Languages invariant under more symmetries: overlapping factors versus palindromic richness

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Abstract

Factor complexity $\mathcal{C}$ and palindromic complexity $\mathcal{P}$ of infinite words with language closed under reversal are known to be related by the inequality

$$\mathcal{P}(n) + \mathcal{P}(n + 1) \leq 2 + \mathcal{C}(n + 1) - \mathcal{C}(n)$$

for any $n \in \mathbb{N}$. Words for which the equality is attained for any $n$ are usually called rich in palindromes. We show that rich words contain infinitely many overlapping factors. We study words whose languages are invariant under a finite group $G$ of symmetries. For such words we prove a stronger version of the above inequality. We introduce the notion of $G$-palindromic richness and give several examples of $G$-rich words, including the Thue-Morse word as well.

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1. Introduction

In the last decade, a broad interest in the study of palindromes can be observed. Attention to palindromes was brought on one hand by the article [14] where a bound on the number of distinct palindromes occurring in a finite word was given, and on the other hand by the role played by palindromes in the spectral theory of Schrödinger operators with aperiodic potential [17]. The fact that the existence of many palindromes in an infinite word $u$ is connected with its factor complexity $\mathcal{C}$ was for the first time recognized in the article [1]. Its authors proved that

$$\mathcal{P}(n) \leq \frac{16}{n} \left( \mathcal{C}(n) + \mathcal{C}(\lfloor \frac{n}{4} \rfloor) \right)$$

(1)

where $\mathcal{P}$ counts the number of distinct palindromes of given length occurring in $u$. A special case of this inequality for fixed points of primitive morphisms was already proven in [13]. In [3], a relation between palindromic complexity and increment of factor complexity was established for infinite words whose language is closed under reversal:

$$\mathcal{P}(n) + \mathcal{P}(n + 1) \leq 2 + \mathcal{C}(n + 1) - \mathcal{C}(n).$$

(2)
In [3], the relation is stated under an additional hypothesis of uniform recurrence. However, this hypothesis is not used in the proof and the claim is valid without it. The words for which the equality in (2) is attained for any \( n \in \mathbb{N} \) are called rich or full and there exists extensive literature on the topic, see [14, 16]. The most famous examples of rich words are episturmian words (see [14]), which include Sturmian and Arnoux-Rauzy words, and words coding interval exchange transformations determined by a symmetric permutation, see [3].

Palindromes are formally defined as fixed points of the reversal mapping. We may replace the reversal mapping by another involutive antimorphism \( \Theta \) to obtain a generalization of the notion of a palindrome. A fixed point of such mapping \( \Theta \) is called a \( \Theta \)-palindrome. In [21], the second author showed that the inequality (2) is still valid if one substitutes palindromic complexity \( P(n) \) by \( \Theta \)-palindromic complexity \( P_{\Theta}(n) \). Moreover, many properties known to be possessed by rich words are possessed by their generalizations - \( \Theta \)-rich words - as well.

In this article we introduce the new concept of \( G \)-rich words, where \( G \) is a finite group generated by more antimorphisms. Motivation for this new notion is a property of \( \Theta \)-rich words we show in Theorem 4, i.e., that any \( \Theta \)-rich word contains infinitely many overlapping factors. Therefore, words without large overlaps have no chance to be rich in \( \Theta \)-palindromes. But paradoxically, the language of the most prominent word without overlaps - the Thue-Morse word - is closed simultaneously under two antimorphisms and contains infinitely many palindromes and \( \Theta \)-palindromes.

For an infinite word whose language is closed under two commuting antimorphisms \( \Theta_1 \) and \( \Theta_2 \) we deduce a new inequality relating factor complexity \( C(n) \) with \( P_{\Theta_1}(n) \) and \( P_{\Theta_2}(n) \), see Theorem 9. We also show that for the Thue-Morse word the equality in Theorem 9 holds for each \( n \), i.e., that the Thue-Morse word is saturated by palindromes and \( \Theta \)-palindromes together up to the highest possible level. Therefore, in Section 5, we propose to adopt a new definition of richness for words whose language is closed under all elements of a finite group \( G \) of symmetries (formed by morphisms and antimorphisms). To such a word we assign a graph of symmetries \( \Gamma_n \) for any \( n \in \mathbb{N} \). The connectedness of \( \Gamma_n \) implies an inequality between factor complexity and \( \Theta \)-palindromic complexities, see Theorem 22. Nevertheless, the definition of \( G \)-richness is not based on an inequality, but on the structure of the graph of symmetries. Let us stress that in the case when the group \( G \) is generated only by one antimorphism the new definition of richness and the old one coincide. In Section 6, we provide some examples of \( G \)-rich words.

The new point of view on richness of infinite words triggers many questions on properties of rich words. Some of these open questions are collected in the last section.

2. Preliminaries

An alphabet \( A \) is a finite set, its elements are usually called letters. By a finite word over an alphabet \( A \) we understand a finite string \( w = w_1w_2 \ldots w_n \) of letters \( w_i \in A \). Its length \( n \) is denoted by \( |w| \). The set of all finite words over \( A \) equipped with the operation of concatenation is the free monoid \( A^* \), its neutral element is the empty word \( \varepsilon \). A word
2.1. Antimorphisms and their fixed points

A mapping \( \phi \) on \( A^* \) is

- a morphism if \( \phi(vw) = \phi(v)\phi(w) \) for any \( v, w \in A^* \);
- an antimorphism if \( \phi(vw) = \phi(w)\phi(v) \) for any \( v, w \in A^* \).

We denote the set of all morphisms and antimorphisms on \( A^* \) by \( AM(A^*) \). Together with composition, it forms a monoid with the identity mapping \( \text{Id} \) as the unit element. The set of all morphisms, denoted by \( M(A^*) \), is a submonoid of \( AM(A^*) \). The reversal mapping \( R \) defined by

\[
R(w_1w_2\ldots w_n) = w_nw_{n-1}\ldots w_2w_1, \quad \text{where } w_i \in A,
\]

is an involutive antimorphism, i.e., \( R^2 = \text{Id} \). It is obvious that any antimorphism is a composition of \( R \) and a morphism. Thus

\[
AM(A^*) = M(A^*) \cup R(M(A^*)).
\]

A morphism or antimorphism \( \nu \in AM(A^*) \) is non-erasing if for all \( a \in A \) we have \( |\nu(a)| > 0 \).

A fixed point of a given antimorphism \( \Theta \) is a \( \Theta \)-palindrome, i.e., a word \( w \) is a \( \Theta \)-palindrome if \( w = \Theta(w) \). If \( \Theta \) is the reversal mapping \( R \), we say palindrome or classical palindrome instead of \( R \)-palindrome. If a non-erasing antimorphism \( \Theta \) has a fixed point \( w \) containing all letters of \( A \), then, since \( \Theta^2 \) is a non-erasing morphism with a fixed point \( w \) containing all letters of \( A \), we have \( \Theta^2(a) = a \) for all \( a \in A \). It means that \( \Theta \) is an involution, and thus a composition of \( R \) and an involutive permutation of letters.

Suppose \( \Theta \) is an involutive antimorphism until stated otherwise. The set of \( \Theta \)-palindromic factors of a word \( w \) is denoted by \( \text{Pal}_\Theta(w) \). The cardinality of \( \text{Pal}_\Theta(w) \) is bounded by

\[
\#\text{Pal}_\Theta(w) \leq |w| + 1 - \gamma_\Theta(w),
\]

where \( \gamma_\Theta(w) := \#\{\{a, \Theta(a)\} \mid a \in A, a \text{ occurs in } w \text{ and } a \neq \Theta(a)\} \), see [14] for classical palindromes and [21] for generalized palindromes.

2.2. Factor and palindromic complexities

An infinite word \( u \) over an alphabet \( A \) is a sequence \( u = (u_n)_{n \in \mathbb{N}} \in A^\mathbb{N} \). We will always implicitly suppose that \( A \) is the smallest possible alphabet for \( u \), i.e., any letter of \( A \) occurs at least once in \( u \).

A finite word \( w \) is a factor of \( u \) if there exists an index \( i \in \mathbb{N} \), called occurrence of \( w \), such that \( w = u_iu_{i+1}\ldots u_{i+|w|-1} \). The set of all factors of \( u \) of length \( n \) is denoted \( \mathcal{L}_n(u) \). The language of an infinite word \( u \) is the set of all its factors \( \mathcal{L}(u) = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n(u) \). An infinite word \( u \) is recurrent if every its factor has at least two occurrences in \( u \). A factor
\( v \in \mathcal{L}(u) \) is a complete return word of a factor \( w \) if \( w \) occurs in \( v \) exactly twice, as a suffix and a prefix of \( v \). A complete return word \( v \) of \( w \) can be written as \( v = qw \) for some factor \( q \) which is called a return word of \( w \). If every factor \( w \) of \( u \) occurs infinitely many times and has only finitely many return words, then \( u \) is uniformly recurrent. Equivalently, \( u \) is uniformly recurrent if every its factor occurs infinitely many times and the gaps between its consecutive occurrences are bounded.

The factor complexity of \( u \) is the mapping \( \mathcal{C} : \mathbb{N} \mapsto \mathbb{N} \) defined by
\[
\mathcal{C}(n) = \#\mathcal{L}_n(u).
\]
To evaluate the factor complexity of an infinite word one may consider possible extensions of factors. A letter \( a \in A \) is a left extension of a factor \( w \) in \( u \) if \( aw \) belongs to \( \mathcal{L}(u) \). The set of all left extensions of \( w \) is denoted \( \text{Lext}(w) \). A factor \( w \in \mathcal{L}(u) \) is left special (LS), if \( \#\text{Lext}(w) \geq 2 \). Analogously, we define right extension, the set \( \text{Rext}(w) \), and right special (RS). If \( w \) is both right and left special, it is bispecial (BS). The first difference of the factor complexity of a recurrent word \( u \) satisfies
\[
\Delta \mathcal{C}(n) = \mathcal{C}(n + 1) - \mathcal{C}(n) = \sum_{w \in \mathcal{L}_n(u)} (\#\text{Lext}(w) - 1) = \sum_{w \in \mathcal{L}_n(u)} (\#\text{Rext}(w) - 1). \tag{4}
\]
The second difference of the factor complexity can be expressed using the bilateral order of a factor. It is the quantity \( b(w) = \#\{ awb | awb \in \mathcal{L}(u) \} - \#\text{Lext}(w) - \#\text{Rext}(w) + 1 \).
In [12], the formula
\[
\Delta^2 \mathcal{C}(n) = \Delta \mathcal{C}(n + 1) - \Delta \mathcal{C}(n) = \sum_{w \in \mathcal{L}_n(u)} b(w) \tag{5}
\]
is deduced.

The \( \Theta \)-palindromic complexity of \( u \) is the mapping \( \mathcal{P}_{\Theta}(n) : \mathbb{N} \mapsto \mathbb{N} \) defined by
\[
\mathcal{P}_{\Theta}(n) = \#\{ w \in \mathcal{L}_n(u) | w = \Theta(w) \}.
\]

2.3. \( \Theta \)-richness

A finite word \( w \) is \( \Theta \)-rich if the equality in (3) holds. An infinite word is \( \Theta \)-rich if all its factors are \( \Theta \)-rich.

As we already mentioned, in [3], an inequality involving factor and \( R \)-palindromic complexities of infinite words with languages closed under reversal was shown. It was generalized for an arbitrary involutive antimorphism in [21]. In particular, it is shown that if an infinite word has its language closed under \( \Theta \), then the following inequality holds
\[
\Delta \mathcal{C}(n) + 2 \geq \mathcal{P}_{\Theta}(n) + \mathcal{P}_{\Theta}(n + 1) \quad \text{for all } n \geq 1. \tag{6}
\]
The difference between the left and the right side in fact decides about \( \Theta \)-richness. Let us denote - in accordance with the notation introduced in [9] - the quantity \( T_{\Theta}(n) \) as
\[
T_{\Theta}(n) = \Delta \mathcal{C}(n) + 2 - \mathcal{P}_{\Theta}(n + 1) - \mathcal{P}_{\Theta}(n).
\]
The following theorem was shown in [10] for \( R \) and in [21] for an arbitrary antimorphism.

**Theorem 1.** If \( u \) is an infinite word with language closed under \( \Theta \), then \( u \) is \( \Theta \)-rich if and only if \( T_{\Theta}(n) = 0 \) for all \( n \geq 1 \).
2.4. Θ-palindromic defect

The Θ-palindromic defect of a finite word \( w \), denoted \( D_\Theta(w) \), is defined as

\[
D_\Theta(w) = |w| + 1 - \gamma_\Theta(w) - \#\text{Pal}_\Theta(w).
\]

It directly follows that \( w \) is Θ-rich if and only if \( D_\Theta(w) = 0 \). The Θ-palindromic defect of an infinite word \( u \) is defined as

\[
D_\Theta(u) = \sup\{D_\Theta(w) \mid w \in \mathcal{L}(u)\}.
\]

Again, \( u \) is Θ-rich if and only if \( D_\Theta(u) = 0 \). We say that \( u \) is almost Θ-rich if its defect \( D_\Theta(u) \) is finite. A close relation between Θ-rich and almost Θ-rich words is explained in [19].

The following proposition is a direct consequence of the definition, cf. [19, 21]. It is analogous to the case of finite \( R \)-defect treated in [16].

**Proposition 2.** If \( u \) is an infinite word, then the Θ-defect of \( u \) is finite if and only if there exists an integer \( H \) such that the longest Θ-palindromic suffix of any prefix \( w \) of \( u \) such that \( |w| \geq H \) occurs in \( w \) exactly once, except for prefixes having the form \( w = pa \) with \( a \in A \) such that \( \gamma_\Theta(p) \neq \gamma_\Theta(w) \), i.e., the letter \( a \) or \( \Theta(a) \), \( a \neq \Theta(a) \), does not occur in \( p \).

In [4] and [16], various properties of words with finite \( R \)-defect were shown. The \( R \)-defect of periodic words was studied in [8]. The following proposition stated for any involutive antimorphism is an analogue of one of these properties stated for the reversal mapping. We provide a short proof.

**Proposition 3.** If \( u \) is an infinite word with finite Θ-defect, then there exists an integer \( H \) such that all complete return words of any Θ-palindrome of length at least \( H \) are Θ-palindromes.

**Proof.** Let \( H \) be the constant from Proposition 2. Suppose there exists a Θ-palindrome \( p \in \mathcal{L}(u) \) such that \( |p| \geq H \) and \( p \) has a non-Θ-palindromic complete return word. Let \( r \) denote the non-Θ-palindromic complete return word of \( p \) that occurs in \( u \) before any other non-Θ-palindromic complete return word of \( p \). Let \( q \) be the prefix of \( u \) ending with the first occurrence of \( r \) in \( u \), i.e., \( q = tr \) for some word \( t \) and \( r \) is unioccurrent in \( q \). Denote by \( s \) the longest Θ-palindromic suffix of \( q \). Since \( p \) is a Θ-palindromic suffix of \( q \), it is clear that \( |s| \geq |p| \). If \( |s| = |p| \), then we have a contradiction to the unioccurrence of \( s \). If \( |r| > |s| > |p| \), then we can find at least 3 occurrences of \( p \) in \( r \) which is a contradiction to \( r \) being a complete return word of \( p \). The equality \( |r| = |s| \) contradicts the fact that we supposed \( r \) to be non-Θ-palindromic. Finally, if \( |r| < |s| \), then we can find an occurrence of \( \Theta(r) \) which is a non-Θ-palindromic complete return word of \( p \) and we have a contradiction to the choice of \( r \). \( \square \)
3. Overlapping factors in infinite words with finite $\Theta$-defect

Infinite words with finite $\Theta$-defect contain a lot of $\Theta$-palindromes. We show that they are very rich in overlapping factors as well.

**Theorem 4.** If $u$ is a recurrent word with finite $\Theta$-defect, then $u$ contains infinitely many overlapping factors, i.e., the set

$$\{www' \in \mathcal{L}(u) \mid w' \text{ is a non-empty prefix of } w\}$$

is infinite.

**Proof.** According to Proposition 3 there exists an integer $H$ such that

- there exists a $\Theta$-palindrome $p_0 \in \mathcal{L}(u)$ of length $|p| > H$,
- any complete return word of a $\Theta$-palindromic factor $v \in \mathcal{L}(u)$ of length $|v| \geq H$ is a $\Theta$-palindrome.

We use the $\Theta$-palindrome $p_0$ as the starting element of a sequence of factors $(p_n)$ constructed in the following way:

for any $n \in \mathbb{N}$, $p_{n+1}$ is a complete return word of $p_n$.

Since the word $u$ is recurrent, any factor has at least one complete return word and therefore our construction is correct.

Let $L \geq H$ be an arbitrary integer. We are going to find a factor $w$, $|w| \geq L$, such that $www' \in \mathcal{L}(u)$, with $w'$ being a non-empty prefix of $w$.

Since the set $\{v \in \mathcal{L}_{2L}(u) \cup \mathcal{L}_{2L+1}(u) \mid v = \Theta(v)\}$ is finite, there are indices $k$ and $\ell$, $k < \ell$, such that $p_k$ and $p_\ell$ have the same central $\Theta$-palindromic factor of length $2L$ or $2L+1$. Let $p$ be that central factor. Let us recall that $p \in \mathcal{L}(u)$ is a central factor of a $\Theta$-palindrome $v$ if $v = wp\Theta(w)$ for some finite word $w$.

According to the construction of the sequence $(p_n)$, $p$ occurs in $p_\ell$ at least 3 times. Let $r$ denote the return word of $p$ occurring at the rightmost occurrence of $p$ before the central occurrence of $p$ in $p_\ell$. Since $p_\ell$ and $rp$ are $\Theta$-palindromes, it is clear that $rp\Theta(r) = rrp$ is a central factor of $p_\ell$.

If $|r| \geq |p|$, then $p$ is a prefix of $r$. Since $rrp \in \mathcal{L}(u)$, we set $w = r$ and we have directly $|w| \geq 2L$.

If $|r| < |p|$, then there exist an integer $j \geq 3$ and a factor $y$ such that $0 < |y| \leq |r|$, $rrp = r^jy$, and $y$ is a prefix of $r$. Set $w = r^{\frac{j}{2}}$. It is clear that $wwy \in \mathcal{L}(u)$. The length of $w$ satisfies $|w| > \frac{1}{2}(j-1)|r| \geq \frac{1}{2}|p| \geq L$.

**Remark 5.** Existence of squares in almost rich words was an important ingredient in proving the Brlek-Reutenauer conjecture for uniformly recurrent words in [5], where the authors together with L. Balková deduced a weaker form of Theorem 4 for uniformly recurrent words only. Let us mention that in [9] Brlek and Reutenauer stated the conjecture for any word with language closed under reversal. It is yet to be proved.
Remark 6. Theorem 4 implies that any infinite Θ-rich word contains an infinity of squares. One can look for the longest finite words that are Θ-rich and do not contain a square. For instance take Θ = R. On a binary alphabet those longest words are clearly 010 and 101. On a ternary alphabet there are exactly two words, up to a permutation of letters, that satisfy these conditions. Namely 0102010 and 0121012. Let \( r(n) \) denote the length of such a word on an alphabet of \( n \) letters. The sequence \( (r(n))_{n=1}^{+\infty} \) begins with

\[ 1, 3, 7, 15, 33, 67 \ldots \]

To find an explicit formula for \( r(n) \) remains an open question.

It is widely accepted that combinatorics on words has started its own life with the discovery (or rediscovery) of an overlap-free word by Axel Thue in 1912. This word, today called Thue-Morse (or Prouhet-Thue-Morse), is the fixed point \( \lim_{n \to \infty} \varphi^n(0) \) of the morphism

\[ \varphi(0) = 01 \quad \text{and} \quad \varphi(1) = 10. \]

The Thue-Morse word

\[ u_{TM} = 0110100110110100110100110110010110100110100101 \ldots \]

has language closed under reversal and contains infinitely many (classical) palindromes. Moreover, \( \mathcal{L}(u_{TM}) \) is closed under permutation of letters 0 and 1 and thus under a second antimorphism \( \Theta \) defined by \( \Theta(0) = 1 \) and \( \Theta(1) = 0 \). The Thue-Morse word contains infinitely many \( \Theta \)-palindromes as well. Nevertheless, absence of overlapping factors in \( u_{TM} \) implies the following corollary, which is a rephrasing of a result in [6].

**Corollary 7.** The Thue-Morse word is not almost \( \Theta \)-rich for any antimorphism \( \Theta \).

**Example 8.** Let us consider the periodic word \( u = (01)\omega \) and denote by \( \Theta \) the antimorphism on \( \{0, 1\}^* \) defined by \( \Theta(0) = 1 \) and \( \Theta(1) = 0 \). Obviously

- \( C(n) = 2 \) for any \( n \geq 1 \);
- \( P_R(2n) = 0 \) and \( P_R(2n - 1) = 2 \) for any for any \( n \geq 1 \);
- \( P_{\Theta}(2n) = 2 \) and \( P_{\Theta}(2n - 1) = 0 \) for any for any \( n \geq 1 \).

Therefore, the periodic word \( u = (01)\omega \) is \( R \)-rich and \( \Theta \)-rich simultaneously.

**4. Words with language closed under two antimorphisms**

The inequality (6) can be interpreted as a lower bound on the increment \( \Delta C(n) \) of the factor complexity. The more palindromes of length \( n \) and \( n + 1 \) are contained in \( u \), the higher the value \( \Delta C \). This bound is weak when the language of a word \( u \) is closed under two antimorphisms. We show that in this case \( \Delta C(n) \) can be estimated more effectively.
Theorem 9. Let $Θ_1$ and $Θ_2$ be two distinct commuting involutive antimorphisms on $A^*$. If $u$ is an infinite word with language closed under $Θ_1$ and $Θ_2$, then we have for all $n \in \mathbb{N}^+$,
\[
\Delta c(n) + 4 \geq \mathcal{P}_{Θ_1}(n) + \mathcal{P}_{Θ_2}(n) - \mathcal{P}_{Θ_1,Θ_2}(n) + \mathcal{P}_{Θ_1}(n + 1) + \mathcal{P}_{Θ_2}(n + 1) - \mathcal{P}_{Θ_1,Θ_2}(n + 1),
\]
where $\mathcal{P}_{Θ_1,Θ_2}(k) = \#\{w \in \mathcal{L}(u) \mid w = Θ_1(w) = Θ_2(w)\}$.

Proof. We write $w \sim v$ if $w$ is equal to $v$ or $Θ_1(v)$ or $Θ_2(v)$ or $Θ_1Θ_2(v)$. Since the antimorphisms $Θ_1$ and $Θ_2$ are commuting, it is easy to see that $\sim$ is an equivalence relation on $\mathcal{L}(u)$. An equivalence class containing a factor $w$ is denoted by $[w]$.

Note that
- $\#[w] = 1$, if $w$ is simultaneously a $Θ_1$-palindrome and $Θ_2$-palindrome;
- $\#[w] = 2$, if $w$ is a $Θ_1$-palindrome or a $Θ_2$-palindrome but not both;
- $\#[w] = 4$, otherwise.

A factor $w$ is RS or LS if and only if any factor from $[w]$ is RS or LS.

Fix $n$. We are going to construct a directed graph $\overrightarrow{Γ} = (V, E)$ with multiple edges and loops allowed. The set $V$ of vertices of $\overrightarrow{Γ}$ is the set
\[
V = \{[w] \mid w \in \mathcal{L}(u), w \text{ is special} \}.
\]

There is an edge labeled $e \in \mathcal{L}(u)$ going from $[v]$ to $[w]$ if the prefix of $e$ of length $n$ belongs to $[w]$, the suffix of $e$ of length $n$ belongs to $[v]$, and $e$ contains exactly two special factors of length $n$. For the number of edges in $\overrightarrow{E}$ we have
\[
\#\overrightarrow{E} = \sum_{w \in \mathcal{L}(u), w \text{ special}} \#\text{Lex}(w).
\]

Obviously $Θ_1(\overrightarrow{E}) = Θ_2(\overrightarrow{E}) = \overrightarrow{E}$.

Note that if $e$ is an edge between $[v]$ and $[w]$, and $\#[v] > 1$ or $\#[w] > 1$, then $Θ_1(e)$, $Θ_2(e)$ and $Θ_1Θ_2(e)$ are also edges between $[v]$ and $[w]$ and are distinct. In the case $\#[v] = 1$ and $\#[w] = 1$, there are at least two edges $e$ and $Θ_1(e)$ between $[v]$ and $[w]$.

Let $α$ denote the number of vertices $[w]$ such that $\#[w] = 2$, $β$ the number of vertices $[w]$ such that $\#[w] = 4$ and $ζ$ the number of vertices $[w]$ such that $\#[w] = 1$.

All edges in $\overrightarrow{Γ}$ can be divided into two disjoint parts. We put all loops, i.e., edges starting and ending in the same vertex, into one part. Let us denote their number by $A$. We put all edges connecting distinct vertices into the second part. Their number is denoted by $B$. Obviously, $\#\overrightarrow{E} = A + B$.

Estimate of $B$ We exploit the connectivity of the graph $\overrightarrow{Γ}$ to give a lower bound on $B$. We have to take into consideration that distinct connected vertices are connected either by four or two edges, as discussed above.
If \((\alpha + \beta) > 0\) and \(\zeta > 0\), then \(B \geq 4(\alpha + \beta - 1) + 4 + 2(\zeta - 1)\).

- If \((\alpha + \beta) = 0\), then \(B \geq 2(\zeta - 1)\).
- If \(\zeta = 0\), then \(B \geq 4(\alpha + \beta - 1)\).

Altogether,

\[B \geq 4(\alpha + \beta) + 2\zeta - 4. \quad (7)\]

**Estimate of \(A\)**  
If an edge \(e\) contains a \(\Theta_1\)-palindrome \(p\) which is neither a prefix nor a suffix of \(e\) of length \(n\), then \(\Theta_1(e) = e\). Such an edge is a loop in the graph \(\overrightarrow{\Gamma}\), and the \(\Theta_1\)-palindrome \(p\) is centered in \(e\).

On the other hand, any \(\Theta_1\)-palindrome of length \(n + 1\) lies on a unique edge \(e\). Similarly, any \(\Theta_1\)-palindrome of length \(n\) which is not a special factor lies on a unique edge \(e\). We may conclude that

\[A \geq \#\{w \in L_n(u) \mid w \text{ is not special}, w = \Theta_1(w) \text{ or } w = \Theta_2(w)\} + \#\{w \in L_{n+1}(u) \mid w = \Theta_1(w) \text{ or } w = \Theta_2(w)\}\]

and thus

\[A + 2\alpha + \zeta \geq P_{\Theta_1}(n) + P_{\Theta_2}(n) - P_{\Theta_1,\Theta_2}(n) + P_{\Theta_1}(n+1) + P_{\Theta_2}(n+1) - P_{\Theta_1,\Theta_2}(n+1). \quad (8)\]

To complete the proof, we have to realize that

\[\#\overrightarrow{E} = \sum_{\substack{w \in L_n(u) \text{ w special} \\#|w|=4}} \#\text{Lext}(w) = \sum_{\substack{w \in L_n(u) \text{ w special} \\#|w|=2}} \#\text{Lext}(w) + \sum_{\substack{w \in L_n(u) \text{ w special} \\#|w|=1}} \#\text{Lext}(w)\]

and thus

\[\Delta C(n) = \sum_{\substack{w \in L_n(u) \text{ w special} \\#|w|=1}} (#\text{Lext}(w) - 1) = \#\overrightarrow{E} - 4\beta - 2\alpha - \zeta = A + B - 4\beta - 2\alpha - \zeta.\]

Using the estimates (8) and (7), we get the inequality announced by the Theorem. \(\square\)

**Corollary 10.** If \(u\) is a uniformly recurrent infinite word with language closed under two distinct commuting involutive antimorphisms \(\Theta_1\) and \(\Theta_2\), then there exists an integer \(N\) such that

\[\Delta C(n) + 4 \geq P_{\Theta_1}(n) + P_{\Theta_2}(n) + P_{\Theta_1}(n+1) + P_{\Theta_2}(n+1) \quad \text{for all } n > N.\]
Proof. Since $Θ_1$ and $Θ_2$ are two distinct commuting antimorphisms, their composition $Θ_1Θ_2$ is a non-identical morphism which permutes letters. Let $a \in A$ be a letter such that $Θ_1Θ_2(a) \neq a$. Since $u$ is uniformly recurrent we can find an integer $N$ such that $a$ occurs in any factor longer than $N$. The equation $Θ_1(w) = Θ_2(w)$ implies $Θ_1Θ_2(w) = w$, which cannot be satisfied for words longer than $N$. Thus for all $n > N$

$$\{w \in L(u) \mid w = Θ_1(w) = Θ_2(w) \text{ and } |w| = n\} = \emptyset \implies P_{Θ_1,Θ_2}(n) = 0.$$  

Corollary 11. If $Θ_1$ and $Θ_2$ are two distinct commuting involutive antimorphisms and $u$ is a uniformly recurrent infinite word such that $u$ is simultaneously $Θ_1$-rich and $Θ_2$-rich, then $u$ is periodic.

Proof. The $Θ_1$-richness and recurrence imply that $L(u)$ is closed under $Θ_1$ and for all $n \geq 1$ we have $ΔC(n) + 2 = P_{Θ_1}(n) + P_{Θ_1}(n + 1)$. Analogously, for $Θ_2$ we can write $ΔC(n) + 2 = P_{Θ_2}(n) + P_{Θ_2}(n + 1)$ for all $n \geq 1$.

Let $N$ be the integer from Corollary 10. Adding the two previous equalities and using Corollary 10, we obtain $2ΔC(n) + 4 ≤ ΔC(n) + 4$, i.e., $ΔC(n) = 0$ for $n > N$ and thus $u$ is periodic.

Corollary 12. Let $Θ_1$ and $Θ_2$ be two distinct commuting involutive antimorphisms such that their composition $Θ_1Θ_2$ has no fixed letter, i.e., it is a derangement when restricted to $A$. If $u$ is an infinite word with language closed under $Θ_1$ and $Θ_2$, then

$$ΔC(n) + 4 ≥ P_{Θ_1}(n) + P_{Θ_2}(n) + P_{Θ_1}(n + 1) + P_{Θ_2}(n + 1) \text{ for all } n \geq 1.$$  

Example 13. The reversal mapping $R$ on $\{0,1\}^*$ and the antimorphism $Θ$ determined by exchange of letters 0 and 1 used in Example 8 satisfy the assumption of the previous corollary. By an argument similar to the one used in the proof of Corollary 11, we deduce that $ΔC(n) = 0$ for all $n ≥ 1$, thus $C(2) = C(1) = \#A = 2$. Therefore, the only infinite words on the alphabet $\{0,1\}$ which are simultaneously $R$-rich and $Θ$-rich are $u = (01)^ω$ and $u = (10)^ω$.

The previous considerations justify the modification of the notion of $Θ$-palindromic richness for words whose languages have more symmetries. Before proceeding, we need to show that there exist words for which the inequality given in Theorem 9 is in fact an equality. Let us show that such a suitable example is the Thue-Morse word.

Let $Θ$ be again the antimorphism on $\{0,1\}^*$ defined by $Θ(0) = 1$ and $Θ(1) = 0$. This antimorphism commutes with the reversal mapping $R$ and $RΘ$ has no fixed letter. Consider the morphism $0 \mapsto 01$ and $1 \mapsto 10$ which generates the Thue-Morse word. From the form of the morphism one can easily see that the language of its fixed points is invariant under $Θ$ and $R$. In the article [6], the classical palindromic complexity and the $Θ$-palindromic complexity of the Thue-Morse word is described.
Proposition 14. The palindromic complexity of the Thue-Morse word is

\[ P_R(n) = \begin{cases} 
1 & \text{if } n = 0, \\
2 & \text{if } 1 \leq n \leq 4, \\
0 & \text{if } n \text{ is odd and } n \geq 5, \\
4 & \text{if } n \text{ is even and } 4^k < n \leq 3 \cdot 4^k, \text{ for } k \geq 1, \\
2 & \text{if } n \text{ is even and } 3 \cdot 4^k < n \leq 4^{k+1}, \text{ for } k \geq 1.
\end{cases} \]

The \(\Theta\)-palindromic complexity of the Thue-Morse word is

\[ P_\Theta(n) = \begin{cases} 
1 & \text{if } n = 0, \\
2 & \text{if } n = 2, \\
0 & \text{if } n \text{ is odd and } , \\
4 & \text{if } n \text{ is even and } \frac{1}{2} \cdot 4^k < n \leq \frac{3}{2} \cdot 4^k, \text{ for } k \geq 1, \\
2 & \text{if } n \text{ is even and } \frac{3}{2} \cdot 4^k < n \leq \frac{5}{2} \cdot 4^k+1, \text{ for } k \geq 1.
\end{cases} \]

The factor complexity of the Thue-Morse word was described in 1989 independently in [7] and [18].

Proposition 15. The first difference of factor complexity of the Thue-Morse word is

\[ \Delta C(n) = \begin{cases} 
1 & \text{if } n = 0, \\
4 & \text{if } 2^k < n \leq 3 \cdot 2^{k-1}, \text{ for } k \geq 1, \\
2 & \text{otherwise.}
\end{cases} \]

Using these results on complexities we can show that the Thue-Morse word is saturated by classical palindromes and \(\Theta\)-palindromes up to the highest possible level given by the inequality in Corollary 12.

Corollary 16. For the Thue-Morse word we have

\[ \Delta C(n) + 4 = P_R(n) + P_R(n+1) + P_\Theta(n) + P_\Theta(n+1) \]

for all \(n \geq 1\).

Proof. The result follows immediately from Propositions 15 and 14. For reader’s convenience we report the values of \(R(n) = P_R(n) + P_R(n+1) + P_\Theta(n) + P_\Theta(n+1)\) and \(\Delta C(n)\) in Table 1.

In fact, the equality is also trivially attained for \(n = 0\) while considering the inequality in its general form in Theorem 9.
Table 1: Values $R(n)$ and $\Delta C(n)$ for the Thue-Morse word.

| $n$ | $R(n)$ | $\Delta C(n)$ |
|-----|--------|---------------|
| 1   | $2 + 2 + 0 + 2$ | 2 |
| 2   | $2 + 2 + 2 + 0$ | 2 |
| 3   | $2 + 2 + 0 + 4$ | 4 |
| $4^k < n \leq \frac{3}{2}4^k$ | $4 + 4$ | 4 |
| $\frac{3}{2}4^k < n \leq 2 \cdot 4^k$ | $4 + 2$ | 2 |
| $2 \cdot 4^k < n \leq 3 \cdot 4^k$ | $4 + 4$ | 4 |
| $3 \cdot 4^k < n \leq 4^{k+1}$ | $2 + 4$ | 2 |

5. Words with language closed under all elements of a group of symmetries

If a finite set $G$ is a submonoid of $AM(A^*)$ such that its elements are non-erasing, then, since it is finite, any of its elements restricted to the set of words of length one is just a permutation on $A$, and one can easily see that $G$ is a group. If an antimorphism $\Theta$ is involutive, then the corresponding permutation has cycles of length at most 2. In this section, we consider all antimorphisms with finite order, not only of order 2.

Example 17. The Champernowne word

$$12345678910111213141516171819202122232425262728293031 \ldots$$

over the alphabet $\{0, 1, 2, \ldots, 9\}$ arises by writing decimal representations of all positive integers in increasing order. The factor complexity of the Champernowne word is $C(n) = 10^n$ and its language is invariant under any element of the group $S_{10} \cup RS_{10}$, where $S_{10}$ is the group of permutations on a 10-element set.

The Champernowne word has maximal factor complexity and its group of symmetries is huge. The opposite extreme is a periodic word. We shall see that its group of symmetries is much more restricted.

Proposition 18. If $w \in A^*$ is the shortest possible period of the periodic word $u = w^\omega$ whose language is closed under a non-erasing antimorphism $\Theta : A^* \leftrightarrow A^*$ of finite order, then

1. $\Theta$ is an involution;

2. $w = ps$, where $p$ and $s$ are $\Theta$-palindromes;
3. \( \mathcal{P}_\Theta(n) + \mathcal{P}_\Theta(n+1) = 2 \) for any \( n \geq |w| \).

Proof. According to the convention we made in Preliminaries, \( w \) contains all letters from \( A \). If \( \Theta(w) = w \), then \( \Theta^2(w) = w \). As \( \Theta \) is non-erasing and its order is finite, \( \Theta^2 \) is a morphism which just permutes the letters. The equality \( \Theta^2(w) = w \) implies that \( \Theta^2 = \text{Id} \) and we can put \( p = w \) and \( s = \varepsilon \).

If \( \Theta(w) \neq w \), then \( \Theta(w) \) is a factor of the word \( ww \). Let \( p \) and \( s \) be factors such that \( ww = p\Theta(w)s \). It is easy to see that in fact \( w = ps \) since \( |w| = |w| \). Therefore, \( ww = p\Theta(s)\Theta(p)s \) and thus \( w = p\Theta(s) = \Theta(p)s \). In other words, \( s \) and \( p \) are \( \Theta \)-palindromes and both contain all letters of \( A \). Analogously to the previous case, this already implies involutivity of \( \Theta \).

To show the last item, observe that any \( \Theta \)-palindrome \( q \in \mathcal{L}(u) \) of length at least \( |w| \) is a central factor of a palindrome \( (sp)^ks(ps)^k \) or \( (ps)^kp(sp)^k \) for some \( k \in \mathbb{N} \). By minimality of the period \( w \), the central factors of \( (sp)^ks(ps)^k \) and \( (ps)^kp(sp)^k \) of length \( n \geq |w| \) are distinct. If \( |p| \) and \( |s| \) have opposite parities, then we have one \( \Theta \)-palindrome of length \( n \) and one \( \Theta \)-palindrome of length \( n + 1 \). If \( |p| \) and \( |s| \) have the same parities, then we have two \( \Theta \)-palindromes of length \( n \) and none of length \( n + 1 \) or vice versa. \( \square \)

A crucial role for the newly proposed definition of richness with respect to more antimorphisms is played by graphs of symmetries. We can assign such a graph to any infinite word \( u \) whose language is invariant under a group \( G \subset AM(A^*) \). For the first time, the most simple variant of this graph for \( G = \{\text{Id}, R\} \) appeared in the proof of the main theorem of the article [3]. See also [10]. In the previous section, we used this graph for \( G = \{\text{Id}, \Theta_1, \Theta_2, \Theta_1\Theta_2\} \). Both examples involve simple groups containing only antimorphisms of order 2. In fact, no such restriction is necessary.

Let us consider a finite group \( G \subset AM(A^*) \). We define a relation on \( A^* \) by

\[
v \sim w \iff v = \Theta(w) \text{ for some } \Theta \in G.
\]

It is obvious that \( \sim \) is an equivalence relation and that any equivalence class, again denoted by \([w]\) for \( w \in A^* \), has at most \( |G| \) elements.

**Definition 19.** Let \( G \subset AM(A^*) \) be a finite group, \( u \) be an infinite word with language closed under each \( \Theta \in G \) and \( n \in \mathbb{N} \).

1) The directed graph of symmetries of the word \( u \) is \( \Gamma_n^\Theta(u) = (V, E) \) with the set of vertices

\[
V = \{[w] \mid w \in \mathcal{L}_n(u), w \text{ is LS or RS}\}
\]

and an edge \( e \in \overrightarrow{E} \subset \mathcal{L}(u) \) starts in a vertex \([w]\) and ends in a vertex \([v]\), if

- the prefix of \( e \) of length \( n \) belongs to \([w]\),
- the suffix of \( e \) of length \( n \) belongs to \([v]\),
- \( e \) has exactly two occurrences of special factors of length \( n \).
2) The graph of symmetries of the word $u$ is $\Gamma_n(u) = (V, E)$ with the same set of vertices as $\overrightarrow{\Gamma}_n(u)$ and for any $e \in \mathcal{L}(u)$ we have

$$[e] \in E \iff e \in \overrightarrow{E}.$$
In [8], the $R$-defect of periodic words is studied. It is shown that $R$-defect is finite if and only if the language of the word is closed under $R$. Let us recall that words with finite $R$-defect were called almost rich in [16]. Therefore, our definition of almost $G$-rich periodic words does not contradict the old one in the case $G = \{\text{Id}, R\}$.

- Even for aperiodic words our definition applied to the group $G = \{\text{Id}, R\}$ is equivalent to the classical definition of richness and almost richness on the set of words with language closed under reversal, see Theorem 1.1, Proposition 1.2 and 3.5 in [10]. The same is valid for the group $G = \{\text{Id}, \Theta\}$, where $\Theta$ is an involutive antimorphism, see Theorem 2 and Corollary 7 in [21].

Although we allowed the group $G$ to have antimorphisms of higher order, in fact only the involutive antimorphisms $\Theta$ can have a fixed point $w$ containing all letters from the alphabet. Therefore, the notion of $\Theta$-palindromic complexity $P_\Theta$ makes sense only for involutions. Let $G^{(2)}$ be the set $G^{(2)} = \{\Theta \in G \mid \Theta \text{ is an antimorphism and } \Theta^2 = \text{Id}\}$, i.e., the set of involutive antimorphisms of $G$.

**Theorem 22.** Let $G \subset AM(A^*)$ be a finite group containing an antimorphism and let $u$ be an infinite word whose language is invariant under all elements of $G$. If there exists an integer $N \in \mathbb{N}$ such that in any factor of $u$ of length at least $N$ all letters of $A$ occur, then

$$\Delta C(n) + \#G \geq \sum_{\Theta \in G^{(2)}} \left( P_\Theta(n) + P_\Theta(n + 1) \right) \quad \text{for any } n \geq N. \quad (9)$$

**Proof.** Let $\Psi$ be an antimorphism of $G$. The mapping $\Theta \mapsto \Psi \Theta$ is a bijection on $G$, satisfying

$$\phi \in G \text{ is a morphism } \iff \Psi \phi \in G \text{ is an antimorphism.}$$

This means that $G$ has an even number of elements, say $\#G = 2k$.

Let us fix $n \geq N$. First, suppose that the graph $\Gamma_n(u)$ is nonempty.

Since each factor of $u$ longer than $N$ contains all letters, for any two antimorphisms $\Theta_1, \Theta_2$ of $G$ we have:

$$\Theta_1 \neq \Theta_2 \implies \Theta_1(v) \neq \Theta_2(v) \quad \text{for any } v \text{ such that } |v| \geq N. \quad (10)$$

And similarly, for any two morphisms $\varphi_1$ and $\varphi_2$ of $G$ we have

$$\varphi_1 \neq \varphi_2 \implies \varphi_1(v) \neq \varphi_2(v) \quad \text{for any } v \text{ such that } |v| \geq N. \quad (11)$$

On the other hand, if $v$ is a $\Theta$-palindrome for an antimorphism $\Theta \in G$, then for any antimorphism $\Theta_t \in G$, the word $\Theta_t(v)$ is a $(\Theta_t \Theta \Theta_t^{-1})$-palindrome. We may conclude the following for the directed graph $\Gamma_n(u)$ with $n \geq N$. 

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• A vertex $[w]$ has $\#G = 2k$ elements if $w$ is not a $\Theta$-palindrome for any antimorphism $\Theta \in G$. Denote the number of such vertices by $\beta$.

• A vertex $[w]$ has $k$ elements if there exists an antimorphism $\Theta \in G$ such that $w = \Theta(w)$. Denote the number of such vertices by $\alpha$.

• If two distinct vertices $[w]$ and $[v]$ are connected by an edge $e$ starting in $w$ and ending in $v$, then there exist at least $2k$ edges between these two vertices, namely $k$ edges $\varphi(e)$ having the same orientation as $e$ and $k$ edges $\Theta(e)$ having the opposite orientation, for any morphism $\varphi \in G$ and any antimorphism $\Theta \in G$.

• No factor of length at least $n$ is a $\Theta$-palindrome simultaneously for two distinct antimorphisms $\Theta$.

Using the property mentioned in the last item, the proof is now easier than the proof of Theorem 9.

As all special factors of $L_n(u)$ belong to the classes forming vertices in $\Gamma_n(u)$, any factor $f \in L_{n+1}(u)$ as well as any non-special factor in $f \in L_n(u)$ is a factor (which is neither the prefix of length $n$, nor a suffix of length $n$) of exactly one edge $e$ in $\Gamma_n(u)$. Clearly $\Theta(\overrightarrow{E}) = \overrightarrow{E}$. If such a factor $f$ is a $\Theta$-palindrome, then necessarily $\Theta(e) = e$ and $e$ is a loop in $\Gamma_n(u)$. For the number of edges in the directed graph $\Gamma_n(u)$ we have the lower bound:
\[
\# \overrightarrow{E} \geq 2k(\alpha + \beta - 1) + A
\]  
where
\[
A \geq \sum_{\Theta \in G^{(2)}} \left( P_{\Theta}(n) + P_{\Theta}(n + 1) \right) - k\alpha.
\]  
(13)

Again, as in the proof of Theorem 9, the number of edges $\# \overrightarrow{E}$ can be written as
\[
\# \overrightarrow{E} = \sum_{w \in L_n(u), w \text{ special}} \# \text{Ext}(w) = \sum_{w \in L_n(u), w \text{ special}} \# \text{Ext}(w) + \sum_{w \in L_n(u), w \text{ special}} \# \text{Ext}(w).
\]

Since the first sum on the right side has $2k\beta$ summands and the second sum has $k\alpha$ summands, we obtain
\[
\# \overrightarrow{E} = \sum_{w \in L_n(u), w \text{ special}} \left( \# \text{Ext}(w) - 1 \right) + 2k\beta + k\alpha = \Delta C(n) + 2k\beta + k\alpha.
\]
This together with (12) and (13) implies the theorem in case that $\Gamma_n(u)$ is nonempty.

If $\Gamma_n(u)$ is empty, then $u$ is periodic and, according to the point 1. in Proposition 18, the left side of the inequality (9) is twice the number of involutive antimorphisms in $G$. According to the point 3. in Proposition 18, the right side of the inequality has the same value.
Remark 23. The previous proof enables us to test \( G \)-richness by verifying the equality in (9) instead of looking at the graph \( \Gamma_n \) for \( n \geq N \). For \( n < N \) we still have to check that \( \Gamma_n \) is a tree.

Remark 24. The assumption on the integer \( N \) in Theorem 22 can be replaced by the following weaker assumption: there exists an integer \( N \) such that for any two antimorphisms \( \Theta_1, \Theta_2 \in G \) it holds

\[
\Theta_1 \neq \Theta_2 \implies \Theta_1(v) \neq \Theta_2(v) \quad \text{for any } v \text{ with } |v| \geq N,
\]

and for any two morphisms \( \varphi_1, \varphi_2 \in G \) it holds

\[
\varphi_1 \neq \varphi_2 \implies \varphi_1(v) \neq \varphi_2(v) \quad \text{for any } v \text{ with } |v| \geq N.
\]

Since the existence of \( N \) required in Theorem 22 is trivially satisfied for uniformly recurrent words, the proof of Theorem 22 gives a simple criterion for almost \( G \)-richness.

Corollary 25. Let \( G \subset AM(A^*) \) be a finite group containing at least one antimorphism. If an infinite uniformly recurrent word \( u \) has its language invariant under all elements of \( G \), then \( u \) is almost \( G \)-rich if and only if there exists \( N \in \mathbb{N} \) such that

\[
\Delta C(n) + \#G = \sum_{\Theta \in G^{(2)}} \left( \mathcal{P}_\Theta(n) + \mathcal{P}_\Theta(n + 1) \right) \quad \text{for any } n \geq N.
\]

6. Examples of \( G \)-rich words

6.1. The Thue-Morse word and its generalizations

To demonstrate \( G \)-richness of the Thue-Morse word, we use Remark 23. From Corollary 16 and the shape of \( \Gamma_2(u_{TM}) \) in Figure 1b we can deduce the following statement.

Corollary 26. The Thue-Morse word is \( G \)-rich, where \( G \) is the group generated by the reversal mapping and the involutive antimorphism determined by the exchange of letters.

The following generalization of the Thue-Morse word was considered for instance in [2]. Let \( s_b(n) \) denote the sum of digits in the base-\( b \) representation of the integer \( n \). The infinite word \( t_{b,m} \) is defined as

\[
t_{b,m} = (s_b(n) \mod m)^{+\infty}_{n=0}.
\]

Using this notation, the famous Thue-Morse word equals \( t_{2,2} \). The word \( t_{b,m} \) is over the alphabet \( \{0, 1, \ldots, m - 1\} \) and is also a fixed point of a primitive morphism, as already mentioned in [2]. To abbreviate the formulas, let us denote \( i \oplus_m j = i + j \mod m \) for any \( i, j \in \{0, 1, \ldots, m - 1\} \). It is easy to see that the morphism defined by

\[
\varphi(k) = k(k \oplus_m 1)(k \oplus_m 2) \ldots (k \oplus_m (b - 1)) \quad \text{for any } k \in \{0, 1, \ldots, m - 1\}
\]

fixes the word \( t_{b,m} \).
Table 2: Values $R(n)$ and $\Delta C(n)$ for $t_{4,2}$, $\#G = 4$.

| $n$       | $R(n)$ | $\Delta C(n)$ |
|-----------|--------|---------------|
| $0 < n < 17$ | 6      | 2             |
| $17 \leq n < 29$ | 8      | 4             |
| $29 \leq n < 65$ | 6      | 2             |
| $65 \leq n < 113$ | 8      | 4             |
| 113       | 6      | 2             |

Example 27. For parameters $(b, m) \neq (2, 2)$ an explicit description of the group $G$ under which the language of $t_{b,m}$ is invariant and values of palindromic complexities $P_\Theta$ is not available. A known fact which can be easily seen from the morphism $\varphi$ is that if $m \mid (b-1)$, then $t_{b,m}$ is periodic. We first present some periodic examples.

The periodic case $m \mid (b-1)$ In this case, one can see that from the morphism that $t_{b,m} = (01 \ldots (m-1))^\omega$. For example, if $b = 5$ and $m = 2$, then the morphism has the form $0 \mapsto 01010$ and $1 \mapsto 10101$.

For $m = 2$, the word $(01)^\omega$ trivially satisfies the equality in Corollary 12 for the reversal mapping and the antimorphism determined by the exchange of 0 and 1. Thus $t_{2k+1,2}$ is $G$-rich with the same group $G$ as the Thue-Morse word $t_{2,2}$.

For $m = 3$, the language of the word $(012)^\omega$ is closed under all elements of a group generated by the 3 involutive antimorphisms on $\{0, 1, 2\}^*$ different from the reversal mapping. The proof of richness is left to the reader.

For other values of $b$ and $m$, we used computer resources, namely the open-source mathematical software Sage [22], to look for candidates exhibiting richness.

The word $t_{2k,2}$ From the shape of the morphism it follows that the language of the word $t_{2k,2}$ is invariant under the same group $G$ as $u_{TM}$. For these words our computer experiments suggest that equality also holds. Table 2 shows some values for $\Delta C(n)$ and $R(n) = \sum_{\Theta \in G^2} \left( P_\Theta(n) + P_\Theta(n+1) \right)$ for $t_{4,2}$.

The word $t_{2,4}$ This word is a fixed point of the morphism

$0 \mapsto 01$, $1 \mapsto 12$, $2 \mapsto 23$, and $3 \mapsto 30$.  

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A computer test on factors of length 100 of the prefix of \( t_{2,4} \) of length 30000 suggests that its language is invariant under four antimorphisms:

\[
\begin{align*}
\Theta_1 : & \quad 0 \rightarrow 0, 1 \rightarrow 3, 2 \rightarrow 2, 3 \rightarrow 1, \\
\Theta_2 : & \quad 0 \rightarrow 1, 1 \rightarrow 0, 2 \rightarrow 3, 3 \rightarrow 2, \\
\Theta_3 : & \quad 0 \rightarrow 2, 1 \rightarrow 1, 2 \rightarrow 0, 3 \rightarrow 3, \\
\Theta_4 : & \quad 0 \rightarrow 3, 1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 0.
\end{align*}
\]

The group \( G \) generated by those 4 antimorphisms has 8 elements. It seems that the word \( t_{2,4} \) is \( G \)-rich as well.

Our only conclusion is that identification of \( G \)-rich words among \( t_{b,m} \) requires a further study.

All examples of words we proved to be \( G \)-rich or we suspect to be \( G \)-rich have a common property: any antimorphism in \( G \) has order two. Proposition 18 says that for periodic words this property is necessary.

6.2. A \( G \)-rich word with irrational densities of letters

If the language of an infinite word with well-defined densities of letters is invariant under an antimorphism \( \Theta \) of finite order and \( \Theta \) is not the reversal mapping \( R \), then the density vector is invariant under the permutation corresponding to \( \Theta \). For example, the densities of both letters 1 and 0 in the Thue-Morse word are necessarily \( \frac{1}{2} \), the densities of all letters of the Champernowne word from Example 17 are \( \frac{1}{10} \). Nevertheless, the densities of letters need not be rational.

In this section we describe a \( G \)-rich word with irrational densities of letters.

**Example 28.** Let \( \varphi \) be the morphism on \( \{0, 1, 2, 3\}^* \) defined as

\[
\varphi : \begin{cases} 
0 \mapsto 0130 \\
1 \mapsto 1021 \\
2 \mapsto 102 \\
3 \mapsto 013
\end{cases}
\]

and let \( u \) be a fixed point of \( \varphi \). The matrix of this morphism \( M_\varphi \) and the eigenvector \( x_\Lambda \) corresponding to the dominant eigenvalue \( \Lambda = 2 + \sqrt{3} \) are

\[
M_\varphi = \begin{pmatrix}
2 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
x_\Lambda = \begin{pmatrix}
\frac{\sqrt{3} - 1}{2} \\
\frac{\sqrt{3} - 1}{2} \\
\frac{2 - \sqrt{3}}{2} \\
\frac{2 - \sqrt{3}}{2}
\end{pmatrix}.
\]

The matrix \( M_\varphi \) is primitive and therefore our morphism is primitive as well (see [15]). It is known that the components of the eigenvector corresponding to the dominant eigenvalue are proportional to the densities of letters. In our case, the letters 0 and 1 have density \( \frac{\sqrt{3} - 1}{2} \). The letters 2 and 3 have density \( \frac{2 - \sqrt{3}}{2} \).

We show the following properties of the word \( u \):
1. Language $\mathcal{L}(u)$ is closed under two involutive antimorphisms $\Theta_1$ and $\Theta_2$, where

$$
\Theta_1 : 0 \mapsto 1, 1 \mapsto 0, 2 \mapsto 2, 3 \mapsto 3 \quad \text{and} \quad \Theta_2 : 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 2.
$$

2. The first increment of factor complexity satisfies

$$
\Delta C(n) = 2 \quad \text{for any } n \in \mathbb{N}^+.
$$

3. Only $\Theta_1$-palindromes of length 1 and 2 occurring in $\mathcal{L}(u)$ are factors 2, 3, 10 and 01; only $\Theta_2$-palindromes of length 1 and 2 are factors 0 and 1.

4. Any $\Theta_i$-palindrome $w \in \mathcal{L}(u)$ has a unique $\Theta_i$-palindromic extension, i.e., there exists a unique letter $a \in \mathcal{A}$ such that $aw\Theta_i(a) \in \mathcal{L}(u)$.

**Proof of the properties of $u$ defined in Example 28.**

**Property 3.** To check this property is an easy task since $\mathcal{L}_2(u) = \{02, 21, 13, 30, 01, 10\}$.

**Property 1.** As $u$ is a fixed point of the primitive morphism $\varphi$, the word $u$ is uniformly recurrent. To prove the invariance of $\mathcal{L}(u)$ under $\Theta_i$, it is sufficient to show that $u$ contains infinitely many $\Theta_i$-palindromes. We give a construction producing from a $\Theta_i$-palindrome a longer $\Theta_i$-palindrome.

Let $a \in \mathcal{A}$. Denote by $p_a$ the unique factor of $u$ of length 5 such that $\varphi(a)p_a$ is a factor of $u$. The correctness of this definition can be seen by looking at the images by $\varphi$ of the factors of $u$ of length 2, as listed above. One can show that $p_0 = p_2 = 10210$ and $p_1 = p_3 = 01301$.

Fix $i \in \{1, 2\}$. Let $ab \in \mathcal{L}_2(u)$. It is easy to show that

$$
\Theta_i(p_a) = \varphi(\Theta_i(a))p_{\Theta_i(a)}.
$$

We now prove the following claim. If $w = w_1 \ldots w_n \in \mathcal{L}(u)$, then

$$
\Theta_i(\varphi(w)p_{w_n}) = \varphi(\Theta_i(w))p_{\Theta_i(w_1)}.
$$

The claim can be shown by induction on $|w|$. Indeed, one can easily verify that for $a \in \mathcal{A}$ we have

$$
\Theta_i(\varphi(a)p_a) = \varphi(\Theta_i(a))p_{\Theta_i(a)}.
$$

Suppose now the claim holds for the word $w = w_1 \ldots w_n \in \mathcal{L}(u)$. We show that it holds also for $aw \in \mathcal{L}(u), a \in \mathcal{A}$. We have

$$
S = \Theta_i(\varphi(aw)p_{w_n}) = \Theta_i(\varphi(w)p_{w_n})\Theta_i(\varphi(a)) = \varphi(\Theta_i(w))p_{\Theta_i(w_1)}\Theta_i(\varphi(a)).
$$

Since $aw_1 \in \mathcal{L}(u)$, using (14) we have

$$
p_{\Theta_i(w_1)} = \Theta_i(p_a).
$$
We continue to manipulate the equation using (16)

\[ S = \varphi(\Theta_i(w))\Theta_i(p_a)\Theta_i(\varphi(a)) = \varphi(\Theta_i(w))\Theta_i(\varphi(a))p_a = \varphi(\Theta_i(w))\varphi(\Theta_i(a))p_{\Theta_i(a)} \]

and finally

\[ S = \varphi(\Theta_i(w)\Theta_i(a))p_{\Theta_i(a)} = \varphi(\Theta_i(aw))p_{\Theta_i(a)} \]

as we claimed.

Let \( w \) be a \( \Theta_i \)-palindrome. Using (15) we have

\[ \Theta_i(\varphi(w)p_{w_n}) = \varphi(\Theta_i(w))p_{\Theta_i(w_n)} = \varphi(w)p_{w_n}. \]

Therefore, \( \varphi(w)p_{w_n} \) is a \( \Theta_i \)-palindrome as well.

**Property 2.** We evaluate \( \Delta C(n) \). As we have explained in the Preliminaries, we need to look at left special factors and their extensions. In our word \( u \), there are only two left special factors of length one: the letter 0 with \( \text{Lext}(0) = \{1, 3\} \) and the letter 1 with \( \text{Lext}(1) = \{0, 2\} \). From the shape of the morphism we see that for any LS factor \( w \) its image \( \varphi(w) \) is LS as well, and moreover, \( \text{Lext}(w) = \text{Lext}(\varphi(w)) \). Thus the factors \( \varphi^k(0) \) and \( \varphi^k(1) \) are LS factors both with two left extensions for any \( k \in \mathbb{N} \). Since any prefix of LS factor is a LS factor as well, we can deduce, that any prefix of a fixed point \( \lim_{k \to \infty} \varphi^k(0) \) or \( \lim_{k \to \infty} \varphi^k(1) \) is a left special factor.

For any length \( n \) we have two LS factors of length \( n \) each with two left extensions. To show that there are no other LS factors in \( u \), one has to show that any left special factor \( w \) longer than one is a prefix of \( \varphi(v) \) where \( v \) is a LS factor with the same left extension. A proof of this part is left to the reader.

Using the equation (4) we can conclude that \( \Delta C(n) = 2 \) for all \( n \geq 1 \).

**Property 4.** The invariance of \( u \) under \( \Theta_i \) implies that \( \Theta_i(\text{Lext}(v)) = \text{Rext}(\Theta_i(v)) \) for any \( v \in \mathcal{L}(u) \). Analogously, for bilateral orders we have \( b(v) = b(\Theta_i(v)) \).

Thus, our description of LS factors gives immediately that for any \( n \) there exist in \( u \) exactly two RS factors of length \( n \), each with two extensions. Moreover, \( v \) is LS if and only if \( \Theta_1(v) \) is RS if and only if \( \Theta_2\Theta_1(v) \) is LS. Therefore, two LS factors form a pair \( v \) and \( \Theta_2\Theta_1(v) \) and have the same bilateral order.

We show by contradiction that any \( \Theta_1 \)-palindrome has a unique \( \Theta_1 \)-palindromic extension. Consider a factor \( w = \Theta_1(w) \in \mathcal{L}(u) \).

First, suppose that \( w \) has no \( \Theta_1 \)-palindromic extension. Since any factor has at least one left and one right extension, there exist letters \( a \) and \( b \) such that \( awb \in \mathcal{L}(u) \). As \( w \) has no \( \Theta_1 \)-palindromic extensions, we have \( b \neq \Theta_1(a) \). The invariance of language under \( \Theta_i \) implies that \( \Theta_1(b)w\Theta_1(a) \in \mathcal{L}(u) \). It means that \( w \) is a bispecial factor with the left extensions \( a, \Theta_1(b) \) only and with the right extensions \( b, \Theta_1(a) \) only.

Thus \( b(w) = -1 \). As we have mentioned, the second BS factor of the same length is the factor \( \Theta_2(w) \) and for its bilateral order we have \( b(\Theta_2(w)) = b(w) = -1 \).
According to (5) we have $\Delta^2 C(n) = \sum_{w \in \mathcal{L}_u(u)} b(w) = -2$ which is a contradiction with Property 2.

Now suppose that $w$ has two $\Theta_1$-palindromic extensions, i.e., there exist two letters $a \neq b$, such that $aw\Theta_1(a)$ and $bw\Theta_1(b)$ belong to $\mathcal{L}(u)$. We have either $b(w) = -1$ or $b(w) = 1$ and we can repeat the argument to deduce a contradiction $\Delta^2 C(n) = \pm 2$.

Finally, Property 3 implies $P_{\Theta_1}(1) + P_{\Theta_1}(2) + P_{\Theta_2}(1) + P_{\Theta_2}(2) = 6$. Thus by using Property 4, we have for all $n \geq 1$

$$P_{\Theta_1}(n) + P_{\Theta_1}(n + 1) + P_{\Theta_2}(n) + P_{\Theta_2}(n + 1) = 6.$$  

Property 1 means $\#G = 4$. Together with $\Delta C(n) = 2$ from Property 2 it gives for any $n \geq 1$ the equality

$$\Delta C(n) + \#G = P_{\Theta_1}(n) + P_{\Theta_1}(n + 1) + P_{\Theta_2}(n) + P_{\Theta_2}(n + 1)$$

and thus the $G$-richness of $u$ (using Remark 24).

7. Open problems

Several papers were devoted to rich words, stating many results. How fruitful is the new definition of the $G$-richness remains open. Answers to questions listed below may help to clarify that.

1. If the group $G$ only contains the identity and an involutive antimorphism $\Theta$, then the $G$-rich words (i.e. $\Theta$-rich words) can be characterized by return words, see [16], [21]. Is there such a characterization of $G$-rich words for general $G$?
2. Palindromic closure ($\Theta$-palindromic closure) is a very effective tool for constructing rich and almost rich words, see for example [11] and [16]. Can $\Theta$-palindromic closures be used to construct $G$-rich words?
3. Can a reasonable $G$-analogue to $\Theta$-defect be defined for a group of symmetries $G$?
4. How to modify the right side of the inequality in Theorem 22 to obtain an inequality for all $n \in \mathbb{N}$ as it holds in Theorem 9?
5. Is there an inequality analogous to (1) if one replaces the classical palindromic complexity $\mathcal{P}$ by the $\Theta$-palindromic complexity $\mathcal{P}_{\Theta}$ for some antimorphism $\Theta$? Can this inequality be improved if $u$ contains simultaneously infinitely many palindromes and $\Theta$-palindromes?
6. Given a group $G \subset AM(A^*)$, how to find an infinite word $u$ such that it is (almost) $G$-rich?
7. Is there an explicit formula for the sequence $r(n)$ from Remark 6?

During the review process of this article, we answered questions 1 and 3 in [20].
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