Ultimate pressure determination in hydraulic fracturing of boreholes for different rock failure criteria

AM Kovrizhnykh* and MV Kurlenya
Chinakal Institute of Mining, Siberian Branch, Russian Academy of Sciences, Novosibirsk, Russia
E-mail: *amkovr@mail.ru

Abstract. The problem on elastoplastic deformation and rock failure around boreholes under the action of internal pressure is considered. The stress state in the intact rock mass is assumed to be hydrostatic. The hydraulic fracturing pressure is conventionally determined by the maximum normal stress criterion. However, the experimental studies described in scientific articles show that this criterion is not consistent with the results obtained in the course of laboratory tests on failure of cylindrical and spherical cavities made in rock samples. It has been known that test results on complex loading for solid samples of various materials are not confirmed by the theory of maximum normal stress both in terms of ultimate load value and direction of failure surface propagation. In this case, for the complex stress state of rocks, it is proposed to determine the ultimate pressure by the experimentally substantiated fracture criteria which are in good agreement with the results of laboratory tests on hydraulic fracturing of boreholes.

1. Introduction
As it is known, the initial stresses in intact massif affect both the hydraulic fracturing pressure (breakdown pressure) in borehole and the pattern of spalls formed on the surface [1–6]. The study of the formation mechanism of the latter and the breakdown pressure analysis can be used for evaluation of the stress state in rocks. A detailed review of theoretical research and experimental works on measuring hydraulic fracturing of boreholes can be found in [1, 4, 8]. Remarkably, the pressure on the failure plane and its direction are commonly determined using the maximum normal stress criterion due to its simplest mathematical formulation [4]. However, results of the experimental laboratory study of the cylindrical and spherical cavities failure conducted on rock samples have revealed that this criterion can only be applied under low external hydrostatic pressures in hard-brittle rocks, while in the case of soft-plastic rocks, the experimental results are generally found to be in disagreement with the theory of maximum normal stress [5]. It turns out that these experimental data can be explained from the perspective of two different theories of failure. The former is based on the Coulomb – Mohr strength criterion and the assumption that the irreversible deformation that occurs before the failure is insignificant. The latter theory suggests that as the borehole pressure increases, so does the irreversible deformation on its contour. Rock failure on the borehole surface takes place when the maximum plastic displacement reaches a certain value which is inherent in this particular material. Thus, in the second theory, the ultimate hydraulic fracturing pressure is determined based on the maximum plastic displacement criterion.

2. Mathematical model and results
Solution of the problem of the elasticity theory for determining the stress state developing around a borehole is an important practical application to the hydraulic fracturing mechanics. The best known elastic-brittle solution to this problem is widely recognized to be found in [1], where the authors presented a borehole as a very long cylinder with internal pressure \( p \) applied to the surface \( r = a \). The external hydrostatic rock pressure is designated as \( q \). The elastic solution to the problem of loading a cylindrical borehole with an internal pressure \( p \) has the form:

\[
\sigma_r = -q + (q - p) \frac{a^2}{r^2}, \quad \sigma_\theta = -q - (q - p) \frac{a^2}{r^2}.
\]

The brittle rock failure is initiated when the Coulomb-Mohr criterion is fulfilled on the contour \( r = a \). Using this criterion and formula (1), we calculate the hydraulic fracturing pressure

\[
p_\ast = q(1 + \sin \varphi) + \sigma_r (1 + \sin \varphi) / 2.
\]

where \( \varphi \) is the angle of internal friction; \( \sigma_r \) is ultimate tensile strength. The borehole failure plane passes through the forming plane and forms an angle with the \( r \) direction. When \( \varphi = 0 \) we obtain the Tresca criterion \( p_\ast = q + \sigma_s / 2 \). If \( \varphi = \pi / 2 \), then from (2) follows the maximum normal stress criterion, \( p_\ast = 2q + \sigma_s \).

The problem of loading a spherical cavity with internal pressure can be solved in a similar way. In brittle rocks, the failure occurs when the Coulomb-Mohr criterion is fulfilled on the contour \( r = a \). The breakdown pressure for a spherical cavity is determined by the formula

\[
p_\ast = 3q \frac{1 + \sin \varphi}{3 - \sin \varphi} + 2\sigma_r \frac{1 + \sin \varphi}{3 - \sin \varphi}.
\]

If in (3) \( \varphi = 0 \), then \( p_\ast = q + 2\sigma_s / 3 \), if \( \varphi = \pi / 2 \) the breakdown pressure is \( p_\ast = 3q + 2\sigma_s \).

The work [5] presents results of the laboratory experiments on cylindrical and spherical hydraulic fracturing. The experiments were carried out on cylindrical cores (diameter: 10 cm; length: 12 cm) of the four representative types of rocks from the Carthage, Indian, Luders and Austin quarries. The core selection criteria were based on the homogeneity in composition and difference in the mechanical properties of rocks which varied from hard-brittle to soft-plastic. The cylindrical and spherical cavities drilled in the selected cores received fluid under pressure \( p \) via a pressed steel tube. The core material was squeezed under external lateral pressure \( q \) and axial force.

The theoretical dependencies constructed using the experimental data on hydraulic fracturing of a cylindrical cavity were taken as the initial experimental data. In [6], a line consisting of two straight intercepts approximates these data perfectly well. Up to \( q = 27.58 \) MPa was accepted as \( p^* = 2q + k \), where \( k = 28.27 \) MPa. Further, at \( q > 27.58 \) MPa the hydraulic fracturing pressure is \( p_\ast = q + k \), where \( k_1 = 55.85 \) MPa.

The calculated dependence \( p_\ast = p_\ast(q) \) in the problem of a spherical cavity was based on formula (3), in which the strength parameters \( \varphi \) and \( \sigma_s = 2k \) were determined from the experimental data obtained for the cylindrical cavity failure. Concerning hard-brittle rocks, it is shown in [6] that for values of external pressure \( q \leq 27.58 \) MPa, the criterion of maximum normal stress is consistent with the experiment results [5], while for other values \( q \geq 27.58 \) MPa, this criterion fails to be confirmed by experimental studies. Let's consider similar experimental studies on cores of soft-plastic rocks from the Austin quarry [5]. The calculated dependencies for cylindrical and spherical cavities are based on formulas (2) and (3). The Mohr-Coulomb failure envelope curve in this case consisted of a single straight intercept with \( \varphi = 0.2 \), \( \sigma_s = 34.48 \) MPa. The calculation results given in [6] are amply underpinned by the experimental data [5]. Thus, the correct interpretations of the experimental data on hydraulic fracturing and their reliable application for determining stresses in the intact massif requires conducting experiments to determine certificates of strength and plasticity in the form of a Coulomb –
Mohr envelope curve and establish ultimate plastic shear strain at the moment of failure which increases exponentially with increasing hydrostatic pressure. A new deformation failure criterion proposed below allows calculating the ultimate loads during collapse of mine workings and boreholes [7].

The use of a strain based failure criterion for solids of specific geometry suggests that under certain boundary conditions, there can be found either analytical or numerical solution to the problem which allows determining stresses at each point, as well as elastic and irreversible (plastic) deformations. Solving such problems in a spatial or plane formulation provides serious challenges. Complete analytical solutions are obtained only in one-dimensional elastic-plastic problems using the Tresca-Saint-Venan criterion. For these purposes, the problems with cylindrical and axial symmetry [7] are considered to be the most pertinent. This is either a thick-walled cylinder, a sphere, or cylindrical and spherical cavities in an unlimited body. The analytical solution of the elastic-plastic problem presented in [7] allows measuring stresses, plastic deformations, ultimate pressure, and fracture plane, which is an important practical application to the problem of hydraulic fracturing of a borehole.

The problem of loading a thick-walled cylinder with internal and external radii \( r = a \) and \( r = b \) is considered in plain strain conditions [6, 7]. If pressure is \( p < p_0 \), then the pipe is deformed elastically and therefore the dependence of pressure on dimensionless displacement \( u_a = u(a)/a \) will be linear. When \( p = p_0 = k(1-a^2/b^2) \) the inner surface of the cylinder \( r = a \) passes into a plastic state. Let \( c \) be the radius of plastic region. In the elastic \( c \leq r \leq b \) and plastic regions \( a \leq r \leq c \) we have the equilibrium equation and compatibility condition for deformations:

\[
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad \frac{de_\theta}{dr} + \frac{e_\theta - e_r}{r} = 0. \quad (4)
\]

By applying Hooke's law to the elastic region and the boundary conditions when \( r = b \) and \( \sigma_\theta - \sigma_r = 2k \) if \( r = c \), we obtain:

\[
\sigma_\theta = kc^2 \left( \frac{1}{r^2} + \frac{1}{b^2} \right), \quad \sigma_r = -kc^2 \left( \frac{1}{r^2} - \frac{1}{b^2} \right), \quad \frac{u}{r} = k \left( \frac{1}{2} \right) \left( \frac{1}{r^2} - \frac{1}{b^2} \right). \]

It can be seen from the elastic stresses analysis that the inequality \( \sigma_\theta > \sigma_z > \sigma_r \) is true in the zone of irreversible deformations \( a \leq r \leq c \), then we determine the stresses from the equilibrium equation, taking into account the plasticity criterion and the boundary condition for \( \sigma \), on the surface \( r = a \):

\[
\sigma_\theta - \sigma_r = 2k, \quad \sigma_r = -p + 2k \ln \frac{r}{a}, \quad \sigma_z = \nu(\sigma_\theta + \sigma_r). \quad (5)
\]

The continuity condition \( \sigma_r \) in the elastic-plastic boundary leads to the equation:

\[
2 \ln \frac{c}{a} + 1 - \frac{c^2}{b^2} = \frac{p}{k}. \quad (6)
\]

In the irreversible deformations zone, the flow theory equations for the Tresca failure surface represent a displacement \( \gamma \) in the direction of the ultimate shear stress:

\[
\varepsilon_\theta = \frac{\gamma}{2} + \frac{1-\nu}{2\mu} \sigma_\theta - \frac{\nu}{2\mu} \sigma_r, \quad \varepsilon_r = -\frac{\gamma}{2} - \frac{\nu}{2\mu} \sigma_\theta + \frac{1-\nu}{2\mu} \sigma_r. \quad (7)
\]

By substituting the stresses from (5) into (7), and then the deformations \( \varepsilon_r \) and \( \varepsilon_\theta \) into the compatibility condition (4), we obtain:

\[
\frac{d\gamma}{dr} + \frac{2\gamma}{r} + \frac{4(1-\nu)k}{\mu} \frac{1}{r} = 0. \quad (8)
\]

We solve this equation under the condition that for \( r = c \), the plastic shift \( \gamma = 0 \):
Substituting (9) in (7), we determine the movement in the zone of irreversible deformations:

\[
\frac{u}{r} = \frac{(1-\nu)k}{\mu} \frac{c^2}{r^2} \left( 1 - \frac{1-2\nu}{2\mu} \left( p - \frac{2k \ln r}{a} \right) \right). 
\]  

(10)

Failure in the region \(a \leq r \leq c\) occurs when the displacement \(\gamma\) in (9) reaches the limit value \(\gamma^*\). The highest value \(\gamma\) is reached when \(r = a\) and therefore the failure begins from the inner surface of the pipe. Let \(c_\ast\) be the radius of the failure front, then it follows from (9):

\[
\frac{c^2}{c_\ast^2} = 1 + \frac{\mu \gamma^*}{2(1-\nu)k}. 
\]  

(11)

Since the fracture zone has zero shear strength, on the inner surface we have \(\sigma_r = \sigma_\theta = -p\) from the equilibrium equation and the boundary condition for \(\sigma_r\). When applying the elastic volume change hypothesis and the continuity condition of the radial displacement \(u\) at the fracture front, we obtain:

\[
\frac{u}{r} = \frac{(1-\nu)k}{\mu} \frac{c^2}{r^2} \left( 1 - \frac{1-2\nu}{2\mu} p \right). 
\]

Dimensionless displacement on the inner surface of the pipe will be:

\[
u = \frac{(1-\nu)k}{\mu} \frac{c^2}{a^2} \left( 1 - \frac{1-2\nu}{2\mu} p \right). 
\]  

(12)

The solution in the region of irreversible deformations \(c_\ast \leq r \leq c\), given there is a failure zone, is found from (5), (6) and (10) by replacing \(a\) with \(c_\ast\). From the continuity condition of the radial stress at \(r = c_\ast\) it follows:

\[2 \ln \frac{c}{c_\ast} + 1 - \frac{c^2}{b^2} = \frac{p}{k},\]  

(13)

In an unbounded body, when \(b \to \infty\), we calculate from (11) and (13) the ultimate pressure at which the cylindrical cavity failure is initiated:

\[p = p_\infty = k + k \ln \left( 1 + \frac{\mu \gamma^*}{2(1-\nu)k} \right).\]  

(14)

If the displacement is preset in (12) on the contour \(r = a\), this will allow to find radius of the plastic zone \(c\), while the radius of fracture front \(c_\ast\) will be inferred from (11). For materials with \(\gamma^* = \infty\), there is not any point at which the plastic displacement \(\gamma\) reaches the limit value, and therefore, in an unlimited body, an increase in the internal pressure leads to an increase in the plastic region.

In the problem of a thick-walled pipe from (9), it follows that the displacement \(\gamma\) takes the greatest value on the inner surface of the cylinder \(r = a\), when \(c = b\) we have:

\[\gamma_m = \frac{2(1-\nu)k}{\mu} \left( \frac{b^2}{a^2} - 1 \right).\]

If \(\gamma^* > \gamma_m\), then at \(p_0 < p < p_m\) the cross section of the pipe will have elastic and plastic regions. When \(0 \leq \gamma^* \leq \gamma_m\), at certain values of pressure \(p\), the cross section of the pipe may have three regions: elastic, plastic regions and a failure zone. If the latter is present in the cross section of the pipe, i.e. \(c_\ast \geq a\), then, taking into account (11), (12) and (13), we obtain a linear relationship between \(u_\ast\) and \(p\):
\[
\begin{align*}
    u_a &= A_a - B_a p,
    \quad A_a = \frac{(1-\nu)b^2}{\mu a^2} p_\infty, \quad B_a = \frac{(1-\nu)b^2}{\mu a^2} + \frac{1-2\nu}{2\mu}.
\end{align*}
\]

Each value \( \gamma^* \) has corresponding values of \( p^* \) and displacement \( u^* \), starting from which the pressure \( p \) declines according to the linear law, as the displacement \( u_a \) increases. The pressure drop is initiated from the moment when the irreversible shear strain on the inner contour of the pipe reaches its ultimate value. When \( c^* = a \), then from (11) we calculate \( c \) and, by substituting the found values in (13), we obtain:

\[
p^* = p_\infty - \frac{a^2}{b^2} \left( k + \frac{\mu \gamma^*}{2(1-\nu)} \right).
\]

The above allows us to determine the hydraulic fracturing pressure in the borehole at zero external pressure at infinity or in the outer boundary between the core sample and cylindrical cavity, when \( r = b \). The influence of the external pressure \( q \) on the breakdown pressure \( p^* \) in this solution is taken into account through the boundary condition \( \sigma_r = q \) at \( r = b \).

Proceeding from the scheme of an ideal elastic-plastic body, a method for solving the problem of deformation and failure of rock around cylindrical and spherical cavities under loads of internal pressure and hydrostatic external pressure is proposed. When the maximum plastic displacement reaches the limit value, the material failure is initiated, leading thereby to a loss of shear strength in the rock.

3. Conclusions

Results of the laboratory experiment for cylindrical and spherical hydraulic fracturing do not fulfill the maximum normal stress criterion for soft-plastic rocks at all external pressure values. For hard-brittle rocks, this criterion is consistent with experimental results only for at values external hydrostatic pressure.

It is shown that the Coulomb–Mohr failure criterion agrees well with the experimental results for both hard-brittle and soft-plastic rocks given a certain choice of strength parameters of this criterion. The proposed method for solving the problem of deformation and failure of rocks around cylindrical and spherical cavities is based on the maximum average shear stress and the criteria for ultimate plastic shear.

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