Community detection with and without prior information

A. E. Allahverdyan¹, G. Ver Steeg² and A. Galstyan²(a)

¹ Yerevan Physics Institute - Alikhanian Brothers Street 2, Yerevan 375036, Armenia
² Information Sciences Institute, University of Southern California - Marina del Rey, CA 90292, USA

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Abstract – We study the problem of graph partitioning, or clustering, in sparse networks with prior information about the clusters. Specifically, we assume that for a fraction \( \rho \) of the nodes their true cluster assignments are known in advance. This can be understood as a semi-supervised version of clustering, in contrast to unsupervised clustering where the only available information is the graph structure. In the unsupervised case, it is known that there is a threshold of the inter-cluster connectivity beyond which clusters cannot be detected. Here we study the impact of the prior information on the detection threshold, and show that even minute (but generic) values of \( \rho > 0 \) shift the threshold downwards to its lowest possible value. For weighted graphs we show that a small semi-supervising can be used for a non-trivial definition of communities.

Graph partitioning is an important problem with a wide range of applications in circuit design, data mining, social sciences, etc. [1]. In the context of social network analysis, a relaxed version of this problem is known as community detection, where community is loosely defined as a group of nodes so that the link density within the group is higher than across different groups. Many real-world networks have well-manifested community structure [1], which explain the significant attention this problem has received recently. Indeed, much recent research has focused on developing-community detection methods using various approaches. A recent review of existing approaches can be found in [2].

Generally, most algorithms are able to detect communities accurately if the number of inter-community edges is not very large. The detection becomes less accurate as one increases the density of links across the communities. In fact, most community detection algorithms seem to have an intrinsic threshold in inter-community coupling beyond which detection accuracy is very poor [3]. Recently this problem has been studied theoretically by formulating community detection as a minimization of a certain Potts-Ising Hamiltonian [4]. It was shown that the graph partitioning problem is indeed characterized by a phase transition from detectable to undetectable regimes as one increases the coupling strength between the clusters [4]. Specifically, for sufficiently large inter-cluster coupling, the ground-state configuration of the Hamiltonian has random overlap with the underlying community structure.

Most work on community detection so far has considered unsupervised version of clustering, where the only available information is the graph structure. In many situations, however, one might have additional information about possible cluster assignments of certain nodes. Generally speaking, such information can be in form of pairwise constraints (via must-links and cannot-links), or, alternatively, via known cluster assignments for a fraction of nodes. Here we consider the latter scenario, which has attracted recent interest in the context of semi-supervised learning and classification [5–7]. For instance, classification of text documents can be posed as a graph clustering problem, with links based on proximity for some similarity score. In this case, we could ask how picking some small random fraction of documents to be classified by humans will affect our clustering algorithm. Semi-supervised learning falls in between unsupervised (i.e., regression) and totally supervised methods (i.e., clustering). The main premise of the semi-supervised learning is to use prior information about fraction of data points in order to facilitate the classification of the other nodes. Here we are specifically interested in graph-based semi-supervised learning. In this approach, one first maps the data to a (weighted) graph using pairwise similarities between different data points, and then partitions the nodes in this the graph, e.g., using spectral clustering. We note that while most clustering methods have been developed

(a)E-mail: galstyan@isi.edu
for unweighted (homogeneous) networks, generalization to the weighted situation has been suggested as well [2,8].

Despite extensive amount of work in semi-supervised learning in recent years, there is a lack of adequate theoretical development. The purpose of this letter is to present a theoretical analysis of the semi-supervised version of the community detection, and uncover new scenarios of community detection facilitated by semi-supervising. Here we focus on the so called planted bisection graph model [4,9], where the clusters are introduced by hand (implanted), and one checks whether the clustering method under consideration will recognize them. This model is the (supposedly) simplest laboratory for studying the performance of the cluster detection method, which places an Ising spin on each node and lets the spins interact via the network links [4,10].

Our main contributions can be summarized as follows: For unweighted graphs, we show analytically that any small (but finite) amount of prior information destroys the critical nature of cluster detectability, by shifting the detection threshold to its lowest possible value. Furthermore, for graphs where links within and across communities have different weights, we find that the semi-supervision leads to detectable clusters even below the intuitive weigh-balanced value. Note that for weighted graphs the very definition of the communities is rather ambiguous. Our results suggests that the availability of prior information might resolve this ambiguity.

**Model:** Consider an Erdős-Rényi graph where each pair of nodes is linked with probability \( \alpha/N \), and where \( N \) is the number of nodes in the graph. We assume that each link carries a weight \( J > 0 \). Now imagine a pair of such identical Erdős-Rényi graph, which models two clusters (communities). Besides the intra-cluster J-links, each node in one graph is linked with probability \( \gamma/N \) with any node of another Erdős-Rényi graph. These inter-cluster nodes are given weight \( K > 0 \). For clarity, both \( J \) and \( K \) are assumed to be integer numbers.

This **planted bisection** graph model [9] will be employed for studying the performance of the cluster detection method, which places an Ising spin on each node and lets these spins interact via the network links [4,10]:

\[
H = - \sum_{i<j} J_{ij} s_i s_j - \sum_{i<j} J_{ij} s_i s_j - \sum_{i,j} K_{ij} s_i s_j, \tag{1}
\]

where we made the bi-cluster nature of the network explicit by introducing separate spin variables \( s_i = \pm 1 \) and \( s_j = \pm 1 \) \((i = 1, \ldots, N)\) for two clusters. Here \( J_{ij} \) and \( J_{ij} \) are identically and independently distributed random variables which assume zero with probability \( 1 - \hat{J} \) and \( J > 0 \) with probability \( \hat{J} \). Likewise, \( K_{ij} \) and \( K_{ij} \) are identically and independently equal to zero with probability \( 1 - \hat{K} \) and to \( K > 0 \) with probability \( \hat{K} \). In order to enforce equipartition, the Hamiltonian (1) will be studied under the constraint

\[
\sum_{i=1}^{N} s_i + \sum_{i=1}^{N} s_i = 0. \tag{2}
\]

Thus, detecting the sign of a given spin \( s_i \) at zero temperature (so as to exclude all thermal fluctuations) we can conclude to which cluster the corresponding node belongs: all spins having equal signs belong to the same cluster. The error probability for the cluster assignment is \( p_e \), which can also be viewed as the fraction of incorrectly identified spins or one minus the probability of correctly identifying a node’s community. This error is directly related to magnetization,

\[
p_e = (1 - |m|)/2, \tag{3}
\]

\[
m \equiv \langle |s_i| \rangle_{T=0} = -\langle |\bar{s}_i| \rangle_{T=0}, \tag{4}
\]

where \( m \) is the (single-cluster) magnetization, \( \langle \cdots \rangle_{T=0} \) is the zero-temperature Gibbsian average, i.e. the average over all configurations of spins having in the thermodynamic limit the minimal energy given by (1, 2), and where \( \langle \cdots \rangle_{av} \) is the average over the bi-graph structure, i.e., over \( \{J_{ij}\}, \{J_{ij}\} \) and \( \{K_{ij}\} \). As implied by the self-averaging feature, instead of taking \( \langle \cdots \rangle_{av} \), we can evaluate \( \langle s_i \rangle_{T=0} \) on the most probable bi-graph structure(s).

The above formulation refers to the unsupervised community detection. The semi-supervising implies that for some nodes their cluster assignment is known in advance [6,7]. Here we assume that these nodes are distributed randomly over the graph. To account for the prior information, we introduce infinitely strong magnetic fields acting on those node in the appropriate direction. Thus, the Hamiltonian (1) is modified as follows:

\[
H = H - \sum_{i=1}^{N} \hat{J}_i s_i - \sum_{i=1}^{N} \hat{K}_i s_i, \tag{5}
\]

where \( \hat{J}_i \) (resp. \( \hat{K}_i \)) are identically and independently distributed random variables that are equal to 0 with probability \( 1 - \rho \) and to \( \infty \) (respectively, \( -\infty \)) with probability \( \rho \). The constraint (2) is satisfied in the average sense. Thus, with respect of two randomly chosen sets of spins (each containing \( pN \) members) we know exactly to which cluster they belong, since \( \hat{J}_i = \infty \) implies \( s_i = 1 \). Below we study the threshold of cluster detection with and without semi-supervising.

It is known that Ising models on Erdős-Rényi graphs can be efficiently studied via the cavity method; see, e.g., [11–13]. The main object of this method is the probability \( P(h) \) of an internal field acting on one s-spin. The physical order parameters are expressed as moments of \( P(h) \); see (10), (11). As applied to our Hamiltonian (5), the cavity method produces the following equation for \( P(h) \) (the derivation of this equation is fairly similar to that given in [11–13] for the ordinary Ising model on the Erdős-Rényi graph; so we shall repeat it here):

\[
P(h) = \prod_{n=0}^{\infty} \int \frac{\alpha^n e^{-\alpha} \gamma^n e^{-\gamma}}{m!} \frac{1}{m!} \langle \cdots \rangle_{T=0} \times \times h \times f - \sum_{k=1}^{m} \phi[h_k, J] - \sum_{k=1}^{n} \phi[g_k, K], \tag{6}
\]
Here we already assumed the zero-temperature limit of the cavity method denoted \(\phi(a,b)\equiv \text{sign}(a) \text{ min}[|a|,b]\) and \(g_k\) (respectively, \(h_k\)) are the fields acting on the s-spin from s-spin (respectively, from other s-spins). These fields naturally enter with weight \(\frac{2\pi e^{i\pi J} n}{m}\) (respectively, \(\frac{2\pi e^{i\pi J} n}{m}\)), which is the excess degree distribution of the corresponding Erdős-Rényi network.

In (6), the distribution of the frozen (supervising) field acting on s-spins is determined by

\[
\tilde{P}(f) = \rho \delta(f - \infty) + (1 - \rho) \delta(f).
\]

Due to (1)–(5) and the complete inversion symmetry between the two clusters, we can take \(\tilde{P}(g) = P(-g)\), and then (6) is worked out via the Fourier representation of the delta-function yielding

\[
P(h) = \rho \delta(h - \infty) + (1 - \rho) \tilde{P}(h),
\]

where \(P(h)\) refers to those s-spins, which were not directly frozen by infinitely strong random fields.

It can be seen from (6, 8) that \(P(h)\) satisfies the following equation:

\[
\tilde{P}(h) = e^{-\alpha - \gamma} \int \frac{dz}{2\pi} e^{izh} \exp\left[\alpha\rho e^{-izJ} + \gamma\rho e^{izK}ight] + \alpha(1 - \rho) \int dq_1 \tilde{P}(q_1) e^{-iz\text{ sign}(q_1) \text{ min}[q_1,J]}
\]
\[
+ \gamma(1 - \rho) \int dq_2 \tilde{P}(q_2) e^{iz\text{ sign}(q_2) \text{ min}[q_2,K]}.
\]

The physical order parameters are expressed as [11]:

\[
m = \langle \bar{s}_i \rangle_{T=0} \left|_{\text{av}} \right. = \int dh \tilde{P}(h) \text{ sign}(h),
\]

\[
q = \langle \bar{s}_i^2 \rangle_{T=0} \left|_{\text{av}} \right. = \int dh \tilde{P}(h) \text{ sign}^2(h), \text{ sign}(0) = 0,
\]

where \([\ldots]_{\text{av}}\) is now the average over the bi-graph structure and the random fields. Recall that \(m\) defines the error probability according to (4). In (11) \(q\) differs from 1 due to possible contribution \(\alpha \delta(h)\) in \(P(h)\). Thus, \(1 - q\) is the fraction of spins that do not have definite magnetization, since they do not belong to the sub-graph of strongly connected spins (which exists above the percolation threshold), while \(q - m\) is the fraction of spins that do not have definite magnetization, because they are strongly frustrated, though they do belong to the sub-graph of strongly connected spins.

**Unweighted** \((J = K)\) **unsupervised** \((\rho = 0)\) situation: Since at \(T = 0\) only the ratio \(J/K\) matters (and not the absolute values of \(J\) and \(K\)) we assume \(J = K = 1\). Now the local fields can attain only integer values and the solution of (9) is searched for as

\[
P(h) = \sum_{n=-\infty}^{\infty} c_n \delta(h - n),
\]

which upon substituting into (9) and using

\[
e^{\frac{x}{2} (e^{\gamma+i\pi} - e^{-\gamma+i\pi})} = \sum_{n=-\infty}^{\infty} I_n(x) e^{-ny-inz},
\]

where \(I_n(x)\) is the modified Bessel function, produces

\[
c_n = e^{-(\alpha + \gamma) q - \frac{ny}{\alpha + \gamma}} I_n(x),
\]

where

\[
x \equiv \sqrt{(\alpha + \gamma)^2 q^2 - (\alpha - \gamma)^2 m^2},
\]

\[
y \equiv \text{ atanh} \left(\frac{\gamma - \alpha}{\alpha + \gamma} q\right).
\]

This then implies via (10)–(12)

\[
1 - q = e^{-(\alpha + \gamma) q} I_0(x),
\]

\[
m = -2e^{-(\alpha + \gamma) q} \sum_{n=1}^{\infty} I_n(x) \sinh[ny].
\]

Equation (16) predicts a second-order transition, where \(m\) is the order parameter. In the vicinity of the second-order phase transition one can expand (15), (16) over \(m\):

\[
1 - q = e^{-(\alpha + \gamma) q} I_0[(\alpha + \gamma)q],
\]

\[
1 = (\alpha - \gamma)(1 - q) \left(1 + \frac{I_1[(\alpha + \gamma)q]}{I_0[(\alpha + \gamma)q]}\right).
\]

where we employed identities involving Bessel functions.

We have \(m > 0\) \((m = 0)\) if the RHS of (18) is larger \((\text{smaller})\) than its LHS.

Equation (18) determines the detection threshold, above of which the method is capable of detecting clustering with **better than random** probability of error \((4)\). In the \((\alpha, \gamma)\)-plane, the threshold line starts from \((\alpha = 1, \gamma = 0)\), see fig. 1, since (17) predicts a percolation bound for \(q = 0\) \((q > 0)\) for \(\alpha + \gamma < 1\) \((\alpha + \gamma > 1)\). Naturally, close the percolation bound \(\alpha = 1\), even very small inter-cluster coupling \(\gamma\) nullifies \(m\). Figure 1 shows that at the detection threshold \(\alpha > \gamma\); moreover the difference \(\alpha - \gamma\) at the threshold grows as \(\sqrt{2\pi(\alpha + \gamma)}\) for a large \(\alpha + \gamma\); see (17). Thus, the ratio \(\frac{\alpha - \gamma}{\alpha + \gamma}\) converges to zero for a large \(\alpha + \gamma\). In this weak sense, the detection threshold converges to \(\alpha = \gamma\) for large \(\alpha + \gamma\), while for any finite \(\alpha\)
Arbitrary degree distributions can also be considered in this framework. First, note that the excess degree distribution for $n$ intra-cluster and $m$ inter-cluster edges in (6) can also be written in terms of an overall excess degree distribution and a parameter $p_{\text{out}}$ which determines the probability that a given edge connects to a node outside the cluster of the current node.

$$\frac{\alpha^n e^{-\alpha} \gamma^m e^{-\gamma}}{n! m!} = \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{\rho^k}{k!} (1 - p_{\text{out}})^{s-k} \delta_{n,s-k} \delta_{m,k}. \quad (20)$$

In this case, $\rho (s) = e^{-(\gamma + \alpha)} s^s / s!$ and $p_{\text{out}} = \gamma / (\alpha + \gamma)$. Now, we want to consider $q(s)$ to be an arbitrary excess degree distribution, while $p_{\text{out}}$ measures the connection between clusters; $p_{\text{out}} = 0$ indicates disjoint clusters while $p_{\text{out}} = \frac{1}{2}$ indicates no cluster structure. We employ a second trick by rewriting $q(s)$ in terms of a generating function $G(x) = \sum_{s=0}^{\infty} x^s q(s)$ using the inverse formula $q(s) = \lim_{s \to 0} \partial_x^s G(x) / s!$.

This transformation leads to expressions similar to (14). For $\rho = 0$,

$$c_n = \lim_{x \to 0} \left( \frac{q + \bar{m}}{q - \bar{m}} \right)^n I_n \sqrt{\bar{q}^2 - \bar{m}^2} \partial_x \int \frac{e^{(1-q)\partial_x} G(x)}{G(x)},$$

where $\bar{m} = (1 - 2p_{\text{out}}) m$. If we use the generating function for the Poisson distribution $G(x) = e^{-(\alpha + \gamma) x}$ and consider a power series representation of this expression, we see that we should just replace $\partial_x \to (-\alpha - \gamma)$ and we recover (14) exactly.

Rewriting the modified Bessel function in integral form and using the identity $\lim_{x \to 0} e^{\gamma x} G(x) = G(y)$, provides the following succinct expression:

$$1 - q = \frac{1}{\pi} \int_0^\pi \cos \theta \left( \sqrt{\bar{q}^2 - \bar{m}^2} \cos(\theta) + 1 - q \right) \, d\theta,$$

$$m = \frac{\bar{m}}{\pi} \int_0^\pi \cos \theta \left( \sqrt{\bar{q}^2 - \bar{m}^2} \cos(\theta) + 1 - q \right) \, d\theta.$$

However, depending on the integral, it may be more tractable to work with a power series representation of the Bessel functions. For a specific generating function, these integrals play the role of (15), (16), implicitly specifying $m$, $q$. We also recover the result that these equations apply in the supervised case with the substitutions $m \to \rho + (1 - \rho)m$ and $q \to \rho + (1 - \rho)q$ on the RHS. For a power law degree distribution, we get results qualitatively the same as depicted in fig. 2, replacing the $\alpha$ with $p_{\text{out}}$ on the $x$-axis. Specifically, for power law networks, both the analytic results above and numerical experiments using simulated annealing confirm the existence of a detection threshold in the unsupervised case, while the supervised case leads to nonzero magnetization whenever $p_{\text{out}} < \frac{1}{2}$.

The weighted situation $J \neq K$ will be studied via two particular (but important) cases $2J = K = 2$ and $2K = J = 2$, to make them amenable to analytic approach.
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Putting it into (6) and using (13) two times, we see that there are now four order parameters:

\[ m, \quad q, \quad q_1 \equiv c_1 + c_{-1}, \quad m_1 \equiv c_1 - c_{-1}. \] (24)

Note that only \( q \) and \( m \) are observed from the single-spin statistics, see (10), (11); \( q_1 \) and \( m_1 \) can be observed only via measuring the internal field distribution \( \bar{P}(h) \).

For \( 2J = K = 2 \) we introduce the following notations:

\[ C_p(q, m) = a \sum_{n=-\infty}^{\infty} I_n(\bar{u}) I_{p-2n}(\bar{x}) \cosh[2\bar{y}n - \bar{v}n - \bar{y}p], \]

\[ S_p(q, m) = a \sum_{n=-\infty}^{\infty} I_n(\bar{u}) I_{p-2n}(\bar{x}) \sinh[2\bar{y}n - \bar{v}n - \bar{y}p], \]

\[ a = 2e^{-(\alpha + \gamma)q}, \quad z^\pm = q \pm m, \]

\[ z_1^\pm = q \pm m_1, \quad \xi = \frac{2p}{1 - \rho}, \]

\[ \bar{x} = (1 - \rho) \sqrt{\gamma (\alpha z^- + \gamma z_1^+)(\alpha[z^++\xi] + \gamma z_1^-)}, \] (25)

\[ \bar{y} = \frac{1}{2} \ln \frac{\alpha z^- + \gamma z_1^+}{\alpha[z^+ + \xi] + \gamma z_1^-}, \quad \bar{v} = \frac{1}{2} \ln \frac{z^+ - z_1^+ + \xi}{z^- - z_1^-}, \] (26)

\[ \bar{u} = \gamma(1 - \rho)(z^- - z_1^-)(z^+ - z_1^+ + \xi), \] (27)

and write down the order parameter equations:

\[ q = \sum_{p=1}^{\infty} C_p(q, m), \quad q_1 = C_1(q, m), \] (28)

\[ m = \sum_{p=1}^{\infty} S_p(q, m), \quad m_1 = S_1(q, m). \] (29)

Equations (28), (29) apply also for \( 2K = J = 2 \), but now in (25)–(27) we should interchange \( \alpha \) and \( \gamma \), and then substitute \( \bar{y} \rightarrow -\bar{y} \) and \( \bar{v} \rightarrow -\bar{v} \).

**Discussion.** Equations (28), (29) predict a second-order transition over \( m \) and \( m_1 \). Similarly to the unsupervised case, the threshold of this transition is found via expanding (28), (29) over \( m \) and \( m_1 \). But the real qualitative differences between weighted and unweighted situations show up under semi-supervising, which we consider in more details below.

First we focus on \( 2J = K = 2 \) and recall that the clustering threshold is defined via \( m = 0 \). While for the previous unweighted situation, any amount of semi-supervision (as quantified by \( \rho \)) sufficed for shifting the clustering threshold to a \( \rho \)-independent value, here the detection threshold starts to depend on \( \rho \), and the smallest threshold is achieved for \( \rho \rightarrow 0 \); see table 1 for an illustration. To understand this seemingly counterintuitive observation, note that detection threshold is achieved as a balance between the inter-cluster links with the average connectivity \( \gamma \) and weight \( K = 2 \), and intra-cluster links with the average connectivity \( \alpha \) and weight \( J = 1 \). Consider now the impact of the semi-supervision on a test-spin.

| Table 1: Weighted situation: \( 2J = K = 2 \). For \( \gamma = 1 \) and various semi-supervising degrees \( \rho \) we list the clustering threshold \( \alpha \) and the values of \( q \) and \( m_1 \) at this threshold. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \rho \)     | 0.3             | 0.1             | 0.005           | 0               |
| \( \alpha \)   | 1.6812          | 1.5976          | 1.5173          | 4.9122          |
| \( q \)        | 0.86734         | 0.7871          | 0.7087          | 0.8373          |
| \( m_1 \)      | 0.0241          | 0.0109          | 7 \times 10^{-4}| 0               |
| \( m_{\alpha=2} \) | 0.0401         | 0.0182          | 0.0016          | –               |

The inter-cluster links will exert negative fields on this test-spin, while the intra-cluster links will exert positive fields. Since inter-cluster links have twice larger weight, increasing \( \rho \) facilitates the negative fields. This explains why vanishing semi-supervising \( \rho \rightarrow 0 \) results in a lower detection threshold.

Another interesting observation is that — contrary to the unsupervised situation, where \( m \) and \( m_1 \) simultaneously turn to zero at the threshold — we get \( m_1 > 0 \) at the semi-supervised threshold. Thus, some memory about the clustering is conserved despite the fact that \( m = 0 \); see table 1. Since \( m_1 \) cannot be observed via a single spin, this memory is hidden. The reason of \( m_1 > 0 \) is that \( m_1 \) counts the internal fields equal to \( \pm 1 \), and there are more such fields coming from the intra-cluster (connectivity \( \alpha \), weight 1) links that exert positive fields due to the semi-supervised (frozen) spins.

Now consider perhaps the most paradoxical aspect of the semi-supervised detection threshold: it is *smaller* than the value deduced from balancing the cumulative weights of intra-cluster and inter-cluster links, which yields \( \alpha J = \gamma K \). Indeed, according to table 1 (where \( \gamma = 1 \)) we have \( \alpha = 1.5 \) (reached for \( \rho \rightarrow 0 \)) vs. the weight-balancing value \( \alpha \approx 2 \). This result seemingly contradicts the intuition we got so far: i) a rough intuition about Hamiltonian (1) is that it is based on defining a cluster via the intra-cluster weight being larger than the inter-cluster weight. ii) The unsupervised threshold is well above the weight-balancing prediction; see table 1. iii) In the unweighted case (\( J = K \)) the semi-supervising just reduces the detection threshold towards \( \alpha = \gamma \), which coincides with the weight-balancing value.

To understand this effect, we turn to the physical picture of the threshold, where positively and negatively acting links driven by the semi-supervised (frozen) spins compensate each other. At the weight-balancing \( \alpha J = \gamma K \) (with \( J < K \)) fewer (but stronger) inter-cluster links have the same weight as more numerous (but weaker) intra-cluster links. Since the intra-cluster links are more numerous, their overall effect on a (randomly chosen) test spin is more deterministic and hence capable of building up a positive \( m \) at \( \alpha J = \gamma K \). Thus, the actual threshold is reached for \( \alpha J < \gamma K \).

We thus conclude that for weighted graph \( K > J \) a small (but generic) semi-supervising can be employed for defining the very clustering structure. This definition is non-trivial, since it performs better than the weight-balancing...
definition. Indeed, for a weighted network the definition of detection threshold is not clear a priori, in contrast to unweighted networks, where the only possible definition goes via the connectivity balance $\alpha = \gamma$. To illustrate this unclarity, consider a node connected to one cluster via few heavy links, and to another cluster via many light links. To which cluster this node should belong in principle? Our answer is that the proper cluster assignment in this case can be defined via semi-supervising.

It is interesting to calculate $m$ at the weight-balancing value $\alpha J = \gamma K$, since this is the semi-supervising benefit of those who would insist on the weight-balancing definition of the threshold; see table 1. Note finally that for large values of $\gamma$ both unsupervised and semi-supervised thresholds converge to $\alpha J = \gamma K$, since now fluctuations are irrelevant from the outset.

All these effects turn upside-down for $2K = J = 2$; see table 2. Now the threshold is minimized for the maximal semi-supervising $\rho \rightarrow 1$, $m_1$ is negative at the threshold—and thus the memory about the clustering is contained in $m - m_1 > 0$—and the semi-supervised detection threshold $\alpha$ is always larger than the weight-balancing value $\gamma K/J$. These results are explained by “inverting” the above arguments developed for $J < K$.

In conclusion, we analyzed the community detection in semi-supervised settings, where one has prior information about the community assignments of certain nodes. We showed that for the planted bisection graph model with intra-cluster and inter-cluster average connectivities $\alpha$ and $\gamma$, respectively, even a tiny (but finite) semi-supervising shifts the detection threshold to its intuitive value $\alpha = \gamma$. We observed a similar effect of lowered detection threshold for weighted graphs. In contrast to the unweighted case, the shift in this case depends on the degree of supervision. Furthermore, we found that when approaching the unsupervised limit by having $\rho \rightarrow 0$, the detection threshold converges to a value lower (better) from the one obtained via balancing intra-cluster and inter-cluster weights. We suggest that this can serve as an alternative definition of clusters. We also saw that in the semi-supervised case some (hidden) memory on the clustering survives at the detection threshold.

Although this work focused on the analytically simpler case of Erdős-Rényi graphs, we have repeated the analytic and numerical analysis for power law graphs and found similar results, suggesting that the impact of the network topology is quantitative rather than qualitative. A similar picture has been observed in [4].

An interesting generalization is to consider choosing frozen spins more deliberately (e.g., based on node connectivity) as opposed to random selection studied here. While this difference might not be important for ER graphs, it might lead to some significant quantitative changes for power law graphs with a large number of well-connected hubs. Finally, it would be interesting to consider prior information not only about nodes, but also about links in the network. An example is a constraint that two nodes belong to the same community. In our model, this can be incorporated by making the coupling between those pairs sufficiently strong. This scenario resonates well with a recent observation that the links in a network are usually community-specific, while the nodes might participate in different communities [16].

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