The many faces of the stochastic zeta function

Benedek Valkó and Bálint Virág

Abstract

We introduce a framework to study the random entire function $\zeta_\beta$ whose zeros are given by the $\text{Sine}_\beta$ process, the bulk limit of beta ensembles. We present several equivalent characterizations, including an explicit power series representation built from Brownian motion.

We study related distributions using stochastic differential equations. Our function is a uniform limit of characteristic polynomials in the circular beta ensemble; we give upper bounds on the rate of convergence. Most of our results are new even for classical values of $\beta$.

We provide explicit moment formulas for $\zeta$ and its variants, and we show that the Borodin-Strahov moment formulas hold for all $\beta$ both in the limit and for circular beta ensembles. We show a uniqueness theorem for $\zeta$ in the Cartwright class, and deduce some product identities between conjugate values of $\beta$. The proofs rely on the structure of the $\text{Sine}_\beta$ operator to express $\zeta$ in terms of a regularized determinant.

Contents

1 Introduction 3

2 Secular functions of Dirac operators 7
  2.1 A class of Dirac operators ...................................................... 7
  2.2 The inverse of a Dirac operator ................................................ 9
  2.3 Definition of the secular function $\zeta_\tau$ .................................. 10
  2.4 The Taylor expansion of $\zeta$ ................................................... 11
  2.5 ODE representation for $\zeta$ .................................................... 16
  2.6 The structure function $E$ and the functions $A, B$ ......................... 21
  2.7 Infinite product representation .................................................. 22

3 Characteristic polynomials of unitary matrices 27
1 Introduction

The goal of this paper is to study a central object in random matrix theory: the large $n$ limit of characteristic polynomials. These limits are expected to be universal; here we study the universality class of general beta ensembles in the bulk.

Relatively little is known about this limit, while the limit of the eigenvalues, the Sine$_\beta$ process, has been more extensively studied. For $\beta = 2$, Chhaibi, Najnudel and Nikhefgbali [8] introduced the random entire function

$$
\zeta_2(z) = \lim_{r \to \infty} \prod_{\lambda \in \text{Sine}_2, |\lambda| < r} (1 - z/\lambda),
$$

and studied its properties using the special determinantal structure of this case. Our goal is to deepen the understanding of this function and its general $\beta$ versions by introducing a set of new tools. Among others, we give an explicit Taylor series expansion given in terms of Brownian motion.

The Cartwright class of entire functions is defined by the growth conditions $|f(z)| \leq c^{1+|z|}$ and $\int_{\mathbb{R}} \log_+ |f(x)|/(1 + x^2) \, dx < \infty$. It can be thought of as an infinite-dimensional version of the class of polynomials.

Let $b_1, b_2$ be independent copies of two-sided standard Brownian motion, let $y_u = e^{b_2 - u/2}$, let $q$ be an independent Cauchy random variable with density $1/(\pi(1 + x^2))$. Let $A_0 \equiv 1, B_0 \equiv 0$, and define, recursively,

$$
\begin{align*}
B_{n,u} &= y_u \int_{-\infty}^{u} e^{\beta s/4} A_{n-1,s}/y_s \, ds, \\
A_{n,u} &= \int_{-\infty}^{u} e^{\beta s/4} B_{n-1,s} \, ds - \int_{-\infty}^{u} B_{n,s} \, dB_1.
\end{align*}
$$

Theorem 1 (The stochastic zeta function). There exists a unique probability measure on the Cartwright class of entire functions $f$ with $f(0) = 1$, $f(\mathbb{R}) \subset \mathbb{R}$, so that the law of zeros is given by the Sine$_\beta$ process.

The corresponding random entire function, the stochastic zeta function $\zeta_{\beta}$ has several explicit representations. As a power series with infinite radius of convergence,

$$
\zeta_{\beta}(z) = \sum_{n=0}^{\infty} (A_{n,0} - qB_{n,0}) z^n.
$$

As the principal value infinite product

$$
\zeta_{\beta}(z) = \lim_{r \to \infty} \prod_{\lambda \in \text{Sine}_\beta, |\lambda| < r} (1 - z/\lambda).
$$
As $\zeta_\beta = [1, -q]H_0$, where $H_u(z)$ is the unique analytic solution of the system of stochastic differential equations

$$dH = \begin{pmatrix} 0 & -db_1 \\ 0 & -db_2 \end{pmatrix} H - z^2 e^{\beta u/4} JH du, \quad u \in \mathbb{R}, \quad \lim_{u \to -\infty \sup_{z < 1} |H_u(z) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}| = 0. \tag{3}$$

The claims of Theorem 1 are proved in Propositions 58, 47, 43 and in Section 7.1, see equation (161). Most of the underlying theory is developed in Sections 2-5 of this paper.

The name, stochastic zeta function, is motivated by [8], and their analogue of the Montgomery conjecture about the Riemann zeta function $\zeta$. Conjecture 2 (Chhaibi, Najnudel and Nikhegbali [8]). For $\omega$ uniform on $[0, 1]$ as $\nu \to \infty$, the following distributional convergence of random analytic functions holds:

$$\frac{\zeta \left( \frac{1}{2} + i\nu \omega + iz/\nu \right)}{\zeta \left( \frac{1}{2} + i\nu \omega \right)} \xrightarrow{d} \zeta_2(z).$$

Properties of $\zeta$ have been conjectured based on $\zeta_2$, see Sodin [28]. The function $\zeta_2$ is the limit of characteristic polynomials, see [8] and Chhaibi, Hovhannisyan, Najnudel, Nikeghbali, and Rodgers [7]. We show that $\zeta_\beta$ is the limit in the case of circular beta ensembles, Section 4.4, with an explicit rate of convergence.

**Theorem 3.** There is a coupling of the characteristic polynomials

$$p_n(z) = \det(I - zU^{-1}_\beta) = \prod_{i=1}^{n}(1 - z/\lambda_i)$$

of the circular beta ensemble, $\zeta_\beta$, and a random $C$ so that for all $z \in \mathbb{C}$, $n > 1$ we have

$$|p_n(e^{iz/n})e^{-iz/2} - \zeta_\beta(z)| \leq \left( e^{\frac{\log^3 n}{\sqrt{n}}} - 1 \right) C^{|z|^2 + 1}. \tag{4}$$

The starting point of our analysis is the general framework of Dirac differential operators $\tau$, see Section 2. In this setting, we introduce the secular function, the analogue of the characteristic polynomial. We apply this theory to the Sine$_\beta$ operator introduced in [32] and its conjugate $\tau_\beta$, see Section 4. The eigenvalues of these operators are given by the Sine$_\beta$ process.

For trace class $\tau^{-1}$, a natural choice for the secular function would be $\det(I - \tau^{-1}z)$. However, the eigenvalues of $\tau_\beta$ are given by the stationary process Sine$_\beta$, so their inverses are not summable. In this case the regularized determinant $\det_2$ can be used. For trace class operators, the ordinary and regularized determinant are related by

$$\det(I - \tau^{-1}z) = \det_2(I - \tau^{-1}z)e^{ztr\tau^{-1}},$$
which motivates the following theorem. We consider the integral operator \( r \tau_\beta \), a conjugate of the inverse of the Sine_\( \beta \) operator. Interestingly enough, the integral trace \( t r \tau_\beta \) can be defined as the diagonal integral of the kernel, see (24), and it agrees with the principal value trace (71) defined as

\[
\lim_{r \to \infty} \sum_{|\lambda_k| \leq r} \frac{1}{\lambda_k}.
\]

The principal value trace has Cauchy distribution, a general phenomenon for translation-invariant processes, see Theorem 60 and Aizenman and Warzel [2].

**Theorem 4.**

\[
\zeta_\beta(z) = \det_2(I - z r \tau_\beta) e^{z t r \tau_\beta}.
\] (5)

The equivalence to (2) is proved in Proposition 47. In fact, we will use (5) as the definition of the secular function \( \zeta \) for general Dirac operators, see Definition 35. Characterizations similar to Theorem 1 hold in this setting, see Section 2.

The Sine_\( \beta \) operator is built from hyperbolic Brownian motion, which allows us to provide additional characterizations of \( \zeta_\beta \) via a system of stochastic differential equations (3). We use this characterization to compute expectations of quantities related to \( \zeta_\beta \). As an application, we verify a conjecture of Borodin and Strahov [4] in this setting, see Section 8.4.

**Theorem 5.** For all \( \beta > 0 \) we have

\[
E \prod_{j=1}^{k} \frac{\zeta_\beta(z_j)}{\zeta_\beta(w_j)} = \begin{cases} 
  e^{i \sum_{j=1}^{k} \frac{z_j - w_j}{2}} & \text{if } \Im w_j < 0 \text{ for all } j, \\
  e^{-i \sum_{j=1}^{k} \frac{z_j - w_j}{2}} & \text{if } \Im w_j > 0 \text{ for all } j.
\end{cases}
\] (6)

In Theorem 69 we show that the corresponding moment formulas also hold for circular beta ensembles. In fact, Theorem 69 applies in greater generality: for all models with independent, rotationally invariant Verblunsky coefficients. This includes well-studied random Schrödinger models for which few exact formulas are known, see Simon [26], Section 12.6.

Because of the Cauchy variable in the normalization (2), \( \zeta_\beta \) itself does not to have a first moment. A better choice for moments is the function \( \hat{\zeta}_\beta(z) = (1 + q^2)^{-1/2} \zeta_\beta(z) \), with \( q \) as in (2). We have

\[
E \hat{\zeta}_\beta(z) = \frac{2}{\pi} \cos(z/2), \quad \beta > 0, z \in \mathbb{R},
\]

see (171). We show that the moments of \( \hat{\zeta}_\beta \) on the real line exist below \( 1 + \frac{\bar{\beta}}{2} \), which is likely optimal. We have the following explicit formula for the second moment in terms of
the generalized cosine integral:

\[
\tilde{E}_\beta(z)^2 = \frac{1}{2} + \frac{2}{\beta} \int_0^1 t^{-4/\beta - 1} (1 - \cos(zt)) dt, \quad \beta > 2, \ z \in \mathbb{R},
\]

see (173). More generally, the moments satisfy an explicit system of ordinary differential equations, see Section 8. The equations are similar to those obtained by Killip and Ryckman [17] for the circular beta ensemble. Their scaling is different from the one appearing in the distributional convergence of characteristic polynomials.

The law of the function \( \zeta_\beta \) can be represented as the stationary distribution of a system of diffusions, Proposition 44. Surprisingly, the first degree Taylor coefficient \( B_{1,0} \) satisfies the stationary stochastic differential equation appearing in Dufresne’s identity. Its distribution can be computed explicitly:

\[
B_{1,0} \overset{d}{=} \frac{\beta}{4} G,
\]

where \( G \) has Gamma distribution with rate 1 and shape parameter \( 1 + \frac{\beta}{2} \), see Remark 46. Similarly, the higher order coefficients \( B_{n,0}, A_{n,0} \) satisfy closed systems of stochastic differential equations. They can be converted to partial differential equations for the joint densities. For given \( n \), the joint density \( p(x, y) \) of \( B_{1,0}, A_{1,0} \ldots, B_{n,0}, A_{n,0} \) satisfies the following partial differential equation:

\[
\sum_{k=1}^{n} \frac{1}{2} (\partial_{x_k}^2 + \partial_{y_k}^2)(x_k^2 p) - \frac{\beta}{8} \partial_{x_k} ( (y_{k-1} - 2k x_k) p ) - \frac{\beta}{8} \partial_{y_k} ( (x_{k-1} - 2k y_k) p ) = 0,
\]

where \( x_0 = 0, y_0 = 1 \), see Proposition 49.

Especially interesting is the function

\[
B(z) = \sum_{n=0}^{\infty} B_{n,0} z^n.
\]

built from the coefficients \( B_{n,0} \). In [33] we show that the zero distribution of \( B \) is the Palm measure of the \( \text{Sine}_\beta \) process: the process conditioned to have a point at zero. In particular, the intensity of zeros of \( B \) is given by the two-point correlations of \( \text{Sine}_\beta \).

The uniqueness of \( \zeta_\beta \) within the Cartwright class and results of Forrester [11] imply simple product identities connecting various \( \beta \)-s. Let \( k \geq 2 \) an integer. There exists a coupling of \( k \) copies \( \zeta_{2k,1}, \ldots, \zeta_{2k,k} \) of \( \zeta_{2k} \) so that

\[
\zeta_{2k}(z) \overset{d}{=} \prod_{j=1}^{k} \zeta_{2k,j}(z/k),
\]
see Corollary [59]. The way $\zeta_\beta$ is the secular function of the Sine-$\beta$ operator has analogues for more classical differential operators. The sine function and versions of the Bessel functions appear as secular functions. We illustrate the development of $\zeta_\beta$ with a series of concrete, classical examples, see Examples [6, 7, 11, 12, 24, 25] and [30].

Many of our results rely on understanding the so-called de Branges structure function
\[
\mathcal{E}(z) = \sum_{n=0}^{\infty} (A_{n,0} - iB_{n,0}) z^n = [1, -i] \cdot \mathcal{H}_0(z)
\]
an analogue of orthogonal polynomials on the unit circle in the infinite-dimensional setting. For Dirac operators based on discrete circular ensembles, $\mathcal{E}$ is a linear combination of orthogonal polynomials, see Proposition [32] ([53]). The entire function $\mathcal{E}$ belongs to the so-called Pólya class, and all of its zeros are in the upper half plane. In de Branges [9] such functions are used to define Hilbert spaces of entire functions, the starting point of the theory of canonical systems. For us, $A, B, \zeta_\beta$ and $\hat{\zeta}_\beta$ can be expressed in terms of $\mathcal{E}$ and $q$, and the moment formulas are simplest for the function $\mathcal{E}$. In particular, as shown in Proposition [65] for $x_j \in \mathbb{R}$ we have
\[
E \prod_{j=1}^{k} \mathcal{E}(x_j) = \prod_{j=1}^{k} e^{ix_j/2}, \quad k < 1 + \frac{\beta}{2}.
\]

2 Secular functions of Dirac operators

Our approach is to understand the function $\zeta_\beta$ as the “characteristic polynomial” of an “infinite matrix”. More precisely, it is the secular function of a random Dirac operator. We first introduce such secular functions for a general class of deterministic Dirac operators.

2.1 A class of Dirac operators

Let $\sigma > 0$ and $R : (0, \sigma] \to \mathbb{R}^{2 \times 2}$ be a function taking values in nonnegative definite matrices. In this paper, a Dirac operator is defined as
\[
\tau u = R^{-1} Ju', \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
acting on some subset of functions of the form $u : (0, \sigma] \to \mathbb{R}^2$.

We assume the following about $R$.

Assumption 1. $R(t)$ is positive definite for all $t \in (0, \sigma]$, $\|R\|, \|R^{-1}\|$ are locally bounded on $(0, \sigma]$. Moreover, $\det R(t) = 1/4$ for all $t \in (0, \sigma]$.
The assumption $\det R = 1/4$ can be replaced by the more general condition $\int_0^\sigma \det R(s) \, ds < \infty$. This setting is equivalent to ours up to a time change.

**Assumption 2.** There is a nonzero vector $u_0 \in \mathbb{R}^2$ so that with $u_0^\perp = Ju_0$ we have

$$
\int_0^\sigma \|u_0^R\| \, ds < \infty, \quad \int_0^\sigma \int_0^t u_0^R(s)u_0(u_0^\perp)^tR(t)u_0^\perp \, dsdt < \infty.
$$

The function $R$ can be parametrized as

$$
R = \frac{X^tX}{2 \det X}, \quad X = \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix}, \quad y > 0, x \in \mathbb{R},
$$

where $X$ is defined as the unique multiple of the square root of $R$ of the above form. We refer to the group of matrices of such form as the affine group. We record the matrix identities

$$
yX^{-1}J = JX^t, \quad yJX^{-1} = X^tJ, \quad R^{-1}J = 2X^{-1}JX, \quad 2JR = X^{-1}JX,
$$

that hold in this setting.

Let $\text{AC}$ be the set of absolutely continuous functions. Define $L^2_R = L^2_R[0,\sigma]$ as the space with norm defined by

$$
\|f\|_R^2 = \int_0^\sigma f^t(s)R(s)f(s) \, ds.
$$

Fix nonzero $u_0, u_1 \in \mathbb{R}^2$. From this point on, we will consider $\tau$ as the Dirac operator with the domain

$$
\text{dom}_\tau = \{ v \in L^2_R \cap \text{AC} : \tau v \in L^2_R, \lim_{s \downarrow 0} v(s)^tJ u_0 = 0, \, v(\sigma)^tJ u_1 = 0 \}.
$$

We will also use the notation

$$
\tau = \text{Dir}(X, u_0, u_1).
$$

We will often have the following.

**Assumption 3.**

$$
u_0^tJu_1 = 1.
$$

In most of the paper, we will use special cases of $u_0, u_1$ satisfying Assumption 3, namely

$$
u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_1 = \begin{pmatrix} -q \\ 1 \end{pmatrix}, \quad q \in \mathbb{R},
$$

which we call $q$-boundary conditions.
2.2 The inverse of a Dirac operator

Under Assumptions 1-3, the operator $\tau$ has an inverse. $\tau^{-1}$ is an integral operator on $L^2_{R}[0, \sigma]$ with kernel

$$K(s, t) = (u_0 u_1^t 1(s < t) + u_1 u_0^t 1(s \geq t)),$$

and by Theorem 9 of [32] it is Hilbert-Schmidt. Hence $\tau$ has a pure point spectrum. The eigenvalues are real, nonzero and have multiplicity one. We label the eigenvalues in an increasing order by $\lambda_k, k \in \mathbb{Z}$ so that $\lambda_{-1} < 0 < \lambda_0$.

Let $Y = X/\sqrt{\det X}$. The conjugate operator $Y \tau Y^{-1}$ is self-adjoint on $\{v : Yv \in \text{dom}_\tau\}$ with the same spectrum as $\tau$. We denote $(Y \tau Y^{-1})^{-1}$ by $r \tau$. This is an integral operator acting on $L^2[0, \sigma]$ with kernel

$$K_{r \tau}(s, t) = \frac{1}{2} a(s)c(t)^t 1(s < t) + \frac{1}{2} c(s)a(t)^t 1(s \geq t),$$

(16)

where

$$a = \frac{X u_0}{\sqrt{\det X}}, \quad c = \frac{X u_1}{\sqrt{\det X}}.$$  

(17)

The first integral condition of Assumption 2 is equivalent to the following pair of bounds:

$$\int_0^\sigma \|a(s)\|^2 ds < \infty, \quad \int_0^\sigma |a(s)^t c(s)| ds < \infty.$$  

(18)

With the $q$-boundary conditions (15) we have

$$a(s) = \frac{1}{\sqrt{y(s)}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c(s) = \frac{1}{\sqrt{y(s)}} \begin{pmatrix} x(s) - q \\ -y(s) \end{pmatrix},$$

and the pair of bounds in (18) is equivalent to

$$\int_0^\sigma \frac{1 + |x(s)|}{y(s)} ds < \infty.$$  

(19)

For $q$-boundary conditions, the second integral bound in Assumption 2 is given by

$$\int_0^\sigma \int_0^t \frac{x^2(t) + y^2(t)}{y(s)y(t)} ds dt < \infty.$$  

(20)

We introduce two simple examples.

Example 6 (Deterministic Sine operator). The simplest example is the operator

$$\tau : u \rightarrow 2J \frac{d}{dt} u$$

on $(0, \sigma]$. In that case $R(s) = \frac{1}{2} I$, $x(s) = 0$, and $y(s) = 1$. Set

$$u_0 = [1, 0]^t, \quad u_1 = [-q, -1]^t, \quad \text{with } q = \cot(\theta/2).$$
Then we have \( a(s) = [1, 0]^t \), and \( c(s) = [-q, -1]^t \). The eigenvalues of \( \tau \) are given by 
\[
\lambda_k = \frac{2\pi k - \theta}{\sigma}, \quad k \in \mathbb{Z},
\]
and the eigenfunction corresponding to \( \lambda_k \) is
\[
\left[ \cos\left(\frac{\sigma(2k\pi - \theta)}{2\sigma}\right), -\sin\left(\frac{\sigma(2k\pi - \theta)}{2\sigma}\right) \right]^t.
\]

**Example 7** (Deterministic Bessel operator). Assume \( \alpha > 0 \), and set \( x(s) = 0 \) and \( y(s) = s^{-\alpha} \). Then we have
\[
R(s) = \begin{pmatrix} s^\alpha & 0 \\ 0 & s^{-\alpha} \end{pmatrix}, \quad \tau u = 2 \begin{pmatrix} s^{-\alpha} & 0 \\ 0 & s^\alpha \end{pmatrix} \frac{d}{dt} u.
\]
The vector \( u_0 = [1, 0]^t \) satisfies the integral conditions (8), we set the other boundary condition as \( u_1 = [0, -1]^t \).

Then \( a(s) = [s^\alpha, 0]^t \) and \( c(s) = [0, s^{-\alpha}]^t \). Let \( J_p \) be the Bessel function of the first kind with parameter \( p \). Then the eigenvalues of \( \tau \) are given by \( \pm \frac{2\gamma_k}{\sigma} \), \( k \geq 1 \), where \( \gamma_k \) are the positive zeros of \( J_{\frac{\alpha-1}{2}}(\cdot) \). The eigenfunction corresponding to eigenvalue \( \lambda \) is given by
\[
u(s) = \left[ \frac{\lambda s}{4} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \left( J_{\frac{\alpha+1}{2}} \left( \frac{s\lambda}{2} \right), -s^\alpha J_{\frac{\alpha-1}{2}} \left( \frac{s\lambda}{2} \right) \right) \right]^t.
\]
(21)

Note that allowing \( \alpha = 0 \) would give us Example 6.

### 2.3 Definition of the secular function \( \zeta_\tau \)

Our goal is to define the analogue of the characteristic polynomial for the operator \( \tau \). This will be an entire function \( \zeta_\tau \) whose zeros are the eigenvalues of \( \tau \) and can be considered to be a version of \( \det(I - z\tau^{-1}) \). We will call \( \zeta_\tau \) the secular function of \( \tau \). The term secular function has been used in the past for the characteristic polynomial.

For a Hilbert-Schmidt operator \( A \) with eigenvalues \( \nu_k \) the regularized determinant
\[
\det_2(I - zA) = \prod_k (1 - z \nu_k) e^{z \nu_k}.
\]
(22)
is an entire function in \( z \) that vanishes exactly at the eigenvalues, see [27], Chapter 9. The product is uniformly convergent on compact subsets of \( \mathbb{C} \). When \( A \) is also trace class, then
\[
\det(I - zA) = \prod_k (1 - z \nu_k)
\]
is a well defined entire function and the product is absolutely convergent, see [27], Chapter 3. Moreover, one has
\[
\det(I - zA) = \det_2(I - zA) e^{-z \text{Tr } A}.
\]
(23)
Our operator \( \tau^{-1} \) or, equivalently \( r \tau \), is not a trace class operator, but the integral trace

\[
t_{\tau} = \int_{0}^{\sigma} \text{Tr} K_{\tau}(s, s)ds = \frac{1}{2} \int_{0}^{\sigma} a(s)^{t}c(s)ds
\]  

(24)

is finite by Assumption 2. Under the \( q \)-boundary conditions (15) we have

\[
t_{\tau} = \int_{0}^{\sigma} x(s) - \frac{q}{2y(s)} ds.
\]

Note that if an integral operator with kernel of the form (16) is trace class, its trace is equal to its integral trace.

By analogy with formula (23), since \( r \tau \) is Hilbert-Schmidt, we define

**Definition 8** (Secular function). Assume that \( \tau \) satisfies Assumptions 1-3. Then the secular function is defined as

\[
\zeta_{\tau}(z) = e^{-zt_{\tau}} \text{det}_{2}(I - z r \tau) = e^{-\frac{z}{2}} \int_{0}^{\sigma} a(s)^{t}c(s)ds \prod_{k}(1 - z/\lambda_{k})e^{z/\lambda_{k}},
\]

(25)

where \( a \) and \( c \) are defined in (17). The infinite product is uniformly convergent on compact sets of \( z \).

The eigenvalues of \( r \tau \) are \( \frac{1}{\lambda_{k}} \). The infinite product \( \prod_{k}(1 - \frac{z}{\lambda_{k}}) \) is not necessarily absolutely convergent, so \( \text{det}(I - xr \tau) \) might not be defined. In Section 2.7 we show that with some additional assumptions, \( \zeta \) is equal to the principal value product

\[
\lim_{r \to \infty} \prod_{|\lambda_{k}| \leq r} (1 - z/\lambda_{k}).
\]

In the rest of the section we discuss other representations for \( \zeta \): an explicit Taylor expansion and a representation by an ordinary differential equation.

### 2.4 The Taylor expansion of \( \zeta \)

We will work towards a more explicit expression for \( \zeta_{\tau} \) in terms of \( X, u_{1}, u_{2} \). The following proposition gives the Taylor expansion of \( \zeta \) with explicit coefficients.

**Proposition 9.** Let \( \tau \) satisfy Assumptions 1-3. Then the secular function \( \zeta_{\tau} \) (25) has the following Taylor expansion with infinite radius of convergence:

\[
\zeta_{\tau}(z) = 1 + \sum_{n=1}^{\infty} r_{n}z^{n}, \quad r_{n} = -\int\int\int_{0<s_{1}<s_{2}<\cdots<s_{n}\leq\sigma} u_{i_{1}}^{t}R(s_{1})JR(s_{2})J\cdots R(s_{n})u_{1}ds_{1}\cdots ds_{n}.
\]

(27)
The coefficient $r_n$ can be bounded as

$$|r_n| \leq \left( \int_0^\sigma |u_0' R(s) u_1| ds + \int_0^\sigma \int_0^t u_0' R(s) u_0 u_1' R(t) u_1 ds dt \right)^n. \quad (28)$$

We start with a statement about the Fredholm expansion of regularized determinants of Hilbert-Schmidt integral operators with a finite diagonal integral.

**Lemma 10.** Suppose that $G$ is a bounded interval on $\mathbb{R}$, $k : G^2 \to \mathbb{R}$ is a measurable function with $\int_{G^2} |k(s, t)|^2 ds dt < \infty$ and $\int_G |k(s, s)| ds < \infty$. Let $Af(s) = \int_G k(s, t)f(s) ds$ be the Hilbert-Schmidt integral operator acting on real valued $L^2(G)$ functions. Then

$$\det_2 (I + zA)e^{z \int_G k(s, s) ds} = 1 + \sum_{n=2}^\infty \frac{z^n}{n!} \int_{G^n} \det [k(t_i, t_j)]_{i,j=1}^n dt_1 \ldots dt_n, \quad (29)$$

where the $n$-variable integrals on the right are all finite and the series converges on $\mathbb{C}$.

Moreover, we also have the bound

$$\int_{G^n} \left| \det [k(t_i, t_j)]_{i,j=1}^n \right| dt_1 \ldots dt_n \leq n! \left( \int_G |k(s, s)| ds + \|A\|_2 \right)^n \quad (30)$$

Note that we do not require $k$ to be continuous near the diagonal.

**Proof.** Since $A$ is a Hilbert-Schmidt integral operator, the classical theory, see [14], [27], implies that

$$\det_2 (I + zA) = 1 + \sum_{n=2}^\infty \frac{z^n}{n!} \int_{G^n} \det [k(t_i, t_j)1(i \neq j)]_{i,j=1}^n dt_1 \ldots dt_n, \quad (31)$$

with the series on the right converging for all $z \in \mathbb{C}$. In particular we also have that

$$\int_{G^n} \det [k(t_i, t_j)1(i \neq j)]_{i,j=1}^n dt_1 \ldots dt_n < \infty, \quad \text{for } n \geq 2. \quad (32)$$

Note that for any $n \geq 1$, and $t_1, \ldots, t_n \in G$ we have

$$\det [k(t_i, t_j)]_{i,j=1}^n = \sum_{B \subseteq \{1, \ldots, n\} \atop |B| \geq 2} \det [k(t_i, t_j)1(i \neq j)]_{i,j \in B} \prod_{1 \leq \ell \leq n \atop \ell \notin B} k(t_\ell, t_\ell). \quad (33)$$

This identity follows by expanding the determinant on the left and collecting the terms based on the number of fixed points in the permutations. Integrating the quantity (33) on $G^n$ we get

$$\sum_{\ell=2}^n \binom{n}{\ell} \int_{G^n} \det [k(t_i, t_j)1(i \neq j)]_{i,j \leq \ell} dt_1 \ldots dt_\ell \left( \int_G k(t, t) dt \right)^{n-\ell}. \quad (34)$$
This expression is finite by (32), and the assumption \( \int_G |k(s, s)|ds < \infty \). The claim (29) follows after multiplying the Taylor expansions of the entire functions \( \det_2(I + zA) \) and \( e^z \int_G k(s, s)ds \).

To prove the bound (30) we first show that for \( \ell \geq 1 \) one has
\[
\int_{G^n} |k(t_1, t_2)k(t_2, t_3) \cdots k(t_\ell, t_1)|dt_1 \cdots dt_\ell \leq \left( \int_G |k(s, s)|ds + \|A\|_2 \right) \ell.
\]
(35)

For \( \ell = 1 \) this follows from the assumption. Define \( B \) by
\[
Bf(x) = \int_G |k(x, y)|f(y)dy.
\]

For \( \ell \geq 2 \) the integral operator \( B^\ell \) is trace class, and its trace is equal to the left side of (35). Moreover, \( \text{Tr} (B^\ell) \leq \text{Tr} (B^2)^{\ell/2} = \|A\|_2^\ell \). This shows (35) for \( \ell \geq 2 \). The bound (30) follows by expanding the determinant \( \det [k(t_i, t_j)]_{i,j=1}^n \) and applying (35) the cycles of the permutation \( \pi \) in \( \int_{G^n} \prod_{i=1}^n |k(s_i, s_{\pi(i)})|ds_1 \cdots ds_n \).

**Proof of Proposition 9** We start by adapting Lemma 10 to integral operators with matrix valued kernels. We can embed the space of \( \mathbb{R}^2 \)-valued \( L^2(0, \sigma) \) functions into the space of real valued \( L^2(0, 2\sigma) \) functions using the following invertible isometry:
\[
g = [g_1, g_2]^t, \quad \mathcal{J} g(t) = \begin{cases} 
g_1(s), & 0 < s \leq \sigma 
g_2(s - \sigma), & \sigma < s \leq 2\sigma. \end{cases}
\]
(36)

If \( B \) is a Hilbert-Schmidt integral operator acting on \( \mathbb{R}^2 \)-valued functions on \( (0, \sigma] \) with kernel \( K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \) then \( A = \mathcal{J} B \mathcal{J}^{-1} \) is a Hilbert-Schmidt integral operator acting on scalar functions on \( (0, 2\sigma] \), and the integral kernel is
\[
k(s, t) = K_{1+1(s \geq \sigma), 1+1(t \geq \sigma)}(s, t).
\]
(37)

We have \( \int_0^\sigma \|K(s, t)\|^2_2ds\,dt = \int_0^{2\sigma} |k(s, t)|^2ds\,dt \) and \( \int_0^{2\sigma} k(s, s)ds = \int_0^\sigma \text{Tr} K(s, s)ds \). Assuming that both of these integrals are finite we may apply Lemma 10 to the integral operator \( A \). Since the spectrum of \( A \) is the same as the spectrum of \( B \), we have \( \det_2(I - zB) = \det_2(I - zA) \) and hence
\[
det_2(I - zB)e^{-z} \int_0^\sigma \text{Tr} K(s, s) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{n}{n!} \int_{(0, 2\sigma)^n} \det [k(t_i, t_j)]_{i,j=1}^n dt_1 \cdots dt_n.
\]
(38)

From (37) we have
\[
\int_{(0, 2\sigma)^n} \det [k(t_i, t_j)]_{i,j=1}^n dt_1 \cdots dt_n = \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \int_{(0, \sigma)^n} \det(K_{i_a, i_b}(s_a, s_b))_{a,b=1}^n ds_1 \cdots ds_n.
\]
Now set $B = r \tau$. From (16) we get that the entries $K_{ij}$ of the matrix valued kernel are given by

$$K_{ij}(s, t) = \frac{1}{2} (a_i(s)c_j(t)1(s < t) + c_i(s)a_j(t)1(t \leq s)).$$  \quad (39)

Note that $K(s, t)^t = K(t, s)$ and thus $K_{ij}(s, t) = K_{ji}(t, s)$. Because of this we have

$$\frac{1}{n!} \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \int_{(0, \sigma]^n} \det(K_{i_j,i_\ell}(s_j, s_\ell))_{j, \ell=1}^n ds_1 \cdots ds_n = \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \int_{0<s_1<\cdots<s_n\leq \sigma} \int_{(0, \sigma]^n} \det(K_{i_j,i_\ell}(s_j, s_\ell))_{j, \ell=1}^n ds_1 \cdots ds_n.$$

Fix $i_1, \ldots, i_n \in \{1, 2\}$ and $0 < s_1 < \ldots < s_n \leq \sigma$. Introduce the temporary notation

$$p_k = a_{ik}(s_k), \quad q_k = c_{ik}(s_k),$$

then by (39) we have $2K_{ij,i_\ell}(s_j, s_\ell) = p_{\min(j, \ell)} \cdot q_{\max(j, \ell)}$. For example, for $n = 3$ we have

$$(2K_{ij,i_\ell}(s_j, s_\ell))_{j, \ell=1}^n = \begin{pmatrix} p_1q_1 & p_1q_2 & p_1q_3 \\ p_1q_2 & p_2q_2 & p_2q_3 \\ p_1q_3 & p_2q_3 & p_3q_3 \end{pmatrix}.$$

We show that

$$\det(p_{\min(j, \ell)} \cdot q_{\max(j, \ell)})_{j, \ell=1}^n = p_1q_1 \prod_{j=1}^{n-1} (p_{j+1}q_j - p_jq_{j+1}).$$  \quad (40)

Subtract row $n - 1$ times $q_n/q_{n-1}$ from row $n$. Then the last row becomes

$$[0, \ldots, 0, p_nq_n - p_{n-1}q_n^2/q_{n-1}].$$

The identity (40) now follows by induction.

Note that $p_{j+1}q_j - p_jq_{j+1} = [p_j, q_j]J[p_j+1, q_{j+1}]^t$, hence, with $v_k(s_k) = [p_k, q_k]^t$ we have

$$\det(p_{\min(j, \ell)} \cdot q_{\max(j, \ell)})_{j, \ell=1}^n = [v_1(s_1)v_1(s_1)^tJv_{i_2}(s_2)v_{i_2}(s_2)^tJ \cdots v_{i_n}(s_n)v_{i_n}(s_n)^tJ]_{1,1}.$$

Note that

$$v_1(s)v_1(s)^t + v_2(s)v_2(s)^t = 2U^tR(s), \quad U = [u_0, u_1].$$

Summing (41) for all choices of $i_1, \ldots, i_n$ gives

$$2^n [U^tR(s_1)UJU^tR(s_2)UJ \cdots U^tR(s_n)UJ]_{1,1} = -(-2)^n u_0^tR(s_1)JR(s_2)J \cdots R(s_n)u_1.$$

In the last step we used $UJU^t = -J$, which is equivalent to the assumption (14).

The statement of the proposition now follows from (38). \quad \Box
Example 11 (Deterministic Sine operator, continued). Consider \( \tau \) from Example 6. The eigenvalues are \( \frac{2\pi k - \theta}{\sigma}, k \in \mathbb{Z} \), and \( a(s)c(s) = -q = -\cot(\theta/2) \). Definition (25) gives
\[
\zeta(z) = e^{\frac{\sigma z}{2} \cot(\theta/2)} \prod_{k} \left( 1 - \frac{\sigma z}{2 \pi k - \theta} \right) e^{\frac{\sigma z}{2} \cot(\theta/2)}.
\]
It is an exercise in complex analysis to show that
\[
\zeta(z) = \frac{\sin \left( \frac{\theta + \sigma z}{2} \right)}{\sin(\theta/2)}.
\]
(42)
This also follows from the power series representation of Proposition 9:
\[
\zeta(z) = 1 + \sum_{n=1}^{\infty} r_n z^n, \quad r_n = -2^{-n} \int \int \int_{0 < s_1 < \cdots < s_n \leq \sigma} u_0^t J^{n-1} u_1 ds_1 \ldots ds_n.
\]
Using \( J^{2k+1} = (-1)^k J \) and \( J^{2k} = (-1)^k I \) we get
\[
r_{2k+1} = \frac{(\sigma/2)^{2k+1}}{(2k+1)!} (-1)^k q, \quad r_{2k} = \frac{(\sigma/2)^{2k}}{(2k)!} (-1)^k
\]
which give the series expansion of \( \zeta \), and proves (42).

Example 12 (Deterministic Bessel operator, continued). For \( \tau \) from Example 7 the definition (25) gives
\[
\zeta(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{\sigma z^2}{4 \gamma_k^2} \right),
\]
(43)
where \( \gamma_k \) is the \( k \)th root of \( J_{\frac{a-1}{2}} \). The infinite product representation of the Bessel function, see 9.5.10 in [1] gives
\[
\zeta(z) = \frac{\Gamma \left( \frac{a+1}{2} \right) \left( \frac{\sigma}{4} \right)^{\frac{a-1}{2}}}{\sqrt{\pi} \Gamma \left( \frac{a}{2} \right)} J_{\frac{a-1}{2}} \left( \frac{\sigma}{2} \right) = {}_0F_1 \left( ; \frac{a+1}{2} ; -\frac{\sigma z^2}{16} \right),
\]
(44)
where \( {}_0F_1 \left( ; a ; w \right) = \sum_{k=0}^{\infty} \frac{\Gamma(a)}{k! \Gamma(a+k)} w^k \) is the confluent hypergeometric function.

The series representation given by Proposition 9 gives the coefficients
\[
r_{2n+1} = 0, \quad r_{2n} = (-1)^n 2^{-2n} \int \int \int_{0 < s_1 < \cdots < s_{2n} \leq \sigma} \frac{s_1 \cdots s_{2n}}{s_2 s_4 \cdots s_{2n}} ds_1 \ldots ds_{2n}.
\]

The multiple integral evaluates to
\[
r_{2n} = (-1)^n 2^{-4n} \frac{\sigma^{2n} \Gamma \left( \frac{a+1}{2} \right)}{n! \Gamma \left( \frac{a+1}{2} + n \right)},
\]
which agrees with (44) and the series expansion of \( {}_0F_1 \).
2.5 ODE representation for $\zeta$

The function $\zeta$ can also be characterized using the solution of a vector-valued ODE, this is the statement of our next proposition.

**Proposition 13.** Suppose that $R$ and $u_0$ satisfy Assumptions 1 and 2. There is a unique vector-valued function $H : (0, \sigma] \times \mathbb{C} \to \mathbb{C}^2$ so that for every $z \in \mathbb{C}$ the function $H(\cdot, z)$ is the solution of the ordinary differential equation

$$J \frac{d}{dt} H(t, z) = z R(t) H(t, z), \quad t \in (0, \sigma], \quad \lim_{t \to 0} H(t, z) = u_0.$$  \hspace{1cm} (45)

For any $t \in (0, \sigma]$ the function $H(t, z)$ satisfies $\|H(t, z)\| > 0$, and its two coordinates are entire functions of $z$ mapping the reals to the reals.

Moreover for any $u_1$ with $u_1^t J u_1 = 1$ the corresponding $\zeta_\tau(z)$ satisfies

$$\zeta_\tau(z) = H(\sigma, z)^t J u_1.$$  \hspace{1cm} (46)

We prove Proposition 13 in two parts. Proposition 18 shows that there is at most one function $H$ satisfying the conditions, while Proposition 19 provides a power series solution.

**Remark 14.** Our proof is self-contained and relies on the Taylor series expansion of $\zeta$ in Proposition 9. This proposition is a version of standard results in de Branges’ theory of Hilbert spaces of analytic functions adapted to our setting. For example, Theorem 41 of de Branges [9] is a version of the existence and uniqueness part of Proposition 13 with slightly different assumptions. The proof of that theorem relies on a deep understanding of de Branges’ theory.

**Remark 15.** The function $H$ can be considered as the solution of the eigenvalue equation $\tau H = z H$ with initial condition $H(0, z) = u_0$. The expression $H(\sigma, z)^t J u_1$ gives a linear transform of $H(\sigma, z)$ which is zero exactly when $H(\sigma, z) \| u_1$, and it is equal to 1 at $z = 0$.

**Remark 16.** The ODE in (45) is linear, and by Assumption 1 the function $\| R(s) \|$ is bounded for any compact subset of $(0, \sigma]$. Hence for any $0 < a < b \leq \sigma, z \in \mathbb{C}$ and $v \in \mathbb{C}^2$ the ODE

$$J \frac{d}{dt} G(t, z) = z R(t) G(t, z), \quad G(a, z) = v,$$  \hspace{1cm} (47)

has a unique solution in $[a, b]$ by the standard theory of ordinary differential equations, and the solution is analytic in $z$ for any given $t \in [a, b]$. This holds also if we assume that the initial condition $v = v(z)$ is an analytic function of $z$. 

16
Under the additional assumption \( \int_0^\sigma \|R(s)\|ds < \infty \) this extends to the \( a = 0 \) case. Hence in this case existence and uniqueness of \( H(t, z) \) in Proposition 13 follows immediately. If \( \|R(t)\| \) is bounded on \( (0, \sigma] \) as opposed to just locally bounded, then \( \int_0^\sigma \|R(s)\|ds < \infty \). Our assumptions allow \( \|R(t)\| \) to blow up near zero, and most of our applications have this property.

**Remark 17.** Gesztesy and Makarov [13] give an explicit formula for the modified Fredholm determinant of a Hilbert-Schmidt integral operator with a semi-separable kernel

\[
M(s, t) = f_1(s)g_1(t)1(s < t) + f_2(s)g_2(t)1(t \leq s)
\]
on an interval \((a, b)\), assuming that the matrix valued functions \(f_1, g_1, f_2, g_2\) are all in the appropriate \(L^2\) spaces.

Note that \(K_{\tau \tau}\) is a semi-separable kernel by (16), and in the case when \(\int_0^\sigma \|R(s)\|ds < \infty\) the vector valued functions \(a(\cdot)\) and \(c(\cdot)\) are both in \(L^2(0, \sigma]\). Hence in this case the results of [13] apply, and it can be checked that the derived formula leads to (46) again.

**Proposition 18 (Uniqueness of \(H\)).** Suppose that for a given \(z \in C\) the functions \(H_1(t, z), H_2(t, z)\) both satisfy (45). Then \(H_1(t, z) = H_2(t, z)\). Moreover, \(\|H_1(t, z)\| > 0\) for all \(t, z\).

**Proof.** From (45) we get for \(t \in (0, \sigma]\)

\[
\frac{d}{dt} (H_2(t, z)^t JH_1(t, z)) = 0,
\]
so \(H_2(t, z)^t JH_1(t, z)\) is constant in \(t\). Since \(\lim_{t \to 0} H_k(t, z) = u_0\) for \(k = 1, 2\), we see that this constant has to be \(u_0^t J u_0 = 0\). The identity \(H_2(t, z)^t JH_1(t, z) = 0\) implies that \(H_2(t, z) \parallel H_1(t, z)\) for all \(t \in (0, \sigma]\).

By Remark 16 the equation (45) is linear, so if \(H_k(t_0, z) = (0, 0)^t\) for a particular \(t_0 \in (0, \sigma]\) then \(H_k(t, z) = (0, 0)^t\) would hold for all \(t \in (0, \sigma]\), which would contradict the behavior at \(t \to 0\). Hence \(\|H_k(t, z)\|\) cannot be zero. Since \(H_2(t, z) \parallel H_1(t, z)\) for all \(t \in (0, \sigma]\), it follows that there exists a function \(f(t, z)\) so that \(H_2(t, z) = f(t, z)H_1(t, z)\), \(f(t, z) \neq 0\) for \(t \in (0, \sigma]\).

Using the linearity of (45) again we see that \(f(t, z)\) must be a constant in \(t\). But the initial condition in (45) then implies that \(f(t, z) = 1\) and \(H_1 = H_2\). \(\square\)

**Proposition 19 (A power series solution).** For \(t \in (0, \sigma]\), \(z \in C\) define \(H(t, z)\) using the following series expansion:

\[
H(t, z)^t = \sum_{n=0}^{\infty} d_n(t)z^n, \quad d_0(t) = u_0^t
\]

\[
d_n(t) = \iiint_{0<s_1<s_2<\cdots<s_n \leq t} u_0^t R(s_1)J R(s_2)J \cdots R(s_n)J ds_1 \cdots ds_n, \quad n \geq 1.
\]

(48)
The series converges for all \( z \in \mathbb{C} \). For any \( t \in (0, \sigma] \) the two coordinates of the function \( H(t, z) \) are entire functions of \( z \) mapping the reals to the reals. The function \( H(t, z) \) satisfies (45) and (46).

**Proof.** Let \( u \) be any vector not parallel to \( u_0 \). Then \( a = u_0^t J u \neq 0 \). Set \( u_1 = u/a \) so that \( u_0^t J u_1 = 1 \). Set \( r_n = d_n^t J u_1 \). Proposition 9 shows that the series

\[
1 + \sum_{n=1}^{\infty} r_n z^n = \frac{1}{a} \sum_{n=0}^{\infty} d_n^t J u z^n
\]

converges everywhere. Using two linearly independent \( u \)-s we see that \( H \) is well-defined and its two coordinates are analytic in \( z \). Since the coefficients \( d_n \) are real vectors, the coordinates of \( H \) map reals to reals for any \( t \in (0, \sigma] \).

Proposition 9 also shows that the identity (46) holds. The only thing left is to show that \( H \) satisfies the ODE (45) for all \( z \in \mathbb{C} \).

By estimate (28) of Proposition 9 we have

\[
|d_n^t J u_1| \leq r(t)^n, \quad r(t) = \int_0^t |u_0 R(s) u_1| ds + \int_0^\sigma \int_0^t u_0^t R(s) u_0 u_1^t R(t) u_1 ds dt. \tag{49}
\]

By Assumption 2, the function \( r(t) \) is finite, non-decreasing and \( \lim_{t \downarrow 0} r(t) = 0 \). This implies the bound \( \|d_n^t(t)\| \leq cr(t)^n \) for a finite \( c > 0 \) for all \( n \geq 1 \), from which we get \( \lim_{t \downarrow 0} H(t, z) = u_0 \) for all \( z \in \mathbb{C} \).

From (48) it follows that for \( n \geq 1 \) and \( 0 \leq t_1 < t_2 \leq \sigma \) we have

\[
d_n^t(t_2) - d_n^t(t_1) = \int_{t_1}^{t_2} d_{n-1}(s) R(s) J ds.
\]

Now assume \( t_1 > 0 \), then \( R \) is uniformly bounded in \([t_1, t_2] \). Multiplying both sides by \( z^n \) and summing over \( n \) we get that both sides are absolutely convergent for \( |z| < r^{-1}(t_2) \). Thus when this inequality holds, we have

\[
H(t_2, z)^t - H(t_1, z)^t = \int_{t_1}^{t_2} z H(t, z)^t R(t) J ds.
\]

Since \( J^{-1} = J^t = -J \), the fundamental theorem of calculus gives that the differential equation (45) holds for \( z \in \mathbb{C} \), \( t \in (0, \sigma] \) with \( |z| < r(t)^{-1} \).

It suffices to show that \( H(t, z) \) satisfies the ODE for \( |z| \leq b \), \( t \in (0, \sigma] \) for any fixed \( b > 0 \). Given \( b > 0 \), pick \( t_0 \) so that \( b < r(t_0)^{-1} \). Then \( H(t, z) \) is a solution in \((0, t_0] \) for \( |z| \leq b \).

For all \( |z| \leq b \) let \( G(t, z) \) be the solution of the ordinary differential equation (45) on \([t_0, \sigma] \) with initial condition \( H(t_0, z) \). By Remark 16 for \( t_0 \leq t \leq \sigma \), the function \( G(t, z) \) is analytic in \( z \). Moreover, \( G(t, z) = H(t, z) \) when \( |z| < r^{-1}(\sigma) \), since \( H(t, z) \) also solves (45) there, and it agrees with \( G(t, z) \) when \( t = t_0 \), \( |z| \leq r^{-1}(\sigma) \leq b \). Thus for \( t \in [t_0, \sigma] \) the analytic functions \( G(t, z) \) and \( H(t, z) \) must agree for all \( |z| \leq b \), which implies that \( H(t, z) \) solves the ODE for \( |z| \leq b \), \( t \in (0, \sigma] \). 

\[\square\]
The next proposition provides a way to approximate $H$ with the solutions of more regular ODE systems for which the uniqueness of the solution is immediate.

**Proposition 20** (Regular approximation of $H$). Let $0 < \varepsilon < \sigma$, $z \in \mathbb{C}$ and let $H_\varepsilon(t, z)$ be the solution of the ODE

$$J \frac{d}{dt} H_\varepsilon(t, z) = z R(t) H_\varepsilon(t, z), \quad t \in [\varepsilon, \sigma], \quad H_\varepsilon(\varepsilon, z) = u_0. \quad (50)$$

Extend the definition of $H_\varepsilon(t, z)$ for $t \in (0, \varepsilon)$ with $H_\varepsilon(t, z) = u_0$.

Then as $\varepsilon \to 0$ we have $H_\varepsilon(\cdot, z) \to H(\cdot, z)$ uniformly on compact subsets of $(0, \sigma) \times \mathbb{C}$.

**Proof.** By Remark 16 the differential equation (50) has a unique solution.

Moreover, for any $u_1$ with $u_0^t J u_1 = 1$ and $\varepsilon \leq t \leq \sigma$ by Proposition 13 the function $H_\varepsilon(t, z)^t J u_1$ is the secular function of $\tau_{\varepsilon,t,u_1}$, which is $\tau$ restricted to $[\varepsilon, t]$ with boundary condition $u_0$ at $\varepsilon$ and $u_1$ at $t$.

Note that $\tau_{\varepsilon,t,u_1}$ is the integral operator with kernel given in (16), but restricted to $[\varepsilon, t] \times [\varepsilon, t]$. In other words, we have

$$H_\varepsilon(t, z)^t J u_1 = \det_2(I - z \tau_{\varepsilon,t,u_1}) e^{-\frac{z}{2} \int_0^t \sigma_a(s)^t c(s)ds}. \quad (51)$$

For $0 < t \leq \sigma$ we denote by $\tau_{t,u_1}$ the version of $\tau$ on $(0, t)$ with boundary conditions $u_0, u_1$. This operator satisfies Assumptions 1 and 2, and Proposition 13 shows that

$$H(t, z)^t J u_1 = \det_2(I - z \tau_{t,u_1}) e^{-\frac{z}{2} \int_0^t \sigma_a(s)^t c(s)ds}. \quad (52)$$

Note that the Hilbert-Schmidt norm of $\tau_{t,u_1}$ is uniformly bounded in $t$ as

$$\| \tau_{t,u_1} \|_2 \leq \| \tau \|_2 < \infty.$$  

By the triangle inequality we have

$$\left| (H(t, z) - H_\varepsilon(t, z)^t) J u_1 \right| \leq \left| \det_2(I - z \tau_{\varepsilon,t,u_1}) - \det_2(I - z \tau_{t,u_1}) \right| e^{-\frac{z}{2} \int_0^t \sigma_a(s)^t c(s)ds}$$

$$+ \left| \det_2(I - z \tau_{t,u_1}) \right| e^{-\frac{z}{2} \int_0^t \sigma_a(s)^t c(s)ds} \left| e^{-\frac{z}{2} \int_0^t \sigma_a(s)^t c(s)ds} - 1 \right|. \quad (53)$$

If $\kappa_1, \kappa_2$ are Hilbert-Schmidt operators on the same domain then

$$|\det_2(I - z \kappa_1) - \det_2(I - z \kappa_2)| \leq |z| \cdot \|\kappa_1 - \kappa_2\|_2 \exp(c|z|^2(\|\kappa_1\|_2^2 + \|\kappa_2\|_2^2) + c) \quad (54)$$

with an absolute constant $c$, see Theorem 9.2(c) in [27]. In particular, using this for $\kappa_2 = 0$ we get the bound

$$|\det_2(I - z \kappa_1) - 1| \leq |z| \cdot \|\kappa_1\|_2 \exp(c|z|^2\|\kappa_1\|_2^2 + c). \quad (55)$$
This implies that the $\det_2$ part of the second term on the right of (53) is uniformly bounded in compact sets of $t, z$. Since $\int_0^\sigma |a(s)^t c(s)| \, ds < \infty$ by assumption, the entire term converges to 0 as $\varepsilon \to 0$ uniformly.

We will show that the same holds for the other term in (53).

Extend the domain of the integral operator $\tau_{\varepsilon,t,u_1}$ to $(0, t)^2$ by setting the kernel equal to the constant 0 matrix on $(0, t)^2 \setminus (\varepsilon, t)^2$. We use the temporary notation $\kappa_\varepsilon$ for the new integral operator. The spectrum of $\kappa_\varepsilon$ is given by the spectrum of $\tau_{\varepsilon,t,u_1}$ and the value 0 with infinite multiplicity. This means that

$$\det_2(I - z \kappa_\varepsilon) = \det_2(I - z \tau_{\varepsilon,t,u_1}).$$

Hence

$$|\det_2(I - z \tau_{\varepsilon,t,u_1}) - \det_2(I - z \tau_{t,u_1})| = |\det_2(I - z \kappa_\varepsilon) - \det_2(I - z \tau_{t,u_1})|. \quad (56)$$

Since $\tau_{t,u_1}$ is Hilbert-Schmidt and its kernel agrees with that of $\tau_{\varepsilon,t,u_1}$ on $[\varepsilon, t]^2$ we have

$$\lim_{\varepsilon \to 0} \| \tau_{t,u_1} - \kappa_\varepsilon \|_2 = 0 \quad (57)$$

By (54) and (57) the term on the right of (56) converges to 0 as $\varepsilon \to 0$ uniformly for $t, z$ in a compact set.

Collecting all of our estimates, and recalling that $\int_0^\sigma |a(s)^t c(s)| \, ds < \infty$ by assumption, we get that for a given $u_1$ with $u_0^t J u_1 = 1$ we have

$$\lim_{\varepsilon \to 0} H_\varepsilon(t, z) J u_1 = H(t, z) J u_1,$$

and the convergence is uniform on compact sets of $t, z$. This statement is true for any $u_1 \parallel u_0$, which implies the proposition.

The bounds (54, 55) together with the definition (25) of $\zeta$ and the triangle inequality gives the following.

**Proposition 21** (Continuity of $\tau \mapsto \zeta_\tau$). Let $\tau_1, \tau_2$ be two Dirac operators on $(0, \sigma]$ satisfying our assumptions. Denote by $\zeta_i, \tau_i, t_i$ the secular function, resolvent and integral trace of $\tau_i$. Let $\| \cdot \|$ denote the Hilbert-Schmidt norm. Then there is a universal constant $a > 1$ so that for all $z \in \mathbb{C}$

$$|\zeta_1(z) - \zeta_2(z)| \leq \left( e^{\|z\|_1 - |t_2|} - 1 + |z| \|\tau_1 - \tau_2\| \right) a |z|^2 \|\tau_1\|^2 + |\tau_2|^2 + |z| (|t_1| + |t_2|) + 1. \quad (58)$$

Proposition 21 provides a sufficient condition for the convergence of secular functions. It shows that if we have a sequence of Dirac operators on $(0, \sigma]$ for which the resolvents converge in Hilbert-Schmidt norm, and the integral traces converge, then the secular functions converge uniformly on compacts of $\mathbb{C}$. 

20
2.6 The structure function $E$ and the functions $A, B$

From this point we will assume that we are in the case of $q$-boundary conditions \(^{15}\).

Consider the unique vector-valued function $H$ described in Proposition \(^{13}\). Set

\[ A(t, z) := [1, 0] H(t, z), \quad B(t, z) := [0, 1] H(t, z), \]  
\[ E(t, z) := [1, -i] H(t, z) = A(t, z) - iB(t, z). \]  

For $t$ fixed, $z \mapsto E(t, z)$ is called the structure function in the context of de Branges’ theory of Hilbert spaces of analytic functions.

**Lemma 22.** The entire functions $A(t, z), B(t, z)$ are real for real $z$, and have only real zeros. If the entire function $E(t, z)$ has zeros then they are in the open upper half plane. Moreover,

\[ E(t, 0) = 1, \quad \bar{A}(t, z) = A(t, \bar{z}), \quad \bar{B}(t, z) = B(t, \bar{z}), \]  
\[ |E(t, z)| \leq |E(t, \bar{z})|, \quad \Im z > 0. \]  

**Proof.** Proposition \(^{13}\) implies that $E(t, z)$ is an entire function of $z$ for any $t$. The vector valued function $H(t, z)$ is not equal to $[0, 0]$ for any $t, z$, and it takes values from $\mathbb{R}^2$ for $z \in \mathbb{R}$. Hence $E$ is nonzero for real $z$, and $A$ and $B$ are real for $z \in \mathbb{R}$. The statements of (61) follow from $H(t, 0) = u_0 = [1, 0]$ and the reflection principle. Since for every $r \in \mathbb{R}$ the function $A + rB$ is a secular function, it only has real zeros. Moreover, $B = \lim_{r \to \infty} (A + rB)/r$ has only real zeros by Hurwitz’s theorem.

From (45) we get

\[ -i \frac{d}{dt} H(t, z)^t J H(t, z) = 2\Im z \overline{H(t, z)}^t R(t) H(t, z). \]

We also have

\[ -2i \overline{H(t, z)}^t J H(t, z) = |E(t, \bar{z})|^2 - |E(t, z)|^2. \]

Since $\det R(t) = \frac{1}{4}$ and the entries of $R(t)$ are real, one can check that $R(t) + \frac{1}{2} J$ is nonnegative definite, which implies that for $\Im z > 0$ we have

\[ \frac{d}{dt} \left( |E(t, \bar{z})|^2 - |E(t, z)|^2 \right) \geq \Im z \left( |E(t, \bar{z})|^2 - |E(t, z)|^2 \right), \]

which shows (62).

Finally, we note that if $E(t, z) = 0$ for $\Im z < 0$ then (62) would imply $E(t, z) = E(t, \bar{z}) = 0$ and $A(t, z) = B(t, z) = 0$. By Proposition \(^{13}\) this is not possible, hence $E(t, z) = 0$ can only hold for $\Im z > 0$. \qed
For the interested reader, the following proposition summarizes additional properties of \( H \) and the structure function \( E \). The proofs can be found in de Branges \cite{9} with some required exercises. We will not use these results in the present paper.

**Proposition 23.**  
(i) The function \( H(t, z) \) is in \( L^2_R \) for any fixed \( z \in \mathbb{C} \). In particular, if \( \lambda \in \mathbb{C} \) is an eigenvalue of \( \tau \) then the corresponding eigenfunction is \( H(\cdot, \lambda) \).

(ii) The function \( |E(t, x - iy)| \) is non-decreasing in \( y \) for \( y \geq 0 \).

(iii) The structure function \( E = A - iB \) satisfies the following bound with all derivatives with respect to \( z \):

\[
\log |E(t, z)| \leq \Re zA'(t, 0) + \Im zB'(0) + \frac{1}{2} (A'(t, 0) - A''(t, 0) + B'(t, 0))^2 |z|^2. \tag{63}
\]

**Example 24** (Deterministic Sine operator, part 3). Let us return to Example \( \ref{ex:6} \) with boundary conditions \( u_0 = [1, 0]^t, u_1 = [-\cot(\theta/2), -1]^t \). The solution of \( (45) \) is given by

\[
H(t, z) = [\cos(tz/2), -\sin(tz/2)]^t. \tag{64}
\]

Proposition \ref{prop:13} gives

\[
\zeta(z) = H(\sigma, z)J u_1 = \cot(\sigma/2)\sin(\sigma z/2) + \cos(\sigma z/2)
\]

which agrees with the expression in \( (42) \). Note also that

\[
A(t, z) = \cos(\frac{it}{2}), \quad B(t, z) = -\sin(\frac{it}{2}), \quad E(t, z) = e^{\frac{it}{2}}.
\]

**Example 25** (Deterministic Bessel operator, part 3). Now consider Example \( \ref{ex:7} \) with boundary conditions \( u_0 = [1, 0]^t, u_1 = [0, -1]^t \). Then the solution of \( (45) \) is given by

\[
H(t, z) = \left(\frac{zt}{4}\right)^{1-\alpha} \Gamma\left(\frac{\alpha+1}{2}\right) \left[J_{\frac{\alpha-1}{2}}\left(\frac{zt}{2}\right), -it^\alpha J_{\frac{\alpha+1}{2}}\left(\frac{zt}{2}\right)\right]^t. \tag{65}
\]

Proposition \ref{prop:13} leads to the expression \( (44) \). We also have

\[
A(t, z) = \left(\frac{zt}{4}\right)^{1-\alpha} \Gamma\left(\frac{\alpha+1}{2}\right) J_{\frac{\alpha-1}{2}}\left(\frac{zt}{2}\right), \quad B(t, z) = -\left(\frac{zt}{4}\right)^{1-\alpha} \Gamma\left(\frac{\alpha+1}{2}\right) t^\alpha J_{\frac{\alpha+1}{2}}\left(\frac{zt}{2}\right)
\]

\[
E(t, z) = \left(\frac{zt}{4}\right)^{1-\alpha} \Gamma\left(\frac{\alpha+1}{2}\right) \left(J_{\frac{\alpha-1}{2}}\left(\frac{zt}{2}\right) + it^\alpha J_{\frac{\alpha+1}{2}}\left(\frac{zt}{2}\right)\right).
\]

### 2.7 Infinite product representation

An entire function \( f \) is of **finite exponential type** if there exists \( c > 1 \) so that

\[
|f(z)| \leq c^{1+|z|}, \quad \text{for all } z \in \mathbb{C} \tag{66}
\]
If, in addition, the integral condition
\[
\int_{-\infty}^{\infty} \log \frac{|f(x)|}{1 + x^2} \, dx < \infty. \tag{67}
\]
holds, then we say that \( f \) is of **Cartwright class**. In this section we study \( \zeta_\tau \) when it falls in the Cartwright class. The role of the Cartwright class of functions in the world of Dirac operators is similar to the role of polynomials in the world of matrices.

When the secular function \( \zeta_\tau \) is of Cartwright class, it can be represented as a principal value product.

**Proposition 26** (Principle value product). *Under Assumptions 1-3, when the secular function \( \zeta = \zeta_\tau \) is of Cartwright class, we have*
\[
\zeta(z) = \lim_{r \to \infty} \prod_{|\lambda_k| < r} \left(1 - \frac{z}{\lambda_k}\right) \tag{68}
\]
*where the convergence is uniform on compact sets of \( z \in \mathbb{C} \). In particular for \( x \in \mathbb{R} \),
\[
|\zeta(x + iy)| \quad \text{is increasing in } y \geq 0. \tag{69}
\]

*Moreover, the following three quantities are finite and equal:
\[
\limsup_{|z| \to \infty} \frac{\log |\zeta(z)|}{|z|} = \limsup_{y \to \infty} \frac{\log |\zeta(iy)|}{y} = \lim_{|k| \to \infty} \frac{\pi k}{\lambda_k}. \tag{70}
\]

*We also have
\[
\lim_{r \to \infty} \sum_{|\lambda_k| < r} \lambda_k^{-1} = \frac{1}{2} \int_{0}^{\sigma} a(s)c(s)ds. \tag{71}
\]

The final statement of the proposition shows that even though the operator \( \tau \tau \) might not be trace class, the integral trace \( \int_{0}^{\sigma} \text{tr} K_{\tau \tau}(s, s) \, ds \) is equal to the principal value sum of \( \lambda_k^{-1} \).

**Proof.** According to Theorem 11 of Section V.4.4 in Levin [20] for an entire function \( \zeta \) of exponential type with (67)
\[
\zeta(z) = cz^m e^{ibz} \lim_{r \to \infty} \prod_{|\lambda_k| < r} \left(1 - \frac{z}{\lambda_k}\right), \tag{72}
\]
where \( b \) is real, \( m \) is a non-negative integer, and \( \lambda_k, k = 1, 2, \ldots \) are the nonzero zeros of \( \zeta \). We apply this to the secular function \( \zeta \). Since \( \zeta(0) = 1 \), we see that \( m = 0 \) and \( c = 1 \). Since \( \zeta \) maps reals to the reals and has real zeros it follows that \( b = 0 \). This completes the proof.
of (68) for pointwise convergence. Claim (69) holds factor-by-factor and is preserved in the limit.

Recall \( t = \frac{1}{2} \int_0^\sigma a(s)c(s)ds \), (24). By the definition (26) of \( \zeta \), we have

\[
e^{tz} \zeta(z) = \lim_{r \to \infty} e^{z \sum_{|\lambda_k| < r} 1/\lambda_k} \prod_{|\lambda_k| < r} (1 - z/\lambda_k),
\]

(73)

and the convergence is uniform on compacts in \( z \). Choosing a \( z \in \mathbb{C} \) with \( z \neq 0 \), \( \zeta(z) \neq 0 \) now implies (71). As a consequence,

\[
e^{-tz} = \lim_{r \to \infty} e^{-z \sum_{|\lambda_k| < r} 1/\lambda_k}
\]

uniformly on compacts. Multiplying this by (73) we get that the convergence in (68) is uniform on compacts.

To prove (70) we use further results from Levin’s books [20, 21]. The relevant theorems can be difficult to find because this is a simple case of a general theory.

Theorem 11 (2) of Section V.4.4, [20] shows that the limit on the right of (70) exists and equals \( d_h/2 \), where \( d_h \) is the length of the indicator diagram of \( \zeta \). The indicator diagram is a convex set given by a simple geometric transform of the exponential growth rate function (called indicator function)

\[
h(\theta) = \limsup_{r \to \infty} \log |\zeta(e^{i\theta}r)|/r,
\]

(74)

see equation (1.64) for the definition of \( h \) and Section I.19 for a description of the transform. Theorem V.7 identifies \( h(\theta) = k|\sin \theta| \) for some \( k \). By Section I.19 [20] we have \( d_h = 2k \). Taking \( \theta = \pi/2 \) in (74) gives the second equality of (70), and the \( \geq \) part of the first equality is clear.

Since \( h \) is continuous, Theorem 2 of Section 8 on page 56 of [21] shows that for every \( \varepsilon > 0 \) there is \( r_0 \) so that for all \( z = re^{i\theta} \) with \( r \geq r_0 \) we have \( |\zeta(z)| \leq e^{r(h(\theta)+\varepsilon)} \), which shows the \( \leq \) part of the first equality in (70).

The secular function \( \zeta_\tau \) can also be defined as follows. This is analogous to the definition of the characteristic polynomial through its zeros. The proof of the proposition follows the argument in pages 156-157 of Chapter III of Levin, [20].

**Proposition 27.** Suppose that \( \tau \) satisfies Assumptions 1-3, and for some 0 < \( \rho \) and \( \varepsilon < 1 \) the ordered sequence of eigenvalues \( \lambda_n \) satisfies

\[
|\lambda_n - \rho n|n^{-\varepsilon} \to 0 \quad \text{as} \quad |n| \to \infty.
\]

(75)

Assume further that \( \zeta_\tau \) satisfies the integral condition (67). Then \( \zeta_\tau \) is of exponential type, and hence it is of Cartwright class.
Proof. Let \( t_r = \sum_{|\lambda_k| \leq r} \frac{1}{\lambda_k} \), by assumption (75) the limit \( t = \lim_{r \to \infty} t_r \) exists. Then we can write

\[
\zeta_r(z) = e^{(t_r-t)z} \prod_{|\lambda_k| \leq r} (1 - \frac{z}{\lambda_k}) \prod_{|\lambda_k| > r} (1 - \frac{z}{\lambda_k}) e^{z/\lambda_k},
\]

where both products are well defined. There is an absolute constant \( c > 0 \) so that for any \( u \in \mathbb{C} \) we have the bounds

\[
\log|1-u| \leq \log(1+|u|), \quad \log|1-u|e^n| \leq c \frac{|u|^2}{1+|u|}.
\]

Hence

\[
\log|\zeta_r(z)| \leq |t_r-t||z| + \sum_{|\lambda_k| \leq r} \log(1 + \frac{|z|}{|\lambda_k|}) + \sum_{|\lambda_k| > r} c \frac{|z/\lambda_k|^2}{1 + |z/\lambda_k|}.
\]

Introduce \( n(t) = |\{k : |\lambda_k| \leq t\}| \), by our assumptions there are finite positive constants \( c, \delta \) so that \( n(t) \leq \rho t + c \) for \( t > 0 \), \( n(\delta) = 0 \). We set \( r = |z| \). Then

\[
\sum_{|\lambda_k| \leq r} \log(1 + \frac{|z|}{|\lambda_k|}) = \int_0^r \log(1 + \frac{r}{t})dn(t) = r \int_0^r \frac{n(t)}{t(t+r)} dt + n(r) \log(2)
\]

\[
\leq \int_\delta^r \frac{n(t)}{t} dt + n(t) \log(2).
\]

Similarly,

\[
\sum_{|\lambda_k| > r} \frac{|z/\lambda_k|^2}{1 + |z/\lambda_k|} = \int_r^\infty \frac{r^2}{t(r+t)} dn(t) = \int_r^\infty \frac{r^2(r+2t)}{t^2(r+t)^2} n(t) dt - \frac{1}{2} n(r) \leq 3r^2 \int_{r\delta}^\infty \frac{n(t)}{t^3} dt.
\]

From these bounds we get \( \log|\zeta(z)| \leq c_0 + c_1 |z| + c_2 \log(1+|z|) \), which implies the statement of the lemma.

The following folklore proposition implies that when \( \zeta_r \) is of Cartwright class, it is determined by its zeroes in this class.

\[\text{Proposition 28. Assume that } f, g \text{ are of Cartwright class, and have the same zeroes with multiplicities. Assume further that } f(0) = g(0) \neq 0, \text{ and } f, g \text{ map reals to reals. Then } f = g.\]

\[\text{Proof. The function } h = \log(f/g) \text{ is an entire function with at most linear growth, so by Liouville's theorem it is linear with } h(0) = 0. \text{ Since } h(\mathbb{R}) \subset \mathbb{R}, \text{ we have } h(z) = rz \text{ for real } r. \text{ Since } h(x)_+ \leq \log_+ |f| + \log_+ |g|, \text{ the function } h_+(x)/(x^2 + 1) \text{ is integrable by the condition } (67). \text{ Hence } r = 0.\]

\[\text{Proposition 29. Suppose that the assumptions of Proposition 27 hold. Let } N : \mathbb{R} \to \mathbb{R} \text{ be the counting function of the zeroes of } \zeta_r \text{ with } N(0) = 0. \text{ Then for } z \notin \mathbb{R} \text{ we have}
\]

\[
\frac{\zeta'_r(z)}{\zeta_r(z)} = \int_{-\infty}^\infty \frac{1}{(z-\lambda)^2} e^{(\lambda/\rho - N(\lambda))d\lambda + \text{sign}(\Im z)\pi i/\rho.}
\]
Proof. The partial products converge uniformly on compacts in the upper/lower half plane. Because of that, the log-derivatives of the partial products converge as well:

\[
\frac{\zeta'(z)}{\zeta(z)} = \lim_{r \to \infty} \sum_{|k| \leq r} \frac{1}{z - \lambda_k}
\]

(77)

For a fixed \( r \), integration by parts gives

\[
\sum_{|\lambda_k| \leq r} \frac{1}{z - \lambda_k} = - \int_{-r}^{r} \frac{1}{(z - \lambda)^2} N(\lambda) d\lambda + \frac{1}{z - r} N(r) - \frac{1}{z + r} N(-r).
\]

With a compensation term, the right hand side can be written as

\[
\int_{-r}^{r} \frac{1}{(z - \lambda)^2} \left( \frac{\lambda}{\rho} - N(\lambda) \right) d\lambda - \int_{-r}^{r} \frac{1}{(z - \lambda)^2} \cdot \frac{\lambda}{\rho} d\lambda
\]

\[
+ \left( \frac{1}{z - r} (N(r) - r/\rho) - \frac{1}{z + r} (N(-r) + r/\rho) \right) + \frac{r/\rho}{z - r} + \frac{r/\rho}{z + r}.
\]

The terms in the second line vanish as \( r \to \infty \) since \( |N(\lambda) - \lambda/\rho| = O(\lambda^{1-\varepsilon}) \). The first term converges to the first term of our formula, which is absolutely integrable. The claim about the second term follows by residue calculus with a contour integral over a radius \( r \) semicircle \( C_r \) in the closed upper half plane:

\[
\lim_{r \to \infty} \int_{-r}^{r} \frac{\lambda}{\rho(z - \lambda)^2} d\lambda = \lim_{r \to \infty} \int_{C_r} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{(z + \lambda)^2} \right) \frac{\lambda}{2 \rho} d\lambda = \frac{\text{sign}(3z) \pi i}{\rho}.
\]

Example 30 (Deterministic Sine and Bessel operators, part 4). Returning to \( \tau \) from Example 6 we see that \( \zeta(z) = \sin\left(\frac{\pi z + \theta}{2}\right)/\sin\left(\frac{\theta}{2}\right) \) is of Cartwright class with zeros \( \lambda_k = \frac{2\pi k - \theta}{\sigma} \). Proposition 26 leads to the following well-known identities:

\[
\frac{\sin\left(\frac{\pi z + \theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \lim_{r \to \infty} \prod_{|k| \leq r} \left( 1 - \frac{\sigma z}{2\pi k - \theta} \right), \quad -\cot\left(\frac{\theta}{2}\right) = \lim_{r \to \infty} \sum_{|k| \leq r} \frac{2}{2\pi k - \theta}.
\]

For \( \tau \) from Example 6 the well-known asymptotics of the Bessel function show that \( \zeta_\tau \) given in (44) is of Cartwright class. Since the zero set of \( \zeta_\tau \) is symmetric about 0, the identity (68) is equivalent to (43) (which follows from the definition), and (71) becomes trivial.

Proposition 31. Suppose that \( \tau \) satisfies Assumptions 1-3, and fix \( t \in (0, \sigma] \). If the function function \( B(z) = [0,1]H(t, z) \) is of Cartwright class, then it has the product representation

\[
B(z) = -z \int_{0}^{t} \frac{1}{2y_s} ds \lim_{r \to \infty} \prod_{0 < |\lambda_k| < r} \left( 1 - \frac{z}{\lambda_k} \right),
\]

(78)

where \( \lambda_k, k \in \mathbb{Z} \) are the ordered sequence of zeros of \( B(z) \) with \( \lambda_0 = 0 \).
Proof. By Proposition 19 we have $B(0) = 0$ and

$$B'(0) = [0, 1] \int_0^1 J^t R(s) u_0 ds = - \int_0^t \frac{1}{2y_s} ds.$$ 

Since $B$ is of Cartwright class by assumption, it has the product representation (72). By Lemma 22 $B(z)$ is real for $z \in \mathbb{R}$, and it has real zeros. Since $y_s > 0$ we have $B'(0) < 0$. From these it follows that $c = B'(0)$, $m = 1$, $b = 0$ in the product representation (72), which is the statement of the proposition.

3 Characteristic polynomials of unitary matrices

Section 5 of [31] provides a connection between discrete measures on the circle and Dirac operators. Here we explain how this connection ties orthogonal polynomials to the structure function, and the characteristic polynomial to the secular function.

Let $\mu$ be a probability measure whose support is exactly $n$ points $e^{i\lambda_j}, 1 \leq j \leq n$ on the unit circle centered at zero in $\mathbb{C}$, and assume $\mu(\{1\}) = 0$. The characteristic polynomial of $\mu$, normalized at 1, is be defined as

$$p(z) = p_\mu(z) = \prod_{j=1}^{n} \frac{z - e^{i\lambda_j}}{1 - e^{i\lambda_j}}. \quad (79)$$

For $k \leq n$, the $k$th orthogonal polynomial $\psi_k(z)$ normalized at 1, is defined as the unique polynomial with $\psi_k(1) = 1$ of degree $k$ that is orthogonal to $1, \ldots, z^{k-1}$ in $L^2(\mu)$. With this definition, $p = \psi_n$. Let $[\psi_k]$ be the main coefficient of $\psi_k$, and for $0 \leq k \leq n-1$ let $\gamma_k \in \mathbb{C}$, $w_k, v_k \in \mathbb{R}$ be so that

$$\frac{2\gamma_k}{1 - \gamma_k} = w_k - i v_k = - \frac{2\psi_{k+1}(0)}{[\psi_k]}. \quad (80)$$

The $\gamma_k$ are called the modified, or deformed, Verblunsky coefficients of the measure $\mu$ introduced by Bourgade, Najnudel and Rouault [5]. They satisfy $|\gamma_k| < 1$, for $0 \leq k \leq n-2$, and $|\gamma_{n-1}| = 1$.

Let $x_0 = 0$, $y_0 = 1$, and define recursively

$$x_{k+1} = x_k + v_k y_k, \quad y_{k+1} = y_k (1 + w_k). \quad (80)$$

Note that $y_k > 0$ for $1 \leq k \leq n-1$ and $y_n = 0$.

The next proposition shows how the orthogonal polynomials $\psi_k$ can be expressed using a Dirac operator built from the path $x_{\lfloor nt \rfloor} + iy_{\lfloor nt \rfloor}$, $t \in [0, 1]$. 

27
Proposition 32. Set $x(t) + iy(t) = x_{[nt]} + iy_{[nt]}$ for $t \in [0, 1]$. Let

$$
\tau = R^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dt}, \quad R = \frac{X^tX}{2 \det X}, \quad X = \begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix},
$$

(81)

with boundary conditions $u_0 = [1, 0]^t, u_1 = [-x(1), -1]^t$. For $(t, z) \in [0, 1] \times \mathbb{C}$ let $H(t, z) \in \mathbb{C}^2$ be the unique solution of

$$
\tau H = zH, \quad H(0, z) = [1, 0]^t.
$$

(82)

(i) The orthogonal polynomials $\psi_k, 0 \leq k \leq n$ satisfy

$$
\psi_k(e^{iz/n}) = e^{izk/(2n)} \left[ 1, -(x_k + iy_k) \right] H(k/n, z).
$$

(83)

In particular the normalized characteristic polynomial satisfies

$$
p(e^{iz/n}) = \psi_n(e^{iz/n}) = e^{iz/2} \left[ 1, -x_n \right] H(1, z)
$$

(ii) The secular function of $\tau$ is $\zeta_{\tau}(z) = p(e^{iz/n})e^{-iz/2}$.

(iii) The spectrum of $\tau$ is given by the set $\{n\lambda_k + 2\pi nj : 1 \leq k \leq n, j \in \mathbb{Z}\}$ see also Proposition 17 from [31].

(iv) We have

$$
\zeta_{\tau}(z) = \prod_{j=1}^{n} \frac{\sin(\lambda_j/2 - z/(2n))}{\sin(\lambda_j/2)}.
$$

Note that $\|R^{-1}\|$ is bounded on $[0, 1]$, hence the solution of (82) is indeed unique.

Proof. Let $\psi_k^*(u) = u^k \psi_k(1/\bar{u})$ be the reversed polynomials. These polynomials satisfy the modified Szegő recursion, see [31] and also [5]:

$$
\begin{pmatrix} \psi_{k+1} \\ \psi^*_{k+1} \end{pmatrix} = A_k \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_k \\ \psi^*_k \end{pmatrix}, \quad \begin{pmatrix} \psi_0 \\ \psi^*_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad 0 \leq k \leq n - 1,
$$

(84)

with

$$
A_k = \begin{pmatrix} \frac{1}{1-\gamma_k} & -\frac{\gamma_k}{1-\gamma_k} \\ -\frac{1-\gamma_k}{\gamma_k} & \frac{1}{1-\gamma_k} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} w_k - iv_k & -w_k + iv_k \\ -w_k - iv_k & w_k + iv_k \end{pmatrix}.
$$

With $U = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and the notation $Y^X = X^{-1}YX$ we have

$$
A_k^U = \begin{pmatrix} 1 & -w_k \\ 0 & 1 + w_k \end{pmatrix}, \quad 0 \leq k \leq n - 1,
$$

(85)
and we set
\[ X_0 := I, \quad X_k := A_{k-1}^r \cdots A_0^r = \begin{pmatrix} 1 & -x_k \\ 0 & y_k \end{pmatrix}, \quad 1 \leq k \leq n. \] (86)

With \( u = e^{iz/n} \) let
\[ H_k(z) = e^{-izkX_k^{-1}U^{-1}} \begin{pmatrix} \psi_k(u) \\ \psi_k^*(u) \end{pmatrix}, \quad 0 \leq k \leq n-1. \] (87)

The functions \( H_k \) satisfy the recursion
\[ H_{k+1} = X_k^{-1} \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-izk} \end{pmatrix}^U X_k H_k, \quad H_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad 0 \leq k \leq n-1. \] (88)

Define \( H(t, z) \) for \( 0 \leq t \leq 1 \) as follows:
\[ H\left(\frac{k}{n}, z\right) = H_k(z), \quad k = 0, 1, \ldots, n-1, \]
\[ H(t, z) = X_k^{-1} \begin{pmatrix} e^{iz(t-k/n)} & 0 \\ 0 & e^{-iz(t-k/n)} \end{pmatrix}^U X_k H_k, \quad t \in \left[\frac{k}{n}, \frac{k+1}{n}\right]. \]

The function \( H(t, z) \) is continuous in \( t \) and analytic in \( z \). For \( t \) in \( t \in \left[\frac{k}{n}, \frac{k+1}{n}\right] \) differentiating in \( t \) we get
\[ H' = \frac{z}{2} X_k^{-1} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^U X_k H = \frac{z}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X_k H = z \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} R_t H, \] (89)

where we used (81) and (10). This means that \( H \) is the unique solution of (82), and (87) implies (i) for \( 0 \leq k \leq n-1 \). From (84)-(89) and the fact that \( y_n = 0 \) it follows that
\[ \begin{pmatrix} \psi_n \\ \psi_n^* \end{pmatrix} = e^{iz} U X_n H(1, z) = e^{iz} \begin{pmatrix} 1 & -x_n \\ 1 & -x_n \end{pmatrix} H(1, z). \]

This implies (i) for \( k = n \). The operator \( \tau \) with \( u_0 = [1, 0]^t, u_1 = [-x_n, -1]^t \) satisfies Assumptions 1 and 2, hence by Proposition 13 we have (ii). Since the eigenvalues of \( \tau \) are the zeros of \( \zeta_\tau \), we get (iii). Finally, (iv) follows from the identity
\[ \frac{e^{iz/n} - e^{i\lambda}}{1 - e^{i\lambda}} = e^{iz/2n} \frac{\sin(\lambda/2 - z/(2n))}{\sin(\lambda/2)}. \]

4 The stochastic zeta function

The Sine\( \beta \) operator was introduced in [31]. It is a Dirac operator on \([0, 1] \to \mathbb{R}^2 \) functions built from standard hyperbolic Brownian motion. In [31] it was shown that its spectrum
is given by the bulk scaling limit of the Gaussian and circular beta ensembles. In [32] it was shown that the operator can be derived as the limit Dirac operators constructed from finite circular beta ensembles.

Recall the definition of the Sine $\beta$ operator from Theorem 25 of [31].

**Definition 33.** Let $\Xi_u, u \geq 0$ be standard hyperbolic Brownian motion in $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ with $\Xi_0 = i$. Let $\Xi_\infty = \lim_{u \to \infty} \Xi_u$ be the a.s. limit of $\Xi$. Then Sine $\beta$ is the Dirac operator built from the path $\Xi_{-\frac{4}{\beta} \log(1-t)}, t \in [0,1)$ with boundary conditions $u_0 = [1,0]^t$, $u_1 = [-\Xi_\infty, -1]$.

We will study a conjugate of this operator that fits into our framework.

### 4.1 $\tau_\beta$ and the stochastic zeta function

Let $b_1, b_2$ be independent copies of two-sided Brownian motion on $\mathbb{R}$, and set

$$y_u = e^{b_2(u) - u/2}, \quad x_u = \begin{cases} -\int_u^0 e^{b_2(s) - \frac{s}{2}} db_1, & u \leq 0, \\ \int_0^u e^{b_2(s) - \frac{s}{2}} db_1 & u \geq 0. \end{cases}$$

Note that $x_u + iy_u, u \in \mathbb{R}$ is an almost surely continuous process in $\mathbb{H}$.

**Remark 34.** The process

$$X_u = \begin{pmatrix} 1 & -x_u \\ 0 & y_u \end{pmatrix}, \quad u \in \mathbb{R},$$

is two-sided Brownian motion in the affine group of matrices of the form

$$\begin{pmatrix} 1 & -x \\ 0 & y \end{pmatrix}, \quad x, y \in \mathbb{R}, y > 0.$$  

The increments $X_u X_s^{-1}$ are stationary and independent over disjoint time intervals $(s, u)$. In particular, $y$ is two-sided geometric Brownian motion. Moreover, $x_u + iy_u, u \geq 0$ is standard hyperbolic Brownian motion in $\mathbb{H}$ started from $i$, while $x_{-u} + iy_{-u}, u \geq 0$ is the same process conditioned to converge to $\infty$.

Consider the time-change

$$u(t) = \frac{4}{\beta} \log t.$$  

Let $q$ be a standard Cauchy distributed random variable independent of $b_1, b_2$. We consider the random Dirac operator built from $X_{u(t)}, t \in (0,1]$ with $q$-boundary conditions (15) and its secular function.
Definition 35. Let \( u_0 = [1, 0]^t, u_1 = [-q, -1]^t \), and set

\[
\tau_\beta = \text{Dir}(X_{u(\cdot)}, u_0, u_1), \quad \zeta_\beta = \zeta_{\tau_\beta}.
\]

We call \( \zeta_\beta \) the stochastic zeta function.

To see that \( \zeta_\beta \) is well defined, we need to check Assumptions 1 and 2, this will be done in Section 4.3 below. To motivate the definition we first show the \( \tau_\beta \) is orthogonal equivalent to the Sine\( _\beta \) operator.

4.2 The Sine\( _\beta \) operator is orthogonal equivalent to \( \tau_\beta \)

Our goal is to show that Sine\( _\beta \) and \( \tau_\beta \) are orthogonal equivalent operators. To do this we first review how a Dirac operator behaves under simple transformations of its parameters. Then we discuss the relationship between affine and hyperbolic Brownian motion and their time reversal.

We consider three transformations. The time reversal transformation \( \rho \) maps a function from \((0, 1)\) to any space to a function from \([0, 1)\) by reversing its time \( \rho f(t) = f(1-t) \).

The transformation from the affine group (92) to itself \( \iota: X \mapsto SXS, \quad S = S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) (94)

simply reverses the sign of the \((1, 2)\) entry of \( X \). This is just the reflection \( z \to -\overline{z} \) for the corresponding element of \( \mathbb{H} \). It is an automorphism of the group. Finally, given a \( 2 \times 2 \) orthogonal matrix \( Q \) of determinant 1, the corresponding linear fractional transformation \( Q \) maps \( z \in \mathbb{H} \) to the ratio of entries of \( Q[z, 1]^t \). If we identify the matrix (92) with the complex number \( x + iy \), then \( Q \) acts on the affine group. Further, it acts on paths in the affine group pointwise.

The proof of the following lemma is just straightforward arrow chasing and simple calculation, so we omit it.

Lemma 36. Given \( X \) and boundary conditions \( u_0, u_1 \in \mathbb{R} \cup \{\infty\} \), we have the following identities.

\[
\rho^{-1} \text{Dir}(X, u_0, u_1) \rho = -\text{Dir}(\rho X, u_1, u_0),
\]

\[
S \text{Dir}(X, u_0, u_1) S = -\text{Dir}(\iota X, -u_0, -u_1),
\]

\[
Q \text{Dir}(X, u_0, u_1) Q^{-1} = \text{Dir}(QX, Qu_0, Qu_1).
\]

In particular, the operator \( \text{Dir}(X, u_0, u_1) \) is orthogonally equivalent to the operators \( \text{Dir}(QX, Qu_0, Qu_1) \) and \( \text{Dir}(\rho \iota X, -u_1, -u_0) \) in the respective \( L^2 \) spaces, and they have the same integral traces (24) as well.
Lemma 37. Let $X$ be two-sided affine Brownian motion defined via (90)-(91), and let $q$ an independent standard Cauchy random variable. The orthogonal matrix

$$Q = \frac{1}{\sqrt{q^2 + 1}} \begin{pmatrix} q & 1 \\ -1 & q \end{pmatrix}.$$ 

corresponds to a fractional linear transformation $Q$ that maps $q$ to $\infty$. Then

$$u \mapsto QX_u, \quad u \geq 0$$

is hyperbolic Brownian motion started from $i$, and $iQX_u$ has the same law. Moreover, as $u \to \infty$ the hyperbolic Brownian motions $(QX)_u$, $(iQX)_u$ converge to the boundary points $Q(\infty)$, and $-Q(\infty)$, respectively.

Proof. The transformation $Q$ is a hyperbolic rotation of $\mathbb{H}$ about the point $i$, mapping $q$ to $\infty$ when extended to $\bar{\mathbb{H}}$.

The first claim follows from the well-known disintegration theorem about hyperbolic Brownian motion into two independent pieces, Proposition X.3.1 in [12]. The first is the boundary point that it converges to. This has Cauchy distribution. The second is the process rotated to converge to $\infty$, which has the same distribution as hyperbolic Brownian motion conditioned to converge to $\infty$.

The second claim follows from the fact that the law of hyperbolic Brownian motion is invariant under the reflection $z \mapsto -\bar{z}$, $X_{-u}, u \geq 0$ is hyperbolic Brownian motion conditioned to converge to $\infty$, and the limit points follow the transformations, demonstrating the last two claims. \hfill \Box

Proposition 38. Let $\tau_\beta$ be as in Definition 35 and let $Q$ and the corresponding fractional linear transformation $Q$ be defined as in Lemma 37. Then the operator $\rho^{-1}(S\tau_\beta)\tau_\beta(S\rho)^{-1}$ is orthogonal equivalent to $\tau_\beta$ and it has the same distribution as the Sine$_\beta$ operator defined in Definition 33. In particular, its eigenvalues agree with those of $\tau_\beta$ and have the law of the Sine$_\beta$ process.

Proof. By Lemma 36 the operator $\rho^{-1}(S\tau_\beta)(S\rho)^{-1}$ is given by

$$\operatorname{Dir}(\rho t Q(X_{u(t)}), \infty, -Q(\infty)).$$

(95)

By Lemma 37 $(iQX)_u, u \geq 0$ is standard hyperbolic Brownian motion. Hence

$$\rho t Q(X_{u(t)}) \overset{d}{=} \rho \Xi_{-u(t)} = \Xi_{-\frac{4}{\beta} \log(1-\cdot)},$$

where $\Xi$ is standard hyperbolic Brownian motion. \hfill \Box
4.3 The stochastic zeta function and its approximations

The operator $\tau_\beta$ has approximate versions that will be useful in the sequel. They are defined in terms of the increment

$$X_u^\nu = X_u X_\nu^{-1}, \quad u \geq \nu$$

(96)

of the process $X$ on the interval $[\nu, 0]$ with $\nu < 0$. We have

$$X_u^\nu = \begin{pmatrix} 1 - x_u^\nu \\ y_u^\nu \end{pmatrix}, \quad x_u^\nu = \frac{x_u - x_\nu}{y_\nu}, \quad y_u^\nu = \frac{y_u}{y_\nu}.$$  

(97)

We set

$$\tau_{\beta,\nu} = \text{Dir}(X_u^\nu, \infty, q), \quad \zeta_{\beta,\nu} = \zeta_{\tau_{\beta,\nu}}.$$  

Note that the operator $\tau_{\beta,\nu}$ acts on $\mathbb{R}^2$-valued functions on the interval $[t(\nu), 1]$, where

$$t(u) = e^{4u/\beta}$$

(98)

is the inverse of the time-change function $u$ (93).

By convention, the $\nu = -\infty$ case will refer to the undecorated $\tau_\beta$. The finite-$\nu$ operators and the corresponding stochastic zeta functions are better behaved than $\tau_\beta$ and $\zeta_\beta$. See, for example Section 8.

**Proposition 39.** Almost surely the operator $\tau_{\beta,\nu}$ satisfies Assumptions 1-3 for all $\nu \in [-\infty, 0)$. In particular, $\zeta_{\beta,\nu}$, $\nu \in [-\infty, 0)$ are well-defined entire functions with probability one.

**Proof.** Since $x_u + iy_u, u \in \mathbb{R}$ is a continuous process in $\mathbb{H}$ with probability one, this is also true for $x_\nu^\nu + iy_\nu^\nu$ for all $\nu \in (-\infty, 0)$. The continuity of $x + iy$ and $x_\nu^\nu + iy_\nu^\nu$ implies Assumption 1. Assumption 3 is satisfied as we are using $q$-boundary conditions (15).

For $\nu \in (-\infty, 0)$ Assumption 2 follows from the fact that $x_\nu^\nu + iy_\nu^\nu$ is continuous on the compact interval $[e^{4\nu/\beta}, 1]$ with probability one. Hence we only need to verify the assumption for $\nu = -\infty$. For this we need to check (19) and (20) on the interval $(0, 1]$ using the process $x_u(\cdot) + iy_u(\cdot)$. We will show that for a given $\varepsilon > 0$ there is a random constant $C > 0$ so that for all $u \in (-\infty, 0]$ we have

$$C^{-1}e^{-\varepsilon|u|} \leq y_u e^{u/2} \leq C e^{\varepsilon|u|}, \quad |x_u e^{-u/2}| \leq C e^{\varepsilon|u|}.$$  

(99)

From these the integral bounds of (19) and (20) follow almost surely for $x_u(\cdot) + iy_u(\cdot)$.

The first bound in (99) follows from the definition (90) and the law of iterated logarithm for Brownian motion, see Theorem 72 in the appendix. From (90) it follows that
$x-u, u \geq 0$ has the same distribution as a Brownian motion run with the time $u \rightarrow \int_{-u}^{0} y_s^2 ds$. Using the first bound of (99) with the law of iterated logarithm again we obtain the bound on $x$ in (99).

**Remark 40.** We defined $\tau_\beta$ from $X_{u(\cdot), t \in (0, 1]}$. The same argument as in the proof of Proposition 39 shows that for any fixed $\sigma > 1$ the operator defined from $X_{u(\cdot), t \in (0, \sigma]}$ also satisfies Assumptions 1-3, and the same holds for the operators defined from $X_{u(\cdot), t \in (t(\nu), \sigma]}$.

### 4.4 Convergence of random characteristic polynomials

The size $n$ circular beta ensemble is a random vector of unit length complex numbers with joint density proportional to

$$\prod_{1 \leq j < k \leq n} |z_j - z_k|^\beta.$$  

on $\{|z| = 1\}^n$. For $\beta = 2$ this has the same distribution as the eigenvalues of a Haar distributed unitary $n \times n$ matrix. For general $\beta$, there are explicit five-diagonal unitary matrices $U_\beta$ of CMV type [6] with this eigenvalue distribution, see Killip and Nenciu [16]. We denote by

$$p_n(z) = p_{\beta,n}(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - z_j}$$

(100)

the normalized characteristic polynomial of the size $n$ circular beta ensemble, see (79).

The next theorem shows that the stochastic zeta function $\zeta_\beta$ is the limit of the characteristic polynomials of the circular beta ensembles. This restates Theorem 3.

**Theorem 41.** Fix $\beta > 0$. There exists a coupling of the random polynomials $p_{\beta,n}$ and the stochastic zeta function $\zeta_\beta$ together with a random variable $C$ so that for all $z \in \mathbb{C}$ and all $n > 1$

$$|p_n(e^{iz/n})e^{-iz/2} - \zeta_\beta(z)| \leq \left( e^{\frac{1}{|z|\log^2 n \sqrt{n}}} - 1 \right) C^{|z|^2+1}.$$  

(101)

**Proof.** Recall the definition of the modified Verblunsky coefficients of a discrete measure supported on $n$ points on the unit circle from Section 3.

Consider the following random discrete measure $\sigma_{n,\beta}$: the support of the measure is given by a size $n$ circular beta ensemble $z_1, \ldots, z_n$, and the vector of weights $(\sigma_{n,\beta}(z_1), \ldots, \sigma_{n,\beta}(z_n))$ is independent and has Dirichlet distribution with parameter $(\beta/2, \ldots, \beta/2)$.

Bourgade, Najnudel and Rouault [5] – building on the work of Killip and Nenciu in [16] – showed that the modified Verblunsky coefficients corresponding to $\sigma_{n,\beta}$ are independent of each other, and identified their marginal distributions.
The construction given in Proposition 32 defines a Dirac operator \( \text{Circ}_{\beta,n} \) corresponding to \( \sigma_{n,\beta} \) acting on functions \([0,1] \to \mathbb{R}^2\), with eigenvalues given by \( n \Lambda_{n,\beta} + 2\pi n \mathbb{Z} \) where \( e^{i\Lambda_{n,\beta}} \) is a size \( n \) circular beta ensemble. By Proposition 32 the secular function of the \( \text{Circ}_{\beta,n} \) operator is given by \( \zeta_n(z) = p_n(e^{iz/n})e^{-iz/2} \).

In [32] it was shown that there is a coupling of the \( \text{Circ}_{\beta,n} \) operators for \( n \geq 1 \) and the \( \text{Sine}_{\beta} \) operator so that

\[
\| r \text{Sine}_{\beta} - r \text{Circ}_{\beta,n} \|_{\text{HS}}^2 \leq \frac{\log^6 n}{n} \]  

holds for all \( n \geq N \) with a random variable \( N \). In this coupling the operators all share the same starting and end conditions \( u_0 = [1,0]^t \) and \( u_1 = [-q,-1]^t \), where \( q \) is Cauchy distributed.

We will use Proposition 21 to estimate \(|\zeta_n(z) - \zeta_{\beta}(z)|\). To apply the proposition to operators satisfying Assumptions 1-3 we first need to conjugate \( \text{Circ}_{\beta,n}, \text{Sine}_{\beta} \) with the transformations appearing in Proposition 38. However, Lemma 36 shows that this conjugation does not change the Hilbert-Schmidt norm of the resolvent and the integral trace, hence we can estimate the appropriate quantities directly for \( \text{Circ}_{\beta,n}, \text{Sine}_{\beta} \).

By (102) and since \( \| r \text{Sine}_{\beta} \| < \infty \), there is a random \( C_0 \) with

\[
\| r \text{Sine}_{\beta} - r \text{Circ}_{\beta,n} \|_{\text{HS}} \leq C_0 \frac{\log^3 n}{\sqrt{n}}, \quad \| r \text{Sine}_{\beta} \|, \| r \text{Circ}_{\beta,n} \|_{\text{HS}} \leq C_0
\]

for all \( n > 1 \). We need similar bounds for the integral traces of \( r \text{Sine}_{\beta} \) and \( r \text{Circ}_{\beta,n} \), denoted by \( t_\beta \) and \( t_n \), respectively. The tools developed in [32] to prove (102) also imply

\[
|t_\beta - t_n| \leq \frac{\log^3 n}{\sqrt{n}}
\]

for \( n \geq N_1 \), see Proposition 42 below. Since \( t_\beta \) is a.s. finite, this implies the existence of a random \( C_1 \) with

\[
|t_\beta - t_n| \leq C_1 \frac{\log^3 n}{\sqrt{n}}, \quad |t_\beta|, |t_n| \leq C_1
\]

for all \( n > 1 \). The bound (101) now follows from (103), (105) and Proposition 21.

**Proposition 42.** In the coupling of [32] the bound (104) holds for \( n \geq N_1 \) random.

**Proof.** Let \( B_n(t) \in \mathbb{H}, t \in [0,1] \) denote the path corresponding to \( \text{Circ}_{\beta,n} \) built from the random modified Verblunsky coefficients according to (80). Let \( B(t) \in \mathbb{H}, t \in [0,1] \) denote the time-changed hyperbolic Brownian motion in the construction of \( \text{Sine}_{\beta} \), see Definition 33. In [32] the coupling of the \( \text{Circ}_{\beta,n}, \text{Sine}_{\beta} \) operators is constructed in a way that
the processes $B_n, B$ are close to each other in the hyperbolic metric. More precisely, Proposition 13 of [32] states that there is a random $N_0$ so that for all $n \geq N_0$ we have the uniform bounds

\[ d_H(B_n(t), B(t)) \leq \frac{\log^{3-1/8} n}{\sqrt{(1-t)n}}, \quad 0 \leq t \leq T_n = 1 - \frac{1}{n} \log^6 n, \quad (106) \]

\[ d_H(B_n(T_n), B(t)) \leq \frac{144}{\beta} (\log \log n)^2, \quad T_n \leq t < 1. \quad (107) \]

Here $d_H$ denotes the hyperbolic distance.

Proposition 15 of [32] gives a way to estimate the effect of a truncation on the Hilbert-Schmidt norm of an integral operator constructed from a path that converges to a boundary point of the hyperbolic plane. Lemma 21 of [32] shows that $B$ satisfies the conditions of Proposition 15 of [32].

Proposition 16 of [32] gives an estimate on the Hilbert-Schmidt norm of the difference of two integral operators where the paths from which they are constructed are close enough. The bound (102) is proved by putting together the results of Propositions 13, 14, 15 and 16 of [32]. To get the estimate (104) we can use some of the intermediate steps in these propositions. We need to estimate

\[ \left| \int_0^1 (a_n(s)^t c_n(s) - a(s)^t c(s)) ds \right|, \]

where $a_n, c_n, a, c$ are the vector-valued functions (17) defined for $\text{Circ}_{\beta,n}$ and $\text{Sine}_{\beta}$. By the triangle inequality it is enough to bound the following three integrals:

\[ \int_0^{T_n} |a_n(s)^t c_n(s) - a(s)^t c(s)| ds, \quad \int_0^1 |a_n(s)^t c_n(s)| ds, \quad \int_0^1 |a(s)^t c(s)| ds. \quad (108) \]

The proofs of Propositions 14, 15, and Lemma 21 of [32] imply the following bounds on $a, c$:

\[ |a(s)| \leq C(1 - s)^{\frac{1}{\beta} - \varepsilon}, \quad |c(s)| \leq C(1 - s)^{-\frac{1}{\beta} - \varepsilon}, \quad s \in [0, 1), \quad (109) \]

with $\varepsilon > 0$ arbitrarily small, and $C = C_\varepsilon$ a finite random variable.

The arguments in the proof of Proposition 16 of [32] imply the bounds

\[ \frac{|a(s) - a_n(s)|}{|a(s)|}, \frac{|c(s) - c_n(s)|}{|c(s)|} \leq 2 \sinh \left( \frac{1}{2} d_H(B_n(s), B(s)) \right), \quad 0 \leq s < 1, \]

\[ \frac{|a_n(s) - a_n(t)|}{|a_n(s)|}, \frac{|c_n(s) - c_n(t)|}{|c_n(s)|} \leq 2 \sinh \left( \frac{1}{2} d_H(B_n(s), B_n(t)) \right), \quad 0 \leq s < t < 1. \]

36
The coupling bounds (106) and (107) give

\[
\frac{|a(s) - a_n(s)|}{|a(s)|}, \frac{|c(s) - c_n(s)|}{|c(s)|} \leq c \frac{\log^{3-1/8} n}{\sqrt{(1 - s)n}}, \quad 0 \leq s \leq T_n,
\]

\[
\frac{|a_n(T_n) - a_n(t)|}{|a_n(T_n)|}, \frac{|c_n(T_n) - c_n(t)|}{|c_n(T_n)|} \leq e^{c(\log \log n)^2}, \quad T_n < t < 1,
\]

for all \( n \geq N_0 \). Together with (109) these bounds are sufficient to estimate all three terms in (108) with a repeated use of the triangle inequality. Taking \( \varepsilon \) small enough in (109) leads to the bound (104).

5 SDE characterization of \( \zeta_\beta \)

Proposition 13 gives an ordinary differential equation description of \( \zeta_\beta \). In this section we obtain a description using stochastic differential equations.

5.1 Stochastic differential equation description of \( \zeta_\beta \)

Let \( X \) be defined as in (91), and \( u = u(t) \) as in (93). We set

\[
R_t = -\frac{1}{2} JX_u^{-1} JX_u,
\]

according to (9) and (10). Proposition 13 states that for every \( z \in \mathbb{C} \) there is a unique vector-valued solution \( H : (0, 1] \times \mathbb{C} \rightarrow \mathbb{C}^2 \) of the ordinary differential equation

\[
R^{-1} J \frac{d}{dt} H = z H, \quad t \in (0, 1], \quad \lim_{t \to 0} H(t, z) = [1, 0]^t
\]

and

\[
\zeta_\beta(z) = -H(1, z)^t J[q, 1] = [1, -q] H(1, z).
\]

We consider two approximations of \( H \). The first one, \( H_\varepsilon \), for \( 0 < \varepsilon < 1 \) is the approximation introduced in Proposition 20. This is the unique solution of the differential equation (111) on \( [\varepsilon, 1] \) with initial condition \( H_\varepsilon(\varepsilon, z) = [1, 0]^t \).

The second approximation is constructed from the process \( X^\nu \) for \( \nu < 0 \), introduced in (96). Recall the definition of \( t(\cdot) \) from (98), and define

\[
R_t^{(\nu)} = -\frac{1}{2} J(X_u^{(\nu)})^{-1} JX_u^{(\nu)}.
\]
Now define the functions $H^{t(\nu)}$ as the unique solutions of the differential equation
\[
(R^{t(\nu)})^{-1} \frac{d}{dt} H = z H, \quad t \in [t(\nu), 1], \quad H^{t(\nu)}(t(\nu), z) = [1, 0]^t. \tag{114}
\]
Note that we have $\zeta_{\beta, \nu} = [1, -q] H^{t(\nu)}(1, z)$.

The two approximations are connected via the identity
\[
H^{t(\nu)}(t, z) = X_{\nu} H(\varepsilon(t, z)), \quad t \in [t(\nu), 1], \varepsilon = t(\nu). \tag{115}
\]
Define
\[
\mathcal{H}_u(z) = X_{\nu} H(t(u), z), \quad \mathcal{H}'_{u}(z) = X_{\nu} H^{t(\nu)}(t(u), z). \tag{116}
\]
The process $\mathcal{H}$ also describes $\zeta_{\beta}$, since $\mathcal{H}_0 = H(1, \cdot)$, and so
\[
\zeta_{\beta} = [1, -q] \mathcal{H}_0. \tag{117}
\]

**Proposition 43.** Consider the independent copies of two-sided Brownian motion $b_1, b_2$ from (90), and let $\mathcal{F}_u$ be the $\sigma$-field generated by the increments $b_k(u) - b_k(s), s < u, k = 1, 2$.

The processes $X^{\nu}$, $\mathcal{H}^{\nu}$ and $\mathcal{H}$ are all adapted to the filtration $\mathcal{F}_u, u \in \mathbb{R}$, and they satisfy the stochastic differential equations
\[
dX^{\nu} = \begin{pmatrix} 0 & -db_1 \\ 0 & db_2 \end{pmatrix} X^{\nu}, \quad u > \nu, \tag{118}
\]
\[
d\mathcal{H} = \begin{pmatrix} 0 & -db_1 \\ 0 & db_2 \end{pmatrix} \mathcal{H} - z_{\beta}^8 e^{\beta u/4} J\mathcal{H} du, \quad u \in \mathbb{R}. \tag{119}
\]
The processes $\mathcal{H}'^{\nu}$ satisfy the equation (119) on $[\nu, \infty)$.

The processes $\mathcal{H}, \mathcal{H}'^{\nu}$ can also be determined as follows. $\mathcal{H}'^{\nu}$ is the unique strong solution of the SDE family (119) on $[\nu, \infty)$ with boundary condition $\mathcal{H}'^{\nu} = [1, 0]^t$. Moreover, a.s. as $\nu \to -\infty$ we have $\mathcal{H}'^{\nu} \to \mathcal{H}$ uniformly on compacts of $\mathbb{R} \times \mathbb{C}$.

**Proof.** $X^{\nu}_u$ is $\mathcal{F}_u$-measurable by the definitions (91) and (96). Itô’s formula shows that it satisfies (118).

Note that for $u < 0$, $X_u$ is not $\mathcal{F}_u$-measurable, in fact it is independent of $\mathcal{F}_u$ as it is built from $b_1(s), b_2(s), s \in [u, 0]$. This issue already arises with two-sided Brownian motion: $b_1(u), b_2(u)$ are not $\mathcal{F}_u$-measurable either for $u < 0$. It is not a priori clear that $\mathcal{H}$ is $\mathcal{F}$-adapted, because it is defined in terms of $X$. However, $\mathcal{H}'^{\nu}$ in (116) is defined in terms of $X^{\nu}$ (see also (114)), so it is $\mathcal{F}$-adapted. The definition and Itô’s formula implies that it satisfies the SDE (119) with the stated boundary conditions.
By (115) we have
\[ H_\nu(z) = X_\nu H^{\nu}(t(u), z) = X_u H_\varepsilon(t(u), z) \]
where \( H_\varepsilon \) solves (111) on \([t(\nu), 1]\) with \( H_\varepsilon(t(\nu), z) = [1, 0]^t \). Proposition 20 now implies that as \( \nu \to -\infty \), we have \( H_\nu \to H \) uniformly on compact subsets of \( \mathbb{R} \times \mathbb{C} \). In particular, this shows that \( H \) is \( F \)-adapted. Itô’s formula implies that \( H \) satisfies (119).

Note that \( H \) and \( H^{\nu} \) are defined on \((0, 1]\) and \([t(\nu), 1]\), respectively, so \( H, H_\nu \) are only defined on \((\nu, 0]\) and \((\nu, \infty)\) a priori. However, by Remark 40 we can extend the definitions for \((\nu, u(\sigma)], \nu, u(\sigma)]\) for any \( \sigma > 1 \), which allows us to extend the definitions to \( \mathbb{R} \) and \((\nu, \infty)\), respectively.

Equation (119) and Itô’s formula implies the following.

**Proposition 44.** Consider \( H \) defined in (116). The random function
\[ (u, z) \mapsto H_u(z e^{-\beta u/4}) \]
is stationary under \( u \)-shifts. In particular, for each \( u \in \mathbb{R} \) the random analytic function \( z \mapsto H_u(z e^{-\beta u/4}) \) has the same distribution as \( z \mapsto H_0(z) \).

### 5.2 The Taylor expansion of \( H \) and \( \zeta_\beta \)

Proposition 43 gives a characterisation of \( H \) as the uniform on compacts limit of strong solutions (119) on \([\nu, \infty)\) with \( \nu \to -\infty \) with an initial condition \([1, 0]^t\). In Corollary 48 below we show that \( H \) is the unique strong solution of (119) under some additional conditions. The main ingredient for this characterisation is the analysis of the Taylor expansion of \( H \) in the variable \( z \).

Let
\[ H = [A, B]^t, \]
and recall from (117) that \( \zeta_\beta = [1, -q]H = A - qB \), where \( H, A, B \) are evaluated at time \( u = 0 \). Let \( A_{n,s}, B_{n,s} \) be the Taylor coefficients of \( A, B \) at \( z = 0 \) and time \( s \). Then
\[ \zeta_\beta(z) = \sum_{n=0}^{\infty} (A_{n,0} - qB_{n,0}) z^n. \]

Surprisingly, any initial sequence of \( B_1, A_1, B_2, \ldots \) has a closed SDE description! Moreover, the SDEs can be explicitly solved.

To describe the solution we first introduce a sequence of processes in Proposition 45 show that they are well-defined, and provide a.s. growth bounds. Proposition 47 below shows that the introduced processes are actually the Taylor coefficients \( A_n, B_n \).
Proposition 45. Let $b_1, b_2$ be independent copies of two-sided Brownian motion on $\mathbb{R}$, and set $y_u = e^{b_2 - u/2}$ as in (90). The recursive system $A_0 \equiv 1, B_0 \equiv 0$, and

$$B_n = y_u \int_{-\infty}^u \frac{\beta}{8} e^{\beta s/4} A_{n-1,s} y_s^{-1} ds,$$

$$A_n = \int_{-\infty}^u \frac{\beta}{8} e^{\beta s/4} B_{n-1,s} ds - B_{n,s} db_1. \quad (122)$$

is well-defined for $u \leq 0$. Moreover, given $a < 1/4$ and $\beta_0 \in (0, 1]$ there is a random constant $C$, so that for all $\beta \geq \beta_0$ a.s. for all $n \geq 0$ and $u \leq 0$ we have

$$|A_{n,u}|, |B_{n,u}| \leq \frac{C^n}{n!} e^{a\beta nu}. \quad (123)$$

By the scaling properties of the processes $A_k, B_k$, Proposition 45 extends to arbitrary $u \in \mathbb{R}$, and (123) holds for $u \leq u_0$ with $C$ depending on $u_0$.

Remark 46 (Dufresne’s identity for $B_1$). We have

$$B_1(0) = \frac{\beta}{8} \int_{-\infty}^0 e^{-b_2(s) + (\frac{\beta}{4} + \frac{1}{2})s} ds = \frac{\beta}{2} \int_0^\infty e^{2(b_2(s) - (\frac{\beta}{4} + 1)s)} ds = \frac{\beta}{4G},$$

where $G$ has Gamma distribution with rate 1 and shape parameter $1 + \frac{\beta}{2}$. The last step follows from Dufresne’s identity [10].

Proof of Proposition 45. We first show by induction that $A_n, B_n$ are well-defined. Set $a < 1/4$ and $\beta_0 > 0$. Set $\varepsilon = \min(1, (1-a)/\beta_0/8)$ so that

$$-a\beta + \beta/4 - \varepsilon \geq \varepsilon, \quad (124)$$

for $\beta \geq \beta_0$. For $n \geq 1$ define $\bar{A}_n = \sup_{u \leq 0}|A_n|/e^{a\beta nu}$, and define $\bar{B}_n$ similarly. In addition, set $\bar{B}_n = \sup_{u \leq 0}|B_n|/e^{(a\beta n + \varepsilon)u}$, we set these quantities to $\infty$ if the corresponding $A_n, B_n$ is not well-defined.

Brownian motion is sublinear, so for each $\varepsilon > 0$ there is a random variable $C_\varepsilon$ so that

$$C_\varepsilon^{-1} e^{\varepsilon u} \leq e^{u y_u^2} \leq C_\varepsilon e^{-\varepsilon u}, \quad u \leq 0. \quad (125)$$

If $\bar{A}_{n-1} < \infty$ then

$$|B_{n,u}| \leq C_\varepsilon e^{-(1+\varepsilon)/2u} \int_{-\infty}^u \frac{\beta}{8} e^{(\beta/4 + 1/2 - \varepsilon/2)s} |A_{n-1,s}| ds$$

so we conclude that

$$\bar{B}_n \leq \bar{B}_n^\varepsilon \leq \frac{\beta}{8} \frac{C_\varepsilon \bar{A}_{n-1}}{a(n-1)\beta + \beta/4} \leq \frac{C_\varepsilon \bar{A}_{n-1}}{8an}. \quad (126)$$
The first term in the definition (122) of $A_n$ for $u \leq 0$ can be bounded as
\[
\left| \int_{-\infty}^{u} \frac{\beta}{8} e^{\beta s/4} B_{n-1,s} \, ds \right| \leq \bar{A}_{n,1} e^{(\alpha + \beta/4)u}, \quad \bar{A}_{n,1} := \frac{\beta \bar{B}_{n-1}}{8\alpha + 2\beta}. \tag{127}
\]
By Proposition 73 there is a random variable $Z_n$ with rate 1 exponential tails (187) so that
\[
\left| \int_{-\infty}^{u} B_{n,u} \, db \right| \leq \frac{2\bar{B}_n}{\sqrt{a'}} e^{a'u} \left( Z_n + \log(1 + 2a'|u|) + |\log a'| + \log(1 + 2|\log \bar{B}_n|) \right)
\]
with $a' = a'' + \varepsilon$. This and (127) implies that $\bar{A}_n \leq \bar{A}_{n,1} + \bar{A}_{n,2}$ with
\[
\bar{A}_{n,2} = \frac{2\bar{B}_n}{\sqrt{a'}} \left( Z_n + |\log a'| + \log(1 + 2|\log \bar{B}_n|) + \sup_{u \leq 0} e^{\varepsilon u} \log(1 + 2a'|u|) \right).
\]
We conclude that all $\bar{A}_n, \bar{B}_n$ are finite and so all $A_n, B_n$ are well defined.

The tail bound $P(Z_n > y) < ce^{-y}$ and the Borel-Cantelli lemma shows that
\[
Z := \sup_{n \geq 1} (Z_n - 2 \log n) < \infty. \tag{128}
\]
Recall the definition of $C_\varepsilon \geq 1$ from (125), and define
\[
C = \frac{C_\varepsilon^2}{16a^2} + 10^8 \left( Z + \log(1 + \frac{1}{\varepsilon}) + \frac{1}{a\beta_0} \right)^2.
\]
We will show by induction that $\bar{A}_n, \bar{B}_n \leq C^n/n!$, which is equivalent to (123). This holds for $n = 0$ since $C \geq 1$. Assuming the bounds for $n - 1$ by (126) we get
\[
\bar{B}_n \leq \bar{B}_n^\varepsilon \leq \frac{C_n^{-1} C_\varepsilon}{n! 8a \leq \frac{C_n^{-1/2}}{n!}}. \tag{129}
\]
This is stronger than the hypothesis for $B_n$, but we need extra room to bound $A_n$. Next,
\[
\bar{A}_{n,1} = \frac{1}{24a/\beta + 1} \leq \frac{C_n^{-1}}{2(n - 1)!(4a/\beta_0 + 1)} \leq \frac{C_n^{-1}}{2(n - 1)!}.
\]
For $\bar{A}_{n,2}$, change variables $u \to \varepsilon u$ and use $\log(1 + ab) \leq \log(1 + a) + \log(1 + b)$ to get
\[
\left( \sup_{u \leq 0} e^{\varepsilon u} \log(1 + 2a'|u|) \right) - \log(1 + a') - \log(1 + \frac{1}{\varepsilon}) \leq \sup_{u \leq 0} e^{\varepsilon u} \log(1 - 2u) \leq 1.
\]
Use $\log(1 + x) \leq 1 + |\log x|$, (128) and (129) to get
\[
\bar{A}_{n,2} \leq \frac{2C_n^{-1/2}}{n! \sqrt{a'}} \left( 2 + Z + \log(1 + \frac{1}{\varepsilon}) + 2 \log n + 2|\log a'| + \log(1 + 2|\log(C_n^{-1/2}!/n!)) \right).
\]
Finally, use $|\log x| x^{-1/2} \leq 1 + x^{-1}$ and $\log \left( 1 + 2|\log(x^{n^{-1/2}}/n!)| \right) \leq 4x^{1/4}n^{1/2}$ to get
\[
\frac{n! \bar{A}_{n,2}}{2C_n^{-1/2}} \leq \frac{2 + Z + \log(1 + \frac{1}{\varepsilon}) + 4}{\sqrt{a\beta}} + 2 + \frac{4}{\sqrt{an\beta}} + \frac{4C_1^{1/4}}{\sqrt{a\beta}} \leq \frac{C^{1/2}}{4}.
\]
Thus $\bar{A}_n \leq C^n/n!$, closing the induction. \qed
Proposition 47. Consider the Taylor coefficient processes $A_{n,u}, B_{n,u}$ at $z = 0$ for the $A, B$ with $H = [A, B]^t$. They satisfy the following system of stochastic differential equations:

$$
B_0 \equiv 0, \quad A_0 \equiv 1, \quad d\mathbf{B}_n = \mathbf{B}_n db_2 - \frac{\beta}{8} e^{\beta u/4} A_{n-1} du, \\
d\mathbf{A}_n = -\mathbf{B}_n db_1 + \frac{\beta}{8} e^{\beta u/4} B_{n-1} du.
$$

(130)

For any $k$, the equations for the first $k$ elements of the sequence $B_0, A_0, B_1, A_1, \ldots$ form an autonomous system. In fact, $A_n, B_n, n \geq 0$ is the unique solution of this system so that for $n \geq 1$ $\{B_{n,u}, u \leq 0\}$ is a tight family and $A_{n,u} \to 0$ in probability as $u \to -\infty$. Moreover, $A_n = A_n$ and $B_n = B_n$, as defined in (122).

Proof. Since the SDE system (119) depends analytically on its parameter $z$, Itô’s formula can be applied to get SDEs for derivatives in this parameter as well, see Protter [25] Section V.7. Differentiate (119) $n$ times in $z$ and evaluate at $z = 0$ to get

$$
d\mathcal{H}^{(n)} = \left( \begin{array}{cc} 0 & -db_1 \\ 0 & db_2 \end{array} \right) \mathcal{H}^{(n)} - n \frac{\beta e^{\beta u/4}}{8} J\mathcal{H}^{(n-1)} du.
$$

By definition, $\mathcal{H}_{u}^{(n)} = \mathcal{H}_{u}^{(n)}(0) = n! [A_{n,u}, B_{n,u}]^t$. This shows that $A_n, B_n$ satisfy (130).

Itô’s formula shows that $A_n, B_n$ from Proposition [45] solve the equations (130). Our goal is to show that these are equal to $A_n, B_n$.

It follows from Proposition [44] that $\mathcal{H}_{u} e^{-\beta u/4} z$ is time-stationary. As a consequence, for $n \geq 1, B_{n,u} e^{-n\beta u/4}$ is also time-stationary, so $\{B_{n,u}, u \leq 0\}$ is tight. Similarly, $A_{n,u} e^{-n\beta u/4}$ is tight, so $A_{n,u} \to 0$ in probability as $u \to -\infty$. Using the definition of $A_n, B_n$ we check that $B_{n,u} e^{-n\beta u/4}, A_{n,u} e^{-n\beta u/4}$ are time-stationary, hence $\{B_{n,0}, u \leq 0\}$ is tight, and $A_{n,u} \to 0$ in probability as $u \to -\infty$.

So it suffices to show that solutions with the stated tightness and convergence conditions are unique. Consider two such solutions, played by $A_n, B_n$ and $A_n, B_n$ here.

We show first that $Y = B_1 - B_1 \equiv 0$. Indeed, we have $dY = Y db_2$, so $Y = C y$ for some random constant $C$ with $y$ as in (90). But $Y$ is tight and $y$ is not, so $C = 0$. Now let $Z = A_1 - A_1$. Then $dZ = 0$, so $Z$ is constant. But $Z_u \to 0$ in probability, so $Z = 0$. Repeating this argument inductively for each $n$ gives uniqueness. 

\[ \square \]

Corollary 48. Almost surely, for all $(u, z) \in \mathbb{R}_+ \times \mathbb{C}$ we have

$$
\|\mathcal{H}_{u}(z) - [1, 0]^t\| \leq e^{Ca^u|z|} - 1,
$$

(131)

for some fixed $a = a_{\beta} \geq 1$ and a random constant $C > 0$. In particular, we have $\mathcal{H}_{u} \to [1, 0]^t$ a.s. as $u \to -\infty$ uniformly on compact subsets of $\mathbb{C}$.
Conversely, \( \mathcal{H} \) defined in (116) is the unique solution of the system (119) consisting of entire functions in \( z \) so that
\[
\sup_{z \in D} |H_u(z) - [1, 0]^t| \to 0
\]
in probability as \( u \to \infty \) for some open neighborhood \( D \) of 0. In particular, \( \mathcal{H} \) is the unique solution, consisting of entire functions, so that the entire function valued process \( t \mapsto \mathcal{H}_u(ze^{-\beta u/4}) \) is time-stationary.

Proof. The bound (131) follows from writing \( H = [A, B]^t \) as a convergent power series in \( z \) and the inequalities (123):
\[
\|H_u(z) - [1, 0]^t\| \leq \sum_{n=1}^{\infty} |A_n z^n| + |B_n z^n| \leq \sum_{n=1}^{\infty} 2 C^n e^{\beta a u} |z|^n \leq e^{2 C e^{\beta a u}|z|} - 1.
\]
This also implies (132) for any bounded set \( D \).

Now consider an entire function solution of (119) for which \( \sup_{z \in D} |H_u - [1, 0]^t| \to 0 \) in probability. By Cauchy’s identity, the Taylor coefficients converge to those of \( [1, 0] \) in probability, in particular each \( B_n \) is tight and each \( A_n \to 0 \) in probability as \( u \to -\infty \). But (119) implies that the Taylor coefficients satisfy (130), which has a unique solution with these conditions. The Taylor coefficients determine \( \mathcal{H} \), so it is also unique.

For the last statement, time-stationarity implies (132) for any bounded set \( D \).

5.3 Differential equations for the Taylor coefficient densities

By Proposition 44 the random function \( (u, z) \mapsto \mathcal{H}_u(ze^{-\beta u/4}) \) is stationary under \( u \)-shifts. In fact, as a function of \( u \), this process is stationary and measurable with respect to the filtration generated by the increments of \( b_1, b_2 \). This observation allows us to give a finite closed system of partial differential equations for the joint density of the first finitely many Taylor coefficients.

Proposition 49. The stationary Taylor coefficients \( A_n = e^{-n\beta u/4} A_n, B_n = e^{-n\beta u/4} B_n \) satisfy the following system of stochastic differential equations:
\[
\begin{align*}
B_0 & \equiv 0, \quad A_0 \equiv 1, \\
\frac{dB_n}{dt} & = B_n db_2 + \left( \frac{\beta}{8} A_{n-1} - \frac{\beta}{4} B_n \right) du, \\
\frac{dA_n}{dt} & = -B_n db_1 + \left( \frac{\beta}{8} B_{n-1} - \frac{n\beta}{4} A_n \right) du.
\end{align*}
\] (133)
In particular, for a given \( n \geq 1 \) the joint density \( p_n(x_1, y_1, \ldots, x_n, y_n) \) of \( B_1, A_1, \ldots, B_n, A_n \) satisfies the following partial differential equation:

\[
\sum_{k=1}^{n} \left( \frac{1}{2} \left( \partial_{x_k}^2 + \partial_{y_k}^2 \right) (x_k^2 p_n) - \frac{\beta}{8} \partial_{x_k} ((y_{k-1} - 2kx_k)p_n) - \frac{\beta}{8} \partial_{y_k} ((x_{k-1} - 2ky_k)p_n) \right) = 0.
\]  \hspace{1cm} (134)

Here \( x_0 = 0, y_0 = 1, \) and \( x_k \in (0, \infty), y_k \in \mathbb{R}. \) The same PDE with the \( x_n \) terms dropped is satisfied by the joint density of \( B_1, A_1, \ldots, B_n. \)

For \( n = 1 \) we get the following PDE for \( p = p_1. \)

\[
x^2 (p_{xx} + p_{yy}) + \frac{\beta}{2} yp_y + \left( \left( \frac{\beta}{2} + 4 \right)x - \frac{\beta}{4} \right) p_x + (\beta + 2)p = 0.
\]

**Proof.** This follows from Itô’s formula, Proposition 47 and Kolmogorov’s forward equation for Markov processes. \( \square \)

The distribution of \( B_1 \) is given in Remark 46 by the Dufresne identity [10].

6 Moment and tail bounds

The goal of this section is to understand the tails of the structure function. The results here will be used later to show analytic properties of \( \zeta_{\beta}, \) and to rigorously justify moment computations based on SDEs.

The main tool is the phase function \( \alpha_{\lambda, \nu}, \) which, essentially, counts eigenvalues. The bounds will be uniform in the approximation parameter \( \nu \in (-\infty, 0). \) They imply similar bounds for the limiting processes.

6.1 The structure function and the phase function

Recall the definition of \( \mathcal{H} = [A, B]^t \) from (116), and introduce the processes

\[
\mathcal{E} = [1, -i] \cdot \mathcal{H} = A - iB,
\]

\[
\mathcal{E}^* = [1, i] \cdot \mathcal{H} = A + iB.
\]

Note that we have \( \mathcal{E}_0(z) = E(1, z) \) where \( E \) is the structure function defined in (60), we will use the same name for the entire process \( \mathcal{E}. \) The name originates in the theory of canonical systems, and refers to an infinite-dimensional analogue of orthogonal polynomials on the unit circle. Proposition 44 implies that for \( u \in \mathbb{R} \) the function \( z \to \mathcal{E}_u(z) \) has the same distribution as \( z \to E(1, e^{\beta u/4}z). \)
From (117) we have
\[ \zeta_\beta = A - qB = \frac{1 - iq}{2} \mathcal{E} + \frac{1 + iq}{2} \mathcal{E}^*, \]
(135)
where the processes are taken at time \( u = 0 \).

For a given \( \nu \in (\infty, 0) \) we use \( \mathcal{H}^\nu \) in the above definitions to obtain \( \mathcal{E}_\nu \) and \( \mathcal{E}_\nu^* \).

The following is an immediate consequence of Proposition 43.

**Corollary 50.** Let \( \nu \in [-\infty, 0) \), and
\[ f_\beta(u) = \frac{\beta}{4} e^{\frac{\beta}{4} u}. \]
(136)
The processes \( \mathcal{E}_\nu, \mathcal{E}_\nu^* \) satisfy the stochastic differential equations
\[
\begin{align*}
    d\mathcal{E} &= \frac{i}{2} f_\beta \mathcal{E} du + \frac{\mathcal{E}^* - \mathcal{E}}{2} (i db_1 - db_2), \\
    d\mathcal{E}^* &= -\frac{i}{2} f_\beta \mathcal{E}^* du + \frac{\mathcal{E}^* - \mathcal{E}}{2} (i db_1 + db_2).
\end{align*}
\]
(137)
For any fixed time \( u \) the entire function \( z \to \mathcal{E}_{\nu,u}(z) \) has no zeros on the real line, and \( \mathcal{E}^* = \bar{\mathcal{E}} \) there. For \( \Im z < 0, u \in \mathbb{R} \) we have
\[ |\mathcal{E}_{\nu,u}(z)| \geq |\mathcal{E}_{\nu,u}^*(z)|. \]
(138)
For \( \nu \neq -\infty \), the processes \( \mathcal{E}_\nu, \mathcal{E}_\nu^* \) are the unique strong solutions of (137) on \([\nu, \infty)\) with initial condition \( \mathcal{E}_{\nu,0} = \mathcal{E}_{\nu,0}^* = 1 \). The processes \( \mathcal{E}, \mathcal{E}^* \) can be obtained as uniformly on compact limits of \( \mathcal{E}_\nu, \mathcal{E}_\nu^* \) as \( \nu \to -\infty \).

The inequality (138) follows from (62) of Lemma 22.

The real and imaginary parts of \( \log \mathcal{E}_\nu \) for \( \nu \in [-\infty, 0) \) will play an important role in the upcoming analysis. Since \( \mathcal{E}_\nu \) has no zeros on \( \mathbb{R} \) and \( \mathcal{E}_\nu(0) = 1 \), the branch of the logarithm can be chosen as the unique continuous function which takes the value 0 at 0.

**Corollary 51** (The phase function). For \( \nu \in [-\infty, 0) \) and \( \lambda \in \mathbb{R} \) we can write \( 2 \log \mathcal{E}_{\nu,u}(\lambda) = \mathcal{L}_\lambda,\nu(u) + i \alpha \lambda,\nu(u) \), where \( \alpha, \mathcal{L} \in \mathbb{R} \) satisfy the stochastic differential equations
\[
\begin{align*}
    d\alpha &= \lambda f_\beta \, du - \Re \left[ (e^{-i\alpha} - 1)(db_1 + idb_2) \right] \\
    &= \lambda f_\beta \, du + 2 \sin(\alpha/2) dW_1 \\
    d\mathcal{L} &= \Im \left[ (e^{-i\alpha} - 1)(db_1 + idb_2) \right] = 2 \sin(\alpha/2) dW_2.
\end{align*}
\]
(139)
Here the \( W_j \) defined above are independent copies of Brownian motion that depend on \( \lambda \).
For \( \nu \neq -\infty \) the processes \( \alpha_{\lambda,\nu}, \mathcal{L}_{\lambda,\nu} \) are the unique strong solutions of (139) with initial conditions \( \alpha_{\lambda,\nu}(\nu) = \mathcal{L}_{\lambda,\nu}(\nu) = 0 \). As \( \nu \to -\infty \) these processes converge a.s. uniformly on compacts to \( \alpha_{\lambda}, \mathcal{L}_{\lambda} \).

For \( \nu \in [-\infty, 0) \), the eigenvalues of \( \tau_{\beta,\nu} \) are of multiplicity one and are given by

\[
\Lambda = \{ \lambda \in \mathbb{R} : \alpha_{\lambda,\nu}(0) = U \mod 2\pi \},
\]

where \( U \) is a uniform random variable on \([0, 2\pi]\), independent of \( b_1, b_2 \). Moreover, for \( \lambda > 0 \) we have

\[
|\#(\Lambda \cap [0, \lambda]) - \frac{1}{2\pi} \alpha_{\lambda,\nu}(0)| \leq 1. \tag{141}
\]

**Proof.** Proposition 43 and Itô’s formula implies the SDE characterization of \( \alpha \) and \( \mathcal{L} \).

The eigenvalues of \( \tau_{\beta,\nu} \) are given by the zero set of \( A_{\nu,0} - qB_{\nu,0} \). By definition \( A = qB \) if and only if \( \alpha = 2\Im \log(A - iB) = 2\Im \log(q - i) \mod 2\pi \). Since \( q \) has Cauchy distribution, \( 2\Im \log(q - i) = -2 \arccot q \) is uniform on \([-\pi, \pi]\).

Note that \( \alpha = 2\Im \log E(1, \lambda) \). By (45) this is nondecreasing in \( \lambda \), a standard result in oscillation theory. Since \( \alpha \) equals 0 for \( \lambda = 0 \), equation (140) implies (141).

Note that \( \alpha \) is the relative phase function introduced by Killip and Stoiciu [18], see also [32] for the connection to the Brownian carousel [29].

### 6.2 Quadratic variation bounds for the process \( \alpha \)

Our goal is to estimate the exponential moment of the quadratic variation of the phase function \( \alpha_{\lambda,\nu} \) introduced in Corollary 51. This will be needed for bounding moments of \( \zeta \).

In this section \( \lambda \in \mathbb{R}, \nu \in (-\infty, 0) \) are fixed, and we often abbreviate \( \alpha = \alpha_{\lambda,\nu} \).

Our first lemma controls the tails of the time when \( \alpha \) could start collecting significant quadratic variation.

**Lemma 52** (Early behavior of \( \alpha \)). For \( \varepsilon > 0 \) define the stopping time

\[
\tau_\varepsilon = \tau_{\varepsilon,\lambda,\nu} = \inf\{ t \leq 0 : \sin^2(\alpha(t)/2) \geq e^{\varepsilon t} \}. \tag{142}
\]

For all \( \beta > 0, \lambda \in \mathbb{R} \) and \( a < \frac{\beta(\beta+2)}{8} \) there exists \( \varepsilon, c \) so that for all \( \nu \) and all \( t \leq 0 \) we have

\[
P(\tau_\varepsilon < t) \leq ce^{at}.
\]

**Proof.** Note that \( X = \log(\tan(\alpha_{\lambda,\nu}/4)) \) satisfies

\[
dX = \frac{\lambda\beta}{8} e^\frac{4t}{\beta} \cosh X dt + \frac{1}{2} \tanh X dt + dB, \quad X(\nu) = -\infty, \tag{143}
\]
and \( \sin^2(\alpha/2) = \text{sech}^2(X) \leq 4e^{2X} \). (This process is studied in detail in [30].) It suffices to show a bound of the form

\[
P(\tau_X \leq t) \leq ce^{at} \quad \text{for} \quad \tau_X = \tau_{X,e} = \inf \{ t \leq 0 : X(t) \geq \varepsilon t \}.
\] (144)

Moreover, it suffices to show (144) for \( t \leq k \) with a fixed \( k < 0 \), as the general case follows by changing \( c \) accordingly.

Let \( k < 0, \varepsilon < \theta < \beta/4 \). The drift term for \( X \) in the region \( t \leq k \) with \( X(t) \in [\theta t, \varepsilon t] \), satisfies

\[
\frac{\lambda \beta}{8} e^{\frac{a}{2} t} \cosh X + \frac{1}{2} \tanh X \leq \frac{\lambda \beta}{8} e^{\frac{a}{2} (\theta - \varepsilon) k} + \frac{1}{2} (2e^{2\varepsilon k} - 1) \leq -q,
\] (145)

where \( q < 1/2 \) can be made arbitrarily close to \( 1/2 \) for \( \theta, \lambda \) fixed and appropriate choice of \( k \).

For the Brownian motion \( B \) driving \( X \) in (143), let \( Y \) be \( B - (\theta + q)t \) reflected at 0. This is the nonnegative process defined as

\[
Y(t) = \sup_{u \leq t} (B(t) - B(u) - (q + \theta)(t - u)).
\]

We claim that

\[
X(s) \leq Y(s) + \theta s \quad \text{for} \quad s \leq \tau_X \wedge k.
\] (146)

Clearly this holds if \( X(s) \leq \theta s \). Otherwise, since \( X \) starts at \(-\infty\), there is a maximal time \( \sigma \) at most \( s \) so that \( X(\sigma) = \theta \sigma \). By (145) for \( \sigma \leq s \leq \tau_X \wedge k \) we have

\[
X(s) = X(\sigma) + \int_{\sigma}^{s} \frac{\lambda \beta}{8} e^{\frac{a}{2} u} \cosh X_u + \frac{1}{2} \tanh X_u \, du + B(s) - B(\sigma)
\]

\[
\leq \theta \sigma - q(s - \sigma) + B(s) - B(\sigma) \leq Y(s) + \theta s.
\]

The process \( Y \) is a stationary Markov process and \( Y(s) \) has exponential distribution with rate \( 2(q + \theta) \) see, for example [22]. Let \( \tau_Y \) be the first hitting time of the line \((\varepsilon - \theta)s\) by \( Y \).

By (146) we have \( \tau_Y \leq \tau_X \wedge k \).

Let \( m(s) = (\varepsilon - \theta)s - 1 \). For \( t \leq -1 \) we have

\[
1(\tau_Y \leq t)1(Y(s) > m(s) \text{ for all } s \in [\tau_Y, \tau_Y + 1]) \leq \int_{-\infty}^{t+1} 1(Y(s) > m(s))ds.
\]

By the strong Markov property, the conditional expectation of the left hand side given \( \mathcal{F}_{\tau_Y} \) is at least \( r \) \( 1(\tau_Y \leq t) \) with

\[
r = r_{\theta - \varepsilon} = P(B(u) > m(u) \text{ for all } u \in [0, 1]) > 0,
\]

47
where $B$ is a standard Brownian motion. Taking expectations we get

$$rP(\tau_Y \leq t) \leq \int_{-\infty}^{t+1} P(Y(s) > m(s))ds = \int_{-\infty}^{t+1} e^{2(q+\theta)((\theta-\varepsilon)s+1)}ds = \frac{e^{2(q+\theta)}(t+1)}{2(q + \theta)(\theta - \varepsilon)} e^{2(q+\theta)(\theta-\varepsilon)(t+1)}.$$ 

By choosing $q < 1/2, \theta < \beta/4$, and $\varepsilon > 0$ appropriately we can make the coefficient of $t$ in the exponent equal to $a < \frac{\beta(\beta+2)}{8}$. Since $\tau_Y \leq \tau_X \wedge k$ this gives the desired bound (144) on $t \leq k$, and the lemma follows.

The next lemma is about a generalized version of the process $\alpha$. It controls the amount of quadratic variation collected on a finite time interval.

**Lemma 53** (Quadratic variation of $\alpha$ on a finite interval). For $p \geq 0$ there is a constant $c_p$ so that the following holds. Consider the SDE

$$d\alpha = \lambda(t)dt + 2 \sin(\alpha/2)dW$$

on $[0, \sigma]$ with deterministic $0 < \lambda(t) \leq \varepsilon$ and arbitrary initial condition. Then

$$E e^{p\int_0^\sigma \sin^2(\alpha(s)/2)ds} \leq (1 + \frac{1}{\varepsilon})e^{(p - \sqrt{p/2})^+ + \varepsilon c_p}\varepsilon}. \quad (148)$$

**Proof of Lemma 53** First assume $p > 1/2$. Set

$$M_t = \exp \left( \int_0^t g(\alpha_s)ds - vt \right) (f(\alpha_t) + \varepsilon),$$

where

$$v = p - \sqrt{p/2} + \delta, \quad g(\alpha) = p \sin^2(\alpha/2), \quad f(\alpha) = |\sin(\alpha/2)|\sqrt{2p},$$

and $\delta = c_p\varepsilon$. We will show that for an appropriate choice of $c_p$ the process $M_t, t \in [0, \sigma]$ is a supermartingale. If this holds then

$$EM_\sigma \leq EM_0 = f(\alpha_0) + \varepsilon \leq 1 + \varepsilon.$$ 

We also have

$$EM_\sigma = E \exp \left( \int_0^\sigma g(\alpha_s)ds - v\sigma \right) (f(\alpha_\sigma) + \varepsilon) \geq e^{-v\sigma\varepsilon}E \exp \left( \int_0^\sigma p \sin^2(\alpha_s/2)ds \right).$$

This gives

$$E \exp \left( \int_0^\sigma p \sin^2(\alpha_s/2)ds \right) \leq \frac{1}{\varepsilon}e^{(p - \sqrt{p/2} + \delta)\varepsilon} (1 + \varepsilon),$$

which implies the statement of the lemma.
A coupling argument, see e.g. Proposition 9 of [29], shows that if $\tau_k$ is the first hitting time of $2\pi k$, then $\alpha(t) > 2k\pi$ for $t > \tau_k$. This can be seen from the SDE (147): when $\alpha$ is a multiple of $2\pi$, then the noise term vanishes and the drift term is positive.

The function $f$ is twice continuously differentiable apart from the set $2\pi\mathbb{Z}$. Itô’s formula applies to $M$ between times $\tau_k, \tau_{k+1}$ and hence over the whole interval $[0, \sigma]$. Let $\mathcal{G} = 2\sin^2(\alpha/2)\partial^2_{\alpha}$ be the generator of the process $\alpha$ from (147) with no drift. We get

$$dM_t = e^{\int_0^t (g(\alpha_s) - v)(f(\alpha) + \varepsilon) + \lambda(t)f'(\alpha_t))dt + dW_t} \text{ terms.}$$

A nonnegative local supermartingale with constant initial condition is a supermartingale, so it suffices to show that

$$\mathcal{G}f + (g - v)(f + \varepsilon) + \lambda(t)f' \leq 0. \quad (149)$$

Since $(\mathcal{G} + g)f = (p - \sqrt{p/2})f$, the inequality (149) reduces to

$$\left(\sqrt{p/2} - p \cos^2(\alpha/2)\right)\varepsilon + \lambda(t)f' \leq \delta f + \delta\varepsilon. \quad (150)$$

We will show that for $p > 1/2$ with the appropriate choice of $c_p$ so that the following holds:

$$\sqrt{p/2} - p \cos^2(\alpha/2) + |f'| \leq c_p f,$$

this implies (150). We have $|f'(\alpha)| = \sqrt{p/2} |\cos (\alpha/2)| |\sin (\alpha/2)|^{\frac{2-p}{2}}$, so it is enough to show that the function

$$\left(\sqrt{p/2} - p \cos^2(\alpha/2)\right) |\sin(\alpha/2)|^{-\frac{2}{\sqrt{2p}}} + \sqrt{p/2} |\cos(\alpha/2)||\sin(\alpha/2)|^{-1}$$

is bounded from above. By continuity, it suffices to check the behavior of this function near $\alpha \in 2\pi\mathbb{Z}$, and since $p > 1/2$ we see that the function converges to $-\infty$ there. The statement of the lemma follows for $p > 1/2$.

For $p \leq 1/2$, we use that for $x \geq 0$, we have $e^{px} \leq 1 \vee e^{(1/2+\delta)x}$ to reduce to the $p > 1/2$ case.

We now combine the results in the previous lemmas and give a bound on the quadratic variation of $\alpha$.

**Proposition 54** (Quadratic variation of $\alpha$). If $0 < p < \frac{1}{2}(1 + \frac{\beta}{2})^2$ then

$$E e^{p\int_0^\nu \sin^2(\alpha_s, \lambda_s/2)ds} < c_{p,\beta}(1 + |\lambda|)^{\frac{4}{2p}},$$

with a constant $c_{p,\beta}$ which does not depend on $\nu$. 49
Proof of Proposition 54. Let $\varepsilon > 0$, $k < 0$, $\lambda = 1$. For $t \leq \tau = \tau_\varepsilon$ as in (142) we have
\[
sin^2(\alpha_{\nu}/2) \leq e^{\varepsilon t},
\]
so
\[
\int_{\nu}^{0} \sin^2(\alpha/2)ds \leq \frac{1}{\varepsilon} + \int_{\tau \wedge k}^{k} \sin^2(\alpha/2)ds + |k|.
\]
Set $\ell(t) = 2 \arcsin(e^{\varepsilon t/2})$. By Lemma 53 for arbitrarily small $\delta > 0$ if $|k|, \frac{1}{\varepsilon}$ are large enough then
\[
E_{t,\ell(t)} e^{\int_{\nu}^{k} \sin^2(\alpha/2)ds} \leq c e^{-(p-\sqrt{p}/2)^{+} + \delta) t} \text{ for all } t \leq k.
\]
Here $E_{t_0,\ell_0}$ is the expectation for the process $\alpha$ satisfying (139) on $[t_0, \infty)$ with initial condition $\alpha(t_0) = \ell_0$. By the strong Markov property
\[
E e^{\int_{\nu}^{k} \sin^2(\alpha_{\lambda}/2)ds} \leq c E e^{-(p-\sqrt{p}/2)^{+} + \delta) \tau},
\]
where $c$ does not depend on $\nu$. Lemma 52 provides an upper bound on $E e^{-(p-\sqrt{p}/2)^{+} + \delta) \tau}$ if $\frac{\beta(\beta+2)}{8} > p-\sqrt{p}/2$ and $\delta$ is small enough. Hence there is a constant $c_{p,\beta}$ so that for $\lambda = 1$
\[
E e^{\int_{\nu}^{0} \sin^2(\alpha_{\lambda},\nu/2)ds} \leq c_{p,\beta}.
\]
(151)
Now let $0 < \lambda$. Note that by the scaling properties of the drift term of the SDE (139) we have
\[
\{\alpha_{\lambda,\nu}(t), \nu \leq t \leq 0\} \overset{d}{=} \{\alpha_{1,\nu}(t + \frac{4}{\beta} \log \lambda), \nu \leq t \leq 0\}.
\]
where $\nu_{\lambda} = \nu + \frac{4}{\beta} \log \lambda$. This implies
\[
E e^{\int_{\nu}^{0} \sin^2(\alpha_{\lambda}/2)ds} = E e^{\int_{\nu_{\lambda}}^{0} \sin^2(\alpha_{1,\nu_{\lambda}}/2)ds}.
\]
If $0 < \lambda < 1$ then we have
\[
E e^{\int_{\nu_{\lambda}}^{0} \sin^2(\alpha_{1,\nu_{\lambda}}/2)ds} \leq E^{p} f_{\nu_{\lambda}}^{0} \sin^2(\alpha_{1,\nu_{\lambda}}/2)ds,
\]
and the statement follows from the $\lambda = 1$ case (151).
If $1 < \lambda < e^{-\frac{\beta}{4} \nu}$ then $\nu_{\lambda} < 0$ and we have
\[
E e^{\int_{\nu_{\lambda}}^{0} \sin^2(\alpha_{1,\nu_{\lambda}}/2)ds} \leq E e^{p \int_{\nu_{\lambda}}^{0} \sin^2(\alpha_{1,\nu_{\lambda}}/2)ds} = \frac{1}{\lambda^{p}} E e^{p \int_{\nu_{\lambda}}^{0} \sin^2(\alpha_{1,\nu_{\lambda}}/2)ds}
\]
which yields the desired bound.
Finally, if $1 < e^{-\frac{\beta}{4} \nu} \leq \lambda$ then $\nu \geq -\frac{4}{\beta} \log \lambda$ and we get
\[
E e^{p \int_{\nu}^{0} \sin^2(\alpha_{1,\nu}/2)ds} \leq e^{-\nu \lambda} \leq \lambda^{\frac{4}{p}}.
\]
The $\lambda < 0$ case follows by symmetry.  \qed
6.3 Moment bounds for $E$

We can use the quadratic variation bounds of the previous section to bound the moments of $E$.

**Theorem 55.** For $\lambda, \gamma \in \mathbb{R}$, $\nu \in [\infty, 0)$ and $|\gamma| < 1 + \frac{\beta}{\gamma}$, we have

\[
E[|E_{\nu,0}(\lambda)|^\gamma] < c(1 + |\lambda|)^{2\gamma^2/\beta},
\]

\[
E e^{\gamma/2|\alpha_{\lambda,\nu}(0) - \lambda(1 - e^{\beta\nu/4})|} < c(1 + |\lambda|)^{\gamma^2/\beta}.
\]

Here $c$ depends on $\beta, \gamma$, but not on $\nu$. For all $\gamma \in \mathbb{R}$ with $c$ depending on $\beta$ only we have

\[
E e^{\gamma\alpha_{\lambda,\nu}(0)} \leq 2e^{c|\gamma|(1 + |\lambda| + (|\gamma| + \log |\lambda|)^+)}.
\]

**Proof.** First let $\nu \neq -\infty$. Recall $2 \log E = \mathcal{L} + i\alpha$, where $\mathcal{L}, \alpha$ satisfy (139). Thus we have

\[
\frac{1}{2} \mathcal{L}_{\lambda,\nu,0} = \int_0^\nu 2 \sin(\alpha_{\lambda,\nu}/2) dW_2 \overset{d}{=} N \cdot \sqrt{\int_0^\nu \sin^2(\alpha_{\lambda,\nu}/2) ds},
\]

since $W_2$ is independent of $\alpha$. Here $N$ is a standard normal random variable independent of $\alpha$. This shows that

\[
E[|E_{\nu,0}(\lambda)|^\gamma] = E e^{\gamma N \cdot \sqrt{\int_0^\nu \sin^2(\alpha_{\lambda,\nu}/2) ds}} = E e^{\gamma^2 \int_0^\nu \sin^2(\alpha_{\lambda,\nu}/2) ds}.
\]

The bound (152) follows from Proposition 54.

Without loss of generality, we assume $\lambda \geq 0$ for the rest of the proof.

\[
M(t) = \exp \left( \frac{\gamma}{2} \left( \alpha_{\lambda,\nu}(t) - \lambda e^{\frac{\beta}{4} t} + \lambda e^{\frac{\beta}{4} \nu} \right) - \frac{\gamma^2}{2} \int_\nu^t \sin^2(\alpha_{\lambda,\nu}/2) ds \right), \quad t \geq \nu,
\]

is a martingale and $EM(0) = EM(\nu) = 1$. We have

\[
E \exp \left( \frac{\gamma^2}{2} (\alpha_{\lambda,\nu}(0) - \lambda + \lambda e^{\frac{\beta}{4} \nu}) \right) = E \left[ \sqrt{M(0)} \exp \left( \frac{\gamma^2}{4} \int_\nu^0 \sin^2(\alpha_{\lambda,\nu}(s)/2) ds \right) \right]
\]

Since $EM(0) = 1$, Cauchy-Schwarz gives the upper bound

\[
\left( E \exp \left( \frac{\gamma^2}{2} \int_\nu^0 \sin^2(\alpha_{\lambda,\nu}(s)/2) ds \right) \right)^{1/2}.
\]

This inequality applied with $\pm \gamma$ together with Proposition 54 yields (153).

To prove (154) we may assume $\gamma > 0$. A coupling argument, Proposition 9 of [29] shows that for $\lambda > 0$ we have $\alpha_{\lambda,\nu}(t) > 0$ for $t > \nu$. Markov’s inequality implies

\[
P(\alpha_{\lambda,\nu}(t) \geq 2\pi) \leq \frac{\lambda}{2\pi} (e^{\frac{\beta}{4} t} - e^{\frac{\beta}{4} \nu}) \leq \lambda e^{\frac{\beta}{4} t}.
\]

51
Let \( k \) be a positive integer, and let \( \tau \) be the hitting time of \( 2\pi k \) for \( \alpha_{\lambda,\nu} \). The strong Markov property implies that given \( \tau = \nu \), the conditional distribution of \( \alpha_{\lambda,\nu}(t - \tau) - 2k\pi, t \geq \nu \) is the same as that of \( \alpha_{\lambda,\nu}(t), t \geq \nu \). Hence from (155) we get
\[
P\left(\alpha_{\lambda,\nu}(t) \geq 2\pi(k + 1) \mid \tau < t\right) \leq \lambda e^{\frac{\alpha}{\nu}}.
\]
It follows that
\[
P(\alpha_{\lambda,\nu}(t) > 2\pi k) \leq \left(\lambda e^{\frac{\alpha}{\nu}}\right)^k. \tag{156}
\]
By Cauchy-Schwarz for any \( \nu < t_0 < 0 \) we have
\[
E e^{\gamma \alpha_{\lambda,\nu}(0)} \leq \left(E e^{2\gamma \alpha_{\lambda,\nu}(t_0)}\right)^{1/2} \left(E e^{2\gamma (\alpha_{\lambda,\nu}(0) - \alpha_{\lambda,\nu}(t_0))}\right)^{1/2}. \tag{157}
\]
From (156) we get
\[
E e^{2\gamma \alpha_{\lambda,\nu}(t_0)} = \sum_{k=0}^{\infty} E\left[e^{2\gamma \alpha_{\lambda,\nu}(t_0)} 1(2k\pi \leq \alpha_{\lambda,\nu}(t_0) < 2(k + 1)\pi)\right]
\leq \sum_{k=0}^{\infty} e^{4\gamma (k+1)\pi} P(\alpha_{\lambda,\nu}(t_0) \geq 2k\pi) \leq \sum_{k=0}^{\infty} e^{4\gamma (k+1)\pi} \lambda^k e^{\frac{\alpha}{\nu} t_0}.
\]
If \( \frac{\alpha}{\nu} t_0 \leq -4\gamma \pi - \log \lambda - 1 \), then we get the bound
\[
E e^{2\gamma \alpha_{\lambda,\nu}(t_0)} \leq 2e^{4\gamma \pi}. \tag{158}
\]
We now use a variant of the martingale in the first part. For \( \nu \leq t_0 \leq 0 \) the process
\[
M(t) = \exp\left(4\gamma \left(\alpha_{\lambda,\nu}(t) - \alpha_{\lambda,\nu}(t_0) \right) - \lambda e^{\frac{\alpha}{\nu} t} - e^{\frac{\alpha}{\nu} t_0}\right) - 8\gamma \int_{t_0}^{t} \sin^2(\alpha_{\lambda,\nu}/2) ds,
\]
is a martingale with \( EM(0) = EM(t_0) = 1 \). By Cauchy-Schwarz we have
\[
E e^{4\gamma \int_{t_0}^{t} \sin^2(\alpha_{\lambda,\nu}(s)/2) ds} \leq E e^{8\gamma \int_{t_0}^{t} \sin^2(\alpha_{\lambda,\nu}(s)/2) ds} \leq e^{-4\gamma t_0}. \tag{159}
\]
where the last step uses the upper bound 1 on \( \sin(\cdot)^2 \). Now set
\[
t_0 = \begin{cases} 
\nu, & \text{if } -4\gamma \pi - \log \lambda - 1 < \frac{\alpha}{\nu}, \\
-4\gamma (\pi + \log \lambda + 1), & \text{if } \frac{\alpha}{\nu} \nu \leq -4\gamma \pi - \log \lambda - 1 < 0, \\
0, & \text{if } 0 \leq -4\gamma \pi - \log \lambda - 1.
\end{cases}
\]
The estimates (157), (158) and (159) yield \( E e^{\gamma \alpha_{\lambda,\nu}(0)} \leq (2e^{4\gamma \pi})^{1/2} e^{-2\gamma t_0 + \gamma \lambda}, \) and (154) follows.

The \( \nu = \infty \) case of all three bounds follows from Fatou’s lemma. \( \square \)
6.4 Precise growth of the number of zeros

The following corollary provides regularity bounds for the Sine\(_\beta\) process and its approximations. It is closely related to the work of Holcomb and Paquette [15], where the optimal constant in front of the \(\log\) is also determined. Bounds of this type that are less precise can be derived from the variance bounds on the counting function in [19]. Such a bound is explicitly given in [23].

**Corollary 56** (Regularity of the Sine\(_\beta\) process). For \(\nu \in [-\infty, 0)\) let \(N_\nu(\lambda)\) be the counting function of the eigenvalue process of the operator \(\tau_{\beta,\nu}\). Then almost surely, for all large enough \(|\lambda|\) we have

\[
|N_\nu(\lambda) - \frac{\lambda}{2\pi} (1 - e^{\frac{2\nu}{\beta}})| < (1 + \beta^{-1}) \log |\lambda|.
\]

**Proof.** We show the statement for \(\lambda \to \infty\), the \(\lambda \to -\infty\) case follows by symmetry. By (141) we have \(|N_\nu(\lambda) - \frac{1}{2\pi} \alpha_{\lambda,\nu}(0)| \leq 1\). Using (153) with \(\gamma = 1\) and \(\lambda \geq 1\) integer we get

\[
P(|\alpha_{\lambda,\nu}(0) - \lambda(1 - e^{\frac{2\nu}{\beta}})| \geq 6(1 + \beta^{-1}) \log \lambda \leq \frac{E e^{\frac{1}{2} |\alpha_{\lambda,\nu}(0) - \lambda(1 - e^{\frac{2\nu}{\beta}})}|}{e^{\frac{1}{2} (1 + \beta^{-1}) \log \lambda}} \leq \frac{c(1 + \lambda)^{1/\beta}}{\lambda^{3/2(1 + \beta^{-1})}}
\]

by the exponential Markov’s inequality. By the Borel-Cantelli lemma we get that

\[
\frac{1}{2\pi} |\alpha_{\lambda,\nu}(0) - \lambda(1 - e^{\frac{2\nu}{\beta}})| \leq \frac{6}{2\pi} (1 + \beta^{-1}) \log \lambda
\]

if \(\lambda\) is a large enough integer. Since \(N_\nu(\lambda)\) is non-decreasing, the claim follows. \(\square\)

7 Analytic properties of \(\zeta_\beta\)

In this section we show that \(\zeta_\beta\) is of Cartwright class. This allows us to show that logarithmic derivative has exponential moments off the real line.

7.1 Cartwright class, exponential type and uniqueness

Theorem 55 directly implies that the integral condition (67) needed for Cartwright class is satisfied by \(\zeta_\beta\).

**Proposition 57.** With \(\zeta = \zeta_{\beta,\nu}\) we have

\[
E \log^+ |\zeta(x)| \leq c_\beta + \frac{2}{\beta} \log^+ |x|
\]

for all real \(x\), and so

\[
E \int \frac{\log^+ |\zeta(x)|}{1 + x^2} dx < \infty.
\]
Proof. On the real line, we have $2\zeta = (E + \bar{E}) - qi(E - \bar{E})$. Since $\log^+ |a + b| \leq \log^+ |a| + \log^+ |b| + 1$, and $\log^+ |a| \leq \log(1 + |a|)$, we get the bound

$$\log^+ |\zeta| \leq 3 + 4 \log^+ |E| + \log^+ q \leq 3 + 4 \log(1 + |E|) + \log^+ q.$$  

By Theorem 55 $E|E(x)| \leq c(1 + |x|)^{2/\beta}$. Since $\log^+$ of a Cauchy distribution has finite expectation, and $\log$ is concave, Jensen’s inequality gives (160). 

This shows the integral condition for Cartwright class. We have two proof that $\zeta$ is of finite exponential type. First, it follows from the bounds (131) in Corollary (48). It also follows from the regularity of zeros, Corollary 56 and Proposition 27. Hence $\zeta$ is of Cartwright class.

Next, by Proposition 26 it satisfies

$$\zeta(z) = \lim_{r \to \infty} \prod_{|\lambda_k| < r} (1 - z/\lambda_k)$$  

(161)

In particular, $\zeta_2$ agrees with the random analytic function constructed in [8].

By Proposition 26 we also have

$$\limsup_{x \to \infty} \frac{\log |\zeta(ix)|}{x} = \limsup_{|z| \to \infty} \frac{\log |\zeta(z)|}{|z|} = 1/2,$$

so $\zeta_\beta$ has exponential type $1/2$.

Finally, by Proposition 28 we have the following.

**Proposition 58.** The law of $\zeta_\beta$ is the unique distribution on the Cartwright class, satisfying $\zeta_\beta(\mathbb{R}) \subset \mathbb{R}$, $\zeta_\beta(0) = 1$, with zero distribution $\text{Sine}_\beta$ of multiplicity one.

### 7.2 Product identities and Cauchy trace

The simple identity

$$2 \sin(x) = -\prod_{j=1}^{k} 2 \sin((j\pi + x)/k)$$

has the following analogue for $\zeta_\beta$.

**Corollary 59.** Let $k \geq 2$ an integer. There exists a coupling of $k$ copies $\zeta_{2k,1}, \ldots, \zeta_{2k,k}$ of $\zeta_{2k}$ so that

$$\zeta_{2/k}(z) = \prod_{j=1}^{k} \zeta_{2k,j}(z/k).$$
Proof. Let $\lambda_\ell$ be the ordered points of the Sine$_{2/k}$ process. Let $K$ be a uniform random element of $\{1, \ldots, k\}$. Let $\Lambda_j = \{\lambda_\ell : \ell = K + j \text{ mod } k\}$. By a result of Forrester [11], each $\Lambda_j$ has the same distribution as $k$ times a Sine$_{2k}$ process. The product formula (161) applied to each $\Lambda_j$ gives the analytic functions $\zeta_{2k,j}(z/k)$. Their product is a Cartwright function with zeros $\lambda_j$. By Proposition 58 it has the same law as $\zeta_{2/k}$.

By taking limits of Theorem 2.5.17 in [3], we see that the distribution of every second point in the superposition of two independent Sine$_1$ processes is a scaled Sine$_2$ process. It follows that given two independent copies of $\zeta_1$, there exists two dependent copies of $\zeta_2$ so that $\zeta_{1,1}\zeta_{1,2} = \zeta_{2,1}\zeta_{2,2}$.

The next theorem identifies the distribution of the integral trace $t_\beta$ for $\tau_\beta$.

**Theorem 60.** Consider the Sine$_\beta$ process, and denote the points in the process with $\lambda_k$, $k \in \mathbb{Z}$. Then the principal value sum

$$\lim_{r \to \infty} \sum_{|\lambda_k| \leq r} \lambda_k^{-1}$$

has Cauchy distribution with density $1/(2\pi(x^2 + 1/4))$.

Proof. This follows from Theorem 5.5 and the following Remark 2 of Aizenman and Warzel [2]. By (77) the function $\zeta'/\zeta$ is of Herglotz-Pick class: it maps the upper half-plane into itself. Moreover, $\zeta'/\zeta$ is almost surely continuous at 0. Since the intensity of the Sine$_\beta$ process is $1/(2\pi)$, and Sine$_\beta$ has the same distribution as $-\text{Sine}_\beta$, the parameter of the Cauchy distribution is $1/2$.

### 7.3 Exponential moments of $(\log \zeta_\beta)'$

Theorem 55 and Proposition 57 imply uniform exponential moment bounds on the log derivative of $\zeta_{\beta,\nu}$ away from the real line, this is the content of the following proposition. Chhaibi, Najnudel and Nikhegbal [8] show such bounds in the $\beta = 2$ case, and the proof below is inspired by their argument.

**Proposition 61.** Let $K$ be a compact set in the upper or lower open half plane, and let $\zeta = \zeta_{\beta,\nu}$. Then for $\nu \in [-\infty, 0)$ with $c$ depending on $p, \beta$ and $K$, but not on $\nu$ we have

$$E\sup_{z \in K} \exp\left(p \left| \frac{\zeta'(z)}{\zeta(z)} \right| \right) \leq c. \quad (162)$$

Proof. $\zeta_{\beta,\nu}$ is of Cartwright class by Proposition 57, Proposition 29 and Corollary (56) implies

$$\frac{\zeta'(z)}{\zeta(z)} = \int_{-\infty}^{\infty} \frac{1}{(z - \lambda)^2} \left( \frac{\lambda}{2\pi} (1 - e^{\frac{\beta}{4}}) - N(\lambda) \right) d\lambda \mp i(1 - e^{\nu\beta/4})/2.$$
By (141), \(|N(\lambda) - \frac{1}{2\pi}\alpha_{\lambda,\nu}(0)| \leq 1.\) With \(f(\lambda) = |\alpha_{\lambda,\nu}(0) - \lambda(1 - e^{2\lambda})|\) we have the bound

\[
\sup_{z \in K} \left| \frac{\zeta'(z)}{\zeta(z)} \right| \leq \int_{-\infty}^{\infty} \frac{f(\lambda)}{d(\lambda, K)^2} d\lambda + \frac{\pi}{d(\mathbb{R}, K)} + 1/2, \tag{163}
\]

where \(d\) is Euclidean distance.

Since \(d(\lambda, K)^{-2}\) decays like \(\lambda^{-2}\) as \(\lambda \to \pm \infty\), there exists a positive bounded function \(g_\lambda\) so that \(q_\lambda = pg_\lambda^{-1}d(K, \lambda)^{-2}\) is a probability density, and \(0 < \epsilon < g_\lambda < \sqrt{\beta/32}\) outside an interval \(L = [-\ell, \ell]\). Here \(\epsilon, \ell\) only depend on \(p\) and \(K\). By Jensen’s inequality,

\[
\exp \int_{-\infty}^{\infty} p f(\lambda) d(\lambda, K)^2 d\lambda \leq \int_{-\infty}^{\infty} e^{f(\lambda) g_\lambda} q_\lambda d\lambda.
\]

By Theorem 55 applied with \(\gamma = \sqrt{\beta/2} < 1 + \beta/2\) on \(L^c\) we have

\[
E e^{f(\lambda) g_\lambda} < c \frac{(1 + |\lambda|)^{1/2}}{1 + \lambda^2}, \quad E \int_{L^c} e^{f(\lambda) g_\lambda} q_\lambda d\lambda \leq \frac{c}{1 + \ell^{1/2}}. \tag{164}
\]

Let \(c'\) be the supremum of \(\gamma_\lambda\). Then by (154) we get

\[
E \int_L e^{f(\lambda) g_\lambda} q_\lambda d\lambda \leq e^{c'(1+\ell+\log(1+\ell))} \tag{165}
\]

with \(c_1\) depending only on \(c', \beta,\) but not \(\nu\). The claim follows from the bounds (163), (164), and (165).

\[\square\]

### 7.4 Moments of ratios of \(\zeta_\beta\) are finite

The exponential moment bounds for the log derivative of \(\zeta_\beta\) in Proposition 61 can be used to show that ratios have finite moments.

**Proposition 62.** Let \(w_j \in \mathbb{C} \setminus \mathbb{R}\), let \(K \subset \mathbb{C}^k\) be compact and \(\zeta = \zeta_{\beta,\nu}\). Then

\[
\sup_{z \in K, \nu} E \prod_{j=1}^k \left| \frac{\zeta(z_j)}{\zeta(w_j)} \right| < \infty.
\]

**Proof.** By Hölder’s inequality and the reflection symmetry \(\zeta(\bar{z}) = \overline{\zeta(z)}\), it suffices to show that

\[
\sup_{z \in K, \nu} E(|\zeta(z)/\zeta(w)|^k) < \infty \text{ for } \Re w > 0 \text{ and } K \text{ in the closed upper half plane.}
\]

Since for real \(x\), \(|\zeta(x + iy)|\) is increasing in \(y \geq 0\) by (69), we may further assume that \(K\) is in the open upper half plane. We write

\[
\frac{\zeta(z)^k}{\zeta(w)^k} = \exp \left( k \int_\gamma \frac{\zeta'(v)}{\zeta(v)} dv \right),
\]

where \(\gamma\) is the oriented line segment connecting \(w\) and \(z\). The claim follows from Proposition 61 and Jensen’s inequality. \[\square\]
Proposition 63. Fix $w_1, \ldots, w_n \in \mathbb{C} \setminus \mathbb{R}$, and $z_2, \ldots, z_n \in \mathbb{C}$. Then

$$r(z_1) = E \prod_{j=1}^{n} \frac{\zeta(z_j)}{\overline{\zeta(w_j)}}$$

is an entire function.

Proof. Proposition 62 implies that for any closed contour $\gamma$ of finite length

$$\int_{\gamma} E \left| \prod_{j=1}^{n} \frac{\zeta(z_j)}{\zeta(w_j)} \right| dz_1 < \infty.$$

Fubini’s theorem gives

$$\int_{\gamma} r(z_1) dz_1 = E \int_{\gamma} \prod_{j=1}^{n} \frac{\zeta(z_j)}{\zeta(w_j)} dz_1 = 0,$$

since $\zeta$ is an entire function. Morera’s theorem completes the proof. \qed

8 Moment formulas

Using the SDE (137) and the bounds in Section 6 we can set up equations for the expectations of certain functionals of $\mathcal{E}$ and $\zeta$.

8.1 The function $\hat{\zeta}$

The function $\zeta = \zeta_{\beta} = A - qB$ does not have a first moment, since $q$ has Cauchy distribution and is independent of $A$ and $B$. The following simple variant has some moments. Let

$$\hat{\zeta} = \frac{\zeta}{\sqrt{1 + q^2}} = \frac{U \mathcal{E} + \overline{U} \mathcal{E}^*}{2}, \quad U = \frac{1 - iq}{\sqrt{1 + q^2}}.$$ \hspace{1cm} (166)

The first formula follows from (135). The random variable $U$ is uniformly distributed on the unit complex semicircle around 0 with positive real part, $\overline{U} = 1/U$, and $U, \mathcal{E}$ are independent. When the $k + \epsilon$-th absolute moment of $\mathcal{E}, \mathcal{E}^*$ exist, we have

$$E \zeta^k = \frac{1}{2k} \sum_{j=0}^{k} \binom{k}{j} E[U^{2j-k}] E[\mathcal{E}^j (\mathcal{E}^*)^{k-j}], \quad E[U^m] = \frac{\sin(\pi m/2)}{\pi m/2}.$$
For real $z$ and $k < 1 + \beta/2$ these moments exist by Theorem 55. Mixed moments can be expressed similarly. Separating odd and even cases, we get

$$E_{\hat{\zeta}^{2\ell}} = \left(\frac{2\ell}{\ell}\right) \frac{1}{4^\ell} E\left[\mathcal{E}^{\ell}(\mathcal{E}^*)^{\ell}\right],$$

$$E_{\hat{\zeta}^{2\ell+1}} = \frac{1}{2^{2\ell+1} \pi} \sum_{j=0}^{2\ell+1} \binom{2\ell + 1}{j} \frac{(-1)^{1+j-\ell}}{j - \ell - 1/2} E\left[\mathcal{E}^j(\mathcal{E}^*)^{2\ell+1-j}\right].$$

These formulas motivate the study of the moments of the structure function $\mathcal{E}$, which are also interesting on their own right.

### 8.2 Joint moments of the structure function

Our methods give differential equations for the moments of products of the structure function $\mathcal{E}$. Fix $z_1, \ldots, z_n \in \mathbb{C}$, let $\eta \in \{-1, 1\}^k$ be an index set, and let

$$\mathcal{E}_\eta = \prod_{j=1}^{k} \mathcal{E}(z_j) \prod_{j=1}^{k} \mathcal{E}^*(z_j), \quad \eta \cdot z = \sum_{j=1}^{k} z_j \eta_j,$$

with $\mathcal{E}_{\eta,\nu}$ defined similarly using $\mathcal{E}_\nu, \mathcal{E}^*_\nu$. In this section we will omit the time parameter in the notation.

For convenience of notation let $\sigma_j$ be the operator that multiplies the $j$th coordinate of $\eta$ by $-1$, and let $\sigma_j \mathcal{E}_\eta = \mathcal{E}_{\sigma_j \eta}$. For a fixed time $u$, $(\mathcal{E}_\eta, \eta \in \{-1, 1\}^k)$ is a vector in $\mathbb{C}^{2^k}$ indexed by $\eta$. Then $\sigma_j$ is a permutation matrix acting on the vector space $\mathbb{C}^{2^k}$.

Consider the Brownian motion $b_1, b_2$ driving the SDE (137). Let $b^{(\eta)} = \eta_j i b_1 - b_2$. For $k = 1$ the equations (137) can be written as

$$d\mathcal{E}_\eta = \frac{iz \eta_1}{2} f_\beta \mathcal{E}_\eta du + \frac{\sigma_1 - 1}{2} \mathcal{E}_\eta db^{(\eta)}.$$ 

The general $k \geq 1$ case is described by the following proposition.

**Proposition 64.** Fix $k \geq 1$ and consider the operator

$$\Xi = \sum_{j=1}^{k} (\sigma_j - 1) \sum_{\eta_j = -1}^{k} (\sigma_\ell - 1),$$

This is a $2^k \times 2^k$ matrix acting on $\mathbb{C}^{2^k}$. Then we have

$$d\mathcal{E}_{\eta,\nu} = \frac{1}{2} \left( iz \cdot \eta f_\beta du + \Xi dt + \sum_{j=1}^{k} (\sigma_j - 1) db^{(\eta_j)} \right) \mathcal{E}_{\eta,\nu}. \quad (168)$$
For $\nu \in (-\infty, 0)$ the processes $E_{\eta,\nu}, \eta \in \{-1,1\}^k$ provide the unique strong solution of the system (168) on $[\nu, \infty)$ with initial condition 1 at time $\nu$. The processes $E_\eta$ can be obtained as the a.s. uniform on compact limits of $E_{\eta,\nu}$ as $\nu \to -\infty$.

Proof. The proof follows from Itô’s formula applied to the product $E_\eta$. For a fixed $\nu \in (-\infty, 0)$ the system (168) is a linear system, so it has a unique strong solution with a given initial condition. Corollary 50 implies the last statement.

For any $\nu \neq \infty$, the coefficients of the vector valued SDE (168) satisfy the Lipschitz conditions required by the standard existence and uniqueness theorem for SDEs, Theorem 5.2.1 in [24]. This implies that $u \to \int_\nu^u \sum_{j=1}^k (\sigma_j - 1) E_{\eta,\nu} db_j$ is an $L^2$ martingale. Hence for each $u \in [\nu, \infty)$ the expected value

$$r_{\eta,\nu}(u) := E E_{\eta,\nu}(u)$$

is finite, and it satisfies the differential equation system

$$r'_{\eta,\nu} = \frac{1}{2} (iz \cdot \eta f_\beta + \Xi) r_{\eta,\nu}$$

with initial condition $r_{\eta,\nu}(\nu) = 1$. Here we extend the definition $\Xi r_\eta = r_{\Xi \eta}$. Since the time dependent coefficients in the linear system (169) are bounded on compacts, there is a unique solution $r_{\eta,\nu}(z)$, which is analytic in $z$.

For $\eta = \pm(1, \ldots, 1)$, one of the sums in the expression for $\Xi$ vanishes, so $\Xi = 0$. We get $r' = \frac{1}{2} iz \cdot \eta f_\beta r$, and so

$$r_{\eta,\nu} = \exp \left(\frac{i z \cdot \eta}{2} \left( e^{\frac{\eta \nu}{4}} - e^{\frac{-\eta \nu}{4}} \right) \right), \quad \eta = \pm(1, \ldots, 1).$$

In particular, in this case we have $r_{\eta,\nu}(0) = \exp \left(\frac{i z \cdot \eta}{2} \left( 1 - e^{\frac{-\eta \nu}{4}} \right) \right)$.

**Proposition 65.** Assume that $1 + \beta/2 > k$, and let $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Then at time 0 we have

$$E \prod_{j=1}^k E(\lambda_j) = \prod_{j=1}^k e^{i \lambda_j/2}.$$

Proof. With $\eta = (1, \ldots, 1)$ we have $r_{\eta,\nu}(0) \to \prod_{j=1}^n e^{i \lambda_j/2}$ as $\nu \to -\infty$.

By Theorem 55 there is a constant $c, \epsilon > 0$ so that at time 0 we have

$$E[|E_\nu(\lambda)|^{k+\epsilon}] < c(1 + |\lambda|)^{2(k+\epsilon)^2/\beta}, \quad \lambda \in \mathbb{R}.$$

Hölder’s inequality implies that $\prod_{j=1}^k E_\nu(\lambda_j) \cdot \mathbb{1}$ are uniformly integrable as $\nu \to -\infty$. Since $\prod_{j=1}^k E_\nu(\lambda_j)$ converges to $\prod_{j=1}^k E(\lambda_j)$, the statement follows. □
With (167) we get
\[
E \hat{\zeta}(\lambda) = \frac{2}{i} \cos(\lambda/2), \quad \lambda \in \mathbb{R}.
\] (171)

Now consider the case \( n = 2, z_1 = z_2 = \lambda \in \mathbb{R} \). From (170) we get
\[
r_{(1,1),\nu} = r_{-(1,1),\nu}^{-1} = \exp \left( i \lambda \left( e^{\frac{\partial u}{\partial \tau}} - e^{\frac{\partial \nu}{\partial \tau}} \right) \right).
\]

Introduce
\[
r_* = r_{(1,-1),\nu} = r_{(-1,1),\nu} = E|\mathcal{E}_{\nu,u}(\lambda)|^2,
\]
by (169) we have
\[
r_*' = \frac{1}{2} (\sigma_1 - 1)(\sigma_2 - 1)r_{\theta,\nu} = -\cos \left( \lambda \left( e^{\frac{\partial u}{\partial \tau}} - e^{\frac{\partial \nu}{\partial \tau}} \right) \right) + r_*.
\]

This ODE can be solved to give
\[
r_*(u) = 1 + e^u \int_{\nu}^u e^{-s} \left( 1 - \cos \left( \lambda \left( e^{\frac{\partial s}{\partial \tau}} - e^{\frac{\partial \nu}{\partial \tau}} \right) \right) \right) \, ds.
\] (172)

If \( \beta > 2 \) then as \( \nu \to -\infty \) this function converges to
\[
1 + e^u \int_{-\infty}^u e^{-s} \left( 1 - \cos \left( \lambda e^{\frac{\partial s}{\partial \tau}} \right) \right) \, ds.
\]

By Theorem 58 if \( \beta > 2 \) then at time 0, \( E|\mathcal{E}_{\nu}(\lambda)|^{2+\epsilon} < c(1 + |\lambda|^2)^{2(2+\epsilon)/\beta} \) with an absolute constant \( c \) and \( \epsilon > 0 \). By uniform integrability,
\[
E[|\mathcal{E}(\lambda)|^2] = 1 + \int_{-\infty}^0 e^{-s} \left( 1 - \cos \left( \lambda e^{\frac{\partial s}{\partial \tau}} \right) \right) \, ds, \quad \lambda \in \mathbb{R}, \beta > 2.
\]

With a time-change and (167) we get
\[
2E[\hat{\zeta}(\lambda)^2] = E[|\mathcal{E}(\lambda)|^2] = 1 + \frac{4}{\beta} \int_0^1 t^{-4/\beta-1}(1 - \cos(\lambda t)) \, dt.
\] (173)

This can be written as a generalized hypergeometric function
\[
_1F_2 \left( -\frac{2}{\beta}; \frac{1}{2}, 1 - \frac{2}{\beta}; -\lambda^2/4 \right) = \sum_{k=0}^{\infty} \frac{(-2/\beta)_k}{(1/2)_k(1 - 2/\beta)_k} \frac{(-\lambda^2/4)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k}}{(2k)!(1 - \frac{\beta}{2} k)}, \quad (174)
\]
with \((x)_k = x(x+1)\cdots(x+k-1)\).

In the \( \beta = 2 \) case the integral (172) with \( u = 0 \) explodes as \(-\lambda^2 \nu/2 \) as \( \nu \to -\infty \). More precisely, we have
\[
\lim_{\nu \to -\infty} \left( E[|\mathcal{E}_{\nu}(\lambda)|^2] + \lambda^2 \nu/2 \right) = 1 - \frac{3}{2} \lambda^2 + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{(k-1)(2k)!} \lambda^{2k}.
\]
With the cosine integral \( Ci \times = - \int_{x}^{\infty} (\cos t) / t \, ds \) and the Euler constant \( \gamma \) we can write this as
\[
\lambda^2 Ci |\lambda| - \gamma \lambda^2 - \lambda^2 \log |\lambda| - \lambda \sin \lambda + \cos \lambda.
\]

Now consider the more general case \( n = 2, z_1 = \lambda_1, z_2 = \lambda_2 \) with \( \lambda_j \in \mathbb{R} \). Using the previous notation we have
\[
r_* = E \left[ \mathcal{E}_\nu(\lambda_1) \mathcal{E}_{\nu}^*(\lambda_2) \right]
\]
By (169) and (170) we get that \( r_* \) satisfies the ODE
\[
r_*' = \frac{1}{2} i (\lambda_1 - \lambda_2) f_\beta r_* + \frac{1}{2} (r_* + \bar{r}_*) - \cos \left( \frac{\lambda_1 + \lambda_2}{2} \left( e^{\frac{\nu}{4}} - e^{-\frac{\nu}{4}} \right) \right).
\]
Writing \( r_* = 1 + e^t a + ib \) with \( a, b \in \mathbb{R} \) this leads to
\[
a' = -\frac{\lambda_1 - \lambda_2}{2} e^{-t} f_\beta b + e^{-t} \left( 1 - \cos \left( \frac{\lambda_1 + \lambda_2}{2} \left( e^{\frac{\nu}{4}} - e^{-\frac{\nu}{4}} \right) \right) \right),
\]
\[
b' = e^t \frac{\lambda_1 - \lambda_2}{2} f_\beta a + \frac{\lambda_1 - \lambda_2}{2} f_\beta,
\]
with \( a(\nu) = b(\nu) = 0 \). This ODE can be directly solved using the integrating factor method to get a somewhat complicated integral expression in terms of Bessel functions. For \( \beta > 2 \) letting \( \nu \to -\infty \) we get the two point function \( E \left[ \mathcal{E}(\lambda_1) \mathcal{E}_{\nu}^*(\lambda_2) \right] \).

### 8.3 Moments of ratios of the structure function

We consider expectations of products of functions \( \mathcal{E}, \mathcal{E}^{-1} \). Let
\[
\mathcal{G}_t = \frac{\mathcal{E}_{t}^*}{\mathcal{E}_t},
\]
and for \( z_1, \ldots, z_k, \eta \in \{-1, 1\}^k \) fixed let
\[
d\mathcal{E}_t^\eta = \prod_{j=1}^k \mathcal{E}_t^{\eta_j}(z_j).
\]
For \( \nu \in (-\infty, 0) \) we define \( \mathcal{G}_{\nu,t}, \mathcal{E}_t^\eta_{\nu,t} \) similarly.

Itô’s formula together with (137) gives
\[
d\mathcal{E}^\eta = \frac{i}{2} z \cdot \eta f_\beta \mathcal{E}^\eta \, dt + \frac{1}{2} \mathcal{E}^\eta \sum_{j=1}^k \eta_j (\mathcal{G}(z_j) - 1)(idb_1 - db_2). \]
(177)

For \( \nu \in (-\infty, 0) \) the process \( \mathcal{E}^\eta_{\nu,t} \) satisfies (177) with initial condition \( \mathcal{E}^\eta_{\nu,0} = 1 \).
Proposition 66. Let \( z \in \mathbb{C}^k \), with \( \Im z_j < 0 \) for all \( j \), and let \( \eta \in \{-1, 1\}^k \). Then for \(-\infty < \nu < u\) we have

\[
E E_{\nu, u} = \exp \left( \frac{i}{2} z \cdot \eta (e^{\beta u/4} - e^{\beta \nu/4}) \right). \tag{178}
\]

Proof. By (138) of Corollary 50 we have \( |G_{\nu, t}(z)| \leq 1 \) for \( \Im z < 0 \). Hence we can use Lemma 74 of the Appendix for the SDE (177) for \( E_\eta \nu \). With \( r(u) = E E_{\nu, u}, \nu < u \) we get the equation

\[
r(u) = 1 + \frac{i}{2} z \eta \int_\nu^u f_\beta(s) r(s) \, ds.
\]

Solving the corresponding ODE gives (178). \( \square \)

8.4 Borodin-Strahov moment formulas for \( \zeta \)

Borodin and Strahov [4] compute

\[
\lim_{n \to \infty} E \prod_{j=1}^k p_n(z_j/\sqrt{n}) = \begin{cases} 
\prod_{j=1}^k a_j + i b_j & \text{if } \Im w_j > 0, 1 \leq j \leq n, \\
\frac{e^{i \sum_{j=1}^n z_j - w_j}}{e^{-i \sum_{j=1}^n z_j - w_j}} & \text{if } \Im w_j < 0, 1 \leq j \leq n,
\end{cases} \tag{179}
\]

for the characteristic polynomial \( p_n \) of the Gaussian beta ensemble for \( \beta = 1, 2, 4 \). The answer does not depend on the value of \( \beta \) if \( \beta = 1, 2, 4 \). Borodin and Strahov [4] pose the question whether this is true for all \( \beta > 0 \).

We will show that the analogous expectation does not depend on \( \beta \) for the stochastic zeta function. In the next section we also show this for the circular beta ensemble.

Chhaibi, Najnudel and Nikeghbali [8] show the formula analogous to (179) for the characteristic polynomial of Haar unitary matrices and for \( \zeta_2 \). Chhaibi, Hovhannisyan, Najnudel, Nikeghbali, and Rodgers [7] show that the normalized characteristic polynomial of the Gaussian unitary ensemble converges to \( \zeta_2 \).

We will need the following simple lemma.

Lemma 67. Suppose that \( a_j, b_j, c_j, d_j \in \mathbb{C}, 1 \leq j \leq k \). Let \( q \) be a Cauchy distributed random variable. Then

\[
E \prod_{j=1}^k \frac{a_j + q b_j}{c_j + q d_j} = \begin{cases} 
\prod_{j=1}^k \frac{a_j + i b_j}{c_j + i d_j} & \text{if } \Im c_j / d_j > 0 \text{ for all } 1 \leq j \leq k, \\
\prod_{j=1}^k \frac{a_j - i b_j}{c_j - i d_j} & \text{if } \Im c_j / d_j < 0 \text{ for all } 1 \leq j \leq k.
\end{cases}
\]

Proof. For the first case, set

\[
r(x) = \prod_{j=1}^k \frac{a_j + x b_j}{c_j + x d_j}.
\]
We compute

\[
\int_{-\infty}^{\infty} \frac{r(x)}{\pi(1+x^2)} \, dx = \lim_{r \to \infty} \int_{\gamma_r} \frac{r(z)}{\pi(1+z^2)} \, dz,
\]

where \(\gamma_r\) is the counterclockwise oriented curve constructed from the line segment \([-r, r]\) and the corresponding semicircle in the upper half plane. The function \(\frac{r(z)}{\pi(1+z^2)}\) has singularities at \(\pm i\) and \(-\frac{c}{d_j}\), and out of these only \(i\) is in the upper half plane. The residue of \(\frac{r(z)}{\pi(1+z^2)}\) at \(z = i\) is exactly \(\frac{r(i)}{2\pi i}\). This proves the first case, the second follows by conjugation.

Next, we show that Borodin-Strahov conjecture holds for \(\zeta_\beta\), Theorem 5 of the Introduction.

**Proof of Theorem 5** We first consider the first case, when \(\Im w_j < 0\). We first prove the appropriate statement for the approximate versions of \(\zeta\). Let \(\nu \neq -\infty\).

Recall that \(\zeta_\nu(z) = A_{\nu,0}(z) - qB_{\nu,0}(z)\) where \(q\) is a Cauchy distributed random variable independent of \(A, B\). By (138) of Corollary 50 we have

\[
|A_{\nu,u}(z) - iB_{\nu,u}(z)| \geq |A_{\nu,u}(z) + iB_{\nu,u}(z)|, \quad \text{for } \Im z < 0,
\]

which implies \(\Im A_{\nu,0}(z) < 0\). The product \(\prod_{j=1}^{k} \left| \frac{\zeta_\nu(z_j)}{\zeta_\nu(w_j)} \right|\) has finite expectation by Proposition 62. Since \(q\) is independent of \(A, B\), Lemma 62 implies

\[
E \left[ \prod_{j=1}^{k} \frac{\zeta_\nu(z_j)}{\zeta_\nu(w_j)} |A, B| \right] = \prod_{j=1}^{k} \frac{A_{\nu,0}(z_j) - iB_{\nu,0}(z_j)}{A_{\nu,0}(w_j) - iB_{\nu,0}(w_j)} = \prod_{j=1}^{k} \frac{\xi_{\nu,0}(z_j)}{\xi_{\nu,0}(w_j)}, \quad \text{if } \Im w_j < 0 \text{ for all } j,
\]

and that the right hand side has finite expectation.

Assume first that we have \(\Im z_j < 0, \Im w_j < 0\) for all \(j\). Then by Proposition 66 we have

\[
E \prod_{j=1}^{k} \frac{\zeta_\nu(z_j)}{\zeta_\nu(w_j)} = E \prod_{j=1}^{k} \frac{\xi_{\nu,0}(z_j)}{\xi_{\nu,0}(w_j)} = \exp \left( \frac{i}{2} \sum_{k=1}^{n} (z_k - w_k)(1 - e^{\beta \nu/4}) \right).
\]

Proposition 62 shows that the products on the left are uniformly integrable as \(\nu \to -\infty\). This implies the statement of the theorem in the case when \(\Im z_j < 0, \Im w_j < 0\) for all \(j\).

The expected value of the product of the ratios is an entire function in each \(z_j\) variable by Proposition 63. This extends the claim to \(z_j \in \mathbb{C}\), and proves the first case of the theorem. The second case follows by conjugation. \(\square\)

**Remark 68.** If one could evaluate the function

\[
r_{\beta,n}(z_1, \ldots, z_k, w_1, \ldots, w_k) = E \prod_{j=1}^{k} \frac{\zeta(z_j)}{\zeta(w_j)} \quad (180)
\]

63
for all choices of \( z_j, w_j \in \mathbb{C} \) then this would lead to the joint \( n \)-point correlation functions of the Sine_\( \beta \) process. The \( n \)-point correlation function \( \rho_{\beta,n} : \mathbb{R}^n \to \mathbb{R} \) would be given as

\[
\rho_{\beta,n}(\lambda_1, \ldots, \lambda_n) = \left[ \frac{\partial^n}{\partial z_1 \ldots \partial z_n} r_{\beta,n}(z_1, \ldots, z_k, w_1, \ldots, w_k) \right]_{w_1=\lambda_1, \ldots, w_n=\lambda_n} \tag{181}
\]

where \([f(\cdot)]_x = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i}(f(x - i\varepsilon) - f(x + i\varepsilon))\), see [4].

Theorem 5 and (181) give \( \rho_{\beta,1}(\lambda) = \frac{1}{2\pi} \), the intensity of the Sine_\( \beta \) process.

### 8.5 Borodin-Strahov moment formulas for the circular beta ensemble

Let \( \beta > 0 \) and let \( p_n(z) \) be the characteristic polynomial of a size \( n \) circular beta ensemble as in (100). The following theorem shows that the Borodin-Strahov moment formulas hold for this model even before taking the limit.

**Theorem 69** (Borodin-Strahov moment conjecture, circular beta case).

\[
E \prod_{j=1}^{\ell} \frac{p_n(e^{iz_j/n}) e^{iz_j/2}}{p_n(e^{iw_j/n}) e^{iw_j/2}} = \begin{cases} 
& e^{i\sum_{j=1}^{\ell} \frac{z_j-w_j}{2}} \quad \text{if } \Im w_j < 0, 1 \leq j \leq \ell, \\
& e^{-i\sum_{j=1}^{\ell} \frac{z_j-w_j}{2}} \quad \text{if } \Im w_j > 0, 1 \leq j \leq \ell.
\end{cases} \tag{182}
\]

**Proof.** Consider the random discrete measure \( \mu_{n,\beta} \) introduced in the proof of Theorem 41. The support of \( \mu_{n,\beta} \) has circular beta distribution, and the weights are given by an independent Dirichlet distribution with parameter \((\beta/2, \ldots, \beta/2)\).

Killip and Nenciu [16], see also [5], identify the joint distribution of the modified Verblunsky coefficients \( \gamma_0, \ldots, \gamma_{n-1} \) corresponding to \( \mu_{n,\beta} \). They show that \( \gamma_k \) are independent and rotational invariant, and \( |\gamma_k|^2 \) has Beta\((1, \frac{\beta}{2}(n-k-1))\) distribution. In particular, \( \gamma_{n-1} \) is uniform on \( \{|z|=1\} \).

We will use the notation introduced in Section 3. Define \( E_k(z), E_k^*(z) \) for \( 0 \leq k \leq n, z \in \mathbb{C} \) as

\[
\begin{pmatrix} E_k(z) \\ E_k^*(z) \end{pmatrix} = e^{-\frac{i\pi}{2n}} \begin{pmatrix} \psi_k(e^{iz/n}) \\ \psi_k^*(e^{iz/n}) \end{pmatrix} = UX_k H_k.
\]

By (84) we have

\[
\begin{pmatrix} E_{k+1} \\ E_{k+1}^* \end{pmatrix} = A_k \begin{pmatrix} e^{\frac{i\pi}{2n}} & 0 \\ 0 & e^{-\frac{i\pi}{2n}} \end{pmatrix} \begin{pmatrix} E_k \\ E_k^* \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\gamma_k} & -\gamma_k \\ \frac{1}{1-\gamma_k} & \frac{1-\gamma_k}{1-\gamma_k} \end{pmatrix} \begin{pmatrix} e^{\frac{i\pi}{2n}} E_k \\ e^{-\frac{i\pi}{2n}} E_k^* \end{pmatrix},
\]

which leads to

\[
\frac{E_{k+1}(z)}{E_{k+1}(w)} = e^{\frac{i(z-w)}{2n}} \frac{E_k(z) - \gamma_k e^{-\frac{i\pi}{n}} E_k^*(z)}{E_k(w) - \gamma_k e^{-\frac{i\pi}{n}} E_k^*(w)}. \tag{183}
\]
If \( \Im w < 0 \) then \(|[1, -i] H_k(w)| \geq |[1, i] H_k(w)|\) by (62), which implies \(|E_k(w)| \geq |E_k^*(w)|\) as well.

Fix \( w_j, z_j \in \mathbb{C} \) with \( 1 \leq j \leq \ell \), and assume that \( \Im w_j < 0 \) for all \( j \). The random variable \( \gamma_k \) is independent of \( E_j, E_j^*, j \leq k \), and it has rotationally invariant distribution. Hence with \( Q_k := \prod_{j=1}^{k} \frac{\mathcal{E}_j(z_j)}{\mathcal{E}_j(w_j)} \) we have

\[
E \left[ Q_{k+1} \left| Q_j, \gamma_k = r \right. \right] = e^{\frac{i}{2n} \sum_{j=1}^{k} (z_j - w_j)} \frac{1}{2\pi} \int_{0}^{2\pi} \prod_{j=1}^{k} \frac{\mathcal{E}_j(z_j) - re^{it}e^{-iz_j} \mathcal{E}_j^*(z_j)}{\mathcal{E}_j(w_j) - re^{it}e^{-iw_j} \mathcal{E}_j^*(w_j)} dt.
\]

We have \(|E_k(w_j)| \geq |E_k^*(w_j)|\) from \( \Im w_j < 0 \), and thus by Lemma 71 below we get

\[
E \left[ Q_{k+1} \left| Q_j, \gamma_k = r \right. \right] = e^{\frac{i}{2n} \sum_{j=1}^{k} (z_j - w_j)} Q_k.
\]

Taking expectations and using \( Q_0 = 1 \) we get \( E Q_n = e^{\frac{i}{2n} \sum_{j=1}^{n} (z_j - w_j)} \). Since

\[
Q_n = \prod_{j=1}^{k} \frac{\mathcal{E}_j(z_j)}{\mathcal{E}_j(w_j)} = \prod_{j=1}^{k} \frac{\psi_n(e^{iz_j}/n)e^{-iz_j/2}}{\psi_n(e^{iw_j}/n)e^{-iw_j/2}},
\]

and \( \psi_n = p_n \), the first case of (182) follows. The second case follows after conjugation. \(\square\)

**Remark 70.** The same proof works for random measures supported on \( n \) points on the unit circle, with Verblunsky coefficients \( \gamma_0, \ldots, \gamma_{n-1} \) satisfying the following condition. Given \(|\gamma_j|, 0 \leq j \leq n - 1 \) the arguments of \( \gamma_j \) are independent and uniformly distributed on \([0, 2\pi]\).

The following lemma is related to Lemma 67 through a Cayley transform.

**Lemma 71.** Suppose that \( a_j, b_j, c_j, d_j \in \mathbb{C} \) with \( c_j \neq 0 \) and \(|d_j| < |c_j|\). Then

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \prod_{j=1}^{k} \frac{a_j - e^{it}b_j}{c_j - e^{it}d_j} dt = \prod_{j=1}^{k} \frac{a_j}{c_j}.
\]

**Proof.** We can rewrite the left hand side as a complex line integral on the unit circle as

\[
\frac{1}{2\pi i} \oint_{\|} \prod_{j=1}^{k} \frac{a_j - z b_j}{c_j - z d_j} dz.
\]

The integrand has poles at 0 and at \( \frac{c_j}{d_j} \), \( 1 \leq j \leq k \). Because of our conditions the only pole inside the unit circle is 0, and the residue is \( \prod_{j} \frac{a_j}{c_j} \). \(\square\)
A Law of iterated logarithm for Brownian integrals

**Theorem 72.** For every $a > 2, b < 1/2$ there is a constant $c$ so that

$$P\left(\sup_{t>0} B(t)^2/t - a \log(1 + |\log t|) > y\right) \leq ce^{-by} \quad \text{for all } y \geq 0. \quad (184)$$

This is an effective small and large-time version of the upper bound in the law of iterated logarithm. By setting $t = 1$ inside the $\sup$ we get $B(1)^2$, which shows that the rate of the exponential decay cannot be more than $1/2$. The lower bound $2$ on the parameter $a$ is sharp by the usual law of iterated logarithm.

**Proof.** Let $f, g : (-1, \infty) \to \mathbb{R}$ be non-decreasing functions and $\varepsilon \in (0, 1)$. If $f(t) > g(t)$ for $t \geq 0$ then $f(s) \geq g(s - \varepsilon)$ for $s \in [t, t + \varepsilon]$. Hence

$$1(f(t) \geq g(t) \text{ for some } t \geq 0) \leq \frac{1}{\varepsilon} \int_0^\infty 1(f(s) \geq g(s - \varepsilon))ds. \quad (185)$$

Let $\bar{B}_t = \max_{0 \leq s \leq t} |B_s|$, and apply (185) to $f(t) = \bar{B}(e^t)^2$ and $g(t) = e^{t(y + a \log(1 + t))}$. The expectation of the resulting inequality bounds $P(\sup_{t \geq 1} B_t^2/t - a \log(1 + \log t) > y)$ above as

$$P(\bar{B}(e^t)^2 \geq g(t) \text{ for some } t \geq 0) \leq \frac{1}{\varepsilon} \int_0^\infty P(\bar{B}(e^s)^2 \geq e^{s-\varepsilon}(y + a \log(1 + s - \varepsilon)))ds. \quad (186)$$

Since $\max_{0 \leq r \leq 1} B(r)$ is distributed as $|B(1)|$, union and Gaussian tail bounds yield

$$P(\bar{B}(e^s)^2 \geq e^sx) = P(\bar{B}(1)^2 \geq x) \leq 2P(\bar{B}(1)^2 \geq x) \leq 2e^{-x/2}.$$ 

Thus the right hand side of (186) is bounded above by

$$\frac{2}{\varepsilon} \int_0^\infty e^{-\varepsilon y + a \log(1 + s - \varepsilon)/2}ds = \frac{4(1 - \varepsilon)1-e^{-\varepsilon a/2}e^{-\varepsilon y/2}}{(ae^{-\varepsilon} - 2)\varepsilon}.$$ 

To make the last step valid and to get the required bound we need $ae^{-\varepsilon} > 2$ and $e^{-\varepsilon}/2 > b$, so we choose $\varepsilon < \min(1, \log(a/2), \log(1/(2b)))$. Time inversion $B_t \to tB(1/t)$ gives the same bound for the supremum for $0 < t \leq 1$, from which (184) follows. \hfill \blacksquare

We apply Theorem 72 to estimate the growth of Brownian integrals.

**Proposition 73.** Suppose that $B$ is two-sided Brownian motion and $x_u, u \leq 0$ is adapted to the filtration generated by its increments. Assume further that there is a random variable $C$ and a constant $a > 0$ so that $|X_u| \leq Ce^{au}$ for $u \leq 0$. Then

$$|\int_{-\infty}^u X_u dB| \leq \frac{2C}{\sqrt{a}} e^{au} (Z + \log(1 + 2a|u|) + |\log a| + \log(1 + 2|\log C|)) \quad \text{for all } u \leq 0,$$

so the left hand side is well defined. With an absolute constant $c$, the random variable $Z$ satisfies

$$P(Z > y) \leq ce^{-uy}, \quad y \geq 0. \quad (187)$$
Proof. We have
\[
\int_{-\infty}^{u} X_s^2 ds \leq \int_{-\infty}^{u} C^2 e^{2a u} ds = \frac{C^2}{2a} e^{2a u} < \infty,
\]
so the process \( M_u = \int_{-\infty}^{u} X_u dB \) is well defined. Moreover,
\[
[M]_u \leq \frac{C^2}{2a} e^{2a u}, \quad \text{for all } u \leq 0. \tag{188}
\]
By the Dubins-Schwarz theorem there is a Brownian motion \( W(x) \), \( x \geq 0 \) so that \( M_u = W([M]_u) \). Let
\[
Z = \frac{1}{3} \sup_{x>0} (W(x)^2/x - 3 \log(1+|\log x|)).
\]
By Theorem 72 this random variable satisfies (187), and for \( u \leq 0 \) we have
\[
M_u^2 \leq 3[M]_u Z + 3[M]_u (1 + \log(1 + |\log[M]_u|)).
\]
The function \( x(1 + \log(1 + |\log x|)) \) is increasing, so by (188) we also get
\[
M_u^2 \leq \frac{3C^2}{2a} e^{2a u} (Z + 1 + \log(1 + \log C^2 e^{2a u})). \tag{189}
\]
For \( x, y > 0 \) we have \( |\log(xy)| \leq |\log x| + |\log y|, \) \( \log(1 + x + y) \leq \log(1 + x) + \log(1 + y), \) and \( \log(1 + x) \leq x \). Using these bounds repeatedly we get
\[
\log(1 + \log \frac{C^2}{2a} e^{2a u}) \leq \log(1 + 2|\log C| + |\log 2a| + 2a|u|)
\leq \log(1 + 2a|u|) + \log 2 + |\log a| + \log(1 + 2|\log C|).
\]
Take square roots in (189) and use the inequality \( \sqrt{1+y} \leq 1 + y \) for \( y > 0 \). For \( q \) fixed, \( Z + q \) satisfies the same tail bound as \( Z \) with a different \( c \). This implies the claim. \( \square \)

B  Moment bounds for an almost linear SDE

The following lemma is used when calculating moments of ratios of \( \zeta \).

Lemma 74. Consider the diffusion
\[
dX = Y dt + Z dW, \tag{190}
\]
with \( |Y| \leq a|X|, |Z| \leq b|X|, \) and \( a, b \) and \( E|X_0^2| \) finite. Then for any \( t \geq 0 \)
\[
E|X_t|^2 \leq E|X_0|^2 e^{(2a^2+2b^2)t}, \quad EX_t = EX_0 + \int_0^t EY_s ds.
\]
In particular, if \( Y = \eta X \) for \( \eta \in \mathbb{C} \) then \( EX_t = EX_0 e^{\eta t} \).
Proof. Let $\tau_c$ be the first time $|X_t| \geq c$. By Itô’s formula
\[
\int_0^{\tau_c \wedge s} d|X|^2 = \int_0^{\tau_c \wedge s} 2\Re(\overline{X} Z dW) + (2 \Re X \bar{Y} + 2 |Z|^2) dt.
\]
In the interval $[0, \tau_c \wedge s]$ the quadratic variation is bounded, so
\[
M_t = |X_{t \wedge \tau_c}|^2 - \int_0^{t \wedge \tau_c} (2 \Re X_s \bar{Y}_s + 2 Z_s^2) ds
\]
is a martingale, and
\[
E|X_{t \wedge \tau_c}^2| = E \int_0^{t \wedge \tau_c} (2 \Re X_s \bar{Y}_s + 2 Z_s^2) ds \leq (2a + 2b^2) E \int_0^{t \wedge \tau_c} |X|^2 ds. \tag{191}
\]
Since
\[
\int_0^{t \wedge \tau_c} |X|^2 ds = \int_0^t |X|^2 1(s \leq \tau_c) ds \leq \int_0^t |X|^2_{s \wedge \tau_c} ds
\]
we have $E|X_{t \wedge \tau_c}^2| \leq E|X_0^2|e^{(2a+2b^2)t}$ by Gronwall’s inequality. Fatou’s lemma gives
\[
E|X_t^2| \leq E|X_0^2|e^{(2a+2b^2)t}
\]
as well. From this we see that the quadratic variation of $\int_0^t Z dW$ has finite expectation, so it is a martingale. Thus we can take expectations in (190) which gives (191). The last claim follows from solving the equation $EX_t = EX_0 + \eta \int_0^t EX_s ds$.

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Benedek Valkó
Department of Mathematics
University of Wisconsin - Madison
Madison, WI 53706, USA
valko@math.wisc.edu

Bálint Virág
Departments of Mathematics and Statistics
University of Toronto
Toronto ON M5S 2E4, Canada
balint@math.toronto.edu