\textbf{Introduction}

We consider problems of solving nonlinear equation of the form
\begin{equation}
F(x) = 0,
\end{equation}
and optimization problems of the form
\begin{equation}
\min \phi(x) \text{ subject to } F(x) = 0,
\end{equation}
where $\phi : X \to \mathbb{R}$ and $F : X \to Y$ is a sufficiently smooth mapping from a Banach space $X$ to a Banach space $Y$. Nonlinear problems \begin{equation} \text{(1)} \end{equation} and \begin{equation} \text{(2)} \end{equation} can be divided into two classes: regular (nonsingular) and singular depending on whether $F$ is regular or singular. Roughly speaking, regular mappings are those for which implicit function theorem arguments can be applied and singular problems are those for which they cannot, at least not directly.

In this work, we give an overview of methods and tools of the $p$-regularity theory in application to the investigation of singular (irregular, degenerate) mappings and singular equality constrained optimization problems. The purpose of this paper is to present selected works in this area in a coherent way, which have been scattered throughout various references.

1. Essential nonlinearity and singular maps

Fix a point $x^* \in X$ and suppose that $F : X \to Y$ is $C^1(W)$, where $W$ is a neighborhood of $x^*$. The mapping $F$ is regular at $x^*$, if
\begin{equation}
\text{Im } F'(x^*) = Y.
\end{equation}

The following lemma on the local representation of regular mapping holds.
We say the mapping

\[ \varphi(0) = I \]

\[ \text{to be regular}; \] that is, its derivative is not onto:

\[ F \]

On the other hand, \( \varphi \) nondegenerate transformation of coordinates) which cannot be trivialized, i.e. there does not exist any diffeomorphism (i.e. a perturbation of the form \( \tilde{\varphi}(0) = 0 \), in general.

There exist numerous mappings which do not admit local trivialization. The concept of essentially nonlinear mappings defined in [29] formalizes this situation.

**Definition 1.2.** Let \( V \) be a neighborhood of \( x^* \) in \( X \) and \( U \subset X \) be a neighborhood of 0. A mapping \( F : V \to Y, F \in C(V), \) is essentially nonlinear at \( x^* \) if there exists a perturbation of the form \( \tilde{F}(x^* + x) = F(x^* + x) + \omega(x), \) where \( \parallel \omega(x)\parallel = o(\parallel x\parallel) \), which cannot be trivialized, i.e. there does not exist any diffeomorphism (i.e. a nondegenerate transformation of coordinates) \( \varphi(x) : U \to V, \varphi \in C(U), \) such that \( \varphi(0) = x^*, \varphi'(0) = I_X \) and \( \tilde{F} \).

**Definition 1.3.** We say the mapping \( F \) is singular (or degenerate) at \( x^* \) if it fails to be regular; that is, its derivative is not onto:

\[ \text{Im } F'(x^*) \neq Y. \]

Let us note that, if \( F \) is singular at the point \( x^* \), \( F(x^*) = 0 \), i.e., there exists \( 0 \neq \xi \in Y, \parallel \xi\parallel = 1, \)

\[ \xi \notin \text{Im } F'(x^*) \]

then \( F \) must be essentially nonlinear at \( x^* \). Indeed, suppose that \( F \) is not essentially nonlinear at \( x^* \) and define the mapping \( \tilde{F} : V \to Y \) as

\[ \tilde{F}(x^* + x) := F(x^*) + F'(x^*)x + \xi\parallel x\parallel^2. \]

Note that \( \parallel \xi\parallel^2 \notin \text{Im } F'(x^*) \) for any \( x \in V \). By Definition 1.2, there exist a neighborhood \( U \) of 0 and a mapping \( \varphi(x) : U \to V, \varphi \in C(U), \) such that \( \varphi(0) = x^*, \varphi'(0) = I_X \) and

\[ \tilde{F}(\varphi(x)) = \tilde{F}(x^*) + \tilde{F}'(x^*)x = F(x^*) + F'(x^*)x \quad \text{for } x \in U. \]

Since \( F'(x^*)x \in \text{Im } F'(x^*), \) by [3] we have

\[ \tilde{F}(\varphi(x)) \in \text{Im } F'(x^*). \]

On the other hand, \( \varphi(0) = x^* \) and \( \varphi'(0) = I_X, \) and
\[ F(\varphi(x)) = F(x^* + (\varphi(x) - x^*)) \]
\[ = F(x^*) + F'(x^*)(\varphi(x) - x^*) + \xi \|\varphi(x) - x^*\|^2 \]
\[ = F'(x^*)(\varphi(x) - x^*) + \xi \|\varphi(0) + \varphi'(0)x + \omega_1(x) - x^*\|^2 \]
\[ = F'(x^*)(\varphi(x) - x^*) + \xi \|x + \omega_1(x)\|^2, \]
where \(\|\omega_1(x)\| = o(\|x\|).\) Thus, for small \(x,\)
\[ \xi \|x + \omega_1(x)\|^2 \neq 0. \]

Taking into account (6), (10) and the fact that \(F'(x^*)(\varphi(x) - x^*) \in \text{Im } F'(x^*),\) we conclude from this that
\[ \tilde{F}(\varphi(x)) \notin \text{Im } F'(x^*). \]
This contradicts (9) and therefore \(F\) is essentially nonlinear at \(x^*\).

The following theorem (see [29]) establishes the relationship between essential nonlinearity and irregularity.

**Theorem 1.4** ([29]). Suppose \(F : V \to Y\) is \(C^2(V),\) where \(V\) is a neighborhood of \(x^*\) in \(X\) and \(F(x^*) = 0.\) Then \(F\) is essentially nonlinear at the point \(x^*\) if and only if \(F\) is singular at the point \(x^*\).

## 2. Examples of singular problems

### 2.1. Description of the solution set. Lyusternik theorem.

Let \(X, Y\) be Banach spaces. Consider the nonlinear equation (1)
\[ F(x) = 0, \]
where \(F : X \to Y, F \in C^{p+1}(X), p \in \mathbb{N}.\) According to Lyusternik theorem (see [18]), if \(F\) is regular at \(x^*\), then \(T_1M(x^*) = \text{Ker } F'(x^*),\) where \(T_1M(x^*)\) is the tangent cone to the set \(M(x^*) = \{x \in X : F(x) = F(x^*) = 0\}\) at the point \(x^*\). The tangent cone to \(M\) at \(x^*\) is the collection of all tangent vectors to \(M\) at \(x^*\) i.e. \(h\) is a tangent vector to \(M\) at \(x^* \in M\) if there exist \(\varepsilon > 0\) and a function \(r : [0, \varepsilon) \to X\) with the property that for \(t \in [0, \varepsilon]\) we have \(x^* + th + r(t) \in M\) and
\[ \lim_{t \to 0} \frac{\|r(t)\|}{t} = 0. \]

If \(F\) is singular at the solution point \(x^*,\) then \(T_1M(x^*) \neq \text{Ker } F'(x^*).\)

For example, if \(F(x) = x_1^2 - x_2^2 + o(\|x\|^2)\) and \(x^* = 0\) then \(\text{Ker } F'(0) = \mathbb{R}^2\) and moreover \(T_1M(0) = \{(t) : t \in \mathbb{R}\} \cup \{(\frac{1}{2}t) : t \in \mathbb{R}\},\) hence \(\text{Ker } F'(0) \neq T_1M(0).\)

The problem of description of the solution sets in more general situations (e.g. general systems of inequalities) is qualitatively approached by means of metric regularity ([11][13][14]) and via geometrical derivability ([23]).

### 2.2. Optimality conditions. Lagrange multiplier theorem.

Consider the optimization problem (2).
\[ \min \phi(x) \quad \text{subject to } F(x) = 0, \]
where \(\phi : X \to \mathbb{R}, \phi \in C^2(X)\) and \(F : X \to Y, F \in C^{p+1}(X), p \in \mathbb{N}.\)

Let \(x^*\) be a solution to (2). In the regular case, that is if \(F'(x^*) \cdot X = Y,\) there exists \(\lambda^* \in Y^*\) such that \(\phi'(x^*) = F'(x^*)^* \cdot \lambda^*.\)
Let us consider (2), where $X = \mathbb{R}^3$, $Y = \mathbb{R}^2$, $\phi(x) = x_2^2 + x_3$ and $F(x) = \left( \begin{array}{c} x_1^2 - x_2^2 + x_3^2 \\ x_1^2 - x_2^2 + x_2^2 + x_2x_3 \end{array} \right)$. In this case $x^* = (0, 0, 0)^T$ and we can easily obtain that $\phi'(x^*) = (0, 0, 1)^T$, $F'(x^*) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$. However, it is obvious that $\phi'(x^*) \neq F'(x^*)^T \cdot \lambda^*$.

There is a vast literature concerning optimality conditions for general regular (satisfying some constraint qualification condition) constrained optimization problems (see e.g. Chapter 3 of [3]).

2.3. Newton method for singular equations. Consider the problem of solving nonlinear equation (1), i.e. $F(x) = 0$ and let $F$ be singular at $x^*$.

In the finite dimensional case, when $X = \mathbb{R}^n, Y = \mathbb{R}^n$ and $F(x) = (f_1(x), \ldots, f_n(x))^T$, singularity of $F$ at $x^*$ means that the Jacobian $F'(x^*)$ of $F$ at $x^*$ is singular as in the following example.

Example 1 ([24]). Let $F(x) = \left( \begin{array}{c} x_1 + x_2 \\ x_1x_2 \end{array} \right)$, $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $x^* = (0, 0)^T$ is a solution to (1) and $F'(x^*) = \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right)$ is singular (degenerate) at $x^*$.

Let $x_0 = (x_{01}, x_{02})^T$ and $x_0 \in U_{\varepsilon}(0)$, $\varepsilon > 0$ be sufficiently small. Then, for classical Newton method, i.e. (11) $x_{k+1} = x_k - \{F'(x_k)\}^{-1}F(x_k), \ k = 0, 1, 2, 3, \ldots,$ we have $x_1 = \frac{1}{x_{01} - x_{02}} \cdot (-x_{01}x_{02}, x_{01}x_{02})^T$.

If $x_{01} = x_{02}$ then $\{F'(x_0)\}^{-1}$ does not exist, hence (11) is not applicable.

But even ever $\{F'(x_0)\}^{-1}$ exists, e.g. for $x_0 = (t + t^3, t)^T$, we have $x_1 = \left( -\frac{1}{t} - t, \frac{1}{t} + t \right)^T$ and $\|x_1 - 0\| \approx \frac{1}{t} \rightarrow \infty$, when $t \rightarrow 0$. For instance, if $t = 10^{-5}$ then $\|x_1 - 0\| \approx 10^5$ and we have rejecting effect.

Example 2. [22] Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$F(x) := \left( \begin{array}{c} x_1 + x_1x_2 + x_2^2 \\ x_1^2 - 2x_1 + x_2^2 \end{array} \right).$$

The singular root is $x^* = (0, 0)^T$, null space is $\text{Ker} F(x^*) = \text{span}\{(0, 1)\}$ and range space is $\text{Im} F(x^*) = \text{span}\{(1, -2)\}$. The Jacobian $F'(x)$ is singular on the hyperbole given by

$$2x_1 - 2x_1^2 + 6x_2 - 4x_1x_2 + 2x_2^2 = 0.$$

For the overview of the existing approaches to Newton-like methods for singular operators, see e.g. [10].

2.4. Newton method for unconstrained optimization problems. Consider the following problem,

$$\min_{x \in \mathbb{R}^2} \phi(x)$$

and the scheme

$$x_{k+1} = x_k - \{\phi''(x_k)\}^{-1}\phi'(x_k), \quad (12)$$
where \( \phi : \mathbb{R}^2 \to \mathbb{R}, \phi(x) = x_1^2 + x_1^3 x_2 + x_1^4 \) (see [24]).

The solution of the considered problem is \( x^* = (0, 0)^T \). At the initial point, \( x_0 = (x_{01}, x_{02})^T \) where \( x_{01} = x_{02} \sqrt{6(1 + x_{02})} \) we have

\[
\phi''(x_0) = \left( \begin{array}{ccc} 2 + 2x_{02} & 2x_{02} \sqrt{6(1 + x_{02})} & 12x_{02}^2 \end{array} \right)
\]

and \( \det \phi''(x_0) = 0 \), hence does not exist \( \{\phi''(x_0)\}^{-1} \) and it follows that (12) is not applicable.

2.5. Singular problems of calculus of variations. Consider the following Lagrange problem (see [19]):

\[
J_0(x) = \int_{t_1}^{t_2} f(t, x, x') dt \to \min
\]

subject to the subsidiary conditions

\[
G(x) = G(t, x, x') = 0, \quad q(x(t_1), x(t_2)) = 0
\]

where \( x \in C^1([t_1, t_2], \mathbb{R}^n) \), \( G : X \to Y, G \in C^{p+1}(X), X = C^1([t_1, t_2], \mathbb{R}^n), Y = C^1([t_1, t_2], \mathbb{R}^m) \), \( G(t, x, x') = (G_1(t, x, x'), \ldots, G_m(t, x, x')) \), \( q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k \). We assume that all mappings and their derivatives are continuous with respect to the corresponding variables \( t, x, x' \).

In the regular case, if \( \text{Im} G'(x^*) = Y \), where \( x^*(t) \) is a solution to (13–14), then necessary conditions of Euler-Lagrange

\[
f_x + \lambda(t)G_x - \frac{d}{dt}(f_{x'} + \lambda(t)G_{x'}) = 0
\]

are satisfied.

Let \( \lambda := (\lambda_1, \ldots, \lambda_m)^T \), \( \lambda(t)G := \lambda_1(t)G_1 + \cdots + \lambda_m(t)G_m \), \( \lambda(t)G_x := \lambda_1(t)G_{1x} + \cdots + \lambda_m(t)G_{mx} \). In the singular case, when \( \text{Im} G'(x^*) \neq Y \), we can only guarantee that the following equations

\[
\lambda_0 f_x + \lambda(t)G_x - \frac{d}{dt}(\lambda_0 f_{x'} + \lambda(t)G_{x'}) = 0
\]

are satisfied, where \( \lambda_0^2 + \|\lambda(t)\|^2 = 1 \), i.e. \( \lambda_0 \) might be equal to 0 and then we have not got any constructive information on \( f \).

Example 3 ([19]). Consider the following problem

\[
J_0(x) = \int_0^{2\pi} (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2) dt \to \min
\]

subject to

\[
G(x) = \begin{pmatrix} x_1 - x_2 + x_3^2 x_1 + x_1^3 x_2 - x_1^2 (x_1 + x_2) \\ x_2 + x_1 + x_3^2 x_2 - x_1^2 x_3 - x_2^2 (x_2 - x_1) \end{pmatrix} = 0,
\]

\( x_i(0) = x_i(2\pi), \quad i = 1, \ldots, 5 \).

Here \( f(x) := x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \) and \( q_i(x(0), x(2\pi)) := x_i(0) - x_i(2\pi), \quad i = 1, \ldots, 5. \)

The solution of (17–18) is \( x^*(t) = 0 \) and \( G'(x^*(t)) \) is singular.

Indeed, \( G'(0) = \begin{pmatrix} (\cdot)_1' - (\cdot)_2' \\ (\cdot)_1' + (\cdot)_2' \end{pmatrix} \) and \( G'(0)x = \begin{pmatrix} x_1' - x_2 \\ x_3' + x_1 \end{pmatrix} \).

Let \( z(t) := x_1(t) \). Thus, we can consider the following equivalent problem: whether the mapping \( \tilde{G}(z) = z'' + z, \quad z(0) = z(2\pi) \) is surjection or not.
It is obvious that for \( y \in C[0, 2\pi] \), such that
\[
\int_{0}^{2\pi} \sin \tau \ y(\tau) d\tau \neq 0 \text{ or } \int_{0}^{2\pi} \cos \tau \ y(\tau) d\tau \neq 0,
\]
the equation \( z'' + z = y \) does not have a solution.

The corresponding Euler-Lagrange equations in this case are as follows:
\[
\begin{align*}
2\lambda_0 x_1 + \lambda_2 - \lambda_1' x_1^2 + \lambda_1 x_3^2 + \lambda_2 x_5^2 - \lambda_2 x_4^2 &= 0 \\
2\lambda_0 x_2 - \lambda_1 - \lambda_1' x_1^2 + \lambda_1 x_4^2 + \lambda_2 x_3^2 - \lambda_2 x_5^2 &= 0 \\
2\lambda_0 x_3 + 2\lambda_1 x_1 x_3 + 2\lambda_2 x_2 x_3 &= 0 \\
2\lambda_0 x_4 + 2\lambda_1 x_2 x_4 - \lambda_2 x_1 x_4 &= 0 \\
2\lambda_0 x_5 - 2\lambda_1 x_5 x_1 - 2\lambda_1 x_2 x_5 - 2\lambda_2 x_2 x_5 + 2\lambda_2 x_1 x_5 &= 0 \\
\lambda_i(0) &= \lambda_i(2\pi), \ i = 1, 2.
\end{align*}
\]
Unfortunately, we cannot guarantee that \( \lambda_0 \neq 0 \) and for \( \lambda_0 = 0 \) we obtain the series of spurious solutions to the system \((17)-(18)\):
\[
x_1 = a \sin t, \ x_2 = a \cos t, \ x_3 = x_4 = x_5 = 0, \ \lambda_1 = b \sin t, \ \lambda_2 = b \cos t, \ a, b \in \mathbb{R}.
\]

**2.6. Modified Lagrange function method.** Consider the following constrained optimization problem
\[
(19) \quad \min \phi(x) \quad \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m,
\]
where \( \phi : \mathbb{R}^n \to \mathbb{R}, \ g_i : \mathbb{R}^n \to \mathbb{R} \) and the modified Lagrangian function \( L_E(x, \lambda), \ L_E : \mathbb{R}^{n+m} \to \mathbb{R} \) associated with \((19)\) (see e.g. \[7\] \[12\], cf. \[3\]),
\[
L_E(x, \lambda) := \phi(x) + \frac{1}{2} \sum_{i=1}^{m} \lambda_i^2 g_i(x).
\]
This modification allows to replace a nonlinear optimization problem with a system of nonlinear equations. Moreover, let us define the mapping \( G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n+m} \),
\[
(20) \quad G(x, \lambda) := \begin{pmatrix}
\nabla \phi(x) + \frac{1}{2} \sum_{i=1}^{m} \lambda_i^2 \nabla g_i(x) \\
D(\lambda) g(x)
\end{pmatrix},
\]
where \( D(\lambda) := \text{diag}\{\lambda_i\}, \ i = 1, \ldots, m, \ \lambda \in \mathbb{R}^m. \)

Consider the equation,
\[
(21) \quad G(x, \lambda) = 0_{n+m}.
\]
For \( G(x, \lambda) \) the Jacobian matrix \( G'(x, \lambda) \) is given by
\[
G'(x, \lambda) = \begin{pmatrix}
\nabla^2 \phi(x) + \frac{1}{2} \sum_{i=1}^{m} \lambda_i^2 \nabla^2 g_i(x) & (g'(x))^T D(\lambda) \\
D(\lambda) g(x) & D(g(x))
\end{pmatrix}.
\]
If the solution point of \((21)\) is \((x^*, \lambda^*)\), such that \( g_i(x^*) = 0 \) and \( \lambda_i^* = 0 \) then strict complementarity condition (SCQ) defined as \( I_0(x^*) = \{ j = 1, 2, \ldots, m : \lambda_i^* = 0, g_j(x^*) = 0 \} \neq 0 \) fails. Consequently, \( G'(x^*, \lambda^*) \) is a degenerate matrix. The example below illustrates the situation.
Example 4. [7] Consider the following problem
\begin{equation}
\min_{x \in \mathbb{R}^n} (x_1^2 + x_2^2 + 4x_1x_2) \quad \text{subject to} \quad x_1 \geq 0, \ x_2 \geq 0.
\end{equation}

It is easy to see that \( x^* = (0, 0)^T \) is the solution to (22) with the corresponding Lagrange multiplier \( \lambda^* = (0, 0)^T \).

The modified Lagrange function in this case is
\[ L_E = x_1^2 + x_2^2 + 4x_1x_2 - \frac{1}{2} \lambda_1^2 x_1 - \frac{1}{2} \lambda_2^2 x_2 \]
and the Jacobian matrix \( G'(x^*, \lambda^*) \) of
\[ G(x, \lambda) = \begin{pmatrix} 2x_1 + 4x_2 - \frac{1}{2} \lambda_1^2 & 2x_2 + 4x_1 - \frac{1}{2} \lambda_2^2 \\ -\lambda_1 x_1 & -\lambda_2 x_2 \end{pmatrix} \]
is singular.

3. Elements of p-regularity theory

Let us recall the basic constructions of p-regularity theory, whose basic concepts and main results are described e.g. in [15, 28].

Suppose that the space \( Y \) is decomposed into a direct sum
\begin{equation}
Y = Y_1 \oplus \ldots \oplus Y_p,
\end{equation}
where \( Y_1 = \text{Im} \ F'(x^*) \), \( Z_1 = Y \). Let \( Z_2 \) be closed complementary subspace to \( Y_1 \) (we assume that such closed complement exists), and let \( P_{Z_2} : Y \to Z_2 \) be the projection operator onto \( Z_2 \) along \( Y_1 \). By \( Y_2 \) we mean the closed linear span of the image of the quadratic form \( P_{Z_2} F'(x^*) \) \( Y \). More generally, define inductively,
\[ Y_i = \text{span} \ \text{Im} \ P_{Z_i} F'(x^*) \] \( Y_i \subseteq Z_i \), \( i = 2, \ldots, p - 1 \),
where \( Z_i \) is a chosen closed complementary subspace for \( (Y_1 \oplus \ldots \oplus Y_i - 1) \) with respect to \( Y_i \), \( i = 2, \ldots, p \). Finally, \( Y_p = Z_p \).

The order \( p \) is chosen as the minimum number for which (23) holds. Let us define the following mappings
\[ F_i(x) = P_{Y_i} F(x), \quad F_i : X \to Y_i \quad i = 1, \ldots, p, \]
where \( P_{Y_i} : Y \to Y_i \) is the projection operator onto \( Y_i \) along \( (Y_1 \oplus \ldots \oplus Y_i - 1 \oplus Y_{i+1} \oplus \ldots \oplus Y_p) \) with respect to \( Y_i \), \( i = 1, \ldots, p \).

Definition 3.1. The linear operator \( \Psi_p(h) \in \mathcal{L}(X, Y_1 \oplus \ldots \oplus Y_p), \ h \in X, \ h \neq 0 \)
\begin{equation}
\Psi_p(h) = F_1'(x^*) + F_2''(x^*) h + \cdots + F_p''(x^*) h^{p-1},
\end{equation}
is called the \( p \)-factor operator.

Example 5. For \( p = 2 \) the formula (23) takes the form
\begin{equation}
\Psi_2(h) = F_1'(x^*) + F_2''(x^*) h,
\end{equation}
where \( 0 \neq h \in X \).

Consider the operator \( F : \mathbb{R}^2 \to \mathbb{R}^2 \), from the Example[1] where
\[ F(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \end{pmatrix}. \]
It was shown that the Jacobian of $F(x)$ is singular at $x^* = (0, 0)^T$, hence $\text{Im} F'(x^*) = \text{span}\{ (1, 0) \} \neq \mathbb{R}^2$ and hence $Y_1 = \text{span}\{ (1, 0) \}$ and $Y_2 = \text{span}\{ (0, 1) \}$.

To construct 2-factor operator we use the projections

$$P_{Y_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{Y_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and define the operators $F_1 : \mathbb{R}^2 \to Y_1$ and $F_2 : \mathbb{R}^2 \to Y_2$. They are as follows,

$$F_1(x) := \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix}, \quad F_2(x) := \begin{pmatrix} 0 \\ x_1 x_2 \end{pmatrix},$$

Hence, for $h \in \mathbb{R}^2$, the 2-factor operator has the form

$$\Psi_2(h)(x) := \begin{pmatrix} x_1 + x_2 \\ h_2 x_1 + h_1 x_2 \end{pmatrix}, \quad \text{where} \quad h = (h_1, h_2)^T.$$ 

It is easy to see that if $h_1 \neq h_2$ then 2-factor operator is surjective.

**Definition 3.2.** We say that the mapping $F$ is $p$-regular at $x^*$ along an element $h$, if

$$\text{Im} \, \Psi_p(h) = Y.$$ 

As we see from the Example 5, a given mapping $F$ may not be regular with respect to all $0 \neq h \in X$.

**Remark 1.** The condition of $p$-regularity of the mapping $F$ at the point $x^*$ along $h$ is equivalent to the following condition

$$\text{Im} \, F^{(p)}(x^*)[h][p-1] \circ \text{Ker} \, \Psi_{p-1}(h) = Y_p,$$

where $\Psi_{p-1}(h) := F'_1(x^*) + F''_2(x^*) + \cdots + F^{(p-1)}(x^*)[h][p-2].$

**Definition 3.3.** We say that the mapping $F$ is $p$-regular at $x^*$ if it is $p$-regular along any $h \in X$ from the set

$$H_p(x^*) := \left\{ \bigcap_{k=1}^p \text{Ker}^k F^{(k)}(x^*) \right\} \setminus \{0\} \neq \emptyset,$$

where $k$-kernel of the $k$-order mapping $F^{(k)}(x^*)$ is defined as

$$\text{Ker}^k F^{(k)}(x^*) := \{ \xi \in X : F^{(k)}(x^*)(\xi)^k = 0 \}.$$ 

In the Example 3, we have $\text{Ker}^1 F'_1(x^*) = \text{span}\{ (1, -1) \}$ and $\text{Ker}^2 F''_2(x^*) = \text{span}\{ (1, 0) \} \cup \text{span}\{ (0, 1) \}$. It means that $H_2 = \emptyset$. As we see, it may happen that $F$ is $p$-regular along some $h \in X$ but $H_p = \emptyset$. Hence, according to Definition 3.2 $F$ is 2-regular at $x^*$ along any $h \in X$, $h_1 \neq h_2$ and is not 2-regular at $x^*$.

For a linear surjective operator $\Psi_p(h) : X \to Y$ between Banach spaces we denote by $\{ \Psi_p(h) \}^{-1}$ its right inverse. Therefore $\{ \Psi_p(h) \}^{-1} : Y \to 2^X$ and we have

$$\{ \Psi_p(h) \}^{-1}(y) = \{ x \in X : \Psi_p(h)x = y \}.$$ 

We define the norm of $\{ \Psi_p(h) \}^{-1}$ via the formula

$$\| \{ \Psi_p(h) \}^{-1} \| = \sup_{\| y \| = 1} \inf \{ \| x \| : x \in \{ \Psi_p(h) \}^{-1}(y) \}.$$ 

We say that $\{ \Psi_p(h) \}^{-1}$ is bounded if $\| \{ \Psi_p(h) \}^{-1} \| < \infty$. 

4. Singular problems via p-regularity theory

4.1. Generalized Lyusternik theorem. The following theorem gives a description of the solution set in the singular case (for the proof see [26]).

**Theorem 4.1** ([26], Generalized Lyusternik Theorem). Let $X$ and $Y$ be Banach spaces and $U$ be a neighborhood of $x^* \in X$. Assume that $F : X \to Y$, $F \in C^p(U)$ is $p$-regular at $x^*$. Then

$$T_1 M(x^*) = H_p(x^*).$$

The problem of description of the tangent cone to solution set of the operator equation with the singular mappings has been also considered e.g. in [2, 8, 17, 27].

We now give another version of the Theorem 4.1. To state the result, we shall define the tangent cone at a point $x \in X$. Let $h \in X$ be a neighborhood of $x^*$. Then there exist a neighborhood $\delta > 0$ such that $\delta \in \{\Psi_p(h)\}^{-1} \leq \infty$, where

$$H_\delta = \{h \in X : \left\|F_i(x^*)|X|\right\| \leq \alpha \text{ for all } i = 1, \ldots, p, \|h\| = 1\}.$$

**Theorem 4.3** ([20]). Let $X$ and $Y$ be Banach spaces, and $U$ be a neighborhood of a point $x^* \in X$. Assume that $F : X \to Y$ is a $p$-times continuously Fréchet differentiable mapping in $U$ and satisfies the condition of strong $p$-regularity at $x^*$. Then there exist a neighborhood $U' \subseteq U$ of $x^*$, a mapping $\xi \mapsto x(\xi) : U' \to X$, and constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $F(\xi + x(\xi)) = F(x^*)$, and

$$\|x(\xi)\| \leq \delta_1 \sum_{i=1}^p \left\|f_i(\xi) - f_i(x^*)\right\| \leq \alpha \|x^*\|^i$$

for all $\xi \in U'$.

For the proof, see [15] and [20].

Consider the mapping $F(x) = \left(\begin{array}{c} x_1^2 - x_2^2 + x_3^2 \\ x_1^2 - x_2^2 + x_3^2 + x_2 x_3 \end{array}\right)$ from the Section 2.2 and recall that $x^* = (0, 0, 0)^T$. It is easy to see that $F'(x^*) = 0$, $F''(x^*) = \left(\begin{array}{c} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right)$ and $\text{Ker}^2 F''(0) = \text{span} \left\{(1, -1, 0)\right\} \cup \text{span} \left\{(1, 1, 0)\right\}$. The tangent cone at $x^*$ in this case is as follows $T_1 M(0) = \text{span} \left\{(1, -1, 0)\right\} \cup \text{span} \left\{(1, 1, 0)\right\}$. Let $h = (1, 1, 0)^T$ (or $h = (1, -1, 0)^T$) then $\text{Im} F''(0) h = \mathbb{R}^2$. It means that the mapping $F(x)$ is 2-regular at $x^* = 0$ and in this case $\text{Ker}^2 F''(0) = H_2(0) = T_1 M(0)$. 

4.2. Optimality conditions for $p$-regular optimization problems. We define $p$-factor Lagrange function

$$\mathcal{L}_p(x, \lambda, h) := \phi(x) + \left( \sum_{k=1}^{p} F_k^{(k-1)}(x)[h]^{k-1}, \lambda \right),$$

where $\lambda \in Y^*$ and

$$\bar{\mathcal{L}}_p(x, \lambda, h) := \phi(x) + \left( \sum_{k=1}^{p} \frac{2}{k(k+1)} F_k^{(k-1)}(x)[h]^{k-1}, \lambda \right).$$

To derive optimality conditions for $p$-regular problems we use Definition 1.2

**Theorem 4.4** ([27]. Necessary and sufficient conditions for optimality). Let $X$ and $Y$ be Banach spaces, $\phi \in C^2(X)$, $F \in C^{p+1}(X)$, $F: X \to Y$, $\phi: X \to \mathbb{R}$. Suppose that $h \in H_p(x^*)$ and $F$ is $p$-regular along $h$ at the point $x^*$. If $x^*$ is a local solution to the problem (29) then there exist multipliers, $\lambda^*(h) \in Y^*$ such that

$$\mathcal{L}_p^*(x^*, \lambda^*(h), h) = 0.$$

Moreover, if $F$ is strongly $p$-regular at $x^*$, there exist $\alpha > 0$ and a multiplier $\lambda^*(h)$ such that (27) is fulfilled and

$$\bar{\mathcal{L}}_p(x^*, \lambda^*(h), h)[h]^2 \geq \alpha||h||^2.$$

for every $h \in H_p(x^*)$, then $x^*$ is a strict local minimizer to the problem (2).

**Example 6.** Consider the problem from the Section 2.2

$$x_2^2 + x_3 \to \text{min subject to } F(x) = \left( \begin{array}{c} x_1^2 - x_2^2 + x_3^2 \\ x_1^2 - x_2^2 + x_3^2 + x_2x_3 \end{array} \right) = 0.$$

It is easy to verify that the point $x^* = 0$ is a local minimum to (29).

We have shown in the Section 5.1 that $F$ is singular at $x^*$ and for $h = (1, 1, 0)^T$ the mapping $F(x)$ is 2-regular at $x^* = 0$ along $h$. Consider the 2-factor-Lagrange function with $\lambda_0 = 0$. In this case it has the form

$$\mathcal{L}_2(x, \lambda(h), h) = x_2^2 + x_3 + \alpha(x_1 - x_2) + \beta(x_1 - x_2 + x_3),$$

where $\lambda(h) = (\lambda_1(h), \lambda_2(h))$ and $\lambda_1(h) = (0, 0)^T$, $\lambda_2(h) = (\alpha, \beta)^T$. Using the equality $\mathcal{L}_{2,2}^*(x^*, \lambda(h), h) = 0$ we obtain $\alpha = 1$ and $\beta = -1$. Putting the coefficients into we have $\bar{\mathcal{L}}_2(x^*, \lambda(h), h) = \frac{2}{5}||h||^2$. Therefore, $\mathcal{L}_{2,2}^*(x^*, \lambda(h), h)[h]^2 = \frac{4}{5} > 0$. It means that $x^*$ is a strict local minimizer to (29).

4.3. $p$-factor Newton method. Based on the $p$-factor operator construction we describe a method for solving nonlinear equations of the form (1), where $F: \mathbb{R}^n \to \mathbb{R}^n$ and the matrix $F'(x^*)$ is singular at the solution point $x^*$ (see [7, 24]).

Let $Y_1 = \text{Im} F'(x^*)$, $P_1 = P_{Y_1}^\perp$, $Y_2 = \text{Im} (F''(x^*) + P_1 F'(x^*)h)$, $P_2 = P_{Y_2}^\perp$,

$$Y_{k+1} = \text{Im} \left( F'(x^*) + \sum_{i=1}^{k} \hat{P}_i F''(x^*)h + \sum_{i_2 > 1} \hat{P}_i \hat{P}_i F^{(3)}(x^*)[h]^2 + \cdots + \right.$$

$$\left. + \sum_{i_2 > 1} \hat{P}_i \cdots \hat{P}_i F^{(k)}(x^*)[h]^{(k-1)} \right)$$

and $\hat{P}_{k+1} = P_{Y_{k+1}}^\perp$, $k = 2, p - 1$. 

Then the principal scheme of the $p$-factor Newton method is as follows
\[
x_{k+1} = x_k - \left\{ F'(x_k) + P_1 F''(x_k)h + \ldots + P_{p-1} F^{(p-1)}(x_k)h^{p-1} \right\}^{-1} \cdot \left\{ F(x_k) + P_1 F'(x_k)h + \ldots + P_{p-1} F^{(p-1)}(x_k)h^{p-1} \right\},
\]
(30)
where $P_k = \sum_{i=1}^{p-1} \bar{P}_i$, $P_2 = \sum_{i_2 > i_1} \bar{P}_{i_2} \bar{P}_{i_1}$, $P_{k+1} = \sum_{i_k > \ldots > i_1} \bar{P}_{i_k} \ldots \bar{P}_{i_1}$,
\[
k = \frac{2}{p-1}
\]
and $h$ such that $\|h\| = 1$ is fixed. $P_1, i = 1, p-1$ are matrices of orthoprojection at the solution point $x^*$. Note that for
\[
F(x^*) + P_1 F'(x^*)h + \ldots + P_{p-1} F^{(p-1)}(x^*)h^{p-1} = 0
\]
the $p$-factor matrix
\[
F'(x^*) + P_1 F''(x^*)h + \ldots + P_{p-1} F^{(p)}(x^*)h^{p-1}
\]
is not singular (it follows from the $p$-regularity along $h$). It means that $\bar{P}_p = 0$, $Y_p = \mathbb{R}^n$.
Consider the case $p = 2$ for the Example 1
\[
x_{k+1} = x_k - \left\{ F'(x_k) + P_1 F''(x_k)h \right\}^{-1} \cdot \left\{ F(x_k) + P_1 F'(x_k)h \right\}
\]
where $P_1$ is orthoprojection onto $\text{Im}(F'(x^*))$ and we choose element $h$ ($\|h\| = 1$), such that 2-factor matrix
\[
F'(x^*) + P_1 F''(x^*)h
\]
is not singular (in fact, it means that $F$ is 2-regular along $h$). Then at the solution point the formula $F(x^*) + P_1 F'(x^*)h = 0$ is satisfied, hence we can solve the equation $F(x) + P_1 F'(x)h = 0$ and by virtue of (33), $x^*$ is a locally unique solution.

**Theorem 4.5 (24).** Let $F \in C^p(\mathbb{R}^n)$ and there exists $h$, $\|h\| = 1$ such that $p$-factor matrix (31) is not singular. Then for any $x_0 \in U_{\varepsilon}(x^*)$ ($\varepsilon > 0$ sufficiently small) and for the scheme (30) the inequality
\[
\|x_{k+1} - x^*\| \leq c \|x_k - x^*\|^2, k = 0, 1, 2, \ldots
\]
holds for some constant $c > 0$.

**Example 7 (24).** Let $F(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \end{pmatrix}$, $x^* = (0, 0)^T$. It was shown in the Example 1 that $F$ is singular at $x^* = (0, 0)^T$. The scheme of 2-factor Newton method is as follows (32), where $P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $h = (1, -1)^T$. Then
\[
F'(x_k) + P_1 F''(x_k)h = \begin{pmatrix} 1 \\ x_k^2 - 1 \\ x_k^1 + 1 \end{pmatrix}
\]
and the formula (32) has the form
\[
x_{k+1} = x_k - \begin{pmatrix} 1 \\ x_k^2 - 1 \\ x_k^1 + 1 \end{pmatrix}^{-1} \begin{pmatrix} x_k^1 + x_k^2 \\ x_k^1 x_k^2 + x_k^2 - x_k^1 \end{pmatrix} = \begin{pmatrix} 1 \\ x_k^2 - 1 \\ x_k^1 + 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ x_k^1 x_k^2 \end{pmatrix}.
\]
It means, that $\|x_{k+1} - 0\| \leq c \|x_k - 0\|^2$. 

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Example 8 ([24]). Consider the following problem
\[
\min_{x \in \mathbb{R}^2} \phi(x),
\]
where \( \phi : \mathbb{R}^2 \to \mathbb{R} \) is defined by \( \phi(x) := x_1^2 + x_2^2 + x_4^4 \). Moreover, let \( F(x) := \phi'(x) \) where \( \phi'(x) = \left( \frac{2x_1 + 2x_4 x_2}{x_1^2 + 4x_2^2} \right) \), \( x^* = (0,0)^T \). It is easy to see that \( F \) is 3-regular at \( x^* \) along \( h = (1,1)^T \) and
\[
F'(0) + P_1 F''(0) h + P_2 F^{(3)}(0)[h]^2 = \phi''(0) + P_1 \phi^{(3)}(0) h + P_2 \phi^{(4)}(0)[h]^2 = \left( \begin{array}{c} 2 \\ 2 \\ 11 \end{array} \right)
\]
and this matrix is nonsingular. Here \( P_1 = \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right), \overline{P}_2 = \frac{1}{2} \left( \begin{array}{ccc} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \), \( P_2 = \overline{P}_2 P_1 = \frac{1}{2} \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{array} \right) \).

Consider the 3-factor scheme
\[
x_{k+1} = x_k - \left( \phi''(0) + P_1 \phi^{(3)}(0)[h] + P_2 \phi^{(4)}(0)[h]^2 \right)^{-1} \cdot \left( \phi'(x_k) + P_1 \phi''(x_k)[h] + P_2 \phi^{(3)}(x_k)[h]^2 \right).
\]

Let us denote \( x_k = (x_1, x_2)^T \). Then
\[
\|x_{k+1} - 0\| = \left\| \left( \begin{array}{c} 2 \\ 2 \\ 11 \end{array} \right)^{-1} \left( \begin{array}{c} 2x_1 - 11x_2 + 2x_1 x_2 - 6x_2^2 \\ 2x_1 + 11x_2 + x_1^2 + 18x_2^2 + 4x_2^2 \end{array} \right) \right\| = \frac{1}{11} \left\| \left( \begin{array}{c} 11x_1^2 + 132x_2^2 + 22x_1 x_2 + 44x_2^3 \\ 2x_1^2 + 148x_2^2 - 4x_1 x_2 + 8x_2^2 \end{array} \right) \right\| \leq 10 \|x_k - 0\|^2.
\]

### 4.4. Optimality conditions for \( p \)-regular problems of calculus of variations.

To formulate optimality conditions for singular problems of the form \([13] - [14]\) we define \( p \)-factor Euler-Lagrange function
\[
S(x) := f(x) + \lambda(t) G^{(p-1)}(x)[h]^{p-1},
\]
where \( G^{(p-1)}(x)[h]^{p-1} := g_1(x) + \ldots + g_p(x)[h]^{p-1}, \lambda(t) G^{(p-1)}(x)[h]^{p-1} = \left\langle \lambda(t), \left( g_1(x) + \ldots + g_p(x)[h]^{p-1} \right) \right\rangle, \lambda(t) = (\lambda_1(t), \ldots, \lambda_m(t))^T \) and \( g_i(x), i = 1, \ldots, p \) are determined for the map \( G(x) \) in similar way as \( F_i(x), i = 1, \ldots, p \) for the mapping \( F(x) \), in the Section 4, i.e. \( g_k(x) = \overline{P}_k G(x), k = 1, \ldots, p \).

Let
\[
g_k^{(k-1)}(x)[h]^{k-1} : = \sum_{i+j=k-1} C_{k-1}^k g_{x^i(x')^j}(x)[h[^i][h[^j]]^j, k = 1, \ldots, p,
\]
where \( g_{x^i(x')^j}(x) = g_{x^i(x')^j}(x) \).

**Definition 4.6.** We say that the problem \([13] - [14]\) is \( p \)-regular at \( x^* \) along \( h \in \bigcap_{k=1}^p \text{Ker} \overline{g}_k^{(k)}(x^*), \|h\| \neq 0 \) if
\[
\text{Im} \left( g_1^{(p)}(x^*) + \ldots + g_p^{(p)}(x^*)[h]^{p-1} \right) = \mathcal{C}_{m}[t_1, t_2].
\]
Theorem 4.7 ([19]). Let $x^*(t)$ be a solution of ([13]–[14]) and assume that this problem is $p$-regular at $x^*$ along $h \in \bigcap_{k=1}^{p} \ker g_k^{(k)}(x^*)$. Then there exist a multiplier $\lambda(t) = (\lambda_1(t), \ldots, \lambda_m(t))^T$ such that the following $p$-factor Euler-Lagrange equation

$$S_x(x^*) - \frac{d}{dt} S_{x'}(x^*) = f_x(x^*) + \left( \lambda \sum_{k=1}^{p} \sum_{i+j=k-1} C_{k-1}^{i} g_{x^*(x')}^{(i+j)} (x^*) h^{(i+j)} \right)_x$$

holds.

The proof of the above theorem is similar to the one for singular isoperimetric problem in [1] or [16].

Consider Example 3. The mapping $G$ is 2-regular (it means that in this case $p = 2$) at $\bar{x} = (a \sin t, a \cos t, 0, 0)^T$ along $h = (a \sin t, a \cos t, 1, 1)^T$.

Consider the following equation

$$f_x(x) + (G'(x) + P_k G''(x) h) \lambda = 0$$

which is equivalent to the system of equations

$$\begin{align*}
2x_1 - \lambda_1' + \lambda_2 &= 0 \\
2x_2 - \lambda_2' - \lambda_1 &= 0 \\
2x_3 + 2\lambda_1 \sin t + 2\lambda_2 \cos t &= 0 \\
2x_4 + 2\lambda_1 \cos t - 2\lambda_2 \sin t &= 0 \\
2x_5 + 2\lambda_1 (\cos t - \sin t) + 2\lambda_2 (\sin t - \cos t) &= 0.
\end{align*}$$

One can verify that the false solutions of ([17]–[18]), that is

$x_1 = a \sin t$, $x_2 = a \cos t$, $x_3 = x_4 = x_5 = 0$

do not satisfy the system ([38]) if $a \neq 0$. It means that $x_1 = a \sin t$, $x_2 = a \cos t$, $x_3 = x_4 = x_5$ do not satisfy 2-factor Euler-Lagrange equation ([35]) from Theorem 4.7.

The only solution to the Example 3 is $x^* = (0, 0, 0, 0)^T$. Indeed, 2-factor Euler-Lagrange equation in this case for $x^* = (0, 0, 0, 0)^T$ has the following form

$$\begin{align*}
-\lambda_1' + \lambda_2 &= 0 \\
-\lambda_2' - \lambda_1 &= 0 \\
2\lambda_1 \sin t + 2\lambda_2 \cos t &= 0 \\
2\lambda_1 \cos t - 2\lambda_2 \sin t &= 0 \\
2\lambda_1 (\cos t - \sin t) + 2\lambda_2 (\sin t - \cos t) &= 0.
\end{align*}$$

where the solution is $\lambda_i^*(t) = 0$, $i = 1, 2$.

4.5 Modified Lagrange function method for 2-regular problems. Consider the constrained optimization problem ([19]),

$$\min \phi(x) \quad \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m$$
and the modified Lagrangian function $L_E(x, \lambda)$ defined in Sec. [2.6]

$$L_E(x, \lambda) := \phi(x) + \frac{1}{2} \sum_{i=1}^{m} \lambda_i^2 g_i(x).$$

According to Sec. [2.6] the matrix

$$G'(x, \lambda) = \begin{pmatrix} \nabla^2 \phi(x) + \frac{1}{2} \sum_{i=1}^{m} \lambda_i \nabla^2 g_i(x) & (g'(x))^T D(\lambda) \\ D(\lambda) g(x) & D(g(x)) \end{pmatrix}$$

is singular at the solution $(x^*, \lambda^*)$ of (21) such that $g_i(x^*) = 0$ and $\lambda_i^* = 0$.

We show that the mapping $G(x, \lambda)$ defined by (20) is 2-regular at $(x^*, \lambda^*)$.

Define the set $I(x^*, \lambda^*) := \{ j = 1, 2, \ldots, m : g_j(x^*) = 0 \}$ of active constraints, the set $I_0(x^*) := \{ j = 1, 2, \ldots, m : \lambda^*_j = 0, g_j(x^*) = 0 \}$ of weakly active constraints, and the set $I_+(x^*) := I(x^*) \setminus I_0(x^*)$ of strongly active constraints.

Since $\lambda^*_j = 0$ and $g_j(x^*) = 0$ for all $j \in I_0(x^*) := \{ 1, \ldots, s \}$ the rows $(n + 1)$ to $(n + s)$ of $G'(x^*, \lambda^*)$ contain only zeros. Define the vector $h \in \mathbb{R}^{n+s}$ as follows

$$h^T := (0^T_n, 1^T_s, 0^T_{m-s})$$

and the mapping $\Phi : \mathbb{R}^n \times \mathbb{R}^m$

$$\Phi(x, \lambda) := G(x, \lambda) + G'(x, \lambda) h,$$

with $h$ given by (37).

The following fact is well known

**Lemma 4.8** ([7]). Let an $n \times n$ matrix $V$ and an $n \times p$ matrix $Q$ be such that $Q$ has linearly independent columns and $\langle Vx, x \rangle > 0 \ \forall x \in \ker Q^T \setminus \{0\}$. Assume moreover that $D_N$ is a full rank diagonal $l \times l$ matrix. Then

$$\bar{A} := \begin{pmatrix} V & Q & 0 \\ Q^T & 0 & 0 \\ 0 & 0 & D_N \end{pmatrix}$$

is a nonsingular matrix.

Let $D(\lambda)$ be the diagonal matrix with $\lambda_j$ as the $j$-th diagonal entry. We say that the constraint qualification condition (CQC) is fulfilled if the gradients of active constraints are linearly independent. The second order sufficient optimality condition holds if there exist $\alpha > 0$ such that

$$z^T \cdot \nabla^2_{zz} L_E(x^*, \lambda^*) z \geq \alpha \|z\|^2$$

for all $z \in \mathbb{R}^n$ satisfying the conditions

$$\langle \nabla g_j(x^*), z \rangle \leq 0, \ j \in I(x^*).$$

**Lemma 4.9** ([7]). Let $\varphi, g_i \in C^3(\mathbb{R}^n)$ $(i = 1, \ldots, m)$. Assume that the CQC and the second order sufficient optimality conditions are fulfilled at the solution $(x^*, \lambda^*)$ and $\Phi$ is a mapping given by (38). Then the 2-factor operator $\Phi'(x, \lambda) = G'(x, \lambda) + G''(x, \lambda) h$ is nonsingular at the point $(x^*, \lambda^*)$.

This assertion is obtained if in Lemma 4.8 we set $V = \nabla^2_{xx} L_E(x^*, \lambda^*)$, $D_N = D(g_N(x^*))$, where $g_N(x) := (g_{p+1}(x), \ldots, g_m(x))^T$ and

$$Q = [\nabla g_1(x^*), \ldots, \nabla g_s(x^*), \lambda^*_{s+1} \nabla g_{s+1}(x^*), \ldots, \lambda^*_{p} \nabla g_p(x^*)].$$

Then $\Phi'(x^*, \lambda^*) = \bar{A}$. 
Lemma (4.9) implies that 2-factor Newton method
\begin{equation}
 w_{k+1} = w_k - \left[ G'(w_k) + G''(w_k)h \right]^{-1} \left( G(w_k) + G'(w_k)h \right), \quad k = 1, 2, \ldots
\end{equation}
can be applied to solve (21) and we have the following proposition

**Theorem 4.10** (\cite{7}). Let \( x^* \) be a solution to (19). Assume that \( \varphi, g_i(x) \in C^3(\mathbb{R}^n), \quad i = 1, \ldots, m \), and the constraint regularity condition CRC and the second order sufficient optimality conditions (39) are fulfilled at the point \( x^* \). Then there exists a sufficiently small neighborhood \( U_{\varepsilon}(w^*) \) of \( w^* = (x^*, \lambda^*) \) such that the estimation
\begin{equation}
 ||w_{k+1} - w^*|| \leq \beta||w_k - w^*||^2,
\end{equation}
holds for the method (40), where \( w_0 \in U_{\varepsilon}(w^*) \) and \( \beta > 0 \) is an independent constant.

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