The Einstein-Hilbert action of the space of holomorphic maps from $S^2$ to $\mathbb{C}P^k$

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Abstract

Let $\mathcal{H}_{n,k}(\Sigma)$ be the space of degree $n \geq 1$ holomorphic maps from a compact Riemann surface $\Sigma$ to $\mathbb{C}P^k$. In the case $\Sigma = S^2$ and $n = 1$, the $L^2$ metric on $\mathcal{H}_{1,k}(S^2)$ was computed exactly by Speight. In this paper, the Ricci curvature tensor and the scalar curvature on $\mathcal{H}_{1,k}(S^2)$ are determined explicitly for $k \geq 2$. An exact direct computation of the Einstein-Hilbert action with respect to the $L^2$ metric on $\mathcal{H}_{1,k}(S^2)$ is made and shown to coincide with a formula conjectured by Baptista.

1 Introduction

Let $\Sigma$ be a compact Riemann surface equipped with a Riemannian metric $g$ and let $h$ be the Fubini-Study metric on $\mathbb{C}P^k$. Let $\phi$ be a holomorphic map from $\Sigma$ to $\mathbb{C}P^k$ of degree $n \geq 1$ defined as

$$n = \int_{\Sigma} \phi^* \omega_0,$$

where $\omega_0$ is the normalized Kähler form with respect to $h$. Consider the space of degree $n$ holomorphic maps $\Sigma \to \mathbb{C}P^k$, denoted $\mathcal{H}_{n,k}(\Sigma)$. There is a natural Riemannian metric on $\mathcal{H}_{n,k}(\Sigma)$ defined by the metrics $g$ and $h$ on $\Sigma$ and $\mathbb{C}P^k$ as

$$\gamma_{L^2}(X,Y) = \int_{\Sigma} h(X,Y) \text{vol}_g,$$

for $X,Y \in T_{\phi} \mathcal{H}_{n,k}(\Sigma) \subset \Gamma(\phi^* T \mathbb{C}P^k)$. This is called the $L^2$ metric on $\mathcal{H}_{n,k}(\Sigma)$.

In the physics literature, the degree $n$ holomorphic map $\phi$ is regarded as a $\mathbb{C}P^k$ lump of charge $n$ on $\Sigma$, that is, a degree $n$ minimal energy static solution of the field equations of the $\mathbb{C}P^k$ model on $\Sigma$. Hence, the degree $n$ moduli space $\mathcal{M}_n$ of the $\mathbb{C}P^k$ model on $\Sigma$ is $\mathcal{H}_{n,k}(\Sigma)$. The low energy dynamics of $\mathbb{C}P^k$ lumps is conjecturally approximated by geodesic motion on $\mathcal{M}_n$ with respect to the $L^2$ metric $\gamma_{L^2}$. A precise version of this conjecture is proved for $\Sigma = T^2$ and $n \geq 2$ by Speight. 

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With respect to the $L^2$ metric, Baptista [11] has given conjectural formulae for the volume and the Einstein-Hilbert action of $\mathcal{H}_{n,k}(\Sigma)$, provided $\Sigma$ has genus $g \leq n/2$,

$$\text{Vol}(\mathcal{H}_{n,k}(\Sigma), \gamma_{L^2}) = \frac{(k + 1)^g}{m!} \left( \pi \text{Vol}(\Sigma, g) \right)^{m},$$

$$H(\mathcal{H}_{n,k}(\Sigma), \gamma_{L^2}) = \frac{2\pi (k + 1)^g [m - 2g + 1]}{(m - 1)!} \left( \pi \text{Vol}(\Sigma, g) \right)^{m-1},$$

where $m = (k + 1)(n + 1 - g) + g - 1$ and $\text{Vol}(\Sigma, g)$ is the volume of $\Sigma$. This conjecture is based on a singular limit relating the $\mathbb{C}P^k$ model on $\Sigma$ with a gauged sigma model whose fields take values in $\mathbb{C}^{k+1}$ [11]. More precisely, a one parameter family of metrics on the $n$-vortex moduli space, which is a compact Kähler manifold, are conjectured to converge, in a certain limit, to the $L^2$ metric on $\mathcal{H}_{n,k}(\Sigma)$. Such convergence has recently been established rigorously by Lui [6] in the sense of Cheeger-Gromov, that is on each open set in some locally finite open cover of $\mathcal{H}_{n,k}(\Sigma)$. This convergence does not directly imply Baptista’s conjectured formulae for the volume and the Einstein-Hilbert action of $\mathcal{H}_{n,k}(\Sigma)$, however.

In the case $n = 1$ and $\Sigma = S^2$, Speight [7, 8] has determined an explicit formula for the $L^2$ metric on $\mathcal{H}_{1,k}(S^2)$, and then computed the volume of $\mathcal{H}_{1,k}(S^2)$ for $k \geq 2$ finding agreement with the conjectural formula (3). In this paper, an explicit formula for the Ricci curvature tensor, and then the scalar curvature on $(\mathcal{H}_{1,k}(S^2), \gamma_{L^2})$ have been determined for $k \geq 2$, by exploiting the Kähler property of the $L^2$ metric. The Einstein-Hilbert action of $\mathcal{H}_{1,k}(S^2)$ with respect to the $L^2$ metric is computed for $k \geq 2$ confirming the formula (4).

\section{Degree 1 Holomorphic Maps $S^2 \to \mathbb{C}P^k$}

This section reviews the geometric structure of $\mathcal{H}_{1,k}(S^2)$ introduced in [6]. Let $S^2$ be the 2-sphere equipped with the standard round metric and let $\phi$ be a degree 1 holomorphic map $S^2 \to \mathbb{C}P^k$. Introducing homogeneous coordinates $[z_0, z_1]$ on $\mathbb{C}P^1 \cong S^2$, then such degree 1 map has the form

$$\phi([z_0, z_1]) = [a_0z_0 + b_0z_1, \ldots, a_kz_0 + b_kz_1],$$

where $(a_0, \ldots, a_k)$ and $(b_0, \ldots, b_k)$ are linearly independent in $\mathbb{C}^{k+1}$. Since the elements $(\xi a_0, \xi b_0, \ldots, \xi a_k, \xi b_k) \in \mathbb{C}^{2k+2}$, where $\xi \in \mathbb{C}^\times$, determine the same holomorphic map $\phi$, then there is an open inclusion $\mathcal{H}_{1,k}(S^2) \hookrightarrow \mathbb{C}P^{2k+1}$ which is used to equip $\mathcal{H}_{1,k}(S^2)$ with a topology, differentiable and complex structures.

The isometry groups $U(2)$ and $U(k + 1)$ of $\mathbb{C}P^1$ and $\mathbb{C}P^k$ respectively build an isometric action of $G = U(k + 1) \times U(2)$ on $\mathcal{H}_{1,k}(S^2)$, that is, $\phi \to \sigma_2 \circ \phi \circ \sigma_1^{-1}$ where $\sigma_1$ and $\sigma_2$ are isometries of $\mathbb{C}P^1$ and $\mathbb{C}P^k$. Generically, each orbit of $G$ on $\mathcal{H}_{1,k}(S^2)$ is a real codimension 1 submanifold of $\mathcal{H}_{1,k}(S^2)$ and has a unique element $\phi_\mu$ given by
An exceptional orbit of real codimension 3 occurs when $\mu = 1$. This action decomposes $\mathcal{H}_{1,k}(S^2)$ into a one parameter family of orbits parametrized by $\mu \in [1, \infty)$. For $\mu > 1$, the isotropy group of the orbit $G_{\mu}$ of $\phi_{\mu}$ is

$$K = \left\{ \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & U \end{pmatrix} : \alpha, \beta, \delta \in \mathbb{R}, U \in U(k-1) \right\}.$$  

By the Orbit-Stabilizer Theorem, each orbit $G_{\mu}$ is diffeomorphic to $G/K$.

Now, let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G$ and $K$ respectively and $\langle , \rangle$ be the $Ad(G)$ invariant inner product on $\mathfrak{g}$,

$$\langle (M_1, m_1), (M_2, m_2) \rangle = -\frac{1}{2}(\text{tr} M_1 M_2 + \text{tr} m_1 m_2),$$  

where $M_i \in \mathfrak{u}(k+1)$ and $m_i \in \mathfrak{u}(2)$. The tangent space of $\mathcal{H}_{1,k}(S^2)$ at $\phi_{\mu}$ is

$$V_{\mu} := T_{\phi_{\mu}} \mathcal{H}_{1,k}(S^2) = \left( \frac{\partial}{\partial \mu} \right) \oplus \mathfrak{p},$$  

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle , \rangle$. The space $\mathfrak{p}$ can be decomposed into $Ad(K)$ invariant subspaces

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{\mu} \oplus \hat{\mathfrak{p}}_{\mu} \oplus \hat{\mathfrak{p}} \oplus \tilde{\mathfrak{p}},$$  

where $\mathfrak{p}_0$ is a 1-real-dimensional space, $\mathfrak{p}_{\mu}$, $\hat{\mathfrak{p}}_{\mu}$ are 1-complex dimensional subspaces depending on $\mu$, and $\hat{\mathfrak{p}}$ and $\tilde{\mathfrak{p}}$ are $(k-1)$ complex dimensional subspaces. The definitions of these subspaces are included in the Appendix. It was shown in [8] that

**Proposition 1.** Let $\gamma$ be a $G$ invariant Kähler metric on $\mathcal{H}_{1,k}(S^2)$. Then, for $k \geq 2$, $\gamma$ is uniquely determined by the one parameter family of symmetric bilinear forms $\gamma_{\mu} : V_{\mu} \times V_{\mu} \to \mathbb{R}$ given by

$$\gamma_{\mu} = A_0(\mu) d\mu^2 + 8\mu^2 A_0(\mu)\langle , \rangle_{\mathfrak{p}_0} + A_1(\mu) \langle , \rangle_{\mathfrak{p}_{\mu}} + A_2(\mu) \langle , \rangle_{\hat{\mathfrak{p}}_{\mu}} + A_3(\mu) \langle , \rangle_{\hat{\mathfrak{p}}} + A_4(\mu) \langle , \rangle_{\tilde{\mathfrak{p}}} ,$$  

where $A_0, \ldots, A_4$ are smooth positive functions of $\mu$ defined by a single function $A(\mu)$ and a positive constant $B$ as follows

$$A_0(\mu) = \frac{1}{4\mu} A'(\mu), \quad A_1(\mu) = A_2(\mu) = \frac{\mu^2 - 1}{\mu^2 + 1} A(\mu), \quad A_3(\mu) = B + \frac{A(\mu)}{2}, \quad A_4(\mu) = B - \frac{A(\mu)}{2} ,$$  

and $\langle , \rangle_{\mathfrak{p}_i}$ denote the induced inner products of $\langle , \rangle$ on the $Ad(K)$ invariant subspaces, given in [10].
For the $L^2$ metric $\gamma_{L^2}$ on $H_{1,k}(S^2)$, the function $A(\mu)$ and the constant $B$ are

$$A_{L^2}(\mu) = \frac{16\pi}{c_1c_2} \frac{\mu^4 - 4\mu^2 \log \mu - 1}{(\mu^2 - 1)^2}, \quad B_{L^2} = \frac{8\pi}{c_1c_2},$$

(13)

where $c_1$ and $c_2$ are the constant holomorphic sectional curvatures of $g$ and $h$ respectively.

Another $G$ invariant Kähler metric on $H_{1,k}(S^2)$ is the induced metric defined by the inclusion $H_{1,k}(S^2) \hookrightarrow \mathbb{C}P^{2k+1}$, where $\mathbb{C}P^{2k+1}$ is given the Fubini-Study metric (of constant holomorphic sectional curvature $c$, say). We call this the Fubini-Study metric on $H_{1,k}(S^2)$, denoted $\gamma_{FS}$. It is determined by

$$A_{FS}(\mu) = \frac{4}{c} \frac{\mu^2 - 1}{\mu^2 + 1}, \quad B_{FS} = \frac{2}{c},$$

(14)

The volume form of a $G$ invariant Kähler metric $\gamma$, determined as in (11) by the function $A(\mu)$ and the constant $B$, on $H_{1,k}(S^2)$ is

$$\text{vol}_\gamma = V(\mu) \, d\mu \wedge \text{vol}_{G/K},$$

(15)

where

$$V(\mu) = \frac{1}{\sqrt{2}} A(\mu)^2 \left( B^2 - \frac{A(\mu)^2}{4} \right)^{k-1} A'(\mu),$$

(16)

and $\text{vol}_{G/K}$ is the volume form of $G/K$ with respect to the inner product $\langle , \rangle$, defined in (8). It was shown that for $k \geq 2$, every $G$ invariant Kähler metric $\gamma$ on $H_{1,k}(S^2)$ has finite volume\textsuperscript{8}. In fact, if $\lim_{\mu \to \infty} A(\mu) = 2B$, this volume is

$$\text{Vol}(H_{1,k}(S^2), \gamma) = \sqrt{2} (2B)^{2k+1} \frac{(k-1)!k!}{(2k+1)!} \text{Vol}(G/K) = \frac{(2B\pi)^{2k+1}}{(2k+1)!},$$

(17)

where $\text{Vol}(G/K)$ is the volume of $G/K$ with respect to $\langle , \rangle$.

### 3 Ricci Curvature Tensor

With respect to any $G$ invariant Kähler metric $\gamma$, determined as in Proposition 1 on $H_{1,k}(S^2)$, we determine an explicit formula for the Ricci curvature tensor $\rho$ as follows

**Proposition 2.** Let $\gamma$ be a $G$ invariant Kähler metric on $H_{1,k}(S^2)$, determined as in (11) by the function $A(\mu)$ and the constant $B$. Then, the Ricci curvature tensor $\rho$ on $(H_{1,k}(S^2), \gamma)$ with $k \geq 2$ is uniquely determined by the one parameter family of symmetric bilinear forms $\rho_\mu : V_\mu \times V_\mu \to \mathbb{R}$, given by

$$\rho_\mu = C_0(\mu)d\mu^2 + 8\mu^2 C_0(\mu)\langle , \rangle_{p_0} + C_1(\mu)\langle , \rangle_{p_\mu} + C_2(\mu)\langle , \rangle_{\bar{p}_\mu} + C_3(\mu)\langle , \rangle_{\bar{p}} + C_4(\mu)\langle , \rangle_{\bar{p}},$$

(18)

where $C_0, \ldots, C_4$ are smooth functions of $\mu$, determined as in (12), by the function $C(\mu)$ and the constant $D$ given by
\[ C(\mu) = 4(k + 1) \frac{\mu^2 - 1}{\mu^2 + 1} - 2\mu \frac{F'(\mu)}{F(\mu)}, \quad D = 2(k + 1), \]  
(19)

where

\[ F(\mu) = \frac{A(\mu)^2 A'(\mu)}{A_{FS}(\mu)^2 A_{FS}'(\mu)} \left( B^2 - \frac{A(\mu)^2}{4} \right)^{k-1} \left( B_{FS}^2 - \frac{A_{FS}(\mu)^2}{4} \right)^{-(k-1)}. \]  
(20)

**Proof:** The Ricci curvature tensor \( \rho \) on \( (H_{1,k}(S^2), \gamma) \) is a \( G \) invariant symmetric \((0,2)\) tensor which is Hermitian and its associated 2-form \( \hat{\rho} = \rho(J,.) \) is closed. Hence, \( \rho \) has the same structure as \( \gamma \), that is, it is uniquely determined by the one parameter family of symmetric bilinear forms \( \rho_\mu : V_\mu \times V_\mu \to \mathbb{R} \), given as in[11]. Since the coefficients \( C_0(\mu), \ldots, C_4(\mu) \) are defined as in[12] by a single function \( C(\mu) \) and a constant \( D \), then we only need to determine \( C(\mu) \) and \( D \). By Proposition 1, we have

\[ C(\mu) = C_3(\mu) - C_4(\mu), \quad D = \frac{1}{2} \left[ C_3(\mu) + C_4(\mu) \right]. \]  
(21)

To compute \( C(\mu) \) and \( D \), we need first an orthonormal basis for \( p \) with respect to the inner product \( \langle . \rangle_p \). We shall use the orthonormal basis \{\( Y_i, Y_j : i = 0, \ldots, 4, j = 1, \ldots, 2k-2 \}\) introduced in[8]. The structure of this basis is included in the Appendix. Hence, the functions \( C_3(\mu) \) and \( C_4(\mu) \) can be given, for example, by

\[ C_3(\mu) = \rho_\mu(Y_1, Y_1) = -\rho_\mu(JY_2, Y_1) = \hat{\rho}_\mu(Y_1, Y_2), \]
\[ C_4(\mu) = \rho_\mu(Y_1, Y_1) = -\rho_\mu(JY_2, Y_1) = \hat{\rho}_\mu(Y_1, Y_2). \]  
(22)

Now, the volume form, given in[15], of any \( G \) invariant Kähler metric \( \gamma \) on \( H_{1,k}(S^2) \) can be written as

\[ \text{vol}_\gamma = F(\mu) \text{vol}_{\gamma_{FS}}, \]  
(23)

where

\[ F(\mu) = \frac{A(\mu)^2 A'(\mu)}{A_{FS}(\mu)^2 A_{FS}'(\mu)} \left( B^2 - \frac{A(\mu)^2}{4} \right)^{k-1} \left( B_{FS}^2 - \frac{A_{FS}(\mu)^2}{4} \right)^{-(k-1)}. \]  
(24)

Hence, the Ricci form \( \hat{\rho} \) with respect to \( \gamma \) is[2],

\[ \hat{\rho} = \hat{\rho}_{FS} - i\partial\bar{\partial}f, \quad f(\mu) := \log F(\mu), \]  
(25)

where \( \hat{\rho}_{FS} \) is the Ricci form with respect to \( \gamma_{FS} \), \( \partial : \Omega^{(p,q)} \to \Omega^{(p+1,q)} \), and \( \overline{\partial} : \Omega^{(p,q)} \to \Omega^{(p,q+1)} \) are the partial exterior derivatives on the space of \((p,q)\)-forms \( \Omega^{(p,q)} \) on \( H_{1,k}(S^2) \). Using (25) in (22), we have

\[ C(\mu) = \rho_{FS}(Y_1, Y_2) - \rho_{FS}(Y_1, Y_2) - i\left[ (\partial\bar{\partial}f)_\mu(Y_1, Y_2) - (\partial\bar{\partial}f)_\mu(Y_1, Y_2) \right], \]
\[ = C_{FS}(\mu) - i\left[ (\partial\bar{\partial}f)_\mu(Y_1, Y_2) - (\partial\bar{\partial}f)_\mu(Y_1, Y_2) \right], \]  
(26)
and

\[ 2D = \hat{\rho}_{FS}(\hat{Y}_1, \hat{Y}_2) + \rho_{FS}(\hat{Y}_1, \hat{Y}_2) - i(\partial \bar{\partial} f)(\mu)(\hat{Y}_1, \hat{Y}_2) + (\partial \bar{\partial} f)(\mu)(\hat{Y}_1, \hat{Y}_2)] = 2D_{FS} - i[(\partial \bar{\partial} f)_{\mu}(\hat{Y}_1, \hat{Y}_2) + (\partial \bar{\partial} f)_{\mu}(\hat{Y}_1, \hat{Y}_2)]. \]  

(27)

Since \((\mathcal{H}_{1, k}(S^2), \gamma_{FS})\) is a \((2k + 1)\) complex dimensional Kähler-Einstein manifold, then

\[ \hat{\rho}_{FS} = c (k + 1) \omega_{FS}, \]  

(28)

where \(\omega_{FS}\) is the Kähler form of \(\gamma_{FS}\). Hence, the function \(C_{FS}(\mu)\) and the constant \(D_{FS}\) are

\[ C_{FS}(\mu) = c(k + 1)A_{FS}(\mu) = 4(k + 1)\frac{\mu^2}{\mu^2 + 1}, \quad D_{FS} = c(k + 1)B_{FS} = 2(k + 1). \]  

(29)

It remains to compute \((\partial \bar{\partial} f)_{\mu}(\hat{Y}_1, \hat{Y}_2)\) and \((\partial \bar{\partial} f)_{\mu}(\hat{Y}_1, \hat{Y}_2)\). Let \(\xi_0 = -Y_0/(2\sqrt{2} \mu)\), then the Hermiticity of \(\gamma\) implies that \(J\xi_0 = -\partial/\partial \mu\), and so,

\[ (J^* d\mu)(\xi_0) = d\mu(J\xi_0) = d\mu(-\frac{\partial}{\partial \mu}) = -1, \]  

(30)

where \(J^*\) is the induced almost complex structure on \(V^*_\mu\). This means that \(\eta_0 = -J^* d\mu\) is the covector of \(\xi_0\), that is, \(\eta_0(\xi_0) = 1\). The exterior derivative of \(f\) is

\[ df = \frac{1}{2} f'(\mu) \left[(d\mu + i\eta_0) + (d\mu - i\eta_0)\right] = \frac{1}{2} f'(\mu) \left[(d\mu - iJ^* d\mu) + (d\mu + iJ^* d\mu)\right]. \]  

(31)

This implies that the \((1, 0)\)-part \(\partial f\) and the \((0, 1)\)-part \(\bar{\partial} f\) of the 1-form \(df\) are

\[ \partial f = \frac{1}{2} f'(\mu) \ (d\mu + i\eta_0), \quad \bar{\partial} f = \frac{1}{2} f'(\mu) \ (d\mu - i\eta_0). \]  

(32)

Since \(d = \partial + \bar{\partial}\) and \(\bar{\partial}^2 = 0\), then

\[ \partial \bar{\partial} f = d\bar{\partial} f = -\frac{i}{2} f''(\mu) \ d\mu \wedge \eta_0 - \frac{i}{2} f'(\mu) d\eta_0, \]  

(33)

where \(d\eta_0\) is a 2-form on \(\mathcal{H}_{1, k}(S^2)\) given for any vector fields \(X, Y\) on \(\mathcal{H}_{1, k}(S^2)\) by

\[ d\eta_0(X, Y) = X[\eta_0(Y)] - Y[\eta_0(X)] - \eta_0([X, Y]). \]  

(34)

Let \(\xi_1, \xi_2\) be the extension of \(\hat{Y}_1\) and \(\hat{Y}_2\) as Killing vector fields on \(\mathcal{H}_{1, k}(S^2)\). Then, from (33) and (34), we have

\[ (\partial \bar{\partial} f)_{\mu}(\hat{Y}_1, \hat{Y}_2) = \frac{i}{2} f'(\mu) \eta_0(\xi_1, \xi_2)_{\phi = \phi_\mu}. \]  

(35)

The Lie bracket of Killing vector fields on \(\mathcal{H}_{1, k}(S^2)\) can be defined by the Lie algebra bracket \([, ]_\theta\) of \(\mathfrak{g}\) as follows [3]
\[ [\xi_1, \xi_2] \big|_{\phi=\phi_\mu} = -P_p([\hat{\xi}_1, \hat{\xi}_2]_\phi), \]  \tag{36}

where \( P_p \) is the projection of \( g \) to \( p \). Since
\[
\hat{\xi}_1 = (-E_{13} + E_{31}, 0), \quad \hat{\xi}_2 = i(E_{13} + E_{31}, 0),
\]  \tag{37}
as in the Appendix. Then, we have

\[
[\hat{\xi}_1, \hat{\xi}_2]_g = -2i(E_{13}E_{31} - E_{31}E_{13}, 0),
\]
\[
= -i(2E_{11} - 2E_{33}, 0),
\]
\[
= -\frac{i}{2}(3E_{11} + E_{22} - 2E_{33}, e_{11} - e_{22}) + \frac{i}{2}(E_{11} - E_{22}, -e_{11} + e_{22}),
\]
\[
= -\frac{i}{2}(3E_{11} + E_{22} - 2E_{33}, e_{11} - e_{22}) + \frac{1}{\sqrt{2}}Y_0,
\]  \tag{38}

where \( E_{\alpha\beta} \) and \( e_{\alpha\beta} \) denote \((k + 1) \times (k + 1)\) and \(2 \times 2\) matrices respectively whose element \((\alpha, \beta)\) is 1, and the others being zero. Since the element \(i(3E_{11} + E_{22} - 2E_{33}, e_{11} - e_{22})/2 \in \mathfrak{t}\), then it vanishes under \( P_p \), and so

\[
[\xi_1, \xi_2] \big|_{\phi=\phi_\mu} = -\frac{1}{\sqrt{2}}Y_0.
\]  \tag{39}

Substituting (39) in (35), we get
\[
(\partial \bar{\partial} f)_\mu(\hat{\xi}_1, \hat{\xi}_2) = if'(\mu).
\]  \tag{40}

Similarly, one can find that
\[
(\partial \bar{\partial} f)_\mu(\hat{\xi}_1, \hat{\xi}_2) = -if'(\mu).
\]  \tag{41}

Substituting (29), (40) and (41) in (26) and (27), we obtain the function \( C(\mu) \) and the constant \( D \) as in (19).

\[ \square \]

4 Scalar Curvature

An orthonormal basis for \((V_\mu, \gamma_\mu)\) can be defined as follows [8],
The Einstein-Hilbert action of a Riemannian manifold \((M, g)\) has the formula (43).

Using (18) in (44), we get

\[
\kappa(\mu) = 2 \left[ 2 \frac{C(\mu)}{A(\mu)} + \frac{C'(\mu)}{A'(\mu)} \right] + 2(k - 1) \left[ \frac{4(k + 1) + C(\mu)}{2B + A(\mu)} + \frac{4(k + 1) - C(\mu)}{2B - A(\mu)} \right].
\]

**Proof:** The scalar curvature of a \(G\) invariant Kähler metric \(\gamma\), determined as in (11), with respect to the orthonormal basis (12) is

\[
\kappa(\mu) = \rho_\mu(X, X) + \sum_{i=0}^{4} \rho_\mu(X_i, X_i) + \sum_{j=1}^{2k-2} \left[ \rho_\mu(\tilde{X}_j, \tilde{X}_j) + \rho_\mu(\tilde{X}_j, \tilde{X}_j) \right],
\]

\[
= \frac{1}{A_0(\mu)} \rho_\mu(\frac{\partial}{\partial \mu}, \frac{\partial}{\partial \mu}) + \frac{1}{8\mu^2 A_0(\mu)} \rho_\mu(Y_0, Y_0) + \frac{1}{A_1(\mu)} \sum_{i=1}^{4} \rho_\mu(Y_i, Y_i)
\]

\[
+ \frac{1}{A_3(\mu)} \sum_{j=1}^{2k-2} \rho_\mu(\tilde{Y}_j, \tilde{Y}_j) + \frac{1}{A_4(\mu)} \sum_{j=1}^{2k-2} \rho_\mu(\tilde{Y}_j, \tilde{Y}_j).
\]

Using (18) in (44), we get

\[
\kappa(\mu) = 2 \frac{C_0(\mu)}{A_0(\mu)} + 4 \frac{C_1(\mu)}{A_1(\mu)} + 2(k - 1) \left[ \frac{C_3(\mu)}{A_3(\mu)} + \frac{C_4(\mu)}{A_4(\mu)} \right].
\]

Using the relations between the functions \(A_i(\mu)\) and \(C_i(\mu)\) with \(A(\mu)\) and \(C(\mu)\) respectively, as in (12), we obtain that the scalar curvature of a \(G\) invariant Kähler metric \(\gamma\) on \(\mathcal{H}_{1,k}(S^2)\) has the formula (43).

\(\square\)

5 Einstein-Hilbert Action of \(\mathcal{H}_{1,k}(S^2)\)

The Einstein-Hilbert action of a Riemannian manifold \((M, g)\) is defined by the integral
\[ H(M, g) = \int_M \kappa \text{vol}_g, \quad (46) \]

where \( \kappa \) and \( \text{vol}_g \) are the scalar curvature and the volume form respectively with respect to the Riemannian metric \( g \) on \( M \).

**Theorem 1.** The Einstein-Hilbert action of \( \mathcal{H}_{1,k}(S^2) \) with respect to the \( L^2 \) metric \( \gamma_{L^2} \) is

\[ H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = \frac{2^{2k+2}\pi^{2k+1}(k+1)B^{2k}}{(2k)!}, \quad \forall \ k \geq 2. \quad (47) \]

**Proof:** In this proof, and for the rest of the paper, we will desist from denoting \( \mu \) dependence explicitly in the functions \( A(\mu) \) and \( C(\mu) \).

The Einstein-Hilbert action of \( \mathcal{H}_{1,k}(S^2) \) with respect to any \( G \) invariant Kähler metric \( \gamma \) is

\[ H(\mathcal{H}_{1,k}(S^2), \gamma) = \int_{\mathcal{H}_{1,k}(S^2)} \kappa(\mu) V(\mu) \, d\mu \wedge \text{vol}_{G/K}, \]

\[ = \text{Vol}(G/K) \int_{1}^{\infty} \kappa(\mu) V(\mu) \, d\mu, \quad (48) \]

The scalar curvature of \( (\mathcal{H}_{1,k}(S^2), \gamma) \), given in (43), can be written as

\[ \kappa(\mu) = \frac{2}{AA'} [2CA' + AC''] + (k - 1) \left( B^2 - \frac{A^2}{4} \right)^{-1} [4(k + 1)B - AC], \quad (49) \]

and then, by (16), we have

\[ \kappa(\mu) V(\mu) = \frac{2}{\sqrt{2}} [2CA' + AC''] \left( B^2 - \frac{A^2}{4} \right)^{k-1} \]

\[ + \frac{(k-1)}{\sqrt{2}} A^2 A' \left[ 4(k + 1)B - AC \right] \left( B^2 - \frac{A^2}{4} \right)^{k-2}, \]

\[ = \frac{2}{\sqrt{2}} \left( B^2 - \frac{A^2}{4} \right)^{k-1} \frac{d}{d\mu} (A^2 C) - \frac{(k-1)}{\sqrt{2}} CA'^2 \left( B^2 - \frac{A^2}{4} \right)^{k-2} \]

\[ + \frac{4(k^2 - 1)B}{\sqrt{2}} A^2 A' \left( B^2 - \frac{A^2}{4} \right)^{k-2}. \quad (50) \]

Since

\[ \frac{d}{d\mu} \left[ \left( B^2 - \frac{A^2}{4} \right)^{k-1} \right] = -\frac{(k-1)}{2} A A' \left( B^2 - \frac{A^2}{4} \right)^{k-2}, \quad (51) \]

then,
\[ \kappa(\mu) V(\mu) = \frac{2}{\sqrt{2}} \frac{d}{d\mu} \left[ A^2 C \left( B^2 - \frac{A^2}{4} \right)^{k-1} \right] + 2\sqrt{2}(k^2 - 1)BA^2 A' \left( B^2 - \frac{A^2}{4} \right)^{k-2}. \]  

(52)

Hence, the Einstein-Hilbert Action \( H(\mathcal{H}_{1,k}(S^2), \gamma) \) is

\[ H(\mathcal{H}_{1,k}(S^2), \gamma) = \frac{2}{\sqrt{2}} \text{Vol}(G/K) \left[ A^2 C \left( B^2 - \frac{A^2}{4} \right)^{k-1} \right] \]  

\[ + 2\sqrt{2}(k^2 - 1)B^{2k-3}\text{Vol}(G/K) \int_{A(1)}^{A(\infty)} A^2 \left( 1 - \frac{A^2}{4B} \right)^{k-2} dA. \]  

(53)

For the \( L^2 \) metric on \( \mathcal{H}_{1,k}(S^2) \), the following limits follow from (13),

\[ \lim_{\mu \to 1} A_{L^2} = 0, \quad \lim_{\mu \to \infty} A_{L^2} = 2B_{L^2}, \]
\[ \lim_{\mu \to 1} C_{L^2} = 0, \quad \lim_{\mu \to \infty} C_{L^2} = 4(k + 1), \]  

(54)

and so,

\[ \lim_{\mu \to 1} \left[ A_{L^2}^2 C_{L^2} \left( B_{L^2}^2 - \frac{A_{L^2}^2}{4} \right)^{k-1} \right] = \lim_{\mu \to \infty} \left[ A_{L^2}^2 C_{L^2} \left( B_{L^2}^2 - \frac{A_{L^2}^2}{4} \right)^{k-1} \right] = 0. \]  

(55)

Thus, the Einstein-Hilbert Action with respect to the \( L^2 \) metric \( \gamma_{L^2} \) on \( \mathcal{H}_{1,k}(S^2) \) is

\[ H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = 2\sqrt{2} (k^2 - 1)B_{L^2}^{2k-3}\text{Vol}(G/K) \int_{A_{L^2}(1)}^{A_{L^2}(\infty)} A_{L^2}^2 \left( 1 - \frac{A_{L^2}^2}{4B_{L^2}} \right)^{k-2} dA_{L^2}. \]  

(56)

To compute the integral above, let \( t = A_{L^2}/2B_{L^2} \), then

\[ H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = 2^4 \sqrt{2} (k^2 - 1)B_{L^2}^{2k} \text{Vol}(G/K) \int_0^1 t^2 \left( 1 - t^2 \right)^{k-2} dt. \]  

(57)

The integral in (57) is finite for all \( k \geq 2 \). In fact

\[ \int_0^1 t^2 \left[ 1 - t^2 \right]^{k-2} dt = \frac{2^{2k-2}(k - 2)! k!}{(2k)!}, \quad \forall k \geq 2. \]  

(58)

The volume of \( G/K \) can be extracted from the formula of \( \text{Vol}(\mathcal{H}_{1,k}(S^2), \gamma) \) in (17), that is,

\[ \text{Vol}(G/K) = \frac{1}{\sqrt{2}} \frac{\pi^{2k+1}}{(k - 1)! k!}. \]  

(59)

Substituting (58) and (59) in (57), we get
\[
H(H_{1,k}(S^2), \gamma_{L^2}) = \frac{2^{2k+2} \pi^{2k+1}(k+1)B_{L^2}^{2k}}{(2k)!}. \tag{60}
\]

By taking the holomorphic sectional curvatures \( c_1 = c_2 = 4 \), then the constant \( B_{L^2} = \frac{\pi}{2} \), and so the Einstein-Hilbert action of \( H_{1,k}(S^2) \) with respect to the \( L^2 \) metric is

\[
H(H_{1,k}(S^2), \gamma_{L^2}) = \frac{2^{2k+4}(k+1)}{(2k)!}, \tag{61}
\]

which confirms Baptista’s conjectured formula \(^\square\). 

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Appendix

The orthogonal complement \( p \) of the Lie algebra \( \mathfrak{t} \) in \( \mathfrak{g} \) decomposes into the \( \text{Ad}(K) \) invariant subspaces \(^8\)

\[
p = p_0 \oplus p_\mu \oplus \tilde{p}_\mu \oplus \hat{p} \oplus \check{p}, \tag{62}
\]

where

\[
p_0 = \{ (\lambda \text{diag}(i, -i, 0, \ldots, 0, \text{diag}(-i, i)) : \lambda \in \mathbb{R} \} \equiv \mathbb{R}, \tag{63}
\]

\[
p_\mu = \left\{ \begin{pmatrix} 0 & x & 0 & \cdots \\ -\bar{x} & 0 & 0 & \cdots \\ 0 & 0 & & \\ \vdots & \vdots & & \ddots \\ \end{pmatrix}, \begin{pmatrix} 0 & \mu x \\ -\mu \bar{x} & 0 \\ \end{pmatrix} : x \in \mathbb{C} \right\} \equiv \mathbb{C}, \tag{64}
\]

\[
\tilde{p}_\mu = \left\{ \begin{pmatrix} 0 & -\mu \bar{y} & 0 & \cdots \\ \mu y & 0 & 0 & \cdots \\ 0 & 0 & & \\ \vdots & \vdots & & \ddots \\ \end{pmatrix}, \begin{pmatrix} 0 & -\bar{y} \\ y & 0 \\ \end{pmatrix} : y \in \mathbb{C} \right\} \equiv \mathbb{C}, \tag{65}
\]

\[
\hat{p} = \left\{ \begin{pmatrix} 0 & 0 & -u^\dagger \\ 0 & 0 & \cdots \\ u & \cdots & \ddots \\ \end{pmatrix}, 0 \} : u \in \mathbb{C}^{k-1} \right\} \equiv \mathbb{C}^{k-1}, \tag{66}
\]

\[
\check{p} = \left\{ \begin{pmatrix} 0 & 0 & -v^\dagger \\ 0 & 0 & \cdots \\ v & \cdots & \ddots \\ \end{pmatrix}, 0 \} : v \in \mathbb{C}^{k-1} \right\} \equiv \mathbb{C}^{k-1}. \tag{67}
\]
The almost complex structure $J$ acts on $\mathfrak{p}$ as

$$J : (\lambda, x, y, u, v) \mapsto 4\mu\lambda \frac{\partial}{\partial \mu} + (0, ix, iy, iu, iv). \tag{68}$$

An orthonormal basis for $\mathfrak{p}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\mathfrak{p}$, defined by $\boxdot$, is given as follows

$$Y_0 = \frac{i}{\sqrt{2}} (E_{11} - E_{22}, -e_{11} + e_{22}), \quad Y_1 = (E_{12} - E_{21}, 0), \quad Y_2 = i(E_{12} + E_{21}, 0),$$

$$Y_3 = (0, -e_{12} + e_{21}), \quad Y_4 = i(0, e_{12} + e_{21}),$$

$$\hat{Y}_{2i-1} = (-E_{1,i+2} + E_{i+2,1}, 0), \quad Y_{2i} = i(E_{1,i+2} + E_{i+2,1}, 0), \quad i = 1, \ldots, k - 1$$

$$\hat{Y}_{2i-1} = (-E_{2,i+2} + E_{i+2,2}, 0), \quad Y_{2i} = i(E_{2,i+2} + E_{i+2,2}, 0), \quad i = 1, \ldots, k - 1, \tag{69}$$

where $E_{\alpha\beta}$ and $e_{\alpha\beta}$ denote $(k+1) \times (k+1)$ and $2 \times 2$ matrices respectively whose element $(\alpha, \beta)$ is 1, and the others being zero.

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