POINT PROCESSES AND
THE INFINITE SYMMETRIC GROUP.
PART I: THE GENERAL FORMALISM AND
THE DENSITY FUNCTION

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ABSTRACT. We study a 2-parametric family of probability measures on an infinite-dimensional simplex (the Thoma simplex). These measures originate in harmonic analysis on the infinite symmetric group (S. Kerov, G. Olshanski and A. Vershik, Comptes Rendus Acad. Sci. Paris I 316 (1993), 773-778). Our approach is to interpret them as probability distributions on a space of point configurations, i.e., as certain point stochastic processes, and to find the correlation functions of these processes.

In the present paper we relate the correlation functions to the solutions of certain multidimensional moment problems. Then we calculate the first correlation function which leads to a conclusion about the support of the initial measures. In the appendix, we discuss a parallel but more elementary theory related to the well-known Poisson-Dirichlet distribution.

The higher correlation functions are explicitly calculated in the subsequent paper (A. Borodin). In the third part (A. Borodin and G. Olshanski) we discuss some applications and relationships with the random matrix theory.

The goal of our work is to understand new phenomena in noncommutative harmonic analysis which arise when the irreducible representations depend on countably many continuous parameters.

CONTENTS

Introduction
§1. Coherent systems of distributions on the Young graph
§2. The coherent z-systems
§3. Controlling measures
§4. Point processes
§5. The density function
§6. An application
§7. Appendix: Correlation functions of the Poisson-Dirichlet processes
§8. Appendix (A. Borodin): A proof of Theorem 2.1
References

In this paper, we begin study of a remarkable family of stochastic point processes. These processes live on the punctured interval $[-1, 1] \setminus \{0\}$ and depend on two real

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parameters. Our purpose is to calculate their correlation functions which supply substantial information about the processes. The present paper is the first one in a series of papers. It contains introduction to the subject, description of the method, calculation of the first correlation function (also called the density function) and an application concerning the ‘support’ of the processes. The higher correlation functions are calculated in the subsequent paper [B] by Alexei Borodin. In the third paper [BO] we discuss certain applications.

The point processes in question originated from harmonic analysis on the infinite symmetric group [KOV]: they govern decomposition of the so-called generalized regular representations. I shall briefly discuss the link with representation theory, as this is the main motivation of the work. I believe that this new kind of a relationship between representations and probabilities is interesting. However, in the body of the paper, we are not dealing with representations, and I tried to make the exposition formally independent of [KOV] and accessible to non–experts in representation theory.

Starting with the infinite symmetric group $S(\infty) = \lim_{n\to\infty} S(n)$ (the union of the finite symmetric groups), we form a ‘$(G, K)$-pair’, where $G$ is the product $S(\infty) \times S(\infty)$ and $K$ is the diagonal subgroup in $G$. (Let us emphasize at once that irreducible representations of $G$, generally speaking, are not tensor products of two irreducible representations of $S(\infty)$ as it would be for a ‘tame’ group in place of $S(\infty)$; but $S(\infty)$ is not tame.) A spherical representation of $(G, K)$ is a couple $(T, v)$, where $T$ is a unitary representation of $G$ and $v$ is a distinguished $K$-invariant unit vector in the space of $T$. Note that if $T$ is irreducible then $v$ is unique up to a scalar factor. Note also that the spherical representations are a particular case of more general ‘admissible representations’ of $(G, K)$, see [O1, Ok].

There exists a parametrization $\omega \leftrightarrow (T(\omega), v(\omega))$ of irreducible spherical representations by points $\omega$ of an infinite–dimensional simplex $\Omega$; the latter is called the Thoma simplex, see [T, VK, O1, Ok]. Any cyclic representation $(T, v)$ can be decomposed into a direct integral,

$$T = \int_{\Omega} T(\omega) P(d\omega), \quad v = \int_{\Omega} v(\omega) P(d\omega). \quad (0.1)$$

Here $P$ is a probability measure on $\Omega$, which is uniquely defined; it is called the spectral measure for $(T, v)$.

In [KOV], we constructed a family $\{T_z\}$ of admissible representations of $(G, K)$ depending on a complex parameter $z$. Each $T_z$ is realized in a $L^2$ space on a compactification $X$ of the discrete space $S(\infty)$. When $z$ tends to infinity, $T_z$ approaches the conventional biregular representation $T_\infty$ of $G$ in the $l^2$ space on $S(\infty)$, so that the representations $T_z$ form a deformation of $T_\infty$; we call them the generalized regular representations.

As is well known, the representation $T_\infty$ is irreducible. However, the representations $T_z$ are highly reducible (as “true” regular representations should be). Their decomposition can be viewed as a model problem of noncommutative harmonic analysis in the situation when the irreducible representations depend on infinitely many parameters.

The construction of [KOV] shows that each $T_z$ possesses a distinguished $K$-invariant vector $v$; moreover, $v$ is cyclic provided that $z$ is not integral. Applying the abstract decomposition (0.1), we arrive to a family $\{P_z\}$ of spectral measures...
on the simplex $\Omega$. The probability measures $P_z$ are the main object of the paper.\footnote{Actually, we also consider a “complementary series” of spectral measures. In the text, we use the notation $P_{zz'}$, where either $z' = \bar{z}$ (the “principal series”) or both $z$ and $z'$ are real and satisfy some extra condition (the “complementary series”).}

To study the measures $P_z$ we propose the following approach:

1) We define an embedding of the Thoma simplex $\Omega$ to the space $\Xi$ of configurations in the locally compact space $I = [-1, 1] \setminus \{0\}$ (a configuration in $I$ is an unordered collection of points which can accumulate only at 0). Then any probability measure on $\Omega$ will define a random configuration in $I$, i.e., a stochastic point process. Thus, the measures $P_z$ can be interpreted as certain point processes $P_z$.

2) As a characteristic of the point processes $P_z$ we choose the correlation functions. Let $n = 1, 2, \ldots$ and $x_1, \ldots, x_n$ be an arbitrary $n$-tuple of points in $I$. Given a point process, the probability of the event that a random configuration intersects each of the infinitely small intervals $x_1 + dx_1, \ldots, x_n + dx_n$ has the form $\rho_n(x_1, \ldots, x_n)dx_1 \ldots dx_n$, and the density $\rho_n$ is called the $n$th correlation function. All point processes originated from probability measures on $\Omega$ are uniquely determined by their correlation functions.

3) We show that the correlation functions of $P_z$ can be obtained from a sequence $\sigma_1, \sigma_2, \ldots$ of probability measures called the controlling measures. The $n$th controlling measure lives on the $n$-dimensional cube $[-1, 1]^n$, and one can write down all its moments. This reduces the problem of calculating the functions to a certain multidimensional moment problem.

The paper is organized as follows.

§1 contains preliminaries on symmetric functions, the Thoma simplex, coherent systems of distributions on the Young graph, and their spectral measures. Using this formalism, we introduce in §2 the so-called coherent $z$-systems on the Young graph, by means of which we define the spectral measures $P_z$. In §3 we discuss the controlling measures and their moments, and in §4 we pass to the point processes.

The technical part of the work begins in §5. Here we solve a moment problem and calculate the density function of $P_z$. We present an integral representation of the density function, Theorem 5.8, and an explicit expression in terms of a multivariate hypergeometric function (the Lauricella function $F_B$ in three variables), Theorem 5.12.

In §6 we show that the first controlling measure has no atom at 0, which implies that the measures $P_z$ are concentrated on a distinguished face of the simplex $\Omega$ (this result was announced in [KOV]).

There are two appendices.

In the first appendix (§7) we discuss the Poisson–Dirichlet distributions $PD(t)$. These distributions were intensively studied in literature and they play an important role in the construction [KOV] of the representations $T_z$. Our purpose is to derive the Watterson [W] formula for the correlation functions of $PD(t)$ by employing the general formalism of §§1–4.

The second appendix (§8) contains a simple direct proof, due to A. Borodin, of Theorem 2.1 asserting the existence of the coherent $z$-systems.

The results of the present paper were obtained, in the main, in 1992. Then Alexei Borodin succeeded to calculate the higher correlation functions; his results constitute the second part of the work, see [B]. I am very grateful to him for
numerous discussions which exerted a substantial influence on the final version of the paper. In particular, one of the devices of [B] allowed me to simplify the derivation of Theorem 5.2.

§1. Coherent systems of distributions on the Young graph

Symmetric functions [M]. Let \( \Lambda \) denote the algebra of symmetric functions over the base field \( \mathbb{R} \). Formally, \( \Lambda \) may be defined as \( \mathbb{R}[p_1, p_2, \ldots] \), the algebra of polynomials over infinitely many indeterminates \( p_1, p_2, \ldots \), called the power sums. Another system of generators of \( \Lambda \) is formed by the complete symmetric functions \( h_1, h_2, \ldots \), which are expressed through \( p_1, p_2, \ldots \) via the following relation of generating series with formal parameter \( t \):

\[
1 + \sum_{n \geq 1} h_n t = \exp \left( \sum_{n \geq 1} \left( \frac{p_n}{n} \right) t^n \right).
\]

A partition is a weakly decreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of nonnegative integers with finitely many nonzero terms. The number of nonzero terms is called the length of \( \lambda \) and denoted as \( \ell(\lambda) \). Partitions are also viewed as Young diagrams. By \( |\lambda| \) we denote the sum \( \lambda_1 + \lambda_2 + \ldots \) or, equivalently, the number of boxes in the diagram \( \lambda \). The zero partition (or the empty Young diagram) is denoted as \( \emptyset \).

The elements

\[
p_\rho = p_{\rho_1} p_{\rho_2} \cdots p_{\rho_1}, \quad l = \ell(\rho),
\]

where \( \rho \) ranges over the set of partitions, form a basis in \( \Lambda \). Another distinguished basis in \( \Lambda \) is formed by the Schur functions, which are also indexed by arbitrary partitions and can be defined, in terms of \( h_1, h_2, \ldots \), by the Jacobi–Trudi formula

\[
s_\lambda = \det[h_{\lambda_i - i + j}],
\]

where the order of determinant is \( \ell(\lambda) \) and it is assumed that \( h_0 = 1 \) and \( h_n = 0 \) when \( n < 0 \).

We shall need two important formulas:

\[
s_\mu \cdot p_1 = \sum_{\lambda: \lambda \triangleright \mu} s_\lambda, \quad (1.1)
\]

\[
p_\rho = \sum_{\lambda: |\lambda| = |\rho|} \chi_\rho^\lambda s_\lambda, \quad (1.2)
\]

see [M, I.5.16 and I.7.8]. Here and in what follows the notation \( \lambda \triangleright \mu \) (or, equivalently, \( \mu \triangleright^\lambda \lambda \)) means that the diagram \( \lambda \) contains the diagram \( \mu \) and differs from it by a single box (in particular, we have \( |\lambda| = |\mu| + 1 \)). By \( \chi^\lambda \) we denote the irreducible character indexed by \( \lambda \) (it is a character of the symmetric group of degree \( |\lambda| \)), and \( \chi_\rho^\lambda \) is the value of \( \chi^\lambda \) on the conjugacy class indexed by \( \rho \), see [M, §I.7].

In the customary realization of the algebra \( \Lambda \), its generators \( p_n \) are identified with the expressions \( x_1^n + x_2^n + \ldots \) in indeterminates \( x_1, x_2, \ldots \), so that elements of \( \Lambda \) become symmetric functions in \( x_1, x_2, \ldots \). But we shall need another realization,
obtained by specializing the generators \( p_n \in \Lambda \) to the following expressions in the indeterminates \( \alpha = (\alpha_1, \alpha_2, \ldots) \), \( \beta = (\beta_1, \beta_2, \ldots) \), and \( \gamma \):

\[
p_1 \mapsto \tilde{p}_1(\alpha, \beta, \gamma) := \sum_{i \geq 1} \alpha_i + \sum_{i \geq 1} \beta_i + \gamma
\]

\[
p_n \mapsto \tilde{p}_n(\alpha, \beta, \gamma) := \sum_{i \geq 1} \alpha_i^n + (-1)^{n-1} \sum_{i \geq 1} \beta_i^n, \quad n \geq 2,
\]

which is equivalent to

\[
1 + \sum_{n \geq 1} h_n t^n \mapsto e^{\gamma t} \prod_{i \geq 1} \frac{1 + \beta_i t}{1 - \alpha_i t}.
\]

This is a generalization of the well–known “super” realization of \( \Lambda \); indeed, setting \( \gamma = 0 \) converts the above expressions to “supersymmetric” functions in \( \alpha \) and \( -\beta \), see [S] and [M, §I.3, Ex. 23].

**The Thoma simplex** [VK, KV, KOO]. We shall abbreviate

\[
\alpha = (\alpha_1, \alpha_2, \ldots), \quad \beta = (\beta_1, \beta_2, \ldots).
\]

Let \( \Omega \) be the set of the triples \( \omega = (\alpha, \beta, \gamma) \) such that

\[
\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \quad \beta_1 \geq \beta_2 \geq \cdots \geq 0, \quad \gamma \geq 0,
\]

\[
\sum_{i \geq 1} (\alpha_i + \beta_i) + \gamma = 1. \tag{1.4}
\]

Since \( \gamma \) is determined by \( \alpha \) and \( \beta \), we shall sometimes omit \( \gamma \) and write \( \omega = (\alpha, \beta) \).

The set \( \Omega \) is an infinite–dimensional simplex; its faces of codimension 1 have the form

\[
\Omega_i = \{ \omega \in \Omega \mid \alpha_i = \alpha_{i+1} \}, \quad \Omega_{-i} = \{ \omega \in \Omega \mid \beta_i = \beta_{i+1} \}, \quad i \geq 1, \tag{1.5}
\]

\[
\Omega_0 = \{ \omega \in \Omega \mid \gamma = 0 \}. \tag{1.6}
\]

The simplex \( \Omega \) is called the **Thoma simplex** in connection with the pioneering Thoma’s work [T]. We equip \( \Omega \) with the weakest topology in which the coordinates \( \alpha_i \) and \( \beta_i \) (but not \( \gamma \)) are continuous functions. In this topology, \( \Omega \) is a metrizable compact space, and the face \( \Omega_0 \) is a dense subset.

We define the functions \( \tilde{p}(\omega) \) on \( \Omega \) by setting \( \omega = (\alpha, \beta, \gamma) \in \Omega \) in the formulas (1.3). Note that \( \tilde{p}_1(\omega) \equiv 1 \). It is readily verified ([KOO, Lemma 5.2]) that these functions are continuous on \( \Omega \). Consequently, any element \( f \in \Lambda \) can be converted to a continuous function \( \tilde{f} \) on \( \Omega \) by writing \( f \) as a polynomial in the generators \( p_n \) and replacing then each \( p_n \) by \( \tilde{p}_n \). We shall call \( \tilde{f} \) the **extended version** of \( f \), cf. [KV, KOO]. In particular, we shall deal with the **extended power sums** \( \tilde{p}_n \) and the **extended Schur functions** \( \tilde{s}_\lambda \).

Note that the functions \( \tilde{f} \), where \( f \) ranges over \( \Lambda \), form a dense subalgebra in the Banach algebra \( C(\Omega) \) of (real) continuous functions on the compact space \( \Omega \), see [KOO, Lemma 5.3].

\(^2\)Observe that the expression \( \sum (\alpha_i + \beta_i) \) is not continuous on \( \Omega \).
The algebra $\Lambda$ possesses an involutive automorphism (denoted as $\omega$ in $[M]$) such that

$$p_n \mapsto (-1)^{n-1}p_n, \quad s_\lambda \mapsto s_{\lambda^t},$$

where $\lambda^t$ means the transposed diagram. In terms of the realization (1.3) this involution exactly corresponds to the symmetry $\alpha \leftrightarrow \beta$. The latter symmetry also defines a symmetry of the Thoma simplex, which will be denoted as $\omega \mapsto \omega^t$. It follows that

$$\tilde{s}_\lambda(t) = \tilde{s}_{\lambda^t}(\omega).$$

**Harmonic functions and coherent systems of distributions on the Young graph** [V, VK, KV, K, KOO]. By definition, the vertices of the Young graph $\mathcal{Y}$ are arbitrary Young diagrams (including $\emptyset$), and its (oriented) edges are formed by the couples $\mu, \lambda$ such that $\mu \rightarrow \lambda$. The number of oriented paths from $\emptyset$ to $\lambda$ is called the dimension of $\lambda$ and denoted as $\dim \lambda$ (we agree that $\dim \emptyset = 1$). The function $\dim$ on the vertices of $\mathcal{Y}$ satisfies the recurrence relation

$$\dim \lambda = \sum_{\mu: \mu \rightarrow \lambda} \dim \mu.$$

These definitions are inspired by the Young branching rule for the characters of the finite symmetric groups $S(n)$, $n = 1, 2, \ldots$,

$$\chi^\lambda \mid_{S(n-1)} = \sum_{\mu: \mu \rightarrow \lambda} \chi^\mu, \quad n = |\lambda|.$$

(See [V, JK, OV].) The above two relations show that the dimension of a diagram $\lambda$ coincides with the dimension of the character $\chi^\lambda$, i.e., with the number $\chi^\lambda_{(1^n)}$, $n = |\lambda|$.

A (real) function $\varphi(\lambda)$ on the vertices of $\mathcal{Y}$ is called harmonic if it satisfies the “harmonicity condition”

$$\varphi(\mu) = \sum_{\lambda: \lambda \rightarrow \mu} \varphi(\lambda)$$

for each diagram $\mu$. Let $\mathcal{Y}_n$ denote the set of Young diagrams with $n$ boxes (equivalently, the set of partitions of the number $n$). The sets $\mathcal{Y}_n$ define a grading of the graph $\mathcal{Y}$. Clearly, the knowledge of a harmonic function on a “floor” $\mathcal{Y}_n$ determines it on the preceding floors $\mathcal{Y}_{n-1}, \mathcal{Y}_{n-2}, \ldots$.

Let $M(\lambda)$ be a function on the vertices of $\mathcal{Y}$ and $M_n$ denote its restriction to the $n$th floor $\mathcal{Y}_n$, $n = 0, 1, 2, \ldots$. We call $M = (M_n)$ a coherent system of distributions on the Young graph (coherent system, for short) if $M$ is nonnegative, normalized at $\lambda = \emptyset$ (i.e., $M(\emptyset) = M_0(\emptyset) = 1$), and the function $M(\lambda)/\dim \lambda$ is harmonic, i.e.,

$$M(\mu) = \sum_{\lambda: \lambda \rightarrow \mu} \frac{\dim \mu}{\dim \lambda} M(\lambda).$$

The latter formula and the recurrence relation for the dimension function imply that the $M_n$’s are probability distributions on the floors of the Young graph (see [KOO, Lemma 8.1]).

By a measure on $\Omega$ we shall always mean a Borel measure with respect to the canonical Borel structure of the topological space $\Omega$. According to [KOO, Theorem
there is a bijective correspondence $M \leftrightarrow P$ between the coherent systems $M$ on the Young graph and the probability measures $P$ on the Thoma simplex. This correspondence is characterized by the relation

$$\frac{M(\lambda)}{\dim \lambda} = \int_{\Omega} \tilde{s}_{\lambda}(\omega) P(d\omega),$$

(1.7)

where $\lambda$ is an arbitrary Young diagram and $\tilde{s}_{\lambda}$ is the extended Schur function as defined above. Moreover, as is shown in the proof of this result, the measure $P$ is approached, in a certain sense, by the finite probability distributions $M_n$ as $n \to \infty$. We shall call $P$ the spectral measure of $M$.

If $M$ is a coherent system on $\mathbb{Y}$ then the function $M^t(\lambda) = M(\lambda^t)$ is a coherent system, too (indeed, this follows from the fact that the involution $\lambda \mapsto \lambda^t$ is a symmetry of the Young graph). On the other hand, if $P$ is a measure on $\Omega$, let $P^t$ denote the image of $P$ under the symmetry $\omega \mapsto \omega^t$ defined above. Now, we have

$$\text{if } M \leftrightarrow P \text{ then } M^t \leftrightarrow P^t.$$

(1.8)

Indeed, this follows from (1.7) and the equality $\tilde{s}_{\lambda^t}(\omega) = \tilde{s}_{\lambda}(\omega^t)$ mentioned above.

**Connection with representations** [T, VK, KV, O1, O2]. The constructions described above are inspired by the representation theory of the infinite symmetric group $S(\infty) = \lim_{n \to \infty} S(n)$. Let $M$ be a coherent system on $\mathbb{Y}$ and $\varphi(\lambda) = M(\lambda)/\dim \lambda$ the corresponding harmonic function. For each $n$, the linear combination of irreducible characters

$$\chi_n = \sum_{\lambda : |\lambda| = n} \varphi(\lambda) \chi^\lambda$$

is a central positive definite function on $S(n)$, normalized at the unit element. By the harmonicity condition, the functions $\chi_1, \chi_2, \ldots$ are compatible with the embeddings $S(n-1) \hookrightarrow S(n)$. Consequently, they define a central, positive definite, normalized function $\chi$ on the group $S(\infty)$. Let $X$ denote the set of all such functions. This is a convex set; its extreme points are called the characters of the group $S(\infty)$ (in the sense of von Neumann).

Via the Gelfand–Naimark–Segal construction, characters generate finite factor representations of the group $S(\infty)$. They also correspond to irreducible unitary spherical representations of the Gelfand pair $(G, K)$, where $G$ stands for the “bisymmetric group” $S(\infty) \times S(\infty)$ and $K$ is the diagonal subgroup of $G$, see [O1, O2].

The integral representation (1.7) has the following meaning. First of all, via the correspondence $M \leftrightarrow \chi$, (1.7) implies that the characters are parametrized by the points $\omega \in \Omega$ (we shall write them as $\chi^{(\omega)}$).

This result is known as *Thoma’s theorem* [T]; see also [VK]. The extreme coherent system $M^{(\omega)} \leftrightarrow \chi^{(\omega)}$ is given by the formula

$$M^{(\omega)}(\lambda) = \dim \lambda \cdot \tilde{s}_{\lambda}(\omega),$$

which is equivalent to *Thoma’s formula* [T]

$$\chi^{(\omega)}(\rho) = \tilde{p}_{\rho_1}(\omega) \tilde{p}_{\rho_2}(\omega) \ldots .$$

\[\text{Thus, the Thoma simplex may be viewed as the spherical dual of the Gelfand pair } (G, K).\]
In the latter expression, the left-hand side is the value of $\chi^{(\omega)}$ at the conjugacy class in $S(\infty)$ of an arbitrary cycle-type $\rho = (\rho_1, \rho_2, \ldots, 1, 1, \ldots)$, and the right-hand side is correctly defined, because almost all cycles have length 1 and $\tilde{p}_1 \equiv 1$.

Next, (1.7) implies that any function $\chi \in X$ is uniquely decomposed into a continual convex combination of the characters $\chi^{(\omega)}$. As $\chi$ generates a cyclic spherical representation $T$ of the pair $(G, K)$, this also means that the spectral measure $P$ effectues the decomposition of $T$ in a direct integral of irreducible spherical representations.

§2. The coherent $z$-systems

Set

$$M_{zz'}(\lambda) = \prod_{(ij) \in \lambda} \frac{(z + j - i)(z' + j - i)}{(t)_{|\lambda|}} \frac{\dim \lambda}{|\lambda|!}. \tag{2.1}$$

Here $\lambda$ is an arbitrary Young diagram, $z$ and $z'$ are complex parameters, $(ij) \in \lambda$ is an arbitrary box of $\lambda$ ($i$ and $j$ are the numbers of the row and the column containing the box), $t = zz'$, and an expression of type $(t)_n$ means the Pohhammer symbol:

$$(t)_n = \Gamma(t + n)/\Gamma(t) = t(t + 1) \ldots (t + n - 1);$$

(2.1) is correctly defined if $t \neq 0, -1, -2, \ldots$. We also agree that $M_{zz'}(\emptyset) = 1$.

**Theorem 2.1.** The function $M_{zz'}(\lambda) / \dim \lambda$ is harmonic.

**Proof.** This follows from [KOV, Theorem 3.1]. A direct combinatorial argument is given in Kerov’s paper [Ke2] (actually, [Ke2] contains a more general result). Another combinatorial proof was proposed by Postnikov [P]. In §8 below we present one more direct proof, due to Borodin. □

**Proposition 2.2.** Assume that $z, z'$ are not integers and are such that $t = zz'$ is not equal to 0, $-1, -2, \ldots$. Then $M_{zz'}(\lambda) \neq 0$ for any $\lambda$. Moreover, $M_{zz'}(\lambda)$ is strictly positive if and only if the parameters $z, z'$ satisfy one of the following conditions:

(i) $z' = \bar{z}$ and $z \notin \mathbb{Z}$;

(ii) $z$ and $z'$ are real and are both contained in an open interval of the form $(m, m+1)$ with $m \in \mathbb{Z}$.

**Proof.** The first claim is obvious from (2.1), let us check the second claim. Assume that (i) or (ii) holds. Then $t > 0$ and $(z + k)(z' + k) > 0$ for any $k \in \mathbb{Z}$, which implies $M_{zz'}(\lambda) > 0$.

Conversely, assume that $M_{zz'}(\lambda) > 0$ for any $\lambda$. Let $\mu \not\sim \lambda$ be an edge of $\Upsilon$, $n = |\mu|$, $(ij)$ be the box $\lambda \setminus \mu$, and $k = j - i$. Comparing $M_{zz'}(\mu)$ and $M_{zz'}(\lambda)$, we see that

$$\frac{(z + k)(z' + k)}{(t + n)} > 0.$$

Clearly, this inequality holds for any numbers $k \in \mathbb{Z}$ and $n = 1, 2, \ldots$ that correspond to edges of $\Upsilon$, that is to say, for any $k, n$ such that either $k \neq 0$ and $n \geq |k|$, or $k = 0$ and $n \geq 3$.

Now, we fix $k$ and let $n \to \infty$. From the above inequality we conclude that the numerator $(z + k)(z' + k)$ must be real and strictly positive. Since this holds for any $k \in \mathbb{Z}$, both $zz'$ and $z + z'$ are real. It follows that either $z, z'$ are complex...
conjugate or both real. It remains to examine the latter possibility. Using the fact that \((z+k)(z'+k)\) is not only real but strictly positive, we get condition (ii) of the proposition. □

**Corollary 2.3.** Under assumptions (i) or (ii) of Lemma 2.2, \(M_{zz'}\) is a coherent system of distributions on the Young graph. □

From now on we shall assume that the parameters \(z, z'\) satisfy one of the conditions (i), (ii) of Proposition 2.2 (note that \(t \neq 0, -1, -2, \ldots\) holds automatically then). We shall call the coherent systems \(M_{zz'}\) the (coherent) \(z\)-systems. To distinguish between (i) and (ii), we shall speak about the principal series and complementary series of \(z\)-systems, respectively. A motivation for such a terminology is that the difference \(z - z'\) ranges, respectively, over the imaginary axis and the open interval \((-1, 1)\), like the parameters of the principal or the complementary series for \(SL(2, \mathbb{R})\).

The principal series of \(z\)-systems first appeared in [KOV]: in that form we described the spherical functions of the generalized regular representations of the pair \((G, K)\). The existence of the complementary series was observed in 1995 by Borodin. The \(z\)-systems can be characterized as the only coherent systems of distributions on the Young graph satisfying a “multiplicativity condition”, see [R].

In the present paper we do not deal with a “degenerate series” of coherent systems which arises when one of the parameters \(z, z'\) is integral. The ”degenerate” coherent systems live, in essence, on truncated versions of the Young graph. About them, see [KOV] and [K].

**Remark 2.4.** The expression (2.1) for \(M_{zz'}\) has two evident symmetries, each of which has a representation–theoretic meaning. First, (2.1) does not change under the transposition \(z \leftrightarrow z'\), which leads to certain intertwining operators for the generalized regular representations. Second, we have

\[
M_{zz'}(\lambda^t) = M_{-z,-z'}(\lambda),
\]

where \(\lambda \mapsto \lambda^t\) denote transposition of Young diagrams. This symmetry reflects the well–known fact that, for irreducible characters \(\chi^\lambda\), transposing \(\lambda\) is equivalent to tensoring \(\chi^\lambda\) with the one–dimensional sign character.

In the next two propositions, we rewrite the expression (2.1) by making use of two explicit formulas for \(\dim \lambda\).

**Proposition 2.5.** The expression (2.1) can be written as

\[
M_{zz'}(\lambda) = \frac{|\lambda|!}{(t|\lambda|)} \prod_{i=1}^{l} (z - i + 1)\lambda_i (z' - i + 1)\lambda_i \\
\times \prod_{1 \leq i,j \leq l} \left(\lambda_i - \lambda_j + j - i\right)^2 \\
\times \prod_{1 \leq i \leq l} \left((\lambda_i + l - i)!\right)^2, \quad \text{where } l \geq \ell(\lambda) \text{ may be taken arbitrarily.}
\]

**Proof.** The first product is exactly the product over the boxes \((ij) \in \lambda\) in formula (2.1), and the remaining terms come from the formula

\[
\frac{\dim \lambda}{|\lambda|!} = \prod_{1 \leq i,j \leq l} \frac{(\lambda_i - \lambda_j + j - i)}{\prod_{1 \leq i \leq l} (\lambda_i + l - i)!}, \quad l \geq \ell(\lambda).
\]
It is readily seen that the right–hand side is stable on \( l \geq \ell(\lambda) \). Hence, it suffices to check it for a particular value of \( l \). But for \( l = |\lambda| \), this coincides with the formula of [M, §I.7, Ex. 6]. □

The explicit expression (2.3) is not quite satisfactory, because it does not reflect the symmetry (2.2). A symmetric expression can be obtained with the help of the Frobenius notation for Young diagrams [M, §I.1]:

\[
\lambda = (p_1, \ldots, p_d \mid q_1, \ldots, q_d),
\]

where \( d \) is the length of the diagonal in \( \lambda \), and

\[
p_i = \lambda_i - i, \quad q_i = (\lambda^t)_i - i, \quad 1 \leq i \leq d,
\]

are the Frobenius coordinates of \( \lambda \). Note that

\[
|\lambda| = |p| + |q| + d,
\]

where

\[
|p| = p_1 + \cdots + p_d, \quad |q| = q_1 + \cdots + q_d.
\]

**Proposition 2.6.** In the Frobenius notation (2.5), the expression (2.1) can be written as follows

\[
M_{zz'}(\lambda) = \frac{(|p| + |q| + d)! t^d}{(t)^{|p|+|q|+d}} \prod_{i=1}^{d} \left( \frac{(z + 1)p_i(z' + 1)p_i(-z + 1)q_i(-z' + 1)q_i}{(p_i!)^2(q_i!)^2} \right) \prod_{i=1}^{d} \prod_{j=1}^{d} (p_i + q_j + 1)^{-2} \prod_{1 \leq i < j \leq d} (p_i - p_j)^2(q_i - q_j)^2.
\]

**Proof.** Given a box \((ij) \in \lambda\), its hook is defined as the shape formed by the boxes \((k l) \in \lambda\) such that either \( k = i, l \geq j \) or \( k > i, l = j \). The total number of boxes in the hook is called the hook–length and denoted as \( h(i, j) \).

Let us represent the shape \( \lambda \) as the union of the diagonal hooks. Then the contribution of the \( k \)th diagonal hook \((k = 1, \ldots, d)\) to the product

\[
\prod_{(ij) \in \lambda} (z + j - i)(z' + j - i)
\]

is equal to

\[
(z - q_k) \cdots (z - 1)z(z + 1) \cdots (z + p_k) \\
\times (z' - q_k) \cdots (z' - 1)z'(z' + 1) \cdots (z' + p_k) \\
= z z'(z + 1)_p (z' + 1)_p (-z + 1)_q (-z' + 1)_q \\
= t(z + 1)_p (z' + 1)_p (-z + 1)_q (-z' + 1)_q,
\]

which explains the term \( t^d \) and the first product in (2.6). The remaining terms in (2.6) come from the following formula expressing \( \dim \lambda \) in the Frobenius notation:

\[
\frac{\dim \lambda}{|\lambda|!} = \prod_{1 \leq i, j \leq d} (p_i - p_j)(q_i - q_j) \prod_{1 \leq i \leq d} (p_i + q_i + 1) \prod_{1 \leq i \leq d} (p_i! q_i!).
\]
To check (2.7) we start with the well–known hook formula

\[
\frac{\dim \lambda}{|\lambda|!} = \prod_{(ij) \in \lambda} h(i,j)^{-1},
\]  

(2.8)

which, by virtue of formula (2.4), is equivalent to the identity

\[
\prod_{(ij) \in \lambda} h(i,j)^{-1} = \prod_{1 \leq i, j \leq l} \frac{(\lambda_i - \lambda_j + j - i)}{(\lambda_i + l - i)!}, \quad l \geq \ell(\lambda),
\]  

(2.9)

see M, §I.1, Ex. 1.

Now, let us divide the shape \( \lambda \) into three pieces: the square shape of size \( d \times d \), the diagram \( \lambda^+ \) formed by the boxes \( (ij) \) with \( j > d \), and the diagram \( \lambda^- \) formed by the boxes \( (ij) \) with \( i > d \). The hook–length of a box \( (ij) \) from the square shape is equal to \( p_i + q_j + 1 \). Consequently, the product of the hook–lengths over the boxes entering the square shape is equal to the double product

\[
\prod_{1 \leq i \leq d} \prod_{1 \leq j \leq d} (p_i + q_j + 1)
\]

in the denominator of (2.7). To explain the remaining terms in (2.7), we express the products of the hook–lengths in the diagrams \( \lambda^+ \) and \( \lambda^- \) via the identity (2.9), where we substitute \( l = d \) and \( \lambda = \lambda^+ \) or \( \lambda = (\lambda^-)^t \). It should also be noted that for any box in \( \lambda^\pm \), its hook with respect to \( \lambda^\pm \) is the same as the hook with respect to \( \lambda^\mp \).

Formula (2.6) will be used for calculations in §5 below and in [B].

§3. Controlling measures

Given a point \( \omega = (\alpha, \beta, \gamma) \in \Omega \), we define the corresponding Thoma measure on \([-1, 1]\) as

\[
\nu^{(\omega)} = \sum_{i \geq 1} \alpha_i \delta(\alpha_i) + \sum_{i \geq 1} \beta_i \delta(-\beta_i) + \gamma \delta(0),
\]

where \( \delta(x) \) stands for the Dirac mass at \( x \in [-1, 1] \). Clearly, \( \nu^{(\omega)} \) is a probability measure.

Proposition 3.1. The moments of \( \nu^{(\omega)} \) are given by the formula

\[
\int_{-1}^{1} x^l \, \nu^{(\omega)}(dx) = \tilde{p}_{l+1}(\omega), \quad l = 0, 1, 2, \ldots,
\]

where \( \tilde{p}_{l+1} \) are the extended power sums as defined in §1.

Proof. This is a direct consequence of the definition of the Thoma measure and that of \( \tilde{p}_n \). \( \square \)

Let \( \text{Prob}[-1, 1] \) denote the set of probability Borel measure on \([-1, 1]\); this set has a natural Borel structure [DVJ]. The map

\[
\Omega \to \text{Prob}[-1, 1], \quad \omega \mapsto \nu^{(\omega)},
\]


is Borel–measurable (this is a routine exercise). Therefore, equipping \( \Omega \) with a probability measure \( P \), we obtain a \textit{random measure} on \([-1, 1]\). Since the above map is injective, we may interpret probability measures \( P \) on \( \Omega \) as random measures on \([-1, 1]\).

The next construction looks rather natural from the point of view of the theory of random measures (see, e.g., [DVJ]) or the exchangeability theory [A]. We take the infinite product

\[
\nu^{(\omega)}_\infty = \nu^{(\omega)} \times \nu^{(\omega)} \times \cdots, \quad \omega \in \Omega,
\]

which is a probability measure on the infinite–dimensional cube

\[
[-1, 1]_\infty = [-1, 1] \times [-1, 1] \times \cdots,
\]

and we average \( \nu^{(\omega)}_\infty \) with respect to a given probability measure \( P \) on \( \Omega \):

\[
\sigma = \int_{\Omega} \nu^{(\omega)}_\infty P(d\omega).
\]

In other words, viewing \((\Omega, P)\) as a probability space, we take the expectation of the random measure \( \nu^{(\omega)}_\infty \). We shall \( \sigma \) the \textit{controlling measure} (of infinite order) for the measure \( P \). Clearly, \( \sigma \) is a symmetric probability measure on the infinite–dimensional cube. Its projection on the \( n \)-dimensional cube \([-1, 1]^n\) will be called the \textit{nth controlling measure} of \( P \) and denoted as \( \sigma_n \):

\[
\sigma_n = \int_{\Omega} \underbrace{\nu^{(\omega)} \times \cdots \times \nu^{(\omega)}}_{n \text{ times}} P(d\omega).
\]

For a Young diagram \( \lambda \), let \( d(\lambda) \) be the number of diagonal boxes. By \( \chi^{\lambda}_{(r_1, \ldots, r_n)} \), where \( r_1, \ldots, r_n \) are (non necessarily decreasing) numbers \( \geq 1 \) such that \( r_1 + \cdots + r_n = |\lambda| \), we denote the value of the irreducible character at any permutation with \( n \) cycles of length \( r_1, \ldots, r_n \).

**Proposition 3.2.** Let \( M \) be a coherent system on the Young graph, \( P \) be its spectral measure on \( \Omega \), and \( \sigma_n \) be the \( nth \) controlling measure of \( P \).

(i) The moments of \( \sigma_n \) satisfy the relations

\[
\int_{[-1,1]^n} x_1^{l_1} \cdots x_n^{l_n} \sigma_n(dx_1 \cdots dx_n) = \int_{\Omega} \tilde{p}_{l_1+1}(\omega) \cdots \tilde{p}_{l_n+1}(\omega) P(d\omega) \quad (3.1a)
\]

\[
= \sum_{\lambda: d(\lambda) \leq n \atop \dim \lambda = l_1 + \cdots + 1 + n} \chi^{\lambda}_{(l_1+1, \ldots, l_n+1)} \frac{M(\lambda)}{\dim \lambda}, \quad (3.1b)
\]

where \( l_1, \ldots, l_n = 0, 1, 2, \ldots \).

**Proof.** By definition of \( \sigma_n \), the left–hand side is the expectation (with respect to \((\Omega, P)\)) of the integral

\[
\int_{[-1,1]^n} x_1^{l_1} \cdots x_n^{l_n} \nu^{(\omega)}(dx_1) \cdots \nu^{(\omega)}(dx_n).
\]
By Proposition 3.1, this integral is equal to

\[ \tilde{p}_{l_1 + 1}^{n+1}(\omega) \]

Integrating over \( \omega \in \Omega \) with respect to \( P \), we get (3.1a).

According to the identity (1.2), the right-hand side of (3.1a) is equal to

\[ \int_{\Omega} \sum_{|\lambda|=l_1+\cdots+l_n+n} \chi_{(l_1+1,\ldots,l_n+1)}^{\lambda}(\omega) P(d\omega). \]

The Murnaghan–Nakayama rule ([M, §I.7, Ex. 5]) implies that \( \chi_{(l_1+1,\ldots,l_n+1)}^{\lambda} \) vanishes when \( d(\lambda) > n \), so we may introduce the supplementary requirement \( d(\lambda) \leq n \) into the above sum.

Finally, transposing integration and summation and applying (1.7) we get (3.1b).

□

Thus, when \( M \) is known, we can, in principle, find the controlling measures \( \sigma_n \) from the moment problem (3.1). The latter has a unique solution, because the support of \( \sigma_n \) is bounded.

The restriction \( d(\lambda) \leq n \) that appears in formula (3.1) will play a crucial role in what follows. It means that the \( n \)th controlling measure is completely determined by the restriction of \( M \) to the set of diagrams \( \lambda \) contained in the \( \Gamma \)-like shape formed by the boxes \((ij)\) with \( \min(i,j) \leq n \). In particular, for calculating \( \sigma_n \), it suffices to know the values of \( M \) on the hook diagrams.

Let \( P_{zz'} \) be the spectral measure of \( M_{zz'} \); we shall refer to the family \( \{P_{zz'}\} \) as to that of spectral \( z \)-measures. The \( n \)th controlling measure of \( P_{zz'} \) will be denoted by \( \sigma_n^{(zz')} \).

**Proposition 3.3.** The moments of the measure \( \sigma_n^{(zz')} \) are given by the formula

\[
\int_{[-1,1]^n} x_1^{l_1} \cdots x_n^{l_n} \sigma_n^{(zz')} (dx_1 \ldots dx_n) = \sum_{d=1}^{n} \sum_{p_1 > \cdots > p_d \geq 0} \sum_{q_1 > \cdots > q_d \geq 0} \chi_{(l_1+1,\ldots,l_n+1)}^{(p_1,\ldots,p_d \mid q_1,\ldots,q_d)} \times \prod_{i=1}^{d} (z + 1)_{p_i} (z' + 1)_{p_i} (-z + 1)_{q_i} (-z' + 1)_{q_i} \times \prod_{i,j=1}^{d} (p_i + q_j + 1)^{-1} \prod_{1 \leq i,j \leq d} (p_i - p_j)(q_i - q_j) \times \frac{t^d}{(t)^{|p|+|q|+d}} \prod_{i=1}^{d} (p_i + q_j + 1)^{-1} \prod_{1 \leq i,j \leq d} (p_i - p_j)(q_i - q_j)
\]

where \( l_1, \ldots, l_n = 0, 1, 2, \ldots \) and

\[
|p| = p_1 + \cdots + p_d, \quad |q| = q_1 + \cdots + q_d, \quad |l| = l_1 + \cdots + l_n + n.
\]

**Proof.** This follows from Proposition 3.2 and Proposition 2.6.

We close the section by indicating first applications of the controlling measures: we shall see that \( \sigma_1 \) and \( \sigma_2 \) control the location of the spectral measure \( P \) with respect to the faces (1.5)–(1.6) of the Thoma simplex.
Proposition 3.4. A spectral measure $P$ is concentrated on the face $\Omega_0$ (see (1.6)) if and only if the first controlling measure $\sigma_1$ has no atom at 0.

Proof. Let us regard the parameter $\gamma$ as a function $\gamma(\omega)$ on $\Omega$. This function is lower semicontinuous, hence a Borel function. By definition of $\sigma_1$ and of $\nu(\omega)$,

$$\sigma(\{0\}) = \int_{\Omega} \gamma(\omega) P(d\omega).$$

Since $\gamma(\omega)$ is nonnegative, this expression is equal to 0 if and only if $\gamma(\omega) = 0$ almost surely with respect to $(\Omega, P)$, and the latter happens if and only $P$ is concentrated on the face $\Omega_0$. □

Proposition 3.5. Let $P$ be a spectral measure and $\sigma_1, \sigma_2$ be its first two controlling measures. Consider the subset

$$\Delta = \{(x, x) \mid x \neq 0\} \subset [-1,1]^2$$

and identify it with the punctured interval $I = [-1,1] \setminus \{0\}$.

We always have

$$\sigma_2|_\Delta (dx) \geq |x| \cdot \sigma_1|_I (dx),$$

(the former measure majorates the latter), and both measures coincide if and only if, for $P$-almost all points $\omega \in \Omega$, there is no repetitions of the type $\alpha_i = \alpha_{i+1} \neq 0$ or $\beta_i = \beta_{i+1} \neq 0$.

Note that if such repetitions occur with a nonzero probability, then at least one of the faces $\Omega_{\pm i}, i \geq 1$, is not a $P$-negligible set.

Proof. For any $\omega \in \Omega$ we have

$$(\nu(\omega) \times \nu(\omega))|_\Delta = \sum_{i,j : \alpha_i = \alpha_j > 0} \alpha_i \alpha_j \delta(\alpha_i) \delta(\alpha_j) + \sum_{i,j : \beta_i = \beta_j > 0} \beta_i \beta_j \delta(-\beta_i) \delta(-\beta_j)$$

$$\geq \sum_{i : \alpha_i > 0} \alpha_i^2 \delta(\alpha_i) + \sum_{i : \beta_i > 0} \beta_i^2 \delta(-\beta_i) = | \cdot | \nu(\omega)|_I,$$

and equality holds if and only there are no couples $i \neq j$ such that $\alpha_i = \alpha_j > 0$ or $\beta_i = \beta_j > 0$.

Integrating over $\omega$ with respect to $P$ gives the desired claim. □

§4. Point processes

We start with generalities about point processes and their correlation functions; our main reference here is the book [DVJ].

Let $X$ be a standard Borel space equipped with a “bornology”. The latter means that we know what subsets of $X$ are “bounded”. We assume that the family of the bounded subsets is closed under taking finite union and passage to a subset, and the whole space $X$ can be represented as countable union of bounded subsets. Bounded Borel subsets will be called test subsets.

A configuration in $X$ is a finite or countable system of points $\xi = (x_1, x_2, \ldots)$ in $I$ such that its intersection with any test set is finite. The word “system” is employed
to emphasize that $\xi$ is neither a subset of $X$ (since repetitions are permitted), nor a sequence (since the points are not ordered); strictly speaking, $\xi$ is a multiset. However, it is occasionally convenient to regard $\xi$ as the image of a sequence of points.

The space of configurations in $X$ will be denoted as $\Xi$. For a configuration $\xi \in \Xi$ and a test set $A$, let $|\xi \cap A|$ denote the number of points of $\xi$ (counted with their multiplicities) that occur in $A$. By the assumption, this number is always finite. We equip $\Xi$ with the Borel structure generated by the functions of the form $\xi \mapsto |\xi \cap A|$.

A point process in $X$ is a (Borel) measurable map from a probability space (the state space) to $\Xi$. Equivalently, a point process defines a random configuration in $X$. A point process is called simple if the random configuration has no multiple points almost surely.

Given a test subset $A$, let $N_A$ be the number of points in $A$ for the random configuration (if the process is not simple, points are counted with their multiplicities); this is a random variable, which is defined on the state space and takes values in $\{0, 1, \ldots\}$.

Assume that for any $A$, all the moments of $N_A$ are finite. Then we assign to our point process a sequence $\rho_1, \rho_2, \ldots$ of measures. The $n$th measure $\rho_n$ lives on the $n$-fold direct product $X^n = X \times \cdots \times X$. In terms of the random configuration $\xi = (x_1, x_2, \ldots)$, $\rho_n$ is defined as the expectation

$$\rho_n = \mathbb{E} \left\{ \sum_{i_1, \ldots, i_n} \delta(x_{i_1}) \times \cdots \times \delta(x_{i_n}) \right\},$$

where summation is taken over all $n$-tuples of pairwise distinct indices. Note that for any test subset $A$,

$$\rho_n(A^n) = \mathbb{E} \left\{ N_A(N_A - 1) \cdots (N_A - n + 1) \right\},$$

the $n$th factorial moment of $N_A$. The measure $\rho_n$ is called the $n$th correlation measure or the $n$th factorial moment measure.

Clearly, the correlation measures are symmetric and take finite values on products of test subsets. The first correlation measure $\rho_1$ is also called the density measure. The value of $\rho_1$ on a test set $A$ is the mean number of points (counted with multiplicities) occurring in $A$.

Assume $X$ is equipped with a “reference” measure $dx$. When the process is simple and the measure $\rho_n$ has a density $\rho_n(x_1, \ldots, x_n)$ with respect to Lebesgue measure on $I^n$, this density is called the $n$th correlation function. Informally, $\rho_n(x_1, \ldots, x_n)$ is equal to the probability that the random configuration intersects each of infinitesimal volumes $dx_1, \ldots, dx_n$ around $x_1, \ldots, x_n$, divided by $dx_1 \cdots dx_n$. When $X$ is a domain in an Euclidean space and $dx$ is Lebesgue measure it is convenient to regard the measure $\rho_n$ as a distribution and call the latter the correlation function, even if we do not know a priori that the measure is absolutely continuous with respect to $dx_1 \cdots dx_n$.

From now on, we take as $X$, the punctured interval $I = [-1, 1] \setminus \{0\}$. This is a locally compact space in the natural topology, the point $0$ playing the role of the infinity. A subset of $I$ will be called “bounded” if it is relatively compact in $I$, i.e., does not intersect a sufficiently small interval $(-\varepsilon, \varepsilon)$. Thus, for a configuration in $I$, the only possible accumulation point in $\mathbb{R}$ is $0$. 
Define a map $\Omega \rightarrow \Xi$ as
\[
\omega \mapsto \xi = (\alpha_1, \alpha_2, \ldots, -\beta_1, -\beta_2, \ldots),
\]
where all the $\alpha_i$ and $\beta_i$ are assumed to be nonzero. In particular, the point $\omega = (\alpha, \beta, \gamma) = (0, 0, 1)$ is represented by the empty configuration. One can verify that $\omega \mapsto \xi$ is a Borel map. Thus, any probability measure $P$ on $\Omega$ defines a point process in $I$ with state space $(\Omega, \Xi)$; we shall denote this process by $\mathcal{P}$.

**Proposition 4.1.** Any point process $\mathcal{P}$ determined by a probability measure $P$ on $\Omega$ has the following special property: all the random variables of type $N_A$ are bounded.

*Proof.* Take $\varepsilon > 0$ such that $A$ is contained in $[-1, -\varepsilon] \cup [\varepsilon, 1]$. Then, for any configuration $\xi$ originating from a point $\omega \in \Omega$,
\[
\sum_{x \in \xi \cap A} |x| \geq \varepsilon |\xi \cap A|.
\]
Since the left-hand side does not exceed 1, we conclude that $|\xi \cap A| \leq \varepsilon^{-1}$. \(\square\)

Consequently, $\mathcal{P}$ possesses correlation measures.

**Proposition 4.2.** Consider a point process in $I$ possessing correlation measures. If the diagonal in $I^2$ is a null set with respect to $\rho_2$ then the process is simple. Conversely, if the process is simple then, for any $n \geq 2$ and any couple $i \neq j$ of indices, the set
\[
\{(x_1, \ldots, x_n) \in I^n \mid x_i = x_j\}
\]
is a null set with respect to $\rho_n$.

*Proof.* This is verified by the same argument as in Proposition 3.5. Actually, the claim holds for general spaces $X$. \(\square\)

**Proposition 4.3.** Let $P$ be a probability measure on $\Omega$, $\sigma_n$ its controlling measures, $\mathcal{P}$ the point process defined by $P$, and $\rho_n$ the correlation measures. Set
\[
(I^n)' = \{(x_1, \ldots, x_n) \in I^n \mid x_i \neq x_j \text{ for } i \neq j\}.
\]

On $(I^n)'$, we have
\[
\rho_n = |x_1 \ldots x_n|^{-1} \sigma_n.
\]

*Proof.* This is a direct consequence of the definitions of the measures $\sigma_n$ and $\rho_n$. \(\square\)

Thus, if $\mathcal{P}$ is simple (which can be tested with the help of Proposition 3.5), then the correlation measures are expressed in a simple way through the controlling measures. Indeed, on the subset $(I^n)' \subset I^n$ we can use the formula of Proposition 4.3, and the complementary subset $I^n \setminus (I^n)'$ is negligible by virtue of Proposition 4.2.

Let $\Pi(n, r)$ be the set of partitions of the set $\{1, \ldots, n\}$ consisting of $r$ nonempty blocs, $1 \leq r \leq n$. An element $\pi \in \Pi(n, r)$ can also be regarded as an equivalence relation on $\{1, \ldots, n\}$ with $r$ equivalence classes. We assign to $\pi$ a subset $I^n_\pi \subset I^n$ obtained by intersecting $I^n$ with all hyperplanes of the form $x_i = x_j$ where $i \equiv j \mod \pi$. Thus, $I^n_\pi$ is a ‘diagonal section’ of $I^n$. Choosing a representative $i$ in each equivalence class $\mod \pi$ we get natural coordinates $\{x_i\}$ in the section $I^n_\pi$ by means of which we can identify $I^n_\pi$ with $I^r$. Then we carry over the correlation measure $\rho_r$ from $I^r$ to $I^n_\pi \subset I^n$ and denote the resulting measure on $I^n$ by $\rho_{r, \pi}$.  

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16
Proposition 4.4. If $\mathcal{P}$ is simple then 

$$ \sigma_n \big|_{I^n} = |x_1 \ldots x_n| \sum_{r=1}^{n} \sum_{\pi \in \Pi(n, r)} \rho_r, \pi \cdot $$

Example. Let $n = 3$. There are 5 partitions $\pi$ of the set $\{1, 2, 3\}$, and we have the following equality on $I^3$ (below $dx = dx_1 dx_2 dx_3$):

$$ \sigma_3(x_1, x_2, x_3) dx = |x_1 x_2 x_3| \{ \rho_3(x_1, x_2, x_3) dx + \rho_2(x_1, x_2) \delta(x_2 - x_3) dx + \rho_2(x_2, x_3) \delta(x_3 - x_1) dx + \rho_2(x_3, x_1) \delta(x_1 - x_2) dx + \rho_1(x_1) \delta(x_1 - x_2) \delta(x_1 - x_3) dx \} $$

Proof. Assume first that $P$ is the delta measure concentrated at a point $\omega = (\alpha, \beta) \in \Omega$ such that both $\alpha$ and $\beta$ have no repetitions. Let $\xi = (-\beta_1 < -\beta_2 < \cdots < \alpha_2 < \alpha_1)$ be the corresponding subset of $I$. By the definition of the controlling measures and the correlation measures,

$$ \sigma_n \big|_{I^n} = \sum_{x_1, \ldots, x_n \in \xi} |x_1 \ldots x_n| \delta(x_1) \otimes \cdots \otimes \delta(x_n), $$

$$ \rho_r = \sum_{y_1, \ldots, y_r \in \xi \atop y_i \neq y_j} \delta(y_1) \otimes \cdots \otimes \delta(y_r). $$

To each $n$-tuple $(x_1, \ldots, x_n)$ one can assign a partition $\pi$ of the set $\{1, \ldots, n\}$ as follows: two distinct indices $i, j$ are in the same bloc of $\pi$ if and only if $x_i = x_j$. Let $r$ be the number of blocs of $\pi$. Then, in the sum over $(x_1, \ldots, x_n)$, we group together summands corresponding to the same bloc $\pi$. To pass from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_r)$ we order the blocs in an arbitrary way and assign to the $i$th bloc $\pi_i$ the $i$th coordinate $y_i$ (in other words, $y_i = x_j$ for any index $j \in \pi_i$). Then we get the desired relation.

It is worth noting that in the above reasoning, enumeration of blocs was used for convenience only: actually, the coordinates on the section $I^n_\pi$ correspond just to blocs of $\pi$.

Thus, we have verified the claim of the proposition in the particular case when $P$ is a delta measure. In the general case, it remains to average over $\omega$ with respect to $P$. □

From Proposition 3.5 or Proposition 4.4 it is clear that for $n \geq 2$, the controlling measure $\sigma_n$ can never be absolutely continuous relative to Lebesgue measure on the cube $[-1, 1]^n$, because the diagonal sections $x_i = x_j$ of the cube always have a nonzero mass. On the contrary, the correlation measures $\rho_n$ can be absolutely continuous, at least, in the interior of $I^n$. It will be shown in [B] that the correlation functions for $P = P_{zz', \cdot}$ have analytic densities.

Remark 4.5. Even if a process of the form $\mathcal{P}$ is not simple, its correlation measures can be expressed through the controlling measures. Conversely, each controlling measure $\sigma_n$ can be expressed through the correlation measures $\rho_i$. It follows, in particular, that the initial measure $P$ is uniquely determined by the
correlation measures of $\mathcal{P}$. Another proof of this fact can be obtained from a general result of the theory of point processes and Proposition 4.1.

Recall (see section 1) that the Thoma simplex possesses a natural symmetry $\Omega \rightarrow \Omega$, \[ \omega = (\alpha, \beta) \mapsto \omega^t = (\beta, \alpha). \] (4.1)

Let $\mathcal{P}_{zz'}$ be the point process defined by the spectral measure $P_{zz'}$ and $\rho_n^{(zz')}$ be its correlation functions.

**Proposition 4.6.** The symmetry map (4.1) takes $P_{zz'}$ to $P_{-z,-z'}$. Likewise, the symmetry $x \mapsto -x$ of $I$ takes the process $\mathcal{P}_{zz'}$ to the process $\mathcal{P}_{-z,-z'}$. In particular, we have

\[ \sigma_n^{(-z,-z')}(x) = \sigma_n^{(zz')}(x), \]
\[ \rho_n^{(-z,-z')}(x) = \rho_n^{(zz')}(x). \]

**Proof.** This follows from (1.8), (2.2) and the definition of the measures $\sigma_n$ and $\rho_n$. \(\square\)

§5. THE DENSITY FUNCTION

The aim of this section is to calculate the first correlation measure (or the density measure) $\rho_1^{(zz')}$ of $\mathcal{P}_{zz'}$, the point process corresponding to $P_{zz'}$. It will be shown that $\rho_1^{(zz')}$ is absolutely continuous with respect to Lebesgue measure on $I$. Hence, one can speak about the first correlation function $\rho_1^{(zz')}(x)$ which is also called the density function.

By Proposition 4.3, we have

\[ \rho_1^{(zz')} = |x|^{-1} \sigma_1^{(zz')} \big|_I, \]

so that it suffices to calculate the measure $\sigma_1^{(zz')}$. on $[-1, 1]$.

**Lemma 5.1.** The measure $\sigma_1^{(zz')}$ is a unique solution of the moment problem

\[ \int_{-1}^{1} x^t \sigma_1^{(zz')} (dx) = \sum_{p,q \geq 0, p+q=l} \frac{(-1)^q t(z+1)p(-z+1)q(z'+1)p(-z'+1)q}{(t)p+q+1(p+q+1)p!q!}, \] (5.1)

where $l = 0, 1, 2, \ldots$.

**Proof.** We apply Proposition 3.3. Since $n = 1$, the parameter $d$ takes the only value 1, and the summation is taken over the hook diagrams $\lambda = (p \mid q)$.

It follows from the Murnaghan–Nakayama rule [M, §I.7, Ex. 5] that

\[ \chi_{(p \mid q)}^{(p+q+1)} = (-1)^q. \]

This yields the desired formula. As was already mentioned, uniqueness holds because the support of the measure in question is bounded. \(\square\)
We shall deal with the distributions
\[ \phi_a(u) = \frac{u^a_+}{\Gamma(a+1)} , \quad u \in \mathbb{R} , \quad a \in \mathbb{C} , \]
concentrated on the right semiaxis \( \mathbb{R}_+ \). Here the numerator \( u^a_+ \) coincides with the function \( u^a \) when \( u > 0 \) and vanishes when \( u < 0 \). When \( \Re a > -1 \), \( \phi_a \) is an integrable function, and when \( \Re a \leq -1 \), it is defined via analytic continuation.\(^4\) For any \( a \in \mathbb{C} \), the only singularity of the distribution \( \phi_a \) may be at 0.

The product
\[ \phi_{ab}(u) = \phi_a(u) \phi_b(1-u) , \quad a,b \in \mathbb{C} \]
is correctly defined because the possible singularities of the factors are at different points (\( u = 0 \) and \( u = 1 \), respectively). The result is a distribution concentrated on \([-1,1]\).

The following formula holds:
\[ \int u^p(1-u)^q \phi_{ab}(u) = \frac{(a+1)_p(b+1)_q}{\Gamma(a+b+p+q+2)} , \quad p,q = 0,1,2,\ldots \quad (5.2) \]
Indeed, when \( \Re a > -p - 1 \) and \( \Re b > -q - 1 \), this formula is equivalent to the classical Euler beta integral formula, and for arbitrary \( a,b \in \mathbb{C} \) the result holds by analytic continuation.

Set
\[ \Phi(a+1; \zeta) = \int e^{\zeta u} \phi_a,-a(u)du , \quad \zeta \in \mathbb{C} . \]
I.e., \( \Phi(a+1; \cdot) \) is the Laplace transform of \( \phi_a,-a \). Expanding the exponential function and using the beta integral \((5.2)\) with \( q = 0 \), one sees that this is a special case of Kummer’s hypergeometric function \(_1F_1\) (also called the confluent hypergeometric function):
\[ \Phi(a+1; \zeta) = \sum_{k=0}^{\infty} \frac{(a+1)_k}{(k+1)!k!} \zeta^k = _1F_1(a+1; 2; \zeta) , \]
see [E, ch. 6]. Clearly, \( \Phi(a+1; \cdot) \) is an entire function in \( \zeta \).

**Theorem 5.2.** The density measure satisfies the following equation
\[ \int_{-1}^{1} \Phi(t+1; \zeta x) \sigma_1^{(zz')}(dx) = \Phi(z+1; \zeta) \Phi(-z'+1; -\zeta) \]
\[ = \Phi(z'+1; \zeta) \Phi(-z+1; -\zeta) , \quad \zeta \in \mathbb{C} . \quad (5.3) \]

**Proof.** We rewrite formula \((5.1)\) as
\[ \frac{\zeta^l(t+1)_l}{(l+1)!l!} \int_{-1}^{1} x^{\zeta'} \sigma_1^{(zz')} (dx) \]
\[ = \sum_{p,q \geq 0 \atop p+q=l} \frac{(z+1)_p(-z+1)_q}{(p+q+1)!} \zeta^p(-\zeta)^q . \]

\(^4\)Here and in what follows, to simplify the notation, we shall write distributions as if they were ordinary functions.
Next, we replace the first and the second ratios on the right by the corresponding beta integrals (5.2) and sum over \( l \). The result will be as follows

\[
\sum_{l \geq 0} \int_{0}^{1} \frac{(\zeta x)^l(t + 1)_l}{(l + 1)!} \sigma_1^{(z')}(dx)
\]

\[
= \sum_{p, q \geq 0} \int \int \phi_{z, -z}(u) \phi_{z', -z'}(v) (\zeta uv)^p (\zeta(1 - u)(1 - v))^q \frac{1}{p!} \frac{1}{q!} dudv.
\]

Interchanging summation and integration, we get

\[
\int_{0}^{1} \Phi(t + 1; \zeta x) \sigma_1^{(z')}(dx) = \int \int e^{\zeta(uv - (1 - u)(1 - v))} \phi_{z, -z}(u) \phi_{z', -z'}(v) dudv.
\]

Since

\[ uv - (1 - u)(1 - v) = u + v - 1 = u - (1 - v) = v - (1 - u), \]

the double integral on the right factorizes into a product of two one–dimensional integrals. Making a change of a variable, \((1 - v) \mapsto v\) or \((1 - u) \mapsto u\), and using the definition of the function \( \Phi \), we obtain the first and the second variants of formula (5.3), respectively.

Note that the equivalence of both variants of formula (5.3) also follows from the identity

\[ \Phi(a + 1; \zeta) = e^\zeta \Phi(-a + 1; \zeta), \]

which is a particular case of Kummer’s transform for \(_1F_1\), see [E, §6.3, (21)]. □

**Remark 5.3.** The idea to use the beta integral is due to Borodin. My initial proof of Theorem 5.2 was more complicated: I transformed the right–hand side of (5.1) to the form

\[
\frac{l!(l + 1)!}{(t)_{l+1}} \sum_{p+q=l} \frac{(-z + 1)_q(z' + 1)_p}{p!(p + 1)!q!(q + 1)}
\]

with the help of a Leibniz–type formula for the difference operator \( f(z) \mapsto f(z) - f(z - 1) \). □

Set

\[ \phi_{ab}^{(-)}(u) = \phi_{ab}(-u); \]

this is a distribution concentrated on \([-1, 0]\).

**Corollary 5.4.** If \( t = 1 \) then the measure \( \sigma_1^{(z')}(z') \) is the convolution product of two distributions, concentrated on \([0, 1]\) and \([-1, 0]\):

\[ \sigma_1^{(z', z')} = \phi_{z, -z} \ast \phi_{-z', -z'}^{(-)}. \]

**Proof.** When \( t = 1 \), the function \( \Phi(t + 1; \zeta) \) degenerates to the exponential \( e^\zeta \). It follows that the left–hand side of (5.3) reduces to the Laplace transform of the
measure $\sigma_1^{(zz')}$.

On the other hand, the right-hand side of (5.3) is the product of the Laplace transforms of the distributions $\phi_{z,-z}$ and $\phi_{-z',z'}^\prime$. □

In the general case, to extract from the equation (5.3) an expression for $\sigma_1^{(zz')}$, we need a little formalism which will also be employed in §6.

Define an operation $\odot$ on distributions by the rule

$$(A \odot B)(x) = \int \int \delta(x - y_1 y_2) A(y_1) B(y_2) dy_1 dy_2.$$ 

Or, in terms of a test function $\psi$,

$$\int (A \odot B)(x) \psi(x) dx = \int \int A(y_1) B(y_2) \psi(y_1 y_2) dy_1 dy_2.$$ 

The operation $\odot$ may be viewed as the convolution product on the semigroup $(\mathbb{R}, \cdot)$ of real numbers under multiplication; for this reason we shall call it pseudoconvolution.

Of course, in order that the pseudoconvolution $A \odot B$ be correctly defined, $A$ and $B$ must satisfy appropriate conditions. For example, it suffices that they would be compactly supported; then $A \odot B$ will be compactly supported, too; moreover, $(A \odot B) \subseteq \text{supp } A \cdot \text{supp } B$. We shall employ this operation for distributions concentrated on $[-1,1]$; since $[-1,1]$ is a subsemigroup of $(\mathbb{R}, \cdot)$, the result will always be a distribution of the same kind.

**Remark 5.5.** (i) Assume that $A$ and $B$ are integrable functions or (complex) measures of finite variance with no atom at 0. Then the point 0 may be neglected, and the pseudoconvolution reduces, in essence, to the conventional convolution on the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

(ii) The same is also true if both $A$ and $B$ are distributions whose supports do not contain 0. However, in the general case, the point 0 can cause complications.

(iii) Here is an illustrative example of what can happen in the extreme case when both $A$ and $B$ are supported at 0: Denoting by $\delta_0$ the delta function and by $\delta_0^{(m)}$ its derivative of order $m$, we have

$$\delta_0^{(m)} \odot \delta_0^{(n)} = \begin{cases} m! \delta_0^{(m)} , & m = n \\ 0 , & m \neq n . \end{cases}$$

**Lemma 5.6.** Formula (5.3) of Theorem 5.2 is equivalent to

$$\sigma_1^{(zz')} \odot \phi_{t,-t} = \phi_{z,-z} \ast \phi_{-z',z'}^\prime ,$$

(5.4)

**Proof.** We claim that (5.3) coincides with the Laplace transform of the latter formula. Indeed, for the right-hand side this was already noted in the proof of Corollary 5.3, and for the left-hand side, this is readily verified by substituting the integral representation of the function $\Phi(t + 1; \cdot)$ to the left-hand side of (5.3). □
Lemma 5.7. Denoting by $\delta_1$ the Dirac mass at the point 1, we have

$$\phi_{t,-t} \odot \phi_{1,t-2} = \frac{1}{\Gamma(t + 1)} \delta_1.$$ 

Proof. Let us check that the $n$th moment of left–hand side (where $n = 0, 1, 2, \ldots$) is equal to $(\Gamma(t + 1))^{-1}$. By the definition of $A \odot B$, the $n$th moment of $A \odot B$ is equal to the product of the $n$th moments of $A$ and $B$. Next, observe that

$$\int x^n \phi_{ab}(x) dx = \frac{(a + 1)_n}{\Gamma(a + b + n + 2)}.$$

Therefore, the $n$th moment in question is equal to

$$\frac{(t + 1)_n}{\Gamma(n + 2)} \frac{(2)_n}{\Gamma(t + n + 1)} = \frac{1}{\Gamma(t + 1)},$$

as was to be shown. □

Theorem 5.8. We have

$$\sigma_1^{(zz')} = \Gamma(t + 1) (\phi_{z,-z} * \phi_{-z',z'}) \odot \phi_{1,t-2}, \quad (5.5)$$

so that

$$\rho_1^{(zz')}(x) = \frac{\Gamma(t + 1)}{|x|} \{(\phi_{z,-z} * \phi_{-z',z'}) \odot \phi_{0,t-2}\}, \quad x \neq 0. \quad (5.6)$$

Proof. Take the pseudoconvolution of the both sides of formula (5.4) of Lemma 5.6 with $\phi_{1,t-2}$ and apply then Lemma 5.7. □

The expression of Theorem 5.8 is a two–dimensional integral representation of the measure $\sigma_1^{(zz')}$. It can also be derived from the moment formula (5.1) by the same method as that we employed in the proof of Theorem 5.2.

Remark 5.9. There is a somewhat different formula for $\sigma_1^{(zz')}$: set $D = x \frac{d}{dx}$; then

$$\sigma_1^{(zz')}(x) = \Gamma(t + 1) (-D) [(\phi_{z,-z} * \phi_{-z',z'}) \odot \phi_{0,t-1}].$$

Indeed, by virtue of Proposition 6.8 (see below), the right–hand side is equal to

$$\Gamma(t + 1) (\phi_{z,-z} * \phi_{-z',z'}) \odot (-D \phi_{0,t-1}).$$

It is readily verified that

$$-D \phi_{0,t-1} = \phi_{1,t-1}.$$

So, the formula in question is equivalent to (5.5).
Theorem 5.10. The restriction of the density measure $\rho_{1}^{(zz')}$ to $(-1,0) \cup (0,1)$ is absolutely continuous with respect to Lebesgue measure $dx$:

$$\rho_{1}^{(zz')}(dx) = \rho_{1}^{(zz')}(x)dx,$$

and the density function $\rho_{1}^{(zz')}(x)$ is real analytic on $(0,1)$ and on $(-1,0)$.

When $x \in (0,1)$, the density function can be written in the following form

$$\rho_{1}^{(zz')}(x) = \Gamma(t+1) \frac{\Gamma(z+1)\Gamma(z'+1)}{\Gamma(z+1)\Gamma(z'+1)} (1-x)^{-z-z'+t} \langle A, \Psi \rangle, \quad x \in (0,1), \quad (5.7)$$

where $A$ is a two–dimensional distribution concentrated on the triangle $u,v \geq 0$, $u+v \leq 1$, and $\Psi$ is a test function, which is correctly defined and smooth in a neighborhood of that triangle:

$$A = A(u,v) = \phi_{-z}(u)\phi_{-z'}(v)\phi_{t-2}(1-u-v),$$

$$\Psi = \Psi(u,v) = (1-(1-x)u)^{z}(1-(1-x)v)^{z'}(1-(1-x)(u+v))^{-t}.$$

When $x \in (-1,0)$, the density function can be written in the same form, with the only modification: $x$ is replaced by $|x|$ and the parameters $z,z'$ are replaced by $-z,-z'$.

Proof. We start with a formal transformation of the expression (5.6) given by Theorem 5.8.

Assume that $x \in (0,1)$. By (5.6) and the definition of $\odot$,

$$\sigma_{1}^{(zz')}(x) = \Gamma(t+1) \int\int\int \delta(x-w(u+v-1))\phi_{z,-z}(u)\phi_{z'-,-z'}(v)\phi_{1,t-2}(w)dw\,dudv,$$

where $\delta(\cdot)$ is the delta function. We can exclude the integration over $w$ using the formula

$$\frac{1}{x} \delta(x-w(u+v-1))\phi_{1,t-2}(w)dw = \frac{1}{x(u+v-1)} \delta(w-x(u+v-1))\phi_{1,t-2}(w)dw = \frac{1}{x(u+v-1)} \phi_{1,t-2}(x(u+v-1)) = \frac{1}{(u+v-1)^t} \phi_{t-2}(u+v-1-x).$$

This gives

$$\rho_{1}^{(zz')}(x) = \Gamma(t+1) \int\int \phi_{z}(u)\phi_{-z}(1-u)\phi_{z'}(v)\phi_{-z'}(1-v)$$

$$\times \phi_{t-2}(u+v-1-x) \frac{dudv}{(u+v-1)^t}.$$ 

Make a change of variables,

$$1-u = (1-x)u_{1}, \quad 1-v = (1-x)v_{1}.$$
Then
\[ dudv = (1 - x)^2 du_1 dv_1, \]
\[ u = 1 - (1 - x)u_1, \quad v = 1 - (1 - x)v_1, \]
\[ u + v - 1 = 1 - (1 - x)(u_1 + v_1), \quad u + v - 1 - x = (1 - x)(1 - u_1 - v_1). \]

Substituting these expressions into the integral and renaming then the variables \( u_1, v_1 \) to \( u, v \), we come to formula (5.7). In the case \( x \in (-1, 0) \) one can use exactly the same argument; an alternative possibility is to use the symmetry property indicated in Lemma 4.6.

To justify these formal transformations from (5.6) to (5.7), let us, for a moment, interpret \( z, -z, z', -z', t - 2 \) as five independent complex parameters. If the real parts of these variables are strictly positive, then all the distributions become ordinary continuous functions, and our transformations are readily justified. On the other hand, the expressions (5.6) (with \( x > 0 \)) and (5.7) are both correctly defined distributions in \( x \), which depend holomorphically on the parameters. Hence, by the principle of analytic continuation, they are equivalent.

Finally, the expression (5.7) is real analytic, because the test function \( \Psi \) is correctly defined for complex values of the parameters \( x \) in the strip \( 0 < \Re x < 1 \) and depends on \( x \) analytically. \( \Box \)

**Remark 5.11.** By Lemma 6.10 (see below), the density measure \( \rho_{1}^{(zz')} \) has no atom at 0, and one can show that \( \rho_{1}^{(zz')} \) has no atoms at the end points \( \pm 1 \). So, it is completely determined by the density function on the open intervals \((0, 1)\) and \((-1, 0)\).

We shall express the density function through a multivariate hypergeometric series. Let
\[ a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n), \quad c \]
be complex parameters and
\[ y = (y_1, \ldots, y_n) \]
be \( n \) complex variables. The \( n \)-dimensional *Lauricella hypergeometric function* of type \( B \) is defined by the series
\[ F_{B}^{[n]}(a, b; c | y) = \sum_{m_1, \ldots, m_n \geq 0} \frac{(a_1)_{m_1}(b_1)_{m_1} \cdots (a_n)_{m_n}(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n} m_1! \cdots m_n!} y_1^{m_1} \cdots y_n^{m_n} \]
where the series is absolutely convergent for \( |y_1| < 1, \ldots, |y_n| < 1 \), see [AK], [Ex]. When \( n = 1 \), this is Gauss’ hypergeometric function, and when \( n = 2 \), this is Appell’s hypergeometric function \( F_3 \). Note also that the function \( F_B \) remains invariant when the couples \((a_1, b_1), \ldots, (a_n, b_n)\) are permuted or the parameters in couples are interchanged.

**Theorem 5.12.** Let
\[ a = (-z, -z', t - 1), \quad b = (-z + 1, -z' + 1, t), \]
\[ c = -z - z' + t + 1 = (-z + 1)(-z' + 1). \]
For $0 < x \leq 1$, we have
\[ \rho_1^{(zz')} (x) = \frac{\Gamma(t + 1)}{\Gamma(z + 1) \Gamma(z' + 1)} \phi_{c-1} (1 - x) F_B^{[3]} (a, b; c | 1 - x, 1 - x, 1 - x). \]  
(5.8)

The same expression holds for $-1 \leq x < 0$ provided that $x$ is replaced by $|x|$ and $z, z'$ are multiplied by $-1$.

Proof. First of all, note that the parameter $c$, as defined above, is a strictly positive real number (this follows from the fundamental assumptions on the parameters $z, z'$). It follows that $\phi_{c-1} (1 - x)$ is integrable at $x = 1$. Consequently, the whole expression is integrable at $x = 1$, as it should be.

As before, the case of negative $x$ can be reduced to that of positive $x$ by symmetry. Next, by Remark 5.11, we may suppose $0 < x < 1$.

Set
\[ u_1 = u, \quad u_2 = v, \quad u_3 = u + v, \]
\[ d_1 = -z, \quad d_2 = -z', \quad d_3 = t - 2, \]
\[ y_1 = y_2 = y_3 = 1 - x. \]

By the binomial expansion,
\[ \Psi = \sum_{m_1, m_2, m_3 \geq 0} \frac{(-z)^{m_1} (-z')^{m_2} (t)^{m_3}}{m_1! m_2! m_3!} (y_1 u_1)^{m_1} (y_2 u_2)^{m_2} (y_3 u_3)^{m_3} \]

Substituting this in (5.7) and employing the Euler beta integral
\[ \int_{u_1 + u_2 + u_3 = 1} \frac{\phi_{d_1} (u_1) \phi_{d_2} (u_2) \phi_{d_3} (u_3) u_1^{m_1} u_2^{m_2} u_3^{m_3}}{u_1, u_2, u_3 \geq 0} du \]
\[ = \frac{(d_1 + 1)^{m_1} (d_2 + 1)^{m_2} (d_3 + 1)^{m_3}}{\Gamma(d_1 + d_2 + d_3 + 3) \Gamma(d_1 + d_2 + d_3 + 3) m_1 + m_2 + m_3} \]
we obtain the desired formula. \( \square \)

§6. AN APPLICATION

In this section we shall prove the following result.\(^5\)

Theorem 6.1. All the spectral measures $P_{zz'}$ are concentrated on the face $\Omega_0$ defined in (1.5).

By Proposition 3.4, this is equivalent to the fact that $\sigma_1^{(zz')}$ has no atom at 0. The proof of the latter claim is divided into a series of lemmas.

Let $C_0 (\mathbb{R})$ be the space of continuous compactly supported functions on $\mathbb{R}$ and $C_0^\infty (\mathbb{R})$ be its subspace consisting of smooth compactly supported functions. Let $a, b$ be complex parameters and $x$ be the coordinate on $\mathbb{R}$. Let
\[ \mathcal{F}^+_{a} = \phi_a \cdot C^\infty_0 (\mathbb{R}) \]
denote the space of distributions formed by the products $\phi_a \cdot f$ with $f \in C_0^\infty (\mathbb{R})$, and let $\mathcal{F}^-_a$ denote the image of $\mathcal{F}^+_a$ under the reflection $x \mapsto -x$.

\(^5\)I am grateful to Jean–Louis Clerc and Jacques Faraut for discussions related to the proof of Theorem 6.1.
Lemma 6.2. We have
\[ \frac{d}{dx} \mathcal{F}_a^\pm \subset \mathcal{F}_{a-1}^\pm + \mathcal{F}_a^\pm. \]

Proof. Indeed, this follows from the well–known formula
\[ \frac{d}{dx} \phi_a(x) = \phi_{a-1}(x). \]

\[ \square \]

Lemma 6.3. We have
\[ \mathcal{F}_a^\pm \subset \frac{d}{dx} \mathcal{F}_{a+1}^\pm + \mathcal{F}_{a+1}^\pm. \]

Proof. Indeed, for any \( f \in C_0^\infty(\mathbb{R}) \),
\[ \phi_a f = (\phi_{a+1} f)' - \phi_{a+1} f' \in \left( \frac{d}{dx} \mathcal{F}_{a+1}^+ + \mathcal{F}_{a+1}^+ \right). \]
The same argument works for the sign “−”. \( \square \)

Lemma 6.4. We have
\[ \mathcal{F}_a^+ * \mathcal{F}_b^- \subset (\mathcal{F}_{a-1}^+ * \mathcal{F}_{b+1}^-) + (\mathcal{F}_a^+ * \mathcal{F}_{b+1}^-). \]

Proof. By Lemma 6.3,
\[ \mathcal{F}_a^+ * \mathcal{F}_b^- \subset \frac{d}{dx} \mathcal{F}_{a-1}^- + \mathcal{F}_{a-1}^- \subset \mathcal{F}_a^+ * \left( \frac{d}{dx} \mathcal{F}_{b+1}^- + \mathcal{F}_{b+1}^- \right) \]
\[ \subset (\mathcal{F}_a^+ * \mathcal{F}_{b+1}^-) + (\mathcal{F}_a^+ * \mathcal{F}_{b+1}^-) \]
\[ = \left( \frac{d}{dx} \mathcal{F}_a^- \right) * \mathcal{F}_{b+1}^- + (\mathcal{F}_a^+ * \mathcal{F}_{b+1}^-). \]
Next, we apply Lemma 6.2. \( \square \)

Lemma 6.5. Assume that one of the following conditions holds:
(i) \( \Re a > 0, \Re b < 1 \);
(ii) \( \Re a = \Re b = 0 \).
Then \( \mathcal{F}_a^+ * \mathcal{F}_{-b}^- \subset C_0(\mathbb{R}) \).

Proof. (i) In this case, the elements of \( \mathcal{F}_a^+ \) are continuous functions with compact support while the elements of \( \mathcal{F}_{-b}^- \) are integrable functions with compact support. Therefore, the result of convolution are continuous functions.
(ii) In this case, the elements of the both spaces are bounded measurable functions with compact support. So, they are square integrable functions. Therefore, the result of convolution are again continuous functions. \( \square \)
Lemma 6.6. Assume that the parameters $a, b$ satisfy the following condition: there exist integers $m \geq n \geq 0$ such that either

$$m < Ra < m + 1, \quad n < Rb < n + 1$$

or

$$Ra = m, \quad Rb = n.$$  

Then

$$\mathcal{F}_a^+ \ast \mathcal{F}_b^- \subset C_0(\mathbb{R}).$$

Actually, we need only the case $m = n$, but it will be convenient to check a slightly more general claim with $m \geq n$.

Proof. Assume first that $n \geq 1$. By lemma 6.4,

$$\mathcal{F}_a^+ \ast \mathcal{F}_b^- \subset (\mathcal{F}_{a-1}^+ \ast \mathcal{F}_{b-1}^-) + (\mathcal{F}_a^+ \ast \mathcal{F}_{b-1}^-),$$

which enables one to reduce the claim of the lemma for a given couple $m \geq n \geq 1$ to the same claims with $(m,n)$ replaced by $(m-1,n-1)$ or by $(m,n-1)$. The reduction stops when $n = 0$, but then we can apply Lemma 6.5. □

Lemma 6.7. Near 0, the distribution

$$\phi_{z,-z} \ast \phi_{-z',z'}^{(-)}$$

is given by a continuous function.

Proof. By making use of an appropriate partition of unity one can represent the both factors in the form

$$\phi_{z,-z} = A_0 + A + A_1, \quad \phi_{-z',z'}^{(-)} = B_{-1} + B + B_0,$$

where $A, B$ are smooth functions and $A_0, A_1, B_{-1}, B_0$ are certain distributions concentrated near the points 0, 1, −1, 0, respectively (they are obtained by multiplying the initial distributions $\phi_{z,-z}$ and $\phi_{-z',z'}^{(-)}$ by appropriate functions from $C_0^\infty(\mathbb{R})$). It follows that

$$\phi_{z,-z} \ast \phi_{-z',z'}^{(-)} = (A_0 \ast B_0) + (A_1 \ast B_{-1}) + (A_0 \ast B_{-1}) + (A_1 \ast B_0) + (\ldots),$$

where $(\ldots)$ is a smooth function. Since the distributions $A_0 \ast B_{-1}$ and $A_1 \ast B_0$ are concentrated near the points $-1$ and $1$, respectively, it remains to check that the distributions $A_0 \ast B_0$ and $A_1 \ast B_{-1}$ are actually continuous functions.

We have $A_0 \in \mathcal{F}_z^+$ and $B_0 \in \mathcal{F}_{-z'}^-$, so that

$$A_0 \ast B_0 \in (\mathcal{F}_z^+ \ast \mathcal{F}_{-z'}^-).$$

Next, the distribution $A_1 \ast B_{-1}$ will not change if we shift the both factors by 1, to the left and to the right, respectively. The resulting distributions will lie in $\mathcal{F}_{-z}^-$ and $\mathcal{F}_{z'}^-$, respectively, whence

$$A_1 \ast B_{-1} \in (\mathcal{F}_{z'}^+ \ast \mathcal{F}_{-z}^-).$$

Now, let us compare our fundamental assumptions on the parameters $z, z'$ with the assumptions on the parameters $a, b$ in Lemma 6.6. Without loss of generality, we may assume that $Rz \geq 0$ and $Rz' \geq 0$ (otherwise we may apply Proposition 4.6). Then the couples $(a,b) = (z,z')$ and $(a,b) = (z',z)$ will satisfy the assumptions of Lemma 6.6 with $m = n$. Application of this lemma concludes the proof. □
Lemma 6.8. Set \( D = x \frac{d}{dx} \). For any compactly supported distributions \( A, B \), we have

\[
(DA) \ast B = D(A \ast B) = A \ast (DB).
\]

Note that the claim is obvious when the supports of \( A \) and \( B \) do not contain 0, because then, replacing \( x \) by \( s = \log |x| \), we can reduce the pseudoconvolution to the ordinary convolution and the operator \( D \) — to \( \frac{d}{ds} \).

Proof. Set \( D' = \frac{d}{dx} \circ x = D + 1 \). For an arbitrary smooth test function \( \psi \), we have

\[
\langle (DA) \ast B, \psi \rangle = \int \int DA(y_1)B(y_2)\psi(y_1y_2)dy_1dy_2 \\
= \int B(y_2)dy_2 \int DA(y_1)\psi(y_1y_2)dy_1 \\
= \int B(y_2)dy_2 \int A(y_1)(-y_1 \frac{d}{dy_1} - 1)\psi(y_1y_2)dy_1 \\
= \int B(y_2)dy_2 \int A(y_1)(-D'\psi)(y_1y_2)dy_1 \\
= \langle A \ast B, -D'\psi \rangle = \langle D(A \ast B), \psi \rangle.
\]

This proves the first equality, and the second one is verified similarly. \( \square \)

Lemma 6.9. Assume that \( A \) is a distribution with a compact support not containing 0, and \( B \) belongs to a class \( \mathcal{F}_a^+ \) with \( \Re a > -1 \). Then the distribution \( A \ast B \) is a function.

Proof. Since the support of \( A \) is separated from 0, \( A \) can be represented in the form \( A = f(D)\tilde{A} \), where \( f \) is a polynomial, \( D \) is as in Lemma 6.8, and \( \tilde{A} \) is an ordinary (say, integrable) function. By Lemma 6.8,

\[
A \ast B = (f(D)\tilde{A}) \ast B = \tilde{A} \ast (f(D)B).
\]

On the other hand, we have

\[
D\phi_a = a\phi_a, \quad a \in \mathbb{C},
\]

which implies

\[
D\mathcal{F}_a^+ \subseteq \mathcal{F}_a^+, \quad a \in \mathbb{C},
\]

so that \( f(D)B \) lies in \( \mathcal{F}_a^+ \) together with \( B \). When \( \Re a > -1 \), the distributions of class \( \mathcal{F}_a^+ \) are ordinary (integrable) functions. Consequently, the pseudoconvolution of \( \tilde{A} \) and \( f(D)B \) is, in essence, the ordinary convolution product on the multiplicative group \( \mathbb{R}^* \) (see Remark 5.5), and its result is an ordinary function. \( \square \)

The next lemma concludes the proof of Theorem 6.1.
Lemma 6.10. The measure $\sigma_1^{(zz')} \text{ has no atom at } 0$.

Proof. We shall use different arguments when $t < 1$, $t = 1$, and $t > 1$.

When $t = 1$, the measure $\sigma_1^{(zz')}$ is given by the expression of Corollary 5.4, and the claim follows from Lemma 6.7.

Assume that $t < 1$ and examine the expression (5.4). Since $0 < t < 1$, the distribution $\phi_{t,-t}$ is actually a nonnegative integrable function. We can decompose the measure $\sigma_1^{(zz')}$ into the sum of two components: one is $\text{const} \cdot \delta_0$ (a multiple of the Dirac mass at 0) and another is a measure $A$ with no atom at 0. The measure $A \circ \phi_{t,-t}$ cannot have an atom at 0 (see Remark 5.5). On the other hand,

$$\text{const} \delta_0 \circ \phi_{t,-t} = \text{const} \left( \int \phi_{t,-t}(y)dy \right) \delta_0,$$

where the integral is strictly positive. But the right-hand side of formula (5.4) cannot have an atom at 0, by virtue of Lemma 6.7. Hence, $\text{const} = 0$, so that $\sigma_1^{(zz')}$ has no atom at 0, too.

Finally, assume that $t > 1$ and look at formula (5.5). By Lemma 6.7, one can write

$$\Gamma(t + 1) \phi_{z,-z} \ast \phi_{z',z'}^{(-)} = A_0 + A_1,$$

where $A_0$ is a continuous function and $A_1$ is a distribution whose support does not contain 0. Next, since $t > 1$, the distribution $\phi_{1,t-2}$ can be written as

$$\phi_{1,t-2} = B_0 + B_1,$$

where $B_0 \in F_1^+$ and $B_1$ is an integrable function concentrated near 1. Then we have

$$\sigma_1^{(zz')} = (A_0 + A_1) \circ (B_0 + B_1) = (A_0 \circ B_0) + (A_1 \circ B_0) + (A_0 \circ B_1) + (A_1 \circ B_1).$$

Let examine the four summands of the latter expression. Since $A_0$, $B_0$, $B_1$ are ordinary (integrable) functions, the terms $A_0 \circ B_0$ and $A_0 \circ B_1$ are ordinary functions according to Remark 5.5. The term $A_1 \circ B_0$ is an ordinary function by Lemma 6.9. The term $A_1 \circ B_1$ is a distribution concentrated outside a neighborhood of 0. We conclude that the whole expression cannot have an atom at 0. \qed

7. Appendix: Correlation functions of Poisson–Dirichlet processes

In this appendix, we briefly discuss a parallel (but more simple) theory. It is related to the Poisson–Dirichlet distributions, a remarkable one-parametric family of probability measures $\{PD(t) \mid t > 0\}$ which live on an infinite-dimensional subsimplex $\Delta \subset \Omega$. The measures $PD(t)$ determine point processes $\mathcal{PD}(t)$ on $(0,1]$, called the Poisson–Dirichlet processes, and we shall calculate the correlation functions of $\mathcal{PD}(t)$ by using the general formalism of §§1–4.

The Poisson–Dirichlet distributions were studied in many papers from different points of view, see, e.g., [Ki2, W]. Our interest in them is caused by the fact that they are one of the basic elements in the construction [KOV] of the generalized regular representations $T_z$. So, both kind of measures, the $PD(t)$'s and the $P_{zz'}$'s,
are connected with the same construction — that of the representations $T_z$. But they appear at different levels of that construction: the former — at the ‘group’ level (as they are responsible for certain quasiinvariant measures on $\widetilde{G}/K$, see the Introduction), and the later — at the ‘dual’ level (the representation level). However, both $PD(t)$ and $P_{zz'}$ are spectral measures in the sense that they govern decomposition of certain objects into indecomposable ones\(^6\), and both $PD(t)$ and $P_{zz'}$ can be interpreted as point processes.

For these reasons, it seems interesting to compare the point processes $PD(t)$ and $P_{zz'}$, and the main purpose of the present appendix is to prepare a foundation for such a comparison (we postpone the discussion to the third article [BO]). Another purpose is to illustrate the formalism of sections 1–4 on a simpler material.

We define the simplex $\Delta$ as the closed subset of $\Omega$ determined by $\beta_1 = \beta_2 = \cdots = 0$. In other words, $\Delta$ is the set of sequences $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0)$ such that $\sum_{i=1}^{\infty} \alpha_i \leq 1$. The specialization (1.3) is replaced by the following one:

$$\begin{align*}
p_1 \mapsto \widetilde{p}_1 &= \widetilde{p}_1(\alpha; \gamma) := \sum_{i=1}^{\infty} \alpha_i + \gamma \\
p_n \mapsto \widetilde{p}_n &= \widetilde{p}_n(\alpha; \gamma) := \sum_{i=1}^{\infty} \alpha_i^n, \quad n \geq 0,
\end{align*}$$

which is equivalent to

$$1 + \sum_{n=1}^{\infty} h_n u^n \mapsto 1 + \sum_{n=1}^{\infty} \widetilde{h}_n u^n := e^{\gamma u} \prod_{i=1}^{\infty} \frac{1}{1 - \alpha_i u}.$$ 

In what follows, we shall assume that $\alpha$ is a point of $\Delta$ and

$$\gamma = 1 - \sum_{i=1}^{\infty} \alpha_i.$$ 

Then all $\widetilde{p}_n$ turn into continuous functions functions on $\Delta$ (note that $\widetilde{p}_1 \equiv 1$). Consequently, any element $f \in \Lambda$ is converted into a continuous function $\tilde{f} = \tilde{f}(\alpha)$ on $\Delta$.

In place of the Schur functions $s_\mu$ we shall deal with the monomial symmetric functions $m_\lambda \in \Lambda$. Recall that in the standard realization of $\Lambda$ as the algebra of symmetric functions in variables $x_1, x_2, \ldots$, the function $m_\lambda$ is the sum of all distinct monomials obtained from $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ by permuting variables, see [M].

The Pieri formula (1.3) for the Schur functions is replaced by its counterpart for the monomial symmetric functions,

$$m_\mu \cdot p_1 = \sum_{\lambda: \lambda \triangleright \mu} \kappa_0(\mu, \lambda) m_\lambda,$$

where the coefficients $\kappa_0(\mu, \lambda)$ are positive integers defined as follows. Given $\lambda \triangleright \mu$, there exists a unique $i \in \{1, 2, \ldots\}$ such that $\lambda_i = \mu_i + 1$ (and $\lambda_j = \mu_j$ for all $j \neq i$). Then $\kappa_0(\mu, \lambda)$ is the multiplicity of the part $\lambda_i$ in the partition $\lambda$.

\(^6\)At the ‘group’ level, the ‘objects’ are $K$-invariant probability measures on $\widetilde{G}/K$ and the ‘indecomposable objects’ are ergodic ones, see [Ki1, KOV].
In ‘exponential notation’ for partitions,
\[ \mu = (1^{r_1(\mu)}2^{r_2(\mu)}\ldots), \quad \lambda = (1^{r_1(\lambda)}2^{r_2(\lambda)}\ldots), \]
\(\lambda \searrow \mu\) means that there exists a unique \(k\) (equal to \(\lambda_i\)) such that
\[ r_k(\lambda) = r_k(\mu) + 1, \quad r_{k-1}(\lambda) = r_{k-1}(\mu) - 1, \quad r_l(\lambda) = r_l(\mu) \quad \text{for} \ l \neq k, \]
and then
\[ \kappa_0(\mu, \lambda) = r_k. \]

The role of Young graph \(\mathbb{Y}\) is played by the Kingman graph \(\mathbb{K} = (\mathbb{Y}, \kappa_0)\): its vertices are the same as for \(\mathbb{Y}\) (arbitrary partitions) but each edge \(\mu \searrow \lambda\) turns into \(\kappa_0(\mu, \lambda)\) edges with the endpoints \(\mu\) and \(\nu\). The grading \(\mathbb{K} = \cup\mathbb{K}_n\) of the vertices remains unchanged. That is to say, the \(n\)th level \(\mathbb{K}_n\) consists of partitions of \(n\).

A new dimension function \(\dim_0\) then arises: \(\dim_0 \lambda\) is still defined as the number of oriented paths from \(\emptyset\) to \(\lambda\) but we take into account edge multiplicities. The recurrence relation is modified as follows:
\[ \dim_0 \lambda = \sum_{\mu: \mu \searrow \lambda} \dim_0 \mu \kappa_0(\mu, \nu). \]
By the very definition, the numbers \(\kappa_0(\mu, \lambda)\) are nothing but the coefficients in the expansion
\[ p^n_1 = \sum_{\lambda: \lambda \in \mathbb{K}_n} \dim_0 \lambda \cdot m_\lambda. \]

The harmonicity condition is written as follows:
\[ \varphi(\mu) = \sum_{\lambda: \lambda \searrow \mu} \kappa_0(\mu, \lambda) \varphi(\lambda). \]

A coherent system of distributions on the graph \(\mathbb{K}\) is a sequence \(M = (M_n)\) of probability distributions on the finite sets \(\mathbb{K}_n\) (partitions of \(n\)), subject to the coherence condition
\[ M_n(\mu) = \sum_{\lambda: \lambda \searrow \mu} \frac{\dim_0 \mu \cdot \kappa_0(\mu, \lambda)}{\dim_0 \lambda} M_{n+1}(\lambda), \quad \mu \in \mathbb{K}_n, \]
equivalent to harmonicity of the function \(M(\lambda)/\dim_0 \lambda\) (here it is convenient to write \(M(\mu)\) in place of \(M_n(\mu)\)). This concept is equivalent to that of partition structure in the sense of Kingman [Ki1].

The next relation establishes a bijective correspondence \(M \leftrightarrow P\) between coherent systems \(M\) on Kingman’s graph and probability measures \(P\) on the simplex \(\Delta:\)
\[ \frac{M(\lambda)}{\dim_0 \lambda} = \int_{\Delta} \tilde{m}_\lambda(\alpha) P(d\alpha) \quad (7.1) \]
(recall that \(\tilde{m}_\lambda\) is the image of \(m_\lambda \in \Lambda\) under the specialization \(f \mapsto \tilde{f}\) defined above). This result is Kingman’s theorem [Ki1, Ke1, KOO].
The coherent $t$-systems $M^{(t)} = (M^{(t)}_n)$ of distributions on Kingman’s graph are defined as follows: $t > 0$ is a parameter and

$$M^{(t)}_n(\lambda) = \frac{t^{\ell(\lambda)}n!}{(t)_n z_\lambda}, \quad \lambda \in \mathbb{K}_n,$$

where $\ell(\lambda)$ is the length of the partition $\lambda$ (number of nonzero parts) and, in ‘exponential notation’,

$$z_\lambda = \prod_i \lambda_i \cdot \prod_k r_k(\lambda)! = \prod_k k^{\ell_k(\lambda)} r_k(\lambda)!.$$  \hspace{1cm} \text{\eqref{eq:z_lambda}}

The fact that $M^{(t)}$ is indeed a coherent system on $\mathbb{K}$ is verified directly using the above formulas for $\dim_0 \lambda$ and $\kappa_0(\mu, \lambda)$.

The systems $M^{(t)}$ are known as Ewens partition structures [Ki1, Ki2]. According to Kingman’s theorem, they determine certain probability measures on $\Delta$. The last are called the Poisson–Dirichlet distributions, see [Ki2], and denoted as $PD(t)$. So, the link between $M^{(t)}$ and $PD(t)$ is as follows:

$$M^{(t)}(\lambda) = \int_{\Delta} \tilde{m}_\lambda(\alpha)(PD(t))(d\alpha).$$

Given a probability measure $P$ on $\Delta$, we may regard it as a measure on $\Omega$. Hence, the definitions of the control measures $\sigma_n$, of the point process $\mathcal{P}$ attached to $P$, and of the correlation measures $\rho_n$ of $\mathcal{P}$ make sense. As in the present situation there is no $\beta$’s, the measure $\sigma_n$ is concentrated on the cube $[0,1]^n$ contained in $[-1,1]^n$, the state space of the process $\mathcal{P}$ is the semiopen interval $(0,1]$, and the $n$th correlation function $\rho_n$ lives on $(0,1]^n$.

Assume that $P$ is related to $M$ by (7.1). According to (3.1a), the moments of $\sigma_n$ are given by

$$\int_{[0,1]^n} x_1^{l_1} \cdots x_n^{l_n} \sigma_n(dx_1 \cdots dx_n) = \int_{\Delta} \tilde{p}_{l_1+1}(\alpha) \cdots \tilde{p}_{l_n+1}(\alpha) P(d\alpha),$$

which implies

$$\int_{[0,1]^n} x_1^{l_1} \cdots x_n^{l_n} \sigma_n(dx_1 \cdots dx_n) = \sum_{\lambda} [p_{l_1+1+\cdots+l_n+1} : m_\lambda] \frac{M(\lambda)}{\dim_0 \lambda}. \hspace{1cm} \text{\eqref{eq:moment_relations}}$$

Here $\lambda$ ranges over $\mathbb{K}_{l_1+\cdots+l_n+n}$ and, for $f \in \Lambda$, the symbol $[f : m_\lambda]$ denotes the coefficient of $m_\lambda$ in the expansion of $f$ into linear combination of monomial symmetric functions. It is worth noting that only partitions $\lambda$ with $\ell(\lambda) \leq n$ really contribute to this formula, cf. (3.1b).

Consequently, given $M$, we have again a collection of moment problems to determine the control measures $\sigma_n$, and from the measures $\sigma_n$ one can get the correlation functions $\rho_n$. The procedure is exactly the same as for the Young graph. We shall apply it to $M = M^{(t)}$. The corresponding control measures will be denoted as $\sigma^{(t)}_n$ and the correlation measures as $\rho^{(t)}_n$. The point process attached to $PD(t)$ will be called the Poisson–Dirichlet process and denoted as $\mathcal{PD}(t)$. 32
We shall see that in the present situation, calculation of the correlation functions turns out to be much easier than in the case of the Young graph. A formal explanations is that the coefficients in the expansion of \( p_{t_{1+1}} \cdots p_{t_{n+1}} \) on monomial functions \( m_\lambda \) are given by much simpler expressions than when expanding on Schur functions \( s_\lambda \).

To state the result we need some notation related to set partitions. Recall (see §4) that partitions of a set are the same thing as equivalence relations. We shall also need \textit{ordered} partitions, i.e., partitions with a fixed enumeration of the blocs. As in Proposition 4.4, we denote the set of partitions of \( \{1, \ldots, n\} \) with \( r \) (nonempty) blocs by \( \Pi(n, r) \); the set of ordered partitions with \( r \) blocs will be denoted as \( \tilde{\Pi}(n, r) \).

There is a natural projection \( \tilde{\Pi}(n, r) \to \Pi(n, r) \) — forgetting enumeration. For each \( \pi \in \Pi(n, r) \), there are \( r! \) ordered partitions \( \tilde{\pi} \) over \( \pi \).

Given \( \pi \in \Pi(n, r) \), we define a ‘diagonal section’ \([0, 1]^n_\pi \) of the cube \([0, 1]^n \) as intersection with all hyperplanes of the form \( x_i = x_j \) where \( i \sim j \mod \pi \). To an ordered partition \( \tilde{\pi} = \tilde{\pi}_1 \sqcup \cdots \sqcup \tilde{\pi}_r \) over \( \pi \) we assign a bijective map

\[
[0, 1]^r \to [0, 1]^n_\pi \subset [0, 1]^n, \quad y = (y_1, \ldots, y_r) \mapsto x = (x_1, \ldots, x_n), \quad \text{(7.5)}
\]

where \( x_j = y_i \) provided that \( j \in \tilde{\pi}_i \).

Define a measure \( \hat{\sigma}_{r, \tilde{\pi}}^{(t)} \) on the cube \([0, 1]^r \) as follows:

\[
\hat{\sigma}_{r, \tilde{\pi}}^{(t)}(dy) = t^r \cdot \prod_{i=1}^r y_i^{[\tilde{\pi}_i]-1} \cdot (1 - \sum_{i=1}^r y_i)^{t-1} dy, \quad \text{(7.6)}
\]

and let \( \sigma_{n, r, \tilde{\pi}}^{(t)} \) be its image under the above map \( y \mapsto x \). The latter measure lives on \([0, 1]^n_\pi \) and does not depend on the choice of \( \tilde{\pi} \) over \( \pi \), whence we may denote is as \( \sigma_{n, r, \pi}^{(t)} \).

**Example.** For \( n = 3 \) the list of partitions and measures is as follows.

- \( \Pi(3, 1) \):
  \[
  \pi = \{1, 2, 3\}, \quad \sigma_{3,1,\pi}^{(t)} = t^3 (1 - x_1 - x_2 - x_3)^{t-1} dx
  \]

- \( \Pi(3, 2) \):
  \[
  \pi = \{1\} \sqcup \{2, 3\}, \quad \sigma_{3,2,\pi}^{(t)} = t^2 x_2 (1 - x_1 - x_2)^{t-1} \delta(x_2 - x_3) dx
  \]

  \[
  \pi = \{2\} \sqcup \{3, 1\}, \quad \sigma_{3,2,\pi}^{(t)} = t^2 x_3 (1 - x_2 - x_3)^{t-1} \delta(x_3 - x_1) dx
  \]

  \[
  \pi = \{3\} \sqcup \{1, 2\}, \quad \sigma_{3,2,\pi}^{(t)} = t^2 x_2 (1 - x_2 - x_3)^{t-1} \delta(x_1 - x_2) dx
  \]

- \( \Pi(3, 3) \):
  \[
  \pi = \{1\} \sqcup \{2\} \sqcup \{3\}, \quad \sigma_{3,3,\pi}^{(t)} = t^3 x_1^2 (1 - x_1)^{t-1} \delta(x_1 - x_2) \delta(x_1 - x_3) dx.
  \]
Theorem 7.1. In the above notation,

\[ \sigma_n^{(t)} = \sum_{r=1}^{n} \sum_{\pi \in \Pi(n, n, r)} \sigma_{n, r, \pi}^{(t)} . \]

Proof. We apply the general formula (7.4) to \( M = M^{(t)} \). Let us abbreviate

\[ |l| = l_1 + \ldots + l_n, \quad r_k = r_k(\lambda). \]

Using the expressions for \( M^{(t)} \) and \( z_\lambda \) given in (7.2) and (7.3), we get

\[
\int_{[0,1]^n} x_1^{l_1} \ldots x_n^{l_n} \sigma_n^{(t)}(x) \, dx = \sum_{r=1}^{n} \sum_{\lambda: |\lambda|=|l|+n, \ell(\lambda)=r} [p_{l_{i+1}} \ldots p_{l_n} : m_\lambda] \frac{t^r \prod_{i=1}^{r} (\lambda_i - 1)!}{(t)^{|l|+n} \prod_{k \geq 1} r_k!}. \quad (7.7a)
\]

Remark that for any \( f \in \Lambda \) and any \( \lambda \) with \( \ell(\lambda) = r \),

\[ [f : m_\lambda] = [f : x_1^{\lambda_1} \ldots x_r^{\lambda_r}] , \]

where the square brackets on the right denote the coefficient of \( x_1^{\lambda_1} \ldots x_r^{\lambda_r} \) in the expansion of \( f \) as linear combination of monomials. Therefore, we may rewrite (7.7a) as

\[
\int_{[0,1]^n} x_1^{l_1} \ldots x_n^{l_n} \sigma_n^{(t)}(x) \, dx

= \sum_{r=1}^{n} \sum_{\lambda: |\lambda|=|l|+n, \ell(\lambda)=r} [p_{l_{i+1}} \ldots p_{l_n} : x_1^{\lambda_1} \ldots x_r^{\lambda_r}] \frac{t^r \prod_{i=1}^{r} (\lambda_i - 1)!}{(t)^{|l|+n} \prod_{k \geq 1} r_k!}. \quad (7.7b)
\]

Next, write \( \lambda \) as the \( n \)-tuple \( \lambda_1 \geq \cdot \cdot \cdot \geq \lambda_r > 0 \) and remark that each summand in the right–hand side of (7.7b) makes sense for any \( n \)-tuple of positive integers \( \lambda_1, \ldots, \lambda_r \) and is symmetric with respect to their permutations. Moreover, remark that there are exactly \( \frac{r!}{r_1! r_2! \ldots} \) distinct permutations of the numbers \( \lambda_1, \ldots, \lambda_r \). It follows that we may drop the restriction \( \lambda_1 \geq \cdot \cdot \cdot \geq \lambda_r \) and at the same time replace \( \prod r_k! \) by \( r! \). Thus, we can transform (7.7b) to

\[
\int_{[0,1]^n} x_1^{l_1} \ldots x_n^{l_n} \sigma_n^{(t)}(x) \, dx

= \sum_{r=1}^{n} \sum_{\lambda_1 > 0, \ldots, \lambda_r > 0, \sum \lambda_i = |l|+n} [p_{l_{i+1}} \ldots p_{l_n+1} : x_1^{\lambda_1} \ldots x_r^{\lambda_r}] \frac{t^r \prod_{i=1}^{r} (\lambda_i - 1)!}{(t)^{|l|+n} \prod_{k \geq 1} r_k!}. \quad (7.8)
\]

Write

\[ p_{l_{i+1}} \ldots p_{l_n+1} = (x_1^{l_{i+1}} + x_2^{l_{i+1}} + \ldots) \ldots (x_n^{l_n+1} + x_1^{l_n+1} + \ldots) \]
and remove the parentheses. Then we will get a sum of monomials, each of which corresponds to a choice of a summand from the first, second, \ldots, \(n\)th parentheses. We are interested only in monomials of the form \(x_1^{\lambda_1} \cdots x_r^{\lambda_r}\) with strictly positive \(\lambda_1, \ldots, \lambda_r\). There is a bijective correspondence between such monomials and ordered partitions \(\pi \in \Pi(n, r)\). Specifically, an index \(j\) belongs to the \(i\)th bloc of \(\pi\) if in the \(j\)th parentheses, the \(i\)th summand was chosen. This yields a correspondence

\[
\pi \mapsto \lambda = \lambda(\pi), \quad \lambda(\pi)_i = \sum_{j \in \pi_i} (l_j + 1) = \left(\sum_{j \in \pi_i} l_j\right) + |\pi_i|.
\]

Now we shall split the moment problem (7.8) into a collection of moment problems corresponding to various \(\pi\). Specifically, assume that for any \(r = 1, \ldots, n\) and any \(\pi \in \Pi(n, r)\) we dispose of a measure \(\sigma^{(t)}_{n, r, \pi}\) on \([0, 1]^n\) which solves the moment problem

\[
\int_{[0,1]^n} x_1^{l_1} \cdots x_n^{l_n} \sigma^{(t)}_{n, r, \pi}(x) \, dx = \frac{t^r \prod_{i=1}^r (\lambda(\pi)_i - 1)!}{(t)^{|l|+n}}
\]  

(7.9)

Then the measure

\[
\sigma_n^{(t)} := \frac{1}{r!} \sum_{\pi \in \Pi(n, r)} \sum_{\pi \in \Pi(n, r)} \sigma^{(t)}_{n, r, \pi}
\]  

(7.10)

will solve the moment problem (7.8).

Looking at (7.9) we remark that the \((l_1, \ldots, l_n)\)-moment depends only of the sums

\[
m_i := \sum_{j \in \pi_i} l_j, \quad i = 1, \ldots, r.
\]

This indicates that the desired measure should live on the section \([0, 1]^n_{\pi} \cap \sigma\), where \(\pi\) stands for the (unordered) partition corresponding to \(\pi\). Then we identify \([0, 1]^n_{\pi}\) with \([0, 1]^r\) via the map \(y \mapsto x\) defined in (7.5) and rewrite the moment problem (7.9) in terms of the coordinates \(y_1, \ldots, y_n:\)

\[
\int_{[0,1]^r} y_1^{m_1} \cdots y_r^{m_r} \tilde{\sigma}_r(y) \, dy = \frac{t^r \prod_{i=1}^r \left(\sum_{i=1}^n (m_i + |\pi_i| - 1)!\right)}{(t)^{\sum_{i=1}^r (m_i + |\pi_i|)}}
\]  

(7.11)

where \(\tilde{\sigma}_r\) stands for the unknown measure on \([0, 1]^r\) and we have used the identity

\[
|l| + n = \sum_{i=1}^r (m_i + |\pi_i|)
\]

On the other hand, consider the measure \(\tilde{\sigma}_{r, \pi} = \tilde{\sigma}_{r, \pi}^{(t)}\) as defined in (7.6). This is a Dirichlet measure on a simplex (see [Ki2]) whose moments can be readily calculated (this is a multivariate version of the classical Euler beta–integral). One verifies that this measure solves the moment problem (7.11).

Thus, we have shown that the measure (7.10) indeed solves the initial moment problem (7.7). Finally, we remark that the measure \(\sigma^{(t)}_{n, r, \pi}\) in (7.10) actually depends on the image \(\pi \in \Pi(n, r)\) of \(\pi\), which concludes the proof. \(\square\)

Theorem 7.1 yields new proofs of certain well–known properties of the Poisson–Dirichlet distributions \(PD(t)\).

Consider the face \(\Delta_0 = \Omega_0 \cap \Delta\) of the simplex \(\Delta\), i.e.,

\[
\Delta_0 = \{ (\alpha; \gamma) \mid \gamma = 1 - (\alpha_1 + \alpha_2 + \ldots) = 0 \}.
\]
Corollary 7.2 (cf. [Ki, 9.4–9.5]). The Poisson–Dirichlet distribution $PD(t)$ is concentrated on the face $\Delta_0$ of $\Delta$.

Proof. Applying Theorem 7.1 with $n = 1$, we get

$$\sigma_1^{(t)}(dx) = t(1-x)^{t-1}dx, \quad 0 \leq x \leq 1.$$ 

As this measure has no atom at zero, we conclude, by Proposition 3.4, that $PD(t)$ lives on the face $\Omega_0$ and hence on $\Delta_0$. $\square$

Corollary 7.3. The Poisson–Dirichlet process $PD(t)$ is simple (see the definition at the beginning of §4).

Proof. Apply Theorem 7.1 with $n = 2$. Both $\Pi(2,2)$ and $\Pi(2,1)$ consist of a single element $\pi$: specifically, $\pi = \{1\} \sqcup \{2\}$ and $\pi = \{1,2\}$, respectively. So, we have

$$\sigma_{2}^{(t)} = \sigma_{2,2,\{1\} \sqcup \{2\}}^{(t)} + \sigma_{2,1,\{1,2\}}^{(t)}.$$ 

The first component on the right is an absolutely continuous measure on the square $[0,1]^2$, while the second one is a singular measure supported by the diagonal $x_1 = x_2 = y$ of $[0,1]^2$. According to Proposition 3.5, we have to examine only the second component. It is equal to

$$tx_1(1-x_1)^{t-1}\delta(x_1-x_2)dx_1dx_2$$

or, in terms of the coordinate $y$, to $ty(1-y)^{t-1}dy$. The latter expression is just $\sigma_1^{(t)}(dy)$ multiplied by $y$. From Proposition 3.5 we conclude that the process is simple. $\square$

Note that this fact is evident from the construction of $PD(t)$ via a subordinator, see [Ki].

Corollary 7.4. The correlation functions of the Poisson–Dirichlet process $PD(t)$ are given by the formula

$$\rho_n^{(t)}(x_1, \ldots, x_n) = \frac{t^n(1-x_1 - \cdots - x_n)^{t-1}}{x_1 \cdots x_n} \quad n = 1, 2, \ldots.$$ 

This result is due to Watterson [W] but our approach differs from that of [W]. One more proof can be obtained by making use of the fact that the so-called biased sampling from $PD(t)$ yields a sequence of independent random variables [Ki2, 9.6].

Proof. Since the process is simple (Corollary 7.3), all diagonal sections $[0,1]_\pi^n$ with $\pi \neq \{1\} \sqcup \cdots \sqcup \{n\}$ are negligible sets with respect to the $n$th correlation measure $\rho_n^{(t)}$ (Proposition 4.3). Let

$$([0,1]^n)' = [0,1]^n \cap (I^n)' = \{x \in [0,1]^n \mid x_i \neq x_j, \quad i < j\}$$

be the complement to all proper diagonal sections. By Theorem 7.1, all the components $\sigma_{n,r,\pi}^{(t)}$ with $r < n$ are concentrated outside $([0,1]^n)'$, and the only component with $r = n$ is absolutely continuous with respect to $dx$ with density $t^n(1-x_1 - \cdots - x_n)^{t-1}$.

According to Proposition 4.3, the $n$th correlation measure is obtained from the latter measure by dividing it by $|x_1 \cdots x_n| = x_1 \cdots x_n$. $\square$

Remark 7.6. Note that the structure of the $n$th controlling measure, as described by Theorem 7.1, is in perfect accordance with the decomposition given in Proposition 4.4.
8. Appendix (A. Borodin): A proof of theorem 2.1

In this Appendix we present a simple direct proof of Theorem 2.1, which is due to A. Borodin.

We shall use the Frobenius notation $\lambda = (p_1, \ldots, p_d | q_1, \ldots, q_d)$ for a Young diagram $\lambda$, see (2.5). We start with the observation that the function $\varphi := M_{zz'}/\dim \lambda$ can be written in the determinantal form

$$\varphi(\lambda) = \frac{\det [m_{pq}]}{(t)_n}, \quad n = |\lambda| = \sum (p_i + q_i + 1), \quad (8.1)$$

where

$$m_{pq} = \frac{(z + 1)p(z' + 1)p(-z + 1)q(-z' + 1)q}{p!q!(p + q + 1)}; \quad t = zz'.$$

Indeed, this easily follows from the formulas (2.6), (2.7), and the Cauchy formula

$$\prod_{1 \leq i,j \leq d} (p_i - p_j)(q_i - q_j) \prod_{1 \leq i \leq d} (p_i + q_j + 1) = \det \left[ \frac{1}{p_i + q_j + 1} \right]. \quad (8.2)$$

Let us introduce a class of functions on the Young graph. We fix a number $t \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, a sequence $\{m_{pq}\}_{p,q=0}^\infty$ and set

$$\varphi(\lambda) = \frac{\det [m_{pq}]_{i,j=1}^d}{t(t + 1) \cdots (t + n - 1)} \quad (8.3)$$

**Theorem 8.1.** If a sequence $\{m_{pq}\}_{p,q=0}^\infty$ satisfies the relations

$$m_{p+1,q} + m_{p,q+1} - (p + q + 1)m_{pq} = m_{p,0}m_{0,q}, \quad p, q = 0, 1, \ldots, m_{0,0} = t \quad (8.4)$$

then the function $\varphi$ defined by (8.1) is harmonic. In other words,

$$\varphi(\lambda) = \sum_{\nu \lessdot \lambda} \varphi(\nu), \quad \lambda \in \mathbb{Y}. \quad (8.5)$$

**Proof.** For a $l \times l$ matrix $A = (a_{ij})$ we shall denote by $A_{\{i_1 \ldots i_k\}, \{j_1 \ldots j_k\}}$ the determinant of the submatrix of $A$ formed by the intersections of rows with numbers $i_1 \ldots i_k$ and columns with numbers $j_1 \ldots j_k$. We shall also denote by $A_{ij}$ the cofactor of $a_{ij}$. That is,

$$A_{ij} = (-1)^{i+j} A_{\{1 \ldots \hat{i} \ldots \hat{j} \ldots l\} \{1 \ldots \hat{i} \ldots \hat{j} \ldots l\}}.$$

The transposed matrix to $(A_{ij})$ is equal to the inverse matrix of $A$, multiplied by $\det A$, so that

$$\sum_j a_{ij} A_{ij} = \sum_i a_{ij} A_{ij} = \det A.$$
It follows that for any two sequences of numbers \( v_1, \ldots, v_l \) and \( w_1, \ldots, w_l \)

\[
\sum_{i,j=1}^{l} (v_i + w_j) a_{ij} A_{ij} = \sum_{i=1}^{l} (v_i + w_i) \cdot \det A.
\]  

(8.6)

We proceed to verify the harmonicity relation (8.5). The Frobenius coordinates of a diagram \( \nu \searrow \lambda \) in (8.5) are obtained from the Frobenius coordinates of the diagram \( \lambda \) by applying one of the following three operations:

1) \( p_i \rightarrow p_i + 1 \) for a certain \( i = 1, \ldots, d \), which corresponds to creating a new box in the \( i \)th row above the diagonal;

2) \( q_j \rightarrow q_j + 1 \) for a certain \( j = 1, \ldots, d \), which corresponds to creating a new box in the \( j \)th column below the diagonal;

3) adding a couple of coordinates \( p_{d+1} = 0, q_{d+1} = 0 \), which corresponds to creating a new box on the diagonal.

It may happen that creating a new box in a certain position is forbidden, because the resulting shape \( \nu \) is not a Young diagram: this occurs exactly when the set of the coordinates for \( \nu \) contains two equal \( p \)-coordinates or two equal \( q \)-coordinates. However, in such a case the formal application of formula (8.3) will give \( \varphi(\nu) = 0 \) as the determinant in the numerator of (8.3) will vanish. This makes it possible to sum up over all the operations of type 1), 2), 3), irrespective to whether the corresponding shape \( \nu \) is a Young diagram. Then the harmonicity relation (8.5) can be rewritten in the following form (below we set \( M = (m_{pq}) \))

\[
\left( t + \sum_{i=1}^{d} (p_i + q_i + 1) \right) M \left( \begin{array}{c} p_1 \cdots p_d \\ q_1 \cdots q_d \end{array} \right) = \sum_{i=1}^{d} M \left( \begin{array}{c} p_1 \cdots p_i + 1 \cdots p_d \\ q_1 \cdots q_d \end{array} \right) + \sum_{j=1}^{d} M \left( \begin{array}{c} p_1 \cdots p_d \\ q_1 \cdots q_j + 1 \cdots q_d \end{array} \right) + M \left( \begin{array}{c} p_1 \cdots p_d 0 \\ q_1 \cdots q_d 0 \end{array} \right)
\]

Expansion along the \( i \)th row gives

\[
M \left( \begin{array}{c} p_1 \cdots p_i + 1 \cdots p_d \\ q_1 \cdots q_d \end{array} \right) = \sum_{j=1}^{d} (-1)^{i+j} m_{p_i+1,q_j} M \left( \begin{array}{c} p_1 \cdots \hat{p_i} \cdots p_d \\ q_1 \cdots q_j \cdots q_d \end{array} \right).
\]

Similarly, expanding along the \( j \)th column, we get

\[
M \left( \begin{array}{c} p_1 \cdots p_d \\ q_1 \cdots q_j + 1 \cdots q_d \end{array} \right) = \sum_{i=1}^{d} (-1)^{i+j} m_{p_i,q_j+1} M \left( \begin{array}{c} p_1 \cdots \hat{p_i} \cdots p_d \\ q_1 \cdots q_j \cdots q_d \end{array} \right).
\]

Finally, expanding \( M \left( \begin{array}{c} p_1 \cdots p_d 0 \\ q_1 \cdots q_d 0 \end{array} \right) \) along the last row and column we get

\[
M \left( \begin{array}{c} p_1 \cdots p_d 0 \\ q_1 \cdots q_d 0 \end{array} \right) = m_{0,0} M \left( \begin{array}{c} p_1 \cdots p_d \\ q_1 \cdots q_d \end{array} \right) + \sum_{i,j=1}^{d} (-1)^{i+j+1} m_{p_i,0} m_{0,q_j} M \left( \begin{array}{c} p_1 \cdots \hat{p_i} \cdots p_d \\ q_1 \cdots \hat{q_j} \cdots q_d \end{array} \right).
\]
Adding everything up, employing the assumption (8.4) and applying the relation (8.6), we get
\[
\sum_{i=1}^{d} M\left(p_1 \ldots p_i + 1 \ldots p_d \quad q_1 \ldots q_d\right) + \sum_{j=1}^{d} M\left(p_1 \ldots p_d \quad q_1 \ldots q_j + 1 \ldots q_d\right) + M\left(p_1 \ldots p_d \quad 0 \quad q_1 \ldots q_d \quad 0\right)
\]
\[
= tM\left(p_1 \ldots p_d \quad q_1 \ldots q_d\right)
\]
\[
+ \sum_{i,j=1}^{d} (-1)^{i+j} (m_{p_i+1,q_j} + m_{p_i,q_j+1} - m_{p_i,0,q_j}) M\left(p_1 \ldots \hat{p}_i \ldots p_d \quad q_1 \ldots \hat{q}_j \ldots q_d\right)
\]
\[
= tM\left(p_1 \ldots p_d \quad q_1 \ldots q_d\right) + \sum_{i,j=1}^{d} (-1)^{i+j} (p_i + q_j) m_{p_i,q_j} M\left(p_1 \ldots \hat{p}_i \ldots p_d \quad q_1 \ldots \hat{q}_j \ldots q_d\right)
\]
\[
= \left(t + \sum_{i=1}^{d} (p_i + q_i + 1)\right) M\left(p_1 \ldots p_d \quad q_1 \ldots q_d\right),
\]
which concludes the proof. □

**Corollary 8.2.** The claim of Theorem 2.1 holds.

**Proof.** It suffices to check that the sequence (8.2) satisfies the assumption (8.4) of Theorem 8.1. But this is easily verified. □

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