Fuzzy circle and new fuzzy sphere through confining potentials and energy cutoffs

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Abstract

Guided by ordinary quantum mechanics we introduce new fuzzy spheres $S^d_\Lambda$ of dimensions $d = 1, 2$: we consider an ordinary quantum particle in $D = d + 1$ dimensions subject to a rotation invariant potential well $V(r)$ with a very sharp minimum on a sphere of unit radius. Imposing a sufficiently low energy cutoff to ‘freeze’ the radial excitations makes only a finite-dimensional Hilbert subspace accessible and on it the coordinates noncommutative `a la Snyder; in fact, on it they generate the whole algebra of observables. The construction is equivariant not only under rotations - as Madore’s fuzzy sphere -, but under the full orthogonal group $O(D)$. Making the cutoff and the depth of the well dependent on (and diverging with) a natural number $\Lambda$, and keeping the leading terms in $1/\Lambda$, we obtain a sequence $S^d_\Lambda$ of fuzzy spheres converging to the sphere $S^d$ in the limit $\Lambda \to \infty$ (whereby we recover ordinary quantum mechanics on $S^d$). These models may be useful in condensed matter problems where particles are confined on a sphere by an (at least approximately) rotation-invariant potential, beside being suggestive of analogous mechanisms in quantum field theory or quantum geometry.

1 Introduction

In 1947 Snyder proposed [1] the first example of noncommutative spacetime hoping that nontrivial (but Poincaré equivariant) commutation relations among the coordinates, acting as a fundamental regularization procedure, could cure ultraviolet (UV) divergencies in quantum field theory (QFT)[2]. He dubbed as distasteful arbitrary the UV regularization based on momentum (or equivalently energy) cutoff, which had just been proposed in the literature

\[\text{1} \text{The idea had originated in the '30s from Heisenberg, who proposed it in a letter to Peierls [2]; the idea propagated via Pauli to Oppenheimer, who asked his student Snyder to develop it.}\]
at the time, presumably as it broke manifest Lorentz equivariance and looked *ad hoc*. Ironically, shortly afterwards this and other more sophisticated regularization procedures found widespread application within the renormalization method; as known, the latter has proved to be extremely successful in extracting physically correct predictions from quantum electrodynamics, chromodynamics, and more generally the Standard Model of elementary particle physics. The proposal of Snyder was thus almost forgotten for decades (exceptions are e.g. [3, 4]). On the other hand, it is believed that any consistent quantum theory of gravitation will set fundamental bounds of the order of Planck length $l_p = \sqrt{\hbar G/c^3} \sim 10^{-33}\text{cm}$ on the accuracy $\Delta x$ of localization measurements. The arguments, which in qualitative form go back at least to [5, 6, 7], are based on a cutoff on the concentration of energy $\hbar$; they were made more precise and quantitative by Doplicher, Fredenhagen, Roberts [8], who also proposed that such a bound could follow from appropriate noncommuting coordinates (for a review of more recent developments see e.g. [9]). More generally, Connes’ Noncommutative Geometry framework [10] allows not only to replace the commutative algebra $\mathcal{A}$ of functions on a manifold $M$ by a noncommutative one, but also to develop on it the whole machinery of differential geometry [10, 11]. Often one deals with a family of noncommutative deformations $\mathcal{A}_\lambda$ of $\mathcal{A}$ that become commutative in some limit of the family’s ruling parameter(s) $\lambda$, exactly as the algebra of observables in ordinary quantum mechanics becomes the algebra of functions on phase space as $\hbar \to 0$.

Fuzzy spaces are particular examples parametrized by a positive integer $n$, so that the algebra $\mathcal{A}_n$ is a finite-dimensional matrix algebra with dimension which increases and diverges with $n$ while $\mathcal{A}_n \to \mathcal{A}$ (in a suitable sense). Since their introduction they have raised big interest in the high energy physics community as a non-perturbative technique in QFT (or string theory) based on a finite-discretization of space(time) alternative to the lattice one: the advantage is that the algebras $\mathcal{A}_n$ can carry representations of Lie groups (not only of discrete ones). The first and seminal fuzzy spaces are the Fuzzy Sphere (FS) of Madore and Hoppe [12, 13] and the noncommutative torus [14, 15] parametrized by a root of unity (this is often called fuzzy torus by theoretical physicists, see e.g. [16]) the first applications to QFT models of the FS are in [17, 18]. The FS is a sequence of $SO(3)$-equivariant, finite noncommutative $\ast$-algebras $\mathcal{A}_n$ isomorphic to $M_n$ (the algebra of $n \times n$ matrices); each matrix represents the expansion in spherical harmonics of an element of $C(S^2)$ truncated at level $n$. $\mathcal{A}_n$ is generated by hermitean noncommutative coordinates $x^i$ fulfilling

$$[x^i, x^j] = \frac{2i}{\sqrt{n^2-1}} \varepsilon^{ijk} x^k, \quad r^2 := x^i x^i = 1, \quad n \in \mathbb{N} \setminus \{1\}$$

(here and below sum over repeated indices is understood). The Hilbert space is chosen as

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2In fact, by Heisenberg uncertainty relations to reduce the uncertainty $\Delta x$ of the coordinate $x$ of an event one must increase the uncertainty $\Delta p_x$ of the conjugated momentum component by use of high energy probes; but by general relativity the associated concentration of energy in a small region would produce a trapping surface (event horizon of a black hole) if it were too large; hence the size of this region, and $\Delta x$ itself, cannot be lower than the associated Schwarzschild radius, $l_p$.

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$\mathcal{H} \simeq \mathbb{C}^n$ so that it carry an irreducible representation of $Uso(3)$, and the square distance $r^2$ from the origin - which is central - be identically equal to 1. We note that however are equivariant only under $SO(3)$, not $O(3)$; in particular not under parity $x^i \mapsto -x^i$. Fuzzy spaces can be used also in extra dimensions to account for internal (e.g. gauge) degrees of freedom, see e.g. [19].

As the arguments leading to $\Delta x \geq l_p$ suggest, imposing an energy cutoff $\mathcal{E}$ on an existing theory can be physically justified by two reasons, at least. It may be a necessity when we believe that $\mathcal{E}$ represents the threshold for the onset of new physics not accountable by that theory. Or it may serve to yield an effective description of the system when we, as well as the interactions with the environment, are not able to bring its state to higher energies; this leads also to a lower bound in the accuracy with which our apparatus can measure some observables (position, momentum, ...) of the system, which corresponds to the maximum energy transferable to the system by the apparatus in the measurement process, or by the environment during the interaction time. (Of course, the two reasons may co-exist.) Mathematically, the cutoff is imposed by a projection on the Hilbert subspace characterized by energies $E \leq \mathcal{E}$. If the Hamiltonian is invariant under some symmetry group, then the projection is invariant as well, and the projected theory will inherit that symmetry.

That imposing such a cutoff can modify a quantum mechanical model by converting its commuting coordinates into non-commuting ones is simply illustrated by the well-known Landau model (see e.g. [20, 21, 22, 23]), which describes a charged quantum particle in 2D interacting only with an uniform magnetic field (in the orthogonal direction) $B$. The energy levels are $E_n = \frac{h}{me} B n$, if we fix the additive constant so that the lowest level is $E_0 = 0$; choosing $\mathcal{E} \leq \frac{h}{me} B$ (this may be physically justified e.g. by a very strong $B$), then the Hilbert space of states is projected to the subspace $\mathcal{H}_0$ of zero energy, and $eB x, y$ become canonically conjugates, i.e. have a non-zero (but constant) commutator. The dimension of $\mathcal{H}_0$ is approximately proportional to the area of the surface, hence is finite (resp. infinite) if the area is.

Inspired by the projection mechanism in the Landau model, here we consider a quantum particle in dimension $D = 2$ or $D = 3$ with a Hamiltonian consisting of the standard kinetic term and a rotation invariant potential energy $V(r)$ with a very deep minimum (a well) respectively on a circle or on a sphere of unit radius; $k \equiv V''(1)/4 > 0$ plays the role of confining parameter. Imposing an energy cutoff $\mathcal{E}$ makes only a finite-dimensional Hilbert subspace $\mathcal{H}_E$ accessible and the projected coordinates noncommutative on $\mathcal{H}_E$. We choose $\mathcal{E} < 2\sqrt{2k}$ so that $\mathcal{H}_E$ does not contain excited radial modes, and on it the Hamiltonian reduces to the square angular momentum; this can be considered as a quantum version of the constraint $r = 1$. It turns out that the coordinates generate the whole algebra $A_E := End(\mathcal{H}_E)$ of observables on $\mathcal{H}_E$. Their commutators are of Snyder type, i.e. proportional to the angular momentum components $L_{ij}$ (apart from a small correction - depending only on the square angular momentum - on the highest energy states), rather than some function of the coordinates. Moreover, $(A_E, \mathcal{H}_E)$ is equivariant under the full group $O(D)$ of orthogonal transformations, because both the starting quantum mechanical model on $L^2(\mathbb{R}^D)$ and the cut-off procedure are. Actually, we prove the realization $A_E = \pi_E[Uso(D+1)]$, with $\pi_E$ a suitable irreducible unitary representation of $Uso(D+1)$ on $\mathcal{H}_E$; as a consequence, $\mathcal{H}_E$
carries a reducible representation of the subalgebra $Uso(D)$ generated by the angular momentum components $L_{ij}$, more precisely the direct sum of all irreducible representations fulfilling $E \leq \overline{E}$; in the $\overline{E} \to \infty$ limit this becomes the decomposition of the Hilbert space $L^2(S^d)$, $d = D - 1$. This welcomed property is not shared by the FS \cite{12, 13}. Similarly, the decomposition of the subspace $C_{\overline{E}} \subset A_{\overline{E}}$ of completely symmetrized polynomials in the non-commutative coordinates into irreducible $Uso(D)$-components becomes the decomposition of the commutative algebra $C(S^d)$ which acts on $L^2(S^d)$ and has the same decomposition.\footnote{In fact, spherical harmonics make up a basis for both $L^2(S^d)$ and $C(S^d)$.}

On $\mathcal{H}_{\overline{E}}$ the square distance $R^2$ from the origin is not identically 1, but a function of the square angular momentum such that nevertheless its spectrum is very close to 1 and collapses to 1 in the $k \to \infty$ limit of an infinitely narrow and deep well; the latter limit is automatic as we have to set $k \sim \overline{E}^2$ for consistency. Thus the confining parameter $k$, or equivalently the energy cutoff $\overline{E}$, or a suitable natural number $\Lambda$ which we shall adopt to discretize both, will also parametrize the noncommutativity of the coordinates. Finally, there are natural embeddings $\mathcal{H}_\Lambda \hookrightarrow L^2(S^d)$, $\mathcal{C}_\Lambda \hookrightarrow B(S^d)$, $C(S^d)$, and in a suitable sense $\mathcal{H}_\Lambda \to L^2(S^d)$, $\mathcal{C}_\Lambda \to C(S^d)$ and $A_{\Lambda}$ goes to the whole algebra of observables on $L^2(S^d)$, as $\Lambda \to \infty$.

We think that our models are interesting not only as new toy-models of fuzzy geometries in QFT and quantum geometry, but also in view of potential applications to quantum models in condensed matter physics with an effective one- or two-dimensional configuration space in the form of a circle, a cylinder or a sphere\footnote{For instance, circular quantum waveguides, graphene nanotubes and fullerene. These are very thin wires or layers of matter where electrons are confined by potential energies with very deep minima there and steep gradients in the normal direction(s). The Hamiltonian on a cylinder can be written as the sum of the one on the transverse section circle and of the kinetic term $-\hbar^2(\partial/\partial z)^2/2m$ in the direction of the axis.} because they respect parity, and the restriction to the circle, cylinder or sphere is an effective one obtained “a posteriori” from the exact dynamics in higher dimension. Moreover, our procedure can be generalized in a straightforward manner to $D > 3$, as well as to other confining potentials; the dimension of the accessible Hilbert space $\mathcal{H}_{\overline{E}}$ will be approximately $B/h^D$, where $h, B$ are the Planck constant and the volume of the classically allowed region in phase space (i.e. the one characterized by $E \leq \overline{E}$). All features of these new fuzzy geometries deserve investigations, in particular their metric aspects, as done e.g. in \cite{24} for the FS.

Fuzzy spheres based on some Snyder-type commutation relations have already been proposed for $d = 4$ in \cite{25} (see e.g. also \cite{26, 27}) and for all $d \geq 3$ in \cite{28, 29}. In section 5 we sketch how we expect the results based on our approach would be related to the latter.

The plan of the paper is as follows. In section 2 we introduce the framework valid for any $D$. In sections 3, 4 we treat the cases $D = 2, 3$ leading to $S^1_{\Lambda}, S^2_{\Lambda}$ respectively. Section 5 contains final remarks, outlook and conclusions. In the Appendix (section 6) we have concentrated lengthy computations and proofs.
2 General setting

As said, we consider a quantum particle in $\mathbb{R}^D$ configuration space with Hamiltonian

$$H = -\frac{1}{2} \Delta + V(r),$$

(2)

where $r^2 := x^i x^i$, $\Delta := \partial_i \partial_i$ (sum over repeated indices understood), $\partial_i := \partial/\partial x^i$, $i = 1, \ldots, D$; the cartesian coordinates $x^i$, the momentum components $-i \partial_i$, $H$ itself are normalized so as to be dimensionless. $x^i, -i \partial_i$ generate the Heisenberg algebra $\mathcal{O}$ of observables. The canonical commutation relations

$$[x^i, x^j] = 0, \quad [-i \partial_i, -i \partial_j] = 0, \quad [x^i, -i \partial_j] = i \delta^i_j$$

(3)

as well as the Hamiltonian are invariant under all orthogonal transformations

$$x^i \mapsto x'^i = Q^{ij} x^j, \quad Q^{-1} = Q^T$$

(4)

(including parity $Q = -I$). This implies $[H, L_{ij}] = 0$, where $L_{ij} := ix^j \partial_i - ix^i \partial_j$ are the angular momentum components. We shall assume that $V(r)$ has a very sharp minimum at $r = 1$ with very large $k = V''(1)/4 > 0$, and fix $V_0 := V(1)$ so that the ground state has zero energy, i.e. $E_0 = 0$. We choose an energy cutoff $\bar{E}$ fulfilling first of all the condition

$$V(r) \simeq V_0 + 2k(r - 1)^2 \quad \text{if } r \text{ fulfills } V(r) \leq \bar{E}$$

(5)

so that we can neglect terms of order higher than two in the Taylor expansion of $V(r)$ around 1 and approximate the potential as a harmonic one in the classical region $v_{\bar{E}}$ compatible with the energy cutoff $V(r) \leq \bar{E}$. We are interested in finding the eigenfunctions of $H$

$$H \psi = E \psi, \quad \psi \in L^2(\mathbb{R}^D)$$

(6)

with eigenvalues $E \leq \bar{E}$ and restricting quantum mechanics to the finite-dimensional Hilbert subspace $\mathcal{H}_{\bar{E}}$ spanned by them. This means that we shall replace every observable $A$ by $\overline{A} := P_{\bar{E}} A P_{\bar{E}}$, where $P_{\bar{E}}$ is the projection on $\mathcal{H}_{\bar{E}}$, and give to $\overline{A}$ the same physical interpretation. In particular, $\overline{\pi^i}$ will be interpreted as the observable associated to the measurement of the $i$-th coordinate of the particle; $\overline{H} = H$ will still appear as the Hamiltonian in the original Schrödinger equation. We shall also replace any Schrödinger equation $i \partial_r \psi = H_\epsilon \psi$, with some extended Hamiltonian $H_\epsilon = H + H'$ (containing a “small” extra term $H'$ representing some additional interaction), with the finite-dimensional one $i \partial_r \psi = \overline{H_\epsilon} \psi$ within $\mathcal{H}_{\bar{E}}$.

By (5), $v_{\bar{E}} \subset \mathbb{R}^D$ is approximately the shell $|r - 1| \leq \sqrt{\frac{\bar{E} - V_0}{2k}}$; in the limit in which both $k, \bar{E}$ diverge, but the right-hand side goes to zero, $v_{\bar{E}}$ reduces to the unit sphere $S^{D-1}$. We expect that in this limit the dimension of $\mathcal{H}_{\bar{E}}$ diverges, and we recover standard quantum mechanics on the sphere $S^{D-1}$. As we shall see, this is the case.

$P_{\bar{E}}$ commutes not only with $H$, but also with the $L_{ij} := ix^j \partial_i - ix^i \partial_j$, which are vector fields tangent to every sphere $r =$const. The $D$ derivatives $\partial_i$ make up a globally defined basis for the linear space of smooth vector fields. The set $B = \{ \partial_r, L_{ij} \mid i < j \} \ (\partial_r := \partial/\partial r)$
is an alternative complete set that is singular for \( r = 0 \), but globally defined elsewhere; for \( D = 2 \) it is a basis, for \( D > 2 \) it is redundant, because of the relations

\[
\varepsilon^{i_1 i_2 i_3 \ldots i_D} x^{i_1} L^{i_2 i_3} = 0.
\]  

(7)

This redundancy is unavoidable. In \( D = 3 \) there are no two independent globally defined vector fields that are tangent to the sphere (\( S^2 \) is not parallelizable); for instance, the derivatives \( \partial_\varphi, \partial_\theta \) with respect to the polar angles are singular at the north and south poles. One needs all three angular momentum components, which however are constrained by \( \varepsilon^{ijk} x^i L^{jk} = 0 \).

Of course, the eigenfunctions of \( H \) can be more easily determined in terms of polar coordinates \( r, \varphi, \ldots \), recalling that the Laplacian in \( D \) dimensions decomposes as follows

\[
\Delta = \partial_r^2 + (D - 1) \frac{1}{r} \partial_r - \frac{1}{r^2} L^2, 
\]  

(8)

where \( \partial_r := \partial/\partial r \) and \( L^2 = \ell_i \ell_j / 2 \) is the square angular momentum (in normalized units), i.e. the Laplacian on the sphere \( S^{D-1} \). We know from the \( D \)-dimensional theory of angular momentum that the eigenvalues of \( L^2 \) are \( j(j + D - 2) \); then replacing the Ansatz \( \psi = \tilde{f}(r) Y(\varphi, \ldots) \) (\( Y \) are eigenfunctions of \( L^2 \) and of the elements of a Cartan subalgebra of \( so(D) \); \( r, \varphi, \ldots \) are polar coordinates) transforms the PDE \( H \psi = E \psi \) into this auxiliary ODE in the unknown \( \tilde{f}(r) \)

\[
\left[ -\partial_r^2 - (D - 1) \frac{1}{r} \partial_r + \frac{1}{r^2} j(j + D - 2) + V(r) \right] \tilde{f}(r) = E \tilde{f}(r).
\]  

(9)

If \( V(r) \) keeps bounded or grows at most as \( \beta/r^2 \) (with some \( \beta \geq 0 \)) as \( r \to 0 \), then in the same limit \( \tilde{f}(r) \) vanishes as \( \tilde{f}(r) = O(r^\alpha) \), with \( \alpha = \sqrt{\beta + j(j + D - 2)} \). In fact, by Fuchs theorem every solution of (9) is a combination of the two independent ones with \( r \to 0 \) asymptotic behaviour \( r^\alpha, r^{-\alpha} \); but the coefficient of the second must vanish in order that \( \psi \in L^2(\mathbb{R}^D) \). Hence \( \tilde{f}(0) = 0 \). On the other hand, \( \psi \in L^2(\mathbb{R}^D) \) implies also \( \tilde{f}(r) \overset{r \to +\infty}{\to} 0 \). Actually, condition (5) implies that \( \tilde{f}, \psi \) become negligibly small outside the thin shell region \( V(r) \leq E \) (around \( r = 1 \)), and that at leading order the lowest eigenvalues \( E \) are those of the 1-dimensional harmonic oscillator approximation of (9).

### 3 \( D = 2 \): \( O(2) \)-equivariant fuzzy circle

We fix the notation as follows: \( x \equiv x^1 = r \cos \varphi, \ y \equiv x^2 = r \sin \varphi, \ x^\pm := (x \pm iy) / \sqrt{2} = re^{\pm i\varphi} / \sqrt{2} \); we abbreviate \( u := e^{i\varphi} \) (whence \( u^* = e^{-i\varphi} \)), \( \partial^x \equiv \partial_\pm \equiv \partial / \partial x^\pm, \ \partial_\varphi \equiv \partial / \partial \varphi \); the angular momentum \( L \equiv L_{12} \) can be expressed in the form \( L = -i \partial_\varphi = x^+ \partial_+ - x^- \partial_- \), and

\[
[L, x^\pm] = \pm x^\pm, \quad [L, \partial_\pm] = \mp \partial_\pm,
\]

i.e. the generators \( x^\pm, \partial_\pm \) of the Heisenberg algebra \( \mathcal{O} \) (of observables) are eigenvectors under the adjoint action of \( L \) with eigenvalues \( \pm 1 \). We look for \( \psi \) of the form \( \tilde{\psi}_m(r, \varphi) = \tilde{f}(r)e^{im\varphi} \),
with eigenvalues \( m \in \mathbb{Z} \) and we can use the \( L \)-eigenvalue as a \( \mathbb{Z} \)-grading for both \( \mathcal{H} \) and \( \mathcal{O} \), in a compatible way:

\[
\mathcal{H} = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}^m, \quad \mathcal{O} = \bigoplus_{m \in \mathbb{Z}} \mathcal{O}^m, \quad \mathcal{O}^m \mathcal{O}^{m'} = \mathcal{O}^{m+m'}, \quad \mathcal{O}^m \mathcal{H}^{m'} = \mathcal{H}^{m+m'}
\]

(the last relation must be understood modulo domain restrictions); clearly \( L, r, h(r), \partial_r, \Delta \in \mathcal{O}^0 \) [for any function \( h(r) \)]. Equation (9) becomes

\[
\tilde{f}''(r) + \frac{1}{r} \tilde{f}'(r) + \left[ E - V(r) - \frac{m^2}{r^2} \right] \tilde{f}(r) = 0.
\]

We change the radial variable \( r \mapsto \rho := \ln r \) and set \( f(\rho) := \tilde{f}(r) = \tilde{f}(e^\rho) \), whereby the previous equation is transformed into \( f''(\rho) + \{ e^{2\rho} [E - V(e^\rho)] - m^2 \} f(\rho) = 0 \). By condition (5), in the region \(|r-1| \leq \sqrt{\frac{E-V_0}{2k}} \) we can neglect the terms of order higher than two in the Taylor expansions \( e^{2\rho} = 1 + 2\rho + 2\rho^2 + ..., V(e^\rho) = V_0 + 2k\rho^2 + ... \) and thus approximate the above equation by

\[
\tilde{\mathcal{H}} f(\rho) = e_m f(\rho), \quad \tilde{\mathcal{H}} := -\partial_\rho^2 + k_m (\rho - \bar{\rho}_m)^2,
\]

where \( k_m = 2(k-E') \), \( E' = E-V_0 \), \( \bar{\rho}_m = \frac{E'}{k_m} \), \( e_m = \frac{E'^2}{k_m} + E' - m^2 \), \( m \in \mathbb{Z} \).

which has the form of the eigenvalue equation for a harmonic oscillator in 1 dimension with equilibrium position at \( \rho = \bar{\rho}_m \). In order that \( \psi \) be square-integrable it must be \( f(\rho) \xrightarrow{\rho \to \infty} 0 \), \( f(\rho) \xrightarrow{\rho \to -\infty} 0 \), what selects as eigenfunctions of the auxiliary operator \( \tilde{\mathcal{H}} \)

\[
f_{n,m}(\rho) = \exp \left[ -\frac{(\rho - \bar{\rho}_m)^2 \sqrt{k_m}}{2} \right] H_n \left( \frac{\rho - \bar{\rho}_m}{\sqrt{k_m}} \right),
\]

(here \( H_n \) is the Hermite polynomial of order \( n \), and as corresponding “eigenvalues” \( e_{m,n} = (2n+1)\sqrt{k_m} \). By (10) this implies that \( E' \) must fulfill the equation

\[
\frac{E'^2}{2(k-E')} + E' - m_0^2 = (2n+1)\sqrt{2(k-E')}
\]

(11)

Squaring both sides of (11) and multiplying them by \( k_m^2 \) one obtains a fourth degree equation which determines \( E' \), and therefore \( E \), in terms of \( V_0, m, n \). As said, we fix \( V_0 \) requiring that the lowest energy level, which corresponds to \( n = m = 0 \), be \( E_0 = 0 \). This implies that \( V_0 \) must fulfill the equation

\[
\frac{V_0^2}{2(k+V_0)} - V_0 = \sqrt{2(k+V_0)} \quad \Leftrightarrow \quad -\sqrt{\frac{1}{2k}} V_0 - \left( \sqrt{\frac{1}{2k}} \right)^3 V_0^2 = \left( 1 + \frac{V_0}{k} \right)^{\frac{3}{2}}.
\]

Looking for the solution in the form \( V_0 = \sum_{n=1}^\infty v_n \left( \frac{1}{2k} \right)^n \) we can determine the coefficients \( v_n \), and therefore \( V_0 \), solving the latter equation order by order in \( \sqrt{\frac{1}{2k}} \). The solution is
\[ V_0 = -\sqrt{2k} + 2 - \frac{7}{2} \sqrt{2k} + O(1/k) \] and Figure 1 shows the appearance of the resulting potential. Replacing this result in (11) one finds that at leading order \( E \) is given by \( E_{n,m} = m^2 + 2n\sqrt{2k} - 2n + O \left( \frac{1}{\sqrt{k}} \right) \). The term \( m^2 \) gives exactly what we wish, (part of) the spectrum of \( L^2 \) (the Laplacian on the circle). To eliminate the subsequent, undesired terms we fix the energy cutoff \( \bar{E} < 2\sqrt{2k} - 2 \), so as to exclude all the states with \( n > 0 \). Physically, this means that radial oscillations are “frozen”, \( n = 0 \), so that all corresponding classical trajectories are circles. The energies \( E \) below \( \bar{E} \) will therefore depend only on \( m \), and will be denoted as \( E_m \). Consequently, \( k_m, \tilde{\rho}_m \) will be determined by (10). Then at leading orders in \( 1/\sqrt{k} \) (11) yields as eigenvalues of \( L, H \) and corresponding eigenfunctions

\[
L = m, \quad H = E_m = m^2 + O \left( \frac{1}{\sqrt{k}} \right)
\]

\[
\psi_m(\rho, \varphi) = N_m e^{im\varphi} \exp \left[ -\frac{(\rho - \tilde{\rho}_m)^2}{2} \sqrt{k_m} \right],
\]

\[
\frac{k_m}{2k} = 1 - \frac{2}{\sqrt{2k}} + \frac{2 - m^2}{k} + O \left( \frac{1}{k^{3/2}} \right), \quad \tilde{\rho}_m = \frac{1}{\sqrt{2k}} + \frac{m^2}{2k} + O \left( \frac{1}{k^{3/2}} \right)
\]

we fix the normalization factor \( N_m \) so that \( N_m > 0 \) and all \( \psi_m \) have unit norm. The condition \( E \leq \bar{E} \) is fulfilled if we project the theory onto the Hilbert subspace \( \mathcal{H}_\Lambda \equiv \mathcal{H}_{\bar{E}} \) spanned by the \( \psi_m \) with \( |m| \leq \Lambda := \lceil \sqrt{\bar{E}} \rceil \) (here \( \lceil a \rceil \) stands for the integer part of \( a > 0 \)). For consistency we must choose

\[
\Lambda^2 < 2\sqrt{2k} - 2
\]

so that all \( E_m \) are smaller than the energy levels corresponding to \( n > 0 \), as we can see from Figure 2; this is also sufficient to guarantee that \( k_m \gg 1 \) for all \( |m| \leq \Lambda \) (by the way, \( k_m > 0 \) is a necessary condition for \( \bar{H} \) to be the Hamiltonian of a harmonic oscillator). The spectrum of \( \bar{H} \) becomes the whole spectrum \( \{m^2\}_{m\in\mathbb{N}_0} \) of \( L^2 \) in the limit \( \Lambda, k \to \infty \) respecting (15).

\[ \mathcal{A}_\Lambda := \mathcal{P}_\Lambda \mathcal{O} \mathcal{P}_\Lambda \text{ and } \mathcal{H}_{\bar{E}} \equiv \mathcal{H}_\Lambda \text{ inherit the grading from } \mathcal{O}, \mathcal{H}. \text{ For any } A^h \in \mathcal{O}^h, |m| \leq \Lambda, \]

\[ \overline{A^h} \psi_m = \sum_{m' = -\Lambda}^{\Lambda} \psi_{m'} \langle \psi_{m'}, A^h \psi_m \rangle = \left\{ \begin{array}{ll} \psi_{m+h} \langle \psi_{m+h}, A^h \psi_m \rangle & \text{if } |m|, |m + h| \leq \Lambda, \\ 0 & \text{otherwise.} \end{array} \right. \]

In formula (69) in the appendix we compute the matrix element \( \langle \psi_{m+h}, A^h \psi_m \rangle \) for \( A \) of the form \( A = f(\rho) e^{ih\varphi} = f(\rho) u^h \) [if \( A \) contains also derivatives \( \partial_\pm \) the result can be expressed as a combination of matrix elements of the same type because of (23)]. In particular, up to terms \( O(1/k^{3/2}) \)

\[ \overline{u^h} \psi_m = \left\{ \begin{array}{ll} \psi_{m+1} & \text{if } -\Lambda \leq m \leq \Lambda - 1, \\ 0 & \text{otherwise,} \end{array} \right. \]

where \( a := 1 + \frac{9}{4} \frac{1}{\sqrt{2k}} + \frac{137}{64k} \). Clearly \( \sqrt{2} \, \overline{x^+} = \pi a \sqrt{1 + \frac{2 (\xi + 1)}{k}} \). As \( u^+, x^- \) are the adjoints of \( u, x^+ \), so are \( \overline{u}, \overline{x}^- \) respectively the adjoints of \( \overline{u}, \overline{x}^+ \). We can get rid of the \( m \)-independent
factor $a$ reabsorbing it in the redefinitions

$$\xi^\pm = \frac{x^\pm}{a}.$$  

Therefore we find

$$\mathcal{L}\psi_m = m\psi_m, \quad \mathcal{H} = \mathcal{L}^2, \quad \xi^\pm \psi_m = \begin{cases} \frac{1}{\sqrt{2}} \sqrt{1 + \frac{m(m+1)}{k^2}} \psi_{m\pm 1} & \text{if } -\Lambda \leq \pm m \leq \Lambda - 1 \\ 0 & \text{otherwise}, \end{cases}$$  

(17)  

[the second, third relations hold up to terms $O(1/k^{1/2}), O(1/k^{3/2})$, respectively]. Eq. (17)
implies at leading order

\[
\begin{align*}
[\xi^+, \xi^-] \psi_m &= \begin{cases} 
-\frac{m}{k} \psi_m & \text{if } |m| \leq \Lambda - 1, \\
\pm \frac{1}{2} \left[ 1 + \frac{\Lambda(\Lambda-1)}{k} \right] \psi_m & \text{if } m = \pm \Lambda, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

(18)

\[
\begin{align*}
\mathcal{R}^2 \psi_m &= \begin{cases} 
\left( 1 + \frac{m^2}{k} \right) \psi_m & \text{if } |m| \leq \Lambda - 1, \\
\frac{1}{2} \left[ 1 + \frac{\Lambda(\Lambda-1)}{k} \right] \psi_m & \text{if } m = \pm \Lambda, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

(19)

in (19) we have introduced the square distance from the origin \( \mathcal{R}^2 := \xi^+ \xi^- + \xi^- \xi^+ \), in analogy with the classical definition. We see that \( \mathcal{R}^2 \) is not identically equal to 1 on all of \( \mathcal{H}_\Lambda \) (as in the standard quantization on the unit circle), but the \( \psi_m \) are eigenvectors of \( \mathcal{R}^2 \) with eigenvalues depending only on \( m^2 \) and growing with \( m^2 \) (with the exception of the states \( \psi_\Lambda, \psi_{-\Lambda} \) closest to the energy cutoff, which however play no role at lower energies); physically this is to be expected because higher square angular momentum \( m^2 \) is equivalent to a larger centrifugal force, which classically yields a slightly more external circular trajectory. Moreover, all these eigenvalues are close to 1 and go to 1 as \( k \to \infty \), while the eigenfunctions factorize as \( \psi_m(r, \varphi) \to \delta(r-1)e^{im\varphi} \).

If we now adopt (17) as exact definitions of \( L, \overline{H}, \xi^+, \xi^- \), then (18-19) are exact as well, and we easily find

**Proposition 3.1.** The \( \xi^+, \xi^- \) defined by (17) generate the *-algebra \( \mathcal{A}_\Lambda = \text{End}(\mathcal{H}_\Lambda) \simeq M_{2\Lambda+1}(\mathbb{C}) \) of observables on \( \mathcal{H}_\Lambda \); they fulfill \( (\xi^+)^{2\Lambda+1} = (\xi^-)^{2\Lambda+1} = 0 \), \( \prod_{m=0}^{\Lambda} (\overline{L} - mI) = 0 \),

\[
\begin{align*}
\xi^+ \dagger = \xi^-, \\
\mathcal{L} \dagger = \mathcal{L}, \\
[\mathcal{L}, \xi^+] = \pm \xi^+, \\
[\xi^+, \xi^-] = -\frac{1}{k} + \frac{\overline{P}_{\Lambda} - \overline{P}_{-\Lambda}}{2},
\end{align*}
\]

(20)

where \( \overline{P}_m \) stands for the projector on the 1-dimensional subspace spanned by \( \psi_m \) (clearly \( P_\Lambda = \sum_{m=-\Lambda}^{\Lambda} \overline{P}_m \)) and \( \mathcal{L} := P_\Lambda L P_\Lambda \) is the projection on \( \mathcal{H}_\Lambda \) of the angular momentum operator. All \( \overline{P}_m \) can be expressed as polynomials in \( \mathcal{L} \) using the spectral decomposition of \( \mathcal{L} \). Moreover, the square distance from the origin can be expressed as the function of \( \mathcal{L}^2 \)

\[
\mathcal{R}^2 := \xi^+ \xi^- + \xi^- \xi^+ = 1 + \frac{\mathcal{L}^2}{k} - \frac{\overline{P}_{\Lambda} + \overline{P}_{-\Lambda}}{2}, \quad \mu := 1 + \frac{\Lambda(\Lambda+1)}{k}.
\]

(21)

Of course, relations (20), (21) only hold at leading order in \( 1/\sqrt{k} \) if also (17) do.

To obtain a fuzzy space depending only on one integer \( \Lambda \) we can choose \( k \) as a function of \( \Lambda \) fulfilling (15); the commutative limit will be simply \( \Lambda \to \infty \) (which implies \( k \to \infty \)).
One possible choice is
\[ k = \Lambda^2(\Lambda+1)^2; \] then (20) becomes
\[ [\xi^+,\xi^-] = \frac{-\mathcal{L}}{\Lambda^2(\Lambda+1)^2} + \left[ 1 + \frac{1}{\Lambda(\Lambda+1)} \right] \frac{\tilde{P}_\Lambda - \tilde{P}_{-\Lambda}}{2}. \] (22)

Summarizing our results so far, we see that the combined effect of the confining potential and of the energy cutoff is a non-vanishing commutator between the coordinates. Note that relations (20), (22) are invariant not only under rotations (this includes parity \( \xi^a \mapsto -\xi^a \), \( \mathcal{L} \mapsto \mathcal{L} \) in \( \mathcal{D} = 2 \)), but also under orthogonal transformations with determinant \(-1\), e.g. \( \xi^1 \mapsto \xi^1 \), \( \xi^2 \mapsto -\xi^2 \), \( \mathcal{L} \mapsto -\mathcal{L} \), i.e. under the whole group \( O(2) \), as in the ordinary theory without cutoff. This had to be expected, because both the commutation relations in the original infinite-dimensional model and the Hamiltonian \( (\text{H}) \) (hence also the projectors \( P_{\mathcal{E}} \)) are \( O(2) \)-invariant. Apart from the sign and from the last term containing the projections, which plays no role far from the cutoff \( \Lambda \), relations (20), (22) are of the Snyder’s Lie algebra type \( \mathbb{I} \), because the commutator of the coordinates is a generator of rotations. \( \xi^+,\xi^- \) (or equivalently \( \pi^+,(\pi^-) \)) generate the whole \( \ast \)-algebra \( \mathcal{A}_\Lambda \) (also \( \mathcal{L} \) can be expressed as a non-ordered polynomial in \( \xi^+,\xi^- \)).

To compute \( \overline{\partial}_\pm \) it is convenient to use polar coordinates. We find
\[ \partial_\pm = \frac{1}{2\pi \pm} (\partial_\rho \mp i \partial_\phi), \quad \partial_\rho \psi_m = -(\rho - \tilde{\rho}_m) \sqrt{k_m} \psi_m \propto \psi_m, \] (23)
and \( \overline{\partial}_\rho = 0 \), because of \( P_\Lambda \psi_m = 0 \). Similarly, \( \overline{\partial}_x, \frac{1}{2\pi \pm} \partial_\rho \) go to zero in norm as \( k \to \infty \). As consequences, neither \( \partial_\pm - \overline{\partial}_\pm \) nor the commutator \( [\partial_\pm,\overline{\partial}_\pm] \) vanish as \( k \to \infty \), as expected. Only the vector field \( \mathcal{L} \) tangent to the circle survives with the correct classical limit, as desired. For completeness, the actions of \( \overline{\partial}_+,\overline{\partial}_- \) are reported in (79-83).

## 3.1 Realization of the algebra of observables through \( \text{Uso}(3) \)

For every \( n \in \mathbb{N} \) the \( \ast \)-algebra \( M_n(\mathbb{C}) \) of endomorphisms of \( \mathbb{C}^n \) can be realized as the \( n = (2\Lambda+1) \)-dimensional unitary representation \( \pi_\Lambda \) of \( \text{so}(3) \simeq \text{su}(2) \). In fact, the operators on \( \mathcal{H}_\Lambda \), and in particular \( \overline{\partial},\xi^\pm \), are naturally realized in \( \pi_\Lambda \), \( \text{Uso}(3) \), identifying \( \psi_m \) as the vectors \( |m\rangle \) of the canonical basis, in standard ket notation. We denote as \( E^+, E^-, E^0 \) the Cartan-Weyl basis of \( \text{so}(3) \),
\[ [E^+,E^-] = E^0, \quad [E^0,E^\pm] = \pm E^\pm, \quad E^\pm \dagger = E^\mp, \quad E^0 \dagger = E^0, \] (24)
as \( \dagger \) (with an abuse of notation also) its real structure, and as \( C \) the Casimir,
\[ C = E^a E^{-a} = 2E^+ E^- + E^0 (E^0 - 1) = 2E^- E^+ + E^0 (E^0 + 1). \] (25)

The representation \( \pi_\Lambda \) is characterized by the Casimir eigenvalue \( \pi_\Lambda(C) = \Lambda(\Lambda + 1) \). We identify \( L = \pi_\Lambda(E^0) \), which will still determine the grading. To simplify the notation in the sequel we drop \( \pi_\Lambda \). For every \( A^h \in \mathcal{O}^h \) one can determine a function of one variable \( f_A(s) \) such that
\[ \overline{A^h} = f_A(E^0) E^h, \quad \text{where} \quad E^h = \begin{cases} (E^+)^h & \text{if } h > 0, \\ 1 & \text{if } h = 0, \\ (E^-)^{-h} & \text{if } h < 0, \end{cases} \]
by requiring that \( \langle \psi_{m+h}, A^h \psi_m \rangle = \langle \psi_{m+h}, f_A(E^0) E^h \psi_m \rangle \) and using (69). In particular it is an easy exercise to check that from (16) and the adjoint relations it follows
\[
\bar{L} = E^0, \quad \xi^\pm = f^\pm(E^0)E^\pm
\]
and \( \pi = f_u(E^0)E^+, \) where
\[
f_u(s) = \frac{1}{\sqrt{\Lambda(\Lambda + 1)-s(s-1)}}, \quad f^+(s) = \frac{1}{\sqrt{2}} \sqrt{\frac{1+s(s-1)/k}{\Lambda(\Lambda + 1)-s(s-1)}} = f_-(s-1) = f_-(s).
\]
Therefore (26) fulfill (20). The inverse of the change of generators (26) is clearly
\[
E^0 = \bar{L}, \quad E^\pm = \left[f^\pm(\bar{L})\right]^{-1} \xi^\pm.
\]
The eigenvalue condition \( C = \Lambda(\Lambda + 1) \) can be put more explicitly in either form
\[
2E^- E^+ = \Lambda(\Lambda + 1) - E^0(E^0 + 1) = 2\xi^- \xi^+ \left[f_+(\bar{L}+1)\right]^{-2}.
\]
Summarizing, we have almost completely shown

**Proposition 3.2.** Formulas (26) provide a \( O(2) \)-equivariant \( \ast \)-algebra isomorphism between the algebra \( \mathcal{A}_\Lambda = \text{End}(\mathcal{H}_\Lambda) \) of observables (endomorphisms) on \( \mathcal{H}_\Lambda \) and that on the \( C = \Lambda(\Lambda + 1) \) irreducible representation of \( U\text{so}(3) \):
\[
\mathcal{A}_\Lambda := \text{End}(\mathcal{H}_\Lambda) \simeq M_N(\mathbb{C}) \simeq \pi_\Lambda[U\text{so}(3)], \quad N := 2\Lambda+1.
\]

(The \( O(2) \)-equivariance of this realization is shown below.) Note also that every function of \( E^0 = \bar{L} \), including \( f^\pm(E^0) \), can be expressed in polynomial form by spectral decomposition. As consequences, the generators \( E^+, E^- \) of \( U\text{so}(3) \) [which are characterized by relations (24), where \( E^0 \) has to be understood as an abbreviation for the commutator of \( E^+, E^- \)] further constrained by (27)\textsubscript{1}, or alternatively \( \xi^+, \xi^- \) fulfilling (20) and (27)\textsubscript{2}, generate all the algebra \( \mathcal{A}_\Lambda \); while ordered polynomials in \( E^+, E^-, E^0 \), or alternatively in \( \xi^+, \xi^- \), \( \bar{L} \), span \( \mathcal{A}_\Lambda \). Therefore the above results for the action of the operators \( \xi^+, \xi^- \),... on \( \mathcal{H}_\Lambda \) can be recovered determining the unique unitary representation of the \( \ast \)-algebra generated by \( \xi^+, \xi^- \) fulfilling relations (20), (27)\textsubscript{2}, or more simply setting (26) and using our knowledge on the representation \( \pi_\Lambda \) of \( \text{so}(3) \).

As known, the group of \( \ast \)-automorphisms of \( M_N(\mathbb{C}) \simeq \mathcal{A}_\Lambda \) is inner and isomorphic to \( SU(N) \), i.e.
\[
a \mapsto g a g^{-1}, \quad a \in \mathcal{A}_\Lambda, \tag{29}
\]
with \( g \) an unitary \( N \times N \) matrix with unit determinant. A special role is played by the subgroup \( SO(3) \) acting through the representation \( \pi_\Lambda \), namely \( g = \pi_\Lambda[e^{ia}] \), where \( a \in \text{so}(3) \),

\textsuperscript{6}Note that \( f_u(E^0), f^+_u(E^0) \) are singular on \( \psi_{-\Lambda} \), while \( f^-_u(E^0) \) is singular on \( \psi_{\Lambda} \), but since their action follows that of \( E^+ \) or \( E^- \) they can never act on such vectors, and the products at the right-hand side of (26) are well-defined on all of \( \mathcal{H}_\Lambda \).
i.e. is a combination with real coefficients of \( E^0, E^+ + E^-, i(E^+ - E^-) \). In particular, choosing \( \alpha = \theta E^0 \) [i.e. in the adopted \( so(2) \) Cartan subalgebra of \( so(3) \)] the automorphism amounts to a rotation in the \( \mathbb{R}^1 \mathbb{R}^2 \) plane by an angle \( \theta \), i.e. \( E^0 \mapsto E^0 \) and \( E^\pm \mapsto e^{\pm i \theta} E^\pm \), or equivalently \( \mathcal{T} \mapsto \mathcal{T} \) and

\[
\mathbb{R}^\pm \mapsto \mathbb{R}^\pm = e^{\pm i \theta} \mathbb{R}^\pm \quad \iff \quad \begin{cases} \mathbb{R}^1 = \mathbb{R}^1 \cos \theta + \mathbb{R}^2 \sin \theta, \\ \mathbb{R}^2 = -\mathbb{R}^1 \sin \theta + \mathbb{R}^2 \cos \theta; \end{cases}
\]

this a \( SO(2) \) transformation in the \( \mathbb{R}^1 \mathbb{R}^2 \) plane. Setting \( \alpha = \pi(E^+ + E^-)/\sqrt{2} \) we obtain a \( O(2) \) transformation with determinant \(-1\) in such a plane; this amounts to a rotation about \( E^1 := (E^+ + E^-)/\sqrt{2} \) by an angle \( \pi \), i.e. to \( E^0 \mapsto -E^0 \), \( E^\pm \mapsto E^\mp \). As the functions \( f_\pm \) fulfill \( f_\pm(-s) = f_\pm(1+s) = f_\mp(s) \), this is equivalent to \( \mathbb{R}^1 \mapsto \mathbb{R}^1, \mathbb{R}^2 \mapsto -\mathbb{R}^2, \mathcal{T} \mapsto -\mathcal{T} \). All other \( O(2) \) transformations with determinant \(-1\) in the \( \mathbb{R}^1 \mathbb{R}^2 \) plane can be obtained by composition with a \( SO(2) \) transformation. \( O(2) \) will play the role of isometry group of the fuzzy circle.

### 3.2 Convergence to \( O(2) \)-equivariant quantum mechanics on \( S \)

Here we explain in which sense our model converges to \( O(2) \)-equivariant quantum mechanics on the circle as \( \Lambda \to \infty \).

The \( \psi_m \in \mathcal{H}_\Lambda \) are the fuzzy analogs of the \( u^m \) considered just as elements of an orthonormal basis of the Hilbert space \( \mathcal{L}^2(S) \). Consider the \( O(2) \)-equivariant embedding \( \mathcal{I} : \mathcal{H}_\Lambda \hookrightarrow \mathcal{L}^2(S) \) defined by

\[
\mathcal{I}\left( \sum_{m=-\Lambda}^{\Lambda} \phi_m \psi_m \right) = \sum_{m=-\Lambda}^{\Lambda} \phi_m u^m.
\]

Below we shall drop the symbol \( \mathcal{I} \) and simply identify \( \psi_m = u^m \). For all \( \phi \in \mathcal{L}^2(S) \) let \( \phi_\Lambda := \sum_{m=-\Lambda}^{\Lambda} \phi_m u^m \) \( \) where \( \{ \phi_m \}_{m \in \mathbb{Z}} \) are the Fourier coefficients of \( \phi \) \( \) be its projection on \( \mathcal{H}_\Lambda \); clearly \( \phi_\Lambda \to \phi \) in the \( \mathcal{L}^2(S) \)-norm \( \| \cdot \| \). In this sense \( \mathcal{H}_\Lambda \) invades \( \mathcal{L}^2(S) \) as \( \Lambda \to \infty \).

The embedding \( \mathcal{I} \) induces the one \( \mathcal{J} : \mathcal{A}_\Lambda \hookrightarrow B[\mathcal{L}^2(S)] \); by definition, \( \mathcal{A}_\Lambda \) annihilates \( \mathcal{H}_\Lambda \).

The operators \( L, \mathcal{T} \) coincide on \( \mathcal{H}_\Lambda \), and we easily check that on the domain \( D(L) \subset \mathcal{L}^2(S)^7 \mathcal{T} \to L \) strongly as \( \Lambda \to \infty \). Similarly, \( f(\mathcal{T}) \to f(L) \) strongly on \( D[f(L)] \) for all measurable functions \( f(s) \).

Bounded (in particular, continuous) functions \( f \) on the circle, acting as multiplication operators \( f \cdot : \phi \in \mathcal{L}^2(S) \mapsto f \phi \in \mathcal{L}^2(S) \), make up a subalgebra \( B(S) \) [resp. \( C(S) \)] of \( B[\mathcal{L}^2(S)] \). An element of \( B(S) \) is actually an equivalence class \( [f] \) of bounded functions differing from \( f \) only on a set of zero (Lebesgue-)measure, because for any \( f_1, f_2 \in [f] \) and \( \phi \in \mathcal{L}^2(S) \) \( f_1 \phi, f_2 \phi \) differ only on a set of zero measure, and therefore are two equivalent representatives of the same element of \( \mathcal{L}^2(S) \). Since \( f \) belongs also to \( \mathcal{L}^2(S) \), by Carleson's theorem \[ f_N(\varphi) := \sum_{m=-N}^{N} f_m e^{im\varphi} \] converges to \( f(\varphi) \) as \( N \to \infty \) for almost all \( \varphi \),

\[ ^7L \text{ is unbounded. } \phi \in D(L) \text{ amounts to } \sum_{m \in \mathbb{Z}} m^2|\phi_m|^2 < \infty. \]
implying that \( f_N(\varphi)\phi(\varphi) \rightarrow f(\varphi)\phi(\varphi) \) almost everywhere; in other words \( f_\infty \in [f] \), where we have abbreviated

\[
f_\infty (\varphi) := \lim_{N \to \infty} f_N (\varphi) = \lim_{N \to \infty} \sum_{m=-N}^{N} f_m e^{im\varphi},
\]

and each class can be identified by the corresponding Plancherel-Fourier series (30).

The natural fuzzy analog of the vector space \( B(S) \) is the vector space of polynomials in \( \xi^+ \) (or \( \xi^- \)) of degree \( 2\Lambda \) at most, or equivalently

\[
C_\Lambda := \left\{ \sum_{h=-2\Lambda}^{2\Lambda} f_h \eta^h, f_h \in \mathbb{C} \right\} \subset A_\Lambda \subset B[\mathcal{L}^2(S)],
\]

where we have abbreviated \( \eta^\pm := \sqrt{2}\xi^\pm \) (so that \( \eta^+ \rightarrow u^+ \)), \( \eta^h := (\eta^\pm)^h \) if \( h \geq 0 \), \( \eta^h := (\eta^-)^h \) if \( h < 0 \). In other words \( \eta^h \) are the fuzzy analogs of the \( u^h \) considered as operators acting by multiplication on \( \phi \in \mathcal{L}^2(S) \).

The operators \( \eta^\pm \) converge strongly to \( u^\pm \), because

\[
(\eta^+-u)\phi = (\eta^+-u) \sum_{m \in \mathbb{Z}} \phi_m u^m = \sum_{m=-\Lambda}^{\Lambda-1} \left[ \sqrt{1+\frac{m(m+1)}{k}} - 1 \right] \phi_m u^{m+1} - \sum_{m<-\Lambda,m\geq \Lambda} \phi_m u^{m+1} \Rightarrow \|

\|(\eta^+-u)\phi\|^2 \leq \sum_{m=-\Lambda}^{\Lambda-1} \frac{m^2(m+1)^2}{4k^2} |\phi_m|^2 + \sum_{|m|\geq \Lambda} |\phi_m|^2 \leq \frac{\Lambda^2(\Lambda+1)^2}{4k^2} \|\phi\|^2 + \sum_{|m|\geq \Lambda} |\phi_m|^2,
\]

and by (15) the rhs goes to zero as \( \Lambda \rightarrow \infty \); the first inequality follows from

\[
|m| \leq \Lambda \Rightarrow 0 \leq m(m+1) \leq \Lambda(\Lambda+1), \quad \varepsilon > 0 \Rightarrow \sqrt{1+\varepsilon} - 1 < \varepsilon/2.
\]

Similarly one shows that \( \eta^- \rightarrow u^- \). Since for all \( \Lambda > 0 \) \( \eta^\pm \) vanish on \( \mathcal{H}_k^\Lambda \), then choosing \( \phi = u^{\Lambda+1} \) one finds \( \eta^\pm \phi = 0 \), \( \|(\eta^+-u)\phi\| = \|u^{\Lambda+2}\| = 1 \), implying \( \|\eta^+-u\| \geq 1 \) for all \( \Lambda \). This prevents \( \eta^\pm \) to converge to \( u^\pm \) in operator norm.

The previous result extends to all \( f \in B(S) \); in particular, to \( f \in C(S) \). Let

\[
\hat{f}_\Lambda := \sum_{h=-2\Lambda}^{2\Lambda} f_h \eta^h \in A_\Lambda \subset B[\mathcal{L}^2(S)].
\]

**Proposition 3.3.** If we choose \( k(\Lambda) \geq 2\Lambda(\Lambda+1)(2\Lambda+1)^2 \), then for all \( f,g \in B(S) \) the following strong limits as \( \Lambda \rightarrow \infty \) hold: \( \hat{f}_\Lambda \rightarrow f \cdot, \) \( \hat{f} \hat{g}_\Lambda \rightarrow fg \cdot \), and \( \hat{f}_\Lambda \hat{g}_\Lambda \rightarrow fg \cdot \).

(the proof is in appendix 6.3). The last statement says that the product in \( A_\Lambda \) of the approximations \( \hat{f}_\Lambda, \hat{g}_\Lambda \) goes to the product in \( B[\mathcal{L}^2(S)] \) of \( f \cdot, g \cdot \).
4 \(O(3)\)-equivariant fuzzy sphere

When \(D = 3\) one can associate a pseudovector \(L_i = \frac{1}{2} \varepsilon^{ijk} L_{jk}\) to the antisymmetric matrix \(L_{ij}\) of the angular momentum components. For all vectors \(v\) depending on \(x, i \nabla\) we shall use either the components \(v^i\) or the ones \(v^a, a \in \{-, 0, +\}\), defined by

\[
\begin{pmatrix}
v^+ \\
v^-\\
v^0
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
v^1 \\
v^2 \\
v^3
\end{pmatrix}.
\]  
(34)

\((U\) is a unitary matrix\) which fulfill

\[
[L_a, v^a] = 0, \quad [L_0, v^0] = \pm v^0, \quad [L_\pm, v^\mp] = \pm v^0, \quad [L_\pm, v^0] = \mp v^\pm.
\]

(35)

In particular, \(x^0 \equiv z, \quad x^\pm = x^1 \mp ix^2 = \frac{x^1 \pm iy}{\sqrt{2}} \). We set \(t^a := \frac{x^a}{r}\). Correspondingly, the metric matrix \(\eta_{ij} = \delta_{ij}\) becomes \(\tilde{\eta}_{ab} = (U \eta U^T)_{ab} = \delta_{-ab}\). Moreover, \((8)\) becomes \(\Delta = \frac{1}{r^2} \partial_r^2 r - \frac{1}{r^2} L^2\). We look for the form \(\psi_i^m(r, \theta, \phi) = \frac{f(r)}{r} Y_i^m(\theta, \phi)\), where \(Y_i^m(\theta, \phi)\) are the spherical harmonics,

\[
L^2 Y_i^m(\theta, \phi) = (l+1)Y_i^m(\theta, \phi), \quad L_3 Y_i^m(\theta, \phi) = m Y_i^m(\theta, \phi).
\]

Equation \((9)\) becomes \(-f''(r) + \left[V(r) + \frac{l(l+1)}{r^2} - E\right] f(r) = 0\). By condition \((5)\), in the region \(|r-1| \leq \sqrt{\frac{E-V_0}{2k}}\) we can neglect the terms of order higher than two in the Taylor expansion of \(\frac{1}{r^2} = 1 - 2(r-1) + 3(r-1)^2 + ...\), \(V(r) = V(1) + 2k(r-1)^2 + ...\) and thus approximate this equation by the eigenvalue equation for a 1–dimensional harmonic oscillator, that is

\[
-f''(r) + k_l(r - \tilde{r}_l)^2 f(r) = \tilde{E} f(r) \quad \text{(36)}
\]

with \(k_l := 2k + 3l(l+1)\), \(\tilde{r}_l := \frac{2k + 4l(l+1)}{2k + 3l(l+1)} = 1 + \frac{l(l+1)}{2k} - \frac{3l^2(l+1)^2}{4k^2} + O(k^{-3})\), \(\tilde{E} := E - V(1) - l(l+1) + \frac{l^2(l+1)^2}{2k + 3l(l+1)}\).

The square-integrable solutions of \((36)\) have the form

\[
f_{n,l}(r) = N_l e^{-\frac{r-\tilde{r}_l}{2} \sqrt{\tilde{r}_l}} H_n \left(\sqrt{\frac{r-\tilde{r}_l}{\tilde{r}_l}}\right), \quad n = 0, 1, ..., \]

and \(\tilde{E} = (2n + 1) \sqrt{\tilde{r}_l}\). Choosing \(V(1)\) such that in the lowest level (characterized by \(n = l = 0\) \(E = 0\), one has \(V(1) = -\sqrt{k_0} = -\sqrt{2k}\) and, consequently,

\[
E \equiv E_{n,l} := l(l+1) + (2n+1) \sqrt{2k + 3l(l+1)} + V(1) - \frac{l^2(l+1)^2}{2k + 3l(l+1)} = l(l+1) + 2n \sqrt{2k + O\left(\frac{1}{\sqrt{k}}\right)}
\]

\[\text{15}\]
The term \( l(l+1) \) gives exactly what we wish, (part of) the spectrum of \( L^2 \) (the Laplacian on the sphere). To eliminate the subsequent, undesired term we fix the energy cutoff \( E < 2\sqrt{2k} \) so as to exclude all the states with \( n > 0 \), i.e. “freeze” radial oscillations: \( n = 0 \). Therefore we impose an energy cut-off \( E = \Lambda(\Lambda + 1) \), that is we project the theory to the finite-dimensional subspace \( \mathcal{H}_E \equiv \mathcal{H}_\Lambda \subset \mathcal{H} \) spanned by the \( \psi_l^m := \psi_{0,l,m} \) with \( |m| \leq l \) and \( l \leq \Lambda \). We denote as \( P_\Lambda \) the projection over \( \mathcal{H}_\Lambda \) and abbreviate \( E_l = E_{0,l} \). For consistency one must choose

\[
\Lambda(\Lambda + 1) \leq 2\sqrt{2k}. \tag{37}
\]

The spectrum of \( \mathcal{H}_E \) becomes the whole spectrum \( \{l(l+1)\}_{l \in \mathbb{N}_0} \) of \( L^2 \) in the limit \( \Lambda, k \to \infty \) respecting (37). The eigenfunction of \( L^2, L_3, H \) with eigenvalues respectively \( l(l+1), m, E_l = l(l+1) + O(\frac{1}{\sqrt{k}}) \) is

\[
\psi_l^m(r, \theta, \varphi) = \frac{N_l}{r} e^{-\frac{(r-r_0)^2}{2}} Y_l^m(\theta, \varphi) \tag{38}
\]
at the leading order in \( k \). The actions of the \( \mathcal{L}_a (a \in \{-1, 0, +1\}) \) and \( \mathcal{H} \) are therefore

\[
\mathcal{L}_0\psi_l^m = m\psi_l^m, \quad \mathcal{L}_\pm\psi_l^m = \frac{\sqrt{(l\pm m)(l\pm m+1)}}{\sqrt{2}} \psi_{l+1}^{m\pm 1} =: \gamma_l^\pm m\psi_{l+1}^{m\pm 1}, \quad \mathcal{H}\psi_l^m = l(l+1)\psi_l^m \tag{39}
\]
[the last relation holds up to \( O(1/\sqrt{k}) \)]. In the appendix we compute the normalization factor \( N_l \); moreover, we show that the action of \( x^a \) on the vectors \( \psi_l^m \) reads

\[
\pi^a\psi_l^m = \begin{cases} 
ct A_l^{a,m} \psi_{l-1}^{m+a} + c_{l+1} B_l^{a,m} \psi_{l+1}^{m+a} & \text{if } l < L, \\
c_l A_l^{a,m} \psi_{L-1}^{m+a} & \text{if } l = L, \\
0 & \text{otherwise,}
\end{cases} \tag{40}
\]

where \( A_l^{a,m}, B_l^{a,m} \) are the coefficients involved in the formula

\[
t^a Y_l^m = A_l^{a,m} Y_{l-1}^{m+a} + B_l^{a,m} Y_{l+1}^{m+a}, \tag{41}
\]

which are explicitly reported in (93), while, up to terms \( O(1/k^{3/2}) \),

\[
c_l = \sqrt{1 + \frac{l^2}{k}} \quad 1 \leq l \leq \Lambda, \quad c_0 = c_{\Lambda+1} = 0. \tag{42}
\]

We now adopt (39-42) as exact definitions of \( \pi^a, \mathcal{L}_i, \mathcal{H} \). The \( \pi^a, \mathcal{L}_i \) can be obtained by the inverse transformation of (34). In the appendix we prove

**Proposition 4.1.** The \( \pi^a, \mathcal{L}_i \) defined by \( \pi^a, \mathcal{L}_i, \mathcal{H} \) generate the *-algebra \( \mathcal{A}_\Lambda := \text{End}(\mathcal{H}_\Lambda) \simeq M_{(\Lambda+1)^2}(\mathbb{C}) \) of observables on \( \mathcal{H}_\Lambda \). They fulfill

\[
\prod_{l=0}^{\Lambda} \left( \mathcal{L}_l^2 - l(l+1)L \right) = 0, \quad \prod_{m=-l}^{l} (\mathcal{L}_3 - mL) \tilde{P}_l = 0, \quad (\pi^\pm)^{2\Lambda+1} = 0, \tag{43}
\]

\[
\pi^a\mathcal{L}_i = \mathcal{L}_i\pi^a, \quad [\mathcal{L}_i, \pi^a] = i\epsilon^{ijk} \mathcal{L}_j, \quad [\mathcal{L}_i, \mathcal{L}_j] = i\epsilon^{ijk} \mathcal{L}_k, \tag{44}
\]

\[
\pi^a \mathcal{L}_i = 0, \quad [\pi^a, \pi^b] = i\epsilon^{ijk} \left( \frac{I}{k} + L \right) \mathcal{L}_h, \quad i, j, h \in \{1, 2, 3\} \tag{45}
\]
where $\tilde{L}^2 := \tilde{L}_i \tilde{L}^a = \tilde{L}_a \tilde{L}^a$ is $L^2$ projected on $\mathcal{H}_\Lambda$, $\tilde{P}_l$ is the projection on its eigenspace with eigenvalue $l(l+1)$, and $K = \frac{1}{k} + \frac{\Lambda^2}{2 \Lambda + 1}$. Moreover, the square distance from the origin is

$$\mathcal{R}^2 := \bar{\pi}^i \bar{\pi}^i = \bar{\pi}^a \bar{\pi}^a = 1 + \frac{\tilde{L}^2 + 1}{k} - \left[ \frac{1}{k} + \frac{1}{2 \Lambda + 1} \right] \frac{\Lambda + 1}{2 \Lambda + 1} \tilde{P}_\Lambda.$$  

(46)

By the last equation, again $\mathcal{R}^2$ can be expressed as a function of $\tilde{L}^2$ only, grows with the latter, and its spectrum collapses to 1 (apart from the highest eigenvalue) as $k \to \infty$.

Of course, relations (45), (46) hold only at leading order in $1/\sqrt{k}$ if also (39), (42) do.

To obtain a fuzzy space depending only on one integer $\Lambda$ we can choose $k$ as a function of $\Lambda$ fulfilling (37); the commutative limit will be simply $\Lambda \to +\infty$ (what implies $k \to +\infty$). One possible choice is $k = \Lambda^2(\Lambda + 1)^2$; then (45) becomes

$$\left[ \bar{\pi}^i, \bar{\pi}^j \right] = i \varepsilon^{ijk} \left[ -\frac{I}{\Lambda^2(\Lambda + 1)^2} + \left( \frac{1}{\Lambda^2(\Lambda + 1)^2} + \frac{1}{2 \Lambda + 1} \right) \tilde{P}_\Lambda \right] \tilde{L}_k$$

and $\mathcal{R}^2 \to 1$ as well.

We note that relations (44), (45), (47) are similar to those defining the Snyder’s Lie algebra, because the commutator of the coordinates is a polynomial in the generator of rotations $\tilde{L}_i$, more precisely proportional to the $\tilde{L}_i$ apart on $\mathcal{H}_\Lambda$, and therefore are invariant under parity $\bar{\pi}^a \to -\bar{\pi}^a$ (because $L_3$ and $L^2$ are), contrary to the fuzzy sphere of Madore.

The operators $\partial_a = P_\Lambda \frac{\partial}{\partial \bar{\pi}^a} P_\Lambda$ are such that

$$\langle \psi_{\Lambda'}^m, \partial_a \psi_\Lambda^m \rangle \neq 0 \quad \Rightarrow \quad l' = l \pm 1, \ m' = m - a;$$

the explicit actions of the $\partial_a$ and of the commutator $[\partial_a, \partial_b]$ are in the appendix. As with $D = 2$, the action of the $\partial_a$ on the eigenfunction $\psi_{\Lambda}^m$ gives a vector which has a non trivial projection on the Hilbert subspace corresponding to $n = 1$. Consequently, neither $\partial_a - \partial_a$ nor the commutator $[\partial_a, \partial_b]$ vanish as $k \to +\infty$, i.e. $\partial_a$ has not the usual commutative limit.

### 4.1 Realization of the algebra of observables through $U\, so(4)$

As the Lie algebra $su(2)$ is spanned by $\{ E_i \}_{i=1}^3$ fulfilling

$$[E_i, E_j] = i \varepsilon^{ijk} E_k,$$

(48)

$so(4) \simeq su(2) \oplus su(2)$ is spanned by $\{ E_i^1, E_i^2 \}_{i=1}^3$, where we have abbreviated $E_i^1 := E_i \otimes 1$, $E_i^2 := 1 \otimes E_i$, and

$$[E_i^1, E_j^1] = 0, \quad [E_i^1, E_j^1] = i \varepsilon^{ijk} E_k^1, \quad [E_i^2, E_j^2] = i \varepsilon^{ijk} E_k^2.$$  

(49)

$L_i = E_i^1 + E_i^2$ and $X_i = E_i^1 - E_i^2$ make up an alternative basis of $so(4)$ and fulfill

$$[L_i, L_j] = i \varepsilon^{ijk} L_k, \quad [L_i, X_j] = i \varepsilon^{ijk} X_k, \quad [X_i, X_j] = i \varepsilon^{ijk} L_k.$$  

(50)
The \( L_i \) close another \( su(2) \) Lie algebra. Applying the transformation (34) we obtain alternative generators labelled by \( a \in \{-, 0, +\} \), fulfilling

\[
[L_+, L_-] = L_0, \quad [L_0, L_\pm] = \pm L_\pm = [X_0, X_\pm], \quad [X_+, X_-] = L_0, \quad [L_\pm, X_\mp] = \pm X_0, \quad [L_0, X_\mp] = \pm X_\pm = [X_0, L_\pm], \quad [L_a, X_a] = 0
\]  

(no sum over \( a \)), where we have abbreviated \( L^2 = L_i L_i = L_a L_{-a} \), \( X^2 = X_i X_i = X_a X_{-a} \).

Let \( \pi \) be the unitary irreducible representation of \( Usu(2) \) on the \((2j+1)\)-dimensional Hilbert space \( V_j \); this is characterized by the eigenvalue \( j(j+1) \) of the Casimir \( C := E_i E_i \). The tensor product representation \( \pi_\Lambda := \pi_\Lambda \otimes \pi_\Lambda \) of \( Uso(4) \cong Usu(2) \otimes Usu(2) \) on the Hilbert space \( V_\Lambda := V_{\frac{\Lambda}{2}} \otimes V_{\frac{\Lambda}{2}} \) is characterized by the conditions \( C^1 := E_i^1 E_i^1 = \frac{\Lambda+1}{2}(\frac{\Lambda+1}{2} + 1) = E_i^2 E_i^2 =: C^2 \), or equivalently

\[
X \cdot L = L \cdot X = 0, \quad X^2 + L^2 = \Lambda(\Lambda+2);
\]

here and below we drop the symbol \( \pi_\Lambda \). \( V_\Lambda \) admits an orthonormal basis consisting of common eigenvectors of \( L^2 \) and \( L_0 \):

\[
L_0 |l, m\rangle = m |l, m\rangle, \quad L^2 |l, m\rangle = l(l+1) |l, m\rangle \quad \text{with} \quad 0 \leq l \leq \Lambda \quad \text{and} \quad |m| \leq l,
\]

in standard ket notation. \( V_\Lambda, H_\Lambda \) have the same dimension \((\Lambda+1)^2\) and the same decomposition in irreducible representations of the \( L_i \) subalgebra, and will be eventually identified.

We determine the action of the \( X_a \) on the \(|l, m\rangle \). Because of the commutation relations \([L_0, X_a] = aX_a \) it must be \( X_a |l, m\rangle = \sum_{j=0}^{\Lambda} \alpha_{\pm}^{a,m} |j, m+a\rangle \). In the appendix we show that \( \alpha_{\pm}^{a,m} = 0 \) unless \( j = \pm l \), and more precisely that the previous relations are fulfilled by

\[
X_a |l, m\rangle = d_l A_l^{a,m} |l-1, m+a\rangle + d_{l+1} B_l^{a,m} |l+1, m+a\rangle, \quad d_l := \sqrt{(\Lambda+1)^2 - l^2}.
\]  

The operators on \( H_\Lambda \), and in particular \( \overline{T}_a, \overline{X}^a \), are naturally realized in \( \pi_\Lambda \) on \( Usu(2) \otimes Usu(2) \), identifying \( \psi^{\Lambda}_i \) as the vectors of the canonical basis \(|l, m\rangle \). For simplicity, we introduce the operator \( \lambda := [\sqrt{4L^2} + 1 - 1]/2; |l, m\rangle \) is an eigenvector with eigenvalue \( l \). The Ansatz

\[
\overline{T}_a = L_a, \quad \overline{X}^a = g^*(\lambda) X_a g(\lambda),
\]

automatically fulfills the hermiticity relations \( \overline{X}^{a\dagger} = \overline{X}^{-a} \). Applying \( \overline{X}^a \) to \(|l, m\rangle \) we find

\[
\overline{X}^a |l, m\rangle = g(l) g^*(l-1) d_l A_l^{a,m} |l-1, m+a\rangle + g^*(l+1) g(l) d_{l+1} B_l^{a,m} |l+1, m+a\rangle;
\]  

this agrees with (40) if and only if for \( l > 1 \)

\[
g^*(l-1) g(l) = \frac{c_l}{d_l} = \frac{\sqrt{1+\frac{l^2}{4}}}{\sqrt{(\Lambda+1-l)(\Lambda+1+l)}},
\]

which is solved by

\[
g(l) = \sqrt{\prod_{h=0}^{l-1}(\Lambda+l-2h) \prod_{j=0}^{\lfloor \frac{l-1}{2} \rfloor} \frac{1 + \frac{(l-2j)^2}{k}}{1 + \frac{(l-2j)^2}{k}}},
\]  

(59)
where again \([b]\) stands for the integer part of \(b\). Alternatively, using the basic property \(\Gamma(z + 1) = z\Gamma(z)\) of the Euler gamma function we can express a solution in the form

\[
g(l) = \sqrt{\frac{\Gamma\left(\frac{L+1}{2}\right) \Gamma\left(\frac{L+1}{2} + 1\right)}{\Gamma\left(\frac{L+1}{2} + 1\right) \Gamma\left(\frac{L+1}{2}\right)}} \frac{\Gamma\left(\frac{l}{2} + 1 + \frac{i\sqrt{2}}{2}\right) \Gamma\left(\frac{l}{2} + 1 - \frac{i\sqrt{2}}{2}\right)}{\sqrt{\Gamma\left(\frac{l+1}{2} + \frac{i\sqrt{2}}{2}\right) \Gamma\left(\frac{l+1}{2} - \frac{i\sqrt{2}}{2}\right)}}
\]  

(60)

(see the Appendix); this makes sense also for generic complex argument \(l\). The inverse of the transformation (56) is clearly \(X_\alpha = [g^*(\lambda)]^{-1} \Phi_\alpha [g(\lambda)]^{-1}\).

We have thus proved by an explicit construction

**Proposition 4.2.** Formulas (56), (60) define a \(O(3)\)-equivariant \(*\)-algebra isomorphism between the algebra \(A_\Lambda = \text{End}(H_\Lambda)\) of observables (endomorphisms) on \(H_\Lambda\) and the \(C_1 = C_2 = \frac{3}{2} \left(\frac{3}{2} + 1\right)\) irreducible representation of \(USO(4) \simeq USU(2) \otimes USU(2)\):

\[
A_\Lambda := \text{End}(H_\Lambda) \simeq M_N(\mathbb{C}) \simeq \pi_\Lambda[USO(4)], \quad N := (\Lambda + 1)^2.
\]

(61)

As already recalled, the group of \(*\)-automorphisms of \(M_N(\mathbb{C}) \simeq A_\Lambda\) is inner and isomorphic to \(SU(N)\), i.e. of the type (29) with \(g\) an unitary \(N \times N\) matrix with unit determinant. A special role is played by the subgroup \(SO(4)\) acting in the representation \(\pi_\Lambda\), namely \(g = \pi_\Lambda[\exp\alpha]\), where \(\alpha \in \text{so}(4)\). In particular, choosing \(\alpha = \alpha_i L_i\) (\(\alpha_i \in \mathbb{R}\)) the automorphism amounts to a \(SO(3) \subset SO(4)\) transformation (a rotation in 3-dimensional space). Parity \((L_i, X_i) \mapsto (L_i, -X_i)\), or equivalently \(E_1^1 \leftrightarrow E_2^2\) [the only automorphism of \(so(4)\) corresponding to the exchange of the two nodes in the Dynkin diagram], is a \(O(3) \subset SO(4)\) transformation with determinant \(-1\) in the \(X_1X_2X_3\) space, and therefore also in the \(\bar{x}_1 \bar{x}_2 \bar{x}_3\) space. This shows that (56) is equivariant under \(O(3)\), which plays the role of isometry group of this fuzzy sphere.

### 4.2 Convergence to \(O(3)\)-equivariant quantum mechanics on \(S^2\)

Here we explain in which sense our model converges to \(O(3)\)-equivariant quantum mechanics on the sphere as \(\Lambda \to \infty\).

The \(\psi^m_l \in H_\Lambda\) are the fuzzy analogs of the spherical harmonics \(Y^m_l\) considered just as elements of an orthonormal basis of the Hilbert space \(L^2(S)\). The decomposition of \(H_\Lambda\) into irreducible components under \(O(3)\) reads

\[
H_\Lambda = \bigoplus_{l=0}^{\Lambda} V_l, \quad V_l := \left\{ \sum_{m=-l}^{l} \phi^m_l \psi^m_l, \quad \phi^m_l \in \mathbb{C} \right\}.
\]

(62)

(62) becomes the decomposition of \(L(S^2)\) in the limit \(\Lambda \to \infty\). Consider the \(O(3)\)-covariant embedding \(I : H_\Lambda \hookrightarrow L^2(S)\) defined by

\[
I \left( \sum_{l=0}^{\Lambda} \sum_{m=-l}^{l} \phi^m_l \psi^m_l \right) = \sum_{l=0}^{\Lambda} \sum_{m=-l}^{l} \phi^m_l Y^m_l
\]
Below we shall drop the symbol $I$ and simply identify $\psi_i^m = Y_i^m$. For all $\phi \in L^2(S^2)$ let

$$\phi_\Lambda := \sum_{l=0}^\Lambda \sum_{m=-l}^l \phi_i^m Y_i^m,$$

where $\{\phi_i^m\}_{l,m}$ are the coefficients of the decomposition of $\phi$ in the orthonormal basis of spherical harmonics; clearly $\phi_\Lambda \to \phi$ in the $L^2(S^2)$-norm $\|\|$. In this sense $H_\Lambda$ invade $L^2(S^2)$ as $\Lambda \to \infty$.

The embedding $I$ induces the one $J : A_\Lambda \to B[L^2(S^2)]$; by construction, $A_\Lambda$ annihilates $H_\Lambda^\Lambda$.

The operators $L_i, \bar{L}_i$ coincide on $H_\Lambda$, and we can easily check that on the domain $D(L_i) \subset L^2(S^2)$ $[L_i] \to L_i$ strongly as $\Lambda \to \infty$. Similarly, $f(\bar{L}_i) \to f(L_i)$ strongly on $D[f(L_i)]$ for all measurable function $f(s)$.

Bound (in particular, continuous) functions $f$ on the sphere, acting as multiplication operators $f \cdot : \phi \in L^2(S^2) \mapsto f(\phi) \in L^2(S^2)$, make up a subalgebra $B(S^2)$ [resp. $C(S^2)$] of $B[L^2(S^2)]$. An element of $B(S^2)$ is actually an equivalence class $[f]$ of bounded functions differing from $f$ only on a set of zero measure, because for any $f_1, f_2 \in [f]$ and $\phi \in L^2(S^2)$ $f_1 \phi, f_2 \phi$ differ only on a set of zero measure, and therefore are two equivalent representatives of the same element of $L^2(S^2)$. Since $f$ belongs also to $L^2(S^2)$, then $f_N(\theta, \varphi) := \sum_{l=0}^N \sum_{m=-l}^l f_i^m Y_i^m(\theta, \varphi)$ converges to $f(\theta, \varphi)$ in the $L^2(S^2)$ norm as $N \to \infty$.

To introduce the fuzzy analogs of $B(S^2)$ and $f$ we need to introduce first the fuzzy analogs $\hat{Y}_i^m$ of the spherical harmonics $Y_i^m$ seen as elements of $B(S^2)$ (acting by multiplication $Y_i^m : \psi \mapsto Y_i^m \psi$). We recall that the $Y_i^m$ are trace-free, homogenous polynomials of degree $l$ in the $t_i$; as the $Y_i^m$ with fixed $l$ span the $(2l+1)$-dim irreducible representation $V_l$ of $so(3)$, they can be obtained recursively from the highest weight $Y_i^1 = M_l(t^+)^l$ (the absolute value of the normalization factor is $|M_l| = \sqrt{(2l+1)!} \leq 1/\sqrt{4\pi}$) applying powers of $L_-$, $\sqrt{2}L_- Y_i^m = \sqrt{(l+m)(l-m+1)} Y_i^{m-1}$, implying

$$Y_i^m = M_l \sqrt{(l+m)!2^{l-m}} (2l)!(l-m)! L^{-m}_{-}(t^+)^l.$$

We therefore define the $\hat{Y}_i^m$ by the formulae

$$\hat{Y}_i^m := M_l \sqrt{(l+m)!2^{l-m}} (2l)!(l-m)! L^{-m}_{-}(\bar{x}^+)^l.$$  \hspace{1cm} (63)

By use of $L_- \bar{x}^+ = \bar{x}^0, L_- \bar{x}^0 = -\bar{x}^-, L_- \bar{x}^0 = 0$ and of the Leibniz rule for $L_-$ we find that

$$L_-(\bar{x}^+)^l = \bar{x}^0 (\bar{x}^+)^{l-1} + \bar{x}^+ \bar{x}^0 (\bar{x}^+)^{l-2} + \cdots + (\bar{x}^+)^{l-1} \bar{x}^0,$$

for $l$ monomials in $\bar{x}^+, \bar{x}^0, \bar{x}^-$. and more generally that $L^h(\bar{x}^+)^l$ can be written as the sum $\sum'_{n \leq l^h}$ (not necessarily distinct) monomials in $\bar{x}^+, \bar{x}^0, \bar{x}^-$ with coefficients $\pm 1$. This homogenous polynomial, and therefore also $\hat{Y}_i^m$, is completely symmetric with respect to permutations of the factors $\bar{x}^a$,
because both the monomial \((x^+)^l\) and the Leibniz rule for \(L^-\) are. The same occurs with \(L^h(t^+)^l\). Hence

\[
Y^m_l - \hat{Y}^m_l = R^m_l \sum_{i} \pm (t^{a_1}t^{a_2} \ldots t^{a_l} - x^{a_1}x^{a_2} \ldots x^{a_l})^{\pm}, \quad R^m_l := M^l \sqrt{(l+m)!2^{l-m}} 
\]

As a fuzzy analog of the vector space \(B(S^2)\) we adopt

\[
\mathcal{C}_\Lambda := \left\{ \sum_{l=0}^{2\Lambda} f^m_l \hat{Y}^m_l, f^m_l \in \mathbb{C} \right\} \subset \mathcal{A}_\Lambda \subset B[\mathcal{L}^2(S^2)];
\]

here the highest \(l\) is \(2\Lambda\) because \((x^+)^{2\Lambda} \propto \hat{Y}^{2\Lambda}\) is the highest power of \(x^+\) acting nontrivially on \(\mathcal{H}_\Lambda\) (it does not annihilate \(\psi_{-\Lambda}\)). By construction

\[
\mathcal{C}_\Lambda = \bigoplus_{l=0}^{2\Lambda} \mathcal{V}_l, \quad \mathcal{V}_l := \left\{ \sum_{m=-l}^{l} f^m_l \hat{Y}^m_l, f^m_l \in \mathbb{C} \right\}
\]

is the decomposition of \(\mathcal{C}_\Lambda\) into irreducible components under \(O(3)\). \(\mathcal{V}_l\) is trace-free for all \(l > 0\), i.e. its projection on the singlet component \(\mathcal{V}_0\) is zero. \((66)\) becomes the decomposition of \(B(S^2), C(S^2)\) in the limit \(\Lambda \rightarrow \infty\). As a fuzzy analog of \(f \in B(S)\) we adopt

\[
\hat{f}_\Lambda := \sum_{l=0}^{2\Lambda} \sum_{|m| \leq l} f^m_l \hat{Y}^m_l \in \mathcal{A}_\Lambda \subset B[\mathcal{L}^2(S^2)].
\]

In appendix \(6.11\) we prove first that the operators \(x^a\) converge strongly to \(t^a\) as \(\Lambda \rightarrow \infty\) if we choose \(k(\Lambda)\) fulfilling \((37)\). Again, since for all \(\Lambda > 0\) the operator \(x^a\) annihilates \(\mathcal{H}_\Lambda\), \(x^a\) does not converge to \(t^a\) in operator norm. Then we prove the more general

**Proposition 4.3.** If we choose \(k(\Lambda) \geq 2^{3\Lambda+3}\Lambda^{\Lambda+5}(\Lambda+1)\), then for all \(f, g \in B(S^2)\) the following strong limits as \(\Lambda \rightarrow \infty\) hold: \(\hat{f}_\Lambda \rightarrow f \cdot, (fg)_\Lambda \rightarrow fg\) and \(\hat{f}_\Lambda \hat{g}_\Lambda \rightarrow fg\).

The last statement says that the product in \(\mathcal{A}_\Lambda\) of the approximations \(\hat{f}_\Lambda, \hat{g}_\Lambda\) goes to the product in \(B[\mathcal{L}^2(S)]\) of \(f \cdot, g\).

The above dependence of \(k\) on \(\Lambda\) is by no means optimal, i.e. better estimates will presumably allow to prove the same result with a function \(k(\Lambda)\) growing much less rapidly.

**5 Final remarks, outlook and conclusions**

For both dimensions \(d = 1, 2\) we have introduced a finite-dimensional approximation of quantum mechanics on the sphere \(S^d_\Lambda\) by projecting below a suitable energy cutoff \(E\) quantum mechanics of a particle in \(\mathbb{R}^D\) \((D = d+1)\) configuration space subject to a rotation invariant
potential $V(r)$ with a very sharp minimum on the sphere of radius $r = 1$. By parametrizing both the confining parameter $k$ and $E$ by a positive integer $\Lambda$ we have obtained a sequence of $O(D)$-equivariant such approximations. The following common features emerge. The algebra of observables $A_\Lambda$ on the Hilbert space $H_\Lambda$ is isomorphic to $\pi_\Lambda[Uso(D+1)]$, where $\pi_\Lambda$ is a suitable irreducible unitary representation of $Uso(D+1)$ on $H_\Lambda$. On the other hand, $H_\Lambda$ carries a reducible representation of the subalgebra $Uso(D)$ generated by the projected angular momentum components $L_{ij}$, more precisely the direct sum of all irreducible representations fulfilling the cutoff condition $L^2 \leq \Lambda(A + d - 1)$; a similar decomposition holds for the subspace $C_A \subset A_\Lambda$ of completely symmetrized polynomials in the projected coordinates $\pi^i$. In the $\Lambda \to \infty$ limit these become the decompositions of the Hilbert space $L^2(S^d)$ and of the algebra of operators $C(S^d)$ acting on $L^2(S^d)$, respectively. The $\pi^i$, or alternatively the elements $X^i$ of a corresponding basis of $so(D+1) \setminus so(D)$, generate the algebra of observables $A_\Lambda \cong \pi_\Lambda[Uso(D+1)]$ [the relation between them is of the form $\pi^i = g(L^2)X^i g(L^2)$]; their commutators span $so(D)$, the Lie algebra of the angular momentum components $L_{ij}$, as in the Snyder algebra; a basis of $A_\Lambda$ is made up of $\pi_\Lambda$-images of monomials - with a fixed ordering - in the elements of a basis of $so(D+1)$, by the Poincaré-Birkhoff-Witt theorem. The square distance $R^2 = \mathfrak{p}^i \mathfrak{p}^i$ from the origin is not identically 1, but a function of $L^2$ with spectrum close to 1. The whole construction is $SO(D+1)$- and $O(D) \subset SO(D+1)$-equivariant. We naturally embed $H_\Lambda$ in $L^2(S^d)$ and $A_\Lambda$ in the algebra $O(S^d)$ of operators on $L^2(S^d)$. In the $\Lambda \to \infty$ limit we recover ordinary $O(D)$-equivariant quantum mechanics on $L^2(S^d)$, with the observables $L_{ij}$, $\pi^i$ going to the angular momentum components $L_{ij}$ and to the coordinates $x^i$ of $S^d$ configuration space, because $R^2 \to 1$ and $H_\Lambda, A_\Lambda$ respectively “invade” $L^2(S^d)$ and the whole $O(S^d)$. In particular, every element $f$ of $B(S^d)$ or $C(S^d)$ is the strong limit of a sequence $\{f_\Lambda\}$ of elements $\hat{f}_\Lambda \in C_\Lambda$.

Our approach seems easily applicable with the same features to higher dimensions, where comparison with previous proposals is possible. The fuzzy spheres of dimension $d \geq 3$ of [25, 28, 29] are based on the algebra $\text{End}(V)$ of endomorphisms of the carrier space $V$ of a particular irreducible representation of $SO(d+1)$, so that the square distance from the origin $R^2$ be central and can be set strictly equal to 1. The commutation relations are also of the Snyder type (although presumably slightly different from ours), hence equivariant with respect to the full group $O(d+1)$. In [26, 27] Steinacker and Sperling consider the possibility of a fuzzy 4-sphere $S^d_N$ with a reducible representation of $Uso(5)$ on a Hilbert space $V$ obtained decomposing an irreducible representation $\pi$ of $Uso(6)$ characterized by a triple of highest weights $(N, n_1, n_2)$; so $\text{End}(V) \simeq \pi[Uso(6)]$, in analogy with our scheme. The elements $X^i$ of a basis of $so(6) \setminus so(5)$ play the role of noncommutative cartesian coordinates. As a consequence the $O(5)$-scalar $R^2 = X^i X^i$ is no longer central, but its spectrum is still very close to 1 if $N \gg n_1, n_2$ [because then the decomposition of $V$ contains few irreducible representations under $SO(5)$]; note that in our approach this is guaranteed by adopting suitable $\pi^i = g(L^2)X^i g(L^2)$ rather than $X^i$ as noncommutative cartesian coordinates. If $n_1 = n_2 = 0$ then $R^2 \equiv 1$, and one recovers the fuzzy 4-sphere [25]. Their physical interpretation of $\text{End}(V)$ is that it represents a fuzzy approximation of some fibre bundle on a sphere $S^d$ (rather than of the algebra of observables of a quantum particle on a $S^d$). If one wishes to describe a scalar field on such fuzzy $S^d$, or more generally $S^d$ with $d \geq 3$, one can project out the unwanted modes of $\{\pi\}$, but this makes the product of spherical harmonics non-
associative. Alternatively, in [31, 29] unwanted modes are only suppressed in probability (not completely eliminated) in the path-integral of the quantum field theory by adding suitable kinetic terms in the action. Starting from fields on fuzzy $\mathbb{C}P^1 \simeq S^2$, in [32] this idea is used also to introduce an effective quantum field theory on a fuzzy $S^1$ by adding suitable kinetic terms in the action that suppress the modes $Y^l_m$ with $l < \Lambda$; the remaining $Y^\Lambda_m$, $|m| \leq \Lambda$, play the role of $u^m = e^{i m \varphi}$ as elements of a basis of a fuzzy circle.

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6 Appendix

We shall repeatedly use the formula

$$
\int_{-\infty}^{+\infty} e^{-a \rho^2 + b \rho} \rho^2 d\rho = e^{b^2/4a} \sum_{h=0}^{n/2} \left( \begin{array}{c} n/2 \\ h \end{array} \right) \left( \frac{b}{2a} \right)^{n-2h} \frac{(2h-1)!!}{(2a)^h} (68)
$$

(for all $a > 0$ and $b \in \mathbb{R}$), which can be easily derived from $\int_{-\infty}^{+\infty} e^{-z^2} dz = \sqrt{\pi}$ through integration by parts and a linear change of integration variable.

6.1 Calculation of a rather general scalar product in $D = 2$

As a preliminary step, we prove a formula regarding a matrix element of a general form.

**Proposition 6.1.** For every entire function $f(\rho)$ not depending on $k$ and $h \in \mathbb{Z}$ the following asymptotic expansion in $1/\sqrt{k}$ holds

$$
T^f_{m,m'} := \langle \psi_{m'}, f(\rho) e^{ih \varphi} \psi_m \rangle = \delta^h_{m'-m} K_{m,m'} \left[ \exp \left( \frac{\partial^2_{\rho}}{4c_{m,m'}} \right) f(\rho) \right]_{\rho=\rho_{m,m'}} (69)
$$

$$
c_{m,m'} := \frac{\sqrt{k_m} + \sqrt{k_{m'}}}{2} = \frac{1}{2} \sqrt{2k} \left[ 1 - \frac{1}{\sqrt{2k}} + \frac{3-m^2-m'^2}{4k} \right] + O \left( \frac{1}{k} \right),
$$

where

$$
\rho_{m,m'} := \frac{2 + \sqrt{k_{m'}} \rho_m + \sqrt{k_m} \rho_{m'}}{2c_{m,m'}} = \frac{2}{\sqrt{2k}} + \frac{m^2 + m'^2 + 2}{4k} + O \left( \frac{1}{k^{3/2}} \right), (70)
$$

$$
K_{m,m'} := \sqrt{\frac{4\pi^3}{c_{m,m'}}} N_m N_{m'} e^{-\frac{4}{\sqrt{2k}} N_m N_{m'} e^{-\frac{2}{\sqrt{2k} \sqrt{k_m} \rho_m}}} \frac{\sqrt{k_{m'}} \rho_{m'} + \sqrt{k_m} \rho_m}{\sqrt{k_{m'}} \rho_{m'} + \sqrt{k_m} \rho_m} = 1 + O \left( \frac{1}{k^2} \right),
$$

where $N_m = \sqrt{\frac{4\pi^3}{4\pi^3}} e^{-\frac{4}{\sqrt{2k} \sqrt{k_m} \rho_m}}$ is the normalization of $\psi_m$ (up to a phase). The powers in
$1/\sqrt{k}$ arise from the Taylor expansion of the exponential and the definition of $c_{m,m'}$, e.g.

$$T_{m,m'}^f = f(\rho_{m,m'}) + \frac{f''(\rho_{m,m'})}{4c_{m,m'}} + \frac{f^{(4)}(\rho_{m,m'})}{32(c_{m,m'})^2} + O \left( \frac{1}{k^{3/2}} \right).$$

If $f(\rho)$ has continuous derivatives up to order $2h + 1$ the above asymptotic expansion holds up to order $2h$.

As consequences, setting $a := 1 + \frac{9}{4} \frac{1}{\sqrt{2k}} + \frac{137}{64k}$, we find up to terms $O \left( 1/k^{3/2} \right)$

$$\sqrt{2} \langle \psi_{m'}, \psi_{m}^{m'-m} \rangle = \langle \psi_{m'}, e^{i(m'-m)\varphi} \psi_{m} \rangle = K_{m,m'} = 1,$$  \hspace{1cm} (71)

$$\langle \psi_{m'}, e^{i\rho+i(m'-m)\varphi} \psi_{m} \rangle = K_{m,m'} e^{\frac{\rho_{m,m'}+\pi n}{2}}, \hspace{1cm} (72)$$

$$\langle \psi_{m+1}, x^+ \psi_{m} \rangle = \frac{\langle \psi_{m+1}, e^{i\varphi} \psi_{m} \rangle}{\sqrt{2}} = \frac{K_{m,m+1}}{\sqrt{2}} e^{\frac{\rho_{m,m'+1}+\pi n_{m+1}}{2}} = \frac{1}{\sqrt{2}} \left[ 1 + \frac{9}{4\sqrt{2k}} \right] \sqrt{2} \frac{m(m+1)}{k}, \hspace{1cm} (73)$$

$$\langle \psi_{m-1}, x^- \psi_{m} \rangle = \frac{\langle \psi_{m}, x^\dagger \psi_{m-1} \rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left[ 1 + \frac{9}{4\sqrt{2k}} \frac{m(m-1)+137/32}{2k} \rho_{m,0} \right], \hspace{1cm} (74)$$

$$|\langle \psi_{m+1}, x^\pm \psi_{m} \rangle|^2 = \frac{a^2}{2} \left( 1 + \frac{m(m+1)}{k} \right). \hspace{1cm} (75)$$

**Proof of the proposition.** From [13] we find

$$T_{m,m'}^f = \int_0^{2\pi} d\varphi \int_0^{+\infty} dr \, r \, \overline{\psi_{m'}}(r, \varphi) f(\rho) e^{i\varphi} \psi_{m}(r, \varphi) \hspace{1cm} (76)$$

in the last step we have changed the integration variable $\rho \rightarrow z = \rho - \rho_{m,m'}$. Using the Taylor
expansion of \( f(\rho_m,\rho_{m'}) + z \) and the vanishing of integrals of odd functions over \( \mathbb{R} \) we find
\[
\int_{-\infty}^{+\infty} dz \ e^{-z^2c_{m,m'}} f(\rho_{m,m'} + z) = \int_{-\infty}^{+\infty} dz \ e^{-z^2c_{m,m'}} \sum_{n=0}^{\infty} \frac{f^{(n)}(\rho_{m,m'})}{n!} z^n
\]
\[
= \sum_{n=0}^{\infty} f^{(2n)}(\rho_{m,m'}) \int_{-\infty}^{+\infty} dz \ e^{-z^2c_{m,m'}} \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{f^{(2n)}(\rho_{m,m'})}{(4c_{m,m'})^n n!} \int_{-\infty}^{+\infty} dy e^{-y^2} y^{2n} \frac{(2n)!}{(2n)!}
\]
\[
= \sqrt{\frac{\pi}{c_{m,m'}}} \sum_{n=0}^{\infty} f^{(2n)}(\rho_{m,m'}) \frac{n!}{(4c_{m,m'})^n n!} \left[ \left. \frac{\partial^2}{\partial \rho^2} f(\rho) \right|_{\rho=\rho_{m,m'}} \right]
\]

here we have used the identity \( \int_{-\infty}^{+\infty} dy \ e^{-y^2} y^{2n} = \sqrt{\pi} (2n)!/2^n \), which can be proved iterating integration by parts. Replacing in (76) we find (69), with
\[
K_{m,m'} := \sqrt{\frac{4\pi^3}{c_{m,m'}}} \left[ \frac{\sum_{n=0}^{\infty} f^{(2n)}(\rho_{m,m'})}{(4c_{m,m'})^n n!} \frac{n!}{(2n)!} \right] \left. \frac{\partial^2}{\partial \rho^2} f(\rho) \right|_{\rho=\rho_{m,m'}}
\]

In particular choosing \( h = 0, f \equiv 1 \) and recalling the normalization condition \( T^{1}_{m,0} = ||\psi_m||^2 = 1 \) we determine the normalization factors \( N_m \):
\[
|N_m|^2 \frac{4\pi^3}{\sqrt{k_m}} e^{\frac{1}{2\sqrt{k_m} + \tilde{\rho}_m}} = 1 \quad \Rightarrow \quad N_m = \sqrt{\frac{4\pi^3}{\sqrt{k_m}}} e^{-\frac{1}{2\sqrt{k_m} + \tilde{\rho}_m}}
\]

Hence,
\[
\sqrt{\frac{4\pi^3}{c_{m,m'}}} \left[ \frac{\sum_{n=0}^{\infty} f^{(2n)}(\rho_{m,m'})}{(4c_{m,m'})^n n!} \frac{n!}{(2n)!} \right] \left. \frac{\partial^2}{\partial \rho^2} f(\rho) \right|_{\rho=\rho_{m,m'}}
\]
\[
= \frac{\sqrt{2}}{\sqrt{4k_m/k_{m'}} + \sqrt{k_m}} e^{-\frac{1}{2\sqrt{k_m} + \tilde{\rho}_m} - \frac{1}{2\sqrt{k_{m'}} + \tilde{\rho}_{m'}}}
\]

which replaced in (77) gives (70). We now determine \( c_{m,m'}, \rho_{m,m'}, K_{m,m'} \) at lowest order. By (14), up to \( O(1/k^2) \),
\[
\sqrt{k_m} = \sqrt{2k} \left[ 1 - \frac{1}{\sqrt{2k}} + \frac{3 - m^2}{2k} \right], \quad \sqrt{k_m} \tilde{\rho}_m = 1 + \frac{m^2 - 1}{\sqrt{2k}} + \frac{3 - 4m^2}{4k}
\]
\[
\frac{1}{\sqrt{k_m}} = \frac{1}{\sqrt{2k}} + \frac{1}{\sqrt{2k}} \sqrt{k_m} + \sqrt{k_{m'}} = \frac{1}{\sqrt{2k}} \left( \frac{1}{2} + \frac{1}{\sqrt{2k}} \right),
\]
\[
c_{m,m'} = \sqrt{k_m + \sqrt{k_{m'}}} = \sqrt{2k} \left[ 1 - \frac{1}{\sqrt{2k}} + \frac{3 - m^2}{4k} \right],
\]
\[
\rho_{m,m'} = \frac{2 + \sqrt{k_m \tilde{\rho}_m + \sqrt{k_{m'} \tilde{\rho}_{m'}}}}{2c_{m,m'}} = \frac{2}{\sqrt{2k}} + \frac{m^2 + m^2 + 2}{4k}.
\]
then using (10) and by algebraic manipulations one has

\[
K_{mm'} = \sqrt{\frac{2}{k_m + k_{m'}}} \cdot e^{\frac{2+\sqrt{k_m\tilde{\rho}_m}+\sqrt{k_{m'}\tilde{\rho}_{m'}}}{2(\sqrt{k_m}+\sqrt{k_{m'}})}} - \frac{\sqrt{k_m\tilde{\rho}_m}^2+\sqrt{k_{m'}\tilde{\rho}_{m'}}^2}{2\sqrt{k_m} - 2\sqrt{k_{m'}} - \tilde{\rho}_m - \tilde{\rho}_{m'}}
\]

and since

\[
\frac{\sqrt{k_m k_{m'}}}{2(\sqrt{k_m} + \sqrt{k_{m'}})} \left( \tilde{\rho}_m + \frac{1}{\sqrt{k_m}} - \frac{1}{\sqrt{k_{m'}}} \right)^2 = 0,
\]

one has \( K_{mm'} = 1 \), up to \( O\left( k^{-\frac{7}{2}} \right) \).

\[\square\]

### 6.2 Calculation of the action of operators \( \overline{\partial}_\pm \) in \( D = 2 \)

We first compute \( \overline{\partial}_+ \psi_m \):

\[
\overline{\partial}_+ \psi_m = \sum_h \psi_h \langle \psi_h, \overline{\partial}_+ \psi_m \rangle = \begin{cases} 
\psi_{m-1} \langle \psi_{m-1}, \partial_+ \psi_m \rangle & \text{if } -\Lambda + 1 \leq m \leq \Lambda \\
0 & \text{otherwise}. 
\end{cases}
\]

By (23) we can calculate this scalar product using (69) with \( h = -1 \) and

\[
f(\rho) = \frac{e^{-\rho}}{2} \left[ m - (\rho - \tilde{\rho}_m) \sqrt{k_m} \right];
\]

consequently,

\[
\langle \psi_{m-1}, \partial_+ \psi_m \rangle = \frac{1}{\sqrt{2}} \left( m - \frac{1}{2} - \frac{3}{8} m - \frac{3}{8} m^2 - \frac{m^3 - \frac{3}{2} m^2 + \frac{79}{32} m - \frac{63}{64}}{2k} \right) + O\left( \frac{1}{k^{\frac{7}{2}}} \right).
\]

From (78) we have

\[
\overline{\partial}_+ \psi_m = \begin{cases} 
\frac{1}{\sqrt{2}} \left( b + m - \frac{3m}{4\sqrt{2k}} - \frac{m^3 - \frac{3}{2} m^2 + \frac{27}{8} m}{2k} \right) \psi_{m-1} & \text{if } -\Lambda + 1 \leq m \leq \Lambda \\
0 & \text{otherwise}. 
\end{cases}
\]
where \( b = -\frac{1}{2} + \frac{3}{8\sqrt{2}k} + \frac{63}{128k} \). Moreover

\[
\partial_+ \psi_{m-1} = \sum_h \psi_h \langle \psi_h, \partial_- \psi_{m-1} \rangle = \begin{cases} \psi_m \langle \psi_m, \partial_- \psi_{m-1} \rangle & \text{if } -\Lambda + 1 \leq m \leq \Lambda \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\langle \psi_m, \partial_- \psi_{m-1} \rangle = \langle \partial_+ \psi_m, \psi_{m-1} \rangle = \langle \psi_{m-1}, \partial_+ \psi_m \rangle = \frac{1}{\sqrt{2}} \left( b + m - \frac{3m}{4\sqrt{2}k} - \frac{m^3 - \frac{3}{2} m^2 + \frac{79}{32} m}{2k} \right) + O \left( \frac{1}{k^3} \right)
\]

whence (replacing \( m \) with \( m + 1 \))

\[
\partial_+ \psi_m = \begin{cases} \frac{1}{\sqrt{2}} \left( b + m + 1 - \frac{3(m+1)}{4\sqrt{2}k} - \frac{(m+1)^3 - \frac{3}{2} (m+1)^2 + \frac{79}{32} (m+1)}{2k} \right) \psi_{m+1} & \text{if } -\Lambda \leq m \leq \Lambda - 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Eq. (79), (80) imply at leading order

\[
\partial_+ \partial_- \psi_m = \begin{cases} \frac{1}{2} \left( b + m + 1 - \frac{3(m+1)}{4\sqrt{2}k} - \frac{(m+1)^3 - \frac{3}{2} (m+1)^2 + \frac{79}{32} (m+1)}{2k} \right)^2 \psi_m & \text{if } -\Lambda \leq m \leq \Lambda - 1, \\ 0 & \text{otherwise,} \end{cases}
\]

whence

\[
[\partial_+, \partial_-] \psi_m = \begin{cases} \left( m - \frac{3m}{2\sqrt{2}k} - \frac{4m^3 + 47m}{2k} \right) \psi_m & \text{if } |m| \leq \Lambda - 1, \\ -\frac{1}{2} \left( b + m - \frac{3m}{4\sqrt{2}k} - \frac{m^3 - \frac{3}{2} m^2 + \frac{79}{32} m}{2k} \right)^2 \psi_m & \text{if } m = \Lambda, \\ \frac{1}{2} \left( b + m + 1 - \frac{3(m+1)}{4\sqrt{2}k} - \frac{(m+1)^3 - \frac{3}{2} (m+1)^2 + \frac{79}{32} (m+1)}{2k} \right)^2 \psi_m & \text{if } m = -\Lambda, \end{cases}
\]

and the analogous of laplacian

\[
(\partial_+ \partial_- + \partial_- \partial_+ \psi_m = \begin{cases} \left( m^2 + \frac{1}{4} - \frac{3m^2 + 47m}{4\sqrt{2}k} - \frac{2m^4 + 47m^2 + 27}{2k} \right) \psi_m & \text{if } |m| \leq \Lambda - 1, \\ \frac{1}{2} \left( b + m - \frac{3m}{4\sqrt{2}k} - \frac{m^3 - \frac{3}{2} m^2 + \frac{79}{32} m}{2k} \right)^2 \psi_m & \text{if } m = \Lambda, \\ \frac{1}{2} \left( b + m + 1 - \frac{3(m+1)}{4\sqrt{2}k} - \frac{(m+1)^3 - \frac{3}{2} (m+1)^2 + \frac{79}{32} (m+1)}{2k} \right)^2 \psi_m & \text{if } m = -\Lambda, \end{cases}
\]

Then one can conclude that there exists 4 polynomials \( P_1, P_2, Q_1, Q_2 \) such that

\[
[\partial_+, \partial_-] = P_1(\mathcal{L}) + Q_1(\mathcal{L}) \left( \tilde{P}_\Lambda - \tilde{P}_{-\Lambda} \right) \quad \text{and} \quad \bar{\partial}_+ \partial_- + \partial_- \partial_+ = P_2(\mathcal{L}) + Q_2(\mathcal{L}) \left( \tilde{P}_\Lambda - \tilde{P}_{-\Lambda} \right).
\]
6.3 Proof of proposition 3.3

Since

\[ \eta^h u_m = u^{m+h} \alpha_{m,m+h}, \text{ where } \alpha_{m,n} := \begin{cases} 
\prod_{j=m}^{n-1} \sqrt{1 + \frac{j(j+1)}{k}} & \text{if } \Lambda \geq n > m \geq -\Lambda, \\
1 & \text{if } \Lambda \geq n = m \geq -\Lambda, \\
\prod_{j=m}^{n-1} \sqrt{1 + \frac{j(j+1)}{k}} & \text{if } -\Lambda \leq n < m \leq \Lambda, \\
0 & \text{otherwise}, 
\end{cases} \]  

(84)

then, more explicitly,

\[ \hat{f}_{\Lambda} \phi = \sum_{h=-2\Lambda}^{2\Lambda} f_h \eta^h \sum_{m=-\Lambda}^{\Lambda} \phi_m u^m = \sum_{n=-\Lambda}^{\Lambda} u^n (\hat{f}_{\Lambda} \phi)_n, \text{ where } (\hat{f}_{\Lambda} \phi)_n := \sum_{m=-\Lambda}^{\Lambda} f_{n-m} \phi_m \alpha_{m,n}, \]

\[ (f - \hat{f}_{\Lambda}) \phi = \sum_{n=-\Lambda}^{\Lambda} u^n \chi_n + \sum_{|n|>\Lambda} u^n (f \phi)_n, \quad \chi_n := (f \phi)_n - (\hat{f}_{\Lambda} \phi)_n \]  

(85)

[here \((f \phi)_n\) is the \(n\)-th Fourier coefficient of \(f \phi \in L^2(S)\)], implying

\[ \|(f - \hat{f}_{\Lambda}) \phi\|^2 = \sum_{n=-\Lambda}^{\Lambda} |\chi_n|^2 + \sum_{|n|>\Lambda} |(f \phi)_n|^2. \]  

(86)

The second sum vanishes as \(\Lambda \to \infty\). To show that the first sum does as well we decompose

\[ \chi_n = \sigma_n - \tau_n, \quad \sigma_n := (f \phi)_n - \sum_{m=-\Lambda}^{\Lambda} f_{n-m} \phi_m, \quad \tau_n := (\hat{f}_{\Lambda} \phi)_n - \sum_{m=-\Lambda}^{\Lambda} f_{n-m} \phi_m \]

\[ \Rightarrow \sum_{n=-\Lambda}^{\Lambda} |\chi_n|^2 \leq 2 \sum_{n=-\Lambda}^{\Lambda} (|\sigma_n|^2 + |\tau_n|^2) \]  

(87)

But

\[ \sigma_n = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-in\varphi} f(\varphi) \phi(\varphi) - \sum_{m=-\Lambda}^{\Lambda} \phi_m \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i(n-m)\varphi} f(\varphi) \]

\[ = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-in\varphi} f(\varphi) \phi(\varphi) - \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-in\varphi} f(\varphi) \sum_{m=-\Lambda}^{\Lambda} \phi_m e^{im\varphi} \]

\[ = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-in\varphi} f(\varphi) [\phi(\varphi) - \phi_\Lambda(\varphi)] = (f[\phi - \phi_\Lambda])_n \Rightarrow \]

\[ \sum_{n=-\Lambda}^{\Lambda} |\sigma_n|^2 \leq \sum_{n \in \mathbb{Z}} |(f[\phi - \phi_\Lambda])_n|^2 = \|f[\phi - \phi_\Lambda]\|^2 \leq \|f\|_\infty^2 \|\phi - \phi_\Lambda\|^2 \xrightarrow{\Lambda \to \infty} 0 \]  

(88)
(we have used \(\|\phi - \phi_\Lambda\| \to 0\) as \(\Lambda \to \infty\), and on the other hand

\[
|\tau_n| \leq \sum_{m=-\Lambda}^{\Lambda} |f_{m-n}| |\phi_m| [\alpha_{m-n}-1] \leq \sum_{m=-\Lambda}^{\Lambda} F\Phi \left( 1 + \frac{\Lambda(\Lambda+1)}{k} \right)^{-\frac{|n-m|}{2}} - 1
\]

\[
\leq F\Phi \sum_{m=-\Lambda}^{\Lambda} \left( 1 + \frac{\Lambda(\Lambda+1)}{k} \right)^{\Lambda} - 1 = F\Phi(2\Lambda+1) \left( 1 + \frac{\Lambda'(\Lambda+1)}{k} \right)^{\Lambda} - 1
\]

\[
\leq F\Phi(2\Lambda+1) \left[ e^{\frac{\Lambda^2(\Lambda+1)}{k}} - 1 \right]
\]

where \(F = \max_{m \in \mathbb{Z}} |f_m|, \Phi = \max_{m \in \mathbb{Z}} |\phi_m|\), and we have used the inequality \((1+y)^{\Lambda} < e^{y\Lambda}\) for all \(y, \Lambda > 0\). Provided we choose \(k(\Lambda)\) sufficiently large, e.g. \(k \geq 2\Lambda(\Lambda+1)(2\Lambda+1)^2\), and note that \(e^y-1 < 2y\) if \(0 < y < 1/2\), we thus find \(|\tau_n| \leq F\Phi/(2\Lambda+1)\) and

\[
\sum_{n=-\Lambda}^{\Lambda} |\tau_n|^2 \leq \frac{F^2\Phi^2}{2\Lambda+1} \xrightarrow{\Lambda \to \infty} 0. \tag{89}
\]

By (86), (88), (89) we find

\[
\|(f - f_\Lambda)\phi\|^2 \leq \sum_{|n|>\Lambda} |(f\phi)_n|^2 + 2\|f\|_\infty^2 \|\phi - \phi_\Lambda\|^2 + \frac{2F^2\Phi^2}{2\Lambda+1} \xrightarrow{\Lambda \to \infty} 0, \tag{90}
\]

i.e. \(\hat{f}_\Lambda \to f\cdot\) strongly for all \(f \in B(S)\), as claimed. Replacing \(f \to fg\), we find also that \((fg)_\Lambda \to (fg)\cdot\) (strongly) for all \(f, g \in B(S)\). On the other hand, relation (90) implies also

\[
\|(f - f_\Lambda)\phi\|^2 \leq \|f\phi\|^2 + 2\|f\|_\infty^2 \|\phi - \phi_\Lambda\|^2 + \frac{2\|f\|_\infty^2 \|\phi\|^2}{2\Lambda+1} < 4\|f\|_\infty^2 \|\phi\|^2,
\]

\[
\|\hat{f}_\Lambda\phi\| \leq \|(\hat{f}_\Lambda-f)\phi\| + \|f\phi\| \leq \|(\hat{f}_\Lambda-f)\phi\| + \|f\|_\infty \|\phi\| \leq 3\|f\|_\infty \|\phi\|
\]

i.e. the operator norms \(\|\hat{f}_\Lambda\|_\text{op}\) of the \(\hat{f}_\Lambda\) are uniformly bounded:

\[
\|\hat{f}_\Lambda\|_\text{op} \leq 3\|f\|_\infty. \tag{91}
\]

Therefore (90) implies also, as claimed,

\[
\|(fg - \hat{f}_\Lambda\hat{g}_\Lambda)\phi\| \leq \|(f - \hat{f}_\Lambda)g\phi\| + \|\hat{f}_\Lambda(g - \hat{g}_\Lambda)\phi\|
\]

\[
\leq \|(f - \hat{f}_\Lambda)(g\phi)\| + \|\hat{f}_\Lambda\|_\text{op} \|\hat{g}_\Lambda\| \|\phi\| \xrightarrow{\Lambda \to \infty} 0. \tag{92}
\]

### 6.4 Spherical Harmonics

Let

\[
\begin{aligned}
x &= r \sin \theta \cos \varphi \\
y &= r \sin \theta \sin \varphi \\
z &= r \cos \theta
\end{aligned}
\]

\[
\approx \begin{cases}
x^+ = \frac{x^+iy}{\sqrt{2}} = \frac{r \sin \theta e^{i\varphi}}{\sqrt{2}} \\
x^- = \frac{x^iy}{\sqrt{2}} = \frac{r \sin \theta e^{-i\varphi}}{\sqrt{2}} \\
x^0 = z = r \cos \theta
\end{cases}
\]

\[
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\]
\[ t^0 = \frac{\dot{z}}{r}, \quad t^1 = \frac{x + iy}{\sqrt{2}r}, \quad \text{and} \quad t^- = \frac{x - iy}{\sqrt{2}r}, \] then one has the following recurrence relations:

\[
t^0 Y_l^m = \cos \theta Y_l^m = \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}} Y_{l-1}^m + \sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}} Y_{l+1}^m, \]
\[
t^1 Y_l^m = \sin \theta e^{i\varphi} Y_l^m = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{(l-m)(l-m-1)}{(2l+1)(2l-1)}} Y_{l-1}^{m+1} - \sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}} Y_{l+1}^{m+1} \right), \quad (93)
\]
\[
t^{-1} Y_l^m = \sin \theta e^{-i\varphi} Y_l^m = \frac{1}{\sqrt{2}} \left( -\sqrt{\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}} Y_{l-1}^{m-1} + \sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}} Y_{l+1}^{m-1} \right).
\]

The coefficients \( A_l^{a,m}, B_l^{a,m} \) are related by

\[
A_l^{a,m} = \langle Y_{l-1}^m, t^a Y_l^m \rangle = \langle t^{-a} Y_{l-1}^m, Y_l^m \rangle = \langle Y_l^m, t^{-a} Y_{l-1}^m \rangle, \quad B_l^{-a,m+a} = \langle Y_l^m, t^{-a} Y_{l-1}^m \rangle\quad (94)
\]

and fulfill the properties (for all \( l \geq 0, |m| \leq l \))

\[
A_l^{b,m} A_l^{-a,m+b+a} + A_l^{b,m+b} A_l^{-a,m+b} = A_l^{a,m} A_l^{-b,m+a+b} + A_l^{-a,m+a} A_l^{-b,m+a},
\]
\[
A_{l+1}^{a,m} A_{l+1}^{-a,m-b} = A_{l+1}^{a,m+b} A_{l+1}^{-a,m+b}, \quad \sum_a A_l^{a,m} A_l^{-a,m+a} = 0, \quad (95)
\]
\[
\sum_a (A_l^{a,m})^2 = \frac{l}{2l+1}, \quad \sum_a (A_{l+1}^{a,m-a})^2 = \frac{l+1}{2l+1}, \quad \sum_a (A_l^{a,m})^2 + \sum_a (A_{l+1}^{a,m-a})^2 = 1.
\]

Actually, the latter are equivalent to the identities \([t^a, t^b] = 0, \quad t^a t^{-a} = 1\) applied to \( Y_l^m \).

### 6.5 Calculation of \(|N_l|\) in \( D = 3 \)

\(|N_l|\) can be determined setting \( \langle \psi_l^m, \psi_l^m \rangle = 1 \) and we will choose \( N_l = |N_l|; \) using the orthonormality of the spherical harmonics, one obtains

\[
1 = |N_l|^2 \int_0^{+\infty} e^{-(r-\bar{r})^2 \sqrt{\pi} k_l} dr \simeq |N_l|^2 \int_{-\infty}^{+\infty} e^{-(r-\bar{r})^2 \sqrt{\pi} k_l} dr = |N_l|^2 \frac{\sqrt{\pi}}{\sqrt{k_l}} \Rightarrow |N_l| = \frac{\sqrt{k_l}}{\sqrt{\pi}} \quad (96)
\]

and the meaning of the approximation symbol \( \simeq \) is explained in subsection 6.10.

### 6.6 Calculation of a rather general scalar product in \( D = 3 \)

We compute the useful scalar product

\[
\langle \psi_{L}^{m'}, g(r) t^{a} \psi_{L}^{m} \rangle = \langle Y_{L}^{m'}, t^{a} Y_{L}^{m} \rangle \int_{0}^{+\infty} f_{L}^{a}(r) f_{L}(r) g(r) dr \quad (97)
\]
with a generic \(g(r)\) not depending on \(k\); here we have used the decomposition \(\psi^m_i(r, \theta, \varphi) = Y^m_i(\theta, \varphi) \frac{L(r)}{l} \) and factorized the integral over \(\mathbb{R}^3\) into an integral over the angle variables and the integral on the radial one. By (41) one finds

\[
\langle Y^m_L, t^a Y^m_i \rangle \neq 0 \quad \iff \quad L = l \pm 1 \text{ and } m' = m + a
\]

The asymptotic expansion of the radial integral can be obtained from the general formula

\[
\int_0^{+\infty} f_i(r)f_L(r)g(r)dr = e^{-\frac{\sqrt{k_l k_L}(r_i-r_l)^2}{2(\sqrt{k_l} + \sqrt{k_L})}} \sum_{n=0}^{+\infty} \frac{g^{(2n)}(\tilde{r}_{i,L})}{(2n)!!(\sqrt{k_l} + \sqrt{k_L})^{n}}, \tag{98}
\]

which we now prove:

\[
\int_0^{+\infty} f_i(r)f_L(r)g(r)dr = N_i N_L \int_0^{+\infty} e^{-r^2 \left( \sqrt{k_l} + \sqrt{k_L} \right)^2} g(r)dr
\]

\[
= N_i N_L e^{-\frac{\sqrt{k_l} \sqrt{k_L}(r_i-r_l)^2}{2(\sqrt{k_l} + \sqrt{k_L})}} \int_0^{+\infty} e^{-\frac{\sqrt{k_l} \sqrt{k_L}}{2}(r-\tilde{r}_{i,L})^2} g(r)dr
\]

\[
= N_i N_L e^{-\frac{\sqrt{k_l} \sqrt{k_L}(r_i-r_l)^2}{2(\sqrt{k_l} + \sqrt{k_L})}} \int_0^{+\infty} e^{-\frac{\sqrt{k_l} \sqrt{k_L}}{2}(r-\tilde{r}_{i,L})^2} g(r)dr.
\]

with \(\tilde{r}_i, k_l \) as defined in (36) and \(\tilde{r}_{i,L} := \frac{\sqrt{k_l} \sqrt{k_L}(r_i-r_l)^2}{\sqrt{k_l} + \sqrt{k_L}}\). By Taylor expansion of \(g(r)\) around \(\tilde{r}_{i,L}\),

\[
\int_0^{+\infty} e^{-\frac{\sqrt{k_l} \sqrt{k_L}}{2}(r-\tilde{r}_{i,L})^2} g(r)dr \approx \int_{-\infty}^{+\infty} e^{-\frac{\sqrt{k_l} \sqrt{k_L}}{2}(r-\tilde{r}_{i,L})^2} g(r)dr
\]

\[
= \int_{-\infty}^{+\infty} e^{-\frac{\sqrt{k_l} \sqrt{k_L}}{2}z^2} \left[ \sum_{h=0}^{+\infty} \frac{g^{(h)}(\tilde{r}_{i,L}) z^h}{h!} \right] dz
\]

\[
= \sqrt{\frac{2\pi}{\sqrt{k_l} + \sqrt{k_L}}} \sum_{n=0}^{+\infty} \frac{g^{(2n)}(\tilde{r}_{i,L})}{(2n)!!(\sqrt{k_l} + \sqrt{k_L})^{n}}, \tag{99}
\]

the equality \(\approx\) holds up to terms vanishing exponentially as \(k \to +\infty\), as explained in subsection 6.10. Now,

\[
N_i = \frac{\sqrt{k_l}}{\sqrt{\pi}}, \quad k_l = \frac{2k + 4l(l+1)}{2k + 3l(l+1)} = 1 + \frac{l(l+1)}{2k} - \frac{3l^2(l+1)^2}{(2k)^2} + O(k^{-3})
\]

\[
\sqrt{k_l} = 2k + \frac{3l(l+1)}{2\sqrt{2k}} - \frac{9l^2(l+1)^2}{8(2k)^{3/2}} + O\left(k^{-5/2}\right),
\]

\[
\sqrt{k_l} \tilde{r}_i = 2k + \frac{5l(l+1)}{2\sqrt{2k}} - \frac{21l^2(l+1)^2}{8(2k)^{3/2}} + O\left(k^{-5/2}\right),
\]

\[
\sqrt{k_l k_L}(\tilde{r}_i - \tilde{r}_L)^2 = \frac{l(l+1) - L(L+1)}{4(2k)^{3/2}} + O\left(k^{-3}\right)
\]

\[
\tilde{r}_{i,L} = 1 + \frac{l(l+1) + L(L+1)}{4k} - \frac{3}{4}l(l+1)L(L+1) + \frac{9}{8} \frac{[l^2(l+1)^2 + L^2(L+1)^2]}{4k^2} + O\left(k^{-3}\right)
\]
From (101) and (102) one has
\[ \sqrt{\frac{2\pi}{\sqrt{k_l + \sqrt{k_L}}}} \frac{\sqrt{k_l k_L}}{\sqrt{\pi}} = \frac{\sqrt{2}}{\sqrt{\frac{k_l}{k_l}} + \sqrt{\frac{k_L}{k_l}}} = 1 + O \left( k^{-2} \right), \]
whence (98). Using relations (97), (98), one finds, for example
\[
\langle \psi_{l-1}^{m+a}, g(r) t^a \psi_l^m \rangle = A_{l+1}^{a,m} \left( 1 - \frac{l^2}{(2k)^{\frac{3}{2}}} + \cdots \right) \left( g(\tilde{r}_{l,l-1}) + \frac{g''(\tilde{r}_{l,l-1})}{2(\sqrt{k_l + \sqrt{k_{l-1}}})^2} + \cdots \right), \quad (101)
\]
\[
\langle \psi_{l+1}^{m+a}, g(r) t^a \psi_l^m \rangle = B_{l+1}^{a,m} \left( 1 - \frac{(l+1)^2}{(2k)^{\frac{3}{2}}} + \cdots \right) \left( g(\tilde{r}_{l,l+1}) + \frac{g''(\tilde{r}_{l,l+1})}{2(\sqrt{k_l + \sqrt{k_{l+1}}})^2} + \cdots \right), \quad (102)
\]

6.7 Proof of (40) and of proposition 4.1

Using the decomposition \( x^a = rt^a \) and the relations (41) one finds
\[
\bar{C}^a \psi_l^m = \sum_{h,k} \langle \psi_{l,h}^k, x^a \psi_l^m \rangle \psi_k^h = \begin{cases} 
\langle \psi_{l-1}^{m+a}, rt^a \psi_l^m \rangle \psi_{l-1}^{m+a} + \langle \psi_{l+1}^{m+a}, rt^a \psi_l^m \rangle \psi_{l+1}^{m+a} & \text{if } l < \Lambda, \\
\langle \psi_{\Lambda-1}^{m+a}, rt^a \psi\Lambda^m \rangle \psi_{\Lambda-1}^{m+a} & \text{if } l = \Lambda, \\
0 & \text{otherwise.}
\end{cases} \quad (103)
\]
From (101) and (102) one has
\[
\langle \psi_{l-1}^{m+a}, rt^a \psi_l^m \rangle = c_l A_{l+1}^{a,m}, \quad \langle \psi_{l+1}^{m+a}, rt^a \psi_l^m \rangle = c_{l+1} B_{l+1}^{a,m}, \quad (104)
\]
where, up to \( O \left( k^{-\frac{3}{2}} \right) \), \( c_l = 1 + \frac{l^2}{2k} = \sqrt{1 + \frac{l^2}{k}} \quad 1 \leq l \leq \Lambda, \quad c_0 = c_{\Lambda+1} = 0. \)

\( c_l \) is an integral over the \( r \) variable, while \( A_{l+1}^{a,m} \) is an integral over the angle variables; replacing in (103) we find (40). The factorizations (104) are manifestations of the Wigner-Eckart theorem.

Using (40) one can calculate the action of the commutator on \( \psi_l^m \). For all \( l < \Lambda \) and \( m \) with \( |m| \leq l \), one has
\[
[\bar{C}^a, \psi_l^m] = \left[ (c_l)^2 - (c_{l+1})^2 \right] \frac{L_0}{2l + 1} \psi_l^m, \quad [\bar{C}^0, \psi_l^m] = \left[ (c_l)^2 - (c_{l+1})^2 \right] \frac{L_0}{2l + 1} \psi_l^m \]
Since
\[
\left[ (c_l)^2 - (c_{l+1})^2 \right] = -\frac{2l+1}{k} + O \left( \frac{1}{k^{\frac{3}{2}}} \right) \quad \text{if } l < \Lambda \text{ and } c_\Lambda^2 = 1 + \frac{\Lambda^2}{k} + O \left( \frac{1}{k^{\frac{3}{2}}} \right)
\]
we find at leading order in \( 1/\sqrt{k} \)
\[
[\bar{C}^a, \psi_l] = -\frac{L_0}{k} + C \bar{P}_\Lambda L_0, \quad [\bar{C}^0, \psi_l] = \pm \frac{L_0}{k} \pm C \bar{P}_\Lambda L_\pm, \quad C := \frac{1}{k} + \frac{1 + \Lambda^2}{2\Lambda + 1}, \quad (105)
\]
Going back to the hermitean coordinates $\overline{x} \equiv \overline{x}^i = \frac{x^i + y^i}{\sqrt{2}}$ and $\overline{y} \equiv \overline{x}^2 = \frac{x^i - y^i}{\sqrt{2}}$, one finds

$$[\overline{x}^i, \overline{x}^j] = i\varepsilon^{ijk} \left(-\frac{I}{k} + C \overline{P}_\Lambda\right) \overline{L}_h$$

In this model the commutator is invariant under parity because the components of the angular momentum are pseudo-vectors and they do not change sign under parity, while $P_{E_i}$ is a projector, then a scalar.

Similarly, one has

$$\mathcal{R}^2 \psi^m_l = \left\{ \begin{array}{ll} \left[ 1 + \frac{1}{k} (l^2 + l + 1) + O\left(\frac{1}{k^2}\right) \right] \psi^m_l & \text{if } l < \Lambda \\ \left[ 1 + \frac{\Lambda^2}{k} \right] \psi^m_l & \text{if } l = \Lambda \end{array} \right. \quad (106)$$

or equivalently, at leading order in $1/\sqrt{k}$,

$$\mathcal{R}^2 := \sum_{l=0}^{\Lambda-1} \left[ 1 + \frac{1}{k} (l^2 + l + 1) \right] \overline{P}_l + \left( 1 + \frac{\Lambda^2}{k} \right) \left( \frac{\Lambda}{2\Lambda + 1} \right) \overline{P}_\Lambda. \quad (107)$$

We see that the square distance from the origin is not central and equal to 1 on all the representation (as in the standard quantization on the unit sphere), but its eigenvalues go all to 1 as $k \to +\infty$, except when $l = \Lambda$. The last relation and (105) are exact if (39-42) are adopted as exact definitions.

In order to prove that $\overline{x} \cdot \overline{L} = 0$, we must recall equation (34) and since (by definition) $L^j = \frac{i}{2} \varepsilon^{jkh} x^k \partial_h$ one has $x \cdot L = \frac{i}{2} \varepsilon^{jkh} x^j x^k \partial_h$, but $\varepsilon^{jkh}$ is anti-symmetric and $x^j x^k$ is symmetric, then $x \cdot L = 0$. If one projects the previous operators on the Hilbert space $\mathcal{H}_\Lambda$, since when $L^j$ acts on $\psi^m_l$ it preserves the square of the angular moment (because $L^j$ doesn’t increase the value of $l$), that is $P_{E_i} L^j P_{E_i}$ then one has $P_{E_i} x^j L^j P_{E_i} = P_{E_i} x^j P_{E_i} L^j$; in conclusion, the condition $x \cdot L = 0$ implies $\overline{x} \cdot \overline{L} = 0$. One can also verify this condition in this way, from $L_\pm = \frac{L_\pm + iL_u}{\sqrt{2}}$, then one has

$$\overline{x} \cdot \overline{L} = \overline{x}^i L^j \eta_{ij} = \overline{x}^a L^b \overline{\eta}_{ab} = (\overline{x}^+ L_- + \overline{x}^- L_+ + \overline{x}^0 L_0) \psi^m_l \quad$$

$$+c_t \left( \frac{1}{2} \sqrt{l(l+1) - m(m-1)} A_l^{+m-1} + \frac{1}{2} \sqrt{l(l+1) - m(m+1)} A_l^{-m+1} + m A_l^{0,m} \right) \psi_{l-1,m} \quad$$

$$+c_{t+1} \left( \frac{1}{2} \sqrt{l(l+1)-m(m-1)} B_l^{+m-1} + \frac{1}{2} \sqrt{l(l+1)-m(m+1)} B_l^{-m+1} + m B_l^{0,m} \right) \psi_{l+1,m}$$

and using (93) one obtains $\overline{x} \cdot \overline{L} = 0$.

### 6.8 Action and commutators of the $\overline{\partial}_a$

In order to calculate the action of the derivation operators on the Hilbert space $\mathcal{H}_\Lambda$, we first note that $Y^{r^l}_l$ is a homogeneous polynomial of degree $l$ in the $x^a$ variables. This implies $\partial_a (Y^{r^l}_l) = C^{0,m}_l (Y^{r^a}_{l-1} x^a l^{-1})$, with some coefficients $C^{0,m}_l$. From the identities

$$[\partial_a, \partial_b] (Y^{r^l}_l) = 0, \quad ([\partial_a, x^b] - \delta_a^b) (Y^{r^l}_l) = 0$$

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one finds
\[ C_{l,m}^{b,m} C_{l-1}^{a,m} - C_{l}^{a,m} C_{l-1}^{b,m} = 0, \]
\[ A_{l}^{b,m} C_{l-1}^{a,m} + 2 A_{l}^{b,m} A_{l-1}^{a,m} + C_{l}^{a,m} A_{l-1}^{b,m} = 0, \]
\[ 2 A_{l}^{b,m} A_{l-1}^{a,m} + A_{l+1}^{b,m} C_{l+1}^{a,m} - C_{l}^{a,m} A_{l-1}^{b,m} = \delta^b_a. \]  

By explicit calculations, one can conclude that \( C_{l}^{a,m} = (2l + 1) A_{l}^{a,m} \). On the other hand, using
\[ \partial_a B_{l+1}^m = \frac{1}{(2l+1)^{1/2}} \left[ \frac{f_1(r)}{r} \right] t^{-a}, \]
and the results of section 6.6 we find
\[ \partial_a \psi^m_l (r, \theta, \varphi) = \partial_a \left( \frac{f_1(r)}{r} \right) r Y_{l}^m (\theta, \varphi) + \frac{f_1(r)}{r^2} C_{l}^{a,m} Y_{l-1}^{m-a} (\theta, \varphi) \]
\[ = \left[ \frac{f_1(r)}{r} - \frac{l+1}{r^2} f_1(r) \right] l^{-a} Y_{l}^m (\theta, \varphi) + \frac{f_1(r)}{r^2} C_{l}^{a,m} Y_{l-1}^{m-a} (\theta, \varphi) \]
\[ = \left[ \frac{f_1(r)}{r} - (l+1) \frac{f_1(r)}{r^2} \right] B_{l+1}^{a,m} Y_{l-1}^{m-a} + \left[ \frac{f_1(r)}{r} + l \frac{f_1(r)}{r^2} \right] A_{l}^{a,m} Y_{l-1}^{m-a}. \]

Hence
\[ \bar{\partial}_a \psi^m_l = \psi^m_{l-1} [M_l + l J_l] A_{l}^{a,m} - \psi^m_{l+1} [M_{l+1} + (l+1) J_{l+1}] B_{l+1}^{a,m}, \]

where
\[ J_l := \int_0^{\infty} \frac{f_1(r) f_{l-1}(r)}{r} dr = 1 + \frac{1}{\sqrt{8k}} - \frac{l^2}{2k} + O \left( \frac{1}{k^2} \right), \]
\[ M_l := \int_0^{\infty} f_{l-1}(r) f_1(r) dr = - \frac{l}{2k} + \frac{18 l^3}{8k \sqrt{2k}} + O \left( \frac{1}{k^2} \right); \]

\( J_l, M_l \) have been evaluated using (98). The commutator \( [\bar{\partial}_a, \bar{\partial}_b] \) on \( \psi^m_l \) is
\[ [\bar{\partial}_a, \bar{\partial}_b] \psi^m_l = \left[ (J_{l+1})^2 \frac{(l+1)^2}{2l+1} - (J_l)^2 \frac{l^2}{2l+1} + 2(l+1) J_{l+1} M_{l+1} - 2l J_l M_l \right. \]
\[ + (M_{l+1})^2 \frac{1}{2l+1} - (M_l)^2 \frac{1}{2l+1} \] \( \alpha_{a,b} \bar{L}_{a-b} \psi^m_l \]

with
\[ \alpha_{a,b} = - \alpha_{b,a} = \begin{cases} 0 & \text{if } a = b \\ -1 & \text{if } a = 0, b = +1 \\ 1 & \text{if } a = 0, b = -1 \\ 1 & \text{if } a = -1, b = +1 \end{cases} \]

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6.9 Proof of proposition 4.2 and other results of subsection 4.1

Let \( \chi_i := i \varepsilon^{ijk} X_j L_k \). Using (50) we easily find

\[
L^2 X_i = X_i L^2 + 2(X_i + \chi_i), \quad L^2 \chi_i = (\chi_i + 2X_i)L^2.
\] (112)

This suggests to look for “eigenvectors” of \( L^2 \) (and therefore of \( \lambda = \frac{\sqrt{4L^2 + 1} - 1}{2} \)), \( L^2 \vartheta_i = \vartheta_i \nu(\lambda) \), in the form \( \vartheta_i = X_i a(\lambda) + \chi_i b(\lambda) \). The compatibility condition is a second degree equation with the “right eigenvalue” \( \nu(\lambda) \) as the unknown. Up to \( \lambda \)-dependent factors the solutions are

\[
\vartheta_i^- = [X_i \lambda - \chi_i] (2\lambda-1)(\lambda-1), \quad \nu^- := (\lambda-1)\lambda,
\]

\[
\vartheta_i^+ = [X_i(\lambda+1) + \chi_i] \lambda(2\lambda+3) \quad \nu^+ := (\lambda+1)(\lambda+2)
\] (113)

(here we have chosen the coefficients \( a(\lambda), b(\lambda) \) so that \( \vartheta_i^{\pm \dagger} = \vartheta_i^{\mp} \)); this implies

\[
\lambda \vartheta_i^\pm = \vartheta_i^\mp (\lambda \pm 1).
\] (114)

Therefore \( \vartheta_a^{\pm} |l,m\rangle \propto |l \pm 1, m + a\rangle \). Inverting (113) one can express the \( X_a \) (as well as the \( \chi_a \)) as linear combinations of \( \vartheta_a^{\pm} \) with \( \lambda \)-dependent coefficients; hence \( a_{i,a}^{\pm} = 0 \) unless \( j = l \pm 1 \), and the Ansatz (55), as anticipated.

On the other hand, using (94-95) and the equalities

\[
A_i^{-m} B_{l-1}^{+m+1} - A_i^{+m} B_{l-1}^{-m+1} = \frac{m}{2l+1}, \quad A_i^{\pm,m} B_{l-1}^{0,m \pm 1} - A_i^{0,m} B_{l-1}^{\pm,m} = \pm \frac{m}{2l+1},
\]

\[
d_i^2 - d_{i+1}^2 = 2l + 1,
\]

\[
A_i^{a,m} \gamma_i^{a,m+a} = A_i^{a,m+a} \gamma_i^{a,m}, \quad A_i^{0,m} \gamma_i^{\pm,m} - A_i^{0,m \pm 1} \gamma_i^{\pm,m} = \mp A_i^{\pm,m},
\]

\[
B_i^{0,m} \gamma_i^{\pm,m} - B_i^{0,m \pm 1} \gamma_i^{\pm,m} = \mp B_i^{\pm,m}, \quad A_i^{\pm,m} \gamma_i^{\pm,m \pm 1} - A_i^{\pm,m \pm 1} \gamma_i^{\pm,m} = \mp A_i^{0,m},
\]

\[
B_i^{\pm,m} \gamma_i^{\pm,m \pm 1} - B_i^{\pm,m \pm 1} \gamma_i^{\pm,m} = \mp B_i^{0,m}.
\]
we obtain, for example, 

\[ [X_+, X_-] |l, m \rangle = d_{l-1} (A_{l-1}^{-m} A_{l-1}^{+m} - A_{l-1}^{+m} A_{l-1}^{-m}) |l - 2, m \rangle + \]

\[ d_l^2 (A_l^{-m} B_{l-1}^{+m} - A_l^{+m} B_{l-1}^{-m}) |l, m \rangle + \]

\[ d_{l+1}^2 (B_l^{+m} A_{l+1}^{-m} - B_l^{-m} A_{l+1}^{+m}) |l, m \rangle + \]

\[ d_{l+1} d_{l+2} (B_l^{+m} B_{l+1}^{+m} - B_l^{-m} B_{l+1}^{-m}) |l + 2, m \rangle = \]

\[ (d_l^2 - d_{l+1}^2) (A_l^{-m} B_{l-1}^{+m} - A_l^{+m} B_{l-1}^{-m}) |l, m \rangle = \]

\[ 2l + 1 \frac{m}{2l + 1} |l, m \rangle = m |l, m \rangle = L_0 |l, m \rangle , \]

\[ [X_+, X_0] |l, m \rangle = d_{l-1} (A_{l-1}^{0m} A_{l-1}^{+m} - A_{l-1}^{+m} A_{l-1}^{0m}) |l - 2, m + 1 \rangle + \]

\[ d_l^2 (A_l^{0m} B_{l-1}^{+m} - A_l^{+m} B_{l-1}^{0m}) |l, m + 1 \rangle + \]

\[ d_{l+1}^2 (B_l^{0m} A_{l+1}^{+m} - B_l^{+m} A_{l+1}^{0m}) |l, m + 1 \rangle + \]

\[ d_{l+1} d_{l+2} (B_l^{0m} B_{l+1}^{+m} - B_l^{+m} B_{l+1}^{0m}) |l + 2, m + 1 \rangle = \]

\[ (d_l^2 - d_{l+1}^2) (A_l^{0m} B_{l-1}^{+m} - A_l^{+m} B_{l-1}^{0m}) |l, m + 1 \rangle = \]

\[ 2l + 1 \frac{-\gamma_{l+1}^{-m} m}{2l + 1} |l, m + 1 \rangle = -\gamma_{l+1}^{+m} |l, m \rangle = -L_+ |l, m \rangle . \]

Since the Ansatz \((55)\) differs from \((41)\) only by the \(l\)-dependent coefficients \(d_l\), and \(l\) is not changed by the action of the \(L_0\), the fact that \((35)\) holds for \(v^a = t^a\) guarantees that it holds also for \(v^a = X_a\), i.e. proves the relations of the form \([L_a, X_b] = f_{ab}^c X_c\) in \((52)\).

In addition, because of

\[
A_l^{+m - 1} \gamma_{l-1}^{-m} + A_l^{-m - 1} \gamma_{l-1}^{+m} + A_l^{0m} m = B_l^{+m - 1} \gamma_{l}^{-m} + B_l^{-m - 1} \gamma_{l}^{+m} + B_l^{0m} m = 0, \quad (115)
\]

\[
A_l^{+m} \gamma_{l-1}^{-m+1} + A_l^{-m} \gamma_{l-1}^{+m+1} + A_l^{0m} m = B_l^{+m} \gamma_{l+1}^{-m+1} + B_l^{-m} \gamma_{l+1}^{+m+1} + B_l^{0m} m = 0 \quad (116)
\]

and

\[
A_l^{0m} B_l^{0m} + A_l^{-m} B_l^{-m+1} + A_l^{+m} B_l^{-m-1} = \frac{l}{2l + 1} \quad (117)
\]

we obtain

\[
X^2 |l, m \rangle = \left[ d_{l+1}^2 + (d_l^2 - d_{l+1}^2) \frac{l}{2l + 1} \right] |l, m \rangle = [L(L + 2) - l(l + 1)] |l, m \rangle \quad (118)
\]

whence we can easily derive \((53)\).

To prove \((60)\) we first note that, using the basic property \(\Gamma(z + 1) = z \Gamma(z)\),

\[
\prod_{h=0}^{l-1} (\Lambda + l - 2h) = 2^l \prod_{h=0}^{l-1} \left( \frac{\Lambda + l}{2} - h \right) = 2^l \Gamma \left( \frac{\Lambda + l + 1}{2} \right) =: h_1(l)
\]

\[
\prod_{h=0}^{l} (\Lambda + l + 1 - 2h) = 2^{l+1} \prod_{h=0}^{l} \left( \frac{\Lambda + l + 1}{2} - h \right) = 2^{l+1} \Gamma \left( \frac{\Lambda + l + 1 + 1}{2} \right) =: h_2(l).
\]
Hence $h_1(l)/h_2(l)$ gives the first ratio under the square root in (59), implying
\[
\frac{h_1(l)}{h_2(l)} = \frac{\Gamma\left(\frac{1}{2}\Lambda_0 + \frac{1}{2}\right)}{4 \Gamma\left(\frac{1}{2}\Lambda_0 + 1\right)} = \frac{1}{(\Lambda + 1 + l)(\Lambda + 1 - l)}.
\] (119)

On the other hand, it is $1 + \frac{l^2}{k} = \frac{1}{k}(l + i\sqrt{k})(l - i\sqrt{k})$. Setting $h_\pm(l) := \sqrt{2\Gamma\left(\frac{1}{2} + \frac{i}{2}\sqrt{k}\right)}$, we find
\[
h_\pm(l)h_\pm(l - 1) = 2 \frac{\Gamma\left(\frac{1}{2} + \frac{i}{2}\sqrt{k}\right)}{\Gamma\left(\frac{1}{2} + \frac{i}{2}\sqrt{k}\right)} = l \pm i\sqrt{k} \Rightarrow 1 + \frac{l^2}{k} = f(l)f(l - 1)
\] (120)
where $f(l) := h_+(l)h_-(l)/\sqrt{k}$. Therefore $g(l) = \sqrt{h_1(l)f(l)/h_2(l)}$, i.e. (60), solves (58).

### 6.10 Shifting the lower extreme of integration over $r$

Here we justify the approximation used in (96) and (99):
\[
\int_0^{+\infty} e^{-(r - \bar{r})^2\sqrt{k}} g(r) dr \simeq \int_{-\infty}^{+\infty} e^{-(r - \bar{r})^2\sqrt{k}} g(r) dr.
\] (121)

We first consider $g(r) \equiv 1$ and estimate for $b > 0$ and $a \to +\infty$ the following difference:
\[
\left(\int_{-\infty}^{+\infty} - \int_0^{+\infty}\right) e^{-a(r-b)^2} dr = \int_0^{+\infty} e^{-a(r+b)^2} dr = \int_{b\sqrt{a}}^{+\infty} e^{-z^2} \frac{dz}{\sqrt{a}} = \sqrt{\pi} \text{erf}(b\sqrt{a})
\] (122)
\[
= \frac{e^{-ab^2}}{2ba} \left[ 1 - \frac{1}{2ab^2} + \cdots \right].
\]

Here we have used: the changes of integration variables $r \mapsto -r$, $r \mapsto z = \sqrt{a}(y + b)$ in the first, second equalities; the definition erfc($x$) = $\frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-z^2} dz$ and the $x \to \infty$ asymptotic expansion erfc($x$) = $\frac{e^{-x^2}}{x\sqrt{\pi}} \left[ 1 - \frac{1}{2x^2} + \cdots \right]$ of the complementary error function in the third and fourth. We can apply the above formula to the integral (96) setting $a = \sqrt{2k}+\cdots$, $b = 1+\cdots$; consequently, the error made shifting from 0 to $-\infty$ the lower extreme in integrating over $r$ is of the order $1/\left(2\sqrt{2ke^{\sqrt{2k}}\right)}$, which has zero asymptotic expansion in $1/\sqrt{2k}$, hence does not contribute to the expansion of the integral: here are the meaning and the justification of the symbol $\simeq$. One obtains a similar result after a suitable number of integration by parts also if $g(r)$ is polynomial (or more generally analytic); this justifies (99).

### 6.11 Proof of proposition 4.3 and other results of section 4.2

We first show that the operators $\mathfrak{a}^a$ converge strongly to $t^a$. From (40-41) we find
\[
(\mathfrak{a}^a - t^a) \phi = (\mathfrak{a}^a - t^a) \sum_{l \in \mathbb{N}_0} \sum_{m = -l}^{l} \phi^m_1 Y^m_l = \sum_{l = 0}^{\infty} \sum_{m = -l}^{l} \phi^m_1 \left[(c_l - 1)A^a_l Y^m_{l-1} + (c_{l+1} - 1)B^a_l Y^m_{l+1}\right],
\]
37
with \( c_l \equiv 0 \) for \( l > \Lambda \). This implies

\[
\| (\bar{\mathcal{A}}^a - t^a) \phi \|^2 = \sum_{l=0}^{\Lambda} \sum_{m=-l}^{l} \left\{ |\phi_l^m|^2 \left[ (c_l - 1)^2 |A_l^{a,m}|^2 + (c_{l+1} - 1)^2 |B_l^{a,m}|^2 \right] + A_l^{a,m} B_l^{a,m} (c_{l+1} - 1) \right\}
\]

\[
\leq \sum_{l=0}^{\Lambda} \sum_{m=-l}^{l} \left\{ \frac{|\phi_l^m|^2}{2} \left[ (c_l - 1)^2 + (c_{l+1} - 1)^2 \right] + \frac{1}{2} (c_{l+2} - 1) (c_{l+1} - 1) \left[ |\phi_{l+1}^m|^2 + |\phi_l^m|^2 \right] \right\}
\]

\[
\leq \sum_{l=0}^{\Lambda} \sum_{m=-l}^{l} \frac{\Lambda^4}{2k^2} 2 l \leq \frac{\Lambda^4}{2k^2} \sum_{l=0}^{\Lambda} \sum_{m=-l}^{l} |\phi_l^m|^2 \| \phi \|^2 + 2 \sum_{l \geq \Lambda} \sum_{m=-l}^{l} |\phi_l^m|^2 ; \quad (123)
\]

here we have used the inequalities

\[
|A_l^{a,m}|, |B_l^{a,m}| \leq \frac{1}{\sqrt{2}}, \quad (c_l - 1) \leq \frac{l^2}{2k} \leq \frac{\Lambda^2}{2k} \quad \text{if } l \leq \Lambda. \quad (124)
\]

(the second one follows from \( \sqrt{1+\varepsilon} < \varepsilon/2 \) for \( \varepsilon > 0 \). By (15) the right-hand side of (123) goes to zero as \( \Lambda \to \infty \), as claimed.

Since \( \mathcal{H}^a \), \( \mathcal{H}^a \) cannot converge to \( t^a \) in operator norm. In fact, the square norm e.g. of \( (t^0 - \bar{\mathcal{A}}^a)^{\pm(A+1)} = B_{A+1}^{\pm(A+1)} Y_{A+1}^{\pm(A+2)} \) is \( \left| B_{A+1}^{\pm(A+1)} \right|^2 = \frac{2A+4}{(2A+5)^2} \geq 3/7 \), implying \( \| \bar{\mathcal{A}} - t^0 \|^2 \geq \sqrt{3/7} \) for all \( A \). Similarly, the square norm of \( (t^0 - \bar{\mathcal{A}}^a) Y_{A+1}^{0,0} = A_{A+1}^{0,0} Y_{A+1}^{0,0} + B_{A+1}^{0,0} Y_{A+2}^{0,0} \) is \( |A_{A+1}^{0,0}|^2 + |B_{A+1}^{0,0}|^2 = \frac{(A+2)^2(A+1)^2}{(2A+5)(2A+6)} \geq 1/3 \), implying \( \| \bar{\mathcal{A}} - t^a \|^2 \geq \sqrt{1/3} \) for all \( \Lambda \).

We now prove 4.3. Since

\[
(f - \hat{f}_A) \phi = \sum_{l=0}^{\Lambda} \sum_{|m| \leq l} Y_l^m \chi_l^m + \sum_{l > \Lambda} \sum_{|m| \leq l} Y_l^m (f \phi)_l^m,
\]

where \( \chi_l^m := (f \phi)_l^m - (\hat{f}_A \phi)_l^m \), \( (f \phi)_l^m = \langle Y_l^m, f \phi \rangle, \left( \hat{f}_A \phi \right)_l^m = \langle Y_l^m, \hat{f}_A \phi \rangle \), we find

\[
\| (f - \hat{f}_A) \phi \|^2 = \sum_{l=0}^{\Lambda} \sum_{|m| \leq l} |\chi_l^m|^2 + \sum_{l > \Lambda} \sum_{|m| \leq l} |(f \phi)_l^m|^2.
\]

As the second sum goes to zero as \( \Lambda \to \infty \), it remains to show that the first sum does as well. We find

\[
\chi_l^m = \langle Y_l^m, (f - \hat{f}_A) \phi \rangle = \langle Y_l^m, \sum_{j=0}^{2\Lambda} \sum_{|s| \leq j} f_j^s (Y_j^s - \hat{Y}_j^s) \phi \rangle
\]

\[
= \sum_{j=0}^{2\Lambda} \sum_{|s| \leq j} f_j^s R_j^s \sum_{l'} \langle Y_l^m, \pm (t^{a_1} \cdots t^a_{l')} - \bar{\mathcal{A}}^{a_1} \cdots \bar{\mathcal{A}}^{a_{l'}} \phi \rangle
\]

\[
= \sum_{j=0}^{2\Lambda} \sum_{|s| \leq j} f_j^s R_j^s \sum_{l'} \sum_{m' = -l'}^{l'} \phi_{p'}^{m'} \langle Y_l^m, \pm (t^{a_1} \cdots t^{a_{l'}} - \bar{\mathcal{A}}^{a_1} \cdots \bar{\mathcal{A}}^{a_{l'}}) Y_m^{p'} \rangle, \quad (127)
\]

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where $\sum'$ is the sum \(^{64}\) of \(n \leq j^2-8\) (not necessarily distinct) monomials in \(x^+, x^0, x^{-1}\) with coefficients $\pm 1$. The computation when $j = 2$ is instructive for the generic situation:

\[
(x^A - t^{a'b}) Y^m_l = (c_l c_{l-1} - 1) A_{l}^{b,m} A_{l-1}^{a,m+b} Y^{m+a+b}_{l-2} + (\sum_{h} c_{l+1} c_{l+1} - 1) B_{l+1}^{b,m} A_{l+1}^{a,m+b} Y^{m+a+b}_{l+2} + \left( c_{l}^{2} - 1 \right) A_{l}^{b,m} B_{l-1}^{a,m+b} + (\sum_{h} c_{l+1} - 1) B_{l+1}^{b,m} A_{l+1}^{a,m+b} Y^{m+a+b}.
\]

More generally

\[
(t^{a_1} \ldots t^{a_j} - x^{a_1} \ldots x^{a_j}) Y^m_{l'} = \sum_{h=0}^{j} A_{l', m', h}^{a_1, \ldots, a_j} Y^{m'+s}_{l'-j+2h}, \tag{128}
\]

where $A_{l', m', h}^{a_1, \ldots, a_j}$ is a sum of at most \(\left[ \frac{j+1}{2} \right]\) terms of the form

\[
\prod_{h=0}^{j} A_{l', h}^{a'_h, m'_h} \left( 1 - \prod_{h'=0}^{j} c_{l, h'} \right), \tag{129}
\]

where $a'_h = a_h$ when the factor comes from a coefficient $A$, while $a'_h = -a_h$ when it comes from a $B$. By \(^{124}\) the factors in \(129\) satisfy the inequalities

\[
\left| \prod_{h=0}^{j} A_{l', h}^{a'_h, m'_h} \right| \leq \frac{1}{2^j}, \quad \left( 1 - \prod_{h'=0}^{j} c_{l, h'} \right) \leq \left( 1 + \frac{\Lambda^2}{k} \right)^{\frac{j}{2}} - 1 \leq e^{\frac{j\Lambda^2}{2k}} - 1.
\]

\[
\Rightarrow \quad |A_{l, m, h}^{a_1, \ldots, a_j}| \leq \left( \left[ \frac{j+1}{2} \right] \right) e^{\frac{j\Lambda^2}{2k}} - 1. \tag{130}
\]

If we replace \(128\) into \(127\) and use $\langle Y^m_l, Y^{m'}_{l'} \rangle = \delta_{l l'} \delta_{m m'}$ we obtain

\[
\chi^m_l = \sum_{j=0}^{2\Lambda} \sum_{|s| \leq j} f^s_j R^s_j \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \phi^{m'}_{l'} \sum_{h=0}^{j} \pm \left( Y^m_l, A_{l', m', h}^{a_1, \ldots, a_j} Y^{m'+s}_{l'-j+2h} \right)
\]

\[
= \sum_{j=0}^{2\Lambda} \sum_{|s| \leq j} f^s_j R^s_j \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} \phi^{m'}_{l'} \sum_{h=0}^{j} \pm A_{l', m', h}^{a_1, \ldots, a_j} \delta_{l l'} \delta_{m, m'} \delta_{m', s} + \phi^{m-s}_{l+j-2h} A_{l+j-2h, m-s, h}^{a_1, \ldots, a_j}
\]

and, by \(127, 130\),

\[
|\chi^m_l| \leq \sum_{j=0}^{2\Lambda} \sum_{|s| \leq j} f^s_j \left( \left[ \frac{j+1}{2} \right] \right) \frac{1}{2^j} \left( e^{\frac{j\Lambda^2}{2k}} - 1 \right) R^s_j \sum_{0 \leq h \leq \min\left\{ \frac{j}{2}, \frac{j+j-|m-s|}{2} \right\}} \phi^{m-s}_{l+j-2h}.
\]

Using

\[
|f^s_j| \leq \|f\|, \quad \|\phi^{m'}_{l'}\| \leq \|\phi\|, \quad \min\left\{ \frac{j}{2}, \frac{j+j-|m-s|}{2} \right\} \leq j \leq 2\Lambda
\]

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and (64), (130) we get

\[ |x_i^n| \leq \|f\| \|\phi\| 2\Lambda \sum_{j=1}^{2\Lambda} T_j \left( \frac{j}{[j+1/2]} \right) \frac{1}{2^j} \left( e^{\frac{4\Lambda^2}{2\pi}} - 1 \right), \quad \text{where } T_j := \sum_{|s| \leq j} \frac{(j+s)!2^{j-s}}{(2j)!(j-s)!} j^{j-s}. \]

Let us now prove that for all \( j \geq 1 \)

\[ T_j \leq 4j^j, \quad \left( \frac{j}{[j+1/2]} \right) \leq 2^j. \tag{131} \]

By straightforward computations we find \( T_1 = 3 \), what fulfills (131). For \( j \geq 2 \) we use the following inequality

\[ \sqrt{(j+s)!2^{j-s}} j^{j-s} \leq j^2 \tau^j, \tag{132} \]

which can be proved \( \forall j \in \mathbb{N} \) and \( \forall |s| \leq j \) iteratively. In fact, it is trivial when \( s = 0 \) and (132) implies

\[ T_j = \sum_{|s| \leq j} \frac{(j+s)!2^{j-s}}{(2j)!(j-s)!} j^{j-s} \leq \sum_{|s| \leq j} \frac{2j^j}{(2j)!} \sum_{s=0}^{2j} \sqrt{j}^s = \frac{j^{j+1}}{\sqrt{j} - 1} = \frac{j^j - 1}{1 - j^{-1/2}} \leq 4j^j; \]

the last inequality holds for \( j \geq 2 \). The proof of the second inequality in (131) is straightforward. Applying (131) we find for all \( |m| \leq l \leq \Lambda \)

\[ |x_i^n| \leq \|f\| \|\phi\| 8\Lambda U_{\Lambda}, \quad \text{where } U_{\Lambda} := \sum_{j=1}^{2\Lambda} j^j 2^{\frac{j}{2}} \left( e^{\frac{4\Lambda^2}{2\pi}} - 1 \right) \]

\[ Q_{\Lambda} := \sum_{l=0}^{\Lambda} \sum_{|m| \leq l} |x_i|^2 \leq \|f\|^2 \|\phi\|^2 64\Lambda^2(\Lambda + 1)^2 U_{\Lambda}^2. \tag{133} \]

Setting \( \sigma := 2^{3/2} \Lambda, \tau := e^{\frac{2\pi}{2\lambda}} \) we find

\[ U_{\Lambda} \leq \sum_{j=1}^{2\Lambda} (2\Lambda)^{2j} \left( e^{\frac{4\Lambda^2}{2\pi}} - 1 \right) = \sum_{j=1}^{2\Lambda} (\sigma \tau)^j - \sum_{j=1}^{2\Lambda} \sigma j = \frac{(\sigma \tau)^{2\Lambda+1} - 1}{\sigma \tau - 1} - \sigma^{2\Lambda+1} - 1 \]

\[ = \frac{\sigma^{2\Lambda+2} \tau (\tau^{2\Lambda} - 1) - \sigma^{2\Lambda+1} (\tau^{2\Lambda+1} - 1) + \sigma (\tau - 1)}{(\sigma \tau - 1)(\sigma - 1)} \leq 2 \frac{\sigma^{2\Lambda+2} \tau (\tau^{2\Lambda} - 1)}{(\sigma \tau - 1)(\sigma - 1)} \]

For \( \Lambda \geq 3 \) we easily show

\[ \frac{1}{\sigma - 1} < \frac{1}{2\Lambda}, \quad \frac{1}{\sigma \tau - 1} < \frac{1}{2\Lambda \tau} \quad \Rightarrow \quad U_{\Lambda} < \frac{1}{2} \sigma^{2\Lambda} (\tau^{2\Lambda} - 1) = 2^{3\Lambda-1} \Lambda^3 \left( e^{\frac{3\Lambda^3}{2\pi}} - 1 \right). \]
Replacing into (133) we obtain

\[ Q_\Lambda \leq \|f\|^2 \|\phi\|^2 (\Lambda + 1)^2 2^{6\Lambda+4} \Lambda^{2\Lambda+2} \left( e^{\frac{\Lambda^3}{k}} - 1 \right)^2 < \|f\|^2 \|\phi\|^2 (\Lambda + 1)^2 2^{6\Lambda+6} \frac{\Lambda^{2\Lambda+8}}{k^2}. \]  

(134)

The last inequality holds for sufficiently small \( \Lambda^3/k \), e.g. \( \Lambda^3/k < 1/2 \), because \( e^x - 1 < 2x \) for \( 0 < x < 1/2 \). Finally, by (126), (134) the choice \( k(\Lambda) = 2^{3\Lambda+3} \Lambda^5(\Lambda+1) \) implies

\[ \|(f - \hat{f}_\Lambda)\phi\|^2 \leq \|f\|^2 \|\phi\|^2 \frac{1}{\Lambda^2} + \sum_{l>\Lambda} \sum_{|m| \leq l} |(f\phi)_l^m|^2 \xrightarrow{\Lambda \to \infty} 0, \]  

(135)
i.e. \( \hat{f}_\Lambda \to f \cdot \) strongly for all \( f \in B(S^2) \), as claimed. Replacing \( f \mapsto fg \), we find also that \( (fg)_\Lambda \to (fg) \cdot \) strongly for all \( f, g \in B(S^2) \). On the other hand, relation (135) implies also

\[ \|(f - \hat{f}_\Lambda)\phi\|^2 = \|f\|^2 \|\phi\|^2 \frac{1}{\Lambda^2} + \|f\phi\|^2 \leq \left( \frac{\|f\|^2}{\Lambda^2} + \|f\|_\infty^2 \right) \|\phi\|^2, \]  

\[ \|\hat{f}_\Lambda\phi\| \leq \|(f - \hat{f}_\Lambda)\phi\| + \|f\phi\| \leq \|(f - \hat{f}_\Lambda)\phi\| + \|f\|_\infty \|\phi\| \leq (\|f\| + 2\|f\|_\infty) \|\phi\|, \]
i.e. the operator norms \( \|\hat{f}_\Lambda\|_{op} \) of the \( \hat{f}_\Lambda \) are bounded uniformly in \( \Lambda \): \( \|\hat{f}_\Lambda\|_{op} \leq \|f\| + 2\|f\|_\infty \).

Therefore, as claimed, (135) implies again also

\[ \|(fg - \hat{f}_\Lambda\hat{g}_\Lambda)\phi\| \leq \|(f - \hat{f}_\Lambda)g\phi\| + \|\hat{f}_\Lambda(g - \hat{g}_\Lambda)\phi\| \leq \|(f - \hat{f}_\Lambda)(g\phi)\| + \|\hat{f}_\Lambda\|_{op} \|(g - \hat{g}_\Lambda)\phi\| \xrightarrow{\Lambda \to \infty} 0. \]  

(136)

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