TIME-FREQUENCY PARTITIONS AND CHARACTERIZATIONS OF MODULATION SPACES WITH LOCALIZATION OPERATORS

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ABSTRACT. We study families of time-frequency localization operators and derive a new characterization of modulation spaces. This characterization relates the size of the localization operators to the global time-frequency distribution. As a by-product, we obtain a new proof for the existence of multi-window Gabor frames and extend the structure theory of Gabor frames.

1. Introduction

A time-frequency representation transforms a function $f$ on $\mathbb{R}^d$ into a function on the time-frequency space $\mathbb{R}^d \times \mathbb{R}^d$. The goal is to obtain a description of $f$ that is local both in time and in frequency [3, 20]. The standard time-frequency representations, such as the short-time Fourier transform and its various modifications known as Wigner distribution, radar ambiguity function, Gabor transform, all encode time-frequency information. However, the pointwise interpretation of such a time-frequency representation meets difficulties because, by the uncertainty principle, a small region in the time-frequency plane does not possess a physical meaning. Therefore the question arises in which sense the short-time Fourier transform describes the local properties of a function and its Fourier transform.

Following Daubechies [10], we use time-frequency localization operators to give meaning to the local time-frequency content. By investigating a whole family of localization operators and gluing together the local pieces, we are able to characterize the global time-frequency distribution of a function. In more technical terms, our main result provides a new characterization of modulation spaces.

We define the short-time Fourier transform (STFT) of a function $f \in L^2(\mathbb{R}^d)$ with respect to a window function $\varphi \in L^2(\mathbb{R}^d)$ as

$$V_{\varphi}f(x, \omega) = \int_{\mathbb{R}^d} f(t) \tilde{\varphi}(t-x)e^{-2\pi i \omega \cdot t} dt,$$

for all $z = (x, \omega) \in \mathbb{R}^{2d}$.

The STFT $V_{\varphi}f(z)$ is a measure of the time-frequency content near the point $z$ in the time-frequency plane $\mathbb{R}^{2d}$. However, the STFT cannot be supported on a set of finite measure by results in [28, 30, 38]. This fact complicates the interpretation.

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of local information obtained from the STFT. In particular, it is impossible to construct a projection operator that satisfies $V_\varphi(P_\Omega f) = \chi_\Omega \cdot V_\varphi f$. As a remedy one resorts to the following definition of localization operators.

We denote translation operators by $T_x f(t) = f(t-x)$ and time-frequency shifts by $\pi(z) f(t) = e^{2\pi i \omega t} f(t-x)$ for $x, \omega, t \in \mathbb{R}^d$. Fix a non-zero function $\varphi \in L^2(\mathbb{R}^d)$ (a so-called window function) and a symbol $\sigma \in L^1(\mathbb{R}^d)$. Then the time-frequency localization operator $H_\sigma$ acting on a function $f$ is defined as

$$H_\sigma f = \int_{\mathbb{R}^2d} \sigma(z) V_\varphi f(z) \pi(z) \varphi \, dz = V_\varphi^* \sigma V_\varphi f.$$ 

The integral is defined strongly on many function spaces, in particular on $L^2(\mathbb{R}^d)$. A useful alternative definition of $H_\sigma$ is the weak definition

$$\langle H_\sigma f, g \rangle_{L^2(\mathbb{R}^d)} = \langle \sigma V_\varphi f, V_\varphi g \rangle_{L^2(\mathbb{R}^d)}.$$ 

This definition can be easily extended to distributional symbols $\sigma \in \mathcal{S}'(\mathbb{R}^d)$. The subtleties of the definition and boundedness properties between various spaces have been investigated in many papers, see [7, 37, 39] for a sample of results.

If $\sigma$ is non-negative and has compact support in $\Omega \subseteq \mathbb{R}^d$, then $H_\sigma f$ can be interpreted as the part of $f$ that lives essentially on $\Omega$ in the time-frequency plane, and so $H_\sigma$ may be taken as a substitute for the non-existing projection onto the region $\Omega$ in the time-frequency plane.

In this paper we investigate the behavior of an entire collection of localization operators. Namely, given a lattice $\Lambda \subseteq \mathbb{R}^2d$ of the time-frequency plane, we consider the collection of operators $\{H_{T_\lambda \varphi} : \lambda \in \Lambda\}$ and the mapping $f \rightarrow \{H_{T_\lambda \varphi} f\}$. If the supports of $T_\lambda \varphi$ cover $\mathbb{R}^2d$, then $\{H_{T_\lambda \varphi} f, \lambda \in \Lambda\}$ should contain enough information to recover $f$ from its local components. In particular, the set $\{H_{T_\lambda \varphi} f : \lambda \in \Lambda\}$ should carry the complete information about the global time-frequency properties of $f$. We make this intuition precise and derive a new characterization of modulation spaces from it. Similar to Besov spaces, modulation spaces are smoothness spaces, but the smoothness is measured by means of time-frequency distribution rather than differences and derivatives. Here, we establish a correspondence between the behavior of the sequence $\|H_{T_\lambda \varphi} f\|_2, \lambda \in \Lambda$, and the membership of $f$ in a modulation space.

As a special case of our main theorem we formulate the following result.

**Theorem 1.** Fix a non-zero function $\varphi$ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and a weight function $m$ on $\mathbb{R}^2d$ that satisfies $m(z_1 + z_2) \leq C(1 + |z_1|)^N m(z_2)$ for some constants $C, N \geq 0$ and all $z_1, z_2 \in \mathbb{R}^2d$. Then a tempered distribution $f$ satisfies

$$\left( \int_{\mathbb{R}^2d} |V_\varphi f(z)|^p m(z)^p \, dz \right)^{1/p} < \infty,$$

if and only if

$$\left( \sum_{\lambda \in \Lambda} \|H_{T_\lambda \varphi} f\|^p_2 m(\lambda)^p \right)^{1/p} < \infty.$$
The expression in (3) is just the norm of \( f \) in the modulation space \( M^p_m(\mathbb{R}^d) \). Our main result shows that the expression in (4) (using the time-frequency components of \( f \)) is an equivalent norm on the modulation space \( M^p_m(\mathbb{R}^d) \).

In pseudodifferential calculus one often defines spaces by conditions on their time-frequency components. For instance, Bony, Chemin, and Lerner [3, 4] introduced a Sobolev-type space \( H(m) \) by using Weyl operators instead of localization operators. For the (extremely simplified) case of a constant Euclidean metric on the time-frequency plane, a distribution \( f \) belongs to \( H(m) \), whenever for some test function \( \psi \) on \( \mathbb{R}^2 \)

\[
\| f \|_{H(m)}^2 = \int_{\mathbb{R}^2} \| (T_Y \psi)^w f \|_2^2 m(Y) \, dY ,
\]

is finite, where \( \sigma^w \) is the Weyl operator corresponding to the symbol \( \sigma \). The only difference between (5) and (4) is the use of Weyl calculus instead of time-frequency localization operators and a continuous definition instead of a discrete one. It was understood only recently that \( H(m) \) coincides with the modulation space \( M^2_m(\mathbb{R}^d) \) and that (5) is an equivalent norm on \( M^2_m(\mathbb{R}^d) \) [26]. Thus Theorem 1 can be interpreted as an extension of [3] to \( L^p \)-like spaces.

Let us also mention that in the language of [36], the operators \( \{H_{T_\lambda \sigma}, \lambda \in \Lambda\} \) form a g-frame for \( L^2(\mathbb{R}^d) \). Our construction seems to be one of the few non-trivial examples of g-frames that are not frames.

In this paper we prove the norm equivalence of Theorem 1 for a large class of modulation spaces and arbitrary time-frequency lattices. For a rather restricted class of lattices, namely lattices with integer oversampling, an analogous result was derived in [12] for unweighted modulation spaces. The main arguments for the integer lattice were based on Zak transform methods and interpolation. For a general lattice, these methods are no longer available, and we have to develop a completely new approach to some of the key arguments.

As a by-product of the new techniques we have found several results of independent interest.

- We formulate several structural results and characterizations of Gabor frames for multi-window Gabor frames.
- We prove a finite intersection property for time-frequency invariant subspaces of the distribution space \( M^\infty(\mathbb{R}^d) \). This property resembles the finite intersection property that characterizes compact sets.
- We give a new, independent proof for the existence of multi-window Gabor frames with well-localized windows. Previous proofs were based on coorbit theory [15] and the theory of projective modules [33]. Our proof provides additional insight how the windows can be chosen.
- We derive precise estimates for the localization of the eigenfunctions of a localization operator.

This paper is organized as follows. In Section 2 we recall necessary facts from time-frequency analysis. On the one hand, we introduce modulation spaces and explain their characterization by means of multi-window Gabor frames. On the other hand, we state and prove several properties of localization operators. In
Section 3 we formulate and prove our main result (Theorem 8). In Section 3.4 we analyze some of the consequences of Theorem 8 and its proof. In the appendix we collect and sketch the proofs of some of the structural results on Gabor frames.

2. Time-Frequency Analysis of Functions and Operators

2.1. Modulation Spaces. Modulation spaces are a class of function spaces associated to the short-time Fourier transform (1). Note that for a suitable test function \( \varphi \), the short-time Fourier transform can be extended to distribution spaces by duality and \( \mathcal{V}_\varphi f(z) = \langle f, \pi(z) \varphi \rangle \).

For the standard definition of modulation spaces, we fix a non-zero “window function” \( g \in S(\mathbb{R}^d) \) and consider moderate weight functions \( m \) of polynomial growth, i.e., \( m(z_1 + z_2) \leq C(1 + |z_1|)^s m(z_2) \), \( z_1, z_2 \in \mathbb{R}^{2d} \) for some \( C, s \geq 0 \). Given a moderate weight \( m \) and \( 1 \leq p, q \leq \infty \), the modulation space \( M_{m}^{p,q}(\mathbb{R}^d) \) is defined as the space of all tempered distributions \( f \in S'(\mathbb{R}^d) \) with \( \mathcal{V}_g f \in L_m^{p,q}(\mathbb{R}^{2d}) \), with norm

\[
\| f \|_{M_{m}^{p,q}(\mathbb{R}^d)} = \| \mathcal{V}_g f \|_{L_m^{p,q}(\mathbb{R}^{2d})}.
\]

If \( p = q \), we write \( M_{m}^{p}(\mathbb{R}^d) \).

For weight functions of faster growth we have to resort to different spaces of test functions and distributions. Let \( g(t) = e^{-\pi t^2} \) be the Gaussian window and \( \mathcal{H}_0 = \text{span} \{ \pi(z) g : z \in \mathbb{R}^{2d} \} \) be the linear space of all finite linear combinations of time-frequency shifts of the Gaussian. Let \( \nu \) be a submultiplicative even weight function on \( \mathbb{R}^{2d} \) and \( m \) be a \( \nu \)-moderate function; this means that \( \nu(z_1 + z_2) \leq \nu(z_1)\nu(z_2) \), \( \nu(z_1) = \nu(-z) \) and \( m(z_1 + z_2) \leq \nu(z_1)m(z_2) \) for all \( z, z_1, z_2 \in \mathbb{R}^{2d} \). For \( 1 \leq p, q < \infty \) the modulation space \( M_{m}^{p,q}(\mathbb{R}^d) \) is then defined as the closure of \( \mathcal{H}_0 \) in the norm \( \| f \|_{M_{m}^{p,q}(\mathbb{R}^d)} \) as in (3). If \( p = \infty \) or \( q = \infty \), we take a weak-*-closure of \( \mathcal{H}_0 \). These general modulation spaces possess the following properties. Assume that \( m \) is \( \nu \)-moderate and \( 1 \leq p, q \leq \infty \), then

\[
M_{\nu}^{1}(\mathbb{R}^d) \subseteq M_{m}^{p,q}(\mathbb{R}^d) \subseteq M_{1/\nu}(\mathbb{R}^d) = M_{\nu}^{1}(\mathbb{R}^d)^{\ast}.
\]

Further, if \( \varphi \in M_{\nu}^{1}(\mathbb{R}^d) \), then

\[
\| \mathcal{V}_\varphi f \|_{L_m^{p,q}} \asymp \| \mathcal{V}_g f \|_{L_m^{p,q}} = \| f \|_{M_{m}^{p,q}}.
\]

Thus different windows in \( M_{\nu}^{1}(\mathbb{R}^d) \) yield equivalent norms on \( M_{m}^{p,q} \).

The embedding (7) says that \( M_{\nu}^{1}(\mathbb{R}^d) \) may serve as a space of test functions and \( M_{1/\nu}(\mathbb{R}^d) \) as a space of distributions for all modulation spaces \( M_{m}^{p,q} \) with a \( \nu \)-moderate weight \( m \).

If \( \nu_s(z) = (1 + |z|)^s \), \( s \geq 0 \) and \( m \) is \( \nu_s \)-moderate, then we have

\[
S(\mathbb{R}^d) \subseteq M_{\nu_s}^{1}(\mathbb{R}^d) \subseteq M_{m}^{p,q}(\mathbb{R}^d) \subseteq M_{1/\nu_s}(\mathbb{R}^d) \subseteq S'(\mathbb{R}^d),
\]

in agreement with the standard definition, but for \( \nu(z) = e^{a|z|^b} \) with \( a > 0 \) and \( 0 < b \leq 1 \) we have

\[
M_{\nu}^{1}(\mathbb{R}^d) \subseteq S(\mathbb{R}^d) \subseteq S'(\mathbb{R}^d) \subseteq M_{\nu_s}^{\infty}(\mathbb{R}^d).
\]
In the sequel we will start with a submultiplicative weight \( \nu \) and take \( M_{1,\nu}^\infty(\mathbb{R}^d) \) as the appropriate distribution space. Our results hold for arbitrary submultiplicative weights \( \nu \).

For the detailed theory of modulation spaces we refer to [21, Ch. 11–13], for a discussion of weights and possible distribution spaces see [23].

**Sequence space norms.** Recall that a time-frequency lattice \( \Lambda \) is a discrete subgroup of \( \mathbb{R}^{2d} \) of the form \( \Lambda = AZ^{2d} \) for some invertible real-valued \( 2d \times 2d \)-matrix \( A \).

Given a lattice \( \Lambda \subseteq \mathbb{R}^{2d} \) with relatively compact fundamental domain \( Q \), the discrete space \( \ell_{p,q}^m(\Lambda) \) consists of all sequences \( a = (a_\lambda)_{\lambda \in \Lambda} \) for which the norm
\[
\|a\|_{\ell_{p,q}^m} = \left( \sum_{\lambda \in \Lambda} |a_\lambda|^q \left( \int |\varphi_{A^{-1}A \lambda} + Q| \right)^p \right)^{1/q}.
\]
is finite. If \( \Lambda = aZ^d \times bZ^d \), then this definition reduces to the usual mixed-norm space \( \ell_{p,q}^m(Z^{2d}) \) with norm
\[
\|a\|_{\ell_{p,q}^m} = \left( \sum_{n \in Z^d} \left( \sum_{k \in Z^d} |a_{km}|^p m(k, b n)^p \right)^{q/p} \right)^{1/q}.
\]

As a technical tool we will need amalgam spaces (in one place only). A measurable function \( F \) on \( \mathbb{R}^{2d} \) belongs to the (Wiener) amalgam space \( W(\mathbb{L}_{p,q}^m) \), if the sequence of local suprema
\[
a_{kn} = \operatorname{esssup}_{x, \omega \in [0,1]^d} |F(x + k, \omega + n)| = \|F \cdot T_{(k, n)} \varphi\|_\infty
\]
belongs to \( \ell_{p,q}^m(Z^{2d}) \). The norm on \( W(\mathbb{L}_{p,q}^m) \) is \( \|F\|_{W(\mathbb{L}_{p,q}^m)} = \|a\|_{\ell_{p,q}^m} \). See [27] for an introductory article. We need their behavior under convolution and their properties under sampling.

(a) Convolution in Wiener amalgam spaces: Let \( 1 \leq p, q \leq \infty \) and let \( m \) be a \( \nu \)-moderate weight. Then
\[
\|F * G\|_{W(\mathbb{L}_{p,q}^m)} \leq C\|F\|_{W(\mathbb{L}_{p,q}^m)} \|G\|_{\mathbb{L}_1}.
\]

(b) Sampling in Wiener amalgam spaces: For \( F \in W(\mathbb{L}_{p,q}^m) \) the following sampling property holds:
\[
\|F\|_{\Lambda} \leq C_\Lambda \|F\|_{W(\mathbb{L}_{p,q}^m)}.
\]
These statements are proved in [27] or [21, Prop. 11.1.4., Thm. 11.1.5.].

**2.2. Gabor frames.** Gabor frames are closely linked to modulation spaces. They constitute “basis-like” sets for modulation spaces and are used to characterize the membership in a modulation space by the magnitude of coefficients in the corresponding series expansion.

For a given lattice \( \Lambda \subseteq \mathbb{R}^{2d} \) and a window function \( \varphi \in \mathbb{L}^2(\mathbb{R}^d) \), let \( G(\varphi, \Lambda) \) denote the set of functions \( \{\pi(\lambda)\varphi : \lambda \in \Lambda\} \) in \( \mathbb{L}^2(\mathbb{R}^d) \). The operator
\[
S_\varphi f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\varphi \rangle \pi(\lambda)\varphi
\]
is the frame operator corresponding to \( \mathcal{G}(\varphi, \Lambda) \). If \( S_\varphi \) is bounded and invertible on \( \mathbf{L}^2(\mathbb{R}^d) \), then \( \mathcal{G}(\varphi, \Lambda) \) is called a Gabor frame for \( \mathbf{L}^2(\mathbb{R}^d) \). This property is equivalent to the existence of two constants \( A, B > 0 \) such that

\[
(12) \quad A\|f\|^2 \leq \sum_{\Lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 = \langle S_\varphi f, f \rangle \leq B\|f\|^2 \quad \text{for all } f \in \mathbf{L}^2(\mathbb{R}^d).
\]

Using several windows \( \varphi = (\varphi_1, \ldots, \varphi_n) \), we say that the union \( \bigcup_{j=1}^n \mathcal{G}(\varphi_j, \Lambda) \) is a multi-window Gabor frame, if the associated frame operator given by

\[
(13) \quad S_\varphi f = \sum_{j=1}^n \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\varphi_j \rangle \pi(\lambda)\varphi_j = \sum_{j=1}^n S_{\varphi_j} f
\]

is invertible on \( \mathbf{L}^2(\mathbb{R}^d) \). The frame operator can be expressed as the composition of the analysis operator \( C_{\varphi, \Lambda} \) defined by

\[
C_{\varphi, \Lambda}(f)(\lambda) = \langle f, \pi(\lambda)\varphi_j \rangle, \quad \lambda \in \Lambda, j = 1, \ldots, n.
\]

and the synthesis operator \( D_{\varphi, \Lambda} \) defined by

\[
D_{\varphi, \Lambda}(c) = \sum_{\lambda \in \Lambda} \sum_{j=1}^n c_{\lambda,j} \pi(\lambda)\varphi_j.
\]

Then \( S_{\varphi, \Lambda} = D_{\varphi, \Lambda} \circ C_{\varphi, \Lambda} \).

2.3. Characterization of Modulation Spaces with Gabor Frames. The following characterization of modulation spaces by means of multi-window Gabor frames is a central result in time-frequency analysis and useful in many applications. It is crucial for the proof of our main theorem (Theorem 3).

**Theorem 2.** Let \( \nu \) be a submultiplicative weight on \( \mathbb{R}^{2d} \) satisfying the condition \( \lim_{n \to \infty} \nu(nz)^{1/n} = 1 \) for all \( z \in \mathbb{R}^{2d} \) and let \( m \) be a \( \nu \)-moderate weight and \( 1 \leq p, q \leq \infty \). Assume further that \( \bigcup_{j=1}^n \mathcal{G}(\varphi_j, \Lambda) \) is a multi-window Gabor frame and that \( \varphi_j \in \mathbf{M}_m^p(\mathbb{R}^d) \) for \( j = 1, \ldots, n \).

(i) A distribution \( f \) belongs to \( \mathbf{M}_m^p(\mathbb{R}^d) \), if and only if \( C_{\varphi_j} f \in \ell_m^p \) for \( j = 1, \ldots, n \). In this case there exist constants \( A, B > 0 \), such that, for all \( f \in \mathbf{M}_m^p(\mathbb{R}^d) \),

\[
A\|f\|_{\mathbf{M}_m^p} \leq \left( \sum_{\lambda \in \Lambda} \left( \sum_{j=1}^n |\langle f, \pi(\lambda)\varphi_j \rangle|^2 \right)^{p/2} m(\lambda)^p \right)^{1/p} \leq B\|f\|_{\mathbf{M}_m^p}.
\]

(ii) Assume in addition that \( \Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d \) is a separable lattice. Then a distribution \( f \) belongs to \( \mathbf{M}_m^{p,q}(\mathbb{R}^d) \) if and only if each sequence \( C_{\varphi_j} f(ak, bl) = \langle f, \pi(ak, bl)\varphi_j \rangle \) belongs to \( \ell_m^{p,q}(\mathbb{Z}^{2d}) \). In this case there exist constants \( A, B \) depending on \( p, q, m \) such that, for all \( f \in \mathbf{M}_m^{p,q} \)

\[
(14) \quad A\|f\|_{\mathbf{M}_m^{p,q}} \leq \left( \sum_{l \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \left( \sum_{j=1}^n |\langle f, \pi(ak, bl)\varphi_j \rangle|^2 \right)^{p/2} m(ak, bl)^p \right)^{q/p} \right)^{1/q} \leq B\|f\|_{\mathbf{M}_m^{p,q}}.
\]

(iii) Let \( \Lambda \subseteq \mathbb{R}^{2d} \) be an arbitrary lattice and \( Q \) be a relatively compact fundamental domain of \( \Lambda \). Then a distribution \( f \) belongs to \( \mathbf{M}_m^{p,q}(\mathbb{R}^d) \), if and only if the function \( \sum_{\lambda \in \Lambda} \left( \sum_{j=1}^n |\langle f, \pi(\lambda)\varphi_j \rangle|^2 \right)^{1/2} \chi_{\lambda+Q} \) belongs to \( \mathbf{L}_m^{p,q}(\mathbb{R}^{2d}) \). In this case
there exist constants $A, B > 0$, such that, for all $f \in M_{m}^{p,q}(\mathbb{R}^{2d})$,

$$A\|f\|_{M_{m}^{p,q}} \leq \left\| \sum_{\lambda \in \Lambda} \left( \sum_{j=1}^{n} |\langle f, \pi(\lambda)\varphi_{j} \rangle|^2 \right)^{1/2} \chi_{\lambda + Q} \|L_{\ell}^{p,q} \right\| \leq B\|f\|_{M_{m}^{p,q}}.$$  

Note that (ii) follows from (iii), since for $Q = [0,a]^{d} \times [0,b]^{d}$ the norm equivalence

$$\| \sum_{k,l \in \mathbb{Z}^{d}} a_{kl} \chi(ak,bl) + Q \|_{L_{\ell}^{p,q}} \simeq \|a\|_{\ell_{1}^{p,q}}$$

holds.

Theorem 2 has a long history. It extends the basic characterizations of modulation spaces by Gabor frames to multi-window Gabor frames. For Gabor frames with a single window and lattices of the form $\Lambda = a\mathbb{Z}^{d} \times b\mathbb{Z}^{d}$ with $ab \in \mathbb{Q}$ Theorem 2 was proved in [16]. For general lattices it follows from the main result in [24] and the techniques in [16]. See also the discussion in [21, Ch. 13]. The proofs for multi-window Gabor frames require only few modifications, we therefore postpone a discussion to the appendix.

2.4. A New Characterization of Multi-Window Gabor Frames. The proof of our main statement relies on a characterization of multi-window Gabor frames without using inequalities. The following lemma is a generalization of [22] from Gabor frames to multi-window Gabor frames.

**Lemma 3.** Assume that $\varphi_{j} \in M_{1}(\mathbb{R}^{d})$ for $j = 1, \ldots, n$. Then the following properties are equivalent.

(i) $\bigcup_{j=1}^{n} G(\varphi_{j}, \Lambda)$ is a multi-window Gabor frame for $L^{2}(\mathbb{R}^{d})$.

(ii) The analysis operator $C_{\varphi, \Lambda}$ is one-to-one from $M_{\infty}(\mathbb{R}^{d})$ to $\ell_{\infty}(\Lambda, \mathbb{C}^{n})$.

The idea of the proof will be given in Appendix A where we will also list many more equivalent conditions.

2.5. Properties of Localization Operators. We next recall some elementary properties of the localization operators $H_{T, \sigma}$. Time-frequency localization operators have been introduced and studied by Daubechies [11, 10] and Ramanathan and Topiwala [34], and are also called STFT multipliers, time-frequency Toeplitz operators, Wick operators, time-frequency filters, etc. They are a popular tool in signal analysis for time-frequency filtering or nonstationary filtering [32, 35], in quantization procedures in physics [1], or in the approximation of pseudodifferential operators [9, 31]. For a detailed account of the early theory we refer to Wong’s book [39], for a study of boundedness and Schatten class properties to [7, 8, 18, 37].

**Lemma 4** (Intertwining property). If $\sigma \in L^{\infty}(\mathbb{R}^{2d})$, $\varphi \in L^{2}(\mathbb{R}^{d})$, and $\lambda \in \Lambda$, then

$$\pi(\lambda) H_{\sigma} \pi(\lambda)^{*} = H_{T, \sigma}.$$  

The proof consists of a simple calculation, see [12, Lemma 2.6].

For estimates of the STFT of $H_{\sigma}f$ we introduce the formal adjoint of $V_{\varphi}$, namely

$$V_{\varphi}^{*} F = \int_{\mathbb{R}^{2d}} F(z) \pi(z) \varphi dz,$$
which maps functions on $\mathbb{R}^{2d}$ to functions or distributions on $\mathbb{R}^d$. With this notation we can write the localization operator $H_\sigma$ as

$$H_\sigma f = \mathcal{V}_\phi^*(\sigma \mathcal{V}_\phi f).$$

The STFT of $\mathcal{V}_\phi^* F$ satisfies a fundamental pointwise estimate [21, Proposition 11.3.2]:

$$|\mathcal{V}_\phi(\mathcal{V}_\phi^* F)(z)| \leq \left(|\mathcal{V}_\phi \sigma| * |F|\right)(z) \quad \forall z \in \mathbb{R}^{2d}.\tag{15}$$

We note that for $F = \sigma \mathcal{V}_\phi f$ this estimate becomes

$$|\mathcal{V}_\phi(H_\sigma f)(z)| = |\mathcal{V}_\phi(\mathcal{V}_\phi^*(\sigma \mathcal{V}_\phi f))(z)| \leq \left(|\mathcal{V}_\phi \sigma| * (\sigma |\mathcal{V}_\phi f|)\right)(z).\tag{16}$$

Thus the short-time Fourier transform of $H_\sigma$ is a so-called product-convolution operator. The standard boundedness results for localization operators can be easily deduced from the well established results for product convolution operators [9].

Estimate (16) is quite useful for the derivation of norm estimates. In the following, we fix a non-negative symbol $\sigma$ and investigate the set of operators $\{H_{\lambda, \sigma} : \lambda \in \Lambda\}$. To simplify notation we will write $H_\lambda$ instead of $H_{\lambda, \sigma}$, and sometimes $H_0 = H_\sigma$ by some abuse of notation.

**Lemma 5.** (i) Assume that $\sigma \in L^1(\mathbb{R}^{2d})$, $\sigma \geq 0$ and that $\phi \in L^2(\mathbb{R}^d)$. Then each $H_\lambda$, $\lambda \in \Lambda$, is a positive trace-class operator.

(ii) If, in addition, $\phi \in M^1_\nu(\mathbb{R}^d)$ and $\sigma \in M^1_\nu(\mathbb{R}^{2d})$, then each $H_\lambda$ is bounded from $M^\infty(\mathbb{R}^d)$ into $M^1_\nu(\mathbb{R}^d)$. In particular, all eigenfunctions $\phi_j$ of $H_\sigma$ belong to $M^1_\nu(\mathbb{R}^d)$.

(iii) Furthermore, if $\phi \in M^1_\nu(\mathbb{R}^d)$ and $\sigma \in L^1(\mathbb{R}^{2d})$, then each $H_\lambda$ is bounded from $M^\infty_{1/\nu}(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$.

**Proof.** Statement (i) is well-known, see, e.g., [2] [17] [39].

To show (ii), we use (16) to obtain, for $f \in M^\infty(\mathbb{R}^d)$,

$$\|H_\sigma f\|_{M^1_\nu} = \|\mathcal{V}_\phi(H_\sigma f)\|_{L^1_\nu} = \|\mathcal{V}_\phi \mathcal{V}_\phi^*(\sigma \mathcal{V}_\phi f)\|_{L^1_\nu} \leq \|\|\mathcal{V}_\phi \sigma| * |\mathcal{V}_\phi f|\|_{L^1_\nu} \leq \|\mathcal{V}_\phi \sigma\|_{L^1_\nu} \|\mathcal{V}_\phi f\|_{L^\infty},\tag{17}$$

where we have used Young’s inequality. Since $\phi \in M^1_\nu(\mathbb{R}^d)$ if and only if $\mathcal{V}_\phi \phi \in L^1(\mathbb{R}^d)$ by [21, Prop. 12.1.2], we find that

$$\|H_\sigma f\|_{M^1_\nu} \leq \|\mathcal{V}_\phi \phi\|_{L^1_\nu} \|\sigma\|_{L^1_\nu} \|\mathcal{V}_\phi f\|_{L^\infty} \leq C\|\sigma\|_{L^1_\nu} \|f\|_{M^\infty},$$

and thus $H_\sigma$ is bounded from $M^\infty(\mathbb{R}^d)$ to $M^1_\nu(\mathbb{R}^d)$.

The proof of (iii) is similar. Again, we apply (16) to obtain for $f \in M^\infty_{1/\nu}(\mathbb{R}^d)$:

$$\|H_\sigma f\|_{L^2} \leq \|\|\mathcal{V}_\phi \sigma| * |\mathcal{V}_\phi f|\|_{L^2} \leq \|\mathcal{V}_\phi \phi\|_{L^2} \|\sigma \mathcal{V}_\phi f\|_{L^1_\nu}.$$

Hence, the result follows from

$$\|\sigma \mathcal{V}_\phi f\|_{L^1} = \int_{\mathbb{R}^{2d}} \sigma(z) |\mathcal{V}_\phi f(z)| \frac{1}{\nu(z)} dz \leq \|\sigma\|_{L^1_\nu} \|f\|_{M^\infty_{1/\nu}}.$$

□
The spectral theorem for compact self-adjoint operators provides the following spectral representation of $H_\lambda$.

**Corollary 6.** Assume $\varphi \in M^1_1(\mathbb{R}^d)$ and $\sigma \in L^1_1(\mathbb{R}^{2d})$. Then there exists a positive sequence of eigenvalues $c = (c_j) \in \ell^1$ and an orthonormal system of eigenfunctions $\varphi_j \in M^1_1(\mathbb{R}^d)$, such that

$$H_\sigma f = \sum_{j=1}^\infty c_j \langle f, \varphi_j \rangle \varphi_j. \quad (18)$$

It follows that

$$H_\lambda f = H_{T_\lambda} f = \pi(\lambda) H_\sigma \pi(\lambda)^* f = \sum_{j=1}^\infty c_j \langle f, \pi(\lambda) \varphi_j \rangle \pi(\lambda) \varphi_j, \quad (19)$$

and $\{\pi(\lambda) \varphi_j, j \in \mathbb{N}\}$ is an orthonormal system of eigenfunctions of $H_\lambda$.

A priori, the spectral representation of $H_\lambda$ holds only for $f \in L^2(\mathbb{R}^d)$. The next corollary extends the spectral representation to all of $M^\infty_{1/\nu}(\mathbb{R}^d)$.

**Corollary 7.** The expansion for $H_\lambda f$ given in (19) is well-defined on $M^\infty_{1/\nu}(\mathbb{R}^d)$ and converges to $H_\lambda f$ in $L^2$ for all $f \in M^\infty_{1/\nu}(\mathbb{R}^d)$.

**Proof.** Without loss of generality, we assume $\lambda = 0$ and set $H = H_\sigma$. Since $H f \in L^2(\mathbb{R}^d)$ for every $f \in M^\infty_{1/\nu}(\mathbb{R}^d)$ by Lemma 3(iii), we can expand $H f$ with respect to the orthonormal system of eigenfunctions of $H$ and obtain that

$$H f = \sum_{j=1}^\infty \langle H f, \varphi_j \rangle \varphi_j + r \quad (20)$$

for some $r \in L^2(\mathbb{R}^d)$ in the orthogonal complement of $\text{span}\{\varphi_j : j \in \mathbb{N}\}$. As $H$ is self-adjoint on $L^2(\mathbb{R}^d)$, we also have $\langle H f, \varphi_j \rangle = \langle f, H \varphi_j \rangle = c_j \langle f, \varphi_j \rangle$, and consequently

$$H f = \sum_{j=1}^\infty c_j \langle f, \varphi_j \rangle \varphi_j + r. \quad (21)$$

We need to show that $r = 0$. Since $r \in L^2(\mathbb{R}^d)$ is orthogonal to all eigenfunctions $\varphi_j$, we find that $\langle H f, r \rangle = \|r\|^2_2$.

To show $r = 0$, we first observe that $\langle H h, r \rangle = 0$ for all $h \in L^2(\mathbb{R}^d)$ by (18). Since $L^2(\mathbb{R}^d)$ is $w^*$-dense in $M^\infty_{1/\nu}(\mathbb{R}^d)$, we may choose an approximating sequence $f_n \in L^2(\mathbb{R}^d)$ such that $f_n \overset{w^*}{\to} f \in M^\infty_{1/\nu}(\mathbb{R}^d)$. For instance, $f_n$ may be chosen as

$$f_n = \int_{\mathbb{R}^d} \chi_{B_n}(z) V_g f(z) \pi(z) g \, dz,$$

where $B_n$ is the ball with radius $n$ and centered at 0. Furthermore, since $f_n \overset{w^*}{\to} f$, we obtain in particular that $\mathcal{V}_\varphi f_n$ converges to $\mathcal{V}_\varphi f$ uniformly on compact sets [13].
Theorem 4.1. Consequently

$$0 = \langle Hf_n, r \rangle = \int_{\mathbb{R}^d} \sigma(z) \mathcal{V}_r f_n(z) \mathcal{V}_r(z) \, dz \to \int_{\mathbb{R}^d} \sigma(z) \mathcal{V}_r f(z) \mathcal{V}_r(z) \, dz = \langle Hf, r \rangle = \|r\|^2_{2\nu}.$$  

This shows that $r = 0$ and so the series (19) represents $Hf$ for all $f \in M^{1/\nu}_{\infty}(\mathbb{R}^d)$. □

### 3. From Local Information to Global Information

We first state and prove the main result for the modulation spaces $M^{p}_{m}(\mathbb{R}^d)$. The generalizations to $M^{p,q}_{m}(\mathbb{R}^d)$ will be discussed later. As always, $\nu$ denotes a submultiplicative, even weight function on $\mathbb{R}^{2d}$ satisfying the condition $\lim_{n \to \infty} \nu(nz)^{1/n} = 1$ for all $z \in \mathbb{R}^{2d}$.

**Theorem 8.** Let $\sigma \in L^{1}_{\nu}(\mathbb{R}^{2d})$ be a non-negative symbol satisfying the condition

$$A \leq \sum_{\lambda \in \Lambda} T_{\lambda} \sigma \leq B,$$  

for two constants $A, B > 0$. Assume that $\varphi \in M^{1}_{\nu}(\mathbb{R}^d)$. Then for every $\nu$-moderate weight $m$ and $1 \leq p < \infty$ the distribution $f \in M^{\infty}_{1/\nu}(\mathbb{R}^d)$ belongs to $M^{p}_{m}(\mathbb{R}^d)$, if and only if

$$\left( \sum_{\lambda \in \Lambda} \| H_{\lambda} f \|^p_{2} m(\lambda)^p \right)^{1/p} < \infty,$$

and the expression in (23) is an equivalent norm on $M^{p}_{m}(\mathbb{R}^d)$.

Similarly, for $p = \infty$ we obtain the norm equivalence

$$\| f \|_{M^{\infty}_{m}} \simeq \sup_{\lambda \in \Lambda} \| H_{\lambda} f \|_{2} m(\lambda).$$

The norm equivalence supports the interpretation that $H_{\lambda} f$ carries the local time-frequency information about $f$ near $\lambda \in \mathbb{R}^{2d}$. By combining the local pieces $H_{\lambda} f$, one obtains the global time-frequency information as it is measured by modulation space norms.

The proof of Theorem 8 requires some preparations. We first show that finitely many eigenfunctions of $H_{0} = \mathcal{V}_{\varphi}^{*} \sigma \mathcal{V}_{\varphi}$ generate a multi-window Gabor frame for $L^{2}(\mathbb{R}^d)$. With this crucial step in place, Theorem 8 can then be deduced from the characterization of modulation spaces by means of Gabor frames.

#### 3.1. Multi-Window Gabor Frames.

**Lemma 9.** Assume that $\sigma \in L^{1}(\mathbb{R}^{2d})$ and $\sum_{\lambda \in \Lambda} T_{\lambda} \sigma \asymp 1$, and that $\varphi \in M^{1}_{\nu}(\mathbb{R}^d)$. Let $\{ \varphi_{j} : j \in \mathbb{N} \}$ be the orthonormal system of eigenfunctions of $H_{0}$. Then there exists $n \in \mathbb{N}$, such that the finite union $\bigcup_{j=1}^{n} G(\varphi_{j}, \Lambda)$ is a multi-window Gabor frame for $L^{2}(\mathbb{R}^d)$.

An analogous statement was proved and used in [12] for the lattice $\Lambda = \mathbb{Z}^{2d}$ and rational lattices by means of Zak transform methods. In the case of general lattices we cannot apply Zak-transform methods. As a substitute, we will use a finite intersection property for $\Lambda$-invariant subspaces of $M^{\infty}$. The following statement may be of interest in its own right.
Lemma 3. By construction, the function consisting of the first $f$ such that $w$ be the kernel of the coefficient operator $C$.

Proof of Lemma 9. To prove that finitely many eigenfunctions generate a multi-window Gabor frame with respect to the lattice $\Lambda$, we assume on the contrary that $A$ will derive a contradiction to the assumption that $\Lambda = A[0,1]^{2d}$. We first choose a sequence $h_n \in \mathcal{W}_n$ with $\|h_n\|_{\mathcal{M}^\infty} = \sup_{z \in \mathbb{R}^{2d}} |\mathcal{V}_\varphi h_n(z)| = 1$. Then there exists a sequence of points $\lambda_n$ in $\Lambda$, such that

$$\sup_{z \in Q} |\mathcal{V}_\varphi (\pi(\lambda_n)h_n)(z)| = 1.$$ 

Since $\mathcal{W}_n$ is invariant under all $\pi(\lambda), \lambda \in \Lambda$, the distribution $f_n = \pi(\lambda_n)h_n$ is in $\mathcal{W}_n$.

Next we show that the set of restrictions $\{\mathcal{V}_\varphi f_n|_Q\}$ is equicontinuous. We have

$$|\mathcal{V}_\varphi f_n(z) - \mathcal{V}_\varphi f_n(\xi)| = |\langle f_n, (\pi(z) - \pi(\xi))\varphi \rangle| \leq \|f_n\|_{\mathcal{M}^\infty} \cdot \|(\pi(z) - \pi(\xi))\varphi\|_{\mathcal{M}^1}.$$ 

Since $\|f_n\|_{\mathcal{M}^\infty} = \|\pi(\lambda_n)h_n\|_{\mathcal{M}^\infty} = 1$, the equicontinuity follows from the strong continuity of time-frequency shifts on $\mathcal{M}^1(\mathbb{R}^{2d})$.

We next choose $z_n \in Q$ with $|\mathcal{V}_\varphi f_n(z_n)| \geq \frac{1}{2}$. Since the unit ball in $\mathcal{M}^\infty(\mathbb{R}^{2d})$ is $w^*$-compact, there exists a subsequence $f_{n_k}$ that converges to some $f \in \mathcal{M}^\infty(\mathbb{R}^{2d})$ in the $w^*$-sense. Furthermore, by compactness of $Q$, there also exists a subsequence $z_{\ell}$ of $z_{n_k}$, such that $z_{\ell} \to z \in Q$. Hence, by equicontinuity,

$$\mathcal{V}_\varphi f_{\ell}(z_{\ell}) \to \mathcal{V}_\varphi f(z).$$ 

Since $|\mathcal{V}_\varphi f(z_{\ell})| \geq 1/2$, we conclude that also $|\mathcal{V}_\varphi f(z)| \geq 1/2$, and consequently $f \neq 0$.

By construction, $f_\ell \in \mathcal{W}_m$ for every $\ell \geq m$, hence we obtain $f = w^* \lim_{\ell \to -\infty} f_\ell \in \mathcal{W}_m$ for all $m$, because $\mathcal{W}_m$ is $w^*$-closed. To summarize, we have constructed a non-zero $f \in \mathcal{M}^\infty(\mathbb{R}^{2d})$ that is in $\mathcal{W}_m$ for all $m$.

Proof of Lemma 10. To prove that finitely many eigenfunctions generate a multi-window Gabor frame with respect to the lattice $\Lambda$, we assume on the contrary that $\bigcup_{j=1}^n G(\varphi_j, \Lambda)$ is not a frame for every $n \in \mathbb{N}$. Using Lemma and Lemma we will derive a contradiction to the assumption that $A \leq \sum_{\lambda \in \Lambda} T_{\lambda, \sigma} \leq B$.

We use the criterion of Lemma 2. Let $\varphi_n = (\varphi_1, \ldots, \varphi_n)$ be the vector-valued function consisting of the first $n$ eigenfunctions of $H_0$, and

$$\mathcal{W}_n = \ker(C_{\varphi_n, \Lambda}) = \{f \in \mathcal{M}^\infty(\mathbb{R}^{2d}) : \langle f, \pi(\lambda)\varphi_j \rangle = 0, \forall \lambda \in \Lambda, j = 1, \ldots, n\}$$

be the kernel of the coefficient operator $C_{\varphi_n, \Lambda}$ in $\mathcal{M}^\infty(\mathbb{R}^{2d})$.

If $\bigcup_{j=1}^n G(\varphi_j, \Lambda)$ is not a frame, then $\mathcal{W}_n$ is a non-trivial subspace of $\mathcal{M}^\infty(\mathbb{R}^{2d})$ by Lemma 3. By construction, the $\mathcal{W}_n$’s form a nested sequence of $w^*$-closed subspaces of $\mathcal{M}^\infty(\mathbb{R}^{2d})$, and they are also invariant under $\pi(\lambda), \lambda \in \Lambda$. Thus the assumptions
of Lemma 10 are satisfied, and we conclude that \( \bigcap_{n=1}^{\infty} W_n \neq \{0\} \). This means that there exists a non-zero \( f \in M^\infty(\mathbb{R}^d) \), such that

\[
(26) \quad \langle f, \pi(\lambda) \varphi_j \rangle = 0 \quad \text{for all } \lambda \in \Lambda \text{ and all } j \in \mathbb{N}.
\]

We now consider \( H \lambda f \). Since \( H \lambda f \in M^1(\mathbb{R}^d) \) by Lemma 5, the bracket \( \langle H \lambda f, f \rangle \) is well-defined and given by

\[
(27) \quad \langle H \lambda f, f \rangle = \int_{\mathbb{R}^{2d}} \sigma(z - \lambda) |\mathcal{V}_\varphi f(z)|^2 dz.
\]

On the other hand, the extended spectral representation of Lemma 7 and (26) imply that

\[
(28) \quad H \lambda f = \sum_{j=1}^{\infty} c_j \langle f, \pi(\lambda) \varphi_j \rangle \pi(\lambda) \varphi_j = 0.
\]

Consequently \( \langle H \lambda f, f \rangle = 0 \) for all \( \lambda \in \Lambda \), and \( |\mathcal{V}_\varphi f(z)|^2 \) vanishes on \( \bigcup_{\lambda \in \Lambda} \text{supp} T_\lambda \sigma \).

According to the crucial assumption (22) we have \( \sum_{\lambda \in \Lambda} T_\lambda \sigma \geq A > 0 \) almost everywhere, and thus \( \bigcup_{\lambda \in \Lambda} \text{supp}(T_\lambda \sigma) = \mathbb{R}^{2d} \). Therefore, (27) and (28) imply that \( \mathcal{V}_\varphi f = 0 \), from which \( f = 0 \) follows. This is a contradiction to \( f \) being a non-zero element in \( \bigcap_{n=1}^{\infty} W_n \).

This contradiction shows that there exists an \( n \in \mathbb{N} \), such that \( \bigcup_{j=1}^n \mathcal{G}(\varphi_j, \Lambda) \) is a multi-window Gabor frame, and we are done. \( \square \)

Remark 1. Note that for finite-rank operators \( H_0 \), it can be seen directly that the finite set of eigenvectors generates a multi-window Gabor frame for \( \Lambda \).

3.2. Proof of Theorem 8. We are now ready to prove the main theorem. We observe that for \( f \in M^\infty_{1/\nu}(\mathbb{R}^d) \), \( H \lambda f \in L^2(\mathbb{R}^d) \) by Lemma 5(iii). Thus the terms in (23) are well-defined.

First assume that \( p < \infty \) and \( f \in M^p_m(\mathbb{R}^d) \subseteq M^\infty_{1/\nu}(\mathbb{R}^d) \). Using the embedding \( M^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \) and the estimate (17) with \( \nu \equiv 1 \), we majorize \( \|H \lambda f\|_2 \) as follows:

\[
(29) \quad \|H \lambda f\|_2 \leq C_\varphi \|H \lambda f\|_{M^1} \leq C_\varphi \|(T_\lambda \sigma) \cdot \mathcal{V}_\varphi f\|_1 \|\mathcal{V}_\varphi \varphi\|_1 \leq C_\varphi C \int_{\mathbb{R}^{2d}} |\sigma(z - \lambda)| \cdot |\mathcal{V}_\varphi f(z)| \, dz \leq C_\varphi C \|(|\mathcal{V}_\varphi f| \ast \sigma^\vee)(\lambda)\|,
\]

where \( \sigma^\vee(z) = \sigma^\vee(-z) \). Thus \( \|H \lambda f\|_2 \) is majorized by a sample of \( |\mathcal{V}_\varphi f| \ast \sigma^\vee \).

To proceed further, we use the fact that \( \mathcal{V}_\varphi f \in W(L_m^p) \) and \( \|\mathcal{V}_\varphi f\|_{W(L_m^p)} \leq C_0 \|\varphi\|_{M^1} \|f\|_{M^p_m} \) for \( \varphi \in M^1_m(\mathbb{R}^d) \) and \( f \in M^p_m(\mathbb{R}^d) \) by [21, Thm. 12.2.1]. Now
The same argument yields sup_{λ ∈ Λ} \|H_λ f\|_{2 \ell m} \leq C_φ C_λ \|\sigma \ast |\mathcal{V}_φ f|\|_{L^p_{\ell m}}
\leq C_φ C_λ \|\sigma\|_{L^p_{\ell m}} \|\mathcal{V}_φ f\|_{L^p_{\ell m}} = C_φ C_λ \|\sigma\|_{L^p_{\ell m}} \|f\|_{M^p_{\ell m}}.

The same argument yields sup_{λ ∈ Λ} \|H_λ f\|_{2 m(λ)} \leq C \|f\|_{M^p_{\ell m}}.

Hence, for 1 \leq p \leq ∞, the mapping f → (\|H_λ f\|_2)_{λ ∈ Λ} is bounded from \mathcal{M}^p_m(\mathbb{R}^d) to \ell^p_m(Λ).

Conversely, assume that p < ∞ and
\[ \sum_λ \|H_λ f\|_{2 m(λ)}^p < ∞. \]

We need to show that f ∈ \mathcal{M}^p_m(\mathbb{R}^d). Since \|H_λ f\|_2 = sup_{\|g\|_2 = 1} |\langle H_λ f, g \rangle|, we have the inequality
\[ \sum_λ |\langle H_λ f, g_λ \rangle|^p m(λ)^p \leq \sum_λ \|H_λ f\|_{2 m(λ)}^p < ∞ \]
for arbitrary sequences g_λ ∈ L^2(\mathbb{R}^d) with \|g_λ\|_2 = 1. Applying the eigenfunction expansion of Corollary 9, we obtain
\[ \sum_λ \left| \sum_{j=1}^∞ c_j \langle f, π(λ)φ_j \rangle \langle π(λ)φ_j, g_λ \rangle \right|^p m(λ)^p \leq \sum_λ \|H_λ f\|_{2 m(λ)}^p < ∞. \]

Now fix j_0 ∈ \mathbb{N} and set g_λ = π(λ)φ_{j_0} for λ ∈ Λ. Since the eigenfunctions of H_λ are orthonormal, the sum over j collapses to a single term, and (31) becomes
\[ \sum_λ |\langle H_λ f, g_λ \rangle|^p m(λ)^p = \sum_λ |c_{j_0} \langle f, π(λ)φ_{j_0} \rangle|^p m(λ)^p \leq \sum_λ \|H_λ f\|_{2 m(λ)}^p < ∞. \]

The last inequality holds for every j_0 ∈ \mathbb{N}. After summing over finitely many j_0 and switching to the \ell^2-norm on \mathbb{C}^n, we obtain the inequality
\[ \sum_λ \left( \sum_{j=1}^n |\langle f, π(λ)φ_j \rangle|^2 \right)^{1/2} m(λ)^p \leq \sum_{j=1}^n \sum_λ |\langle f, π(λ)φ_j \rangle|^p m(λ)^p \]
\[ \leq \left( \sum_{j=1}^n \frac{1}{c_{j_0}} \right) \sum_λ \|H_λ f\|_{2 m(λ)}^p < ∞. \]

We now apply Lemma 12 and choose an n ∈ \mathbb{N}, such that \bigcup_{j=1}^n \mathcal{G}(φ_j, Λ) is a multi-window Gabor frame for L^2(\mathbb{R}^d). Since all φ_j are in \mathcal{M}^p_m(\mathbb{R}^d), the fundamental characterization of modulation spaces (Section 2.3) is valid. Thus Theorem 2(i) implies that f ∈ \mathcal{M}^p_m(\mathbb{R}^d).

If p = ∞ and sup_{λ ∈ Λ} \|H_λ f\|_2 m(λ) < ∞, then, by choosing g_λ as before, we find
\[ c_{j_0} \sup_λ |\langle f, π(λ)φ_{j_0} \rangle| m(λ) \leq \sup_λ \|H_λ f\|_2 m(λ) < ∞ \]
for every j_0.
Arguing as above, Theorem 2 says that
\[ \|f\|_{M_m^\infty} \leq C \max_{j=1,...,n} \sup_\lambda |\langle f, \pi(\lambda)\varphi_j \rangle| m(\lambda) \leq \left( \max_{j=1,...,n} \frac{1}{c_j} \right) \sup_\lambda \|H_\lambda f\|_2 m(\lambda) < \infty, \]
and \( f \in M_m^\infty(\mathbb{R}^{2d}) \).

Combining (30) and (32), we have shown that \( \|f\|_{M_m^\infty} \) and \( \left( \sum_{\lambda \in \Lambda} \|H_\lambda f\|_2^p m(\lambda)^p \right)^{1/p} \) for \( 1 \leq p < \infty \) (or \( \sup_{\lambda \in \Lambda} \|H_\lambda f\|_2 m(\lambda) \) for \( p = \infty \) are equivalent norms on \( M_m^p(\mathbb{R}^{2d}) \). \( \square \)

3.3. Variations of Theorem 8 In order to formulate our main result for mixed-norm spaces and arbitrary lattices, we have to resort to the theory of coorbit spaces, as introduced in [13, 14]. In particular, for arbitrary lattices, a sequence \( (c_\lambda)_{\lambda \in \Lambda} \) is in the sequence spaces associated with \( L_m^{p,q}(\mathbb{R}^{2d}) \), if \( \sum_{\lambda \in \Lambda} c_\lambda \chi_{\lambda+Q} \) is in \( L_m^{p,q}(\mathbb{R}^{2d}) \) for some fundamental domain \( Q \) of \( \Lambda \). With this definition, we may give the following characterization.

**Theorem 11.** Let \( \Lambda \) be an arbitrary lattice in \( \mathbb{R}^{2d} \) and \( Q \) be a relatively compact fundamental domain \( Q \). Assume the same conditions on \( \sigma \) and \( \varphi \) as in Theorem 8. Then a distribution \( f \in M_{1/\nu}^\infty(\mathbb{R}^{2d}) \) belongs to \( M_m^{p,q}(\mathbb{R}^{2d}) \), \( 1 \leq p, q \leq \infty \), if and only if

\[ \sum_{\lambda \in \Lambda} \|H_\lambda f\|_2 \chi_{\lambda+Q} \in L_m^{p,q}(\mathbb{R}^{2d}), \]
and \( \| \sum_{\lambda \in \Lambda} \|H_\lambda f\|_2 \chi_{\lambda+Q} \|_{L_m^{p,q}} \precsim \|f\|_{M_m^{p,q}} \).

**Proof.** The proof is almost identical to the proof of Theorem 8. The only modifications occur in (30), which has to be replaced by

\[ \| \sum_{\lambda \in \Lambda} \|H_\lambda f\|_2 \chi_{\lambda+Q} \|_{L_m^{p,q}} \leq \| \sum_{\lambda \in \Lambda} |\mathcal{V}_\varphi f * \tilde{\sigma}(\lambda)| \chi_{\lambda+Q} \|_{L_m^{p,q}} \leq C \|\mathcal{V}_\varphi f * \tilde{\sigma}\|_{W(L_m^{p,q})}. \]

Likewise, in (32) we replace the weighted \( L_m^p \)-norm by the general \( L_m^{p,q} \)-norm. \( \square \)

For a separable lattice \( \Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d \) the norm in (33) is just the \( l_m^{p,q} \)-norm on \( \mathbb{Z}^{2d} \) with \( \tilde{m}(k, n) = m(ak, bn) \). In this case, \( \lambda = (ka, nb) \), \( k, n \in \mathbb{Z}^d \) and we may write \( H_\lambda f = H_{k,n} f \).

**Corollary 12.** Let \( \Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d \) be a separable lattice and assume the same conditions on \( \sigma \) and \( \varphi \) as in Theorem 8. Then a distribution \( f \in M_{1/\nu}^\infty(\mathbb{R}^{2d}) \) belongs to \( M_m^{p,q}(\mathbb{R}^{2d}) \) for \( 1 \leq p, q \leq \infty \), if and only if

\[ \left( \sum_{n \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \|H_{k,n} f\|_2^p m(ka, nb)^p \right)^{q/p} \right)^{1/q} < \infty, \]
and (34) defines an equivalent norm on \( M_m^{p,q}(\mathbb{R}^{2d}) \). The result holds for \( p = \infty \) or \( q = \infty \) with the usual modifications.
3.4. Existence of multi-window Gabor frames and properties of the eigenfunctions $\varphi_j$. We finally point out some immediate consequences of our results and methods.

The intermediate results leading to Theorem 8 also imply the existence of multi-window Gabor frames for general lattices.

**Theorem 13.** Let $\Lambda$ be an arbitrary lattice and $\nu$ a submultiplicative weight on $\mathbb{R}^{2d}$. Then there exist finitely many functions $\varphi_j \in M^1_{\nu}(\mathbb{R}^d)$, such that $\bigcup_{j=1}^n \mathcal{G}(\varphi_j, \Lambda)$ is a multi-window Gabor frame for $L^2(\mathbb{R}^d)$.

**Proof.** Choose $\sigma \in L^1_{\nu}(\mathbb{R}^{2d})$ such that $\sum_{\lambda \in \Lambda} T_{\lambda} \sigma \asymp 1$ and fix a window $\varphi \in M^1_{\nu}(\mathbb{R}^d)$. For instance, one may choose the characteristic function $\chi_Q$ of a (relatively compact) fundamental domain of $\Lambda$ and the Gaussian window $\varphi(t) = e^{-\pi t \cdot t}$.

Now consider the localization operator $H_0 = V_\varphi^* \sigma V_\varphi$. According to Lemma 5(ii), all eigenfunctions $\varphi_j$ of $H_0$ belong to $M^1_{\nu}(\mathbb{R}^d)$. Lemma 9 states that for some finite $n \in \mathbb{N}$ the set $\bigcup_{j=1}^n \mathcal{G}(\varphi_j, \Lambda)$ is a multi-window Gabor frame for $L^2(\mathbb{R}^d)$. □

The existence of multi-window Gabor frames for general lattices was known before. On the one hand, it is an immediate consequence of coorbit theory applied to the Heisenberg group. To be more precise, according to [15, Thm. 7] for every lattice $\Lambda$ and every non-zero $g \in M^1_{\nu}(\mathbb{R}^d)$ there exists $n \in \mathbb{N}$, such that the set $\mathcal{G}(g, \frac{1}{n} \Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Using a coset decomposition $\frac{1}{n} \Lambda = \bigcup (\mu + \Lambda)$ for suitable $\mu \in \Lambda$, one sees that $\mathcal{G}(g, \frac{1}{n} \Lambda) = \bigcup \mathcal{G}(\pi(\mu) g, \Lambda)$ is a multi-window Gabor frame with all windows $\pi(\mu) g$ derived from a single window $g$. Recently Luef [33] proved the existence of multi-window Gabor frames by exploiting a connection between Gabor analysis and non-commutative geometry. Our methods provide a third, independent proof for this interesting result.

The construction of multi-window Gabor frames in Proposition 13 yields more detailed information about the frame generators, since they are eigenfunctions of a localization operator. Intuitively the eigenfunctions corresponding to the largest eigenvalues of a localization operator concentrate their energy on the essential support of the symbol $\sigma$ of $H_0$. For the special case of compactly supported $\sigma$, this intuition is made precise by the following result.

**Proposition 14.** Let the non-negative function $\sigma \in L^1(\mathbb{R}^{2d})$ be supported in a compact set $\Omega$ in $\mathbb{R}^{2d}$ with $0 \leq \sigma(z) \leq C_\sigma < \infty$ for $z \in \Omega$. Consider the localization operator given by $H_\sigma f = V_\varphi^* \sigma V_\varphi f$ with $\varphi \in M^1(\mathbb{R}^d)$, $\|\varphi\|_2 = 1$ and spectral representation as in Corollary 4. Then the eigenfunctions $\varphi_j$ of $H_\sigma$ satisfy the following time-frequency concentration

$$\int_{\Omega} |V_\varphi \varphi_j(z)|^2 \, dz \geq \frac{c_j}{C_\sigma}.$$  

Equality holds, if and only if $\sigma(z)/C_\sigma = \chi_\Omega(z)$ is the characteristic function of $\Omega$. 

Proof. Using the weak interpretation of $H_\sigma$ from (2), we obtain
\[
\int_\Omega |V_\phi \varphi_j(z)|^2 \, dz \geq \frac{1}{C_\sigma} \int_\Omega \sigma(z) |V_\phi \varphi_j(z)|^2 \, dz
= \frac{1}{C_\sigma} \langle H_\sigma \varphi_j, \varphi_j \rangle = \frac{c_j}{C_\sigma} \| \varphi_j \|^2 = \frac{c_j}{C_\sigma}.
\]
\[\square\]

Appendix A. Characterizations of Modulation Spaces and Multi-Window Gabor Frames

In the appendix, we will sketch the proof of Theorem 2 and formulate a series of new characterizations of multi-window Gabor frames. These statements generalize well-known facts from Gabor analysis and the results about Gabor frames without inequalities in [22].

For the investigation of multi-window Gabor frames we need the dual concept of vector-valued Gabor systems. In this case we consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C}^n)$ consisting of all vector-valued functions $f(t) = (f_1(t), \ldots, f_n(t))$ with the inner product

\[
\langle f, \varphi \rangle_{L^2(\mathbb{R}^d, \mathbb{C}^n)} = \sum_{j=1}^n \int f_j(t) \overline{\varphi_j(t)} \, dt = \sum_{j=1}^n \langle f_j, \varphi_j \rangle_{L^2(\mathbb{R}^d)}.
\]

Time-frequency-shifts act coordinate-wise on $f$. The vector-valued Gabor system $\mathcal{G}(\varphi, \Lambda) = \{ \pi(\lambda) \varphi : \lambda \in \Lambda \}$ is a Riesz sequence in $L^2(\mathbb{R}^d, \mathbb{C}^n)$, if there exist constants $0 < A, B < \infty$ such that for all finitely supported sequences $c$,

\[
A \| c \|_2^2 \leq \| \sum_{\lambda \in \Lambda} c_\mu(\lambda) \varphi \|_{L^2(\mathbb{R}^d, \mathbb{C}^n)}^2 \leq B \| c \|_2^2.
\]

We now proceed to the proof of Theorem 2. The crucial step is to show the invertibility of the frame operator on $M_1\nu(\mathbb{R}^d)$. This step requires a special representation of the frame operator due to Janssen [29] and at its core uses “Wiener’s lemma for twisted convolution” [25].

For $\varphi_j, \psi_j$ in $M^1(\mathbb{R}^d), j = 1, \ldots, n$, we denote frame-type operators by

\[
S_{\varphi, \psi} f = \sum_{\lambda \in \Lambda} \sum_{j=1}^n \langle f, \pi(\lambda) \varphi_j \rangle \pi(\lambda) \psi_j = \sum_{j=1}^n \langle f, \varphi_j \rangle \psi_j.
\]

The frame operator of the Gabor system $\bigcup_{j=1}^n \mathcal{G}(\varphi_j, \Lambda)$ is $S = S_{\varphi, \varphi}$. We usually omit the reference to the lattice $\Lambda$ and the windows $\varphi_j$.

The volume $s(\Lambda)$ of a lattice $\Lambda = A \mathbb{Z}^{2d}$ is defined as the measure of a fundamental domain of $\Lambda$ and is $|\det(A)|$. The adjoint lattice of $\Lambda$ is $\Lambda^0 = \{ \mu \in \mathbb{R}^{2d} : \pi(\lambda) \pi(\mu) = \pi(\mu) \pi(\lambda) \text{ for all } \lambda \in \Lambda \}$.

**Lemma 15** (Janssen’s representation). Assume that $\varphi_j, \psi_j \in M^1(\mathbb{R}^d)$ for all $j = 1, \ldots, n$. Then the frame type operator associated to $\bigcup_{j=1}^n \mathcal{G}(\varphi_j, \Lambda)$ and $\bigcup_{j=1}^n \mathcal{G}(\psi_j, \Lambda)$
Lemma 16. The following statement is a generalization of [25, Thm. 9] to multi-window Gabor frames. As a general principle the localization of a frame is inherited by the dual frame [19].

\[ S_{\varphi,\psi} f = s(\Lambda)^{-1} \sum_{\mu \in \Lambda^o} \sum_{j=1}^{n} \langle \varphi_j, \pi(\mu) \psi_j \rangle \pi(\mu) f \]

with unconditional convergence in the operator norm on \( L^2 \).

Proof. By Janssen’s result [20] the representation holds for a single \( S_{\varphi_j,\psi_j} \) and (38) follows by taking a sum.

The canonical dual frame is defined to be \( \gamma_{j,\lambda} = \pi(\lambda)S^{-1}\varphi_j \). Since the frame operator \( S = S_{\varphi,\varphi} \) commutes with time-frequency shifts on \( \Lambda \), we obtain the reconstruction formulas

\[ f = S^{-1}S f = \sum_{\lambda \in \Lambda} \sum_{j=1}^{n} \langle f, \pi(\lambda) \varphi_j \rangle \pi(\lambda) \gamma_j \]
\[ = SS^{-1} f = \sum_{\lambda \in \Lambda} \sum_{j=1}^{n} \langle f, \pi(\lambda) \gamma_j \rangle \pi(\lambda) \varphi_j \]
\[ = D_{\varphi,\Lambda} C_{\gamma,\Lambda} f = D_{\gamma,\Lambda} C_{\varphi,\Lambda} f \]

As a general principle the localization of a frame is inherited by the dual frame [19]. The following statement is a generalization of [25, Thm. 9] to multi-window Gabor frames on general lattices.

Lemma 16. Assume that \( \nu \) is a submultiplicative, even weight on \( \mathbb{R}^{2d} \) satisfying \( \lim_{n \to \infty} \nu(nz)^{1/n} = 1 \) for all \( z \in \mathbb{R}^{2d} \). Assume further that \( \bigcup_{j=1}^{n} G(\varphi_j,\Lambda) \) is a frame for \( L^2(\mathbb{R}^d) \) and that \( \varphi_j \in M^1_{\nu}(\mathbb{R}^d) \). Then the frame operator \( S \) is invertible on \( M^1_{\nu}(\mathbb{R}^d) \) and \( \gamma_j = S^{-1}\varphi_j \in M^1_{\nu}(\mathbb{R}^d) \) for \( j = 1, \ldots, n \).

Proof. Janssen’s representation (38) implies that

\[ S = S_{\varphi,\varphi} = s(\Lambda)^{-1} \sum_{\mu \in \Lambda^o} c_\mu \pi(\mu), \]

with a coefficient sequence \( c_\mu = \sum_{j=1}^{n} \langle \varphi_j, \pi(\mu) \varphi_j \rangle \). The hypothesis \( \varphi_j \in M^1_{\nu}(\mathbb{R}^d) \) guarantees that \( \sum_{\mu \in \Lambda^o} |\langle \varphi_j, \pi(\mu) \varphi_j \rangle| \nu(\mu) < \infty \) for each \( j \), see [21, Cor.12.1.12], and therefore the coefficient sequence \( (c_\mu) \) is in \( \ell^1(\Lambda^o) \). Since \( \bigcup_{j=1}^{n} G(\varphi_j,\Lambda) \) is a frame, the frame operator \( S_{\varphi,\varphi} \) is invertible on \( L^2(\mathbb{R}^d) \). It follows from [25, Theorem 3.1] that the inverse frame operator \( S^{-1} \) is again of the form \( S^{-1} = \sum_{\mu \in \Lambda^o} d_\mu \pi(\mu) \) with a coefficient sequence \( d \) in \( \ell^1(\Lambda^o) \). This representation implies that \( S^{-1} \) is bounded on \( M^1_{\nu}(\mathbb{R}^d) \) and that

\[ \| \gamma_j \|_{M^1_{\nu}} = \| S^{-1}\varphi_j \|_{M^1_{\nu}} \leq C \| \varphi_j \|_{M^1_{\nu}}. \]

Therefore the dual windows \( \gamma_j, j = 1, \ldots, n \) are in \( M^1_{\nu}(\mathbb{R}^d) \) as claimed.

Once the invertibility of the multi-window frame operator on \( M^1_{\nu}(\mathbb{R}^d) \) is established, the proof of Theorem 2 is straight-forward by using the following boundedness properties of the coefficient operator \( C_{\varphi,\Lambda} \) and \( D_{\varphi,\Lambda} \) from [21, Theorem 12.2.3. and 12.3.4.]. If \( \varphi_j \in M^1_{\nu}(\mathbb{R}^d) \) and \( \gamma_j \in M^1_{\nu}(\mathbb{R}^d) \), then both \( C_{\varphi,\Lambda} \) and \( C_{\gamma,\Lambda} \)
are bounded from $\mathbf{M}_{m,q}^p(\mathbb{R}^d)$ into $\ell_{m,q}^p(\Lambda, \mathbb{C}^n)$ for $1 \leq p, q \leq \infty$ and for every $\nu$-moderate weight $m$. Likewise $D_{\varphi, \Lambda}$ and $D_{\gamma, \Lambda}$ are bounded from $\ell_{m,q}^p(\Lambda, \mathbb{C}^n)$ into $\mathbf{M}_{m,q}^p(\mathbb{R}^d)$. For the $\ell_{m,q}^p(\Lambda, \mathbb{C}^n)$-norm we use the Euclidean norm on $\mathbb{C}^n$, so that $\|c\|_{\ell_{m,q}^p(\Lambda, \mathbb{C}^n)} = \sum_{\lambda \in \Lambda} \left( \sum_{j=1}^m |c_{\lambda,j}|^2 \right)^{1/2} \chi_{\lambda+q} \|L_{m,q}^p\).

As a consequence, the reconstruction formula $f = D_{\varphi, \Lambda} C_{\gamma, \Lambda} f = D_{\gamma, \Lambda} C_{\varphi, \Lambda} f$ holds for $f \in \mathbf{M}_{m,q}^p(\mathbb{R}^d)$ with the correct norm estimates. The norm equivalence stated in Theorem 2 then follows from

$$\|f\|_{\mathbf{M}_{m,q}^p(\mathbb{R}^d)} = \|D_{\varphi, \Lambda} C_{\gamma, \Lambda} f\|_{\mathbf{M}_{m,q}^p(\mathbb{R}^d)} \leq \|D_{\gamma, \Lambda}\|_{op} \|C_{\varphi, \Lambda} f\|_{\ell_{m,q}^p(\Lambda, \mathbb{C}^n)}$$

$$\leq \|D_{\gamma, \Lambda}\|_{op} \|C_{\varphi, \Lambda}\|_{op} \|f\|_{\ell_{m,q}^p(\Lambda, \mathbb{C}^n)}.$$

Next we come to the characterization of multi-window Gabor frames (Lemma 3) and extend the list of equivalent conditions. For the formulation of the dual conditions on the adjoint lattice $\Lambda^\circ$ we need the vector-valued versions of the analysis and synthesis operators. For $f = (f_1, \ldots, f_n) \in \mathbf{M}_{\infty}^1(\mathbb{R}^d, \mathbb{C}^n)$ and $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbf{M}_1^1(\mathbb{R}^d, \mathbb{C}^n)$ the coefficient operator is defined to be $\tilde{C}_{\varphi, \Lambda^\circ}(f)(\mu) = (\langle f, \pi(\mu) \varphi \rangle, \mu \in \Lambda^\circ$, and the synthesis operators is $\tilde{D}_{\varphi, \Lambda^\circ}(c) = \sum_{\mu \in \Lambda^\circ} c(\mu) \varphi$. The Gramian operator $G_{\varphi, \Lambda^\circ} = \tilde{C}_{\varphi, \Lambda^\circ} \tilde{D}_{\varphi, \Lambda^\circ}$ is defined on sequences indexed by $\Lambda^\circ$.

**Lemma 17.** Assume that $\varphi_j \in \mathbf{M}_1^1(\mathbb{R}^d)$ for $j = 1, \ldots, n$. The following are equivalent for the multi-window Gabor system $\bigcup_{j=1}^n G(\varphi_j, \Lambda)$:

(i) $\bigcup_{j=1}^n G(\varphi_j, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.

(ii) Wexler-Raz biorthogonality: There exist $\gamma_j \in M_1^1(\mathbb{R}^d), j = 1, \ldots, n$, such that

$$s(\Lambda)^{-1} \sum_{j=1}^n \langle \varphi_j, \pi(\mu) \gamma_j \rangle = \delta_{\mu,0} \text{ for } \mu \in \Lambda^\circ.$$  

(iii) Ron-Shen duality: $G(\varphi, \Lambda^\circ)$ is a Riesz sequence in $L^2(\mathbb{R}^d, \mathbb{C}^n)$.

(iv) $S_{\varphi, \varphi}$ is invertible on $M_1^1(\mathbb{R}^d)$.

(v) $S_{\varphi, \varphi}$ is invertible on $M_{\infty}^1(\mathbb{R}^d)$.

(vi) $S_{\varphi, \varphi}$ is one-to-one on $M_{\infty}^1(\mathbb{R}^d)$.

(vii) The analysis operator $C_{\varphi, \Lambda} : M_{\infty}^1(\mathbb{R}^d) \rightarrow \ell_{\infty}(\Lambda, \mathbb{C}^n)$ is one-to-one from $M_{\infty}^1(\mathbb{R}^d)$ to $\ell_{\infty}(\Lambda, \mathbb{C}^n)$.

(viii) The synthesis operator $D_{\varphi, \Lambda}$ defined on $\ell_1^1(\Lambda, \mathbb{C}^n)$ has dense range in $M_1^1(\mathbb{R}^d)$.

(ix) $D_{\varphi, \Lambda}$ is surjective from $\ell_1^1(\Lambda, \mathbb{C}^n)$ onto $M_1^1(\mathbb{R}^d)$.

(x) The synthesis operator $\tilde{D}_{\varphi, \Lambda^\circ}$ defined on $\ell_{\infty}(\Lambda^\circ)$ is one-to-one from $\ell_{\infty}(\Lambda^\circ)$ to $M_{\infty}^1(\mathbb{R}^d, \mathbb{C}^n)$.

(xi) The analysis operator $\tilde{C}_{\varphi, \Lambda^\circ}$ defined on $M_1^1(\mathbb{R}^d, \mathbb{C}^n)$ has dense range in $\ell_1^1(\Lambda^\circ)$.

(xii) $\tilde{C}_{\varphi, \Lambda^\circ}$ is surjective from $M_1^1(\mathbb{R}^d, \mathbb{C}^n)$ onto $\ell_1^1(\Lambda^\circ)$.

(xiii) $G_{\varphi, \Lambda^\circ}$ is invertible on $\ell_1^1(\Lambda^\circ)$.

(xiv) $G_{\varphi, \Lambda^\circ}$ is invertible on $\ell_{\infty}(\Lambda^\circ)$.

(xv) $G_{\varphi, \Lambda^\circ}$ is one-to-one on $\ell_1^1(\Lambda^\circ)$.  


The equivalence \((i) \Leftrightarrow (vi)\) is claimed in Lemma 3 and is all we need for the main results of our paper.

Proof. The implication \((i) \Rightarrow (iv)\) was sketched in Lemma 16. 

\((i) \Leftrightarrow (ii)\): Time-frequency shifts on a lattice are linearly independent in the following sense: if \(c = (c_\mu)_{\mu \in \Lambda^o} \in \ell^\infty \) and \( \sum_{\mu \in \Lambda^o} c_\mu \pi(\mu) = 0 \) (as an operator from \(M^1(\mathbb{R}^d) \) to \(M^{\infty}(\mathbb{R}^d) \)), then \(c_\mu = 0\) for all \(\mu \in \Lambda^o\), see [22]. Now, if \(f = S_{\varphi, \gamma} f\) for all \(f \in M^1(\mathbb{R}^d)\), then by Janssen's representation (38) we have

\[
 f = s(\Lambda)^{-1} \sum_{\mu \in \Lambda^o} \sum_{j=1}^n \langle \varphi_j, \pi(\mu)\gamma_j \rangle \pi(\mu)f .
\]

The linear independence of time-frequency shifts implies (11). The converse is obvious.

\((ii) \Leftrightarrow (iii)\): Assume first that \(\bigcup_{j=1}^n G(\varphi_j, \Lambda)\) is a multi-window Gabor frame for \(L^2(\mathbb{R}^d)\). The upper bound in (37) follows from the boundedness of the synthesis operator \(\tilde{D}_\varphi\) on \(L^2(\mathbb{R}^d)\). To show the existence of a lower bound, we apply the Wexler-Raz relations. Since \(\bigcup_{j=1}^n G(\varphi_j, \Lambda)\) is a frame with dual \(\bigcup_{j=1}^n G(\gamma_j, \Lambda)\) and \(\gamma_j \in M^1(\mathbb{R}^d)\) for all \(j\), we have \(\langle \varphi, \pi(\mu)\gamma \rangle = \sum_{j=1}^n \langle \varphi_j, \pi(\mu)\gamma_j \rangle = s(\Lambda)\delta_{\mu,0}\), and \(G(\varphi, \Lambda^o)\) and therefore \(G(\gamma, \Lambda^o)\) are biorthogonal systems in \(L^2(\mathbb{R}^d, \mathbb{C}^n)\). If \(f = \sum_{\mu \in \Lambda^o} c_\mu \pi(\mu)\varphi\), then \(c_\mu = s(\Lambda)^{-1} \langle f, \pi(\mu)\gamma \rangle_{L^2(\mathbb{R}^d, \mathbb{C}^n)}\) and

\[
 c = s(\Lambda)^{-1} \tilde{C}_{\varphi, \Lambda^o} f ,
\]

from which the lower bound in (37) follows.

Conversely, assume that \(G(\varphi, \Lambda^o)\) is a Riesz sequence in \(L^2(\mathbb{R}^d, \mathbb{C}^n)\). Then there exists a biorthogonal basis of the form \(\{\pi(\mu)\gamma : \mu \in \Lambda^o\}\) contained in \(K = \text{span}(G(\varphi, \Lambda^o))\). It can be shown that \(\gamma \in M^1(\mathbb{R}^d, \mathbb{C}^n)\). The frame property of \(G(\varphi_j, \Lambda)\) follows from the Wexler-Raz relations (11).

With three classical statements (11) and (ii), (iii) for multi-window Gabor frames the remaining equivalences follow exactly as in [22].

\[\square\]

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