A Variational Tate Conjecture in crystalline cohomology

Matthew Morrow

Abstract

Given a smooth, proper family of varieties in characteristic $p > 0$, and a cycle $z$ on a fibre of the family, we formulate a Variational Tate Conjecture characterising, in terms of the crystalline cycle class of $z$, whether $z$ extends cohomologically to the entire family. This is a characteristic $p$ analogue of Grothendieck’s Variational Hodge Conjecture. We prove the conjecture for divisors, and an infinitesimal variant of the conjecture for cycles of higher codimension.

This reduces the $\ell$-adic Tate conjecture for divisors over finite fields to the case of surfaces.

CONTENTS

0 Introduction and statement of main results 1
1 Conjecture 0.1 4
2 Crystalline Théorème de la Partie Fixe 8
3 Local and infinitesimal forms of Conjecture 0.1 13
  3.1 Some remarks on crystalline cohomology 13
  3.2 Proofs of Theorems 0.5 and 0.6 via topological cyclic homology 14
  3.3 Line bundles with $\mathbb{Q}_p$-coefficients 21
  3.4 Cohomologically flat families over $k[[t]]$ and Artin’s theorem 24
4 An application to the Tate conjecture 26

0 Introduction and statement of main results

Let $f : X \to S$ be a smooth, proper morphism of smooth varieties over a field $k$, and let $s \in S$ be a closed point. Grothendieck’s Variational Hodge or $\ell$-adic Tate Conjecture [22, pg. 359] gives conditions on the cohomology class of an algebraic cycle on $X$, under which the cycle conjecturally extends cohomologically to the entire family $X$. According as $k$ has characteristic 0 or $p > 0$, the cohomology theory used to formulate Grothendieck’s conjecture is de Rham or $\ell$-adic étale ($\ell \neq p$). The Variational Hodge conjecture is easily proved for divisors using the exponential map; in contrast, it seems little is known about the $\ell$-adic Variational Tate conjecture.

Suspecting that deformation problems in characteristic $p$ are best understood $p$-adically, we formulate in this article a Variational Tate Conjecture in crystalline cohomology. To state it, we now fix some notation for the rest of the article. Unless stated
otherwise, \( k \) is a perfect field of characteristic \( p > 0 \), and \( W = W(k) \), \( K = \text{Frac} W \). Given any reasonable (not necessarily of finite type) scheme \( X \) over \( k \) of characteristic \( p \), let \( H^{n}_{\text{crys}}(X) = H^{n}_{\text{crys}}(X/W) \otimes W K \) denote its rational crystalline cohomology. Assuming now that \( X \) is a smooth \( k \)-variety, let \( c_i : CH^i(X)_{Q} \rightarrow H^{2i}_{\text{crys}}(X) \otimes W K \) denote the crystalline cycle class map for each \( i \geq 0 \); these land in the \( p^i \)-eigenspace of the absolute Frobenius \( \phi : x \mapsto x^p \).

Our proposed Variational Tate Conjecture in crystalline cohomology is as follows:

**Conjecture 0.1** (Crystalline Variational Tate Conjecture). Let \( f : X \rightarrow S \) be a smooth, proper morphism of smooth \( k \)-varieties, \( s \in S \) a closed point, and \( z \in CH^i(X_s)_{Q} \). Let \( c := cl_i(z) \in H^{2i}_{\text{crys}}(X_s) \). Then the following are equivalent:

1. **(deform)** There exists \( \tilde{z} \in CH^i(X)_{Q} \) such that \( cl_i(\tilde{z})|_{X_s} = c \).
2. **(crys)** \( c \) lifts to \( H^{2i}_{\text{crys}}(X) \).
3. **(crys-\( \phi \))** \( c \) lifts to \( H^{2i}_{\text{crys}}(X)^{\phi = p^i} \).
4. **(flat)** \( c \) is flat, i.e., it lifts to \( H^0_{\text{crys}}(S, R^{2i}f_*O_{X/K}) \).

A more detailed discussion of Conjecture 0.1, including an equivalent formulation via rigid cohomology and an explanation of the condition (flat), is given in Section 1.

Our first main result is the proof of Conjecture 0.1 for divisors, at least assuming that the family is projective:

**Theorem 0.2** (See Thm. 1.4). Conjecture 0.1 is true for divisors, i.e., when \( i = 1 \), if \( f \) is projective.

Theorem 0.2 remains true for divisors with \( \mathbb{Q}_p \)-coefficients rather than \( \mathbb{Q} \)-coefficients. This is important for the following application\(^1\), which follows from a hyperplane pencil argument:

**Corollary 0.3** (See Thm. 4.2). Assume that the Tate conjecture for divisors is true for all smooth, projective surfaces over a finite field \( k \). Then the Tate conjecture for divisors is true for all smooth, projective varieties over \( k \).

Regarding Corollary 0.3, we remark that the “Tate conjecture for divisors” is independent of the chosen Weil cohomology theory (see Proposition 4.1); in particular, the corollary may be stated in terms of \( \ell \)-adic étale cohomology, though the proof is crystalline in nature.

We also prove a variant of Theorem 0.2 for line bundles on smooth, projective schemes over the spectrum of a power series ring \( k[[t_1, \ldots, t_m]] \); see Theorem 3.5. Combining this with N. Katz’ results on slope filtrations of \( F \)-crystals over \( k[[t]] \) yields the following consequence:

**Corollary 0.4** (See Thm. 3.12). Let \( X \) be a smooth, proper scheme over \( k[[t]] \), where \( k \) is an algebraically closed field of characteristic \( p \). Assume that \( R^n f_* O_{X/W} \) is locally-free for all \( n \geq 0 \) and is a constant \( F \)-crystal for \( n = 2 \). Then the cokernel of the restriction map \( \text{Pic}(X) \rightarrow \text{Pic}(X \times_k [t]) \) is killed by a power of \( p \).

\(^1\)I recently learned of an unpublished note of A. J. de Jong in which Corollary 0.3 is proved.
Corollary 0.4 applies in particular to supersingular families of K3 surfaces over $k[[t]]$, thereby reproving a result of M. Artin [2].

We now explain our other variational results while simultaneously indicating the main ideas of the proofs. Firstly, Section 2 is devoted to the proof of a crystalline analogue of Deligne’s Théorème de la Partie Fixe, stating that the crystalline Leray spectral sequence for the morphism $f$ degenerates in a strong sense; see Theorem 2.6. This implies, in the situation of Conjecture 0.1, that conditions (crys), (crys-$\phi$), and (flat) are in fact equivalent (assuming $f$ is projective).

Hence, to prove Theorem 0.2, it is enough to show that (crys-$\phi$) implies (deform) for divisors. By standard arguments, we may base change by $k^{\text{alg}}$, replace $S$ by $\text{Spec} \hat{\mathcal{O}}_{S,s}$, and, identifying divisors with line bundles, then prove the following, in which it is only necessary to invert $p$:

**Theorem 0.5** (See Corol. 3.4). Let $X$ be a smooth, proper scheme over $A = k[[t_1, \ldots, t_m]]$, where $k$ is an algebraically closed field of characteristic $p$, and let $L \in \text{Pic}(X \times_A k)[\frac{1}{p}]$. Then the following are equivalent:

(i) There exists $\tilde{L} \in \text{Pic}(X)[\frac{1}{p}]$ such that $\tilde{L}|_{X \times_A k} = L$.

(ii) The first crystalline Chern class $c_1(L) \in H^2_{\text{crys}}(X \times_A k)$ lifts to $H^2_{\text{crys}}(X)^{\phi=p}$.

We finally state our infinitesimal version of Conjecture 0.1 which holds in all codimensions; it provides a necessary and sufficient condition under which the $K_0$ class of a vector bundle on the special fibre admits infinitesimal extensions of all orders:

**Theorem 0.6** (See Thm. 3.3). Let $X$ be a smooth, proper scheme over $A = k[[t_1, \ldots, t_m]]$, where $k$ is a finite or algebraically closed field of characteristic $p$; let $Y$ denote the special fibre and $Y_r := X \times_A A/\langle t_1, \ldots, t_m \rangle^r$ its infinitesimal thickenings. Then the following are equivalent for any $z \in K_0(Y)[\frac{1}{p}]$:

(i) $z$ lifts to $(\lim_{\leftarrow r} K_0(Y_r))[\frac{1}{p}]$.

(ii) The crystalline Chern character $\text{ch}(z) \in \bigoplus_{i \geq 0} H^{2i}_{\text{crys}}(Y)$ lifts to $\bigoplus_{i \geq 0} H^{2i}_{\text{crys}}(X)^{\phi=p^i}$.

Theorem 0.6 is an exact analogue in characteristic $p$ of a deformation result in mixed characteristic due to S. Bloch, H. Esnault, and M. Kerz [8, Thm. 1.3]. Analogues in characteristic zero have also been established by them [9] and the author [34].

We finish this introduction with a brief discussion of the proofs of Theorems 0.5 and 0.6. The new input which makes these results possible is recent work of the author joint with B. Dundas [35], which establishes that topological cyclic homology is continuous under very mild hypotheses. In particular, in the framework of Theorem 0.6, our results yield a weak equivalence

$$TC(X;p) \sim \text{holim}_r TC(Y_r;p)$$

between the $p$-typical topological cyclic homologies of $X$ and the limit of those of all the thickenings of the special fibre. Moreover, R. McCarthy’s theorem and the trace map describe the obstruction to infinitesimally lifting elements of $K$-theory in terms of $TC(Y_r;p)$ and $TC(Y;p)$, while results of T. Geisser and L. Hesselholt describe $TC(X;p)$
and $\text{TC}(Y; p)$ in terms of logarithmic de Rham–Witt cohomology since $X$ and $Y$ are regular. Combining these two descriptions with the weak equivalence (1) yields a preliminary form of Theorem 0.6 phrased in terms of Gros’ logarithmic crystalline Chern character; see Proposition 3.1.

To deduce Theorem 0.6 we then analyse the behaviour of the Frobenius on the crystalline cohomology of the $k[[t_1, \ldots, t_m]]$-scheme $X$; see Proposition 3.2. Theorem 0.5 then follows from Theorem 0.6 using Grothendieck’s algebrisation isomorphism $\text{Pic} X \cong \lim_{\leftarrow r} \text{Pic} Y_r$.

Acknowledgments

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1 Conjecture 0.1

Let $k$ be a perfect field of characteristic $p > 0$, and write $W := W(k)$, $K := \text{Frac} W(k)$. For any $k$-variety $X$, we denote by $H^n_{\text{crys}}(X/W)$ and $H^n_{\text{crys}}(X) := H^n_{\text{crys}}(X/W) \otimes_W K$ its integral and rational crystalline cohomology groups. The crystalline cohomology $H^n_{\text{crys}}(X)$ is naturally acted on by the absolute Frobenius $\phi : x \mapsto x^p$, whose $p^i$-eigenspaces we will denote by adding the traditional superscript $\phi = p^i$. There is a theory of crystalline Chern classes, cycle classes, and the associated Chern character, constructed originally by P. Berthelot and L. Illusie [6], A. Ogus [37], and H. Gillet and W. Messing [19]:

\[
\begin{align*}
    c_i(E) &= c_i^{\text{crys}}(E) \in CH^i(X)_{\mathbb{Q}} \quad (E \text{ a vector bundle on } X), \\
    cl_i &= cl_i^{\text{crys}} : CH^i(X)_{\mathbb{Q}} \rightarrow H^{2i}_{\text{crys}}(X), \\
    ch &= ch^{\text{crys}} : K_0(X)_{\mathbb{Q}} \rightarrow \bigoplus_{i \geq 0} H^{2i}_{\text{crys}}(X).
\end{align*}
\]

The cycle classes and Chern character land in the $p^i$-eigenspace of $\phi$ acting on $H^{2i}_{\text{crys}}(X)$.

For the reader’s convenience, we restate the main conjecture from the Introduction:

Conjecture 0.1 (Crystalline Variational Tate Conjecture). Let $f : X \rightarrow S$ be a smooth, proper morphism of smooth $k$-varieties, $s \in S$ a closed point, and $z \in CH^i(X_s)_{\mathbb{Q}}$. Let $c := cl_i(z) \in H^{2i}_{\text{crys}}(X_s)$. Then the following are equivalent:

(deform) There exists $\tilde{z} \in CH^i(X)_{\mathbb{Q}}$ such that $cl_i(\tilde{z})|_{X_s} = c$.  


ful thanks to K. Kedlaya [29], and so

However, the conditions (flat) and (rig-flat) are in fact equivalent for any

More generally, we will discuss the conditions (crys), (crys-\phi), and (flat) for arbitrary cohomology classes \(c \in H^{2i}_{\text{crys}}(X_s)\).

**Remark 1.1** (The condition (flat)). The flatness condition, involving the global sections of Ogus’ convergent \(F\)-isocrystal \(R^{2i}f_*O_{X/K}\) [37, §3], may require further explanation. The restriction map \(H^{2i}_{\text{crys}}(X) \to H^{2i}_{\text{crys}}(X_s)\) may be factored as

\[
H^{2i}_{\text{crys}}(X) \to H^0_{\text{crys}}(S, R^{2i}f_*O_{X/K}) \to H^{2i}_{\text{crys}}(X_s),
\]

where the first arrow is an edge map in the Leray spectral sequence

\[
E_2^{ab} = H^a_{\text{crys}}(S, R^b f_*O_{X/K}) \Longrightarrow H^{a+b}_{\text{crys}}(X).
\]

We call an element \(c \in H^{2i}_{\text{crys}}(X_s)\) flat if and only if it lifts to \(H^0_{\text{crys}}(S, R^{2i}f_*O_{X/K})\); the lift, if it exists, is actually unique, assuming \(S\) is connected [37, Thm. 4.1]. In particular, the lift of a flat cycle class automatically lies in the eigenspace \(H^{2i}_{\text{crys}}(S, R^{2i}f_*O_{X/K})^\phi = p^i\), so there is no need to introduce a (flat-\phi) condition.

**Remark 1.2** (Rigid cohomology 1). In this remark we explain why we have chosen not to include Berthelot’s rigid cohomology [3] in the statement of Conjecture 0.1, even though it is \textit{a priori} reasonable to consider the following conditions:

(crys) \(c\) lifts to \(H^{2i}_{\text{crys}}(X)\).

(crys-\phi) \(c\) lifts to \(H^{2i}_{\text{crys}}(X)^\phi = p^i\).

Remark 1.3, the validity of Conjecture 0.1 is unchanged by the addition of conditions (rig) and (rig-\phi).

The rigid analogue of condition (flat) is more subtle, but also redundant, as we now explain. Let \(j^*: F\text{-Isoc}^1(S/K) \to F\text{-Isoc}(S/K)\) denote the forgetful functor from overconvergent \(F\)-isocrystals on \(S\) to convergent \(F\)-isocrystals on \(S\). It is an open conjecture of Berthelot [3, §4.3] that Ogus’ convergent \(F\)-isocrystal \(R^{2i}f_*O_{X/K}\) admits an overconvergent extension; that is, there exists \(R^{2i}f_*O^1_{X/K} \in F\text{-Isoc}^1(S/K)\) such that \(j^*(R^{2i}f_*O^1_{X/K}) = R^{2i}f_*O_{X/K}\). Moreover, the functor \(j^*\) is now known to be fully faithful thanks to K. Kedlaya [29], and so \(R^{2i}f_*O^1_{X/K}\) is unique if it exists.

Assuming for a moment the validity of Berthelot’s conjecture, the following rigid analogue of (flat) could be considered:

(rig-flat) \(c\) lifts to \(H^0_{\text{rig}}(S, R^{2i}f_*O^1_{X/K})\).

However, the conditions (flat) and (rig-flat) are in fact equivalent for any \(c \in H^{2i}_{\text{crys}}(X_s)^\phi = p^i\). Firstly, the implication \(\Leftarrow\) is trivial. Secondly, assuming \(c\) lifts to \(\tilde{c} \in H^0_{\text{crys}}(S, R^{2i}f_*O_{X/K})\),

\[
\tilde{c} \in H^{2i}_{\text{crys}}(X_s)^\phi = p^i,
\]

Further details on this can be found in Section 2.2. It is an open problem whether or not this is true in general. However, it is known to be true when \(S\) is a perfect ring of characteristic \(p\) and \(X_s\) is a smooth \(p\)-adic formal scheme.
Remark 1.1. But, in the following diagram,
\[
\begin{array}{c}
H^0_{\text{rig}}(S, R^{2i} f_* \mathcal{O}_{X/K})^{\phi=p^i} \\
\text{Hom}_{F-\text{Isoc}(S/K)}(O_{S/k}(i), R^{2i} f_* \mathcal{O}_{X/K}) \\
\end{array}
\]
where (i) denotes Tate twists and all other notation should be clear, the bottom horizontal arrow is an isomorphism by Kedlaya’s aforementioned fully faithfulness result. Therefore \(\tilde{c}\) lifts uniquely to \(H^0_{\text{rig}}(S, R^{2i} f_* \mathcal{O}_{X/K})^{\phi=p^i}\), proving the implication \((\text{rig-flat})\). So assume that \((\text{crys})\) lifts to \(H^2_{\text{crys}}(X)^{\phi=p}\) factors through \(H^2_{\text{rig}}(X)^{\phi=p}\) by [39].

Our main result is the proof of Conjecture 0.1 for divisors (henceforth identified with line bundles), assuming \(f\) is projective. We finish this section by proving this, assuming the main results of later sections:

**Theorem 1.4.** Let \(f : X \to S\) be a smooth, projective morphism of smooth \(k\)-varieties, \(s \in S\) a closed point, and \(L \in \text{Pic}(X_s)\). Let \(c := c_1(L) \in H^2_{\text{crys}}(X_s)\). Then the following are equivalent:

- **(deform)** There exists \(\tilde{L} \in \text{Pic}(X)\) such that \(c_1(\tilde{L})|_{X_s} = c\).
- **(crys)** \(c\) lifts to \(H^2_{\text{crys}}(X)\).
- **(crys-\(\phi\))** \(c\) lifts to \(H^2_{\text{crys}}(X)^{\phi=p}\).
- **(flat)** \(c\) is flat, i.e., it lifts to \(H^0_{\text{crys}}(S, R^2 f_* \mathcal{O}_{X/K})\).

**Proof (assuming Corol. 3.4 and Thm. 2.1).** We may assume \(S\) is connected. By Theorem 2.1 below and the obvious implication (deform)\(\Rightarrow\)(crys), it is enough to prove the implication (crys-\(\phi\))\(\Rightarrow\)(deform). So assume that \(c\) lifts to \(\tilde{c} \in H^2_{\text{crys}}(X)^{\phi=p}\). Let \(A := \mathcal{O}_S^{\text{sh}}\) be the strict Henselisation of \(\mathcal{O}_{s,s}\), and let \(L^{\text{alg}}\) denote the pullback of \(L\) to \(X \times_S k(s)^{\text{alg}}\).
Applying Corollary 3.4 to $\hat{X} := X \times_S \hat{A}$ we see that there exists $L_1 \in \text{Pic}(\hat{X})[\frac{1}{p}]$ such that of $L_1|_{X \times_S k(s)} = L^{\text{alg}}$. The rest of the proof consists of descending $L_1$ from $\hat{X}$ to $X$.

By Neron–Popescu desingularisation [40, 41], we may write $\hat{A}$ as a filtered colimit of smooth, local $A$-algebras. Therefore there exist a smooth, local $A$-algebra $A'$, a morphism of $A$-algebras $A' \to \hat{A}$, and a line bundle $L_2 \in \text{Pic}(X \times_S A')[\frac{1}{p}]$ such that $L_2$ pulls back to $L_1$ via the morphism $\hat{X} \to X \times_S A'$. As usual, the composition $A \to A' \to \hat{A}$ induces an isomorphism of residue fields, so that $A \to A'$ has a section at the level of residue fields; but $A$ is Henselian and $A \to A'$ is smooth, so this lifts to a section $\sigma : A' \to A$. Then, by construction, the restriction of $L_3 := \sigma^* L_2 \in \text{Pic}(X \times_S A)[\frac{1}{p}]$ to $X \times_S k(s) = L^{\text{alg}}$ is $L^{\text{alg}}$.

But $A$ is the filtered colimit of the connected étale neighbourhoods of $\text{Spec } k(s) \to S$; so there exists an étale morphism $U \to S$ (with $U$ connected), a closed point $s' \in U$ sitting over $s$, and a line bundle $L_4 \in \text{Pic}(X \times_S U)[\frac{1}{p}]$ such that the restriction of $L_4$ to $X \times_S k(s')$ coincides with the pullback of $L$ to $X \times_S k(s')$.

Let $S'$ be the normalisation of $S$ inside the function field of $U$, and let $L_5 \in \text{Pic}(X \times_S S')[\frac{1}{p}]$ be any extension of $L_4$, which exists by normality of $X \times_S S'$. By de Jong’s theory of alterations [12], there exists a generically étale alteration $\pi'' : S'' \to S'$ with $S''$ connected and smooth over $k$. Let $L_6 := \pi''^* L_5 \in \text{Pic}(X \times_S S'')[\frac{1}{p}]$, and also let $s'' \in S''$ be any closed point sitting over $s'$. To summarise, we have a commutative diagram

\[
\begin{array}{ccc}
S & \xymatrix{X \ar[r]^{\pi_X} & X' \ar[r]^{\pi'} & X'' \\
S' & \xymatrix{S \ar[r]^\pi & S' \ar[r]^\pi' & S''} \\
& \text{Spec } k(s) \ar[u] & \text{Spec } k(s') \ar[u] & \text{Spec } k(s'') \ar[u]
\end{array}
\]

where:

- $X' := X \times_S S'$ and $X'' := X \times_S S''$;
- $\pi := \pi' \circ \pi''$ is a generically étale alteration;
- the restriction of $L_6$ to $X'' \times_{S''} k(s'') = X \times_S k(s'')$ coincides with the pullback to $L$ to $X \times_S k(s'')$.

By studying crystalline Chern classes, we can now complete the proof. It will be convenient to denote by $e : H^2_{\text{crys}}(X) \to H^0_{\text{crys}}(S, R^2 f_* \mathcal{O}_{X/K})$ the edge map in the Leray spectral sequence (abusing notation, we also use the notation $e$ for the families $X'$, $X''$, etc.; in particular, set $\overline{e} := e(\overline{c})$).

Let $V \subseteq X$ be a nonempty open subscheme such that $\pi^{-1}(V) \to V$ is finite étale (this exists since $\pi$ is proper and generically étale), and let $L_7 \in \text{Pic}(X \times_S V)[\frac{1}{p}]$ be the pushforward of $L_6|_{X \times_S \pi^{-1}(V)}$. We claim that $e(c_1(L_7)) = n \overline{e}|_V$, where $n$ is the generic degree of $\pi$.  

7
First we note that \( e(c_1(L_6)) = \pi^*(\overline{c}) \) in \( H^0_{\text{crys}}(S''', R^2f'''_*\mathcal{O}_{X''/K}) \); the two classes agree at \( s'' \) by construction of \( L_6 \), and the specialisation map \( H^0_{\text{crys}}(S''', R^2f'''_*\mathcal{O}_{X''/K}) \to H^2_{\text{crys}}(X'' \times_{S''} k(s'')) \) is injective by [37, Thm. 4.1]. Restricting to \( \pi^{-1}(V) \), we therefore obtain \( e(c_1(L_6|_{X \times_S \pi^{-1}(V)})) = \pi^* (\overline{c}_V) \). Pushing forwards along the finite étale morphism \( \pi^{-1}(V) \to V \) proves the claim.

Finally, let \( \tilde{L} \in \text{Pic}(X)_\mathbb{Q} \) be any extension of \( L_7^{1/n} \). Then \( e(c_1(\tilde{L})) \) agrees with \( \overline{c} \) on \( V \), hence agrees with \( \overline{c} \) everywhere. In particular, by specialising to \( s \) we obtain that \( c_1(\tilde{L})|_{X_s} = c \); this completes the proof.

**Remark 1.5** (Line bundles with \( \mathbb{Q}_p \)-coefficients). Theorem 1.4 remains true after replacing \( \text{Pic}(X_s)_\mathbb{Q} \) and \( \text{Pic}(X)_\mathbb{Q} \) by \( \text{Pic}(X_s)_{\mathbb{Q}_p} \) and \( \text{Pic}(X)_{\mathbb{Q}_p} \) respectively. The proof is almost exactly the same, replacing Corollary 3.4 by the "\( \mathbb{Q}_p \)-version of Corollary 3.4" or Remark 3.10.

## 2 Crystalline Théorème de la Partie Fixe

The aim of this section is to prove the following equivalences between the conditions appearing in Conjecture 0.1 (\( k \) continues to be a perfect field of characteristic \( p \)):

**Theorem 2.1.** Let \( f : X \to S \) be a smooth, projective morphism of smooth \( k \)-varieties, \( s \in S \) a closed point, and \( c \in H^2_{\text{crys}}(X_s)^{\phi = p^i} \). Then the following are equivalent:

1. **(crys)** \( c \) lifts to \( H^2_{\text{crys}}(X) \).
2. **(crys-ϕ)** \( c \) lifts to \( H^2_{\text{crys}}(X)^{\phi = p^i} \).
3. **(flat)** \( c \) is flat, i.e., it lifts to \( H^0_{\text{crys}}(S, R^{2i}f_*\mathcal{O}_{X/K}) \).

**Remark 2.2.** Theorem 2.1 and its proof via Theorem 2.6 resemble P. Deligne’s Théorème de la Partie Fixe [15, §4.1] for de Rham cohomology in characteristic zero, though the name in our case is a misnomer as we do not consider the action of the fundamental group \( \pi_1(S, s) \).

Deligne extends his result in characteristic zero to smooth, proper morphisms using resolution of singularities. Unfortunately, the standard arguments with alterations appear to be insufficient for us to do the same in characteristic \( p \), and so we are forced to restrict to projective morphisms in some of our main results.

We begin with two preliminary results, Lemmas 2.3 and 2.4, on spectral sequences in an arbitrary abelian category. In these lemmas all spectral sequences are implicitly assumed to start on the \( E_1 \)-page for simplicity. We say that the \( n^{th} \) column of a spectral sequence \( E^a_* \) is stable if and only if \( E^a_{i\ge n} = E^a_{i=n} \) for all \( a \in \mathbb{Z} \); in other words, all differentials into and out of the \( n^{th} \) column are zero. Obvious modification of the terminology, such as stable in columns \( > n \), or \( < n \), will be used in this section.

**Lemma 2.3.** Suppose that \( E^a_* \) and \( F^a_* \) are spectral sequences, that \( n \in \mathbb{Z} \), and that the following conditions hold:

1. **(i)** The spectral sequence \( E^a_* \) is stable in columns \( > n \).
2. **(ii)** The spectral sequence \( F^a_* \) is stable in columns \( < n \).
(iii) There is a map of spectral sequences $f : E^b_{*} \to F^b_{*}$ which is an isomorphism on the $n^{th}$ columns of the first pages, i.e., $f : E^b_1 \cong F^b_1$ for all $b \in \mathbb{Z}$.

Then the $n^{th}$ columns of both spectral sequences are stable, and isomorphic via $f$, i.e.,

$$
\begin{array}{ccc}
E^b_1 & \xrightarrow{f \cong} & F^b_1 \\
\downarrow & & \downarrow \\
E^b_\infty & \xrightarrow{f \cong} & F^b_\infty
\end{array}
$$

Proof. According to assumptions (i) and (ii), all differentials with domain (resp. codomain) in the $n^{th}$ column of any page of the $E$-spectral sequence (resp. $F$-spectral sequence) are zero. So it remains to check that all differentials with codomain (resp. domain) in the $n^{th}$ column of any page of the $E$-spectral sequence (resp. $F$-spectral sequence) are zero. This is an easy induction, using assumption (iii), on the page number of the spectral sequence.

The following technique to check the degeneration of a family of spectral sequences is inspired by [14, Thm. 1.5]:

**Lemma 2.4.** Let $d \geq 0$ and let

$$
\cdots \xrightarrow{u} E^{ab}_*(-4) \xrightarrow{u} E^{ab}_*(-2) \xrightarrow{u} E^{ab}_*(0) \xrightarrow{u} E^{ab}_*(2) \xrightarrow{u} E^{ab}_*(4) \xrightarrow{u} \cdots
$$

be a sequence of right half plane spectral sequences. Make the following assumptions:

(i) For every $n \in 2\mathbb{Z}$, the spectral sequence $E^{ab}_*(n)$ vanishes in columns $> n$.

(ii) For every $n \in \mathbb{Z}$ and $i \geq 0$ such that $i \equiv n \mod 2$, the map of spectral sequences $u^i : E^{ab}_*(n-i) \to E^{ab}_*(n+i)$ is an isomorphism on the $n-2d^{th}$ columns of the first pages.

Then, for every $n \in 2\mathbb{Z}$, the spectral sequence $E^{ab}_*(n)$ degenerates on the $E_1$-page.

Proof. For any $n \in 2\mathbb{Z}$ and any integer $c \leq d+n$, assumption (ii) implies that

$$
E^{ab}_*(2c-n) \xrightarrow{u^{d+n-c}} E^{ab}_*(2d+n)
$$

is an isomorphism on the $c^{th}$ columns of the first pages. In particular, if $c < n$, so that the $c^{th}$ column of the left spectral sequence in (2) vanishes, then the $c^{th}$ column of the right spectral sequence also vanishes. That is, $E^{ab}_*(2d+n)$ vanishes in columns $< n$; or, reindexing, $E^{ab}_*(n)$ vanishes in columns $< n-2d$ for every $n \in 2\mathbb{Z}$.

For each $i = 0, \ldots, d+1$, we now make the following claim: for every $n \in 2\mathbb{Z}$, the spectral sequence $E^{ab}_*(n)$ is stable in columns $> n-i$ and in columns $< n+i-2d$.

The claim is true when $i = 0$, thanks to assumption (i) and our vanishing observation above. Proceeding by induction, assume that the claim is true for some $i \in \{0, \ldots, d\}$. Then, for any $n \in 2\mathbb{Z}$, the map of spectral sequences

$$
E^{ab}_*(n) \xrightarrow{u^{d-i}} E^{ab}_*(n+2d-2i)
$$

is an isomorphism on the $c^{th}$ columns of the first pages. In particular, if $c < n$, so that the $c^{th}$ column of the left spectral sequence in (3) vanishes, then the $c^{th}$ column of the right spectral sequence also vanishes. That is, $E^{ab}_*(2d+n)$ vanishes in columns $< n$; or, reindexing, $E^{ab}_*(n)$ vanishes in columns $< n-2d$ for every $n \in 2\mathbb{Z}$.
is an isomorphism on the \( n - i \) columns of the \( E_1 \)-pages, by assumption (ii); moreover, by the inductive hypothesis, the left spectral sequence in (3) is stable in columns \( > n - i \) and the right spectral sequence in columns \( < n - i \). By Lemma 2.3, both spectral sequences are therefore stable in column \( n - i \), proving the inductive claim for \( i + 1 \).

But this completes the proof, for the inductive claim at \( i = d + 1 \) asserts that the spectral sequence \( E^ab_n(n) \) is stable in columns \( > n - d - 1 \) and \( < n - d + 1 \), hence degenerates on the first page. \( \square \)

Remark 2.5 (Hard Lefschetz for crystalline cohomology). Before proving the main theorems of the section we make some remarks on the Hard Lefschetz theorem for crystalline cohomology. Let \( X \) be a smooth, projective, connected, \( d \)-dimensional variety over \( k \), and let \( L \) be an ample line bundle on \( X \). Let \( u := c_1(L) \in H^2_{\text{cris}}(X) \), and also denote by \( u \) the induced cup product map \( u \cup - : H^\ast_{\text{cris}}(X) \to H^{\ast + 2}_{\text{cris}}(X) \). We understand the Hard Lefschetz theorem as the assertion that

\[
u^i : H^{d-i}_{\text{cris}}(X) \longrightarrow H^{d+i}_{\text{cris}}(X)
\]

is an isomorphism of \( K \)-vector spaces for \( i = 0, \ldots, d \).

Assuming in addition that \( L = \mathcal{O}(D) \) for some smooth hyperplane section \( D \) of \( X \), that \( X \) is geometrically connected over \( k \), and that \( k \) is finite, isomorphism (4) follows from the \( \ell \)-adic case, as explained in [28]; the assumption that \( k \) is finite may be eliminated by a standard spreading out argument [23, §3.8]. We now explain how to eliminate the other additional assumptions.

Firstly, replacing \( L \) by \( L^m \) replaces \( u \) by \( mu \), so we may assume that \( L \) is very ample; that is, there is a closed embedding \( i : X \hookrightarrow \mathbb{P}^N_k \) such that \( L = i^*\mathcal{O}(1) \). Now let \( k' \) be a finite extension of \( k \) with the following properties: the geometric connected components of \( X \) are defined over \( k' \); and there exists a hyperplane \( H \) in \( \mathbb{P}^N_{k'} \) having smooth intersection with \( X' \). Since \( H^\ast_{\text{cris}}(X') = H^\ast_{\text{cris}}(X) \otimes_K \text{Frac} W(k') \), the validity of isomorphism (4) for \( X' \) implies it for \( X \), so we may replace \( k \) by \( k' \); clearly we may also then replace \( X \) by each of its connected components. So we have reduced to the case that \( X \) is geometrically connected and that \( L = i^*\mathcal{O}(1) \) for some closed embedding \( i : X \hookrightarrow \mathbb{P}^N_k \) such that there exists a hyperplane \( H \in |\mathcal{O}(1)| \) having smooth intersection with \( X \); i.e., we have reduced to the case of the previous paragraph, proving isomorphism (4) in general.

The following is our analogue in crystalline cohomology of Deligne’s Théorème de la Partie Fixe [15, §4.1], in which \( \phi \) denotes, as everywhere, the absolute Frobenius:

Theorem 2.6. Let \( f : X \to S \) be a smooth, projective morphism of smooth \( k \)-varieties. Then:

(i) The Leray spectral sequence \( E_2^{ab} = H^\ast_{\text{cris}}(S, R^b f_*\mathcal{O}_{X/K}) \Rightarrow H^{ab}_{\text{cris}}(X) \) degenerates on the \( E_2 \)-page.

(ii) For any \( r \geq 0, s \in \mathbb{Z} \), the Leray spectral sequence in (i) contains a sub spectral sequence \( H^\ast_{\text{cris}}(S, R^b f_*\mathcal{O}_{X/K})^{\phi^s = p^s} \Rightarrow H^{a+b}_{\text{cris}}(X)^{\phi^r = p^r} \) which also degenerates on the \( E_2 \)-page. (The superscripts denote the \( p^s \)-eigenspaces of \( \phi^r \).)
A Variational Tate Conjecture in crystalline cohomology

(iii) For any \( n, r \geq 0, s \in \mathbb{Z} \), the canonical map

\[
H^n_{\text{crys}}(X)^{\phi^r=p^s} \to H^0_{\text{crys}}(S, R^n f_* \mathcal{O}_{X/K})^{\phi^r=p^s}
\]

is surjective.

Proof. We may assume that \( S \) and \( X \) are connected, and we let \( d \) denote the relative dimension of \( f \). Let \( L \) be a line bundle on \( X \) which is relatively ample with respect to \( f \), and let \( u := c_1(L) \in H^2_{\text{crys}}(X) \). Also denote by \( u \) the induced cup product morphism of convergent isocrystals \( u : R^i f_* \mathcal{O}_{X/K} \to R^{i+2} f_* \mathcal{O}_{X/K} \), and note that

\[
u^i : R^{d-i} f_* \mathcal{O}_{X/K} \to R^{d+i} f_* \mathcal{O}_{X/K}
\]

is an isomorphism for \( i \geq 0 \); indeed, it is enough to check this isomorphism after restricting to each closed point of \( S \) [37, Lem. 3.17], where it is exactly (using the identification in [37, Rem. 3.7.1]) the Hard Lefschetz theorem for crystalline cohomology which was discussed in Remark 2.5. Note that isomorphism (5) is valid even if \( i > d \), the left side being zero by convention and the right side by relative dimension considerations; this is helpful for indexing.

Claim (i) now follows by applying Deligne’s axiomatic approach to the degeneration of Leray spectral sequences via Hard Lefschetz [14, §1] to \( Rf_* \mathcal{O}_{X/K} \), which lives in the derived category of Ogus’ convergent topos \((S/W)^{\text{conv}} \mathcal{O} \).

(ii): By (i), each group \( H^n := H^n_{\text{crys}}(X) \) (which, for simplicity of indexing, is defined to be zero if \( n < 0 \)) is equipped with a natural descending filtration

\[H^n = \cdots = F_{-1}H^n = F_0H^n \supseteq \cdots \supseteq F_nH^n \supseteq F_{n+1}H^n = F_{n+2}H^n = \cdots = 0 \]

having graded pieces \( g^n_a = F_a H^n/F_{a+1}H^n \cong H^n_{\text{crys}}(S, R^{n-a} f_* \mathcal{O}_{X/K}) \) for \( a \in \mathbb{Z} \). Note that this filtration is respected by \( \phi \). The assertion to be proved is that the induced filtration on the subgroup \((H^n)^{\phi^r=p^s}\) has graded pieces \((g^a)_{n=\phi^r=p^s}\).

In other words, fixing \( r \geq 0 \) but allowing \( s \in \mathbb{Z} \) to vary, we must show that the Ker–Coker spectral sequence for the morphism \( \phi^r - p^s : H^n \to H^n \) of filtered groups, namely

\[
E_{1}^{ab}(n, s) = \begin{cases} \text{Ker}(g^{n}_a \phi^r - p^s \to g^{n}_a) & a + b = 0 \\ \text{Coker}(g^{n}_a \phi^r - p^s \to g^{n}_a) & a + b = 1 \\ 0 & \text{else} \end{cases} \to \begin{cases} \text{Ker}(H^n \phi^r - p^s \to H^n) & a + b = 0 \\ \text{Coker}(H^n \phi^r - p^s \to H^n) & a + b = 1 \\ 0 & \text{else} \end{cases}
\]

degenerates on the first page.

Since \( \phi u = pu \phi \), cupping with \( u \) defines a morphism of spectral sequences \( u : E_{1}^{ab}(n, s) \to E_{1}^{ab}(n + 2, s + 1) \), and we will check that this family of spectral sequences satisfies the hypotheses of Lemma 2.4 (to be precise, we apply Lemma 2.4 to \( E_{1}^{ab}(n) := \bigoplus_{s \in \mathbb{Z}} E_{1}^{ab}(n, s) \), where \( n \in \mathbb{N} \) and with \( u \) defined degree-wise). Firstly, \( E_{1}^{ab}(n, s) \) vanishes in columns \( n \) since the filtration on \( H^n \) has length \( n \). Secondly, the morphism

\[
u^i : E_{1}^{ab}(n - i, s) \to E_{1}^{ab}(n + i, s + i)
\]

is an isomorphism on the \( (n - d)^{\text{th}} \) column of the \( E_1 \)-pages for any \( n \in \mathbb{Z} \) by (5). Hence, by Lemma 2.4, the spectral sequence \( E_{1}^{ab}(n, s) \) degenerates on the first page for every
$n \in 2\mathbb{Z}$ and $s \in \mathbb{Z}$; a minor reindexing treats the case that $n$ is odd, completing the proof of (ii).

Claim (iii) is immediate from (ii), as the canonical map is the edge map. \hfill \Box

**Remark 2.7.** More generally, Theorem 2.6(ii)&(iii) remain true if $\phi^r - p^s$ is replaced by any $K$-linear combination of integral powers of $\phi$.

**Remark 2.8.** Deligne’s result used in the proof of Theorem 2.6(i) states not only that the Leray spectral sequence degenerates, but even that there is a non-canonical isomorphism $Rf_*\mathcal{O}_{X/K} \cong \bigoplus_i R^i f_*\mathcal{O}_{X/K}[-i]$ in $D^b((S/W)_{\text{conv}})$. The content of Lemmas 2.3 and 2.4 is essentially that this isomorphism may be chosen to be compatible with the action of $\phi$.

Theorem 2.6 is evidently sufficient to prove our desired equivalences:

**Proof of Theorem 2.1.** We may assume $S$ is connected. In light of the obvious implications of Remark 1.3, it is enough to prove that (flat)$\Rightarrow$(crys-$\phi$); so assume that $c$ lifts to some $\tilde{c} \in H^0_{\text{crys}}(S, R^{2i}f_*\mathcal{O}_{X/K})$. As discussed in Remark 1.1, the canonical map $H^0_{\text{crys}}(S, R^{2i}f_*\mathcal{O}_{X/K}) \to H^2_{\text{crys}}(X_s)$ is injective by [37, Thm. 4.1], and hence $\tilde{c} \in H^0_{\text{crys}}(S, R^{2i}f_*\mathcal{O}_{X/K})^{\phi^r=p^s}$. Theorem 2.6(iii) implies that $\tilde{c}$ lifts to $H^{2i}_{\text{crys}}(X)^{\phi^r=p^s}$, completing the proof. \hfill \Box

**Remark 2.9 (Rigid cohomology 2).** We finish this section by making further remarks on rigid cohomology, continuing Remark 1.2. We assume throughout this remark that $f : X \to S$ is a smooth, proper morphism, where $S$ is a smooth, affine curve over $k = k_{\text{alg}}$, and that $f$ admits a semi-stable compactification $\overline{f} : \overline{X} \to \overline{S}$, where $\overline{S}$ and $\overline{X}$ are smooth compactifications of $S$ and $X$. We will sketch a proof of the following strengthening (which we do not need) of Theorems 2.1 and 2.6 in this special case:

The composition

$$H^{2i}_{\text{rig}}(X)^{\phi^r=p^s} \longrightarrow H^{2i}_{\text{crys}}(X)^{\phi^r=p^s} \longrightarrow H^0_{\text{crys}}(S, R^{2i}f_*\mathcal{O}_{X/K})^{\phi^r=p^s}$$

is surjective.

Since $S$ is an affine curve, Berthelot’s conjecture discussed in Remark 1.2 is known to have an affirmative answer by S. Matsuda and F. Trihan [31]: a unique overconvergent extension $R^i f_*\mathcal{O}^1_{X/K}$ of $R^i f_*\mathcal{O}_{X/K}$ exists for each $i \geq 0$. It is then not unreasonable to claim that there “obviously” exists a Leray spectral sequence in rigid cohomology, namely

$$E_2^{ab} = H^a_{\text{rig}}(S, R^b f_*\mathcal{O}_{X/K}^1) \implies H^{a+b}_{\text{rig}}(X),$$

but this turns out to be highly non-trivial. Assuming for a moment that this spectral sequence exists, it must degenerate for dimension reasons and thus yield short exact sequences

$$0 \longrightarrow H^1_{\text{rig}}(S, R^{n-1}f_*\mathcal{O}_{X/K}^1) \longrightarrow H^1_{\text{rig}}(X) \longrightarrow H^0_{\text{rig}}(S, R^n f_*\mathcal{O}_{X/K}^1) \longrightarrow 0.$$
A Variational Tate Conjecture in crystalline cohomology

By finiteness of rigid cohomology these groups are all finite dimensional $K$-vector spaces, and so by Dieudonné–Manin the sequence remains exact after restricting to Frobenius eigenspaces. In particular, the edge maps

$$H_{\text{rig}}^{2i}(X)^{\phi=p^i} \to H_{\text{rig}}^0(S, R^{2i}f_*\mathcal{O}_{X/K})^{\phi=p^i} = H_{\text{cris}}^0(S, R^{2i}f_*\mathcal{O}_{X/K})^{\phi=p^i}$$

(the final equality was explained in Remark 1.2) are surjective, as desired.

It remains to show that the rigid Leray spectral sequence (6) really exists; the proof uses results from log crystalline cohomology due to A. Shiho. Let $M = S \setminus S$ and $D = X \setminus X$, let $(S, M)$ and $(X, D)$ denote the associated log schemes, and view $f : (X, D) \to (S, M)$ as a log smooth morphism of Cartier type. By the general theory of sites there is an associated Leray spectral sequence in log-convergent cohomology

$$E_2^{ab} = H^a((S, M)_{\text{logconv}}, R^b f_*\mathcal{K}) \implies H^{a+b}((X, D)_{\text{logconv}}, \mathcal{K}),$$

(7)

where $\mathcal{K}$ is the structure sheaf on $(X, D)_{\text{logconv}}$, the log-convergent site of $(X, D)$. However, [43, Corol. 2.3.9 & Thm. 2.4.4] state that $H^{a+b}((X, D)_{\text{logconv}}, \mathcal{K}) \cong H_{\text{rig}}^{a+b}(X)$. It can moreover be shown, using Shiho’s results on relative log-convergent cohomology [45, 46, 47], that the terms on the $E_2$-page of (7) are naturally isomorphic to $H_{\text{rig}}^a(S, R^b f_*\mathcal{O}_{X/K})$, thereby completing the proof of the existence of (6). This completes the sketch of the proof of the strengthening of Theorems 2.1 and 2.6 in this special case.

3 Local and infinitesimal forms of Conjecture 0.1

The aim of this section is to prove Theorems 0.5 and 0.6 from the Introduction, which are local and infinitesimal analogues of Conjecture 0.1. As always, $k$ is a perfect field of characteristic $p$. We will study a smooth, proper scheme $X$ over Spec $A$, where $A := k[[t_1, \ldots, t_m]]$, and we adopt the following notation: the special fibre and its infinitesimal thickenings inside $X$ are always denoted by

$$Y := X \times_A k, \quad Y_r := X \times_A A/(t_1, \ldots, t_m)^r.$$

Some of the deformation results of this section only require $p$-torsion to be neglected, so we write $M[\frac{1}{p}] := M \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$ for any abelian group $M$.

3.1 Some remarks on crystalline cohomology

The proofs of Theorems 0.5 and 0.6 require us to work with crystalline cohomology and the de Rham–Witt sheaves $W_r\Omega^n_X$ of S. Bloch, P. Deligne, and L. Illusie, in greater generality than can be found in Illusie’s treatise [24]; some of the generalisations we need can be readily extracted from his proofs and have already been presented in [44], while others follow via a filtered colimit argument to reduce to the smooth, finite type case.

Firstly, if $X$ is any regular (not necessarily of finite type) $k$-scheme and $n \geq 0$, then the sequence of pro étale sheaves

$$0 \rightarrow \{W_r\Omega^n_X, \log\}_r \rightarrow \{W_r\Omega^n_X\}_r \xrightarrow{1-F_r} \{W_r\Omega^n_X\}_r \rightarrow 0$$

(8)
is known to be short exact by [44, Cor. 2.9], where \( W_r \Omega_{X, log}^n \) denotes the étale subsheaf of \( W_r \Omega_X^\bullet \) generated étale locally by logarithmic forms.

The continuous cohomology [25] of a pro étale sheaf such as \( \{ W_i \Omega_{X, log}^n \}_r \) will be denoted by \( H^*_\cont(X, \{ W_i \Omega_{X, log}^n \}_r) \), or simply by \( H^*_\cont(X, W_r \Omega_X^\bullet) \) when there is no chance of confusion; similar notation is applied for continuous hypercohomology of pro complexes of étale sheaves, and in other topologies (if we do not specify a topology, it means Zariski). A more detailed discussion of such matters may be found in [16, §1.5.1].

If \( X \) is any \( k \)-scheme, then its crystalline cohomology groups \( H^n_{\text{crys}}(X/W) \) and \( H^n_{\text{crys}}(X) := H^n_{\text{crys}}(X/W) \otimes_W K \) are defined using the crystalline site as in [7]; this does not require \( X \) to satisfy any finite-type hypotheses. Illusie’s comparison theorem [24, §2.1] states that there is a natural isomorphism

\[
H^n_{\text{crys}}(X/W) \xrightarrow{\sim} \mathbb{H}^n_{\text{cont}}(X, W \Omega_X^\bullet) \tag{9}
\]

for any smooth variety \( X \) over \( k \), but this remains true for any regular \( k \)-scheme \( X \).

(Proof: It is enough to show that \( H^n_{\text{crys}}(X/W_r) \xrightarrow{\sim} \mathbb{H}^n(X, W_r \Omega_X^\bullet) \) for all \( n \geq 0, r \geq 1 \); by the Mayer–Vietoris property of both sides we may assume \( X = \text{Spec} C \) is affine. By Neron–Popescu desingularisation [40, 41] we may write \( C = \lim \alpha C_{\alpha} \) as a filtered colimit of smooth \( k \)-algebras; since the de Rham–Witt complex commutes with filtered colimits, it is now enough to prove that \( \lim \alpha H^n_{\text{crys}}(C_{\alpha}/W_r) \xrightarrow{\sim} H^n_{\text{crys}}(C/W_r) \) for all \( n \geq 0, r \geq 1 \). This is easily seen to be true if we pick compatible representations \( C_{\alpha} = P_{\alpha}/J_{\alpha} \) of \( C_{\alpha} \) as quotients of polynomial algebras over \( W_r(k) \) and compute the crystalline cohomology in the usual way via the integrable connection arising on the divided power envelope of \( (W_r(k), pW_r(k), \gamma) \to (P_{\alpha}, J_{\alpha}) \). In particular, all proofs in Section 3.2 will use \( \mathbb{H}^n_{\text{cont}}(X, W \Omega_X^\bullet) \), but we will identify it with \( H^n_{\text{crys}}(X/W) \) when stating our results.

### 3.2 Proofs of Theorems 0.5 and 0.6 via topological cyclic homology

We begin with a preliminary version of Theorem 0.6, phrased in terms of M. Gros’ [20] logarithmic crystalline Chern character \( ch_{\text{log}} : K_0(Y) \otimes_\mathbb{Q} \to \bigoplus_{i \geq 0} H^i_{\text{cont}}(Y, W \Omega^i_{Y, log}) \otimes_\mathbb{Q} \). This is the most fundamental result of the article, depending essentially on a recent continuity theorem in topological cyclic homology.

**Proposition 3.1.** Let \( X \) be a smooth, proper scheme over \( k[[t_1, \ldots, t_m]] \), and let \( z \in K_0(Y)|_{\frac{1}{p}} \). Then the following are equivalent:

\( \begin{align*}
(i) \ z \text{ lifts to } (\lim \alpha K_0(Y_{eta}))|_{\frac{1}{p}}.
(ii) \ ch_{\text{log}}(z) \in \bigoplus_{i \geq 0} H^i_{\text{cont}}(Y_{et}, W \Omega^i_{Y, log}) \otimes_\mathbb{Q} \text{ lifts to } \bigoplus_{i \geq 0} H^i_{\text{cont}}(X_{et}, W \Omega^i_{X, log}) \otimes_\mathbb{Q}.
\end{align*} \)

**Proof.** The proof is an application of a continuity theorem in topological cyclic homology due to the author and B. Dundas; for a summary of topological cyclic homology and its notation, we refer the reader to, e.g., [16] or [35]. Indeed, according to [35, Thm. 5.8], the canonical map \( TC(X; p) \to \text{holim}_r TC(Y_r; p) \) is a weak equivalence, where \( TC(-; p) \) denotes the \( p \)-typical topological cyclic homology spectrum of a scheme; since the homotopy fibre of the trace map \( tr : K(-) \to TC(-; p) \) is nilinvariant by McCarthy’s theorem [33] (or rather the scheme-theoretic version of McCarthy’s theorem coming from Zariski...
A Variational Tate Conjecture in crystalline cohomology

descent [17]), it follows that there is a resulting homotopy cartesian square of spectra

\[
\begin{array}{ccc}
\text{holim}_r K(Y_r) & \longrightarrow & K(Y) \\
\downarrow & & \downarrow \text{tr} \\
TC(X; p) & \longrightarrow & TC(Y; p)
\end{array}
\]

In other words, there is a commutative diagram of homotopy groups with exact rows:

\[
\begin{array}{ccc}
\cdots & \longrightarrow & \pi_n \text{holim}_r K(Y_r) \\
\downarrow & & \downarrow \\
\pi_n K(Y) & \longrightarrow & \pi_{n-1} \text{holim}_r K(Y_r, Y) & \longrightarrow & \cdots
\end{array}
\]

\[
\begin{array}{ccc}
\cdots & \longrightarrow & TC_n(X; p) \\
\downarrow & & \downarrow \\
TC_n(Y; p) & \longrightarrow & TC_{n-1}(X, Y; p) & \longrightarrow & \cdots
\end{array}
\]

Inverting \( p \), and noting that the natural map \( \pi_n \text{holim}_r K(Y_r) \rightarrow K_n(Y) \) factors through the surjection \( \pi_n \text{holim}_r K(Y_r) \rightarrow \lim_{\leftarrow r} K_n(Y_r) \), one deduces from the diagram that the following are equivalent for any \( z \in K_n(Y)\frac{1}{p} \):

(i) \( z \) lifts to \( (\lim_{\leftarrow r} K_n(Y_r))\frac{1}{p} \).

(ii) \( \text{tr}(z) \) lifts to \( TC_n(X; p)\frac{1}{p} = TC_n(X; p)\mathbb{Q} \).

Moreover, since \( Y \) and \( X \) are regular \( \mathbb{F}_p \)-schemes, there are natural decompositions

\[
\begin{align*}
TC_n(Y; p)\mathbb{Q} &= \bigoplus_i H^{i-n}_{\text{cont}}(Y_{\text{ét}}, W\Omega^i_{Y, \log})\mathbb{Q}, \\
TC_n(X; p)\mathbb{Q} &= \bigoplus_i H^{i-n}_{\text{cont}}(X_{\text{ét}}, W\Omega^i_{X, \log})\mathbb{Q},
\end{align*}
\]

by T. Geisser and L. Hesselholt [17, Thm. 4.1.1].

Taking \( n = 0 \), the proof of the theorem will be completed as soon as we show that the rationalised trace map of topological cyclic homology

\[
\text{tr} : K_0(Y)\mathbb{Q} \longrightarrow TC_0(Y; p)\mathbb{Q} = \bigoplus_i H^{i}_{\text{cont}}(Y_{\text{ét}}, W\Omega^i_{Y, \log})\mathbb{Q}
\]

is equal to Gros’ logarithmic crystalline Chern character. This reduces via the usual splitting principle to the case of a line bundle \( L \in H^1(Y, \mathcal{O}^\times_Y) \), where it follows from the fact that both \( \text{tr}(L) \) and the log crystalline Chern class \( c^\log_1(L) \) are induced by the dlog map \( \mathcal{O}^\times_Y \rightarrow W\Omega^1_Y; \) see [17, Lem. 4.2.3] and [20, §1.2] respectively.

To transform Proposition 3.1 into Theorem 0.6, we must study the relationship between the cohomology of the logarithmic de Rham–Witt sheaves and the eigenspaces of Frobenius acting on crystalline cohomology. For smooth, proper varieties over \( k \), this follows from the general theory of slopes, but we require such comparisons also for the smooth, proper scheme \( X \) over \( k[[t_1, \ldots, t_m]] \). To be more precise, for any regular \( k \)-scheme \( X \) we denote by

\[
\varepsilon : W^r \Omega^i_{X, \log}[-i] \longrightarrow W^r \Omega^i_X
\]

the canonical map of complexes obtained from the inclusion \( W^r \Omega^i_{X, \log} \subseteq W^r \Omega^i_X \), and we will consider the induced map on continuous cohomology. Note that \( W^r \Omega^i_X \) has the
same cohomology in the Zariski and étale topologies, since it is a quasi-coherent sheaf on the scheme $W_r(X)$ (e.g., by [44, Prop. 2.18]); so we obtain an induced map
\[
\varepsilon : H^i_{\text{cont}}(X_{\text{ét}}, W\Omega^i_{X,\log}) \longrightarrow \mathbb{H}^{2i}_{\text{cont}}(X_{\text{ét}}, W\Omega^i_X) = \mathbb{H}^{2i}_{\text{cont}}(X, W\Omega^i_X) = H^{2i}_{\text{crys}}(X/W),
\]
(the final equality follows from line (9)), which we study rationally in the following proposition:

**Proposition 3.2.** Let $X$ be a regular k-scheme and $i \geq 0$; consider the above map rationally:
\[
\varepsilon_Q : H^i_{\text{cont}}(X_{\text{ét}}, W\Omega^i_{X,\log})_Q \longrightarrow H^{2i}_{\text{crys}}(X).
\]
Then:

(i) The image of $\varepsilon_Q$ is $H^{2i}_{\text{crys}}(X)^{p_0 \neq p}$.

(ii) If $X$ is a smooth, proper variety over a finite or algebraically closed field $k$, then $\varepsilon_Q$ is injective.

**Proof.** Let $W_r\Omega^{\geq 1}_X$ and $W_r\Omega^{< 1}_X$ denote the naive upwards and downwards truncations of $W_r\Omega^i_X$ at degree $i$. Recalling the well-known de Rham–Witt identities $dF = pFd$ and $Vd = pdV$, we may define morphisms of complexes
\[
\mathcal{F} : W_r\Omega^{\geq 1}_X \longrightarrow W_{r-1}\Omega^{\geq 1}_X, \quad \forall : W_r\Omega^{< 1}_X \longrightarrow W_{r+1}\Omega^{< 1}_X
\]
degree-wise as
\[
p^\lambda F : W_r\Omega^{i+\lambda}_X \longrightarrow W_{r-1}\Omega^{i+\lambda}_X, \quad p^{i-1-\lambda}V : W_r\Omega^{i-1-\lambda}_X \longrightarrow W_{r+1}\Omega^{i-1-\lambda}_X
\]
for all $\lambda \geq 0$. We claim that the resulting morphisms of pro complexes of sheaves
\[
1 - \mathcal{F} : \{W_r\Omega^{\geq 1}_X\}_r \longrightarrow \{W_r\Omega^{\geq 1}_X\}_r, \quad 1 - \forall : \{W_r\Omega^{< 1}_X\}_r \longrightarrow \{W_r\Omega^{< 1}_X\}_r
\]
are isomorphisms.

To prove this claim, it is sufficient to show that $1 - p^\lambda F : \{W_r\Omega^{\geq 1}_X\}_r \rightarrow \{W_r\Omega^{\geq 1}_X\}_r$ and $1 - p^{\lambda-1}V : \{W_r\Omega^{< 1}_X\}_r \rightarrow \{W_r\Omega^{< 1}_X\}_r$ are isomorphisms of pro sheaves for all $\lambda \geq 1$. This follows from the fact that $p^\lambda F$ and $p^{\lambda-1}V$ are contracting operators; more precisely, inverses are provided by the maps
\[
\sum_{i=0}^{r-1} p^i F R^i : W_{2r-1}\Omega^n_X \rightarrow W_r\Omega^n_X, \quad \sum_{i=0}^{r-1} (p^{\lambda-1}V) R^i : W_r\Omega^n_X \rightarrow W_r\Omega^n_X.
\]

Next, as recalled at line (8), the sequence of pro étale sheaves
\[
0 \longrightarrow \{W_r\Omega^{i, \log}_X\}_r \longrightarrow \{W_r\Omega^i_X\}_r \xrightarrow{1-F} \{W_r\Omega^i_X\}_r \longrightarrow 0
\]
is short exact; combining this with our observation that $1 - \mathcal{F}$ is an isomorphism of $\{W_r\Omega^{\geq 1}_X\}_r$, we arrive at a short exact sequence of pro complexes of sheaves
\[
0 \longrightarrow \{W_r\Omega^{i, \log}_X\}_r \longrightarrow \{W_r\Omega^i_X\}_r \xrightarrow{1-F} \{W_r\Omega^{\geq 1}_X\}_r \longrightarrow 0,
\]
(10)
A Variational Tate Conjecture in crystalline cohomology

which is well-known for smooth varieties over a perfect field [24, §I.3.F]. Recalling that \( p^n F = \phi \) on \( W_r \Omega^n_X \), we see that \( p^i F = \phi \), and so we obtain from (10) an exact sequence of rationalised continuous cohomology groups

\[
\begin{align*}
H^i_{\text{cont}}(X_{\acute{e}t}, W \Omega^i_{X, \log})_Q & \longrightarrow H^{2i}_{\text{cont}}(X, W \Omega^{2i}_X)_Q \\
& \longrightarrow H^{2i}_{\text{cont}}(X, W \Omega^{2i}_X)_{p^i} \\
& \longrightarrow H^i_{\text{cont}}(X_{\acute{e}t}, W \Omega^i_{X, \log})_{p^i}.
\end{align*}
\] (11)

This proves that the canonical map

\[
H^i_{\text{cont}}(X_{\acute{e}t}, W \Omega^i_{X, \log})_Q \longrightarrow H^{2i}_{\text{cont}}(X, W \Omega^{2i}_X)_{p^i} 
\] (12)

is surjective.

Next, the short exact sequence of pro complexes of sheaves

\[
0 \longrightarrow \{W_r \Omega^{2i}_X\}_r \longrightarrow \{W_r \Omega^i_r\}_r \longrightarrow \{W_r \Omega^{2i}_X\}_r \longrightarrow 0
\]

gives rise to an exact sequence of rationalised continuous cohomology

\[
\begin{align*}
H^{2i-1}_{\text{cont}}(X, W \Omega^{2i}_X)_Q & \longrightarrow H^{2i}_{\text{cont}}(X, W \Omega^{2i}_X)_Q \\
& \longrightarrow H^{2i}_{\text{crys}}(X) \\
& \longrightarrow H^{2i}_{\text{cont}}(X, W \Omega^{2i}_X)_Q.
\end{align*}
\] (13)

Recalling that \( VF = FV = p \) on \( W_r \Omega^n_X \), we see that \( \phi \mathcal{V} = \mathcal{V} \phi = p^i \) on \( W_r \Omega^{2i}_X \); since we have shown that \( 1 - \mathcal{V} \) is an automorphism of \( H^n_{\text{cont}}(X, W \Omega^n_X) \) for all \( n \geq 0 \), it follows that \( p^i - \phi \) is an is automorphism of \( H^n_{\text{cont}}(X, W \Omega^n_X) \) for all \( n \geq 0 \), so that in particular \( p^i - \phi \) is an automorphism of the outer terms of the exact sequence (13). By elementary linear algebra, the map of eigenspaces

\[
H^{2i}_{\text{cont}}(X, W \Omega^{2i}_X)_Q \longrightarrow H^{2i}_{\text{crys}}(X)^{\phi=p^i}
\] (14)

is therefore surjective.

Composing our two surjections proves (i). Now assume that \( X \) is a smooth, proper variety over the perfect field \( k \). Then the map of line (14) is injective, hence an isomorphism, by degeneration modulo torsion of the slope spectral sequence [24, §II.3.A]. So, to prove (ii), we must show that map (12) is injective whenever \( k \) is finite or algebraically closed. We begin with some general observations. Firstly, in (13) we may replace \( 2i \) by \( 2i - 1 \) and then apply the same argument as immediately above to deduce that the map

\[
H^{2i-1}_{\text{cont}}(X, W \Omega^{2i-1}_X)^{\phi=p^i} \longrightarrow H^{2i-1}_{\text{crys}}(X)^{\phi=p^i} 
\] (15)

is an isomorphism. Secondly, continuing (11) to the left as a long exact sequence, we see that the kernel of (12) is isomorphic to the cokernel of

\[
H^{2i-1}_{\text{cont}}(X, W \Omega^{2i-1}_X)_Q \longrightarrow H^{2i-1}_{\text{cont}}(X, W \Omega^{2i-1}_X)_Q, 
\] (16)

so we must show that this map is surjective.

Now suppose that \( k = \mathbb{F}_q \) is a finite field. According to the crystalline consequences of the Weil conjectures over finite fields [28], the eigenvalues of the relative Frobenius \( \phi_q : x \mapsto x^q \) acting on the crystalline cohomology \( H^i_{\text{crys}}(X) \) are algebraic over \( \mathbb{Q} \) and all have complex absolute value \( q^{n/2} \). It follows that \( p^i \) cannot be an eigenvalue for the action of \( \phi \) on \( H^{2i-1}_{\text{crys}}(X) \); so the right, hence the left, side of (15) vanishes, and so map
(16) is injective. But (16) is a $\mathbb{Q}_p$-linear endomorphism of a finite-dimensional $\mathbb{Q}_p$-vector space, so it must also be surjective, as desired.

Secondly suppose that $k = k^\text{alg}$. Then $V := \mathbb{H}^{2i−1}(X, W\Omega_X^i)_{\mathbb{Q}}$, equipped with operator $\mathcal{F}$, is an $F$-isocrystal over $k$. Since $k$ is algebraically closed, it is well-known that $1 − \mathcal{F} : V \to V$ is therefore surjective. Indeed, by the Dieudonné–Manin slope decomposition, we may suppose that $V = V_{r/s}$ is purely of slope $r/s$, where $r/s \in \mathbb{Q}_{\geq 0}$ is a fraction written in lowest terms, i.e., $V = K^r$ and $\mathcal{F}^r = p^s \phi^r$: if $r ≠ 0$ it easily follows that $1 − \mathcal{F} : V \to V$ is an automorphism; and if $r = 0$ then $V = K$ and $\mathcal{F} = \phi$, whence $1 − \mathcal{F} : V \to V$ is surjective since $k$ is closed under Artin–Schreier extensions.

We may now prove Theorem 0.6; recall that $\text{ch} = \text{ch}^{\text{crys}}$ denotes the crystalline Chern character:

**Theorem 3.3.** Let $X$ be a smooth, proper scheme over $k[[t_1, \ldots, t_m]]$, where $k$ is a finite or algebraically closed field of characteristic $p$, and let $z \in K_0(Y)\left[\frac{1}{p}\right]$. Then the following are equivalent:

(i) $z$ lifts to $(\varprojlim_r K_0(Y_r))\left[\frac{1}{p}\right]$.

(ii) $\text{ch}(z) \in \bigoplus_{i \geq 0} H^{2i}_{\text{crys}}(Y)$ lifts to $\bigoplus_{i \geq 0} H^{2i}_{\text{crys}}(X)^{\phi = p^i}$.

**Proof.** The proof is a straightforward diagram chase combining Propositions 3.1 and 3.2 using the following diagram, in which we have deliberately omitted an unnecessary arrow:

\[
\begin{array}{ccc}
\left(\varprojlim_r K_0(Y_r)\right)\left[\frac{1}{p}\right] & \to & K_0(Y)\left[\frac{1}{p}\right] \\
\oplus_i H^{i}_{\text{cont}}(X_{\text{ét}}, W\Omega^i_{X,\text{log}})_{\mathbb{Q}} & \to & \oplus_i H^{i}_{\text{cont}}(Y_{\text{ét}}, W\Omega^i_{Y,\text{log}})_{\mathbb{Q}} \\
\oplus_i H^{2i}_{\text{crys}}(X)^{\phi = p^i} & \cong & \oplus_i H^{2i}_{\text{crys}}(Y)^{\phi = p^i} \\
\end{array}
\]

Given $z \in K_0(Y)\left[\frac{1}{p}\right]$, Proposition 3.1 states that $z$ lifts to $(\varprojlim_r K_0(Y_r))\left[\frac{1}{p}\right]$ if and only if $\chi^{\text{log}}(z)$ lifts to $\bigoplus_i H^{i}_{\text{cont}}(X_{\text{ét}}, W\Omega^i_{X,\text{log}})_{\mathbb{Q}}$. However, Proposition 3.2 states that the bottom left (resp. right) vertical arrow is a surjection (resp. an isomorphism); so the latter lifting condition is equivalent to $\varepsilon_{\mathbb{Q}}(\chi^{\text{log}}(z)) = \chi(z)$ lifting to $\bigoplus_i H^{2i}_{\text{crys}}(X)^{\phi = p^i}$, as required.

In the case of a line bundle, Grothendieck's algebrization theorem allows us to prove a stronger result, thereby establishing Theorem 0.5:

**Corollary 3.4.** Let $X$ be a smooth, proper scheme over $k[[t_1, \ldots, t_m]]$, where $k$ is a finite or algebraically closed field of characteristic $p$, and let $L \in \text{Pic}(Y)\left[\frac{1}{p}\right]$. Then the following are equivalent:

(i) There exists $\tilde{L} \in \text{Pic}(X)\left[\frac{1}{p}\right]$ such that $\tilde{L}|_Y = L$.

(ii) $c_1(L) \in H^{2}_{\text{crys}}(Y)$ lifts to $H^{2}_{\text{crys}}(X)^{\phi = p}$. 

18
A Variational Tate Conjecture in crystalline cohomology

Proof. The result follows from Theorem 3.3 and two observations: firstly, Grothendieck’s algebrization theorem [21, Thm. 5.1.4] that \( \text{Pic} X = \lim_{\text{crys}} \text{Pic} Y \); secondly, that \( c_1(L) \) lifts to \( H_{\text{crys}}^2(X)^{\phi=p} \) if and only if \( ch(L) = \exp(c_1(L)) \) lifts to \( \bigoplus_{i \geq 0} H_{\text{crys}}^{2i}(X)^{\phi=p} \).

The final main result of this section is a modification of Corollary 3.4 which extends Theorem 1.4 to the base scheme \( S = \text{Spec} k[[t_1, \ldots, t_m]] \); we do not provide all details of the proof:

**Theorem 3.5.** Let \( f : X \to S = \text{Spec} k[[t_1, \ldots, t_m]] \) be a smooth, projective morphism, where \( k \) is a perfect field of characteristic \( p > 0 \), let \( L \in \text{Pic}(Y)_\mathbb{Q} \), and let \( c := c_1(L) \in H_{\text{crys}}^2(Y) \). Then the following are equivalent:

1. (deform) There exists \( \tilde{L} \in \text{Pic}(X)_\mathbb{Q} \) such that \( \tilde{L}|_Y = L \).
2. (crys) \( c \) lifts to \( H_{\text{crys}}^2(X) \).
3. (crys−φ) \( c \) lifts to \( H_{\text{crys}}^2(X)^{\phi=p} \).
4. (flat) \( c \) is flat, i.e., it lifts to \( H_{\text{crys}}^0(S, R^2 f_* \mathcal{O}_{X/W})_\mathbb{Q} \).

Proof. We first claim that the analogue of Theorem 2.1 is true in this setting; that is, that the conditions (crys), (crys−φ), and (flat) are equivalent for our local family \( f : X \to \text{Spec} k[[t_1, \ldots, t_m]] \). The technical obstacle is that the theories of the convergent site and of isocrystals for non-finite-type schemes such as \( X \) do not appear in the literature, though there is no doubt that the majority of these theories extend verbatim. To be precise, the key result we need is the following: if \( d = \dim Y \) and \( u \in H_{\text{crys}}^2(X/W) \) denotes the Chern class of an ample line bundle, then the induced morphism of \( \mathcal{O}_{S/W} \)-modules \( u^*: R^{d-i} f_* \mathcal{O}_{X/W} \to R^{d+i} f_* \mathcal{O}_{X/W} \) is an isomorphism up to a bounded amount of \( p \)-torsion.

Using the arguments of [37, Thm. 3.1] and [37, Lem. 3.17] (which work in much greater generality than stated, since the base change theorem of crystalline cohomology [7, Corol. 7.12] does not require the schemes to be of finite type over \( k \)), this isomorphism mod \( p \)-torsion may be checked on the special fibre \( Y \). As in the proof of Theorem 2.6(i), this then follows from the Hard Lefschetz theorem for crystalline cohomology.

Deligne’s axiomatic approach to the degeneration of Leray spectral sequences now shows that the rationalised Leray spectral sequence \( E_2^{ab} = H_{\text{crys}}^a(S, R^b f_* \mathcal{O}_{X/W})_\mathbb{Q} \Rightarrow H_{\text{crys}}^{a+b}(X) \) degenerates at the \( E_2 \)-page, just as in the proof of Theorem 2.6(i). Verbatim repeating the rest of the proof of Theorem 2.6, and of Theorem 2.1, shows that (crys), (crys−φ), and (flat) are equivalent.

To complete the proof of the theorem, it remains to show that (crys−φ) implies (deform); so assume that \( c_1(L) \) lifts to \( H_{\text{crys}}^2(X)^{\phi=p} \). Write \( A = k[[t_1, \ldots, t_m]] \), whose strict Henselisation is \( A^{sh} = A \otimes_k k^{alp} \), whose completion is \( \widehat{A^{sh}} = k^{alp}[[t_1, \ldots, t_m]] \).

By applying Corollary 3.4 and the same Neron–Popescu argument as in the proof of Theorem 1.4 (whose indexing convention on line bundles we will follow), we find a line bundle \( L_3 \in \text{Pic}(X \times_A A^{sh})[\frac{1}{p}] \) whose restriction to \( Y \times_k k^{alp} \) coincides with the pullback of \( L \) to \( Y \times_k k^{alp} \). Evidently there therefore exists a finite extension \( k' \) of \( k \) and a line bundle \( L_5 \in \text{Pic}(X')[\frac{1}{p}] \), where \( X' := X \times_A k'[\frac{1}{p}] \), such that the restriction of \( L_3 \) to \( Y \times_k k' \) coincides with the pullback of \( L \) to \( Y \times_k k' \).
Let $L_7 \in \text{Pic}(X)[\frac{1}{p}]$ be the pushforward of $L_4$ along the finite étale morphism $X' \to X$. Evidently $L_7|_Y = L^n$, where $n := |k' : k|$, and thus $\widetilde{L} := L_7^{1/n} \in \text{Pic}(X)_\mathbb{Q}$ lifts $L$, as desired to prove (deform).

**Remark 3.6 (Boundedness of $p$-torsion).** The amount of $p$-torsion which must be neglected in Proposition 3.1 – Corollary 3.4 is bounded, i.e., annihilated by a large enough power of $p$ which depends only on $X$. For example, in Corollary 3.4 there exists $\alpha \geq 0$ with the following property: if $L \in \text{Pic}(X)$ is such that $c_1(L) \in H^2_{\text{crys}}(Y/W)$ lifts to $H^2_{\text{crys}}(X/W)^{p = p}$, then $L^{p^\alpha}$ lifts to $\text{Pic}(X)$.

To prove this boundedness claim, it is enough to observe that only a bounded amount of $p$-torsion must be neglected in both Proposition 3.2 and the Geisser–Hesselholt decomposition of Proposition 3.1. The former case is clear from the proof of Proposition 3.2 and the finite generation of the $W$-modules $H^n_{\text{crys}}(Y/W)$. The latter case is a consequence of the fact that the decomposition arises from a spectral sequence $E_2^{a,b} = H^a_{\text{cont}}(X_{et}, W\Omega^{-b}_{X,\log}) \Rightarrow TC_{-a-6}(X;p)$ (and similarly for $Y$); this spectral sequence is compatible with $\phi$, which acts as multiplication by $p^{-b}$ on $E_2^{a,b}$, hence it degenerates modulo a bounded amount a $p$-torsion depending only on $\dim X$; see [17, Thm. 4.1.1]

**Remark 3.7 (Lifting $L$ successively).** Suppose that $X$ is a smooth, proper scheme over $k[[t_1, \ldots, t_m]]$, and let $L \in \text{Pic}(Y)$. Assuming that $L$ lifts to $L_r \in \text{Pic}(Y_r)$ for some $r \geq 1$, then there is a tautological obstruction in coherent cohomology to lifting $L_r$ to $\text{Pic}(Y_{r+1})$, which lies in $H^2(Y, \mathcal{O}_Y)$ if $m = 1$. The naive approach to prove Theorem 3.5 is to understand these tautological obstructions and thus successively lift $L$ to $\text{Pic}(Y_2)$, $\text{Pic}(Y_3)$, etc. (modulo a bounded amount of $p$-torsion). The proofs in this section have not used this approach, and in fact we strongly suspect that this naive approach does not work in general. We attempt to justify this suspicion in the rest of the remark.

If one unravels the details of the proof of Proposition 3.1 in the case of line bundles using the pro isomorphisms appearing in [35], one can prove the following modification of Corollary 3.4:

Given $r \geq 1$ there exists $s \geq r$ such that for any $L \in \text{Pic}(Y)$ the following implications hold (mod a bounded amount of $p$-torsion independent of $r, s, L$):

$L$ lifts to $\text{Pic}(Y_s) \implies c_1(L)$ lifts to $H^2_{\text{crys}}(Y_s/W)^{p = p} \implies L$ lifts to $\text{Pic}(Y_r)$

We stress that $s$ is typically strictly bigger than $r$. In particular, it appears to be impossible to give a condition on $c_1(L)$, defined only in terms of $Y_r$, which ensures that $L$ lifts to $\text{Pic}(Y_r)$.

Next, let us assume that $L$ is known to lift to $\widetilde{L} \in \text{Pic}(X)$ (we continue to ignore a bounded amount of $p$-torsion). Then the tautological coherent obstruction to lifting $L$ to $\text{Pic}(Y_2)$ must vanish, so we may choose a lift $L_2 \in \text{Pic}(Y_2)$. Then it is entirely possible that $L_2 \neq \widetilde{L}|_{Y_2}$ and that $L_2$ does not lift to $\text{Pic}(Y_3)$ (more precisely, this phenomenon would first occur at $Y_r$, for some $r \geq 1$ depending on the amount of $p$-torsion neglected). However, see Remark 3.8.

For these reasons it appears to be essential to prove the main results of this section by considering all infinitesimal thickenings of $Y$ at once, not one at a time.
Remark 3.8 (de Jong’s result). The main implication (crys) ⇒ (deform) of Theorem 3.12 was proved by A. J. de Jong [13] for smooth, proper $X$ over $k[[t]]$ under the following conditions: assumption (18) in Section 3.4 holds, and $H^1(Y, \mathcal{O}_Y) = H^0(Y, \Omega^1_{Y/k}) = 0$.

In contrast to Remark 3.7, de Jong proved his result by lifting the line bundle $L$ successively to $\text{Pic}(Y_2)$, $\text{Pic}(Y_3)$, etc. Indeed, the vanishing assumption $H^1(Y, \mathcal{O}_Y) = 0$ implies that all of the restriction maps $\text{Pic}(X) \to \cdots \to \text{Pic}(Y_3) \to \text{Pic}(Y_2) \to \text{Pic}(Y)$ are injective, and therefore the problem discussed in the penultimate paragraph of Remark 3.7 cannot occur: the arbitrarily chosen lift $L_2$ of $L$ must equal $\tilde{L}|_{Y_2}$, and hence $L_2$ lifts to $\text{Pic}(Y_2)$, etc.

3.3 Line bundles with $\mathbb{Q}_p$-coefficients

With the Tate conjecture in mind it is natural to work with $\text{Pic}(Y)_{\mathbb{Q}_p} = \text{Pic}(Y) \otimes \mathbb{Q}_p$ rather than $\text{Pic}(Y)_{\mathbb{Q}}$, especially if $k$ is finite; for example, this will be required in Section 4. This difference is not formal, and so the goal of this section is to refine some of the arguments of Section 3.2 to allow this change in coefficients.

Throughout this section $k$ is a finite field (though this is unnecessary for most of the arguments; see Remark 3.10), and $X$ is a smooth, proper scheme over $k[[t_1, \ldots, t_m]]$, with special fibre $Y$; we will continue to use other notation from Section 3.2. The $p$-adic completion of a group $M$ is denoted by $\hat{M} = \lim_{\leftarrow} M/p^n M$. We enumerate the results of this section according to their analogues in Section 3.2, apart from the following preliminary lemma:

Lemma 3.9. The canonical map $TC_n(Y; p) \to TC_n(Y; p)^{\hat{p}}$ is an isomorphism for all $n \geq 0$.

Proof. The proof amounts to establishing suitable finite generation claims for topological cyclic homology to show that it has no infinitely divisible part. Since $Y$ is smooth, this essentially reduces to finite generation of logarithmic de Rham–Witt cohomology, via the Geisser–Hesselholt spectral sequence which appeared in Remark 3.6.

However, we prefer to give a more general proof, using finite generation results in topological cyclic homology, which works for an arbitrary proper $k$-scheme $Y$. We once again refer the reader to [16] or [35] for basic properties and notation of topological cyclic homology.

Since $TC(Y; p) = \text{holim}_s TC^s(Y; p)$, where $TC^s(Y; p)$ is the $p$-typical topological cyclic homology of level $s$, there is an exact sequence

$$0 \to \lim_{\leftarrow} TC_{n+1}^s(Y; p) \to TC_n(Y; p) \to \lim_{\leftarrow} TC_n^s(Y; p) \to 0.$$ 

Each group $TC^s_n(Y; p)$ is a $W_s(\mathbb{F}_p)$-module, which we claim is finitely generated. Letting $TR^s_n(Y; p)$ denote the fixed point spectrum for the action of the cyclic group $C_{p^{n+1}}$ on topological Hochschild homology $THH(A)$, there are long exact sequences

$$\cdots \to TC^s_n(Y; p) \to TR^s_n(Y; p) \xrightarrow{1-F} TR^s_{n-1}(Y; p) \to \cdots$$
Matthew Morrow

of $W_s(F_p)$-modules. Moreover, according to [35, Corol. 5.9], the group $TR^s_n(Y; p)$ is finitely generated as a $W_s(k)$-module, hence also as a $W_s(F_p)$-module since $k$ is finite, for all $n \geq 0$, $s \geq 1$.

This proves the finite generation claim, which implies that $\lim_{s \to +\infty} TC_{n+1}(Y; p) = 0$. Hence $TC_n(Y; p)$ is an inverse limit over $s$ of $W_s(F_p)$-modules, which is sufficient to complete the proof.

The follow analogue of Proposition 3.1 is the main technical step in this section:

**Q_p-version of Proposition 3.1.** For any $z \in K_0(Y)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, the following are equivalent:

(i) $z$ lifts to $(\varprojlim_{s \to r} K_0(Y_r))_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

(ii) $ch^{log}(z) \in \bigoplus_{i \geq 0} H^i_{cont}(Y_{\text{ét}}, W\Omega_{Y, log}^1)_{\mathbb{Q}}$ lifts to $\bigoplus_{i \geq 0} H^i_{cont}(X_{\text{ét}}, W\Omega_{X, log}^i)_{\mathbb{Q}}$.

**Proof.** Since the spectra $TC(Y; p)$, $TC(Y_r; p)$, and $TC(Y, Y_r; p) \simeq K(Y, Y_r)$ are all $p$-complete, the homotopy cartesian square in the proof of Proposition 3.1 remains valid if $K(Y)$ and $K(Y_r)$ are replaced by their $p$-completions $K(Y; \mathbb{Z}_p)$ and $K(Y_r; \mathbb{Z}_p)$. Hence the following are equivalent for any $z \in K_0(Y; \mathbb{Z}_p)$:

(i') $z$ lifts to $\varprojlim_{s \to r} K_0(Y_r; \mathbb{Z}_p)$. (ii') $tr(z) \in TC_0(Y; p)$ lifts to $TC_0(X; p)$. (17)

Next recall that for any abelian group $M$, there is a natural short exact sequence $0 \to [\lim_{s \to} M[p^s]] \to \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M) \to M_p \to 0$, where the derived inverse limit is taken over the $p^s$-torsion of $M$, with transition maps $M[p^{s+1}] \xrightarrow{\times p} M[p^s]$. We use these sequences to compare topological cyclic homology and $p$-completed $K$-theory via the trace map:

$$
0 \to \lim_{s \to} K_0(Y_r)[p^s] \to K_0(Y_r; \mathbb{Z}_p) \cong \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, K_0(Y_r)) \to K_0(Y_r)_p \to 0
$$

$$
0 \to \lim_{s \to} K_0(Y)[p^s] \to K_0(Y; \mathbb{Z}_p) \cong \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, K_0(Y)) \to K_0(Y)_p \to 0
$$

$$
0 \to \lim_{s \to} TC_0(Y, Y_r; p)[p^s] \to TC_0(Y; p) \cong \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, TC_0(Y; p)) \to TC_0(Y; p)_p \to 0
$$

$$
0 \to \lim_{s \to} TC_0(X; p)[p^s] \to TC_0(X; p) \cong \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, TC_0(X; p)) \to TC_0(X; p)_p \to 0
$$

The central column of isomorphisms in this diagram must be justified. Firstly, there is a short exact sequence

$$
0 \to \text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, K_0(\{\})) \to K_0(\{\}; \mathbb{Z}_p) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, K_{-1}(\{\})) \to 0
$$

for $\{\} = Y$ and $Y_r$. But smoothness of $Y$ implies that $K_{-1}(Y) = 0$; hence $K_{-1}(Y_r)$ is a quotient of $K_{-1}(Y_r, Y)$, which is an abelian group killed by a power of $p$ by [18, Thm. A]. Hence $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, K_{-1}(\{\})) = 0$ for $\{\} = Y$ and $Y_r$, and so $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, K_0(\{\})) \cong$
Implication for any $z$ inverse limit over a countable system of abelian groups is $\lim_{r}$, which is true because the obstruction to exactness is $\lim_{r} \lim_{s} K_{0}(Y_{r})[p^{s}]$, which vanishes by Roos’ spectral sequence $E^{ab}_{2} = \lim_{r} \lim_{s} K_{0}(Y_{r})[p^{s}] \Rightarrow \lim_{r} a+b K_{0}(Y_{r})[p^{s}]$ and the fact that the only non-zero derived inverse limit over a countable system of abelian groups is $\lim_{r} [42].$

From the first claim and equivalence (17), a simple diagram chase proves the following implication for any $z \in K_{0}(Y)^{p}_{p}$:

$tr(z) \in TC_{0}(Y;p) \implies z \text{ lifts to } \lim_{r} K_{0}(Y_{r})^{p}_{p}.$

The converse implication easily follows from the second claim, the fact that $TC(X;p) \simeq \text{holim}_{r} TC(Y_{r};p)$, and functoriality of $tr$.

Identifying $TC_{0}(?;q)_{q}$ with $\bigoplus_{i \geq 0} H_{\text{cont}}^{i}(?) \Omega^{j}_{? \log} q$ for $? = Y$ and $X$, and identifying $tr$ with $ch^{log}$, exactly as in Proposition 3.1, completes the proof.

Q$^{p}$-version of Theorem 3.3. For any $z \in K_{0}(Y)^{p}_{p} \otimes_{Z_{p}} Q_{p}$, the following are equivalent:

1. $z$ lifts to $(\lim_{r} K_{0}(Y_{r}))^{p}_{p} \otimes_{Z_{p}} Q_{p}$.
2. $ch(z) \in \bigoplus_{i \geq 0} H_{\text{crys}}^{2i}(Y)^{p}$ lifts to $\bigoplus_{i \geq 0} H_{\text{crys}}^{2i}(X)^{p}.$

Proof. Repeat the proof of Theorem 3.3 using the previous proposition in place of Proposition 3.1.

Q$^{p}$-version of Corollary 3.4. For any $L \in \text{Pic}(Y)_{Q_{p}}$, the following are equivalent:

1. There exists $\hat{L} \in \text{Pic}(X)_{Q_{p}}$ such that $\hat{L}|_{Y} = L.$
2. $c_{1}(L) \in H_{\text{crys}}^{2}(Y)$ lifts to $H_{\text{crys}}^{2}(X)^{p}.$

Proof. Repeating the proof of Corollary 3.4 using the previous theorem in place of Theorem 3.3 shows that the following are equivalent for any $L \in \text{Pic}(Y)^{p}_{p} \otimes_{Z_{p}} Q_{p}$:

1. There exists $\hat{L} \in \text{Pic}(X)^{p}_{p} \otimes_{Z_{p}} Q_{p}$ such that $\hat{L}|_{Y} = L.$
2. $c_{1}(L) \in H_{\text{crys}}^{2}(Y)$ lifts to $H_{\text{crys}}^{2}(X)^{p}.$

The exact sequence $\text{Pic}(X) \to \text{Pic}(Y) \to \text{Coker}(\text{Pic}(X) \to \text{Pic}(Y)) \to 0$ may be both tensored by $Z_{p}$ and $p$-adically completed:
The bottom row is exact. Furthermore, \( \text{Pic}(Y) \), and hence also \( \text{Coker}(\text{Pic}(X) \to \text{Pic}(Y)) \), is a finitely generated abelian group by the Néron–Severi theorem (see, e.g., [26, Lem. 4] for the argument), and so the central and right vertical arrows are isomorphisms. It follows that the top row is also exact, and that any \( L \in \text{Pic}(Y)_{\mathbb{Z}} = \text{Pic}(Y) \otimes \mathbb{Z}_p \) lifts to \( \text{Pic}(X)_{\mathbb{Z}} \) if and only if it lifts to \( \text{Pic}(X) \otimes \mathbb{Z}_p \). Inverting \( p \) completes the proof. □

Remark 3.10 (The case \( k = k_{\text{alg}} \)). At the risk of overindulging in variations on a theme, we remark that the following \( \mathbb{Q}_p \)-coefficient version of Corollary 3.4 can be proved over an algebraically closed, rather than finite, field:

Let \( X \) be a smooth, proper scheme over \( k[[t_1, \ldots, t_m]] \), where \( k \) is an algebraically closed field of characteristic \( p \), and let \( L \in \text{NS}(Y)_{\mathbb{Q}_p} \). Then the following are equivalent:

(i) There exists \( \tilde{L} \in \text{NS}(X)_{\mathbb{Q}_p} \) such that \( \tilde{L}|_Y = L \).
(ii) \( c_1(L) \in H^2_{\text{crys}}(Y) \) lifts to \( H^2_{\text{crys}}(X)^{\phi=p} \).

Indeed, once Lemma 3.9 has been established over an algebraically closed field (which is a subtle verification of certain Mittag–Leffler conditions, which we omit), the above proofs of the \( \mathbb{Q}_p \)-versions of Proposition 3.1 and Theorem 3.3 remain valid. The proof of the \( \mathbb{Q}_p \)-version of Corollary 3.4 then also holds after replacing Pic by the Néron–Severi group \( \text{NS} \), since \( \text{NS}(Y) \) is a finitely generated abelian group.

3.4 Cohomologically flat families over \( k[[t]] \) and Artin’s theorem

The aim of this section is to further analyse Corollary 3.4 in a special case, to show that there is no obstruction to lifting line bundles in “cohomologically constant” families.

Let \( k \) be an algebraically closed field of characteristic \( p \), and let \( f : X \to S = \text{Spec} k[[t]] \) be a smooth, proper morphism. Throughout this section we impose the following additional assumption on \( X \):

The coherent \( \mathcal{O}_{S/W} \)-modules \( R^n f_* \mathcal{O}_{X/W} \) are locally free, for all \( n \geq 0 \). (18)

The assumption (and base change for crystalline cohomology: see especially [7, Rmk. 7.10]) implies that each \( \mathcal{O}_{S/W} \)-module \( R^n f_* \mathcal{O}_{X/W} \) is in fact an \( \mathbb{F}_p \)-crystal on the crystalline site \( (S/W)_{\text{crys}} \); henceforth we simply say “\( \mathbb{F}_p \)-crystal over \( k[[t]] \)”.

We briefly review some of the theory of \( \mathbb{F}_p \)-crystals. Let \( \sigma : W[[t]] \to W[[t]] \) be the obvious lifting of the Frobenius on \( k[[t]] \) which satisfies \( \sigma(t) = t^p \). In the usual way [27, §2.4], we identify the category of \( \mathbb{F}_p \)-crystals over \( k[[t]] \) with the category of free, finite rank \( W[[t]] \)-modules \( M \) equipped with a connection \( \nabla : M \to M dt \) and a compatible \( k[[t]] \)-linear isogeny \( F : \sigma^* M \to M \). The crystalline cohomology of the \( F \)-crystal is then equal to \( H^*(M \sum \nabla) \). An \( F \)-crystal \( M = (M, F, \nabla) \) is constant if and only if it has the form \( (M \otimes W[[t]], F \otimes \sigma, \frac{d}{dt}) \) for some \( F \)-crystal \((M, F)\) over \( k \).

We require the following simple lemma:

Lemma 3.11. Let \( (M, F, \nabla) \) be an \( F \)-crystal over \( k[[t]] \). Then:

(i) The operator \( p - F \) is an isogeny of \( M dt \).
(ii) If \((M, F, \nabla)\) is constant then the composition \(\text{Ker} \nabla \to M \to M/tM\) is an isomorphism of \(W\)-modules.

**Proof.** (i): Since \(F(m \ dt) = pt^{p-1}F(m) \ dt\), the operator \(\frac{1}{p}F\) is well-defined on \(M \ dt\) and is contracting. Therefore \(1 - \frac{1}{p}F\) is an automorphism of \(M \ dt\), and so \(p - F\) is injective with image \(pM \ dt\).

(ii): Write \((M, F, \nabla) = (M \otimes_W W[[t]], \mathcal{F} \otimes \sigma, \frac{d}{dt})\) for some \(F\)-crystal \((M, \mathcal{F})\) over \(k\). Since \(W[[t]] = \text{Ker} \frac{d}{dt} \oplus tW[[t]]\), it is easy to see that \(M = \text{Ker} \nabla \oplus tM\), as desired. \(\square\)

The following is the main theorem of the section, in which \(k\) continues to be an algebraically closed field of characteristic \(p\):

**Theorem 3.12.** Let \(f : X \to S = \text{Spec} k[[t]]\) be a smooth, proper morphism satisfying (18), and assume that the \(F\)-crystal \(R^2 f_* \mathcal{O}_{X/W}\) on \(k[[t]]\) is isogenous to a constant \(F\)-crystal. Then the cokernel of the restriction map \(\text{Pic}(X) \to \text{Pic}(Y)\) is killed by a power of \(p\).

**Proof.** We will prove this directly from Corollary 3.4, deliberately avoiding use of the stronger Theorem 3.5, at the expense of slightly lengthening the proof. The first half of the proof is similar to de Jong’s [13, Thm. 1].

By assumption \(R^2 f_* \mathcal{O}_{X/W}\) is isogenous to a constant \(F\)-crystal \((M, F, \nabla)\), and Lemma 3.11(i) implies that the canonical map \(\text{Ker} \nabla \to M/tM\) is an isomorphism. But, up to isogeny, the left side of this isomorphism is \(H^0_{\text{crys}}(S, R^2 f_* \mathcal{O}_{X/W})\) and the right side is \(H^2_{\text{crys}}(Y/W)\). In conclusion, the canonical map

\[
H^0_{\text{crys}}(S, R^2 f_* \mathcal{O}_{X/W}) \to H^2_{\text{crys}}(Y)
\]

is an isogeny, and hence induces an isogeny on eigenspaces of the Frobenius.

Next note that \(H^1_{\text{crys}}(S, R^n f_* \mathcal{O}_{X/W}) = 0\) for all \(n \geq 0\) and \(i \geq 2\), since the crystalline cohomology of the crystal \(R^n f_* \mathcal{O}_{X/W}\) is computed using a two-term de Rham complex, as mentioned in the above review. Therefore the Leray spectral sequence for \(f\) degenerates to short exact sequences

\[
0 \to H^1_{\text{crys}}(S, R^1 f_* \mathcal{O}_{X/W}) \to H^1_{\text{crys}}(X/W) \to H^0_{\text{crys}}(S, R^2 f_* \mathcal{O}_{X/W}) \to 0.
\]

Letting \((M', F', \nabla')\) be the \(F\)-crystal \(R^1 f_* \mathcal{O}_{X/W}\), Lemma 3.11 implies that the operator \(p - F'\) is an isogeny of \(M \ dt\), hence is surjective modulo a bounded amount of \(p\)-torsion on its quotient \(\text{Coker}(M \to M \ dt)\), which is isogenous to \(H^1_{\text{crys}}(S, R^1 f_* \mathcal{O}_{X/W})\). It follows from elementary linear algebra that the surjection \(H^2_{\text{crys}}(X/W) \to H^0_{\text{crys}}(S, R^2 f_* \mathcal{O}_{X/W})\) remains surjective modulo a bounded amount of \(p\)-torsion after restricting to \(p\)-eigenspaces of the Frobenius.

Combining the established surjection and isogeny proves that the canonical map \(H^2_{\text{crys}}(Y/W)^{>p} \to H^2_{\text{crys}}(Y/W)^{>p}\) is surjective modulo a bounded amount of \(p\)-torsion. Corollary 3.4, or rather its improvement explained in Remark 3.6, completes the proof. \(\square\)

**Remark 3.13.** Let \(f : X \to S = \text{Spec} k[[t]]\) satisfy the assumptions of Theorem 3.12; then a consequence of the theorem is that the cokernel of the Néron–Severi specialisation map [5, Exp. X, §7]

\[
s_p : \text{NS}(X \times_S k((t))^\text{alg}) \to \text{NS}(Y)
\]

25
is a finite $p$-group. Indeed, $sp$ arises as a quotient of the colimit of the maps

$$\text{Pic}(X \times_S F) \cong \text{Pic}(X \times_S A) \to \text{Pic}(Y),$$

(19)

where $F$ varies over all finite extensions of $k((t))$ inside $k((t))^{\text{alg}}$ and $A$ is the integral closure of $k[[t]]$ inside $F$. But the hypotheses of Theorem 3.12 are satisfied for each morphism $X \times_S A \to \text{Spec } A$, by base change for crystalline cohomology, and hence the cokernel of (19) is killed by a power of $p$. Passing to the colimit over $F$ we deduce that the cokernel of $sp$ is a $p$-torsion group; it is moreover finite since $\text{NS}(Y)$ is finitely generated.

**Example 3.14.** Let $f : X \to S = \text{Spec } k[[t]]$ be a smooth, proper morphism. Then assumption (18) is known to be satisfied in each of the following cases:

(i) $X$ is a family of K3 surfaces over $S$.

(ii) $X$ is an abelian scheme over $S$ [4, Corol. 2.5.5].

Moreover, under assumption (18), $R^2 f_* \mathcal{O}_{X/W}$ is isogenous to a constant $F$-crystal if the geometric generic fibre of $X$ is supersingular in degree 2, i.e., the crystalline cohomology $H^2_{\text{cryst}}(X \times_k k((t))^{\text{perf}}/W(k((t))^{\text{perf}}))$ is purely of slope 1. Indeed, supersingularity forces the Newton polygon of $R^2 f_* \mathcal{O}_{X/W}$ to be purely of slope 1 at both geometric points of $S$, and so $R^2 f_* \mathcal{O}_{X/W}$ is isogenous to a constant $F$-crystal by [27, Thm. 2.7.1].

In particular, Theorem 3.12 and Remark 3.13 apply to any smooth, proper family of K3 surfaces over $k[[t]]$ whose geometric generic fibre is supersingular; this reproves a celebrated theorem of M. Artin [2, Corol. 1.3]. We believe that the analogous assertion for abelian schemes with supersingular geometric generic fibre is new.

### 4 An application to the Tate conjecture

In this section we apply Theorem 0.2 to the study of the Tate conjecture. We begin with the following folklore result that all formulations of the Tate conjecture for divisors are equivalent:

**Proposition 4.1.** Let $X$ be a smooth, projective variety over a finite field $k$ of characteristic $p$, and let $\ell \neq p$ be a prime number. Then the following are equivalent:

(i) The $\ell$-adic Chern class map $c_1^\ell \otimes \mathbb{Q}_\ell : \text{Pic}(X)_{\mathbb{Q}_\ell} \to H^2_\text{ét}(X \times_k k^{\text{alg}}, \mathbb{Q}_\ell(1))^{\text{Gal}(k^{\text{alg}}/k)}$ is surjective.

(ii) The crystalline Chern class map $c_1^{\text{cryst}} \otimes \mathbb{Q}_p : \text{Pic}(X)_{\mathbb{Q}_p} \to H^2_{\text{cryst}}(X)^{\phi=p}$ is surjective.

(iii) The order of the pole of the zeta function $\zeta(X, s)$ at $s = 1$ is equal to the dimension of $A_{\text{num}}(X)_{\mathbb{Q}}$, the group of rational divisors modulo numerical equivalence.

**Proof.** We closely follow arguments from [50], though Tate considered only the case of étale cohomology. We may assume $X$ is connected of dimension $d$. Deviating slightly
from the statement of the proposition, we let $\ell$ denote any prime number, possibly equal to $p$, and we let

$$c_I^{(\ell)} : CH^i(X)_{\mathbb{Q}} \to H_{(\ell)}^{2i}(X) := \begin{cases} H_{\text{ct}}^{2i}(\overline{X}, \mathbb{Q}_\ell(i)) & \ell \neq p \\ H_{\text{crys}}^{2i}(X)(i) & \ell = p \end{cases}$$

denote the associated cycle class maps. Here $\overline{X} := X \times_k k^{\text{alg}}$, and $H_{\text{crys}}^{2i}(X)(i)$ denotes the vector space $H_{\text{crys}}^{2i}(X)$ with Frobenius operator $F = p^{-i}\phi$. To give uniform statements, also denote by $F$ a generator of $\text{Gal}(k^{\text{alg}}/k)$ in the case $\ell \neq p$.

The Chow group of rational divisors modulo homological equivalence is

$$A_{\text{et}}(X)_{\mathbb{Q}} := CH^1(X)_{\mathbb{Q}}/ \text{Ker} \ c_I^{(\ell)},$$

and to keep notation clear we denote by $c_I^{Q\ell} : A_{\text{et}}(X)_{\mathbb{Q}} \otimes \mathbb{Q}_\ell \to H_{(\ell)}^{2i}(X)^{F=1}$ the induced cycle class map. We may uniformly formulate statements (i) and (ii) as the assertion that $c_I^{Q\ell}$ is surjective.

Now we exploit the fact that we are only interested in the case of divisors. Homological equivalence (for either $\ell$-adic étale cohomology, $\ell \neq p$, or crystalline cohomology) and numerical equivalence for divisors agree rationally [1, Prop. 3.4.6.1]; this implies that $A_{\text{et}}^1(X)_{\mathbb{Q}}$ is independent of the prime $\ell$ and that the intersection pairing

$$A_{\text{et}}^1(X)_{\mathbb{Q}} \times A_{\text{et}}^{d-1}(X)_{\mathbb{Q}} \to A_{\text{et}}^d(X)_{\mathbb{Q}} \xrightarrow{\deg} \mathbb{Q}$$

is non-degenerate on the left. Combining these observations, we obtain a commutative diagram

$$
\begin{array}{cccccccc}
H_{(\ell)}^2(X)_{\text{F=1}} & \xrightarrow{j} & H_{(\ell)}^2(X)_{\text{gen}} & \xrightarrow{\gamma} & H_{(\ell)}^2(X)_{\text{F=1}} & \xrightarrow{\approx} & \text{Hom}_{\mathbb{Q}_\ell}(H_{(\ell)}^{2d-2}(X)_{\text{F=1}}, \mathbb{Q}_\ell) \\
\downarrow c_I^{Q\ell} & & & & & & \downarrow \text{dual of } c_I^{Q\ell}_{d-1} \\
A_{\text{et}}^1(X)_{\mathbb{Q}} \otimes \mathbb{Q}_\ell & \xLeftarrow{\text{Hom}} & \text{Hom}_{\mathbb{Q}}(A_{\text{et}}^{d-1}(X)_{\mathbb{Q}} \otimes \mathbb{Q}_\ell, \mathbb{Q}_\ell) & \xLeftarrow{\text{Hom}} & \text{Hom}_{\mathbb{Q}}(A_{\text{et}}^{d-1}(X)_{\mathbb{Q}} \otimes \mathbb{Q}_\ell, \mathbb{Q}_\ell) \\
\end{array}
$$

where:

- The top right (resp. bottom left) horizontal arrow arises from the pairing on cohomology groups (resp. Chow groups); the top right horizontal arrow is an isomorphism by the compatibility of Poincaré duality with the action of $F$.
- $H_{(\ell)}^2(X)_{\text{gen}} \subseteq H_{(\ell)}^2(X)$ is the generalised eigenspace for $F$ of eigenvalue 1.
- $H_{(\ell)}^2(X)_{\text{F=1}} := H_{(\ell)}^2(X)/\text{Im}(1-F)$.
- $\gamma$ is the composition $H_{(\ell)}^2(X)_{\text{F=1}} \hookrightarrow H_{(\ell)}^2(X) \to H_{(\ell)}^2(X)_{\text{F=1}}$.

It follows from the diagram that $c_I^{Q\ell}$ is injective. The proof can now be quickly completed.

According to the Lefschetz trace formula (for either $\ell$-adic étale cohomology, $\ell \neq p$, or crystalline cohomology [28, Thm. 1]), the order of the pole of interest in (iii) is equal to $\dim_{\mathbb{Q}_\ell} H_{(\ell)}^2(X)_{\text{F=1}}$. The equality $A_{\text{et}}^1(X)_{\mathbb{Q}} = A_{\text{num}}^1(X)_{\mathbb{Q}}$ and the injectivity of $c_I^{Q\ell}$ and $j$ therefore immediately imply:
Statement (iii) is true \(\iff cl_1^{Q_t} \text{ and } j \text{ are surjective.}\)

Moreover, by elementary linear algebra and diagram chasing:

\[
\begin{align*}
cl_1^{Q_t} \text{ is surjective} & \iff cl_1^{Q_t} \text{ is an isomorphism} \\
& \iff \gamma j \text{ is injective, i.e., } \text{Ker}(1- F) \cap \text{Im}(1- F) = 0 \\
& \iff j \text{ is surjective, i.e., } \text{Ker}(1- F) = \text{Ker}(1- F)^N \text{ for } N \geq 1
\end{align*}
\]

In conclusion, statement (iii) is equivalent to the surjectivity of \(cl_1^{(f)}\), as required to complete the proof. 

We will refer to the equivalent statements of Proposition 4.1 as “the Tate conjecture for divisors for \(X\)” or, following standard terminology, simply as \(T_1(X)\). The main theorem of this section is the reduction of the Tate conjecture for divisors to the case of surfaces:

**Theorem 4.2.** Let \(k\) be a finite field, and assume that \(T^1(X)\) is true for every smooth, projective surface over \(k\). Then \(T^1(X)\) is true for every smooth, projective variety over \(k\).

**Proof.** We will prove by induction on \(d \geq 0\) that the assumption of the theorem implies that \(T^1(X)\) is true for every smooth, connected, projective, \(d\)-dimensional variety \(X\) over \(k\). If \(d = 0\) or \(1\) then \(T^1(X)\) is true for trivial reasons. If \(d = 2\) then \(T^1(X)\) is true by assumption. Now let \(X\) be a smooth, connected, projective variety over \(k\) of dimension \(d \geq 3\) and proceed by induction.

Possibly after replacing \(X\) by each connected component of \(X \times_k k'\) for some finite extension \(k'\) of \(k\) (and noting that \(T^1(X \times_k k') \Rightarrow T^1(X)\)), we may assume that \(X\) admits a Lefschetz pencil of hyperplane sections \(\{X_t : t \in \mathbb{P}_k^1\}\), whose base locus we denote by \(Y \hookrightarrow X\). As usual, this results in a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X' \\
\downarrow & & \downarrow f \\
\mathbb{P}_k^1 & \xrightarrow{i} & \mathbb{P}_k^1
\end{array}
\]

where:

- \(\pi\) is the blow-up of \(X\) along its smooth subvariety \(Y\).
- \(f\) is projective and flat, and is smooth over a non-empty open \(V \subseteq \mathbb{P}_k^1\).
- For each closed point \(t \in \mathbb{P}_k^1\), the fibre \(X'_t\) is isomorphic via \(\pi\) to the hyperplane section \(X_t\) of \(X\).

In particular, for any closed point \(t \in V\) the morphisms \(X'_t \xrightarrow{i} X' \xrightarrow{\pi} X\) gives rise to maps

\[
H^2_{\text{crys}}(X) \xrightarrow{\pi^*} H^2_{\text{crys}}(X') \xrightarrow{i^*} H^2_{\text{crys}}(X'_t),
\]

and the composition \(i^* \pi^*\) is injective by the Weak Lefschetz theorem for crystalline cohomology [23, §3.8] and the identification \(X'_t = X_t\).

Now let \(c \in H^2_{\text{crys}}(X)^{\phi=p}\) and fix any closed point \(t \in U\). Assuming the validity of \(T^1(X'_t)\), we will construct a line bundle \(L' \in \text{Pic}(X')_{Q_p}\) such that

(i) \(i^*(c_1(L')) = i^* \pi^*(c)\), and
A Variational Tate Conjecture in crystalline cohomology

(ii) \( L' = \pi^*(L'') \) for some \( L'' \in \text{Pic}(X) \).

Firstly, by \( T^1(X'_t) \) we may write \( i^* \pi^*(c) = c_1(L) \) for some \( L \in \text{Pic}(X'_t) \). Noting that \( f \) restricts to a smooth, projective morphism \( U := f^{-1}(V) \to V \), the \( \mathbb{Q}_p \)-coefficient version of Theorem 0.2 stated in Remark 1.5 implies the existence of a line bundle \( \pi L \in \text{Pic}(U) \) such that \( c_1(L)|X'_t = c_1(L) \). Then \( \pi L \) may be spread out to some \( L' \in \text{Pic}(X') \), which evidently satisfies \( i^*(c_1(L')) = i^* \pi^*(c) \).

Secondly we must show that \( L' \) can be further assumed to satisfy condition (ii). Let \( E := \pi^{-1}(Y) \) denote the exception divisor of the blow-up \( \pi \). By the standard formula for the Picard group of a blow-up along a regularly embedded subvariety, we may write \( L' = \pi^*(L'') \otimes \mathcal{O}(E)^a \) for some \( L'' \in \text{Pic}(X) \) and \( a \in \mathbb{Q}_p \). But the line bundles \( \mathcal{O}(E) \) and \( \pi^*(\mathcal{O}(X'_t)) \) have the same restriction to \( X'_t = X'_t \), namely \( \mathcal{O}(Y) \). So replacing \( L'' \) by \( L'' \otimes \mathcal{O}(X'_t)^{-a} \) and \( L' \) by \( \pi^*(L'') \) ensures that condition (ii) is satisfied without sacrificing condition (i).

In conclusion, we have constructed \( L'' \in \text{Pic}(X) \) such that \( i^* \pi^*(c_1(L'')) = i^* \pi^*(c) \). By the aforementioned injectivity of \( i^* \pi^* \) we deduce that \( c_1(L'') = c \), and this completes the inductive step of the proof.

\section*{Remark 4.3} (Classical reduction to 3-folds). In the notation of the proof of Theorem 4.2, note that if \( \dim X_t \geq 3 \) then \( H^2_{\text{cris}}(X) \xrightarrow{\sim} H^2_{\text{cris}}(X_t) \) by Weak Lefschetz and \( \text{Pic}(X) \xrightarrow{\sim} \text{Pic}(X_t) \) by Grothendieck–Lefschetz. This immediately reduces the Tate conjecture for divisors to 3-folds and surfaces.

\section*{Example 4.4.} Suppose for example that \( X \) is a smooth, connected, projective 3-fold over a finite field \( k \) such that \( X \) contains a smooth hyperplane section \( S \) which is a rational, ruled, abelian, or K3 surface (assuming \( p \neq 2 \) in the K3 case). Then \( T^1(S) \) is true (by Tate’s original results in the first three cases [48, 49, 50], and by Artin [2], Nygaard–Ogus [36], Maulik [32], Charles [10], and Madapusi Pera [30] in the K3 case) and so the argument of Theorem 4.2 shows that \( T^1(X) \) is true. Unfortunately, the class of 3-folds \( X \) admitting such a hyperplane section does not appear to be very interesting.

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A Variational Tate Conjecture in crystalline cohomology

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Matthew Morrow
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn, Germany

morrow@math.uni-bonn.de
http://www.math.uni-bonn.de/people/morrow/