Localization of a pair of bound particles in a random potential

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Abstract

We study the localization length \( l_c \) of a pair of two attractively bound particles moving in a one-dimensional random potential. We show in which way it depends on the interaction potential between the constituents of this composite particle. For a pair with many bound states \( N \) the localization length is proportional to \( N \), independently of the form of the two particle interaction. For the case of two bound states we present an exact solution for the corresponding Fokker-Planck equation and demonstrate that \( l_c \) depends sensitively on the shape of the interaction potential and the symmetry of the bound state wave functions.

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1 Introduction

Within the single-parameter scaling hypothesis [1] the dimensionless conductance \( g(L) \) as a function of system size \( L \) determines the transport properties of noninteracting electrons in a random potential (for a review, see Kramer and MacKinnon [2]). In one-dimensional disordered systems all states are localized [3] and the conductance \( g(L) \) decreases exponentially [4] with increasing sample size \( L \). The localization length \( l_c \) can be derived from the asymptotic
behavior of the conductance $g(L)$ using the relation

$$l_c^{-1} = - \lim_{L \to \infty} \frac{d}{dL} \ln g(L).$$

(1)

The localization length $l_c$ and the Lyapunov exponent $\gamma \sim l_c^{-1}$ are self-averaging, i.e. non-random quantities [5] whereas the distribution function of the conductance is approximately log normal [6]. In the weak disorder limit characterized by a small Fermi wavelength compared to the scattering length $l$ the localization length $l_c$ is given by the mean free path $l$ for backward scattering [5].

In this paper we study the localization of a composite particle with internal degrees of freedom. As a model we consider a single pair of two particles in a weak one-dimensional white noise potential. We restrict our considerations to the limit of two tightly bound constituents so that the pair does not decay during its motion through the disorder potential. Considering the semiclassical limit the kinetic energy of the center of mass motion is assumed to be larger than the typical energy of the disorder potential. Typical length scales of our model include the mean free paths $l_b$ and $l_f$ of the pair for backward and forward scattering, respectively. The scattering processes cause a transfer of kinetic energy of the center of mass motion to the relative motion within the pair and vice versa. Therefore, the scattering lengths $l_b$ and $l_f$ depend on the structure of the pair, i.e. on the interaction potential between the constituents. Thus one expects that the pair localization length $l_c$ also depends on the shape of the attractive pair potential.

As another but different example for the interplay between interaction and disorder many recent papers study the coherent propagation of two interacting particles (TIP) in a one-dimensional random potential (see [9] and the literature cited there). It is worth mentioning that the TIP model considers the propagation of two repulsing or weakly attracting particles which form a pair due to the scattering by the disordered potential. The size of the pair is of the order of the one particle localization length $l$. As Shepelyansky [10] and Imry [11] argued, attractive as well as repulsive interaction between the particles cause a localization length of the pair which is much larger than the single particle localization length. In the present work, we study two particles with strong interaction which form bound states and have a size that is much smaller than the one particle localization length. For that reason the TIP model and the composite particle model studied in this work consider different limiting cases, which are not directly comparable. Moreover, in the mentioned TIP approach a lattice model of two particles interacting at one lattice site is considered. Generally the coherent motion of the interacting pair is studied only in the middle of the band ($E = 0$). However, we study a continuum model of two interacting particles in the semiclassical regime with a Fermi wavelength much smaller than the scattering length ($\lambda_F \ll l$).
A composite particle with \( N \) bound states may be considered as an \( N \)-channel problem for the center of mass motion. On the other hand the motion of a single electron in a thick wire is also a multichannel problem since the transversal motion is quantized. In the thick wire the scattering processes between different channels are random and of the same strength. The localization length \( l_c \) of the wire increases with the channel number \( N \) as \( l_c \sim N l \) because effectively only one of these channels gives rise to coherent backward scattering and produces localization [7,8]. In our composite particle model the scattering processes between different bound states are also random. However, in contrast to the model of a thick wire, the scattering probabilities depend strongly on the channel numbers. Therefore it is interesting to investigate the localization length in the composite particle model and compare the results for \( l_c \) of both different models.

The model of two particles with strong attractive interaction might be relevant for the Coulomb-correlated electron-hole pairs in disordered semiconductors. Recent work [12] has dealt with the dynamics of the electron-hole pair in a 1D model. Another physical problem which can be related to the model we consider below is the superconductor-isolator transition. In order to study the effects of interaction and disorder, Lages and Shepelyansky [13] investigated numerically the Cooper problem of two quasiparticles with attractive interaction above a frozen Fermi sea in the presence of disorder.

The basic ideas of the model we consider were introduced by Dorokhov [14]. He studied a pair of particles bound by a harmonic oscillator potential in a weak random disorder potential. For this very special interaction he found a dominant forward scattering in the case of many relevant bound states \( N \) \( (l_b/l_f \sim \ln N) \) employing the specific properties of the harmonic oscillator eigenfunctions. Moreover, the pair scattering length \( l_b \) was shown to be on the order of the single particle mean free path \( l \) and independent of the strength of the oscillator potential. Using the fact that the forward scattering is the dominant process the pair localization length \( l_c = N l_b / 2 \sim N l \) was calculated.

Our results show that Dorokhov’s conclusions are generic for any attractive interaction potential in the case of many bound states \( N \gg 1 \) regardless of its shape. We extend the method [14] from the harmonic oscillator potential to an arbitrary attractive interaction potential. This can be accomplished since the excited states of the relative motion within the pair can be described in the semiclassical approximation. We show that the specific properties of the harmonic oscillator wave functions (in [14] the result was obtained using the recurrence relations of the Hermitian polynomials) are of no importance in the multi-channel case. Using WKB wave functions for the bound states of the pair we estimate the scattering matrix elements and the mean free paths \( l_b \) and \( l_f \) in the case \( N \gg 1 \). We find that \( l_b \gg l_f \) for any interaction potential. We also show that \( l_b \) is qualitatively independent of the interaction. For the localization
length $l_c$ we obtain $l_c \sim Nl_b/2$ which implies that $l_c$ is also independent of the interaction.

In addition to the considerations for many bound states $N \gg 1$ we also investigate the first nontrivial case of a small number of bound states, namely $N = 2$. For a single particle in a random white noise potential the mean free paths $l_b$ and $l_f$ are equal. We show that already for a pair with only two bound states, the forward scattering is enhanced relative to the backward scattering and that the ratio $l_f/l_b$ depends sensitively on the energy. Finally we derive an exact expression for the localization length of the pair with two accessible bound states. In contrast to earlier treatments [7] of this problem we do not have to impose any additional restrictions to the scattering matrix elements. Therefore, our solution allows to study the influence of the two particle interaction on the localization directly. The result indicates that no simple relation exists between $l_c$ and the mean free path of the pair $l_b$ as it was in the limit of many channels $N \gg 1$. As an application of our general result we present numerical calculations showing that the localization length $l_c$ depends sensitively on the interaction potential, the symmetry of the bound state wave functions and the total energy of the pair.

The paper is organized as follows. In Section 2 we describe the model and briefly summarize the transfer matrix approach applied to the composite particle model with arbitrary interaction potential between the two constituents. In Section 3 we consider a pair with high enough total energy allowing for many bound states. We determine the scattering lengths $l_b$ and $l_f$ as well as the pair localization length $l_c$. In Section 4 we derive an exact expression for the localization length of a pair with two bound states ($N = 2$). Furthermore, we present numerical results for the localization length $l_c$ as well as for the mean free paths $l_b$ and $l_f$ for different interaction potentials.

### 2 Model and method of calculations

We consider two interacting particles in a one-dimensional random potential $V(x)$. The Hamiltonian is given by

$$\hat{H} = -\frac{1}{4} \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + u(y) + V(x + y/2) + V(x - y/2),$$

(2)

where the units are chosen in a way that $\hbar = 1$ and the single electron mass $m = 1$. The variable $x$ denotes the center of mass coordinate of the pair while $y$ represents the relative coordinate. The symmetric interaction potential $u(y)$ between the two particles is assumed to be attractive. The bound states $\phi_n(y)$
and the corresponding energy eigenvalues $\epsilon_n$ are given by
\[
\left[ -\frac{\partial^2}{\partial y^2} + u(y) \right] \phi_n(y) = \epsilon_n \phi_n(y). \tag{3}
\]

In the case of two identical particles the states $\phi_n(y)$ of even parity describe two bosons or a singlet state of fermions while the wave functions of odd parity correspond to a triplet state of fermions. The random potential $V(x)$ is nonzero in the finite interval $0 < x < L$. This region of length $L$ is enclosed by two ideal leads without any disorder.

The scattering of the center of mass of the pair with $N$ bound states can be mapped onto an $N$-channel problem of a single free particle. The stationary scattering states of the pair in the leads are given by
\[
\Psi(x, y) = \sum_{n=0}^{N-1} \phi_n(y) \left[ A_n \frac{\exp(ik_n x)}{\sqrt{k_n}} + B_n \frac{\exp(-ik_n x)}{\sqrt{k_n}} \right]. \tag{4}
\]

Here, the momentum of the center of mass is denoted by $k_n$. The total energy of the pair $E = k_n^2/4 + \epsilon_n$ determines the number of bound states (channels) $N$ which are included in the sum in Eq. (4). For a given total energy $E$ this number $N$ can be obtained from the relation $\epsilon_{N-1} < E < \epsilon_N$.

As far as the coefficients $A_n$ and $B_n$ are concerned one has to distinguish between the left lead and the right lead. They can be written as vectors $(\vec{A}^L, \vec{B}^L)$ and $(\vec{A}^R, \vec{B}^R)$ for the left and right one, respectively. The transfer matrix $\tau$ describes the scattering in the region between the two leads. It relates the states in the right lead to the states in the left lead [8]:
\[
\begin{pmatrix} \vec{A}^L \\ \vec{B}^L \end{pmatrix} = \tau \begin{pmatrix} \vec{A}^R \\ \vec{B}^R \end{pmatrix}. \tag{5}
\]

The transfer matrix $\tau$ is a $2N \times 2N$ symplectic matrix which can be expressed by the $N \times N$ reflection matrix $r$ and the transmission matrix $t$. These matrices describe the scattering between the channels $n$ and $m$ which correspond to the associated bound states of the pair. The dimensionless conductance $g$ can be expressed [15] in terms of the $N$ eigenvalues $T_n$ of the Hermitian matrix $t^\dagger t$ as $g = \sum T_n$. Therefore, it is convenient to employ the Hermitian matrix $M \equiv \tau^\dagger \tau$ which can be expressed by the polar decomposition as [7]
\[
M = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \cosh 2\Gamma & -\sinh 2\Gamma \\ -\sinh 2\Gamma & \cosh 2\Gamma \end{pmatrix} \begin{pmatrix} u^\dagger & 0 \\ 0 & u^T \end{pmatrix}. \tag{6}
\]

In this parameterization $u$ is a $N \times N$ unitary matrix and $\Gamma$ is a $N \times N$ diagonal matrix. The transmission eigenvalues $T_n$ are then related to the matrix...
elements $\Gamma_n$ by $T_n = 2 / \cosh(2\Gamma_n) + 1$.

To evaluate the transmission eigenvalues $T_n(L)$ for a given size $L$ we now consider the matrix $M$ as a function of the sample size $L$. As was shown in [14] the evolution of $M(L)$ with increasing $L$ can be described by a set of Langevin equations. The random potential $V(L)$ enters these equations as a multiplicative factor. In order to derive these equations the following assumptions were made:

- The potential $V(x)$ is a weak Gaussian white noise potential with the correlation function

$$\langle V(x)V(x') \rangle = 2D \delta(x-x'),$$

where $\langle \ldots \rangle$ denotes the ensemble average. The mean free path $l$ of the pair in comparison to the inverse wave numbers $k_n^{-1}$ is the appropriate quantity to describe the strength of the disorder. The semiclassical limit is considered where $k_n l \gg 1$. The mean free path $l$ of a single particle with Fermi energy $E_F$ is given by $l = E_F / D$.

- The mean free path $l$ is large compared to the typical width $b_n$ of the bound states $\phi_n(y)$: $l \gg b_n$. This assumption ensures the concept of mean free path for the center of mass motion.

The localization length $l_c$ of the pair is studied by the asymptotic behavior of the transmission eigenvalues $T_n(L)$ for large system sizes $L \to \infty$. These coefficients $T_n(L)$ decrease exponentially with the system size $L$. The corresponding length scales are defined by

$$l_n^{-1} \equiv - \lim_{L \to \infty} \frac{d}{dL} \langle \ln T_n \rangle.$$  

(8)

The transmission eigenvalues can be arranged in a hierarchical order $T_{N-1} \ll T_{N-2} \ll \cdots \ll T_0 \ll 1$. The localization length $l_c$ is then given by the largest length $l_0$. All lengths $l_n$ averaged over the $N$ channels just give the mean free path $l_b$ of the center of mass for backward scattering [7]:

$$\frac{1}{N} \sum_{n=0}^{N-1} l_n^{-1} = l_b^{-1}$$  

(9)

(see Eq. (14) below for the backward scattering length). Taking into account the exponential smallness of all transmission eigenvalues and their different order of magnitude, the Langevin equations for the matrix elements $\Gamma_n$ and $u_{nm}$ are given by
\[
\frac{\partial \Gamma_n}{\partial L} = i (\beta^* u - \beta u)_{nm} \cdot V(L)
\]
\[
\frac{\partial u_{nm}}{\partial L} = -2i \sum_{l=0}^{N-1} [\alpha_{nl} u_{lm} + u_{nl}(\beta^* u)_{lm} \Theta(l - m)
+ u_{nl}(\beta u)_{lm} \Theta(m - l)] \cdot V(L)
\]

where the abbreviation \( \beta_u \) stands for \( \beta_u \equiv u^\dagger \beta u^* \). The value of the step function \( \Theta(k) \) for \( k = 0 \) is defined as \( \Theta(0) \equiv 1/2 \). The matrix elements \( \alpha_{nm} \) and \( \beta_{nm} \) in Eq. (11) describe the forward scattering and the backward scattering of the pair, respectively. They depend on the two-particle interaction and can be written as

\[
\alpha_{nm} = \frac{e^{i(k_n - k_m)L}}{\sqrt{k_n k_m}} W_{nm}(k_n - k_m),
\]
\[
\beta_{nm} = \frac{e^{-i(k_n + k_m)L}}{\sqrt{k_n k_m}} W_{nm}(k_n + k_m),
\]

where \( W_{nm}(k) \) is

\[
W_{nm}(k) \equiv 2 \int_{-\infty}^{\infty} dy \phi_n(y) \phi_m(y) \cos \left( \frac{ky}{2} \right).
\]

Within the Born approximation, the mean free paths \( l_{b,f} \) of the pair for backward and forward scattering are given by [14]:

\[
l_{b}^{-1} = \frac{8D}{N} \sum_{n,m} |\beta_{nm}|^2 \quad \text{and} \quad l_{f}^{-1} = \frac{8D}{N} \sum_{n,m} |\alpha_{nm}|^2.
\]

Note that one has to consider bound states of a given parity only. Due to the finite width of the pair we therefore expect that the forward scattering is enhanced relative to the backward scattering although the scattering of a single particle is isotropic. This circumstance allows to solve the many channel problem [14] with \( N \gg 1 \) (see section 3).

Using the fact that the transport is effectively determined by just one channel (\( T_0 \gg T_{n>0} \)) Eq. (8) results in the following expression for the pair localization length [14]

\[
l_{c}^{-1} = 16D \sum_{n,m=0}^{N-1} \langle |u_{n0}|^2 |u_{m0}|^2 \rangle (1 - \frac{1}{2} \delta_{nm}) |\beta_{nm}|^2.
\]

Hence, the localization length \( l_c \) depends on the correlation function \( \langle |u_{n0}|^2 |u_{m0}|^2 \rangle \) which has to be evaluated using the nonlinear Langevin Eqs. (11).
For $N = 1$ the unitary matrix $u$ is simply a phase factor with $|u_{00}| = 1$. The localization length $l_c$ is then given by the mean free path $l_b$ for backward scattering

$$l_c^{-1} = l_b^{-1} = 8D|\beta_{00}|^2$$

and the interaction potential $u(y)$ enters via the ground state wave function in the matrixelement (13). In the next section we consider the multi-channel case $N \gg 1$ and show that the localization length depends only weakly on the interaction.

### 3 Multi-channel localization length

In order to evaluate the localization length (15) in the multi-channel case $N \gg 1$ one has to compute the correlation function $\langle |u_{n0}|^2 |u_{m0}|^2 \rangle$ using the nonlinear Langevin equations (11). For the harmonic oscillator interaction it was shown in [14] that the forward scattering dominates in Eq. (11). Therefore the nonlinear terms due to the backward scattering processes were neglected in (11) and it was found that the correlation function is properly described by the invariant unitary ensemble

$$\langle |u_{n0}|^2 |u_{m0}|^2 \rangle = (1 + \delta_{nm})/N^2.$$  

(17)

In the following we proof that the same result is valid for any attractive interaction potential. The order of magnitude of forward and backward scattering processes is determined by the corresponding mean free paths $l_f$ and $l_b$ given by (14). These mean free paths depend on the pair interaction potential $u(y)$ via the bound state wave functions $\phi_n(y)$ in Eq. (13). We calculate the matrix elements $W_{nm}(k)$ using WKB wave functions and evaluate the integral (13) in the saddle point approximation. This approximation is justified by the semiclassical approximation and the assumption $N \gg 1$, which implies that the vast majority of bound states is properly described by the WKB approximation. In this approximation the following expression can be derived from (13)

$$W_{nm}(k) = 4 (-1)^{(n+m)} \left( \frac{2\pi}{\mathcal{T}_n \mathcal{T}_m |u'(y_0) \cdot k|} \right)^{1/2} \times \cos \left[ \pm \frac{1}{2} \int_{k_{2,1}}^{k_0} \frac{d \tilde{k} y_0(\tilde{k}) - \frac{\pi}{4}}{k} \right].$$

(18)

The time $T_n$ denotes the period of the semiclassical motion of the bound state $n$. The two saddle points $\pm y_0$ are functions of $k$ and follow from the implicit
Fig. 1. Comparison of the exact function $W_{22}(k)$ and the calculated WKB estimate in the case of the harmonic oscillator interaction. The WKB approximation is excellent already for small quantum numbers in the wave function (in this example $\phi_{n=2}$) as long as $k$ lies in the oscillating regime. The WKB approximation clearly breaks down at $k_1\xi = 0$ and $k_2\xi = 12$.

In Eq. (18) $u'(y_0)$ stands for the derivative of the potential evaluated at position $y_0$. It results from the Gaussian integral at the saddle point. Calculating the matrix elements $W_{nm}(k)$ for backward or forward scattering we set $k = k_n \pm k_m$. In this relation as well as in Eq. (18) the upper sign and $k_2$ characterize the backward scattering while the lower sign and $k_1$ represent the forward scattering. The expression (18) is valid within the interval $k_1 < |k| < k_2$ where Eq. (19) has a real solution $y_0(k)$. The limits of the interval are given by

$$k_{2,1} = 2(\sqrt{\epsilon_n - u_0} \pm \sqrt{\epsilon_m - u_0})$$

with the potential minimum $u_0 \equiv u(y = 0)$. Outside this $k$ region the saddle point is shifted to the complex $y$-plane and the matrix element $W_{nm}(k)$ decreases exponentially [17]. Calculating the mean free paths (14) we will neglect these exponentially small contributions.

To demonstrate the accuracy of the WKB wave functions together with the saddle point approximation we show an example for a matrix element in Fig. 1. We have calculated the exact function $W_{22}(k)$ for the harmonic oscillator potential $u_{osc}(y) = \xi^{-2}(y/\xi)^2$ using the analytical expression for the third symmetric bound state $\phi_2^s(y)$. This exact result is compared with our WKB result (18). One clearly sees that the oscillatory part is very well described by the approximation. Of course, close to $k = k_{1,2}$ the saddle point approximation fails because $u'(y_0)$ goes to zero and the Gaussian approximation is not valid. Furthermore, the diagonal matrix elements $W_{nn}$ cannot be calculated from Eq. (18) for $k = 0$. However, definition (13) immediately yields $W_{nn}(k = 0) = 2$.

The scattering lengths are then evaluated by replacing the sum in (14) by an
integral (with $k_n \to k$ and $k_m \to \tilde{k}$). We obtain

$$l_{b,f}^{-1} = \frac{8D}{4\pi N} \int \frac{dk \, d\tilde{k}}{(k \pm k) \cdot |u'(y_0)|}. \tag{21}$$

This result may be understood as follows. For $k^2 = (k_n \pm k_m)^2$ the saddle point equation (19) can be rewritten as

$$u(y_0) = E - \frac{k_n^2}{4} - \frac{k_m^2}{4} = \epsilon_n - \frac{k_n^2}{4} = \epsilon_m - \frac{k_m^2}{4}. \tag{22}$$

This means that the kinetic energy of the relative motion $\epsilon_n - u(y_0)$ for the initial channel $n$ transforms into the kinetic energy $\frac{k_m^2}{4}$ of the center of mass in the final channel $m$ and vice versa. The transition rate into the final channel $m$ is therefore proportional to the probability density $\rho_n(k_m)$ of finding the pair in the initial channel $n$ with the appropriate relative momentum. The expression (21) for the mean free paths follows then immediately from $\rho_n(k_m) = \frac{4}{(T_n |u'(y_0)|)}$.

We will now show that Eq. (21) implies a mean free path for the backward scattering that is almost independent of the channel number $N$. Using Eq. (22) we write $u'(y_0)$ formally as

$$u'(y_0) = F(u(y_0)) = F\left( E - \frac{k_n^2}{4} - \frac{\tilde{k}_n^2}{4} \right). \tag{23}$$

This is always a unique representation if the interaction potential is assumed to be monotonic. Introducing polar coordinates the angular integration in (21) can be performed and we obtain the expression for the backward scattering length

$$l_b^{-1} = \sqrt{2 \ln \left[ \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right]} \int_0^1 \frac{dt \, (1 - t)^{-1/2} \, [F(Et)]^{-1}}{1 \int_0^1 dt \, (1 - t)^{1/2} \, [F(Et)]^{-1}} l^{-1}. \tag{24}$$

The denominator in (24) results from the number $N$ of channels with given parity below the energy $E$. Within the WKB approximation the relation between the number $N$ and the energy $E$ can be expressed by using expression (23) once more:

$$N(E) = \frac{1}{2\pi} \int dy \sqrt{E - u(y)} = \frac{1}{\pi} E^{3/2} \int_0^1 dt \, (1 - t^2)^{1/2} \, [F(Et)]^{-1}. \tag{25}$$

From Eq. (24) follows that $l_b \sim l$ with a prefactor of order one. This prefactor depends only weakly on the interaction potential $u(y)$ and the number of
accessible bound states $N$. To demonstrate this we consider two examples of interaction potentials.

In the case of a scale invariant potential with $u(y) \sim y^\nu$ and $F(u) \sim u^{(\nu-1)/\nu}$ one immediately realizes that $l_b/l$ is completely independent of the energy as it was the case for the harmonic oscillator ($\nu = 2$). From (24) follows

$$l_b^{-1} = l^{-1/4} \sqrt{2} \ln \left[ \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right] \cdot \left( \frac{1}{2} + \frac{1}{\nu} \right). \quad \text{(26)}$$

This expression includes the result for the harmonic oscillator presented in [14] which was found by calculating the matrix elements $W_{nm}$ exactly. For any interaction potential $u(y)$ with many bound states we expect only a weak dependence of $l_b$ on the energy $E$ or on the number of channels $N$ which are below this pair breaking energy. As a further example we give the result for the Pöschel-Teller interaction [16] $u_{\text{PT}}(y) = \xi^{-2}(a^2 - a(a + 1)/\cosh^2[y/\xi])$ where $a$ is an integer parameter. In contrast to the scale invariant potentials this interaction has a maximal number $N_{\text{max}}$ of bound states given by the parameter $a$. The pair breaking energy is $\sim \xi^{-2}a^2$. The semiclassical limit can be achieved for a large parameter $a$ implying $N_{\text{max}} \gg 1$. The backward scattering length (24) can then be written as

$$l_b^{-1} = l^{-1/4} \sqrt{2} \ln \left[ \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right] \cdot \frac{1}{1 - (N/N_{\text{max}})^2}. \quad \text{(27)}$$

If the total energy of the pair is much smaller than the pair breaking energy then $N \ll N_{\text{max}}$ follows and again the ratio $l_b/l$ is only very weakly energy dependent.

Using the same approach for the forward scattering length $l_f$ one realizes that the main contribution in Eq. (21) is due to the scattering between adjacent channels with $k \approx \tilde{k}$. However, the saddle point approximation fails for $k = \tilde{k}$ and the integral in (21) diverges. Therefore, one has to exclude a small strip of order $|k - \tilde{k}| \sim 1/N$ in the integral and calculate the in-channel scattering separately. We find in accordance with the earlier result for the harmonic oscillator interaction [14] that

$$l_f^{-1} \sim l^{-1} \ln N \quad \text{(28)}$$

for $N \gg 1$. From this estimate (28) follows $l_f \ll l_b$ for the multi-channel case. This means that the forward scattering is dominant and the relation (17) can be applied for any interaction potential $u(y)$. The Eqs. (24, 28) are thus the first major result of this work.
Using the correlation function (17) we then obtain the localization length

\[ l_c = \frac{N}{2} l_b \]  

(29)

from expression (15). Thus we find a delocalization proportional to the number of open channels \( N \) compared to the one particle localization length which equals \( l \). This shows that the localization effect is qualitatively independent of the shape of the attractive interaction potential \( u(y) \). The interaction only gives rise to a weak energy dependence and a numerical prefactor in the backward scattering length of the pair \( l_b \) compared to the single particle scattering length \( l \).

Our result (29) has the same structure as in the different model of a thick wire [8]. For that case the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation describes the evolution of the distribution function for the transmission eigenvalues \( T_n \) (or \( \Gamma_n \)) with increasing length of the wire. Compared to our model an important simplification there is the isotropy assumption for the scattering between the channels on average. Using this assumption the Fokker-Planck equation for the transmission eigenvalues \( T_n \) decouples from the distribution function of the matrix elements \( u_{nm} \). At least in the absence of time-reversal symmetry the DMPK equation could be solved exactly [18]. In our model the dominance of the forward scattering processes is the important property which yields the correlation function (17) and leads with (15) to the final result (29).

### 4 Two channel system

As yet another application of the general result (15) we now derive the exact solution for the case of two open channels \( N = 2 \). Here, the matrix \( u_{nm} \) is a \( 2 \times 2 \) unitary matrix. Thus only the expression \( \zeta \equiv |u_{00}|^2 - |u_{10}|^2 \) occurs in Eq. (15) for the localization length. One can rewrite (15) in terms of this variable and obtains

\[ l_c^{-1} = 8D \left[ (\beta^+ - |\beta_{10}|^2)(\zeta^2) + 2\beta^- \langle \zeta \rangle + (\beta^+ + |\beta_{10}|^2) \right] \]  

(30)

with \( \beta^\pm = (|\beta_{00}|^2 \pm |\beta_{11}|^2)/4 \). This reduces the ensemble average over combinations of matrix elements \( u_{nm} \) to the moments \( \langle \zeta \rangle \) and \( \langle \zeta^2 \rangle \). In order to find the stationary distribution function \( P(\zeta) \) we use the parameterization \( \zeta = \cos 2\theta \) and

\[
\begin{pmatrix}
\cos \theta - \sin \theta \\
\sin \theta \cos \theta
\end{pmatrix}
\begin{pmatrix}
e^{-i\phi_0} & 0 \\
0 & e^{-i\phi_1}
\end{pmatrix}
\begin{pmatrix}
e^{-i\gamma} & 0 \\
0 & e^{i\gamma}
\end{pmatrix}
\]

(31)

for the unitary matrix \( u_{nm} \). The stationary probability distribution can then be obtained from the associated Fokker-Planck equation. Analyzing this equa-
We find that the exponentials $e^{i\phi_{0,1}}$ and $e^{i\gamma}$ always occur together with the matrix elements $\alpha$ and $\beta$, respectively. If one neglects the rapidly oscillating contributions in this Fokker-Planck equation then only those terms which contain the magnitudes $|\alpha_{nm}|^2$ and $|\beta_{nm}|^2$ remain. For that reason, the stationary probability distribution is independent of the phases $\phi_{1,2}$ and $\gamma$. The solution $P(\zeta)$ of the stationary Fokker-Planck equation follows from the first order differential equation

$$0 = \left[ \frac{\partial}{\partial \zeta} (1 + A \zeta^2) + B \right] P(\zeta)$$

with

$$A \equiv \frac{|\beta_{10}|^2 - \beta^+}{|\alpha_{10}|^2 + \beta^+} \quad \text{and} \quad B \equiv \frac{2\beta^-}{|\alpha_{10}|^2 + \beta^+}. \quad (33)$$

It is of the type $P(\zeta) \sim \exp[-B \int^\zeta dx (1 + Ax^2)^{-1}]/(1 + A \zeta^2)$. Thus we arrive at the following expression for the two channel localization length:

$$l_c^{-1} = 8E l^{-1} (|\alpha_{10}|^2 + \beta^+) \left[ 1 + \frac{|\beta_{10}|^2 + \beta^+}{|\alpha_{10}|^2 + \beta^+} - \frac{|\alpha_{10}|^2 + |\beta_{10}|^2}{|\alpha_{10}|^2 + \beta^+} P^+ \right]$$

with $P^+ \equiv |P(1) + P(-1)|$. The length scale $l$ is the mean free path of a single particle having the same energy as the pair, i.e. $E = 2E_F$. The probabilities $P(\pm 1)$ for $\zeta = \pm 1$ can easily be obtained from the solution of Eq. (32). It yields

$$P^+ = \frac{B}{1 + A} \coth \left[ \frac{B}{\sqrt{A}} \arctan \sqrt{A} \right] \quad \text{for} \quad A > 0$$

$$P^+ = \frac{B}{1 + A} \coth \left[ \frac{B}{\sqrt{-A}} \arctanh \sqrt{-A} \right] \quad \text{for} \quad A < 0.$$ 

Expression (34) provides a general solution for the localization length in the two channel regime without any further restrictions concerning the matrix elements $\alpha_{nm}$ and $\beta_{nm}$. It is therefore the main result regarding the two channel case.

Some special cases of (34) may be of interest. For isotropic backward scattering $|\beta_{00}|^2 = |\beta_{11}|^2 = 2|\beta_{10}|^2$ follows $A = 0$ and $B = 0$ in (32) and the distribution function is independent of the forward scattering: $P(\zeta) = 1/2$. This case corresponds to the invariant distribution of an ensemble of unitary $2 \times 2$ matrices. If the two channels are equivalent, so that $|\beta_{00}|^2 = |\beta_{11}|^2$ and thus $B = 0$, the distribution function $P(\zeta) \sim (1 + A \zeta^2)^{-1}$ follows from (32).
The two-channel case $N = 2$: The two interaction potentials under consideration are $u_{\text{osc}}(y)$ (thick solid line) and $u_{\text{box}}(y)$ (thick dashed line). The width of the box and the absolute position of the oscillator potential have been chosen so that the energy levels $\epsilon_{0,1}$ (thin lines) of the first two symmetric bound states coincide.

The special case where $|\beta_{00}|^2 = |\beta_{11}|^2 = |\beta_{10}|^2$ was already discussed in [7] and is correctly reproduced by our general solution.

We complete our discussion of the $N = 2$ case with the presentation of some numerical results for two specific examples of interaction potentials. Using the result (34) one can analyze the influence of the interaction potential $u(y)$ on the localization length $l_c$. The requirement of exactly two open channels restricts the energy range to $\epsilon_1 < E < \epsilon_2$ where $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ are the energies corresponding to the bound states with given parity. In order to demonstrate how the localization length depends on the structure of the pair, we present a comparison between the harmonic oscillator interaction $u_{\text{osc}}(y)$ and the interaction where the two particles are bound together by a box-like potential $u_{\text{box}}(y)$. While $u_{\text{osc}}(y)$ is a smooth function of the relative coordinate $y$ the second interaction potential $u_{\text{box}}(y)$ is constant for $|y|$ being smaller than some fixed value $y_s$ and jumps to infinity outside this region, see Fig. 2.

We first discuss the case of bound states with even parity. The size of the box and a total energy shift for the harmonic oscillator potential were adjusted so that the energy levels $\epsilon_{0,1}$ entering (34) are at the same position. This approach ensures that all direct energy dependencies of the localization length have the same structure for either interaction type. The energy window for the $N = 2$ case is then given by $4.5 < E\xi^2 < 8.5$ ($4.5 < E\xi^2 < 12.5$) for the harmonic oscillator (box-like potential), Fig. 2.

The next step is to evaluate the matrix elements $\alpha$ and $\beta$ as given by (12) and (13). Inserting these matrix elements in (34) gives the result for the localization length $l_c(E)$ which is presented in Fig. 3. The interaction between the constituents of the composed particle clearly causes a delocalization compared to the single particle with no internal degrees of freedom. The reason is that the finite size of the pair effectively smoothes out the disorder. However, this delocalization effect sensitively depends on the total energy $E$ and the type
Fig. 3. Two channel case $N = 2$: The pair localization length $l_c$ compared to the single particle localization length $l$ as a function of the dimensionless energy $E\xi^2$ (even parity of the bound state wave functions $\phi_n$).

Fig. 4. Two channel case $N = 2$: The ratio between forward scattering length $l_f$ and backward scattering length $l_b$ as a function of the dimensionless energy $E\xi^2$ (even parity of the wave functions $\phi_n$).

of the two particle interaction. The resonance in the case of the box-like potential is clearly more pronounced. If the energy is close to this resonance, i.e. $E\xi^2 \approx 6.6$, the delocalization effect differs by a factor $\sim 5$ due to the different interaction potentials.

Furthermore, we have examined the ratio between forward and backward scattering $l_f/l_b$ as a function of the energy. As Fig. 4 indicates, the forward scattering relative to the backward scattering is already for just two channels enhanced. Although the ratio $l_f/l_b$ depends on the energy this effect holds true for the entire energy window accessible for the two channel case. The suppression of the backward scattering is because the finite size of the pair reduces scattering processes with large momentum transfer as can be concluded from Eqs. (12, 13). This effect is even more pronounced if many channels are open, a fact that is of importance for the solution of the multi-channel case, as discussed in Sec. 3.

Using the sum rule (9) we calculate the length scale $l_1$ of the transmission eigenvalue of the second internal channel. It is given by $l_1 = (2l_b^{-1} - l_c^{-1})^{-1}$, see Fig. 5. We find that the ratio $l_1/l_c$ is smaller than 0.6 over the entire energy range for either interaction potential. This justifies the assumption that the
transmission is mainly due to one internal channel if the system is large, i.e. 
$L \gg l$, because of the exponential dependence of the transmission amplitudes 
on the system size.

Finally we compute the localization length for the situation where the two 
bound states are of odd parity. Again, the parameters of the interaction poten-
tials are fixed such that the energies of the two involved energy levels coin-
cide for the different interaction potentials. The resulting localization length 
is plotted in Fig. 6. There is a significant difference between the localization 
lengths in the case of wave functions with even parity compared to odd parity. 
For the harmonic oscillator potential there is a sharp resonance near the lower 
limit $\epsilon_1$ of the energy window in the second case (Fig. 6) while the first case 
shows a smooth behavior with a weak resonance in the middle of the energy 
range (Fig. 3). For the $u_{\text{box}}(y)$ interaction the resonance position and height 
also changes significantly. Since the energy levels are shifted to the same po-
sition for either case the differences result only from the different structure 
of the bound state wave functions. The largest delocalization effect occurs for 
$u_{\text{box}}(y)$ for the states of odd parity and is about $l_c/l \approx 25$ at the resonance.
5 Conclusion

In this work we have studied the localization of a pair of two bound particles in a one-dimensional system with weak disorder. The interaction potential within this composite particle is fully taken into account within the semiclassical approximation. Our results are based on a method introduced by Dorokhov for two particles bound by a harmonic oscillator potential. We have generalized the method from this special interaction potential to an arbitrary attractive interaction. This allows us to analyze the dependence of the pair localization on the interaction. We find an enhancement of the pair localization length \( l_c \) in comparison to the single particle mean free path \( l \). This enhancement is independent of the form of the pair interaction potential in the limit of many bound states \( N \gg 1 \). It is given by \( l_c/l \sim N \) as can be seen from the central result (29) in conjunction with (24). Furthermore we derived an exact solution for the two channel case \( N = 2 \). For a bound pair with \( N = 1, 2 \) we observe a sensitive dependence of the localization length (16), (34) on the shape of the interaction potential, the parity of the involved bound states and the kinetic energy of the pair. We expect that such a behavior is typical for bound pair with a small number of bound states.

The approach described in this work is valid for two identical particles. Nevertheless, it is possible to extend the method without difficulties to a pair of different particles (e.g. electron-hole pair). In this case, the expression for the matrix elements (13) has to be modified taking into account that the random potential and the mass may be different for the two particles. However, the described method cannot be extended to a repulsing interaction because the assumption that the pair size is smaller than the mean free path breaks down.

References

[1] E. Abrahams, P.W. Anderson, D.C. Licciardello and T.V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979)
[2] B. Kramer and A. MacKinnon, Rep. Prog. Phys. 56, 1469 (1993)
[3] N.F. Mott and W.D. Twose, Adv. Phys. 10, 107 (1961)
[4] P.W. Anderson, D.J. Thouless, E. Abrahams and D.S. Fisher, Phys. Rev. B 22, 3519 (1980)
[5] I.M. Lifshitz, S.A. Gredeskul and L.A. Pastur, Introduction to the theory of disordered systems, (Wiley, New York, 1988)
[6] A.A. Abrikosov, Solid State Comm. 37, 997 (1981)
[7] O.N. Dorokhov, Zh. Eksp. Teor. Fiz. 85, 1040 [Sov. Phys. JETP 58, 606] (1983)
[8] C.W.J. Beenakker, Rev. Mod. Phys. 69, 731 (1997)
[9] R.A. Römer, M. Schreiber and T. Vojta, Phys. Stat. Sol. (b) 211, 681 (1999); O. Halfpap, Ann. Phys. (Leipzig) 10, 623 (2001)
[10] D.L. Shepelyansky, Phys. Rev. Lett. 73, 2607 (1994)
[11] Y. Imry, Europhys. Lett. 30, 405 (1995)
[12] D. Brinkmann, J.E. Golub, S.W. Koch, P. Thomas, K. Maschke and I. Varga, Eur. Phys. J. B 10, 145, (1999)
[13] J. Lages and D. Shepelyansky, Phys. Rev. B 62, 8665, (2000); J. Lages and D. Shepelyansky, Eur. Phys. J. B 21, 129, (2001)
[14] O.N. Dorokhov, Zh. Eksp. Teor. Fiz. 98, 646 [Sov. Phys. JETP 71, 360] (1990)
[15] Y. Imry, Introduction to mesoscopic physics, (Oxford University Press, 1997)
[16] G. Junker, Supersymmetric Methods in Quantum and Statistical Physics, (Springer Verlag Berlin Heidelberg, 1996)
[17] L.D. Landau and E.M. Lifschitz, Quantenmechanik, (Akademie-Verlag Berlin, 1988)
[18] C.W.J. Beenakker and B. Rejaci, Phys. Rev. B 49, 7499, (1994)