Relativistic fluid mechanics,
Kähler manifolds and supersymmetry

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Abstract

We propose an alternative for the Clebsch decomposition of currents in fluid mechanics, in terms of complex potentials taking values in a Kähler manifold. We reformulate classical relativistic fluid mechanics in terms of these complex potentials and rederive the existence of an infinite set of conserved currents. We perform a canonical analysis to find the explicit form of the algebra of conserved charges. The Kähler-space formulation of the theory has a natural supersymmetric extension in 4-D space-time. It contains a conserved current, but also a number of additional fields complicating the interpretation. Nevertheless, we show that an infinite set of conserved currents emerges in the vacuum sector of the additional fields. This sector can therefore be identified with a regime of supersymmetric fluid mechanics. Explicit expressions for the current and the density are obtained.

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1 Introduction

Relativistic fluid mechanics has applications in the laboratory, e.g. in plasma physics and heavy ion collisions, as well as in astrophysics and cosmology [1, 2]. In recent times various extensions and reformulations of the theory have been proposed. The inclusion of non-zero vorticity and the role of non-trivial vortex topology in passing from the hamiltonian to the lagrangean description have been studied, e.g. in refs. [3]-[5]. Non-abelian extensions of the theory have been found in [6]-[9], and supersymmetric models of fluid dynamics have been proposed in [10]-[14]. The role of space and space-time symmetries has been investigated in [15, 16] and references therein. A rather remarkable result is the existence of an infinite set of conserved currents in 4-D space-time, related to the reparametrization invariance in the space of potentials [17, 18]. This seems to offer an important key to identifying fluid-dynamical phases of 4-D relativistic field theory. In this paper we use the existence of such an infinite system of currents to find a fluid-dynamical regime in a class of supersymmetric models.

This paper is organized as follows. In sect. 2 we first recall the basic facts about non-dissipative relativistic fluid mechanics. We propose an alternative to the standard Clebsch parametrization, based on complex potentials taking values in a Kähler manifold, and we rederive the fluid equations in this formalism. In sect. 3 we show the existence of a topological invariant (the vortex linking number), and an infinite set of divergence-free currents. This is followed by a discussion of the canonical structure of the theory in terms of Dirac-Poisson brackets in sect. 4. We also compute the algebra of the conserved charges. The constructions are illustrated with a simple model for a fluid with an extremal value of the state parameter \( \eta = p/\varepsilon = 1 \). In sect. 5 we introduce 4-D supersymmetry into the structure, by showing how the potentials and the fluid current can be naturally incorporated in \( N = 1 \) superfields. We propose a superfield action and present its component form, which is a generalization of the model proposed in ref. [13]. In sect. 6 we study currents and their conservation laws in the supersymmetric model. We show that there exists a regime in which an infinite number of currents is reobtained; this regime we interpret as the description of a supersymmetric fluid. We finish with a discussion of our results and possible extensions.

2 Relativistic fluid mechanics

The equations of motion of a perfect (dissipationless) relativistic fluid can be expressed in terms of a conserved and symmetric energy-momentum tensor \( T_{\mu\nu} \), derived from Poincaré invariance by Noether’s theorem. The general form of the energy-momentum tensor of a relativistic perfect fluid is (see, e.g. [1, 2]):

\[
T_{\mu\nu} = pg_{\mu\nu} + (\varepsilon + p)u_\mu u_\nu,
\]

(1)

where \( p \) is the pressure, \( \varepsilon \) is the energy-density and \( u^\mu \) is the velocity four-vector, which in natural units \((c = 1)\) is a time-like unit vector: \( u_\mu^2 = -1 \). Local energy-
momentum conservation is expressed by the vanishing of the four-divergence of the energy-momentum tensor
\[ \partial \mu T_{\mu \nu} = 0. \]  
(2)
The conserved energy-momentum four-vector is then given in a laboratory inertial frame by
\[ P_\mu = \int_{t=t_0} d^3 x T_{\mu 0}, \quad \frac{dP_\mu}{dt} = 0. \]  
(3)
In addition to the conservation of energy and momentum, the fluid density is conserved during ordinary flow as well. This is expressed by the vanishing divergence of the fluid density current \( j^\mu \):
\[ \partial \mu j^\mu = 0, \quad j^\mu = \rho u^\mu, \]  
(4)
where \( \rho \) represents the local fluid density in the local instantaneous rest frame; the normalization of the four velocity then implies that the current satisfies
\[ -j_\mu ^2 = \rho^2 \geq 0. \]  
(5)
Thus the local fluid density is defined in a Lorentz-invariant manner. In a space-plus-time formulation, equation (4) is seen to imply the equation of continuity
\[ \partial \cdot j = \partial_t (\rho \gamma) + \nabla_i (\rho \gamma v^i) = 0, \quad \gamma = \left(1 - v^2\right)^{-1/2}. \]  
(6)
Because of the vanishing divergence, for general fluid flow the current has three independent components. A standard way to express this is to write the current in terms of three scalar potentials \( (\theta, \alpha, \beta) \); they are introduced as lagrange multipliers combined in an auxiliary vector field \( a_\mu \), with the Clebsch decomposition
\[ a_\mu = \partial_\mu \theta + \alpha \partial_\mu \beta. \]  
(7)
In this formalism the component \( \theta \) describes pure potential flow, whilst \( \alpha \) and \( \beta \) are necessary to include non-zero vorticity; for a review, see \[5\].

In the present paper we advocate an alternative to the Clebsch decomposition, which is mathematically equivalent but has several advantages: it gives insight into the construction of an infinite set of conserved currents \[17, 18\], and it allows a straightforward supersymmetric generalization.

Our approach consists in replacing the real Clebsch potentials \( (\theta, \alpha, \beta) \) by one real potential \( \theta \) and one complex potential \( z \), with its conjugate \( \bar{z} \). In terms of these potentials we propose a general Lagrange density for a relativistic fluid, reproducing the conserved energy-momentum tensor \[1\]. It is given by the expression
\[ \mathcal{L}[j^\mu, \theta, \bar{z}, z] = -j^\mu a_\mu - f(\rho) \]
\[ = -j^\mu (\partial_\mu \theta + iK_z \partial_\mu z - iK_{\bar{z}} \partial_\mu \bar{z}) - f\left(\sqrt{-j^2}\right). \]  
(8)
Here \( K(z, \bar{z}) \) is a real function of the complex potentials, which we refer to as the Kähler potential, \( K_z \) and \( K_{\bar{z}} \) are its partial derivatives w.r.t. \( z \) and \( \bar{z} \), and \( f \) is a function of \( \rho = \sqrt{-j^2} \) only.
The equations motion derived from (8) are

\[
\frac{j'_\mu}{\sqrt{-j^2}} = \partial_\mu \theta + iK_z \partial_\mu z - iK_\bar{z} \partial_\mu \bar{z}, \quad \partial \cdot j = 0,
\]
\[
-2iK_z \bar{z} \partial z = 2iK_\bar{z} j \cdot \partial \bar{z} = 0.
\]

Translation invariance of the action constructed from \( \mathcal{L} \) implies the conservation of the energy-momentum tensor

\[
T_{\mu\nu} = g_{\mu\nu} \left( f' \sqrt{-j^2} - f \right) + f' \frac{j_{\mu}j_{\nu}}{\sqrt{-j^2}}, \quad \partial^\mu T_{\mu\nu} = 0,
\]

where \( f' \) is the derivative of \( f(\rho) \) w.r.t. its argument \( \rho = \sqrt{-j^2} \). Writing \( j_\mu = \rho u_\mu \), this energy-momentum tensor is of the form (10) with

\[
\varepsilon = f(\rho), \quad p = \rho f'(\rho) - f(\rho).
\]

Hence the pressure is the negative of the Legendre transform of the specific energy w.r.t. the density \( \rho \). Observe, that linear relations between pressure and specific energy correspond to power-law specific energies:

\[
\varepsilon = f(\rho) = \alpha \rho^{(1+\eta)} \Rightarrow p = \eta \varepsilon.
\]

### 3 Conservation laws

The essential elements of the class of fluid-dynamical models presented above are the existence of a divergence-free density current \( j_\mu \) and a divergence-free energy-momentum tensor \( T_{\mu\nu} \). We now show that there exist still other divergence-free currents, connected with conserved charges in the models defined above.

First we recall the construction of a conserved topological charge, related to the linking number of vortices. Following Carter we define the momentum density

\[
\pi_\mu = \frac{\delta \mathcal{L}}{\delta u^\mu}_\rho = \rho \left( \partial_\mu \theta + iK_z \partial_\mu z - iK_\bar{z} \partial_\mu \bar{z} \right).
\]

Observe that the auxiliary vector potential \( a_\mu \) is related to the momentum density by \( \pi_\mu = \rho a_\mu \):

\[
a_\mu = \partial_\mu \theta + iK_z \partial_\mu z - iK_\bar{z} \partial_\mu \bar{z} = f' \frac{j_\mu}{\sqrt{-j^2}} = f' u_\mu,
\]
\[
\pi_\mu = \rho f' u_\mu = (p + \varepsilon) u_\mu.
\]

From the definition it follows, that the axial current defined by \( k^\mu = \varepsilon^{\mu\nu\kappa\lambda} a_\nu \partial_\kappa a_\lambda \) is divergence-free:

\[
\partial_\mu k^\mu = \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu a_\nu \partial_\kappa a_\lambda = 0.
\]
The conserved charge
\[ \omega = \int d^3x \kappa^0 = \int d^3x \varepsilon^{ijk} a_i \partial_j a_k, \] (16)
is a topological quantity (the linking number of vortices), given by a pure surface term
\[ \omega = \int d^3x \partial_j \left[ i\varepsilon^{ijk} \theta \partial_j (K_{\bar{z}} a_k \bar{z} - K_z a_k \bar{z}) \right] = -2i \int d^3x \partial_j \left[ \varepsilon^{ijk} \theta K_{\bar{z}} a_k \bar{z} \partial_k \bar{z} \right]. \] (17)
Next we show that there is an infinite set of conserved charges related to the reparametrization of the potentials [17]. As a first step observe that whenever \( K_{z\bar{z}} \neq 0 \), the equations of motion for the complex potentials \( z \) and \( \bar{z} \) reduce to
\[ j \cdot \partial z = 0, \quad j \cdot \partial \bar{z} = 0. \] (18)
It follows, that any current
\[ J_\mu[G] = -2G(\bar{z}, z) j_\mu, \] (19)
is divergence-free:
\[ \partial \cdot J[G] = -2 \left( G_z j \cdot \partial z + G_{\bar{z}} j \cdot \partial \bar{z} \right) = 0. \] (20)
which allows the construction of infinitely many conserved charges of the form
\[ Q[G] = \int d^3x J^0[G]. \] (21)
The non-singularity of the Kähler potential is satisfied in all cases where \( K_{z\bar{z}} \) is the metric of a geodesically complete complex manifold. The simplest example is the complex plane with \( K(z, \bar{z}) = \bar{z} z \) and hence \( K_{z\bar{z}} = 1 \). Another example is the sphere \( S^2 = CP^1 \), with the Kähler potentials \( K^{(\pm)}(z_\pm, \bar{z}_\pm) = \ln(1 + \bar{z}_\pm z_\pm) \) to be used on the northern and southern hemisphere, respectively, related up to the real part of a holomorphic function by the analytic co-ordinate transformation \( z_- = 1/z_+ \); in this case
\[ K^{(\pm)}_{z\bar{z}} = \frac{1}{(1 + \bar{z}_\pm z_\pm)^2}. \] (22)

4 Canonical structure

We now pass to the canonical formulation of the theory. First we define the canonical momenta
\[ \pi_\theta = \frac{\partial L}{\partial \dot{\theta}} = j_0, \quad \pi_z = \frac{\partial L}{\partial \dot{z}} = iK_z j_0, \quad \pi_{\bar{z}} = \frac{\partial L}{\partial \dot{\bar{z}}} = -iK_{\bar{z}} j_0. \] (23)
With \( \rho = \sqrt{\pi_\theta^2 - j^2} \) the hamiltonian density and spatial current components are
\[ H = \frac{f'(\rho)}{\rho} j^2 + f(\rho), \quad \frac{f'(\rho)}{\rho} j = \nabla \theta + iK_z \nabla z - iK_{\bar{z}} \nabla \bar{z}. \] (24)
Obviously, the last two equations (23) are second-class constraints, expressing \((\pi_z, \pi_{\bar{z}})\) in terms of the other phase-space variables \((z, \bar{z}, \pi_\theta)\):

\[
\chi_z = \pi_z - iK_{\bar{z}}\pi_\theta = 0, \quad \chi_{\bar{z}} = \pi_{\bar{z}} + iK_z\pi_\theta = 0.
\]

To describe the canonical dynamics on the reduced phase-space determined by these equations, we introduce Poisson-Dirac brackets

\[
\{A, B\}^* = \{A, B\} - \{A, \chi_i\} C^{-1}_{ij} \{\chi_j, B\},
\]

where \(C^{-1}\) is the inverse of the matrix of constraint brackets

\[
C_{ij} = \{\chi_i(r, t), \chi_j(r', t)\} = \begin{pmatrix} 0 & -2iK_{\bar{z}}\pi_\theta \\ 2iK_{z}\pi_\theta & 0 \end{pmatrix} \delta(r - r').
\]

From the definition (23), it follows, that in the reduced phase space spanned by \((z, \bar{z}, \theta, \pi_\theta)\) the canonical Poisson-Dirac brackets are

\[
\{z(r, t), \bar{z}(r', t)\}^* = \frac{-i}{2K_{z}\pi_\theta} \delta(r - r'), \quad \{\theta(r, t), \pi_\theta(r', t)\}^* = \delta(r - r'),
\]

\[
\{z(r, t), \theta(r', t)\}^* = \frac{K_{\bar{z}}}{2K_z\pi_\theta} \delta(r - r'), \quad \{\bar{z}(r', t), \theta(r, t)\}^* = \frac{K_z}{2K_{\bar{z}}\pi_\theta} \delta(r - r').
\]

With the help of these rules we can determine the algebra of the conserved charges. It is useful to revert to a geometrical notation in terms of a simple Kähler manifold with metric \(g_{zz} = g_{\bar{z}\bar{z}} = K_{\bar{z}\bar{z}}\) and its inverse \(g^{zz} = g^{\bar{z}\bar{z}} = 1/K_{zz}\). The action of the \(Q[G]\) on the potentials then is:

\[
\delta_G \theta = \{Q[G], \theta\}^* = 2G - g^{\bar{z}\bar{z}} (K_{\bar{z}}G_z + K_zG_{\bar{z}}),
\]

\[
\delta_G z = \{Q[G], z\}^* = -i g^{\bar{z}\bar{z}}G_{\bar{z}},
\]

\[
\delta_G \bar{z} = \{Q[G], \bar{z}\}^* = i g^{zz}G_z.
\]

If \(G(\bar{z}, z)\) is taken to transform as a scalar on the complex manifold, the transformations \(\delta_G\) are seen to take a covariant form and represent a reparametrization of the complex target manifold of the potentials \((\bar{z}, z)\). These transformations have the property that they leave the auxiliary vector potential (one-form) invariant:

\[
a = dx^\mu a_\mu = d\theta + iK_zdz - iK_{\bar{z}}d\bar{z} \quad \Rightarrow \quad \delta_G a = 0.
\]

As \(a_\mu^2 = -f^{\prime 2}(\rho)\), it follows that also \(\delta_G \rho = 0\) and \(\delta_G j_\mu = 0\). It is clear, that the transformations \(\delta_G(\theta, z, \bar{z})\) in eq. (29) together with \(\delta_G j_\mu = 0\) define an infinite set of global symmetries of the lagrangean (8) and the hamiltonian (24):

\[
\{Q[G], H\}^* = 0, \quad H = \int d^3 x H.
\]
These symmetries imply the reparametrization invariance of the equations of motion of the auxiliary vector potential.

A particularly simple instance is the flat complex plane with $K = \bar{z}z$. Then

$$\{Q[G], \theta\}^* = 2G - (zG_z + \bar{z}G_{\bar{z}}),$$

$$\{Q[G], z\}^* = -iG_{\bar{z}}, \quad \{Q[G], \bar{z}\}^* = iG_z. \quad (32)$$

For this special choice we then find

$$\{Q[1], \theta\}^* = 2, \quad \{Q[K], \theta\}^* = 0,$$

$$\{Q[1], z\}^* = 0, \quad \{Q[K], z\}^* = iz,$$

$$\{Q[1], \bar{z}\}^* = 0, \quad \{Q[K], \bar{z}\}^* = -i\bar{z}. \quad (33)$$

Returning to the case of general $K(z, \bar{z})$, we finally notice the closure of the algebra of conserved charges:

$$\{Q[G^{(1)}], Q[G^{(2)}]\}^* = Q[G^{(3)}], \quad (34)$$

with

$$G^{(3)} = ig^{zz} \left( G^{(1)}_z G^{(2)}_{\bar{z}} - G^{(1)}_{\bar{z}} G^{(2)}_z \right). \quad (35)$$

This expression has itself the structure of a Poisson bracket on the 2-d manifold spanned by $(\bar{z}, z)$.

An example of a simple model relevant for later discussions is defined by

$$f(\rho) = \frac{\lambda}{2} \rho^2 \quad \rightarrow \quad p = \varepsilon = \frac{\lambda}{2} \rho^2. \quad (36)$$

The hamiltonian density then becomes

$$\mathcal{H} = \frac{\lambda}{2} \left( j^2 + \pi^2_{\theta} \right), \quad (37)$$

where the current is

$$j = \frac{1}{\lambda} \left( \nabla \theta + iK_z \nabla z - iK_{\bar{z}} \nabla \bar{z} \right). \quad (38)$$

In this case the brackets with the current components become

$$\{z(r, t), j(r', t)\}^* = \frac{\nabla z}{\lambda \pi_{\theta}} \delta(r - r'), \quad \{\bar{z}(r, t), j(r', t)\}^* = \frac{\nabla \bar{z}}{\lambda \pi_{\theta}} \delta(r - r'),$$

$$\{\theta(r, t), j(r', t)\}^* = \frac{i}{\lambda \pi_{\theta}} \left( K_z \nabla \bar{z} - K_{\bar{z}} \nabla z \right) \delta(r - r'). \quad (39)$$

Upon the identification $j_0 = \pi_{\theta}$, the brackets of the fields $\Phi = (\theta, z, \bar{z})$ with the hamiltonian can easily be checked to reproduce the field equations: \varepsilon:

$$\dot{\Phi} = \{\Phi, H\}^*. \quad (40)$$
5 Supersymmetry

The decomposition of the auxiliary vector in terms of real and complex scalar potentials has a natural supersymmetric extension in 4-d Minkowski space-time. This leads to a proposal for a supersymmetric version of relativistic fluid dynamics in 4-d space-time. The supersymmetric extension is obtained by identifying the current \( j_\mu \) and the auxiliary vector \( a_\mu \) with the vector components \( V_\mu \) and \( A_\mu \) of two real superfields \( V \) and \( A \), with a general superfield action of the form

\[
S[V, A] = \int d^4x \mathcal{L}[V, A] = \int d^4x \int d^2\theta_+ \int d^2\theta_- \left( \frac{1}{4} VA - F(V) \right). \quad (41)
\]

Here \( \theta_\pm \) are the positive/negative chirality components of the spinor coordinates of superspace. In terms of the multiplets of components \( V = (C, \psi, M, V_\mu, \lambda, D) \) and \( A = (B, \chi, N, A_\mu, \omega, G) \) the component lagrangean reads

\[
\mathcal{L}[V, A] = CG + BD + 2(M\bar{N} + N\bar{M}) - V \cdot A - \partial C \cdot \partial B - F'(C)D
\]

\[
- \bar{\lambda}_+\chi_+ - \bar{\lambda}_-\chi_- - \bar{\omega}_+\psi_+ - \bar{\omega}_-\psi_- - \frac{1}{2} \bar{\psi}_+ \gamma_+ \psi_- - \frac{1}{2} \bar{\chi}_+ \gamma_+ \psi_-
\]

\[
- F''(C) \left[ 2|M|^2 - \frac{1}{2} V_\mu^2 - \frac{1}{2} (\partial_\mu C)^2 - \bar{\lambda}_+\psi_+ - \bar{\lambda}_-\psi_- - \frac{1}{2} \bar{\psi}_+ \gamma_+ \psi_- \right]
\]

\[
+ \frac{1}{2} F'''(C) \left[ M\bar{\psi}_-\psi_- + \bar{M}\psi_+\psi_+ - i\bar{\psi}_+\psi_- \right] - \frac{1}{8} F''''(C) \bar{\psi}_+\psi_+\psi_-\psi_-
\]

The decomposition (8), (9) of a vector field in terms of potentials can be extended to supersymmetric models by writing the auxiliary vector superfield in terms of two sets of complex chiral superfields \((\Lambda, \bar{\Lambda}, \Phi, \bar{\Phi})\), as follows:

\[
A = \Lambda + \bar{\Lambda} + K(\bar{\Phi}, \Phi), \quad (43)
\]

where \( K(\Phi, \bar{\Phi}) \) is a real function of its superfield arguments; below it will become clear that its lowest bosonic component \( K(\bar{z}, z) \) is the Kähler potential for the complex potentials \((\bar{z}, z)\). We wish to point out, that a slight refinement of the theory can be obtained by taking \( \Lambda \) to be defined in terms of a real (vector) superfield \( W \) by

\[
\Lambda = \frac{1}{4} \bar{D}_+D_+W, \quad \bar{\Lambda} = \frac{1}{4} \bar{D}_-D_-W, \quad (44)
\]

with \( D_\pm \) the usual covariant superspace derivatives. However, the equations for the physical degrees of freedom are the same as in the case of \( \Lambda \) being a fundamental superfield, except for one mass parameter which is forced to vanish in the case of the decomposition (13). In this paper we treat only this simpler case.

We label the components of the chiral superfield potentials by \( \Lambda = (v, \varphi, E) \) and \( \Phi = (z, \eta, H) \). Then the components of the auxiliary superfield \( A \) are replaced by
To make the connection with fluid mechanics, cf. eq.(8), we have introduced the expressions

\[ \chi_+ = \psi_+ + Kz \eta_+, \quad \chi_- = \psi_+ + Kz \eta_- \]

\[ N = \frac{1}{2} \left[ E + Kz \left( H - \frac{1}{2} \Gamma_{zz}^z \eta_+ \eta_+ \right) \right], \quad \bar{N} = \frac{1}{2} \left[ \bar{E} + Kz \left( H - \frac{1}{2} \Gamma_{zz}^z \eta_- \eta_- \right) \right], \]

\[ A_\mu = -i \partial_\mu (v - \bar{v}) + i (Kz \partial_\mu z - Kz \partial_\mu \bar{z}) - \frac{i}{2} g_{z z} \bar{\eta}_+ \gamma_\mu \eta_- , \]

\[ \omega_+ = \frac{1}{2} g_{z z} \left( H - \frac{1}{2} \Gamma_{zz}^z \eta_- \eta_- \right) \eta_+ - g_{z z} \bar{\eta}_+ \bar{\eta}_- \]

\[ \omega_- = \frac{1}{2} g_{z z} \left( H - \frac{1}{2} \Gamma_{zz}^z \eta_+ \eta_+ \right) \eta_- - g_{z z} \bar{\eta}_- \bar{\eta}_+ , \]

\[ G = -2 g_{z z} \partial \bar{z} \cdot \partial z - \frac{1}{2} g_{z z} \bar{\eta}_+ \bar{\eta}_+ \psi \psi - \frac{1}{8} R_{z z z z} \bar{\eta}_+ \eta_+ \bar{\eta}_- \eta_- \]

\[ + \frac{1}{2} g_{z z} \left( H - \frac{1}{2} \Gamma_{zz}^z \eta_+ \eta_+ \right) \left( H - \frac{1}{2} \Gamma_{zz}^z \eta_- \eta_- \right) . \]

Here \( g, \Gamma \) and \( R \) denote the metric, connection and curvature constructed from the Kähler potential \( K \), respectively; moreover, the covariant derivative of the chiral spinors \( \eta_\pm \) are defined by

\[ \bar{\psi} \eta_+ = \partial \eta_+ + \bar{\eta}_+ \Gamma_{zz}^z \eta_+ , \quad \bar{\psi} \eta_- = \partial \eta_- + \partial z \Gamma_{zz}^z \eta_- . \]

Upon substitution of these expressions into eq.(42) and elimination of the auxiliary fields, we then find the effective component lagrangian

\[ \mathcal{L}_{\text{eff}} = -V^\mu \left( \partial_\mu \theta + i Kz \partial_\mu z - i Kz \partial_\mu \bar{z} - \frac{i}{2} g_{z z} \bar{\eta}_+ \gamma_\mu \eta_- \right) \]

\[ - \frac{1}{2} F''(C) \left[ -V^2 + (\partial C)^2 + \bar{\psi}_+ \imath \partial \psi_- \right] - \frac{i}{2} F'''(C) \bar{\psi}_+ \nabla \psi_- - C \left( 2 g_{z z} \partial z \cdot \partial z + \frac{1}{2} g_{z z} \bar{\eta}_+ \bar{\psi}_+ \partial z \eta_- + \frac{1}{8} R_{z z z z} \bar{\eta}_+ \eta_+ \bar{\eta}_- \eta_- \right) \]

\[ - \frac{1}{4c} g_{z z} \bar{\eta}_+ \gamma_\mu \eta_- \bar{\psi}_+ \gamma_\mu \psi_- + g_{z z} \left( \bar{\psi}_+ \bar{\eta}_- \psi_- + \bar{\psi}_+ \bar{\psi}_- \bar{\eta}_+ \right) \]

\[ - \frac{1}{8} F'''(C) \bar{\psi}_+ \psi_+ \bar{\psi}_- \psi_- . \]

To make the connection with fluid mechanics, cf. eq.(3), we have introduced the notation \( \theta = 2 \text{Im} \, v \). It is obvious, that in the absence of fermions \( \psi_\pm = \eta_\pm = 0 \) and for \( C = 0 \) we reobtain the lagrangian (3) with

\[ f(\rho) = \frac{1}{2} F''(0) \rho^2 . \]
This is of the type (36) with $\lambda = F''(0)$. The scalar $C$ and the spinor fields $\psi$ and $\eta$ describe additional dynamical fields. Being the co-efficient of the kinetic terms of the fields $(\bar{z}, z)$ and $\eta_{\pm}$, physics requires the scalar field $C$ to be non-negative. This can easily be achieved, for example by replacing the real superfield $V$ by another real superfield $W$ such that $V = e^W$. Thus we can take the condition $C \geq 0$ for granted.

A simpler version of this action with only a single real scalar potential $\theta$ (hence $\bar{z} = z = \eta_{\pm} = 0$) was discussed in [13]. In the absence of the complex potentials its bosonic reduction in the hydrodynamical regime describes only potential flow; therefore in this model vorticity arises only by the presence of fermions.

6 Supersymmetric fluid dynamics

The supersymmetric extension of the action for fluid dynamics constructed above generally goes at the expense of most of the infinitely many conservation laws related to reparametrizing the potential, eqs.(19), (21). This can already be inferred from the bosonic part of the theory. Consider the bosonic terms in the equations of motion for the current and the potentials:

$$\partial \cdot V = 0, \quad 2D \cdot (C \partial z) - 2iV \cdot \partial z = 2D \cdot (C \partial \bar{z}) + 2iV \cdot \partial \bar{z} = 0,$$

(49)

Here $D_{\mu}$ denotes a covariant derivative containing the Kähler connection, e.g.

$$D_{\mu}(\partial_{\nu}z) = \partial_{\mu} \partial_{\nu} z + \Gamma_{zz}^{\nu} \partial_{\mu} z \partial_{\nu} z.$$

(50)

Now construct the currents

$$J_{\mu}[G] = -2G(\bar{z}, z) V_{\mu} - 2iC(G_{\bar{z}} \partial_{\mu} z - G_{z} \partial_{\mu} \bar{z}),$$

(51)

where $G(\bar{z}, z)$ is a real function of the complex scalar fields. Using eqs. (49) it can be seen to satisfy

$$\partial \cdot J[G] = -2iC \left( G_{\bar{z}z} (\partial z)^2 - G_{\bar{z}\bar{z}} (\partial \bar{z})^2 \right).$$

(52)

It follows that the divergence of the current vanishes identically only for functions $G(\bar{z}, z)$ such that the homogeneous second derivatives w.r.t. $\bar{z}$ and $z$ vanish:

$$G_{\bar{z}z} = G_{\bar{z}\bar{z}} = 0.$$  

(53)

This holds if and only if the gradients of $G(\bar{z}, z)$ represent holomorphic Killing vectors $(R(z), R(\bar{z}))$, generating isometries of the Kähler manifold; a proof is presented in appendix A. As the number of independent isometries of a finite-dimensional manifold is finite, no infinite set of conserved currents can be generated by Killing vectors.

Still, as anticipated an infinite set of conserved currents $J_{\mu}[G]$ is obtained for all models (47) under the restriction $C = 0$. Therefore we identify the manifold
of states with $C = 0$ as the hydrodynamical regime of the supersymmetric models constructed here.

In the fully supersymmetric case it is easy to prove that holomorphic Killing vectors generate conserved currents. It can largely be done directly in superspace, as follows: let $R(\Phi)$ and $\bar{R}(\bar{\Phi})$ be holomorphic functions of the chiral superfields $\Phi$ and $\bar{\Phi}$, respectively, defining Killing vectors of the Kähler metric $g_{\Phi \bar{\Phi}} = K_{\Phi \bar{\Phi}}$; the essential element is, that under these transformations the Kähler potential transforms non-homogeneously into the real part of a holomorphic function:

$$\delta \Phi = R(\Phi), \quad \delta \bar{\Phi} = \bar{R}((\bar{\Phi}), \quad \delta K(\Phi, \bar{\Phi}) = F_R(\Phi) + \bar{F}_R((\bar{\Phi}). \quad (54)$$

The precise form of the holomorphic function $F_R(\Phi)$ depends on the Killing vector $R(\Phi)$. Then defining the superfield transformations

$$\delta V = 0, \quad \delta \Lambda = -F_R(\Phi), \quad \delta \bar{\Lambda} = -\bar{F}_R(\bar{\Phi}), \quad (55)$$

the superspace action (41) with $A$ given by (43) is seen to be invariant. Noether’s theorem then guarantees the existence of a conserved current for each of the Killing vectors. In components they take the form

$$J_{\mu}[G] = -2GV_\mu - 2iCG_{zz} \left( \partial_\mu z - \frac{1}{2C} \bar{\psi}^+ \gamma_\mu \eta_- \right) + 2iCG_{\bar{z} \bar{z}} \left( \partial_\mu \bar{z} - \frac{1}{2C} \psi^- \gamma_\mu \eta_+ \right)$$

$$- iCG_{\bar{z} z} \bar{\eta}_- \gamma_\mu \eta_+, \quad (56)$$

where $G(\bar{z}, z)$ is the Killing potential for the isometries $\delta z = R(z)$ and their complex conjugates (see appendix A). For a generic real function $G(\bar{z}, z)$ which is not a Killing potential, the current $J_{\mu}[G]$ is not conserved, unless one takes the limit $(C, \eta) \to 0$, such that the spinor field $\eta$ vanishes as fast as $C$. Solutions of the model with this property we interpret as a supersymmetric fluid.

To analyse this regime, we rescale the fermion fields as follows

$$\psi_\pm = \frac{1}{\sqrt{F''(C)}} \Psi_\pm, \quad \eta_\pm = C\Omega_\pm. \quad (57)$$

Then the lagrangean (47) becomes

$$\mathcal{L}_{\text{eff}} = -V^{\mu} \left( \partial_\mu \theta + iK_z \partial_\mu z - iK_{\bar{z}} \partial_\mu \bar{z} - \frac{i}{2} C^2 g_{\bar{z} z} \bar{\Omega}_+ \gamma_\mu \Omega_- + \frac{iF''(C)}{2F'(C)} \bar{\Psi}^+ \gamma_\mu \Psi_\pm \right)$$

$$+ \frac{1}{2} F''(C) \left( V^{\mu} - \left( \partial_\mu C \right)^2 \right) - \frac{1}{2} \bar{\Psi}^+ \bar{\tilde{\Theta}} \Psi_- - \frac{F''(C)}{8F'(C)^2} \bar{\Psi}^+ \Psi_+ \bar{\Psi}^- \Psi_-$$

$$- 2C g_{\bar{z} z} \partial_\bar{z} \cdot \partial z - \frac{1}{2} C^2 g_{\bar{z} z} \bar{\Omega}_+ \bar{\tilde{\Theta}} \Omega_- + \frac{1}{8} C^5 R_{\bar{z} z z z} \bar{\Omega}_+ \Omega_+ \Omega_- \Omega_-$$

$$+ g_{\bar{z} z} \frac{C}{\sqrt{F''(C)}} \left( \bar{\Psi}^+ \partial_\bar{z} \Omega_- + \bar{\Psi}^- \partial z \Omega_+ \right) - g_{\bar{z} z} \frac{C}{4F'(C)} \bar{\Omega}_+ \gamma_\mu \Omega_- \bar{\Psi}^+ \gamma_\mu \Psi_-. \quad (58)$$
We observe that in the limit $C = 0$ divergent terms can be avoided, provided $F''(0) \neq 0$. Then we can always normalize $F(C)$ such that $F''(0) = 1$; with this choice the quadratic vector term and the kinetic term of the real scalar $C$ have the canonical normalization.

Next we observe, that there exist many choices of the function $F(C)$ such that also the coefficients of the bilinear and quartic terms in $\Psi$ are finite. Indeed, any function such that the second derivative has the expansion

$$F''(C) = 1 + \lambda_1 C + \lambda_2 C^2 + O(C^3)$$

satisfies the conditions

$$F''(0) = 1, \quad F'''(0) = \lambda_1, \quad F''''(0) = 2\lambda_2,$$

and makes the lagrangean finite in the hydrodynamical regime. Having established the existence of regular configurations with $C = 0$, the expression for the current in this regime becomes

$$V_\mu[C = 0] = \partial_\mu \theta + i K_z \partial_\mu z - i K_\bar{z} \partial_\mu \bar{z} + \frac{i\lambda_1}{2} \Psi + \gamma_\mu \Psi_-.$$

The physical interpretation of these equations is implicit in their bosonic terms. For a hydrodynamical current

$$V_{\mu} = \rho u_\mu \quad \Rightarrow \quad V_{\mu}^2 = -\rho^2.$$

The bosonic part of the first equation becomes

$$\rho^2 = -\frac{4}{\lambda_1} g_{zz} \partial \bar{z} \cdot \partial z \geq 0.$$
which implies that apart from fermionic contributions, the spatial gradient of the complex scalar field determines the fluid density $\rho$. Similarly, for $\lambda_1 > 0$ the time rate of change of $z$ determines $\rho$. With the identification (63), the bosonic part of the energy-momentum (62) is of the form (1). The corresponding energy and pressure densities are given by

$$\varepsilon = p = \frac{1}{2} \rho^2 = \frac{2}{|\lambda_1|} g_{zz} \left( |\nabla z|^2 - |\dot{z}|^2 \right). \tag{66}$$

It is not difficult to check, that the equation for the complex scalar fields ($\bar{z}, z$) for $C = 0$ reproduce the conditions

$$V \cdot \partial z = V \cdot \partial \bar{z} = 0, \tag{67}$$

cf. eqs. (9). This establishes the existence of a regime $C = 0$ in which the supersymmetric model allows a fluid-mechanical interpretation.

In the following, we show that the full energy-momentum tensor (62) takes the standard form (1), under particular conditions specified by

$$\partial_{\mu} \Psi_{\pm} = \pm \frac{i}{2} \lambda_1 V_{\mu} \Psi_{\pm} - \frac{\lambda_2}{8} \gamma_{\mu} \Psi_{\mp} \bar{\Psi} \Psi_{\pm}, \quad \lambda_1^2 = \lambda_2. \tag{68}$$

Indeed upon substitution of these expressions into (62), the energy-momentum becomes

$$T_{\mu\nu} = W_{\mu} W_{\nu} - \frac{1}{2} g_{\mu\nu} W^2, \quad W_{\mu} = V_{\mu} - \frac{i}{2} \lambda_1 \Psi_{\mp} \gamma_{\mu} \Psi_{\pm}, \quad \partial \cdot W = 0. \tag{69}$$

The vanishing divergence of the current $W_{\mu}$ follows upon using the field equations

$$\partial \Psi_{\pm} = \pm \frac{i}{2} \lambda_1 V_{\Psi_{\pm}} - \frac{\lambda_2}{2} \Psi_{\mp} \bar{\Psi}_{\pm} \Psi_{\pm}, \quad \partial \cdot V = 0. \tag{70}$$

Therefore, we can reinterpret the $W_{\mu}$ as the hydrodynamical current $W_{\mu} = \rho u_{\mu}$ with the equation of state:

$$\varepsilon = p = \frac{1}{2} \rho^2, \quad \rho^2 = -\frac{1}{\lambda_1} \left[ 4 g_{zz} \partial \bar{z} \cdot \partial z + \frac{1}{2} \left( 3 \lambda_3 - 2 \lambda_1^3 \right) \bar{\Psi}_{\mp} \Psi_{\mp} \bar{\Psi}_{\pm} \Psi_{\pm} \right].$$

Next, we investigate the properties of non-trivial solutions to the Ansatz (68). Since (68) can be written as

$$D_{\mu} \Psi_{\pm} = \left( \partial_{\mu} + \frac{i \lambda_1}{2} W_{\mu} \right) \Psi_{\pm} = 0, \tag{71}$$

with the axial covariant derivative $D_{\mu} = \partial_{\mu} - \frac{i \lambda_1}{2} W_{\mu} \gamma_5$ using a Fierz identity, the ansatz has the formal solution

$$\Psi_{\pm}(x) = \mathcal{P} \exp \left( \pm \frac{i \lambda}{2} \int_0^x W \cdot dx \right) \Psi_{\pm}(0). \tag{72}$$
where, $\mathcal{P}$ is the path-ordering operator. In fact, this solution shows that the fermion bilinear
\[
\bar{\Psi}(x)\gamma_\mu \Psi(x) = \bar{\Psi}(0)\gamma_\mu \Psi(0)
\] (73)
is constant. On the other hand, from (72) one infers that the field strength associated with the covariant derivative $\mathcal{D}_\mu$ vanishes. This implies that $W_\mu$ has to be pure gauge for non–trivial fermion solutions to exist. And because the fermion bilinear is constant, we conclude that $V_\mu$ is pure gauge. This shows that the system in this regime is described by potential flow.

Finally, we return to the identically conserved currents constructed from Killing potentials. For the simplest Kähler geometry, the complex plane, the potential is
\[
K(\bar{z}, z) = \bar{z}z.
\] (74)
The plane admits the following holomorphic Killing vectors, related to translations and rotations in the plane:
\[
\delta z = R^z(z) = \epsilon + i\alpha z, \quad \delta \bar{z} = \bar{R}^\bar{z}(\bar{z}) = \bar{\epsilon} - i\alpha \bar{z}.
\] (75)
with the property
\[
\delta K = \epsilon \bar{z} + \bar{\epsilon}z.
\] (76)
The corresponding Killing potentials are
\[
G(\bar{z}, z) = i(\bar{\epsilon}z - \epsilon \bar{z}) + \alpha \bar{z}z.
\] (77)
If we insert these expressions into the brackets (29), (32) we obtain the transformations generated by the conserved Noether charges
\[
Q[G] = \int d^3x J_0[G],
\] (78)
then we indeed reobtain the isometries:
\[
\delta_G z = R^z(z), \quad \delta_G \bar{z} = \bar{R}^\bar{z}(\bar{z}),
\] (79)
\[
\delta_G \theta = i(\epsilon \bar{z} - \bar{\epsilon}z).
\]
By including the fermionic contributions to the conserved Noether charges [50], one finds in addition to (79) the transformations of other fields
\[
\delta_G \eta_- = \{Q[G], \eta_-\}^* = R^z_\bar{z}(z)\eta_-,
\]
\[
\delta_G \eta_+ = \{Q[G], \eta_+\}^* = \bar{R}^{\bar{z}}\bar{z}(\bar{z})\eta_+, \quad \delta_G V_\mu = \{Q[G], V_\mu\}^* = 0
\] (80)
\[
\delta_G \psi_{\pm} = \{Q[G], \psi_{\pm}\}^* = 0, \quad \delta_G C = \{Q[G], C\}^* = 0.
\]
For the details see ref. [19].
7 Conclusions

In this paper we have investigated the geometrical structure of relativistic fluid mechanics and its supersymmetric extension.

Conventionally, a relativistic fluid with non-vanishing vorticity is described by the Clebsch decomposition of the current. We propose a different parameterization in terms of one real and one complex degree of freedom. The complex degree of freedom \( z \) takes its values on a Kähler manifold. In this formulation the infinite set of conserved currents \( J_\mu[G] \) of fluid dynamics are associated with the set of functions \( G \) of this complex variable \( z \) and its conjugate \( \bar{z} \). A canonical analysis using the Poisson-Dirac bracket showed, that the closure of the algebra of conserved currents leads to a Poisson bracket structure for these functions on the Kähler manifold. In eqs. (35) this Poisson structure is given in terms of the inverse Kähler metric.

However, the main advantage of our Kähler parameterization of the fluid current is, that it allows for a rather straightforward supersymmetric completion. The lagrangean for the supersymmetric system (31) has been expressed in terms of superfields, which allows both an on- and off-shell supersymmetric formulation. The only restriction which our formulation seems to impose is, that the energy density is proportional to the square of the current density. Contrary to a fluid mechanical system, the supersymmetric model does not possess an infinite set of conserved: In general, the currents \( J_\mu[G] \) are only conserved, if the function \( G \) is a Killing potential associated to an isometry of the Kähler manifold. However, the theory does contain a regime, in which the infinite number of currents are conserved. This regime arises in a supersymmetric background in which expectation values of the fermions and some bosonic fields are zero.

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A Killing vectors and their currents

A Kähler manifold is a complex manifold with a curl-free metric; the latter condition implies that locally one can write the metric as the mixed second derivative of a real scalar function $K(\bar{z}, z)$, the Kähler potential: $g_{\bar{z}z} = K_{\bar{z}z}$. This metric is invariant under infinitesimal holomorphic co-ordinate transformations $\delta z = R^z(z)$, $\delta \bar{z} = \bar{R}^\bar{z}(\bar{z})$ if and only if

$$ R_{\bar{z};z} + R_{z;\bar{z}} = 0. \quad (81) $$

The solutions of these equations are holomorphic Killing vectors. The invariance of the metric implies that the Kähler potential can only change by the real part of a holomorphic function:

$$ \delta K(\bar{z}, z) = R^z K_{,z} + \bar{R}^\bar{z} K_{,\bar{z}} = F(z) + \bar{F}(\bar{z}) \quad (82) $$

From this we infer the existence of a real scalar function $G(\bar{z}, z)$ such that

$$ R^z K_{,z} - F = -\bar{R}^\bar{z} K_{,\bar{z}} + \bar{F} = iG. \quad (83) $$

As both $R^z$ and $F$ are holomorphic functions, and the metric is covariantly constant, it follows that

$$ G_{,\bar{z}} = -i g_{\bar{z}z} R^z = -i R_{\bar{z}} \quad \Leftrightarrow \quad G_{,\bar{z};\bar{z}} = -i R_{\bar{z};\bar{z}} = 0. \quad (84) $$

Because of this relation $G$ is called a Killing potential. The above argument runs both ways, because first the condition $G_{,\bar{z};\bar{z}} = 0$ can be integrated to give

$$ i G_{,\bar{z}} g^{\bar{z}z} = R^z(z), \quad (85) $$

a holomorphic function; and second because this implies that the function $F$ defined by

$$ F = R^z K_{,z} - iG, \quad (86) $$

then is also holomorphic: $F_{,\bar{z}} = 0$. Therefore under a transformation $\delta z = R^z(z)$ and its complex conjugate the Kähler potential changes only by the real part of a holomorphic function, and the metric is invariant.
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