The Gravitational Action in
Asymptotically AdS and Flat Spacetimes

Per Kraus\textsuperscript{1}, Finn Larsen\textsuperscript{1} and Ruud Siebelink\textsuperscript{1,2} \textsuperscript{*}

\textsuperscript{1} the Enrico Fermi Institute
University of Chicago
5640 S. Ellis Ave., Chicago, IL-60637, USA

\textsuperscript{2} Instituut voor Theoretische Fysica
Katholieke Universiteit Leuven
Celestijnenlaan 200D
B-3001 Leuven, Belgium

E-mail: pkraus,flarsen,siebelin@theory.uchicago.edu

Abstract

The divergences of the gravitational action are analyzed for spacetimes that
are asymptotically anti-de Sitter and asymptotically flat. The gravitational
action is rendered finite using a local counterterm prescription, and the relation
of this method to the traditional reference spacetime is discussed. For AdS, an
iterative procedure is devised that determines the counterterms efficiently. For
asymptotically flat space, we use a different method to derive counterterms
which are sufficient to remove divergences in most cases.

\textsuperscript{*}Post-doctoraal Onderzoeker FWO, Belgium
1 Introduction

The concepts of action and energy-momentum play central roles in theories with gravity, but they are surprisingly difficult to define (see, e.g., [1]), and they are often laborious to compute. A well known obstacle to the straightforward definition of the gravitational action in a non-compact space is that the sum of the Einstein-Hilbert and Gibbons-Hawking terms diverges. There is a standard remedy for this calamity: first regulate the divergence by restricting the spacetime to the interior of some bounding surface, then subtract the (similarly infinite) action of some reference spacetime with the same boundary geometry [2]. For an appropriately chosen reference spacetime, the resulting action will be finite as the boundary is taken to infinity. The energy-momentum tensor of the spacetime is then related to the variation of the total action with respect to the boundary metric [3].

The above procedure suffers from a number of important drawbacks. On a conceptual level, it is not satisfying since it relies on the introduction of a spacetime which is auxiliary to the problem. It is sometimes said that this merely corresponds to defining the overall zero of energy, but in fact the procedure also affects relative energies, because different reference spacetimes are needed for different boundary geometries. A more glaring defect is that the procedure is generally ill-defined, since it is not possible to embed an arbitrary boundary geometry in the reference spacetime. One is forced to resort to an approximate embedding, and this often leads to confusion and ambiguity; good examples of this are Taub-NUT and Taub-bolt spacetimes [4, 6, 5, 7, 8].

Recently, a preferable alternative procedure has been proposed [9]. For a manifold with boundary, the only way to modify the gravitational action without disturbing the equations of motion or the symmetries is to add a coordinate invariant functional of the intrinsic boundary geometry. By choosing a functional — which we refer to as a counterterm — which cancels the divergences, one arrives at well defined expressions for the action and energy-momentum of the spacetime. The procedure is satisfying since it is intrinsic to the spacetime of interest, and unambiguous once the counterterm is specified.

The new prescription is particularly elegant in the case of asymptotically AdS
In these cases the structure of the divergences is such that they can be fully removed by adding a finite polynomial in the boundary curvature and its derivatives. Moreover, the counterterm method is precisely analogous to the standard removal of divergences in quantum field theory by adding finite polynomials in the fields; indeed, the AdS/CFT correspondence [10] asserts that they are the very same. The equivalence between the gravitational action and that of a CFT on the boundary is illustrated by the agreement between trace anomalies [11, 12, 13, 14] and Casimir energies [9, 15] obtained from the two descriptions (see [16] for a different appearance of divergences in the AdS/CFT correspondence).

Counterterms for low dimensional AdS spacetimes were obtained in [9, 17]. Our focus in the first half of this paper is to develop an algorithm for generating counterterms for arbitrary dimensions. We begin by analyzing the structure of divergences in more detail. An important tool is the interplay between the bulk and boundary geometries, as embodied in the Gauss-Codazzi equations. This formulation of the problem leads to an iterative process that generates the counterterms as an expansion in the radius of the anti-de Sitter space, denoted by $\ell$:

\[
\tilde{S} = \frac{1}{\ell} \tilde{S}^{(0)} + \ell^2 \tilde{S}^{(1)} + \cdots
\]  

The coefficients $\tilde{S}^{(n)}$ are increasingly complicated polynomials of the boundary curvature tensor and its derivatives. The iterative process is manifestly covariant and quite efficient; we compute the four leading orders explicitly, and show that our algorithm defines a series of local counterterms to all orders. The trace anomalies of the boundary theory play a prominent role in the discussion; indeed, there is a close connection to the anomaly computation of [11].

Crucial to the success of the counterterm prescription is that the divergences are universal, so that a single choice of counterterms suffices to render finite the action of all asymptotically AdS solutions\(^1\). The finite terms are non-universal, though, and so the counterterm subtraction leaves a finite remainder in general. We use the AdS-Schwarzschild spacetimes to illustrate these properties of the action. An important

\(^1\)The action will sometimes have logarithmic divergences which remain uncancelled by the counterterms. However, these divergences are well understood physically as arising from the trace anomaly.
feature of the counterterm prescription is that it provides unambiguous results even for nontrivial boundary topologies, such as Taub-NUT-AdS [18, 17].

It is of obvious interest to apply the counterterm method to spacetimes that are asymptotically flat, rather than asymptotically AdS. One approach is to try to determine these by solving the flat space versions of the Gauss-Codazzi equations. However, the resulting equation is highly non-linear and we have had no progress with this strategy. An alternative strategy is to define flat space as the limit where the AdS curvature \( \ell \to \infty \). But to take the limit one first has to determine the counterterms to all orders in \( \ell \), which seems prohibitively difficult. We pursue a third approach, which can be understood as a refinement of the reference spacetime prescription. We derive the action of a particular spacetime which asymptotes to the solutions of interest, and write the result in terms of intrinsic invariants of the boundary. The resulting counterterm action is then expected to share the divergences of spacetimes which look sufficiently like the chosen spacetime near infinity. Unlike in the standard reference spacetime prescription, once we have obtained the counterterm we can forget about the reference spacetime altogether.

Recently Lau [19] and Mann [18] proposed the remarkably simple counterterm for spaces that are asymptotically \( \text{AdS}_4 \):

\[
8 \pi G \tilde{S} = -\frac{2}{\ell} \int d^3x \sqrt{-\gamma} \sqrt{1 + \frac{\ell^2}{2} R},
\]

which has a smooth limit as \( \ell \to \infty \). Mann further showed that, in many explicit examples, this removes all divergences and gives a finite part that agrees with the reference spacetime procedure [18, 20]. By following the strategy of the previous paragraph we derive the \( d \)-dimensional generalization of the Lau-Mann formula. However, the assumptions of the derivation are quite strong, and there are simple examples where divergences are not removed. We give a more general counterterm that removes the divergences for asymptotically flat space in more cases, though not in general. Our examples suggest that a counterterm capable of removing the divergences from arbitrary asymptotically flat spacetimes would be quite complicated. However, we stress that such an expression is not needed under normal circumstances — the counterterms we define provide well defined actions for the most common class of spacetimes. A more general result would only be needed if one wished to consider spacetimes

4
which deviate strongly from these.

All of the counterterms that we derive will be coordinate invariant, intrinsic to the boundary, and local. The property of locality is not \textit{a priori} mandatory, since adding non-local intrinsic counterterms would not disturb the equations of motion and so cannot be excluded on such grounds. As we will describe below, for asymptotically AdS spaces the divergences are always local polynomials in the boundary fields and their derivatives, as was to be expected given the AdS/CFT correspondence. In the case of asymptotically flat spaces it is less clear what to expect, since we do not know whether the putative holographically dual theories will have only local divergences. Still, in the simple asymptotically flat examples we consider below, it suffices to use local counterterms only in order to remove the divergences.

The flat space limit of AdS space has recently been discussed in the context of the complete, dynamical string theory \cite{[21, 22]}. It was argued that, in the appropriate limit, the AdS/CFT correspondence constitutes a suitable starting point for non-perturbative M-theory in flat space. The divergences in the gravitational action which we study give nontrivial information concerning a possible holographic description of flat space. The correct understanding of the flat space counterterm may ultimately be interlinked with these far-reaching perspectives.

The paper is organized as follows. In section 2 we develop our algorithm for generating AdS counterterms, and give examples. We turn to flat space in section 3, and give two examples of counterterms. The counterterms are seen to lead to the standard results for the actions of black hole spacetimes, provided that appropriate coordinates are chosen. We show in an example how more general coordinates may lead to ambiguities, whose nature we explain.
2 Counterterms and the Gauss-Codazzi equation

We write the standard action for the gravitational field as

\[ S = \frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{-g} \left( R + \frac{d(d-1)}{\ell^2} \right) - \frac{1}{8\pi G} \int_{\partial M} d^d x \sqrt{-\gamma} \Theta. \]  

(3)

Variation of this action with respect to the geometry of the boundary \( \partial M \) gives the energy-momentum tensor \[ \Pi^{ab} = \Theta^{ab} - \Theta \gamma^{ab}, \]  

(4)

where \( \gamma^{ab} \) is the metric of the boundary. Concrete computations show that in most spacetimes both the action integral (3) and the energy-momentum tensor (4) actually diverge as the boundary \( \partial M \) is taken to infinity. We therefore think of these as the unrenormalized quantities.

The divergences must be cancelled in order to achieve physically meaningful expressions; i.e. some counterterm action:

\[ \tilde{S} = \frac{1}{8\pi G} \int d^d x \sqrt{-\gamma} \tilde{\mathcal{L}}, \]  

(5)

must be added, along with the corresponding counterterm energy-momentum tensor:

\[ \tilde{\Pi}^{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma^{ab}} \int d^d x \sqrt{-\gamma} \tilde{\mathcal{L}}. \]  

(6)

The counterterms by definition contain the divergent part of the corresponding unrenormalized quantities, but finite terms may depend on the details of the renormalization.

Now, recall that the Gibbons-Hawking boundary term in (3) has been determined precisely such that the combined action satisfies a well-defined variational principle, giving the correct bulk equation of motion. The counterterm will ruin this property unless it is a function of the boundary geometry only. Additionally, suppose the counterterm is an analytical function of the boundary geometry, and expand it as a power series in the metric and its derivatives. Dimensional analysis shows that in

This fixes our conventions for the Riemann curvature to be \( R_{\mu\nu\lambda\sigma} = -2\partial_{[\mu} \Gamma_{\nu]\lambda}^{\sigma} + 2\Gamma_{\lambda\nu}^\rho \Gamma_{\nu\rho}^{\sigma} \), where the antisymmetrization is defined with strength one, i.e. \( [\mu\nu] = \frac{1}{2}(\mu\nu - \nu\mu) \). Also \( R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda \). With these conventions spheres have a positive scalar curvature. The cosmological constant is written as \( \Lambda = -d(d-1)/2\ell^2 \); in this notation pure \( AdS_{d+1} \) has radius \( \ell \).
AdS\(_{d+1}\) only terms of order \(n < d/2\) contribute to the divergent part of the action. (By terms of order \(n\) we mean terms containing \(2^n\) derivatives.) Therefore one may truncate the series at this order and obtain a finite polynomial \([9]\). This agrees with the expectations from the interpretation of the divergences in terms of a dual boundary theory that obeys the usual axioms of quantum field theory, including locality. In most of this paper we will treat the dimension \(d\) as a free parameter which can be made very large and so think of the counterterm as a \(d\)-dependent power series with an arbitrarily high number of terms. Of course, when restricting the attention to a particular value of \(d\), the general result should be truncated.

### 2.1 The Counterterm Generating Algorithm

The structure of divergences is tightly constrained by the Gauss-Codazzi equations. These are covariant expressions of the bulk Einstein tensor \(G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\) in terms of the boundary Einstein tensor \(G_{ab}(\gamma)\) (which only depends on the induced metric \(\gamma_{ab}\)) and the extrinsic curvature \(\Theta_{ab}\) (which characterizes the embedding of the boundary surface into the bulk geometry). After using (4) they read \([1]\):

\[
G_{ab} = G_{ab}(\gamma) + \hat{n}^\mu \nabla_\mu \Pi_{ab} - \frac{1}{2} \gamma_{ab} \left( \frac{\Pi^2}{d-1} - \Pi_{cd} \Pi^{cd} \right) + \frac{1}{d-1} \Pi_{ab} \Pi , \quad (7)
\]

\[
G_{a\mu} \hat{n}^\mu = 0 , \quad (8)
\]

\[
G_{\mu\nu} \hat{n}^\mu \hat{n}^\nu = \frac{1}{2} \left( \frac{1}{d-1} \Pi^2 - \Pi_{ab} \Pi^{ab} - R(\gamma) \right) , \quad (9)
\]

where \(\hat{n}^\mu\) is an outward pointing unit vector normal to the boundary. We will always consider solutions of the bulk equations of motion so:

\[
G_{ab} = \frac{1}{2} \frac{d(d-1)}{\ell^2} \gamma_{ab} ,
\]

\[
G_{a\mu} \hat{n}^\mu = 0 ,
\]

\[
G_{\mu\nu} \hat{n}^\mu \hat{n}^\nu = \frac{1}{2} \frac{d(d-1)}{\ell^2} , \quad (10)
\]

determines the left hand side of the Gauss-Codazzi equations.

In principle, one could solve the Gauss-Codazzi equations \((7)-(9)\) for the unrenormalized energy-momentum tensor \(\Pi_{ab}\), and then identify its divergent part with \(-\tilde{\Pi}_{ab}\). However, this strategy is rather complicated due to the presence of the normal derivatives in \((7)\). The appearance of these normal derivatives expresses the intuitive fact
that, to determine the solution throughout, both the boundary values and their derivatives are needed. However, the counterterm should be determined independently of data that is extrinsic to the boundary, such as the normal derivative.

Now, there exists a set of coordinates for which the bulk Einstein equations in \(AdS_{d+1}\) (which are equivalent to the Gauss-Codazzi equations) can be solved in a perturbative fashion \[23\]. Explicit computations in this coordinate system show that the divergent part of the normal derivatives can be expressed in terms of the intrinsic boundary data \[23\]. We implement this observation covariantly, as follows. We impose the constraint equation (11):

\[
\frac{1}{d-1} \tilde{\Pi}^2 - \tilde{\Pi}_{ab} \tilde{\Pi}^{ab} = \frac{d(d-1)}{\ell^2} + R ,
\]

and further insist that the counterterm energy-momentum tensor must derive from a counterterm action, which is itself intrinsic to the boundary:

\[
\tilde{\Pi}^{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma_{ab}} \int d^dx \sqrt{-\gamma} \tilde{L} .
\]

As we will show, the conditions (11) and (12) fully determine the counterterm. The form of (12) ensures that the counterterm energy-momentum is conserved, which in turn implies (8). It is important to stress that the remaining Gauss-Codazzi equations (7) are also satisfied: they can be viewed as expressions for the normal derivatives specified implicitly in our construction. We note that the normal derivatives thus determined do not in general vanish.

There is another implicit definition of the normal derivative which is important in the AdS/CFT correspondence: this is the requirement of regularity in the bulk of spacetime \[24\]. For Euclidean \(AdS_{d+1}\) with a boundary sufficiently close to the round sphere \(S^d\), it is known that, for given intrinsic boundary data, there exists a unique solution to the Gauss-Codazzi equations which is regular in the interior of \(AdS\) \[25\]. This is not in general the solution we consider. That our solution may become singular when expanded to all orders is of no concern because, for a specific value of the boundary dimension \(d\), we always truncate to a finite number of terms.

We are now prepared to describe an algorithm that determines the counterterm as an expansion in the parameter \(\ell\). The leading order term scale as \(\ell^{-1}\) and terms at a given order \(\ell^{2n-1}\) with \(n \geq 0\) are denoted by \(\tilde{\Pi}^{(n)}_{ab}\) and \(\tilde{L}^{(n)}\). The starting point
is to note that the curvature term in (11) can be neglected to the leading order in $\ell$, so that the metric is the only tensor characterizing the boundary geometry to the leading order. The $\bar{\Pi}^{(0)}_{ab}$ are therefore proportional to the metric, with the overall numerical factor determined by (11). This gives:

$$\bar{\Pi}^{(0)}_{ab} = -\frac{d-1}{\ell} \gamma_{ab}.$$  \hspace{1cm} (13)

The sign was determined using positivity conditions on the energy-momentum tensor.

Higher order counterterms are now given by induction. Assuming that $\bar{\Pi}_{ab}$ is known up to and including order $n-1$, the following three steps determine $\bar{\Pi}^{(n)}_{ab}$:

**step 1:** Insert the known terms in (11); the resulting expression is a linear equation with the trace $\bar{\Pi}^{(n)}$ as the only unknown.

**step 2:** With the trace $\bar{\Pi}^{(n)}$ in hand, integrate (12) and find $\bar{L}^{(n)}$. This step is purely algebraic, as discussed in the following subsection.

**step 3:** Finally, take the functional derivative of $\bar{L}^{(n)}$ with respect to $\gamma_{ab}$, and so find the full tensor $\bar{\Pi}^{(n)}_{ab}$ from (12).

The fact that $\bar{\Pi}^{(0)}_{ab}$ is proportional to the metric $\gamma_{ab}$ is crucial to make step 1 possible. We stress that higher orders of $\bar{\Pi}_{ab}$ in general will depend also on other tensor structures.

### 2.2 Some Comments on Weyl Rescalings

The integration in step 2 is interesting and deserves comment. It is related to the behavior of the various terms under the local Weyl variations which transform the metric as:

$$\delta_W \gamma_{ab} = \sigma \gamma_{ab},$$  \hspace{1cm} (14)

where $\sigma$ is an arbitrary function. Consider the counterterm action at the $n$th order and note that dimensional analysis gives the behavior under a *global* Weyl rescaling. The result of a *local* Weyl variation can therefore be written in the form:

$$\delta_W \int d^d x \sqrt{-\gamma} \bar{L}^{(n)} = \int d^d x \sqrt{-\gamma} \sigma \left( \frac{d-2n}{2} \bar{L}^{(n)} + \nabla_a X^a(n) \right),$$  \hspace{1cm} (15)
where $X^a(n)$ is some unspecified expression (involving $2n + 1$ derivatives).

However, it follows from (12) that:

$$
\delta_W \int d^d x \sqrt{-\gamma} \hat{L}^{(n)} = \frac{1}{2} \int d^d x \sqrt{-\gamma} \sigma \tilde{\Pi}^{(n)} ,
$$

so:

$$(d - 2n) \hat{\mathcal{L}}^{(n)} = \tilde{\Pi}^{(n)} ,
$$

up to a total derivative term. Now, recall that counterterm Lagrangians are in fact only defined up to total derivatives; a total derivative term can be added without changing the action. We can therefore freely choose a scheme where (17) is exact, without need for total derivatives. The practical significance of this identity is that it renders the integration in step 2 almost trivial. We also note that:

$$
\delta_W \int d^d x \sqrt{-\gamma} \Pi^{(n)} = \frac{d - 2n}{2} \int d^d x \sqrt{-\gamma} \sigma \Pi^{(n)} ,
$$

so that $\sqrt{-\gamma} \Pi^{(n)}$ transforms as a conformal density with Weyl weight $\frac{1}{2} (d - 2n)$, up to total derivatives [26, 27]. This constrains the form of the counterterms.

In even dimensions it is clear that (17) prevents $\tilde{\Pi}^{(d/2)}$ from being obtained as the variation of any local action. This is the origin of trace anomalies. For a given even dimension $d$ the trace $\tilde{\Pi}^{(d/2)}$ is therefore identified with the trace anomaly of the dual boundary theory. This result for the anomaly agrees with that of [11], as may be verified by looking at the explicit expressions given below.

2.3 Explicit Computations of Counterterms

At this point we evaluate the counterterms explicitly, to the first few orders.

The leading order was given in (13). For completeness we give its trace and the corresponding counterterm, computed using (17):

$$
\tilde{\Pi}^{(0)} = - \frac{d(d - 1)}{\ell} ,
$$

$$
\hat{\mathcal{L}}^{(0)} = - \frac{d - 1}{\ell} .
$$

\footnote{It is important that the Lagrangian $\hat{\mathcal{L}}^{(n)}$ is assumed to be local, otherwise (13) need not be true. For example, $\delta_W \int d^d x \sqrt{-\gamma} R \nabla_R R = \int d^d x \sqrt{-\gamma} (\frac{d - 2}{d - 1} R \nabla_R R - 2(d - 1) R)$.}
At the first non-trivial order we insert this in (11) and find:

$$\tilde{\Pi}^{(1)} = -\frac{\ell}{2} R .$$

(20)

Now (17) gives:

$$\tilde{\mathcal{L}}^{(1)} = -\frac{\ell}{2(d - 2)} R ,$$

(21)

and the variation (12) yields:

$$\tilde{\Pi}^{(1)}_{ab} = \frac{\ell}{d - 2} \left( R_{ab} - \frac{1}{2} \gamma_{ab} R \right) .$$

(22)

We have used the algorithm to generate a few more orders in the expansion, finding

the trace of the energy-momentum tensor:

$$\tilde{\Pi} = -\frac{d(d - 1)}{\ell} - \frac{\ell}{2} R - \frac{\ell^3}{2(d - 2)^2} \left( R_{ab} R^{ab} - \frac{d}{4(d - 1)} R^2 \right)$$

$$+ \frac{\ell^5}{(d - 2)^3(d - 4)(d - 6)} \left\{ \frac{3d + 2}{4(d - 1)} R R_{ab} R^{ab} - \frac{d(d + 2)}{16(d - 1)^2} R^3 - 2 R^{ab} R_{acbd} R^{cd} \right\}$$

$$+ \frac{d - 2}{2(d - 1)} R^{ab} \nabla_a \nabla_b R - R^{ab} \Box R_{ab} + \frac{1}{2(d - 1)} R \Box R \right\} + \cdots ,$$

(23)

the counterterm Lagrangian:

$$\tilde{\mathcal{L}} = -\frac{d - 1}{\ell} - \frac{\ell}{2(d - 2)} R - \frac{\ell^3}{2(d - 2)^2(d - 4)} \left( R_{ab} R^{ab} - \frac{d}{4(d - 1)} R^2 \right)$$

$$+ \frac{\ell^5}{(d - 2)^3(d - 4)(d - 6)} \left\{ \frac{3d + 2}{4(d - 1)} R R_{ab} R^{ab} - \frac{d(d + 2)}{16(d - 1)^2} R^3 - 2 R^{ab} R_{acbd} R^{cd} \right\}$$

$$+ \frac{d - 2}{2(d - 1)} R^{ab} \nabla_a \nabla_b R - R^{ab} \Box R_{ab} + \frac{1}{2(d - 1)} R \Box R \right\} + \cdots ,$$

(24)

and the full energy-momentum tensor:

$$\tilde{\Pi}_{ab} = -\frac{d - 1}{\ell} \gamma_{ab} + \frac{\ell}{d - 2} \left( R_{ab} - \frac{1}{2} \gamma_{ab} R \right)$$

$$+ \frac{\ell^3}{(d - 2)^2(d - 4)} \left\{ -\frac{1}{2} \gamma_{ab} \left( R_{cd} R^{cd} - \frac{d}{4(d - 1)} R^2 \right) - \frac{d}{2(d - 1)} R R_{ab} \right\}$$

$$+ 2 R^{cd} R_{cd} - \frac{d - 2}{2(d - 1)} \nabla_a \nabla_b R + \Box R_{ab} - \frac{1}{2(d - 1)} \gamma_{ab} \Box R \right\} + \cdots .$$

(25)

The most laborious step is to find the full energy momentum-tensor from the counterterm. Accordingly, we have resisted carrying out this computation to the fourth order.

The first three orders of (23) agree with the results previously deduced from explicit examples [9, 17]. All four terms may be obtained from the results in [11] (see in particular the Appendix of the second reference.)
2.4 Explicit examples in AdS

We now consider a few simple examples that illustrate the general results.

2.4.1 Euclidean $AdS_{d+1}$ with a $S^d$ boundary

Consider pure $AdS_{d+1}$ with the metric:

$$ds^2 = \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\Omega_d^2 .$$  \hspace{1cm} (26)

Using the definition (4) of the unrenormalized energy-momentum tensor, one can straightforwardly compute the density:

$$\sqrt{\gamma} \Pi = \frac{d(d-1)}{\ell} r_0^d \sqrt{1 + \frac{\ell^2}{r_0^2} \sqrt{g_{\Omega_d}}}$$

$$= \frac{d(d-1)}{\ell} \left( r_0^d + \frac{1}{2} \ell^2 r_0^{d-2} - \frac{1}{8} \ell^4 r_0^{d-4} + \cdots \right) \sqrt{g_{\Omega_d}} .$$  \hspace{1cm} (27)

It clearly diverges for $r_0 \to \infty$, so one needs to add counterterms in order to cancel the terms of order $n < d/2$. The Ricci scalar on the boundary is expressed in terms of the position $r_0$ of the boundary through:

$$R = \frac{d(d-1)}{r_0^2} .$$  \hspace{1cm} (28)

Inserting this in the formula for the counterterm $\bar{\Pi}$ given in (23), and exploiting that the space is maximally symmetric, we find terms that are precisely equal to (27), with the opposite sign. Thus the divergences cancel.

However, as emphasized above, the counterterm (23) should be truncated so that only the divergences cancel. For even dimension this implies that the finite term in (27) is not subtracted. This residual term is the anomaly. That there is a genuine obstruction that precludes the cancellation of this term is seen by inspecting the would-be counterterm action (24) at order $d = n/2$. After evaluation on $S^d$ this term remains ill-defined due to the $1/(d - 2n)$ factor.

For odd-dimensional boundaries there are no anomalies and the renormalized energy-momentum tensor vanishes. However, this does not imply that the renormalized action vanishes. To see this, consider the contribution to the action from the bulk part of (3). This term has the form of a radial integral $\int_0^r dr$ leading to
a term at \( r = r_0 \), which is in fact cancelled by the counterterms, but also a term at \( r = 0 \) which does not get subtracted off. Interestingly, the resulting uncancelled action implies negative entropy for the case of \( AdS_4 \) with an \( S^3 \) boundary \([17]\).

### 2.4.2 The AdS-Schwarzschild solution

Next, we consider the AdS-Schwarzschild metric:

\[
\begin{align*}
    ds^2 &= -\left(1 + \frac{r^2}{\ell^2} - \frac{\mu}{r^{d-2}}\right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{\ell^2} - \frac{\mu}{r^{d-2}}\right)} + r^2 d\Omega_{d-1}^2, \\
    \ell &= \sqrt{\frac{d}{d-2}}, \\
    \approx &= \ell_0, \\
    m &= \mu r_0^2/\ell_0^d + \mathcal{O}(r_0^{-d-1}).
\end{align*}
\]

which for large values of \( r \) asymptotes to a pure \( AdS_{d+1} \) solution with a \( \mathbb{R} \times S^{d-1} \) boundary. The corresponding unrenormalized energy-momentum tensor (4) can be expanded as:

\[
\begin{align*}
    \Pi_{\hat{a}\hat{b}} &= \gamma \frac{\ell}{\ell_0} \left\{ (d-1) + \frac{d-3}{2} \frac{\ell^2}{r_0^2} - \frac{d-5}{8} \frac{\ell^4}{r_0^4} + \cdots + \frac{\mu \ell^2}{2 r_0^d} + \mathcal{O}(r_0^{-d+1}) \right\}, \\
    \Pi_{tt} &= \gamma \frac{\ell}{\ell_0} \left\{ (d-1) - \frac{d+1}{2} \frac{\ell^2}{r_0^2} - \frac{d+1}{8} \frac{\ell^4}{r_0^4} + \cdots - (d-1) \frac{\mu \ell^2}{2 r_0^d} + \mathcal{O}(r_0^{-d+1}) \right\}, \\
    \Pi &= \frac{(d-1)}{\ell} \left\{ d + \frac{d-2}{2} \frac{\ell^2}{r_0^2} - \frac{d-4}{8} \frac{\ell^4}{r_0^4} + \cdots + \mathcal{O}(r_0^{-d+1}) \right\},
\end{align*}
\]

where the index \( \hat{a} \) labels the \( S^{d-1} \) directions. In order to see which of the terms in (30) correspond to divergent, finite or vanishing physical quantities in the \( r_0 \to \infty \) limit, one must convert the above expressions to the appropriate proper densities. One finds that the proper scaling of the quantities in (30) is determined by multiplying the formulae inside the brackets by \( r_0^d \); thus the \( \mathcal{O}(r_0^{-d+1}) \) terms vanish in the limit.

It is very important that the divergent terms depend on the boundary position \( r_0 \) only. This ensures that these terms are \textit{intrinsic} to the boundary, as they should be. (That the divergent terms are intrinsic can be made manifest by expressing \( r_0 \) in terms of the Ricci scalar \( R \), using \( R = (d-1)(d-2)/r_0^2 \).) The mass parameter \( \mu \) is an \textit{extrinsic} quantity from the boundary point of view, so it is important that the \( \mu \)-dependence appears only at the finite level. Because of its extrinsic nature, the \( \mu \)-dependence can never be subtracted off.

Generically \([17]\) forces a \( 1/(d-2n) \) divergence in the counterterm action at order \( n = d/2 \), rendering the corresponding subtraction impossible. The present example is special because the trace \( \Pi \) contains a \( (d-2n) \) factor at each order \( n \) so that a finite
counterterm at order $n = d/2$ is viable. The physical significance of this possibility is that we can choose to include the order $d/2$ counterterm as well, such that the entire $\mu$-independent part of the energy-momentum tensor is cancelled. It was shown in [3] that the $\mu$-independent finite part of the energy-momentum tensor can be identified with the Casimir energy in the dual conformal field theory. The ability to cancel this part of the energy-momentum tensor is equivalent to the option of choosing a renormalization scheme where the Casimir energy of $\mathbb{R} \times S^{d-1}$ is set to zero.

3 Counterterms for Asymptotically Flat Space

We now turn to defining the gravitational action in asymptotically flat space (AFS). At first glance, AFS may seem like just a special case of AdS, since it can be obtained by taking the limit $\ell \to \infty$. This analogy leads us to consider (11), which in the limit reads:

$$\frac{1}{d-1} \tilde{\Pi}^2 - \tilde{\Pi}_{ab} \tilde{\Pi}^{ab} = R ,$$

and we should further impose (12) on the solution. The problem is now nonlinear and does not appear to allow a perturbative expansion; a direct computation is therefore impractical. We might instead try to apply the limit $\ell \to \infty$ to our perturbative expansion of the counterterm action. However, the large $\ell$ limit can only be taken after summing the infinite series, which is clearly a difficult task. It is also doubtful whether the sum exists, for the following reasons: As we mentioned before, the counterterm action (24) implicitly defines a bulk solution which in general need not be regular. Therefore, if the sum did exist, it would generically assign some finite action to singular solutions, which seems unphysical. Most glaringly, for even $d$ the coefficients of individual terms diverge.

We will take an alternative approach to define the AFS counterterms. We will start with some particular solution, work out its action, and then express the result in terms of intrinsic invariants of the boundary. The counterterm action is then defined as minus this expression. To the extent that divergences are universal, this counterterm action will remove the divergences of solutions which asymptote to the particular solution used in the construction. The counterterm is not uniquely defined, since choosing different solutions or different curvature invariants will yield inequivalent...
results. As we will discuss, this does not appear to be a drawback of the procedure. An added benefit is that this method exhibits the relation between the counterterm method and the reference spacetime approach clearly.

3.1 A Counterterm for AFS

We first consider the most common class of metrics, those having boundary topology $S^{d-1} \times \mathbb{R}$. In this case we can work out a simple closed form counterterm for AdS with finite $\ell$, and then take the flat space limit $\ell \to \infty$. To do so we consider $AdS_{d+1}$ in global coordinates:

$$ds^2 = -(1 + r^2/\ell^2)dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\Omega_{d-1}^2 .$$

(32)

To evaluate the gravitational action (3) we use that $R - 2\Lambda = -2d/\ell^2$ for pure $AdS_{d+1}$, and also the general expression:

$$S_{GH} = -\frac{1}{8\pi G} \int d^d x \sqrt{-\gamma} \Theta = \frac{1}{8\pi G} \int d^d x \frac{\partial}{\partial \hat{n}} \sqrt{-\gamma} ,$$

(33)

where $\hat{n}$ is the unit normal to the boundary. Then the action becomes:

$$S_{\text{bulk}} + S_{GH} = \frac{d - 1}{8\pi G \ell} \int d^d x \sqrt{-\gamma} \sqrt{1 + \ell^2/r_0^2} .$$

(34)

To express this in terms of intrinsic invariants we use:

$$R = \frac{(d - 1)(d - 2)}{r_0^2} ,$$

(35)

for a $d - 1$ sphere of radius $r_0$. This leads to the definition:

$$\tilde{S}^A = -(S_{\text{bulk}} + S_{GH}) = -\frac{d - 1}{8\pi G \ell} \int d^d x \sqrt{-\gamma} \sqrt{1 + \frac{\ell^2 R}{(d - 1)(d - 2)} .}$$

(36)

By definition, $\tilde{S}^A$ will assign vanishing total action to AdS in global coordinates. We further expect it to give finite action for solutions which asymptote to (32). In the flat space limit we find:

$$\tilde{L}^A = -\sqrt{\frac{d - 1}{d - 2}} \sqrt{R} .$$

(37)

\footnote{The case of $d = 2$ requires special considerations, so we assume $d > 2$. It is interesting that the $d = 2$ case is exactly solvable for all boundary geometries.}
It is instructive to compare (36) with the power series (24). Choosing coordinates \((\hat{a}, \tau)\) on \(S^{d-1} \times \mathbb{R}\) we have:

\[
R_{\hat{a}\hat{b}\hat{c}\hat{d}} = \frac{1}{(d-1)(d-2)}(g_{\hat{a}\hat{c}}g_{\hat{b}\hat{d}} - g_{\hat{a}\hat{d}}g_{\hat{b}\hat{c}})R, \quad R_{\hat{a}\hat{b}} = \frac{1}{d-1}g_{\hat{a}\hat{b}}R, \tag{38}
\]

as well as \(R_{\hat{a}\hat{b}\hat{c}\hat{d}} = R_{\tau\tau} = 0\). The power series (24) becomes:

\[
\tilde{\mathcal{L}} = -\frac{d-1}{\ell} - \frac{1}{2(d-2)}R + \frac{\ell^3}{8(d-1)(d-2)^2}R^2 - \frac{\ell^5}{16(d-1)^2(d-2)^3}R^3 + \cdots. \tag{39}
\]

This precisely corresponds to the expansion of \(\tilde{S}^A\) in (36). So it appears that for the \(S^{d-1} \times \mathbb{R}\) class of boundary geometries we have summed the series (24). Note that the potentially divergent factors of \(1/(d-4), 1/(d-6), \ldots\) were cancelled in the process, leading to a well defined counterterm for arbitrary \(d > 2\). Since (38) only relies on \(S^{d-1}\) being a maximally symmetric space, this computation further indicates that (36) also cancels the divergences when \(S^{d-1}\) is replaced by a space with constant negative curvature.

For \(d = 3\) the counterterm (36) is closely related to an expression obtained by Lau through different means [19]. It agrees precisely with the counterterm used by Mann [20]. Indeed, the present reasoning provides a simple derivation of Mann’s proposal, and its generalization to \(d\) dimensions.

### 3.2 Counterterms and Black Hole Thermodynamics

Before continuing the main argument we pause to derive a constraint on the counterterms from Smarr’s formula, and also comment on the (absent) effect of counterterms on the black hole entropy. A useful reference for the formulas in this section is [28].

Upon continuation to imaginary time, the action of a Euclidean black hole solution represents the thermodynamic free energy of the system:

\[
S = \beta M - \mu J - S_{\text{ent}}, \tag{40}
\]

where we denote entropy by \(S_{\text{ent}}\), and inverse temperature — or equivalently, the periodicity of imaginary time — by \(\beta\). \(J\) represents the angular momentum of the horizon and \(\mu J\) is its conjugate potential. Given \(S\) as a function of \(\beta\) and \(\mu J\), the mass and angular momentum follow from:

\[
M = \frac{\partial S}{\partial \beta}, \quad J = -\frac{\partial S}{\partial \mu J}. \tag{41}
\]
The Bekenstein-Hawking formula $S_{\text{ent}} = A/4G$ and the generalized Smarr formula:

$$(d - 2)M\beta = (d - 1)(\mu J + \frac{A}{4G}) ,$$

(42)

give the simple expression for the action:

$$S = \frac{\beta M}{d - 1} .$$

(43)

A notable point is that $\mu J$ and $J$ do not appear explicitly in the result, but only via $\beta$. In sec. 3.4 we will compute the action for a general rotating solution and see explicitly that it only receives contributions from the leading long-range part of the metric, which is equivalent to that of a spherically symmetric solution.

As a simple example, we work out the action of the $d + 1$ dimensional Euclidean Schwarzschild solution using the counterterm action (37):

$$ds^2 = \left(1 - \frac{\mu}{r^{d-2}}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{\mu}{r^{d-2}}\right)} + r^2d\Omega^2_{d-1} ,$$

(44)

where $\mu$ is related to the ADM mass by:

$$\mu = \frac{16\pi GM}{(d - 1)\omega_{d-1}} .$$

(45)

We find (after rotating to Euclidean time):

$$S = S_{\text{GH}} + \tilde{S}^A = - \lim_{r_0 \to -\infty} \frac{\omega_{d-1}\beta}{8\pi G} \left[ (d - 1)r_0^{d-2} - \frac{d}{2}\mu - (d - 1)r_0^{d-2} \sqrt{1 - \frac{\mu}{r_0^{d-2}}} \right]$$

$$= \frac{\omega_{d-1}\beta\mu}{16\pi G} = \frac{\beta M}{d - 1} ,$$

(46)

in agreement with (43).

As we have discussed and will see again explicitly, one may sometimes have a choice between several counterterm actions which all subtract off the infinities but lead to different finite results. It is important to stress that in the case of black holes these choices affect only the definition of energy and not the entropy; the latter is always given by the Bekenstein-Hawking formula. This can be seen as follows (for simplicity we consider the spherically symmetric case). Consider restricting the imaginary time integration region in (46) to some duration $\Delta\tau$. Let $W$ be the wedge shaped region in the $r - \tau$ plane given by restricting $\tau$ to the range $\Delta\tau$. The tip of the wedge lies at the horizon. To compute the action of $W$ we need to include
a boundary term at the tip. Here only the Gibbons-Hawking term can contribute, since the vanishing of \( g_{\tau\tau} \) at the horizon causes intrinsic invariants to vanish upon integration. Now, from the Hamilton-Jacobi equation it follows that the action of a static (\( \tau \) independent) solution obeys:

\[
S = \Delta \tau M . \tag{47}
\]

This equation defines the mass of the spacetime, and so is independent of choice of counterterms at infinity, although the actual value of \( M \) can be. Next, consider evaluating the action for the full Euclidean black hole manifold. We then set \( \Delta \tau = \beta \), and omit the Gibbons-Hawking term at the horizon, since the only boundary is at \( r = r_0 \). Therefore the action is:

\[
S = \beta M - S_{\text{hor}}^{\text{GH}} . \tag{48}
\]

Finally, a simple computation yields that \( S_{\text{hor}}^{\text{GH}} = S_{\text{ent}} = A/4G \). We have thus established that the entropy is insensitive to the addition of boundary terms at infinity. The entropy is in this sense renormalization scheme independent.

### 3.3 More Divergences and Another Counterterm for AFS

The counterterm \( \tilde{S}^A \) was designed to remove divergences for boundaries of the form \( S^{d-1} \times \mathbb{R} \), so there is no guarantee that it will continue to do so for other boundary topologies. As a simple example, take flat Euclidean space in spherical coordinates:

\[
ds^2 = dr^2 + r^2 d\Omega_d^2 . \tag{49}
\]

Then:

\[
S_{\text{GH}} = \frac{d\omega_d}{8\pi G} r_0^{d-1} , \quad \tilde{S}^A = -\frac{(d-1)\omega_d}{8\pi G} \sqrt{\frac{d}{d-2}} r_0^{d-1} . \tag{50}
\]

The sum of the two terms is not finite as the boundary is taken to infinity \( r_0 \to \infty \); so the divergences are not removed by \( \tilde{S}^A \). Note that this problem persists in spacetimes with four bulk dimensions, i.e. \( d = 3 \).

By generalizing the derivation that led to \( \tilde{S}^A \) we can derive a counterterm which will remove divergences for a larger class of spacetimes, including the example just given. Write flat \( d + 1 \) dimensional space in a form with boundary \( S^n \times \mathbb{R}^{d-n} \):

\[
ds^2 = (dt^2 + dx_1^2 + \cdots + dx_{d-1-n}^2) + dr^2 + r^2 d\Omega_n^2 . \tag{51}
\]
The action is:
\[ S_{GH} = \frac{1}{8\pi G} \int d^d x \sqrt[4]{\gamma} \frac{n}{r_0}, \]  
\( (52) \)

where \( \sqrt[4]{\gamma} = r_0^n \sqrt{g_{tt}} \). We wish to write a counterterm that will remove the divergence at large \( r_0 \) for arbitrary \( n \). To do so it is necessary to use more invariants than just \( R \). We take also \( R_{\mu\nu} R_{\mu\nu} \), and use that for \( S \times R d - n \):
\[ R = \frac{n(n-1)}{r_0^2}, \quad R_{\mu\nu} R^{\mu\nu} = R^2 / n. \]  
\( (53) \)

Then we arrive at:
\[ \tilde{S}^B = - S_{GH} = - \frac{1}{8\pi G} \int d^d x \sqrt[4]{\gamma} \frac{R^{3/2}}{\sqrt{R^2 - R_{\mu\nu} R^{\mu\nu}}}. \]  
\( (54) \)

This counterterm reverts to \( \tilde{S}^A \) in the special case of \( n = d - 1 \) for which \( R_{\mu\nu} R^{\mu\nu} = R^2 / (d - 1) \). But \( \tilde{S}^B \) is more general than \( \tilde{S}^A \) since it will remove divergences from metrics that are asymptotic to any of \( (51) \).

\[ \text{3.4 Discussion} \]

In cases for which both \( \tilde{S}^A \) and \( \tilde{S}^B \) remove divergences one can ask whether they will also agree on the finite part. An interesting feature is that the answer depends on the coordinates used to describe a given spacetime — by using a preferred class of coordinates the two counterterms will agree in the main cases of interest.

\[ \text{3.4.1 Counterterms and the ADM prescription} \]

It is important to verify that under suitable conditions we obtain results for the mass and angular momentum which agree with the standard ADM definitions. To show this, we use that any AFS can be cast in the form:
\[ ds^2 = - \left( 1 - \frac{\mu}{r^{d-2}} + O \left( \frac{1}{r^{d-1}} \right) \right) dt^2 - \left( \frac{A_{ij} x^i}{r^d} + O \left( \frac{1}{r^d} \right) \right) dx^i dt \]
+ \[ \left[ \left( 1 + \frac{\mu}{r^{d-2}} + O \left( \frac{1}{r^{d-1}} \right) \right) \delta_{ij} + \frac{e_{ij}}{r^{d-2}} + O \left( \frac{1}{r^{d-1}} \right) \right] dx^i dx^j. \]  
\( (55) \)

The boundary is taken to be at fixed \( r_0^2 = x^i x^i \). \( A_{ij} \) is proportional to the angular momentum of the spacetime, and the symmetric, traceless tensor \( e_{ij} \) represents gravitational radiation. We will restrict attention to isolated systems, for which \( \mu \) and \( A_{ij} \) are constants; the time dependent case requires a more detailed analysis.
Now, the important point is that upon evaluating the action we find that the angular momentum and gravitational radiation terms in the metric make no contribution as \( r_0 \to \infty \). First note that the \( r_0 \) dependence is such that only terms in the action linear in \( A^{ij} \) or \( e_{ij} \) can potentially contribute. But \( A^{ij} \) cannot appear linearly in the action due to time reversal symmetry. And \( e_{ij} \) can only appear in the rotationally invariant combinations \( e_{ii}, e_{ij} x^i x^j \) (sum on \( i, j \)); but for \( e_{ij} \) traceless, the former vanishes identically, while the latter vanishes upon integration over the sphere. Therefore, in the action, only the \( \mu \) dependent terms survive, and the calculation effectively reduces to that for the Schwarzschild metric, with boundary \( S^{d-1} \times \mathbb{R} \). However, we already know from our previous computation (46) that the counterterms \( \tilde{S}^A \) and \( \tilde{S}^B \) agree for boundaries of this form, and indeed by direct calculation we find:

\[
S = S_{GH} + \tilde{S}^A = S_{GH} + \tilde{S}^B = - \frac{\omega_{d-1} \beta \mu}{16 \pi G} .
\]  

(56)

This is the correct result, as we discussed in the derivation of (46). Demanding regularity of the Euclidean black hole metric fixes the relation between \( \mu \) and \( \beta \). Then, by differentiating the action with respect to \( \beta \) we can read off the mass of the solutions. In so doing, we find agreement with the ADM definition (45).

It is satisfying that our counterterms reproduce the ADM definitions for the general class of metrics (55), since these follow from quite general considerations. For instance, the ADM mass is the unique (up to an overall constant) quantity which is conserved, transforms as the time component of a Lorentz vector, and is additive for distant subsystems. If one is willing to relax one or more of these conditions then other results are possible, and we will see an explicit example of this in the following subsection.

### 3.4.2 An Example: Spheroidal Coordinates

To get agreement between \( \tilde{S}^A, \tilde{S}^B \), and the standard result for the action (56) it was important to use the preferred coordinates (55). This fact is illustrated by considering flat space in non-standard coordinates. We know that both counterterms assign zero action to flat space in the form:

\[
ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-1}^2 .
\]  

(57)
Now instead use spheroidal coordinates:

\[
\begin{align*}
    ds^2 &= -dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 + \sin^2 \theta (r^2 + a^2) d\phi^2 + r^2 \cos^2 \theta d\Omega^2_{d-3}
\end{align*}
\]

(58)

This is in fact the form of the metric one finds upon setting the mass parameter to zero in the D-dimensional generalization of the Kerr metric written in Boyer-Lindquist coordinates \[28\]. In these coordinates, the action (33) yields:

\[
S_{GH} = \frac{1}{8\pi G} \int d^d x \left[ d - 1 + (d - 3) \frac{a^2}{r_0^2} + \frac{a^2 \sin^2 \theta}{r_0^2 + a^2 \cos^2 \theta} \right] r_0^{d-2} \sin \theta \cos^{d-3} \theta \sqrt{g_{d-3}}
\]

(59)

where the measure is \( d^d x = dt d\theta d\phi d\Omega_{d-3} \). The expression in the square bracket forms a kernel with the expansion:

\[
K = (d - 1) + (d - 2 - \cos^2 \theta) \frac{a^2}{r_0^2} - (\cos^2 \theta - \cos^4 \theta) \frac{a^4}{r_0^4} + \cdots.
\]

(60)

After computing the curvature tensors of the boundary geometry the counterterm (52) can be written as:

\[
\tilde{S}^A = -\frac{d - 1}{8\pi G} \int d^d x \frac{(r^2 + a^2)^{1/2}}{(r^2 + a^2 \cos^2 \theta)^{1/2}} \left[ 1 + \frac{(d - 3)}{(d - 1)} (1 + \cos^2 \theta) \frac{a^2}{r_0^2} \right]
\]

\[
+ \frac{(d - 3)(d - 4)}{(d - 1)(d - 2)} \cos^2 \theta \frac{a^4}{r_4} \right]^{1/2} r_0^{d-2} \sin \theta \cos^{d-3} \theta \sqrt{g_{d-3}},
\]

(61)

corresponding to the expansion:

\[
\tilde{K}^A = -(d - 1) - (d - 2 - \cos^2 \theta) \frac{a^2}{r_0^2}
\]

\[
- \frac{1}{2(d - 1)} \left( -1 - \frac{2(d^2 - 3d + 1)}{d - 2} \cos^2 \theta + (2d - 3) \cos^4 \theta \right) \frac{a^4}{r_0^4} + \cdots
\]

(62)

in the same normalization as (60). The general expression for the alternative counterterm (54) is quite lengthy, so we give just the expansion:

\[
\tilde{K}^B = -(d - 1) - (d - 2 - \cos^2 \theta) \frac{a^2}{r_0^2} + \frac{1}{(d - 1)(d - 2)^2} \left( \frac{2d - 5}{2} + \frac{13 - 22d + 18d^2 - 7d^3 + d^4}{d - 2} \cos^2 \theta + \frac{3 - 14d + 10d^2 - 2d^3}{2} \cos^4 \theta \right) \frac{a^4}{r_0^4} + \cdots
\]

(63)

These expressions show that both counterterms correctly capture the divergences to the first two leading orders. The renormalized action therefore vanishes as \( r_0 \to \infty \).

21
for $d < 6$, it is finite for $d = 6$, and for $d > 6$ the divergences of flat space in spheroidal coordinates are not cancelled by either counterterm. After integration over $\theta$ we find that, for $d = 6$, the finite action assigned to flat space is nonvanishing, but different for the two counterterms.

On a technical level, note that the metrics (57), (58) are equivalent to leading order for large $r$, with both boundaries asymptoting to $S^{d-1} \times \mathbb{R}$. It is the subleading $a$ dependent terms which lead to the finite terms in the action.

This discussion was for flat space in spheroidal coordinates. It is straightforward to compute the expression analogous to (59) for the Kerr black hole in $D$ dimensions [28]. This yields a structure of divergences that departs from the flat space expression (60) only by terms of order $a^4/r_0^4$ and higher. The renormalized action is therefore finite for $d < 6$, with the expected value. For $d = 6$ the action is finite and nonvanishing, but different for the two counterterms; and for $d > 6$ the divergences are not cancelled. This indicates that for $d > 6$ the spheroidal boundary deviates too strongly from the round sphere (for which both counterterms $\tilde{S}^A$ and $\tilde{S}^B$ were designed). Presumably there exists another more sophisticated counterterm which subtracts off all divergences for $d > 6$ spheroidal boundaries as well. We have not tried to construct such a counterterm.

3.4.3 Concluding Remarks

The fact that flat space in spheroidal coordinates is assigned a nonzero action in $d = 6$ is at first surprising, but becomes less so when we recall the analogous situation in AdS. There we know that the simplest choice of counterterms assign vanishing action to AdS$_{d+1}$ in Poincaré coordinates, but finite action to AdS$_{d+1}$ in global coordinates for $d$ even. This fact has a nice interpretation in terms of the AdS/CFT correspondence: the boundary of AdS in global coordinates is $S^{d-1} \times \mathbb{R}$, upon which the CFT can have a nonzero action due to the Casimir effect. The ability of the counterterm prescription to assign nonzero action to AdS in various coordinates is crucial to the consistency of the AdS/CFT correspondence. Although we do not currently have access to a holographic description of AFS, we should not be surprised that the action behaves in a way analogous to AdS.

Since $\tilde{S}^A$ and $\tilde{S}^B$ can lead to different finite terms in the action, one can ask
whether this implies that one, both, or neither of them is in some sense “correct”. We believe that both are valid expressions for the action, and should be thought of as the results in different renormalization schemes. The important criterion is that the counterterm $\tilde{S}$ can be written in terms of intrinsic invariants of the boundary, and that it removes the divergences of the action. In this sense, $\tilde{S}^A$ and $\tilde{S}^B$ are both valid for boundaries of topology $S^{d-1} \times \mathbb{R}$, while only $\tilde{S}^B$ is valid for the more general case of $S^n \times \mathbb{R}^{d-n}$. Future work may identify the most general counterterm that subtracts the divergences of any regularization of AFS. This would be interesting for several purposes, including the general understanding of the asymptotic symmetry group (the Spi group). However, from a practical point of view, this development is unnecessary: our work shows that, when the divergences of the action are cancelled for some counterterm, the finite part has necessarily been correctly identified.

Acknowledgments: We thank V. Balasubramanian for collaboration in the initial stages of this work and for further discussions. We also thank A. Ashtekar, R. Mann and M. Taylor-Robinson for helpful discussions. This work was supported by DOE Grant no. DE FG0290ER40560 (PK, FL), by NSF Grant no. PHY 9600697 (PK), by a Robert R. McCormick fellowship (FL). RS thanks the Enrico Fermi Institute and the Particle Theory Group of the University of Chicago for support. FL also thanks the Theory Groups at Harvard University and at the University of Michigan in Ann Arbor for hospitality.

References

[1] R. Wald. General Relativity. University of Chicago press, 1984.

[2] G. W. Gibbons and S. W. Hawking. Action integrals and partition functions in quantum gravity. Phys. Rev., D15:2752–2756, 1977.

[3] J. D. Brown and J. W. York Jr. Quasilocal energy and conserved charges derived from the gravitational action. Phys. Rev., D47:1407–1419, 1993.

[4] C. J. Hunter. The action of instantons with nut charge. Phys. Rev., D59:024009, 1999.
[5] S. W. Hawking and C. J. Hunter. Gravitational entropy and global structure. *Phys. Rev.*, D59:044025, 1999. hep-th/9808083.

[6] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Phys. Rev.*, D59, 064010, 1999. hep-th/9808177.

[7] S. W. Hawking, C. J. Hunter, and D. N. Page. Nut charge, anti-de Sitter space and entropy. *Phys. Rev.*, D59:044033, 1999. hep-th/9809035.

[8] M. Taylor-Robinson. Instanton symmetries and the entropy of compact manifolds. hep-th/9809040.

[9] V. Balasubramanian and P. Kraus. A stress tensor for Anti-de Sitter gravity. hep-th/9902121.

[10] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz. Large N field theories, string theory and gravity. hep-th/9905111.

[11] M. Henningson and K. Skenderis. The holographic Weyl anomaly, hep-th/9806087; Holography and the Weyl anomaly, hep-th/9812032.

[12] S. Hyun, W. T. Kim, and J. Lee. Statistical entropy and AdS/CFT correspondence in BTZ black holes. *Phys. Rev.*, D59:084020, 1999.

[13] S. Nojiri and S. D. Odintsov. Conformal anomaly for dilaton coupled theories from AdS/CFT correspondence. *Phys. Lett.*, B444:92, 1998.

[14] W. Muck and K. S. Viswanathan. Counterterms for the Dirichlet prescription of the AdS/CFT correspondence. hep-th/9905046.

[15] R. C. Myers. Stress tensors and Casimir energies in the AdS / CFT correspondence. hep-th/9903203.

[16] G. Chalmers and K. Schalm. Holographic normal ordering and multiparticle states in the AdS/CFT correspondence. hep-th/9901144.

[17] R. Emparan, C. V. Johnson, and R. C. Myers. Surface terms as counterterms in the AdS/CFT correspondence. hep-th/9903238.
[18] R. B. Mann. Misner string entropy. hep-th/9903229.

[19] S. R. Lau. Light cone reference for total gravitational energy. gr-qc/9903038.

[20] R. B. Mann. Entropy of rotating Misner string space-times. hep-th/9904148.

[21] J. Polchinski. S-matrices from AdS space-time. hep-th/9901076.

[22] L. Susskind. Holography in the flat space limit. hep-th/9901079.

[23] C. Fefferman and C. R. Graham. Conformal invariants. In *Elie Cartan et les Mathématiques d’aujourd'hui*, page 95. Asterisque, 1985.

[24] E. Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253, 1998.

[25] R. Graham and J. Lee. Einstein metrics with prescribed metrics on the ball. *Adv. Math.*, 87:186, 1991.

[26] L. Bonora, P. Pasti, and M. Bregola. Weyl cocycles. *Class. Quant. Grav.*, 3:635, 1986.

[27] S. Deser and A. Schwimmer. Geometric classification of conformal anomalies in arbitrary dimensions. *Phys. Lett.*, B309:279–284, 1993. hep-th/9302047.

[28] R. C. Myers and M. J. Perry. Black holes in higher dimensional space-times. *Ann. Phys.*, 172:304, 1986.

[29] S. W. Hawking. The path integral approach to quantum gravity. In *General Relativity, an Einstein Centenary Survey*, pages 746–789. Cambridge, 1979.