ON FLOER-TYPE NUMERICAL INVARIANTS, GIT QUOTIENTS, AND ORBIT BIFURCATIONS OF REAL-LIFE PLANETARY SYSTEMS

(WITH NUMERICAL EXPLORATIONS OF THE JUPITER-EUROPA AND SATURN-ENCELADUS SYSTEMS)

URS FRAUENFELDER, DAYUNG KOH, AGUSTIN MORENO

ABSTRACT. The intention of this article is to illustrate the use of invariants coming from Floer-type theories, as well as global topological methods, for practical purposes. Our intended audience is scientists interested in orbits of Hamiltonian systems (e.g. the three-body problem). In this paper, we illustrate the use of the GIT sequence introduced in [10] by the first and third authors, consisting of a sequence of spaces and maps between them. Roughly speaking, closed orbits of an arbitrary Hamiltonian system induce points in these spaces, and so their topology imposes restrictions on the existence of regular orbit cylinders, as well as encodes information on all types of bifurcations. We also consider the notion of the SFT-Euler characteristic, as the Euler characteristic of the local Floer homology groups associated to a closed orbit of a Hamiltonian system. This number stays invariant before and after a bifurcation, and can be computed in explicit terms from the eigenvalues of the reduced monodromy matrix. This makes it useful for practical applications, e.g. for predicting the existence of orbits for optimization and space mission design. For the case of symmetric orbits (i.e. preserved by an antisymplectic involution), which may be thought both as open or closed strings, we further consider the notion of the real Euler characteristic, as the Euler characteristic of the corresponding Lagrangian Floer homology group. We illustrate the practical use of these invariants, together with the GIT sequence, via numerical work based on the cell-mapping method as described in [16], for the Jupiter-Europa and the Saturn-Enceladus systems. These are currently systems of interest, as they fall in the agenda of space agencies like NASA, as these icy moons are considered candidates for harbouring conditions suitable for extraterrestrial life.

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1. Introduction

The study of closed orbits of Hamiltonian systems and their bifurcations, is one of the central topics of Floer theory, as introduced by Andreas Floer in a series of papers [4–9] in order to address the Arnold conjecture; and of Symplectic Field Theory (SFT), as proposed by Eliashberg–Givental–Hofer in [3]. Formidable in their depth and scope, both theories underlie many of the powerful methods of modern symplectic geometry. On the other hand, the search of orbits entails significant practical interest. For instance, the restricted three-body problem, concerning the gravitational motion of a negligible mass around two larger masses, is an interesting problem in astronomy and is relevant for space mission design. In this context, the influence on a satellite of a planet which comes with an orbiting moon (e.g. the Jupiter-Europa or the Saturn-Enceladus systems) can be approximated by a three-body problem of restricted type. Finding families of orbits for placing such a satellite around the target moon, which is natural with less orbit corrections and less risks of collisions, is then of central importance for space exploration. The purpose of this article is then to explore the interaction and marriage between theory and practice, and to popularize ideas that come from symplectic geometry to an audience with more applied background. Namely, drawing from the rich algebraic structure of Floer homology, we will extract simpler invariants which help predict the existence of closed orbits before and after a bifurcation, as well as providing a theoretical guide to help organize the numerical work underlying the search of orbits. These invariants, the Euler characteristic of suitable Floer groups, are numbers which stay invariant under bifurcations, and can be described in concrete terms, making them amenable to practical implementation and numerics. This provides a test: after the numerical work has produced a list of orbits before and after a bifurcation, one can explicitly check whether this number remains constant; if it does not, this is an indication that the algorithm has not produced all the relevant orbits, and one may keep on looking.

In practice, one would like to compare the orbits at hand with those that have been found before. The natural notion of equivalence of two orbits is to say that they are qualitatively the same, provided one can find a path of orbits joining them, which is regular, i.e. it does not undergo bifurcation (the parameter for the path is usually the energy or e.g. a mass parameter). For this purpose, we will illustrate the use of global topological methods, via the GIT sequence introduced in [10] by the first and third authors. This is a sequence of three spaces (“top”, “middle” and “base”) consisting of equivalence classes of symplectic matrices, and concrete maps between them, which are also explicitly computable. A closed orbit of an arbitrary Hamiltonian system induces a point in the “base” and “middle” spaces of this sequence. The “base” (in the dimension we will care about) is a copy of the plane $\mathbb{R}^2$, split into components labelled according to the eigenvalue decomposition of the monodromy matrix of the orbit in question; see Figure 1. The resulting diagram, as we learned after rediscovering it in the context of the GIT sequence, was originally introduced by Broucke in [2], and it is sometimes referred to as “Broucke’s stability diagram” in the engineering literature (see also Howard–Mackay [15] for higher-dimensional versions, also incorporating the notion of Krein signature for the study of linear stability).

If the Hamiltonian is moreover preserved by a symmetry, the orbit is symmetric, and we choose one of its symmetric points, then there is also a unique induced point in the “top” space. We remark that many orbits which have been found via numerical explorations of classical problems, such as the three-body problem, are actually symmetric, and therefore are natural objects of study. A family of orbits induces a path in the corresponding space of the sequence. These spaces also contain bifurcation loci, i.e. subsets corresponding to bifurcations of orbits (which look like a pencil
of lines tangent to a parabola; see Figure 2. A family of orbits which bifurcates induces a path in the GIT spaces which crosses the component of the bifurcation loci corresponding to the type of bifurcation. Therefore if two orbits correspond to points that lie in different regular components (each of the seven of Figure 1), there is no regular family that joins them. In other words, the topology of these quotient spaces can be used as an obstruction to the existence of regular families of orbits, as well as the study of bifurcations, in a concrete, visual manner. What is more, one can refine the obstructions provided by the bifurcation loci, by attaching to the point corresponding to an orbit, an ordered pair of sign labels (i.e. “plus” or “minus”), the $B$-signature. Even if two orbits induce points lying in the same regular component of the plane, there is no regular family between them if they have different pair of sign labels.

The topological approach also serves the purpose of providing a data base for orbits, in the form of a cloud of dots in the plane with labels attached. Moreover, the amount of data to be stored is relatively cheap, which makes the approach resource-efficient. To sum up, in this article we will address the following objectives:

- **(Classification)** Given two orbits, decide when they are they qualitatively different.
- **(Catalogue)** Construct an efficient “data base” of known orbits as a cloud of dots in the plane, together with sign labels. If they are qualitatively the same, record regular cylinder of orbits between them. If they are qualitatively different, record bifurcations between them, if any.
- **(Practical tests)** Introduce Floer-type numerical invariants to test the accuracy of the algorithms, and to guide/organize the numerical work.

In what follows, we shall illustrate these ideas in a number of examples that arise in models for real-life planetary systems, via numerical plots. Let us now introduce the main tools in more detail.

**Symmetric orbits.** Consider a Hamiltonian $H : \mathbb{R}^{2n+2} \to \mathbb{R}$, defined on the phase-space $\mathbb{R}^{2n+2}$ of position-momentum pairs $(q, p) \in \mathbb{R}^{2n+2}$, which comes with the standard symplectic form $\omega = \sum\limits_{j} dq_{j} \wedge dp_{j}$ (we choose $2n + 2$ instead of $2n$ to simplify notation in what follows). Consider also an antisymplectic involution $\rho$, i.e. a map of $\mathbb{R}^{2n+2}$ satisfying $\rho^{2} = id, \rho^{*} \omega = -\omega$. An example of such a map is

$$\rho : \mathbb{R}^{6} \to \mathbb{R}^{6}, \ (q_{1}, q_{2}, q_{3}, p_{1}, p_{2}, p_{3}) \mapsto (q_{1}, -q_{2}, -q_{3}, -p_{1}, p_{2}, p_{3}).$$

Assume that $H$ is invariant under $\rho$, i.e. $H \circ \rho = H$. A prototypical example is the Hamiltonian for the restricted three-body problem; see Section 5 below.

A periodic orbit $x : S^{1} \to \mathbb{R}^{2n+2}$ of $H$ is symmetric if it satisfies $x(t) = \rho(x(-t)), t \in S^{1}$, so in particular $x(0), x\left(\frac{1}{2}\right) \in L := \text{Fix}(\rho) = \{(q, p) \in \mathbb{R}^{2n} : \rho(q, p) = (q, p)\}$ lie in the fixed-point set of $\rho$, a Lagrangian in $\mathbb{R}^{2n+2}$ (i.e. $\omega$ vanishes along $L$). After fixing the value of the energy $H$, and projecting out the direction of the Hamiltonian flow, the linearization of the dynamics along a symmetric orbit gives a time-dependent family of $2n \times 2n$-symplectic matrices (the reduced monodromy matrices), all related to each other by symplectic conjugation. If we linearize precisely at an intersection point $l$ of $x$ with $L$ (a symmetric point), the corresponding matrix has the special form

$$M = M_{A,B,C} = \left( \begin{array}{cc} A & B \\ C & A^{T} \end{array} \right) \in M_{2n \times 2n}(\mathbb{R}),$$

where $A, B, C$ are $n \times n$-matrices that satisfy the equations

$$B = B^{T}, \quad C = C^{T}, \quad AB = BA^{T}, \quad A^{T}C = CA, \quad A^{2} - BC = I,$$

(2)
which ensure that $M$ is symplectic. We will denote the space of such symplectic matrices by $Sp^2(2n)$. In order to write down the matrix, we have implicitly chosen a basis for the tangent space of $L$ at $l$. Choosing a different basis amounts to replacing $M_{A,B,C}$ with $M_{R,(A,B,C)}$, where

$$R_* (A, B, C) = \left( R A R^{-1}, R B R^T, (R^T)^{-1} C R^{-1} \right),$$

and $R \in GL_n(\mathbb{R})$ is the invertible matrix corresponding to the change of basis.

Let us look at the case $n = 2$, the dimension relevant for e.g. the spatial three-body problem. Note that the pair $p = (\text{tr}(A), \det(A)) \in \mathbb{R}^2$, where $A$ is the first block of $M_{A,B,C}$, is independent on the choice of basis, i.e. they are the same for $M_{A,B,C}$ and $M_{R,(A,B,C)}$. This point $p$ in the plane (the “base”) is the one which we will associate to a symmetric orbit having monodromy matrix $M_{A,B,C}$.

If we now have a family of symmetric orbits $t \mapsto \gamma_t$ where $t$ is a parameter (usually the energy or a mass parameter), and $M_t$ is the associated monodromy matrix, this induces a path $t \mapsto p_t = (\text{tr}(A_t), \det(A_t)) \in \mathbb{R}^2$. All our plots of families of orbits will then be sequences of points in the plane (representing paths), which is well-suited for visualization; see Figure 2 for a sketch, and Section 5 for numerical plots.

Moreover, as advertised, the data of the point $p$ can be refined, via the $B$-sign for each of the eigenvalues of $A$, which we now discuss.

**B-signature for symmetric orbits.** We now explain the notion of $B$-sign, introduced in [10]. For the case $n = 2$, the following construction will assign a pair $\epsilon = (\epsilon_1, \epsilon_2)$, the $B$-signature, where $\epsilon_i = \pm$ is a “plus” or a “minus” label for $i = 1, 2$, to the eigenvalues of the $2 \times 2$-block $A$ of $M = M_{A,B,C}$, assuming that they are real and different. We do not assign a sign to the remaining cases, i.e. when they coincide or are complex, which correspond respectively to $p$ lying in $\Gamma_d$ or $N$ in Figure 1. If $\mu_1 < \mu_2$ are these eigenvalues, let $v_i$ be an eigenvector of $A^T$ with eigenvalue $\mu_i$, i.e. $A^T v_i = \mu_i v_i$. The $B$-sign $\epsilon_i$ of $\mu_i$ is then defined as

$$\epsilon_i = \text{sign}(v_i^T B v_i) \in \{ \pm \},$$

where we use the $B$-block of $M$. We remark that $\epsilon_i$ is independent of the choice of basis: indeed, replacing $A$ by $R A R^{-1}$, and $B$ by $R B R^T$, we let $w_i = R^{-T} v_i$. Note that

$$(R A R^{-1})^T w_i = (R^{-T} A^T R^{-T}) R^{-T} v_i = R^{-T} A^T v_i = \mu_i w_i,$$

and so $w_i$ is an eigenvector $(R A R^{-1})^T$; moreover,

$$\text{sign}(v_i^T R B R^T w_i) = \text{sign}(w_i^T R B R^T w_i) = \text{sign}(v_i^T R^{-1} R B R^T R^{-T} v_i) = \text{sign}(v_i^T B v_i),$$

as desired.

The “data point” that is then associated to a symmetric orbit with monodromy matrix $M_{A,B,C}$ is the tuple $(p = (\text{tr}(A), \det(A)), \epsilon = (\epsilon_1, \epsilon_2))$, which is independent of all choices, except perhaps the choice of symmetric point at which we linearize (here, $\epsilon$ is empty for the complex or double-eigenvalue case). It turns out that the $B$-signature $\epsilon$ can detect whether there are chord bifurcations at the symmetric point $l$, which happens when the $B$-signs do not jump; see Section 5.

**Remark 1.1.** Alternatively to the $B$-sign, one could also consider the “$C$-sign”, obtained by replacing the $B$-block, with the $C$-block of $M$, and $A^T$, by $A$. They give equivalent information; although do not necessarily agree (they agree in the elliptic case, i.e. $|\mu_i| < 1$, and differ in the hyperbolic case, i.e. $|\mu_i| > 1$). It turns out that the $C$-sign does jump at chord bifurcations, as opposed to the $B$-sign.
Non-symmetric orbits and Krein theory. In practice, it may not be apparent whether a given orbit is symmetric, and hence the above considerations need not apply. However, in order to refine the data proved by the point \( p \), we can still appeal to classical Krein theory, as proposed by Krein \cite{18,19,20,21} and rediscovered by M"oser \cite{22}, which associates a sign only to the elliptic eigenvalues. It turns out that, in the elliptic case, the notion of \( B \)-signature coincides with the more classical one of Krein signature; see \cite{10}. Before introducing this notion, first, recall that the reduced monodromy matrix of any orbit at any point is a symplectic matrix \( M \) in \( \text{Sp}(4) \), and changing the base point only changes the matrix up to symplectic conjugation. Now, by a symmetric point, it may be "lifted" uniquely from the "middle" space to the top one. One can do precisely the Krein sign over elliptic components. Whenever an orbit is symmetric and we choose the topology of the configuration space consisting of the collection of all possible pairs \((p, \epsilon)\) that may arise from the above procedure, together with the structure of the projection \((p, \epsilon) \mapsto p \in \mathbb{R}^2\). This is illustrated in Figure 3, where the configuration space lies on the top, and has different "branches" corresponding to different \( B \)-signature \( \epsilon \), which get collapsed on top of each other under the projection. The plus/minus labels in the branches of the "middle" space of Figure 3 records precisely the Krein sign over elliptic components. Whenever an orbit is symmetric and we choose a symmetric point, it may be "lifted" uniquely from the "middle" space to the top one. One can do

\[
\det(A) = \frac{a}{4} - \frac{1}{2} \cdot \text{tr}(A) = \frac{b}{2},
\]

where, if \( \lambda_1^p, \lambda_2^p, \lambda_1^s, \lambda_2^s \) are the eigenvalues of \( M \) (satisfying \( \lambda_1^p \lambda_2^p = \lambda_1^s \lambda_2^s = 1 \)), then \( a, b \) are given by the symmetric polynomials

\[
a = \lambda_1^p \lambda_1^s + \lambda_2^p \lambda_2^s + \lambda_1^s \lambda_1^p + \lambda_2^s \lambda_2^p + 2,
\]

and

\[
b = \lambda_1^p + \lambda_2^p + \lambda_1^s + \lambda_2^s = \text{tr}(M).
\]

See Section 4 for a derivation of these expressions, which can be computed directly from the eigenvalues of \( M \) without need of conjugating \( M \) to be of the form \( M_{A,B,C} \).

If one or both pairs of the eigenvalues are elliptic (and distinct), i.e. \( p \) lies in a region of Figure 1 with at least one \( E \) label on it, we attach a sign to each of them as follows. We first complexify, i.e. we now work over \( \mathbb{C} \). Consider the matrix

\[
G = -iJ = \begin{pmatrix} 0 & -iId \\ iId & 0 \end{pmatrix}
\]

which we think of as acting on \( \mathbb{C}^4 \) by left-multiplication (each block is \( 2 \times 2 \)). Let \( \lambda = e^{i\theta}, \bar{\lambda} = e^{-i\theta} \) be an elliptic pair of eigenvalues of \( M \), with \( \text{im}(\lambda) > 0 \). Given \( v \in \mathbb{C}^4 \) an eigenvector for \( \lambda \), the Krein-sign of \( \lambda \) is

\[
\kappa(\lambda) = \text{sign}(v^t \bar{G} \bar{v}).
\]

It is easy to check that \( v^t \bar{G} \bar{v} \) is a non-zero real number, whose sign is independent of \( v \) (indeed, since \( \lambda \) is simple, any other vector is of the form \( w = \mu \cdot v \) with \( \mu \neq 0 \), and so \( w^t \bar{G} \bar{w} = |\mu|^2 v^t \bar{G} \bar{v} \)). We have the property \( \kappa(\bar{\lambda}) = -\kappa(\lambda) \).

**GIT sequence.** We now explain the notion of the GIT sequence. The rough idea is to understand the topology of the configuration space consisting of the collection of all possible pairs \((p, \epsilon)\) that may arise from the above procedure, together with the structure of the projection \((p, \epsilon) \mapsto p \in \mathbb{R}^2\). This is illustrated in Figure 3, where the configuration space lies on the top, and has different "branches" corresponding to different \( B \)-signature \( \epsilon \), which get collapsed on top of each other under the projection. The plus/minus labels in the branches of the "middle" space of Figure 3 records precisely the Krein sign over elliptic components. Whenever an orbit is symmetric and we choose a symmetric point, it may be "lifted" uniquely from the "middle" space to the top one. One can do
this more formally, as follows. The treatment will assume some mathematical background, and is included for completeness. It may be skipped on a first read.

**Remark 1.2 (GIT quotient).** To give the definition of the GIT sequence, we need to introduce some terminology. Recall that if a group $G$ acts on a topological space $X$, the geometric quotient $X/G$ is the space of $G$-orbits, i.e. a point in $X/G$ is a set of the form \( \{ g \cdot x : x \in X \} \subset X \). In general, this space might not be Hausdorff, i.e. there might be points which cannot be separated from other points. To fix this, one considers the GIT quotient, the space $X//G$ obtained by identifying two points of $X$ if the closures of their $G$-orbits intersect (and this space is indeed Hausdorff). We shall consider only GIT quotients in what follows, although the reader might choose to ignore this technicality.

The GIT sequence consists of the sequence of maps

\[
\text{Sp}^T(2n)/\text{GL}_n(\mathbb{R}) \to \text{Sp}(2n)/\text{Sp}(2n) \to M_{n \times n}(\mathbb{R})//\text{GL}_n(\mathbb{R}) \cong \mathbb{R}^n,
\]

given by

\[
[M_{A,B,C}] \mapsto [\begin{bmatrix} A & B & C \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}] \mapsto [A].
\]

Here, $\text{Sp}(2n)$ is the symplectic group, which acts on itself by conjugation, i.e. via $A \cdot B = ABA^{-1}$; and $\text{GL}_n(\mathbb{R})$ also acts by conjugation on the space of matrices $M_{n \times n}(\mathbb{R})$. Above we denote by $[M_{A,B,C}]$ the equivalence class of the matrix $M_{A,B,C} \in \text{Sp}^T(2n)$ in the GIT quotient $\text{Sp}^T(2n)/\text{GL}_n(\mathbb{R})$, by $[[M_{A,B,C}]]$ the equivalence class in the GIT quotient $\text{Sp}(2n)/\text{Sp}(2n)$, and by $[A]$ the equivalence class of the first block $A \in M_{n \times n}(\mathbb{R})$ in $M_{n \times n}(\mathbb{R})//\text{GL}_n(\mathbb{R})$. We have used the fact that mapping the equivalence class of a matrix $A \in M_{n \times n}(\mathbb{R})$ to the coefficients of its characteristic polynomial, we get an identification $M_{n \times n}(\mathbb{R})//\text{GL}_n(\mathbb{R}) \cong \mathbb{R}^n$; see [10] Appendix A.

In the examples above, where the spaces consists of matrices, the transition from to the geometric quotient to the GIT quotient basically means, in practice, to ignore Jordan factors, replacing them with diagonal blocks. The resulting matrices, while not necessarily equivalent in the original quotient, become so in the GIT one, see [10] Appendix A. In [10], the cases $n = 1$ and $n = 2$ (for instance, relevant for the planar and the spatial three-body problems, respectively) are studied in detail. In particular, the topology of these GIT quotients is fully determined, as well as the maps. In this article, we will make use of the case $n = 2$, where the base of the GIT sequence is the plane $\mathbb{R}^2$, together with the structure of its bifurcation loci, as shown in Figure 2. The maps of the sequence are also very concrete and therefore simple to implement, i.e. given by

\[
\text{Sp}^T(4)/\text{GL}_2(\mathbb{R}) \to \text{Sp}(4)/\text{Sp}(4) \to \mathbb{R}^2
\]

\[
[M_{A,B,C}] \mapsto [\begin{bmatrix} A & B \\ 0 & I \end{bmatrix}] \mapsto (\text{tr}(A), \det(A)) = p,
\]

which motivates the construction of $p$ that we explained above. Indeed, as follows from [10], the GIT quotient $\text{Sp}^T(4)/\text{GL}_2(\mathbb{R})$ is precisely the configuration space for the pairs $(p, \epsilon)$.

**Local Floer homology and its Euler characteristic.** At a bifurcation, the so-called local Floer homology group of the orbit in question stays invariant (the interested reader can consult e.g. [13] Section 3, and references therein, for a definition of this group, which lies beyond the scope of this article). A particular instance of this fact is that its Euler characteristic (here called the SFT-Euler characteristic), before and after a bifurcation, does not change. This means that the difference between the number of (good) periodic orbits of Conley-Zehnder index of even parity and that of the periodic orbits of Conley-Zehnder index of odd parity stays the same, before and after. As mentioned above, this helps as a practical test while searching for orbits. We shall focus only in dimensions four and six, by first explaining how to compute this invariant in explicit terms.
In the case where the orbit is symmetric, we will also consider the real Euler characteristic, which is the Euler characteristic of the associated local Lagrangian Floer homology. This invariant serves as a further test, as it can detect when the orbit bifurcates as a chord, even when it does not bifurcate as an orbit. We shall illustrate this in concrete examples.

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2. THE SFT-EULER CHARACTERISTIC OF A PERIODIC ORBIT

The SFT-Euler characteristic of a closed orbit can be defined in any dimension, as the Euler characteristic of its so-called local Floer homology (which we will not treat here). If the orbit is degenerate and good (i.e. not an even multiple cover of a negative hyperbolic orbit), then this invariant is $\pm 1$, depending on the parity of its so-called Conley-Zehnder index; if it is non-degenerate and bad, it is zero (we shall not need the definition of the Conley-Zehnder index, but the interested reader can consult e.g. [11]). The interesting case is when it is degenerate, and hence may undergo bifurcation. If one adds a small perturbation, it might bifurcate into a finite collection of other non-degenerate

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{The picture shows $\mathbb{R}^2 \cong M_{2 \times 2}(\mathbb{R})/\text{GL}_2(\mathbb{R})$, the base of the GIT sequence for $n=2$. There are seven regions, each labelled by the eigenvalue classification of the corresponding matrices that lie above them. Namely, $N$ corresponds to non-real quadruples $\lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda}$, whereas $H^\pm$, to a positive/negative hyperbolic pair $\lambda, 1/\lambda$, and $E$, to an elliptic pair $e^{\pm 2\pi i \theta}$. For instance, if $M_{A,B,C}$ maps to the region $E^2$, it means that it has two pairs $e^{\pm 2\pi i \theta_1}, e^{\pm 2\pi i \theta_2}$ of elliptic eigenvalues. This diagram is sometimes referred to as “Broucke stability diagram”.
}
\end{figure}
Figure 2. The picture shows again the base $\mathbb{R}^2$, for $n = 2$. The parabola $\Gamma_d = \{y = 1/4x^2\}$ corresponds to matrices with double eigenvalues. The locus of matrices with a fixed eigenvalue is a line tangent to $\Gamma_d$. On the left, we have the elliptic pencil of lines $\{\Gamma_\theta, \theta \in [0, 2\pi]\}$, where $\Gamma_\theta$ has slope $\cos(2\pi\theta)$ and corresponds to the eigenvalue $e^{2\pi i \theta}$. On the right, the complete pencil, also containing the hyperbolic pencil $\{\Gamma_\lambda : \lambda \in \mathbb{R}\setminus[-1, 1]\}$, where $\Gamma_\lambda$ has slope $a(\lambda) = \frac{1}{2}(\lambda + \frac{1}{\lambda})$ and corresponds to real (i.e. hyperbolic) eigenvalue $\lambda$. A family of orbits bifurcating induces a family of monodromy matrices whose eigenvalues crosses 1, and hence is seen as a path in $\mathbb{R}^2$ which crosses $\Gamma_1$; we sketch such a path in the picture below, where one eigenvalue pair goes from elliptic to positive hyperbolic, while the other stays elliptic. We also sketch an example of a period-doubling bifurcation, where one eigenvalue pair goes from elliptic to negative hyperbolic. Similarly, a $k$-fold bifurcation, where $\lambda = e^{2\pi i \frac{k}{L}}$ (with $\lambda^k = 1$) is being crossed, induces a path crossing $\Gamma_{l/k}$, for some $l$. We will plot numerical examples below.
orbits. One then defines the SFT-Euler characteristic as the number of good orbits with even Conley-Zehnder index, minus the number of good orbits with odd Conley-Zehnder index, where the orbits are taken among the orbits that appear after perturbation. The remarkable fact is that this number is independent on the perturbation. In particular, it remains invariant before and after a bifurcation. In what follows, we shall explain this in more concrete terms, in dimensions 4 and 6.

2.1. The four-dimensional case. In the four-dimensional case, the reduced monodromy matrix $M$ is a $2 \times 2$-matrix. Since it is symplectic its determinant is 1 and therefore its spectrum is completely determined by its trace. The trace of the reduced monodromy matrix can as well be obtained from the trace of the nonreduced monodromy matrix $\tilde{M}$ by the formula

$$\text{tr}(M) = \text{tr}(\tilde{M}) - 2.$$ 

We distinguish the following cases.

**Negative hyperbolic case:** In this case

$$\text{tr}(M) \leq -2.$$ 

If $\text{tr}(M) < -2$ the spectrum of $M$ contains two negative real eigenvalues which are inverse to each other. If $\text{tr}(M) = -2$, then the only eigenvalue of $M$ is $-1$, having algebraic multiplicity two.

**Elliptic case:** In this case

$$-2 < \text{tr}(M) < 2.$$ 

The spectrum of $M$ contains two nonreal eigenvalues on the unit complex circle which are inverse to each other.

**Degenerate case:** In this case

$$\text{tr}(M) = 2.$$ 

The spectrum of $M$ only contains 1.

**Positive hyperbolic case:** In this case

$$\text{tr}(M) > 2.$$ 

The spectrum of $M$ contains two positive real eigenvalues different from one which are inverse to each other.

We can now explain the parity of the Conley-Zehnder index, which is roughly speaking a rotation number associated to $M$, and its relationship to good/bad orbits. In the negative hyperbolic as well as the elliptic case the parity of the Conley-Zehnder index is odd, while in the positive hyperbolic case, it is even. In the degenerate case bifurcation occurs, and we do not define the parity of the Conley-Zehnder index. To define the SFT-Euler characteristic we additionally need the distinction of periodic orbits into good and bad ones. In the four-dimensional case, periodic orbits whose Conley-Zehnder index has odd parity are always good, i.e. negative hyperbolic and elliptic orbits are always good. On the other hand, positive hyperbolic ones can be bad. To explain what this means we need to recall that a periodic orbit gives rise to multiple covers of itself. The monodromy matrix of the $k$-fold cover is then $M^k$ and if $\lambda$ is an eigenvalue of $M$, then $\lambda^k$ is an eigenvalue of $M^k$. In particular, if $M^k$ is negative hyperbolic or elliptic, the same is true for $M$. On the other hand, nondegenerate even covers of negative hyperbolic ones are positive hyperbolic. If a positive hyperbolic orbit is an even cover of a negative hyperbolic one it is called bad. Therefore, in dimension
four, the SFT-Euler characteristic of an orbit $x$ is defined by

$$
\chi_{\text{SFT}}(x) = \# \{ \text{good orbits with even CZ-index} \} - \# \{ \text{orbits with odd CZ-index} \} = \# \{ \text{good positive hyperbolic orbits} \} - \# \{ \text{elliptic and negative hyperbolic orbits} \}.
$$

That the local SFT-Euler characteristic before and after a bifurcation does not change follows from the invariance of local Floer homology. However, for the generic bifurcations in dimension four, we can check this directly, as follows.

2.2. **Generic four-dimensional bifurcations.** We follow the cases as listed in the book by Abraham and Marsden [1].

**Creation [1, p. 598]:** In this case initially there was no periodic orbit at all. Hence

$$
\chi_{\text{SFT}} = 0.
$$

After the creation there is a simple elliptic and positive hyperbolic orbit. In particular, the SFT-Euler characteristic stays zero.

**Subtle division [1, p. 599]:** In this case the double cover of an elliptic orbit bifurcates. It is interesting to look at the SFT-Euler characteristic for the simple orbit as well as for the double cover. We remark that “locality” of the Euler-characteristic refers to the loop space, in which space a periodic orbit and its double cover are far from each other. Therefore we should look at the local SFT-Euler characteristic for the simple orbit, which we denote by $\chi_{\text{SFT}}^1$, as well as the one for the double cover which we denote by $\chi_{\text{SFT}}^2$. We first discuss $\chi_{\text{SFT}}^1$. Before the transition there is one simple elliptic orbit. Therefore we have

$$
\chi_{\text{SFT}}^1 = -1.
$$

After the transition the simple orbit becomes negative hyperbolic. There is no bifurcation of the simple periodic orbit, just its double cover bifurcates. Hence $\chi_{\text{SFT}}^1$ stays minus one.

We next discuss the invariance of $\chi_{\text{SFT}}^2$. The double cover of an elliptic orbit is elliptic as well. Therefore the SFT-Euler characteristic for the double cover satisfies as well

$$
\chi_{\text{SFT}}^2 = -1.
$$

After the transition the simple elliptic orbit becomes negative hyperbolic. Its double cover is therefore a bad positive hyperbolic orbit and does not contribute to the SFT-Euler characteristic. The orbit which bifurcates is elliptic and hence the SFT-Euler characteristic stays $-1$.

**Murder [1, p. 600]:** In this case it is again interesting to look at the SFT-Euler characteristic $\chi_{\text{SFT}}^1$ of the simple orbit as well as the one for the double cover $\chi_{\text{SFT}}^2$. The case for the simple orbit is completely analogous as in the subtle division. An elliptic periodic orbit becomes negative hyperbolic and therefore

$$
\chi_{\text{SFT}}^1 = -1.
$$

However, the case of the double cover is different. Here before bifurcation we have a double covered elliptic one and a simple positive hyperbolic one. A simple positive hyperbolic orbit is good
and the double cover of an elliptic orbit is elliptic as well. Therefore
\[ \chi^2_{SFT} = 0. \]

After bifurcation just the double cover of the negative hyperbolic orbit is left. This is a bad positive hyperbolic orbit and therefore does not contribute to the SFT-Euler characteristic. We see that \( \chi^2_{SFT} \) stays zero after the transition.

**Phantom kiss** [1, p. 602]: We discuss the 3-kiss. In this case it is interesting to look at the SFT-Euler characteristic of the 3-fold cover \( \chi^3_{SFT} \). Before the bifurcation we have a 3-fold covered elliptic orbit and a simple positive hyperbolic one. Therefore
\[ \chi^3_{SFT} = 0. \]

After bifurcation we still have a 3-fold covered elliptic orbit and a positive hyperbolic one, so that the SFT-Euler characteristic does not change for obvious reasons. The discussion for the 4-kiss is similar, when one looks there at the SFT-Euler characteristic of the 4-fold cover \( \chi^4_{SFT} \).

**Emission** [1, p. 603]: We discuss here the case \( p = 4 \) as illustrated in the figure in [1, p. 603]. In this case it is interesting to look at the SFT-Euler characteristic for the 4-fold cover \( \chi^4_{SFT} \). Before bifurcation there is one 4-fold covered elliptic orbit. Therefore we have
\[ \chi^4_{SFT} = -1. \]

After bifurcation there is a 4-fold covered elliptic orbit, a simple elliptic orbit and a simple positive hyperbolic orbit. We see again that the SFT-Euler characteristic does not change.

2.3. **Non-generic four-dimensional bifurcations.** There are relevant problems in celestial mechanics where there are bifurcations which do not fall in the generic classification. We now discuss some of them.

**Hénon families in Hill’s lunar problem.** Although in theory the probability to have a non-generic bifurcation is basically zero, in practice non-generic bifurcations occur quite often. The reason is that the Hamiltonians one usually considers (e.g. for the restricted three-body problem) are invariant under various symmetries. If one would choose a Hamiltonian at random then the probability that it is invariant under a symmetry is basically zero as well. Therefore it is not surprising that for symmetric Hamiltonians new phenomena occur.

A nongeneric bifurcation was for instance described by Hénon in [14]. In this paper, Hénon was studying Hill’s lunar problem. This is a limit case of the restricted three-body problem. Namely, while in the restricted three-body problem the masses of the primaries are comparable, this is not the case anymore in Hill’s lunar problem, although the massless body is supposed to be very close to the small primary. Hill’s lunar problem can therefore be considered as an approximation to the Jupiter-Europa or Saturn-Enceladus systems, when one lets the mass of Europa, respectively Enceladus, go to zero. A surprising property of Hill’s lunar problem is that it is invariant under an additional symmetry. While the potential of the restricted-three body problem is invariant under reflection at the \( x \)-axis, i.e. the axis on which the two primaries lie, Hill’s lunar problem is additionally invariant under reflection at the \( y \)-axis.

The family of the direct or prograde periodic orbit is traditionally referred to as family \( g \). In Hill’s lunar problem the direct orbit is invariant under reflection at the \( x \)-axis as well as under reflection
at the $y$-axis. For small energy the direct orbit is elliptic. However, for higher energy it becomes positive hyperbolic. At the bifurcation point two new families referred to as $g'$ appear. These two families are still invariant under reflection at the $x$-axis but not anymore under reflection at the $y$-axis. Instead of that, reflection at the $y$-axis maps one branch of the $g'$-family to the other branch. As explained by Hénon [14] at their birth the two $g'$-branches are elliptic. We can now check the invariance of the SFT-Euler characteristic for this nongeneric bifurcation. Before the bifurcation the direct orbit was elliptic. Therefore the SFT-Euler characteristic is minus one. After the bifurcation the direct periodic orbit is positively hyperbolic. Since it is simple it is a good positive hyperbolic orbit and therefore contributes $+1$ to the SFT-Euler characteristic. However, after bifurcation we have to take into account in addition the two $g'$-periodic orbits which are both elliptic and therefore contribute each $-1$ to the SFT-Euler characteristic. So their sum $1 - 1 - 1 = -1$ stays minus one in accordance with the invariance of the SFT-Euler characteristic under bifurcation.

2.4. The six-dimensional case. In the six-dimensional case the reduced monodromy matrix $M$ is a $4 \times 4$-matrix. Since it is symplectic its characteristic polynomial

$$p(x) = x^4 + c_3x^3 + c_2x^2 + c_1x + 1$$

is a palindrom, i.e. $c_1 = c_3$. In particular, there exists a quadratic matrix

$$q(y) = y^2 + b_1y + b_0$$

such that

$$p(x) = x^2q(x + \frac{1}{2}).$$

Let $\mu_1$ and $\mu_2$ be the roots of the quadratic polynomial $q$. Since the polynomial $q$ is a real polynomial its roots are both real or complex conjugate to each other. We now distinguish several cases with the help of the roots of $q$.

**Nonreal case ($N$):** The two roots $\mu_1$ and $\mu_2$ are not real. In this case the eigenvalues of the monodromy matrix $M$ are neither real nor lie on the unit circle. They appear as a quadruple $(\lambda, \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda})$.

The real case has to be subdivided into several subcases. If $\mu_1$ and $\mu_2$ are real and distinct then maybe after a symplectic change of coordinates the reduced monodromy matrix splits as

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

where $M_1$ and $M_2$ are symplectic $2 \times 2$-matrices satisfying

$$\text{tr}(M_1) = \mu_1, \quad \text{tr}(M_2) = \mu_2.$$ 

The roots $\mu_1$ and $\mu_2$ hence determine if $M_1$ respectively $M_2$ is elliptic, negative hyperbolic, positive hyperbolic or degenerate. In the real case we order the roots such that

$$\mu_1 \leq \mu_2.$$

**Doubly negative hyperbolic case $\mathcal{H}^-$:** In this case we have

$$\mu_1 \leq -2, \quad \mu_2 \leq -2.$$

**Elliptic/ negative hyperbolic case $\mathcal{E}\mathcal{H}^-$:** In this case we have

$$\mu_1 \leq -2, \quad -2 < \mu_2 < 2.$$
Negative/ positive hyperbolic case $\mathcal{H}^{-+}$: In this case we have
\[ \mu_1 \leq -2, \quad \mu_2 > 2. \]

Doubly elliptic case $\mathcal{E}^2$: In this case we have
\[ -2 < \mu_1 < 2, \quad -2 < \mu_2 < 2. \]

Elliptic/ positive hyperbolic case $\mathcal{E}\mathcal{H}^+$: In this case we have
\[ -2 < \mu_1 < 2, \quad \mu_2 > 2. \]

Doubly positive hyperbolic case $\mathcal{H}^{++}$: In this case we have
\[ \mu_1 > 2, \quad \mu_2 > 2. \]

Degenerate case $\mathcal{D}$: In this case we have $\mu_1 = 2$ or $\mu_2 = 2$.

The parity of the Conley-Zehnder index is additive. For example since the parity of the Conley-Zehnder index in the elliptic case is odd and in the positive hyperbolic case is even it follows that in the $\mathcal{E}\mathcal{H}^+$-case it is odd again. The following table displays the parity in the various cases.

| Type     | Parity of Conley-Zehnder index |
|----------|---------------------------------|
| $\mathcal{H}^{-+}$ | even                           |
| $\mathcal{E}\mathcal{H}^+$ | even                           |
| $\mathcal{E}\mathcal{H}^+$ | odd                            |
| $\mathcal{E}^2$     | even                           |
| $\mathcal{E}\mathcal{H}^+$ | odd                            |
| $\mathcal{H}^{++}$ | even                           |
| $\mathcal{N}$       | even                           |

The only periodic orbits which can be bad are the ones of $\mathcal{E}\mathcal{H}^+$ and $\mathcal{H}^{++}$ type. Namely a periodic orbit of $\mathcal{E}\mathcal{H}^+$ type is bad, if it is an even cover of one of $\mathcal{E}\mathcal{H}^-$ type. Similarly, a periodic orbit of $\mathcal{H}^{++}$-type is bad, if it is an even cover of an $\mathcal{H}^{-+}$-orbit. Otherwise orbits are good. For example if an $\mathcal{H}^{++}$-orbit is an even cover of a $\mathcal{H}^{-+}$ orbit it is good. The SFT-Euler characteristic of the orbit $x$ is now
\[
\chi_{\text{SFT}}(x) = \# \{ \text{good orbits with even CZ-index} \} - \# \{ \text{good orbits with odd CZ-index} \}
= \# \{ \mathcal{H}^{-+}, \mathcal{E}\mathcal{H}^-, \mathcal{E}^2, \text{ good } \mathcal{H}^{++}, \mathcal{N} \} - \# \{ \mathcal{H}^{-+}, \text{ good } \mathcal{E}\mathcal{H}^+ \}.
\]

3. The Real Euler Characteristic of a Symmetric Orbit

In this section, we focus on the particular case of symmetric orbits, i.e. orbits which are invariant under an antisymplectic involution which preserves the Hamiltonian, as defined in the Introduction. We will follow the exposition of [12], where the Hörmander index is introduced.

A symmetric periodic $x$ can be seen both as a periodic orbit, as well as a chord between the Lagrangian fixed-point locus of the involution. Therefore it has a Conley-Zehnder index $\mu_{\text{CZ}}(x)$, and a Lagrangian Maslov index $\mu_L(x)$, which is a half-integer, i.e. takes values in $\frac{1}{2}\mathbb{Z}$ (again, its definition
will not be needed, but can be found e.g. in [23]). The difference of these two, as introduced in [12], is the Hörmander index

\[ s(x) = \mu_{CZ}(x) - \mu_L(x) \in \frac{1}{2} \mathbb{Z}, \]

also a half-integer. One can use this index to detect when \( x \) bifurcates as a chord, even when it doesn’t bifurcate as an orbit; we will explain this in the next section via concrete examples.

We note that the iterates of a symmetric periodic orbit \( x^k \) for \( k \in \mathbb{N} \) are symmetric orbits as well. We say that a symmetric periodic orbit is nondegenerate if for any \( k \in \mathbb{N} \) we have \( \det(\Phi^k - I) \neq 0 \), where \( \Phi \) is the monodromy matrix of \( x \), i.e. 1 is not an eigenvalue of any iterate of \( \Phi \). Moreover, the Chebyshev polynomials of the first kind are recursively defined by

\[ T_0(x) = 1 \]
\[ T_1(x) = x \]
\[ T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x). \]

The Chebyshev polynomials of the second kind are similarly defined by

\[ U_0(x) = 1 \]
\[ U_1(x) = 2x \]
\[ U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x). \]

The following gives a formula for computing the Hörmander index of the iterates of a symmetric orbit, in terms of the monodromy matrix, which in particular is easy to implement numerically, and does not make use of the definition of the Conley-Zehnder index nor the Lagrangian Maslov index.

**Theorem 3.1.** [12] Let \( x \) be a nondegenerate, symmetric periodic orbit, with monodromy matrix

\[ M = M_{A,B,C} = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}, \]

satisfying Equations (2) given in the Introduction. Then the Hörmander indices of its iterates are given by

\[ s(x^k) = \frac{1}{2} \text{sign} \left((I - T_{k}(A))U_{k-1}(A)^{-1}C^{-1}\right), \]

(5)

for \( k \in \mathbb{N} \). For \( k = 1 \), we have in particular that

\[ s(x) = \frac{1}{2} \text{sign} \left((I - A)C^{-1}\right). \]

(6)

Here, \text{sign} denotes the signature of a matrix (the number of positive eigenvalues, minus the number of negative eigenvalues). We have also used the fact that \( C \) is invertible if \( x \) is non-degenerate [12, Lemma 3.2].

**Real Euler characteristic.** Similarly as with the Conley-Zehnder index, which induces the SFT-Euler characteristic, one can consider the Euler characteristic of the so-called local Lagrangian Floer homology of a symmetric orbit \( x \), when viewed as a chord. We call the resulting quantity, the real Euler characteristic \( \chi_L(x) \). More concretely, this works as follows. Before or after a bifurcation, one obtains a collection of non-degenerate symmetric orbits, for which computes the parity of the Maslov index \( \mu_L(x) \). By this, if \( \mu_L(x) = \frac{1}{2}m_L(x) \), we mean the parity of \( m_L(x) \in \mathbb{Z} \); note that \( m_L(x) \) is even if and only if \( \mu_L(x) \) is an integer. Note that, in practice, without needing to know the definition of this index, its parity can be determined from the following:
• The monodromy matrix;
• the formula \( \mu_L(x) = \mu_{CZ}(x) - s(x) \);
• Formula (3) (or (6)) of Theorem 3.1, which in particular gives the parity of \( s \);
• the table above giving the parity of the CZ-index (which is always an integer) in terms of the eigenvalue classification of the monodromy matrix.

The real Euler characteristic \( \chi_L(x) \) is then defined as

\[
\chi_L(x) = \sum_j (-1)^{\mu_L(x_j)} = \sum_i (-1)^{\mu_{CZ}(x_j)}(-1)^{-s(x_j)} = \sum_i i^{\mu_L(x_j)} \in \mathbb{C},
\]

where the sum runs over the collection \( x_j \) of non-degenerate chords arising after perturbation of \( x \). Note that by definition, \( \chi_L(x) \) is complex-valued. Its invariance under bifurcation follows from invariance of the local Lagrangian Floer homology of \( x \).

4. Non-symmetric orbits and branching structure

In this section, we derive the equations for plotting examples of non-symmetric orbits, and explain in more detail the relationship of the signs to the branching structure of the GIT quotients and the maps in the GIT sequence.

Recall the notion of \( \tilde{B} \)-signature, as explained in the Introduction. The sign of the two eigenvalues of \( A \) is what determines what branch of the “top” space \( Sp^T(4)/GL_2(\mathbb{R}) \) of the GIT sequence the monodromy matrix lies in. The branching structure of such space was completely determined in [10], and is depicted in Figure 3.

**Non-symmetric case.** As explained in the Introduction, the reduced monodromy matrix of any orbit at any point is a symplectic matrix \( M \) in \( Sp(4) \), and changing the base point only changes the matrix up to symplectic conjugation. This means that the element \( [[M]] \in Sp(4)//Sp(4) \) is independent on the choice of base point. Now, every symplectic matrix \( M \) is symplectically conjugated to some matrix of the form \( M_{A,B,C} \); this is a theorem of Wonenburger [24]. This means that \( [[M]] = [[M_{A,B,C}]] \) as points in the GIT quotient \( Sp(4)//Sp(4) \), and therefore \( [[M]] \) can be lifted to \( Sp^T(4)//GL_2(\mathbb{R}) \) as the point \( [M_{A,B,C}] \), i.e. the first map in the GIT sequence is surjective. However, the choice of \( M_{A,B,C} \) is in general not unique: there are as many as the covering degree of the map \( Sp^T(4)//GL_2(\mathbb{R}) \to Sp(4)//Sp(4) \) (which depends on the component of \( Sp(4)//Sp(4) \) containing \( [[M]] \); see Figure 4).

If the orbit is symmetric, finding a representative matrix \( M_{A,B,C} \) for the linearization \( M \) is easy: just linearize at a symmetric point. In general, if the orbit is non-symmetric, it might be difficult to do this. However, we only want to compute the determinant and trace of the first block of a representative \( M_{A,B,C} \) for \( M \) (and these are independent on the choice of \( M_{A,B,C} \)). To do this, we can appeal to [10, Lemma 3.2], which gives a relationship between the characteristic polynomial of \( M_{A,B,C} \) (which coincides with that of \( M \)), and that of \( A \). For the case \( n = 2 \), denoting \( p(t) = \det(M - tI) \) the characteristic polynomial of \( M \), this boils down to the relation

\[
p(t) = t^4 - 2\text{tr}(A)t^3 + 2(1 + 2\det A)t^2 - 2\text{tr}(A)t + 1.
\]

(7)

On the other hand, if \( \lambda_1, \lambda_2, \lambda_1^p, \lambda_2^p \) denote the pairs of eigenvalues of \( M \), these satisfy

\[
\lambda_1 \lambda_2 = \lambda_1^p \lambda_2^p = 1,
\]

(8)

and we have

\[
p(t) = (\lambda_1^p - t)(\lambda_2^p - t)(\lambda_1 - t)(\lambda_2 - t).
\]
Figure 3. This picture shows the branches of $\text{Sp}^I(4)/\text{GL}_2(\mathbb{R})$ and $\text{Sp}(4)/\text{Sp}(4)$, which are 2-dimensional “sheets” covering the different regions of the plane depicted in Figure 1 (we drop one dimension for visualization). The signs on each branch correspond to $B$-positivity/negativity of the corresponding eigenvalues (a priori there are 4 possibilities, since there are two eigenvalues). The first vignette shows how they come together when crossing from $E^2$ to $N$. On the second, when crossing from $H^-$ to $N$; the picture is the same for $H^{++}$ to $N$, and so on. All branches come together to a single point along each of the three singular points $2, 1), (0, -1), (-2, 1)$. The map $\text{Sp}^I(4)/\text{GL}_2(\mathbb{R}) \to \text{Sp}(4)/\text{Sp}(4)$ in the GIT sequence collapses branches together, as shown. For example, $B$-positivity/negativity over the hyperbolic eigenspace of matrices of type $EH^+$ is not invariant under symplectic conjugation, and hence the corresponding branches come together in $\text{Sp}(4)/\text{Sp}(4)$. 
Expanding, we obtain
\[ p(t) = t^4 - bt^3 + at^2 - bt + 1, \]
where \(a, b\) are elementary symmetric polynomials on the eigenvalues. Using Equations (7) and (8), we find that
\[ a = \lambda_1^2 \lambda_1^2 + \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_1^2 + \lambda_2^2 \lambda_2^2 + 2 = 2(1 + 2 \det(A)) \]
and
\[ b = \lambda_1^2 + \lambda_2^2 + \lambda_1^s + \lambda_2^s = \text{tr}(M) = 2\text{tr}(A). \]
We conclude that
\[ \det(A) = \frac{a}{4} - \frac{1}{2} \cdot \text{tr}(A) = \frac{b}{2} \]
which can be computed directly from the eigenvalues of \(M\) without need of conjugating \(M\) to be of the form \(M_{A,B,C}\), and may be plotted numerically.

Figure 4. The covering degree of the map \([M_{A,B,C}] \mapsto [[M_{A,B,C}]]\), i.e. the number of preimages of a point in \(\text{Sp}(4) // \text{Sp}(4)\), varies from component to component, as indicated in the figure. For example, if \(M\) is of type \(H^{++}\) or \(H^{--}\), then there are 4 choice of matrices \(M_{A,B,C}\) which are symplectically conjugated to \(M\), but which are distinct in \(\text{Sp}^T(4) // \text{GL}_2(\mathbb{R})\). Each branch has a sign attached to it; the positive/negative labels in our plots in the plane will record in which branch the orbits are moving.
Figure 5. (A) A simple symmetric orbit; we indicate the intersection with $L = \text{Fix}(\rho)$ (i.e. the symmetric points) with right angles. (B) A period doubling bifurcation, only intersecting $L$ near $x(0)$ (we also indicate the “fake” intersection points near $x(1/2)$). (C) The alternative version of (B), intersecting $L$ only near $x(1/2)$.

Remark 4.1. For numerical implementations, it is sometimes easier to work with the non-reduced $6 \times 6$ monodromy matrix, which always has 1 as an eigenvalue with multiplicity 2; see Appendix A below. Therefore its characteristic polynomial is of the form $q(t) = (1 - t)^2p(t)$, where $p$ is that of the reduced monodromy matrix.

Krein sign for non-symmetric orbits. For non-symmetric orbits, we do not work on the GIT quotient $Sp^+(4) // GL_2(\mathbb{R})$ (the “top” space) anymore, but we can still work with the GIT quotient $Sp(4) // Sp(4)$ (the “middle”). While we cannot assign a meaningful sign to every eigenvalue, we can still consider the signs coming from Krein theory for elliptic eigenvalues. The plus/minus labels in Figure 3 record precisely this sign, for the branches of $Sp(4) // Sp(4)$ lying over components of the plane with at least one $E$ label. Note that the first map in the sequence collapses together precisely those branches corresponding to hyperbolic eigenvalues, for which we cannot define a Krein sign.

5. Symmetric subtle division

Let us now discuss the interesting example of symmetric subtle division (i.e. period-doubling), in dimension four, in order to illustrate the use of the invariants and Krein-type signatures that we defined in the previous sections.

Symmetric subtle division. We study the case corresponding to [1, p. 599], but in the symmetric case. Consider a simple symmetric periodic orbit $x$ which intersects the Lagrangian fixed-point locus at time $t = 0$ and $t = 1/2$, and which belongs to a family whose reduced monodromy matrix goes from elliptic to negative hyperbolic; see Figure 5(A). As a simple orbit there is no bifurcation since there is no eigenvalue 1 in the reduced monodromy matrix. However, we can interpret this orbit as a chord from the Lagrangian to itself, where $x(0)$ happens to agree with $x(1)$; now, as a chord, it might bifurcate. While the Conley-Zehnder index does not jump (the eigenvalue 1 is not being crossed), such a chord bifurcation takes place if and only if the Maslov index jumps, which happens if and only if the Hörmander index jumps (being the difference of the Conley-Zehnder index and the Maslov index). If this happens, we can apply the symmetry again to the red chord in Figure 5(B)/(C) to obtain the green chord in the same figure. So, two chords bifurcate. This
is compatible with the real Euler characteristic. Indeed, the Maslov index of \( x \) before and after bifurcation (thought of as a chord) differ by one. The green and red chords have the same Maslov index, say \( k \), as they are symmetric to each other, and this coincides with that of \( x \) before bifurcation. This makes sure that the real Euler characteristic stays invariant. Indeed, before bifurcation we have \( \mu_{\text{CZ}}(x) = k \) and so \( \chi(x) = (-1)^k \); and after bifurcation, \( \mu_{\text{CZ}}(x) = k + 1 \), so \( \chi(x) = 2(-1)^k + (-1)^{k+1} = 2(-1)^k - (-1)^k = (-1)^k \), which explicitly shows invariance. Note that travelling through the red chord and then through the green chord gives another symmetric periodic orbit of double period (as expected in a period-doubling bifurcation, in which the double cover \( x^2 \) of \( x \) bifurcates); call it \( y \).

This can be understood within the framework provided by the GIT sequence. All monodromy matrices for different base-points along a periodic orbit are symplectically conjugated, and hence the two reduced monodromy matrices of \( x \) at \( t = 0 \) and \( t = 1/2 \) induce the same element in the GIT quotient \( \text{Sp}(2)/\text{Sp}(2) \). However, in \( \text{Sp}(2)^2/\text{GL}_1(\mathbb{R}) \) they differ, since not both Hörmander indices can jump. Therefore we can apply the refined Krein-type invariants from [10] that we discussed above, to decide in practice whether the period doubling bifurcations happen at \( t = 0 \) or \( t = 1/2 \), which is of help for the numerical analysis of these bifurcations. Namely, the \( B \)-signature jumps either at \( t = 0 \) or at \( t = 1/2 \). The period doubling bifurcations will be symmetric at the point where the \( B \)-sign did not jump (or equivalently, where the “\( C \)-sign” jumps; see Remark [11]). In particular, the \( B \)-signs of the negative hyperbolic critical point at the two different symmetric points of \( x \) have to differ after bifurcation, precisely since only one of the points of the double cover can be symmetric, while the other one is fake symmetric (analogously, the \( C \)-signs will also differ). Below, we will illustrate this with a numerical example, in the Jupiter-Europa system.

6. Numerics

In this section, we give examples of orbit bifurcations found numerically, and illustrate the use of the various invariants discussed above. The numerical method used is the cell-mapping method, as discussed at length in [16]. We will focus on the (circular, restricted) three-body problem, whose Hamiltonian is given by

\[
H : T^*\mathbb{R}^3 \setminus \{(M, P)\} = (\mathbb{R}^3 \setminus \{M\}) \times \mathbb{R}^3 \to \mathbb{R},
\]

\[
H(q, p) = \frac{1}{2} \|p\|^2 - \frac{\mu}{\|q - M\|} - \frac{1 - \mu}{\|q - P\|} + p_1q_2 - p_2q_1,
\]

where \( q = (q_1, q_2, q_3) \) is the position of the Satellite, \( p = (p_1, p_2, p_3) \) is its momentum, \( \mu \in (0, 1) \) is the mass of the secondary body \( M \), which is fixed at \( M = (\mu - 1, 0, 0) \), and \( 1 - \mu \) is the mass of the primary body \( P = (\mu, 0, 0) \). The Jacobi constant is then a fixed value \( C \) for \( H \). The Hamiltonian \( H \) is invariant under the anti-symplectic involutions

\[
\rho : (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, -q_2, -q_3, -p_1, p_2, p_3),
\]

\[
\tilde{\rho} : (q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (q_1, -q_2, q_3, -p_1, p_2, -p_3),
\]

with corresponding Lagrangian fixed-point loci given by

\[
\mathcal{L} = \text{Fix}(\rho) = \{q_2 = q_3 = p_1 = 0\},
\]

\[
\tilde{\mathcal{L}} = \text{Fix}(\tilde{\rho}) = \{q_2 = p_3 = p_1 = 0\}.
\]

We will then study orbits symmetric under these two symmetries, for specific cases of parameter \( \mu \).
Figure 6. The prograde planar orbit \( \gamma \) (before bifurcation) is the dotted orange line; the Jacobi constant is \( c = 3.00357414 \), and its period is \( T_0 = 2.1215 \). The spatial orbit \( \beta \) of double period (after bifurcation) is the blue one; the Jacobi constant is now \( c = 3.003571774 \), with period \( T_1 = 4.245 = 2T_0 \) (up to small error). We call this the “snitch” configuration.

We will look at two relevant systems: the Jupiter-Europa system, which corresponds to a circular restricted three-body problem with mass ratio \( \mu = 2.5266448850435e^{-5} \), and the Saturn-Enceladus system, corresponding to \( \mu = 1.9002485658670e^{-7} \).

Remark 6.1. In what follows, the numerics were carried out in long format in MATLAB, which is higher precision as that shown here, where for readability we have truncated to 6 digits after the decimal.

Jupiter-Europa system: period-doubling of prograde orbits \((H_2 \text{ family})\) [16]. As the Jacobi constant \( C \) decreases, the \( H_2 \) family orbit depicted in Figure 6 undergoes period-doubling, i.e. a spatial prograde orbit of double the period appears. We denote by \( \gamma_{bef} \) and \( \gamma_{aft} \) the simple orbit before and after bifurcation, and by \( \beta \) the orbit with double period appearing after bifurcation. This is a doubly symmetric subtle division, where all orbits are invariant under \( \rho \) and \( \tilde{\rho} \). We have, for each orbit, two \( \rho \)-symmetric points, where the orbit intersects \( L \); similarly, two \( \tilde{\rho} \)-symmetric ones, where the orbit intersects \( \tilde{L} \) (see Figure 7).

For \( \gamma_{bef} \), the symmetric points are numerically found to be

\[
P_1(\gamma_{bef}) = (1.016776, 0, 0, 0.0130372, 0),
\]

\[
P_2(\gamma_{bef}) = (0.997370, 0, 0, -0.125493, 0).
\]
Note that $P_1(\gamma_{bef})$, $P_2(\gamma_{bef}) \in L \cap \tilde{L}$, i.e. they are both $\rho$-symmetric and $\tilde{\rho}$-symmetric. The (non-reduced) monodromy matrix of $\gamma_{bef}$ at $P_1(\gamma_{bef})$, is numerically computed to be:

$$M_1(\gamma_{bef}) = \begin{pmatrix}
2.930464 & 1.567115 & 0 & 0.416572 & -0.859311 & 0 \\
-3.982191 & -2.232667 & 0 & -0.859311 & 1.772599 & 0 \\
0 & 0 & -0.999948 & 0 & 0 & 0.000320 \\
17.398921 & 6.866933 & 0 & 2.930464 & -3.982191 & 0 \\
6.866933 & 2.056350 & 0 & 1.567115 & -2.232667 & 0 \\
0 & 0 & -0.326763 & 0 & 0 & -0.999948
\end{pmatrix}$$

Up to small numerical rounding errors, $M_1(\gamma_{bef})$ is of the form $M_{A,B,C}$. The eigenvalues different from 1 (which always has with multiplicity 2), denoted $\lambda_p(\gamma_{bef})$ for planar, and $\lambda_s(\gamma_{bef})$ for spatial (cf. Appendix B), are

$$\lambda_p(\gamma_{bef}) = -0.302203 + i 0.953244, \quad \overline{\lambda_p(\gamma_{bef})} = -0.302203 - i 0.953244,$$

$$\lambda_s(\gamma_{bef}) = -0.999948 + 0.010225, \quad \overline{\lambda_s(\gamma_{bef})} = -0.999948 - 0.010225.$$

Both come in elliptic conjugate pairs. Similarly, the monodromy matrix of $\gamma_{bef}$ at $P_2(\gamma_{bef})$ is

$$M_2(\gamma_{bef}) = \begin{pmatrix}
-286.401882 & -9.995795 & 0 & 0.342004 & -9.788843 & 0 \\
8226.036901 & 287.100358 & 0 & -9.788864 & 280.176894 & 0 \\
0 & 0 & -0.999948 & 0 & 0 & 0.001776 \\
266456.859258 & 9329.998723 & 0 & -286.402124 & 8226.024432 & 0 \\
9329.998751 & 326.685501 & 0 & -9.995804 & 287.099922 & 0 \\
0 & 0 & -0.058878 & 0 & 0 & -0.999948
\end{pmatrix}$$

Again one sees that up small errors, this is of the form $M_{A,B,C}$. By construction, $M_2(\gamma_{bef})$ is symplectically conjugated to $M_1(\gamma_{bef})$, and hence their eigenvalues need agree (we have checked that this is indeed the case, again up to small error).

After bifurcation, the symmetric points of $\gamma_{aft}$ are

$$P_1(\gamma_{aft}) = (1.016787, 0, 0, 0, 0.013014, 0).$$
\[ P_2(\gamma_{aft}) = (0.997377, 0, 0, -0.125701, 0). \]

Again, \( P_1(\gamma_{aft}), P_2(\gamma_{aft}) \in L \cap \tilde{L} \) are doubly symmetric. The non-reduced monodromy matrix of \( \gamma_{aft} \) at \( P_1(\gamma_{aft}) \) is:

\[
M_1(\gamma_{aft}) = \begin{pmatrix}
2.921879 & 1.570836 & 0 & 0.412068 & -0.847786 \\
-3.954059 & -2.231824 & 0 & -0.847786 & 1.744227 \\
0 & 0 & -1.000378 & 0 & 0 \\
17.374784 & 6.880740 & 0 & 2.921879 & -3.954059 \\
6.880740 & 2.066818 & 0 & 1.570836 & -2.231824
\end{pmatrix},
\]

and that at \( P_2(\gamma_{aft}) \) is

\[
M_2(\gamma_{aft}) = \begin{pmatrix}
-290.249559 & -10.091019 & 0 & 0.343062 & -9.857004 \\
8368.287012 & 290.938771 & 0 & -9.856979 & 283.215067 \\
0 & 0 & -1.000378 & 0 & 0 \\
271816.526762 & 9480.654623 & 0 & -290.249258 & 8368.302559 \\
9480.654594 & 330.669844 & 0 & -10.091008 & 290.939312
\end{pmatrix}.
\]

The eigenvalues of \( M_1(\gamma_{aft}) \), which up to small error coincide with that of \( M_2(\gamma_{aft}) \), are

\[
\lambda_p(\gamma_{aft}) = -0.309945 + i0.950755, \quad \lambda_p(\gamma_{aft}) = -0.309945 - i0.950755,
\]

\[
\lambda_s(\gamma_{aft}) = -0.972874, \quad \frac{1}{\lambda_s(\gamma_{aft})} = -1.027883.
\]

Note that the planar eigenvalues stay elliptic, but the spatial ones are now a negative hyperbolic pair, as expected in a planar-to-spatial subtle division (this is explained in Appendix B). The spatial
The corresponding points: L lie in ones and which ones are the self-intersections of used, rather than look for intersections of this case we have two symmetries, things become rather interesting. So how can we tell? Note that it is unclear just by looking at the plot in Figure 8. However, since in this case we have two symmetries, things become rather interesting.

We can do the same analysis for the orbit \( \beta \) as we did for \( \gamma_{bef} \) and \( \gamma_{ aft} \). Now, the algorithm used, rather than look for intersections of \( \beta \) with the fixed-point loci, was implemented to look for self-intersections of \( \beta \). These are:

\[
P_1(\beta) = (0.997372, 2.557023 \times 10^{-17}, 0.000126, 2.09223 \times 10^{-9}, -0.125462, -2.29831 \times 10^{-9}),
\]

\[
P_2(\beta) = (0.997372, 1.768865 \times 10^{-15}, -0.000126, 2.230434 \times 10^{-10}, -0.125462, 1.327282 \times 10^{-8}),
\]

\[
P_3(\beta) = (1.016772, -3.693284 \times 10^{-16}, -3.907861 \times 10^{-10}, 2.316050 \times 10^{-9}, 0.013029, -0.001706),
\]

\[
P_4(\beta) = (1.016772, -4.489275 \times 10^{-20}, 1.590107 \times 10^{-9}, -1.587443 \times 10^{-9}, 0.013029, 0.001706).
\]

But note that one cannot tell simply by inspection which of the above points are the \( \rho \)-symmetric ones and which ones are the \( \rho \)-fake ones (and similarly for \( \tilde{\rho} \)), as they are very close to points which lie in \( L \) (resp. \( \tilde{L} \), and there is numerical error involved. For this, we compute the linearizations at the corresponding points:

\[
M_1(\beta) = \begin{pmatrix}
-395.432864 & -13.811767 & 17.771296 & -0.211344 & 6.057227 & 0.014285 \\
11341.304295 & 396.130733 & -508.406671 & 6.057163 & -173.608235 & -0.414298 \\
12.283601 & 0.426822 & 0.450370 & 0.014285 & -0.414300 & -0.004380 \\
862435.117023 & 30173.612241 & -38842.035995 & -395.434620 & 11341.21522 & 12.283685 \\
30173.612275 & 1055.671763 & -1358.951208 & -13.811828 & 396.127616 & 0.426825 \\
-38842.035970 & -1358.951205 & 1749.703885 & 17.771376 & -508.402659 & 0.450367
\end{pmatrix}
\]

\[
M_2(\beta) = \begin{pmatrix}
-395.431847 & -13.811731 & -17.771251 & -0.211345 & 6.05724 & -0.014285 \\
11341.356017 & 396.132542 & 508.409001 & 6.057139 & -173.607555 & 0.414297 \\
-12.283564 & -0.426821 & 0.450372 & -0.014285 & 0.414301 & 0.004380 \\
862435.117287 & 30173.612173 & 38842.036019 & -395.434622 & 11341.215308 & 12.283680 \\
30173.612368 & 1055.671764 & 1358.951212 & -13.811828 & 396.127619 & -0.426825 \\
38842.035946 & 1358.951201 & 1749.703883 & -17.771376 & 508.402662 & 0.450367
\end{pmatrix}
\]
Now, \( M_1(\beta), M_2(\beta) \) are, up to small error, of the form \( M_{A,B,C} \); whereas \( M_3(\beta), M_4(\beta) \) are not (note e.g. that the \( A \)-blocks and the \( D \)-blocks are not the transpose of each other, as there are differences in sign). We then conclude that \( P_1(\beta), P_2(\beta) \) are the \( \rho \)-symmetric points, whereas \( P_3(\beta), P_4(\beta) \) are \( \rho \)-fake ones. However, these matrices implicitly assume the choice of basis, and we have chosen the basis so that \( \rho \) is the standard antisymplectic involution. The roles are reversed after a change of basis for which \( \tilde{\rho} \) becomes the standard such involution (as explained in Appendix C). After this change, one sees that \( P_1(\beta), P_2(\beta) \) are \( \tilde{\rho} \)-fake ones, and \( P_3(\beta), P_4(\beta) \) are the \( \tilde{\rho} \)-symmetric ones. So, for the perspective of \( \tilde{\rho} \), bifurcation happened at \( P_1(\beta), P_2(\beta) \), whereas for the perspective of \( \rho \), it happened at \( P_3(\beta), P_4(\beta) \). This rather cumbersome situation is an artifact of the fact that \( \gamma \) is doubly symmetric.

The eigenvalues of \( M_1(\beta) \) (which agree with those of \( M_j(\beta) \) up to small error for all \( j \)) are the two elliptic conjugate pairs

\[
\lambda_p(\beta) = 0.965396 + i0.260789, \quad \bar{\lambda}_p(\beta) = 0.965396 - i0.260789,
\]

\[
\lambda_s(\beta) = -0.819634 + i0.572887, \quad \bar{\lambda}_s(\beta) = -0.819634 - i0.572887.
\]

This is of course compatible with the general discussion of symmetric subtle division of Section 6.

**Jupiter-Europa: Period tripling of prograde orbit (H2 family) [16].** The following is an example of a period tripling bifurcation of a \( H_2 \) family orbit \( \gamma \), which again is doubly symmetric; see Figure 10. The 3-fold cover \( \gamma^3 \) is of type \( E^2 \), and bifurcates into four orbits \( \gamma_1, \ldots, \gamma_4 \), related by the symmetries \( \bar{\rho}(\gamma_1) = \gamma_2, \bar{\rho}(\gamma_3) = \gamma_4, \rho(\gamma_1) = \gamma_3, \rho(\gamma_2) = \gamma_4 \). The orbits \( \gamma_1, \gamma_2 \) are in \( E^2 \), and bifurcate from the symmetric point of \( \gamma \) corresponding to \( \tilde{\rho} \); the orbits \( \gamma_3, \gamma_4 \), in \( E^H \), and bifurcate from the symmetric point corresponding to \( \rho \). This is compatible with the SFT-Euler characteristic: indeed, before bifurcation there is only the 3-fold cover and so we have \( \chi_{SFT}(\gamma^3) = 1 \). After bifurcation, the contributions of \( \gamma_1, \gamma_2 \) is 2, which cancels that of \( \gamma_3, \gamma_4 \), which is \(-2\); and we still have the contribution of the 3-fold cover, which is 1. So we again see that \( \chi_{SFT}(\gamma^3) = 1 \) after bifurcation. This is also compatible with the real Euler characteristic: none of the \( \gamma_i \) are symmetric, while \( \gamma^3 \) is; therefore \( \chi_L(\gamma^3) = (-1)^{\mu_L(\gamma^3)} \) before and after.

6.1. Numerical plots in the GIT quotient. In the following, we illustrate the numerical use of the GIT quotients via numerical plots.

**Snitch configuration.** We again consider the snitch configuration in the Jupiter-Europa system. Figure 11 shows a numerical plot of this period-doubling bifurcation, as seen in the base \( \mathbb{R}^2 \) of
Figure 10. Left: the first period-tripling family $\gamma_1$ after bifurcation. Applying $\tilde{\rho}$, we obtain the family $\gamma_2$. Right: The second period-tripling family $\gamma_3$ after bifurcation. Applying $\rho$, we obtain its symmetric version $\gamma_4$.

Figure 11. GIT pot of the period-doubling bifurcation of the snitch configuration.

The GIT sequence, in three different scales. The time parameter is the Jacobi constant. Red dots correspond to $\gamma_{bef}$, and blue dots, to $\gamma_{aft}$. The bifurcation takes place when the period-doubling branch locus separating the doubly-elliptic region $E^2$ and the elliptic-negative hyperbolic region $EH^-$ is crossed. The plot also contains the $B$-signature of the simple orbit, before and after the bifurcation, which were computed as follows.
The $A^T$-block of $M_1(\gamma_{bf})$, at $P_1(\gamma_{bf})$, is given by

$$A_1^T(\gamma_{bf}) = \begin{pmatrix} 2.930464 & -3.982191 & 0 \\ 1.567115 & -2.232667 & 0 \\ 0 & 0 & -0.999948 \end{pmatrix}$$

Its two non-trivial eigenvalues are

$$\mu_1(\gamma_{bf}) = -0.999948, \quad \mu_2(\gamma_{bf}) = -0.302203,$$

ordered so that $\mu_1(\gamma_{bf}) < \mu_2(\gamma_{bf})$, with corresponding eigenvectors

$$v_1(P_1(\gamma_{bf})) = (0, 0, 1), \quad v_2(P_1(\gamma_{bf})) = (0.776387, 0.630256, 0).$$

Using the $B$-block of $M_1(\gamma_{bf})$, given by

$$B_1(\gamma_{bf}) = \begin{pmatrix} 0.416572 & -0.859311 & 0 \\ -0.859311 & 1.772599 & 0 \\ 0 & 0 & 0.00032 \end{pmatrix},$$

we simply compute

$$\epsilon_1(P_1(\gamma_{bf})) = \text{sign}(v_1^T(P_1(\gamma_{bf})) \cdot B_1(\gamma_{bf}) \cdot v_1(P_1(\gamma_{bf}))) = \text{sign}(0.00032) = +,$$

$$\epsilon_2(P_1(\gamma_{bf})) = \text{sign}(v_2^T(P_1(\gamma_{bf})) \cdot B_1(\gamma_{bf}) \cdot v_2(P_1(\gamma_{bf}))) = \text{sign}(0.114256) = +,$$

and so the $B$-signature before bifurcation at the symmetric point $P_1(\gamma_{bf})$ is $\epsilon(P_1(\gamma_{bf})) = (+, +)$ (as depicted in Figure 11). The same procedure applied to $M_1(\gamma_{af})$ gives

$$\mu_1(P_1(\gamma_{af})) = -1.00038, \quad \mu_2(P_1(\gamma_{af})) = -0.309942,$$

with corresponding eigenvectors

$$v_1(P_1(\gamma_{af})) = (0, 0, 1), \quad v_2(P_1(\gamma_{af})) = (0.774275, 0.632849, 0),$$

and corresponding $B$-signs

$$\epsilon_1(P_1(\gamma_{af})) = \text{sign}(v_1^T(P_1(\gamma_{af})) \cdot B_1(\gamma_{af}) \cdot v_1(P_1(\gamma_{af}))) = \text{sign}(-0.00245) = -, $$

$$\epsilon_2(P_1(\gamma_{af})) = \text{sign}(v_2^T(P_1(\gamma_{af})) \cdot B_1(\gamma_{af}) \cdot v_2(P_1(\gamma_{af}))) = \text{sign}(0.114766) = +,$$

and so the $B$-signature after bifurcation at the symmetric point $P_1(\gamma_{af})$ is $\epsilon(P_1(\gamma_{af})) = (-, +)$ (also depicted in Figure 11).

We can also check, explicitly in this example, the fact, alluded to in the general discussion of Section 5, that the $B$-signature at different symmetric points of will differ after bifurcation in a symmetric subtle division. Indeed, replacing $P_1$ by $P_2$, the eigenvalues of the corresponding $A^T$-blocks $A_2^T(\gamma_{bf})$, $A_2^T(\gamma_{af})$ need respectively coincide with the $\mu_i(\gamma_{bf}), \mu_i(\gamma_{af})$ for $i = 1, 2$ (checked up to numerical error), and the corresponding eigenvectors are

$$v_1(P_2(\gamma_{bf})) = (0, 0, 1), \quad v_2(P_2(\gamma_{bf})) = (-0.999396, -0.0347591, 0),$$

$$v_1(P_2(\gamma_{af})) = (0, 0, 1), \quad v_2(P_2(\gamma_{af})) = (-0.9994, -0.0346264, 0),$$

with associated $B$-signs

$$\epsilon_1(P_2(\gamma_{bf})) = \text{sign}(v_1^T(P_2(\gamma_{bf})) \cdot B_2(\gamma_{bf}) \cdot v_1(P_2(\gamma_{bf}))) = \text{sign}(0.001776) = +,$$

$$\epsilon_2(P_2(\gamma_{bf})) = \text{sign}(v_2^T(P_2(\gamma_{bf})) \cdot B_2(\gamma_{bf}) \cdot v_2(P_2(\gamma_{bf}))) = \text{sign}(6.86473 \times 10^{-6}) = +,$$

$$\epsilon_1(P_2(\gamma_{af})) = \text{sign}(v_1^T(P_2(\gamma_{af})) \cdot B_2(\gamma_{af}) \cdot v_1(P_2(\gamma_{af}))) = \text{sign}(0.001672) = +,$$

$$\epsilon_2(P_2(\gamma_{af})) = \text{sign}(v_2^T(P_2(\gamma_{af})) \cdot B_2(\gamma_{af}) \cdot v_2(P_2(\gamma_{af}))) = \text{sign}(7.10883 \times 10^{-6}) = +.$$

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We see that the $B$-signatures are $\epsilon(P_2(\gamma_{be})) = \epsilon(P_2(\gamma_{af})) = (+,+)$, and thererfore the $B$-sign of the eigenvalue $\mu_1$ (the one undergoing bifurcation) indeed differs after bifurcation, for different choices of symmetric point. As pointed out in the general case, the fact that there is a sign jump at $P_1$ and not at $P_2$ indicates that $P_2$ gives rise to the $\rho$-symmetric points, and not $P_1$, where $\rho$ is the involution which is standard in the current choice of basis.

We conclude this section with a series of plots, including examples in the Saturn-Enceladus system, some of which are also discussed in [16] and [17].
FIGURE 12. Jupiter-Europa system: a symmetric planar to planar period-tripling bifurcation of DRO family. Above: the planar simple orbit at bifurcation, the distant retrograde orbit, $c = 2.9999$, $T_0 = 2.504$. Middle: the planar triple period orbit after bifurcation $c \gtrsim 2.9999$, $T = 7.3 \approx 3T_0$. Below: GIT plot, including $B$-signs.
Figure 13. Jupiter-Europa system: a symmetric planar to spatial 5-fold bifurcation of DRO family. Above: the planar simple orbit at bifurcation, a distant retrograde orbit in the same family of Figure 12 but with $c = 3.0005$, $T_0 = 1.705$. Middle: the spatial 5-fold period orbit after bifurcation, $c \gtrsim 3.0005$, $T = 8.52 \approx 5T_0$. Below: GIT plot, including $B$-signs.
Figure 14. Jupiter-Europa system: a symmetric spatial to spatial period-doubling bifurcation. Above: the spatial simple orbit at bifurcation, $c = 3.0028$, $T_0 = 4.62$. Middle: the spatial period-doubling orbit after bifurcation $c \gtrsim 3.0028$, $T = 9.23 \approx 2T_0$. Below: GIT plot, including $B$-signs.
FIGURE 15. Saturn-Enceladus system: two symmetric planar to spatial bifurcations of the same family of planar orbits, one period-doubling, and one period-tripling. Above: $T_1 = 1.2, T_2 = 1.6, T_3 = 2$ respectively. Middle: $T = 4.2 \approx 2T_3$ after bifurcation. Below: $T = 4.85 \approx 3T_2$ after bifurcation.
APPENDIX A. REDUCED VS NONREDUCED MONODROMY MATRICES

In this appendix, we explain how to go from the nonreduced monodromy matrix to its reduced version, and vice versa, for the purposes of plotting in the GIT quotient. In practice, it is much simpler and computationally cheaper to work with the nonreduced matrix, at the time of producing plots.

Let us consider the unreduced monodromy matrix \( M \) of a symmetric periodic orbit at a symmetric point. This is a \( 6 \times 6 \) symplectic matrix. We denote its left upper \( 3 \times 3 \) block by \( A \). The reduced monodromy matrix is a \( 4 \times 4 \) symplectic matrix \( M_r \), whose left upper \( 2 \times 2 \) block we denote by \( A_r \).

If \( \mu_1 \) and \( \mu_2 \) are the eigenvalues of \( A_r \), the eigenvalues of \( A \) are \( \mu_1, \mu_2 \) and 1. In particular, we have the following relation between the determinants and traces:

\[
\det A = 1 \cdot \mu_1 \cdot \mu_2 = \mu_1 \cdot \mu_2 = \det A_r, \quad \text{tr}(A) = 1 + \mu_1 + \mu_2 = 1 + \text{tr}(A_r).
\]

The parameters in our GIT quotient are then

\[
\delta = \det(A_r) = \det(A), \quad \tau = \text{tr}(A_r) = \text{tr}(A) - 1.
\]

The eigenvalues \( \mu_1 \) and \( \mu_2 \) of the reduced monodromy matrix are determined by its trace and determinant via

\[
\mu_{1,2} = \frac{1}{2} \left( \tau \pm \sqrt{\tau^2 - 4\delta} \right).
\]

We can express them as well using the determinant and trace of the unreduced monodromy matrix as follows

\[
\mu_{1,2} = \frac{1}{2} \left( \text{tr}(A) - 1 \pm \sqrt{(\text{tr}(A) - 1)^2 - 4\det(A)} \right).
\]

Note that it is rather remarkable that it is possible to express the eigenvalues of \( A \) just in terms of its trace and determinant, since this is in general not possible for a \( 3 \times 3 \) matrix. However, in our case we are taking advantage of the fact that we already know that 1 is another eigenvalue of \( A \).

The eigenvalues of the reduced monodromy matrix \( M_r \) can be obtained from \( \mu_1 \) and \( \mu_2 \) as follows

\[
\mu_1 \pm \sqrt{\mu_1^2 - 1}, \quad \mu_2 \pm \sqrt{\mu_2^2 - 1}.
\]

Combining with the expressions for \( \mu_{1,2} \), this allows us to compute the eigenvalues of the reduced monodromy matrix directly from the trace and determinant of the matrix \( A \). The unreduced monodromy matrix \( M \) has then

\[
1 \pm \sqrt{1^2 - 1} = 1
\]

as an additional eigenvalue of algebraic multiplicity two.

APPENDIX B. PLANAR ORBITS AND SPATIAL VS. PLANAR BIFURCATIONS

In this appendix, we document the standard fact of how to understand planar-to-planar or planar-to-spatial bifurcations in terms of the spectrum of the relevant matrix.

Recall that the planar problem \( \{ p_3 = q_3 = 0 \} \) is an invariant subset for the restricted three-body problem. Given a point \( (q, p) \) in the 4-dimensional phase-space \( \mathbb{R}^4 \) corresponding to the planar problem, one can view this point as sitting in the 6-dimensional phase-space \( \mathbb{R}^6 \) corresponding to the spatial problem, by adding two zeroes. One then can choose a symplectic basis of the tangent space of \( \mathbb{R}^6 = \mathbb{R}^4 \oplus \mathbb{R}^2 \) along this point, consisting of vectors spanning the first 4-dimensional factor, together with two extra vectors which complete the basis and span the symplectic complement \( \mathbb{R}^2 \) to the planar problem.
This implies the following: given a planar orbit, the linearization of the spatial problem along this orbit, with respect to a choice of basis of $\mathbb{R}^6$ as explained above, gives a $6 \times 6$ symplectic matrix $M$ which splits into a block form of a $4 \times 4$ symplectic matrix $M_p$ (the planar part), and a $2 \times 2$ one $M_s$ (the spatial part). Note that $M_p$ has 1 as an eigenvalue with multiplicity two. We call the remaining eigenvalues of $M_p$, the planar ones, denoted $\lambda_p, \lambda_p^{-1}$, and those of $M_s$, the spatial ones, denoted $\lambda_s, \lambda_s^{-1}$. The spectrum of $M$ is then $\{1, 1, \lambda_p, \lambda_p^{-1}, \lambda_s, \lambda_s^{-1}\}$, where $\{\lambda_p, \lambda_p^{-1}, \lambda_s, \lambda_s^{-1}\}$ is the spectrum of the reduced symplectic matrix $M_r$. Bifurcations of the planar eigenvalues (i.e. crossings of the eigenvalue 1) correspond to bifurcation of the orbit through planar orbits, i.e. planar-to-planar bifurcations; bifurcations of the spatial ones, correspond to planar-to-spatial bifurcations.

APPENDIX C. BASIS CHANGES

In this appendix, we review some basic conventions concerning the Lagrangian and Hamiltonian coordinates systems, and the change of basis relating each other. These conventions are useful to keep in mind when checking the accuracy of numerical work.

Consider the restricted three-body problem in a rotating frame, where the primaries are at rest. We assume that in the original inertial frame the primaries were rotating clockwise around their common center of mass, so that our frame is rotating counterclockwise. We choose our time unit such that the angular frequency of the rotation is equal to one. We consider the monodromy matrix of a periodic orbit. While the similarity class of the monodromy matrix only depends on the periodic orbit the monodromy matrix itself depends additionally on the starting point on the periodic orbit as well as the choice of coordinates. In symplectic coordinates the monodromy matrix is a symplectic matrix. In the unregularized spatial restricted three-body problem we have two natural sets of global coordinates. The first set are the following Lagrangian coordinates

$$\mathcal{L} = [x, y, z, \dot{x}, \dot{y}, \dot{z}],$$

while the second one or the following Hamiltonian coordinates

$$\mathcal{H} = [x, y, z, p_x, p_y, p_z].$$

The coordinates $\mathcal{H}$ are symplectic while the coordinates $\mathcal{L}$ are not. Therefore the monodromy matrix $M_{\mathcal{L}}$ with respect to the Lagrangian coordinates is in general not symplectic. However, it is conjugated to a symplectic matrix and therefore shares many properties of a symplectic matrix, like having determinant one and the fact that if $\lambda$ is an eigenvalue than so are $\lambda^{-1}$, $\overline{\lambda}$, and $\overline{\lambda}^{-1}$. By our assumptions on the rotating frame, the two coordinate systems are related to each other by the following linear transformation

$$p_x = \dot{x} - y, \quad p_y = \dot{y} + x, \quad p_z = \dot{z}.$$

The basis change matrix from the coordinates $\mathcal{H}$ to the coordinates $\mathcal{L}$ is therefore

$$P_{\mathcal{L}\mathcal{H}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$
The coordinate change matrix from the Lagrangian coordinates $L$ to the Hamiltonian coordinates $H$ is then given by

$$P^H_L = (P^L_H)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$ 

Therefore the monodromy matrix $M_L$ with respect to the coordinates $L$ and the monodromy matrix $M_H$ with respect to the coordinates $H$ are related to each other by

$$M_H = P^H_L M_L P^L_H.$$ 

In particular, $M_H$ and $M_L$ have the same characteristic polynomial, but $M_H$ is symplectic while $M_L$ in general is not.

If one writes a $2n \times 2n$-symplectic matrix $M$ into four blocks of $n \times n$-matrices

$$M = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix},$$

then the blocks satisfy the equations

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I$$

where $I$ is the $n \times n$-identity matrix. Moreover, the inverse of $M$ is given by

$$M^{-1} = \begin{pmatrix}
D^T & -B^T \\
-C^T & A^T
\end{pmatrix}.$$ 

In particular, the monodromy matrix $M_H$ has to satisfy this for $n = 3$, which gives the opportunity to double check the accuracy of numerical computations.

The restricted three-body problem is invariant under various symmetries. Given an orbit of the restricted three-body problem one obtains another orbit by combining one of the following transformations

$$(x, y, z, \dot{x}, \dot{y}, \dot{z}) \rightarrow (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}), \quad (x, y, z, \dot{x}, \dot{y}, \dot{z}) \rightarrow (x, -y, -z, -\dot{x}, \dot{y}, \dot{z})$$

with time-reversal. The above two transformation give rise to the following two linear antisymplectic involutions on phase space

$$\rho_1: T^*\mathbb{R}^3 \rightarrow T^*\mathbb{R}^3, \quad (x, y, z, p_x, p_y, p_z) \mapsto (x, -y, z, -p_x, p_y, -p_z)$$

and

$$\rho_2: T^*\mathbb{R}^3 \rightarrow T^*\mathbb{R}^3, \quad (x, y, z, p_x, p_y, p_z) \mapsto (x, -y, -z, -p_x, p_y, p_z).$$

The fixed point sets

$$L_1 = \text{Fix}(\rho_1) = \{(x, y, z, p_x, p_y, p_z) \in T^*\mathbb{R}^3 : y = p_x = p_z = 0\}$$

and

$$L_2 = \text{Fix}(\rho_2) = \{(x, y, z, p_x, p_y, p_z) \in T^*\mathbb{R}^3 : y = z = p_x = 0\}$$

are Lagrangian subspaces of $T^*\mathbb{R}^3$. They coincide with the eigenspaces of the two antisymplectic involutions with respect to the eigenvalue one. Moreover, the eigenspaces of the two antisymplectic
involutions to the eigenvalue minus one are Lagrangian subspaces of $T^*\mathbb{R}^3$ as well. They are given by

$$L_1^\perp = \{(x, y, z, p_x, p_y, p_z) \in T^*\mathbb{R}^3 : x = z = p_y = 0\}$$

and

$$L_2^\perp = \{(x, y, z, p_x, p_y, p_z) \in T^*\mathbb{R}^3 : x = p_y = p_z = 0\}.$$

In particular, we have the following two Lagrangian splittings

$$T^*\mathbb{R}^3 = L_1 \oplus L_1^\perp, \quad T^*\mathbb{R}^3 = L_2 \oplus L_2^\perp.$$

In the restricted three-body problem we have two kinds of symmetric periodic orbits, namely periodic orbits which are invariant under the composition of the antisymplectic involution $\rho_1$ and time reversal and periodic orbits which are invariant under the composition of $\rho_2$ with time reversal. We begin with the first case. Such periodic orbits will pass at two different points through the fixed point set $L_1$ of $\rho_1$. The consecutive times where the orbit passes the fixed point set differ by half the period of the periodic orbit. We choose one of the intersection points of the periodic orbit with $L_1$ and consider the monodromy matrix at this point. The basis $\mathcal{B}$ is not so well compatible with the symmetry. To make the monodromy matrix more compatible with the symmetry we consider a different symplectic basis, namely

$$\mathcal{G}_1 = [x, p_y, z, p_x, -y, p_z].$$

Different from the basis $\mathcal{B}$, the first three basis vectors of $\mathcal{G}_1$ build a basis of $L_1$, while the last three basis vectors build a basis of $L_1^\perp$. The basis change matrix from the basis $\mathcal{G}_1$ to the basis $\mathcal{B}$ is the symplectic matrix

$$P_{\mathcal{G}_1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

and the basis change matrix from $\mathcal{B}$ to $\mathcal{G}_1$ reads

$$P_{\mathcal{B}}^{\mathcal{G}_1} = (P_{\mathcal{B}}^{\mathcal{G}_1})^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

With the help of these matrices, the monodromy matrix $M_\mathcal{B}$ with respect to the basis $\mathcal{B}$ is related to the monodromy matrix $M_{\mathcal{G}_1}$ with respect to the basis $\mathcal{G}_1$ by

$$M_{\mathcal{G}_1} = P_{\mathcal{B}}^{\mathcal{G}_1} M_\mathcal{B} P_{\mathcal{G}_1}^{\mathcal{B}}.$$

The monodromy matrix $M = M_{\mathcal{G}_1}$ is symplectic again, since the basis $\mathcal{G}_1$ is still symplectic. However, in the basis $\mathcal{G}_1$, the antisymplectic involution $\rho_1$ is represented by the matrix

$$R = \begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix}.$$
The fact that \( M \) is the monodromy matrix of a symmetric periodic orbit at a fixpoint of the anti-symplectic involution \( \rho_1 \) translates into

\[
RM^R = M^{-1},
\]

which in view of (10) translates into

\[
\begin{pmatrix}
A & -B \\
-C & D
\end{pmatrix} =
\begin{pmatrix}
D^T & -B^T \\
-C^T & A^T
\end{pmatrix}.
\]

Therefore we have

\[
D = A^T, \quad B = B^T, \quad C = C^T,
\]

i.e., the matrices \( B \) and \( C \) are symmetric and \( D \) is just the transpose of \( A \). In particular, (11) simplifies to

\[
AB = BA^T, \quad CA = A^TC, \quad A^2 - BC = I.
\]

A similar phenomenon happens for periodic orbits which are invariant under the composition of the antisymplectic involution \( \rho_2 \) and time reversal. The only difference is that in this case the required basis change has to be adjusted. We assume that \( M_H \) is the monodromy matrix for such a symmetric periodic orbit at a point in \( L_2 \) the fixed point set of the antisymplectic involution \( \rho_2 \). As a new symplectic basis compatible with the antisymplectic involution \( \rho_2 \) we choose

\[
\mathcal{S}_2 = [x, p_y, p_z, p_x, -y, -z].
\]

Note that the first three basis vectors build a basis of the Lagrangian subspace \( L_2 \) and the last three basis vectors build a basis of the Lagrangian subspace \( L_2^\perp \). As basis change matrix from the basis \( \mathcal{S}_2 \) to the basis \( \mathcal{H} \) we obtain the symplectic matrix

\[
P_{\mathcal{H}}^{\mathcal{S}_2} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

with inverse

\[
P_{\mathcal{S}_2}^{\mathcal{H}} = (P_{\mathcal{H}}^{\mathcal{S}_2})^{-1} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}.
\]

If we consider the monodromy matrix with respect to the basis \( \mathcal{S}_2 \)

\[
M_{\mathcal{S}_2} = P_{\mathcal{H}}^{\mathcal{S}_2} M_H P_{\mathcal{S}_2}^{\mathcal{H}}
\]

its blocks again satisfy (11) and (12).
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(Urs Frauenfelder) INSTITUT FÜR MATHEMATIK, AUGSBURG UNIVERSITÄT, AUGSBURG, GERMANY
Email address: urs.frauenfelder@math.uni-augsburg.de

(Dayung Koh) PASADENA, CALIFORNIA, USA
Email address: dayung.koh@gmail.com

(Agustin Moreno) SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, USA,
MATHEMATISCHES INSTITUT, UNIVERSITÄT HEIDELBERG, HEIDELBERG, GERMANY
Email address: agustin.moreno2191@gmail.com