SPECTRUM OF THE $\bar{\partial}$-NEUMANN LAPLACIAN ON POLYDISCS

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Abstract. The spectrum of the $\bar{\partial}$-Neumann Laplacian on a polydisc in $\mathbb{C}^n$ is explicitly computed. The calculation exhibits that the spectrum consists of eigenvalues, some of which, in particular the smallest ones, are of infinite multiplicity.

1. Introduction

The $\bar{\partial}$-Neumann Laplacian $\square_q$ on a bounded domain $\Omega$ in $\mathbb{C}^n$ is (a constant multiple of) the usual Laplacian acting diagonally on $(0,q)$-forms subjected to the non-coercive $\bar{\partial}$-Neumann boundary conditions. It is a densely defined, non-negative, and self-adjoint operator. As such, its spectrum is a non-empty closed subset of the non-negative real axis. Unlike the usual Dirichlet Laplacian, its spectrum need not be purely discrete. (See [FS01] for a discussion on related subjects.) Spectral behavior of the $\bar{\partial}$-Neumann Laplacian is more sensitive to the boundary geometry of the domain than the Dirichlet/Neumann Laplacians. (See [Fu05a, Fu05b] and references therein for related discussions.)

Spectral behavior of the $\bar{\partial}$-Neumann Laplacian on special domains often serves as a model for the general theory. One certainly cannot expect to explicitly calculate the spectrum for wide classes of domains. The spectrum for the ball and annulus was explicitly computed by Folland [Fo72]. In this note, we compute the spectrum for the polydiscs. Our computation exhibits that the spectrum of the $\bar{\partial}$-Neumann Laplacian on a polydisc consists of eigenvalues, some of which, in particular the smallest ones, are of infinite multiplicity. That the essential spectrum of the $\bar{\partial}$-Neumann Laplacian is non-empty is consistent with, in fact, equivalent to, the well-known fact that the $\bar{\partial}$-Neumann operator (the inverse of the $\bar{\partial}$-Neumann Laplacian) is non-compact (e.g., [K88]). It is noteworthy that for a polydisc, the bottom of the spectrum is always in the essential spectrum—a phenomenon not stipulated in the general operator theory.

2. Preliminaries

We first recall the setup for the $\bar{\partial}$-Neumann Laplacian. We refer the reader to [FoK72, CS99] for an in depth treatise of the $\bar{\partial}$-Neumann problem.

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. For $1 \leq q \leq n$, let $L^2_{(0,q)}(\Omega)$ denote the space of $(0,q)$-forms with square integrable coefficients and with the standard Euclidean inner product whose norm is given by

$$\|\sum' a_J d\bar{z}_J\|^2 = \sum' \int_{\Omega} |a_J|^2 dV(z),$$

where the prime indicates the summation over strictly increasing $q$-tuples $J$. (We consider $a_J$ to be defined on all $q$-tuples, antisymmetric with respect to $J$.) For $0 \leq q \leq n-1$, let

$$\bar{\partial}_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$$

be the $\bar{\partial}$-operator defined in the sense of distribution. This is

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a closed and densely defined operator. Let $\overline{\partial}_q^*$ be its adjoint. Then $\overline{\partial}_q^*$ is also a closed and densely defined operator with domain

$$\text{Dom}(\overline{\partial}_q^*) = \{u \in L^2_{(0,q+1)}(\Omega) \mid \exists C > 0 \text{ such that } |\langle u, \overline{\partial} v \rangle| \leq C\|v\|, \forall v \in \text{Dom}(\overline{\partial}_q)\}.$$  

For $1 \leq q \leq n - 1$, let

$$Q_q(u, v) = (\overline{\partial}_q u, \overline{\partial}_q v) + (\overline{\partial}_{q-1} u, \overline{\partial}_{q-1} v)$$

be the sesquilinear form on $L^2_{(0,q)}(\Omega)$ with $\text{Dom}(Q_q) = \text{Dom}(\overline{\partial}_q) \cap \text{Dom}(\overline{\partial}_{q-1})$. It is evident that $Q_q$ is non-negative, densely defined, and closed. The $\overline{\partial}$-Neumann Laplacian $\square_q = \overline{\partial}_q \overline{\partial}_q + \overline{\partial}_{q-1} \overline{\partial}_{q-1} : L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega)$ is the associated self-adjoint operator with domain $\text{Dom}(\square_q) = \{u \in L^2_{(0,q)}(\Omega) \mid u \in \text{Dom}(\overline{\partial}_q) \cap \text{Dom}(\overline{\partial}_{q-1}), \overline{\partial} u \in \text{Dom}(\overline{\partial}_q), \overline{\partial}_{q-1} u \in \text{Dom}(\overline{\partial}_{q-1})\}$.

For the reader’s convenience, we also briefly review relevant facts of the Bessel functions. Extensive treatment of the Bessel functions can be found, for example, in [W48]. The Bessel functions of integer orders are given via the following Laurent expansion:

$$e^{\pm (t - \frac{1}{2})} = \sum_{m=-\infty}^{\infty} t^m J_m(z).$$

Evidently, $J_{-m}(z) = (-1)^m J_m(z)$ and when $m \geq 0$,

$$J_m(z) = \sum_{l=0}^{\infty} \frac{(-1)^l (z/2)^{2l+m}}{l!(l+m)!}.$$

Hence $J_m(z)$ is an entire function with zero of order $|m|$ at the origin. By differentiating both sides of (2.1) with respect to $t$ and with respect to $z$, we have the recurrence formulas:

$$mJ_m(z) = \frac{z}{2}(J_{m+1}(z) + J_{m-1}(z)), \quad J'_m(z) = \frac{1}{2}(J_{m-1}(z) - J_{m+1}(z)).$$

Therefore,

$$zJ_{m-1}(z) = zJ'_m(z) + mJ_m(z), \quad zJ_{m+1}(z) = -zJ'_m(z) + mJ_m(z).$$

It follows that $J_m(z)$ satisfies the Bessel equation:

$$J''_m(z) + \frac{1}{z} J'_m(z) + (1 - \frac{m^2}{z^2})J_m(z) = 0.$$  

Thus $J_m(z)$ has only simple zeroes. On the other hand, by multiplying both sides of (2.1) by $t^{-m-1}$ then integrating on $|t| = 1$, we obtain the following integral representation of the Bessel functions:

$$J_m(z) = \frac{1}{2\pi} \int_0^{2\pi} \cos(m\theta - z\sin\theta) \, d\theta.$$  

From this integral representation, we know that $J_0(x)$ is positive on the interval $[k\pi, (k + 1/2)\pi]$ when $k$ is even and negative on the interval when $k$ is odd. It follows that $J_0(x)$ has infinitely many of zeroes on the positive real axis and all of these zeroes are on the intervals $((k + 1/2)\pi, (k + 1)\pi)$. From (2.2), we know that

$$J_{m-1}(z) = z^{-m} \frac{d}{dz}(z^m J_m(z)), \quad J_{m+1}(z) = -z^{-m} \frac{d}{dz}(z^{-m} J_m(z)).$$

Therefore,

$$zJ_{m-1}(z) = zJ'_m(z) + mJ_m(z), \quad zJ_{m+1}(z) = -zJ'_m(z) + mJ_m(z).$$

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$$J_{m-1}(z) = z^{-m} \frac{d}{dz}(z^m J_m(z)), \quad J_{m+1}(z) = -z^{-m} \frac{d}{dz}(z^{-m} J_m(z)).$$
It follows that $J_m(z)$ also has infinite many zeroes on the positive real axis. Furthermore, the zeroes of $J_m(z)$ and those of $J_{m+1}(z)$ interlace. Let $\lambda_{m,j}, j = 1, 2, \ldots,$ be the positive zeroes of $J_m(z)$, arranged in increasing order. Then it follows from (2.4) that

$$
\int_0^1 rJ_m(\lambda_{m,j}r)J_m(\lambda_{m,k}r) \, dr = \begin{cases} 
0, & j \neq k; \\
\frac{1}{2}J_{m+1}(\lambda_{m,j}), & j = k.
\end{cases}
$$

Furthermore, for any given integer $m$, $\{\sqrt{r}J_m(\lambda_{m,j}r)\}_{j=1}^{\infty}$ forms a complete orthogonal basis for $L^2(0,1)$. Moreover, it follows from (2.4) that for $m \geq 0$, $\{r^{1/2+m}\} \cup \{\sqrt{r}J_m(\lambda_{m+1,j}r)\}_{j=1}^{\infty}$ forms a complete orthogonal basis for $L^2(0,1)$ and so does $\{\sqrt{r}J_m(\lambda_{m-1,j}r)\}_{j=1}^{\infty}$ for $m > 0$.

3. The Computations

Let $P = P(a_1, \ldots, a_n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \|z_1\| < a_1, \ldots, \|z_n\| < a_n\}$. Write $\rho_j(z) = |z_j|^2 - a_j^2$. Then $P = \{z \in \mathbb{C}^n \mid \rho_j(z) < 0, j = 1, \ldots, n\}$. Suppose that

$$
u = \sum_{|J| = q} u_J dz J \in C^\infty(P),$$

For any integer $q$ between 1 and $n - 1$, we now solve the $\bar{\partial}$-Neumann boundary value problem:

\begin{align}
\Box_q u &= \lambda u; \\
u &\in \text{Dom}(\bar{\partial}^*_q); \\
\bar{\partial} q u &\in \text{Dom}(\bar{\partial}^q).
\end{align}

It follows from an easy integration by parts argument that $u \in \text{Dom}(\bar{\partial}^*_q)$ provided $u_{JK}(z) = 0$ when $|z_j| = a_j$ for any $(q - 1)$-tuple $K$ and $j \in \{1, \ldots, n\}$. Write $z_j = r_j e^{i\theta_j}$. Using separation of variables, we write

$$u_J(z) = \prod_{k=1}^n u^k_j(z_k).$$

Then $u \in \text{Dom}(\bar{\partial}^*_q)$ provided

\begin{equation}
 u^k_j(a_ke^{i\theta_k}) = 0, \quad \text{when } k \in J.
\end{equation}

For any $K = (k_1, \ldots, k_{q+1})$, write

$$v_K = \sum_{l=1}^{q+1} (-1)^l \frac{\partial u_{K\setminus k_l}}{\partial \bar{z}_{k_l}},$$

where $K \setminus k_l$ means the deletion of the $k_l$ entry from $K$. Then

$$\bar{\partial} u = \sum_{|K| = q+1} v_K d\bar{z}_K.$$ 

Thus $\bar{\partial} q u \in \text{Dom}(\bar{\partial}^q)$ if $v_{jJ}(z) = 0$ whenever $|z_j| = a_j$ for any $j \in \{1, \ldots, n\}$ and $q$-tuple $J$. Using the separation of variables (3.4), we have that $\bar{\partial} q u \in \text{Dom}(\bar{\partial}^q)$ provided, in addition to (3.3), $u_J$ also satisfies

\begin{equation}
\frac{\partial u^k_j}{\partial \bar{z}_k}(a_k e^{i\theta_k}) = 0, \quad \text{when } k \notin J.
\end{equation}
Recall that $\Box_q = (-1/4)\Delta$ where $\Delta$ is the usual Laplacian acting diagonally. Denote by $\Delta_k = 4(\partial^2/\partial z_k \partial \bar{z}_k)$ the Laplacian in the $z_k$-variable. Then, with the separation of variables (3.4), the boundary value problem (3.1)-(3.3) is reduced to:

\[(3.7) \quad \Delta_k u^k_j(z_k) = -\lambda_k u^k_j, \quad u^k_j(a_k e^{i\theta}) = 0, \quad \text{for } k \in J,\]

and

\[(3.8) \quad \Delta_k u^k_j(z_k) = -\lambda_k u^k_j(z_k), \quad \frac{\partial u^k_j}{\partial \bar{z}_k}(a_k e^{i\theta}) = 0, \quad \text{for } k \notin J,\]

with

\[(3.9) \quad \lambda = \frac{1}{4} \sum_{k=1}^{n} \lambda_k.\]

The boundary value problem (3.7) gives the eigenvalues for the Dirichlet Laplacian on the disc $|z_k| < a_k$. It is well known (and easy to see) that these eigenvalues are

\[(3.10) \quad \left(\frac{\lambda_{mk,jk}}{a_k}\right)^2\]

and the associated eigenfunctions are

\[(3.11) \quad J_{mk}(\lambda_{mk,jk} r_k/a_k) e^{im_k \theta_k},\]

for $m_k \in \mathbb{Z}$ and $j_k \in \mathbb{N}$.

To solve the boundary value problem (3.8), we separate the variables in polar coordinates:

\[(3.12) \quad \frac{\Theta''}{\Theta} = -\mu, \quad \Theta(\theta_k + 2\pi) = \Theta(\theta_k),\]

and

\[(3.13) \quad \frac{R''}{R} + \frac{1}{r_k} \frac{R'}{R} = -\lambda_k, \quad \frac{R'}{R}(a_k) = -\frac{i}{a_k} \frac{\Theta'}{\Theta}.\]

From (3.12), we know that $\mu = m_k^2$, $m_k \in \mathbb{Z}$, with the associated eigenfunctions $e^{im_k \theta_k}$. We first consider the case when $\lambda_k = 0$. In this case, we know from $\Theta = e^{im_k \theta_k}$ and (3.14) that $R = r_k^{m_k}$. Since by interior elliptic regularity, the eigenfunctions must be smooth at the origin, we know that 0 is an eigenvalue of the boundary value problem (3.8) with the associated eigenfunctions $z_k^{m_k}$. Now we consider the case when $\lambda_k > 0$. Using the substitution $r = \sqrt{\lambda_k} r_k$, we reduce (3.13) to

\[(3.14) \quad R'' + \frac{1}{r} R' + (1 - \frac{m_k^2}{r^2})R = 0, \quad \sqrt{\lambda_k} a_k R'(\sqrt{\lambda_k} a_k) = 0.\]

From (3.14), we know that $R = J_{mk}(r)$, and from (3.8), we know that $J_{mk+1}(\sqrt{\lambda_k} a_k) = 0$. In summary, from the boundary value problem (3.8), we obtain the eigenvalues

\[(3.15) \quad \left(\frac{\lambda_{mk+1,jk}}{a_k}\right)^2\]

with the associated eigenfunctions

\[(3.16) \quad J_{mk}(\lambda_{mk+1,jk} r_k/a_k) e^{im_k \theta_k},\]

for $m_k \in \mathbb{Z}$ and $j_k \in \mathbb{N}$.
From the above computations, we now know that the spectrum of $\Box_q$ on the polydisc $P$ contains the eigenvalues

\begin{equation}
\frac{1}{4} \sum_{k \in J} \left( \frac{\lambda_{m_k,j_k}}{a_k} \right)^2
\end{equation}

of infinite multiplicity with the associated eigenforms

\begin{equation}
\prod_{k \in J} \left( J_{m_k} \left( \frac{\lambda_{m_k,j_k} r_k}{a_k} \right) e^{i m_k \theta_k} \right) \prod_{k \notin J} \left( J_{m_k-1} \left( \frac{\lambda_{m_k,j_k} r_k}{a_k} \right) e^{i (m_k-1) \theta_k} \right) d\bar{z}_J,
\end{equation}

and eigenvalues

\begin{equation}
\frac{1}{4} \sum_{k=1}^{n} \left( \frac{\lambda_{m_k,j_k}}{a_k} \right)^2
\end{equation}

with the associated eigenforms

\begin{equation}
\prod_{k \in J} \left( J_{m_k} \left( \frac{\lambda_{m_k,j_k} r_k}{a_k} \right) e^{i m_k \theta_k} \right) \prod_{k \notin J} \left( J_{m_k-1} \left( \frac{\lambda_{m_k,j_k} r_k}{a_k} \right) e^{i (m_k-1) \theta_k} \right) d\bar{z}_J,
\end{equation}

for any strictly increase $q$-tuple $J$, $m_k \in \mathbb{Z}$, and $j_k \in \mathbb{N}$.

It remains to show that the spectrum of $\Box_q$ consists of nothing else but the eigenvalues listed in (3.17) and (3.19). To do this, we use the following well known fact from the general operator theory (e.g., [Dav95], Lemma 1.2.2): Let $T$ be a symmetric operator on a complex Hilbert space $H$. If there exists a complete orthonormal basis $\{f_j\}_{j=1}^{\infty}$ and $\lambda_j \in \mathbb{R}$ such that $T f_j = \lambda_j f_j$, then $T$ is essentially self-adjoint and the spectrum of $\overline{T}$ is the closure of $\{\lambda_j\}_{j=1}^{\infty}$ in $\mathbb{R}$. It follows from facts about the Bessel functions stated in the last paragraph of Section 2 that for each $q$-tuple $J$, the coefficients of $d\bar{z}_J$ in (3.18) and (3.20) form a complete orthogonal basis for $L^2(P)$. Thus the spectrum of $\Box_q$ contains nothing else but eigenvalues listed in (3.17) and (3.19) with associated eigenforms listed in (3.18) and (3.19) respectively. The bottom of the spectrum is

$$\min_{|J|=q} \left\{ \frac{\lambda_{0,1}^2}{4} \sum_{k \in J} \frac{1}{a_k^2} \right\},$$

which is always of infinite multiplicity.

Since we now know explicitly the spectrum and the associated eigenforms, it is then easy to explicitly express the $\overline{\partial}$-Neumann operator as an infinite sum of projections onto the eigenspaces. We left this to the interested reader.

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