A note on the duality principle and Osserman condition

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Abstract
In this note we prove that for a Riemannian manifold the Osserman pointwise condition is equivalent to the Rakić duality principle.

1 Introduction
Let $\mathcal{R}$ be an algebraic curvature tensor on a Euclidean space $\mathbb{R}^n$ and let for $X \in \mathbb{R}^n$, $\mathcal{R}_X : Y \mapsto \mathcal{R}(Y, X)X$ be the corresponding Jacobi operator. An algebraic curvature tensor $\mathcal{R}$ is called Osserman, if the spectrum of the Jacobi operator $\mathcal{R}_X$ does not depend on the choice of a unit vector $X \in \mathbb{R}^n$.

Let $M^n$ be a Riemannian manifold, $R$ its curvature tensor and $R_X$ the corresponding Jacobi operator. It is well known that the properties of $R_X$ are intimately related with the underlying geometry of the manifold. The manifold $M^n$ is called pointwise Osserman if $R$ is Osserman at every point $p \in M^n$, and is called globally Osserman if the spectrum of $R_X$ is the same for all $X$ in the unit tangent bundle of $M^n$. Locally two-point homogeneous spaces are globally Osserman, since the isometry group of each of these spaces is transitive on its unit tangent bundle. Osserman conjectured that the converse is also true. This gives a very nice characterisation of local two-point homogeneous spaces in terms of the geometry of the Jacobi operator.

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At present, the Osserman Conjecture is almost completely solved by the results of Chi [C], who proved the Conjecture in dimensions \( n \neq 4k, k > 1 \) and \( n = 4 \), and the first author [N1, N2, N3], who proved it in all the remaining cases, except for some cases in dimension \( n = 16 \).

One of the crucial steps in the existing proofs of the Osserman Conjecture is the following Rakić duality principle [R]:

\[
\text{Suppose } R \text{ is an Osserman algebraic curvature tensor and } X, Y \text{ are unit vectors. Then } Y \text{ is an unit eigenvector of } R_X \text{ if and only if } X \text{ is an unit eigenvector of } R_Y \text{ (with the same eigenvalue).}
\]

The duality principle is extended to the pseudo-Riemannian settings in [AR].

2 Equivalence of duality principle and Osserman pointwise condition

Recently, for an algebraic curvature tensor in Riemannian signature, M. Brozos-Vázquez and E. Merino [BM] proved the equivalence of the Osserman condition and the duality principle for spaces of dimension less than 5. We show that this holds in an arbitrary dimension.

**Theorem.** The following two conditions for an algebraic curvature tensor \( R \) in Riemannian signature are equivalent:

(a) \( R \) satisfies the duality principle;

(b) \( R \) is Osserman.

**Proof.** The implication (b) \( \Rightarrow \) (a) is proved in [R].

To establish the converse, consider the characteristic polynomial \( \chi_X(t) \) of the Jacobi operator \( R_X \), where \( X \) is a unit vector. As the coefficients of \( \chi_X \) are analytic function on the unit sphere \( S \subset \mathbb{R}^n \), there is an open and dense subset \( S' \subset S \) such that for all \( X \in S' \) the number and the multiplicity of the eigenvalues of \( R_X \) are constant, the eigenvalues are analytic functions of \( X \), and the eigendistributions of \( J_X \) are analytic (viewed as the curves in the appropriate Grassmannians), see [Re], [K].

Let \( X \in S' \) and let \( Y \in S \) be orthogonal to \( X \). Suppose \( \lambda_0 \) is an eigenvalue of \( R_X \) with a unit eigenvector \( e_0 \). For small \( \phi \), the vector \( \cos \phi X + \sin \phi Y \) belongs to \( S' \), so there exist a differentiable (in fact, analytic) eigenvalue function \( \lambda(\phi) \) of the operator \( R_{\cos \phi X + \sin \phi Y} \) such that \( \lambda(0) = \lambda_0 \) and a differentiable unit vector function \( e(\phi) \), a section of the \( \lambda(\phi) \)-eigenspace of \( R_{\cos \phi X + \sin \phi Y} \) such that \( e(0) = e_0 \). Differentiating the equation

\[
R(\cos \phi X + \sin \phi Y, e(\phi), \cos \phi X + \sin \phi Y, e(\phi)) = \lambda(\phi)
\]
at $\phi = 0$ we obtain

$$2\mathcal{R}(Y, e_0, X, e_0) + 2\mathcal{R}(X, e_0, X, e'(0)) = \lambda'(0).$$

But $\mathcal{R}(X, e_0, X, e'(0)) = \lambda_0\langle e_0, e'(0) \rangle = 0$ and also $\mathcal{R}(Y, e_0, X, e_0) = \lambda_0\langle X, Y \rangle = 0$, by duality. It follows that the eigenvalues of $\mathcal{R}_X$ are constant on every connected component of $S'$. Then the coefficients of $\chi_X(t)$ are constant on the whole unit sphere $S$, which implies that $\mathcal{R}$ is Osserman.

\[\square\]

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