Arrays and the octahedron recurrence

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1 Introduction

Recently, in [12, 9, 14] several interesting bijections have been constructed. In [12, 9] bijections relate special sets of discretely concave functions (hives) on triangular grids and the octahedron recurrence (OR) plays the main role for these bijections. Bijections in [14] relate special sets of Young tableaux and constructions of these bijections based on standard algorithms in this theory, jeu de taquen, Schutzenberger involution, tableaux switching etc.

The transparency of the octahedron recurrence is a undoubtable advantage, but a rationale why the OR provides natural bijections (and even bijections) was rather obscured.

In this paper we investigate these constructions from the third point of view, combinatorics of arrays, theory worked out by the authors in [4]. Arrays naturally related as well to functions on the lattice of integers as to Young tableaux, and have some advantages comparing to functions and tableaux. For example, Young tableaux are nothing but integer-valued $D$-tight arrays. In the tensor category of arrays, the bijections of associativity and commutativity arise naturally. We establish coincidence of these bijections with that defined in [12, 9, 14].

In order to relate different approaches and to reveal combinatorics of the octahedron recurrence, we, first, show that the octahedron recurrence agrees with discrete convexity and, second, we construct another bijection using the OR, the functional form of the RSK correspondence.

The paper is organized as follows: after a brief introduction to the octahedron recurrence and discrete concave functions, in Section 4 we state Theorem 1 on heredity of discrete concavity under propagation due to octahedron recurrence. In Section 5 we recall definitions and facts from theory of

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arrays, which are of use in this paper. In Theorem 2 we establish a relation between the OR and the operation of condensation of arrays. This might be seen as a functional form of the RSK correspondence (more precisely, modified RSK, see [4]). In Section 7 we recall the definitions of the associativity bijection for arrays and the associativity bijection from [12]. In Theorem 3 (Section 8) we state that these bijections coincide. More subtle constructions are used for proving Theorem 4, in which we establish coincidence of our bijection of commutativity with the functional commutativity bijection in [9] and with two fundamental symmetries of Pak and Vallexo ([14], Conjecture 1).

### 2 Octahedron recurrence

The main idea of the octahedron recurrence is rather transparent. Specifically, consider the octahedron

![Octahedron Diagram](https://example.com/octahedron.png)

Picture 1.

with the vertexes \(0, a, a', b, b'\) and \(1\). Let \(f\) be a real-valued function given at the points \(0, a, a', b, b'\). Then we can *propagate* \(f\) to the point \(1\) by the following rule

\[
f(1) = \max(f(a) + f(a'), f(b) + f(b')) - f(0).
\]

We refer to [15] for justification of this rule and its interesting appearances in combinatorics. Rather unexpectedly this related to flips in [7]. We want to point out a relation of this rule to concavity. Specifically, suppose \(f(b) + f(b') = \max(f(a) + f(a'), f(b) + f(b'))\). Then, we have \(f(0) + f(1) = f(b) + f(b')\). This means that the restriction of the function to the rhombus \(0, b, 1, b'\) coincides with the restriction of an affine function \(h\). Moreover,

\[
h(a) + h(a') = 2h((a + a')/2) = 2h((b + b')/2) = 2f((b + b')/2) = f(b) + f(b') \geq f(a) + f(a').
\]
We can choose $h$ such that $f(a) \leq h(a)$ and $f(a') \leq h(a')$ hold true. That means that the function $f$ is sub-affine on the octahedron, i.e. $f$ looks alike a concave function. Moreover, the rhombus $0, b, 1, b'$ is an affinity set of $f$.

In other words, we propagate the function $f$ to the point $1$ in order to get a concave (discretely) function on the octahedron, such that an affinity area (the convex hull of the affinity set) has to contain the vector $01$, a propagation vector.

Now, using this rule, which is called the octahedron recurrence (OR), we can propagate a function given at some domain to a large domain. Here is one of possible initial domains (see [15]). Let us consider the set $L$ of points $(n, i, j)$ with integers $i, j, n$, $n \geq 0$ and $n = i + j \pmod{2}$. Suppose a function $f$ is given at a subset of $L$ constituted from points of the form $(\cdot, \cdot, 0)$ and $(\cdot, \cdot, 1)$. Then using the octahedron recurrence with the propagation vector $(0, 0, 2)$ we can propagate the function to points of $L$ of the form $(\cdot, \cdot, 3)$ and so on to the whole $L$.

Of course, the initial data can be given at more sophisticated subsets, see [15] and [9].

We will display the stuff in a slightly different manner. Specifically, we consider the integer orthant $\mathbb{Z}_+^3$ (with coordinates $x, y, z$). The propagation vector is $(1, 0, 1)$, that is proportional to the vector $OD$ see Picture 2. On Picture 2 with $n = m$, we can see modular and non-modular flats: the modular flats are parallel to the faces of the tetrahedron $OEAB$, and the non-modular flats parallel to the face $OEDC$ and the plane passing through $OAB$.

We locate the initial data of functions on $Oxz$, $Oyz$ (typically equal 0 on $OEA$ and $OEDC$) and on the plane $z = y$ (more precisely at integer points of the rectangle $OABC$, and here are the main data).

1Sometimes it is convenient to draw pictures for the OR with different propagation vectors. In order to set the octahedron recurrence, we have to choose a unimodular set in the lattice $\mathbb{Z}^3$, say, $\{e_1, e_2, e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3\}$ and the propagation vector $e_3 - e_1 + e_2$, where $e_1, e_2, e_3$ is a basis in the lattice $\mathbb{Z}^3$. Then the primitive octahedron becomes the convex hull of the points $0, e_3 - e_1, e_2, e_3, e_2 - e_1, e_3 - e_1 + e_2$. An integer translation of a plane, spanned by a triple of vectors in the unimodular set, is a modular flat. A non-modular flats are parallel to planes spanned either by the pair $(e_3 - e_1, e_2)$ or $(e_2 - e_1, e_3)$. 

3
Due to the octahedron recurrence propagation we get a function on the prism $OEABDC$, and of our particular interest will be the resulting functions at the rectangle $EABD$ and at triangle $BCD$. For $n = m$, we will be also interested for functions at the tetrahedron $OBAE$ and the half-octahedron $OEDCB$.

In this set-up, the unit octahedron if of the form depicted at Picture 3.

Thus, a primitive propagation takes the following form: given values at the points $0$, $a$ and $b$ at the ground flour and two values at the points $a'$ and $b'$ at the first flour, due to the OR we get a value at the third point $1$ at the first flour. If points $0$, $b$, $b'$ and $1$ are located at the quadrant $Oxz$ (that is they have the $y$-th coordinate equals 0), we set the value in $1$ by the rule: $f(1) = f(b) + f(b') - f(0)$ (an instance of the octahedron recurrence for the case $f = -\infty$ for points outside the orhtant $\mathbb{Z}_+^3$).

We claim that functions, which we get as an output of the octahedron recurrence, inherit some concavity properties of input functions. The next two sections are devoted to this issue.

## 3 Discrete concave functions on 2D-grids

We consider functions on $\mathbb{Z}^2$ defined on finite sets of special form. We call such sets grids and they are specified as follows. A finite subset $T \subset \mathbb{Z}^2$ is a
grid if i) $T$ has no holes, i.e. $T = \text{co}(T) \cap \mathbb{Z}^2$, and ii) any edge of the convex hull $\text{co}(T)$ is parallel to one of the vectors $(1,0)$, $(0,1)$, $(1,1)$. (Obviously, a grid has a hexagonal shape, which might degenerate to a pentagon, a trapezoid, a parallelogram or a triangle.)

Let $f : T \to \mathbb{R}$ be a function on a grid $T$. A primitive triangle in $T$ is either a triple $x$, $x + (0,1)$ and $x + (1,1)$ of points of $T$, or a triple $x$, $x + (1,0)$, $x + (1,1)$. Convex hulls of these primitive triangles constitute a simplicial decomposition of $\text{co}(T)$ (if $T$ is not one-dimensional). We uniquely interpolate the function $f$ by affinity to the triangles on this decomposition of $\text{co}T$, and get a function $\tilde{f} : \text{co}(X) \to \mathbb{R}$.

**Definition.** A function $f$ on a grid $T$ is said to be discrete concave, if the interpolation $\tilde{f}$ is a concave function on $\text{co}(T)$.

We can reformulate discrete concavity of a function $f$ without using the interpolation $\tilde{f}$. Namely we have to require validity of three types of “rhombus” inequalities. Consider “primitive” rhombus in $T$ of the form

\[ \square \quad \square \quad \square \]

Then discrete concavity is equivalent to validity of three types of “rhombus” inequalities. The inequalities require that sum at two points of drawn diagonal is greater or equal to the sum at two points of non-drawn diagonal.

(i) $f(i,j) + f(i+1,j+1) \geq f(i+1,j) + f(i,j+1)$

(ii) $f(i,j+1) + f(i+1,j+1) \geq f(i+1,j+2) + f(i,j)$

(iii) $f(i+1,j) + f(i+1,j+1) \geq f(i,j) + f(i+2,j+1)$

Note, that if only the requirement (i) is valid, then a function is called supermodular. If a function is supermodular and the requirement (ii) is valid, then the function is discrete concave on every vertical strip of the unit length, and we call such functions vertically-strip concave ($VS$-concave). Analogously, if (i) and (iii) are valid, a function is called horizontally strip-concave ($HS$-concave).

Mostly, we will be interested in functions on the triangle grid with the vertexes $(0,0)$, $(0,n)$, $(n,n)$; denoted by $\Delta_n$. On the next picture we depicted the grid $\Delta_4$. 

Consider a discrete concave function $f$ on the grid $\Delta_n^2$ and consider its restriction to each side of the triangle: the left-hand side, the top of the triangle and the hypotenuse. Specifically, we orient these sides as depicted on the previous picture and consider increments of the function on each unit segment. Then, increments along the left-hand side constitute an $n$-tuple $\lambda(1) = f(0, 1) - f(0, 0), \lambda(2) = f(0, 2) - f(0, 1), \ldots, \lambda(n) = f(0, n) - f(0, n-1)$.

It is easy follows from the rhombus inequalities of the type (i) and (iii) that $\lambda(1) \geq \lambda(2) \geq \ldots \geq \lambda(n)$.

Analogously, we define $n$-tuple $\mu$ $\mu(i) = f(i, n) - f(i-1, n), i = 1, \ldots, n$ and $\nu$ $\nu(k) = f(k, k) - f(k-1, k-1), k = 1, \ldots, n$, which are also decreasing tuples. We call these $n$-tuples increments of the function $f$ on the corresponding sides of the triangle grid. Obviously, the increments are invariant under adding a constant to $f$. Therefore, we have to consider functions modulo adding a constant or to require $f(0, 0) = 0$.

Let us briefly say about main roles of discrete concave functions in combinatorics and representation theory. We let to denote $\text{DC}_n(\lambda, \mu, \nu)$ the set of discrete concave functions on the grid $\Delta_n$ with increments $\lambda, \mu, \nu$. This set is a polytope (probably empty) in the space of all functions on $\Delta_n$. If this polytope is non-empty, when the $n$-tuples $\lambda, \mu, \nu$ are decreasing and there holds $|\lambda| + |\mu| = |\nu|$. For $n > 2$, we need more relations in order to get a non-empty $\text{DC}_n(\lambda, \mu, \nu)$. The necessary and sufficient conditions for non-emptiness of $\text{DC}_n(\lambda, \mu, \nu)$ (so-called Horn inequalities) are in [11], see also [8, 10, 3]. Moreover, $\text{DC}_n(\lambda, \mu, \nu)$ is non-empty if and only if there exist Hermitian matrices $A$ and $B$, such that $A, B, A + B$ have spectra $\lambda, \mu, \nu$, respectively (a solution to the Horn problem).

We let to denote $\text{DC}_n(\lambda, \mu, \nu)$ the set of integer-valued discrete concave functions on the grid $\Delta_n$, of course the tuples $\lambda, \mu, \nu$ have to be integer-valued as well. The cardinality of this set coincides with the Littlewood-Richardson coefficient, the multiplicity of the irreducible representation $V_\nu$ (of $GL(n)$) in

\footnote{Such a discrete concave function was called a hive in [11, 12].}
the tensor product irreps $V_\lambda \otimes V_\mu$. In Section 7 we will be more specific on this issue.

4 Functions on 3D-grids

For the purpose of this section, it is convenient to consider the octahedron recurrence with the propagation vector $(-1, 1, 1)$ and locate the initial data at the qudrants $OZX$ (ground) and $OXY$ (front wall). The modular flats take the form $x = a, y = b, z = c$ and $x + y + z = d$, where $a, b, c, d \in \mathbb{Z}$. If we cut $\mathbb{R}^3$ by these planes, we get a decomposition of $\mathbb{R}^3$ into primitive tetrahedrons and octahedrons. All octahedrons are parallel, that is one can be obtained by an integer translation of another. Each octahedron has three diagonals parallel to vectors $(1, 1, -1), (1, -1, 1)$ and $(-1, 1, 1)$, respectively, and corresponding three pairs of antipodal vertexes.

The diagonal being parallel to the propagation vector $(-1, 1, 1)$, we call the mail diagonal. The OR leads us to the following notion.

**Definition.** A function $F : \mathbb{Z}^3 \to \mathbb{R} \cup \{-\infty\}$ is said to be polarized, if, for any primitive octahedron, sum of values of $F$ at the vertexes of the main diagonal is equal to the maximum of the sum of values of $F$ at the antipodal vertexes of two others diagonals.

We denote $\Delta_n(OXYZ)$ the three-dimensional grid, constituted from the non-negative integer points $(x, y, z)$, such that $x + y + z \leq n$. It is easy to see that, for any initial data given at the ground $\Delta_n(OXY)$ and the front wall $\Delta_n(OZX)$, there exists a unique polarized function with domain $\Delta_n(OXYZ)$ and these given values. This is done by the OR. However, we can set initial data at the shadow wall $\Delta_n(OYZ)$ and the slope wall $\Delta_n(XYZ)$ and get a polarized function. In that case, we have to apply the OR with the reverse propagation vector $(1, -1, -1)$. On the next Picture we depicted the tetrahedron $co\Delta_n(OXYZ)$ with the direction of the OR propagation.
The fundamental property of the octahedron recurrence is that if the initial data (at the ground and the front wall) are discrete concave function, then the corresponding polarized function on the grid $\Delta_n(OXYZ)$ is a kind of three-dimensional discrete concave function. Without going in details of discrete concave functions in $\mathbb{Z}^n$, we give notions appropriate for this paper.

Discrete concavity on $2\text{D}$-grids is equivalent to fulfill three kinds of rhombus inequalities. In dimension 3, we have four kinds of modular flats. In each such a 2-dimensional flat we have rhombuses, which corresponds to triangular decomposition of the flat by cutting it by three others kinds of modular flats. We have to require validity of rhombus inequality for each such a rhombus: the sum of values at the “short” diagonal is greater or equal to the sum at the “long” diagonal.

**Definition.** A function $F : \mathbb{Z}^3 \to \mathbb{R} \cup \{-\infty\}$ is a polarized discrete concave function if $F$ is polarized and all kinds of rhombus inequalities in each modular flat are fulfilled.

Let us denote by $PDC_n$ the set of polarized discrete concave functions on the three-dimensional grid $\Delta_n(OXYZ)$.

**Theorem 1.** Let $F$ be a polarized function on the three-dimensional grid $\Delta_n(OXYZ)$. Suppose the restriction of $F$ to the ground face $\Delta_n(OXY)$ and to the front wall face $\Delta_n(OXZ)$ are 2-dimensional discrete concave functions. Then $F \in PDC_n$.

For a proof see [6]. Note, that this theorem is equivalent to the following corollary (a sketch of proof of which is also in [9]).
Corollary 1. If the restrictions of a polarized function to the ground and the front wall faces are discrete concave, then the restriction to the shadow wall and the slope wall are also discrete concave.

Proof. In fact, any rhombus located on the slope or shadow wall is also a rhombus for three-dimensional grid, and therefore, the corresponding rhombus inequality is valid. □

Corollary 2. Let the restriction of a polarized function to the ground be discrete concave and the restriction to the front wall be HS-concave. Then the restrictions to two other faces are HS-concave.

Proof. In fact, we can add to $F$ an appropriate function $\varphi(z)$ of the vertical variable $z$, in order to get a discrete concave function on the front wall. $F + \varphi(z)$ is not changed on the ground, therefore by Corollary 1, $F + \varphi(z)$ is discrete concave at the other two wall, therefore $F$ is HS-concave on these walls. □

Corollary 3. Suppose the restriction to the ground of a polarized function is HS-concave and the restriction to the front wall is VS-concave (here we consider horizontal being parallel to the segment $XY$). Then $F$ is VS-concave on the shadow wall.

Proof. As above, having add to $F$ an appropriate separable function on variables $x$ and $y$, we get a polarized function $G = F + \varphi(x) + \psi(y)$, which will be discrete concave on the ground and the front wall. By Corollary 1, $G$ is discrete concave on the shadow wall. Therefore, $F$ is VS-concave on this wall. □

Corollary 4. Suppose a polarized function $F$ is discrete concave on the ground and VS-concave on the front wall. Then $F$ is discrete concave on the shadow wall.

Proof. In fact, having add an appropriate function on $x$ to $F$, we get a discrete concave function on the front wall. On the ground this function will be also discrete concave. But this function remains the same on the shadow wall, and by Corollary 1 the function on this wall is discrete concave. □

Now let us consider the polarized functions (or the octahedron recurrence) on the prism $\Delta_n(OXY) \times \{0, 1, ..., m\}$ (see next Picture).
We have to set functions equals $-\infty$ at points outside the prism. Therefore, on the non-modular face $\Delta_n(XY) \times \{0,1,...,m\}$, a polarized function $F$ has to be a separable function (on variables $x+y$ and $z$). In other words, for any “primitive” quadrat on this wall, the sum of values at the opposite pairs of vertexes coincide.

**Corollary 5** Let $F$ be a polarized function on the prism $\Delta_n(OXY) \times \{0,1,...,m\}$. Suppose the restriction of $F$ to the ground face $\Delta_n(OXY) \times \{0\}$ and the restriction to the front wall $\Delta_n(OX) \times \{0,1,...,m\}$ are discrete concave functions. Then $F$ is polarized discrete concave function on the prism (and, in particular, $F$ is discrete concave on the shadow wall $\Delta_n(OY) \times \{0,1,...,m\}$ and on the ceiling $\Delta_n(OXY) \times \{m\}$).

**Proof.** It is easy to see that it suffices to prove the corollary in the case $m = 2$.

In the beginning we consider the case $m = 1$. Let us extend the ground to the size of $n+1$, that is we add to $\Delta_n(OXY)$ new points $(n+1,0,0),..., (0,n+1,0)$. Let us extend $F$ to these points such that we get a discrete concave function on the extended ground $\Delta_{n+1}(OXY) \times \{0\}$ and a discrete concave function on the “extended” front wall. We can always do that by setting small values ($<<0$) to these points. Let us denote $\tilde{F}$ such an extension. By Theorem 1, the function $\tilde{F}$ is a polarized discrete concave function. We claim, that the restriction of this function to the prism is a polarized discrete concave function. In fact, it suffices to check that $\tilde{F}$ coincides with $F$ on the non-modular face $\Delta_n(XY) \times \{0,1\}$. But this holds since we assigned small
values to the new points. Thus $F$ and $\tilde{F}$ coincide on the prism. Since $\tilde{F}$ is discrete concave function, $F$ is discrete concave too.

Let us move to the case $m = 2$. We have to check all rhombus inequalities for all rhombuses in the prism $\Delta_n(OXY) \times \{0, 2\}$. Let us first consider the rhombuses of the vertical size 2. It is easy to see that these rhombuses belong to the tetrahedron of size $n + 1$. Then the corresponding rhombus inequality is valid, since they are valid for $\tilde{F}$.

Other rhombuses are located either in the prism $\Delta_n(OXY) \times \{0, 1\}$, or in the prism $\Delta_n(OXY) \times \{1, 2\}$. For the first prism, the corresponding inequality follows due to the above case with $m = 1$. Moreover, we get that $F$ is discrete concave on the triangle $\Delta_n(OXY) \times \{1\}$.

Now, again applying the case $m = 1$ to the prism $\Delta_n(OXY) \times \{1, 2\}$, we get validity of rhombus inequalities in this prism. $\square$

Using similar reasonings one can get the following

Corollary 6 Suppose a polarized function $F$ on the prism $\Delta_n(OXY) \times \{0, 1, ..., m\}$ has discrete concave restrictions to the ceiling $\Delta_n(OXY) \times \{m\}$ and the shadow wall $\Delta_n(OY) \times \{0, 1, ..., m\}$. Then $F$ is a polarized discrete concave function and its restrictions to the front wall $\Delta_n(OX) \times \{0, 1, ..., m'\}$ and the ground $\Delta_n(OXY) \times \{0\}$ are discrete concave functions.

5 Arrays

In this section, we introduce another key player of a game – arrays. Consider a rectangle $[0, n] \times [0, m]$ on the plane with natural $n$ and $m$, constituted from unit squares with the centers at the points $(i - 1/2, j - 1/2)$, $i = 1, ..., n$, $j = 1, ..., m$, we call such squares boxes. An array is a filling of each box $(i, j)$ with a non-negative “mass” $a(i, j)$.

To each array $a$ we associate a function $f = f_a$ on the rectangular grid $\{0, 1, ..., n\} \times \{0, 1, ..., m\}$ by setting to the point $(i, j)$ the value

$$ f_a(i, j) = \sum_{i' \leq i, j' \leq j} a(i', j'). $$

In other words, this value is equal to the mass of all boxes to the south-west from the point $(i, j)$. This is a reason to denote by $\iint a$ the function $f_a$. On the bottom and the left boundary of the rectangle the function equals 0. For other $(i, j)$, we obviously have

$$ f(i, j) - f(i - 1, j) - f(i, j - 1) + f(i - 1, j - 1) = a(i, j). $$

From this $a(i, j)$ might be understand as the mixed derivative of $f$ ($a = \partial \partial f$), or as a break of $f$ along the common edge $[(i - 1, j - 1), (i, j)]$ of two affinity areas. Since $a(i, j) \geq 0$, the function $\iint a$ is supermodular.
Here we collect some notions and results on arrays which are of use in the paper (for details see [4]).

1. For each $j = 1, \ldots, m - 1$, there is an operation $D_j$ on the set of arrays, which acts on a given array by moving down (vertically) an amount of mass ($\leq 1$) from a box $(i, j)$ to $(i, j - 1)$ (for a definition such a $i$ see [4] (see also [5])). For any $a$, starting from some power $\epsilon_j(a)$, there holds $D_j^{\epsilon_j(a)+1}(a) = D_j^{\epsilon_j(a)}(a)$; we denote $D_j = D_j^{+\infty}$, that is $D_j(a) = D_j^{\epsilon_j(a)}(a)$.

2. If, for an array $a$, there holds $D_j(a) = a$ (or $D_j a = a$), then $a$ is called $D_j$-tight. Equivalently, this means that the function $f = \iint a$ satisfies the inequalities
   
   \[ f(i - 1, j) + f(i, j) - f(i - 1, j - 1) - f(i, j + 1) \geq 0 \]

   for all $i = 1, \ldots, n$. In other words, the rhombus inequalities of type (ii) (see Section 3) hold true for the rhombuses which are cut by the line $y = j$.

3. If an array $a$ is $D_j$-tight for all $j = 1, \ldots, m - 1$, then $a$ is said to be $D$-tight. For a $D$-tight array $a$, the corresponding function $f_a = \iint a$ is a $VS$-concave function.

   It is clear that we can condense any array to a $D$-tight (for example, by applying $(D_1 \ldots D_{m-1})^m$, but this is only one of ways). Moreover, for each $a$, such a $D$-tight array is defined uniquely and we let to denote it by $Da$. Since, descending massed due to the operations $D_j$, does not change the vector of column sums (masses), the values of the functions $\iint Da$ and $\iint a$ coincide at the top boundary of the rectangle, i.e. at the points with $y = m$. At the right boundary, i.e., for $x = n$, the values are different (for non $D$-tight arrays). The increments of the function $\iint Da$ along the right side we let to denote by $\lambda_1, \ldots, \lambda_m$. Due to $VS$-concavity of the function $\iint Da$, we have the inequalities $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$. This m-tuple $\lambda$ we call $D$-shape of $a$.

   Obviously, for a $D$-tight array $a$, the $D$-shape of $a$ coincides with the vector of its row sums.

   Integer-valued $D$-tight arrays are in a natural bijection with semistandard Young tableaux. For details see [4], and here we explain this bijection by an example.

**Example.** Consider the following $D$-tight $4 \times 3$ array

\[
\begin{pmatrix}
0 & 0 & 0 & 3 \\
0 & 4 & 0 & 4 \\
5 & 1 & 2 & 4 \\
\end{pmatrix}
\]
To get the corresponding semi-standard Young tableaux we have to read this array from left to right and from bottom to top. Reading a row gives us filling of the corresponding row in the Young tableau, the mass $a(i, j)$ exhibits the multiplicity of repetitions of the letters $i$ in the $j$-th row of the Young tableaux (we consider the French style of drawing Young diagrams and tableaux, that is the Young diagram for a partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ is a collection of boxes in the grid $\mathbb{N} \times \mathbb{N}$ with north-east corners $(i, j)$ such that $j \leq n, i \leq \lambda_j$, and a Young tableau is a filling of the diagram from some alphabet increasingly along each row (from left to right) and strictly increasing from bottom to top). Thus, for the above array, we get

\[
\begin{array}{cccccccc}
4 & 4 & 4 \\
2 & 2 & 2 & 4 & 4 & 4 & 4 \\
1 & 1 & 1 & 1 & 2 & 3 & 4 & 4 & 4 \\
\end{array}
\]

4. Using the transposition (with respect to the diagonal $a^T(i, j) = a(j, i)$) we define the operations $L_i L_i$ (which translate masses to the left along a row), $L_i(a) = (D_i(a^T))^T$. Using the operations $L_i$, $i = 1, \ldots, n - 1$, we can condense to the left any array $a$, and get $L$-tight array $La$. The function $\mathcal{L}a$ is an $HS$-concave function. In particular, the increments of this function along the top side is a decreasing tuple, the column sum of the array $La$, that is the $L$-shape of $a$.

5. The key claim of theory of arrays is that the operations $L_i$ and $D_j$ commute for any $i, j$ ([4], Theorem 4.2). Here we present some consequences of this commutation property

a) For any array $a$, the $D$-shape of $a$ coincides with the $L$-shape of $a$, and, thus, this tuple is the shape of $a$.

b) The bijection theorem (or modified $RSK$ correspondence, [4], Theorem 6.2): suppose we are given a $D$-tight array $d$ and an $L$-tight array $l$, such that $d$ and $l$ have the same shape, then there exists a unique array $a$, such that $d = Da$ and $l = La$ hold true.

Our next task is to obtain this modified $RSK$ bijection using the octahedron recurrence.
6 Functional form of RSK

Let us turn back to Picture 2 and locate the zero function at the face $OEA$; at the slope rectangle $OABC$ we locate the function $f_a = \iint a$. Specifically, we assign the value $\iint a(i,j)$ to the point $(i,j)$. Now, we propagate these data by the octahedron recurrence to the prism. From Corollary 5 (Section 4), we get a VS-concave function at the top face rectangle $EABD$ and HS-concave function at the right face triangle $CDB$ (the vertical is $y$-axe in the first case and $z$-axe in the second case). Moreover, we get the function $\iint Da$ at the tope face and the function $\iint La$ at the right face of the prism (specifically, the restriction of this function to this triangle).

Namely, we state

**Theorem 2.** Let $F$ denote a function on the prism obtained by the octahedron recurrence from the following initial data: the zero values at the faces $OEDC$ and $OEA$, and $\iint a$ at $OABC$. Then $F(i,j,m) = (\iint Da)(i,j)$ for all $i,j$, and $F(n,j,k) = (\iint La)(j,k)$ for $0 \leq j \leq k \leq m$.

Let us note, that the latter two functions coincide at the edge $DB$. That is $(\iint Da)(n,j) = (\iint La)(j,m)$ for all $j$. In fact, this is $\int$ from the shape of $a$.

**Remark.** Of course, we can consider a propagation in the reverse direction. Specifically, assume we are given a function $f$ on the top face $EABD$ and a function $g$ on the triangle $CDB$. Suppose there hold

a) $f$ is VS-concave and equals 0 at the edges $EA$ and $ED$;

b) $g$ is HS-concave and equals 0 at the edge $CD$;

b) the functions $f$ and $g$ coincide at the edge $DB$.

Then having apply the OR (with the propagation vector $(-1,0,-1)$) for these data, we get a pair of functions on the triangle $OEA$ and the slope rectangle $OABC$. Due to Corollary 6, we get a discrete concave function on $OEA$ and a supermodular function on $OABC$. Moreover, we get the identically zero function on the triangle $OEA$. This is because this function equals 0 at the edge $EA$ (from the item a)) and at the edge $OE$ (this follows from b) and separability of the OR on the non-modular face). But these boundary values force nullity of the discrete concave function.

Now, we get an array $a$ as the mixed derivatives of the supermodular function on $OABC$. It is clear that $f = \iint a$ and $g = \iint a$. Thus, this octahedron recurrence provides us with a functional form of the modified RSK (item 5 of Section 5). An advantage of this form of RSK is that the direct and inverse bijections are done symmetrically.

**Proof of Theorem 2.** The main case of the proof is the case of an array with two rows, that is $m = 2$. We denote the masses in the bottom row by...
a(i, 1), and in the top row by a(i, 2), i = 1, \ldots, n. Therefore, for i = 1, \ldots, n, we have f(i, 0) = 0,
\begin{align*}
f(i, 1) &= a(1, 1) + \ldots + a(i, 1), \\
f(i, 2) &= f(i, 1) + a(1, 2) + \ldots + a(i, 2).
\end{align*}
This follows from the definition of $\iint a$. Denote $a' = Da$ and $f' = \iint a'$. Then $f'$ coincides with $f$ for $j = 0$ (both functions = 0) and for $j = 2$. Differences could occur only at points with $j = 1$, because some mass moves from the first level to the ground level. We get in [4], formula (∗∗∗), the following $f'(i, 1) = f(i, 1) + \max(\beta_1, \ldots, \beta_i)$, where
\begin{align*}
\beta_i &= a(1, 2) + \ldots + a(i, 2) - a(1, 1) - \ldots - a(i - 1, 1) = \\
f(i, 2) - f(i, 1) - f(i - 1, 1) + f(i - 1, 0).
\end{align*}
Now, taking into account $f(i, 2) = f'(i, 2)$, and
\begin{align*}
\max(\beta_1, \ldots, \beta_i) &= \max(\max(\beta_1, \ldots, \beta_{i-1}), \beta_i),
\end{align*}
and
\begin{align*}
\max(\beta_1, \ldots, \beta_{i-1}) &= f'(i - 1, 1) - f(i - 1, 1),
\end{align*}
we obtain
\begin{align*}
f'(i, 1) &= \\
f(i, 1) + \max[f'(i - 1, 1) - f(i - 1, 1), f'(i, 2) - f(i, 1) - f(i - 1, 1) + f(i - 1, 0)] = \\
&= \max[f'(i - 1, 1) + f(i, 1), f'(i, 2) + f(i - 1, 0)] - f(i - 1, 1).
\end{align*}
That is the octahedron recurrence indeed.

We claim that in the prism $OEACDB$ at the height $z = k$ is located the function $\iint D(a_k)$, where the array $a_k$ is obtained from $a$ by omitting the rows with $j = k + 1, \ldots, m$. In fact, suppose this claim is true for some $k$, and let us check it for $k + 1$. So, we are given a $D$-tight array $d_k = D(a_k)$, and the row $a(\cdot, k+1)$ from the array $a$. From the above case with $m = 2$ follows that the octahedron recurrence lifting, from the height $k$ to the height $k+1$ in the prism, corresponds to the product of the condensation operations $D_1 \ldots D_k$. The formula 6.5 in [4] demonstrates exactly this claim.

Thus at the top face, we get the function $\iint D(a)$.

Now, we have to show that at the right side face we get the function $\iint L(a)$. In fact, from the above claim, on the segment $\{(n, 0, k), \ldots, (n, k, k)\}$, we have the function $\iint d_k = \iint D(a_k)$. That is the integral $\int$ of the shape of the array $d_k$, or, equivalently, the integral of the shape of the array $a_k$ (see
But the shape of $a_k$ equals the shape of $L(a_k) = L(a)$. For any L-tight array, its shape coincides with the column sum vector. This completes the proof. □

Let us illustrate this theorem by an example. Consider the following array

$$a = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}.$$  

The corresponding supermodular function $f_a = \begin{bmatrix} 4 & 3 & 7 \\ 3 & 2 & 5 \\ 18 & 13 & 6 \end{bmatrix}$ is located at the face $OABC$; the values of the polarized function $F$ are depicted on the next Picture.

The values of the function $F$ at the integer points of top face $EABD$ are

$$4 \quad 10 \quad 18$$  
$$4 \quad 10 \quad 17$$  
$$4 \quad 7 \quad 11$$

and the corresponding SSYT is

$$\begin{bmatrix} 3 \\ 2 \quad 2 \quad 3 \quad 3 \quad 3 \\ 1 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 3 \quad 3 \quad 3 \quad 3 \end{bmatrix}.$$

The values of $F$ at the face $CBD$ are

$$18 \quad 13 \quad 17$$  
$$6 \quad 8 \quad 11$$

and the corresponding
SSYT is

\[
\begin{array}{cccccc}
3 & 2 & 2 & 2 & 2 & 3 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3
\end{array}
\]

Now we give consequences of this theorem for the associativity and commutativity bijections.

## 7 Associativity bijection

Let us briefly recall the matter. Pick an integer \( n \geq 1 \), let to denote by small Greek letters \( \lambda, \mu, \nu \) etc. partitions with \( n \)-parts (that is an \( n \)-tuple \( \lambda = (\lambda_1, ..., \lambda_n) \) of integers such that \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0 \)). To each partition \( \lambda \) is associated the Schur function \( s_\lambda \) (see [13][14]). While \( \lambda \) runs over the set of all \( n \)-partitions, the Schur functions constitute an additive basis of the ring of symmetric functions on \( n \) variables. Therefore, the product \( s_\lambda s_\mu \) of the Schur functions can be presented of the form

\[
s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.
\]

The structure constants \( c^\nu_{\lambda,\mu} \) are called the \textit{Littlewood-Richardson coefficients}. Since the ring of symmetric functions is commutative and associative, the Littlewood-Richardson coefficients satisfy the commutativity

\[
c^\nu_{\lambda,\mu} = c^\nu_{\mu,\lambda}
\]

and associativity

\[
\sum_\sigma c^\sigma_{\lambda,\mu} c^\tau_{\sigma,\nu} = \sum_\tau c^\tau_{\mu,\nu} c^\sigma_{\lambda,\tau}.
\]

Combinatorics learns us to seek for numbers underlying finite sets, and for equalities look for natural bijections between the corresponding sets. Littlewood and Richardson were the first who give a combinatorial rule for computing LR-coefficients as the cardinality of a set of special semistandard skew Young tableaux (for example, from this, follows that these coefficients are non-negative). Recently ([1][2][3][12]), these special semistandard Young tableaux have been identified with integer-valued discrete concave functions. Here we will give one more interpretation of \( c^\nu_{\lambda,\mu} \) as the cardinality of the set of standard pairs of arrays \( SP_Z(\lambda, \mu, \nu) \). In the array language, the commutativity means equal cardinality of the sets \( SP_Z(\lambda, \mu, \nu) \) and \( SP_Z(\mu, \lambda, \nu) \). Moreover, we construct a natural bijection between these sets. Regarding
the associativity, we will provide a natural bijection between the following sets
\[ \prod_\sigma (SP_\mathbb{Z}(\lambda, \mu, \sigma) \times SP_\mathbb{Z}(\sigma, \nu, \pi)) \]
and
\[ \prod_\tau (SP_\mathbb{Z}(\mu, \nu, \tau) \times SP_\mathbb{Z}(\lambda, \tau, \pi)). \]

The associativity bijection in [12] was constructed in terms of hives (discrete concave functions) and the octahedron recurrence played the main role. However, despite on an elegance of the construction, a reason why it is natural (and even why it is a bijection) was obscured. Recall that this bijection, which we will call the functional associativity bijection provides a bijection between sets
\[ \prod_\sigma (DC_\mathbb{Z}^n(\lambda, \mu, \sigma) \times DC_\mathbb{Z}^n(\sigma, \nu, \pi)) \text{ and } \prod_\tau (DC_\mathbb{Z}^n(\mu, \nu, \tau) \times DC_\mathbb{Z}^n(\lambda, \tau, \pi)), \]
and it takes the following form. We have to locate a pair of discrete concave functions \((f,g)\) from the first set at the faces \(OEA\) and \(OAB\), respectively, of the tetrahedron \(OEAB\) (see Picture 2. \(m = n\)). Then, due to the OR, we get a pair of discrete concave functions \((p,q)\) on the other two faces from the second set (due to Theorem 1). The mapping \((f,g) \to (p,q)\) provides the functional associativity bijection.

Using a relation between arrays and functions, and a natural associativity bijection in terms of arrays, we provide a justification for this construction. Namely, we prove that the associativity bijection in the arrays terms coincides with this functional bijection. Analogously, for the case of the commutativity bijection, we prove that the commutativity bijection for arrays coincides with commutativity bijection in \([9]\) and with two fundamental symmetries due to Pak and Valleilo ([14], Conjecture 1).

Here we consider square arrays of size \(n \times n\). Let us pick an array \(a\), and consider the collection of arrays of the form \(Ta\), where \(T\) is an arbitrary word in non-commutative variables \(D_j\) and \(U_j\), \(j = 1, \ldots, n-1\). This set constitute the orbit \(O(a)\) of this array under action of the semi-group, spanned by \(D_j\) and \(U_j\), \(j = 1, \ldots, n-1\). In particular, \(Da \in O(a)\). It is not difficult to check (see [4]), that each orbit contains a unique \(D\)-tight array among its elements. Thus, to set an orbit is equivalent to pick a \(D\)-tight array.

Furthermore, if we take two \(D\)-tight arrays \(d\) and \(d' = R_i d\) with some \(i\) (or a pair of \(D\)-tight arrays which belong to an orbit under the action of the operations \(R_i\) and \(L_i\), \(i = 1, \ldots, n-1\)), then the orbits, corresponding to these arrays are isomorphic (in some sense). Thus, if we are interested in orbits by
modulo isomorphisms, we can consider the orbit with simultaneously $D$-tight and $L$-tight array $d$ (or $D$-tight and $R$-tight, which is useful sometimes). Such a bi-tight array is a partition $\lambda$ indeed. Specifically, for an $n$-tuple partition $\lambda = (\lambda_1 \geq \ldots)$, we denote by $\text{diag}(\lambda)$ an array which contains the mass $\lambda_i$ at the diagonal box $(i, i)$, $i = 1, \ldots, n$, and zero masses at all other boxes. This array $\text{diag}(\lambda)$ is a bi-tight array, and any bi-tight array takes such a form.

We let to call such an orbit $O(\lambda) = O(\text{diag}(\lambda))$ the standard orbit of shape $\lambda$.

(Let us note, that irreducible (and polynomial) representations of $GL(n)$ are also indexed by $n$-tuple partitions. Moreover, the dimension of such a representation $V_\lambda$ coincides with the cardinality of the orbit $O(\lambda)$. Therefore, one can imagine the orbit $O(\lambda)$ as a skeleton of the irreducible representation. Decomposition of an invariant set of arrays (under the $DU$-action) into orbits corresponds to decomposition of a representation into irreducibles. Moreover such a decomposition of any invariant set is multiplicity-free.)

Let us consider two standard orbits $O(\lambda)$ and $O(\mu)$. We can form their tensor product $O(\lambda) \otimes O(\mu)$. As a set it is constituted of concatenations of arrays $a \otimes b$, $a \in O(\lambda)$, $b \in O(\mu)$, (that is an array of size $2n \times n$; the first $n$ columns come from $a$, and than come columns of $b$). Since $\lambda$ and $\mu$ are $L$-tight arrays, the arrays $a$ and $b$ are also $L$-tight (commuting $L$-operations and $U$-operations). The decomposition of $O(\lambda) \otimes O(\mu)$ into orbits consists in distinguishing $D$-tight arrays of the form $a \otimes b$. Thus, we come to the following

**Definition**. A standard pair is a pair of arrays $(a, b)$ such that there holds 1) the arrays $a$ and $b$ are $L$-tight, and 2) the array $a \otimes b$ is $D$-tight. The shape of $a$ is the starting shape of the pair, the shape of $b$ is the intermediate shape of the pair, and the shape of the array $a \otimes b$, that is the vector of its row sums, is the final shape.

We let to denote $SP(\lambda, \mu, \nu)$ the set of standard pair with the starting shape $\lambda$, intermediate shape $\mu$ and the final shape $\nu$. The subset of integer valued standard pair we mark with $\mathbb{Z}$.

Thus, the set of orbits in $O(\lambda) \otimes O(\mu)$, which are isomorphic to $O(\nu)$, is identified to the set $SP_{\mathbb{Z}}(\lambda, \mu, \nu)$. In [4], 12.4, we established that $c^\nu_{\lambda, \mu}$ is equal to the cardinality of the finite set $SP_{\mathbb{Z}}(\lambda, \mu, \nu)$.

Analogously, we can decompose the triple product $O(\lambda) \otimes O(\mu) \otimes O(\nu)$ into the disjoint union of orbits. An orbit of this decomposition is identified to a standard triple $(a, b, c)$ of arrays, such that $a$, $b$ and $c$ are $L$-tight arrays (of shapes $\lambda$, $\mu$ and $\nu$, respectively), and the concatenated array $a \otimes b \otimes c$ is $D$-tight.
To a standard triple we can correspond a couple of standard pairs in
two different ways, respectively to parenthesizes $O(\lambda) \otimes O(\mu) \otimes O(\nu)$. The
parenthesizes $(O(\lambda) \otimes O(\mu)) \otimes O(\nu)$, provides us with the pair $(a, b)$ and
the pair $(L(a \otimes b), c)$. Let us note, that the final shape of the first pair
coincides with the starting shape of the second pair. Another parenthesizes
$O(\lambda) \otimes (O(\mu) \otimes O(\nu))$ yields the pair $(a, L(b \otimes c))$ and the pair $D(b, c)$ (the
latter is a $(b', c')$, such that there holds $b' \otimes c' = D(b \otimes c)$). It is not difficult
to check (see, for example [4]) that these pairs are standard indee d. (For
example, because $b$ and $c$ are $L$-tight, and since $D$ and $L_i$ commute, we
get that $b'$ and $c'$ are $L$-tight.) Let us also note that, in the latter case,
the intermediate shape of the first pair coincides with the fin al shape of the
second pair. We call such couples of standard pairs compatible couples.

From the bijection theorem (see above the item 5b) follows that such a compatible
couple determines a standard triple $(a, b, c)$. In fact, assume a compatible
couple of standard pairs $(a, l)$ and $(b', c')$ is given, and, in particular, there
holds $Dl = L(b', c')$. According to the bijection theorem there exists a pair
of arrays $b$ and $c$, such that $l = L(b \otimes c)$ and $b' \otimes c' = D(b \otimes c)$. Thus, we
get a natural associativity bijection for arrays

$$
\prod_{\sigma}(SP(\lambda, \mu, \sigma) \times SP(\sigma, \nu, \pi)) \sim \prod_{\tau}(SP(\mu, \nu, \tau) \times SP(\lambda, \tau, \pi)),
$$

$((a, b), (L(a \otimes b), c)) \mapsto ((a, L(b \otimes c)), D(b, c))$.

Let us note, that in this construction the integer-validness of $\lambda$, $\mu$, $\nu$, .... and
the arrays do not play any role, all is correct for arbitrary arrays.

8 Comparing of two associativity bijections

In the beginning, we present a natural bijection between the sets $SP(\lambda, \mu, \nu)$
and $DC_n(\lambda, \mu, \nu)$, where $n$-tuples $\lambda, \mu, \nu$ are partitions. Namely, to a pair of
array $(a, b)$ we assign the restriction of the function $\int\int (a \otimes b)$ to the grid
$n \leq i \leq n + j, 0 \leq j \leq n$.

Proposition 2. The above defined assignment provides a bijection be-tween sets $SP(\lambda, \mu, \nu)$ and $DC(\lambda, \mu, \nu)$. This assignment sends integer arrays
into integer-valued functions and vice versa.

Proof. Let $(a, b)$ be a standard pair. Since the array $a \otimes b$ is $D$-tight, the
function $\int\int (a \otimes b)$ is $VS$-concave on the rectangle $2n \times n$. Since the array $b$ is
$L$-tight, this function is $HS$-concave on the square $[n, 2n] \times [0, n]$. Therefore,
on this square, the function $\int\int a \otimes b$ is discrete concave. Furthermore, the
increments of this function on the left side of the square coincide with the

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vector of row sums of the array $a$. Since, $a$ is a $D$-tight array, these increments equal to the shape of $a$, that is $\lambda$. The increments on the top side equal the column sums of $b$. Since $b$ is a $L$-tight array, these sums equal the shape of $b$, that is $\mu$. Finally, increments on the right side ("hypotenuse") coincide with the row sums of the array $a \otimes b$, that is $\nu$, because $a \otimes b$ is $D$-tight. Thus, we get indeed a function in $DC(\lambda, \mu, \nu)$.

Vice versa, suppose we are given a function $f \in DC(\lambda, \mu, \nu)$. Then we set the array $b$ as the array of mixed derivatives $\partial \partial f$. The array $a$ is just $\text{diag}(\lambda)$. It is clear that we obtain a standard pair of type $(\lambda, \mu, \nu)$. It is also clear that the above constructions are invertible. □

We claim that, in the course of this bijection between standard pairs and discrete concave functions, the associativity bijection for arrays coincides with the functional associativity bijections [12]. For it suffices to clarify the construction of the second couple of standard pairs in terms of discrete concave functions (for the first pair it is clear).

For the pair of arrays $(b, c)$, we consider the corresponding function $\int \int (b \otimes c)$ on the rectangle $2n \times n$. We locate this function on the slope rectangular face $OABC$ (see Picture 5) of the prism (with $b = (2n, n, n)$) and apply the octahedron recurrence (with the propagation vector $(1, 0, 1)$ and zero values at the faces $OEA$ and $OEDC$, as we did in Section 2). Due to the functional form of RSK (Theorem 2), we obtain the function $\int \int D(b \otimes c)$ on top face $EABD$ and the function $\int \int L(b \otimes c)$ on the right triangle $CDB$.

Thus, we have to recognize the functions which we get on the triangles (grids) $C'D'B'$, $D'B'B$, $B'BC'$ and $BC'D'$ (these triangles are the faces of the tetrahedron $C'D'B'B$). Recall that the functional associativity bijection is obtained via the octahedron recurrence for this tetrahedron [12] (with the same propagation vector and modular flats).

![Picture 5.](image-url)
1. By Theorem 2, on the first face $C'D'B'$ we obtain $\int\int$ of the array $Lb = b$.

2. On the second face $D'B'B$ we get the restriction of the function $\int\int (b' \otimes c')$ to this triangle (or the square $D'B'BD$). Recall that we denoted by $b' \otimes c'$ the standard pair $D(b \otimes c)$. Thus, by modulo adding the function, $\int\int c'$ is located on this face.

3. On the third (slope) face $B'BC'$ of the tetrahedron is located the restriction of function $\int\int (b \otimes c)$ to this triangle. It is easy to see (as in the item 2), that this restriction is equal to $\int\int c$ plus the function of one variable $\int$ (row sums of $b$). For what follows it is worth to note that the row sums of $b$ equals the row sums of $a \otimes b$ minus the row sums of $a$, or, equivalently, the shape of $L(a, b)$ minus the shape of $a$.

4. The only non-trivial function is located on the forth face $BC'D'$ of the tetrahedron. Namely, we claim that the function on the face $BC'D'$ is “almost” coincided with the function on the face $BCD$, that is the value of the function at the $(n+i, i, k)$ is equal to the value of the function $\int\int L(b \otimes c)$ in the point $(i, k)$.

Let us postpone proving this claim, and conclude that we get finally. On the faces 1 and 3 we have the initial data, the function $\int\int b$ on the face 1 and the function $\int\int c$ (modulo adding a function of one variable) on the face 3. At the output faces, that is faces 2 and 4, we have $\int\int c' + \int b'$ and $\int\int L(b \otimes c)$. In order to get the exact case of [12], we have slightly modify our functions in order to get discrete concave functions on the faces 1 and 3. For this, we have to add to the polarized function on the prism $OEACDB$ the “one-dimensional” function of the z-coordinate (vertical axe) equals $\int a$, or, equivalently, $\int \lambda$. Thus, we get discrete concave function $\int \lambda + \int\int b$ on the face 1 with increments $\lambda, \mu$ and $\sigma$, where $\sigma$ denotes the shape of the array $a \otimes b$. On the face 3, we get the function $\int\int c + (\int l - \int a) + \int a = \int\int c + \int l$. Since $l = L(a \otimes b)$, there holds $\int l = \int \sigma$. Therefore, on the face 3 is located a discrete concave function with the increments $\sigma, \nu$ and $\pi$, where $\pi$ denotes the shape of the array $a \otimes b \otimes c$. These are the input functions. The output functions are: on the face 2 is located the function $\int\int c' + \int b' = \int\int c' + \int \mu$, since the shape of $b' = Db$ is $\mu$. That is a discrete concave function with increments $\mu, \nu, \tau$, where $\tau$ denotes the shape of the array $b \otimes c$. On the face 4 is located the function $\int\int L(b \otimes c) + \int \lambda$, discrete concave function corresponding to the standard pair $(a, L(b \otimes c))$, with the increments $\lambda, \tau$.

Finally, we note that adding the function $\int \lambda$ of the vertical variable does not affect on the octahedron recurrence, and therefore, we get the coincidence of the array associativity bijection and that is in [12] (the functional form).
To prove the claim, we observe that after adding the function of the vertical variable $\int \lambda$, the polarized function on the tetrahedron $C'D'DC'B$ becomes discrete concave (Corollary 5). Moreover, it is a constant on each segment in the ground of the octahedron, which is parallel to the axe $x$. Therefore, the function is constant on each segment parallel to the axe $x$.

Thus, we have prove the following proposition.

**Theorem 3.** Under the above isomorphism between $SP$ and $DC$, the associativity bijection for array (Section 7) coincides with the functional associativity bijection constructed in [12] using the OR.

## 9 Commutativity bijection

Now we turn to the commutativity bijection. Namely, we claim existence of a natural commutativity bijection between the sets $SP(\lambda, \mu, \nu)$ and $SP(\mu, \lambda, \nu)$. Here, as usual, $n$-tuples $\lambda, \mu, \nu$ denote partitions (not necessary integer-valued). Thus, we have to associate to a given standard pair $(a, b)$, of type $(\lambda, \mu, \nu)$, a standard pair $(b', a')$ of type $(\mu, \lambda, \nu)$. In [12], we proposed such a bijection, and called it the **commuter**.

Here is convenient to use *anti-standard pairs* $(a, b)$, that is an $\mathbf{R}$-tight array $a$ and an $\mathbf{L}$-tight array $b$ such that $a \otimes b$ is a $\mathbf{D}$-tight array. The type of such an array is defined similarly as for standard pair, the starting shape equals the shape of $a$, the intermediate shape equals the shape of $b$, and the final shape is the shape of $a \otimes b$.

For example, the following concatenated array $(a, b)$

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 2 & 1 \\
0 & 0 & 2 & 1 & 2 & 0 \\
0 & 2 & 1 & 3 & 0 & 0 \\
\end{array}
\]

is an anti-standard array of the type $\lambda = (3, 2, 0), \mu = (5, 4, 1), \nu = (6, 5, 4)$.

The set of anti-standard of type $(\lambda, \mu, \nu)$ we let to denote by $ASP(\lambda, \mu, \nu)$. There is a canonical bijection

\[ SP(\lambda, \mu, \nu) \rightarrow ASP(\lambda, \mu, \nu), \quad (a, b) \mapsto (Ra, b); \]

(and the reverse mapping is $(a, b) \mapsto (La, b)$).

Now, we define the commutativity bijection, commuter,

\[ Com : ASP(\lambda, \mu, \nu) \rightarrow ASP(\mu, \lambda, \nu). \]
Let \((a, b)\) be an anti-standard pair. We define the commuter from the rule
\[
Com(a, b) = D(*(a, b)) = D(*b, *a).
\]
(Here and in what follows, \(*\) denotes the central symmetry of an array, \(*a(i, j) = a(n - i + 1, m - j + 1)\).) It is easy to check that the pair \((b', a') = D(*b, *a)\) is an anti-standard of the required type. For example, consider \(b' = D(*b)\). Since \(b\) is a \(L\)-tight, that is \(b = Lb\), we get that \(*b = R(*b)\) is \(R\)-tight. Due to commuting \(D\) and \(R\), the array \(b'\) is \(R\)-tight. Its shape is equal to the shape of \(*b\) and is equal to the shape of \(b\), that is nothing but \(\mu\). Similarly, one can check that we get the correct type.

**Example.** Let \((a, b)\) be a standard pair from the previous example. The inverse assay (with respect to \(*\)) \(*a \otimes b = *b \otimes *a\) takes the form
\[
\begin{bmatrix}
0 & 0 & 3 & 1 & 2 & 0 \\
0 & 2 & 1 & 2 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]
It is easy to check that \(D\)-condensation of this array equals
\[
\begin{bmatrix}
0 & 0 & 1 & 1 & 2 & 0 \\
0 & 1 & 3 & 1 & 0 & 0 \\
1 & 3 & 1 & 1 & 0 & 0
\end{bmatrix}.
\]
It is easy to check that if we centrally symmetrically reverse the array and then condensate it, we get the initial array
\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 2 & 1 \\
0 & 0 & 2 & 1 & 2 & 0 \\
0 & 2 & 1 & 3 & 0 & 0
\end{bmatrix}.
\]
Of course, this coincidence has a reason and there holds

**Lemma.** The commuter \(Com\) is an involution.

That is the composition
\[
ASP(\lambda, \mu, \nu) \xrightarrow{Com} ASP(\mu, \lambda, \nu) \xrightarrow{Com} ASP(\lambda, \mu, \nu)
\]
is the identical mapping.

**Proof.** In fact, let \((b', a') = Com(a, b) = D(*a \otimes b)\). Then \(Com(b', a') = D(*b' \otimes a') = D*D(*a \otimes b) = D(U(a \otimes b))\). According to [4], Section 5.10, the latter equals \(D(a \otimes b)\) and this equals to \(a \otimes b\), since \(a \otimes b\) is \(D\)-tight. \(\square\)
The corresponding mapping $SP(\lambda, \mu, \nu) \to SP(\mu, \lambda, \nu)$ we also denote by $Com$. We claim that, in integer-valued case, this mapping $Com$ coincides with the mapping, which is called the first fundamental symmetry in [14] is denoted by $\rho_1$. To clarify this claim, we have to translate the stuff from the array language in the language of Young tableaux, because the mapping $\rho_1$ is defined in [14] in this language. (Note that this language forces to restrict ourselves to integer arrays and partitions.) For this translation we use the following natural bijection

$$SP_Z(\lambda, \mu, \nu) \to LR(\nu \setminus \lambda, \mu),$$

where $LR(\nu \setminus \lambda, \mu)$ denotes the set of Littlewood-Richardson tableaux of the skew shape $\nu \setminus \lambda$ and weight $\mu$ ([13] [14]).

Let $(a, b)$ be a standard pair of integer-valued arrays from $SP_Z(\lambda, \mu, \nu)$. Since the array $a \otimes b$ is $D$-tight, there is a corresponding semistandard Young tableau of shape $\nu$ (see [1] and end of the item 4 in Section 5). Moreover, the $D$-tight array $a$ defines a sub-tableau of shape $\lambda$. The complement to the latter tableau gives a semistandard skew Young tableau of shape $\nu \setminus \lambda$ and weight $\mu$ (=shape of $b$), and, finally, due to $L$-tightness of the array $b$, this filling gives a reverse lattice (or dominated or Yamanuchi) word (see [1], (9.6)). Let us note, that the skew tableau is filled from the alphabet $n + 1, \ldots, 2n$, and, in order to get a tableau, filled from $1, \ldots, n$, we have to subtract $n$ from each letter of the tableau.

The reverse mapping is as follows. Let we have a tableau from $LR(\nu \setminus \lambda, \mu)$ filled from the alphabet $1, \ldots, n$. Then we add $n$ to each letter of the tableau and fill the “empty part” of shape $\lambda$ as the Yamanuchi tableau (i.e. the $j$-th row is constituted only of the letters $j$, $j = 1, \ldots, m$). Thus, we get a semistandard tableau of shape $\nu$, and the corresponding $D$-tight array $a \otimes b$. The array $a$ correspond to the Yamanuchi tableau and, therefore, is $L$-tight. Since the word $w(b)$ is a reverse lattice word, the array $b$ is $L$-tight.

For example, for the pair $(a, b)$ from the above example (specifically, for the standard pair $(L(a), b)$) we obtain the following tableau

$$\begin{array}{cccccc}
4 & 5 & 5 & 6 \\
2 & 2 & 4 & 5 & 5 \\
1 & 1 & 4 & 4 & 4 & 4
\end{array}$$

One can check that the word $w(b) = 4556455444$ is a Yamanuchi word indeed (in the alphabet $\{4, 5, 6\}$).

Recall that in [14] the first fundamental symmetry is defined using the tableaux-switching algorithm. In this example, we have to transport the
letters 1, 2 and 3 through the letters 4, 5, 6. As a result, we get a tableau with the letters order: 4 < 5 < 6 < 1 < 2 < 3).

\[
\begin{array}{cccc}
6 & 1 & 2 & 2 \\
5 & 5 & 5 & 5 \\
4 & 4 & 4 & 4 \\
\end{array}
\]

This tableau corresponds to the pair \((b', a')\) of the above-mentioned example, and this is not a curious. Specifically, we claim that there holds the following

**Proposition 3.** The above defined bijection between \(SP_Z(\lambda, \mu, \nu)\) and \(LR(\nu \setminus \lambda, \mu)\) makes the following diagram commutative (that is coincidence \(Com\) and \(\rho_1\))

\[
\begin{array}{ccc}
SP_Z(\lambda, \mu, \nu) & \rightarrow & LR(\nu \setminus \lambda, \mu) \\
Com \downarrow & & \rho_1 \downarrow \\
SP_Z(\mu, \lambda, \nu) & \rightarrow & LR(\nu \setminus \mu, \lambda)
\end{array}
\]

In fact, the tableaux-switching might be obtained ([14], 3.1) of the form of the composition of three Schützenberger involutions \(S_1S_12S_1\). That is, for a standard pair \((a, b)\), we, first, apply \(S\) to the array \(a\), then we apply \(S\) to the array \(Sa \otimes b\), and finally, we apply \(S\) to the first \(n \times n\) part of the array \(S(Sa \otimes b)\).

According to [4], the array \(Sa\) is equal to \(D(*a)\). Since \(a\) is bi-tight array (\(D-\) and \(L-\)tight), the array \(*a\) is \(R\)-tight. Therefore \(D(*a)\) is a (unique) \(D\) and \(R\)-tight array of shape \(\lambda\). But the array \(Ra\) is also \(D\)- and \(R\)-tight of shape \(\lambda\). Thus, we get \(Sa = D(*a) = Ra\), and the application of \(S_1\) is nothing but sending a standard pair to the antistandard one.

Now, \(S = S_{12}(Ra \otimes b)\) is equal to \(D(* (Ra \otimes b))\), and finally, applying \(S_1\) sends the antistandard pair \(D(* (Ra \otimes b))\) to a standard one. This composition does exactly that \(Com\) does. □

### 10 Functional form of the commutativity bijection

Now we translate the commutativity bijection into the language of functions. Recall (see Section 8) that the set \(SP(\lambda, \mu, \nu)\) (and also \(ASP(\lambda, \mu, \nu)\)) is bijective to the set \(DC(\lambda, \mu, \nu)\) of discrete concave functions on the triangle grid with the increments \(\lambda, \mu, \nu\) (increments along the left-hand side constitute an \(n\)-tuple \(\lambda\), along the top of the triangle constitute \(\mu\), and \(\nu\) along the hypotenuse).
Using these bijections and the bijection

\[ \text{Com} : ASP(\lambda, \mu, \nu) \to ASP(\mu, \lambda, \nu), \]

we obtain (from the commutative diagram) a bijection \( \text{Com}' \) between the set \( DC(\lambda, \mu, \nu) \) and \( DC(\mu, \lambda, \nu) \). Here we transform this definition of \( \text{Com}' \) in more direct and transparent form.

Let \( f \in DC(\lambda, \mu, \nu) \). Consider the corresponding \( L \)-tight array \( b = \partial \partial f \); the vector of its column sums (\( I \)-weight) is equal to \( \mu \). The vector of row sums (\( J \)-weight) is equal to \( \nu - \lambda \). Let us pick the \( DR \)-tight array \( a = R(\text{diag}(\lambda)) \) of shape \( \lambda \). The pair \( (a, b) \) is anti-standard and corresponds to \( f \).

The commuter \( \text{Com} \) sends the anti-standard pair \( (a, b) \) to the anti-standard pair \( (b', a') = D(*b, *a) \). (Of course, we know that \( b' \) is the \( DR \)-tight array of shape \( \mu \), and we know that the vector of column sums of \( a' \) is equal to that of \( *a \), and that is indeed \( \lambda \). In fact, the vector of column sums of \( a \) is equal to \( \lambda^{\text{op}} = (\lambda_n, \ldots, \lambda_1) \), therefore, that of \( *a \) is equal to \( \lambda \).) Now, from this pair we have to return to a discrete concave function, and, by the definition, we get

\[ \text{Com}'(f) = \int \int a' + \int \mu. \]

To understand better the pair \( (b', a') \) we exploit Theorem 2. Pick the supermodular function \( g = \int \int (*b \otimes *a) \), and locate it on the slope face \( OABC \) (see Picture 6), and then apply the octahedron recurrence in the prism as in Section 6. Then on the top face \( EABD \) we get the function which corresponds to \( (b', a') \). Specifically, the function \( \text{Com}'(f) \) is located on the triangle \( D'B'B \) (the increments along the left side \( D'B' \) constitute \( \mu \), \( \lambda \) along the top side \( B'B \), and \( \nu \) along the hypotenuse \( D'B \)).

Now we show that the function \( g = \int \int (*b \otimes *a) \) is very simple related to the function \( f \). Specifically, we explain a relation with the function \( \int \int (a \otimes b) \)
on the rectangle $2n \times n$. Since the restriction of this function to the triangle $C'B'B$ is exactly $f$, we also denote the function $\int\int (a \otimes b)$ by $f$. Its boundary increments we depicted on Picture 7.

\[
\begin{array}{c|c|c}
\lambda^{op} & f & \nu \\
\hline
0 & 0 & 0
\end{array}
\]

Picture 7. Function $f$.

If we “turn over” the function $f$, that is if we consider the function $*f$, given by the rule $(*f)(i, j) = f(2n - i, n - j)$, then we will almost get the function $g$. More precisely, $*f$ and $g$ have the same mixed derivatives $\partial \partial$, but are differ in the boundary increments and have different values at $(0, 0)$ (see the next picture).

\[
\begin{array}{c|c|c}
0 & 0 & 0 \\
\hline
-\nu^{op} & *f & -\lambda \\
-\mu^{op} & -\lambda
\end{array}
\]

Picture 8. Function $*f$.

Changing of boundary values (with preserving mixed derivatives) can be made by adding an appropriate separable function (of variables $x$ and $z = y$). Doing this, we obtain

\[
g = *f + \int_x (\mu^{op}, \lambda) + \int_z \nu^{op} - f(0, 0) = *f + \int_x (\mu^{op}, \lambda) + \int_z \nu^{op} - |\nu|.
\]

Here $(\mu^{op}, \lambda)$ denotes the tuple $(\mu_n, ..., \mu_1, \lambda_1, ..., \lambda_n)$.

\[
\begin{array}{c|c|c}
\mu^{op} & g & \nu^{op} \\
\hline
0 & 0 & 0
\end{array}
\]

Picture 9. Function $g$.

For what follows it is worth to note that the function $g$ is constant on the segments $[(n + j, j), (2n, j)]$, because the array $*a$ takes zero values at these segments.
Thus, let us conclude this part with recalling what we have done. We locate the function $g$ on the face $OABC$ of the prism, using the OR, we obtain a polarized function $G$ on the prism. The restriction of $G$ to the triangle $D'B'B$ is the function of our interest $Com'(f)$. Now, we are interested in the restriction of the function $G$ to the vertical cross connected wall $C'D'B'$. By Theorem 2, we know that this function is $\iint L(*b)$ with boundary increments $0$, $\mu$ and $\nu^{op} - \lambda^{op} = (\nu - \lambda)^{op}$. Let us add to this function the “one-dimensional” function $\int z (-\nu^{op})$. Thus, the function

$$h = \iint L(*b) - \int z \nu^{op} + |\nu|.$$  

is discrete concave and has increments $-\nu^{op}, \mu$ and $-\lambda^{op}$.

![Picture 10. Function $h$.](image)

The inter-relation between the function $h$ and the function $Com'(f)$ is established in the following proposition.

**Proposition 4.** 1) The octahedron recurrence on the quarter of the octahedron $OD'B'C'$ (the Henriques-Kamnitzer construction) sends the function $*f$ (on the face $OB'C'$) to the function $h$, that is $h = HK(*f)$ up to adding a constant.

2) $h$ almost coincides with $Com'(f)$. Specifically, if we consider $Com'(f)$ as a function on the triangle $D'B'B$, then there holds $h(i, j) = Com'(f)(n - j, n - j + i, n)$.

In other words, in order to get $Com'(f)$, we have to rotate the function $h$ “counter-clockwise on 120°”, see Picture 11.

![Picture 11.](image)

**Corollary.** In the language of discrete concave functions the commuter $Com'$ coincides with the OR commuter due to Henriques and Kamnitzer.
In remark 4.5 in [9] is noted, that in order to get the HK-commuter, one has to apply the OR on the quarter of the octahedron, and then to rotate the picture counter-clockwise on 120° as in Picture 11.

Let us illustrate this one the example from the beginning of this section. It is easy to see, that the corresponding function \( g \) (which we locate on the slope face of the prism) takes the form

\[
\begin{array}{cccccccc}
0 & 1 & 5 & 10 & 13 & 15 & 15 \\
0 & 1 & 5 & 7 & 9 & 9 & 9 \\
0 & 1 & 3 & 4 & 4 & 4 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

The restriction of the function \( G \) to the tope face is equal to the following function

\[
\begin{array}{cccccccc}
0 & 1 & 5 & 10 & 13 & 15 & 15 \\
0 & 1 & 5 & 9 & 11 & 11 & 11 \\
0 & 1 & 4 & 5 & 6 & 6 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

and, in particular, we get \( \text{Com}^\prime(f) \)

\[
\begin{array}{cccc}
10 & 13 & 15 & 15 \\
9 & 11 & 11 \\
5 & 6 \\
0 \\
\end{array}
\]

The following function is equal to the restriction of \( G \) to the triangle \( C'D'B' \):

\[
\begin{array}{cccc}
0 & 5 & 9 & 10 \\
0 & 5 & 7 \\
0 & 4 \\
0 \\
\end{array}
\]

And after adding to this function \( z \)-function \(- \int \nu + |\nu|\), we get the function \( h \)

\[
\begin{array}{cccc}
0 & 5 & 9 & 10 \\
6 & 11 & 13 \\
11 & 15 \\
15 \\
\end{array}
\]

From these computations one can see that \( h \) coincides with \( \text{Com}^\prime(f) \) after 120° counter-clockwise rotation.

Proof of Proposition 4. 1) By Theorem 2, we get the function \( \int \int L(*b) \) (on \( C'D'B' \)) from the restriction of \( g \) to \( OAB'C' \) and the OR. Since the OR commutes with adding (and subtracting) of separable functions of variables
x and z), we subtract from g the separable function \( \int_x (\mu^{op}, \lambda) + \int_z \nu^{op} - |\nu| \). Then on OAB'C' we get the function \(*f\). And we get the function h (up to constant) on the wall C'D'B', since we have to subtract \( \int_z \nu^{op} \) from \( \iint L(\ast b) \).

Thus on the quarter of the octahedron, the octahedron recurrence sends \(*f\) into h.

2) Adding a function of the z-variable to the function G does not affect on the top face EABD. Therefore, we add the function \( -\int_z \nu^{op} + |\nu| \) to the function G.

Let us consider the restriction of the function \( G - \int_z \nu^{op} + |\nu| \) to the tetrahedron C'D'B'B. It is clear that this is a polarized function. Thereupon, its restriction to the face C'D'B', being the function \( \iint L(\ast b) - \int_z \nu^{op} + |\nu| = h \), is a discrete concave function. In fact, by modulo of adding an affine function, this function corresponds to the standard pair \((\text{diag}(-\nu^{op}), L(\ast b))\), and, thus, is discrete concave. Finally, the restriction to the face C'B'B is also discrete concave, and, moreover, it is constantly equal \(|\nu|\) on the edge C'B. Let us verify this claim.

The restriction of g to the square C'B'BC is equal to the function \( \iint \ast a \) plus the restriction of g to the edge C'B', that is \( \iint (\nu^{op} - \lambda^{op}) \). Of our interest is the function \( g - \int_z \nu^{op} \), that is the function \( \iint \ast a - \int_z \lambda^{op} \). Now we have honestly deduce the claim from the LU-tightness of \( \ast a \) (DR-tightness of a).

Note that it suffices to check the claim for \( \lambda \) of the form \((1,\ldots,1,0,\ldots,0) = (1^k,0^{n-k})\). In this case, one can find values of the function \( \iint \ast a - \int_z \lambda^{op} \) at a point \((i,j)\), that is \( \min(k+i-j,0) \). From this discrete concavity is obvious. On the edge C'B, we have \( i = j \), and our function is equal to \( \min(k,0) = 0 \), since \( k \geq 0 \).

Thus, the polarized function \( G - \int_z \nu^{op} + |\nu| \) has discrete concave restrictions to the faces C'D'B' and C'B'B of the tetrahedron C'D'B'B. By Theorem 1 \( G \) is polarized discrete concave function. Since \( G \) is a constant on the edge C', \( G \) is a constant function on any segment, parallel to \((1,1,1)\), of the tetrahedron. In particular, the values in the points \((n,i,j)\) and \((n,i,j) + (n-j)(1,1,1) = (2n-j, n+i-j, n)\) coincide. But we claimed exactly that in the item 2. □

11 Commuter and the second fundamental symmetry

Here we prove that, for the integer-valued set-up, the commuter \( \text{Com} \) coincides with the second fundamental symmetry due to Pak and Vallexo. This bijection (we will consider \( \rho_2' \)) is defined as the following composition (for
array in the previous Section! Recall that the array \( \text{diag}(-b) \) makes transposition of \( SL \) that a tableau of \( D \) (see [14]).

By Proposition 4, this function is easily recalculated from the function \( T \). Here \( T \) the arrays \( \partial \partial \partial \) and \( \partial \partial \partial \) means of \( \partial \partial \partial \partial \partial \partial \partial \partial \partial \partial \partial \). Thus, we get

\[
\tilde{b}(i, j) = h(i, j) - h(i - 1, j) - h(i, j - 1) + h(i - 1, j - 1).
\]

Recalling the relation between \( h \) and \( \tilde{h} \), we get

\[
\tilde{b}(i, j) = \tilde{h}(n-j, n-j+i) - \tilde{h}(n-j, n-j+i+1) - \tilde{h}(n-j+1, n-j+i+1) + \tilde{h}(n-j+1, n-j+i).
\]

The first two summands in the above expression give

\[
\mu_{n-j+i} + a'(1, n-j+i) + a'(2, n-j+i) + ... + a'(n-j, n-j+i).
\]

Analogously, one can find the difference of the third and the forth summands. Thus, we get

\[
\tilde{b}(i, j) = [\mu_{n-j+i} + a'(1, n-j+i) + a'(2, n-j+i) + ... + a'(n-j, n-j+i)] - [\mu_{n-j+i+1} + a'(1, n-j+i+1) + a'(2, n-j+i+1) + ... + a'(n-j+1, n-j+i+1)].
\]

This is exactly the definition of the mapping \( \gamma \) in [14]. Thus, we get that, under bijection between \( LR(\nu \setminus \lambda, \mu) \) and \( SP_Z(\lambda, \mu, \nu) \), the commuter \( \text{Com} \) coincides with the mapping \( \rho'_2 \). Due to Lemma the commuter \( \text{Com} \) is an involution, that is it coincides with its reversion. In [14] is shown that \( \rho'_2 \) is the reversion to \( \rho_2 \). Thus \( \rho_2 = \rho'_2 \) and we have proved the following theorem.
Theorem 4. The commuter Com (defined in terms of arrays) coincides with the Henriques-Kamnitzer commuter (defined in terms of discrete concave functions), and coincides with the Pak-Vallejo fundamental symmetries $\rho_1$, $\rho_2$ and $\rho'_2$ (defined in terms of Young tableaux).

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