FROM MONGE-AMPERE EQUATIONS TO ENVELOPES AND GEODESIC RAYS IN THE ZERO TEMPERATURE LIMIT

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Abstract. Let $(X, \theta)$ be a compact complex manifold $X$ equipped with a smooth (but not necessarily positive) closed $(1, 1)$–form $\theta$. By a well-known envelope construction this data determines a canonical $\theta$–psh function $u_\theta$ which, in the case when the cohomology class $[\theta]$ is Kähler is in the Hölder space $C^{1,\alpha}$ for any $\alpha \in (0,1)$ (but, typically, $u_\theta$ is not $C^2$–smooth). We introduce a family $u_\beta$ of regularizations of $u_\theta$, parametrized by a positive number $\beta$, where $u_\beta$ is defined as the unique smooth solution of a complex Monge-Ampère equation of Aubin-Yau type. It is shown that, as $\beta \to \infty$, the functions $u_\beta$ converge to the envelope $u_\theta$ uniformly on $X$ in the strongest possible Hölder sense. More generally, a generalization of this result to the case of a nef and big cohomology class is obtained. Application to the regularization problem for geodesic rays in the space of Kähler metrics are given. As briefly explained there is a statistical mechanical motivation for this regularization procedure, where $\beta$ appears as the inverse temperature. This point of view also leads to an interpretation of $u_\beta$ as a “transcendental” Bergman metric.

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1. Introduction

Let $X$ be a compact complex manifold equipped with a smooth closed $(1, 1)$–form $\theta$ on $X$ and denote by $[\theta]$ the corresponding class in the cohomology group $H^{1,1}(X, \mathbb{R})$. There is a range of positivity notions for such cohomology classes, generalizing the classical positivity notions in algebraic geometry. The algebro-geometric situation concerns the special case when $X$ is projective variety and the cohomology class in question has integral periods, which equivalently means that the class may be realized as the first Chern class $c_1(L)$ of a line bundle $L$ over $X$ [19, 20, 21].
Accordingly, general cohomology classes in $H^{1,1}(X, \mathbb{R})$ are some times referred to as transcendentals classes and the corresponding notions of positivity may be formulated in terms of the convex subspace of positive currents in the cohomology class \(-\) the strongest notion of positivity is that of a Kähler class, which means that the class contains a Kähler metric, i.e. a smooth positive form (see [21] for equivalent numerical characterizations of positivity). In general, once the reference element $\theta$ in the cohomology class in question has been fixed the subspace of positive forms may be identified (mod $R$) with the space $PSH(X, \theta)$ of all $\theta$--plurisubharmonic function ($\theta$--psh, for short), i.e. all integrable upper semi-continuous functions $u$ on $X$ such that
\[
\theta + dd^c u \geq 0, \quad dd^c := i\partial \bar{\partial}
\]
holds in the sense of currents (in the integral case the space $PSH(X, \theta)$ may be identified with the space of all singular positively curved metrics on the corresponding line bundle $L$). When the class $[\theta]$ is pseudo-effective, i.e. it contains a positive current, there is a canonical element in $PSH(X, \theta)$ defined as the following envelope:
\[
u_\theta(x) := \sup\{u(x): u \leq 0, \; u \in PSH(X, \theta)\},
\]
defining a $\theta$--plurisubharmonic function with minimal singularities in the sense of Demailly [19, 17].

In this paper we introduce a natural family of regularizations $u_\beta$ of the envelope $u_\theta$, indexed by a positive real parameter $\beta$, where $u_\beta$ is determined by an auxiliary choice of volume form $dV$. To be a bit more precise, the functions $u_\beta$ will be defined as solutions to certain complex Monge-Ampère equations, parametrized by $\beta$.

Several motivations for studying the functions $u_\beta$ and their asymptotics as $\beta \to \infty$, will be given below. For the moment we just mention that $u_\beta$ can, in a certain sense, be considered as a “transcendental” analog of the Bergman metric for a high power of a line bundle $L$ over $X$ and moreover from a statistical mechanical point of view the limit $\beta \to 0$ appears as a zero-temperature limit.

In order to introduce the precise setting and the main results we start with the simplest case of a Kähler class $[\theta]$. First note that the envelope construction above can be seen as a generalization of the process of replacing the graph of a given smooth functions with its convex hull. By this analogy it is already clear from the one-dimensional case that $u_\theta$ will almost never by $C^2$--smooth even if the class $[\theta]$ is Kähler (unless $\theta$ is semi-positive, so that $u_\theta = 0$). However, by the results in [15] the complex Hessian of the function $u_\theta$ is always locally bounded and in particular $u_\theta$ is in the Hölder space $C^{1,\alpha}(X)$ for any $\alpha \in [0, 1]$ (see also [2] for a slightly more precise result in the case of a class with integral periods). Fixing a volume form $dV$ we consider, for $\beta$ a fixed positive number, the following complex Monge-Ampère equations for a smooth function $u_\beta$ :
\[
(\theta + dd^c u_\beta)^n = e^{\beta u_\beta} dV
\]
By the seminal results of Aubin [11] and Yau [47] there exists indeed a unique smooth solution $u_\beta$ to the previous equation. In fact, any smooth solution is automatically $\theta$--psh and the form $\omega_\beta := \theta + dd^c u_\beta$ defines a Kähler metric in $[\theta]$.

**Theorem 1.1.** Let $\theta$ be a smooth $(1, 1)$--form on a compact complex manifold $X$ such that $[\theta]$ is a Kähler class. Denote by $u_\theta$ the corresponding $\theta$--psh envelope and by $u_\beta$ the unique smooth solution of the complex Monge-Ampère equations [14].
determined by \( \theta \) and a fixed volume form \( dV \) on \( X \). Then, as \( \beta \rightarrow \infty \), the functions \( u_\beta \) converge to \( u_\theta \) in \( C^{1,\alpha}(X) \) for any \( \alpha \in ]0,1[ \).

More generally, the proof reveals that the result remains valid if \( dV \) is replaced by any \( C^2 \)-bounded family \( dV_\beta \) of volume forms (in the sense that \( \log(dV_\beta/dV_1) \) is a family of \( C^2 \)-bounded functions). As a consequence the convergence result above admits the following geometric formulation: let \( \omega_\beta \) be a family of Kähler metrics in \([\theta]\) satisfying the following asymptotic twisted Kähler-Einstein equation:

\[
\text{Ric} \omega_\beta = -\beta(\omega_\beta + \theta) + O(1),
\]

where \( \text{Ric} \omega_\beta \) denotes the form representing the Ricci curvature of the Kähler metric \( \omega_\beta \) and \( O(1) \) denotes a family of forms on \( X \) which is bounded as \( \beta \rightarrow \infty \). Then the previous theorem says that \( \omega_\beta \) is uniformly bounded and converges to \( \theta + dd^c u_\theta \) in the sense of currents and the normalized potentials of \( \omega_\beta \) converge in \( C^{1,\alpha}(X) \) to \( u_\theta \).

More generally, we will consider the case when the cohomology class \([\theta]\) is merely assumed to be big; this is the most general setting where complex Monge-Ampère equations of the form make sense \cite{17}. The main new feature in this general setting is the presence of \( -\infty \)– singularities of all \( \theta \)--psh functions on \( X \). Such singularities are, in general, inevitable for cohomological reasons. Still, by the results in \cite{17}, the corresponding complex Monge-Ampère equations admit a unique \( \theta \)--psh function \( u_\beta \) with minimal singularities; in particular its singularities can only appear along a certain complex subvariety of \( X \), determined by the class \([\theta]\), whose complement is called the Kähler locus \( \Omega \) of \([\theta]\) (or the ample locus) introduced in \cite{10} (which in the algebro-geometric setting corresponds to the complement of the augmented base locus of the corresponding line bundle). Moreover, in the case when the class \([\theta]\) is also assumed to be nef the solution \( u_\beta \) is known to be smooth on \( \Omega \), as follows from the results in \cite{17}. In this general setting our main result may be formulated as follows:

**Theorem 1.2.** Let \( \theta \) be a smooth \((1,1)\)--form on a compact complex manifold \( X \) such that \([\theta]\) is a big class. Then, as \( \beta \rightarrow \infty \), the functions \( u_\beta \) converge to \( u_\theta \) uniformly, in the sense that \( \|u_\beta - u_\theta\|_{L^\infty(X)} \rightarrow 0 \). Moreover, if the class \([\theta]\) is also assumed to be nef, then the convergence holds in \( C^{1,\alpha}_{loc}(\Omega) \) on the Kähler locus \( \Omega \) of \( X \).

Some remarks are in order. First of all, as pointed out above, it was previously known that the norm \( \|u_\beta - u_\theta\|_{L^\infty(X)} \) is finite for any fixed \( \beta \) (since \( u_\beta \) and the envelope \( u_\theta \) both have minimal singularities) and the thrust of the first statement in the previous theorem is thus that the norm in fact tends to zero. This global uniform convergence on is considerably stronger than a a local uniform convergence on \( \Omega \). Secondly, it should be stressed that, as shown in \cite{15}, the complex Hessian of the envelope \( u_\theta \) is locally bounded on \( \Omega \) for any big class \([\theta]\) and hence it seems natural to expect that the local convergence on \( \Omega \) in the previous theorem always holds in the \( C^{1,\alpha}_{loc}(\Omega) \)--topology, regardless of the nef assumption. However, the smoothness on \( \Omega \) of solutions of complex Monge-Ampère equations of the form \cite{17} is an open problem; in fact, it even seems to be unknown whether there always exists a \( \theta \)--psh functions with minimal singularities, which is smooth on \( \Omega \). On the other hand, for special big classes \([\theta]\), namely those which admit an appropriate Zariski decomposition on some resolution of \( X \), the regularity and convergence problem
can be reduced to the nef case (in the line bundle case this situation appears if the corresponding section ring is finitely generated). Let us also point out that in the case of a semi-positive and big class $\theta$ complex Monge-Ampère equations of the form \cite{[12]} were studied in \cite{[22]} using viscosity techniques and it was shown that the corresponding solution is continuous on all of $X$. In particular, by letting $\beta \to 0$ the global continuity on $X$ was also obtained for $\beta = 0$ (even for degenerate volume forms).

1.1. Degenerations induced by a divisor and applications to geodesic rays. Note that in the case of a Kähler class and when $\theta$ is positive, i.e. $\theta$ is Kähler form, it follows immediately from the definition that $u_\theta = 0$ and in this case the convergence in Theorem 1.1 holds in the $C^\infty$-sense, as is well-known (see section 2.3). However, as shown in \cite{[35],[33]} in the integral case $[\omega] = c_1(L)$, a non-trivial variant of the previous envelopes naturally appear in the geometric context of test configurations for the polarized manifold $(X, L)$, i.e. $\mathbb{C}^*$-equivariant deformations of $(X, L)$ and they can be used to construct *geodesic rays* in the space of all Kähler metrics in $[\omega]$. Such test configurations were introduced by Donaldson in his algebro-geometric definition of K-stability of a polarized manifold $(X, L)$, which according to the the Yau-Tian-Donaldson is equivalent to the existence of a Kähler metric in the class $c_1(L)$ with constant scalar curvature. Briefly, K-stability of $(X, L)$ amounts to the negativity of the so called Donaldson-Futaki invariant for all test configurations, which in turn is closely related to the large time asymptotics of Mabuchi’s K-energy functional along the corresponding geodesic rays (see \cite{[31]} and references therein).

Let us briefly explain how this fits into the present setup in the special case of the test configurations defined by the deformation to the normal cone of a divisor $Z$ in $X$ (e.g. a smooth complex hypersurface in $X$). First we consider the following complex Monge-Ampère equations degenerating along the divisor $Z$,

\[(\omega - \lambda \theta_L + dd^c u)^n = e^{\beta u} ||s||^{2\lambda \beta} dV,\]

where we have realized $Z$ as the zero-locus of a holomorphic section $s$ of a line bundle $L$ over $X$ equipped with a fixed Hermitian metric $||\cdot||$ with curvature form $\theta_L$ and where $\lambda \in [0, \infty]$ is an additional fixed parameter. As is well-known, for $\lambda$ sufficiently small ($\lambda \leq \epsilon$) there is, for any $\beta > 0$, a unique continuous $\omega - \lambda \theta_L$-psh solution $u_{\beta, \lambda}$ to the previous equation, which is smooth on $X - Z$. We will show that, when $\beta \to \infty$, the solutions $u_{\beta, \lambda}$ converge in $C^{1,\alpha}(X)$ to a variant of the envelope $u_\theta$, that we will (abusing notation slightly) denote by $u_\lambda$:

\[u_\lambda(x) := \sup \{ u(x) : \ u \leq -\lambda \log ||s||^2 \ \ u \in PSH(X, \omega - \lambda \theta_L) \}\]

(see section 3) It may identified with the envelopes with prescribed singularities introduced in \cite{[2]} in the context of Bergman kernel asymptotics for holomorphic sections vanishing to high order along a given divisor (see \cite{[33]} for detailed regularity results for such envelopes and \cite{[22]} for related asymptotic results in the toric case).

Remarkably, as shown in \cite{[35],[33]} (in the line bundle case) taking the Legendre transform of the envelopes $u_\lambda + \lambda \log ||s||^2$ with respect to $\lambda$ produces a geodesic ray in the closure of the space of Kähler potentials in $[\omega]$, which coincides with the $C^{1,\alpha}$-geodesic constructed by Phong-Sturm \cite{[29],[30]} (in general, the geodesics are not $C^2$-smooth). Here, building on \cite{[35],[33]}, we show that the logarithm of the Laplace transform, with respect to $\lambda$, of the Monge-Ampère measures of the envelopes $u_\lambda$ defines a family of subgeodesics in the space of Kähler potentials.
converging to the corresponding geodesic ray (see Cor 4.4). In geometric terms the result may be formulated as follows

**Corollary 1.3.** Let $\omega$ be a Kähler form, $\lambda \in [0, \epsilon]$, where $\epsilon$ is the Seshadri constant of $Z$ with respect to $[\omega]$, and let $\omega_{\beta, \lambda}$ be a family of (singular) Kähler metrics in $[\omega] - \lambda[Z]$ (whose potentials are bounded on $X$) satisfying

$$\text{Ric} \, \omega_{\beta, \lambda} = -\beta(\omega_{\beta, \lambda} - \omega + \lambda[Z]) + O(1)$$

Then

$$\varphi^t_{\beta} := \frac{1}{\beta} \log \int_{[0, \epsilon]} d\lambda e^{\beta(\lambda-\epsilon)\rho_{\beta, \lambda}} \frac{\omega_{\beta, \lambda}}{\omega^n}$$

defines a family of subgeodesics converging in $C^0(X \times [0, \infty)$ to a geodesic ray $\varphi^t$ associated to the test configuration defined by the deformation to the normal cone.

This can be seen as a “transcendental” analogue of the approximation result of Phong-Sturm [32], which uses Bergman geodesic rays. However, while the latter convergence result holds point-wise almost everywhere and for $t$ fixed, an important feature of the convergence in the previous corollary is that it is uniform, even when $t$ ranges in all of $[0, \infty]$. See the end of section 4 for a discussion of how to extend the previous corollary to general test configurations.

The motivation for considering this “transcendental” approximation scheme for geodesic rays is two-fold. First, as is well-known, recent examples indicate that a more “transcendental” notion of K-stability is needed for the validity of the Yau-Tian-Donaldson conjecture, obtained by relaxing the notion of a test configuration. One such notion, called analytic test configurations, was introduced in [35] and as shown in op. cit. any such test configuration determines a weak geodesic ray, which a priori has very low regularity. However, the approximation scheme above could be used to regularize the latter weak geodesic rays, which opens the door for defining a notion of generalized Donaldson-Futaki invariant by studying the large time asymptotics of the K-energy functional along the corresponding regularizations (as in the Bergman metrics approach in [32]). In another direction, the approximation scheme above should be useful when considering the analog of K-stability for a non-integral Kähler class $[\omega]$ (compare section 4).

### 1.2. Relation to free boundaries and Hele-Shaw type growth processes.

Coming back to the envelope $u_\theta$ let us briefly point out that it can be seen as a solution to a free boundary value problem for the complex Monge-Ampère operator $MA(u) := (\theta + dd^c u)^n$. Indeed, as follows from the locally boundedness of the complex Hessian $dd^c u_\theta$ (compare [13]) we have that

$$MA(u_\theta) = 1_D \theta^n,$$

where $D$ is the closed set defined as the zero locus of $u_\theta$. In particular, the function $u_\theta$ is a non-positive $\theta$-psh solution to the following boundary value problem for the “homogenous” Monge-Ampère equation on the open domain $M := X - D(= \{u_\theta < 0\})$:

$$\begin{align*}
MA(u_\theta) &= 0 & \text{on } M \\
\theta u_\theta &= 0 & \text{on } \partial M \\
du_\theta &= 0 & \text{on } \partial M
\end{align*}$$

According to Theorem [13] the family $u_\beta$ thus gives approximations to the solution of the previous problem with a uniform convergence up to the boundary (including the
first derivatives $du_\beta$). Moreover, we will also show that the Monge-Ampère measures $MA(u_\beta)$ converges exponentially to zero on compact subsets of $M$. Of course, for a given domain $M$ the boundary value problem above is overdetermined, but the point is that the boundary of $M$ is “free”, in the sense that it is part of the solution of the problem. This picture becomes particularly striking in the setting of envelopes associated to a divisor $Z$; for $\lambda = 0$ the corresponding domain $M_\lambda$ is empty and for small $\lambda$ the domain $M_\lambda$ forms a small neighborhood of $Z$. In particular, if the class $[\omega]$ is cohomologous to the divisor $Z$ the family $M_\lambda$ of domains exhaust all of $X$ as $\lambda$ moves from 0 to 1, “interpolating” between the divisor $Z$ and the whole manifold $X$. In the Riemann surface case the evolution $\lambda \mapsto \partial M_\lambda$ of the moving boundaries is precisely the Laplace growth (Hele-Shaw flow) defined by the metric $\omega$ (compare [33, 34, 25, 41, 3] and references therein), which is closely related to various integrable systems of Toda type [41].

On the proofs. Next, let us briefly discuss the proofs of the previous theorems, starting with the case of a Kähler class. First, the weak convergence of $u_\beta$ towards $u_\theta$ (i.e. convergence in $L^1(X)$) is proved using variational arguments (building on [13, 7]) and thus to prove the previous theorem we just have to provide uniform a priori estimates on $u_\beta$, which we deduce from a variant of the Aubin-Yau Laplacian estimates. In particular, this implies convergence in $L^\infty(X)$. However, in the case of a general big class, in order to establish the global $L^\infty$–convergence, we need to take full advantage of the variational argument, namely that the argument shows that $u_\beta$ converges to $u_\theta$ in energy and not only in $L^1(X)$. This allows us to invoke the $L^\infty$–stability results in [24]. Briefly, the point is that convergence in energy implies convergence in capacity, which together with an $L^p$–control on the corresponding Monge-Ampère measures opens the door for Kolodziej type $L^\infty$–estimates.

An intriguing aspect of the proof of the $L^1$–convergence result (already in the Kähler case) is that it needs some a priori regularity information about $u_\theta$, namely that its Monge-Ampère measure has finite entropy. This is a weaker property then having a bounded Laplacian and it is thus a consequence of the results in [13]. But it would certainly be interesting to have a direct proof of the $L^1$–convergence, which does not invoke the regularity results in [13]. This would then yield a new proof, in the case of a Kähler class, of the regularity result for $u_\theta$, based on a priori estimates, while the proof in [13] uses completely different pluripotential theoretic arguments. These latter argument involve Demailly’s extension of the Kiseiman technique for attenuating singularities (compare [20]) and they have the virtue of applying in the general setting of a big class. Conversely, it would be very interesting if a similar pluripotential theoretic argument could be used to establish the conjectural smoothness of $u_\beta$ on the Kähler locus $\Omega$, thus avoiding the difficulties which appear when trying to use a priori estimates in the setting of a big class. There are certainly strong indications that this can be done (see for example Remark 2.15), but we shall leave this problem for the future.

1.3. **Further background and motivation.** Before turning to the proofs of the results introduced above it may be illuminating to place the result into a geometric and probabilistic context.

*Kähler-Einstein metrics and the continuity method.* First of all we recall that the main geometric motivation for studying complex Monge-Ampère equations of the form [1.1] comes from *Kähler-Einstein geometry* and goes back to the seminal works
algebraic variety, i.e. the canonical line bundle \( K_X := \Lambda^n T^* X \) of \( X \) is ample. If the form \( \theta \) is taken as a Kähler metric \( \omega \) on \( X \) in the first Chern class \( c_1(K_X) \) of \( K_X \) and \( dV \) is chosen to be depend on \( \omega \) in a suitable sense (i.e. \( dV = e^{h_\omega} \omega^n \), where \( h_\omega \) is the Ricci potential of \( \omega \)), then the corresponding solution \( u_\beta \) of the equation \( (1.1) \) for \( \beta = 1 \) is the Kähler potential of a Kähler-Einstein metric \( \omega_{KE} \) on \( X \) with negative curvature. In other words, \( \omega_{KE} := \omega + dd^c u_\beta \) is the unique Kähler metric in \( c_1(K_X) \) with constant (negative) Ricci curvature. Similarly, in the case of \( \beta = -1 \) the equation \( (1.1) \) corresponds to the Kähler-Einstein equation for a positively curved Kähler-Einstein equation in \( c_1(-K_X) \) on a Fano manifold \( X \). For a general value on the parameter \( \beta \) in the equation appears in the continuity method for the Kähler-Einstein equation. Indeed, for \( L = \pm K_X \) the equation \( (1.1) \) is equivalent to the following equation for \( \omega_\beta \) in \( c_1(L) \)

\[
(1.3) \quad \text{Ric} \ \omega_\beta = - \beta \omega_\beta + (\beta - \pm 1) \theta,
\]

which, for \( \beta \) negative, is precisely Aubin’s continuity equation for the Kähler-Einstein problem on a Fano manifold (when \( \theta \) is taken as Kähler form in \( c_1(\pm K_X) \)).

In the present setting, where \( c_1(\pm K_X) \) is replaced by a Kähler (or big) cohomology class \([\theta]\) there is no canonical volume form \( dV \) attached to \( \omega \) and we thus need to work with a general volume form \( dV \), but this only changes the previous equation with a term which is independent of \( \beta \) and which, as we show, becomes negligible as \( \beta \to \infty \).

Interestingly, as observed in [35] the equation \( (1.3) \) can also be obtained from the Ricci flow via a backwards Euler discretization. Accordingly, the corresponding continuity path is called the Ricci continuity path in the recent paper [27], where it (or rather its “conical” generalization) plays a crucial role in the construction of Kähler-Einstein metrics with edge/cone singularities, by deforming the “trivial” solution \( \omega_\beta = \theta \) at \( \beta = \infty \) to a Kähler-Einstein metric at \( \beta = \pm 1 \) (compare section 2.3 below). It should however be stressed that the main point of the present paper is to study the case of a non-positive form \( \theta \) which is thus different from the usual settings appearing in the context of Kähler-Einstein geometry and where, as we show, the limit as \( \beta \to \infty \) is a canonical positive current associated to \( \theta \).

**Cooling down: the zero temperature limit.** In [6] a probabilistic approach to the construction of Kähler-Einstein metrics, was introduced, using certain \( \beta \)-deformations of determinantal point processes on \( X \) (which may be described in terms of “free fermions” [3]). The point is that if \( \theta \) is the curvature form of a given Hermitian metric \(||\cdot||\) on a, say ample, line bundle \( L \to X \), then

\[
(1.4) \quad \mu^{(N_k, \beta)} := \frac{\left|\det S^{(k)}\right|(x_1, x_2, \ldots, x_{N_k})}{Z_{k, \beta}} 2^{\beta/k} dV \otimes N_k
\]

defines a random point process on \( X \), i.e. probability measure on the space \( X^{N_k} \) (modulo the permutation group) of configurations of \( N_k \) points on \( X \), where \( N_k \) is dimension of the vector space \( H^0(X, L^0) \) of global holomorphic sections of \( L^0 \) and \( \det S^{(k)} \) is any fixed generator in the top exterior power \( \Lambda^{N_k} H^0(X, L^0) \), identified with a holomorphic section of \((L^0)^{\otimes N_k} \to X^{N_k} \).

From a statistical mechanical point of view the parameter \( \beta \) appears as the “thermodynamical \( \beta \)”, i.e. \( \beta = 1/T \) is the inverse temperature of the underlying
statistical mechanical system and the complex Monge-Ampère equations above appear as the mean field type equations describing the macroscopic equilibrium state of the system at inverse temperature $\beta$. More precisely $\mu_{\beta} := MA(u_{\beta})$ describes the expected macroscopic distribution of a single particle when $k$ and (hence also the number of particles $N_k$) tends to infinity,

$$\int_{X^{N_k-1}} \mu^{(N_k; \beta)} \to \mu_{\beta}$$

A formal proof of this convergence was outlined in [6] and a rigorous proof appears in [10] (in fact, a much stronger convergence result holds, saying that the convergence towards $\mu_{\beta}$ holds exponentially in probability in the sense of large deviations with a rate functional which may be identified with the twisted K-energy functional). Anyway, here we only want to provide a statistical motivation for the large $\beta$-limit, which thus corresponds to the zero-temperature limit, where the system is slowly cooled down. From this point of view the convergence result in Theorem 1.1 can then be interpreted as a second order phase transition for the corresponding equilibrium measures $\mu_{\beta}$. Briefly, the point is that while the support of $\mu_{\beta}$ is equal to all of $X$ for any finite $\beta$ the limiting measure $\mu_{\infty} (= MA(u_{\theta}))$ is supported on a proper subset $S$ of $X$ as soon as $\theta$ is not globally positive (compare formula 1.2 where $S = D \cap \{ \theta^n > 0 \}$). The formation of a limiting ordered structure (here $MA(u_{\theta})$ and its support $S$) in the zero-temperature limit is typical for second order phase transitions in the study of disordered systems. In fact, in many concrete examples the limiting support $S$ is a domain with piece-wise smooth boundary, but it should be stressed that there are almost no general regularity results for the boundary of $S$ (when $n > 1$). In the one-dimensional case of the Riemann sphere the support set $S$ appears as the “droplet” familiar from the study of Coulomb gases and normal random matrices (see [41, 25] and references therein).

As recently shown in [9] there is also a purely “real” analogue of this probabilistic setting, involving real Monge-Ampère equations where the determinantal random point processes get replaced by permanental random point process.

Transcendental Bergman metric asymptotics. Consider, as before, an ample line bundle $L \to X$ and a pair $(\|\cdot\|, dV)$ consisting of an Hermitian metric $\|\cdot\|$ on $L$ and a volume form $dV$ on $X$ (where the curvature form of $\|\cdot\|$ is denoted by $\theta$). The corresponding Bergman function $\rho_k$ (also called the density of states function), at level $k$, may be defined

$$\rho_k(x) = \sum_{i=1}^{N_k} \left\| s_i^{(k)}(x) \right\|^2,$$

in terms of any fixed basis $s_i^{(k)}$ in $H^0(X, L^{\otimes k})$. The function $v_k := \frac{1}{k} \log \rho_k$ is often referred to as the Bergman metric (potential) at level $k$, determined by $(\|\cdot\|, dV)$ (geometrically, $\|\cdot\| e^{-kv_k}$ is the pull-back of the Fubini-Study metric on the projective space $\mathbb{P}H^0(X, L^{\otimes k})$ under the corresponding Kodaira embedding). As shown in [2] the corresponding Bergman measures $\nu_k := \frac{1}{N_k} \rho_k(x) dV$ converge weakly to $MA(u_{\theta})$ and $v_k$ converges uniformly to $u_{\theta}$. In particular,

$$MA(v_k) \approx e^{kv_k} dV$$

in the sense that both measures have the same weak limit (namely $MA(u_{\theta})$). We can thus view the Bergman metric $v_k$ as an approximate solution to the equation 1.1.
for \( \beta = k \). This motivates thinking of a general family \( u_\beta \) of exact solutions, defined with respect to a general smooth closed \((1,1)\)-form \( \theta \) (not necessarily corresponding to a line bundle) as a transcendental Bergman metric, in the sense that it behaves (at least asymptotically as \( \beta \to \infty \)) as a Bergman metric associated to an Hermitian line bundle.

Finally, let us explain how this fits into the previous statistical mechanical setup. The point is that one can let the inverse temperature \( \beta \), defining the probability measures 1.4, depend on \( k \). In particular, for \( \beta = k \) one obtains a determinantal random point process. A direct calculation (compare [5]) reveals that the corresponding one point correlation measure \( \chi_{N_k - 1} \mu_\beta \) then coincides with the Bergman measure \( \nu_k \) defined above. This means that the limit \( k \to \infty \) which appears in the “Bergman setting” can - from a statistical mechanical point of view - be seen as a limit where the number \( N_k \) of particles and the inverse temperature \( \beta \) jointly tend to infinity.

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1.3.1. Organization. After having setup the general framework in Section 2 we go on to first prove the main result (Theorem 1.1) in the case of Kähler class and then its generalization to big classes (Theorem 1.2). The section is concluded with a discussion about the special case of a positive reference form \( \theta \). Then in Section 3 we consider the singular version of the previous setup which appears in the presence of a divisor \( Z \) on \( X \). Finally, the results in the latter section are applied in Section 4 to the construction of geodesic rays.

2. From Monge-Ampère equations to \( \theta \)--psh envelopes

Let \( X \) be a compact complex manifold equipped with a smooth closed \((1,1)\)-form \( \theta \) and denote by \([\theta]\) the corresponding (Bott-Chern) cohomology class of currents:

\[
[\theta] := \{ \theta + dd^c u : u \in L^1(X) \}
\]

The space of all \( \theta \)--functions, denoted by \( PSH(X, \theta) \), is the convex subspace of \([\theta]\) consisting of all \( u \in L^1(X) \) which are upper semi-continuous (usc) and such that \( \theta + dd^c u \geq 0 \) in the sense of currents. We equip, as usual, the space \( PSH(X, \theta) \) with its \( L^1 \)--topology. The class \([\theta]\) is said to be pseudo-effective if \( PSH(X, \theta) \) is non-empty. There is then a canonical element \( u_\theta \) in the space \( PSH(X, \theta) \) defined as the following envelope:

\[
(2.1) \quad u_\theta(x) := \sup \{ u(x) : \ u \leq 0, \ u \in PSH(X, \theta) \}
\]

Given a smooth function \( u \) we will write

\[
MA(u) := (\theta + dd^c u)^n
\]

for the corresponding Monge-Ampère operator. In the case when the class \([\theta]\) is a Kähler class, i.e. \([\theta]\) contains a smooth and strictly positive form \( \omega \) (i.e. a Kähler
form) we will, also fixing volume form $dV$ on $X$, denote by $u_\beta$ the unique solution to the following complex Monge-Ampère equation:

$$MA(u_\beta) = e^{\beta u_\beta} dV$$

(the solution is automatically $\theta$-psh). More generally, the previous equation makes sense as long as the class $[\theta]$ is big (see section 2.2 below), but in general the unique $\theta$-psh solution $u_\beta$ will have $-\infty$-singularities (even if the singularities are always minimal [17]). We will be relying on the following regularity result:

**Theorem 2.1.** [15]. Let $\theta$ be a smooth $(1,1)$-form on a compact complex manifold $X$ such that $[\theta]$ defines a big cohomology class. Then the Laplacian of $u_\theta$ is locally bounded on a Zariski open subset $\Omega$ of $X$ (which can be taken as the Kähler locus of $[\theta]$). As a consequence, $MA(u_\theta)$ has an $L^\infty$-density, or more precisely:

$$(\theta + dd^c u_\theta)^n = 1_D \theta^n,$$

where $D := \{ u_\theta = 0 \}$.

Note that it follows immediately from the previous theorem that the following “orthogonality relation” holds

$$\int_X u_\theta MA(u_\theta) = 0$$

(which can be proved directly, only using that $\theta$ has lower semi-continuous potentials, using basic properties of free envelopes [12]). In particular, we have that $\sup_X u_\theta = 0$.

2.0.2. *An alternative formulation in the Kähler case.* It may be worth pointing out that, in the Kähler case, the following equivalent formulation of the previous setup may be given, where the role of smooth form $\theta$ is played by a smooth function $\varphi_0$. We start by fixing a Kähler form $\omega$ on $X$ and consider the corresponding Kähler class $[\omega]$. We can then define a projection operator $P_\omega$ from $C^\infty(X)$ to $PSH(X, \omega)$ by setting

$$(P_\omega \varphi_0)(x) := \sup\{ \varphi(x) : \varphi \leq \varphi_0, \varphi \in PSH(X, \omega) \}$$

Setting $\theta := \omega + dd^c \varphi_0$ we see that $u_\theta = P_\omega \varphi_0 - \varphi_0$. Similarly, given a volume form $dV$ on $X$ we denote by $\varphi_\beta$ the unique smooth solution to

$$(\omega + dd^c \varphi_\beta)^n = e^{\beta (\varphi_\beta - \varphi_0)} dV$$

so that $u_\beta = \varphi_\beta - \varphi_0$. One advantage of this new formulation is that it allows one to consider case where $\varphi_0$ is allowed to have $+\infty$-singularities, leading to degeneracies in the rhs of the previous Monge-Ampère equation. In particular, this will allow us to consider a framework of complex Monge-Ampère equations degenerating along a fixed divisor $Z$ in $X$. Interestingly, this latter framework can, from the analytic point view, be seen as a variant of the setting of a big class within a Kähler framework.

We will be interested in the limit when $\beta \to \infty$. In order to separate the different kind of analytical difficulties which appear in the case when $[\theta]$ is Kähler from those which appear in the general case when $[\theta]$ is big, we will start with the Kähler case, even though it can be seen as a special case of the latter.

2.1. *The case of a Kähler class (Proof of Theorem 1.1).* In this section we will assume that $[\theta]$ is a Kähler class, i.e. there exists some smooth function $v \in PSH(X, \theta)$ such that $\omega := \theta + dd^c v > 0$, i.e. $\omega$ is a Kähler form.
2.1.1. Convergence in energy. For a given smooth function \( u \) we will write

\[
E(u) := \frac{1}{n+1} \int_X \sum_{j=0}^n u(\theta + dd^c u)^j \wedge \theta^{n-j}
\]

More generally, the functional \( E(u) \) extends uniquely to the space \( PSH(X, \theta) \), by demanding that it be increasing and usc [13]. Following [13] we will say that a sequence \( u_j \) in \( PSH(X, \theta) \) converges to \( u \) in energy if \( u_j \to u \) in \( L^1(X) \) and \( E(u_j) \to E(u) \).

We recall that the functional \( E \) restricted to the convex space \( PSH(X, \theta) \cap L^\infty(X) \) (or more generally, to the finite energy space \( \{ E > \infty \} \)) may be equivalently defined as a primitive for the Monge-Ampère operator, viewed as a one-form on the latter space, in the sense that

\[
dE_{\mu} = MA(u)
\]

(i.e. \( dE(u + tv)/dt = \int MA(u)v \) at \( t = 0 \)). This is the starting point for the variational approach to the “inhomogeneous” complex Monge-Ampère equation \( MA(u) = \mu \) developed in [13], which gives a variational principle for the latter equation, that we will have great use for in the proof of the following theorem.

The next theorem ensures that the solutions \( u_\beta \) of the Monge-Ampère equations 2.2 converge to \( u_\theta \) in energy. More generally, the theorem concerns the following setting: given a measure \( \mu_0 \) on \( X \) we denote by \( u_\beta \) the solution to the equations 2.2 obtained by replacing \( dV \) with \( \mu_0 \) (the existence of a solution with full Monge-Ampère mass is equivalent to \( \mu_0 \) not charging pluripolar subsets of \( X \)). Recall that a measure \( \mu \) on \( X \) is said to have finite entropy wrt a measure \( \mu_0 \) if \( \mu \) is absolutely continuous wrt \( \mu_0 \) and \( \log(\mu/\mu_0) \) in \( L^1(X, \mu) \). On then defines the entropy \( D_{\mu_0}(\mu) \) of \( \mu \) relative to \( \mu_0 \) by \( D_{\mu_0}(\mu) := \int_X \log(\mu/\mu_0)\mu \) (in particular, if \( \mu_0 = dV \) then a convenient sufficient condition for \( \mu \) to have finite entropy wrt \( dV \) is that \( \mu \) is in \( L^p(dV) \) for some \( p > 1 \).

**Theorem 2.2.** Denote by \( u_\beta \) the solution to the complex Monge-Ampère equation determined by the data \( (\theta, \mu_0, \beta) \). Assume that \( MA(u_\theta) \) has finite entropy relative to \( \mu_0 \). Then \( u_\beta \) converges to \( u_\theta \) in energy. In particular, if \( \mu_0 = dV \) then \( u_\beta \) converges to \( u_\theta \) in energy.

**Proof.** Without loss of generality we may assume that the volume \( V \) of the class \( [\theta] \) is equal to one (by a trivial scaling).

Consider the following functional:

\[
\mathcal{G}_\beta(u) := E(u) - \mathcal{L}_\beta(u), \quad \mathcal{L}_\beta(u) := \frac{1}{\beta} \log \int_X e^{\beta u_\beta} \mu_0,
\]

which is invariant under the additive action of \( \mathbb{R} \). Its critical point equation is the “normalized” equation \( MA(u) = e^{\beta u_\mu}/\int_X e^{\beta u} \mu_0 \), whose unique sup-normalized solution is given by \( U_\beta := u_\beta - \sup_X u_\beta \), where, as before \( u_\beta \) denotes the unique solution of the corresponding “non-normalized” equation. We will use that \( U_\beta \) is a maximizer of \( \mathcal{G}_\beta \), as follows from a concavity argument [17].

**Step 1:** Convergence of \( U_\beta \) towards \( u_\theta \)

First, if \( \mu \) is a probability measure on \( X \) then Jensen’s inequality gives

\[
- \int_X U_\beta \mu + \frac{1}{\beta} D_{\mu_0}(\mu) \geq - \frac{1}{\beta} \mathcal{L}_\beta(U_\beta),
\]
where $D_{\mu_0}(\mu)$ is the entropy of $\mu$ relative to $\mu_0$. Hence, setting $\mu := MA(u_0)$ and defining the constant $D := D_{\mu_0}(u_0)$ (which is indeed finite, by the regularity results in [15]) gives

$$\mathcal{E}(U_\beta) - \int_X U_\beta M A(u_0) + \frac{D}{\beta} \geq \mathcal{E}(U_\beta) - \frac{1}{\beta} \mathcal{L}_\beta(U_\beta) \geq \mathcal{E}(u_0) - \frac{1}{\beta} \mathcal{L}_\beta(u_0)$$

using, in the last inequality that $U_\beta$ maximizes the functional $\mathcal{G}_\beta$. Now, since $\mathcal{L}_\beta(u_0) \leq \sup_X u_0 + C/\beta$ and $\sup_X u_0 = 0$ we thus get

$$\liminf_{\beta \to \infty} \mathcal{E}(U_\beta) - \int_X U_\beta M A(u_0) \geq \mathcal{E}(u_0) - \int_X u_0 M A(u_0)$$

using the orthogonality relation [2.4], saying that the second term in the rhs vanishes. But then it follows from the variational principle in [13] that $U_\beta \to u_\theta$ in $L^1$ (and even in energy) and that [2.9] is actually an equality when $\liminf$ is replaced by $\lim$.

**Step two: Convergence of $u_\beta$**

By the asymptotic equality referred to above combined with the fact that $U_\beta \to u_\theta$ and the orthogonality relation we get the following convergence in energy:

$$\mathcal{E}(U_\beta) \to \mathcal{E}(u_\theta)$$

Hence, using the orthogonality relation [2.4] again the inequalities [2.8] force

$$-\frac{1}{\beta} \mathcal{L}_\beta(U_\beta) \to 0$$

i.e. $u_\beta = U_\beta - \frac{1}{\beta} \mathcal{L}_\beta(U_\beta)$ has the same limit as $U_\beta$, as desired. \qed

**Remark 2.3.** It can be shown that the finite entropy assumption is in fact also necessary for the convergence.

**Example 2.4.** One interesting example where a finite entropy reference form $\mu_0$ appears, a part from the case of a volume form studied here, is when $\mu_0$ is the volume form of a metric with conical singularities with klt (Kawamata Log Terminal) singularities along a simple normal crossings divisor $\Delta$ in $X$ [7, 27]. More generally, starting with a log pair $(X, \Delta)$ with klt singularities on a (possibly singular) normal variety $X$ one gets a natural class of measures $\mu_0$ with density in $L^p_{loc}$ for some $p > 1$ on any smooth resolution of $X$ (see [14]).

In the case when $[\theta]$ is a Kähler class we will only need the $L^1$–convergence implicit in the previous theorem. But it should be stressed that when we go on to the case of a big class the convergence in energy will be crucial in order to establish the convergence in $L^\infty$–norms.

2.1.2. $L^\infty$– estimates. We start with the following basic

**Lemma 2.5.** Assume that $u$ and $v$ are (say, bounded) $\theta$– psh functions such that $MA(v) \geq e^{\beta v}dV$ and $MA(u) \leq e^{\beta u}dV$. Then $v \leq u$.

**Proof.** In the smooth case this follows immediately from the maximum principle and in the general case we can apply the comparison principle (which will be important in the setting of big class considered below). Indeed, according to the comparison principle $\int_{\{u \leq v\}} MA(v) \leq \int_{\{u \leq v\}} MA(u)$ and hence $\int_{\{u \leq v\}} e^{\beta v}dV \leq \int_{\{u \leq v\}} e^{\beta u}dV$. But then it must be that $v \leq u$ a.e. on $X$ and hence everywhere. \qed
The previous lemma allows us to construct “barriers” to show that $u_\beta$ is uniformly bounded:

**Lemma 2.6.** There exists constant $C$ such that $\sup_X |u_\beta| \leq C$.

**Proof.** Let us start with the proof of the lower bound on $u_\beta$. Since $[\theta]$ is a Kähler class there is a smooth $\theta$–psh function $v$ such that $MA(v) \geq e^{-C}dV$ for some constant $C$. After shifting $v$ by a constant we may assume that $v \leq -C/\beta$. But then $MA(v) \geq e^{-C}dV \geq e^{\beta v}$ and hence by the previous lemma $v \leq u_\beta$ which concludes the proof of the lower bound. Similarly, taking $v$ to be a smooth $\theta$–psh function $v$ such that $MA(v) \leq e^{C}dV$ and shifting $v$ so that $C/\beta \leq v$ proves that $u_\beta \leq v$, which concludes the proof of the lemma. □

2.1.3. *The Laplacian estimate.*** Next we will establish the following key Laplacian estimate:

**Proposition 2.7.** Fix a Kähler form $\omega$ in $[\theta]$. Then there exists a constant $C$ such that

$$-C \leq \Delta_\omega u_\beta \leq C$$

**Proof.** The lower bound follows immediately from $\theta + dd^c u_\beta \geq 0$. To prove the upper bound we first recall the following variant of the Aubin-Yau Laplacian estimate in this context due to Siu (compare page 99 in [13] and Prop 2.1 in [18]): given two Kähler forms $\omega'$ and $\omega$ such that $\omega' = e^f \omega$ we have that

$$\Delta_{\omega'} \log tr_{\omega'} \omega' \geq \Delta_{\omega'} f - Btr_{\omega'} \omega,$$

where the constant $-B$ is a lower bound on the holomorphic bisectional curvatures of $\omega$. Fixing $\beta > 0$ and setting $\omega' := \theta + dd^c u$ for $u := u_\beta$ we have, by the MA-equation for $u_\beta$, that $f = \beta u$ and hence

$$Btr_{\omega'} \omega + \Delta_{\omega'} \log tr_{\omega'} \omega' \geq \beta \frac{\Delta_{\omega'} u}{tr_{\omega'} \omega'}$$

Next, we note that $\Delta_{\omega'} u = tr_{\omega'} \omega' - tr_{\omega'} \theta$. Moreover, writing $\omega = \omega' - dd^c (u - v)$, where $v$ is a smooth function such that

$$\omega' = \theta + dd^c v,$$

also gives $tr_{\omega} \omega = 1 - \Delta_{\omega'} (u - v)$. Accordingly, the previous inequality may be reformulated as follows:

$$B + \Delta_{\omega'} (\log tr_{\omega'} \omega' - B(u - v)) \geq \beta \frac{tr_{\omega'} \omega' - tr_{\omega'} \theta}{tr_{\omega'} \omega'},$$

and hence (letting $C$ be the sup of $tr_{\omega} \theta$)

$$(C\beta + Btr_{\omega'} \omega') e^{-B(u-v)} + \Delta_{\omega'} \log (tr_{\omega'} \omega' - B(u - v)) tr_{\omega'} \omega' e^{-B(u-v)} \geq \beta tr_{\omega'} \omega' e^{-B(u-v)}$$

Thus, setting $s := \sup_X e^{-B(u-v)} tr_{\omega'} \omega'$ and taking the maximum over $X$ in the previous inequality gives

$$\beta s \leq 0 + B + s + \beta \sup_X Ce^{-B(u-v)}$$
Finally, by the previous lemma \( u := u_\beta \) is uniformly bounded in \( x \) and \( \beta \) and since, by definition \( v \) is bounded, it follows that \( tr_\omega \omega' \) is uniformly bounded from above, as desired. More precisely, the previous argument gives the estimate

\[
tr_\omega \omega' \leq \frac{1}{1 - 1/\beta} e^{B(u-v)} \left( B/\beta + \sup_X (tr_\omega \theta) e^{-\inf_X B(u-v)} \right)
\]

Remark 2.8. Note that the Laplacian estimate (2.12) breaks down when \( \beta = 1 \), which is precisely the case appearing in the Aubin-Yau theorem for the existence of Kähler-Einstein metrics with negative Ricci curvature. The point is that the in latter case one also has to exploit some extra information, in particular a uniform upper bound on \( \text{Ric} \omega' \). But, unless \( \theta \geq 0 \), such a bound does not hold in the present setting for \( \beta \) large (since \( \text{Ric} \omega' \) contains a term of the form \(-\beta \theta\)) and hence the main point with the proof above is the observation that for \( \beta > 1 \) the latter extra assumption on the Ricci curvature is not needed.

2.1.4. End of proof of Theorem 1.1. By Lemma 2.6 \( u_\beta \) is uniformly bounded and by the Laplacian estimate in Prop 2.7 combined with Green’s formula the gradients of \( \omega \) classes. and the cone of Kähler classes may be defined as the interior of the cone of nef \( [1] \) metric so that

\[
H
\]

Example 2.9. Let \( Y \) be a singular algebraic variety in complex projective space \( \mathbb{P}^N \) and \( \omega \) a Kähler form on \( \mathbb{P}^n \) (for example, \( \omega \) could be taken as the Fubini-Study metric so that \( [\omega_Y] \) is the first Chern class of \( \mathcal{O}_X(1) \)). If now \( X \to Y \) is a smooth resolution of \( Y \), which can be taken to invertible over the regular locus of \( Y \); then the pull-back of \( \omega \) to \( X \) defines a class which is nef and big and such that its Kähler locus corresponds to the regular part of \( Y \).
We will denote by $MA$ the Monge-Ampère operator on $PSH(X, \theta)$ defined by replacing wedge products of smooth forms with the *non-pluripolar product* of positive currents introduced in [17]. The corresponding operator $MA$ is usually referred to as the *non-pluripolar Monge-Ampère operator*. For example, if $u$ has minimal singularities, then $MA(u) = 1_\Omega MA(u|_\Omega)$ on the Kähler locus $\Omega$, where $MA(u|_\Omega)$ may be computed locally using the classical definition of Bedford-Taylor. We let $V$ stand for the volume of the class $[\theta]$, which may be defined as the total mass of $MA(u)$ for any function $u$ in $PSH(X, \theta)$ with minimal singularities. By [17] there exists a unique solution $u_\beta$ to the equations 2.2 in $PSH(X, \theta)$ with minimal singularities. Moreover, by [17] the solution is smooth on the Kähler locus in the case when $[\theta]$ is nef and big (which is expected to be true also without the nef assumption; compare the discussion in [17]).

2.2.1. Convergence in energy. In the case of a big class one first defines, following [13], the following functional on the space of all functions in $n PSH(X, \theta)$ with minimal singularities:

\[(2.13) \quad \mathcal{E}(u) := \frac{1}{n+1} \int_X \sum_{j=0}^n (u - u_\theta)(\theta + dd^c u)^j \wedge (\theta + dd^c u_\theta)^{n-j}\]

(the point is that we needs to subtract $u_\theta$ to make sure that the integral is finite). Equivalently, $\mathcal{E}$ may be defined as the primitive of the Monge-Ampère operator on the the space of all finite energy functions in $PSH(X, \theta)$, normalized so that $\mathcal{E}(u_\theta) = 0$. We then define convergence in energy as before.

**Remark 2.10.** Strictly speaking, in the case of a Kähler class the definition 2.13 of $\mathcal{E}$ only coincides with the previous one (formula 2.6) in the case when $\theta$ is semi-positive (since the definition in formula 2.6 corresponds to the normalization condition $\mathcal{E}(0) = 0$). But the point is that, in the Kähler case, different normalizations gives rise to functionals which only differ up to an overall additive constant and hence the choice of normalization does not effect the notion of convergence in energy.

The proof of Theorem 1.1 can now be adapted word for word (using that the finite entropy property also holds in the big case, by Theorem 2.1) to give the following

**Proposition 2.11.** Suppose that $\theta$ is a smooth form such that the class $[\theta]$ is big. Then $u_\beta$ converges to $u_\theta$ in energy.

2.2.2. $L^\infty$-estimates. We will need the following upper bound on $u_\beta$, which refines the upper bound implicit in Lemma 2.6

**Lemma 2.12.** There exists a constant $C$ such that

\[u_\beta \leq u_\theta + C/\beta\]

(the constant $C$ may be taken as $\sup_D \log(\theta^n / V dV)$.

**Proof.** First note that, by the domination principle (which holds since $u_\theta$ has minimal singularities ref) it is enough to prove that $u_\beta \leq u_\theta + C_2/\beta$ wrt $MA(u_\theta)$. By the regularity results in [15], or more precisely formula 2.3

\[MA(u_\theta) \leq 1_D e^{C_2} dV\]
for some constant $C_2$. Applying the domination principle (which applies to any pair of $\theta$–psh functions with minimal singularities \cite{17}) thus gives

$$
\int_{\{u_\theta + C_2/\beta \leq u_\theta\}} e^{\beta u_\theta} dV \leq \int_{\{u_\theta + C_2/\beta \leq u_\theta\} \cap D} e^{C_2} dV
$$

In particular, intersecting the region of integration in the lhs above with the set $D = \{u_\theta = 0\}$ gives

$$
\int_{\{C_2/\beta \leq u_\theta\} \cap D} e^{\beta u_\theta} dV \leq \int_{\{u_\theta + C_2/\beta \leq u_\theta\} \cap D} e^{C_2} dV
$$

But then it must be that $u_\beta \leq C_2/\beta$ a.e. on $D$ and hence $u_\beta \leq u_\theta + C_2/\beta$ ae wrt $MA(u_\theta)$ as desired. \hfill \Box

**Proposition 2.13.** Suppose that $\theta$ is a smooth form such that the class $[\theta]$ is big. Then $u_\beta$ converges uniformly to $u_\theta$ on $X$, i.e.

$$
\lim_{\beta \to 0} \|u_\beta - u_\theta\|_{L^\infty(X)} = 0
$$

*Proof.* According to the previous lemma we have that $u_\beta \leq u_\theta + C/\beta$ and hence $MA(u_\beta)/dV \leq e^C$. Moreover, by Prop 2.11 $u_\beta$ converges to $u$ in energy. As will be next explained these properties are enough to conclude that $u_\beta$ converges uniformly to $u$. Indeed, it is well-known that if $u_j$ is a sequence in $PSH(X, \theta)$ converging in capacity to $u_\infty$ with a uniform bound $L^p$–bound on $MA(u_j)/dV$, then $\|u_j - u_\infty\|_{L^\infty(X)} \to \infty$, as follows from a generalization of Kolodziej’s $L^\infty$–estimates to the setting of a big class (see \cite{17, 24} and references therein). Finally, as shown in \cite{13}, convergence in energy implies convergence in capacity, which thus concludes the proof of the previous proposition. In fact, using the stability results in \cite{24} a more quantitative convergence result can be given. Indeed, according to Prop 4.2 in \cite{24} the following holds: assume that $\varphi$ and $\psi$ are functions in $PSH(X, \theta)$ normalized so that $\sup \varphi = \sup \psi = 0$ and such that $MA(\varphi) \leq f dV$, where $f \in L^p(X, dV)$. Then, for any sufficiently small positive number $\gamma$ (see \cite{24} for the precise condition) there exists a constant $M$, only depending on $\gamma$ and an upper bound on $\|f\|_{L^p(dV)}$, such that

$$
\sup_{X} (\psi - \varphi)^+ \leq M \left\| (\psi - \varphi)^+ \right\|^\gamma_{L^1(X, MA(\varphi))}
$$

Setting $\varphi := u_\beta - \epsilon_\beta$, where $\epsilon_\beta = \sup u_\beta$ and $\psi := u_\theta$ thus gives, for $\gamma$, fixed

$$
\sup_{X} (u_\theta - u_\beta - \epsilon_\beta)^+ \leq M \left( \int |u_\theta - u_\beta - \epsilon_\beta| MA(u_\beta) \right)^\gamma
$$

Now, by the convergence in energy and the $L^1$–convergence in Prop 2.11 we have

$$
\int (u_\beta - u_\theta) MA(u_\beta) \to 0
$$

and since $\int |u_\theta - u_\beta - \epsilon_\beta| MA(u_\beta) \leq \int (u_\theta - u_\beta - C/\beta)MA(u_\beta) + C/\beta + \epsilon_\beta$ we deduce that $\sup_{X} (u_\theta - u_\beta - \epsilon_\beta)^+ \to 0$, i.e. $u_\theta \leq u_\beta + \epsilon_\beta$, which concludes the proof. \hfill \Box
2.2.3. Laplacian estimates. For the Laplacian estimate we will have to assume that
the big class $[\theta]$ is nef.

Proposition 2.14. Suppose that the class $[\theta]$ is nef and big. Then the Laplacian
of $u_\beta$ is locally bounded wrt $\beta$ on the Zariski open set $\Omega \subset X$ defined as the Kähler
locus of $X$.

Proof. We will assume that $X$ is a Kähler manifold, i.e. $X$ admits some Kähler
form $\omega_0$ (not necessarily cohomologous to $\theta$). Then $\theta$ is nef precisely when the class
$[\theta] + \epsilon [\omega_0]$ is Kähler for any $\epsilon > 0$. Setting $\theta_\epsilon := \theta + \epsilon \omega_0$ and fixing $\epsilon > 0$ and
$\beta > 0$ we denote by $u_{\beta,\epsilon}$ the solutions of the Monge-Ampère equations obtained by
replacing $\theta$ with $\theta_\epsilon$. Then it follows from well-known results [17] that, as $\epsilon \to 0$,
\[
 u_{\beta,\epsilon} \to u_\beta \text{ in } C^\infty_{0,\loc}(\Omega).
\]
Moreover, since $[\theta]$ is assumed big there exists a positive current $\omega$ in $[\theta]$ such that
the restriction of $\omega$ to $\Omega$ coincides with the restriction of a Kähler form on $X$. More
precisely, we can take $\omega$ to be a Kähler current on $X$ such that $\omega = dd^c v + \theta$ for a
function $v$ on $X$ such that $v$ is smooth on $\Omega$ and $u - v \to -\infty$ at the “boundary”
of $\Omega$ (using that $u$ has minimal singularities; compare [17]). Setting $u := u_{\beta,\epsilon}$ the
inequality 2.13 still applies on $\Omega$. Moreover, since $u - v \to -\infty$ at the boundary of $\Omega$ the sup $s$ defined above is attained at some point of $\Omega$ and $\sup_X e^{-B(u-v)} \leq C'$. Accordingly, we deduce that
\[
 s := \sup_X e^{-B(u-v)}tr_\omega \omega' \leq C''
\]
precisely as before, which in particular implies that $tr_\omega (\theta + dd^c u_{\beta,\epsilon})$ is locally
bounded from above (wrt $\beta$ and $\epsilon$). Finally, letting $\epsilon \to 0$ concludes the proof. □

In the special case when $\theta$ is semi-positive and big (the latter condition then
simply means that $V > 0$) it follows from the results in [21] that $u_\beta$ is continuous
on all of $X$ and hence Prop 2.13 then says that $u_\beta \to u_\theta$ in $C^0(X)$.

Remark 2.15. The precise Laplacian estimate obtained in the previous proof may,
for $v$ and $\omega$ as in the proof above may be formulated as
\begin{equation}
 tr_\omega \omega_{u_\beta} \leq \frac{1}{1 - 1/\beta} e^{B(u_\beta - v)} \left( B/\beta + \sup_X (tr_\omega \theta) e^{-\inf_X B(u_\beta - v)} \right)
\end{equation}
In particular, normalizing $v$ so that $\sup_X v = 0$ gives
\[
 tr_\omega \omega_{u_\beta} \leq e^{\sup_X u_\beta - \inf_X u_\beta} \left( B/\beta + \sup_X (tr_\omega \theta) \right) e^{-Bv}
\]
By the $L^\infty$-estimates above $\sup_X u_\beta - \inf_X u_\beta$ is uniformly bounded in terms of
$\sup_X |\theta^n/dV|$. In particular, letting $\beta \to \infty$ gives the following a priori estimate for the Laplacian of the envelope $u_\theta$ :
\begin{equation}
 tr_\omega \omega_{u_\theta} \leq Ce^{-Bv},
\end{equation}
where the constant $C$ only depends on an upper bound on $|\theta|_\omega$. Interestingly, the
estimate 2.15 is essentially of the same form as the one obtained in [15], by a
completely different method, in the more general setting of a big class.

2.2.4. End of the proof of Theorem 1.2 in the big case. This is proved exactly as in
the case of a Kähler class, given the convergence results established above.
2.3. The case when \( \theta \) is positive (comparison with \cite{[18]} \cite{[27]}). In the case when \( \theta \) is semi-positive it follows immediately that \( u_\theta = 0 \). If moreover \( \theta \) is positive (i.e. \( \theta > 0 \)) a much more direct proof of Theorem \cite{[1]4} can be given (replacing the variational argument with the comparison principle) and it also leads to a stronger statement.

Proposition 2.16. Suppose that \( \theta \) is a smooth positive form. Then \( u_\beta \to 0 \) in \( \mathcal{C}^\infty(X) \). More precisely, \( u_\beta = \beta^{-1} \log(\frac{\nu}{\beta}) + o(1) \), where \( o(1) \) denotes a term tending to zero in \( \mathcal{C}^\infty(X) \).

Proof. The point is that, since we now have a two-side bound \( e^{C_-} dV \leq MA(0) \leq e^{C_+} dV \), we can apply the comparison principle to the pairs \((u_\beta, u_\pm)\) where \( u_\pm = C_\pm/\beta \) and deduce the two-side bound \( C_-/\beta \leq u_\beta \leq C_+/\beta \) and hence \( u_\beta \to 0 \). The Laplacian estimates then show that the convergence holds in \( \mathcal{C}^{1,\alpha}(X) \) for any \( \alpha \in [0, 1[ \). Moreover, the lower bound \( C_-/\beta \leq u_\beta \) gives, by the very definition of \( u_\beta \), that \( MA(u_\beta) \geq e^{C_-} dV \). Hence, it follows from the \( \mathcal{C}^{1,\alpha} \)-convergence combined with Evans-Krylov theory that \( u_\beta \to 0 \) in \( \mathcal{C}^\infty(X) \). Finally, since this implies that \( MA(u_\beta)/\theta^n \to 1 \) in \( \mathcal{C}^\infty(X) \) it also follows that \( u_\beta = \beta^{-1} \log(\frac{\nu}{\beta}) + o(1) \), as desired.

The previous proposition is a special case of a result in \cite{[27]}, concerning the case when \( \theta \) is a Kähler form with conical singularities and the volume form \( dV \) is replaced by a reference measure \( \mu_0 \) with conical singularities. The proof in \cite{[27]} is based on a Newton iteration argument, which is an adaptation of an argument of Wu previously used in a different setting (see \cite{[46]}, Prop 7.3). Note that in the notation of \cite{[27]} \( \beta = -s \), where \( s \) is the time parameter in the Ricci continuity path used in op. cit.

3. Degenerations induced by a divisor

Let now \((X, \omega)\) be a compact Kähler manifold with a fixed divisor \( Z \), i.e. \( Z \) is cut out by a holomorphic section \( s \) of a line bundle \( L \to X \). We identify the divisor \( Z \) with the corresponding current of integration \([Z] := [s = 0]\). Let us also fix a smooth Hermitian metric \( \| \cdot \| \) on \( L \) and denote by \( \theta_L \) its normalized curvature form. Fixing a parameter \( \lambda \in [0, 1[ \) we set

\begin{equation}
\varphi_\lambda := \sup\{\varphi : \varphi \leq 0, \varphi \leq \log \|s\|^2 + O(1)\}
\end{equation}

The upper bound on \( \varphi \) is equivalent to demanding that \( \nu_Z(\varphi) \geq \lambda \), where \( \nu_Z(\varphi) \) denotes the Lelong number of \( \varphi \) along \( Z \). We will assume that \( \lambda \in [0, \epsilon[ \), where \( \epsilon \) is the Seshadri constant of \( Z \) with respect to \( [\omega] \), i.e. the sup over all \( \lambda \) such that \([\omega] - \lambda[Z] \) is a Kähler class. We set \( u_\lambda := \varphi_\lambda - \lambda \log \|s\|^2 \), defining a function in \( PSH(X, \theta) \), where \( \theta := \omega - \lambda \theta_L \). Equivalently,

\begin{equation}
u_\lambda := P_\theta(-\lambda \log \|s\|^2)
\end{equation}

in the sense of formula \cite{[25]}. This is equivalent to the construction of envelopes of metrics with prescribed singularities out-lined in the introduction of \cite{[2]} (see also \cite{[33]} where it is shown that \( u_\lambda \) is in \( \mathcal{C}^{1,\alpha}_{loc}(X - Z) \) in the case of an integral class). Note that it follows immediately from the definition that \( u_\lambda \) has minimal singularities and is thus bounded. In fact, \( u_\lambda \) is even continuous. The point is that, as long as the function \( \varphi_0 \) is lower semi-continuous the corresponding envelope \( P_\theta(\varphi_0) \) will also be continuous. Indeed, it follows immediately that \( P_\theta(\varphi_0)^* \leq \varphi_0 \) and hence
$P_0(\varphi_0)^* = P_0(\varphi_0)$, showing upper-semi continuity. The lower semi-continuity is then a standard consequence of Demailly’s approximation theorem.

The following lemma extends Theorem 2.1 to the present singular setting, at least when the class $[\theta]$ is semi-positive (i.e. the class contains a smooth semi-positive current):

**Lemma 3.1.** Let $[\theta]$ be a semi-positive class and $\varphi_0$ a lower-semi continuous function such that $\Sigma := \{ \varphi_0 = \infty \}$ is closed and $\varphi_0$ is smooth on $X - \Sigma$. Then the corresponding envelope $P_{\varphi_0}$ has a Laplacian which is locally bounded on the Kähler locus $\Omega$ of the class and moreover its Monge-Ampère measure has a density which is uniformly bounded on $X$ (wrt a given volume form).

**Proof.** By Theorem 2.1 it will be enough to show that $P_{\varphi_0} = P_\psi$ for some smooth function $\psi$ on $X$. We will prove this using a regularization argument. First observe that $D := \{ P_{\varphi_0} = \varphi_0 \}$ is compact. Indeed, by the assumption on the class $[\theta]$ the envelope $P_{\varphi_0}$ is uniformly bounded and hence $D$ is compactly included in $X - \Sigma$. For future reference we also recall that by general properties of free envelopes $MA(P_{\varphi_0}) = 0$ on $X - D$ (see for example (14)). Now set $\varphi^{(j)} := \min \{ \varphi_0, j \}$. In particular, $\varphi^{(j)} \leq \varphi_0$ and hence $P_{\varphi^{(j)}} \leq P_{\varphi_0}$. Moreover, since $P_{\varphi_0} \leq C$ it follows that $P_{\varphi^{(j)}} \leq j$ for $j$ sufficiently large and since, by definition, $P_{\varphi_0} \leq \varphi_0$ we deduce that $P_{\varphi_0} \leq P_{\varphi^{(j)}}$ for $j$ sufficiently large, i.e. $P_{\varphi_0} = P_{\varphi^{(j)}}$. Let us fix such a large index and define $\varphi^{(j)}_\epsilon := \min_{\epsilon} \{ \varphi_0, j \}$ as a regularized version of $\min \{ \varphi_0, j \}$ in the sense that $\varphi^{(j)}_\epsilon$ is smooth and $\varphi^{(j)}_\epsilon = \varphi_0$ if $\varphi_0 \geq j - 1$ and $\sup_X |\varphi^{(j)}_\epsilon - \varphi^{(j)}| \leq \epsilon$, We claim that $P_{\varphi^{(j)}_\epsilon} = P_{\varphi_0}$ for $\epsilon$ a sufficiently small number in $[0, 1]$ (so that we can take $\psi$ above as $\varphi^{(j)}_\epsilon$). To see this first observe that since $D := \{ P_{\varphi_0} = \varphi_0 \}$ is compact we may assume that $\varphi^{(j)}_\epsilon = \varphi_0$ on $D$ and hence, by the domination principle applied to the pair $(\varphi^{(j)}_\epsilon, P_{\varphi_0})$ we have $P_{\varphi^{(j)}_\epsilon} \leq P_{\varphi_0}$ on $X$ (recall that the domination principle says that if $u \leq v$ a.e. wrt $MA(v)$ then $u \leq v$ everywhere, if $u, v \in PSH(X, \theta)$ and $v$ has minimal singularities (17)). To prove the converse inequality it will, using the domination principle again, be enough to show that $\varphi_0 \leq \varphi^{(j)}_\epsilon$ on $D_{\epsilon, \epsilon} := \{ \varphi^{(j)}_\epsilon = \varphi^{(j)} \}$ But, by construction, on $D_{\epsilon, \epsilon}$ we have $\varphi^{(j)}_\epsilon = P_{\varphi^{(j)}_\epsilon} \leq P_{\varphi^{(j)}} + \epsilon \leq C + \epsilon$ and hence $\min \{ \varphi_0, j \} := \varphi^{(j)} \leq C + 2\epsilon \leq C + 2$ on $D_{\epsilon, \epsilon}$. In particular, for $j$ large $\varphi_0 \leq C + 2$ and hence, by definition, $\varphi^{(j)}_\epsilon = \varphi_0$ there, as desired.

We can now prove the following convergence result:

**Theorem 3.2.** Let $(X, \omega)$ be a Kähler manifold and $Z$ a divisor on $X$ and fix a positive number $\lambda$ such that $[\omega] - \lambda[Z]$ is a Kähler class. Let $(L, s)$ be a line bundle over $X$ with a holomorphic section $s$ cutting out $Z$ and fix a smooth metric $\| \cdot \|$ on $L$ with curvature form $\theta_L$. Setting $\theta := \omega - \lambda \theta_L$, let $u_{\beta, \lambda}$ be the unique bounded $\theta$-psh solution of

$$(\theta + dd^c u)^n = e^{\beta u} \| s \|^{2\lambda^2} dV$$

(which is automatically smooth on $X - Z$). Then $u_{\beta, \lambda}$ converges, as $\beta \to \infty$, to the envelope $u_\lambda$ in $C^{1, \alpha}(X)$, for any $\alpha \in [0, 1]$. Moreover, the convergence in $C^0(X)$ holds as long as the class $[\omega] - \lambda[Z]$ is semi-positive and big and it is uniform wrt $\lambda$ for any fixed $c$ such that $[\omega] - c[Z]$ is semi-positive and big, i.e.

$$\sup_X |u_{\beta, \lambda} - u_\lambda| \leq \epsilon \beta$$
for some family of positive numbers $\epsilon_{\beta}$ (independent of $\lambda$) tending to 0 as $\beta \to \infty$.

**Proof.** The convergence in energy is proved essentially as before, using the previous lemma, which furnishes the required finite entropy property and also using that the orthogonality relation still holds (for the same reason is before). The $C^0$–convergence then follows essentially as before. The uniform wrt $\lambda$ follows from tracing through the argument in the proof of the latter convergence, using the uniform bound $MA(u_\lambda) \leq C dV$, which in turn follows from the regularity result in the previous lemma, which gives $MA(u_\lambda) = 1_{D_\lambda} \theta^n$, where $D_\lambda := \{ u_\lambda = -\lambda \log \| s \|_x^2 \}$.

The Laplacian estimate:

We will write $\Omega := X - Z$. Set $w_\lambda := u + \lambda \log \| s \|_x^2$ so that $d\bar{d} w_\lambda + \omega = d\bar{d} u_\lambda + \theta$ on $\Omega$ and hence $w_\lambda$ satisfies the following equation on $\Omega$:

$$(\omega + d\bar{d} w_\lambda)^n = e^{\beta w_\lambda} dV,$$

Set $\omega' := \omega + d\bar{d} w_\lambda$ on $\Omega$ and fix $\omega_\lambda$ a Kähler current in $[\omega]$ on $X$ which is sufficiently singular along all of $Z$ and smooth on $X - Z$. More precisely, we can arrange that $\omega_\lambda = \omega + d\bar{d} v_\lambda$ where

$$v_\lambda \leq (\lambda + \delta) \log \| s \|_x^2 + C$$

for some small positive number $\delta$ (just using that $[\omega] - (\lambda + \delta)[Z]$ is a Kähler class for $\delta$ small; we will make this more precise below). We can now apply the estimates in the proof of Prop [2.7] with $u$ replaced by $w_\lambda$ and $v$ by $v_\lambda$ to get the following inequality on $\Omega$:

$$\beta \text{tr}_{\omega_\lambda} \omega' e^{-B(w_\lambda - v_\lambda)} \leq (C\beta + B \text{tr}_{\omega_\lambda} \omega') e^{-B(u_\lambda - v_\lambda)} + \Delta \omega' \log(\text{tr}_{\omega_\lambda} \omega' - B(w_\lambda - v_\lambda)) \text{tr}_{\omega_\lambda} \omega' e^{-B(w_\lambda - v_\lambda)},$$

But, since $|u| \leq C$ on $X$ we have $w_\lambda - v_\lambda \to \infty$ at $\partial \Omega$ and hence if we knew that $\omega'$ were smooth then we could apply the maximum principle, just as before, to conclude that $s := \sup_X e^{-B(u_\lambda - v_\lambda)} \text{tr}_{\omega_\lambda} \omega'$ is bounded, which would give the estimate

$$\text{tr}_{\omega_\lambda} \omega' \leq \frac{1}{1 - 1/\beta} e^{B(u_\lambda - v_\lambda)} \left( B/\beta + C e^{-\sup_X B(w_\lambda - v_\lambda)} \right)$$

To get around the regularity issue pointed out above we can simply apply a regularization argument: fix $\epsilon > 0$ and replace $\log \| s \|_x^2$ with $\log(\| s \|_x^2 + \epsilon)$. Then the corresponding solution $u^{(\epsilon)}$ is smooth by Yau’s theorem [47] and $u^{(\epsilon)} \to u$ in $C^0(\Omega)$.

We can then apply the previous argument to $\omega^{(\epsilon)} := \theta + d\bar{d} u^{(\epsilon)}$ and obtain a bound on $s^{(\epsilon)} := \sup_X e^{-B(u^{(\epsilon)}_\lambda - v^{(\epsilon)}_\lambda)} \text{tr}_{\omega^{(\epsilon)}} \omega'$. Finally, letting $\epsilon \to 0$ proves the desired estimates on $u_\lambda$ which then proves the local $C^{1,\alpha}$–convergence on $X - Z$ in the usual way.

Finally, to get the global $C^{1,\alpha}$–convergence on $X$ we have to check that the Laplacian estimates above can be made uniform up to the boundary of $X - Z$. To this end we note that $v_\lambda$ above can be constructed as follows. Let us first set $\theta_0 := \omega - \mu \theta_1$ (so that $\theta_\lambda = \theta$ in our previous notation). Since, by assumption, $[\theta_\lambda]$ is Kähler there exists a smooth function $U_\lambda$ on $X$ such that $\omega_{\lambda,0} := d\bar{d} U_\lambda + \theta_\lambda > 0$ (i.e. $\omega_{\lambda,0}$ is a Kähler form on $X$). Hence $d\bar{d} U_\lambda + \theta_{\lambda,\delta} > 0$ for any sufficiently small positive number $\delta$. Let us now define

$$v_{\lambda,\delta} := U_\lambda + (\lambda + \delta) \log \| s \|_x^2.$$
Then, on $X - Z$, $\omega_{\lambda,\delta} := \omega + dd^c v_{\lambda,\delta} = dd^c U_\lambda + \theta_{\lambda+\delta} > 0$ and, as $\delta \to 0$, we have that $\omega_{\lambda,\delta} \to \omega_{\lambda,0}$ uniformly on $X - Z$ with all derivatives. In particular, the corresponding constants $B = B_\delta$, i.e. the lower bounds on the bisectional curvatures of $\omega_{\lambda,\delta}$ are uniformly bounded (as they converge to $B_0$). Moreover, $v_{\lambda,\delta} - \omega_{\lambda} \leq \delta \log \|s\|^2 \leq C$ (using that $|u_\lambda| \leq C$ ) and $v_{\lambda,\delta} - \omega_{\lambda} \to U_\lambda - u_\lambda$ point-wise on $X - Z$. Hence, taking $\nabla = v_{\lambda,\delta}$ and $\omega = \omega_{\lambda,\delta}$ in the estimate\ref{eq:conv} and letting $\delta \to 0$ gives

$$tr_{\omega_{\lambda,\delta}} \omega' \leq \frac{1}{1 - 1/\beta} e^{B_0\beta} (B_0/\beta + C)$$

which is uniformly bounded on $X - Z$, as $\beta \to \infty$ and that concludes the proof. \(\Box\)

Note that $\varphi_{\lambda,\beta} := u_\lambda + \lambda \log \|s\|^2$ is uniquely determined by the following equation on $X - Z$ :

\begin{equation}
(\omega + dd^c \varphi_{\lambda,\beta})^n = e^{\beta \varphi_{\lambda,\beta}} dV
\end{equation}

together with the asymptotics $\varphi_{\lambda,\beta} = \lambda \log \|s\|^2 + O(1)$ close to $Z$.

\textbf{Remark} 3.3. More generally, it is enough to assume that $[\omega - \lambda[Z]$ is nef and big so obtain the previous $C^{1,\alpha}$—convergence results on the complement of $Z$ in the Kähler locus.

4. APPLICATIONS TO GEODESIC RAYS AND TEST CONFIGURATIONS

Let us start by briefly recalling the notions of geodesic rays and test configurations in Kähler geometry (see\cite{21} and references therein). Given an $n$—dimensional Kähler manifold $(X, \omega)$ we denote by $K_\omega$ the space of all $\omega$—Kähler potentials $\varphi$ on i.e. $\varphi$ is smooth and $\omega + dd^c \varphi > 0$ (which equivalently means that $\varphi$ is in the interior of the space $PSH(X, \omega) \cap C^\infty(X)$). The infinite dimensional space $K_\omega$ comes with a canonical Riemannian metric, the Mabuchi-Semmes-Donaldson metric. The corresponding geodesics rays $\varphi^t(x)$ satisfy a PDE on $X \times [0, \infty]$ which, upon complexification of $t$ (where $t := - \log |\tau|^2$) is equivalent to an $S^1$—invariant smooth solution to the Dirichlet problem for the Monge-Ampère equation on the product $X \times \Delta^*$ of $X$ with the punctured unit-disc in the one-dimensional complex torus $\mathbb{C}^*$. In other words, $\varphi(x, \tau) := \varphi^t(x)$ satisfies

$$(dd^c \varphi^t + \pi^* \omega)^n+1 = 0, \text{ on } X \times \Delta^*$$

and $\varphi^t$ is called a \emph{subgeodesic} if $dd^c \varphi + \pi^* \omega \geq 0$. In the case of an integral class $[\omega]$, i.e. when the class is equal to the first Chern class $c_1(L)$ of a line bundle $L$, there is a particularly important class of (weak) geodesics which are associated to so called \emph{test configurations} for $(X, L)$. This is an algebro-geometric gadget which gives an appropriate $\mathbb{C}^*$—equivariant polarized closure $X'$ of $X \times \mathbb{C}^*$ over $\mathbb{C}$. More precisely, the data defining a test configuration $(X', L)$ for $(X, L)$ consists of

- A normal variety $X'$ with a $\mathbb{C}^*$—action and flat equivariant map $\pi : X' \to \mathbb{C}$
- A relatively ample line bundle $L$ over $X'$ equipped with an equivariant lift $\rho$ of the $\mathbb{C}^*$—action on $X$
- An isomorphism of $(X, L)$ with $(X', L)$ over $1 \in \mathbb{C}$

Here, we note that a “transcendental” analog of a test configuration can be defined in the setting of non-integer classes.

\textbf{Definition} 4.1. Let $(X, [\omega])$ be a complex manifold equipped with a Kähler class $[\omega]$. A test configuration for $(X, [\omega])$ consists of the following data:
A normal Kähler space \( \mathcal{X} \) equipped with a holomorphic \( S^1 \)-action and a flat holomorphic map \( \pi : \mathcal{X} \to \mathbb{C} \).

An \( S^1 \)-equivariant embedding of \( X \times \mathbb{C}^* \) in \( \mathcal{X} \) such that \( \pi \) commutes with projection onto the second factor of \( X \times \mathbb{C}^* \).

A \((1,1)\)-cohomology Kähler class \([\Omega]\) on \( \mathcal{X} \) whose restriction to \( X \times \{1\} \) may be identified with \([\omega]\) under the previous embedding.

In particular, a test configuration \( (\mathcal{X}, \mathcal{L}) \) for a polarized variety \((X,L)\) induces a test configuration for \((X, c_1(L))\). The point is that the \( \mathbb{C}^* \)-action on \((\mathcal{X}, \mathcal{L})\) induces the required isomorphism between \( \mathcal{X} \) and \( X \times \mathbb{C}^* \) over \( \mathbb{C}^* \).

Next, we explain how to obtain geodesic rays from a test configuration. Given a test configuration \( (\mathcal{X}, [\Omega]) \) for \((X, [\omega])\), we fix a smooth representative form \( \Omega \) which is \( S^1 \)-invariant. For the sake of notational simplicity we also assume that \( \Omega \) coincides with \( \omega \) on \( X \times \{1\} \).

First we let \( \Phi \) be the unique bounded \( \Omega \)-psh function on \( \mathcal{M} := \pi^{-1}(\Delta) \subset \mathcal{X} \) satisfying the Dirichlet problem
\[
(dd^c \Phi + \Omega)^{n+1} = 0, \quad \text{on} \, \text{int} (\mathcal{M})
\]
with vanishing boundary values (in the sense that \( \Phi(p) \to 0 \) as \( p \) approaches a point in \( \partial \mathcal{M} \)). In fact, it can be shown, that \( \Phi \) is automatically continuous up to the boundary (see below). Next, we fix an \( S^1 \)-invariant function \( F \) on \( X \times \mathbb{C}^* \) such that
\[
\Omega = \pi^* \omega + dd^c F
\]
and set \( \varphi := \Phi + F \), which gives a correspondence
\[
PSH(X \times \mathbb{C}^*, \Omega) \leftrightarrow PSH(X \times \mathbb{C}^*, \pi^* \omega), \quad \Phi \leftrightarrow \varphi
\]

Setting \( \varphi^t(x) := \varphi(x, \tau) \) for \( \varphi \) corresponding to the solution \( \Phi \) of the Dirichlet problem \(4.1\) then defines the geodesic ray in question.

Let us also recall that the solution \( \Phi \) of the Dirichlet problem \(4.1\) may alternatively be defined as the following envelope:
\[
\Phi(x) := \sup \{ \Psi(x) : \Psi \in PSH(\mathcal{M}, \Omega) : \Psi_{\partial \mathcal{M}} \leq 0 \}
\]
As shown in \([35]\), in the line bundle case, the geodesic ray \( \varphi^t \) may be realized as a Legendre transform of certain envelopes determined by the test configuration. Here we note that the latter result may be generalized to the “transcendental” setting. To this end first observe that a test configuration \( (\mathcal{X}, [\Omega]) \) for \((X, [\omega])\) determines a concave family
\[
\mathcal{F}^\mu(X, \omega) \subset PSH(X, \omega)
\]
of convex subspaces indexed by \( \mu \in \mathbb{R} \), defined as follows: the subspace \( \mathcal{F}^\mu(X, \omega) \) consists of all \( \varphi \) in \( PSH(X, \omega) \) such that, setting \( \tilde{\varphi}(x, t) := \varphi(x) \), the current
\[
dd^c (\tilde{\varphi} - \mu \log |\tau|^2) + \pi^* \omega
\]
on \( X \times \mathbb{C}^* \) extends to a positive current on \( \mathcal{X} \) in \([\Omega]\). In other words, we demand that the current \( dd^c \tilde{\varphi} + \pi^* \omega \) extends to current on \( \mathcal{X} \) in \([\Omega]\) with Lelong number at least \( \lambda \) along the central fiber of \( \mathcal{X} \) (in a generalized sense, as we are allowing negative Lelong numbers). The family \( \mathcal{F}^\mu(X, \omega) \), thus defined, is clearly a concave family of convex subspaces (it is the “psh analogue” of the filtrations of \( H^0(X, kL) \) defined in \([35, 33]\)). Next, to the family \( \mathcal{F}^\mu(X, \omega) \) we associate the following family of envelopes \( \psi_\mu \) in \( PSH(X, \omega) \):
\[
\psi_\mu(x) := \sup_{\psi \in \mathcal{F}^\mu(X, \omega)} \{ \psi(x), \psi \leq 0 \},
\]
Proposition 4.2. Let \((\mathcal{X}, [\Omega])\) be a test configuration for \((X, [\omega])\). Then the corresponding geodesic ray \(\varphi^t\) in \(PSH(X, \omega)\) may be realized as the Legendre transform (wrt \(t\)) of the envelopes \(\psi_\mu\), i.e.

\[
\varphi^t(x) = \sup_{\mu \in \mathbb{R}} \{\psi_\mu(x) + \mu t\}
\]

Proof. By the definition of the envelopes it is equivalent to prove that

\[
\varphi^t(x) = \sup_{\psi_\mu} \{\psi_\mu(x) + \mu t\}
\]

where the sup ranges over all \(\psi \in \psi \leq 0\) on \(X\). Using the correspondence \([4.2]\) we may identify \(\psi_\mu(x) + \mu t\) with a function \(\Phi_\mu\) in \(PSH(X \times \mathbb{C}^*, \Omega)\), which, by the extension assumption for the elements in the subspace \(\mathcal{F}_\mu(X, \Omega)\), extends uniquely to define an element in \(PSH(X, \Omega)\) (which by construction vanishes on the boundary of \(M\)). But then \(\Phi_\mu \leq \Phi\), the envelope defining the geodesic ray \(\varphi^t\). This proves the lower bound on \(\varphi^t(x)\). To prove the upper bound we note that, by the convexity in \(t\), we may write

\[
\varphi^t(x) = \sup_{\mu \in \mathbb{R}} \{\phi^*_\mu(x) + \mu t\},
\]

where \(\phi^*_\mu\) is the Legendre transform, wrt \(t\), of \(\varphi^t\) (with our sign conventions \(\phi^*_\mu\) is thus concave wrt \(\mu\)):

\[
\phi^*_\mu(x) = \inf_{t} \{\mu t + \varphi^t(x)\}
\]

In particular, \(\phi^*_\mu(x) + \mu t \leq \varphi^t\) and moreover, by Kiselman’s minimum principle, \(\phi^*_\mu(x)\) is \(\omega\)-psh on \(X\). Identifying \(\phi^*_\mu(x) + \mu t\) with a function \(\Phi_\mu\) in \(PSH(X \times \mathbb{C}, \Omega)\), as before, it thus follows that \(\Phi_\mu \leq \Phi\). In particular, \(\Phi_\mu\) is bounded from above and thus extends to define an element in \(PSH(\mathcal{X}, \Omega)\), i.e. the corresponding curvature current is positive. But this means that \(\phi^*_\mu(x) \in \mathcal{F}_\mu(X, \omega)\) which concludes the proof of the upper bound.

Example 4.3. (deformation to the normal cone; compare \([37, 36]\)). Any given (say reduced) divisor \(Z\) in \(X\) determines a special test configuration whose total space \(\mathcal{X}\) is the deformation to the normal cone of \(Z\). In other words, \(\mathcal{X}\) is the blow-up of \(X \times \mathbb{C}\) along the subscheme \(Z \times \{0\}\). Denote by \(\pi\) the corresponding flat morphism \(\mathcal{X} \to \mathbb{C}\) which factors through the blow-down map \(p\) from \(\mathcal{X}\) to \(X \times \mathbb{C}\). This construction also induces a natural embedding of \(X \times \mathbb{C}^*\) in \(\mathcal{X}\). Given a Kähler class \([\omega]\) on \(X\), which we may identify with a class on \(X \times \mathbb{C}\) and a positive number \(e\) we denote by \([\Omega_e]\) the corresponding class \([p^*\omega] - eE\) on \(\mathcal{X}\), where \(E\) is the exceptional divisor and we are assuming that \(c < e\), where \(e\) is defined as the sup over all positive numbers \(c\) such that the class \([\Omega_c]\) is Kähler (i.e. \(e\) is the Seshadri constant of \(Z\) wrt \([\omega]\)). In this setting it is not hard to check that \(\varphi \in \mathcal{F}_\mu(X, \omega)\) iff \(\nu_Z(\varphi) \geq \mu + c\), where \(\nu_Z(\varphi)\) denotes the Lelong number of \(\varphi\) along the divisor \(Z\) in \(X\). The point is that \([p^*\omega] - eE\) may be identified with the subspace of currents in \([p^*\omega]\) with Lelong number at least \(c\) along the divisor \(E\) in \(\mathcal{X}\) which in this case is equivalent to having Lelong number at least \(c\) along the central fiber \([\lambda_0]\), which in turn is equivalent to \(\varphi\) having Lelong number at least \(c\) along \(Z\) in \(X\). In particular, setting \(\mu = \lambda - c\) we have \(\varphi_\lambda = \psi_\mu\), where \(\varphi_\lambda\) is the envelope defined by formula \([3.1]\) i.e. \(u_\lambda = \psi_\mu - \lambda \log \|s\|^2\), where \(u_\lambda\) is defined by \([3.2]\).

Combining Theorem \([3.2]\) with the previous proposition we now arrive at the following
Corollary 4.4. Let $[\omega]$ be a Kähler class on $X$ and $Z$ a divisor in $X$ and fix a positive number $c \in [0, \epsilon]$. Then the following family of curves in the (closure) of the space of Kähler potentials in $[\omega]$

$$\varphi^t_\beta := \frac{1}{\beta} \log \int_{[0,c]} d\lambda e^{\beta(\lambda-c)t} \frac{MA(u_{\beta,\lambda})}{dV}$$

define a family of smooth subgeodesics converging, as $\beta \to \infty$, to the geodesic ray $\varphi^t$ associated to the test configuration $(X, [\omega] - cE)$, defined by the deformation along the normal cone of $Z$. More precisely, the convergence holds in $C^0(X \times [0, \infty[)$.

Proof. As follows immediately from the definitions

$$\varphi^t_\beta = \frac{1}{\beta} \log \int_{[0,A]} d\lambda e^{\beta(\lambda-c)t+\varphi_{\beta,\lambda}}, \quad \varphi_{\beta,\lambda} := u_{\beta,\lambda} + \lambda \log \|s\|^2$$

and hence, by Theorem 3.2

$$\varphi^t_\beta = \frac{1}{\beta} \log \int_{[0,A]} d\lambda e^{\beta(\lambda-c)t+\psi_{\mu}} + o(1), \quad \varphi_{\lambda} := u_{\theta,\lambda} + \lambda \log \|s\|^2,$$

where the $o(1)$-term is independent of $t$ and converges uniformly to 0 on $X \times [0,c]$ as $\beta \to \infty$. As a consequence, for $t \in [0,T]$ we clearly have

$$\varphi^t_\beta = \sup_{\mu \in [-c,0]} (\mu t + \psi_{\mu}) + o(1)$$

(where, as explained in the previous example, $\psi_{\mu} = \varphi_{\lambda}$ for $\mu = \lambda - c$) and by Prop 4.2 the first term above defines the desired geodesic ray $\varphi^t$. Finally, we need to show that the error term above is uniform at $T \to \infty$. To this end we will use a compactification argument. Set, as before $t = -\log |\tau|^2$, where $\tau \in \mathbb{C}^*$. By the definition of the deformation to the normal cone $X$ (see the previous example) the function $\Phi_{\mu}$ defined in the proof of Prop 4.2 defines an $\Omega$-psh function on $X$. We thus a get a family of functions on $X$ defined by

$$\Psi^t_\beta := \frac{1}{\beta} \log \int_{[-c,0]} d\mu e^{\beta \Phi_{\mu}}$$

and such that $\Psi^t_\beta$ increases (by Hölder’s inequality) to the function $\Psi_\infty := \sup_{\mu} \Phi_{\mu}$, which, according to the proof of Prop 4.2 coincides with the envelope $\Phi$ defined by formula 4.3. But the latter envelope is continuous (up to the boundary) on $\mathcal{M}$ and hence it follows from Dini’s lemma that $\Psi^t_\beta$ converges to $\Psi$ uniformly, as desired. The continuity of the envelope $\Phi$ follows from standard arguments in the case when $\mathcal{M}$ is smooth and the back-ground form $\eta$ is Kähler. We recall that the argument just uses that any sequence of $\eta$–psh functions may be approximated by a decreasing sequence of continuous $\eta$–psh functions, as follows from the approximation results in [20] (see for example [15] for a similar situation). Recently, the latter approximation property has been generalized to the case when $\eta$ is merely assumed to be semi-positive (and big) [22] and hence the proof of the continuity still applies in the present situation (strictly speaking the results in op. cit. apply to compact complex manifolds, but we can simply pass to a resolution of the the $\mathbb{C}^*$–equivariant compactification of $X$ fibered over the standard $\mathbb{P}^1$–compactification of $\mathbb{C}$ and adopt the argument using barriers in [8]).

We note that the previous corollary gives a simple proof of the following formula established in [35] (in a very general setting):

□
Corollary 4.5. Denote by $\varphi^t$ the geodesic associated to $Z$ appearing in the previous corollary. Then

$$E_\omega(\varphi^t) = t \int_0^c (\lambda - c)(\omega - \lambda[Z])^n d\lambda$$

for any $t \in [0, \infty]$.

**Proof.** First, let us take $dV = \omega^n$. It then follows immediately from the definition of $\varphi^t$ and the property 2.7 of $E$ that

$$\frac{dE_\omega(\varphi^t)}{dt} \bigg|_{t=0} = \frac{dE_\omega(\varphi^t)}{dt} \bigg|_{t=0} = \lim_{\beta \to \infty} \frac{dE_\omega(\varphi^t)}{dt} \bigg|_{t=0}$$

using, in the last equality, that the integral over $X$ of $MA(u_{\beta, \lambda})$ is equal to the top intersection number of the class $[\omega - \lambda[Z]]$. Next, we recall the well-known fact that $E_\omega$ is affine along (weak) geodesic rays and convex along (weak) subgeodesics ref. But then it follows from basic one variable convex analysis that

$$\frac{dE_\omega(\varphi^t)}{dt} \bigg|_{t=0} = \lim_{\beta \to \infty} \frac{dE_\omega(\varphi^t)}{dt} \bigg|_{t=0}$$

which concludes the proof. Finally, since changing the volume form $dV$ in the definition of $\varphi^t$ only introduces an extra term of the form $f(x)/\beta$ the same argument also proves the general case. \qed

Of course, the test configurations defined by the deformation to the normal cone of a divisor are very special ones. But the convergence result in Cor 4.4 can be extended to general test configurations for a polarized manifold $(X, L)$ (by replacing $MA(u_{\beta, \lambda})$ with $MA(\varphi_{\beta, \mu})$ where $\varphi_{\beta, \mu} \in F_{\mu}(X, \omega)$ satisfies the equation 3.5). The argument uses Odaka’s generalization of the Ross-Thomas slope theory [28] defined in terms of a flag of ideals on $X$. The point is that by blowing up the corresponding ideals one sees that the pullback of the corresponding envelopes $\psi^\mu$ have divisorial singularities (compare Prop 3.22 in [26]) so that the previous convergence argument can be repeated (as they apply also when $L$ is merely semi-ample and big, which is the case on the blow-up).

More generally, an analytic generalization of test configurations for a polarization $(X, L)$ was introduced in [35]. Similarly, an **analytic test configuration** for a Kähler manifold $(X, \omega)$ may be defined as a concave family $[\psi^\mu]$ of singularity classes in $PSH(X, \omega)$. The corresponding space $F^\mu(X, \omega)$ may then be defined as all elements $\psi$ in such that $[\psi] = [\psi^\mu]$. To any such family one associates a family of envelopes $\psi^\mu$ defined by formula 4.4. As shown in [35] taking the Legendre transform of $\psi^\mu$ wrt $\mu$ gives a curve $\varphi^t$ in $PSH(X, \omega)$ which is a weak geodesic. The regularization scheme introduced in this paper could be adapted to this general framework by first introducing suitable algebraic regularizations of the singularity classes and using blow-ups (as in [28]). But we leave these developments and their relation to K-stability and the Yau-Tian-Donaldson conjecture for the future. For the moment we just observe that the latter conjecture admits a natural generalization to transcendental classes.
4.0.1. A generalization of the Yau-Tian-Donaldson conjecture to transcendental classes.

Using Wang’s intersection formula \[44\] there is a natural generalization of the notion of K-stability of a polarization \((X,L)\): by definition, a Kähler class \([\omega]\) on \(X\) is \(K\)-stable if, for any test configuration \((\mathcal{X},[\Omega])\) for \((X,[\omega])\) the corresponding Donaldson-Futaki invariant satisfies \(DF(\mathcal{X},[\Omega]) \geq 0\) with equality iff \(\mathcal{X}\) is equivariantly isomorphic to a product. Similarly, \(K\)-polystability is defined by not requiring that the isomorphism be equivariant. Here \(DF(\mathcal{X},[\Omega])\) is defined as the following sum of intersection numbers

\[
DF(\mathcal{X},[\Omega]) := a[\Omega]^{n+1} + (n+1)K_{X/P^1} \cdot [\Omega]^n, \quad a := n(-K_X) \cdot [\omega]^{n-1}/[\omega]^n
\]

where we have replaced \(\mathcal{X}\) with its equivariant compactification over \(P^1\) and \([\Omega]\) with the corresponding class on the compactification and the intersection numbers are computed on the compactification. The transcendental version of the Yau-Tian-Donaldson conjecture may then be formulated as the conjecture that \([\omega]\) admits a constant scalar curvature metric iff \((X,[\omega])\) is \(K\)-polystable. It is interesting to compare this generalization with Demailly-Paun’s generalization of the Nakai-Moishezon criterium for ample line bundles \[21\], which in the case when \(X\) is a projective manifold says that if a \((1,1)\)-class \([\theta]\) has positive intersections with all \(p\)-dimensional subvarieties of \(X\) then \([\theta]\) contains a Kähler form \(\omega\). The difference is thus that in order to draw the considerably stronger conclusion that \(\omega\) can be chosen to have constant scalar curvature one needs to impose conditions on “secondary” intersection numbers as well, i.e. intersection numbers defined over all suitable degenerations of \((X,[\theta])\). Finally, it should be pointed out that it may very well be that the notion of (transcendental) test configuration above has to be generalized a bit further in order for the previous conjecture to stand a chance of being true (compare the discussion in the introduction of the paper).

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