DOMINO TABLEAUX, SCHÜTZENBERGER INVOLUTION, 
AND THE SYMMETRIC GROUP ACTION

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ABSTRACT

We define an action of the symmetric group $S_n$ on the set of domino tableaux, and prove that the number of domino tableaux of weight $\beta'$ does not depend on the permutation of the weight $\beta'$. A bijective proof of the well-known result due to J. Stembridge that the number of self–evacuating tableaux of a given shape and weight $\beta = (\beta_1, \ldots, \beta_{\lfloor \frac{n+1}{2} \rfloor}, \beta_{\lfloor \frac{n}{2} \rfloor}, \ldots, \beta_1)$, is equal to that of domino tableaux of the same shape and weight $\beta' = (\beta_1, \ldots, \beta_{\lfloor \frac{n+1}{2} \rfloor})$ is given.

0. Introduction

Domino tableaux play a significant role in the representation theory of the symmetric group $S_n$, and the theory of symmetric polynomials. The shortest definition of semi-standard domino tableaux is given in [10], p. 139. In particular, the generating function of the number of domino tableaux of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$, and weight $\beta' = (\beta_1, \ldots, \beta_{\lfloor \frac{n+1}{2} \rfloor})$ is given by specializing the Schur polynomial $s_\lambda(x)$ at $x \mapsto y$ where $y_n = -y_{n-1}, y_{n-2} = -y_{n-3}, \ldots$:

$$\pm s_\lambda(y) = \sum_{\beta'} K_{\lambda, \beta'}^{(2)} y_{n}^{2\beta_1} y_{n-2}^{2\beta_2} \cdots , \quad (0.1)$$

where the sign $\pm$ depends only on $\lambda$.

Recently they were studied in connection with the representation theory of $GL_n$. In particular, in [12] John Stembridge proved the following conjecture by Richard Stanley (who stated it for $\beta' = (1, 1, \ldots, 1)$, that is, for standard tableaux).

Theorem 0.1 ([12]). The number of self-evacuating tableaux of shape $\lambda$ and weight $\beta = (\beta_1, \ldots, \beta_{\lfloor \frac{n+1}{2} \rfloor}, \beta_{\lfloor \frac{n}{2} \rfloor}, \ldots, \beta_1)$, equals the number of domino tableaux $K_{\lambda, \beta'}^{(2)}$ of shape $\lambda$ and weight $\beta'$.
The self-evacuating tableaux are defined as fixed points of the Schützenberger evacuation involution $S$ on the semi-standard tableaux (see [11], [12]). The proof of Theorem 0.1 in [12] uses properties of the canonical basis for the $GL_n$-module $V_\lambda$. A key ingredient of the proof is the result of [3], Theorem 8.2, which states that $S$ is naturally identified with the longest element of the symmetric group $S_n$ acting on the canonical basis of $V_\lambda$.

There are combinatorial algorithms (see e.g., [5] or [9]) which lead to a bijective proof of Theorem 0.1 for the standard tableaux. For semi-standard tableaux and even $n$ a similar algorithm appeared in [7], remark on the page 399.

In the present paper we give a bijective proof of Theorem 0.1 for any $n$ (Theorem 1.2). The main idea of the bijection constructed in this paper is that (semi-standard) domino tableaux are naturally identified with those ordinary (semi-standard) tableaux which are fixed under a certain involution $D$ (see the Appendix).

We discover that the involutions $D$ and $S$ are conjugate as automorphisms of all tableaux, say $S = PDP^{-1}$ (see Lemma 1.3). Thus, the conjugating automorphism $P$ bijects the domino tableaux onto the self-evacuating tableaux.

The main difference of our construction and those given in [5], [9] and [7] is that instead of a description of combinatorial algorithms as it was done in the papers mentioned, we give a direct algebraic formula for the bijection under consideration. Our approach is based on results from [2].

It follows from (0.1) that the number $K^{(2)}_{\lambda, \beta, \beta'}$ does not depend on the permutation of the weight $\beta'$. This is analogous to the property of the ordinary Kostka numbers $K_{\lambda, \beta}$. Recently, Carre and Leclerc constructed in [4] an appropriate action of the symmetric group $S_2$ (for even $n$) which realizes this property. Their construction used a generalized RSK-type algorithm ([4], Algorithm 7.1, and Theorem 7.8).

The second result of the present paper is an algebraic construction of action of the symmetric group $S_{[2]}$ on $\text{Tab}_n^D$. This action is parallel to the action of $S_n$ on the ordinary tableaux studied in [2], and has a natural interpretation in terms of self-evacuating tableaux (Theorem 1.8). One can conjecture that for even $n$ our action coincides with the Carre–Leclerc’s action.

The material of the paper is arranged as follows. In Section 1 we list main results and construct the bijection.

Section 2 contains proofs of Theorems 1.6 and 1.8 on the action of the symmetric group $S_{[2]}$.

For the reader’s convenience, in Appendix we remind definitions of the Bender–Knuth’s involution and domino tableaux.

Acknowledgements. The authors are grateful to George Lusztig for fruitful discussions. Special thanks are due to Mark Shimozono for an inestimable help in the preparation of the manuscript. Both authors are grateful to CRM and the Department of Mathematics of University of Montreal, for hospitality and support during this work.
1. Main results

Following [10], denote by $\text{Tab}_n$ the set of all (semi-standard Young) tableaux with entries not exceeding $n$. For a partition $\lambda$ with at most $n$ rows, and for an integer vector $\beta = (\beta_1, \ldots, \beta_n)$ denote by $\text{Tab}_\lambda(\beta)$ the set of $T \in \text{Tab}_n$ of shape $\lambda$ and weight $\beta$ (for the reader’s convenience we collect all necessary definitions in the Appendix).

Let $t_i : \text{Tab}_n \to \text{Tab}_n$, $(i = 1, \ldots, n - 1)$ be the Bender-Knuth automorphisms which we define in the Appendix (see also [1],[2] and [6]).

Define the automorphism $D : \text{Tab}_n \to \text{Tab}_n$ by the formula:

$$D = D_n := t_{n-1}t_{n-3} \cdots ,$$

i.e., $D = t_{n-1}t_{n-3} \cdots t_2$ for odd $n$, and $D = t_{n-1}t_{n-3} \cdots t_{1}$ for even $n$.

Denote by $\text{Tab}_n^D$ the set of tableaux $T \in \text{Tab}_n$ such that $D(T) = T$. For every shape $\lambda$ and weight $\beta' = (\beta_1, \ldots, \beta_{\frac{n+1}{2}})$ denote $\text{Tab}_\lambda^D(\beta') = \text{Tab}_\lambda(\beta) \cap \text{Tab}_n^D$, where $\beta = (\beta_{\frac{n+1}{2}}, \ldots, \beta_k, \beta_k, \ldots, \beta_2, \beta_2, \beta_1, \beta_1)$.

The semi-standard domino tableaux of shape $\lambda$ and weight $\beta'$ were studied by several authors (see, e.g., [10], p. 139). We also define them in the Appendix. Our first ”result” is another definition of domino tableaux.

**Proposition-Definition 1.1.** For every $\lambda$ and $\beta'$ as above the domino tableaux of shape $\lambda$ and weight $\beta'$ are in a natural one-to-one correspondence with the set $\text{Tab}_\lambda^D(\beta')$.

We prove this proposition in the Appendix. From now on we identify the domino tableaux with the elements of $\text{Tab}_n^D$.

Let us introduce the last bit of notation prior to the next result.

Let $p_i := t_1t_2 \cdots t_i$ for $i = 1, \ldots, n - 1$ (this $p_i$ is the inverse of the $i$-th promotion operator defined in [11]). Define the automorphism $P : \text{Tab}_n \to \text{Tab}_n$ by the formula:

$$P = P_n := p_{n-1}p_{n-3} \cdots ,$$

so $P = p_{n-1}p_{n-3} \cdots p_2$ for odd $n$, and $P = p_{n-1}p_{n-3} \cdots p_1$ for even $n$.

It is well-known (see, e.g., [2],[6]) that the Schützenberger evacuation involution factorizes as follows.

$$S = S_n = p_{n-1}p_{n-2} \cdots p_1$$

We regard (1.3) as a definition of $S$.

Denote by $\text{Tab}_n^S \subset \text{Tab}_n$ the set of the self-evacuating tableaux, that is all the tableaux $T \in \text{Tab}_n$ fixed under $S$. Similarly, denote $\text{Tab}_\lambda^S(\beta) = \text{Tab}_\lambda(\beta) \cap \text{Tab}_n^S$. It is well-known that this set is empty unless $\beta_i = \beta_{n+1-i}$ for $i = 1, 2, \ldots, \lceil \frac{n+1}{2} \rceil$. 


Theorem 1.2. The above automorphism $P$ of $\text{Tab}_D^D_n$ induces the bijection

$$\text{Tab}_D^D_n \cong \text{Tab}_S^n.$$ (1.4)

More precisely, $P$ induces the bijection $\text{Tab}_D^D_\lambda(\beta') \cong \text{Tab}_S_\lambda(\beta)$ for any shape $\lambda$ and the weight $\beta' = (\beta_1, \ldots, \beta_{n+1})$, where $\beta = (\beta_1, \ldots, \beta_{n+1}^1, \beta_2, \ldots, \beta_1)$.

Remark. Theorem 1.2 is the ”realization” of the theorem in Section 0.

The proof of this theorem is so elementary that we present it here.

Proof of Theorem 1.2.

The statement (1.4) immediately follows from the surprisingly elementary lemma below.

Lemma 1.3. $S = PDP^{-1}$.

Indeed, if $T \in \text{Tab}_D^D_n$ then $T = D(T) = P^{-1}SP(T)$ by Lemma 1.3, so $S(P(T)) = P(T)$, that is, $P(T) \in \text{Tab}_S^n$. This proves the inclusion $P(\text{Tab}_D^D_n) \subset \text{Tab}_S^n$. The opposite inclusion also follows.

Remark. By definition (see the Appendix), the involutions $t_1, \ldots, t_{n-1}$ involved in Lemma 1.3, satisfy the following obvious relations (see e.g. [2]):

$$t_i^2 = \text{id}, \quad t_it_j = t_jt_i, \quad 1 \leq i, j \leq n-1, \quad |i - j| > 1. \quad (1.5)$$

In fact, the factorization in Lemma 1.3 is based only on this relations (as we can see from the proof below), and, hence, makes sense in any group with the relations (1.5).

It also follows from the definition of $t_i$ that

$$t_i : \text{Tab}_\lambda(\beta) \cong \text{Tab}_\lambda((i, i + 1)(\beta)).$$

This proves the second assertion of Theorem 1.2 provided that Lemma 1.3 is proved.

Proof of Lemma 1.3. We will proceed by induction on $n$. If $n = 2$ then $S = t_1$ while $P = t_1$, and $D = t_1$. If $n = 3$ then $S_3 = t_1t_2t_1$ while $D = t_2$, and $P = t_1$. Then, for any $n > 3$ the formula (1.3) can be rewritten as $S_n = p_{n-1}S_{n-1}$. It is easy to derive from (1.5) that $S_n = p_{n-1}S_{n-1}$. Applying both of the above relations, we obtain

$$S_n = p_{n-2}S_{n-2}p_{n-1}^{-1}. \quad (1.6)$$

Finally, by (1.5) and the inductive assumption for $n - 2$,

$$S_n = p_{n-2}S_{n-2}p_{n-1}^{-1} = p_{n-2}P_{n-2}D_{n-2}^{-1}P_{n-2}^{-1} = p_{n-2}P_{n-2}t_{n-1}D_nP_{n-2}^{-1}p_{n-1}^{-1}$$

since $D_{n-2} = t_{n-1}D_n$. Finally, $P_{n-2}^{-1} = p_{n-1}^{-1}$, and $p_{n-2}P_{n-2}t_{n-1} = p_{n-2}t_{n-1}P_{n-2} = p_{n-1}P_{n-2}P_n$.

Lemma 1.3 is proved.

Theorem 1.2 is proved.

Recall (see the Appendix) that the weight of a domino tableau $T \in \text{Tab}_D^D_n$ is the vector $\beta' = (\beta_1, \ldots, \beta_{n+1})$, where $\beta_1$ is the number of occurrences of $n$, $\beta_2$ is that of $n - 2$, and so on. Denote by $\text{Tab}_D^D(\beta')$ the set of all $T \in \text{Tab}_D^D_n$ of weight $\beta'$. Our next main result is the following.
Theorem 1.4. There exists a natural faithful action of the symmetric group $S_{[2]}$ on $Tab_n^D$ preserving the shape and acting on weight by permutation.

In order to define the action precisely, let us recall one of the results of [2].

Theorem 1.5 ([2]). There is an action of $S_n$ on $Tab_n$ given by $(i, i+1) \mapsto s_i$, where

$$s_i = S_it_1S_i, \quad i = 1, \ldots, n,$$

(1.7)

that is, the $s_i$ are involutions, and $(s_j s_{j+1})^3 = id, s_is_j = s_js_i$ for all $i, j$ such that $|j-i| > 1$. (Here $S_i$ given by (1.3) acts on $Tab_n$).

Definition. For $n \geq 4$ define the automorphisms $\sigma_i : Tab_n \to Tab_n, i = 2, \ldots, n-2$ by the formula

$$\sigma_i := t_is_{i-1}s_{i+1}t_i.$$  

(1.8)

Remark. The automorphisms $s_i$ were first defined by Lascoux and Schützenberger in [8] in the context of plactique monoid theory. The definition (1.7) of $s_i$ first appeared in [2] in a more general situation. As a matter of fact, for the restriction of $s_i$ to the set of tableaux, (1.7) is implied in [8].

Theorem 1.4 follows from a much stronger result, namely the following theorem:

Theorem 1.6.

(a) $D \sigma_i = \sigma_i D$ if and only if $i \equiv n \pmod{2}$.

(b) The automorphisms $\sigma_2, \sigma_3, \ldots, \sigma_{n-2}$ are involutions satisfying the Coxeter relations $(\sigma_i \sigma_j)^{n_{ij}} = id$ for $i, j = 2, \ldots, n-2$, where

$$n_{ij} = \begin{cases} 
3 & \text{if } |j-i| = 1 \text{ or } 2 \\
6 & \text{if } |j-i| = 3 \\
2 & \text{if } |j-i| > 3
\end{cases}.
$$

(1.9)

We will prove Theorem 1.6 in Section 2.

Remark. One can conjecture that the relations (1.8) (together with $\sigma_i^2 = id$) give a presentation of the group $\Sigma_n$ generated by the $\sigma_i$.

Indeed, the action of $S_{[2]}$ on $Tab_n$ from Theorem 1.4 can be defined by

$$(i, i+1) \mapsto \sigma_{n-2i}, \quad i = 2, \ldots, n-2.$$ 

According to Theorem 1.6(a), this action preserves $Tab_n^D$. By definition (1.7), this action preserves shape of tableaux and permutes the weights as follows:

$$\sigma_{n-2i} : Tab_n^D(\beta') \cong Tab_n^D((i, i+1)(\beta')).$$  

(1.10)

Let us try to get the action of $S_{[2]}$ on $Tab_n^S$ using the following nice property of the involutions $s_1, \ldots, s_{n-1}$ (which can be found, e.g., in [2], Proposition 1.4):

$$SS_iS = s_{n-i}, \quad i = 1, \ldots, n-1.$$ 

(1.11)

Let $\tau_1, \ldots, \tau_{[2]-1}$ be defined by $\tau_k := s ks_{n-k}$. The following proposition is obvious.
Proposition 1.7.
(a) $S \tau_k = \tau_k S$ for all $k$.
(b) The group generated by the $\tau_k$ is isomorphic to $S_{[n/2]}$ via

$$(k, k + 1) \mapsto \tau_k, \quad k = 1, \ldots, \left[ \frac{n}{2} \right] - 1.$$  \hfill (1.12)

Restricting the action (1.12) to $Tab^S_n$ we obtain the desirable action of $S_{[n/2]}$ on $Tab^S_n$.

Our last main result compares of the latter action with that on $Tab^D_n$. The following theorem confirms naturality of our choice of $\sigma_j$ in (1.8).

Theorem 1.8. For $k = 1, \ldots, \left[ \frac{n}{2} \right] - 1$, we have $P^{-1} \tau_k P = \sigma_{n-2k}$.

2. Proof of Theorems 1.6, 1.8 and related results

We start with a collection of formulas relating $s_i$ and $t_j$. First of all,

$$s_i t_j = t_j s_i$$ \hfill (2.1)

for all $i, j$ with $|j - i| > 1$ (This fact follows from [8] and is implied in [2], Theorem 1.1). Then, by definition (1.8) of $\sigma_i$,

$$\sigma_i t_j = t_j \sigma_i$$ \hfill (2.2)

whenever $|j - i| > 2$. Furthermore, (1.3) and (1.7) imply that

$$s_j = p_j s_j^{-1} p_j^{-1}, \quad j = 1, \ldots, n - 1.$$ \hfill (2.3)

More generally,

$$s_i = p_j s_i^{-1} p_j^{-1}, \quad 1 \leq i \leq j < n.$$ \hfill (2.4)

Finally, (2.3) and (2.1) imply that

$$t_{j-1} s_j t_{j-1} = t_j s_{j-1} t_j.$$ \hfill (2.5)

**Proof of Theorem 1.6(a).** Let $i$ be congruent $n$ modulo 2. Then by (2.2) and the definition (1.1) of $D$,

$$D \sigma_i D = t_{i+1} t_{i-1} \sigma_i t_{i+1} t_{i-1}.$$ \hfill (2.6)

We now prove that (2.6) is equal to $\sigma_i$. Indeed,

$$D \sigma_i D = t_{i+1} t_{i-1} \sigma_i t_{i+1} t_{i-1} = t_{i+1} t_{i-1} t_i s_{i-1} s_i t_i t_{i+1} t_{i-1}.$$
Now we will apply the relations (2.5) in the form $t_{i-1}t_is_{i-1} = s_it_{i-1}t_i$, and $s_{i+1}t_{i+1} = t_it_{i+1}s_i$. We obtain

$$D\sigma_iD = t_{i+1}(s_it_{i-1}t_i)(t_is_{i-1}t_i)s_{i-1} = t_{i+1}s_it_{i-1}t_is_{i-1} = t_{i+1}s_it_{i+1}t_is_{i-1}.$$ 

Applying (2.5) with $j = i-1, i$ once again, we finally obtain

$$D\sigma_iD = (t_is_{i+1}t_i)(t_is_{i-1}t_i)t_is_{i-1}t_i = t_is_{i-1}t_is_{i+1}t_i = \sigma_i$$

by Theorem 1.5, which proves that $D\sigma_i = \sigma_iD$. The implication "if" of Theorem 1.6(a) is proved. Let us prove the "only if" implication. Let $\sigma$ be the canonical homomorphism from the group generated by $t_1, \ldots, t_{n-1}$ to $S_n$ defined by $t_j \mapsto (j, j+1)$. Then $\rho(\sigma_i) = (i, i+1)(i-1, i)(i+1, i+2)(i, i+1)$ for $i = 2, \ldots, n-2$, and $\rho(D) = (n-1, n)(n-3, n-2) \cdots$. If $i \not\equiv n \pmod{2}$ then $\rho(\sigma_i)$ and $\rho(D)$ do not commute. Hence $\sigma_iD \neq D\sigma_i$ for such $i$. Thus Theorem 1.6(a) is proved.

**Proof of Theorem 1.6(b).** We are going to consider several cases when $j - i = 1, 2, 3$ and $> 3$. In each case we compute the product $\sigma_i\sigma_j$ up to conjugation, and express it as a product of several $s_k$. These products are always of finite order, because, according to Theorem 1.5, these $s_k$'s generate the group isomorphic to $S_n$.

The easiest case is $j - i > 3$. Then all the ingredients of $\sigma_j$ in (1.7) (namely $t_j, s_{j-1}, s_{j+1}$) commute with those of $\sigma_i$ (namely $t_i, s_{i-1}, s_{i+1}$). Thus $\sigma_j$ commutes with $\sigma_i$, that is, $(\sigma_i\sigma_j)^2 = id$, and we are done in this case.

In what follows we will write $a \equiv b$ if $a$ is conjugate to $b$.

**Case** $j - i = 3$, i.e., $j = i + 3$. We have

$$\sigma_i\sigma_{i+3} = t_is_{i-1}s_{i+1}t_is_{i+3}s_{i+2}s_{i+4}t_{i+3} \equiv s_{i-1}s_{i+1}s_{i+2}s_{i+4}$$

by the above commutation between remote $s_k$ and $t_i$.

By Theorem 1.5, $(\sigma_i\sigma_{i+3})^6 \equiv (s_{i-1})^6(s_{i+1}s_{i+2})^6(s_{i+4})^6 = id$. So $(\sigma_i\sigma_{i+3})^6 = id$ and we are done with this case.

**Case** $j - i = 2$, i.e., $j = i + 2$. We have

$$\sigma_i\sigma_{i+2} = t_is_{i-1}s_{i+1}t_is_{i+2}s_{i+3}t_{i+2} \equiv t_is_{i-1}s_{i+1}t_is_{i+2}s_{i+3}$$

$$= t_is_{i-1}t_is_{i+1}s_{i+3} = t_is_{i-1}t_is_{i+1}s_{i+3}$$

by (2.5) with $j = i + 2$. Conjugating the latter expression with $t_i$, we obtain,

$$\sigma_i\sigma_{i+2} \equiv s_{i-1}t_is_{i+1}s_{i+2}t_is_{i+1}s_{i+3} = s_{i-1}t_is_{i+1}s_{i+2}t_is_{i+1}s_i$$

$$= s_{i-1}t_is_{i+1}s_{i+2}t_is_{i+1}s_{i+3} = s_{i-1}t_is_{i+1}s_{i+2}t_is_{i+1}s_{i+3}$$
\[ \equiv t_{i+1}s_{i-1}t_{i+1}s_{i+2}s_{i+3} = s_{i-1}s_{i+2}s_{i+3} = s_{i-1}s_{i+2}s_{i+3} . \]

Thus, \((\sigma_i \sigma_{i+2})^3 \equiv (s_{i-1}s_{i+2}s_{i+3})^3 = (s_{i-1}s_i)^3(s_{i+2}s_{i+3})^3 = \text{id}, \) and we are done with this case too.

**Case** \( j - i = 1, \) i.e., \( j = i + 1. \) We have

\[
\sigma_i \sigma_{i+1} = t_i s_{i-1}s_{i+1}t_i t_{i+1}s_{i+2}t_{i+1} = t_i s_{i-1}s_{i+1}t_i t_{i+1}s_{i+2}t_{i+1}
= t_i s_{i-1}s_{i+1}t_i t_{i+1}s_{i+2}t_{i+1} = t_i s_{i-1}s_{i+1}t_i t_{i+1}s_{i+2}t_{i+1} = t_i s_{i+1} .
\]

Thus, \((\sigma_i \sigma_{i+1})^3 \equiv (s_{i+1})^3 = \text{id}, \) and we are done with this final case.

This finishes the proof of **Theorem 1.6.**

**Proof of Theorem 1.8.** First of all we prove Theorem 1.8 with \( k = 1, \) that is,

\[
P^{-1}_n s_{n-1}P_n = \sigma_{n-2} \tag{2.7}
\]

Since \( P_n = p_n P_{n-2}, \) the left hand side of (2.7) equals

\[
P^{-1}_{n-2} p^{-1}_{n-1} s_{1} p^{-1}_{n-1} s_{n-1} p^{-1}_{n-2} P_{n-2} = P^{-1}_{n-2} (p^{-1}_{n-1} s_{1} p_{n-1}) s_{n-2} P_{n-2} . \tag{2.8}
\]

Now we prove by induction on \( n \) that

\[
p^{-1}_{n-1} s_{1} p_{n-1} = p_{n-2} s_{n-1} p^{-1}_{n-2} \tag{2.9}
\]

Indeed, if \( n = 2, \) (2.9) is obvious (we agree that \( p_0 = t_0 = \text{id}. \) Then using (2.9) as inductive assumption, we obtain

\[
p^{-1}_{n-1} s_{1} p_{n-1} = t_n (p^{-1}_{n-1} s_{1} p_{n-1}) t_n = t_n (p_{n-2} s_{n-1} p^{-1}_{n-2}) t_n = p_{n-2} t_{n-1} s_{n-1} p^{-1}_{n-2} = p_{n-1} s_{n} p^{-1}_{n-1}
\]

by (2.5), and we are done with (2.9).

Finally, in order to finish the proof of (2.7), let us substitute (2.9) into (2.8). Thus, taking into account that \( P_{n-2} = p_{n-3} P_{n-4} = p_{n-2} t_{n-2} P_{n-2}, \) (2.8) is equal to

\[
P^{-1}_{n-2} (p_{n-2} s_{n-1} p^{-1}_{n-2}) s_{n-2} P_{n-2} = P^{-1}_{n-4} t_{n-2} s_{n-2} p^{-1}_{n-2} (p_{n-2} s_{n-1} p^{-1}_{n-2}) s_{n-2} p_{n-2} t_{n-2} P_{n-4}
= P^{-1}_{n-4} t_{n-2} s_{n-1} (p^{-1}_{n-2} s_{n-2} p_{n-2}) t_{n-2} P_{n-4} = P^{-1}_{n-4} t_{n-2} s_{n-1} s_{n-3} t_{n-2} P_{n-4}
= t_{n-2} s_{n-1} s_{n-3} t_{n-2} = \sigma_{n-2} ,
\]

which proves **Theorem 1.8** for \( k = 1. \)
Now using induction on \( n \) we will prove the general case, namely, the formula

\[
P_n^{-1}s_{k}s_{n-k}P_n = \sigma_{n-2k}, \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.
\]  

(2.10)

Indeed,

\[
P_n^{-1}s_{k}s_{n-k}P_n = P_{n-2}^{-1}(p_{n-1}^{-1}s_{k}s_{n-k}p_{n-1})P_{n-2}.
\]  

(2.11)

But \( p_{n-1}^{-1}s_{k}s_{n-k}p_{n-1} = (p_{n-1}^{-1}s_{k}p_{n-1})(p_{n-1}^{-1}s_{n-k}p_{n-1}) = s_{k-1}s_{n-k-1} \) by (2.4) applied with \( j := n - 1, i = k, \) and \( i = n - k. \) Thus (2.11) is equal to

\[
P_{n-2}^{-1}s_{k-1}s_{n-k-1}P_{n-2} = \sigma_{n-2-2(k-1)} = \sigma_{n-2k}
\]

by the inductive assumption (2.10) with \( n - 2. \) Theorem 1.8 is proved.

3. Appendix. Definitions of domino tableaux

By definition from [10], a (semi-standard Young) tableau \( T \) with entries not exceeding \( n \) is an ascending sequence of Young diagrams \( T = (\{\emptyset\} = \Lambda_0 \subset \Lambda_1 \subset \ldots \subset \Lambda_n) \) such that \( \Delta_i = \Lambda_i \setminus \Lambda_{i-1} \) is a horizontal strip for \( i = 1, \ldots, n \) (we allow \( \Lambda_{i+1} = \Lambda_i \)). Denote by \( Tab_n \) the set of all (semi-standard Young) tableaux with entries not exceeding \( n. \) Define the shape of \( T \) by \( \lambda = \Lambda_n, \) and the weight \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) of \( T \) by \( \beta_i = |\Delta_i| \) for \( i = 1, \ldots, n. \) Denote by \( Tab_n(\beta) \) the set of all \( T \in Tab_n \) having weight \( \beta. \)

One visualizes the tableau \( T \) as filling of each box of \( \Delta_i \) with the number \( i \) for \( i = 1, \ldots, n. \)

Let \( T = (\Lambda_0 \subset \Lambda_1 \subset \Lambda_2) \) be a skew 2-tableau, that is, \( \Lambda_1 \setminus \Lambda_0 \) and \( \Lambda_2 \setminus \Lambda_1 \) both are horizontal strips. For such a skew 2-tableau \( T \) define the skew 2-tableau \( t(T) = T = (\Lambda_0 \subset \Lambda_1' \subset \Lambda_2) \) as follows. First of all we visualize \( T \) as the shape \( D = \Lambda_2 \setminus \Lambda_0 \) filled with the letters \( a \) and \( b \) where the letters \( a \) fill the horizontal strip \( \Lambda_1 \setminus \Lambda_0, \) and the letters \( b \) fill the remaining horizontal strip \( \Lambda_2 \setminus \Lambda_1. \) Then \( t(T) \) will be a new filling of \( D \) with the same letters \( a \) and \( b \) which we construct below.

Let \( D_k \) be the longest connected sub-row of the \( k \)-th row of \( D, \) whose boxes have no horizontal edges in common with other boxes of \( D. \) \( D_k \) contains say, \( l \) boxes on the left filled with \( a, \) and \( r \) boxes (on the right) filled with \( b. \) We will replace this filling of \( D_k \) with the \( r \) letters \( a \) on the left, and \( l \) letters \( b \) on the right. We do this procedure for every \( k, \) and leave unchanged the filling of the complement of the union of all the \( D_k \) in \( D. \) It is easy to see that the new filling of \( D \) is a skew 2-tableau of the form \( t(T) = T = (\Lambda_0 \subset \Lambda_1' \subset \Lambda_2). \) Clearly the correspondence \( T \mapsto t(T) \) is an involutive automorphism of the set of skew 2-tableaux.

The following lemma is obvious.
Lemma-Definition A1. Let $D$ be a skew Young diagram. Then the following are equivalent:

(i) There is a (unique) skew 2-tableau $T = (\Lambda_0 \subset \Lambda_1 \subset \Lambda_2)$ with $D = \Lambda_2 \setminus \Lambda_0$ such that

$$t(T) = T$$

(ii) There is a (unique) covering of $D$ by (non-overlapping) dominoes such that every domino has no common horizontal edges with other boxes of $D$.

Define the Bender-Knuth involution ([1], see also [2]) $t_i : \text{Tab}_n \to \text{Tab}_n$ as follows. Let $T = (\cdots \subset \Lambda_{i-1} \subset \Lambda_i \subset \Lambda_{i+1} \subset \cdots) \in \text{Tab}_n$. Define $T_i(T) \in \text{Tab}_n$ by

$$t_i(T) = (\cdots \subset \Lambda_{i-1} \subset \Lambda_i' \subset \Lambda_{i+1} \subset \cdots)$$

where $\Lambda_i'$ is defined by $(\Lambda_{i-1} \subset \Lambda_i' \subset \Lambda_{i+1}) = t(\Lambda_{i-1} \subset \Lambda_i \subset \Lambda_{i+1})$, and other $\Lambda_j$ remain unchanged.

Denote the resulting tableau by $t_i(T)$. It is easy to see that the correspondence $T \mapsto t_i(T)$ is a well-defined involutive automorphism $t_i : \text{Tab}_n \to \text{Tab}_n$, and $t_it_j = t_jt_i$ whenever $|j-i| > 1$.

A domino tableau $T$ with entries not exceeding $\frac{n+1}{2}$ is an ascending sequence of the Young diagrams $T = (\Lambda_\varepsilon \subset \Lambda_{\varepsilon+2} \subset \cdots \subset \Lambda_{n-2} \subset \Lambda_n)$ with $\varepsilon = n - 2[\frac{n}{2}]$, such that

(i) $\Lambda_\varepsilon = \emptyset$ if $n$ is even, (i.e., $\varepsilon = 0$), and $\Lambda_\varepsilon$ is a connected horizontal strip if $n$ is odd, (i.e., $\varepsilon = 1$);

(ii) Each difference $D_k = \Lambda_{n-2k} \setminus \Lambda_{n-2k+2}$ is subject to condition (ii) of Lemma A1.

The shape of $T$ is defined to be $\Lambda_n$. We also define the weight $\beta' = (\beta_1, \ldots, \beta_{\lfloor \frac{n+1}{2} \rfloor})$ of $T$ by $\beta_k = \frac{|D_k|}{2}$ for $k = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$.

Thus the above definitions together with Lemma-Definition A1 make Proposition-Definition 1.1 obvious.
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