Singular Fano threefolds of genus 12

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Abstract. We study singular Fano threefolds of type $V_{22}$.
Bibliography: 31 titles.

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§ 1. Introduction

All Fano varieties in this paper are assumed to be three-dimensional, to have at worst terminal Gorenstein singularities and to be defined over an algebraically closed field $k$ of characteristic 0. A Fano threefold $X$ is said to be of the main series if the canonical class $K_X$ generates the Picard group $\text{Pic}(X)$. For a Fano threefold $X$ of the main series we can write $-K_X^3 = 2g(X) - 2$, where $g(X)$ is an integer which is called the genus of $X$. It is known that $g(X)$ takes the following values: $2, 3, \ldots, 10, 12$ (see [1] and [2]). Nonsingular Fano threefolds of the main series and genus 12 were described by Iskovskikh [3] and Mukai [4]. In this paper we study singular Fano threefolds of genus 12. An important invariant of a Fano variety is $r(X) := \text{rk Cl}(X)$, the rank of the Weil divisor class group. The case $r(X) = 1$ is already known:

Theorem 1.1 (see [5] and [6]). Let $X$ be a $\mathbb{Q}$-factorial Fano threefold of the main series with $g(X) = 12$. Then $X$ is nonsingular.

In § 5 we prove the following result.

Theorem 1.2. Let $X$ be a Fano threefold of the main series with $g(X) = 12$. Assume that $X$ satisfies one of the following equivalent conditions:

(i) the singular locus of $X$ consists of a single ordinary double point,
(ii) $r(X) = 2$.

Then $X$ is the midpoint of a Sarkisov link, that is, it fits in the following commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\chi} & Y^+ \\
\pi & \downarrow & \pi^+ \\
Z & \xleftarrow{f} & X & \xrightarrow{f^+} & Z^+
\end{array}
\]

(1.1)

where $\pi$ and $\pi^+$ are small $\mathbb{Q}$-factorializations and $\chi$ is a flop. The morphisms $f$ and $f^+$ are extremal Mori contractions described in the following table, where $Q$ is

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a nonsingular quadric in $\mathbb{P}^4$ and $V_5 \subset \mathbb{P}^6$ is a nonsingular del Pezzo threefold of degree 5:

| No | $Z$       | $f$                                                                 | $Z^+$                          | $f^+$                                                                 |
|----|-----------|----------------------------------------------------------------------|--------------------------------|------------------------------------------------------------------------|
| (I)| $\mathbb{P}^3$ | The blowup of a nonsingular rational quintic curve $B \subset \mathbb{P}^3$ which is not contained in a quadric | $\mathbb{P}^3$ | The blowup of a nonsingular rational quintic curve which is not contained in a quadric |
| (II)| $Q$ | The blowup of a nonsingular rational quintic curve $B \subset Q$ which is not contained in a hyperplane section | $\mathbb{P}^2$ | A conic bundle with discriminant curve of degree 3 |
| (III)| $V_5$ | The blowup of a nonsingular rational quartic curve $B \subset V_5$ | $\mathbb{P}^1$ | A del Pezzo fibration of degree 6 |
| (IV)| $\mathbb{P}^2$ | $\mathbb{P}^2(\mathcal{E}) \to \mathbb{P}^2$, where $\mathcal{E}$ is a stable rank-2 vector bundle on $\mathbb{P}^2$ with $c_1 = 0$ and $c_2 = 4$ | $\mathbb{P}^1$ | A del Pezzo fibration of degree 5 |

Note that in the general case the rank of the group $\text{Cl}(X)$ for a Fano threefold of genus 12 can be quite large: in Example 6.5 we have $r(X) = 8$ and, moreover, $r(X) = 10$ for some toric varieties. Proposition 8.1 proves that $r(X) \leq 10$ in the case where $X$ does not contain planes. A sharp bound of the rank of the group $\text{Cl}(X)$ is not known.

In principle, the proof of Theorem 1.2 can be extracted from the series of papers [7]–[10]. However, these all consider the case when the $Q$-factorializations in (1.1) are nonsingular. We prefer to give a relatively short self-contained proof in the general case$^1$. Note that all the varieties in Theorem 1.2 admit a moving decomposition (see [5], §7). Thus we cannot expect Mukai’s techniques (see [4] and [5]) to work in our case.

In §6 and §7 we study cases (I)–(IV) in detail. In particular, we show that all these cases occur and describe general members of the families. According to Mukai the moduli space $\mathcal{M}_{\text{Fano}}^{12}$ of nonsingular Fano threefolds is 6-dimensional and birational to the moduli space $\mathcal{M}_3$ of curves of genus 3. Our four families (I)–(IV) are parametrized by varieties $\mathcal{M}_{\text{Fano}}^{(I)}, \ldots, \mathcal{M}_{\text{Fano}}^{(IV)}$ of dimension 5.

The original motivation for this work was to study $G$-varieties and finite subgroups of the Cremona group (see [11] and [12]). An algebraic variety is said to be a $G$-variety, if it is equipped with an action $G \to \text{Aut}_k(X)$ of a finite group $G$. A projective $G$-variety $X$ is called a $G$-Fano variety if $X$ has at worst terminal singularities, $-K_X$ is an ample Cartier divisor and the rank of the invariant part of the Weil divisor class group, $\text{Cl}(X)^G$, equals 1. $G$-Fano threefolds which are not of the main series were classified in [13] and [14].

The main result of this paper is the following.

**Theorem 1.3.** Let $X$ be a $G$-Fano threefold of the main series with $g(X) = 12$. Then either $X$ is nonsingular or its singular locus consists of a single ordinary double point. In the latter case $X$ is the anticanonical image of the blowup of $\mathbb{P}^3$ along a nonsingular rational curve of degree 5 (see Theorem 1.2, (I)).

$^1$The variety (IV) was erroneously omitted in [7].
In particular, in case (I) diagram (1.1) gives a well-known Cremona map $\mathbb{P}^3 \rightarrow \mathbb{P}^3$ of degree $(3, 3)$ (see [15], Ch. VIII, Example 8, p. 185).

The definition of $G$-varieties ($G$-Fano varieties, etc.) can be adapted to the arithmetic case: the variety $X$ is defined over a nonclosed field $k$ and $G$ is the absolute Galois group acting on $\overline{X} = X \otimes \overline{k}$ through the second factor. With minor modifications our result can be applied in this situation.

Another motivation for studying singular Fano threefolds comes from some questions about affine and Moishezon non-projective varieties (cf. [16]). In §6 we construct a new compactification of the affine space $\mathbb{A}^3$ as a variety of type (III) (see Theorem 6.6).

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§2. Preliminaries

We use the following standard notation:

- $\text{Cl}(X)$ is the Weil divisor class group of a variety $X$;
- $r(X) = \text{rk} \text{Cl}(X)$;
- $\text{Pic}(X)$ is the Picard group of a variety $X$;
- $\rho(X)$ is the Picard number of $X$;
- $Q = Q_d \subset \mathbb{P}^{d+1}$ is a nonsingular quadric of dimension $d$;
- $Q' = Q'_d \subset \mathbb{P}^{d+1}$ is a singular irreducible quadric of dimension $d$;
- $V_5 \subset \mathbb{P}^6$ is a nonsingular del Pezzo threefold of degree 5;
- $\mathcal{N}_{C/V}$ is the normal bundle of $C$ in $V$;
- $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ is the rational ruled surface (Hirzebruch surface);
- $\iota(X)$ is the Fano index of a (generalized) Fano variety (see [1], §2.1);
- $\langle B \rangle \subset \mathbb{P}^n$ is the linear span of $B \subset \mathbb{P}^n$.

A contraction is a morphism with connected fibres of normal varieties.

Throughout this paper all (generalized) Fano varieties are assumed to have at worst terminal Gorenstein singularities. A Fano threefold $X$ of Fano index 2 is called a del Pezzo threefold (for instance, see [1] and [13]).

We will use the following very useful observation systematically.

**Proposition 2.1** (see [17], Lemma 5.1). Let $U$ be a three-dimensional variety with terminal Gorenstein singularities. Then any $\mathbb{Q}$-Cartier divisor on $U$ is Cartier.

**Proposition 2.2** (see [17], Corollary 4.5). Let $U$ be a three-dimensional variety with terminal singularities. Then there exists a small morphism $\pi: \widetilde{U} \rightarrow U$ such that $\widetilde{U}$ has only terminal $\mathbb{Q}$-factorial singularities and $K_{\widetilde{U}} = \pi^* K_U$.

If in the above notation the variety $U$ is Gorenstein, then $\widetilde{U}$ has only terminal factorial singularities. In this case we say that $\pi$ is a factorialization of $U$.

The classification of extremal contractions of terminal Gorenstein threefolds is almost the same as in the smooth case:
Theorem 2.3 (see [18]). Let $V$ be a three-dimensional variety with terminal factorial singularities and let $f: V \to W$ be an extremal Mori contraction with $\dim W > 0$. Then one of the following holds:

(e$_1$) $f$ is birational and contracts a surface $E$ to an irreducible curve $B$. Then $B$ has only locally planar singularities and is contained in the nonsingular locus of $W$. The contraction $f$ is the blowup of the ideal of the curve $B \subset W$ and the variety $W$ also has only terminal factorial singularities.

(e$_2$)–(e$_5$) $f$ is birational and contracts a surface $E$ to a point $P$. Then $f$ is the blowup of the maximal ideal of $P \in W$. The following cases can occur:

| Case | $f(E)$ | $E$ | $\Theta_E(E)$ | $\alpha_E$ | $\delta$ | $K_V^2 \cdot E$ |
|------|--------|-----|---------------|------------|---------|---------------|
| (e$_2$) | nonsingular | $\mathbb{P}^2$ | $\Theta(-1)$ | 2 | 8 | 4 |
| (e$_3$), (e$_4$) | cA-point | $Q_2$ or $Q_2^t$ | $\Theta(-1)$ | 1 | 2 | 2 |
| (e$_5$) | $\frac{1}{2}(1,1,1)$ | $\mathbb{P}^2$ | $\Theta(-2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |

where $\delta := (-K_W)^3 - (-K_V)^3$ and $\alpha_E$ is the discrepancy of $E$.

(d) $W$ is a nonsingular curve and $f$ is a del Pezzo fibration.
(c) $W$ is a nonsingular surface and $f$ is a conic bundle.

2.1. In case (c) there exists a reduced curve $\Delta_f \subset W$ such that $f$ is a smooth morphism over $W \setminus \Delta_f$ and any fibre $V_w := f^{-1}(w)$ over $w \in \Delta_f$ is a degenerate (nonreduced or reducible) conic. We say that $\Delta_f$ is the discriminant curve of $f$. If $\Delta_f = \emptyset$, then $f$ is a $\mathbb{P}^1$-bundle, in particular, $V$ is nonsingular (for instance, see [6], Proposition 5.2). Moreover, if the surface $W$ is rational, then this bundle is locally trivial.

2.2. In case (d) a general fibre $V_\nu$ is a nonsingular del Pezzo surface of degree $d = K_{V_\nu}^2$ with $1 \leq d \leq 9$, $d \neq 7$. If $d = 9$, then $f$ is a locally trivial $\mathbb{P}^2$-bundle. If $d = 8$, then a general fibre $V_\nu$ is a nonsingular quadric.

2.3. There exists the following natural exact sequence

$$0 \to \text{Pic}(W) \xrightarrow{f^*} \text{Pic}(V) \xrightarrow{l} \mathbb{Z},$$

where $l$ is any curve contracted by $f$. To describe the cokernel of the right-hand map we set

$$\mu_f := \min \{-K_V \cdot l \mid l \subset V \text{ is a curve contracted by } f\}. \quad (2.2)$$

Then $\mu_f = 1$ apart from in the following cases:

- $\mu_f = 2$: types (e$_2$), (c) with $\Delta_f = \emptyset$ and type (d) with $K_{V_\nu}^2 = 8$;
- $\mu_f = 3$: type (d) with $K_{V_\nu}^2 = 9$.

Lemma 2.4 (see [19], Lemma 3.1 and [6], Proposition 5.1). Let $f: V \to W$ be the blowup of a reduced (possibly, reducible) locally planar curve $B$ (as in case (e$_1$) in Theorem 2.3). Then

$$(-K_V)^3 = (-K_W)^3 + 2K_W \cdot B + 2p_a(B) - 2,$$

$$(-K_V)^2 \cdot E = -K_W \cdot B - 2p_a(B) + 2,$$

$$(-K_V) \cdot E^2 = 2p_a(B) - 2. \quad (2.3)$$
Therefore,
\[(−K_W)^3 − (−K_V)^3 = 2K_V^2 · E + 2p_a(B) − 2.\]  (2.4)

**Lemma 2.5.** Let \( V \) be a three-dimensional projective variety with terminal \( \mathbb{Q} \)-factorial singularities such that the divisor \( −K_V \) is nef. Let \( f: V → W \) be a birational contraction such that \( −K_W \) is a nef \( \mathbb{Q} \)-Cartier divisor. Let \( S ⊂ V \) be an irreducible surface which is not \( f \)-exceptional. Then
\[ K_V^2 · S ≤ K_W^2 · f(S). \]

**Proof.** Write
\[ K_V ∼_Q f^∗K_W + \sum \alpha_i E_i, \]
where \( E_i \)'s are prime exceptional divisors and \( \alpha_i ∈ \mathbb{Q} \). Since \( −K_V \) is \( f \)-nef, we have \( \alpha_i > 0 \). Set \( E = \sum \alpha_i E_i \). Then
\[
K_V^2 · S = K_V · (f^∗K_W + E) · S ≤ K_V · f^∗K_W · S
= (f^∗K_W + E) · f^∗K_W · S ≤ (f^∗K_W)^2 · S = K_W^2 · f(S).
\]

**Definition 2.6.** A **generalized Fano variety** is a projective variety \( X \) with terminal Gorenstein singularities such that the anticanonical divisor \( −K_X \) is nef and big, and the natural morphism
\[ Φ: X → \overline{X} := \text{Proj} \bigoplus_{n ≥ 0} H^0(X, −nK_X) \]
to the anticanonical model contracts no divisors.

Note that in this situation \( \overline{X} \) is a Fano threefold (with terminal Gorenstein singularities). Conversely, if \( \overline{X} \) is a Fano threefold (as above) and \( Φ: X → \overline{X} \) is its small factorialization (see Proposition 2.2), then \( X \) is a generalized Fano threefold.

**2.4.** Let \( X \) be a generalized Fano threefold. Put \( g(X) := −K_X^3/2 + 1 \). By the Riemann-Roch formula and the Kawamata-Viehweg vanishing theorem
\[ \dim |−K_X| = g(X) + 1. \]

Hence \( g(X) \) is an integer. It is called the **genus of** \( X \). The Picard group \( \text{Pic}(X) \) and the Weil divisor class group \( \text{Cl}(X) \) are finitely generated and torsion free (see [1], Proposition 2.1.2). Moreover, there exists a natural embedding \( \text{Pic}(X) ⊆ \text{Cl}(X) \) as a primitive sublattice (see Proposition 2.1). The **Fano index** of \( X \) is the maximal integer \( i = i(X) \) such that \( K_X \) is divisible by \( i \) in \( \text{Pic}(X) \). The **degree** of a surface \( S \) in a generalized Fano variety \( X \) is the degree with respect to the anticanonical divisor, that is, the (positive) number \( (−K_X)^2 · S \). We say that a surface \( S ⊂ X \) is a **plane** if its degree is 1. Note that for a Fano threefold \( X \) of the main series with \( g(X) ≥ 4 \) the linear system \( |−K_X| \) is base point free (see [20]). Hence in this situation any surface of degree 1 is isomorphic to \( \mathbb{P}^2 \), so that it is a plane in the usual sense.
Lemma 2.7. Let $\mathcal{E}$ be a rank-2 vector bundle over a nonsingular surface $Z$, let $Y := \mathbb{P}_Z(\mathcal{E})$, and let $M$ be the tautological divisor on $Y$. Then the following relations hold:

\[-K_Y \sim 2M + f^*(-K_Z - c_1(\mathcal{E})), \tag{2.5}\]

\[M^2 = M \cdot f^*c_1(\mathcal{E}) - f^*c_2(\mathcal{E}), \quad M^3 = c_1(\mathcal{E})^2 - c_2(\mathcal{E}), \tag{2.6}\]

\[-K_Y^3 = 6K_Z^2 + 2c_1(\mathcal{E})^2 - 8c_2(\mathcal{E}). \tag{2.7}\]

The proof uses the relative Euler exact sequence and Hirsch’s formula.

§ 3. The minimal model program on generalized Fano threefolds

Our proof of Theorem 1.3 makes key use of the following result.

Theorem 3.1 (see [6]). Let $X$ be a $G$-Fano threefold of the main series with $g(X) \geq 6$. Then $X$ contains no planes.

It turns out that the absence of planes is important for applying the minimal model program (MMP) in our situation. We describe the steps of the MMP explicitly on generalized Fano threefolds. This technique was developed in [21] and [19].

Notation 3.2. Let $X$ be a Fano threefold. Assume that $X$ contains no planes and that $r(X) > 1$. Let $\pi: \tilde{X} \to X$ be a small factorialization (see Proposition 2.2). Then

\[K_{\tilde{X}} = \pi^*K_X,\]

and so $\tilde{X}$ is a generalized Fano threefold with $\rho(\tilde{X}) = r(X) > 1$.

The following important lemma shows that the class of generalized $\mathbb{Q}$-factorial Fano threefolds that do not contain planes is closed under the MMP.

Lemma 3.3 (see [21] and [19]). Let $V$ be a generalized $\mathbb{Q}$-factorial Fano threefold. Assume that $V$ contains no planes. Let $\varphi: V \rightarrow W$ be a divisorial Mori contraction or a flop. Then $W$ is also a generalized $\mathbb{Q}$-factorial Fano threefold containing no planes.

Proof. The assertion is obvious if $f$ is a flop. So we assume that $f$ is a divisorial Mori contraction. Since $V$ contains no planes, the contraction $f$ cannot be of type $(e_5)$. Then by Theorem 2.3 the variety $W$ has only factorial terminal singularities. Moreover, $-K_W$ is nef and big by [21], Proposition 4.5 (note that $f$ cannot be a ‘bad’ contraction of type $(2,0)_o^-$ or $(2,1)_o^-$ (loc. cit.) because the exceptional divisor is not a plane). Finally, $W$ contains no planes by Lemma 2.5.

3.1. Let $\Theta$ be an arbitrary divisor on $\tilde{X}$. According to [22], 2.6–2.8 we may run the $\Theta$-MMP:

\[\varphi: \tilde{X} = \tilde{X}_0 \dashrightarrow \tilde{X}_1 \dashrightarrow \cdots \dashrightarrow \tilde{X}_N\]

By Lemma 3.3, on each step $\varphi_k: \tilde{X}_k \dashrightarrow \tilde{X}_{k+1}$, the following assertions hold:

(i) $\tilde{X}_k$ has only factorial terminal singularities;

(ii) $\tilde{X}_k$ is a generalized Fano threefold;

(iii) $\tilde{X}_k$ contains no planes;

(iv) $\varphi_k$ is either a flop or an extremal Mori contraction of type $(e_1)-(e_4)$.
Note that in the ‘classical’ case $\Theta = K_X$ all the steps $\varphi_k$ must be divisorial contractions of type $(e_1)$–$(e_4)$, in particular, $\varphi$ is a morphism.

3.2. We get the following diagram:

$$
\varphi: \tilde{X} = \tilde{X}_0 \xrightarrow{\varphi_1} \tilde{X}_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_N} \tilde{X}_N
$$

$$
\begin{array}{ccc}
X = X_0 & \xrightarrow{\pi} & X = X_1 & \xrightarrow{\pi_1} & \cdots & \xrightarrow{\pi_N} & X = X_N \\
\end{array}
$$

(3.1)

where $X_k$ is the pluri-anticanonical image of $\tilde{X}_k$. Denote by $\Theta^{(k)}$ the proper transform\(^2\) of $\Theta$ on $\tilde{X}_k$. At the final step we get one of the following possibilities.

- There exists a $\Theta^{(N)}$-negative Mori contraction $\upsilon: \tilde{X}_N \to Z$ to a variety $Z$ of dimension $< 3$. In particular, $\rho(\tilde{X}_N/Z) = 1$ and $-K_{\tilde{X}_N}$ is $\upsilon$-ample.

- The divisor $\Theta^{(N)}$ is nef.

If $\varphi_{k+1}$ is a divisorial contraction, then we denote the $\varphi_{k+1}$-exceptional divisor by $E_k$ and set $B_{k+1} := \varphi_{k+1}(E_k)$.

**Definition 3.4.** A **weak del Pezzo surface** is a projective surface whose singularities are at worst Du Val and such that the anticanonical divisor $-K_X$ is nef and big.

**Remark 3.5.** It is easy to see that the class of weak del Pezzo surfaces is closed under birational contractions.

**Lemma 3.6.** Let $X$ be a generalized Fano threefold and let $\upsilon: X \to Y$ be a contraction to a surface. Assume that $X$ contains no planes.

Then the following assertions hold.

(i) $Y$ is a weak del Pezzo surface.

(ii) If $X$ is $\mathbb{Q}$-factorial and $\upsilon$ is an extremal Mori contraction, then $Y$ is a nonsingular del Pezzo surface.

(iii) If in the assumptions in (ii) the surface $Y$ contains a $(-1)$-curve $\Gamma$, then there is the following diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\chi} & X' \\
\downarrow{\upsilon} & & \downarrow{\upsilon'} \\
Y & \xrightarrow{} & Z
\end{array}
$$

where $Z$ is a nonsingular surface, $Y \to Z$ is a contraction of $\Gamma$, $\chi$ is either an isomorphism or a flop, $\varphi$ is an extremal divisorial contraction and $\upsilon'$ is an extremal contraction of type (c).

\(^2[19]\) asserts that $\rho(X_i) = 1$ for all $i$. This is wrong in general. Indeed, let $X$ be a del Pezzo threefold of degree 5 with $r(X) = 4$ (see [13], §7). For a suitable choice of factorialization, we have $N = 3$, $\tilde{X}_3 = X_3 = \mathbb{P}^3$, and all the $\varphi_i$’s are contractions of type $(e_2)$. Then $X_1$ is a del Pezzo threefold of degree 6 with $r(X) = 3$ and $\rho(X_1) = 2$. 
Proof. (i) As in (3.1), run the $K$-MMP on $X$ over $Y$:

$$
\begin{array}{ccc}
X & \xleftarrow{\pi} & \tilde{X} \\
\downarrow{\psi} & & \downarrow{\tau} \\
Y & \xleftarrow{\chi} & \tilde{Y}
\end{array}
$$

Here $\pi$ is a suitable factorialization, $\psi$ is a composition of divisorial contractions and $\tau$ is an extremal contraction of type (c). By Lemma 3.3 $\tilde{X}$ is a generalized Fano threefold. Then, by Proposition 5.2, (i) in [21], $\tilde{Y}$ is a nonsingular weak del Pezzo surface. This implies that $Y$ is a weak del Pezzo surface (see Remark 3.5).

We now prove (ii) and (iii). Again by Theorem 2.3, (c) and [21], Proposition 5.2, (i) we see that $Y$ is a nonsingular weak del Pezzo surface. Let $\Gamma \subset Y$ be a curve with $\Gamma^2 < 0$ and let $\tau : Y \to U$ be its contraction. It is clear that $\rho(X/U) = 2$, and so the relative Mori cone $\overline{NE}(X/U)$ has two extremal rays, say $\mathcal{R}_1$ and $\mathcal{R}_2$. Set $F := \nu^*(\Gamma)$. We can assume that $F$ is generated by the fibres of $\nu$ and so $F \cdot \mathcal{R}_1 = 0$. Since $\Gamma^2 < 0$, for any curve $\Gamma' \subset X$ dominating $\Gamma$ we have $F \cdot \Gamma' < 0$. Hence the divisor $F := \nu^*(\Gamma)$ is not nef and $F \cdot \mathcal{R}_2 < 0$. Thus we may run the $F$-MMP over $U$ starting with $\mathcal{R}_2$:

$$
\begin{array}{ccc}
X & \xleftarrow{\chi} & X' \\
\downarrow{\psi} & & \downarrow{\nu'} \\
Y & \xleftarrow{\tau} & U' \xleftarrow{\tau'} Y'
\end{array}
$$

Here $\chi$ is either an isomorphism or a flop in the ray $\mathcal{R}_2$ and $\psi$ is either an isomorphism or a divisorial contraction. By Lemma 3.3 $X'$ is a generalized Fano threefold. Thus $\nu'$ is an extremal contraction to a nonsingular surface $Y'$ (see Theorem 2.3, (c)). Note that $\rho(X') = \rho(X) = \rho(U) + 2$.

Assume that $U$ is singular. Then $\rho(Y') > \rho(U)$ and so $\rho(X') = \rho(Y') = \rho(Y') + 1$, that is, $\psi$ is an isomorphism. Moreover, $Y'$ is the minimal resolution of $U$ and so $Y \simeq Y'$. Since the divisors $-K_X$ and $-K_{X'}$ are ample over $Y = Y'$, the map $\psi \circ \chi$ must be an isomorphism, a contradiction.

Thus $\Gamma$ is a $(-1)$-curve and the surface $U$ is nonsingular. In particular, this implies that $Y$ is a del Pezzo surface. If $\psi$ is an isomorphism, then $\tau'$ is the blowup of a (nonsingular) point $\tau(\Gamma) \in U$ and $Y' \simeq Y$. We obtain a contradiction as above. Hence $\psi$ is a (single) divisorial contraction.

**Lemma 3.7.** The MMP (3.1) can be run so that $\rho(Z) \leq 2$.

**Proof.** Assume that $\rho(Z) \geq 3$. Then $Z$ is a nonsingular rational surface (see Lemma 3.6, (ii)) containing a $(-1)$-curve. Then by Lemma 3.6, (iii) we can run the MMP until we obtain a surface $Z_M$ with $\rho(Z_M) \leq 2$.

**Lemma 3.8.** Let $X$ be a generalized Fano threefold and let $f : X \to Y$ be a birational contraction. Assume that $X$ contains no planes. Then $Y$ is also a generalized Fano threefold.
Proof. As in (3.1), we run the K-MMP on X over Y: we start with a suitable factorialization \( \pi : \tilde{X} \to X \) and after a number of divisorial contractions obtain a minimal model \( \tilde{Y} \) over Y. By Lemma 3.3 \( \tilde{Y} \) is a generalized Fano threefold. Hence the morphism \( \tilde{Y} \to Y \) is small and crepant. Thus Y is also a generalized Fano threefold.

Corollary 3.9. In the notation of Lemma 3.8

\[
\rho(X) - \rho(Y) \leq \frac{1}{2} K_X^2 \cdot E,
\]

where E is the f-exceptional divisor.

Proof. Since X contains no planes, the divisor E has at most \((K_X^2 \cdot E)/2\) components.

§ 4. Deformations of Fano threefolds

Theorem 4.1 (see [2]). Let X be a generalized Fano threefold. Then X is smoothable. This means that there exists a flat family \( \mathfrak{X} \to (\mathfrak{U} \ni 0) \) over a small disc \( \mathfrak{U} \ni 0 \subseteq \mathbb{C} \) such that \( \mathfrak{X}_0 \simeq X \) and a general fibre \( \mathfrak{X}_u, u \in \mathfrak{U} \setminus \{0\} \) is a nonsingular generalized Fano threefold.

Theorem 4.2 (see [23]). Let X be a generalized Fano threefold and let \( \mathfrak{X} \to (\mathfrak{U} \ni 0) \) be its smoothing as above. Then \( \mathfrak{X} \) is normal and has at worst isolated terminal factorial singularities. Moreover, there are natural identifications

\[
\text{Pic}(X) = \text{Pic}(\mathfrak{X}_u) = \text{Pic}(\mathfrak{X})
\]

so that \( K_{\mathfrak{X}_u} = K_X \).

Remark 4.3. If in the above notation, the fibre \( \mathfrak{X}_u \) contains a plane, for \( u \neq 0 \), then the same holds for the central fibre.

Corollary 4.4. Let X be a Fano threefold with terminal Gorenstein singularities such that \( \rho(X) = 1 \) and \(-K_X^3 > 22\). Then \( \iota(X) > 1 \). If moreover \(-K_X^3 \geq 40\), then X is either isomorphic to \( \mathbb{P}^3 \), to a quadric in \( \mathbb{P}^4 \) or to a del Pezzo threefold of degree 5 in \( \mathbb{P}^6 \). If \(-K_X^3 \geq 40\) and X is \( \mathbb{Q} \)-factorial, then X is nonsingular.

Proof. Let \( \mathfrak{X} \to (\mathfrak{U} \ni 0) \) be a smoothing as above. A general fibre \( \mathfrak{X}_u \) is a nonsingular Fano threefold with \( \rho(\mathfrak{X}_u) = \rho(X) = 1 \) and \(-K_{\mathfrak{X}_u}^3 = -K_X^3 > 22\). By the classification of nonsingular Fano threefolds with \( \rho = 1 \) we have \( \iota(\mathfrak{X}_u) > 1 \) (see [1], §12.2) and by Theorem 4.2 \( \iota(X) = \iota(\mathfrak{X}_u) > 1 \). The case \( \iota(X) \geq 3 \) is well known (for instance, see [1], §3.1). For the case \( \iota(X) = 2 \) we refer the reader to [13], Corollary 8.7.

Notation 4.5. In this section, from now on we will assume that X is a Fano threefold with terminal Gorenstein singularities containing no planes. Let \( \mathfrak{X} \to (\mathfrak{U} \ni 0) \) be its one-parameter smoothing as above.

Proposition 4.6 (see [14], Proposition 6.3). In the notation of 4.5, let \( \mathfrak{L} \) be a divisor on \( \mathfrak{X} \). Then \( \mathfrak{L} \) is nef (ample) if and only if its restriction \( \mathfrak{L}|_{\mathfrak{X}_u} \) is nef (ample, respectively) for some \( u \in \mathfrak{U} \).
Proposition 4.7 (see [14], Corollary 6.4). In the notation 4.5 let \( f_u : \mathcal{X}_u \to \mathcal{Y}_u \) be an extremal contraction. Then there exists an extremal contraction \( f : \mathcal{X} \to \mathcal{Y} \) over \( U \) such that the restriction \( f|_{\mathcal{X}_u} \) coincides with \( f_u \), where the variety \( \mathcal{Y} \) is \( \mathbb{Q} \)-factorial. Let \( f : X = \mathcal{X}_0 \to Y = \mathcal{Y}_0 \) be the restriction of \( f \) to \( X = \mathcal{X}_0 \). Then \( Y \) is normal, \( \mathbb{Q} \)-Gorenstein and, \( f \) has connected fibres. Moreover, the following assertions hold.

(i) If \( \dim \mathcal{Y} = 3 \), then \( Y \) is a weak del Pezzo surface.

(ii) If \( f \) is birational, then \( Y \) is a generalized Fano threefold. If, moreover, \( \mathcal{Y}_u \) is nonsingular, then there are natural identifications \( \text{Pic}(Y) = \text{Pic}(\mathcal{Y}_u) \) so that \( K_{\mathcal{Y}_s} = K_Y \); in particular, \( \rho(Y) = \rho(\mathcal{Y}_s) \) and \( \iota(Y) = \iota(\mathcal{Y}_s) \).

Proof. The existence of \( f \) follows immediately from Proposition 4.6 (see [14], Corollary 6.4). By our assumption \( \mathcal{X} \) contains no planes. Hence by [24], Theorem 1.1 there exist no flipping contractions on \( \mathcal{X} \). This shows that the variety \( \mathcal{Y} \) is \( \mathbb{Q} \)-factorial (see [25], Lemma 5.1.5 and Proposition 5.1.6) and so \( Y = \mathcal{Y}_0 \) is \( \mathbb{Q} \)-Gorenstein. By the projection formula and the Kawamata-Viehweg vanishing theorem

\[
R^1 f_* \mathcal{O}_\mathcal{X}(-X) = R^1 f_* \mathcal{O}_\mathcal{X} \otimes f^* \mathcal{O}_\mathcal{Y}(-Y) = R^1 f_* \mathcal{O}_\mathcal{X} \otimes \mathcal{O}_\mathcal{Y}(-Y) = 0.
\]

Applying \( f_* \) to the exact sequence

\[
0 \to \mathcal{O}_\mathcal{X}(-X) \to \mathcal{O}_\mathcal{X} \to \mathcal{O}_X \to 0,
\]
we obtain

\[
0 \to \mathcal{O}_\mathcal{Y}(-Y) \to \mathcal{O}_\mathcal{Y} \to f_* \mathcal{O}_X \to 0.
\]

Therefore, \( f_* \mathcal{O}_X = \mathcal{O}_\mathcal{Y} \), \( Y \) is normal, and \( f \) has connected fibres, that is, \( f \) is a contraction. Finally, we apply Lemmas 3.6 and 3.8 and Theorem 4.2.

Lemma 4.8. In the notation 4.5 let \( f : \mathcal{X} \to \mathcal{Y} \) be an extremal contraction of relative dimension 1. Assume that a general fibre \( \mathcal{X}_u \) has also a contraction \( f'_u : \mathcal{X}_u \to \mathcal{Y}_u' \simeq \mathbb{P}^1 \) which does not pass through \( \mathcal{Y}_u \). Then \( f \) has no two-dimensional fibres.

Proof. Assume that \( f_0 : X = \mathcal{X}_0 \to \mathcal{Y}_0 \) has a two-dimensional fibre \( F \subset X \). According to Proposition 4.6 there exists a contraction \( f' : \mathcal{X} \to \mathcal{Y}' \) that extends \( f'_u \). It is clear that \( \mathcal{Y}' \to U \) is a \( \mathbb{P}^1 \)-bundle. We take ample divisors \( \mathcal{H} \) and \( \mathcal{H}' \) on \( \mathcal{Y} \) and \( \mathcal{Y}' \), respectively. Let \( \mathcal{L} := f^* \mathcal{H} \) and \( \mathcal{L}' := f^* \mathcal{H}' \). Then the divisor \( \mathcal{L} + \mathcal{L}' \) is ample by our assumption. According to Proposition 4.6 the same holds for \( \mathcal{L} + \mathcal{L}' \). On the other hand, \( \mathcal{L}|_F = 0 \) and \( \dim f'(F) = 1 \). Hence the linear system \( |\mathcal{L} + \mathcal{L}'| \) (that is, the corresponding morphism) contracts \( F \), a contradiction.

Lemma 4.9. Using the notation 4.5, let \( f : \mathcal{X} \to \mathcal{Y} \) be a contraction (over \( U \ni 0 \)) with one-dimensional fibres. Then \( \mathcal{Y} \) is nonsingular, \( f \) is a conic bundle, and its discriminant locus \( \mathcal{E} \subset \mathcal{Y} \) is either empty or flat of relative dimension 1 over \( U \). Moreover, the fibre \( \mathcal{Y}_0 \) is a nonsingular weak del Pezzo surface of degree \( K_{\mathcal{Y}_0}^2 = K_{\mathcal{Y}_u}^2 \), and the discriminant locus of \( f_0 \) coincides with \( \mathcal{E}_0 \subset \mathcal{Y}_0 \). If the set \( \mathcal{E} \subset \mathcal{Y} \) is empty, then \( X \) is nonsingular and \( f_0 \) is a \( \mathbb{P}^1 \)-bundle.

Proof. Similarly to [18], Theorem 7 we can show that \( \mathcal{Y} \) is nonsingular and \( f \) is a conic bundle. By Lemma 3.6 \( \mathcal{Y}_0 \) is a nonsingular weak del Pezzo surface. The rest is obvious.
Lemma 4.10. In the notation 4.5 assume that a general fibre $X_u$ is isomorphic to a product $X_u \simeq Y_u \times P^1$. Then the special fibre $X$ is nonsingular.

Proof. Consider an extremal contraction $f: X \to Y$ corresponding to the projection $f_u: X_u \simeq Y_u \times P^1 \to Y_u$. By Lemma 4.8 the contraction $f: X \to Y$ has no two-dimensional fibres and by Lemma 4.9 the variety $X$ is nonsingular.

Lemma 4.11. Using the notation 4.5, assume that a general fibre $X_u$ is isomorphic to a divisor of bidegree $(1, 2)$ in $P^2 \times P^2$. Then the special fibre $X$ has the same form and $r(X) \leq 5$.

Proof. Two projections $p_{u,i}: X_u \to P^2$ induce two contractions $p_i: X \to P^2$. The divisors $H_i := p_i^* O_{P^2}(1)$ generate the Picard group $Pic(X)$. Moreover,

$$-K_X = 2H_1 + H_2, \quad H_1^3 = H_2^3 = 0, \quad H_1 \cdot H_2 = 2, \quad H_1 \cdot H_1 = 1.$$

Since $r(X) = 2$, the product map $\pi = p_1 \times p_2: X \to P^2 \times P^2$ is finite. It is easy to see that $\pi$ is birational and its image $Y := \pi(X)$ is a divisor of bidegree $(1, 2)$. Then $K_X = \pi^* K_Y$ by the adjunction formula. Hence the map $\pi$ is an isomorphism onto its image. If the projection $p_2: X \to P^2$ has a two-dimensional fibre, then its anticanonical image must be a plane. This contradicts our assumption. Hence $p_2$ is an equidimensional conic bundle with discriminant curve $\Delta_f \subset P^2$ of degree 3. Then $\Delta_f$ has at most three components and so $r(X) \leq r(P^2) + 1 + 3 = 5$.

Lemma 4.12. Let $X$ be a Fano threefold. Assume that $X$ contains no planes, has no birational contractions and $-K_X^3 > 30$. Then either $\iota(X) \geq 2$ or $X \simeq P^1 \times P^2$.

Proof. Assume that $\iota(X) = 1$. Let $X \to (U \ni 0)$ be a smoothing as in Theorem 4.1. Then, for $0 \neq u \in U$, the fibre $X_u$ is a nonsingular Fano threefold with $\rho(X_u) = \rho(X)$, $\iota(X_u) = 1$, and it has no birational contractions. Then by [26] there is only one possibility: $X_u \simeq P^1 \times P^2$. By Lemma 4.10 $X \simeq P^1 \times P^2$.

Proposition 4.13. Let $X$ be a Fano threefold and let $-K_X^3 = 24$. Assume that $X$ contains no planes. Then $r(X) \leq 9$.

Proof. If $\rho(X) = 1$, then $\iota(X) > 1$ and $X$ is a del Pezzo threefold (see Corollary 4.4). In this case $r(X) \leq 6$ by [13], Corollary 3.13. Thus we assume that $\rho(X) > 1$. Consider a one-parameter smoothing $\{X_u \mid u \in U\}$ as in Theorem 4.1. A general fibre $X_u$ is a nonsingular Fano threefold with $\rho(X_u) = \rho(X) > 1$ and $-K_{X_u}^3 = -K_X^3 = 24$. Next we apply the classification from [26] to $X_u$.

If $\rho(X_u) > 4$, then $X_u$ is the product $X_u \simeq Y_u \times P^1$, where $Y_u$ is a del Pezzo surface of degree 4 (see [26], Table 5). Then by Lemma 4.10 the variety $X$ has the same form and so $r(X) = 7$. Thus we can assume that $2 \leq \rho(X_u) \leq 4$.

If $X_u$ is not isomorphic to the blowup of a nonsingular Fano threefold, then $X_u$ is a double cover of $P^1 \times P^2$ whose branch locus is a divisor of bidegree $(2, 2)$ (see [26], Table 2, No 18c). In this case $X_u$ has two extremal contractions: a conic bundle $f_u: X_u \to P^2$ with discriminant curve $D_u \subset P^2$ of degree 4 and a del Pezzo fibration $f_u': X_u \to P^1$. Consider an extremal contraction $f: X \to Y$ that extends $f_u$. Then $Y = Y_0$ is a del Pezzo surface with at worst Du Val singularities and $\rho(Y) = 1$. Since $K_Y^2 = K_{Y_u}^2 = 9$, we have $Y \simeq P^2$. By Lemma 4.8 the contraction $f: X \to Y$ has no two-dimensional fibres and $f$ is a conic bundle.
Let $\mathcal{O} \subset \mathcal{Y}$ be the discriminant locus of $\mathcal{X}$. Thus, $D_u = \mathcal{Y}_u \cap \mathcal{O}$. Moreover, $D_0 = \mathcal{Y}_0 \cap \mathcal{O}$ is the discriminant curve of

$$\mathcal{X}_0 \to \mathcal{Y}_0 \simeq \mathbb{P}^2, \quad \deg D_0 = \deg D_u = 4.$$ 

Then, as in the proof of Lemma 4.11, we have

$$r(\mathcal{X}) \leq r(\mathcal{Y}) + 1 + 4 \leq 6.\quad (4.1)$$

In the remaining cases $\mathcal{X}_u$ is isomorphic to the blowup of a nonsingular Fano threefold $\mathcal{Y}_u$ along the nonsingular curve $B_u$. Consider an extremal contraction $f: \mathcal{X} \to \mathcal{Y}$ that extends $\mathcal{X}_u \to \mathcal{Y}_u$. Let $Y := \mathcal{Y}_0$. According to Proposition 4.7, $Y$ is a generalized Fano threefold with $\rho(Y) = \rho(\mathcal{Y}_u)$, $-K^3_Y = -K^3_{\mathcal{Y}_u}$ and $\iota(Y) = \iota(\mathcal{Y}_u)$. Let $\mathcal{E} \subset \mathcal{X}$ be the exceptional divisor, let $E_u := \mathcal{E} \cap \mathcal{X}_u$ and let $E := \mathcal{E}_0$. It is clear that the morphism $\mathcal{E} \to \mathcal{U}$ is flat and so

$$K^2_{\mathcal{X}} \cdot E = K^2_{\mathcal{X}_u} \cdot E_u.$$

By Corollary 3.9

$$r(\mathcal{X}) \leq r(\mathcal{Y}) + \frac{1}{2} K^2_{\mathcal{X}_u} \cdot E_u.$$  

According to [26] there are the following possibilities:

| No. | $\rho(\mathcal{X}_u)$ | $\mathcal{Y}_u$ | $r(\mathcal{Y})$ | $p_a(B_u)$ | $K^2_{\mathcal{X}_u} \cdot E_u$ |
|-----|----------------------|-----------------|-----------------|-----------|-----------------|
| 1°  | 2                    | a quadric $Q \subset \mathbb{P}^4$ | $\leq 2$ | 1 | 15 |
| 2°  | 3                    | the del Pezzo threefold $V_6 \subset \mathbb{P}^7$ | $\leq 3$ | 1 | 12 |
| 3°  | 3                    | $Y_{2,1} \subset \mathbb{P}^2 \times \mathbb{P}^2$ | $\leq 5$ | 0 | 4 |
| 4°  | 4                    | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 3 | 1 | 12 |

where $Y_{2,1} \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a divisor of bidegree $(2,1)$. In the above table the values of $K^2_{\mathcal{X}_u} \cdot E_u$ are computed using equality (2.4). To estimate $r(Y)$ we note that $Y$ is a (possibly singular) quadric in case $1°$, $r(Y) \leq 3$ in cases $2°$ and $4°$ by [13], Corollary 3.13, and $r(Y) \leq 5$ in case $3°$ by Lemma 4.11. Comparing this table with (4.1) we obtain the desired estimate $r(X) \leq 9$.

§5. Proof of Theorem 1.2

In this section we prove Theorem 1.2.

**Proposition 5.1.** Let $U$ be a Fano threefold whose singular locus consists of a single ordinary double point. Then $r(U) \leq \rho(U) + 1$.

**Proof.** We can assume that $U$ is not $\mathbb{Q}$-factorial. Let $\pi: \widetilde{U} \to U$ be a small factorialization (see Proposition 2.2). Then $r(\widetilde{U}) = r(U)$. On the other hand, $\rho(\widetilde{U}/U) = 1$ because $\pi$ contracts a single irreducible curve.

Thus, in Theorem 1.2, the implication $(i) \implies (ii)$ holds and furthermore we can assume that $r(X) = 2$.

Let $X$ be a Fano threefold with $g(X) = 12$, $\rho(X) = 1$, and $r(X) = 2$. Let $\pi: Y \to X$ be a small factorialization (see Proposition 2.2). Under our assumption,
\( \pi \) is not an isomorphism and \( Y \) is a generalized Fano threefold with \( \rho(Y) = 2 \). That diagram (1.1) exists is a standard fact (see [1], §4.1, for instance). Consider the possibilities for the contraction \( f \) in Theorem 2.3. Note that our diagram (1.1) is symmetric. So we can interchange \( f \) and \( f^+ \). In particular, if one of the contractions \( f \) and \( f^+ \) is birational, then we can assume that this also holds for \( f \).

**Lemma 5.2.** Neither \( f \) nor \( f^+ \) is of type \((e_2), (e_3) \) or \((e_4)\).

**Proof.** Assume that \( f \) is of type \((e_2)\), \((e_3) \) or \((e_4)\). Let \( E \subset Y \) be the exceptional divisor. Then \( Z \) is a \( \mathbb{Q} \)-factorial Fano threefold with \( \rho(Z) = 1 \). By (2.1) the degree \(-K_Z^3\) equals 30 or 24. Hence \( \iota(Z) \geq 2 \). Note that \(-K_Z^3\) must be divisible by \( \iota(Z)^3\). The only possibility is as follows: \( \iota(Z) = 2 \) and \(-K_Z^3 = 24\), that is, \( Z \) is a del Pezzo threefold of degree 3 and \( f \) is a contraction of type \((e_3)\) or \((e_4)\). In other words, \( Z = Z_3 \subset \mathbb{P}^4 \) is a three-dimensional cubic and \( f(E) \in Z \) is a \( cA \)-point. In this case \( Z^+ \simeq \mathbb{P}^3 \), the map \( Z \dashrightarrow Z^+ \) is the usual projection from \( f(E) \in Z \) and \( Y \) is a Fano threefold (see [26], Table 2, \( N_{15} \)), a contradiction.

### 5.1. Type \((e_1)\).

Let \( E \subset Y \) be the exceptional divisor and let \( E^+ := \chi_* E \). Then \( Z \) is a \( \mathbb{Q} \)-factorial Fano threefold with \( \rho(Z) = 1 \) and \( B := f(E) \subset Z \) is an irreducible curve. Let \( H \) be the ample generator of \( \text{Pic}(Z) \), let \( H^* := f^* H \), and \( H^+ := \chi_* H^* \).

**Lemma 5.3.** In the notation of §5.1 one of following cases occurs:

(I) \( Z \simeq \mathbb{P}^3 \), \( p_a(B) = 0 \) and \( \deg B = 5 \);

(II) \( Z \simeq Q \subset \mathbb{P}^4 \), \( p_a(B) = 0 \) and \( \deg B = 5 \);

(III) \( Z \simeq V_5 \subset \mathbb{P}^6 \), \( p_a(B) = 0 \) and \( \deg B = 4 \).

**Proof.** By Lemma 2.4

\[
(-K_Z^3) = 22 + 2K_Y^2 \cdot E + 2p_a(B) - 2 > 24,
\]

\[
(-K_Z^3) = 22 - 2K_Z \cdot B - 2p_a(B) + 2.
\]

Hence \( \iota(Z) \geq 2 \) and by Corollary 4.4 we have the following three cases.

**The case \( \iota(Z) = 4 \).** Then \( Z \simeq \mathbb{P}^3 \) and (5.1) gives us

\( p_a(B) = 4k \), \( \deg B = 5 + k \), \( K_Y^2 \cdot E = 22 - 4k \).

If \( k = 1 \), then the divisor \(-K_Y \) is in fact ample (see [26], Table 2, \( N_{15} \)) and \( \pi \) is an isomorphism, a contradiction. Thus we can assume that \( k \geq 2 \). Since the linear system \(-K_Y \) is base point free, the curve \( B \) is an intersection of quadrics. In particular, \( B \) is not contained in a plane. The Castelnuovo genus bound gives a contradiction (for instance, see [27], Ch. 4, Theorem 6.4).

**The case \( \iota(Z) = 3 \).** Then \( Z \) is a nonsingular quadric \( Q \subset \mathbb{P}^4 \). As above, we have

\( p_a(B) = 3k \), \( \deg B = 5 + k \), \( K_Y^2 \cdot E = 17 - 3k \), \( 0 \leq k \leq 5 \).

If \( k = 0 \), then we obtain case (II). Let \( k \geq 1 \). Then according to the Castelnuovo genus bound the curve \( B \) is contained in hyperplane. Since \(-K_Y \) is base point free, the curve \( B \) is an intersection of cubics. Hence \( \deg B \leq 6 \). Then \( k = 1 \), \( \deg B = 6 \), and \( B \) is a complete intersection of a quadric and a cubic. But in this case \( p_a(B) = 4 \), a contradiction.
The case \( \nu(Z) = 2 \). Then \( Z = Z_d \subset \mathbb{P}^{d+1} \) is a del Pezzo threefold of degree \( d = 4 \) or \( d = 5 \). Note that a \( \mathbb{Q} \)-factorial del Pezzo threefold of degree 5 is nonsingular (and isomorphic to \( V_5 \); see [13], Corollary 5.4). As above,

\[
p_a(B) = 2k, \quad \deg B = 2d - 6 + k, \quad K_Y^2 \cdot E = 4d - 10 - 2k, \quad 0 \leq k < 2d - 5.
\]

If \( k = 0 \) and \( d = 5 \), then we obtain case (III). If \( k = 0 \) and \( d = 4 \), then \( Y \) is a Fano threefold, as in [26], Table 2, \#16. It does not admit small contractions, a contradiction. Let \( k \geq 1 \). Then \( p_a(B) \geq 2 \) and \( \deg B \geq 3 \). Note that the curve \( B \) is an intersection of quadrics because \( | -K_Y | \) is base point free. Hence \( \dim \langle B \rangle \geq 4 \). Moreover, if \( \dim \langle B \rangle = 4 \), then \( \deg B = 5 \), \( d = 5 \) and \( B = V_5 \cap \langle B \rangle \) is a curve of arithmetic genus 1. This contradicts our relation \( p_a(B) = 2k \geq 2 \). Therefore, \( \dim \langle B \rangle \geq 5 \). Then the Castelnuovo genus bound gives a contradiction.

**Lemma 5.4.** Neither \( f \) nor \( f^+ \) is of type \( (e_5) \).

**Proof.** Assume the contrary. By symmetry we may assume that \( f \) is of type \( (e_5) \). Let \( E \subset Y \) be the exceptional divisor. We have

\[
(-K_Y)^2 \cdot E = 1, \quad (-K_Y) \cdot E^2 = -2, \quad E^3 = 4. \tag{5.2}
\]

The variety \( Z \) is non-Gorenstein and \( \mathbb{Q} \)-factorial with \( \rho(Z) = 1 \), and \( f(E) \in Z \) is a point of type \( \frac{1}{2}(1,1,1) \). (Thus \( Z \) is a so-called \( \mathbb{Q} \)-Fano threefold.) According to (2.1) we have \( -K_Z^3 = -K_Y^3 + 1/2 = 45/2 \). Assume that \( \text{Cl}(Z) \) contains a torsion element, say \( T \). Since \( f(E) \) is the only non-Gorenstein point \( Z \) and it is of type \( \frac{1}{2}(1,1,1) \), in its neighborhood we have \( K_Z + T \sim 0 \). This means that \( K_Z + T \) is a Cartier divisor. On the other hand, \( (K_Z + T)^3 = K_Z^3 \) is not an integer, a contradiction. Therefore, the group \( \text{Cl}(Z) \) is torsion free. Let \( A \) be the ample generator of \( \text{Cl}(Z) \). We can write \( -K_Z \sim qA \), \( q \in \mathbb{Z} \). Then \( 45/2 = -K_Z^2 = q^3A^3 \). According to Proposition 2.2, \( 2A \) is a Cartier divisor. Therefore, \( 2A^3 \) is an integer and so \( q = 1 \), that is, \( K_Z \) generates the group \( \text{Cl}(Z) \). Since \( f_*K_Y = K_Z \), the group \( \text{Pic}(Y) \) is generated by \( K_Y \) and \( E \) (see §2.3). Thus for any effective divisor \( D \neq E \) on \( Y \) we can write

\[
D \sim -\alpha K_Y - \beta E, \quad \alpha \in \mathbb{Z}_{>0}, \quad \beta \in \mathbb{Z}_{\geq 0}.
\]

Then

\[
(-K_Y) \cdot D^2 = 22\alpha^2 - 2\alpha\beta - 2\beta^2.
\]

We may assume that the contraction \( f^+ \) is of type \( (c) \), \( (d) \) or \( (e_5) \). In these cases let \( D^+ \subset Y^+ \) be the pull-back of a line, a fibre or the exceptional divisor, respectively. Let \( D := \chi_*^{-1}D^+ \). Then \( (-K_Y) \cdot D^2 \) equals 2, 0 or \(-2\) respectively. Hence,

\[
11\alpha^2 - \alpha\beta - \beta^2 = \delta, \quad \delta \in \{1, 0, -1\}. \tag{5.3}
\]

This equation has solutions only for \( \delta = -1 \), that is, when \( f^+ \) is of type \( (e_5) \). But then, as above, the group \( \text{Pic}(Y) \) is generated by \( K_Y \) and \( D \). Hence \( \beta = 1 \). In this case (5.3) has no solutions, a contradiction.

**Lemma 5.5.** Both contractions \( f \) and \( f^+ \) cannot be of type \( (d) \).
Proof. Assume that both contractions \( f \) and \( f^+ \) are of type (d). Let \( F \) be a fibre of \( f \). It is clear that
\[
(-K_Y) \cdot F^2 = 0, \quad (-K_Y)^2 \cdot F = K_F^2.
\]
Let \( L^+ \) be a fibre of \( f^+ \) and let \( L \) be the proper transform of \( L^+ \) under the flopping map. Write \( L \sim -\alpha K_Y - \beta F, \alpha, \beta \in \frac{1}{2}\mathbb{Z} \cup \frac{1}{3}\mathbb{Z} \) (see §2.3). Then
\[
0 = (-K_Y) \cdot L^2 = 22\alpha^2 - 2(K_F^2)\alpha\beta,
\]
\[
K_{F^+}^2 = (-K_Y)^2 \cdot L = 22\alpha - (K_F^2)\beta.
\]
This gives
\[
11\alpha = (K_F^2)\beta = K_{F^+}^2.
\]
But then the degree \( K_{F^+}^2 \) of the del Pezzo surface \( F^+ \) must be divisible by 11, a contradiction.

Lemma 5.6. If neither \( f \) nor \( f^+ \) is birational, then up to permutations \( f \) can be taken to be a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^2 \) and \( f^+ \) a del Pezzo fibration of degree 5.

Proof. We can assume that \( f \) is of type (c). Let \( \Delta_f \subset Z \) be the discriminant curve of our conic bundle \( f \). Let \( l \subset Z = \mathbb{P}^2 \) be a general line and let \( F := f^{-1}(l) \). Then \( F \) is a nonsingular rational surface admitting a conic bundle structure with degenerate fibres over the points of \( l \cap \Delta_f \). Hence,
\[
F^3 = 0, \quad K_Y \cdot F^2 = -2, \quad K_Y^2 \cdot F = 12 - \deg \Delta_f. \quad (5.4)
\]
We can assume that the contraction \( f^+ \) is of type (c) or (d). Consider these cases separately.

\( f^+ \) is of type (c). Let \( \Delta_{f^+} \subset Z^+ \) be the discriminant curve of \( f^+ \) and let \( L^+ := f^+^{-1}(l^+) \), where \( l^+ \subset Z^+ = \mathbb{P}^2 \) is a line. Let \( L \) be the proper transform of \( L^+ \) under the flopping map. By §2.3 we can write
\[
L \sim -\alpha K_Y - \beta F, \quad \alpha, \beta \in \frac{1}{2}\mathbb{Z}, \quad \alpha, \beta > 0.
\]
As in the proof Lemma 5.5 we have
\[
2 = (-K_{Y^+}) \cdot L^+^2 = (-K_Y) \cdot L^2 = 22\alpha^2 - 2(12 - \deg \Delta_f)\alpha\beta + 2\beta^2. \quad (5.5)
\]
If \( \Delta_f = \emptyset \), then the latter relation can be rewritten as follows
\[
1 + 25\alpha^2 = (6\alpha - \beta)^2.
\]
It is easy to see that this equation has no solutions among the numbers \( \alpha, \beta \in \frac{1}{2}\mathbb{Z} \). Therefore, we can assume that \( \Delta_f \neq \emptyset \) and \( \Delta_{f^+} \neq \emptyset \). In this case the divisors \( K_Y \) and \( F \) generate \( \text{Pic}(Y) \). Hence \( \alpha, \beta \in \mathbb{Z} \). By symmetry \( K_Y \) and \( L \) generate \( \text{Pic}(Y) \). Hence, \( \beta = 1 \). Then (5.5) has the form
\[
12 = 11\alpha + \deg \Delta_f.
\]
From this we obtain $\deg \Delta_f = 1$, so that $\Delta_f$ is a line on $Z = \mathbb{P}^2$. In this case $f^{-1}(\Delta_f)$ must be a reducible surface, which contradicts the extremal property of $f$.

$f^+$ is of type (d). Let $L^+$ be a fibre of $f^+$ and let $L$ be the proper transform of $L^+$ under the flopping map. Write $L \sim -\alpha K_Y - \beta F$, $\alpha, \beta \in \frac{1}{2}\mathbb{Z}$. Then

$$0 = (-K_Y) \cdot L^2 = 22\alpha^2 - 2(12 - \deg \Delta_f)\alpha\beta + 2\beta^2,$$

$$K_{L^+}^2 = (-K_Y)^2 \cdot L = 22\alpha - (12 - \deg \Delta_f)\beta.$$

The first equation has a rational solution only for $\deg \Delta_f = 0$ and then

$$K_{L^+}^2 = 2\beta\left(\frac{11\alpha}{\beta} - 6\beta\right), \quad \frac{\alpha}{\beta} = 1, \quad K_{L^+}^2 = 10\beta = 5.$$ 

Since $\Delta_f = \emptyset$, the variety $Y$ is nonsingular and $f$ is a $\mathbb{P}^1$-bundle (see §2.2). According to Lemma 5.7 below we obtain case (IV).

**Lemma 5.7.** Let $\mathcal{E}$ be a rank-2 vector bundle on $\mathbb{P}^2$ such that $Y := \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$ is a generalized Fano threefold with $g(Y) = 12$. Then the vector bundle $\mathcal{E}$ is stable and $c_1(\mathcal{E})^2 + 16 = 4c_2(\mathcal{E})$.

**Proof.** Apply Lemma 2.7. It follows from (2.6) that $c_1(\mathcal{E})$ is even and up to twisting $\mathcal{E}$ by a line bundle we may assume that $c_1(\mathcal{E}) = 0$. Moreover, $c_2(\mathcal{E}) = 4$. Thus, $-K_Y \sim 2M + 3F$, where $F$ is the pull-back of a line on $\mathbb{P}^2$. Since

$$2(-K_Y)^2 \cdot M = (-K_Y)^3 - 3(-K_Y)^2 \cdot F = 22 - 36 < 0,$$

we have $|M| = \emptyset$ and so $H^0(\mathcal{E}) = 0$. Then $\mathcal{E}$ must be stable (for instance, see [28], Ch. 4, Proposition 14).

To complete the proof of Theorem 1.2, it remains to describe the right-hand side of diagram (1.1) in cases (I), (II) and (III).

**Notation 5.8.** Let $Z$ be a nonsingular Fano threefold with $\rho(Z) = 1$ and let $B \subset Z$ be a nonsingular curve. Let $\iota := \iota(Z)$ be the Fano index of $Z$ and $d := \iota - 1$. Let $H = -\frac{1}{\iota}K_Z$ be the ample generator of Pic($Z$). Let $f: Y \to Z$ be the blowup of $B$, let $E \subset Y$ be the $f$-exceptional divisor and $H^* := f^*H$.

We assume that the pair $(Z, B)$ is one of those described in Lemma 5.3. Thus $(Z, B)$ appears on the left-hand side of (1.1) in cases (I)–(III). Denote the subsystem of the linear system $|dH|$ consisting of all divisors passing through $B$ by $|dH - B|$.

**Lemma 5.9.** In the notation 5.8, $\dim |dH - B| \geq d$.

**Proof.** It is clear that $\dim |dH - B| = h^0(\mathcal{I}_B(d)) = 1$. On the other hand, $h^0(\mathcal{I}_B(d)) \geq h^0(\mathcal{O}_Z(d)) - h^0(\mathcal{O}_B(d)) = d + 1$.

**Lemma 5.10.** If $Y$ is a generalized Fano threefold, then any member $\widetilde{S} \in |dH^* - E|$ is irreducible.

**Proof.** Let $\widetilde{S}_1 \subsetneq \widetilde{S}$ be an irreducible component other than $E$. Then $\widetilde{S}_1 \sim d'H^* - kE,$ where $d' \leq d$, $k \geq 1$ and at least one of these inequalities is strict. Then easy computations give $(-K_Y)^2 \cdot \widetilde{S}_1 \leq 0$. This contradicts our assumptions.
Corollary 5.11. Under the assumptions of Lemma 5.10 any member $S \in |dH - B|$ is irreducible. In particular, $B$ is not contained in a quadric when $Z = \mathbb{P}^3$ and $B$ is not contained in a hyperplane section when $Z = Q$.

5.2. Now suppose that $Y$ is a generalized Fano threefold, that is, the divisor $-K_Y$ is nef and the morphism defined by the linear system $|-nK_Y|$ contracts no divisors. Let $\chi: Y \to Y^+$ be a flop, let $H^+ := \chi_*H^*$ and $E^+ := \chi_*E$. The description of the right-hand side of the diagram (1.1) in cases (I), (II) and (III) is an immediate consequence of the following proposition.

Proposition 5.12. In the notation of § 5.2 $dH^+ - E^+$ is a supporting divisor of the contraction $f^+$, so that the morphism $f^+$ is defined by the linear system $|n(dH^+ - E^+)|$, $n \gg 0$.

Proof. Let $\widetilde{S} \in |dH^* - E|$ and $\widetilde{S}^+ := \chi_*\widetilde{S}$. Since $-\widetilde{S} \sim K_Y + H^*$ is $\pi$-ample, $\widetilde{S}^+$ must be $\pi^+$-ample. Since the linear system $|\widetilde{S}^+|$ has no fixed components and the contraction $f^+$ is not small, the divisor $\widetilde{S}^+$ is nonnegative on the fibres of $f^+$. Hence it is nef. It remains to show that $\widetilde{S}^+$ is not ample. Assume the contrary. Let $l$ be a curve in a fibre of $f^+$ that delivers the minimum of $\mu_{f^+}$ (see (2.2)). Since $|H^+|$ has no fixed components and $H^+$ is negative on flopped curves, $H^+ \cdot L > 0$. Then

$$\mu_{f^+} = -K_{Y^+} \cdot l = \widetilde{S}^+ \cdot l + H^+ \cdot l \geq 2.$$ 

Moreover, if $\mu_{f^+} = 2$, then $\widetilde{S}^+ \cdot l = H^+ \cdot l = 1$. In this case the supporting divisor of $f^+$ has the form $\widetilde{S}^+ - H^+$ and so $|\widetilde{S} - H^*| \neq \emptyset$. On the other hand, $(-K_Y)^2 \cdot (\widetilde{S} - H^*) \leq 0$. This contradicts our assumption that $Y$ is a generalized Fano threefold. Hence $\mu_{f^+} = 3$. Then $Z^+ \simeq \mathbb{P}^1$ and $f^+$ is a $\mathbb{P}^2$-bundle. As above, the only possibility is the following: $\widetilde{S}^+ \cdot l = 1$, $H^+ \cdot l = 1$ and the supporting divisor of $\varphi^+$ is equal to $2\widetilde{S}^+ - H^+$. Since $\dim Z^+ = 1$, we have $(2\widetilde{S}^+ - H^+)^2 = 0$. On the other hand,

$$(2\widetilde{S}^+ - H^+)^2 \cdot K_{Y^+} = (2\widetilde{S} - H^*)^2 \cdot K_Y < 0$$

by (2.3), giving a contradiction.

This completes the proof of Theorem 1.2.

§ 6. Fano threefolds of types (I), (II) and (III)

First, we show that the possibilities (I), (II) and (III) in Theorem 1.2 really occur. More precisely, we show that any pair $(Z, B)$ satisfying conditions (I)–(III) in Theorem 1.2 generates diagram (1.1).

6.1. Let the pair $(Z, B)$ be as given in the notation 5.8. By Lemma 5.9 we have $\dim |dH - B| > 0$. Assume, in addition, that $B$ is not contained in a quadric in the case $Z = \mathbb{P}^3$ and $B$ is not contained in a hyperplane section in the case $Z = Q$. This implies that any member $S \in |dH - B|$ is irreducible.

Lemma 6.1. Any member $S \in |dH - B|$ is a del Pezzo surface of degree $d$ with at worst Du Val singularities.
Proof. First, we claim that the surface $S$ is normal. Suppose that $S$ is singular along a curve $J$. Then $J$ is contained in the base locus $B_s|dH - B|$ by Bertini’s theorem. The intersection of two members $S, S' \in |(\iota - 1)H - B|$ contains $J$ with multiplicity $\geq 4$ and the curve $B$. Hence,

$$(\iota - 1)^2H^3 = S \cdot S' \cdot H \geq H \cdot B + 4H \cdot B.$$ 

In cases (II) and (III) this gives a contradiction. In case (I) we have the equality $S \cdot S' = B + 4J$. Since in this case $\dim |(\iota - 1)H - B| = 3$, we can take $S$ and $S'$ so that they have a common point $P \notin B \cap J$. Then $S \cdot S' \supseteq B + 4J$. This means that $S$ and $S'$ have a common component and so $S$ is reducible. This contradiction proves our claim.

Further, by the adjunction formula the divisor $-K_S = H|_S$ is ample, that is, $S$ is a del Pezzo surface. If the singularities $S$ are worse than Du Val, then $S$ is a cone over an elliptic curve (see [29]). But this is impossible because $S$ contains a rational curve $B$ of degree $> 1$.

**Corollary 6.2.** In the notation of § 6.1, any member $\tilde{S} \in |dH^* - E|$ is a del Pezzo surface of degree $d$ with at worst Du Val singularities and the restriction map $f_S: \tilde{S} \to f(\tilde{S}) = S$ is crepant.

**Proof.** By the adjunction formula the divisor $-K_{\tilde{S}} = f^*H|_{\tilde{S}} = -f^*K_S$ is nef and big.

**Proposition 6.3.** In the notation 5.8, let $S \in |dH - B|$ be a general member and let $\tilde{S} \subset Y$ be the proper transform of $S$. Then the surface $S$ is nonsingular and $f_S: \tilde{S} \to S$ is an isomorphism. The blowup $f: Y \to Z$ can be completed to the diagram (1.1). The $\pi$-exceptional locus consists of a single nonsingular rational curve $Y$ which is a $(-1)$-curve on $\tilde{S}$ and $\mathcal{N}_{Y/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

**Proof.** We claim that $Y$ is a generalized Fano threefold. Assume that $K_Y \cdot Y \geq 0$ for some irreducible curve $Y$. It is clear that $Y$ is not a fibre of the ruling $E \to B$. Hence, $H^* \cdot Y > 0$ and $\tilde{S} \cdot Y < 0$. In particular, $Y \subset \tilde{S}$, $Y \subset B_s|dH^* - E|$ and there is at most a finite number of such curves. On the minimal resolution $S_{\text{min}}$ of the surface $\tilde{S}$, the Mori cone is generated by $(-1)$- and $(-2)$-curves (because the divisor $-K_{S_{\text{min}}}$ is nef and big). Therefore, we may assume that $-K_{\tilde{S}} \cdot Y = 1$ and so $H \cdot f(Y) = -K_{\tilde{S}} \cdot f(Y) = 1$, that is, $f(Y)$ is a line on $Z$. Since $K_Y \cdot Y \geq 0$, we have $E \cdot \tilde{Y} \geq d + 1$ and so $f(Y)$ is a $(d + 1)$-secant line of the curve $B \subset S \subset \mathbb{P}^d$. It is easy to see that $B$ has no $(d + 2)$-secant lines. Therefore, $E \cdot Y = d + 1$ and $K_Y \cdot Y = 0$, that is, $-K_Y$ is nef. It is clear that the divisor $-K_Y$ is big. This proves our claim.

Thus, the blowup $f: Y \to Z$ can be completed to the diagram (1.1) and by Theorem 1.2 we have cases (I)–(III). Let $\tilde{S}^+ = \chi_*\tilde{S} \subset Y^+$ be the proper transform of $\tilde{S}$. Then $\tilde{S}^+$ is a nonsingular del Pezzo surface of degree $d + 1$. Since the linear system $|\tilde{S}^+|$ is base point free, $\tilde{S}^+$ meets the flopped curves $Y_i^+$ transversely. Hence $\tilde{S}$ contains all the flopping curves $Y_i$. By Zariski’s main theorem $\chi$ induces a morphism $\chi_S: \tilde{S} \to \tilde{S}^+$ which contracts $\bigcup Y_i$. By Noether’s formula $\rho(\tilde{S}) \leq 10 - d$ and $\rho(\tilde{S}^+) = 9 - d$. Since there exists at least one flopping curve, $\rho(\tilde{S}) = \rho(\tilde{S}^+) + 1$ and $\rho(\tilde{S}) = 10 - d$. Hence $\tilde{S}$ is nonsingular and $\chi_S: \tilde{S} \to \tilde{S}^+$ is the blowup of a single point. Thus the $\pi$-exceptional locus consists of a single nonsingular rational
curve $\mathcal Y = \mathcal Y_1$ which is a $(-1)$-curve on $\widetilde S$. Then $\mathcal{N}_{\mathcal Y/\mathcal S}$ contains a subbundle $\mathcal{N}_{\mathcal Y/\mathcal S} \simeq \mathcal{O}_{\mathbb P^1}(-1)$. Since $\deg \mathcal{N}_{\mathcal Y/\mathcal S} = -2$, we obtain the desired splitting. If the restriction $f_\mathcal Y: \widetilde S \to S$ is not an isomorphism, then the intersection $\widetilde S \cap E$ is reducible: it contains a section and a fibre of the ruling $E \to B$. But this contradicts Bertini’s theorem applied to $|\mathcal O_{\mathcal E}(-1)|_E$.

**Corollary 6.4.** In cases (I)–(III) of Theorem 1.2 the singular locus of $X$ consists of a single ordinary double point.

**6.2.** Let $Z$ be either the projective space $\mathbb P^3$, a quadric in $\mathbb P^4$ (possibly, singular) or a (possibly, singular) del Pezzo threefold of degree 5 in $\mathbb P^6$. Put $d := \nu(Z) - 1$. Let $H$ be the ample generator of $\text{Pic}(Z)$ and let $S = S_d \in |dH|$ be a nonsingular member. Thus, $\mathcal S = S_d \subset \mathbb P^d$ is a nonsingular del Pezzo surface of degree $d = 3, 4, 5$. Regard $S$ as the blowup $\sigma: S \to \mathbb P^2$ at $9 - d$ points in general position. Let $h$ be the class of a line on $\mathbb P^2$, $h^* := \sigma^* h$ and let $e_1, \ldots, e_{9-d}$ be the $\sigma$-exceptional divisors. Furthermore, let $B \subset S$ be a reduced curve such that $B \sim 2h^* - e_1$ in the cases $d = 3, 4$ and $B \sim 2h^* - e_1 - e_2$ in the case $d = 5$. Then $p_a(B) = 0$. Moreover, $-K_S \cdot B = 5$ in the cases $d = 3, 4$ and $-K_S \cdot B = 4$ in the case $d = 5$.

Let $f: Y \to Z$ be the blowup of $B$ and let $S \subset Y$ be the proper transform of $S$. Then the singularities of $Y$ are at worst terminal Gorenstein (cf. Theorem 2.3, (e1)). It is clear that $\widetilde S \simeq S$. Up to this identification we can write

$$-K_Y|_{\widetilde S} \sim -(d + 1)K_S - B.$$ 

It is easy to check that this divisor is nef and big. Moreover, $-K_Y|_{\widetilde S}$ has positive intersection number with all curves except for one. Denote it by $\Lambda$. Then $\Lambda$ is a line on $S$. If $-K_Y \cdot \mathcal Y \leq 0$ for some curve $\mathcal Y$, then $\widetilde S \cdot \mathcal Y < 0$. Hence $\mathcal Y \subset \widetilde S$ and $\mathcal Y = \Lambda$. This shows that the divisor $-K_Y$ is nef and $\Lambda$ is the only $K$-trivial curve on $Y$. Using (2.3), we can show that $-K_Y^3 = 22$. Thus $Y$ is a generalized Fano threefold of genus 12.

Thus possibilities (I)–(III) in Theorem 1.2 really occur.

In the construction in §6.2 the curve $B$ can be reducible and then $r(Y) - r(Z)$ is equal to the number of components of $B$.

**Example 6.5.** Let $Z = V_5^*$ be a del Pezzo threefold of degree 5 with $r(Z) = 4$ (see [13]) and let $B$ be a combinatorial chain of four lines contained in a nonsingular hyperplane section $S$. Then the construction in §6.2 gives us a generalized Fano threefold $Y$ with $r(Y) = 8$.

Another application of our construction 6.2 is the following.

**Theorem 6.6** (cf. [16]). There exists a Fano threefold $X = X_{22} \subset \mathbb P^{13}$ of type (III) and a reducible hyperplane section $A = A_1 \cup A_2$ such that

$$X \setminus A \simeq \mathbb A^3.$$ 

**Proof.** Recall that the normal bundle of a line on $V_5$ has the form $\mathcal O_{\mathbb P^1}(-a) \oplus \mathcal O_{\mathbb P^1}(a)$ with $a = 0$ or $a = 1$ (see [1], Lemma 4.2.1 or [30], for instance). Let $\Lambda \subset V_5$ be a line with $\mathcal N_{\Lambda/V_5} \simeq \mathcal O_{\mathbb P^1}(-1) \oplus \mathcal O_{\mathbb P^1}(1)$ and let $R$ be the ruled surface swept out by lines meeting $\Lambda$. Then $R$ is a hyperplane section of $V_5$ and its normalization $R'$
is isomorphic to $\mathbb{P}_3$ (see [30]). Moreover, $V_5 \setminus R \cong \mathbb{A}^3$ (loc. cit.). The map $\nu: R' \to R$ is an isomorphism outside $\Lambda$ and $\nu^{-1}(\Lambda)$ is the union of the negative section $\Sigma \subset R' = \mathbb{P}_3$ and a fibre $\Gamma_1$. Furthermore, the map $\nu: R' \to R \subset \mathbb{P}^5$ is given by a codimension 1 subsystem of the linear system $|\Sigma + 4\Gamma_1|$. Let $B' \subset R'$ be a general member of $|\Sigma + 3\Gamma_1|$. Then $B := \nu(B')$ is a nonsingular rational curve of degree 4 and the line $\Upsilon := \nu(\Gamma_0)$ is a 2-secant of $B$ for some fibre $\Gamma_0 \subset R'$. Thus, $V_5 \cap \langle B \rangle = B \cup \Upsilon$. Then we can apply Proposition 6.3 and get a Fano threefold $X_{22}$. By our construction

$$X_{22} \setminus (\pi(f^{-1}(R))) \cong V_5 \setminus R \cong \mathbb{A}^3.$$  

§ 7. Fano threefolds of type (IV)

In this section we investigate case (IV) in Theorem 1.2. The relation between singular Fano threefolds $X_{22}$ and certain rank-2 vector bundles was noticed by Mukai (see [4], Remark 5). This observation is based on the explicit description of stable rank-2 vector bundles on $\mathbb{P}^2$ with even $c_1$ (see [31]).

7.1. Let $\mathcal{E}$ be a stable rank-2 vector bundle on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = 4$ (see [31]). Let $Y := \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$ and let $f: Y \to \mathbb{P}^2$ be the projection. Let $F$ be the pull-back of a line $l \subset \mathbb{P}^2$ and let $M$ be the tautological divisor. Then

$$M^3 = -4, \quad M^2 \cdot F = 0, \quad M \cdot F^2 = 1, \quad F^3 = 0 \quad (7.1)$$

(see (2.6)). For $n \leq 2$ by the Serre duality we have

$$H^2(\mathcal{E}(n)) = H^0(\mathcal{E}(n) \otimes \det \mathcal{E}(n)^\vee \otimes \omega_{\mathbb{P}^2})^\vee = H^0(\mathcal{E}(n-3))^\vee = 0.$$  

Then by the Riemann-Roch formula

$$h^0(\mathcal{E}(n)) \geq \frac{1}{2}(c_1(\mathcal{E}(n))^2 - 2c_2(\mathcal{E}(n)) - K_{\mathbb{P}^2} \cdot c_1(\mathcal{E}(n))) + 2 = n^2 + 3n - 2.$$  

In particular,

$$\dim |M + F| = h^0(\mathcal{E}(1)) - 1 \geq 1,$$

$$\dim |M + 2F| = h^0(\mathcal{E}(2)) - 1 \geq 7. \quad (7.2)$$

Since $\mathcal{E}$ is stable, $H^0(\mathcal{E}) = 0$ (for instance, see [28], Ch. 4, Proposition 14) and so $|M| = \emptyset$. Therefore, any member $S \in |M + F|$ is irreducible. Let $S, S' \in |M + F|$ be general members and let $\Gamma := S \cap S'$ (in the scheme sense).

Claim 7.1. In the notation of § 7.1, $-K_Y \cdot \Gamma = 0$. In particular, $Y$ is not a Fano threefold. Moreover, the image $f(\Gamma)$ is a conic (possibly, degenerate).

Proof. Using (7.1) we obtain

$$(M + F)^2 \cdot K_Y = 0 \quad \text{and} \quad (M + F)^2 \cdot F = 2.$$
Proposition 7.2. In the notation of §7.1 the following conditions are equivalent:

(i) \( Y \) is a generalized Fano threefold;

(ii) the vector bundle \( \mathcal{E}(2) \) is ample (or, equivalently, \( M + 2F \) is ample);

(iii) the curve \( \Gamma \) is reduced and irreducible;

(iv) \( \Gamma \) is a nonnegative rational curve;

(v) a general member \( S \in |M + F| \) is a nonsingular del Pezzo surface of degree 4.

Proof. (i) \( \implies \) (ii). Since \( \rho(Y) = 2 \), the Mori cone \( \overline{NE}(Y) \) is generated by two extremal rays \( \mathcal{R}_0 \) and \( \mathcal{R} \). We can assume that \( \mathcal{R}_0 \) is generated by curves in the fibres of the projection \( f: Y \to \mathbb{P}^2 \). It is clear that all the effective divisors are nonnegative on \( \mathcal{R}_0 \). Since \( Y \) is a generalized Fano threefold and \( -K_Y \cdot \Gamma = 0 \), the ray \( \mathcal{R} \) is generated by the class of \( \Gamma \). Hence \( (M + 2F) \cdot \mathcal{R} > 0 \). By Kleiman’s ampleness criterion the divisor \( M + 2F \) is ample and so \( \mathcal{E}(2) \) is too.

(ii) \( \implies \) (iii). By assumption \( M + 2F \) is ample. Since \( (M + 2F) \cdot \Gamma = (M + 2F) \cdot (M + F)^2 = 1 \), \( \Gamma \) is reduced and irreducible.

(iii) \( \implies \) (i). Assume that \( -K_Y \cdot C \leq 0 \) for some irreducible curve \( C \). Then \( (M + 2F) \cdot C < 0 \). Hence \( C \) is contained in the base locus of \( |M + F| \). Since \( \Gamma \) is reduced and irreducible, \( C = \Gamma \). In this case \( -K_Y \cdot C = 0 \) and so the divisor \( -K_Y \) is nef. Since \( -K_Y^3 = 22 \), it is big. Moreover, \( \Gamma \) is the only curve contracted by \( |\Gamma| \).

The implication (iii) \( \implies \) (iv) follows by the adjunction formula, the implication (iv) \( \implies \) (iii) is obvious, and the implication (iv) \& (ii) \( \implies \) (v) follows by Bertini’s theorem and the adjunction formula. We will prove the implication (v) \( \implies \) (iii). By the adjunction formula on \( S \) have

\[
-K_S \cdot \Gamma = (M + 2H) \cdot (M + H)^2 = 1.
\]

Since \( -K_S \) is ample, the curve \( \Gamma \) is reduced and irreducible.

The proposition is proved.

7.2. From now on we assume that \( \mathcal{E} \) satisfies the equivalent conditions of Proposition 7.2. Let \( \pi: Y \to X = X_{22} \subset \mathbb{P}^{13} \) be the anticanonical morphism. It follows from the proof of Proposition 7.2 that \( \Gamma \) is the only curve contracted by the linear system \( |\Gamma| \) (that is, by the corresponding morphism). Therefore, the singular locus of \( X \) consists of a single point \( P := \pi(\Gamma) \). Let \( \Gamma^+ \subset Y^+ \) be the flopped curve. Thus, \( \pi^+(\Gamma^+) = P \). Then we can restore diagram (1.1) with the map \( f^+ \) being a degree 5 del Pezzo fibration by Theorem 1.2. The right-hand side of (1.1) was explicitly described by Takeuchi (see [8]).

Remark 7.3. (i) It follows from Claim 7.1 and Proposition 7.2, (iv) that the image \( \Omega := f(\Gamma) \) is a nonsingular conic.

(ii) Since \( \mathcal{N}_{\Gamma/Y} \) contains a subbundle \( \mathcal{N}_{\Gamma/S} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \), \( \mathcal{N}_{\Gamma/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \). In particular, the image \( \pi(\Gamma) \in X \) is an ordinary double point.

(iii) By the Kawamata-Viehweg vanishing theorem, \( H^i(\mathcal{O}_Y(M + F)) = 0 \) for \( i > 0 \). Hence \( \dim |M + F| = 1 \).

(iv) Since the base locus of the pencil \( |M + F| \) is a nonsingular curve \( \Gamma \), any member \( S \in |M + F| \) has at most isolated singularities and \( \text{Sing}(S) \cap \Gamma = \emptyset \). Moreover, \( \Gamma \) is a \((−1)\)-curve on \( S \) and the singularities of \( S \) are at worst Du Val (see [29]).
7.3. The map \( f_S : S \to \mathbb{P}^2 \) is the blowup of the ideal sheaf of some zero-dimensional scheme \( \Xi \subset \Omega \subset \mathbb{P}^2 \) of degree 5. If \( S \in |M + F| \) is a general member, then \( S \) is a nonsingular del Pezzo surface of degree 4 and \( f_S : S \to \mathbb{P}^2 \) is the blowup of five points in general position.

Denote \( T := f^{-1}_S(\Omega) \). It is easy to show that \( T \cong \mathbb{F}_6 \) and \( \Gamma \subset T \) is the negative section. Since \((M + F)^2 \cdot T = 4\), any member of the restriction \(|M + F|_T\) has the form \( \Gamma + \sum_i \Upsilon_i \), where the \( \Upsilon_i \) are fibres. In other words,

\[
|M + F|_T = \Gamma + f^*_T \mathfrak{g},
\]

where \( \mathfrak{g} = \mathfrak{g}_5^1 \) is a pencil (linear series) of degree 5 on \( \Omega \). Thus \( \mathcal{E} \) defines \( \mathfrak{g}_5^1 \) on \( \Omega \cong \mathbb{P}^1 \).

7.4. In general case a stable rank-2 vector bundle \( \mathcal{E} \) on \( \mathbb{P}^2 \) with \( c_1(\mathcal{E}) = 0 \) and \( c_2(\mathcal{E}) = 4 \) fits in the following (non-unique) exact sequence:

\[
0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E}(1) \to \mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{I}_\Xi \to 0, \tag{7.3}
\]

where \( \mathcal{I}_\Xi \) is the ideal sheaf of a zero-dimensional subscheme \( \Xi \subset \mathbb{P}^2 \) of degree 5 which is not contained in a line (for instance, see [28], Ch. 4, Proposition 14). Conversely, for \( \Xi = \{P_1, \ldots, P_5\} \) the extension (7.3) corresponds to an element \( s \) of the five-dimensional vector space

\[
\text{Ext}^1(\mathcal{I}_\Xi(2), \mathcal{O}_{\mathbb{P}^2}) \cong \mathcal{O}_{\Xi}(1) = \bigoplus_{i=1}^5 \mathcal{O}_{P_i}(1),
\]

and for a general choice of \( s \) the sheaf \( \mathcal{E}(1) \) is locally free (see [31], § 5). This shows that for a general choice of \( \mathcal{E} \) (in the sense of moduli) the vector bundle \( \mathcal{E} \) satisfies the conditions of Proposition 7.2.

7.5. We say that an irreducible curve \( \Lambda \) on a generalized Fano variety \( U \) is a line if \( -K_U \cdot \Lambda = 1 \). Denote by \( \mathfrak{L}(U) \) the union of those components of the Hilbert scheme \( \text{Hilb}(U) \) that contain classes of lines. Roughly speaking, we can regard \( \mathfrak{L}(U) \) as the scheme parametrizing lines on \( U \).

**Lemma 7.4.** Let \( \Lambda \subset Y \) be a line. Then \( F \cdot \Lambda = 1 \) and \( \Gamma \cap \Lambda = \emptyset \).

**Proof.** Since \( M + 2F \) is ample, we have \((M + 2F) \cdot \Lambda = 1 \) and \((M + F) \cdot \Lambda = 0 \). Hence \( \Lambda \) is disjoint from a general member of the pencil \(|M + F|\).

**Corollary 7.5.** Any line on \( X \) is contained in the nonsingular locus and so \( \mathfrak{L}(Y) \cong \mathfrak{L}(X) \cong \mathfrak{L}(Y^+) \). In particular, \( \mathfrak{L}(X) \) parametrizes lines in the fibres of \( f^+ \).

**Corollary 7.6.** For a general line \( \Lambda \subset X \) there exist exactly three other lines meeting \( \Lambda \).

Note that the same holds also for a general nonsingular Fano threefold of genus 12 (see the proof of Theorem 6.1 in [3]).
7.6. Now we give another description of varieties $X_{22}$ of type (IV) which does not use vector bundle techniques.

**Lemma 7.7.** Let $g: \hat{X} \to X$ be the blowup of the singular point $P \in X$. Then $\hat{X}$ is a (nonsingular) Fano threefold with $\rho(\hat{X}) = 3$ and $(-K_{\hat{X}})^3 = 20$.

**Proof.** By Corollary 7.5 the variety $X = X_{22} \subset \mathbb{P}^{13}$ contains no lines passing through $P$ and so $T_{P,X} \cap X = \{P\}$.

It is clear that the exceptional divisor $R \subset \hat{X}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and $\hat{X}$ admits two contractions $\tau: \hat{X} \to Y$ and $\tau^+: \hat{X} \to Y^+$ so that $\tau(R) = \Gamma$ and $\tau^+(R) = \Gamma^+$. Thus we have the product map

$$h = (f \circ \tau) \times (f^+ \circ \tau^+): \hat{X} \to \mathbb{P}^2 \times \mathbb{P}^1.$$  

We get the following extension of diagram (1.1):

$$\begin{array}{c}
\mathbb{P}^2 \\
\uparrow \pi \\
\downarrow \tau \\
\hat{X} \quad \tau^+ \\
\downarrow \Rightarrow \Rightarrow \\
\leftarrow Y \\
\mathbb{P}^2 \\
\uparrow f \\
\downarrow \pi \\
X \quad g \\
\downarrow \Rightarrow \Rightarrow \\
\leftarrow \mathbb{P}^1
\end{array}$$  

(7.4)

Since $\hat{X}$ is a Fano threefold, $h$ is an extremal contraction. Since $b_3(\hat{X}) = 0$ and $(-K_{\mathbb{P}^2 \times \mathbb{P}^1})^3 - (-K_{\hat{X}})^3 = 34$, this contraction $h$ is the blowup of a nonsingular rational curve $C \subset \mathbb{P}^2 \times \mathbb{P}^1$ with $-K_{\mathbb{P}^2 \times \mathbb{P}^1} \cdot C = 16$ (see (2.3)). Then it is easy to see that $C$ is a curve of bidegree $(2,5)$ such that the restriction $p_2|_C$ is an embedding (see [26], Table 3, № 5°). It is clear that $p_2(C) = \Omega$ and $h(R) = p_2^{-1}(\Omega)$. The $h$-exceptional divisor is the proper transform of $f^{-1}(\Omega)$. Let $G := p_2^{-1}(\Omega) = h(R)$. Then $G \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $C \subset G$ is a curve of bidegree $(5,1)$. The projection $C \to \mathbb{P}^1$ onto the first factor is defined by a linear system $g_1^5$. Conversely, a linear series $g_1^5$ on $\mathbb{P}^1$ defines an embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 = G$ as a divisor of bidegree $(5,1)$ and therefore it defines the whole diagram (7.4), where $X$ is a Fano threefold with $g(X) = 12$.

\section*{§ 8. Proof of Theorem 1.3}

We use the notation of Theorem 1.3. To show that the case $r(X) > 2$ does not occur, we need some upper bounds for the rank $r(X)$. We start with the following result.

**Proposition 8.1.** Let $X = X_{22} \subset \mathbb{P}^{13}$ be a Fano threefold of the main series with $g(X) = 12$. Assume that $X$ contains no planes. Then $r(X) \leq 10$.

Note that the proofs of similar estimates for $r(X)$ in [19], § 3.2 contain some gaps (see the footnote in § 3.2).
Proof of Proposition 8.1. Assume that \( r(X) \geq 11 \). Let \( \pi: \tilde{X} \to X \) be a factorialization. Apply K-MMP to \( \tilde{X} \). We obtain diagram (3.1), where all the maps \( \varphi_i \) are divisorial contractions.

Claim 8.2. The morphism \( \varphi \) contracts a surface of degree 2.

Proof. Let the minimum degree of a surface contracted by \( \varphi \) be equal to \( d \). Assume that \( d \geq 3 \). We can also assume that \( \tilde{X}_N \) does not admit birational contractions and \( \rho(\tilde{X}_N) \leq 3 \) by Lemma 3.7. Hence \( N \geq r(X) - 3 \geq 8 \). By (2.4)

\[
-K^3_{\tilde{X}_N} \geq 22 + (2d - 2)N \geq 54. \tag{8.1}
\]

If \(-K^3_{\tilde{X}_N} = 54\), then by Lemma 4.12 we have \( \rho(\tilde{X}_N) \leq 2 \) and so \( N \geq 9 \). In this case \(-K^3_{\tilde{X}_N} \geq 22 + 4N \geq 56\), a contradiction. Therefore, \(-K^3_{\tilde{X}_N} > 54\) and again by Lemma 4.12 there is only one possibility: \( \tilde{X}_N \cong \mathbb{P}^3 \). Then \( N = 10 \) and \( d = 3 \). Moreover, similarly to (8.1) we have \(-K^3_{\tilde{X}_{N-1}} \geq 58\). According to the classification in [26] the only possibility for the smoothing of \( X_{N-1} \) is \( X_{36^0} \) in [26], Table 2. But in this case \( X_{N-1} \) must contain a plane, a contradiction.

Let \( S \subset \tilde{X} \) be an irreducible surface of degree 2 contracted by \( \varphi \). Apply S-MMP to \( \tilde{X} \). We claim that \( S \) is contracted during this process. If not, after a number of flops we get a model \( \tilde{X}' \) such that the proper transform \( S' \subset \tilde{X}' \) of the divisor \( S \) is nef. Since \( \tilde{X}' \) is a generalized Fano threefold, some multiple of \( S' \) must be movable. Hence the image \( S \subset X \) of the divisor \( S' \) under the anticanonical morphism is also movable. On the other hand, \( S \) is a quadric and its intersection \( S \cap H \) with a general member \( H \in |-K_X| \) is a nonsingular rational curve. Since \( H \) is a nonsingular K3 surface, the curve \( S \cap H \) cannot be movable. This contradiction shows that after a number of flops we must contract the proper transform of \( S \). Replacing \( \tilde{X} \) with another factorialization, we can assume that \( \varphi_1 \) contracts a surface \( E_0 = S \) of degree 2. We claim that \( B_1 = \varphi_1(E_0) \) is either a point or a nonsingular rational curve. If this is not so, \( B_1 \) is a rational curve which is singular at some point \( P \). Since \( \varphi_1 \) is the blowup of a locally planar curve \( B_1 \) contained in the smooth locus of \( \tilde{X}_1 \) (see Theorem 2.3), easy local computations show that the exceptional divisor \( E_1 \) is singular along the fibre \( C := \varphi_1^{-1}(P) \). Since the quadric \( \pi(E_1) \) is normal, \( C \) must be contracted by \( \pi \). This contradicts the fact that \( \varphi_1 \) is a Mori contraction and proves our claim. Thus by (2.1) and (2.4) we obtain \(-K^3_X = 24\). Taking Proposition 4.13 into account we obtain

\[
r(X) = \rho(\tilde{X}) = \rho(\tilde{X}_1) + 1 = r(X_1) + 1 \leq 10.
\]

This proves Proposition 8.1.

Lemma 8.3. Let \( X = X_{22} \subset \mathbb{P}^{13} \) be a Fano threefold of the main series with \( g(X) = 12 \). Assume that \( r(X) > 2 \). Then \( X \) contains a surface of degree \( d \) with \( d \neq 0 \mod 11 \).

Proof. Assume that the degree of any surface \( S \subset X \) is divisible by 11 (in particular, \( X \) does not contain planes). Apply the construction (3.1). By Lemma 2.5 on each step the variety \( \tilde{X}_k \) contains no surfaces of degree \( \leq 10 \). Since the degree of the
exceptional divisor of contractions of types $(e_2)–(e_4)$ is at most 4 by (2.1), all the birational contractions in (3.1) are of type $(e_1)$. We can assume that $\tilde{X}_N$ does not admit birational contractions.

First, consider the case $\rho(\tilde{X}_N) = 1$. Then $N \geq 2$ and by (2.4) we have

$$64 \geq -K^3_{\tilde{X}_N} \geq 22 + 20N.$$  

Hence $N = 2$ and $X_2 = \tilde{X}_2 \simeq \mathbb{P}^3$. The morphism $X_1 \simeq \tilde{X}_1 \to \tilde{X}_2 \simeq \mathbb{P}^3$ is the blowup of an irreducible curve $B_2 \subset \mathbb{P}^3$. By (2.3) the number

$$K^2_{\tilde{X}_1} \cdot E_1 = -K_{\mathbb{P}^3} \cdot B_2 + 2 - 2p_a(B_2)$$

is even. Hence $K^2_{\tilde{X}_1} \cdot E_1 \geq 12$ and

$$42 = (-K_{\mathbb{P}^3})^3 - (K_{X_0})^3 \geq 20 + 2K^2_{\tilde{X}_1} \cdot E_1 + 2p_a(B_2) - 2.$$  

We note that the above inequality is in fact an equality and so $p_a(B_2) = 0$, $K^2_{\tilde{X}_1} \cdot E_1 = 12$ and

$$-K_{\mathbb{P}^3} \cdot B_2 = 10 \not\equiv 0 \mod 4,$$

a contradiction.

Therefore, $\rho(\tilde{X}_N) \geq 2$ and $Z$ is not a point. If $\tilde{X}_N$ admits a contraction to a curve, then a general fibre $F$ is a del Pezzo surface. Hence

$$(-K_{\tilde{X}})^2 \cdot F = K^2_F \leq 9.$$  

This contradicts our assumption. Thus we can assume that $Z$ is a surface with $\rho(Z) = 1$, that is, $Z \simeq \mathbb{P}^2$. Then $\rho(\tilde{X}_N) = 2$ and so $N \geq 1$. By (2.4)

$$-K^3_{\tilde{X}_N} \geq 22 + 2 \cdot K^2_{\tilde{X}} \cdot E_0 - 2 \geq 42. \quad (8.2)$$

Since $\tilde{X}_N$ has no contractions to a curve and $\rho(\tilde{X}_N) \geq 2$, the variety $X_N$ is not a quadric. By Lemma 4.12, $\iota(\tilde{X}_N) = 2$. In this case the (anticanonical) degree of any curve on $\tilde{X}_N$ is even. Hence

$$(-K_{\tilde{X}})^2 \cdot E_0 \equiv 0 \mod 22$$

by (2.3). This contradicts (8.2). The lemma is proved.

**Proposition 8.4.** Let $X = X_{22} \subset \mathbb{P}^{13}$ be a $G$-Fano threefold of the main series with $g(X) = 12$ and $r(X) > 2$. Let $G_\bullet \subset G$ be an 11-Sylow subgroup. Then $G_\bullet$ acts nontrivially on the lattice $\Cl(X)$. In particular, $G_\bullet$ is nontrivial and $r(X) \geq 11$.

**Proof.** Let $S_1 \subset X$ be a surface of degree $d$. By Lemma 8.3 we can take $S_1$ such that $d \not\equiv 0 \mod 11$. Let $O = G \cdot S_1$ be a $G$-orbit of $S_1$ and let $n := \text{card} \ O$. Set $D := \sum_{S \in O} S$. We can write $D \sim -aK_X$ for some $a$. Comparing the degrees we get

$$nd = 22a, \quad n \equiv 0 \mod 11.$$
Consider the following natural homomorphism of $G$-modules:

$$\varsigma: \bigoplus_{S \in O} Z \cdot S \to \text{Cl}(X).$$

It is clear that the set $\varsigma(O)$ is a $G$-orbit. Let $\Theta_1, \ldots, \Theta_m$ be all the elements of $\varsigma(O)$. Take their representatives $S_i \in \varsigma^{-1}(\Theta_i) \cap O$ and put $D' := S_1 + \cdots + S_m$. Since all the elements in the preimage $\varsigma^{-1}(\Theta) \cap O$ are linearly equivalent, $m$ divides $n$ and $D \sim \frac{n}{m} D'$. Hence

$$22D' = \frac{nd}{a} D' \sim \frac{md}{a} D \sim -mdK_X.$$

Since $\iota(X) = 1$, we have $m \equiv 0 \mod 11$ and so $G_*$ acts nontrivially on $\text{Cl}(X)$. The proposition is proved.

Now we are in a position to prove Theorem 1.3.

First, assume that $r(X) = 2$. Since $\text{rk} \text{Cl}(X) G = 1$, the action of $G$ on $\text{Cl}(X) \cong \mathbb{Z}^2$ is nontrivial. Then some element $\tau \in G$ switches two extremal rays $\mathcal{R}$ and $\mathcal{R}^+$ in the cone of movable divisors $\text{Mov}(X)$. Hence the two corresponding rational maps $\Phi: X \dashrightarrow \mathbb{P}^1$ and $\Phi^+: X \dashrightarrow \mathbb{P}^1$ are also switched. In particular, $\dim Z = \dim Z^+$. This only happens in case (I) of Theorem 1.2.

Now assume that $r(X) > 2$. Then according to Proposition 8.4 we have $r(X) \geq 11$. On the other hand, the variety $X$ contains no planes (see Theorem 3.1) and so we obtain a contradiction by Proposition 8.1.

Theorem 1.3 is proved.

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