THE STEKLOV AND LAPLACIAN SPECTRA OF RIEMANNIAN
MANIFOLDS WITH BOUNDARY

BRUNO COLBOIS, ALEXANDRE GIROUARD, AND ASMA HASSANNEZHAD

Abstract. Given two compact Riemannian manifolds with boundary $M_1$ and $M_2$ such that their respective boundaries $\Sigma_1$ and $\Sigma_2$ admit neighborhoods $\Omega_1$ and $\Omega_2$ which are isometric, we prove the existence of a constant $C$, which depends only on the geometry of $\Omega_1 \cong \Omega_2$, such that $|\sigma_k(M_1) - \sigma_k(M_2)| \leq C$ for each $k \in \mathbb{N}$. This follows from a quantitative relationship between the Steklov eigenvalues $\sigma_k$ of a compact Riemannian manifold $M$ and the eigenvalues $\lambda_k$ of the Laplacian on its boundary. Our main result states that the difference $|\sigma_k - \sqrt{\lambda_k}|$ is bounded above by a constant which depends on the geometry of $M$ only in a neighborhood of its boundary. The proofs are based on the Rellich-Pohozaev identity and on comparison geometry for principal curvatures. In several situations, the constant $C$ is given explicitly in terms of bounds on the geometry of $\Omega_1 \cong \Omega_2$.

1. Introduction

Let $M$ be a smooth compact Riemannian manifold of dimension $n + 1 \geq 2$, with nonempty boundary $\Sigma$. The Steklov eigenvalue problem on $M$ is to find all numbers $\sigma \in \mathbb{R}$ for which there exists a nonzero function $u \in C^\infty(M)$ which satisfies

$$\begin{cases}
\Delta u = 0 & \text{in } M, \\
\frac{\partial u}{\partial n} = \sigma u & \text{on } \Sigma.
\end{cases}$$

Here $n$ is the outward unit normal along $\Sigma$ and $\Delta = \Delta_M$ is the Laplace operator defined from the Riemannian structure of $M$. The Steklov problem has a discrete unbounded spectrum

$$0 = \sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \cdots \nearrow +\infty,$$

where each eigenvalue is repeated according to its multiplicity. For background on this problem, see \cite{11, 16} and references therein.

Key words and phrases. Steklov eigenvalues, Rellich-Pohozaev identity, comparison geometry, Riccati equation, Dirichlet-to-Neumann map.
1.1. The Dirichlet-to-Neumann map and spectral asymptotics. Let \( H : C^\infty(\Sigma) \to C^\infty(M) \) be the harmonic extension operator: the function \( u = Hf \) satisfies \( u = f \) on \( \Sigma \) and \( \Delta u = 0 \) in \( M \). The Steklov eigenvalues of \( M \) are the eigenvalues of the Dirichlet-to-Neumann (DtN) map \( D : C^\infty(\Sigma) \to C^\infty(\Sigma) \), which is defined by

\[
Df = \frac{\partial}{\partial n} Hf.
\]

The DtN map is a first order elliptic pseudodifferential operator \(^{21, p. 37-38}\). Its principal symbol is given by \( p(x, \xi) = |\xi| \), while that of the Laplace operator \( \Delta_\Sigma : C^\infty(\Sigma) \to C^\infty(\Sigma) \) is \( |\xi|^2 \). Standard elliptic theory \(^{12, 20}\) then implies that

\[
\sigma_k \sim \sqrt{\lambda_k} \sim 2\pi \left( \frac{k}{\omega_n |\Sigma|} \right)^{1/n} \quad \text{as } k \to \infty,
\]

where \( \omega_n \) is the volume of the unit ball \( B(0, 1) \subset \mathbb{R}^n \). It follows that manifolds \( M_1 \) and \( M_2 \) which have isometric boundaries satisfy \( \sigma_k(M_1) \sim \sigma_k(M_2) \) as \( k \to \infty \).

It was also proved in \(^{17}\) that the full symbol of the DtN map is determined by the Taylor series of the Riemannian metric of \( M \) along its boundary \( \Sigma \) (see also \(^{19}\)). Consider a closed Riemannian manifold \( (\Sigma, g_\Sigma) \) and two compact Riemannian manifolds \( (M_1, g_1) \) and \( (M_2, g_2) \) with the same boundary \( \Sigma = \partial M_1 = \partial M_2 \) such that \( g_1|_\Sigma = g_2|_\Sigma = g_\Sigma \). If the metrics \( g_1 \) and \( g_2 \) have the same Taylor series on \( \Sigma \), then \( M_1 \) and \( M_2 \) have asymptotically equivalent\(^1\) Steklov spectra:

\[
\sigma_k(M_1) = \sigma_k(M_2) + O(k^{-\infty}) \quad \text{as } k \to \infty.
\]

See \(^{10}\) Lemma 2.1. In particular, \( \lim_{k \to \infty} \sigma_k(M_1, g_1) - \sigma_k(M_2, g_2) = 0 \). Nevertheless, the asymptotic behavior described by \(^1\) or \(^2\) does not contain any information regarding an individual eigenvalue \( \sigma_k \). In order to obtain such information, stronger hypothesis are needed. Indeed on any smooth compact Riemannian manifold \( (M, g_0) \) with boundary, there exists a family of Riemannian metrics \( (g_\epsilon)_{\epsilon \in \mathbb{R}_+} \) such that \( g_\epsilon = g_0 \) on a neighborhood \( \Omega_\epsilon \) of \( \Sigma = \partial M \), while for each \( k \in \mathbb{N} \),

\[
\lim_{\epsilon \to \infty} \sigma_k(M, g_\epsilon) = 0.
\]

See \(^1\) and \(^5\) for two such constructions. If \( n \geq 2 \), there also exists a family of Riemannian metrics such that \( g_\epsilon = g_0 \) on a neighborhood \( \Omega_\epsilon \) of \( \Sigma \) and

\[
\lim_{\epsilon \to \infty} \sigma_2(M, g_\epsilon) = +\infty.
\]

\(^1\)The notation \( O(k^{-\infty}) \) designates a quantity which tends to zero faster than any power of \( k \).
See \cite{1} for the construction of these families. In this last construction, the neighborhoods $\Omega_\epsilon$ are shrinking to $\Sigma$ as $\epsilon \to \infty$:

$$
\bigcap_{\epsilon \in (0, \infty)} \Omega_\epsilon = \Sigma.
$$

However, if the manifolds $M_1$ and $M_2$ are uniformly isometric near their boundary the situation is completely different.

**Theorem 1.** Given two compact Riemannian manifolds with boundary $M_1$ and $M_2$ such that their respective boundaries $\Sigma_1$ and $\Sigma_2$ admit neighborhoods $\Omega_1$ and $\Omega_2$ which are isometric, there exists a constant $C$ such that $|\sigma_k(M_1) - \sigma_k(M_2)| \leq C$ for each $k \in \mathbb{N}$. The constant $C$ only depends on the geometry of $\Omega_1 \cong \Omega_2$.

This is a manifestation of the principle stating that Steklov eigenvalues are mostly sensitive to the geometry of a manifold near its boundary. From the above discussion, the following picture emerges: the more information we have on the metric $g$ on, and near, the boundary $\Sigma$, the more we can say about the Steklov spectrum of $M$. As a first step, knowing the metric on $\Sigma$ leads to the asymptotic formula (1), then knowing its Taylor series leads to the refined asymptotic formula (2) and finally the knowledge of the metric on a neighborhood of the boundary provides the extra information needed to bound individual eigenvalues in Theorem 1.

**Remark 2.** Let $b$ be the number of connected components of the boundary $\Sigma$. Under the hypothesis of Theorem 1, it is well known that for $k \geq b$ the following holds:

$$
C^{-1} \leq \frac{\sigma_k(M_1)}{\sigma_k(M_2)} \leq C.
$$

This was used for instance in \cite{1, 6}. Theorem 1 is stronger since it implies

$$
\frac{1}{1 + \frac{C}{\sigma_k(M_1)}} \leq \frac{\sigma_k(M_1)}{\sigma_k(M_2)} \leq 1 + \frac{C}{\sigma_k(M_2)}.
$$

It is also known that $\sigma_k(M_i) \geq \sigma_k^N \geq \sigma_k^{N_b}$, where $\sigma_k^N$ refer to the mixed Neumann-Steklov problem on $\Omega_1 \cong \Omega_2$. See \cite{1} for details.

### 1.2. Main result.

Theorem 1 follows from the main result of this paper, which is a quantitative comparison between the Steklov eigenvalues of $M$ and the eigenvalues of the Laplace operator $\Delta_\Sigma$ on its boundary $\Sigma$, which are denoted $0 = \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$. Let $n \in \mathbb{N}$ and let $\alpha, \beta, \kappa_-, \kappa_+, h \in \mathbb{R}$ be such that $\alpha \leq \beta$ and $\kappa_- \leq \kappa_+$. Consider the class $\mathcal{M} = \mathcal{M}(n, \alpha, \beta, \kappa_-, \kappa_+, h)$ of smooth compact Riemannian manifolds of dimension $n$ with nonempty boundary $\Sigma$, which satisfy the following hypothesis:
(H1) The rolling radius\(^2\) of \(M\) satisfies \(\bar{h} := \text{roll}(M) \geq h\).

(H2) The sectional curvature \(K\) satisfies \(\alpha \leq K \leq \beta\) on the tubular neighborhood 
\[ M_{\bar{h}} = \{ x \in M : d(x, \Sigma) < \bar{h} \} . \]

(H3) The principal curvatures of the boundary \(\Sigma\) satisfies \(\kappa_- \leq \kappa_i \leq \kappa_+\).

The main result of this paper is the following.

**Theorem 3.** There exist explicit constants \(A = A(n, \alpha, \beta, \kappa_-, \kappa_+, h)\) and \(B = B(n, \alpha, \beta, \kappa_-, \kappa_+, h)\) such that each manifold \(M\) in the class \(\mathcal{M}\) satisfies the following inequalities for each \(k \in \mathbb{N}\),

\[ \lambda_k \leq \sigma_k^2 + A \sigma_k, \]

\[ \sigma_k \leq \frac{B}{2} + \sqrt{\left( \frac{B}{2} \right)^2 + \lambda_k}. \]

In particular, for each \(k \in \mathbb{N}\), \(|\sigma_k - \sqrt{\lambda_k}| < \max\{A, B\}\).

This theorem is in the spirit of the very nice result of [19] where a similar statement is proved for Euclidean domains. The precise definition of the constants \(A\) and \(B\) will be given in (24). Note that the dependence of \(A\) on the dimension \(n\) is necessary. Indeed, it was observed in [19] that on the ball \(B(0, R) \subset \mathbb{R}^n\) the following holds for each \(k\):

\[ \lambda_k = \sigma_k^2 + n - 2 \frac{R}{\sigma_k}. \]

We do not know if the dependence of \(B\) on \(n\) is necessary in general. However, see Theorem 6 for situations where \(B\) does not depend on \(n\).

**Remark 4.** Inequality (4) also holds when (H2) and (H3) are replaced by the following weaker hypothesis:

(H2') The Ricci curvature satisfies \(\text{Ric} \geq n \alpha\) on \(M_{\bar{h}}\).

(H3') The mean curvature of \(\Sigma\) satisfies \(H \geq \kappa_-\).

1.3. Discussion and remarks. Theorem 3 is a generalization of the fundamental result of [19], and also of [23]. For bounded Euclidean domains \(\Omega \subset \mathbb{R}^{n+1}\) with connected boundary \(\Sigma = \partial \Omega\), Provenzano and Stubbe [19] proved a comparison result similar to Theorem 3 with the constants \(A\) and \(B\) replaced by a constant \(C_{\Omega}\) depending on the dimension, the maximum of the mean of the absolute values of the principal curvatures on \(\Sigma = \partial \Omega\) and the rolling radius of \(\Omega\). Their main

\(^2\)The rolling radius will be defined in Section 2.
insight was to use a generalized Pohozaev identity\(^3\) in order to compare the Dirichlet energy of an harmonic function \(u \in C^\infty(\Omega)\) with the tangential Dirichlet energy of the restriction \(u|_{\Sigma}\). In [23], Xiong extended the results of [19] to the Riemannian setting under rather stringent hypothesis. Indeed he considered simply-connected domains \(\Omega\) with convex boundary in a complete Riemannian manifold \(X\) with either nonpositive \((K_X \leq 0)\) or strictly positive \((K_X > 0)\) sectional curvature, with some hypothesis on the shape operator (or second fundamental form).

Theorem 3 improves these results in several ways. First we consider compact Riemannian manifolds with boundary. This is more general than bounded domains in a complete manifold, since they are not necessarily domains in a complete manifold with the same bounds on curvature (see [18] for a discussion of this question). Another strength of Theorem 3 is that we require geometric control of \(M\) only in the neighborhood \(M_{\bar{h}}\) of the boundary. Finally, we are not assuming the boundary \(\Sigma\) to be connected.

Remark 5. Let \(\ell\) be the number of connected components of \(\Sigma\). Then \(\lambda_i = 0\), for \(i = 1, \ldots, \ell\). Thus, inequality (3) is of interest only for \(k \geq \ell + 1\). Furthermore inequality (4) becomes \(\sigma_k \leq B\) for \(k = 1, \ldots, \ell\). There are indeed examples where \(\sigma_\ell\) can become arbitrarily small. See Section 6.

Previously, quantitative estimates relating individual Steklov eigenvalues to eigenvalues of the tangential Laplacian \(\Delta_\Sigma\) had already been studied in [22, 6, 1, 3, 15, 24]. They are relatively easy to obtain if the manifold \(M\) is isometric (or quasi-isometric with some control) to a product near its boundary. See for example [3, Lemma 2.1]. In this context however, it is usually the quotient \(\sqrt{\lambda_k}/\sigma_k\) which is controlled.

1.4. Signed curvature and convexity. In [23], Xiong extended the results of [19] to simply-connected domains \(\Omega\) with convex boundary in a complete Riemannian manifold \(X\) with either nonpositive \((K_X \leq 0)\) or strictly positive \((K_X > 0)\) sectional curvature, with some hypothesis on the shape operator. This lead to explicit values for the constants \(A\) and \(B\). This work was enlightening for us, and lead us to Theorem 6 below.

Let us specialize Theorem 3 to various geometric settings where the constants \(A\) and \(B\) can be computed explicitly.

Theorem 6. Let \(M\) be a smooth compact manifold of dimension \(n+1\) with nonempty boundary \(\Sigma\). Let \(h = \text{roll}(M)\). Let \(\lambda, \kappa_+ > 0\).

\(^3\)See Section 3 for details on the Pohozaev identity.
(1) Suppose that $\Sigma$ is totally geodesic. If $|K| \leq \lambda^2$, then (3) and (4) hold with

$$B = A = (n + 1) \max \{\lambda, \frac{\pi}{2h}\}.$$ 

Knowing the sign of $K$ leads to slightly more precise bounds.

a) If $-\lambda^2 < K < 0$, then (3) and (4) hold with

$$B = A = \frac{1}{h} + n\lambda.$$ 

b) $0 < K < \lambda^2$, then (3) and (4) hold with

$$A = (1 + n) \max \{\frac{1}{h}, \frac{2\lambda}{\pi}\} \quad \text{and} \quad B = \max \{\frac{1}{h}, \frac{2\lambda}{\pi}\}.$$ 

(2) If $-\lambda^2 \leq K \leq 0$ and each principal curvature satisfies $\lambda < \kappa_i < \kappa_+$, then (3) and (4) hold with

$$B = A = (n + 1) \max \{\kappa_+, \frac{1}{h}\}.$$ 

(3) If $0 < K < \lambda^2$ and each principal curvature satisfies $0 < \kappa_i < \kappa_+$, then (3) and (4) hold with

$$A = (n + 1) \max \{\frac{1}{h}, \sqrt{\frac{\lambda^2 + \kappa_+^2}{\kappa_i^2}}\} \quad \text{and} \quad B = \max \{\frac{1}{h}, \sqrt{\frac{\lambda^2 + \kappa_+^2}{\kappa_i^2}}\}.$$ 

(4) If $\Sigma$ is a minimal hypersurface, then (3) and (4) hold with

$$B = A = (n + 1) \max \{\lambda, \frac{\pi}{2h}\}.$$ 

Theorem 6 should be compared with [23, Theorem 1]. Its proof is presented in Section 5. Note that in point 1 (b) and point 3 of Theorem 6, the constant $B$ does not depend on the dimension $n$.

Let us conclude with the situation where $K \equiv 0$, which is motivated by the Euclidean case from [19].

**Theorem 7.** Let $M$ be a smooth compact manifold with nonempty boundary $\Sigma$. Suppose that $M$ is flat ($K \equiv 0$).

- If $0 \leq \kappa_i < \kappa_+$, then $A = (n + 1) \max \{\frac{1}{h}, \kappa_+\}$ and $B = \max \{\frac{1}{h}, \kappa_+\}$.
- If $\kappa_- < \kappa_i \leq 0$, then $B = A = \frac{1}{h} - n|\kappa_-|$.

The proof of Theorem 7 is presented in Section 5.5.
One could also consider manifolds whose boundary admits a neighborhood which is isometric to the Riemannian product \([0, L) \times \Sigma\) for some \(L > 0\), in which case one can take \(B = A = 1/L\). This is discussed in Section (5.6).

**Plan of the paper.** The proof of our main result (Theorem 3) is based on the comparison geometry of principal curvatures of parallel hypersurfaces to the boundary. This is presented in Section 2 following a review of relevant Jacobi fields and Riccati equations. The Pohozaev identity is used in Section 3 to relate the Dirichlet energy of an harmonic function to that of its restriction to the boundary. The proof of Theorem 3 is presented in Section 4. In Section 5 we specialize to various geometrically rigid settings and prove Theorem 6. Here, we give precise values of the constants \(A, B\) occurring in the estimates. In order to do this, we need some 1-dimensional calculations on some specific Riccati equations, which are treated in an appendix, at the end of the paper. In Section 6, we give various examples which are used to illustrate the necessity of the geometric hypothesis occurring in Theorem 3 and Theorem 6.

2. Preliminaries from Riemannian geometry

Let \(M\) be a smooth compact Riemannian manifold of dimension \(n + 1\) with boundary \(\Sigma\). The distance function \(f : M \to \mathbb{R}\) to the boundary \(\Sigma\) is given by

\[
f(x) = \text{dist}(x, \Sigma).
\]

Any \(s \geq 0\) that is small enough is a regular value of \(f\), so that the level sets \(\Sigma_s := f^{-1}(s)\) are submanifolds of \(M\), which are called parallel hypersurfaces. They are the boundary of \(\Omega_s := \{x \in M : f(x) \geq s\}\). For \(x \in \Sigma_s\), the gradient of the distance function \(\nabla f(x)\), is the inward normal vector to \(\Sigma_s = \partial \Omega_s\):

\[
\nu(x) := \nabla f(x).
\]

In particular, for \(x \in \Sigma\), \(\nu(x) = \nabla f(x) = -n(x)\). The distance function \(f\) satisfies \(|\nabla f| = 1\), whence the integral curves of \(\nabla f\) are geodesics. That is,

\[
\nabla_{\nabla f} \nabla f = 0.
\]

2.1. Cut locus and rolling radius. Given \(p \in \Sigma\), the exponential map defines a normal geodesic curve \(\gamma_p : \mathbb{R}_+ \longrightarrow M\),

\[
\gamma_p(s) := E_s(p) = \exp_p(s\nu).
\]
The cut point \( \text{cut}_\Sigma(p) \) of \( p \in \Sigma \) is the point \( E_{s_0}(p) \), where \( s_0 > 0 \) is the first time that \( E_s(p) \) stops minimizing the distance to \( \Sigma \). The cut locus of \( \Sigma \) is
\[
\mathcal{C}_\Sigma := \{ \text{cut}_\Sigma(p) : p \in \Sigma \}.
\]
The distance between \( \Sigma \) and \( \mathcal{C}_\Sigma \) is called the rolling radius\(^4\) of \( M \):
\[
\text{roll}(M) = \text{dist}(\Sigma, \mathcal{C}_\Sigma).
\]
Given \( h \in (0, \text{roll}(M)] \), define the tubular neighborhood
\[
M_h = \{ x \in M : f(x) < h \}.
\]
For each \( x \in M_h \), there is exactly one nearest point to \( x \) in \( \Sigma \) and the exponential map \( E_{s_0}(p) = \exp_p(s\nu(p)) \) defines a diffeomorphism between \([0, h) \times \Sigma \) and the tubular neighborhood \( M_h \). Moreover, for each \( s \in [0, \text{roll}(M)) \), the map \( E_s : \Sigma \to \Sigma_s \) is a diffeomorphism.

\textbf{Remark 8.} For each point \( p \in \Sigma \) the focal distance of \( p \), denoted by \( \text{focal}_\Sigma(p) \), is the smallest number \( s_0 > 0 \) such that \( p \) is not a local minimum of the function \( \text{dist}(E_{s_0}(p), .) : \Sigma \to \mathbb{R} \). The focal distance of \( \Sigma \) is
\[
\text{focal}(\Sigma) := \min \{ \text{focal}_\Sigma(p) : p \in \Sigma \}.
\]
Note that it follows from their definitions that \( \text{roll}(M) \leq \text{focal}(\Sigma) \). It is often easier to estimate \( \text{focal}(\Sigma) \) in terms of bounds on the curvature. For results on when \( \text{focal}(\Sigma) = \text{roll}(M) \) see \cite{13}.

\subsection{2.2. The principal curvatures of parallel hypersurfaces.}

For each \( s \in [0, \text{roll}(M)) \), the shape operator associated to the parallel hypersurface \( \Sigma_s \) is the endomorphism \( S_s : T\Sigma_s \to T\Sigma_s \) defined by
\[
S_s(X) = \nabla_X \nabla f.
\]
For each tangent vectors \( X, Y \in T\Sigma_s \), the following holds:
\[
\langle S_s(X), Y \rangle = \langle \nabla_X \nabla f, Y \rangle = \nabla^2 f(X, Y) = \langle X, S_s(Y) \rangle,
\]
where \( \langle X, Y \rangle \) is the Riemannian product and \( \nabla^2 f \) is the Hessian of \( f \). In other words, the Hessian \( \nabla^2 f \) restricted to \( \Sigma_s \) is its second fundamental form. The principal curvatures \( \kappa_i(x) \), \( i = 1, \ldots, n \), of \( \Sigma_s \) at point \( x \) with respect to outward normal vector \( n(x) := -\nabla f(x) \) are the eigenvalues of
\[
-S_s : T_x \Sigma_s \to T_x \Sigma_s.
\]
In other words, the eigenvalues of the Hessian \( \nabla^2 f \) at \( x \in \Sigma_s \), are
\[
0, -\kappa_1(x), \ldots, -\kappa_n(x).
\]
\flushright{\( ^4 \)It is called the rolling radius because any open ball of radius \( \leq \text{roll}(M) \) can roll along \( \Sigma \) while always remaining a subset of \( M \).}
Note that the choice of sign is such that the principal curvatures of the boundary of Euclidean balls are positive.

Our main geometric tools is a comparison estimate for the principal curvatures $\kappa_i$ of the parallel hypersurfaces $\Sigma_s$. This is adapted from the main theorem in \cite{9} and of \cite[Proposition 2.3]{8}.

**Theorem 9** (Principal curvature comparison theorem I). Let $M$ be a smooth compact manifold with nonempty boundary $\Sigma$. Suppose the sectional curvature satisfies $\alpha \leq K$ and that the principal curvatures of the boundary satisfies $\kappa_- \leq \kappa_i$. Let $a : [0, m(\alpha, \kappa_-)) \to \mathbb{R}$ be the solution of

$$a' + a^2 + \alpha = 0, \quad a(0) = -\kappa_-,$$

with maximal existence time $m(\alpha, \kappa_-)$. Then $m(\alpha, \kappa_-) \geq \text{roll}(M)$ and for each $\delta \in (0, \bar{h})$ the shape operator $S_\delta$ associated to the parallel hypersurface $\Sigma_\delta$ satisfies

$$S_\delta < a(\delta)I.$$

That is, each principal curvature of $\Sigma_\delta$ satisfies $\kappa_i(\delta) \geq -a(\delta)$.

**Remark 10.** Theorem 9 gives an immediate upper bound for $\text{roll}(M)$:

$$\text{roll}(M) \leq m(\alpha, \kappa_-).$$

In Lemma 17 below, an explicit formula for $m(\beta, \kappa_-)$ is given. One can compare it with the result by Donnelly and Lee in \cite[Theorem 3.1]{7} on estimates for $\text{roll}(M)$ on strictly convex domains, i.e. all eigenvalues of $-S$ are positive.

We also need a lower bound for the shape operators $S_\delta$ of parallel hypersurfaces in a neighborhood of the boundary $\Sigma$. This requires further constraints on the neighborhood.

**Theorem 11** (Principal curvature comparison theorem II). Let $M$ be a smooth compact manifold with nonempty boundary $\Sigma$. Suppose the sectional curvature satisfies $K \leq \beta$ and that the principal curvatures of the boundary satisfies $\kappa_i \leq \kappa_+$. Let $\beta_+ = \max\{0, \beta\}$, and let $b : [0, m(\beta_+, \kappa_+)) \to \mathbb{R}$ be the solution of

$$b' + b^2 + \beta_+ = 0, \quad b(0) = -\kappa_+,$$

with maximal existence time $m(\beta_+, \kappa_+)$. Let

$$\bar{h} = \min\{m(\beta_+, \kappa_+), \text{roll}(M)\}.$$

Then for each $\delta \in (0, \bar{h})$ the shape operator $S_\delta$ associated to the parallel hypersurface $\Sigma_\delta$ satisfies

$$S_\delta > b(\delta)I.$$

That is, each principal curvatures of $\Sigma_\delta$ satisfies $\kappa_i \leq -b(\delta)$. 
Remark 12. Note that Theorem 11 also holds with $\beta_+$ replaced by $\beta$, so that $\tilde{h} = \min\{m(\beta, \kappa_+), \text{roll}(M)\}$. However in our applications of Theorem 11, it is required that we use comparison with a space of non-negative sectional curvature. See Lemma 21 and Lemma 18.

Remark 13. Theorems 9 and 11 compare the evolution of the maximal and minimal principal curvatures of parallel hypersurfaces to those of umbilical hypersurfaces in model spaces of constant curvature $K \equiv \alpha, \beta$. Indeed, the functions $-a$ represent the principal curvatures of parallel umbilical hypersurfaces in a space form of constant curvature $K \equiv \alpha$, and similarly for the functions $-b$, for constant curvature $K \equiv \beta$.

In our proofs, we will use mostly the following corollary in combination with explicit formulas for $a(s)$ and $b(s)$.

Corollary 14. Let $M$ be a smooth compact manifold with nonempty boundary $\Sigma$. Suppose the sectional curvature satisfies $\alpha < K < \beta$ and that each principal curvatures along the boundary satisfies $\kappa_ \leq \kappa_i \leq \kappa_+$. Let $\tilde{h} = \min\{\text{roll}(M), m(\beta_+, \kappa_+)\}$, where $\beta_+ = \max\{0, \beta\}$. Then for each $\delta \in (0, \tilde{h})$ the principal curvatures of the parallel hypersurface $\Sigma_\delta$ satisfies

$$|\kappa_i| < \max\{|a(\delta)|, |b(\delta)|\}.$$ 

Theorem 9 and Theorem 11 are sufficient to prove a version of Theorem 3. Nevertheless, the following mean curvature comparison theorem leads to better bounds on the constant $B$.

Theorem 15 (Mean curvature comparison theorem). Let $M$ be a smooth compact manifold with nonempty boundary $\Sigma$. Suppose the Ricci curvature satisfies $\text{Ric} \geq n\alpha$ and that the mean curvature $H(= -\frac{\text{tr}S}{n})$ of the boundary satisfies $H \geq H_{\text{min}}$. Let $\mu : [0, m(\alpha, \kappa_-)) \rightarrow \mathbb{R}$ be the solution of

$$\mu' + \mu^2 + \alpha = 0, \quad \mu(0) = -H_{\text{min}}, \quad (7)$$

with maximal existence time $m(\alpha, \kappa_-)$. Then $m(\alpha, \kappa_-) \geq \text{roll}(M)$ and for each $\delta \in (0, \tilde{h})$ the mean curvature of the parallel hypersurface $\Sigma_\delta$ satisfies

$$H(\delta) > -\mu(\delta).$$

2.3. Jacobi fields and Riccati equations. For the convenience of the reader, we explain briefly in this section how Theorem 9 and Theorem 11 are direct consequences of $[8]$ and $[9]$. 
Given \( v \in T_{p_0} \Sigma \), let \( p : (-\epsilon, \epsilon) \rightarrow \Sigma \) be a smooth curve with \( p(0) = p_0 \) and \( p'(0) = v \). Then

\[
J(s) := \frac{d}{dt} E_s(p(t))|_{t=0} \in T \Sigma
\]

defines a Jacobi field along the normal geodesic \( \gamma(s) := E_s(p_0) \). This means that \( J(s) \) satisfies the Jacobi equation

\[
J''(s) + R_\nu(J(s)) = 0,
\]

(8)

where \( R_\nu(J(s)) := R(J(s), \nu)\nu \) is the curvature tensor in the normal direction \( \nu = \nabla f \) to \( \Sigma \). The field \( J \) satisfies the initial conditions

\[
J(0) = v, \quad J'(0) = S(v).
\]

In fact, one can check that the shape operator satisfies

\[
S_s(J(s)) = J'(s).
\]

(9)

Note that the shape operator is a tensor in \( T \Sigma \otimes T^* \Sigma \). Its covariant derivative is therefore given

\[
(D_X S)V = D_X(SV) - SD_XV.
\]

(10)

Differentiating (9) and using (10) and the Jacobi equation (8) leads to the Riccati equation

\[
S'_s(J(s)) + S^2_s(J(s)) + R_\nu(J(s)) = 0.
\]

(11)

Given \( p_0 \in \Sigma \), consider the integral geodesic curve \( \gamma : [0, \text{cut}_\Sigma(p_0)] \rightarrow M \) of \( \nabla f \) starting at \( \gamma(0) = p_0 \). Identifying vectors in \( T_{\gamma(s)}M \) with \( T_{p_0}M \) via parallel transport along \( \gamma \), we can consider \( S_s \) and \( R_\nu \) as endomorphisms on a single vector space \( T_{p_0} \Sigma \). They satisfy the following matrix-valued Riccati equation in \( \text{End}(T_{p_0} \Sigma) \):

\[
S'(s) + S^2(s) + R(s) = 0.
\]

(12)

More generally, let \( E \) be a finite-dimensional real vector space with an euclidean inner product \( \langle \cdot, \cdot \rangle \). Let \( S(E) \) be the space of self-adjoint endomorphisms over \( E \). For \( A, B \in S(E) \), we say \( A \leq B \) if \( B - A \) is positive semi definite. Then, we have

**Theorem 16** (Riccati comparison theorem [9]). Let \( R_1, R_2 : \mathbb{R} \rightarrow S(E) \) be smooth curves with \( R_1 \geq R_2 \). Let \( S_i : [s_0, s_i) \rightarrow S(E) \) be a solution of

\[
S'_i + S^2_i + R_i = 0, \quad i = 1, 2
\]

(13)

with maximal existence time \( s_i \in (s_0, \infty] \). Assume that \( S_1(s_0) \leq S_2(s_0) \). Then \( s_1 \leq s_2 \) and \( S_1(s) \leq S_2(s) \) on \( (s_0, s_1) \).
Proof of Theorem 9 and Theorem 11. Let $M^{n+1}$ be a space form of sectional curvature $\alpha$. Then $R = \alpha I_n$ in (12), where $I_n$ is the identity matrix. Moreover, if we assume that $M$ is simply connected and $\Sigma$ is an umbilical hypersurface i.e. $S(0) = -\kappa I_n$, then the Riccati equation (12) reduces to the one dimensional problem
\[
\begin{aligned}
a' + a^2 + \alpha &= 0 \\
a(0) &= -\kappa.
\end{aligned}
\] (14)
and its solutions describe the shape operator of a family of parallel umbilical hypersurfaces in $M$. The idea of the proof is to compare the situation of the manifold $M$ with the situation of an umbilical hypersurface on a space form, where it is possible to do exact calculations. Theorems 9 and 11 say that we can estimate the situation on $M$ in comparison with umbilical hypersurfaces on a space form, and in the sequel (Lemma 17) we will produce exact calculations in these particular situations.

For the proof of Theorem 9, we apply Theorem 16 with $R_1 = R$, where $R$ is given in (12), $R_2 = \alpha I$, $S_1(0) = S(0)$ and $S_2(0) = -\kappa I$. By hypothesis, we have $R_1 \geq R_2$ and $S_1(0) \leq S_2(0)$, so that we can apply Theorem 16. Moreover, $s_1 \leq s_2$ implies that the principal curvature of the hypersurfaces level degenerate before $s_2$ so that the rolling radius has to be less than $s_2 = m(\alpha, \kappa)$.

For the proof of Theorem 11, we apply Theorem 16 with $R_1 = \beta I$, $R_2 = R$, $S_1(0) = -\kappa_+ I$ and $S_2(0) = S(0)$. By hypothesis, we have $R_1 \geq R_2$ and $S_1(0) \leq S_2(0)$, so that we again apply Theorem 16. However the results of Theorem 16 are available only when the solution on $M$ is defined: we have to be less than $s_1 = m(\beta, \kappa_+)$ but also less than roll($M$). □

Proof of Theorem 15. Taking the trace of the Riccati equation (12) and using the Schwarz inequality for endomorphisms $n \operatorname{tr}S^2 \geq (\operatorname{tr}S)^2$ leads to the following Riccati inequality:
\[
\frac{\operatorname{tr}(S(s))'}{n} + \left(\frac{\operatorname{tr}(S(s))}{n}\right)^2 + \frac{\operatorname{tr}R}{n} \leq 0.
\]
This can be compared with the one dimensional Riccati problem
\[
\begin{aligned}
\mu' + \mu^2 + \alpha &= 0 \\
\mu(0) &= -H_{\min}
\end{aligned}
\]
using [14, Corollary 1.6.2] to get the desired inequality. See [14, Pages 181-182] and [8] for details. □
2.4. **The Riccati equation for umbilic hypersurfaces in spaceforms.** In order to apply Corollary \[14\] it will be useful to know the solutions of the one-dimensional Riccati equation explicitly. The following Lemma is adapted from \[8\].

**Lemma 17.** Let $\alpha, \kappa \in \mathbb{R}$. Consider the Riccati initial value problem \([14]\):

\[
\begin{align*}
\frac{d a}{d s} + a^2 + \alpha &= 0 \\
a(0) &= -\kappa.
\end{align*}
\]

Let $\lambda = \sqrt{|\alpha|} \geq 0$. The solutions of \((14)\), together with maximal existence time $m(\alpha, \kappa)$, are as follows:

a) In Euclidean spaces $\alpha = 0$. The umbilical hypersurfaces are given by spheres and hyperplanes.

| Description                  | Solution $a(s)$ | $m(\alpha, \kappa)$ |
|------------------------------|-----------------|----------------------|
| Contracting spheres $\kappa > 0$ | $\frac{1}{s - \frac{\kappa}{\lambda}}$ | $1/\kappa$ |
| Expanding spheres $\kappa < 0$      | $\frac{1}{s - \frac{\kappa}{\lambda}}$ | $\infty$ |
| Parallel hyperplanes $\kappa = 0$    | 0               | $\infty$ |

b) In the sphere $\mathbb{S}_\alpha$ of constant curvature $K = \alpha = \lambda^2$, $\lambda > 0$, the umbilical hypersurfaces are geodesic spheres.

$$a(s) = -\lambda \tan \left( \lambda s + \arctan \left( \frac{\kappa}{\lambda} \right) \right) \quad \text{with} \quad m(\alpha, \kappa) = \frac{1}{\lambda} \left( \frac{\pi}{2} - \arctan \left( \frac{\kappa}{\lambda} \right) \right) = \frac{1}{\lambda} \arccot \left( \frac{\kappa}{\lambda} \right)$$

c) In the hyperbolic space $\mathbb{H}_\alpha$ of constant curvature $K = \alpha = -\lambda^2 < 0$, the umbilical hypersurfaces are geodesic balls, horospheres and hyperbolic subspaces.

| Description                  | Solution $a(s)$ | Maximal existence $m(\alpha, \kappa)$ |
|------------------------------|-----------------|--------------------------------------|
| Hyperbolic subspace $\kappa = 0$ | $\lambda \tanh \left( \lambda s \right)$ | $\infty$ |
| $|\kappa| < \lambda$           | $\lambda \tanh \left( \lambda s - \operatorname{arctanh} \left( \frac{\kappa}{\lambda} \right) \right)$ | $\infty$ |
| Expanding spheres $\kappa < -\lambda$ | $\lambda \coth \left( \lambda s - \operatorname{arcoth} \left( \frac{\kappa}{\lambda} \right) \right)$ | $\infty$ |
| Contracting spheres $\kappa > \lambda$ | $\lambda \coth \left( \lambda s - \operatorname{arcoth} \left( \frac{\kappa}{\lambda} \right) \right)$ | $\frac{1}{\lambda} \arcoth \left( \frac{\kappa}{\lambda} \right)$ |
| Horospheres $|\kappa| = \lambda$ | $-\kappa$ | $\infty$ |

The proof of Lemma \[17\] is presented in an appendix. Let us conclude this paragraph with a technical lemma which will allow the control in the situation near the cut locus, where the principal curvatures of the level hypersurfaces could degenerate.
Lemma 18. Let $\alpha \in \mathbb{R}_{\geq 0}$ and $\kappa \in \mathbb{R}$. Let $a : I \subset \mathbb{R} \to \mathbb{R}$ be a solution of the Riccati equation

$$a' + a^2 + \alpha = 0, \quad a(0) = -\kappa,$$

with maximal interval $I$. Then $-(h-s)a(s) \leq 1$ for any $0 \leq s < h \in I$.

The proof of Lemma 18 will be presented in the appendix.

3. Pohozaev identity and its application

Let $u \in C^\infty(M)$ be an harmonic function. The main goal of this section is to obtain a quantitative comparison inequality relating the norms $\|\nabla_{\Sigma} u\|_{L^2(\Sigma)}$ and $\|\frac{\partial u}{\partial n}\|_{L^2(\Sigma)}$. Here $\nabla_{\Sigma} u$ denotes the tangential gradient of a function $u \in H^1(\Sigma)$ which is the gradient of $u$ on $\Sigma$. To achieve this goal, we need the generalized Pohozaev identity for harmonic functions on $\Sigma$.

**Lemma 19** (Generalized Pohozaev identity). Let $F : M \to TM$ be a Lipschitz vector field. Let $u \in C^\infty(M)$ with $\Delta u = 0$ in $M$. Then

$$0 = \int_{\Sigma} \frac{\partial u}{\partial n} \langle F, \nabla u \rangle dV_{\Sigma} - \frac{1}{2} \int_{\Sigma} |\nabla u|^2 \langle F, n \rangle dV_{\Sigma} + \frac{1}{2} \int_{M} |\nabla u|^2 \text{div} F dV_{M} - \int_{M} \langle \nabla_{\Sigma} u F, \nabla u \rangle dV_{M},$$

where $\nabla F$ denotes the covariant derivative of $F$, $n$ is the unit outward normal vector field along $\Sigma$, and $dV_{M}$ and $dV_{\Sigma}$ are Riemannian volume elements of $M$ and $\partial M$ respectively.

The proof of Lemma 19 was provided first in [19, Lemma 3.1] for Euclidean domains. For simply-connected domains in a complete Riemannian manifold a proof was provided in [23, Lemma 9]. For the sake of completeness, we also include a proof here, which works for each compact manifold with boundary.

**Proof of Lemma 19.** Working in normal coordinates at a point $p \in M$, we can proceed exactly as in $\mathbb{R}^n$ since only the value of the metric and of its first order derivative at $p$ are involved in the following local computation. It follows from $\Delta u = 0$ that

$$\text{div} (\langle F, \nabla u \rangle \nabla u) = \langle \nabla \langle F, \nabla u \rangle, \nabla u \rangle = \langle \nabla_{\Sigma} u F, \nabla u \rangle + \nabla^2 u(F, \nabla u).$$

Moreover,

$$\text{div} |\nabla u|^2 F = 2\nabla^2 u(F, \nabla u) + |\nabla u|^2 \text{div} F.$$
Therefore
\[
\text{div} \left( (F, \nabla u) \nabla u - \frac{1}{2} |\nabla u|^2 F \right) = (\nabla u F, \nabla u) - \frac{1}{2} |\nabla u|^2 \text{div} F.
\]
Integrating this identity on \( M \) and using the divergence theorem completes the proof. \( \Box \)

3.1. **Analytic estimate using distance function.** Recall that \( f : M \to \mathbb{R} \) is defined by \( f(x) = \text{dist}(x, \Sigma) \). Given \( h \in [0, \text{roll}(M)) \) let \( \tilde{f} : M_h \to \mathbb{R} \) be the distance function to \( \Sigma_h \). That is

\[ \tilde{f}(x) := h - f(x) = \text{dist}(x, \Sigma_h). \]

**Lemma 20.** Let \( u \in C^\infty(M) \) be such that \( \Delta u = 0 \) and let \( \eta = \frac{1}{2} \tilde{f}^2 \). Then

\[
\int_\Sigma \left( |\nabla \Sigma u|^2 - \left( \frac{\partial u}{\partial n} \right)^2 \right) dV_\Sigma = \frac{1}{h} \int_{M_h} \left( \Delta \eta |\nabla u|^2 - 2\nabla^2 \eta (\nabla u, \nabla u) \right) dV_M. \tag{18}
\]

**Proof.** Define the vector field \( F : M \to TM \) by

\[
F(x) := \begin{cases} 
\nabla \eta(x) & \text{if } x \in M_h, \\
0 & \text{if } x \in M \setminus M_h.
\end{cases} \tag{19}
\]

Because \( \nabla \eta = \tilde{f} \nabla \tilde{f} \) on \( M_h \), the vector field \( F \) is Lipschitz on \( M \). Note that for \( u \in C^\infty(M) \) and any \( p \in \Sigma \),

\[
|\nabla u(p)|^2 = |\nabla \Sigma u(p)|^2 + \left( \frac{\partial u}{\partial n}(p) \right)^2.
\]

It follows from the Pohozaev identity (17) that

\[
2 \int_\Sigma \frac{\partial u}{\partial n} \langle \nabla \eta, \nabla u \rangle dV_\Sigma - \int_\Sigma |\nabla \Sigma u|^2 \langle \nabla \eta, n \rangle dV_\Sigma
= \int_\Sigma \left( \frac{\partial u}{\partial n} \right)^2 \langle \nabla \eta, n \rangle dV_\Sigma + \int_\Omega |u|^2 \Delta \eta dV_M
- 2 \int_M \langle \nabla_\Sigma u \nabla \eta, \nabla u \rangle dV_M = 0.
\]
Since $\nabla \eta|_{\Sigma} = h \mathbf{n}$, we get

\[
2h \int_{\Sigma} \left( \frac{\partial u}{\partial n} \right)^2 dV_{\Sigma} - h \int_{\Sigma} |\nabla_{\Sigma} u|^2 dV_{\Sigma} - h \int_{\Sigma} \left( \frac{\partial u}{\partial n} \right)^2 dV_{\Sigma} + \int_M |\nabla u|^2 \Delta \eta dV_M - 2 \int_M \langle \nabla_{\nabla u} \nabla \eta, \nabla u \rangle dV_M = 0.
\]

3.2. Geometric control on an admissible tubular neighborhood. Let $M$ be a smooth compact Riemannian manifold with nonempty boundary $\Sigma$ and rolling radius $\bar{h} = \text{roll}(M)$. Suppose the sectional curvature satisfies $\alpha \leq K \leq \beta$ on $M$ and suppose the principal curvatures of $\Sigma$ satisfy $\kappa_- \leq \kappa_i \leq \kappa_+$. The admissible neighborhood of $\Sigma$, is the set of points of $M$ whose distance to $\Sigma$ is less than $\tilde{h}$, where

\[
\tilde{h} := \min\{\text{roll}(M), m(\beta^+, \kappa_+)\},
\]

for $\beta^+ = \max\{\beta, 0\}$. Notice that $m(\beta^+, \kappa_+) \leq m(\beta, \kappa_+)$.

The link between $\eta = \frac{1}{2} \tilde{f}^2$ and the local geometry of parallel hypersurfaces is established in the following lemma.

**Lemma 21.** Let $h \in (0, \tilde{h})$ and let $\delta \in (0, h)$. Then the eigenvalues of the Hessian $\nabla^2 \eta$ at $y \in \Sigma_\delta$ are

\[
\rho_1 = (h - \delta)\kappa_1(y) \leq \ldots \leq \rho_n = (h - \delta)\kappa_n(y) \leq \rho_{n+1} = 1.
\]

**Proof.** The Hessian of $\eta$ is

\[
\nabla^2 \eta = \nabla \tilde{f} \otimes \nabla \tilde{f} + \tilde{f} \nabla^2 \tilde{f}.
\]

The eigenvalues of $\nabla^2 \tilde{f}(y)$ are $0, \kappa_1(y), \ldots, \kappa_n(y)$. Moreover, for each $V, W \in T\Sigma_\delta$,

\[
\nabla \tilde{f} \otimes \nabla \tilde{f}(V, W) = 0.
\]

It follows from

\[
\Sigma_\delta = \tilde{f}^{-1}(h - \delta)
\]

that for each $i = 1, \ldots, n$ the number $\rho_i := (h - \delta)\kappa_i(y)$ is an eigenvalue of $\nabla^2 f(y)$. Finally, it follows from Corollary [14] that $(h - \delta)\kappa_i(y) \leq -(h - \delta)b(\delta)$ and this quantity is bounded above by 1, from Lemma [18].
Lemma 22. For each smooth function $v \in C^{\infty}(M)$, the following pointwise estimate holds on $M_h$,

$$-(1 - \sum_{i=1}^{n} \rho_i)|\nabla v|^2 \leq |\Delta \eta| |\nabla v|^2 - 2\nabla^2 \eta(\nabla v, \nabla v) \leq (1 + n \max\{-\rho_1, \rho_n\}) |\nabla v|^2.$$  \hfill (20)

Proof. Let $A = \text{diag}(\rho_i)$ be the diagonal matrix representing the Hessian of $\eta$ at $x$ in an orthonormal frame. Let $w = \nabla v(x)$. Observe that

$$\rho_1|w|^2 \leq Aw \cdot w \leq \rho_n + 1 |w|^2$$

It follows that

$$(\rho_1 + \cdots + \rho_n - \rho_{n+1})|w|^2 \leq |w|^2 \text{tr}(A) - 2Aw \cdot w \leq (-\rho_1 + \rho_2 + \cdots + \rho_{n+1})|w|^2,$$

with $\rho_{n+1} = 1$. \hfill \Box

3.3. Estimate on admissible tubular neighborhood.

Lemma 23. Let $M$ be a smooth compact Riemannian manifold with nonempty boundary $\Sigma$ and rolling radius $\bar{h} = \text{roll}(M)$. Suppose the sectional curvature satisfies $\alpha \leq K \leq \beta$ on $M_h$. Suppose the principal curvatures of $\Sigma$ satisfy $\kappa_\leq \kappa_i \leq \kappa_\geq$. Let $\beta_+ = \max\{0, \beta\}$ and let

$$\tilde{h} := \min\{\text{roll}(M), m(\beta_+, \kappa_+)\}.$$  

Then there exist constants $\overline{C}_M$ and $\overline{c}_M$ depending on $\alpha$, $\beta$, $\tilde{h}$, $\kappa_\leq$, $\kappa_\geq$ and $\beta_+$ such that every $v \in H^1(M)$ satisfies the following inequality for each $h \in (0, \tilde{h})$.

$$-(1 + n\overline{B}) \int_M |\nabla v|^2 dV_M \leq \int_{M_h} |\nabla v|^2 \Delta \eta - 2\nabla^2 \eta(\nabla v, \nabla v) dV_M \leq (1 + n\overline{A}) \int_M |\nabla v|^2 dV_M.$$ \hfill (21)

Proof. It follows directly from Lemma 22, Lemma 21, Corollary 14 and Theorem 15 that one can take

$$\overline{A} = \max_{0 \leq \delta < \tilde{h}} \left\{ (\tilde{h} - \delta) \max\{|a(\delta)|, |b(\delta)|\} \right\},$$  \hfill (22)

and

$$\overline{B} = \max_{0 \leq \delta < \tilde{h}} -(\tilde{h} - \delta)\mu(\delta).$$ \hfill (23)  \hfill \Box
Theorem 24. The assumptions are the same as in Lemma 23.

Let

$$A = \frac{1}{\hat{h}}(1 + n\mathbf{A}), \quad \text{and} \quad B = \frac{1}{\hat{h}}(1 + n\mathbf{B}).$$

(24)

Let $v \in C^\infty(M)$ be such that $\Delta v = 0$ in $M$ and normalized such that $\int_\Sigma v^2 dV_\Sigma = 1$. Then the following holds

i) $$\int_\Sigma |\nabla \Sigma v|^2 dV_\Sigma \leq \int_\Sigma \left( \frac{\partial v}{\partial \nu} \right)^2 dV_\Sigma + A \left( \int_\Sigma \left( \frac{\partial v}{\partial \nu} \right)^2 d\sigma \right)^{1/2}. \quad (25)$$

ii) $$\left( \int_\Sigma \left( \frac{\partial v}{\partial \nu} \right)^2 d\sigma \right)^{1/2} \leq \frac{B}{2} + \sqrt{\frac{1}{4} B^2 + \int_\Sigma |\nabla \Sigma v|^2 d\sigma}. \quad (26)$$

Proof. The proof of Theorem 24 is identical to the proof of Theorem 3.18 in [19]. For the sake of completeness we recall the proof.

We start with the proof of part i). By (18) and Lemma (23) we have

$$\int_\Sigma |\nabla \Sigma u|^2 dV_\Sigma = \int_\Sigma \left( \frac{\partial u}{\partial \nu} \right)^2 dV_\Sigma$$

$$+ \frac{1}{\hat{h}} \left( \int_{M_{\hat{h}}} |\nabla u|^2 \Delta \eta dV_M - 2 \int_{M_{\hat{h}}} \nabla^2 \eta (\nabla u, \nabla u) dV_M \right)$$

$$\leq \int_\Sigma \left( \frac{\partial u}{\partial \nu} \right)^2 dV_\Sigma + A \int_M |\nabla v|^2 dV_M$$

$$= \int_\Sigma \left( \frac{\partial u}{\partial \nu} \right)^2 dV_\Sigma + \int_\Sigma v \frac{\partial v}{\partial \nu} dV_\Sigma$$

$$\leq \int_\Sigma \left( \frac{\partial u}{\partial \nu} \right)^2 dV_\Sigma + \left( \int_\Sigma \left( \frac{\partial u}{\partial \nu} \right)^2 dV_\Sigma \right)^{1/2}$$

where in the last inequality we use the Cauchy-Schwarz inequality and the fact that $\int_\Sigma v^2 dV_\Sigma = 1$ by assumption. This completes the proof of part i).
We now prove part ii). We repeat the calculation above using identity (18) and Lemma (23).

\[\int_{\Sigma} |\nabla u|^2 dV = \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 dV\]
\[+ \frac{1}{h} \left( \int_{M_h} |\nabla u|^2 \Delta \eta dV_M - 2 \int_{M_h} \nabla^2 \eta (\nabla u, \nabla u) dV_M \right)\]
\[\geq \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 dV \Sigma - B \int_{M} |\nabla v|^2 dx\]
\[= \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 dV \Sigma - B \int_{\Sigma} v \frac{\partial v}{\partial \nu} dV \Sigma\]
\[\geq \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 dV \Sigma - B \left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 dV \Sigma \right)^{1/2}.\]

We solve this inequality in terms of the unknown \( \left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 dV \Sigma \right)^{1/2} \) and get

\[\left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 dV \Sigma \right)^{1/2} \leq \frac{1}{2} B + \sqrt{\frac{1}{4} B^2 + \int_{\Sigma} |\nabla u|^2 dV \Sigma}.\]

□

4. PROOF OF THE MAIN RESULT

We are now ready for the proof of the main result. The proof follows the same lines of argument as in [19, Theorem 1.7]. For the sake of completeness we give the proof here.

Proof of Theorem 3. Let us first recall the variational characterizations of the eigenvalues of Dirichlet-to-Neumann operator \( \sigma_k \) and of the Laplacian \( \lambda_k \). For each \( k \in \mathbb{N} \),

\[\sigma_k = \inf_{V \subseteq H^1(M), \, \dim V = k, \, \delta_{\Sigma} v^2 d\sigma = 1} \sup_{0 \neq v \in V} \int_M |\nabla v|^2 dV_M;\]  
\[\lambda_k = \inf_{V \subseteq H^1(\Sigma), \, \dim V = k, \, \int_{\Sigma} v^2 d\sigma = 1} \sup_{0 \neq v \in V} \int_{\Sigma} |\nabla v|^2 dV \Sigma.\]
**Part a.** Let \( \{ \psi_i \}_{i=1}^k \) be an orthonormal (in \( L^2(\partial M) \)) basis of the Steklov eigenfunctions associated with \( \{ \sigma_j \}_{j=1}^n \). We use the space generated by \( \{ \psi_i|_{\partial M} \}_{i=1}^k \) as the test space \( V \) in (28). Every \( v \in V \) with \( \int_{\Sigma} v^2 dV_{\Sigma} = 1 \) can be written as \( v = \sum c_i \psi_i|_{\partial M} \) with \( \sum c_i^2 = 1 \). Note that

\[
\frac{\partial \psi_i}{\partial \nu} = \sigma_i \psi_i, \quad i = 1, \ldots, k.
\]

Hence, using inequality (25) we get

\[
\lambda_k \leq \sup_{0 \neq u \in V, \int_{\Sigma} u^2 dV_{\Sigma} = 1} \int_{\Sigma} |\nabla_{\Sigma} u|^2 dV_{\Sigma}
\]

\[
\leq \sup_{0 \neq u \in V, \int_{\Sigma} u^2 dV_{\Sigma} = 1} \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 dV_{\Sigma} + A \left( \int_{\Sigma} \left( \frac{\partial u}{\partial \nu} \right)^2 d\sigma \right)^{\frac{1}{2}}
\]

\[
= \sup_{(c_1, \ldots, c_k) \in S^{k-1}} \left( \sum_{i=1}^k c_i^2 \sigma_i^2 + A \left( \sum_{i=1}^k c_i^2 \sigma_i^2 \right)^{\frac{1}{2}} \right) = \sigma_k^2 + A \sigma_k.
\]

This proves inequality (3).

**Part b.** We prove inequality (4) in an analogous way. Let \( \{ \varphi_i \}_{i=1}^k \) be an orthonormal basis of eigenfunctions corresponding to the eigenvalues \( \{ \lambda_i \}_{i=1}^k \).

Consider the solutions \( u_i, i = 1, \ldots, k \) to the following problem

\[
\begin{aligned}
\Delta u_i &= 0, \quad \text{in } M, \\
u_i &= \varphi_i, \quad \text{on } \Sigma.
\end{aligned}
\]

(29)

Let \( V \subset \bar{H}^1(M) \) be the space generated by \( u_1, \ldots, u_k \). Any function \( \phi \in V \) with \( \int_{\Sigma} \phi^2 d\sigma = 1 \) can be written as \( \phi = \sum c_i \varphi_i = \sum c_i u_i|_{\Sigma} \) with \( (c_1, \ldots, c_k) \in S^{k-1} \). Moreover \( \Delta \phi = 0 \) for all \( \phi \in V \). Using the min-max principle and inequality (26)
leads to
\[
\sigma_k \leq \sup_{0 \neq \phi \in W} \int_{\Sigma} |\nabla \phi|^2 dV_M = \sup_{(c_1, \ldots, c_k) \in \mathbb{R}^{k-1}} \int_M \left| \nabla \left( \sum_{i=1}^k c_i u_i \right) \right|^2 dV_M
\]
\[
\leq \max_{(c_1, \ldots, c_k) \in \mathbb{R}^{k-1}} \left( \int_{\Sigma} \left( \frac{\partial}{\partial \nu} \left( \sum_{i=1}^j c_i \varphi_i \right) \right)^2 dV_{\Sigma} \right)^{\frac{1}{2}}
\]
\[
\leq \frac{B}{2} + \left( \frac{1}{4} B^2 + \sup_{(c_1, \ldots, c_k) \in \mathbb{R}^{k-1}} \int_{\Sigma} \left| \nabla_{\Sigma} \left( \sum_{i=1}^k c_i \varphi_i \right) \right|^2 dV_{\Sigma} \right)^{\frac{1}{2}}
\]
\[
\leq \frac{B}{2} + \left( \frac{1}{4} B^2 + \sum_{i=1}^k \frac{\lambda_i}{c_i^2} \right)^{\frac{1}{2}} = \frac{B}{2} + \sqrt{\frac{1}{4} B^2 + \lambda_k}.
\]
This proves (4).

Part c. It remains to prove that \(|\sigma_k - \sqrt{\lambda_k}| \leq A\). Indeed,
\[
\sigma_k - \sqrt{\lambda_k} \leq \frac{B}{2} + \sqrt{\frac{B^2}{4} + \lambda_k} - \sqrt{\lambda_k}
\]
\[
= \frac{B}{2} + \frac{B^2/4}{\sqrt{B^2/4 + \lambda_k} + \sqrt{\lambda_k}} \leq B.
\]

Similarly,
\[
\sqrt{\lambda_k} - \sigma_k = \frac{\lambda_k - \sigma_k^2}{\sqrt{\lambda_k + \sigma_k}} \leq \frac{A \sigma_k}{\sqrt{\lambda_k + \sigma_k}} \leq A.
\]

We conclude \(|\sigma_k - \sqrt{\lambda_k}| \leq \max\{B, A\} \). \(\square\)

The proof of Theorem 4 is now an easy consequence.

**Proof of Theorem 4** Let \(\Omega\) be a neighborhood of the boundary \(\Sigma\) and let \(g_1, g_2\) be two Riemannian metrics which coincide on \(\Omega\). Let \(M_{\hat{h}}\) be the admissible neighborhood of \(\Sigma\) for the metric \(g_1\). Let \(\hat{h} \in (0, \hat{h})\) be the largest number such that \(M_{\hat{h}} \subset \Omega\). One can then define \(C > 0\) using formula (22) and formula (24), for the metric \(g_1\), with \(\hat{h}\) replaced by \(\hat{h}\). Substituting \(C\) for \(\max\{B, A\}\) in the proof of Theorem 3 leads to \(|\sigma_k(g_i) - \sqrt{\lambda_k}| < C\) for \(i = 1, 2\). This implies that \(|\sigma_k(g_1) - \sigma_k(g_2)| \leq 2C\). \(\square\)
5. Signed curvature and convexity

In this section we will give precise bounds on the constants $A$ and $B$ defined in (24) and prove Theorem 6. In the situation where the sectional curvature is constrained to have a constant sign and where we impose convexity assumptions on the boundary $\Sigma$, our results are closely related those of Xiong [23].

Our strategy is to estimate the quantities $(\tilde{h} - \delta) \max\{|a(\delta)|, |b(\delta)|\}$ and also $-(\tilde{h} - \delta)\mu(\delta)$ which appear in the definitions (22) and (23) of the constants $\overrightarrow{A}$ and $\overrightarrow{B}$, and thus also in the definitions (24) of $A$ and $B$. This is achieved by using the explicit formulas for $a(\delta)$, $b(\delta)$ and $\mu(\delta)$ provided by Lemma 17 together with some technical lemmas regarding the solutions of Riccati equation in dimension 1. These are stated in the next paragraph.

5.1. Technical lemmas regarding Riccati equation in dimension 1. In this paragraph we state a technical lemma which will be proved in the appendix. Its goal is to estimate the function $f(x) = (h - x)a(x)$ for solutions $a(x)$ of the Riccati equation.

Lemma 25. Let $\alpha, \kappa \in \mathbb{R}$. Let $a : [0, m) \to \mathbb{R}$ be a solution to the Riccati equation

$$a' + a^2 + \alpha = 0, \quad a(0) = -\kappa,$$

with maximal interval $I$. Let $h \in I$ and define the function $f : I \to \mathbb{R}$ by $f(x) = -(h - x)a(x)$. The following inequalities hold:

| Condition on $\alpha$ | Condition on $\kappa$ | Inequality |
|------------------------|------------------------|------------|
| $\alpha \geq 0$        | $\kappa \in \mathbb{R}$| $\min\{0, h\kappa\} \leq f(x) \leq 1$ |
| $\alpha < 0$          | $\kappa \geq \sqrt{|\alpha|}$ | $0 \leq f(x) \leq h\kappa$ |

5.2. Totally geodesic boundary. This corresponds to the part (1) of Theorem 6.

Proposition 26. Let $M$ be a compact manifold with nonempty totally geodesic boundary $\Sigma$. Let $\bar{h} = \text{roll}(M)$ be its rolling radius. Assume that $|K| < \lambda^2$, where $\lambda > 0$, on $M_{\bar{h}}$. Then, (3) and (4) hold with

$$B = A = (1 + n) \max\{\lambda, \frac{\pi}{2\bar{h}}\}.$$

Moreover, if $-\lambda^2 < K < 0$, then (3) and (4) hold with

$$A = \frac{1}{\bar{h}} + n\lambda.$$

On the other hand, if $0 < K < \lambda^2$, then (3) and (4) hold with
\[ A = (1 + n) \max\{h^{-1}, \frac{2\lambda}{\pi}\} \quad \text{and} \quad B = \max\{h^{-1}, \frac{2\lambda}{\pi}\}. \]

**Proof.** It follows from Corollary 14 and Lemma 17 that
\[ \tilde{h} = \min\{\text{roll}(M), m(\lambda^2, 0)\} = \min\{\frac{\pi}{2\lambda}, \tilde{h}\}. \]
Moreover, the corresponding comparison functions are
\[ \mu(\delta) = a(\delta) = \lambda \tanh(\lambda \delta) \quad \text{and} \quad b(\delta) = -\lambda \tan(\lambda \delta). \]
It follows from Lemma 25 that \((h - \delta)|b(\delta)| \leq 1\). Moreover, because \(\delta < \frac{\pi}{2\lambda}\),
\[ (h - \delta)\lambda \tanh(\lambda \delta) < \frac{\pi}{2}. \]
Whence,
\[ \mathcal{B}, \mathcal{A} \leq \frac{\pi}{2}. \]
And
\[ \mathcal{B}, \mathcal{A} \leq \frac{1}{h}(1 + \frac{n\pi}{2}) \leq (1 + n) \max\{\lambda, \frac{\pi}{2h}\}. \]

In the situation where \(K < 0\),
\[ \tilde{h} = \min\{\text{roll}(M), m(0, 0)\} = \tilde{h}. \]
The comparison functions are
\[ \mu(\delta) = a(\delta) = \lambda \tanh(\lambda \delta) \quad \text{and} \quad b(\delta) = 0. \]
We have \((h - \delta)|a(\delta)| \leq \tilde{h}\lambda\) and \(\mathcal{B}, \mathcal{A} \leq \tilde{h}\lambda\). Thus,
\[ \mathcal{B}, \mathcal{A} \leq \frac{1}{h} + n\lambda. \]
In the case \(K > 0\) the comparison functions are
\[ \mu(\delta) = a(\delta) = 0 \quad b(\delta) = -\lambda \tan(\lambda \delta). \]
and as mentioned above \((h - \delta)|b(\delta)| \leq 1\). Therefore,
\[ A = (1 + n) \max\{h^{-1}, \frac{2\lambda}{\pi}\}, \quad \text{and} \quad B = \max\{h^{-1}, \frac{2\lambda}{\pi}\}. \]
5.3. **Horoconvex boundary.** This corresponds to the part (2) of Theorem 6. Our hypothesis on the principal curvature implies local convexity as in [23], but we do not require $M$ to be simply connected.

**Proposition 27.** Let $M$ be a compact manifold with nonempty boundary $\Sigma$. Let $\bar{h} = \text{roll}(M)$ be its rolling radius. Assume that $-\lambda^2 \leq K \leq 0$ on $M_{\bar{h}}$ (where $\lambda > 0$) and assume that each of the principal curvatures of $\Sigma$ satisfies $\lambda < \kappa_\iota \leq \kappa_+$. Then, one can take

$$B = A = (n + 1) \max\{\kappa_+\frac{1}{\bar{h}}\}.$$ 

**Proof.** It follows from Corollary 14 and Lemma 17 that

$$\bar{h} = \min\{\text{roll}(M), m(0, \kappa_+)\} = \min\{1/\kappa_+, \bar{h}\}.$$ 

Moreover, the corresponding comparison functions satisfy

$$\mu(\delta) = a(\delta) = \lambda \coth(\lambda \delta - \lambda \phi) \quad \text{and} \quad |b(\delta)| = \frac{1}{-\delta + 1/\kappa_+},$$

where $\phi = \frac{1}{\lambda} \text{arcoth}(\frac{\kappa_+}{\lambda})$. It follows from Lemma 25 that $(\bar{h} - \delta)|a(\delta)| \leq \kappa_+ \bar{h} \leq 1$.

Moreover,

$$(h - \delta)|b(\delta)| \leq \left(1 - \frac{1}{\lambda}\right)\left(\frac{1}{-\delta + 1/\lambda}\right) = 1.$$ 

Whence, $B, A \leq 1$, and

$$B, A \leq \frac{1 + n}{\bar{h}}.$$ 

5.4. **Positive sectional curvature.** This corresponds to the part (3) of Theorem 6.

**Proposition 28.** Let $M$ be a compact manifold with nonempty boundary $\Sigma$. Let $\bar{h} = \text{roll}(M)$ be its rolling radius. Assume that $0 < K < \lambda^2$, where $\lambda > 0$, on $M_{\bar{h}}$. Also assume that each principal curvatures satisfy $0 \leq \kappa_\iota \leq \kappa_+$ on $\Sigma$. Then, one can take

$$A = (n + 1) \max\{\frac{1}{\bar{h}}, \sqrt{\lambda^2 + \kappa_+^2}\}, \quad \text{and} \quad B \leq \max\{\frac{1}{\bar{h}}, \sqrt{\lambda^2 + \kappa_+^2}\}.$$ 

**Proof.** It follows from Corollary 14 and Lemma 17 that

$$\bar{h} = \min\{\text{roll}(M), m(\lambda^2, \kappa_+)\} = \min\{\frac{1}{\lambda} \left(\frac{\pi}{2} - \arctan(\kappa_+)\right), \bar{h}\}.$$
Let us show that
\[
\frac{\lambda}{\frac{\pi}{2} - \arctan\left(\frac{\kappa}{\lambda}\right)} \leq \sqrt{\lambda^2 + \kappa^2_+}.
\] (30)

Indeed this is equivalent to
\[
\frac{\lambda}{\sqrt{\lambda^2 + \kappa^2_+}} \leq \frac{\pi}{2} - \arctan\left(\frac{\kappa}{\lambda}\right),
\]
Using that \(\sin\) is increasing on \([0, \pi/2]\) this leads to the equivalent
\[
\sin\left(\frac{\lambda}{\sqrt{\lambda^2 + \kappa^2_+}}\right) \leq \sin\left(\frac{\pi}{2} - \arctan\left(\frac{\kappa}{\lambda}\right)\right) = \cos\left(\arctan\left(\frac{\kappa}{\lambda}\right)\right) = \frac{\lambda}{\sqrt{\lambda^2 + \kappa^2_+}}.
\]
Which is true.

Moreover, the corresponding comparison functions satisfy
\[
\mu(\delta) = a(\delta) = 0 \quad \text{and} \quad |b(\delta)| = \lambda \tan\left(\lambda \delta + \frac{1}{\lambda} \arctan\left(\frac{\kappa_+}{\lambda}\right)\right)
\]
It follows from Lemma 25 that \((h - \delta)|b(\delta)| \leq \kappa_+ h \leq 1\). Whence, \(\overline{A} \leq 1\), and
\[
A \leq \frac{n + 1}{\bar{h}} \leq (n + 1) \max\left\{\frac{1}{\bar{h}}, \sqrt{\lambda^2 + \kappa^2_+}\right\}\quad \text{and} \quad B \leq \max\left\{\frac{1}{\bar{h}}, \sqrt{\lambda^2 + \kappa^2_+}\right\}.
\]

5.5. **Flat manifolds.** The goal is to prove Theorem 7.

**Proof.** Case 1: \(0 \leq \kappa_i < \kappa_+\). It follows from Corollary 14 and Lemma 17 that
\[
\bar{h} = \min\{\text{roll}(M), m(0, \kappa_+)\} = \min\{1/\kappa_+, \bar{h}\}.
\]
Moreover, the corresponding comparison functions are
\[
\mu(\delta) = a(\delta) = 0, \quad \text{and} \quad b(\delta) = \frac{1}{\delta - 1/\kappa_+}.
\]
Whence,
\[
(h - \delta)|b(\delta)| \leq 1.
\]
Thus, \(\overline{A} \leq 1\), and \(\overline{B} = 0\) and
\[
A \leq \frac{n + 1}{\bar{h}} = (n + 1) \max\{\bar{h}^{-1}, \kappa_+\}, \quad \text{and} \quad B \leq \max\{\bar{h}^{-1}, \kappa_+\}.
\]
Case 2: $\kappa_- < \kappa \leq 0$. It follows from Corollary 14 and Lemma 17 that
\[
\tilde{h} = \min\{\text{roll}(M), m(0, \kappa_-)\} = \bar{h}.
\]
Moreover, the corresponding comparison functions are
\[
\mu(\delta) = a(\delta) = \frac{1}{\delta - 1/\kappa_-}, \quad \text{and} \quad b(\delta) = 0.
\]
It follows from Lemma 25 that $(h - \delta) \frac{1}{\delta - 1/\kappa_-} \leq h\kappa$ for $0 < h < \bar{h}|\kappa_-|$, so that
\[
B, A \leq \bar{h}^{-1} + n|\kappa_-|.
\]

\[\square\]

Remark 29. In all estimates above for $B$, one can consider $\kappa_-$ as a lower bound for the mean curvature of $\Sigma$ and $\alpha$ as a lower bound on the Ricci curvature. The same estimates remain true. See also Remark 4.

5.6. Cylindrical boundaries. Let assume that a neighborhood of each boundary component $\Sigma_j$ is isometric to $\Sigma_j \times [0, L]$. Hence the sectional curvature along the normal geodesic is zero. Note that in the Riccati equation (11) only the sectional curvature along the normal geodesic appears. Moreover, $\Sigma_j$’s are totally geodesic, i.e., $\kappa_i = 0, i = 1, \ldots, n$. Thus we have $B = A(h) = 0$ and roll$(M) \leq L$. Therefore (3) and (4) hold with $B = A = \frac{1}{L}$.

6. Examples and remarks

In this section, we discuss the necessity of the hypothesis of Theorem 3 and give different kinds of examples to illustrate this.

Example 30. The condition on the rolling radius is clearly a necessary condition. An easy example, with two boundary components, is given as follow: Take $M = T^n \times [0, L]$ where $T^n$ is a $n$-dimensional flat torus. The sectional curvature of $M$ is 0, the principal curvatures of $\partial M = T^n$ are 0. As $L \to 0$, the rolling radius tends to 0, and $\sigma_k \to 0$ for all $k$ (2). However $\lambda_3(\partial M)$ is fixed and strictly positive. This contradicts Inequality (3), and shows that we need a lower bound on the rolling radius.

We can construct an example with one boundary component in the spirit of [4, Example 2]. The length of the boundary is bounded, and this insures that $\lambda_2$ is bounded away from 0. The curvature of the boundary is bounded, but as the boundary becomes “close to itself”, this allow to construct small $\sigma_2$. 
Example 31. The condition on the curvature of the ambient space is also a necessary condition: in Theorem 3.5 of [1], we have constructed an example where the boundary is fixed (which implies that all $\lambda_i$ are fixed), its principal curvatures equal to 0 and the rolling radius uniformly bounded from below. However, $\sigma_2$ becomes arbitrarily large. This contradicts Inequality (4) and this comes from the fact that the curvature of the ambient space goes to $\infty$.

We can also adapt this example in order to show that without constraint on the curvature, $\sigma_3$ may be small and not $\lambda_3$, even with a control of the rolling radius. We take $\Sigma$ of dimension $n \geq 3$, and consider a Riemannian metric $g_\epsilon = h_2^2(t)g_0 + dt^2$ which coincides with $g$ in a $\epsilon$-neighborhood of $\Sigma$ in $M$ and such that $h_2^2(t)$ is $\leq \epsilon$ outside a neighborhood of radius $2\epsilon$ of $\Sigma$. A direct and simple calculation shows that if $f$ is a function of norm 1 on $\Sigma$, then the Rayleigh quotient of $f$ viewed as a function on $M$ (that is $F(p,t) = f(p)$) has a Rayleigh quotient converging to 0 with $\epsilon$. All the Steklov spectrum of $(M,g_\epsilon)$ goes to 0.

Example 32. One can use Example 6.3 given in the proof of theorem 7 in [6] in order to construct an example with $\sigma_2$ bounded from below by a positive constant, but $\lambda_2$ arbitrarily small. Example 6.3 consist in gluing together a finite number of fundamental pieces, so that we get a family of example with uniformly bounded geometry (curvature, curvature of the boundary, rolling radius). Moreover, we have shown that $\sigma_2$ was uniformly bounded below. At the same time, the boundary is a curve of length going to $\infty$, so that $\lambda_2$ goes to 0.

7. Appendix: the Riccati equation in dimension one

Let $\alpha, \kappa \in \mathbb{R}$. Let $M_\alpha$ be the space-form of constant sectional curvature $\alpha$. Let $\Omega \subset M_\alpha$ be an open connected set with connected smooth umbilical boundary $\Sigma = \partial \Omega \subset M_\alpha$ with principal curvature $\kappa$. We recall that, in this situation, we have to
investigate the equation \[ \begin{cases} a' + a^2 + \alpha = 0, \\ a(0) = -\kappa. \end{cases} \]

Our goal is to compute the solution \( a(s) \) explicitly and to determine the maximal \( m > 0 \) such that \( a \) exists on \([0, m]\) in order to proof Lemma 17 and 18.

**Proof of Lemma 17**

*Proof.* In the particular situation where \( \kappa^2 + \alpha = 0 \), the solution is the constant function \( a : \mathbb{R} \to \mathbb{R} \) defined by \( a(t) = -\kappa \). In all other situations, the solution satisfies

\[-\frac{a'}{a^2 + \alpha} = 1 \quad \text{on } I.\]

Integration from 0 to \( x \in I \) leads to

\[- \int_0^x \frac{a'(t)}{a^2(t) + \alpha} \, dt = x. \tag{31}\]

The explicit representation of this integral by elementary functions depend on the sign of \( \alpha \).

**Case \( \alpha = 0 \).** It follows from

\[- \int_0^x \frac{a'(t)}{a^2(t)} \, dt = \frac{1}{a(x)} - \frac{1}{a(0)} = \frac{1}{a(x)} + \frac{1}{\kappa}\]

that

\[a(x) = \frac{1}{x - \frac{1}{\kappa}} = \frac{\kappa}{\kappa x - 1}\]

The maximal existence time therefore is

\[m(0, \kappa) = \begin{cases} \frac{1}{\kappa} & \text{if } \kappa > 0, \\ \infty & \text{if } \kappa \leq 0. \end{cases}\]
Case $\alpha > 0$. Let $\lambda > 0$ be such that $\alpha = \lambda^2$, and observe that

$$- \int_0^x \frac{a'(t)}{a^2(t) + \alpha} dt = -\frac{1}{\lambda} \int_0^x \frac{a'(t)}{\lambda (a(t) \lambda)^2 + 1} dt$$

$$= -\frac{1}{\lambda} \left( \arctan \left( \frac{a(x)}{\lambda} \right) - \arctan \left( \frac{a(0)}{\lambda} \right) \right)$$

$$= \frac{1}{\lambda} \left( - \arctan \left( \frac{a(x)}{\lambda} \right) - \arctan \left( \frac{\kappa}{\lambda} \right) \right)$$

Together with (31) this implies that

$$a(x) = -\lambda \tan \left( \lambda x + \arctan \left( \frac{\kappa}{\lambda} \right) \right)$$

For $x > 0$, this solution is well-defined as long as

$$\lambda x + \arctan \left( \frac{\kappa}{\lambda} \right) < \pi/2.$$

That is,

$$m(\alpha, \kappa) = \frac{1}{\lambda} \left( \frac{\pi}{2} - \arctan \left( \frac{\kappa}{\lambda} \right) \right).$$

Case $\alpha < 0$. Let $\lambda > 0$ be such that $\alpha = -\lambda^2$. The left-hand-side of (31) is

$$- \int_0^x \frac{a'(t)}{a^2(t) + \alpha} dt = \frac{1}{\lambda} \int_0^x \frac{a'(t)}{\lambda (a(t) \lambda)^2 + 1} dt = \frac{1}{\lambda} \int_{a(0)/\lambda}^{a(x)/\lambda} \frac{1}{1 - u^2} du = \frac{1}{\lambda} \int_{-\kappa/\lambda}^{a(x)/\lambda} \frac{1}{1 - u^2} du.$$

Now if $-\kappa/\lambda \in (-1, 1)$ then

$$\frac{1}{\lambda} \int_{-\kappa/\lambda}^{a(x)/\lambda} \frac{1}{1 - u^2} du = \frac{1}{\lambda} \left( \text{arctanh} \left( \frac{a(x)}{\lambda} \right) - \text{arctanh} \left( \frac{-\kappa}{\lambda} \right) \right).$$

On the other hand, if $|\kappa/\lambda| > 1$ then

$$\frac{1}{\lambda} \int_{-\kappa/\lambda}^{a(x)/\lambda} \frac{1}{1 - u^2} du = \frac{1}{\lambda} \left( \text{arcoth} \left( \frac{a(x)}{\lambda} \right) - \text{arcoth} \left( \frac{-\kappa}{\lambda} \right) \right).$$

It follows that

$$a(x) = \begin{cases} 
\lambda \tanh \left( \lambda x - \text{arctanh} \left( \frac{\kappa}{\lambda} \right) \right) & \text{if } |\kappa| < \lambda; \\
\lambda \coth \left( \lambda x - \text{arcoth} \left( \frac{\kappa}{\lambda} \right) \right) & \text{if } |\kappa| > \lambda. 
\end{cases}$$

(32)
The maximal time of existence is

\[ m(\alpha, \kappa) = \begin{cases} 
\frac{1}{\lambda} \arcoth(\frac{\kappa}{\lambda}) & \text{if } \kappa > \lambda \\
+\infty & \text{if } |\kappa| < \lambda, \\
+\infty & \text{if } \kappa < -\lambda.
\end{cases} \]

\[ \square \]

Proof of Lemma 18.

Proof. We use the formula for \( a(s) \) given in Lemma 17 to verify the statement in different cases. The proof is a straightforward calculation.

Case 1. \( \alpha = \lambda^2, \lambda > 0 \). Here we use the fact that

\[ x \cot(x) \leq 1, \quad \forall x \in [0, \pi), \quad (33) \]

with equality at \( x = 0 \).

Recall from Lemma 17 that for \( \alpha = \lambda^2 \) we have

\[ -a(s) = \lambda \tan(\lambda s + \arctan(\frac{\kappa}{\lambda})) = \lambda \cot \left( \arctan(\frac{\kappa}{\lambda}) - \lambda s \right) \]

for any \( 0 \leq s \leq \frac{1}{\lambda} \arctan(\frac{\kappa}{\lambda}) \). Notice that we used the identity \( \arctan(x) + \arctan(1/x) = \frac{\pi}{2} \).

Thus

\[ -(h - s)a(s) \leq \left( \arctan(\frac{\kappa}{\lambda}) - \lambda s \right) \cot \left( \arctan(\frac{\kappa}{\lambda}) - \lambda s \right) \leq 1. \]

Case 2. \( \alpha = 0 \). If \( \kappa \leq 0 \), then \( -a(s) \leq 0 \) on \( I \). If \( \kappa > 0 \), then \( -(h - s)a(s) = (h - s)\frac{\kappa}{1 - \kappa s} \leq 1 \) if and only if \( h < \frac{1}{\kappa} \). Hence, it is clear that in both situations the statement of the lemma holds.

\[ \square \]

One last proof is still needed.

Proof of Lemma 25. Case 1: \( \alpha \geq 0 \). From Lemma 18 we know that \( f(x) \leq 1 \) for \( \alpha \geq 0 \). It remains to show that \( f(x) \geq \min\{0, h\kappa\} \).

Case 1(a). \( \alpha = 0 \). If \( \kappa = 0 \), then \( a \equiv 0 \) and \( f \equiv 0 \) and there is nothing more to prove. If \( \kappa > 0 \), then \( I = (0, 1/\kappa) \) and \( f(x) = -(h - x)\frac{1}{x - 1/\kappa} > 0 \). If \( \kappa < 0 \), then \( I = (0, \infty) \) and \( f(x) = -(h - x)\frac{1}{x - 1/\kappa} \). It follows from \( (h - x) \leq h \) and \( (x - 1/\kappa) \geq 1/|\kappa| \) that

\[ -(h - x)\frac{1}{x - 1/\kappa} \geq h\kappa. \]
Case 1(b). $\alpha > 0$.

From Lemma 18 and its proof we know
\[ f(x) = \lambda(h - s) \cot \left( \arccot \left( \frac{K}{\lambda} \right) - \lambda s \right), \]
with $I = \left( 0, \frac{1}{\lambda} \arccot \left( \frac{K}{\lambda} \right) \right)$. If $\arccot \left( \frac{\kappa}{\lambda} \right) - \lambda s \leq \frac{\pi}{2}$, then $f(x) \geq 0$. If $\arccot \left( \frac{\kappa}{\lambda} \right) - \lambda s \geq \frac{\pi}{2}$, then $\cot \left( \arccot \left( \frac{\kappa}{\lambda} \right) - \lambda s \right) \leq 0$ and
\[ f'(x) = -\lambda \cot \left( \arccot \left( \frac{K}{\lambda} \right) - \lambda s \right) + \frac{\lambda^2(h - s)}{\sin^2(\arccot (\frac{\kappa}{\lambda}) - \lambda s)} \geq 0. \]
Therefore, $f(x) \geq f(0) = h\kappa$.

Case 2: $\alpha < 0$ and $\kappa \geq \lambda := \sqrt{|\alpha|}$.

The maximal existence interval is $I = (0, \frac{1}{\lambda} \arccot (\kappa/\lambda))$ and
\[ f(x) = -(h - x)\lambda \coth(\lambda x - \arccot(\kappa/\lambda)). \]
Note that for any $\gamma \geq 0$ the function $\psi_\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by
\[ \psi_\gamma(t) = t \coth(t + \gamma) \]
is increasing. Let $\phi = \frac{1}{\lambda} \arccot(\kappa/\lambda)$. Observe that for $\gamma = \lambda \phi - \lambda h$
\[ 0 \leq f(x) = \psi_\gamma(\lambda h - \lambda x). \]
It follows that
\[ 0 \leq f(x) = (\lambda h - \lambda x) \coth(\lambda \phi - \lambda x) \]
\[ = (\lambda h - \lambda x) \coth(\lambda h - \lambda x + \lambda \phi - \lambda h) \leq \lambda h \coth(\lambda \phi) = h\kappa. \]

\[ \square \]

Acknowledgment

Part of this work was done while BC was visiting Université Laval. He thanks the personnel from the Département de mathématiques et de statistique for providing good working conditions. Part of this work was done while AG was visiting Neuchâtel. The support of the Institut de Mathématiques de Neuchâtel is warmly acknowledged. AH would like to thank the Mittag–Leffler Institute for the support and for an excellent working conditions during the starting phase of the project.
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Université de Neuchâtel, Institut de Mathématiques, Rue Emile-Argand 11, CH-2000 Neuchâtel, Switzerland

E-mail address: bruno.colbois@unine.ch

Département de mathématiques et de statistique, Université Laval, Pavillon Alexandre-Vachon, 1045, av. de la Médecine, Québec QC G1V 0A6, Canada

E-mail address: alexandre.girouard@mat.ulaval.ca

University of Bristol, School of Mathematics, University Walk, Bristol BS8 1TW, UK

E-mail address: asma.hassannezhad@bristol.ac.uk