Gauged Duality, Conformal Symmetry and Spacetime with Two Times

I. Bars, C. Deliduman, O. Andreev

Department of Physics and Astronomy
University of Southern California
Los Angeles, CA 90089-0484

ABSTRACT

We construct a duality between several simple physical systems by showing that they are different aspects of the same quantum theory. Examples include the free relativistic massless particle and the hydrogen atom in any number of dimensions. The key is the gauging of the Sp(2) duality symmetry that treats position and momentum \((x, p)\) as a doublet in phase space. As a consequence of the gauging, the Minkowski space-time vectors \(x^\mu, p^\mu\) get enlarged by one additional space-like and one additional time-like dimensions to \((x^M, p^M)\). A manifest global symmetry \(SO(d, 2)\) rotates \((x^M, p^M)\) like \((d + 2)\) dimensional vectors. The \(SO(d, 2)\) symmetry of the parent theory may be interpreted as the familiar conformal symmetry of quantum field theory in Minkowski spacetime in one gauge, or as the dynamical symmetry of a totally different physical system in another gauge. Thanks to the gauge symmetry, the theory permits various choices of “time” which correspond to different looking Hamiltonians, while avoiding ghosts. Thus we demonstrate that there is a physical role for a spacetime with two times when taken together with a gauged duality symmetry that produces appropriate constraints.

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b Permanent address: Landau Institute, Moscow.
1 Introduction

The purpose of this paper is to introduce some new points of views on duality as a gauge symmetry and to connect duality to the concept of a spacetime with two time-like dimensions. This is an attempt at finding a physical role for the idea that there may be more than one timelike dimension to describe our universe at the fundamental level. We will show that certain familiar physical systems, such as the free massless relativistic particle, Hydrogen atom, harmonic oscillator, and others, do fit such a concept, as reported in this paper and in a companion paper [1]. We will show that these and other apparently different physical systems correspond to the same quantum Hilbert space characterized by a unique unitary representation of the conformal group SO(d,2). We will argue that the presence of conformal symmetry or dynamical symmetry in these special cases is the evidence for the presence of two timelike coordinates. The physics looks different because the choice of “time” is not unique and hence the Hamiltonians look different, although they describe the same parent system for which we present an action. These special physical systems are related to each other by a duality that is a gauge symmetry. Thanks to the gauge symmetry ghosts are eliminated from the two time Hilbert space.

Clues for two or more timelike dimensions have been emerging from various points of view, including the brane-scan [2], the structure of extended supersymmetry of p-branes [3], extensions of M-theory [4] to F-theory [5] and S-theory [6]-[7], (1,2) strings [8], 12D super Yang-Mills & supergravity theories in backgrounds of constant lightlike vectors [9], and finally the discovery of models of multi-superparticles that are fully covariant in (10,2) and (11,3) dimensions [10]-[13].

Two or more timelike dimensions are possible only with appropriate gauge symmetry and constraints that reduce the theory to an effective theory with a single timelike dimension and no ghosts. The gauged Sp(2) duality symmetry suggested here is an evolution of the local bosonic symmetry introduced in [10]-[12] for the same purpose. The difference is that we apply the concept to the phase space doublet \((X^M, P^M)\) for a single particle rather than to a multiplet of the positions of several particles \((X_1^M, X_2^M, \cdots)\). We suggest an action principle in phase space, including invariant interactions with background fields, with and without supersymmetry.

We have suggestively named our local symplectic symmetry Sp(2) “dual-
ity” because we see signs that our duality is related to the generalized concept of electric-magnetic duality in super Yang-Mills theories and M-theory. However, this connection remains to be established by further detailed study.

2 Gauging duality

The quantization rules of quantum mechanics are symmetric under the interchange of coordinates and momenta. This is known as the symplectic symmetry $\text{Sp}(2)$ that transforms $(x, p)$ as a doublet. Maxwell’s equations for electricity and magnetism are symmetric under the interchange of electricity and magnetism in the absence of sources. The electric and magnetic fields are generalized coordinates and momenta. In the presence of particles with quantized electric and magnetic charges the symmetry is a discrete version of $\text{Sp}(2)$. This symmetry, known as “electric-magnetic duality”, is apparently broken in our part of the universe by the absence of magnetic monopoles and dyons. The idea of electric-magnetic duality symmetry has been generalized in recent non-perturbative studies of supersymmetric field theory \[14\] and string theory \[15\], which are now believed to be only some aspect of a larger, duality invariant, mysterious theory (M-theory, F-theory, S-theory, U-theory, · · ·). In the context of the mysterious theory, “duality”, which is a much larger symmetry than $\text{Sp}(2)$, but containing it, is believed to be a gauge symmetry.

In this paper we study an elementary system with local continuous $\text{Sp}(2)$ duality symmetry. We start by reformulating the worldline description of the standard free massless relativistic point particle by gauging the $\text{Sp}(2)$ duality symmetry. What we find in doing so is a more general theory capable of describing not only the free particle but other physical systems dual to it, such as the hydrogen atom, harmonic oscillator, and others.

To remove the distinction between $x$ and $p$ we will rename them $X_1^M \equiv X^M$ and $X_2^M \equiv P^M$ and define the doublet $X_i^M = (X_{1i}^M, X_{2i}^M)$. The local $\text{Sp}(2)$ acts as follows

$$
\delta_\omega X_i^M (\tau) = \varepsilon_{ik} \omega^{kl} (\tau) X_{k}^M (\tau).
$$

Here $\omega^{ij} (\tau) = \omega^{ji} (\tau)$ is a symmetric matrix containing three local parameters, and $\varepsilon_{ij}$ is the Levi-Civita symbol that is invariant under $\text{Sp}(2, R)$ and
serves to raise or lower indices. We also introduce an $\text{Sp}(2, \mathbb{R})$ gauge field $A^{ij}(\tau)$ which is symmetric in $(ij)$ which transforms in the standard way

$$\delta \omega A^{ij} = \partial_\tau \omega^{ij} + \omega^{ik} \varepsilon_{kl} A^{lj} + \omega^{jk} \varepsilon_{kl} A^{il}. \quad (2)$$

The covariant derivative is

$$D_\tau X^M_i = \partial_\tau X^M_i - \varepsilon_{ik} A^{kl} X^N_l. \quad (3)$$

An action that is invariant under this gauge symmetry is

$$S_0 = \frac{1}{2} \int_0^T d\tau \left( D_\tau X^M_i \varepsilon^{ij} X^N_j \eta_{MN} \right) \quad (4)$$

$$= \int_0^T d\tau \left( \partial_\tau X^M_1 X^N_2 - \frac{1}{2} A^{ij} X^M_i X^N_j \right) \eta_{MN},$$

Here $\eta_{MN}$ is a flat metric in $d + 2$ dimensions and a total derivative has been dropped in rewriting the first term. The signature of the metric $\eta_{MN}$ is not specified at this stage, but we will see that it will be imposed on us that it must have signature for two timelike dimensions. From the second form of the action one may identify the canonical conjugates as $X^M_1 = X^M$ and $\partial S/\partial \dot{X}^M_1 = X^M_2 = P^M$, so that the action is consistent with the idea that $(X^M_1, X^M_2)$ is the doublet $(X^M, P^M)$ rather than describing two particles.

If instead of the full $\text{Sp}(2)$ group we had gauged a triangular abelian subgroup containing only $\omega^{22}(\tau)$, and kept only the gauge potential $A^{22}(\tau)$, then the resulting action would have been the free massless particle action in the first order formalism, with $\eta_{\mu\nu}$ the standard Minkowski metric. Thus $\omega^{22}$ is closely related to $\tau$ reparametrization invariance, but $\omega^{12}, \omega^{11}$ are new local symmetry parameters that permit the removal of redundant gauge degrees of freedom. In the presence of the gauge degrees of freedom we are able to see the structure of duality and the role it plays in exhibiting higher symmetries in higher dimensions.

In addition to the local $\text{Sp}(2, \mathbb{R})$ symmetry there is a manifest global symmetry $\text{SO}(d, 2)$ (assuming signature $(d, 2)$) acting on the space-time $X^M_i$ with $d$-spacelike and 2-timelike dimensions labelled by the index $M$. This symmetry contains the $d$-dimensional Poincaré symmetry $\text{ISO}(d-1, 1)$ as a subgroup, but there is no translation symmetry in $d + 2$ dimensions. Using Noether’s theorem one finds the generators of the symmetry $\text{SO}(d, 2)$

$$L^{MN} = \varepsilon^{ij} X^M_i X^N_j = X^M P^N - X^N P^M. \quad (5)$$
They are manifestly *gauge invariant* under the local $Sp(2, R)$ transformations.

Supersymmetry on the worldline is introduced using the Neveu-Schwarz approach but only for zero modes. To do so, phase space is enlarged by the addition of fermionic degrees of freedom $\psi^M(\tau)$ which are their own canonical conjugates (i.e. they form a Clifford algebra when quantized). The $Sp(2)$ doublet is enlarged to an $OSp(1/2)$ triplet $(\psi^M, X^M_1, X^M_2)$ and the supergroup $OSp(1/2)$ is gauged by adding two fermionic gauge potentials $F_i$ in addition to the three bosonic gauge potentials $A^i$. The action is the direct generalization of (4) to a gauge theory based on $OSp(1/2)$. In a particular gauge, the degrees of freedom reduce correctly to the free Dirac particle in Minkowski space. This scheme can be enlarged to $N$ supersymmetries by gauging $OSp(N/2)$. Like the bosonic case, the supersymmetric case also has multiple physical sectors as seen from the point of view of various gauge choices for “time”. The supersymmetric case will be discussed in more detail in another paper \[16\].

Interactions with gravitational fields $G_{MN}(X_1, X_2)$ and gauge fields $A^N_j(X_1, X_2)$ in a way that respects the $Sp(2)$ duality symmetry are possible (of course, also in the supersymmetric case)

$$S_{G,A} = \frac{1}{2} \int_0^T d\tau \left[ \left( D_\tau X^M_i \right) \varepsilon^{ij} X^N_j G_{MN}(X_1, X_2) + \left( D_\tau X^M_i \right) \varepsilon^{ij} A_{jN}(X_1, X_2) \right].$$

$G_{MN}$ is a scalar under $Sp(2)$ and a symmetric traceless tensor in $d+2$ dimensions. Similarly $A^M_j$ is a doublet under $Sp(2)$ and a vector in $d+2$ dimensions. It is tempting to suggest that the $Sp(2)$ doublet of electromagnetic fields $A^M_j$ are related to the electric-magnetic dual potentials of Maxwell’s theory and its Yang-Mills generalizations. For the local invariance to hold, there must be restrictions on the functional forms of both $G_{MN}(X_1, X_2)$ and $A^N_j(X_1, X_2)$ since the arguments $(X_1, X_2)$ also transform under $Sp(2)$. These amount to a set of differential equations that restrict the functional forms of $G_{MN}(X_1, X_2)$ and $A^N_j(X_1, X_2)$. One automatic solution is to take any functions $G_{MN}(L), A^N_j(L)$ where $L^{MN}$ is the gauge invariant combination of $(X_1, X_2)$ given in (5). In the presence of the background fields the global symmetry $SO(d, 2)$ is replaced by the Killing symmetries of the background fields. We see that, for consistency with the local symmetry, gravity and gauge interactions are more conveniently expressed in terms of bi-local fields $G_{MN}(X_1, X_2)$ and $A^N_j(X_1, X_2)$ in $d + 2$ dimensions. Bi-local fields were advocated in \[6\] as a
means of extending supergravity and super Yang-Mills theory to (10,2) dimensions based on clues from the BPS solutions of extended supersymmetry.

We refer to the forms of the actions $S_0, S_{G,A}$ above as the first order formalism. Although not necessary, a second order formalism is obtained if $X^M_2$ is integrated out in the path integral (or eliminated semi-classically through one of the equations of motion). Eliminating $X^M_2$ is not easy for the interacting case, but for the free action $S_0$ the result is

$$S_0 = \int d\tau \left[ \frac{1}{2A^{22}} \left( \partial_\tau X^M - A^{12}X^M \right)^2 - \frac{A^{11}}{2} X \cdot X \right], \quad (7)$$

This form of the action may be thought of as “conformal gravity” on the worldline, with the conformal group $SO(1,2) = Sp(2)$.

In this paper we will mainly analyze the simplest case $S_0$. The configuration space version of $S_0$ was previously obtained with different reasoning and motivation \[17\], and without the concept of duality. Our solutions to both the classical and quantum problems go well beyond previous discussion of this system \[18]-\[20\]. More importantly, our interpretation of the system and its scope as a theory for duality and two times, and the applications to physical situations are new.

### 3 Classical solutions and dual sectors

The equations of motion for $(X_1, X_2)$ in the case of $S_0$ are

$$\begin{pmatrix} \partial_\tau X^M \\ \partial_\tau P^M \end{pmatrix} = \begin{pmatrix} A_{12} & A_{22} \\ -A_{11} & -A_{12} \end{pmatrix} \begin{pmatrix} X^M \\ P^M \end{pmatrix}. \quad (8)$$

In addition, the equations of motion for the $A_{ij}$ produces the constraints

$$X \cdot X = 0, \quad X \cdot P = 0, \quad P \cdot P = 0. \quad (9)$$

At least two timelike dimensions are required to obtain non-trivial solutions to the constraints \[17\], and our gauge symmetry does not allow more than two timelike dimensions without running into problems with ghosts. Thus our system exists physically only with the signature $(d,2)$.

\[1\] We thank K. Pilch for discovering this reference at the time of publication.
To show that the massless Minkowski particle is one of the classical solutions of our system, we may choose the gauge \( A^{12} = A^{11} = 0 \) and \( A^{22} = 1 \), solve the equations \( X^M = Q^M + P^M \tau \), and obtain the constraints \( Q^2 = P^2 = Q \cdot P = 0 \). There is a remaining gauge symmetry

\[
\begin{align*}
\omega^{11} (\tau) &= \omega_0^{11}, \\
\omega^{12} (\tau) &= -\omega_0^{11} \tau + \omega_0^{12}, \\
\omega^{22} (\tau) &= \omega_0^{11} \tau^2 - 2 \omega_0^{12} \tau + \omega_0^{22},
\end{align*}
\]

where \( \omega_{ij} \) are \( \tau \)-independent constants. Next define the basis \( Q^M = (Q^+, Q^-, q^\mu) \), \( P^M = (P^+, P^-, p^\mu) \), where \( \pm \) indicate a lightcone type basis for the extra \((1,1)\) dimensions with metric \( \eta^{+,-} = -1 \). Using two parameters of the remaining gauge freedom choose \( Q^+ = 1, P^+ = 0 \), and solve the two constraints \( Q^2 = P^2 = Q \cdot P = 0 \), so that the solution takes the form

\[
\begin{align*}
X^+ (\tau) &= 1, \\
X^- (\tau) &= \frac{q^2}{2} + q \cdot p \tau, \\
X^\mu (\tau) &= q^\mu + p^\mu \tau, \quad p^2 = 0 \text{ massless.}
\end{align*}
\]

There remains one free gauge parameter \( \omega_0^{22} \) and one constraint \( P^2 = p^2 = 0 \), which is also what follows from \( \tau \) reparametrizations on the worldline. The motion in \( d \)-dimensional Minkowski subspace \( x^\mu (\tau) \) is the same as the standard massless particle. Furthermore, the motion in the remaining two coordinates \( X^+, X^- \) is fully determined by the position and momentum \((q^\mu, p^\mu)\) in Minkowski space.

The free massless particle is not the only classical solution. For example, in the gauge \( A^{12} = 0, A^{11} = A^{22} = \omega \) the solution is

\[
\begin{align*}
X_M &= a_M e^{i \omega \tau} + a_M^\dagger e^{-i \omega \tau}, \\
a \cdot a &= a^\dagger \cdot a = a^\dagger \cdot a^\dagger = 0.
\end{align*}
\]

This is an oscillatory motion with a different physical interpretation than the free relativistic particle. As we will see, our system has dual sectors that include the H-atom and harmonic oscillator, which evidently are periodic systems. Some previously known solutions include a massive particle in Minkowski space \([17]\), a massless particle in deSitter space \([17]\), etc. Thus, there are classical solutions of the same system with various physical meanings.

What is going on is that choosing “time” is tricky in our system since there is more than one timelike dimension. The dynamics of the system
is arranged to evolve according to some gauge choice of “time” which is not unique in the system. For each such choice there is a corresponding canonical conjugate Hamiltonian which looks like different physics. However, there really is one single overall theory that follows from our action. It has various physical interpretations that are dual to each other, where duality is the Sp(2) gauge symmetry that we have introduced. Under Sp(2) transformations every classical solution which has a different physical interpretation in some gauge can be mapped to the free massless particle by a gauge transformation and a different choice of “time”.

There is a gauge invariant way to characterize the overall system at the classical as well as quantum levels. The SO($d, 2$) global symmetry generators $L^{MN}$ are gauge invariant, as well as constants of motion with respect to the “time” $\tau$. Using the constraints, it is straightforward to compute that all the Casimirs of SO($d, 2$) vanish at the classical level

$$C_n (SO(d, 2)) = \frac{-1}{n!} Tr (L)^n = 0, \quad \text{classical.} \quad (13)$$

For a non-compact group such a representation is non-trivial. For example the free particle is such a representation. This can be verified by inserting the free particle gauge of eq.(11) into (5). As we will see, the Casimirs $C_n$ will not all be zero at the quantum level, when ordering of operators are taken into account. We will find very specific values in the quantum gauge invariant sector, in particular $C_2 (SO(d, 2)) = 1 - d^2 / 4$. Both at the classical and quantum levels, the Casimir invariants specify a unique unitary representation of SO($d, 2$) which fully characterizes the gauge invariant physical space of the system. This approach does not involve a choice of “time” or Hamiltonian or effective Lagrangian in a fixed gauge.

Having realized this important observation one may now understand more generally that in a special gauge we find a rather non-trivial classical and quantum solution of our system, namely the Hydrogen atom in any dimension (the non-relativistic central force problem with the $1/r$ potential). The essential reason for its existence is that all the levels of the H-atom taken together form a single irreducible representation of the conformal group SO($d, 2$), in accord with the observation above. In fact, the representation is precisely the unique one that emerges from quantum ordering (next section), with specific values of the Casimirs. It was known that the H-atom in three dimensions ($d - 1 = 3$) forms a single irreducible representation of SO($4, 2$) \[21\]. The well
known SO(4) symmetry is the subgroup of SO(4, 2). This solution will be fully explained and generalized to any dimension at the classical and quantum levels in a separate paper [1] (with quantum ordering and other technical aspects that differ from the old literature [21]). It will also be shown there that the harmonic oscillator in \((d - 2)\) dimensions, with its mass equal to a lightcone momentum in an additional dimension, is also a solution of the system. As for all solutions, the H-atom or harmonic oscillator are Sp(2) dual to the free massless relativistic particle!

To close this section we provide a general parametrization of classical solutions in any gauge. We take advantage of the fact that the SO\((d, 2)\) generators are constants of motion \(\partial_\tau L^M N = 0\) with respect to the “time” \(\tau\). A general classical solution in any gauge may be given in various bases \(M = (+', -', \mu)\), \(M = (0', 1', \mu)\), \(M = (0', 0, I)\). The first is a lightcone type basis in the extra dimensions \(X_+ = (X_0' + X_1')\), \(X_- = \frac{1}{2}(X_0' - X_1')\), and the last distinguishes the two timelike coordinates from the spacelike ones. The first two are covariant under SO\((1, 1) \otimes S(d - 1, 1)\) and the last is covariant under SO\((2) \otimes SO(d)\). The general solution is

\[
M = \begin{bmatrix} +', & -', & \mu \\ a, & b, & \frac{-aL^+_{-\mu} + bL^+_{'+\mu}}{ad - bc} \\ c, & d, & \frac{-cL^-_{-\mu} + dL^-_{'+\mu}}{ad - bc} \end{bmatrix} \tag{14}
\]

with

\[
\begin{pmatrix} A_{12} & A_{22} \\ -A_{11} & -A_{12} \end{pmatrix} = \begin{pmatrix} \partial_\tau a & \partial_\tau b \\ \partial_\tau c & \partial_\tau d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}, \tag{15}
\]

where the matrix \((a(\tau), b(\tau), c(\tau), d(\tau))\) is a group element of GL\((2, R)\). It can be checked that by inserting this form into (2) that the constants \(L^\pm_{\mu}\) that appear in (14) are consistent with their definitions. Another constant of motion is the determinant of the matrix

\[
L^{+_{-'}} = ad - bc. \tag{16}
\]

So, effectively the local gauge group is Sp\((2)\) as parametrized by \((a, b, c, d)\). The remaining generators \(L^\mu_{\nu}\) which are also constants of motion, are now written in terms of the constants \(L^{+_{-'}}\), \(L^\pm_{\mu}\)

\[
L^\mu_{\nu} = X^\mu P^\nu - X^\nu P^\mu = \frac{1}{L^{+_{-'}}} \left( L^{+_{+\mu}} L^{-_{-\nu}} - L^{-_{-\mu}} L^{+_{+\nu}} \right). \tag{17}
\]
We may forget completely about the gauge potentials $A^{ij}$ and concentrate instead on the local group element $(a, b, c, d)$ and the global symmetry generators $L^{MN}$. The constraints (9) become conditions to be satisfied by the $L^{+\mu}, L^{+\nu}, L^{\mu\nu}$ without any condition on the group element

$$L^{+\mu} L^{+\nu} \eta_{\mu\nu} = L^{-\mu} L^{-\nu} \eta_{\mu\nu} = 0,$$

$$\frac{1}{2} (L^{\mu\nu})^2 = L^{+\mu} L^{-\nu} \eta_{\mu\nu} = - \left( L^{+\nu} \right)^2.$$   \hfill (18)

With these conditions the Casimir for $\text{SO}(d, 2)$ becomes

$$C_2 = \frac{1}{2} (L^{MN})^2 = 0,$$ \hfill (19)

at the classical level, and similarly for all higher Casimirs. But we will see below that in the quantum theory, when we watch the orders of the operators, the quadratic Casimir will be $C_2 = 1 - \frac{d^2}{4}$. Similarly all higher Casimirs of $\text{SO}(d, 2)$ vanish at the classical level, but not at the quantum level.

The same arguments may be repeated in the other bases. For example, in the basis $M = (0', 0, I)$ we have

$$M = [0', 0, I]$$

$$X^M = [a, b, \frac{aL^{0I} + bL^{0I'}}{ad - bc}]$$

$$P^M = [c, d, \frac{cL^{0I} + dL^{0I'}}{ad - bc}]$$ \hfill (20)

with

$$L^{00} = ad - bc,$$

$$L^{IJ} = \frac{1}{L^{00}} \left( L^{0I} L^{0J} - L^{0J} L^{0I} \right),$$ \hfill (21)

and

$$\left( L^{0I} \right)^2 = \left( L^{0I} \right)^2 = - \frac{1}{2} (L^{IJ})^2 = - \left( L^{00} \right)^2,$$ \hfill (22)

so that again (19), and the same conditions on the higher order Casimirs hold.
4 Quantum theory

In any gauge the naive quantum rules that follows from the action $S_0$ are

$$\left[ X^M_i, X^N_j \right] = i\varepsilon_{ij} \eta^{MN}.$$  

These are subject to the constraints $X_i \cdot X_j = 0$. We will rewrite these in any gauge as

$$\left[ X^M_i, P^N_j \right] = i\eta^{MN}, \quad X^2 = P^2 = X \cdot P = 0.$$  \hspace{1cm} (23)

As usual one may approach the problem of quantization in a covariant formalism or in a non-covariant formalism.

In a covariant formalism one may apply the constraints on states constructed in a Hilbert space that obeys the naive quantization rules above. This approach would be manifestly covariant under both the duality symmetry $Sp(2, R)$ as well as the $SO(d, 2)$ symmetry. But is does not seem to give direct insight into the physical content of the theory since “time” or “Hamiltonian” is not specified. In this paper we will obtain one crucial result on the values of the Casimirs $C_n (SO (d, 2))$ that follow from covariant quantization.

In a non-covariant formalism both the duality symmetry as well as the manifest $SO(d, 2)$ symmetry is broken by the choice of gauges and solution of the constraints. One must then verify that the quantization procedure respects the gauge invariant algebra of the global $SO(d, 2)$ generators $L^{MN}$ in eq. (5). In the gauge fixed formalism these generators incorporate the naive global transformation on the $d + 2$ space-time coordinates as well as the duality transformations $Sp(2, R)$. This is because after an $SO(d, 2)$ transformation one goes out of the gauge slice, and a gauge transformation must be applied to go back to the gauge slice. Thus, the details of the $SO(d, 2)$ conformal generators in the fixed gauge provide the information on the duality transformations. In a fixed gauge some of the $L^{MN}$ require normal ordering of the canonical degrees of freedom and therefore there are anomaly coefficients. The closure of the algebra can fix some of these coefficients, but it turns out this is not so in every gauge. It turns out that imposing the eigenvalues of the Casimirs $C_n$ obtained in the covariant quantization must be used to fully determine the anomaly coefficients. In particular for the Hydrogen atom this additional constraint is needed. In this paper we will treat only the free particle in two different gauges and verify that we have the correct representation.
4.1 SO\((d, 2)\) and Sp\((2)\) covariant quantization

The hermitian quantum generators of Sp\((2, R)\) are

\[
J_0 = \frac{1}{4} \left( P^2 + X^2 \right), \quad J_1 = \frac{1}{4} \left( P^2 - X^2 \right),
\]

\[
J_2 = \frac{1}{4} \left( X \cdot P + P \cdot X \right).
\]

The Lie algebra that follows from the quantum rules is

\[
[J_0, J_1] = iJ_2, \quad [J_0, J_2] = -iJ_1, \quad [J_1, J_2] = -iJ_0.
\]

The quadratic Casimir operator \(C_2(Sp(2))\) takes the hermitian form (watching the orders of operators)

\[
C_2(Sp(2)) = \frac{1}{4} \left[ X^M P^2 X_M - (X \cdot P) (P \cdot X) + \frac{d^2}{4} - 1 \right]
\]

where the constant term arises from re-ordering the operators \((d + 2)^2 - 4 (d + 2) = d^2 - 4\). The gauge invariant SO\((d, 2)\) Lorentz generators given in eq.(2) are used to compute the quadratic Casimir operator for SO\((d, 2)\). One finds that the quadratic Casimirs of the two groups are related

\[
C_2(SO(d, 2)) = \frac{1}{2} L_{MN} L^{MN} = \left[ C_2(Sp(2)) + 1 - \frac{d^2}{4} \right],
\]

where \(C_2(Sp(2))\) is given by eq.(27). Since \(L_{MN}\) is gauge invariant, both \(C_2(SO(d, 2))\) and \(C_2(Sp(2))\) must have the same spectrum in any quantization scheme in any gauge.

We will describe the general properties of the covariant Hilbert space we should find. The “physical” states form a subset of the Hilbert space for which the matrix elements of Sp\((2, R)\) generators vanish weakly

\[
<\text{phys}| J_{0,1,2}|\text{phys}' > \sim 0.
\]

For SL\((2, R) = Sp(2, R)\) all the unitary representations are labelled by \(|jm\rangle\). Within this space the singlet state \(C_2(Sp(2)) = j(j + 1) = 0\) and \(m = 0\), satisfy the physical requirements. This is the module with only one state from the point of view of Sp\((2, R)\). However, there can be an infinite number
of such gauge invariant states which are classified by the global symmetry $\mathrm{SO}(d,2)$. This must be the case since we already know that there is a non-trivial solution of the constraints in the classical limit when the signature of $\eta^{MN}$ is $(d,2)$. Thus, we have argued that for non-trivial states we must have

$$C_2(\mathrm{SO}(d,2)) = 1 - \frac{d^2}{4}, \quad C_2(\mathrm{Sp}(2)) = 0. \quad (30)$$

This will be confirmed by the non-covariant quantization below. To compute the eigenvalues of all the Casimir operators $C_n$ we use the same approach. We find that all $C_n$ at the quantum level can first be written in terms of $C_2(\mathrm{Sp}(2))$ plus normal ordering constants that depend on $d$. Once the general expression is obtained we set $C_2(\mathrm{Sp}(2)) = 0$ and obtain the eigenvalues of $C_n(\mathrm{SO}(d,2))$ for the gauge invariant states. This procedure uniquely determines the physical space content of our theory as a unique unitary representation of the conformal group $\mathrm{SO}(d,2)$. We only need the quadratic Casimir in the present paper.

Although we have identified the physical representation of $\mathrm{Sp}(2,R)$ and $\mathrm{SO}(d,2)$, building explicitly the $\mathrm{Sp}(2,R)$ and $\mathrm{SO}(d,2)$ fully covariant Hilbert space in terms of the covariant canonical variables $X^M, P^M$ remains as an open problem. For a physical interpretation this is desirable. A natural approach to study the general problem covariantly is in terms of bi-local fields $\phi(X^M_1, X^M_2)$. Recall that bi-local fields are also relevant as background fields in the general action $S_{G,A}$.

### 4.2 Fully gauge fixed quantization

In the non-covariant approach we choose a gauge and solve all the constraints at the classical level, and then quantize the remaining degrees of freedom.\(^2\)

\(^2\)There is another trivial state that satisfies the physical state conditions with some modification in the weak condition. This is the Fock vacuum if one uses a harmonic oscillator representation with $X_M = (a_M + a_M^\dagger) / \sqrt{2}$ and $P_M = (a_M - a_M^\dagger) / \sqrt{2}$. Taking into account operator ordering, then one finds $J_0 = \frac{1}{2}a^\dagger \cdot a + \frac{1}{4}(d+2)$ and compute that the Fock vacuum has $j(j+1) = -1 + d^2/4$ and $m_0 = \frac{1}{4}(d+2)$. The physical state condition gets modified to $J_0 = \frac{1}{4}(d+2)$ instead of zero. This state is the lowest state of the non-trivial discrete series representation of $\mathrm{Sp}(2)$. However, it is the trivial singlet state from the point of view of $\mathrm{SO}(d,2)$ since $L_{MN} = a_M^\dagger a_N - a_N a_M$ annihilates it. This is the only state that would exist in the theory if the signature were $(d + 2, 0)$ or $(d + 1, 1)$.\(^3\)

\(^3\)This procedure uniquely determines the physical space content of our theory as a unique unitary representation of the conformal group $\mathrm{SO}(d,2)$.
The advantage of this approach is that unitarity is manifest and we work
directly with the physical states. The disadvantage is that by choosing a
gauge we hide the duality properties. We will discuss here only the free
massless particle interpretation of the Hilbert space. In another paper we will
show that the same Hilbert space is dual the H-atom and harmonic oscillator.
We fix three gauges that make evident the free particle interpretation as in
the classical solution (11) $X^+ = 1, P^{+'} = 0, X^+ = p^+ \tau$. Since we will
express the commutation rules at $\tau = 0$, we have, in a lightcone basis $M = (+', -', +, -, i)$

$$X^M = \left(1, x^-, 0, x^i; x^i\right), \quad P^M = \left(0, p^-, p^+, p^i; p^i\right)$$

(31)

where the transverse vectors $x^i, p^i$ are in $(d - 2)$ dimensions. Inserting this
form in the constraints gives

$$x^- = \frac{x^2}{2}, \quad p^- = \left(x \cdot p - x^p \right), \quad p^- = \frac{p^2}{2p^+}$$

(32)

where we have used $\eta^{+'-'} = \eta^{+-} = -1$. The canonical pairs are

$$\left[x, p\right], \quad \left[x^-, p^+\right], \quad [x^+, 0], \quad p^- = \frac{p^2}{2p^+},$$

(33)

The ones in the first line $[x, p], [x^-, p^+]$ are the true canonical operators for
the relativistic particle, which are quantized according to the usual canonical
rules

$$\left[x^i, p^j\right] = i\delta^{ij}, \quad \left[x^-, p^+\right] = i\eta^{+-} = -i.$$

(34)

On the other hand, $x^+ = 0, x^+ = 1, p^+ = 0$ are gauge choices and $p^-, p^-, x^-$ are dependent operators which must be replaced by the given
expressions in all gauge invariant observables.

Recall that the Lorentz generators $L^{MN}$ are gauge independent and com-
mutate with the Sp(2, R) generators. Therefore they can be expressed in any
gauge, consistently with the constraints, by simply replacing our gauge choice
(31,32) into (5). Thus, we obtain

$$L^{ij} = x^i p^j - x^j p^i$$

(35)
where operators are ordered to insure that all components of $L^{MN}$ are hermitian. All possible ordering constants are uniquely fixed by hermiticity except for the parameter $\alpha$ in $L^{-i}$. Our aim is to show that these operators form the correct commutation rules for SO($d,2$), namely

$$[L_{MN}, L_{PQ}] = i\eta_{MP} L_{NQ} + i\eta_{NQ} L_{MP} - i\eta_{NP} L_{MQ} - i\eta_{MQ} L_{NP}. \quad (42)$$

This requirement fixes the parameter $\alpha = -1$. In particular

$$[L^{-i}, L^{-j}] = i\delta^{ij} L^{-i}, \quad \rightarrow \quad \alpha = -1. \quad (43)$$

In a laborious calculation it can be verified that our construction satisfies the correct commutation rules. The structure of the algebra may be described as follows. First note that $L^\mu = (L^i, L^x, L^z)$ form the SO($d-1,1$) Lorentz algebra, and that $p^\mu = (L^{+i}, L^{-i}, L^{ij})$ are the generators of translations. The set $(L^{\mu}, p^\mu)$ forms the Poincaré algebra ISO($d-1,1$) in the massless sector $p^2 = 0$. The operators $K^\mu = (L^{-i}, L^{-i}, L^{-i})$ are the special conformal transformations and finally $D = L^{+i}$ is the dilatation operator.

It is also useful to note that the subset $(L^{\pm}, L^{\pm}, L^{\pm}, L^{\pm})$ form the algebra of SO($2,2$). Since SO($2,2$) = SL($2, R)_L \otimes$ SL($2, R)_R, it is convenient to identify the SL($2, R)_L \otimes$ SL($2, R)_R combinations as

$$G^L_2 = \frac{1}{2} (L^{t-i} + L^{t-}) \quad G^R_2 = \frac{1}{2} (L^{t-i} - L^{t-}) \quad \pm G^L_1 = L^{x \pm}, \quad G^R_0 \pm G^R_1 = L^{x \mp}. \quad (44, 45)$$
which satisfy \([G^L_{a}, G^R_{b}] = 0\) and

\[
\begin{align*}
[G^L_{0,R}, G^L_{1,R}] &= iG^L_{2,R}, \quad [G^L_{0,R}, G^L_{2,R}] = -iG^L_{1,R}, \quad (46) \\
[G^L_{1,R}, G^L_{2,R}] &= -iG^L_{0,R}, \quad (47)
\end{align*}
\]

In our case we found the representation

\[
\begin{align*}
G^L_2 &= \frac{1}{4} (\mathbf{x} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{x}) - \frac{1}{2} \left( x^- p^+ + p^+ x^- \right) \quad (48) \\
G^L_0 + G^L_1 &= p^+, \quad (49) \\
G^L_0 - G^L_1 &= \left[ \frac{1}{8p^+} (\mathbf{x}^2 \mathbf{p}^2 + \mathbf{p}^2 \mathbf{x}^2 - 2\alpha) - \frac{x^-}{2} (\mathbf{x} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{x}) + x^- p^+ x^- \right] \quad (50)
\end{align*}
\]

and

\[
\begin{align*}
G^R_2 &= \frac{1}{4} (\mathbf{x} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{x}) , \quad (51) \\
G^R_0 + G^R_1 &= \frac{\mathbf{p}^2}{2p^+} , \quad (52) \\
G^R_0 - G^R_1 &= \frac{1}{2} \mathbf{x}^2 p^+. \quad (53)
\end{align*}
\]

These structures do indeed correctly form the Lie algebras of \(SO(2,2) = SL(2,R)_L \otimes SL(2,R)_R\). We compute the quadratic Casimir operator of each \(SL(2,R)\) by \(j(j+1) = G^2_0 - G^2_1 - G^2_2\). We find

\[
\begin{align*}
j_R (j_R + 1) &= \frac{1}{4} \mathbf{L}^2 + \frac{1}{16} (d - 2)^2 - \frac{1}{4} (d - 2) \quad (54) \\
j_L (j_L + 1) &= j_R (j_R + 1) - \frac{1 + \alpha}{4} \quad (55)
\end{align*}
\]

where \(\mathbf{L}^2 = \frac{1}{2} L_{ij} L^{ij}\) is the Casimir operator for the orbital rotation subgroup \(SO(d-2)\)

\[
\mathbf{L}^2 = \mathbf{p}^i \mathbf{x}^2 \mathbf{p}^i - \mathbf{p} \cdot \mathbf{x} \cdot \mathbf{p} . \quad (56)
\]

Note that for \(\alpha = -1\) we have \(j_L = j_R\). The overall quadratic Casimir for \(SO(d,2)\) of eq.(28) takes the form

\[
C_2 = \{L^{++}, L^{-}\} + \{L^{'+}, L'^+\} - (L'^+)^2 - (L^+)^2
\]

15
\[ -\{L^+i, L^{-i}\} - \{L^+i, L^{-i}\} + \frac{1}{2}L_{ij}L^{ij} \tag{57} \]
\[ = L^2 + \frac{1}{4}(d-2)^2 - (d-2) \tag{58} \]
\[ -2L^2 - \frac{1}{2}(d-2)^2 + L^2 \]
\[ = \frac{d^2}{4} + 1 \tag{59} \]

As expected, the “orbital” part involving the canonical pairs \((x^-, p^+)\) and \((x, p)\) dropped out. By comparison to the covariant form (28) we have verified that \(C_2(Sp(2)) = 0\). This makes sense since we have enforced the constraints at the classical level and thereby guaranteed that the \(Sp(2, R)\) generators vanish in the physical sector.

### 4.3 Lorentz covariant quantization and field theory

We may choose the gauge for the free particle partially to the following form in the basis \(M = (+', -, \mu)\)

\[ X^M (\tau) = \begin{pmatrix} 1, x^2 (\tau), x^\mu (\tau) \end{pmatrix}, \tag{60} \]
\[ P^M (\tau) = (0, p (\tau) \cdot x (\tau), p^\mu (\tau)). \]

There remains the gauge degree of freedom that corresponds to \(\tau\)-reparametrization \(\omega^{22} (\tau)\) and the corresponding constraint \(p^2 (\tau) = 0\). The independent canonical pairs are quantized as \([x^\mu, p^\nu] = i\eta^\mu\nu\), which is Lorentz covariant. Physical states \(|\phi>\) must satisfy the \(p^2|\phi>=0\) condition weakly. The well known solution may be given in \(x\)-space \(\phi (x) = <x|\phi>\), where \(<x|p_\mu = -i\frac{\partial}{\partial x^\mu} <x|\)

\[ \Box \phi (x) = 0. \tag{61} \]

The field theory “effective action” that gives this equation is

\[ S_{eff} = \frac{1}{2} \int d^d x \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} \eta^{\mu\nu}. \tag{62} \]

The solutions of the constraint are well known

\[ \phi (x) = \int \frac{d^d k}{(2\pi)^{d-1}} \frac{\theta (k^d)}{\delta (k^2)} \left[ a (k) \ e^{ik \cdot x} + a^\dagger (k) \ e^{-ik \cdot x} \right]. \tag{63} \]
The states $|\phi> \rangle$ have the Lorentz invariant positive norm defined by
\[
< \phi | \phi >= -\frac{i}{2} \int d^{d-1}x \ (\phi^* \partial_0 \phi - \partial_0 \phi^* \phi), \tag{64}
\]
which is independent of the time component $x^0$ even though $x^0$ is not integrated. The Lorentz invariance of this norm is well known from the study of the Klein-Gordon equation, and can be seen by writing it in the form $\int dx \wedge \cdots \wedge dx \wedge J$ where $J_\mu = -\frac{i}{2} (\phi^* \partial_\mu \phi - \partial_\mu \phi^* \phi)$. If one wishes one may rewrite the norm by choosing to fix the lightcone time $x^+$ instead of the ordinary time $x^0$.

Since SO($d, 2$) is not manifest, we must check that the gauge invariant conserved symmetry generators for SO($d, 2$) have the correct commutation rules (42). We must first compute the gauge invariant $L_{MN}$ in terms of $(x^\mu, p^\mu)$ by inserting our gauge choice and solutions of constraints given in (60). We find
\[
L^{+*} = \frac{1}{2} (p \cdot x + x \cdot p) + i \tag{65}
\]
\[
L^{+\mu} = p^\mu \tag{66}
\]
\[
L^{-\mu} = \frac{1}{2} x^\lambda p^\mu x^\lambda - \frac{1}{2} x^\mu p \cdot x - \frac{1}{2} x \cdot px^\mu - ix^\mu \tag{67}
\]
\[
L^{\mu\nu} = x^\nu p^\mu - x^\mu p^\nu \tag{68}
\]
where operators are ordered. The commutation rules for SO($d, 2$) are satisfied. All ordering ambiguities are uniquely determined by hermiticity
\[
< \phi_1 | L^{MN} \phi_2 > =< L^{MN} \phi_1 | \phi_2 >, \tag{69}
\]
relative to the non-trivial Lorentz invariant norm in (64). This is the reason for the appearance of the anomalous corrections proportional to $i$ in $L^{+*}, L^{-\mu}$. Without these anomaly pieces the generators are not hermitian. As a check that we have correctly ordered our operators we compute the dimension of the scalar field by applying $L^{+*}$ on it
\[
iL^{+*} \phi (x) = < x | iL^{+*} | \phi >
\]
\[
\begin{align*}
\frac{1}{2} x \cdot \partial \phi + \frac{1}{2} \partial \cdot (x \phi) - \phi \\
&= x \cdot \partial \phi + \left( \frac{d}{2} - 1 \right) \phi.
\end{align*}
\] (70)

The dimension \( \left( \frac{d}{2} - 1 \right) \) is the correct dimension of the scalar field in the effective field theory action (62). We also see that by replacing \( p_\mu = -i \partial/\partial x^\mu \) we arrive at the well known construction of the conformal group in terms of differential operators as known in field theory. The effective field theory \( S_{\text{eff}} \) as well as the dot product are invariants under these \( \text{SO}(d, 2) \) conformal transformations applied on the field

\[
\delta \phi(x) = i \varepsilon_{MN} L^{MN} \phi(x), \quad \delta S_{\text{eff}} = 0 = \delta (\langle \phi_1 \phi_2 \rangle).
\] (71)

Thus \( L^{\mu' - \mu} \) is the dimension operator, \( L^{+\mu} \) is the translation operator, \( L^{-\mu} \) is the generator of special conformal transformations and \( L^{\mu\nu} \) is the generator of Lorentz transformations.

We can now compute the quadratic Casimir operator for \( \text{SO}(d, 2) \) in this gauge. As we have argued in the previous section, its value is gauge invariant, therefore it can be computed in any gauge. We find that it reduces to a number

\[
C_2 = -\left( L^{\mu' - \mu} \right)^2 - \left\{ L^{+\mu}, L^{-\nu} \right\} \eta_{\mu\nu} + \frac{1}{2} L_{\mu\nu} L^{\mu\nu}
\] (72)

\[
= -\frac{d^2}{4} + 1.
\] (73)

where all \( (x, p) \) dependence has dropped out. The value of the gauge invariant quadratic Casimir is again the same. This fixes the \( \text{Sp}(2, R) \) representation uniquely to \( C_2(\text{Sp}(2)) = 0 \) in agreement with the previous sections.

\footnote{The dropping out of the orbital part is a phenomenon that occurs more generally for any Casimir operator in a more general construction available for \( \text{any group} \) \cite{22}. For example, a more general construction for \( \text{SO}(d, 2) \) including the spin operator \( s^{\mu\nu} \) and the anomalous dimension operator \( d_0 \)

\[
J^{\mu' - \mu} = L^{\mu' - \mu} + id_0, \quad J^{+\mu} = L^{+\mu}, \quad J^{-\mu} = L^{-\mu} - id_0 x^\mu - s^{\mu\lambda} x_\lambda, \quad J^{\mu\nu} = L^{\mu\nu} + s^{\mu\nu}
\]

also has the property that all Casimir operators do not depend on the “orbital” operators \( (x, p) \) contained in the \( L^{MN} \). In particular the quadratic casimir is \( C_2 = -\frac{d^2}{4} + (d_0 + 1)^2 + \frac{1}{2} s^{\mu\nu} s_{\mu\nu} \).}
One may be puzzled by questions such as follows: Originally the operator $L^{a' - b'}$ was a transformation that acted purely in the extra dimensions $X^{\pm'}$ while leaving Minkowski space $x^{\mu}$ untouched; how can it now act like the scale transformations in Minkowski space? The answer is that we chose the gauge $X^{\pm'} = 1$ that fixed a scale. However, linear transformation in global $\text{SO}(d, 2)$ transforms $X^M$ out of this gauge slice. To come back to the same gauge one must apply also a duality gauge transformation on $X^M(\tau)$. The duality gauge transformation that corresponds to a rescaling of $X^{+'}$ also rescales the rest of the components. This is precisely what the operator $L^{a'-b'}$ does on Minkowski space. The structure of the gauge invariant operator $L^{a'-b'}$ “knows” that this gauge transformation must be performed on $X^M(\tau)$.

Through our construction, the conformal group of massless field theories has now acquired the new meaning of being the Lorentz-like group in an actual space-time with two timelike dimensions $X^M$. The conformal field theory $S_{\text{eff}}$ has been expressed in a fixed gauge of the larger $d+2$ dimensional space. There should exist a fully covariant effective field theory corresponding to the $\text{SO}(d, 2) \otimes \text{Sp}(2, R)$ covariant quantization. The fully covariant action in $d+2$ dimensions would collapse to the effective action of a massless particle given above upon gauge fixing. Such a field theory may be formulated in terms of a bi-local field $\phi(X^M_1, X^M_2)$.

5 Outlook

We have seen that the familiar free massless particle in $d$-dimensional Minkowski space-time may be viewed as living in a larger space-time of $d+2$ dimensions. The higher space-time includes gauge degrees of freedom, but in their presence the full $\text{SO}(d, 2)$ conformal invariance takes the new meaning of being the linear “Lorentz symmetry” in a space-time that includes two timelike dimensions. Which of the two “times” $x^{0'}, x^0$ is the familiar time coordinate? For the gauge choice we have made, time is $x^0$ and with it we have described the dynamics a free particle. However, there are other choices of time as we have demonstrated in the classical solutions here, and quantum solutions in another paper [1]. For other choices of time the Hamiltonian is different and the physics looks different (such as H-atom), even though we are describing the same overall system that corresponds to a single unique representation of the conformal group $\text{SO}(d, 2)$. So, the concept of “time”
seems to be more general, and both of our two times play a physical role. We may say that for the free massless particle the appearance of conformal symmetry is the manifestation of a larger space-time that includes two timelike coordinates. Similarly, for the H-atom and other dual systems, the presence of the conformal symmetry is part of the evidence of the presence of two timelike dimensions.

Duality and the concept of two times are meshed together in our theory. The duality we found is in the same spirit of the duality symmetry of M-theory, but its realization requires two timelike dimensions in target space $X^M(\tau)$. This is more in line with the ideas of S-theory and F-theory. In our case, we have actually constructed an action for a miniature s-theory, which should serve as a guide for constructing a full fledged S-theory in $(10,2)$ and perhaps even in $(11,3)$ dimensions.

It may be interesting to view our theory as conformal gravity on the worldline as noted earlier in the paper. We may then regard the gauge fields $(A^{22}, A^{12}, A^{11})$ as the gauge fields for translations, dilatations, and special conformal transformations respectively. Our theory may be used as a guide for generalizations from the worldline to the worldsheet or worldvolume for various $p$-branes. Although conformal gravity on the worldsheet has been considered before, our approach in phase space is somewhat different and may yield a new and different action. Such a reformulation of $p$-brane actions would permit the introduction of two timelike dimensions in $X^M(\tau, \sigma_1, \cdots, \sigma_p)$ just as in the particle case $p = 0$.

The present paper, as well as some of our previous papers, are attempts to take the concept of two or more timelike dimensions seriously. We may ask: are there more observable effects of two timelike dimensions besides the conformal invariance and the duality connections we have suggested? To answer such questions it would be useful to study interacting theories that are consistent with the gauge duality symmetries. This is essential in order to avoid ghosts. As a first step one may explore the interacting theory $S_G$ that would result from a curved background in $(d,2)$ dimensions. This is formulated by taking a curved metric $G_{MN}(X_1, X_2)$ instead of $\eta_{MN}$ in the action. One way to maintain the local $\text{Sp}(2,\mathbb{R})$ symmetry is to take $G_{MN}$ as a function of only the gauge invariant combination $X_i^M X_j^N \varepsilon^{ij}$. It is also possible to study interactions using $S_A$ in the presence of background gauge fields $A_M^i(X_1, X_2)$ that couple in a gauge invariant way to $D_\tau X_i^M(\tau)$ in $(d+2)$ dimensions. Here it would be interesting to explore the possible
relation between our $\text{Sp}(2)$ doublet $A^i_M$ and the electric-magnetic dual potentials of Maxwell’s theory and its generalizations [14]. One thing that is becoming clearer is that bi-local fields $\phi(X_1, X_2)$ are probably going to be very useful for writing down the low energy effective theories consistently with the local $\text{Sp}(2, R)$ invariance.

The idea of bi-local fields also emerged before as a means of displaying the hidden timelike dimensions in certain BPS sectors which provide short representations of the superalgebra of S-theory [8]. It was emphasized that such BPS sectors, which reveal extra timelike dimensions in black holes [24], must be considered dual sectors to other BPS solutions of M-theory. Progress along these and other directions for interacting theories will be reported in the future. We hope that such interacting theories would provide the means to discuss how to probe the higher hidden dimensions and perhaps find some additional measurable consequences and tests.

We would like to think that the presence of duality [1] and conformal symmetry [25] in M-theory, as well as in special super Yang-Mills theories under current consideration, are also signs of the presence of higher dimensions, and in particular of extra timelike dimensions. Indeed various signs that the mysterious theory may actually have 12 dimensions with signature $(10, 2)$ has been accumulating. It has also been argued that a fundamental theory that is manifestly covariant under both duality and supersymmetry requires 14 dimensions with signature $(11, 3)$ to display the covariance (in the spirit of the current paper), and it must have certain “BPS” constraints that are due to gauge invariances [7]. The various ideas outlined in this paper may be regarded as a small step toward a formulation of such a theory.

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