SEMIGROUP-THEORETIC APPROACH TO IDENTIFICATION OF LINEAR DIFFUSION COEFFICIENTS

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Abstract. Let $X$ be a complex Banach space and $A : D(A) \to X$ a quasi-$m$-sectorial operator in $X$. This paper is concerned with the identification of diffusion coefficients $\nu > 0$ in the initial-value problem:

$$\frac{d}{dt}u(t) + \nu Au(t) = 0, \quad t \in (0, T), \quad u(0) = x \in X,$$

with additional condition $\|u(T)\| = \rho$, where $\rho > 0$ is known. Except for the additional condition, the solution to the initial-value problem is given by $u(t) := e^{-t \nu A}x \in C([0, T]; X) \cap C^1((0, T]; X)$. Therefore, the identification of $\nu$ is reduced to solving the equation $\|e^{-\nu T A}x\| = \rho$. It will be shown that the unique root $\nu = \nu(x, \rho)$ depends on $(x, \rho)$ locally Lipschitz continuously if the datum $(x, \rho)$ fulfills the restriction $\|x\| > \rho$. This extends those results in Mola [6] (2011).

1. Introduction. As is well-known, the abstract parabolic Cauchy problem

$$(d/dt)u(t) + \nu Au(t) = 0, \quad t \in (0, T), \quad u(0) = x, \quad (1.1)$$

admits global well-posedness in the sense of strong topology for all initial data $x$ in a Banach space $X$, under the assumptions that for some $\alpha \geq 0$, the linear operator $A - \alpha : D(A) \to X$ is $m$-sectorial in $X$. More precisely, it is possible to prove that the problem (1.1) has a unique solution of regularity

$$u \in C([0, T]; X) \cap C^1((0, T]; X) \cap C((0, T]; D(A)),$$

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which depends continuously on the initial datum \(x\), and whose energy \(\|u(t)\|\) is non-increasing on the trajectories (here, \(\|\cdot\|\) is the norm in \(X\)). This result holds for all positive values of the diffusion coefficient \(\nu\), which, for the sake of our investigation, will be always considered as a mere constant quantity.

The sectoriality (or sectorial-valuedness) was first introduced in the Hilbert space framework (see Kato [4, Section V.3.10]) as an extension of the notion of nonnegative selfadjointness. Then the notion was generalized to the Banach space case. Let \(F\) be the (single-valued) duality mapping on \(X\) with uniformly convex adjoint \(X^\ast\) and \(\langle f,g \rangle_{X,X^\ast}\) denote the pairing between \(f \in X\) and \(g \in X^\ast\). We define a sector by

\[
S(\tan \theta) := \begin{cases} 
[0, \infty) & \text{if } \theta = 0, \\
\{ z \in \mathbb{C}; |\Im z| \leq (\tan \theta) \Re z \} & \text{if } 0 < \theta < \pi/2, \\
\{ z \in \mathbb{C}; \Re z \geq 0 \} & \text{if } \theta = \pi/2.
\end{cases}
\]  

(1.2)

Then a linear operator \(A\) with domain \(D(A)\) and range \(R(A)\) in \(X\) is said to be sectorial of type \(S(\tan \theta)\) if \(\langle Au, F(u) \rangle_{X,X^\ast} \in S(\tan \theta)\), \(\theta \in [0, \pi/2)\), for \(u \in D(A)\). Note that an accretive operator can be regarded as sectorial of type \(S(\pi/2)\).

In particular, a sectorial operator \(A\) is \(m\)-sectorial (or regularly \(m\)-accretive) if, additionally, \(R(1 + A) = X\). It should be noted that a linear operator \(A\) is the generator of an analytic contraction semigroup \(\{e^{-tA}; t \geq 0\}\) on \(X\) if and only if \(-A\) is an \(m\)-sectorial operator in (reflexive) \(X\). In this case, if \(t > 0\), then \(Ae^{-tA}\) is bounded on \(X\) (smoothing effect).

Back to our problem, we stress that in the basic case when \(A = -\Delta\) on the domain \(D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\) in \(L^p(\Omega)\) \((1 < p < \infty)\) (that is, \(A\) is the \(L^p\)-realization of the Laplace operator on a domain \(\Omega \subset \mathbb{R}^N\)), then, as it is well-known, \(A\) is \(m\)-sectorial of type \(S(c_p)\) in \(L^p\), as a consequence of the inequality

\[
|\Im (-\Delta u, |u|^{p-2}u)_{L^p,L^p'}| \leq c_p \Re ((-\Delta - \alpha)u, |u|^{p-2}u)_{L^p,L^p'},
\]

which holds for all \(u \in D(A)\), where \(\alpha\) is the first eigenvalue of \(A\) and we have defined \(c_p := |p - 2|/(2\sqrt{p-1})\) (see, e.g., Okazawa [7] and Voigt [10]). Moreover, in the Hilbert case \(p = 2\), from \(c_2 = 0\) it follows that \(A = -\Delta\) on the domain \(D(A) = H^2(\Omega) \cap H_0^1(\Omega)\) is nonnegative and selfadjoint in \(L^2(\Omega)\).

The coefficient \(\nu\) accounts for important features of the diffusion phenomena associated with the Cauchy problem (1.1). For example (see [9]), in the case where \(u\) represents the concentration of a chemical substance subjected to diffusion, then (1.1) may be regarded as Fick’s second law, and \(\nu\) reflects the mobility of the diffusing species in the given environment (accordingly, it is expected to assume larger values in gases, smaller ones in liquids, and extremely small ones in solids). Nevertheless, as it is understood, the measurement of such an intrinsic quantity as \(\nu\) can be extremely difficult to be performed. Thus, it is natural to think of \(\nu\) as an unknown of our problem, as well as the “solution” \(u\). Therefore, in order to recast a proper determination, it is necessary to feed the problem with additional measurements on the accessible parameters (overdeterminating conditions). Among all the possible choices, we shall study the inverse problem or identification of the diffusion constant \(\nu\) under the “final-time” energy measurement

\[
\|u(T)\| = \rho,
\]

where \(\rho\) is a given positive constant, to be considered from now on as a datum of our problem, together with the initial value \(x \in X\).
The inverse problem we presented has been studied by the first author in [6], under the assumption that $A$ is positive-definite and self-adjoint. The main idea therein introduced is the one to use the fundamental energy identity at the final time $t = T$

$$\|u(T)\|^2 + 2\nu \int_0^T \|A^{1/2}u(\tau)\|^2 d\tau = \|u(0)\|^2$$

to deduce the following relationship between the unknowns

$$\nu = \mathcal{N}(u) := \frac{\|u(0)\|^2 - \|u(T)\|^2}{2\int_0^T \|A^{1/2}u(t)\|^2 dt},$$

where $A^{1/2}$ denotes the square root of $A$. In other words, the unknown constant $\nu$ can be computed in terms of $u$ as a nonlinear functional $\mathcal{N}(u)$ which, consequently, entails a modification of equation (1.1) as

$$(d/dt)u(t) + \mathcal{N}(u)Au(t) = 0, \quad u(0) = x.$$ (1.3)

It is important to stress that, strictly speaking, (1.3) is not a differential (forward or backward) equation, due to the nonlocal and noncausal nature of the functional $\mathcal{N}$, which requires the knowledge of the global dynamics on the whole interval $[0, T]$. Thus, usual abstract techniques for quasilinear equations cannot be applied to our case. The results provided in [6] concern existence and uniqueness of a weak solution $(u, \nu)$ as well as its continuous dependence on the data $(x, \rho)$, and have been achieved by adapting a finite-dimensional Faedo-Galerkin approximation scheme to the inverse problem. This requires a deep use of the real spectral decomposition, which restricts the application of the abstract result to the realization of second-order differential operators only on bounded domains, and can be performed only in the Hilbert case.

The aim of the present paper is to study the same inverse problem as in [6] by means of deeper techniques in nonlinear analysis in Banach spaces (with uniformly convex adjoint). The main advantages of this alternative approach are twofold. First, for the sake of applications, such an approach may apply to the realization of second-order differential operators also on unbounded domains, and extends the previous results even to complex Banach spaces. Second, and more importantly, such a formulation opens new scenarios towards more general equations (i.e. fully nonlinear) in which the identification problem can be stated. Those features will be object of further investigations in the next future.

2. Preliminaries. Let $F$ be the duality mapping on a general Banach space $X$ to its adjoint $X^*$:

$$F(v) := \{ f \in X^*; \langle v, f \rangle_{X, X^*} = \|v\|^2 = \|f\|^2 \} \quad \forall \ v \in X.$$ 

Here we denote by $\langle v, f \rangle_{X, X^*}$ the pairing between $v \in X$ and $f \in X^*$. Assume that $X^*$ is strictly convex. Then $F$ is single-valued. Now we consider the “tangent functional” of the unit ball in $X$:

$$\tau_+(y, x) := \lim_{s \downarrow 0} \frac{\|x + sy\| - \|x\|}{s} \quad (x, y \in X).$$ (2.1)

Put

$$\varphi(x) := (1/2)\|x\|^2 = (1/2)\langle x, F(x) \rangle_{X, X^*}.$$
Then, since 
\[ \|x + sy\|^2 - \|x\|^2 = (\|x + sy\| + \|x\|)(\|x + sy\| - \|x\|), \]
we see from (2.1) that
\[ \lim_{s \downarrow 0} \frac{\varphi(x + sy) - \varphi(x)}{s} = \|x\|\tau(y, x) = \text{Re} \langle y, F(x)X, x^* \rangle. \]
(see, e.g., Miyadera [5, Section 1, Chapter 2]). In other words, \( \varphi \) is Gâteaux differentiable. Thus \( \varphi \) is Fréchet differentiable, with Fréchet gradient \( F(x) \in X^* = \mathcal{L}(X, \mathbb{C}) \) if \( F: X \to X^* \) is continuous on the whole of \( X \) (see Zeidler [11, Proposition 4.8 (c)]).

Assume further that \( X^* \) is uniformly convex. Then \( F \) is uniformly continuous on bounded subsets of \( X \) (see Zeidler [12, Proposition 32.22 (e)]). This implies that \( \varphi \in C^1(X; \mathbb{R}) \) with \( \varphi' = I \) (the identity) if \( X \) is a Hilbert space (cf. Section 3) or with \( \varphi' = F \) if \( X \) has the uniformly convex \( X^* \) (cf. Section 4).

As employed by Goldstein [3, Definition 1.5.8] (see also Ouhabaz [8, p. 97]), an important role in semigroup theory is played by the notion of sectorial operator in a general Banach space (more precisely, in the case where \( F \) is not single-valued). That is, a linear operator \( A \in X \) is said to be sectorial of type \( S(\tan \theta) \) if for every \( u \in D(A) \) there is \( f \in F(u) \) such that \( \langle Au, f \rangle_{X, X^*} \in S(\tan \theta), \theta \in \left[0, \pi/2\right) \), where \( S(\tan \theta) \) is the sector defined by (1.2). A sectorial operator \( A \) is called \( m \)-sectorial (of type \( S(\tan \theta) \)) if \( R(1 + A) = X \) additionally. Recall that \( A \) is \( m \)-accretive if and only if \( A \) is \( m \)-sectorial of type \( S(\pi/2) \). The analytic case of the Hille-Yosida theorem states that a linear operator \( A \) is the generator of an analytic contraction semigroup \( \{e^{-tA}; t \geq 0\} \) on \( X \) if and only if \( A \) is an \( m \)-sectorial operator with domain \( D(A) \) dense in \( X \) (see [3, Theorem 1.5.9 and Proposition 1.3.9]); in this case there exists a constant \( c > 0 \) such that
\[ \|A e^{-tA}x\| \leq \frac{c}{t} \|x\|, \quad t > 0, \quad x \in X. \]

Now let \( A \) be a linear quasi-\( m \)-accretive operator in a Banach space \( X \), that is, for any \( u \in D(A) \) there is \( f \in F(u) \) such that
\[ \text{Re} \langle Au, f \rangle_{X, X^*} \geq \alpha \|u\|^2 \quad \forall \ u \in D(A), \]
(2.2)
where \( \alpha > 0 \) is a constant. In other words, \( A - \alpha \) is \( m \)-accretive in \( X \). A quasi-\( m \)-sectorial operator in \( X \) satisfies the stronger estimate:
\[ \text{Re} \langle Au, f \rangle_{X, X^*} - \alpha \|u\|^2 \geq \frac{1}{\tan \theta} \text{Im} \langle Au, f \rangle_{X, X^*} \quad \forall \ u \in D(A), \ \theta \in (0, \pi/2). \]
(2.3)
In these cases we have \( e^{\alpha t} \|e^{-tA}x\| = \|e^{-t(A-\alpha)}x\| \leq 1 \) \( (t \geq 0) \). Therefore (2.2) implies that the semigroup generated by \( -A \) is of negative type:
\[ \|e^{-tA}x\| \leq e^{-\alpha t} \|x\| \quad (t \geq 0). \]
(2.4)

On account to the functional setting above introduced, we can state the abstract formulation of the inverse problem we aim at investigating.

**Problem P.** Given \( (x, \rho) \in H \times \mathbb{R}_+ \), find a vector-valued function \( u : [0, T] \to H \) and a real number \( \nu > 0 \) fulfilling the equation
\[ u'(t) + \nu Au(t) = 0, \quad t \in (0, T), \]
(2.5)
with the initial datum
\[ u(0) = x, \]
(2.6)
and the additional constraint
\[ \|u(T)\| = \rho. \]
(2.7)
First of all, we know that for every \( \nu > 0 \), \( u(t) = e^{-\nu AT}x \) is a unique solution to the initial-value problem (2.5)–(2.6) of suitable regularity (see [1, Chapter 7]). So it remains to verify additional condition (2.7), that is, the one-to-one correspondence between \( \rho \) and \( \nu \) when \( x \) is fixed. To this end, for \( \nu \geq 0 \) put

\[
\phi(\nu) := 2^{-1}\|e^{-\nu AT}x\|^2.
\]

Then we have \( \phi(0) = 2^{-1}\|x\|^2 > 2^{-1}\rho^2 \) and \( \phi(\nu) \rightarrow 0 \ (\nu \rightarrow \infty) \) by virtue of (2.4). More precisely, \( \phi \) is (strictly) monotone decreasing. In fact, since \( x \in D(A) \), it follows that \( \phi \in C^1(\mathbb{R}_+) \), with

\[
(d/d\nu)\phi(\nu) = -T\text{Re} \langle Ae^{-\nu AT}x, f_\nu \rangle_{X, X^*} \leq -\alpha T\|x\|^2 < -\alpha T\rho^2 < 0,
\]

where \( f_\nu \) is an arbitrary element in \( F(e^{-\nu AT}x) \) (see [5, Lemma 2.8]). Thus we see from the intermediate theorem that for every \( \nu \in (0, \|x\|) \) there exists a unique \( \nu > 0 \) satisfying (2.7). Therefore, choosing \( \nu > 0 \) as the unique root of \( \phi(\nu) - \rho = 0 \), \( u(t) = e^{-\nu AT}x \) is a unique solution to Problem \( P \). Moreover, uniqueness of the root immediately shows that the map \( (x, \rho) \mapsto (\nu, u) \) is continuous.

Generally speaking, \( \nu \) is expected to be determined by the pair \( (x, \rho) \) as above and depends continuously on the data \( (x, \rho) \). One can expect more regularity of the solution map. In order to establish this assertion we shall use the Banach space version of the implicit function theorem applied to the map \( \psi \in C^1(Y \times \mathbb{R}_+ \times \mathbb{R}_+) \) defined by

\[
\psi(x, \rho, \nu) := 2^{-1}\|e^{-\nu TA}x\|^2 - 2^{-1}\rho^2, \quad (x, \rho, \nu) \in D(A) \times \mathbb{R}_+ \times \mathbb{R}_+,
\]

where \( Y = (D(A) \setminus B(\rho)) \) or \( (X \setminus B(\rho)) \), with \( B(\rho) := \{ x \in X \mid \|x\| \leq \rho \} \).

**Remark 2.1.** Recalling Feller’s renorming trick, the above-mentioned assertion can be extended to general \( C_0 \)-semigroups of negative type, at the expense of enlarging the lower bound on \( \|x_0\| \). In fact, replacing the norm by the Feller norm

\[
\|x\| = \sup_{t>0} \|e^{-tA}x\|, \quad x \in X,
\]

we have \( \|x\| \leq \|x\| \leq M\|x\| \), with \( M = \sup_{t>0} \|e^{-At}\| \). Then the condition \( \|x\| > \rho \) has to be replaced with \( \|x\| > M\rho \).

3. **Identification of diffusion coefficients in Hilbert spaces.** Let \( H \) be a Hilbert space with inner product \( (\cdot, \cdot) \). Then the duality mapping \( F \) is the identity. We denote by \( \mathcal{G}(\theta, -\alpha) \), \( (\theta, \alpha) \in [0, \pi/2] \times \mathbb{R}_+ \), the set of all linear operators \( A \) such that \( A - \alpha \) are \( m \)-sectorial of type \( S(\tan \theta) \) in \( H \). Namely, when \( \theta \in (0, \pi/2) \), then

\[
\text{Re} (Au, u) - \alpha\|u\|^2 \geq \frac{1}{\tan \theta} |\text{Im} (Au, u)| \quad \forall \ u \in D(A)
\]

(in this case \( D(A) \) is dense in \( H \)). In particular, when \( \theta = \pi/2 \), then \( A \) is nothing but quasi-\( m \)-accretive in \( H \); note that \( \mathcal{G}(\theta, -\alpha) \subset \mathcal{G}(\pi/2, -\alpha) \), \( \theta \in (0, \pi/2) \). In addition, when \( \theta = 0 \), then \( A \) is positive-definite selfadjoint operator in \( H \). The symbol \( \mathcal{G}(\theta, -\alpha) \) is a modification of \( \mathcal{G}(M, \beta) \) which has been introduced by Kato [4, Section IX.4].

Now we define the set of *admissible data* as

\[
\mathcal{A}(Y) := \{(x, \rho) \in Y \times \mathbb{R}_+ \mid \|x\| > \rho\},
\]

where \( Y = H \) or \( D(A) \). Then the identification of diffusion coefficients for abstract parabolic problem is stated as follows:
Problem \((P)_H\) \textbf{(parabolic case).} Given \(A \in \mathcal{B}(\theta, -\alpha)\) and \((x, \rho) \in \mathcal{A}(H)\) find the pair

\[
(\nu, u) \in \mathbb{R}_+ \times (C([0, T]; H) \cap C^1((0, T]; H))
\]

fulfilling equation (2.5) (with \(t > 0\)), initial datum (2.6) and additional condition (2.7).

In order to differentiate \(e^{-\nu TA}x\) with respect to \(\nu\), we first consider an approximate problem, that is, a problem with initial value \(x \in H\) replaced with \(x \in D(A)\) (we need this process only when \(A\) is quasi-\(m\)-accretive). To this end let \(D(A)\) be a Hilbert space with inner product and norm:

\[
(x, y)_{D(A)} := (x, y) + (Ax, Ay), \quad \|x\|^2_{D(A)} := \|x\|^2 + \|Ax\|^2.
\]

Then we can introduce

\textbf{Problem \((P)_{D(A)}\) (non-parabolic case).} Given \(A \in \mathcal{B}(\theta, -\alpha)\) and \((x, \rho) \in \mathcal{A}(D(A))\) find the pair

\[
(\nu, u) \in \mathbb{R}_+ \times C^1([0, T]; H)
\]

fulfilling equation (2.5) (with \(t \geq 0\)), initial datum (2.6) and additional condition (2.7).

We are now in a position to state the main result in the Hilbert space case.

\textbf{Theorem 3.1.} Let \(A \in \mathcal{B}(\theta, -\alpha)\), \(\theta \in [0, \pi/2]\). Then

\textbf{(I)} Problem \((P)_{D(A)}\) is uniquely solvable for \(\theta \in [0, \pi/2]\).

\textbf{(II)} Problem \((P)_H\) is uniquely solvable for \(\theta \in [0, \pi/2]\).

To solve Problem \((P)_{D(A)}\) we apply the \textit{implicit function theorem} to the function of three variables:

\[
\psi(x, \rho, \nu) := \frac{1}{2}\|e^{-\nu TA}x\|^2 - \frac{1}{2}\rho^2, \quad (x, \rho, \nu) \in D(A) \times \mathbb{R}_+ \times \mathbb{R}_+.
\]

The \(\rho\)-dependence of \(\psi\) is not so complicated. To simplify the notation we also use the function of two variables:

\[
\phi(x, \nu) := \frac{1}{2}\|e^{-\nu TA}x\|^2, \quad (x, \nu) \in D(A) \times \mathbb{R}_+
\]

so that we have \(\psi(x, \rho, \nu) = \phi(x, \nu) - \rho^2/2\).

Now we can show that

\textbf{Lemma 3.2.} Let \(A\) be quasi-\(m\)-accretive in \(H\): Re \((Au, u) \geq \alpha\|u\|^2\), \(u \in D(A)\), \(\alpha > 0\). Then \(\psi \in C^1((D(A) \setminus B(\rho)) \times \mathbb{R}_+ \times \mathbb{R}_+)\):

\textbf{(i) (\(\rho\)-derivative).} Let \(x \in D(A)\). Then one has

\[
\frac{\partial \psi}{\partial \rho}(x, \rho, \nu) = -\rho, \quad (\rho, \nu) \in \mathbb{R}_+ \times \mathbb{R}_+.
\]

\textbf{(ii) (\(x\)-derivative).} Let \(x \in D(A)\). Then one has

\[
\nabla_x \psi(x, \rho, \nu) = e^{-\nu TA}e^{-\nu TA} \in \mathcal{L}(D(A), X), \quad (\rho, \nu) \in \mathbb{R}_+ \times \mathbb{R}_+.
\]

Here \(A^*\) is the adjoint operator of \(A\).

\textbf{(iii) (\(\nu\)-derivative).} Let \(x \in (D(A) \setminus B(\rho)), \rho \geq \rho_0 > 0\). Then one has

\[
\frac{\partial \psi}{\partial \nu}(x, \rho, \nu) = -T \text{Re} \left( Ae^{-\nu TA}x, e^{-\nu TA}x \right) < -\alpha T \rho_0^2 < 0, \quad (\rho, \nu) \in [\rho_0, \infty) \times \mathbb{R}_+.
\]
Proof. It suffices to prove (ii) and (iii).

(ii) Let \( x \in D(A) \). Since \( \psi(x, \rho, \nu) = \phi(x, \nu) - \rho^2/2 \), we can show that \( \phi \) is Fréchet differentiable with respect to \( x \):

\[
\forall \ \varepsilon > 0 \ \exists \ \delta > 0; \quad \|y - x\| < \delta \ \Rightarrow \ |\phi(y, \nu) - \phi(x, \nu) - \operatorname{Re}(\nabla_x \phi(x, \nu), y - x)| \leq \varepsilon \|y - x\|, \tag{3.1}
\]

with \( \nabla_x \phi(x, \nu) := e^{-\nu T^*A}e^{-\nu T A}x \). In fact, we can write as

\[
\phi(y, \nu) - \phi(x, \nu) = \frac{1}{2}\|e^{-\nu T A}(x + (y - x))\|^2 - \frac{1}{2}\|e^{-\nu T A}x\|^2 = \operatorname{Re}(e^{-\nu T A}x, e^{-\nu T A}(y - x)) + (1/2)\|e^{-\nu T A}(y - x)\|^2 = \operatorname{Re}(\nabla_x \phi(x, \nu), y - x) + (1/2)\|e^{-\nu T A}(y - x)\|^2.
\]

Now, put \( \delta := 2 \varepsilon \). If \( \|y - x\| < \delta \), then we have the implication of (3.1):

\[
|\phi(y, \nu) - \phi(x, \nu) - \operatorname{Re}(\nabla_x \phi(x, \nu), y - x)| \leq 2^{-1}e^{-2\alpha T}\|y - x\|^2 \leq 2^{-1}\delta \|y - x\| = \varepsilon \|y - x\|.
\]

It remains to show that \( \nabla_x \phi(\cdot, \nu) \) in (3.1) is the Fréchet derivative of \( \phi \) with respect to \( x \). Since \( A - \alpha \) is m-accretive in \( H \) if and only if so is \( A^* - \alpha \), \( e^{-\nu TA} \) makes sense, with the equality \( \|e^{-\nu TA}\| = \|e^{-\nu T A}\| \). Therefore it follows from (3.1) that \( \nabla_x \phi(\cdot, \nu) : D(A) \to X' \) is a bounded linear operator, with norm-estimate

\[
\|\nabla_x \phi(x, \nu)\| \leq \|e^{-\nu T^*A}\| \cdot \|e^{-\nu T A}\| \cdot \|x\| \leq e^{-2\alpha T}\|x\| \leq \|x\|_{D(A)}; \tag{3.2}
\]

note that \( X' = \mathcal{L}(X, \mathbb{C}) \) is identified as \( X \).

(iii) Let \( x \in D(A) \setminus B(\rho) \), \( \rho \geq \rho_0 > 0 \). Then it follows that

\[
\frac{\partial \phi}{\partial \nu}(x, \nu) = -T \operatorname{Re}(Ae^{-\nu TA}x, e^{-\nu TA}x) \leq -\alpha T \|e^{-\nu TA}x\|^2 \leq -\alpha T \|x\|^2 < -\alpha T \rho_0^2 < 0 \quad \forall \ \nu \in \mathbb{R}_+; \tag{3.3}
\]

note that \( (Ae^{-\nu TA}x, e^{-\nu TA}x) = (e^{-\nu TA}Ax, e^{-\nu TA}x) \) is continuous with respect to \( (x, \nu) \in (D(A) \setminus B(\rho)) \times \mathbb{R}_+ \).

Consequently, we can apply the implicit function theorem (cf. Drabek-Milota [2] for its infinite dimensional statement).

Proof of Theorem 3.1. Part (I) For every pair of \( (x, \rho) \) there is a unique function \( \nu = \nu(x, \rho) \in C^1((D(A) \setminus B(\rho)) \times \mathbb{R}_+) \) such that

\[
\nabla_x \nu(x, \rho) = -\frac{\nabla_x \phi(x, \nu)}{(\partial/\partial \nu)\phi(x, \nu)} = \frac{e^{-\nu T^*A^*}e^{-\nu T A}x}{T \operatorname{Re}(A e^{-\nu T^*A}x, e^{-\nu T A}x)}, \tag{3.4}
\]

\[
\frac{\partial \nu}{\partial \rho}(x, \rho) = -\frac{(\partial/\partial \rho)\psi(x, \rho, \nu)}{(\partial/\partial \nu)\psi(x, \nu)} = \frac{\rho}{T \operatorname{Re}(A e^{-\nu T^*A}x, e^{-\nu T A}x)}. \tag{3.5}
\]

Noting that

\[
\|\nabla_x \nu(x, \rho)\| \leq \frac{1}{\alpha T \|x\|^2} < \frac{1}{\alpha T \rho^2}, \quad \left| \frac{\partial \nu}{\partial \rho}(x, \rho) \right| \leq \frac{\rho}{\alpha T \|x\|^2} < \frac{1}{\alpha T \rho},
\]
we have the Lipschitz continuity of $\nu(x, \rho)$:

$$
|\nu(x_1, \rho_1) - \nu(x_2, \rho_2)| \leq \frac{1}{\alpha T \rho_0^2} \text{dist}_{D(A) \setminus B(\rho_0)}(x_1, x_2) + \frac{1}{\alpha T \rho_0} |\rho_1 - \rho_2| \quad (3.6)
$$

for $\|x_1\|, \|x_2\| \geq \rho_0$ and $\rho_1, \rho_2 \geq \rho_0$ for some fixed $\rho_0 > 0$. Here $\text{dist}_{D(A) \setminus B(\rho_0)}(x_1, x_2)$ denotes the geodesic distance from $x_1$ to $x_2$; note that

$$
\text{dist}_{D(A) \setminus B(\rho_0)}(x_1, x_2) = \|x_1 - x_2\|
$$

if $\|x_1 - x_2\|$ is sufficiently small.

Part (II) can be proven as Part (I) with minor modifications. \hfill \Box

**Remark 3.3.** Now fix $\rho \geq \rho_0$ in the proof of Part (I). Then, since $D(A)$ is dense in $H$, for every $x \in H \setminus B(\rho)$ there is a sequence $\{x_n\}$ in $D(A) \setminus B(\rho)$ such that $x_n \to x$ ($n \to \infty$). Put $\nu_n := \nu(x_n, \rho)$ and $u_n(t) := e^{-\nu_n t} A x_n$. Then it follows from (3.6) that

$$
|\nu_n - \nu_m| = |\nu(x_n, \rho) - \nu(x_m, \rho)| \leq \frac{1}{\alpha T \rho_0^2} \|x_n - x_m\|
$$

for sufficiently large $n, m$. Therefore for every $x \in H \setminus B(\rho_0)$ we may define

$$
\nu = \nu(x, \rho) := \lim_{n \to \infty} \nu(x_n, \rho) \quad \forall \rho \geq \rho_0,
$$

$$
u(t) := \lim_{n \to \infty} u_n(t).
$$

Since $\rho_0$ is arbitrary, we may define $\nu \in C((H \setminus B(\rho)) \times \mathbb{R}_+)$, satisfying

$$
|\nu(x_1, \rho_1) - \nu(x_2, \rho_2)| \leq \frac{1}{\alpha T \rho_0^2} \text{dist}_{H \setminus B(\rho_0)}(x_1, x_2) + \frac{1}{\alpha T \rho_0} |\rho_1 - \rho_2|.
$$

Note that in the latter non-parabolic case, the unique solution $u$ to the initial value problem (2.5)–(2.6) loses the differentiability. Thus, solutions are to be intended in a weak sense as in [6].

4. **Identification of diffusion coefficients in Banach space $X$ with uniformly convex $X^\ast$.** Let $X$ be a Banach space with uniformly convex adjoint $X^\ast$, with duality mapping $F$:

$$
\|F(u)\|_{X^\ast} = \|u\|_X \quad \forall u \in X.
$$

A typical example is the Lebesgue space $L^p$ ($1 < p < \infty$) with $F : L^p \to L^{p'}$ given by

$$
F(u) = \|u\|_{L^{p'}}^2 |u|^{p-2}u, \quad u \in L^p, \quad \frac{1}{p} + \frac{1}{p'} = 1.
$$

**Lemma 4.1.** Put $\varphi(u) := (1/2)\|u\|^2_X$ ($u \in X$). Then $\varphi$ is Fréchet differentiable, with Fréchet derivative $F(u) : \nabla_u \varphi(u) = F(u), \; u \in X$. In other words, one has

$$
\forall \varepsilon > 0 \exists \delta > 0; \quad \|v - u\|_X < \delta \Rightarrow |\varphi(v) - \varphi(u) - \text{Re} \langle v - u, F(u) \rangle_{X,X^\ast}| \leq \varepsilon \|v - u\|_X. \quad (4.1)
$$

As in Section 3 we denote by $\mathcal{G}(\theta, -\alpha), \; (\theta, \alpha) \in [0, \pi/2] \times \mathbb{R}_+$, the set of all linear operators $A$ such that $A - \alpha$ are $m$-sectorial of type $S(\tan \theta)$ in $X$. For example, when $\theta \in (0, \pi/2)$, then we have

$$
\text{Re} \langle Au, F(u) \rangle_{X,X^\ast} - \alpha \|u\|^2_X \geq \frac{1}{\tan \theta} |\text{Im} \langle Au, F(u) \rangle_{X,X^\ast}| \quad \forall u \in D(A); \quad (4.2)
$$
note that if \( A \in \mathcal{G}(\theta,-\alpha) \), then \( D(A) \) is dense in \( X \). The set of admissible data is also defined as in Section 3:

\[
\mathcal{A}(Y) := \{(x, \rho) \in Y \times \mathbb{R}_+; \|x\|_X > \rho\},
\]

where \( Y = X \) or \( D(A) \).

Now we consider the further problems

**Problem \( (P)_{D(A)} \) (non-parabolic case).** Given \( A \in \mathcal{G}(\theta,-\alpha) \) and \( (x, \rho) \in \mathcal{A}(D(A)) \) find the pair

\[
(\nu, u) \in \mathbb{R}_+ \times C^1([0,T]; X)
\]

fulfilling equation (2.5) (with \( t > 0 \)), initial datum (2.6) and additional condition (2.7).

**Problem \( (P)_X \) (parabolic case).** Given \( A \in \mathcal{G}(\theta,-\alpha) \) and \( (x, \rho) \in \mathcal{A}(X) \) find the pair

\[
(\nu, u) \in \mathbb{R}_+ \times (C([0,T]; X) \cap C^1((0,T]; X))
\]

fulfilling equation (2.5) (with \( t > 0 \)), initial datum (2.6) and additional condition (2.7).

We are in a position to state the main result in this section.

**Theorem 4.2.** Let \( A \in \mathcal{G}(\theta,-\alpha) \), \( (\theta, \alpha) \in [0, \pi/2] \times \mathbb{R}_+ \). Then one has the following assertions:

(I) **Problem \( (P)_{D(A)} \)** is uniquely solvable for \( \theta \in [0, \pi/2] \).

(II) **Problem \( (P)_X \)** is uniquely solvable for \( \theta \in [0, \pi/2] \).

To solve Problem \( (P)_X \) we apply the implicit function theorem to the function of three variables:

\[
\psi(x, \rho, \nu) := \frac{1}{2} \| e^{-\nu TA} x \|_X^2 - \frac{1}{2} \rho^2, \quad (x, \rho, \nu) \in Q(\rho_0),
\]

where

\[
Q(\rho_0) := \{(x, \rho, \nu); (x, \rho) \in \mathcal{A}(X), \rho \geq \rho_0 > 0, \nu \in \mathbb{R}_+\}.
\]

The \( \rho \)-dependence of \( \psi \) is not so complicated. To simplify the notation we also use the function of two variables:

\[
\phi(x, \nu) := \frac{1}{2} \| e^{-\nu TA} x \|_X^2, \quad \|x\|_X > \rho \geq \rho_0 > 0, \quad \nu \in \mathbb{R}_+,
\]

so that we have

\[
\psi(x, \rho, \nu) = \phi(x, \nu) - \rho^2 / 2.
\]

**Lemma 4.3.** Let \( \mathcal{G}(\theta,-\alpha) \), \( (\theta, \alpha) \in [0, \pi/2] \times \mathbb{R}_+ \). Then \( \psi \in C^1(Q(\rho_0)) \):

(i) **(\( \rho \)-derivative).** Let \( \rho > 0 \). Then one has

\[
\frac{\partial \psi}{\partial \rho}(x, \rho, \nu) = -\rho, \quad (x, \rho, \nu) \in Q(\rho_0).
\]

(ii) **(\( x \)-derivative).** Let \( x \in X \). Then \( \nabla_x \psi(\cdot, \rho, \nu) \in \mathcal{L}(X, X^*) \), with

\[
\nabla_x \psi(x, \rho, \nu) = e^{-\nu TA^*} F(e^{-\nu TA} x), \quad (x, \rho, \nu) \in Q(\rho_0).
\]

Here \( A^* \) is the adjoint operator of \( A \) in \( X^* \).

(iii) **(\( \nu \)-derivative).** Let \( \nu > 0 \). Then one has

\[
\frac{\partial \psi}{\partial \nu}(x, \rho, \nu) = -T \text{Re} (A e^{-\nu TA} x, e^{-\nu TA} x) < -\alpha T \rho_0^2 < 0, \quad (x, \rho, \nu) \in Q(\rho_0).
\]
Proof. It suffices to prove (ii) and (iii).

(ii) Put
\[ \phi(x, \nu) := \frac{1}{2} \| e^{-\nu TA} x \|^2. \]
Then we shall show that \( \phi(x, \nu) \) is Fréchet differentiable with respect to \( x \), with Fréchet derivative
\[ \nabla_x \phi(., \nu) := e^{-\nu TA^*} F(e^{-\nu TA} x). \]  
(4.3)
Since \( X \) is reflexive, \( A - \alpha \) is \( m \)-accretive in \( X \) if and only if \( A^* - \alpha = (A - \alpha)^* \) is \( m \)-accretive in \( X^* \). Therefore \( e^{-\nu TA^*} \) makes sense, and the next equality holds
\[ \| e^{-\nu TA^*} \| = \| e^{-\nu TA} \| = \| e^{-\nu T} \|. \]
In the \( \varepsilon\)-\( \delta \) argument the assertion is expressed as
\[ \forall \varepsilon > 0 \exists \delta > 0; \]
\[ \| y - x \|_X < \delta \]
\[ \Rightarrow | \phi(y, \nu) - \phi(x, \nu) - \text{Re} \langle y - x, \nabla_x \phi(x, \nu) \rangle_{X,X^*} | \leq \varepsilon \| y - x \|_X. \]  
(4.4)
To see this we can employ (4.1) with \( u := e^{-\nu TA} x \) and \( v := e^{-\nu TA} y \), that is, if \( \| y - x \|_X < \delta \), then
\[ | \phi(y, \nu) - \phi(x, \nu) - \text{Re} \langle e^{-\nu TA} (y - x), F(e^{-\nu TA} x) \rangle_{X,X^*} | \leq \varepsilon \| e^{-\nu TA} (y - x) \|_X. \]
This is nothing but (4.4), with \( \nabla_x \phi(x, \nu) \) given by (4.3).
It remains to show that \( \nabla_x \phi(., \nu) \) in (4.4) is the Fréchet derivative of \( \phi \) with respect to \( x \). But, it follows from (4.3) that \( \nabla_x \phi(., \nu) : X \rightarrow X^* \) is a bounded linear operator, with norm-estimate
\[ \| \nabla_x \phi(x, \nu) \|_{X^*} \leq \| e^{-\nu TA^*} \| \cdot \| e^{-\nu TA} \| \cdot \| x \|_X \]
\[ \leq e^{-\alpha \| x \|} \| x \|_X \]
\[ \leq \| x \|_X. \]  
(4.5)
(iii) Let \( x \in X \setminus B(\rho), \rho \geq \rho_0 > 0 \). Then it follows from [5, Lemma 2.8] that
\[ \frac{\partial \phi}{\partial \nu}(x, \nu) = - T \text{Re} \langle A e^{-\nu TA} x, F(e^{-\nu TA} x) \rangle_{X,X^*} \]
\[ \leq - \alpha T \| e^{-\nu TA} x \|^2_X \]
\[ \leq - \alpha T \| x \|^2_X \]
\[ < - \alpha T \rho_0^2 < 0 \quad \forall \nu \in \mathbb{R}_+. \]  
(4.6)
Since \( \nu T > 0 \), \( A e^{-\nu TA} \) is bounded on \( X \) and hence \( \langle A e^{-\nu TA} x, F(e^{-\nu TA} x) \rangle_{X,X^*} \) is continuous with respect to \( (x, \nu) \in (X \setminus B(\rho)) \times \mathbb{R}_+, \rho \geq \rho_0 > 0 \).
This completes the proof of \( \psi \in C^1(Q(\rho_0)). \)

Using Lemma 4.3 instead of Lemma 3.2, we can prove Theorem 4.2 in the same way as in Section 3. Of course Theorem 3.1 is contained in Theorem 4.2.

5. Applications in \( L^p \)-space. In this last section we shall display concrete applications of the abstract results stated in the previous sections to a family of initial-boundary value problems. We stress the fact that all the function spaces we mention can be complex-valued, thus extending analogous results in [6], where the underlying theory was only real.
5.1. **Realization on bounded domains.** Let \( \Omega \subset \mathbb{R}^N \) \((N \in \mathbb{N})\) be a bounded domain with smooth boundary \( \partial \Omega \). Denote by \( \Delta = \sum_{i=1}^{N} (\partial / \partial x_i)^2 \) the Laplace operator in the space \( L^p = L^p(\Omega) \) \((1 < p < \infty)\). Then we define in \( L^p \) the negative Dirichlet-Laplace operator by means of

\[
A := -\Delta \quad \text{with} \quad D(A) := W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega).
\]

We recall that, in this case \(-\Delta - \alpha\) is \( m \)-sectorial in \( L^p \) and the dissipation constant \( \alpha \) can be chosen to be the first eigenvalue of \( A \).

Thus, the first inverse problem we want to study reads

**Problem 1.** Find the function \( u : \Omega \times [0, T] \to \mathbb{C} \) and the real constant \( \nu > 0 \) such that the following parabolic initial-boundary value problem is satisfied:

\[
\begin{aligned}
(\partial / \partial t)u(x, t) - \nu \Delta u(x, t) &= 0, \quad (x, t) \in \Omega \times (0, T), \\
u(x, 0) &= u_0(x), \quad x \in \Omega, \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
\|u(\cdot, T)\|_{L^p(\Omega)} &= \rho,
\end{aligned}
\]

where \( u_0 \in L^p(\Omega) \) and \( \rho > 0 \) are given.

Then the abstract Theorem 4.2 yields

**Theorem 5.1.** Let \( \rho > 0 \) and \( u_0 \in L^p(\Omega) \) with \( \|u_0\|_{L^p(\Omega)} > \rho \). Then there exists a unique pair

\[
(u, \nu) \in \left( C([0, T]; L^p(\Omega)) \cap C^1((0, T]; L^p(\Omega)) \right) \times (0, +\infty)
\]

such that

(i) \( u(t) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \quad \forall \ t \in (0, T], \)

(ii) there holds \( \|\Delta u(\cdot, t)\|_{L^p(\Omega)} \leq \frac{c}{t}\|u_0\|_{L^p(\Omega)} \) for some \( c > 0, \)

solving Problem 1. In particular, if \( p = 2 \), then

(iii) \( \nu = N(u) := \frac{\|u_0\|^2_{L^2(\Omega)} - \rho^2}{2 \int_0^T \|\nabla u(\cdot, t)\|^2_{L^2(\Omega)} \, dt} > 0, \)

(iv) \( \sqrt{t} \cdot u \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)). \)

5.2. **Realization on unbounded domains.** Now, denote by \( \Delta = \sum_{i=1}^{N} (\partial / \partial x_i)^2 \) the Laplace operator in the space \( L^p = L^p(\mathbb{R}^N) \) of the complex-valued \( p \)-th-power-summable functions. Then we define in \( L^p \) the realization of negative Laplace operator:

\[
A := -\Delta \quad \text{with} \quad D(A) := W^{2,p}(\mathbb{R}^N).
\]

We recall that, in this case \( \alpha - \Delta \) is \( m \)-sectorial, with dissipation constant \( \alpha > 0. \)

Thus, the second inverse problem we want to study reads
Problem 2. Find the function $u : \mathbb{R}^N \times [0,T] \to \mathbb{C}$ and the real constant $\nu > 0$ such that the following parabolic initial value problem is satisfied:

\[
\begin{aligned}
(\partial/\partial t)u(x,t) + \nu (\alpha - \Delta)u(x,t) &= 0, \\
\begin{cases}
u u(x,t) &= u_0(x), \\
\|u(\cdot,T)\|_{L^p(\mathbb{R}^N)} &= \rho.
\end{cases}
\end{aligned}
\]

where $u_0 \in L^p(\mathbb{R}^N)$ and $\rho > 0$ are given.

Once more, abstract Theorem 4.2 yields

Theorem 5.2. Let $\rho > 0$ and $u_0 \in L^p(\mathbb{R}^N)$ with $\|u_0\|_{L^p(\mathbb{R}^N)} > \rho$. Then there exists a unique pair

\[
(u, \nu) \in (C([0,T]; L^p(\mathbb{R}^N)) \cap C^1((0,T]; L^p(\mathbb{R}^N))) \times (0, +\infty)
\]

such that

(i) $u(t) \in W^{2,p}(\mathbb{R}^N)$ for all $t \in (0,T)$,

(ii) there holds $\|\Delta u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \leq \tilde{c} \|u_0\|_{L^p(\mathbb{R}^N)}$ for some $c > 0$,

(iii) $\nu = \mathcal{N}(u) := \frac{\|u_0\|^2_{L^2(\mathbb{R}^N)} - \rho^2}{2 \int_0^T \left( \alpha \|u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} + \|\nabla u(\cdot,t)\|^2_{L^2(\mathbb{R}^N)} \right) dt} > 0$,

(iv) $\sqrt{\nu} \cdot u \in L^2(0,T; H^2(\mathbb{R}^N))$.

Appendix. Let $X := L^p = L^p(\Omega) = L^p(\Omega; \mathbb{C})$, where $\Omega \subset \mathbb{R}^N$ is a domain $(1 < p < \infty, N \in \mathbb{N})$, with norm $\| \cdot \|_{L^p}$. Then $F(u) := \|u\|^2_{L^p} |u|^{p-2} u$ is the (normalized) duality mapping on $L^p(\Omega)$ to its adjoint $L^{p'}(\Omega)$, $p^{-1} + p'^{-1} = 1$, with

$\|F(u)\|_{L^{p'}} = \|u\|_{L^p}.$

Hereby, we shall show that $\varphi(u) := (1/2)\|u\|^2_{L^p}$ ($p \geq 4$) is Fréchet differentiable, with Fréchet derivative $F(u)$ by means of a direct computation because, as it appears, no direct proof is displayed in the literature.

Proposition A1. $\nabla_u \varphi(u) = F(u) = \|u\|^2_{L^p} |u|^{p-2} u$, $0 \neq u \in L^p(\Omega)$.

Applying the chain rule for composite functions, we have

\[
\nabla_u \left( \frac{1}{2} \|u\|_{L^p}^2 \right) = \frac{1}{2} \nabla_u \left( \|u\|_{L^p}^{2/p} \right) = \frac{\|u\|^{2-p}_{L^p} \nabla_u \left( \frac{1}{p} \|u\|_{L^p}^p \right)}{p}, \quad u \neq 0.
\]

Therefore it suffices to prove the next

Lemma A2. Let $p \geq 4$. Then $\nabla_u (p^{-1} \|u\|_{L^p}^p) = |u|^{p-2} u$, $u \in L^p(\Omega)$.

The proof of Lemma A2 is accomplished by the following computation.

Lemma A3. Let $u, h \in L^p(\Omega)$ with $p \geq 4$. Then one has

\[
\Delta_0 := \frac{\|u+h\|_{L^p}^p}{p} - \frac{\|u\|_{L^p}^p}{p} - \int_{\Omega} \text{Re} \left\{ u(x) \overline{h(x)} \right\} \left( \int_0^1 |u(x) + s h(x)|^{p-2} ds \right) dx
\]

\[
= \int_{\Omega} |h(x)|^2 \left( \int_0^1 s |u(x) + s h(x)|^{p-2} ds \right) dx,
\]

(A2)
with estimate
\[ |\Delta_0| \leq \int_{\Omega} |h(x)|^2 (|u(x)| + |h(x)|)^{p-2} \, dx \leq \|h\|_{L^p}^2 \|u\| + |h|_{L^p}^{p-2}. \quad (A3) \]

**Proof.** Let \( q \in (1, \infty) \) and \( s \in [0, 1] \). Then the proof is based on the equality:
\[ \frac{1}{q} \frac{d}{ds} \left( |u(x) + sh(x)|^q \right) = |u(x) + sh(x)|^{q-2} \text{Re} \left\{ \frac{u(x) + sh(x)}{h(x)} \right\} = |u(x) + sh(x)|^{q-2} \left[ \text{Re} \left\{ \frac{u(x)}{h(x)} \right\} + s|h(x)|^2 \right]. \quad (A4) \]
In fact, applying (A4) with \( q = p \), we can obtain (A2):
\[ \frac{1}{p} \int_{\Omega} |u(x) + h(x)|^p \, dx - \frac{1}{p} \int_{\Omega} |u(x)|^p \, dx = \frac{1}{p} \int_{\Omega} \frac{d}{ds} \left( |u(x) + s h(x)|^p \right) \, ds \]
\[ = \int_{\Omega} \left( \int_{0}^{1} |u(x) + s h(x)|^{p-2} \left[ \text{Re} \left\{ \frac{u(x)}{h(x)} \right\} + s|h(x)|^2 \right] \, ds \right) \, dx. \]
Since \( s|u(x) + sh(x)|^{p-2} \leq (|u(x)| + |h(x)|)^{p-2} \) (\( s \in [0, 1] \)), the Hölder inequality applies to give (A3).

**Proof of Lemma A2.** Put
\[ \Delta := \frac{1}{p} \|u + h\|_{L^p}^p - \frac{1}{p} \|u\|_{L^p}^p - \text{Re} \left\langle h, |u|^{p-2} u \right\rangle_{L^p, L^{p'}}. \]
Then it follows from (A2) that
\[ \Delta - \Delta_0 = \int_{\Omega} \text{Re} \left\{ \overline{u(x)} h(x) \right\} \left( \int_{0}^{1} |u(x) + s h(x)|^{p-2} - |u(x)|^{p-2} \right) \, ds \, dx \]
\[ = \int_{\Omega} \text{Re} \left\{ \overline{u(x)} h(x) \right\} \left( \int_{0}^{1} \int_{0}^{s} \frac{d}{d\sigma} |u(x) + \sigma h(x)|^{p-2} \, d\sigma \right) \, ds \, dx. \]
Applying (A4) with \( q := p - 2 \geq 2 \), we have
\[ \frac{\Delta - \Delta_0}{p - 2} = \int_{\Omega} \left| \text{Re} \left\{ \overline{u(x)} h(x) \right\} \right|^2 \left( \int_{0}^{1} \left[ \int_{0}^{s} |u(x) + \sigma h(x)|^{p-4} \, d\sigma \right] \, ds \right) \, dx \]
\[ + \int_{\Omega} |h(x)|^2 \text{Re} \left\{ \overline{u(x)} h(x) \right\} \left( \int_{0}^{1} \int_{0}^{s} \sigma |u(x) + \sigma h(x)|^{p-4} \, d\sigma \, d\sigma \right) \, dx. \]
Thus we obtain
\[ |\Delta - \Delta_0| \leq (p - 2) \int_{\Omega} |h(x)|^2 (|u(x)|^2 + |u(x)| \cdot |h(x)|) (|u(x)| + |h(x)|)^{p-4} \, dx \]
\[ \leq (p - 2) \int_{\Omega} |h(x)|^2 (|u(x)| + |h(x)|)^{p-2} \, dx. \]
Setting \( v := u + h \), we see from (A3) that
\[ |\Delta| = \left| \frac{\|v\|_{L^p}^{p}}{p} - \frac{\|u\|_{L^p}^{p}}{p} - \text{Re} \left\langle v - u, |u|^{p-2} u \right\rangle_{L^p, L^{p'}} \right| \]
\[ \leq (p - 1) \|v - u\|_{L^p}^2 \|u\| + |v - u|_{L^p}^{p-2}, \]
that is, \( \nabla_u (p-1)\|u\|_{L^p}^p = |u|^{p-2} u \).

Taking (A1) into account, we have finished the proof of Proposition A1. \( \square \)
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