C*-ALGEBRAS FROM k GROUP REPRESENTATIONS

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Abstract. We introduce certain C*-algebras and k-graphs associated to k finite dimensional unitary representations ρ1, ..., ρk of a compact group G. We define a higher rank Doplicher-Roberts algebra Oρ1,...,ρk, constructed from intertwiners of tensor powers of these representations. Under certain conditions, we show that this C*-algebra is isomorphic to a corner in the C*-algebra of a row finite rank k graph Λ with no sources. For G finite and ρi faithful of dimension at least 2, this graph is irreducible, it has vertices Ĝ and the edges are determined by k commuting matrices obtained from the character table of the group. We illustrate with some examples when Oρ1,...,ρk is simple and purely infinite, and with some K-theory computations.

1. INTRODUCTION

The study of graph C*-algebras was motivated among other reasons by the Doplicher-Roberts algebra Oρ associated to a group representation ρ, see [19, 17]. It is natural to imagine that a rank k graph is related to a fixed set of k representations ρ1, ..., ρk satisfying certain properties.

Given a compact group G and k finite dimensional unitary representations ρi on Hilbert spaces H_i of dimensions d_i for i = 1, ..., k, we first construct a product system E indexed by the semigroup (N^k, +) with fibers E_n = H_1^⊗n_1 ⊗ · · · ⊗ H_k^⊗n_k for n = (n_1, ..., n_k) ∈ N^k. Using the representations ρi, the group G acts on each fiber of E in a compatible way, so we obtain an action of G on the Cuntz-Pimsner algebra O(E). This action determines the crossed product O(E) ∗ G and the fixed point algebra O(E)^G.

Inspired from Section 7 of [17] and Section 3.3 of [1], we define a higher rank Doplicher-Roberts algebra Oρ1,...,ρk associated to the representations ρ1, ..., ρk. This algebra is constructed from intertwiners Hom(ρ^n, ρ^m), where ρ^n = ρ_1^⊗n_1 ⊗ · · · ⊗ ρ_k^⊗n_k acting on H^n =
\( \mathcal{H}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}_k^{\otimes n_k} \) for \( n = (n_1, \ldots, n_k) \in \mathbb{N}^k \). We show that \( \mathcal{O}_{\rho_1, \ldots, \rho_k} \) is isomorphic to \( \mathcal{O}(\mathcal{E})^G \).

If the representations \( \rho_1, \ldots, \rho_k \) satisfy some mild conditions, we construct a \( k \)-coloured graph \( \Lambda \) with vertex space \( \Lambda^0 = \hat{G} \), and with edges \( \Lambda^{\varepsilon_i} \) given by some matrices \( M_i \) indexed by \( \hat{G} \). Here \( \varepsilon_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^k \) with 1 in position \( i \) are the canonical generators. The matrices \( M_i \) have entries

\[
M_i(w, v) = |\{ e \in \Lambda^{\varepsilon_i} : s(e) = v, r(e) = w \}| = \dim \text{Hom}(v, w \otimes \rho_i),
\]

the multiplicity of \( v \) in \( w \otimes \rho_i \) for \( i = 1, \ldots, k \). The matrices \( M_i \) commute because \( \rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i \) for all \( i, j = 1, \ldots, k \) and therefore

\[
\dim \text{Hom}(v, w \otimes \rho_i \otimes \rho_j) = \dim \text{Hom}(v, w \otimes \rho_j \otimes \rho_i).
\]

By a particular choice of isometric intertwiners in \( \text{Hom}(v, w \otimes \rho_i) \) for each \( v, w \in \hat{G} \) and for each \( i \), we can choose bijections

\[
\lambda_{ij} : \Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \rightarrow \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_i},
\]

obtaining a set of commuting squares for \( \Lambda \). For \( k \geq 3 \), we need to check the associativity of the commuting squares, i.e.

\[
(id_\ell \times \lambda_{ij})(\lambda_{i\ell} \times id_j)(id_i \times \lambda_{j\ell}) = (\lambda_{j\ell} \times id_i)(id_j \times \lambda_{i\ell})(\lambda_{ij} \times id_\ell)
\]

as bijections from \( \Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_\ell} \) to \( \Lambda^{\varepsilon_\ell} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_i} \) for all \( i < j < \ell \), see [13]. If these conditions are satisfied, we obtain a rank \( k \) graph \( \Lambda \), which is row-finite with no sources, but in general not unique.

In many situations, \( \Lambda \) is cofinal and it satisfies the aperiodicity condition, so \( C^*(\Lambda) \) is simple. For \( k = 2 \), the \( C^* \)-algebra \( C^*(\Lambda) \) is unique when it is simple and purely infinite, because its \( K \)-theory depends only on the matrices \( M_1, M_2 \). It is an open question what happens for \( k \geq 3 \).

Assuming that the representations \( \rho_1, \ldots, \rho_k \) determine a rank \( k \) graph \( \Lambda \), we prove that the Doplicher-Roberts algebra \( \mathcal{O}_{\rho_1, \ldots, \rho_k} \) is isomorphic to a corner of \( C^*(\Lambda) \), so if \( C^*(\Lambda) \) is simple, then \( \mathcal{O}_{\rho_1, \ldots, \rho_k} \) is Morita equivalent to \( C^*(\Lambda) \). In particular cases we can compute its \( K \)-theory using results from [10].

2. The product system

Product systems over arbitrary semigroups were introduced by N. Fowler [12], inspired by work of W. Arveson, and studied by several authors, see [23, 4, 1]. In this paper, we will mostly be interested in
product systems $\mathcal{E}$ indexed by $(\mathbb{N}^k, +)$, associated to some representations $\rho_1, \ldots, \rho_k$ of a compact group $G$. We remind some general definitions and constructions with product systems, but we will consider the Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{E})$ and we will mention some properties only in particular cases.

**Definition 2.1.** Let $(P, \cdot)$ be a discrete semigroup with identity $e$ and let $A$ be a $C^*$-algebra. A **product system** of $C^*$-correspondences over $A$ indexed by $P$ is a semigroup $E = \bigsqcup_{p \in P} E_p$ and a map $E \to P$ such that

- for each $p \in P$, the fiber $E_p \subset E$ is a $C^*$-correspondence over $A$ with inner product $\langle \cdot, \cdot \rangle_p$;
- the identity fiber $E_e$ is $A$ viewed as a $C^*$-correspondence over itself;
- for $p, q \in P \setminus \{e\}$ the multiplication map $M_{p,q}: E_p \times E_q \to E_{pq}$, $M_{p,q}(x, y) = xy$

induces an isomorphism $M_{p,q}: E_p \otimes_A E_q \to E_{pq}$;
- multiplication in $E$ by elements of $E_e = A$ implements the right and left actions of $A$ on each $E_p$. In particular, $M_{e,p}$ is an isomorphism.

Let $\phi_p: A \to \mathcal{L}(E_p)$ be the homomorphism implementing the left action. The product system $E$ is said to be **essential** if each $E_p$ is an essential correspondence, i.e. the span of $\phi_p(A)E_p$ is dense in $E_p$ for all $p \in P$. In this case, the map $M_{e,p}$ is also an isomorphism.

If the maps $\phi_p$ take values in $\mathcal{K}(E_p)$, then the product system is called **row-finite** or **proper**. If all maps $\phi_p$ are injective, then $E$ is called **faithful**.

**Definition 2.2.** Given a product system $E \to P$ over $A$ and a $C^*$-algebra $B$, a map $\psi: E \to B$ is called a **Toeplitz representation** of $E$ if

- denoting $\psi_p := \psi|_{E_p}$, then each $\psi_p: E_p \to B$ is linear, $\psi_e: A \to B$ is a $\ast$-homomorphism, and
  $$\psi_e(\langle x, y \rangle_p) = \psi_p(x)^*\psi_p(y)$$
  for all $x, y \in E_p$;
- $\psi_p(x)\psi_q(y) = \psi_{pq}(xy)$ for all $p, q \in P, x \in E_p, y \in E_q$.

For each $p \in P$ we write $\psi^{(p)}$ for the homomorphism $\mathcal{K}(E_p) \to B$ obtained by extending the map $\theta_{\xi, \eta} \mapsto \psi_p(\xi)\psi_p(\eta)^*$, where

$$\theta_{\xi, \eta}(\zeta) = \langle \eta, \zeta \rangle.$$  

The Toeplitz representation $\psi: E \to B$ is **Cuntz-Pimsner covariant** if $\psi^{(p)}(\phi_p(a)) = \psi_e(a)$ for all $p \in P$ and all $a \in A$ such that $\phi_p(a) \in \mathcal{K}(E_p)$.  

There is a $C^*$-algebra $\mathcal{T}_A(\mathcal{E})$ called the Toeplitz algebra of $\mathcal{E}$ and a representation $i_{\mathcal{E}}: \mathcal{E} \to \mathcal{T}_A(\mathcal{E})$ which is universal in the following sense: $\mathcal{T}_A(\mathcal{E})$ is generated by $i_{\mathcal{E}}(\mathcal{E})$ and for any representation $\psi: \mathcal{E} \to B$ there is a homomorphism $\psi_*: \mathcal{T}_A(\mathcal{E}) \to B$ such that $\psi_* \circ i_{\mathcal{E}} = \psi$.

There are various extra conditions on a product system $\mathcal{E} \to P$ and several other notions of covariance, which allow to define the Cuntz-Pimsner algebra $\mathcal{O}_A(\mathcal{E})$ or the Cuntz-Nica-Pimsner algebra $\mathcal{N}\mathcal{O}_A(\mathcal{E})$ satisfying certain properties, see [12, 23, 4, 1, ?] among others. We mention that $\mathcal{O}_A(\mathcal{E})$ (or $\mathcal{N}\mathcal{O}_A(\mathcal{E})$) comes with a covariant representation $j_{\mathcal{E}}: \mathcal{E} \to \mathcal{O}_A(\mathcal{E})$ and is universal in the following sense: $\mathcal{O}_A(\mathcal{E})$ is generated by $j_{\mathcal{E}}(\mathcal{E})$ and for any covariant representation $\psi: \mathcal{E} \to B$ there is a homomorphism $\psi_*: \mathcal{O}_A(\mathcal{E}) \to B$ such that $\psi_* \circ j_{\mathcal{E}} = \psi$.

Under certain conditions, $\mathcal{O}_A(\mathcal{E})$ satisfies a gauge invariant uniqueness theorem.

Example 2.3. For a product system $\mathcal{E} \to P$ with fibers $\mathcal{E}_p$ nonzero finitely dimensional Hilbert spaces, in particular $A = \mathcal{E}_e = \mathbb{C}$, let us fix an orthonormal basis $\mathcal{B}_p$ in $\mathcal{E}_p$. Then a Toeplitz representation $\psi: \mathcal{E} \to B$ gives rise to a family of isometries $\{\psi(\xi) : \xi \in \mathcal{B}_p\}_{p \in P}$ with mutually orthogonal range projections. In this case $\mathcal{T}(\mathcal{E}) = \mathcal{T}_c(\mathcal{E})$ is generated by a collection of Cuntz-Toeplitz algebras which interact according to the multiplication maps $\mathcal{M}_{p,q}$ in $\mathcal{E}$.

A representation $\psi: \mathcal{E} \to B$ is Cuntz-Pimsner covariant if

$$\sum_{\xi \in \mathcal{B}_p} \psi(\xi) \psi(\xi)^* = \psi(1)$$

for all $p \in P$. The Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{E}) = \mathcal{O}_c(\mathcal{E})$ is generated by a collection of Cuntz algebras. N. Fowler proved in [11] that if the function $p \mapsto \dim \mathcal{E}_p$ is injective, then the algebra $\mathcal{O}(\mathcal{E})$ is simple and purely infinite. For other examples of multidimensional Cuntz algebras, see [3].

Example 2.4. A row-finite $k$-graph with no sources $\Lambda$ (see [16]) determines a product system $\mathcal{E} \to \mathbb{N}^k$ with $\mathcal{E}_0 = A = C_0(\Lambda^0)$ and $\mathcal{E}_n = \overline{C_c(\Lambda^n)}$ for $n \neq 0$ such that we have a $\mathbb{T}^k$-equivariant isomorphism $\mathcal{O}_A(\mathcal{E}) \cong C^*(\Lambda)$. Recall that the universal property induces a gauge action on $\mathcal{O}_A(\mathcal{E})$ defined by $\gamma_z(\psi(\xi)) = z^n \psi(\xi)$ for $z \in \mathbb{T}^k$ and $\xi \in \mathcal{E}_n$.

The following two definitions and two results are taken from [7], see also [15].

Definition 2.5. An action $\beta$ of a locally compact group $G$ on a product system $\mathcal{E} \to P$ over $A$ is a family $(\beta^p)_{p \in P}$ such that $\beta^p$ is an action
of $G$ on each fiber $E_p$ compatible with the action $\alpha = \beta^G$ on $A$, and furthermore, the actions $(\beta^p)_{p \in P}$ are compatible with the multiplication maps $M_{p,q}$ in the sense that
\[
\beta^p_g(M_{p,q}(x \otimes y)) = M_{p,q}(\beta^p_g(x) \otimes \beta^q_g(y))
\]
for all $g \in G$, $x \in E_p$ and $y \in E_q$.

**Definition 2.6.** If $\beta$ is an action of $G$ on the product system $E \to P$, we define the crossed product $E \rtimes_{\beta} G$ as the product system indexed by $P$ with fibers $E_p \rtimes_{\beta^p} G$, which are $\mathcal{C}^*$-correspondences over $A \rtimes_{\alpha} G$.

For $\zeta \in C_c(G,E_p)$ and $\eta \in C_c(G,E_q)$, the product $\zeta \eta \in C_c(G,E_{pq})$ is defined by
\[
(\zeta \eta)(s) = \int_G M_{p,q}(\zeta(t) \otimes \beta^q_t(\eta(t^{-1}s)))dt.
\]

**Proposition 2.7.** The set $E \rtimes_{\beta} G = \bigsqcup_{p \in P} E_p \rtimes_{\beta^p} G$ with the above multiplication satisfies all the properties of a product system of $\mathcal{C}^*$-correspondences over $A \rtimes_{\alpha} G$.

**Proposition 2.8.** Suppose that a locally compact group $G$ acts on a row-finite and faithful product system $E$ indexed by $P = (\mathbb{N}^k, +)$ via automorphisms $\beta^p_g$. Then $G$ acts on the Cuntz-Pimsner algebra $\mathcal{O}_A(E)$ via automorphisms denoted by $\gamma_g$. Moreover, if $G$ is amenable, then $E \rtimes_{\beta} G$ is row-finite and faithful, and
\[
\mathcal{O}_A(E) \rtimes_{\gamma} G \cong \mathcal{O}_{A \rtimes_{\alpha} G}(E \rtimes_{\beta} G).
\]

Now we define the product system associated to $k$ representations of a compact group $G$. We limit ourselves to finite dimensional unitary representations, even though the definition makes sense in greater generality.

**Definition 2.9.** Given a compact group $G$ and $k$ finite dimensional unitary representations $\rho_i$ of $G$ on Hilbert spaces $H_i$ for $i = 1, \ldots, k$, we construct the product system $E = E(\rho_1, \ldots, \rho_k)$ indexed by the commutative monoid $(\mathbb{N}^k, +)$, with fibers
\[
E_n = H^n = H_1^\otimes n_1 \otimes \cdots \otimes H_k^\otimes n_k
\]
for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, in particular, $A = E_0 = \mathbb{C}$. The multiplication maps $M_{n,m} : E_n \times E_m \to E_{n+m}$ in $E$ are defined using repeatedly the standard isomorphisms $\rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i$ for all $i < j$. The associativity in $E$ follows from the fact that
\[
M_{n+m,p} \circ (M_{n,m} \times id) = M_{n,m+p} \circ (id \times M_{m,p})
\]
as maps from $\mathcal{E}_n \times \mathcal{E}_m \times \mathcal{E}_p$ to $\mathcal{E}_{n+m+p}$. Then $\mathcal{E} = \mathcal{E}(\rho_1, \ldots, \rho_k)$ is called the product system of the representations $\rho_1, \ldots, \rho_k$.

**Remark 2.10.** Similarly, a semigroup $P$ of unitary representations of a group $G$ would determine a product system $\mathcal{E} \to P$.

**Proposition 2.11.** With notation as in Definition 2.9, assume $d_i = \dim \mathcal{H}_i \geq 2$. Then the Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{E})$ associated to the product system $\mathcal{E} \to \mathbb{N}^k$ described above is isomorphic with the $C^*$-algebra of a rank $k$ graph $\Gamma$ with a single vertex and with $|\Gamma^{e_i}| = d_i$. This isomorphism is equivariant for the gauge action. Moreover,

$$\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_{d_1} \otimes \cdots \otimes \mathcal{O}_{d_k},$$

where $\mathcal{O}_n$ is the Cuntz algebra.

**Proof.** Indeed, by choosing a basis in each $\mathcal{H}_i$, we get the edges $\Gamma^{e_i}$ in a $k$-coloured graph $\Gamma$ with a single vertex. The isomorphisms $\rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i$ determine the factorization rules of the form $ef = fe$ for $e \in \Gamma^{e_i}$ and $f \in \Gamma^{e_j}$ which obviously satisfy the associativity condition. In particular, the corresponding isometries in $C^*(\Gamma)$ commute and $\mathcal{O}(\mathcal{E}) \cong C^*(\Gamma) \cong \mathcal{O}_{d_1} \otimes \cdots \otimes \mathcal{O}_{d_k}$, preserving the gauge action. \(\square\)

**Remark 2.12.** For $d_i \geq 2$, the $C^*$-algebra $\mathcal{O}(\mathcal{E}) \cong C^*(\Gamma)$ is always simple and purely infinite since it is a tensor product of simple and purely infinite $C^*$-algebras. If $d_i = 1$ for some $i$, then $\mathcal{O}(\mathcal{E})$ will contain a copy of $C(\mathbb{T})$, so it is not simple. Of course, if $d_i = 1$ for all $i$, then $\mathcal{O}(\mathcal{E}) \cong C(\mathbb{T}^k)$. For more on single vertex rank $k$ graphs, see [3, 6].

**Proposition 2.13.** The compact group $G$ acts on each fiber $\mathcal{E}_n$ of the product system $\mathcal{E}$ via the representation $\rho^n = \rho^1_{e_1} \otimes \cdots \otimes \rho^k_{e_k}$. This action is compatible with the multiplication maps and commutes with the gauge action of $\mathbb{T}^k$. The crossed product $\mathcal{E} \rtimes G$ becomes a row-finite and faithful product system indexed by $\mathbb{N}^k$ over the group $C^*$-algebra $C^*(G)$. Moreover,

$$\mathcal{O}(\mathcal{E}) \rtimes G \cong \mathcal{O}_{C^*(G)}(\mathcal{E} \rtimes G).$$

**Proof.** Indeed, for $g \in G$ and $\xi \in \mathcal{E}_n = \mathcal{H}^n$ we define $g \cdot \xi = \rho^n(\xi)$ and since $\rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i$, we have $g \cdot (\xi \otimes \eta) = g \cdot \xi \otimes g \cdot \eta$ for $\xi \in \mathcal{E}_n, \eta \in \mathcal{E}_m$. Clearly,

$$g \cdot \gamma_z(\xi) = g \cdot (z^n \xi) = z^n(g \cdot \xi) = \gamma_z(g \cdot \xi),$$

so the action of $G$ commutes with the gauge action. Using Proposition 2.7, $\mathcal{E} \rtimes G$ becomes a product system indexed by $\mathbb{N}^k$ over $C^*(G) \cong C \rtimes G$ with fibers $\mathcal{E}_n \rtimes G$. The isomorphism $\mathcal{O}(\mathcal{E}) \rtimes G \cong \mathcal{O}_{C^*(G)}(\mathcal{E} \rtimes G)$ follows from Proposition 2.8. \(\square\)
Corollary 2.14. Since the action of $G$ commutes with the gauge action, the group $G$ acts on the core algebra $F = \mathcal{O}(\mathcal{E})^{\mathbb{T}^k}$.

3. The Doplicher-Roberts algebra

The Doplicher-Roberts algebras $O_\rho$, denoted by $O_G$ in [8], were introduced to construct a new duality theory for compact Lie groups $G$ which strengthens the Tannaka-Krein duality. Here $\rho$ is the $n$-dimensional representation of $G$ defined by the inclusion $G \subseteq U(n)$ in some unitary group $U(n)$. Let $\mathcal{T}_G$ denote the representation category whose objects are tensor powers $\rho^p = \rho \otimes \rho$ for $p \geq 0$, and whose arrows are the intertwiners $\text{Hom}(\rho^p, \rho^q)$. The group $G$ acts via $\rho$ on the Cuntz algebra $O_n$ and $O_G = O_\rho$ is identified in [8] with the fixed point algebra $O_n^G$. If $\sigma$ denotes the restriction to $O_\rho$ of the canonical endomorphism of $O_n$, then $\mathcal{T}_G$ can be reconstructed from the pair $(O_\rho, \sigma)$. Subsequently, Doplicher-Roberts algebras were associated to any object $\rho$ in a strict tensor $C^\ast$-category, see [9].

Given finite dimensional unitary representations $\rho_1, \ldots, \rho_k$ of a compact group $G$ on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_k$ we will construct a Doplicher-Roberts algebra $O_{\rho_1, \ldots, \rho_k}$ from intertwiners

$$\text{Hom}(\rho^n, \rho^m) = \{ T \in \mathcal{L}(\mathcal{H}^n, \mathcal{H}^m) \mid T \rho^n(g) = \rho^m(g)T \quad \forall \ g \in G \},$$

where for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$ the representation $\rho^n = \rho_1^{\otimes n_1} \otimes \cdots \otimes \rho_k^{\otimes n_k}$ acts on $\mathcal{H}^n = \mathcal{H}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}_k^{\otimes n_k}$. Note that $\rho^0 = \iota$ is the trivial representation of $G$, acting on $\mathcal{H}_i^0 = \mathbb{C}$. This Doplicher-Roberts algebra will be a subalgebra of $O(\mathcal{E})$ for the product system $\mathcal{E}$ as in Definition 2.9.

Lemma 3.1. Consider

$$\mathcal{A}_0 = \bigcup_{m,n \in \mathbb{N}^k} \mathcal{L}(\mathcal{H}^n, \mathcal{H}^m).$$

Then the linear span of $\mathcal{A}_0$ becomes a $\ast$-algebra $\mathcal{A}$ with appropriate multiplication and involution. This algebra has a natural $\mathbb{Z}^k$-grading coming from a gauge action of $\mathbb{T}^k$. Moreover, the Cuntz-Pimsner algebra $O(\mathcal{E})$ of the product system $\mathcal{E} = \mathcal{E}(\rho_1, \ldots, \rho_k)$ is equivariantly isomorphic to the $C^\ast$-closure of $\mathcal{A}$ in the unique $C^\ast$-norm for which the gauge action is isometric.
Proof. Recall that the Cuntz algebra $O_n$ contains a canonical Hilbert space $H$ of dimension $n$ and it can be constructed as the closure of the linear span of $\bigcup_{p,q \in \mathbb{N}} \mathcal{L}(H^p, H^q)$ using embeddings

$$\mathcal{L}(H^p, H^q) \subseteq \mathcal{L}(H^{p+1}, H^{q+1}), \ T \mapsto T \otimes I$$

where $H^p = H^\otimes p$ and $I : H \to H$ is the identity map. This linear span becomes a $*$-algebra with a multiplication given by composition and an involution (see [8] and Proposition 2.5 in [18]).

Similarly, for all $r \in \mathbb{N}^k$, we consider embeddings $\mathcal{L}(H^n, H^m) \subseteq \mathcal{L}(H^{n+r}, H^{m+r})$ given by $T \mapsto T \otimes I_r$, where $I_r : H^r \to H^r$ is the identity map, and endow $A$ with a multiplication given by composition and an involution. More precisely, if $S \in \mathcal{L}(H^n, H^m)$ and $T \in \mathcal{L}(H^q, H^p)$, then the product $ST$ is

$$(S \otimes I_{p \vee m - n}) \circ (T \otimes I_{p \vee m - p}) \in \mathcal{L}(H^{p+q \vee n - p}, H^{m+p \vee n - n}),$$

where we write $p \vee n$ for the coordinatewise maximum. This multiplication is well defined in $A$ and is associative. The adjoint of $T \in \mathcal{L}(H^n, H^m)$ is $T^* \in \mathcal{L}(H^m, H^n)$.

There is a natural $\mathbb{Z}^k$-grading on $A$ given by the gauge action $\gamma$ of $\mathbb{T}^k$, where for $z = (z_1, \ldots, z_k) \in \mathbb{T}^k$ and $T \in \mathcal{L}(H^n, H^m)$ we define

$$\gamma_z(T)(\xi) = z_1^{m_1-n_1} \cdots z_k^{m_k-n_k} T(\xi).$$

Adapting the argument in Theorem 4.2 in [9] for $\mathbb{Z}^k$-graded $C^*$-algebras, the $C^*$-closure of $A$ in the unique $C^*$-norm for which $\gamma_z$ is isometric is well defined. The map

$$(T_1, \ldots, T_k) \mapsto T_1 \otimes \cdots \otimes T_k,$$

where

$$T_1 \otimes \cdots \otimes T_k : H^n \to H^m, \ (T_1 \otimes \cdots \otimes T_k)(\xi_1 \otimes \cdots \otimes \xi_k) = T_1(\xi_1) \otimes \cdots \otimes T_k(\xi_k) \text{ for } T_i \in \mathcal{L}(H_i^{m_i}, H_i^{m_i}) \text{ for } i = 1, \ldots, k$$

preserves the gauge action and it can be extended to an equivariant isomorphism from $O(E) \cong O_{d_1} \otimes \cdots \otimes O_{d_k}$ to the $C^*$-closure of $A$. Note that the closure of $\bigcup_{n \in \mathbb{N}^k} \mathcal{L}(H^n, H^n)$ is isomorphic to the core $F = O(E)^{\mathbb{Z}^k}$, the fixed point algebra under the gauge action, which is a UHF-algebra.

To define the Doplicher-Roberts algebra $O_{\rho_1, \ldots, \rho_k}$, we will again identify $Hom(\rho^n, \rho^m)$ with a subset of $Hom(\rho^{n+r}, \rho^{m+r})$ for each $r \in \mathbb{N}^k$, via $T \mapsto T \otimes I_r$. After this identification, it follows that the linear span $^0O_{\rho_1, \ldots, \rho_k}$ of $\bigcup_{m, n \in \mathbb{N}^k} Hom(\rho^n, \rho^m) \subseteq A_0$ has a natural multiplication.
and involution inherited from $\mathcal{A}$. Indeed, a computation shows that if $S \in \text{Hom}(\rho^n, \rho^m)$ and $T \in \text{Hom}(\rho^p, \rho^q)$, then $S^* \in \text{Hom}(\rho^m, \rho^n)$ and

$$(S \otimes I_{p\vee n-n}) \circ (T \otimes I_{p\vee n-p})\rho^{p+n-p}(g) =$$

$$= \rho^{m+p+n-p}(g) (S \otimes I_{p\vee n-n}) \circ (T \otimes I_{p\vee n-p}),$$

so $(S \otimes I_{p\vee n-n}) \circ (T \otimes I_{p\vee n-p}) \in \text{Hom}(\rho^{p+n-p}, \rho^{m+p+n-p})$ and $^0\mathcal{O}_{r_1,...,r_k}$ is closed under these operations. Since the action of $G$ commutes with the gauge action, there is a natural $\mathbb{Z}^k$-grading of $^0\mathcal{O}_{r_1,...,r_k}$ given by the gauge action $\gamma$ of $\mathbb{T}^k$ on $\mathcal{A}$.

It follows that the closure $^0\mathcal{O}_{r_1,...,r_k}$ of $^0\mathcal{O}_{r_1,...,r_k}$ in $\mathcal{O}(\mathcal{E})$ is well defined, obtaining the Doplicher-Roberts algebra associated to the representations $\rho_1,...,\rho_k$. This $C^*$-algebra also has a $\mathbb{Z}^k$-grading and a gauge action of $\mathbb{T}^k$. By construction, $^0\mathcal{O}_{r_1,...,r_k} \subseteq \mathcal{O}(\mathcal{E})$.

**Remark 3.2.** For a compact Lie group $G$, our Doplicher-Roberts algebra $^0\mathcal{O}_{r_1,...,r_k}$ is Morita equivalent with the higher rank Doplicher-Roberts algebra $\mathcal{D}$ in [11]. It is also the section $C^*$-algebra of a Fell bundle over $\mathbb{Z}^k$.

**Theorem 3.3.** Let $\rho_i$ be finite dimensional unitary representations of a compact group $G$ on Hilbert spaces $\mathcal{H}_i$ of dimensions $d_i \geq 2$ for $i = 1,...,k$. Then the Doplicher-Roberts algebra $^0\mathcal{O}_{r_1,...,r_k}$ is isomorphic to the fixed point algebra $\mathcal{O}(\mathcal{E})^G \cong (\mathcal{O}_{d_1} \otimes \cdots \otimes \mathcal{O}_{d_k})^G$, where $\mathcal{E} = \mathcal{E}(\rho_1,...,\rho_k)$ is the product system described in Definition 2.9.

**Proof.** We known from Lemma 3.1 that $\mathcal{O}(\mathcal{E})$ is isomorphic to the $C^*$-algebra generated by the linear span of $\mathcal{A}_0 = \bigcup_{m,n \in \mathbb{N}^k} \mathcal{L}(\mathcal{H}_n, \mathcal{H}_m)$. The group $G$ acts on $\mathcal{L}(\mathcal{H}_n, \mathcal{H}_m)$ by

$$(g \cdot T)(\xi) = \rho^n(g)T(\rho^m(g^{-1})\xi)$$

and the fixed point set is $\text{Hom}(\rho^n, \rho^m)$. Indeed, we have $g \cdot T = T$ if and only if $T\rho^n(g) = \rho^m(g)T$. This action is compatible with the embeddings and the operations, so it extends to the $*$-algebra $\mathcal{A}$ and the fixed point algebra is the linear span of $\bigcup_{m,n \in \mathbb{N}^k} \text{Hom}(\rho^n, \rho^m)$.

It follows that $^0\mathcal{O}_{r_1,...,r_k} \subseteq \mathcal{O}(\mathcal{E})^G$ and therefore its closure $^0\mathcal{O}_{r_1,...,r_k}$ is isomorphic to a subalgebra of $\mathcal{O}(\mathcal{E})^G$. For the other inclusion, any element in $\mathcal{O}(\mathcal{E})^G$ can be approximated with an element from $^0\mathcal{O}_{r_1,...,r_k}$, hence $^0\mathcal{O}_{r_1,...,r_k} = \mathcal{O}(\mathcal{E})^G$. \qed

**Remark 3.4.** By left tensoring with $I_r$ for $r \in \mathbb{N}^k$, we obtain some canonical unital endomorphisms $\sigma_r$ of $^0\mathcal{O}_{r_1,...,r_k}$. 


In the next section, we will show that in many cases, $O_{\rho_1,\ldots,\rho_k}$ is isomorphic to a corner of $C^*(\Lambda)$ for a rank $k$ graph $\Lambda$, so in some cases we can compute its $K$-theory. It would be nice to express the $K$-theory of $O_{\rho_1,\ldots,\rho_k}$ in terms of the endomorphisms $\pi \mapsto \pi \otimes \rho_i$ of the representation ring $R(G)$.

4. The rank $k$ graphs

For convenience, we first collect some facts about higher rank graphs, introduced in [16]. A rank $k$ graph or $k$-graph $(\Lambda, d)$ consists of a countable small category $\Lambda$ with range and source maps $r$ and $s$ together with a functor $d : \Lambda \to \mathbb{N}^k$ called the degree map, satisfying the factorization property: for every $\lambda \in \Lambda$ and all $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu \nu$ and $d(\mu) = m$, $d(\nu) = n$. For $n \in \mathbb{N}^k$ we write $\Lambda^n := d^{-1}(n)$ and call it the set of paths of degree $n$. The elements in $\Lambda^e$ are called edges and the elements in $\Lambda^0$ are called vertices.

A $k$-graph $\Lambda$ can be constructed from $\Lambda^0$ and from its $k$-coloured skeleton $\Lambda^e_1 \cup \cdots \cup \Lambda^e_k$ using a complete and associative collection of commuting squares or factorization rules, see [22].

The $k$-graph $\Lambda$ is row-finite if for all $n \in \mathbb{N}^k$ and all $v \in \Lambda^0$ the set $v\Lambda^n := \{ \lambda \in \Lambda^n : r(\lambda) = v \}$ is finite. It has no sources if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. A $k$-graph $\Lambda$ is said to be irreducible (or strongly connected) if, for every $u, v \in \Lambda^0$, there is $\lambda \in \Lambda$ such that $u = r(\lambda)$ and $v = s(\lambda)$.

Recall that $C^*(\Lambda)$ is the universal $C^*$-algebra generated by a family $\{S_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

- $\{S_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections,
- $S_{\lambda \mu} = S_\lambda S_\mu$ for all $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$,
- $S^*_\lambda S_\lambda = S_{s(\lambda)}$ for all $\lambda \in \Lambda$,
- for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ we have $S_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S^*_\lambda$.

A $k$-graph $\Lambda$ is said to satisfy the aperiodicity condition if for every vertex $v \in \Lambda^0$ there is an infinite path $x \in v\Lambda^\infty$ such that $\sigma^m x \neq \sigma^n x$ for all $m \neq n$ in $\mathbb{N}^k$, where $\sigma^m : \Lambda^\infty \to \Lambda^\infty$ are the shift maps. We say that $\Lambda$ is cofinal if for every $x \in \Lambda^\infty$ and $v \in \Lambda^0$ there is $\lambda \in \Lambda$ and $n \in \mathbb{N}^k$ such that $s(\lambda) = x(n)$ and $r(\lambda) = v$. 
Assume that \( \Lambda \) is row finite with no sources and that it satisfies the aperiodicity condition. Then \( C^*(\Lambda) \) is simple if and only if \( \Lambda \) is cofinal (see Proposition 4.8 in [16] and Theorem 3.4 in [20]).

We say that a path \( \mu \in \Lambda \) is a loop with an entrance if \( s(\mu) = r(\mu) \) and there exists \( \alpha \in s(\mu)\Lambda \) such that \( d(\mu) \geq d(\alpha) \) and there is no \( \beta \in \Lambda \) with \( \mu = \alpha\beta \). We say that every vertex connects to a loop with an entrance if for every \( v \in \Lambda^0 \) there is a loop with an entrance \( \mu \in \Lambda \) and a path \( \lambda \in \Lambda \) with \( r(\lambda) = v \) and \( s(\lambda) = r(\mu) = s(\mu) \). If \( \Lambda \) satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, then \( C^*(\Lambda) \) is purely infinite (see Proposition 4.9 in [16] and Proposition 8.8 in [21]).

Given finitely dimensional unitary representations \( \rho_i \) of a compact group \( G \) on Hilbert spaces \( \mathcal{H}_i \) for \( i = 1, \ldots, k \), we want to construct a rank \( k \) graph \( \Lambda = \Lambda(\rho_1, \ldots, \rho_k) \). Let \( R \) be the set of equivalence classes of irreducible summands \( \pi : G \to U(\mathcal{H}_\pi) \) which appear in the tensor powers \( \rho^n = \rho_1^\otimes m_1 \otimes \cdots \otimes \rho_k^\otimes m_k \) for \( n \in \mathbb{N}^k \) as in [19]. Take \( \Lambda^0 = R \) and for each \( i = 1, \ldots, k \) consider the set of edges \( \Lambda^{\varepsilon_i} \) which are uniquely determined by the matrices \( M_i \) with entries

\[
M_i(w, v) = |\{ e \in \Lambda^{\varepsilon_i} : s(e) = v, r(e) = w \}| = \dim \text{Hom}(v, w \otimes \rho_i),
\]

where \( v, w \in R \). The matrices \( M_i \) commute since \( \rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i \) and therefore

\[
\dim \text{Hom}(v, w \otimes \rho_i \otimes \rho_j) = \dim \text{Hom}(v, w \otimes \rho_j \otimes \rho_i)
\]

for all \( i < j \). This will allow us to fix some bijections

\[
\lambda_{ij} : \Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \to \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_i}
\]

for all \( 1 \leq i < j \leq k \), which will determine the commuting squares of \( \Lambda \). As usual,

\[
\Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} = \{(e, f) \in \Lambda^{\varepsilon_i} \times \Lambda^{\varepsilon_j} : s(e) = r(f)\}.
\]

For \( k \geq 3 \) we also need to verify that \( \lambda_{ij} \) can be chosen to satisfy the associativity condition, i.e.

\[
(id_{\ell} \times \lambda_{ij})(id_{\ell} \times \lambda_{ij})(id_{\ell} \times \lambda_{ij}) = (\lambda_{ij} \times id_{i})(id_{j} \times \lambda_{ij})(\lambda_{ij} \times id_{i})
\]

as bijections from \( \Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \) to \( \Lambda^{\varepsilon_j} \times_{\Lambda^0} \Lambda^{\varepsilon_i} \times_{\Lambda^0} \Lambda^{\varepsilon_j} \) for all \( i < j < \ell \).

Remark 4.1. Many times \( R = \hat{G} \), so \( \Lambda^0 = \hat{G} \), for example if \( \rho_i \) are faithful and \( \rho_i(G) \subseteq SU(\mathcal{H}_i) \) or if \( G \) is finite, \( \rho_i \) are faithful and \( \dim \rho_i \geq 2 \) for all \( i = 1, \ldots, k \), see Lemma 7.2 and Remark 7.4 in [17].
Proposition 4.2. Given representations $\rho_1, \ldots, \rho_k$ as above, assume that $\rho_i$ are faithful and that $R = \hat{G}$. Then each choice of bijections $\lambda_{ij}$ satisfying the associativity condition determines a rank $k$ graph $\Lambda$ which is cofinal and locally finite with no sources.

Proof. Indeed, the sets $\Lambda^{\varepsilon_i}$ are uniquely determined and the choice of bijections $\lambda_{ij}$ satisfying the associativity condition will be enough to determine $\Lambda$. Since the entries of the matrices $M_i$ are finite and there are no zero rows, the graph is locally finite with no sources. To prove that $\Lambda$ is cofinal, fix a vertex $v \in \Lambda^0$ and an infinite path $x \in \Lambda^\infty$. Arguing as in Lemma 7.2 in [17], any $w \in \Lambda^0$, in particular $w = x(n)$ for a fixed $n$ can be joined by a path to $v$, so there is $\lambda \in \Lambda$ with $s(\lambda) = x(n)$ and $r(\lambda) = v$. See also Lemma 3.1 in [19]. \hfill \Box

Remark 4.3. Note that the entry $M_i(w, v)$ is just the multiplicity of the irreducible representation $v$ in $w \otimes \rho_i$ for $i = 1, \ldots, k$. If $\rho_i^* = \rho_i$, the matrices $M_i$ are symmetric since

$$\dim \text{Hom}(v, w \otimes \rho_i) = \dim \text{Hom}(\rho_i^* \otimes v, w).$$

Here $\rho_i^*$ denotes the dual representation, defined by $\rho_i^*(g) = \rho_i(g^{-1})^t$, and equal in our case to the conjugate representation $\bar{\rho}_i$.

For $G$ finite, these matrices are finite, and the entries $M_i(w, v)$ can be computed using the character table of $G$. For $G$ infinite, the Clebsch-Gordan relations can be used to determine the numbers $M_i(w, v)$. Since the bijections $\lambda_{ij}$ in general are not unique, the rank $k$ graph $\Lambda$ is not unique, as illustrated in some examples. It is an open question how the $C^*$-algebra $C^*(\Lambda)$ depends in general on the factorization rules.

To relate the Doplicher-Roberts algebra $O_{\rho_1, \ldots, \rho_k}$ to a rank $k$ graph $\Lambda$, we mimic the construction in [19]. For each edge $e \in \Lambda^{\varepsilon_i}$, choose an isometric intertwiner

$$T_e : \mathcal{H}_{s(e)} \to \mathcal{H}_{r(e)} \otimes \mathcal{H}_i$$

in such a way that

$$\mathcal{H}_\pi \otimes \mathcal{H}_i = \bigoplus_{e \in \pi \Lambda^{\varepsilon_i}} T_e T_e^*(\mathcal{H}_\pi \otimes \mathcal{H}_i)$$

for all $\pi \in \Lambda^0$, i.e. the edges in $\Lambda^{\varepsilon_i}$ ending at $\pi$ give a specific decomposition of $\mathcal{H}_\pi \otimes \mathcal{H}_i$ into irreducibles. When $\dim \text{Hom}(s(e), r(e) \otimes \rho_i) \geq 2$ we must choose a basis of isometric intertwiners with orthogonal ranges, so in general $T_e$ is not unique. In fact, specific choices for the isometric intertwiners $T_e$ will determine the factorization rules in $\Lambda$ and whether they satisfy the associativity condition or not.
Given \( e \in \Lambda^{\varepsilon_i} \) and \( f \in \Lambda^{\varepsilon_j} \) with \( r(f) = s(e) \), we know how to multiply \( T_e \in \text{Hom}(s(e), r(e) \otimes \rho_i) \) with \( T_f \in \text{Hom}(s(f), r(f) \otimes \rho_j) \) in the algebra \( \mathcal{O}_{\rho_1, \ldots, \rho_k} \), by viewing \( \text{Hom}(s(e), r(e) \otimes \rho_i) \) as a subspace of \( \text{Hom}(\rho^m, \rho^n) \) for some \( m, n \) and similarly for \( \text{Hom}(s(f), r(f) \otimes \rho_j) \). We choose edges \( e' \in \Lambda^{\varepsilon_i}, f' \in \Lambda^{\varepsilon_j} \) with \( s(f) = s(e'), r(e) = r(f'), r(e') = s(f') \) such that \( T_e T_f = T_{e'} T_{f'} \), where \( T_{f'} \in \text{Hom}(s(f'), r(f') \otimes \rho_j) \) and \( T_{e'} \in \text{Hom}(s(e'), r(e') \otimes \rho_i) \). This is possible since

\[
T_e T_f = (T_e \otimes I_j) \circ T_f \in \text{Hom}(s(f), r(e) \otimes \rho_i \otimes \rho_j),
\]

\[
T_{f'} T_{e'} = (T_{f'} \otimes I_i) \circ T_{e'} \in \text{Hom}(s(e'), r(f') \otimes \rho_j \otimes \rho_i),
\]

and \( \rho_i \otimes \rho_j \cong \rho_j \otimes \rho_i \). In this case we declare that \( ef = f'e' \). Repeating this process, we obtain bijections \( \lambda_{ij} : \Lambda^{\varepsilon_i} \times \Lambda^0 \rightarrow \Lambda^{\varepsilon_j} \times \Lambda^0 \). Assuming that the associativity conditions are satisfied, we obtain a \( \Lambda \)-graph \( \Lambda \).

We write \( T_{ef} = T_e T_f = T_{f'} T_{e'} = T_{f'e'} \). A finite path \( \lambda \in \Lambda^n \) is a concatenation of edges and determines by composition a unique intertwiner

\[
T_\lambda : \mathcal{H}_{s(\lambda)} \rightarrow \mathcal{H}_{r(\lambda)} \otimes \mathcal{H}^n.
\]

Moreover, the paths \( \lambda \in \Lambda^n \) with \( r(\lambda) = \iota \), the trivial representation, provide an explicit decomposition of \( \mathcal{H}^n = \mathcal{H}^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}^{\otimes n_k} \) into irreducibles, hence

\[
\mathcal{H}^n = \bigoplus_{\lambda \in \Lambda^n} T_\lambda^* T_\lambda^* (\mathcal{H}^n).
\]

**Proposition 4.4.** Assuming that the choices of isometric intertwiners \( T_e \) as above determine a \( \Lambda \)-graph \( \Lambda \), then the family

\[
\{ T_\lambda^* T_\mu^* : \lambda \in \Lambda^m, \mu \in \Lambda^n, r(\lambda) = r(\mu) = \iota, s(\lambda) = s(\mu) \}
\]

is a basis for \( \text{Hom}(\rho^m, \rho^n) \) and each \( T_\lambda^* T_\mu^* \) is a partial isometry.

**Proof.** Each pair of paths \( \lambda, \mu \) with \( d(\lambda) = m, d(\mu) = n \) and \( r(\lambda) = r(\mu) = \iota \) determines a pair of irreducible summands \( T_\lambda(\mathcal{H}_{s(\lambda)}), T_\mu(\mathcal{H}_{s(\mu)}) \) of \( \mathcal{H}^m \) and \( \mathcal{H}^n \) respectively. By Schur’s lemma, the space of intertwiners of these representations is trivial unless \( s(\lambda) = s(\mu) \) in which case it is the one dimensional space spanned by \( T_\lambda T_\mu^* \). It follows that any element of \( \text{Hom}(\rho^m, \rho^n) \) can be uniquely represented as a linear combination of elements \( T_\lambda T_\mu^* \) where \( s(\lambda) = s(\mu) \). Since \( T_\mu \) is isometric, \( T_\mu^* \) is a partial isometry with range \( \mathcal{H}_{s(\mu)} \) and hence \( T_\lambda T_\mu^* \) is also a partial isometry whenever \( s(\lambda) = s(\mu) \). \( \square \)

**Theorem 4.5.** Consider \( \rho_1, \ldots, \rho_k \) finite dimensional unitary representations of a compact group \( G \) and let \( \Lambda \) be the \( k \)-coloured graph with \( \Lambda^0 = R \subseteq \hat{G} \) and edges \( \Lambda^{\varepsilon_i} \) determined by the incidence matrices \( M_i \)
defined above. Assume that the factorization rules determined by the choices of \( T_e \in \text{Hom}(s(e), r(e) \otimes \rho_i) \) for all edges \( e \in \Lambda^e \) satisfy the associativity condition, so \( \Lambda \) becomes a rank \( k \) graph. If we consider \( P \in C^*(\Lambda) \),

\[
P = \sum_{\lambda \in \Lambda(1, \ldots, 1)} S_{\lambda}S^*_\lambda,
\]

where \( \iota \) is the trivial representation, then there is a \( * \)-isomorphism of the Doplicher-Roberts algebra \( O_{\rho_1, \ldots, \rho_k} \) onto the corner \( PC^*(\Lambda)P \).

**Proof.** Since \( C^*(\Lambda) \) is generated by linear combinations of \( S_{\lambda}S^*_\mu \) with \( s(\lambda) = s(\mu) \) (see Lemma 3.1 in [16]), we first define the maps

\[
\phi_{n,m} : \text{Hom}(\rho^n, \rho^m) \to C^*(\Lambda), \quad \phi_{n,m}(T_{\lambda}T_{\mu}^*) = S_{\lambda}S^*_\mu
\]

where \( s(\lambda) = s(\mu) \) and \( r(\lambda) = r(\mu) = \iota \). Since \( S_{\lambda}S^*_\mu = PS_{\lambda}S^*_\mu P \), the maps \( \phi_{n,m} \) take values in \( PC^*(\Lambda)P \). We claim that for any \( r \in \mathbb{N}^k \) we have

\[
\phi_{n+r,m+r}(T_{\lambda}T_{\mu}^* \otimes I_r) = \phi_{n,m}(T_{\lambda}T_{\mu}^*).
\]

This is because

\[
\mathcal{H}_{s(\lambda)} \otimes \mathcal{H}^r = \bigoplus_{\nu \in s(\lambda)\Lambda^r} T_{\nu}T_{\nu}^*(\mathcal{H}_{s(\lambda)} \otimes \mathcal{H}^r),
\]

so that

\[
T_{\lambda}T_{\mu}^* \otimes I_r = \sum_{\nu \in s(\lambda)\Lambda^r} (T_{\lambda} \otimes I_r)(T_{\nu}T_{\nu}^*)(T_{\mu}^* \otimes I_r) = \sum_{\nu \in s(\lambda)\Lambda^r} T_{\lambda\nu}T_{\mu}^*
\]

and

\[
S_{\lambda}S^*_\mu = \sum_{\nu \in s(\lambda)\Lambda^r} S_{\lambda}(S_{\nu}S^*_\nu)S^*_\mu = \sum_{\nu \in s(\lambda)\Lambda^r} S_{\lambda\nu}S^*_\mu.
\]

The maps \( \phi_{n,m} \) determine a map \( \phi : \mathcal{O}_{\rho_1, \ldots, \rho_k} \to PC^*(\Lambda)P \) which is linear, \( * \)-preserving and multiplicative. Indeed,

\[
\phi_{n,m}(T_{\lambda}T_{\mu}^*) = (S_{\lambda}S^*_\mu)^* = S_{\mu}S^*_\lambda = \phi_{m,n}(T_{\mu}T_{\lambda}^*).
\]

Consider now \( T_{\lambda}T_{\mu}^* \in \text{Hom}(\rho^n, \rho^m) \), \( T_{\nu}T_{\omega}^* \in \text{Hom}(\rho^\theta, \rho^\nu) \) with \( s(\lambda) = s(\mu), s(\nu) = s(\omega), r(\lambda) = r(\mu) = r(\nu) = r(\omega) = \iota \). Since for all \( n \in \mathbb{N}^k \)

\[
\sum_{\lambda \in \Lambda^n} T_{\lambda}T_{\lambda}^* = I_n,
\]

we get

\[
T_{\mu}^*T_{\nu} = \begin{cases} T_{\beta}^* & \text{if } \mu = \nu\beta \\ T_{\alpha} & \text{if } \nu = \mu\alpha \\ 0 & \text{otherwise}, \end{cases}
\]
hence
\[ \phi((t_\lambda t_\mu^*)(t_\nu t_\omega^*)) = \begin{cases} 
\phi(t_\lambda t_\omega^*) &= S_\lambda s_\omega^* \text{ if } \mu = \nu \\
\phi(t_\lambda t_\omega^*) &= S_\lambda s_\omega^* \text{ if } \nu = \mu \\
0 & \text{otherwise.}
\end{cases} \]

On the other hand, from Lemma 3.1 in [16],
\[ S_\lambda s_\mu^* s_\nu s_\omega^* = \begin{cases} 
S_\lambda s_\omega^* \text{ if } \mu = \nu \\
S_\lambda s_\omega^* \text{ if } \nu = \mu \\
0 & \text{otherwise,}
\end{cases} \]

hence
\[ \phi((t_\lambda t_\mu^*)(t_\nu t_\omega^*)) = \phi(t_\lambda t_\mu^*) \phi(t_\nu t_\omega^*). \]

Since \( PS_\lambda s_\mu^* P = \phi_{\eta,m}(t_\lambda t_\mu^*) \) if \( r(\lambda) = r(\mu) = \iota \) and \( s(\lambda) = s(\mu) \), it follows that \( \phi \) is surjective. Injectivity follows from the fact that \( \phi \) is equivariant for the gauge action. \( \square \)

**Corollary 4.6.** If the \( k \)-graph \( \Lambda \) associated to \( \rho_1, \ldots, \rho_k \) is cofinal, it satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, then the Doplicher-Roberts algebra \( \mathcal{O}_{\rho_1, \ldots, \rho_k} \) is simple and purely infinite, and is Morita equivalent with \( C^*(\Lambda) \).

**Proof.** This follows from the fact that \( C^*(\Lambda) \) is simple and purely infinite and because \( PC^*(\Lambda)P \) is a full corner. \( \square \)

**Remark 4.7.** There is a groupoid \( \mathcal{G}_\Lambda \) associated to a row-finite rank \( k \) graph \( \Lambda \) with no sources, see [16]. By taking the pointed groupoid \( \mathcal{G}_\Lambda(\iota) \), the reduction to the set of infinite paths with range \( \iota \), under the same conditions as in Theorem 4.5 we get an isomorphism of the Doplicher-Roberts algebra \( \mathcal{O}_{\rho_1, \ldots, \rho_k} \) onto \( C^*(\mathcal{G}_\Lambda(\iota)) \).

5. Examples

**Example 5.1.** Let \( G = S_3 \) be the symmetric group with \( \hat{G} = \{ \iota, \epsilon, \sigma \} \) and character table

|       | (1) | (12) | (123) |
|-------|-----|------|------|
| \( \iota \) | 1   | 1    | 1    |
| \( \epsilon \) | 1   | -1   | 1    |
| \( \sigma \) | 2   | 0    | -1   |
Here \( \iota \) denotes the trivial representation, \( \epsilon \) is the sign representation and \( \sigma \) is an irreducible 2-dimensional representation, for example

\[
\sigma((12)) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \sigma((123)) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.
\]

By choosing \( \rho_1 = \sigma \) on \( H_1 = \mathbb{C}^2 \) and \( \rho_2 = \iota + \sigma \) on \( H_2 = \mathbb{C}^3 \), we get a product system \( E \to \mathbb{N}^2 \) and an action of \( S_3 \) on \( \mathcal{O}(E) \cong \mathcal{O}_2 \otimes \mathcal{O}_3 \) with fixed point algebra \( \mathcal{O}(E)^{S_3} \cong \mathcal{O}_{\rho_1, \rho_2} \) isomorphic to a corner of the \( \mathcal{C}^* \)-algebra of a rank 2 graph \( \Lambda \). The set of vertices is \( \Lambda^0 = \{ \iota, \epsilon, \sigma \} \) and the edges are given by the incidence matrices

\[
M_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.
\]

This is because

\[
\iota \otimes \rho_1 = \sigma, \quad \epsilon \otimes \rho_1 = \sigma, \quad \sigma \otimes \rho_1 = \iota + \epsilon + \sigma,
\]

\[
\iota \otimes \rho_2 = \iota + \sigma, \quad \epsilon \otimes \rho_2 = \epsilon + \sigma, \quad \sigma \otimes \rho_2 = \iota + \epsilon + 2\sigma.
\]

We label the blue edges by \( e_1, \ldots, e_5 \) and the red edges by \( f_1, \ldots, f_8 \) as in the figure.

The isometric intertwiners are

\[
T_{e_1} : H_\iota \to H_\sigma \otimes H_1, \quad T_{e_2} : H_\sigma \to H_\epsilon \otimes H_1, \quad T_{e_3} : H_\epsilon \to H_\sigma \otimes H_1,
\]

\[
T_{e_4} : H_\sigma \to H_\iota \otimes H_1, \quad T_{e_5} : H_\sigma \to H_\epsilon \otimes H_1,
\]

\[
T_{f_1} : H_\iota \to H_\iota \otimes H_2, \quad T_{f_2} : H_\epsilon \to H_\epsilon \otimes H_2, \quad T_{f_3} : H_\sigma \to H_\iota \otimes H_2,
\]

\[
T_{f_4} : H_\iota \to H_\sigma \otimes H_2, \quad T_{f_5} : H_\sigma \to H_\iota \otimes H_2, \quad T_{f_6} : H_\iota \to H_\sigma \otimes H_2,
\]

\[
T_{f_7}, T_{f_8} : H_\sigma \to H_\sigma \otimes H_2
\]

such that

\[
T_{e_1} T_{e_1}^* + T_{e_3} T_{e_3}^* + T_{e_5} T_{e_5}^* = I_\sigma \otimes I_1, \quad T_{e_2} T_{e_2}^* = I_\iota \otimes I_1, \quad T_{e_4} T_{e_4}^* = I_\iota \otimes I_1,
\]

\[
T_{f_1} T_{f_1}^* + T_{f_3} T_{f_3}^* = I_\iota \otimes I_2, \quad T_{f_2} T_{f_2}^* + T_{f_5} T_{f_5}^* = I_\iota \otimes I_2,
\]

\[
T_{f_4} T_{f_4}^* + T_{f_6} T_{f_6}^* + T_{f_7} T_{f_7}^* + T_{f_8} T_{f_8}^* = I_\sigma \otimes I_2.
\]
Here $I_\pi$ is the identity of $\mathcal{H}_\pi$ for $\pi \in \hat{G}$ and $I_i$ the identity of $\mathcal{H}_i$ for $i = 1, 2$. Since

$$M_1 M_2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

and

$$T_{e_2} T_{f_4}, T_{f_5} T_{e_1} \in \text{Hom}(\iota, \iota \otimes \rho_1 \otimes \rho_2),$$

$$T_{e_2} T_{f_6}, T_{f_7} T_{e_3} \in \text{Hom}(\iota, \iota \otimes \rho_1 \otimes \rho_2),$$

$$T_{e_2} T_{f_8}, T_{f_1} T_{e_2}, T_{f_3} T_{e_5} \in \text{Hom}(\sigma, \iota \otimes \rho_1 \otimes \rho_2),$$

$$T_{e_4} T_{f_1}, T_{f_5} T_{e_1} \in \text{Hom}(\iota, \epsilon \otimes \rho_1 \otimes \rho_2),$$

$$T_{e_4} T_{f_0}, T_{f_5} T_{e_3} \in \text{Hom}(\epsilon, \epsilon \otimes \rho_1 \otimes \rho_2),$$

$$T_{e_4} T_{f_7}, T_{e_4} T_{f_6}, T_{f_2} T_{e_4}, T_{f_5} T_{e_5} \in \text{Hom}(\sigma, \epsilon \otimes \rho_1 \otimes \rho_2),$$

$$T_{e_5} T_{f_1}, T_{e_5} T_{f_4}, T_{f_5} T_{e_1}, T_{f_8} T_{e_1} \in \text{Hom}(\iota, \sigma \otimes \rho_1 \otimes \rho_2),$$

$$T_{e_6} T_{f_2}, T_{e_5} T_{f_6}, T_{f_5} T_{e_3} \in \text{Hom}(\epsilon, \sigma \otimes \rho_1 \otimes \rho_2),$$

$$T_{e_5} T_{f_7}, T_{e_5} T_{f_8}, T_{e_3} T_{f_5}, T_{e_1} T_{f_3}, T_{f_6} T_{e_4}, T_{f_4} T_{e_2}, T_{f_7} T_{e_5}, T_{f_8} T_{e_5} \in \text{Hom}(\sigma, \sigma \otimes \rho_1 \otimes \rho_2),$$

a possible choice of commuting squares is

$$e_2 f_4 = f_3 e_1, \ e_2 f_6 = f_3 e_3, \ e_2 f_7 = f_1 e_2, \ e_2 f_8 = f_3 e_5, \ e_4 f_4 = f_5 e_1, \ e_4 f_6 = f_5 e_3$$

$$e_4 f_7 = f_2 e_4, \ e_4 f_8 = f_5 e_5, \ e_1 f_1 = f_7 e_1, \ e_5 f_4 = f_8 e_1, \ e_3 f_2 = f_7 e_3, \ e_5 f_6 = f_8 e_3,$$

$$e_5 f_7 = f_6 e_4, \ e_5 f_8 = f_4 e_2, \ e_3 f_5 = f_7 e_5, \ e_1 f_3 = f_8 e_5.$$  

This data is enough to determine a rank 2 graph $\Lambda$ associated to $\rho_1, \rho_2$.

But this is not the only choice, since for example we could have taken

$$e_2 f_4 = f_3 e_1, \ e_2 f_6 = f_3 e_3, \ e_2 f_7 = f_1 e_2, \ e_2 f_8 = f_3 e_5, \ e_4 f_4 = f_5 e_1, \ e_4 f_6 = f_5 e_3$$

$$e_4 f_8 = f_2 e_4, \ e_4 f_7 = f_5 e_5, \ e_1 f_1 = f_7 e_1, \ e_5 f_4 = f_8 e_1, \ e_3 f_2 = f_7 e_3, \ e_5 f_6 = f_8 e_3,$$

$$e_5 f_7 = f_6 e_4, \ e_5 f_8 = f_4 e_2, \ e_3 f_5 = f_7 e_5, \ e_1 f_3 = f_8 e_5,$$

which will determine a different 2-graph.

A direct analysis using the definitions shows that in each case, the 2-graph $\Lambda$ is cofinal, it satisfies the aperiodicity condition and every vertex connects to a loop with an entrance. It follows that $C^*(\Lambda)$ is simple and purely infinite and the Doplicher-Roberts algebra $O_{\rho_1, \rho_2}$ is Morita equivalent with $C^*(\Lambda)$.

The $K$-theory of $C^*(\Lambda)$ can be computed using Proposition 3.16 in [10] and it does not depend on the choice of factorization rules. We have

$$K_0(C^*(\Lambda)) \cong \ker[I - M_1^t - I - M_2^t] \oplus \ker \left[ \begin{array}{ccc} M_2^t - I \\ I - M_1^t \end{array} \right] \cong \mathbb{Z}/2\mathbb{Z},$$

$$K_1(C^*(\Lambda)) \cong \ker[I - M_1^t - I - M_2^t]/\im \left[ \begin{array}{ccc} M_2^t - I \\ I - M_1^t \end{array} \right] \cong 0.$$
In particular, $\mathcal{O}_{\rho_1,\rho_2} \cong \mathcal{O}_3$.

On the other hand, since $\rho_1, \rho_2$ are faithful, both $\mathcal{O}_{\rho_1}, \mathcal{O}_{\rho_2}$ are simple and purely infinite with

$$K_0(\mathcal{O}_{\rho_1}) \cong \mathbb{Z}/2\mathbb{Z}, \quad K_1(\mathcal{O}_{\rho_1}) \cong 0, \quad K_0(\mathcal{O}_{\rho_2}) \cong \mathbb{Z}, \quad K_1(\mathcal{O}_{\rho_2}) \cong \mathbb{Z},$$

so $\mathcal{O}_{\rho_1,\rho_2} \not\cong \mathcal{O}_{\rho_1} \otimes \mathcal{O}_{\rho_2}$.

\textbf{Example 5.2.} With $G = S_3$ and $\rho_1 = 2\iota, \rho_2 = \iota + \epsilon$, then $R = \{\iota, \epsilon\}$ so $\Lambda$ will have two vertices and incidence matrices

$$M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which give

Again, a corresponding choice of isometric intertwiners will determine some factorization rules, for example

$$e_1f_1 = f_1e_2, \quad e_2f_1 = f_1e_1, \quad e_1f_3 = f_3e_3, \quad e_2f_3 = f_3e_4,$$

$$e_3f_2 = f_2e_1, \quad e_4f_2 = f_2e_2, \quad e_3f_4 = f_4e_4, \quad e_4f_4 = f_4e_3.$$  

Even though $\rho_1, \rho_2$ are not faithful, the obtained 2-graph is cofinal, satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, so $\mathcal{O}_{\rho_1,\rho_2}$ is simple and purely infinite with trivial $K$-theory. In particular, $\mathcal{O}_{\rho_1,\rho_2} \cong \mathcal{O}_2$.

Note that since $\rho_1, \rho_2$ have kernel $N = \langle (123) \rangle \cong \mathbb{Z}/3\mathbb{Z}$, we could replace $G$ by $G/N \cong \mathbb{Z}/2\mathbb{Z}$ and consider $\rho_1, \rho_2$ as representations of $\mathbb{Z}/2\mathbb{Z}$.

\textbf{Example 5.3.} Consider $G = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ with $\hat{G} = \{\iota, \chi\}$ and character table

|    | 0 | 1 |
|----|---|---|
| $\iota$ | 1 | 1 |
| $\chi$ | 1 | −1 |
Choose the 2-dimensional representations
\[ \rho_1 = \iota + \chi, \quad \rho_2 = 2\iota, \quad \rho_3 = 2\chi, \]
which determine a product system \( \mathcal{E} \) such that \( \mathcal{O}(\mathcal{E}) \cong \mathcal{O}_2 \otimes \mathcal{O}_2 \otimes \mathcal{O}_2 \) and a Doplicher-Roberts algebra \( \mathcal{O}_{\rho_1,\rho_2,\rho_3} \cong \mathcal{O}(\mathcal{E})^\mathbb{Z}/2\mathbb{Z} \).

An easy computation shows that the incidence matrices of the blue, red and green graphs are
\[
M_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}.
\]

With labels as in the figure, we choose the following factorization rules
\[
e_1f_1 = f_2e_1, \quad e_1f_2 = f_1e_1, \quad e_2f_1 = f_4e_2, \quad e_2f_2 = f_3e_2,
\]
\[
e_3f_3 = f_2e_3, \quad e_3f_4 = f_1e_3, \quad e_4f_4 = f_3e_4, \quad e_4f_3 = f_4e_4,
\]
\[
f_1g_1 = g_2f_3, \quad f_1g_2 = g_1f_3, \quad f_2g_1 = g_2f_4, \quad f_2g_2 = g_1f_4,
\]
\[
f_3g_3 = g_4f_1, \quad f_3g_4 = g_3f_1, \quad f_4g_3 = g_4f_2, \quad f_4g_4 = g_3f_2,
\]
\[
e_1g_1 = g_2e_4, \quad e_1g_2 = g_1e_4, \quad e_2g_1 = g_3e_3, \quad e_2g_2 = g_4e_3,
\]
\[
e_3g_3 = g_1e_2, \quad e_3g_4 = g_2e_2, \quad e_4g_3 = g_4e_1, \quad e_4g_4 = g_3e_1.
\]

A tedious verification shows that all the following paths are well defined
\[
e_1f_1g_1, \quad e_1f_1g_2, \quad e_1f_2g_1, \quad e_1f_2g_2, \quad e_2f_1g_1, \quad e_2f_1g_2, \quad e_2f_2g_1, \quad e_2f_2g_2,
\]
\[
e_3f_3g_3, \quad e_3f_3g_4, \quad e_3f_4g_3, \quad e_3f_4g_4, \quad e_4f_3g_3, \quad e_4f_3g_4, \quad e_4f_4g_3, \quad e_4f_4g_4,
\]
so the associativity property is satisfied and we get a rank 3 graph \( \Lambda \) with 2 vertices. It is not difficult to check that \( \Lambda \) is cofinal, it satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, so \( C^*(\Lambda) \) is simple and purely infinite.

Since \( \partial_1 = [I - M_1^t I - M_2^t I - M_3^t] : \mathbb{Z}^6 \rightarrow \mathbb{Z}^2 \) is surjective, using Corollary 3.18 in [10], we obtain
\[
K_0(C^*(\Lambda)) \cong \ker \partial_2 / \im \partial_3 \cong 0, \quad K_1(C^*(\Lambda)) \cong \ker \partial_1 / \im \partial_2 \oplus \ker \partial_3 \cong 0,
\]
where
\[
\partial_2 = \begin{bmatrix}
M_2 - I & M_3 - I & 0 \\
I - M_1 & 0 & M_2 - I \\
0 & I - M_1 & I - M_2
\end{bmatrix}, \quad \partial_3 = \begin{bmatrix}
I - M_3 \\
M_2 - I \\
I - M_2
\end{bmatrix},
\]
in particular \(\mathcal{O}_{\rho_1,\rho_2,\rho_3} \cong \mathcal{O}_2\).

**Example 5.4.** Let \(G = \mathbb{T}\). We have \(\hat{G} = \{\chi_k : k \in \mathbb{Z}\}\), where \(\chi_k(z) = z^k\) and \(\chi_k \otimes \chi_\ell = \chi_{k+\ell}\). The faithful representations
\[
\rho_1 = \chi_{-1} + \chi_0, \quad \rho_2 = \chi_0 + \chi_1
\]
of \(\mathbb{T}\) will determine a product system \(\mathcal{E}\) with \(\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_2 \otimes \mathcal{O}_2\) and a Doplicher-Roberts algebra \(\mathcal{O}_{\rho_1,\rho_2} \cong \mathcal{O}(\mathcal{E})^\mathbb{T}\) isomorphic to a corner in the \(C^*\)-algebra of a rank 2 graph \(\Lambda\) with \(\Lambda^0 = \hat{G}\) and infinite incidence matrices, where
\[
M_1(\chi_k, \chi_\ell) = \begin{cases}
1 & \text{if } \ell = k \text{ or } \ell = k - 1 \\
0 & \text{otherwise},
\end{cases}
\]
\[
M_2(\chi_k, \chi_\ell) = \begin{cases}
1 & \text{if } \ell = k \text{ or } \ell = k + 1 \\
0 & \text{otherwise}.
\end{cases}
\]
The skeleton of \(\Lambda\) looks like
\[
\cdots \chi_{-1} \chi_0 \chi_1 \chi_2 \cdots
\]
and this 2-graph is cofinal, satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, so \(C^*(\Lambda)\) is simple and purely infinite.

**Example 5.5.** Let \(G = SU(2)\). It is known (see p.84 in [2]) that the elements in \(\hat{G}\) are labeled by \(V_n\) for \(n \geq 0\), where \(V_0 = \iota\) is the trivial representation on \(\mathbb{C}\), \(V_1\) is the standard representation of \(SU(2)\) on \(\mathbb{C}^2\), and for \(n \geq 2\), \(V_n = S^n V_1\), the \(n\)-th symmetric power. In fact, \(\dim V_n = n + 1\) and \(V_n\) can be taken as the representation of \(SU(2)\) on
the space of homogeneous polynomials $p$ of degree $n$ in variables $z_1, z_2$, where for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(2)$ we have

$$(g \cdot p)(z) = p(az_1 + cz_2, bz_1 + dz_2).$$

The irreducible representations $V_n$ satisfy the Clebsch-Gordan formula

$$V_k \otimes V_\ell = \bigoplus_{j=0}^{q} V_{k+\ell-2j}, \ q = \min\{k, l\}.$$

If we choose $\rho_1 = V_1, \rho_2 = V_2$, then we get a product system $\mathcal{E}$ with $\mathcal{O}(\mathcal{E}) \cong \mathcal{O}_2 \otimes \mathcal{O}_3$ and a Doplicher-Roberts algebra $\mathcal{O}_{\rho_1, \rho_2} \cong \mathcal{O}(\mathcal{E})^{SU(2)}$ isomorphic to a corner in the $C^*$-algebra of a rank 2 graph with $\Lambda^0 = \hat{G}$ and edges given by the matrices

$$M_1(V_k, V_\ell) = \begin{cases} 
1 & \text{if } k = 0 \text{ and } \ell = 1 \\
1 & \text{if } k \geq 1 \text{ and } \ell \in \{k - 1, k + 1\} \\
0 & \text{otherwise},
\end{cases}$$

$$M_2(V_k, V_\ell) = \begin{cases} 
1 & \text{if } k = 0 \text{ and } \ell = 2 \\
1 & \text{if } k = 1 \text{ and } \ell \in \{1, 3\} \\
1 & \text{if } k \geq 2 \text{ and } \ell \in \{k - 2, k, k + 2\} \\
0 & \text{otherwise}.
\end{cases}$$

The skeleton looks like

$$V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_4 \rightarrow V_5 \cdots$$

and this 2-graph is cofinal, satisfies the aperiodicity condition and every vertex connects to a loop with an entrance, in particular $\mathcal{O}_{\rho_1, \rho_2}$ is simple and purely infinite.

**References**

[1] S. Albandik, R. Meyer, *Product systems over Ore monoids*, Documenta Math. **20** (2015), 1331–1402.
[2] T. Bröcker, T. tom Dieck, *Representations of compact Lie groups*, Springer GTM 98 1985.
[3] B. Burgstaller, *Some multidimensional Cuntz algebras*, Aequationes Math. 76 (2008), no. 1-2, 19–32.
[4] T.M. Carlsen, N. Larsen, A. Sims, S.T. Vittadello, *Co-Universal Algebras Associated To Product Systems and Gauge-Invariant Uniqueness Theorems*, Proc. London Math. Soc. (3) 103 (2011), no. 4, 563–600.
[5] K. R. Davidson, D. Yang, *Periodicity in rank 2 graph algebras*, Canad. J. Math. Vol. 61 (6) 2009, 1239–1261.
[6] K. R. Davidson, D. Yang, *Representations of higher rank graph algebras*, New York J. Math. 15(2009), 169–198.
[7] V. Deaconu, L. Huang, A. Sims, *Group Actions on Product Systems and K-Theory*, work in progress.
[8] S. Doplicher, J.E. Roberts, *Duals of compact Lie groups realized in the Cuntz algebras and their actions on C*-algebras*, J. of Funct. Anal. 74 (1987) 96–120.
[9] S. Doplicher, J.E. Roberts, *A new duality theory for compact groups*, Invent. Math. 98 (1989) 157–218.
[10] D.G. Evans, *On the K-theory of higher rank graph C*-algebras*, New York J. Math. 14 (2008), 1–31.
[11] N. J. Fowler, *Discrete product systems of finite dimensional Hilbert spaces and generalized Cuntz algebras*, preprint 1999.
[12] N. J. Fowler, *Discrete product systems of Hilbert bimodules*, Pacific J. Math. 204 (2002), 335–375.
[13] N. J. Fowler, A. Sims, *Product systems over right-angled Artin semigroups*, Trans. Amer. Math. Soc. 354 (2002), 1487–1509.
[14] G. Hao, C.-K. Ng, *Crossed products of C*-correspondences by amenable group actions*, J. Math. Anal. Appl. 345 (2008), no. 2, 702–707.
[15] E. Katsoulis, *Product systems of C*-correspondences and Takai duality*, arXiv: 1911.12265.
[16] A. Kumjian, D. Pask, *Higher rank graph C*-algebras*, New York J. Math. 6(2000), 1–20.
[17] A. Kumjian, D. Pask, I. Raeburn, J. Renault, *Graphs, Groupoids and Cuntz-Krieger algebras*, J. of Funct. Anal. 144 (1997) No. 2, 505–541.
[18] T. Kajiwara, C. Pinzari, Y. Watatani, *Ideal structure and simplicity of the C*-algebras generated by Hilbert bimodules*, J. of Funct. Anal. 159 (1998), No. 2, 295–322.
[19] M. H. Mann, I. Raeburn, C.E. Sutherland, *Representations of finite groups and Cuntz-Krieger algebras*, Bull. Austral. Math. Soc. 46(1992), 225–243.
[20] D. Robertson and A. Sims, *Simplicity of C*-algebras associated to row-finite locally convex higher- rank graphs*, Israel J. Math. 172 (2009), 171–192.
[21] A. Sims, *Gauge-invariant ideals in C*-algebras of finitely aligned higher-rank graphs*, Canad. J. Math. vol. 58 (6) 2006, 1268–1290.
[22] A. Sims, *Lecture notes on higher-rank graphs and their C*-algebras*, 2010.
[23] A. Sims, T. Yeend, *C*-algebras associated to product systems of Hilbert bimodules*, J. Operator Theory, 64 no. 2 (Fall 2010), 349–376.