On Locating Paths in Compressed Cardinal Trees

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Abstract. A compressed index is a data structure representing a text within compressed space and supporting fast count and locate queries: given a pattern, count/return all positions where the pattern occurs. The first compressed indexes date back twenty years and operate within a space bounded by the text’s entropy. Entropy, however, is insensitive to long repetitions. For this reason, in recent years more powerful compressed indexes have emerged; these are based on the Lempel-Ziv factorization, the run-length Burrows-Wheeler Transform (BWT), context-free grammars and, more recently, string attractors (combinatorial objects generalizing those compressors). Labeled trees add a whole new dimension to the problem: one needs not only to compress the labels, but also the tree’s topology. On this side, less is known. Jacobson showed how to represent the topology of a tree with \( n \) nodes in \( 2^n + o(n) \) bits of space (succinct) while also supporting constant-time navigation queries. Ferragina et al. presented the first entropy-compressed labeled tree representation (the XBWT) able to count, but not locate, paths labeled with a given pattern. Grammars and the Lempel-Ziv factorization have been extended to trees, but those representations do not support indexing queries. In this paper, we show for the first time how to support the powerful locate queries on compressed trees. We start by proposing suitable generalizations of run-length BWT, high-order entropy, and string attractors to cardinal trees (tries). We show that the number \( r \leq n \) of XBWT-runs upper-bounds the size of the smallest tree attractor and lower-bounds the trie’s high-order worst-case entropy \( H_{wc}^{\infty} \). We finally present the first trie index able to locate the pre-order identifier of all nodes reached by a path labeled with a given pattern. Our index locates path occurrences in constant time each and takes \( O(r) \subseteq O(H_{wc}^{\infty}) \) words of space on top of the full-fledged succinct topology of \( 2n + o(n) \) bits. The locate time per occurrence is optimal and the topology further supports advanced navigation queries in constant time.

Keywords: Tree Compression · Tree Indexing · Burrows-Wheeler Transform
1 Introduction

A compressed index is a data structure representing a text $T$ within compressed space and supporting fast *count* and *locate* queries: given a pattern, count/return all positions in $T$ where the pattern occurs $[28]$. The first compressed indexes date back twenty years and operate within a space bounded by the text’s empirical entropy $[7, 14]$. Entropy, however, is insensitive to long repetitions: entropy-compressing $T \cdot T$ yields an archive twice as big as the entropy-compressed $T$ $[22]$. For this reason, in recent years more powerful compressed indexes have emerged; these are based on the Lempel-Ziv factorization $[22]$, the run-length Burrows-Wheeler Transform (BWT)$[11, 25, 32]$, context-free grammars $[5]$ and, more recently, string attractors $[20, 29]$ (combinatorial objects generalizing those compressors). Compressed text indexes have had a dramatic impact in domains such as bioinformatics $[23, 24]$. The recent rise of massive repetitive datasets, however, advocates for compressed indexes able to handle even more structured data such as labeled (repetitive) graphs $[6]$. Already on trees, the state of the art is not as mature as on strings. While some of the above compression techniques have been extended to trees $[3, 12, 18]$, much less is known about tree indexing within compressed space. Jacobson $[16]$ showed how to represent the topology of a tree with $n$ nodes in worst-case optimal $2n + o(n)$ bits of space while also supporting basic navigation queries in constant time. Subsequent works $[30, 31]$ added many navigation functionalities to this (as well as other) succinct tree representation. Ferragina et al. $[8]$ have been the first to tackle the tree indexing problem: their *XBW Transform* (XBWT in the following) stores any labeled tree within entropy-compressed space while also supporting fast *count* queries on it. Crucially, they did not discuss how to *locate* paths labeled with a given pattern. Subsequent works extended this technique to de Bruijn graphs $[1]$, variable-order de Bruijn graphs $[33]$ and, finally, Wheeler graphs $[9]$ (labeled graphs whose nodes can be prefix-sorted). Among these works, Sirén et al. $[33]$ have been the only ones to consider *locate* queries on structures more complex than strings. However, their solution samples the identifiers of nodes with more than one successor. In the worst case, this is asymptotically equivalent to sampling the identifier of every node using $\Theta(n \log n)$ bits of space.

**Our Contributions** In this paper, we show for the first time how to support the powerful locate queries on *compressed* trees: to return the pre-order identifier of all nodes reached by a path labeled with a given pattern $P$. To begin with, we extend well-established tools for string compression to cardinal trees (tries). We first generalize the notion of run-length encoding to the XBWT of a trie. We proceed by showing that the number $r$ of runs in the XBWT is a strong compressibility measure as it lower bounds the $k$-th order worst-case entropy $H_{wc}^k$ of the trie. As observed above, however, entropy does not capture long repetitions. We therefore extend the concept of string attractors $[20]$ to labeled trees, and show that the XBWT induces a tree attractor of size $r$. We furthermore relate $r$ with the size $\omega$ of the smallest equivalent Wheeler automaton $[1, 9]$ by showing $r \leq \omega$. We finally turn our attention to locate queries on compressed tries, motivated by the fact that the XBWT of Ferragina et al. $[8]$ cannot locate. We first observe that the standard sampling mechanism of compressed suffix arrays can easily be extended to the XBWT. This simple solution, however, requires also the succinct tree topology of $2n + o(n)$ bits and a sampling of $O((n/t) \log n)$ bits *on top of the XBWT* to support $O(t)$-time locate queries. Our main contribution is to show how to fit a more advanced locate machinery, as well as the trie’s labels, within $O(r) \subseteq O(H_{wc}^k)$ words of space *on top of the full-fledged succinct topology representation of $2n + o(n)$ bits* of Sadakane and Navarro $[30]$. Our index locates path occurrences in optimal constant time each.

The first ten pages of this manuscript contain a concise description of all our contributions and can be followed by non-specialists. The proofs of all claims can be found in the appendix.
2 Definitions

We work with edge-labeled cardinal trees \(T = (V, E)\) with \(n\) nodes and labels from alphabet \(\Sigma = \{1, \ldots, \sigma\}\) of size \(\sigma\) ordered by a total order \(<\). We extend \(<\) to \(\Sigma^*\) using the co-lexicographic order (i.e. the strings’ characters are compared right-to-left). Given a string \(S\), the number \(\text{rle}(S)\) of equal-letter runs of \(S\) is the number of maximal unary substrings of \(S\) (for example, \(\text{rle}(aaabbccaaa) = 4\)).

We identify tree nodes by their pre-order identifier \(\hat{u}\); node 1 is the root. Function \(\pi(\hat{u})\) returns the parent of node \(\hat{u}\), and \(\lambda(\hat{u})\) indicates the label of the edge \((\pi(\hat{u}), \hat{u})\). For the root, we take \(\lambda(1) = \#\), where \# is the lexicographically-smallest character in \(\Sigma\), not labeling any edge. Notation \(\lambda(\Pi)\) denotes the string \(\lambda(\hat{u}) \cdots \lambda(\hat{v})\) labeling path \(\Pi = \hat{u} \rightsquigarrow \hat{v}\). We assume the alphabet to be effective: for each \(c \in \Sigma\), there exists \(\hat{u}\) such that \(\lambda(\hat{u}) = c\). Function \(\text{child}_c(\hat{u})\) returns the child of \(\hat{u}\) reached by following the edge labeled \(c\). If \(\hat{u}\) does not have such a child, then \(\text{child}_c(\hat{u}) = \perp\). The children of each node are implicitly sorted according to their incoming labels. Function \(\text{out}(\hat{u})\) returns the (possibly empty) set \(\{c: \text{child}_c(\hat{u}) \neq \perp\}\) of the characters labeling the outgoing edges of \(\hat{u}\). Let \(U \subseteq V\). The forest \(\mathcal{T}(U)\) is the set of the subtrees of \(\mathcal{T}\) induced by \(U\). We say that \(\mathcal{T}(U)\) is a subtree if it is connected. A subtree \(\mathcal{T}(U)\) with root \(\hat{u}\) is complete if \(U\) contains all descendants of \(\hat{u}\) in \(\mathcal{T}\). The equivalence relation \(\approx\) denotes isomorphism between (the complete subtrees rooted in) two nodes: \(\hat{u} \approx \hat{v}\) if and only if, for each \(c \in \Sigma\), \(\text{child}_c(\hat{u}) \approx \text{child}_c(\hat{v})\), where \(\hat{u} \approx \perp\) if and only if \(\hat{u} = \perp\). In some of our results we will treat trees as deterministic finite state automata (DFA), with the root being the initial state and all states being final. We work in the word RAM model with words of size \(w = \Theta(\log n)\) bits. The space of our data structures will be given either in words or bits; in all cases we will clearly specify which unit of measurement we use.

3 Trie Compression

In this section we introduce generalizations of the run-length encoded Burrows-Wheeler transform and of string attractors to tries and relate these combinatorial objects with the trie’s high-order worst-case entropy and the smallest Wheeler automaton [1,9] equivalent to the trie.

3.1 The Run-Length encoded XBWT

The XBWT is based on the idea of sorting the \(n\) tree’s nodes by the unique order \(<\) such that the following two Wheeler properties hold: (i) if \(\lambda(\hat{u}) < \lambda(\hat{v})\) then \(\hat{u} < \hat{v}\), and (ii) if \(\lambda(\hat{u}) = \lambda(\hat{v})\) and \(\pi(\hat{u}) < \pi(\hat{v})\), then \(\hat{u} < \hat{v}\). Wheeler graphs [9] generalize this idea to labeled graphs admitting such an ordering. The set of Wheeler graphs is a strict superset of the set of all labeled trees. We shall call \(<\) the Wheeler order of the nodes, which for trees always exists [8]: \(\hat{u} < \hat{v}\) iff \(\lambda(1 \rightsquigarrow \hat{u}) < \lambda(1 \rightsquigarrow \hat{v})\). Let \(\hat{u}_1 < \ldots < \hat{u}_n\) be the sorted sequence of nodes. With \(<_{\text{pred}}\) we denote the predecessor relation with respect to the Wheeler order: \(\hat{u}_i <_{\text{pred}} \hat{u}_j\) if and only if \(j = i + 1\).

The subscripts in nodes \(\hat{u}_1 < \ldots < \hat{u}_n\) are the second node representation we will use in the paper: the Wheeler-order representation \(\hat{u}\) of (pre-order) node \(\hat{v}\) is precisely \(\hat{u} = i\).

We now give a definition of the XBW Transform that (on tries) is completely equivalent to the original one given by Ferragina et al. [8]. See Figures 1 and 2 for a running example.

**Definition 1** ([8]). \(XBWT(\mathcal{T}) = \text{out}(\hat{u}_1), \text{out}(\hat{u}_2), \ldots, \text{out}(\hat{u}_n)\).

For brevity, we shall simply write \(XBWT\) instead of \(XBWT(\mathcal{T})\). The original trie \(\mathcal{T}\) can be reconstructed from \(XBWT\) [8].
Fig. 1. Running example used throughout the paper. This repetitive cardinal tree has \( n = 26 \) nodes (numbered in pre-order) and labels from the alphabet \( \Sigma = \{a, b, c\}\). The trie’s topology, the colored nodes and the orange dashed edges are a concise representation of our compressed trie index discussed in Section 4. These components are discussed more in detail in the caption of Figure 2.

Fig. 2. XBWT (Subsection 3.1). First four rows: (1) the Wheeler order and (2) the pre-order identifiers of the nodes of the tree in Figure 1 (3) the incoming label \( \lambda(\hat{u}) \) of each node \( \hat{u} \in V \), and (4) for each node, the characters labeling its outgoing edges. Row (4) is the XBWT of the tree. RL-XBWT (Subsection 3.1). Fifth row: the transform has \( r' = 8 \) blocks and \( r = 8 \) runs. For each block, we store (1) the set \( \text{DEL} \) of characters deleted w.r.t. the previous block (colored in red in the fourth row: these are the \( c \)-runs), (2) the set \( \text{ADD} \) of characters added w.r.t. the previous block, and (3) the length \( \ell \) (number of nodes) of the block. Wheeler Automata (Subsection 3.3). In the last three rows of the table, we show the three quotients \( V/\equiv_r \) (RL-XBWT blocks), \( V/\approx_r \) (convex isomorphism), and \( V/\equiv_{\approx_c} \) (states of the minimum equivalent WDFA, by 1 Thm 4.1). The notation \( \hat{j} \) indicates the equivalence class of node \( j \). Tree attractors (Subsection 3.4). By Theorem 3 the red edges (fourth row) form a tree attractor: every subtree has an isomorphic occurrence crossing a red edge. Locate (Section 4). A red node has different outgoing labels w.r.t. its Wheeler-successor. A blue node has a different incoming label w.r.t. its Wheeler-successor. Orange dashed arrows in Figure 1 represent the sampled values of the Wheeler-successor function: \( \phi(\hat{u}_i) = \hat{u}_{i+1} \). These arrows depart from red nodes and from nodes reached by red edges.
The locate problem can be naturally generalized from strings to labeled trees as follows: given a pattern $P$, return the pre-order identifier $\hat{u}$ of all nodes such that $\lambda(1 \sim \hat{u})$ is suffixed by $P$. In such a case, we will say that $\hat{u}$ is reached by a path labeled $P$. Plugging up-to-date data structures [2] in the XBWT of Ferragina et al. [8], this structure takes $2n + o(n)$ bits on top of the entropy-compressed labels and counts nodes reached by a path labeled with a pattern $P \in \Sigma^m$ in $O(n \log \log n, \sigma)$ time. We observe that it is straightforward to support also locate queries on the XBWT by extending the standard solution (based on sampling) used in compressed suffix arrays:

**Lemma 1.** For any $1 \leq t \leq o(n/\log n)$, the XBWT can be augmented with additional $2n + o(n) + O((n/t) \log n)$ bits so that, after counting, the pre-order identifiers of all occ nodes reached by a path labeled with a pattern $P \in \Sigma^m$ can be returned in $O(occ \cdot t)$ time.

The above simple solution has two issues. First, it represents the topology twice. The reason is that the topology stored in the XBWT works in Wheeler order rather than pre-order, and we need the latter to locate. Second, the trade-off $t$ allows obtaining either a fast but large index or a slow and small index. The goal of this paper is to solve both these issues.

Consider Definition 1. We say that $1 \leq i < n$ is a c-run break, with $c \in \Sigma$, if $c \in out(\hat{u}_i)$ and either (i) $i = n$ or (ii) $c \notin out(\hat{u}_{i+1})$. When $c$ is not specified, we simply say that $i$ is a run-break (for some $c$). Let $r_c(\mathcal{T})$ be the number of c-run breaks. We define the number $r(\mathcal{T})$ of XBWT runs as $r(\mathcal{T}) = \sum_{c \in \Sigma} r_c(\mathcal{T})$. For brevity, in the following we will omit $\mathcal{T}$ and simply write $r_c$ and $r$. The fourth row of Figure 2 shows run breaks in red. In the figure, we have $r = 8$. If $\mathcal{T}$ is a path (that is, a string), then $r$ coincides with the number of equal-letter runs in the BWT of $\mathcal{T}$.

In the following definition we present the lun-length (RL) encoded XBWT. See Figure 2 for a running example. Importantly, note the distinction between XBWT runs and blocks.

**Definition 2.** The RL-XBWT of a trie $\mathcal{T}$ is the sequence of $r'$ triples $\langle (ADD_1, DEL_1, \ell_1) \rangle_{i=1}^{r'}$ obtained as follows. Break the sequence $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n$ into maximal blocks such that all nodes in the same block $\hat{u}_i, \hat{u}_{i+1}, \ldots, \hat{u}_{i+\ell}$ have the same set $out(\hat{u}_j)$ of outgoing labels, for $i \leq j < i + \ell$. Only for the sake of the following definition, let $out(\hat{u}_0) = \emptyset$. The $q$-th block, starting with node $\hat{u}_q$ is then encoded with the triple $(ADD_q, DEL_q, \ell_q)$, where $ADD_q = out(\hat{u}_q) - out(\hat{u}_{q-1})$, $DEL_q = out(\hat{u}_{q-1}) - out(\hat{u}_q)$, and $\ell_q$ is the length (number of nodes) of the block.

The representation of Definition 2 is sufficient to reconstruct the XBWT: $out(\hat{u}_q) = (out(\hat{u}_{q-1}) - DEL_q) \cup ADD_q$. In the next lemma we show that our representation can be stored in $O(r)$ space:

**Lemma 2.** The RL-XBWT representation takes $O(r)$ words to be stored.

### 3.2 Relation with the Trie’s Entropy

Several notions of empirical entropy for trees have been considered in the literature so far. Ferragina et al. [8] define the high-order empirical entropy of the tree’s labels. This notion, however, does not take into account the tree’s topology and is defined for arbitrary labeled trees. Jansson et al. [17] and Hucke et al. [15] define tree entropy measures taking into account also the topology. Also their notions, however, work for arbitrary trees. The worst-case entropy $C(n, \sigma)$ of a trie considered by Raman et al. [31] is the measure we consider as starting point in this section. This quantity is defined as $C(n, \sigma) = \log_2(|U_{n,c}|)$, where $U_{n,c}$ is the universe containing all tries with $n$ nodes on an alphabet of cardinality $\sigma$. It is well known that the worst-case entropy has a strong relation with the
The notion of empirical entropy \cite{21}. For example, on binary strings the two measures differ at most by an additive \(O(\log n)\) term \cite{27}. Measure \(C(n, \sigma)\) is still too weak for our purposes: ideally, we would like to model also character frequencies (that is, zero-order compression) and, ultimately, high-order compression. We now show how to strengthen \(C(n, \sigma)\) by taking into account these aspects.

As in previous works \cite{8, 13}, we assume a model where the string \(\pi_k[\hat{u}]\) of the last \(k\) labels seen on the path connecting the root to a node \(\hat{u}\) is a good predictor for the set \(\text{out}(\hat{u})\). More formally, \(\pi_0[\hat{u}] = \epsilon\) (empty string), \(\pi_1[\hat{u}] = \lambda(\hat{u})\) and \(\pi_k[\hat{u}] = \pi_{k-1}[\pi(\hat{u})] \cdot \lambda(\hat{u})\) for \(k > 1\). For this to be well-defined, we also set \(\pi(1) = 1\) (1 is the root) to pad with \(\lambda(1) = \#\) the contexts of nodes at depth less than \(k\). Let \(X = X_1, \ldots, X_{n'}\) be a sequence of \(n'\) subsets of \(\Sigma\) such there are \(n'_c = \sum_{i=1}^{n'} |X_i \cap \{c\}|\) occurrences of character \(c\) in the sequence, for all \(c \in \Sigma\). The worst-case entropy \(H_{wc}(X)\) of \(X\) is defined as the logarithm of the size of the universe containing all set sequences of length \(n'\) having the same characters’ frequencies as \(X\) (see Navarro \cite{27}): \(H_{wc}(X) = \log_2 \left( \prod_{c \in \Sigma} \binom{n'}{n'_c} \right) = \sum_{c \in \Sigma} \log_2 \left( \frac{n'_c}{n'} \right) \). In the following we will simply write \(H_{wc}\) when the characters’ frequencies are clear from the context. Note that \(H_{wc}(\text{XBWT}) \leq C(n, \sigma)\), since the former fixes the frequencies of each character while the latter allows any frequency combination summing up to \(n - 1\).

At this point, we adopt the approach of Ferragina et al. \cite{8}. We model high-order compression by defining the sequence of sets \(\text{cover}(\rho) = \langle \text{out}(\hat{u}_i) \rangle_i : \pi_k[\hat{u}_i] = \rho\) and define:

**Definition 3.** \(H_{wc}^k(\mathcal{T}) = \sum_{\rho \in \Sigma^k} H_{wc}(\text{cover}(\rho))\)

In the following we will simply write \(H_{wc}^k\) when \(\mathcal{T}\) is clear from the context. Clearly, \(H_{wc}^k \leq H_{wc}\) since the former fixes the characters’ frequencies for each context \(\rho\).

The next step is to relate \(r\) with \(H_{wc}^k\). On strings, it is well known that \(r\) lower-bounds the \(k\)-th order empirical entropy \cite{25}. We show that this is the case also for the worst-case entropy of tries.

**Theorem 1.** The number \(r\) of XBWT runs is always at most \(H_{wc}^k + \sigma^{k+1}\) for any \(k \geq 0\).

As a consequence, we have (see Appendix \[D\] for a proof):

**Corollary 1.** For any \(0 < \alpha < 1\) and \(0 \leq k \leq \max\{0, \alpha \log_\sigma n - 1\}\) it holds \(r \leq 2H_{wc}^k + o(n/\log^c n)\) for any constant \(c > 0\).

We will use Corollary \[1\] to express the size of our structures as a function of the trie’s worst-case entropy. However, we stress out that measure \(r\) captures large self-repetitions \cite{32} and is therefore a better measure for repetitive tries. The empirical high-order entropy (per symbol) \(H_k\) does not capture long repetitions: on a text \(T\), the relation \(H_k(T \cdot T) \geq H_k(T)\) always holds \cite{22}.

### 3.3 Relations with Wheeler Automata

The smallest Wheeler Deterministic Finite-state Automaton (WDFA) \[1\] equivalent to \(\mathcal{T}\) can also be considered as a compressed representation of the trie. In particular, it is the smallest automaton equivalent to the trie for which the Wheeler properties (i) and (ii) hold \[1\, 9\]. In this section we show that these combinatorial objects and the XBWT are deeply related. We start by introducing two equivalence relations between nodes that will play a fundamental role throughout the paper. We write \(\hat{u} \equiv^\pi \hat{v}\) if and only if \(\text{out}(\hat{u}) = \text{out}(\hat{v})\). Note that the following property holds: for \(i < n\), we have \(\hat{u}_i \not\equiv^\pi \hat{u}_{i+1}\) if and only if \(i\) is a run break. The second equivalence relation is a refinement of \(\equiv^\pi\) and captures a slightly stronger relation than isomorphism: we write \(\hat{u} \equiv \hat{v}\) if and only if
\( \lambda(\hat{u}) = \lambda(\hat{v}) \) and \( \hat{u} \approx \hat{v} \). Clearly, \( \equiv \) is a refinement of \( \equiv^r \): if \( \hat{u} \equiv \hat{v} \), then \( \hat{u} \equiv^r \hat{v} \). The convex closure \( \equiv< \) of \( \equiv \) with respect to the Wheeler order \( < \) is defined as follows: \( \hat{u}_i \equiv< \hat{u}_j \) if and only if \( \hat{u}_i \equiv \hat{u}_j \) and \( \forall k (\min\{i, j\} < k < \max\{i, j\} \Rightarrow \hat{u}_k \equiv \hat{u}_i) \). The convex closures \( \equiv< \) and \( \equiv< \) of \( \equiv^r \) and \( \approx \) are defined analogously. Note that the equivalence classes of \( \equiv< \) correspond to the RL-XBWT blocks. Note also that (see Figure 2) \( \equiv<, \equiv<, \) and \( \equiv< \) are refinements of \( \equiv, \equiv^r \) and \( \approx \), respectively, and \( \equiv< \) is a refinement of \( \approx< \), which in turn is a refinement of \( \equiv< \). Relation \( \equiv< \) has been introduced for the first time (with the symbol \( \equiv_w \)) by Alanko et al. \( \textbf{[1]} \), who prove (\([1, \text{Thm 4.1}]\)) that the quotient automaton \( T/_{\equiv<} \) is the minimum WDFA equivalent to \( T \). We show that \( r \) is a lower bound to the size (number of edges) \( \omega \) of such automaton (proof in Appendix \( \textbf{E} \)):

**Theorem 2.** \( r \leq \omega \).

An intriguing consequence of Theorem 2 is that one can reduce the problem of indexing any acyclic Wheeler automaton \( A \) to the problem of indexing (the run-length XBWT of) the equivalent tree within \( O(r) \) words of space: the resulting index will not be larger than \( A \).

### 3.4 Tree Attractors

Let \( S \in \Sigma^n \) be a string. A string attractor \( \textbf{[20]} \) is a set \( \Gamma \subseteq [1, n] \) of the string’s positions such that any substring \( S[i, j] \) has at least one occurrence \( S[i', j'] = S[i, j] \) such that \( \Gamma \cap [i', j'] \neq \emptyset \). String attractors generalize most known dictionary compressors (for example, the run-length BWT, Lempel-Ziv 77, and straight-line programs), in the sense that a compressed representation of size \( \alpha \) can be turned into a string attractor of size \( O(\alpha) \). Conversely, most compressibility measures can be upper-bounded by \( O(\gamma \cdot \text{polylog} \ n) \), where \( \gamma \) is the size of the smallest string attractor \( \textbf{[10, 19, 20]} \).

Since string attractors capture the repetitiveness of a string, it is natural to try to generalize them to trees. We now propose such a generalization and exhibit a tree attractor of size \( r \).

**Definition 4.** Let \( \mathcal{T} = (V, E) \). A tree attractor is a subset \( \Gamma \subseteq E \) such that any subtree \( \mathcal{T}(U) \), with \( U \subseteq V \), has at least one isomorphic occurrence \( \mathcal{T}(U') = (U', E') \) such that \( \Gamma \cap E' \neq \emptyset \).

Let \( \mathcal{T} = (V, E) \). We define \( \Gamma^r = \{(\hat{u}_i, \hat{v}) \in E : \exists c \in \Sigma \mid i \text{ is a } c-\text{run break and } \lambda(\hat{v}) = c\} \).

In Figure 1 the edges of \( \Gamma^r \) are colored in red. We now show that \( \Gamma^r \) is a tree attractor. Before proving our result, we need the following Lemma (that will turn out very useful also later):

**Lemma 3.** If \( \hat{u} \prec_{\text{pred}} \hat{v} \) then \( \text{child}_{c}(\hat{u}) \prec_{\text{pred}} \text{child}_{c}(\hat{v}) \) for all \( c \in \text{out}(\hat{u}) \cap \text{out}(\hat{v}) \).

**Theorem 3.** \( \Gamma^r \) is a tree attractor of size \( |\Gamma^r| = r \).

Let \( \gamma \) be the size of the smallest tree attractor and \( \omega \) be the number of edges of the smallest Wheeler DFA equivalent to \( \mathcal{T} \). By Theorems 2 and 3 we obtain \( \gamma \leq r \leq \omega \).

### 4 Locating Paths on Compressed Tries

In this section we present indexing data structures based on the run-length encoded XBWT. We start with navigation operations that will be needed in our index.

1. Child rank \( cr(\hat{u}, c) \). Given a Wheeler-order node \( \hat{u} = i \) and a label \( c \in \text{out}(\hat{u}_i) \), return the integer \( k \) such that the edge connecting \( \hat{u} \) with its \( k \)-th child is labeled with character \( c \).
2. **Depth** $\text{depth}(\hat{u})$. Return the depth of pre-order node $\hat{u}$ (where the root has depth 0).
3. **Child by rank** $\text{cbr}(\hat{u}, k)$. Return the $k$-th child of pre-order node $\hat{u}$.
4. **Sibling rank** $\text{sr}(\hat{u})$. Return the integer $k$ such that $\hat{u}$ is the $k$-th child of its parent.
5. **Lowest Common Ancestor** $\text{LCA}(\hat{u}, \hat{v})$ of two pre-order nodes $\hat{u}$ and $\hat{v}$.
6. **Level Ancestor Queries** $\text{LAQ}(\hat{u}, \ell)$. Given $\ell \geq 1$, return $\pi(\ell)(\hat{u})$, that is, the parent function $\pi$ applied $\ell$ times to pre-order node $\hat{u}$.
7. **Isomorphic Descendant** $\text{ISD}(\hat{u}, \hat{v}, \hat{u}')$. Let $\hat{v}$ be a descendant of $\hat{u}$ reached by following a path $\hat{u} \rightarrow \hat{w} \sim \hat{v}$ with $\alpha = \lambda(\hat{w} \sim \hat{v})$, and let $\hat{u}' \approx \hat{u}$ be a node isomorphic to $\hat{u}$. This operation returns the descendant $\hat{v}'$ of $\hat{u}'$ reached by following the path $\hat{u}' \rightarrow \hat{w}' \sim \hat{v}'$ with $\lambda(\hat{w}' \sim \hat{v}') = \alpha$.
8. **Isomorphic Child** $\text{ISC}(\hat{u}_i, k)$. Given a pre-order node $\hat{u}_i$, $i < n$, such that $\text{out}(\hat{u}_i) \neq \text{out}(\hat{u}_{i+1})$ and given an integer $1 \leq k \leq |\text{out}(\hat{u}_i)|$, let $c = \lambda(\text{cbr}(\hat{u}_i, k))$ be the $k$-th smallest label in $\text{out}(\hat{u}_i)$. Assuming that $c \in \text{out}(\hat{u}_{i+1})$, this function returns the integer $t$ such that $c = \lambda(\text{cbr}(\hat{u}_{i+1}, t))$.

**Lemma 4.** There is a data structure taking $O(r \log n) + o(n)$ bits of space and supporting operation $\text{cr}(\hat{u}, c)$ in $O(\log \sigma)$ time.

For the remaining operations, we need to store explicitly the topology. Navarro and Sadakane [30] show how to support operations 2-6 in $O(1)$ time using $2n + o(n)$ bits of space. We show:

**Lemma 5.** The structure of Navarro and Sadakane [30] supports also $\text{ISD}(\hat{u}, \hat{v}, \hat{u}')$ in $O(1)$ time.

**Lemma 6.** Operation $\text{ISC}(\hat{u}_i, k)$ can be supported in $O(1)$ time and $O(r \log n) + o(n)$ bits of space.

Our strategy for supporting efficient locate queries on the XBWT is a nontrivial generalization to tries of the $r$-index data structure [11] (a locate machinery on strings). Let $[\ell, \bar{r}]$ be the range of Wheeler-order nodes reached by a path labeled $P$. We divide the problem of answering locate queries into two sub-problems. (1) **Toehold:** compute $[\ell, \bar{r}]$ and $\hat{u}_\ell$. (2) **Climb:** evaluate function $\phi(\hat{u}_i) = \hat{u}_{i+1}$ for any $i < n$. The combination of (1) and (2) yields $\hat{u}_\ell, \ldots, \hat{u}_\bar{r}$.

The **Toehold** step requires navigating the tree using both node representations. We show:

**Lemma 7.** There is a data structure taking $O(r \log n) + o(n)$ bits of space on top of the succinct tree topology of Navarro and Sadakane [30] that, given a pattern $P \in \Sigma^m$, returns the range $[\ell, \bar{r}]$ of Wheeler-order nodes reached by a path labeled $P$, as well as $\hat{u}_\ell$, in $O(m \log \sigma)$ time.

We now show how to implement the **Climb** step with a constant number of jumps (from $\hat{u}_i$ to $\hat{u}_{i+1}$) on the tree, each taking constant time. We mark nodes in blue, red, or both (colors are not exclusive). A node $\hat{u}_i$, $i < n$, is red if it does not have the same outgoing labels as its Wheeler-successor: $\text{out}(\hat{u}_i) \neq \text{out}(\hat{u}_{i+1})$ (equivalently, $\hat{u}_i \not\equiv_r \hat{u}_{i+1}$). A node $\hat{u}_i$, $i < n$, is blue if it does not have the same incoming label as its Wheeler-successor: $\lambda(\hat{u}_i) \neq \lambda(\hat{u}_{i+1})$. Since $r' \leq 3r$ (Lemma 2) and $\sigma \leq r$, there are $O(r)$ marked nodes in total. Our running example in Figures 1 and 2 shows how nodes are colored according to the above definitions. Let $i < n$. By recursively applying Lemma 3 to the descendants of a node, one can easily see the following (see Appendix L):

**Corollary 2.** $\hat{u}_i \not\equiv \hat{u}_{i+1}$ if and only if the complete subtree rooted in $\hat{u}_i$ contains a red node.

The following lemma shows that we can find colored descendants and ancestors in $O(1)$ time:

**Lemma 8.** There is a data structure taking $O(r \log n) + o(n)$ bits of space on top of the succinct tree topology of Navarro and Sadakane [30] and answering the following queries in $O(1)$ time. Given a pre-order node $\hat{u}_i$ with $i < n$:
(a) If \( \hat{u}_i \) is not colored, find a colored node \( \hat{u}_j \neq \hat{u}_i \) in the complete subtree rooted in \( \hat{u}_i \) such that no node on the path from \( \hat{u}_i \) to \( \hat{u}_j \) is colored (except \( \hat{u}_j \)), or report that \( \hat{u}_j \) does not exist.
(b) Find the lowest ancestor \( \hat{u}_j \) of \( \hat{u}_i \) such that the complete subtree rooted in \( \hat{u}_j \) contains a colored node. Note that such a node always exists, since the root is always blue.

Lemma 9. In Lemma 8 (a), if \( \hat{u}_j \) exists then \( \hat{u}_j \) must be red and not blue.

We introduce the notion of adjacent paths:

Definition 5. We say that two paths \( \hat{u}_{i_1} \rightarrow \hat{u}_{i_2} \sim \hat{u}_{i_k} \) and \( \hat{u}_{j_1} \rightarrow \hat{u}_{j_2} \sim \hat{u}_{j_k} \) of the same length \( k \) are adjacent if it holds that \( j_t = i_t + 1 \) for all \( 1 \leq t \leq k \).

In Appendix \( \square \) we prove that uncolored paths have always an adjacent path:

Lemma 10. Let \( II = \hat{u}_{i_1} \rightarrow \hat{u}_{i_2} \sim \hat{u}_{i_k} \), with \( i_j < n \) for some \( 1 \leq j \leq k \), be a path of length \( k \) without blue nodes other than (possibly) \( \hat{u}_{i_j} \) and without red nodes other than (possibly) \( \hat{u}_{i_k} \). Then, \( \hat{u}_{i_{t+1}} \rightarrow \hat{u}_{i_{t+2}} \sim \hat{u}_{i_{t+k}} \) is a path in the tree (adjacent to \( II \)).

We furthermore explicitly store (sample) the value of function \( \phi \) on the following nodes: (1) on each colored node \( \hat{u}_i \), we explicitly store \( \phi(\hat{u}_i) = \hat{u}_{i+1} \). We call these \( \phi \)-samples of type 1. (2) Let \( \hat{u}_i \) be a red node, and let \( c \) be such that \( c \in \text{out}(\hat{u}_i) \) and \( c \notin \text{out}(\hat{u}_{i+1}) \). Let moreover \( \hat{u}_j = \text{child}_c(\hat{u}_i) \). If \( j < n \), then we explicitly store \( \phi(\hat{u}_j) = \hat{u}_{j+1} \) on node \( \hat{u}_j \). We call these \( \phi \)-samples of type 2. Note that a \( \phi \)-sample could be both of type 1 and 2 (for example, see Figure \( \square \) node 7). Since samples of type 1 are stored only on colored nodes and samples of type 2 correspond to run breaks, in total we explicitly store \( O(r) \) \( \phi \)-samples. Figure \( \square \) depicts these samples as orange dashed arrows.

We are now ready to show how to compute \( \phi(\hat{u}_i) = \hat{u}_{i+1} \) for any \( 1 \leq i < n \). We break our algorithm in cases. In Appendix \( \square \) we discuss examples of all cases based on the trie of Figure \( \square \).

**Case 1: the complete subtree rooted in \( \hat{u}_i \) contains colored nodes.** See Figure \( \square \) (left). If \( \hat{u}_i \) is colored (red, blue, or both), then \( \phi(\hat{u}_i) \) is explicitly stored. Otherwise, we use Lemma 8 (a) to find a colored node \( \hat{u}_j \neq \hat{u}_i \) in the complete subtree rooted in \( \hat{u}_i \) such that no node other than \( \hat{u}_j \) on the path \( II = \hat{u}_i \sim \hat{u}_j \) is colored. In particular, by Lemma 9 \( \hat{u}_j \) must be red and no node on the path is blue. Since \( II \) enjoys this property and \( i < n \), we can apply Lemma 10 to it and obtain that \( \hat{u}_{i+1} \sim \hat{u}_{j+1} \) is a valid path with \( t = \text{depth}(\hat{u}_j) - \text{depth}(\hat{u}_i) \) edges and it is adjacent to \( II \). We find \( \hat{u}_{j+1} = \phi(\hat{u}_j) \), which is stored explicitly since \( \hat{u}_j \) is red. Finally, we jump to \( \hat{u}_{i+1} \) with a level ancestor query by \( t \) levels from \( \hat{u}_{j+1} \). More formally, we obtain:

\[
\phi(\hat{u}_i) = \hat{u}_{i+1} = \text{LAQ}(\phi(\hat{u}_j), \text{depth}(\hat{u}_j) - \text{depth}(\hat{u}_i)).
\]

We note that on trees that are simple paths (i.e. strings) it is always the case that \( \hat{u}_i \) has a red descendant (that is, the unique leaf). It follows that the above equation is always applied when the tree is a string. In fact, in this case the equation reduces to what is implemented in the \textit{r-index} data structure \( \square \). On trees, however, things are more complicated: it is not always the case that \( \hat{u}_i \) has colored descendants. This case is treated below.

**Case 2: the complete subtree rooted in \( \hat{u}_i \) does not contain any colored node.** See Figures \( \square \) (right) and \( \square \). The idea is to navigate upwards instead of downwards as done in Case 1. We first find, using Lemma 8 (b), the lowest ancestor \( \hat{u}_j \) of \( \hat{u}_i \) such that the complete subtree rooted in \( \hat{u}_j \) contains a colored node. We further distinguish two sub-cases, depending on whether \( \hat{u}_j \) is red or not.
\textbf{Case 2.1:} \( \hat{u}_j \) is not red. See Figure 3 (right). Consider the path \( \hat{u}_j \rightarrow \hat{u}_k \sim \hat{u}_i \), where \( \hat{u}_k \) is child of \( \hat{u}_j \) on the path. Index \( k \) might coincide with \( i \); in this case, the path is simply \( \hat{u}_j \rightarrow \hat{u}_i \). Note that \( i < n \) and that no node \( \hat{v} \) in this path is colored except, possibly, \( \hat{u}_j \) (which might be blue), otherwise we would have chosen \( \hat{v} \) in place of \( \hat{u}_i \). Then, we can apply Lemma 10 and obtain that \( \hat{u}_j + 1 \rightarrow \hat{u}_k + 1 \sim \hat{u}_i + 1 \) must also be a path. In particular, \( j < n \).

Next, we show that we can retrieve \( \phi(\hat{u}_j) = \hat{u}_j + 1 \) in constant time. Since \( \hat{u}_j \) is either not colored or blue, then (by definition of \( \hat{u}_j \)) there must be a colored node (possibly \( \hat{u}_j \) itself) in the subtree rooted in \( \hat{u}_j \). Then, since \( j < n \) we can apply Case 1 and find \( \phi(\hat{u}_j) = \hat{u}_j + 1 \) in constant time.

Let \( t = \text{depth}(\hat{u}_i) - \text{depth}(\hat{u}_j) \). We compute \( \hat{u}_k = \text{LAQ}(\hat{u}_i, t - 1) \) and get its rank \( q \) among its siblings with \( q = \text{sr}(\hat{u}_k) \). Since \( \hat{u}_j \) is not red, we have that \( \hat{u}_j \equiv \hat{u}_j + 1 \): the two nodes have the same outgoing labels. Since \( \hat{u}_k \) is not blue, we have \( \lambda(\hat{u}_k) = \lambda(\hat{u}_k + 1) \). These two observations imply that \( \hat{u}_k + 1 \) is the \( q \)-th children of \( \hat{u}_j + 1 \) as well: we compute it as \( \hat{u}_k + 1 = \text{cbr}(\hat{u}_j + 1, q) \). If \( t = 1 \) then \( \hat{u}_k + 1 \) coincides with \( \hat{u}_i + 1 \) and we are done. Otherwise, since \( \hat{u}_j \) is the lowest ancestor of \( \hat{u}_i \) such that the complete subtree rooted in \( \hat{u}_j \) contains a colored node, the subtree rooted in \( \hat{u}_k \) does not contain any colored node. By Corollary 2 we obtain \( \hat{u}_i \approx \hat{u}_k + 1 \): the two complete subtrees are isomorphic. But then, we can finally find \( \hat{u}_i + 1 \) with an isomorphic descendant query: \( \hat{u}_i + 1 = \text{ISD}(\hat{u}_k, \hat{u}_i, \hat{u}_k + 1) \).

\textbf{Case 2.2:} \( \hat{u}_j \) is red. See Figure 4. Consider the path \( \hat{u}_j \rightarrow \hat{u}_k \sim \hat{u}_i \), where \( \hat{u}_k \) is child of \( \hat{u}_j \) on the path (\( k \) might coincide with \( i \)); in this case, the path is simply \( \hat{u}_j \rightarrow \hat{u}_k \). Let \( t = \text{depth}(\hat{u}_i) - \text{depth}(\hat{u}_j) \). We find \( \hat{u}_k = \text{LAQ}(\hat{u}_i, t - 1) \). We distinguish two sub-cases.

\textbf{Case 2.2.1:} \( \phi(\hat{u}_k) \) is a \( \phi \)-sample of type 2. See Figure 4 (left). Then, we retrieve \( \hat{u}_k + 1 = \phi(\hat{u}_k) \) in constant time. Since \( \hat{u}_j \) is the lowest ancestor of \( \hat{u}_i \) such that the complete subtree rooted in \( \hat{u}_j \) contains a colored node, the subtree rooted in \( \hat{u}_k \) does not contain any colored node. By Corollary 2 this implies that \( \hat{u}_k \approx \hat{u}_k + 1 \): the two complete subtrees are isomorphic. But then, we can find \( \hat{u}_i + 1 \) with an isomorphic descendant query: \( \hat{u}_i + 1 = \text{ISD}(\hat{u}_k, \hat{u}_i, \hat{u}_k + 1) \).

\textbf{Case 2.2.2:} \( \phi(\hat{u}_k) \) is not a \( \phi \)-sample of type 2. See Figure 4 (right). Since \( i < n \) and \( \hat{u}_j \sim \hat{u}_i \) does not contain colored nodes then by Lemma 10 \( \hat{u}_k + 1 \sim \hat{u}_i + 1 \) is a path in the tree. Since \( \hat{u}_k \prec \hat{u}_k + 1 \) and \( \hat{u}_k \) is not blue, we have that \( \lambda(\hat{u}_k) = \lambda(\hat{u}_k + 1) \). In particular, \( \lambda(\hat{u}_k + 1) \neq \# \) so \( \hat{u}_k + 1 \) is not the root. Let \( \hat{u}_j' \) be the parent of \( \hat{u}_k + 1 \). By Wheeler property (ii), \( \lambda(\hat{u}_k) = \lambda(\hat{u}_k + 1) \) and \( \hat{u}_k < \hat{u}_k + 1 \) imply \( \hat{u}_j < \hat{u}_j' \) (otherwise, Wheeler (ii) would force \( \hat{u}_k + 1 < \hat{u}_k \) a contradiction). We can say more: since
φ(ûk) is not a φ-sample of type 2, then it must be the case that ûj ≺predûj′ = ûj+1. Assume, for contradiction, that this were not true, i.e. that there existed a node ˆv such that ûj ≺pred ˆv < ûj′. Let c = λ(ûk). The cases are two: (a) c ∈ out( ˆv). Then, by Wheeler property (iii) it must be the case that ûk < childc( ˆv) < ûk+1, a contradiction. (b) c /∈ out( ˆv). Then, ˆj would be a c-run break and φ(ûk) would be a φ-sample of type 2, a contradiction.

Since j′ = j + 1, i < n, and since none of the nodes in ûk ≺iûi are colored, Lemma 10 implies that ûj → ˆûk ≺û and ˆûj+1 → ˆûk+1 ≺û are (adjacent) paths. Since ûj is red, φ(ûj) = ûj+1 is a φ-sample of type 1 and we can retrieve it in constant time.

Let q = sr(ûk): node ûk is the q-th among the children of its parent ˆûj. Since ûj is red, then out(ûj) /≠ out(ûj+1). This and the fact that λ(ûk) = λ(ûk+1) imply that we can find ˆûk+1 with an isomorphic child operation (Operation 3): ˆûk+1 = cbr(ûj+1, ISC(ûj, q)).

Since ˆûj is the lowest ancestor of ˆûi such that the complete subtree rooted in ˆûj contains a colored node, the subtree rooted in ûk does not contain any colored node. By Corollary 2, this implies that ûi ≈ ûk+1: the two complete subtrees are isomorphic. But then, we can finally find ûi+1 with an isomorphic descendant query: ûi+1 = ISD(ûk, ûi, ûk+1).

Plugging the bound of Corollary 1, we obtain our final result:

**Theorem 4.** Let T be a trie with n nodes whose XBWT has r runs, and let \( H_k^{wc} \) be the k-th order worst-case entropy of T for any \( 0 \leq k \leq \max\{0, a \log \sigma n - 1\} \) and \( 0 < \alpha < 1 \). Our index takes \( 2n + O(n) + O(r \log n) \leq 2n + O(n) + O(H_k^{wc} \log n) \) bits of space and locates the pre-order identifiers of the occ nodes reached by a path labeled with \( P \in \Sigma^m \) in \( O(m \log \sigma + \text{occ}) \) time.

Note that the whole locate machinery, as well as the edges’ labels, fits within compressed space on top of the succinct topology. Moreover, the topology is stored using Navarro and Sadakane’s representation [30], which supports much more advanced navigation queries than the XBWT [8]. We note that improvements in navigation queries [2, 8] on compressed trees will have a direct impact on our index. We leave as an exciting open question whether it is possible to support those queries within \( O(r) \) words of space, thus reducing the size of our index to \( O(r) \subseteq O(H_k^{wc}) \) words in total.
A Proof of Lemma 1

Claim. For any $1 \leq t \leq o(n / \log n)$, the XBWT can be augmented with additional $2n + o(n) + O((n/t) \log n)$ bits so that, after counting, the pre-order identifiers of all $occ$ nodes reached by a path labeled with a pattern $P \in \Sigma^m$ can be returned in $O(occ \cdot t)$ time.

Proof. We exploit the fundamental property (used also in count queries) that characters occur in the same relative order in XBWT and in the sequence $\Lambda = \lambda(\hat{u}_1), \ldots, \lambda(\hat{u}_n)$ (see also Figure 2): the $i$-th occurrence of character $c \in \Sigma$ in the sequence $XBWT = \text{out}((\hat{u}_1), \ldots, \text{out}((\hat{u}_n))$ corresponds to the same edge associated with the $i$-th occurrence of character $c \in \Sigma$ in the sequence $\Lambda$. Using up-to-date rank and select data structures [2], this property allows navigating to the parent of a node (in Wheeler order) in constant time: given $\hat{u}_j$, count the number $q$ of occurrences of $\lambda(\hat{u}_j)$ occurring in $\lambda(\hat{u}_1), \ldots, \lambda(\hat{u}_j)$ (one constant-time rank query on the bitvector representing $\Lambda$, see [8]), and jump to the $q$-th occurrence of $\lambda(\hat{u}_j)$ in XBWT (one constant-time select query using the structures of [2]). See Ferragina et al. [8] for a more detailed discussion of this operation.

Fix a parameter $1 \leq t \leq n$. We use the tree covering procedure described in [13, Sec. 2.1] to decompose $\mathcal{T}$ in $\Theta(n/t)$ sub-trees containing $O(t)$ nodes each. Two subtrees are either disjoint or intersect only at their common root. Each time $\hat{u}_i$ is the root of a sub-tree, we store the pre-order identifier $\hat{u}_i$ at position $i$ in the XBWT. Positions not containing sampled pre-order nodes can be marked using a zero-order compressed bitvector supporting constant-time rank and select queries and taking $o(n) + O((n/t) \log n)$ bits of space [31]. Overall, our sampling takes $o(n) + O((n/t) \log n)$ bits of space. We also keep the Balanced Parentheses Sequence (BPS) representation of the tree’s topology of Navarro and Sadakane [30]. This structure takes $2n + o(n)$ bits of space and supports constant-time navigation operations between nodes represented in pre-order.

Let $i = \hat{u}$. To retrieve $\hat{u}_i$, we take the parent of $\hat{u}$ until finding a sampled node $j$ (that is, $\hat{u}_j$ is explicitly stored). Since the XBWT supports parent operations in constant time and each subtree has size $O(t)$, node $\hat{u}_j$ is found in $O(t)$ time. While navigating from node $i$ to node $j$, we also collect the rank of each node among its siblings in a sequence $R$ of $O(t)$ integers. This operation also takes constant time per node using the XBWT [8] (since each last sibling is marked in a bitvector supporting constant-time rank and select queries). Since we assume $t \leq o(n / \log n)$, sequence $R$ takes $o(n)$ bits to be stored. Finally, we use $R$ to navigate the BPS representation from $\hat{u}_j$ until finding $\hat{u}_i$, in constant time per step [30]. Overall, the process takes $O(t)$ time. \hfill \Box

B Proof of Lemma 2

Claim. The RL-XBWT representation takes $O(r)$ words to be stored.

Proof. Let $A = \sum_{q=1}^{r'} |ADD_q|$ and $D = \sum_{q=1}^{r'} |DEL_q|$. Note that the union of all sets $DEL_q$ contains the labels of all run breaks except the ones in the last position $n$: $D \leq r$ (see also Figure 2). The first occurrence of a character in $XBWT(\mathcal{T})$ appears in the set $ADD_q$ of the corresponding block. These characters contribute $\sigma \leq r$ to the total size $A$ of these sets. Furthermore, each other element $c \in ADD_q$ is charged to the previous $c$-run break in the XBWT. It follows that $A \leq 2r$. Finally, note that $ADD_q \cup DEL_q \neq \emptyset$ must hold for every $q$, since otherwise the outgoing labels of the $q$-th block would coincide with those of the $(q-1)$-th block. Since $A + D \leq 3r$, this implies that there are also at most $r' \leq 3r$ blocks. Our thesis follows. \hfill \Box
C Proof of Theorem 1

**Claim.** The number $r$ of XBWT runs is always at most $\mathcal{H}_k^{uc} + \sigma^{k+1}$ for any $k \geq 0$, where $\mathcal{H}_k^{uc}$ is the trie’s $k$-th order worst-case entropy (Definition 3).

**Proof.** Let $OUT_i = out(\hat{u}_i)$, and let $RLE_c(i, j)$ be the number of $c$-run breaks in the sequence $X = OUT_i, \ldots, OUT_j$: $RLE_c(i, j)$ increases by one unit for every $i \leq t \leq j$ such that $c \in OUT_i$ and either $t = j$ or $c \notin OUT_{t+1}$. We denote $RLE(i, j) = \sum_{c \in \Sigma} RLE_c(i, j)$. Note that $r_c = RLE_c(1, n)$ and $r = RLE(1, n)$. For any partition $[i_1, i_2], [i_2 + 1, i_3], \ldots, [i_m + 1, i_{m+1}]$ of the interval $[1, n]$ into $m$ sub-intervals, it is easy to see that

$$r = RLE(1, n) \leq \sum_{j=1}^{m} RLE(i_j, i_{j+1}) \quad (1)$$

since the right-hand side has the same run breaks as the left-hand side, plus one more run break for the last occurrence of each character in each sub-interval. We now consider the partition into sub-intervals induced by the contexts of length $k$: we put in the same interval $cover(\rho) = OUT_i, \ldots, OUT_j$ the outgoing labels of all nodes $\hat{u}_i$ having the same context $\rho = \pi_k[\hat{u}_i]$ (note that, by definition of Wheeler order, such nodes form a consecutive range). To prove our thesis, we are going to show that $RLE(i, j) \leq \mathcal{H}_k^{uc}(cover(\rho)) + \sigma$ for any such interval $[i, j]$ corresponding to context $\rho$. Let $n' = j - i + 1$ and $n'_c$ be the number of occurrences of $c \in \Sigma$ in the sequence of sets $cover(\rho) = OUT_i, \ldots, OUT_j$. We first prove $RLE_c(i, j) \leq n'_c \log_2(n'/n'_c) + 1$ for any character $c$ such that $n'_c > 0$ (note: if $n'_c = 0$ then $c$ does not contribute to $RLE(i, j)$ nor to the worst-case entropy of the interval).

Build a binary sequence $S[1, n']$ such that $S[i] = 1$ if and only if $c \in OUT_{i+t-1}$. Letting $r_x(S)$ be the number of equal-letter maximal runs of symbol $x \in \{0, 1\}$ in $S$, by definition we have $r_1(S) = RLE_c(i, j)$. Note that $r_1(S) \leq r_0(S) + 1$. Note also that $r_1(S) \leq n'_c \leq \min\{n' - n'_c, n'_c\}$ and $r_0(S) \leq n' - n'_c \leq \min\{n'_c, n' - n'_c\}$. From these inequalities we obtain that $RLE_c(i, j) = r_1(S) \leq \min\{n'_c, n' - n'_c\} + 1$ always holds.

The next step is to prove $\min\{n'_c, n' - n'_c\} \leq n'_c \log_2(n'/n'_c)$. We are going to prove this analytically by extending the domain of $n'_c$ and $n'$ to the whole $\mathbb{R}^+$, with the constraint $1 \leq n'_c \leq n'$. If $n'_c < n'/2$, the inequality reduces to $n'_c \leq n'_c \log_2(n'/n'_c)$ which is obviously true in the considered range. If $n'_c \geq n'/2$, the inequality reduces to $n' - n'_c \leq n'_c \log_2(n'/n'_c)$. Let us define $\epsilon = n'_c/n'$. The inequality further simplifies to $f(\epsilon) = \epsilon - \epsilon \log_2 \epsilon - 1 \geq 0$ for $0.5 \leq \epsilon \leq 1$. The derivative $f'(\epsilon) = 1 - \log_2 \epsilon - \log_2 \epsilon$ goes to zero in $f'(2/e) = 0$, is positive for $\epsilon < 2/e$ and negative for $\epsilon > 2/e$. Since $0.5 \leq 2/e \leq 1$, we obtain our claim: first, $f(0.5) = 0$, then $f(\epsilon)$ is increasing until $\epsilon = 2/e$, and finally it decreases until reaching $f(1) = 0$.

From the above, we obtain that $RLE_c(i, j) \leq 1 + n'_c \log_2(n'/n'_c)$ for any character $c$ such that $n'_c > 0$. Since $(n'/n'_c)^{n'_c} \leq (n'_c/n'_c)$, we obtain $RLE_c(i, j) \leq 1 + \log_2(n'_c/n'_c)$. Summing both sides for all $c \in \Sigma$, we obtain $RLE(i, j) \leq \sigma + \sum_{c \in \Sigma} \log_2(n'_c/n'_c) = \sigma + \mathcal{H}_k^{uc}(cover(\rho))$. On the trie’s paths there are in total at most $\sigma^k$ different contexts $\rho \in \Sigma^k$. Summing both sides of the inequality for all possible (at most) $\sigma^k$ contexts $\rho$ and applying Definition 3 and Inequality \cite{3} we obtain $r \leq \mathcal{H}_k^{uc} + \sigma^{k+1}$. □

D Proof of Corollary 1

**Claim.** For any $0 < \alpha < 1$ and $0 \leq k \leq \max\{0, \alpha \log_\sigma n - 1\}$ it holds $r \leq 2\mathcal{H}_k^{uc} + o(n/\log^c n)$ for any constant $c > 0$. 

12
**Proof.** If $0 < \alpha < 1$ and $0 \leq k \leq \alpha \log_{\sigma} n - 1$, then the corollary follows immediately from Theorem 1: \[ r \leq H_{kwc}^c + \sigma^c \log_{\sigma} n = H_{kwc}^c + n^\alpha \leq H_{kwc}^c + o(n/\log^c n) \] for any constant $c > 0$.

However, for large $\sigma$ the interval $[0, \alpha \log_{\sigma} n - 1]$ could be empty. To prove the claim, we have to give a useful bound in the case $k = 0$. In this case, Theorem 1 yields $r \leq H_{0wc}^c + \sigma$. The problem is that $\sigma$ could be $O(n)$; the solution is to note that, in this case, also $H_{0wc}^c$ must be large. In the following we prove that the bound $r \leq 2H_{0wc}^c + 1$ holds. This will prove the claim.

We can assume the number of nodes to be $n \geq 2$, otherwise the tree is either empty or composed of the root only and both $r$ and $H_{0wc}^c$ are equal to 0. We can moreover assume $\sigma \geq 2$, since character $\# \text{ does not label any edge and there are at least 2 nodes.}$

Let $n_c$ be the number of edges labeled $c$. Note that $n_{\#} = 0$ since $\#$ does not label any edge. By definition, $H_{0wc}^c = \sum_{c \in \Sigma} \log_2 \left( \frac{n-1}{n_c} \right) = \sum_{c \in \Sigma - \{\#\}} \log_2 \left( \frac{n-1}{n_c} \right)$.

If $\sigma = 2$, then the tree is a unary path, $r = 1$, and $H_{0wc}^c = 0$. The claim follows. We can therefore assume $\sigma \geq 3$. Since we assume the alphabet to be effective we have then $n - 1 \geq 2$ and $1 \leq n_c < n - 1$, therefore $\left( \frac{n-1}{n_c} \right) \geq 2$ for every $c \neq \#$. It follows that $H_{0wc}^c = \sum_{c \in \Sigma - \{\#\}} \log_2 \left( \frac{n-1}{n_c} \right) \geq \sigma - 1$. Re-arranging terms, this becomes $\sigma \leq H_{0wc}^c + 1$. Plugging this into the bound $r \leq H_{0wc}^c + \sigma$ of Theorem 1 we obtain our claim.

\[ \square \]

E Proof of Theorem 2

**Claim.** Let $\omega$ be the number of edges of the minimum WDFA recognizing the same language of $T$. Then, $r \leq \omega$.

**Proof.** Let $T = (V, E)$. Consider any equivalence class $[u]_{\equiv^c \ast} = \{ \hat{u}_i, \hat{u}_{i+1}, \ldots, \hat{u}_j \}$. By definition of $\equiv^c \ast$, all nodes in this class have the same children labels. It follows that the only run break in this class can be $\hat{u}_j$. This shows that $r \leq \sigma \cdot |V|_{\equiv^c \ast}$, because $r$ can increase by at most $\sigma$ only between two adjacent $\equiv^c \ast$-classes. We can say more: between $[\hat{u}]_{\equiv^c}$ and the class immediately succeeding it in the ordering of the nodes, $r$ can increase at most by the number of children of $\hat{u}$ (since $\hat{u}_j$ can be a $c$-run only if $c$ is the label of a child of $\hat{u}_j$). It follows that $r$ can be upper-bounded as follows:

$$r \leq \sum_{U \in V|_{\equiv^c \ast}} |\text{out}(\max(U))|$$

where $\max(U)$ returns the largest $\hat{u} \in U$ (by the ordering $<$). Now, since $\equiv \ast$ is a refinement of $\equiv^c \ast$ we have that

$$\sum_{U \in V|_{\equiv^c \ast}} |\text{out}(\max(U))| \leq \sum_{U \in V|_{\equiv \ast}} |\text{out}(\max(U))| = \omega$$

from which the thesis follows.

\[ \square \]

F Proof of Lemma 3

**Claim.** If $\hat{u} <_{\text{pred}} \hat{v}$ then $\text{child}_c(\hat{u}) <_{\text{pred}} \text{child}_c(\hat{v})$ for all $c \in \text{out}(\hat{u}) \cap \text{out}(\hat{v})$.

**Proof.** Let $c \in \text{out}(\hat{u}) \cap \text{out}(\hat{v})$, $\hat{u}' = \text{child}_c(\hat{u})$, and $\hat{v}' = \text{child}_c(\hat{v})$. Suppose, by contradiction, that there exists $\hat{w}$ such that $\hat{u}' < \hat{w} < \hat{v}'$. By Wheeler property (i), it must be the case that $c = \lambda(\hat{u}') = \lambda(\hat{v}') = \lambda(\hat{w})$. Then, we have two cases. (a) $\pi(\hat{w}) < \hat{u} <_{\text{pred}} \hat{v}$, which by Wheeler property (ii) implies $\hat{w} < \hat{u}'$, a contradiction. (b) $\hat{u} <_{\text{pred}} \hat{v} < \pi(\hat{w})$, which by Wheeler property (ii) implies $\hat{v}' < \hat{w}$, a contradiction.

\[ \square \]
G  Proof of Theorem \[3\]

**Claim.** \(\Gamma^r\) is a tree attractor of size \(|\Gamma^r| = r\).

**Proof.** The fact that \(|\Gamma^r| = r\) follows from the very definitions of \(\Gamma^r\) and \(r\). Let \(\mathcal{T}(U) = (U, E^r)\), with \(U \subseteq V\), be a subtree of \(\mathcal{T}\). If \(E^r \cap \Gamma^r \neq \emptyset\) then we obtain our claim. Similarly, if the root of \(\mathcal{T}(U)\) is \(\hat{u}_n\) (the last node in the Wheeler order of \(\mathcal{T}\)) then \(n\) is a run break and all edges leaving \(\hat{u}_n\) are in \(\Gamma^r\). It follows that \(E^r \cap \Gamma^r \neq \emptyset\) holds and we are done.

Let us therefore assume that \(E^r \cap \Gamma^r = \emptyset\) and that the root of \(\mathcal{T}(U)\) is \(\hat{u}_i\), with \(i < n\). Since no edge from \(E^r\) leaving \(\hat{u}_i\) belongs to \(\Gamma^r\), We have that \(c \in \text{out}(\hat{u}_i) \Rightarrow c \in \text{out}(\hat{u}_{i+1})\). But then, since \(\hat{u}_i <_{\text{pred}} \hat{u}_{i+1}\) by Lemma \[3\] it must be the case that \(\text{child}_c(\hat{u}_i) <_{\text{pred}} \text{child}_c(\hat{u}_{i+1})\) for all \(c = \lambda(\hat{v})\), where \((\hat{u}_i, \hat{v}) \in E^r\); the children of \(\hat{u}_i\) and \(\hat{u}_{i+1}\) reached by following label \(c\) must be adjacent in the Wheeler order of the tree. It is clear that we can repeat the above reasoning to each such node \(\hat{v} = \text{child}_c(\hat{u}_i)\) since, by assumption, no edge from \(E^r\) leaving \(\hat{v}\) belongs to \(\Gamma^r\). This procedure can be repeated until we visit the whole \(\mathcal{T}(U)\). As a consequence, we obtain that \(\mathcal{T}(U)\) has an isomorphic occurrence \(\mathcal{T}(U') = (U', E')\) with root \(\hat{u}_{i+1} \in \mathcal{T}\). If \(E' \cap \Gamma^r \neq \emptyset\), we are done. Otherwise, we can repeat the whole reasoning to \(\mathcal{T}(U')\), finding another isomorphic occurrence (rooted in \(\hat{u}_{i+2}\)). Note that the roots of this sequence of isomorphic trees are \(\hat{u}_i <_{\text{pred}} \hat{u}_{i+1} <_{\text{pred}} \hat{u}_{i+2}, \ldots\) By the finiteness of \(\mathcal{T}\) and by the totality of \(<\), this sequence cannot be infinite, therefore at some point we must stop finding a subtree \((\hat{U}, \hat{E}) = \mathcal{T}(U) \approx \mathcal{T}(U)\) such that \(\hat{E} \cap \Gamma^r \neq \emptyset\). \(\square\)

H  Proof of Lemma \[4\]

**Claim.** There is a data structure taking \(O(r \log n) + o(n)\) bits of space and supporting operation \(cr(\bar{u}, c)\) in \(O(\log \sigma)\) time.

**Proof.** Consider our RL-XBWT representation of Definition \[2\] \((\text{ADD}_q, \text{DEL}_q, \ell_q)_{q=1, \ldots, r'}\). Let moreover \(\bigcirc\) be the concatenation operator between strings, and let \(\not\in \Sigma\) be a new character lexicographically *larger* than all other characters in \(\Sigma\). We define the following two sequences on \(\Sigma \cup \{\not\}\), obtained by simply concatenating all characters in the sets \(\text{ADD}_q\) and \(\text{DEL}_q\) and separating different sets with the symbol `'`

\[
S_{\text{ADD}} = \bigcirc_{i=1}^{r'} \left( \left( \bigcirc_{c \in \text{ADD}_i} c \right) \bigcirc \not\right)
\]

and

\[
S_{\text{DEL}} = \bigcirc_{i=1}^{r'} \left( \left( \bigcirc_{c \in \text{DEL}_i} c \right) \bigcirc \not\right)
\]

where the operator \(\bigcirc\) concatenates characters from a set in any order (for example, lexicographic). By Lemma \[2\] the length of \(S_{\text{ADD}}\) and \(S_{\text{ADD}}\) is upper-bounded by \(O(r)\). See Figure \[5\] for a running example.

First, note that by definition all nodes within the same RL-XBWT block have the same answers to operation \(cr(\bar{u}, c)\). We mark all nodes at the end of a block (in Wheeler order) in an entropy-compressed bitvector supporting constant-time predecessor queries \[31\]. Since the total number of
blocks is $r' \leq 3r$ (Lemma 2), this bitvector takes $O(r \log n) + o(n)$ bits of space. With a constant-time predecessor, we can therefore reduce $cr(\hat{u}, c)$ to the analogous operation $cr'(i, c)$ on blocks, where this time $1 \leq i \leq r'$ is the index of the RL-XBWT block the node $\hat{u}$ belongs to and $cr'(i, c)$ is the answer to $cr(\bar{v}, c)$ for any node $\bar{v}$ in the $i$-th block. Now, let $i$ be a block number, and $j_{ADD} = S_{ADD}.select_j(i)$ be the position in $S_{ADD}$ containing the $i$-th occurrence of $\slash$. Similarly, let $j_{DEL} = S_{DEL}.select_j(i)$ be the position in $S_{DEL}$ containing the $i$-th occurrence of $\slash$. Then, it is easy to see that the following holds:

**Lemma 11.** $S_{ADD}.rank_c(j_{ADD}) - S_{DEL}.rank_c(j_{DEL})$ is equal to the number of edges labeled $c$ exiting any node in the $i$-th block (in particular, it is always either 0 or 1).

Let $S.rank_{\leq c}(i) = \sum_{d \leq c} S.rank_d(i)$ be the number of character lexicographically equal to or smaller than $c$ in $S[1, i]$. A direct consequence of Lemma 11 is the following:

**Corollary 3.** $S_{ADD}.rank_{\leq c}(j_{ADD}) - S_{DEL}.rank_{\leq c}(j_{DEL})$ is equal to the number of edges labeled with all characters smaller than or equal to $c$ exiting any node in the $i$-th block.

Operations $S.rank_c(j)$, $S.select_j(i)$, and $S.rank_{\leq c}(i)$ on a string $S$ can be implemented in $O(\log \sigma)$ time and $O(|S| \log \sigma)$ bits of space using wavelet trees [26]. Corollary 3 solves precisely query $cr'(i, c)$, so we obtain our claim.

Even if we will not need it in our index, we note that binary search on Corollary 3 can be used to solve also the following operation in $O(\log^2 \sigma)$ time: **child label cl($\hat{u}, k$),** which returns the label of the edge connecting $\hat{u}$ with its $k$-th (in lexicographic order) child. This operation could be useful, for example, to list the (labels of the) children of any $\hat{u}$ within $O(r \log n)$ bits of space.

**I Proof of Lemma 5**

**Claim.** The tree representation [30] supports also operation $ISD(\hat{u}, \hat{v}, \hat{u}')$ in $O(1)$ time at no additional space usage.

**Proof.** The tree representation [30] stores the Balanced Parentheses Sequence (BPS) representation of the tree topology, augmented with additional (light) structures. Let $i_{\bar{u}}$, $i_\hat{v}$, $i_{\hat{u}}'$ be the positions of the open parentheses corresponding to nodes $\hat{u}$, $\hat{v}$, and $\hat{u}'$ in the BPS representation of the tree topology. Since $\hat{u} \approx \hat{u}'$ the parentheses substring representing $\hat{u}$ and its descendants is equal to the one representing $\hat{u}'$ and its descendants. Then, it must be the case that $i_{\bar{u}} - i_{\hat{u}} = i_{\hat{u}'} - i_{\hat{u}'}$, therefore $i_{\hat{u}'} = i_{\bar{u}} - i_{\hat{u}} + i_{\hat{u}'}$. The representation [30] allows moving between positions in the BPS sequence and pre-order ranks in constant time, so our thesis follows.

**J Proof of Lemma 6**

**Claim.** Operation $ISC(\hat{u}, k)$ can be supported in $O(1)$ time and $O(r \log n) + o(n)$ bits of space.

**Proof.**
\textbf{Proof.} For brevity, let $OUT_k = out(\hat{u}_k)$. For each node $\hat{u}_i$ such that $i < n$ is a run break (i.e. $out(\hat{u}_i) \neq out(\hat{u}_{i+1})$), we build the following two bitvectors:

$$S^1_i = \bigcup_{c \in OUT_i} c \in OUT_{i+1}$$

and

$$S^2_i = \bigcup_{c \in OUT_{i+1}} c \in OUT_i$$

where the operator $\bigcup$ visits characters in lexicographic order and where $c \in A$ equals the symbol '1' if $c \in A$ and '0' otherwise. In other words, $S^1_i$ marks an outgoing label of $\hat{u}_i$ with a bit 1 if it is also an outgoing label of $\hat{u}_{i+1}$ and with a bit 0 otherwise (similar for $S^2_i$). We concatenate these two bit-sequences and further concatenate all such $S^1_iS^2_i$ in pre-order (that is, according to the pre-order number $\hat{u}_i$, rather than on $i$) in a single sequence $S$ of length $|S| \leq n$. We furthermore use a bitvector $B_1$ of length $n$ to mark in pre-order the nodes that are run-breaks (i.e. nodes $\hat{u}_i$ for which we built $S^1_iS^2_i$), and a bitvector $B_2$ of length $|B_2| = |S| \leq n$ to mark the boundaries of each $S^1_i$ and $S^2_i$ inside sequence $S$. We build on the two bitvectors the entropy-compressed representation of Raman et al. \cite{Raman2007}, which answers \textit{rank} and \textit{select} queries in constant time. Since those bitvectors have length at most $n$ and have $O(r)$ bits set, the entropy-compressed representation of Raman et al. uses $o(n) + O(r \log n)$ bits \cite{Raman2007}. Using $S$, $B_1$, and $B_2$, we can retrieve in constant time the (boundaries in $S$ of the) two sequences $S^1_i$ and $S^2_i$ associated with any pre-order node $\hat{u}_i$ that is a run-break. We use Raman et al.'s representation \cite{Raman2007} to represent also sequence $S$. Note that $S$ has one bit equal to 0 for each $c \not\in OUT_i$ such that $c \not\in OUT_{i+1}$ and for each $c \in OUT_{i+1}$ such that $c \not\in OUT_i$. It follows that $S$ has at most $O(r)$ bits equal to 0, therefore the entropy-compressed data structure \cite{Raman2007} uses $o(n) + O(r \log n)$ bits to represent it.

We now show how to answer $ISC(\hat{u}_i,k)$. Let $c$ be the $k$-th (in lexicographic order) outgoing label of $\hat{u}_i$. We first retrieve in constant time the (boundaries in $S$ of) $S^1_i$ and $S^2_i$. Note that we can assume $S^1_i[k] = 1$ since, by assumption in our query definition, $c$ is an outgoing label of $\hat{u}_{i+1}$. Let $S^1_i[k]$ be the $j$-th bit equal to '1' in $S^1_i$ (we can find $j$ in constant time with a \textit{rank} query). Then, it must be the case that the $j$-th bit equal to '1' $S^2_i[t]$ is such that the $t$-th outgoing label of $\hat{u}_{i+1}$ is equal to $c$ (note: by the way we constructed those two sequences, the corresponding bits set in $S^1_i$ and $S^2_i$ correspond to the same labels). We can find $t$ in constant time with a \textit{select} operation on $S^2_i$. Finally, we return $t$. \hfill \Box

K \quad \textbf{Proof of Lemma 7}

\textbf{Claim.} There is a data structure taking $O(r \log n) + o(n)$ bits of space on top of the succinct tree topology of Navarro and Sadakane \cite{Navarro2003} that, given a pattern $P \in \Sigma^m$, returns the range $[\bar{l}, \bar{r}]$ of Wheeler-order nodes reached by a path labeled $P$, as well as $\hat{u}_{\bar{r}}$, in $O(m \log \sigma)$ time.

\textbf{Proof.} Finding the range of nodes $[\bar{l}, \bar{r}]$ reached by a pattern requires, as building block, being able to count the number of occurrences of a character $c$ in a prefix $out(\hat{u}_1), \ldots, out(\hat{u}_i)$ of the XBWT, an operation we denote as $\text{rank}_c(i)$ \cite{Beller2014}. Moreover, in order to find node $\hat{u}_{\bar{r}}$ we will need to find, given an index $i$ and a character $c \in \Sigma$, the minimum $i'$ such that $i' \geq i$ and $c \in out(\hat{u}_{i'})$. If such $i'$ does not exist, we simply return $\perp$. We denote this operation as $\text{successor}_c(i) = i'$. Next, we show how to solve these operations.
**rank.** We first show how to support \( \text{rank}_c(i) \) in \( O(r \log n) + o(n) \) bits of space and \( O(\log \sigma) \) time (it is actually possible to improve upon this running time, but for us \( O(\log \sigma) \) will be sufficient due to the complexity of operation \( cr(\tilde{u}, c) \), Lemma [4]). We mark in an entropy-compressed bitvector supporting constant-time rank and select queries \( [3] \) all nodes (in Wheeler order) that are the first in their XBWT block. Since the total number of XBWT blocks is \( r' \leq 3r \) (Lemma [2]), the bitvector takes \( O(r \log n) + o(n) \) bits of space \( [3] \). Let \( \Sigma' = \{/\} \cup \{c^-, c^+ : c \in \Sigma\} \) be a new alphabet. We build a sequence \( S' \) over \( \Sigma' \) by concatenating the characters of all sets \( ADD_i \) and \( DEL_i \) of our RL-XBWT, separating each block with a special symbol ‘/’ (\( \bigcirc \) is the concatenation operator between strings):

\[
S' = \bigcirc_{i=1}^{r'} \left( \bigcirc_{c \in ADD_i} c^+ \right) \bigcirc \left( \bigcirc_{c \in DEL_i} c^- \right) \bigcirc / \right)
\]

Figure [6] shows a running example.

\[
S' = a^+ b^+ c^+ / a^- c^- / b^- / a^+ c^+ / a^- c^- / b^+ c^+ / b^- c^- / a^+ /
\]

**Fig. 6.** Sequence \( S' \) obtained from the example of Figures [1] and [2]

Clearly, \( S' \) has \( O(r) \) characters over an alphabet of size \( \sigma' \in O(\sigma) \). We build over \( S' \) a wavelet tree \( [26] \), taking \( O(|S'| \log \sigma') \subseteq O(r \log n) \) bits of space and supporting rank and select operations in \( O(\log \sigma) \) time. Consider any occurrence of a character \( c^+ \) in \( S' \), with \( c \in \Sigma \), belonging to XBWT block \( j \) (that is, between the \((j-1)\)-th and \( j \)-th occurrence of \( / \)), and let \( \hat{u}_{j'} \) be the first node in the \( j \)-th XBWT block. We explicitly store \( \text{rank}_c(j'-1) \) in correspondence to this occurrence of \( c^+ \) (if \( j' = 1 \), then we take \( \text{rank}_c(j'-1) = 0 \)). Storing all these partial ranks takes \( O(r) \) words of space in total.

Now, it is not hard to see that all these structures allow us to compute \( \text{rank}_c(i') \) in \( O(\log \sigma) \) time for any \( c \in \Sigma \) and \( 1 \leq i' \leq n \). First, we find the XBWT block \( i \) containing node \( \hat{u}_{i'} \) (constant time on the bitvector marking the first node of each block). Then, we find in \( S' \) the occurrences of \( c^- \) and of \( c^+ \) that immediately precede the \( i \)-th symbol ‘/’ \( (O(\log \sigma) \) time using rank and select operations). If there are no such occurrences of \( c^+ \), then \( \text{rank}_c(i') = 0 \). We consider two other cases.

(A) The occurrence found of \( c^+ \) is to the right of that of \( c^- \), or there are no such occurrences of \( c^- \). Let \( j' \) be the the XBWT block containing such occurrence of \( c^+ \). This means that all nodes contained in the XBWT blocks from the \( j' \)-th to the \( i \)-th (included) have an outgoing edge labeled \( c \). Let \( \hat{u}_{j'} \) be the first node of the \( j' \)-th XBWT block (found in constant time using our bitvector). Then, \( \text{rank}_c(j'-1) \) is explicitly stored and we obtain \( \text{rank}_c(i') = \text{rank}_c(j'-1) + (i' - j') + 1 \).

(B) The other case to be considered is the one where the occurrence found of \( c^- \) is to the right of that of \( c^+ \). Let \( j^- \) be the XBWT block containing such occurrence of \( c^- \). Then, all XBWT blocks from the \( j^- \)-th to the \( i \)-th (included) do not have an outgoing edge labeled with \( c \). However, the nodes in the \((j^- - 1)\)-th block do have such an outgoing label. Let \( \hat{u}_{j'} \) be the first node in the \( j^- \)-th block (found in constant time using our bitvector). Then, \( \text{rank}_c(i') = \text{rank}_c(j'-1) \), which reduces to case (A).

**successor.** We show how to solve \( \text{successor}_c(i') \). If \( \text{rank}_c(i') = \text{rank}_c(i'-1) \) (where \( \text{rank}_c(0) = 0 \)), then node \( \hat{u}_i \) has an outgoing edge labeled \( c \) and we return \( i' \). Otherwise, we need to find the next XBWT block containing nodes that have an outgoing edge labeled \( c \). Let \( i \) be the XBWT block
computing \([\bar{\ell}, \bar{r}]\). We show how to find the range of Wheeler-order nodes reached by a given pattern. The algorithm (known as backward search) is based on the observation that labels occur in the same order in the XBWT and in the sequence \(\lambda(\hat{u}_1), \ldots, \lambda(\hat{u}_n)\)\ [9], see Figure 1. Moreover, the nodes reached by a path labeled \(P \in \Sigma^*\) always form a consecutive range with respect to the Wheeler order \([9]\). These observations lead to the following algorithm, first described in \([8]\) (on trees). First, note that characters in \(\lambda(\hat{u}_1), \ldots, \lambda(\hat{u}_n)\) are sorted (i.e. clustered in increasing order). We store in an array \(C\) a total of \(\sigma \leq r\) integers recording the starting point of every distinct character in this sequence. At this point, given the range \([\bar{\ell}, \bar{r}]\) of Wheeler-order nodes reached by a path labeled \(P \in \Sigma^*\), to extend it with character \(c \in \Sigma\) we map the characters equal to \(c\) contained in \(\text{out}(\hat{u}_{\bar{\ell}}), \ldots, \text{out}(\hat{u}_{\bar{r}})\) to the corresponding range \(\lambda(\hat{u}_{\bar{\ell}}), \ldots, \lambda(\hat{u}_{\bar{r}})\) using just two rank queries and one access to array \(C\). The result \([\bar{\ell}', \bar{r}']\) is the range of nodes reached by a path labeled \(P \cdot c\). At the beginning, the algorithm starts with \(P = \epsilon\) (empty pattern) and \([\bar{\ell}, \bar{r}] = [1, n]\). Crucially, note that this procedure returns only the range of ranks (in Wheeler order) \([\bar{\ell}, \bar{r}]\) of the nodes reached by a path labeled \(P\). To obtain their pre-order identifiers \(\hat{u}_{\bar{\ell}}, \ldots, \hat{u}_{\bar{r}}\) we will need the more complex \textit{locate} queries, discussed in Section \([3]\).

computing \(\hat{u}_{\bar{\ell}}\). We show how to extend the above procedure in order to also compute \(\hat{u}_{\bar{\ell}}\). At the beginning, we start with an empty pattern \(P = \epsilon\) and its range \([1, n]\). Then, \(\hat{u}_1 = 1\) is the root. Assume now that we have computed the range \([\bar{\ell}, \bar{r}]\) of a pattern \(P\), and that we know the pre-order node \(\hat{u}_{\bar{\ell}}\). We extend \(P\) with letter \(c\) and obtain the range \([\bar{\ell}', \bar{r}']\) of \(P \cdot c\) with an extension step described above (assume that the range is not empty, otherwise the search stops). Then, we find in \(O(\log \sigma)\) time with a successor query (read above) the smallest \(i\) in the range \([\bar{\ell}, \bar{r}]\) such that \(c \in \text{out}(\hat{u}_i)\). If \(i = \bar{\ell}\), then we simply descend to the corresponding child of \(\hat{u}_{\bar{\ell}}\) with \(\hat{u}_{\bar{r}} = \text{cbr}(\hat{u}_{\bar{\ell}}, \text{cr}(\bar{\ell}, c))\) in \(O(\log \sigma)\) time and \(2n + o(n) + O(r \log n)\) bits of space (by Operations \([3]\) and \([1]\)). Otherwise, \(i > \bar{\ell}\). But then, Wheeler-order node \(i\) is the first in a run of nodes having an outgoing edge labeled \(c\) (that is, Wheeler-order node \(i - 1\) does not have an outgoing edge labeled \(c\)). We can therefore explicitly store all those pre-order nodes, since there are at most \(O(r)\) of them, and retrieve \(\hat{u}_i\) in constant time. Finally, we descend to the edge labeled \(c\) of \(\hat{u}_i\) with \(\hat{u}_{\bar{r}} = \text{cbr}(\hat{u}_i, \text{cr}(i, c))\) in \(O(\log \sigma)\) time and \(2n + o(n) + O(r \log n)\) bits of space (by Operations \([3]\) and \([1]\)). \(\square\)

L Proof of Corollary \([2]\)

\textbf{Claim.} Let \(i < n\), \(\hat{u}_i \neq \hat{u}_{i+1}\) if and only if the complete subtree rooted in \(\hat{u}_i\) contains a red node.

\textbf{Proof.} Assume that the complete subtree rooted in \(\hat{u}_i\) does not contain any red node. Since \(\hat{u}_i\) is not red and \(i < n\), then (by definition of red node) \(\text{out}(\hat{u}_i) = \text{out}(\hat{u}_{i+1})\). But then, by Lemma \([3]\) \(\text{child}_c(\hat{u}_i) \prec_{\text{pred}} \text{child}_c(\hat{u}_{i+1})\) for all \(c \in \text{out}(\hat{u}_i) = \text{out}(\hat{u}_{i+1})\). The reasoning can be repeated
inductively to the children of \( \hat{u}_i \) until reaching the leaves, since the complete subtree rooted in \( \hat{u}_i \) does not contain any red node. As a consequence, we obtain \( \hat{u}_i \approx \hat{u}_{i+1} \).

Conversely, assume that the complete subtree rooted in \( \hat{u}_i \) contains a red node. If \( \hat{u}_i \) is red, then (by definition of red node) \( out(\hat{u}_i) \neq out(\hat{u}_{i+1}) \) and therefore \( \hat{u}_i \neq \hat{u}_{i+1} \). Otherwise, \( \hat{u}_i \) is not red and we can repeat the reasoning to the children of \( \hat{u}_i \) and \( \hat{u}_{i+1} \) (as seen above). Since the complete subtree rooted in \( \hat{u}_i \) contains a red node, at some point we will find a red descendant \( \hat{u}_j \) of \( \hat{u}_i \) such that \( out(\hat{u}_j) \neq out(\hat{u}_{j+1}) \), where \( \hat{u}_{j+1} \) is the corresponding descendant of \( \hat{u}_{i+1} \). As a consequence, \( \hat{u}_i \neq \hat{u}_{i+1} \).

\( \square \)

**M  Proof of Lemma 8**

**Claim.** There is a data structure taking \( O(r \log n) + o(n) \) bits of space on top of the succinct tree topology of Navarro and Sadakane [30] and answering the following queries in \( O(1) \) time. Given a pre-order node \( \hat{u}_i \) with \( i < n \):

(a) If \( \hat{u}_i \) is not colored, find a colored node \( \hat{u}_j \neq \hat{u}_i \) in the complete subtree rooted in \( \hat{u}_i \) such that no node on the path from \( \hat{u}_i \) to \( \hat{u}_j \) is colored (except \( \hat{u}_j \)), or report that \( \hat{u}_j \) does not exist.

(b) Find the lowest ancestor \( \hat{u}_j \) of \( \hat{u}_i \) such that the complete subtree rooted in \( \hat{u}_j \) contains a colored node. Note that such a node always exists, since the root is always blue.

**Proof.** Consider the Balanced Parentheses Sequence (BPS) representation of the tree. To answer (a), it is sufficient to mark in a bitvector \( B \) all open parentheses corresponding to a colored node (note: we mark \( O(r) \) parentheses). By the definition of BPS, \( I = \hat{u}_i \) corresponds to the \( I \)-th open parentheses in the sequence. If the \( I \)-th open parentheses is marked, then we return \( \hat{u}_i \); otherwise, let \( J \) be the position of the marked open parentheses immediately following the \( I \)-th. If the position of \( J \) falls inside the BPS range of node \( \hat{u}_i \) (that is, between its corresponding open and close parentheses), then \( J = \hat{u}_j \) is the descendant of \( \hat{u}_i \) that we are looking for. Otherwise, the complete subtree rooted in \( \hat{u}_i \) does not contain colored nodes and we report that \( \hat{u}_j \) does not exist. By using Raman et al.’s entropy-compressed representation, bitvector \( B \) takes \( O(r \log n) + o(n) \) bits and answers successor queries in constant time. All operations on the BPS representation (in particular, finding matching pairs of open/close parentheses) take constant time [30].

We now show how to answer (b). Consider again the \( I \)-th open parentheses, with \( I = \hat{u}_i \). The idea is to find the \( K \)-th open parentheses that immediately precedes the \( I \)-th and that is also marked. Let \( \hat{u}_k = K \), and let \( \hat{u}_l = LCA(\hat{u}_i, \hat{u}_k) \). Then, if another node \( \hat{u}_v \neq \hat{u}_l \) on the path \( \hat{u}_l \leadsto \hat{u}_i \) is such that the complete subtree rooted in \( \hat{u}_v \) contains a colored node \( \hat{u}_k' \), it must be the case that \( \hat{u}_k' \) appears after \( \hat{u}_l \) in pre-order (otherwise we would have found the rightmost such node in place of \( \hat{u}_k \)). To complete the procedure we must therefore also find the \( K' \)-th open parentheses that immediately succeeds the closing parentheses of \( \hat{u}_l \) and that is also marked. Let \( \hat{u}_{k'} = K' \), and let \( \hat{u}_{v'} = LCA(\hat{u}_i, \hat{u}_{k'}) \). The answer to our query is the deepest node between \( \hat{u}_l \) and \( \hat{u}_{v'} \) (this requires computing \( depth(\hat{u}_l) \) and \( depth(\hat{u}_{v'}) \)). Note that all operations take constant time and that we use the same structures defined for query (a). Again, all operations on the BPS representation (in particular: matching parentheses, LCA, depth) take constant time [30].

\( \square \)

**N  Proof of Lemma 9**

**Claim.** In Lemma 8(a), if \( \hat{u}_j \) exists then \( \hat{u}_j \) must be red and not blue.
**Proof.** Assume that the complete subtree rooted in \( \hat{u}_i \) contains a colored node \( \hat{u}_j \neq \hat{u}_i \) such that no node other than \( \hat{u}_j \) on the path \( \hat{u}_i \leadsto \hat{u}_j \) of length (number of nodes) \( k \geq 2 \) is colored. We are going to prove that \( \hat{u}_j \) is red. By assumption in Lemma 3(a), we have \( i < n \). We prove the property inductively on the length \( k \) of the path. Assume \( k = 2 \) (that is, \( \hat{u}_j \) is child of \( \hat{u}_i \)), and let \( c = \lambda(\hat{u}_j) \). Since by assumption \( \hat{u}_i \) is not red and \( i < n \), then \( c \in \text{out}(\hat{u}_{i+1}) \). By Lemma 3 we have that \( \hat{u}_{j+1} = \text{child}_c(\hat{u}_{i+1}) \). But then, \( c = \lambda(\hat{u}_{j+1}) = \lambda(\hat{u}_j) \), therefore \( \hat{u}_j \) cannot be blue. Since \( \hat{u}_j \) is colored, it must be the case that \( \hat{u}_j \) is red (and not blue).

Let \( k > 2 \), and let \( \hat{u}_i \to \hat{u}_j \leadsto \hat{u}_j \) be the path from \( \hat{u}_i \) to \( \hat{u}_j \), where \( \hat{u}_i \) is child of \( \hat{u}_i \). Let \( c = \lambda(\hat{u}_j) \). Since \( i < n \) and \( \hat{u}_i \) is not red, we conclude that \( c \in \text{out}(\hat{u}_{i+1}) \). By Lemma 3 we have that \( \hat{u}_{i+1} = \text{child}_c(\hat{u}_{i+1}) \). Then, this implies that \( i' < n \) therefore we can apply our inductive hypothesis to the path \( \hat{u}_{i'} \leadsto \hat{u}_j \) of length \( k - 1 \) and conclude that \( \hat{u}_j \) is red and not blue. \( \square \)

O  **Proof of Lemma 10**

**Claim.** Let \( \Pi = \hat{u}_{i_1} \to \hat{u}_{i_2} \leadsto \hat{u}_{i_k} \), with \( i_j < n \) for some \( 1 \leq j \leq k \), be a path of length \( k \) without blue nodes other than (possibly) \( \hat{u}_{i_1} \) and without red nodes other than (possibly) \( \hat{u}_{i_k} \). Then, \( \hat{u}_{i_1+1} \to \hat{u}_{i_2+1} \leadsto \hat{u}_{i_k+1} \) is a path in the tree (adjacent to \( \Pi \)).

**Proof.** Let us break the path into two subpaths, overlapping by node \( \hat{u}_{i_j} \): \( \Pi' = \hat{u}_{i_1} \leadsto \hat{u}_{i_j} \) and \( \Pi'' = \hat{u}_{i_j} \leadsto \hat{u}_{i_k} \).

Note that the following properties hold on the two individual subpaths: (1) in both \( \Pi' \) and \( \Pi'' \), only the first node might be blue and only the last node might be red. (2) the last node \( \hat{u}_{i_{z'}} \) of \( \Pi' \) is such that \( z' < n \), and the first node \( \hat{u}_{i_{z''}} \) of \( \Pi'' \) is such that \( z'' < n \). Note also that \( i_j \) might coincide with \( i_1, i_k \), or both. In this case, one of the two subpaths (or both) reduces to a single node. We prove the lemma separately for these two subpaths.

(Subpath \( \Pi' \)) We prove the property by induction on the number \( t \) of nodes in the subpath. If \( t = 1 \) the claim is immediate, since by assumption the only node \( \hat{u}_{i_{z'}} \) in the subpath is such that \( z' < n \), thus \( \hat{u}_{i_{z'+1}} \) exists.

Let therefore \( \Pi' = \hat{u}_{j_1} \leadsto \hat{u}_{j_{t-1}} \to \hat{u}_{j_t} \) have length \( t \geq 2 \). By assumption, \( j_t < n \) and no node other than (possibly) \( \hat{u}_{j_1} \) is blue: it follows that \( \lambda(\hat{u}_{j_1}) = \lambda(\hat{u}_{j_{t+1}}) \). Consider the parents of these two nodes, \( \pi(\hat{u}_{j_t}) = \hat{u}_{j_{t-1}} \) and \( \pi(\hat{u}_{j_{t+1}}) = \hat{v} \). Since \( \lambda(\hat{u}_{j_t}) = \lambda(\hat{u}_{j_{t+1}}) \), by Wheeler property (ii) it must be the case that \( \hat{u}_{j_{t-1}} < \hat{v} \). We can say more: since by assumption \( \hat{u}_{j_{t-1}} \) is not red, it must be the case that \( \hat{u}_{j_{t-1}} \prec_{\text{pred}} \hat{v} \), i.e. that \( \hat{v} = \hat{u}_{j_{t-1}+1} \). Assume, for contradiction, that there exists a node \( \hat{w} \) such that \( \hat{u}_{j_{t-1}} \prec_{\text{pred}} \hat{w} \prec \hat{v} \). Let \( c = \lambda(\hat{u}_{j_t}) = \lambda(\hat{u}_{j_{t+1}}) \). We have two cases. If \( c \in \text{out}(\hat{w}) \), then by Wheeler (ii) it must be the case that \( \hat{u}_{j_t} <_{\text{pred}} \text{child}_c(\hat{w}) < \hat{u}_{j_{t+1}} \), a contradiction. If \( c \notin \text{out}(\hat{w}) \), then \( \text{out}(\hat{u}_{j_{t-1}}) \neq \text{out}(\hat{w}) \), therefore \( \hat{u}_{j_{t-1}} \) is red: also a contradiction. We obtained that \( \hat{u}_{j_{t-1}+1} \to \hat{u}_{j_{t+1}} \) is an edge in the tree and, in particular, \( j_{t-1} < n \). We can therefore apply the inductive hypothesis to the subpath \( \hat{u}_{j_1} \leadsto \hat{u}_{j_{t-1}} \) of length \( t - 1 \) and obtain that \( \hat{u}_{j_{t-1}+1} \leadsto \hat{u}_{j_{t+1}} \) is a path in the tree. Merging these two results, we obtain that \( \hat{u}_{j_{t+1}} \leadsto \hat{u}_{j_{t-1}+1} \to \hat{u}_{j_{t+1}} \) is a path in the tree (adjacent to \( \Pi' \)).

(Subpath \( \Pi'' \)) We prove the property by induction on the number \( t \) of nodes in the subpath. If \( t = 1 \) the claim is immediate, since by assumption the only node \( \hat{u}_{i_{z'}} \) in the subpath is such that \( z' < n \), thus \( \hat{u}_{i_{z'+1}} \) exists.

Let therefore \( \Pi'' = \hat{u}_{j_1} \to \hat{u}_{j_2} \leadsto \hat{u}_{j_t} \) have length \( t \geq 2 \). By assumption, \( j_1 < n \) and no node other than (possibly) \( \hat{u}_{j_1} \) is red.
Let $c = \lambda(\hat{u}_j)$. Since by assumption $\hat{u}_j$ is not red and $j_1 < n$, then $c \in \text{out}(\hat{u}_j + 1)$. Then, by Lemma 3 we have that $\hat{u}_{j_2 + 1} = \text{child}_c(\hat{u}_{j_1 + 1})$, thus $\hat{u}_{j_1 + 1} \to \hat{u}_{j_2 + 1}$ is an edge in the tree. In particular, $j_2 < n$. By inductive hypothesis, $\hat{u}_{j_2 + 1} \sim \hat{u}_{j_1 + 1}$ is a path in the tree. Merging these two results, we obtain that $\hat{u}_{j_1 + 1} \to \hat{u}_{j_2 + 1} \sim \hat{u}_{j_k + 1}$ is a path in the tree (adjacent to $\Pi''$).

To conclude, we merge the two results obtained for $\Pi'$ and $\Pi''$ and obtain our claim: $\hat{u}_{i_1 + 1} \to \hat{u}_{i_2 + 1} \sim \hat{u}_{i_k + 1}$ is a path in the tree (adjacent to $\Pi$). □

P Examples of Climb, Section 4

Example of Case 1 Consider Figure 1 and suppose we want to compute $\phi(2)$. First, we find a (any) red descendant of 2: let’s say we pick node 14 (the same reasoning holds with red node 3). Note that we have explicitly stored (orange dashed arrow) $\phi(14) = 6$. Thus, moreover, that the path connecting 2 and 14 has length 1 and is labeled with string $S = b$. Lemma 10 tells us that, along the path labeled $S$ connecting $\phi(2) = 3$ and $\phi(14) = 6$, the nodes are always adjacent in Wheeler order with the relative nodes in the path 2 $\sim$ 14. By applying our formula, we obtain

$$\phi(2) = \text{LAQ}(\phi(14), \text{depth}(14) - \text{depth}(2)) = \text{LAQ}(6, 1) = 3$$

Example of Case 2.1 Consider Figure 1 and suppose we want to compute $\phi(\hat{u}_i) = \phi(24)$. Node $\hat{u}_j = 1$ is the lowest ancestor of 24 such that the complete subtree rooted in 1 contains colored nodes. In this particular case, 1 is blue so we follow the explicit edge $\phi(1) = 2 = \hat{u}_{j+1}$. Moreover, we find the successor $\hat{u}_k$ of 1 in $\Pi = 1 \to 22 \sim 24$ with $t = \text{depth}(24) - \text{depth}(1) = 3$ and $\hat{u}_k = \text{LAQ}(24, t - 1) = 22$. Since 1 is not red and 22 is the second child of 1, nodes 1 and 2 have the same outgoing labels and therefore the node 14 on the path 2 $\sim$ 16 $\sim$ 1 must be the second child of 2. By definition of $\hat{u}_j = 1$, no node in the complete subtree rooted in 22 is colored: this subtree is therefore isomorphic with the complete subtree rooted in 14. It follows that the relative position of $\hat{u}_{i+1} = 16$ in the subtree rooted 14 is the same as that of $\hat{u}_i = 24$ in the subtree rooted 22: we can therefore find node 16 with an isomorphic descendant query.

Example of Case 2.2.1 Consider Figure 1 and suppose we want to compute $\phi(\hat{u}_i) = \phi(5)$. Node $\hat{u}_j = 3$ is the lowest ancestor of 5 such that the complete subtree rooted in 3 contains a colored node. Let $t = \text{depth}(\hat{u}_i) - \text{depth}(\hat{u}_j) = 2$. We find $\hat{u}_k = \text{LAQ}(\hat{u}_i, t - 1) = 4$. Node $\hat{u}_k = 4$ is a $\phi$-sample of type 2. Then, $\phi(\hat{u}_k) = \hat{u}_{k+1} = 11$ is stored explicitly and we retrieve it in constant time. By definition of $\hat{u}_j$, no node in the subtree rooted in 4 is colored. Then, this subtree and the one rooted in 11 are isomorphic and we can find $\phi(\hat{u}_{i+1}) = 12$ with an isomorphic descendant query.

Example of Case 2.2.2 Consider Figure 1 and suppose we want to compute $\phi(\hat{u}_i) = \phi(6)$. Node $\hat{u}_j = 3$ is the lowest ancestor of 6 such that the complete subtree rooted in 3 contains a colored node. In this particular case, $\hat{u}_k$ coincides with $\hat{u}_i$ and $\phi(\hat{u}_k)$ is not a $\phi$-sample of type 2. In fact, as proved above, $\phi(\hat{u}_j) = \hat{u}_{j+1} = 4$ (which we retrieve in constant time, being it a $\phi$-sample of type 1) is adjacent in Wheeler order to node 3. Now, $\hat{u}_k = 6$ and $\hat{u}_{k+1} = 5$ are both reached by following label b from $\hat{u}_j = 3$ and $\hat{u}_{j+1} = 4$, respectively. Since 3 is red and 3 $<_{\text{pred}}$ 4, we can find $\hat{u}_{k+1} = 5$ with an isomorphic child operation. Finally, as noted in the previous examples the subtrees rooted in 5 and 6 are isomorphic, so we can find $\phi(\hat{u}_i) = \phi(6) = 5$ in constant time with an isomorphic descendant query.
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