Nodal Domain Statistics for Quantum Maps, Percolation and SLE

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We develop a percolation model for nodal domains in the eigenvectors of quantum chaotic torus maps. Our model follows directly from the assumption that the quantum maps are described by random matrix theory. Its accuracy in predicting statistical properties of the nodal domains is demonstrated by numerical computations for perturbed cat maps and supports the use of percolation theory to describe the wave functions of general hamiltonian systems, where the validity of the underlying assumptions is much less clear. We also demonstrate that the nodal domains of the perturbed cat maps obey the Cardy crossing formula and find evidence that the boundaries of the nodal domains are described by SLE with κ close to the expected value of 6, suggesting that quantum chaotic wave functions may exhibit conformal invariance in the semiclassical limit.

One of the central problems in the field of quantum chaos is to understand the morphology of quantum eigenfunctions in classically chaotic systems. In time-reversal-symmetric systems one can always find a basis in which these eigenfunctions are real. They can thus be divided into nodal domains - connected regions of the same sign, separated by nodal lines on which the eigenfunctions vanish. The statistical properties of these nodal domains then constitute a natural way to characterize the morphology of the eigenfunctions.

Nodal domain statistics were studied for separable billiards in [1], where it was shown that if νn is the number of nodal domains in the nth energy eigenstate then \( \chi_n = \nu_n/n \) has a limiting distribution as \( n \to \infty \) with a square-root singularity at a system-dependent maximum value \( \chi_{\text{max}} < 1 \).

In chaotic systems the eigenfunctions may be modeled statistically, far from boundaries and turning points, by random superpositions of plane waves [2]:

\[
 u(x) = \sqrt{\frac{2}{J}} \sum_{j=1}^{J} \cos(k x \cos \theta_j + k y \sin \theta_j + \phi_j) \tag{1}
\]

where \( \theta_j \) and \( \phi_j \) are random phases. This is known as the random wave model. Since plane waves are solutions of the Schrödinger equation for a free particle, \( \nabla^2 \psi = -k^2 \psi \), the maxima of any superposition are positive and the minima are negative. Hence the nodal domains correspond to groups of either maxima or minima. A given pair of adjacent maxima (minima) lie in the same nodal domain if the saddle point between them is positive (negative). The density of saddles in the random wave model is asymptotically twice the density maxima or minima. This would be exactly the case, for example, if the maxima and minima lay on alternate sites of a square lattice and the saddles on the corresponding dual lattice, e.g. midway between diagonally adjacent maxima (or minima) (although it is important to note that in the random wave model typical realizations of the wave function are in fact highly irregular). The saddles may be thought of as lying at the midpoints of bonds of the dual lattice connecting the maxima, for example. If the saddle height is positive, then the corresponding maxima are connected and the bond may be thought of as 'open'; if it is negative, the maxima are not directly connected, and the bond may be thought of as 'closed'. This was the basis of the very interesting suggestion put forward by Bogomolny and Schmit [3] that statistical properties of nodal domains in the random wave model, and hence in quantum chaotic eigenfunctions, correspond to those in critical percolation. Specifically, Bogomolny and Schmit assumed that the heights of the saddles are uncorrelated and have equal probability of being positive or negative, and proposed bond percolation on a square lattice as a model for nodal domain statistics. This implies that \( \chi_n \) is Gaussian distributed as \( n \) varies in the semiclassical limit. Moreover, it leads to the conclusion that the scaling exponents associated with critical percolation also characterize properties of the nodal domains in quantum chaotic eigenfunctions, for example their area distribution and fractal dimension.

The predictions of the percolation model are consistent with numerical computations [4] and experimental measurements [5] of the fluctuation statistics for the nodal domains of quantum billiards, but the data do not provide conclusive verification. This is important, because the model has been the subject of considerable debate. Foltin has shown that the heights of the saddles in the random wave model exhibit long range correlations [5], contradicting one of the key assumptions of the percolation model. Bogomolny has argued that oscillations in the two-point correlation function are sufficient to ensure the applicability of the Harris criterion [6] and so guarantee that the scaling exponents are unaffected [7], but the issue awaits a more systematic investigation. Moreover, Foltin, Gnutzmann and Smilansky have devised a particular statistical measure for which the percolation model fails [8]. The range of validity of the model and the precise assumptions upon which it relies thus remain to be determined.
Our purpose here is to establish a percolation model for quantum torus maps. These are some of the most important examples of quantum chaotic systems, because one can find maps that are fully chaotic and quantum mechanically they are finite dimensional and so easily tractable. We will show that for these systems there is a critical percolation model that follows directly from the Bohigas-Giannoni-Schmit (BGS) conjecture, which asserts that local quantum fluctuation statistics in classically chaotic systems are modeled by Random Matrix Theory. This model corresponds to site percolation on a triangular lattice, which falls into the same universality class as bond percolation on a square lattice and so has the same critical exponents. The advantages of investigating the percolation model for maps are, first, that the assumptions underlying it are very much more straightforward – one only has to assume the BGS conjecture, and there are no problems analogous to those relating to the slow decay of correlations in billiard eigenfunctions – and, second, that one can perform more extensive and controlled numerical computations, leading to significantly more precise tests of the predictions.

We find that the percolation model for maps is extremely accurate in that the critical scaling exponents associated with the nodal domains are very close to those predicted by percolation theory. Moreover, the agreement goes beyond scaling laws: the nodal domains of the quantum maps we study also obey the Cardy crossing formula and its generalizations. We also verify that Cardy’s formula is satisfied within the random wave model. This suggests that both linear superpositions of random waves and quantum chaotic eigenfunctions may exhibit conformal invariance in the semiclassical limit. Finally, the link between processes governed by Stochastic Loewner Evolution (SLE) and statistical models has recently been the focus of considerable attention. Critical percolation is believed to relate to SLE with diffusion constant $\kappa = 6$. For percolation on a triangular lattice this has been established rigorously. On the basis of the percolation model one would expect nodal lines to behave like processes governed by SLE$_6$. We find evidence that this is the case for the boundaries of the nodal domains in the case of quantum maps.

The systems we study correspond to chaotic symplectic maps acting on the unit $2L$-dimensional torus, which is viewed as their phase space. Such maps can be quantized using an approach introduced by Hannay and Berry. The Hilbert space has finite dimension $N^L$, where $N$ plays the role of the inverse of Planck’s constant. Quantum maps correspond to unitary matrices $U$ acting on wave functions in this Hilbert space so as to generate their (discrete) time evolution. In the position representation these wave functions take values on an $L$-dimensional lattice. For example, when $L = 1$ the wave functions take values $(c_1, c_2, \ldots, c_N)$ at positions $q = Q/N$, $0 \leq Q < N$; and when $L = 2$ they take values $(c_1, c_2, \ldots, c_{N^2})$ at positions on the square lattice $q = (Q_1/N, Q_2/N)$, $0 \leq Q_1, Q_2 < N$. We shall be concerned with the quantum map eigenvectors. If the map is time-reversal symmetric (and so $U$ is symmetric), the components of the eigenvectors are real. For a given eigenvector, we can thus split the quantum lattice into regions such that the components associated with neighbouring sites have the same sign. These regions then correspond to nodal domains.

When $L = 1$ this can be done straightforwardly: if sites lying next to each other on the 1-dimensional lattice have eigenvector components $c_j$ with the same sign then they constitute part of the same nodal domain. When $L = 2$ the situation requires more careful consideration, because one needs a convention for deciding whether lattice points that are diagonal neighbours and have eigenvector components with the same sign lie in the same nodal domain or not. Consider, for example, when the eigenvector components associated with a group of four lattice points which form a square have signs in a checkerboard arrangement, e.g. on the top row + -, and underneath - +. Are the pluses automatically part of the same nodal domain, or the minuses? We take as our convention that lattice points are connected along diagonals running from the top left to the bottom right; so in the example just given it is the pluses that are connected. This takes us from the original square lattice to the triangular lattice. Nodal domains then correspond to regions on this triangular lattice in which connected points have the same sign. Our convention is, of course, one of many possibilities. However, we note that it is necessary to incorporate diagonal neighbours for the definition of nodal domains to be consistent with that in billiards, and that all of the conventions we have tested which do this lead to the same results.

In order to develop a statistical model for the nodal domains we now need to introduce a statistical ansatz for the signs. According to the BGS conjecture, for generic, classically chaotic, time-reversal-symmetric systems statistical properties of the matrix $U$ should coincide with those of random matrices taken from the Circular Orthogonal Ensemble of Random Matrix Theory. The joint probability density for the eigenvector components $(c_1, c_2, \ldots, c_{N^L})$ is then

$$P(c_1, c_2, \ldots, c_{N^L}) = \delta \left( \sum_{j=1}^{N^L} c_j^2 - 1 \right)$$

that is, the eigenvectors are uniformly distributed on the unit hypersphere. Crucially for us, it follows immediately that the sign of a given component is equally likely to be positive or negative and that these signs are independent of each other at different sites, i.e. they are uncorrelated.

When $L = 1$ this model was explored in [12]. When $L = 2$ it corresponds directly to site percolation on a triangular lattice, which falls into the same universality
class as the Bogomolny-Schmit model. This means that the critical exponents associated with the nodal domain statistics will be the same. We note that in our case these have been established rigorously for percolation \[17\].

We now test the percolation model for a particular family of quantum torus maps. In essence, we are seeing whether this family is described sufficiently accurately by RMT for the model to apply. Linear maps are not sufficient for our purpose: because of non-generic arithmetic symmetries they are not described by RMT \[13\]. Instead, we take a linear map composed with a nonlinear perturbation. Specifically, we use $M = \rho \circ A \circ \rho$ with

$$A: \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -2 & -2 & -1 \\ -2 & 6 & -1 & 0 \\ 16 & -39 & 2 & -2 \\ -39 & 94 & -2 & 6 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \mod 1$$

and $\rho$ a nonlinear periodic shear in momentum: $p_1 \rightarrow p_1 + \frac{k}{\pi} \cos(2\pi q_1)$, $p_2 \rightarrow p_2 + \frac{k}{\pi} \cos(2\pi q_2)$. The map $M$ is time-reversal-symmetric and, for sufficiently small values of the perturbation parameters, completely chaotic. The corresponding quantum map, a unitary matrix of dimension $N^2$ can be written down easily using the prescriptions in \[11\] and \[14\] (the formula is long and so we do not give it here, see \[15\] for details). We now compare statistical properties of the nodal domains of this map with those of percolation clusters.

Consider first the distribution of the number $n$ of nodal domains. For percolation on $N^2$ sites this should be a Gaussian with mean $n_c N^2 + b + o(1)$ and variance $c N^2$, where Monte Carlo simulations give $n_c = 0.0176 \ldots$, $b = 0.878 \ldots$ and $c = 0.0309 \ldots \[10\]$. For the quantum map we find a Gaussian distribution with a mean and variance consistent with the percolation formulae. Specifically, when $k_1 = 0.04$ and $k_2 = 0.01$, a best fit gives $n_c = 0.0176$, $b = 0.902$ and $c = 0.0297$. The data are shown in Figure 1. For the distribution of areas $a$ of the nodal domains, the percolation model predicts a scaling law $a^{-\tau}$, with $\tau = 187/91$. A log-log plot of the data for the eigenvectors of the quantum map is shown in Figure 2. The percolation model also implies that the nodal domains should have a fractal dimension $D = 91/48 = 1.89 \ldots$. Data for the quantum map, obtained using a box-counting algorithm and shown as a log-log plot in Figure 3 are consistent with this in that the best fitting straight line has a gradient 1.8774.

One of the key features of critical percolation is that it has a conformally invariant limit. This underlies the use of conformal field theory, in deriving the Cardy crossing formula for example, and the link with SLE. Given the success of the percolation model in describing scaling exponents associated with their nodal domains, it is natural to ask whether random waves and quantum eigenfunctions are also conformally invariant in the semiclassical limit. In percolation, the Cardy crossing formula gives the probability of there being a cluster spanning the system between fixed sections of the boundary \[17\]. For the eigenvectors of the quantum map, we can determine the probability that a nodal domain spans the lattice, connecting a fixed section of length $x$ on the left to any region on the right-hand side. We also test the nodal domains of realizations of the random wave model using the same geometry. The results are shown in Figure 4 where they are compared to the crossing formula. To explore the connection with SLE we use an idea introduced in \[18\] to test conformal invariance in 2-D turbulence. The SLE$_\kappa$ process generates a trace from a stochastic differential equation dependent on a driving function $\xi(t)$, which is proportional to a Brownian motion. The trace is obtained for each eigenvector by following a nodal line while keeping the positive points to the right. Upon hitting the

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with SLE holds is evidence of conformal invariance. It process even in this restricted geometry. If it touches it. This allows us to use the half-plane SLE of kappa close to 6. SLE This method is self consistent because we obtain a value motion. The best fit to the diffusion constant is Gaussian distribution (inset) is consistent with Brownian section in turn. This gives a sequence of times and driv-

is natural to conjecture that this will extend to generic quantum chaotic eigenfunctions in the semiclassical limit.

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