Positive Stable Realisation of Fractional Electrical Circuits Consisting of $n$ Subsystem

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Abstract. This paper presents a method of the determination of a positive stable realisation of the fractional continuous-time positive system consisting of $n$ subsystems with one fractional order and with different fractional orders. For the proposed method, a digraph-based algorithm was constructed. In this paper, we have shown how we can realise the transfer matrix using electrical circuits consisting of resistances, inductances, capacitances and source voltages. The proposed method was discussed and illustrated with some numerical examples.

1. Introduction

In recent years, many researchers have been interested in positive linear systems. In this type of the system, state variables and outputs take only non-negative values. Analysis of positive one-dimensional (1D) systems is more difficult than of standard systems. Examples of positive systems include industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. An overview of the state of the art in the positive systems theory is given in [1], [2], [3], [4].

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century. Mathematical fundamentals of fractional calculus are given in the monographs [5], [6], [7], [8], [9], [10]. The notion of positive fractional discrete-time linear systems have been introduced in [6].

The realisation problem is a very difficult task. In many research studies, we can find the canonical form of the system, i.e. constant matrix form, which satisfies the system described by the transfer function. With the use of this form, we are able to write only one realisation (or some by the transformation matrices) of the system, while there exist many sets of matrices which fit into the system transfer function. The realisation problem for positive discrete-time systems without and with delays was considered in [2], [11], [12], [13], while in [14], [15] a solution for finding a set of possible realisations of the characteristic polynomial was proposed, that allows for finding many sets of matrices. The proposed method for finding minimal positive realisations is an extension of the method for finding a realisation of the characteristic polynomial. The optimisation of the proposed algorithm is presented in the paper [21] and [22].

The digraphs theory was applied to the analysis of dynamical systems. The use of the multidimensional theory was proposed for the first time in the paper [23], [24], [25] for analysis of...
positive two-dimensional systems.

In this paper, a new method of determination positive minimal realisation for the fractional continuous one-dimensional system in the form of the electrical circuit will be proposed and the procedure for computation of the minimal realisation will be given. The procedure will be illustrated with a numerical example.

This work has been organised as follows: Section 1 presents some notations and basic definitions of one-dimensional digraph theory. Section 2 presents basics of fractional order systems theory. In this section fractional continuous-time system is defined as the state-space representation of the electrical circuit consisting of $n$ subsystems with one and different fractional order. In Section 3, we construct an algorithm for the determination of a positive minimal realisation of the fractional continuous-time system as the electrical circuit. Finally, we demonstrate the workings of the algorithm on two numerical examples in Section 4, and at the end we present some concluding remarks, open problems and bibliography positions.

**Notion:** In this paper, the following notion will be used. The matrices will be denoted by a bold font (for example $\mathbf{A}$, $\mathbf{B}$, ...), the sets by a double line (for example $\mathbb{A}$, $\mathbb{B}$, ...), lower/upper indices and polynomial coefficients will be written as a small font (for example $a$, $b$, ... ) and digraphs will be denoted using a mathfrak font $\mathfrak{D}$. The set $n \times m$ real matrices will be denoted by $\mathbb{R}^{n \times m}$ and $\mathbb{R}^n = \mathbb{R}^{n \times 1}$. If $\mathbf{G} = [g_{ij}]$ is a matrix, we write $\mathbf{G} \succ 0$ (matrix $\mathbf{G}$ is called strictly positive), if $g_{ij} > 0$ for all $i, j$; $\mathbf{G} \succeq 0$ (matrix $\mathbf{G}$ is called positive), if $g_{ij} \geq 0$ for all $i, j$; $\mathbf{G} \preceq 0$ (matrix $\mathbf{G}$ is called non-negative), if $g_{ij} \leq 0$ for all $i, j$. The set of $n \times m$ real matrices with non-negative entries will be denoted by $\mathbb{R}^{n \times m}_+ \quad \text{and} \quad \mathbb{R}^n_+ = \mathbb{R}^{n \times 1}_+$. The set of $n \times n$ real matrices with off-diagonal non-negative entries will be denoted by $\mathbb{M}^{n \times n}_+$. The $n \times n$ identity matrix will be denoted by $\mathbf{I}_n$. For more information about matrix theory, an interested reader is referred, for instance, to: [26], [27].

**Digraphs:** A directed graph (or just digraph) $\mathfrak{D}$ consists of a non-empty finite set $\mathcal{V}(\mathfrak{D})$ of elements called vertices and a finite set $\mathcal{A}(\mathfrak{D})$ of ordered pairs of distinct vertices called arcs ( [28], [29]). We call $\mathcal{V}(\mathfrak{D})$ the vertex set and $\mathcal{A}(\mathfrak{D})$ the arc set of $\mathfrak{D}$. We will often write $\mathfrak{D} = (\mathcal{V}, \mathcal{A})$ which means that $\mathcal{V}$ and $\mathcal{A}$ are the vertex set and arc set of $\mathfrak{D}$, respectively. The order of $\mathfrak{D}$ is the number of vertices in $\mathfrak{D}$. The size of $\mathfrak{D}$ is the number of arc in $\mathfrak{D}$. For an arc $(v_1, v_2)$ the first vertex $v_1$ is its tail and the second vertex $v_2$ is its head.

There exists an $\mathfrak{A}$-arc from vertex $v_1$ to vertex $v_2$ if and only if the $(i, j)$-th entry of the matrix $\mathbf{A}$ is non-zero. There exists a $\mathfrak{B}$-arc from source $s$ to vertex $v_j$ if and only if the $l$-th entry of the matrix $\mathbf{B}$ is non-zero.

**Example 1** Let us be given the positive system single input described by the following matrices

$$
\begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
$$

we can draw a one-dimensional digraph $\mathfrak{D}^{(1)}$ consisting of vertices $v_1, v_2, v_3$ and source $s_1$. A one-dimensional digraph corresponding to the system (1) is presented in Figure 1.

We present below some basic notions from the graph theory which are used in further considerations. A walk in a digraph $\mathfrak{D}^{(1)}$ is a finite sequence of arcs in which every two vertices $v_i$ and $v_j$ are adjacent or identical. A walk in which all of the arcs are distinct is called a path. The path that goes through all vertices is called a finite path. If the initial and the terminal vertices of the path are the same, then the path is called a cycle.

More information about use digraph theory in positive system is given in [23], [25].
2. Fractional order system

2.1. Model and Representation

The equation for a continuous-time dynamic system of the fractional-order can be written as follows:

\[ H(\mathcal{D}^{\alpha_0\alpha_1\ldots\alpha_m})(y_1, y_2, \ldots, y_l) = G(\mathcal{D}^{\beta_0\beta_1\ldots\beta_m})(u_1, u_2, \ldots, y_k), \]

where \( y_i, u_i \) are the function of time and \( H(\cdot), G(\cdot) \) are the combinations of the fractional-order derivative operator. If we describe the linear time-invariant single-variable case we obtain the following equation:

\[ H(\mathcal{D}^{\alpha_0\alpha_1\ldots\alpha_m})(y_t) = G(\mathcal{D}^{\beta_0\beta_1\ldots\beta_m})(u_t) \tag{2} \]

with

\[ H(\mathcal{D}^{\alpha_0\alpha_1\ldots\alpha_n}) = \sum_{k=0}^{n} a_k \mathcal{D}^{\alpha_k}, \quad a_k \in \mathbb{R}; \quad G(\mathcal{D}^{\beta_0\beta_1\ldots\beta_m}) = \sum_{k=0}^{m} b_k \mathcal{D}^{\beta_k}, \quad b_k \in \mathbb{R}. \]

or

\[ a_n \mathcal{D}^{\alpha_n} y(t) + a_{n-1} \mathcal{D}^{\alpha_{n-1}} y(t) + \cdots + a_0 \mathcal{D}^{\alpha_0} y(t) = b_m \mathcal{D}^{\beta_m} u(t) + b_{m-1} \mathcal{D}^{\beta_{m-1}} u(t) + \cdots + b_0 \mathcal{D}^{\beta_0} u(t). \tag{3} \]

After applying the Laplace transform to (3) with zero initial conditions, the input-output representation of fractional-order system can be obtained. The fractional-order system as the transfer function has the following form:

\[ G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}. \tag{4} \]

2.2. State-Space Representation

In this paper, the following Caputo definition of the fractional derivative will be used:

\[ C^\alpha_a D^\alpha_t = \frac{d^\alpha}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{5} \]

where \( \alpha \in \mathbb{R} \) is the order of fractional derivative, \( f^{(n)}(\tau) \) and \( \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \) is the gamma function.

The Laplace transform [6] of the derivative-integral (5) has the following form:

\[ L \left[ C^\alpha_a D^\alpha_t \right] = s^\alpha F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0^+). \tag{6} \]

Let the current \( i_C(t) \) in a supercondensator with the capacity \( C \) be the \( \alpha \)-order derivative:

\[ i_C(t) = C \frac{d\alpha u_C(t)}{dt^\alpha}. \tag{7} \]
where $\frac{d^\alpha}{dt^\alpha}$ is the $\alpha$–order derivative defined by (6) and $u_C(t)$ is the voltage on the condensator. Similarly, let the voltage $u_L(t)$ on the coil with the inductance $L$ be the $\beta$–order derivative

$$u_L(t) = L \frac{d^\beta i_L(t)}{dt^\beta}, \quad (8)$$

where $\frac{d^\beta}{dt^\beta}$ is the $\beta$–order derivative defined by (6) and $i_L(t)$ is the current in the coil.

Let us consider electrical circuit composed of resistance, capacitances, coils and voltage sources. As the state variable, we choose the voltages on the capacitors and the currents in the coils. Using (7) and (8) and Kirhoff’s laws we can write

$$\begin{bmatrix} \frac{d^\alpha x_C(t)}{dt^\alpha} \\ \frac{d^\beta x_L(t)}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_C(t) \\ x_L(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \quad (9a)$$

$$y(t) = Cx(t) + Du(t), \quad (9b)$$

where $x_C(t) \in \mathbb{R}^{n_1}$ are voltages on the condensators, $x_L(t) \in \mathbb{R}^{n_2}$ are currents in the coil, $u(t) \in \mathbb{R}^n$ are the voltage or current source and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m}$ for $i, j = 1, 2$.

After using the Laplace transform to (9a)–(9b) and taking into account

$$\mathcal{L} \left[ \frac{d^\alpha x_C}{dt^\alpha} \right] = s^\alpha X_C(s) - s^{\alpha-1} x_C(0),$$

$$\mathcal{L} \left[ \frac{d^\beta x_L}{dt^\beta} \right] = s^\beta X_L(s) - s^{\beta-1} x_L(0),$$

$$U(s) = \mathcal{L} [u(t)], \quad (10)$$

we obtain:

$$\begin{bmatrix} X_C(s) \\ X_L(s) \end{bmatrix} = \begin{bmatrix} I_{n_1} s^\alpha - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} s^\beta - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} s^{\alpha-1} x_C(0) \\ s^{\beta-1} x_L(0) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(s), \quad (11)$$

$$Y(s) = CX(s) + DU(s).$$

After using (11) with zero initial conditions, we can determine transfer matrix of the system in the following form:

$$T(s) = C \begin{bmatrix} I_{n_1} s^\alpha - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} s^\beta - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D. \quad (12)$$

**Definition 1** The fractional system (9a)–(9b) is called the internally positive fractional system if and only if $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^{n_2}_+$ for $t \geq 0$ for any initial conditions $x_0 \in \mathbb{R}^n_+$ and all inputs $u(t) \in \mathbb{R}^m_+$ for $t \geq 0$.

**Definition 2** A square real matrix $A = [a_{ij}]$ is called the Metzler matrix if its off-diagonal entries are non-negative, i.e. $a_{ij} \geq 0$ for $i \neq j$.

**Definition 3** The fractional system (9a)–(9b) is positive if and only if

$$A \in \mathbb{M}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}, \quad C \in \mathbb{R}^{n_2 \times n}, \quad D \in \mathbb{R}^{n_2 \times m}. \quad (13)$$
Matrices (13) are called a positive realisation of the transfer function $T(s)$ if they satisfy the equality (12). The realisation is called minimal if the dimension of the state matrix $A$ is minimal among all possible realisation of (12).

Let us consider single-input single-output multi-order fractional continuous-time linear system:

$$
\begin{bmatrix}
D^\alpha_1 x_1(t) \\
\vdots \\
D^\alpha_n x_n(t)
\end{bmatrix} =
\begin{bmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
\vdots \\
x_n(t)
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
\vdots \\
B_n
\end{bmatrix} u(t),
$$

$$
y(t) =
\begin{bmatrix}
C_1 & \cdots & C_n
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
\vdots \\
x_n(t)
\end{bmatrix} + Du(t),
$$

where $0 < \alpha_i < 1$, $i = 1, \ldots, n$ $x_i(t) \in \mathbb{R}^{n_i}$, $i = 1, \ldots, n$ is the state vector, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are the input and output of the system, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B \in \mathbb{R}^{n_i}$, $C \in \mathbb{R}^{1 \times n_i}$, $i, j = 1, \ldots, n$ and $D \in \mathbb{R}$. The transfer function of the system (14) has the form

$$
T(s) = [ C_1 \ C_2 \ \cdots \ C_n ]
\begin{bmatrix}
I_{n_1} s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n} \\
-A_{21} & I_{n_2} s^{\alpha_2} - A_{22} & \cdots & -A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-A_{n1} & -A_{n2} & \cdots & I_{n_n} s^{\alpha_n} - A_{nn}
\end{bmatrix}^{-1}
\begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_n
\end{bmatrix} + D(15)
$$

where:

$$
A =
\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{bmatrix}, \quad B =
\begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_n
\end{bmatrix}, \quad C = [ C_1 \ C_2 \ \cdots \ C_n ], \quad D.
$$

Matrices (16) are called a realisation of the transfer matrix $T(s)$ if they satisfy the equality (15).

**Task:** For the given transfer matrix (12) (or (15)), determine a minimal positive realisation as a electrical circuit of the system (9a)–(9a) using the one-dimensional $D(1)$ digraphs theory. The dimension of the system must be the minimal among possible.

3. **Problem Solution**

Let us be given the matrix $A$ in the following form

$$
A =
\begin{bmatrix}
A_{11} & 0 & \cdots & 0 \\
0 & A_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{nn}
\end{bmatrix}
$$

then the strictly proper transfer matrix can be written in the following form:

$$
T_{sp}(s) = [ C_1 \ C_2 \ \cdots \ C_n ]
\begin{bmatrix}
I_{n_1} s^{\alpha_1} - A_{11} & 0 & \cdots & 0 \\
0 & I_{n_2} s^{\alpha_2} - A_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_{n_n} s^{\alpha_n} - A_{nn}
\end{bmatrix}^{-1}
\begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_n
\end{bmatrix}.
$$
The strictly proper transfer matrix (18) can be considered as a pseudo-rational function of the variable $\lambda_1 = s^{\alpha_1}, \lambda_2 = s^{\alpha_2}, \ldots, \lambda_n = s^{\alpha_n}$ in the form:

$$T_{sp}(\lambda) = [C_1 \ C_2 \cdots C_n] \begin{bmatrix} I_{n_1} \lambda_1 - A_{11} & 0 & \cdots & 0 \\ 0 & I_{n_2} \lambda_2 - A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_n} \lambda_n - A_{nn} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = (19)$$

where

$$d(\lambda) = \frac{\det(I_{n_1} \lambda_1 - A_{11}) \det(I_{n_2} \lambda_2 - A_{22}) \cdots \det(I_{n_n} \lambda_n - A_{nn})}{d_{11}(\lambda) d_{22}(\lambda) \cdots d_{kk}(\lambda)}, \quad k = 1, 2, \ldots, n.$$ (20)

In the first step, we must find matrices $A_{kk} \in \mathbb{R}^{n \times n}_+, \ k = 1, 2, \ldots, n$; using decomposition characteristic polynomial (20). We decompose each simple polynomial into a set of simple monomials

$$d_{kk}(s) = \left(1 - d_{kk,M_1}(\lambda)\right) \cup \left(1 - d_{kk,M_2}(\lambda)\right) \cup \cdots \cup \left(1 - d_{kk,M_p}(\lambda)\right).$$ (21)

where $p$ is number of simple monomials $d_{kk,M_p}$ in the simple polynomial $d_{kk}(\lambda)$. For each simple monomial, we create digraphs representations. Then we can determine all possible simple polynomial realisations using all combinations of the digraphs monomial representations. Finally, we combine received digraphs in one digraph which is corresponding to a characteristic polynomial $d(s)$.

**Theorem 1** There exists a stable positive fractional continuous-time linear system described by the equation (14) if a multi-dimensional digraph corresponding to matrix $A \in \mathbb{R}_+^{n \times n}$, satisfies the following conditions:

(C1) the sets $D_{kkM_1} \cap D_{kkM_2} \cap \cdots \cap D_{kkM_p}$, where $k = 1, 2, \ldots, n$; and $p$ is a number of simple monomials in simple polynomial $d_{kk}(s)$ corresponding to multi-dimensional digraphs are not disjoint.

(C2) the obtained multi-dimensional digraph does not have additional cycles.

(C3) the poles of the characteristic polynomial are distinct real and negative.

**Proof:** Condition (C1): The sets $D_{kkM_1} \cap D_{kkM_2} \cap \cdots \cap D_{kkM_p}$, $k = 1, 2, \ldots, n$; where $p$ is a number of simple monomials in a simple polynomial $d(s)$ are disjoint if $D_{kkM_1} \cap D_{kkM_2} \cap \cdots \cap D_{kkM_p} = \emptyset$ then we have a digraph whose vertices can be divided into two disjoint sets. It means that we obtain an additional simple monomial in a simple characteristic polynomial $d_{kk}(s)$. In this situation, we obtain a new characteristic polynomial $\bar{d}(s)$, $d(s) \neq \bar{d}(s)$. Condition (C2): Each monomial is represented by one cycle. If after combining all digraphs (each corresponding to one simple monomial in every simple polynomial) we obtain an additional cycle, this means that in the polynomial an additional simple monomial appears. Condition (C3): If the state
matrix \( A \) has the structure as (17) then the roots of the characteristic polynomial are real negative and the matrix \( A \) is stable and is a Metzler matrix. \( \square \)

Using the Theorem 1 we can construct the Algorithm 1.

**Algorithm 1**

**DetermineMinimalRealisation()**

1: \( \text{monomial} = 1; \)
2: Determine number of \( \text{simple \_polynomial} \) in characteristic polynomial (20);
3: for \( \text{simple \_polynomial} = 1 \) to \( k \) do
4: Determine number of cycles in simple polynomial;
5: for \( \text{monomial} = 1 \) to cycles do
6: Determine one-dimensional digraph \( D^{(1)} \) for all monomial;
7: \( \text{MonomialRealisation(monomial)}; \)
8: end for
9: for \( \text{monomial} = 1 \) to cycles do
10: Determine digraph as a combination of the digraph monomial representation;
11: \( \text{SimplePolynomialRealisation(monomial)}; \)
12: if \( \text{SimplePolynomialRealisation} != \text{cycles} \) then
13: Digraph contains additional cycles or digraph contains disjoint union;
14: BREAK
15: else if \( \text{SimplePolynomialRealisation} == \text{cycles} \) then
16: Digraph satisfies characteristic polynomial;
17: Determine weights of the arcs in digraph;
18: Write state matrix \( A_{kk} | k=1,\ldots,n \);
19: end if
20: end for
21: return \( (\text{PolynomialRealisation}, \ A = \text{diag}(A_{kk} | k=1,\ldots,n)); \)
22: end for

Let us assume that the matrix \( B_k, k = 1, 2, \ldots, n \) and matrix \( C \) have the following form:

\[
B_k | k=1,2,\ldots,n = \begin{bmatrix}
  b_{k1} & b_{k2} & \ldots & b_{km} \\
b_{21} & b_{22} & \ldots & b_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & \ldots & b_{nm}
\end{bmatrix}, \quad C = \begin{bmatrix}
  c_{11} & c_{12} & \ldots & c_{1n} \\
c_{21} & c_{22} & \ldots & c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
c_{p1} & c_{p2} & \ldots & c_{pn}
\end{bmatrix}.
\]

(22)

After determining the state matrix \( A_{kk} \in \mathbb{R}^{n \times n}, k = 1, 2, \ldots, n \) from the Algorithm 1 and inserting matrices (22) to the equation (18) we obtain the polynomial \( \tilde{n}_{ij}(s) \). After comparing variables with the same power of \( s \) polynomials \( \tilde{n}_{ij}(s) = n_{ij}(s) \) we receive the set of equations. After solving the equation, we obtain matrices (13).

**Remark 1** If there exists the characteristic polynomial in the form (20), then we can always draw an electrical circuit consist from: resistances, inductances and source voltages; resistances and capacitances; resistances, inductances and capacitances.

4. Numerical Example

4.1. Example 1 (one fractional order).

For the given strictly proper transfer matrix

\[
T_{sp}(s) = \frac{1}{s^{3\alpha} + 34s^{2\alpha} + 245s^\alpha + 500} \begin{bmatrix}
  0 & 25s^{2\alpha} + 225s^\alpha + 500 \\
4s^{2\alpha} + 120s^\alpha + 500 & 0
\end{bmatrix},
\]

(23)
find a minimal positive stable realisation as an electrical circuit using the one-dimensional digraph theory for \(0 < \alpha < 1\).

**Solution.** In the first step, using a pseudo-rational function of the variable \(\lambda = s^\alpha\) write transfer matrix (23) in the following form:

\[
T_{sp}(\lambda) = \frac{1}{\lambda^3 + 34\lambda^2 + 245\lambda + 500} \begin{bmatrix} 0 & 25\lambda^2 + 225\lambda + 500 \\ 4\lambda^2 + 120\lambda + 500 & 0 \end{bmatrix}.
\] (24)

After multiplying the nominator and denominator of the transfer function (24) by \(\lambda^{-3}\), we obtain:

\[
T_{sp}(\lambda) = \frac{1}{d(\lambda)} \frac{n_{12}(\lambda)}{n_{21}(\lambda)} = \frac{1}{1 + 34\lambda^{-1} + 245\lambda^{-2} + 500\lambda^{-3}} \begin{bmatrix} 0 & 25\lambda^{-1} + 225\lambda^{-2} + 500\lambda^{-3} \\ 4\lambda^{-1} + 120\lambda^{-2} + 500\lambda^{-3} & 0 \end{bmatrix}.
\] (25)

We can write the characteristic polynomial in the following form

\[
d(\lambda) = 1 + 34\lambda^{-1} + 245\lambda^{-2} + 500\lambda^{-3} = (1 + 4\lambda^{-1})(1 + 5\lambda^{-1})(1 + 25\lambda^{-1}).
\] (26)

In the next step, we decompose characteristic polynomial (26) into a set of simple polynomials \(d_{11}(\lambda) = 1 + 4\lambda^{-1}\), \(d_{22}(\lambda) = 1 + 5\lambda^{-1}\) and \(d_{33}(\lambda) = 1 + 25\lambda^{-1}\). In the next step, for each simple monomial we write initial conditions. For a simple polynomial \(d_{11}(\lambda)\), we have following conditions: number of vertices in digraph: \(\text{vertices} = 1\); possible weights from which we will build digraphs: \((v_i, v_j)_{A_{11}}\lambda^{-1}\); monomials: \(M_1 = 1 + 4\lambda^{-1}\).

Then, we determine all possible realisations of the simple polynomial \(d_{11}(\lambda)\). In the considered example, we have only one realisation presented in Figure 2(a). The realisation meets conditions (C1) and (C2) of the Theorem 1. Finally, we must verify the third condition. In the considered simple polynomial, the poles are real and negative. Described realisation satisfies the condition (C3). **The realisation does satisfy all conditions and is correct.** In this same way, we can determine realisations of the simple polynomial \(d_{22}(\lambda)\) presented in Figure 2(b) and \(d_{33}(\lambda)\) presented in Figure 2(c).

**Figure 2.** Multi-dimensional digraphs corresponding to a simple polynomial: 2(a)–\(d_{11}(\lambda)\); 2(b)–\(d_{22}(\lambda)\); 2(c)–\(d_{33}(\lambda)\).

From the obtained digraphs, we can write a state matrix \(A\) in the form:

\[
A = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} = \begin{bmatrix} w(v_1, v_1)_{A_{11}} & 0 & 0 \\ 0 & w(v_1, v_1)_{A_{22}} & 0 \\ 0 & 0 & w(v_1, v_1)_{A_{33}} \end{bmatrix}.
\] (27)

After using a characteristic polynomial (26) and obtaining matrix (27), we can draw an electrical circuit, presented in Figure 3(a), which consists of three loops: Loop \(L_1\) corresponding
to matrix $A_{11}$ and polynomial $d_{11}(\lambda)$; Loop $L_2$ corresponding to matrix $A_{22}$ and polynomial $d_{2}(\lambda)$; and Loop $L_3$ corresponding to matrix $A_{33}$ and polynomial $d_{33}(\lambda)$. In every loop, we have resistances and inductances.

After using (27), Figure 3(a) and the value of the electrical circuit: $R_1 = 12\Omega$, $R_2 = 20\Omega$, $R_3 = 25\Omega$, $L_1 = 3H$, $L_2 = 4H$ and $L_3 = 1H$, we can write a state matrix in the following form:

$$A = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & 0 \\ 0 & -\frac{R_2}{L_2} & 0 \\ 0 & 0 & -\frac{R_3}{L_3} \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -25 \end{bmatrix}$$

(28)

In the next step, we must add source voltages for example to the Loop $L_1$ and Loop $L_3$ (Figure 3(a)). After this operation, we will obtain the electrical circuit presented in Figure 3(b).

Now, we can write matrix $B$ in the following form:

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.333 & 0 \\ 0 & 1 \end{bmatrix}.$$

(29)

After inserting matrices (28), (29) and (22) to the equation (20), we obtain the polynomial $	ilde{n}_{ii}(\lambda)$, $i = 1, 2$. In the next step, we compare one of the coefficients of the same power of $\lambda$ polynomials $	ilde{n}_{ii}(\lambda) = n_{ii}(\lambda)$ for $i = 1, 2$, and we receive the set of the equations. After solving them, we obtain the following matrix:

$$C = \begin{bmatrix} 0 & 0 & R_3 \\ R_1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 25 \\ 12 & 0 & 0 \end{bmatrix}.$$

(30)

The obtained realisation is stable, as can be seen in Figure 4, showing the step response of the system for $\alpha$ equal to: 0.3, 0.5, 0.7 and 0.9. In Figure 5(a), we can see the Nyquist characteristic for: $\alpha = 0.3$ (Figure 5(a)); $\alpha = 0.5$ (Figure 5(b)); $\alpha = 0.7$ (Figure 5(c)); $\alpha = 0.9$ (Figure 5(d)). Simulations were performed using the Matlab toolbox FOMCON (Fractional-order Modeling and Control) presented in the paper [30] and [31].

4.2. Example 2 (three fractional order).

For the given transfer matrix

$$T_{sp}(s) = \begin{bmatrix} 0 & 25s^{1.2} + 100s^{0.9} + 125s^{0.3} + 500 \\ 4s^{1.3} + 100s^{0.9} + 20s^{0.4} + 500 & 0 \\ s^{1.6} + 4s^{1.3} + 25s^{1.2} + 100s^{0.9} + 5s^{0.7} + 20s^{0.4} + 125s^{0.3} + 500 \end{bmatrix},$$

(31)
find a minimal positive stable realisation as an electrical circuit using the one-dimensional digraph theory for $0 < \alpha < 1$.

**Solution.** In the first step using pseudo-rational function of the variable $\lambda_1 = s^{0.3}$, $\lambda_2 = s^{0.9}$, $\lambda_3 = s^{0.4}$ we can write transfer matrix (31) in the following form:

$$T_{sp}(\lambda) = \begin{bmatrix} 0 & 25\lambda_1\lambda_2 + 100\lambda_2 + 125\lambda_1 + 500 \\ 4\lambda_2\lambda_3 + 100\lambda_2 + 20\lambda_3 + 500 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1\lambda_2\lambda_3 + 4\lambda_2\lambda_3 + 25\lambda_1\lambda_2 + 5\lambda_1\lambda_3 + 100\lambda_2 + 20\lambda_3 + 125\lambda_1 + 500 \\ 0 \end{bmatrix}$$

(32)
After multiplying the nominator and denominator of the transfer function (24) by $\lambda^{-1}\lambda_2^{-1}\lambda_3^{-1}$, we obtain:

$$T_{sp}(\lambda^{-1}) = \frac{1}{d(\lambda)} \begin{bmatrix} 0 & n_{12}(\lambda) \\ n_{21}(\lambda) & 0 \end{bmatrix} = \begin{bmatrix} 25\lambda_2^{-1} + 100\lambda_1^{-1}\lambda_3^{-1} + 125\lambda_2^{-1}\lambda_3^{-1} + 500\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1} \\ 1 + 4\lambda_1^{-1} + 25\lambda_3^{-1} + 5\lambda_2^{-1} + 100\lambda_1^{-1}\lambda_3^{-1} + 20\lambda_1^{-1}\lambda_2^{-1} + 125\lambda_2^{-1}\lambda_3^{-1} + 500\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1} \\ 0 \end{bmatrix}$$

The characteristic polynomial can be written in the following form:

$$d(\lambda) = 1 + 4\lambda_1^{-1} + 25\lambda_3^{-1} + 5\lambda_2^{-1} + 100\lambda_1^{-1}\lambda_3^{-1} + 20\lambda_1^{-1}\lambda_2^{-1} + 125\lambda_2^{-1}\lambda_3^{-1} + 500\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1} = \frac{25\lambda_2^{-1} + 100\lambda_1^{-1}\lambda_3^{-1} + 125\lambda_2^{-1}\lambda_3^{-1} + 500\lambda_1^{-1}\lambda_2^{-1}\lambda_3^{-1}}{d_{11}(\lambda) d_{22}(\lambda) d_{33}(\lambda)}$$

After using Algorithm 1, we determine the matrices $A$, $B$, $C$ in the form (28), (29), (30) and electrical circuit presented in Figure 3(b). The obtained realisation is stable, as can be seen in Figure 6, showing the step response of the system for $\alpha_1 = 0.34$; $\alpha_2 = 0.9$; $\alpha_3 = 0.4$. Simulations were performed using the Matlab toolbox FOMCON (Fractional-order Modeling and Control) presented in the paper [30] and [31].

5. Concluding Remarks
The paper presents a method, based on the one-dimensional digraph theory, for finding the realisations of the one-dimensional continuous-time fractional system. The difference between the proposed algorithm in this paper and currently used methods based on canonical forms of the system (i.e. constant matrix forms) is the creation of not one (or few) minimal realisations, but a set of every possible minimal realisation. In this paper, we have shown how we can realise the transfer matrix using electrical circuits. Also a method for determining a realisation using a system consisting of $n$ subsystems with one fractional order and with many different fractional orders has been shown.

Further work includes extension of the algorithm to find the broader class of electrical circuits corresponding to the transfer matrix.

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