Abstract

Building on the Utiyama principle we formulate an approach to Lagrangian field theory in which exterior covariant differentials of vector-valued forms replace partial derivatives, in the sense that they take up the role played by the latter in the usual jet bundle formulation. Actually a natural Lagrangian can be written as a density on a suitable “covariant prolongation bundle”; the related momenta turn out to be natural vector-valued forms, and the field equations can be expressed in terms of covariant exterior differentials of the momenta. Currents and energy-tensors naturally also fit into this formalism. The examples of bosonic fields and spin one-half fields, interacting with non-Abelian gauge fields, are worked out. The “metric-affine” description of the gravitational field is naturally included, too.

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Introduction

Let $E \to M$ be a vector bundle. The notion of \textit{[exterior] covariant differential} of $E$-valued exterior forms on $M$, defined by means of a linear connection of $E$, can be viewed as a generalization of the standard covariant derivative; it has been variously present in the literature for several years, possibly in relation to ideas by Koszul \cite{21,22}. More recently, the realization that the Frölicher-Nijenhuis bracket yields a natural framework for dealing with such notions has suggested further generalizations and a systematic study by various authors \cite{9,10,33,29,30,32,31,20,14}. In particular, it has been observed that the curvature tensor of any connection can be regarded as the covariant differential of the connection with respect to itself.

Thus, in consideration of the \textit{Utiyama principle} \cite{38,20,18}, covariant differentials may provide a convenient setting for natural gauge field theories. Actually one finds, in the General Relativistic literature, a formulation of Lagrangian field theory on a gravitational background in which covariant derivatives with respect to the spacetime connection replace partial derivatives \cite{27,15}. While that formulation explicitly considers only fields with spacetime indices, it is not difficult to become convinced that, even when the fields have further “internal degrees of freedom”, labeled by fiber indices that are not “soldered” to spacetime, such limitation does not invalidate the essential results regarding the properties of the stress-energy tensor of matter fields (the right-hand side of the Einstein equation).

The main purpose of this paper is to present a generalization of that approach, and to explore the extent up to which the role of partial derivatives can be taken up by covariant differentials. The field of an essential gauge field theory is to be described as a couple consisting of a \textit{matter field} and a \textit{gauge field}, namely $(\phi, \kappa) : M \to F \equiv E \times_M C$ where $C \to M$ is a bundle of linear connections of $E$. Denoting by $d_\kappa$ the covariant differential with respect to $\kappa$, we get a “covariant prolongation”

$$(d_\kappa \phi, d_\kappa \kappa) : M \to (T^*M \otimes E) \times_M (\wedge^2 T^*M \otimes \text{End } E)$$

that is suitable for taking up the role of the fields’ jet prolongations used in the standard formulation of Lagrangian field theory. Indeed, the Lagrangian density can be expressed as a function on a “covariant prolongation space” $\mathcal{DF}$ (rather than a function on the first jet prolongation space $J^1M$), and we show that various fundamental constructions and results of the standard theory have a counterpart in this new formulation.

In particular, momenta turn out to be geometrically well-defined objects on $\mathcal{DF}$. Infinitesimal variations can be introduced as morphisms on $\mathcal{DF}$, and the field equations can be expressed in terms of covariant differentials of the momenta evaluated through the covariant prolongations of the fields (theorem 2.1). Thus, the field equations can be explicitely written in coordinate-free form. Currents and canonical energy-tensors can also be revisited in terms of objects defined on $\mathcal{DF}$.

Above, the base $M$ can be a generic manifold. Then (§3) we consider the case when $M$ is a Lorentzian spacetime—whose structure is regarded as a fixed gravitational background—and the matter field can have spacetime or spinor indices. These behave differently from the other internal degrees of freedom, but the formalism can be naturally adapted to this situation and yields analogous results. In particular the field equations can be written again in terms of covariant differentials of the momenta. Moreover a \textit{generalized replacement theorem} holds, so that the covariant differentials can be replaced by covariant divergences (up to torsion terms).

The usual argument regarding the stress-energy tensor is then reviewed in hopefully clearer terms from a geometric point of view. Finally we consider concrete examples, giving the
explicit coordinate-free expressions and the coordinate expressions of the momenta, the field equations and the energy-tensors for a field of arbitrary integer spin and for a field of spin one-half, both interacting with non-Abelian gauge fields. We also show how the “metric-affine” treatment of the gravitational field fits into the covariant-differential formulation.

1 Covariant differential

1.1 Frölicher-Nijenhuis bracket and covariant differential

Let $M$ be a sufficiently regular real manifold. A tangent-valued (t.v.) $r$-form on $M$ is a section $M \to \wedge^r T^*M \otimes T^*M$. If $\Phi$ is a t.v. $r$-form and $\Psi$ is a t.v. $s$-form then their Frölicher-Nijenhuis bracket $[\Phi, \Psi]$ is a t.v. $(r+s)$-form [29, 30, 31, 20].

Considering the FN-bracket on a fibered manifold $p : E \to M$, we are mostly interested in “basic” t.v. forms $E \to \wedge^r T^*M \otimes T^*E$. In particular, a connection of $E \to M$ can be regarded as a special, basic t.v. 1-form $\kappa : E \to T^*M \otimes T^*E$. The covariant differential [29, 30, 32] associated with $\kappa$ acts on a t.v. $r$-form $\Phi$ as $d_{\kappa} \Phi \equiv [\kappa, \Phi]$, which turns out to be a basic t.v. $(r+1)$-form if $\Phi$ is basic.

In this paper we will mostly deal with the case when $E \to M$ is a vector bundle. We will explicitly indicate the base manifold in a fiber tensor product only when it is not $M$. We will use the shorthand $\Omega^rE \equiv \wedge^r T^*M \otimes E$, and note that

$$E \times_M \Omega^rE \cong \wedge^r T^*M \otimes V_E \to E$$

can be regarded as the bundle of vertical-valued basic $r$-forms on $E$ (as now the vertical subspace $V_E \subset T^*E$ can be identified with $E \times_M E$). In particular, a section $\phi : M \to E$ yields the section

$$\tilde{\phi} : E \to \Omega^0E \equiv V_E : y \mapsto (y, \phi(p(y))$$

and $d_{\kappa} \tilde{\phi}$ is essentially $\nabla_{\kappa} \phi$, namely the covariant differential can be regarded as a natural generalization of the usual covariant derivative.

The latter statement can be intended in a broader sense by observing that the curvature tensor associated with $\kappa$ is the section

$$\rho \equiv -d_{\kappa} \kappa \equiv -[\kappa, \kappa]$$

which in turn can be regarded [2] as the covariant derivative of $\kappa$ with respect to a certain “overconnection”, i.e. a connection (associated with $\kappa$ itself) of the bundle of linear connections of $E$.

Let $(x^a, y^i)$ be linear fibered coordinates on $E$. We denote the induced fiber coordinates on $\otimes^r T^*M$ and $\otimes^r T^*M \otimes E$ by the shorthands

$$z_{a_1 \ldots a_r} \equiv \partial x_{a_1} \otimes \cdots \otimes \partial x_{a_r} , \quad z_{a_1 \ldots a_r}^i \equiv z_{a_1 \ldots a_r} \otimes y^i ,$$

and use the same symbols for their restrictions to $\wedge^r T^*M$ and $\Omega^rE$. Moreover we set

$$dx_{a_1 \ldots a_r} \equiv z_{a_1 \ldots a_r} | d^m x \quad \text{(interior product)},$$
where \( d^m \mathbf{x} \equiv dx^1 \wedge \cdots \wedge dx^m \), \( m \equiv \dim \mathbf{M} \). We obtain a handy “complementary notation” for basic forms of degree \( m-r \), namely if \( \xi : E \to \Omega^{m-r}E \) then we write

\[
\xi = \xi^{a_1 \ldots a_r}_i \, dx_{a_1 \ldots a_r} \otimes \partial y_i = \xi_{a_{r+1} \ldots a_m} \, i \, dx^{a_{r+1}} \wedge \cdots \wedge dx^{a_m} \otimes \partial y_i ,
\]

\[
\xi^{a_1 \ldots a_r}_i = \frac{1}{r!} \varepsilon^{a_1 \ldots a_r a_{r+1} \ldots a_m} \xi_{a_{r+1} \ldots a_m} \, i , \quad \xi_{a_{r+1} \ldots a_m} = \frac{1}{(m-r)!} \varepsilon_{a_1 \ldots a_r a_{r+1} \ldots a_m} \xi^{a_1 \ldots a_r} .
\]

It is not difficult to show that

\[
dx^b \wedge dx_{a_1 \ldots a_r} = \frac{1}{(r-1)!} \delta^b_{[a_r} \, dx_{a_1 \ldots a_{r-1}]} ,
\]

where the above anti-symmetrization over the indices \( a_1 \ldots a_r \) does not include normalizing factorials. Using this identity one computes the “complementary notation” coordinate expression of \( d_\kappa \xi \) for \( \xi : E \to \Omega^{m-r}E \), obtaining

\[
d_\kappa \xi = r \left( \partial_{a_r} \xi^{a_1 \ldots a_r-1 a_r}_i + \partial_j \xi^{a_1 \ldots a_r-1 a_r}_i \kappa^{j}_{a_r} - \xi^{a_1 \ldots a_r-1 a_r}_i \partial_j \kappa^{j}_{a_r} \right) dx_{a_1 \ldots a_{r-1}} \otimes \partial y_i .
\]

### 1.2 Generalized replacement principle

In the sequel we will often deal with the special situation when \( \kappa \) is a linear connection and \( \xi : \mathbf{M} \to \Omega^{m-r}E \) (the condition that the “source” manifold be \( \mathbf{M} \) typically arises when \( \xi \) is obtained by evaluating some object defined on \( E \) through a field). Then we get the simplified expression

\[
d_\kappa \xi = r \left( \partial_{a_r} \xi^{a_1 \ldots a_r-1 a_r}_i - \xi^{a_1 \ldots a_r-1 a_r}_i \kappa^{j}_{a_r} \right) dx_{a_1 \ldots a_{r-1}} \otimes \partial y_i .
\]

Let now \( \Gamma \) be a linear connection of \( \mathbf{T} \mathbf{M} \to \mathbf{M} \). The covariant derivative \( \nabla \xi \) with respect to the couple \((\Gamma, \kappa)\)—see \S \!3.1—is a section

\[
\mathbf{M} \to \mathbf{T}^* \mathbf{M} \otimes_{\mathbf{M}} \wedge^{m-r} \mathbf{T}^* \mathbf{M} \otimes E .
\]

Under **antisymmetrization**, this yields the “covariant divergence”

\[
\nabla \xi = r \nabla_{a_r} \xi^{a_1 \ldots a_r-1 a_r}_i \, dx_{a_1 \ldots a_{r-1}} \otimes \partial y_i : \mathbf{M} \to \wedge^{m-r+1} \mathbf{T}^* \mathbf{M} \otimes E \equiv \Omega^{m-r+1} \mathbf{E} ,
\]

which we want to compare with \( d_\kappa \xi \).

The relation between \( \nabla \xi \) and \( d_\kappa \xi \) turns out to be governed by the torsion \( T \) of \( \Gamma \), with components \( T^a_{bc} \equiv \Gamma^a_{eb} - \Gamma^a_{eb} \). Consider the sections

\[
\tau \wedge \xi , \ T \wedge \xi : \mathbf{M} \to \Omega^{m-r+1} \mathbf{E} ,
\]

where \( \tau \) is the **torsion 1-form**, with components \( \tau_a = T^b_{ab} \), and the symbol \( \wedge \) stands for exterior product followed by interior product. We get the coordinate expressions

\[
\tau \wedge \xi = r \xi^{a_1 \ldots a_r}_i \tau_{a_r} \, dx_{a_1 \ldots a_{r-1}} \otimes \partial y_i ,
\]

\[
T \wedge \xi = r (r-1) T^a_{bc} \xi^{a \ldots a_r-2 b c i} \, dx_{a_1 \ldots a_{r-1}} \otimes \partial y_i .
\]

A coordinate computation then yields:

**Proposition 1.1 (generalized replacement principle).** We have

\[
\nabla \xi = d_\kappa \xi - \tau \wedge \xi - \frac{1}{2} T \wedge \xi .
\]
Thus we see that $\nabla \xi$ depends on $\Gamma$ only via its torsion, and coincides with $d_\kappa \xi$ if $T = 0$. In particular:
- if $r = 0$, $\xi = \xi^i d^m x \otimes \partial y_i$, then $\nabla \xi = 0$;
- if $r = 1$, $\xi = \xi^{ai} dx_a \otimes \partial y_i$, then $\nabla \xi = (\partial_a \xi^{ai} - \kappa_{aj}^i \xi^{aj} - \tau_a \xi^{ai}) d^4 x \otimes \partial y_i = d_\kappa \xi - \tau \wedge \xi$;
- if $r = 2$, $\xi = \xi^{ab i} dx_{ab} \otimes \partial y_i$, then
  \[
  \nabla \xi = 2 \left( \partial_a \xi^{ba i} - \xi^{ba j} \kappa_{aj}^i - \frac{1}{2} \xi^{ac i} \tau_{ac}^b - \xi^{ba i} \tau_a \right) dx_b \otimes \partial y_i.
  \]

We will be specially involved (§3.3) in the situation when $M$ is a pseudo-metric manifold and $\xi \equiv *\xi$, where $*$ is the Hodge isomorphism and $\xi : M \rightarrow \Omega^* E$. Then
\[
\xi^{a_1 \ldots a_r i} = \sqrt{|g|} g^{a_1 b_1} \ldots g^{a_r b_r} \xi_{b_1 \ldots b_r i}.
\]

Remark. The above introduced covariant divergence is somewhat different from the analogous operation considered in most physics texts, which is defined as the contraction of the covariant derivation index with a contravariant index of the tensor acted upon. For a comparison, consider the interior product $\iota_\xi \equiv (\eta^\# | \xi) : M \rightarrow \wedge TM \otimes E$, where $\eta^\#$ is the inverse of a covariantly constant volume form $\eta$. Then we can introduce the covariant divergence $\nabla \tilde{\xi}$ by interior product in $\nabla \tilde{\xi} : M \rightarrow T^* M \otimes \wedge TM \otimes E$; a coordinate computation yields
\[
\nabla \tilde{\xi} = (-1)^{m(r-1)} \nabla \xi.
\]

1.3 Covariant prolongations in gauge field theory

The geometric setting underlying an essential gauge field theory is constituted by a vector bundle $E \rightarrow M$, possibly with some added fiber structure. For the moment we are not making any special assumption about the base manifold $M$.

The construction of bundles of linear connections of $E$ can be summarized as follows [2]. The first jet prolongation $J^1 E \rightarrow E$ is an affine bundle, while $J^2 E \rightarrow M$ inherits the vector bundle structure. The affine sub-bundle $C_{all} \subset J^1 E \otimes E^*$ over $M$, constituted by all elements that project onto the identity $\mathbb{I}_E : M \rightarrow E \otimes E^*$, can be regarded as the bundle of all linear connections of $E$. Its ‘derived’ vector bundle is $T^* M \otimes \text{End } E$. We also note that $\text{End } E \equiv E \otimes E^*$, the bundle of all linear fiber endomorphisms of $E$, has a Lie-algebra bundle structure determined by the ordinary commutator.

If the fibers of $E$ are endowed with a more specialized algebraic structure, then the linear connections preserving it can be regarded as sections of an affine sub-bundle $C \subset C_{all}$, with derived vector bundle $DC = T^* M \otimes \mathfrak{L}$ where $\mathfrak{L} \subset \text{End } E$ is a Lie-subalgebra bundle. The curvature tensor of a connection can be regarded as a section $M \rightarrow \wedge^3 T^* M \otimes \mathfrak{L}$. Linear fiber coordinates $(y^i)$ on $E$ also determine linear fiber coordinates $(y^i_j) \equiv (y^i \otimes y_j)$ on $\text{End } E$. If we deal with a Lie-subalgebra bundle $\mathfrak{L}$ then we may choose ‘adapted’ coordinates $(l^i) = (l^i_j y_j^i)$, by which we essentially recover the familiar principal-bundle formalism. We will use this notation just occasionally, as a shorthand, but it is easy to realize that all our results remain valid in a restricted setting anyhow.

Remark. Any section $\xi : M \rightarrow \wedge^r T^* M \otimes \mathfrak{L}$ can be also regarded as a basic vertical-valued form $E \rightarrow \wedge^r T^* M \otimes \mathfrak{L}$. Accordingly, its covariant differential $d_\kappa \xi$ with respect to a linear connection $\kappa$ could be, in principle, intended in two different ways, since $\kappa$ also determines a linear connection of $\mathfrak{L} \rightarrow M$. It is not difficult to check, however, that these two points of view essentially yield the same $d_\kappa \xi$, with the coordinate expression
\[
d_\kappa \xi = \left( \partial_{a_1} \xi_{a_2 \ldots a_{r+1} i} - [\kappa_{a_1}, \xi_{a_2 \ldots a_{r+1}}] \right) dx^{a_1} \wedge \ldots \wedge dx^{a_{r+1}} \otimes l_i,
\]
where the bracket means ordinary commutator. A similar observation holds about equivalent ways of regarding a linear connection and its curvature tensor.

Altogether, the ‘configuration bundle’ of an essential gauge field theory is $F \equiv E \times_M C$, with sections $\phi : M \to E$ and sections $\kappa : M \to C$ playing the role of ‘matter fields’ and ‘gauge fields’, respectively. A couple $(\phi, \kappa)$ yields the covariant differentials

$$d\kappa \phi : M \to \Omega^1 E, \quad d\kappa \kappa : M \to \Omega^2 L,$$

so that we are led to consider the covariant prolongation bundle

$$\mathcal{DF} \equiv F \times_{M} \Omega^1 E \times_{M} \Omega^2 L$$

as an analogous of the first jet prolongation $J_F$. We have the morphism

$$\mathcal{D} : J_F \to \mathcal{DF}$$

classified by

$$\mathcal{D} \circ (j_\phi, j_\kappa) = (\phi, \kappa, d\kappa \phi, d\kappa \kappa), \quad \forall (\phi, \kappa) : M \to F,$$

where $j_\phi$, $j_\kappa$ denote the first jet prolongations of sections.

We denote the induced fiber coordinates on $C$ by $(k^i_a)$, and the induced fiber coordinates on $\Omega^1 E \times_{M} \Omega^2 L$ by $(x^a, y^i, z^i_a, z^{i}_{abj})$. We get the coordinate expression

$$(x^a, y^i, k^i_{aj}, z^i_a, z^{i}_{abj}) \circ \mathcal{D} = (x^a, y^i, k^i_{aj} - y^a_j, -k^{i}_{aj,b} + k^{i}_{bj,a} - [k^i_a, k^i_b]_j),$$

$$[k^i_a, k^i_b]_j \equiv k^i_{ah} k^h_{bj} - k^i_{bh} k^h_{aj}.$$

By a straightforward computation one proves:

**Proposition 1.2** Let $f : \mathcal{DF} \to \mathbb{R}$. Then there is a unique morphism $d_H f : J_2 F \to T^* M$ such that

$$d_H (f \circ \mathcal{D}) = d_H f \circ J_2 F \to T^* M,$$

where the left-hand side is the standard horizontal differential of the function $f \circ \mathcal{D} : J_2 F \to \mathbb{R}$ (here $J_2 F$ is intended as restricted to holonomic jets). We have the coordinate expression $d_H f = d_a f \, dx^a$ with

$$d_a f = \partial_a f + (z^i_a + k^i_{aj} y^j) \partial_i f + z^{i}_{baj} \partial^b_j f + z^{i}_{abj} \partial^{b} f.$$

## 2 Covariant-differential Lagrangian field theory

### 2.1 Summary of Lagrangian field theory in jet space

For a reference, we sketch the standard geometric formulation of Lagrangian field theory, which is widely discussed in the literature [37, 12, 11, 26, 17, 19, 39, 40, 4, 23, 25, 24]. A first-order Lagrangian density on a fibered manifold $E \to M$ is defined to be a totally horizontal $m$-form

$$\mathcal{L} : JE \to \wedge^m T^* M \subset \wedge^m T^* JE, \quad m \equiv \dim M.$$

We write its coordinate expression as $\mathcal{L} = \ell \, d^m x$, with $\ell : JE \to \mathbb{R}$. 
Consider a morphism \( v = v^a \partial x_a + v^i \partial y_i : J_1 E \to T E \) over \( E \). Taking its jet prolongation restricted to holonomic jets, and composing it with the natural morphism JT \( F \to TJ F \), we get the natural prolongation [29, 2]

\[
v_{(1)} = v^a \partial x_a + v^i \partial y_i + v^j_i \partial y_i : J_2 E \to T J E ,
\]

\[
v^j_a = d_a v^i - d_a v^b y^i_b = (\partial_i v^i + \partial_j v^i y^i_j + \partial_j^b v^j_i - (\partial_i v^b + \partial_j v^b y^j_a + \partial^j x^b y^j_{ac}) y^i_b .
\]

Though \( v \) is not a vector field on \( J E \), we can introduce a generalized Lie derivative

\[
L_v \mathcal{L} \equiv d(v|\mathcal{L}) + v_{(1)}|d\mathcal{L} : J_2 E \to \wedge^m T^* E .
\]

(This kind of order-raising procedure is used in some literature, in particular by authors working on infinite jets [39, 36, 40].) We obtain the coordinate expression

\[
L_v \mathcal{L} = (\partial_i \ell v^a) + \partial_i \ell v^i + \partial_i^a \ell (d_a v^i - d_a v^b y^i_b) d^m x + \ell (\partial_i v^a dy^i + \partial^b \partial_i v^a dy^i_b) \wedge dx_a .
\]

We then see that there is a unique morphism \( \delta_v \mathcal{L} : J_2 E \to \wedge^m T^* M \) over \( M \), called the (infinitesimal) variation of \( \mathcal{L} \) determined by \( v \), which is characterized by

\[
\delta_v \mathcal{L} \circ j_2 \phi = j^\phi L_v \mathcal{L} \quad \forall \phi : M \to E .
\]

If \( v \) is vertical \( (v^a = 0) \) then we get

\[
L_v \mathcal{L} = \delta_v \mathcal{L} = (\partial_i \ell v^i) d^m x = \equiv \mathcal{E}_j v + d_h (D \mathcal{L}|v) : J_2 E \to \wedge^m T^* M ,
\]

where \( \mathcal{E} : J_2 E \to \wedge^m T^* M \otimes_E V^* E \) is the Euler-Lagrange operator, and

\[
D \mathcal{L} = \partial_i^a \ell dx_a \otimes dy^i : J E \to \wedge^{m-1} T^* M \otimes V^* E ,
\]

is the fiber derivative of \( \mathcal{L} \).

We can now express the “Principle of Least Action” as the following definition: a section \( \phi : M \to E \) is called a critical field if for any regular compact subset \( K \subset M \) and for any vector field \( v : E \to VE \) vanishing on \( \bar{p}(\partial K) \subset E \) one has

\[
\int_K \delta_v \mathcal{L} \circ j_2 \phi \equiv \int_K j^\phi L_v \mathcal{L} = 0 .
\]

Critical fields then fulfill the EL field equation

\[
\mathcal{E}_i \circ j_2 \phi \equiv (\partial_i \ell - d_a \partial_i^a \ell) \circ j_2 \phi = 0 .
\]

The above introduced fiber derivative \( D \mathcal{L} \), via natural operations and identifications, yields the momentum form

\[
\mathcal{P} = \mathcal{P}_i^a (dy^i - y^i_b dx_b) \wedge dx_a : J E \to \wedge^m T^* J E , \quad \mathcal{P}_i^a \equiv \partial_i^a \ell ,
\]

and the Poincaré-Cartan form \( \mathcal{C} \equiv \mathcal{L} + \mathcal{P} : J E \to \wedge^m T^* J E \). By straightforward computations one finds

\[
\delta_v \mathcal{P} = 0 \quad \Rightarrow \quad \delta_v \mathcal{C} = \delta_v \mathcal{L} .
\]
If $\phi : M \to E$ is any section then $j\phi^*(dy^i - y^i_1 dx^k) = 0$, whence

$$j\phi^*(v_{(1)}|C) = \left[ (\ell v^a + \mathcal{P}^a_i (v^i_1 - y^i_1 v^b_i)) \circ j\phi \right] dx_a .$$

Thus the first-order horizontal form

$$\mathcal{J}_\nu = (\ell v^a + \mathcal{P}^a_i (v^i_1 - y^i_1 v^b_i)) dx_a : JE \to \wedge^{m-1}T^*M$$

is characterized by $\mathcal{J}_\nu \circ j\phi = j\phi^*(v_{(1)}|C)$ for all sections.

One says that the morphism $\nu$ is an on-shell (infinitesimal) symmetry of the field theory under consideration if $0 = \delta_\nu C \circ j_2 \phi = \delta_\nu \mathcal{L} \circ j_2 \phi$ for any critical section $\phi$. Moreover, a form $\Phi : JE \to \wedge^{m-1}T^*JE$ is called a conserved current if $j\phi^* d\Phi \equiv d_i \Phi \circ j_2 \phi = 0$ for any critical section $\phi$.

By straightforward computations one finds that the condition that the section $\phi : M \to E$ be critical is equivalent to $j\phi^*(v_{(1)}|dC) = 0$ for any morphism $v : JE \to TE$. The following generalized version of the Noether theorem then holds: if $v : JE \to TE$ is an on-shell symmetry then $v_{(1)}|C : JE \to \wedge^{m-1}T^*JE$ is a conserved current. More generally, if there exists an $m-1$-form $\varphi : JE \to \wedge^{m-1}T^*JE$ such that $j\phi^* \mathcal{L} = j\phi^* d\varphi$ for all critical sections, then $v_{(1)}|C - \varphi$ is a conserved current.

### 2.2 Covariant Lagrangian density and momentum

According to the Utiyama principle [38, 20, 18], in a gauge theory’s natural Lagrangian the matter field’s derivatives appear through covariant derivatives, and a gauge field’s derivatives appear through its curvature tensor. Thus all derivatives appear through the covariant differentials of the fields.

Consider the configuration bundle $F \equiv E \times_M C$ described in §1.3. Then

$$\nabla\phi \equiv d_\kappa \phi : M \to \Omega^1E , \quad \rho \equiv -d_\kappa \kappa : M \to \Omega^2\mathcal{L} ,$$

so that the Lagrangian density can be expressed as a morphism

$$\Lambda : \xi F \equiv F \times_M \Omega^1E \times_M \Omega^2\mathcal{L} \to \wedge^mT^*M ,$$

related to the usual Lagrangian density $\mathcal{L} : JF \equiv JE \times_M JC \to \wedge^mT^*M$ by $\mathcal{L} = \Lambda \circ \mathcal{D}$, that is

$$\mathcal{L} \circ (j\phi, j\kappa) = \Lambda \circ (\phi, d_\kappa \phi, d_\kappa \kappa) \equiv \Lambda \circ \mathcal{D} \circ (j\phi, j\kappa) .$$

Then we regard $\xi F$ as playing a role analogous to that of $JF$ in the standard presentation of Lagrangian field theory, while $(\phi, \kappa, d_\kappa \phi, d_\kappa \kappa)$ plays a role analogous to the first-jet prolongations of sections. We remark, however, that $\kappa$ appears in $\Lambda$ only through $d_\kappa \phi$ and $d_\kappa \kappa$. Thus if we write the local expression $\Lambda = \lambda d^m x^a$ we have $\partial \lambda / \partial k_{aj}^i = 0$.

We also introduce the “covariant momentum” morphism over $M$

$$\Pi \equiv (\Pi^{(0)}, \Pi^{(1)}, \Pi^{(2)}) : \xi F \to \Omega^m E^* \times_M \Omega^m T^*E^* \times_M \Omega^{m-2} \mathcal{L}^*$$

obtained by fiber derivative of $\Lambda$ followed by a natural contraction. Using linear fiber coordinates $(y^i)$ on $E$, and recalling the coordinate notations and conventions introduced in §1.1, we get

$$\Pi^{(0)} = \Pi_i d^m x \otimes y^i , \quad \Pi^{(1)} = \Pi_i^a dx_a \otimes y^i , \quad \Pi^{(2)} = \Pi_i^{abj} dx_{ab} \otimes y^i ,$$

$$\Pi_i = \frac{\partial \lambda}{\partial y^i} , \quad \Pi_i^a = \frac{\partial \lambda}{\partial z^a_i} , \quad \Pi_i^{abj} = \frac{\partial \lambda}{\partial z_{abj}^i} .$$
Moreover we have
\[ \mathrm{d} \Lambda = \mathrm{d} \lambda \wedge \mathrm{d}^m x = (\Pi^i \mathrm{d} y^i + \Pi^{a i} \mathrm{d} z^i + \Pi^{a b i} \mathrm{d} z_{a b i}) \wedge \mathrm{d}^m x . \]

Setting \( \ell = \lambda \circ \partial \) we also get
\[
\frac{\partial \ell}{\partial \xi^a} = \frac{\partial \lambda}{\partial \xi^a} \circ \partial , \quad P_i = \frac{\partial \ell}{\partial y^i} = (\Pi_i - \Pi^a_{i j} k^j_{a i}) \circ \partial , \quad P^a_i = \frac{\partial \ell}{\partial y^i_a} = \Pi^a_{i} \circ \partial ,
\]
\[
P_{a j}^i = -2 \Pi^{a b i}_{j} \circ \partial = 2 \Pi^{b a j}_{i} \circ \partial .
\]

2.3 Variations and field equations

If \( v = v^a \partial x^a + v^i \partial y^i + v^i_j \partial k^a_{i j} : J F \rightarrow T F \) then the corresponding infinitesimal variation of \( \mathcal{L} \) is (§2.1) \( \delta \mathcal{L} = \delta \ell \mathrm{d}^m x \), with
\[
\delta \ell = (P_i - \partial_a P^a_i) v^i + (P^b_{i j} - \partial_a P^{a b j}_{i}) v^i_{b j} + d_a (P^a_i v^i + P^{a b j}_{i} v^i_{b j}) + v^a \partial_a \ell + \partial_a v^a - (y^i_c P^a_i + k^a_{b j c} P^{a b j}_{i}) d_a c .
\]

Let moreover \( v = u \circ \partial \) with \( u : \mathcal{OD} \rightarrow T F \). Then, recalling proposition 1.2, we obtain
\[
\delta \ell = \left( (\Pi_i - \Pi^a_{i j} k^j_{a i}) u^i + (\Pi^b_{i j} k^j_{a i}) u^i_{b j} + d_a (\Pi^a_{i} u^i + \Pi^{a b j}_{i} u^i_{b j}) + d_a (\Pi^a_{i} u^i + 2 \Pi^{b a j}_{i} u^i_{b j}) + u^a \partial_a \lambda + \partial_a u^a - (y^i_c \Pi^a_i + 2 k^a_{b j c} \Pi^{a b j}_{i}) d_a c \right) \circ J \partial
\]

In the sequel we will denote the covariant prolongation of \( (\phi, \kappa) : M \rightarrow F \) by the shorthand
\[
\partial (\phi, \kappa) \equiv \partial \circ (j \phi, j \kappa) \equiv (\phi, \kappa, \kappa \partial \phi, \kappa \partial \kappa) : M \rightarrow \mathcal{OD} .
\]

Evaluating the “momenta” \( \Pi^{(r)} : \mathcal{OD} \rightarrow \Omega^r E^* \) through field prolongation we obtain sections
\[
\Pi^{(r)} \circ \partial (\phi, \kappa) : M \rightarrow \Omega^r E^* , \quad r = 0, 1, 2 .
\]

We can now apply the results of §1.1 and of §1.2 to these objects, with the only adjustment that, since \( E \) is here replaced by \( E^* \), their covariant differentials are
\[
d_{\kappa} \left( \Pi^{(r)} \circ \partial (\phi, \kappa) \right) = [\kappa, \Pi^{(r)} \circ \partial (\phi, \kappa)] : M \rightarrow \Omega^{r+1} E^* ,
\]
where \( \kappa \) is the dual connection of \( \kappa \). We obtain the coordinate expressions
\[
d_{\kappa} (\Pi^{(1)} \circ \partial (\phi, \kappa)) = (\Pi^a_{i j} k^j_{a i} + d_a \Pi^a_{i}) \circ j \partial (\phi, \kappa) =
\]
\[
= (\Pi^a_{i} \circ \partial (\phi, \kappa)) \kappa^a_{i} + \partial_a (\Pi^a_{i} \circ \partial (\phi, \kappa)) ,
\]
\[
d_{\kappa} (\Pi^{(2)} \circ \partial (\phi, \kappa)) = -2 (d_a \Pi^{a b j}_{i} k^j_{a i} \Pi^{a b h}_{i} + k^a_{i} \Pi^{a b j}_{i}) \circ j \partial (\phi, \kappa) =
\]
\[
= -2 \partial_a (\Pi^{a b j}_{i} \circ \partial (\phi, \kappa)) - 2 \kappa^a_{i} \Pi^{a b j}_{i} \circ \partial (\phi, \kappa) + 2 \kappa^a_{i} \Pi^{a b j}_{i} \circ \partial (\phi, \kappa) ,
\]
which yield a natural interpretation of the coefficients of \( v^i \) and of \( v^i_{b j} \) in \( \delta \ell \).

For the sake of simplicity we will now employ an abuse of language which is common in physics texts: when the context is clear we may write \( \Pi^{(r)} \) as a shorthand for \( \Pi^{(r)} \circ \partial (\phi, \kappa) \). Similarly we may write \( \Pi^a_{i} \) for \( \Pi^a_{i} \circ \partial (\phi, \kappa) \) and the like. From the above results we obtain:
Theorem 2.1 Let \( (\phi, \kappa) : M \to F \). The condition that \( \int_K \delta \varLambda \circ j \varphi(\phi, \kappa) \) vanishes for any sufficiently regular compact subset \( K \subset M \) and for any vertical-valued morphism \( v : \mathcal{J}F \to \mathcal{V}F \) is equivalent to the validity of the field equations

\[
\begin{align*}
\Pi^{(0)} - d_a \Pi^{(1)} &= 0, \\
\Pi^{(1)} \otimes \phi - d_a \Pi^{(2)} &= 0.
\end{align*}
\]

In simplified coordinate form, the above field equations can be written as

\[
\begin{align*}
\Pi_i - \partial_a \Pi^a_i - \kappa^j_{a i} \Pi^a_j &= 0, \\
\Pi^b_i \phi^j + 2 (\partial_a \Pi^{abj} - \kappa^j_{ah} \Pi^{abh} + \kappa^j_{ai} \Pi^{abj}) &= 0.
\end{align*}
\]

Remarks.

\( a) \) Theorem 2.1 may remain valid if its statement is modified by making certain assumptions about the type of the arbitrary morphism \( v \), e.g. by restricting it to be a vertical vector field on \( F \) or even a section \( M \to \mathcal{V}F \).

\( b) \) It is not difficult to check that the above field equations for the couple \( (\phi, \kappa) \), when written explicitly in coordinates, do coincide with the Euler-Lagrange field equations derived from the Lagrangian density \( \varLambda \) in the standard theory.

\( c) \) We can formulate a theory of the field \( \phi \) alone, in which the gauge field \( \kappa \) is a fixed background structure; then the field equation is just the first of the equations derived in theorem 2.1.

2.4 Further remarks about variations

In the usual formulation of Lagrangian field theory on jet space (§2.1), an “infinitesimal variation” of the field \( \phi : M \to E \) can be described as a section \( \delta \phi \equiv v : M \to \mathcal{J}E \). One avails of the natural isomorphism \( \mathcal{J}E \cong \mathcal{V}\mathcal{J}E \), which allows the identification of the first jet prolongation \( \mathcal{J}\delta \phi : M \to \mathcal{J}E \) as the variation \( \delta \mathcal{J} \phi \); this is needed in the derivation of the usual Euler-Lagrange equations.

Commutation between field variation and field derivation is not valid when the role of derivation is taken up by the covariant differential, so we get a slightly more involved situation. In the context of the previously described essential gauge theory, an infinitesimal variation of the field \( (\phi, \kappa) \) in the above sense can be represented as a couple

\[
(\delta \phi, \delta \kappa) : M \to E \times \mathcal{J}M
\]

(since \( \mathcal{J}E \cong E \times_M E \) and \( \mathcal{J}C \cong C \times_M \mathcal{J}M \)). It is then natural to set

\[
\delta d_a \phi \equiv \delta [\kappa, \phi] = \delta \kappa, \phi + \kappa, \delta \phi : M \to \mathcal{J}^1 E,
\]

\[
\delta d_a \kappa \equiv \delta [\kappa, \kappa] = 2 [\kappa, \delta \kappa] : M \to \mathcal{J}^2 \mathcal{L}.
\]

The reader may wish to compare these expressions to the Lie derivatives of a linear connection of the tangent bundle of a manifold and its curvature tensor [41].

We can now recover the field equations by a procedure similar to the usual one. In fact, using again the notational simplification of dropping the explicit evaluation of the involved objects through the fields, we get \( \delta \varLambda = \delta \lambda d^m x : M \to \wedge M \mathcal{T} \mathcal{M} \) with

\[
\delta \lambda = \langle d \lambda, \delta \varphi(\phi, \kappa) \rangle = \Pi_i \delta \phi^i + \Pi_i \delta [\kappa, \phi] + 2 \Pi^{abj} [\kappa, \delta \phi]_{abj} = \Pi_i \delta \phi^i + \Pi_i \delta \phi^i - \kappa^i_{aj} \delta \phi^j - \kappa^i_{aj} \phi^j + 2 \Pi^{abj} (\partial_a \delta \kappa^i_{bj} + \delta \kappa^i_{aj} \kappa^i_{kj} - \kappa^i_{aj} \delta \kappa^i_{bj}) =
\]

\[
= \Pi_i \delta \phi^i + \Pi_i (\partial_a \delta \phi^i - \kappa^i_{aj} \delta \phi^j - \kappa^i_{aj} \phi^j) + 2 \Pi^{abj} (\partial_a \delta \kappa^i_{bj} + \delta \kappa^i_{aj} \kappa^i_{kj} - \kappa^i_{aj} \delta \kappa^i_{bj}) =
\]
2.5 Currents and energy-tensors

\[
\begin{align*}
&= (\Pi_i - \partial_a \Pi^a_i - \Pi^a_i \kappa^i_a \phi^i) \delta \phi^i - 2 \left( \partial_a \Pi^{abj}_i - \Pi^{abj}_i \kappa^j_a + \Pi^{aj}_h \kappa^j_a h + \frac{1}{2} \Pi^i_j \phi^j \right) \delta \kappa^i_j + \\
&\quad + \partial_a (\Pi^a_i \delta \phi^i + 2 \Pi^{abj}_i \delta \kappa^i_j).
\end{align*}
\]

**Remark.** An *infinitesimal gauge transformation* can be obtained as a different type of variation, determined by a section \( l : M \to \mathcal{L} \). This can be regarded as a vertical vector field \( v : E \to VE \) via the rule \( v(y) \equiv (y, l(y)) \), yielding

\[
\delta \phi \equiv v \circ \phi = l(\phi), \quad \delta \kappa = [\kappa, v] = [\kappa, l].
\]

Clearly this transformation is not suitable for deriving field equations, as a natural Lagrangian has to be invariant with respect to it.

2.5 Currents and energy-tensors

The Poincaré-Cartan form for an essential gauge field theory of matter and gauge fields is \( \mathcal{C} \equiv L + \mathcal{P} : JF \to \Lambda^m T^* JF \) where

\[
\mathcal{P} = \left( \mathcal{P}^a_i (dy^i - y^i_j dx^j) + \mathcal{P}^{abj}_i (d\kappa^j_a - k^j_a c dx^c) \right) \wedge dx_a.
\]

It is apparent that, in general, \( \mathcal{C} \) cannot be completely recovered only in terms of covariant prolongations. Nevertheless, the current associated with a morphism \( v : JF \to TF \) can be actually seen as an object “living” on \( \mathcal{D}F \), provided that \( v \) is of a suitably restricted type.

Let \( v = u \circ \partial : JF \to TF \) with \( u : \mathcal{D}F \to TF \). We have the current associated with \( v \), that is the horizontal form \((\S 2.1)\)

\[
\mathcal{J}^a = \ell v^a + \mathcal{P}^a_i (v^i - y^i_j v^j) + \mathcal{P}^{abj}_i (v^j_a - k^j_{b,c} v^c) .
\]

We would like to write \( \mathcal{J}^a = \mathcal{I}_u \circ \partial \) with \( \mathcal{I}_u : \mathcal{D}F \to \Lambda^{m-1} T^* M \). Since

\[
\mathcal{J}^a = \left( \lambda u^a + \Pi^a_i (u^i - (z^i_b + k^i_j y^j) u^b) + 2 \Pi^{abj}_i (u^j_a - k^j_{b,c} u^c) \right) \circ \partial,
\]

we see that the obstruction to doing so lies in the need for expressing \( k_{b,c}^i \) via a function on \( \mathcal{D}F \); the obstruction disappears, in particular, when \( u \) is vertical-valued.

Another special case is that of the “horizontal lift” of a vector field \( \overline{u} = u^a \partial x_a \) on \( M \). Actually a natural generalization of the usual notion of horizontal lift of a basic vector field via a connection, exploiting the notion of “overconnection” [2], yields the morphism

\[
\overline{u}^\tau : E \times JC \to TF
\]

with the coordinate expression

\[
\overline{u}^\tau = u^a \left( \partial x_a + k_{aj}^i y^j \partial y_i + (k^i_{aj,b} - k^i_{aj,b} k^j_{bh} + k^j_{b,c} k^i_{ah}) \partial k^i_{bj} \right).
\]

Inserting \( \overline{u}^\tau \) into \( \mathcal{J}^a \) in the place of \( v \), by a short computation we obtain

\[
\mathcal{J}^a = u^b \left( \lambda \delta^b_a - \Pi^a_i z^j_b - 2 \Pi^{acj}_i z^j_{bc} \right) \circ \partial .
\]

Summarizing, we can express the above discussion as follows.
Proposition 2.1 Let $u : M \to TM$, $w = w^i \partial y_i + w^i_{aj} \partial k^a_j$ : $\mathcal{DF} \to \nabla F$, and set

$$v \equiv u^+ + w \circ d : JF \to TF.$$ 

Then there exists $I_{u,w} : \mathcal{DF} \to \wedge^{m-1} T^*M$ such that $J_v = I_{u,w} \circ d$, with the components

$$I^a = u^b \left( \lambda \delta^a_b - \Pi^a_i z^i_b - 2 \Pi^{acj}_i z^c_{bj} \right) + \Pi^a_i w^i + 2 \Pi^{abj}_i w^i_{bj}.$$ 

The notion of “canonical energy-tensor” [5, 6, 7, 8, 28, 34, 35] is essentially about relating conserved currents with vector fields on the base manifold. Though the expression $\ell \delta^a_b - \phi^i_{,a} \partial_i \ell$ found in the literature is usually recognized to be devoid of geometric meaning in general, its truly covariant modification introduced by Hermann [16] is still not very well-known. The construction requires a connection of the theory’s configuration bundle, as this is the most natural way to lift basic vector fields; in terms of the tensor’s coordinate expression, it amounts to replacing the field’s ordinary derivative $\phi^i_{,a}$ with its covariant derivative with respect to the assumed connection [3, 2].

The Hermann construction is nicely suited to be extended to a theory of coupled matter and gauge fields ($\phi, \kappa$). In this case no extra structure is needed, as one avails of the covariant differentials $(d_\kappa \phi, d_\kappa \kappa)$; one obtains the joint canonical energy-tensor

$$U : JF \to \wedge^{m-1} T^*M \otimes T^*M,$$

which has the coordinate expression

$$U^a_b = \ell \delta^a_b - P^a_i \left( y^i_b - k^i_{bj} y^j \right) - P^{acj}_i \left( -k^i_{aj,b} + k^i_{bj,a} - k^i_{aj,b} \right) =$$

$$= \left( \lambda \delta^a_b - \Pi^a_i z^i_b - 2 \Pi^{acj}_i z^c_{bj} \right) \circ d \equiv \Upsilon^a_b \circ d.$$ 

We then see that there exists a unique “covariant canonical energy-tensor”

$$\Upsilon : \mathcal{DF} \to \wedge^{m-1} T^*M \otimes T^*M$$

such that $U = \Upsilon \circ d$. Moreover for any basic vector field $u$ we have $I_u = \Upsilon u$.

3 Field theory in spacetime

3.1 Gauge field theory on a General Relativistic background

We consider an essential gauge field theory setting as in §2.2, but now the base manifold $M$ is assumed to be an oriented Lorentz spacetime ($m = 4$). We will denote the metric, the spacetime connection and the unit volume form as $g$, $\Gamma$ and $\eta$, respectively.

We allow the matter field $\phi$ to have spacetime indices, namely the interacting matter and gauge fields constitute a section

$$(\phi, \kappa) : M \to F \equiv (E \otimes Y) \times C$$

where $Y \subset (\otimes TM) \otimes (\otimes T^*M)$ is a vector sub-bundle of the tensor algebra of $TM$. Thus $\phi$ can be regarded as a charged bosonic field of possibly non-zero spin (in §3.3 we will also consider fermionic fields).
3.1 Gauge field theory on a General Relativistic background

We indicate fiber coordinates of $E \otimes Y$ as $y^A$, where $A$ represents the appropriate set of spacetime indices. Accordingly, the coefficients of the connection of $Y$ determined by the spacetime connection will be denoted as $\Gamma^A_{\,\,\,\,B}$. We have the covariant differential

$$\nabla \phi \equiv d_{\kappa \otimes \Gamma} \phi \equiv [\kappa \otimes \Gamma, \phi]$$

where the “tensor product connection” $\kappa \otimes \Gamma$ is the induced connection of $E \otimes Y$, namely

$$\nabla_a \phi^{jA} = \partial_a \phi^{jA} - \kappa^i_{\,\,a} \phi^{jA} - \Gamma^A_{\,\,\,\,B} \phi^{jB},$$

and the field $(\phi, \kappa) : M \to F$ has the covariant prolongation

$$\mathfrak{d}(\phi, \kappa) \equiv d \circ (j \phi, j \kappa) \equiv (d_{\kappa \otimes \Gamma} \phi, d_{\kappa \kappa}).$$

The basic setting laid out in §2.2 must be now adjusted. On $\mathcal{DF} \rightarrow M$ we have fiber coordinates $(y^A, k^i_{\,\,a}, z^i_{\,a}, z_{\,a}^i)$. The “covariant” momentum morphism

$$\Pi \equiv (\Pi^{(0)}, \Pi^{(1)}, \Pi^{(2)}): \mathcal{DF} \rightarrow (\Omega^{m}E^{*} \otimes Y^{*}) \times (\Omega^{m-1}E^{*} \otimes Y^{*}) \times \Omega^{m-2}L^{*}$$

has components $(\Pi_{\,iA}, \Pi^{a}_{\,iA}, \Pi^{abj}_{\,i})$, and the components of the standard momentum $P$ are related to these by

$$P_{\,iA} = (\Pi_{\,iA} - \Pi^{a}_{\,jB}(k_{\,a}^{\,j} \delta_{\,A}^B + \delta_{\,B}^j \Gamma_{\,a \,A}^B)) \circ \mathfrak{d}, \quad P^{a}_{\,iA} = \Pi^{a}_{\,iA} \circ \mathfrak{d}, \quad P^{b}_{\,iaj} = 2 \Pi^{b}_{\,iaj} \circ \mathfrak{d}.$$

The procedure which led to theorem 2.1 can be adapted to this situation without difficulty, and employing again the notational simplification used there—$\Pi^{(r)}$ for $\Pi^{(r)} \circ \mathfrak{d}(\phi, \kappa)$ and the like for the momentum components—we find the field equations

$$\begin{cases} 
\Pi^{(0)} - d_{\kappa \otimes \Gamma} \Pi^{(1)} = 0, \\
\Pi^{(1)} \otimes \phi - d_{\kappa} \Pi^{(2)} = 0,
\end{cases}$$

that is, in coordinate form,

$$\begin{cases} 
\Pi_{\,iA} - \Pi^{a}_{\,jB}(k_{\,a}^{\,j} \delta_{\,A}^B + \delta_{\,B}^j \Gamma_{\,a \,A}^B) - \partial_a \Pi^{a}_{\,iA} = 0, \\
\Pi^{b}_{\,iaj} \phi^{jA} + 2(\partial_a \Pi^{abj}_{\,i} + \Pi^{abj}_{\,h} \kappa_{\,a}^{\,h} - \Pi^{abj}_{\,h} \kappa_{\,a}^{\,h}) = 0.
\end{cases}$$

**Remark.** By contraction with the inverse $\eta^{\#}$ of $\eta$ we get (§1.2) the “contravariant momentum”

$$\tilde{\Pi} \equiv (\tilde{\Pi}^{(0)}, \tilde{\Pi}^{(1)}, \tilde{\Pi}^{(2)}): \mathcal{DF} \rightarrow (E^{*} \otimes Y^{*}) \times (\mathcal{T}M \otimes E^{*} \otimes Y^{*}) \times (\wedge^2 \mathcal{T}M \otimes L^{*})$$

with the components

$$\tilde{\Pi}_{\,iA} = \Pi_{\,iA} / \sqrt{|g|}, \quad \tilde{\Pi}^{a}_{\,iA} = \Pi^{a}_{\,iA} / \sqrt{|g|}, \quad \tilde{\Pi}^{abj}_{\,i} = \Pi^{abj}_{\,i} / \sqrt{|g|}.$$

The replacement principle for $\Pi^{(1)}$ holds, in the present case, in the modified form

$$(\nabla \tilde{\Pi}^{(1)})_{iA} = \frac{1}{\sqrt{|g|}} (\nabla \Pi^{(1)})_{iA} = \frac{1}{\sqrt{|g|}} (d_{\kappa \otimes \Gamma} \Pi^{(1)} - \tau \wedge \Pi^{(1)})_{iA} =$$

$$= \frac{1}{\sqrt{|g|}} (\partial_a \Pi^{a}_{\,iA} + \kappa_{\,a}^{\,j} \Pi^{a}_{\,jA} + \Gamma_{\,a \,A}^B \Pi^{a}_{\,iB} - \tau_a \Pi^{a}_{\,i}),$$

while it is unchanged for $\Pi^{(2)}$. Hence the field equations can be recast in terms of covariant divergences in the form

$$\begin{cases} 
\tilde{\Pi}^{(0)} - \nabla \tilde{\Pi}^{(1)} = \text{torsion terms}, \\
\tilde{\Pi}^{(1)} \otimes \phi - \nabla \tilde{\Pi}^{(2)} = \text{torsion terms}.
\end{cases}$$
3.2 Stress-energy tensor

We now revisit, in the context introduced in §3.1, the usual approach to stress-energy tensors in Einstein spacetime. This notion develops from considering infinitesimal deformations determined by a “basic” vector field $\mu : M \to TM$, so it should be related to the procedure yielding the “canonical” energy-tensor (§2.5); it is actually well-known [13] that in standard theories the two objects are nearly the same.

The argument under consideration is based on the notion of Lie derivative of the fields with respect to $\mu$; the usual formulation [27, 15] explicitly considers fields with spacetime indices only. If the fields also have indices of other kinds then their Lie derivatives are not well-defined in general, but the argument can be successfully carried on by employing a suitable extension.

One such extension can be introduced in terms of a possibly local, linear “reference connection” $\kappa$ of $E$. For any $\xi : E \to \wedge^r TM \otimes_{k} TE$ we consider the “covariant Lie derivative”

$$\delta \xi \equiv L_{\kappa^{\mu}} \xi \equiv i_{\mu} d_{\kappa} \xi + d_{\kappa}(i_{\mu} \xi) : E \to \wedge^{r+1} TM \otimes V E .$$

Then by comparing coordinate expressions it is not difficult to show that

$$L_{\kappa^{\mu}} \xi = (L_{\mu^{\mu}} \xi)_{\omega_{\kappa}} ,$$

where $L_{\mu^{\mu}} \xi$ is the ordinary Lie derivative of $\xi$, seen as a tensor field on $E$, with respect to the horizontal lift $\mu^{\mu} : E \to TE$, and $\omega_{\kappa} : TE \to VE$ is the vertical projection form associated with $\kappa$. (See Kolar-Michor-Slovak [20] for a discussion of Lie derivatives from a general point of view.)

For $\xi : M \to \Omega^{r} E$ we then get $d_{\kappa} \xi : M \to \Omega^{r} E$. If $\sigma : M \to E$ is a section and $\kappa$ is a linear connection of $E$ then we easily find

$$\delta d_{\kappa} \sigma = d_{\kappa} \delta \sigma - (\delta \kappa)_{\sigma} .$$

For $t : M \to Y$ let us now define $\delta t \equiv L_{\mu^{\mu}} t$ to be the standard Lie derivative. Then it is not difficult to see that the operator $\delta$ can be naturally extended, via linearity and the Leibnitz rule, to act on sections $\phi : M \to E \otimes Y$. Similarly we set

$$\delta(\kappa \otimes \Gamma) \equiv \delta \kappa \otimes \Gamma + \kappa \otimes \delta \Gamma ,$$

where $\delta \Gamma \equiv L_{\mu^{\mu}} \Gamma$ is the standard Lie derivative of the (tensor extension of the) spacetime connection [41]. We obtain

$$\delta \nabla \phi = \nabla \delta \phi - \delta (\kappa \otimes \Gamma)_{\phi} \quad \text{i.e.} \quad \delta \nabla \phi = \nabla_{\phi} \delta \phi - \delta \kappa_{\phi} = - \delta \Gamma_{\phi} + \kappa \phi - \Gamma_{\phi} \phi ,$$

$$\delta [\kappa, \kappa] = 2 [\kappa, \delta \kappa] ,$$

$$\delta [\kappa \otimes \Gamma, \kappa \otimes \Gamma]_{ab} = 2 \delta \kappa_{ab} ^{i} + 2 \delta \Gamma_{ab} ^{i} ,$$

$$\delta \kappa_{ab} ^{i} = \partial_{a} \delta \kappa_{i}^{ab} - \kappa_{ah} ^{i} \delta \kappa_{b}^{h} + \kappa_{bh} ^{i} \kappa_{a}^{h} .$$

Remark. We may choose $\kappa$ to be curvature-free (a local gauge); then we may work in local charts such that the coefficients of $\kappa$ vanish, getting

$$\delta s^{i} = u^{a} \partial_{a} s^{i} \quad \text{and} \quad \delta \kappa_{bj} ^{i} = \partial_{b} u^{a} \kappa_{aj} ^{i} + u^{a} \partial_{a} \kappa_{bj} ^{i} .$$

Such choice is then equivalent to using standard expressions for Lie derivatives by ignoring the fiber indices of $E$; namely, these indices are now seen as mere labels, so that e.g. $(s^{i})$ is treated as a collection of scalar functions and $(\kappa_{bj} ^{i})$ is treated as a collection of 1-forms on $M$. Or, we could use $\kappa$ itself as the reference connection, obtaining in particular $\delta \kappa = i_{\mu} d_{\kappa} \kappa$. 
3.3 Gauge field theory examples

Expressing the Lagrangian density as $\Lambda = \tilde{\lambda} \eta$ we get

$$\delta \Lambda = (\delta \tilde{\lambda}) \eta + \tilde{\lambda} \delta \eta, \quad \delta \eta = \frac{1}{2} g^{ab} \delta g_{ab} \eta \quad \text{with} \quad \delta g_{ab} \equiv L_{ab} g_{ab},$$

$$\delta \tilde{\lambda} = \frac{\partial \tilde{\lambda}}{\partial g_{ab}} \delta g_{ab} + \tilde{\Pi}_{iA} \delta \phi^{iA} + \tilde{\Pi}_{iA} (\delta \nabla \phi)^{iA} + 2 \tilde{\Pi}^{abj} \left[ \kappa, \delta \eta \right]_{abj} =$$

$$= \frac{\partial \tilde{\lambda}}{\partial g_{ab}} \delta g_{ab} - \tilde{\Pi}^{a}_{iA} \delta \Gamma^{A}_{AB} \phi^{iB} +$$

$$+ \left( \tilde{\Pi}^{a}_{iA} - \nabla_{a} \tilde{\Pi}^{a}_{iA} \right) \delta \phi^{iA} + \nabla_{a} (\tilde{\Pi}^{a}_{iA} \delta \phi^{iA}) + 2 \nabla_{a} (\tilde{\Pi}^{abj}_{i} \delta \kappa_{bj}) +$$

$$+ \left( -\tilde{\Pi}^{b}_{iA} \phi^{iA} - 2 \partial_{a} \tilde{\Pi}^{abj}_{i} + \Gamma^{c}_{ac} \tilde{\Pi}^{abj}_{i} - 2 \kappa_{a}^{b} \tilde{\Pi}^{b}{}_{ih} + 2 \tilde{\Pi}^{abh} \kappa_{ah}^{j} \right) \delta \kappa_{bj},$$

where we assumed that $\tilde{\lambda}$ depends on the base coordinates only through the metric and its derivatives.

We now make the further assumption that $\Gamma$ is the the Levi-Civita connection, namely it is torsionless in addition to being metric. Then by standard computations one shows that the second term in the above expression of $\delta \tilde{\lambda}$ can be expressed in the form

$$-\tilde{\Pi}^{a}_{iA} \delta \Gamma^{A}_{AB} \phi^{iB} = S^{ab} \delta g_{ab} + \text{a divergence},$$

where $S$ is a symmetric tensor field. Moreover we observe that the vanishing of the torsion, taking the replacement principle (§1.2) into account, implies that the coefficients of $\delta \phi^{iA}$ and $\delta \kappa_{bj}$ in $\delta \tilde{\lambda}$ vanish when $(\phi, \kappa)$ obeys the field equations. Hence, picking out those terms in $\delta \Lambda$ that are not divergences and do not vanish on-shell, we prove:

**Theorem 3.1** Let $(M, g)$ be a Lorentz spacetime and $\Gamma$ the related Levi-Civita connection. Let $\Lambda = \tilde{\lambda} \eta : D\Phi \to \Lambda^{Q} T^{*} M$ be such that $\tilde{\lambda}$ explicitly depends on spacetime coordinates only through $g$ and its derivatives. There there exists a unique morphism

$$\mathcal{T} : JD\Phi \to T^{*} M \otimes T^{*} M,$$

called the stress-energy tensor, with the following property: if the vector field $\delta : M \to TM$ vanishes on the boundary of the compact subset $D \subset M$ and $(\phi, \kappa)$ obeys the field equations, then

$$\int_{D} \delta \Lambda \circ \delta(\phi, \kappa) = \int_{D} (\mathcal{T}^{ab} \circ J\delta(\phi, \kappa)) \delta g_{ab} \eta.$$

Moreover $\mathcal{T}$ turns out to be symmetric, as we obtain the coordinate expression

$$\mathcal{T}^{ab} = \frac{\partial \tilde{\lambda}}{\partial g_{ab}} + \frac{1}{2} \tilde{\lambda} g^{ab} + S^{ab},$$

where $S$ is the symmetric tensor arising from the dependence of $\Lambda$ from the derivatives of $g$ through the spacetime connection.

By a standard argument then one also proves:

**Theorem 3.2** The stress-energy tensor is “on-shell” divergence-free, namely for any critical field $(\phi, \kappa) : M \to F$ we have

$$\nabla(\mathcal{T} \circ J\delta(\phi, \kappa)) = 0.$$
Bosonic field

As for the matter field we consider a complication with respect to the general scheme, namely it now consists of a couple

\[(\phi, \bar{\phi}) : M \to (E \otimes Y) \times (E^\ast \otimes Y^\ast)\]

of mutually independent fields valued into mutually dual bundles. In the usual formulations \(\phi\) and \(\bar{\phi}\) are often regarded as mutually adjoint fields through some Hermitian fiber structure, but that particularization is not needed here.

It is not difficult to see that the field equations (theorem 2.1) must now be rewritten in the adapted form

\[
\begin{align*}
\Pi^{(0)} - d_{\kappa \otimes \bar{\kappa}} \Pi^{(1)} &= 0 , \\
\bar{\Pi}^{(0)} - d_{\kappa \otimes \bar{\kappa}} \bar{\Pi}^{(1)} &= 0 , \\
\Pi^{(1)} \otimes \phi - \bar{\phi} \otimes \bar{\Pi}^{(1)} - d_\kappa \Pi^{(2)} &= 0 ,
\end{align*}
\]

that is

\[
\begin{align*}
\Pi_{iA} - \Pi^a_{jB} (\kappa^i_{aj} \delta^B_A + \delta^i_A \Gamma^{Bij}_A) - d_a \Pi^a_{iA} &= 0 , \\
\bar{\Pi}^A + \Pi^{ajB} (\kappa^{aj}_{B} \delta^A_j + \delta^A_j \Gamma^{ajB}) - d_a \Pi^{ajA} &= 0 , \\
\Pi^b_{jA} \phi^i_{A} - \bar{\Pi}^{bjA} \phi_{jA} + 2 (\partial_a \Pi^{abj}_i + \Pi^{abj}_i \kappa^{ah}_j - \Pi^{abj}_i \kappa^{ah}_j) &= 0 ,
\end{align*}
\]

where \(\Pi^{(0)}\) and \(\bar{\Pi}^{(1)}\), with components \(\Pi^{AJ}_{iA}\) and \(\bar{\Pi}^{ajA}\), denote the momenta related to the dual sector, and compositions of the momenta by \(\bar{\phi}(\phi, \bar{\phi}, \kappa)\) are intended. Note that the same gauge field interacts with \(\phi\) and \(\bar{\phi}\). On the other hand we could consider independent gauge fields \(\kappa\) and \(\bar{\kappa}\), getting one more field equation; then by identifying \(\bar{\kappa}\) as the dual of \(\kappa (\bar{\kappa}^i_{aj} = -\kappa^{aj}_{i})\) we obtain the above field equation for \(\kappa\).

We set \(\Lambda = \Lambda_{\text{matter}} + \Lambda_{\text{gauge}}\) with \(\Lambda_{\text{matter}} = \lambda_{\text{boson}} d^4x\) and

\[
\lambda_{\text{boson}} = \frac{1}{2} (g^{ab} z_{aA} z_{b}^{iA} - m^2 y_{iA} y^{iA}) .
\]

The explicit derivation of the field equations is now a straightforward task (maybe somewhat simpler than their usual derivation as the Euler-Lagrange equations). The momenta can be immediately expressed in coordinate-free form as

\[
\begin{align*}
\Pi^{(0)} &= -\frac{1}{2} m^2 \eta \otimes \bar{\phi} , \\
\Pi^{(1)} &= \frac{1}{2} \ast d_{\kappa \otimes \bar{\kappa}} \phi \equiv \frac{1}{2} \ast \nabla \phi , \\
\Pi^{(2)} &= \frac{1}{2} \ast d_\kappa \kappa ,
\end{align*}
\]

where \(\ast\) denotes the Hodge isomorphism. Hence the field equations can be cast (up to obvious transpositions) in the coordinate-free form

\[
\begin{align*}
\nabla(\ast \nabla \phi) + m^2 \eta \otimes \bar{\phi} &= 0 , \\
\nabla(\ast \nabla \phi) + m^2 \eta \otimes \phi &= 0 , \\
(\ast \nabla \phi) \otimes \phi - \bar{\phi} \otimes (\ast \nabla \phi) - d_\kappa (\ast \nabla \phi) &= 0 .
\end{align*}
\]
Using shorthands $\rho \equiv -d_x \kappa$, $\rho^{ab\beta\gamma} \equiv g^{ac} g^{bd} \rho_{cdj}$, we find the coordinate expressions

\[
\frac{1}{\sqrt{|g|}} \partial_a (g^{ab} \nabla_b \phi_{i\alpha} \sqrt{|g|}) + m^2 \phi_{i\alpha} + g^{ab} \nabla_b \phi_{j\beta} (\kappa_{ai}^{j\beta} + \delta^j_i \Gamma_{\alpha\beta\gamma}^{a\beta\gamma} - d^{a\beta\gamma}) = 0 ,
\]

\[
\frac{1}{\sqrt{|g|}} \partial_a (g^{ab} \nabla_b \phi^i A \sqrt{|g|}) + m^2 \phi^i A - g^{ab} \nabla_b \phi^j B (\kappa_{ai}^{j\beta} \delta^\beta_{\alpha\gamma} + \delta^j_i \Gamma_{\alpha\beta\gamma}^{a\beta\gamma} - d^{a\beta\gamma}) = 0 ,
\]

\[
\frac{1}{\sqrt{|g|}} \partial_a (\rho^{ab\beta\gamma} \sqrt{|g|}) + \rho^{ab\beta\gamma} h_{a\beta}^j - \kappa_{a\beta}^i \rho^{ab\beta\gamma} h_{\beta j} + \frac{1}{2} g^{ab} (\bar{\phi}_j \nabla_a \phi^j - \nabla_a \bar{\phi}_j \phi^j) = 0 .
\]

By virtue of the replacement principle ($\S$1.2), the field equations can also be written in terms of covariant divergences.

**Spin-$\frac{1}{2}$ field**

The geometric setting for Dirac spinors in curved spacetime has finer points, not examined here, that are widely discussed in the literature. My own view about this subject has been expressed in previous papers [1, 2].

The 4-spinor bundle $W \to M$ is endowed with a linear morphism $\gamma : TM \to \text{End} W$ whose components (the “gamma matrices”) are constant in a suitable frame. Allowing further internal degrees of freedom besides spin, the matter field can be described as a section

\[ (\psi, \bar{\psi}) : M \to (W \otimes E) \times (W^* \otimes E^*) . \]

Besides the gauge field we have to deal with a spinor connection $\Gamma$, that is a linear connection of $W$; in the present context it is considered as a fixed structure, related to the gravitational background. The tensor product connection $\Gamma \otimes \kappa$ of $W \otimes E$ has then the components

\[ (\Gamma \otimes \kappa)_{\alpha i}^{a\beta j} = \Gamma_{a\beta}^{\alpha i} \delta_j^i + \delta^\alpha_{\beta} \kappa_{ai}^j . \]

The field equations are now

\[
\begin{align*}
\Pi^{(0)} - [\Gamma \otimes \kappa, \Pi^{(1)}] &= 0 , \\
\Pi^{(0)} - [\Gamma \otimes \kappa, \Pi^{(1)}] &= 0 , \\
\Pi^{(1)} \otimes \psi - \bar{\psi} \otimes \Pi^{(1)} - [\kappa, \Pi^{(2)}] &= 0 ,
\end{align*}
\]

that is

\[
\begin{align*}
\Pi_{\alpha i} - \Pi_{\beta j}^a (\Gamma \otimes \kappa)_a^{\beta j} \delta_{\alpha i}^j - \partial_a \Pi_{\alpha i}^a &= 0 , \\
\Pi_{\alpha i}^a + \Pi_{\beta j}^a (\Gamma \otimes \kappa)_a^{\alpha i} \delta_{\beta j}^i - \partial_a \Pi_{\alpha i}^a &= 0 , \\
\Pi_{\alpha j}^b \psi_{\alpha i} - \bar{\psi}_{\alpha j} \Pi^b_{\alpha i} + 2 (\partial_a \Pi_{\alpha i}^a + \Pi_{\alpha j}^a \kappa_{\alpha h j}^i - \Pi_{\alpha h j}^a \kappa_{h ah j}^i) &= 0 .
\end{align*}
\]

Comparing these to the generic field equations in a gravitational background ($\S$3.1) one notes that here the spinor indices take up the role of the spacetime indices there.

The gauge sector Lagrangian is the same as in the boson case. The matter Lagrangian is the Dirac Lagrangian $\lambda_{\text{Dirac}} d^4x$, with

\[ \lambda_{\text{Dirac}} = (i \frac{1}{2} g^{ab} (y_{ai} \gamma_{a\beta}^\alpha z_b^\beta - z_{ai} \gamma_{\beta\alpha}^\beta y^\alpha) - m y_{ai} y_{\alpha i}) \sqrt{|g|} , \]
We also note that through critical fields one gets

\[ \Pi^{(0)} = (\frac{1}{2} \nabla \psi - m \bar{\psi}) \otimes \eta \equiv (\frac{1}{2} \nabla \psi - m \bar{\psi}) , \]

\[ \bar{\Pi}^{(0)} = (\frac{1}{2} \nabla \bar{\psi} - m \psi) \otimes \eta \equiv (\frac{1}{2} \nabla \bar{\psi} - m \psi) , \]

\[ \Pi^{(1)} = \frac{1}{2} * (\bar{\psi} \gamma) , \quad \bar{\Pi}^{(1)} = -\frac{1}{2} * (\gamma \psi) . \]

The field equations for the matter field then become

\[
\begin{cases}
\frac{1}{2} \left[ \mathbf{F} \otimes \kappa , * (\bar{\psi} \gamma) \right] + * (\frac{1}{2} \nabla \bar{\psi} + m \bar{\psi}) = 0 , \\
\frac{1}{2} \left[ \mathbf{F} \otimes \kappa , * (\gamma \psi) \right] - * (\frac{1}{2} \nabla \psi - m \psi) = 0 .
\end{cases}
\]

By some elaboration, these can be set in the usual form of the Dirac equations, namely

\[
\begin{cases}
-i g^{ab} \nabla_{a} \bar{\psi} \gamma_{b} \gamma_{\alpha} \bar{\psi}_{\alpha} - m \bar{\psi}_{\alpha} - \frac{i}{2} g^{ab} \tau_{a} \bar{\psi}_{\beta} \gamma_{b} \gamma_{\alpha} = 0 , \\
i g^{ab} \gamma_{a} \gamma_{\alpha} \nabla_{b} \bar{\psi}^{\beta} - m \psi^{\alpha} + \frac{i}{2} g^{ab} \tau_{a} \gamma_{b} \gamma_{\beta} \psi^{\beta} = 0 ,
\end{cases}
\]

where \( \tau_{a} \) is the torsion 1-form (§1.2).

As for the gauge field, we get the field equation

\[ 0 = -\frac{1}{2} d_{a} * d_{a} \kappa + i * (\gamma , \bar{\psi} \otimes \psi) , \]

with the coordinate expression

\[ \frac{1}{\sqrt{|g|}} \partial_{a} \left( \rho^{\alpha i j} \bar{\psi}_{\alpha} \gamma_{d} \psi^{i j} \right) + \rho^{\alpha h} \kappa_{a j} - \kappa^{i} \rho_{a h}^{j} - \frac{i}{2} g^{ab} \bar{\psi}_{\beta} \gamma_{a} \psi^{\alpha} = 0 . \]

**Canonical energy-tensors**

The canonical energy tensors (§2.5) for the considered boson, fermion and gauge sectors have, respectively, the expressions

\[
(U_{\text{boson}})_{b}^{a} = \lambda_{\text{boson}} \delta^{a}_{b} - \Pi^{a_{i}A} \nabla_{b} \phi^{i A} - \Pi^{aiA} \nabla_{b} \bar{\phi}_{i A} =
\]

\[ = \left( \frac{i}{2} \left( g^{cd} \nabla_{c} \bar{\phi}_{i A} \nabla_{d} \phi^{i A} \right) \delta^{a}_{b} - m^{2} \bar{\phi}_{i A} \phi^{i A} \delta^{a}_{b} - g^{ac} \left( \nabla_{c} \bar{\phi}_{i A} \nabla_{b} \phi^{i A} + \nabla_{b} \bar{\phi}_{i A} \nabla_{c} \phi^{i A} \right) \right) \sqrt{|g|} , \]

\[
(U_{\text{Dirac}})_{b}^{a} = \lambda_{\text{Dirac}} \delta^{a}_{b} - \Pi^{a_{i}A} \nabla_{b} \bar{\phi}_{i A} - \Pi^{aiA} \nabla_{b} \phi_{i A} =
\]

\[ = \frac{i}{2} \left( \bar{\psi} \gamma_{\alpha \beta} \nabla_{d} \psi^{\beta} \gamma_{c} \bar{\psi} \gamma_{d} \phi^{i A} \right) \left( g^{cd} \delta^{a}_{b} - g^{ca} \delta^{d}_{b} \right) \sqrt{|g|} - m \bar{\psi}_{\alpha} \psi^{\alpha} \delta^{a}_{b} \sqrt{|g|} , \]

\[
(U_{\text{gauge}})_{b}^{a} = \lambda_{\text{gauge}} \delta^{a}_{b} + 2 \Pi_{i}^{aj} \rho_{bcj}^{i} = \left( \frac{i}{4} \rho^{cdi} \jmath \rho_{cdi}^{j} \delta^{a}_{b} - \rho^{aci} \jmath \rho_{bcj}^{i} \right) \sqrt{|g|} . \]

Then it is not difficult to check that in all cases the canonical energy-tensor and the stress-energy tensor are related by

\[ U_{ab} + U_{ba} = -4 \mathcal{T}_{ab} \]

(the symmetrization in the indices \( a \) and \( b \) is only required for the Dirac field), and that by evaluation through critical fields one gets

\[ \nabla (\mathcal{T}_{\text{matter}} + \mathcal{T}_{\text{gauge}}) = \text{torsion terms}. \]

We also note that \( \mathcal{T}_{\text{matter}} \) and \( \mathcal{T}_{\text{gauge}} \) are not separately divergence-free: their sum is such.
Gravitational field

The “metric-affine” approach to gravity can be treated in the covariant-differential formalism, too. Let the gravitational field be represented by the couple \((g, \Gamma)\) constituted by a spacetime metric and a linear spacetime connection, and \(\Lambda_{\text{grav}} = \lambda_{\text{grav}} d^4x\) with

\[
\lambda_{\text{grav}} = g^{ad} \delta^b_c R_{abcd} \sqrt{|g|}, \quad R \equiv -d\Gamma.
\]

Then we find \(\Pi^{(0)} = G \otimes \eta\), where \(G\) is the Einstein tensor, \(\Pi^{(1)} = 0\), and

\[
\Pi_{e}^{abd} \equiv (\Pi^{(2)}_{e})^{abd} = \frac{1}{2} (g^{bd} \delta^a_c - g^{ad} \delta^b_c) \sqrt{|g|}.
\]

Accordingly, the field equations (theorem 2.1) turn out to be the Einstein equation for the \(g\)-sector and the equation

\[
d\Gamma (\Pi^{(2)} \circ (g, \Gamma)) = 0
\]

i.e.

\[
\partial_a \Pi_{e}^{abd} - \Gamma^{d}_{ae} \Pi_{e}^{abc} + \Gamma^{e}_{ae} \Pi_{e}^{abd} = 0,
\]

for the \(\Gamma\)-sector. After some elaboration this can be written in the form

\[
\nabla_{c}(g^{bd} \sqrt{|g|}) = \text{torsion terms}.
\]

If the torsion is assumed to vanish then this is equivalent to \(\nabla_{c}g^{bd} = 0\).

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