Free-Fermionic $SO(8)$ And tri($\mathbb{O}$)

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Abstract

In this note, we speculate about the fundamental role being played by the $SO(8)$ group representations displaying the triality structure that necessarily arise in models constructed under the free fermionic methodology as being remnants of the higher-dimensional triality algebra

$$\text{tri}(\mathbb{O}) = \mathfrak{so}(8).$$

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1 Introduction

The heterotic $E_8 \times E_8$ and the heterotic $SO(32)$ hold a special place when it comes to relating string vacua to experimental phenomena. In this note, we speculate about the fundamental role being played by the $SO(8)$ group representations, displaying the triality structure which is the four dimensional manifestation of the twisted generation of gauge groups already noticed in the ten dimensional case, that necessarily arise in models constructed following the free fermionic methodology being remnants of the higher-dimensional triality algebra

$$\text{tri}(0) = so(8).$$

2 The Free-Fermionic Methodology

For each consistent heterotic string model, there exists a partition function defined by a set of vectors with boundary conditions and a set of coefficients associated to each pair of these vectors. It will be shown that for each set of boundary conditions basis vectors and the set of associated coefficients, a set of general rules can be summarized for any model realized in the free fermionic formalism. These rules, originally derived by Antoniadis, Bachas, Kounnas in [10], are known as the ABK rules. For further convenience, the vectors containing the boundary conditions used to define a model are called the basis vectors and the associated coefficients are called the one-loop phases that appear in the partition function.

2.1 The ABK Rules

One of the key elements is the set of basis vectors that defines $\Xi$, the space of all the sectors. For each sector $\beta \in \Xi$ there is a corresponding Hilbert space of states. Each basis vector $b_i$ consists of a set of boundary conditions for each fermion denoted by

$$b_i = \{ \alpha(\psi_{1,2}^\mu), \ldots, \alpha(\omega^6)|\alpha(\overline{\psi}^1), \ldots, \alpha(\overline{\omega}^8) \}$$

where $\alpha(f)$ is defined by

$$f \rightarrow -e^{i\pi\alpha(f)} f.$$

The $b_i$ have to form an additive Abelian group and satisfy the constraints. If $N_i$ is the smallest positive integer for which $N_i b_i = 0$ and $N_{ij}$ is the least common multiple of $N_i$ and $N_j$ then the rules for the basis vectors, known popularly as the ABK rules,

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2 These rules were also developed with a different formalism by Kawai, Lewellen and Tye in [11].
are given as

\[(1) \quad \sum m_i \cdot b_i = 0 \iff m_i = 0 \mod N_i \forall i \quad (2.1)\]
\[(2) \quad N_{ij} \cdot b_i \cdot b_j = 0 \mod 4 \quad (2.2)\]
\[(3) \quad N_i \cdot b_i \cdot b_i = 0 \mod 8 \quad (2.3)\]
\[(5) \quad b_1 = 1 \iff 1 \in \Xi \quad (2.4)\]
\[(4) \quad \text{Even number of real fermions} \quad (2.5)\]

where

\[b_i \cdot b_j = \left( \frac{1}{2} \sum_{\text{left real}} + \sum_{\text{left complex}} - \frac{1}{2} \sum_{\text{right real}} - \sum_{\text{right complex}} \right) b_i(f) \times b_j(f).\]

### 2.2 Rules for the One-Loop Phases

The rules for the one-loop phases are

\[C\left( b_i \right) = \delta_{b_i} e^{\frac{2\pi}{N_j} n} = \delta_{b_i} e^{\frac{2\pi}{N_i} m} e^{\frac{\pi}{N_j} b_i b_j} \quad (2.6)\]
\[C\left( b_i \right) = -e^{\frac{\pi}{N_j} b_i b_j} C\left( b_i \right) \quad (2.7)\]
\[C\left( b_i \right) = e^{\frac{\pi}{N_j} b_i b_j} C\left( b_i \right) \quad (2.8)\]
\[C\left( b_i \right) C\left( b_j \right) = \delta_{b_i} C\left( b_i \right) C\left( b_j \right) \quad (2.9)\]

where the spin-statistics index is defined as

\[\delta_{\alpha} = e^{i\alpha(\psi^\mu)\pi} = \begin{cases} 1, & \alpha(\psi_{1,2}) = 0 \\ -1, & \alpha(\psi_{1,2}) = 1 \end{cases}.\]

### 2.3 The GGSO Projections

To complete this construction, we have to impose another set of constraints on the physical states called the GGSO projections. The GGSO projection selects the states \(|S\rangle_{\alpha}\) belonging to the \(\alpha\) sector satisfying

\[e^{\pi b_i \cdot F_{\alpha}} |S\rangle_{\alpha} = \delta_{\alpha} C\left(\frac{\alpha}{b_i}\right)^* |S\rangle_{\alpha} \quad \forall b_i \quad (2.10)\]
where
\[ b_i \cdot F_\alpha = \left( \frac{1}{2} \sum_{\text{left real}} + \sum_{\text{left complex}} - \frac{1}{2} \sum_{\text{right real}} - \sum_{\text{right complex}} \right) b_i(f) \times F_\alpha(f) \] (2.11)

where \( F_\alpha(f) \) is the fermion number operator given by
\[ F_\alpha(f) = \begin{cases} +1, & \text{if } f \text{ is a fermionic oscillator}, \\ -1, & \text{if } f \text{ is the complex conjugate}. \end{cases} \]

### 2.4 The Massless String Spectrum

As we are interested in low-energy physics, we are only interested in the massless states. The physical states in the string spectrum satisfy the level matching condition
\[ M^2_L = -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + N_L = -1 + \frac{\alpha_R \cdot \alpha_R}{8} + N_R = M^2_R \] (2.12)

where \( \alpha = (\alpha_L; \alpha_R) \in \Xi \) is a sector in the additive group, and
\[ N_L = \sum_f (\nu_L); \quad N_R = \sum_f (\nu_R); \] (2.13)

The frequencies of the fermionic oscillators depending on their boundary conditions is taken to be
\[ f \rightarrow -e^{i\pi \alpha(f)} f, \quad f^* \rightarrow -e^{-i\pi \alpha(f)} f^*. \]

The frequency for the fermions is given by
\[ \nu_{f,f^*} = \frac{1 \pm \alpha(f)}{2}. \]

Each complex fermion \( f \) generates a \( U(1) \) current with a charge with respect to the unbroken Cartan generators of the four dimensional gauge group given by
\[ Q_{\nu}(f) = \nu - \frac{1}{2} = \frac{\alpha(f)}{2} + F \]
for each complex right-moving fermion \( f \).
2.5 The Enhancements

Extra space-time vector bosons may be obtained from the sectors satisfying the conditions:

\[ \alpha^2_L = 0, \quad \alpha^2_R \neq 0. \]

There are three possible types of enhancements:

- Observable for example \( x \),
- Hidden for example \( z_1 + z_2 \),
- Mixed for example \( \alpha \).

3 The Free-Fermionic 4D Models

The phenomenological free fermionic heterotic string models were constructed following two main routes, the first are the so-called NAHE–based models. This set of models utilise a set of eight or nine boundary condition basis vectors. The first five consist of the so-called NAHE set \([1]\) and are common in all these models. The basis vectors underlying the NAHE–based models therefore differ by the additional three or four basis vectors that extend the NAHE set.

The second route follows from the classification methodology that was developed in \([5]\) for the classification of type II free fermionic superstrings and adopted in \([2, 4, 6]\) for the classification of free fermionic heterotic string vacua with \( SO(10) \) GUT symmetry and its Pati–Salam \([4]\) and Flipped \( SU(5) \) \([3]\) subgroups. The main difference between the two classes of models is that while the NAHE–based models allow for asymmetric boundary conditions with respect to the set of internal fermions \( \{ y, \omega|\bar{y}, \bar{\omega} \} \), the classification method only utilises symmetric boundary conditions. This distinction affects the moduli spaces of the models \([8]\), which can be entirely fixed in the former case \([9]\) but not in the later. On the other hand the classification method enables the systematic scan of spaces of the order of \( 10^{12} \) vacua, and led to the discovery of spinor–vector duality \([2, 7]\) and exophobic heterotic string vacua \([4]\).
3.1 The Classification Methodology

A subset of basis vectors that respect the $SO(10)$ symmetry is given by the set of 12 boundary condition basis vectors $V = \{v_1, v_2, \ldots, v_{12}\}$

$v_1 = 1 = \{\psi^{1,2}_\mu, \chi^{1,6}_1, y^{1,6}_1, \omega^{1,6}_5, \bar{y}^{1,6}_1, \bar{\omega}^{1,6}_5, \bar{\psi}^{1,6,5}_1, \bar{\phi}^{1,6,5}_1, \}$
$v_2 = S = \{\psi^{12}_\mu, \chi^{34}, \chi^{56}_1\},

\begin{align*}
v_2_+i &= e_i = \{y^i, \omega^i, \bar{y}^i, \bar{\omega}^i\}, \quad i = 1, \ldots, 6,

v_9 &= b_1 = \{\chi^{34}, \chi^{56}, y^{34}, y^{56}, \bar{y}^{34}, \bar{y}^{56}, \bar{\psi}^{1,5}\},

v_{10} &= b_2 = \{\chi^{12}, \chi^{56}, y^{12}, y^{56}, \bar{y}^{12}, \bar{y}^{56}, \bar{\psi}^{1,5}\},

v_{11} &= z_1 = \{\bar{\phi}^{1,4}\},

v_{12} &= z_2 = \{\bar{\phi}^{5,8}\}
\end{align*}

where the basis vectors 1 and $S$, generate a model with the $SO(44)$ gauge symmetry and $N = 4$ space–time SUSY with the tachyons being projected out of the massless spectrum. The next six basis vectors: $e_1, \ldots, e_6$ all correspond to the possible symmetric shifts of the six internal coordinates thus breaking the $SO(44)$ gauge group to $SO(32) \times U(1)^6$ but keeping the $N = 4$ SUSY intact. The vectors $b_i$ for $i = 1, 2$ correspond to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold twists. The vectors $b_1$ and $b_2$ play the role of breaking the $N = 4$ down to $N = 1$ whilst reducing the gauge group to $SO(10) \times U(1)^2 \times SO(18)$. The states coming from the hidden sector are produced by $z_1$ and $z_2$ left untouched by the action of previous basis vectors. These vectors together with the others generate the following adjoint representation of the gauge symmetry: $SO(10) \times U(1)^3 \times SO(8) \times SO(8)$ where $SO(10) \times U(1)^3$ is the observable gauge group which gives rise to matter states from the twisted sectors charged under the $U(1)s$ while $SO(8) \times SO(8)$ is the hidden gauge group gives rise to matter states which are neutral under the $U(1)s$.

3.2 The Various $SO(10)$ Subgroups

The $SO(10)$ GUT models generated can be broken to one of its subgroups by the boundary condition assignment on the complex fermion $\psi^{1,5}$. For the Pati-Salam and the Flipped $SU(5)$ case, one additional basis vector is required to break the $SO(10)$ GUT symmetry. However, in order to construct the $SU(4) \times SU(2) \times U(1)$, the Standard-Like models and the Left-Right Symmetric models, the Pati-Salam breaking is required along with an additional $SO(10)$ breaking basis vector. The following boundary condition basis vectors can be used to construct the necessary gauge groups:
3.2.1 The Pati-Salam Subgroup
\[ v_{13} = \alpha = \{ \psi^{4,5}, \phi^{1,2} \} \]

3.2.2 The Flipped SU(5) Subgroup
\[ v_{13} = \alpha = \{ \eta^{1,2,3}, \psi^{1,5}, \phi^{1,4} \} = \frac{1}{2} \phi^5 \]
\[ v_{14} = \beta = \{ \eta^{1,2,3}, \psi^{3}, \phi^{1,6} \} = \frac{1}{2} \phi^5 \]

3.2.3 The SU(4) × SU(2) × U(1) Subgroup
\[ v_{13} = \alpha = \{ \psi^{4,5}, \phi^{1,2} \} \]
\[ v_{14} = \beta = \{ \eta^{1,2,3}, \psi^{1,5}, \phi^{1,6} \} = \frac{1}{2} \phi^5 \]

3.2.4 The Left-Right Symmetric Subgroup
\[ v_{13} = \alpha = \{ \eta^{4,5}, \phi^{1,2} \} \]
\[ v_{14} = \beta = \{ \eta^{1,2,3}, \psi^{1,5}, \phi^{1,6} \} = \frac{1}{2} \phi^5 \]

3.2.5 The Standard-Like Model Subgroup
\[ v_{13} = \alpha = \{ \eta^{4,5}, \phi^{1,2} \} \]
\[ v_{14} = \beta = \{ \eta^{1,2,3}, \psi^{1,5}, \phi^{1,6} \} = \frac{1}{2} \phi^5 \]

4 The SU421 And LRS Models

In [12], the fact was highlighted that the LRS and SU421 models are not viable as these models circumvent the \( E_6 \rightarrow SO(10) \times U(1)_\zeta \) symmetry breaking pattern with the price that the \( U(1)_\zeta \) charges of the SM states do not satisfy the \( E_6 \) embedding necessary for unified gauge couplings to agree with the low energy values of \( \sin^2 \theta_W(M_Z) \) and \( \alpha_s(M_Z) \) [13, 14].

While the statement is true for the SU421 models [16, 17], we introduce LRS models which are constructed in the free fermionic formalism to verify that the \( U(1)_\zeta \) charges of the SM states satisfy the \( E_6 \) embedding. We begin by assuming that \( U(1)_\zeta \) charges admit the \( E_6 \) embedding. In this case the heavy Higgs states consists of the pair \( \mathcal{N}(1, \frac{3}{2}, 1, 2, \frac{1}{2}), \mathcal{N}(1, -\frac{3}{2}, 1, 2, -\frac{1}{2}) \). The VEV along the electrically neutral component leaves unbroken the SM gauge group and the \( U(1)_{Z'} \) combination
\[ U(1)_{Z'} = \frac{1}{2} U(1)_{B-L} - \frac{2}{3} U(1)_{T_{3R}} - \frac{5}{3} U(1)_{\zeta} \notin SO(10) \]
where \( U(1)_\zeta = \sum_{i=1}^{3} U(1)_i \) is anomaly free which may remain unbroken down to low scales. We remark, however, that in the NAHE-based free fermionic LRS models [15] the \( U(1)_\zeta \) charges do not admit the \( E_6 \) embedding and go on to show that the same is true for free fermionic models constructed by utilizing the classification methodology [5].

4.1 The Non-Viable \( SU(4) \times SU(2) \times U(1) \)

In this section, we briefly consider the model presented in [17] which was obtained using the classification methodology. The set of basis vectors that generate the \( SU(4) \times SU(2) \times U(1) \) heterotic string model are given by

\[
\begin{align*}
v_1 &= 1 = \{ \psi^{1,2}_\mu, \chi^{1,...,6}, y^{1,...,6}, \omega^{1,...,6}, \bar{\omega}^{1,...,6}, \bar{\psi}^{1,...,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,...,8} \}, \\
v_2 &= S = \{ \psi^\mu, \chi^{12}, \chi^{34}, \chi^{56} \}, \\
v_{2+i} &= e_i = \{ y^i, \omega^i | \bar{y}^i, \bar{\omega}^i \}, \quad i = 1, \ldots, 6, \\
v_9 &= b_1 = \{ \chi^{34}, \chi^{56}, y^{34}, y^{56}, \bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^{1}, \bar{\psi}^{1,...,5} \}, \\
v_{10} &= b_2 = \{ \chi^{12}, \chi^{56}, y^{12}, y^{56}, \bar{y}^{12}, \bar{y}^{56}, \bar{\eta}^{2}, \bar{\psi}^{1,...,5} \}, \\
v_{11} &= z_1 = \{ \bar{\phi}^{1,...,4} \}, \\
v_{12} &= z_2 = \{ \bar{\phi}^{5,...,8} \}, \\
v_{13} &= \alpha = \{ \bar{\psi}^{4,5}, \bar{\phi}^{1,2} \}, \\
v_{14} &= \beta = \{ \bar{\psi}^{4,5} = \frac{1}{2}, \bar{\phi}^{1,...,6} = \frac{1}{2} \}
\end{align*}
\]

where the space-time vector bosons are obtained solely from the untwisted sector and generate the observable and hidden gauge symmetries, given by:

\[
\begin{align*}
\text{observable :} \quad & SU(4) \times SU(2) \times U(1) \times U(1)^3 \\
\text{hidden :} \quad & SU(2) \times U(1) \times SU(2) \times U(1) \times SU(2) \times U(1) \times SO(4)
\end{align*}
\]

In order to preserve the aforementioned observable and hidden gauge groups, all the additional spacetime vector bosons need to be projected out which can arise from
the following 36 sectors as enhancements:

$$\begin{align*}
&z_1, \quad z_1 + \beta, \quad z_1 + 2\beta, \\
z_1 + \alpha, \quad z_1 + \alpha + \beta, \quad z_1 + \alpha + 2\beta, \\
z_2, \quad z_2 + \beta, \quad z_2 + 2\beta, \\
z_2 + \alpha, \quad z_2 + \alpha + \beta, \quad z_2 + \alpha + 2\beta, \\
z_1 + z_2, \quad z_1 + z_2 + \beta, \quad z_1 + z_2 + 2\beta, \\
z_1 + z_2 + \alpha, \quad z_1 + z_2 + \alpha + \beta, \quad z_1 + z_2 + \alpha + 2\beta, \\
\beta, \quad 2\beta, \quad \alpha, \\
\alpha + \beta, \quad \alpha + 2\beta, \quad x, \\
z_1 + x + \beta, \quad z_1 + x + 2\beta, \quad z_1 + x + \alpha, \\
z_1 + x + \alpha + \beta, \quad z_2 + x + \beta, \quad z_2 + x + \alpha + \beta, \\
z_1 + z_2 + x + \beta, \quad z_1 + z_2 + x + 2\beta, \quad z_1 + z_2 + x + \alpha + \beta, \\
x + \beta, \quad x + \alpha, \quad x + \alpha + \beta,
\end{align*}$$

where $x = \{\psi^{1,2,3}, \eta^{1,2,3}\}$. The conclusion was reached that the $SU(421)$ class of models is the only class that is excluded in vacua with symmetric internal boundary conditions.

### 4.2 The Free Fermionic LRSz Model Gauge Group

In this section, we present the LRS model constructed using the free-fermionic construction with one $z$ basis vector. This model is generated by the following set of basis vectors:

$$v_1 = S = \{\psi^\mu, \chi^{12}, \chi^{34}, \chi^{56}\},$$

$$v_{1+i} = e_i = \{y^i, \omega^i, |\eta, \overline{\omega}\rangle\}, \quad i = 1, \ldots, 6,$$

$$v_8 = b_1 = \{\chi^{34}, \chi^{56}, y^{34}, y^{56}, g^{34}, g^{56}, \eta^1, \psi^{1,2,3}\},$$

$$v_9 = b_2 = \{\chi^{12}, \chi^{56}, y^{12}, y^{56}, g^{12}, g^{56}, \eta^2, \psi^{1,2,3}\},$$

$$v_{10} = z = \{\phi^{1,2,3}\},$$

$$v_{11} = \alpha = \{\psi^{4,5}, \phi^{1,2}\},$$

$$v_{12} = \beta = \{\eta^{1,2,3} = \frac{1}{2}, \psi^{1,2,3} = \frac{1}{2}, \phi^{1,2} = \frac{1}{2}, \phi^{3,4}\}$$

where

$$1 = S + \sum_{i=1}^{6} e_i + \alpha + 2\beta + z,$$

$$x = \alpha + 2\beta,$$

$$b_3 = b_1 + b_2 + x.$$
The space-time vector bosons are obtained solely from the untwisted sector and generate the following observable gauge symmetries, given by:

observable : \( SU(3) \times SU(2)_L \times SU(2)_R \times U(1) \times U(1)^3 \)

hidden : \( SU(2) \times U(1) \times SO(4) \times SO(8) \)

### 4.3 The Free Fermionic LRS2z Model Gauge Group

In this section, we present the LRS model constructed using the free-fermionic construction where \( z_i \) basis vectors are utilized for \( i = 1, 2 \). This model is generated by the following set of basis vectors:

\[
\begin{align*}
v_1 &= 1 = \{ \psi_\mu^{1,2}, \chi^{1,...,6}, y^{1,...,6}, \omega^{1,...,6}, \bar{\psi}^{1,...,5}, \bar{\eta}^{1,2,3}, \bar{\phi}^{1,...,8} \}, \\
v_2 &= S = \{ \psi^\mu, \chi^{12}, \chi^{34}, \chi^{56} \}, \\
v_{2+i} &= e_i = \{ y^i, \omega^i, \bar{y}^i, \bar{\omega}^i \}, \quad i = 1, \ldots, 6, \\
v_9 &= b_1 = \{ \chi^{34}, \chi^{56}, y^{34}, y^{56}, \bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^{1,...,5} \}, \\
v_{10} &= b_2 = \{ \chi^{12}, \chi^{56}, y^{12}, y^{56}, \bar{y}^{12}, \bar{y}^{56}, \bar{\eta}^{2,...,5} \}, \\
v_{11} &= z_1 = \{ \bar{\phi}^{1,...,4} \}, \\
v_{12} &= z_2 = \{ \bar{\phi}^{5,...,8} \}, \\
v_{13} &= \alpha = \{ \bar{\psi}^{4,5}, \bar{\phi}^{1,2} \}, \\
v_{14} &= \beta = \{ \bar{\eta}^{1,2,3} = \frac{1}{2}, \bar{\psi}^{1,...,3} = \frac{1}{2}, \bar{\phi}^{1,2} = \frac{1}{2}, \bar{\phi}^{3,4} \}
\end{align*}
\]

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hidden : \( SU(2) \times U(1)^3 \times SO(8) \)

### 5 Descending To D=2

In this section, compactifying the heterotic string to two dimensions, we find that the two dimensional free fermions in the light-cone gauge are the real left-moving fermions

\( \chi^i, y^i, \omega^i, \quad i = 1, ..., 8, \)

the real right-moving fermions

\( \bar{\eta}^i, \bar{\phi}^i, \quad i = 1, ..., 8 \)
and the complex right-moving fermions
\[ \overline{\psi}^A, \quad A = 1, \ldots, 4, \quad \overline{\eta}^B, \quad B = 0, \ldots, 3, \quad \overline{\phi}^\alpha, \quad \alpha = 1, \ldots, 8. \]

The class of models we consider will be generated by a maximal set of 7 basis vectors defined as
\[
\begin{align*}
v_1 &= 1 = \{ \chi^i, y^i, \omega^i | \psi^A, \overline{\eta}^B, \overline{\phi}^\alpha \}, \\
v_2 &= H_L = \{ \chi^i, y^i, \omega^i \}, \\
v_3 &= z_1 = \{ \overline{\phi}^{5 \ldots 8} \}, \\
v_4 &= z_2 = \{ \overline{\phi}^{5 \ldots 8} \}, \\
v_5 &= z_3 = \{ \overline{\psi}^A \}, \\
v_6 &= z_4 = \{ \overline{\eta}^B \}, \\
v_7 &= z_5 = \{ y^{1 \ldots 4}, \overline{\omega}^{1 \ldots 4} \}
\end{align*}
\]

where
\[ z_6 = 1 + H_L + \sum_{i=1}^{5} z_i = \{ \overline{\eta}^{5 \ldots 8}, \overline{\omega}^{5 \ldots 8} \} = \{ \overline{\phi}^{5 \ldots 8} \}. \]

The set of GGSO phases is given by
\[
\begin{pmatrix}
1 & H_L & z_1 & z_2 & z_3 & z_4 & z_5 \\
1 & -1 & -1 & +1 & +1 & +1 & +1 \\
-1 & +1 & +1 & -1 & -1 & -1 & -1 \\
-1 & -1 & +1 & -1 & -1 & -1 & -1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1 \\
+1 & +1 & +1 & +1 & +1 & +1 & +1
\end{pmatrix}
\]

or simply
\[
-C\left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = -C\left( \begin{pmatrix} 1 \\ H_L \end{pmatrix} \right) = C\left( \begin{pmatrix} 1 \\ z_i \end{pmatrix} \right) = -C\left( \begin{pmatrix} H_L \\ H_L \end{pmatrix} \right) = -C\left( \begin{pmatrix} H_L \\ z_i \end{pmatrix} \right) = C\left( \begin{pmatrix} z_i \\ z_j \end{pmatrix} \right) = 1
\]

yielding the untwisted symmetry
\[ SO(8)_1 \times SO(8)_2 \times SO(8)_3 \times SO(8)_4 \times SO(8)_5 \times SO(8)_6. \]

Here our focus was on the \( SO(48) \) and the dedicated GGSO phases were chosen appropriately as the following table highlights:
6 Normed Division Algebras

In this section, we briefly discuss the normed division algebras. An algebra $A$ is a vector space equipped with a bilinear multiplication rule and a unit element. We call $A$ a division algebra if, given $x, y \in A$ with $xy = 0$, then either $x = 0$ or $y = 0$. A normed division algebra is an algebra $A$ equipped with a positive-definite norm satisfying the condition

$$||xy|| = ||x|| ||y||$$

which also implies $A$ is a division algebra. There is a remarkable theorem due to Hurwitz [18], which states that there are only four normed division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. The algebras have dimensions $n = 1, 2, 4$ and 8, respectively. They can be constructed, one-by-one, by use of the Cayley-Dickson doubling method, starting with the reals; the complex numbers are pairs of real numbers equipped with a particular multiplication rule, a quaternion is a pair of complex numbers and an octonion is a pair of quaternions.

There is a Lie algebra associated with the division algebras [19] known as the triality algebra of $A$ defined as follows

$$\text{tri}(A) = \{(A, B, C) | A(xy) = B(x)y + xC(y), \ A, B, C \in \mathfrak{so}(A), \ x, y \in A\}$$

where $\mathfrak{so}(A)$ is the norm-preserving algebra isomorphic to $\mathfrak{so}(n)$ where $n = \dim A$. We are interested primarily in the case where

$$\text{tri}(\mathbb{O}) = \mathfrak{so}(8).$$

The division algebras subsequently can be used to describe field theory in Minkowski space using the Lie algebra isomorphism

$$\mathfrak{so}(1, 1 + n) \cong \mathfrak{sl}(2, A)$$

particularly

$$\mathfrak{so}(1, 9) \cong \mathfrak{sl}(2, \mathbb{O}).$$

Table 1: The configuration of the symmetry groups.

| $C(H_L)$ | $C(z_i)$ | Gauge Group |
|---------|---------|-------------|
| $-$     | $+$     | $SO(48)$   |
7 Discussion

In the free fermionic methodology the equivalence of the $8_v$, $8_s$ and $8_c$, $SO(8)$ representations is referred to as the triality structure. This equivalence then enables twisted constructions of the $E_8 \times E_8$ or $SO(32)$ gauge groups. The root lattice of $SO(8)$ has a quaternionic description given by the set

$$ V = \left\{ \pm 1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \right\} $$

which give the required 24 roots. Alternatively, the root lattice of $SO(8)$ could have been composed from $SU(2)^4$. On the other hand, the decomposition of the adjoint representation of $E_8$ under $SO(8) \times SO(8)$ is given by

$$ 248 = (28,1) + (1,28) + (8_v,8_v) + (8_s,8_s) + (8_c,8_s). $$

The weights of the vectorial representation $8_v$ are

$$ V_1 = \left\{ \frac{1}{2}(\pm 1 \pm e_1), \frac{1}{2}(\pm e_2 \pm e_3) \right\}, $$

the weights of the conjugate spinor representation $8_c$ are

$$ V_2 = \left\{ \frac{1}{2}(\pm 1 \pm e_2), \frac{1}{2}(\pm e_3 \pm e_1) \right\}, $$

and the weights of the spinor representation $8_s$ are

$$ V_3 = \left\{ \frac{1}{2}(\pm 1 \pm e_3), \frac{1}{2}(\pm e_1 \pm e_2) \right\}. $$

This description makes the triality of $SO(8)$ manifest. It can be easily seen that permutations of the three imaginary elements $e_1$, $e_2$ and $e_3$ will map the representations $V_1 \rightarrow V_2 \rightarrow V_3$. In [20] an explicit correspondence between simple super-Yang-Mills and classical superstrings in dimensions 3, 4, 6, 10 and the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ was established.
Here, we speculated about the fundamental role being played by the $SO(8)$ group representations, displaying the triality structure, which necessarily arise in models constructed under the free fermionic methodology being remnants of the higher-dimensional triality algebra, namely

$$\text{tri}(\mathcal{O}) = \mathfrak{so}(8).$$

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