Universal Risk Budgeting

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Abstract

I juxtapose Cover’s vaunted universal portfolio selection algorithm (Cover 1991) with the modern representation (Qian 2016; Roncalli 2013) of a portfolio as a certain allocation of risk among the available assets, rather than a mere allocation of capital. Thus, I define a Universal Risk Budgeting scheme that weights each risk budget (instead of each capital budget) by its historical performance record (à la Cover). I prove that my scheme is mathematically equivalent to a novel type of Cover and Ordentlich 1996 universal portfolio that uses a new family of prior densities that have hitherto not appeared in the literature on universal portfolio theory. I argue that my universal risk budget, so-defined, is a potentially more perspicuous and flexible type of universal portfolio; it allows the algorithmic trader to incorporate, with advantage, his prior knowledge (or beliefs) about the particular covariance structure of instantaneous asset returns. Say, if there is some dispersion in the volatilities of the available assets, then the uniform (or Dirichlet) priors that are standard in the literature will generate a dangerously lopsided prior distribution over the possible risk budgets. In the author’s opinion, the proposed “Garivaltis prior” makes for a nice improvement on Cover’s timeless expert system (Cover 1991), that is properly agnostic and open (from the very get-go) to different risk budgets. Inspired by Jamshidian 1992, the universal risk budget is formulated as a new kind of exotic option in the continuous time Black and Scholes 1973 market, with all the pleasure, elegance, and convenience that that entails.

Keywords: Risk Decomposition; Risk Contributions; Risk Budgeting; Risk Parity; On-Line Portfolio Selection; Portfolio Construction; Universal Portfolios; Correlation Options, Expert Systems; Multi-Armed Bandit Problem.

JEL Classification Codes: D80; D81; D83; G11; G13; G17.

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“Balance may be achieved with equal risk allocation to equities, commodities, Treasuries, and TIPS.”

—RPAR Risk Parity ETF, fund literature

“HEDGEFUNDIE’S excellent adventure [risk parity strategy using 3x leveraged ETFs].”

—Title of a famous discussion thread (3300+ posts) on the Bogleheads investment forum

1 Introduction.

This paper contains a novel combination of some of the most important and perspicuous ideas in all of financial economics. We construct an exciting extension of Cover’s universal portfolios (Cover 1991; Cover and Ordentlich 1996; Cover and Thomas 2012) by viewing his universal schemes under the lens of the modern notion of a portfolio (Maillard, Roncalli, and Teiletche 2010; Roncalli 2013) as a certain distribution (or budgeting) of risk among the available assets, instead of the classical definition (from time immemorial, e.g., Markowitz 1952; Markowitz 1959) of a portfolio as a mere distribution of the investor’s capital.

The discrete time universal portfolio (and its continuous time cousin, Jamshidian 1992) is analytically constructed as a performance-weighted average of constant-rebalanced portfolios, or CRPs. This is an ingeniously robust procedure that satisfactorily deals with the classical problem (cf. with Merton and Samuelson 1992) that the drift vector of a log-normal random walk requires many years (decades) of observation in order to estimate an accurate value. It turned out that the univer-
sal portfolio has very strong asymptotic optimality properties (as proved by Cover and Ordentlich 1996) uniformly for all possible individual sequences of asset prices. Roughly speaking, this means that, regardless of the actual (realized) path of stock prices, the trading strategy is guaranteed to achieve an acceptable percentage of the final wealth of the best CRP determined in hindsight (such determination, of course, being made in the distant future).

Working in continuous time, we study a variant of the Cover/Jamshidian asymptotically optimal portfolio that has the following (novel) pair of advantages over the approach that is taken in the existing literature about on-line portfolio selection (OPS). Specifically:

1. Our universal risk budgeting scheme explicitly incorporates the investor’s prior knowledge (or beliefs) about the covariance structure of instantaneous financial returns. This is sensible, since empirical estimates of volatilities and correlations are well known to be much more reliable than estimates of expected returns;

2. We replace Cover’s uniform prior over capital budgets (i.e., he makes a uniform distribution of the intial dollar over all the points of the portfolio simplex) with a uniform prior over risk budgets (as defined in Maillard, Roncalli, and Teïletche 2010; Roncalli 2013; Bruder and Roncalli 2012). Thus, we reject the prior belief that one capital distribution is as good as any other, in favor of the initial hypothesis that one risk distribution is as good as any other, until proven otherwise (by exponential capital growth).

Thus, the present paper constructs a certain performance-weighted average of risk budgets, and proves a theorem to the effect that our suggested Universal Risk Budget is equivalent to a Cover and Ordentlich 1996 universal portfolio that uses
a special weighting scheme (to be derived below) that was not considered by Or- dendtlich and Cover, who (with good reason) confined their study to the family of Dirichlet densities. The simplest examples will show (viz., our Figure 4 below) that the Cover 1991 uniform prior over capital budgets will induce (under very general conditions) a dangerously lopsided prior weighting over the possible risk budgets that are possible for the institutional investor. Thus, our Universal Risk Budget is in some sense guaranteed to start its life with a well-diversified exposure to the available sources of risk (and return) that exist in the financial markets. In the long run, of course, Cover’s brilliantly simple concept of the running, performance-weighted average (with all its ramifications) will guarantee that the investor’s risk (and his capital) will grow increasingly condensed and concentrated among the best (read: high-growth) continuously-rebalanced portfolios. In the context of the Black and Scholes 1973 market, this means that the portfolio that is selected by the algorithm will ultimately converge to the log-optimal portfolio, or Kelly rule (cf. with MacLean, Thorp, and Ziemba 2011). Accordingly, we take the point of view that our universal risk budgeting scheme has a certain superiority over the popular (and ingenious) risk parity portfolio (Qian 2011; Anderson, Bianchi, and Goldberg 2012), that was popularized and used to great effect by Ray Dalio and his All Weather Fund.

2 Risk Budgets.

We take up a classical Black and Scholes 1973 market in continuous time that features a pair of risk assets \( i = 1, 2 \) whose price processes \( S(t) := (S_1(t), S_2(t))' \).
follow correlated geometric Brownian motions

\[
\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW_i(t),
\]

(1)

where \(\mu_i\) and \(\sigma_i\) denote the drift and diffusion parameters for asset \(i\), respectively. \(W_1(t)\) and \(W_2(t)\) denote unit Brownian motions whose instantaneous changes are correlated, that is, \(\rho := \text{Corr}(dW_1(t), dW_2(t)) = \text{Corr}(dS_1(t)/S_1(t), dS_2(t)/S_2(t)) = \mathbb{E}[dW_1(t)dW_2(t)]/dt\). Thus, by Itô’s Lemma (cf. with Wilmott 2007) the instantaneous change in the log-price of asset \(i\) amounts to

\[
d (\log S_i(t)) = \nu_i dt + \sigma_i dW_i(t),
\]

(2)

where the parameter \(\nu_i := \mu_i - \sigma_i^2/2 = \mathbb{E}[d (\log S_i(t))]/dt\) denotes the exponential growth rate of asset \(i\). Indeed, upon integration of (2), we have the formula (i.e., Wilmott 1998)

\[
S_i(t) = S_i(0) \times \exp (\nu_i t + \sigma_i W_i(t)).
\]

(3)

The variance per unit time of the vector of instantaneous percentage asset price changes will be denoted by

\[
\Sigma := \frac{1}{dt} \text{Var} \left( \begin{bmatrix} dS_1(t)/S_1(t) \\ dS_2(t)/S_2(t) \end{bmatrix} \right) = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}.
\]

(4)

In our context, a portfolio (that will be continuously-rebalanced, i.e., as in Cover 1991 and Jamshidian 1992) is a vector \(\mathbf{w} := (w, 1 - w)'\), where the numerical weight parameter \(w \in [0, 1]\) denotes the percentage of capital that will be invested in asset 1 for the duration of the differential timestep \([t, t + dt]\). In accordance with the (good) notation established by Ordentlich and Cover 1998, we let the value process
\((V_t(w))_{t \geq 0}\) denote the bankroll of the continuously-rebalanced portfolio \(w\) at time \(t\), where we normalize the investor’s initial capital to \(V_0(w) := 1\). Thus, \(V_t(w)\) is the cumulative capital growth factor (or multiplier) that \(w\) has achieved over the time interval \([0, t]\). The instantaneous percent change in the gambler’s fortune consists in the expression \(dV_t(w)/V_t(w) = w \cdot dS_1(t)/S_1(t) + (1 - w) \cdot dS_2(t)/S_2(t)\). At such point in time, of course, the investor must transact in the two asset markets in order to restore the target weight \(w\), on account of the price fluctuation \(dS(t) = (dS_1(t), dS_2(t))'\).

The variance per unit time of the instantaneous percentage change in portfolio value, which will be denoted \(\sigma^2(w)\) (as in the fundamental risk parity paper by Maillard, Roncalli, and Teiletche 2010), corresponds to the quadratic form

\[
\sigma^2(w) := \frac{1}{dt} \text{Var} \left[ \frac{dV_t(w)}{V_t(w)} \right] = \mathbf{w}' \Sigma \mathbf{w} = \sigma_1^2 w^2 + \sigma_2^2 (1 - w)^2 + 2\sigma_{12} w(1 - w) \\
= \left( \sigma_1^2 + \sigma_2^2 - 2\sigma_{12} \right) w^2 + 2 \left( \sigma_{12} - \sigma_2^2 \right) w + \sigma_2^2.
\tag{5}
\]

Thus, when regarded as a function of the vector \(\mathbf{w} = (w, 1 - w)'\), the instantaneous portfolio risk \(\sigma(w)\) is homogeneous of degree one; we have, by Euler’s theorem on homogeneous functions (cf. with Widder 1989), the representation as an inner product

\[
\sigma(w) = \langle \mathbf{w}, \nabla \sigma(w) \rangle = w \frac{\partial \sigma(w)}{\partial w} + (1 - w) \frac{\partial \sigma(w)}{\partial (1 - w)},
\tag{6}
\]

where \(\nabla \sigma(\mathbf{w}) = \Sigma \mathbf{w}/\sigma(w)\), by the chain rule. This expansion leads to the portfolio
Figure 1: Total instantaneous portfolio risk, $\sigma(w)$, for different capital budgets $w \in [0, 1]$, and different correlation coefficients $\rho \in \{\pm 100\%, \pm 75\%, \pm 50\%, \pm 25\%, 0\%\}$. Here, we have used the asset volatilities $\sigma_1 := 40\%$ per year and $\sigma_2 := 60\%$ per year.

**Risk Decomposition** (Roncalli 2013; Qian 2016)

\[
\frac{w (\Sigma w)_1}{\sigma^2(w)} + \frac{(1 - w) (\Sigma w)_2}{\sigma^2(w)} = 100\%,
\]  

\[\text{where} \quad (\Sigma w)_i \text{ denotes the } i^{th} \text{ coordinate of the column vector } \Sigma w. \text{ The first term of equation (7) is regarded as the percentage of portfolio risk that has been budgeted to asset 1, and similarly for the second term of (7). Thus, the continuously-rebalanced portfolio } w \text{ has been re-conceptualized as a distribution of risk, rather than a distribution of capital, among the available assets (Bruder and Roncalli 2012). Accordingly, the investor’s risk budget will be denoted by the vector } b := (b, 1 - b)^t, \text{ where the} \]
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Figure 2: The risk budget \( b = w \left[ \left( \sigma_1^2 - \sigma_{12} \right) w + \sigma_{12} \right] / \sigma^2(w) \) that corresponds to each capital budget \( w \in [0, 1] \), for the parameters \( \rho := 50\% \) and \( \lambda \in \{1/4, 1/3, 1/2, 1, 2, 3, 4\} \). Here, \( \lambda := \sigma_2/\sigma_1 \) denotes the volatility ratio for the two assets.

\[
b = w \frac{\left( \sigma_1^2 - \sigma_{12} \right) w + \sigma_{12}}{\sigma^2(w)}
\]

is the percentage of total risk that has been allocated to asset 1.

3 Learning to Allocate Risk.

Cover’s universal portfolio (Cover 1991), which was extended to continuous time by Jamshidian 1992, is constructed analytically as a performance-weighted average of all continuously-rebalanced portfolios (CRPs). In practice, this amounts to the following (very attractive) interpretation. Assuming no prior knowledge as to what is the correct rebalancing rule (from the standpoint of asymptotic capital growth, e.g., as discussed in Luenberger 1997), we elect to make a uniform distribution
of the initial dollar over all portfolios \( w \in [0, 1] \). Each CRP is then presumed to manage its money and transact independently of all the others; thus, the CRPs in the differential vicinity of \( w \) will grow their initial grubstake of \( dw \) dollars into \( V_i(w)dw \) dollars at time \( t \).

The wisdom of this scheme is that, over time, the most successful CRPs (from the standpoint of exponential capital growth) will tend to control an ever increasing share of the investor’s overall capital (of \( \hat{S}_i := \frac{1}{0} \int V_i(w)dw \) dollars). Note well that this process of wealth condensation happens continuously and automatically; the scheme manages to ultimately concentrate its capital in the best possible location (viz., the log-optimal portfolio, or Kelly 1956 rule, as discussed in MacLean, Thorp, and Ziemba 2011) \( w^* \in [0, 1] \), regardless of the fact that \( w^* \) is not actually known in advance (cf. with Cover and Ordentlich 1996).

Put differently, let us imagine that our hedge fund (or pension fund, family of-
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A firm, university endowment, etc.) employs a continuum of experts, or traders, called \( w \in [0, 1] \). Each such expert is continuously transacting in the financial markets. The magnitude and direction of these trades will vary over time in accordance with:

- His initial capital allocation, \( dw \);
- The particular weighting \((w, 1 - w)'\) that he is using on behalf of the parent institution;
- The relative performance of the two assets over \([t, t + dt]\);
- The cumulative capital growth factor \( V_t(w) \) that he has achieved to date.

Thus, there will always be internal disagreement among the experts \( w \in [0, 1] \); some will sell asset 2 in order to buy asset 1, and some will do the exact opposite. Accordingly, a certain quantity of the trading by the expert fiduciaries can be considered to have been netted out internally; then, we are left with a list of net internal transactions, which is the ultimate output of the algorithm. Hence, in the long run, the ill-advised activity of the less successful “experts” will be drowned out by the massive transaction volume of the genuinely skillful traders, viz., those who have achieved a very high capital growth factor, \( V_t(w) \).

Now, given some knowledge of (or beliefs about) the instantaneous covariance matrix, \( \Sigma \) (which, in practice, is far easier to estimate than the drift vector \( \mu \)), one can object that a uniform prior density over capital budgets \( w \in [0, 1] \) will typically generate a very uneven initial density over risk budgets. Given the fact that financial markets are structured so as to reward investors for their actual risk taking (rather than just the sheer quantity of their savings) it is seems far more attractive to use a uniform prior over risk budgets, instead of a uniform distribution over capital budgets.

To this end, note that (8) amounts to a quadratic equation in \( w \) that has the
solution (cf. with Bruder and Roncalli 2012)

\[
w = F(b) := \frac{-\sigma_{12}/2 + (\sigma_{12} - \sigma_2^2)b + \sigma_1\sigma_2\sqrt{\rho^2 (b - 1/2)^2 + b(1 - b)}}{(2\sigma_{12} - \sigma_1^2 - \sigma_2^2) b + \sigma_1^2 - \sigma_{12}}
\]

\[
= \lambda \frac{-\rho/2 + (\rho - \lambda)b + \sqrt{\rho^2 (b - 1/2)^2 + b(1 - b)}}{(2\rho\lambda - 1 - \lambda^2) b + 1 - \rho\lambda},
\]

where \(\lambda := \sigma_2/\sigma_1\) denotes the volatility ratio for the pair of assets. Now, for a uniform density over capital budgets (Cover 1991; Jamshidian 1992), the allocation of \(dw\) dollars to the differential vicinity of \(w\) amounts to an allocation of \(dw = F'(b)db\) dollars to the vicinity of the corresponding risk budget, \(b\). That is, \(f(b) := F'(b)\) is the prior density over risk budgets that is induced by a uniform prior over capital budgets. By logarithmic differentiation, we obtain the relation

\[
\frac{F'(b)}{F(b)} = \frac{\rho - \lambda + (b - 1/2)(\rho^2 - 1)\left[\rho^2 (b - 1/2)^2 + b(1 - b)\right]^{-1/2}}{-\rho/2 + (\rho - \lambda)b + \left[\rho^2 (b - 1/2)^2 + b(1 - b)\right]^{1/2}} - \frac{2\rho\lambda - 1 - \lambda^2}{(2\rho\lambda - 1 - \lambda^2) b + 1 - \rho\lambda}.
\]

(10)

Now, we proceed to calculate the prior density \(g(w)\) over capital budgets \(w\) that induces a uniform density over risk budgets. Under a uniform distribution, the risk budgets in the interval \([b, b + db]\) will be allocated \(db\) dollars to manage, where

\[
db = \frac{db}{dw} = \frac{dw}{F'(F^{-1}(w))} = g(w)dw,
\]

(11)

so that \(g(w) := 1/F'(F^{-1}(w))\) is the unique prior density over capital budgets that
induces a uniform density over risk budgets. This density corresponds to the CDF

\[ G(w) := b = w \frac{(1 - \lambda \rho)w + \lambda \rho}{(1 + \lambda^2 - 2\lambda \rho)w^2 + 2\lambda(\rho - \lambda)w + \lambda^2}, \]  

where \( \lambda \) is the systematic over- or under-weighting of extreme or unbalanced risk budgets (e.g., risk budgets near the endpoints of \([0, 1]\)), even when the two assets have equal volatilities (viz., \( \lambda := 1 \)).

On the other hand, if we apply the quotient rule directly, and (carefully) simplify,
then we get the expression

\[
g(w) = \lambda \left( \rho (\lambda^2 + 1) - 2\lambda \right) w^2 + 2\lambda (1 - \lambda \rho) w + \rho \lambda^2 \left\{ (\lambda^2 + 1 - 2\rho \lambda) w^2 + 2\lambda (\rho - \lambda) w + \lambda^2 \right\}.
\]

(14)

The Maple computer algebra system (cf. with Garvan 2001; Heck 2003) suggests the most compact representation

\[
g(w) = \lambda \frac{\rho [\lambda (1 - w)]^2 + 2\lambda w (1 - w) + \rho w^2}{\{ [\lambda (1 - w)]^2 + 2\lambda \rho w(1 - w) + w^2 \}^2}.
\]

(15)

**Definition 1.** The Universal Risk Budget is the (unique) trading strategy that generates...
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\[ V_t := \int_0^1 V_t(w) g(w) dw, \]  

(16)

where \( g(\bullet) \) is the density defined by (13).

The (multi-underlying) option payoff (16) is a very pleasurable extension of the Jamshidian 1992 asymptotically optimal portfolio; its novelty becomes readily apparent after searching through standard textbooks on exotic option pricing (Zhang 1997; Wilmott et al. 1995; Haug 2007). Accordingly, we proceed to express the payoff (16) in the standard (path-independent form), \( \Pi(S_1, S_2, t; \sigma_1, \sigma_2, \sigma_{12}) \) that solves the multi-dimensional Black and Scholes 1973 equation (cf. with Wilmott 2007):

\[
\frac{\partial \Pi}{\partial t} + \frac{1}{2} \left( \sigma_1^2 S_1^2 \frac{\partial^2 \Pi}{\partial S_1^2} + \sigma_2^2 S_2^2 \frac{\partial^2 \Pi}{\partial S_2^2} + 2 \sigma_{12} S_1 S_2 \frac{\partial^2 \Pi}{\partial S_1 \partial S_2} \right) + r \left( S_1 \frac{\partial \Pi}{\partial S_1} + S_2 \frac{\partial \Pi}{\partial S_2} \right) = r \Pi, \]

(17)

where \( r \) is the risk-free rate of (continuously-compounded) interest\(^1\). We proceed briefly to evaluate the bankroll of the continuously-rebalanced portfolio \( V_t(w) \) (which itself is a solution of the Black and Scholes 1973 equation; cf. with Ordentlich and Cover 1998; Garivaltis 2019; Garivaltis 2021) in terms of the state vector \((S, t) = (S_1, S_2, t)\). Starting with the fact that

\[
\frac{dV_t(w)}{V_t(w)} = w \frac{dS_1(t)}{S_1(t)} + (1 - w) \frac{dS_2(t)}{S_2(t)},
\]

(18)

basic operations of the Itô calculus (i.e., Björk 2009; Mikosch 1998) yield the calcu-

\(^1\)The universal risk budget was defined without any usage of cash or a risk-free bond. Indeed, it will transpire that \( \Pi(\bullet) \) is linearly homogeneous in prices \( S = (S_1, S_2)' \), and the Black and Scholes 1973 term that references the interest rate, \( r (S_1 \partial \Pi / \partial S_1 + S_2 \partial \Pi / \partial S_2 - \Pi) \), nets out to zero on account of Euler’s theorem for homogeneous functions.
Figure 6: Payoff diagram for the continuously-rebalanced portfolio value $V_t(w)$ that $w$ achieves over $[0, t]$, illustrated for the parameter values $(t, w, \sigma_1, \sigma_2, \rho) := (10 \text{ years}, 40\%, 70\% \text{ per year}, 60\% \text{ per year}, 25\%)$. The initial (time-zero) asset prices have both been normalized to $\$1$.

\[
d (\log V_t(w)) = \frac{dV_t(w)}{V_t(w)} - \frac{1}{2} \left( \frac{dV_t(w)}{V_t(w)} \right)^2 = (w\mu_1 + (1 - w)\mu_2) \, dt + w\sigma_1 dW_1(t) + (1 - w)\sigma_2 dW_2(t) - \frac{1}{2} \left( w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_{12} \right) \, dt
\]

\[
= w \cdot d (\log S_1(t)) + (1 - w) \cdot d (\log S_2(t)) + \left( \frac{\sigma_1^2 + \sigma_2^2}{2} - \sigma_{12} \right) w(1 - w)dt. \quad (19)
\]

Thus, upon integration of (19), we obtain the formula

\[
V_t(w) = \left( \frac{S_1(t)}{S_1(0)} \right)^w \left( \frac{S_2(t)}{S_2(0)} \right)^{1-w} \times \exp \left\{ \left( \frac{\sigma_1^2 + \sigma_2^2}{2} - \sigma_{12} \right) w(1 - w)t \right\}. \quad (20)
\]

Accordingly, we have just proved
Theorem 1. The Universal Risk Budget (that parameterizes internal expert portfolios by their risk budgets $b \in [0, 1]$, and makes a uniform distribution of the initial dollar among these experts) generates a path-independent, multi-asset (correlation) option that pays off

$$\Pi(S_1, S_2, t) := \int_0^1 \left( \frac{S_1(t)}{S_1(0)} \right)^w \left( \frac{S_2(t)}{S_2(0)} \right)^{1-w} \exp(\theta w(1 - w)t) g(w)dw,$$

(21)

where the positive\(^2\) parameter $\theta := (\sigma_1^2 + \sigma_2^2)/2 - \sigma_{12} = (1/2) \text{Var} \left[ dS_1(t)/S_1(t) - dS_2(t)/S_2(t) \right]$ is equal to half the variance of the instantaneous excess percent return of asset 1 (over asset 2).

Now, we proceed to manipulate of the option payoff $\Pi(S_1, S_2, t)$ in order to work out the implied Black and Scholes 1973 (replicating) trading strategy.

\(^2\)If the excess percent return $dS_1(t)/S_1(t) - dS_2(t)/S_2(t)$ had no variance, then it would imply that the percentage returns of the two assets are exact linear functions of each other. In order to explicitly disavow this (degenerate) case, we will assume that $\sigma_1^2 + \sigma_2^2 > 2\sigma_{12}$. 

**Theorem 2.** The Universal Risk Budget amounts to using the dynamic portfolio weight process \( \hat{w}(S_1, S_2, t) \), where

\[
\hat{w}(S_1, S_2, t) = \frac{1}{\int_0^1 (S_1(t)/S_1(0))^w (S_2(t)/S_2(0))^{1-w} \exp(\theta w(1-w)t) g(w)dw} \int_0^1 w (S_1(t)/S_1(0))^w (S_2(t)/S_2(0))^{1-w} \exp(\theta w(1-w)t) g(w)dw,
\]

where the investor will have the percentage \( \hat{w}(S_1, S_2, t) \) of his capital invested in asset 1 and \( 1 - \hat{w}(S_1, S_2, t) \) invested in asset 2; the induced Black and Scholes 1973 \( \Delta \)-hedging strategy will never take a position in the risk-free bond (neither long nor short). Thus, the universal risk budget can be realized as a (weighted) universal portfolio that uses a novel initial weighting scheme that was not studied by Cover and Ordentlich 1996, which was exclusively concerned with Dirichlet (or Beta) densities.

**Proof.** The (unique) replicating strategy (cf. with Björk 2009) for the option payoff \( \Pi(S_1, S_2, t) \) must hold \( \Delta_1 := \partial \Pi/\partial S_1 \) shares of asset 1 in state \((S_1, S_2, t)\), for a value of \( \Delta_1 S_1 \) dollars. On the other hand, the replicating strategy (by the very definition of ‘replication’) has a portfolio value of \( \Pi(S_1, S_2, t) \) dollars in state \((S_1, S_2, t)\). Thus, it holds the dynamic percentage \( \Delta_1 S_1/\Pi(S_1, S_2, t) \) of wealth in asset 1. Accordingly, we differentiate under the integral sign in (21), and obtain (22). Now, since \( \Pi(\bullet) \) is linearly homogeneous in prices, we have the expansion

\[
\Pi(S_1, S_2, t) = S_1 \frac{\partial \Pi}{\partial S_1} + S_2 \frac{\partial \Pi}{\partial S_2} = \Delta_1 S_1 + \Delta_2 S_2,
\]

where \( \Delta_2 := \partial \Pi/\partial S_2 \) is the number of shares of asset 2 that are owned by the replicating strategy. Notice that the left-hand side of (23) is the total bankroll of the replicating strategy; on the other hand, the right-hand side of (23) represents the
Figure 8: The portfolio $\omega(S_1, S_2)$ used by the Universal Risk Budget, assuming the parameter values $(t, \sigma_1, \sigma_2, \rho) := (10 \text{ years}, 40\% \text{ per year}, 60\% \text{ per year}, 35\%)$. The initial (time-zero) asset prices have both been normalized to $\$1$.

The combined dollar value of the strategy’s holdings in the two risky assets. Thus, the net cash position of the Black and Scholes 1973 replicating strategy is $\Pi - \Delta_1 S_1 - \Delta_2 S_2 \equiv 0$, which is the desired result.

Note well the enormous practical value of the weight formula (22). It requires no knowledge of the asset price history except for the cumulative capital growth factors $S_1(t)/S_1(0)$ and $S_2(t)/S_2(0)$ that have been achieved to date. There is no need to deal with the interest rate $r$, and, crucially, there is no reference to the drift vector $\mu = (\mu_1, \mu_2)'$, which is perennially hard to estimate. We simply need to know how much time $t$ has elapsed since the reference date, along with an estimate of the covariance matrix, $\Sigma$. The pair of uni-dimensional integrals (22) are easily implemented in numerical software.
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The option theta (e.g., the time sensitivity) is given by

\[ \Theta := \frac{\partial \Pi}{\partial t} = \left( \frac{\sigma_1^2 + \sigma_2^2}{2} - \sigma_{12} \right) \times \int_0^1 w(1 - w)V_i(w)g(w)dw > 0, \]  

(24)

i.e., the mere passage of time (whilst holding constant the stock prices \(S_1\) and \(S_2\)) will serve to increase the bankroll of the universal risk budget. The interpretation of \(\Theta > 0\) is just this: as time goes on, the “sideways” motion of the pair of asset prices (which is assumed by this comparative static analysis) creates profitable, ongoing opportunities to execute rebalancing trades for the sake of “harvesting,” or “pumping,” volatility (Luenberger 1997; Bouchey et al. 2012). Every single continuously-rebalanced portfolio \(w \in [0, 1]\) has the property that it is continuously selling shares of the outperforming asset (that is, the asset that is outperforming \(w\)) and using the proceeds to buy more shares of the asset that underperformed the portfolio over \([t, t + dt]\). Note well that \(\Theta\) is directly proportional to the variance of the difference in instantaneous percent returns, \(\text{Var} [dS_1/S_1 - dS_2/S_2]\); the greater this variance, the greater the potential for “living off the fluctuations” of the two underlyings (cf. with Cover 1987).

As discussed above, the option Rho is zero, e.g., \(\partial \Pi/\partial r \equiv 0\), since \(r\) is absent from the definition of the universal risk budget. Next, we have the Gammas,

\[ \Gamma_i := \frac{\partial \Delta_i}{\partial S_i} = \frac{\partial^2 \Pi}{\partial S_i^2} = \frac{1}{S_i^2} \int_0^1 w(w - 1)V_i(w)g(w)dw < 0. \]  

(25)

Thus, all else equal, a marginal increase in the price of asset \(i\) will cause the universal risk budget so sell some shares of this (outperforming) asset. By the way, all of the investor’s internal “experts” \(w \in [0, 1]\) will sell some shares of asset \(i\), if it rises
suddenly without a corresponding increase in the price of the other asset. Here, our option Gammas indicate the aggregate external volume of these transactions. To this point, we have the cross Gamma

\[
\frac{\partial \Delta_1}{\partial S_2} = \frac{\partial \Delta_2}{\partial S_1} = \frac{\partial^2 \Pi}{\partial S_1 \partial S_2} = \frac{1}{S_1 S_2} \int_0^1 w(1 - w)V_i(w)g(w)dw > 0. \tag{26}
\]

This means that we will buy more shares of asset \( i \) in response to a sudden rise in the price of the other asset; of course, this is an obvious intuitive consequence of the fact that \( \Gamma_i < 0 \): asset \( i \) is the only remaining place to put the proceeds, since the universal risk budget has no cash position (long or short).

### 4 Conclusion.

This paper combined Thomas Cover’s concept of a universal portfolio (Cover 1991) with the revolutionary insight (Qian 2016; Maillard, Roncalli, and Teiletche 2010; Roncalli 2013) that a portfolio is most properly construed as an allocation of risk rather than an allocation of capital. Uniform distributions of capital among several risk assets or asset classes, while mathematically robust and attractive from a certain point of view (e.g., for the sake of volatility harvesting, or “volatility pumping,” as discussed in Luenberger 1997), can lead to unintentionally lopsided allocations of risk between assets, ETFs, or asset classes. Say, if the set of assets contains a multiplicity of double leveraged or triple leveraged ETFs that cover various sectors\(^3\), then an equal dollar allocation to each product will generate an insufficiently diversified portfolio vis-à-vis the various sources of risk and return that are available in the marketplace.

\(^3\)For example, we have the SOXL 3x semiconductor ETF and the UPW 2x utilities ETF.
The Cover 1991 concept of weighting the various capital distributions $w$ by their historical performance is susceptible of the following (very attractive) interpretation: distribute the investor’s initial dollar uniformly among all the constant-rebalanced portfolios (CRPs) in the portfolio simplex. Then, we just “let it ride,” allowing this continuum of “experts” to manage their share of the investor’s capital completely independently of one another. The disagreement between these fictitious experts (one for each capital distribution, $w$) will ultimately be resolved by their success (or failure) in compounding their initial (uniform) allocation of capital. In the long run, the transaction activity of the most successful experts (in the sense of the exponential growth rate of their capital, à la Walter White and Cathie Wood) will dwarf all the others, who will tend to hold a less and less relevant share of the investor’s overall capital as time goes on. The ongoing buy and sell orders of the various expert portfolios (some of which will be netted out internally) will result in a net external trade that the investor’s institution will make with the rest of the world. Thus, the user of the Cover 1991 universal portfolio algorithm will tend to transact in ever more sophisticated and perspicacious ways as time goes by.

In these pages, we added to the academic literature concerning on-line portfolio selection (OPS) by studying what happens if the Cover 1991 uniform prior over distributions of capital is replaced by a uniform prior over distributions of risk. This variation was shown to generate a more perspicious type of universal portfolio that is able to incorporate the covariance structure of instantaneous asset returns (which, after all, is much easier to estimate than the drift vector). Thus, we protect the universal investor from making a (potentially wrongheaded) prior distribution of capital that would tend to overweight and over-allocate to constant-rebalanced portfolios that feature uneven or excessive risk taking.

Our main theorem says that a performance-weighted averaging scheme for risk
budgets is mathematically equivalent to an Cover and Ordentlich 1996 universal portfolio that is generated by a special type of prior weighting scheme. Whereas Cover and Ordentlich 1996 only considered Dirichlet (e.g., Beta) distributions over the portfolio simplex, we derived a new and interesting family of “Garivaltis priors” that are parameterized by the volatilities and correlations of all risk assets. Naturally, the bivariate scheme was found to be homogeneous of degree zero in the pair of asset volatilities, resulting in a correlation option that essentially depends only on the volatility ratio $\lambda := \sigma_2/\sigma_1$. Thus, in keeping with modern attitudes of asset management, we find it stylistically preferable to just lever up the universal risk budget, rather than use a universal capital budget that (in its understandable reach for yield) is clearly biased in favor of certain modes of risk budgeting.

We constructed our risk budgeting scheme in the continuous time Black and Scholes 1973 market, which has the pleasant feature that the universal wealth process $(\nabla_t)_{t \geq 0}$ can be construed as the payoff $\Pi(S_1, S_2, t)$ of a certain (novel) multi-asset option (i.e., a correlation option). In spite of continuous trading by our various internal “experts” $b \in [0, 1]$ (who are now recommending risk budgets, rather than capital budgets), the final payoff is shown to depend only on the terminal asset prices $S_i(t)$ and the covariance structure of instantaneous financial returns. Thus, we have contributed to the literature of quantitative finance by proposing and studying a most interesting extension of the Jamshidian 1992 asymptotically optimal portfolio; our Universal Risk Budget is incredibly easy to operationalize in numerical software, and it takes its cue from the historical record of realized returns $S_i(t)/S_i(0)$ in a continuously robust and sensible way.
Disclosures.

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