A note on $N = 8$ counterterms

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Abstract

The most difficult counterterms to construct in any supersymmetric theory are those that cannot be written as full superspace integrals of gauge-invariant integrands. In $D = 4$ maximal supergravity it has been known for some time that there are just three of these at the linearised level. In this article we discuss these counterterms again from the point of view of representations of the superconformal group. In particular, we show that the only independent invariants constructed from shortened superconformal multiplets in $D = 4$ are BPS.

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Introduction

Higher-order invariants in supersymmetric theories are important as possible field-theoretic counterterms and as higher-derivative terms in effective actions for strings or branes. These invariants fall into two classes, which we might call long and short, and which generalise D and F terms in $N = 1, D = 4$ supersymmetry. A D, or long, term corresponds to an integral over the full superspace of some gauge-invariant superfield, while an F, or short, term is a chiral integral in $N = 1$ or a suitable generalisation for $N > 1$. There are many D terms but rather few F terms. In fact, in $D = 4, N = 8$ supergravity it was shown in [1] that there are only three of these in the linearised theory, each with four points ($d^{2k} R^4$ integrals in spacetime with $k = 0, 2, 3$.)

The existence of the three-loop $R^4$ counterterm was established (at the linearised level) for $N = 1, 2$ supersymmetry in [2, 3], where they are D-type invariants. In $N = 8$ on the other hand, the simplest D-type counterterm does not occur until seven loops [4], and the $R^4$ invariant turns out to be a short F-term invariant. It was first constructed in [5], a manifestly covariant formulation making use of the notion of a superaction was given in [6], and it was shown in [7] that this can be written very simply in harmonic superspace. In [1] a study was made of all possible counterterms in harmonic superspaces and it was found that there are only two other F-terms, corresponding to counterterms at the five and six loop levels ($d^4 R^4$ and $d^6 R^4$). It is now known that $d = 4, N = 8$ supergravity is finite at three loops [8], and more recently it has been shown that maximal supergravity is finite at four loops in $D = 5$ [9], where the relevant invariant is $d^6 R^4$. Although the $D = 4, N = 8$ case has not been explicitly checked at five and six loop order, it would no longer be regarded as a surprise if it turned out to be finite here as well.

In this article we revisit the $D = 4, N = 8$ F-term invariants, but from a slightly different point of view to that adopted in [1]. We shall still make use of the fact that the linearised field strength superfield is superconformal, but we avoid the use of harmonic superspace. Instead we approach the problem by directly determining which superconformal representations possess suitable singlet top components. Such top components automatically give rise to supersymmetric integrals in spacetime. In particular, this enables us to confirm very easily that there are no F-term invariants other than the three we have referred to above. More generally, we show that there are no independent superactions with measures that are not Lorentz scalars for massless supersymmetric theories in $D = 4$.

Before going into the details, we would like to comment on two features of our analysis: the use of on-shell supersymmetry and the linearisation of supergravity. No off-shell formulations are known for either maximal super Yang-Mills (SYM) or maximal supergravity. In order to construct counterterms or higher-order terms in effective actions it is therefore necessary to start with the on-shell supersymmetry transformations of the original, lowest-order Lagrangian. The addition of any on-shell deformation to the original action will then be invariant up to terms proportional to the original equations of motion. This procedure is perfectly satisfactory since such terms can then be compensated by amending the supersymmetry transformations. This will in turn induce higher-order terms in the action, and the iteration of this procedure gives rise to a perturbative method for handling the modified supersymmetry transformations. An example of this is provided by the $F^4$ invariant in $D = 10$ SYM. In the abelian case this gives rise to an $F^6$ contribution at the next order and eventually to the full Born-Infeld series of terms. In the non-abelian case there is a similar single-trace $F^4$ deformation that gives rise to many more terms than just $F^6$ at the next order [10], a result that has recently been confirmed.
using only supersymmetry [11]. Of course, one would also want to know that the modified
supersymmetry transformations do indeed satisfy the supersymmetry algebra. A convenient
way of doing this, particularly relevant in the quantum-mechanical context, is to introduce
ghosts and make use of the full BRST/BV formalism, see, for example, [12, 13, 1, 14]. It is
possible there might be an obstruction to the extension of a deformation to all orders, which
would indicate a supersymmetry anomaly of an unusual type, although this would be unexpected
from a string theory perspective.

In the case of maximal supergravity we are also forced to deal with linearised superfields when
looking at possible F-terms. This is legitimate for making comparisons with graviton scattering
amplitudes, but one would also eventually like to understand what the full non-linear expressions
are that correspond to the linearised ones. In supergravity, therefore, we have to cope not only
with the non-linearities induced by the on-shell nature of the supersymmetry transformations,
but also the non-linearities of the full classical theory. It is not easy to see how to generalise
arbitrary F-terms to the full theory because the linearised superspace measures do not have
obvious non-linear counterparts. Indeed, it could be the case that some, or even all, of these
invariants do not admit non-linear extensions, and this might be the reason for the unexpected
finiteness results. It has been shown that $E_7$ symmetry can be maintained in perturbation
theory [15], at the cost of manifest Lorentz invariance, and this would be a further constraint
that would need to be satisfied. In a recent paper a string theory based argument has been
given which shows that the full $R^4$ invariant is not $E_7$ invariant [16].

In this note we shall not discuss these important issues further; instead we restrict our discussion
to the linearised level and use on-shell supersymmetry. In this way we can be certain that we
have all the allowed invariants (i.e. we have not missed any) although there remains the
possibility that some of them will not extend to genuine invariants of the full non-linear theory.

**Linearised $N = 8$ supergravity.**

The spectrum of supergravity consists of the graviton, 8 gravitinos, 28 vector fields, 56 spin one-
half fermions and 70 scalars. The whole set of component field strengths can be assembled into an
$N = 8$ scalar superfield $W_{ijkl}$, $i, j = 1 \ldots 8$, that transforms under the 70-dimensional representation
of $SU(8)$. It is therefore totally antisymmetric and self-dual. It depends on $x^a, a = 0, 1, 2, 3$ and
8 two-component fermionic coordinates and their complex conjugates $(\theta^{\alpha i}, \bar{\theta}^{\dot{\alpha} i})$. $W_{ijkl}$ lives in flat $N = 8$ superspace which is equipped with the supersymmetric invariant derivatives $(\partial_a, D_{\alpha i}, \bar{D}_{\dot{\alpha}})$ where

$$[D_{\alpha i}, \bar{D}_{\dot{\beta}}] = i\delta_{\beta}^{\alpha} \partial_{\alpha \dot{\beta}}$$

is the only non-trivial graded commutator, and where we have replaced the vector index on the
spacetime derivative by a pair of spinor indices. The superfield $W_{ijkl}$ is constrained to satisfy

$$D_{\alpha i} W_{jklm} = D_{\alpha [i} W_{jklm]}$$

$$\bar{D}_{\dot{\alpha}}^i W_{jklm} = \frac{4}{5} \delta_{ij} \bar{D}_{\dot{\alpha}}^k W_{klm|n}$$

$$W^{ijkl} = \frac{1}{4!} \varepsilon^{ijklmnopq} W_{mnopq}. \quad (2)$$

$^1E_7$ symmetry has also been invoked as a constraint in the context of light-cone superfields [17, 18]
The third of these is the $SU(8)$ self-duality condition; it implies that the first two are equivalent under complex conjugation. The differential constraints may be interpreted as stating that the supersymmetry variation of the scalars gives 56 spin one-half fields, i.e. the physical fields at this level. One can easily check that the remaining independent components of $W_{ijkl}$ are the field strengths of the fields listed above, for example, the graviton field strength is the (linearised) Weyl tensor, $C_{\alpha\beta\gamma\delta} \sim D_{\alpha i} \ldots D_{\delta l} W^{ijkl}$. All of the component fields obey the free-field equations of motion, which is necessary in order for the number of bosonic and fermionic degrees of freedom to match.

A third important feature of our analysis is the use of superconformal representation theory. At first sight this might seem strange since maximal supergravity is certainly not superconformal and we are interested in objects invariant under supersymmetry, not superconformal symmetry, so we briefly explain why we can do this.

Firstly, the component-field equations derived from (2) are all conformal, since all of the fields are massless (and free) while $W_{ijkl}$ itself transforms as a primary superfield under the superconformal group $SU(2,2|8)$.

Secondly, as in $N=1$ supersymmetry, any invariant, whether of D- or F-term type, comes from taking the top component of a supermultiplet and integrating it over space-time. Now in the $N=8$ case any such supermultiplet will be constructed from products of $W$s and (super)derivatives acting on $W$s and hence transforms in a well-defined manner under the superconformal group (which can be easily determined from the group’s action on $W_{ijkl}$). Any such supermultiplet can be decomposed into irreducible supermultiplets that transform under primary or descendant representations of the superconformal group. Since a descendant supermultiplet will have the same top component as the primary from which it is descended (or is a spacetime derivative in which case it can be ignored in an integral), it follows that we need only consider top components that are contained in primary representations.

Therefore, in order to classify the possible integral invariants, we only need to classify all of the possible primaries that can be constructed from $W_{ijkl}$ and determine which ones can contain suitable top components. The short invariants will come from short primary multiplets, or atypical superconformal representations, while the long invariants, D terms, correspond to typical representations which are also singlets under the Lorentz group and $SU(8)$, i.e. unconstrained scalar superfields. Reducing the problem of finding invariants to that of finding suitable superconformal representations is extremely useful since there is much known about the classification of superconformal representations which we can straightforwardly exploit. We review this now.

**Superconformal representations.**

The generators of the superconformal algebra of $SU(2,2|N)$ are $(L, R, K, M, N | Q, S)$, corresponding to dilations, $U(1)$ R-symmetry, conformal boosts, Lorentz transformations, $SU(N)$, supersymmetry and special supersymmetry respectively. Representations of $N$-extended superconformal symmetry in $D = 4$ are specified by $N+3$ quantum numbers $(L, R, J_1, J_2, a_1, \ldots a_{N-1})$, where $L$ is the dilation weight, $R$ is the R-charge (for these we use the same labels for both the charge and the generator hopefully without confusion), $J_1$ $J_2$ are the two spin quantum numbers, and the $a_i$s are the Dynkin labels of an irreducible internal $SU(N)$ representation [19]. The unitary representations have to satisfy certain unitarity bounds which can be one of three types:
Here $m = \sum k a_k$ is the total number of boxes in the Young tableau corresponding to the $SU(N)$ representation $(a_1, \ldots a_{N-1})$, and $m_1 = \sum a_k$ is the number of boxes in the first row. Representations in series B and C are always short, while there are some series A representations that are short, although they are not BPS.

These representations are all of highest weight type, and all the states in a given irreducible module are determined from the highest weight state by operating on it with the lowering operators. The highest weight, $\mathcal{O}$ say, is annihilated by all $K$ and $S$ operators, and we can ignore these from now on. In addition, the dilation operator just gives the dimension of each state which we need not bother with since we know this from the dimension of the highest weight. The momentum operator $P_a$ essentially generates the spacetime-dependence of the module; we can instead replace the highest weight state by a highest weight field depending on $x$, which we shall also denote by $\mathcal{O}$. The supersymmetry operators, $Q, \bar{Q}$ generate all of the components of the superfield associated with the module, while the internal symmetry operators generate the internal symmetry modules at each level in the supersymmetry generators. A top component of a multiplet is one that is annihilated by all $Q$s and $\bar{Q}$s up to a total derivative. Since we are interested in integrating top components over spacetime, we can, when operating on $\mathcal{O}$ with the supersymmetry operators, assume that they all anticommute, since any spacetime derivative terms will integrate to zero.

In order to understand the irreducible representations more easily we will construct any representation $\mathcal{O}$ out of three building-block representations (see [20, 21] for a similar approach)

$$\mathcal{O} = \mathcal{O}^{(1)} \mathcal{O}^{(C)} \mathcal{O}^{(2)}.$$  \hspace{1cm} (3)

$\mathcal{O}^{(1)}$ will provide the left-handed spin quantum number $J_1$, $\mathcal{O}^{(2)}$ the right-handed spin quantum number $J_2$ and $\mathcal{O}^{(C)}$ the $SU(N)$ quantum numbers. Any shortening conditions on $\mathcal{O}$ will be a consequence of the shortening conditions of the building block representations which we will give shortly. For series B representations one of either $\mathcal{O}^{(1)}$ or $\mathcal{O}^{(2)}$ will be trivial, i.e. 1, and for series C representations both $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$ will be trivial leaving $\mathcal{O} = \mathcal{O}^{(C)}$. The building blocks for any given irreducible representation are unique and all irreducible representations can be so constructed.

The building block $\mathcal{O}^{(C)}$, if non-trivial, has non-zero quantum numbers, $a_1 \ldots a_{N-1}$ together with $L = m_1$ and $R = 2m/N - m_1$. It corresponds to a series C representation. The highest weight state of such a representation is annihilated by the $su(N)$ raising operators $N_{ij}^j$, $j > i$. It is also annihilated by (both components of) $Q_r, r = 1 \ldots p$ and $\bar{Q}^{r'}, r' = (N - q) \ldots N$, where $a_p$ is the left-most non-zero $su(N)$ label and $a_{N-q}$ the right-most non-zero one. It is not difficult to see that these constraints are consistent with this state’s being annihilated by the $su(N)$ raising operators. Moreover, this set of $Q$s and $\bar{Q}$s is anticommutative so that no spacetime constraint is generated.

Series A : $L \geq 2 + 2J_2 - R + \frac{2m}{N}$, $L \geq 2 + 2J_1 + R + 2m_1 - \frac{2m}{N}$

Series B : $L = -R + \frac{2m}{N}$, $L \geq 1 + m_1 + J_1$, $J_2 = 0$

or : $L = R + 2m_1 - \frac{2m}{N}$, $L \geq 1 + m_1 + J_2$, $J_1 = 0$

Series C : $L = m_1$, $R = \frac{2m}{N} - m_1$, $J_1 = J_2 = 0$
The building block $\mathcal{O}^{(1)}$ has non-zero quantum numbers $J_1$ and $L = -R \geq 1 + J_1$. It is chiral, i.e. annihilated by all of the $Q$s, and if the bound on the dilation weight $L$ is saturated it obeys a further $Q$ constraint (a divergence constraint if $J_1 > 0$ or a second-order constraint if $J_1 = 0$). In this case $\mathcal{O}^{(1)}$ corresponds to an on-shell chiral massless multiplet. The building block $\mathcal{O}^{(2)}$ is the conjugate of an $\mathcal{O}^{(1)}$ representation for which the $Q$ and $\bar{Q}$ constraints are interchanged. All of this is summarised in the table.

| non-vanishing quantum numbers | $Q, \bar{Q}$ constraints |
|------------------------------|--------------------------|
| $\mathcal{O}^{(C)}$         | $L = m_1, \quad R = 2m/N - m_1, a_p, \ldots, a_{N-q}$, $Q^*_r \mathcal{O}^{(C)} = 0, \quad r = 1 \ldots p$ |
| $\mathcal{O}^{(1)}_{\text{long}}$ | $L = -R > 1 + J_1, \quad J_1$ | $\bar{Q}^i \mathcal{O}^{(1)} = 0$ |
| $\mathcal{O}^{(1)}_{\text{short}}$ | $L = -R = 1 + J_1, \quad J_1 \neq 0$ | $\bar{Q}^i \mathcal{O}^{(1)} = 0, \quad Q^*_r \mathcal{O}^{(1)} = 0$ |
| $\mathcal{O}^{(2)}_{\text{long}}$ | $L = R > 1 + J_2, \quad J_2$ | $Q_i \mathcal{O}^{(2)} = 0$ |
| $\mathcal{O}^{(2)}_{\text{short}}$ | $L = R = 1 + J_2, \quad J_2 \neq 0$ | $Q_i \mathcal{O}^{(2)} = 0, \quad (\bar{Q}^{(2)ij})^2 = 0$ |
| $\mathcal{O}^{(2)}_{\text{short}}$ | $L = R = 1, \quad J_2 \neq 0$ | $Q_i \mathcal{O}^{(2)} = 0, \quad (\bar{Q}^{(2)ij})^2 = 0$ |

Table 1: Summary of the quantum numbers (only those which may be non-zero) together with the shortening conditions of the building block representations, $\mathcal{O}^{(1)}$, $\mathcal{O}^{(C)}$ and $\mathcal{O}^{(2)}$. One can easily see from these quantum numbers that the quantum number of the representation $\mathcal{O} = \mathcal{O}^{(1)} \mathcal{O}^{(C)} \mathcal{O}^{(2)}$ will satisfy both series A bounds (3). Similarly $\mathcal{O} = \mathcal{O}^{(C)} \mathcal{O}^{(2)}$ and $\mathcal{O} = \mathcal{O}^{(1)} \mathcal{O}^{(C)}$ are series B representations and $\mathcal{O} = \mathcal{O}^{(C)}$ is a series C representation. $Q_{ij} := Q_i^a Q_{aj} = Q_{ji}^2$

Invariants from irreducible representations

In order to find invariants we need to find multiplets whose top components are both Lorentz scalars and $SU(N)$ singlets. In this section we will rule out certain classes of representations on the grounds that they cannot have scalar top components. Specifically, any representation which contains either a chiral or anti-chiral massless multiplet, $\mathcal{O}^{(1)}_{\text{short}}$ or $\mathcal{O}^{(2)}_{\text{short}}$, as one of its building blocks, can never have a scalar top component.

To prove this, consider first a representation constructed as $\mathcal{O} = \mathcal{O}^{(1)}_{\text{short}} \mathcal{O}^{(C)} \mathcal{O}^{(2)}$ where $\mathcal{O}^{(2)}$ can be short or long (or indeed trivial). The $Q$-constraints on this operator can be read off from the constraints on the building block operators from the table. If the representation is a scalar, so that $J_1 = 0$, the highest weight will satisfy a second-order constraint with respect to the subset of the $Q$s that annihilate the series C representation, $Q^2_{rs} \mathcal{O} = 0, \quad r, s = 1 \ldots p$. If the representation has non-zero spin on the other hand, the highest weight state will satisfy a divergence constraint with respect to the subset of $Q$s that are annihilated by the series C centre, $Q_r^a \mathcal{O}_{\alpha \beta \ldots} = 0, \quad r, s = 1 \ldots p$.

We can now prove very easily that there are no invariants that can be constructed from operators which have $\mathcal{O}^{(1)}_{\text{short}}$ as a building block (and by conjugation therefore the same is true for operators which have $\mathcal{O}^{(2)}_{\text{short}}$ as a building block). These correspond to series A and B representations which
saturate one or both of the unitarity bounds.

This assertion rests on the simple fact that the top components of such multiplets cannot be Lorentz scalars. To see this it is enough to consider the top $Q$-component (i.e. the component obtained by applying as many $Q$s as possible to the highest weight state) of $O^{(1)}_{\text{short}}O^{(C)}O^{(2)}$ where $O^{(2)}$ is kept general (although it is anti-chiral, i.e. annihilated by all the $Q$s). On considering the various constraints, it is clear that the top $Q$-component has left spin $J_1 + p/2$ and is explicitly given by

$$Q_1^{(\alpha_1} ... Q_p^{\alpha_p}[Q_{p+1} ... Q_N]^2 \left( O^{(1)}_{\alpha_{p+1} ... \alpha_{p+2J_1}} O^{(C)} O^{(2)} \right).$$

Since $p \geq 1$ it immediately follows that the top component of any multiplet involving $O^{(1)}_{\text{short}}$ can never be a Lorentz scalar. Similar conclusions regarding the top components of series A and B representations in the case of $\mathcal{N} = 4$ SYM can be read off from the results of [22].

We conclude that there can be no scalar top components of series A or B representations, saturating a unitary bound. The only remaining possibilities are therefore $O = O^{(1)}_{\text{long}}O^{(C)}O^{(2)}_{\text{long}}$ (i.e. long series A representations) $O = O^{(C)}O^{(2)}_{\text{long}}$ or $O = O^{(1)}_{\text{long}}O^{(C)}$ “long” series B representations, or $O = O^{(C)}$ series C representations.

The series C representations and long series A representations can give rise to invariants; the series C case was analysed in a number of different theories including $\mathcal{N} = 8$ in [1].

The simplest representations to consider from the point of view of integral invariants are the long multiplets, corresponding to series A reps with the unitary bound unsaturated. The highest weight field $O$ must be a Lorentz and $SU(N)$ scalar, satisfying no supersymmetry constraints, and then the top component is given by $[Q_1 ... Q_N \tilde{Q}^1 ... \tilde{Q}^N]^2O$ (each $Q$ and $\tilde{Q}$ is a two-component spinor). This is also a Lorentz and $SU(N)$ scalar, and hence defines a supersymmetric invariant integral. It may carry dilation and R weights, but this does not spoil things. There will be an infinite number of such long invariants and in the $\mathcal{N} = 8$ context they first appear at seven loops [4].

As a simple example of series C consider the multiplet in $N = 2$ with non-zero quantum numbers $L = 4, a_1 = 4$, i.e. the only non-zero super-Dynkin label is $n_3 = 4$. This corresponds to a one-half BPS scalar superfield $O_{ijkl}$ in the 5 of $SU(2)$. The highest weight field is $O_{1111}(x)$; it is annihilated by $Q_1$ and $\tilde{Q}^2$. The top component of this multiplet is therefore $[Q_2\tilde{Q}^1]^2O_{1111}$. This clearly determines a supersymmetric invariant since operating on this with either $Q_1$ or $\tilde{Q}^2$ is zero up to a spacetime derivative. Notice that this expression is invariant under $su(2)$, even though each factor has a particular numerical index. The raising operator $N_1^2$ annihilates $O_{1111}$, and commuted with $Q_2$ or $\tilde{Q}^1$ gives either $Q_1$ or $\tilde{Q}^2$ both of which also annihilate $O_{1111}$. Moreover, it has zero charge under the $u(1)$ sub-algebra of $su(2)$. Since it is a highest weight state under $su(2)$ and has charge zero, it must be a singlet and therefore annihilated by $N_2^3$. The fact that the top component is an internal singlet is important. There is an infinite number of one-half BPS supermultiplets, but only the one with $a_1 = 4$ gives rise to an $SU(2)$-invariant supersymmetric integral.

\[\text{Note that to keep things simple we have kept } O^{(2)} \text{ general and have ignored the action of } \tilde{Q}. \text{ Since dotted indices can not be contracted with undotted indices the top component has the same left spin as the top } Q\text{-component.}\]
\[ N = 8 \text{ invariants.} \]

Now let us focus on the theory of interest, \( N = 8 \) supergravity. First we consider the series C multiplets, for which the highest weight state is annihilated by some consistent subset of the supersymmetry generators. The field strength itself is one of these, with \( L = 1 \) and \( a_4 = 1 \) being the only non-zero quantum numbers. The highest weight is \( \mathcal{O}_{1234} = W_{1234}(\theta = 0) \). The shortest series C representations are one-half BPS for which the highest weight state is annihilated by half of the supersymmetry generators. One can see that this set has to consist of four \( Q \)s and four \( \bar{Q} \)s owing to the fact that the candidate BPS multiplets must be products of \( W \)s. Multiplets of this sort therefore have highest weight states of the form \( \mathcal{O} = (W_{1234})^p \) for some \( p \), and the top component of such a multiplet is \( [Q_5 \ldots Q_8 Q^1 \ldots Q^4]^2 \mathcal{O} \), for \( p \geq 4 \). There is clearly only one choice of \( p \), \( p = 4 \), for which this top component is a singlet. This is the three-loop \( R^4 \) invariant.

Our construction is rather similar to the original one [5], but makes it clearer that the integral is \( SU(8) \) invariant. One can investigate systematically all the other possibilities. This was done in [1] and we do not repeat it here. There are just two, \( d^4 R^4 \) and \( d^6 R^4 \), with highest weight states that are annihilated by two \( Q \)s and \( \bar{Q} \)s and one \( Q \) and one \( \bar{Q} \) respectively.

As an example of a vanishing theorem, we prove that there are no loop invariants. The only possibility is a highest weight state that is annihilated by \( Q_r, r = 1, 2, 3 \) and \( \bar{Q}^{s'}, s' = 6, 7, 8 \). The top component will therefore be \([Q_4 \ldots Q_8 Q^1 \ldots Q^4]^2 \mathcal{O} \). In order for this to be a singlet \( \mathcal{O} \) must be of the form \( \mathcal{O}_{1111222233334455} \) (corresponding to the \( \mathfrak{su}(8) \) Dynkin labels \( (0, 0, 2, 0, 2, 0, 0) \)). This would have to be \( (W_{1234})^2 (W_{1235})^2 \), but it is easy to see that this is not a highest weight state because it is not annihilated by \( N_4^5 \). This explicit example also makes it clear why the highest weight state for any putative BPS invariant can only have four fields. The state must be annihilated by at least one \( Q \), say \( Q_1 \), and at least one \( \bar{Q} \), \( \bar{Q}^8 \) say, and this means that \( \mathcal{O} \) must have exactly four 1s, together with various other numerical indices. The only way of achieving this is to have four \( W \)s each having one 1 index. (A factor of \( W \) without a 1 index is of course not annihilated by \( Q_1 \).)

We now turn to series A and B. As we have seen above, the only such representations that can give rise to integral invariants are “long” series B. These are partially chiral. A study of the possible multiplets of this type for \( N = 8 \) supergravity was made in [1]. It was found that there is only one possibility, at three points, that has the right quantum numbers to be a candidate partially chiral integrand. However, it turns out that this multiplet satisfies the series B unitarity bound and so satisfies a second-order constraint as well as being partially chiral. It therefore integrates to zero. There are examples of series B multiplets that do not satisfy a second-order constraint, for example, one can simply take the square of this three-point candidate, but none of these have the right quantum numbers to be integrands.

**The helicity structure of counterterms.**

A different proposal for classifying counterterms to that initiated in [1] and developed further here was put forward in [23]. It is based on a study of the corresponding amplitudes. Here, we explicitly connect the two approaches, show how the helicity structure of amplitudes is related to the particular superconformal operators we consider, and give a very simple explanation for a bound on the helicity structure of counterterms conjectured in reference [23].

Let us now consider the helicity structure of the amplitudes the possible counterterms can contribute to. Let us recall that the on-shell \( N = 8 \) supergravity multiplet is CPT self-conjugate.
and contains all particles from the negative helicity graviton (with helicity $-2$) to the positive helicity graviton (with helicity 2). Writing it as an on-shell superfield in light-cone momentum superspace we have

$$\Phi(\eta) = g^{++} + \eta^i \Gamma^i + \ldots + \eta^{i_1} \ldots \eta^{i_8} \epsilon^{i_1 \ldots i_8} \Gamma^8 + (\eta)^8 g^{--}. \quad (5)$$

Here $\eta^i$ is a Grassmann variable transforming in the fundamental representation of the $SU(8)$ R-symmetry group. We have included only the first two and last two terms in the superfield expansion for brevity. The expansion begins with the positive helicity graviton and gravitino and ends with the corresponding negative helicity states. The dots stand for all other on-shell states that appear between.

On-shell $N=8$ supersymmetry dictates that the sum of the helicities of the particles in a given amplitude must lie between $8 - 2n$ and $2n - 8$. An amplitude with total helicity $8 - 2n + 4k$ is called an $N_k$MHV amplitude and when $k = n - 4$ (its maximum value) the amplitude is often referred to as an $\overline{\text{MHV}}$ amplitude.

The three $F$ terms which arise as possible counterterms are all four-point invariants. As such the total helicity must be zero (and the amplitude is both MHV and $\overline{\text{MHV}}$). The pure-graviton amplitudes which these counterterms contribute to therefore involve two negative and two positive helicity gravitons. Let us split the on-shell curvature tensor into its chiral and anti-chiral Weyl curvatures, $R = (C, \bar{C})$, containing the negative helicity graviton and positive helicity graviton respectively,

$$C_{\alpha\beta\gamma\delta} = \lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta g^{++}, \quad \bar{C}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \bar{\lambda}^{\dot{\alpha}} \bar{\lambda}^{\dot{\beta}} \bar{\lambda}^{\dot{\gamma}} \bar{\lambda}^{\dot{\delta}} g^{--}. \quad (6)$$

Here we have introduced the spinor helicity variables describing the on-shell momentum of the particle $p^{\alpha\dot{\alpha}} = \lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$. The three $F$ terms thus have the pure-gravity structure, $C^2 \bar{C}^2$, $d^4 C \bar{C}$ and $d^6 C \bar{C}^2$, corresponding to the three-loop, five-loop and six-loop counterterms respectively.

When we consider long multiplets, many more counterterms can be constructed. The simplest suitable long multiplet is the one whose highest weight state is a product of four scalars. This four-point counterterm occurs at seven loops and has the pure gravity structure $d^8 C \bar{C}^2$. As a full superspace integral it can be written

$$\int d^{32} \theta W^4 = D^{16} \bar{D}^{16} W^4 = d^8 C \bar{C}^2 + \ldots \quad (7)$$

where the dots refer to terms involving fields other than the curvature. As it is a four-point counterterm it is again of MHV type (and $\overline{\text{MHV}}$ type).

Let us generalise this example to other long multiplets. We will consider multiplets whose top components can be written

$$D^p \bar{D}^q W^n = d^w C^u \bar{C}^v + \ldots \quad (8)$$

Here $n$ is the number of fields in the linearised counterterm. There must be at least 16 superderivatives of either chirality, $p \geq 16$, $q \geq 16$ as this is the top component of a long multiplet. Since the counterterm contains $v$ anti-chiral Weyl tensors $C$ it contributes to $N^k$MHV amplitudes for $k = v - 2$. From counting the dimensions we see that it arises at $l = \frac{w}{2} + u + v - 1$

\footnote{There is a simple relation between the light-cone chiral superfield and $W_{ijkl}$, see [26].}
loops. Since $d = D\tilde{D}$, $C = D^4W$ and $\tilde{C} = \tilde{D}^4W$ we can see that there is a simple relation between $(p, q, n)$ and $(w, u, v)$. Indeed we have

$$u = \frac{1}{8}(p - q + 4n), \quad (9)$$

$$v = \frac{1}{8}(q - p + 4n), \quad (10)$$

$$w = \frac{1}{2}(p + q - 4n). \quad (11)$$

Thus we find the loop order $l = \frac{1}{8}(p + q) - 1$ and the MHV degree $k = v - 2 = \frac{1}{8}(q - p + 4n) - 2$. Rearranging we find that

$$k = \frac{1}{2}(q/2 + n - 5 - l) \quad (12)$$

and since $q \geq 16$ we find a bound on the chirality of a given counterterm,

$$k \geq \frac{3 + n - l}{2}. \quad (13)$$

By parity there is an equivalent bound coming from $p \geq 16$. The bound (13) agrees with that conjectured in [23]. Thus at a given number of points and a given loop order a counterterm can only violate helicity by a certain amount as dictated by the above bound. Note that this analysis straightforwardly rules out the seven loop counterterms of the type $d^6R^5$ and $d^2R^7$ since for any helicity configuration they would require fewer than 16 superderivatives of one or other type. The absence of these counterterms was shown in [23] where explicit examples of non-vanishing MHV and NMHV matrix elements were also constructed corresponding to the long multiplets considered here. It is also simple to see that there will be a non-vanishing $N^2$MHV counterterm of the type $R^8$. In general, beyond seven loops, one would expect that counterterms of any pure gravity type can exist as long as the helicity structure respects the bound (13) and the corresponding parity conjugate bound.

We have not by any means given an exhaustive list of all possible long multiplets which give counterterms of a given MHV type at a given loop order. One could go on to count all possible ways such counterterms can be constructed, which amounts to counting all possible long multiplets with a given dimension, given total number of fields and a given chirality. Counting operators in superconformal theories can be done using partition functions and supercharacters. This was carried out for $\mathcal{N}=4$ super Yang-Mills in [24], and would be fairly straightforward to generalise to $\mathcal{N}=8$.

We emphasise that the above analysis holds for the linearised theory. In the full theory, full superspace integrals of terms involving arbitrary functions of the scalar fields are unlikely to be $E_7$ invariant.

**Concluding remarks**

In this note we have given a very simple discussion of the F terms that are allowed in $D = 4, N = 8$ supergravity based on the observation that the linearised field strength superfield is a superconformal primary field. The fact that there are only three such invariants was derived in our earlier paper [1] but we did not give the details of the impossibility of constructing invariants from non-BPS short primaries there. In [23], along with other results, the fact that there are only three counterterms below the full superspace threshold, i.e. at six loops or fewer, was derived by a completely different method making use of scattering amplitude techniques.
The argument given here is more general, as regards the F term issue, in that it shows that there are no independent non-Lorentz invariant superactions for any supersymmetric theory in $D = 4$ built from multiplets that are superconformal, a category that includes super Yang-Mills theories, linearised supergravity and massless Wess-Zumino and hypermultiplets.

This result does not necessarily hold in other dimensions. For example, consider (1, 1) supersymmetry in $D = 6$. There the left and right chiral spinors can be contracted so a series A type supermultiplet could in principle have a scalar top component.

We conclude by remarking on the significance of these counter terms for the ultra-violet properties of $N = 8$ supergravity. It has been known for some years that the theory is three-loop finite [8], a result that is in line with expectations from considerations in field theory [25, 26] and string theory [27]. More recently, it has been established that the theory is finite at four loops [9] in $D = 5$ where the relevant invariant is $d^5 R^4$. As we showed in [1] the $D = 4$ theory is finite at four loops, owing to the absence of an invariant. However, given the $D = 5$ result it would not be a surprise if the $D = 4$ theory turned out to be finite at five and six loops as well, even though there are linearised counterterms. One possible explanation for this could be, as we mentioned earlier, that the linearised counterterms do not admit duality invariant extensions in the full theory.

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