Topological duals of Banach function spaces

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August 21, 2020

Abstract

This paper studies topological duals of Banach function spaces (BFS). We assume a finite measure but our arguments extend to general locally convex function spaces whose topology is generated by seminorms that satisfy the usual BFS axioms. The dual is identified with the direct sum of another space of random variables (Köthe dual), a space of purely finitely additive measures and the annihilator of $L^\infty$. In the special case of rearrangement invariant spaces, the second component in the dual vanishes and we obtain various classical as well as new duality results e.g. on Lebesgue, Orlicz, Lorentz-Orlicz spaces and spaces of finite moments. Beyond rearrangement invariant spaces, we find the topological duals of Musielak-Orlicz spaces and those associated with general convex risk measures.

Keywords. Banach function spaces, topological duals, finitely additive measures

AMS subject classification codes. 46E30, 46A20, 28A25

1 Introduction

Banach function spaces (BFS) provide a convenient set up for functional analysis in spaces of measurable functions. Many well known properties of e.g. Lebesgue spaces and Orlicz spaces extend to BFS with minor modifications; see e.g. [13, 23, 11, 10]. Much of the theory focuses on rearrangement invariant (ri) spaces where the value of the norm a function only depends on the distribution of the function. Such spaces are arguably the most important among BFS but they do exclude some interesting cases such as Musielak-Orlicz spaces and spaces of random variables that arise in the theory of risk measures; see Section 5 below.

This paper studies the topological duals of locally convex spaces of random variables where the topology is generated by an arbitrary collection of seminorms that satisfy the usual properties of BFS-norms; see Section 4 below.

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Building on the classical result of Yosida and Hewitt [22, Section 2] on the dual of $L^\infty$, we identify the topological dual as the direct sum of another space of random variables (Köthe dual), a space of purely finitely additive measures and the annihilator of $L^\infty$. The last two components have a singularity property that has been found useful in the analysis of convex integral functionals by Rockafellar [19] in the case of $L^\infty$ and by Kozek [8] in the case of Orlicz spaces. In the case of $L^\infty$, the last component in the dual vanishes while in other Orlicz spaces, the second one vanishes; see [17]. Our result thus unifies the two seemingly complementary cases.

The main result is illustrated by simple derivations of various existing duality results in Musielak-Orlicz, Marcinkiewich, Lorentz and Orlicz-Lorentz spaces. In the last case, we also obtain an expression for the dual norm which seems to be new. We go beyond the existing BFS settings by identifying topological duals of the space of random variables with finite moments, generalized Orlicz spaces as well as spaces of random variables associated with general convex risk measures. Such spaces have attracted attention in the recent literature on insurance and financial mathematics; see e.g. [16], [10] and [7].

The last section establishes the necessity of our axioms for locally convex spaces of random variables that are in separating duality with another one. More precisely, a complete solid decomposable space in separating duality with another solid decomposable space has a compatible topology generated by a collection of seminorms satisfying the usual BFS-properties.

The rest of the paper is organized as follows. Section 2 reviews the duality theory for $L^\infty$. Section 3 extends the notion of an integral with respect to a finitely additive measure to measurable not necessarily bounded random variables. Section 4 defines a general locally convex space of random variables and gives the main result of the paper by characterizing the topological dual of a space. Section 5 applies the main result to characterize the topological dual in various known and new settings. Section 6 concludes by illustrating the necessity of the employed axioms.

## 2 Topological dual of $L^\infty$

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a $\sigma$-algebra $\mathcal{F}$ and a countably additive probability measure $P$. This section gives a quick review of the Banach space $L^\infty$ of equivalence classes of essentially bounded measurable functions on a probability space $(\Omega, \mathcal{F}, P)$. We consider $\mathbb{R}^n$-valued functions and endow $L^\infty$ with the norm

$$
\|u\|_{L^\infty} := \sqrt[n]{\|u_1\|_{L^\infty}, \ldots, \|u_n\|_{L^\infty}},
$$

where $|\cdot|$ is a norm on $\mathbb{R}^n$. The dual norm on $\mathbb{R}^n$ is denoted by $|\cdot|^*$. Let $\mathcal{M}^1$ be the set of $P$-absolutely continuous finitely additive $\mathbb{R}^n$-valued measures on $(\Omega, \mathcal{F})$ and let $\mathcal{M}^{1s}$ be set of those $m^s \in \mathcal{M}^1$ which are singular ("purely finitely additive" in the terminology of [22]; see [22, Theorem 1.22]) in the sense that there is a decreasing sequence $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ with $P(A^\nu) \setminus 0$ and $\lim_{\nu \to \infty} P(A^\nu) = 0$.
Given $m \in \mathcal{M}^1$, the set function $|m|^*: \mathcal{F} \to \mathbb{R}$ is defined by

$$|m|^*(A) := |m^+(A) + m^-(A)|^*,$$

where $i$th components of $m^+ \in \mathcal{M}^1$ and $m^- \in \mathcal{M}^1$ are the positive and negative parts, respectively, of the $i$th component $m_i$ of $m$; see [22, Theorem 1.12].

Recall that the space $\mathcal{E}$ of $\mathbb{R}^n$-valued simple random variables (i.e. piecewise constant with a finite range) is dense in $L^\infty$. Given $m \in \mathcal{M}^1$, the integral of a $u \in \mathcal{E}$ is defined by

$$\int_{\Omega} u \, dm := \sum_{j=1}^J \alpha_j m(A^j),$$

where $A^j \in \mathcal{F}$ and $\alpha^j \in \mathbb{R}^n$, $j = 1, \ldots, m$ are such that $u = \sum_{j=1}^m \alpha^j 1_{A^j}$. On $L^\infty$, the integral is defined as the unique norm continuous linear extension from $\mathcal{E}$ to $L^\infty$.

The following is from [22, Theorem 2.3] except that we do not assume that the underlying measure space is complete. The proof uses [5, Theorem 20.35] which does not rely on the completeness but identifies the dual of $L^\infty$ with the space of finitely additive measures that are absolutely continuous with respect to $P$. Combined with the results of [22, Section 1] on decomposition of finitely additive measures then completes the proof. The extension to spaces of $\mathbb{R}^n$-valued random variables is straightforward.

**Theorem 1** (Yosida–Hewitt). The topological dual $(L^\infty)^*$ of $L^\infty$ can be identified with $\mathcal{M}^1$ in the sense that for every $u^* \in (L^\infty)^*$ there exist unique $m \in \mathcal{M}^1$ such that

$$\langle u, u^* \rangle = \int_{\Omega} u \, dm,$$

where the integral is defined componentwise. The dual norm is given by

$$\|m\|^*_{L^\infty} = |m|^*(\Omega).$$

Moreover, $\mathcal{M}^1 = L^1 \oplus \mathcal{M}^1_s$ in the sense that for every $m \in \mathcal{M}^1$ there exist unique $y \in L^1$ and $m^s \in \mathcal{M}^1_s$ such that

$$\int_{\Omega} u \, dm = E[u \cdot y] + \int_{\Omega} u \, dm^s.$$

We have $m^s = 0$ if and only if $\langle u 1_{A^\nu}, u^* \rangle \to 0$ for every $u \in L^\infty$ and every decreasing $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ such that $P(A^\nu) \searrow 0$.

**Proof.** Assume first that $n = 1$. By [5, Theorem 20.35], the dual of $L^\infty$ can be identified with the linear space of finitely additive $P$-absolutely continuous measures $m$ in the sense that every $u^* \in (L^\infty)^*$ can be expressed as

$$\langle u, u^* \rangle = \int_{\Omega} u \, dm$$

Moreover,
and, conversely, any such integral belongs to \((L^\infty)^*\). By [22, Theorem 1.24], there is a unique decomposition \(m = m^a + m^s\), where \(m^a\) is countably additive and \(m^s\) is purely finitely additive. The construction in [22] also shows that \(m^a\) and \(m^s\) are absolutely continuous with respect to \(m\) and thus, absolutely continuous with respect to \(P\) as well. By [22, Theorem 1.22], there is a decreasing sequence \((A^\nu)_{\nu=1}^{\infty} \subset \mathcal{F}\) such that \(P(A^\nu) \downarrow 0\) and \(m^s(\Omega \setminus A^\nu) = 0\). The functional \(y^s \in (L^\infty)^*\) given by

\[
\langle u, y^s \rangle := \int_\Omega u \, dm^s
\]

then has the property in the statement. By Radon-Nikodym, there exists a \(y \in L^1\) such that

\[
\langle u, u^* \rangle := E[u \cdot y] + \int_\Omega u \, dm^s.
\]

To prove the last claim, let \(u^* \in (L^\infty)^*\) and consider the representation in terms of \(y \in L^1\) and \(m^s \in \mathcal{M}^{1,s}\) given by the second claim. Let \(A^*\) be the sets in the characterization of the singularity of \(m^s\). By [22 Theorems 1.12 and 1.17], \(m^s = m^{s+} - m^{s-}\) for nonnegative purely finitely additive \(m^{s+}\) and \(m^{s-}\). Given \(\epsilon > 0\), [22 Theorem 1.21] gives the existence of \(A \in \mathcal{F}\) such that \(m^{s+}(\Omega \setminus A) < \epsilon\) and \(m^{s-}(A) < \epsilon\). We have

\[
\langle u1_{A^*}, u^* \rangle = E[1_A 1_{A^*} u \cdot y] + m^s(A \cap A^*) \to m^s(A) > m^{s+}(\Omega) - 2\epsilon.
\]

By assumption, the left side converges to zero. Since \(\epsilon > 0\) was arbitrary, \(m^{s+} = 0\). By symmetry, we must have \(m^{s-} = 0\) so that \(m^s = 0\) which means that \(u^*\) is \(\tau(L^\infty, L^1)\)-continuous.

By [22 Theorem 2.3], the dual norm of \(\| \cdot \|_{L^\infty}\) is given by \(\|m\|_{TV} := m^{+}(\Omega) + m^{-}(\Omega)\). This completes the proof of the case \(n = 1\). The general case follows from the fact that the dual of a Cartesian product of Banach spaces is the Cartesian product of the dual spaces with the norm

\[
\|u\|_{L^\infty}^* = \left(\|m_1\|_{TV}, \ldots, \|m_n\|_{TV}\right)^*,
\]

which completes the proof.

\[\Box\]

3 Extension of the integral

In [22] and in Section 2, integrals with respect to an \(m \in \mathcal{M}^1\) were defined only for elements of \(L^\infty\) as norm-continuous extensions of integrals of simple functions. Weakening the topology, it is possible to extend the definition of the integral to a larger space of measurable functions using Daniell’s construction much as in [2, Chapter II] who considered countably additive integrals of arbitrary (not necessarily \(\mathcal{F}\)-measurable) functions.

Another approach to integration of unbounded functions with respect to finitely additive measures is that of Dunford; see Dunford and Schwartz [3] or
Luxemburg [12]. A benefit of the Daniell extension adopted here is that it gives rise to a simpler definition of integrability that is easier to verify for larger classes of measurable functions.

Given $m \in \mathcal{M}$, we define $\rho_m : L^1 \to \mathbb{R}$ by

$$\rho_m(u) := \sup_{u' \in L^\infty} \left\{ \int_{\Omega} u' \, dm \left| |u'_j| \leq |u_j| \right. \forall j = 1, \ldots, n \right\}.$$ 

We denote $\text{dom} \rho_m := \{ u \in L^1 \mid \rho_m(u) < \infty \}$.

**Lemma 2.** The function $\rho_m$ is a seminorm on $\text{dom} \rho_m$ and

$$|\int_{\Omega} u \, dm| \leq \rho_m(u)$$

for all $u \in L^\infty$. For every $u \in \text{dom} \rho_m$ and $\epsilon > 0$, there exists a $u' \in L^\infty$ such that $\rho_m(u - u') \leq \epsilon$.

**Proof.** We have

$$\rho_m(u) := \sum_{j=1}^{n} \rho_{m_j}(u_j)$$

where

$$\rho_{m_j}(u_j) = \sup_{u' \in L^\infty(\mathbb{R})} \left\{ \int_{\Omega} u' \, dm_j \left| |u'_j| \leq |u_j| \right. \right\}.$$ 

Thus we may assume that $n = 1$ and the claims from Theorem 27 in the appendix.

By Lemma 2 the integral is $\rho_m$-continuous on $L^\infty$ and $L^\infty$ is $\rho_m$-dense in $\text{dom} \rho_m$. Thus the integral has a unique $\rho_m$-continuous linear extension to $\text{dom} \rho_m$. We call the extension the $m$-integral of $u$ and denote it by

$$\int_{\Omega} u \, dm.$$ 

The elements of $\text{dom} \rho_m$ will be said to be $m$-integrable. If $m$ is countably additive, then, e.g., by the interchange rule [21, Theorem 14.60],

$$\rho_m(u) = \sum_{j=1}^{n} \int_{\Omega} |u_j| \, |m_j| = \sum_{j=1}^{n} E[|u_j||y_j|],$$

where $y$ is the density of $m$, and thus,

$$\text{dom} \rho_m = \{ u \mid u_j \in L^1(\Omega, \mathcal{F}, |m_j|) \forall j = 1, \ldots, n \}.$$ 

In this case, the integral is the Lebesgue integral.
4 Topological duals of spaces of random variables

This section presents the main results of the paper. The setup extends that of Banach function spaces by replacing the norm by an arbitrary collection of seminorms thus covering more general locally convex spaces of random variables. The main result identifies the topological dual of the space with the direct sum of the Köthe dual and two spaces of singular functionals, the first of which is represented by finitely additive measures while the second is the orthogonal complement of $L^\infty$.

Let $\mathcal{P}$ be a collection of sublinear symmetric functions $p : L^1 \to \mathbb{R}$, define

$$\mathcal{U} := \bigcap_{p \in \mathcal{P}} \text{dom } p,$$

and endow $\mathcal{U}$ with the locally convex topology generated by $\mathcal{P}$. Our aim is to characterize the topological dual $\mathcal{U}^*$ of $\mathcal{U}$. To this end, we will assume that

(A1) the topology of $\mathcal{U}$ is no weaker than the relative $L^1$-topology,

and that each $p \in \mathcal{P}$ satisfies

(A2) there exists a constant $c$ such that $p(u) \leq c\|u\|_{L^\infty}$ for all $u \in L^1$,

(A3) $p(u') \leq p(u)$ for every $u \in \mathcal{U}$ and $u' \in L^1$ with $|u'| \leq |u|$.

Occasionally, we will also assume the following

(A4) $p(u1_{A}) \searrow 0$ for all $u \in L^\infty$ and decreasing sequence $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ with $P(A^\nu) \searrow 0$.

(A5) $p(u1_{A}) \searrow 0$ for all $u \in \mathcal{U}$ and decreasing sequence $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ with $P(A^\nu) \searrow 0$.

It is clear that (A4) and (A5) are implied by the following order continuity properties:

(A4') $p(u^\nu) \searrow 0$ for all $(u^\nu) \in L^\infty$ such that $|u^\nu| \searrow 0$.

(A5') $p(u^\nu) \searrow 0$ for all $(u^\nu) \in \mathcal{U}$ such that $|u^\nu| \searrow 0$.

When $\mathcal{P}$ is a singleton, $\mathcal{U}$ is a normed space and we are in the setting of normed Köthe function spaces; see e.g. [23]. If, in addition, $p$ is lower semicontinuous in $L^1$, then $\mathcal{U}$ is a Banach function space; see Remark 3 below. In the Banach space setting, (A1) and (A2) hold under (A5') if $\mathcal{U}$ has a weak unit; see e.g. [11] Theorem 1.b.14. Necessity of the axioms will be discussed in more detail in Section 6.
Remark 3. Given \( p \in \mathcal{P} \), we define,
\[
\hat{\phi}_p(t) := \sup_{A \in \mathcal{F}} \{ p(1_A) \mid P(A) \leq t \},
\]
\[
\tilde{\phi}_p(t) := \inf_{A \in \mathcal{F}} \{ p(1_A) \mid P(A) \geq t \}.
\]
Since we assume \([A2]\) and \([A3]\), the condition \([A4]\) is equivalent to
\[
\lim_{t \searrow 0} \tilde{\phi}_p(t) = 0.
\]
If \( \lim_{t \searrow 0} \tilde{\phi}_p(t) > 0 \), then \( \mathcal{U} = L^\infty \).

Assume now that \( p \) is rearrangement invariant in the sense that \( p(u) = p(\hat{u}) \)
whenever \( u \) and \( \hat{u} \) have the same distribution. Then, for any \( A \in \mathcal{F} \) with
\( P(A) = t \),
\[
\hat{\phi}_p(t) = \tilde{\phi}_p(t) = p(1_A)
\]
where the common value is known as the fundamental function. In particular, \( \text{dom } p = L^\infty \) if \([A4]\) does not hold while \([A4]\) is equivalent to
\( \lim_{t \searrow 0} \tilde{\phi}_p(t) = 0 \).

Proof. Assuming \([A4]\) let \( t^\nu \searrow 0 \). There exists \( (A^\nu)^{\nu=1}_\nu \) such that
\( P(A^\nu) \leq t^\nu \) and \( \hat{\phi}_p(t^\nu) \leq p(1_{A^\nu}) + 1/\nu \). Passing to a subsequence if necessary, \( 1_{A^\nu} \to 0 \)
almost surely. Defining \( A^\nu := \bigcup_{\nu \geq \nu'} A^\nu, (A^\nu)^{\nu=1}_\nu \) is decreasing with \( A^\nu \subset A^\nu' \)
and \( P(A^\nu) \searrow 0 \), so, by \([A3]\)[A4]
\[
\hat{\phi}(t^\nu) \leq p(1_{A^\nu}) + 1/\nu \searrow 0.
\]
For the converse, let \( u \in L^\infty \) and \((A^\nu)^{\nu=1}_\nu \subset \mathcal{F} \) with \( t^\nu := P(A^\nu) \searrow 0 \). By \([A3]\)
\( p(u 1_{A^\nu}) \leq ||u||_{L^\infty} \hat{\phi}(t^\nu) \searrow 0 \).

If \( \lim_{t \searrow 0} \tilde{\phi}_p(t) > \delta \) for some \( \delta > 0 \), then \( p(u) \geq p(\nu 1_{|u| \geq \nu}) \geq \delta \nu \) whenever
\( P(\{|u| \geq \nu\}) \to 0 \), so \( p(u) = +\infty \) if \( u \notin L^\infty \).

Remark 4. As soon as \([A4]\) holds, (relative) weak compactness and sequential
(relative) weak compactness on \( \mathcal{U} \) are equivalent (Eberlein–Smulian property).
If, in addition, \( p \) are lower semicontinous on \( L^1 \), then \( \mathcal{U} \) is complete. In this case, \( \mathcal{U} \) is a Banach/Frégel space if \( \mathcal{P} \) is a singleton/countable.

Denoting \( p(u) = \rho(|u|) \), the function \( p \) is lsc in \( L^1 \) if and only if \( \rho \) has the Fatou property: for any sequence \( (\eta^\nu)^{\nu=1}_\nu \subset L^1_+ \) with \( \eta \wedge \eta \in L^1_+ \), \( \lim \rho(\eta^\nu) = \rho(\eta) \).

Proof. The first claim follows from the Theorem on p. 31 and Remark (2) on
p. 39 in [3]. If \((u^\nu)\) is a Cauchy net in \( \mathcal{U} \), it is Cauchy also in \( L^1 \) so it \( L^1 \)-
converges to an \( u \in L^1 \). Being Cauchy in \( \mathcal{U} \) means that for every \( \epsilon > 0 \) and
\( p \in \mathcal{P} \), there is a \( \bar{\nu} \) such that
\[
p(u^\nu - u) \leq \epsilon \quad \forall \nu \geq \bar{\nu}.
\]
The lower semicontinuity then gives
\[
p(u^\nu - u) \leq \epsilon \quad \forall \nu \geq \bar{\nu}.
\]
so $u \in \mathcal{U}$, by triangle inequality, and $(u^\nu)$ converges in $\mathcal{U}$ to $u$. Thus $\mathcal{U}$ is complete.

If $p$ is lsc, $\liminf \rho(\eta^\nu) \geq \rho(\eta)$ while (A3) gives $\limsup \rho(\eta^\nu) \leq \rho(\eta)$. If Fatou property holds and $u^\nu \to u$ in $L^1$, then, passing to a subsequence if necessary, $\bar{\eta}^\nu := \inf_{\nu \geq \nu} |u^\nu|$ increases pointwise to $|u|$, so $p(u) = \liminf \rho(\bar{\eta}^\nu) \leq \liminf \rho(u^\nu)$.

**Remark 5.** Under (A2) and (A3), $\mathcal{U}$ is solid and decomposable. Solidity means that $u \in \mathcal{U}$, $u' \in L^\infty$ and $|u'| \leq |u|$ imply $u' \in \mathcal{U}$. Decomposability means that $u \chi_A + \bar{u} \chi_{\Omega \setminus A} \in \mathcal{U}$ for every $u \in \mathcal{U}$, $\bar{u} \in L^\infty$ and $A \in \mathcal{F}$.

**Proof.** Assumption (A2) implies that $L^\infty \subset \mathcal{U}$ while (A3) gives $u \chi_A \in \mathcal{U}$ whenever $A \in \mathcal{F}$ and $u \in \mathcal{U}$. Since $\mathcal{U}$ is a linear space, the claim follows.

For each $p \in \mathcal{P}$, we define a sublinear symmetric function $p^\circ$ on $\mathcal{M}^1$ by

$$p^\circ(m) := \sup_{u \in L^\infty} \left\{ \int_\Omega ud\nu : p(u) \leq 1 \right\}.$$ 

**Lemma 6.** Let $p \in \mathcal{P}$. For each $m \in \text{dom } p^\circ$, every $u \in \text{dom } p$ is $m$-integrable and

$$\int_\Omega ud\nu \leq (p(u)) p^\circ(m).$$

For every $m \in \text{dom } p^\circ$, there exist unique $y \in L^1 \cap \text{dom } p^\circ$ and $m^s \in \mathcal{M}^{1s} \cap \text{dom } p^\circ$ such that

$$\int_\Omega ud\nu = E[u \cdot y] + \int_\Omega udm^s \quad \forall u \in \text{dom } p.$$

Given $m^s \in \mathcal{M}^{1s} \cap \text{dom } p^\circ$, there exists a decreasing $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ such that $P(A^\nu) \downarrow 0$ and

$$\int u \chi_{\Omega \setminus A^\nu} dm^s = 0$$

for every $u \in \text{dom } p$. Under (A4), $\mathcal{M}^{1s} \cap \text{dom } p^\circ = \{0\}$.

**Proof.** Lemma 2 and (A3) give

$$\int_\Omega ud\nu \leq \rho_m(u) \leq \sup_{u' \in L^\infty} \left\{ \int_\Omega u'd\nu : |u'| \leq |u| \right\} \leq (p(u)) p^\circ(m).$$

By Theorem 1 there exist $y \in L^1$ and $m^s \in (L^\infty)^s$ such that $m = yP + m^s$. Let $\alpha < p^\circ(y)$ and $\alpha^s < p^\circ(m^s)$ and $u, u^s \in L^\infty$ such that $p(u), p(u^s) \leq 1$ and

$$\int udP \geq \alpha \quad \text{and} \quad \int u^sdm^s \geq \alpha^s.$$  

Let $(A^\nu)_{\nu=1}^\infty \subset \mathcal{F}$ be decreasing with $P(A^\nu) \downarrow 0$ and $m^s(\Omega \setminus A^\nu) = 0$ and let $u^\nu = \lambda u \chi_{\Omega \setminus A^\nu} + (1 - \lambda) u^s \chi_{A^\nu}$, where $\lambda \in (0, 1)$. By convexity and (A3)

$$p(u^\nu) \leq \lambda p(u \chi_{\Omega \setminus A^\nu}) + (1 - \lambda) p(u^s \chi_{A^\nu}) \leq \lambda p(u) + (1 - \lambda) p(u^s) \leq 1.$$
while
\[ \limsup \int_{\Omega} u^\nu dm \geq \lambda \alpha + (1 - \lambda) \alpha^s. \]

Thus, \( p^\rho(m) \geq \lambda \alpha + (1 - \lambda) \alpha^s \). Since \( \alpha < p^\rho(y) \) and \( \alpha^s < p^\rho(m^s) \) were arbitrary, \( p^\rho(m) \geq \lambda p^\rho(y) + (1 - \lambda) p^\rho(m^s) \). Since \( \lambda \in (0, 1) \) was arbitrary, we get \( p^\rho(y) \leq p^\rho(m) \) and \( p^\rho(m^s) \leq p^\rho(m) \). Thus, \( y \in \text{dom } p^\rho \) and \( m^s \in \text{dom } p^\rho \).

To prove the last claim, let \( m^s \in \mathcal{M}^{1s} \cap \text{dom } p^\rho \). By the first claim,
\[ \int_{\Omega} u_1 \, dm^s \leq p(u_1) p^\rho(m^s) \quad \forall u \in L^\infty, A \in \mathcal{F} \]
so, by the last claim of Theorem 1, condition \( \text{[A4]} \) implies \( m^s = 0 \).

Let \( \mathcal{M} \) be the set of \( P \)-absolutely continuous finitely additive measures \( m \) such that \( p^\rho(m) < \infty \) for some \( p \in \mathcal{P} \). The set of purely finitely additive elements of \( \mathcal{M} \) will be denoted by \( \mathcal{M}^s \). The set of densities \( y = dm/dP \) of countably additive \( m \in \mathcal{M} \) will be denoted by \( \mathcal{Y} \).

The following is the main result of this section. It identifies the topological dual of \( \mathcal{U} \) with the direct sum of the Köthe space, purely finitely additive measures \( \mathcal{M}^s \) and the annihilator \( (L^\infty)^\perp := \{ w \in \mathcal{U}^* | \langle u, w \rangle = 0 \ \forall u \in L^\infty \} \) of \( L^\infty \).

**Theorem 7.** We have
\[ \mathcal{U}^* = \mathcal{Y} \oplus \mathcal{M}^s \oplus (L^\infty)^\perp \]
in the sense that for every \( u^* \in \mathcal{U}^* \) there exist unique \( y \in \mathcal{Y}, m^s \in \mathcal{M}^s \) and \( w \in (L^\infty)^\perp \) such that
\[ \langle u, u^* \rangle = E[u \cdot y] + \int_{\Omega} u \, dm^s + \langle u, w \rangle. \]

For every \( u \in \mathcal{U} \) and \( m \in \mathcal{M} \),
\[ \int_{\Omega} u \, dm \leq p(u) p^\rho(m). \]

Given \( w \in (L^\infty)^\perp \) and \( u \in \mathcal{U} \), there exists a decreasing sequence \( (A^\nu)_{\nu=1}^\infty \subset \mathcal{F} \) with \( P(A^\nu) \searrow 0 \) and
\[ \langle u, w \rangle = \langle u_1 A^\nu, w \rangle \quad \forall \nu = 1, 2, \ldots. \]

Under \( \text{[A4]} \), \( \mathcal{M}^s = \{0\} \) and under \( \text{[A5]} \), \( (L^\infty)^\perp = \{0\} \).

**Proof.** By Lemma 6, \( \mathcal{M} \subset \mathcal{U}^* \), so \( \mathcal{M} \oplus (L^\infty)^\perp \subseteq \mathcal{U}^* \). To prove the opposite inclusion, let \( u^* \in \mathcal{U}^* \). There exists \( p \in \mathcal{P} \) and \( \gamma > 0 \) such that \( u^* \leq \gamma p \). Assumption \( \text{[A2]} \) implies that \( u^* \) is continuous in \( L^\infty \). By Theorem 1, there exists a unique \( m \in \mathcal{M}^1 \) such that \( \langle u, u^* \rangle = \int_{\Omega} u \, dm \) for all \( u \in L^\infty \). Since
\[ u^* \leq \gamma \rho, \text{ so } m \in \text{dom } p^\circ, \text{ so } m \text{ is continuous on } \mathcal{U} \] by Lemma 6. Now \[ w := u^* - m \] belongs to \( (L^\infty)^\perp \), so \( u^* \) has the required decomposition. Given another decomposition \( u^* = \tilde{m} + \tilde{w} \) with \( \tilde{w} \in (L^\infty)^\perp \) and \( \tilde{m} \in \mathcal{M} \), we have \( (m - \tilde{m}) + (w - \tilde{w}) = 0 \). Thus \( \int_\Omega ud(m - \tilde{m}) = 0 \) for all \( u \in L^\infty \), so \( m - \tilde{m} = 0 \) and hence also \( w - \tilde{w} = 0 \), so the decomposition is unique.

The inequality follows directly from that of Lemma 6. Let \( u \in \mathcal{U} \) and \( A^\nu := \{|u| > \nu\} \). Clearly \( P(A^\nu) \searrow 0 \) and \( u1_{\Omega \setminus A^\nu} \in L^\infty \), so \( \langle u1_{\Omega \setminus A^\nu}, w \rangle = 0 \) and thus \( w \) is singular. That \( \mathcal{M} = \mathcal{Y} \) under \([A4]\) is the last claim of Lemma 6. Under \([A5]\) the truncations \( u^\nu := u1_{\{|u| \geq \nu\}} \) of any \( u \in \mathcal{U} \) converge to \( u \) so \( L^\infty \) is dense in \( \mathcal{U} \) and thus, \((L^\infty)^\perp = \{0\}\).

Applications of Theorem 7 are given in Section 5. When \( P \) is a singleton, we are in the setting of [23], where the dual of \( \mathcal{U} \) is decomposed into a direct sum of \( \mathcal{Y} \) and "singular elements". Theorem 7 gives a more precise description of the singular elements as a direct sum of \( \mathcal{M}^s \) and \((L^\infty)^\perp \).

Note that the inequality in Theorem 7 implies that \( p^\circ \) coincides on \( \mathcal{M} \) with the polar (i.e., the dual seminorm) of \( p \). An application of Theorem 7 and the Hahn-Banach theorem gives the following result, where \( \tilde{\mathcal{U}} \) is the closure of \( L^\infty \) in \( \mathcal{U} \).

**Corollary 8.** We have
\[ \tilde{\mathcal{U}}^* = \mathcal{M} \]
in the sense that for every \( \tilde{u}^* \in \tilde{\mathcal{U}}^* \) there exist unique \( m \in \mathcal{M} \) such that
\[ \langle \tilde{u}, \tilde{u}^* \rangle = \int_\Omega \tilde{u} dm. \]
In particular, if \([A4]\) holds, then \( \tilde{\mathcal{U}}^* = \mathcal{Y} \) and if \([A5]\) holds, then \( \tilde{\mathcal{U}} = \mathcal{U} \).

The following lists some basic properties of the Köthe dual \( \mathcal{Y} \).

**Lemma 9.** We have
1. \( L^\infty \subseteq \mathcal{Y} \)
and, for each \( p \in \mathcal{P} \)
2. there is a constant \( c \) such that \( c\|y\|_{L^1} \leq p^\circ(y) \) for all \( y \in L^1 \),
3. \( p^\circ(y') \leq p^\circ(y) \) for every \( y', y \in L^1 \) with \( |y'| \leq |y| \).
4. We have the “Hölder’s inequality”
\[ E[u \cdot y] \leq p(u)p^\circ(y) \]
and, conversely, if there is \( c > 0 \) such that \( c\|u\|_{L^1} \leq p(u) \) for all \( u \) and \( p \)
is lsc in \( L^1 \), then \( p^\circ(y) < \infty \) whenever \( E[u \cdot y] < \infty \) for all \( u \in \text{dom } p \).

In particular, \( \mathcal{Y} \) is solid and decomposable.
Proof. Assumption \((A1)\) implies 1, and \((A2)\) implies 2. By \((A3)\)
\[
p^\circ(y') = \sup_{u' \in L^\infty, u \in L^\infty} \{ E[u' \cdot y'] | |u'| \leq |u|, \ p(u) \leq 1 \}
\]
\[
= \sup_{u \in L^\infty} \{ E[|u||y'|] | p(u) \leq 1 \}
\]
\[
\leq \sup_{u \in L^\infty} \{ E[|u||y|] | p(u) \leq 1 \}
\]
\[
= p^\circ(y),
\]
so 3 holds. To prove 4, the inequality in Lemma 6 gives the H"{o}lder’s inequality.
Assume now that \(p^\circ(y) = +\infty\). Let \(\alpha > 0\) be such that \(\sum \alpha^\nu = 1\). There exists \(u^\nu\) with \(p(u^\nu) \leq 1\), \(u^\nu \cdot y \geq 0\) and \(E[u^\nu \cdot y] \geq 1/\alpha^\nu\). We have that \(\sum_{\nu=1}^\nu \alpha^\nu u^\nu\) converges to \(u := \sum \alpha^\nu u^\nu\) in \(L^1\) and, since \(p\) is lsc in \(L^1\), \(u \in \text{dom} \ p\). By monotone convergence,
\[
E[u \cdot y] = \sum_{\nu=1}^\infty \alpha^\nu E[u^\nu \cdot y] = +\infty,
\]
which completes the proof.

\[\square\]

5 Examples

The following example is a direct application of Corollary 8.

Example 10 (The space of finite moments). The \(L^p\)-norms with \(p = 1, 2, \ldots\) satisfy \((A1)-(A5)\), so the space
\[
U := \bigcap_{p \geq 1} L^p
\]
of measurable functions with finite moments is a Fréchet space and its dual may be identified with
\[
Y := \bigcup_{p > 1} L^p
\]
under the bilinear form \((u, y) = E[u \cdot y]\).

Spaces with finite moments strictly less that \(p\)?

Given a set \(C\) in a linear space, we will use the notation
\[
\text{pos} \ C := \bigcup_{\alpha > 0} (\alpha C) \quad \text{and} \quad C^\infty := \bigcap_{\alpha > 0} (\alpha C).
\]
The following construction, inspired by the Luxemburg norm in the theory of Orlicz spaces, turns out to be convenient.

Example 11. Let \(H : L^1 \rightarrow \bar{\mathbb{R}}_+\) be lsc convex such that \(H(0) = 0\) and
(H1) there is a constant \(c > 0\) such that \(H(u) \leq 1\) implies \(\|u\|_{L^1} \leq c\),

(H2) \(L^\infty \subset \text{pos}(\text{dom} \ H)\),

(H3) \(H(u_1) \leq H(u_2)\) whenever \(|u_1| \leq |u_2|\).

The function

\[
p(u) := \inf\{\beta > 0 \mid H(u/\beta) \leq 1\}
\]

is lsc, symmetric and sublinear. Let \(P = \{p\}\) and \(U = \text{dom} \ p\). Assumptions \([A1][A3]\) hold and, in particular, \(U\) is a Banach space with dual

\[
U^* = \mathcal{M} \oplus (L^\infty)^{\perp},
\]

where

\[
\mathcal{M} = \text{pos} \text{ dom} \ H^*
\]

with \(H^* : \mathcal{M}^1 \to \mathbb{R}\) given by

\[
H^*(m) := \sup_{u \in L^\infty} \left\{ \int_\Omega u \, \text{d}m - H(u) \right\}.
\]

For any \(m \in \mathcal{M}^1\),

\[
p^\circ(m) = \sup_{u \in L^\infty} \left\{ \int_\Omega u \, \text{d}m \mid H(u) \leq 1 \right\} = \inf_{\beta > 0} \{\beta H^*(m/\beta) + \beta\},
\]

restriction of \(p^\circ\) to \(\mathcal{M}\) is the polar of \(p\) and

\[
\|m\|_{H^*} \leq p^\circ(m) \leq 2\|m\|_{H^*},
\]

where

\[
\|m\|_{H^*} := \inf\{\beta > 0 \mid H^*(m/\beta) \leq 1\}.
\]

Assume now that \(L^\infty \subseteq \text{dom} \ H\). If

(H4) \(H(u^\nu) \searrow 0\) whenever \((u^\nu)_{\nu=1}^\infty \subseteq L^\infty\) with \(|u^\nu| \searrow 0\) almost surely,

then \([A4]\) holds so \(\mathcal{M}^* = \{0\}\) and the dual of the closure \(\hat{U}\) of \(L^\infty\) in \(U\) can be identified with \(\mathcal{Y}\). If

(H5) \(H(u^\nu) \searrow 0\) whenever \((u^\nu)_{\nu=1}^\infty \subseteq \text{dom} \ H\) with \(|u^\nu| \searrow 0\) almost surely,

then \(\hat{U} = (\text{dom} \ H)^\infty\). In particular, \(U = \hat{U}\) if \(\text{dom} \ H\) is a cone.

Proof. Let \(u^\nu \to u\) in \(L^1\) be such that \(p(u^\nu) \leq \alpha\) or, in other words, \(H(u^\nu/\alpha) \leq 1\). Thus lower semicontinuity of \(H\) implies that of \(p\). It is clear that \([H1]\) implies \([A1]\). By \([H2]\) \(p\) is finite on \(L^\infty\). Since \(p\) is lsc on \(L^1\), it is lsc on \(\sigma(L^\infty, L^1)\). Thus, by \([20]\) Corollary 8B, \(p\) is continuous in \(L^\infty\) and thus \([A2]\) holds. Assumption \([A3]\) is clear from \([H3]\).
Let \( m \in M^1 \). Since the infimum in the definition of the Luxemburg norm is attained,

\[
p^\circ(m) = \sup_{u \in \mathcal{L}^\infty} \left\{ \int_{\Omega} u \, dm \mid p(u) \leq 1 \right\} = \sup_{u \in \mathcal{L}^\infty} \left\{ \int_{\Omega} u \, dm \mid H(u) \leq 1 \right\}.
\]

Lagrangian duality gives

\[
p^\circ(m) = \inf_{\beta > 0} \sup_{u \in \mathcal{L}^\infty} \left\{ \int_{\Omega} u \, dm - \beta H(u) + \beta \right\} = \inf_{\beta > 0} \left\{ \beta \mathcal{H}(m/\beta) + \beta \right\}.
\]

Clearly,

\[
p^\circ(m) \leq \inf_{\beta > 0} \left\{ \beta \mathcal{H}(m/\beta) + \beta \mid \mathcal{H}(m/\beta) \leq 1 \right\} \leq 2 \inf_{\beta > 0} \left\{ \beta \mid \mathcal{H}(m/\beta) \leq 1 \right\}.
\]

On the other hand, we have

\[
p^\circ(m) = \inf_{\beta > 0} \left\{ \beta \mathcal{H}(m/\beta) + \beta \right\} = \inf_{\alpha > 0} \frac{g(\alpha m)}{\alpha},
\]

where \( g(m) = \mathcal{H}(m) + 1 \). Since \( \mathcal{H}^* \geq 0 \), we have \( g \geq \| \cdot \|_{\mathcal{H}^*} \) when \( \| m \|_{\mathcal{H}^*} \leq 1 \). When \( \| m \|_{\mathcal{H}^*} > 1 \), convexity and the fact that \( \mathcal{H}^*(0) = 0 \) give

\[
\mathcal{H}(m/\| m \|_{\mathcal{H}^*}) \leq \mathcal{H}((m/\| m \|_{\mathcal{H}^*})/\beta) \leq 0.
\]

By definition of \( \| m \|_{\mathcal{H}^*} \), the left side equals 1 so \( \| m \|_{\mathcal{H}^*} \leq \mathcal{H}(m/\| m \|_{\mathcal{H}^*}) \leq g(m) \).

Thus,

\[
p^\circ(m) \geq \inf_{\alpha > 0} \frac{\| m \|_{\mathcal{H}^*}}{\alpha} = \| m \|_{\mathcal{H}^*}.
\]

If \( (H4) \) holds and \( |u^\nu| \searrow 0 \) almost surely in \( \mathcal{L}^\infty \), then for all \( \beta > 0 \),

\[
H((u^\nu)/\beta) \nearrow 0
\]

so \( p(u^\nu) \searrow 0 \). In particular, \( (A4) \) holds.

To prove the last claim, let \( u \in (\text{dom } H)^\infty \), \( u^\nu := u1_{|u| \leq \nu} \) and \( \beta > 0 \). By \( (H3) \) \( u - u^\nu = u1_{\Omega \setminus \{ |u| \leq \nu \}} \in \beta \text{ dom } \mathcal{H} \) so \( (H5) \) implies

\[
H((u - u^\nu)/\beta) \nearrow 0.
\]

Since \( \beta > 0 \) was arbitrary, we get \( p(u - u^\nu) \searrow 0 \) so \( \text{dom } H)^\infty \subseteq \tilde{U} \). To prove the converse, it remains to show that \( \text{dom } H)^\infty \) is closed in \( \mathcal{U} \). If \( (u^\nu) \) is in \( \text{dom } H)^\infty \) and converges to \( u \in \tilde{U} \), we have for any \( \beta > 0 \),

\[
H(u/(2\beta)) \leq \frac{1}{2} H(u^\nu/\beta) + \frac{1}{2} H((u - u^\nu)/\beta) \leq \frac{1}{2} H(u^\nu/\beta) + \frac{1}{2}
\]

for \( \nu \) large enough, so \( H(u/(2\beta)) < \infty \) and thus \( u \in (\text{dom } H)^\infty \). \( \square \)
Musielak–Orlicz spaces are generalizations of Orlicz spaces where the associated Young function $\Phi$ is allowed to be random in the sense that it is a function on $\mathbb{R} \times \Omega$ such that

$$\omega \mapsto \{(\xi, \alpha) \mid \Phi(\xi, \omega) \leq \alpha\}$$

is a convex-valued measurable mapping; see [21, Chapter 14]. If $\Phi$ only takes finite real values, this happens exactly when $\Phi(\xi, \cdot)$ is measurable for every $\xi \in \mathbb{R}$ and $\Phi(\cdot, \omega)$ is convex for every $\omega \in \Omega$. The dual of a Musielak–Orlicz space can be characterized in terms of the conjugate function defined by

$$\Phi^*(\eta, \omega) = \sup_{\xi \in \mathbb{R}} \{\xi \eta - \Phi(\xi, \omega)\}.$$ 

The measurability condition on $\Phi$ implies the same property for $\Phi^*$; see [21, Theorem 14.50].

**Example 12** (Musielak-Orlicz spaces). Let $\Phi: \mathbb{R} \times \Omega \to \mathbb{R}_+$ be nonzero random symmetric convex function with $\Phi(0) = 0$ and such that $\Phi(a, \cdot), \Phi^*(a, \cdot) \in L^1$ for some constant $a > 0$. Endowed with the Luxemburg norm

$$\|u\|_{\Phi} := \inf\{\beta > 0 \mid E\Phi(|u|/\beta) \leq 1\},$$

$L^\Phi := \{u \in L^1 \mid \|u\|_{\Phi} < \infty\}$ is a Banach space. The dual of $L^\Phi$ is

$$(L^\Phi)^* = L^\Phi^* \oplus M^* \oplus (L^\infty)^\perp,$$

where

$$M^* = \{m \in M^{1s} \mid \sigma_\Phi(m) < \infty\}$$

with $\sigma_\Phi(m) := \sup_{u \in L^\infty} \left\{\int_{\Omega} udm \mid E\Phi(|u|) < \infty\right\}$. For any $y + m^* \in L^\Phi^* \oplus M^*$, the dual norm can be expressed as

$$\|y + m^*\|^*_\Phi = \sup_{u \in L^\infty} \{E[u \cdot y] + \int_{\Omega} udm^* \mid E\Phi(|u|) \leq 1\} = \inf_{\beta > 0} \{\beta E\Phi^*(|y^*|/\beta) + \beta + \sigma_\Phi(m^*)\},$$

we have

$$\|y\|_{\Phi^*} \leq \|y\|^*_\Phi \leq 2\|y\|_{\Phi^*} \quad \forall y \in L^\Phi^*,$$

and the dual of the closure $M^\Phi$ of $L^\infty$ in $L^\Phi$ is

$$(M^\Phi)^* = L^\Phi^* \oplus M^*.$$ 

Assume now that $\Phi(a, \cdot) \in L^1$ for all $a > 0$. Then, $M^* = \{0\}$, $M^\Phi$ coincides with the Morse heart

$$(\text{dom } E\Phi)^\infty = \{\xi \in L^1 \mid E\Phi(|\xi|/\beta) < \infty \quad \forall \beta > 0\},$$

and, in particular, $L^\Phi = M^\Phi$ if $\text{dom } E\Phi$ is a cone.
Proof. We apply Example 11 to $H(u) := E\Phi(|u|)$. By [21, Theorem 14.60],

$$H(u) = \sup_{\eta \in L^\infty} E\{[|u|\eta] - \Phi^*(\eta)\},$$

so $H$ is $L^1$-lsc. This also gives

$$H(u) \geq a\|u\|_{L^1} - E\Phi^*(a)$$

so $\Phi^*(a) \in L^1$ implies [H1]. The assumption $\Phi(a) \in L^1$ implies that $H(u) < \infty$ when $\|u\|_{L^\infty} \leq a$ so [H2] holds. Property [H3] holds since $\Phi$ is increasing. By [19, Theorem 1] and [18, Theorem 15.3],

$$H^*(m) = \sup_{u \in L^\infty} \{ \int u dm - Eh(u) \} = E\Phi^*(|y|^*) + \sigma\Phi(m^*).$$

If $\Phi(a) \in L^1$ for all $a > 0$, then $L^\infty \subset \text{dom}\ S$ and [H4] and [H5] hold by monotone convergence theorem. Thus all the claims follow from Example 11.

In [14], the assumption $\Phi(a, \cdot) \in L^1$ for all $a > 0$ is called "local integrability". Thus we recover [14, Theorem 2.4.4] for probability spaces. Our characterization of the dual without local integrability seems new.

Example 13 (Risk measures). Let $\rho : L^1 \to \mathbb{R}$ be a "convex risk measure" in the sense that it is convex, nondecreasing, $\rho(0) = 0$ and $\rho(\xi + \alpha) = \rho(\xi) + \alpha$ for all $\xi \in L^1$ and $\alpha \in \mathbb{R}$. Assume that $n = 1$, $\rho$ is $L^1$-lsc and that there is a constant $c > 0$ such that $\rho(|u|) \leq 1$ implies $\|u\|_{L^1} \leq c$.

Endowed with the norm

$$\|u\|_\rho := \inf\{\beta > 0 \mid \rho(|u|/\beta) \leq 1\},$$

$L^\rho := \{u \in L^1 \mid \rho(|u|) < \infty\}$ is a Banach space whose dual can be identified with $\mathcal{M} \oplus (L^\infty)^\perp$, where

$$\mathcal{M} = \{m \in \mathcal{M}^1 \mid \exists \beta > 0 : \alpha(|m|/\beta) < \infty\}$$

with $\alpha : \mathcal{M}^1 \to \mathbb{R}$ defined by

$$\alpha(m) := \sup_{\xi \in L^+_\infty} \{ \int_\Omega \xi dm - \rho(\xi) \}.$$ 

For any $m \in \mathcal{M}$, the dual norm can be expressed as

$$\|m\|_\rho^* = \sup_{u \in L^\infty} \{ \int_\Omega u dm \mid \rho(u) \leq 1 \} = \inf_{\beta > 0} \{ \beta \alpha(|m|/\beta) + \beta \},$$

and

$$\|m\|_\alpha \leq \|m\|_\rho^* \leq 2\|m\|_\alpha,$$

where

$$\|m\|_\alpha := \inf\{\beta > 0 \mid \alpha(|m|/\beta) \leq 1\}.$$
1. If \( \rho \) has the Lebesgue property on \( \mathbb{L}^\infty \): \( \rho(\xi^n) \rightarrow 0 \) for any decreasing sequence \( (\xi^n) \subset \mathbb{L}^\infty \) with \( \xi^n \rightarrow 0 \) almost surely, then the dual of the closure \( \hat{\mathbb{L}}^\rho \) of \( \mathbb{L}^\infty \) in \( \mathbb{L}^\rho \) can be identified with

\[
\mathbb{L}^\rho := \{ y \in \mathbb{L}^1 \mid \exists \beta > 0 : \alpha(|y|/\beta) < \infty \}.
\]

2. If \( \rho \) has the Lebesgue property on \( \text{dom} \rho \): \( \rho(\xi^n) \rightarrow 0 \) for any decreasing sequence \( (\xi^n) \subset \text{dom} \rho \) with \( \xi^n \rightarrow 0 \) almost surely, then

\[
\hat{\mathbb{L}}^\rho = \{ u \in \mathbb{L}^1 \mid \rho(|u|/\beta) < \infty \forall \beta > 0 \},
\]

and, in particular, \( \mathbb{L}^\rho = \hat{\mathbb{L}}^\rho \) if \( \text{dom} \rho \) is a cone.

**Proof.** We apply Example 11 to the function \( H(u) := \rho(|u|) \). By assumption, \( (H1) \) and \( (H3) \) hold. By monotonicity and translation invariance, \( \rho(|u|) \leq \rho(\|u\|_{L^\infty}) = \|u\|_{L^\infty} \), so \( \mathbb{L}^\infty \subset \text{dom} H \). In particular, \( (H2) \) holds. The conditions \( (H4) \) and \( (H5) \) in Example 11 translate directly to those of 1 and 2. Thus the claims follow from Example 11 since here

\[
H^*(m) := \sup_{u \in \mathbb{L}^\infty} \left\{ \int u dm - \rho(|u|) \right\}
= \sup_{u \in \mathbb{L}^\infty, \xi \in \mathbb{L}^\infty_+} \left\{ \int u \xi dm - \rho(\xi) \mid |u| = 1 \right\}
= \sup_{\xi \in \mathbb{L}^\infty_+} \left\{ \int \xi |m| - \rho(\xi) \right\}
= \alpha(|m|),
\]

where the second last equality follows from [22, Theorem 2.3] and the fact that \( \nu(A) := \int_A \xi dm \) is a finitely additive measure with \( |\nu(A)| = \int_A \xi dm \).

Given \( u \in \mathbb{L}^1 \), let

\[ n_u(\tau) := E1_{\{|u| > \tau\}} \]

and

\[ q_u(t) := \inf\{ \tau \in \mathbb{R} \mid n_u(\tau) \leq t \}. \]

Note that \( \tau \mapsto 1 - n_u(\tau) \) is the cumulative distribution function of \( |u| \) and that \( q_u \) is an inverse of \( n_u \). Both \( n_u \) and \( q_u \) are nonincreasing.

**Lemma 14.** We have

\[
\int_0^t q_u(t) dt = \inf_{s \in \mathbb{R}^+} \{ ts + E[|u| - s]^+ \}.
\]

**Proof.** By Theorems 23.5 and 24.2 of [13], the functions

\[ f(t) := \int_0^t q_u(s) ds \]
and
\[ f^*(s) = \int_0^s n_u(\tau)d\tau - \int_0^\infty n_u(\tau)d\tau = -\int_s^\infty n_u(\tau)d\tau \]
are concave and conjugate to each other. By Fubini,
\[ f^*(s) = -E\int_s^\infty 1_{\{|u|>\tau\}}d\tau = -E[|u| - s]^+ \]
so
\[ \int_0^t q_u(s)ds = \inf_{s\in \mathbb{R}^+} \{ts + E[|u| - s]^+\}, \]
by the biconjugate theorem (see e.g. [18, Theorem 12.2]).

Recall that a probability space is resonant if it is atomless or completely atomic with all atoms having equal measure.

**Example 15** (Lorentz and Marcinkiewicz spaces). Assume that \((\Omega, \mathcal{F}, P)\) is resonant. Given a nonnegative concave increasing function \(\phi\) on \([0,1]\) with \(\phi(0) = 0\), the associated Marcinkiewicz space is the linear space \(M_\phi\) of \(u \in L^1\) with
\[ \|u\|_\phi := \sup_{t \in [0,1]} \left\{ \frac{1}{\phi(t)} \int_0^t q_u(s)ds \right\} < \infty. \]
The function \(\| \cdot \|_\phi\) is a norm and \(M_\phi\) is a Banach space. If \(\lim_{t \to 0} t/\phi(t) > 0\), we have \(M_\phi = L^\infty\). Assume now that \(\lim_{t \to 0} t/\phi(t) = 0\). The topological dual of \(M_\phi\) is
\[ M_\phi^* = \Lambda_\phi \oplus (L^\infty)^\perp, \]
where \(\Lambda_\phi\) is the Lorentz space
\[ \Lambda_\phi := \{ y \in L^1 \mid \|y\|_\phi^* < \infty \}, \]
where
\[ \|y\|_\phi^* := \int_0^1 q_y(t)d\phi(t). \]
The closure of \(L^\infty\) in \(M_\phi\) can be expressed as
\[ \tilde{M}_\phi = \{ u \in L^1 \mid \lim_{t \to 0} \frac{1}{\phi(t)} \int_0^t q_u(s)ds = 0 \}. \]
The topological dual of \(\tilde{M}_\phi\) is \(\Lambda_\phi\) and the topological dual of \(\Lambda_\phi\) is \(M_\phi\).

**Proof.** By Lemma [14]
\[ u \mapsto \int_0^t q_u(t)dt \]
is the infimal projection of a sublinear function of \(s\) and \(u\) and thus, sublinear in \(u\). It is also continuous in \(L^1\). It follows that \(\| \cdot \|_\phi\) is sublinear, symmetric and lsc in \(L^1\).
Since
\[ \|u\|_\phi \geq \phi(1) \int_0^1 q_u(s)\,ds = \phi(1)E[|u|], \]
(A1) holds. By Remark 2, \( M_\phi \) is Banach. Since \( q_u \leq \|u\|_{L^\infty} \), we have
\[ \|u\|_\phi \leq \sup_{t \in (0, 1]} \frac{t}{\phi(t)} \|u\|_{L^\infty}, \]
where \( \sup_{t \in (0, 1]} \frac{t}{\phi(t)} < \infty \) since \( \phi \) is concave and strictly positive for \( t > 0 \). Thus, (A2) holds. Property (A3) is clear. Given \( A \in F \),
\[ \|1_A\|_\phi = \sup_{t \in (0, 1]} \frac{1}{\phi(t)} \min\{t, P(A)\} = \frac{P(A)}{\phi(P(A))}, \]
since \( t \mapsto \frac{t}{\phi(t)} \) is increasing by concavity. Thus \( \hat{\phi}_p(t) := \frac{t}{\phi(t)} \) is the fundamental function of \( M_\phi \). By Remark 3, \( M_\phi = L^\infty \) if \( \lim_{t \to 0^+} t/\phi(t) > 0 \) while (A4) holds if \( \lim_{t \to 0^+} t/\phi(t) = 0 \). We have
\[ \|y\|_{\phi}^* = \sup_{u \in L^1} \{ E[uy] \mid \|u\|_\phi \leq 1 \} \]
= \( \sup_{u \in L^1} \left\{ \int_0^1 q_u(t)q_y(t)\,dt \mid \int_0^t q_u(s)\,ds \leq \phi(t) \ \forall t \in [0, 1] \right\} \]
= \( \int_0^1 q_y(t)\phi'(t)\,dt \]
= \( \int_0^1 q_y(t)\,d\phi(t) \),
where the second equality follows from [1 Corollary 2.4.4] and the third from Hardy’s lemma [1 Proposition 2.3.6]. The representation of the topological dual of \( M_\phi \) now follows from Theorem 7.

If \( u \in L^\infty \), \( q_u \) is bounded, so
\[ \lim_{t \to 0^+} \frac{1}{t}\int_0^t q_u(s)\,ds = \lim_{t \to 0^+} t \int_0^t \frac{1}{t} q_u(s)\,ds = 0, \]
by assumption. Thus, \( L^\infty \subset \tilde{M}_\phi \). Let \( u \in M_\phi \) and \( \tilde{M}_\phi \). We have \( q_{u+\tilde{u}}(s^1+s^2) \leq q_u(s^1) + q_{\tilde{u}}(s^2) \), so
\[ \lim_{t \to 0^+} \frac{1}{\phi(t)} \int_0^t q_u(s)\,ds \leq \lim_{t \to 0^+} \frac{1}{\phi(t)} \int_0^t (q_u(s^1) + q_{\tilde{u}}(s^2))\,ds \]
= \( \lim_{t \to 0^+} \frac{1}{\phi(t)} \int_0^t q_u(s)\,ds \]
= \( \lim_{t \to 0^+} \frac{2}{\phi(2t)} \int_0^{2t} q_u(s)\,ds \]
\leq \lim_{t \to 0^+} \frac{1}{\phi(2t)} \int_0^{2t} q_u(s)\,ds \]
\leq \|u - \tilde{u}\|_\phi,
where the second last inequality follows from concavity of $\phi$. Thus, $\tilde{M}_\phi$ is closed in $M_\phi$ so $\tilde{M}_\phi$ contains the closure of $L^\infty$. To prove the converse, let $u \in M_\phi$ and $u^\nu = u_1\{|u| \leq \nu\}$. We have $q_{u-u^\nu}(t) = 0$ for $t \geq t^\nu := P(|u| \geq \nu)$ while $q_{u-u^\nu}(t) = q_u(t)$ for $t < t^\nu$. Thus,

$$\|u - u^\nu\|_\phi = \sup_{t \in [0,1]} \left\{ \frac{1}{\phi(t)} \int_0^t q_{u-u^\nu}(s) \, ds \right\}. $$

Since $u \in \tilde{M}_\phi$, this converges to 0 as $\nu \to \infty$. Thus, $\tilde{M}_\phi$ is the closure of $L^\infty$ in $M_\phi$.

By Lemma 9, the Lorentz seminorm satisfies (A1)-(A3). If $y^\nu \rightharpoonup 0$ with $\|y^\nu\|_\phi^* < \infty$, we have $q_{y^\nu} \rightharpoonup 0$, so by monotone convergence, $\|y^\nu\|_\phi^* \to 0$. Thus, the Lorenz norm satisfies (A5). The fact that the topological dual of $\Lambda_\phi$ is $M_\phi$ now follows from Theorem 7 and the fact that, by the bipolar theorem, $p$ is the polar of $p^\circ$.

Example 16 (Generalized Orlicz-spaces). Let $\Phi$ be as in Example 12 with $\text{dom } \Phi = \mathbb{R}$ and let $r$ be a sublinear symmetric lsc function on $L^1$ satisfying (A1)-(A4). Endowed with the norm

$$\|u\|_{\Phi,r} := \inf_{\beta > 0} \{\beta > 0 | r(\Phi(|u|/\beta) \leq 1)\},$$

$\mathcal{U} := \{u \in L^1 | \|u\|_{\Phi,r} < \infty\}$ is a Banach space with dual

$$\mathcal{U}^* = \mathcal{Y} \oplus (L^\infty)^\perp,$$

where $\mathcal{Y} := \{y \in L^1 | \|y\|_{\Phi,r}^* < \infty\}$ with

$$\|y\|_{\Phi,r}^* = \inf_{v \in \mathcal{L}^1} \{E[v\Phi^*(y/v)] + r^\circ(v)\}.$$

Moreover,

$$\|y\|_{H}^* \leq \|y\|_{\Phi,r}^* \leq 2\|y\|_{H},$$

where

$$\|y\|_{H}^* = \inf_{\beta > 0} \{\beta > 0 | H^*(y/\beta) \leq 1\} = \inf_{v \in \mathcal{L}^1} \max\{r^\circ(v), E[v\Phi^*(y/v)]\}.$$ 

If $r$ satisfies (A5) then the closure of $L^\infty$ in $\mathcal{U}$ has the expression

$$\tilde{\mathcal{U}} = \{u \in L^1 | r(\Phi(|u|/\beta)) < \infty \forall \beta > 0\}.$$ 

In this case, $\tilde{\mathcal{U}} = \mathcal{U}$ if $\text{dom } H$ is a cone. In particular, $\text{dom } H$ is a cone if $\Phi$ satisfies $\Delta_2$-condition: there exists $K > 0$ and $x_0$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq x_0$. 


Proof. This fits Example 11 with

\[ H(u) := \begin{cases} r(\Phi(|u|)) & \text{if } \Phi(|u|) \in L^1, \\ +\infty & \text{otherwise.} \end{cases} \]

For every \( u \in L^1 \),

\[ H(u) = \sup_{\eta \in L^\infty} \{ E[\eta \Phi(|u|)] - r^*(\eta) \}, \]

so \( H \) is lsc in \( L^1 \). Since \( r \) satisfies (A1)–(A4), \( H \) satisfies (H1)–(H4).

We compute the conjugate of \( H \) by employing conjugate duality; see [20]. Let \( F(x, u) := r(\Phi(u) + x) \) be defined on \( L^\infty \times L^\infty \). The conjugate \( F^* \) on \( L^1 \times L^1 \) has the expression

\[ F^*(v, y) := \sup_{u, x \in L^\infty} \{ E[xv + uy] - r(\Phi(u) + x) \} \]
\[ = \sup_{u, x \in L^\infty} \{ E[vx - v\Phi(u) + uy] - r(x) \} \]
\[ = E[v\Phi^*(y/v)] + \delta_{B^*}(v), \]

where the last equality comes from the interchange rule [21, Theorem 14.60] and \( B^* := \{ v \in L^1 \mid r^o(v) \leq 1 \} \).

Since \( r \) satisfies \([A4]\) it is \( r(L^\infty, L^1)\)-continuous?. By [20] Theorem 17, this implies

\[ H^*(y) = \inf_{v \in L^1} F^*(y, v) = \inf_{v \in L^1} \{ E[v\Phi^*(y/v)] \mid r^o(v) \leq 1 \} \]

so, by Example 11

\[ \|y\|_{\Phi, r} = \inf_{\beta > 0} \{ \beta H^*(y/\beta) + \beta \} \]
\[ = \inf_{\beta > 0, v \in L^1} \{ E[\beta v\Phi^*(y/(\beta v))] \mid r^o(v) \leq 1 \} \]
\[ = \inf_{\beta > 0, v \in L^1} \{ E[v\Phi^*(y/v)] \mid r^o(v) \leq \beta \} \]
\[ = \inf_{v \in L^1} \{ E[v\Phi^*(y/v)] \mid r^o(v) \}. \]

The claims concerning the dual space and its norm follow from Example 11. We have

\[ \|y\|_{H^*} := \inf \{ \beta > 0 \mid H^*(y/\beta) \leq 1 \} \]
\[ = \inf \{ \beta > 0 \mid \exists v \in L^1 : r^o(v) \leq 1, E[v\Phi^*(y/(\beta v))] \leq 1 \} \]
\[ = \inf \{ \beta > 0 \mid \exists v \in L^1 : r^o(v) \leq \beta, E[v\Phi^*(y/v)] \leq \beta \} \]
\[ = \inf_{v \in L^1} \max \{ r^o(v), E[v\Phi^*(y/v)] \}. \]
Assume now that \( r \) satisfies (A5). Then \( H \) satisfies (H5), so Example 11 gives
\[
\tilde{U} = (\text{dom } H)^\infty.
\]
The set on the right can be written as \( \{u \in L^1 \mid r(\Phi(u/\beta) < \infty \ \forall \beta > 0 \} \).

Note that if \( r \) is the \( L^\infty \)-norm, we simply have \( U = L^\infty \) and \( Y = L^1 \) while if \( r \) is the \( L^1 \)-norm, then we are back in Musielak-Orlicz spaces of Example 12. If \( \Phi \) is nonrandom and \( r \) is the Lorentz-norm associated with a concave function \( \phi \) (see Example 15), \( U \) becomes the Orlicz-Lorentz-space studied e.g. in [6]. In this case the above expressions for the dual norm seem new. One could also take \( r \) the Marcinkiewicz norm in which case \( r^o \) is the Lorentz-norm. This setting seems new.

6 On necessity of the assumptions

This section goes beyond Banach and Fréchet spaces. We assume that \( U \) and \( Y \) are solid decomposable spaces (see Remark 5) of random variables in separating duality under the bilinear form
\[
\langle u, y \rangle := E[u \cdot y].
\]

Clearly, solid spaces are decomposable but there are decomposable spaces that are not solid.

Example 17. Let \((\Omega, \mathcal{F}) := ([0,1], \mathcal{B}([0,1]))\), \( u(\omega) := \omega^{-\frac{1}{4}} + \omega^{-\frac{1}{2}} \) and \( U := L^\infty + \text{Lin}(u1_A \mid A \in \mathcal{F}) \). Then \( U \) is decomposable, by construction, but not solid, since it does not contain \( \bar{u}(\omega) = \omega^{-\frac{1}{4}} \) for which \( 0 < \bar{u} < u \).

The following two lemmas do not require solidity of \( U \) or \( Y \). The first one is Lemma 6 from [15].

Lemma 18. We have \( L^\infty \subseteq U \subseteq L^1 \) and
\[
\sigma(L^1, L^\infty)|_U \subseteq \sigma(U, Y), \quad \sigma(U, Y)|_{L^\infty} \subseteq \sigma(L^\infty, L^1),
\]
\[
\tau(L^1, L^\infty)|_U \subseteq \tau(U, Y), \quad \tau(U, Y)|_{L^\infty} \subseteq \tau(L^\infty, L^1).
\]

Lemma 19. The following are equivalent:

1. \( U \) is solid,
2. \( y \mapsto u \cdot y \) is continuous from \((Y, \sigma(Y, U))\) to \((L^1, \sigma(L^1, L^\infty))\),
3. \( \eta \mapsto \eta u \) is continuous from \((L^\infty, \tau(L^\infty, L^1))\) to \((U, \tau(U, Y))\).

Proof. For any \( u \in U, \ y \in Y \) and \( \eta \in L^\infty \),
\[
E[(u \cdot y)\eta] = E[(\eta u) \cdot y].
\]
This is $\sigma(\mathcal{Y}, \mathcal{U})$-continuous in $y$ if and only if there is a $u' \in \mathcal{U}$ such that $E[(\eta y) \cdot y] = E[u' \cdot y]$ for all $y \in \mathcal{Y}$. Since $L^\infty \subset \mathcal{Y}$ separates the elements of $L^1$, we get that $y \mapsto E[(u \cdot y)\eta]$ is continuous if and only if $\eta u \in \mathcal{U}$. This proves the equivalence of 1 and 2.

Assume 2 and let $K \subset \mathcal{Y}$ be $\sigma(\mathcal{Y}, \mathcal{U})$-compact. We have
\[
\sup_{y \in K} \langle y, \eta u \rangle = \sup_{y \in K} \langle u \cdot y, \eta \rangle_{L^\infty} = \sup_{\xi \in D} \langle \xi, \eta \rangle_{L^\infty},
\]
where $D = \{u \cdot y \mid y \in K\}$ is $\sigma(L^1, L^\infty)$-compact since $y \mapsto u \cdot y$ is continuous. \hfill \Box

**Corollary 20.** In the setting of Corollary 8 [A4] holds if and only if $\tilde{U}^* = \mathcal{Y}$.

**Proof.** By Lemma 19 [A4] implies $\mathcal{M}^* = 0$, so $\tilde{U}^* = \mathcal{Y}$ by Corollary 8. On the other hand, if $\tilde{U}^* = \mathcal{Y}$, the topology of $\tilde{U}$ cannot be stronger than $\tau(\tilde{U}, \mathcal{Y})$. In that case, Lemma 19 implies that $p(u\eta^\nu) \to 0$ if $\eta^\nu \to 0$ in $\tau(L^\infty, L^1)$. Since $1_{A^\nu} \to 0$ in $\tau(L^\infty, L^1)$ if $P(A^\nu) \to 0$, assumption [A4] holds. \hfill \Box

**Lemma 21.** A convex set $C \subset \mathcal{U}$ is $\sigma(\mathcal{U}, \mathcal{Y})$-compact if and only if, for every $y \in \mathcal{Y}$, the set $\{u \cdot y \mid u \in C\}$ is weakly compact in $L^1$.

**Proof.** Since continuous images of compact sets are compact, Lemma 19 gives the necessity. Let $(u^\nu)$ be a net in $C$. Letting $y$ range over unit constant vectors, we see that $C$ is $\sigma(L^1, L^\infty)$-compact. Thus there is a subnet and $u \in C$ such that $u^\nu \to u$ in $\sigma(L^1, L^\infty)$. Let $y \in \mathcal{Y}$ and $\epsilon > 0$. Since $\{u \cdot y \mid u \in C\}$ is weakly compact in $L^1$, it is uniformly integrable, so there exists $n$ such that $|E[(u^\nu - u) \cdot y1_{|y| > n}]| < \epsilon$ for every $\nu$. Since $u^\nu \to u$ in $\sigma(L^1, L^\infty)$, there exists $\nu'$ such that $|E[(u^\nu - u) \cdot y1_{|y| \leq n}]| < \epsilon$ for all $\nu \geq \nu'$. Thus, for all $\nu \geq \nu'$,
\[
|E[(u^\nu - u) \cdot y]| \leq 2\epsilon,
\]
which proves that $u^\nu \to u$ in $\sigma(\mathcal{U}, \mathcal{Y})$. \hfill \Box

**Corollary 22.** Given $\tilde{u} \in \mathcal{U}$, the set
\[
C := \{u \in \mathcal{U} \mid |u| \leq |\tilde{u}|\}
\]
is $\sigma(\mathcal{U}, \mathcal{Y})$-compact.

**Proof.** By Lemma 21 it suffices to show that
\[
C_y := \{u \cdot y \mid u \in \mathcal{U}, |u| \leq |\tilde{u}|\}
\]
is $\sigma(L^1, L^\infty)$-compact for every $y \in \mathcal{Y}$. The set $C_y$ is uniformly integrable, so, by Dunford-Pettis, it suffices to show that $C_y$ is $\sigma(L^1, L^\infty)$-closed. Since $\mathcal{U}$ is solid,
\[
C_y = \{u \cdot y \mid u \in L^1, |u| \leq |\tilde{u}|\}.
\]
Let $u^\nu \cdot y \to \xi$ in $L^1$, where $|u^\nu| \leq |\tilde{u}|$. Passing to convex combinations, we may assume, by Komlos lemma, that $u^\nu \to u$ almost surely for some $u$ with $|u| \leq |\tilde{u}|$. By dominated convergence, $u^\nu \cdot y \to u \cdot y$ in $L^1$, so $C_y$ is closed. \hfill \Box
Theorem 23. If $\mathcal{U}$ is $\tau(\mathcal{U}, \mathcal{Y})$-complete, then there exists a collection $\mathcal{P}$ of lsc sublinear symmetric functions $p : L^1 \to \mathbb{R}$ such that the topology generated by $\mathcal{P}$ on $\mathcal{U}$ is compatible with the duality,

$$\mathcal{U} = \bigcap_{p \in \mathcal{P}} \text{dom } p, \quad \mathcal{Y} = \bigcup_{p \in \mathcal{P}} \text{dom } p^\circ$$

and each $p \in \mathcal{P}$ satisfies $[A1] \quad [A5]$.

Proof. Let $\mathcal{C}$ be the collection of $\sigma(\mathcal{Y}, \mathcal{U})$-compact solid convex subsets of $\mathcal{Y}$ and let $\mathcal{P}$ the collection of the functions $p : L^1 \to \mathbb{R}$ of the form

$$p(u) = \sup_{y \in C} E[u \cdot y],$$

where $C \in \mathcal{C}$. Each $p \in \mathcal{P}$ is convex and positively homogeneous. Since the unit ball of $L^\infty$ is in $C$, $[A1]$ holds. The topology generated by $\mathcal{P}$ is weaker than the Mackey-topology which is generated by all $\sigma(\mathcal{Y}, \mathcal{U})$-compact sets. By Lemma 18, $[A2]$ holds. Given $\bar{y} \in \mathcal{Y}$, $\{y \in \mathcal{Y} \mid |y| \leq |\bar{y}|\}$ is compact by Corollary 22. It is also solid and convex, so the topology generated by $\mathcal{P}$ is no weaker than $\sigma(\mathcal{U}, \mathcal{Y})$.

The topology generated by $\mathcal{P}$ is thus compatible with the duality.

Solidity of $C$ and the interchange rule [21, Theorem 14.60] give

$$p(u) = \sup_{y \in C, y' \in L^1} \{E[u \cdot y'] \mid |y'|^* \leq |y|^* \}$$

$$= \sup_{y \in C} E[|u||y|^*],$$

so $p$ is lower semicontinuous in $L^1$ and satisfies $[A3]$.

By Lemma 21, the set $\{u \cdot y \mid y \in C\}$ is uniformly integrable so $p(u1_{A^\nu}) \downarrow 0$ whenever $(A^\nu)_{\nu=1}^\infty$ is a decreasing sequence with $P(A^\nu) \downarrow 0$. Thus, $[A5]$ holds. This also implies that $L^\infty$ is $\mathcal{P}$-dense in $\text{dom } p$.

Any $C \in \mathcal{C}$ is $\sigma(L^1, L^\infty)$-compact so an application of bipolar theorem in the duality pairing $(L^1, L^\infty)$ gives

$$p^\circ(y) = \inf\{\gamma > 0 \mid y/\gamma \in C\}.$$ 

Thus $\text{dom } p^\circ \subset \mathcal{Y}$. As noted earlier, any $y \in \mathcal{Y}$ belongs to some $C \in \mathcal{C}$ so $\mathcal{Y} = \cup_{p \in \mathcal{P}} \text{dom } p^\circ$.

The $\sigma(\mathcal{Y}, \mathcal{U})$-compactness of $C \in \mathcal{C}$ implies $\sigma_C(u) < \infty$ for any $u \in \mathcal{U}$. Thus, $\mathcal{U} \subset \cap_{p \in \mathcal{P}} \text{dom } p$. On the other hand, $\cap_{p \in \mathcal{P}} \text{dom } p$ is complete in the $\mathcal{P}$-topology (see Remark 4) so it is complete also in the topology generated by $\sigma(\mathcal{Y}, \mathcal{U})$-compact convex sets. Since $L^\infty$ is dense in $\text{dom } p$, we have that $\mathcal{U}$ is dense in $\cap_{p \in \mathcal{P}} \text{dom } p$ and thus, $\mathcal{U} = \cap_{p \in \mathcal{P}} \text{dom } p$. \hfill \square

Theorem 23 puts us in the setting of Remark 4. Combined with Lemma 19 we thus get the following two results.

Corollary 24. If $\mathcal{U}$ is $\tau(\mathcal{U}, \mathcal{Y})$-complete, then it is sequentially $\sigma(\mathcal{U}, \mathcal{Y})$-complete.
Proof. Let \((u^\nu)_{\nu=1}^\infty\) be a \(\sigma(\mathcal{U}, \mathcal{Y})\)-Cauchy sequence. Since \(\sigma(\mathcal{U}, \mathcal{Y})\) is stronger than \(\sigma(L^1, L^\infty)\) which, by Theorem IV.8.6, is sequentially complete, there exists \(u \in L^1\) such that \(u^\nu \to u\) in \(\sigma(L^1, L^\infty)\). Since \(\sigma(\mathcal{U}, \mathcal{Y})\)-Cauchy sequences are bounded in any topology compatible with the pairing, the sequence is also bounded in the \(\mathcal{P}\)-topology of Theorem 23. Thus, for any \(p \in \mathcal{P}\), there exist \(\gamma\) such that \(p(u^\nu) \leq \gamma\). Since level-sets of \(p\) are closed in \(L^1\) and \(\mathcal{U} = \bigcap p\), we get \(u \in \mathcal{U}\). It suffices to show that \(u^\nu \to u\) in \(\sigma(\mathcal{U}, \mathcal{Y})\).

By Lemma 19 for any \(y \in \mathcal{Y}\), \((u^\nu \cdot y)_{\nu=1}^\infty\) is Cauchy in \(\sigma(L^1, L^\infty)\), so by sequential closedness of \(L^1\) again, it converges in \(\sigma(L^1, L^\infty)\) to some \(\xi \in L^1\). By Mazur’s theorem, there is a subsequence of convex combinations \(\bar{u}^\nu\) such that \(\bar{u}^\nu \to u\) in \(L^1\)-norm, and thus \(\bar{u}^\nu \cdot y \to u \cdot y\) in probability. Clearly, \(\bar{u}^\nu \cdot y \to \xi\) in \(\sigma(L^1, L^\infty)\), so we must have \(\xi = u \cdot y\).

When \(\mathcal{U}\) is \(\tau(\mathcal{U}, \mathcal{Y})\)-complete, we get the following version of Lemma 21.

Corollary 25. Assume that \(\mathcal{U}\) is \(\tau(\mathcal{U}, \mathcal{Y})\)-complete. A convex set \(C \subset \mathcal{U}\) is relatively \(\sigma(\mathcal{U}, \mathcal{Y})\)-compact if and only if, for every \(y \in \mathcal{Y}\), the set \(\{u \cdot y \mid u \in C\}\) is relatively \(\sigma(L^1, L^\infty)\)-compact in \(L^1\).

Proof. Since continuous images of relatively compact sets are relatively compact, Lemma 19 gives the necessity. For the sufficiency, it suffices, by Theorem 23 and Remark 4 to show sequential relative compactness. Let \((u^\nu)\) be a sequence in \(C\). As in the proof of Lemma 21 we get that there is \(u \in L^1\) such that, for every \(y \in \mathcal{Y}\) and \(\epsilon > 0\),

\[|E[(u^\nu - u) \cdot y]| \leq 2\epsilon,\]

for \(\nu\) large enough, so \((u^\nu)\) is Cauchy in \(\sigma(\mathcal{U}, \mathcal{Y})\). By Corollary 24 \((u^\nu)\) converges to \(u\).

Appendix

This appendix studies integration of measurable not-necessarily bounded functions with respect to a real-valued finitely additive measure \(m\). Define \(r_m : L^+_1 \to \overline{\mathbb{R}}\) by

\[r_m(\eta) := \sup_{u' \in L^\infty} \left\{ \int_{\Omega} u' dm \mid |u'| \leq \eta \right\}.\]

Lemma 26. For any real-valued finitely additive measure \(m\),

1. Relative to \(L^\infty\),

\[r_m(\eta) = \sup_{u' \in L^\infty} \left\{ \int_{\Omega} \eta(u' dm) \mid |u'| \leq 1 \right\} \leq ||\eta||_{L^\infty} ||m||_{TV}.

In particular, \(r_m\) is \(L^\infty\)-norm continuous and sublinear relative to \(L^+_1\).

2. For every \(\eta \in L^+_1\),

\[r_m(\eta) = \lim_{\nu \to \infty} r_m(\eta \wedge \nu)\]
3. \( r_m \) is positively homogeneous and subadditive and \( r_m(\eta') \leq r_m(\eta) \) whenever \( \eta' \leq \eta \).

Proof. The expression in \( \Pi \) follows from the change of variables \( \tilde{u} = \eta u' \). To prove \( \Pi \) the inequality \( r_m(\eta) \geq \lim_\nu r_m(\eta \land \nu) \) is clear. To prove the opposite inequality, let \( \alpha \in \mathbb{R} \) with \( r_m(\eta) > \alpha \). There exists \( u' \in L^\infty \) with \( |u'| \leq \eta \) and \( r_m(|u'|) > \alpha \). Then \( |u'| \land \nu \to |u'| \) in \( L^\infty \)-norm, so monotonicity and \( \Pi \) give

\[
\lim r_m(\eta \land \nu) \geq \lim r_m(|u'| \land \nu) > \alpha.
\]

In \( \Pi \) only subadditivity requires a proof. Given \( \eta_1, \eta_2 \in \text{dom } p \), we have \( (\eta_1 + \eta_2) \land \nu \leq \eta_1 \land \nu + \eta_2 \land \nu \). Indeed, a concave function vanishing at the origin is subadditive on the positive reals. Thus, by \( \Pi \) and \( \Pi \)

\[
r_m(\eta_1 + \eta_2) = \limsup \sup r_m((\eta_1 + \eta_2) \land \nu)
\leq \limsup \sup (r_m(\eta_1 \land \nu) + r_m(\eta_2 \land \nu))
\leq \limsup \sup r_m(\eta_1 \land \nu) + \limsup \sup r_m(\eta_2 \land \nu)
= r_m(\eta_1) + r_m(\eta_2),
\]

which proves the subadditivity.

Define \( \rho_m : L^1 \to \overline{\mathbb{R}} \) by

\[
\rho_m(u) := r_m(|u|).
\]

Theorem 27. For any real-valued finitely additive measure \( m \),

1. \( \rho_m \) is symmetric and sublinear, and \( \rho_m(u') \leq \rho_m(u) \) whenever \( |u'| \leq |u| \),
2. for any \( u \in \text{dom } \rho_m \) and \( \epsilon > 0 \), there exists \( u' \in L^\infty \) with \( \rho_m(u - u') < \epsilon \),
3. \( \int \Omega u \, d\rho_m \) has a unique \( \rho_m \)-continuous linear extension from \( L^\infty \) to \( \text{dom } \rho_m \),
4. if \( m \) is purely finite additive, there exists a decreasing \( (A^\nu)_{\nu=1}^\infty \subset \mathcal{F} \) with \( P(A^\nu) \searrow 0 \) and \( \int \Omega u 1_{\Omega \setminus A^\nu} \, d\rho_m = 0 \) for all \( u \in \text{dom } \rho_m \).

Proof. Properties in 1 are clear. To prove 2 assume first that \( m \) is nonnegative. Given \( u' \in \text{dom } \rho_m \cap L^1_+ \) and \( \epsilon > 0 \), let \( \tilde{u} \in L^\infty \) be such that \( 0 \leq \tilde{u} \leq u' \) and \( \rho_j(u') \leq \langle \tilde{u}, m \rangle + \epsilon \). Then \( \tilde{u}^1 + \tilde{u}^2 \leq u^1 + u^2 \) and

\[
\rho_m(u^1) + \rho_m(u^2) \leq \langle \tilde{u}^1 + \tilde{u}^2, m \rangle + 2\epsilon \leq \rho_m(u^1 + u^2) + 2\epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary, \( \rho_m \) is superlinear on \( \text{dom } \rho_m \cap L^1_+ \). Given \( u \in \text{dom } \rho_m \) and \( \epsilon > 0 \), Lemma 26 gives \( \rho_m(u^+) \leq \rho_m(u^+ \land \nu) + \epsilon \) for \( \nu \) large enough. By superlinearity,

\[
\rho_m(u^+ - u^+ \land \nu) + \rho_m(u^+ \land \nu) \leq \rho_m(u^+) \leq \rho_m(u^+ \land \nu) + \epsilon.
\]
Similarly, \( \rho_m(u^- - u^- \wedge \nu) \leq \epsilon \), so \( \rho_m(u - \pi_B u) \leq 2\epsilon \) by sublinearity of \( \rho_m \).

By [22, Theorem 1.12], general \( m \in \mathcal{M}^1 \) can be written as \( m = m^+ - m^- \) for nonnegative \( m^+, m^- \in \mathcal{M}^1 \), so

\[
\rho_m(u - \pi_B u) \leq \rho_m^+(u - \pi_B u) + \rho_m^-(u - \pi_B u) \leq 4\epsilon
\]

for \( \nu \) large enough.

We have \( \int_{\Omega} u \, dm \leq \rho_m(u) \) on \( L^\infty \), so, by Hahn-Banach, there exists a \( \rho_m \)-continuous linear extension of \( m \) to \( \text{dom} \rho_m \). Since \( L^\infty \) is dense in \( \text{dom} \rho_m \), the extension is unique. If \( m \) is purely finitely additive, there exists \( (A^\nu)^{\nu=1}_\infty \subset \mathcal{F} \) with \( P(A^\nu) \searrow 0 \) and \( \int_{\Omega} u 1_{\Omega \setminus A^\nu} \, dm = 0 \) for all \( u \in L^\infty \). Note that \( r_m \) inherits this property so that \( \rho_m \) and the integral does as well.

\[ \Box \]

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