Research Article

Toeplitz Operators whose Symbols Are Borel Measures

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In this paper, we are concerned with Toeplitz operators whose symbols are complex Borel measures. When a complex Borel measure \( \mu \) on the unit circle is given, we give a formal definition of a Toeplitz operator \( T_\mu \) with symbol \( \mu \), as an unbounded linear operator on the Hardy space. We then study various properties of \( T_\mu \). Among them, there is a theorem that the domain of \( T_\mu \) is represented by a trichotomy. Also, it was shown that if the domain of \( T_\mu \) contains at least one polynomial, then \( T_\mu \) is densely defined. In addition, we give evidence for the conjecture that \( T_\mu \) with a singular measure \( \mu \) reduces to a trivial linear operator.

1. Introduction

A classical Toeplitz operator is the compression of a multiplication operator on the Lebesgue space \( L^2(\mathbb{T}) \) of the unit circle \( \mathbb{T} \) to the Hardy space \( H^2(\mathbb{T}) \). The study of Toeplitz operators seems to have originated from the paper of Toeplitz [1]. In the paper [2], he used Toeplitz matrices to characterize non-negative continuous functions on the unit circle in terms of their Fourier coefficients. The remarkable paper of Brown and Halmos [3] started the systematic study of spectral properties of Toeplitz operators. Since then, the theory of Toeplitz operators has been studied in various ways. Recently, the theory of Toeplitz operators has been studied in a variety of settings and connections with other fields. One direction is to deal with Toeplitz operators on reproducing kernel spaces like Bergman spaces, Dirichlet spaces, or Fock spaces (cf. [4–8]). Another direction is to study Toeplitz operators with operator-valued symbols (cf. [9–11]). Also, truncated Toeplitz operators have attracted attention. A systematic approach on truncated Toeplitz operators can be found in the paper of Sarason in 2007 [12]. In that paper, he has used “compatible” measures to describe bounded truncated Toeplitz operators. The boundedness of infinite Hankel matrices is also related to the compatibility of measures: the infinite Hankel matrix of the moment of a nonnegative Carleson measure is bounded and vice versa [13]. (For related recent studies, see [14,15].) These works inspired us to consider Toeplitz operators whose symbols are measures. The Toeplitz operators whose symbols are measures have been studied in the setting of Bergman spaces and other spaces (cf. [15], chapter 7).

In this paper, we consider Toeplitz operators on the Hardy space, whose symbols are measures. In this study, unbounded Toeplitz operators arise naturally. When studying unbounded Toeplitz operators, it was usually considered that the symbols come from \( L^2(\mathbb{T}) \). In 2008, Sarason [16] treated not only the case of \( L^2(\mathbb{T}) \)-symbols but the case of analytic functions on the open unit disk \( D \). It is natural to attempt to extend the symbols of Toeplitz operators to measures, because the initial research for them was related to the moment problem. As mentioned before, Toeplitz and Hankel operators associated with measures can be seen in the papers [13] and [12]. In this paper, we provide an explicit definition of Toeplitz operators whose symbols are complex Borel measures and then consider their unbounded operator theory. As the study on Toeplitz operators whose symbols are functions shows the interplay between function theory and operator theory, the study on Toeplitz operators whose symbols are measures is also expected to show the interplay between measure theory and operator theory.

Our consideration for the symbol of a Toeplitz operator, denoted by \( T_\mu \), is a complex Borel measure \( \mu \) on the unit cir-
2. Preliminaries

Let \( T \) be the unit circle in the complex plane. Let \( m(\mathbb{T}) = 1 \). For \( 1 \leq p \leq \infty \), we write \( L^p(T) = L^p(T, m) \) for the Lebesgue space on \( T \) and \( H^p(T) \) for the Hardy space on \( T \). Note that \( H^p(T) \) is a closed subspace of \( L^p(T) \).

Let \( D \) be the open unit disk and let \( \overline{D} \) be the closed unit disk in the complex plane. Let \( C(D) \) denote the disk algebra, i.e., the set of all continuous functions on \( \overline{D} \) which is analytic on \( D \).

For \( 1 \leq p \leq \infty \), we write \( H^p(D) \) for the Hardy space on \( D \). Two spaces \( H^p(D) \) and \( H^p(T) \) are identified via nontangential limits and Poisson integral. Thus, we often write \( H^p \) to denote the both of them. The norm in \( L^p(T) \) (or \( H^p(D) \)) will be denoted by \( \| \cdot \|_p \), and the inner product in \( L^2(T) \) (or \( H^2(D) \)) will be denoted by \( \langle \cdot, \cdot \rangle \). We refer the reader to the texts [17–19] and [20] for details of Hardy spaces.

The shift operator and its adjoint are one of the most interesting operators on the Hardy space. For convenience, we define them on \( H(D) \), the class of all analytic functions on \( D \). For \( f \in H(D) \), define

\[
Sf(z) = zf(z) (z \in D), \quad S^* f(z) = \frac{f(z) - f(0)}{z} (z \in D). \tag{1}
\]

The operators \( S \) and \( S^* \) are often called the unilateral shift and the backward shift, respectively. We refer the reader to the text [21] which treats the shift operator in great detail.

One of the most remarkable theorems in analysis is Beurling’s theorem (cf. [18, 20, 22]), which characterizes all \( S \)-invariant subspaces of \( H^2 \). (We use the term "subspace" for a closed linear subspace.) For a nonzero subspace \( M \) of \( H^2 \), \( M \) is \( S \)-invariant if and only if

\[
M = \Theta H^2 = \{ \theta f : f \in H^2 \}, \tag{2}
\]

for some inner function \( \theta \in H^\infty \). A bounded analytic function \( \theta \) on \( D \) is called an inner function if its radial limit \( \lim_{r \to 1} \theta(re^{i\theta}) \) has a unit modulus for almost all \( e^{i\theta} \in \mathbb{T} \). If an inner function has no zero in \( D \), we call it a singular inner function.

Let \( M(T) \) be the set of all complex (finite) Borel measures on \( T \). Note that \( M(T) \) is a Banach space with the total variation norm \( \| \mu \| = \| \mu \|_1 \), where \( \| \mu \| \) is the total variation measure of \( \mu \). We may regard the normalized Lebesgue measure \( m \) as a finite positive Borel measure. Hence, \( m \in M(T) \). We write \( \mathcal{B}_T \) for the \( \sigma \)-algebra of all Borel sets in \( T \). We say \( \mu \) is singular if \( \mu \perp m \).

Suppose that \( \mu \in M(T) \). For any function \( f \in L^1(\mathbb{T}, \mu|_T) \), let \( f \cdot \mu \) denote the complex Borel measure on \( T \) defined by

\[
(f \cdot \mu)(E) = \int_E f \, d\mu(E \in \mathcal{B}_T). \tag{3}
\]

Then, \( \| f \cdot \mu \| = \| f \|_1 \cdot | \mu | \). Hence, \( \| f \cdot \mu \| = \| f \|_1 \cdot | \mu | \cdot \| \mu \|_1 \) holds. In particular, for every \( f \in C(\mathbb{T}) \), the measure \( f \cdot \mu \) is defined and \( \| f \cdot \mu \| \leq \| f \|_\infty \cdot \| \mu \|_1 \).

Let \( \mu \in M(T) \), the \( n \)th Fourier–Stieltjes coefficient of \( \mu \) is given by

\[
\hat{\mu}(n) = \int_\mathbb{T} e^{i\zeta n} \, d\mu(\zeta) (n \in \mathbb{Z}). \tag{4}
\]

For any \( \mu \in M(T) \), the bilateral sequence \( \hat{\mu} = \{ \hat{\mu}(n) \}_{n \in \mathbb{Z}} \) is bounded and the mapping \( \mu \mapsto \hat{\mu} \) is a bounded linear transformation from \( M(T) \) into \( \ell^\infty(\mathbb{Z}) \). Note that the mapping \( \mu \mapsto \hat{\mu} \) is one-to-one, and hence, a measure \( \mu \in M(T) \) is completely determined by its Fourier–Stieltjes coefficients. By the theorem of F. and M. Riesz, if \( \mu \in M(T) \) is analytic, i.e., \( \hat{\mu}(n) = 0 \) for all \( n \leq 0 \), then \( \mu \ll m \) and \( d\mu = dm \) in \( H^1(T) \); in other words, \( \mu = f \cdot m \) for some \( f \in H^1(T) \).

For the definition of Toeplitz operators whose symbols are measures, we use the Cauchy transform as the “projection” of measures. For this reason, we use the notation \( P\mu \) instead of \( K\mu \) for the Cauchy transform of \( \mu \). We refer the reader to the text [23] for thorough treatments of the Cauchy transform. For \( \mu \in M(T) \), the analytic function \( P\mu \) on \( D \), given by

\[
(P\mu)(z) = \int_\mathbb{T} \frac{1}{1 - \zeta z} \, d\mu(\zeta) = \sum_{n=0}^{\infty} \hat{\mu}(n) z^n (z \in D), \tag{5}
\]

is called the Cauchy transform of \( \mu \). Clearly, the mapping \( P \) is a linear transformation from \( M(T) \) into \( H(D) \). We may regard \( f \in L^1(T) \) as the absolutely continuous measure \( f \cdot m \in M(T) \). Hence, we denote \( P(f \cdot m) \) by \( Pf \), i.e.,

\[
(Pf)(z) = \int_\mathbb{T} \frac{f(\zeta)}{1 - \zeta z} \, dm(\zeta) = \sum_{n=0}^{\infty} \hat{f}(n) z^n (z \in D). \tag{6}
\]

(Clearly, \( f \cdot m(n) = \hat{f}(n) \).) As we have identified \( H^2(D) \) with \( H^2(T) \), the mapping \( P \) may be regarded as the
orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ (the so-called Riesz projection).

Let $\varphi \in L^2(\mathbb{T})$. The *Toeplitz operator* $T_\varphi$ with symbol $\varphi$ is the linear operator on $H^2$ with domain
\[ \mathcal{D}(T_\varphi) = \{ f \in H^2(\mathbb{D}) : \varphi(f) \in H^2(\mathbb{D}) \}, \tag{7} \]
given by
\[ T_\varphi f = \varphi(f) f \in \mathcal{D}(T_\varphi). \tag{8} \]

(Recall that every function in $H^2(\mathbb{D})$ may be identified with its nontangential limit function which belongs to $H^2(\mathbb{T})$.) Clearly, $C_A(\mathbb{D}) \subseteq \mathcal{D}(T_\varphi)$. Hence, $T_\varphi$ is densely defined. Also, $T_\varphi$ is closed. Observe that
\[ \varphi^{z_i^j} = \varphi(z_i^j) = \overline{\varphi}(i - j), \tag{9} \]
for every $i, j \in \mathbb{N} \cup \{0\}$. Hence, the matrix representation of $T_\varphi$ with respect to the orthonormal basis \{1, $z$, $z^2$, \ldots\} is
\[ \begin{bmatrix} \varphi(0) & \varphi(-1) & \varphi(-2) & \cdots \\ \varphi(1) & \varphi(0) & \varphi(-1) & \cdots \\ \varphi(2) & \varphi(1) & \varphi(0) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{10} \]

A matrix of this form is called a *Toeplitz matrix*; in other words, an infinite matrix $\{a_{ij}\}_{i,j \geq 0}$ is called a Toeplitz matrix if
\[ a_{ij} = a_{i+j,i}, \tag{11} \]
for every $i, j \in \mathbb{N} \cup \{0\}$.

For a bilateral sequence $s = \{s_n\}_{n \in \mathbb{Z}}$ of complex numbers, we denote by $T(s)$ the infinite Toeplitz matrix corresponding to $s$, i.e., $T(s)$ is the infinite matrix whose $(i,j)$-entry is $s_{i-j}$. Note that if $\varphi \in L^2(\mathbb{T})$, then the matrix representations of $T_\varphi$ is $T(\overline{\varphi})$. For $n \in \mathbb{N} \cup \{0\}$, we denote by $T_n(s)$ the $(n+1) \times (n+1)$ Toeplitz matrix corresponding to $s$, i.e.,
\[ T_n(s) = \begin{bmatrix} s_0 & s_{-1} & \cdots & s_{-n} \\ s_1 & s_0 & \cdots & s_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n-1} & \cdots & s_0 \end{bmatrix}. \tag{12} \]

3. The Main Results

Let $\mu$ be a complex Borel measure on $\mathbb{T}$. For any function $f \in C_A(\mathbb{D})$, $f \cdot \mu$ is a complex Borel measure on $\mathbb{T}$, and hence, the Cauchy transform $P(f \cdot \mu)$ is an analytic function on $\mathbb{D}$. Define
\[ \mathcal{D}(T_\mu) = \{ f \in C_A(\mathbb{D}) : P(f \cdot \mu) \in H^2(\mathbb{D}) \}. \tag{13} \]

It is easy to show that $\mathcal{D}(T_\mu)$ is a linear manifold of $H^2(\mathbb{D})$. Now define
\[ T_\mu f = P(f \cdot \mu)(f \in \mathcal{D}(T_\mu)). \tag{14} \]

Then, $T_\mu$ is a linear operator on $H^2(\mathbb{D})$ with domain $\mathcal{D}(T_\mu)$.

**Definition 1.** The operator $T_\mu$ is called the *Toeplitz operator* with symbol $\mu$.

We begin with the following:

**Proposition 2.** Suppose that $\mu \ll m$ and the Radon–Nikodym derivative $\varphi = d\mu/dm$ belongs to $L^2(\mathbb{T})$. Then, $\mathcal{D}(T_\mu) = C_A(\mathbb{D})$ and
\[ T_\mu f = T_\varphi f, \tag{15} \]
for every $f \in C_A(\mathbb{D})$.

**Proof.** Suppose that $\mu = \varphi \cdot m$, where $\varphi \in L^2(\mathbb{T})$. Let $f$ be an arbitrary function in $C_A(\mathbb{D})$. Then,
\[ P(f \cdot \mu)(z) = \int_\mathbb{T} \frac{f(\xi)\varphi(\xi)}{1 - \overline{\xi}z} \, d\mu(\xi) = \int_\mathbb{T} \frac{f(\xi)\overline{\varphi}(\xi)}{1 - \overline{\xi}z} \, dm(\xi) = P(\varphi f)(z), \tag{16} \]
for every $z \in \mathbb{D}$, and so, $P(f \cdot \mu) = P(\varphi f)$. Since $\varphi f \in L^2(\mathbb{T})$, it follows that $P(f \cdot \mu) \in H^2(\mathbb{D})$. Hence, $f \in \mathcal{D}(T_\mu)$ and
\[ T_\mu f = T_\varphi f = P(\varphi f) = T_\varphi f. \tag{17} \]

This completes the proof.

Proposition 2 shows that the notion of $T_\mu$ is a kind of generalization of the Toeplitz operators whose symbols are $L^2$-functions.

**Remark 3.**

(a) *Toeplitz operators with $L^1$–symbols*: every function $\varphi \in L^1(\mathbb{T})$ would be regarded as the absolutely continuous measure $\varphi \cdot m \in M(\mathbb{T})$. Hence, we may use Definition 1 to define Toeplitz operators with $L^1$-symbols: if $\varphi \in L^1(\mathbb{T})$ and $\mu = \varphi \cdot m$, then
\[ \mathcal{D}(T_\mu) = \{ f \in C_A(\mathbb{D}) : P(\varphi f) \in H^2(\mathbb{D}) \}, \tag{18} \]
\[ T_\mu f = P(\varphi f), \tag{19} \]
for $f \in \mathcal{D}(T_\mu)$. 
(b) **Toeplitz operators with \( H^1 \) -symbols:** let \( \varphi \in H^1(\mathbb{T}) \) and put \( \mu = \varphi \cdot m \in M(\mathbb{T}) \). For every \( f \in C_A(\mathbb{D}) \), \( \varphi f \in H^1(\mathbb{T}) \). Hence, \( P(\varphi f) = \varphi f \) (if we view \( \varphi \) in the right-hand side as a function in \( H^1(\mathbb{D}) \)). It follows that

\[
\mathcal{D}(T_\mu) = \{ f \in C_A(\mathbb{D}) : \varphi f \in H^2(\mathbb{D}) \}\text{,} \tag{20}
\]

\[
T_\mu f = \varphi f, \tag{21}
\]

for \( f \in \mathcal{D}(T_\mu) \). This shows that a Toeplitz operator with \( H^1 \)-symbol behaves as a multiplication. Notice that the action of \( T_\mu \) is the same as that of \( T_\varphi \) defined in (116), Section 5). (In that paper, the domain of \( T_\varphi \) is given by \( \mathcal{D}(T_\varphi) = \{ f \in H^2(\mathbb{D}) : \varphi f \in H^2(\mathbb{D}) \} \).) Moreover, since \( \varphi \) is of Smirnov class, \( \varphi = b/a \) for some \( a, b \in H^\infty(\mathbb{D}) \) such that \( a \) is an outer function, \( a(0) > 0 \), and \( |a|^2 + |b|^2 = 1 \) on \( \mathbb{T} \). In this case, \( \mathcal{D}(T_\varphi) = aH^2(\mathbb{D}) \) (cf. [16]). It follows that

\[
\mathcal{D}(T_\mu) = \mathcal{D}(T_\varphi) \cap C_A(\mathbb{D}) = aH^2(\mathbb{D}) \cap C_A(\mathbb{D}). \tag{22}
\]

Since \( a \) is an outer function, it follows that \( aH^2(\mathbb{D}) \) is dense in \( H^2(\mathbb{D}) \).

**Question:** is \( aH^2 \cap C_A(\mathbb{D}) \) dense in \( H^2 \)?

We give some concrete examples.

**Example 4.**

(a) Let \( \varphi \) be the analytic function on \( \mathbb{D} \) such that \( (\varphi(z))^2 = (1 - z)^{-1} \) and \( \varphi(0) = 1 \). Then, \( \varphi \in H^1(\mathbb{D}) \) but \( \varphi \notin H^2(\mathbb{D}) \). Put \( \mu = \varphi \cdot m \). By Remark 3, (b), we have

\[
\mathcal{D}(T_\mu) = \{ f \in C_A(\mathbb{D}) : \varphi f \in H^2(\mathbb{D}) \}\text{.} \tag{23}
\]

How large is the domain \( \mathcal{D}(T_\mu) \)? Suppose that \( g \in C_A(\mathbb{D}) \) and \( g(1) \neq 0 \). Then, there exists a constant \( c > 0 \) such that \( |g| \geq c \) on a neighborhood of \( \zeta = 1 \). It follows that \( \varphi g \notin H^2(\mathbb{D}) \). Hence, \( g \notin \mathcal{D}(T_\mu) \). This shows that

\[
\mathcal{D}(T_\mu) \subseteq \{ f \in C_A(\mathbb{D}) : f(1) = 0 \}. \tag{24}
\]

On the other hand, if \( r > 0 \) and if \( \psi_r \) is the function in \( C_A(\mathbb{D}) \) which satisfies \( (\psi_r(z))^2 = 1 - z \) and \( \psi_r(0) = 1 \), then, for every \( g \in C_A(\mathbb{D}) \),

\[
\| \varphi \psi_r g \|_2^2 = \int_0^\infty |\varphi(\zeta)|^2 |\psi_r(\zeta)|^2 |g(\zeta)|^2 \, d\zeta \leq \int_0^\infty |g|_2^2 \sum_{n=0}^\infty \int_0^\infty |1 - e^{it\zeta}|^2 |dt| \tag{25}
\]

and hence, \( \varphi \psi_r g \in H^2(\mathbb{D}) \), i.e., \( \psi_r g \in \mathcal{D}(T_\mu) \). It follows that

\[
\bigcup_{r>0} \mathcal{D}(T_\mu) \subseteq \mathcal{D}(T_\mu). \tag{26}
\]

Since \( \psi_1 = 1 - z \), we have

\[
(1 - z) \cdot C_A(\mathbb{D}) \subseteq \mathcal{D}(T_\mu). \tag{27}
\]

In particular, \( \mathcal{D}(T_\mu) \) contains all polynomials vanishing at \( \zeta = 1 \).

(b) Let \( \mu = \delta_1 \) be the unit point mass concentrated at \( \zeta = 1 \). Note that the measure \( \mu \) is discrete. Observe that, for \( f \in C_A(\mathbb{D}) \),

\[
P(f \cdot \mu)(z) = \int_0^\infty f(\zeta) \, d\mu(\zeta) = \frac{f(1)}{1 - z} \quad (z \in \mathbb{D}). \tag{28}
\]

Since \( 1/(1 - z) = \sum_{n=0}^\infty z^n \), the function \( 1/(1 - z) \) does not belong to \( H^2(\mathbb{D}) \). It follows that \( P(f \cdot \mu) \in H^2(\mathbb{D}) \) if and only if \( f(1) = 0 \). Therefore,

\[
\mathcal{D}(T_\mu) = \{ f \in C_A(\mathbb{D}) : f(1) = 0 \}. \tag{29}
\]

Also, we have

\[
T_\mu f = 0, \tag{30}
\]

for all \( f \in \mathcal{D}(T_\mu) \). Hence, \( T_\mu \) is trivial, i.e., \( T_\mu f = 0 \) for all \( f \in \mathcal{D}(T_\mu) \). Consequently, \( T_\mu \) is bounded (on \( \mathcal{D}(T_\mu) \)). Notice that \( \mathcal{D}(T_\mu) \) does not contain the constant function 1. We show later (see Remark 11) that \( \mathcal{D}(T_\mu) \) is dense in \( H^2(\mathbb{D}) \).

(c) **The Cantor middle-third measure:** let \( C \) denote the Cantor ternary set and let \( \varphi \) be the Cantor function, i.e., for \( x = \sum_{j=1}^{\infty} (a_j / 3^j) \in C \),
\[ \varphi(x) = \sum_{j=1}^{\infty} \frac{a_j}{2^j}, \]

and \( \varphi(x) = \sup \{ \varphi(y); y < x, y \in C \} \) for \( x \notin C \). Then, \( \varphi \) is continuous and monotonically increasing. Hence, there exists a positive Borel measure \( \mu \) on \( \mathbb{T} \) such that
\[ \mu\left( \left\{ e^{2\pi \theta}; 0 \leq \theta < t \right\} \right) = \varphi(t) (0 \leq t \leq 1). \]

The measure \( \mu \) (the so-called Cantor middle-third measure) is a typical example of a singular continuous measure. We refer the reader to the papers [24] and [25] which treat measures of the Cantor type. It is known that
\[ \tilde{\mu}(n) = (-1)^n \prod_{j=1}^{\infty} \cos \frac{2\pi n}{3^j} (n \in \mathbb{Z}). \]

Hence,
\[ |\mu \wedge (n)|^2 = \prod_{j=1}^{\infty} \left( 1 - \sin^2 \frac{2\pi n}{3^j} \right) (n \in \mathbb{Z}). \]

Since \( 0 \leq \sin^2(2\pi n/3^j) < 1 \) for each \( j \) and \( \sum_{j=1}^{\infty} \sin^2(2\pi n/3^j) < \infty \), it follows that \( \tilde{\mu}(n) \neq 0 \). Note also that \( \tilde{\mu}(n) = \tilde{\mu}(3n) = \tilde{\mu}(n) \) for every \( n \in \mathbb{Z} \). We may here ask the following questions:

(a) What is \( \mathcal{D}(T_\mu) \)? Is \( \mathcal{D}(T_\mu) \) dense in \( H^2(\mathbb{D}) \)?

(b) What is \( T_\mu \)? Is \( T_\mu \) trivial?

We next ask: when is the domain \( \mathcal{D}(T_\mu) \) dense in \( H^2(\mathbb{D}) \)? It does not seem easy to answer this question in general. The following lemma is used to derive some properties of \( \mathcal{D}(T_\mu) \) which are helpful to determine the density of \( \mathcal{D}(T_\mu) \) in \( H^2(\mathbb{D}) \). Recall that \( S \) is the shift operator on \( H(\mathbb{D}) \), i.e., if \( f \in H(\mathbb{D}) \), then \( Sf(z) = zf(z) \) for \( z \in \mathbb{D} \).

We then have the following:

**Lemma 5.** For every \( \mu \in M(\mathbb{T}) \) and \( f \in C_A(\mathbb{D}) \),
\[ P(Sf \cdot \mu) = SP(f \cdot \mu) + P(Sf \cdot \mu)(0). \]

**Proof.** For each \( z \in \mathbb{D}, \)
\[ P(Sf \cdot \mu)(z) - P(Sf \cdot \mu)(0) = \int_T \frac{\xi f(\xi)}{1 - \xi z} d\mu(\xi) - \int_T f(\xi) d\mu(\xi) \]
\[ = \int_T \frac{f(\xi)}{1 - \xi z} d\mu(\xi) \]
\[ = zP(f \cdot \mu)(z) \]
\[ = SP(f \cdot \mu)(z). \]

The following proposition gives an important information for the domain of \( T_\mu \).

**Proposition 6.** Let \( \mu \in M(\mathbb{T}) \) and let \( a \) be a complex number such that \( |a| \neq 1 \). Then, the following statements hold:

(a) For \( f \in C_A(\mathbb{D}) \), \( f \in \mathcal{D}(T_\mu) \) if and only if \( (S-a)f \in \mathcal{D}(T_\mu) \)

(b) For \( f \in H^2(\mathbb{D}) \), \( f \in cl_{H^2}(\mathcal{D}(T_\mu)) \) if and only if \( (S-a)f \in cl_{H^2}(\mathcal{D}(T_\mu)) \)

**Proof.** (a) Suppose that \( f \in C_A(\mathbb{D}) \). Then, by Lemma 5,
\[ P((S-a)f \cdot \mu) = P(Sf \cdot \mu) - P(af \cdot \mu) \]
\[ = SP(f \cdot \mu) + P(Sf \cdot \mu)(0) - aP(f \cdot \mu) \]
\[ = (S-a)P(f \cdot \mu) + P(Sf \cdot \mu)(0). \]

Hence, \( P((S-a)f \cdot \mu) \in H^2(\mathbb{D}) \) if and only if \( (S-a)P(f \cdot \mu) \in H^2(\mathbb{D}) \). Since \( P(f \cdot \mu) \in H^2(\mathbb{D}) \) and \( |a| \neq 1 \), it follows that \( (S-a)P(f \cdot \mu) \in H^2(\mathbb{D}) \). Therefore, \( f \in \mathcal{D}(T_\mu) \) if and only if \( (S-a)f \in \mathcal{D}(T_\mu) \). This proves (a).

(b) Suppose that \( f \in H^2(\mathbb{D}) \) and \( f \in cl_{H^2}(\mathcal{D}(T_\mu)) \). Then, there exists a sequence \( \{f_j\} \) in \( \mathcal{D}(T_\mu) \) such that \( \|f - f_j\|_2 \longrightarrow 0 \). Since \( S - a \) is a bounded operator on \( H^2(\mathbb{D}) \), we have
\[ \|(S-a)f - (S-a)f_j\|_2 = \|(S-a)(f-f_j)\|_2 \longrightarrow 0. \]

By (a), each \( (S-a)f_j \) belongs to \( \mathcal{D}(T_\mu) \). It follows that \( (S-a)f \in cl_{H^2}(\mathcal{D}(T_\mu)) \).

Conversely, suppose that \( f \in H^2(\mathbb{D}) \) and \( (S-a)f \in cl_{H^2}(\mathcal{D}(T_\mu)) \). Then, there exists a sequence \( \{g_j\} \) in \( \mathcal{D}(T_\mu) \) such that
\[ \|(S-a)f - g_j\|_2 \longrightarrow 0. \]

We want to show that \( f \in cl_{H^2}(\mathcal{D}(T_\mu)) \). To see this we consider two cases.

**Case 1.** \((|a| < 1)\). Assume first that \( g_j(a) = 0 \) for all \( j \). Then,
\[ g_j = (S-a)f_j, \]
where \( f_j \in C_A(\mathbb{D}) \). Since \( g_j \in \mathcal{D}(T_\mu) \), it follows from (a) that \( f_j \in \mathcal{D}(T_\mu) \). Note that the approximate point spectrum of the operator \( S \) on \( H^2(\mathbb{D}) \) is \( \sigma_{ap}(S) = \mathbb{T} \) (cf. [26]). Since \( a \) does not belong to \( \mathbb{T} \), the operator \( S - a \) is bounded below on \( H^2(\mathbb{D}) \). It follows that there exists a constant \( c > 0 \) such that
\[ \|(S-a)f - g_j\|_2 = \|(S-a)(f-f_j)\|_2 \geq c \cdot \|f - f_j\|_2 \]
for all \( j \). This implies that \( \|f - f_j\|_2 \longrightarrow 0 \). Therefore, \( f \in cl_{H^2}(\mathcal{D}(T_\mu)) \).
In the case that \(g_j(\alpha) \neq 0\) for some \(j\), we may assume that \(g_1(\alpha) \neq 0\). Note that \(g_j \rightarrow (S - \alpha)f\) weakly. Hence, \(g_j(z) \rightarrow ((S - \alpha)f)(z)\) for each \(z \in \mathbb{D}\). In particular, we have

\[
g_j(\alpha) \rightarrow 0. \tag{42}
\]

Now put

\[
h_j = g_j - \frac{g_j(\alpha)}{g_1(\alpha)}g_1 \quad (j = 1, 2, 3, \ldots). \tag{43}
\]

Then, \(h_j \in \mathcal{D}(T_\mu)\) and \(h_j(\alpha) = 0\) for all \(j\). Observe that

\[
\| (S - \alpha) f - h_j \|_2 \leq \| (S - \alpha) f - g_j \|_2 + \| \frac{g_j(\alpha)}{g_1(\alpha)} \| \| g_1 \|_2. \tag{44}
\]

It follows that

\[
\| (S - \alpha) f - h_j \|_2 \rightarrow 0. \tag{45}
\]

Hence, by the preceding paragraph, we conclude that \(f \in \text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu))\).

Case 2. \((\alpha | > 1)\). The operator \(S - \alpha\) on \(H^2(\mathbb{D})\) is invertible. Hence,

\[
\| f - (S - \alpha)^{-1} g_j \|_2 \rightarrow 0. \tag{46}
\]

Since \((S - \alpha)^{-1} = -\sum_{n=0}^{\infty} S^n/\alpha^{n+1}\) and \(\mathcal{D}(T_\mu)\) is \(S\)-invariant by (a), each \((S - \alpha)^{-1} g_j\) belongs to \(\text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu))\). It follows that \(f \in \text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu))\), and the proof is complete.

**Remark 7.** If we take \(\alpha = 0\) in Proposition 6, then the linear subspaces \(\mathcal{D}(T_\mu)\) and its closure \(\text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu))\) are \(S\)-invariant. Also, the equality in Lemma 5 can be rewritten as \(S^p (Sf \cdot \mu) = P(f \cdot \mu)\). Consequently, we have \(S^p T_\mu Sf = T_\mu f\) for every \(f \in \mathcal{D}(T_\mu)\).

As a consequence of Proposition 6, we derive the following theorem which describes the domain \(\mathcal{D}(T_\mu)\). Recall that an inner function is said to be singular if it has no zero in the unit disk.

**Theorem 8.** Let \(\mu \in M(\mathbb{T})\). Then, one of the following holds:

(i) \(\mathcal{D}(T_\mu) = \{0\}\)

(ii) \(\mathcal{D}(T_\mu)\) is dense in \(H^2(\mathbb{D})\)

(iii) \(\text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu)) = \theta H^2(\mathbb{D})\), where \(\theta\) is a singular inner function.

**Proof.** By Proposition 6, \(\text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu))\) is an \(S\)-invariant subspace of \(H^2(\mathbb{D})\). It follows from Beurling’s theorem that

\[
\text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu)) = \theta H^2(\mathbb{D}), \tag{47}
\]

where \(\theta\) is an inner function or \(\theta = 0\). If \(\theta = 0\), then the case (i) occurs. If \(\theta\) is a nonzero constant function, case (ii) occurs. Now, suppose that \(\theta\) is nonconstant. We show that \(\theta\) has no zero in \(\mathbb{D}\). To see this, choose any nonzero function \(f\) in \(\mathcal{D}(T_\mu)\). Fix an arbitrary point \(\alpha\) of \(\mathbb{D}\) and let \(n\) be the multiplicity of the zero of \(f\) at \(\alpha\). Then,

\[
f(z) = (z - \alpha)^n g(z) (z \in \mathbb{D}), \tag{48}
\]

where \(g \in C_\alpha(\mathbb{D})\) and \(g(\alpha) \neq 0\). Hence, by a repeated application of Proposition 6(a), we have

\[
g \in \mathcal{D}(T_\mu) \subseteq \theta H^2(\mathbb{D}). \tag{49}
\]

It follows that \(g = \theta h\) for some \(h \in H^2(\mathbb{D})\). Thus, \(\theta(\alpha)\) cannot be 0. Since \(\alpha\) was arbitrary, we conclude that \(\theta\) has no zero in \(\mathbb{D}\). Therefore \(\theta\) is a singular inner function.

**Remark 9.** Unfortunately, we cannot find a concrete example for the third case. It would be possible that the third case never occurs.

The following proposition is another consequence of Proposition 6 which gives a sufficient condition for the domain \(\mathcal{D}(T_\mu)\) to be dense in \(H^2(\mathbb{D})\).

**Proposition 10.** If \(\text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu))\) contains a polynomial, then \(\mathcal{D}(T_\mu)\) is dense in \(H^2(\mathbb{D})\).

**Proof.** Suppose that \(\text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu))\) contains a polynomial. Then, by Proposition 6, (b), there exists a polynomial \(p \in c l_{\mathcal{H}}(\mathcal{D}(T_\mu))\), all of whose zeros are in \(T\), such that \(p(0) = 1\). Let \(\zeta_1, \ldots, \zeta_N \in \mathbb{T}\) be the zeros of \(p\), listed according to their multiplicities. Then,

\[
p(z) = \left(1 - \zeta_1 z\right) \cdots \left(1 - \zeta_N z\right). \tag{50}
\]

Choose a sequence \(\{k_n\}\) in \(\mathbb{N}\) such that \(k_{n+1} > Nk_n\) (e.g., \(k_n = (N + 1)^n\)). For each \(n \in \mathbb{N}\), define

\[
p_n(z) = \frac{1}{n} \sum_{j=1}^{n} \left(1 - \left(\zeta_1 z\right)^{k_j}\right) \cdots \left(1 - \left(\zeta_N z\right)^{k_j}\right). \tag{51}
\]

All of them are polynomials, divisible by \(p\). Since \(\text{cl}_{\mathcal{H}}(\mathcal{D}(T_\mu))\) is \(S\)-invariant, the polynomials \(p_n\) belong to \(\mathcal{D}(T_\mu)\). It follows by a direct computation that

\[
\| 1 - p_n \|_2^2 \leq \frac{n}{n^2} \left(\begin{array}{c} N \\ 1 \end{array}\right) \left(\begin{array}{c} N \\ 1 \end{array}\right) \cdots \left(\begin{array}{c} N \\ N \end{array}\right) \left(\begin{array}{c} N \\ N \end{array}\right), \tag{52}
\]

for every \(n \in \mathbb{N}\). This implies that \(p_n \rightarrow 1\) in \(H^2(\mathbb{D})\).
Therefore, the constant function 1 belongs to \( cl_{HF}(D(T_{\mu})) \). Since \( cl_{HF}(D(T_{\mu})) \) is \( S \)-invariant, we conclude that \( cl_{HF}(D(T_{\mu})) = H^2(\mathbb{D}) \); in other words, \( D(T_{\mu}) \) is dense in \( H^2(\mathbb{D}) \).

**Remark 11.** Proposition 10 shows that the domains \( D(T_{\mu}) \), presented in (a) and (b) of Example 4, are dense in \( H^2(\mathbb{D}) \), because they contain the polynomial \( p(z) = 1 - z \). The proof of Proposition 10 shows that every polynomial, all of whose zeros are in \( \mathbb{T} \), is an outer function.

In order to consider the matrix representation of a linear operator on \( H^2(\mathbb{D}) \), it is necessary that its domain contains all polynomials. Let us interpret the condition that \( D(T_{\mu}) \) contains all polynomials. Note that this is equivalent to the condition that \( D(T_{\mu}) \) contains any polynomial which does not vanish on \( \mathbb{T} \), by Proposition 6, (a).

**Lemma 12.** Let \( \mu \in M(\mathbb{T}) \). Then, the following are equivalent:

(i) \( D(T_{\mu}) \) contains all polynomials, or equivalently, \( D(T_{\mu}) \) contains the constant function 1

(ii) \( \mu \ll m \) and \( d\mu / dm \in H^2(\mathbb{T}) + H^2_{0}(\mathbb{T}) \)

**Proof.** (i) \( \Rightarrow \) (ii): suppose that the constant function 1 belongs to \( D(T_{\mu}) \). Then, \( \mu \mu = P(1 \cdot \mu) \in H^2(\mathbb{D}) \). Let \( \psi \) denote the nontangential limit function of \( \mu \mu \). Since \( \mu \mu = \sum_{n=0}^{\infty} \tilde{\mu}(n)z^n \), it follows that \( \tilde{\psi}(n) = \tilde{\mu}(n) \) for all \( n \in \mathbb{N} \cup \{0\} \). Put \( \nu = \mu - \psi \cdot m \). Then, \( \nu \in M(\mathbb{T}) \) and

\[
\tilde{\psi}(n) = \tilde{\mu}(n) - \tilde{\psi}(n) = 0, \quad (53)
\]

for all \( n \in \mathbb{N} \cup \{0\} \). It follows from the F. and M. Riesz theorem that \( \nu \ll m \) and \( \nu = \chi \cdot m \) for some \( \chi \in H^2_{0}(\mathbb{T}) \). Thus, we have \( \mu = \nu + \psi \cdot m = (\chi + \psi) \cdot m \). This proves (ii).

(ii) \( \Rightarrow \) (i): suppose that (ii) holds so that \( \mu = (\psi + \chi) \cdot m \) for some \( \psi \in H^2(\mathbb{T}) \) and \( \chi \in H^2_{0}(\mathbb{T}) \). Then, \( \tilde{\mu}(n) = \tilde{\psi}(n) \) for all \( n \in \mathbb{N} \cup \{0\} \). Hence, we have

\[
\sum_{n=0}^{\infty} |\mu \lambda(n)|^2 < \infty. \quad (54)
\]

Since \( \mu \mu = \sum_{n=0}^{\infty} \tilde{\mu}(n)z^n \), it follows that \( \mu \mu \in H^2(\mathbb{D}) \). Clearly, the constant function 1 belongs to \( C_A(\mathbb{D}) \). Therefore, \( 1 \in D(T_{\mu}) \). Now, Proposition 6, (a), implies that \( D(T_{\mu}) \) contains all polynomials.

**Corollary 13.** Let \( \mu \in M(\mathbb{T}) \) be a real measure. Then, \( D(T_{\mu}) = C_A(\mathbb{D}) \) if and only if \( \mu \ll m \) and \( d\mu / dm \in L^2(\mathbb{T}) \).

**Proof.** Suppose that \( D(T_{\mu}) = C_A(\mathbb{D}) \). Then, \( \mu \ll m \) and \( \mu = (\psi + \chi) \cdot m \) for some \( \psi \in H^2(\mathbb{T}) \) and \( \chi \in H^2_{0}(\mathbb{T}) \) by

\[
\begin{align*}
\mu(-n) &= \int_{\mathbb{T}} \bar{z}^{-n} d\mu = \int_{\mathbb{T}} \bar{z}^{-n} d\mu = \tilde{\mu}(n), \\
\text{for every } n \in \mathbb{Z}. \quad (55)
\end{align*}
\]

It follows that \( \chi \in H^2_{0}(\mathbb{T}) \). Therefore, \( d\mu / dm = \psi + \chi \in L^2(\mathbb{T}) \). The converse is a part of Proposition 2.

On the other hand, we would like to conjecture the following:

**Conjecture 14.** Every Toeplitz operator with a singular symbol is trivial.

We give evidence for Conjecture 14 by using the known fact about the Cauchy transform. Let \( E \) be a closed subset of \( \mathbb{T} \) and let

\[
F(E) = \{ g \in H^2(\mathbb{D}) : g = \mu \text{ for some } \mu \in M(E) \}. \quad (57)
\]

Then, it is known that \( F(E) = \{0\} \) if and only if \( m(E) = 0 \) (cf. [23], Theorem 5.5.2).

We then have the following:

**Theorem 15.** If \( \mu \in M(\mathbb{T}) \) is singular and \( m(\sup \mu) = 0 \), then \( T_{\mu} \) is trivial.

**Proof.** Let \( E = \sup \mu \). By assumption, \( m(E) = 0 \). Thus, \( F(E) = \{0\} \). Suppose that \( f \in D(T_{\mu}) \), i.e., \( f \in C_A(\mathbb{D}) \) and \( P(f \cdot \mu) \in H^2(\mathbb{D}) \). Note that \( \sup P(f \cdot \mu) \leq \sup \mu = E \). Hence, \( f \cdot \mu \in M(E) \). So the function \( P(f \cdot \mu) \in H^2(\mathbb{D}) \) belongs to \( F(E) = \{0\} \). It follows that \( P(f \cdot \mu) = 0 \). We have shown that \( P(f \cdot \mu) \in H^2(\mathbb{D}) \) implies \( P(f \cdot \mu) = 0 \). In other words,

\[
f \in D(T_{\mu}) T_{\mu} f = 0. \quad (58)
\]

Therefore \( T_{\mu} \) is trivial (on its domain).

**Remark 16.** Conjecture 14 seems to be known when \( \mu \) is a positive singular measure. Indeed, if \( \mu \) is a positive singular measure, then its Poisson integral is the real part of \( (1 + \theta)/(1 - \theta) \) for some inner function \( \theta \) (cf. [23], Remark 9.1.4).

Now, if \( f \in C_A(\mathbb{D}) \) and \( P(f \cdot \mu) \in H^2(\mathbb{D}) \), then the function
Suppose that each support is a finite set. Then, the Toeplitz operator \( T_\mu \), in Example 4, is densely defined. But it does not seem easy to determine whether \( T_\mu \) is trivial. Nevertheless, often, information about \( T_\mu \) gives information about \( T(\hat{\mu}) \). The following is one of such examples.

**Example 18.** Let \( \mu \in M(\mathbb{T}) \) be a discrete measure whose support contains a nonzero function by Fatou’s theorem for \( C_A(\mathbb{D}) \). Then, \( \mu \in M(\mathbb{T}) \) is dense in \( H^2(\mathbb{D}) \). Hence, \( T_\mu \) is trivial. This proves (62). In particular, \( \mathcal{D}(T_\mu) \) contains the polynomial \( p(z) = (z - \zeta_1) \cdots (z - \zeta_N) \). Hence, by Proposition 10, \( \mathcal{D}(T_\mu) \) is dense in \( H^2(\mathbb{D}) \).

Equations (62) and (63) imply that \( T_\mu f = 0 \) for all \( f \in \mathcal{D}(T_\mu) \), i.e., \( T_\mu \) is trivial. This completes the proof.

**Proposition 17.** Let \( \mu \in M(\mathbb{T}) \) be a discrete measure whose support contains a nonzero function by Fatou’s theorem for \( C_A(\mathbb{D}) \). Then, the Toeplitz operator \( T_\mu \) is densely defined. Therefore, \( P(f \cdot \mu) = 0 \).

The Cantor-middle-third measure \( \mu \) in Example 4, (c), is a singular continuous measure, and its support is the Cantor set (in \( \mathbb{T} \)) whose Lebesgue measure is 0. Hence, Theorem 15 implies that \( T_\mu \) is trivial.

We have seen that the Toeplitz operator \( T_\mu \) in Example 4, (b), is a densely defined trivial linear operator. This result can be extended to the case that \( \mu \) has a finite support. In this case, the fact that \( T_\mu \) is trivial may follow from Theorem 15. However, we give a direct proof and also show that \( T_\mu \) is densely defined.

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Corollary 19. Let \( \mu \in M(\mathbb{T}) \) be a discrete measure whose support consists of \( N \) points of \( \mathbb{T} \). Then,
\[
\det T_n(\tilde{\mu}) = 0,
\]
for all \( n \geq N \).

Proof. Suppose that \( \mu \) is the discrete measure given by (61). Then, the domain \( \mathcal{D}(T_{\mu}) \) is given by (62). Choose any polynomial \( p \) in \( \mathcal{D}(T_{\mu}) \) whose degree is \( N \) (e.g., \( p(z) = (z - \zeta_1) \cdots (z - \zeta_N) \)). Write \( p = \sum_{k=0}^{N} a_k z^k \). Since \( T_{\mu} z^k = \sum_{n=0}^{\infty} \tilde{\mu}(n-k) z^n \), it follows that
\[
0 = T_{\mu} p = \sum_{k=0}^{N} a_k T_{\mu} z^k = \sum_{k=0}^{N} a_k \sum_{n=0}^{\infty} \tilde{\mu}(n-k) z^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{N} a_k \tilde{\mu}(n-k) \right) z^n.
\]
Hence, we have
\[
\sum_{k=0}^{N} a_k \tilde{\mu}(n-k) = 0,
\]
for all \( n \geq 0 \). Now, let \( n \geq N \) and put
\[
x = [a_0 \cdots a_N 0 \cdots 0]^T \in \mathbb{C}^{n+1}.
\]
Then, by (73), \( T_n(\tilde{\mu}) x = 0 \), i.e., \( x \in \ker T_n(\tilde{\mu}) \). Since \( x \neq 0 \), the square matrix \( T_n(\tilde{\mu}) \) is not invertible, or equivalently, \( \det T_n(\tilde{\mu}) = 0 \).

Lastly, we may ask: what is the adjoint of \( T_{\mu} \)? To answer this question, we need the following:

Lemma 20. Let \( \mu \in M(\mathbb{T}) \). Then,
\[
\langle T_{\mu} f, g \rangle = \int_T f(\xi) \overline{g}(\xi) d\mu(\xi),
\]
for every \( f \in \mathcal{D}(T_{\mu}) \) and \( g \in C_A(\mathbb{D}) \).

Proof. Suppose that \( f \in \mathcal{D}(T_{\mu}) \) and \( g \in C_A(\mathbb{D}) \). Then, \( T_{\mu} f \in H^2(\mathbb{D}) \). Write \( T_{\mu} f = \sum_{n=0}^{\infty} a_n z^n \) and \( g = \sum_{n=0}^{\infty} b_n z^n \). Then,
\[
\langle T_{\mu} f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b}_n.
\]
Observe that, for each \( z \in \mathbb{D} \),
\[
(T_{\mu} f)(z) = \int_T \frac{f(\xi)}{1 - \xi z} d\mu(\xi) = \int_T f(\xi) \sum_{n=0}^{\infty} \zeta^n z^n d\mu(\xi)
= \sum_{n=0}^{\infty} \left[ \int_T f(\xi) \zeta^n d\mu(\xi) \right] z^n.
\]
Hence, we have
\[
a_n = \int_T f(\xi) \overline{\zeta^n} d\mu(\xi).
\]
Observe that, for each \( 0 < r < 1 \),
\[
g_r = \sum_{n=0}^{\infty} b_n r^n z^n \in C_A(\mathbb{D}).
\]
It follows that
\[
\langle T_{\mu} f, g_r \rangle = \sum_{n=0}^{\infty} a_n \overline{b}_n r^n = \sum_{n=0}^{\infty} \int_T f(\xi) \overline{\zeta^n} b_n r^n d\mu(\xi)
= \int_T f(\xi) \sum_{n=0}^{\infty} \overline{b}_n r^n \zeta^n d\mu(\xi) = \int_T f(\xi) \overline{g_r} d\mu(\xi).
\]
If we let \( r \to 1 \), then \( \|g - g_r\|_{\infty} \to 0 \), and hence, \( \langle T_{\mu} f, g \rangle \to \langle T_{\mu} f, g \rangle \) and \( \int_T f(\xi) \overline{g_r} d\mu \to \int_T f(\xi) \overline{g} d\mu \). This proves (75).

Assume that \( \mu \in M(\mathbb{T}) \) and \( \mathcal{D}(T_{\mu}) \) is dense in \( H^2(\mathbb{D}) \). Then, the adjoint \( T_{\mu}^{\ast} \) of \( T_{\mu} \) can be defined; the domain of \( T_{\mu}^{\ast} \) is
\[
\mathcal{D}(T_{\mu}^{\ast}) = \{ g \in H^2(\mathbb{D}) : \exists h \in H^2(\mathbb{D}) s.t. \langle T_{\mu} f, g \rangle = \langle f, h \rangle \forall f \in \mathcal{D}(T_{\mu}) \},
\]
and, for each \( g \in \mathcal{D}(T_{\mu}^{\ast}) \), \( T_{\mu}^{\ast} g \) is the (unique) element of \( H^2(\mathbb{D}) \) such that
\[
\langle T_{\mu}^{\ast} g, f \rangle = \langle f, T_{\mu} g \rangle.
\]
for every \( f \in \mathcal{D}(T_{\mu}) \).

If \( \varphi \in L^{\infty}(\mathbb{T}) \), then \( T_{\mu}^{\ast} \varphi = \varphi \cdot \overline{\mu} \cdot \hat{\varphi} \). Hence, it is reasonable to expect that the adjoint of \( T_{\mu} \) is the Toeplitz operator induced by the “complex conjugation” of \( \mu \). For \( \mu \in M(\mathbb{T}) \), define
\[
\hat{\mu}(E) = \mu(E) \overline{E} \in \mathcal{D}_T.
\]
Then, \( \mu \in M(\mathbb{T}) \). Of course, \( \mu \in M(\mathbb{T}) \) is a real measure if and only if \( \mu = \mu \cdot \overline{\mu} \). Note that
\[
\hat{\mu}(n) = \hat{\mu}(-n),
\]
for every \( n \in \mathbb{Z} \).

We now have the following:

Proposition 21. Let \( \mu \in M(\mathbb{T}) \). Assume that \( \mathcal{D}(T_{\mu}) \) is dense in \( H^2(\mathbb{D}) \). Then,
\[
T_{\mu} \subseteq T_{\mu}^{\ast},
\]
that is \( \mathcal{D}(T_{\mu}) \subseteq \mathcal{D}(T_{\mu}^{\ast}) \) and \( T_{\mu} = T_{\mu}^{\ast} \) on \( \mathcal{D}(T_{\mu}) \).
for all $\zeta \in T(a)$ Suppose that $f \in D(T_\mu)$. By Lemma 20, it follows that
\[
\langle T_\mu f, g \rangle = \int_\Gamma f \overline{g} \, d\mu = \int_\Gamma g \overline{f} \, d\mu = \langle f, T_\mu g \rangle,
\]
for every $f \in D(T_\mu)$. It follows that $g \in D(T_\mu^*)$ and $T_\mu^* g = T_\mu g$. Therefore, we conclude that
\[
D(T_\mu) \subseteq D(T_\mu^*),
\]
and $T_\mu^* g = T_\mu g$ for every $g \in D(T_\mu)$. This completes the proof.

If $\mu \in M(\mathbb{T})$, and $T$ is the restriction of the Toeplitz operator $T_\mu$ to $cl_{H^2}(D(T_\mu))$, then $T$ is a densely defined linear operator. In this case, $T^*$ is a linear operator from $H^2(D)$ onto $cl_{H^2}(D(T_\mu))$. By the same argument as the proof of Proposition 21, we have $D(T_\mu) \subseteq D(T^*)$ and $T^* g = T_\mu g$ for $g \in D(T_\mu)$.

We also have the following:

**Proposition 22.** Let $\mu \in M(\mathbb{T})$ be positive. Then, the following hold:

(a) $T_\mu$ is positive, i.e., $\mu f, f \geq 0$ for all $f \in D(T_\mu)$

(b) $T_\mu = \{ f \in C_A(D) : f(\zeta) = 0 \text{ for every } \zeta \in \text{supp } \mu \}$

Proof. (a) Suppose that $\mu \geq 0$. Then, by Lemma 20, we have
\[
\langle T_\mu f, f \rangle = \int_\Gamma |f|^2 \, d\mu \geq 0,
\]
for every $f \in D(T_\mu)$.

(b) Suppose that $\mu \in M(\mathbb{T})$ is positive. If $f \in \text{ker } T_\mu$, then \[
\int_\Gamma |f|^2 \, d\mu = \langle T_\mu f, f \rangle = 0.
\]
Hence, $f = 0 \mu$-a.e. on $\mathbb{T}$. We show that $f = 0$ on supp $\mu$. Assume to the contrary that $f(\zeta_0) \neq 0$ for some $\zeta_0 \in \text{supp } \mu$. Since $f \in C_A(D)$, there exist a constant $\epsilon > 0$ and an open arc $I \subseteq \mathbb{T}$ with center $\zeta_0$ such that $|f(\xi)| \geq \epsilon$ for all $\xi \in I$. Since $\zeta_0 \in \text{supp } \mu$, we have $\mu(I) > 0$. It follows that
\[
\int_I |f|^2 \, d\mu \geq \int_I |f|^2 \, d\mu \geq \epsilon \cdot \mu(I) > 0,
\]
which is a contradiction. Hence, $f(\zeta) = 0$ for all $\zeta \in \text{supp } \mu$. Therefore, $\text{ker } T_\mu \subseteq \{ f \in C_A(D) : f = 0 \text{ on supp } \mu \}$.

The reverse inclusion is trivial.

The operator $T_\mu$ may be positive even though $\mu$ is complex. For example, for any complex number $\alpha$, the measure $\alpha \cdot \delta_1$ is trivial, and hence, it is positive.

We conclude with a remark on the boundedness of $T_\mu$. It is well known (cf. [3]) that $f \in L^2(\mathbb{T})$, $T_\mu f$ is bounded if and only if $f \in L^{\infty}(\mathbb{T})$, in which case, $\|T_\mu f\| = \|f\|_{\infty}$. If $\mu \geq 0$ and $T_\mu$ is bounded, then
\[
\int_\Gamma |f|^2 \, d\mu \leq c \cdot \|f\|_{\infty}^2 \quad (f \in D(T_\mu)).
\]

Let us call a positive measure $\mu \in M(\mathbb{T})$ a compatible measure if $\mu$ satisfies (91) for all $f \in C_A(D)$. The word “compatible” comes from the paper [12]. One can show that the following statements are equivalent:

(i) $\mu$ is a compatible measure

(ii) $\mu \ll m$ and $d\mu/dm \in L^{\infty}(\mathbb{T})$

(iii) $D(T_\mu)$ contains all polynomials and $T_\mu$ is bounded

If these conditions are satisfied and if $f = d\mu/dm$, then $D(T_\mu) = C_A(D)$ and
\[
T_\mu f = T \varphi f,
\]
for every $f \in C_A(D)$. In (iii), we cannot reduce the condition that $D(T_\mu)$ contains all polynomials to the condition that $D(T_\mu)$ is dense in $H^2(D)$; there is a measure $\mu \in M(\mathbb{T})$ which is not compatible such that $T_\mu$ is densely defined and bounded (see Example 4, (b)).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares there are no conflicts of interest.

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