Towards a Ryll-Nardzewski-type theorem for weakly oligomorphic structures

Christian Pech\textsuperscript{1} and Maja Pech\textsuperscript{1,2}\textsuperscript{*}

\textsuperscript{1} Institute of Algebra, Technische Universität Dresden, 01062, Dresden, Germany
\textsuperscript{2} Department of Mathematics and Informatics, University of Novi Sad, Trg Đositeja Obradovića 3, 21000, Novi Sad, Republic of Serbia

Received 17 July 2014, revised 4 March 2015, accepted 5 March 2015
Published online 17 February 2016

A structure is called weakly oligomorphic if its endomorphism monoid has only finitely many invariant relations of every arity. The goal of this paper is to show that the notions of homomorphism-homogeneity, and weak oligomorphy are not only completely analogous to the classical notions of homogeneity and oligomorphy, but are actually closely related. We first prove a Fraïssé-type theorem for homomorphism-homogeneous relational structures. We then show that the countable models of the theories of countable weakly oligomorphic structures are mutually homomorphism-equivalent (we call first order theories with this property weakly $\omega$-categorical). Furthermore we show that every weakly oligomorphic homomorphism-homogeneous structure contains (up to isomorphism) a unique homogeneous, homomorphism-homogeneous core, to which it is homomorphism-equivalent. As a consequence we obtain that every countable weakly oligomorphic structure is homomorphism-equivalent to a finite or $\omega$-categorical structure. As a corollary we obtain a characterization of positive existential theories of weakly oligomorphic structures as the positive existential parts of $\omega$-categorical theories.

\textsuperscript{*} Corresponding author: e-mail: maja@dmi.uns.ac.rs

1 Introduction

The notion of oligomorphic permutation groups goes back to Peter Cameron, who introduced it in the 1970s. They create a bridge between such diverse fields of mathematics as permutation group theory, enumerative combinatorics, and model theory (cf. [4]). A permutation group $G \leq \text{Sym}(A)$ is called oligomorphic if it has only finitely many orbits on $n$-tuples for every $n$. The Ryll-Nardzewski-Theorem characterizes countable structures with an oligomorphic automorphism group—a countable structure has an oligomorphic automorphism group if and only if its elementary theory is $\omega$-categorical.

There is a close relationship between oligomorphic permutation groups and homogeneous structures. A structure is called homogeneous if every isomorphism between finitely generated substructures can be extended to an automorphism. For instance, every countable homogeneous relational structure over a finite signature has an oligomorphic automorphism group, and is hence $\omega$-categorical. On the other hand, every countable structure that has an oligomorphic automorphism group can be expanded by first order definable relations to a homogeneous structure (though, not necessarily over a finite signature).

In their seminal paper [5] Peter Cameron and Jaroslav Nešetřil introduced several variations to the concept of homogeneity, one of them being homomorphism-homogeneity, which states that every homomorphism between finitely generated substructures of a given structure extends to an endomorphism of that structure.

Weak oligomorphy is a phenomenon that arises naturally in the context of homomorphism-homogeneity. A relational structure is weakly oligomorphic if its endomorphism monoid is oligomorphic, i.e., it has of every arity only finitely many invariant relations. It is not hard to see that every homomorphism-homogeneous relational structure over a finite signature is weakly oligomorphic. Moreover, every countable weakly oligomorphic relational structure has a positive existential expansion that is homomorphism-homogeneous (cf. [11, Corollary 4.4], [12, Corollary 6.9, Theorem 6.1]). Clearly, every oligomorphic structure is weakly oligomorphic but the...
reverse does not hold—e.g., there exist countably infinite homomorphism-homogeneous graphs that have a trivial automorphism group (cf. [5, Corollary 2.2]). However, the graph signature is finite and hence such graphs are weakly oligomorphic.

Cameron and Nešetřil in [5] posed the problem to understand the phenomenon of weak oligomorphy. Of particular interest is to characterize the (countable) structures with an oligomorphic endomorphism monoid. First steps in solving this problem were done in [11] and in [12]. The main goal of this paper is to understand further the nature of weakly oligomorphic structures, and thus to come closer to a satisfying answer to the problem by Cameron and Nešetřil. To this end, we shall create cross-links between the theory of weakly oligomorphic structures and the theory of oligomorphic structures, as well as between the theory of homomorphism-homogeneous structures and the theory of homogeneous structures.

In § 3 we give a characterization of the ages of homomorphism-homogeneous structures in the vein of Fraïssé’s Theorem by showing that a class of finite structures is the age of a homomorphism-homogeneous structure if and only if it is a homo-amalgamation class. An analogous result for monomorphism-homogeneous structures was proved by Cameron and Nešetřil in [5, Proposition 4.1] (there the term homo-amalgamation class needs to be replaced by the term mono-amalgamation class).

In § 4, we define the notion of weak oligomorphy and point out some connections with the notion of homomorphism-homogeneity.

In § 5, we show that the existence of a homomorphism between countable weakly oligomorphic structures depends only on a relation between their ages. This observation is used to show that the countable models of the first order theories of countable weakly oligomorphic structures are all mutually homomorphism-equivalent.

In § 6, we show that every countable weakly oligomorphic homomorphism-homogeneous structure has a unique (up to isomorphism) homomorphism-equivalent substructure that is oligomorphic, homogeneous, and a core. We conclude from this that every countable weakly oligomorphic structure is homomorphism-equivalent to a finite or \( \omega \)-categorical structure. This result is combined with a result by Bodirsky [3, Theorem 16] that states that every \( \omega \)-categorical structure is homomorphism-equivalent to a model complete core that is unique up to isomorphism and that is finite or \( \omega \)-categorical. We conclude that the positive existential theories of countable weakly oligomorphic structures are exactly the positive existential parts of \( \omega \)-categorical theories.

### 2 Preliminaries

The main objects of study in this paper are relational structures. As a basis for our notions and notations we use Hodges [9]. A relational signature is a model-theoretic signature without constant and function symbols. A model over a relational signature will be called a relational structure. Note that throughout this paper we make no other assumptions about the signatures. In particular, if not stated otherwise, we allow signatures of any cardinality. Relational structures will be denoted by bold capital letters \( \mathbf{A}, \mathbf{B} \), etc., while their universes will be denoted by usual capital letters \( A, B \), etc., respectively.

As usual, a homomorphism between two relational structures is a function between the universes that preserves all relations. We shall use the notation \( \mathbf{A} \rightarrow \mathbf{B} \) as a way to say that there exists a homomorphism from \( \mathbf{A} \) to \( \mathbf{B} \). If \( \mathbf{A} \rightarrow \mathbf{B} \) and \( \mathbf{B} \rightarrow \mathbf{A} \), then we call \( \mathbf{A} \) and \( \mathbf{B} \) homomorphism-equivalent. Moreover, if \( \mathbf{A}, \mathbf{B} \) are classes of relational structures over a common signature, then we say that \( \mathbf{A} \) projects onto \( \mathbf{B} \) (written \( \mathbf{A} \rightarrow \mathbf{B} \)) if for every \( \mathbf{A} \in \mathbf{A} \) there exists a \( \mathbf{B} \in \mathbf{B} \) such that \( \mathbf{A} \rightarrow \mathbf{B} \).

If \( f : \mathbf{A} \rightarrow \mathbf{B} \), then we call \( \mathbf{A} \) the domain and \( \mathbf{B} \) the codomain of \( f \), respectively. Moreover, the structure induced on \( f(\mathbf{A}) \) is called the image of \( f \).

**Monomorphisms** are injective homomorphisms, and isomorphisms are bijective homomorphisms whose inverse is a homomorphism, too. **Embeddings** are monomorphisms that not only preserve relations but also reflect them. That is, a monomorphism is an embedding if and only if it is an isomorphism to its image. We say that \( \mathbf{A} \) is a substructure of \( \mathbf{B} \) (and write \( \mathbf{A} \leq \mathbf{B} \)) if \( \mathbf{A} \subseteq \mathbf{B} \), and if the identical embedding of \( \mathbf{A} \) into \( \mathbf{B} \) is an embedding from \( \mathbf{A} \) into \( \mathbf{B} \).

As a final note, in this paper by a countable set we understand a finite or countably infinite set.
3 Homomorphism-homogeneity and amalgamation

Recall that the age of a relational structure is the class of finite structures embeddable into it. Recall also that a class $C$ of relational structures has the amalgamation property (AP) if for all $A, B_1, B_2 \in C$, and for all embeddings $f_1 : A \hookrightarrow B_1$, $f_2 : A \hookrightarrow B_2$ there exist $C \in C$, and embeddings $g_1 : B_1 \hookrightarrow C$ and $g_2 : B_2 \hookrightarrow C$ such that $g_1 \circ f_1 = g_2 \circ f_2$. In the following, we shall work with an adaptation of the amalgamation property—the homo-amalgamation property:

Definition 3.1 Let $C$ be a class of finite relational structures over the same signature. We say that $C$ has the homo-amalgamation property (HAP) if for all $A, B_1, B_2 \in C$, for all homomorphisms $f_1 : A \rightarrow B_1$, and for every embedding $f_2 : A \hookrightarrow B_2$ there exist $C \in C$, an embedding $g_1 : B_1 \hookrightarrow C$, and a homomorphism $g_2 : B_2 \rightarrow C$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Note the asymmetry in the definition of the HAP between $f_1$ and $f_2$. The embedding $f_2$ describes an extension of $A$, and the HAP essentially states that the homomorphism $f_1$ can be extended to $B_2$ with respect to $f_2$. The necessity of such an asymmetry was first noted by Cameron and Nešetřil in [5] in connection with the monoamalgamation property.

The HAP made its first appearance in literature in 2010 in a preprint by Dolinka about the Bergman property for endomorphism monoids of homogeneous structures [7]. He used it to show that a homogeneous structure is homomorphism-homogeneous if and only if its age has the HAP (cf. [7, Proposition 3.8]).

A central result of Fraïssé is the characterization of the ages of homogeneous structures. We quote the formulation due to Cameron (cf. [4, (2.12-13)]):

Theorem 3.2 (Fraïssé; [8]) A class $C$ of finite relational structures is the age of some countable homogeneous relational structure if and only if it is closed under isomorphism, it has only countably many isomorphism types, and it has the hereditary property (HP) and the amalgamation property (AP). Moreover, any two countable homogeneous relational structures with the same age are isomorphic.

A class of finite relational structures over the same signature, that is isomorphism-closed and that has the HP, and the AP, is called a Fraïssé-class.

In [5], Peter Cameron and Jaroslav Nešetřil introduced the notion of homomorphism-homogeneous structure.

A local homomorphism of a structure $A$ is a homomorphism from a finite substructure of $A$.

Definition 3.3 (Cameron, Nešetřil) A relational structure $A$ is called homomorphism-homogeneous if every local homomorphism of $A$ can be extended to an endomorphism of $A$.

As a straightforward adaptation of the notion of weakly homogeneous structure (cf. [9, § 6.1]) it will be useful to introduce the notion of a weakly homomorphism-homogeneous structure:

Definition 3.4 A relational structure $A$ is called weakly homomorphism-homogeneous if whenever $B \leq C$ are finite substructures of $A$, then every homomorphism $f : B \rightarrow A$ extends to $C$.

Clearly, a countable relational structure is weakly homomorphism-homogeneous if and only if it is homomorphism-homogeneous.

We continue with an analogue of Fraïssé’s Theorem for homomorphism-homogeneous structures:

Theorem 3.5

(a) The age of any homomorphism-homogeneous structure has the HAP.
(b) If a class $C$ of finite relational structures is isomorphism-closed, has only a countable number of isomorphism types, and has the HP, the joint embedding property (JEP), and the HAP, then there is a countable homomorphism-homogeneous structure $H$ whose age is equal to $C$.

Proof. (a) is clear.

The following simple observation is part of what is known as the Rasiowa-Sikorski-Lemma, a basic tool in set theory (cf. [10, Lemmas 16.1, 23.2], [13]). It will be the main tool in proving part (b):
Let \((P, \leq)\) be a non-empty poset and let \((D_i)_{i<\omega}\) be a sequence of cofinal subsets\(^1\) of \(P\). Then there exists a sequence \((p_i)_{i<\omega}\) in \((P, \leq)\) such that \(p_i \leq p_j\) whenever \(i \leq j\) and \(p_i \in D_i\) for all \(i \in \omega\).

We shall consider the set \(\mathbb{P}\) of all structures \(A \in \mathfrak{C}\) such that \(A = \{0, \ldots, |A| - 1\}\), together with the substructure-relation \(\leq\). Whenever \(X \in \mathbb{P}\) and \(n \leq |X|\), by \(X|_n\), we shall denote the substructure of \(X\) induced by \(\{0, \ldots, n-1\}\).

Now, for every \(A \in \mathbb{P}\) we define \(D_A := \{X \in \mathbb{P} | A \hookrightarrow X\}\). Since \(\mathcal{C}\) has the JEP, it follows that \(D_A\) is cofinal in \((\mathbb{P}, \leq)\). Moreover, for all structures \(A, B \in \mathbb{P}\) with \(A \leq B\) and for every \(n < \omega\) we define
\[
D_{A,B,n} := \{X \in \mathbb{P} | |X| \geq n, \forall f : A \rightarrow X|_n \exists g : B \rightarrow X : g|_A = f\}
\]
Since \(\mathcal{C}\) has the HAP, we have that \(D_{A,B,n}\) is cofinal in \((\mathbb{P}, \leq)\).

Altogether we defined a countable family of cofinal subsets of \((\mathbb{P}, \leq)\). Thus, by the Rasiowa-Sikorski-Lemma, there exists a sequence \((A_i)_{i<\omega}\) of elements of \(\mathbb{P}\) such that

1. \(A_i \leq A_j\) whenever \(i \leq j\),
2. for every \(A \in \mathbb{P}\) there exists an \(i < \omega\) such that \(A_i \in D_A\),
3. for all \(A \leq B \in \mathbb{P}\) and for every \(n < \omega\) there exists some \(i < \omega\) such that \(A_i \in D_{A,B,n}\).

Let us define \(H := \bigcup_{i<\omega} A_i\). By construction we have that \(\text{Age}(H) = \mathcal{C}\) and that \(H\) is weakly homomorphism-homogeneous. Moreover, as \(H\) is a union of finite structures it is countable. Hence, \(H\) is homomorphism-homogeneous. \(\square\)

It may seem odd that in the formulation of Theorem 3.5, the JEP is required while in Theorem 3.2 it is not. The reason is that the JEP follows from the HP together with the AP. However, the HAP is too weak to allow a similar simplification.

A class of finite relational structures over the same signature, that is closed under isomorphism and that has the HP, the JEP, and the HAP will be called a \textit{homo-amalgamation class}.

Note that, in contrast to Fraïssé’s construction, the construction of a homomorphism-homogeneous structure from an age does not guarantee the uniqueness of the result up to isomorphism. Indeed, all countably infinite linear orders (with reflexive order relations) are homomorphism-homogeneous and have as the age the class of all finite linear orders. In general we have:

\textbf{Lemma 3.6} Let \(A\) be countable, and let \(B\) be homomorphism-homogeneous, such that \(\text{Age}(A) \rightarrow \text{Age}(B)\). Then \(A \rightarrow B\).

In particular, any two countable homomorphism-homogeneous structures of the same age are homomorphism-equivalent.

\section{Weakly oligomorphic structures}

A structure \(A\) is called \textit{oligomorphic} if its automorphism group is oligomorphic.

In [11], oligomorphic transformation monoids were introduced: Let \(A\) be a set and let \(M \subseteq A^A\) be a transformation monoid. \(M\) acts naturally on \(A^n\) according to \((a_1, \ldots, a_n)^h := (h(a_1), \ldots, h(a_n))\). For \(\bar{a} \in A^n\) we define \(\bar{a}^M := \{a^h | h \in M\}\). Finally, we define an equivalence relation \(\sim_M\) on \(A^n\) according to \(\bar{a} \sim_M \bar{b}\) whenever \(\bar{a}^M = \bar{b}^M\). The equivalence classes of \(\sim_M\) are called the \(n\)-orbits of \(M\). Now \(M\) is called \textit{oligomorphic} if it has only finitely many \(n\)-orbits for every \(n > 0\). If the endomorphism monoid of a structure \(A\) is oligomorphic, then we call \(A\) \textit{weakly oligomorphic}. Clearly, if a structure is oligomorphic, then it is also weakly oligomorphic.

In [11], a theory of weakly oligomorphic structures was developed in analogy with the theory of oligomorphic structures. As we are going to use and extend some results from there, let us introduce the key notions: Let \(\Sigma\) be a relational signature, and let \(L(\Sigma)\) be the language of first order logic with respect to \(\Sigma\). Let \(A\) be a \(\Sigma\)-structure. For a formula \(\varphi(\bar{x})\) (where \(\bar{x} = (x_1, \ldots, x_n)\)) we define \(\varphi^A \subseteq A^n\) as the set of all \(n\)-tuples \(\bar{a}\) over \(A\) such that \(A \models \varphi(\bar{a})\). More generally, for a set \(\Phi\) of formulae from \(L\) with free variables from \(\{x_1, \ldots, x_n\}\) we define \(\Phi^A\) as

\(^1\) Recall that a subset \(D\) of a poset \((P, \leq)\) is called \textit{cofinal} if for all \(p \in P\) there is a \(d \in D\) such that \(p \leq d\).
the intersection of all relations $\varphi^A$ where $\varphi$ ranges through $\Phi$. If $\Phi^A \neq \emptyset$, then we say that $A$ realizes $\Phi$. If there is a structure that realizes $\Phi$, then we call $\Phi$ an $n$-type. If $n$ is clear from the context, then we call $\Phi$ just a type. We call $\Phi$ positive if it consists just of positive existential formulae$^2$.

For a non-empty relation $\varrho \subseteq A^n$ by $\text{Tp}_A^+ (\varrho)$ we denote the set of all positive existential formulæ $\varphi (\vec{x})$ such that $\varrho \subseteq \varphi^A$. This is the positive type defined by $\varrho$ with respect to $A$. Instead of $\text{Tp}_A^+ ([\vec{a}])$, we shall write $\text{Tp}_A^+ (\vec{a})$.

Such positive types are called complete for $A$. More generally, if $T$ is a theory, then a positive type $\Phi$ is called complete for $T$ if it is complete for some model of $T$. The set of all complete positive $n$-types for $T$ is denoted by $S_n^+ (T)$.

At this point let us mention some known results that will be used routinely later on in the paper:

**Proposition 4.1** (Mašulović & Pech; [11, Lemma 3.4, 12, Theorem 6.3.4]) Let $A$ be a weakly oligomorphic structure. Then $A$ realizes just finitely many complete positive $n$-types for every $n \in \mathbb{N}$.

If $A$ is countable and realizes just finitely many complete positive $n$-types for every $n \in \mathbb{N}$, then it is weakly oligomorphic.

An immediate consequence is:

**Corollary 4.2** In every weakly oligomorphic structure $A$ there are of every arity just finitely many relations definable by positive types.

From the following result from [11] we recall only the parts that are used later on in this paper:

**Theorem 4.3** (Mašulović & Pech; [11, Theorem 3.5]) Let $T$ be a complete theory over a relational signature $\Sigma$. Then the following are equivalent:

1. Every countable model of $T$ is weakly oligomorphic.
2. There exists a countable model of $T$ which is weakly oligomorphic.
3. For every $n \geq 1$, $S_n^+ (T)$ is finite.

Let us conclude this section with some observations about the relationship between weak oligomorphy and homomorphism-homogeneity:

**Lemma 4.4** Let $A$ be a relational structure. Then the following are true:

(a) If $A$ is weakly oligomorphic, then for every $n \in \mathbb{N}$ the class $\text{Age}(A)$ contains up to isomorphism only finitely many structures of cardinality $n$.
(b) If for every $n \in \mathbb{N}$ the class $\text{Age}(A)$ contains up to isomorphism only finitely many structures of cardinality $n$, and $A$ is homomorphism-homogeneous, then $A$ is weakly oligomorphic.

An immediate consequence of the previous Lemma is, that if $A$ is a homomorphism-homogeneous relational structure over a finite signature, then $A$ is weakly oligomorphic. This together with the characterization of the ages of countable homomorphism-homogeneous structures gives a rich source of weakly oligomorphic structures since any reduct of a countable weakly oligomorphic structure will again be weakly oligomorphic.

The following characterization of homomorphism-homogeneity for countable weakly oligomorphic structures will be of help in later proofs.

**Proposition 4.5** (Pech; [12, Theorem 6.1]) Let $A$ be a countable weakly oligomorphic structure. Then $A$ is homomorphism-homogeneous if and only if every positive existential formula in the language of $A$ is equivalent in $A$ to a quantifier free positive formula.

## 5 Homomorphisms between weakly oligomorphic structures

In this section we collect some results about the existence of homomorphisms between weakly oligomorphic structures.

---

$^2$ Recall that a formula is called positive existential if it can be built using existential quantification together with positive logical connectives $\land$ and $\lor$. It is called positive primitive if it is positive existential and does not contain $\lor$. 

---
Clearly, for two relational structures \(A\) and \(B\) the condition \(\text{Age}(A) \rightarrow \text{Age}(B)\) is necessary for \(A \rightarrow B\). In the following, we shall prove that if \(A\) is countable, and \(B\) is weakly oligomorphic, then this condition is also sufficient:

**Proposition 5.1** Let \(A, B\) be relational structures over the same signature such that \(\text{Age}(A) \rightarrow \text{Age}(B)\), and suppose that \(A\) is countable and that \(B\) is weakly oligomorphic. Then \(A \rightarrow B\).

**Proof.** If \(A\) is finite, then nothing needs to be proved. So we assume that \(A\) is countably infinite. In that case we can write the universe \(A\) as \(A = \{a_0, a_1, a_2, \ldots\}\). Define \(A_n := \{a_0, \ldots, a_{n-1}\}\), and let \(A_n\) be the substructure of \(A\) that is induced by \(A_n\).

Let \(M\) be the endomorphism monoid of \(B\). Define \(\sigma_n := \{(b_0, \ldots, b_{n-1}) \in B^n \mid \exists f : A_n \rightarrow B : f(a_i) = b_j\) for \(0 \leq i < n\)\). On \(\sigma_n\) we define \(\bar{b} \rightarrow \bar{c}\) if \(\bar{c} \in \bar{b}^M\). By definition, if \(\bar{b} \rightarrow \bar{c}\) and \(\bar{c} \rightarrow \bar{b}\), then \(\bar{b} \sim_M \bar{c}\). Since \(B\) is weakly oligomorphic, we have that \(\sigma_n/\sim_M\) is finite.

Next we define a tree \(T\) whose nodes on level \(n\) are the elements of \(\sigma_n/\sim_M\). If \(\bar{b} = (b_0, \ldots, b_n) \in \sigma_{n+1}\), then the unique lower neighbor of \([\bar{b}]_{\sim_M}\) in \(T\) is \([\{b_0, \ldots, b_{n-1}\}]_{\sim_M}\). By the definition of \(\sim_M\) the choice of the lower neighbor of \([\bar{b}]_{\sim_M}\) is independent of the choice of the representative \(\bar{b}\). Observe that \(T\) is finitely branching since it has only finitely many nodes on each level. Moreover, since \(\text{Age}(A) \rightarrow \text{Age}(B)\), we have that \(T\) has nodes on every level. Hence, by König’s tree-lemma, \(T\) has a branch \((\{\bar{b}_i\}_{i \in \mathbb{N}}\)

Our next goal is to construct a sequence \((d_i)_{i \in \mathbb{N}}\) such that for every \(i \in \mathbb{N}\) we have that \(\bar{b}_i \rightarrow d_i = (d_0, \ldots, d_{i-1})\), and such that whenever \(i \leq j\), then \(d_j\) is a prefix of \(d_i\). Indeed, if such a sequence can be constructed, then its limit \((d_i)_{i \in \mathbb{N}}\) defines a homomorphism \(h : A \rightarrow B\) according to \(h(a_i) = d_i\).

We proceed by induction: Define \(\bar{d}_0 := \bar{b}_0 = (\varepsilon)\). If \(\bar{d}_i\) is already defined, consider \(\bar{b}_{i+1} = (b_0, \ldots, b_j)\). Set \(\bar{b}_i \in [\bar{b}_i]_{\sim_M}\). By the induction hypothesis there exists an endomorphism \(h\) of \(B\) such that \((\bar{b}_i)^h = \bar{d}_j\). Now we may define \(d_{i+1} := (\bar{b}_{i+1})^h\). This finishes the construction.

Let us immediately apply Proposition 5.1: By the Ryll-Nardzewski Theorem two countable oligomorphic Structures are isomorphic if and only if they have the same first order theory. The following is an analogous statement for weakly oligomorphic structures. Recall that the positive existential theory of a structure \(A\) is the set of all positive existential sentences that hold in \(A\).

**Theorem 5.2** Let \(A\) and \(B\) be two countable weakly oligomorphic structures. Then the following are equivalent:

1. \(A\) and \(B\) are homomorphism-equivalent,
2. \(A\) and \(B\) have the same positive existential theory,
3. \(\text{Age}(A) \leftrightarrow \text{Age}(B)\),
4. \(\text{CSP}(A) = \text{CSP}(B)\).

Here \(\text{CSP}(A) := \{C \mid C\text{ finite, } C \rightarrow A\}\).

Before coming to the proof of Theorem 5.2 we need to prove some additional auxiliary results:

**Lemma 5.3** Let \(A\) be a weakly oligomorphic structure over the signature \(R\), and let \(\Psi\) be a positive type. If every finite subset of \(\Psi\) is realized in \(A\), then \(\Psi\) is realized in \(A\).

**Proof.** Suppose that every finite subset of \(\Psi\) is realized in \(A\) but \(\Psi\) is not. In the following, we shall define a sequence \((\varphi_i)_{i \in \mathbb{N}}\) of formulae from \(\Psi\), and a sequence \((d_i)_{i \in \mathbb{N}}\) such that \(d_i \in \varphi_i^A\) for \(1 \leq j \leq i\), but \(d_i \notin \varphi_i^{A\bullet}\).

Let \(\varphi_0 \in \Psi\). Then there exists a \(\bar{d}_0 \in A^m\) such that \(\bar{d}_0 \in \varphi_0^A\). Suppose that \(\varphi_i\) and \(d_i\) are defined already. By assumption \(\bar{d}_i\) does not realize \(\Psi\). Let \(\varphi_{i+1} \in \Psi\) such that \(\bar{d}_i \notin \varphi_{i+1}^A\). Again, by assumption the set \(\{\varphi_0, \ldots, \varphi_{i+1}\}\) is realized in \(A\). Define \(\bar{d}_{i+1} \in A^m\) to be a tuple that realizes \(\varphi_{i+1}\).

By construction the sets \(\Psi_i := \{\varphi_0, \ldots, \varphi_i\}\) are positive types that define distinct non-empty relations in \(A\). However, this is in contradiction with the assumption that \(A\) is weakly oligomorphic (cf. Corollary 4.2).

We conclude that \(\Psi\) is realizable.

**Lemma 5.4** Let \(A\) and \(B\) be relational structures over the same signature. Suppose that \(B\) is weakly oligomorphic and that \(A\) and \(B\) have the same positive existential theories. Then \(\text{Age}(A) \rightarrow \text{Age}(B)\).
6 Cores of homomorphism-homogeneous structures

A finite relational structure is called a core if each of its endomorphisms is an automorphism. There are many ways
to generalize the definition of a core to infinite structures. Several possibilities were explored in [1]. The following
definition of a core was used successfully by Bodirsky in [3] for studying cores of \( \omega \)-categorical structures. It
turns out that this definition is good for our purposes, too.

**Definition 6.1** A relational structure \( \mathcal{C} \) is called a core if every endomorphism of \( \mathcal{C} \) is an embedding.

We say \( \mathcal{C} \) is a core of \( \mathcal{A} \) (or \( \mathcal{A} \) has a core \( \mathcal{C} \)) if \( \mathcal{C} \) is a core, \( \mathcal{C} \subseteq \mathcal{A} \), and there is an endomorphism \( f \) of \( \mathcal{A} \) whose
image is contained in \( \mathcal{C} \).

Note that in [3] it is defined that \( \mathcal{C} \) is a core of \( \mathcal{A} \) if \( \mathcal{C} \) is a substructure of \( \mathcal{A} \) and if there exists an endomorphism of \( \mathcal{A} \) whose
image is equal to \( \mathcal{C} \), rather than just contained in \( \mathcal{C} \). Hence our definition is slightly weaker than
Bodirsky’s. Note also that [3, Theorem 16] shows the existence of cores of \( \omega \)-categorical structures in our (weaker)
sense.

For finite relational structures the core always exists and is unique up to isomorphism. For infinite structures a
core may exist or may not exist. Moreover, if it exists, it may not be unique up to isomorphism.

In this section, we shall employ the machinery that was developed in the previous section in order to study
cores of homomorphism-homogeneous structures. The crucial definition in this section is that of a hom-irreducible
element in some class of structures:

**Definition 6.2** Let \( \mathcal{C} \) be a class of relational structures over the same signature and let \( \mathcal{A} \in \mathcal{C} \). We say that \( \mathcal{A} \) is
hom-irreducible in \( \mathcal{C} \) if for every \( \mathcal{B} \in \mathcal{C} \) and every homomorphism \( f : \mathcal{A} \rightarrow \mathcal{B} \) we have that \( f \) is an embedding.

**Lemma 6.3** Let \( \mathcal{C} \) be a homo-amalgamation class, and let \( \mathcal{D} \) be the class of all structures from \( \mathcal{C} \) that are
hom-irreducible in \( \mathcal{C} \). If \( \mathcal{C} \rightarrow \mathcal{D} \), then \( \mathcal{D} \) is a Fraïssé class, i.e., it has the HP, and the AP.
Proof. (HP): Let $A \in D$, and let $B$ be a substructure of $A$ (in particular, $B \in C$). Let $C \in C$, and let $f : B \to C$ be any homomorphism. Then, since $C$ has the HAP and is isomorphism-closed, we have that there exist a $D \in C$ with $C \subseteq D$, and a homomorphism $g : A \to D$ such that $g |_B = f$. Since $A$ is hom-irreducible in $C$, it follows that $g$ is an embedding, and that $f$, being a restriction of $g$ to $B$, is an embedding, too. We conclude now that $B$ is hom-irreducible in $C$, and hence that $D$ has the HP.

(AP): Let $A, B, C \in D \subseteq C$ and let $f_1 : A \hookrightarrow C$ and $f_2 : A \hookrightarrow B$ be embeddings. Then, by the HAP of $C$, there exists a $D \in C$, an embedding $g_1 : B \hookrightarrow D$ and a homomorphism $g_2 : C \to D$ such that $g_2 \circ f_1 = g_1 \circ f_2$. Since $C \in D$, it follows that $g_2$ is an embedding, too. Using that $C \hookrightarrow D$, we obtain that there exists a structure $D \subseteq D$ and a homomorphism $h : D \to D$. Then $D$ will be the amalgam with $h \circ g_1$ and $h \circ g_2$ as embeddings.

For a relational structure $A$ we denote by $C_A$ the class of all finite structures of the same type as $A$ that are hom-irreducible in the age of $A$.

Lemma 6.4 Let $A$ be a weakly oligomorphic relational structure. Then $Age(A) \hookrightarrow C_A$.

Proof. By Lemma 4.4 we have that $Age(A)$ contains up to isomorphism just finitely many structures of every given cardinality. Let $C \in Age(A)$, and let $D \in Age(A)$ such that $C \to D$, and such that $|D|$ is minimal—say $|D| = n$. Let $D_1, \ldots, D_k$ be a system of distinct representatives of the isomorphism classes of structures $B$ of cardinality $n$ from $Age(A)$ such that $C \to B$. Without loss of generality $D = D_1$. The set $\{D_1, \ldots, D_k\}$ equipped with the relation $\to$ is a poset. By finiteness, this poset contains a maximal element $D_i$, above $D_1$ (that is, $D_1 \to D_i$). It is easy to see that $D_i \in C_A$. Moreover, $C \to D_i$.

Proposition 6.5 Let $A$ be a countable homomorphism-homogeneous relational structure such that $Age(A) \to C_A$. Then $A$ has a core that is isomorphic to the Fraïssé-limit $F$ of $C_A$. Moreover, $F$ is homomorphism-homogeneous.

Proof. The restriction of any endomorphism of $F$ to a finite substructure is an embedding. Thus it is immediate that $F$ is a core. Since every local homomorphism of $F$ is an embedding, from the homogeneity of $F$ follows the homomorphism-homogeneity of $F$. From Lemma 3.6 and from $Age(A) \to Age(F)$ follows $A \to F$. Since $C_A \subseteq Age(A)$ it follows that $Age(F) \to Age(A)$. Thus, again using Lemma 3.6 we have $F \to A$. Let $h : F \to A$ be a homomorphism. Then all restrictions of $h$ to finite substructures of $F$ are embeddings. Hence $h$ itself is an embedding. So without loss of generality we may assume that $F$ is a substructure of $A$. This shows that $F$ is a core of $A$.

Lemma 6.6 Let $C$ be a homogeneous core. Then $C$ is weakly oligomorphic if and only if it is oligomorphic.

Proof. Obviously, if a structure is oligomorphic, then it is weakly oligomorphic, too.

Suppose that $C$ is weakly oligomorphic. Take $\bar{a}, \bar{b} \in C^n$ such that there exists $f \in End(C)$ with $\bar{a}^f = \bar{b}$. Since $C$ is a core, we conclude that $f$ is an embedding, and therefore $a_i \mapsto b_i$ for $i = 1, \ldots, n$ is a local isomorphism. Since $C$ is homogeneous, it follows that there is $g \in Aut(C)$ with $\bar{a}^g = \bar{b}$. Hence $\bar{b}$ is in the $n$-orbit of $Aut(C)$ generated by $\bar{a}$. This implies that $C$ is oligomorphic.

Corollary 6.7 Every countable, weakly oligomorphic, homomorphism-homogeneous structure $A$ has up to isomorphism a unique homogeneous core $F$. Moreover, $F$ is oligomorphic and homogeneous.

Proof. Let $F$ be the Fraïssé-limit of $C_A$. Since $A$ is weakly oligomorphic and homomorphism-homogeneous, it follows from Lemma 6.4 that $Age(A) \to C_A$. Thus, from Proposition 6.5 we obtain that $A$ has a core isomorphic to $F$, and that $F$ is homomorphism-homogeneous.

Suppose that $F'$ is another homomorphism-homogeneous core of $A$. Then $C_A \subseteq Age(F')$. On the other hand, from $Age(A) \to C_A$ it follows that $Age(F') \to C_A$. Hence, any finite substructure of $F'$ that is not hom-irreducible in $Age(A)$ homomorphically maps to an element of $C_A$. Thus, if $Age(F') \neq C_A$, then $F'$ has a local homomorphism that is not an embedding. By the homomorphism-homogeneity of $F'$ this extends to an endomorphism that is not an embedding, a contradiction. Thus $Age(F') = C_A$, and every local homomorphism is an embedding. From this it follows that $F'$ is weakly homogeneous, and hence homogeneous. From Fraïssé’s Theorem we conclude that $F' \cong F$.

It remains to show that $F$ is oligomorphic: Since $A$ is weakly oligomorphic, from Lemma 4.4 it follows that $Age(A)$ contains up to isomorphism only finitely many structures of every cardinality. Since $C_A \subseteq Age(A)$, and since $F$ is homomorphism-homogeneous, it follows from Lemma 4.4 that $F$ is weakly oligomorphic. Finally, from Lemma 6.6 it follows that $F$ is oligomorphic.
Example 6.8 Let $G$ be a countable homomorphism-homogeneous graph. Then all hom-irreducible graphs in $\text{Age}(G)$ are complete graphs. Indeed, if $G$ is itself a complete graph, then nothing need to be proved. So suppose that $G$ is not complete. Let $A \subseteq G$ be a finite non-complete subgraph, and suppose that $a, b \in A$ induce a non-edge in $A$. Consider the function $h : \{a, b\} \to G$ defined by $h : a \mapsto b, b \mapsto b$. Clearly, $h$ is a local homomorphism of $G$. Since $G$ is homomorphism-homogeneous, $h$ extends to an endomorphism $\hat{h}$ of $G$. Since $\hat{h} |_A$ is non-injective, we conclude that $A$ is not hom-irreducible in $\text{Age}(G)$. We conclude that every countable homomorphism-homogeneous graph has a core that is a complete graph.

Note that the countable homomorphism-homogeneous graphs are still not completely classified. The finite and the disconnected homomorphism-homogeneous graphs are known [5, Proposition 1.1]. Moreover, we know that every graph that contains the Rado graph as a spanning subgraph is homomorphism-homogeneous [5, Proposition 2.1]. However, it was shown by Rusinov and Schweitzer [14] that there are connected countable homomorphism-homogeneous graphs that do not have the Rado graph as a spanning subgraph.

Previously we linked weakly oligomorphic homomorphism-homogeneous structures with oligomorphic homogeneous structures. The following theorem makes a similar link between weakly oligomorphic structures and $\omega$-categorical structures.

Theorem 6.9 Let $A$ be a countable weakly oligomorphic relational structure. Then $A$ is homomorphism-equivalent to a finite or $\omega$-categorical structure $F$. Moreover, $F$ embeds into $A$.

Proof. Let $\hat{A}$ be the structure that is obtained by expanding $A$ by all positive existentially definable relations over $A$. Clearly, $\hat{A}$ is weakly oligomorphic, too.

In $\hat{A}$ every positive existential formula is equivalent to a positive quantifier-free formula. Hence, by Proposition 4.5 $\hat{A}$ is homomorphism-homogeneous. With Corollary 6.7 we conclude that $A$ has a homomorphism-homogeneous core $\hat{F}$ that is oligomorphic, and homogeneous.

Let $F$ be the reduct of $\hat{F}$ to the signature of $A$. Then $F$ is oligomorphic, and since $\hat{A}$ and $A$ have the same endomorphisms, $F$ is still homomorphism-equivalent to $A$.

If $F$ is not finite and hence countably infinite, then, by the Ryll-Nardzewski Theorem, it is $\omega$-categorical.

The previous result may be combined with a result by Bodirsky, in order to obtain a general result about the existence of cores in countable weakly oligomorphic structures:

Theorem 6.10 (Bodirsky; [3, Theorem 16]) Every $\omega$-categorical structure has a model-complete\(^3\) core which is unique up to isomorphism. Moreover, this core is $\omega$-categorical, too.

Corollary 6.11 Every countable weakly oligomorphic structure has a model-complete core that is unique up to isomorphism. Moreover, this core is $\omega$-categorical or finite.

In Corollary 6.7, the uniqueness of the homomorphism-homogeneous core follows from homogeneity. In Theorem 6.10 (and hence also in Corollary 6.11) the uniqueness of the core follows from model-completeness.

The Ryll-Nardzewski Theorem can also be understood as a characterization of the first order theories of countable oligomorphic structures. Using Theorem 6.9 we can give a similar characterization of the positive existential theories of weakly oligomorphic structures.

Theorem 6.12 Let $T$ be a set of positive existential sentences. Then the following are equivalent:

1. $T$ is the positive existential theory of a countable weakly oligomorphic structure.
2. $T$ is the positive existential part of an $\omega$-categorical theory.
3. $T$ is the positive existential theory of a countable oligomorphic structure.

Proof. From the Ryll-Nardzewski-Theorem it follows immediately that statements (2) and (3) are equivalent. Obviously, from (3) follows (1). So, to complete the proof it is left to show that from (1) follows (3): Let $T$ be the positive existential theory of a countable weakly oligomorphic structure $A$. Then by Theorem 6.9 $A$ is homomorphism-equivalent to a finite or $\omega$-categorical structure $F$. If $F$ is finite, then it is homomorphism-equivalent to an $\omega$-categorical structure $\hat{F}$ (take $\omega$ disjoint copies of $F$; this structure surely is oligomorphic and

\(^3\) Recall that a structure $A$ is called model-complete if every embedding between models of $\text{Th}(A)$ is elementary.
hence ω-categorical; moreover, F is a retract of ˆF). Clearly, two homomorphism-equivalent structures have the same positive existential theories.

□

7 Concluding remarks

In [2], Bodirsky, Hils and Martin, using maximal positive types, characterize all countable structures whose positive existential theory coincides with the positive existential theory of an ω-categorical structure. While Theorem 6.12 can be proved using their results, we decided against this way since our techniques are on the one hand more elementary and on the other hand they give a better idea of the ω-categorical structure inside of a weakly oligomorphic structure. Indeed, in many cases (in particular for homomorphism-homogeneous structures) our technique not only yields the existence of an ω-categorical substructure but also gives a concrete description (cf. Example 6.8).

We conclude this paper with an open problem:

Problem 6.13 Give an algebraic characterization of all transformation monoids that occur as the endomorphism monoids of countable (relational) structures whose first order theory is weakly ω-categorical.

Acknowledgements The second author was supported by the Ministry of Education and Science of the Republic of Serbia through Grant No.174018, by the grant (Contract 114-451-1901/2011) of the the Secretariat of Science and Technological Development of the Autonomous Province of Vojvodina, and a DAAD reinvitation scholarship.

References

[1] B. Bauslaugh, Core-like properties of infinite graphs and structures, Discrete Math. 138(1-3), 101–111 (1995).
[2] M. Bodirsky, M. Hils, and B. Martin, On the scope of the universal-algebraic approach to constraint satisfaction, Log. Methods Comput. Sci. 8(3:13), 1–30 (2012).
[3] M. Bodirsky, Cores of countably categorical structures, Log. Methods Comput. Sci. 3(1), 1:2 (2007).
[4] P. J. Cameron, Oligomorphic permutation groups, London Mathematical Society Lecture Note Series Vol. 152 (Cambridge University Press, Cambridge, 1990).
[5] P. J. Cameron and J. Nešetřil, Homomorphism-homogeneous relational structures, Combin. Probab. Comput. 15(1-2), 91–103 (2006).
[6] A. K. Chandra and P. M. Merlin, Optimal implementation of conjunctive queries in relational data bases, in: Proceedings of the 9th Annual ACM Symposium on Theory of Computing, May 4-6, 1977, Boulder, Colorado, USA, edited by J. E. Hopcroft, E. P. Friedman, M. A. Harrison, (ACM, 1977), pp. 77–90.
[7] I. Dolinka, The Bergman property for endomorphism monoids of some Fraïssé limits, Forum Math. 26(2), 357–376 (2014).
[8] R. Fraïssé, Sur certaines relations qui généralisent l’ordre des nombres rationnels, C. R. Acad. Sci. Paris 237, 540–542 (1953).
[9] W. Hodges, A shorter model theory (Cambridge University Press, Cambridge, 1997).
[10] T. Jech, Set theory, Pure and Applied Mathematics (Academic Press, 1978).
[11] D. Mašulović and M. Pech, Oligomorphic transformation monoids and homomorphism-homogeneous structures, Fundam. Math. 212(1), 17–34 (2011).
[12] M. Pech, Endolocality meets homomorphism-homogeneity - a new approach in the study of relational algebras, Algebra Univers. 66(4), 355–389 (2011).
[13] H. Rasiowa and R. Sikorski, A proof of the completeness theorem of Gödel, Fundam. Math. 37, 193–200 (1950).
[14] M. Rusinov and P. Schweitzer, Homomorphism-homogeneous graphs, J. Graph Theory 65(3), 253–262 (2010).