spekkens representable. We will also talk about *QCSI-minimal* non-contextual subtheories of SQM meaning those subtheories that can no longer be used for QCSI schemes after the removal of just one object from them.

## 2 Wigner functions for SS

We now identify the functions \( \gamma \) defining the Wigner functions in equation (1) that allow us to show that the known examples of QCSI with contextuality as a resource, [15] and [16], fit into the framework depicted in figure 1, i.e. that the cheap parts of those schemes are SS.

### 2.1 Qudit case

In the case of qudits of odd dimensions Gross’ theorem [17] guarantees that there is a non-negative Wigner representation of all stabiliser states. This Wigner function is covariant and also Clifford transformations and Pauli measurements are non-negatively represented. Thus Gross’ Wigner function proves the op-
eral equivalence, in odd dimensions, between stabiliser quantum mechanics and the whole Spekkens’ toy theory, as shown in [2] and [3].

Gross’ Wigner function for odd dimensional systems (qudits) is defined according to equation (1), where \( \chi(a) = e^{i2\pi a} \), for \( a \in \mathbb{Z}_d \), and \( w^{\gamma}(\lambda) = \chi(2^{-1}\gamma(\lambda)) \).

The function \( \gamma \) is given by \( \gamma(\lambda) = \lambda X \cdot \lambda Z \), where the “\( \cdot \)” denotes the inner product, and the generalised Pauli operators are defined as

\[
X(\lambda X) = \sum_{\lambda X \in \mathbb{Z}_d} |\lambda X^\prime - \lambda X^\prime X^\prime \rangle \langle \lambda X^\prime |, \tag{11}
\]

\[
Z(\lambda Z) = \sum_{\lambda X \in \mathbb{Z}_d} \chi(\lambda X \cdot \lambda Z) |\lambda X^\prime \rangle \langle \lambda X| \tag{12}.
\]

In the scheme of Howard et al. [15] where they prove that contextuality is a resource for UQC, the cheap part of the computation is given by SQM in odd prime dimensions, which, by Gross’ Wigner functions, is a maximal Spekkens’ subtheory.

### 2.2 Qubit case

In this case of qubits an analogue of Gross’ Wigner function for SQM does not exist [18,19], as some negative stabiliser states are present for any possible choice of Wigner functions. This is related to the cont extuality shown by qubit SQM (see, for example, the Peres-Mermin square [11]). Nevertheless it is possible to state a similar result to Howard et al.’s by restricting the cheap part of the computation to a strict non-contextual and positive subtheory of qubit SQM, as shown by the result of Delfosse et al. in 2015 [16].

We start by describing the set of allowed states/gates/observables (\( \mathcal{S}_r, \mathcal{T}_r, \mathcal{M}_r \)) considered by Delfosse et al. The set \( \mathcal{S}_r \), a subset of the stabiliser states, is composed by CSS states, i.e. stabiliser states \( \{|\psi\rangle\} \), whose corresponding stabiliser group \( S(|\psi\rangle) \) decomposes into an \( X \) and a \( Z \) part; i.e. \( S(|\psi\rangle) = S_X(|\psi\rangle) \times S_Z(|\psi\rangle) \), where all elements of \( S_X(|\psi\rangle) \) and \( S_Z(|\psi\rangle) \) are of the form \( X(q) \) and \( Z(p) \), respectively, where \( q,p \in \mathbb{Z}_2^n \). CSS states are the eigenstates of the allowed observables belonging to the set \( \mathcal{M}_r \).

\[
\mathcal{M}_r = \{X(q), Z(p)|q,p \in \mathbb{Z}_2^n \}. \tag{13}
\]

The set of allowed transformations is composed by the CSS preserving gates, subset of the Clifford group \( C_n \) (which is the group of unitaries that maps Pauli operators to Pauli operators by conjugation),

\[
\mathcal{T}_r = \{ g \in C_n | |\psi\rangle \in \mathcal{S}_r, \forall |\psi\rangle \in \mathcal{S}_r \}
= \left( \bigotimes_{i=1}^{n} H_i, CNOT(i,j), X_i, Z_i \right), \tag{14}
\]

where \( i,j \in \{1,2,\ldots,n\} \) and \( i \neq j \). The UQC is reached by injecting two particular magic states to the cheap subtheory of CSS rebits just described [16].

The Wigner function used by Delfosse et al. to prove their result is given by

\[
A_r(\lambda) = \frac{1}{2^n} \sum_{T(\lambda^\prime) \in \mathcal{A}} (-1)^{\langle \lambda, X^\prime \rangle} T(\lambda^\prime), \tag{15}
\]

where \( T(\lambda) = Z(p)X(q), \lambda = (q,p), \) and \( \mathcal{A} = \{T(\lambda)|q \cdot p = 0 \mod 2\} \), where \( q \cdot p \) denotes the inner product. The set \( \mathcal{A} \) is the set of inferred observables. "Inferred" means that these observables may not be directly measurable, but they can be inferred by multiple measurements. For example in the case of two qubits, the set \( \mathcal{M}_r \) and \( \mathcal{A} \) are \( \mathcal{M}_r = \{I, IX, IZ, XI, ZI, XX, ZZ\}, \mathcal{A} = \{I, IX, IZ, XI, ZI, XX, ZZ, ZX, ZY, YY\} \), i.e. the set of all rebits observables. Notice that we removed the " \( \otimes \) " symbol for the tensor product in order to soften the notation. This Wigner function is always non-negative for CSS states ([16]) and it is covariant. In terms of the definition provided in equation (1), the function \( \gamma(\lambda) = 0, \chi(a) = (-1)^a \) and \( w^{\gamma}(\lambda) = 1 \).

This choice guarantees that the phase point operators are Hermitian. However the price to pay for the Hermitian in this case is the non-factorisability of the Wigner function, i.e. the Wigner function is composed by phase-point operators of \( n \) qubits that are not given by the tensor products of the ones for the single qubit, e.g. \( A_r((0,0), (0,0)) \neq A_r(0,0) \otimes A_r(0,0) \).

Before we proceed, one may wonder whether the non-factorisability of the Wigner function is necessary to treat the CSS case and preserve the non-negativity and covariance. Here we show that it is not. We define a Wigner function that, we argue, is more in line with the construction of Spekkens’ model, where the ontic space of \( n \) systems is made by the cartesian products of individual systems’ subspaces. The non-negative, covariant and factorisable Wigner function for the CSS theory is built out from the single-qubit phase-point operators

\[
A_f(0,0) = I + X + Z + iY. \tag{16}
\]

The phase point operators \( A_f(0,1), A_f(1,0), A_f(1,1) \) are given by applying the Pauli \( X, Y, Z \) respectively by conjugation on \( A_f(0,0) \). The phase point operators of

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\(^3\)SQM in odd dimensions is the unique maximal Spekkens’ subtheory, since it coincides with the whole Spekkens’ theory.
many qubits are given by tensor products of the ones for single qubits $A_f(0,0), A_f(0,1), A_f(1,0), A_f(1,1)$. Notice that the phase point operators are not Hermitian; however the allowed observables are only present in their Hermitian part (e.g. for the single qubit in $I+X+Z$). We now need to prove the following lemma.

**Lemma 2.** The Wigner function of Delfosse et al. $W_f(\lambda) = \text{Tr}(\rho A_f(\lambda))$, given by (15), is equivalent to the factorisable Wigner function $W_f(\lambda) = \text{Tr}(\rho A_f(\lambda))$, given by (16), for any $\rho \in \mathcal{S}_r$.

Proof. What we need to prove is actually that $A_f(\lambda) = \mathcal{H}(A_f(\lambda))$, where $\mathcal{H}(A_f(\lambda))$ indicates the Hermitian part of the phase point operator $A_f(\lambda)$. The non-Hermitian part of $A_f(\lambda)$ has zero contribution to the Wigner function. It is always composed by tensors of mixtures of Pauli operators with an odd number of $Y$’s, that never form allowed observables and so are never in the stabiliser group of any $\rho \in \mathcal{S}_r$. This implies that the non-Hermitian part of $A_f(\lambda)$ has no contribution to the Wigner function as Pauli operators (apart from the identity) are traceless. However the non-Hermitian part of $A_f(\lambda)$ is important since when its operators compose into phase point operators for multiple qubits, they sometimes provide Hermitian operators that contribute to the Wigner function. We know that $A_f(\lambda)$ is defined as the sum of observables $T(\lambda)$, where $\lambda = (q,p)$ such that $q \cdot p = 0 \mod 2$. We can now see that also $\mathcal{H}(A_f(\lambda))$ is given by the sum of observables subjected to the same condition of having zero inner product between the components. This condition indeed singles out all the rebit observables, which are the only ones we are interested in. Given an observable $T(\lambda) = Z(p)X(q)$ in $A_f(\lambda)$, with $\lambda = (q,p)$ and $q, p \in \mathbb{Z}_2$, it is Hermitian if and only if $T(\lambda) = T(\lambda)^\dagger$. This means that

$$T(\lambda)^\dagger = X(q)Z(p) = (-1)^{q \cdot p}T(\lambda),$$

which holds if and only if $q \cdot p = 0 \mod 2$. 

In conclusion, by using one of the above Wigner functions, (15) or (16), given the duality between states and measurement elements, the covariance and lemma 1, we can conclude that CSS rebits subtheory is Spekkens representable. Moreover the definition of CSS-preserving transformations guarantees the closure property and the discrete Hudson’s theorem for rebits (i.e. non-negativity of the Wigner function of a state if and only if it is a CSS state [16]) guarantees that it is maximal. Therefore the CSS rebit subtheory of QM is a maximal SS.

## 3 SS as toolboxes for QCSI

We now prove that qubit SQM can be reduced to a SS, in the sense that within SS it is possible to build a state-injection scheme that injects all the objects of qubit SQM that are not in SS. We need to understand which objects do we actually need to inject to reach the full qubit SQM from a SS. Let us start by stating the list of instructions to construct the maximal SS that corresponds to a given choice of $\gamma$ (in analogy with the framework of [13]).

1. The function $\gamma$ uniquely defines the set of allowed observables, $\mathcal{M} = \{T(\lambda) | \beta(\lambda, \lambda') = 0 \ \forall \ \lambda' \ s.t. [\lambda, \lambda'] = 0\}$. The function $\beta(\lambda, \lambda')$ is such that $T(\lambda)T(\lambda') = w^\beta(\lambda,\lambda')T(\lambda + \lambda')$. It results that the observable $T(\lambda)$ preserves positivity iff $\beta(\lambda, \lambda') = 0 \ \forall \ \lambda' \ s.t. [\lambda, \lambda'] = 0$, as proven in [13].

2. The set of allowed states $\mathcal{S}$ is given by the states corresponding to common eigenstates of $d^n$ commuting observables in $\mathcal{M}$.

3. The set of allowed gates is $\{U \ U\rho U^\dagger = \rho' \in \mathcal{S} \ \forall \ \rho \in \mathcal{S}, \text{ and } W_{U\rho T}(\lambda) = W_\rho(S\lambda + a) \ \forall \ \lambda \in \Omega\}$. This is a subset of the Clifford unitaries.

Let us point out that the above construction differs from [13] in that it does not require tomographical completeness and it does require covariance of the Wigner functions. With respect to the Wallman-Bartlett 8-state model [13] the difference holds for analogous reasons. The 8-state model consists of a measurement non-contextual ontological model for one qubit SQM, where the one-qubit quantum states (and measurement elements) are represented as uniform probability distributions over an ontic space of dimension 8 and the Clifford transformations (generated by the Hadamard $H$ and phase gate $S$) are represented by permutations over the ontic space. In the definition of the one qubit stabiliser distributions both the possible phase-point operators for a Wigner function are considered, i.e. both the one with an even number of minuses $A_+(0,0) = 1 + X + Y + Z$, and the one with an odd number $A_-(0,0) = 1 + X + Y - Z$. See [19] for a more extensive description of these two possible pairs of single qubit Wigner functions.

In addition to this, the proposed and straightforward generalisation of the 8-state model to more than one qubit, consists of considering the distributions built from the tensor products of the phase-point operators

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4 Notice that with the following construction a given $\gamma$ provides the maximal SS, but the Wigner function from the same $\gamma$ can, obviously, be used to represent a smaller SS.
of the single qubit. The resulting subtheory of qubit SQM described by the model then becomes the one composed by all the product states of tensors of Pauli \(X, Y, Z\) observables and all the local Clifford unitary gates (generated by local \(H\) and \(S\)). No entanglement is present. Nevertheless it is possible to reach UQC from this subtheory by performing measurement-based quantum computation [6] with a particular entangled cluster state [12] [13]. Unlike Raussendorf et al.’s framework [13] that also includes cluster state computation, we consider only standard QCSI schemes in the present work [5]; let us precisely define what we mean by that.

**Definition 1 (QCSI scheme).** A standard scheme of quantum computation state injection (QCSI) is a scheme where a diagonal unitary gate \(U\) is injected in the circuit by making use of the following objects:

- The injected state \(|+\rangle\).
- The \(CNOT\) gate.
- The Pauli \(Z\) measurement.
- The correction gate \(UXU\dagger\) (its application is conditioned on the outcome of the measurement of \(Z\)).

If the gate \(U\) is over multiple qubits, then also the \(|+\rangle\), the \(CNOT, Z\) observable and the gate \(X\) are over multiple qubits. The task of QCSI is to inject the gates needed to boost the computation to quantum universality.

Figure 2 depicts the QCSI scheme just defined. Notice that the above gates and observables composing the QCSI scheme would work the same up to a unitary gate applied to all of them.

We have seen that the presence of non-covariant Clifford gates in the framework of Raussendorf et al. and the 8-state model is the main difference with respect to SS. Even more, the injection of all the non-covariant Clifford unitaries would boost SS, composed by covariant Clifford gates, to the full qubit SQM.

**Theorem 1.** Qubit SQM can be reduced to a SS: all the possible non-covariant Clifford gates can be obtained by standard QCSI via a circuit made of objects in SS.

**Proof.** In order to prove that qubit SQM can be reduced to SS we need to show that all the objects needed for injecting any non-covariant Clifford gate are present in at least one Spekkens’ subtheory. We recall that in order to generate the whole Clifford group we need, in addition to the \(CNOT\), also the generators of the local single gates, e.g. the usual phase and Hadamard gates, \(S, H\). Let us consider the following subtheory:

- The allowed observables are non-mixing tensors of \(X\) and \(Z\) Pauli operators, \(\mathcal{M} = \{X(q), Z(p) | q, p \in \mathbb{Z}_2^d\}\).
- The allowed gates are the ones generated by the \(CNOT\) and the Pauli rotations \(X, Z\), i.e. \(\{CNOT(i, j), X_i, Z_i\}\).
- The allowed states are, as usual, the eigenstates of the allowed observables.

This is a smaller subtheory than CSS rebit (the difference being the absence of the global Hadamard gate). It possesses all the objects needed for state injection of non-covariant gates. The \(Z\) observables and the \(CNOT\) gate are present. The correction gates are always Pauli gates, as for any injected Clifford unitary \(U\), even when \(U\) is non-covariant, \(UX\otimes nU\dagger\), by definition of a Clifford gate, gives back a Pauli gate. All the objects of this subtheory can be non-negatively represented by the Wigner functions for the CSS rebit theory of the previous section and also in Spekkens’ toy model, as shown in figure 3. Therefore this subtheory is closed and Spekkens representable, i.e. a SS, and it is possible for it to reach UQC via state injection, as proven in details in the next section.

We are here interested in showing that we can obtain the whole Clifford group. Once we have it, we can map any of the allowed states and observables to any other in qubit SQM. The whole Clifford group can be achieved by first injecting the \(CZ\) gate, as shown in figure 4, that also provides a construction for the Hadamard gate (figure 5) and then the phase gate \(S\). The correction gate for the QCSI scheme of the \(S\) gate is given by the Pauli \(Y\) gate, which is present, up to

Figure 2: **Standard QCSI scheme.** The diagonal gate \(U\) can be injected in the circuit by using objects that are allowed in the cheap part of the computation. The injected state is \(|+\rangle\), which is subjected to a controlled not with the input state \(|\psi\rangle\). Conditioned on the outcome of the measurement of the Pauli \(Z\) on the state \(|\psi\rangle\) after the \(CNOT\), the correction \(UXU\dagger\) is applied to the state \(|+\rangle\). At the end we obtain the gate \(U\) applied to the input state \(|\psi\rangle\).
obtaining the Bell state

In the figure above the allowed pure states 3a, observ-

cional subtheory of qubit SQM in Spekkens’ toy

X,Y

with

3c, 3d we have considered the scenarios corresponding to acting

states and how the gates act on them. In the examples of figures

dicated the probability distributions associated to the epistemic

Spekkens’ toy model. In figure 3a, 3c and 3d we have also in-

ables 3b and gates 3c, 3d of the non-contextual subtheory of qubit

SQM considered in the proof of theorem 1 are represented in

Representation of a QCSI-minimal non-

contextual subtheory of qubit SQM in Spekkens’ toy model. In the figure above the allowed pure states 3a, observ-

ables 3b and gates 3c, 3d of the non-contextual subtheory of qubit

SQM considered in the proof of theorem 1 are represented in

Spekkens’ toy model. In figure 3a, 3c and 3d we have also in-

dicated the probability distributions associated to the epistemic

states and how the gates act on them. In the examples of figures

3c, 3d we have considered the scenarios corresponding to acting

with X, Y on |0\rangle, with Z on |+\rangle and with CNOT on |+0\rangle (thus

obtaining the Bell state \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).
all the tools to do state-injection of non-covariant Cliff-
ford gates that appear in qubit SQM and also in its
subtheories bigger than SS as formalised in [13]. Notice
that the current approach looks at at the minimal
non-contextual subtheories of SQM such that we can
still reach universal quantum computation via state-
injection. All the times these QCSI-minimal subthe-
ories are Spekkens’ subtheories. In the next section
we describe how the QCSI-minimal subtheory of rebit
SQM used in the proof of theorem 1 can reach UQC
through the injection of CCZ magic states.

4 QCSI with CCZ states

Reaching fault tolerant UQC by exploiting Toffoli
gates, or CCNOT - control control X, goes back to
Peter Shor in 1997 [20]. It is known that the Toffoli
gate (enough for universal classical computation) and
Hadamard gate allow to reach UQC [21, 22], and, in a
sense, this is the most natural universal set of gates,
since the Toffoli allows to do all the classical opera-
tions, and just by adding the Hadamard gate, they lead
to UQC. The same result holds if we use the related
CCZ, control control Z, gate instead of the Toffoli gate.
Other examples of fault tolerant UQC involving Toffoli
state distillation have been proposed [23–26]. We here
propose a scheme of CCZ injection with the fewest
possible objects in the cheap part of the computation.
We reach that by also injecting the CCZ state before the
CCZ. The state-injection scheme is depicted in figure
6.

The cheap part is composed, similarly to the one de-
scribed in the proof of theorem 1, by observables that
are tensors of non-mixing Pauli I, X, Z, i.e. \( M_r =
\{X(q), Z(p)\}_{q, p \in Z_2^2} \). The allowed gates are the
ones generated by the CCNOT and the Pauli rotations

\( X, Z, i.e. \langle CCNOT(i, j), X_t, Z_t \rangle \). This is the subtheory
of CSS rebits with no global Hadamard, thus it is a
SS. The QCSI scheme works with two state-injections:
first the state CCZ \(|++\rangle\) (figure 4), where the correction
is given by the CCZ \cdot X^a X^b \cdot CCZ conditioned on ob-
taining \( x, y, z \) outcomes from the measurements of Z’s,
where \((-1)^a = x \) and \((-1)^b = y \). Just to give an
example, for outcomes \( x = 1, y = -1 \) the correction is
CCZ \cdot X \cdot CCZ = XZ. Secondly, the injection of the state
CCZ \(|++\rangle\) (figure 7), where the correction is given by
CCZ \cdot X^a X^b X^c \cdot CCZ, with outcomes \( x, y, z \) of the
measurements of Z’s such that \((-1)^a = x \), \((-1)^b = y \)
and \((-1)^c = z \), e.g. CCZ \cdot X \cdot CCZ = X \cdot CZ if the
outcomes are \(-1, 1, 1 \). Notice that the injection of the
CCZ allows also to obtain the Hadamard gate, as shown
in figure 5. With Hadamard and CCZ gates we then
have a universal set for quantum computation. The
contextuality, which is not present in the subtheory
of CSS rebits, is clearly present after the two injections
that lead to UQC.

Few comments on the cheap part of the scheme are
needed. As already said, it is QCSI-minimal, in the
sense that it is not possible to remove any object from
the cheap part of the computation without denying the
possibility of obtaining UQC via state-injection. Also
it is a strict subtheory of the CSS rebit, where we allow
all the same objects apart from the global Hadamard.
We argue that this is desirable, since in principle the
Hadamard gate is a local gate; we want to keep only
the entangling gates to have a global nature. Lastly, notice
that the resource states, unlike the previous example
based on cluster state computation, where the bigger
the input the bigger is the resource state, are fixed in
sizes, so they do not grow with the number of qubits.
5 Proofs of Contextuality and State-Injections

The Spekkens’ subtheory used for the CCZ state-injection scheme of the previous section allows us to establish a relation between the different resources injected and different proofs of contextuality. It is well known that within qubit SQM we can obtain the Peres-Mermin square proof of contextuality and the GHZ paradox [9–11]. These proofs are not present within the Spekkens’ subtheory, which, as we know, always witnesses the absence of any form of contextuality. We now explicitly show how the Peres-Mermin square and GHZ cases are obtained after the injections that bring in either the CZ gate or the S gate. We also show that the popular non-Clifford gate $\frac{T}{2}$ in addition to the Peres-Mermin square and GHZ (provided that we can apply the $T$ gate at least two times, as $T^2 = S$), also provides the argument that brings to the CHSH inequality maximum violation.

- **Peres-Mermin square.** Let us consider the cheap SS of the CCZ injection scheme supplemented with the injection of the CZ gate. It allows us to construct a circuit to perform the Peres-Mermin square argument [10]. Let us first remind that the Peres-Mermin square (shown below) is one of the most intuitive and popular way to express the notion of Kochen-Specker contextuality.

![Peres-Mermin Square](image)

The square is composed by nine Pauli observables on a two-qubit system. Each row and each column is composed by commuting (simultaneously measurable) observables. With the assumption that the functional relation between observables is preserved in terms of their outcomes (e.g. if an observable $A$, $B$, also its outcome $c$ is the product of the the outcomes $a, b$ of $A, B$) and the outcome of each observable does not depend on which other commuting observables are performed with it (non-contextuality), the square shows that it is impossible to predict the outcome of each observable among all the rows and columns without falling into contradiction. For example, if we start by assigning values, say $\pm 1$, to the observables starting from the first (top left) row on, the contradiction can be easily seen when we arrive at the last column and last row (red circles), that bring different results to the same observable YY, as witnessed by the following simple calculation, $(XZ) \cdot (ZX) = YY$, and $(XX) \cdot (ZZ) = -(YY)$. Kochen-Specker contextuality refers to the fact that the outcome of a measurement does depend on the other compatible measurements that we perform with it (i.e. on the contexts).

While in our original SS we are only allowed to perform the observables in the first two rows of the square, with the presence of the CZ we can obtain the last row too, since $CZ \cdot XI \cdot CZ = XZ$, $CZ \cdot IX \cdot CZ = ZX$ and $CZ \cdot XX \cdot CZ = YY$. Figure 8 shows a circuit where we can perform all the contexts of the Peres-Mermin square on an arbitrary input state $|\psi\rangle$ by just using objects belonging to the SS and CZ injections.

We can obtain the Peres-Mermin square argument also with the injection of the S gate. This time the observables considered in the square are $I, XLX, XX, YI, IY, YX, XY, ZZ$. The ones containing $Y$ can be obtained by applying $S$ to the $X$ observable, while the others are already present in our SS.

- **GHZ paradox.** In order to obtain the GHZ paradox we need to be able to implement the GHZ state $\frac{|000\rangle + |111\rangle}{\sqrt{2}}$, already present in our SS, and the mutually commuting observables $XXX, YYY, YXY, YYX$. The GHZ state is the common eigenstate of these four operators, with the eigenvalues being $+1, -1, -1, -1$ respectively. While the first observable $XXX$ is already present in our SS, the others can be obtained either by local $S$ gate or CZ gate on two of the three single Pauli operators composing each observable. By considering these observables, the quantum predictions are in conflict with any non-contextual hidden variable model that assigns definite pre-existing values, $+1$ and $-1$, to the local Pauli observables $X,Y$. Let us denote these definite values as $\lambda_{x1}, \lambda_{x2}, \lambda_{x3}, \lambda_{y1}, \lambda_{y2}, \lambda_{y3}$ in correspondence of each local Pauli X and Y. The product of the three observables $YYY, YXY, YYX$, that must yield the outcome $-1$, in the hidden variable model means the following expression $\lambda_{x1} \cdot \lambda_{x2} \cdot \lambda_{x3} \cdot \lambda_{y1} \cdot \lambda_{y2} \cdot \lambda_{y3} = -1$. However this is in neat contradiction with the outcome of $XXX$ which is

\[\text{Notice that we will not write again the } \otimes \text{ symbol for the tensor product in order to soften the notation.}\]
Figure 8: Peres-Mermin square via SS and CZ gates. The above circuit provides a way of implementing all the contexts of the Peres-Mermin square. Each block denoted by CZ corresponds to the injection scheme of figure 4 endowed also with a swap gate (which is present in our SS as it can be made of a series of three alternated CNOT gates) in order to set the output state $CZ\ket{\psi}$ as a precise modification of the input state $\ket{\psi}$ (and not of the ancillary resource state $CZ\ket{++}$). Each context can be selected according to some combinations of the classical control bits $a, b, c, d, e, \alpha, \beta, \gamma$ that can take values in $\{0, 1\}$. The value 0 indicates that the corresponding gate is not performed, while the value 1 that it is performed. At the end of every grey block (labelled by numbers) we assume that we can read the output outcome. The three row contexts of the Peres Mermin square are indicated by $(\alpha, \beta, \gamma, d, e)$ assuming value one, respectively. The three column contexts are identified by $(a, b, \gamma)$ $(a, \beta, \gamma)$ and $(a, d, \gamma)$ $(b, e, \gamma)$ and $(c, d, e, \gamma)$. Notice that in the last case where we implement the context $XX, ZZ, YY$, the measurement of $XX$ is implemented by performing $I$ first and then $X$, and in this case we consider the outputs related to the blocks labelled by 3, 5, 6.

+1, and corresponds to $\lambda_2\lambda_2\lambda_3 = +1$.

- CHSH argument. If we consider our SS with the addition of the $T$ gate we can obtain the maximum violation of the CHSH inequality. In the CHSH game a referee asks questions $x, y \in \{0, 1\}$ to Alice and Bob respectively, who agree on a strategy beforehand to then answer $a, b \in \{0, 1\}$ respectively. They win the game if $xy = a \oplus b$, where the sum is meant to be modulo 2. The best classical strategy for them consists of always answering $a = b = 0$, which means winning the game with a probability of 75%. By exploiting quantum states and measurements they can do better than that. It results that if they share the Bell state $\ket{+i, +i, +i, -i}/\sqrt{2}$, which is a $-1$ eigenstate of $XX$ and +1 eigenstate of $YY$, and they perform the appropriate observables $A_q, B_q$ (depending on which question $q$, 0 or 1, the referee asks them) they can win the game with the maximum quantum probability of about 85%. Notice that the Bell state that we consider is available if we apply twice $T$ to the Bell state $\ket{+i, +i, +i, -i}/\sqrt{2}$, which is present in our SS, as $T^2 = S$. The observables, two of which are provided by the presence of the $T$ gate, are $A_0 = Y, A_1 = X, B_0 = TYT^\dagger = \frac{Y+X}{\sqrt{2}}, B_1 = TXY^\dagger = \frac{X+Y}{\sqrt{2}}$. The probability that Alice and Bob win minus the probability that they lose is $\frac{1}{4}\langle A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \rangle = \frac{1}{2\sqrt{2}}$, as $\langle A_0 B_0 \rangle = \langle A_0 B_1 \rangle = \langle A_1 B_0 \rangle = -\langle A_1 B_1 \rangle = \frac{1}{2\sqrt{2}}$. Therefore the probability of winning is $\frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85$.

6 Conclusions

In this work we define the subtheories of Spekkens’ theory that are compatible with quantum mechanics. They are the closed subtheories of ST that have non-negative and covariant Wigner function representations. Stabiliser quantum mechanics is the maximal Spekkens’ subtheory in odd dimensions, as it corresponds to the full Spekkens’ theory. This is not true for qubits, as SQM is contextual and ST does not reproduce its statistics. We use SS as a unifying framework for known examples of standard QCSI schemes with contextuality as an injected resource [15, 16], in
the sense that they fit into the scheme of figure 1, i.e. \( SS + \text{Magic states} \rightarrow UQC \), where SS represent non-contextuality and the contextuality arises in the injection of the magic states. Even more, we show that SS contain the toolbox to perform any standard QCSI. Theorem 1 proves that all qubit SQM can be reduced to a SS, as all the objects which do not belong to SS but are present in qubit SQM (namely non-covariant gates) can be obtained in the injection stage by a circuit made of objects in SS. This means that for standard QCSI schemes we only need to study SS, i.e. the part of Spekkens’ theory that coincides with QM, because we can always generate the whole Clifford group by injection and therefore all the other QCSI schemes can be mapped to our framework by injection. In order to prove theorem 1 we construct a SS which is a strict sub-theory of the CSS rebit theory (used in [16]) and provides a novel QCSI scheme that allows to reach UQC by injections of \( CZ \) and \( CCZ \) states. This subtheory is QCSI-minimal, meaning that it is not possible to remove any object from it without denying the possibility of reaching UQC via state-injection. By analysing the different injection processes in the above scheme we also associate different proofs of contextuality to specific state-injections of non-covariant gates. In particular we explicitly show how the \( CZ \) and \( S \) gates are resources for the Peres-Mermin proof of contextuality and the GHZ paradox, and how the \( T \) gate, used in the most popular QCSI schemes [7], is a resource also for the CHSH argument.

We believe that the results presented here suggest some related future projects. The importance of the property of covariance highlighted by our result questions its relationship with non-contextuality. A recent work on contextuality in the cohomological framework could give the right tools to address this question [28].

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