Exact solutions of the mKdV equation

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Abstract. Applying the first integral method, and combining the computation software, Maple, we find exact traveling wave solutions of the mKdV equation. The method is based on the ring theory of commutative algebra.

Keywords: The first integral method; The mKdV equation; Exact solutions.

1. Introduction
Most of the problems arising in science and engineering are described by nonlinear partial differential equations (NPDEs). Investigating exact solutions of such equations are of essential importance. In recent years, a number of methods have been proposed, for solving such equations, Extended trial equation method [1], \( G'/G \)-expansion method [2-3], Tanh [4-5], Homogeneous balance method [6-7], Exp-function method [8], Hirota’s bilinear methods [9], First integral method [10-12], and so on.

In this paper, we shall study the mKdV equation and find its exact solution. The rest of the Paper is organized as follows. In the following Section 2, the description of the proposed method will be given. The applications of the proposed method to the mKdV equation are illustrated in Section 3. Finally, some conclusions are given.

2. Description of the first integral method
Consider the general form of the nonlinear partial differential equation on the form:

\[
F(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0,
\]

Applying the transformation \( u = u(\xi) \) which \( \xi = x - ct \), and the use of some mathematical operations, converts (1) into a second-order nonlinear ordinary differential equation as the following form

\[
F(u, u', u'', \ldots) = 0,
\]

Where prime denotes the derivative with respect to \( \xi \). By introducing new variables \( X(\xi) = u(\xi) \) and \( Y(\xi) = u'(\xi) \), (2) changes into a system of ordinary differential equations as
Now, if the solution of the system (3) is obtained, the solution of (2) will be in hand, since ordinary differential (2) is equivalent to the system (3) which is not easy to solve and there is not any systematic theory to find the first integral of this system (3). It is known that division theorem which is based on the ring theory of commutative algebra will be helpful to obtain a first integral of (3), and so the solution of equation (1).

Division Theorem Suppose that \( P(w, z) \), \( Q(w, z) \) are polynomials defined on the complex field \( \mathbb{C} \), which \( P(w, z) \) is irreducible in the complex field \( \mathbb{C} \). If \( Q(w, z) \) vanishes at all zero points of \( P(w, z) \), then there exists a polynomial \( G(w, z) \) in \( \mathbb{C} \), such that

\[
Q(w, z) = P(w, z) \cdot G(w, z).
\] (4)

3. Applications of method

Let us consider the following mKdV equation:

\[
u_t + \alpha u^2 \nu_x + \beta \nu_{xxx} = 0,
\] (5)

Where \( \alpha, \beta \) are nonzero constants. Applying the transformation \( u(x, t) = u(\xi) \), which \( \xi = x - ct \), converts (5) into an ODE as

\[-cu' + \alpha u^2 u' + \beta u''' = 0.
\] (6)

Taking anti-derivative of (6) with integral constant zero yields

\[
u'' = -\frac{\alpha}{3\beta} u^3 + \frac{c}{\beta} u.
\] (7)

Using (3), to get the following system

\[
\begin{align*}
X'(\xi) &= Y(\xi) \\
Y'(\xi) &= -\frac{\alpha}{3\beta} X^3(\xi) + \frac{c}{\beta} X(\xi),
\end{align*}
\] (8)

Now, the Division Theorem is implemented to seek the first integral to (8). Suppose that \( X = X(\xi) \) and \( Y = Y(\xi) \) are the nontrivial solutions to (8) and \( p(X, Y) = \sum_{i=0}^{m} a_i(X)Y' \) is an irreducible polynomial in \( \mathbb{C}[w, z] \) such that

\[
p(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y'(\xi) = 0,
\] (9)
\(a_i(X(\xi))(i=0,1,\ldots,m)\) are polynomials of \(X\) and \(a_m(X(\xi)) \neq 0\). Equation (9) is also
Called the first integral to (8). Due to the Division Theorem, there exists a polynomial
\(q(X(\xi), Y(\xi)) = g(X(\xi)) + h(X(\xi))Y(\xi)\) in \(C[w, z]\) such that

\[
\frac{dp}{d\xi} = q(X(\xi), Y(\xi))\sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi).
\]

(10)

In this example, Let’s consider special cases, \(m = 1\), and \(m = 2\), in (9).
Case A: \(m = 1\)
Comparing the coefficients of \(Y^i (i=0,1,2)\) in both sides of (10), leads to

\[
a'_1(X) = h(X)a_1(X),
\]

(11)

\[
a'_0(X) = g(X)a_1(X) + h(X)a_0(X),
\]

(12)

\[
a_1(X)(-\frac{\alpha}{3\beta}X^3 + \frac{c}{\beta}X) = g(X)a_0(X).
\]

(13)

Since \(a_i(X)(i=0,1)\) are polynomials, then from (11) one can deduce that \(a_i(X)\) is a
Constant and \(h(X) = 0\). For convenience, we take \(a_1(X) = 1\). Now, by balancing the
Degrees of \(a_0(X)\) and \(g(X)\), we can conclude that \(\text{deg}[g(X)] = 1\). Suppose that
\(g(X) = A_0X^2 + B_0, (A_0 \neq 0)\), then \(a_0(X)\) will be computed as

\[
a_0(X) = \frac{1}{2}A_1X^2 + B_0X + A_0,
\]

(14)

Where \(A_0\) is an integrating constant. Substituting (14) into (13) and setting all the coefficients of
\(X\) to zero, the following two sets of solutions will be obtained by Maple.

\[
B_0 = 0, \quad A_1 = \frac{1}{3}\sqrt{-\frac{6\alpha}{\beta}}, \quad c = \frac{1}{3}\beta A_0 \sqrt{-\frac{6\alpha}{\beta}}, \quad (15)
\]

\[
B_0 = 0, \quad A_1 = -\frac{1}{3}\sqrt{-\frac{6\alpha}{\beta}}, \quad c = -\frac{1}{3}\beta A_0 \sqrt{-\frac{6\alpha}{\beta}}, \quad (16)
\]

Setting (15) in (9) gives

\[
Y(\xi) = -\frac{1}{6}\sqrt{-\frac{6\alpha}{\beta}}X^2(\xi) - A_0
\]

(17)

Substituting (17) into (8), an exact solution of Eq. (5) are given as
\[ u_i(x,t) = -\frac{\sqrt{A_0} \sqrt{\frac{6\alpha}{\beta}}}{\sqrt{\frac{\alpha}{\beta}}} \tan \left[ \sqrt{\frac{6}{A_0}} \sqrt{\frac{6\alpha}{\beta}} (x + \frac{1}{3} \beta A_0 \sqrt{\frac{6\alpha}{\beta} t} + 1) - C_i \right] \]  

(18)

Where \( C_i \) is an arbitrary constant. Similarly, setting (16) in (9) gives

\[ Y(\xi) = \frac{1}{6} \sqrt{\frac{6\alpha}{\beta}} X^2 (\xi) - A_0 \]  

(19)

Substituting (19) into (8), an exact solution of Eq. (5) are given as

\[ u_2(x,t) = -\frac{\sqrt{A_0} \sqrt{\frac{6\alpha}{\beta}}}{\sqrt{\frac{\alpha}{\beta}}} \tanh \left[ \sqrt{\frac{6}{A_0}} \sqrt{\frac{6\alpha}{\beta}} (x + \frac{1}{3} \beta A_0 \sqrt{\frac{6\alpha}{\beta} t} + 1) - C_i \right] \]  

(20)

Where \( C_i \) is an arbitrary constant.

**Fig. 1:** 3D graphs of \( u_i(x,t) \) and \( u_2(x,t) \) for \( A_0 = 2, \alpha = -2, \beta = 2, C_1 = 1 \) in the intervals \( x, t \in (-100,100) \).

Case A: \( m = 2 \)

Comparing the coefficients of \( Y^i (i = 0,1,2,3) \) in both sides of (10), leads to

\[ a'_2(X) = h(X)a_2(X), \]  

(21)

\[ a'_1(X) = g(X)a_2(X) + h(X)a_1(X), \]  

(22)
\[ a'(X) + 2a_2(X)(-\frac{\alpha}{3\beta}X^3 + \frac{c}{\beta}X) = g(X)a_1(X) + h(X)a_0(X) \] (23)

\[ a_1(X)(-\frac{\alpha}{3\beta}X^3 + \frac{c}{\beta}X) = g(X)a_0(X) \] (24)

Since \(a_i(X) (i = 0, 1, 2)\) are polynomials, then from (21) one can deduce that \(a_2(X)\) is a constant and \(h(X) = 0\). For convenience, we take \(a_2(X) = 1\). Now, by balancing the

Degrees of \(a_0(X)\), \(a_1(X)\) and \(g(X)\), we can conclude that \(\deg g(X) = 1\). Suppose that \(g(X) = A_1X + B_0, (A_1 \neq 0)\), then \(a_0(X)\) and \(a_1(X)\) will be computed as

\[ a_1(X) = \frac{1}{2}A_1X^2 + B_0X + A_0, \] (25)

\[ a_0(X) = \left(\frac{1}{8}A_1^2 + \frac{\alpha}{6\beta}\right)X^4 + \frac{1}{2}B_0A_1X^3(\xi) + \left(-\frac{c}{\beta} + \frac{1}{2}B_0^2 + \frac{1}{2}A_1A_0\right)X^2(\xi) + B_0A_0X(\xi) + D \] (26)

Where \(D\) is an integrating constant. Substituting \(a_0(X)\), \(a_1(X)\) and \(g(X)\) into (24) and setting all the coefficients of \(X\) to zero, the following two sets of solutions will be obtained by Maple.

\[ B_0 = 0, A_0 = -\frac{c}{\alpha}\sqrt{-\frac{6\alpha}{\beta}}, A_1 = \frac{2}{3}\sqrt{-\frac{6\alpha}{\beta}}, D = -\frac{3c^2}{2\alpha\beta} \] (27)

\[ B_0 = 0, A_0 = \frac{c}{\alpha}\sqrt{-\frac{6\alpha}{\beta}}, A_1 = -\frac{2}{3}\sqrt{-\frac{6\alpha}{\beta}}, D = -\frac{3c^2}{2\alpha\beta} \] (28)

Setting (27) in (9) gives

\[ Y(\xi) = -\frac{1}{6}\sqrt{-\frac{6\alpha}{\beta}X^2(\xi)} - A_0 \] (29)

Substituting (29) into (8), an exact solution of Eq. (5) are given as

\[ u_3(x,t) = \frac{\sqrt{3ac}}{\alpha}\tanh\left[\sqrt{-\frac{2\alpha}{\beta}\sqrt{\frac{ac}{2\alpha}}(x-\alpha t-C_1)}\right], \] (30)

Where \(C_1\) is an arbitrary constant. Similarly, setting (28) in (9) gives
Substituting (31) into (8), an exact solution of Eq. (5) are given as

\[ u_4(x,t) = -\frac{\sqrt{3\alpha c}}{\alpha} \tanh\left(\sqrt{\frac{-2\alpha}{\beta}} \frac{\sqrt{\alpha c}}{2\alpha} (x - ct - C_1)\right) \]  

(32)

Where \( C_1 \) is an arbitrary constant.

**Fig. 2:** 3D graphs of \( u_5(x,t) \) and \( u_4(x,t) \) for \( A_0 = 2, \alpha = -2, \beta = 2, C_1 = 1 \) in the intervals \( x, t \in (-100,100) \).

4. Conclusion

In this manuscript, the first integral method has been applied successfully for solving the mKdV equation. This method has been led to exact solutions. And \( u_5(x,t) \) and \( u_4(x,t) \) are new solutions not mentioned in other literatures. Therefore, we can conclude that the method is very effective for solving some nonlinear partial differential equations.

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