Temperature Expansions for Magnetic Systems

Daniel Cangemi

Department of Physics, University of California, Los Angeles, CA 90095-1547

Gerald Dunne

Department of Physics, University of Connecticut, Storrs, CT 06269-3046

Abstract

We derive finite temperature expansions for relativistic fermion systems in the presence of background magnetic fields, and with nonzero chemical potential. We use the imaginary-time formalism for the finite temperature effects, the proper-time method for the background field effects, and zeta function regularization for developing the expansions. We emphasize the essential difference between even and odd dimensions, focusing on $2 + 1$ and $3 + 1$ dimensions. We concentrate on the high temperature limit, but we also discuss the $T = 0$ limit with nonzero chemical potential.

I. INTRODUCTION

The study of fermion systems in the presence of external electromagnetic fields has applications in diverse areas of physics, including astrophysics, solid state, condensed matter, plasma and particle physics. Indeed, astrophysical considerations led to the first systematic study of relativistic noninteracting fermion systems \cite{1}. The existence of very high intensity magnetic fields in gravitationally collapsed objects motivated further investigations of the energy-momentum tensor and equation of state for a degenerate electron gas in a strong uniform magnetic field \cite{2}. More recently, this problem has been addressed in the framework
of finite temperature quantum field theory \cite{3-5}, using the fact that many of the thermodynamic properties may be derived from the corresponding finite temperature effective action.

In this paper we present a pedagogical discussion of the finite temperature ($T$) and finite chemical potential ($\mu$) effective action for fermions in an external static magnetic field. Our aim is to review and unify the imaginary-time approach \cite{6-8}, developed initially for finite temperature free systems, with the Fock-Schwinger proper-time approach \cite{9-11} developed for systems interacting with external electromagnetic fields but at zero temperature. Formally, these two approaches fit together beautifully for the case of static external magnetic fields; in fact, the finite $T$ and finite $\mu$ effects are completely separate from the computation required to compute the effects of the external static magnetic field. Thus, knowledge of the zero $T$ and zero $\mu$ proper-time effective action is completely sufficient to write down a corresponding expression at nonzero temperature and nonzero chemical potential. However, this expression for the effective action is formal and the separation between $T$ and $\mu$ effects and those of the external field may become blurred when one tries to make various approximate asymptotic expansions, such as for high or low temperature. Such expansions are necessarily complicated due to the proliferation of energy scales: thermal energy $kT$, fermion mass $m$, chemical potential $\mu$, cyclotron energy $B/m$, and also possible momentum scales associated with spatial variations of the external field. We concentrate mainly on the high temperature limit, in which $T$ (we use units in which Boltzmann’s constant $k = 1$) is the dominant energy scale. However, we also present a simple approach to the other extreme: the $T = 0$ limit. This complements Refs. \cite{3-5}, which have focussed on the low $T$ limit, and which have primarily used the real-time formalism for discussing the finite temperature effects.

Throughout our analysis, we treat both $2+1$ dimensional and $3+1$ dimensional theories. This is motivated by the known profound differences between $2+1$ and $3+1$ dimensions (and in general between odd and even dimensional space-times) for free fermion systems at finite $T$ and $\mu$ \cite{12,13}, and for zero $T$ fermions in external fields \cite{14}. We find that a consistent treatment of both cases requires careful use of zeta function regularization, which has been used previously for free systems \cite{12,13} (and for systems with constant external $A_0$ \cite{14}). For the magnetic backgrounds we also find that the $2+1$ and $3+1$ cases involve
very different expansions at high temperature. The high temperature behavior of QED$_{2+1}$ is also of interest for studying questions of spontaneous symmetry breaking [15–19].

In Section II we review the structure of the zero temperature effective action with nonzero chemical potential for $2 + 1$ and $3 + 1$ dimensional fermions in external static magnetic fields. Finite temperature is introduced in Section III using the imaginary-time formalism. In Section IV this is combined with the proper-time formalism to provide a general formal expression for the finite $T$ and $\mu$ effective action. In Section V we apply this to the high temperature limit of the free fermionic theories, and in Section VI to the high temperature limit of fermions in a static magnetic field. The zero temperature limit is examined in Section VII, and we conclude in Section VIII with some comments regarding possible further extensions of this approach.

II. EFFECTIVE ACTION FOR MAGNETIC SYSTEMS

The basic object of interest in this paper is the effective action

$$iS_{\text{eff}} = \text{Log Det} \left( i\not{D} - m - \mu \gamma^0 \right)$$

(1)

where $\not{D} = D_\nu \gamma^\nu = (\partial_\nu + ieA_\nu) \gamma^\nu$, and we choose Minkowski gamma matrices $\gamma^\nu$ satisfying

$$\{\gamma^\nu, \gamma^\sigma\} = -2\gamma^{\nu\sigma} = 2 \text{diag}(1, -1, -1, \ldots, -1)$$

(2)

The term $\mu \gamma^0$ in the Dirac operator in (1) reflects the presence of a chemical potential $\mu$, corresponding to a term $-\mu \psi^\dagger \psi$ in the Lagrangian.

In $2 + 1$ dimensions, the irreducible gamma matrices may be chosen to be

$$\gamma^0 = \sigma^3 \quad \gamma^1 = i\sigma^1 \quad \gamma^2 = i\sigma^2$$

(3)

where the $\sigma^i$ are the $2 \times 2$ Pauli matrices. Note that an alternative choice, $\gamma^0 = -\sigma^3$, $\gamma^1 = -i\sigma^1$, $\gamma^2 = -i\sigma^2$, corresponds to changing the sign of the mass, $m \rightarrow -m$, in the Dirac operator appearing in the effective action (1). The system with effective action (1) is not parity invariant since a fermion mass term breaks parity in $2+1$ dimensions [15–18]. However, a parity invariant model may be constructed by considering two species of fermions, one of
mass $m$ and the other of mass $-m$ (see Footnote 11 in Ref. [15]). This may be achieved by choosing a reducible set of $4 \times 4$ gamma matrices

$$\Gamma^\nu = \begin{pmatrix} \gamma^\nu & 0 \\ 0 & -\gamma^\nu \end{pmatrix}$$

in which case the Dirac operator is block diagonal:

$$i D^\nu \Gamma^\nu - m - \mu \Gamma^0 = \begin{pmatrix} i\not{D} - m - \mu \gamma^0 & 0 \\ 0 & -i\not{D} - m + \mu \gamma^0 \end{pmatrix}$$

Thus, the effective action for this parity invariant system may be written as

$$iS_{\text{eff}} = \log \det_{2+1} \left( i D^\nu \Gamma^\nu - m - \mu \Gamma^0 \right)$$

$$= \log \det_{2+1} \left( - (iD_0 - \mu)^2 + m^2 + (\not{D} \cdot \gamma)^2 \right)$$

(6)

In 3 + 1 dimensions parity symmetry is not an issue, but the effective action (1), which involves the first-order Dirac operator, may still be written in the same form as (6), which involves a second order operator. We use the fact that there exists an additional gamma matrix $\gamma^5$ satisfying $\{\gamma^\nu, \gamma^5\} = 0$ and $(\gamma^5)^2 = 1$. Then

$$\det_{3+1} \left( i\not{D} - m - \mu \gamma^0 \right) = \det_{3+1} \left( \gamma^5 \left( i\not{D} - m - \mu \gamma^0 \right) \gamma^5 \right) = \det_{3+1} \left( -i\not{D} - m + \mu \gamma^0 \right)$$

(7)

Therefore, the effective action (1) may be expressed as

$$iS_{\text{eff}} = \frac{1}{2} \log \det_{3+1} \left[ \left( i\not{D} - m - \mu \gamma^0 \right) \left( -i\not{D} - m + \mu \gamma^0 \right) \right]$$

$$= \frac{1}{2} \log \det_{3+1} \left( - (iD_0 - \mu)^2 + m^2 + (\not{D} \cdot \gamma)^2 \right)$$

(8)

Note that the spatial operator $(\not{D} \cdot \gamma)^2$ which appears in (1) and in (8) is a positive operator. In 2 + 1 dimensions it reduces to

$$\left( \not{D} \cdot \gamma \right)^2 = -\not{D}^2 + eB \gamma^0 = - \begin{pmatrix} D_- D_- & 0 \\ 0 & D_+ D_+ \end{pmatrix}$$

(9)

where the magnetic field is $B = F_{12} = \partial_1 A_2 - \partial_2 A_1$, and

$$D_\pm = D_1 \pm iD_2$$

(10)

In 3 + 1 dimensions the operator $(\not{D} \cdot \gamma)^2$ reduces to
\[
\left( \vec{D} \cdot \vec{\gamma} \right)^2 = -\vec{D}^2 + i\frac{1}{4} [\gamma^i, \gamma^j] F_{ij}
\]

(11)

If we choose the external magnetic field to be directed along the \(x^3\) direction and to be independent of \(x^3\), then (with a suitable choice of gamma matrices) this may be simplified further to

\[
\left( \vec{D} \cdot \vec{\gamma} \right)^2 = -\partial_3^2 - \begin{pmatrix}
D_- & 0 & 0 & 0 \\
0 & D_+ & 0 & 0 \\
0 & 0 & D_- & 0 \\
0 & 0 & 0 & D_+ \\
\end{pmatrix}
\]

(12)

where \(D_{\pm}\) are as defined for the 2 + 1 dimensional system in (10).

Thus, in each case, the spectrum of the spatial operator \(\left( \vec{D} \cdot \vec{\gamma} \right)^2\) is determined by the spectrum of the 2-dimensional Schrödinger-like operators \(D_{\pm} D_{\mp}\). The operator \(\left( \vec{D} \cdot \vec{\gamma} \right)^2\) is effectively diagonal and we may write

\[
m^2 + \left( \vec{D} \cdot \vec{\gamma} \right)^2 \equiv \mathcal{E}^2
\]

(13)

For static magnetic backgrounds \(A_0 = 0\) and we can replace \((iD_0 - \mu)\) in (11) by \(\omega - \mu\), where \(\omega\) is an energy eigenvalue. Therefore

\[
iS_{\text{eff}} = \int \frac{d\omega}{2\pi} \text{Tr} \log \left[ - (\omega - \mu)^2 + \mathcal{E}^2 \right]
\]

(14)

Here, the trace operation Tr is understood to mean a summation over the eigenvalues \(\mathcal{E}^2\) of both the positive operators \(m^2 - D_+ D_-\) and \(m^2 - D_- D_+\) in 2 + 1 dimensions, resp. \(m^2 - \partial_3^2 - D_+ D_-\) and \(m^2 - \partial_3^2 - D_- D_+\) in 3 + 1 dimensions.

III. FESTINE TEMPERATURE FORMULATION

It is clear from (14) that the effects of the external static magnetic field are contained solely within \(\mathcal{E}^2\), and are clearly separated from the chemical potential \(\mu\) and the energy trace over \(\omega\). We can therefore pass to a finite temperature formulation just as in the free case \([8\,9\,13]\), by replacing the energy integration with a discrete summation:

\[
\int \frac{d\omega}{2\pi} \rightarrow \frac{i}{\beta} \sum_{n = -\infty}^{\infty} \\
\omega \rightarrow \omega_n = \frac{2\pi i}{\beta} \left( n + \frac{1}{2} \right)
\]

(15)
Here $\beta = 1/T$, where $T$ is the temperature and Boltzmann’s constant $k$ has been absorbed into $T$. The transition to finite temperature would not be so straightforward if there were external electric fields, but here we consider only external static magnetic fields. Also note that in the zero temperature expression (14) it looks as though the dependence on the chemical potential $\mu$ may be formally eliminated through a naive shift of the integration variable $\omega$. However, such a shift would violate the boundary conditions used to compute the trace, and a proper treatment at zero temperature leads to the appearance of non-analytic behavior in $\mu$, corresponding to sharp cut-offs in the energy spectrum [7,20]. At finite temperature, these sharp cut-offs are smoothed out, and the dependence on $\mu$ is correspondingly smooth, as we shall see below.

The effective action (14) may now be expressed as

$$S_{\text{eff}} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \text{Tr} \log \left[ - \left( \omega_n + (|E| - \mu) \right) \left( \omega_n - (|E| + \mu) \right) \right]$$

$$= \frac{1}{\beta} \text{Tr} \log \left( \prod_{n=0}^{\infty} \left[ (|E| - \mu)^2 + (\pi (2n + 1)/\beta)^2 \right] \left( (|E| + \mu)^2 + (\pi (2n + 1)/\beta)^2 \right) \right)$$

$$= \frac{1}{\beta} \text{Tr} \log \left[ \cosh \frac{\beta}{2} (|E| - \mu) \cosh \frac{\beta}{2} (|E| + \mu) \right]$$

where in the last step we have used the infinite product representation of the cosh function

$$\cosh(x) = \prod_{n=0}^{\infty} \left( 1 + \frac{4x^2}{\pi^2(2n + 1)^2} \right)$$

and we have dropped an infinite contribution that is independent of $E^2$.

It is a simple matter to re-write (16) as (dropping again an irrelevant constant)

$$S_{\text{eff}} = \text{Tr} \left[ |E| + \frac{1}{\beta} \log \left( 1 + e^{-\beta(|E| - \mu)} \right) + \frac{1}{\beta} \log \left( 1 + e^{-\beta(|E| + \mu)} \right) \right]$$

This expression for the effective action generalizes an analogous expression (with $\mu = 0$) derived in [3] for the free case. The only effect of the chemical potential is to shift the ‘energy’ eigenvalue $|E|$ by $\mp \mu$, which corresponds to a shift in the threshold energies for particles and antiparticles. The only effect of the external static magnetic field is to modify the spectrum of the ‘energy’ eigenvalue $|E|$ from the free spectrum to a spectrum involving dependence on the external $B$ field. Therefore, at least in principle, we can now compute
the effective action for any external static magnetic field for which we know the spectrum of the operators $D_\pm D_\mp$ appearing in (11) and (12).

When $\mu = 0$ and we take the zero temperature limit ($\beta \to \infty$) then the effective action in (18) reduces to

$$S_{\text{eff}} |_{\mu=0 ; \beta \to \infty} = \text{Tr} (|\mathcal{E}|)$$

which is the familiar $T = 0, \mu = 0$ effective action in a static magnetic background [10].

When $\mu \neq 0$ and $\beta \to \infty$ we must distinguish between the cases $\mu < m$ and $\mu > m$. When $\mu < m$, all low temperature thermal excitations are exponentially suppressed because $|\mathcal{E}| + \mu > 0$ (since $|\mathcal{E}| \geq m$). Therefore, in this case the effective action reduces just as in (19). However, if $\mu > m$, then in the infinite $\beta$ limit the first logarithmic term in (18) contributes for the portion of the spectrum for which $|\mathcal{E}| - \mu < 0$. Thus, we have

$$S_{\text{eff}} |_{\mu \neq 0 ; \beta \to \infty} = \text{Tr} \left[ |\mathcal{E}| + (\mu - |\mathcal{E}|) \theta (\mu - |\mathcal{E}|) \right]$$

$$= \text{Tr} \left[ \mu \theta (\mu - |\mathcal{E}|) + |\mathcal{E}| \theta (|\mathcal{E}| - \mu) \right]$$

(20)

where $\theta$ is the step function. This is the standard expression for the effective action at zero temperature and with nonzero chemical potential [7,20]. The first equality in (20) emphasizes the correction from the zero $T$ and zero $\mu$ expression (19), while the second emphasizes the physical content of the effective action with zero $T$ and nonzero $\mu$ as $\mu$ times the number of occupied particle states plus the trace of the energy eigenvalues above the threshold $\mu$ [20]. In Section VII we examine this $T = 0$ limit in detail for fermions in a static magnetic background.

The form of the effective action (18) is reminiscent of the grand partition function in non-relativistic statistical mechanics. Indeed, the non-relativistic limit corresponds to the situation in which the rest mass energy $m$ is the dominant contribution to $|\mathcal{E}|$,

$$|\mathcal{E}| = \sqrt{m^2 + (\vec{D} \cdot \vec{\gamma})^2} = m + \frac{1}{2m} (\vec{D} \cdot \vec{\gamma})^2 + \ldots$$

(21)

In this limit, with $\mu > m$ and $m \to \infty$, the first logarithmic term in (18) dominates over the second (i.e. antiparticles are suppressed), and we are left with
\[ S_{\text{eff}} \rightarrow \frac{1}{\beta} \text{Tr} \log \left( 1 + e^{\beta \mu_{\text{NR}} e^{-\beta (\vec{D} \cdot \vec{\gamma})^2/2m}} \right) \] (22)

where we have identified \( \mu_{\text{NR}} = \mu - m \) as the non-relativistic chemical potential. The expression (22) is \( 1/\beta \) times the logarithm of the grand partition function for the corresponding non-relativistic fermion system.

**IV. PROPER TIME FORMULATION**

The proper time formulation provides an efficient method for computing the effective action at zero temperature [9–11], and furthermore has the virtue that the generalization to finite temperature and nonzero chemical potential naturally separates out the influence of a static background magnetic field. Using an integral representation of the logarithm to define the logarithm of an operator, we may express the finite temperature version of the effective action (14) as

\[ S_{\text{eff}} = -\frac{1}{\beta} \int_0^\infty \frac{ds}{s} \text{Tr} \left[ \exp \left( -\frac{(|\mathcal{E}|^2 - \mu^2)}{m^2} s - \frac{|\mathcal{E}|^2}{m^2} \right) \right] \] (23)

where we have chosen to refer all energy scales to \( m \) in order to have dimensionless operators in the exponent. For massless theories one must choose a different reference energy scale, as discussed in Section VI.

Expression (23) may be re-cast in terms of the second elliptic theta function \( \theta_2(u|\tau) \),

\[ S_{\text{eff}} = -\frac{1}{\beta} \int_0^\infty \frac{ds}{s} \theta_2 \left( \frac{2e^{\pi i s}}{\beta m^2} \frac{4\pi i s}{\beta^2 m^2} \right) \text{Tr} \left[ \exp \left( -\frac{(|\mathcal{E}|^2 - \mu^2)}{m^2} s \right) \right] \] (24)

where [21][22]

\[ \theta_2(u|\tau) = 2 \sum_{n=0}^\infty e^{i\pi r(n+1/2)^2} \cos \left((2n + 1)u\right) \] (25)

A Poisson summation formula converts the second theta function into a fourth theta function according to the identity [21]:

\[ \theta_4 \left( \frac{u}{\tau} \bigg| -\frac{1}{\tau} \right) = \left( \frac{i}{\tau} \right)^{-1/2} e^{iu^2/(4\pi)} \theta_2(u|\tau) \] (26)
This converts the expression (24) for the effective action into a form involving $\theta_4$:

$$ S_{\text{eff}} = -\frac{m}{2\sqrt{\pi}} \int_0^{\infty} \frac{ds}{s^{3/2}} \theta_4 \left( \frac{i\beta \mu}{2} \frac{i\beta^2 m^2}{4\pi s} \right) \text{Tr} \left[ \exp \left( -\frac{|E|^2}{m^2 s} \right) \right] \quad (27) $$

where the fourth theta function is

$$ \theta_4(u|\tau) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{i\pi n^2} \cos(2nu) \quad (28) $$

This expression (27) for the effective action is particularly useful as it shows clearly the separation between the effects of the static background magnetic field, which appear solely in the trace factor, and the effects of finite temperature and nonzero chemical potential, which appear solely in the $\theta_4 \left( \frac{i\beta \mu}{2} \frac{i\beta^2 m^2}{4\pi s} \right)$ factor. Indeed, when $\mu \equiv 0$ and $\beta \to \infty$, the $\theta_4$ factor reduces to 1 and we are left with the standard proper time expression for the zero temperature effective action in a static system. This corresponds to keeping just the term “1” in the expansion (28) of the $\theta_4$ function, so the remaining summation over $n$ in (28) represents the nonzero temperature correction. The utility of elliptic theta functions in the computation of finite temperature effective actions has been noted previously in [23] for fermions (without chemical potential) and in [24] for bosonic systems.

All information about the static magnetic background is neatly encapsulated in the proper time propagator

$$ \text{Tr} \left[ \exp \left( -\frac{|E|^2}{m^2 s} \right) \right] = e^{-s} \text{Tr} \left[ \exp \left( -\frac{(\vec{D} \cdot \vec{\gamma})^2}{m^2 s} \right) \right] \quad (29) $$

which is computed independent of any reference to temperature or chemical potential. Thus, if one computes the zero temperature effective action using the proper time method it is completely straightforward to then write down an expression for the effective action at finite temperature and at nonzero chemical potential simply by inserting the $\theta_4$ factor as in (27).

However, while the expression (27) illustrates the separate roles of finite $\beta$, $\mu$ and the external static field, it is not so straightforward to use it to obtain useful numerical estimates. This is because of the wildly oscillatory behavior of the $\theta_4$-function in (27) for large values of the proper time parameter $s$. This oscillatory behavior is also sensitive to the magnitude of the dimensionless parameter $\beta \mu$ which appears in the first argument of the $\theta_4$ function.
These difficulties are further complicated by the proliferation of energy scales - for zero $T$ and $\mu$ the only scales are the fermion mass $m$, the characteristic strength scale $B$ (with dimensions of $m^2$) of the external magnetic field, and possibly also characteristic length scales associated with spatial variations in the magnetic field. The generalization to nonzero temperature and chemical potential introduces two further energy scales: $\beta$ and $\mu$. In this paper, we concentrate mainly on high temperature expansions (in which $T = 1/\beta$ is the dominant energy scale), although we also discuss the zero temperature limit (with nonzero chemical potential) in Section VII.

To conclude this brief review of the finite temperature formalism for fermionic systems, we show how the general expression (27) relates to the previous expression (18), for which our statistical mechanics intuition is most direct. Using (28) and rescaling the proper time variable $s$ in (27), $s/m^2 \rightarrow \beta^2 s/4$, we obtain

$$S_{\text{eff}} = -\frac{1}{\beta \sqrt{\pi}} \int_0^\infty \frac{ds}{s^{3/2}} \theta_4 \left( \frac{i\beta \mu}{2} \left| \frac{i}{\pi s} \right| \right) \text{Tr} \left[ e^{-\beta^2 \mathcal{E}^2 s/4} \right]$$

$$= -\frac{1}{\beta \sqrt{\pi}} \int_0^\infty \frac{ds}{s^{3/2}} \left[ 1 + 2 \sum_{n=1}^\infty (-1)^n e^{-n^2/s} \cosh(n\beta \mu) \right] \text{Tr} \left[ e^{-\beta^2 \mathcal{E}^2 s/4} \right]$$

(30)

The integrations over $s$ may now be performed, with those in the summation term requiring the identity (see [25] Eqs. 3.471.9 and 8.469.3)

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{ds}{s^{3/2}} \exp \left( -\frac{n^2}{s} - \frac{\beta^2 \mathcal{E}^2}{4} s \right) = \frac{e^{-\beta n|\mathcal{E}|}}{n}, \quad n > 0$$

(31)

Thus

$$S_{\text{eff}} = \text{Tr} |\mathcal{E}| + \frac{2}{\beta} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} e^{-\beta n|\mathcal{E}|} \cosh(n\beta \mu)$$

(32)

which is just expression (18).

**V. FREE THEORIES**

Before discussing the temperature expansions for fermions in the presence of background magnetic fields, we first describe the high temperature expansions for free theories. This is partly to establish some notation and to introduce some number theoretic functions (the
zeta and eta functions), but also in order to point out some important subtleties that arise even in the free theories and which have implications for the magnetic systems.

In the free fermionic theory in \((d + 1)\)-dimensional spacetime, \(E^2 = m^2 + \vec{p}^2\), where \(\vec{p}\) is a \(d\)-dimensional momentum vector. Thus, the proper-time propagator (29) is simply

\[
\text{Tr} \left[ \exp \left(-\frac{E^2}{m^2} s \right) \right] = \frac{c}{2} \left( \frac{m}{2\sqrt{\pi}} \right)^d \frac{e^{-s}}{s^{d/2}}
\]

where \(c\) is the spinor dimension (the \(d = 2\) and \(d = 3\) cases discussed previously correspond thus to \(c = 4\)). Therefore, the effective action (27) is

\[
S_{\text{eff}} = -\frac{c}{2} \left( \frac{m}{2\sqrt{\pi}} \right)^{d+1} \int_0^\infty \frac{ds}{s^{d/2}} e^{-s} \left[ 1 + 2 \sum_{n=1}^\infty (-1)^n \cosh(\beta \mu n) \exp \left( -\frac{\beta^2 m^2 n^2}{4s} \right) \right]
\]

This expression already illustrates an important difference between the \(3 + 1\) and \(2 + 1\) dimensional cases, for which \((d + 3)/2\) is an integer or half-odd-integer, respectively. For example, in the zero temperature and zero chemical potential case, the square parentheses in (34) reduce to a single term “1”, and so the \(3 + 1\) free case naively leads to a divergent factor \(\Gamma(-2)\) which must be regulated consistently. This may be achieved, for example, by cutting off the lower limit 0 of the \(s\) integration, or by shifting the dimension \(d \to 3 + \epsilon\) and extracting a finite piece. In contrast, in \(2 + 1\) dimensions the corresponding factor is \(\Gamma(-3/2)\), which is finite. These issues are well understood for the zero temperature theories \([9–11]\), but below we illustrate analogous differences between \(d\) odd and \(d\) even for the \(\beta\) and \(\mu\) dependent contribution to the effective action.

The \(s\) integrations in the \(n\) summation in (34) may be performed using the following integral representation of the modified Bessel function (see [21], Eq. 7.12.23)

\[
K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\nu \int_0^\infty \frac{ds}{s} s^{-\nu} \exp \left[ -\left( s + \frac{z^2}{4s} \right) \right], \quad \Re z^2 > 0, \ |\arg z| < \frac{\pi}{2}
\]

This leads to

\[\text{1}\]

As mentioned previously, the 1 in the square bracket in Eq. (34) corresponds to the zero temperature (and zero chemical potential) proper time representation of the effective action, which we denote by \(S_{\text{eff}}^{T=0;\mu=0}\).
\[ S_{\text{eff}} = S_{T=0; \mu=0}^{T=0; \mu=0} - 2c \left( \frac{m}{2\sqrt{\pi}} \right)^{d+1} \sum_{n=1}^{\infty} (-1)^n \cosh(\beta \mu n) K_{d+1} \left( \frac{\beta m}{2} \right)^{-(d+1)/2} \] (36)

For free theories the only energy scales are \( m, \mu \) and \( T = 1/\beta \), so a high temperature expansion corresponds to \( \beta m \ll 1 \) and \( \beta \mu \ll 1 \). Therefore, expression (36) may be converted into a high temperature expansion by using the ascending series expansions of the modified Bessel function \( K_\nu(z) \). The difference between \( 3+1 \) and \( 2+1 \) dimensions is reflected in the fact that the series expansions of \( K_\nu(z) \) are very different for \( \nu \) an integer or \( \nu \) a half-odd-integer.

For \( \nu = \text{integer} = N \), the expansion of \( K_N(z) \) for small \( z \) begins with a \( z^{-N} \) term and also involves a logarithmic piece:

\[
K_N(z) = \frac{1}{2} \left( \frac{z}{2} \right)^{-N} \sum_{j=0}^{N-1} \frac{(N-j-1)!}{j!} \left( -\frac{z^2}{4} \right)^j \tag{37}
\]

\[
= (-1)^N \left( \frac{z}{2} \right)^N \sum_{j=0}^{\infty} \frac{(z^2/4)^j}{j!(N+j)!} \left[ \log \frac{z}{2} - \frac{1}{2} \psi(j+1) - \frac{1}{2} \psi(N+j+1) \right]
\]

where \( \psi(x) = \frac{d}{dx} \log \Gamma(x) \) is the digamma function [21]. On the other hand, for \( \nu = \text{half-odd-integer} = N + \frac{1}{2} \), the corresponding small \( z \) expansion is

\[
K_{N+\frac{1}{2}}(z) = \sqrt{\pi} \left( \frac{z}{2} \right)^{-N+1/2} e^{-z} \sum_{j=0}^{N} \frac{(N+j)!}{j!(N-j)!} \left( \frac{z}{2} \right)^{-j} \tag{38}
\]

\[
= \frac{1}{2} \sum_{j=0}^{\infty} \left[ \frac{\Gamma(N-j+\frac{1}{2})}{j!} \left( \frac{z}{2} \right)^{-N-j} + \frac{\Gamma(-N-j+\frac{1}{2})}{j!} \left( \frac{z}{2} \right)^{N+1/2} \right] \left( -\frac{z^2}{4} \right)^j
\]

which begins with \( z^{-N-1/2} \), but has no logarithmic piece.

These Bessel function expansions (37) and (38) lead to the following explicit high temperature expansions of the free effective action (36). In \( 2+1 \) dimensions, or more generally in \( \text{d + 1} \) dimensions with \( \text{d} \) even,

\[
S_{\text{eff}}^{d \text{ even}} = S_{\text{eff}}^{T=0; \mu=0} - c \left( \frac{m}{2\sqrt{\pi}} \right)^{d+1} \sum_{n=1}^{\infty} (-1)^n \cosh(\beta \mu n) \tag{39}
\]

\[
\times \sum_{j=0}^{\infty} \left[ \frac{\Gamma \left( \frac{d+1}{2} - j \right)}{j!} \left( \frac{\beta m n}{2} \right)^{-(d+1)/2} + \frac{\Gamma \left( -\left( \frac{d+1}{2} \right) - j \right)}{j!} \left( -\frac{\beta^2 m^2 n^2}{4} \right)^j \right]
\]

Notice that the \( \cosh(\beta \mu n) \) term can be expanded in powers of \( n^2 \), in which case the summation over \( n \) has the form of the eta function [21] (an alternating series analogue of the Riemann zeta function).
\[ \eta(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^r} \]  

(40)

For \( \text{Re } r > 1 \) the eta function \( \eta(r) \) is related to the Riemann zeta function, \( \zeta(r) = \sum_{n=1}^{\infty} 1/n^r \), by

\[ \eta(r) = (1 - 2^{1-r})\zeta(r) \]  

(41)

For negative integer values of \( r \), this function takes especially simple values. For even negative integers

\[ \eta(0) = \frac{1}{2}, \quad \eta(2k) = 0 \quad k = 1, 2, 3, \ldots \]  

(42)

while for odd-negative integer arguments

\[ \eta(1 - 2k) = (2^{2k} - 1) \frac{B_{2k}}{2k} \quad k = 1, 2, 3, \ldots \]  

(43)

where \( B_{2k} \) is the \((2k)^{th}\) Bernoulli number. Also note that \( \eta(1) = \log 2 \).

Since \( \eta(-2j) \) vanishes for \( j = 1, 2, \ldots \), only the \( j = 0 \) term survives in the second summation term inside the square brackets in (39), and this surviving term then cancels against the zero \( T \), zero \( \mu \) effective action. In contrast, the first summation term inside the square brackets in (39) involves odd powers of \( n \), and so contributes for all \( j \). Finally, we obtain the following double expansion

\[ S_{\text{eff}}^{d \text{ even}} = c \left( \beta \sqrt{\pi} \right)^{-(d+1)} \sum_{k=0}^{\infty} \frac{(\beta \mu)^{2k}}{(2k)!} \sum_{j=0}^{\infty} \left( -1 \right)^j \frac{\Gamma\left( \frac{d+1}{2} - j \right) \eta(d + 1 - 2k - 2j)}{j!} \left( \frac{\beta m}{2} \right)^{2j} \]  

(44)

which agrees with the corresponding expansion in Eq. (4.24) of Ref. [12], although the sum is organized in a different manner. Notice that the leading term at high temperature is proportional to \( T^{d+1} \), with the remaining terms in the expansion involving higher powers of \( \beta m \) or \( \beta \mu \). For zero chemical potential the expansion simplifies to a single sum

\[ S_{\text{eff}}^{d \text{ even}} |_{\mu=0} = c \left( \beta \sqrt{\pi} \right)^{-(d+1)} \sum_{j=0}^{\infty} \left( -1 \right)^j \frac{\Gamma\left( \frac{d+1}{2} - j \right) \eta(d + 1 - 2j)}{j!} \left( \frac{\beta m}{2} \right)^{2j} \]  

(45)

and in the massless case

\[ S_{\text{eff}}^{d \text{ even}} |_{m=0} = c \left( \beta \sqrt{\pi} \right)^{-(d+1)} \frac{\Gamma\left( \frac{d+1}{2} \right)}{(2k)!} \sum_{k=0}^{\infty} \frac{(\beta \mu)^{2k} \eta(d + 1 - 2k)}{(2k)!} \]  

(46)
In 3+1 dimensions, or more generally for $d+1$ dimensions with $d$ odd, the Bessel function expansion (37) leads to the following high temperature expansion of the free effective action:

$$S_{\text{eff}}^{d \text{ odd}} = S_{\text{eff}}^{T=0; \mu=0} - 2c \left( \frac{m}{2\sqrt{\pi}} \right)^{d+1} \sum_{n=1}^{\infty} (-1)^n \cosh(\beta \mu n)$$

$$\times \left[ \frac{1}{2} \left( \frac{\beta mn}{2} \right)^{(d+1)/2} \sum_{j=0}^{(d+1)/2} \frac{\Gamma\left(\frac{d+1}{2} - j\right)}{j!} \left(-\frac{\beta^2 m^2 n^2}{4}\right)^j \right]$$

$$-(-1)^{(d+1)/2} \sum_{j=0}^{(d+1)/2} \frac{(\beta^2 m^2 n^2/4)^j}{j!(j+(d+1)/2)!} \left( \log \frac{\beta m}{4\pi} - \frac{1}{2} \psi(j+1) - \frac{1}{2} \psi(j+\frac{d+3}{2}) \right)$$

(47)

The first term inside the square brackets produces $T^{d+1}$ times a polynomial $P_{d+1}(\beta \mu, \beta m)$ of order $d+1$ in $\beta \mu$ and $\beta m$, because the expansion involves only even powers of $n$, for which the eta function vanishes if these are positive even powers, but gives a finite contribution while the power is negative or zero. Thus the summation over $j$ and the expansion of $\cosh(\beta \mu n)$ are each truncated. The effective action takes then the form

$$S_{\text{eff}}^{d \text{ odd}} = S_{\text{eff}}^{T=0; \mu=0} - c \left( \frac{m}{2\sqrt{\pi}} \right)^{d+1} \left\{ (\beta m)^{-(d+1)} P_{d+1}(\beta \mu, \beta m) \right.$$ 

$$+ \frac{(-1)^{(d+1)/2}}{\Gamma\left((d+3)/2\right)} \left[ \log \frac{\beta m}{4\pi} - \frac{1}{2} \psi(1) - \frac{1}{2} \psi\left(\frac{d+3}{2}\right) \right] + \text{Tayl}(\beta m, \beta \mu) \right\}$$

(48)

where the Taylor series $\text{Tayl}(\beta m, \beta \mu)$ is determined using the following further summation identity (for details see [12]):

$$\sum_{n=1}^{\infty} (-1)^n n^{2k} \log(2\pi n) = \begin{cases} -\log 2, & k = 0 \\ (-1)^k \frac{1}{2^k} \left(1 - 2^{2k+1}\right) \frac{(2k)!}{(2\pi)^{2k}} \zeta(2k+1), & k \in \mathbb{Z}^+ \end{cases}$$

(49)

VI. CONSTANT MAGNETIC BACKGROUND

As discussed in Section [V], in order to obtain a closed expression of the form (27) for the finite temperature and finite chemical potential effective action for fermions in a static background magnetic field one simply needs the proper time propagator (29). For the case of a uniform background field this is a well-known computation whose exact solubility is due to the fact that the spectra of the operators $-D_+D_\mp$ correspond to one-dimensional harmonic oscillators, with ‘frequency’ $2B$, each energy level having degeneracy given by the (infinite) background magnetic flux [26].
\[ -D_+ D_- = 2Bn \quad n = 1, 2, 3, \ldots \]
\[ -D_- D_+ = 2Bn \quad n = 0, 1, 2, 3, \ldots \]  \hspace{1cm} (50)

Thus, the proper-time propagator is

\[ \text{Tr} \left[ \exp \left( -\frac{\mathcal{L}^2}{m^2} s \right) \right] = \begin{cases} \frac{B}{2\pi} \left( \frac{m}{2 \sqrt{\pi}} \right) \frac{1}{\sqrt{s}} e^{-s} \coth \left( \frac{Bs}{m^2} \right), & \text{3 + 1 dim.} \\ \frac{B}{2\pi} e^{-s} \coth \left( \frac{Bs}{m^2} \right), & \text{2 + 1 dim.} \end{cases} \]  \hspace{1cm} (51)

The extra \( \left( \frac{m}{2 \sqrt{\pi}} \right) \frac{1}{\sqrt{s}} \) factor in 3+1 dimensions is due to the integration over \( k_3 \), the momentum corresponding to the free motion in the \( z \) direction and \( B/2\pi \) is the usual Landau level degeneracy factor [26].

The effective action \( (27) \) is therefore

\[ S_{\text{eff}} = -\left( \frac{m}{2 \sqrt{\pi}} \right)^{d-1} B \frac{2}{2\pi} \int_0^\infty ds \frac{d}{(d+1)/2} e^{-s} \left[ \coth \left( \frac{Bs}{m^2} \right) - \frac{m^2}{Bs} \right] \theta_4 \left( \frac{i\beta\mu}{2} \left| \frac{i\beta^2 m^2}{4\pi s} \right. \right) \]  \hspace{1cm} (52)

where \( d = 2 \) or \( d = 3 \). Note that we have explicitly subtracted the zero \( B \) contribution by subtracting the leading small \( s \) divergence \( m^2/(Bs) \) from the function \( \coth(Bs/m^2) \).

This ensures that the effective action vanishes with vanishing \( B \), and simply corresponds to computing a difference of the effective action with and without the \( B \) field. We could also subtract the \( B^2 \) contribution as this may be absorbed into the original action via a charge renormalization. However, we choose to defer this additional subtraction for two reasons. First, we wish to emphasize that the \( B^2 \) contribution differs in an interesting manner for \( d = 2 \) and \( d = 3 \). Second, it is conventional [9–11] to make this subtraction only for the zero \( T \) and zero \( \mu \) portion of the effective action, since in the 3 + 1 dimensional case this contribution involves a divergence whose regularization involves a charge renormalization.

To develop high temperature expansions from (52), along the lines discussed for the free case in the previous Section, would require expanding the \( \theta_4 \) factor in the integrand and then performing the proper time integrals. However, these integrals cannot be done in closed form (in contrast to the free case where they led to modified Bessel functions - see (35)) because of the \( \coth(Bs/m^2) \) factor. We therefore rescale the proper time parameter \( s \): \( s/m^2 \rightarrow \beta^2 s \).

Then in the high \( T \) limit we expand the \( \coth(\beta^2 Bs) \) factor:

\[ \coth \left( \beta^2 Bs \right) - \frac{1}{\beta^2 Bs} = \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!} \left( \beta^2 Bs \right)^{2k-1} \]  \hspace{1cm} (53)
where $B_{2k}$ are the Bernoulli numbers \[24\]. This leads to the following double expansion

$$
S_{\text{eff}} = S_{\text{eff}}^{T=0;\mu=0} - 8 \left( \beta \sqrt{2\pi} \right)^{-(d+1)} \sum_{k=1}^{\infty} \frac{B_{2k} (\beta^2 B)^{2k}}{(2k)!} \sum_{n=1}^{\infty} (-1)^n \frac{\cos(h \beta m n)}{n^{d+1-4k}} \frac{K_{2k-\frac{d+1}{2}} (\beta mn)^{2k-\frac{d+1}{2}}}{(\beta mn)^{2k-\frac{d+1}{2}}} (54)
$$

Assuming that the temperature $T$ dominates all other energy scales, we can further expand using the series representations \[37\] and \[38\] of the modified Bessel functions $K_{\nu}(z)$. Note that, just as in the free case, there is a significant difference between $d = 2$ and $d = 3$, as the former case involves Bessel functions with half-odd-integer index, while the latter involves Bessel functions with integer index.

For $d = 2$,

$$
S_{\text{eff}}^{d=2} = \frac{4}{(\beta \sqrt{\pi})^3} \sum_{k=1}^{\infty} \frac{B_{2k} (\beta^2 B/2)^{2k}}{(2k)!} \sum_{j=0}^{\infty} \frac{(-\beta^2 m^2)^j}{j!} \frac{\Gamma \left( \frac{3}{2} - j - 2k \right)}{(\beta \mu)^{2j} \eta(3 - 4j - 2k)} (2l)! (55)
$$

Notice that the zero $T$, zero $\mu$ effective action is canceled by a term arising from the expansion of the Bessel functions. Considered as an expansion in the magnetic field, the leading term is of the form

$$
S_{\text{eff}}^{d=2} = \beta B^2 f(\beta \mu, \beta m) + \cdots (56)
$$

where $f(\beta \mu, \beta m)$ is a dimensionless function whose leading term is a (negative) constant.

The expression \[33\] has a well-defined massless limit:

$$
S_{\text{eff}}^{d=2} \big|_{m=0} = \frac{4}{(\beta \sqrt{\pi})^3} \sum_{k=1}^{\infty} \frac{B_{2k} (\beta^2 B/2)^{2k}}{(2k)!} \sum_{l=0}^{\infty} \frac{(\beta \mu)^{2l} \eta(3 - 4k - 2l)}{(2l)!} (57)
$$

And for zero chemical potential,

$$
S_{\text{eff}}^{d=2} \big|_{\mu=0} = \frac{4}{(\beta \sqrt{\pi})^3} \sum_{k=1}^{\infty} \frac{B_{2k} (\beta^2 B/2)^{2k}}{(2k)!} \sum_{j=0}^{\infty} \frac{(-\beta^2 m^2)^j \Gamma \left( \frac{3}{2} - j - 2k \right)}{j!} \eta(3 - 2j - 4k) (58)
$$

For $d = 3$ the situation is somewhat different owing to the appearance of a divergent term in the zero $T$, zero $\mu$ effective action and of logarithmic terms in both the zero $T$, zero $\mu$ piece and in the finite temperature piece of the effective action. From \[34\], using the Bessel function expansion \[37\], we find

$$
S_{\text{eff}}^{d=3} = -\frac{B^2}{12\pi^2} \left( \log \frac{\beta m}{\pi} + 6\pi^2 C - \psi(1) \right) + \frac{64\pi^2}{\beta^4} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left( \frac{\beta^2 B}{8\pi^2} \right)^{2k} \sum_{j=0}^{\infty} \frac{(-\beta^2 m^2)^j}{j!} \sum_{l=0}^{\infty} \frac{(-\beta^2 \mu^2)^l}{(2l)!} (59)
$$

\[
\times \left( 1 - 2^{4k+2j+2l-3} \right) \zeta(4k + 2j + 2l - 3) \frac{\Gamma(4k + 2j + 2l - 3)}{\Gamma(2k + j - 1)} \text{ except the term } \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-\beta^2 \mu^2)^l}{(2l)!}
\]
Notice that the $T = 0, \mu = 0$ effective action cancels against a term

$$
\frac{m^4}{8\pi^2} \sum_{k=2}^{\infty} \frac{B_{2k} (2B/m^2)^{2k} (2k - 3)!}{(2k)!}
$$

(60)

arising from the Bessel function expansions, leaving a term quadratic in the $B$ field with an (infinite) coefficient

$$
C = \frac{1}{12\pi^2} \int_0^\infty ds s^{-1} e^{-m^2 s}
$$

(61)

which may be absorbed in a renormalization of the electric charge [9].

The remaining expansion (59) has no quartic or quadratic term in the temperature but has a logarithmic dependence on the temperature. This agrees with Ref. [3] but disagrees with Ref. [23].

To conclude this discussion of the high temperature limit for fermions in a constant magnetic background, we observe that in the 2 + 1 dimensional case it is possible to give an alternative derivation of the magnetic background effective action beginning instead with the expression (16). For ease of presentation, consider the case $\mu = 0$. Then

$$
S_{\text{eff}} = \frac{2}{\beta} \operatorname{Tr} \log \cosh \left[ \frac{\beta}{2} |\mathcal{E}| \right] = \frac{2}{\beta} \sum_{p=1}^{\infty} \frac{(2^{2p} - 1)B_{2p}^2}{(2p)(2p)!} \beta^{2p} \operatorname{Tr} \left[ |\mathcal{E}|^{2p} \right]
$$

(62)

The trace over the even powers of the energy operator can be expressed in terms of generalized zeta functions. In 2 + 1 dimensions

$$
\operatorname{Tr} \left[ |\mathcal{E}|^{2p} \right] = \frac{B}{2\pi} \left[ 2 \sum_{n=1}^{\infty} (m^2 + 2Bn)^p + m^{2p} \right] = \frac{(2B)^{p+1}}{2\pi} \zeta\left(-p, 1 + \frac{m^2}{2B}\right) + \frac{B}{2\pi} m^{2p}
$$

(63)

where the generalized zeta function is defined as [21]

$$
\zeta(s, q) = \sum_{n=0}^{\infty} \frac{1}{(n + q)^s}
$$

(64)

When the first argument of the generalized zeta function is a negative integer, it reduces to a polynomial (in the second argument) which is in fact proportional to a Bernoulli polynomial:

$$
\zeta(-p, x) = -\frac{B_{p+1}(x)}{p+1}
$$

(65)

The Bernoulli polynomials are given by
\[ B_p(x) = \sum_{r=0}^{p} \binom{p}{r} B_r x^{p-r} \]  

(66)

with expansion coefficients involving the Bernoulli numbers \( B_r \).

Thus, the high temperature expansion (62) may be written as

\[ S_{d^2} |_{\mu=0} = -\frac{1}{2\pi \beta^3} \sum_{p=1}^{\infty} \frac{(2^{2p} - 1)B_{2p}}{p(p+1)(2p)!} (2B\beta^2)^{p+1} \left[ B_{p+1} \left( \frac{m^2}{2B} \right) + \frac{p+1}{2} \left( \frac{m^2}{2B} \right)^p \right] \]  

(67)

Remarkably, this expansion (67) is simply another way of expressing the double expansion in (58), up to a \( B \) independent term.

VII. ZERO TEMPERATURE LIMIT

Having considered the high temperature behavior of the effective action, we now turn to the other extreme: \( T = 0 \). With zero chemical potential this is just the usual field theoretic effective action, which may be computed using the proper-time formalism [9–11]. With nonzero chemical potential, the zero \( T \) limit is more subtle due to nonanalytic contributions from the sharp cut-off of the energy spectrum which is no longer smoothed out by thermal fluctuations. The low \( T \) effective action with nonzero chemical potential and background magnetic field in 3 + 1 dimensions has been studied recently [3,4], primarily using the real-time formalism. Here we discuss a somewhat different (and more direct) approach to the low temperature limit, based on the expression (20). According to (20), the correction \( \Delta S \) to the zero \( T \) and zero \( \mu \) effective action when the chemical potential is nonzero may be written as

\[ \Delta S |_{T=0; \mu \neq 0} = \text{Tr} \left[ (\mu - |\mathcal{E}|) \theta(\mu - |\mathcal{E}|) \right] \]  

(68)

Since \( |\mathcal{E}| \geq m \) it is clear that this correction to the usual zero temperature effective action vanishes unless \( \mu > m \). For the remainder of this Section we assume that \( \mu \) is indeed greater than \( m \).

The free case is well known (see for example [20]) and we turn directly to the case of a magnetic background. We consider the 2 + 1 and 3 + 1 dimensional cases separately. In 2 + 1
dimensions, to compute the correction term (68) we simply fill the Landau levels, according to the Landau level spectrum (50), up to just below the level of the chemical potential

\[ \Delta S_{d=2}^{(s)} |_{T=0; \mu \neq 0} = \frac{B}{2\pi} \left\{ 2 \sum_{n=1}^{\left[ \frac{m\mu_{NR}}{B} \right]} (\mu - \sqrt{m^2 + 2Bn}) + (\mu - m) \right\} \]

\[ = \frac{Bm}{\pi} \left\{ \frac{\mu}{m} \left[ \frac{\mu^2 - m^2}{2B} \right] + \sqrt{\frac{2B}{m^2}} \zeta\left( -\frac{1}{2}, 1 + \frac{m^2}{2B} + \left[ \frac{\mu^2 - m^2}{2B} \right] \right) \right\} \]

\[ + \frac{\mu - m}{2m} - \sqrt{\frac{2B}{m^2}} \zeta\left( -\frac{1}{2}, 1 + \frac{m^2}{2B} \right) \] (69)

The last two terms are in fact canceled by the zero chemical potential part of the effective action. We note the appearance of a nonanalytic dependence on the integer part \( \frac{m\mu_{NR}}{B} \) of the dimensionless ratio \( \frac{\mu^2 - m^2}{2B} \). In the nonrelativistic limit, this leads to oscillatory behavior of the effective action and related thermodynamics quantities (such as the magnetization) as the strength \( B \) of the external magnetic field is varied.

Specifically, in the nonrelativistic, weak field limit, \( m^2 \gg (\mu^2 - m^2) \gg B \), the 2 + 1 dimensional correction term is

\[ \Delta S_{d=2}^{NR} |_{T=0; \mu \neq 0} = \frac{B}{2\pi} \left\{ 2 \sum_{n=1}^{\left[ \frac{m\mu_{NR}}{B} \right]} \left( \mu_{NR} - \frac{B}{m}n \right) + \mu_{NR} \right\} \] (70)

where \( \mu_{NR} = \mu - m \) is the nonrelativistic chemical potential. This may be re-expressed as

\[ \Delta S_{d=2}^{NR} |_{T=0; \mu \neq 0} = \frac{B^2}{\pi m} \left( 1 + \left[ \frac{m\mu_{NR}}{B} \right] \right) \left( \frac{m\mu_{NR}}{B} - \frac{1}{2} \left[ \frac{m\mu_{NR}}{B} \right] \right) - \frac{B\mu_{NR}}{2\pi} \] (71a)

\[ = \frac{B^2}{\pi m} \left\{ \zeta\left( -1, \frac{m\mu_{NR}}{B} \right) - \left[ \frac{m\mu_{NR}}{B} \right] \right\} - \zeta\left( -1, 1 + \frac{m\mu_{NR}}{B} \right) \right\} - \frac{B\mu_{NR}}{2\pi} \]

\[ = \frac{B^2}{4\pi^3 m} \sum_{k=1}^{\infty} \sin \left( 2\pi k \left( \frac{m\mu_{NR}}{B} - \frac{m\mu_{NR}}{B} \right) - \frac{\pi}{2} \right) - \frac{B^2}{\pi m} \zeta\left( -1, 1 + \frac{m\mu_{NR}}{B} \right) - \frac{B\mu_{NR}}{2\pi} \] (71b)

where we have used the zeta function identity (see [21] 1.10 Eq. (6))

\[ \zeta(s, v) = 2(2\pi)^{s-1} \Gamma(1 - s) \sum_{k=1}^{\infty} k^{s-1} \sin(2\pi kv + \pi s/2), \quad 0 < v \leq 1, \text{Re } s < 0 \] (72)

Thus the effective action includes a term which as a function of \( 1/B \) is periodic with period \( 1/(m\mu_{NR}) \), the characteristic period of the de Haas - van Alphen oscillations [27]. It is
interesting to notice that (in contrast to 3+1 dimensions) there is a explicit algebraic expression (71a) for the effective action that shows these oscillations.

In 3 + 1 dimensions, the relativistic correction term (68) is

$$\Delta S^d_{3|T=0; \mu \neq 0} = \frac{B}{2\pi} \left\{ 2 \sum_{n=1}^{[\frac{\mu^2-m^2}{2B}]} \int_0^{\sqrt{\mu^2-m^2-2Bn}} \frac{dp}{2\pi} (\mu - \sqrt{p^2 + m^2 + 2Bn}) + 2 \int_0^{\sqrt{\mu^2-m^2-2Bn}} \frac{dp}{2\pi} (\mu - \sqrt{p^2 + m^2}) \right\}$$

$$= \frac{B}{(2\pi)^2} \left\{ \mu \sqrt{\mu^2-m^2} - m^2 \log \left( \frac{\mu + \sqrt{\mu^2-m^2}}{m} \right) \right\} + 2 \sum_{n=1}^{[\frac{\mu^2-m^2}{2B}]} \left( \mu \sqrt{\mu^2-m^2 - 2Bn} - (m^2 + 2Bn) \log \left( \frac{\mu + \sqrt{\mu^2-m^2 - 2Bn}}{\sqrt{m^2 + 2Bn}} \right) \right) \right\}$$

(73)

This expression agrees with a corresponding one in [4], derived from a zero temperature limit in the real-time formalism. Expanding the logarithmic term in the weak field limit (in which $B \ll \mu^2$) we obtain

$$\Delta S^d_{3|T=0; \mu \neq 0} = \frac{Bm^2}{(2\pi)^2} \left\{ \frac{\mu}{m} \sqrt{\frac{\mu^2-m^2}{m^2}} - \log \left( \frac{\mu + \sqrt{\mu^2-m^2}}{m} \right) \right\} + \frac{2B^2}{\pi^2} \sum_{l=0}^{\infty} \frac{(2B)^{l+1/2}}{(2l+1)(2l+3)} \left\{ \zeta (-l - \frac{3}{2}, \frac{\mu^2-m^2}{2B} - \left[ \frac{\mu^2-m^2}{2B} \right]) - \zeta (-l - \frac{3}{2}, \frac{\mu^2-m^2}{2B}) \right\}$$

(74)

This expression involves a term depending on the nonanalytic fractional part of $\frac{\mu^2-m^2}{2B}$. Again oscillations arise when we represent the first zeta function with the help of the zeta function identity (72)

$$\zeta (-l - \frac{3}{2}, \frac{\mu^2-m^2}{2B} - \left[ \frac{\mu^2-m^2}{2B} \right]) = 2\Gamma(l + 5/2) \sum_{k=1}^{\infty} \sin \left( \frac{2\pi k}{m} \right) \left( \frac{m^2 - \mu^2}{2B} - \left[ \frac{m^2 - \mu^2}{2B} \right] \right)$$

(75)

These oscillatory components reveal themselves in a nonrelativistic limit $m^2 \gg (\mu^2 - m^2) \gg B$ where only the first term $l = 0$ survives, being responsible for the usual nonrelativistic de Haas - van Alphen effect [27]:

$$\Delta S^d_{3|oscill \, NR\, T=0; \mu \neq 0} = \frac{B^{5/2}}{4m\pi^4} \sum_{k=1}^{\infty} \sin \left( \frac{2\pi k}{m} \frac{m^2 - \mu^2}{B} - \frac{3\pi}{4} \frac{k^{5/2}}{k^{5/2}} \right)$$

(76)
VIII. CONCLUSIONS

In this paper we have shown how to compute finite temperature expansions of the effective action for fermions in the presence of a static background magnetic field. The computational techniques of the imaginary time formalism (for the finite temperature effects) and the proper time method (for the background field effects) combine elegantly to produce a simple formal expression (27) for the finite \( T \), and nonzero chemical potential, effective action. Explicit high (and low) temperature expansions may then be obtained given the explicit functional form of the proper time propagator (29). Generically, high temperature expansions involve logarithmic terms in \( d + 1 \) dimensions with \( d \) odd, with these terms being absent if \( d \) is even. These logarithmic terms may be traced to corresponding terms appearing in the series expansions of the modified Bessel function \( K_N(z) \). For a uniform magnetic field background, the zero temperature limit (with nonzero chemical potential) may also be evaluated in a straightforward manner in terms of generalized zeta functions. These zero temperature effective actions involve oscillatory pieces which reflect the Landau level structure of the underlying single particle spectrum.

The formalism developed here is motivated in part by the fact that the proper time method provides a powerful technique for analyzing non-constant background fields. For example, one could generalize from a uniform static background magnetic field to a static magnetic field with spatial dependence. Such a generalization has been considered recently for the zero temperature (and zero chemical potential) effective action in \( 2 + 1 \) dimensions, with one particular nonuniform background field leading to an exact closed-form expression for the effective action \([28]\). It would be interesting to incorporate the finite temperature and finite chemical potential effects into such a model. Since expression (27) completely separates the finite temperature (and nonzero chemical potential) effects from those of the static background, it is very tempting to make high (and low) temperature expansions at this level, without having an explicit expression for the proper time propagator. However, even in the free case this leads to subtleties which can be missed if one is not careful. For example, even in the free case, one might think to make a high temperature expansion by
expanding the exponential term inside the proper time integral in the effective action (34). This amounts to deriving the series expansions (37) and (38) for the modified Bessel function \( K_\nu(z) \) directly from the integral representation (35) by expanding in powers of \( z^2 \) inside the integral. But this type of expansion does not produce the necessary logarithmic terms when \( \nu \) is an integer - instead, to obtain the correct series expansion one must use a different integral representation based on a contour integral \([29]\). For fermions in the presence of some general background gauge field, such naive expansions, made inside the proper time integral, must also be treated with care, particularly in 3 + 1 dimensions.

This work is supported in part by the NSF under contract PHY-92-18990 and by the DOE under grant DE-FG02-92ER40716.00. We thank the Aspen Center for Physics for hospitality during the initial stages of this work.
REFERENCES

∗ cangemi@physics.ucla.edu

† dunne@hep.phys.uconn.edu

[1] S. Chandrasekhar, *Introduction to the Study of Stellar Structure* (Univ. of Chicago Press, 1949); S. Shapiro and S. Teukolsky, *Black Holes, White Dwarfs and Neutron Stars* (Wiley, 1983).

[2] H-Y. Chiu and V. Canuto, “Properties of High-Density Matter in Intense Magnetic Fields”, *Phys. Rev. Lett.* 21 (1968) 110, “Quantum Theory of an Electron Gas in Intense Magnetic Fields”, *Phys. Rev.* 173 (1968) 1210, “Thermodynamic Properties of a Magnetized Fermi Gas”, *Phys. Rev. Lett.* 23 (1968) 1220, “Magnetic Moment of a Magnetized Fermi Gas”, *Phys. Rev.* 173 (1968) 1229; H-Y. Chiu, V. Canuto and L. Fassio-Canuto, “Quantum Theory of an Electron Gas with Anomalous magnetic Moment in Intense Magnetic Fields”, *Phys. Rev.* 176 (1969) 1438; H. J. Lee, V. Canuto, H-Y. Chiu and C. Chiuderi, “New State of Ferromagnetism in Degenerate Electron Gas and Magnetic Fields in Collapsed Bodies”, *Phys. Rev. Lett.* 23 (1969) 390.

[3] P. Elmfors, D. Persson and B-S. Skagerstam, “The QED Effective Action at Finite Temperature and Density”, *Phys. Rev. Lett.* 71 (1993) 480, “Real-Time Thermal Propagators and the QED Effective Action for an External Magnetic Field”, *Astropart. Phys.* 2 (1994) 299.

[4] D. Persson and Vad. Zeitlin, “A Note on QED with Magnetic Field and Chemical Potential”, *Phys. Rev. D* 51 (1995) 2026; Vad. Zeitlin, “Low-Temperature QED with External Magnetic Field”, Lebedev preprint FIAN/TD/94-10, hep-ph/9412204. “Effective Lagrangian of Thermal QED with External Magnetic Field and the Static Limit of the Polarization Operator”, Lebedev preprint FIAN/TD/95-11, hep-ph/9507404.

[5] U. Danielsson and D. Grasso, “Polarization of a QED Plasma in a Strong Magnetic Field”, *Phys. Rev. D* 52 (1995) 2533.
[6] L. Dolan and R. Jackiw, “Symmetry Behavior at Finite Temperature”, *Phys. Rev. D* 9 (1974) 3320.

[7] E. Shuryak, “Quantum Chromodynamics and the Theory of Superdense Matter”, *Phys. Rep.* 61 (1980) 71.

[8] J. Kapusta, *Finite-Temperature Field Theories* (Cambridge Univ. Press, Cambridge, 1989).

[9] J. Schwinger, “On Gauge Invariance and Vacuum Polarization”, *Phys. Rev. D* 82 (1951) 664.

[10] A. Salam and J. Strathdee, “Transition Electromagnetic Fields in Particle Physics”, *Nucl. Phys. B* 90 (1975) 203.

[11] S. Blau, M. Visser and A. Wipf, “Analytic Results for the Effective Action”, *Int. J. Mod. Phys.* A6 (1991) 5409.

[12] A. Actor, “Zeta Function Regularization of High-Temperature Expansions in Field Theory”, *Nucl. Phys. B* 265 (1986) 689.

[13] H. Weldon, “Proof of Zeta Function Regularization of High-Temperature Expansions”, *Nucl. Phys. B* 270 (1986) 79.

[14] A. Actor, “Chemical Potentials in Gauge Theories”, *Phys. Lett. B* 157 (1985) 53.

[15] R. Jackiw and S. Templeton, “How Super-renormalizable Interactions Cure Their Infrared Divergences”, *Phys. Rev. D* 23 (1981) 2291.

[16] R. Pisarski, “Chiral Symmetry Breaking in Three-Dimensional Electrodynamics”, *Phys. Rev. D* 29 (1984) 2423.

[17] T. Appelquist, M. Bowick, D. Karabali and L. C. R. Wijewardhana, “Spontaneous Breaking of Parity in (2 + 1)-dimensional QED” *Phys. Rev. D* 33 (1986) 3704, “Spontaneous Chiral-Symmetry Breaking in Three-Dimensional QED”, *Phys. Rev. D* 33 (1986) 3774.
[18] A. Polychronakos, “Symmetry Breaking Patterns in (2 + 1) Dimensional Gauge Theories”, Phys. Rev. Lett. 60 (1988) 1920.

[19] V. Gusynin, V. Miransky and I Shovkovy, “Catalysis of Dynamical Flavor Symmetry Breaking by a Magnetic Field in 2 + 1 Dimensions”, Phys. Rev. Lett. 73 (1994) 3499.

[20] A. Chodos, K. Everding and D. Owen, “QED with a Chemical Potential: the Case of a Constant Magnetic Field”, Phys. Rev. D 42 (1990) 2881.

[21] A. Erdélyi et al, Higher Transcendental Functions, The Bateman Manuscript Project, Vols. I and II, (Krieger, Florida, 1981).

[22] R. Bellman, A Brief Introduction to Theta Functions, (Holt, Rinehart and Winston, New York, 1961).

[23] W. Dittrich, “Effective Lagrangians at Finite Temperature”, Phys. Rev. D 19 (1979) 2385; W. Dittrich and M. Reuter, Effective Lagrangians in Quantum Electrodynamics, Lecture Notes in Physics, Vol. 220 (Springer, Berlin, 1985).

[24] H. Braden, “Expansions for Field Theories on $S^1 \times \Sigma$”, Phys. Rev. D 25 (1982) 1028.

[25] I. Gradshteyn and I. Ryzhik, Tables of Integrals, Series and Products (Academic, New York, 1980).

[26] L. Landau, “Diamagnetism of Metals”, Z. Phys. 64 (1930) 629, reprinted in English in L. D. Landau, by D. ter Haar (Pergamon 1965).

[27] R. Pathria, Statistical Mechanics, (Pergamon, New York, 1972).

[28] D. Cangemi, E. D’Hoker and G. Dunne, “Derivative Expansion of the Effective Action and Vacuum Instability for QED in 2 + 1 Dimensions”, Phys. Rev. D 51 (1995) R2513, “Effective Energy for QED$^{2+1}$ with Semi-localized Static Magnetic Field: A Solvable Model”, Phys. Rev. D, 52 (1995) R3163.

[29] N. McLachlan and A. Meyers, “Contour Integral Expressions for Bessel Functions”, Philos. Mag. 23 (1937) 762.