FLAT RELATIVE MITTAG-LEFFLER MODULES AND APPROXIMATIONS

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Abstract. The classes $D_Q$ of flat relative Mittag-Leffler modules are sandwiched between the class $FM$ of all flat (absolute) Mittag-Leffler modules, and the class $F$ of all flat modules. Building on the works of Angeleri Hügel, Herbera, and Šaroch, we give a characterization of flat relative Mittag-Leffler modules in terms of their local structure, and show that Enochs’ Conjecture holds for all the classes $D_Q$. In the final section, we apply these results to the particular setting of $f$-projective modules.

Introduction

For a ring $R$, denote by $P$, $F$, and $FM$ the classes of all projective, flat, and flat Mittag-Leffler (right $R$-) modules, respectively. We always have the inclusions $P \subseteq FM \subseteq F$. The equality $P = FM = F$ holds, if and only if $R$ is a right perfect ring. By a classic result of Bass, in this case $P$ is a covering class consisting of modules isomorphic to direct sums of (indecomposable projective) modules generated by primitive idempotents of the ring $R$.

If $R$ is not right perfect, then $P \subsetneq FM \subsetneq F$. In fact, though the classes $P$ and $FM$ contain the same countably generated modules, there always exist $\aleph_1$-generated modules in $FM$ that are not projective, cf. [3, §VII.1]. Moreover, there exist countably presented modules $N \in F \setminus FM$. Each such module $N$ is called a Bass module [19].

By a classic theorem of Kaplansky, each projective module is a direct sum of countably generated projective modules, so the class $P$ is $\aleph_1$-decomposable. If $\kappa = \text{card } R + \aleph_0$, then each flat module is known to be a transfinite extension of $\leq \kappa$-presented flat modules, so the class $F$ is $\kappa^*$-deconstructible. The class $P$ is easily seen to be precovering, while $F$ is a covering class by [2] (see Section 1 for unexplained terminology).

The intermediate class $FM$ can be described as the class of all ‘locally projective’, or better $\aleph_1$-projective modules [9]. Its global structure over non-right perfect rings is known to be quite complex: there is no cardinal $\lambda$ such that $FM$ is $\lambda$-deconstructible [9]; moreover, the class $FM$ is not precovering [17].

In this note, we will deal with classes of flat relative Mittag-Leffler modules, or more precisely, flat $Q$-Mittag-Leffler modules for a class of left $R$-modules $Q$.

The notion of an (absolute) Mittag-Leffler module was introduced already in the seminal paper by Raynaud and Gruson [14], and studied in a number of sequel works revealing its many facets. Relative Mittag-Leffler modules appeared much later, in the Habilitationsschrift of Rothmaler [15]. Rothmaler has further pursued the model theoretic point of view in [16], where he proved that if $Q$ is a definable class of left $R$-modules and $D(Q)$ is its dual definable class of (right $R$-) modules, then $Q$-Mittag-Leffler modules are exactly the $D(Q)$-atomic modules, [16 Theorem 3.1]. For a very recent application of relative atomic modules to a description of Ziegler spectra of tubular algebras, we refer to [12].

A detailed algebraic study of relative Mittag-Leffler modules was performed in [1] (see also [8] and [9]); notably, it discovered their role in (infinite dimensional) tilting theory.

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Following [3], we denote the class of all flat $Q$-Mittag-Leffler modules by $\mathcal{D}_Q$. Thus $\mathcal{F} \mathcal{M} \subseteq \mathcal{D}_Q \subseteq \mathcal{F}$. Notice that $\mathcal{D}_{\text{R-Mod}} = \mathcal{F} \mathcal{M}$ and $\mathcal{D}_{\{0\}} = \mathcal{F}$, while $\mathcal{D}_{\{R\}}$ is the class of all f-projective modules studied by Goodearl et al. in [3], [4], etc.

Our goal here is to investigate the structure and approximation properties of the class $\mathcal{D}_Q$ in dependence on $Q$. In Theorem 2.5, we prove that the classes $\mathcal{D}_Q$ are determined by their countably presented modules, while Theorem 2.6 shows that approximation properties depend completely on whether there exists a Bass module $N \notin \mathcal{D}_Q$. In the final part, we apply these results to the particular setting of $Q = \{R\}$, i.e., to the f-projective modules.

1. Preliminaries

For a ring $R$, we denote by $\text{Mod–R}$ the class of all (right $R$-) modules, and by $R–\text{Mod}$ the class of all left $R$-modules.

1.1. Filtrations and deconstructible classes. Let $R$ be a ring, $M$ a module, and $\mathcal{C}$ a class of modules. A family of submodules, $M = (M_\alpha \mid \alpha \leq \sigma)$, of $M$ is called a continuous chain in $M$, provided that $M_0 = 0$, $M_\alpha \subseteq M_{\alpha+1}$ for each $\alpha < \sigma$, and $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$.

A continuous chain $M$ in $M$ is a $\mathcal{C}$-filtration of $M$, provided that $M = M_\sigma$, and of each of the modules $M_{\alpha+1}/M_\alpha (\alpha < \sigma)$ is isomorphic to an element of $\mathcal{C}$. $M$ is called $\mathcal{C}$-filtered, provided that $M$ possesses at least one $\mathcal{C}$-filtration. We will use the notation $\text{Filt}(\mathcal{C})$ for the class of all $\mathcal{C}$-filtered modules. The modules $M \in \text{Filt}(\mathcal{C})$ are also called transfinite extensions of the modules in $\mathcal{C}$. A class $\mathcal{A}$ is said to be closed under transfinite extensions provided that $\mathcal{A} = \text{Filt}(\mathcal{A})$. Clearly, this implies that $\mathcal{A}$ is closed under extensions and arbitrary direct sums.

Given a class $\mathcal{C}$ and a cardinal $\kappa$, we use $\mathcal{C}^{<\kappa}$ and $\mathcal{C}^{\leq\kappa}$ to denote the subclass of $\mathcal{C}$ consisting of all $\leq \kappa$-presented and $< \kappa$-presented modules, respectively.

Let $\kappa$ be an infinite cardinal. A class of modules $\mathcal{C}$ is $\kappa$-deconstructible provided that $\mathcal{C} \subseteq \text{Filt}(\mathcal{C}^{<\kappa})$. If moreover each module $M \in \mathcal{C}$ is a direct sum of modules from $\mathcal{C}^{<\kappa}$, then $\mathcal{C}$ is called $\kappa$-decomposable. For example, the class $\mathcal{P}$ of all projective modules is $\aleph_1$-deconstructible by a classic theorem of Kaplansky. A class $\mathcal{C}$ is deconstructible in case it is $\kappa$-deconstructible for some infinite cardinal $\kappa$.

1.2. Approximations. A map $f \in \text{Hom}_R(C, M)$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-precover of $M$, if the abelian group homomorphism $\text{Hom}_R(C', f) : \text{Hom}_R(C', C) \rightarrow \text{Hom}_R(C', M)$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $f \in \text{Hom}_R(C, M)$ of $M$ is called a $\mathcal{C}$-cover of $M$, provided that $f$ is right minimal, that is, provided $fg = f$ implies that $g$ is an automorphism for each $g \in \text{End}_R(\mathcal{C})$.

$\mathcal{C} \subseteq \text{Mod–R}$ is a precovering class (covering class) provided that each module has a $\mathcal{C}$-precover ($\mathcal{C}$-cover).

1.3. (Relative) Mittag-Leffler modules. Let $R$ be a ring. A module $M$ is Mittag-Leffler provided that the canonical group homomorphism

$$\varphi : M \otimes_R \prod_{i \in I} N_i \rightarrow \prod_{i \in I} M \otimes_R N_i$$

defined by

$$\varphi(m \otimes_R (n_i)_{i \in I}) = (m \otimes_R n_i)_{i \in I}$$

is monic for each family $(N_i \mid i \in I)$ of left $R$-modules.

Let $M \in \text{Mod–R}$ and $Q \subseteq R–\text{Mod}$. Then $M$ is $Q$-Mittag-Leffler, provided that the canonical morphism $M \otimes \prod_{i \in I} Q_i \rightarrow \prod_{i \in I} M \otimes_R Q_i$ is injective for any family $(Q_i \mid i \in I)$ consisting of elements of $Q$. 
1.4. **Direct limits and add(M).** Let \( \mathcal{C} \) be any class of modules and \( \mathcal{D} = (C_i, f_{ij} \mid i \leq j \in I) \) a direct system of modules in \( \mathcal{C} \). Viewing \( \mathcal{D} \) as a diagram in the category \( \text{Mod–}R \), we can form its colimit, \((M, f_i \mid i \in I)\). In particular, \( M \) is a module, and \( f_i \in \text{Hom}_R(M_i, M) \) satisfy \( f_i = f_{ij}f_j \) for all \( i \leq j \in I \).

This colimit (or sometimes just the module \( M \) itself) is called the **direct limit** of the direct system \( \mathcal{D} \). It is denoted by \( \lim_{i \in I} M_i \) (or just \( \lim \mathcal{D} \)).

Let \( Q \subseteq R–\text{Mod} \). We denote by \( \lim Q \) the class of all modules \( N \) such that \( N = \lim Q_i \), where \((Q_i, f_{ij} \mid i \leq j \in I)\) is a direct system of modules from \( Q \).

Let \( R \) be a ring, \( M \) be a module. We define \( \text{add}(M) \) to be the class of all modules isomorphic to direct sums of copies of \( M \).

1.5. **Bass modules.** Given a class \( \mathcal{C} \) of finitely generated free modules, we call a module \( M \) a **Bass module** over \( \mathcal{C} \), provided that \( M \) is the direct limit of a direct system

\[
\begin{align*}
C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots & \xrightarrow{f_{n-1}} C_n \xrightarrow{f_n} C_{n+1} \xrightarrow{f_{n+1}} \cdots
\end{align*}
\]

where \( C_n \in \mathcal{C} \) for each \( n < \omega \).

If \( \mathcal{C} \) is the class of all finitely generated free modules, then the Bass modules over \( \mathcal{C} \) are just called the (unadorned) **Bass modules**; they are exactly the countably presented flat modules.

For basic properties of the notions defined above, we refer the reader to [5].

2. **Flat relative Mittag-Leffler modules**

We record the following well-known properties of the class \( \mathcal{D}_Q \) of all flat \( Q \)-Mittag-Leffler modules (cf. [11] SS1 and 5], [9] §4 or [5] 3.20(a)):

**Lemma 2.1.** Let \( Q \subseteq R–\text{Mod} \).

(i) The class \( \mathcal{D}_Q \) is closed under pure submodules, extensions, and unions of pure chains. Hence \( \mathcal{D}_Q \) is closed under transfinite extensions.

(ii) \( \mathcal{D}_Q \) is a resolving subcategory of \( \text{Mod–}R \) (i.e., \( \mathcal{D}_Q \) contains all projective modules, and it is closed under extensions and kernels of epimorphisms).

**Remark 1.** Clearly, \( \mathcal{D}_Q \) is closed under direct limits, iff \( \mathcal{D}_Q = \mathcal{F} \). This case will be examined in more detail in Theorems 2.5(ii) and 2.6 below. The closure of the class \( \mathcal{D}_Q \) under products was studied in [9] §4: if \( Q \) is the limit closure of a class of finitely presented left \( R \)-modules, then \( \mathcal{D}_Q \) is closed under products, iff \( R^\mathcal{E} \in \mathcal{D}_Q \) (see [9] Theorem 4.6).

Another basic property of the classes \( \mathcal{D}_Q \) is that in their study, one can restrict to definable classes of left \( R \)-modules. Recall that a class of modules is **definable** provided that it is closed under direct limits, direct products and pure submodules. For each class of left \( R \)-modules \( Q \) there is a least definable class \( \text{Def}(Q) \) in \( R–\text{Mod} \) containing \( Q \); it is obtained by closing \( Q \) first by direct products, then direct limits, and finally by pure submodules, cf. [8] Lemma 2.9 and Corollary 2.10].

**Lemma 2.2.** Let \( Q \subseteq R–\text{Mod} \). Then \( \mathcal{D}_Q = \mathcal{D}_{\text{Def}(Q)} \).

Definable classes are parametrized by the subset of the set of all indecomposable pure-injective modules which they contain. So though \( R–\text{Mod} \) is a proper class, there is only a set of classes of relative Mittag-Leffler modules. Note however, that it may still happen that \( \mathcal{D}_{\text{Def}(Q)} = \mathcal{D}_{\text{Def}(Q')} \) even if \( \text{Def}(Q) \neq \text{Def}(Q') \): just take a right noetherian ring \( R \) which is not completely reducible, and consider the following two definable classes of left \( R \)-modules: \( Q = \{0\} \) and \( Q' = \mathcal{F} \) (the class of all flat left \( R \)-modules). Then \( \mathcal{D}_Q = \mathcal{D}_{Q'} = \mathcal{F} \) by Proposition 2.4(i) below. In Theorem 2.5 we will give a different parametrization of the classes \( \mathcal{D}_Q \), by their countably presented modules.

Our next prerequisite was proved in [9] 2.2 (see also [5] 3.11):
Lemma 2.3. Let $R$ be a ring, $M$ be a module, and $Q$ be a class of left $R$-modules. Assume that $M = \varinjlim_{\alpha \in A} M_\alpha$ where $(M_\alpha, f_{\alpha \beta} \mid \alpha < \beta \in L)$ is a direct system of $Q$-Mittag-Leffler modules. Moreover, assume that $M' = \varinjlim_{n<\omega} M_{\alpha_n}$ is $Q$-Mittag-Leffler for each countable chain $\alpha_0 < \cdots < \alpha_n < \alpha_{n+1} < \cdots$ in $L$.

Then $M$ is $Q$-Mittag-Leffler.

For all rings, flat relative Mittag-Leffler modules include the flat (absolute) Mittag-Leffler modules, and for some rings, even all the flat modules (see Section 3 below). So the following description of the local structure of flat relative Mittag-Leffler modules extends simultaneously the ‘local projectivity’ of flat Mittag-Leffler modules from [9, Theorem 2.10(i)] and the deconstructibility, and hence abundance of small pure flat submodules, of flat modules from [5, Lemma 6.17 and Theorem 7.10] (cf. also [11, Theorem 5.1], [9, Theorem 2.6], [16, Corollary 6.5], and [18, Lemma 2.5]):

Proposition 2.4. Let $R$ be a ring, $M$ be a module, and $Q$ be a class of left $R$-modules. Let $\kappa = \text{card } (R) + \aleph_0$. Then the following conditions (i)-(iv) are equivalent:

(i) $M$ is a flat $Q$-Mittag-Leffler module.

(ii) For each subset $C$ of $M$ of cardinality $\leq \kappa$, there exists a pure flat $Q$-Mittag-Leffler submodule $N$ of $M$ such that $C \subseteq N$, and $N$ has cardinality $\leq \kappa$.

(iii) There exists a system $S$ consisting of pure flat $Q$-Mittag-Leffler submodules of $M$ of cardinality $\kappa$, such that for each subset $C$ of $M$ of cardinality $\leq \kappa$ there is $N \in S$ containing $C$, and $S$ is closed under unions of well-ordered chains of length $\leq \kappa$.

(iv) $M$ is a directed union of a direct system $T$ consisting of flat $Q$-Mittag-Leffler submodules of $M$, such that $T$ is closed under unions of countable chains.

Consider also the following condition:

(v) There exists a system $U$ consisting of countably presented flat $Q$-Mittag-Leffler submodules $N$ of $M$ such that the inclusion $N \hookrightarrow M$ remains injective when tensored by any left $R$-module $Q \in Q$, and satisfying the following two conditions:

(a) for each countable subset $C$ in $M$ there is $N \in U$ containing $C$, and (b) $U$ is closed under unions of countable chains.

Then (v) implies (i), and if $R \in Q$, then (v) is equivalent to (i).

Proof. (i) $\Rightarrow$ (ii). Let $C$ be a subset of $M$ with $\text{card } C \leq \kappa$. Then there is a pure submodule $P \subseteq^* M$ such that $C \subseteq P$ and $\text{card } P \leq \kappa$ (see e.g. [5, 2.25(i)]). By Lemma 2.1 $P$ is flat and $Q$-Mittag-Leffler, whence (ii) holds.

(ii) $\Rightarrow$ (iii). We will prove that the set $S$ consisting of all pure flat $Q$-Mittag-Leffler submodules of $M$ of cardinality $\leq \kappa$ has the required two properties. The first one is just a restatement of (ii). For the second (closure under unions of well-ordered chains of length $\leq \kappa$), let $(N_\alpha \mid \alpha < \kappa)$ be a such a chain in $S$. Let $N = \bigcup_{\alpha < \kappa} N_\alpha$. Since $N_\alpha$ is pure in $M$ for each $\alpha < \kappa$, $N$ is pure in $M$, too, by [5, Lemma 2.25(d)]. Since $\text{card } N \leq \kappa$, (ii) yields existence of $N' \in S$ such that $N \subseteq N'$. Finally, $N \subseteq^* M$ implies $N \subseteq^* N'$. As $N'$ is flat and $Q$-Mittag-Leffler, Lemma 2.1(i) gives $N \in S$.

(iii) $\Rightarrow$ (iv) This is clear - just take $T = S$.

(iv) $\Rightarrow$ (i) First, $M$, being a directed union of flat modules, is flat. In view of (iv), we can apply Lemma 2.3 to the presentation of $M$ as the directed union of the elements of $T$; thus, $M$ is $Q$-Mittag-Leffler.

Assume (v). Then $M$ is a directed union of the modules in $U$, and Lemma 2.3 applies, showing that $M$ is a flat $Q$-Mittag-Leffler module.

Finally, let $R \in Q$. Assume $M$ is a flat $Q$-Mittag-Leffler module, and let $U'$ be the set consisting of all countably presented flat $Q$-Mittag-Leffler submodules $N$ of $M$ such that the inclusion $N \hookrightarrow M$ remains injective when tensored by any left $R$-module $Q \in Q$. 

Since $R \in Q$, the implication $(1) \Rightarrow (4)$ of [1] Theorem 5.1] (for $S =$ the class of all finitely generated free modules) yields condition (a). Let $N'$ be the union of a countable chain $(N_i \mid i < \omega)$ of modules from $\mathcal{U}$. Then $N'$ is flat, and each finite subset of $N'$ is contained in some term of the chain, so by the implication $(4) \Rightarrow (1)$ of [1] Theorem 5.1], $N'$ is a $Q$-Mittag-Leffler module. Since the inclusion $\nu : N' \hookrightarrow M$ is a direct limit of the inclusions $\nu_i : N_i \hookrightarrow M$ ($i < \omega$), $\nu \otimes_R Q$ is the direct limit of the monomorphisms $\nu_i \otimes_R Q : N_i \otimes_R Q \hookrightarrow M \otimes_R Q$ ($i < \omega$), hence $\nu \otimes_R Q$ is injective, for each $Q \in Q$. Thus $N' \in \mathcal{U}$, and condition (b) holds, too. □

Remark 2.1. If $R \notin Q$, then (i) need not imply (v). For a simple counter-example, consider a von Neumann regular ring $R$ such that there exists a simple module $M$ which is not countably presented (i.e., $M = R/I$ where $I$ is a maximal right ideal of $R$ which is not countably generated). Examples of such rings $R$ include infinite products of fields, or endomorphism rings of infinite dimensional linear spaces. Let $Q = \{0\}$. Then $M$ is flat (= flat $Q$-Mittag-Leffler), as all modules over von Neumann regular rings are flat, but (v) fails, because the only countably presented submodule of $M$ is 0.

2. If we remove the assumption of flatness from conditions (i)-(v), then the result still holds true, cf. [3] Theorem 2.6.

3. The system $S$ in (iii) consists of 'big' (= of cardinality $\leq \kappa$) pure submodules of $M$ and it is closed under unions of 'long' (= of length $\leq \kappa$) well-ordered chains, while the system $\mathcal{U}$ in (v) consists of 'small' (= countably presented), but possibly non-pure, submodules of $M$, and it is closed under unions of 'short' (= countable) chains.

It may even happen that no non-zero module in the system $\mathcal{U}$ is pure in $M$: for an example, consider the polynomial ring $R = \mathbb{C}[x]$, let $Q = \{R\}$, and let $M$ be the quotient field of $R$ viewed as an (uncountably generated) $R$-module. Then $M \in \mathcal{D}_Q$, but $M$ has no non-zero countably generated pure submodules. So in this setting, there are only the trivial choices for a system $S$ satisfying condition (iii) (namely, $S = \{M\}$, and $S = \{0, M\}$), while $\mathcal{U}$ from (v) must be uncountable - e.g., $\mathcal{U}$ can be taken as the set of all countably generated submodules of $M$.

We arrive at a simple test of coincidence for various classes $\mathcal{D}_Q$ – one only has to check their countably presented modules:

**Theorem 2.5.** Let $R$ be a ring.

(i) Let $Q$ and $Q'$ be classes of left $R$-modules containing $R$. Then $\mathcal{D}_Q = \mathcal{D}_{Q'}$, iff $\mathcal{D}_Q$ and $\mathcal{D}_{Q'}$ contain the same countably presented modules.

(ii) Let $Q$ be an arbitrary class of left $R$-modules. Then $\mathcal{D}_Q = \mathcal{F}$, iff each countably presented flat module is $Q$-Mittag-Leffler.

(iii) Let $Q$ be a class of left $R$-modules containing $R$. Then $\mathcal{D}_Q = \mathcal{F} M$, iff each countably presented flat $Q$-Mittag-Leffler module is projective.

**Proof.** (i) Assume there is a module $M \in \mathcal{D}_Q \setminus \mathcal{D}_{Q'}$. Consider the system $\mathcal{U} \subseteq \mathcal{D}_Q$ provided by Proposition 2.3 (v) for the class $Q$. Then $M$ is the directed union of the modules in $\mathcal{U}$, but $M \notin \mathcal{D}_{Q'}$. By Lemma 2.3] there is a (countably presented) module $N \in \mathcal{U}$ such that $N \notin \mathcal{D}_{Q'}$.

(ii) Assume there is a module $M \in \mathcal{F} \setminus \mathcal{D}_Q$. Being flat, $M$ is a direct limit of a direct system $\mathcal{D}$ of finitely generated free modules. By Lemma 2.3] there exists a Bass module $N$ over $\mathcal{D}$ such that $N \notin \mathcal{D}_Q$.

(iii) is a special instance of (i) for $Q' = R$-Mod, since countably presented flat Mittag-Leffler modules are projective. □

Note that in [16] Theorem 7.1], countably generated $Q$-Mittag-Leffler modules were characterized using $D(\text{Def}(Q))$-pure chains of finitely presented modules.
Our next theorem says that precovers (right approximations) by modules in the class $\mathcal{D}_Q$ exist only in the threshold case of $\mathcal{D}_Q = \mathcal{F}$. The theorem also confirms Enochs’ Conjecture (that covering classes of modules are closed under direct limits) for all the classes $\mathcal{D}_Q$:

**Theorem 2.6.** Let $R$ be a ring and $Q$ be a class of left $R$-modules. Then the following conditions are equivalent:

1. Each Bass module (= countably presented flat module) is $Q$-Mittag-Leffler.
2. $\mathcal{D}_Q = \mathcal{F}$.
3. $\mathcal{D}_Q$ is covering.
4. $\mathcal{D}_Q$ is precovering.
5. $\mathcal{D}_Q$ is deconstructible.
6. $\mathcal{D}_Q$ is closed under direct limits.

**Proof.** (i) $\Rightarrow$ (ii) by Theorem 2.5(ii).

(ii) $\Rightarrow$ (iii), (v), and (vi): This follows from the fact that for any ring $R$, the class of all flat modules is a deconstructible covering class closed under direct limits, cf. [2].

(iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i): Assume (i) fails, so there is a Bass module $N \in \mathcal{F} \setminus \mathcal{D}_Q$. Let $f : A \to N$ be a (surjective) $\mathcal{D}_Q$-precover of $N$ and $M = \ker f$. Let $\kappa$ be an infinite cardinal such that $\card R \leq \kappa$ and $\card M \leq 2^\kappa = \kappa^\omega$. By [19] Lemma 5.6, there are a ‘tree module’ $L$ and an exact sequence $0 \to D \to L \to N' \to 0$, where $D$ is a direct sum of $\kappa$ finitely generated free modules and $L$ is flat and Mittag-Leffler. Proceeding as in the proof of [17] Lemma 3.2, we infer that $f$ splits, whence $N \in \mathcal{D}_Q$, a contradiction.

(v) $\Rightarrow$ (i): This has been proved in [2] Corollary 7.2(ii).

(vi) $\Rightarrow$ (i): This holds since $\mathcal{F} = \lim \mathcal{P}$, whence $\lim \mathcal{D}_Q = \mathcal{F}$. □

**Remark 3.** If $Q$ is a class of left $R$-modules such that $\mathcal{D}_Q = \mathcal{F}$, then $\mathcal{D}_Q = \Filt \mathcal{D}_Q^{\leq \kappa}$ for any infinite cardinal $\kappa \geq \card R$.

In contrast, if $Q$ is a class of left $R$-modules such that $\mathcal{D}_Q \neq \mathcal{F}$, then the classes $\Filt \mathcal{D}_Q^{\leq \kappa}$, where $\kappa$ runs over all infinite cardinals $\geq \card R$, form a strictly increasing ‘chain’ – a proper class of subclasses of $\mathcal{D}_Q$ – consisting of classes closed under transfinite extensions, whose union is $\mathcal{D}_Q$.

Indeed, the existence of a Bass module $N \in \mathcal{F} \setminus \mathcal{D}_Q$ makes it possible to construct for each infinite cardinal $\kappa \geq \card R$ a $\kappa^+$-generated flat Mittag-Leffler module $M_\kappa^+$ such that $M_\kappa^+$ is not $\mathcal{D}_Q^{\leq \kappa}$ filtered, cf. [9] §5. Thus $\Filt \mathcal{D}_Q^{\leq \kappa} \subseteq \Filt \mathcal{D}_Q^{\leq \kappa^+} \subseteq \mathcal{D}_Q$, and $\mathcal{D}_Q = \bigcup_{\kappa \geq \card R} \Filt \mathcal{D}_Q^{\leq \kappa}$. Moreover, though $\mathcal{F}M \subseteq \mathcal{D}_Q$, there is no cardinal $\kappa$ such that $\mathcal{F}M \subseteq \Filt \mathcal{D}_Q^{\leq \kappa}$.

3. **$f$-projective modules**

In this section, we will consider a particular kind of relative Mittag-Leffler modules, the $f$-projective ones. Their original definition is as follows:

**Definition 3.1.** A module $M$ is said to be $f$-projective if for every finitely generated submodule $C$ of $M$, the inclusion map factors through a (finitely generated) free module $F$.

\[
\begin{array}{ccc}
C & \rightarrow & M \\
\uparrow & & \downarrow \\
F & & \\
\end{array}
\]

So every projective module is $f$-projective, and every finitely generated $f$-projective module is projective. Since flat modules are characterized as the direct limits of finitely generated free modules, each $f$-projective module is flat by the following lemma due to Lenzing, cf. [5] Lemma 2.13:
Lemma 3.2. Let $R$ be a ring and $C$ be a class of finitely presented modules closed under finite direct sums. Then the following are equivalent for a module $M$.

(i) Every homomorphism $\varphi : G \to M$, where $G$ is finitely presented, has a factorisation through a module in $C$.

(ii) $M \in \lim_{\rightarrow} C$.

The fact that $f$-projective modules are a particular kind of flat relative Mittag-Leffler modules goes back to Goodearl [6], see also [4, Proposition 2.7]:

Proposition 3.3. A module $M$ is $f$-projective if and only if it is flat and $(R)$-Mittag-Leffler. In particular, each countably generated $f$-projective module is countably presented, and hence of projective dimension $\leq 1$.

Proof. Let $F'$ denote the class of all flat left $R$-modules. By Lemma 2.2 (or [6, Theorem 1]), $\mathcal{D}(R) = \mathcal{D}_{F'}$. By [6, Theorem 1], for each module $M$, $M \in \mathcal{D}_{F'}$ iff $M$ is flat and for each finitely generated submodule $F$ of $M$, the inclusion $F \hookrightarrow M$ factors through a finitely presented module. By Lemma 3.2 this is further equivalent to the $f$-projectivity of $M$.

The final claim follows from [1, Corollary 5.3]. □

We also note the following corollary of [1, Theorem 5.1] (and Theorem 2.4):

Corollary 3.4. Let $R$ be a ring and $M \in \text{Mod-}R$. Then $M$ is $f$-projective, if and only if $M$ possesses a system of submodules, $U$, consisting of countably presented $f$-projective modules such that each countable subset of $M$ is contained in an element of $U$ (and $U$ is closed under unions of countable chains).

We will denote by $FP$ the class of all flat $f$-projective modules. So $FP = \mathcal{D}(R) = \mathcal{D}_{F'}$.

There is an interesting relation between $f$-projectivity and coherence. Here, we will call a module $M$ coherent, if all its finitely generated submodules are finitely presented. (Thus a ring $R$ is right coherent, if the regular right module is coherent.)

Lemma 3.5. Let $R$ be a ring.

(i) Let $M$ be a flat coherent module. Then $M$ is $f$-projective.

(ii) The ring $R$ is right coherent, iff $FP$ coincides with the class of all flat coherent modules.

Proof. (i) Since $M$ is flat, $M = \lim F_i$ where the modules $F_i$ are finitely generated and free. If $C$ is a finitely generated submodule of $M$, then $C$ is finitely presented, so the inclusion $C \subseteq M$ factors through some $F_i$ by Lemma 3.2 whence $M$ is $f$-projective.

(ii) In view of part (i) and Lemma 3.2 in order to prove the only-if part, we have to prove that $f$-projectivity implies coherence for any module $M$. Let $F$ be a finitely generated submodule of $M$. By $f$-projectivity, $F$ is a submodule of a finitely generated free module. Since $R$ is right coherent, $F$ is finitely presented.

The if part is clear, since the regular module $R$ is always $f$-projective. □

Note that the situation simplifies for coherent domains:

Lemma 3.6. Let $R$ be a coherent domain. Then $FP = F$.

Proof. Let $M$ be a flat module. By Lemma 3.5(i), it suffices to prove that $M$ is coherent. Let $F$ be a finitely generated submodule of $M$. Then $M$ and $F$ are torsion-free, so by a classic result of Cartan and Eilenberg [5, 16.1], $F$ embeds into a finitely generated free module. By the coherence of $R$, $F$ is finitely presented, proving that $M$ is coherent. □

Further instances of the coincidence $FP$ with $F$ (i.e., of the case when $FP$ is a covering class, see Theorem 2.6) appear in part (i) of the following proposition:

Proposition 3.7. Let $R$ be a ring.
Assume that $R$ is right noetherian or $R$ is right perfect. Then $\mathcal{FP} = \mathcal{F}$ is a covering class.

(ii) If $R$ is right non-singular, then all $\mathcal{F}$-projective modules are non-singular.

(iii) If $R$ is von Neumann regular, then $\mathcal{FP} = \mathcal{FM}$. Hence $\mathcal{FP}$ is covering, only if $R$ is completely reducible.

(iv) Assume $R$ is von Neumann regular and right self-injective. Then $\mathcal{FP}$ coincides with the class of all non-singular modules.

**Proof.** (i) If $R$ is right noetherian, then each finitely generated module is finitely presented, so Lemma 3.2 yields that all flat modules are $\mathcal{F}$-projective. If $R$ is right perfect, then all relative Mittag-Leffler modules are projective ($= \text{flat}$).

(ii) Let $M$ be an $\mathcal{F}$-projective module. By Definition 3.1 each finitely generated submodule $C$ of $M$ embeds into a finitely generated free module $F$. By assumption, $F$ is non-singular, hence so are $C$ and $M$.

(iii) This is clear, since von Neumann regularity of $R$ is equivalent to the property that all left $R$-modules are flat.

(iv) The non-singularity of all $\mathcal{F}$-projective modules follows from (ii). Since $R$ is right self-injective, [7 9.2] shows that all finitely generated submodules of non-singular modules are projective, hence all non-singular modules are $\mathcal{F}$-projective by Definition 3.1. Alternatively, we can use (iii) and the fact that under the assumptions of (iv), flat Mittag-Leffler modules coincide with the non-singular ones by [9] 6.8. □

For semihereditary rings, we have a fully ring theoretic characterization:

**Proposition 3.8.** (i) The following conditions are equivalent for a ring $R$:

1. $R$ is right semihereditary.
2. $\mathcal{FP}$ coincides with the class of all modules $M$ such that each finitely generated submodule of $M$ is projective.
3. The class $\mathcal{FP}$ is closed under submodules.

(ii) Assume $R$ is right semihereditary. Then the following conditions are equivalent:

1. $\mathcal{FP}$ is a covering class.
2. Each finitely generated flat module is projective.
3. For each $n > 0$, the full matrix ring $M_n(R)$ contains no infinite sets of non-zero pairwise orthogonal idempotents.

**Proof.** (i) By [10] Theorem 2.29, $R$ is right semihereditary, iff all finitely generated submodules of projective modules are projective. So the implication (1) implies (2) is immediate from Definition 3.1. (2) implies (3) is trivial, and (3) implies (1) because projective modules are $\mathcal{F}$-projective, and finitely generated $\mathcal{F}$-projective modules are projective.

(ii) The implication (1) $\Rightarrow$ (2) holds for any ring: By Theorem 2.6 (1) implies $\mathcal{FP} = \mathcal{F}$, so each finitely generated flat module is $\mathcal{F}$-projective, hence projective, and (2) holds. If $R$ is right semihereditary, then $R$ has flat dimension $\leq 1$, i.e., submodules of flat modules are flat. So if each finitely generated flat module is projective, then by part (i), each flat module is $\mathcal{F}$-projective. This proves (2) $\Rightarrow$ (1).

Assume (2) holds, and there is an $n > 0$ such that the full matrix ring $S = M_n(R)$ contains an infinite set $\{e_i \mid i < \omega\}$ of non-zero pairwise orthogonal idempotents. Then $M = S/\oplus_{i<\omega} e_iS$ is a direct limit of the projective modules $S/\oplus_{i \in X} e_iS$, where $X$ runs over all finite subsets of $\omega$. So $M$ is a cyclic flat right $S$-module which is not projective.

Further, $S$ is Morita equivalent to $R$. If $(F, G)$ is a pair of functors realizing this equivalence, then $F(M)$ is a finitely generated non-projective flat module, in contradiction with (2).

The implication (3) $\Rightarrow$ (2) is proved e.g., in [13] Proposition 4.10. □

On the one hand, if $\mathcal{FP} \subseteq \mathcal{F}$, then there is always a finitely generated projective module $M \in \mathcal{FP}$ such that $\lim \text{add} M \nsubseteq \mathcal{FP}$ (just take $M = R$). On the other hand, we have the following result that applies, e.g., to any simple projective module $M$:
Proposition 3.9. Let $R$ be a ring and $M$ be an $f$-projective module. Let $S = \text{End } M$. Assume that $S$ is right noetherian and $M$ is a flat left $S$-module. Then $\lim \text{ add } M \subseteq TP$.

Proof. By [11, Theorem 2.5], $\lim \text{ add } M = \{ F \otimes_S M \mid F \text{ a flat right } S\text{-module} \}$. So we have to prove that $F \otimes_S M$ is a flat and $(R)$-Mittag-Leffler module, for each flat right $S$-module $F$. Flatness of $F \otimes_S M$ is clear since $M$ is $f$-projective, hence flat.

Let $I$ be a set. By assumption, the canonical map $M \otimes_R R^I \to M^I$ is an injective homomorphism of left-$S$-modules, whence the map $(F \otimes_S M) \otimes_R R^I \to F \otimes_S M^I$ is monic. Since $S$ is right noetherian, $F$ is an $f$-projective right $S$-module by Proposition 3.7(i). Since $M$ is a flat left $S$-module, the canonical map $F \otimes_S M^I \to (F \otimes_S M)^I$ is monic by Proposition 3.3. Thus, the module $F \otimes_S M$ is $(R)$-Mittag-Leffler. □

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