Structural Solutions For Additively Coupled Sum Constrained Games

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Abstract

We propose and analyze a broad family of games played by resource-constrained players, which are characterized by the following central features: 1) each user has a multi-dimensional action space, subject to a single sum resource constraint; 2) each user’s utility in a particular dimension depends on an additive coupling between the user’s action in the same dimension and the actions of the other users; and 3) each user’s total utility is the sum of the utilities obtained in each dimension. Familiar examples of such multi-user environments in communication systems include power control over frequency-selective Gaussian interference channels and flow control in Jackson networks. In settings where users cannot exchange messages in real-time, we study how users can adjust their actions based on their local observations. We derive sufficient conditions under which a unique Nash equilibrium exists and the best-response algorithm converges globally and linearly to the Nash equilibrium. In settings where users can exchange messages in real-time, we focus on user choices that optimize the overall utility. We provide the convergence conditions of two distributed action update mechanisms, gradient play and Jacobi update.

Index Terms

Game theory, multi-user communications, Nash equilibrium, best-response dynamics, Gradient play, Jacobi update, pricing mechanism.

I. INTRODUCTION

Game theory provides a formal framework for describing and analyzing the interactions of multiple decision-makers. Recently, there has been a surge in research activities that adopt game theoretic tools to investigate a wide range of modern communications and networking problems, such as flow and congestion control, network routing, load balancing, power control, peer-to-peer content sharing, etc [1]-[5]. In resource-constrained communication networks, a user’s utility is usually not only affected by
its own action but also by the actions taken by all the other users sharing the same resources. Due to
the mutual coupling among users, the performance optimization of multi-user communication systems is
challenging. Depending on the characteristics of different applications, numerous game-theoretical models
and solution concepts have been proposed to characterize the multi-user interactions and optimize the
users’ decisions in communication networks. A variety of game theoretic solutions have been developed
to characterize the resulting performance of the multi-user interactions, including Nash equilibrium (NE)
and Pareto optimality [6].

The majority of the existing game theoretic research works in communication networking applications
usually depend on the specific structures and inter-user coupling of their action sets and utility functions.
By considering or even architecting these specific structures, the associated games become analytically
tractable and possess various important convergence properties. For instance, if users cannot exchange
messages with each other and choose to individually maximize their utilities, to show the existence of and
the convergence to a pure NE, several well-investigated classes of game models, such as concave games,
supermodular games, and potential games, have been extensively applied in various communication
scenarios [6] - [13]. When real-time information exchange is possible, various mechanisms have also
been proposed to enable collaborative users to jointly improve their performance and find the optimum
joint policy. A well-known example is the framework of network utility maximization (NUM) started by
Kelly etc. [14] [15], which has recently been widely adopted to analyze the problems related to fairness
and efficiency in communication networks. Moreover, various distributed resource allocation algorithms
have been developed to implement the NUM framework in an informationally-decentralized manner.
In particular, if a convex NUM problem can be decomposed into several subproblems by introducing
Lagrange multipliers associated with different resource constraints, the global optimum can be computed
using distributed algorithms by deploying message passing mechanisms [16].

Power control is one of the first few communication problems in which researchers started to apply
game theoretic tools to formalize the multi-user interaction and characterize its properties. An interesting
and important topic that has been extensively investigated recently is how to optimize multiple devices’
power allocation when sharing a common frequency-selective interference channel. In [17], Yu et. al.
first defined such a power control game from a game-theoretic perspective, proposed a best-response
algorithm in which all users iteratively update their power allocations using the water-filling solution,
and proved several sufficient conditions under which the algorithm globally converges to a unique pure
NE. Many follow-up papers further establish various sufficient convergence conditions with or without
real-time information exchange for power control in communication networks [18] - [22]. The purpose of
this paper is to introduce and analyze a general framework that abstracts the common characteristics of this family of multi-user interaction scenarios, which includes, but is not limited to, the power control scenario. In particular, the main contributions of this paper are as follows.

First of all, we define the class of Additively Coupled Sum Constrained Games (ACSCG), which captures and characterizes the key features of several communication and networking applications. In particular, the central features of ACSCG are: 1) each user has a multi-dimensional strategy that is subject to a single sum resource constraint; 2) each user’s payoff in each dimension is impacted by an additive combination of its own action in the same dimension and a function of the other users’ actions; 3) users’ utilities are separable across different dimensions and each user’s total utility is the sum of the utilities obtained within each dimension.

Second, based on the feasibility of real-time information exchange, we provide the convergence conditions of various generic distributed algorithms in different scenarios. When no message exchanges between users are possible and every user maximizes its own utility, it is essential to determine whether a NE exist and if yes, how to achieve such an equilibrium. In ACSCG, a pure NE exists in ACSCG because ACSCG belongs to concave games [6] [7]. Our key contribution in this context is that we investigate the uniqueness of pure NE and consider the best response dynamics to compute the NE. We explore the properties of the additive coupling among users given the sum constraint and provide several sufficient conditions under which best response dynamics converges linearly \(^1\) to the unique NE, for any set of feasible initialization with either sequential or parallel updates. We also explain the relationship between our results and the conditions previously developed in the game theory literature [7] [24]. When users can collaboratively exchange messages with each other in real-time, we present the sufficient convergence conditions of two alternative distributed pricing algorithms, including gradient play and Jacobi update, to coordinate users’ action and improve the overall system efficiency. The proposed convergence conditions generalize the results that have been previously obtained in [17]- [22] for the multi-user power control problem and they are immediately applicable to other multi-user applications in communication networks that fulfill the requirements of ACSCG.

The rest of this paper is organized as follows. Section II defines the model of ACSCG. For ACSCG models, Sections III and IV present several distributed algorithms without and with real-time information exchanges, respectively, and provide sufficient conditions that guarantee the convergence of the proposed

\(^1\)A sequence \(x^{(k)}\) with limit \(x^*\) is linearly convergent if there exists a constant \(c \in (0, 1)\) such that \(|x^{(k)} - x^*| \leq c|x^{(k-1)} - x^*|\) for \(k\) sufficiently large [23].
algorithms. Section V presents the numerical examples and conclusions are drawn in Section VI.

II. GAME MODEL

In this section, we introduce some basic definitions from the theory of strategic games to characterize the multi-user interaction, define the model of ACSCG, and present some illustrative examples for the class of ACSCG.

A. Strategic Games, Nash equilibrium, and Pareto Optimality

A strategic game is a suitable model for the analysis of a game where all users act independently and simultaneously according to their own self-interests and with no or limited a priori knowledge of the other users’ strategies. This can be formally defined as a tuple $\Gamma = (\mathcal{N}, \mathcal{A}, u)$. In particular, $\mathcal{N} = \{1, 2, \ldots, N\}$ is the set of decision makers. Define $\mathcal{A}$ to be the joint action set $\mathcal{A} = \times_{n \in \mathcal{N}} A_n$, with $A_n \subseteq \mathcal{R}^K$ being the action set available for user $n$. The vector utility function $u = \times_{n \in \mathcal{N}} u_n$ is a mapping from the individual users’ joint action set to real numbers, i.e. $u : \mathcal{A} \rightarrow \mathcal{R}^N$. In particular, $u_n(a) : \mathcal{A} \rightarrow \mathcal{R}$ is the utility of the $n$th user that generally depends on the strategies $a = (a_n, a_{-n})$ of all users, where $a_n \in A_n$ denotes a feasible action of user $n$, and $a_{-n} = \times_{m \neq n} a_m$ is a vector of the actions of all users except $n$. We also denote by $\mathcal{A}_{-n} = \times_{m \neq n} A_m$ the joint action set of all users except $n$. To capture the multi-user performance tradeoff, the utility region is defined as $\mathcal{U} = \{(u_1(a), \ldots, u_N(a)) | \exists a \in \mathcal{A}\}$. Various game theoretic solutions, such as NE and Pareto optimality, were developed to characterize the resulting performance [6]. Significant research efforts have been devoted in the literature to constructing operational algorithms in order to achieve NE and Pareto optimality in various games with special structures of action set $\mathcal{A}_n$ and utility function $u_n$.

1) Nash equilibrium: definition, existence, and convergence: To avoid the overhead associated with exchanging information in real-time, network designers may prefer fully decentralized solutions in which the participating users simply compete against other users by choosing actions $a_n \in A_n$ to selfishly maximize their individual utility functions $u_n(a_n, a_{-n})$, given the actions $a_{-n} \in \mathcal{A}_{-n}$. Most of these approaches focus on investigating the existence and properties of NE. NE is defined to be an action profile $(a_1^*, a_2^*, \ldots, a_N^*)$ with the property that for every player, it satisfies $u_n(a_n^*, a_{-n}^*) \geq u_n(a_n, a_{-n}^*)$ for all $a_n \in A_n$, i.e. given the other users’ actions, no user can increase its utility alone by changing its action. For an extensive discussion of the methodologies studying the existence, uniqueness, and convergence of various equilibria in communication networks, we refer the readers to [25]. Many of the well-known results rely on specific structural properties of action set $\mathcal{A}$ and utility function $u$ in the investigated
multi-user interactions. For example, to establish the existence of and convergence to a pure NE, we can examine whether $A$ and $u$ satisfy the conditions of concave games, supermodular game, potential game, etc. Specifically, to apply the existence result of a pure NE in concave games [6] [7], we need to check the following conditions: i) each player’s action set $A_n$ is convex and compact; and ii) the utility function $u_n(a_n, a_{-n})$ is continuous in $a$ and quasi-concave in $a_n$ for any fixed $a_{-n}$. As additional examples of games that guarantee the convergence to NE, it is well-known that, in supermodular games [8] [10] and potential games [12] [13], the best response dynamics can be used to search for a pure NE. Suppose that utility function $u_n$ is twice continuously differentiable, $\forall n \in \mathcal{N}$. If $A_n$ is a compact subset of $\mathcal{R}$ (or more generally $A_n$ is a nonempty and compact sublattice of $\mathcal{R}^K$), $\forall n \in \mathcal{N}$, establishing that game $\Gamma$ is a supermodular game is equivalent to showing that $u_n$ satisfies

$$\forall (m, n) \in \mathcal{N}^2, m \neq n, \frac{\partial^2 u_n}{\partial a_n \partial a_m} \geq 0. \quad (1)$$

If action set $A$ in game $\Gamma$ is an interval of real numbers, we can show that game $\Gamma$ is a potential game by verifying

$$\forall (m, n) \in \mathcal{N}^2, m \neq n, \frac{\partial^2 (u_n - u_m)}{\partial a_n \partial a_m} = 0. \quad (2)$$

2) Pareto optimality and network utility maximization: It is important to note that operating at a Nash equilibrium will generally limit the performance of the user itself as well as that of the entire network, because the available network resources are not always effectively exploited due to the conflicts of interest occurring among users. As opposed to the NE-based approaches, there exists a large body of literature that focuses on studying how users can jointly improve the system performance by optimizing a certain common objective function $f(u_1(a), u_2(a), \ldots, u_N(a))$. This function represents the fairness rule based on which the system-wide resource allocation is performed. Different objective functions, e.g. sum utility maximization in which $f(u_1(a), u_2(a), \ldots, u_N(a)) = \sum_{n=1}^{N} u_n(a)$, can provide reasonable allocation outcomes by jointly considering fairness and efficiency. A profile of actions is Pareto optimal if there is no other profile of actions that makes every user at least as well off and at least one user strictly better off.

The majority of these approaches focus on studying how to efficiently or distributedly find the optimum joint policy. There exists a large body of literature that investigates how to compute Pareto optimal

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2 A real-valued function $f$ is quasi-concave if dom $f$ is convex and $\{x \in \text{dom } f | f(x) \geq \alpha\}$ is convex for all $\alpha$.

3 A real $K$-dimensional set $\mathcal{V}$ is a sublattice of $\mathcal{R}^K$ if for any two elements $a, b \in \mathcal{V}$, the component-wise minimum, $a \land b$, and the component-wise maximum, $a \lor b$, are also in $\mathcal{V}$. 

solutions in large-scale networks where centralized solutions are infeasible. Numerous convergence results have been obtained for various generic distributed algorithms. An important example is the NUM framework that develops distributed algorithms to solve network resource allocation problems [15]. The majority of the results in the existing NUM literature are based on convex optimization theory, in which the investigated problems share the following structures: the objective function \( f(u_1(a), u_2(a), \ldots, u_N(a)) \) is convex, inequality resource constraint functions are convex, and equality resource constraint functions are affine. It is well-known that, for convex optimization problems, users can collaboratively exchange price signals that reflect the “cost” for consuming the constrained resources and the Pareto optimal allocation that maximizes the network utility can be determined in a fully distributed manner [16].

Summarizing, these general structural results without and with real-time message exchange turn out to be very useful when analyzing various multi-user interactions in communication networks. Numerous existing works are devoted to constructing or shaping the multi-user coupling such that it fits into these frameworks and the corresponding generic solutions can be directly applied. In the remaining part of this paper, we will derive several structural results for a particular type of multi-user interaction scenario.

B. Additively Coupled Sum Constrained Games

In this subsection, we present the definition of ACSCG and subsequently, we present several exemplary multi-user scenarios which appertain to this new class of game.

Definition 1: A multi-user interaction \( \Gamma = \langle N, A, u \rangle \) is a ACSCG if it satisfies the following assumptions:

A1: \( \forall n \in N, \) action set \( A_n \subseteq \mathcal{R}^K \) is defined to be
\[
A_n = \left\{ (a_1^n, a_2^n, \ldots, a^K_n) \mid a_k^n \in [a_{n,k}^{\min}, a_{n,k}^{\max}] \text{ and } \sum_{k=1}^{K} a_k^n \leq M_n \right\}, \tag{3}
\]

A2: There exist \( h_k^n : \mathcal{R} \to \mathcal{R}, f_k^n : A_{-n} \to \mathcal{R}, \) and \( g_k^n : A_{-n} \to \mathcal{R}, k = 1, \ldots, K, \) such that
\[
u_n(a) = \sum_{k=1}^{K} \left[ h_k^n(a_k^n + f_k^n(a_{-n})) - g_k^n(a_{-n}) \right], \tag{4}
\]

for all \( a \in A \) and \( n \in N. \) \( h_k^n(\cdot) \) is an increasing, twice differentiable, and strictly concave function and \( f_k^n(\cdot) \) and \( g_k^n(\cdot) \) are both twice differentiable.

\(^4f : \mathcal{R}^n \to \mathcal{R} \) is convex if \( \text{dom} f \) is a convex set and \( f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y), \forall x, y \in \text{dom} f, 0 \leq \theta \leq 1. \)

\(^5\)We consider a sum constraint throughout the paper rather than a weighted-sum constraint, because a weighted-sum constraint can be easily converted to a sum constraint by rescaling \( A_n. \) Besides, we nontrivially assume that \( \sum_{k=1}^{K} a_{n,k}^{\max} \geq M_n. \)
The ACSCG model defined by assumptions A1 and A2 covers a broad class of multi-user interactions. Assumption A1 indicates that each player’s action set is a $K$-dimensional vector set and its action vector is sum-constrained. This represents the communication scenarios in which each user needs to determine its multi-dimensional action in various channels or networks while the total amount of resources it can consume is constrained. Assumption A2 implies that each user’s utility is separable and can be represented by the summation of concave functions $h^k_n$ minus “penalty” functions $g^k_n$ across the $K$ dimensions. In particular, within each dimension, the input of $h^k_n$ is an additive combination of user $n$’s action $a^k_n$ and function $f^k_n(a_{-n})$ that depends on the remaining users’ joint action $a_{-n}$. Since $a^k_n$ only appears in the concave function $h^k_n$, it implies that each user’s utility is concave in its own action, i.e. diminishing returns per unit of user $n$’s invested action $a_n$, which is common for many application scenarios in communication networks.

Summarizing, the key features of the game model defined by A1 and A2 include: each user’s action is subject to a sum constraint; users’ utilities are impacted by additive combinations of $a^k_n$ and $f^k_n(a_{-n})$ through concave functions $h^k_n$. Therefore, we term the game $\Gamma$ that satisfies assumptions A1 and A2 as ACSCG. In the following section, we present several illustrative multi-user interaction examples that belong to ACSCG.

C. Examples of ACSCG

We present four examples that satisfy assumptions A1 and A2 and belong to ACSCG. The details of functions $h^k_n(\cdot)$, $f^k_n(\cdot)$ and $g^k_n(\cdot)$ in each example are summarized in Table I. For each example, Table I also summarizes the applicable convergence conditions that will be provided in the remaining parts of the paper.

Example 1: We first consider a simple two-user game with two-dimension action spaces, i.e. $N = K = 2$. The utility functions are given by

$$u_n(a) = -\exp\left\{-a^1_n - \sqrt{(a^1_{-n})^2 + 1} + \sqrt{(a^2_{-n})^2 + 1}\right\} - \exp\left\{-a^2_n + \sqrt{(a^1_{-n})^2 + 1} - \sqrt{(a^2_{-n})^2 + 1}\right\},$$

for $n = 1, 2$. The resource constraints are $\sum_{k=1}^{2} a^k_n \leq M_n$ in which $M_n > 0$ and $a^k_n \geq 0$ for $\forall n, k$.

Example 2: (Power control in frequency-selective Gaussian interference channel [17] [20]) There are $N$ transmitter and receiver pairs in the system. The entire frequency band is divided into $K$ frequency bins. In frequency bin $k$, the channel gain from transmitter $i$ to receiver $j$ is denoted as $H^k_{ij}$, where $k =$

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6In this example, since there are only two users, the subindex $-n$ denotes the user but $n$. 
TABLE I
EXAMPLES [1-4] AS ACSCG.

| Examples | $f_n^k(a_\sim n)$ | $h_n^k(x)$ | $g_n^k(a_\sim n)$ | Convergence conditions |
|----------|--------------------|-------------|-------------------|-----------------------|
| Example 1 | $f_n^1(a_\sim n) = \sqrt{(a^2_{1,n})^2 + 1 - \sqrt{(a^2_{2,n})^2 + 1}} - e^{-x}$ | $-e^x$ | 0 | (C4) |
| Example 2 | $\sum_{m \neq n} \frac{H_{mn}^k}{H_{mm}^k} P_m^k$ | $\log_2(\sigma_n^k + H_{mn}^k x)$ | $\log_2(\sigma_n^k + \sum_{m \neq n} H_{mn}^k P_m^k)$ | (C1)-(C8) |
| Example 3 | $\sum_{m \neq n} \frac{1}{\psi_n^k} \psi_m^k$ | $-\frac{1}{\sqrt{\psi_n^k} - \psi_m^k}$ | 0 | (C1)-(C8) |
| Example 4 | $\sum_{m \neq n} \left( \sum_{j=1}^K \frac{a_{j,n} \cdot (k-j) H_{mn}^k P_m^k}{H_{mm}^k} \right)$ | $\log_2(\sigma_n^k + H_{mn}^k x)$ | $\log_2(\sigma_n^k + H_{mn}^k P_m^k)$ | (C1)-(C8) |

1, 2, · · · , $K$. Similarly, denote the noise power spectral density (PSD) that receiver $n$ experiences as $\sigma_n^k$ and player $n$’s transmit PSD as $P_n^k$. The action of user $n$ is to select its transmit power $P_n = [P_n^1 P_n^2 \cdots P_n^K]$ subject to its power constraint: $\sum_{k=1}^K P_n^k = \rho_n^\max$. For a fixed $P$, if treating its interference as noise, user $n$ can achieve the following data rate:

$$
 r_n(P) = \sum_{k=1}^K \log_2 \left( 1 + \frac{H_{mn}^k P_m^k}{\sigma_n^k + \sum_{m \neq n} H_{mn}^k P_m^k} \right)
$$

$$
 = \sum_{k=1}^K \left( \log_2(\sigma_n^k + \sum_{m=1}^N H_{mn}^k P_m^k) - \log_2(\sigma_n^k + \sum_{m \neq n} H_{mn}^k P_m^k) \right). \quad (5)
$$

**Example 3:** (Delay minimization in Jackson Networks [26]) As an additional example, we consider a network of $N$ nodes. A Poisson stream of external packets arrive at node $n$ with rate $\psi_n$ and the input stream is split into $K$ traffic classes, which are individually served by exponential servers. Denote node $n$’s input rate and service rate for class $k$ as $\psi_n^k$ and $\mu_n^k$ respectively. Therefore, the action of node $n$ is to determine the rates for different traffic classes $\Psi_n = [\psi_1^k \psi_2^k \cdots \psi_N^k]$ and the total rate is subject to the minimum rate constraint: $\sum_{k=1}^K \psi_n^k \geq \psi_n^\min$. The packets of the same traffic class constitute a Jackson network in which Markovian routing is adopted: packets of class $k$ completing service at node $m$ are routed to node $n$ with probability $r_{mn}^k$ or exit the network with probability $r_{m0}^k = 1 - \sum_{n=1}^N r_{mn}^k$. Denote the arrival rate for class $k$ at node $n$ as $\eta_n^k$. By Jackson’s Theorem, we have $\eta_n^k = \psi_n^k + \sum_{m=1}^N \eta_m^k r_{mn}^k$, $n = 1, 2, \cdots , K$. Denote $[R^k]_{mn} = r_{mn}^k$, $[\Upsilon^k] = (I - R^k)^{-1}$, and $\psi_m^k = [\Upsilon^k]_{mn}$. Equivalently, we have $\eta_n^k = \sum_{m=1}^N \psi_m^k \psi_n^k$. Each node aims to minimize its total M/M/1
queueing delay incurred by accommodating its traffic:

\[ d_n(\Psi) = \sum_{k=1}^{K} \frac{\mu_n^k}{\mu_n^k - \sum_{m=1}^{N} v_m^k \psi_m^k}. \quad (6) \]

Example 3 can be shown to be a special case of ACSCG by slightly transforming the action sets and utilities. We can define user \( n \)'s action as \(-\Psi_n\). For user \( n \), the sum constraint becomes \( \sum_{k=1}^{K} -\psi_n^k \leq -\psi_{\text{min}}^n \) and minimizing \( d_n(\Psi) \) is equivalent to maximizing \(-d_n(\Psi)\).

**Example 4:** (Asynchronous transmission in digital subscriber lines network [19]) The basic setting of this example is similar to that of Example 2 except that inter-carrier interference (ICI) exist among different frequency bins. Due to the loss of the orthogonality, the interference that user \( n \) experiences in frequency bin \( k \) is

\[ f_n^k(P_{-n}) = \sum_{m \neq n} \left( \sum_{j=1}^{K} \gamma(k-j)H_{mn}^jP_m^j \right), \quad (7) \]

in which \( \gamma(j) \) is the ICI coefficient that represents the relative interference transmitted signal in a particular frequency bin generates to its \( j \)th neighbor bin. In particular, it takes the form

\[ \gamma(j) = \begin{cases} 1, & \text{if } j = 0 \\ \frac{2}{K^2 \sin^2(\frac{\pi j}{K})}, & -\frac{K}{2} \leq j \leq \frac{K}{2}, \ j \neq 0. \end{cases} \quad (8) \]

It satisfies the symmetric and circular properties, i.e. \( \gamma(-j) = \gamma(j) = \gamma(K-j) \). User \( n \)'s achievable rate in the presence of ICI is given by

\[ r_n(P) = \sum_{k=1}^{K} \log_2 \left[ 1 + \frac{H_{mn}^kP_n^k}{\sigma_n^k + \sum_{m \neq n} \left( \sum_{j=1}^{K} \gamma(k-j)H_{mn}^jP_m^j \right)} \right]. \quad (9) \]

**D. Issues related to ACSCG**

Since the ACSCG model represents a good abstraction of numerous multi-user resource allocation problems, we aim to investigate the convergence properties of various distributed algorithms in ACSCG without and with real-time message passing.

ACSCG is a concave game [6] [7] and therefore, it admits at least one pure NE. In practice, we want to provide the sufficient conditions under which best response dynamics provably and globally converges to a pure NE. However, the existing literature, e.g. the diagonal strict concavity (DSC) conditions in [7] and the supermodular game theory [8]- [10], does not provide such convergence conditions for the general ACSCG model. For example, the DSC conditions developed for general concave games do not guarantee the convergence of best response dynamics [7]. Even if the utility functions in ACSCG possess the supermodular type structure, due to the sum constraint, the action set of each user is generally not a
Therefore, the convergence results based on supermodular games cannot be directly applied in ACSCG. On the other hand, if we want to maximize the sum utility by enabling real-time message passing among users, we also note that, the utility \( u_n \) is not necessarily jointly concave in \( a \) because of the existence of \( g^k_n(\cdot) \). Therefore, the existing algorithms developed for the convex NUM are not immediately applicable either.

In fact, a unique feature of the ACSCG is that different users’ actions are *additively coupled* in \( h^k_n(\cdot) \) and each user’s action space is *sum-constrained*. In the following sections, we will fully explore these specific structures and address the convergence properties of various distributed algorithms in two different scenarios. Specifically, Section III investigates the scenarios in which each user \( n \) can only observe \( \{f^k_n(a_{-n})\}_{k=1}^K \) and cannot exchange any information with any other user. Section IV focuses on the scenarios in which each user \( n \) is able to announce and receive information in real-time to and from the remaining users about \( \frac{\partial u_n(a)}{\partial a_k} \) and \( \frac{\partial u_m(a)}{\partial a_k} \), \( \forall m \neq n, k = 1, \ldots, K \).

### III. Scenario I: No Message Exchange Among Users

In communication scenarios where users cannot exchange messages to achieve coordination, the participating users can simply choose actions to selfishly maximize their individual utility functions \( u_n(a) \) without taking into account the utility degradation caused to the other users. In particular, each user individually solves the following optimization program:

\[
\max_{a_n \in \mathcal{A}_n} u_n(a).
\]  

The steady state outcome of such a multi-user interaction is usually characterized as a NE, at which given the other users’ actions, no user can increase its utility alone by unilaterally changing its action. It is worth pointing out that, since there is no coordination signal among users, NE generally does not lead to a Pareto-optimal solution. Section IV will discuss distributed algorithms in which users exchange coordination signals in order to improve the system efficiency.

#### A. Properties of Best Response Dynamics in ACSCG

To better understand the key properties of the ACSCG, in this subsection, we first focus on the scenarios in which \( f^k_n(a_{-n}) \) is the linear combination of the remaining users’ action in the same dimension \( k \), i.e.

\[
f^k_n(a_{-n}) = \sum_{m \neq n} P_{mn}^k a^k_m
\]  

\(^7\)In supermodular games, for each player, the action set is a nonempty and compact sublattice of \( \mathcal{R}^K \). We can verify that with the sum constraint, \( \mathcal{A}_n \) is usually not a sublattice of \( \mathcal{R}^K \) by taking the component-wise maximum.
and $F^{k}_{mn} \in \mathcal{R}$, $\forall m,n,k$. Specifically, both Example 2 and 3 in Table I belong to this category. In Section III-B, we will extend the results derived for the functions $f^{k}_{n}(a_{-n})$ defined in (11) to general $f^{k}_{n}(a_{-n})$.

Since $h^{k}_{n}(\cdot)$ is concave, the objective in (10) is a concave function in $a^{k}_{n}$ when the other users’ actions $a_{-n}$ are fixed. To find the globally optimal solution of the problem in (10), we can first form its Lagrangian

$$L_{n}(a_{n}, \lambda) = u_{n}(a) + \lambda(M_{n} - \sum_{k=1}^{K} a^{k}_{n}),$$

in which $a^{k}_{n} \in [a^{\text{min}}_{n,k}, a^{\text{max}}_{n,k}]$. By taking the first derivatives of (12), we have

$$\frac{\partial L_{n}(a_{n}, \lambda)}{\partial a^{k}_{n}} = \frac{\partial h^{k}_{n}(a^{k}_{n} + \sum_{m \neq n} F^{k}_{mn} a^{k}_{m})}{\partial a^{k}_{n}} - \lambda = 0.$$  

Denote

$$l^{k}_{n}(a_{-n}, \lambda) \triangleq \left[ \frac{\partial h^{k}_{n}}{\partial x} \right]^{-1}(\lambda) - \sum_{m \neq n} F^{k}_{mn} a^{k}_{m} \frac{a^{\text{max}}_{n,k}}{a^{\text{min}}_{n,k}},$$

in which $\left[ \frac{\partial h^{k}_{n}}{\partial x} \right]^{-1}$ is the inverse function of $\frac{\partial h^{k}_{n}}{\partial x}$ and $[x]_{\theta} = \max\{\min\{x,a\}, b\}$. The optimal solution of (10) is given by $a^{*k}_{n} = l^{k}_{n}(a_{-n}, \lambda^{*})$, where the Lagrange multiplier $\lambda^{*}$ is chosen to satisfy the sum constraint $\sum_{k=1}^{K} a^{*k}_{n} = M_{n}$.

We define the best response operator $B^{k}_{n}(\cdot)$ as

$$B^{k}_{n}(a_{-n}) = l^{k}_{n}(a_{-n}, \lambda^{*}).$$ (15)

We consider the dynamic adjustment process in which users revise their actions over time based on their observations about their opponents. A well-known candidate for such adjustment processes is the so-called best response dynamics. In the best response algorithm, each user updates its action using the best response strategy that maximizes its utility function in (4). We consider two types of update orders, including sequential update and parallel update. Specifically, in sequential update, individual players iteratively optimize in a circular fashion with respect to their own actions while keeping the actions of their opponents fixed. Formally, at stage $t$, user $n$ chooses its action according to

$$a^{k,t}_{n} = B^{k}_{n}([a^{1}_{1}, \ldots, a^{t-1}_{n-1}, a^{t-1}_{n+1}, \ldots, a^{t-1}_{N}]).$$ (16)

On the other hand, players adopting the parallel update revise their actions at stage $t$ according to

$$a^{k,t}_{n} = B^{k}_{n}(a_{-n}^{t-1}).$$ (17)

We obtain several sufficient conditions under which best response dynamics converges. Similar convergence conditions are proved in [18]- [20] for Example 2 in which $h^{k}_{n}(x) = \log_{2}(\sigma^{k}_{n} + H^{k}_{nn}x)$. We

\[8\] If $\exists x = x^{*}$ such that $\frac{\partial h^{k}_{n}}{\partial x}|_{x=x^{*}} = \lambda$, we let $\left[ \frac{\partial h^{k}_{n}}{\partial x} \right]^{-1}(\lambda) = -\infty$. 

TABLE II
COMPARISON AMONG CONDITIONS (C1)-(C6).

| Conditions | Assumptions about $f_k^n(a_{-n})$ | Measure of residual error $a_{n+1}^{t} - a_{n}^{t}$ | Contraction factor |
|------------|-----------------------------------|-----------------------------------------------|-------------------|
| (C1)       | (11)                              | $\ell_1$-norm                                | $2\rho(T_{\text{max}})$ |
| (C2)       | (11) and $F_{mn}^k$ have the same sign for $\forall k, m \neq n$ | $\ell_1$-norm                                | $\rho(T_{\text{max}})$ |
| (C3)       | (11)                             | $\ell_1$-norm weighted Euclidean norm | $\rho(S_{\text{max}})$ |
| (C4)       | (20)                             | $\ell_1$-norm weighted Euclidean norm | $\rho(S_{\text{max}})$ |
| (C5)       | (20)                             | $\ell_1$-norm weighted Euclidean norm | $\rho(T_{\text{max}})$ |
| (C6)       | (20)                             | $\ell_1$-norm weighted Euclidean norm | $\rho(S_{\text{max}})$ |

consider more general functions $h_k^n(\cdot)$ and further extend the convergence conditions in [18]-[20]. The key differences among all the sufficient conditions which will be provided in this section are summarized in Table II.

1) General $h_k^n(\cdot)$: The first sufficient condition is developed for the general cases in which the functions $h_k^n(\cdot)$ in the utilities $u_n(\cdot)$ are specified in assumption A2. Define

$$\max_k |F_{mn}^k|, \quad \text{if } m \neq n$$

and let $\rho(T_{\text{max}})$ denote the spectral radius of the matrix $T_{\text{max}}$.

**Theorem 1:** If

$$\rho(T_{\text{max}}) < \frac{1}{2},$$

then there exists a unique NE in game $\Gamma$ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $\mathcal{A}$ with either sequential or parallel updates.

**Proof:** This theorem is proved by showing that the best response dynamics defined in (16) and (17) is a contraction mapping under (C1). See Appendix A for details.

In multi-user communication applications, it is common to have games of strategic complements (or strategic substitutes), i.e. the marginal returns to any one component of the player’s action rise with increases (or decreases) in the components of the competitors’ actions [27]. For instance, in Examples 2 and 4 increasing user $n$’s transmitted power creates stronger interference to the other users and decreases their marginal achievable rates. Similarly, in Example 3 increasing node $n$’s input traffic rate congests all the servers in the network and increases the marginal queueing delay. Mathematically, if $u_n$ is twice
differentiable, strategic complementarities (or strategic substitutes) can be described as
\[
\frac{\partial^2 u_n(a_n, a_{-n})}{\partial a^2_k (a_n, a_{-n})} \geq 0, \quad \forall m \neq n, j, k, \quad \text{(or)} \quad \frac{\partial^2 u_n(a_n, a_{-n})}{\partial a^2_n \partial a^2_k (a_n, a_{-n})} \leq 0, \quad \forall m \neq n, j, k. \tag{19}
\]
We can verify that Examples 2, 3, and 4 are games with strategic substitutes. For the ACSCG models that exhibit strategic complementarities (or strategic substitutes), the following theorem further relaxes condition (C1).

**Theorem 2:** Let \( \Gamma \) be an ACSCG with strategic complementarities (or strategic substitutes), i.e. \( F^k_{mn} \leq 0, \forall k, m \neq n \), (or \( F^k_{mn} \geq 0, \forall k, m \neq n \)). If
\[
\rho(T^{\text{max}}) < 1,
\]
then there exists a unique NE in game \( \Gamma \) and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to \( \mathcal{A} \) with either sequential or parallel updates.

**Proof:** This theorem is proved by adapting the proof of Theorem 1. See Appendix B. \( \square \)

**Remark 1:** (Implications of conditions (C1) and (C2)) Theorem 1 and Theorem 2 give sufficient conditions for best response dynamics to globally converge to a unique fixed point. Specifically, \( \max_k |F^k_{mn}| \) can be regarded as a measure of the strength of the mutual coupling between user \( m \) and \( n \). The intuition behind (C1) and (C2) is that, the weaker the coupling among different users is, the more likely that best response dynamics converges. Consider the extreme case in which \( F^k_{mn} = 0, \forall k, m \neq n \). Since each user’s best response is not impacted by the remaining users’ action \( a_{-n} \), the convergence is immediately achieved after a single best-response iteration. If no restriction is imposed on \( F^k_{mn} \), Theorem 1 specifies a mutual coupling threshold under which best response dynamics provably converge. The proof of Theorem 1 can be intuitively interpreted as follows. We regard every best response update as the users’ joint attempt to approach the NE. Due to the linear coupling structure in (11), user \( n \)’s best response in (14) contains a term \( \sum_{m \neq n} F^k_{mn} a^k_m \) that is a linear combination of \( a_{-n} \). As a result, the residual error \( |a_i^{t+1} - a_i^t| \), which is the 1-norm distance between the updated action profile \( a_i^{t+1} \) and the current action profile \( a_i^t \), can be upper-bounded using linear combinations of \( |a_i^t - a_i^{t-1}| \) in which \( m \neq n \). Recall that \( F^k_{mn} \) can be either positive or negative. We also note that, if \( a_i^t \neq a_i^{t-1} \), \( a_m^t - a_m^{t-1} \) contains both positive and negative terms due to the sum-constraint. In the worst case, the distance \( |a_i^{t+1} - a_i^t| \) is maximized if \( \{ F^k_{mn} \} \) and \( \{ a_i^{k,t} - a_i^{k,t-1} \} \) are co-phase multiplied and additively summed, i.e. \( F^k_{mn} (a_i^{k,t} - a_i^{k,t-1}) \geq 0 \), for \( \forall k = 1, \ldots, K, m \neq n \). After an iteration, all users except \( n \) contributes to user \( n \)’s residual error at stage \( t + 1 \) up to \( \sum_{m \neq n} 2 \max_k |F^k_{mn}||a_i^t - a_i^{t-1}|_1 \). Under condition (C1), it is guaranteed that the residual error contracts with respect to the special norm defined in (67). Theorem 2 focuses on the situations in
which the signs of \( F_{mn}^k \) are the same, \( \forall m \neq n, k \). In this case, \( \{ F_{mn}^k \} \) and \( \{ a_m^k - a_m^{k-1} \} \) cannot be co-phase multiplied. Therefore, the region of convergence enlarges and hence, condition (C2) stated in Theorem 2 is weaker than condition (C1) in Theorem 1.

Remark 2: (Relation to the results in references [18]-[20]) Similar to [18] [19], our proofs choose 1-norm as the distance measure for the residual errors \( a_n^{t+1} - a_n^t \) after each best-response iteration. However, by manipulating the inequalities in a different way, condition (C2) is more general than the results in [18] [19], where they require \( \max_k F_{mn}^k < \frac{1}{n-1} \). Interestingly, condition (C2) recovers the result obtained in [20] where it is proved by choosing the Euclidean norm as the distance measure for the residual errors \( a_n^{t+1} - a_n^t \) after each best-response iteration. However, the approach in [20] using the Euclidean norm only applies to the scenarios in which \( h_n^k(\cdot) \) is a logarithmic function. We prove that condition (C2) applies to any \( h_n^k(\cdot) \) that is increasing and strictly concave.

2) A special class of \( h_n^k(\cdot) \): In addition to conditions (C1) and (C2), we also develop a sufficient convergence condition for a family of utility functions parameterized by a negative number \( \theta \). In particular, \( h_n^k(\cdot) \) satisfies

\[
h_n^k(x) = \begin{cases} 
\log(\alpha_n^k + F_{mn}^k x), & \text{if } \theta = -1, \\
\frac{(\alpha_n^k + F_{mn}^k x)^{\theta+1}}{\theta+1}, & \text{if } -1 < \theta < 0 \text{ or } \theta < -1.
\end{cases} \tag{20}
\]

and \( \alpha_n^k \in \mathcal{R} \) and \( F_{mn}^k > 0 \). The interpretation of this type of utilities has been addressed in [28]. It is shown that varying the parameter \( \theta \) leads to different types of fairness across \( \alpha_n^k + F_{mn}^k (a_n^k + \sum_{m \neq n} F_{mn}^k a_m^k) \) for all \( k \). In particular, \( \theta = -1 \) corresponds to the proportional fairness; if \( \theta = -2 \), then harmonic mean fairness; and if \( \theta = -\infty \), then max-min fairness. We can see that, Examples 2 and 3 are special cases of this type of utility functions. In these cases, best response dynamics in equation (14) is reduced to

\[
h_n^k(a_{-n}, \lambda) = \left[ \frac{1}{F_{nn}^k} \right]^{1+\frac{1}{\theta}} \lambda + \frac{\alpha_n^k}{F_{nn}^k} - \sum_{m \neq n} F_{mn}^k a_m^k \right] \max_{\lambda} \triangleq \left\{ \begin{array}{ll}
\sum_{k=1}^K (F_{nn}^k)^{1+\frac{1}{\theta}} \max_k \left\{ \frac{F_{mn}^k}{F_{nn}^k} \right\}^{1+\frac{1}{\theta}}, & \text{if } m \neq n \\
0, & \text{otherwise}.
\end{array} \right. \tag{21}
\]

Define

\[
[S^{\max}]_{mn} \triangleq \left\{ \begin{array}{ll}
\sum_{k=1}^K (F_{mn}^k)^{1+\frac{1}{\theta}} \max_k \left\{ \frac{F_{mn}^k}{F_{nn}^k} \right\}^{1+\frac{1}{\theta}}, & \text{if } m \neq n \\
0, & \text{otherwise}.
\end{array} \right. \tag{22}
\]

For the class of utility functions in (20), Theorem 3 gives a sufficient condition that guarantees the convergence of the best response dynamics defined in (21).

\[\text{If } \alpha_n^k + F_{mn}^k x \leq 0, \text{ we let } h_n^k(x) = -\infty. \text{ We assume for this class of } h_n^k(\cdot) \text{ that for } \forall a_{-n} \in \mathcal{A}_{-n}, \text{ there exists } a_n \in \mathcal{A}_n \text{ such that } \alpha_n^k + F_{mn}^k x > 0 \text{ for } \forall n, k.\]
Theorem 3: For $h^k_n(\cdot)$ defined in (20), if

$$\rho(S_{\text{max}}^{\text{max}}) < 1,$$  \hspace{1cm} (C3)

then there exists a unique NE in game $\Gamma$ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $A$ and with either sequential or parallel updates.

Proof: It can be proved by showing that the best response dynamics defined in (21) is a contraction mapping with respect to the weighted Euclidean norm. See Appendix C for details. $\blacksquare$

Remark 3: (Relation between conditions (C3) and the results in reference [20]) For aforementioned Example 2, Scutari et al. established in [20] a sufficient condition under which the iterative water-filling algorithm converges. The iterative water-filling algorithm essentially belongs to best response dynamics. Specifically, in [20], Shannon’s formula leads to $\theta = -1$ and cross channel coefficients satisfy $F_{mn}^k \geq 0, \forall k, m \neq n$. Equation (21) reduces to the water-filling formula

$$l^k_n(a_{-n}, \lambda) = \left[ \frac{1}{\lambda} - \frac{\alpha^k_n}{F_{mn}^k} - \sum_{m \neq n} F_{mn}^k a^k_m \right] a_{mn}^{\max},$$  \hspace{1cm} (23)

and $[S_{\text{max}}]_{mn} = \max_k F_{mn}^k$. By choosing the weighted Euclidean norm as the distance measure for the residual errors $a_{n}^{t+1} - a_{n}^t$ after each best-response iteration, Theorem 3 generalizes the results in [20] for the family of utility functions defined in (20).

Remark 4: (Relation between conditions (C1), (C2) and (C3)) The connections and differences between conditions (C1), (C2) and (C3) are summarized in Table II. We have addressed the implications of (C1) and (C2) in Remark 1. Now we discuss their relation with (C3). First of all, condition (C1) is proposed for general $h^k_n(\cdot)$ and condition (C3) is proposed for the class of utility functions defined in (20). However, Theorem 1 and Theorem 3 individually establish the fact that best response dynamics is a contraction map by selecting different vector and matrix norms. Therefore, in general, (C1) and (C3) do not immediately imply each other. Note that $[S_{\text{max}}]_{mn} \leq \zeta_{mn} \cdot \max_k |F_{mn}^k|$ in which $\zeta_{mn}$ satisfies

$$\zeta_{mn} = \frac{\sum_{k=1}^{K}(F_{mn}^k)^{1+\frac{1}{\theta}} \cdot \max_k (F_{nn}^k)^{1+\frac{1}{\theta}}}{\sum_{k=1}^{K}(F_{mn}^k)^{1+\frac{1}{\theta}} \cdot \min_k (F_{nn}^k/F_{mn}^k)^{1+\frac{1}{\theta}}} \in \left[ 1, \max_k (F_{nn}^k/F_{mn}^k)^{1+\frac{1}{\theta}} \right].$$  \hspace{1cm} (24)

The physical interpretation of $\zeta_{mn}$ is the similarity between the preferences of user $m$ and $n$ across the total $K$ dimensions of their action spaces. Recall that both $S_{\text{max}}$ and $T_{\text{max}}$ are non-negative matrices and $S_{\text{max}}$ is element-wise less than or equal to $\max_{m \neq n} \zeta_{mn} T_{\text{max}}$. By the property of non-negative matrix and condition (C1), we can conclude $\rho(S_{\text{max}}) \leq \rho(\max_{m \neq n} \zeta_{mn} T_{\text{max}}) < \max_{m \neq n} \frac{\zeta_{mn}}{2}$. The relation between (C1) and (C3) is pictorially illustrated in Fig. 1. Specifically, if users have similar preference in their available actions and the upper bound of $\zeta_{mn}$ that measures the difference of their preferences is
In general

\[
\frac{\max_{k,m \neq n} (F_{mn}^k F_{mm}^k)^{1+\frac{1}{\theta}}} {\min_{k,m \neq n} (F_{mn}^k F_{mm}^k)^{1+\frac{1}{\theta}}} < 2
\]

we know that (C1) implies (C3) in this situation because

\[
\rho(S_{\text{max}}^\text{m,n}) < \max_{m,n} \zeta_{mn} \cdot \rho(T_{\text{max}}^\text{m,n}) < 2 \cdot \frac{1}{2} = 1.
\]

We also would like to point out that, the LHS of (25) is a function of \(\theta\) and the LHS \(\equiv 1\) if \(\theta = -1\).

When \(\theta = -1\), \(T_{\text{max}}\) coincides with \(S_{\text{max}}\). Mathematically, in this case, (C3) is actually more general

\[\text{than (C2), because it still holds even if coefficients } F_{mn}^k \text{ have different signs.} \]

\[\text{B. Extensions to General } f_k^\cdot(\cdot)\]

As a matter of fact, the results above can be extended to the more general situations in which \(f_k^\cdot(\cdot)\) is

\[\text{a nonlinear differentiable function, } \forall n,k \text{ and its input } a_{-n} \text{ consists of the remaining users' action from all the dimensions. Accordingly, equation (14) becomes} \]

\[
l_{k}^{n}(a_{-n}, \lambda) \triangleq \left\{ \frac{\partial h_{k}^{n}}{\partial x} \right\}^{-1}(\lambda) - f_{k}^{n}(a_{-n}) \bigg|_{a_{-n}^{\text{max}}}^{a_{-n}^{\text{min}}}.
\]

The conclusions in Theorem 1, 2, and 3 can be further extended as Theorem 4 and 5, 6 that are listed below. We only provide the proof of Theorem 4 in Appendix D. The detailed proofs of Theorem 5 and 6 are omitted because they can be proven similarly as Theorem 4.
For general $f^k_n(\cdot)$, we denote

$$[T^{\text{max}}]_{mn} \triangleq \begin{cases} \max_{a \in A, k'} \sum_{k=1}^{K} |\frac{\partial f^k_n(a_{-n})}{\partial a^m_m}|, & \text{if } m \neq n \\ 0, & \text{otherwise.} \end{cases} \quad (27)$$

Besides, for $h^k_n(\cdot)$ defined in (20), we define

$$[S^{\text{max}}]_{mn} \triangleq \begin{cases} \frac{\sum_{k=1}^{K}(F^k_{mm})^{1+\frac{\theta}{2}}}{\sum_{k=1}^{K}(F^k_{nn})^{1+\frac{\theta}{2}}} \max_{a \in A, k'} \left\{ \sum_{k=1}^{K} \left| \frac{\partial f^k_n(a_{-n})}{\partial a^m_m} \right| \left( \frac{F^k_{mn}}{F^k_{mm}} \right)^{1+\frac{\theta}{2}} \right\}, & \text{if } m \neq n \\ 0, & \text{otherwise.} \end{cases} \quad (28)$$

**Theorem 4:** If

$$\rho(T^{\text{max}}) < \frac{1}{2}, \quad (C4)$$

then there exists a unique NE in game $\Gamma$ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $A$ with either sequential or parallel updates.

**Proof:** This theorem can be proved by combining the proof of Theorem 1 and the mean value theorem for vector-valued functions. See Appendix D for details. ■

Similarly as in Theorem 2 for the general ACSCCG models that exhibit strategic complementarities (or strategic substitutes), we can further relax condition (C4).

**Theorem 5:** For $\Gamma$ with strategic complementarities (or strategic substitutes), i.e. $\frac{\partial f^k_n(a_{-n})}{\partial a^m_m} \geq 0, \forall m \neq n, k, k', a \in A$, (or $\frac{\partial f^k_n(a_{-n})}{\partial a^m_m} \leq 0, \forall m \neq n, k, k', a \in A$), if

$$\rho(T^{\text{max}}) < 1, \quad (C5)$$

then there exists a unique NE in game $\Gamma$ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $A$ with either sequential or parallel updates.

**Theorem 6:** For $h^k_n(\cdot)$ defined in (20), if

$$\rho(S^{\text{max}}) < 1, \quad (C6)$$

then there exists a unique NE in game $\Gamma$ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $A$ with either sequential or parallel updates.

**Remark 5:** (Implications of conditions (C4), (C5), and (C6)) Based on the mean value theorem, we know that the upper bound of the additive sum of first derivatives $\sum_{k=1}^{K} \left| \frac{\partial f^k_n(a_{-n})}{\partial a^m_m} \right|$ governs the maximum impact that user $m$’s action can make over user $n$’s utility. As a result, Theorem 4, Theorem 5 and Theorem 6 indicate that $\sum_{k=1}^{K} \left| \frac{\partial f^k_n(a_{-n})}{\partial a^m_m} \right|$ can be used to develop similar sufficient conditions for the global convergence of best response dynamics. Table III summarizes the connections and differences.
among all the aforementioned conditions from (C1) to (C6). We can verify that, for the linear function 
\( f^k_n(\cdot) \) that is defined in (11) and studied in Section III-A, \( \forall a \in A, m \neq n \), it satisfies

\[
\frac{\partial f^k_n(a - n)}{\partial a^k_m} = \begin{cases}
F^k_{mn}, & \text{if } k' = k \\
0, & \text{otherwise}
\end{cases}
\] (29)

In addition, we can see that, in Example 4, 
\( f^k_n(\cdot) \) is actually an affine function with

\[
\frac{\partial f^k_n(P - n)}{\partial P^k_m} = \begin{cases}
\gamma(k - k')H^k_{mn}, & \text{if } k' = k \\
0, & \text{otherwise}
\end{cases}
\] (30)

and \( \bar{S}^{\text{max}} \) is reduced to

\[
[\bar{S}^{\text{max}}]_{mn} \triangleq \begin{cases}
\max_{k'} \sum_{k=1}^{K} \gamma(k - k')H^k_{mn}, & \text{if } m \neq n \\
0, & \text{otherwise}
\end{cases}
\] (31)

As an immediate result of Theorem 6, we have the following corollary which specifies a sufficient condition that guarantees the convergence of the iterative water-filling algorithm for asynchronous transmissions in multi-carrier systems [19].

**Corollary 1:** In Example 4, if the matrix \( \bar{S}^{\text{max}} \) defined in (31) satisfies

\[
\rho(\bar{S}^{\text{max}}) < 1,
\] (32)

then there exists a unique NE in game \( \Gamma \) and the iterative water-filling algorithm converges linearly to the NE, for any set of initial conditions belonging to \( A \) and with either sequential or parallel updates.

**Remark 6:** (Impact of sum constraints) An interesting phenomenon that can be observed from the analysis above is that, the convergence condition may depend on the maximum constraints \( \{M^k_n\}_{n=1}^{N} \). This differs from the observation in [20] that the presence of the transmit power and spectral mask constraints does not affect the convergence capability of the iterative water-filling algorithm. This is because when functions \( f^k_n(a - n) \) are affine, e.g. in Example 2, 3, and 4, the elements in \( \bar{T}^{\text{max}} \) and \( \bar{S}^{\text{max}} \) are independent of the values of \( \{M^k_n\}_{n=1}^{N} \). Therefore, (C1)-(C6) are independent of \( M_n \) for affine \( f^k_n(a - n) \). However, for non-linear \( f^k_n(a - n) \), the values of \( \{M^k_n\}_{n=1}^{N} \) specify the range of users’ joint feasible action set \( A \), and this will affect \( \bar{T}^{\text{max}} \) and \( \bar{S}^{\text{max}} \) accordingly. In other words, in the presence of non-linearly coupled \( f^k_n(a - n) \), convergence may depend on the players’ maximum sum constraints \( \{M^k_n\}_{n=1}^{N} \).

**C. Connections to the Results of Rosen [7] and Gabay [24]**

In [7], Rosen proposed a continuous-time gradient projection based iterative algorithm to obtain a pure NE under the assumption of DSC conditions. Here we present a discrete version of the algorithm in [7].
named “gradient play”. Specifically, at stage $t$, each user first determines the gradient of its own utility function $u_n(a_n, a_{-n}^{t-1})$. Then each user updates its action $a_n^t$ using gradient projection according to

$$a_n^{k,t} = a_n^{k,t-1} + \kappa_n \frac{\partial u_n(a_n, a_{-n}^{t-1})}{\partial a_n^k}$$

and

$$a_n^t = [a_1^{1,t} a_2^{2,t} \ldots a_K^{K,t}] = \left[ a_1^{1,t} a_2^{2,t} \ldots a_K^{K,t} \right]_{\|\cdot\|_2} \bigg|_{A_n},$$

where $\kappa_n$ is the stepsize and $[v]_{\|\cdot\|_2}$ denotes the projection of the vector $v$ onto user $n$’s action set $A_n$ with respect to the Euclidean norm $\| \cdot \|_2$. If $\kappa_n$ is chosen to be sufficiently small, gradient play approximates the continuous-time gradient projection algorithm. For each nonnegative vector $\kappa = [\kappa_1 \ldots \kappa_N]$, define

$$g(a, \kappa) = [\kappa_1 \nabla_1 u_1(a) \kappa_2 \nabla_2 u_2(a) \ldots \kappa_N \nabla_N u_N(a)]^T.$$ 

The definition of DSC in [7] is that, for fixed $\kappa > 0$ and every $a^0, a^1 \in \mathcal{A}$, we have

$$(a^1 - a^0)^T g(a^0, \kappa) + (a^0 - a^1)^T g(a^1, \kappa) > 0.$$ 

A sufficient condition for DSC is that the symmetric matrix $G(a, \kappa) + G^T(a, \kappa)$ be negative definite for $a \in \mathcal{A}$, where $G(a, \kappa)$ is the Jacobian with respect to $a$ of $g(a, \kappa)$.

However, when using gradient play to search for a pure NE, the stepsize $\kappa_n$ needs to be carefully chosen and set to be sufficiently small, which usually slows down the rate of convergence. As an alternative distributed algorithm, for concave games with $A_n = \mathbb{R}^+$, $\forall n \in \mathcal{N}$, Gabay and Moulin provided in [24] a dominance solvability condition under which best response dynamics globally converges to a unique NE. Specifically, the dominance solvability condition is given by

$$-\frac{\partial^2 u_n}{\partial^2 a_n} \geq \sum_{m \neq n} \left| \frac{\partial^2 u_n}{\partial a_n \partial a_m} \right|.$$ 

The sufficient conditions provided in this section and Gabay’s dominance solvability condition specify the convergence conditions of best response dynamics in different subclasses of concave games. Specifically, our results are developed for concave games in which every user has a multi-dimensional action space subject to a single sum-constraint and Gabay’s dominance solvability condition is proposed for concave games with single dimensional strategy.

D. Connections to Linearly Coupled Communication Games

We investigated in [29] the convergence properties in certain communication scenarios, namely linearly coupled communication games (LCCG), in which each user has a convex action set $A_n \subseteq \mathbb{R}^+$ and the
TABLE III
A SUMMARY OF VARIOUS CONVERGENCE CONDITIONS IN CONCAVE GAMES.

| Algorithms       | Sufficient conditions and the applicable games                                                                 |
|------------------|-------------------------------------------------------------------------------------------------------------|
| Gradient play    | Rosen’s DSC conditions for concave games [7]                                                                   |
| Best response    | Gabay’s dominance solvability condition for concave games with $\mathcal{A}_n = \mathbb{R}_+$ [24], conditions (C1)-(C6) for ACSCG |

Utility functions take the form

$$u_n(a) = a_{n}^{\beta_n} \cdot (\mu - \sum_{m=1}^{N} \tau_m a_m).$$  \hspace{1cm} (38)

It has been used to model the flow control mechanism in communication networks [30]. In best response dynamics, at stage $t$, user $n$ chooses its action according to

$$B_n(a^{t-1}) = \frac{\beta_n (\mu - \sum_{m \in \mathcal{N} \setminus \{n\}} \tau_m a_m^{t-1})}{\tau_n (1 + \beta_n)}.$$  \hspace{1cm} (39)

We can see that, LCCG is similar to ACSCG in the sense that the best response iterations at stage $t$ in (14) and (39) both contain the linear combinations of $a^{t-1}$. However, since $\mathcal{A}_n \subseteq \mathbb{R}$ in LCCG, we can explicitly derive the Jacobian matrix for best response dynamics and determine the exact locations of all its eigenvalues. Consequently, we are able to develop the necessary and sufficient condition that ensures the spectral radius of the Jacobian matrix to be less than 1 and best response dynamics globally converges. However, in ACSCG, due to the sum-constraint, there exists a non-linear operation $[x]_0^p$ in equation (14). This complicates the analysis of the Jacobian matrix’s eigenvalues. Therefore, we usually choose various appropriate matrix norms to bound the spectral radius of the Jacobian matrix and ensure the best response iteration to converge under these matrix norms. This approach generally results in various sufficient, but not necessary, conditions.

IV. SCENARIO II: MESSAGE EXCHANGE AMONG USERS

In this section, our objective is to coordinate the users’ actions in ACSCG to maximize the overall performance of the system, measured in terms of their total utilities, in a distributed fashion. Specifically, the optimization problem we want to solve is

$$\max_{a \in \mathcal{A}} \sum_{n=1}^{N} u_n(a).$$  \hspace{1cm} (40)

We will study two distributed algorithms in which the participating users exchange price signals that indicate the “cost” or “benefit” that its action causes to the other users. Allocating network resources via
pricing has been well-investigated for convex NUM problems [15], where the original NUM problem can be decomposed into distributedly solvable subproblems by setting price for each constraint resource, and each subproblem has to decide the amount of resources to be used depending on the charged price. However, unlike in the conventional convex NUM, pricing mechanisms may not be immediately applicable in ACSCG if the objective in (40) is not jointly concave in $a$. Therefore, we are interested in characterizing the convergence condition of different pricing algorithms in ACSCG.

We know that for any local maximum $a^*$ of problem (40), there exist Lagrange multipliers $\lambda_n, \nu^1_n, \ldots, \nu^N_n$ and $\nu'^1_n, \ldots, \nu'^N_n$ such that the following Karush-Kuhn-Tucker (KKT) conditions hold for all $n \in N$:

\[
\frac{\partial u_n(a^*)}{\partial a^k_n} + \sum_{m \neq n} \frac{\partial u_m(a^*)}{\partial a^k_n} = \lambda_n + \nu^k_n - \nu'^k_n, \; \forall n \quad (41)
\]

\[
\lambda_n \left( \sum_{k=1}^K a^k_n - M_n \right) = 0, \; \lambda_n \geq 0 \quad (42)
\]

\[
\nu^k_n (a^k_n - a^\text{max}_n) = 0, \; \nu'^k_n (a^\text{min}_n - a^k_n) = 0, \; \nu^k_n, \nu'^k_n \geq 0. \quad (43)
\]

Denote $\pi^k_{mn}$ user $m$’s marginal fluctuation in utility per unit decrease in user $n$’s action $a^k_n$ within the $k$th dimension

\[
\pi^k_{mn}(a^k_m, a^k_n) = - \frac{\partial u_m(a^*)}{\partial a^k_n}, \quad (44)
\]

which is announced by user $m$ to user $n$ and can be viewed as the cost charged (or compensation paid) to user $n$ for changing user $m$’s utility. Using (44), equation (41) can be rewritten as

\[
\frac{\partial u_n(a^*)}{\partial a^k_n} - \sum_{m \neq n} \pi^k_{mn}(a^k_m, a^k_n) = \lambda_n + \nu^k_n - \nu'^k_n. \quad (45)
\]

If we assume fixed prices $\{\pi^k_{mn}\}$ and action profile $a^k_{-n}$, condition (45) gives the necessary and sufficient KKT condition of the following problem:

\[
\max_{a_n \in A_n} u_n(a) - \sum_{k=1}^K a^k_n \cdot \left( \sum_{m \neq n} \pi^k_{mn} \right). \quad (46)
\]

At an optimum, a user behaves as if it maximizes the differences between its utility minus its payment to the other users in the network due to its impact over the other users’ utilities. Different distributed pricing mechanisms can be developed based on the individual objective function in (46) and the convergence conditions may also vary based on the specific action update equation.

When optimization program (40) is not convex, the pricing algorithms developed for convex NUM, e.g. gradient and subgradient algorithms, cannot be directly applied. In the next two subsections, we will investigate two distributed pricing mechanisms for non-convex ACSCG and provide two sufficient
conditions that guarantee their convergence. Specifically, under these sufficient conditions, both algorithms guarantee that the total utility is monotonically increasing until it converges to a feasible operating point that satisfies the KKT conditions. Similarly as in Section III-A we first assume \( f^k_n(a_{-n}) \) takes the form in (11) and users update their actions in parallel.

### A. Gradient Play

The first distributed pricing algorithm that we consider is gradient play. The update iterations of gradient play need to be properly redefined in presence of real-time information exchange. Specifically, at stage \( t \), users adopting this algorithm exchange price signals \( \{\pi^k_{mn}^{t-1}\} \) using the gradient information at stage \( t - 1 \). Within each iteration, each user first determines the gradient of the objective in (46) based on the price vectors \( \{\pi^k_{mn}^{t-1}\} \) and its own utility function \( u_n(a_n, a_{-n}^{t-1}) \). Then each user updates its action \( a^t_n \) using gradient projection algorithm according to

\[
a^{'k,t}_n = a^{k,t-1}_n + \kappa \left( \frac{\partial u_n(a_n, a_{-n}^{t-1})}{\partial a^k_n} - \sum_{m \neq n} \pi^k_{mn}^{t-1} \right). \tag{47}
\]

and

\[
a^t_n = [a^{'1,t}_n, a^{'2,t}_n, \ldots, a^{'K,t}_n] = \left[ a^{'1,t}_n, a^{'2,t}_n, \ldots, a^{'K,t}_n \right] \| \cdot \|_2 \in A_n. \tag{48}
\]

in which the stepsize \( \kappa > 0 \). The following theorem provides a sufficient condition under which gradient play will converge monotonically provided that we choose small enough constant stepsize \( \kappa \).

**Theorem 7:** If \( \forall n, k, x, y \in A_{-n} \),

\[
\inf_x \frac{\partial^2 h^k_n(x)}{\partial^2 x} > -\infty, \text{ and } \left\| \nabla g^k_n(x) - \nabla g^k_n(y) \right\| \leq L'\|x - y\|, \tag{C7}
\]

gradient play converges for a small enough stepsize \( \kappa \).

**Proof:** This theorem can be proved by showing the gradient of the objective function in (40) is Lipschitz continuous and applying Proposition 3.4 in [31]. See Appendix E for details. \( \blacksquare \)

**Remark 7:** (Application of condition (C7)) A sufficient condition that guarantees the convergence of distributed gradient projection algorithm is the Lipschitz continuity of the gradient of the objective function in (40). For example, in the power control problem in multi-channel networks [21], we have \( h^k_n(x) = \log_2(\alpha^k_n + H^k_{mn}x) \) and \( g^k_n(P_{-n}) = \log_2(\alpha^k_n + \sum_{m \neq n} H^k_{mn}P_{mn}) \). For this configuration, we can immediately verify that condition (C7) is satisfied. Therefore, gradient play can be applied. Moreover, as in [21], if we can further ensure that the problem in (40) is convex for some particular utility functions, gradient play converges to the unique optimal solution of (40) at which achieving KKT conditions implies global optimality.
B. Jacobi Update

We consider another alternative strategy update mechanism called Jacobi update \[32\]. In Jacobi update, every user adjusts its action gradually towards the best response strategy. Specifically, the maximizer of problem (46) takes the following form

\[
B_n^k(a_{-n}) = \left( \frac{\partial h_n^k}{\partial x} \right)^{-1} \left( \lambda_n + \nu_n^k - \nu_n + \sum_{m \neq n} \pi_{mn}^k \right) - \sum_{m \neq n} F_{mn}^k a_{nm}^k,
\]

in which \( \lambda_n, \nu_n^k, \) and \( \nu_n^k \) are the Lagrange multipliers that satisfy complementary slackness in (42) and (43), and \( \pi_{mn}^k \) is defined in (44). In Jacobi update, at stage \( t \), user \( n \) chooses its action according to

\[
a_n^{k,t} = a_n^{k,t-1} + \kappa \left[ B_n^k(a_{-n}^{t-1}) - a_n^{k,t-1} \right],
\]

in which the stepsize \( \kappa \in (0, 1] \). The following theorem establishes a sufficient convergence condition for Jacobi update.

**Theorem 8:** If \( \forall n,k,x,y \in A_{-n} \),

\[
\inf_x \frac{\partial^2 h_n^k(x)}{\partial^2 x} > -\infty, \sup_x \frac{\partial^2 h_n^k(x)}{\partial^2 x} < 0, \quad \| \nabla g_n^k(x) - \nabla g_n^k(y) \| \leq L' \| x - y \|,
\]

Jacobi update converges if the stepsize \( \kappa \) is sufficiently small.

**Proof:** This can be proved using the descent lemma and the mean value theorem. The details of the proof are provided in Appendix F. ■

**Remark 8:** (Relation between condition (C8) and the result in [22]) Shi et al. considered the power allocation for multi-carrier wireless networks with non-separable utilities. Specifically, \( u_n(\cdot) \) takes the form

\[
u_n(P) = r_i \left( \sum_{k=1}^K \log_2 \left( 1 + \frac{H_{nn}^k P_n^k}{\sigma_n^k + \sum_{m \neq n} H_{mn}^k P_m^k} \right) \right),
\]

in which \( r_i(\cdot) \) is an increasing and strictly concave function. Since the utilities are non-separable, the distributed pricing algorithm proposed in [22], which in fact belongs to Jacobi update, requires only one user to update its action profile at each stage while keeping the remaining users’ action fixed. The condition in (C8) gives the convergence condition of the same algorithm in ACSCG. We prove in Theorem 7 that, if the utilities are separable, convergence can still be achieved even if these users update their actions at the same time. Therefore, we do not need an arbitrator to select the single user that updates its action at each stage.

**Remark 9:** (Complexity of signaling) The complexity of message exchange measured in terms of the number of price signals to update in (44) is generally of the order of \( O(KN^2) \). It is worth mentioning that the amount of signaling can be further reduced to \( O(KN) \) in the scenarios where \( g_n^k(\cdot) \) are functions
of $\sum_{m \neq n} F_{mn}^k a_m^k$. In this case, each user only needs to announce one price signal $\pi_n^k$ for each dimension of its action space:

$$\pi_n^k(a_n^k, a_{-n}^k) = -\frac{\partial u_n(a)}{\partial \left( \sum_{m \neq n} F_{mn}^k a_m^k \right)} \quad (52)$$

Consequently, $\pi_{mn}$ can be determined based on $\pi_{mn} = F_{nm}^k a_n^k$, which greatly reduces the overhead of signaling requirement. It is straightforward to check that only $O(KN)$ messages need to be generated and exchanged per iteration in both utility functions (5) and (6).

Remark 10: (Extension to general cases) As a matter of fact, conditions (C7) and (C8) apply to a broader class of multi-user interaction scenarios, including the general model defined in (4). Specifically, as addressed in Remark 7, the Lipschitz continuity of the gradient of $\sum_{n=1}^N u_n(a)$ is sufficient to guarantee that gradient play with a small enough stepsize achieves an operating point at which KKT conditions are satisfied. In addition, we can use the same technique in Appendix F to show the convergence of Jacobi update given that $\sup_x \frac{\partial^2 h^k(x)}{\partial^2 x} < 0$, $\forall n, k$, and the gradient of $\sum_{n=1}^N u_n(a)$ is Lipschitz continuous.

V. Numerical Examples

In Section II-C we present several illustrative examples of ACSCG. This section uses Examples 1 and 3 to illustrate the various distributed algorithms discussed in the paper.

We start with Example 1 to verify the proposed convergence conditions of best response dynamics. Even though it is a simple two-user game with $A_n \subseteq \mathcal{R}^2$, existing results in the literature cannot immediately determine whether or not the best response dynamics in this simple game can globally converge to a NE. Specifically, in Example 1 we have

$$\frac{\partial f_n^1(a_{-n})}{\partial a_{-n}^1} = \frac{a_{-n}}{\sqrt{(a_{-n}^1)^2 + 1}}, \quad \frac{\partial f_n^1(a_{-n})}{\partial a_{-n}^2} = -\frac{a_{-n}^2}{\sqrt{(a_{-n}^2)^2 + 1}},$$

$$\frac{\partial f_n^2(a_{-n})}{\partial a_{-n}^1} = -\frac{a_{-n}^1}{\sqrt{(a_{-n}^1)^2 + 1}}, \quad \frac{\partial f_n^2(a_{-n})}{\partial a_{-n}^2} = \frac{a_{-n}^2}{\sqrt{(a_{-n}^2)^2 + 1}} \quad (53)$$

According to the definition of (27), we have

$$[T_{\text{max}}]_{12} = \max \left\{ \max_{a \in A} \sum_{k=1}^K \left| \frac{\partial f_n^k(a_{-n})}{\partial a_1^k} \right|, \max_{a \in A} \sum_{k=1}^K \left| \frac{\partial f_n^k(a_{-n})}{\partial a_2^k} \right| \right\}$$

$$= \max \left\{ \max_{a \in A_1, a_1^2} \frac{2a_1}{\sqrt{(a_1^1)^2 + 1}}, \max_{a \in A_1, a_2^2} \frac{2a_1^2}{\sqrt{(a_2^1)^2 + 1}} \right\} = \frac{2M_1}{\sqrt{M_1^2 + 1}}, \quad (54)$$

Similarly, we can obtain $[T_{\text{max}}]_{21} = \frac{2M_2}{\sqrt{M_2^2 + 1}}$. Therefore, $\rho(T_{\text{max}}) = \sqrt{\frac{4M_1 M_2}{M_1^2 + 1}}$. It is easy to show that $\rho(T_{\text{max}}) < 1 \iff (M_1^2 - \frac{1}{2})(M_2^2 - \frac{1}{2}) < \frac{1}{4}$. By condition (C4), we know that if $(M_1^2 - \frac{1}{2})(M_2^2 - \frac{1}{2}) < \frac{1}{4}$, the best response dynamics is guaranteed to converge to a unique NE. We numerically simulate a scenario
with parameters $M_1 = \frac{2}{3}$ and $M_2 = 1$ in which condition (C4) holds. We generate multiple initial action profiles of $a^0_1$ and $a^0_2$, iterate the best response dynamics, and obtain the action sequences $a^t_1$ and $a^t_2$. Fig. 2 shows the trajectories of $a^t_1$ and $a^t_2$ for different realizations. We can see that, best response dynamics converges to a unique NE. If we set $M_1 = 2$ and $M_2 = 1$, condition (C4) does not hold any more. We observe from simulations that in many circumstances the best response dynamics will not converge, which agrees with our analysis in Remark 6.

Now we consider Example 3 which is the problem of minimizing queueing delays in a Jackson network. In particular, we consider a network with $N = 5$ nodes and $K = 3$ traffic classes. The total routing probability $1 - r^k_{m0}$ that node $m$ will route packets of class $k$ completing service to other nodes is the same for $\forall m \in \mathcal{N}$. We varied the total routing probability $1 - r^k_{m0}$ and generated multiple sets of network parameters in which $r^k_{mn}$ are uniformly distributed for $n = 1, 2, \cdots, N$, $\mu^k_n$ are uniformly selected in $[4, 5]$ for $\forall n, k$, and $\psi^\text{min}_n$ are uniformly chosen in $[0.6, 1]$ for $n = 1, 2, \cdots, N$.

First of all, we compare the range of validity of the proposed convergence conditions. As we mentioned before, we have $F^k_{mn} = \frac{([I - R^k]^{-1})_{mn}}{([I - R^k]^{-1})_{nn}}$ in this example. Note that $(I - R^k)^{-1} = I + \sum_{i=1}^{\infty} (R^k)^i$ and $R^k$ is a non-negative matrix. Therefore, we can conclude $F^k_{mn} \geq 0, \forall m \neq n, k$. Moreover, since $h^k_n(x) = \frac{1}{\mu^k_n - \psi^\text{min}_n x}$, we choose to compare conditions (C2) and (C3). In Fig. 3 we plot the probability that conditions (C2) and (C3) are satisfied versus the total routing probability $1 - r^k_{m0}$. From Fig. 3 we
can see that the probability of guaranteeing convergence decreases as the routing probability \(1 - r_{m0}^k\) increases and condition (C3) shows a similar but slightly broader validity than (C2). Fig. 4 shows the delay trajectories of three nodes using both sequential and parallel updates in a certain network realization in which (C2) and (C3) are satisfied. We can see that, the parallel update converges faster than the sequential update.

In Fig. 3 we also note that the probability that (C2) or (C3) is satisfied transits very quickly from the almost certain convergence to the non-convergence guarantee as \(1 - r_{m0}^k\) varies from 0.5 to 0.58. Similar observations have been drawn in the multi-channel power control problem [20], where \(\theta = -1\) in (20) and the probability that condition (C3) is satisfied exhibits a neat threshold behavior as the ratio between the source-interferer distance and the source-destination distance varies. In Jackson networks, this threshold can be roughly estimated. Define \([S^k]_{mn} = F^k_{mn}\) for \(m \neq n\) and \([S^k]_{nn} = 0\) for \(n \in \mathcal{N}\). If we fix \(1 - r_{m0}^k\) for \(\forall m, k\), we prove in Appendix G that \(\rho(S^k) \leq \frac{1}{1 - r_{m0}^k} - 1\) for \(\forall k\). Therefore, \(\rho(S^k) < 1\) when \(r_{m0}^k > 0.5\).

We would like to estimate \(\rho(T^{\max})\) and \(\rho(S^{\max})\) based on \(\rho(S^k)\). Note that \(T^{\max}\) defined in (18) is the element-wise maximum over \(S^k\) for \(k = 1, 2, \ldots, K\). Since \(T^{\max}\) and \(S^k\) are all non-negative matrices, we know that \(\rho(T^{\max}) \geq \max_k \rho(S^k)\). In addition, recall the effect of \(\max_{m,n} \zeta_{mn}\) discussed in Remark 4. We can approximate \(\rho(S^{\max})\) defined in (22) using \(\rho(S^{\max}) \approx \max_{m,n} \zeta_{mn} \max_k \rho(S^k)\). Therefore, we expect that \(\rho(T^{\max})\) and \(\rho(S^{\max})\) exceeds 1 for \(r_{m0}^k < 0.5\), which agrees with our observation from
The physical interpretation is that, if the packets exit the network with a probability less than 50% after completing its service, i.e. more than half of the served packets will be routed to other nodes, the strength of the mutual coupling among users becomes too strong and the multi-user interaction in Jackson networks will gradually lose its convergence guarantee.

In addition, we numerically compare two distributed algorithms in which users pass coordination messages in real time, including Jacobi update and gradient play. Fig. 5 shows the delay evolution of both distributed solutions for a particular simulated network in which we set $\kappa = 0.2$. We initialize the system parameters such that $\inf_{n,k} \mu_k - \sum_{m=1}^N r_{mn}^k v_m^k > 0$ and both conditions (C7) and (C8) are satisfied. We can verify that for Example 3, problem (40) is in fact a convex program. Therefore, there exists a unique operating point at which KKT conditions (41)-(43) are satisfied. We can see that, both algorithms cause the total delay to monotonically decrease until it reaches the same performance limit that is strictly better than NE. Using the same stepsize $\kappa$, Jacobi update converges more quickly than gradient play in this example. Similar observations are drawn in the other simulated examples. This is because the update directions of these two algorithms are different. Jacobi update moves directly towards the optimal solution of (46), which is a local approximation of the original optimization program in (40), whereas the gradient play algorithm simply updates the actions along the gradient direction of (40).
VI. CONCLUSION

In this paper, we propose and investigate a new game model, which we refer to as additively coupled sum constrained games, in which each player is subject to a sum constraint and its utility is additively impacted by the remaining users’ actions. The convergence properties of various generic distributed adjustment algorithms, including best response, gradient play, and Jacobi update, have been investigated. The sufficient conditions obtained in this paper generalize the existing results developed in the multi-channel power control problem and can be extended to other applications that belong to ACSCG.

APPENDIX A

PROOF OF THEOREM 1

The following lemma is needed to prove Theorem 1.

Lemma 1: Consider any non-decreasing function \( p(x) \) and non-increasing function \( q(x) \). If there exists a unique \( x^* \) such that \( p(x^*) = q(x^*) \), and the functions \( p(x) \) and \( q(x) \) are strictly increasing and strictly decreasing at \( x = x^* \) respectively, then \( x^* = \arg \min_{x} \{ \max \{ p(x), q(x) \} \} \).

Proof of Lemma 1: See Lemma 1 in [19]. ■

Denote \( a_{n,t}^{k,t} \) as the action of user \( n \) in the \( k \)th dimension after iteration \( t \). Recall that \( [h_{n}^{k}]'(-) > 0 \), for \( \forall n, k \). Therefore, \( \sum_{k=1}^{K} a_{n,t}^{k,t} = M_{n} \) is satisfied at the end of any iteration \( t \) for any user \( n \). Define
\[ [x]^+ = \max\{x, 0\} \text{ and } [x]^- = \max\{-x, 0\}. \] It is straightforward to see that
\[
\sum_{k=1}^{K} [a_n^{k,t+1} - a_n^{k,t}]^+ = \sum_{k=1}^{K} [a_n^{k,t} - a_n^{k,t-1}]^+, \forall n, t, t'.
\] (55)

We also define
\[
p^{n,t}(x) \triangleq \sum_{k=1}^{K} [l_n^k(a_{n}^{t-n}, x) - a_n^{k,t}]^-
\] (56)

and
\[
q^{n,t}(x) \triangleq \sum_{k=1}^{K} [l_n^k(a_{n}^{t-n}, x) - a_n^{k,t}]^+, \forall n, t
\] (57)
in which \(l_n^k(\cdot)\) is defined in (14). Since \(h_n^k(\cdot)\) is a continuous increasing and strictly concave function, it is clear that \(\{\frac{\partial h_n^k}{\partial x}\}^{-1}(\cdot)\) is a continuous decreasing function. If \(p^{n,t}(\lambda_n^{t+1}) \neq 0\) (i.e. it has not converged), \(p^{n,t}(x)\) \((q^{n,t}(x)\), respectively\) is non-decreasing (non-increasing) in \(x\), and strictly increasing (strictly decreasing) at \(x = \lambda_n^{t+1}\). From (55) it is always true that \(p^{n,t}(\lambda_n^{t+1}) = q^{n,t}(\lambda_n^{t+1})\). We first prove the convergence of the parallel update case in (17). For \(\forall n\), we have
\[
\sum_{k=1}^{K} [a_n^{k,t+1} - a_n^{k,t}]^+
\]
\[
= \max\left\{ \sum_{k=1}^{K} [a_n^{k,t+1} - a_n^{k,t}]^+, \sum_{k=1}^{K} [a_n^{k,t+1} - a_n^{k,t}]^- \right\}
\] (58)

\[
= \max\{p^{n,t}(\lambda_n^{t+1}), q^{n,t}(\lambda_n^{t+1})\}
\] (59)

\[
\leq \max\{p^{n,t}(\lambda_n^{t+1}), q^{n,t}(\lambda_n^{t+1})\}
\] (60)

\[
\leq \max\left\{ \sum_{k=1}^{K} \left[ \sum_{m \neq n} F_{mn}^k(a_m^{k,t} - a_m^{k,t-1}) \right]^+, \sum_{k=1}^{K} \left[ \sum_{m \neq n} F_{mn}^k(a_m^{k,t} - a_m^{k,t-1}) \right]^-ight\}
\] (61)

\[
\leq \max\left\{ \sum_{k=1}^{K} \sum_{m \neq n} \left[ F_{mn}^k(a_m^{k,t} - a_m^{k,t-1}) \right]^+, \sum_{k=1}^{K} \sum_{m \neq n} \left[ F_{mn}^k(a_m^{k,t} - a_m^{k,t-1}) \right]^-ight\}
\] (62)

\[
= \max\left\{ \sum_{m \neq n} \sum_{k=1}^{K} \left[ F_{mn}^k(a_m^{k,t} - a_m^{k,t-1}) \right]^+, \sum_{m \neq n} \sum_{k=1}^{K} \left[ F_{mn}^k(a_m^{k,t} - a_m^{k,t-1}) \right]^-ight\}
\] (63)

\[
\leq \sum_{m \neq n} \max_k |F_{mn}^k| \cdot \left\{ \sum_{k=1}^{K} \left[ a_m^{k,t} - a_m^{k,t-1} \right]^+ + \sum_{k=1}^{K} \left[ a_m^{k,t} - a_m^{k,t-1} \right]^+ \right\}
\] (64)

\[
= \sum_{m \neq n} 2 \max_k |F_{mn}^k| \cdot \sum_{k=1}^{K} \left[ a_m^{k,t} - a_m^{k,t-1} \right]^+, \forall n
\] (65)

where (58) and (65) follows from (55), (59) follows from the definition of \(p^{n,t}\) and \(q^{n,t}\) in (56) and (57), (60) is due to Lemma 1 in which \(x = \lambda_n^{t}\), (61) follows from the definition of \(p^{n,t}\) and \(q^{n,t}\), the
expression of $a^k_t$ in (17), and the fact that $[[x^b_a - y^b_a]+ \leq [x - y]^+$ and $[[x^b_a - y^b_a]- \leq [x - y]^-$, (62) is due to the fact that $[x + y]^+ \leq [x]^+ + [y]^+$ and $[x + y]^- \leq [x]^+ + [y]^-$, (64) follows by using $\sum_k x_k y_k^+ \leq \sum_k x_k \|y_k\| = \sum_k x_k \max_k ([y_k]^+ + [y_k]^-) \leq \max_k \sum_k ([y_k]^+ + [y_k]^-)$. For user $n$, we define that $e^t_n = \left[a^k_t - a^k_{t-1}\right]^+$. Inequality (65) can be written as $e^t_{n+1} \leq \sum_{m \neq n} T^m e^t_m$ in which $T^m$ is defined in (18).

Since $T^m$ is a nonnegative matrix, by the Perron-Frobenius Theorem [31], there exists a positive vector $\bar{w} = [\bar{w}_1 \ldots \bar{w}_N]$ such that

$$\|T^m\|_{\infty, \text{mat}} = \rho(T^m),$$

where $\| \cdot \|_{\infty, \text{mat}}$ is the weighted maximum matrix norm defined as

$$\|A\|_{\infty, \text{mat}} = \max_{i=1,2,\ldots,N} \frac{1}{\bar{w}_i} \sum_{j=1}^N |A|_{ij} \bar{w}_j, \quad A \in \mathbb{R}^{N \times N}.$$

Define the vectors $e^{t+1} = [e^{t+1}_1, e^{t+1}_2, \ldots, e^{t+1}_N]^T$ and $e^t = [e^t_1, e^t_2, \ldots, e^t_N]^T$. The set of inequalities in (65) can be expressed in the vector form as $0 \leq e^{t+1} \leq T^m e^t$. By choosing the vector $\bar{w}$ that satisfies $\|T^m\|_{\infty, \text{mat}} = \rho(T^m)$ and applying the infinity norm $\| \cdot \|_{\infty}$, we obtain the following

$$\|e^{t+1}\|_{\infty} \leq 2 \|T^m e^t\|_{\infty} \leq 2 \|T^m\|_{\infty, \text{mat}} \|e^t\|_{\infty},$$

Finally, based on (65) and (68), it follows that

$$\max_{n \in N} \frac{e^t_{n+1}}{\bar{w}_n} = \|e^{t+1}\|_{\infty} \leq 2 \|T^m\|_{\infty, \text{mat}} \|e^t\|_{\infty} \leq 2 \|T^m\|_{\infty, \text{mat}} \cdot \max_{n \in N} \frac{e^t_n}{\bar{w}_n} = 2 \rho(T^m) \cdot \max_{n \in N} \frac{e^t_n}{\bar{w}_n}$$

Therefore, if $\|T^m\|_{\infty, \text{mat}} = \rho(T^m) < \frac{1}{2}$, the best response dynamics in (17) is a contraction with the modulus $\|T^m\|_{\infty, \text{mat}}$ with respect to the norm $\max_{n \in N} \frac{\|e^t_n\|_{\infty}}{\bar{w}_n}$. We can conclude that, the best response dynamics has a unique fixed point $a^*$ and, given any initial value $a^0$, the update sequence $\{a^t\}$ converges to the fixed point $a^*$.

In the sequential update case, the convergence result can be established by using the proposition 1.4 in [31]. The key step is to obtain

$$\max_{n \in N} \frac{e^t_{n+1}}{\bar{w}_n} \leq 2 \rho(T^m) \cdot \max \left\{ \max_{j < n} \frac{e^t_{j+1}}{\bar{w}_j}, \max_{j \geq n} \frac{e^t_j}{\bar{w}_j} \right\}.$$ A simple induction on $n$ yields

$$\max_{n \in N} \frac{e^t_{n+1}}{\bar{w}_n} \leq 2 \rho(T^m) \cdot \max_{n \in N} \frac{e^t_n}{\bar{w}_n}$$

for all $n$. Therefore, inequality (65) also holds for the sequential update and the contraction iteration globally converges to a unique equilibrium. ■
APPENDIX B

PROOF OF THEOREM 2

If $F_{mn}^k \geq 0, \forall m \neq n, k$, the inequalities after (63) become

$$\max \{ \sum_{m \neq n} \sum_{k=1}^{K} F_{mn}^k (a_m^{k,t} - a_m^{k,t-1})^+, \sum_{m \neq n} \sum_{k=1}^{K} F_{mn}^k (a_m^{k,t} - a_m^{k,t-1})^- \} \leq \sum_{m \neq n} \max_k F_{mn}^k \cdot \max \{ \sum_{k=1}^{K} (a_m^{k,t} - a_m^{k,t-1})^+, \sum_{k=1}^{K} (a_m^{k,t} - a_m^{k,t-1})^- \}$$

(72)

$$= \sum_{m \neq n} \max_k F_{mn}^k \cdot \sum_{k=1}^{K} (a_m^{k,t} - a_m^{k,t-1})^+.$$  

(74)

Similarly, for $F_{mn}^k \leq 0, \forall m \neq n, k$, we have

$$\max \{ \sum_{m \neq n} \sum_{k=1}^{K} F_{mn}^k (a_m^{k,t} - a_m^{k,t-1})^+, \sum_{m \neq n} \sum_{k=1}^{K} F_{mn}^k (a_m^{k,t} - a_m^{k,t-1})^- \} \leq \sum_{m \neq n} \max_k (-F_{mn}^k) \cdot \max \{ \sum_{k=1}^{K} (a_m^{k,t} - a_m^{k,t-1})^+, \sum_{k=1}^{K} (a_m^{k,t} - a_m^{k,t-1})^- \} \leq \sum_{m \neq n} \max_k (-F_{mn}^k) \cdot \sum_{k=1}^{K} (a_m^{k,t} - a_m^{k,t-1})^+.$$  

(77)

Therefore, if $F_{mn}^k \geq 0, \forall m \neq n, k$ or $F_{mn}^k \leq 0, \forall m \neq n, k$, given (C2), the sequence $\{a_n^t\}$ contracts with the modulus $\rho(T^\text{max}) < 1$ under the norm $\max_{n \in \mathcal{N}} \sum_i |x_i^n|^2 / \sum_i |x_i^n|$ and the convergence follows readily.

APPENDIX C

PROOF OF THEOREM 3

Let $\| \cdot \|^w_2$ denote the weighted Euclidean norm with weights $w = [w_1 \ldots w_K]^T$, i.e. $\|x\|^w_2 \triangleq (\sum_i w_i |x_i|^2)^{1/2}$.

Define the simplex

$$\mathcal{S} \triangleq \left\{ x \in \mathbb{R}^K : \frac{1}{K} \sum_{k=1}^{K} x_k = 1, x_k^\text{min} \leq x_k \leq x_k^\text{max}, \forall k = 1, 2, \ldots, K \right\},$$

(78)

in which $\sum_k x_k^\text{max} \geq 1$. The following lemma is needed to prove Theorem 3.

Lemma 2: The projection with respect to the weighted Euclidean norm with weights $w$, of the $K$-dimensional real vector $-x_0 \triangleq -[x_{0,1}, \ldots, x_{0,K}]^T$ onto the simplex $\mathcal{S}$ defined in (78), denoted by $[-x_0]^w_\mathcal{S}$, is the optimal solution to the following convex optimization problem:

$$[-x_0]^w_\mathcal{S} \triangleq \arg \min_{x \in \mathcal{S}} \|x - (-x_0)\|^w_2$$

(79)
and takes the following form:

\[ x_k^* = \left[ \frac{\lambda}{w_k} - x_{0,k} \right] x_{0,k}^{\max} = 1, \ldots, K \]  
(80)

where \( \lambda > 0 \) is chosen in order to satisfy the constraint \( \frac{1}{K} \sum_{k=1}^{K} x_k^* = 1 \).

**Proof of Lemma 2**  See Corollary 2 in [20]. ■

For \( h^k_n(\cdot) \) defined in (20), user \( n \) updates its action according to

\[ a_{nk}^* = l^k_n(a_{-n}, \lambda^*) = \left[ \left( \frac{1}{F_{kn}} \right)^{1 + \frac{1}{\alpha}} \cdot (\lambda^*)^\frac{1}{\alpha} - \frac{\alpha_n}{F_{kn}} - \sum_{m \neq n} F_{mn} a_{mk} \right] a_{nk}^{\max}. \]  
(81)

and \( \lambda^* \) is chosen to satisfy \( \sum_{k=1}^{K} a_{nk}^* = M_n \). Define the vector update operator as \( [BR(a_{-n})]_k \triangleq a_{nk}^* \) and the coupling vector as

\[ [C_n(a_{-n})]_k \triangleq \frac{\alpha_n}{F_{kn}} + \sum_{m \neq n} F_{mn} a_{mk} \]  
(82)

with \( k \in \{1, \ldots, K\} \). We also define

\[ F'_{mn} \triangleq \text{diag} \left( F_{1mn}, F_{2mn}, \ldots, F_{Kmn} \right) \]  
(83)

and

\[ \alpha'_n \triangleq \left[ \alpha_n^1, \alpha_n^2, \ldots, \alpha_n^K \right]^T. \]  
(84)

Therefore, the coupling vector can be alternatively rewritten as

\[ C_n(a_{-n}) = \alpha'_n + \sum_{m \neq n} F'_{mn} a_m. \]  
(85)

Define a weight matrix \( W = [w_1 \ldots w_N] \) in which the element \( [W]_{kn} \) is chosen according to

\[ [W]_{kn} = [w_n]_k = \left( \frac{F_{kn}}{F_{mn}} \right)^{1 + \frac{1}{\alpha}}. \]  
(86)

By Lemma 2, we know that the vector update operator \( BR_n(a_{-n}) \) in (21) can be interpreted as the projection of the coupling vector \( -C_n(a_{-n}) \) onto user \( n \)'s action set \( A_n \) with respect to \( \| \cdot \|_{w_n} \), i.e.

\[ BR_n(a_{-n}) = [-C_n(a_{-n})]_{w_n}. \]  
(87)

Given any \( a^{(1)}, a^{(2)} \in A \), we define respectively, for each user \( n \), the weighted Euclidean distances between these two vectors and their projected vectors using (87) as \( e_n = \| a^{(2)} - a^{(1)} \|_{w_n} \) and \( e_{BR_n} = \| BR_n(a_{-n}) - BR_n(a_{-n}) \|_{w_n} \). Again, we first prove the convergence of the parallel update case in (17).
We have \( \forall n \in \mathcal{N} \),

\[
e_{BR_n} = \left\| \left[ -C_n(a_{-n}^{(1)}) \right]_{A_n} - \left[ -C_n(a_{-n}^{(2)}) \right]_{A_n} \right\|_{w_n}^{w_n}
\]

\[
\leq \left\| C_n(a_{-n}^{(2)}) - C_n(a_{-n}^{(1)}) \right\|_{w_n}
\]

\[
= \left\| \sum_{m \neq n} F'_{mn} a_m^{(2)} - \sum_{m \neq n} F'_{mn} a_m^{(1)} \right\|_{w_n}^{w_n} = \left\| \sum_{m \neq n} F'_{mn} (a_m^{(2)} - a_m^{(1)}) \right\|_{w_n}^{w_n}
\]

(88)

\[
\leq \sum_{m \neq n} \left\| F'_{mn} (a_m^{(2)} - a_m^{(1)}) \right\|_{2} = \sum_{m \neq n} \sqrt{\sum_{k=1}^{K} \left[ w_n k \right] \left( \left| \left[ F'_{mn} kk \right] \right| \right) \left( a_m^{(2)k} - a_m^{(1)k} \right)^2}
\]

(90)

\[
= \sum_{m \neq n} \sqrt{\sum_{k=1}^{K} \left[ w_m k \right] \left( \left| \left[ F'_{mn} kk \right] \right| \right) \left( a_m^{(2)k} - a_m^{(1)k} \right)^2}
\]

(91)

\[
\leq \sum_{m \neq n} \max_k \left( \left| \left[ F'_{mn} kk \right] \right| \left( \left[ w_n k \right] \right) \right) \sqrt{\sum_{k=1}^{K} \left[ w_m k \right] \left( a_m^{(2)k} - a_m^{(1)k} \right)^2}
\]

(92)

\[
= \sum_{m \neq n} \max_k \left( \left| \left[ F'_{mn} kk \right] \right| \left( \left[ w_m k \right] \right) \right) \left\| a_m^{(2)} - a_m^{(1)} \right\|_{w_n}^{w_n}
\]

(93)

\[
= \sum_{m \neq n} \left[ S_{\max}^m \right]_{mn} e_m,
\]

(94)

where (88) follows from the non-expansion property of the projector \( \left[ \cdot \right]_{A_n} \) in the norm \( \left\| \cdot \right\|_{w_n}^{w_n} \) (See Proposition 3.2(c) in [31]), (90) follows from the triangle inequality [33], and \( S_{\max}^m \) in (94) is defined according to (22).

The rest of the proof is similar as the proof after equation (65) in Appendix A. Details are omitted due to space limitations. ■
APPENDIX D

PROOF OF THEOREM 4

The beginning part of the proof is the same as the proof of Theorem 1. For any user \( n \) with general \( f_n^k(\cdot) \), the inequalities after (59) become

\[
\sum_{k=1}^{K} |a_n^{k,t+1} - a_n^{k,t}|^+ \leq \max \{ p_n^t(\lambda_n^t), q_n^t(\lambda_n^t) \}
\]

\[
= \max \left\{ \sum_{k=1}^{K} \left[ f_n^k(a_{-n}^t) - f_n^k(a_{-n}^{t-1}) \right]^+, \sum_{k=1}^{K} \left[ f_n^k(a_{-n}^t) - f_n^k(a_{-n}^{t-1}) \right]^- \right\}
\]

\[= \max \left\{ \sum_{k=1}^{K} \left[ \sum_{m \neq n} \sum_{k'=1}^{K} \frac{\partial f_n^k}{\partial a_m^k}(\xi_{-n}^t)(a_m^{k',t} - a_m^{k',t-1}) \right]^+, \sum_{k=1}^{K} \left[ \sum_{m \neq n} \sum_{k'=1}^{K} \frac{\partial f_n^k}{\partial a_m^k}(\xi_{-n}^t)(a_m^{k',t} - a_m^{k',t-1}) \right]^- \right\}
\]

\[\leq \max \left\{ \sum_{m \neq n} \sum_{k' = 1}^{K} \sum_{k=1}^{K} \frac{\partial f_n^k}{\partial a_m^k}(\xi_{-n}^t)(a_m^{k',t} - a_m^{k',t-1}) \right]^+, \sum_{m \neq n} \sum_{k' = 1}^{K} \sum_{k=1}^{K} \frac{\partial f_n^k}{\partial a_m^k}(\xi_{-n}^t)(a_m^{k',t} - a_m^{k',t-1}) \right]^-ight\}
\]

\[= \max \left\{ \max_{k'} \sum_{k=1}^{K} \left| \frac{\partial f_n^k}{\partial a_m^k}(\xi_{-n}^t) \right| \cdot \sum_{k=1}^{K} \left[ (a_m^{k',t} - a_m^{k',t-1}) \right]^+ + \sum_{k' = 1}^{K} \left[ (a_m^{k',t} - a_m^{k',t-1}) \right]^-. \right\}
\]

\[= \sum_{m \neq n} 2 \cdot \left\{ \max_{k'} \sum_{k=1}^{K} \left| \frac{\partial f_n^k}{\partial a_m^k}(\xi_{-n}^t) \right| \cdot \sum_{k'=1}^{K} \left[ (a_m^{k',t} - a_m^{k',t-1}) \right]^+ \right\}
\]

where (95) follows from the definition of \( p_n^t(\cdot) \) and \( q_n^t(\cdot) \) and the expression of \( a_n^{k,t} \) and \( B_n^k(a_{-n}, \lambda) \) in (17) and (26). (96) follows from the mean value theorem for vector-valued functions with \( \xi^t = \alpha a^t + (1 - \alpha) a^{t-1} \) and \( \alpha \in [0, 1] \). By (C4), it is straightforward to show that the iteration is a contraction by following the same arguments in Appendix A. The rest of the proof is omitted. ■

APPENDIX E

PROOF OF THEOREM 6

The gradient play algorithm in (46) is in fact a gradient projection algorithm with constant stepsize \( \kappa \). In order to establish its convergence, we first need to prove that the gradient of the objective in (40) is
Lipschitz continuous, with a Lipschitz constant given by $L > 0$, i.e.
\[
\left\| \nabla \left( \sum_{n=1}^{N} u_n(x) \right) - \nabla \left( \sum_{n=1}^{N} u_n(y) \right) \right\| \leq L \|x - y\|, \ \forall x, y \in \mathcal{A}.
\]  
(101)

It is known that it has the property of Lipschitz continuity if it has a Hessian bounded in the Euclidean norm.

The Hessian matrix $\mathbf{H}$ of $\sum_{n=1}^{N} u_n(\mathbf{a})$ can be decomposed into two matrices: $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$, in which the elements of matrix $\mathbf{H}_1$ are
\[
\frac{\partial^2}{\partial a^m_i \partial a^j_l} \left[ \sum_{n=1}^{N} \sum_{k=1}^{K} h_n^k (a_n^k + \sum_{m \neq n} F_{mn}^k a_{m}^k) \right] = \begin{cases} \sum_{n=1}^{N} \frac{\partial^2 h_n^k}{\partial x^2} (a_n^k + \sum_{m \neq n} F_{mn}^k a_{m}^k) F_{mn}^k F_{ln}^k, & \text{if } k = j \\ 0, & \text{otherwise.} \end{cases}
\]  
(102)

with $F_{mn}^k = 1$ and the elements of matrix $\mathbf{H}_2$ are
\[
- \frac{\partial^2}{\partial a^m_i \partial a^j_l} \left[ \sum_{n=1}^{N} \sum_{k=1}^{K} g_n^k(\mathbf{a}_{-n}) \right] = - \sum_{n=1}^{N} \sum_{k=1}^{K} \frac{\partial^2 g_n^k(\mathbf{a}_{-n})}{\partial a^m_i \partial a^j_l}.
\]  
(103)

Recall that $g_n^k(\cdot)$ is Lipschitz continuous and it satisfies
\[
\left\| \nabla g_n^k(x) - \nabla g_n^k(y) \right\| \leq L' \|x - y\|, \ \forall n, k, x, y \in \mathcal{A}_{-n}.
\]

Consequently, we have $\|\mathbf{H}_2\|_2 \leq NK L'$. As a result, we can estimate the Lipschitz constant $L$ using the following inequalities
\[
\|\mathbf{H}\|_2 \leq \|\mathbf{H}_1\|_2 + \|\mathbf{H}_2\|_2 \leq \sqrt{\|\mathbf{H}_1\|_1 \|\mathbf{H}_1\|_\infty} + NK L' \leq \sup_{x,n,k} \left| \frac{\partial^2 h_n^k}{\partial x^2} \right| \cdot \max_{k,l} \sum_{m=1}^{N} \sum_{m=1}^{N} |F_{mn}^k F_{ln}^k| + NK L'.
\]  
(104)

We can choose the RHS of (104) as the Lipschitz constant $L$ . By Proposition 3.4 in [31], we know that if $0 < \kappa < 2/L$, the sequence $\mathbf{a}^t$ generated by the gradient projection algorithm in (47) and (48) converges to a limiting point at which the KKT conditions in (41)–(43) are satisfied. ■

APPENDIX F

PROOF OF THEOREM 7

We know from the proof of Theorem 6 that, under Condition (C7), $\sum_{n=1}^{N} u_n(\mathbf{a})$ is Lipschitz continuous and the inequality in (101) holds. Recall that $\sum_{n=1}^{N} u_n(x)$ is continuously differentiable. Therefore, by the descent lemma [31], we have
\[
\sum_{n=1}^{N} u_n(x) \geq \sum_{n=1}^{N} u_n(y) + (x - y)^T \cdot \nabla \left( \sum_{n=1}^{N} u_n(y) \right) - \frac{L}{2} \|x - y\|^2, \ \forall x, y \in \mathcal{A}.
\]  
(105)
Therefore, in order to prove \( \sum_{n=1}^{N} u_n(a^t) \geq \sum_{n=1}^{N} u_n(a^{t-1}) \), we only need to show that
\[
(a^t - a^{t-1})^T \cdot \nabla \left( \sum_{n=1}^{N} u_n(a^{t-1}) \right) \geq \frac{L}{2} \|a^t - a^{t-1}\|_2^2
\] (106)
for sufficiently small \( \kappa \). Substituting (50) into (106), we can see that it is equivalent to
\[
\sum_{n=1}^{N} \sum_{k=1}^{K} (B_{kn}^k(a_{kn}^{t-1}) - a_{kn}^{k,t-1}) \cdot \frac{\partial}{\partial a_{kn}^{k,t-1}} \sum_{n=1}^{N} u_n(a^{t-1}) \geq \kappa \cdot \frac{L}{2} \sum_{n=1}^{N} \sum_{k=1}^{K} (B_{kn}^k(a_{kn}^{t-1}) - a_{kn}^{k,t-1})^2.
\] (107)

By equation (49), we have
\[
B_{kn}^k(a_{kn}^{t-1}) - a_{kn}^{k,t-1} = \left\{ \frac{\partial h_n^k}{\partial x} \right\}^{-1} (\lambda_n + \nu_k - \nu_n') - \sum_{m \neq n}^N \sum_{m \neq n} (F_{mn}^k a_{mn}^k) - \sum_{m \neq n}^N \sum_{m \neq n} \pi_{mn} k_{mn}^{k,t-1} - a_{kn}^{k,t-1}
\] (108)
and
\[
\frac{\partial}{\partial a_{n}^{k,t-1}} \sum_{n=1}^{N} u_n(a^{t-1}) = \frac{\partial h_n^k}{\partial x} (a_{n}^{k,t-1} + \sum_{m \neq n} F_{mn}^k a_{mn}^k) - \sum_{m \neq n} \pi_{mn} k_{mn}^{k,t-1}.
\] (109)

By the mean value theorem, there exists \( \xi_n^k \in \mathcal{R} \) such that
\[
\frac{\partial h_n^k}{\partial x} (a_{n}^{k,t-1} + \sum_{m \neq n} F_{mn}^k a_{mn}^k) - \sum_{m \neq n} \pi_{mn} k_{mn}^{k,t-1} = \frac{\partial h_n^k}{\partial x} (a_{n}^{k,t-1} + \sum_{m \neq n} F_{mn}^k a_{mn}^k) - \sum_{m \neq n} \pi_{mn} k_{mn}^{k,t-1} = \frac{\partial^2 h_n^k}{\partial x^2} (\xi_n^k) \cdot \left\{ a_{n}^{k,t-1} + \sum_{m \neq n} F_{mn}^k a_{mn}^k - \left\{ \frac{\partial h_n^k}{\partial x} \right\}^{-1} (\lambda_n + \nu_k - \nu_n') + \sum_{m \neq n} \pi_{mn} k_{mn}^{k,t-1} \right\} + \lambda_n + \nu_k - \nu_n'.
\]

Multiplying (108) and (109) leads to
\[
\sum_{n=1}^{N} \sum_{k=1}^{K} (B_{kn}^k(a_{kn}^{t-1}) - a_{kn}^{k,t-1}) \cdot \frac{\partial}{\partial a_{n}^{k,t-1}} \sum_{n=1}^{N} u_n(a^{t-1})
\]
\[
= - \sum_{n=1}^{N} \sum_{k=1}^{K} \frac{\partial^2 h_n^k}{\partial x^2} (\xi_n^k) \cdot \left\{ \left( \frac{\partial h_n^k}{\partial x} \right)^{-1} (\lambda_n + \nu_k - \nu_n') + \sum_{m \neq n} \pi_{mn} k_{mn}^{k,t-1} - a_{kn}^{k,t-1} - \sum_{m \neq n} F_{mn}^k a_{mn}^k \right\}^2
\]
\[
+ \sum_{n=1}^{N} \sum_{k=1}^{K} (B_{kn}^k(a_{kn}^{t-1}) - a_{kn}^{k,t-1}) \cdot (\lambda_n + \nu_k - \nu_n')
\] (110)

In the following, we differentiate two cases in which the Lagrange multipliers \( \lambda_n, \nu_n^k, \nu_n'^k \) take different values.
First of all, if \( \lambda_n = \nu_n^k = \nu_n^\prime = 0 \) for all \( k, n \), equation (110) can be simplified as
\[
\sum_{n=1}^{N} \sum_{k=1}^{K} \left( B_n^k(a_{-n}^{t-1}) - a_{n,t-1}^k \right) \cdot \frac{\partial}{\partial a_{n,t-1}^k} \sum_{n=1}^{N} u_n(a_{-n}^{t-1}) = -\sum_{n=1}^{N} \sum_{k=1}^{K} \frac{\partial^2 h_n^k(x)}{\partial^2 x} \cdot \left\{ \left( \frac{\partial h_n^k(x)}{\partial x} \right)^{-1} \left( \lambda_n + \nu_n^k - \nu_n^\prime + \sum_{m \neq n} \pi_{mn}^k \right) - a_{n,t-1}^k - \sum_{m \neq n} F_{mn}^k a_{m,t-1}^k \right\}^2.
\]
(111)

On the other hand, if \( \lambda_n > 0 \), \( \nu_n^k > 0 \), or \( \nu_n^\prime > 0 \) for some \( k, n \). Due to complementary slackness in (42) and (43), We know that
\[
\lambda_n > 0 \Rightarrow \sum_{k=1}^{K} B_n^k(a_{-n}^{t-1}) = \lambda_n \geq \sum_{k=1}^{K} a_{n,t-1}^k,
\]
\[\nu_n^k > 0 \Rightarrow B_n^k(a_{-n}^{t-1}) = \nu_n^k \geq a_{n,t-1},
\]
\[\nu_n^\prime > 0 \Rightarrow B_n^k(a_{-n}^{t-1}) = \nu_n^k \leq a_{n,t-1}.
\]

As a result, the last term in (110) satisfy
\[
\sum_{n=1}^{N} \sum_{k=1}^{K} \left( B_n^k(a_{-n}^{t-1}) - a_{n,t-1}^k \right) \cdot \left( \lambda_n + \nu_n^k - \nu_n^\prime + \sum_{m \neq n} \pi_{mn}^k \right) - a_{n,t-1}^k - \sum_{m \neq n} F_{mn}^k a_{m,t-1}^k \geq 0.
\]
(112)

Therefore, in both cases, the following inequality holds
\[
\sum_{n=1}^{N} \sum_{k=1}^{K} \left( B_n^k(a_{-n}^{t-1}) - a_{n,t-1}^k \right) \cdot \frac{\partial}{\partial a_{n,t-1}^k} \sum_{n=1}^{N} u_n(a_{-n}^{t-1}) \geq -\sum_{n=1}^{N} \sum_{k=1}^{K} \sup_x \left( \frac{\partial^2 h_n^k(x)}{\partial^2 x} \right) \cdot \left\{ \left( \frac{\partial h_n^k(x)}{\partial x} \right)^{-1} \left( \lambda_n + \nu_n^k - \nu_n^\prime + \sum_{m \neq n} \pi_{mn}^k \right) - a_{n,t-1}^k - \sum_{m \neq n} F_{mn}^k a_{m,t-1}^k \right\}^2.
\]
(113)

Finally, we can conclude that the inequality in (107) holds for \( \kappa \leq \frac{2}{T} \cdot \left( -\max_{n,k} \sup_x \frac{\partial^2 h_n^k(x)}{\partial^2 x} \right) \). Recall that Jacobi update requires \( \kappa \in (0, 1] \). The stepsize \( \kappa \) can be eventually chosen as \( 0 < \kappa \leq \min\{ \frac{2}{T} \cdot \left( -\max_{n,k} \sup_x \frac{\partial^2 h_n^k(x)}{\partial^2 x} \right) \}, 1 \} \]

**APPENDIX G**

**Upper bound of \( \rho(S^k) \)**

Denote \( 1^T = [1 \ 1 \cdots 1]^T \). If we fix \( 1 - r_{m0}^k \) for \( \forall m, k \), we have \( 1^T R^k = (1 - r_{m0}^k)1^T \). Note that \( \Upsilon^k = (1 - R^k)^{-1} = 1 + \sum_{i=1}^{\infty} (R^k)^i \). We have \( 1^T \Upsilon^k = 1^T (1 + \sum_{i=1}^{\infty} (R^k)^i) = 1^T + (1 - r_{m0}^k)1^T \Upsilon^k \) and
$\mathbf{1}^T \Upsilon^k = \frac{1}{1 - r_{m0}} \mathbf{1}^T$. Therefore, $|\Upsilon^k|_1 = \frac{1}{r_{m0}}$. Since $P^k_{mn} = \frac{[\Upsilon^k]_{mn}}{[\Upsilon^k]_{nn}}$ and $\Upsilon^k = I + \sum_{i=1}^{\infty} (R^k)^i$, we know $[\Upsilon^k]_{nn} \geq 1$ for $\forall n$. Denote a diagonal matrix $\text{diag}(\Upsilon^k)$ with the entries of $\Upsilon^k$ on the diagonal. Recall that $[S^k]_{mn} = P^k_{mn}$ for $m \neq n$, and $[S^k]_{nn} = 0$ for $n \in \mathcal{N}$. We can conclude that $\rho(S^k) \leq |S^k|_\infty \leq |(\Upsilon^k)^T - \text{diag}(\Upsilon^k)|_\infty \leq |(\Upsilon^k)^T|_\infty - 1 = |\Upsilon^k|_1 - 1 = \frac{1}{r_{m0}} - 1$.

REFERENCES

[1] E. Altman, T. Boulogne, R. El-Azouzi, T. Jimenez, and L. Wynter, “A survey on networking games in telecommunications,” *Computer Operation Research*, vol. 33, pp. 286-311, Feb. 2006.

[2] A. MacKenzie and S. Wicker, “Game theory and the design of self-configuring, adaptive wireless networks,” *IEEE Commun. Magazine*, vol. 39, pp. 126-131, Nov. 2001.

[3] M. Felegyhazi and J. P. Hubaux, “Game theory in wireless networks: a tutorial.” *EPFL Technical Report*, LCA-REPORT-2006-002, Feb., 2006.

[4] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani, *Algorithmic Game Theory*. Cambridge University Press, Sep., 2007.

[5] E. A. Jorswieck, E. G. Larsson, M. Luise, and H. V. Poor, “Game theory in signal processing and communications,” *IEEE Signal Process. Magazine*, vol. 26, no. 5, Sep. 2009.

[6] D. Fudenberg and J. Tirole, *Game Theory*. Cambridge, MA: MIT Press, 1991.

[7] J. Rosen, “Existence and uniqueness of equilibrium points for concave n-person games,” *Econometrica*, vol. 33, pp. 520-534, July 1965.

[8] D. M. Topkis, *Supermodularity and Complementarity*. Princeton, NJ: Princeton Univ. Press, 1998.

[9] D. Yao, “S-modular games with queueing applications,” *Queueing Syst.*, vol. 21, pp. 449-475, 1995.

[10] E. Altman and Z. Altman, “S-modular games and power control in wireless networks,” *IEEE Trans. Automatic Control*, vol. 48, no. 5, pp. 839-842, May 2003.

[11] R. Rosenthal, “A class of games possessing pure-strategy Nash equilibria,” *International Journal of Game Theory*, vol. 2, pp. 65-67, 1973.

[12] D. Monderer and L. S. Shapley, “Potential games,” *Games Econ. Behav.*, vol. 14, no. 1, pp. 124-143, May 1996.

[13] G. Scutari, S. Barbarossa, D. P. Palomar, “Potential games: A framework for vector power control problems with coupled constraints,” *Proc. IEEE ICASSP*, Toulouse, May 2006.

[14] F. Kelly, A. K. Maulloo, and D. K. H. Tan, “Rate control in communication networks: shadow prices, proportional fairness and stability,” *Journal of the Operational Research Society*, vol. 49, pp. 237-252, 1998.

[15] M. Chiang, S. H. Low, A. R. Calderbank, and J. C. Doyle, “Layering as optimization decomposition,” *Proceedings of the IEEE*, vol. 95, pp. 255-312, Jan 2007.

[16] D. P. Palomar and M. Chiang, “A tutorial on decomposition methods for network utility maximization,” *IEEE J. Sel. Areas Commun.*, vol. 24, no. 8, pp. 1439-1451, 2006.

[17] W. Yu, G. Ginis, and J. Cioffi, “Distributed multiuser power control for digital subscriber lines,” *IEEE J. Sel. Areas Commun.*, vol. 20, no. 5, pp. 1105-1115, June 2002.

[18] S. T. Chung, J. L. Seung, J. Kim, and J. Cioffi, “A game-theoretic approach to power allocation in frequency-selective Gaussian interference channels,” *Proc. IEEE Int. Symp. on Inform. Theory*, p. 316, June 2003.
[19] R. Cendrillon, J. Huang, M. Chiang, and M. Moonen, “Autonomous spectrum balancing for digital subscriber lines,” *IEEE Trans. on Signal Process.*, vol. 55, no. 8, pp. 4241-4257, Aug. 2007.

[20] G. Scutari, D. P. Palomar, and S. Barbarossa, “Optimal linear precoding strategies for wideband noncooperative systems based on game theory - Part II: Algorithms,” *IEEE Trans. Signal Process.*, vol. 56, no. 3, pp. 1250-1267, Mar. 2008.

[21] J. Huang, R. Berry, and M. Honig, “Distributed interference compensation for wireless networks,” *IEEE J. Sel. Areas Commun.*, vol. 24, no. 5, pp. 1074-1084, May 2006.

[22] C. Shi, R. Berry, and M. Honig, “Distributed interference pricing for OFDM wireless networks with non-separable utilities,” *Proc. of Conference on Information Sciences and Systems (CISS)*, pp. 755-760, Mar. 19-21, 2008.

[23] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[24] D. Gabay and H. Moulin, “On the uniqueness and stability of Nash equilibria in noncooperative games,” *Applied Stochastic Control in Econometrics and Management Science*, North-Holland, Amsterdam, Holland, pp. 271-293, 1980.

[25] S. Lasaulce, M. Debah, E. Altman, “Methodologies for analyzing equilibria in wireless games,” *IEEE Signal Process. Magazine*, pp. 41-52, vol. 26, no. 5, Sep. 2009.

[26] H. Chen and D. Yao, *Fundamentals of Queueing Networks: Performance, Asymptotics, and Optimization*. Springer, 2001.

[27] J. Bulow, J. Geanakoplos, and P. Klemperer, “Multimarket oligopoly: strategic substitutes and strategic complements,” *Journal of Political Economy*, vol. 93, pp. 488-511, 1985.

[28] J. Mo and J. Walrand, “Fair end-to-end window-based congestion control,” *IEEE Trans. on Networking*, vol. 8, no. 5, pp. 556-567, Oct. 2000.

[29] Y. Su and M. van der Schaar, “Linearly coupled communication games”, UCLA Technical Report, Aug. 2009.

[30] Z. Zhang and C. Douligeris, “Convergence of synchronous and asynchronous greedy algorithm in a multiclass telecommunications environment,” *IEEE Tran. Commun.*, vol. 40, pp. 1277-1281, 1992.

[31] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation*. Englewood Cliffs, New Jersey: Prentice Hall, 1997.

[32] R. La and V. Anantharam, “Utility based rate control in the internet for elastic traffic,” *IEEE/ACM Trans. Networking*, vol. 10, no. 2, pp. 271-286, Apr 2002.

[33] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.