Why is the mission impossible? – Decoupling the mirror Ginsparg-Wilson fermions in the lattice models for two-dimensional abelian chiral gauge theories

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Abstract: It has been known that the four-dimensional abelian chiral gauge theories of an anomaly-free set of Weyl fermions can be formulated on the lattice preserving the exact gauge invariance and the required locality property in the framework of the Ginsparg-Wilson relation. This holds true in two dimensions. However, in the related formulation including the mirror Ginsparg-Wilson fermions and therefore having the simpler fermion path-integral measure, it has been argued that the mirror fermions do not decouple: in the 345 model with Dirac- and Majorana-Yukawa couplings to XY-spin field, the two-point vertex function of the (external) gauge field in the mirror sector shows a singular non-local behavior in the PMS phase. We re-examine why the attempt seems a “Mission Impossible” in the 345 model. We point out that the effective operators to break the fermion number symmetries (’t Hooft operators plus others) in the mirror sector do not have sufficiently strong couplings even in the limit of large Majorana-Yukawa couplings. We also observe that the type of Majorana-Yukawa term considered there is singular in the large limit due to the nature of the chiral projection of the Ginsparg-Wilson fermions, but a slight modification without such singularity is allowed by virtue of the very nature. We then consider a simpler four-flavor axial gauge model, the $1^4(-1)^4$ model, in which the $U(1)_A$ gauge and $\text{Spin}(6)(\cong SU(4))$ global symmetries prohibit the bilinear terms, but allow the quartic terms to break all the other continuous mirror-fermion symmetries. We formulate the model so that it is well-behaved and simplified in the strong-coupling limit of the quartic operators. Through Monte-Carlo simulations in the weak gauge coupling limit, we show a numerical evidence that the two-point vertex function of the gauge field in the mirror sector shows a regular local behavior, and we still argue that all you need is killing the continuous mirror-fermion symmetries with would-be gauge anomalies non-matched. Finally, by gauging a $U(1)$ subgroup of the $U(1)_A \times \text{Spin}(6)(SU(4))$ of the previous model, we formulate the $21(-1)^3$ chiral gauge model and argue that the induced fermion measure term satisfies the required locality property and provides a solution to the reconstruction theorem. This gives us “A New Hope” for the mission to be accomplished.

Keywords: lattice gauge theory, chiral symmetry
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1 Introduction

Chiral gauge theories have several interesting possibilities in their own dynamics: fermion number non-conservation due to chiral anomaly[1, 2], various realizations of the gauge symmetry and global flavor symmetry[3, 4], the existence of massless composite fermions suggested by ’t Hooft’s anomaly matching condition[5], the classical scale invariance and the vanishing vacuum energy[6, 7] and so on. Unfortunately, little is known so far about the actual behavior of chiral gauge theories beyond perturbation theory. It is desirable to develop a formulation to study the non-perturbative dynamics of chiral gauge theories.

Lattice gauge theory can provide a framework for non-perturbative formulation of chiral gauge theories, despite the well-known problem of the species doubling [8–11]. A clue to this development is the construction of local and gauge-covariant lattice Dirac operators satisfying the Ginsparg-Wilson relation[16–21].

\[ \gamma_5 D + D \gamma_5 = 2aD \gamma_5 D. \] (1.1)

An explicit example of such lattice Dirac operator is given by the overlap Dirac operator [17, 19], which was derived by Neuberger from the overlap formalism [22–35]. By the Ginsparg-Wilson relation, it is possible to realize an exact chiral symmetry on the lattice[44] in the manner consistent with the no-go theorem.[45–49] It is also possible to introduce Weyl fermions on the lattice and this opens the possibility to formulate anomaly-free chiral gauge theories on the lattice[50–68]. In the case of U(1) chiral gauge theories, Lüscher[50] proved rigorously that it is possible to construct the fermion path-integral measure which depends smoothly on the gauge field and fulfills the fundamental requirements such as

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1 See [13, 15, 52, 53] for the recent reviews on this subject.
2 The overlap formula was derived from the five-dimensional approach of domain wall fermion proposed by Kaplan[36]. In the vector-like formalism of domain wall fermion by Shamir[37–40], the local low energy effective action of the chiral mode is precisely given by the overlap Dirac operator [41–43].
locality, gauge-invariance and lattice symmetries. This gauge-invariant construction holds true in two dimensions.

In two dimensions, the target theories are nothing but chiral Schwinger models of the sets of left- and right-handed Weyl fermions satisfying the anomaly-free conditions on the U(1) charges, \( \sum_\alpha (q^\alpha_L)^2 = \sum_{\alpha'} (q^\alpha_R)^2 \). The models are super-renormalizable and essentially solvable in the continuum limit\(^7\). The effective action induced by the Weyl fermions can be obtained exactly in the continuum limit (i.e. taking the infinite UV cutoff limit \( \Lambda \to \infty \) and neglecting the irrelevant higher-order terms) up to a regularization-dependent and gauge-noninvariant relevant term. The total effective action of the U(1) gauge field is obtained exactly (in the Euclidean spacetime) as

\[
S_{\text{eff}} = \int d^2x \left\{ \frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \frac{1}{2} \sum_\alpha (q^\alpha_L)^2 e^2 \frac{c}{4\pi} A_\mu \left[ c \delta_{\mu \nu'} - (\delta_{\mu \nu'} + i\epsilon_{\mu \nu'}) \frac{\partial \nu'}{\Box} (\delta_{\nu \nu'} - i\epsilon_{\nu \nu'}) \right] A_{\nu'} + \frac{1}{2} \sum_{\alpha'} (q^\alpha_R)^2 e^2 \frac{c}{4\pi} A_\mu \left[ c \delta_{\mu \nu'} - (\delta_{\mu \nu'} - i\epsilon_{\mu \nu'}) \frac{\partial \nu'}{\Box} (\delta_{\nu \nu'} + i\epsilon_{\nu \nu'}) \right] A_{\nu'} \right\}, \tag{1.2}
\]

where \( c \) is the regularization-dependent constant and, when \( c = 1 \), the effective action is gauge-invariant. Due to the one-loop correction of the massless fermions, the U(1) gauge boson acquires the mass (square) as

\[
m^2 = \left[ \sum_\alpha (q^\alpha_L)^2 + \sum_{\alpha'} (q^\alpha_R)^2 \right] \frac{e^2}{2\pi}.
\]

In this respect, it is known in the continuum theory that there is no gauge-invariant regularization method for chiral gauge theories in general and even in two-dimensions. Then, these two-dimensional theories are nice testing grounds for the attempts/approaches to seek the exactly gauge-invariant and nonperturbative formulation of chiral gauge theories, where one can compare the outcomes of the attempts/approaches with the exact results of the target continuum theories.

The above lattice construction\(^5\) gives a gauge-invariant non-perturbative regularization of the chiral Schwinger models in the framework of lattice gauge theory for all possible topological sectors of the U(1) gauge field in two-dimensions. One can verify in the (classical) continuum limit \( a (\equiv \pi/\Lambda) \to 0 \) that this lattice formulation reproduces the exact results of the target continuum theories, eq. (1.2), with \( c = 1 \). And there are good reasons to believe that the lattice model (i.e. keeping the lattice spacing \( a \) finite) has a simple phase-structure of the single phase with the massive U(1) gauge boson, where the second-order critical point is given at the vanishing gauge-coupling constant, \( ga = 0 \) (in unit of the lattice spacing \( a \)).

\(^3\)For generic non-abelian chiral gauge theories, the construction in all orders of the weak gauge-coupling expansion was given by Suzuki\(^57, 58\) and by Lüscher\(^59\).

\(^4\)In this formulation of U(1) chiral lattice gauge theories\(^50\), although the proof of the existence of the fermion measure is constructive, the resulted formula of the fermion measure turns out to be rather complicated for the case of the finite-volume lattice. It also relies on the results obtained in the infinite lattice. Therefore it does not provide a formulation which is immediately usable for numerical applications. See \[64–67\] for a simplified formulation toward a practical implementation.

\(^5\)This construction was extended to the SU(2)×U(1) chiral gauge theory of the Glashow-Weinberg-Salam model\(^69–71\) based on the pseudo reality and anomaly-free condition of SU(2) by Kadoh and the author\(^68\).
However, in the related formulation by Poppitz and his collaborators\cite{73–80} which includes the mirror degrees of freedom\cite{81–91} in terms of Ginsparg-Wilson fermions and therefore has the simpler fermion path-integral measure, it was argued that the mirror fermions do not decouple: in the two-dimensional 345 model with Dirac- and Majorana-Yukawa couplings to XY-spin field, the two-point vertex function of the (external) $U(1)$ gauge field in the mirror sector shows a singular non-local behavior in the paramagnetic strong-coupling phase\cite{74, 92, 93}.\(^6\) The singular non-local term turns out to be same as the contribution of the massless Weyl fermions of the target sector. It implies that the $U(1)$ gauge boson acquires twice as large as the mass square expected in the target chiral Schwinger model. This result seems puzzling because the Dirac- and Majorana-Yukawa couplings can break two “would-be anomalous” global $U(1)$ symmetries in the mirror sector, that is the required condition for the decoupling of the mirror fermions, as claimed by Eichten and Preskill \cite{73, 94, 107, 108}. In their numerical simulations, though, a specific limit of large Majorana-Yukawa couplings was taken to tame the sign problem of their lattice model.

On the other hand, this question of decoupling the mirror degrees of freedoms in the 345 model was also studied by Wang and Wen\cite{110} from the point of view of the Hamiltonian construction based on Topological Insulators/Superconductors\cite{109–113}: based on the effective bosonic (bosonized) description of the 2D Chern Insulator by the bulk Chern-Simon gauge theory and the boundary chiral-boson theory with sine-Goldon couplings, it was shown that the boundary phase can be fully gapped in the 345(0) model by the two sine-Goldon couplings required precisely to break the two “would-be anomalous” global $U(1)$ symmetries in the mirror sector. This result suggests that the mirror fermions can be decoupled by the suitable choice of the coupling strengths of the symmetry-breaking interactions.

In this paper, we re-examine why the attempt seems a “mission impossible” for the 345 model in the mirror-fermion approach with Ginsparg-Wilson fermions\cite{73–80}. We point out that the effective operators to break the fermion number symmetries (’t Hooft operators plus others) in the mirror sector do not have sufficiently strong couplings even in the limit of large Majorana-Yukawa couplings. We also observe that the type of Majorana-Yukawa term considered there is singular in the large limit due to the nature of the chiral projection of the Ginsparg-Wilson fermions, but a slight modification without such singularity is allowed by virtue of the very nature. Based on these results, we argue that one can attribute the failure of decoupling to the singular Majorana-Yukawa terms of the lattice model and may expect a better result by modifying the Majorana-Yukawa terms so that they are well-behaved in the strong-coupling limit.

We then consider a simpler four-flavor axial gauge model, the $1^4(-1)^4$ model, in which the $U(1)_{A}$ gauge and Spin(6)(SU(4)) global symmetries prohibit bilinear mass terms, but allow the quartic terms to break the other continuous mirror-fermion symmetry $U(1)_{V}$. We formulate the model so that it is well-behaved and simplified in the strong-coupling limit.

\(^6\)See \cite{94–106} for the former attempts to decouple the species-doubler/mirror-fermions by strong Yukawa and multi-fermion interactions.
of the quartic operators. Through Monte-Carlo simulations in the weak gauge coupling limit, we show a numerical evidence that the two-point vertex function of the $U(1)_A$ gauge field in the mirror sector shows a regular local behavior, consistently with the decoupling of the mirror-fermions, and we argue that still all you need is killing the (continuous) mirror-fermion symmetries with would-be gauge anomalies non-matched, as originally claimed by Eichten and Preskill[73, 94, 110].

Finally, we formulate the $21(-1)^3$ chiral gauge model by gauging a $U(1)$ subgroup of the $U(1)_A \times \text{Spin}(6)(SU(4))$ of the previous model. We show again a numerical evidence that the two-point vertex function of the $U(1)$ gauge field in the mirror sector shows a regular local behavior through Monte-Carlo simulations in the weak gauge coupling limit. We then deduce a definition of the (target) Weyl-field measure of the $21(-1)^3$ chiral gauge model, where the mirror-fermion part of the Dirac-field measure is just saturated by the suitable products of the 't Hooft vertices in terms of the mirror-fermion fields. Based on the results of Monte-Carlo simulations, we argue that the induced fermion measure term satisfies the required locality property and provides a solution to the reconstruction theorem of the Weyl field measure in the framework of the Ginsparg-Wilson relation[50]. This result gives us a new hope for the mission to be accomplished.

The mirror-fermion models formulated with overlap fermions in this paper, the $1^4(-1)^4$- and $21(-1)^3$-models, can be also constructed through the 2+1D vector-like domain wall fermion by adding suitable boundary interaction terms. We give the explicit form of the boundary terms which precisely reproduce the $U(1)_A \times \text{Spin}(6)(SU(4))$-invariant multi-fermion interaction in the mirror sector without the singularity in the large-coupling limit (cf. [114]).

The four-flavor model with the $U(1)_A \times \text{Spin}(6)(SU(4))$-invariant multi-fermion interaction, which we adopt for the mirror-fermion sector, is closely related to the eight-flavor 1D Majorana chain with the SO(7)-invariant quartic interaction (1D TSC with time-reversal symmetry; class BDI in 1D classified by $Z_8(\leftarrow Z))$[115–117]. It also resembles the SU(4)/SO(4)-invariant reduced-staggered-fermion models in 3+1, 2+1, 1+1D, which are used in the recent studies of “mass without symmetry breaking” [118–125]. We clarify this relation by formulating the quantum 1D Majorana chain as the classical 1+1D Majorana-fermion model in Euclidean metric through the path-integral quantization. By this relation, the rigorous result about the mass gap of the eight-flavor 1D Majorana chain with the SO(7)-invariant quartic interaction by Fidkowski and Kitaev[115] and its extension to the model with the reduced SO(6) symmetry by Y.-Z. You and C. Xu [112] suggest strongly that the four-flavor axial model with $U(1)_A \times \text{Spin}(6)(SU(4))$ symmetry is indeed gapped. And vise versa: our numerical-simulation results that the correlation lengths of the mirror-sector fields are of order multiple lattice spacings provide a numerical evidence for the mass gap of the eight-flavor 1D Majorana chain based on the framework of 1+1D Euclidean path-integral quantization.

\footnote{We hope that the formulation of the four-flavor model with the $U(1)_A \times \text{Spin}(6)(SU(4))$-invariant multi-fermion interaction in terms of overlap fermions and its use for the construction of chiral lattice gauge theories given in this paper provides an answer to the question raised by Catterall and his collaborators[123, 125].}
Since the 2+1D domain wall fermion is nothing but 2+1D classical formulation of a 2D Topological Insulator\[126–128\] (Chern Insulator/IQHE without time-reversal symmetry; class A in 2D classified by $\mathbb{Z}$), our result here provides the explicit procedure to bridge between the two constructions for 1+1D chiral gauge theories, the 2+1D classical construction of domain wall fermion with boundary interactions to decouple the mirror-modes[114] and the 2D quantum Hamiltonian construction of TI/TSC with gapped boundary phases[109–113]. And the mirror-fermion model in terms of overlap fermions is obtained precisely as the 1+1D low-energy effective local lattice theory, and it can describe directly the gapless/gapped boundary phases. We illustrate this relation for the case of the eight-flavor 2D chiral p-wave TSC with time-reversal and $\mathbb{Z}_2$ symmetries (class D'/DIII+R in 2D classified by $\mathbb{Z}_8(\leftarrow \mathbb{Z})$)[129–132]. This connection should hold true in lower and higher dimensions. In particular, it would be useful to examine the Hamiltonian constructions of 3+1D chiral gauge theories based on the 4D TI/TSC with the “proposed” gapped boundary phases[109, 111, 112] from the point of view of the 3+1D/4+1D Euclidean construction based on the overlap/domain wall fermions.

This paper is organized as follows. In section 2, we first review the construction of U(1) chiral lattice gauge theories based on the Ginsparg-Wilson relation, adapted for two-dimensional theories. In section 3, we next review the mirror-fermion approach with Ginsparg-Wilson fermions to two-dimensional U(1) chiral gauge theories. Section 4 is devoted to the re-examination of the 345 model with Dirac- and Majorana-Yukawa couplings to XY-spin field. In section 5, we introduce the $1^4(-1)^3$ model and discuss its properties in detail. In section 6, we formulate the $21(-1)^3$ model and discuss its properties in relation to the reconstruction theorem reviewed in section 2. In section 7, the mirror-fermion models introduced in sections 5 and 6, the $1^4(-1)^4$- and $21(-1)^3$- models, are constructed through the 2+1D vector-like domain wall fermion by adding the suitable boundary interaction terms. In section 8, we discuss the relations of the 1+1D/2+1D Euclidean formulation of mirror-fermion/domain wall-fermion models to the 1D/2D quantum Hamiltonian construction of TI/TSC with gapped boundary phases. In the final section 9, we conclude with a few discussions.

2 Two-dimensional abelian chiral gauge theories on the lattice in the framework of the Ginsparg-Wilson relation

In this section, we review the construction of U(1) chiral lattice gauge theories based on the Ginsparg-Wilson relation [50], adapted for the two-dimensional theories.

We consider U(1) gauge theories where the gauge field couples to $N$ left-handed Weyl fermions with charges $q_\alpha$ and $N'$ right-handed Weyl fermions with charges $q'_\alpha$ satisfying the anomaly cancellation condition,

$$\sum_{\alpha=1}^{N} (q_\alpha)^2 - \sum_{\alpha'=1}^{N'} (q'_{\alpha'})^2 = 0. \quad (2.1)$$

We assume the two-dimensional lattice of the finite size $L$ and choose lattice units,

$$\Gamma = \{ x = (x_1, x_2) \in \mathbb{Z}^2 \mid 0 \leq x_{\mu} < L (\mu = 1, 2) \}, \quad (2.2)$$
and adopt the periodic boundary condition for both boson fields and fermion fields.

### 2.1 Gauge fields

We adopt the compact formulation of U(1) gauge theory on the lattice. U(1) gauge fields on $\Gamma$ then are represented by link fields, $U(x, \mu) \in U(1)$. We require the so-called admissibility condition on the gauge fields:

$$|F_{\mu\nu}(x)| < \epsilon \quad \text{for all } x, \mu, \nu,$$

where the field tensor $F_{\mu\nu}(x)$ is defined from the plaquette variables,

$$F_{\mu\nu}(x) = \frac{1}{i} \ln P_{\mu\nu}(x), \quad -\pi < F_{\mu\nu}(x) \leq \pi,$$

and $\epsilon$ is a fix number in the range $0 < \epsilon < \pi$. This condition ensures that the overlap Dirac operator\cite{17, 19} is a smooth and local function of the gauge field if $|e_\alpha|\epsilon < 2/5$ for all $\alpha$ and $|e'_\alpha|\epsilon < 2/5$ for all $\alpha'$ \cite{21}. The admissibility condition may be imposed dynamically by choosing the following action,

$$S_G = \frac{1}{4g_0^2} \sum_{x \in \Gamma} \sum_{\mu,\nu} L_{\mu\nu}(x),$$

where

$$L_{\mu\nu}(x) = \begin{cases} [F_{\mu\nu}(x)]^2 \left\{ 1 - \frac{|F_{\mu\nu}(x)|^2}{\epsilon^2} \right\}^{-1} & \text{if } |F_{\mu\nu}(x)| < \epsilon, \\ \infty & \text{otherwise}. \end{cases}$$

(2.7)

The admissible U(1) gauge fields can be classified by the magnetic fluxes,

$$m_{\mu\nu} = \frac{1}{2\pi} \sum_{s,t=0}^{L-1} F_{\mu\nu}(x + s\hat{\mu} + t\hat{\nu}),$$

(2.8)

which are integers independent of $x$. We denote the space of the admissible gauge fields with a given magnetic flux $m_{\mu\nu}$ by $\mathcal{U}[m]$. As a reference point in the given topological sector $\mathcal{U}[m]$, one may introduce the gauge field which has the constant field tensor equal to $2\pi m_{\mu\nu}/L^2(\epsilon)$ by

$$V_{[m]}(x, \mu) = e^{-\frac{2\pi i}{L}[L\tilde{x}_\mu - \sum_{\nu > \mu} m_{\mu\nu} \tilde{x}_\nu + \sum_{\nu < \mu} m_{\mu\nu} \tilde{x}_\nu]} \quad (\tilde{x}_\mu = x_\mu \mod L).$$

(2.9)

Then any admissible U(1) gauge field in $\mathcal{U}[m]$ may be expressed as

$$U(x, \mu) = \tilde{U}(x, \mu) V_{[m]}(x, \mu),$$

(2.10)

where $\tilde{U}(x, \mu)$ stands for the dynamical degrees of freedom.
2.2 Weyl fields

Weyl fermions are introduced based on the Ginsparg-Wilson relation. We first consider Dirac fields $\psi(x)$ which carry a Dirac index $s = 1, 2$ and a flavor index $\alpha = 1, \cdots, N$ and $\psi'(x)$ which carry a Dirac index $s' = 1, 2$ and a flavor index $\alpha' = 1, \cdots, N'$. Each components $\psi_\alpha(x)$ and $\psi'_{\alpha'}(x)$ couple to the link fields, $U(x, \mu)q_\alpha$ and $U(x, \mu)q'_{\alpha'}$, respectively. We assume that the lattice Dirac operators $D$ and $D'$ acting on $\psi(x)$ and $\psi'(x)$, respectively satisfy the Ginsparg-Wilson relation\footnote{In this paper, we adopt the normalization of the lattice Dirac operator so that the factor 2 appears in the right-hand-side of the Ginsparg-Wilson relation: $\gamma_3 D + D\gamma_3 = 2D\gamma_3 D$.},

$$\gamma_3 D + D\gamma_3 = 0, \quad \gamma_3 \equiv \gamma_5(1 - 2D),$$

(2.11)

$$\gamma_3 D' + D'\gamma'_3 = 0, \quad \gamma'_3 \equiv \gamma_3(1 - 2D'),$$

(2.12)

and we define the projection operators as

$$P_\pm = \left( \frac{1 \pm \gamma_3}{2} \right), \quad \hat{P}_\pm = \left( \frac{1 \pm \gamma'_3}{2} \right), \quad \hat{P}'_\pm = \left( \frac{1 \pm \gamma'_3}{2} \right).$$

(2.13)

The left-handed and right-handed Weyl fermions can be defined by imposing the constraints,

$$\psi_-(x) = \hat{P}_- \psi(x), \quad \bar{\psi}_-(x) = \bar{\psi}(x) P_+, \quad \psi'_+(x) = \hat{P}'_+ \psi'(x), \quad \bar{\psi}'_+(x) = \bar{\psi}'(x) P_-.$$ 

(2.14)

(2.15)

The action of the left-handed Weyl fermions is then given by

$$S_W = \sum_{x \in \Gamma} \bar{\psi}_-(x) D\psi_-(x) + \sum_{x \in \Gamma} \bar{\psi}'_+(x) D'\psi'_+(x).$$

(2.16)

The kernel of the lattice Dirac operator in finite volume, $D (D')$, may be represented through the kernel of the lattice Dirac operator in infinite volume, $D_\infty$, as follows:

$$D(x, y) = D(x, y)_\infty + \sum_{n \in \mathbb{Z}^4, n \neq 0} D_\infty(x, y + nL),$$

(2.17)

where $D_\infty(x, y)$ is defined with a periodic link field in infinite volume. We assume that $D_\infty(x, y)$ possesses the locality property given by

$$\|D_\infty(x, y)\| \leq C(1 + \|x - y\|^p) e^{-\|x-y\|/\varrho}$$

(2.18)

for some constants $\varrho > 0$, $C > 0$, $p \geq 0$, where $\varrho$ is the localization range of the lattice Dirac operator.

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(2.18)

for some constants $\varrho > 0$, $C > 0$, $p \geq 0$, where $\varrho$ is the localization range of the lattice Dirac operator.
2.3 Path-integral measure of the Weyl fermions

The path-integral measure of the Weyl fermions may be defined by the Grassmann integrations,

\[ D[\psi_-]D[\bar{\psi}_-]D[\psi_+]D[\bar{\psi}_+] = \prod_j dc_j \prod_k d\bar{c}_k \prod_j dc'_j \prod_k d\bar{c}'_k, \]  

(2.19)

where \( \{c_j\}, \{\bar{c}_k\} \) and \( \{c'_j\}, \{\bar{c}'_k\} \) are the Grassmann coefficients in the expansion of the Weyl fields,

\[ \psi_-(x) = \sum_j v_j(x)c_j, \quad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x) \]  

(2.20)

\[ \psi'_+(x) = \sum_j u'_j(x)c'_j, \quad \bar{\psi}'_+(x) = \sum_k \bar{c}'_k \bar{u}'_k(x) \]  

(2.21)

in terms of the chiral (orthonormal) bases defined by

\[ \hat{P}_-v_j(x) = v_j(x), \quad \bar{v}_k(x)P_+ = \bar{v}_k(x), \]  

(2.22)

\[ \hat{P}'_+u'_j(x) = u'_j(x), \quad \bar{u}'_k(x)P_- = \bar{u}'_k(x). \]  

(2.23)

Since the projection operators \( \hat{P}_- \) and \( \hat{P}'_+ \) depend on the gauge field through \( D \) and \( D' \), respectively, the fermion measure also depends on the gauge field. In this gauge-field dependence of the fermion measure, there is an ambiguity by a pure phase factor, because any unitary transformations of the bases,

\[ \hat{v}_j(x) = \sum_l v_l(x) (Q^{-1})_{lj}, \quad \hat{\bar{c}}_j = \sum_l Q_{jl} \bar{c}_l, \]  

(2.24)

\[ \hat{u}'_j(x) = \sum_l u'_l(x) (Q'^{-1})_{lj}, \quad \hat{\bar{c}}'_j = \sum_l Q'^{jl} \bar{c}'_l, \]  

(2.25)

induces a change of the measure by the pure phase factor \( \det Q \times \det Q' \). This ambiguity should be fixed so that it fulfills the fundamental requirements such as locality, gauge-invariance, integrability and lattice symmetries.

2.4 Reconstruction theorem of the fermion measure

The effective action induced by the path-integration of the Weyl fermions is given by

\[ \Gamma[U] = \ln \{\det(\bar{v}_kDv_j) \det(\bar{u}'_kD'u'_j)\}. \]  

(2.26)

Its variation with respect to the gauge field, \( \delta_\eta U(x, \mu) = i\eta_\mu(x)U(x, \mu) \), reads

\[ \delta_\eta \Gamma[U] = \text{Tr} \{P_+\delta_\eta D D^{-1} \} + \sum_j (v_j, \delta_\eta v_j) \]

\[ + \text{Tr} \{P_-\delta_\eta D' D'^{-1} \} + \sum_j (u'_j, \delta_\eta u'_j). \]  

(2.27)

Then the properties of the fermion measure can be characterized by the so-called measure term which is given in terms of the chiral basis and its variation with respect to the gauge field as

\[ \mathcal{L}_\eta = i \sum_j (v_j, \delta_\eta v_j) + i \sum_j (u'_j, \delta_\eta u'_j). \]  

(2.28)
The reconstruction theorem given in [50] asserts that if there exists a local current $j_\mu(x)$ which satisfies the following four properties, it is possible to reconstruct the fermion measure (the bases $\{v_j(x)\}, \{u'_j(x)\}$) which depends smoothly on the gauge field and fulfills the fundamental requirements such as locality, gauge-invariance, integrability and lattice symmetries:

**Theorem** Suppose $j_\mu(x)$ is a given current with the following properties:

1. $j_\mu(x)$ is defined for all admissible gauge fields and depends smoothly on the link variables.
2. $j_\mu(x)$ is gauge-invariant and transforms as an axial vector current under the lattice symmetries.
3. The linear functional $\Sigma_\eta = \sum_{x \in \Gamma} \eta_\mu(x) j_\mu(x)$ is a solution of the integrability condition
   \[ \delta_\eta \Sigma_\zeta - \delta_\zeta \Sigma_\eta = i \text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} + i \text{Tr} \left\{ \hat{P}_+ [\delta_\eta \hat{P}_+, \delta_\zeta \hat{P}_+] \right\} \quad (2.29) \]
   for all periodic variations $\eta_\mu(x)$ and $\zeta_\mu(x)$.
4. The anomalous conservation law holds:
   \[ \partial^\ast j_\mu(x) = \text{tr} \{ Q \gamma_5 (1 - D)(x,x) \} - \text{tr} \{ Q' \gamma_5 (1 - D')(x,x) \}, \]
   where $Q = \text{diag}(q_1, \cdots, q_N)$ and $Q' = \text{diag}(q'_1, \cdots, q'_N)$.

Then there exists a smooth fermion integration measure in the vacuum sector such that the associated current coincides with $j_\mu(x)$. The same is true in all other sectors if the number of fermion flavors with $|q_\alpha| = q$ (or $|q'_\alpha| = q$) is even for all odd $q$. In each case the measure is uniquely determined up to a constant phase factor.

In [50], it is proved constructively that there exists a local current $j_\mu(x)$ which satisfies the properties required in the reconstruction theorem. In fact, the construction of the current is not straightforward by two reasons. The first reason is that the locality property of the current must be maintained. The second reason is that the measure term must be smooth w.r.t. the gauge field, but the topology of the space of the admissible gauge fields in finite volume is not trivial. To take these points into account, the construction in [50] is made by separating the part definable in infinite volume from the part of the finite volume corrections. Then, the explicit formula of the measure term turns out to be complicated. Therefore it does not provide a formulation which is immediately usable for practical numerical applications.

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9 We adopt the generalized notion of locality on the lattice given in [21, 46, 50] for Dirac operators and composite fields. See also [64] for the case of the finite volume lattice.

10 The lattice symmetries mean translations, rotations, reflections and charge conjugation.

11 Throughout this paper, $\text{Tr}\{ \cdots \}$ stands for the trace over the lattice index $x$, the flavor index $\alpha(=1, \cdots, N)$ and the spinor index, while $\text{tr}$ stands for the trace over the flavor and spinor indices only. $\text{Tr}_L\{ \cdots \}$ stands for the trace over the finite lattice, $x \in \Gamma$. 

– 9 –
In [67], by formulating the procedure to solve the local cohomology problem of the U(1) gauge (chiral) anomaly within finite volume, a rather explicit formula of the local current $j_\mu(x)$ is derived as

$$\sum_{x \in \Gamma} \eta_\mu(x) j_\mu(x) \equiv \frac{1}{2^{2d}} \sum_{R \in O(2,\mathbb{Z})} \det R \mathcal{L}_\eta^0 |_{U \to \{U\}^{R^{-1}}, \eta_\mu \to \{\eta_\mu\}^{R^{-1}}}, \quad (2.31)$$

where

$$\mathcal{L}_\eta^0 = i \int_0^1 ds \Tr \left\{ \hat{P}_- [\delta_s \hat{P}_-, \delta_\eta \hat{P}_-] \right\} \bigg|_{\hat{A}_\mu \to s \hat{A}_\mu} + \delta_\eta \int_0^1 ds \sum_{x \in \Gamma} \left\{ \tilde{A}_\mu^T(x) k_\mu(x) \right\} + \mathcal{M}_\eta |_{U = U[w] V[m], \eta = \eta[w]}, \quad (2.32)$$

In this formula, the link field $U(x, \mu)$ in $\mathfrak{U}[m]$ is represented as

$$U(x, \mu) = e^{i A^T_\mu(x)} U[w](x, \mu) \Lambda(x) \Lambda(x + \hat{\mu})^{-1} V[m](x, \mu), \quad (2.33)$$

where $A^T_\mu(x)$ is the transverse vector potential in satisfying

$$\partial^\ast_\mu A^T_\mu(x) = 0, \quad \sum_{x \in \Gamma} A^T_\mu(x) = 0, \quad (2.34)$$

$$\partial_\mu A^T_\nu(x) - \partial_\nu A^T_\mu(x) + 2\pi m_{\mu\nu}/L^2 = F_{\mu\nu}(x), \quad (2.35)$$

$U[w](x, \mu)$ represents the degrees of freedom of the Wilson lines,

$$U[w](x, \mu) = \begin{cases} w_\mu & \text{if } x_\mu = L - 1, \\ 1 & \text{otherwise}, \end{cases} \quad (2.36)$$

with the phase factor $w_\mu \in U(1)$ and $\Lambda(x)$ is the gauge function satisfying $\Lambda(0) = 1$. $\tilde{A}_\mu(x)$ is then defined by

$$\tilde{A}_\mu(x) = A^T_\mu(x) - \frac{1}{i} \partial_\mu \left[ \ln \Lambda(x) \right]; \quad \frac{1}{i} \ln \Lambda(x) \in (-\pi, \pi]. \quad (2.37)$$

$k_\mu(x)$ is the gauge-invariant local current which satisfies

$$\partial^\ast_\mu k_\mu(x) = \text{tr}\{Q\gamma_5 (1 - D)(x, x)\} - \text{tr}\{Q'\gamma_5 (1 - D')(x, x)\} \quad (2.38)$$

and transforms as an axial vector field under the lattice symmetries. $\mathcal{M}_\eta |_{U = U[w] V[m], \eta = \eta[w]}$ is the additional measure term at the gauge field $U(x, \mu) = U[w](x, \mu) V[m](x, \mu)$ with the variational parameters in the directions of the Wilson lines, $\eta_\mu[w](x) = \sum_\nu \delta_{\mu\nu} \delta_{x_\nu, L-1} \eta(\nu)$

Using these formula, it is indeed feasible to compute numerically the gauge-field dependence of the Weyl fermion measure in two-dimensions. [66]
3 Mirror-fermion approach with the Ginsparg-Wilson fermions

In this section, we review the mirror-fermion approach[81–91] with the Ginsparg-Wilson fermions[73–80] to lattice models of two-dimensional abelian chiral gauge theories.

In the mirror fermion approach in the framework of the Ginsparg-Wilson fermions, the opposite chirality fields $\psi_+(x)$ and $\psi'_-(x)$ are also considered:

$$\psi_+(x) = \bar{P}_+ \psi(x), \quad \bar{\psi}_+(x) = \bar{\psi}(x) P_-, \quad (3.1)$$

$$\psi'_-(x) = \bar{P}'_- \psi'(x), \quad \bar{\psi}'_-(x) = \bar{\psi}'(x) P_+. \quad (3.2)$$

These fields, which are referred as mirror fermions, are assumed to be dynamical, but to be decoupled by acquiring the masses of order the inverse lattice spacing $1/a$ through the dynamical effect of certain (gauge-invariant) local interactions among the mirror fermion fields and additional auxiliary boson fields. The action of the mirror sector is then given in the form

$$S_M = \sum_{x \in \Gamma} \{ \bar{\psi}_+(x) D \psi_+(x) + \bar{\psi}'_-(x) D' \psi'_-(x) \} + \sum_{x \in \Gamma} V(\psi_+(x), \bar{\psi}_+(x), \psi'_-(x), \bar{\psi}'_-(x), \Phi(x), U(x, \mu)) + \sum_{x \in \Gamma} \kappa |\nabla \Phi(x)|^2, \quad (3.3)$$

where $\Phi(x)$ stands for the additional boson fields collectively, and the total action of the lattice model is assumed to be

$$S_{\text{mirror}} = S_G + S_W + S_M. \quad (3.4)$$

The path-integral measures of the mirror fermion fields may be defined by

$$D[\psi_+] D[\bar{\psi}_+] D[\psi'_-] D[\bar{\psi}'_-] = \prod_j db_j \prod_k db_k \prod_j db'_k \prod_k db''_k, \quad (3.5)$$

where $\{b_j\}$, $\{\bar{b}_k\}$ and $\{b'_k\}$, $\{\bar{b}''_k\}$ are the grassman coefficients in the expansion of the mirror fermion fields,

$$\psi_+(x) = \sum_j u_j(x) b_j, \quad \bar{\psi}_+(x) = \sum_k \bar{b}_k \bar{u}_k(x) \quad (3.6)$$

$$\psi'_-(x) = \sum_j v'_j(x) b'_j, \quad \bar{\psi}'_-(x) = \sum_k \bar{b}'_k \bar{v}'_k(x) \quad (3.7)$$

in terms of the chiral (orthonormal) bases defined by

$$\bar{P}_+ u_j(x) = u_j(x), \quad \bar{u}_k(x) P_- = \bar{u}_k(x), \quad (3.8)$$

$$\bar{P}'_- v'_j(x) = v'_j(x), \quad \bar{v}'_k(x) P_+ = \bar{v}'_k(x). \quad (3.9)$$

On the other hand, since the target Weyl fermions and the mirror fermions now consist the Dirac pairs (in the sense of the Ginsparg-Wilson fermions) as $\psi = \psi_- + \psi_+$, $\psi' = \psi'_+ + \psi'_-$, the path-integral measures of the fermion fields can be defined simply by

$$D[\psi] D[\bar{\psi}] D[\psi'] D[\bar{\psi}'] = \prod_{x,s,\alpha} d\psi_{sa}(x) d\bar{\psi}_{sa}(x) \prod_{x,s',\alpha'} d\psi'_{s'a'}(x) d\bar{\psi}'_{s'a'}(x), \quad (3.10)$$
which are independent of the gauge fields and are manifestly gauge invariant. This fact implies that one can always choose the bases of the Dirac fields, \{u_j(x), v_j(x)\} and \{u'_j(x), v'_j(x)\}, so that the Jacobian factors, \(\det(u_j(x), v_j(x))\) and \(\det(u'_j(x), v'_j(x))\), are independent of the gauge fields:

\[
\sum_j (u_j, \delta_\eta u_j) + \sum_j (v_j, \delta_\eta v_j) = 0,
\]

\[
\sum_j (u'_j, \delta_\eta u'_j) + \sum_j (v'_j, \delta_\eta v'_j) = 0.
\] (3.11)

Adjusting the overall constant phase factors of the Jacobians, one obtain

\[
\mathcal{D}[\bar{\psi}_-] \mathcal{D}[\bar{\psi}_-] \mathcal{D}[\bar{\psi}_+'] \mathcal{D}[\bar{\psi}_+] \times \mathcal{D}[\bar{\psi}_+] \mathcal{D}[\bar{\psi}_-'] \mathcal{D}[\bar{\psi}_-'] = \mathcal{D}[\bar{\psi}] \mathcal{D}[\bar{\psi}] \mathcal{D}[\bar{\psi}'] \mathcal{D}[\bar{\psi}'].
\] (3.12)

This factorization of the path-integral measure, as well as the action, into the target and mirror sectors is the characteristic feature of the mirror Ginspger-Wilson fermion approach.\(^{[75]}\) For later convenience, we introduce the abbreviations for the path-integrations of the parts of the target-sector and the mirror-sector fields as follows:

\[
\langle \mathcal{O}_W \rangle_W \equiv \int \mathcal{D}[\bar{\psi}_-] \mathcal{D}[\bar{\psi}_-] \mathcal{D}[\bar{\psi}_+'] \mathcal{D}[\bar{\psi}_+] e^{-S_W} \mathcal{O}_W,
\] (3.13)

\[
\langle \mathcal{O}_M \rangle_M \equiv \int \mathcal{D}[\bar{\psi}_+] \mathcal{D}[\bar{\psi}_-] \mathcal{D}[\bar{\psi}_-'] \mathcal{D}[\bar{\psi}_-'] e^{-S_M} \mathcal{O}_M,
\] (3.14)

and

\[
\langle \mathcal{O}_{WM} \rangle_{WM} \equiv \int \mathcal{D}[\bar{\psi}] \mathcal{D}[\bar{\psi}] \mathcal{D}[\bar{\psi}'] \mathcal{D}[\bar{\psi}'] e^{-S_W - S_M} \mathcal{O}_{WM}.
\] (3.15)

In the last formula, the result of the path-integration is independent of the choice of the chiral bases. To make this fact clear, \(S_W\) and \(S_M\) may be represented simply with the Dirac fields \(\bar{\psi}(x), \psi'(x)\) as

\[
S_W = \sum_{x \in \Gamma} \bar{\psi}(x) P_+ D \psi(x) + \sum_{x \in \Gamma} \bar{\psi}'(x) P_- D' \psi'(x),
\] (3.16)

\[
S_M = \sum_{x \in \Gamma} \{ \bar{\psi}(x) P_- D \psi(x) + \bar{\psi}'(x) P_+ D' \psi'(x) \}
\]

\[
+ \sum_{x \in \Gamma} \nabla(\hat{P}_+ \psi(x), \bar{\psi}(x) P_-, \hat{P}_- \psi'(x), \bar{\psi}'(x) P_+, \Phi(x), U(x, \mu)) + \sum_{x \in \Gamma} \kappa |\nabla \Phi(x)|^2.
\] (3.17)

With these abbreviations, the factorization means \(\langle \mathcal{O}_W \mathcal{O}_M \rangle_{WM} = \langle \mathcal{O}_W \rangle_W \langle \mathcal{O}_M \rangle_M\).

The effective action induced by the path-integration of the mirror-sector fields as well as the target Weyl fermions is then represented by

\[
\Gamma_{\text{mirror}}[U] = \ln \{ \langle \mathcal{O}_{WM} \rangle \},
\] (3.18)

\(^{[75]}\)These sectors are not completely independent each other with respect to the coupling to the gauge link fields because of the constraints, eqs. (3.11).
Its variation with respect to the gauge field reads

$$
\delta_\eta \Gamma_{\text{mirror}}[U] = \{\langle -\delta_\eta S_W \rangle_{WM} + \langle -\delta_\eta S_M \rangle_{WM} \} / \langle 1 \rangle_{WM} \\
= \langle -\delta_\eta S_W \rangle_{W} / \langle 1 \rangle_{W} + \langle -\delta_\eta S_M \rangle_{M} / \langle 1 \rangle_{M} \\
= \text{Tr}\{P_+ \delta_\eta DD^{-1}\} + \text{Tr}\{P_- \delta_\eta D'D'^{-1}\} + \langle -\delta_\eta S_M \rangle_{M} / \langle 1 \rangle_{M}.
$$

(3.19)

By comparing this result with Eqs. (2.27) and (2.28), one can see that the contribution of the mirror sector, $\langle -\delta_\eta S_M \rangle_{M} / \langle 1 \rangle_{M}$, should play the role of the measure term $\mathcal{L}_\eta$.

If the mirror-sector fields could successfully decouple by acquiring the masses of order $1/a$, all these fields should have the short range correlation lengths of order multiple lattice spacings. Moreover, these fields should leave only local terms of the gauge fields in the induced effective action, according to the decoupling theorem (and from the more general point of view of the Wilsonian renormalization group). This implies that the contribution of the mirror sector, $\langle -\delta_\eta S_M \rangle_{M} / \langle 1 \rangle_{M}$, should be a local function of the gauge fields. In the weak gauge-coupling expansion, the vertex functions are derived from this contribution as

$$
\langle -\delta_\eta S_M \rangle_{M} / \langle 1 \rangle_{M} = \sum_{m=0}^{\infty} \frac{1}{L^{2+2m}} \frac{1}{m!} \sum_{k,p_1,\ldots,p_m} \bar{\eta}_\mu (-k) \Gamma'_{\mu \nu_1,\ldots,\nu_m}(k,p_1,\ldots,p_m) \tilde{A}_{\nu_1}(p_1) \cdots \tilde{A}_{\nu_m}(p_m)
$$

(3.20)

and they should be regular (analytic) w.r.t. the external momenta. These conditions are indeed consistent with the requirement of the locality properties of the measure-term in the reconstruction theorem.

In order to achieve the above situation, one important requirement about the fermion symmetries of the mirror-sector action follows from the consideration of 't Hooft anomaly matching condition.[73, 94, 110] If there exists a global continuous fermion symmetry in $S_M$, it must be free from the “would-be gauge anomaly”, i.e. that global symmetry can be gauged successfully without encountering gauge anomalies. This is because the “would-be gauge anomaly” implies an IR singularity in the (gauge-invariant) correlation function of the symmetry currents and it in turn implies certain massless states in the spectrum of the model, so that they can saturate the IR singularity. This contradicts the required situation.

4 345 model in the mirror-fermion approach

In this section, we first review the 345 model in the mirror fermion approach with the Ginsparg-Wilson fermions[73–80], which is formulated by introducing all possible Dirac- and Majorana-Yukawa couplings with XY-spin field to break the global continuous symmetries of the mirror sector. We next examine why the attempt seems a “Mission impossible” in the 345 model. We point out that the effective fermionic operators to break the symmetries $U(1)_f$ and $U(1)_{f'}$ in the mirror sector do not have sufficiently strong couplings even in the limit of large Majorana-Yukawa couplings. We observe also that the type of
Majorana mass term considered there is singular in the large limit due to the nature of the chiral projection of the Ginsparg-Wilson fermions, but a slight modification without such singularity is allowed by virtue of the very nature.

4.1 345 model

The 345 model is defined by the charge assignment of the U(1) gauge symmetry as

\[ Q = \text{diag}(q_1, q_2) = \text{diag}(3, 4), \quad Q' = \text{diag}(q'_1, q'_2) = \text{diag}(5, 0). \]  

(4.1)

The neutral fermion is introduced as a spectator. Let us index the components of the Weyl fields by their charges, \( q, q' \) as

\[ \psi^- = (\psi_3^-, \psi_4^-), \quad \bar{\psi}^- = (\bar{\psi}_3^-, \bar{\psi}_4^-), \]  

(4.2)

\[ \psi^+ = (\psi_{5^+}, \psi_{0^+}), \quad \bar{\psi}^+ = (\bar{\psi}_{5^+}, \bar{\psi}_{0^+}). \]  

(4.3)

We also specify the representations of the gamma matrices by the Pauli matrices as \( \gamma_1 = \sigma_1, \gamma_2 = \sigma_2, \gamma_3 = \sigma_3 \), and of the charge conjugation operator as \( c_D = i\gamma_2 \).

Accordingly, let us index the components of the Mirror fermion fields as

\[ \psi^+_+ = (\psi_{3^+}, \psi_{4^+}), \quad \bar{\psi}^+_+ = (\bar{\psi}_{3^+}, \bar{\psi}_{4^+}), \]  

(4.4)

\[ \psi'^-_+ = (\psi_{5^-}, \psi_{0^-}), \quad \bar{\psi}'^-_+ = (\bar{\psi}_{5^-}, \bar{\psi}_{0^-}). \]  

(4.5)

Without interaction, the fermionic symmetries of the mirror-sector are as listed in the table 1.

| | + | + | - | - | gauge anomaly | chiral anomaly |
|---|---|---|---|---|---|---|
| U(1)_g | 3 | 4 | 5 | 0 | matched (gauged) | — |
| U(1)_b | 2 | 1 | 2 | 1 | matched (can be gauged) | anomaly free |
| U(1)_f | 1 | 1 | 1 | 0 | not matched | anomalous |
| U(1)_{f'} | 0 | 0 | 0 | 1 | not matched | anomaly free (can be anomalous) |

Table 1. Fermionic continuous symmetries in the mirror sector of the 345 model and their would-be gauge anomalies

The two types of gauge-invariant local operators

\[ O_f = (\psi_{3^+})^1 (\psi_{4^+})^3 (\bar{\psi}_{5^-})^3 \psi_{0^-}, \quad O_{f'} = (\psi_{3^+})^2 \psi_{4^+} (\bar{\psi}_{5^-})^2 \bar{\psi}_{0^-}, \]  

(4.6)

can break the symmetries, U(1)_f and U(1)_{f'}. The product of these operators,

\[ O_T = O_f O_{f'} = (\psi_{3^+})^3 (\psi_{4^+})^4 (\bar{\psi}_{5^-})^5 \bar{\psi}_{0^-} \psi_{0^-}, \]  

(4.7)

---

\[ ^{13}\text{One should note that this charge assignment does not satisfy the assumption of the reconstruction theorem that the number of fermion flavors with } |q_{\alpha}| = q \text{ (or } |q'_{\alpha'}| = q \text{) is even for all odd } q. \]
involves the 't Hooft vertex which can induced by the U(1) instantons in two-dimensions. To define the actual operators, one needs the point-splitting procedure because of the fermi statistics. A possible choice is the following:

\[ O_f(x) = \psi_0(x)c_D\psi_3(x)\bar{\psi}_5(x)\psi_4(x) \{ \Box (\bar{\psi}_5(x)\psi_4(x)) \}^2, \]  
\[ O_f(x) = \bar{\psi}_0(x)\psi_3(x)\bar{\psi}_5(x)\psi_4(x) \Box (\bar{\psi}_5(x)\psi_3(x)), \]  
\[ O_f(x) = \bar{\psi}_0(x)\psi_3(x)\bar{\psi}_5(x)\psi_4(x) \times \right] \] 
\[ \Box (\bar{\psi}_5(x)\psi_3(x)) \{ \Box (\bar{\psi}_5(x)\psi_4(x)) \}^3 \Box (\bar{\psi}_0(x)c_D\psi_3(x)), \]  

where

\[ \Box O_q(x) \equiv \sum_{\mu} (U(x, \mu)^qO_q(x + \mu) + U(x - \mu, \mu)^{-q}O_q(x - \mu) - 2O_q(x)). \]  

### 4.2 Mirror sector of the 345 model with Dirac- and Majorana-type Yukawa couplings to XY spin field

In [73–80], the 345 model is formulated by introducing all possible Dirac- and Majorana-type Yukawa couplings to the XY spin field in order to break the global continuous symmetries of the mirror sector.

\[ S_M = \sum_{x \in \Gamma} z \left\{ \bar{\psi}_+(x)D\psi_+(x) + \bar{\psi}_-(x)D^\dagger\psi_-(x) \right\} \]  
\[ + \sum_{x \in \Gamma} \sum_{q, q'} \left\{ y_{qq'} \bar{\psi}_+(x)\psi_{q'-}(x) \phi(x)^{q-q'} + y_{q'q} \bar{\psi}_{q'}(x)\psi_{q}(x) \phi(x)^{q'-q} \right\} \]  
\[ + \sum_{x \in \Gamma} \sum_{q, q'} \left\{ \tilde{h}_{qq'} \bar{\psi}_+(x)c_D\psi_{q'}(x) \phi(x)^{q+q'} + h_{qq'} \bar{\psi}_+(x)c_D\psi_{q}(x) \phi(x)^{-q-q'} \right\} \]  
\[ + \sum_{x \in \Gamma, \mu} \frac{\kappa}{2} \left\{ 2 - \phi(x)^{-1}U(x, \mu)\phi(x + \mu) - \phi(x + \mu)^{-1}U(x, \mu)^{-1} \phi(x) \right\}, \]  

where \( y_{qq'} = \gamma_{qq'} \) and \( h_{qq'} = \tilde{h}_{qq'} \) for hermiticity. The path-integral measure of the XY spin field is defined by

\[ \mathcal{D}[\phi] = \prod_x \delta(|\phi(x)| - 1) d\phi(x) d\phi^*(x)/2i. \]  

A comment is in order about our conventions. There are several differences in the conventions from those in the original works[73–80]. First of all, the definition of the chiral projection of the Ginsparg-Wilson fermions is opposite: here \( \hat{\gamma}_3 = \gamma_3(1 - 2D) \) is used for the field and \( \gamma_3 \) for the anti-field (as usual). Secondly, the Majorana-Yukawa couplings are defined here with \( c_D = i\gamma_2 \), but not \( \gamma_2 \). (Our choice of the representation of the Dirac gamma matrices in the Euclidean metric is specified as \( \gamma_0 = \sigma_1, \gamma_1 = \sigma_2, \gamma_3 = \sigma_3 \).) Therefore, it is \( ih_{qq'}(-i\tilde{h}_{qq'}) \) which corresponds to the coupling \( h_{qq'} \) in the original works. Thirdly, the chirality assignments of the target Weyl fermions and the mirror fermions is

\[ ^{14} \text{One can also include another operator, } O_{f'} = (\psi_{++})^1 (\bar{\psi}_{++})^2 (\bar{\psi}_{--})^1 (\bar{\psi}_{--}^2, \text{ which breaks only } U(1)_{f'}. \]
opposite: here $(3-, 4-, 5+, 0+)$ for the target Weyl fermions, while $(3+, 4+, 5-, 0-)$ for the mirror fermions.

The model was studied in the parameter regions where $\kappa$ is small and $y_{qq'}, \ h_{qq'}$ are large compared to $z$, so that the model is within the so-called PMS (paramagnetic strong-coupling) phase where the XY spin field is disordered, and the fermion fields form certain bound states (with the XY spin field or among the fermion fields) and acquire masses in the manner consistent with the chiral gauge invariance. The typical values of the coupling constants are listed in table 2, which are the values used in the latest numerical study in [79, 80]. It was then claimed through the Monte-Carlo studies that, although the XY spin field and the mirror fermion fields both have short correlation lengths indeed in the parameter region of their choice, the two-point vertex function of the (external) gauge field in the mirror sector $\tilde{\Pi}_{\mu\nu}(k)$, which is defined by

$$
\frac{1}{L^2} \sum_k \tilde{\eta}_{\mu}(-k) \tilde{\Pi}_{\mu\nu}(k) \tilde{\zeta}_{\nu}(k) = \delta \left( \langle -\delta S_M \rangle_M / \langle 1 \rangle_M \right) \bigg|_{U(x, \mu) \rightarrow 1},
$$

shows a singular non-local behavior there. It was also shown that the normalization of the singular term matches well with that of the target Weyl fermion fields. This singularity implies that there remains certain massless states in the mirror sector which are charged under the gauged $U(1)$, and the model looks vector-like, where both the target Weyl fermions and the mirror fermions remain massless and couples to the $U(1)$ gauge field.

### 4.3 Why is the mission impossible in the 345 model with Dirac- and Majorana-type Yukawa couplings to XY spin field?

We now examine why the attempt seems a “Mission impossible” in the 345 model with the Dirac- and Majorana-Yukawa couplings with XY-spin field. We will point out that the effective operators to break the fermion number symmetries $U(1)_f$ and $U(1)_{f'}$ in the mirror sector do not have sufficiently strong couplings even in the limit of large Majorana-Yukawa couplings. We will observe also that the type of Majorana mass term considered there is singular in the large limit due to the nature of the chiral projection of the Ginsparg-Wilson fermions, but a slight modification without such singularity is allowed by virtue of the very nature.

#### 4.3.1 Strength of the effective fermionic operators to break $U(1)_f$ and $U(1)_{f'}$

Let us consider to evaluate the partition function of the mirror fermion sector by the weak-coupling expansion w.r.t. to $\kappa$, assuming $\kappa \ll 1$,

$$
\langle 1 \rangle_M \equiv \int \mathcal{D}[\psi_+] \mathcal{D}[\bar{\psi}_+] \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] \mathcal{D}[\phi] e^{-S_M},
$$

| $\kappa$ | $z$ | $y_{35}$ | $y_{30}$ | $y_{45}$ | $y_{40}$ | $ih_{35}$ | $ih_{30}$ | $ih_{45}$ | $ih_{40}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 3.00278 | 30.3214 | 23.7109 | 3.08123 |

Table 2. The values of the coupling constants.
where the original action of the mirror fermion sector, $S_M$, is given by

$$ S_M = \sum_x \mathcal{L}(x) + \kappa S_B, \quad (4.16) $$

$$ \mathcal{L}(x) = z \{ \bar{\psi}_3^+(x) D_3 \psi_3^+(x) + \bar{\psi}_4^+(x) D_4 \psi_4^+(x) + \bar{\psi}_5^-(x) D_3 \psi_5^-(x) + \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \} 
+ \{ y_{35} \bar{\psi}_3^+(x) \psi_5^-(x) \phi(x)^{-2} + y_{53} \bar{\psi}_5^-(x) \psi_3^+(x) \phi(x)^2 
+ y_{30} \bar{\psi}_3^+(x) \psi_0^-(x) \phi(x)^3 + y_{03} \bar{\psi}_0^-(x) \psi_3^+(x) \phi(x)^{-3} 
+ y_{45} \bar{\psi}_4^+(x) \psi_5^-(x) \phi(x)^{-1} + y_{54} \bar{\psi}_5^-(x) \psi_4^+(x) \phi(x)^1 
+ y_{40} \bar{\psi}_4^+(x) \psi_0^-(x) \phi(x)^4 + y_{04} \bar{\psi}_0^-(x) \psi_4^+(x) \phi(x)^{-4} \} 
+ \{ \bar{h}_{35} \bar{\psi}_3^+(x) c_D \bar{\psi}_5^-(x) \phi(x)^8 + h_{35} \psi_3^+(x) c_D \psi_5^-(x) \phi(x)^{-8} 
+ \bar{h}_{30} \bar{\psi}_3^+(x) c_D \bar{\psi}_0^-(x) \phi(x)^3 + h_{30} \psi_3^+(x) c_D \psi_0^-(x) \phi(x)^{-3} 
+ \bar{h}_{45} \bar{\psi}_4^+(x) c_D \bar{\psi}_5^-(x) \phi(x)^9 + h_{45} \psi_4^+(x) c_D \psi_5^-(x) \phi(x)^{-9} 
+ \bar{h}_{40} \bar{\psi}_4^+(x) c_D \bar{\psi}_0^-(x) \phi(x)^4 + h_{40} \psi_4^+(x) c_D \psi_0^-(x) \phi(x)^{-4} \}, \quad (4.17) $$

$$ S_B = \sum_{x, \mu} \frac{1}{2} \{ 2 - \phi(x)^{-1} U(x, \mu) \phi(x + \mu) - \phi(x + \mu)^{-1} U(x, \mu)^{-1} \phi(x) \}, \quad (4.18) $$

using the abbreviation for the overlap Dirac operator which act on the Dirac field of charge $q$ as $D_q = D[U(x, \mu)^\eta]$. In the hopping parameter expansion, one can perform the path-integration of the XY-spin field first and formulate the fermionic effective action by the relation,

$$ \langle 1 \rangle_M = \int \mathcal{D}[\psi_+] \mathcal{D}[\bar{\psi}_+] \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] e^{-\sum_x \mathcal{L}(x) \sum_{k=0}^\infty \frac{1}{k!} (-\kappa S_B)^k} \quad (4.19) $$

where the fermionic effective action, $S'_M$, is defined by the expansion w.r.t. $\kappa$ as

$$ S'_M = \sum_{k=0}^\infty \kappa^k S'_M^{(k)}. \quad (4.20) $$

In this respect, we note that the path-integration measure of the chiral anti-fields, $\bar{\psi}_3^+(x)$, $\bar{\psi}_4^+(x)$, $\bar{\psi}_5^-(x)$ and $\bar{\psi}_0^-(x)$, can be defined as

$$ \prod_x d\bar{\psi}_3^+(x) d\bar{\psi}_4^+(x) d\bar{\psi}_5^-(x) d\bar{\psi}_0^-(x) \quad (4.21) $$

and it projects out the local composite operators which include each chiral anti-fields, $\bar{\psi}_3^+(x)$, $\bar{\psi}_4^+(x)$, $\bar{\psi}_5^-(x)$, $\bar{\psi}_0^-(x)$, just once from the products of the Lagrangian density $\{ -\sum_x \mathcal{L}(x) \}^l / l! (l = 4, \cdots)$. Furthermore, the path-integration of the XY-spin field, $\phi(x)$, projects out the composite operators which do not include $\phi(x)$ and are neutral w.r.t. the U(1) charge.
In the leading order, the fermionic effective action is given by

\[
S_M^{(0)} = \sum_x Z \{ \bar{\psi}_3(x)D_3\psi_3(x) + \bar{\psi}_4(x)D_4\psi_4(x) + \bar{\psi}_5(x)D_5\psi_5(x) + \bar{\psi}_0(x)D_0\psi_0(x) \}
- \sum_x \{ \bar{G}_{35}\psi_3(x)\bar{\psi}_5(x)\bar{\psi}_4(x) + G_{30}\bar{\psi}_3(x)\bar{\psi}_0(x)\psi_4(x) \\
+ G_{45}\bar{\psi}_4(x)\bar{\psi}_5(x)\bar{\psi}_0(x) + G_{40}\bar{\psi}_4(x)\bar{\psi}_0(x)\psi_4(x) \}
- \sum_x \{ G_{3450}\bar{\psi}_3(x)\bar{\psi}_5(x)\bar{\psi}_4(x)\psi_0(x) + G_{30}\bar{\psi}_3(x)\bar{\psi}_0(x)\psi_5(x) + G_{3450}\bar{\psi}_3(x)\bar{\psi}_5(x)\psi_4(x) + \bar{\psi}_0(x)\psi_3(x) \}, \tag{4.22}
\]

where the effective couplings are determined by the following matching conditions,

\[
(-Z)^4 = (-z)^4 \\
- (-z)^2 \bar{h}_{35}h_{35} - (-z)^2 \bar{h}_{30}h_{30} - (-z)^2 \bar{h}_{45}h_{45} - (-z)^2 \bar{h}_{40}h_{40} \\
+ \bar{h}_{35}h_{35}h_{30}h_{40} + \bar{h}_{30}h_{30}h_{45}h_{45} \\
- \bar{h}_{35}h_{40}h_{30}h_{45} - \bar{h}_{30}h_{45}h_{35}h_{40} \tag{4.23}
\]

\[
G_{35}(-Z)^2 = (y_{35}y_{35} - \bar{h}_{35}h_{35}) (-z)^2 \\
- (y_{35}y_{35} - \bar{h}_{35}h_{35}) \bar{h}_{40}h_{40} \\
- \bar{h}_{30}h_{45}h_{45}h_{40} \\
- \bar{h}_{35}h_{40}y_{45}y_{53} \tag{4.24}
\]

\[
G_{30}(-Z)^2 = (y_{30}y_{30} - \bar{h}_{30}h_{30}) (-z)^2 \\
- (y_{30}y_{30} - \bar{h}_{30}h_{30}) \bar{h}_{45}h_{45} \\
- \bar{h}_{35}h_{40}h_{30}h_{45} \\
- \bar{h}_{30}h_{45}y_{40}y_{53} \tag{4.25}
\]

\[
G_{45}(-Z)^2 = (y_{45}y_{45} - \bar{h}_{45}h_{45}) (-z)^2 \\
- (y_{45}y_{45} - \bar{h}_{45}h_{45}) \bar{h}_{30}h_{30} \\
- \bar{h}_{35}h_{40}h_{30}h_{45} \\
- \bar{h}_{30}h_{45}y_{35}y_{40} \tag{4.26}
\]

\[
G_{40}(-Z)^2 = (y_{40}y_{40} - \bar{h}_{40}h_{40}) (-z)^2 \\
- (y_{40}y_{40} - \bar{h}_{40}h_{40}) \bar{h}_{35}h_{35} \\
- \bar{h}_{30}h_{45}h_{35}h_{40} \\
- \bar{h}_{35}h_{40}y_{30}y_{54} \tag{4.27}
\]

and

\[
G_{3450} - G_{35}G_{40} - G_{30}G_{45} = (y_{35}y_{40}y_{30}y_{35} + \bar{h}_{35}h_{40}h_{30}h_{45}) \\
+ (y_{35}y_{40}y_{30}y_{45} + \bar{h}_{30}h_{45}h_{35}h_{40}) \\
- (y_{35}y_{35} - \bar{h}_{35}h_{35}) (y_{40}y_{40} - \bar{h}_{40}h_{40}) \\
- (y_{30}y_{30} - \bar{h}_{30}h_{30}) (y_{45}y_{45} - \bar{h}_{45}h_{45}) \\
+ \bar{h}_{30}h_{45}y_{40}y_{53} + \bar{h}_{30}h_{45}y_{35}y_{40}, \tag{4.28}
\]
This result can be obtained by noting first that in the limit $\kappa = 0$, $S'_M$ is obtained explicitly as

$$S'_M = \sum_x \left\{ \bar{\psi}_{3+}(x) D_3 \psi_{3+}(x) + \bar{\psi}_{4+}(x) D_4 \psi_{4+}(x) + \bar{\psi}_{5-}(x) D_5 \psi_{5-}(x) + \bar{\psi}_0-(x) D_0 \psi_0-(x) \right\}$$

$$- \sum_x \left\{ y_{35} y_{53} \bar{\psi}_{3+}(x) \psi_{5-}(x) \bar{\psi}_{5-}(x) \psi_{3+}(x) \right.$$ 

$$+ y_{30} y_{30} \bar{\psi}_{3+}(x) \psi_0-(x) \bar{\psi}_0-(x) \psi_{3+}(x)$$

$$+ y_{45} y_{54} \bar{\psi}_{4+}(x) \psi_{5-}(x) \bar{\psi}_{5-}(x) \psi_{4+}(x)$$

$$+ y_{40} y_{04} \bar{\psi}_{4+}(x) \psi_0-(x) \bar{\psi}_0-(x) \psi_{4+}(x) \right\}$$

$$- \sum_x \left\{ y_{35} y_{40} y_{03} y_{54} \bar{\psi}_{3+}(x) \psi_{5-}(x) \bar{\psi}_{4+}(x) \psi_{5-}(x) \psi_{3+}(x) \psi_0-(x) \bar{\psi}_{5-}(x) \psi_{4+}(x) \right.$$ 

$$+ y_{40} y_{04} y_{40} y_{06} y_{54} \left( \bar{\psi}_{3+}(x) \psi_{5-}(x) \psi_{4+}(x) \psi_{5-}(x) \psi_{3+}(x) \psi_0-(x) \bar{\psi}_{4+}(x) \psi_{5-}(x) \right)$$

$$- \sum_x \left\{ \bar{h}_{35} h_{35} y_{45} \bar{\psi}_{3+}(x) c_D \bar{\psi}_{5-}(x) \psi_{3+}(x) c_D \psi_{5-}(x) \right.$$ 

$$+ \bar{h}_{30} h_{30} \bar{\psi}_{3+}(x) c_D \psi_0-(x) \psi_{3+}(x) c_D \psi_0-(x)$$

$$+ \bar{h}_{45} h_{45} \bar{\psi}_{4+}(x) c_D \bar{\psi}_{5-}(x) \psi_{4+}(x) c_D \psi_{5-}(x)$$

$$+ \bar{h}_{40} h_{40} \bar{\psi}_{4+}(x) c_D \psi_0-(x) \psi_{4+}(x) c_D \psi_0-(x) \right\}$$

$$- \sum_x \left\{ \bar{h}_{35} h_{40} y_{45} y_{03} \bar{\psi}_{3+}(x) c_D \bar{\psi}_{5-}(x) \psi_{4+}(x) c_D \psi_0-(x) \psi_{3+}(x) c_D \psi_{5-}(x) \psi_{4+}(x) \psi_0-(x) \right.$$ 

$$+ \bar{h}_{30} h_{45} \bar{\psi}_{3+}(x) \psi_0-(x) \bar{\psi}_{4+}(x) \psi_{3+}(x) c_D \psi_{5-}(x) \psi_{4+}(x) \psi_0-(x) \right\}$$

$$- \sum_x \left\{ \bar{h}_{35} h_{40} y_{45} y_{03} \bar{\psi}_{3+}(x) c_D \bar{\psi}_{5-}(x) \psi_{4+}(x) \psi_0-(x) \psi_{4+}(x) \psi_0-(x) \psi_{3+}(x) \psi_0-(x) \psi_{5-}(x) \psi_{4+}(x) \psi_0-(x) \right.$$ 

$$+ \bar{h}_{30} h_{45} y_{30} y_{04} \bar{\psi}_{3+}(x) c_D \bar{\psi}_{5-}(x) \psi_{4+}(x) \psi_{5-}(x) \psi_{3+}(x) c_D \psi_{5-}(x) \psi_{4+}(x) \psi_{5-}(x) \psi_{4+}(x) \psi_0-(x) \right\}$$

$$(4.29)$$

We next remind the fact that the chiral fields, $\psi_{3+}(x)$, $\psi_{4+}(x)$, $\psi_{5-}(x)$, $\psi_0-(x)$, have two components and the bilinear operators of the kinetic, Dirac- and Majorana-type Yukawa-coupling terms have the following structures in the components,

$$\bar{\psi}_{q+}(x) D_q \psi_{q+}(x) = \bar{\psi}_{q+}(x) \psi_{q+}(x)^{(2)}$$

$$\bar{\psi}_{q-}(x) D_q \psi_{q-}(x) = \bar{\psi}_{q-}(x) \psi_{q-}(x)^{(1)},$$

$$\bar{\psi}_{q+}(x) \psi_{q-}(x) = \bar{\psi}_{q+}(x) \psi_{q-}(x)^{(2)},$$

$$\bar{\psi}_{q-}(x) \psi_{q+}(x) = \bar{\psi}_{q-}(x) \psi_{q+}(x)^{(1)},$$

and

$$\psi_{q+}(x) c_D \psi_{q-}(x) = + \psi_{q+}(x)^{(1)} \psi_{q-}(x)^{(2)} - \psi_{q+}(x)^{(2)} \psi_{q-}(x)^{(1)},$$

$$\bar{\psi}_{q+}(x) c_D \bar{\psi}_{q-}(x) = - \bar{\psi}_{q+}(x) \bar{\psi}_{q-}(x),$$

$$(4.32)$$

$$(4.33)$$

$$(4.34)$$

$$(4.35)$$
respectively. Then the bilinear operators of the Majorana-type Yukawa-coupling terms can be eliminated and the fermionic effective action can be rewritten with only the bilinear operators of the kinetic and Dirac-type Yukawa-coupling terms as follows.

\[ S'_M = \sum_x z \left\{ \bar{\psi}_3^+(x) D_3 \psi_3^+(x) + \bar{\psi}_4^+(x) D_4 \psi_4^+(x) + \bar{\psi}_5^-(x) D_5 \psi_5^-(x) + \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \right\} \]

\[ - \sum_x \left\{ (y_{35} y_{53} - h_{35} h_{35}) \bar{\psi}_3^+(x) \psi_5^-(x) \bar{\psi}_5^-(x) \psi_3^+(x) \right. \]

\[ + (y_{30} y_{30} - h_{30} h_{30}) \bar{\psi}_3^+(x) \psi_0^-(x) \bar{\psi}_0^-(x) \psi_3^+(x) \]

\[ + (y_{45} y_{45} - h_{45} h_{45}) \bar{\psi}_4^+(x) \psi_5^-(x) \bar{\psi}_5^-(x) \psi_4^+(x) \]

\[ + (y_{40} y_{40} - h_{40} h_{40}) \bar{\psi}_4^+(x) \psi_0^-(x) \bar{\psi}_0^-(x) \psi_4^+(x) \} \]

\[ - \sum_x \left\{ (y_{35} y_{40} y_{54} + h_{35} h_{40} h_{30} h_{45}) \times \right. \]

\[ \bar{\psi}_3^+(x) \psi_5^-(x) \bar{\psi}_4^+(x) \psi_0^-(x) \bar{\psi}_0^-(x) \psi_3^+(x) \psi_4^+(x) \]

\[ + (y_{53} y_{30} y_{45} + h_{30} h_{45} h_{35} h_{40}) \times \]

\[ \bar{\psi}_5^-(x) \psi_3^+(x) \bar{\psi}_0^-(x) \psi_4^+(x) \bar{\psi}_4^+(x) \psi_5^-(x) \psi_0^-(x) \psi_3^+(x) \psi_4^+(x) \} \]

\[ - \sum_x \left\{ - h_{35} h_{35} \bar{\psi}_3^+(x) D_3 \psi_3^+(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \right. \]

\[ - h_{30} h_{30} \bar{\psi}_3^+(x) D_3 \psi_0^-(x) \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \]

\[ - h_{45} h_{45} \bar{\psi}_4^+(x) D_4 \psi_5^-(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \]

\[ - h_{40} h_{40} \bar{\psi}_4^+(x) D_4 \psi_0^-(x) \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \} \]

\[ - \sum_x \left\{ - h_{35} h_{40} h_{30} h_{45} \bar{\psi}_3^+(x) D_3 \psi_3^+(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_4^+(x) D_4 \psi_4^+(x) \right. \]

\[ - h_{30} h_{45} h_{35} h_{40} \bar{\psi}_3^+(x) D_3 \psi_0^-(x) \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_4^+(x) D_4 \psi_4^+(x) \}

\[ - \sum_x \left\{ - h_{35} h_{40} h_{30} h_{45} \bar{\psi}_4^+(x) D_3 \psi_3^+(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_4^+(x) D_4 \psi_4^+(x) \right. \]

\[ - h_{35} h_{40} h_{30} h_{45} \bar{\psi}_3^+(x) D_3 \psi_0^-(x) \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_4^+(x) D_4 \psi_4^+(x) \}

\[ - \sum_x \left\{ - h_{35} h_{40} h_{30} h_{45} \bar{\psi}_5^+(x) \psi_5^-(x) \bar{\psi}_5^-(x) \psi_3^+(x) \psi_0^-(x) \psi_0^-(x) \psi_4^+(x) \right. \]

\[ - h_{35} h_{40} h_{40} h_{45} \bar{\psi}_3^+(x) D_3 \psi_0^-(x) \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_4^+(x) D_4 \psi_4^+(x) \}

\[ - \sum_x \left\{ - h_{35} h_{40} h_{30} h_{45} \bar{\psi}_4^+(x) \psi_5^-(x) \bar{\psi}_5^-(x) \psi_3^+(x) \psi_0^-(x) \psi_0^-(x) \psi_4^+(x) \right. \]

\[ - h_{35} h_{40} h_{40} h_{45} \bar{\psi}_3^+(x) D_3 \psi_0^-(x) \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_4^+(x) D_4 \psi_4^+(x) \}

\[ - \sum_x \left\{ - h_{35} h_{40} h_{30} h_{45} \bar{\psi}_5^+(x) \psi_5^-(x) \bar{\psi}_5^-(x) \psi_3^+(x) \psi_0^-(x) \psi_0^-(x) \psi_4^+(x) \right. \]

\[ - h_{35} h_{40} h_{40} h_{45} \bar{\psi}_3^+(x) D_3 \psi_0^-(x) \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_4^+(x) D_4 \psi_4^+(x) \}

\[ - \sum_x \left\{ - h_{35} h_{40} h_{30} h_{45} \bar{\psi}_4^+(x) \psi_5^-(x) \bar{\psi}_5^-(x) \psi_3^+(x) \psi_0^-(x) \psi_0^-(x) \psi_4^+(x) \right. \]

\[ - h_{35} h_{40} h_{40} h_{45} \bar{\psi}_3^+(x) D_3 \psi_0^-(x) \bar{\psi}_0^-(x) D_0 \psi_0^-(x) \bar{\psi}_5^-(x) D_5 \psi_5^-(x) \bar{\psi}_4^+(x) D_4 \psi_4^+(x) \}

\[ - \sum_x \left\{ - h_{35} h_{40} h_{30} h_{45} \bar{\psi}_5^+(x) \psi_5^-(x) \bar{\psi}_5^-(x) \psi_3^+(x) \psi_0^-(x) \psi_0^-(x) \psi_4^+(x) \right. \]
This result can be matched with $S_M^{(0)}$ in eq. (4.22) with the given conditions eq. (4.23).

In higher orders, the effective fermionic operators $O_f(x)$, $O_{f'}(x)$ and $O_T(x)$, which break the fermion number symmetries $U(1)_f$ and $U(1)_{f'}$ in the mirror sector, are indeed generated. $O_f(x)$ and $O_{f'}(x)$ appear at the second order,

$$
\Delta S_M^{(2)} = \sum_x \{ G_f O_f(x) + G_{f'} O_{f'}(x) \} + c.c. \quad (\subset S_M^{(2)}),
$$

$$
G_f = c_{30} h_{30} y_{543}^3 + c_{40} h_{40} y_{53} y_{54}^2,
$$

$$
G_{f'} = c_{03} y_{03} y_{54} + c_{04} y_{04} y_{53}^2.
$$

$O_T(x)$ appears at the eighth order,

$$
\Delta S_M^{(8)} = \sum_x G_T O_T(x) + c.c. \quad (\subset S_M^{(8)}),
$$

$$
G_T = c_{30,03} h_{30} y_{03} y_{53} y_{54}^4 + c_{30,04} h_{30} y_{04} y_{53}^2 y_{54}^3
+ c_{40,03} h_{40} y_{03} y_{53}^2 y_{54}^3 + c_{40,04} h_{40} y_{04} y_{53}^3 y_{54}^2.
$$

Here $c$'s are certain numerical coefficients.

To evaluate the strength of the fermionic operators, $O_f(x)$, $O_{f'}(x)$ and $O_T(x)$, we first need to renormalize the fermionic field variables by $Z^{-1/2}$ so that the kinetic terms of these fields become canonical. We then assume that $z = 0$, $y_{qq'} = y_{q'q} \approx 1$ ($q = 3, 4, q' = 5, 0$), $h_{30} = h_{30} = h_{45} = h_{45} = h$, $h_{35} = h_{35} = h_{40} = h_{40} = h'$, and $h \gg h' \approx 1$. In this case, the renormalized coupling constants, defined by $g_{qq'} = Z^{-2} g_{qq'}$ ($q = 3, 4, q' = 5, 0$), $g_{3450} = Z^{-4} g_{3450}$, $g_f = Z^{-4} g_f$, $g_{f'} = Z^{-4} g_{f'}$ and $g_T = Z^{-4} g_T$, are evaluated as $g_{30} = g_{45} = 1$, $g_{35} = g_{40} = -h'^2/h^2$, $g_{3450} = 6h'^2/h^2$, $g_f = c_{30}/h^3$, $g_{f'} = (c_{03} + c_{04})/h^3$, and $g_T = (c_{30,03} + c_{30,04})/h^6$ up to the corrections of the fraction $O(h'^2/h^2, y^2/h^2)$. And the effective fermionic actions reads

$$
S_M^{(0)} = \sum_x \{ \bar{\psi}_3(x) D_3 \psi_3(x) + \bar{\psi}_4(x) D_4 \psi_4(x) + \bar{\psi}_5(x) D_5 \psi_5(x) + \bar{\psi}_0(-x) D_0 \psi_0(-x) \}
$$

$$
- \sum_x \{ \bar{\psi}_3(x) \psi_0(-x) \bar{\psi}_0(-x) \psi_3(x) + \bar{\psi}_4(x) \psi_5(x) \bar{\psi}_5(x) \psi_4(x) \},
$$

and

$$
\Delta S_M^{(2)} = \sum_x \frac{1}{h^3} \{ c_{30} O_f(x) + (c_{o3} + c_{o4}) O_{f'}(x) \} + c.c. \quad (\subset S_M^{(2)}),
$$

$$
\Delta S_M^{(8)} = \sum_x \frac{1}{h^6} (c_{30,03} + c_{30,04}) O_T(x) + c.c. \quad (\subset S_M^{(8)}),
$$

up to the corrections of order $O(h'^2/h^2, y^2/h^2)$, respectively. We can see that the leading effective fermionic action is symmetric under $U(1)_f$ and $U(1)_{f'}$, while the symmetry-breaking operators $O_f(x)$ and $O_{f'}(x)$ are suppressed by the factor $\kappa^2/h^3$ and $O_T(x)$ by $\kappa^8/h^6$. In the limit $h \to \infty$, the 345 model turns out to be massless Thirring model of the two Dirac pairs $\{\psi_3(x), \psi_0(-x)\}$ and $\{\psi_4(x), \psi_5(-x)\}$ in the framework of the overlap/Ginsparg-Wilson fermions. In the continuum limit, the massless Thirring model is known to be
equivalent to the model of free massless bosons. When coupled to the (external) gauge field, these massless degrees of freedom can produce singular and non-local terms in the two-point vertex function $\tilde{\Pi}_\mu^\nu(k)$ of the (external) gauge field. It is suspected that the similar result holds true in the lattice model and this explains the numerical result observed in the works[79, 80].

4.3.2 Limit of the large Majorana-type Yukawa-coupling/Mass terms

From the result of the effective fermionic action in the hopping parameter expansion, we can see that the kinetic terms are not suppressed in the limit, $z \to 0$ and $h, h' \to \infty$. To make the effective kinetic terms vanish, one should choose $h, h'$ and $z$ so that they satisfy the condition,

$$Z^4 = z^4 - 2z^2(h^2 + h'^2) + (h^2 - h'^2)^2 = 0.$$  \hspace{1cm} (4.45)

Namely,

$$z = |h + h'|, \quad |h - h'|.$$ \hspace{1cm} (4.46)

We note that this is the common property of the mass-like terms of the Ginsparg-Wilson fermion. For the Dirac mass term, it is usually formulated as

$$S_D = \sum_x \{ \bar{\psi}(x)D\psi(x) + m_D\bar{\psi}(1 - D)\psi(x)\},$$ \hspace{1cm} (4.47)

because the scalar and pseudo scalar operators, $\bar{\psi}(1-D)\psi(x)$ and $\bar{\psi}\gamma_3(1-D)\psi(x)$, have the good transformation properties under the chiral transformation, $\delta\psi(x) = \gamma_3(1 - 2D)\psi(x)$, $\delta\bar{\psi}(x) = \bar{\psi}(x)\gamma_3$. However, this choice makes the limit of the large mass parameter $m_D$ singular, because the factor $(1 - D)$ projects out the modes with the momenta $p_\mu = (\pi, 0), (0, \pi), (\pi, \pi)$. The maximal value of the mass is given at $m_D = 1$, where the kinetic term $\bar{\psi}D\psi$ cancels out in the action and the simple bilinear operator $\bar{\psi}(x)\psi(x)$ saturates the path-integral measure of the Dirac field completely. To make the limit of the large mass parameter well-defined, we should write the action as

$$S_D = \sum_{x \in \Lambda} \{ z\bar{\psi}(x)D\psi(x) + m\bar{\psi}(x)\psi(x)\},$$ \hspace{1cm} (4.48)

where $z = 1 - m_D$ and $m = m_D$ and should take the limit $z/m = (1 - m_D)/m_D \to 0$.

As for the Majorana mass term, it is often formulated as

$$S_M = \sum_x \{ \bar{\psi}(x)D\psi(x) + M'(\bar{\psi}_+(x)^Tc_D\psi_-(x) + \bar{\psi}_+(x)c_D\psi_-(x)^T)\}.$$ \hspace{1cm} (4.49)

Indeed, this type of the Majorana-Yukawa couplings are used in the formulation of the 345 model by Bhattacharya, Chen, Giedt, Poppitz and Shang in consideration.[73–80] However, again, the limit of the large Majorana mass parameter $M'$ is singular. In fact, in the chiral basis, the Majorana mass term has the matrix elements as

$$M'\bar{u}_j^Tc_Dv_k = -M'\delta_{p+p',0} \frac{b(p')}{\omega(p')} \quad (j = \{p\}, k = \{p'\}),$$ \hspace{1cm} (4.50)

$$M'\bar{u}_j c_D\bar{v}_k^T = -M'\delta_{x,x'} \quad (j = \{x\}, k = \{x'\}).$$ \hspace{1cm} (4.51)
where $b(p) = \sum \mu (1 - \cos p_\mu) - m_0$ and $\omega(p) = \sqrt{\sum \mu \sin^2 p_\mu + \{\sum \mu (1 - \cos p_\mu) - m_0\}^2}$. And the first matrix is singular because its determinant has the factor $\prod_{p'} (b(p')/\omega(p'))$ and $b(p')$ can vanish for $0 < m_0 < 2$.

Instead, one can formulate the action as

$$S_M = \sum_x \{z \bar{\psi}(x)D\psi(x) + M(\psi_+(x)^T i\gamma_3 C D \psi_-(x) + \bar{\psi}_+(x)i\gamma_3 C D \bar{\psi}_-(x)^T)\},$$

(4.52)

where the matching conditions are given by $(-z)^2 = 1 - 2M'^2$ and $M = M'$. Then the limit $z/M = \sqrt{1 - 2M'^2}/M' \to 0$ is well-defined. In fact, in the chiral basis, the Majorana mass term has the matrix elements as

$$Mu^T_j i\gamma_3 C D v_k = iM\delta_{p+p',0} \quad (j = \{p\}, k = \{p'\}),$$

(4.53)

and the determinants of these matrices are both unity. Therefore the bilinear operator $M(\psi_+(x)^T i\gamma_3 C D \psi_-(x) + \bar{\psi}_+(x)i\gamma_3 C D \bar{\psi}_-(x)^T)$ saturates the path-integral measure of the Dirac field completely.

### 5 14(-1)4 axial gauge model in the mirror-fermion approach

In this section, we consider a simpler four-flavor axial gauge model, a 14(-1)4 model, and clarify the effect of the symmetry-breaking operators like the ’t Hooft vertices in the mirror fermion sector to the behavior of the correlation functions of the (external) gauge field.

#### 5.1 14(-1)4 axial gauge model with Spin(6)(SU(4)) symmetry

We consider the axial gauge model with Spin(6)(SU(4)) flavor symmetry, which is defined by the charge assignment of the U(1) gauge symmetry as

$$Q = \text{diag}(q_1, q_2, q_3, q_4) = \text{diag}(+1, +1, +1, +1),$$

(5.1)

$$Q' = \text{diag}(q'_1, q'_2, q'_3, q'_4) = \text{diag}(-1, -1, -1, -1).$$

(5.2)

The left- and right-handed Weyl fermions, $\psi_-(x)$ and $\psi'_-(x)$, are assumed in 4, the four-dimensional irreducible spinor representation of SO(6). The generators of the spinor representation of SO(6), i.e. Spin(6)(= SU(4)), are defined by

$$\Sigma^{ab} = -\frac{i}{4} [\Gamma^a, \Gamma^b] \quad (a, b = 1, \cdots, 6),$$

(5.3)
where $\Gamma^a$ are the eight-dimensional representation of the Clifford algebra $\Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2\delta^{ab}$ ($a, b = 1, \cdots, 6$) specified by

\[
\begin{align*}
\Gamma^1 &= \sigma_1 \times \sigma_1 \times \sigma_1, \\
\Gamma^2 &= \sigma_2 \times \sigma_1 \times \sigma_1, \\
\Gamma^3 &= \sigma_3 \times \sigma_1 \times \sigma_1, \\
\Gamma^4 &= \mathbb{I} \times \sigma_2 \times \sigma_1, \\
\Gamma^5 &= \mathbb{I} \times \sigma_3 \times \sigma_1, \\
\Gamma^6 &= \mathbb{I} \times \mathbb{I} \times \sigma_2,
\end{align*}
\]

and the Weyl fields $\psi_-(x)$ and $\psi'_+ (x)$ satisfy the constraints,

\[
\begin{align*}
P_+ \psi_-(x) &= +\psi_-(x), \\
P_+ \psi'_+(x) &= +\psi'_+(x),
\end{align*}
\]

(5.4)

\[
\begin{align*}
P_+ \bar{\psi}^- (x) &= +\bar{\psi}^- (x), \\
P_+ \bar{\psi}'_- (x) &= +\bar{\psi}'_- (x),
\end{align*}
\]

(5.5)

where

\[
P_\pm = \frac{1 \pm \Gamma^7}{2}, \quad \Gamma^7 = i\Gamma^1 \cdots \Gamma^6.
\]

(5.6)

The U(1) gauge and Spin(6)(SU(4)) global symmetries prohibit the Dirac- and Majorana-type bilinear mass terms for these fermions.

We assume accordingly that the right- and left-handed mirror fermions, $\psi_+(x)$ and $\psi'_-(x)$, are in 4, the same four-dimensional irreducible spinor representation of SO(6).

\[
\begin{align*}
P_+ \psi_+ (x) &= +\psi_+ (x), \\
P_+ \psi'_- (x) &= +\psi'_- (x),
\end{align*}
\]

(5.7)

\[
\begin{align*}
P_+ \bar{\psi}_+ (x) &= +\bar{\psi}_+ (x), \\
P_+ \bar{\psi}'_- (x) &= +\bar{\psi}'_- (x),
\end{align*}
\]

(5.8)

Then, as shown in table 3, the remaining continuous symmetry in the mirror sector is the vector U(1) symmetry, U(1)$_V$.

|   | + | - | gauge anomaly          | chiral anomaly    |
|---|---|---|------------------------|-------------------|
| U(1)$_g$ | 1 | -1 | matched (gauged)        |                  |
| Spin(6)/SU(4) | 4 | 4 | matched (can be gauged) | anomaly free |
| U(1)$_V$ | 1 | 1 | not matched | anomalous |

Table 3. Fermionic continuous symmetries in the mirror sector of the 1$^4$(-1)$^4$ model and their would-be gauge anomalies

The U(1) gauge and Spin(6)(SU(4)) global symmetries prohibit the bilinear terms, but allow the following quartic terms to break the U(1)$_V$ in the mirror sector,

\[
O_V (x) = \frac{1}{2} \psi_+ (x)^T i \gamma_3 c_D T^a \psi'_-(x) \psi_+ (x)^T i \gamma_3 c_D T^a \psi'_-(x),
\]

(5.9)

\[
\bar{O}_V (x) = \frac{1}{2} \bar{\psi}_+ (x) i \gamma_3 c_D T^a \bar{\psi}'_- (x)^T \bar{\psi}_+ (x) i \gamma_3 c_D T^a \bar{\psi}'_- (x)^T,
\]

(5.10)
where $T^a (a = 1, \ldots , 6)$ are defined by $T^a = CT^a (a = 1, \ldots , 6)$ and satisfying $\{ T^a \}^T = -T^a$. $C$ is the charge-conjugation operator satisfying $CT^a C^{-1} = - (\Gamma_a)^T$, $CT^7 C^{-1} = -\Gamma_7$, $C^T = -C^{-1} = -C^\dagger = C$. The explicit representations of $T^a (a = 1, \ldots , 6)$ and $C$ are given as follows.

$$
T^1 = (+1) \sigma_3 \times \sigma_2 \times \sigma_3,
T^2 = (+i) \ I \times \sigma_2 \times \sigma_3,
T^3 = (-1) \sigma_1 \times \sigma_2 \times \sigma_3,
T^4 = (-i) \sigma_2 \times \sigma_1 \times \sigma_3,
T^5 = (+1) \sigma_2 \times I \times \sigma_3,
T^6 = (+i) \sigma_2 \times \sigma_3 \times I,
C = (+i) \sigma_2 \times \sigma_3 \times \sigma_2.
$$

The squares of these operators are nothing but the 't Hooft vertices which can be induced by the U(1) instantons in two-dimensions.

$$
O_T(x) = \frac{1}{2} O_V(x) O_V(x), \quad \bar{O}_T(x) = \frac{1}{2} \bar{O}_V(x) \bar{O}_V(x).
$$

5.2 Mirror sector of the $1^4(-1)^4$ model with the Majorana-type Yukawaa-coupling to SO(6)-vector spin fields

We formulate the mirror fermion sector of the $1^4(-1)^4$ model with the Majorana-type Yukawa-couplings to the auxiliary SO(6)-vector spin fields, $E^a(x)$, $\bar{E}^a(x) (a = 1, \ldots , 6)$ with the unit lengths $E^a(x)E^a(x) = 1$, $\bar{E}^a(x)\bar{E}^a(x) = 1$ as follows.

$$
S_M = \sum_x z \left\{ \bar{\psi}_+(x) D_{+1} \psi_+(x) + \bar{\psi}_-(x) D_{-1} \psi_-(x) \right\}
+ \sum_x \hbar \left\{ \psi_+(x)^T i \gamma_3 c_D T^a E^a(x) \psi_-'(x) + \bar{\psi}_+(x) i \gamma_3 c_D T^a \bar{E}^a(x) \bar{\psi}_-'(x)^T \right\}
+ \sum_{x, \mu} \kappa \left\{ E^a(x) E^a(x + \hat{\mu}) + \bar{E}^a(x) \bar{E}^a(x + \hat{\mu}) \right\}.
$$

The path-integral measure of the SO(6)-vector spin fields are defined by

$$
\mathcal{D} [E^a] = \prod_x \left[ (\pi^3)^{-1} \prod_{a=1}^6 dE^a(x) \delta(|E(x)| - 1) \right],
$$

$$
\mathcal{D} [\bar{E}^a] = \prod_x \left[ (\pi^3)^{-1} \prod_{a=1}^6 d\bar{E}^a(x) \delta(|\bar{E}(x)| - 1) \right].
$$

Note that we adopt the type of the Majorana-Yukawa coupling with the factor $i \gamma_3 c_D$ instead of $c_D$, trying to make the large coupling limit $z/\hbar \to 0$ well-defined.

We then consider the limit $z/\hbar \to 0$ and $\kappa \to 0$ in the mirror fermion sector of the $1^4(-1)^4$ model defined by eq. (5.12), where the kinetic terms of the mirror fermion and the spin fields are both suppressed. In this limit, the partition function of the mirror sector is
obtained by performing the path-integration of the mirror fermion fields in the chiral basis as

$$\langle 1 \rangle_M = \int \mathcal{D}[\psi_+]\mathcal{D}[\bar{\psi}]\mathcal{D}[\bar{\psi}]D[\bar{v}^\dagger]D[\bar{E}]D[E] e^{-S_M} \quad \text{(5.15)}$$

$$= \int \mathcal{D}[E] \det(u^T i\gamma_3 c_D \bar{T}^a E^a v') \int \mathcal{D}[\bar{E}] \det(\bar{u} i\gamma_3 c_D \bar{T}^a \bar{E}^a \bar{v}'^T), \quad \text{(5.16)}$$

where \((u^T i\gamma_3 c_D \bar{T}^a E^a v')\) and \((\bar{u} i\gamma_3 c_D \bar{T}^a \bar{E}^a \bar{v}'^T)\) are the complex matrices given by

\[
(u^T i\gamma_3 c_D \bar{T}^a E^a v')_{ij} = u_i^T i\gamma_3 c_D \bar{T}^a E^a v_j', \quad \text{(5.17)}
\]

\[
(\bar{u} i\gamma_3 c_D \bar{T}^a \bar{E}^a \bar{v}'^T)_{kl} = \bar{u}_k i\gamma_3 c_D \bar{T}^a \bar{E}^a \bar{v}_l'^T. \quad \text{(5.18)}
\]

The chiral basis for the anti-fields can be chosen as \(\bar{u}_k(x) = (0, 1)\delta_{k,s'}\delta_{x,x'}\) for \(k = (s', x')\) and \(\bar{v}_l(x) = (1, 0)\delta_{ls''}\delta_{x,x''}\) for \(l = (s'', x'')\). Then the second matrix eq. (5.18) is given by

\[
\bar{u}_k i\gamma_3 c_D \bar{T}^a \bar{E}^a \bar{v}_l'^T = i\{\bar{T}^a\}_{s's''} \bar{E}^a(x') \delta_{x,x''},
\]

where \(\bar{T}^a (a = 1, \ldots, 6)\) are 4 \times 4 matrices defined as \(T^a = \bar{T}^a \otimes \sigma_3 (a' = 1, \ldots, 5)\), \(T^6 = \bar{T}^6 \otimes I\). And its determinant turns out to be unity,

\[
\det(\bar{u} i\gamma_3 c_D \bar{T}^a \bar{E}^a \bar{v}'^T) = 1. \quad \text{(5.20)}
\]

Therefore the partition function is given simply by

\[
\langle 1 \rangle_M = \langle 1 \rangle_E, \quad \text{(5.21)}
\]

where \(\langle \cdots \rangle_E\) is the abbreviation for the path-integration of the spin-fields \(E^a(x)\):

\[
\langle \mathcal{O} \rangle_E \equiv \int \mathcal{D}[E] \det(u^T i\gamma_3 c_D \bar{T}^a E^a v') \mathcal{O}[E^a]. \quad \text{(5.22)}
\]

We note that the above formula of the partition function of the mirror sector makes sense in all topological sectors of the admissible U(1) link fields, \(\mathcal{U}[m]\). This is because the excess (or decrease) in the number of the right-handed basis vectors \(\{u_j(x)\}\) due to the topologically non-trivial link fields is always equal to that of the left-handed ones \(\{v_j'(x)\}\) thanks to the axial assignment of the U(1) charges, and the matrix \((u^T i\gamma_3 c_D \bar{T}^a E^a v')\) remains to be a square matrix.

The chiral basis for the fields, on the other hand, can be chosen so that the basis vectors satisfy the relation

\[
u^T_1 \gamma_3 c_D C T^6 = C_{ij} v'^{i'}_j, \quad C^\dagger = C^{-1}, \quad \text{(5.23)}
\]

because the chiral projectors commute with the Gamma matrices,

\[\Gamma^a \hat{P}_+ \Gamma^a = \hat{P}_+, \quad \Gamma^a \hat{P}_- \Gamma^a = \hat{P}_- \quad (a = 1, \ldots, 6) \quad \text{(5.24)}\]

and satisfy the charge-conjugation relation,

\[C^{-1}(\gamma_3 c_D)^{-1} \hat{P}_+[U]^T(\gamma_3 c_D)C = \hat{P}_- [U^*] = \hat{P}_- [U]. \quad \text{(5.25)}\]
Then the first matrix eq. (5.17) is given by

\[(u^T i\gamma_3 c D^a v') = C \times (v'^\dagger i\Gamma^6 E^a v')\]  \hspace{1cm} (5.26)

\[= (u^T i\Gamma^a E^a v^*) \times C\]  \hspace{1cm} (5.27)

where \(C = (u^T i_{\gamma_3 c D^a} \Gamma^6 v').\) And its determinant can be written as

\[\det(u^T i\gamma_3 c D^a E^a v') = \det C \det(v'^\dagger i\Gamma^6 E^a v')\]  \hspace{1cm} (5.28)

\[= \det C \det(u^\dagger i\Gamma^6 E^a u).\]  \hspace{1cm} (5.29)

5.3 Properties in the weak gauge-coupling limit

We examine the properties of the \(1^4(-1)^4\) model in the weak gauge-coupling limit, where the U(1) link variables are set to unity, \(U(x,\mu) = 1.\)

5.3.1 Positive semi-definite mirror-fermion determinant

In the weak gauge-coupling limit, one can choose the chiral basis of the fields so that the basis vectors satisfy the relations,

\[u'_j(x) = u_j(x), \quad v'_j(x) = v_j(x),\]  \hspace{1cm} (5.30)

\[u^T_j(x)\gamma_3 c D^a \Gamma^6 = v_j^\dagger(x).\]  \hspace{1cm} (5.31)

And the basis vectors \(\{u_j(x)\} (j = \{p_\mu, t\}, t = 1, \cdots, 4)\) can be chosen explicitly as

\[u_j(x) = \frac{1}{\sqrt{L^2}} e^{ipx} u_\alpha(p) \delta_{s,t} \quad (j = \{p, t\}),\]  \hspace{1cm} (5.32)

where \(\{u_\alpha(p)\}\) are the two-spinor eigenvectors of the free hermitian Wilson-Dirac operator \(H_w = \gamma_3 (D_w - m_0) (0 < m_0 < 2)\) with the negative eigenvalues in the plane-wave basis given by

\[u_\alpha(p) = \begin{cases} \begin{pmatrix} -c(p) \\ (\omega(p) + b(p)) \end{pmatrix} / \sqrt{2\omega(p)(\omega(p) + b(p))} & (p \neq 0) \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & (p = 0) \end{cases}\]  \hspace{1cm} (5.33)

and

\[b(p) = \sum_\mu (1 - \cos p_\mu) - m_0,\]  \hspace{1cm} (5.34)

\[c(p) = i \sin p_0 + \sin p_1,\]  \hspace{1cm} (5.35)

\[\omega(p) = \sqrt{\sum_\mu \sin^2 p_\mu + \left\{\sum_\mu (1 - \cos p_\mu) - m_0\right\}^2}.\]  \hspace{1cm} (5.36)

The two-momentum \(p_\mu\) is given by \(p_\mu = 2\pi n_\mu / L (n_\mu \in \mathbb{Z})\) for the periodic boundary condition and \(p_\mu = 2\pi (n_\mu + 1/2) / L (n_\mu \in \mathbb{Z})\) for the anti-periodic boundary condition.
The zero modes with \( p_\mu = 0 \) in eq. (5.33) exist only for the periodic boundary condition. (See the appendix A for detail.) In this basis, the matrix elements of \( (u^\dagger i\Gamma^6\Gamma^aE^a_u) \) are given by

\[
(u^\dagger i\Gamma^6\Gamma^aE^a_u)_{ij} = i\{\Gamma^6\Gamma^a\}_tt\bar{E}^a(k) \times \begin{pmatrix}
\frac{c(p')^2 - c(p')^2 - (\omega + \beta)(\omega + \beta)}{L^2} \delta_{p,p' + k} & (p \neq 0, p' \neq 0) \\
\frac{L^2 - 2(\omega + \beta)(\omega + \beta)}{2L^2} \delta_{0,p' + k} & (p = 0, p' \neq 0) \\
\frac{L^2 - 2(\omega + \beta)(\omega + \beta)}{2L^2} \delta_{p,k} & (p \neq 0, p' = 0) \\
1 \delta_{0,k} & (p = 0, p' = 0)
\end{pmatrix},
\]

(5.37)

where \( \bar{E}^a(k) \) is the Fourier components of \( E^a(x) \) defined by \( \bar{E}^a(k) \equiv \sum_x e^{-ikx} E^a(x) \) with the constraints, \( \sum_{k_\mu} \bar{E}^a(k)^* \bar{E}^a(k) = \tilde{L}^4 \) and \( \sum_{k_\mu} \bar{E}^a(k)^* \bar{E}^a(k + p) = 0 \) \( (p \neq 0) \).

Moreover, the matrices \( (u^\dagger i\Gamma^6\Gamma^aE^a_u) \) and \( (v^\dagger i\Gamma^6\Gamma^aE^a_v) \) are the block-diagonal parts of the matrix

\[
U = \begin{pmatrix}
(u^\dagger i\Gamma^6\Gamma^aE^a_u) & (v^\dagger i\Gamma^6\Gamma^aE^a_v) \\
(v^\dagger i\Gamma^6\Gamma^aE^a_u) & (v^\dagger i\Gamma^6\Gamma^aE^a_v)
\end{pmatrix},
\]

(5.38)

which is unitary \( U^\dagger = U^{-1} \) and has the unit determinant \( \det U = 1 \). Then, as long as these matrices are not singular, they satisfy the relation

\[
\det(u^\dagger i\Gamma^6\Gamma^aE^a_u) = \det(v^\dagger i\Gamma^6\Gamma^aE^a_v)^*.
\]

(5.39)

From these results, it follows that

\[
\det(u^T \gamma_3 C_D T^a v') = \det(u^T \gamma_3 C_D T^a v) = 1,
\]

(5.40)

\[
\det(u^T i\gamma_3 C_D T^a E^a v') = \det(u^\dagger i\Gamma^6\Gamma^aE^a_u) = \det(u^\dagger i\Gamma^6\Gamma^aE^a_u)^*.
\]

(5.41)

And it turns out that the determinant \( \det(u^T i\gamma_3 C_D T^a E^a v') \) is real for any spin-field configuration \( E^a(x) \),

\[
\det(u^T i\gamma_3 C_D T^a E^a v') = \det(u^\dagger i\Gamma^6\Gamma^aE^a_u) \in \mathbb{R},
\]

(5.42)

and, in particular, it is unity for the constant configuration \( E^a_0(x) = \delta^{a,6} \),

\[
\det(u^T i\gamma_3 C_D T^a E^a_0 v') = 1.
\]

(5.43)

On the other hand, by inspecting the matrix elements of \( (u^\dagger i\Gamma^6\Gamma^aE^a_u) \), one can see that the zero modes with \( p_\mu = 0 \) mix with linear-combinations of the modes with \( p'_\mu \neq 0 \) for which \( -c(p') \delta_{0,p' + k} \bar{E}^a(k) \neq 0 \), but they decouple completely from the modes with the momenta \( p'_\mu = \pi_\mu^{(A)} \) \( (A = 1, 2, 3) \) where \( \pi^{(1)} \equiv (\pi, 0), \pi^{(2)} \equiv (0, \pi), \pi^{(3)} \equiv (\pi, \pi) \). This implies that the mixing of the zero modes is completely suppressed for the following class of the spin configurations,

\[
E^a(x) = \frac{1}{V} \sum_{A=1,2,3} \cos(\pi(A)x) \bar{E}^a(\pi(A)),
\]

(5.44)

\[
\sum_{A=1,2,3} \bar{E}^a(\pi(A)) \bar{E}^a(\pi(A)) = V^2,
\]

(5.45)

\[
\sum_{A \neq B} \bar{E}^a(\pi(A)) \bar{E}^a(\pi(A) + \pi(B)) = 0 \quad (B = 1, 2, 3).
\]

(5.46)
For these configurations, zero eigenvalues appear in the eigenspectrum of \((u^\dagger i\Gamma^6\Gamma^aE^u)\) and the multiplicity of the zero eigenvalues is at least eight.

It is instructive to verify the above results numerically. For randomly generated spin-field configurations, we found that the eigenvalues of \((u^T i\gamma_3\gamma_D\Gamma^aE^v)\) and \((u^\dagger\Gamma^6\Gamma^aE^u)\) are all non-zero, and the determinants \(\det(u^T i\gamma_3\gamma_D\Gamma^aE^v)\) and \(\det(u^\dagger\Gamma^6\Gamma^aE^u)\) are both real and positive. We also observed that the eigenvalue spectra of \((u^\dagger\Gamma^6\Gamma^aE^u)\) have the structure like \(\{(\lambda_i, -\lambda_i^*) | i = 1, \cdots, L^2/2\}\) approximately. The typical examples of the eigenvalue spectra are shown in fig. 1 for \(L = 12\) with the periodic boundary condition.

For comparison, the eigenvalue spectrum of the chiral Dirac matrix \((\bar{u}Du)\) is shown in fig. 2 for \(L = 12\) with the same periodic boundary condition.

\begin{figure}[h]
\begin{center}
\includegraphics[width=0.4\textwidth]{fig1_left}
\includegraphics[width=0.4\textwidth]{fig1_right}
\caption{Eigenvalue spectra of \((u^T i\gamma_3\gamma_D\Gamma^aE^v)\) [left] and \((u^\dagger\Gamma^6\Gamma^aE^u)\) [right] for a randomly generated spin-field configuration \(E^a(x)\). The lattice size is \(L = 12\) and the periodic boundary condition is imposed on the fermion fields.}
\end{center}
\end{figure}

\begin{figure}[h]
\begin{center}
\includegraphics[width=0.4\textwidth]{fig2}
\caption{Eigenvalue spectrum of \((\bar{u}Du)\). The lattice size is \(L = 12\) and the periodic boundary condition is imposed on the fermion fields.}
\end{center}
\end{figure}

For the spin field configurations \(E^a(x)\), we found eight zero eigenvalues for the periodic boundary condition, but none for the anti-periodic boundary condition. The typical examples of the eigenvalue spectra are shown in fig. 3 for \(L = 12\) with the periodic and anti-periodic boundary conditions. We also observed that the degeneracy of the eight zero modes are resolved by small randomly-generated perturbations to \(E^a(x)\) in the structure like \(\{(\lambda_i, -\lambda_i^*) | i = 1, \cdots, 4\}\) approximately, as shown in fig. 4, and the determinant
det \((u^\dagger \Gamma^a E^a u)\) remains real and positive.

Figure 3. Eigenvalue spectra of \((u^\dagger \Gamma^a E^a u)\) for a spin-field configuration of the class \(E_a^a(x)\) with the periodic b.c. [left] and the anti-periodic b.c. [right]. The lattice size is \(L = 12\).

Figure 4. Eigenvalue spectra of \((u^\dagger \Gamma^a E^a u)\) for the spin-field configuration \(E_a^a(x)\) plus a small randomly-generated perturbation with the periodic b.c. [left]. The range of the would-be zero eigenvalues are zoomed up in the [right] figure. The lattice size is \(L = 12\).

Based on the above analytical results and numerical observations, we can argue that the determinant \(\det(u^\dagger i\gamma_\beta c_D \Gamma^a E^a v')\) (= \(\det(u^\dagger i\Gamma^a E^a u)\)) is positive semi-definite for any spin-field configuration \(E^a(x)\) in the weak gauge-coupling limit:

\[
\det(u^\dagger i\gamma_\beta c_D \Gamma^a E^a v') = \det(u^\dagger \Gamma^a E^a u) \geq 0 \quad (g_0 = 0).
\]  

We first note that the space of the SO(6)-vector spin field configurations, which we denote with \(\mathcal{V}_E\), is the direct product of multiple \(S^5\) and is pathwise connected. Then any configuration of the spin field \(E^a(x)\) can be reached from the constant configuration \(E_0^a(x) = \delta^{a,6}\) through a continuous deformation. Since it is unity for the constant configuration, the determinant of \((u^\dagger i\Gamma^a E^a u)\) should be positive for a given configuration \(E^a(x)\) as long as there exists a path to \(E^a(x)\) from \(E_0^a(x)\)(= \(\delta^{a,6}\)) such that the determinant never vanish along the path. On the other hand, for the spin configurations with which the determinant is zero, a certain subset in the eigenvalue spectrum of \((u^\dagger i\Gamma^a E^a u)\) should be zero. Along the path which goes through such a spin configuration, the eigenvalue spectrum flow and the subset of would-be zeros pass the origin in the complex plane. Then the determinant as
the product of the eigenvalues, \( \det(u^\dagger i\Gamma^aE^a u) = \prod_{j=1}^{4L^2} \lambda_j \), can change discontinuously in its signature (phase). Since the signature (phase) of the determinant stays constant as far as the determinant is nonzero, this could happen if and only if the subspace of the configurations with the vanishing determinant, which we denote with \( \mathcal{V}_E^0 \), can divide the entire space of the spin configurations \( \mathcal{V}_E \) into the subspaces which are disconnected each other. And the divided disconnected space, \( \mathcal{V}_E \setminus \mathcal{V}_E^0 \), should be classified by the values of the signature (phase) of the determinant. In this respect, however, one notes that \( \pi_k(S^6) = 0 \) \((k < 6)\) and any topological obstructions and the associated topological terms are not known in the continuum limit for the SO(6)-vector spin field \( E^a(x) \) on the two-dimensional space-time \( S^2 \) or \( T^2 \). In particular, any topologically non-trivial configurations/defects of the SO(6)-vector spin field and the associated fermionic massless excitations are not known in the continuum limit. Then it seems reasonable to assume that \( \mathcal{V}_E^0 \) consists of lattice artifacts and in particular it is given solely by the subspace of the configurations \( E^a_*(x) \), which we denote with \( \mathcal{V}_E^* \). If one assumes that \( \mathcal{V}_E^0 = \mathcal{V}_E^* \), the multiplicity of the zero eigenvalues are eight and the would-be zero eigenvalues have the approximate structure \( \{ (\lambda_i, -\lambda_i^*) \mid i = 1, \ldots, 4 \} \). Then the signature (phase) of the determinant does not change in passing \( \mathcal{V}_E^0 (= \mathcal{V}_E^*) \). Therefore the determinant \( \det(u^\dagger i\Gamma^aE^a u) \) is positive semi-definite.

It then follows that the partition function of the mirror fermion sector is real and positive in the weak gauge-coupling limit:

\[
\langle 1 \rangle_M = \langle 1 \rangle_E = \int \mathcal{D}[E^a] \det(u^T i\gamma_3 c_D T^a E^a u') = \int \mathcal{D}[E^a] \det(u^\dagger i\Gamma^a E^a u) > 0 \quad (g_0 = 0).
\]

### 5.3.2 Monte Carlo simulation of the SO(6)-vector spin field

From the positive semi-definiteness of the determinant \( \det(u^T i\gamma_3 c_D T^a E^a u') \), it also follows that the Monte Carlo method can be applied to the path-integration of the SO(6)-vector spin field \( E^a(x) \) in the weak gauge-coupling limit, using the effective action

\[
S_E[E^a] = -\ln \det(u^T i\gamma_3 c_D T^a E^a u') = -\ln \det(u^\dagger i\Gamma^a E^a u).
\]

We have applied a hybrid Monte Carlo method to this spin model and have performed simulations for the range of lattice sizes \( L = 4, 8, 12 \). The examples of the histories of the effective action \( S_E[E^a] \) are shown in fig 5 for the various lattice sizes. The trajectory length is 0.05 and the average acceptance ratio is 0.5. We have used the spin-field configurations generated by these simulations to compute the observables of the mirror sector such as the correlation functions of the mirror-sector fields and the two-point vertex function of the (external) gauge fields. These results are shown and discussed in the following sections.

### 5.3.3 Short-ranged correlation functions

We first examine the correlation functions of the fields of the mirror fermion sector in the weak gauge coupling limit. We consider the following two-point correlation functions in
Figure 5. Monte Carlo histories of the effective action $S_E(E^a)$ for the lattice sizes $L = 4, 8, 12$. The periodic boundary condition is used for the fermion fields.

The channels of 6 and 4 representations of SO(6) and Spin(6).

$$G_E(x, y)_{ab} = \langle E^a(x)E^b(y) \rangle_M / \langle 1 \rangle_M$$

$$= \langle E^a(x)E^a(y) \rangle_E / \langle 1 \rangle_E$$

$$G_{\psi'\psi E}(x, y) = \langle \psi'_-(x)\psi'_+(y)i\gamma_3cD^aE^a(y) \rangle_M / \langle 1 \rangle_M,$$

$$G_{\psi\psi' E}(x, y) = \langle \psi_+(x)\psi'_-(y)i\gamma_3cD^aE^a(y) \rangle_M / \langle 1 \rangle_M.$$

The fermionic correlation functions above satisfy the Schwinger-Dyson equations given as follows:

$$\{ G_{\psi'\psi E} \hat{P}_- \}(x, y) = \hat{P}_-(x, y),$$

$$\{ G_{\psi\psi' E} \hat{P}_+ \}(x, y) = \hat{P}_+(x, y).$$

And these equations can be solved as

$$G_{\psi'\psi E}(x, y) = \hat{P}_-(x, y) + \{ G_{\psi'\psi E} \hat{P}_+ \}(x, y),$$

$$G_{\psi\psi' E}(x, y) = \hat{P}_+(x, y) + \{ G_{\psi\psi' E} \hat{P}_- \}(x, y).$$

Therefore, non-trivial parts of the fermionic correlation functions are given by $\{ G_{\psi'\psi E} \hat{P}_+ \}(x, y)$ and $\{ G_{\psi\psi' E} \hat{P}_- \}(x, y)$, which may be expressed explicitly in terms of the chiral basis as follows.

$$\{ G_{\psi'\psi E} \hat{P}_+ \}(x, y) = \langle v'(x)(u^T\mathcal{M}_E v')^{-1}(u^T\mathcal{M}_E u)u(y)\rangle_E / \langle 1 \rangle_E,$$

$$\{ G_{\psi\psi' E} \hat{P}_- \}(x, y) = \langle u(x)(v'^T\mathcal{M}_E v')^{-1}(v'^T\mathcal{M}_E v')v'(y)\rangle_E / \langle 1 \rangle_E,$$

where $\mathcal{M}_E = i\gamma_3cD^aE^a$. As to the similar correlation functions which are related to the anti-fields,

$$G_E(x, y)_{ab} = \langle E^a(x)\overline{E}^b(y) \rangle_M / \langle 1 \rangle_M.$$
\[ \tilde{G}_{\psi\psi E}(x, y) = \langle \bar{\psi}'_-(x)^T \tilde{\psi}_+(y) i\gamma_3 c_D T^a \bar{E}^a(y) \rangle_M / \langle 1 \rangle_M, \]  
\[ \tilde{G}_{\tilde{\psi}\tilde{\psi} E}(x, y) = \langle \tilde{\psi}_+(x)^T \bar{\psi}'_-(y) i\gamma_3 c_D T^a \bar{E}^a(y) \rangle_M / \langle 1 \rangle_M, \]

they are obtained exactly and have the short-ranged property as follows.

\[ G_{E}(x, y)^{ab} = \delta_{xy} \delta^{ab}, \]  
\[ G_{\psi\psi E}(x, y) = P_+^T \delta_{xy} \delta_{st}, \]  
\[ G_{\tilde{\psi}\tilde{\psi} E}(x, y) = P_-^T \delta_{xy} \delta_{st}. \]

In fig. 6, we show the numerical-simulation results of \( G_E(x, y)^{ab} \) on the lattice with \( L = 12 \). 1,100 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones and the translational invariance is assumed. We found that the simulation results respect the SO(6) symmetry, i.e. the diagonal components with \( a = b \) are equal to each other and the off-diagonal components with \( a \neq b \) are vanishing up to the statistical errors. We can see that the correlation length is of order the lattice spacing and the spin field \( E^a(x) \) is disordered almost completely just like \( \bar{E}^a(x) \).

In figs. 7, 8 and 9, we show the numerical-simulation results of \( \{ G_{\psi\psi E P_+} \}(x, y) \) for the lattice size \( L = 12 \). 1,100 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones. Again, we found that the simulation results respect the Spin(6) symmetry, i.e. the diagonal components with \( s = t \) are equal to each other and the off-diagonal components with \( s \neq t \) are vanishing within the statistical errors. We can see that these correlation functions are short-ranged and the correlation length is estimated as \( \xi \simeq 12 / \ln 10^4 \simeq 1.30 \).

From these results, we can see that the fields of the mirror fermion sector in the representations 6, 4 and 4* of SO(6) and Spin(6) have the short-range correlation lengths of order the lattice spacing.
Figure 6. $G_E(x)$ [left] and $\ln |G_E(x)|$ [right] vs. $|x|_1 \equiv |x_0| + |x_1|$, where $G_E(x) \equiv \sum_{a=1}^{6} G_E(x)^{a \alpha}$. The lattice size is $L = 12$. The square-symbol (blue) plot is along the temporal axis ($x_0 = 0$), while the triangle-symbol (light blue) plot is along the diagonal axis ($x_0 = x_1$). 1,100 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones and the translational invariance is assumed.
Figure 7. $\sum_{s=1}^{4} \{ G_{\psi E}^{\psi} \hat{P}_+ \}_{0,s,s}(x)$ vs. $x = (x_0, x_1)$ [top] and $|x| \equiv |x_0| + |x_1|$ [middle, bottom]. The lattice size is $L = 8$. The blue-symbol and black-symbol plots are along the spacial axis ($x_0 = 0$) and temporal axis ($x_1 = 0$), respectively, while the lightblue-symbol plot is along the diagonal axis ($x_0 = x_1$). 1,100 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.
Figure 8. The real [left] and imaginary [right] parts of $\sum_{s=1}^{4}\{G_{\psi\psi}E\hat{P}_{s}\}_{01,ss}(x)$ vs. $x = (x_0, x_1)$ [top] and $|x|_1 \equiv |x_0| + |x_1|$ [middle, bottom]. The lattice size is $L = 12$. The blue-symbol and black-symbol plots are along the spacial axis ($x_0 = 0$) and temporal axis ($x_1 = 0$), respectively, while the light-blue-symbol plot is along the diagonal axis ($x_0 = x_1$). 1,100 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.
Figure 9. The real [left] and imaginary [right] parts of $\sum_{s=1}^{4}\{G_{\psi\psi E}\tilde{P}_s\}_{10,ss}(x)$ vs. $x = (x_0, x_1)$ [top] and $|x| \equiv |x_0| + |x_1|$ [middle, bottom]. The lattice size is $L = 12$. The blue-symbol and black-symbol plots are along the spacial axis ($x_0 = 0$) and temporal axis ($x_1 = 0$), respectively, while the light-blue-symbol plot is along the diagonal axis ($x_0 = x_1$). 1,100 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

5.3.4 Regular two-point vertex function of the U(1) gauge field

We next examine the two-point vertex function of the U(1) gauge field in the mirror fermion sector, $\tilde{\Pi}_{\mu\nu}(k)$, which is defined by eq. (4.14),

$$\frac{1}{L^2} \sum_{k} \tilde{\eta}_\mu(\textbf{-}k) \tilde{\Pi}_{\mu\nu}(k) \tilde{\zeta}_\nu(k) = \delta_\zeta \left( \langle \delta_{0} S_{M} \rangle_{M} / \langle 1 \rangle_{M} \right) \left| U(x,\mu) \rightarrow 1 \right. . \tag{5.66}$$
In the $1^4(-1)^4$ axial gauge model in consideration, it is given by

$$
\frac{1}{T^2} \sum_k \tilde{\eta}_\mu(-k) \Pi'_{\mu\nu}(k) \tilde{\zeta}_\nu(k)
= \delta_\zeta \left[ \left\langle \text{Tr}\left\{ (u^T \delta_{\eta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E \right]_{U(x, \mu) \rightarrow 1}
= \left[ \left\langle \text{Tr}\left\{ (u^T \delta_{\eta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} \hat{P}_-^T \mathcal{P}_-^T v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E \right.
- \left\langle \text{Tr}\left\{ (u^T \delta_{\eta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E
+ \left\langle \text{Tr}\left\{ (u^T \delta_{\eta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E
\times \left\langle \text{Tr}\left\{ (u^T \delta_{\zeta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E
- \left\langle \text{Tr}\left\{ (u^T \delta_{\eta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E
\times \left\langle \text{Tr}\left\{ (u^T \delta_{\zeta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E \right]_{U(x, \mu) \rightarrow 1}
= \left[ \left\langle \text{Tr}\left\{ \hat{P}_+^T \delta_{\zeta} \hat{P}_+^T \delta_{\eta} \hat{P}_-^T \right\} + \text{Tr}\left\{ \hat{P}_-^T \delta_{\eta} \hat{P}_-^T \delta_{\zeta} \hat{P}_-^T \right\} \right\rangle / \langle 1 \rangle_E \right.
+ \left\langle \text{Tr}\left\{ (u^T \{ \hat{P}_+^T \delta_{\eta} \hat{P}_+^T \} \hat{P}_-^T v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E
- \left\langle \text{Tr}\left\{ (u^T \delta_{\eta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E
+ \left\langle \text{Tr}\left\{ (u^T \delta_{\eta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E
\times \left\langle \text{Tr}\left\{ (u^T \delta_{\zeta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E
- \left\langle \text{Tr}\left\{ (u^T \delta_{\eta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E
\times \left\langle \text{Tr}\left\{ (u^T \delta_{\zeta} \{ \hat{P}_+^T \mathcal{M}_E \hat{P}_-^T \} v') (u^T \mathcal{M}_E v')^{-1} \right\} \right\rangle / \langle 1 \rangle_E \right]_{U(x, \mu) \rightarrow 1}.
(5.67)

It satisfies the Ward-Takahashi relation,

$$
\Pi'_{\mu\nu}(k) 2 \sin \left( \frac{k_\nu}{2} \right) = 0,
$$
(5.68)
because

$$
\left\langle -\delta_{\eta} S_M \right\rangle_M / \langle 1 \rangle_M = \left\langle -\delta_{\eta} S_M \right\rangle_{W_M} / \langle 1 \rangle_{W_M}
$$
(5.69)
is gauge invariant. In the weak gauge-coupling limit in particular, it also satisfies the relation,

$$
2 \sin \left( \frac{k_\mu}{2} \right) \Pi''_{\mu\nu}(k) = 0,
$$
(5.70)
because one can show for the gauge-variation $\eta_{\mu}(x) = -\partial_{\mu} \omega(x)$ that

$$
\delta_\zeta \left[ \left\langle -\delta_{\eta} S_M \right\rangle_M / \langle 1 \rangle_M \right]_{U(x, \mu) \rightarrow 1} = \delta_\zeta \left[ \left\langle \text{Tr}\{ i \omega \hat{P}_+^T \} + \text{Tr}\{ -i \omega \hat{P}_+ \} \right\rangle / \langle 1 \rangle_E \right]_{U(x, \mu) \rightarrow 1}
= \left[ \left\langle \text{Tr}\{ i \omega \delta_\zeta \hat{P}_+ \} + \text{Tr}\{ -i \omega \delta_\zeta \hat{P}_+ \} \right\rangle / \langle 1 \rangle_E \right]_{U(x, \mu) \rightarrow 1}
= \left[ \left\langle \text{Tr}\{ i \omega \delta_\zeta \hat{P}_+ \} + \text{Tr}\{ -i \omega (-\delta_\zeta \hat{P}_-) \} \right\rangle / \langle 1 \rangle_E \right]_{U(x, \mu) \rightarrow 1}
= 0.
$$
(5.71)
In figs. 10 and 11, we show the numerical-simulation results of $\tilde{\Pi}_{00}^\prime(k)$ and $\tilde{\Pi}_{01}^\prime(k)$ for the lattice size $L = 8$ and for both the periodic and anti-periodic boundary conditions of the mirror fermion fields. 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones. It was verified that the Ward-Takahashi relations eqs (5.68) and (5.70) are satisfied up to the machine precision (double precision) of order $10^{-16}$.

The above result of the contribution of the mirror fermions $\tilde{\Pi}_{\mu\nu}^\prime(k)$ should be compared with that of the (target) Weyl fermions $\tilde{\Pi}_{\mu\nu}(k)$, which is given by

$$
\frac{1}{L^2} \sum_k \tilde{\eta}_\mu(-k) \tilde{\Pi}_{\mu\nu}(k) \tilde{\zeta}_\nu(k) = \delta_\zeta \left[ \text{Tr}\{P_+\delta_\eta DD^{-1}\} + \text{Tr}\{P_-\delta_\eta D'D'^{-1}\} \right] \bigg|_{U(x,\mu)\to 1} \\
= \left[ \text{Tr}\{\delta_\zeta\delta_\eta DD^{-1}\} - \text{Tr}\{\delta_\eta DD^{-1}\delta_\zeta DD^{-1}\} \right] \bigg|_{U(x,\mu)\to 1},
$$

It shows a singular non-local behavior due to the massless singularities of the Weyl fermion.
propagators $D^{-1}P_-$ and $D'^{-1}P_+$. For small momentum region $|k| \ll \pi$ (in the thermodynamic limit $L = \infty$), it is given as

$$
\tilde{\Pi}_{\mu\nu}(k) \simeq [4 \times 1^2 + 4 \times (-1)^2] \frac{1}{2\pi} \frac{\delta_{\mu\nu} k^2 - k_\mu k_\nu}{k^2} \quad (|k| \ll \pi).
$$

(5.73)

Then it shows the non-uniform behavior depending on how the limit $|k| \to 0$ is approached as follows[73–80].

$$
\tilde{\Pi}_{00}(k) \simeq \frac{4}{\pi} \frac{k_1^2}{k_0^2 + k_1^2} \rightarrow \frac{4}{\pi} \times \begin{cases}
0 & k_\mu = (|k|, 0) \\
1/2 & k_\mu = (|k|, |k|)/\sqrt{2} \\
1 & k_\mu = (0, |k|)
\end{cases}
$$

(5.74)

$$
\tilde{\Pi}_{01}(k) \simeq \frac{4}{\pi} \frac{-k_0 k_1}{k_0^2 + k_1^2} \rightarrow \frac{4}{\pi} \times \begin{cases}
0 & k_\mu = (|k|, 0) \\
-1/2 & k_\mu = (|k|, |k|)/\sqrt{2} \\
0 & k_\mu = (0, |k|)
\end{cases}
$$

(5.75)
This singular behavior of $\tilde{\Pi}^{\mu\nu}(k)$ can be verified by numerical computation. In fig. 12, we show the numerical-computation result of $\tilde{\Pi}_{00}(k)$ and $\tilde{\Pi}_{01}(k)$ for the lattice size $L = 8$ and the anti-periodic boundary conditions of the (target) Weyl fermion fields. We can see rather clearly the non-uniform limits to $|k| = 0$. The normalizations of $(1/4)L^2 \tilde{\Pi}_{00}(k)$ and $(1/4)L^2 \tilde{\Pi}_{01}(k)$ at the singularity point shown in fig. 12 are also consistent with the result in the thermodynamic limit $L = \infty$: $L^2 \times \frac{\pi}{2} \times \{0, \pm 1/2, 1\} \simeq 20.47 \times \{0, \pm 1/2, 1\}$.

By comparing the numerical result of the contribution of the mirror fermions $\tilde{\Pi}_{\mu\nu}(k)$ with that of the (target) Weyl fermions $\tilde{\Pi}_{\mu\nu}(k)$, we can see that the mirror fermion contribution does not show any evidence of the singularities due to charged massless excitations. It behaves like a regular function of momentum $k_{\mu}$. This result is consistent with the fact that the mirror fermions decouple by acquiring the masses of order the inverse lattice spacing and leave only local terms in the effective action.

5.3.5 Regular two-point vertex function of the Spin(6) (SU(4)) vector field

In the $1^4(-1)^4$ axial gauge model in consideration, the global Spin(6)(SU(4)) symmetry can be gauged consistently. Then it is instructive to examine the vertex functions of the (external) Spin(6) gauge field in the mirror fermion sector, which can be defined in the
similar manner as those of the U(1) gauge field. Let us denote the link field of the (external) \( \text{Spin}(6) \) gauge field with \( V(x, \mu) \) and its variation with \( \delta \eta V(x, \mu) = i \eta \mu(x) V(x, \mu) \) where \( \eta \mu(x) = (1/2) \eta^{ab}(x) \Sigma^{ab} \). Then the two-point vertex function is given by

\[
\frac{1}{L^2} \sum_k \tilde{\eta}^A_{\mu \nu}(-k) \tilde{\Pi}^{AB}_{\mu \nu}(k) \tilde{\zeta}^B(k) = \delta \zeta \left[ \left\langle \left\{ - \delta \eta S_M \right\}_M / \left\langle 1 \right\>_M \right\rangle_{V(x, \mu) \rightarrow 1} \right],
\]

where \( A, B \) stand for the anti-symmetrized indices \( A = [ab], B = [cd] \). The link field of the U(1) gauge field can be set to unity, \( U(x, \mu) = 1 \), from the beginning in the weak gauge-coupling limit.

The two-point vertex function satisfies the Ward-Takahashi relations,

\[
\begin{align*}
2 \sin \left( \frac{k_\mu}{2} \right) \tilde{\Pi}^{AB}_{\mu \nu}(k) &= 0, \quad (5.77) \\
\tilde{\Pi}^{AB}_{\mu \nu}(k) 2 \sin \left( \frac{k_\nu}{2} \right) &= 0.
\end{align*}
\]

For the gauge-variation \( \eta \mu(x) = -D_\mu \omega(x) \) with \( \omega(x) = \Sigma^A \omega^A(x) \), it gives

\[
\begin{align*}
\delta \zeta \left[ \left\langle \left\{ - \delta \eta S_M \right\}_M / \left\langle 1 \right\>_M \right\rangle_{V(x, \mu) \rightarrow 1} \right] &= \delta \zeta \left[ \text{Tr} \left( i \omega^T \hat{P}^T \right) + \text{Tr} \left( i \omega \hat{P} \right) \right. \\
&\left. + \langle \text{Tr} \left\{ (u^T \{ \hat{P}^+ \left[ \omega^T \hat{M}_E + \hat{M}_E \omega \right] \hat{P} \} v') (u^T \hat{M}_E v')^{-1} \right\} \rangle_{E} / \left\langle 1 \right\>_E \right]_{V(x, \mu) \rightarrow 1} \\
&= \delta \zeta \left[ \langle \text{Tr} \left\{ (u^T \{ \omega^T \hat{M}_E + \hat{M}_E \omega \} v') \langle u^T \hat{M}_E v' \rangle^{-1} \right\} \rangle_{E} / \left\langle 1 \right\>_E \right]_{V(x, \mu) \rightarrow 1},
\end{align*}
\]

(5.79)

And the term in the square bracket \( \cdots \) vanishes identically because of the Schwinger-Dyson equation w.r.t. the spin field \( E^a(x) \),

\[
\langle \text{Tr} \left\{ (u^T \{ \omega^T \hat{M}_E + \hat{M}_E \omega \} v') \langle u^T \hat{M}_E v' \rangle^{-1} \right\} \rangle_{E} / \left\langle 1 \right\>_E = 0,
\]

(5.80)

which holds true with a nontrivial external \( \text{Spin}(6) \) vector field, \( V(x, \mu) \neq 1 \). For the gauge-variation \( \zeta \mu(x) = -D_\mu \omega(x) \), on the other hand, \( \left\langle \left\{ - \delta \eta S_M \right\}_M / \left\langle 1 \right\>_M \right\rangle_{W,M} = \left\langle - \delta \eta S_M \right\rangle_{W,M} / \left\langle 1 \right\>_W \) is gauge covariant and it gives

\[
\begin{align*}
\delta \zeta \left[ \left\langle \left\{ - \delta \eta S_M \right\}_M / \left\langle 1 \right\>_M \right\rangle_{V(x, \mu) \rightarrow 1} \right] &= \delta \zeta \left[ \langle \text{Tr} \left\{ (u^T \{ \hat{P}^+ \left[ \omega^T \hat{M}_E \hat{P} \} v') \langle u^T \hat{M}_E v' \rangle^{-1} \right\} \rangle_{E} / \left\langle 1 \right\>_E \right]_{V(x, \mu) \rightarrow 1} \\
&= \delta \eta \left[ \langle \text{Tr} \left\{ (u^T i \left[ \omega^T \hat{M}_E + \hat{M}_E \omega \right] v') \langle u^T \hat{M}_E v' \rangle^{-1} \right\} \rangle_{E} / \left\langle 1 \right\>_E \right]_{V(x, \mu) \rightarrow 1} \\
&\quad + \langle \text{Tr} \left\{ (u^T \delta \eta \omega \left[ \hat{P}^+ \hat{M}_E \right] v') \langle u^T \hat{M}_E v' \rangle^{-1} \right\} \rangle_{E} / \left\langle 1 \right\>_E \right]_{V(x, \mu) \rightarrow 1}.
\end{align*}
\]

(5.81)

Again, the first term vanishes identically because of the Schwinger-Dyson equation w.r.t. the spin field \( E^a(x) \), eq. (5.80), while the second term is the one-point vertex function and vanishes identically.
Figure 13. $L^2 \tilde{\Pi}_{00}^{AA}(k) \ [\text{left}]$ and $L^2 \tilde{\Pi}_{01}^{AA}(k) \ [\text{right}]$ vs. $|k|_2 \equiv \sqrt{k_0^2 + k_1^2}$ where $A = [12]$. The lattice size is $L = 8$. The periodic boundary condition is assumed for the fermion fields. The black-, blue-, red-symbol plots are along the spacial momentum axis ($k_0 = 0$), the temporal momentum (energy) axis ($k_1 = 0$) and the diagonal momentum axis ($k_0 = k_1$), respectively. 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

In fig. 13, we show the numerical-simulation results of $\tilde{\Pi}_{00}^{AA}(k)$ and $\tilde{\Pi}_{01}^{AA}(k)$ with $A = [12]$ for the lattice size $L = 8$ and the periodic boundary condition of the mirror fermion fields. 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones. In fig. 14, one can see that the Ward-Takahashi relations eqs. (5.77) and (5.78) are satisfied only up to the statistical error in this case. This is because the Schwinger-Dyson equation w.r.t. the spin field $E^a(x)$, eq. (5.80), holds true at most in the same numerical precision.

From these results, we can see that the mirror fermion contribution, $\tilde{\Pi}_{\mu\nu}^{AB}(k)$, does not show any evidence of the singularities due to Spin(6)-charged massless excitations. It behaves like a regular function of momentum $k_\mu$. This result is again consistent with the fact that the mirror fermions decouple by acquiring the masses of order the inverse lattice spacing and leave only local terms in the effective action.
\[2 \sin(k_\mu / 2) \times L^2 \tilde{\Pi}_{\mu A}^A(k) \text{ [left]} \quad \text{and} \quad 2 \sin(k_\mu / 2) \times L^2 \tilde{\Pi}_{\mu A}^{A'}(k) \text{ [right]} \quad \text{vs.} \quad |k|_2 \equiv \sqrt{k_0^2 + k_1^2}\]

where \(A = [12]\). The lattice size is \(L = 8\). The periodic boundary condition is assumed for the fermion fields. The black-, blue-, red-symbol plots are along the spacial momentum axis \((k_0 = 0)\), the temporal momentum (energy) axis \((k_1 = 0)\) and the diagonal momentum axis \((k_0 = k_1)\), respectively. 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

### 6 21(-1)³ chiral gauge model – A solution to the reconstruction theorem

In this section, we consider the 21(-1)³ chiral gauge model, which is obtained from the previous 1⁴(-1)³ axial model by modifying the gauge group from U(1)ₐ to a U(1) subgroup of U(1)ₐ × Spin(6) (SU(4)). We first formulate the model in the mirror fermion approach with the Ginsparg-Wilson fermions. We then deduce a definition of the path-integral measure for the (target) Weyl fermions of the 21(-1)³ chiral gauge model, and argue that the induced measure-term current fulfills the requirement of the original reconstruction theorem of the Weyl fermion measure.

#### 6.1 21(-1)³ chiral gauge model

We consider the 21(-1)³ chiral gauge model which is defined by the charge assignment of the U(1) gauge symmetry as

\[
Q = \text{diag}(q_1, q_2, q_3, q_4) = \text{diag}(+2, 0, 0, 0), \quad (6.1)
\]

\[
Q' = \text{diag}(q'_1, q'_2, q'_3, q'_4) = \text{diag}(+1, -1, -1, -1). \quad (6.2)
\]
We note that $Q$ and $Q'$ can be regarded as the linear combinations of the axial charge of $U(1)_A$ and the Cartan subalgebra of $Spin(6)$, $\{\Sigma^{12}, \Sigma^{34}, \Sigma^{56}\}$, in the previous four-flavor axial gauge model as follows,

$$Q = +\frac{1}{2} + \Sigma^{12} + \Sigma^{34} + \Sigma^{56},$$

(6.3)

$$Q' = -\frac{1}{2} + \Sigma^{12} + \Sigma^{34} + \Sigma^{56},$$

(6.4)

assuming that in the weak gauge-coupling limit the left- and right-handed Weyl fermions, $\psi_-(x)$ and $\psi_+^c(x)$, are in $\mathbf{4}$, the four-dimensional irreducible (spinor) representation of $Spin(6)$ ($SO(6)$).

A comment is in order about the relation with $21^4$ model. In the $21^4$ model, the anomaly matching condition for the flavor chiral $SU(4)$ symmetry can be saturated by a Majorana-Weyl field in $\mathbf{6}$ of $SU(4)$, and it predicts the appearance of such excitation as a composite state\cite{33–35}. However, it is known to be difficult to formulate the local lattice action of Majorana-Weyl fermions without species doubling (even with overlap fermions)\cite{133}. Then, it seems difficult to formulate the three neutral spectator Weyl fields, $0^3$, into the Majorana-Weyl field in $\mathbf{6}$ of $SU(4)$ to saturate the anomaly. In the case of the $21(-1)^3$ model, on the other hand, the anomaly matching condition for the flavor chiral $SU(3)$ symmetry can be saturated by a Weyl field in $\mathbf{3}$ of $SU(3)$, and the three neutral spectator Weyl fields, $0^3$, can do the job.

### 6.2 21(-1)$^3$ chiral gauge model in the mirror-fermion approach

To formulate the $21(-1)^3$ model in the mirror fermion approach, we introduce that the (four-flavor) right- and left-handed mirror fermions, $\psi_+(x)$ and $\psi_+^c(x)$. Then, as shown in table 4, the remaining continuous symmetry in the mirror sector is the vector flavor symmetry $SU(3)$, the vector and axial $U(1)$ symmetries $U(1)_b$ and $U(1)_a$ acting on the flavor $SU(3)$ sector, and another vector $U(1)$ symmetry $U(1)_{b-3l}$. For $SU(3)$ and $U(1)_{b-3l}$, the would-be gauge anomalies are matched. $U(1)_b$ and $U(1)_a$ are anomalous and should be broken explicitly.

|       | + | + | - | - | (mixed) gauge anomaly | chiral anomaly |
|-------|---|---|---|---|-----------------------|---------------|
| $U(1)_g$ | 2 | 0 | 1 | -1 | matched (gauged) | —             |
| SU(3)   | 1 | 3 | 1 | 3  | matched (can be gauged) | anomaly free |
| $U(1)_b$ | 0 | 1 | 0 | 1  | not matched | anomalous       |
| $U(1)_a$ | 0 | 1 | 0 | -1 | not matched | anomalous       |
| $U(1)_{b-3l}$ | -3 | 1 | -3 | 1  | matched (can be gauged) | anomaly free |

Table 4. Fermionic continuous symmetries in the mirror sector of the $21(-1)^3$ model and their would-be gauge anomalies

Then we can formulate the mirror fermion sector of the $21(-1)^3$ model in the same manner as that of the $1^4(-1)^4$ model given by eq. (5.12), using the Majorana-type Yukawa-couplings to the auxiliary $SO(6)$-vector spin fields, $E^a(x), E^a(x)$ $(a = 1, \cdots , 6)$ with the
unit lengths $E^a(x)E^a(x) = 1$, $\bar{E}^a(x)\bar{E}^a(x) = 1$. And we can consider the limit $z/h \to 0$ and $\kappa \to 0$ in the mirror fermion sector, where the kinetic terms of the mirror fermion and the spin fields are both suppressed.

The mirror fermion sector of the $21(-1)^3$ model so defined shares almost all the properties in the weak gauge-coupling limit with that of the $1^4(-1)^4$ model. The only non-trivial one is the behavior of the vertex functions of the U(1) gauge field. In fig. 15, we show the numerical-simulation results of $\tilde{\Pi}_{00}^Q(k)$ and $\tilde{\Pi}_{01}^Q(k)$ for the lattice size $L = 8$ and the periodic boundary condition of the mirror fermion fields. ($\hat{Q}$ is the abbreviation for $Q, Q' = \pm 1/2 + \Sigma^{12} + \Sigma^{34} + \Sigma^{56}$ on $\psi_+, \psi'_-$, respectively.) 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones. In fig. 16, it is verified that the Ward-Takahashi relations are satisfied up to the statistical error. These results should be compared with that of the (target) Weyl fermions $\Pi_{\mu\nu}(k)$ shown in fig. 12. From these results, we can see that the mirror fermion contribution of the $21(-1)^3$ model, $\tilde{\Pi}_{\mu\nu}^Q(k)$, does not show any evidence of the singularities due to charged massless excitations. It behaves like a regular function of momentum $k_\mu$. This result is again consistent with the fact that the mirror fermions decouple by acquiring the masses of order the inverse lattice spacing and leave only local terms in the effective action.

6.3 Weyl-field measure through the saturation of the mirror-fermion part of Dirac-field measure by the ’t Hooft vertices

Let us recall the fermion path-integral of the $21(-1)^3$ chiral gauge model in the mirror-fermion approach. In particular, in the limit where the kinetic terms of the mirror fermions and the spin fields are both suppressed ($z/h \to 0$ and $\kappa \to 0$), it is formulated as follows.

$$e^{\Gamma_{\text{mirror}}[U]} = \langle 1 \rangle_W \langle 1 \rangle_M = \int \mathcal{D}[\psi_-] \mathcal{D}[\bar{\psi}_-] \mathcal{D}[\psi'_+] \mathcal{D}[\bar{\psi}'_+] e^{-S_W} \times$$

$$\int \mathcal{D}[\psi_+] \mathcal{D}[\bar{\psi}_+] \mathcal{D}[\psi'_-] \mathcal{D}[\bar{\psi}'_-] \mathcal{D}[E^a] \mathcal{D}[\bar{E}^a] e^{-S_M}$$

$$= \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[\psi'] \mathcal{D}[\bar{\psi}'] \mathcal{D}[E^a] \mathcal{D}[\bar{E}^a] e^{-S_W - S_M}, \quad (6.5)$$

where

$$S_W = \sum_{x \in \Gamma} \bar{\psi}(x)P_+ D\psi(x) + \sum_{x \in \Gamma} \bar{\psi}'(x)P_- D'\psi'(x), \quad (6.6)$$

$$S_M = \sum_{x \in \Gamma} \left\{ \psi_+(x)^T i\gamma_3 c_D T^a E^a(x) \psi'_-(x) + \bar{\psi}_+(x)i\gamma_3 c_D T^{a\dagger} \bar{E}^a(x) \bar{\psi}'_-(x)^T \right\}. \quad (6.7)$$

This formula can be rewritten further through the path-integration of $E^a(x)$ and $\bar{E}^a(x)$ using the integral,

$$(\pi^3)^{-1} \int \prod_{a=1}^6 de^a \delta(\sqrt{e^a e^b} - 1) e^{e^a u^a} = 2! \sum_{k=0}^{\infty} \frac{u^k}{k!(k + 2)!} \bigg|_{u=(1/2) u^a u^a}. \quad (6.8)$$
Figure 15. $L^2 \tilde{\Pi}^\circ\breve{Q}^Q_0(k)$ [left] and $L^2 \tilde{\Pi}^\circ\breve{Q}^Q_1(k)$ [right] vs. $|k|_2 \equiv \sqrt{k_0^2 + k_1^2}$. The lattice size is $L = 8$. The periodic boundary condition is assumed for the fermion fields. The black-, blue-, red-symbol plots are along the spacial momentum axis ($k_0 = 0$), the temporal momentum (energy) axis ($k_1 = 0$) and the diagonal momentum axis ($k_0 = k_1$), respectively. 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

The result is given by

$$e^{\Gamma_{\text{mirror}}[U]} = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[\psi'] \mathcal{D}[\bar{\psi}'] \prod_x F(O_V(x)) \prod_x F(\bar{O}_V(x)) \ e^{-S_W},$$

(6.9)

where the function $F(\omega)$ is defined by

$$F(\omega) \equiv 2! \sum_{k=0}^{\infty} \frac{\omega^k}{k!(k+2)!} = 2! \left(\frac{z}{2}\right)^{-2} I_2(z) \bigg|_{(z/2)^2 = w}$$

(6.10)

and $I_\nu(w)$ is the modified Bessel function of the first kind. $O_V(x)$ and $\bar{O}_V(x)$ are given by eqs. (5.9) and (5.10),

$$O_V(x) = \frac{1}{2} \psi_+(x)^T i\gamma_3 c_D T^a \psi'_+(x) \psi_+(x)^T i\gamma_3 c_D T^a \psi'_-(x),$$

(6.11)

$$\bar{O}_V(x) = \frac{1}{2} \bar{\psi}_+(x)i\gamma_3 c_D T^a \bar{\psi}'_+(x)^T \bar{\psi}_+(x)i\gamma_3 c_D T^a \bar{\psi}'_-(x)^T.$$

(6.12)
Figure 16. $2 \sin(k_\mu/2) \times L^2 \tilde{\Pi}^{QQ}_0(k)$ [left] and $2 \sin(k_\mu/2) \times L^2 \tilde{\Pi}^{QQ}_{11}(k)$ [right] vs. $|k|_2 \equiv \sqrt{k_{0}^2 + k_{1}^2}$. The lattice size is $L = 8$. The periodic boundary condition is assumed for the fermion fields. The black-, blue-, red-symbol plots are along the spatial momentum axis ($k_0 = 0$), the temporal momentum (energy) axis ($k_1 = 0$) and the diagonal momentum axis ($k_0 = k_1$), respectively. 5,000 configurations are sampled with the interval of 20 trajectories. The errors are simple statistical ones.

We note that in this model the counterparts of the ’t Hooft vertices in the mirror sector, which are induced by the U(1) instantons in two-dimensions, can be written as

$$O_T(x) = \frac{1}{2} O_V(x)O_V(x), \quad \bar{O}_T(x) = \frac{1}{2} O_V(x)\bar{O}_V(x)O_V(x).$$  \hspace{1cm} (6.13)

Therefore $O_V(x)$ and $\bar{O}_V(x)$ give essentially the ’t Hooft vertices in the mirror sector.

The above result implies that the path-integral measure of the (target) Weyl fields $\psi_-(x), \bar{\psi}_-(x)$ and $\psi'_+(x), \bar{\psi}'_+(x)$ in the $21(-1)^3$ model can be defined simply through the saturation of the mirror-fermion part of the Dirac field measure by the suitable products of the ’t Hooft vertices in terms of the mirror-fermion fields. Namely, one can define the path-integral measure of the (target) Weyl fields $\psi_-(x), \bar{\psi}_-(x)$ and $\psi'_+(x), \bar{\psi}'_+(x)$ as

$$D_\ast[\psi_-]D_\ast[\bar{\psi}_-]D_\ast[\psi'_+]D_\ast[\bar{\psi}'_+] \equiv \prod_{x} \prod_{s=1}^{4} \prod_{\alpha=1}^{2} \{d\psi_{s\alpha}(x) d\bar{\psi}_{s\alpha}(x) d\psi'_{s\alpha}(x) d\bar{\psi}'_{s\alpha}(x)\} \times \prod_{x} F(O_V(x)) \prod_{x} F(\bar{O}_V(x)).$$  \hspace{1cm} (6.14)
It is manifestly gauge-invariant, but it depends on the gauge (link) field through the chiral projectors $\hat{P} +$ and $\hat{P}' -$ which necessarily appear in the definitions of $O^V(x)$ in terms of the mirror-fermion fields $\psi_+(x) = \hat{P} \psi(x)$ and $\psi'_-(x) = \hat{P}' \psi'(x)$. It applies to all topological sectors of the admissible $U(1)$ link field $\mathfrak{U}[m]$, because the matrix $(u^T i\gamma_3 c_D T^a E^a v')$ can change in size depending on the topological charge, but not in shape (square).

6.4 A solution to the measure term current required for the reconstruction theorem

As discussed in section 3, the variation of the effective action $\Gamma_{\text{mirror}}[U]$ with respect to the $U(1)$ link field is given by

$$\delta_\eta \Gamma_{\text{mirror}}[U] = \langle -\delta_\eta S_W \rangle_W / \langle 1 \rangle_W + \langle -\delta_\eta S_M \rangle_M / \langle 1 \rangle_M$$

$$= \text{Tr} \{ P_+ \delta_\eta D D^{-1} \} + \text{Tr} \{ P_- \delta_\eta D' D'^{-1} \} + \langle -\delta_\eta S_M \rangle_M / \langle 1 \rangle_M,$$

(6.15)

and the contribution of the mirror sector, $\langle -\delta_\eta S_M \rangle_M / \langle 1 \rangle_M$, should play the role of the measure term $\mathcal{L}_\eta$. In the weak gauge-coupling expansion, the vertex functions are derived from this contribution as

$$\langle -\delta_\eta S_M \rangle_M / \langle 1 \rangle_M = \sum_{m=0}^{\infty} \frac{1}{2^{2m+1} m!} \sum_{k,p_1,\ldots,p_m} \tilde{\eta}_\mu(-k) \Gamma_{\mu\nu_1,\ldots,\nu_m}(k,p_1,\ldots,p_m) \tilde{A}_\nu(p_1) \cdots \tilde{A}_{\nu_m}(p_m).$$

(6.16)

The simulation results about the leading two-point vertex function shown in fig. 15 provide a numerical evidence that $\langle -\delta_\eta S_M \rangle_M / \langle 1 \rangle_M$ is a local functional of the $U(1)$ link field. Then it (or its axial part) can provide a solution to the local current required in the reconstruction theorem of the Weyl fermion measure reviewed in section 2.

Thus the Weyl field measure defined by eq. (6.14) can provide a solution to the gauge-invariant and local construction of the fermion path-integral measure in the $21(-1)^3$ chiral lattice gauge theory.

7 The mirror-fermion models through three-dimensional domain wall fermions with boundary interaction terms

The mirror-fermion models formulated with overlap fermions in sections 5 and 6, the $1^4(-1)^4$- and $21(-1)^3$- models, can be also constructed through the three-dimensional vector-like domain wall fermions by adding suitable boundary interaction terms[114]. The following action defines the explicit form of the boundary terms which precisely reproduce the $U(1)_A \times \text{Spin}(6)(\text{SU}(4))$-invariant multi-fermion interaction in the mirror sector.
without the singularity in the large-coupling limit.

\[
S_{DW} = \sum_{t=1}^{L_3} \sum_{x \in \Lambda} \bar{\psi}(x, t) \{ [1 + a'_3(D_{2w} - m_0)] \delta_{tt'} - P_- \delta_{t+1,t'} - P_+ \delta_{t,t'+1} \} \psi(x, t'),
\]

\[
S'_{DW} = \sum_{t=1}^{L_3} \sum_{x \in \Lambda} \bar{\psi}'(x, t) \{ [1 + a'_3(D'_{2w} - m_0)] \delta_{tt'} - P_+ \delta_{t+1,t'} - P_- \delta_{t,t'+1} \} \psi'(x, t'),
\]

\[
S_{bd} = \sum_{x \in \Lambda} (z - 1) \bar{\psi}(x, L_3) P_- [1 + a'_3(D_{2w} - m_0)] \psi(x, L_3)
+ \sum_{x \in \Lambda} (z - 1) \bar{\psi}'(x, L_3) P_+ [1 + a'_3(D'_{2w} - m_0)] \psi'(x, L_3)
+ \sum_{x \in \Lambda} h \{ \bar{\psi}^T(x, L_3) i\gamma_3 c_D T^a E^a(x) \psi'(x, L_3) + \bar{\psi}(x, L_3) P_- i\gamma_3 c_D T^a \bar{E}^a(x) \psi'(x, L_3) \}
+ \sum_{x,\mu} \kappa \{ E^a(x) E^a(x + \mu) + \bar{E}^a(x) \bar{E}^a(x + \mu) \},
\]

where the Dirichlet b.c. is assumed,

\[
P_+ \psi(x, 0) = 0, \quad \bar{\psi}(x, 0) P_- = 0 ; \quad P_- \psi(x, L_3 + 1) = 0, \quad \bar{\psi}(x, L_3 + 1) P_+ = 0,
\]

\[
P_- \psi'(x, 0) = 0, \quad \bar{\psi}'(x, 0) P_+ = 0 ; \quad P_+ \psi'(x, L_3 + 1) = 0, \quad \bar{\psi}'(x, L_3 + 1) P_- = 0.
\]

$D_{2w}$ and $D'_{2w}$ are the two-dimensional Wilson-Dirac operators and $a'_3 (= a_3/a)$ is the lattice spacing of extra third dimension in the lattice unit. The three-dimensional Dirac fields $\psi(x, t)$ and $\psi'(x, t)$ are assumed in 4, the four-dimensional irreducible spinor representation of SO(6), satisfying the constraints

\[
P_+ \psi(x) = +\psi(x), \quad \bar{\psi}(x) P_+ = +\bar{\psi}(x), \quad P_+ \psi'(x) = +\psi'(x), \quad \bar{\psi}'(x) P_+ = +\bar{\psi}'(x).
\]

In the boundary action $S_{bd}$, the first and second terms in the r.h.s. are introduced so that all the terms which involve the fields $\bar{\psi}(x, L_3) P_-$ and $\bar{\psi}'(x, L_3) P_+$ in the original bulk actions of the domain wall fermions $S_{DW}$ and $S'_{DW}$ can be rescaled by the factor $z$ and made vanished in the limit $z \to 0$. Then the second part of the third term of the Majorana-Yukawa couplings is required so that it saturates the path-integral measure of those fields $\bar{\psi}(x, L_3) P_-$ and $\bar{\psi}'(x, L_3) P_+$. On the other hand, the fields $\psi(x, L_3)$ and $\psi'(x, L_3)$ are related to the (truncated) overlap fermion fields $\psi(x)$ and $\psi'(x)$ by the relations $\psi(x, L_3) = (-\gamma_5)(1 + T L_3)^{-1} \psi(x)$ and $\psi'(x, L_3) = (-\gamma_5)(1 + T' L_3)^{-1} \psi'(x)$, respectively in the usual subtraction scheme with the anti-periodic b.c.\cite{41, 42}. They are projected to the right- and left-handed Weyl fields $\psi_+(x) = \hat{P}_+ \psi(x)$ and $\psi_-'(x) = \hat{P}_- \psi'(x)$, respectively in the limit $L_3 \to \infty$ and $a'_3 \to 0$. See [43] for detail.
Then one can show the equality of the fermion partition functions\[41–43\],

\[
\langle 1 \rangle_{w_m} = \int \mathcal{D}[E^a] \mathcal{D}[\bar{E}^a] \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[\psi'] \mathcal{D}[\bar{\psi}'] e^{-S_w - S_m} = \lim_{a' \to 0} \lim_{L_3 \to \infty} \int \mathcal{D}[E^a] \mathcal{D}[\bar{E}^a] \int \prod_{x,t} d\bar{\psi}d\psi d\bar{\psi}'d\psi'(x,t) e^{-S_{dw} - S_{dw}' - S_{sd}} \bigg|_{\text{Dir}},
\]

(7.5)

where \( S_w \) and \( S_m \) are given by eq. (3.16) and eq. (5.12), respectively.

From the above equality, we can see that the limits \( z/h \to 0 \) and \( \kappa \to 0 \) are both well-defined in the domain wall formulation. In this respect, we note that if one uses the boundary field variables introduced by Furmam and Shamir,

\[
q(x) = \psi_-(x,1) + \psi_+(x,L_3), \quad \bar{q}(x) = \bar{\psi}_-(x,1) + \bar{\psi}_+(x,L_3)
\]

(7.6)

\[
q'(x) = \psi_+(x,1) + \psi_-(x,L_3), \quad \bar{q}'(x) = \bar{\psi}_+(x,1) + \bar{\psi}_-(x,L_3),
\]

(7.7)

and formulate the boundary interaction terms\[114\] as

\[
\sum_{x \in \Lambda} h \{ q^T(x)i\gamma_3cD T^a E^a(x)P_-q'(x) + \bar{q}(x)P_-i\gamma_3cD T^a \bar{E}^a(x)q'(x)^T \},
\]

(7.8)

it is singular in the large-coupling limit \( z/h \to 0 \). This is because \( q(x) \) and \( q'(x) \) can be related to the overlap Dirac fields as \( q(x) = (1 - D)\psi(x) \) and \( q'(x) = (1 - D')\psi'(x) \)[43] and the factors \( (1 - D) \) and \( (1 - D') \) project out the modes with the momenta \( p_\mu = (\pi,0),(0,\pi),(\pi,\pi) \) in the interaction terms.

8 Relations to 1D/2D Topological Insulators/Superconductors with Gapped Boundary Phases

8.1 Eight-flavor 1D Majorana chain with SO(7)-invariant quartic interaction

--- A 1+1D classical formulation on Euclidean lattice

Let us recall the fact that the mirror-fermion sectors of the \( 1^4(-1)^4 \) axial gauge model and the \( 21(-1)^3 \) chiral gauge model both consist of the four-flavor right- and left-handed Weyl fields, \( \psi_+(x) \) and \( \psi'_-(x) \). In the weak gauge-coupling limit, these Weyl fields can be combined into four Dirac fields \( \psi(x) = \psi_+(x) + \psi'_-(x) \). Then the action of the mirror sector eq. (5.12) may be written for \( z = 1 \) and \( \kappa = 0 \) as follows

\[
S_M = \sum_x \{ \bar{\psi}(x) D \psi(x) + h (\bar{\psi}_+^T i\gamma_3 cD \bar{T}^a E^a(x) \psi_-(x) + \bar{\psi}_+(x) i\gamma_3 cD T^a \bar{E}^a(x) \psi_-^T(x)^T) \},
\]

(8.1)

After the path-integration of the spin fields \( E^a(x) \) and \( \bar{E}^a(x) \), if one keeps only the leading terms of the multi-fermion interactions, the effective action may be given by

\[
S'_M = \sum_x \{ \bar{\psi}(x) D \psi(x) - \frac{h^2}{6} \left( [\bar{\psi}_+(x)T i\gamma_3 cD \bar{T}^a \psi_-(x)]^2 + [\bar{\psi}_+(x)i\gamma_3 cD T^a \bar{E}^a(x) \psi_-^T(x)^T]^2 \right) \},
\]

(8.2)
This model with $U(1)_A \times \text{Spin}(6)(\text{SO}(6))$ symmetry has a close relation with the eight-flavor 1D Majorana chain with $\text{SO}(7)$-invariant quartic interaction, which was examined by Fidkowski and Kitaev\cite{115,116}, when it is formulated in two-dimensional Euclidean spacetime.

The eight-flavor 1D Majorana chain with $\text{SO}(7)$-invariant quartic interaction\cite{115,116} is defined by the following quantum Hamiltonian.

$$\hat{H} = \sum_{\alpha=1}^{8} \hat{H}_\alpha + \hat{V},$$

where

$$\hat{H}_\alpha = \frac{i}{2} \left( u \sum_{l=1}^{n} c^\alpha_{2l-1} c^\alpha_{2l} + v \sum_{l=1}^{n-1} c^\alpha_{2l} c^\alpha_{2l+1} \right),$$

$$\hat{V} = \sum_{l=1}^{n} (\hat{W}_{2l-1} + \hat{W}_{2l})$$

and

$$\hat{W} = c^1 c^2 c^3 c^4 + c^5 c^6 c^7 c^8 + c^1 c^2 c^5 c^6 + c^3 c^4 c^7 c^8 - c^2 c^3 c^5 c^7 - c^1 c^4 c^5 c^8 + c^1 c^3 c^5 c^7 + c^3 c^4 c^6 c^8 + c^1 c^2 c^7 c^8 - c^2 c^3 c^5 c^8 - c^1 c^4 c^6 c^7 - c^1 c^4 c^6 c^8 - c^1 c^3 c^5 c^8 - c^2 c^3 c^5 c^7,$$

$$\hat{W}_m = \hat{W}_{|_{c^\alpha \rightarrow c^\alpha_m}}.$$  \hspace{1cm} (8.7)

The quartic interaction $\hat{W}$ is invariant under the $\text{SO}(7)$ transformation which acts on the operators $c^\alpha (\alpha = 1, \cdots, 8)$ so that they are in $\mathbf{8}$, the irreducible spinor representation of $\text{SO}(7)$.\footnote{The eight-component Majorana operator $c^\alpha$ can be regarded as a real vector or a real spinor of $\text{SO}(8)$, thanks to the triality of $\text{SO}(8)$.} In fact, as shown by Y.-Z. You and C. Xu \cite{112}, the operator $\hat{W}$ can be written into the form which is manifestly $\text{SO}(7)$-invariant:

$$\hat{W} = -\frac{1}{4!} \left( \sum_{a=1}^{7} c^T a \hat{c} c^T a \hat{c} - 16 \right),$$

where $\{ \gamma^a | a = 1, \cdots, 7 \}$ is the set of the gamma matrices for $\text{SO}(7)$ given explicitly as

$$\gamma^1 = I \otimes I \otimes \sigma_2,$$  \hspace{1cm} (8.9)

$$\gamma^2 = \sigma_3 \otimes \sigma_2 \otimes \sigma_3,$$  \hspace{1cm} (8.10)

$$\gamma^3 = I \otimes \sigma_2 \otimes \sigma_1,$$  \hspace{1cm} (8.11)

$$\gamma^4 = \sigma_2 \otimes I \otimes \sigma_3,$$  \hspace{1cm} (8.12)

$$\gamma^5 = \sigma_2 \otimes \sigma_3 \otimes \sigma_1,$$  \hspace{1cm} (8.13)

$$\gamma^6 = \sigma_1 \otimes \sigma_2 \otimes \sigma_3,$$  \hspace{1cm} (8.14)

$$\gamma^7 = \sigma_2 \otimes \sigma_1 \otimes \sigma_1.$$  \hspace{1cm} (8.15)
which are pure imaginary and anti-symmetric, $\gamma^a T = -\gamma^a$. This Hamiltonian can be rewritten further using the two-component Majorana field operator, $\hat{\psi}^\dagger = (\hat{c}_{2\ell}^\dagger, \hat{c}_{2\ell-1}^\dagger)^T$ as

$$\hat{H}_a = \sum_{l=1}^n \frac{\hat{\gamma}}{2} \hat{\psi}^\dagger \{ \alpha \frac{1}{2\ell} (\nabla - \nabla^\dagger) + \beta \frac{1}{2\ell} \nabla \nabla^\dagger + \beta m_0 \} \hat{\psi}^\dagger, \tag{8.16}$$

$$\hat{V} = -\frac{1}{4!} \sum_{l=1}^n \left\{ \frac{7}{a=1} (\hat{\psi}^\dagger \hat{P}_+ \gamma^a \hat{\psi})^2 + \sum_{a=1}^7 (\hat{\psi}^\dagger \hat{P}_- \gamma^a \hat{\psi})^2 - 32 \right\}, \tag{8.17}$$

where $\alpha = -\sigma_1, \beta = \sigma_2, \hat{P}_\pm = (1 \pm i\beta\alpha)/2 = (1 \pm \sigma_3)/2$ and the matching condition of the couplings are given by $v = 2\pi$ and $u/v = 1 + m_0$. We note that the Majorana condition for the eight-flavor Majorana field $\psi^a(x)$ is formulated in general by $\psi(x) = \psi(x)^\dagger \gamma^0 = \psi(x)^T d C$. The choice of the representation in the above case is understood as follows: $\gamma^0 = \beta = \sigma_2, d_C = i\sigma_2$, and $C = -i$. The time-reversal transformation acts on $\hat{\psi}^\dagger \hat{\psi}$ as $\hat{T} \hat{\psi}^\dagger \hat{T}^{-1} = \sigma_3 \hat{\psi}^\dagger \hat{\psi}$ and $\hat{H}^0, \hat{V}$ are both invariant.

The 1D quantum lattice model of the eight-flavor Majorana chain defined with $\hat{H}$ may be formulated as a 1+1D classical lattice model in the Euclidean metric within the framework of the path-integral quantization[135, 136]. The action can be chosen as

$$S_{SMC} = \sum_x \left\{ \frac{\nu}{2} \psi_M(x)^T d C (D_w + m_0) \psi_M(x) \right.$$  

$$-\frac{1}{4!} \sum_{a=1}^7 (\psi_M(x)^T d C \left( \frac{\gamma^0 - i\gamma_3}{2} \right) C \Gamma^a \psi_M(x))^2 \right.$$  

$$\left. -\frac{1}{4!} \sum_{a=1}^7 (\psi_M(x)^T d C \left( \frac{\gamma^0 + i\gamma_3}{2} \right) C \Gamma^a \psi_M(x))^2 \right\}, \tag{8.18}$$

where $D_w$ is the two-dimensional massless Wilson-Dirac operator, $D_w = \sum_{\mu} \{ \gamma_{\mu} (\nabla_{\mu} - \nabla^\dagger_{\nu}/2 + \nabla_{\mu} \nabla^\dagger_{\nu}/2) \}$. $\psi_M(x)$ is the Grassmann-number field, obeying the constraint $\psi_M(x)^\dagger = \psi_M(x) d C$, if it is taken as complex. One may assume generic representations for the Dirac- and SO(7)-gamma matrices. (Our choice of the representation of the Dirac gamma matrices in the Euclidean metric is specified as $\gamma^0 = \sigma_1, \gamma^1 = \sigma_2, \gamma^3 = \sigma_3$.) The action is invariant under the parity and charge conjugation transformations, $P : \psi_M(x) \rightarrow i\gamma_0 \psi_M(x_P)$ where $x_P = (x_0, -x_1)$ and $C : \psi_M(x) \rightarrow \psi_M(x)$. We note that the SO(7)-invariant quartic interaction terms possess the $Z_2$ symmetry under the discrete chiral transformation, $Z_2 : \psi_M(x) \rightarrow \gamma_3 \psi_M(x)$, but it is broken by the mass and Wilson terms. We also note that the quartic interaction terms do not respect the covariance w.r.t. 2 dim. (hyper-cubic) rotation in the case of Euclidean metric nor Lorentz transformation in the case of Minkowski metric by the terms with $\gamma_0$.

The eight-flavor Majorana field in of SO(7), $\psi_M(x)$, can be composed into the four-flavor Dirac pairs of left- and right-handed Weyl fields in of SO(6), $\psi(x) = \psi_+(x) + \psi_-(x)$. In the representation of the SO(7) gamma matrices specified in section 5, we have $C = \tilde{C} \otimes \sigma_2, \Gamma^a = T^d \otimes \sigma_3 (a' = 1, \cdots, 5), \Gamma^6 = T^d \otimes I, \Gamma^7 = T^5 = \overline{C} \otimes (i\sigma_1)$ and $(\Gamma^a)^T = \tilde{C}^T T^a C (a' = 1, \cdots, 5), (T^6)^T = -T^6$. Therefore the Majorana field $\psi_M(x)$
can be parametrized as
\[ \psi_M(x) = \begin{pmatrix} \psi(x) \\ -ic_D \bar{\psi}(x)^T \end{pmatrix}. \] (8.19)

And the bilinear operators in the quartic interaction can be rewritten as
\[ (\psi_M(x)^T c_D (\mp i \gamma_3) C T^a \psi_M(x)) = \pm \left\{ \psi(x) i \gamma_3 c_D \bar{T}^a \psi(x) - \bar{\psi}(x) i \gamma_3 c_D \bar{T}^a \bar{\psi}(x)^T \right\}, \]
\[ = \pm 2 \left\{ \psi_+ (x) i \gamma_3 c_D \bar{T}^a \psi_- (x) - \bar{\psi}_+ (x) i \gamma_3 c_D \bar{T}^a \bar{\psi}_- (x)^T \right\} \] (8.20)
\[ (\psi_M(x)^T c_D (\mp i \gamma_3) C T^7 \psi_M(x)) = \mp 2 \bar{\psi}(x) i \gamma_3 \psi(x) \]
\[ = \mp 2 \left\{ \psi_- (x) i \gamma_3 \psi_+ (x) + \bar{\psi}_+ (x) i \gamma_3 \bar{\psi}_- (x) \right\} \] (8.21)
\[ (\psi_M(x)^T c_D (\gamma_0) C T^a \psi_M(x)) = \psi(x) c_D \gamma_0 \bar{T}^a \psi(x) - \bar{\psi}(x) c_D \gamma_0 \bar{T}^a \bar{\psi}(x)^T \]
\[ = \psi_+(x) c_D \gamma_0 \bar{T}^a \psi_+(x) - \bar{\psi}_+(x) c_D \gamma_0 \bar{T}^a \bar{\psi}_+(x)^T \]
\[ + \psi_- (x) c_D \gamma_0 \bar{T}^a \psi_- (x) - \bar{\psi}_- (x) c_D \gamma_0 \bar{T}^a \bar{\psi}_- (x)^T, \] (8.22)
\[ (\psi_M(x)^T c_D (\gamma_0) C T^7 \psi_M(x)) = +2 \bar{\psi}(x) \gamma_0 \psi(x) \]
\[ = +2 \left\{ \psi_+ (x) \gamma_0 \psi_+ (x) + \bar{\psi}_- (x) \gamma_0 \bar{\psi}_- (x) \right\}, \] (8.23)
for \( a = 1, \ldots, 6 \). The four quartic terms, which are obtained with the above bilinear operators squared, compose the original SO(7)-invariant interaction terms. \( U(1)_V \) is broken by the first and third ones. \( U(1)_A \) is broken by the second and third ones, but its \( Z_2 \) subgroup is preserved.

One can reduce the SO(7) symmetry of the model to SO(6) by restricting the summations of the group index in the quartic interaction \( V \) to \( a = 1, \ldots, 6 \) without affecting the non-degenerate gapped ground state[112, 115]. Then, the action of 1+1D lattice model of the eight-flavor Majorana chain with the reduced SO(6) symmetry can be given by
\[ S_{SMC/SO(6)} = \sum_x \left\{ \frac{1}{2} \sum_{a=1}^{6} \left( \psi_+ (x) i \gamma_3 c_D \bar{T}^a \psi_- (x) - \bar{\psi}_+ (x) i \gamma_3 c_D \bar{T}^a \bar{\psi}_- (x)^T \right)^2 \right. \]
\[ - \frac{1}{12} \sum_{a=1}^{6} \left( \psi_+ (x) c_D \gamma_0 \bar{T}^a \psi_+ (x) + \bar{\psi}_+ (x) c_D \gamma_0 \bar{T}^a \bar{\psi}_+ (x)^T \right. \]
\[ - \psi_- (x) c_D \gamma_0 \bar{T}^a \psi_- (x) - \bar{\psi}_- (x) c_D \gamma_0 \bar{T}^a \bar{\psi}_- (x)^T \right)^2 \]. (8.24)

This action is invariant under the parity and charge conjugation transformations, \( P : \psi(x) \rightarrow i \gamma_0 \psi(x), \bar{\psi}(x) \rightarrow -i \bar{\psi}(x) \gamma_0 \) where \( x_P = (x_0, -x_1) \) and \( C : \psi(x) \rightarrow c_D \bar{C} \bar{\psi}(x)^T, \bar{\psi}(x) \rightarrow -\psi(x)^T c_D \bar{C}^T \).

In this model with the reduced SO(6) symmetry, the axial \( U(1)_A \) symmetry is broken by the bilinear mass- and Wilson-terms and also by the quartic, but non-covariant interaction.
term, although the $Z_2$ subgroup of $U(1)_A$ is intact by the latter. One can expect that the $Z_2$ symmetry is restored in the chiral limit $m_0 \to m_c = \pm 0 + \delta m (1/z)$ at least in the weak quartic-coupling region $1/z^2 \ll 1^{16}$. If the non-covariant quartic terms are irrelevant in this region of the couplings, then one can further expect the restoration of the covariance and the full $U(1)_A$ symmetry. To make this chiral limit clear and manifest, one can use the Ginsparg-Wilson fermion instead of the Wilson fermion, making the replacements, $D_w \to D$ and $\psi_\pm(x) = P_\pm \psi(x) \to \tilde{P}_\pm \psi(x)$. Neglecting the non-covariant terms of the quartic interaction, we obtain

$$S'_{8MC/\text{SO}(6)} = \sum_x \left\{ \bar{\psi}(x) (D + m_0) \psi(x) - \frac{1}{12} \sum_{a=1}^6 \left( \psi_+(x)i\gamma_3 c_D \tilde{T}^a \psi_-(x) - \psi_+(x)i\gamma_3 c_D \tilde{T}^a \tilde{\psi}_-(x) \right)^2 \right\}. \quad (8.25)$$

Omitting further the cross term of the quartic interaction in eq. (8.25), which is irrelevant in breaking the $U(1)_V$ symmetry, we end up with the action eq. (8.2) with the matching condition $h^2 = 1/(2z^2)^{17}$. In this case, however, we note that the charge conjugation invariance is not manifest, but the chiral projection operators are interchanged as follows,$^{[137–139]}$

$$\psi_\pm(x) = \tilde{P}_\pm \psi(x) \to \psi_\pm = P_\pm \psi(x), \quad (8.26)$$
$$\tilde{\psi}_\pm(x) = \tilde{\psi}(x) P_\pm \to \tilde{\psi}_\pm = \tilde{\psi} \{ \gamma_3 \tilde{P}_\pm \gamma_3 \}(x). \quad (8.27)$$

Thus our four-flavor axial model with $U(1)_A \times \text{Spin}(6)(\text{SU}(4))$ symmetry can be regarded as an effective model for the chiral limit of the eight-flavor 1D Majorana chain with the reduced $\text{SO}(6)$ symmetry$^{[112, 115]}$. The rigorous result about the mass gap of the eight-flavor 1D Majorana chain with the $\text{SO}(7)$-invariant quartic interaction by Fidkowski and Kitaev$^{[115]}$ and its extension to the model with the reduced $\text{SO}(6)$ symmetry by Y.-Z. You and C. Xu$^{[112]}$ therefore suggest strongly that the four-flavor axial model with $U(1)_A \times \text{Spin}(6)(\text{SU}(4))$ symmetry is indeed gapped. On the other hand, our numerical-simulation results that the correlation lengths of the mirror-sector fields are of order multiple lattice spacings provide a numerical evidence for the mass gap of the eight-flavor 1D Majorana chain based on the framework of 1+1D Euclidean path-integral quantization. What is actually new in our numerical-simulation results is the finding that the two-point vertex functions of the $U(1)_A$ and $\text{Spin}(6)(\text{SU}(4))$ gauge fields are regular and local, which implies that the gapped eight-flavor 1D Majorana chain with the reduced $\text{SO}(6)$ symmetry is indeed robust against the couplings of (external) gauge fields to its continuous symmetries.$^{16}$

$^{16}$This point deserves further studies, because this question is related to Aoki phase.

$^{17}$In the original work by Fidkowski and Kitaev$^{[115]}$, they simply neglected the non-cross terms of the quartic interaction (as well as the non-covariant terms) in order to respect the vector $U(1)_V$ symmetry for their continuum-theory analysis of the phase transition at the chiral limit $m_0 \to \pm 0$. In this case, the effective model is given by the Gross-Neveu-type model with $\text{SO}(7)$ symmetry.
8.2 Eight-flavor 2D Topological Superconductor with Gapped Boundary Phase
— A description of gapped boundary phase in terms of overlap fermions

The 2+1D domain wall fermion (using Wilson fermions) is nothing but a classical Euclidean formulation of the 2D Topological Insulator (Chern Insulator/IQHE without time-reversal symmetry: class A in 2D classified by $Z$)[126–128]. Then our result in section 7 provides the explicit procedure to bridge between the two constructions for 1+1D chiral gauge theories, the 2+1D classical construction of the domain wall fermion with boundary interactions to decouple the mirror-modes[114] and the 2D quantum Hamiltonian construction of TI/TSC with gapped boundary phases[109–113]. The 1+1D mirror-fermion model in terms of overlap fermions is derived precisely as a low-energy effective local lattice theory of the domain wall fermion, and the lattice theory can describe directly the gapless/gapped boundary phases of the TI/TSC.

To illustrate the above point, it is instructive to consider the eight-flavor 2D chiral p-wave TSC with the time-reversal and $Z_2$ symmetries (class D'/DIII+R in 2D classified by $Z_8$)[129–132]. In this class of TSC, the edge modes are 1+1D Majorana fermions those are protected from acquiring mass terms by the discrete chiral symmetry, $\psi_M(x) \rightarrow \gamma_3 \psi_M(x)$ in the continuum limit. When these Majorana fermions are described by the lattice theory of overlap fermions, however, the asymmetric assignment of the chiral operators $\hat{\gamma}_3$ and $\gamma_3$ to the fields and the anti-fields as $\psi(x) \rightarrow \hat{\gamma}_3 \psi(x)$ and $\bar{\psi}(x) \rightarrow \bar{\psi}(x)(-\gamma_3)$, respectively contradicts the Majorana condition $\bar{\psi}_M(x) = \psi_M(x)^T e_D C$. Then one needs to understand the eight-flavor Majorana field in $8$ of SO(7) in terms of the four-flavor Dirac field in $4$ of SO(6) through the relation

$$
\psi_M(x) = \begin{pmatrix}
\psi(x) \\
-ic_D \bar{\psi}(x)^T
\end{pmatrix},
$$

(8.28)

and the discrete chiral transformation should be defined as

$$
\psi_M(x) \rightarrow (\hat{\gamma}_3 P_+ + \gamma_3 P_-) \psi_M(x),
$$

(8.29)

where $P_\pm = (1 \pm \Gamma_7)/2$. This means that the SO(7) symmetry must be reduced to SO(6), while the Majorana field are protected from acquiring the mass terms by the discrete chiral symmetry based on the Ginsparg-Wilson relation.

With this understanding, we can see that the domain wall fermion model with $U(1)_A \times \text{Spin}(6) (\text{SU}_4)$ symmetry discussed in section 7 is indeed defining a 2+1D classical Euclidean lattice model of the 2D quantum TSC. In fact, the action given by eq. (7.1) can be rewritten as follows:

$$
S_{\text{DW}} = \sum_{t=1}^{L_3} \sum_{x \in \Lambda} \frac{1}{2} \bar{\psi}_M(x,t)^T \bar{c}_D C \left\{ [1 + a'_3(\tilde{D}_{2w} - m_0)]\delta_{tt'} + \tilde{P}_- \delta_{t+1,t'} - \tilde{P}_+ \delta_{t,t+1} \right\} \tilde{\psi}_M(x,t'),
$$

$$
S_{\text{bd}} = \sum_{x \in \Lambda} \frac{1}{2} (z - 1) \bar{\psi}_M(x,L_3)^T \bar{c}_D C \tilde{P}_- [1 + a'_3(\tilde{D}_{2w} - m_0)] \tilde{\psi}_M(x,L_3)
$$

$$
+ \sum_{x \in \Lambda} \frac{h}{2} \bar{\psi}_M(x,L_3)^T i\tilde{\gamma}_5 \tilde{\gamma}_3 \tilde{c}_D (P_+ + \tilde{P}_-) T^a E^a(x) \tilde{\psi}_M(x,L_3).
$$

(8.30)
Here $\tilde{\psi}_M(x)$ is the eight-flavor four-component Majorana field given in terms of the four-flavor
four-component Dirac field as

$$\tilde{\psi}_M(x) = \begin{pmatrix} \psi(x) \\ -i\hat{c}_D C \tilde{\psi}(x) \end{pmatrix}; \quad \tilde{\psi}(x) = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}, \quad \tilde{\psi}(x) = (\tilde{\psi}(x), \tilde{\psi}'(x)). \quad (8.31)$$

Dirac gamma matrices are defined as $\gamma = \tilde{\gamma}_0 = \sigma_1 \otimes \sigma_3, \tilde{\gamma}_1 = \sigma_2 \otimes \sigma_3, \tilde{\gamma}_3 = \sigma_3 \otimes \sigma_3, \tilde{\gamma}_4 = I \otimes \sigma_1$. The charge conjugation operator $\hat{c}_D$ is given as $\hat{c}_D = i\sigma_2 \otimes \sigma_3$. The 2+1D
Dirac operator is defined with 1+1D Wilson-Dirac operator $\tilde{D}_{2w} = \sum_{\mu=0}^{1} \{ \tilde{\gamma}_\mu (\nabla_\mu - \nabla_\mu^\dag) / 2 + \nabla_\mu \nabla_\mu^\dag / 2 \}$ and $\tilde{P}_\pm = (1 \mp \tilde{\gamma}_3) / 2$, and $a'_3 (= a_3 / a)$ is the lattice spacing of the third dimension in the lattice unit. The Dirichlet b.c. is imposed as

$$\tilde{P}_+ \tilde{\psi}_M(x,0) = 0, \quad \tilde{P}_- \tilde{\psi}_M(x, L_3 + 1) = 0. \quad (8.32)$$

The action is invariant under the parity and charge conjugation transformations, $\mathcal{P}' : \tilde{\psi}_M(x) \rightarrow \tilde{\gamma}_5 \tilde{\gamma}_1 \tilde{\psi}_M(x\nu); E^a(x) \rightarrow - E^a(x\nu)$ where $x\nu = (x_0, -x_1, x_2)$ and $\mathcal{C} : \tilde{\psi}_M(x) \rightarrow \tilde{\psi}_M(x)$. It is also invariant under the $Z_2$ transformation, $Z_2 : \tilde{\psi}_M(x) \rightarrow i\tilde{\gamma}_4 \tilde{\gamma}_5 \tilde{\psi}_M(x) = (I \otimes \sigma_3) \tilde{\psi}_M(x)$, but not invariant under the discrete chiral transformation, $\tilde{\psi}_M(x) \rightarrow \tilde{\gamma}_3 \tilde{\psi}_M(x) = (a_3 \otimes \sigma_3) \tilde{\psi}_M(x)$ by the mass and Wilson terms.

As for the 1+1D mirror-fermion model in terms of overlap fermions, which describes the gapless/gapped boundary phases, the action can be rewritten as

$$S_W = \sum_x \left\{ \frac{1}{2} \psi'_M(x) T c_D C D \psi'_M(x) \right\}, \quad (8.33)$$

$$S_M = \sum_x \left\{ \frac{z}{2} \psi'_M(x) T c_D C D \psi'_M(x) + \frac{\hbar}{2} \psi'_M(x) X_E \psi'_M(x) \right\}. \quad (8.34)$$

where

$$X_E = \tilde{P}_+ T \tilde{P}_+ i\gamma_3 c_D T^a E^a(x) \tilde{P}_+ \tilde{P}_- + \tilde{P}_+ T \tilde{P}_+ i\gamma_3 c_D T^a E^a(x) \tilde{P}_+ \tilde{P}_+.$$

Here $\psi'_M(x)$ and $\psi_M(x)$ are the eight-flavor two-component Majorana fields given in terms of the four-flavor two-component Dirac fields $\psi_<(x)$ and $\psi_>(x)$ as

$$\psi'_M(x) = \begin{pmatrix} \psi_<(x) \\ -i\hat{c}_D C \tilde{\psi}_<(x) \end{pmatrix}, \quad \psi_M(x) = \begin{pmatrix} \psi_>(x) \\ -i\hat{c}_D C \tilde{\psi}_>(x) \end{pmatrix}. \quad (8.36)$$

$$\psi_<(x) = \tilde{P}_+ \psi'_>(x) + \tilde{P}_- \psi'_>(x), \quad \tilde{\psi}_<(x) = \tilde{\psi}'(x) P_+ + \tilde{\psi}'(x) P_. \quad (8.37)$$

$$\psi_>(x) = \tilde{P}_+ \psi(x) + \tilde{P}_- \psi(x), \quad \tilde{\psi}_>(x) = \tilde{\psi}'(x) P_+ + \tilde{\psi}'(x) P_. \quad (8.38)$$

Note that $\psi_<(x)$ and $\psi_>(x)$ consist of the fields of the boundaries at $t = 0$ and $t = L_3$, respectively. $S_W$ and $S_M$ stand for the actions of the boundary phases at $t = 0$ and $t = L_3$, respectively. The precise relation between the bulk TSC(domain wall fermion) and the
boundary phases (overlap fermions) are given by the following identity\cite{41–43} \cite{134}.

\[
\lim_{a' \to 0} \lim_{L_3 \to \infty} \int D[\bar{E}^a] \prod_{x,t} d\tilde{\psi}_M(x,t) e^{-S_{DW} - S_{mL}} |_{\text{Dir}}
\]

\[
= \lim_{a' \to 0} \lim_{L_3 \to \infty} \int \prod_{x,t} d\tilde{\psi}_M(x,t) e^{-S_{DW}} \int D[\psi'_M] e^{-S_W} \int D[\bar{E}^a] D[\psi_M] e^{-S_M}.
\]

(8.39)

In terms of the functional pfaffians, it is given by

\[
\lim_{a' \to 0} \lim_{L_3 \to \infty} \int D[\bar{E}^a] \text{pf}(\tilde{c}_D C[\tilde{D}_{3w}^{\prime} - m_0 + hi\beta_3 \bar{E}^a \delta_{tL_3} \delta_{t'L,3}]) |_{\text{Dir}}.
\]

\[
= \lim_{a' \to 0} \lim_{L_3 \to \infty} \text{pf}(\tilde{c}_D C[\tilde{D}_{3w}^{\prime} - m_0]) \text{pf}(c_D C D) \int D[\bar{E}^a] \text{pf}(\bar{E}^a C D + hX_E),
\]

(8.40)

where \(\{\tilde{D}_{3w}^{\prime}\}_tt' = \{\tilde{D}_{3w}\}_tt' + \delta_{tL_3} \delta_{t'L,3} (z-1) \bar{P}_{-}[1 + a'_3(\tilde{D}_{2w} - m_0)]\).

The boundary phase at \(t = L_3\), which is supposed to be gapped, is now described by the 1+1D lattice model of the eight-flavor two-component overlap Majorana field \(\psi_M(x)\) with the action \(S_M\) of eq. (8.34). As argued in section 5, the limit of the large Majorana-Yukawa coupling, \(z/h \to 0\), is well-defined in this formulation. Then the pfaffian factorizes as

\[
\text{pf}(X_E) = \det (u^T i\gamma_3 c_D \bar{T}^a E^a v) \det (\bar{u} i\gamma_3 c_D \bar{T}^a E^a \bar{v}^T),
\]

(8.41)

where the first determinant is positive semi-definite and the second one is unity. Therefore the partition function of the boundary phase is positive-definite in this limit:

\[
\langle 1 \rangle_M = \int D[\bar{E}^a] \text{pf}(\bar{E}^a C D + hX_E)
\]

\[
\to \int D[\bar{E}^a] \det (u^T i\gamma_3 c_D \bar{T}^a E^a v) > 0 \quad (z/h \to 0).
\]

(8.42)

And the numerical-simulation results in section 5 show that the correlation lengths of the mirror-sector fields are of order multiple lattice spacings and that the two-point vertex functions of the U(1)_{A} and Spin(6)(SU(4)) gauge fields are regular and local. These results provide a numerical evidence in the framework of 2+1D path-integral quantization that the boundary phase of the eight-flavor 2D chiral p-wave TSC with the time-reversal and Z_{2} symmetries (class D'/DIII + R in 2D) is indeed gapped by the SO(6)-invariant multi-fermion interaction, shown originally in \cite{129–132}.

The above connection should hold true in lower and higher dimensions. It is straightforward to extend the above discussion to the case of the eight-flavor 1D TSC with time-reversal symmetry (class BDI in 1D classified by Z_{8} (+ Z)) \cite{115} through dimensional reduction. It would be also useful to examine the Hamiltonian constructions of 3+1D chiral gauge theories based on the 4D TI/TSC with the proposed gapped boundary phases \cite{109, 111, 112} from the point of view of the 3+1D/4+1D Euclidean construction based on the overlap/domain wall fermions. These topics will be discussed elsewhere \cite{140, 141}.
9 Discussions

In this paper, we addressed the basic question how to decouple the mirror Ginsparg-Wilson fermions in lattice models for two-dimensional abelian chiral gauge theories. After we investigated why the mirror-fermion approach seems to fail for the 345-model with Dirac- and Majorana-Yukawa couplings to XY-spin field\cite{73–80}, we proposed the two mirror-fermion models with $U(1)_A \times \text{Spin}(6)(SU(4))$-invariant Majorana-Yukawa couplings to SO(6)-vector spin field: $1^4\!(-1)^4$ axial gauge model and $21(-1)^3$ chiral gauge model. These models are well-defined and simplified in the limit of the large Majorana-Yukawa couplings. We examined their properties in the weak gauge-coupling limit through Monte Carlo simulations and provided numerical evidences that the mirror-fermions are indeed decoupled in these models. For the $21(-1)^3$ chiral gauge model, we deduced a definition of the (target) Weyl-field measure, in which the mirror-fermion part of the Dirac-field measure is just saturated by the suitable products of the 't Hooft vertices in terms of the mirror-fermion fields. Based on the results of Monte Carlo simulations, we argued that the induced fermion measure term satisfies the required locality property and provides a solution to the reconstruction theorem of the Weyl field measure in the framework of the Ginsparg-Wilson relation\cite{50}.

As to the properties of these models in the weak gauge-coupling limit, we have argued that the functional determinant of the mirror-fermions, $\text{det} (u^T i \gamma_3 c_D T^a E^a \nu')$, is positive semi-definite based on the analytical and numerical results. We have also argued that the induced measure term of the mirror sector, $\langle (\delta \eta S_M)_M \rangle / \langle 1 \rangle_M$, is a local functional of the (external) U(1) and Spin(6) link fields by computing the two-point vertex functions numerically. It is highly desirable to establish these properties rigorously, if possible. The verification by numerical simulations should be also extended for larger lattice sizes with higher statistics.

The deduced definition of the Weyl-field measure of the $21(-1)^3$ chiral-gauge model seems generic, although another constraint on the charge assignment, $\text{Tr} Q + \text{Tr} Q' = 0$, is actually required so that the matrix $(u^T i \gamma_3 c_D T^a E^a \nu')$ remains square in all topological sectors $\mathcal{U}[m]$.\cite{18} We will discuss the application of this definition to SO(10) chiral gauge theory in four-dimensions elsewhere\cite{140}.

We find it interesting that the gauge/global symmetries of the $21(-1)^3$ chiral gauge model shown in table 4 mimics the standard model, where the chiral U(1) gauge interaction plays the role of SU(2)$_L \times$ U(1)$_Y$ Electroweak gauge interaction. By introducing an Abelian Higgs field and its Yukawa couplings to “quarks” and “leptons”, the model may be used as a toy model to study/simulate the baryon-number non-conservation in the standard model (cf. \cite{142–154}).

It is known that a chiral lattice gauge theory is a difficult case for numerical simulations because the effective action induced by Weyl fermions has a non-zero imaginary part. But, in view of the recent studies of the simulation methods based on the complex Langevin dynamics\cite{155–190} and the complexified path-integration on Lefschetz thimbles\cite{191–233}, one may consider to apply these methods to chiral lattice gauge theories. In particu-
lar, it seems feasible to apply the generalized Lefschetz thimble method\cite{220, 228, 229}, which uses the integration contours followed from the holomorphic gradient flow with various flow-times and the exchange Monte Carlo (parallel tempering) algorithm, to the two-dimensional abelian chiral lattice gauge theories discussed in this paper. Analytical study of the Lefschetz-thimble structures of these chiral lattice gauge theories would be also interesting and useful.\cite{226} The tensor renormalization group method\cite{234–247} is another option. It seems also feasible to apply the method to these two-dimensional theories.

The recent proposal by Grabowska and Kaplan\cite{248–255} is “orthogonal” to the mirror-fermion approach with Ginsparg-Wilson fermions discussed in this paper. It is based on the original domain wall fermion by Kaplan\cite{36}, but coupled to the “five-dimensional” link field which is obtained from the dynamical four-dimensional link field at the target wall by the gradient flow toward the mirror wall. This choice of the “five-dimensional” link field makes possible a chiral gauge coupling for the target and mirror walls, while keeping the system four-dimensional and gauge-invariant.\footnote{In the weak gauge-coupling region of the topologically trivial sector, the condition that the form factor of the mirror-modes is soft enough to suppress the (transverse) gauge-coupling is given by $\sum_\mu 4 \sin^2 (p_\mu/2) t \gg 1$ for all possible momenta above a certain IR cutoff. If one assumes $|p_\mu| \geq \pi/L$, the condition reads $\sqrt{8t} \gg \sqrt{8/\pi^2}L$, which apparently contradicts the other condition $1 \ll \sqrt{8t} \ll L$ for that local composite operators of the flowed five-dimensional link field is local w.r.t. the original four-dimensional link field. This implies that the imaginary part of the effective action, which can be written with the local operators of the five-dimensional link field, actually contains the non-local operators w.r.t. the dynamical four-dimensional link field. This is one reason why the gauge-invariance is maintained in this formulation even when any anomalous set of chiral-modes appear in the target wall. (The other reason is that the mirror-modes are never decoupled from the gauge degrees of freedom of the dynamical four-dimensional link field.)}

It is “orthogonal” in the sense that the authors do not try to decouple the massless-modes at the mirror wall, but interpret them as physical degrees of freedom with very soft form factor caused by the gradient flow, and that the authors do not try (do not need) to break explicitly the continuous global symmetries with “would-be gauge anomalies” in the mirror-wall sector, which would be required if one would try to decouple the mirror-modes as claimed by Eichten and Preskill and by the other and present authors\cite{73, 94, 110}.

In this respect, we point out the following fact. If one indeed tries to obtain the non-local/five-dimensional counter terms to subtract the above non-local/five-dimensional contributions (nonperturbatively as it should be), one actually ends up with solving the known local cohomology problem, which was first formulated by Lüscher for Ginsparg-Wilson fermions in the 4-dim. lattice plus 2-dim. continuum space\cite{46, 51} and then extended by the present author for domain wall fermion in the 5-dim. lattice plus 1-dim. continuum space\cite{63}. And if one would include the non-local/five-dimensional counter terms so obtained to the original formulation, one can show that the resulted four-dimensional model is local and does not actually depend on how the dynamical four-dimensional link field is extrapolated to the extra dimension\cite{63}. Then there is no particular reason to choose the method of gradient flow for this purpose.
A chiral basis in the free theory

Dirac gamma matrices:
\[
\begin{align*}
\gamma_0 &= \sigma_1, \quad \gamma_1 = \sigma_2, \quad \gamma_3 = \sigma_3, \\
\{ \gamma_\mu, \gamma_\nu \} &= 2\delta_{\mu\nu}, \quad \gamma_\mu^\dagger = \gamma_\mu, \quad \gamma_3 = -i\gamma_0 \gamma_1, \\
\gamma_0^T &= \gamma_0, \quad \gamma_1^T = -\gamma_1, \quad \gamma_3^T = \gamma_3, \\
c_D &= i\gamma_1 = i\sigma_2, \\
c_D \gamma_\mu c_D^\dagger &= -\gamma_\mu^T, \quad c_D \gamma_3 c_D^{-1} = -\gamma_3; \quad c_D^T = c_D^{-1} = c_D.
\end{align*}
\] (A.1 - A.5)

Kernels of chiral operators \( \hat{\gamma}_3 = \gamma_3(1 - 2D), \gamma_3 \):
\[
\begin{align*}
\hat{\gamma}_3(x,y) &= -\int_{-\pi}^{+\pi} \frac{d^2p}{(2\pi)^2} e^{ip(x-y)} \frac{1}{\omega(p)} \begin{pmatrix} b(p) & c(p) \\ c(p)^\dagger & -b(p) \end{pmatrix}, \\
\gamma_3(x,y) &= +\int_{-\pi}^{+\pi} \frac{d^2p}{(2\pi)^2} e^{ip(x-y)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
\] (A.6 - A.7)

where \( c(p) = i\sin(p_1) + \sin(p_2), b(p) = 2 - \cos(p_1) - \cos(p_2) - m_0, \omega(p) = \sqrt{c^2 + b^2}(p) \).

Orthonormal chiral bases:
The case with \( j = p \)
\[
\begin{align*}
u_j(x) &= e^{ipx} u(p); \quad u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u(p) = \frac{1}{\sqrt{2\omega(\omega + b)}} \begin{pmatrix} -c \\ \omega + b \end{pmatrix} (p) \quad (p \neq 0), \\
v_j(x) &= e^{ipx} v(p); \quad v(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v(p) = \frac{1}{\sqrt{2\omega(\omega + b)}} \begin{pmatrix} \omega + b \\ c^\dagger \end{pmatrix} (p) \quad (p \neq 0),
\end{align*}
\] (A.8 - A.9)

\[
\begin{align*}
\bar{u}_j(x) &= e^{-ipx} \bar{u}(p); \quad \bar{u}(p) = \begin{pmatrix} 0 & 1 \end{pmatrix}, \\
\bar{v}_j(x) &= e^{-ipx} \bar{v}(p); \quad \bar{v}(p) = \begin{pmatrix} 1 & 0 \end{pmatrix}.
\end{align*}
\] (A.10 - A.11)

The case with \( j = x \)
\[
\begin{align*}
u_j(x) &= \int_{-\pi}^{+\pi} \frac{d^2p}{(2\pi)^2} e^{ip(x-x_j)} u(p); \quad u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u(p) = \frac{1}{\sqrt{2\omega(\omega + b)}} \begin{pmatrix} -c \\ \omega + b \end{pmatrix} (p) \quad (p \neq 0), \\
v_j(x) &= \int_{-\pi}^{+\pi} \frac{d^2p}{(2\pi)^2} e^{ip(x-x_j)} v(p); \quad v(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v(p) = \frac{1}{\sqrt{2\omega(\omega + b)}} \begin{pmatrix} \omega + b \\ c^\dagger \end{pmatrix} (p) \quad (p \neq 0),
\end{align*}
\] (A.12 - A.13)

\[
\begin{align*}
\bar{u}_j(x) &= \int_{-\pi}^{+\pi} \frac{d^2p}{(2\pi)^2} e^{-ip(x-x_j)} \bar{u}(p); \quad \bar{u}(p) = \begin{pmatrix} 0 & 1 \end{pmatrix}, \\
\bar{v}_j(x) &= \int_{-\pi}^{+\pi} \frac{d^2p}{(2\pi)^2} e^{-ip(x-x_j)} \bar{v}(p); \quad \bar{v}(p) = \begin{pmatrix} 1 & 0 \end{pmatrix}.
\end{align*}
\] (A.14 - A.15)
Majorana-mass-type inner products:
\[
 u(-p)^T c_D v(p) = \frac{1}{2\omega(\omega + b)}[\bar{c}c - (\omega + b)^2](p) = -\frac{b}{\omega}(p), \quad \text{(A.16)}
\]
\[
 u(-p)^T \gamma_3 c_D v(p) = \frac{1}{2\omega(\omega + b)}[\omega + b]^2 + cc^\dagger](p) = 1 \quad \text{(A.17)}
\]

B \ SO(7) spinor

The Clifford algebra in 7 dimensions (\(a = 1, \cdots, 7\)):

\[
\Gamma^a \Gamma^b + \Gamma^b \Gamma^a = 2\delta^{ab}, \quad \Gamma^a = \Gamma^a, \quad \Gamma^7 = i\Gamma^1 \Gamma^2 \cdots \Gamma^6, \quad \text{(B.1)}
\]
\[
\Gamma^a C C^{-1} = -(\Gamma^a)^T, \quad C^T = -C^{-1} = -C^\dagger = C \quad \text{(B.2)}
\]

\[
\Gamma^1 = \sigma_1 \times \sigma_1 \times \sigma_1, \quad \text{(B.3)}
\]
\[
\Gamma^2 = \sigma_2 \times \sigma_1 \times \sigma_1, \quad \text{(B.4)}
\]
\[
\Gamma^3 = \sigma_3 \times \sigma_1 \times \sigma_1, \quad \text{(B.5)}
\]
\[
\Gamma^4 = I \times \sigma_2 \times \sigma_1, \quad \text{(B.6)}
\]
\[
\Gamma^5 = I \times \sigma_3 \times \sigma_1, \quad \text{(B.7)}
\]
\[
\Gamma^6 = I \times I \times \sigma_2, \quad \text{(B.8)}
\]
\[
\Gamma^7 = I \times I \times \sigma_3, \quad \text{(B.9)}
\]
\[
C = i\sigma_2 \times \sigma_3 \times \sigma_2. \quad \text{(B.10)}
\]

T matrices:

\[
T^a = \Gamma^a, \quad \{T^a\}^T = -T^a \quad \text{(B.11)}
\]

\[
T^1 = i(-i)(+i)(-i) \sigma_3 \times \sigma_2 \times \sigma_3 = \hat{T}^1 \times \sigma_3, \quad \text{(B.12)}
\]
\[
T^2 = i(+1)(+i)(-i) I \times \sigma_2 \times \sigma_3 = \hat{T}^2 \times \sigma_3, \quad \text{(B.13)}
\]
\[
T^3 = i(+i)(+i)(-i) \sigma_1 \times \sigma_2 \times \sigma_3 = \hat{T}^3 \times \sigma_3, \quad \text{(B.14)}
\]
\[
T^4 = i(+1)(-i)(-i) \sigma_2 \times \sigma_1 \times \sigma_3 = \hat{T}^4 \times \sigma_3, \quad \text{(B.15)}
\]
\[
T^5 = i(+1)(+1)(-i) \sigma_2 \times I \times \sigma_3 = \hat{T}^5 \times \sigma_3, \quad \text{(B.16)}
\]
\[
T^6 = i(+1)(+1)(+1) \sigma_2 \times \sigma_3 \times I = T^6 \times I, \quad \text{(B.17)}
\]
\[
T^7 = i(+1)(+1)(+i) \sigma_2 \times \sigma_3 \times \sigma_1 = +i \hat{C} \times \sigma_1. \quad \text{(B.18)}
\]
The Clifford algebra in 5 dimensions \((a = 1, \cdots, 5)\):

\[
\begin{align*}
\tilde{\Gamma}^1 &= \sigma_1 \times \sigma_1, \\
\tilde{\Gamma}^2 &= \sigma_2 \times \sigma_1, \\
\tilde{\Gamma}^3 &= \sigma_3 \times \sigma_1, \\
\tilde{\Gamma}^4 &= I \times \sigma_2, \\
\tilde{\Gamma}^5 &= I \times \sigma_3, \\
\tilde{C} &= i \sigma_2 \times \sigma_3.
\end{align*}
\] (B.19) (B.20) (B.21) (B.22) (B.23) (B.24)

Reduced \(T\) matrices:

\[
\begin{align*}
\tilde{T}^a &= -i \tilde{C} \tilde{\Gamma}^a, \\
\{\tilde{T}^a\}^T &= -\tilde{T}^a, \\
\tilde{T}^6 &= \tilde{C}, \\
\{\tilde{T}^6\}^T &= -\tilde{T}^6.
\end{align*}
\] (B.25) (B.26)

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