TRANSITIONS AND BIFURCATIONS OF DARCY-BRINKMAN-MARANGONI CONVECTION

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Abstract. This study examines dynamic transitions of Brinkman equation coupled with the thermal diffusion equation modeling the surface tension driven convection in porous media. First, we show that the equilibrium of the equation loses its linear stability if the Marangoni number is greater than a threshold, and the corresponding principle of exchange stability (PES) condition is then verified. Second, we establish the nonlinear transition theorems describing the possible transition types associated with the linear instability of the equilibrium by applying the center manifold theory to reduce the infinite dynamical system to a finite dimensional one together with several non-dimensional transition numbers. Finally, careful numerical computations are performed to determine the sign of these transition numbers as well as related transition types. Our result indicates that the system favors all three types of transitions. Unlike the buoyancy forces driven convections, jump and mixed type transition can occur at certain parameter regimes.

1. Introduction. The renowned Rayleigh-benard convection [2] has opened up a series of researches regarding the instability of a horizontal fluid heated from below, and the discovery of the role of surface tension in addition to the buoyancy forces in forming the instabilities has further enriched the mathematical formulation with the so-called Benard-Marangoni convection. It has drawn great attention among researchers, for example [4, 6], or more recently in [7]. For detailed reviews, see [3]. On the other hand, buoyancy driven convection in porous media, or the so-called Darcy-Benard convection has been studied extensively as well, see [14] for a thorough review. In current work, lesser investigated case of surface tension driven convection in porous media, which is important in various applications include material

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science processing, alloy solidification, or the so-called Darcy-Marangoni convection is researched using a Brinkman model.

Experimental evidence of the influence of surface tension on porous media has been observed for example in [16] and [27]. The seminal work of [9] did a thorough linear stability analysis on related models and observed that a Brinkman term is needed for convection instability to occur (though in [15] Nield proposed an alternative two-layer model to explain the onset of the convection). Using the Brinkman model, linear, asymptotic and perturbation analysis and numerical methods are employed in [20, 18, 26, 5, 19, 21] to treat related Marangoni convection phenomena in porous media. Similar analysis has been done with consideration of other various effects, see for example [25, 24, 1, 17].

These fruitful studies paint a rather miscellaneous picture of the convection phenomena related to Marangoni convection in porous media, however, rigorous study of the nonlinear effects, and so forth the precise reduced dynamics related to the onset of the convection is still lacking, in this article we extend the previous researches in this direction.

The main tool used in this article is the dynamic transition theory developed by Ma & Wang [12]. In their work, the onset of instability of a dissipative system is analyzed through a combination of topological and analytical methods, and apart from giving the precise nonlinear dynamics during the onset of instability, the transition involved is categorized into three scenarios: Continuous transition, jump transition and mixed transition. The continuous transition indicates a gradual bifurcation of the steady state into solutions nearby, wherever the jump transition indicates a more drastic change, while the mixed being the mixture of the above two transitions, for details and recent applications of dynamic transition theory see [11, 12, 13]. This classification is intuitively helpful in understanding the onset of instability. In our case, the onset of the Darcy-marangoni convection when the Marangoni number crosses the critical value is found to be of multiple types, including all three of the transition types.

In this article, we first establish the Principle of Exchange of Stabilities of the Darcy-Brinkman-Marangoni system through linear eigenvalue analysis, and the (principle) eigenvalues are found to be all real numerically which indicates a non-oscillating transition during the onset of the convection. Then we make use of the center manifold reduction technique to derive the reduced ODE for original systems for two important cases, the first case is when the critical eigenvalue is of multiplicity one, which is the generic case; the second being the multiplicity two case but with restricted domain size so that a hexagonal pattern may appear. This case might correspond to more observable patterns from experiments. We found the transition types can be both jump and continuous for the single eigenvalue case, which is quite different from the classical Benard convection which only allows continuous transitions [10, 22]; for the double eigenvalue case we found all three type of transitions can occur. During the calculation of the eigensystems and nonlinear reductions, we also devise a numerical scheme which is based on the weak form of the linear problem and techniques from spectral methods [23] to expedite the calculation involved.

The paper is organized as follows: the model and the necessary mathematical adjustments is given in section 2, linear stability analysis and principle of exchange of stabilities are carried out in section 3, in section 4 we reduce the original system into a set of ODEs near critical parameters, in section 5 we give a numerical scheme
to calculate the related transitions and list notable results, in section 6 we summarize
the conclusions and physical significance derived from previous calculations.

2. Mathematical formulation. For a liquid saturated porous media, its evolu-
tion is modeled by the Brinkman model coupled with the thermal diffusion equation
\[14\], given by
\[
\begin{aligned}
\rho_\text{c} a \frac{\partial \mathbf{v}}{\partial t} + \mu_{\text{eff}} \Delta \mathbf{v} - \nabla P - \rho_l g k &= \nabla \cdot (\rho c_m \mathbf{v}) + \rho_l c_l (\mathbf{v} \cdot \nabla) T = k_m \Delta T, \\
\nabla \cdot \mathbf{v} &= 0,
\end{aligned}
\]  
where \( \mathbf{v} = (u, v, w) \) is the seepage velocity, \( \mu_{\text{eff}} \) is the effective saturated porous
medium viscosity, \( P \) is the pressure, \( K \) is the permeability of the porous matrix, \( \rho_l \)
is the liquid density, \( c_a \) is the acceleration coefficient, and where \( \mu \) is the pure liquid
viscosity. Besides,
\[
\begin{aligned}
(\rho c)_m &= (1 - \phi) \rho_s c_s + \phi \rho_l c_l, \\
\kappa_m &= (1 - \phi) \kappa_s + \phi \kappa_l,
\end{aligned}
\]  
in which \( \rho_s \) is the solid density, \( c_s \) is the specific solid capacity, \( c_l \) is the pure liquid
capacity, \( \phi \) is porosity, and where \( \kappa_s \) and \( \kappa_l \) are the thermal conductivities of the
solid matrix and the pure liquid, respectively.

The saturated porous media of interest in present work is confined in the box
\[
\Omega_d = [0, L_x] \times [0, L_y] \times [0, H],
\]  
which then becomes the domain for Eq.(1). For mathematical convenience, we
assume the saturated porous media satisfies the following horizontal boundary conditions
\[
\begin{aligned}
u &= \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial T}{\partial x} = 0, \quad x = 0, L_x, \\
\frac{\partial u}{\partial y} &= v = \frac{\partial w}{\partial y} = \frac{\partial T}{\partial y} = 0, \quad y = 0, L_y.
\end{aligned}
\]  
The vertical boundary (\( z = 0, z = H \)) conditions are given as follows: We suppose
that the lower border of the porous material is an impermeable perfect thermal
conductor. Hence on the boundary \( z = 0 \),
\[
T = T_0 > 0, \quad u = v = w = 0, \quad z = 0,
\]  
which indicates the flow is nonslip and has the fixed temperature \( T_0 \). At the upper
surface, we assume that the gaseous phase is only a heat sink with respect to the
porous matrix and, thus, we have, introducing the heat transfer coefficient \( q \), the
boundary condition of the kind
\[
\begin{aligned}
v &= \kappa_m \frac{\partial T}{\partial z} + q_m T = 0, \quad z = H,
\end{aligned}
\]  
where \( q_s \) and \( q_l \) are the pure solid and pure liquid heat transfer coefficient re-
spectively. Note that \( q_m = 0 \) is the case where the gaseous phase is insulated.
Furthermore at \( z = H \), the Marangoni equation reads
\[
\begin{aligned}
\mu_{\text{eff}} \frac{\partial u}{\partial z} &= -\gamma_m \frac{\partial T}{\partial x}, \quad z = H, \\
\mu_{\text{eff}} \frac{\partial v}{\partial z} &= -\gamma_m \frac{\partial T}{\partial y}, \quad z = H,
\end{aligned}
\]


where \(-\gamma_m = \frac{\partial \sigma_m}{\partial T}\) in which \(\sigma_m\) is the mean surface tension.

The basic state without movement is quiescent, which is given by

\[
\begin{align*}
 u_s &= v_s = w_s = 0, & P_s &= -\rho g z + P_0, \\
 T_s &= T_0 - T_d z, & T_d &= \frac{T_0 q_m}{H q_m + \kappa_m}.
\end{align*}
\]

The perturbation equations for the basic state reads

\[
\begin{align*}
 \rho l c_a \frac{\partial v}{\partial t} + \mu K v &= \mu_{eff} \Delta v - \nabla P, \\
 \frac{(pc)_m}{\kappa_m} \frac{\partial T}{\partial t} + \rho c_l (v \cdot \nabla) T &= k_m \Delta T + \rho c_l T_d w, \\
 \nabla \cdot v &= 0.
\end{align*}
\]

For mathematical convenience, the following scalings are chosen:

\[
\begin{align*}
 (x, y, z) &= H(x', y', z'), & t &= \frac{(pc)_m H^2}{\kappa_m} t', \\
 v &= \frac{\kappa_m}{\rho c_l H} v', & P &= \frac{c_a \kappa_m^2}{c_l H^2} P', & T &= T_d T'.
\end{align*}
\]

Dropping the primes, we arrive at the dimensionless perturbation equations

\[
\begin{align*}
 \frac{\partial v}{\partial t} &= Da \lambda \Delta v - \nabla P - \frac{v}{\lambda}, \\
 \frac{\partial T}{\partial t} &= \Delta T + w - \frac{(v \cdot \nabla) T}{\lambda}, \\
 \nabla \cdot v &= 0.
\end{align*}
\]

in which

\[
\lambda = \frac{c_a \rho l c_m K}{H^2 \mu}, \quad Da = \frac{\mu_{eff} K}{H^2 \mu}.
\]

The dimensionless perturbation equations are defined on the box

\[
\Omega = (0, a) \times (0, b) \times (0, 1), \quad a = L_x / H, \quad b = L_y / H,
\]

and are subjected to the following horizontal boundary conditions

\[
\begin{align*}
 u &= \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial T}{\partial x} = 0, & x &= 0, a, \\
 \frac{\partial u}{\partial y} &= v = \frac{\partial w}{\partial y} = \frac{\partial T}{\partial y} = 0, & y &= 0, b,
\end{align*}
\]

and vertical boundary conditions

\[
\begin{align*}
 u &= \frac{\partial T}{\partial z} + Bi T = 0, & z &= 1, \\
 \frac{\partial u}{\partial z} + Ma \frac{\partial T}{\partial x} = \frac{\partial v}{\partial z} + Ma \frac{\partial T}{\partial y} = 0, & z &= 1, \\
 T &= 0, & u = v = w = 0, & z &= 0,
\end{align*}
\]

where

\[
\begin{align*}
 Bi &= \frac{q_m H}{\kappa_m}, \quad Ma = \frac{2 m \rho l c T_d H}{\mu_{eff} \kappa_m},
\end{align*}
\]

which are the Biot number and the Marangoni number, respectively.

To avoid changing domains for the linear operator as Ma changes, we use a slightly weaker formulation of problem (5). First we define the relevant function
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spaces:
\[ H_1 = \{ v \in H^1(\Omega)^2 : \nabla \cdot v = 0, \quad v \cdot n = 0, \quad v|_{z=0} = 0 \} , \]
\[ H_2 = \{ T \in H^1(\Omega) : T|_{z=0} = 0 \} , \quad H = H_1 \times H_2. \]

The norm \( \| \cdot \|_H \) is naturally defined by
\[ \| \cdot \|_H = \sqrt{\| \cdot \|^2_{H_1} + \| \cdot \|^2_{H_2}}. \]

Thus, we define the bilinear operator \( L_{Ma} \) on \( H \) as follows (\( \Gamma \) is the upper boundary):
\[ L_{Ma}(\psi, \psi') = -\frac{Da}{\lambda}(\nabla v, \nabla v') - \frac{1}{\lambda}(v, v') - (\nabla T, \nabla T') + (w, T') \]
\[ - Ma \int_{\Gamma} \nabla T \cdot v' \, dx \, dy - Bi \int_{\Gamma} TT' \, dx \, dy, \quad \psi = (v, T). \]

Lemma 2.1. The bilinear operator \( L_{Ma} \) defined above is bounded.

Proof. First, note that
\[ \int_{\Gamma} \nabla T \cdot v' \, dx \, dy = -\int_{\Gamma} T \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) \, dx \, dy = \int_{\Gamma} T \frac{\partial w'}{\partial z} \, dx \, dy \]
where we have used the divergence free condition \( \nabla \cdot v = 0 \) and the identity
\[ \int_{\partial \Omega \setminus \Gamma} T \frac{\partial v'}{\partial z} \cdot n \, dS = - \int_{\partial \Omega \setminus \Gamma} \frac{\partial T}{\partial z} v' \cdot n \, dS, \]
due to the boundary condition \( v' \cdot n = 0 \). Hence, there exists a positive constant \( C \) only depending on \( Da, Bi, Ma \) and \( \lambda \) such that
\[ |L_{Ma}(\psi, \psi')| \leq C \| \psi \|_H \| \psi' \|_H. \]

The boundedness of the bilinear operator \( L_{Ma} \) and the Riesz representation theorem imply that there exists a bounded linear operator \( L_{Ma} : H \to H^* \) such that
\[ (L_{Ma} \psi, \psi') = L_{Ma}(\psi, \psi'), \quad \forall \psi \in H. \]  

Define the nonlinear operator \( G \) on \( H \times H \) as follows:
\[ G(\psi, \psi') = \int_{\Omega} (v \cdot \nabla) T \cdot T' \, dx \, dy \, dz. \]

Apparently, for each given \( \psi \in H \), \( G(\psi, \cdot) \) is an operator on \( H \). It is well know that there exists a positive constant \( C_0 \) independent of \( \psi \) and \( \psi' \) such that
\[ |G(\psi, \psi')| \leq C_0 \| (v \cdot \nabla) T \|_{L^1/2} \| T' \|_{L^2} \leq C_0 \| (v \cdot \nabla) T \|_{L^1/2} \| \psi' \|_{L^2}, \]
by which and the Riesz representation theorem we infer that there exists a continuos nonlinear operator \( G : H \to H^* \) such that
\[ (G(\psi), \psi') = G(\psi, \psi'). \]

We now consider the following evolution equation defined on \( H \)
\[ \frac{d\psi}{dt} = L_{Ma} \psi + G(\psi), \quad \psi|_{t=0} = \psi_0 \in H. \]
We call the Eq. (10) the weak form of the system (5) together with boundary conditions (6) and (7). Notice when $\psi$ is regular enough, one can derive that

$$G(\psi) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (v \cdot \nabla)T \end{pmatrix}.$$ 

Occasionally we use the following bilinear representation for $G$:

$$G(\psi_1, \psi_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (v_1 \cdot \nabla)T_2 \end{pmatrix},$$

where $\psi_1 = (v_1, T_1)$ and $\psi_2 = (v_2, T_2)$.

3. Linear stability analysis. In this section we aim to establish the principle of exchange of stabilities for the system (10) as parameter $Ma$ grows.

3.1. Eigenvalue problems. It is not tough to show that the spectrum set of $L_{Ma}$ only consists of eigenvalues. First, we consider the following eigenvalue problem

$$L_{Ma}\psi = \sigma(Ma)\psi, \quad \psi \in H,$$

which is equivalent to the eigenvalue problem

$$\begin{cases} 
\sigma v + \chi = \frac{Da}{\chi} \Delta v - \nabla P; \\
\sigma T = \Delta T + w; \\
\nabla \cdot v = 0,
\end{cases}$$ (12)

subjected to the boundary conditions (6) and (7).

To solve the eigenvalue problem (12), we can use the separation of variables to represent the solutions in the following form:

$$\begin{align*}
u(x, y, z) &= U(z) \sin a_m x \cos b_n y, \\
v(x, y, z) &= V(z) \cos a_m x \sin b_n y, \\
w(x, y, z) &= W(z) \cos a_m x \cos b_n y, \\
T(x, y, z) &= \Theta(z) \cos a_m x \cos b_n y,
\end{align*}$$ (13)

for $(m, n) \in \mathbb{N} \times \mathbb{N}$.

If $(m, n) = (0, 0)$, after some simple calculations,

$$U = V = W = 0, \quad \Theta(z) = \sin \rho z,$$

where the corresponding eigenvalue $\sigma$ is given by

$$\sigma = -\rho^2,$$

in which $\rho > 0$ and solves

$$\rho + Bi \tan \rho = 0.$$

There are a countable number of $\rho$’s that satisfy the above equation, and we denote the corresponding $\sigma$’s as

$$0 > \sigma_{(0, 0), 1} > \sigma_{(0, 0), 2} > \sigma_{(0, 0), 3} \cdots \to -\infty,$$

evidently these modes are always stable.
If \((m, n) \neq (0, 0)\), inserting the expressions (24) into the equation (12), one can derive that

\[
\begin{align*}
\sigma (D^2 - \alpha^2) W + \frac{1}{\lambda} (D^2 - \alpha^2) W &= \frac{Da}{\lambda} (D^2 - \alpha^2)^2 W, \\
\sigma \Theta &= (D^2 - \alpha^2) \Theta + W, \\
\alpha := \alpha(m, n) &= \sqrt{a^2_m + b^2_n},
\end{align*}
\]

equipped with the following boundary conditions

\[
\begin{align*}
W &= D \Theta + Bi \Theta = 0, \quad z = 1, \\
D^2 W + Ma^2 \Theta &= 0, \quad z = 1, \\
\Theta &= 0, \quad DW = W = 0, \quad z = 0.
\end{align*}
\]

(15) will yield a sequence of \(\sigma\)’s with non-zero \((W, \Theta)\)’s as solutions, along with the trivial zero solution.

For \(U(z)\) and \(V(z)\), they solve the following equations

\[
\begin{align*}
\sigma(a_m V - b_n U) + \frac{1}{\lambda} (a_m V - b_n U) &= \frac{Da}{\lambda} (D^2 - \alpha^2)(a_m V - b_n U), \\
a_m U + b_n V &= -DW,
\end{align*}
\]

which are subjected to the boundary conditions

\[
\begin{align*}
DU &= a_m Ma \Theta, \quad DV = b_n Ma \Theta, \quad z = 1, \\
U &= V, \quad DW = W = 0, \quad z = 0.
\end{align*}
\]

Note that for the nonzero \(W\)’s determined from the system (15), the Eq. (17) has unique solutions which must satisfy the condition

\[-DW^2 = a_m DU + b_n DV = M a^2_{m,n} \Theta, \quad z = 1.\]

Thus, it is not too tough to see that \(U\) and \(V\) are given by

\[
\begin{align*}
U(z) &= -\frac{a_m}{\alpha^2} DW, \quad V(z) = -\frac{b_n}{\alpha^2} DW.
\end{align*}
\]

If we choose \(W = \Theta \equiv 0\) in (15), we will obtain modes with horizontal rotation, namely, the modes where \(a_m U + b_n V = 0\) and both \(U\) and \(V\) solves the following (with \(F = U\) or \(V\)):

\[
\sigma F + \frac{F}{\lambda} = \frac{Da}{\lambda} (D^2 - \alpha^2) F \text{ while } F = 0 \text{ at } z = 0, \quad DF = 0 \text{ at } z = 1.
\]

It’s not hard to see these rotation modes are stable.

It’s evident then that for each \(I = (m, n)\) with \(m, n \in \{0, 1, 2, \cdots\}\), the eigenvalue problem (15) has infinitely many eigenvalues (it can be shown numerically that they are all real) which can be ordered as:

\[
\sigma_{1,1} \geq \sigma_{1,2} \geq \sigma_{1,3} \cdots \to -\infty,
\]

and the corresponding eigenvectors \(\psi_{1,i}\) can be written as

\[
\psi_{(m,n),i} = \begin{pmatrix}
-\frac{a_m}{\alpha^2} DW_{(m,n),i}(z) \sin a_m x \cos b_n y \\
-\frac{b_n}{\alpha^2} DW_{(m,n),i}(z) \cos a_m x \sin b_n y \\
W_{(m,n),i}(z) \cos a_m x \cos b_n y \\
\Theta_{(m,n),i}(z) \cos a_m x \cos b_n y
\end{pmatrix}
\]

(20)
or

\[
\psi_{(m,n),i} = \begin{pmatrix}
    b_n F_{\alpha,i}(z) \sin a_m x \cos b_n y \\
    -a_m F_{\alpha,i}(z) \cos a_m x \sin b_n y \\
    0 \\
    0
\end{pmatrix},
\]  

(21)

with \(W_{(m,n),i}(z) = 0\) when \((m, n) = (0, 0)\).

3.2. Dual eigenvalue problem. The adjoint eigenvalue problem associated with (12) reads

\[
\begin{aligned}
\sigma v^* + \nabla^* \lambda^* &= D \Delta v^* - \nabla P^* + T^* k, \\
\sigma T^* &= \Delta T^*, \\
\nabla \cdot v^* &= 0,
\end{aligned}
\]  

(22)

subjected to the horizontal boundary conditions

\[
\begin{aligned}
u^* &= \frac{\partial v^*}{\partial x} = \frac{\partial w^*}{\partial x} = \frac{\partial T^*}{\partial x} = 0, \quad x = 0, a, \\
\frac{\partial u^*}{\partial y} &= v^* = \frac{\partial w^*}{\partial y} = \frac{\partial T^*}{\partial y} = 0, \quad y = 0, b,
\end{aligned}
\]  

(23a, b)

and the following vertical boundary condition

\[
\begin{aligned}
\frac{\partial u^*}{\partial z} &= \frac{\partial v^*}{\partial z} = w^* = 0, \quad z = 1, \\
\frac{\partial T^*}{\partial z} + Bi T^* + \lambda^* - 1 Da Ma \frac{\partial w^*}{\partial z} &= 0, \quad z = 1, \\
T^* &= 0, \quad u^* = v^* = w^* = 0, \quad z = 0.
\end{aligned}
\]  

(24)

Again the eigenvectors can be written as follows with the help of separation of variables:

\[
\begin{aligned}
u(x, y, z) &= U^*(z) \sin a_m x \cos b_n y, \\
v(x, y, z) &= V^*(z) \cos a_m x \sin b_n y, \\
w(x, y, z) &= W^*(z) \cos a_m x \cos b_n y, \\
T(x, y, z) &= \Theta^*(z) \cos a_m x \cos b_n y, \\
a_m &= \frac{m\pi}{a}, \quad b_n = \frac{n\pi}{b}.
\end{aligned}
\]  

(25)

One can thus derive that \((W^*, \Theta^*)\) solves the system of ODEs

\[
\begin{aligned}
\pi (D^2 - \alpha^2) W^* + \frac{1}{3} (D^2 - \alpha^2) W^* &= \frac{Da}{\lambda} (D^2 - \alpha^2)^2 W^* - \alpha^2 \Theta^*, \\
\pi \Theta^* &= (D^2 - \alpha^2) \Theta^*,
\end{aligned}
\]  

(26)

with the boundary conditions

\[
\begin{aligned}
W^* &= D^2 W^* = 0, \quad z = 1, \\
D \Theta^* + Bi \Theta^* + \lambda^* - 1 Da Ma D W^* &= 0, \quad z = 1, \\
\Theta^* &= 0, \quad D W^* = W^* = 0, \quad z = 0.
\end{aligned}
\]  

(27a, b, c)

Totally similar results holds, namely (14) and (20) with variables replaced by conjugates by adding stars.
3.3. **Critical eigenvalue problem.** In the previous section, we have shown that all eigenvalues of the system (12) are determined by the system of ODEs (15)-(16c). To conduct the stability analysis of the basic state and determine the corresponding dynamic transition, the critical eigenvalues and the critical eigenvectors are needed to be derived. By the result of numerical testing, we assume the the eigenvalues are all real.

The critical eigenvalue problem then reads

\[
\begin{cases}
    D^4W - (2\alpha^2 + 1)D^2W + (\alpha^4 + \alpha^2)W = 0, \\
    (D^2 - \alpha^2)\Theta + W = 0,
\end{cases}
\]

(28)

with the boundary conditions (16a)-(16c). Denote

\[ \beta := \beta(m, n) = \sqrt{\alpha^2 + \frac{1}{Da}}, \]

it is not tough to see that the general solution to the equation

\[
\begin{cases}
    D^4W - (2\alpha^2 + 1/\text{Da})D^2W + (\alpha^4 + \alpha^2/\text{Da})W = 0, \\
    W = 0, \quad z = 1; \quad DW = W = 0, \quad z = 0,
\end{cases}
\]

(29)

is given by

\[ W(z) = A\gamma (\beta \sinh(\alpha z) - \alpha \sinh(\beta z)) - A (\cosh(\alpha z) - \cosh(\beta z)), \]

(30)

in which

\[ \gamma = \frac{\cosh(\alpha) - \cosh(\beta)}{\beta \sinh(\alpha) - \alpha \sinh(\beta)}. \]

(31)

For the inhomogeneous equation

\[
\begin{cases}
    (D^2 - \alpha^2)\Theta = W, \\
    \Theta = 0, \quad z = 0,
\end{cases}
\]

(32)

based on the exact expression (30), it has the special solution of form

\[ \Theta(z) = -\frac{A}{2\alpha} z \sinh(\alpha z) - ADa \cosh(\alpha z) + \frac{A\gamma\beta}{2\alpha} z \cosh(\alpha z) - A\gamma\alpha Da \sinh(\beta z) + ADa \cosh(\beta z). \]

(33)

For the homogeneous problem

\[
\begin{cases}
    (D^2 - \alpha^2)\Theta' = 0, \\
    \Theta' = 0, \quad z = 0,
\end{cases}
\]

(34)

its general solution is given by

\[ \Theta'(z) = AB \sinh(\alpha z). \]

Then, we obtain that the solutions of the problem (28) take the form of

\[ W(z) = -A\gamma (\beta \sinh(\alpha z) - \alpha \sinh(\beta z)) + A (\cosh(\alpha z) - \cosh(\beta z)), \]

(35)

\[ \Theta(z) = AB \sinh(\alpha z) - ADa \cosh(\alpha z) - ADa\gamma\alpha \sinh(\beta z) + ADa \cosh(\beta z) - \frac{A}{2\alpha} z \sinh(\alpha z) + \frac{A\gamma\beta}{2\alpha} z \cosh(\alpha z). \]

(36)
Additionally, note that $W$ and $\Theta$ must satisfy the compatibility condition
\begin{align}
\frac{\partial \Theta}{\partial z} + B \Theta &= 0, \quad z = 1, \\
D^2 W + \alpha^2 \Theta &= 0, \quad z = 1,
\end{align}
(37a)
(37b)
hence for $M_a$, we have
\begin{equation}
M_a = -\frac{D^2 W(1)}{\alpha^2 \Theta(1)}, \quad B = \frac{G(\alpha, \Bi)}{\Bi \sinh(\alpha) + \alpha \cosh(\alpha)},
\end{equation}
(38)
in which
\begin{equation}
G(\alpha, \Bi) = \left( \alpha Da + \frac{\gamma \beta}{2} - \frac{\Bi}{2 \alpha} \right) \sinh(\alpha) \\
+ \left( \frac{1}{2} - \frac{\gamma \beta}{2 \alpha} + \Bi Da - \frac{\Bi \gamma \beta}{2 \alpha} \right) \cosh(\alpha) \\
+ (\Bi Da \gamma \alpha - \beta Da) \sinh(\beta) \\
+ (Da \gamma \alpha \beta - \Bi Da) \cosh(\beta).
\end{equation}
(39)

Let us denote $I = (m, n)$ for each $m, n \in \{0, 1, 2, \ldots\}$. Then, we define the function $M$ as
\begin{equation}
M(I) = \frac{F(\alpha)}{\alpha^2 H(\alpha)}, \quad \alpha := \alpha(I) = \sqrt{a_m^2 + b_n^2},
\end{equation}
where
\begin{align}
F(\alpha) &= \gamma \beta \sinh(\alpha) - \cosh(\alpha) - \gamma \alpha \sinh(\beta) + \cosh(\beta), \\
H(\alpha) &= \frac{1}{2} \sinh(\alpha) + \frac{\gamma \beta}{2 \alpha} \cosh(\alpha).
\end{align}
(40)
(41)

Then we can denote the critical Marangoni number as
\begin{equation}
M_a^* = \min_{\substack{I = (m, n) \quad M(I) \quad \alpha(I) = \sqrt{a_m^2 + b_n^2}, \\
1 \leq a \leq 5 \quad 1 \leq b \leq 5 \quad Da = 0.1 \quad Bi = 2 \quad (10) \quad \text{we plot } M_a^* \quad \text{as a function of } a \quad \text{and } b \quad \text{in Fig. 1, we see that } M_a^* \quad \text{increases when } Bi \quad \text{grows.} \\
\text{For illustration purpose, we plot } I_c = (m_c, n_c) \quad \text{as a function of } a \quad \text{in } [3, 4] \quad \text{and } b \quad \text{in } [3, 4] \quad \text{for specified } Bi = 2 \quad \text{and } Da = 0.1, \quad \text{shown in Fig. 2.} \\
\text{In generic case there exists a unique } I_c = (m_c, n_c) \quad \text{such that } M_a^* = M(I_c). \quad \text{However note that for } (a, b) \quad \text{on each black solid line of Fig. 2, there are at least two different } I_c = (m_c, n_c) \quad \text{at which } M_a^* = M(I_c), \quad \text{i.e., there are at least two different eigenvalues become critical simultaneously (the real parts of which become zeros at } Ma = Ma^*). \\
\text{Similar analysis can be done on dual critical eigenvalue problems. We omit the details here.}
\end{equation}

3.4. Principle of exchange stabilities. Based on the preceding analysis, at $Ma = Ma^*$, there exists an eigenvalue 0 for the problem (12). For each $I = (m, n)$ with $m, n \in \{0, 1, 2, \ldots\}$, the eigenvalue problem (15) has infinitely many eigenvalues which can be ordered as:
\begin{equation}
\sigma_{1,1} \geq \sigma_{1,2} \geq \sigma_{1,3} \ldots \rightarrow -\infty.
\end{equation}
(43)
Figure 1. Plot of critical porous Marangoni number $Ma^*$ as a function of $a \in [3, 5]$ and $b \in [3, 5]$ with $Bi = 2$ (left) and $Bi = 10$ (right) and $Da = 0.1$.

Figure 2. Plot of $(m_c, n_c)$ as a function of $a \in [3, 4]$ and $b \in [3, 4]$ such that $Ma^* = \mathcal{M}(m_c, n_c)$, where $Bi = 2$ and $Da = 0.1$.

We denote the corresponding eigenvectors by $(v_{1,k}, T_{1,k})$. Let $\mathcal{C}$ be the index set defined by

$$\mathcal{C} = \{I|\mathcal{M}(I) = Ma^*\}.$$  \hspace{1cm} (44)

One can verify that $\mathcal{C}$ is not empty and finite. Then, we have the following lemma:

**Lemma 3.1.** With the assumption all eigenvalues of system (12) are real, denote the eigenvalues as $\sigma_j$ as above, we have the following principle of exchange of
stabilities:

\[
\sigma_{1,1}(Ma) \begin{cases} < 0, & Ma < Ma^* \\ = 0, & Ma = Ma^* , \ I \in C, \\ > 0, & Ma > Ma^* \end{cases}, \quad (45)
\]

\[
\sigma_{1,k}(Ma^*) < 0, \quad I \notin C. \quad (46)
\]

**Proof.** First, we show the inequality (46) by contradiction. If \( Ma = 0 \), with the help of the first equation of (15), we have

\[
- \sigma \int_0^1 |DW|^2 \, dz - \sigma \alpha^2 \int_0^1 |W|^2 \, dz
= \frac{Da}{\lambda} \int_0^1 |D^2W|^2 \, dz + \frac{2Da\alpha^2 + 1}{\lambda} \int_0^1 |DW|^2 \, dz
+ \frac{Da\alpha^4 + \alpha^2}{\lambda} \int_0^1 |W|^2 \, dz > 0,
\]

which infers that all \( \sigma \) are negative at \( Ma = 0 \). If there exists \( I \in C \) such that \( \sigma_{1,k}(Ma^*) > 0 \). Then, the continuous dependence of \( \sigma_{1,k} \) on \( Ma \) yields that there exists \( Ma_c < Ma^* \) such that \( \sigma_{1,k}(Ma_c) = 0 \), which contradicts (42).

To show (45), we only need to show

\[
\frac{d\sigma_{1,1}}{dMa} \bigg|_{Ma=Ma^*} > 0, \quad I \in C.
\]

Let \( Ma = Ma^* + \delta \) with \( 0 < |\delta| \ll 1 \). Let the eigenvector \((v_{1,1}, T_{1,1})\) at \( Ma \) be denoted by

\[
v_{1,1} = v_c + v_\delta, \quad T_{1,1} = T_c + T_\delta,
\]

where

\[
\lim_{\delta \to 0} v_\delta = 0, \quad \lim_{\delta \to 0} T_\delta = 0,
\]

and \((v_c, T_c)\) be the eigenvector corresponding to \( \sigma_j \) at \( Ma = Ma^* \). That is,\n
\[
0 = - \int_\Omega |v_c|^2 \, dxdydz - Da \int_\Omega |\nabla v_c|^2 \, dxdydz - Ma^* \int_\Gamma \nabla v_c \cdot v_e \, dxdy,
\]

\[
0 = - \int_\Omega |\nabla T_c|^2 \, dxdydz - \int_\Omega w_c T_c \, dxdydz - Bi \int_\Gamma T^2 \, dxdy,
\]

which are derived from (12) by setting \( \sigma = 0 \). Direct calculations give

\[
\sigma(Ma^* + \delta)((v_{1,1}, v_c) + (T_{1,1}, T_c)) - \sigma(Ma^*)((v_c, v_c) + (T_c, T_c))
= - \frac{1}{\lambda}(v_{1,1}, v_c) - \frac{Da}{\lambda}(\nabla v_{1,1}, \nabla v_c) - (\nabla T_{1,1}, T_c) + (w, T)
\]

\[
- Ma \int_\Gamma \nabla T_{1,1} \cdot v_c \, dxdy - Bi \int_\Gamma T_{1,1} T_c \, dxdy
+ \frac{1}{\lambda}(v_c, v_c) + \frac{Da}{\lambda}(\nabla v_c, \nabla v_c) + (\nabla T_c, T_c) - (w_c, T)
\]

\[
+ Ma^* \int_\Gamma \nabla T_c \cdot v_c \, dxdy + Bi \int_\Gamma T_c T_c \, dxdy
= - \frac{1}{\lambda}(v_\delta, v_c) - \frac{Da}{\lambda}(\nabla v_\delta, \nabla v_c) - (\nabla T_\delta, T_c) + (w_\delta, T)
\]

\[
- Ma^* \int_\Gamma \nabla T_\delta \cdot v_c \, dxdy - Bi \int_\Gamma T_\delta T_c \, dxdy
\]
\[-\delta \int_{\Gamma} \nabla T_{\delta} \cdot \mathbf{v}_c \, dxdy - \delta \int_{\Gamma} \nabla T_{c} \cdot \mathbf{v}_c \, dxdy \]

\[= -\delta \int_{\Gamma} \nabla T_{\delta} \cdot \mathbf{v}_c \, dxdy - \int_{\Gamma} \nabla T_{c} \cdot \mathbf{v}_c \, dxdy.\]

where we have used

\[0 = -\frac{1}{\lambda}(\mathbf{v}_{\delta}, \mathbf{v}_c) - \frac{Da}{\lambda} (\nabla \mathbf{v}_{\delta}, \nabla \mathbf{v}_c) - (\nabla T_{\delta}, T_c) - (w_{\delta}, T)\]

\[-Ma^* \int_{\Gamma} \nabla T_{\delta} \cdot \mathbf{v}_c \, dxdy - Bi \int_{\Gamma} T_{\delta} T_c \, dxdy.\]

Further calculations shows

\[\sigma'_{I_1}(Ma^*) = \lim_{\delta \to 0} \frac{\sigma(Ma^* + \delta)}{\delta}\]

\[= -\lim_{\delta \to 0} \frac{\int_{\Gamma} \nabla T_{\delta} \cdot \mathbf{v}_c \, dxdy + \int_{\Gamma} \nabla T_{c} \cdot \mathbf{v}_c \, dxdy}{((\mathbf{v}_j, \mathbf{v}_c) + (T_j, T_c))}.\]

Finally, we find that

\[\sigma'_{I_1}(Ma^*) > 0 \iff -\int_{\Gamma} \nabla T_{c} \cdot \mathbf{v}_c \, dxdy = \int_{\Omega} |\mathbf{v}_c|^2 \, dxdydz + Da \int_{\Omega} |\nabla \mathbf{v}_c|^2 \, dxdydz > 0,\]

where the identity (47) have been utilized.

4. **Nonlinear transition.** We study dynamic transitions of the system in two different scenarios. The first is the case where the critical eigenvalue is simple. That is, the critical index set \(C_i\) has only one element \(I_{c} = (m_{c}, n_{c})\). The second is where the critical eigenvalue is double, and the index set \(C_i\) has two different elements \(I_{i} = (m_{i}, n_{i})\) with \(i = 1, 2\), they are specifically restricted so that a hexagonal pattern may occur during transition.

4.1. **Transitions with multiplicity one.** We denote the eigenvector corresponding to \(\sigma_{I_{1},1}\) at \(Ma = Ma^*\) by

\[\psi_{I_{1},1} = (\mathbf{v}_{I_{1},1}, T_{I_{1},1}), \quad \mathbf{v}_{I_{1},1} = (u_{I_{1},1}, v_{I_{1},1}, w_{I_{1},1}),\]

and denote \(E_u\) as the first unstable space, which is given by

\[E_u = \{\zeta \psi_{I_{1},1} | \zeta \in \mathbb{R}\}.\]

Let \(E_s\) be the standard complement of \(E_u\), i.e., \(E_s = \{\psi | (\psi, \psi^*_{I_{1},1}) = 0\}.\) Then, then the center-unstable invariant manifold function \(\Phi\) associated with the PES condition (45) is a sufficiently smooth mapping from \(E_u \to E_s\) with the property

\[\Phi(0) = 0, \quad D\Phi(0) = 0.\]

The property allows us to turn to the standard approximation procedure of the center-unstable invariant manifold function \(\Phi\). Since the non-linear interactions \((\mathbf{v} \cdot \nabla) T\) at hand are quadratic, the leading order approximation of \(\Phi\) is the bilinear form of

\[\Phi_2 = \frac{1}{2} D^2 \Phi(0)(\psi, \psi), \quad \psi \in E_u.\]

With the approximation formula for \(\Phi_2\), see, [12]:

\[\Phi_2(\zeta \psi_{I_{1},1}, \zeta \psi_{I_{1},1}) = -L^{-1} G(\zeta \psi_{I_{1},1}, \zeta \psi_{I_{1},1}) + o(2), \quad (48)\]
we can obtain a reduced equation which has equivalent dynamics with (10) by projecting it into the first eigenspace as the center manifold is exponentially attracting. Denote \( P \) as the standard projection onto \( E_u \) and the projection of any orbit in the center-unstable manifold of (10) onto \( E_u \) can be written as:

\[
\frac{d\zeta}{dt} = \sigma_{L,1} \zeta + P_1 G(\zeta \psi_{L,1} + \Phi_2(\zeta \psi_{L,1}, \zeta \psi_{L,1}) + o(2)),
\]

which is equivalent to

\[
\frac{d\zeta}{dt} = \sigma_{L,1} \zeta + \frac{\langle G(\zeta \psi_{L,1} + \Phi_2(\zeta \psi_{L,1}, \zeta \psi_{L,1}) + o(2)), \psi_{L,1}^* \rangle}{\langle \psi_{L,1}, \psi_{L,1}^* \rangle}.
\]

(49)

we will see later that (49) can be written as

\[
\frac{d\zeta}{dt} = \sigma_{L,1} \zeta + r \zeta^3 + o(3).
\]

In the following we simplify the notation to describe the non-horizontal-rotational functions with horizontal wave number \( m \) and \( n \):

\[
\psi^{(m,n)}(z) = \begin{pmatrix} W(z) \\ \Theta(z) \end{pmatrix},
\]

(50)

which means simply

\[
\begin{pmatrix}
-\frac{a_m}{\sigma^2} DW(z) \sin a_m x \cos b_n y \\
-\frac{b_n}{\sigma^2} DW(z) \cos a_m x \sin b_n y \\
W(z) \cos a_m x \cos b_n y \\
\Theta(z) \cos a_m x \cos b_n y
\end{pmatrix}.
\]

Then through (42) we can find the critical wave number \( m_c \) and \( n_c \) and the first eigenvector \( (W_{L,1} \Theta_{L,1}) \) solved explicitly in Section 3.3. Then similarly the dual of the first eigenvector \( (W_{L,1} \theta_{L,1}) \) can be solved. Then

\[
\langle \psi_{L,1}, \psi_{L,1}^* \rangle = \begin{pmatrix}
W_{L,1} \\
\Theta_{L,1}
\end{pmatrix} \begin{pmatrix}
W_{L,1}^* \\
\Theta_{L,1}^*
\end{pmatrix}_L
\]

\[
= \begin{cases}
\frac{ab}{2} \int_1^{a_m} \frac{1}{\alpha^2} DW DW^* + WW^* + \Theta \Theta^* dz & \text{if one of } a_m \text{ and } b_n \text{ is } 0,
\frac{ab}{2} \int_0^{1} \int_0^{b_n} DW DW^* + WW^* + \Theta \Theta^* dz & \text{if none of } a_m \text{ and } b_n \text{ is } 0.
\end{cases}
\]

Now notice the nonlinear terms have no horizontal rotational modes, hence it can be written using functions represented by (50). Then the action of \( G \) can be written as:

\[
G(\psi^{(m,n)}) = -\begin{pmatrix}
0 \\
\frac{1}{4} [DW \Theta + WD \Theta]\end{pmatrix}_{(0,0)}
\]

\[
-\begin{pmatrix}
0 \\
\frac{1}{4} \left[ \frac{a_m^2 - b_n^2}{\alpha^2} DW \Theta + WD \Theta \right]\end{pmatrix}_{(0,2b_n)}
\]

\[
-\begin{pmatrix}
0 \\
\frac{1}{4} \left[ b_n^2 - \frac{a_m^2}{\alpha^2} DW \Theta + WD \Theta \right]\end{pmatrix}_{(2a_m,0)}
\]

\[
-\begin{pmatrix}
0 \\
\frac{1}{4} [ -DW \Theta + WD \Theta]\end{pmatrix}_{(2a_m,2b_n)}.
\]

Notice some of these terms may combine when one of \( a_m \) and \( b_n \) is 0. We classify this into three situations:
Case 1. when both \( m_c \) and \( n_c \) are not zero.

Denote \( \psi_{1,1} \) as \( \Psi^{L,1} = (\Theta_{1,1})_{L} \), then (using formula (48) and the exclusion of the rotation modes) representing \( \Phi_2(\zeta \psi_{1,1}, \zeta \psi_{1,1}) \) as

\[
\zeta^2 \left( \Psi_1^{(0,0)} + \Psi_2^{(0, 2b_c)} + \Psi_3^{(2a_c, 0)} + \Psi_4^{(2a_c, 2b_c)} \right) = \zeta^2 \left[ \begin{pmatrix} W_1 \\ \Theta_1 \end{pmatrix}_{(0,0)} + \begin{pmatrix} W_2 \\ \Theta_2 \end{pmatrix}_{(0, 2b_c)} + \begin{pmatrix} W_3 \\ \Theta_3 \end{pmatrix}_{(2a_c, 0)} + \begin{pmatrix} W_4 \\ \Theta_4 \end{pmatrix}_{(2a_c, 2b_c)} \right],
\]

then each of this four vectors will satisfy an ODE as follows:

\[
\begin{cases}
W_1 &= 0, \\
4D^2\Theta_1 &= DW_{1,1}\Theta_{1,1} + W_{1,1}D\Theta_{1,1}, \\
\Theta_1 &= 0 \text{ at } z = 0, Bi\Theta_1 + D\Theta_1 = 0 \text{ at } z = 1.
\end{cases}
\]

(51)

\[
\begin{cases}
(D^2 - 4b_{c}^2)W_2 &= Da(D^2 - 4a_{c}^2)^2 W_2, \\
(D^2 - 4b_{c}^2)\Theta_2 + W_2 &= \frac{1}{4} \left[ -DW_{1,1}\Theta_{1,1} + W_{1,1}D\Theta_{1,1} \right], \\
\Theta_2 &= DW_2 = W_2 = 0 \text{ at } z = 0, \\
Bi\Theta_2 + D\Theta_2 &= D^2W_2 = 0 \text{ at } z = 1.
\end{cases}
\]

(52)

\[
\begin{cases}
(D^2 - 4a_{c}^2)W_3 &= Da(D^2 - 4a_{c}^2)^2 W_3, \\
(D^2 - 4a_{c}^2)\Theta_3 + W_3 &= \frac{1}{4} \left[ -DW_{1,1}\Theta_{1,1} + W_{1,1}D\Theta_{1,1} \right], \\
\Theta_3 &= DW_3 = W_3 = 0 \text{ at } z = 0, \\
Bi\Theta_3 + D\Theta_3 &= D^2W_3 = 0 \text{ at } z = 1.
\end{cases}
\]

(53)

\[
\begin{cases}
(D^2 - 4a_{c}^2 - 4b_{c}^2)W_4 &= Da(D^2 - 4a_{c}^2 - 4b_{c}^2)^2 W_4, \\
(D^2 - 4a_{c}^2 - 4b_{c}^2)\Theta_4 + W_4 &= \frac{1}{4} \left[ -DW_{1,1}\Theta_{1,1} + W_{1,1}D\Theta_{1,1} \right], \\
\Theta_4 &= DW_4 = W_4 = 0 \text{ at } z = 0, \\
Bi\Theta_4 + D\Theta_4 &= D^2W_4 = 0 \text{ at } z = 1.
\end{cases}
\]

(54)

Once we obtain the approximation center manifold, we can plug it in (49). Notice in the expansion of \( G(\zeta \psi_{1,1} + \Phi_2(\zeta \psi_{1,1}, \zeta \psi_{1,1})) \), the only terms that are nontrivial when taking inner product with \( \psi_{1,1} \) are the following:

\[
r_1 = \langle G(\Psi^{L,1}, \Psi_1), \Psi^{L,1} \rangle = -\frac{ab}{4} \int_0^1 D\Theta_1 W_{1,1} \Theta_{1,1}^*, \]

(55)

\[
r_2 = \langle G(\Psi^{L,1}, \Psi_2), \Psi^{L,1} \rangle = \frac{ab}{4} \int_0^1 \left( \frac{b_{c}^2}{\alpha^2} \Theta_2 DW_{1,1} + \frac{1}{2} D\Theta_2 W_{1,1} \right) \Theta_{1,1}^*,
\]

(56)

\[
r_3 = \langle G(\Psi^{L,1}, \Psi_3), \Psi^{L,1} \rangle = -\frac{ab}{4} \int_0^1 \left( \frac{a_{c}^2}{\alpha^2} \Theta_3 DW_{1,1} + \frac{1}{2} D\Theta_3 W_{1,1} \right) \Theta_{1,1}^*,
\]

(57)

\[
r_4 = \langle G(\Psi^{L,1}, \Psi_4), \Psi^{L,1} \rangle
= -\frac{ab}{4} \int_0^1 \left( \frac{1}{2} \Theta_4 DW_{1,1} + \frac{1}{4} D\Theta_4 W_{1,1} \right) \Theta_{1,1}^*,
\]

(58)
and

\[ r_5 = \langle G(\Psi_1, \Psi^{L_1,1}), \Psi^{* L_1,1} \rangle = 0, \]
\[ r_6 = \langle G(\Psi_2, \Psi^{L_1,1}), \Psi^{* L_1,1} \rangle = -\frac{ab}{4} \int_0^1 \left( \frac{b^2}{\alpha^2} DW_2 \Theta_{L_1,1} + \frac{1}{2} W_2 D \Theta_{L_1,1} \right) \Theta^*_L, \]
\[ r_7 = \langle G(\Psi_3, \Psi^{L_1,1}), \Psi^{* L_1,1} \rangle = -\frac{ab}{4} \int_0^1 \left( \frac{a^2}{\alpha^2} DW_3 \Theta_{L_1,1} + \frac{1}{2} W_3 D \Theta_{L_1,1} \right) \Theta^*_L, \]
\[ r_8 = \langle G(\Psi_4, \Psi^{L_1,1}), \Psi^{* L_1,1} \rangle = -\frac{ab}{4} \int_0^1 \left( \frac{1}{2} DW_4 \Theta_{L_1,1} + \frac{1}{4} W_4 D \Theta_{L_1,1} \right) \Theta^*_L. \]

Then \( r \) can be calculated by

\[ r = \frac{\sum_{i=1}^{8} r_i}{\langle \Psi_{L,1}, \Psi^*_{L,1} \rangle}. \]

**Case 2.** when \( m_c = 0 \) \( (n_c = 0 \) will be totally similar), representing \( \Phi_2(\zeta \psi_{L,1}, \zeta \psi_{L,1}) \) as

\[ \zeta^2 (\Psi_{01}^{(0,0)} + \Psi_{02}^{(0,2b)} - \Psi_{02}^{(0,2b) - \Psi_{02}^{(0,2b)}}) = \zeta^2 \left[ \left( W_{01} \right)_{(0,0)} + \left( W_{02} \right)_{(0,2b)} \right], \]

then each of this two vectors will satisfy an ODE as follows:

\[ \begin{cases} W_{01} \equiv 0, \\
2D^2 \Theta_{01} = DW_{L_1,1} \Theta_{L_1,1} + W_{L_1,1} D \Theta_{L_1,1}, \\
\Theta_{01} = 0 \text{ at } z = 0, B_{0} \Theta_{01} + D \Theta_{01} = 0 \text{ at } z = 1.
\end{cases} \]

\[ \begin{cases} (D^2 - 4b_2^2)W_{02} = Da(D^2 - 4b_2^2)W_{02}, \\
(D^2 - 4b_2^2)\Theta_{02} + W_{02} = \frac{1}{2} \left[ -DW_{L_1,1} \Theta_{L_1,1} + W_{L_1,1} D \Theta_{L_1,1} \right], \\
\Theta_{02} = DW_{02} = W_{02} = 0 \text{ at } z = 0, \\
B_{0} \Theta_{02} + D \Theta_{02} = D^2 W_{02} + 4Mab_2^2 \Theta_{02} = W_{02} = 0 \text{ at } z = 1.
\end{cases} \]

Then again we investigate the four terms inside \( \langle G(\zeta \psi_{L,1} + \Phi_2(\zeta \psi_{L,1}, \zeta \psi_{L,1})), \Psi^*_{L,1} \rangle \) that might possibly be none-trivial.

\[ r_{01} = \langle G(\Psi^{L_1,1}, \Psi_{01}), \Psi^*_{L,1} \rangle = -\frac{ab}{2} \int_0^1 D \Theta_{01} \Theta^*_{L,1}, \]
\[ r_{02} = \langle G(\Psi^{L_1,1}, \Psi_{02}), \Psi^*_{L,1} \rangle = -\frac{ab}{2} \int_0^1 \left( \Theta_{02} DW_{L_1,1} + \frac{1}{2} W_{02} D \Theta_{L_1,1} \right) \Theta^*_{L,1}, \]
\[ r_{03} = \langle G(\Psi_{01}, \Psi^{L_1,1}), \Psi^{* L_1,1} \rangle = 0, \]
\[ r_{04} = \langle G(\Psi_{02}, \Psi^{L_1,1}), \Psi^{* L_1,1} \rangle = -\frac{ab}{2} \int_0^1 \left( DW_{02} \Theta_{L_1,1} + \frac{1}{4} W_{02} D \Theta_{L_1,1} \right) \Theta^*_{L,1}. \]
Then \( r \) can be calculated by
\[
r = \left( \sum_{i=1}^{4} r_{0i} \right) / \langle \psi_{1,1}^r, \psi_{1,1}^s \rangle. \tag{68}
\]

Using the dynamic transition theorem developed in [12] we obtain the following result:

**Theorem 4.1.** As parameter \( Ma \) crosses \( Ma^* \) from left, if \( r(Ma^*) > 0 \), the system (10) undergoes a jump type transition. If \( r(Ma^*) < 0 \) the system undergoes a continuous type transition, the steady state will bifurcate into two stable solutions approximated by
\[
\pm \sqrt{-\frac{\sigma_{1,1}}{r}} \psi_{1,1} + o(|\sigma_{1,1}|^{1/2}). \tag{69}
\]

4.2. **Transitions with multiplicity two.** This case will be of great importance when the underlying region is regular, for example square shaped in the horizontal direction, or when \( a, b \) are of certain ratios as the following shows.

For simplicity, we consider only the case where there are possible hexagonal patterns, that is, when
\[
\frac{a}{b} = \frac{m}{n\sqrt{3}}. \tag{70}
\]

In this case the modes with wave numbers \( \mathbf{l} = (m_c, n_c) \) and \( \mathbf{l}_d = (0, 2n_c) \) will have the same \( \alpha \) hence they will be associated with the critical eigenvectors at the same time when appropriate parameters are chosen.

If this is the case then the reduced equation can be written as
\[
\frac{d\zeta_1}{dt} = \sigma_{1,1} \zeta_1 + P_{01} G(\zeta_1 \psi_{1,1} + \zeta_2 \psi_{1,2}) + \Phi_2(\zeta_1 \psi_{1,1} + \zeta_2 \psi_{1,2}, \zeta_1 \psi_{1,1} + \zeta_2 \psi_{1,2}) + o(2),
\]
\[
\frac{d\zeta_2}{dt} = \sigma_{1,1} \zeta_2 + P_{02} G(\zeta_1 \psi_{1,1} + \zeta_2 \psi_{1,2}) + \Phi_2(\zeta_1 \psi_{1,1} + \zeta_2 \psi_{1,2}, \zeta_1 \psi_{1,1} + \zeta_2 \psi_{1,2}) + o(2). \tag{71}
\]

Then the action of \( G \) can be written as (using the fact that \( W_c = W_d, \Theta_c = \Theta_d \)):
\[
G(\zeta_1 \psi_{1,1} + \zeta_2 \psi_{1,2}) = -\left( \zeta_1^2 + 2\zeta_2^2 \right) \left( \begin{array}{c}
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c] \\
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c]
\end{array} \right)_{(0,0)} - \zeta_1^2 \left( \begin{array}{c}
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c] \\
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c]
\end{array} \right)_{(0,2n_c)} - \zeta_1^2 \left( \begin{array}{c}
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c] \\
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c]
\end{array} \right)_{(2m_c,0)} - \zeta_1^2 \left( \begin{array}{c}
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c] \\
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c]
\end{array} \right)_{(2m_c,2n_c)} - \zeta_2^2 \left( \begin{array}{c}
\frac{1}{2} [DW_c \Theta_c + W_c \Theta_c] \\
\frac{1}{2} [DW_c \Theta_c + W_c \Theta_c]
\end{array} \right)_{(0,4n_c)} - \zeta_1 \zeta_2 \left( \begin{array}{c}
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c] \\
\frac{1}{4} [DW_c \Theta_c + W_c \Theta_c]
\end{array} \right)_{(m_c,n_c)} - \zeta_1 \zeta_2 \left( \begin{array}{c}
\frac{1}{2} [DW_c \Theta_c + W_c \Theta_c] \\
\frac{1}{2} [DW_c \Theta_c + W_c \Theta_c]
\end{array} \right)_{(m_c,3n_c)}.
\]

Then if we write
\[
\Phi_2(\zeta_1 \psi_{1,1} + \zeta_2 \psi_{1,2}) = (\zeta_1^2 + 2\zeta_2^2) \psi_{1,1}^{(0,0)} + \zeta_2^2 \psi_{1,2}^{(2m_c,0)},
\]
we obtain the following equations
\[
\begin{cases}
W_1 \equiv 0, \\
4D^2 \Theta_1 = DW_c \Theta_c + W_c D \Theta_c, \\
\Theta_1 = 0 \text{ at } z = 0, \text{Bi} \Theta_1 + D \Theta_1 = 0 \text{ at } z = 1.
\end{cases}
\] (73)

\[
\begin{cases}
(D^2 - 12b_c^2)W_3 = \text{Da}(D^2 - 12b_c^2)^2W_3, \\
(D^2 - 12b_c^2)\Theta_3 + W_3 = \frac{1}{4} \left[- \frac{1}{2} DW_c \Theta_c + W_c D \Theta_c \right], \\
\Theta_3 = DW_3 = W_3 = 0 \text{ at } z = 0, \\
\text{Bi} \Theta_3 + D \Theta_3 = D^2 W_3 + 12 \text{Ma} b_c^2 \Theta_3 = W_3 = 0 \text{ at } z = 1.
\end{cases}
\] (74)

\[
\begin{cases}
(D^2 - 16b_c^2)W_4 = \text{Da}(D^2 - 16b_c^2)^2W_4, \\
(D^2 - 16b_c^2)\Theta_4 + W_4 = \frac{1}{4} \left[- DW_c \Theta_c + W_c D \Theta_c \right], \\
\Theta_4 = DW_4 = W_4 = 0 \text{ at } z = 0, \\
\text{Bi} \Theta_4 + D \Theta_4 = D^2 W_4 + 16 \text{Ma} b_c^2 \Theta_4 = W_4 = 0 \text{ at } z = 1.
\end{cases}
\] (75)

Using similar calculation as in the simple eigenvalue case, we can see the reduced equation can then be written as:
\[
\begin{cases}
\frac{dc_1}{dt} = \sigma_{l,1}c_1 + r_1c_1c_2 + c_1(r_2c_2^2 + r_3c_3^2) + o(3), \\
\frac{dc_2}{dt} = \sigma_{l,1}c_2 + s_1c_1^2 + c_2(s_2c_2^2 + s_3c_3^2) + o(3),
\end{cases}
\] (78)

where
\[
r_1 = 4s_1, \\
s_1 = - \frac{\int_0^1 \left( \frac{1}{2} DW_c \Theta_c + \frac{1}{2} W_c D \Theta_c \right) \Theta_c^* \frac{d\xi}{W_c^*} + W_c W_c^* + \Theta_c \Theta_c^*}{\int_0^1 \frac{d\xi}{W_c^*} DW_c \Theta_c^* + W_c W_c^* + \Theta_c \Theta_c^*},
\]
\[
r_3 = 2s_3, \\
s_3 = - \frac{1}{\int_0^1 \frac{d\xi}{W_c^*} DW_c \Theta_c^* + W_c W_c^* + \Theta_c \Theta_c^*} \\
\int_0^1 \left\{ W_c (D \Theta_3 + D \Theta_1) + \frac{3}{2} DW_c \Theta_3 + \frac{1}{2} \Theta_c DW_3 + D \Theta_c W_3 \right\} \Theta_c^*,
\]
\[
4r_2 = r_3 + s_2, \\
s_2 = - \frac{1}{\int_0^1 \frac{d\xi}{W_c^*} DW_c \Theta_c^* + W_c W_c^* + \Theta_c \Theta_c^*} \\
\int_0^1 \left\{ W_c (2D \Theta_4 + D \Theta_4) + 2 DW_c \Theta_4 + \frac{1}{2} \Theta_c DW_4 + D \Theta_c W_4 \right\} \Theta_c^*.
\]

Rather surprisingly, the relationship of the coefficients of reduced equations obtained above is the same as those obtained in clear fluid marangoni convection case as in [4][7], despite the obvious difference of the nonlinear terms between our model and theirs. Hence we conclude with a similar dynamic transition theorem as in [7]:
**Theorem 4.2.** Assume that the relation (70) holds and the critical index set is
\[ \{ I_c = (m_c, n_c), I_d = (0, 2n_c) \}. \]
Then assume \( r_1 > 0 \) (with an obvious substitution \( \zeta_2 \rightarrow -\zeta_2 \) the case of \( r_1 < 0 \) can be characterized under current case) and
\[
R_{Ma}^i = -\sqrt{\frac{\sigma_{1,1}}{-s_2}} \psi_{I_d,1} + O(\sigma_{1,1}),
\]
\[
H_{Ma}^i = \frac{\sigma_{1,1}}{r_1} (2(-1)^i, -1) + O(\sigma_{1,1}^2).
\]

1. If \( s_2 < 0 \) then the system (10) undergoes a mixed transition at \( Ma = Ma^* \). More precisely, there is a neighborhood \( U \) of \( \psi = 0 \) in \( H \) such that for \( Ma^* < Ma < Ma^* + \epsilon \) with some \( \epsilon > 0 \), \( U \) can be decomposed into two open sets \( U_{Ma}^1 \cup U_{Ma}^2 \) such that
\[
\lim_{Ma \rightarrow Ma^*} \limsup_{t \rightarrow \infty} ||S_{Ma}(t, \psi)||_H = 0 \quad \forall \psi \in U_{Ma}^1,
\]
\[
\limsup_{t \rightarrow \infty} ||S_{Ma}(t, \psi)||_H \geq \delta > 0 \quad \psi \in U_{Ma}^2,
\]
for some \( \delta > 0 \). Here \( S_{Ma} \) denotes the evolution operator associated with system (10) with parameter \( Ma \). The system bifurcates in \( U_{Ma}^1 \) to an attractor \( \Sigma_{Ma} \) which consists of three steady states \( R_{Ma}^i, H_{Ma}^1, H_{Ma}^2 \) and heteroclinic orbits connecting \( H_{Ma}^i \) to \( R_{Ma}^i \) and \( H_{Ma}^2 \) to \( R_{Ma}^i \).

2. If \( s_2 > 0 \) then the system undergoes a jump type transition at \( Ma = Ma^* \).

**Remark 1.** We will have the same topological structure of the transition as in Fig. 3 and 4 in [7].

5. **Numerical investigations.** For the eigenvalue problem (15) subject to the boundary conditions (16a)-(16c), although we have derived the exact expressions of critical eigenfunctions in section 3, it is too complicated to calculate the exact values of these coefficients of the reduced equations derived in section 4. To determine the specific transition types involved the Marangoni instability, we have to count on the numerical approach to compute values of these coefficients. To this end, we introduce a numerical approach solving the eigenvalue problem (15) with the boundary conditions (16a)-(16c).

5.1. **Numerical approach solving weak form eigenvalue problem.** It is not tough to verify if \((W, \Theta)\) is an eigenfunction for the eigenvalue problem (15), then \((W, \Theta) \in H \) solve the following equation
\[
-\sigma \left< (DW, DW') + \alpha^2 (W, W') \right> - \sigma \left< \Theta, \Theta' \right>
= \frac{Da}{\lambda} \left( (D^2W, D^2W') + 2\alpha^2(DW, DW') + \alpha^4(W, W') \right)
+ \frac{1}{\lambda} \left< (DW, D\Theta') + \alpha^2 (W, \Theta') \right> + \frac{DaMa^2}{\lambda} \Theta(1)DW'(1)
+ \left< D\Theta, D\Theta' \right> + \alpha^2 (\Theta, \Theta') - \left< W, \Theta' \right> + Bi(1)\Theta'(1)
\]
for each \( (W', \Theta') \in H \) with \( H = Y \times X \) and

\[
X = \{ u \in H^1(0,1) | u(0) = 0 \}, \\
Y = \{ u \in H^2(0,1) | u'(0) = u(0) = u(1) = 0 \}. \
\]

(82)

Define the following bilinear operators: \( F_1, F_2, F_3 : H \times H \rightarrow \mathbb{R} \) as follows

\[
F_1(\Psi, \Psi') = \frac{Da}{\lambda} \left( \langle D^2W, D^2W' \rangle + 2\alpha^2 \langle DW, DW' \rangle + \alpha^4 \langle W, W' \rangle \right)
+ \frac{1}{\lambda} \left( \langle DW, DW' \rangle + \alpha^2 \langle W, W' \rangle \right)
+ \langle D\Theta, D\Theta' \rangle + \alpha^2 \langle \Theta, \Theta' \rangle
- \langle W, \Theta' \rangle + Bi\Theta(1)\Theta'(1),
\]

(83)

\[
F_2(\Psi, \Psi') = \frac{Da}{\lambda} \alpha^2 \Theta(1)DW'(1), \quad \Psi = (W, \Theta),
\]

\[
F_3(\Psi, \Psi') = \langle DW, DW' \rangle + \alpha^2 \langle W, W' \rangle + \langle \Theta, \Theta' \rangle.
\]

(84)

It follows from the Riesz theorem that there exist bounded linear operators \( A, B \) and \( C \) such that

\[
F_1(\Psi, \Psi') = (A\Psi, \Psi'_H),
\]

\[
F_2(\Psi, \Psi') = (B\Psi, \Psi'_H),
\]

\[
F_3(\Psi, \Psi') = (C\Psi, \Psi'_H).
\]

Hence, the equation (81) can be rewritten as

\[
-\sigma C\Psi = L_{Ma}\Psi,
\]

(85)

where

\[
L_{Ma} = A + MaB.
\]

Given a family of Legendre polynomials \( \{P_n\}_{n=0}^\infty \) and a sufficiently large positive integer \( N \), our aim is to approximate the \( j \)th component of our target function \( \Psi = (W, \Theta) \) solving the equation (81) by using a basis \( \{\phi_n^j\}_{n=0}^{N-1} \) of the form

\[
\phi_n^1 = P_n + \sum_{k=1}^3 c_{n,k}^1 P_{n+k}, \quad n = 0, \ldots, N-3,
\]

\[
\phi_n^2 = P_n + c_{n,1}^2 P_{n+1}, \quad n = 0, \ldots, N-1,
\]

(86)

where, for each \( (n, j) \), the coefficients \( \{c_{n,k}^j\} \) need to be chosen so that \( (\phi_n^1, \phi_n^2) \) \( \in H \).

Performing some calculations, we find that

\[
c_{n,1}^1 = \frac{2n + 3}{2n + 5}, \quad c_{n,2}^1 = -1, \quad c_{n,3}^1 = \frac{2n + 3}{2n + 5}, \quad n = 0, \ldots, N-3,
\]

\[
c_{n,1}^2 = 1, \quad n = 0, \ldots, N-1.
\]

We let

\[
L(\alpha) = \begin{pmatrix} L_{11}^{12} & L_{12}^{12} \\ L_{21}^{12} & L_{22}^{12} \end{pmatrix}, \quad R(\alpha) = \begin{pmatrix} R_{11}^{12} & R_{12}^{12} \\ R_{21}^{12} & R_{22}^{12} \end{pmatrix}.
\]

where

\[
L_{ji}^{11} = \langle (D\phi_i^1, D\phi_j^1) + \alpha^2 (\phi_i^1, \phi_j^1) \rangle,
\]

\[
L_{ji}^{12} = 0, \quad L_{ji}^{21} = 0, \quad L_{ji}^{22} = (\phi_i^2, \phi_j^2),
\]

(86)
Let \( p \) be given by

\[
P = (a_1, \ldots, a_{N-3}, a_0, \ldots, a_{N-1})^T,
\]

then the eigenvalue problem \( (85) \) is then can be numerically solved by considering the following equation

\[
-\sigma L(\alpha)p = R(\alpha)p.
\]

To compute the coefficients of reduced equations, we need to solve the following equation

\[
-\sigma C\Psi = L_{Ma}\Psi + F(\Psi_{1,1})
\]

(90)
to get \((W_i, \Theta_i)\) which solve (15), where \( F(\Psi_{1,1}) \) are determined by the nonlinear part of the equation (10) and the first eigenfunction \( \Psi_{1,1} \) given in Section 3.4. Similarly, the problem (90) can be numerically solved by considering the following equation

\[
-\sigma L(\alpha)p = R(\alpha)p + f,
\]

where

\[
f = (f_0^1, \ldots, f_{N-3}^1, f_0^2, \ldots, f_{N-1}^2)^T,
\]

and

\[
F_1(\Psi_{1,1}^N) = \sum_{j=0}^{N-3} f_j^1 \phi_j^1, \quad F_2(\Psi_{1,1}^N) = \sum_{j=0}^{N-1} f_j^2 \phi_j^2,
\]

\[
\Psi_{1,1}^N = (W^N, \Theta^N), \quad W^N = \sum_{j=0}^{N-3} a_j^1 \phi_j^1, \quad \Theta^N = \sum_{j=0}^{N-1} a_j^2 \phi_j^2.
\]

5.2. Numerical investigation. Relying on the numerical approach introduced in the preceding section, in what follows, we shall give some detailed numerical examples to illustrate our results. To this end, we take \( N = 80 \), where \( N \) is the degree \( N \) of Legendre polynomials. First, we compare numerical estimates on the critical Marangoni number and the exact values given by the formula (42) to show that the numerical approach works well, shown in Table 1.

We have shown that if the first eigenvalue is simple and \( m_c \neq 0, n_c \neq 0 \), then the transition is determined by the sign of a parameter \( r \) given by (63). For specified \( a, b, Da \) and \( \lambda \), numerical values of \( r \) can be derived, which are small, whose signs are shown in the Table 2. From the Table 2 one can see that both continuous and jump transition could occur for the Marangoni convection in porous media, which is quite different from the classical Benard convection.

If the first eigenvalue is simple and \( m_c = 0, n_c \neq 0 \), then the transition is determined by the sign of a parameter \( r \) given by (68). For specified \( a, b, Da \) and \( \lambda \), numerical values of \( r \) can be derived, which are shown in the Table 3. From the Table 3 one can see that both continuous and jump transition could occur for the Marangoni convection in porous medium, which is also different from the classical Benard convection which only allows continuous transition.
| $a, b$ | Exact $Ma^*$ | Numerical prediction | Relative error |
|-------|-------------|---------------------|---------------|
| $a = 3, b = 3$ | 209.82420647 | 209.82420611 | $3.6 \times 10^{-7}$ |
| $a = 3.2, b = 3.2$ | 210.25798681 | 210.25798320 | $3.6 \times 10^{-6}$ |
| $a = 3.4, b = 3.4$ | 208.54692216 | 208.54692182 | $3.4 \times 10^{-7}$ |
| $a = 3.6, b = 3.6$ | 208.51417969 | 208.51417509 | $4.6 \times 10^{-6}$ |

Table 1. Comparison between exact values of Marangoni number and numerical predictions, where $Bi = 2$, $Da = 0.1$, and $\lambda = 1000$.

| $a, b$ | $(m_{c}, n_{c})$ | sign($r$) |
|-------|-----------------|------------|
| $a = 3.3, b = 3.3$ | (3, 1) | 1 |
| $a = 3.35, b = 3.3$ | (3, 1) | 1 |
| $a = 3.4, b = 3.3$ | (1, 3) | $-1$ |
| $a = 3.45, b = 3.3$ | (1, 3) | $-1$ |

Table 2. Numerical predictions of the sign of the transition number $r$, where $Bi = 2$, $Da = 0.1$, and $\lambda = 1000$.

| $a, b$ | $(m_{c}, n_{c})$ | sign($r_1$) | sign($s_2$) |
|-------|-----------------|-------------|------------|
| $a = 1.43300244, b = 2.47683688$ | (1, 1), (0, 2) | 1 | 1 |
| $a = 1.43300244, b = 4.95367570$ | (1, 2), (0, 4) | 1 | 1 |

Table 3. Numerical predictions of the sign of the transition numbers $r_1$ and $s_2$, where $Bi = 2$, $Da = 0.1$, and $\lambda = 1000$.

For the non-general cases $a/b = m/(n_c\sqrt{3})$, the modes with wave numbers $I_c = (m_{c}, n_{c})$ and $I_d = (0, 2n_{c})$ will have the same $\alpha$ hence they will be associated with the critical eigenvectors at the same time when appropriate parameters are chosen, see the Table 4. In this case, to determine the transition types, we need to numerically estimate $r_1 - r_3$ and $s_1 - s_3$. For specified $Da = 0.1$, $Bi = 2$ (10) and $\lambda = 1000$, we find that when $a, b$ satisfy the following condition,

$$\frac{4m_{c}^{2}}{3a^{2}} = \frac{4n_{c}^{2}}{b^{2}} = \frac{m_{c}^{2}}{a^{2}} + \frac{n_{c}^{2}}{b^{2}} = 0.6520 \ (0.8982),$$

the preceding condition holds. Correspondingly, the numerical estimates of $r_1 - r_3$ and $s_1 - s_3$ are shown in the Table 4 and Table 5.

From the Table 4 and Table 5, one can see that both mixed and jump type transitions occur at certain parameter regimes, and the Biot number has a prominent impact on the types of the transition for both cases.
6. **Conclusions.** We have investigated the dynamic transitions for the surface tension driven convection in porous medium based on Brinkman equation coupled with the thermal diffusion equation by employing a hybrid analysis-computation approach. Two different transition theorems with several explicit transition numbers are obtained by reducing the infinite dynamical system onto the center manifold after showing the validity of principles of exchange of stabilities. We show by careful numerical evaluations of these transition numbers that the system favors all three types of transitions. This is a new phenomenon not present in the buoyancy forces driven convection (the classical Benard convection) \[10, 8\].

The results in this article can be extended in several directions which we will pursue in future works. First note that the time derivative of the Darcy velocity is present in our model. However, since the Darcy number is typically small in the regime that we are interested in, we would like to consider the classical Darcy equation neglecting this term. Secondly, we would like to show theoretically that all eigenvalues of the corresponding linear part of the model are real. That is, oscillation convection does not exist in this model (Though numerically true.) Thirdly, it would be interesting to consider the transition types with the addition of buoyancy forces. Finally it would be interesting to extend the current research to the convection problems in two-layer flows and with other physical boundary conditions.

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