A NEW ALGORITHM FOR COMPLEX NON ORTHOGONAL JOINT DIAGONALIZATION BASED ON SHEAR AND GIVENS ROTATIONS

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ABSTRACT

This paper introduces a new algorithm to approximate non orthogonal joint diagonalization (NOJD) of a set of complex matrices. This algorithm is based on the Frobenius norm formulation of the JD problem and takes advantage from combining Givens and Shear rotations to attempt the approximate joint diagonalization (JD). It represents a non trivial generalization of the JDi (Joint Diagonalization) algorithm (Soulloumiac 2009) to the complex case. The JDi is first slightly modified then generalized to the CJDi (i.e. Complex JDi) using complex to real matrix transformation. Also, since several methods exist already in the literature, we propose herein a brief overview of existing NOJD algorithms then we provide an extensive comparative study to illustrate the effectiveness and stability of the CJDi w.r.t. various system parameters and application contexts.

Index Terms— Non orthogonal joint diagonalization, Performance comparison of NOJD algorithm, Givens and Shear rotations.

1. INTRODUCTION

Joint diagonalization problem and its related algorithms are found in various applications, especially in blind source separation (BSS) and independent component analysis (ICA). In such problems, it is desired to diagonalize simultaneously a set of square matrices. These matrices can be covariance matrices estimated on different time windows, intercorrelation matrices with time shifts, fourth or higher order cumulant slice matrices or spatial time-frequency matrices.

Mathematically, the joint diagonalization problem can be stated as follows: Given a set of $K$ square matrices $\{M_k \in \mathbb{C}^{N \times N}, k = 1, \ldots, K\}$, find a matrix $V$ such that the transformed matrices $D_k = VM_kV^H$ are as diagonal as possible.

In the context of BSS, $\{M_k\}$ are complex matrices sharing the same structure defined by $M_k = ADA^H$ where $\{D_k\}$ are diagonal matrices and $A$ is an unknown mixing matrix. The problem consists of finding the diagonalizing matrix $V$ that left inverts $A$ and transforms $VM_kV^H$ into diagonal matrices.

Various algorithms have been developed to solve the JD problem. These algorithms can be classified in two classes, orthogonal joint diagonalization (OJD) and non orthogonal JD (NOJD). The first class imposes $V$ to be orthogonal by transforming the JD problem into an OJD problem using the whitening step. This step can introduce errors which might reduce the diagonalization performance. The most popular algorithm for OJD is JADE which is a Jacobi-like algorithm based on Givens rotations. The NOJD class treats the problem without any whitening step. Among the first NOJD algorithms, one can cite the Iterative Decorrelation Algorithm (IDA) developed for complex valued NOJD in [11] [12] and the AC-DC (Alternating Columns-Diagonal Centers) given in [13]. The latter suffers from slow linear convergence performance. Many other algorithms have been developed by considering specific criteria or constraints in order to avoid trivial and degenerate solutions [14] [15]. These algorithms can be listed as follow: QDiag [16] (Quadratic Diagonalization algorithm) developed by Vollgraf and Obermayer where the JD criterion is rearranged as a quadratic cost function; FAJD [17] (Fast Approximative Joint Diagonalization) developed by Li and Zhang where the diagonalizing matrix is estimated column by column; UWEDGE [14] (UnWeighted Exhaustive joint Diagonalization with Gauss iErations) developed by Tichavsky and Yeredor where numerical optimization is used to get the JD solution; JUST [18] (Joint Unitary and Shear Transformations) developed by Herrardjene, Abed-Meraim and Belouchrani where the algebraic joint diagonalization is considered; CVFFDiag [19] [20] (Complex Valued Fast Frobenius Diagonalization) developed by Xu, Feng and Zheng where first order of Taylor expansion is used to minimize the JD criterion; ALS [21] (Alternating Least Squares) developed by Trainini and Moreau where the mixing and diagonal matrices are estimated alternatively by using least squares criterion and LUCJD [22] [23] (LU decomposition for Complex Joint Diagonalization) developed by Wang, Gong and Lin where the diagonalizing matrix is estimated by LU decomposition.

In this paper, we generalize the JDi algorithm developed by Soulloumiac in [24] for real joint diagonalization by using Shear and Givens rotations in the complex case. We transform the considered complex matrices to real symmetric ones allowing us to apply the JDi algorithm. At the convergence, the diagonalizing (mixing) matrix is retrieved from the real diagonalizing one by taking into account the particular structure of the latter (see subsection 4.4 for more details). The main drawback of this algorithm’s version is that it does not take into consideration the particular structure of the real valued diagonalizing matrix along the iterations which results in a slight performance loss. To avoid this drawback, we propose an improved version which uses explicitly the complex matrices using special structure of Shear and Givens rotations to increase both the convergence rate and the estimation accuracy, while reducing the overall computational cost. Another contribution of this paper is a comparative study of different non orthogonal joint diagonalization algorithms with respect to their robustness in severe JD conditions:

A first attempt to generalize the JDi has been given in [25]. Unfortunately, the latter algorithm has been found to diverge in most simulation contexts considered in section 6, and hence, it has been omitted in our comparative study.
i.e. large dimensional matrices, noisy matrices, ill-conditioned matrices and large valued MOU (modulus of uniqueness [24] [26] [27].

The rest of the paper is organized as follows. In section 2 the problem formulation, mathematical notations and paper’s main objectives are stated. Section 3 introduces a brief overview of major NOJD algorithms and existing JD criteria. Section 4 presents the basic generalization of JD algorithm to the complex case while section 5 is dedicated to the proposed method’s developments. In particular, we present in this section a complex implementation of our method with a computational cost comparison with existing NOJD algorithms. Simulation based performance assessment for exact and approximate joint diagonalizable matrices are provided in section 6. Section 7 is dedicated to the concluding remarks.

2. PROBLEM FORMULATION

Consider a set of $K$ square matrices, $M_k \in \mathbb{C}^{N \times N} \ \ k = 1, \ldots, K$ sharing the following decomposition:

$$M_k = A D_k A^H, \quad k = 1, \ldots, K$$

(1)

where $D_k$ are diagonal matrices and $A$ is the unknown $N \times N$ square complex non-defective matrix known as a mixing matrix. $A^H$ denotes the transpose conjugate of $A$.

The problem of joint diagonalization consists of estimating matrices $A$ and $D_k, k = 1, \ldots, K$ given the observed matrices $M_k, k = 1, \ldots, K$. Equivalently, the JD problem consists of finding the transformation $V$ such that $V M_k V^H$ are diagonal.

Note that JD decomposition given in equation (1) is not unique. Consider a set of $K$ square matrices, $M_k \in \mathbb{C}^{N \times N} \ \ k = 1, \ldots, K$ sharing the following decomposition:

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$$M_k = A D_k A^H + \Xi_k, \quad k = 1, \ldots, K$$

(2)

where $\Xi_k$ are perturbation (noise) matrices.

3. REVIEW OF MAJOR NOJD ALGORITHMS

In this section, we present a brief overview of NOJD algorithms. First, the different JD criteria are presented before giving a brief description for each considered algorithm.

3.1. Joint diagonalization criteria

In this subsection, we present different criteria considered for JD problem. The first one is given in [13] and expressed as follow:

$$C_{DLS}(A_1, D_1, D_2, \ldots, D_K) = \sum_{k=1}^{K} w_k \left\| M_k - A D_k A^H \right\|_F^2$$

(3)

where $\| \cdot \|_F$ refers to the Frobenius norm, $w_k$ are some positive weights and $(A_1, D_1, D_2, \ldots, D_K)$ are the searched mixing matrix and diagonal matrices, respectively. This cost function is called in [23], the Direct Least-Squares (DLS) criterion as it takes into account the mixing matrix rather than the diagonalizing one.

Unlike the previous JD criterion, the second one is called the Indirect Least Squares (ILS) criterion and takes into account the diagonalizing matrix $V$. The latter is expressed as [28]:

$$C_{ILS}(V) = \sum_{k=1}^{K} \left\| V M_k V^H - \text{diag} \left( V M_k V^H \right) \right\|_F^2$$

(4)

This criterion is widely used in numerous algorithms [2] [13] [26]. However, the minimization of (4) might lead to undesired solutions e.g. trivial solution $V = 0$ or degenerate solutions where $\det(V) = 0$. Consequently, the algorithms based on the minimization of (4) introduce a constraint to avoid these undesirable solutions. In [2], the estimated mixing matrix (resp. the diagonalizing matrix) have to be unitary so that undesired solutions are avoided. In our developed algorithm, the diagonalizing matrix is estimated as a product of Givens and Shear rotations where undesired solutions are excluded implicitly. In [23], the diagonalizing matrix $V$ is estimated in the form of LU (or LQ) factorization where $L$ and $U$ are lower and upper triangular matrices with ones at the diagonals and $Q$ is a unitary matrix. These two factorizations impose a unit valued determinant for the diagonalizing matrix. Previous factorizations (Givens, Givens and shear, LU and LQ factorizations) represent the different constraints used to avoid undesired solutions.

In [17], the undesired solutions are excluded by considering the penalization term $-\beta \log(|\det(V)|)$ so that the JD criterion becomes:

$$C_{ILS}(V) = \sum_{k=1}^{K} \left\| V M_k V^H - \text{diag} \left( V M_k V^H \right) \right\|_F^2 - \beta \log(|\det(V)|)$$

(5)

Another criterion has been introduced in [14] taking into account two matrices $(V, A)$ which are the diagonalizing matrix and its residual mixing one, respectively. It is expressed as:

$$C_{LS}(V, A) = \sum_{k=1}^{K} \left\| V M_k V^H - A \text{diag} \left( V M_k V^H \right) A^H \right\|_F^2$$

(6)

The previous criterion $C_{LS}$ fuses the direct and indirect forms by relaxing the dependency between $A$ and $V$ and it is known as least squares criterion. In [1], another criterion is developed for positive definite target matrices as follow:

$$C_{LL}(V) = \sum_{k=1}^{K} \log \frac{\det \left( \text{diag} \left( V M_k V^H \right) \right)}{\det(V M_k V^H)}$$

(7)

This criterion can not be applied for non positive target matrices, thus some real life applications can not be treated by minimizing (7). In [23], a scale invariant criterion in $V$ is introduced as:

$$C_{ILS2}(V) = \sum_{k=1}^{K} \left\| M_k - V^{-1} \text{diag} \left( V M_k V^H \right) V^{-H} \right\|_F^2$$

(8)

Note that $C_{ILS2}(DV) = C_{ILS2}(V)$ for any diagonal matrix $D$. This criterion is used only for real JD.

3.2. NOJD Algorithms

We describe herein the basic principles of each of major NOJD algorithms considered in our comparative study given in section 6.

ACDC [13]: This algorithm is developed by Yeredor in 2002. It
proceeds by minimizing $C_{DLS}$ criterion given in (3) by alternating two steps: The first one is the AC (Alternating Columns) step and the second one is the DC (Diagonal Centers) step. For AC step, only one column in the mixing matrix is updated by minimizing the cited criterion while the other parameters are kept fixed. For the DC step, the diagonal matrices entries are estimated by keeping the mixing matrix fixed. Note that, the DC phase is followed by several AC phases in order to guarantee the algorithm’s convergence.

**FAJD** [17]: This algorithm is developed by Li and Zhang in 2007. It estimates the diagonalizing matrix by minimizing the modified indirect least squares $C^{I}_{ILS}$ criterion given in [5]. At each iteration, the algorithm updates one column of the diagonalizing matrix while keeping the others fixed. This process is repeated until reaching the convergence state. Note that the value assigned to $\beta$ in [17] is one.

**QDiag** [16]: This algorithm is developed by Vollgraf and Obermayer in 2006. It minimizes the indirect least squares criterion given in [4]. At each iteration, the algorithm updates one column of the diagonalizing matrix while the others are kept fixed. This step is repeated until reaching the convergence state. Note that there is no update step for target matrices and the condition to avoid undesired solutions is implicitly included by normalizing the diagonalizing matrix columns.

**WEDGE** [13]: This algorithm is developed by Tichavsky and Yeredor in 2008. It minimizes the $C_{ILS}$ criterion given in [6] and computes in alternative way the residual mixing and diagonalizing matrices. At first, the diagonalizing matrix $V$ is initialized as $M_{1}^{-\frac{1}{2}}$. This value is introduced in the considered criterion to find the mixing matrix $A$. The minimization w.r.t. $A$ is achieved by using numerical Gauss iterations. Once an estimate $A$ of the mixing matrix is obtained, the diagonalizing matrix is updated as $V^{(i)} = A^{-1}V^{(i-1)}$ ($i$ is the iteration index). The previous process is repeated until the convergence is reached.

**JUST** [18]: This algorithm is developed by Iferroudjene, Abed-Meraim and in Belouchrani 2009. It is applied to target matrices sharing the algebraic joint diagonalization structure $M_{k} = A D_{k} A^{H}$. Hence in our context, given the target matrices sharing the decomposition described in (1), the latter are transformed to another set of new target matrices sharing the algebraic joint diagonalization structure by right multiplying them by the inverted first target matrix. Once the new set of target matrices is obtained, JUST algorithm estimates the diagonalizing matrix by successive Shear and Givens rotations minimizing $C_{ILS}$ criterion.

**CV FFDiag** [19]: This algorithm’s idea is given in [11] and it is formulated as an algorithm for real NOJD in [20]. Xu, Feng and Zheng generalized the latter to the complex NOJD in [19]. It minimizes $C_{ILS}$ criterion and estimates the diagonalizing matrix $V$ in an iterative scheme using the following form $V^{(i)} = (I + W^{(i)}) V^{(i-1)}$ where $W^{(i)}$ is a matrix having null diagonal elements. The latter is estimated in each iteration by optimizing the first order Taylor expansion of $C_{ILS}$.

**LUCJD** [22]: This algorithm is developed by Wang, Gong and Lin in 2012. It considers $C_{ILS}$ criterion as the CVFFDiag. It decomposes the mixing matrix in its LU form where $L$ and $U$ are lower and upper triangular matrices with diagonal entries equal to one. This algorithm is developed in [23] for real NOJD and generalized to complex case in [22]. Matrices $L$ and $U$ are optimized in alternating way by minimizing $C_{ILS}$. Note that the entries of $L$ and $U$ are updated one by one (keeping the other entries fixed).

**ALS** [21]: This algorithm is developed by Trainini and Moreau in 2011. It minimizes $C_{DLS}$ criterion as the ACDC and relaxes the relationship between $A^{H}$ and $A$. The algorithm is developed by considering three steps, the first one estimates diagonal matrices $D_{k}$ by keeping $A$ and $A^{H}$ fixed. The second one uses the obtained $D_{k}$ and fixed $A^{H}$ to compute the mixing matrix $A$ and the last step uses the obtained $D_{k}$ and $A$ from the first and second steps, respectively, to estimate $A^{H}$. These steps are realized for each iteration and repeated until the convergence state is reached.

Note that other algorithms exist in the literature, developed for special cases, but are not considered in our study. For example, in [11] the developed algorithm is applied only for positive definite matrices. In [23], the developed algorithm is a direct method (not iterative) which makes it more sensitive to difficult JD problem (the algorithm is not efficient when the number of matrices is less than the matrix dimension).

## 4. BASIC GENERALIZATION OF JDI ALGORITHM

We introduce herein the basic generalization of JDi algorithm, given in [24], from real to complex case. First, the basic idea is to transform hermitian matrices obtained in [24] to real symmetric ones given by [10] to which, we apply the JDi algorithm. Then in section5 by modifying the first approach, we develop the CJDi algorithm which uses the hermitian matrices directly.

### 4.1. Complex to real matrix transformation

The first idea of our approach consists of transforming the original problem of complex matrix joint diagonalization into JD of real symmetric matrices which allows us to apply JDi algorithm. Hence, we transform the $K$ complex matrices into $2K$ hermitian matrices \{$\tilde{M}_{k}$, $k = 1, \ldots, 2K$\} according to:

$$
\tilde{M}_{2k} = \frac{1}{2} (M_{k} + M_{k}^{H}) = A \text{Re}(D_{k}) A^{H}
$$

$$
\tilde{M}_{2k-1} = \frac{1}{2} (M_{k} - M_{k}^{H}) = A \text{Im}(D_{k}) A^{H}.
$$

(9)

where \text{Re}() and \text{Im}() refer to the real part and imaginary part of a complex entity, respectively. Now, the $N \times N$ hermitian matrices are transformed into $2N \times 2N$ real matrices according to:

$$
f(\tilde{M}_{k}) = M_{k}
$$

$$
= \begin{bmatrix}
\text{Re} (\tilde{M}_{k}) & \text{Im} (\tilde{M}_{k}) \\
-\text{Im} (\tilde{M}_{k}) & \text{Re} (\tilde{M}_{k})
\end{bmatrix}, k = 1, \ldots, 2K
$$

(10)
Based on (9) and (10), one can easily see that matrices $\mathcal{M}_k$, $k = 1, \ldots, 2K$ share the appropriate JD structure, i.e.

$$
\mathcal{M}_{2k} = A \left[ \begin{array}{cc} \Re(D_k) & 0 \\ 0 & \Re(D_k) \end{array} \right] A^T
$$

$$
\mathcal{M}_{2k-1} = A \left[ \begin{array}{cc} \Im(D_k) & 0 \\ 0 & \Im(D_k) \end{array} \right] A^T
$$

(11)

where $A = f(\Lambda)$. This property allows us to apply JD algorithm to achieve the desired joint diagonalization.

Like in the JDi method, the real diagonalizing matrix $V$ associated to the complex one $V$, is decomposed as a product of generalized rotation matrices according to:

$$
V = \prod_{# \text{sweeps}} \prod_{1 \leq i < j \leq 2N} \mathcal{H}_j^i(\theta, y)
$$

(12)

where $# \text{sweeps}$ represents the sweeps (iterations) number and $\mathcal{H}_j^i(\theta, y)$ is the generalized rotation matrix given by:

$$
\mathcal{H}_j^i(\theta, y) = S_j^i(y)G_j^i(\theta)
$$

(13)

$G_j^i(\theta)$ and $S_j^i(y)$ being the elementary Givens and Shear rotation matrices which are equal to the identity matrix except for their $(i, i)$$^{th}$, $(i, j)$$^{th}$, $(j, i)$$^{th}$ and $(j, j)$$^{th}$ entries given by:

$$
\begin{bmatrix}
G_j^i(i, i) & G_j^i(i, j) \\
G_j^i(j, i) & G_j^i(j, j)
\end{bmatrix} =
\begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{bmatrix}
$$

(14)

$$
\begin{bmatrix}
S_j^i(i, i) & S_j^i(i, j) \\
S_j^i(j, i) & S_j^i(j, j)
\end{bmatrix} =
\begin{bmatrix}
\cosh(y) & \sinh(y) \\
\sinh(y) & \cosh(y)
\end{bmatrix}
$$

(15)

where $\theta$ and $y$ are the Givens angle and the Shear parameter, respectively. Based on these elementary transformations, we express next the transformed matrices as well as the JD criterion given in (4).

4.2. Matrix transformations

As shown in subsection 4.1 the set of complex matrices is transformed into a set of real symmetric ones, $\{\mathcal{M}_k, k = 1, ..., 2K\}$, to which all Givens and Shear rotations are applied. We denote by $\mathcal{M}_k$, $k = 1, \ldots, 2K$ the updated matrices when using the elementary rotations, i.e.:

$$
\mathcal{M}_k = \mathcal{H}_j^i(\theta, y)\mathcal{M}_k \mathcal{H}_j^i(\theta, y)^T
$$

(16)

Note that only the $i^{th}$ and $j^{th}$ rows and columns of $\mathcal{M}_k$ are transformed so that $(i, j)^{th}$ entries are twice affected by the latter transformation. These entries can be expressed as:

$$
\mathcal{M}_k'(i, j) = v^T\left[ \begin{array}{c}
\frac{M_k(i, i) + M_k(j, j)}{2} \\
\frac{M_k(i, j) - M_k(j, i)}{2}
\end{array} \right]
$$

(17)

where

$$
v = \begin{bmatrix}
\sinh(2y) \\
-\sin(2\theta) \cosh(2y) \\
\cos(2\theta) \cosh(2y)
\end{bmatrix}
$$

(18)

Souloumiac in [24] introduces a simplified JD criterion which is the sum of squares of $(i, j)^{th}$ entries. This JD criterion denoted $C_{ij}'$ can be expressed by using (17) as:

$$
C_{ij}' = \sum_{k=1}^{2K} M_k'(i, j)^2 = v^T R v
$$

(19)

where

$$
R = WW^T
$$

(20)

and

$$
W = \begin{bmatrix}
\frac{M_1(i, i) + M_1(j, j)}{2} & \cdots & \frac{M_1(i, i) + M_2K(j, j)}{2} \\
\frac{M_1(i, j) - M_1(j, i)}{2} & \cdots & \frac{M_2K(i, j) - M_2K(j, i)}{2} \\
M_1(i, j) & \cdots & M_2K(i, j)
\end{bmatrix}
$$

(21)

The results in [24] show that by minimizing this simplified criterion, joint diagonalization can be achieved in few iterations (see [24] for more details).

4.3. Direct generalization of JDi

In JD algorithm, $C_{ij}'$ JD criterion given in (19) is minimized under the hyperbolic normalization as follows:

$$(\theta, y) = \arg \min_{v, J_{1:3}=1} \left\{v^T R v \right\}
$$

(22)

with $J = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$. The solution of (22) is the eigenvector associated to the median generalized eigenvalue of $(R, J)$ denoted as $v = [v_1 \ v_2 \ v_3]^T$. Then, the optimal parameters can be expressed as:

$$
\begin{bmatrix}
\cosh(y) \\
\sinh(y) \\
\cos(\theta)
\end{bmatrix} = 
\begin{bmatrix}
\sqrt{\frac{1}{2}} \sqrt{1 + v_1^2} \\
\frac{1}{2} \cosh y \\
\sqrt{\frac{1}{2} \frac{1}{\sqrt{1 + v_1^2}}}
\end{bmatrix}
$$

(23)

A normalization is introduced in JD algorithm. It ensures that the estimated mixing matrix has columns of equal norm and determinant equal to one. The normalizing matrix $D_a$ can be expressed as:

$$
D_a = \frac{1}{\sqrt{\|a_1\| \cdots \|a_{2N}\|}} \begin{bmatrix}
1 & \cdots & 0 \\
\|a_1\| & \cdots & 0 \\
\|a_2\| & \cdots & 1
\end{bmatrix}
$$

(24)

where $a_1, ..., a_{2N}$ are the columns of the estimated mixing matrix $\tilde{A}$. We propose here to modify the JD algorithm by replacing the previous normalization step by the following eigenvector normalization:

$$
v \leftarrow \frac{v}{\|v^T J v\|}
$$

(25)

The modified JD algorithm is summarized in Table I, where $\tau$ and $M_{\text{th}}$ are a fixed threshold and maximum sweep number respectively, chosen to stop the algorithm.
and hence the \( \tilde{\theta} \) is orthogonal, one can represent it as:

**Proof**

for a given scalar factor matrix vector of solution of discrepancies according to the following lemma:

However, the specific structure of our real valued finalizing matrix matrices. However, the specific structure of our real valued matrices. Indeed, in the basic generalization of JDi algorithm. More precisely, we look for trans-

In this section, we give, first, the algorithm’s development based on real symmetric matrices given in equation (10). Then, direct complex implementation is explained in subsection [4.4].

### 4.4. Complex diagonalizing matrix retrieval

As mentioned in section 2, the JD problem has inherent indeterminacies in the sense that matrix \( V \) is estimated up to permutation and diagonal matrices. However, the specific structure of our real valued matrices \( \{M_{k=1\ldots 2K}\} \) given in (10) leads to extra indeterminacies according to the following lemma:

**Lemma 1** Define vectors \( d_i, i = 1, ..., N \) as

\[
d_i = [D_1 (i, i) \ldots D_K (i, i)]
\]

Under the condition that the \( 1 \times 2K \) dimensional vectors \( \lceil \text{Re} (d_i) \rceil \), \( \lceil \text{Im} (d_i) \rceil \) are pairwise linearly independent, the JD problem’s solution of \( \{M_1, ..., M_{2K}\} \) is such that, there exists a permutation matrix \( P \) satisfying:

\[
\hat{A} P = [A_1, ..., A_N]
\]

with \( A_i = [A(:, i), A(:, i + N)] U_i, A(:, i) \) being the \( i^{th} \) column vector of \( A \) and \( U_i \) is a \( 2 \times 2 \) orthogonal matrix i.e. \( U_i U_i^T = \alpha_i I_2 \) for a given scalar factor \( \alpha_i \).

**Proof:** This result can be deduced directly from Theorem 3 in [12].

To retrieve the original complex matrix \( \hat{A} \) from the estimated matrix \( \hat{A} \), one needs to find the permutation that associates correctly the \( i^{th} \) column of \( \hat{A} \) to its \( (i + N)^{th} \) one. For that, since matrix \( U_i \) is orthogonal, one can represent it as:

\[
U_i = \begin{bmatrix}
    a_i & b_i \\
    -b_i & a_i
\end{bmatrix}
\]

and hence

\[
A_i = \begin{bmatrix}
    \text{Re}(a_i) & \text{Im}(a_i) \\
    -\text{Im}(a_i) & \text{Re}(a_i)
\end{bmatrix} \begin{bmatrix}
    a_i & b_i \\
    -b_i & a_i
\end{bmatrix}
\]

(27)

### Table 1. Modified JDi algorithm

| Step 1 | Transform the complex matrices into real symmetric ones as explained in subsection [4.4]. |
|--------|------------------------------------------------------------------------------------------|
| Step 2 | Apply the modified JDi algorithm as given in Table II. |
| Step 3 | Retrieve the complex diagonalizing matrix as developed in subsection [4.4]. |

Table 2. Basic JDi algorithm generalization

where \( a_i \) is the \( i^{th} \) column vector of \( A \) and \( \hat{a}_i = c_ia_i \) with \( c_i = a_i + j b_i \). From (27), one can observe that two columns of \( A \):

\[
\hat{a}_i = \begin{bmatrix} \hat{a}_{i,1} \\ \hat{a}_{i,2} \end{bmatrix} \quad \text{and} \quad \hat{a}_j = \begin{bmatrix} \hat{a}_{j,1} \\ \hat{a}_{j,2} \end{bmatrix}
\]

can be associated if they satisfy the relation:

\[
\begin{align*}
|\hat{a}_{i,1} - \hat{a}_{j,2}| &= 0 \\
|\hat{a}_{i,2} + \hat{a}_{j,1}| &= 0
\end{align*}
\]

(28)

In practice, we solve equation (28) in the least squares sense to take into account the estimation errors. Once, this pairing process is achieved, the \( i^{th} \) column of matrix \( A \) is estimated (up to a scalar complex valued factor) from the first column of matrix \( \hat{A} \), as \( \hat{a}_i = \hat{a}_{i,1} - \hat{a}_{i,2} \).

Similarly, if \( P \) is the permutation pairing correctly the columns of \( \hat{A} \), then \( P^T \) is the one pairing correctly the rows of \( \hat{V} \); i.e. \( \hat{V} = P^T \hat{V} \).

### 5. CJDI ALGORITHM

In this section, we give, first, the algorithm’s development based on real symmetric matrices given in equation (10). Then, direct complex implementation is explained in subsection [4.4].

#### 5.1. Algorithm’s development based on real matrices

Note that the real symmetric matrices given in (10) have a special structure which can be used to simplify and improve the previous generalization of JDi algorithm. More precisely, we look for transformations that preserve the considered matrix structure along the iterations which allows us to skip the step of complex diagonalizing (resp. mixing) matrix retrieval. Indeed, in the basic generalization, the introduction of the elementary rotation matrix \( H_j^N(\theta, y) \) causes the loss of matrix structure. For example, when rotation indices are \( 1 \leq i < j \leq N \) then the updated entries \( M_{k}(i, j) \) and \( M_{k}(j, i) \) are not anymore equal to \( M_{k}(i + N, i + N) \) and \( M_{k}(j + N, j + N) \), respectively. Hence, to preserve the considered matrix structure, one needs to introduce a second elementary rotation which is \( H_{j+i+N}^N(\theta, y) \). The following lemma provides the solution to the previous problem.

**Lemma 2** 1) If the real symmetric matrices \( \{M_k\} \) are updated by the elementary rotation matrix \( H_j^N(\theta, y) \), \( 1 \leq i < j \leq N \), then the second elementary rotation which preserves the matrix structure in (10) is \( H_{j+i+N}^N(\theta, y) \), i.e. the generalized rotation matrix with same
angle and Shear parameters.

2) If the real symmetric matrices \( \{ \mathcal{M}_k \} \) are updated by the elementary rotation matrix \( \mathcal{H}^i_{i+j} \), then the second elementary rotation which preserves matrix structure in \( \mathcal{H}^i_{i+j} \) is such where the sign of the Shear parameter is inverted.

**Proof:** The proof of this lemma is given in appendix A.

The rotation parameters are now optimized in such a way we minimize the simplified JD criterion given in \( \text{[19]} \) for the transformed matrices:

\[
\mathcal{M}''_k = \mathcal{H}^i_{i+j} \mathcal{M}_k \mathcal{H}^i_{i+j} \mathcal{H}^i_{i+j} (\theta, y)^T \mathcal{H}^i_{i+j} (\theta, y)^T \tag{29}
\]

and

\[
\mathcal{M}'''_k = \mathcal{H}^i_{i+j} (\theta, y) \mathcal{M}_k \mathcal{H}^i_{i+j} (\theta, y)^T \mathcal{H}^i_{i+j} (\theta, y)^T \tag{30}
\]

Interestingly, the estimation of the optimal parameters is the same as the one we obtained before when using the matrix transformation in \( \text{[16]} \).

**Lemma 3** The JD criterion \( C^i_{ij} \) calculated with the matrix transform \( \text{[16]} \) for indices \((i, j)\) (resp. the matrix transform \( \text{[16]} \) for indices \((i, j+N)\)) is equal, up to constant factors, to the one calculated with the matrix transform \( \text{[29]} \) (resp. the matrix transform \( \text{[30]} \)), i.e.

\[
\sum_{k=1}^{2K} \left[ \mathcal{M}''_k (i, j)^2 + \mathcal{M}''_k (i + N, j + N)^2 \right] + \mathcal{M}''_k (i, j + N)^2 + \mathcal{M}''_k (j, i + N)^2 = 2 \sum_{k=1}^{2K} \mathcal{M}_k (i, j)^2 + c_1
\]

\[
\sum_{k=1}^{2K} \left[ \mathcal{M}''_k (i, j)^2 + \mathcal{M}''_k (i + N, j + N)^2 \right] + \mathcal{M}''_k (i, j + N)^2 + \mathcal{M}''_k (j, i + N)^2 = 2 \sum_{k=1}^{2K} \mathcal{M}_k (i, j + N)^2 + c_2
\]

where \( c_1 \) and \( c_2 \) are scalar constants independent from the angle and shear parameters. Consequently, the optimal parameters obtained by minimizing \( \text{[27]} \) with indices \((i, j)\) (resp. \( \text{[22]} \) with indices \((i, j+N)\)) are the same as the one obtained by minimizing \( \text{[19]} \) with indices \((i, j)\) (resp. \( \text{[19]} \) with indices \((i, j+N)\)).

**Proof:** The proof is given in appendix B.

Compared to the previous basic generalization of JD algorithm, the developed algorithm called CJDi preserves our matrix structure and decreases the number of iterations per sweep. In the first generalization, the number of iterations (index pairs) per sweep is \((2N-1)N\) while in CJDi, this number decreases to \(N(\text{N}\,\text{N})\). Also, CJDi takes into account some extra information about the matrix structure which leads to a slight performance improvement.

### 5.2. Complex implementation

The matrix rotations in \( \text{[29]} \) and \( \text{[30]} \) can be rewritten in the complex form by reversing the function in \( \text{[16]} \), i.e. using the real to complex transformation \( \mathcal{M}_k = f^{-1}(\mathcal{M}_k) \). Considering this transformation, one can express the previous rotations as:

\[
\mathcal{H}_{1,i,j}(\theta, y) = f^{-1}\left(\mathcal{H}^i_{i+j}(\theta, y) \mathcal{H}^i_{i+j} (\theta, y)^T \mathcal{H}^i_{i+j} (\theta, y)^T \right) = S_{ij}(0, y) G_{ij}(0, \theta) \tag{33}
\]

---

**Table 3. CJDi algorithm**

| Requirement | Description |
|-------------|-------------|
| \( \mathcal{M}_k \), \( k = 1, \ldots, K \), a fixed threshold \( \tau \) and a maximum sweep number \( M_{it} \). |

**Initialization:** \( \mathbf{V} = I + N \cdot j \mathbf{I} \).

**while** max \( |\text{sin}(h(y))| \), \( |\text{sin}(\theta)| \) \( \geq \tau \) and \(#\text{sweeps} < M_{it} \)

for all \( 1 \leq i < j \leq N \)

\begin{align*}
\text{Estimate } & \mathbf{H}_{1,i,j}(\theta, y) \text{ using } \text{[39]} \text{ and } \text{[22]}. \\
\text{Update matrices } & \{ \mathbf{M}_k \, k = 1, \ldots, 2K \} \text{ as in } \text{[37]}. \\
\mathbf{V} & \leftarrow \mathbf{H}_{1,i,j}(\theta, y) \mathbf{V} \\
\text{Estimate } & \mathbf{H}_{2,i,j}(\theta, y) \text{ using } \text{[40]} \text{ and } \text{[22]}. \\
\text{Update matrices } & \{ \mathbf{M}_k \, k = 1, \ldots, 2K \} \text{ as in } \text{[38]}. \\
\mathbf{V} & \leftarrow \mathbf{H}_{2,i,j}(\theta, y) \mathbf{V} \\
\text{end for} & \text{while}.
\end{align*}

**Remark:** Using the previous equations \( \text{[37]} \) and \( \text{[38]} \), one can express the \( (i, j)^{th} \) entries as:

\[
\tilde{\mathbf{M}}_k(i, j) = \mathbf{V}^T \left[ \begin{array}{c}
\tilde{\mathbf{M}}_k(i, j) \\
\Re(\tilde{\mathbf{M}}_k(i, j))
\end{array} \right] + j 3m(\mathbf{M}_k(i, j)) \tag{39}
\]

\[
\tilde{\mathbf{M}}''_k(i, j) = \Re(\tilde{\mathbf{M}}_k(i, j)) + j \mathbf{V}^T \left[ \begin{array}{c}
\mathbf{M}_k(i, j) + \mathbf{M}_k(j, i) \\
\Re(\mathbf{M}_k(i, j))
\end{array} \right] \tag{40}
\]

Hence, applying transformation \( \text{[37]} \) (resp. \( \text{[38]} \)) updates only the real part (resp. the imaginary part) of the \( (i, j)^{th} \) entries.
5.3. Computational cost

We provide here an evaluation of the computational cost of our algorithm expressed in terms of real flops number (i.e. real multiplication plus real addition) per iteration (sweep). In our evaluation, we took into account the matrices symmetry and the particular structure of the transformation matrices in (33) and (34) which entries are either real or pure imaginary numbers. Taking this into consideration, the matrices product in (37) or (38) cost for each matrix $M_k$, $8N^2 + O(N^2)$ flops (instead of $32N^2$ for a brute force implementation). This numerical cost evaluation is performed similarly for the other considered NOJD algorithms and summarized in Table 4.

| Algorithm | Number of real flops per sweep |
|-----------|--------------------------------|
| CDJi      | $16KN^2 + O(KN^2)$            |
| ACDC      | $20KN^2 + O(N^2)$             |
| FADJ      | $12KN^2 + O(N^2)$             |
| QDiag     | $36KN^2 + O(KN^2)$            |
| UWEDGE    | $8KN^2 + O(KN^2)$             |
| JUST      | $32KN^2 + O(KN^2)$            |
| CVFFDij   | $8KN^2 + O(KN^2)$             |
| LUCJD     | $24KN^2 + O(KN^2)$            |
| ALS       | $20KN^2 + O(N^2) + O(N^2)$    |

Table 4. NOJD Algorithms complexity

6. COMPARATIVE PERFORMANCE ANALYSIS

The aim of this section is to compare CDJi with the different existing NOJD algorithms, cited in section 3 for different scenarios. More precisely, we have chosen to evaluate and compare the algorithms sensitivity to different factors that may affect the JD quality. These different criteria of comparison used in our study, are described in the next subsection.

6.1. Factors affecting NOJD problem

In ‘good JD conditions’ most algorithms perform well, with slight differences in terms of convergence rate or JD quality. However, in adverse JD conditions, many of these algorithms lose their effectiveness or otherwise diverge. In this study, adverse conditions are accessible through the following factors:

- **Modulus of uniqueness (MOU):** defined in [23] as the maximum correlation factor of vectors $d_i = [D_1(i, i), ..., D_K(i, i)]^T$, i.e.

$$MOU = \max_i \left( \frac{|d_i^T \cdot d_i|}{\|d_i\| \cdot \|d_i\|} \right)$$

It is shown in [23] that the JD quality decreases when the MOU get close to 1.

- **Mixing matrix condition number:** The JD quality depends on the good or ill conditioning of mixing matrix $A$. The comparative study reveals the algorithms’ robustness w.r.t. Cond($A$).

- **Diagonal matrices condition number:** In BSS problem, the power range of source signals affects the conditioning of diagonal matrices $D_k$ and hence the source separation quality. By comparing the algorithms sensitiveness w.r.t. this factor, we reveal their potential performance if used for separating sources with high dynamical range.

- **Matrix dimensions:** The JD problem is more difficult for large dimensional matrices ($N >> 1$) and hence we compare the algorithms performance in the following two cases: $N = 5$ (small dimension) and $N = 50$ (large dimension).

- **Noise effect:** In practice, matrices $M_k$ are given by some sample averaged statistics and hence are affected by finite sample size and noise effects. In that case, the exact JD (EJD) becomes approximate JD (AJD) and the algorithms performance is lower bounded by the noise level as shown by our comparative study. The algorithms behaviour and robustness w.r.t. noise effect is investigated in our study.

6.2. Simulation experiment set up

First, we have chosen the classical performance index (PI) to measure the JD quality. It can be expressed as [30]:

$$PI(G) = \frac{1}{2N(N-1)} \sum_{n=1}^{N} \left( \sum_{m=1}^{N} \frac{|m(n,m)|^2}{\max_{m} |G(n,m)|^2} - 1 \right)$$

$$= \frac{1}{2N(N-1)} \sum_{n=1}^{N} \left( \sum_{m=1}^{N} \frac{|m(n,m)|^2}{\max_{m} |G(n,m)|^2} - 1 \right)$$

(42)

where $G(n,m)$ is the $(n,m)$th entry of global matrix $G = VA$, $A$ being the generated mixing matrix and $V$ the estimated diagonalizing matrix. The closer the $PI$ is to zero, the better is the JD quality. Contrary to other existing criteria, the $PI$ is common to all algorithms and allows us to compare them properly.

Figure 1 illustrates the organization of our simulation scenarios where simulations are divided in two sets. The first one is dedicated to the exact joint diagonalization (EJD) case whilst the second one assess the algorithms performance in the AJD case by investigating the noise effect as given by equation 2. Also, for each experiment, we have considered two scenarios namely the small dimensional case ($N = 5$) and the large dimensional case ($N = 50$). Finally, for each of these cases, we run four simulation experiments: (i) a reference simulation of relatively good conditions where $MOU < 0.6$ (resp. $MOU \approx 0.9$ for large dimension), Cond($A$) < 5 (resp. Cond($A$) < 50 for large dimension) and Cond($D_k$) < 10 (resp. Cond($D_k$) < 50 for large dimension); (ii) a simulation experiment with $MOU > 1 - 10^{-6}$; (iii) a simulation experiment with Cond($A$) > 100 and (iv) a simulation experiment with Cond($D_k$) > $10^4$.

The other simulation parameters are as follows: mixing matrix entries are generated as independent and normally distributed variables of unit variance and zero mean. Similarly, the diagonal entries of $D_k$ are independent and normally distributed variables of unit variance and zero mean except in the context where Cond($D_k$) > $10^4$ in which case $D_k(2, 2)$ has a standard deviation of $10^{-4}$ or in the context where $MOU > 1 - 10^{-6}$ in which case $D_k(2, 2) = D_k(1, 1) + \epsilon_k$, $\epsilon_k$ being a random variable of small amplitude generated to tune the value of MOU. Target matrices are computed as in [1] for EJD case. The number of matrices is set to $K = 5$. For AJD case, these matrices are corrupted by additive noise as given in [2].
noise ratio) expressed in dB as:

$$PL(dB) = 10 \log_{10} \left( \frac{\|AD_k A^H\|_F}{\|\Xi_k\|_F} \right)$$ (43)

The error matrix $\Xi_k$ is generated as:

$$\Xi_k = \beta_k N_k$$ (44)

where $N_k$ is a random perturbation matrix (regenerated at each Monte Carlo run) and $\beta_k$ is a positive number allowing us to tune the perturbation level.

The simulation results (i.e. performance index) are averaged over 200 Monte Carlo runs for small dimensional matrices and 20 Monte Carlo runs for large dimensional matrices.

6.3. Exact joint diagonalizable matrices

The exact joint diagonalizable matrices are generated as given in equation (1). This part illustrates the convergence rate of each algorithm where two scenarios are considered. The first one is for small matrix dimension and the second one treats large dimensional matrices. Obtained results are given in the two following points.

6.3.1. Small matrix dimension

For small matrix dimension, four simulations are realized according to the experiments scheme shown in figure 1 and results are given in figures 2, 4, 5 and 6. The first one is the reference case where $MOU < 0.6$ and $\text{Cond}(A) < 5$. In this case, the majority of algorithms converge at different rates. The fastest one is our developed algorithm where it needs less than ten sweeps to converge. It is followed by CVFFDiag, UWEDGE, JUST, LUCJD and FAJD. Note that ACDC, ALS and QDiag diverge in some realizations or need more than 100 sweeps to converge as illustrated in figure 3 where twenty runs are plotted for each algorithm.

Figure 2 represents the simulation results in the case of $\text{Cond}(A) > 100$. We observe that UWEDGE, JUST, CJDi and FAJD keep approximately the same behaviour as before while CVFFDiag and LUCJD became slower in terms of convergence rate.

In the third simulation where $\text{Cond}(D_k) > 10^4$, the same remarks as in the previous one can be observed in figure 5. In this simulation UWEDGE and JUST are slightly faster than our proposed algorithm CJDi.
In the fourth simulation, all parameters are generated as in the reference case except for the diagonal matrices which are generated by keeping the MOU greater than $1 - 10^{-6}$. As shown in figure 6, our proposed algorithm CJDi gives the best results in terms of convergence rate and JD quality and JUST, UWEDGE and CVFFDiag performance is slightly decreased. Otherwise, LUCJD and FAJD diverge in some realization or need more than 100 sweep to converge hence their performance is degraded considerably.

In the second and third simulations from figures 8 and 9 respectively. In the fourth simulation, the results given in figure 10 show that LUCJD and FAJD have lost their JD quality. Note that only our algorithm CJDi keeps its high performance in terms of convergence rate and JD quality.

As a conclusion for EJD case, one can say that our proposed algorithm leads to the best results in major of the simulation cases and presents a good robustness in the considered adverse scenarios.

### 6.4. Approximate joint diagonalizable matrices

In this part, we investigate the algorithms robustness to the noise effect. Simulation scenarios are the same as in exact joint diagonalizable matrices.

In this second experiments set, we have kept fixed the number of matrices as 5 and matrix dimension is increased to 50. Four simulation cases are considered as in the small matrix dimension context and the results are given in figures 7, 8, 9 and 10. The first simulation corresponds to the reference case where $\text{Cond}(A) < 50$, $\text{MOU} \approx 0.9$ and $\text{Cond}(D_k) < 50$. Note that, in addition to ALS, QDiag and ACDC, UWEDGE and CVFFDiag diverge in some realization or need more than 100 sweep to converge. Hence their averaged performance degraded considerably in terms of convergence rate and JD quality. JUST has preserved its convergence rate but it has lost its JD quality. LUCJD and FAJD have convergence rates decreased while our algorithm CJDi has preserved its convergence rate and JD quality.

The same remarks can be observed in the second and third simulations from figures 8 and 9 respectively.

In the fourth simulation, the results given in figure 10 show that LUCJD and FAJD have lost their JD quality. Note that only our algorithm CJDi keeps its high performance in terms of convergence rate and JD quality.

As a conclusion for EJD case, one can say that our proposed algorithm leads to the best results in major of the simulation cases and presents a good robustness in the considered adverse scenarios.
6.4.1. Small matrix dimension

Obtained results from the first simulations are illustrated in figure 11 where plots of mean PI, obtained after 100 sweep, versus perturbation level are represented for each algorithm. In this scenario, the majority of algorithms converge with different JD quality. Note that, similarly to the EJD case, QDiag, ALS and ACDC diverge in some realization or need more than 100 to converge. In that experiment, the CJDi has the best results in terms of JD quality followed by CVFFDiag, LUCJD, FAJD, JUST and UWEDGE.

Results from the second scenario (i.e. Cond(\(A\)) > 100) are illustrated in figure 12. Note that algorithms performance are degraded as compared to the reference case given in figure 11. ACDC and ALS provide the best results when the noise power is high. However, when the latter decreases our proposed algorithm provides the best results.

In the third simulation scenario (Cond(\(D_k\)) > 104), obtained results are presented in figure 13. Note that our algorithm gives the best results especially for low noise power levels.

In the last simulation scenario (MOU > 10^{-6}), figure 14 show that our algorithm and CVFFDiag lead to the best performance.
Fig. 14. Mean performance index versus different perturbation levels (matrix dimension $5$, $\text{MOU} > 1 - 10^{-6}$, $\text{cond}(A) < 5$).

in terms of JD quality.

6.4.2. Large matrix dimension

Here, we consider the JD of $K = 5$, $50 \times 50$ matrices corrupted by additive noise as given in equation (2). The four previously mentioned scenarios are considered and the results are given in figures 15, 16, 17 and 18.

Results from the reference case are given in figure 15. Only LUCJD, CJDi, JUST and FAJD have kept their performance while the others need more than $100$ sweep to converge or diverge in some realizations. Note that LUCJD provides the best results, in that case, followed by CJDi, FAJD and JUST.

Results for the second (resp. third) simulation scenarios ($\text{cond}(A) > 100$ (resp. $\text{cond}(D_k) > 10^4$)) are given in figure 16 (resp. in figure 17). It can be seen that LUCJD still provides the best results followed by CJDi, JUST and FAJD (resp. CJDi, FAJD and JUST) respectively.

In the last scenario, given in figure 18 we observed that LUCJD and FAJD have lost their efficiency and that CJDi leads to the best

Fig. 15. Mean performance index versus perturbation levels (matrix dimension $50$, $\text{MOU} \approx 0.9$, $\text{cond}(A) < 50$).

Fig. 16. Mean performance index versus perturbation levels (matrix dimension $50$, $\text{MOU} \approx 0.9$, $\text{cond}(A) > 100$).

Fig. 17. Mean performance index versus perturbation level (matrix dimension $50$, $\text{MOU} \approx 0.9$, $\text{cond}(D_k) > 10^4$).

Fig. 18. Mean performance index versus different perturbation levels (matrix dimension $50$, $\text{MOU} > 1 - 10^{-6}$, $\text{cond}(A) < 50$).
results followed by JUST in that context where $MOU > 1 - 10^{-6}$.

All the observed results are summarized in Table 5 where we express the algorithms sensitiveness to the different factors considered in our study, namely $MOU$, $Cond(A)$, $Cond(D_k)$, $N$ (matrix size) and noise effects. H, M and L refers to high sensitiveness (i.e. high risk of performance loss), moderate sensitiveness and low sensitiveness respectively.

For better results illustration, Table 5 summarizes all remarks given below.

### 7. CONCLUSION

In this paper, a new NOJD algorithm is developed. First, we have considered a basic generalization of IDi from the real to the complex case. Then, we have proposed our new algorithm referred to as CJDi that takes into account the special structure of the transformed matrices and consequently improves the performance in terms of convergence rate and JD quality. Finally, the comparative study provided in this paper, illustrates the algorithm’s efficiency and robustness in adverse JD conditions based on simulation experiments for both EID and AJD cases. This performance comparison study reveals many interesting features summarized in Tables 4 and 5. Based on this study, one can conclude that the CJDi algorithm has the best performance for a relatively moderate computational cost. On the other hand, ACDC, ALS and QDiag are the most sensitive ones, diverging in most considered realizations. However, when ACDC or ALS converge, they allow to reach the best (lowest performance index) JD quality values in the noisy case.

### A. APPENDIX

Let us consider a real symmetric matrix $M_k$ given in (10) to which the two rotation matrices $H_j^i(\theta, y)$ and $H_j^{i+N}(\theta, y)$ are applied as in (29) leading to matrix $M''_k$.

The objective is to show that $M''_k$ preserves the same structure of $M_k$ given in (10). Using equation (29), only $i^{th}$, $j^{th}$, $(i+N)^{th}$ and $(j+N)^{th}$ rows and columns of $M_k$ are affected according to

$$
M''_k(i, :) = \alpha_{11} M_k(i, :) + \alpha_{12} M_k(j, :)
$$

$$
M''_k(i+N, :) = \alpha_{11} M_k(i+N, :) + \alpha_{12} M_k(j+N, :)
$$

(45)

where $\alpha_{11}$ and $\alpha_{12}$ are given according to Shear parameter $y$ and Givens rotation angle $\theta$ in (26).

$$
\alpha_{11} = \cosh(y) \cos(\theta) - \sinh(y) \sin(\theta)
$$

$$
\alpha_{12} = \cosh(y) \sin(\theta) + \sinh(y) \cos(\theta)
$$

$$
\alpha_{21} = \sinh(y) \cos(\theta) - \cosh(y) \sin(\theta)
$$

$$
\alpha_{22} = \cosh(y) \cos(\theta) + \sinh(y) \sin(\theta)
$$

(46)

Since for all $i$:

$$
M_k(i, 1 : N) = M_k(i+N, N+1 : 2N)
$$

$$
M_k(i, N+1 : 2N) = -M_k(i+N, 1 : N)
$$

(47)

it comes from equations (45) and (47) that the same relations apply for matrix $M''_k$, i.e.

$$
M''_k(i, 1 : N) = M''_k(i+N, N+1 : 2N)
$$

$$
M''_k(i, N+1 : 2N) = -M''_k(i+N, 1 : N)
$$

In the second part of the lemma, we consider the rotation matrices $H_{j+N}^i(\theta, y)$ and $H_{j+N}^i(\theta, y)$ and the transformation given in (30).

In that case, the $i^{th}$ and $(i+N)^{th}$ rows of $M''_k$ can be expressed as:

$$
M''_k(i, :) = \alpha_{11} M_k(i, :) + \alpha_{12} M_k(j+N, :)
$$

$$
M''_k(i+N, :) = -\alpha_{12} M_k(j+N, :) + \alpha_{11} M_k(i+N, :)
$$

(48)

Again, by taking into account the structure in (47) and equation (48), one can obviously observe that $M''_k$ has the same structure as $M_k$.

### B. APPENDIX

Consider transformations (29) and (30) where we get $M''_k$ and $M''''_k$, respectively. Thanks to Lemma 1, these obtained matrices have the structure given in (10).

To compute the simplified JD criterion $C''_j$ for (29) and (30), we consider only, the entries twice affected by the considered rotations which are $(i,j)^{th}$, $(i+N,j+N)^{th}$, $(i,j+N)^{th}$, $(j,i+N)^{th}$ and $(j,j+N)^{th}$ entries. Considering the structure given in (10):

- $(i,i+N)^{th}$ and $(j,j+N)^{th}$ entries are equal to zero;
- $(i+N,j+N)^{th}$ entries are equal to $(i,j)^{th}$ entries.

The development of considered transformations, $(i,j)^{th}$ and $(j,i+N)^{th}$ entries are not changed and we get:

$$
M''_k(i, j+N) = M_k(i, j+N)
$$

$$
M''_k(i+N, j) = M_k(i+N, j)
$$

Combining these results with equation (31) (resp. (32)), the simplified JD criteria $C''_j$ for matrix transforms (16) and (29) considering $(i,j)$ indices (resp. for matrix transforms (16) and (30) considering $(i,j+N)$ indices) are equal up to a constant factor.

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