Nonlocality of the Schrödinger cat

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Abstract. The nonlocality of the Schrödinger cat is established. A correlation measurement that exhibits quantum interference in the form of the Wigner function of the cat state is introduced. The relation of this correlation to quantum interference in phase space and the nonlocal character of the Schrödinger cat are discussed. It is shown that the Schrödinger cat correlation function leads to violation of the Bell inequality for macroscopically distinguishable states of the cat.

1. Introduction

Probability amplitudes are at the core of the probabilistic interpretation of quantum mechanics. Among many representations of the probability amplitudes, the Wigner function offers the appealing possibility of being able to describe quantum phenomena using the classical-like concept of a phase space distribution function.

In this paper quantum nonlocality of the Schrödinger cat in the Wigner representation will be studied. Using the Wigner function, we show that there are links among quantum interference, entangled correlations and quantum nonlocality. An experimental set-up which exhibits the role of the Wigner function of the Schrödinger cat and its relation to Bell inequalities is proposed. We show that the Schrödinger cat violates the Bell inequality, indicating that entangled correlations of such a system cannot be described by a realistic model of hidden variables.

2. Quantum interference in phase space

The fundamental difference between quantum and classical interference of radiation is that the particle–wave duality exhibited by quantum systems leads to interference between probability amplitudes rather than between physical realities such as the electromagnetic waves. We shall study quantum interference effects of probability amplitudes in phase space using the Wigner function, which allows for a classical-like description of quantum phenomena.
function. Such interference for distinguishable states, combined with quantum entanglement, will lead to the Schrödinger cat.

The phase space of a single mode, quantized light, will be labelled by a complex number \( \alpha = (1/\sqrt{2})(q + ip) \), corresponding to the coherent state \(|\alpha\rangle\), which is extensively used in quantum optics. In a coherent state representation the Wigner function can be written in the following form [1]–[3]:

\[
W_\alpha = \frac{2}{\pi} \langle \Psi |\hat{\Pi}(\alpha)|\Psi \rangle
\]

(1)

where

\[
\hat{\Pi}(\alpha) = \hat{D}(\alpha)\hat{\Pi}\hat{D}^\dagger(\alpha)
\]

(2)

is the parity operator \( \hat{\Pi} = (-1)^\hat{a} \), shifted in phase space by \( \alpha \), with the help of the coherent state displacement operator \( \hat{D}(\alpha) \).

Because the Wigner function (1) is bilinear in the wavefunction, it can be used to exhibit quantum interference in phase space transparently. This follows from the fact that, for a linear superposition \(|\Psi\rangle = |\Psi_1\rangle + |\Psi_2\rangle\), the corresponding Wigner function is

\[
W_\Psi = W_{\Psi_1} + W_{\Psi_2} + W_{12}
\]

(3)

where the last term describes the interference between the two probability amplitudes \(|\Psi_1\rangle\) and \(|\Psi_2\rangle\).

The general character and the structure of the quantum interference in phase space can be easily understood if the Wigner function of a linear superposition of two plane waves \( e^{ip_0x} \) and \( e^{-ip_0x} \) (using the convention \( \hbar = 1 \)) is investigated. For the two plane waves the Wigner function is

\[
W(q,p) \simeq \delta(p - p_0) + \delta(p + p_0) + 2\delta(p)\cos(2p_0q).
\]

(4)

The first two terms describe two sharply localized positive distributions located at \( \pm p_0 \), whereas the last term is nonpositive and describes the interference of the two amplitudes. The interference term is located at the mean momentum of the two plane waves and oscillates with a frequency corresponding to the momentum separation \( 2p_0 \) of the two states. This general property will be valid for linear superpositions of arbitrary wavepackets.

For example, we shall study the interference phenomena that can occur for a system radiating semiclassical fields. The best known example of such a semiclassical field is the coherent state of a single mode electromagnetic field. Interesting features occur when interference phenomena for a linear superposition of such semiclassical fields are investigated. The simplest case of such a superposition is given by a linear combination of two ‘mirror-like’ coherent states:

\[
|\Psi\rangle \sim |\alpha_0\rangle + |-\alpha_0\rangle.
\]

(5)

The state (5), called the even coherent state (ECS), exhibits properties such as the reduction of quadrature fluctuations below the vacuum level and the oscillation of the photon number distribution [4, 5]. The appearance of these nonclassical features of the ECS has been attributed to the effect of quantum interference [6].

In configuration space, the parameter \( D = 2\alpha_0 \) plays the role of a ‘distance’ (in suitably selected dimensionless units) between the two coherent states \(|\alpha_0\rangle\) and \(|-\alpha_0\rangle\). The two states are said to be macroscopically distinguishable if the uncertainty regions of the two coherent states do not overlap. This is equivalent to the condition that \( D \gg \sqrt{2} \). The Wigner function for the
ECS given by (5) has been calculated and discussed in several papers [7]. The Wigner function of this state is

$$W(\alpha) = W_0(\alpha - \frac{D}{2}) + W_0(\alpha + \frac{D}{2}) + 2W_0(\alpha) \cos[2D \text{Im}(\alpha)]$$  \hspace{1cm} (6)

where $W_0(\alpha) = \frac{2}{\pi} \exp(-2|\alpha|^2)$ is the Wigner function of the vacuum state $|0\rangle$.

As can be seen in figure 1, the Wigner function of the linear superposition is not positive due to quantum interference. In the limit of $D = 0$, the quantum state (5) is just a harmonic oscillator vacuum state and there are no quantum interference effects. Much has been said about the negative features of this Wigner function. In most cases the nonpositive character of this function has been associated with the nonclassical character of the ECS (5). As we shall see in the following sections, the interference pattern exhibited by the ECS will become accessible to possible experimental tests of the Schrödinger cat.

In recent years linear superpositions of two spatially localized Gaussian wavepackets have become an experimental reality. A double slit experiment involving cold atoms [8] and a fractional revival of atomic wavepackets of highly excited Rydberg atoms [9], provide examples of linear superpositions very similar to expression (5). Such linear superpositions are an interesting laboratory framework in which to study the relation between classical and nonclassical interference effects [10].

3. Entanglement and interference

A striking feature of the superposition principle emerges when two or more different physical systems are investigated. As pointed out by Einstein, Podolsky and Rosen (EPR) [11, 12], an unfactorizable state of two particles exhibits remarkable nonlocal quantum correlations defying the classically intuitive concept of local realities associated with each individual particle. An entangled state of two particles (labelled 1 and 2) can be written in the following form:

$$|\Psi\rangle = |\Psi_1\rangle|\Phi_1\rangle + |\Psi_2\rangle|\Phi_2\rangle.$$  \hspace{1cm} (7)

The quantum interference for an entangled state involving a macroscopic system has been criticized by Schrödinger in his famous cat paradox [13]–[15]. In this thought experiment, a
Schrödinger cat and an unstable atom are placed in a quantum linear superposition involving a living cat with an undecayed atom (↑) and a dead cat with a decayed atom (↓).

Schrödinger stated that the wavefunction of the composed system would express this interference; ‘having in it the living and the dead cat [...] mixed or smeared out in equal parts’, attributed quantum effects to a ‘blurring’ of sharp realities and claimed that no blurring should occur in the macroscopic domain. It has been shown recently that the relation between the interference of probability amplitudes and the quantum blurring of the Schrödinger cat can be derived from quantum mechanical considerations using the coherent state phase space methods [16].

The entanglement of two wavepackets with the atomic system that has been generated in recent experiments [17, 18] provides a very close mesoscopic analogy to the Schrödinger cat state, with the living cat represented by a coherent state |D/2⟩, and the dead cat represented by a mirror coherent state |−D/2⟩. Such states corresponding to the Schrödinger cat are represented by the motional degree of freedom of a trapped ion, or as an electromagnetic field in a microwave cavity. In both cases the atomic states correspond to the internal degrees of freedom of the irradiated atom or ion. The entangled state of the composite system, that can be interpreted as a mesoscopic realization of the Schrödinger cat, can be written in the following form:

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |D/2\rangle \otimes |\uparrow\rangle + |D/2\rangle \otimes |\downarrow\rangle \right). \tag{8}
\]

A different scheme leading to an entangled superposition of macroscopically distinguishable states involves trapping and cooling of atoms. An entangled wavefunction prepared by a process of laser cooling based on velocity-selective coherent population trapping (VSCPT) can be used as a different model of the Schrödinger cat [19]. In such a scheme the atomic external degree of freedom is described as atomic wavepackets moving with momentum p along a given direction. The atom–laser interaction modifies the atomic momentum by the photon momentum ±k projected along the direction of motion. Owing to the VSCPT, the atomic motion becomes entangled with the internal degree of freedom of the atom. As a result a symmetrical dark state is produced. This entangled state has the form

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |p + k\rangle \otimes |\uparrow\rangle + |p - k\rangle \otimes |\downarrow\rangle \right). \tag{9}
\]

where the momentum states |p ± k⟩ are the mesoscopic analogues of the dead and the living cat. The preparation of such a state has been realized for helium atoms [20].

It is generally believed that, due to quantum decoherence, the Schrödinger cat will not exhibit quantum interference effects on time scales larger that the decoherence time of the entangled superposition states. For such times the Schrödinger cat state will become a mixed state. For example, for the state described by equation (8), as a result of decoherence, we ought to have the following density operator:

\[
\hat{\rho} = \frac{1}{2} \left( |D/2\rangle \langle D/2| \otimes |\uparrow\rangle \langle \uparrow| + \frac{1}{2} |D/2\rangle \langle -D/2| \otimes |\downarrow\rangle \langle \downarrow| \right). \tag{10}
\]

The time taken for transition of the state (8) into the coherent mixture (10) for various engineered reservoirs has been investigated recently [21].
4. Quantum correlations of the Schrödinger cat

In this section, we shall prove that the mesoscopic state (8) leads to strong correlations between the cat state and the atomic system. In order to study the nonlocality of the cat state, we shall extend the technique of quantum state reconstruction from [22, 23].

We shall probe the correlated cat state with the help of the set-up presented in figure 2. The harmonic excitation, corresponding to the cat, will be probed by a detection scheme that can resolve the number of simultaneously detected quanta and provides only a binary +1 or −1 outcome, depending on whether an even or an odd number of quanta has been registered. Owing to an additional interaction with a classical pump source, the detected signal becomes shifted by an amount $\alpha$ representing the complex amplitude of the classical pump field. The quantum observable corresponding to such a measurement is given by the shifted parity operator $\hat{\Pi}(\alpha)$ from equation (2). For photons, such a shift can be achieved by mixing the signal field with the pump field on a beam splitter with very high transmission mirrors. For other harmonic excitations, such a shift can be obtained by applying an external classical force to the system.

The state of the atom is detected by measuring the projection $\hat{\sigma}(\theta)$ of the atomic Bloch vector $|\theta\rangle$ onto a direction $\theta$ represented by the vector $|\theta\rangle = \sin(\theta/2)|\uparrow\rangle + \cos(\theta/2)|\downarrow\rangle$.

This joint measurement on the system is described by the following dichotomous observable for the cat and the atom:

$$\hat{\Pi}(\alpha, \theta) = \hat{\Pi}(\alpha) \otimes \hat{\sigma}(\theta).$$

The atom and the cat correlations revealed in such an experiment are given by the following quantum expectation value of this operator:

$$\Pi(\alpha, \theta) \equiv \langle \Psi | \hat{\Pi}(\alpha, \theta) | \Psi \rangle.$$  \hspace{1cm} (12)

For the entangled Schrödinger cat state (8) this correlation is

$$\Pi(\alpha, \theta) = \frac{1}{2} \cos \theta \left( e^{-2|\alpha+D/2|^2} - e^{-2|\alpha-D/2|^2} \right) + e^{-2|\alpha|^2} \cos[2D \text{ Im}(\alpha)] \sin \theta. \hspace{1cm} (13)$$

This correlation can be measured by performing local measurements on the cat and the atomic states. A remarkable property of this correlation is that it involves the Wigner function of the linear superposition (5), weighted by the atomic spin orientations $\sin \theta$ and $\cos \theta$. Indeed, up to a normalization constant, the correlation function (13) is

$$\Pi(\alpha, \theta) \simeq \frac{1}{2} \cos \theta \left[ W_0 \left( \alpha + \frac{D}{2} \right) - W_0 \left( \alpha - \frac{D}{2} \right) \right] + W_0(\alpha) \cos[2D \text{ Im}(\alpha)] \sin \theta. \hspace{1cm} (14)$$
This formula exhibits the interference and the entanglement of the atom and the cat system. This correlation function, bounded by $+1$ and $-1$, is plotted in figure 3. For $D = 0$ the correlation function can be factorized:

$$\Pi(\alpha, \theta) = W_0(\alpha) \sin(\theta)$$

indicating the absence of any correlations of the atomic system to the vacuum state. We shall note that the correlation function (13) exhibits perfect correlations, i.e. correlations such that one can predict with certainty the state of the cat, from the measurement of the atom. As an example let us take $\alpha = 0$ corresponding, in the Wigner representation, to a superposition of a live and a dead cat with equal weights. In this case

$$\Pi(0, \uparrow) = \Pi(0, \downarrow) = 0$$

indicating, in agreement with classical intuition, that, for an undecayed and a decayed atom, it is impossible to see the living and the dead cat mixed in equal parts. On the other hand, such a state becomes a certainty if the atom is blurred in equal parts between the decayed and the undecayed states:

$$\Pi(0, \rightarrow) = \Pi(0, \pi/2) = 1.$$  

5. Violation of local reality by the Schrödinger cat

In the standard description of Bell inequalities, the violation of local realism is associated with the nonlocal character of quantum mechanics. It is worth pointing out an important difference between the nonlocal character of the EPR correlations and the Schrödinger cat state. In the EPR correlations, one performs measurements on one particle, without ever disturbing the second particle of the entangled state located in a remote region of space. Quantum entanglement in such a case involves two different particles. In the cat problem, the measurements are performed in a cavity or an ion trap with the cat and the decaying atom being parts of the same system. The two entangled parts (internal and external degrees of freedom of an atom) do not interact with
each other, because of the proper separation of the variables involved in the preparation of the cat state. Such variables are measured by performing local experiments. A possible violation of the Bell inequality indicates that a local hidden variable model cannot describe these two variables tested by local experiments. Because of this we shall associate the violation of the Bell inequality for the cat state with a failure to describe the experiment, using local measurements and local correlations of objective realities associated with the two subsystems.

Such a theory based on hidden variables for the Schrödinger cat atom correlations is based on the assumption that quantum correlations can be written in the form of a statistical ensemble over local hidden variables $\lambda_c$ and $\lambda_a$ for the cat and the atom, distributed with a positive probability:

$$\Pi(\alpha, \theta) = \int d\lambda_c \int d\lambda_a \Pi(\alpha, \lambda_c) \sigma(\theta, \lambda_a) P(\lambda_a, \lambda_c).$$

(18)

This expression involves local dichotomous realities of the cat $\Pi(\alpha, \lambda_c) = \pm 1$ and the atom $\sigma(\theta, \lambda_a) = \pm 1$ that are supposed in order to replace the quantum observables.

For correlations described by such local hidden variable theories, the Clauser–Horne–Shimony–Holt combination [24, 25]

$$B = \Pi(\alpha'; \theta') + \Pi(\alpha'; \theta) + \Pi(\alpha; \theta') - \Pi(\alpha; \theta)$$

(19)

satisfies the Bell inequality

$$-2 \leq B \leq 2$$

(20)

for any choice of the parameters $\alpha$ and $\alpha'$ and angles $\theta$ and $\theta'$.

The quantum mechanical correlation function (13) has a clear analogy to the hidden variable model described above, or to the Bell inequality for spin-$\frac{1}{2}$ EPR correlations. The parity of absorbed quanta and the projection of the atomic spin both have binary outcomes. The adjustable parameters of the apparatuses are represented by $\alpha$ and $\theta$. As a result the Bell inequality (20) derived for a measurement of local realities bounded by $-1$ and $+1$ can be applied. This provides a test of the nonlocal character of the entangled state, using the Wigner function of the cat state.

In figure 4, the violation of the Bell inequality (20) is exhibited for the case in which the coherent displacements are real with $\alpha' = 0$ and $\alpha = x$ and the distance between the coherent states is $D = 4$. We take the atomic orientations to be $\theta' = \pi/2$ and $\theta$. This plot shows that the correlation function (13) cannot be interpreted as a local average of local realities, so it violates the upper bound (20) imposed by local theories.

Figure 4. Violation of the Bell inequality by $B$. 

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6. Conclusions

We have shown that one can test the nonlocality of the Schrödinger cat, if the parity of emitted quanta is correlated to the direction of the atomic Bloch vector. We have shown that such correlations involve the Wigner function of the linear superposition of the cat state. Such correlations violate the upper bound of the Bell inequality, indicating that no local model of the cat and the atom correlations can describe an entangled Schrödinger cat.

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References

[1] Wigner E 1932 Phys. Rev. 40 749
[2] Tatarski j V I 1983 Sov. Phys.–Usp. 26 311
[3] Hillery M, O’Connell R F, Scully M O and Wigner E P 1984 Phys. Rep. 106 121
[4] Schleich W, Perinío M and Le Kien F 1991 Phys. Rev. A 44 2172
[5] Bužek V, Vidiella-Barranco A and Knight P L 1992 Phys. Rev. A 45 6570
[6] Bužek V and Knight P L 1995 Progress in Optics XXXIV ed E Wolf (Amsterdam: North-Holland)
[7] Garry C C and Knight P L 1997 Am. J. Phys. 65 964
[8] Kurtsiefer Ch, Pfau T and Mlynek J 1997 Nature 386 150
[9] Noel M W and Stroud C R 1996 Phys. Rev. Lett. 77 1913
[10] Wódkiewicz K and Herling G 1998 Phys. Rev. A 57 815
[11] Einstein A, Podolsky B and Rosen N 1935 Phys. Rev. 47 777
[12] Redhead M 1987 Incompleteness, Nonlocality, and Realism (Oxford: Clarendon)
[13] Schrödinger E 1935 Naturwissenschaften 23 807
[14] Schrödinger E 1935 Naturwissenschaften 23 823
[15] Schrödinger E 1935 Naturwissenschaften 23 844
[16] Wódkiewicz K 2000 Opt. Commun. 179 215
[17] Brune M et al 1996 Phys. Rev. Lett. 77 4887
[18] Monroe C, Meekhof D M, King B E and Wineland D J 1996 Science 272 1131
[19] Arimondo E 1995 Progress in Optics XXXV ed E Wolf (Amsterdam: North-Holland) p 257
[20] Bardou F et al 1994 Phys. Rev. Lett. 72 203
[21] Myatt C J et al 2000 Nature 403 269
[22] Banaszek K and Wódkiewicz K 1996 Phys. Rev. Lett. 76 4344
[23] Banaszek K and Wódkiewicz K 1999 Phys. Rev. Lett. 82 2009
[24] Clauser J F, Horne M A, Shimony A and Holt R A 1969 Phys. Rev. Lett. 23 880
[25] Bell J S 1971 Foundations of Quantum Mechanics ed B d’Espagnat (New York: Academic)