Boundaries of Filled Julia Sets in Generalized Jungck Mann Orbit

DONG LI¹, MUHAMMAD TANVEER², WAQAS NAZEER³, AND XIAORUI GUO⁴

¹Zhengzhou Railway Vocational & Technical College, Zhengzhou 451460, China
²Department of Mathematics and Statistics, The University of Lahore, Lahore 54000, Pakistan
³Division of Science and Technology, University of Education, Lahore 54000, Pakistan
⁴Zhengzhou University of Light Industry, Zhengzhou 450002, China

Corresponding authors: Waqas Nazeer (nazeer.waqas@ue.edu.pk) and Xiaorui Guo (496143326@qq.com)

ABSTRACT In this paper, we study the generalized Jungck Mann orbit (GJMO) and prove the converse theorem of results. We develop algorithms for the generation of filled Julia sets and their boundaries in the GJMO. In simple Julia set (i.e., boundary of filled Julia set), we show the internal structure of Julia set and establish the correspondence between Julia points via dark blue lines in graphs. Moreover, we demonstrate the pictorial effects of filled Julia set and its boundary and graphically present the image change with the change of complex parameter \( c \) in the GJMO.

INDEX TERMS Jungck-Mann iteration, filled Julia set, boundary of Julia set.

I. INTRODUCTION

The history of fractal geometry is the benignant of two French mathematicians Gaston Julia and Pierre Fatou. First time in the history of fractals they concentrated on iteration of general complex rational functions and explained all the facts about Julia sets and Fatou domains. However, in those days there were no computers and this made it practically difficult to study the pictorial effects of Julia sets and Fatou domains. Graphical behavior of Julia sets and Fatou domains were studied in 1975 by Mandelbrot [2]. Mandelbrot was working at IBM had begun to study the works from Fatou and Julia, and plotted [3] Julia sets for \( z^2 + c \). He observed that Julia sets have astonishing richness of characteristics. After the achievement of Mandelbrot, authors presented many modifications of his work. Lakhtakia et al. [4] presented the Julia sets of general complex function (i.e. \( f(z) = z^p + c \)). The rational complex and transcendental complex function were used in [5]. The better understandable generalization was the use of quaternions [6], bi [7] and tri [8] complex numbers to visualize the Julia sets and Mandelbrot sets. The use of iterative methods in the generation of fractals is another attractive generalization of fractal geometry. Authors used different iterations to generate beautiful aesthetic patterns. Rani et al. presented some generalized Julia sets and Mandelbrot sets with some nonstandard iterations in [9] and [10]. After the use of iterations from fixed point theory, researcher studied the different explicit iterations to generalize most studied fractals from fractal geometry, especially Julia sets and Mandelbrot sets [11–13] and [14]. Some implicit iterations studied to prove escape radius for Jungck Mann iteration with \( s \)-convexity, Jungck Ishikawa iteration with \( s \)-convexity and Jungck Noor iteration with \( s \)-convexity [1]. Recently, Kwun et al. presented some Mandelrot sets, Julia sets and Biomorphs in Jungck-CR orbit with \( s \)-convexity [16] and in modified Jungck-S orbit [17] respectively. In literature about fractal geometry authors studied filled Julia sets, connected and disconnected Julia sets and their generalizations. They studied the characteristics of Julia sets and the applications of Julia sets in many field of social sciences. They discussed the complex behavior of just filled Julia sets (i.e. may be connected or disconnected). But they did not demonstrated the boundaries (i.e. Simple Julia sets) of filled Julia sets and Fatou domains.

In this article we introduce the Jungck Mann iteration with \( s \)-convexity [1] in the generation of fractals. We develop algorithms for filled Julia sets and their boundaries in GIMO. We also discuss the nature of filled Julia sets, their boundaries and Fatou domains. Moreover, we present some complex graphs to demonstrate internal structure of filled Julia sets.

The Sec. II of article provide preliminaries. In Sec. III we demonstrate the main algorithms in which we use escape radius of generalized Jungck Mann orbit from [1]. We present filled Julia sets, boundaries of corresponding Julia sets and...
Fatou domains in Sec. IV against complex polynomial function \( f_2(x) = x^n - bx + c \) in GJMO. In the Sec. V, we deduce concluding remarks.

II. PRELIMINARIES

Definition 1 (Julia Set [18]): Let \( F_f \) be the set of points in \( \mathbb{C} \) such that \( f : \mathbb{C} \to \mathbb{C} \) is a complex polynomial of degree \( \geq 2 \). The set \( F_f \) is called filled Julia set, when the orbits of \( F_f \to \infty \) as \( p \to \infty \), i.e.,

\[
F_f = \{ z \in \mathbb{C} : |f^p(z)|_{p=0}^{\infty} \text{ is bounded}. \tag{1}
\]

The boundary of filled Julia set is called simply Julia set.

Definition 2 (Mandelbrot Set [19]): The set which consists of all connected Julia sets is called Mandelbrot set \( M \), i.e.,

\[
M = \{ c \in \mathbb{C} : F_f \text{ is connected} \}. \tag{2}
\]

or we can define Mandelbrot set equivalently as follows [20]:

\[
M = \{ c \in \mathbb{C} : f^p(0) \text{ does not tend to } \infty \text{ as } p \to \infty \}. \tag{3}
\]

\( f \) has only critical point 0 (i.e., \( f'(0) = 0 \)). So authors choose 0 as an initial point. In literature authors used different approaches to generate Julia sets. Some popular algorithms to visualize the Julia sets are, distance estimator, escape time and potential function algorithms. To generate filled Julia sets, simply Julia sets and Fatou domains. We use escape time algorithms. The escape time algorithm iterate the function up to the desire number of iterations. The algorithm generate two sets, one is consists of points for which the GJMO does not escape to infinity (i.e. filled Julia set or boundary of Julia set) and the second set consists of points for which the GJMO escape to infinity (i.e. Fatou domains).

Definition 3 (Jungck-Mann Iteration [21]): Let \( S, T : X \to X \) be the two maps such that \( S \) is one to one and \( T \) is differentiable of degree greater and equal to 2. For any \( x_0 \in X \) the Jungck iteration is defined in the following way

\[
S(x_{n+1}) = T(x_n), \tag{4}
\]

where \( n = 0, 1, \ldots \).

Definition 4 (Jungck-Mann Iteration [1]): Let \( S, T : \mathbb{C} \to \mathbb{C} \) be the two complex maps such that \( T \) is a complex polynomial of degree greater and equal to 2, also differentiable and \( S \) is injective. For any \( x_0 \in \mathbb{C} \) the Jungck-Mann iteration defined as:

\[
S(x_{n+1}) = (1 - a)S(x_n) + aT(x_n), \tag{5}
\]

where \( a, s \in (0, 1], n = 0, 1, 2, \ldots \).

Definition 5 (Jungck-Mann Iteration With s-Convexity [1]): Let \( S, T : \mathbb{C} \to \mathbb{C} \) be the two complex maps such that \( T \) is a complex polynomial of degree greater and equal to 2, also differentiable and \( S \) is one to one. For any \( x_0 \in \mathbb{C} \) the Jungck-Mann iteration with s-convexity is defined as:

\[
S(x_{n+1}) = (1 - a)^s S(x_n) + a^s T(x_n), \tag{6}
\]

where \( a, s \in (0, 1], n = 0, 1, 2, \ldots \).

We observed that generalized Jungck Mann orbit change into:
- Picard orbit when \( S(x) = x \) and \( a, s = 1 \),
- Mann orbit when \( S(x) = x \) and \( s = 1 \),
- Jungck Mann orbit when \( s = 1 \).

Therefore, we notice that Jungck Mann orbit is a special case of generalized Jungck Mann orbit. In generalized Jungck Mann orbit we deal with two different mappings, we break \( f \) in two mappings \( S \) and \( T \) in such a way that \( f = T - S \) and \( S \) is one to one. This type of formation of \( f \) restrain us to adopt \( S \) as one to one mapping and \( T \) as analytic mapping. Thus we use escape radius of GJMO proved in [1] and implement in our algorithms to visualize desire Julia sets and Fatou domains.

III. MAIN RESULT

In the ambience’s of complex dynamics, the Julia set and the Fatou set are complement to each other (Julia ‘laces’ and Fatou ‘dusts’) defined from a function. Informally, the Fatou set of the function contains the values having property that all nearby values behave same under repeated iteration of the function, and the Julia set contains the values such that an arbitrarily small change can cause drastic changes in the sequence of iterated function values. Thus the behavior of complex function on the Fatou set is ‘regular’, while on the Julia set its behavior is ‘chaotic’. In this section we prove the converse theorem of results proved in [1]. To analyze the boundary and behavior of Julia set and Fatou domain of complex function \( x^m - bx + c \) we use the established escape radius to develop algorithm 1 for filled Julia set and add a condition \( f(x_0) = \text{Min} \{ |R x_0|, |I x_0| \} \) in filled Julia set algorithm to develop algorithm 2 for the boundary of filled Julia set. In both algorithms 1 and 2 we use a necessary condition

\[
R = \text{Max} \{ |c|, \left( \frac{(1+|b|)}{as} \right)^{\frac{1}{\theta}} \} \text{ and } |x_{n+1}| > R \text{ for the visualization of Julia sets. The main role of this condition is that if the iterations obey this then these will be in Julia set, if not then iterations will be in Fatou domain.}
\]

Theorem 1: Let the sequence \( x_n \) be consists of iterates from generalized Jungck Mann iteration for mth power of \( f(x) \), where \( f \) is complex polynomial and \( m \) is always greater than and equal to 2. If the sequence \( x_n \) \( n \to \infty \), then \( |x_n| \geq |c| > \left( \frac{(1+|b|)}{as} \right)^{\frac{1}{\theta}} \) where \( a, s \in (0, 1] \).

Proof: Since \( x_n \) is the sequence of generalized Jungck Mann iteration with mth degree complex polynomial, where \( m \geq 2 \) such that \( |x_n| \to \infty \) as \( n \to \infty \), then there exist \( \theta > 0 \) such that

\[
|x_n| > (1 + \theta)^s |x|. \tag{7}
\]

For \( n = 1 \), we have

\[
|x_1| \geq (1 + \theta) |x|. \tag{7}
\]

Now let \( f_{1}(x) = x^n - bx + c \), where \( b, c \in \mathbb{C} \) and \( x_0 = x \). Since in generalized Jungck Mann iteration we deal with two maps, so we set \( f = T - S \) in such a way that: \( T(x) = x^n + c \)
and \( S(x) = bx \), then
\[
|S(x_1)| = \left| (1 - a)^2 S(x_0) + a^2 T(x_0) \right|
= \left| (1 - a)^2 bx + (1 - (1 - a))^2 (x^2 + c) \right|
\]
With the expansion of \((1 - a)^2\) and \((1 - (1 - a))^2\) up to first terms and applying condition \( s \leq 1 \) we get
\[
|bx_1| \geq (1 - s(1 - a)) x^2 + c - (1 - sa) bx
\geq (s - s(1 - a))(x^2 + c) - |(1 - sa)bx|.
\]
Since \( |x| \geq |c| \) is necessary condition for fractal generation and \( sa < 1 \) we have
\[
|bx_1| \geq sa \left( x^2 - sa |c| - (1 - sa)bx \right)
= sa \left( x^2 - sa |c| - |bx| + sa |bx| \right)
\geq sa \left( x^2 - |x| - |b||x| \right)
\geq |x| \left( sa \left( x^m - 1 \right) - (1 + |b|) \right).
\]
Thus
\[
|x_1| \geq |x| \left( sa \left( x^m - 1 \right) - (1 + |b|) \right). \tag{8}
\]
From equations (7) and (8), we deduce that
\[
\frac{sa \left( x^m - 1 \right)}{1 + |b|} - 1 = 1 + \theta
\]
\[
\frac{sa \left( x^m - 1 \right)}{1 + |b|} - 1 > 1,
\]
for the reason that \( \theta > 0 \). This provide us
\[
|x| \geq \left( \frac{2(1 + |b|)}{sa} \right)^{\frac{1}{m-1}}.
\]
Wherefore, we get \( |x| \geq \left( \frac{2(1 + |b|)}{sa} \right)^{\frac{1}{m-1}} \) where \( m \geq 2 \) and \( a, s \in (0, 1] \) and also \( |x| \geq |c| \). Therefore for fractal generation in generalized Jungck Mann orbit, we have a sequence \( \{x_n\}_{n \in \mathbb{N}} \) of iteration must obey the above condition otherwise generalized Jungck Mann orbit escape to infinity. Which is our desire result.

IV. GENERATION OF JULIA SETS AND THEIR BOUNDARIES

This section consists of three subsection in which we demonstrate some graphical examples of quadratic, cubic and biquadratic filled Julia sets, simple Julia set and Fatou domains in GJMO. We used mathematics to generate images. The color of images vary with variation in \( a, s, K \).

A. QUADRATIC OF JULIA SETS

In this example we iterate the complex function \( f(x) = x^2 - bx + c \) with the selection of maps as: \( S(x) = bx \) and \( T(x) = x^2 + c \). We set \( K = 30 \) for each quadratic image. The internal part of simple Julia set or boundary of each filled Julia set consists of dark blue lines. These lines show the correspondence of Julia elements in each figure. The figure 1 is a unit quadratic filled Julia set for \( c = -1 \). The figure 2 is boundary of figure 1. The dark red color circular line is represent simple Julia set and remaining region is represent the Fatou domain. The figure 3 is a quadratic filled Julia set for \( c = -2 \). The yellow color in 4 is a boundary for 3, white and light blue regions are Fatou domains in figures 3 and figure 4 respectively. The figure 5 is a quadratic filled Julia set for \( c = -3 \). The red color chaotic closed chain in 6 is the boundary of 5 while the white and light blue regions are Fatou domains in both figures 5 and 6. The figure 7 is another example of quadratic filled Julia set for \( c = -4 \) and the figure 8 is its boundary. The red color chaotic closed chain

|Algorithm 1| Filled Julia set and Fatou Domain
|---|---|
|**Input:**| \( f \): a complex polynomial function, \( b, c \in \mathbb{C} \) – input parameter, \( A \): area occupied by image, \( K \): fixed number of iterations.
|**Output:**| Julia set and Fatou domain in GJMO.
|1| \( R = \max \left[ |c|, \left( \frac{2(1+|b|)}{sa} \right)^{\frac{1}{m-1}} \right] \)
|2| for \( x_0 \in A \) do
|3| \( n = 0 \)
|4| while \( n \leq K \) do
|5| \( x_{n+1} = f(x_n) \)
|6| if \( |x_{n+1}| > R \) then
|7| break
|8| \( n = n + 1 \)
|9| \( i = \lfloor (C - 1)n \rfloor \)\(|R| \)
|10| colour \( x_0 \) with colormap\([i]\)

|Algorithm 2| Boundary of Julia set and Fatou Domain
|---|---|
|**Input:**| \( f \): a complex polynomial function, \( b, c \in \mathbb{C} \) – input parameter, \( A \): area occupied by image, \( K \): fixed number of iterations.
|**Output:**| Simple Julia set and Fatou domain in GJMO.
|1| \( R = \max \left[ |c|, \left( \frac{2(1+|b|)}{sa} \right)^{\frac{1}{m-1}} \right] \)
|2| for \( x_0 \in A \) do
|3| \( n = 0 \)
|4| while \( n \leq K \) do
|5| \( x_{n+1} = f(x_n) \)
|6| if \( |x_{n+1}| > R \) then
|7| break
|8| \( n = n + 1 \)
|9| if \( f(x_n) = \text{Min} \left[ |\Re x|, |\Im x| \right] \), where \( \Re x \) is real and \( \Im x \)
|10| is imaginary part of \( f(x_n) \) then
|11| \( i = \lfloor (C - 1)n \rfloor \)\(|R| \)
|12| colour \( x_0 \) with colormap\([i]\)
|13| else
|14| colour \( x_0 \) with colormap\([0]\)
in figure 8 is a simple Julia set while the sky blue regions are Fatou domains. The figure 9 is also a quadratic filled Julia set for \( c = -5 \) consists of two large bulbs and four small bulbs each bulb behave same as figure 9 whole. The red color chaotic or irregular line in figure 10 is the boundary of 9 while the sky blue regions are Fatou domains. The figure 11 is again a quadratic filled Julia set for \( c = -6 \), also consists of two large bulbs and four small bulbs each bulb behave same as
figure 11 whole. The red color chaotic chain in figure 12 is the boundary of 11 and the sky blue regions are Fatou domain of 11 and 12. The figures 13 to 18 are quadractic Julia sets for $c = -7, -8, -9$ and $-10$ respectively. In each figure the red color irregular boundaries are simple Julia sets while the white, yellow and sky blue regions are Fatou domains.

The input parameters used in Figs. 1–18 were the following:

- Fig. 1: $b = \frac{5}{3}$, $a = 0.1$, $s = 0.1$, $A = [-4.4, 2.4] \times [-3.5, 3.5]$.
- Fig. 2: $b = \frac{5}{3}$, $a = 0.1$, $s = 0.1$, $A = [-4.4, 2.4] \times [-3.5, 3.5]$.
- Fig. 3: $b = \frac{5}{3}$, $a = 0.1$, $s = 0.1$, $A = [-5.4, 3.4] \times [-2.5, 2.5]$.
- Fig. 4: $b = \frac{5}{3}$, $a = 0.1$, $s = 0.1$, $A = [-5.4, 3.4] \times [-2.5, 2.5]$.
- Fig. 5: $b = \frac{5}{3}$, $a = 0.1$, $s = 0.1$, $A = [-5.4, 3.4] \times [-2.5, 2.5]$.
- Fig. 6: $b = \frac{5}{3}$, $a = 0.1$, $s = 0.1$, $A = [-5.4, 3.4] \times [-2.5, 2.5]$.
- Fig. 7: $b = \frac{5}{3}$, $a = 0.1$, $s = 0.1$, $A = [-4.8, 2.8] \times [-2.5, 2.5]$.
In second example the cubic complex polynomial $f(x) = x^3 - bx + c$ where $S(x) = bx$ and $T(x) = x^3 + c$ is analyze. We choose for each figure $K = 15$. In figures figure 19 to figure 26 the red boundaries are connected simple Julia sets, dark blue lines represented the correspondence of Julia points and white, yellow and light blue regions are fatou domains for $c = 0, 1, 1.2, 1.3$ respectively. The input parameters used in in Figs. 19–26 were the following:

- Fig. 19: $b = \frac{5}{3}, a = 0.5, s = 0.5, A = [-1.5, 1.5] \times [-2.5, 2.5]$,
- Fig. 20: $b = \frac{5}{3}, a = 0.5, s = 0.5, A = [-1.5, 1.5] \times [-2.5, 2.5]$,
- Fig. 21: $b = \frac{5}{3}, a = 0.9, s = 0.9, A = [-1.5, 1.5] \times [-1.5, 1.5]$,
- Fig. 22: $b = \frac{5}{3}, a = 0.9, s = 0.9, A = [-1.5, 1.5] \times [-1.5, 1.5]$,
- Fig. 23: $b = \frac{5}{3}, a = 0.9, s = 0.9, A = [-1.5, 1.5] \times [-1.5, 1.5]$,
- Fig. 24: $b = \frac{5}{3}, a = 0.9, s = 0.9, A = [-1.5, 1.5] \times [-1.5, 1.5]$,
- Fig. 25: $b = \frac{5}{3}, a = 0.9, s = 0.9, A = [-1.5, 1.5] \times [-1.5, 1.5]$,
- Fig. 26: $b = \frac{5}{3}, a = 0.9, s = 0.9, A = [-1.5, 1.5] \times [-1.5, 1.5]$.

**B. CUBIC JULIA SETS AND THEIR BOUNDARIES**

In second example the cubic complex polynomial $f(x) = x^3 - bx + c$ with the selection of maps as follows as: $T(x) = bx$ and $T(x) = x^3 + c$. The figures 27, 29 and 31 are filled Julia sets for $c = 0, 1, 2$ and $K = 20, 10, 20$, respectively. While the figures 28, 30 and 32 are boundaries of figures 27, 29 and 31 respectively for same input.
parameters and $K$. The white, light pink, yellow and light blue regions are Fatou domains in all six figures. The generation of filled Julia sets and their boundaries presented in Figs. 27–32 and the parameters used to generate them were the following:
V. CONCLUSIONS

We proved the converse theorem of results in [1] and studied the generalized Jungck Mann orbit (GJMO) in the generation of Julia sets. We developed algorithms for filled Julia sets and boundaries of Julia sets. For the generation of boundary of filled Julia set we added a condition $f(x_n) = \text{Min} \{ |\Re x_n|, |\Im x_n| \}$ in algorithm 2. We visualized the filled Julia sets and their boundaries in the form of some examples. We also discussed the internal structure of Julia set and showed the correspondence between the Julia points via lines. Furthermore, we graphically presented that images change when we changed complex parameter $b$ and $c$. We believe the results of this article will inspire those who are touched with fractal geometry. In future we will try to establish algorithms for all explicit and implicit iterations from fixed point theory.

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**DONG LI** was born in Zhengzhou, Henan, China, in 1971. He received the B.S. degree in physics from Henan University, in 1993, and the M.S. degree in computer application technology from the Huazhong University of Science and Technology, in 2007. Since 2004, he has been a Lecturer in information engineering with the Zhengzhou Railway Vocational & Technical College. His current research interests include pattern recognition and machine learning, intelligent information processing in large data environments, and data fusion technology.

**MUHAMMAD TANVEER** received the M.Sc. degree in mathematics from Government College University Faisalabad, Pakistan, in 2008, and the M.Phil. degree in mathematics from Lahore Leads University, Lahore, Pakistan, in 2014. He is currently pursuing the Ph.D. degree with The University of Lahore, Lahore. He has published over 30 research articles in different international journals. His current research interest includes fixed point results in fractal generation via different Jungck type iterations.

**WAQAS NAZEER** received the Ph.D. degree in mathematics from the Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan. He is currently an Assistant Professor with the University of Education, Lahore. He is also a Mathematician from Pakistan. He has published over 100 research articles in different international journals. His current research interest includes analysis and graph theory. He received the OutStanding Performance Award for the Ph.D. degree. During his studies, he was funded by the Higher Education Commission of Pakistan.

**XIAORUI GUO** was born in Zhengzhou, Henan, China, in 1975. She received the B.S. degree in business management from the Henan University of Economics and Law, in 2000. She has been an Assistant Professor with the Zhengzhou University of Light Industry, since 2015. She has authored more than 15 articles, and she holds one patent. Her current research interests include data mining techniques and massive data analysis.

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