QUASI-PRIME IDEALS

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Abstract. In this paper, the new concept of quasi-prime ideal is introduced which at the same time generalizes the “prime ideal” and “primary ideal” notions. Then a natural topology on the set of quasi-prime ideals of a ring is introduced which generalizes the Zariski topology. The basic properties of the quasi-prime spectrum are studied and several interesting results are obtained. Specially, it is proved that if the Grothendieck t-functor is applied on the quasi-prime spectrum then the prime spectrum is deduced. It is also shown that there are the cases that the prime spectrum and quasi-prime spectrum do not behave similarly.

1. Introduction

In this paper, we first generalize naturally the notion of prime ideal. In fact, a proper ideal \( q \) of a commutative ring \( A \) is said to be a quasi-prime ideal of \( A \) if \( fg \in q \) for some \( f, g \in A \) then either \( f \in \sqrt{q} \) or \( g \in \sqrt{q} \). The set of quasi-prime ideals of \( A \) is denoted by \( Sq_A \). Then we equip this set with a natural topology whose basis opens are of the form \( U_f = \{ q \in Sq_A : q \cap S_f = \emptyset \} \) where \( f \in A \) and \( S_f = \{ 1, f, f^2, \ldots \} \). The space \( Sq_A \) is called the quasi-prime spectrum of \( A \). It is shown that the quasi-prime spectrum satisfies in all of the conditions of being a spectral space except the uniqueness of generic point, see Theorem 2.3 and Proposition 6.1. This topology is a natural generalization of the Zariski topology, i.e. the prime spectrum \( \text{Spec} A \) is a dense subspace of \( Sq_A \). In many cases, this topology behaves completely different from the Zariski topology. For instance, \( Sq \mathbb{Z} \) has no closed point. The basic properties of the quasi-prime spectrum are studied. Indeed, Theorems 2.3, 2.5, 3.1, Proposition 3.3, Theorem 3.7, Propositions 5.1 and 6.1 are the main results on this topology. In Theorem 3.7 the connected components of the quasi-prime spectrum are characterized. In Theorem 4.1 it is shown that the prime spectrum can be canonically recovered.
from the quasi-prime spectrum by applying the Grothendieck \( t \)-functor.

2. QUASI-PRIMES

**Definition 2.1.** A proper ideal \( q \) of a ring \( A \) is called a quasi-prime ideal of \( A \) if \( fg \in q \) for some \( f, g \in A \) then either \( f \in \sqrt{q} \) or \( g \in \sqrt{q} \).

**Lemma 2.2.** An ideal \( q \) of a ring \( A \) is a quasi-prime ideal of \( A \) if and only if \( \sqrt{q} \) is a prime ideal of \( A \).

**Proof.** Easy. \( \square \)

Recall that a proper ideal \( q \) of a ring \( A \) is called a primary ideal of \( A \) if there exist \( f, g \in A \) such that \( fg \in q \) and \( f \notin q \) then \( g \in \sqrt{q} \). It is equivalent to the statement that if there exist \( f, g \in A \) such that \( fg \in q \) and \( f \notin \sqrt{q} \) then \( g \in q \). Note that if \( q \) is a primary ideal of \( A \), \( fg \in q \), \( f \notin q \) and \( g \notin q \) then by definition both of \( f \) and \( g \) must be in \( \sqrt{q} \). Every primary ideal is a quasi-prime ideal but the converse is not necessarily true. As a specific example, let \( A \) be the polynomial ring \( k[x, y, z] \) modulo \( I = (xy - z^2) \) where \( k \) is a domain. Then \( p = (x + I, z + I) \) is a prime ideal of \( A \) since \( A/p \cong k[y] \), but \( q = p^2 \) is a quasi-prime ideal which is not a primary ideal. Because \( (x + I)(y + I) = (z + I)^2 \in q \) but \( x + I \notin q \) and \( y + I \notin \sqrt{q} \).

If \( p \) is a prime ideal of a ring \( A \) then \( p^n \) is a quasi-prime ideal of \( A \) for all \( n \geq 1 \). But there are also non-primary quasi-prime ideals which are not as the power of a prime ideal. As an example, let \( A \) be the polynomial ring \( k[x, y, z, t] \) modulo \( I = (xy - z^2) \) where \( k \) is a domain. Then \( p = (z + I, t + I) \) is a prime of \( A \) since \( A/p \cong k[x, y] \), but \( q = (z + I, t^2 + I) \) is a non-primary quasi-prime of \( A \) which is not also as a power of a prime ideal.

The set of quasi-prime ideals of \( A \) is denoted by \( \text{Sq} A \). For each \( f \in A \) we define \( U_f = \{ q \in \text{Sq} A : q \cap S_f = \emptyset \} \) where \( S_f = \{ f^i : i \geq 0 \} = \{ 1, f, f^2, \ldots \} \). Then clearly \( U_1 = \text{Sq} A \) and \( U_f \cap U_g = U_{fg} \) for all \( f, g \in A \). Thus there exists a (unique) topology over \( \text{Sq} A \) such that the basis opens are precisely of the form \( U_f \) where \( f \in A \). We call \( \text{Sq} A \) the quasi-prime spectrum (or, the space of quasi-primes) of \( A \). Clearly \( \text{Spec} A \) is a subspace of \( \text{Sq} A \) since \( D(f) = U_f \cap \text{Spec} A \) for all \( f \in A \). Note that \( U_f = \{ q \in \text{Sq} A : \sqrt{q} \cap S_f = \emptyset \} \) for all \( f \in A \). It follows that
Spec $A$ is a dense subspace of $\text{Sq} A$.

**Theorem 2.3.** Every basis open $U_f$ is quasi-compact. In particular, $\text{Sq} A$ is quasi-compact.

**Proof.** It suffices to show that every open covering of $U_f$ by the basis opens has a finite refinement. Hence let $U_f = \bigcup_{i \in I} U_{g_i}$ where $g_i \in A$ for all $i$. It follows that $f \in \sqrt{(g_i : i \in I)}$. Thus the exists a finite subset $J$ of $I$ such that $f \in \sqrt{(g_i : i \in J)}$. We show that $U_f = \bigcup_{i \in J} U_{g_i}$. If $q \in U_f$ then there exists some $i \in J$ such that $g_i \notin \sqrt{q}$. It follows that $q \in U_{g_i}$. □

The proof of Theorem 2.3 also shows that $D(f)$ is quasi-compact for all $f \in A$.

**Corollary 2.4.** If $X$ is a subspace of $\text{Sq} A$ such that $\text{Spec} A \subseteq X$ then $X$ is quasi-compact. In particular, the primary spectrum (space of primary ideals) of $A$ is quasi-compact.

**Proof.** If $U$ is an open of $\text{Sq} A$ such that $X \subseteq U$ then $U = \text{Sq} A$. Because if $q$ is a quasi-prime ideal of $A$ then there exists some $f \in A$ such that $\sqrt{q} \in U_f \subseteq U$. It follows that $q \in U_f$. Thus by Theorem 2.3, $X$ is quasi-compact. □

**Theorem 2.5.** If $q$ is a quasi-prime ideal of $A$ then $\overline{\{q\}} = \{p \in \text{Sq} A : q \subseteq \sqrt{p}\}$.

**Proof.** Let $p \in \overline{\{q\}}$ and $f \in q$. If $f \notin \sqrt{p}$ then $p \in U_f$. It follows that $q \in U_f$, a contradiction. Conversely, assume that $q \subseteq \sqrt{p}$. If $p \notin \overline{\{q\}}$ then there exists some $f \in A$ such that $p \in U_f$ but $q \notin U_f$. Hence there exists a natural number $n \geq 1$ such that $f^n \in q$. It follows that $f \in \sqrt{p}$. But this is contradiction since $p \cap S_f = \emptyset$. □

Unlike the prime spectrum, a maximal ideal is not necessarily a closed point of the quasi-prime spectrum. As a specific example, if $p$ is a prime number then $\{p^n \mathbb{Z} : n \geq 1\}$ is the closure of $\{p \mathbb{Z}\}$ in $\text{Sq} \mathbb{Z}$. In fact, $\text{Sq} \mathbb{Z}$ has no closed point.
In the space $\text{Sq} A$ generic points are not unique:

**Corollary 2.6.** If $q$ is a quasi-prime ideal of $A$ then $\overline{\{q\}} = \overline{\sqrt{q}}$. □

**Corollary 2.7.** Let $E$ be a closed subset of $\text{Sq} A$. If $q \in E$ then $\sqrt{q} \in E$. □

It is easy to see that the closed subsets of $\text{Sq} A$ are precisely of the form $\mathcal{V}(I) = \{q \in \text{Sq} A : I \subseteq \sqrt{q}\}$ where $I$ is an ideal of $A$. Clearly $\mathcal{V}(I) \cap \text{Spec } A = \mathcal{V}(I)$. If $f \in A$ then $\mathcal{V}(f) = \{q \in \text{Sq} A : f \in \sqrt{q}\}$. If $p$ is a quasi-prime of $A$ then by Theorem 2.5, $\mathcal{V}(p) = \{p\}$.

### 3. Connected components

A subspace $Y$ of a topological space $X$ is called a retraction of $X$ if there exists a continuous map $\gamma : X \to Y$ such that $\gamma(y) = y$ for all $y \in Y$. Such a map $\gamma$ is called a retraction map.

**Lemma 3.1.** The prime spectrum is a retraction of quasi-prime spectrum.

**Proof.** The map $\gamma : \text{Sq} A \to \text{Spec } A$ given by $q \leadsto \sqrt{q}$ is continuous since $\gamma^{-1}(\mathcal{D}(f)) = U_f$ for all $f \in A$. □

The map $\gamma$ in Lemma 3.1 is an open map and $\gamma^{-1}(\mathcal{V}(I)) = \mathcal{V}(I)$ for all ideals $I$ of $A$.

**Remark 3.2.** There is a fundamental result due to Grothendieck which states that the map $f \leadsto \mathcal{D}(f)$ is a bijection from the set of idempotents of $A$ onto the set of clopen (both open and closed) subsets of $\text{Spec } A$, see [1, Tag 00EE]. Under the light of this result we obtain that:

**Proposition 3.3.** The map $f \leadsto U_f$ is a bijection from the set of idempotents of $A$ onto the set of clopens of $\text{Sq} A$.

**Proof.** By Remark 3.2, it suffices to show that the map $U \leadsto \gamma^{-1}(U)$ is a bijection from the set of clopens of $\text{Spec } A$ onto the set of clopens
of $\text{Sq}A$, for $\gamma$ see Lemma \ref{lema:gamma}. Assume that $\gamma^{-1}(U) = \gamma^{-1}(V)$. If $p \in U$ then $p \in \gamma^{-1}(U)$ and so $\gamma(p) = p \in V$. Hence $U = V$. It remains to show that this map is surjective. If $U$ is a clopen of $\text{Sq}A$ then $U \cap \text{Spec}A$ is a clopen of $\text{Spec}A$. We have $U = \gamma^{-1}(U \cap \text{Spec}A)$. Because if $q \in U$ then by Corollary \ref{cor:gamma} $\gamma(q) = \sqrt{q} \in U \cap \text{Spec}A$. Conversely, if $q \in \gamma^{-1}(U \cap \text{Spec}A)$ then there exists some $f \in A$ such that $\sqrt{q} \in U_f \subseteq U$. It follows that $q \in U$. □

Corollary 3.4. The space $\text{Sq}A$ is connected if and only if $A$ has no nontrivial idempotents. □

Proposition 3.5. If $\varphi : A \to B$ is a morphism of rings then the induced map $\varphi^* : \text{Sq}B \to \text{Sq}A$ given by $q \sim \varphi^{-1}(q)$ is continuous.

Proof. If $q$ is a quasi-prime ideal of $B$ then $\varphi^{-1}(q)$ is a quasi-prime ideal of $A$ because $\varphi^{-1}(\sqrt{q}) = \sqrt{\varphi^{-1}(q)}$. Hence $\varphi^*$ is well-defined. It is continuous since $(\varphi^*)^{-1}(U_f) = U_{\varphi(f)}$ for all $f \in A$. □

Note that if $S$ is a multiplicative subset of $A$ then the map $\pi^* : \text{Sq}(S^{-1}A) \to \text{Sq}A$ induced by the canonical ring map $A \to S^{-1}A$ is not injective and $\text{Im} \pi^* \subseteq \{q \in \text{Sq}A : q \cap S = \emptyset\}$. Specially $\text{Im} \pi^* \subseteq U_f$ where $\pi : A \to A_f$ is the canonical map.

Lemma 3.6. If $I$ is an ideal of a ring $A$ then the map $\pi : \text{Sq}A/I \to \text{Sq}A$ induced by the canonical ring map $A \to A/I$ is injective and $\text{Im} \pi = \mathcal{V}(I)$.

Proof. Easy. □

An ideal of $A$ is said to be a regular ideal of $A$ if it is generated by a subset of idempotent elements of $A$. Each maximal element of the set of proper regular ideals of $A$ (ordered by inclusion) is called a max-regular ideal of $A$. By the Zorn’s Lemma, every proper regular ideal of $A$ is contained in a max-regular ideal of $A$. It is well known that a regular ideal $M$ is a max-regular ideal of $A$ if and only if $A/M$ has no nontrivial idempotents, see \cite[Lemma 3.19]{A}. It is also well known that the connected components of $\text{Spec}A$ are precisely of the form $\mathcal{V}(M)$ where $M$ is a max-regular ideal of $A$, see \cite[Theorem 3.17]{A}. We have then the following result.
Theorem 3.7. The connected components of $\text{Sq} A$ are precisely of the form $V(M)$ where $M$ is a max-regular ideal of $A$.

Proof. If $N$ is a max-regular ideal of $A$. Then by Corollary 3.4, $\text{Sq} A/N$ is connected. Thus by Proposition 3.5 and Lemma 3.6, $V(N)$ is connected. Now let $C$ be a connected component of $\text{Sq} A$. Then $\gamma(C)$ is contained in a connected component of $\text{Spec} A$, for $\gamma$ see Lemma 3.1. Thus there exists a max-regular ideal $M$ of $A$ such that $\gamma(C) \subseteq V(M)$. It follows that $C \subseteq V(M)$. Thus $C = V(M)$ since $\mathcal{V}(M)$ is connected. Conversely, let $N$ be a max-regular ideal of $A$. Then there exists a connected component $C$ of $\text{Sq} A$ such that $\mathcal{V}(N) \subseteq C$. We observed that there exists a max-regular ideal $M$ of $A$ such that $C = V(M)$. It follows that $\sqrt{M} \subseteq \sqrt{N}$. Thus $M \subseteq N$ since $M$ is a regular ideal. This implies that $M = N$ because $M$ is max-regular and $N$ is regular ideal.

\[\square\]

4. **t-functor**

There exists a covariant functor due to Grothendieck from the category of topological spaces to itself. It is called the $t$-functor. This functor has geometric applications and builds a bridge between the classical algebraic geometry and modern algebraic geometry. In what follows we shall introduce this functor. If $X$ is a topological space then the points of $t(X)$ are the irreducible and closed subsets of $X$. Recall that a topological space is said to be an irreducible space if it is non-empty and can not be written as the union of two proper closed subsets. The closed subsets of $t(X)$ are precisely of the form $t(Y)$ where $Y$ is a closed subset of $X$. If $f : X \rightarrow X'$ is a continuous map of topological spaces then the function $t(f) : t(X) \rightarrow t(X')$ given by $Z \rightsquigarrow f(Z)$ is well-defined and continuous. There exists also a canonical continuous map $X \rightarrow t(X)$ defined by $x \rightsquigarrow \{x\}$. We have then the following result.

Theorem 4.1. The space $t(\text{Sq} A)$ is canonically homeomorphic to $\text{Spec} A$.

Proof. Let $Z$ be an irreducible and closed subset of $\text{Sq} A$. By Proposition 6.1 there exists a quasi-prime $q$ of $A$ such that $Z = \{q\}$. We then define $\varphi : t(\text{Sq} A) \rightarrow \text{Spec} A$ as $Z \rightsquigarrow \sqrt{q}$. We show that it is a homeomorphism. The map $\varphi$ is injective, see Corollary 2.6. If $p$ is a prime ideal of $A$ then $Z := \mathcal{V}(p) = \{p\}$ is an irreducible and closed subset of $\text{Sq} A$ and $\varphi(Z) = p$. The map $\varphi$ is continuous because
\[ \varphi^{-1}(D(f)) = t(SQ A) \setminus t(V(f)) \] for all \( f \in A \). It remains to show that \( \varphi \) is a closed map. Let \( Y = V(I) \) be a closed subset of \( SQ A \) where \( I \) is an ideal of \( A \). Then \( \varphi(t(Y)) = V(I) \). \( \Box \)

5. Normality

A topological space is called a normal space if every two disjoint closed subsets admit disjoint open neighborhoods. Clearly a closed subspace of a normal space is a normal space, but an arbitrary subspace is not necessarily a normal space.

**Proposition 5.1.** Let \( A \) be a ring. Then \( SQ A \) is a normal space if and only if \( Spec A \) is a normal space.

**Proof.** Let \( SQ A \) be a normal space. Let \( E = V(I) \) and \( F = V(J) \) be two disjoint closed subsets of \( Spec A \) where \( I \) and \( J \) are ideals of \( A \). It follows that \( I + J = A \). Thus \( V(I) \cap V(J) = \emptyset \). Hence there are disjoint opens \( U \) and \( V \) in \( SQ A \) such that \( V(I) \subseteq U \) and \( V(J) \subseteq V \). It follows that \( E \subseteq U \cap Spec A \) and \( F \subseteq V \cap Spec A \). Conversely, let \( Spec A \) be a normal space. Let \( V(I) \) and \( V(J) \) be two disjoint closed subsets of \( SQ A \). It follows that \( V(I) \cap V(J) = \emptyset \). Thus there are disjoint opens \( U \) and \( V \) in \( Spec A \) such that \( V(I) \subseteq U \) and \( V(J) \subseteq V \). It follows that \( V(I) \subseteq \gamma^{-1}(U) \) and \( V(I) \subseteq \gamma^{-1}(U) \). Hence \( SQ A \) is a normal space. \( \Box \)

It is well known that \( Spec A \) is a normal space if and only if \( A \) is a pm-ring, see [2, Theorem 1.2] or [5, Theorem 4.3].

**Proposition 5.2.** If \( Spec A \) is a normal space then \( Max A \) is a normal space.

**Proof.** It is well known that \( Max A \) is quasi-compact. It suffices to show that \( Max A \) is Hausdorff because it is well known that every compact (quasi-compact and Hausdorff) space is a normal space. Thus let \( m \) and \( m' \) be two distinct maximal ideals of \( A \). The closed points of \( Spec A \) are precisely the maximal ideals. Therefore by the hypothesis, there exist disjoint opens \( U \) and \( V \) in \( Spec A \) such that \( m \in U \) and \( m' \in V \). It follows that \( (U \cap Max A) \cap (V \cap Max A) = \emptyset \). \( \Box \)

The converse of Proposition 5.2 is not necessarily true, see [5, Remark 4.8]. This Remark, as stated there, also shows that the main
result of [3] is not true.

6. Spectrality and Hausdorffness

Proposition 6.1. Every irreducible and closed subset of $\text{SqA}$ has a generic point.

Proof. Let $Z$ be an irreducible and closed subset of $\text{SqA}$. There exists an ideal $J$ of $A$ such that $Z = \mathcal{V}(J)$. By Theorem 2.3, it suffices to show that $J$ is a quasi-prime of $A$. Clearly $J \neq A$ since $Z$ is non-empty. Let $f, g \in A$ such that $fg \in J$. We have $Z = (\mathcal{V}(f) \cap Z) \cup (\mathcal{V}(g) \cap Z)$. It follows that either $Z \subseteq \mathcal{V}(f)$ or $Z \subseteq \mathcal{V}(g)$. Thus either $f \in \sqrt{J}$ or $g \in \sqrt{J}$. □

The irreducible components of $\text{SqA}$ are precisely of the form $\mathcal{V}(p)$ where $p$ is a minimal prime of $A$.

Recall that a topological space $X$ is called a spectral space if it is quasi-compact, its topology has a basis consisting of quasi-compact opens such that every finite intersection of these basis opens is again quasi-compact and every irreducible and closed subset of $X$ has a unique generic point. In this definition, the uniqueness of generic point is a crucial point. For instance, the space $\text{SqA}$, by Theorem 2.3 and Proposition 6.1, satisfies in all of the conditions of a spectral space except the uniqueness of generic point. This leads us to the following result:

Corollary 6.2. $\text{SqA}$ is a spectral space if and only if $\text{SqA} = \text{Spec}A$. □

Remark 6.3. It is well known that for a ring $A$ then $\text{Spec}A$ is Hausdorff if and only if every prime ideal of $A$ is maximal.

Proposition 6.4. $\text{SqA}$ is a Hausdorff space if and only if $\text{SqA} = \text{Max}A$.

Proof. Let $\text{SqA}$ be a Hausdorff space and $q$ a quasi-prime of $A$. There exists a maximal ideal $m$ of $A$ such that $q \subseteq m$. It follows that $m \in \mathcal{V}(q) = \{q\}$ and so $q = m \in \text{Max}A$. The converse implies from
Remark 6.3.

Corollary 6.5. If $\text{Sq} A$ is a Hausdorff space then $\text{Spec} A$ is Hausdorff.

The converse of Corollary 6.5 does not hold. As a specific example, if $A = \mathbb{Z}/8\mathbb{Z}$ then $\text{Spec} A = \{m\}$ but $\text{Sq} A = \{0, m, m^2\}$ where $m = 2\mathbb{Z}/8\mathbb{Z}$.

Corollary 6.6. Let $A$ be a local ring such that the maximal ideal $m$ is a finitely generated ideal. Then the following are equivalent.
(i) $A$ is a field.
(ii) $\text{Sq} A$ is Hausdorff.
(iii) $\text{Sq} A$ is a spectral space.
(iv) $\text{Sq} A$ has a closed point.

Proof. If one of the statements (ii), (iii) and (iv) hold then we have $m = m^2$. Thus by the Nakayama lemma, $m = 0$.

If $(A, m)$ is an Artinian local ring then $\text{Sq} A$ with the operation $m^i * m^j = m^{i+j}$ can be viewed as a cyclic group of order $n$ where $n$ is the least natural number such that $m^n = 0$. In particular, if $p$ is a prime number and $n \geq 1$ then $\text{Sq} A = \{m, ..., m^n\}$ is a cyclic group of order $n$ where $A = \mathbb{Z}/p^n\mathbb{Z}$ and $m = p\mathbb{Z}/p^n\mathbb{Z}$.

References

[1] A. J. de Jong et al., Stacks Project, see http://stacks.math.columbia.edu
[2] G. De Marco and A. Orsatti, Commutative rings in which every prime ideal is contained in a unique maximal ideal, Proc. Amer Math. Soc. 30 (3) (1971) 459-466.
[3] H. Simmons, Reticulated rings, J. Algebra 66 (1980) 169-192.
[4] A. Tarizadeh, Flat topology and its dual aspects, Comm. Algebra (2018), DOI: 10.1080/00927872.2018.1469637.
[5] A. Tarizadeh and M. Aghajani, Notes on pm-rings and mp-rings, submitted, arXiv:1803.04817v2 [math.AC].

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