The quantized Hall conductance of a single atomic wire: A proposal based on synthetic dimensions

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We propose a method by which the quantization of the Hall conductance can be directly measured in the transport of a one-dimensional atomic gas. Our approach builds on two main ingredients: (1) a constriction optical potential, which generates a mesoscopic channel connected to two reservoirs, and (2) a time-periodic modulation of the channel, specifically designed to generate motion along an additional synthetic dimension. This fictitious dimension is spanned by the harmonic-oscillator modes associated with the tightly-confined channel, and hence, the corresponding “lattice sites” are intimately related to the energy of the system. We analyze the quantum transport properties of this hybrid two-dimensional system, highlighting the appealing features offered by the synthetic dimension. In particular, we demonstrate how the energetic nature of the synthetic dimension, combined with the quasi-energy spectrum of the periodically-driven channel, allows for the direct and unambiguous observation of the quantized Hall effect in a two-reservoir geometry. Our work illustrates how topological properties of matter can be accessed in a minimal one-dimensional setup, with direct and practical experimental consequences.

I. INTRODUCTION

Electronic transport in solids plays a fundamental role in our exploration of matter, and it constitutes the basis for innumerable device applications. In fact, the need for smaller and more efficient hardware has naturally led to the development of mesoscopic devices, where the quantum nature of the electron gas becomes relevant [1]. One of the most prominent examples of such quantum phenomena is the quantization of the electrical conductance in mesoscopic channels, which stems from the existence of discrete transverse modes [2–6]. In the 1980’s, studies of the Hall conductance in two-dimensional electron gases subjected to high magnetic fields revealed the quantized Hall effect [7–9], which was later related to the existence of topological invariants in the band structure [10, 11] and chiral edge modes [12, 13]. Moreover, such Hall measurements subsequently revealed the fractional quantum Hall effect [14], a first instance of a strongly-correlated topological phase [15, 16]. More recent transport experiments revealed the existence of topological insulators, such as those realizing the quantum spin Hall effect [17–20], as well as Dirac and Weyl semimetals [21, 22]. In this regard, transport measurements are an important and well-established method for probing and studying the properties of quantum matter [23–24].

In parallel to the exploration of new materials and devices, quantum-engineered systems have been developed in ultracold-atom laboratories in view of offering a novel perspective on transport in quantum matter [25]. In these settings, non-equilibrium dynamics can be probed through different protocols [26], for instance, by suddenly releasing the atomic cloud in an optical lattice and imaging its expansion [27, 28], or by driving the cloud with an external (optical or magnetic) force [29, 30]. A third approach consists in engineering a two-terminal geometry, i.e. a mesoscopic channel for atoms connected to two reservoirs, using a constriction optical potential [31]; this scheme, which reproduces the two-terminal configuration used in electronic transport experiments, allows for a direct evaluation of a neutral gas’ conductance. Importantly, such a setting has demonstrated the quantized conductance of a one-dimensional atomic channel [32], which constitutes a good starting point to study the transport of strongly correlated matter thanks to the ability to tune the interaction strength [33]. A particularly exciting perspective concerns the observation and characterization of fractional quantum Hall states in ultracold atomic gases [34, 35].

Measuring the quantized Hall conductance of a two-dimensional (2D) ultracold Fermi gas, using the engineered-reservoir scheme of Refs. [36, 37], is definitely appealing. However, this challenging goal would a priori require the combination of two main ingredients: (1) the realization of a synthetic gauge field [38, 39, 40] to create a non-trivial topological band structure and reach the quantum-Hall regime [41, 42]; and (b) in direct analogy with the multi-terminal devices (Hall-bar geometries) used in solid-state experiments [43, 44], one would need to connect the 2D Fermi gas to several reservoirs. In this work, we propose that such an apparently complicated setting could in fact be readily designed starting from a single atomic channel [37], by exploiting the concept of synthetic dimensions [14, 33]. As we will show, the use of a synthetic dimension does not only simplify the implementation of the “atomic Hall bar”, but it also allows for a direct read-out of the quantized Hall conductance associated with chiral edge modes using a simple two-
A system is described by a 2D tight-binding model in the synthetic dimension posed in Ref. [53], to generate hopping processes along the dimension. The potential is then periodically shaken, as proposed in Ref. [53], to generate hopping processes along the synthetic dimension \( \lambda \); the corresponding sites are indexed by \( j \) and the hopping amplitude denoted \( J_y \). The harmonic oscillator levels \( \lambda \) are reinterpreted as lattice sites along a synthetic dimension. The potential is then periodically shaken, as proposed in Ref. [53], to generate hopping processes along the synthetic dimension \( \lambda \); the corresponding hopping amplitude is denoted \( J_y = J_1 \sqrt{\lambda} \). For a suitable shaking protocol, the system is described by a 2D tight-binding model in the \( y - \lambda \) plane, which includes the effects of a tunable artificial magnetic field; the corresponding flux per plaquette is denoted by \( \varphi \).

**Reservoir geometry.** This scheme is universal in the sense that it could be applied to study the Hall conductance of a wide variety of atomic states (with potential applications to strongly-correlated states). Besides, we note that the main concepts developed in this proposal could also be used in other physical platforms; see, for instance, the recent proposal in Ref. [53] to braid Majorana fermions in a single superconducting wire extended by a synthetic dimension; and Ref. [54] for a recent implementation of synthetic dimensions in photonics.

### A. The main approach and central results

The aim of this work is to lay out a scheme by which a single atomic wire, connected to two engineered reservoirs [Fig. (a)], can be turned into an “atomic Hall bar”: a 2D atomic system exhibiting the quantum Hall effect and designed so as to extract its (quantized) Hall conductance through transport.

Our approach is based on the observation that the constriction potential used to generate the atomic channel [39] defines a natural synthetic dimension. As illustrated in Fig. (b), the harmonic-oscillator levels associated with the tight confinement form a large set of discrete states, indexed by \( \lambda \in \mathbb{N} \), which can be interpreted as fictitious “lattice sites” along a synthetic dimension. As shown in Ref. [53], motion can be induced along the \( \lambda \) direction by shaking the channel in a time-periodic manner. Hence, within the central region of the constriction potential, atoms are allowed to move along the real direction defined by the channel (denoted “\( y \) direction” in Fig. (c) and hereafter), as well as along the synthetic dimension \( \lambda \). We note that this construction essentially replaces the continuous transverse direction \( x \) [Fig. (a)] by a discrete synthetic dimension \( \lambda \) [Fig. (b)], and that the motion is inhibited along the third spatial direction \( z \); besides this, we will assume that the channel direction \( y \) can also be discretized upon adding an optical lattice, as recently implemented in Ref. [57]. As a final ingredient, we will assume that the phase of the modulation that generates motion along \( \lambda \) can be made \( y \)-dependent, \( \phi(y) \): as previously shown in Ref. [53], this can generate a uniform magnetic flux in the 2D lattice defined in the fictitious \( \lambda - y \) plane [Fig. (c)]; see also Refs. [58–62].

In the following, we will consider that the reservoirs are not subjected to the modulation, and hence, that the corresponding regions do not include a synthetic magnetic field.

This work analyzes the conductance of this hybrid 2D atomic system, as probed by the inherent two-reservoir geometry [Fig. (a)]. At this stage, let us highlight a couple of peculiarities introduced by the synthetic \( \lambda \) dimension. First, we emphasize that this synthetic dimension is intimately related to the energy of the system (each “site” along \( \lambda \) is associated with a harmonic-oscillator level), and hence, it cannot be simply treated as a genuine spatial direction. In particular, there is a built-in chemical-potential bias along the \( \lambda \) direction, in the sense that particles privilege the occupation of low-\( \lambda \) (i.e., low-energy) states; this natural bias leads to a subtle interplay with the overall chemical-potential imbalance that is imposed by the two reservoirs to drive current across the channel [Fig. (a)]. Second, since the system is periodically driven (and thus belongs to the class of Floquet-engineered systems [63]), transport properties rely on the underlying quasi-energy spectrum [67–69].

As explained in more detail in the following Sections, these unusual features lead to an effective (fictitious) multi-terminal geometry, which allows us to substantially improve the conductance measurement stemming from the (real) two-reservoir geometry [Fig. (a)]. As a central result of our work, we demonstrate that a proper state preparation and reservoir configuration can allow for a clear separation of the bulk and edge contributions to the conductance. In particular, our unusual single-channel setup can be designed so as to directly reveal the quantized Hall conductance associated with chiral edge modes (we recall that this measurement requires at least four terminals in conventional static systems [5]). Our work opens new avenues towards the exploration of topo-
logical transport in ultracold-atom experiments, through the development of new probing schemes based on synthetic dimensions.

B. Outline

The paper is organized as follows: The first Sections II A-III C review several notions, which are central in the main part of our study. In Sec. II A, we review the theoretical framework offered by the Landauer-Büttiker formalism [5, 6, 23, 24], and we introduce the quantities that are used in the following sections to describe quantum transport. Section II B discusses the transport properties of a simple 2D quantum Hall system, the Harper-Hofstadter model [70], which will play an important role in the main part of the study; this Section II B also aims to highlight the main differences between the transport measurements that are performed using two-terminal and four-terminal geometries [5]. We then recall in Sec. II C how quantum-transport measurements can be applied to time-periodic systems, based on the Floquet formalism [67–69, 71].

Section II A introduces the shaken-channel scheme; we discuss the emergence of a synthetic dimension [53], derive an effective time-independent model, and propose a possible implementation using the constrictive potential of Fig. 1. The transport properties of the effective time-independent model are studied in two different regimes: in Sec. IIIB we make an approximation and map the model to the standard Harper-Hofstadter Hamiltonian; in Sec. IIIC we relax this approximation, and we show that the signatures of quantized transport survive. We also discuss how the energetic nature of the synthetic dimension naturally leads to an effective multi-terminal geometry, which greatly enriches the measurement based on the (real) two-reservoir geometry.

In Sec. IV we consider the full time-dependent problem and apply the formalism introduced in Sec. II C to the two aforementioned regimes, in Sec. IV A and in Sec. IV B respectively. Experimental considerations are briefly discussed in Sec. V and conclusions are drawn in Sec. VI.

II. QUANTUM TRANSPORT: GENERAL FRAMEWORK

A. Landauer-Büttiker formalism and non-equilibrium Green’s function

We start by reviewing the theoretical framework offered by the Landauer-Büttiker formalism, which was originally developed for calculating the conductance in solid-state systems [5, 6, 23, 24], but which has also been recently applied to describe transport in charge-neutral atomic systems [24]. We first consider the case of a single channel connected to two external reservoirs, which act as contact terminals and are labelled by α and β; throughout, we will assume that particles obey Fermi statistics. The chemical potential in the α [resp. β] reservoir is denoted μ_α [resp. μ_β]; we will assume that these chemical potentials are centered around the Fermi energy E_F and that their differences are small. In this case, and assuming zero temperature for now, the linear d.c. current that flows between the two reservoirs can be expressed as

$$I_{α, β} = G_{α, β}(E_F)(μ_α - μ_β),$$

(1)

where $G_{α, β}(E_F)$ is the conductance of the system at a given Fermi energy $E_F$ [24]. This conductance can be evaluated using the Landauer formula

$$G_{α, β}(E_F) = \frac{e^2}{h} \sum_m T^{(m)}_{α, β}(E_F),$$

(2)

which involves a sum over the $m$ possible transport channels of the transmission probabilities $T^{(m)}_{α, β}(E_F)$ for a particle of charge $e$ to be carried through the system; from now on, we will set $e = 1$ to equally treat charged and neutral particles, and we will maintain Planck’s constant $h$ in our expressions. The transmission probabilities are related to the scattering properties of the system [5]; in particular, if the system has $N$ perfectly transmitting channels, i.e. $T^{(m)} = 1$ for all $m$, each of these will contribute with a quantum of conductance $G_0 = 1/h$ such that the total conductance is quantized according to $G_{α, β} = NG_0$.

For a system connected to many reservoirs, the d.c. current at a given terminal $α$ is generalized by the Landauer-Büttiker formula

$$I_α = \sum_β I_{α, β} = \sum_β G_{α, β}(E_F)(μ_α - μ_β),$$

(3)

where the sum is now taken over all other reservoirs $β$, which are connected to $α$.

In the case of finite temperature, the currents in Eq. (1) and (3) must be weighted with the Fermi-Dirac distributions evaluated at the two reservoirs $f_α ≡ f(E - μ_α)$ and $f_β ≡ f(E - μ_β)$, and then integrated over all energies, which yields the following generalized expression [72]:

$$I_α \equiv \frac{e}{h} \sum_β \int T^{(m)}_{α, β}(E)[f_α - f_β] dE.$$

To evaluate the transmission probabilities in Eq. (2), it is often convenient to use the non-equilibrium Green’s function method, which is mathematically equivalent to the scattering approach in the linear regime [24, 73]. The matrix representation of the retarded Green’s function $G^r$ of a system at Fermi energy $E_F$ is defined through the Hamiltonian matrix $H$ as

$$[E - H] G^r = 1,$$

(4)

where $E = (E_F + i0^+)1$, 1 is the identity matrix and $0^+$ is an infinitesimally small positive quantity.
Without loss of generality, we focus on a channel connected to two reservoirs, which are attached to the left and the right of the system (in this setting, $\alpha, \beta = R, L$ refer to the two reservoirs). The Hamiltonian matrix of the entire scattering system, including the reservoirs, has the following block structure

$$
\mathbf{H}_{\text{tot}} = \begin{pmatrix}
\mathbf{H}_L & \mathbf{H}_{LS} & 0 \\
\mathbf{H}_{LS}^\dagger & \mathbf{H}_S & \mathbf{H}_{RS} \\
0 & 0 & \mathbf{H}_{RR}
\end{pmatrix},
$$

(5)

where $\mathbf{H}_{L,R}$ refers to the Hamiltonians describing the left/right reservoirs, $\mathbf{H}_S$ describes the inner system (the transport channel), and where $\mathbf{H}_{LS}$ and $\mathbf{H}_{RS}$ describe the couplings between the inner system and the left/right reservoirs. Since the size of the reservoirs is typically very large, the size of the Hamiltonian matrices $\mathbf{H}_{L,R}$ is large compared to the size of $\mathbf{H}_S$. From Eq. (5), the Green’s function of the total system is

$$
\begin{pmatrix}
E - \mathbf{H}_L & -\mathbf{H}_{LS} & 0 \\
-\mathbf{H}_{LS}^\dagger & E - \mathbf{H}_S & -\mathbf{H}_{RS} \\
0 & -\mathbf{H}_{RS}^\dagger & E - \mathbf{H}_R
\end{pmatrix}^{-1} = \begin{pmatrix}
\mathbf{g}_L & \mathbf{g}_L^\dagger & 0 \\
\mathbf{g}_L^\dagger & \mathbf{g}_S & \mathbf{g}_R \\
0 & \mathbf{g}_R^\dagger & \mathbf{g}_R^\dagger
\end{pmatrix}.
$$

(6)

From Eq. (6), one can obtain the following relation for the Green’s function of the inner system $\mathbf{g}_S$

$$
[E - \mathbf{H}_S - (\Sigma_L(E_F) + \Sigma_R(E_F))] \mathbf{g}_S = \mathbf{1},
$$

(7)

where $\Sigma_L(E_F) = \mathbf{H}_{LS}^\dagger (E - \mathbf{H}_L)^{-1} \mathbf{H}_{LS}$ and $\Sigma_R(E_F) = \mathbf{H}_{RS} (E - \mathbf{H}_R)^{-1} \mathbf{H}_{RS}^\dagger$. Equation (7) is very similar to Eq. (5), except that the Hamiltonian is now modified with the term $\Sigma(E_F) = \sum_\alpha \Sigma_\alpha(E_F)$, the so-called contact self-energy, which includes the details of the reservoirs. The anti-Hermitian counterpart of the self-energy $\mathbf{g}_S^\alpha(E_F) = i \left( \Sigma_\alpha(E_F) - \Sigma_\alpha^\dagger(E_F) \right)$ defines the linewidth of the $\alpha$ reservoir, and it reflects the fact that particles are lost from the inner system due to leakage into the reservoirs; in this sense, the channel is out of equilibrium [24]. The linewidth of the reservoir and the self-energy contributions can be obtained following standard prescriptions [24] [74].

Using the non-equilibrium Keldysh formalism [75], the transmission can be calculated through the Caroli formula [76], which involves the Green’s functions of the inner system and the self-energies of the reservoirs:

$$
T_{\alpha,\beta}(E_F) = \text{Tr} \left[ \Gamma^\alpha(E_F) \mathbf{g}_S^\dagger(E_F) \Gamma^\beta(E_F) \mathbf{g}_S(E_F) \right],
$$

(8)

where we have omitted the subscript $S$ associated with the Green’s functions of the system, for simplicity of notation; we note that the advanced Green’s function satisfies $\mathbf{g}_\alpha^\dagger = (\mathbf{g}_\alpha^\dagger)^\dagger$. Equation (8) is very convenient for numerical evaluations of the d.c. current that flows between two terminals, when combined with the recursive Green’s function (RGF) method [74] [77] based on the Dyson equation. The RGF method can be generalized to multi-terminal systems [75], which we will use for calculating the Hall conductance later in this article.

1. The wide-band approximation

We have seen that all the details of the reservoirs are included in the self-energy matrix $\Sigma(E_F)$, which is used both for obtaining the linewidth $\Gamma(E_F)$ and the Green’s function of the inner system connected to the terminals. In order to calculate the self-energy $\Sigma(E_F)$, the reservoir is typically assumed to have a large volume and a high density of states. If the density of states of the reservoir is constant over an energy range much larger that the bandwidth of the inner system, the wide-band approximation can be used [79]. Under this approximation, both the self-energy and the linewidth are taken to be energy independent. In particular $\Sigma_\alpha \propto -i\delta_{\ell,\gamma}$, where $\gamma$ is a constant that is of the order of the bandwidth of the inner system, and the $\delta_{\ell,\gamma}$ selects only the sites of the system that belong to the terminal $\alpha$ [69]. Unless otherwise stated, we will use the wide-band approximation for all the results presented in the article.

B. Quantum Hall transport: Application to the Harper-Hofstadter model

We now review the peculiar transport properties associated with the Hall effect, by applying the RGF method to the Harper-Hofstadter model [70]. The main goal of this Section is to distinguish between the transport measurements that result from two-terminal and four-terminal geometries; we will assume zero temperature throughout. We will also discuss how the results of the RGF method can be interpreted in terms of (topological) edge-state transport, based on the Landauer-Büttiker formalism [50] [51]. The results of this Section will constitute a good basis for understanding the transport properties of the time-independent shaken-channel model [Fig. 1], which approximately maps onto the Harper-Hofstadter model (see Section III B).

The Harper-Hofstadter Hamiltonian describes a particle moving in a two-dimensional lattice in the presence of a perpendicular magnetic field [70]

$$
\hat{H} = -J \sum_{i,j} \left( e^{i\varphi} |i,j\rangle \langle i+1,j| + |i,j\rangle \langle i,j+1| + \text{H.c.} \right).
$$

(9)

This Hamiltonian describes hopping processes taking place between nearest-neighboring sites $(i,j)$ of the lattice, where the indices $i$ and $j$ refer to the two directions $(x$ and $y$), respectively. For fractional values of the flux $\varphi = 2\pi p/q$, with $q, p \in \mathbb{Z}$, the spectrum of the Hamiltonian in Eq. (9) depicts $q$ bulk bands, which are connected by topologically-protected chiral edge states [82]. These chiral edge states are responsible for the quantized Hall conductance of the system, whenever the Fermi energy lies in a spectral bulk gap [7] [10] [12] [82]. Specifically, in this quantum Hall regime, the longitudinal conductance vanishes, while the transverse (Hall) conductance exhibits robust plateaus [7], whose values di-
Figure 2. (insets) Example of a two-terminal setup with a square lattice geometry of \( N_x = N_y = 7 \) sites along the \( x \) and \( y \) directions, respectively. The inner system is represented by the black dots, while the red dots identify the sites of the left (L) and right (R) reservoir. The arrows describe the motion of the chiral edge modes present in the model (clockwise at energies \( E < 0 \) and counter-clockwise at \( E > 0 \)). (main) Conductance \( G_{L,R} \) of the Harper-Hofstadter model with magnetic flux \( \varphi = 2\pi/3 \), within the wide-band approximation, for an input bias applied to the left reservoir; the system size is \( N_x = N_y = 30 \). The plateaux are associated with the edge modes illustrated in the insets.

rectly correspond to the number of current-carrying edge states \[82, 83\]. This behaviour can be understood as the bulk of the system being insulating, while chiral edge currents carry the Hall current. Importantly, the chirality (orientation) of these edge modes around the 2D sample determines the sign of the Hall conductance.

We shall now discuss how these considerations apply to the conductance signal that is extracted from transport measurements using two or four terminals.

1. Two-terminal geometry

We consider a square lattice with \( N_x \) and \( N_y \) sites along the \( x \) and \( y \) directions, respectively. This lattice constitutes the inner system to be probed. Then, each site at the left and right end of the inner system is coupled to a reservoir, through a left and right terminal, respectively. An example of such a geometry is shown in Fig. 2 where the inner system is represented in black, while the reservoir sites and the terminal links connecting them to the inner system are indicated in red. For simplicity, we have indicated only a few sites of the reservoirs, as the latter are assumed to be very large and to have translational symmetry (this is schematically indicated by dashed lines). The two reservoirs are labelled with \( L \) and \( R \), respectively, for left and right. We assume that a small bias is applied to the left reservoir and that the conductance \( G_{L,R} \) is measured through a current detected at the right terminal.

The inner system is described by the Harper-Hofstadter model in Eq. (9), and we will fix the flux per plaquette to the value \( \varphi = 2\pi/3 \), for which the spectrum depicts three isolated bulk bands; in this setting, the two spectral gaps host a single chiral edge mode each \[82\]; the bulk gap at \( E > 0 \) [resp. \( E < 0 \)] hosts an edge mode that propagates counter-clockwise [resp. clockwise] around the 2D lattice. When the Fermi energy \( E_F \) lies in the middle of a bulk band, one expects the observation of a metallic behavior; bulk states provide a large set of non-perfectly transmitting channels, which results in a non-quantized conductance across the system. In contrast, when the Fermi energy is set within a band gap, the only channels that are available for transport are provided by the edge modes; in this regime, the conductance is quantized according to the number of edge modes present in the gap (i.e. one in the present model). Importantly, the chirality of the edge modes (and hence, the sign of the Hall conductance) cannot be identified in a two-terminal geometry \[8\].

Particles populating the “clockwise” (resp. “counter-clockwise”) chiral edge mode flow from the left to the right reservoir by following the top (resp. bottom) edge; see insets in Fig. 2. In both cases, this gives rise to a positive (quantized) conductance between the two reservoirs. In this sense, measuring a quantized conductance \( G_{L,R} \) in this two-terminal geometry can only reveal the absolute value of the Hall conductance associated with the underlying 2D lattice system \[8\]. We illustrate this phenomenon in Fig. 2 where we plot the conductance \( G_{L,R} \) resulting from the RGF method, as a function of the Fermi energy \( E_F \) of the inner system (of size \( N_x = N_y = 30 \)). This plot clearly indicates that the conductance \( G_{L,R} \) is quantized and positive whenever the Fermi energy falls within one of the two band gaps of the model [see the plateaus in Fig. 2], in agreement with the discussion above. Conversely, this same conductance is found to be not quantized whenever the Fermi energy hits one of the three bulk bands [see the irregular peaks in Fig. 2].

Summarizing, while this two-terminal measurement cannot capture the chirality of the edge modes (and hence, the sign of the quantized Hall conductance), it does give a clear indication that the system displays perfectly-transmitting channels (i.e. potential chiral edge modes) within well-defined energy ranges.

2. Four-terminal geometry

We now extend the two-terminal geometry by adding two more terminals at the top and at the bottom of the inner system; we label these \( T \) and \( B \), respectively. As a technical note, we impose that the corner sites of the inner system are only coupled to a single reservoir, which allows one to unambiguously define the terminal regions. An example of such a four-terminal geometry is shown in Fig. 3(a). As further illustrated below, this four-terminal configuration allows for a clear and independent identification of the longitudinal and transverse (Hall) conductances of the system; we note that this configuration forms a minimal “Hall-bar” setup, as routinely used in solid-state experiments.

Specifically, the longitudinal conductance \( G_{\text{long}} \equiv G_{L,R} \) is obtained as in Sec. 1B1, namely, by applying a small
biased to the left reservoir and measuring a current at the right terminal.

In order to extract the transverse (Hall) conductance, $G_{\text{Hall}}$, one needs to analyze the transport that takes place between the top and the bottom reservoirs (while a bias is imposed at the left reservoir, as above). Applying the Landauer-Büttiker formulism to the four-terminal configuration [84, 85], one finds that the Hall conductance can be obtained as $G_{\text{Hall}} \equiv G_{LT} - G_{LB}$, which is indeed suitable in the present configuration where the bias is set in the left reservoir. We note that this expression for the Hall conductance is specifically chosen so as to probe the unidirectional transport of chiral edge states, propagating around the 2D quantum Hall system [84].

In Fig. 3(b), we show the longitudinal conductance $G_{\text{long}}$ and the Hall conductance $G_{\text{Hall}} \equiv G_{LT} - G_{LB}$, as obtained using the RGF method. As before, the magnetic flux is $\phi = 2\pi/3$, for an input bias applied to the left reservoir; the system size is $N_x = N_y = 30$.

C. Evaluating the conductance in periodically-driven systems

As already mentioned in Sec. I A our proposal builds on the possibility of addressing a synthetic dimension by applying a time-periodic modulation to an atomic channel [53]. In fact, this proposed scheme belongs to the general class of Floquet-engineered quantum systems, which aims to realize intriguing Hamiltonian models through periodic driving [64–66, 89, 90]. In this context, it is common to derive an effective (Floquet) Hamiltonian that describes the time-long dynamics of the system, and which results from a rich interplay between the time-modulation and the underlying static system [61, 66].

In fact, Floquet engineering can also be exploited to transfigure quantum transport properties, in the sense that applying a temporal modulation can greatly modify the transport channels of a quantum system. In this quantum-transport framework, where the time-modulated system is further connected to reservoirs, it is generally insufficient to simply apply the standard tools of quantum-transport theory to the effective (Floquet) Hamiltonian, which describes the inner system; such a naive approach will be discussed in Sec. III. Instead, a more rigorous approach consists in using a generalization of the non-equilibrium Green’s function method [67–69, 74], which is specifically tailored to treat time-periodic systems, as we now review in this Section: this second approach will be applied in Sec. IV.

Consider a time-dependent Hamiltonian $H(t + T) = H(t)$, where $T = 2\pi/\omega$ is the period of the applied temporal modulation. The Fourier expansion of the Hamiltonian yields

$$H(t) = \sum_{n=-n_F}^{n_F} H_n e^{-in\omega t},$$

where we have truncated the series up to $N_F = 2n_F + 1$ modes. In such a Floquet system, the energy $E$ is only defined up to multiples of the driving frequency $E = E_n \omega$ (hereafter, we set $\hbar = 1$). This leads to the notion of quasi-energies, which can be chosen within the Brillouin zone $\varepsilon \in [-\omega/2, \omega/2]$; see the review [69]. The so-called Floquet spectrum, which is defined in this restricted range, is then periodically repeated for each multiplicity (i.e. around each $\omega$). As discussed in Ref. [91], it is convenient to treat such periodically-driven systems in an extended (Floquet) Hilbert space, of dimension $N_x N_y N_F$, which explicitly takes these multiplicities into account. In this framework, the Hamiltonian is replaced by a so-called “quasienergy operator”, $Q$, whose components $Q_{m,n} = H_{m-n + \Delta m,n \omega}$ act in the original Hilbert space; here $H_n$ refers to the Fourier components introduced in Eq. (10) and $\Delta$ is the $N_x \times N_y$ identity matrix.

Following Refs. [68, 69], one generalizes Eq. (7) in view of defining a Floquet representation for the Green’s function through the relation

$$[E - Q - (\Sigma_F^R + \Sigma_F^I)] G_F = \mathbb{I},$$

with
where all matrices are defined in the extended Floquet Hilbert space of dimension $N_x N_y N_F$, and where $I$ is the identity matrix in this extended space. Specifically, $E$ and $\Sigma_F$ are diagonal matrices whose elements are

\[
E \equiv \text{Diag}\left(\{E_F + i0^+\}_{1}, \ldots, \{E_F + i0^+\}_{N_F}\right),
\]

\[
\Sigma_F \equiv \text{Diag}\left(\Sigma_0(E_F - n_F \omega), \ldots, \Sigma_0(E_F + n_F \omega)\right),
\]

where $\alpha = L, R$ and $\Sigma_0(E_F)$ is defined below Eq. (11).

Solving Eq. (11) yields the Floquet representation for the Green’s function, $G_F$:

\[
G_F^\alpha = \left(\begin{array}{cccc}
G^\alpha_{n_F, n_F, 0} & \cdots & G^\alpha_{n_F, n_F, r_F} \\
\vdots & \ddots & \vdots \\
G^\alpha_{0, n_F, 0} & \cdots & G^\alpha_{0, n_F, r_F}
\end{array}\right).
\]

As in Section IIA, the linewidth can be defined as $\Gamma_n = i \left(\Sigma_n - \Sigma_n^\dagger\right)$, and it can also be represented in the extended Hilbert space (with components denoted $\Gamma_{n,m}^\alpha$).

The Floquet generalization of the Caroli formula in Eq. (8) can then be obtained by treating the Fourier components of the Green’s function and the linewidth individually. The resulting Floquet-Caroli formula for the transmission, at a given Fermi energy $E_F$, reads [69]

\[
T_{\alpha,\beta}(E_F) = \sum_{n=-n_F}^{n_F} \text{Tr} \left[ \Gamma_{0,0}^\alpha(E_F) G_{r,0}^\alpha(E_F) \Gamma_{n,n}^\alpha(E_F) G_{0,n}^\alpha(E_F) \right],
\]

where $\Gamma_{n,m}^\alpha$ denote the components of the linewidth $\Gamma^\alpha$ introduced above.

1. Floquet sum rule

In the present Floquet framework, the calculation of the transmission [Eq. (15)] requires special treatment: indeed, one has to take into account the contribution from all the states with quasi-energy $\varepsilon \approx E_F$, i.e. $T_{\alpha,\beta}(E_F)$, but also consider the many multiplicities corresponding to energies $E_F + n\omega$, with $n \in \mathbb{Z}$.

As elucidated in Refs. [69] [92], an accurate calculation of the conductance is recovered by simply summing the transmissions associated with the individual multiplicities,

\[
T_{\alpha,\beta}(\varepsilon = E_F) = \sum_{n \in \mathbb{Z}} T_{\alpha,\beta}(E_F + n\omega).
\]

As further illustrated below, the sum rule in Eq. (16) is essential to recover a quantized Hall conductance in periodically-driven systems realizing the quantum Hall effect (and Floquet Chern insulators in general [68]).

The sum rule in Eq. (16) also leads to an interesting experimental corollary, which is that the conductance cannot be evaluated based on a single measurement. As discussed in Ref. [69], the transport experiment should be repeated for various values of the reservoirs’ chemical potential, which should be chosen so as to probe the many multiplicities $E_F + n\omega$. The convergence of this approach will be illustrated below in Section IV.

III. TRANSPORT IN A SHAKEN CHANNEL: THE EFFECTIVE HAMILTONIAN APPROACH

A. The model

In this Section, we define the shaken-channel model [Fig. 1], and describe its transport properties using an effective-Hamiltonian approach. We discuss how a natural synthetic dimension emerges in the problem, and elaborate on how this feature affects the coupling to the reservoirs. In particular, this leads to the notion of “effective multi-terminal configurations”, which allows for a clear detection of the quantum Hall effect in the shaken-channel model. The effective time-independent Hamiltonian approach will be further validated in the full-time-dependent approach of Section IV.

1. The shaken channel

We consider a non-interacting gas of ultracold fermions (of mass $M$), which are restricted to move within a single channel aligned along the $y$ direction; we focus our attention on the channel, and disregard the reservoirs for now. This system will be described by a single-particle Hamiltonian of the form

\[
\hat{H}_0 = \frac{\hat{p}_x^2}{2M} + \frac{M\omega^2}{2} \hat{z}^2 - J_y \sum_j \left[ |x,j\rangle \langle x,j+1| + \text{H.c.} \right].
\]

As the two first terms in Eq. (17) describe the motion along the harmonically-confined transverse direction ($x$), with trapping frequency $\omega$, while the last term describes motion along the channel. Here, we have assumed that a deep lattice potential is set along the channel, and that a single-band tight-binding approximation can be made to capture the dynamics along this direction. Then the hopping processes between neighboring orbitals, $|x,j\rangle$ and $|x,j+1\rangle$, are fully characterized by the tunneling parameter $J_y$; here $j = y/a$ refers to the site index along the channel direction, and we set the lattice spacing $a = 1$ in the following.

Inspired by Ref. [63], we subject the tight harmonic confinement to a resonant time-periodic modulation, with frequency $\omega_D \approx \omega$,

\[
\hat{V}(t) = \kappa \hat{z} \cos (\omega_D t + \varphi_j).
\]

which corresponds to shaking the channel along the transverse ($x$) direction. Importantly, the modulation in
Eq. (18) includes a phase $\theta(y) = \varphi j$, which explicitly depends on the channel direction ($y$). In the following, we assume that such a modulation is only active on a (substantial) part of the channel, so that there are intermediate regions that adiabatically connect the non-shaken reservoirs to the shaken channel (i.e. the inner system).

Following Ref. [53], we write the total time-dependent Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$ in the basis formed by the harmonic oscillators states, $|\lambda,j\rangle$, where $\lambda$ refers to the discrete harmonic levels associated with the transverse ($x$) direction:

$$\hat{H}(t) = \sum_{\lambda,j} \left[ \omega \lambda |\lambda,j\rangle \langle \lambda,j| - \left( J_y |\lambda,j\rangle \langle \lambda,j+1| + \text{H.c.} \right) \right]$$

$$+ \sum_{\lambda,j} 2 \cos(\omega_D t + \varphi j) \times$$

$$\left( J_0 |\lambda,j\rangle \langle \lambda+1,j| + J_{\lambda+1} |\lambda+1,j\rangle \langle \lambda,j| \right),$$

where $J_\lambda = \kappa \sqrt{\lambda/8M\omega}$. In the high-frequency regime ($\omega_D \approx \omega \rightarrow \infty$), an effective model can be obtained by invoking the rotating-wave approximation [53, 64, 66]:

$$\hat{H}_{\text{eff}} = \sum_{\lambda,j} \left[ \Delta \lambda |\lambda,j\rangle \langle \lambda,j| - \left( J_y |\lambda,j\rangle \langle \lambda,j+1| + \text{H.c.} \right) \right]$$

$$- \sum_{\lambda,j} \left( J_\lambda e^{i\varphi j} |\lambda,j\rangle \langle \lambda+1,j| + \text{H.c.} \right),$$

where $\Delta = \omega_D - \omega$ is the detuning between the trap and drive frequencies; in the following, we shall consider a resonant drive with $\Delta = 0$. As previously discussed in Ref. [53], the rotating-wave approximation and, hence, the description based on the effective Hamiltonian in Eq. (20), breaks down whenever $|J_\lambda| \gtrsim \omega/4$. This limits the number of states $\lambda$ that are available along the synthetic dimension, for a given ratio $s l_H/\omega$, where $l_H = 1/\sqrt{M\omega}$ is the harmonic length (which sets a natural length scale in the problem). Under these approximations, the effective time-independent Hamiltonian in Eq. (20) corresponds to a 2D tight-binding model, defined on a square lattice in the $\lambda - y$ plane, with homogeneous and anisotropic hopping strengths ($J_\lambda, J_y$). Furthermore, this 2D-lattice model includes a uniform artificial magnetic field, which corresponds to having $\varphi$ quanta of flux per plaquette. Summarizing, the model realizes the Harper-Hofstadter model in Eq. (9), with homogeneous and anisotropic hopping parameters. We note that the optical lattice set along the channel direction is not a crucial ingredient, as removing it would result in a model of “quantum-Hall wires” with similar properties [33, 34].

2. Connecting the shaken channel to reservoirs

Our proposal builds on the observation that the shaken channel described above is naturally connected to two reservoirs, as illustrated in Fig. 1. Specifically, the total system is constituted of an inner 2D system realizing the anisotropic Harper-Hofstadter model (defined in the fictitious $\lambda - y$ plane), which is connected to two non-shaken reservoirs. The latter are attached at both ends of the channel, namely, they connect to the 2D inner system at $y = 0$ and $y = L_y$.

In the following, we will consider that a chemical potential imbalance can be applied between the two reservoirs, so as to force the atoms to move from the left side of the channel to the right side [Sec. II B]; the relative particle difference then provides a measurement of the system’s conductance [40]. Since the effective model describing the 2D inner system corresponds to the Harper-Hofstadter-like model in Eq. (20), one expects that the resulting transport properties should be reminiscent of those presented in Sec. II B1 (which discussed the conductance of the Harper-Hofstadter model as probed by a two-terminal geometry).

This naive prediction is based on the assumption that the synthetic ($\lambda$) direction can be treated as a genuine spatial direction: specifically, it assumes that the reservoirs inject particles in the inner system, at $y = 0$, in a $\lambda$-independent manner. However, the energetic nature of the synthetic dimension (i.e. the fact that each "site" along $\lambda$ is associated with an energy $\lambda \omega$) leads to a more involved particle injection in the $\lambda - y$ plane, and hence, enriches the transport experiment; this crucial aspect will be analyzed in the following Sections.

B. Simple effective model: Homogeneous hopping along the synthetic dimension

For the sake of clarity, let us first simplify the analysis of the time-independent effective model in Eq. (20), by neglecting the inhomogeneity in the hopping along the synthetic dimension $\lambda$; specifically, we substitute $J_\lambda \rightarrow J_1 \equiv \kappa \sqrt{1/8M\omega}$. The corresponding effective model that describes the inner system then reads

$$\hat{H}_{\text{eff}}^{\text{HH}} = - J_y \sum_{\lambda,j} \left( |\lambda,j\rangle \langle \lambda,j+1| + \text{H.c.} \right)$$

$$- J_1 \sum_{\lambda,j} \left( e^{i\varphi j} |\lambda,j\rangle \langle \lambda+1,j| + \text{H.c.} \right),$$

and it exactly maps onto the Harper-Hofstadter model when $J_y = J_1$. We note that the approximation of constant $J_\lambda$ is only valid in the limit of large $\lambda \gg 1$, which is a priori problematic since we anticipate that low-$\lambda$ states should substantially contribute to transport (we remind the reader that transport is dominated by chiral edge modes in the quantum-Hall regime, and that $\lambda = 0$ defines a clear edge along the synthetic dimension). In this sense, the results presented in this Section only aim to provide a general intuition on the transport that takes place in this synthetic 2D system. The analysis of the full (inhomogeneous) effective Hamiltonian will be postponed to
Sec. III C, while the properties of the full time-dependent model (including the application of the Floquet approach to transport) will be presented in Sec. IV.

Although the synthetic direction \( \lambda \) introduced above is semi-infinite, we note that, in practice, the anharmonicity of the confinement realized in ultracold-atom experiments can produce a natural (soft) boundary. Besides, a box-type potential could be applied on top of the harmonic trap to further confine the particles along the transverse direction, which would result in a sharp edge in the \( \lambda \) direction (at a desired \( \lambda_{\text{cut}} \)). In the following, we introduce a cutoff along the synthetic direction \( \lambda_{\text{cut}} \gg 1 \), which is convenient for performing numerical simulations on a finite system.

1. Two-terminal geometry

As a first step, we treat the synthetic dimension as a genuine spatial dimension. Under this assumption, the conductance of the model in Eq. (21) can be calculated as in Section III B 1, namely, by treating the full system (inner part and reservoirs) as a standard two-terminal problem. A sketch of this simple configuration is shown in the inset of Fig. 4(a).

In order to calculate the conductance of the 2D inner system, we now explicitly compute the self energies \( \Sigma_{L,R} \) associated with the left and right reservoirs. This is different from applying the wide-band approximation described in Sec. II A, as was previously considered for the calculation shown in Fig. 2. This approach allows one to check whether the details of the reservoirs influence the calculation of the conductance. As a first assumption, we take the Hamiltonian describing the reservoir as in Eq. (21) but with \( \varphi = 0 \), i.e. the effective magnetic field is assumed to be absent in the reservoirs. Furthermore, we will first (naively) assume that the hopping amplitudes are uniform and isotropic throughout the entire system: \( J_{y}^r = J_y \) and \( J_{r}^l = J_r \), where the superscript \( r \) refers to the reservoirs. The resulting conductance plot is shown in Fig. 4(a), which naturally shares the same features as in Fig. 3. In particular, the plateau at \( E_F > 0 \) in Fig. 4(a) is attributed to the counter-clockwise propagating edge mode, which is associated with the spectrum of the Harper-Hofstadter Hamiltonian in Eq. (21), and which is localized at \( \lambda = 0 \). Conversely, the plateau at \( E_F < 0 \) is associated with the clockwise propagating edge mode localized at \( \lambda = \lambda_{\text{cut}} \); see inset of Fig. 4(a).

We stress that the description used above for the reservoirs (\( J_y^r = J_y \) and \( J_r^l = J_1 \)) is not compatible with the actual scheme described in Section III A. Indeed, since the temporal modulation only acts on the inner part of the system (or more precisely, on a substantial part of the transport channel), a more accurate description consists in setting \( J_r^l = 0 \) in the junction regions connecting the inner system to the reservoirs (noting that the harmonic-oscillator states \( \lambda \) are indeed decoupled in the absence of the time-modulation); see the sketch in the inset of Fig. 3(b). As shown in the conductance plot of Fig. 4(b), the quantized plateaux remain unaffected by this modification of the reservoirs properties [compare Figs. 4(a) and (b)].

As a technical remark, we note that the bulk-band response at \( |E_F|/J_1| > 2 \) is dramatically suppressed in Fig. 4(b), which is due to the breakdown of the wide-band approximation: when setting \( J_r^l = 0 \), the bandwidth associated with the reservoirs is of the order of \( W^r \approx 2J_y \), which is smaller than the bandwidth of the inner system, \( W = 4J_1 = 4J_r^l \), in the situation considered in Fig. 4(b). We have checked that the bulk response of Fig. 2 is indeed recovered when setting \( J_y^r = J_y = J_1 \), i.e. when reaching a regime where the wide-band approximation is again satisfied. Importantly, we verified that the details of the reservoirs do not break the robustness of the quantized plateaux, as soon as the wide-band approximation is fulfilled. In the remainder of the paper, we will always treat the reservoirs in the wide-band approximation.

2. Effective multi-terminal geometry

As a crucial step in the description and understanding of our scheme, we now take the energetic nature of the synthetic dimension into account. To do so, we analyze how particles are injected from the left reservoir into the inner shaken channel by partitioning the whole system into three connected parts: (a) the inner system (2D lattice in the fictitious \( \lambda - y \) plane), (b) the two reservoirs, (c) the two junction regions that connect the inner system to the reservoirs; see Fig. 5. It is reasonable to assume that the junction regions can be treated as discrete harmonic levels of energy \( \hbar \omega \), which are populated following a thermal (Fermi-Dirac) distribution set by the reservoirs: particles are injected such that \( E_F > \lambda \hbar \omega \), and holes are injected in states where \( E_F < \lambda \hbar \omega \). In this sense, the reservoir injects more particles in the low-\( \lambda \) states (bottom of the 2D inner sys-
As before, we still assume that the chemical potentials are set such that transport is driven from the left part to the right part of the system. Besides, in the following paragraphs, we will assume that hopping is allowed in the reservoirs along the synthetic dimension ($J''_1 = J'_1$); this choice can be modified in order to reach an even finer description [i.e. by setting $J''_1 = 0$; see discussion of Section III.B1]. Other transport configurations will be also briefly discussed below.

a. Effective four-terminal geometry As a first approximation, we consider that the initial 2-reservoir configuration can be split into an effective four-terminal geometry: the inner system is coupled to two effective reservoirs on the left (labelled by $L_{1,2}$) and by two effective reservoirs ($R_{1,2}$) on the right; see the sketch in Fig. 5(a). We point out that this setting corresponds to a rearrangement of the more standard four-terminal geometry previously discussed in Sec. III.B2 [Fig. 5(a)]. Importantly, it turns out that this unusual (effective) four-terminal geometry allows for a clear measure of the Hall and longitudinal conductances, as we now explain.

Figure 5(b) shows the conductances, as obtained from the RGF method for two different configurations: the longitudinal conductance was calculated as $G_{\text{long}} \equiv G_{L_1, R_2}$, while the transverse (Hall) conductance was evaluated as $G_{\text{Hall}} \equiv G_{L_2, R_2} - G_{L_1, R_1}$. These choices can be explained based on simple arguments. Firstly, as previously discussed in Sec. III.B2, the longitudinal conductance results from the contribution of the many extended bulk states, and since the transport taking place between the terminals $L_1$ and $R_2$ in Fig. 5(a) necessarily involves bulk states, it is thus legitimate to define the longitudinal conductance as $G_{L_1, R_2}$ in this context. Secondly, in quantum-Hall systems, the Hall conductance can be attributed to the contribution of the edge modes. We note that the transport between the terminals $L_1$ and $R_1$ in Fig. 5(a) can be attributed to the propagation of a counter-clockwise edge mode along the bottom edge as well as to bulk states and similarly that the transport between the terminals $L_2$ and $R_2$ involves the bulk and a clockwise edge state following the top edge. Consequently, the difference $G_{L_2, R_2} - G_{L_1, R_1}$ allows one to reveal the edge-current contribution, and thus, the Hall conductance; a more rigorous derivation can be obtained based on the Landauer-Büttiker formalism [54, 55].

b. Effective six-terminal geometry One can further refine the model by considering an effective six-terminal configuration, where the main (physical) reservoirs are now split into three effective reservoirs each; we will denote these $L_{1,2,3}$ on the left and $R_{1,2,3}$ on the right, respectively; see the sketch in Fig. 6(c). As compared to the four-terminal configuration [Fig. 6(a)], the six-terminal geometry allows for an even more direct detection of the
bulk and edge contributions to transport. Indeed, $G_{L_2,R_2}$ reflects the transport taking place in the bulk, and hence provides an accurate probe of the longitudinal conductance, whereas $G_{L_1,R_1}$ [resp. $G_{L_3,R_3}$] reflects the transport associated with the counter-clockwise [resp. clockwise] edge mode propagating along the bottom [resp. top] edge. This is demonstrated in Fig. 6 (d), which shows the corresponding conductances, and which indeed reproduces the expected features of the Hall and longitudinal conductances. As a technical remark, we note that $G_{L_1,R_1}$ and $G_{L_3,R_3}$ also show a weak contribution of the bulk states, in the vicinity of the band edges.

This construction of an effective multi-terminal geometry can be straightforwardly extended to the limit where each row of “sites” at a given $\lambda$ is connected to a terminal $L_\lambda$ (resp. $R_\lambda$) on the left (resp. right), see Fig. 7 (c). In this extreme case, the conductances $G_{L_{\lambda=\infty},R_{\lambda=0}}$ and $G_{L_{\lambda=0},R_{\lambda=\infty}}$ would isolate the chiral-edge-mode contributions, while the others $G_{L_\lambda,R_\lambda}$ would capture the contributions from the bulk states only.

**c. Other configurations** We point out that other transport configurations can be envisaged. For instance, the motion taking place along the synthetic dimension could be probed by analyzing the conductance associated with two fictitious terminals located on a given side of the channel (e.g. $G_{L_{\lambda=\infty},R_{\lambda=0}}$ or $G_{R_{\lambda=\infty},L_{\lambda=0}}$). Such a motion along $\lambda$ would physically correspond to a heat transport [128] within a given (real) reservoir. Besides, the detuning $\Delta$ defined in Eq. (20) could be used to generate an artificial electric field aligned along the synthetic dimension, hence offering an additional control parameter to the transport experiment.

**C. Complete effective model with $\lambda$-dependent hopping**

So far, we have described the shaken-channel model in terms of the simplified effective Hamiltonian in Eq. (21), namely, the Harper-Hofstadter model [Sec. IIIB] with isotropic and homogeneous hopping ($J_y = J_1$). This allowed us to analyze how conductance measurements are modified as one changes the reservoirs configuration, offering a first important step in our understanding of how transport takes place in the presence of a synthetic dimension [Section IIIB].

We now go beyond these studies, and consider the complete effective model in Eq. (20), by taking the inhomogeneous hopping along the synthetic dimension ($J_\lambda$) into account; as before, we take the resonant-drive limit and set $\Delta = 0$.

As a first step, we calculate the energy spectrum for this effective model, in view of identifying the energy ranges that correspond to the edge modes (and bulk gaps); these ranges will then correspond to the quantized plateaux depicted by the Hall conductance when plotted as a function of the Fermi energy. Following Ref. [53], we diagonalize the Hamiltonian Eq. (20) in a gauge where translational symmetry is recovered along the $y$ direction. Applying periodic conditions along the $y$ direction and setting $\lambda_{\text{cut}} = 30$, one obtains the spectrum shown in Fig. 7 (a); the color scale indicates the mean position ($\lambda$) of the eigenstates along the synthetic dimension. The main effect of the inhomogeneous and anisotropic hopping is to increase the bandwidth of the spectrum, and to reduce the size of the bulk gap that hosts the topological edge modes. The band width of the complete effective model can be compared with the one of the homogeneous and isotropic model, indicated as gray regions in Fig. 7 (a). For our choice of parameters $J_y = 4J_1$, these modifications are not too severe in the sense that chiral edge modes are still present in reasonably large bulk gaps (of order $J_y$); we also note that the group velocity (and chirality) of the edge modes are still preserved. For example, the edge mode located in the upper gap, and which propagates from left to right (positive group velocity along $y$), is localized at $\lambda = 0$, while in the lower gap, this mode is localized at $\lambda = \lambda_{\text{cut}}$.

We now calculate the conductance of this effective-Hamiltonian system, based on a simple two-terminal configuration, which allows for a direct comparison with the results previously presented in Fig. 4 (a); hence, for clarity, we neglect the energetic nature of the synthetic dimension in this part of the study. The results are presented in Fig. 7 (b), which shows the conductance calcu-

![Figure 7](image-url)
lulated using the RGF method. Importantly, the plateaux associated with the chiral edge modes (at $\lambda = 0$ and $\lambda = \lambda_{\text{cut}}$) are still visible in this more realistic (anisotropic) model. We also notice that the size of the plateaux, which is indicative of the band-gaps displayed in Fig. 7(a), is reduced compared to the isotropic case shown in Fig. 4(a).

Finally, one can include the energetic nature of the synthetic dimension into the description, by following the effective-multi-terminal construction of Section IIIIB2 (shown in Fig. 7(c)). The conductance associated with the chiral edge modes (at $\lambda = 0$) is indicative of the band-gaps displayed in Fig. 7(a), is reduced compared to the isotropic case shown in Fig. 4(a).

At this stage, the only approximation that was used concerns the truncation of the Fourier space into $2n_F+1$ modes. In particular, no assumption was made on the energetic nature of the synthetic dimension. In the previous Sections, we have analyzed the conductance of the shaken-channel model using an effective-Hamiltonian approach; there, traditional tools of quantum transport were directly applied to time-independent Hamiltonians, which allowed us to explore the peculiarities introduced by the synthetic dimension. In particular, we discussed how the emergent notion of an “effective multi-terminal configuration” allows for a clear detection of the transverse (Hall) and longitudinal conductances in a single atomic wire, hence revealing the quantum Hall effect in a minimal cold-atom setting.

In this Section, we now build on these results to analyze the full time-dependent problem, using the Floquet-approach to transport introduced in Sec. IIIIC. The aim of this Section is to fully validate the main result of this article, namely, that a quantized Hall conductance associated with chiral edge modes can be extracted from the shaken-channel model displayed in Fig. 1.

We start with the Schrödinger equation associated with the full time-dependent Hamiltonian in Eq. (19), which was expressed in the basis of the harmonic-oscillator-states:

\begin{equation}
\text{i} \partial_t \psi_{\lambda,j}(t) = \sum_{\lambda,j} \left[ \omega \psi_{\lambda,j}(t) - J_y \left( \psi_{\lambda,j+1}(t) + \psi_{\lambda,j-1}(t) \right) \right] + \sum_{\lambda,j} 2 J_1 \cos(\omega t + \varphi_j) \times \left( \sqrt{\lambda} \psi_{\lambda-1,j}(t) + \sqrt{\lambda + 1} \psi_{\lambda+1,j}(t) \right),
\end{equation}

where $\psi_{\lambda,j}(t)$ is the wavefunction of a particle in the state $|\lambda,j\rangle$; as previously, we explicitly set the drive frequency on resonance $\omega_D = \omega$. We then expand the wavefunction into its Fourier components,

\begin{equation}
\psi_{\lambda,j}(t) = \sum_{n=-n_F}^{n_F} \psi_{\lambda,j}^{(n)} e^{in\omega t},
\end{equation}

where we have truncated the series up to $2n_F+1$ modes, and where $\psi_{\lambda,j}^{(n)}$ denotes the time-independent Fourier amplitudes; the number of modes $n_F$ was chosen so as to reach convergence of the numerical observables. Substituting Eq. (23) into Eq. (22) and isolating the components proportional to $e^{in\omega t}$, we obtain the components of the $H_n$ matrix defined in Eq. (10):

\begin{equation}
\omega (\lambda + n) \psi_{\lambda,j}^{(n)} - J_y \left( \psi_{\lambda,j+1}^{(n)} + \psi_{\lambda,j-1}^{(n)} \right) + J_1 e^{i\varphi_j} \left( \sqrt{\lambda} \psi_{\lambda-1,j}^{(n)} + \sqrt{\lambda + 1} \psi_{\lambda+1,j}^{(n)} \right) \equiv H_n \psi_{\lambda,j}^{(n)}.
\end{equation}

The results presented in this Section have been obtained using the Floquet RGF method introduced in Sec. IIIIB2.
in Sec. IIC, namely, by calculating the transmission [Eqs. (15)–(16)],

\[ T_{\alpha,\beta}(E_F) = \sum_{n,n'=-n_F,...,n_F} \text{Tr}\left[ \Gamma_{0,0}^\alpha(E_F + n\omega) \Gamma_{0,0}^\beta(E_F + n\omega) \right] \]

\[ \times G_{n,n'}^{\alpha,\beta}(E_F + n\omega) \text{ as } n \text{ approaches } n_F = \lambda_{\text{cut}}, \]

which explicitly takes the Floquet sum rule [Eq. (16)] into account: in the present model, numerical convergence is reached for \( n_F = \lambda_{\text{cut}} \), where \( \lambda_{\text{cut}} \) denotes the cut-off along the synthetic dimension.

We illustrate the results in Fig. 8(a), which shows the conductance of the shaken-channel model [Eq. (22)] with \( N_y = 12 \) lattice sites along the real direction \( y \), and we have set \( \lambda_{\text{cut}} = 11 \). Here, we have applied the Floquet sum rule by including the contribution of all Fourier modes [Eq. (25)], which, as discussed in Ref. [69], allows for an accurate evaluation of the conductance associated with the underlying effective (time-independent) system; in the present case, this leads to the clear quantized plateaux in Fig. 8(a), in agreement with the effective-Hamiltonian result in Fig. 4.

As a technical remark, we note that the results are obtained within the validity range of the wide-band approximation [see Section IIA.1]. In the context of time-periodic systems, this approximation should be extended by assuming that the self-energy and the linewidth are energy independent for all Floquet bands that contribute to the sum rule in Eq. (16) as discussed in Ref. [69], this is required to accurately capture the conductance.

1. Identification of chiral edge modes using the Floquet sum rule

As illustrated above, the Floquet sum rule [Eq. (16)] allows for a clear detection of the conductance associated with the underlying effective model [69, 92]. In order to test this result, we show in Fig. 8(b) the conductance as obtained by truncating the sum over the Fourier modes up to some critical mode \( n_{\text{cut}} \): by restricting this sum to \( n = -n_F, ..., 5 \), we find that the plateau at \( E_F < 0 \) is drastically reduced, while the quantized plateau at \( E_F > 0 \) survives the truncation [compare the red curve in Fig. 8(b) with Fig. 8(a)]. Besides, we find that this behavior is reversed (i.e. only the plateau at \( E_F < 0 \) survives) upon reversing the sign of the magnetic flux \( (\varphi = -2\pi/3) \). This suggests an interesting interplay between the truncation of the Floquet sum rule, the energetic nature of the synthetic dimension and the detection of edge modes, as we now explain.

From our analysis of the effective Hamiltonian, we know that the quantized plateau at \( E_F > 0 \) [resp. \( E_F < 0 \)] is due to the edge mode localized at \( \lambda = 0 \) [resp. \( \lambda = \lambda_{\text{cut}} \)]. When the Floquet sum rule is truncated up to some mode \( n_{\text{cut}} \ll \lambda_{\text{cut}} \), the total transmission \( \sum_{n=0}^{n_{\text{cut}}} T_F(E_F + n\omega) \) no longer captures the contribution of the edge mode localized at \( \lambda = \lambda_{\text{cut}} \); this explains the absence of the expected plateau at \( E_F < 0 \) in Fig. 8(b) [blue curve]. In contrast, the quantized plateau at \( E_F > 0 \) is still present, since the total transmission still captures the contribution of the edge mode localized at \( \lambda = 0 \). This observation is further validated by reversing the magnetic flux \( (\varphi = -2\pi/3) \): in this case, the propagating edge mode at \( \lambda = 0 \) corresponds to the lower gap \( E < 0 \), and hence, it is the plateau at \( E_F > 0 \) that disappears [red curve in Fig. 8(b)].

This apparent relation between the Fourier modes \( n \) entering the Floquet sum rule and the “site” index \( \lambda \) associated with the synthetic dimension naturally stems from the resonant nature of the time-modulation, which is at the heart of the present proposal. Indeed, the harmonic-oscillator levels (i.e. the “sites” along the synthetic dimension \( \lambda \)) are equispaced according to the energy separation \( \omega \) set by the trap frequency, which also corresponds to the separation between the many multiplicities \( (n) \) associated with the Floquet spectrum (since \( \omega_D = \omega \)). Furthermore, as we have discussed, that the junction regions that connect the reservoirs to the inner system [Fig. 6] can be thought of as uncoupled harmonic-oscillator levels, and that this leads to an effective multi-terminal geometry where each row of “sites” corresponding to a given \( \lambda \) is connected to two individual (fictitious) reservoirs \( (L_\lambda \text{ on the left, and } R_\lambda \text{ on the right}) \); see Sec. III B 2 and Fig. 7c. In this picture, calculating the contribution of the \( n \)-th mode to the conductance, \( G(E_F + n\omega) \), is related to selecting the effective “terminals” \( L_n \) and \( R_n \) that are energetically resonant with the mode, namely, the terminals that are connected to the “sites” \( \lambda = n \).

In this sense, scanning through the Fourier modes is reminiscent of analyzing various transport channels in the effective multi-terminal geometry [see Fig. 9(c)]: in particular, the contribution of the edge modes localized at \( \lambda = 0 \) can be identified through the conductance \( G_{L_0,R_0} \) associated with the lower “terminals”. In practice, measuring the contribution of a given Fourier mode, \( G_{L_n,R_n} \), can be achieved by setting the Fermi energy with
which is due to the fact that the band-gap is slightly
note that the plateaux are slightly larger in Fig. 9(a),
obtained using the effective-Hamiltonian approach. We
reproduces the quantized plateaux of Fig. 7, which was
in Fig. 9(a), which was calculated using the complete
Floquet sum rule (in the wide-band limit). This result
in Sec. IIIC.

Fourier modes \( n \)
channel model defined in Eq. (22), as obtained from the Floq-
et RGF method. Parameters are \( \lambda_{\text{cut}} = 11, N_y = 12, \)
\( \varphi = 2\pi/3, J_y = 4J_1, \kappa = 0.03\omega/\hbar, \) and \( n_F = 11. \) (a) The
conductance is obtained from the complete Floquet sum rule
in Eq. (16); (b) individual contributions to the conductance,
corresponding to \( n = 0 \) (blue), \( n = 1 \) (orange) and \( n = 2 \)
(green). The main contribution to the plateau at \( E_F > 0 \)
stems from the \( n=0 \) component.

respect to the corresponding harmonic oscillator states
(\( \lambda = n \)).

These observations leads to a remarkable corollary: Re-
restricting the Floquet sum rule to a limited number of
modes (set by \( n_{\text{cut}} \)) can be used as a method to isolate
the contribution of individual chiral edge modes. This
particular feature of our synthetic-dimension system al-
allows for the unambiguous detection of the quantized Hall
conductance in a two-reservoir setting. We will further
illustrate this important result in the next Section IVB
based on the complete time-dependent model [Fig. 9].

As a technical remark, we note that the Floquet sum
rule can be restricted to positive modes (\( n \geq 0 \)) in the
present context, as the transmissions \( T(E_F + n\omega) \) as-
associated with negative \( n \)'s are found to have negligible
contributions.

B. Full time-dependent model

We finally discuss the transport properties of the full
time-dependent model in Eq. (22), without neglecting
the anisotropic hopping along the synthetic dimension
\( \lambda \). Here, we chose the hopping parameters \( J_y = 4J_1, \)
such that the band-gaps remain well open for \( \lambda < 30; \) see Sec. IIIC.

We show the corresponding two-terminal conductance
in Fig. 9(a), which was calculated using the complete
Floquet sum rule (in the wide-band limit). This result
reproduces the quantized plateaux of Fig. 7 which was
obtained using the effective-Hamiltonian approach. We
note that the plateaux are slightly larger in Fig. 9(a),
which is due to the fact that the band-gap is slightly
larger in the present configuration where \( \lambda_{\text{cut}} = 11, \) and
\( N_y = 12. \)

In Fig. 9(b), we show individual contributions to the
total conductance \( G(E_F + n\omega) \), considering the first three
Fourier modes \( n = 0, 1, 2; \) these are the first three non-
vanishing contributions to the Floquet sum rule used in
Fig. 9(a). As previously discussed in Sec. IVA the main
contribution to the plateau at \( E_F > 0 \) is due to the edge
mode localized at \( \lambda = 0, \) which is selected by the lowest
\( n \approx 0 \) Fourier component entering the sum rule. As can
be seen in Fig. 9(b), the contribution of the \( n = 1 \) com-
ponent to the edge-mode signal is already substantially
reduced, which is due to the highly localized nature of
the edge mode. Similarly, we have verified that the main
contribution to the plateau at \( E_F < 0 \) mainly comes from
the component \( n_F = \lambda_{\text{cut}}. \)

The result shown in Fig. 9(b) demonstrates how the
quantized conductance associated with a topological edge
mode (here, at \( \lambda = 0 \)) can be unambiguously detected
using a few conductance measurements, in a single atomic
wire.

V. EXPERIMENTAL CONSIDERATIONS

We now discuss the experimental implementation of the
shaken-channel scheme, based on the demonstrated
two-terminal cold-atom setup of Refs. [35, 37]. As shown
in Fig. 4, the quantum wire consists of a region with a
tight harmonic confinement along both \( x \)- and \( z \)-
directions. The propagation of atoms from one reservoir
to the other in the \( y \)-direction is ballistic [23, 36, 37]. We
consider, in line with Refs. [36, 37], that the temperature
is low enough such that individual harmonic-oscillator
states in the wire can be resolved by transport, yield-
ing a quantized conductance upon varying the chemical
potential [see Section IVB].

Using standard high-resolution optical techniques, the
wire can be exposed to a periodic drive, while keeping
the adiabatic connection to non-shaken reservoirs. The
position-dependent phase of the temporal modulation in
Eq. (18) can be realized using Raman transitions between
harmonic-oscillator states along one transverse direction.
Alternatively, a direct time- and space-periodic deforma-
tion of the wire structure could be engineered based on
light shaping techniques. The typical length that can be
achieved in these quantum wires is from ten to twenty
micrometers. The spatial period \( l_B = 2\pi/\varphi \) associated
with the time-dependent potential in Eq. (18) must be
much shorter than the length of the wire itself. The
driving strength \( \kappa \) is controlled by the intensity of the
Raman beams or the amplitude of the deformation and
should satisfy the two following criteria: (i) ensure the
validity of the rotating-wave approximation for the re-
levant, low energy harmonic-oscillator states participating
to transport, which places an upper bound on the driving
strength; (ii) create a topological bulk gap (hosting the
dge modes) larger than temperature in order to be re-
solved by transport measurements, which constrains the
strength from below. While these are required for the
above formalism to apply, we expect the physics to be
robust against moderate deviations from these bounds.

Note that there is no need for the projection of a lattice
structure along the transport direction for the observation of the chiral edge states in the synthetic dimension, even though such a projection has recently been demonstrated [57]. Indeed, without a lattice the model maps onto the coupled-wires model of Refs. [36, 37], known to exhibit the quantum Hall effect.

The natural observable in the experiment is the two-terminal conductance, measured as a function of the chemical potential. By repeating measurements for chemical potentials increased by an integer multiple of $\omega$, the contribution of each Floquet band can be extracted. As discussed in the previous Section, these conductances can be related to the transverse or Hall conductances of the synthetic system.

VI. CONCLUSIONS

We have proposed a scheme by which the quantum Hall conductance of a neutral atomic gas can be detected using a minimal one-dimensional setting: a quantum wire connected to two reservoirs [36, 37]. The two-dimensional nature of the quantum Hall effect is offered by an additional (synthetic) dimension, which is naturally present in the system. Inspired by Ref. [53], we proposed that a Chern-insulating state (realizing the quantum Hall effect) can be realized in this setting upon subjecting the quantum wire to a resonant modulation. Importantly, the resulting quantized Hall conductance can be unambiguously detected in this scheme, by exploiting an unusual feature offered by the synthetic dimension: its energetic structure along the transport direction for the observation of chiral edge modes’ contribution to transport. This appealing result was demonstrated using two complementary approaches, one based on effective (time-independent) Hamiltonians and the other on a Floquet approach, which takes the full time-dependence of the problem into account.

Intriguing perspectives include the study of inter-particle interactions in this synthetic-dimension approach. As discussed in Ref. [53], interactions are long-ranged (but not infinite-range) along the synthetic dimension, and the corresponding phases are still to be elucidated. In particular, it would be interesting to identify regimes where strongly-correlated states with topological features could be stabilized in this setting; the corresponding Hall conductance could then be explored using the schemes and concepts detailed in the present work.

Finally, the notions and results introduced in this work could be applied to other physical platforms. For instance, synthetic dimensions have been proposed in the context of photonics [96–99], and a first experimental realization – reminiscent of the scheme proposed in Ref. [53] – was recently reported in Ref. [56]. In this context, we note that a transport formalism (analogous to the Landauer formalism) has been proposed to describe transport of photons [100]. Altogether, this suggests that the scheme discussed in this work could be directly transposed to the context of topological photonics [101].

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