On the Structure of the Effective Potential for a Spherical Wormhole

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Abstract

The structure of the effective potential \( V \) describing causal geodesics near the throat of an arbitrary spherical wormhole is analyzed. Einstein’s equations relative to a set of regular coordinates covering a vicinity of the throat imply that any spherical wormhole can be constructed from solutions of an effective initial value problem with the throat serving as an initial value surface. The initial data involve matter variables, the area \( A(0) \) of the throat and the gradient \( \Lambda(0) \) of the red shift factor on the throat. Whenever \( \Lambda(0) = 0 \), the effective potential \( V \) has a critical point on the throat. Conditions upon the data are derived ensuring that the critical point is a local minimum (resp. maximum). For particular families of Quasi-Schwarzschild wormholes, \( V \) exhibits a local minimum on the throat independently upon the energy \( E \) and angular momentum \( L^2 \) of the test particles and thus such wormholes admit stable circular timelike and null geodesics on the throat. For families of Chaplygin wormholes, we show that such geodesics are unstable. Based on a suitable power series representation of the metric, properties of \( V \) away from the throat are obtained that are useful for the analysis of accretion disks and radiation processes near the throat of any spherical wormhole.

1 Introduction

Wormholes, like black holes are topologically non trivial solutions of Einstein’s equations. The former do not possess an event horizon and this property combined with their non trivial topology endows them with remarkable properties. If a wormhole is considered as a handle connecting two distant regions of the same universe, it offers the opportunity for interstellar travel [1] and it can act as a time machine [2]. If it connects two “different universes” transfer of matter
and radiation from one “universe” to the other is a possibility. This property lead the authors of [3] to put forward a radical hypothesis: active galactic nuclei and other compact objects may be entrances to wormhole throats. Whether this hypothesis reflects reality, eventually will be decided by observations combined with analysis of astrophysical processes near the wormhole throats. In a recent work [4] (see also [5]) some intriguing properties of wormholes have been pointed out. Many features of black holes can be closely mimicked by a family of Quasi-Schwarzschild wormholes [6]. Those features include quasinormal mode ringing [7], the well known 337 Ohms finite surface resistivity of an event horizon and even some aspects of Hawking radiation [9].

In the works of [3] and [4], the behavior of causal geodesics in the vicinity of the throat plays an important role. In [3] it has been suggested that the existence of geodesics oscillating through the throat, may lead to the observational identification of a wormhole throat. The family of wormholes employed in [4], exhibits two worth noticing properties: They admit stable circular timelike and null geodesics on the throat and furthermore allow geodesics to oscillate with respect to the throat. These properties should be contrasted with the behavior of geodesics on the spherical wormholes [10] admitted by massless K-essence. Spherical wormholes of this theory fall into two families: the so called zero mass family and a second family having the property that the two asymptotically flat ends possess ADM masses of opposite sign. For the later family, the effective potential exhibits no extrema on the throat and its structure shows absence of geodesics oscillating through the throat [12]. These contradictory conclusions that one arrives at from the behavior of geodesics on the family of Quasi-Schwarzschild wormholes considered in [4] and the behavior of geodesics on K-essence wormholes, call for an understanding of the factors influencing the potential near the throat of an arbitrary spherical wormhole.

The present paper analyzes such issue. In the next section, we demonstrate that spherical wormholes can be constructed from solutions of an effective initial value problem (IVP here after) with the throat serving as an initial value surface. In sec. (3) we introduce the effective potential \( V \) and formulate criteria guaranteeing the presence or absence of stable geodesics on the throat while in sec. (4) we discuss the structure of \( V \) for Quasi-Schwarzschild wormholes. A series representation of the metric valid in a vicinity containing the throat is derived in sec. (5). This representation offers insights regarding the structure of \( V \) away from the throat. We finish the paper with a brief summary of the results and open problems.

## 2 Constructing Spherical wormholes

We start with a Gaussian chart covering an open vicinity of the throat so that \( g \) can be written as:

\[
g = -e^{2\Phi(l)} dt^2 + dl^2 + r^2(l) d\Omega^2, \quad l \in (-\alpha, \alpha).
\]  

(1)
The throat is located at \( l = 0 \) so that \( r(l = 0) \neq 0, \frac{dr(l)}{dl}|_0 = 0, \frac{d^2r(l)}{dl^2}|_0 > 0 \) and \( \Phi(l) \) is assumed to be smooth [13]. Relative to this chart, we consider a stress tensor \( T_{\alpha\beta} \) that decomposes according to:

\[
T_{\alpha\beta} = \rho(l)c^2u_\alpha u_\beta - \tau(l)X_\alpha X_\beta + P(l)(Y_\alpha Y_\beta + Z_\alpha Z_\beta),
\]

\[
\begin{align*}
\mathbf{u} &= e^{-\Phi(l)} \frac{\partial}{\partial t}, X = \frac{\partial}{\partial l}, Y = \frac{1}{r(l)} \frac{\partial}{\partial \theta}, \\
Z &= \frac{1}{r(l) \sin \theta} \frac{\partial}{\partial \phi},
\end{align*}
\]

where \( (\rho(l)c^2, \tau(l), P(l)) \) stand for the energy density, tension and tangential pressure as measured by the Killing observers. In the gauge of (1), Einstein’s eqs \( G_{\alpha\beta} = \hat{k}T_{\alpha\beta} \) and \( \nabla_\alpha T^{\alpha\beta} = 0 \) can be cast in the form:

\[
\begin{align*}
\frac{dr(l)}{dl} &= \frac{1}{2}K(l)r(l), \quad (3) \\
\frac{dK(l)}{dl} &= -\frac{3}{4}K^2(l) + \frac{1}{r^2} - \hat{k}\rho(l)c^2, \quad (4) \\
\frac{d\Lambda(l)}{dl} &= -\frac{1}{2}K(l)\Lambda(l) + \frac{1}{2}\hat{k}\rho(l)c^2 - \frac{1}{2r^2(l)} \\
&\quad + \frac{K^2(l)}{8} - \Lambda^2(l) + \hat{k}P(l), \quad (5) \\
\frac{d\tau(l)}{dl} &= [\rho(l)c^2 - \tau(l)]\Lambda(l) - [\tau(l) \\
&\quad + P(l)]K(l), \quad (6) \\
- \frac{1}{r^2(l)} + \frac{K^2(l)}{4} + \hat{k}\tau(l) + \Lambda(l)K(l) &= 0, \quad (7)
\end{align*}
\]

where \( K(l) = \frac{2}{r(l)} \frac{dr(l)}{dl} \) is the trace of the extrinsic curvature of the \( SO(3) \) orbits as embedded in \( t = \text{const} \) hypersurfaces and \( \Lambda(l) = \frac{d\Phi(l)}{dl} \). As a consequence of (3-7), it follows that (7) is a constraint and if it is satisfied at an \( l_0 \), then it is satisfied for all \( l \). Evaluating (7) on the throat we obtain \( \hat{k}\tau(0) = r^{-2}(0) \), while from (4) it follows that \( A(l) = 4\pi r^2(l) \) exhibits a local minimum at \( l = 0 \) provided \( \hat{k}r^2(0)\rho(0)c^2 < 1 \) or equivalently \( (\tau(0) - \rho(0)c^2) > 0 \). Thus any solution of (3-7) subject to:

\[
\begin{align*}
K(0) &= 0, \Lambda(0), \tau(0) = \frac{1}{k r^2(0)}, \quad (8)
\end{align*}
\]

with \( \rho(0) \) chosen so that \( (\tau(0) - \rho(0)c^2) > 0 \) and \( \Lambda(0) \) arbitrary, describes a local wormhole. Notice that the constraint and dynamical eqs (3-7) do not restrict
Λ(0) and thus together with $A(0)$ may be viewed as the degrees of freedom characterizing the throat.

In the absence of equations of state or dependence of $(ρc^2, τ, P)$ upon fundamental fields [14], the system (3-7) simplifies. Two of the unknown functions can be a-priori specified [15] and on grounds of mathematical simplicity we shall specify $(ρc^2, Λ)$, referred here after as the free data. For that case, eq. (6) written in the form:

$$\dot{P}(l) = \frac{dΛ(l)}{dl} + Λ^2(l) + \frac{1}{2} \frac{dK(l)}{dl} + \frac{K^2(l)}{4} + \frac{Λ_2(l)}{2},$$

implies that $P(l)$ is determined once $(r(l), K(l))$ are known. As long as $ρc^2$ is continuous and $Λ$ is continuous differentiable on $(-α, α)$ then Picard-Lindeloff’s theorem [16] applied to (3-7) guarantees the existence of a unique $C^1$ solution defined on $[-b, b] \subset (-α, α)$. In [17] it was shown that under mild restrictions upon $(ρc^2, Λ)$, local solutions of (3-7) can be extended so that they represent smooth, asymptotically flat wormholes [18].

The treatment of (3-7) has to proceed along different lines whenever $T_{αβ}$ described by (2) is attributed to a fluid or defined by an underlying field theory such as $K$-essence [14]. For instance, in the case that $T_{αβ}$ describes a perfect fluid with four velocity $u$ parallel to the Killing field then $(ρc^2, τ, P)$ are linked by equations of state. For a monoparametric family of equations of state: $τ = F(ρ), P = G(ρ)$, eqs (6,7) take the form:

$$\frac{dΛ(l)}{dl} = -\frac{1}{2} K(l)Λ(l) + \frac{1}{2} \dot{K}ρ(l)c^2 - \frac{1}{2r^2(l)} + \frac{K^2(l)}{8} - Λ^2(l) + \dot{K}G(ρ(l)),$$

$$\frac{dF(ρ)}{dρ} \frac{dρ(l)}{dl} = [ρ(l)c^2 - F(ρ(l))]Λ(l) - \dot{K}(ρ(l)) + G(ρ(l))]K(l).$$

Again as long as $F, G$ are smooth functions, Picard-Lindeloff’s theorem applied to (3-7) [18] combined with $[8]$ assures the existence of a local solution. However properties of the maximal solution are by no means obvious. Depending upon the equations of state, (11) can be a highly non linear equation and the possibility that while the data are propagating off the throat a singularity in the solution will develop is not excluded [19]. For the case where $T_{αβ}$ is associated with fields, the situation is not getting any easier. Eq. (7) which determines the tension, will involve second derivatives of the fields and similar comments regarding the maximal solution apply in this case as well [20].
Despite the absence of a global wormhole solution of (3-7), by appealing to the local solutions predicted by the Picard-Lindeloff we can analyze the structure of the effective potential describing geodesics in a vicinity of the throat.

3 Structure of the effective potential on the throat

On the background of (1), any timelike or null geodesic \( x^\mu(\mu) = (t(\mu), l(\mu), \theta(\mu), \phi(\mu)) \) satisfies:

\[
\frac{dt(\mu)}{d\mu} = \frac{E}{g_{00}}, \quad \frac{d\phi(\mu)}{d\mu} = \frac{L}{g_{\phi\phi}}, \quad \theta(\mu) = \frac{\pi}{2},
\]

(12)

\[
\left(\frac{dl(\mu)}{d\mu}\right)^2 = \frac{1}{e^{2\Phi(l)}}[E^2 - V(l, L^2)],
\]

(13)

where \( \mu \) denotes proper time for a timelike and an affine parameter for a null geodesic. \((E, L)\) are constants along a particular geodesic and we have made use of spherical symmetry to set: \( \theta(\mu) = \frac{\pi}{2} \). The function \( V(l, L^2) \) is referred as the effective potential and is defined by:

\[
V(l, L^2) = e^{2\Phi(l)}(k + \frac{4\pi L^2}{A(l)}),
\]

(14)

where \( A(l) = 4\pi r^2(l) \) and \( k = 1 \) ( \( k = 0 \) ) for timelike (null) geodesics. It follows from (14) that:

\[
\frac{\partial V(l, L^2)}{\partial l} = e^{2\Phi(l)} \left[ 2\Lambda(l)(k + \frac{4\pi L^2}{A(l)}) \right] - \frac{4\pi L^2}{A^2(l)} \frac{dA(l)}{dl},
\]

(15)

and thus if \( \Lambda(0) \neq 0 \) then \( V(l, L^2) \) has a regular value on the throat for all \( L^2 \neq 0 \). Geodesics can pass from one asymptotic region to the other or they may have a turning point on the throat. Moreover depending on the maxima and minima of \( V(l, L^2) \) away from the throat, there may exist geodesics oscillating through the throat.

However for \( \Lambda(0) = 0 \), \( V(l, L^2) \) has a critical point on the throat regardless of the value of \( L^2 \) and whether \( k = 1 \) or \( k = 0 \). Evaluating the second derivative of \( V(l, L^2) \) on the throat, we obtain:

\[
\left. \frac{\partial^2 V(l, L^2)}{\partial l^2} \right|_0 = e^{2\Phi(0)} \left[ 2 \frac{d\Lambda(l)}{dl} (k + \frac{4\pi L^2}{A(l)}) \right] - \frac{4\pi L^2}{A^2(l)} \left( k + \frac{4\pi L^2}{A(l)} \right) \frac{dA(l)}{dl^2} \right|_0.
\]

(16)
For the case where \((\rho, \Lambda)\) are free data, it reduces to:

\[
\frac{\partial^2 V(l, L^2)}{\partial l^2} \bigg|_0 = e^{2\Phi(0)} \left[ 2 \frac{d\Lambda(l)}{dl} \left( k + \frac{4\pi L^2}{A(l)} \right) - \frac{4\pi L^2}{A(l)} \left( \frac{1}{r^2(l)} - \hat{k} \rho(l) c^2 \right) \right]_0,
\]

and thus if \(\frac{d\Lambda(l)}{dl} \big|_0 \leq 0\) then \(V(l, L^2)\) possesses a local maximum for all \(L^2\).

In this case any timelike or null geodesic on the throat would be unstable. On the other hand, whenever \(\frac{d\Lambda(l)}{dl} \big|_0 > 0\), the nature of the extremum depends upon \(L^2\) and the form of \(\frac{d\Lambda(l)}{dl} \big|_0\).

For wormholes supported by a perfect fluid, the right hand side of (17) takes the form:

\[
\frac{\partial^2 V(l, L^2)}{\partial l^2} \bigg|_0 = e^{2\Phi(0)} \hat{k} \left[ 2 \left( 2\Phi(0) - a(0) \right) + 8\pi L^2 A(0) (P(0) - a(0)) \right],
\]

where \(a(0) \equiv \tau(0) - \rho(0) c^2\). Therefore, whenever the data satisfy \(P(0) \geq a(0) \equiv \tau(0) - \rho(0) c^2\), the extremum of \(V(l, L^2)\) is a local minimum regardless of the value of \(L^2\) and whether \(k = 1\) or \(k = 0\). For other values of \(P(0)\) the nature of the extremum can be easily inferred from (15).

In the next section we shall consider geodesic motion for the family of Quasi-Schwarzschild wormholes [25].

4 The effective potential for Quasi-Schwarzschild wormholes

A Quasi-Schwarzschild wormhole is a spherical wormhole with the property that the intrinsic geometry of the hypersurfaces orthogonal to the timelike Killing field is isometric to the corresponding geometry of the positive mass Schwarzschild manifold [26]. If \(\gamma\) stands for the induced metric on the \(t = \text{cons}\) hypersurfaces and \(R(\gamma)\) its scalar curvature, the Hamiltonian constraint for Einstein’s equations imply that \(R(\gamma) = 2\hat{k} \rho c^2\) and thus for any Quasi-Schwarzschild wormhole \(\rho \equiv 0\). This property simplifies the system (3-7). In fact if we define \(r(l)\) via:

\[
\frac{dr(l)}{dt} = \pm \sqrt{1 - \frac{r_0}{r(l)}}, \quad r(0) = r_0, \quad l \in (-\infty, \infty),
\]

it can be verified that this \(r(l)\) combined with
\[ K(l) = \frac{2}{r(l)} \frac{dr(l)}{dl} = \pm \frac{2}{r(l)} \left[ 1 - \frac{r_0}{r(l)} \right]^{1/2} , \]

\[ l \in (-\infty, \infty), \quad (20) \]

satisfies (34) and the required initial conditions (we adopt the notation that the (+) sign corresponds to \( l \in (0, \infty) \) and (−) for \( l \in (-\infty, 0) \)). Combining these expressions for \((r(l), K(l))\) with any \(C^1\) function \(\Lambda(l)\) subject to:

\[ \lim_{l \to \pm \infty} \Lambda(l) = O(l^{-2}) \quad [28], \]

eqs (6,7) are satisfied provided:

\[ \hat{k}\tau(l) = \frac{r_0}{r^3(l)} - \frac{2\Lambda(l)}{r(l)} \left[ 1 - \frac{r_0}{r(l)} \right]^{1/2} , \quad l \in [0, \infty), \quad (21) \]

\[ \hat{k}P(l) = \frac{d\Lambda(l)}{dl} + \Lambda^2(l) + \frac{\Lambda(l)}{r(l)} \left[ 1 - \frac{r_0}{r(l)} \right]^{1/2} + \frac{r_0}{2r^3(l)} , \quad l \in [0, \infty), \quad (22) \]

while the corresponding \(\hat{k}\tau(l), \hat{k}P(l)\) for \( l \in (-\infty, 0] \) are obtained from (21,22) by changing the sign in front of the linear term in \(\Lambda\). Finally the metric has the form:

\[ g = -e^{2\Phi(0)} + e^{2\hat{\Phi}(l)} \int_0^l \Lambda(l') dl' + dt^2 + r^2(l)d\Omega^2, \quad l \in (-\infty, \infty), \quad (23) \]

with \(\Phi(0)\) an arbitrary constant. If on the sheet with \( l \in (0, \infty) \) resp. \( l \in (-\infty, 0) \) we eliminate the \( l \) coordinate in favor of \( r \) then the spatial metric takes the form:

\[ \gamma = \frac{dr^2}{1 - \frac{r_0}{r}} + r^2d\Omega^2, \quad r \in (r_0, \infty), \]

showing that the ends \( l \to \infty \) resp. \( l \to -\infty \), are asymptotically flat ends with equal ADM masses. Orbiting test particles in the asymptotic regions are coupled to the so called Komar mass (often referred in the literature as the Schwarzschild mass) which is determined by the asymptotic behavior of \(\Lambda(l)\).

For any Quasi-Schwarzschild wormhole, it follows from (15) that the critical points \( l_c \) of \( V(l, L^2) \) satisfy:

\[ \Lambda(l)(kr^2(l) + L^2) = \pm \frac{L^2}{r(l)} \left( 1 - \frac{r_0}{r(l)} \right)^{1/2} , \quad (24) \]
and using (17) the nature of the critical point on the throat can be determined. It is worth to notice that due to the freedom in the choice of \( \Lambda(l) \), one can give \( V(l, L^2) \) any desirable structure and below we shall discuss a few examples.

Setting \( \Lambda(l) = 0 \) in (21,22,23) we recover the Bronnikov-Kim family of wormholes (derived in [21] via different techniques and motivations). For this family, it follows from (14) or from (24) that \( V(l, L^2) \) has a unique global maximum on the throat for all \( L^2 \) and thus circular timelike of null geodesics on the throat are unstable. The Damour-Solodukhin family [4], mentioned in the introduction section, is generated by:

\[
\Lambda(l) = \frac{\sqrt{f(l) + \lambda^2}}{2r^2(l) (f + \lambda^2)}, \quad f(l) = 1 - \frac{r_0}{r(l)}, \quad \lambda \neq 0, \quad l \in [0, \infty),
\]

and \( \Lambda(l) = -\Lambda(-l) \). For this \( \Lambda(l) \), we obtain from (21,22):

\[
\hat{k}P(l) = \frac{r_0^2 \lambda^2}{4r^4(l) (1 - \frac{r_0}{r(l)} + \lambda^2)^2} - \frac{r_0^2 \lambda^2}{2r^3(l) (1 - \frac{r_0}{r(l)} + \lambda^2)}, \quad l \in (-\infty, \infty),
\]

\[
\hat{k}\tau(l) = \frac{r_0 \lambda^2}{r^3(l) (1 - \frac{r_0}{r(l)} + \lambda^2)}, \quad l \in (-\infty, \infty),
\]

whereas (18) yields:

\[
\left. \frac{\partial^2 V(l, L^2)}{\partial l^2} \right|_0 = e^{2\Phi(0)} \left[ \frac{4\pi L^2}{A(0)r^2(0)} \left( \frac{1}{2\lambda^2} - 1 \right) + \frac{k}{2r_0^2 \lambda^2} \right].
\]

Thus, as long as \( \lambda^2 < \frac{1}{2} \) this family admits stable circular timelike and null geodesics on the throat in agreement with the conclusions of ref.[4]. Away from the throat the effects of non vanishing \( \lambda^2 \) can be easily worked out.

5 On the effective potential near the throat

In an effort to get insights in the properties of \( V(l, L^2) \) away from the throat, in this section we construct a power series solution of (3−7) valid in a vicinity containing the throat. For simplicity we shall treat only wormholes with reflective symmetry [18], although the following analysis can be extended to wormholes lacking this symmetry. We shall treat first the case where \((\rho c^2, \Lambda)\) are smooth free data so that [27]:

8
\[
\rho(l)c^2 = \rho(0)c^2 + \rho(1)l + \rho(2)l^2 + \rho(3)l^3 + O(l^4),
\]
\[
\Lambda(l) = \Lambda(0) + \Lambda(1)l + \Lambda(2)l^2 + \Lambda(3)l^3 + O(l^5).
\]

According to the results of section (2), we choose \( \Lambda(0) = 0 \), \( \hat{k}\tau(0) = r^{-2}(0) \) so that \( \tau(0) > \rho(0)c^2 \) and set: \( \rho(1) = \rho(3) = \Lambda(0) = \Lambda(2) = 0 \). It is a matter of algebra to show that a power series solution of (3-7) has the form:

\[
r(l) = r(0) + \frac{\hat{k}r(0)}{4}(\tau(0) - \rho(0)c^2)l^2 + O(l^4),
\]
\[
K(l) = \hat{k}(\tau(0) - \rho(0)c^2)l + K(3)l^3 + O(l^5),
\]

\[
\hat{k}P(l) = [\Lambda(1) + \frac{\hat{k}}{2}(\tau(0) - \rho(0)c^2)]
\]
\[
+ \left[ \Lambda^2(1) + 2\Lambda(3) - \frac{\hat{k}\rho(2)}{2} + \frac{\hat{k}}{2}(\tau(0) - \rho(0)c^2) \
(\Lambda(1) - \frac{\hat{k}\tau(0)}{2}) - \frac{\hat{k}^2}{8}(\tau(0) - \rho(0)c^2)^2 \right] l^2 + O(l^3),
\]
\[
\hat{k}\tau(l) = \hat{k}\tau(0) - \hat{k}[\tau(0) - \rho(0)c^2][\hat{k}P(0) \
+ \frac{\hat{k}\tau(0) + \hat{k}\rho(0)c^2}{4}]l^2 + O(l^4),
\]

with \( K(3) \) described by:

\[
K(3) = -\frac{\hat{k}^2}{12}[(\tau(0) - \rho(0)c^2)(5\tau(0) - 3\rho(0)c^2)] + \frac{\hat{k}\rho(2)}{3}.
\]

This series implies that the metric \( g \) can be written in the form:

\[
g = -(1 + \Lambda(1))l^2 + \frac{\Lambda(3)}{2}l^4 + O(l^6)]dt^2 + dl^2
+ r^2(0)[1 + \frac{\hat{k}}{4}(\tau(0) - \rho(0)c^2)l^2 + O(l^4)]d\Omega^2,
\]
where in above we have set $\Phi(0) = 0$. From (15), the critical points of $V(l, L^2)$ are identified as the roots of the equation:

$$-L^2 K(l) + 2\Lambda(l)(kr^2(l) + L^2) = 0, \quad (37)$$

and in terms of the series solution it yields a lengthy algebraic cubic equation. One root occurs at $l = 0$ and the other two are either real and of opposite sign or they are imaginary. We have checked that there exist data so that on the non vanishing real roots, $V(l, L^2)$ exhibits local maxima and moreover exist geodesics that oscillate through the throat, a conclusion that verifies the claim in [3].

Finally we shall briefly consider Chaplygin wormholes. Such wormholes are supported by a perfect fluid stress tensor $T_{\alpha \beta}$ so that the isotropic pressure $P$ and energy density $\rho c^2$ obey:

$$P(\rho) = -\frac{\rho c^2}{\rho}. \quad (29)$$

Setting $P = -\tau$, the constraint $\tau(0) - c^2 \rho(0) > 0$ is satisfied provide $A > \rho^2(0)$. Again we assume:

$$c^2 \rho(l) = c^2 \rho(0) + \rho(2)l^2 + O(l^4), \quad (38)$$

and thus

$$\tau(l) = \frac{A}{c^2 \rho(0)} \left[ 1 - \frac{\rho(2)}{c^2 \rho(0)} l^2 + O(l^4) \right]. \quad (39)$$

Substituting these expressions into (34) we obtain:

$$r(l) = r(0) + \frac{1}{4r(0)} [1 - \hat{k} \rho(0)c^2 r^2(0)] l^2 + O(l^4), \quad (40)$$

$$K(l) = \frac{4l}{r(0)} \left[ \frac{1}{4r(0)} [1 - \hat{k} \rho(0)c^2 r^2(0)] + O(l^2) \right], \quad (41)$$

$$\Lambda(l) = -\left[ \frac{1}{2r^2(0)} [1 - \hat{k} c^2 \rho(0)r^2(0)] + \frac{\hat{k} A}{c^2 \rho(0)} \right] l + O(l^2), \quad (42)$$

resulting into:

$$g = -\left[ 1 - \left( \frac{1 - \hat{k} c^2 \rho(0)r^2(0)}{2r^2(0)} + \frac{\hat{k} A}{c^2 \rho(0)} \right) l^2 + O(l^4) \right] dt^2 + dl^2 + r^2(0) \left[ 1 + \frac{1 - \hat{k} \rho(0)c^2 r^2(0)}{2r^2(0)} l^2 + O(l^4) \right] d\Omega^2. \quad (43)$$
For this solution we find:

\[ \frac{\partial^2 V(l, L^2)}{\partial l^2} = -e^{2\phi(0)} \left[ \left( k + \frac{4\pi L^2}{A(0)} \right) \frac{2\hat{k} A}{c^2 \rho(0)} \right. \]

\[ + \left. \frac{1 - \hat{k} c^2 \rho(0) r^2(0)}{r^2(0)} \left( k + \frac{8\pi L^2}{A(0)} \right) \right], \]

and since \( A \) is positive and \( 1 - \hat{k} c^2 \rho(0) r^2(0) > 0 \), all circular timelike or photon orbits on the throat are unstable \[31\]. The behavior of \( V(l, L^2) \) away from the throat can be obtained from \[37\] adapted to the series \[45\].

6 Discussion

In this work a connection between initial data determining a wormhole and properties of the effective potential \( V \) near the throat is established. Our analysis shows that with suitable selection of the data, \( V \) can acquire any desirable structure as illustrated by the Quasi-Schwarzschild and Chaplygin wormholes. In the course of the paper, we have also analyzed the structure of Quasi-Schwarzschild wormholes. Such wormholes have simple geometries and this can be useful. For instance, they can be used as backgrounds where the effects of the wormhole topology upon accretion flows and emergent spectra can be addressed. Of course the stability of Quasi-Schwarzschild wormholes has to be analyzed and we hope to discuss some of these issues in the near future.

7 Acknowledgments

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[20] Spherical wormholes of massless K-essence are described by: $g = e^{f(r)} dt^2 + e^{-f(r)} [dr^2 + r^2 d\Omega^2]$, $r \in (0, \infty)$ with $f(r) = 4\gamma \tan^{-1}(\frac{\lambda}{r}) - 2\gamma \pi$, $g(r) = (\frac{r^2 + \lambda^2}{r^2 \lambda^2})^2$ and the scalar field described by: $\Phi(r) = 2\gamma \tan^{-1}(\frac{\lambda}{r}) - \gamma \pi$, $\gamma = \pm [2(\gamma^2 + 1) / k]^{1/2}$. The throat is located at $r_m = r_0 (\gamma + \sqrt{1 + \gamma^2})$ and $(\gamma, r_0)$ are real parameters with $r_0 > 0$. Causal geodesics are described by: $\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{g(r)} [E^2 - V(r, L^2)]$ and if we employ $\Phi$ as a coordinate via $r(\Phi) = -\frac{r_0}{\tan \alpha} = -\frac{r_0}{\tan \alpha}$, $\alpha = \Phi / 2$, $\alpha \in (-\frac{\pi}{2}, 0)$, then: $V(\alpha, L^2) = ke^{4\gamma \alpha} + L^2 e^{8\gamma \alpha} \left(\frac{\tan \alpha}{1 + \tan^2 \alpha}\right)^2$, $\alpha \in (-\frac{\pi}{2}, 0)$ where $r \to \infty$ is mapped into $\alpha \to 0$, while $r \to 0$ into $\alpha \to -\pi / 2$. The choice $\gamma = 0$ describes the family of zero mass wormholes and for this family $V(\alpha, L^2)$ admits a global maximum occurring on the throat. For $\gamma \neq 0$, $V(\alpha, L^2)$ has a regular value on the throat. The behavior of the geodesics can be obtained by plotting $V(\alpha, L^2)$ for various values of $L^2$ and a detailed analysis can be found in:

N. Montelongo Garcia and T. Zannias: Modeling massive dark objects by wormhole throats, IFM-UMSNH Report (2007) unpublished.
[13] Note that there is an ambiguity in the timelike coordinate $t$ resulting from the inability to fix uniquely the timelike Killing field, implying that $\Phi(0)$ is undetermined. For the present work this freedom is innocuous and we shall gloss it over.

[14] For $K$-essence [11], the Lagrangian is taken in the form: $L = K(\Phi)X(\nabla^a \Phi \nabla_a \Phi)$, implying: $T_{\mu \nu} = \frac{\partial L}{\partial \Phi} \nabla_\mu \Phi \nabla_\nu \Phi + L g_{\mu \nu}$. This $T_{\mu \nu}$, for a static and spherical solution, can be decomposed according to [2] where $(\rho, \tau, P)$ depend upon $\Phi$, and its gradient. For fields lacking spherical symmetry, a Killing observer may perceive “heat flux” and “shear stresses”.

[15] As we have mentioned earlier $\Lambda(0)$ is part of the initial data, but for this particular case the entire $\Lambda(l)$ is a part of the free data. For other systems this freedom is not available.

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[18] It is worth mentioning that solutions of (3-7) fall into two families. The first family contains solutions where $r(l)$ and matter variables obey: $r(l) = r(-l), \rho(l) = \rho(-l), \tau(l) = \tau(-l), P(l) = P(-l), \forall \ l \in (-a,a)$, whereas the second family contains solutions that lack this symmetry. Solutions belonging to the first family can be constructed as follows: For any $(r(l), K(l), \tau(l), P(l))$ generated by $(\rho(l)c^2, \Lambda(l))$ obeying (3-7) on $l \in [0,a]$, we define on $l \in (-a,0)$: $(\tilde{r}(-l), \tilde{K}(-l), \tilde{\Lambda}(-l), \tilde{\tau}(-l)) = (r(l), -K(l), -\Lambda(l), \tau(l))$, $l \in [0,a]$, and simultaneously consider the even extension of $\rho c^2$ on $(-a,0]$, i.e. $\rho(-l)c^2 = \rho(l)c^2, \forall \ l \in [0,a]$. Since $K(0) = 0$, the odd extension of $K(l), l \in [0,a]$ is differentiable at $l = 0$ and coincides with $\tilde{K}(-l)$. If $\Lambda(0) = 0$, then the odd extension of $\Lambda(l), l \in [0,a]$ is differentiable at $l = 0$ and coincides with $\tilde{\Lambda}(-l)$. Furthermore $(\tilde{r}(-l), \tilde{K}(-l), \tilde{\Lambda}(-l), \tilde{\tau}(-l))$ satisfy (3-7) in $(-a,0]$ and join smoothly to $(r(l), K(l), \Lambda(l), \tau(l))$ across $l = 0$. Since $\Phi(l) = \Phi(0) + \int_0^l \Lambda(l')dl'$ it follows that $\Phi(l) = \Phi(-l)$. This, in combination with $r(l) = r(-l)$ implies $g(l) = g(-l), \forall \ l \in (-a,a)$, and this symmetry is shared also by $(\rho(l)c^2, P(l), \tau(l))$. Wormholes possessing this property referred in the text as reflectively symmetric. Based on the results of [17] this conclusion can be extended $\forall \ l \in (-\infty, \infty)$.

[19] Although establishing conditions upon the eqs of state so that the maximal solution is extended $\forall \ l \in (-\infty, \infty)$ is an involved problem, nevertheless the behavior of solutions of (3-7) subject to wormhole initial conditions can be addressed via numerical techniques.

[20] Although in this work wormholes are constructed from solutions of an effective initial value problem, ought to be mentioned that there exist other
techniques generating wormholes. A popular one is the Morris-Thorne tech-
nique discussed in [1]. A different method has been proposed in [21, 22].
It is an algorithmic technique that a-priori prescribes the scalar curvature
of the wormhole which in turn constraints the two degrees of freedom of
the wormhole metric. The rest of Einsteins eqs specify the stress tensor.
Details can be found in: [21, 22].

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[23] An exception corresponds to ’radial’ null geodesics. For such geodesics
$V(l, L^2)$ exhibits a critical point on the throat. However [17] shows that it
is an inflection point.
[24] In the present work any wormhole is considered as connecting two distinct
universes and the possibility of oscillating geodesics through the asymptotic
region is not considered.
[25] Quasi-Schwarzschild wormholes have been considered in M. Visser,
Lorentzian Wormholes From Einstein to Hawking, A. Inst. of Phys. New
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[26] To be precise, it should be isometric to the spacelike hypersurface of the
Schwartzschild black hole that goes through the bifurcation two sphere and
away from this sphere coincides with any $t =$const spacelike hypersurface.
Such hypersurface exist and for a proof see: I. Racz and R.M.Wald; Class.
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coefficients, for the first, second, and third derivatives of $\rho(l)$ evaluated
at $l = 0$.
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[31] Since the Chaplygin equation of state is incompatible with asymptotic flat-
ness, any wormhole of this theory is truncated and as a rule is joined to a
Schwarzschild vacuum (see ref. [30] for details). Therefore the power series
solution discussed in the text may be a reliable approximation, if it is joined
to a Schwarzschild vacuum at an $l$ very close to the throat. The structure
of the resulting surface layer can be easily deduced.