Four coupled SYK models and nearly AdS$_2$ gravities: phase transitions in traversable wormholes and in bra-ket wormholes

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Abstract
We study four coupled Sachdev–Ye–Kitaev (SYK) models and nearly AdS$_2$ gravities. In the SYK model side, we construct a model that couples two copies of two coupled SYK models. In nearly AdS$_2$ gravity side, we entangle matter fields in two copies of traversable wormholes. In both cases, the systems show first order phase transitions at zero temperature by changing couplings, which is understood as the exchange of traversable wormhole configurations. In nearly AdS$_2$ gravity cases, by exchanging the role of space and time the wormholes are interpreted as bra-ket wormholes. In Lorentzian signature, these bra-ket wormholes lead to two closed universes that are entangled with each other as well as matter fields in the flat space where we do not have dynamical gravity. We study the effect of projection or entangling operation for matters on flat spaces and they cause phase transitions in bra-ket wormholes, which leads to the pair annihilation of closed universes. Using these bra-ket wormholes, we discuss the way to embed states in 2D holographic CFTs into Hilbert space of many 2D free fields.

Keywords: wormholes, entropy, black holes, symmetry breaking, low dimensional gravity, SYK model

(Some figures may appear in colour only in the online journal)

1. Introduction
The wormholes are interesting configurations of spacetime. They are closely related to quantum entanglement [1, 2] and moreover the equivalence of entanglement and wormholes is also
conjectured [3, 4]. Understanding this relation is important to understand the nature of spacetime. Studies of traversable wormholes give a shed light on this relation. In quantum mechanics, only having entangle states is not enough to send information. Similarly, only having wormholes does not imply that we can send a message through a wormhole. Rather, the average null energy condition (ANEC) [5, 6] does not allow to make such a traversable wormhole. One way to violate the ANEC is to use quantum effects like the Casimir energy. Actually, some controllable examples of traversable wormholes are found [7–10] recently. For example, we can make traversable wormholes by introducing the double trace deformations that directly couple two asymptotic boundaries of the eternal black holes in AdS [7, 8], which violates the ANEC. Interestingly, we can also find traversable wormhole solutions in four dimensions [9, 10]. In four dimensions, these direct coupling can arise from the local and causal dynamics. The solutions have a nontrivial topology that can be detectable from the observers who are in the asymptotical infinity. These configurations are prohibited from the topological censorship [11–13] at classical level, but quantum effects enable us to construct such a configuration. These considerations on traversable wormholes give an interesting connection among double trace deformations, quantum teleportation and topology of spacetime.

We can also realize a state that is similar to the eternal traversable wormholes [8, 14] in the Sachdev–Ye–Kitaev (SYK) model. The SYK model [15, 16] is a strongly interacting quantum mechanical model but is still solvable in the large $N$ limit. One of the remarkable properties of this model is that at low energies the theory is described by the conformal symmetry that is broken explicitly and spontaneously [16, 17]. The low energy effective action is known as the Schwartzian action [17, 18]. This action also appears as a low energy description of near extremal black holes [18–20]. The theory which gives the low energy description of near extremal black holes is known as the nearly AdS$_2$ gravity [18]. In particular, eternal traversable wormholes in nearly AdS$_2$ can be constructed based on this Schwartzian action [14]. The two cite SYK model with a sort of double trace deformation and its ground state is also analyzed exactly in the same manner with the eternal traversable wormholes in nearly AdS$_2$ [14], which motivates us to call the ground state ‘SYK traversable wormhole’ [21, 22].

Euclidean wormholes, which are other kinds of wormholes in Euclidean signature that connect more than two asymptotic boundaries, also play important roles. Recently Euclidean wormholes in the calculation of entanglement entropy, which are known as replica wormholes [23, 24], play an important role to reproduce the Page curve [25] from semiclassical gravity calculation [26–32, 32, 33]. Euclidean wormholes are sometimes confusing objects in the AdS/CFT correspondence [34] because they give correlations between partition functions and the factorization is not manifest [35, 36]. There is an old discussion on the connection between Euclidean wormholes and ensemble averages [37, 38] and recently it was found that 2D pure dilaton gravity is equivalent to quantum mechanics with random Hamiltonians [39]. There are further recent discussions on the ensemble average and quantum gravity [40–46] and average of conformal field theory [47–49].

In this paper, we study the coupling of two traversable wormholes both in the SYK model and nearly AdS$_2$ gravities. The motivation is to study what happens when we entangle matter fields in different spacetimes. These are important questions to understand the relation between quantum entanglement and the wormhole configurations, in particular in the context of ER = EPR [3, 4]. Actually, it is suggested that entanglement of quantum fields can make geometric connections in string theory [50]. To study similar configurations in the context of the SYK model and the NAdS$_2$ gravity with potential application to four dimensional setups, we consider two traversable wormholes with entangled matters between them. We model the situation of two traversable wormholes in four dimensions both in the SYK model and in Nearly
AdS$_2$ gravity, see figure 1. This is achieved by further introducing the double trace deformation between two traversable wormholes. In nearly AdS$_2$ gravities, we consider the matter fields a part of which is living on a traversable wormhole and another part is living on a different traversable wormhole. In nearly AdS$_2$ gravity setup, by exchanging the role of space and time in this configuration, we can think of the configuration as two bra-ket wormholes, which is introduced in [52]. In Lorentzian signature, two bra-ket wormholes lead to two closed universes that are entangled with each other as well as other matters in flat spaces without dynamical gravity. We vary the pattern of entanglement and study how spacetime changes. As a bonus, the bra-ket wormhole configuration gives a way to embed a 2D holographic CFT state into the Hilbert space of many free CFTs.

Both in the SYK models and Nearly AdS$_2$ gravities, we found first order phase transitions at zero temperature when we change the couplings, which causes the change of entanglement. These first order phase transitions are caused by the exchange of the dominant wormhole configurations. They lead to a $\mathbb{Z}_2$ symmetry breaking by each wormhole configuration at a special point of coupling constants, which is characterized by an order parameter.

After exchanging the role of time and space, the above transitions are interpreted as the transition in bra-ket wormholes. By entangling operations or partial projections, bra-ket wormholes can annihilate in the Euclidean regime before reaching $t = 0$ slice and disappear from the Lorentzian geometries. This disappearance of bra-ket wormholes plays an important role to keep the unitarity for the matter fields in the no gravity regions. These relations between traversable wormholes and bra-ket wormholes give a connection between physics in Euclidean wormholes and in spacial wormholes.

This paper is organized as follows. In section 2, we review traversable wormholes in the SYK model and in nearly AdS$_2$ gravity. In section 3, we construct four coupled SYK models and derive the large $N$ saddle point equations. Using these equations, we study the phase structure at zero temperature. In section 4, we study the property of traversable wormholes coupled to 2D CFT in nearly AdS$_2$ gravity. This section includes the calculation of entanglement...
entropy using the island formula. We also study traversable wormholes with partial couplings. In section 5, we construct four coupled nearly AdS$_2$ gravities and derive solutions for their equation of motion. We study the phase transitions by varying the boundary conditions outside the wormholes. We also interpret traversable wormholes as bra-ket wormholes by exchanging the role of Euclidean time and space. The phase transitions in the context of bra-ket wormholes are studied. Section 6 contains a brief summary and discussion of our results and we also discuss possible future directions. In appendix A, we collect some formulas in 2D CFT which is used in the main parts of our paper.

2. Review of the SYK model and nearly AdS$_2$ gravity

2.1. SYK model and nearly AdS$_2$ gravity

2.1.1. The SYK model. Let us first consider $N$ Majorana fermions in $0+1$ dimensions that obey the anti-commutation relation $\{\psi^i, \psi^j\} = \delta_{ij}$. The Hamiltonian of the SYK model \cite{16, 17, 53} is

$$H_{\text{SYK}} = i^2 \sum_{i_1 < \cdots < i_q} J_{i_1 \cdots i_q} \psi^{i_1} \cdots \psi^{i_q}, \quad (2.1)$$

with mean $\langle J_{i_1 \cdots i_q} \rangle = 0$ and variance $\langle J_{i_1 \cdots i_q}^2 \rangle = \frac{J^2}{N^{q-1}}(q-1)! = \frac{1}{q} \frac{J^2(q-1)!}{(2q)!}$. In the large $N$ limit, we get the Schwinger–Dyson equation

$$\partial_\tau G(\tau, \tau') - \int d\tau'' \Sigma(\tau, \tau'') G(\tau'', \tau') = \delta(\tau - \tau'), \quad \Sigma(\tau, \tau') = \frac{J^2}{q} (G(\tau, \tau'))^{q-1}, \quad (2.2)$$

for the Euclidean correlator

$$G(\tau, \tau') = \frac{1}{N} \sum_{i=1}^N \langle T(\psi^{i}(\tau)\psi^{i}(\tau')) \rangle$$

$$= \frac{1}{N} \sum_{i=1}^N \left( \langle \psi^{i}(\tau)\psi^{i}(\tau') \rangle \theta(\tau - \tau') - \langle \psi^{i}(\tau')\psi^{i}(\tau) \rangle \theta(-\tau + \tau') \right). \quad (2.3)$$

This Schwinger–Dyson equation is obtained as the equation of motion of the large $N$ effective action:

$$Z = \int \mathcal{D}G \mathcal{D}\Sigma e^{-NS_{\text{eff}}(G, \Sigma)}$$

$$= \int \mathcal{D}G \mathcal{D}\Sigma \exp \left[ N \left\{ \log \text{Pf}(\partial_\tau - \Sigma) - \frac{1}{2} \int d\tau' d\tau'' \left[ \Sigma(\tau, \tau') G(\tau, \tau') - \frac{J^2}{q} G(\tau, \tau') \right] \right\} \right]. \quad (2.4)$$
In the long time limit $1 \ll J(\tau - \tau') \ll N$, we can ignore the derivative term $\partial_\tau G(\tau, \tau')$ in (2.2) and can have an analytical solution

$$G(\tau, \tau') \approx G_s(\tau, \tau') \equiv \frac{c_\Delta}{|J(\tau - \tau')|^{2\Delta}} \sgn(\tau - \tau'), \quad c_\Delta = \frac{1}{2} \left[ (1 - 2\Delta) \frac{\tan \frac{\pi \Delta}{\pi \Delta}}{\pi \Delta} \right]^{\Delta}$$  \hspace{1cm} (2.5)

which is scale invariant. Actually in this limit the conformal transformation of the scale invariant solution (2.5)

$$G(\tau, \tau') = [f(\tau)f(\tau')]^{\Delta} G(f(\tau), f(\tau')) \quad \Sigma(\tau, \tau') = [f(\tau)f(\tau')]^{1-\Delta} \Sigma(f(\tau), f(\tau'))$$  \hspace{1cm} (2.6)

is also a solution of the Schwinger–Dyson equation (2.2). Therefore, the system have an emergent conformal symmetry. This conformal symmetry is spontaneously broken by each solution spontaneously and also explicitly broken by the derivative term $\partial_\tau G(\tau, \tau')$. The effect of this symmetry breaking is summarized in the so called Schwartzian action [17, 53]

$$S = -\frac{N G_s}{J} \int d\tau \{f(\tau), \tau\}, \quad \{f(\tau), \tau\} \equiv \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2} \left( \frac{f''(\tau)}{f'(\tau)} \right)^2.$$  \hspace{1cm} (2.7)

The coefficient can be determined numerically. For example, $\alpha_s \approx 0.00709$ for $q = 4$ [54].

For large $q$ limit it scales as $\alpha_s \sim \frac{1}{4q^4}$. At low temperature, the energy and the entropy become

$$E = E_0 + \frac{c}{2} T^2 + \cdots,$$

$$S = S_0 + cT + \cdots.$$  \hspace{1cm} (2.8)

where $c$ is the specific heat. The specific heat is given by the coefficient of the Schwartzian action (2.7) as $c = \frac{\partial^2 S}{\partial T^2}$. Therefore the Schwartzian action captures the corrections to the ground state energy and entropy. The zero temperature entropy $S_0$ is given by

$$S_0/N = \log \text{Pf}(\Sigma(\tau)) = \frac{1}{2} \log 2 - \pi \int_0^\Delta \left( \frac{1}{2} - x \right) \tan \pi x \, dx.$$  \hspace{1cm} (2.9)

On the other hand, the ground state energy is calculated numerically through

$$E/N = \frac{1}{q} \partial_\tau G(\tau) \bigg|_{\tau=0} = \frac{J^2}{2q^2} \int_0^\beta (2G(\tau))^q.$$  \hspace{1cm} (2.10)

For $q = 4$ case, the ground state energy is calculated as $E_0 \approx -0.0574 J$.

2.1.2. Nearly AdS$_2$ gravity. Here we briefly summarize the results in nearly AdS$_2$ gravity. The action of nearly AdS$_2$ gravity [18, 19, 55], or Jackiw–Teitelboim (JT) gravity [56, 57] is given by

$$S = \frac{\phi_0}{16\pi G_N} \int d^2x \sqrt{-g} R + \frac{\phi_0}{8\pi G_N} \int d^2x \sqrt{-h} K$$

$$+ \frac{1}{16\pi G_N} \int d^2x \sqrt{-g}(\phi(R + 2) + \frac{1}{8\pi G_N} \int d\tau \sqrt{-h} \phi(K + 1) + S_{\text{max}}[\chi, g].$$  \hspace{1cm} (2.11)
The first term is the two dimensional Einstein Hilbert action, which is topological. It does not affect the equation of motion of the system but is important to take into account the extremal entropy. We consider the AdS$_2$ with cut off at finite distance and impose the boundary condition

$$d^2s_{\text{bdy}} = -\frac{du^2}{\epsilon^2}, \quad \phi|_{\text{bdy}} = \frac{\bar{\phi}}{\epsilon},$$

(2.12)

and finally we take $\epsilon \to 0$ limit. The equation of motion of this system is derived in a straightforward way by taking the variation with respect to the metric $g_{\mu\nu}$ and the dilaton $\phi$. The result is

$$\frac{\delta S}{\delta \phi} = 0 \quad \Rightarrow \quad R + 2 = 0,$$

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \quad \Rightarrow \quad \nabla_\mu \nabla_\nu \phi - \nabla^2 \phi + g_{\mu\nu} \phi = \frac{1}{8\pi G_N} \langle T_{\mu\nu}^{\text{mat}} \rangle,$$

(2.13)

where $\langle T_{\mu\nu}^{\text{mat}} \rangle$ is the matter stress energy tensor expectation value. The first equation simply set the metric to be that of AdS$_2$. Here we set the matter stress tensor $\langle T_{\mu\nu}^{\text{mat}} \rangle$ to be zero. Finally the system reduces to the motion of the boundary $t_\text{P}(u)$. Here $u$ is the boundary time (2.12) and $t_\text{P}$ is the time in Poincare coordinate in AdS$_2$. The action for this function $t_\text{P}(u)$ is again by the Schwrtzian action [18, 19]

$$S = -\frac{\phi_t}{8\pi G_N} \int du \{ t_\text{P}(u), u \}.$$

(2.14)

Taking a conformal gauge $ds^2 = -e^{2\omega(x^+ - x^-)} dx^+ dx^-$, the equation of motion (2.13) becomes

$$2\partial_+ \partial_- \phi + \phi e^{2\omega} = 16\pi G_N \langle T_{++}^{\text{mat}} \rangle,$$

$$-e^{2\omega} \partial_+ (e^{-2\omega} \partial_+ \phi) = 8\pi G_N \langle T_{++}^{\text{mat}} \rangle,$$

$$-e^{2\omega} \partial_- (e^{-2\omega} \partial_- \phi) = 8\pi G_N \langle T_{--}^{\text{mat}} \rangle.$$

(2.15)

Let us briefly discuss the higher dimensional setup that are described by nearly AdS$_2$ gravity. Nearly AdS$_2$ gravity describes the low-energy dynamics of near extremal charged black holes [9, 18, 58]. The mass and the near extremal entropy are

$$M = \frac{r_e}{G_N} + \frac{2\pi^2 r_e^3}{G_N} T^2 + \cdots,$$

$$S = \frac{\pi r_e^2}{G_N} + \frac{4\pi^2 r_e^3}{G_N} T + \cdots.$$

(2.16)

where $r_e$ is the radius of the $S^2$ in the near horizon geometry AdS$_2 \times S^2$ and $T$ is the Hawking temperature. The parameters in JT gravity are given by [59]

$$\phi_0 = 4\pi r_e^3, \quad \bar{\phi}_e = 8\pi r_e^3.$$

(2.17)
If these extremal black holes are magnetically charged, \( r_e \) is given by

\[ r_e^2 = \frac{\pi q^2 G_N}{g^2}, \tag{2.18} \]

where \( g \) is the \( U(1) \) gauge coupling and \( q \) is the magnetic charge of black holes. In the presence of the magnetic fields, we have \( q \) fermion zero modes on \( S^2 \) from a single massless 4D Dirac fermion\,[60]\), which leads to the \( q \) 2D Dirac fermions \([9, 58]\). In other words, we have the free fermion CFT with central charge \( c = q \). These large number of two dimensional CFTs enhance the quantum effect from the matter fields.

### 2.2. Traversable wormholes in SYK and in nearly AdS\(_2\)

#### 2.2.1. Two coupled SYK model

Here we briefly describe the two coupled model SYK model, which ground state shares many properties with traversable wormholes in nearly AdS\(_2\) gravity. First we prepare two SYK models which is decoupled. We denote the left fermions (right fermions) by \( \psi^L_i \) (\( \psi^R_i \)). The random couplings (2.1) of left and right SYK (\( J^L_{i_1 \cdots i_q} \) and \( J^R_{i_1 \cdots i_q} \)) are the same up to the sign for odd \( q/2 \):

\[ J^L_{j_1 \cdots j_q} = (-1)^{q/2} J^R_{j_1 \cdots j_q}. \]

The Hamiltonian of the two coupled SYK model (Maldacena–Qi model) is

\[ H = H^L_{\text{SYK}} + H^R_{\text{SYK}} + H_{\text{int}}, \quad H_{\text{int}} = i\mu \sum_{i=1}^N \psi^L_i \psi^R_i. \tag{2.19} \]

In \( \mu \ll J \) limit, we can still use the low energy, Schwartzian action and the effective action for the coupled system is

\[ S = N \int du \left\{ -\frac{\alpha_S}{f} \left( \frac{\tan t(u)}{2}, u \right) + \left( \frac{\tan t(u)}{2}, u \right) \right\} + \mu \frac{c_\Delta}{(2f)^{2\Delta}} \left[ \frac{f_i(u)f_i(u)}{\cos^2 \left[ \frac{\tan t(u)}{2}, u \right]} \right]^\Delta. \tag{2.20} \]

Another comment is that we can consider deformations like \( S_{\text{int}} \sim i^p gN^{1-p} \sum (\psi^L_i \cdots \psi^L_j) (\psi^R_i \cdots \psi^R_j) \). This only change the dimension \( \Delta \to p\Delta \). For even \( p \), we can keep the time reversal symmetry that exists for even \( q/2 \). We can also realize a marginal deformation for \( p = q \) case, which is a similar situation to JT gravity with 2D conformal matters.

#### 2.2.2. Traversable wormholes in nearly AdS\(_2\): double trace deformations

We introduce the double trace deformation\,[7, 8] that directly couples the two side of AdS\(_2\):

\[ S_{\text{int}} = g \sum_{i=1}^N \int du O^L_i(u)O^R_i(u), \tag{2.21} \]

where \( O^j \) are a set of \( N \) operators with dimension \( \Delta \). In gravity language, we impose the boundary condition for dual fields \( \chi'(t, z) \) such that the two side are directly correlated. We consider the theory with \( N \) matter fields with \( N \) that is comparable with \( \frac{q}{G_N} \) so that we can balance the matter quantum effect and the classical JT gravity. Such a situation arises from the 4D magnetically charged near extremal black holes.
For small $g$, we can approximate the effect of the double trace deformation (2.21) as

$$\langle \epsilon^b \sum_u \int du O_l^{C}(u) O_l^{(2)}(u) \rangle \sim \epsilon^b \sum_u \int du \langle O_l^{C}(u) O_l^{(2)}(u) \rangle.$$  

(2.22)

This amounts to the resummation of bulk ladder type diagrams, which dominate in the large $N$, small $g$ limit with $Ng$ kept fixed. Then, we couple this to the gravity mode, which is achieved by performing a reparametrization of the left and right times. These reparametrizations are expressed as the map from the proper time $u$ to the global times $t_i(u), t_j(u)$ at the two boundaries of the AdS$_2$. In this way, we finally obtain the effective action

$$S = N \int du \left\{-\frac{N}{\phi_t} \left( \left\{ \tan \frac{t_i(u)}{2}, u \right\} + \left\{ \tan \frac{t_j(u)}{2}, u \right\} \right) + \frac{gN}{2\Delta} \left[ \frac{t_i'(u) t_j'(u)}{\cos^2 \frac{t_j(u) - t_i(u)}{2}} \right]^\Delta \right\}.$$  

(2.23)

Therefore, we obtain the same effective action with that in the two coupled SYK model (2.20). Here we put $8\pi G_N = 1$ for simplicity.

**2.2.3. Low energy analysis.** Starting from both of the SYK model and the nearly AdS$_2$ gravity, we obtain the coupled Schwartzian action

$$S = N \int d\tilde{u} \left\{-\left( \left\{ \tan \frac{t_i(\tilde{u})}{2}, \tilde{u} \right\} + \left\{ \tan \frac{t_j(\tilde{u})}{2}, \tilde{u} \right\} \right) + \eta \left[ \frac{t_i'(\tilde{u}) t_j'(\tilde{u})}{\cos^2 \frac{t_j(\tilde{u}) - t_i(\tilde{u})}{2}} \right]^\Delta \right\},$$  

(2.24)

where the relation between the parameters in the previous actions is

$$\tilde{u} = \frac{J}{\alpha_s} u = \frac{N}{\phi_t} u, \quad \eta \equiv \frac{\mu \alpha_s}{2} \frac{c_\Delta}{(2\alpha_s)^{2\Delta}} = \frac{g}{2^\Delta} \left( \frac{N}{\phi_t} \right)^{2\Delta - 1}. \quad (2.25)$$

This action should be supplemented by SL(2, $\mathbb{R}$) constraints and the total SL(2, $\mathbb{R}$) charges vanish [18, 61]. This system (2.24) has a classical, static solution of the form $t_i(\tilde{u}) = t_j(\tilde{u}) = t' \tilde{u}$ with

$$(t')^{2(1-\Delta)} = \eta \Delta, \quad \left( \frac{1}{J} \frac{dr}{du} \right)^{2(1-\Delta)} = \frac{\mu \Delta}{2} \frac{2c_\Delta}{2^\Delta} \quad (2.26)$$

The parameter $t'$ is important. This is because the time to traverse the interior, or the gap of the system, is determined by this $t'$. The SYK correlation function in the low energy limit, or the boundary propagator of the nearly AdS$_2$, is

$$\langle O(t_1) O(t_2) \rangle = \left[ \frac{1}{\cos \frac{t_1 - t_2}{2}} \right]^{2\Delta} \rightarrow \langle O(\tilde{u}_1) O(\tilde{u}_2) \rangle = \left[ \frac{1}{\cos \frac{\tilde{u}_1 - \tilde{u}_2}{2}} \right]^{2\Delta}. \quad (2.27)$$

In the Euclidean signature $u = -i\tau$, the correlation function in the conformal limit becomes

$$\langle O(\tilde{\tau}_1) O(\tilde{\tau}_2) \rangle = \left[ \frac{1}{\cos \frac{\tilde{\tau}_1 - \tilde{\tau}_2}{2}} \right]^{2\Delta} \quad (2.28)$$
which decays exponentially for large relative Euclidean time $\tau_1 - \tau_2 \gg 1$. From this we can read off the energy gap of the system. This becomes

$$E_{\text{gap}, \bar{t}} = \bar{t}' \Delta, \quad E_{\text{gap}, t} = \frac{J}{\alpha_s} \bar{t}' \Delta = \frac{N}{\phi_r} \bar{t}' \Delta. \quad (2.29)$$

$E_{\text{gap}, \bar{t}}$ is the energy gap with respect to the rescaled time $\bar{t}$ whereas $E_{\text{gap}, t}$ is that for the physical time $t$. In the $q = 4$ model, the energy gap scales as $E_{\text{gap}} \sim \mu^4$.

### 2.2.4 Two coupled SYK beyond small coupling

Here we summarize further on the two coupled SYK model. First, when $q = 4k$ ($k \in \mathbb{N}$) the Hamiltonian (2.19) has a $\mathbb{Z}_4$ symmetry$^1$ which is given by

$$\psi_L \rightarrow \psi_R, \quad \psi_R \rightarrow -\psi_L. \quad (2.30)$$

The mass term is directly related to the thermofield double state. Actually, the SYK thermofield double state is defined as

$$|\text{TFD}(\beta)\rangle = Z_\beta^{-\frac{1}{2}} e^{-\frac{1}{2} (H_L + H_R)} |I\rangle, \quad (2.31)$$

where $|I\rangle$ is the infinite temperature thermofield double state

$$\langle \psi_L^j + i \psi_R^j |I\rangle = 0, \quad \text{for} \quad j = 1, \ldots, N. \quad (2.32)$$

This means that the $|I\rangle$ is characterized as the state that is annihilated by the annihilation operator $f_j = \psi_L^j + i \psi_R^j$. Note that the $\mathbb{Z}_4$ symmetry (2.30) acts on these annihilation operators as $f_j \rightarrow -f_j$ and the thermofield double is invariant under the $\mathbb{Z}_4$ symmetry. By multiplying both sides by $\psi_L^j$, we obtain $S^I|I\rangle = |I\rangle$ for $S^I = -2i \psi_L^j \psi_R^j$. $S^I$ is the spin operator with eigenvalues $\pm 1$ since $(S^I)^2 = 1$. The mass term is actually a sum of these spin operators. Therefore, the infinite temperature thermofield double state is characterized as the ground state of the mass term Hamiltonian, in which all the spins are up. In other words, when $\mu \rightarrow \infty$ limit the ground state of the two coupled model (2.19) is perfectly agrees with the infinite temperature TFD state. On the other hand, in the $\mu \rightarrow 0$ limit the ground state is the two copies of the SYK ground states. This is also the TFD state at zero temperature. Therefore, one can imagine that the ground state $|G(\mu)\rangle$ of (2.19) is always close to the thermofield double state. This suggest that even for generic $\mu$ the ground state $|G(\mu)\rangle$ of (2.19) is close to the thermofield double state. One can directly study the maximal overlaps between $|G(\mu)\rangle$ and $|\text{TFD}(\beta)\rangle$ when we vary the inverse temperature $\beta$ at finite $N$ [62, 63], and they are very close to 1.

It will be useful to study how the states deviate from the maximally entangled state (2.32). Since the maximally entangled state is annihilated by the fermion annihilation operator $f_j$, the expectation value of the occupation number operator

$$\frac{\langle \psi | f_j^\dagger f_j | \psi \rangle}{\langle \psi | \psi \rangle} = \langle |\psi_L^j + i \psi_R^j|^2 \rangle. \quad (2.33)$$

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$^1$ Even when $q = 4k + 2$ ($k \in \mathbb{N}$), the Hamiltonian (2.19) is invariant under the combination of $\mathbb{Z}_2$ with the generator $w: \psi_L \rightarrow \psi_R, \psi_R \rightarrow -\psi_L$, and the time reversal $T$ that satisfies $T \psi_L^j T^{-1} = \psi_R^j$ for $A = L, R$ i.e. $(wT) H (wT)^{-1} = H$. Using this symmetry we can do basically the same arguments with $q = 4k$ cases. For simplicity, we focus on the $q = 4k$ cases with the $\mathbb{Z}_4$ symmetry.
will characterize how the state is close to the maximally entangled state $|f\rangle$. This is related to the spin operator expectation value

$$
\langle |\psi_L + i|\psi_R|^2 \rangle = 1 - \langle f|f\rangle = 1 - \langle S_i \rangle.
$$

(2.34)

When the condensation $\langle S_i \rangle$ is big, the state becomes close to the maximally entangled state.

In this model, we can actually write down the large $N$ effective action in $G, \Sigma$ variables. We introduce the correlators $G_{ab}(\tau_1, \tau_2) = \langle \psi_a(\tau_1)\psi_b(\tau_2) \rangle$ for $a, b = L, R$ that means that the fermion belongs to the left SYK cluster or the right cluster. Similarly, we introduce the self energy variable $\Sigma_{ab}(\tau_1, \tau_2)$. Then, the effective action becomes [14]

$$
-S_E/N = \log \text{Pr}(\partial_t \delta_{ab} - \Sigma_{ab}) - \frac{1}{2} \int d\tau_1 d\tau_2
\times \sum_{a,b} \left[ \Sigma_{ab}(\tau_1, \tau_2)G_{ab}(\tau_1, \tau_2) - s_{ab} \frac{J^2}{2q}[2G_{ab}(\tau_1, \tau_2)]^q \right]
+ \frac{H}{2} \int d\tau[-G_{LR}(\tau_1, \tau_2) + G_{RL}(\tau_1, \tau_2)].
$$

(2.35)

Then, the Schwinger Dyson equation becomes

$$
\partial_t G_{LL}(\tau_1, \tau_2) - \int d\tau_1 \Sigma_{LL}(\tau_1, \tau_3)G_{LL}(\tau_1, \tau_2) - \int d\tau_2 \Sigma_{LR}(\tau_1, \tau_3)G_{LR}(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2),
$$

$$
\partial_t G_{LR}(\tau_1, \tau_2) - \int d\tau_1 \Sigma_{LL}(\tau_1, \tau_3)G_{LR}(\tau_1, \tau_2) - \int d\tau_2 \Sigma_{LR}(\tau_1, \tau_3)G_{RR}(\tau_1, \tau_2) = 0.
$$

$$
\Sigma_{LL}(\tau_1, \tau_2) = \frac{J^2}{q}[2G_{LL}(\tau_1, \tau_2)]^{q-1},
$$

$$
\Sigma_{LR}(\tau_1, \tau_2) = (-1)^{\frac{q}{2}} \frac{J^2}{q}[2G_{LR}(\tau_1, \tau_2)]^{q-1} - i\mu \delta(\tau_1 - \tau_2).
$$

(2.36)

This equation allows us to study the two coupled model beyond the low energy effective action (2.24) and is also useful to check that the Schwartzian analysis gives the correct answer. The comparison of the numerics from the SD equation (2.36) and the results from the conformal limit (2.29) are shown in figure 2. The low energy limit (2.24) shows a good agreement with the exact numerical results from the SD equation (2.36) for small $\mu$.

2.2.5. **Traversable wormholes in JT gravity with conformal matters.** Here we discuss more on the traversable wormholes in nearly AdS$_2$ gravity. We can consider the traversable wormholes in JT gravity, in particular the bulk fields are 2D CFT [14, 64] and gives a direct interaction on them at two asymptotic boundaries. In this case, we can explicitly construct the dilaton profile. Imagine that we have $N$ massless Majorana fermions $\psi(t, \sigma) = (\psi_+, (t, \sigma), \psi_- (t, \sigma))^T$ with the twisted boundary condition

$$
\psi_+(t, 0) = \cos \pi \epsilon \psi_-(t, 0) - \sin \pi \epsilon \psi_-(t, \pi)
$$

$$
\psi_-(t, \pi) = - \cos \pi \epsilon \psi_+(t, \pi) - \sin \pi \epsilon \psi_+(t, 0).
$$

(2.37)

$\epsilon = 0$ corresponds to a usual Majorana fermion on a strip of width $\pi$ with boundary conditions. Non zero $\epsilon$ introduces a non zero transparency and gives a direct coupling between two asymptotic boundaries. $\epsilon = \frac{1}{2}$ corresponds to the anti periodic boundary conditions for a Majorana
We plot the $E_g$ and the spin operator expectation value $\langle S_{LR} \rangle = |\langle \psi_L \psi_R \rangle|$ for several $\mu$ and compare with the conformal limit results (2.29). Here we take the temperature to be $T = 0.001$, which is very low and essentially the system at zero temperature. The conformal limit is a good approximation for small $\mu$. The spin operator, which is given by $\langle S_{LR} \rangle = -i \langle \psi_L(0) \psi_R(0) \rangle$, behaves as $-i G_{LR}(0) \approx c_\Delta (t')^2 \Delta$ in the conformal limit.

The energy per fermion on cylinder is

$$E(\epsilon) = -\frac{1}{48} \left[ 1 + 12 \epsilon (1 - \epsilon) \right].$$

(2.39)

By solving (2.15), with the energy momentum tensor (2.38) gives the dilaton profile under the assumption of $\phi = \phi(\sigma)$

$$\phi = N \epsilon (1 - \epsilon) \left[ \frac{\tilde{\sigma}}{\tan \sigma} + 1 \right] + \frac{N}{48 \pi} \epsilon (1 - \epsilon) \left[ \frac{\tilde{\sigma}}{\tan \sigma} + 1 \right] + \frac{c}{24 \pi}.$$  

(2.40)

To satisfy the boundary condition $\phi_b = \bar{\phi}$, we consider the rescaling

$$ds^2 = \tilde{t}'^2 (dt'^2 + d\sigma^2), \quad \phi(\sigma) = N \epsilon (1 - \epsilon) \left[ \frac{\tilde{\sigma}}{\tan(\tilde{t}' \sigma)} + 1 \right] + \frac{N}{48 \pi},$$

(2.41)

where $0 < \sigma < \frac{\pi}{2}$. Then, near the boundary $\sigma = \epsilon$, we obtain

$$\phi_b \approx \frac{c}{2 \pi} \epsilon (1 - \epsilon) \frac{\tilde{\sigma}}{\tilde{t}' \epsilon} = \frac{c}{4 \epsilon} \epsilon (1 - \epsilon) \frac{1}{\epsilon}.$$  

(2.42)

Matching with the boundary condition $\phi_b = \tilde{\phi}$, we obtain

$$\tilde{t}' = \frac{c}{4} \epsilon (1 - \epsilon) \frac{\tilde{\phi}}{\epsilon}.$$  

(2.43)

Using $\eta = 4 \epsilon \ll 1$ and $c = \frac{N}{2}$, we obtain

$$\tilde{t}' = \frac{N \eta}{2 \epsilon \tilde{\phi}}.$$  

(2.44)
which reproduce the results in the low energy limit (2.26) for $\eta \ll 1$ and $\Delta = \frac{1}{2}$.

The ADM energy for one side is calculated as

$$M = \frac{1}{8\pi G_N} \sqrt{h} [\phi_0 - \partial_\eta \phi] = \lim_{\epsilon \to 0} \frac{1}{8\pi G_N} \frac{\epsilon'}{\sin(\epsilon')} \left( \left[ \frac{2\phi_0 \epsilon'}{\pi} \frac{\tan(\epsilon')}{\tan(\epsilon')} + 1 \right] \right)$$

$$+ \frac{\sin(\epsilon')}{\epsilon'} \partial_\eta \left( \frac{2\phi_0 \epsilon'}{\pi} \frac{\tan(\epsilon')}{\tan(\epsilon')} + 1 \right) \right]_{\sigma = \epsilon}$$

$$= - \frac{\phi_0 \epsilon^2}{16\pi G_N}. \quad (2.45)$$

The total ADM energy is the twice of this, one from the left and the one from the right boundary. This becomes

$$M = M_L + M_R = - \frac{\phi_0 \epsilon^2}{8\pi G_N}. \quad (2.46)$$

To compare with Maldacena–Qi notation, we set $8\pi G_N = 1$ and use $\epsilon' = \frac{N\eta}{2\Delta}$. Then, this becomes

$$\frac{M}{N} = - \frac{\phi_0 \epsilon^2}{N^2 \eta^2} = \frac{\phi_0 \epsilon^2}{N \phi_0} 4 \eta^2 \quad (2.47)$$

which reproduce the Maldacena–Qi results of the ground state energy shift for $\Delta = \frac{1}{2}$.

Beyond small $\epsilon$ limit, we can also consider finite $\epsilon$ in this model. $\epsilon = \frac{1}{2}$ case is specially interesting because this gives a perfectly transparent boundary condition between two side. When fermion boundary conditions are anti periodic, then this becomes

$$\phi(\sigma) = \frac{c}{8\pi} \left[ \frac{\pi - \sigma}{\tan \sigma} + 1 \right] + \frac{c}{24\pi}. \quad (2.48)$$

with $c = N/2$ for fermions. These results for perfectly transparent boundary conditions are written only using the central charge, which is a universal quantity in CFT. Actually we can impose this type of boundary condition in any CFT, even in holographic 2D CFT. Another interesting property is that we can also insert the no gravitating outside region between two boundaries and can study physics on this non gravitating region. In particular, this situation comes from traversable wormholes in four dimensions [9] where the wormhole throat region is described by the JT gravity with CFT described below (2.17). In the four coupled model, we use this setup as well as we analyze some properties in two coupled model through non gravitating region.

### 3. Four coupled SYK models

#### 3.1. The Hamiltonian

Here we consider the model which couples four cites SYK models. First, we prepare four decoupled SYK models. We label these fermions by $\psi_{ij}^\alpha$ with $A = L, R, \alpha = 1, 2$ and $j = 1, \ldots, N$. The Hamiltonian of the four coupled SYK model is

$$H_A = H_{1L} + (-1)^j H_{1R} + (-1)^j H_{2L} + H_{2R} + H_{intLR}^{11} + H_{intLR}^{12} + H_{intLL}^{22} + H_{intRR}^{12}. \quad (3.1)$$

We absorb the constant shift $\frac{1}{\phi_0}$ in the dilaton field $\phi$ to the constant $\phi_0$ which accounts the extremal entropy.
Here $H_{\alpha A}$ is

$$H_{\alpha A} = i \frac{\partial}{\partial t} \sum_{j_1 < \cdots < j_N} J_{j_1 \cdots j_N} \phi_{\alpha A}^{j_1} \cdots \phi_{\alpha A}^{j_N}.$$  

(3.2)

and $H_{\text{int}}^{\alpha \beta}$ is

$$H_{\text{int}}^{\alpha \beta} = i \left[ \delta^{\alpha \beta} J_{\lambda \alpha} \sum_{k=1}^{N} \psi_{\alpha A}^{k} \psi_{\beta B}^{k} + \mu^{\alpha \beta} \sigma_{AB}^{\gamma} \sum_{k=1}^{N} \psi_{\alpha A}^{k} \psi_{\beta B}^{k} \right].$$  

(3.3)

where $\sigma^{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix. Or, more explicitly we can write

$$H_{\text{int}}^{11} = i \mu_{LR} \sum_{k=1}^{N} \psi_{1L}^{k} \psi_{1R}^{k}, \quad H_{\text{int}}^{22} = i \mu_{LR} \sum_{k=1}^{N} \psi_{2L}^{k} \psi_{2R}^{k},$$  

(3.4)

and

$$H_{\text{int}}^{12} = i \mu_{LR} \sum_{k=1}^{N} \psi_{1L}^{k} \psi_{2R}^{k}, \quad H_{\text{int}}^{12} = -i \mu_{LR} \sum_{k=1}^{N} \psi_{1R}^{k} \psi_{2L}^{k}.$$  

(3.5)

Another description of this model is the two coupled Maldacena–Qi Hamiltonian;

$$H_4 = (H_{1L} + (-1)^{\frac{\lambda}{2}} H_{1R} + H_{\text{int}}^{11}) + ((-1)^{\frac{\lambda}{2}} H_{2L} + H_{2R} + H_{\text{int}}^{22}) + H_{\text{int}}^{12} + H_{\text{int}}^{22}$$

$$= H_{\text{MQ}_1} + H_{\text{MQ}_2} + H_{\text{int}}^{11} + H_{\text{int}}^{22}. \quad (3.6)$$

Here we make a pair between L and R and defined $H_{\text{MQ}_1} = H_{1L} + (-1)^{\frac{\lambda}{2}} H_{1R} + H_{\text{int}}^{11}$, $H_{\text{MQ}_2} = (-1)^{\frac{\lambda}{2}} H_{2L} + H_{2R} + H_{\text{int}}^{22}$. We can also make a pair in 1 and 2 direction, and then we can describe as

$$H_4 = (H_{1L} + (-1)^{\frac{\lambda}{2}} H_{2L} + H_{\text{int}}^{12}) + ((-1)^{\frac{\lambda}{2}} H_{1R} + H_{2R} + H_{\text{int}}^{22}) + H_{\text{int}}^{11} + H_{\text{int}}^{22}$$

$$= H_{\text{MQ}_1} + H_{\text{MQ}_2} + H_{\text{int}}^{11} + H_{\text{int}}^{22}. \quad (3.7)$$

The ground state of the mass term Hamiltonian $H_{\text{int}}^{11} + H_{\text{int}}^{22}$ is given by

$$\langle \psi_{1L}^k + i \psi_{1R}^k | I_{LR} \rangle = 0, \quad \langle \psi_{2L}^k + i \psi_{2R}^k | I_{LR} \rangle = 0.$$  

(3.8)

The state satisfies

$$H_{1L} | I_{LR} \rangle = (-1)^{\frac{\lambda}{2}} H_{1R}, \quad H_{2L} | I_{LR} \rangle = (-1)^{\frac{\lambda}{2}} H_{2R}$$  

(3.9)

and

$$H_{\text{int}}^{11} | I_{LR} \rangle = -H_{\text{int}}^{22} | I_{LR} \rangle.$$  

(3.10)

Therefore, the state satisfies

$$H_{\text{MQ}_1} | I_{LR} \rangle = H_{\text{MQ}_2} | I_{LR} \rangle,$$  

(3.11)

3 Here we include the minus sign in $H_{\text{int}}^{11}$ so that $H_{\text{int}}^{11}$ becomes the complex conjugate of $H_{\text{int}}^{11}$.
and |I_{LR}\rangle is interpreted as the infinite temperature thermofield double state of the two coupled SYK model.

Similarly, the ground state of the mass term Hamiltonian $H_{\text{int}}^{12} + H_{\text{int}}^{12}$ is given by

$$
\langle I_{12} | \psi^k_{1L} + i \psi^k_{2L} \rangle = 0, \quad \langle I_{12} | \psi_{1R}^k - i \psi_{2R}^k \rangle = 0.
$$

(3.12)

By changing the sign $\psi^k_{2R} = -\tilde{\psi}^k_{2R}$, we obtain the same relation with the LR direction:

$$
\langle I_{12} | \psi^k_{1L} + i \psi^k_{2L} \rangle = 0, \quad \langle I_{12} | \psi_{1R}^k + i \tilde{\psi}_{2R}^k \rangle = 0.
$$

(3.13)

with the opposite sign in front of the mass term

$$
H_{\text{int}}^{11} = i \mu_{LR} \sum_{k=1}^{N} \psi^k_{1L} \psi^k_{1R}, \quad H_{\text{int}}^{22} = -i \mu_{LR} \sum_{k=1}^{N} \psi^k_{2L} \tilde{\psi}^k_{2R}.
$$

(3.14)

Therefore, by the same calculation with the LR infinite temperature thermofield double case, we obtain

$$
H_{MQ1} |I_{12}\rangle = H_{MQ2} |I_{12}\rangle.
$$

(3.15)

Again, |I_{12}\rangle is the infinite temperature thermofield double state of the two coupled SYK model but in 12 direction.

There is a duality transformation, which plays an important role in our analysis. The transformation is given by

$$
\psi_{1R}^k \to \psi_{2L}^k, \quad \psi_{2L}^k \to \psi_{1R}^k, \quad \psi_{2R}^k \to -\psi_{2R}^k,
$$

(3.16)

that exchange the couplings ($\mu_{LR}, \mu_{12}) \to (\mu_{12}, \mu_{LR})$. Therefore this duality transformation exchanges the role of L–R direction and 1–2 direction. We can think of strong-weak duality that exchange the strongly coupled a two coupled MQ model to a weakly coupled MQ model. Moreover, this becomes a $\mathbb{Z}_2$ symmetry in the four coupled system at the ‘self dual’ point $\mu_{LR} = \mu_{12}$. At the symmetric point, an order parameter is

$$
S_{\text{diff}} = \frac{1}{2} (S_{\text{LL}}^{11} + S_{\text{RR}}^{11}) = \frac{1}{2} (S_{\text{LL}}^{12} - S_{\text{RR}}^{12}).
$$

(3.17)

This operator transforms $S_{\text{diff}} \to -S_{\text{diff}}$ and plays an role of order parameters. If the operator $S_{\text{diff}}$ have an expectation value, the $\mathbb{Z}_2$ symmetry at $\mu_{LR} = \mu_{12}$ point is broken.

### 3.1.1. The analysis of the mass term Hamiltonian for $N = 1$ case.

We analyze the mass term Hamiltonian for $N = 1$ i.e. one fermion per each site. In this case, we can explicitly realize the fermions as

$$
\psi_{1L} = \frac{1}{\sqrt{2}} \sigma_x \otimes \mathbb{1}, \quad \psi_{1R} = \frac{1}{\sqrt{2}} \sigma_y \otimes \mathbb{1}, \quad \psi_{2L} = -\frac{1}{\sqrt{2}} \sigma_z \otimes \sigma_y, \quad \psi_{2R} = \frac{1}{\sqrt{2}} \sigma_z \otimes \sigma_x.
$$

(3.18)
Then, the spin operators $S_{i,j}^{ab} = -2i\psi_{a\alpha}\psi_{b\beta}$ become

$$
S_{LR}^{11} = \sigma_z \otimes 1, \quad S_{LR}^{22} = 1 \otimes \sigma_z, \quad S_{LL}^{12} = \sigma_y \otimes \sigma_z, \quad S_{RR}^{12} = \sigma_x \otimes \sigma_z.
$$

(3.19)

The mass term Hamiltonian is$^4$

$$
H_{\text{int}}^{LR} = -\frac{1}{2}\mu_{LR}(\sigma_z \otimes 1 + 1 \otimes \sigma_z), \quad H_{\text{int}}^{12} = -\frac{1}{2}\mu_{12}(\sigma_y \otimes \sigma_z - \sigma_z \otimes \sigma_z).
$$

(3.20)

We denote the eigenstates of $S_{LR}^{i} (i = 1, 2)$ by $|\uparrow\rangle$ and $|\downarrow\rangle$. Then, the ground state of $H_M = \mu_{LR}H_{\text{int}}^{LR} + \mu_{12}H_{\text{int}}^{12}$ is

$$
|G(\mu_1, \mu_2)\rangle = \cos \frac{\theta}{2} |\uparrow\rangle - \sin \frac{\theta}{2} |\downarrow\rangle, \quad \tan \theta = \frac{\mu_{12}}{\mu_{LR}}.
$$

(3.21)

The full spectrum are given by $-\sqrt{\mu_{LR}^2 + \mu_{12}^2}$, $0$, $\sqrt{\mu_{LR}^2 + \mu_{12}^2}$. There is always a gap between ground state and the first excited state that is given

$$
E_g = \sqrt{\mu_{LR}^2 + \mu_{12}^2}.
$$

(3.22)

When we fix $\mu_{LR}$ and increase $\mu_{12}$ from $0$ to $\infty$, this ground state continuously interpolates $|\uparrow\rangle$ and $|\downarrow\rangle$ without closing a gap. Energy gap monotonically increases as we increase $\mu_{12}$.

If we do not have SYK term in (3.1), the system is just the $N$ copies of the two spin system described above. In this way, the mass term Hamiltonian is completely diagonalized analytically. The energy gap is given by $\sqrt{\mu_{LR}^2 + \mu_{12}^2}$ again.

3.2. The large $N$ effective action

As we did in the two coupled SYK model, we can write down the action for $G, \Sigma$ variables. The effective action in Euclidean signature is

$$
-\frac{S_E}{N} = \log \text{Pf}(\partial_\tau \delta_{\alpha\beta} \delta^{ab} - \Sigma_{AB}^{\alpha\beta}) - \frac{1}{2} \sum_{A,B=L,R} \sum_{\alpha,\beta = 1,2} \int d\tau_1 d\tau_2 \times \left[ G_{AB}^{\alpha\beta}(\tau_1, \tau_2) \Sigma_{AB}^{\alpha\beta}(\tau_1, \tau_2) - \delta_{\alpha\beta} \frac{J^2}{2g'} [2G_{AB}^{\alpha\beta}(\tau_1, \tau_2)]^g \right]
$$

$$
+ i\mu_{LR} \int d\tau_1 [G_{LR}^{11}(\tau_1, \tau_1) + G_{LR}^{22}(\tau_1, \tau_1)]
$$

$$
+ i\mu_{12} \int d\tau_1 [G_{LR}^{12}(\tau_1, \tau_1) - G_{RR}^{12}(\tau_1, \tau_1)].
$$

(3.23)

Here $G_{AB}^{\alpha\beta}(\tau_1, \tau_2)$ stands for the fermion correlation function

$$
G_{AB}^{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{N} \sum_{i=1}^{N} \langle \psi_{A\alpha_i}(\tau_1) \psi_{B\beta_i}(\tau_2) \rangle,
$$

(3.24)

$^4$We can think of this Hamiltonian as the two cite quantum XY model $H_{XY} = J_x \sigma_x^1 \sigma_x^1 + J_y \sigma_y^1 \sigma_y^1 - h(\sigma_y^1 + \sigma_y^1)$ with $J_x = \frac{\mu_{LR}}{2}$, $J_y = -\frac{\mu_{12}}{2}$ and $h = \frac{\mu_{12}}{2}$. 
and $\Sigma_{AB}^{\alpha\beta}(\tau_1, \tau_2)$ is the self energy. $s_{AB}^{\alpha\beta}$ is given by

$$s_{AB}^{\alpha\beta} = \hat{s}_{AB} \tilde{s}_{AB}^{\alpha\beta},$$

where

$$\hat{s}_{11} = \hat{s}_{RR} = 1, \quad \hat{s}_{LR} = \hat{s}_{RL} = (-1)^q,$$

$$\tilde{s}_{11} = \tilde{s}_{22} = 1, \quad \tilde{s}_{12} = \tilde{s}_{21} = (-1)^q.$$

which comes from the factor $(-1)^q$ in (3.1).

By taking the saddle point of (3.23), we obtain the Schwinger–Dyson equation. The Schwinger–Dyson equation is

$$G_{AB}^{\alpha\beta}(\omega) = -\left[(i\omega + \Sigma(\omega))^{-1}\right]_{AB}^{\alpha\beta}. \tag{3.26}$$

and

$$\Sigma_{AB}^{\alpha\beta}(\tau_1, \tau_2) = s_{AB}^{\alpha\beta} \left( G_{AB}^{\alpha\beta}(\tau_1, \tau_2) \right)^{-1} - i\delta_{\alpha\beta} \mu_{AB} \delta(\tau_1 - \tau_2) - i\mu_{AB} \sigma_{\alpha\beta}^{L-R} \delta(\tau_1 - \tau_2). \tag{3.27}$$

The first equation (3.26) is more explicitly written as

$$\begin{pmatrix}
G_{11}^{LL}(\omega_n) & G_{11}^{LR}(\omega_n) & G_{12}^{LL}(\omega_n) & G_{12}^{LR}(\omega_n) \\
G_{11}^{RL}(\omega_n) & G_{11}^{RR}(\omega_n) & G_{12}^{RL}(\omega_n) & G_{12}^{RR}(\omega_n) \\
G_{21}^{RL}(\omega_n) & G_{21}^{RR}(\omega_n) & G_{22}^{RL}(\omega_n) & G_{22}^{RR}(\omega_n) \\
G_{21}^{LL}(\omega_n) & G_{21}^{LR}(\omega_n) & G_{22}^{LL}(\omega_n) & G_{22}^{LR}(\omega_n)
\end{pmatrix}
= -\begin{pmatrix}
\Sigma_{11}^{LL}(\omega_n) & \Sigma_{11}^{LR}(\omega_n) & \Sigma_{12}^{LL}(\omega_n) & \Sigma_{12}^{LR}(\omega_n) \\
\Sigma_{21}^{LL}(\omega_n) & \Sigma_{21}^{LR}(\omega_n) & \Sigma_{22}^{LL}(\omega_n) & \Sigma_{22}^{LR}(\omega_n) \\
\Sigma_{11}^{RR}(\omega_n) & \Sigma_{11}^{RR}(\omega_n) & \Sigma_{12}^{RR}(\omega_n) & \Sigma_{12}^{RR}(\omega_n) \\
\Sigma_{21}^{RR}(\omega_n) & \Sigma_{21}^{RR}(\omega_n) & \Sigma_{22}^{RR}(\omega_n) & \Sigma_{22}^{RR}(\omega_n)
\end{pmatrix}^{-1}.$$

We have the $\mathbb{Z}_4$ symmetry (2.30) in the two coupled SYK. Since the construction of the four coupled SYK model is based on the two coupled SYK, the four coupled Hamiltonian (3.1) also have symmetries that are inherited from the two coupled SYK. There is a $\mathbb{Z}_4^{L-R}$ symmetry

$$\psi_{1L} \rightarrow -\psi_{1R}, \quad \psi_{1R} \rightarrow \psi_{1L}, \quad \psi_{2L} \rightarrow \psi_{2R}, \quad \psi_{2R} \rightarrow -\psi_{2L}. \tag{3.29}$$
which is the $Z_4$ symmetry that exchanges the left SYK models and right SYK models. There is another $Z_4$ symmetry

$$\psi_{1L} \to -\psi_{2L}, \quad \psi_{2L} \to \psi_{1L}, \quad \psi_{1R} \to -\psi_{2R}, \quad \psi_{2R} \to \psi_{1R}. \quad (3.30)$$

which exchanges the system 1 and the system 2. We call this $Z_4$ symmetry $Z_4^{1-2}$. For symmetric configurations, the equation of motion simplifies. The Schwinger Dyson equation reduces to (figure 3)

$$\left( \begin{array}{cccc}
G_{11}^{11}(\omega_n) & G_{11}^{12}(\omega_n) & G_{11}^{1R}(\omega_n) & G_{11}^{1L}(\omega_n) \\
G_{11}^{1R}(\omega_n) & G_{11}^{11}(\omega_n) & G_{11}^{1L}(\omega_n) & G_{11}^{1R}(\omega_n) \\
G_{11}^{1L}(\omega_n) & G_{11}^{1R}(\omega_n) & G_{11}^{11}(\omega_n) & G_{11}^{1L}(\omega_n) \\
G_{11}^{1R}(\omega_n) & G_{11}^{1L}(\omega_n) & G_{11}^{1R}(\omega_n) & G_{11}^{11}(\omega_n)
\end{array} \right)
= \frac{i\omega_n + \Sigma_{1L}^{11}(\omega_n)}{(i\omega_n + \Sigma_{1L}^{11}(\omega_n))^2 + (\Sigma_{1L}^{1R}(\omega_n))^2 + (\Sigma_{1L}^{1L}(\omega_n))^2} \cdot \left( \begin{array}{cccc}
\Sigma_{1L}^{1R}(\omega_n) & \Sigma_{1L}^{1L}(\omega_n) & \Sigma_{1L}^{1R}(\omega_n) & \Sigma_{1L}^{1L}(\omega_n) \\
-\Sigma_{1L}^{1L}(\omega_n) & \Sigma_{1L}^{1R}(\omega_n) & \Sigma_{1L}^{1L}(\omega_n) & -\Sigma_{1L}^{1R}(\omega_n) \\
-\Sigma_{1L}^{1R}(\omega_n) & \Sigma_{1L}^{1L}(\omega_n) & \Sigma_{1L}^{1R}(\omega_n) & \Sigma_{1L}^{1L}(\omega_n) \\
\Sigma_{1L}^{1R}(\omega_n) & \Sigma_{1L}^{1L}(\omega_n) & -\Sigma_{1L}^{1R}(\omega_n) & \Sigma_{1L}^{1L}(\omega_n)
\end{array} \right)^{-1}. \quad (3.31)
$$

If we write down the independent equations explicitly, they become

$$G_{11}^{11}(\omega_n) = \frac{i\omega_n + \Sigma_{1L}^{11}(\omega_n)}{(i\omega_n + \Sigma_{1L}^{11}(\omega_n))^2 + (\Sigma_{1L}^{1R}(\omega_n))^2 + (\Sigma_{1L}^{1L}(\omega_n))^2},$$

$$G_{11}^{1R}(\omega_n) = \frac{\Sigma_{1L}^{1R}(\omega_n)}{(i\omega_n + \Sigma_{1L}^{11}(\omega_n))^2 + (\Sigma_{1L}^{1R}(\omega_n))^2 + (\Sigma_{1L}^{1L}(\omega_n))^2},$$

$$G_{11}^{1L}(\omega_n) = \frac{\Sigma_{1L}^{1L}(\omega_n)}{(i\omega_n + \Sigma_{1L}^{11}(\omega_n))^2 + (\Sigma_{1L}^{1R}(\omega_n))^2 + (\Sigma_{1L}^{1L}(\omega_n))^2},$$

$$G_{11}^{1R}(\omega_n) = \frac{\Sigma_{1L}^{1R}(\omega_n)}{(i\omega_n + \Sigma_{1L}^{11}(\omega_n))^2 + (\Sigma_{1L}^{1R}(\omega_n))^2 + (\Sigma_{1L}^{1L}(\omega_n))^2}. \quad (3.32)$$

and

$$\Sigma_{1L}^{11}(\tau_1, \tau_2) = \frac{\mathcal{J}^2}{q} (2G_{11}^{11}(\tau_1, \tau_2))^{\nu-1},$$

$$\Sigma_{1L}^{1R}(\tau_1, \tau_2) = (-1)^{\mathcal{J}} \frac{\mathcal{J}^2}{q} (2G_{11}^{1L}(\tau_1, \tau_2))^{\nu-1} - i\mu_{1L}\delta(\tau),$$

$$\Sigma_{1L}^{1L}(\tau_1, \tau_2) = (-1)^{\mathcal{J}} \frac{\mathcal{J}^2}{q} (2G_{11}^{1R}(\tau_1, \tau_2))^{\nu-1} - i\mu_{1L}\delta(\tau),$$

$$\Sigma_{1L}^{1R}(\tau_1, \tau_2) = \frac{\mathcal{J}^2}{q} (2G_{11}^{1R}(\tau_1, \tau_2))^{\nu-1}. \quad (3.33)$$
As far as we observed, the solutions that we obtained are symmetric under $\mathbb{Z}_{LR}$ and $\mathbb{Z}_{1-2}$. The equation for symmetric configurations are simpler and they are convenient to reduce the numerical costs.

Figure 4. The phase diagram of the four coupled model for $\mu_{LR} = 0.03$, $T = 0.001$, and varying $\mu_{12}$. 
Figure 5. The phase diagram at zero temperature in the $\mu_{LR} - \mu_{12}$ plane. The diagram is symmetric under the reflection along $\mu_{12} = \mu_{LR}$ line. The blue line and the orange line meet at about $\mu_{12} = \mu_{LR} \approx 0.154$. Beyond this critical point, the different wormhole phases are continuously connected.

3.3. Phases of the four coupled SYK models at zero temperature

The Schwinger–Dyson equations (3.26) and (3.27) can be solved numerically. Using the correlation functions that are numerically found, we can evaluate several observables. We show the results in figure 4, and we explain the results below.

Here we explain the numerical results. Within the range of the interaction $\mu_{12} \in [\mu_{WH12}, \mu_{WHLR}]$, there are two local minima. One solution is understood as the wormhole in L–R direction, and the other is understood as the wormhole in 1–2 direction. The 1–2 wormhole solution disappears at $\mu_{12} = \mu_{WH12}$ whereas the L–R wormhole solution disappears at $\mu_{12} = \mu_{WHLR}$. The phase diagram is shown in figure 5. The dominance of the saddle is exchanged at $\mu_{LR} = \mu_{12}$ as we can see from the behavior of the system energy. At $\mu_{LR} = \mu_{12}$ the expectation value of the spin operators $\langle S_{LR} \rangle$ and $\langle S_{12} \rangle$ jump. At this point the Hamiltonian is $\mathbb{Z}_2$ symmetric and the order parameter $S_{LR} - S_{12}$ has an expectation value, which means the spontaneous symmetry breaking of the $\mathbb{Z}_2$ symmetry.

In the L–R wormhole phase, increasing the strength of the interaction $H_{LL}^1 + H_{RR}^2$ in 1–2 direction actually decreases the energy gap of the system rather than increasing. This is different from what happens without the SYK interaction terms as we saw in section 3.1.1. In that case we observe that the energy gap increase when we increase $\mu_{12}$. Therefore, this decreasing behavior is a strongly coupled phenomenon. In real time, the inverse of the energy gap plays the role of the wormhole length. Therefore, we can interpret that entangling bulk fields in two different wormholes increases the wormhole length. In the bulk language, we expect that entangling bulk fields in different wormholes leads to bulk excitation, which leads to increase of the wormhole length.
Figure 6. The fitting of $\mu_{\text{WHLR}}$ and $\mu_{\text{WH12}}$. We use the data for small $\mu_{12}$ and fit the data assuming power low behavior. The rest of data are well fitted when $\mu_{12}$ is sufficiently small. (Left) The fitting of $\mu_{\text{WHLR}}$. (Right) The fitting of $\mu_{\text{WH12}}$.

Figure 7. A bulk picture of the perturbation in (3.34).

An interesting thing is that at symmetric point we can actually find a symmetric solution numerically imposing the symmetry in (3.16). This symmetric saddle is unstable, which we can confirm numerically\(^5\).

The first order phase transition only exists for small $\mu_{LR}$. We checked this numerically and the phase transition disappears around $\mu_{LR} \sim 0.154$. Beyond this critical value, the phase transition disappears and we only have symmetric solution at $\mu_{LR} = \mu_{12}$ point.

We have also found a power low behavior for $\mu_{\text{WHLR}}$ and $\mu_{\text{WHLR}}$ as functions of $\mu_{LR}$ at small $\mu_{LR}$, see figure 6, $\mu_{\text{WHLR}}$ behaves as $\mu_{\text{WHLR}} \sim (\mu_{LR})^{0.33}$. The power is close to 1/2. This power law that is smaller than 1 means that the maximal value of $\mu_{12}$ with L–R wormhole saddle with fixed $\mu_{LR}$ is parametrically large compared to $\mu_{LR}$ when $\mu$’s are small. This means that even for large $\mu_{12}$, the L–R wormhole saddle exists. Roughly speaking, we can entangle bulk matter fields a lot in different wormholes by these boundary interactions.

\(^5\)For example, we can perturb the solution by some $G^{\alpha\beta}_{\alpha\beta} \rightarrow G^{\alpha\beta}_{\alpha\beta} + \delta G^{\alpha\beta}_{\alpha\beta}$ with small perturbation $\delta G^{\alpha\beta}_{\alpha\beta}$. Then, the solution falls to the LR wormhole or the 12 wormhole depending on the perturbation.
Figure 8. The phase diagram of the four coupled model for $\mu_{LR} + \mu_{12} = 0.03$, $T = 0.001$ with varying $\mu_{12}$. 
3.4. Small $\mu_{12}$ perturbation

When $\mu_{12}$ is small, we can treat this as an perturbation term and use (conformal) perturbation theory. The $G_{LL}^{12}$ correlator becomes

\[ G_{LL}^{12}(\tau_1, \tau_2) = \left\langle \psi_L^1(\tau_1)\psi_L^2(\tau_2) \right\rangle \left( i\mu_{12} \int_{-\infty}^{\infty} d\tau \psi_L^1(\tau)\psi^2_L(\tau) - i\mu_{12} \int_{-\infty}^{\infty} d\tau \psi^1_L(\tau)\psi^2_L(\tau) \right) \]

\[ = i\mu_{12} \int_{-\infty}^{\infty} d\tau \ G_{LL}^{11}(\tau-\tau_1)G_{LL}^{12}(\tau-\tau_2) + i\mu_{12} \int_{-\infty}^{\infty} d\tau \ G_{12}^{11}(\tau-\tau_1)G_{12}^{12}(\tau-\tau_2) \]

\[ = i\mu_{12} \left[ \int_{-\infty}^{\infty} d\tau \ G_{LL}^{12}(\tau-\tau_1)G_{LL}^{12}(\tau-\tau_2) + \int_{-\infty}^{\infty} d\tau \ G_{12}^{12}(\tau-\tau_1)G_{12}^{12}(\tau-\tau_2) \right]. \tag{3.34} \]

We used the symmetry $G_{11}^{11}(-\tau) = -G_{11}^{11}(\tau)$ and $G_{12}^{12}(-\tau) = G_{12}^{12}(\tau)$. In the last line we use $G_{LL}^{12}$ and $G_{12}^{12}$, which is the correlation function of the two coupled model (2.36), since we use perturbation theory around $\mu_{12} = 0$ and at this point we just have two decoupled Maldacena–Qi model. This relation holds whether or not the conformal limit is applicable.

Intuitively, these perturbation corresponds to taking into account the bulk diagrams in figure 7. The first term in (3.34) comes from the direct interactions in the left boundaries and the second term comes from that in the right boundaries. Once we obtain the correlation function $G_{12}^{12}$ in this perturbation, we can calculate the physical quantities in perturbation in $\mu_{12}$. For example, we can calculate the spin operator expectation value $\langle S_{12}^x \rangle = -2i G_{12}^{12}(0)$ perturbatively by

\[ G_{12}^{12}(0) = i\mu_{12} \left[ \int_{-\infty}^{\infty} d\tau \ G_{LL}^{12}(\tau)^2 + \int_{-\infty}^{\infty} d\tau \ G_{12}^{12}(\tau)^2 \right]. \tag{3.35} \]

Then we can also evaluate the change of the ground state energy of the Hamiltonian (3.1). We show this perturbation in figure 4 and for $\mu_{12}$ the results agree with the exact numerical results.

3.5. Varying $\mu_{12}$ with $\mu_{LR} + \mu_{12}$ to be fixed

We can also study the phase diagram when we fix $\mu_{LR} + \mu_{12}$ to be a constant. In other words, we change $\mu_{12}$ with fixing $\mu_{LR} \equiv \mu_{LR} + \mu_{12}$. The results are shown in figure 8. Because we treat $\mu_{LR}$ and $\mu_{12}$ equally, the phase diagram becomes symmetric under the exchange of L–R direction and 1–2 direction. If we compare with the classical Ising model, $\mu_{12} - \mu_{tot}/2$ plays a role of external magnetic field and the order parameter $(S_{LR} - S_{12}^x)$ plays a role of magnetization.

In the analysis in the gravity side, we also study the same type of change of couplings and draw the similar behavior of the system energy.

3.6. Effective potential

Given the $G, \Sigma$ configuration, we can evaluate the effective action (3.23). For the static configuration $G_{ab}^{ij}(\tau_1, \tau_2) = G_{ab}^{ij}(\tau_1 - \tau_2)$, we can consider the effective potential

\[ S_{ef}/N \approx \beta V_{ef}(G, \Sigma). \tag{3.36} \]

Here $\beta$ is the IR cutoff in the Euclidean time. In the four coupled SYK model, we have three saddle points: L–R wormholes ($G_{WH,LR}(\tau), \Sigma_{WH,LR}(\tau)$), 1–2 wormholes ($G_{WH,12}(\tau), \Sigma_{WH,12}(\tau)$),
and the symmetric solutions \((G_{\text{sym}}(\tau), \Sigma_{\text{sym}}(\tau))\)\(^6\). Then, using these saddles we can consider the slice of
\[
G(\tau) = sG_{\text{WHLR}}(\tau) + tG_{\text{WH12}}(\tau) + uG_{\text{sym}}(\tau), \quad s + t + u = 1,
\]
\[
\Sigma(\tau) = J^2G(\tau)^{\gamma-1}.
\]

These constitute a two-dimensional slice in \(G, \Sigma\) configuration spaces. We can parametrize \(s, t, u\) by two parameters \(x\) and \(y\) as
\[
s = x - \frac{y}{2}, \quad t = (1 - x) - \frac{y}{2}, \quad u = y.
\]

The effective potential on in this \((x, y)\) plane can be evaluated numerically and we show the plot of them in figure 9. In this \((x, y)\) plane, we have found that the L–R wormholes and 1–2 wormholes are local minima. On the other hand, the symmetric saddle point is actually only an extremum and a maximum in one direction. Therefore, the plot shows that the symmetric saddle point is unstable. Another interesting thing is that in this slice the L–R wormhole and 1–2 wormhole are actually smoothly connected. We expect that the effective potential are smoothly changed when we move from \(\mu_{\text{LR}} = \mu_{\text{12}}\) point. Then we expect that these unstable saddle points are always exists when the system has both of L–R wormhole and 1–2 wormhole saddles.

### 3.7 \(\mu\) dependence of the symmetric solutions

In this section we study the \(\mu\) dependence of the symmetric saddle points with comparison to wormhole phases at the \(\mu_{\text{12}} = \mu_{\text{LR}} \equiv \mu\) points. Here we focus on the energy gap and the spin operator expectation values.

First we consider the gap. A fitting for the numerical data (the green dashed line in the left of figure 10) gives
\[
E_{\text{gap}} \approx 17.3054\mu^{1.74896},
\]
scaling of which is close to \(E_{\text{gap}} \sim \mu^\gamma\). On the other hand, in both of the L–R wormhole phases and the 1–2 wormhole phases, the energy gap \(E_{\text{gap}}\) agrees with that in the two coupled model which is given by (2.29). This is because the ratio \(\mu/E_{\text{MQ}}\) is small, which controls the strength of the perturbation that connects to wormholes, where \(\mu/E_{\text{MQ}}\) is the gap in the two coupled model at the coupling \(\mu\). Comparing the \(E_{\text{gap}} \sim \mu^{1.74896}\) in the symmetric solution to the scaling of the \(E_{\text{gap}} \sim \mu^\gamma\) in the wormhole phases, the power is greater than 1 and this power low makes the energy gap smaller than the naive gap \(\mu\). If we identify the energy gap with the wormhole length, this suggests that the wormhole length in the symmetric saddle is much larger than that in wormhole phases.

The spin operator expectation value is plotted in figure 10. The power law fitting in the symmetric saddle (the green dashed line in the right of figure 10) is given by
\[
S_{\text{LR}} \sim 2.80592\mu^{0.84502},
\]
which is close to \(S_{\text{LR}} \sim \mu^\gamma\). This power of the scaling in symmetric solutions is much smaller than in L–R wormhole solutions. On the other hand, the scaling in the 1–2 wormhole phase

\(^6\) Here we suppress the index in \(G_{\text{sym}}\) as \(G\).
Figure 9. The plots of the effective potential for $\mu_1 = \mu_2 = 0.075$, $T = 0.001$ on the two dimensional slice (3.37). (Top left) The 3D plot of the effective potential as a function $V_{\text{eff}}(x, y)$. (Top right) The contour plot of the effective potential as a function $V_{\text{eff}}(x, y)$. (Bottom left) The $t = 0$ slice of the $V_{\text{eff}}(x, y)$. In this slice, it is easy to see that the symmetric saddle point is an extremum rather than a minimum. (Bottom middle) The $u = 0$ slice of the $V_{\text{eff}}(x, y)$. In this slice, the symmetry to exchange WHLR and WH12 saddles. (Bottom right) The $x = 1/2$ slice of the $V_{\text{eff}}(x, y)$. Along this direction, the symmetric saddle is a minimum.

Figure 10. The plots of $E_{\text{gap}}$ and $S_{LR}$ for $\mu_{LR} = \mu_{12} \equiv \mu$ cases i.e. the L–R coupling and the 1–2 coupling are the same. We vary $\mu$ and plot $E_{\text{gap}}$ and $S_{LR}$ as a function of $\mu$. (Left) The plot of the $E_{\text{gap}}$. For fitting of the $E_{\text{gap}}$ in symmetric solutions, we use the data below $\mu < 0.01$. (Right) The plot of the spin operator $S_{LR}$. For fitting of the $S_{LR}$ in symmetric solutions, we use the data below $\mu < 0.01$.

is $S_{LR} \sim \mu$ for generic $q$. Therefore, the expectation value of $S_{LR}$ in the symmetric solution is much larger than that in the 1–2 solution. As the plot shows, the spin operator expectation value is between that of L–R wormhole and 1–2 wormhole. The three solutions meet at $\mu \sim 0.154$. Beyond that point, the solutions merges and we only have one solution.
4. Further analysis on traversable wormholes coupled with CFT$_2$

Before going to the construction of four coupled JT gravities, we study the properties of two coupled JT gravities which we will use in the analysis of four coupled models, which include some new results on entanglement entropy and partially coupling in this setup. We consider the models of JT gravity with CFT$_2$ that are coupled to the same CFT$_2$ on with no dynamical gravity [26, 65]. We imagine that JT gravity with CFT$_2$ is dual to a quantum mechanical system. The situation we consider is that CFT$_2$ is coupled to two quantum mechanical systems with dual
description by JT gravity + CFT$_2$ systems. We basically use the saddle point approximation and the results only depend on the central charge of CFT$_2$. Therefore, they can be holographic CFT with dual AdS$_3$ gravity description. In figure 11 we show the setup in this subsection with three descriptions [59, 65]: 2D gravity description, 3D gravity description and the full quantum mechanical description. We can take the 3D gravity interpretation only when the bulk matter fields are holographic 2DCFTs. In figure 11, we only describe the theory that we consider and the IR region is not described since they depend on states.

4.1. The traversable wormhole solutions

We are interested in the ground state of these two coupled models. The ground state is given by traversable wormholes, which is illustrated in figure 12. Now we consider the traversable wormhole solutions with the finite ‘bath length’ $d$ where the CFT$_2$ is sitting on. Here the finite bath length means that the ratio $d/\ell = a$ is a finite constant. In this case, we first put CFT on a cylinder with the length $\pi(1+a)$, which is a flat space. This case, we obtain the stress tensor expectation value

$$\langle T_{\sigma \sigma}^{cyl} \rangle = -\frac{c}{6\pi(1+a)^2}.$$  \hspace{1cm} (4.1)

The notations and the derivation from conformal anomaly are summarized in appendix A. In terms of $T_{y^+y^+}$ and $T_{y^-y^-}$, they become

$$\langle T_{y^+y^+}^{cyl} \rangle = \langle T_{y^-y^-}^{cyl} \rangle = -\frac{c}{12\pi(1+a)^2}.$$  \hspace{1cm} (4.2)

Next, we introduce the AdS$_2$ metric in the region $\sigma \in [0, \pi]$. Then, $T_{y^+y^+}$ and $T_{y^-y^-}$ in the AdS$_2$ region is

$$\langle T_{y^+y^+}^{AdS_2} \rangle = \langle T_{y^-y^-}^{AdS_2} \rangle = -\frac{c}{48\pi} - \frac{c}{12\pi(1+a)^2}.$$  \hspace{1cm} (4.3)

The first term comes from the Weyl anomaly, see appendix A. We can rewrite this as

$$\langle T_{y^+y^+}^{AdS_2} \rangle = \langle T_{y^-y^-}^{AdS_2} \rangle = -\frac{c_{\text{eff}}(a)}{16\pi}.$$  \hspace{1cm} (4.4)

Here we defined an ‘effective central charge’

$$c_{\text{eff}}(a) = c\frac{(3+a)(1-a)}{3(1+a)^2},$$  \hspace{1cm} (4.5)

which is introduced in the wormhole regime by conformal matters. This null energy is also directly calculated using the holographic stress energy tensor [66] when the bulk matter fields are holographic. Then,

$$\phi(\sigma) = \frac{c_{\text{eff}}(a)}{8\pi} \left[ \frac{\sigma}{\tan(\ell/\pi)} - t' - t \sigma \right] + \frac{c}{24\pi} \left[ \frac{\sigma}{\tan(\ell/\pi)} + 1 \right] = \frac{2\phi}{t' \ell} \left[ \frac{\ell}{\tan(\ell/\pi)} - t' - t + 1 \right] + \frac{c}{24\pi}.$$  \hspace{1cm} (4.6)
The inverse of wormhole length is given by \( t' = \frac{c_{\text{eff}}(a) \pi G_N}{2 \phi_t} \). If we restore the \( 8 \pi G_N \) which we put to be 1 above, we obtain

\[
t' = \frac{c \pi G_N (3 + a)(1 - a)}{2 \phi_t 3(1 + a)^2},
\]

\[
M = -\frac{c^3 \pi G_N (3 + a)^2(1 - a)^2}{32 \phi_t 9(1 + a)^4}.
\]

The wormhole length is

\[
\ell = \frac{2 \phi_t}{c \pi G_N} \frac{3(1 + a)^2}{(3 + a)(1 - a)}
\]

and the bath (region without gravity) length is

\[
d = \pi \ell a = \frac{2 \phi_t}{c G_N} \frac{3(1 + a)^2}{(3 + a)(1 - a)} a.
\]

The last equation determines the parameter \( a \) as a function of \( d \). As a function of \( d \), \( a \) becomes

\[
a = F \left( \frac{c G_N}{2 \phi_t} d \right),
\]

where \( F(X) \) is the root of

\[
3a^3 + (X + 6)a^2 + (2X + 3)a - 3 = 0.
\]

as a polynomial of \( a \). The wormhole length is plotted in figure 13.

For large \( d \gg \frac{2 \phi_t}{c G_N} \), \( a \approx 1 \). This means \( \pi \ell \approx d \). It takes almost the same time to go to traversable wormholes with going from the outside. The energy in the bath region is

\[
T_{\text{tt}} \cdot d = -\frac{c}{6 \pi} \frac{t'^2 d}{(1 + a)^2} = -\frac{c}{6 \pi} \frac{d}{(\ell' + a)^2} = -\frac{c}{6 \pi} \frac{\pi d}{(\ell' + d)^2}.
\]

The energy density itself is

\[
T_{\text{tt}} = -\frac{c}{6 \pi} \frac{t^2}{(1 + a)^2} = -\frac{c}{6 \pi} \frac{\pi}{(\ell' + d)^2}.
\]

4.1.1. Variational ansatz. We approximate the geometry by the \( t = 0 \) slice of the eternal black holes (=thermo field double states). The wormhole length is \( \ell = \frac{1}{2 \pi \mathcal{H}} \). The energy par a black hole is

\[
M_{\text{BH}} = \frac{\pi \phi_t}{4G_N} T_{\text{H}}^2 = \frac{\phi_t}{16 \pi G_N} \frac{1}{\ell^2}.
\]

\(^7\)The parameter \( a \) is not a fixed parameter but a parameter which is determined from the saddle point equation of quantum gravity i.e. JT gravity in this setup. This is because the wormhole length \( \ell \) is a dynamical parameter rather than a fixed parameter. The length of the bath region \( d \) is a parameter of the theory because CFT in the bath region there are no dynamical gravity.
The quantum matter contribution is

\[ E_{\text{mat}} = \frac{c^2}{24 \ell} - \frac{c \pi}{6(\pi \ell + d)}. \]  

(4.16)

The total variational energy is

\[ E(\ell) = 2M_{\text{BH}} + E_{\text{mat}} = \frac{\bar{\phi}_r}{8\pi G_N} \frac{1}{\ell^2} + \frac{c}{24\ell} - \frac{c \pi}{6(\pi \ell + d)}. \]  

(4.17)

We minimize this energy

\[ \frac{\partial E}{\partial \ell} = -\frac{\bar{\phi}_r}{4\pi G_N} \frac{1}{\ell^3} - \frac{c}{24\ell^2} + \frac{c \pi^2}{6(\pi \ell + d)^2} = 0. \]  

(4.18)

By setting \( \pi \ell a = d \), this equation becomes

\[ d = \frac{\pi \bar{\phi}_r}{2G_N} \frac{12(1 + a)^2}{(3 + a)(1 - a)} a. \]  

(4.19)

This equation is precisely the same with the former one (4.10) that determines the length of the wormhole as a function of \( d \). Then \( \ell \) becomes

\[ \ell = \frac{2 \bar{\phi}_r}{c G_N} \frac{3(1 + a)^2}{(3 + a)(1 - a)}. \]  

(4.20)

This again reproduces the former result. The approximation by the eternal black holes (TFD state) is a good description.
4.1.2. Entanglement entropy for region with a boundary. In this subsection, we consider entanglement entropy when we divide the system to two parts at point \( x = b \) in no gravity region in a full quantum mechanical description in figure 12. In other words, we consider entanglement entropy for an interval of \([0, b]\) in the notation of [59]. Here the bold notation means that we consider the interval in the full quantum mechanical description, not in the 2D gravity description. In particular, we should notice that in 2D gravity picture we do not have the concept to divide the system to two parts because only specifying one entangling surface does not divide the system to two parts in figure 12. Correspondingly, the configuration without islands [59, 65] does not exist in 2D gravity description in this case.

The general form of the generalized entropy functional is given by [26, 65]

\[
S(A) = \text{Min}_{\mathcal{E}} \left[ S_{\text{eff}}(A \cup \mathcal{I}_g) + \frac{\text{Area}[\partial \mathcal{I}_g]}{4G_N} \right].
\] (4.21)

In our setup, generalized entropy functional for the \([0, b]\) is

\[
S_{\text{gen}}(\sigma) = S_0 + 2\pi \phi(\sigma) + \frac{c}{6} \log \left( \frac{\pi \ell + d}{\ell} \right) \frac{2 \sin^2 \frac{\pi (\sigma + b)}{\pi + d}}{\epsilon_B \ell \sin \frac{\pi}{2}}
\]

\[
= S_0 + \frac{2\phi}{\ell} \left[ \frac{\pi \ell - 2\sigma}{\ell \tan \frac{\pi}{2}} + 1 \right] + \frac{c}{6} \log \left( \frac{\pi \ell + d}{\ell} \right) \frac{2 \sin^2 \frac{\pi (\sigma + b)}{\pi + d}}{\epsilon_B \ell \sin \frac{\pi}{2}}.
\] (4.22)

The first term is the topological contribution from the Einstein–Hilbert term of the JT gravity. \( \epsilon_B \) is a UV cutoff in no gravity regions. The second term is the RT surface contribution. We choose the normalization of \( 8\pi G_N = 1 \) here and in this normalization \( \frac{d\phi}{4G_N} = 2\pi \phi(\sigma) \). The third term is the entanglement entropy of matter fields. Note that \( \frac{\phi}{\ell} \propto c \) and both term can compete.

Entanglement entropy is given by the saddle point of the generalized entropy functional:

\[
\partial_\sigma S_{\text{gen}}(\sigma) = 0.
\] (4.23)

This becomes

\[
\partial_\sigma S_{\text{gen}}(\sigma) = \frac{c\ell + 24\phi}{6\ell^2 \tan \frac{\pi}{2}} \frac{2\phi(\ell \pi - 2\sigma)}{\ell^3 \sin^2 \left( \frac{\pi}{2} \right)} + \frac{\pi c}{3} \frac{1}{\pi \ell + d} \frac{\sin \frac{\pi (\sigma + b)}{\pi + d}}{\sin \frac{\pi}{2}}
\] (4.24)

The quantum extremal surface is given by the saddle point of the generalized entropy:

\[
S_{\text{ext}} = S_{\text{gen}}(\sigma_s).
\] (4.25)

The 2D gravity, 3D gravity and full quantum mechanical description for \( S_{\text{ent}}(\sigma_s) \) are shown in figure 14.

For general \( d \), we can calculate entanglement entropy (4.25) numerically. We show the numerical solutions for \( S_{\text{ent}} \) and the position of the quantum extremal surface \( \sigma_s \) in figure 15. Note that entropy (4.25) is a function of \( d \) and \( b \) because the wormhole length \( \ell \) is also determined as a function of \( d \) in (4.9).

This is related to the factorization problem in [61]. Without assuming that single JT gravity factorizes to two systems, the concept to divide the system two to parts by only specifying entangling surface \( x = b \) does not exist.

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8 This is related to the factorization problem in [61]. Without assuming that single JT gravity factorizes to two systems, the concept to divide the system two to parts by only specifying entangling surface \( x = b \) does not exist.
Figure 14. We describe the three descriptions for the quantum extremal surface which contains the dual of a JT + matter system. (Left) The JT gravity plus matter CFT description. We are computing quantum extremal surfaces. (Middle) We use the 3D gravity description for matter CFT assuming that the matters are holographic. The shaded region (which is denoted by EW) is the entanglement wedge, which is surrounded by the entangling surface and the RT/HRT surfaces. (Right) The full quantum mechanical description. We calculate entanglement entropy of CFT using the large $N$ expansion. This is another candidate saddle of the full answer.

Figure 15. The plots of the location of the quantum extremal surface and entanglement entropy for $b = 0$ as function of $d$. We set $\ell = \frac{16\phi}{r_c} = 1$, and $c = 1$. (Left) The plot of the quantum extremal surface $\sigma_s$. (Right) The plot of entanglement entropy. We also omitted the extremal entropy part $S_0$. We also subtract the divergence $c_6 \log \frac{\epsilon_B}{\ell}$. In this choice, the initial value is $S_0 + \frac{c}{8} + \frac{c}{6} \log \frac{16\phi}{r_c} = \frac{3}{8}$. Finally entanglement entropy approaches to $S_0 + \frac{c}{8} + \frac{c}{6} \log \frac{16\phi}{r_c} = \frac{3}{8} + \frac{c}{6} \log \frac{16\phi}{r_c} = 0.309$.

When $d = 0$ i.e. we can ignore the length of the bath region, the saddle point is $\sigma_s = \frac{\pi}{4} \ell$ with $\ell = \frac{16\phi}{r_c}$, which is the middle of the AdS$_2$. Entanglement entropy for $b = 0$ case becomes

$$S_{\text{gen}}(\pi \ell/2) = S_0 + \frac{2\phi_r}{\ell} + \frac{c}{6} \log \frac{\ell}{\epsilon_B}$$

$$= S_0 + \frac{c}{8} + \frac{c}{6} \log \frac{16\phi_r}{c\epsilon_B}$$

(4.26)
When $\sigma \ll \ell$ and $b \ll \ell$, we can approximate $\tan \frac{\sigma}{\ell} \approx \frac{\sigma}{\ell}$, $\sin \frac{\sigma}{\ell} \approx \frac{\sigma}{\ell}$ and $\sin \frac{\pi(\sigma + b)}{\pi + d} \approx \frac{\pi(\sigma + b)}{\pi + d}$. Then, the generalized entanglement entropy before the minimization is

$$S_{\text{gen}}(\sigma) \approx S_0 + \frac{2\pi \phi_t}{\sigma} + \frac{c}{6} \log \left(\frac{(\sigma + b)^2}{\epsilon_B \sigma}\right),$$

(4.27)

which is exactly the same with the 0 temperature generalized entropy. The saddle point equation becomes

$$\frac{\partial S_{\text{gen}}(\sigma)}{\partial \sigma} = -\frac{2\pi \phi_t}{\sigma^2} + \frac{c}{6} \left[ \frac{2}{(\sigma + b)} - \frac{1}{\sigma} \right] = 0.$$

(4.28)

The solution is given by

$$\sigma_s = \frac{1}{2c} \left( bc + 12\pi \phi_t + \sqrt{b^2c^2 + 72\pi \phi_t bc + 144\pi^2 \phi_t^2} \right).$$

(4.29)

By introducing $\tilde{\sigma}_s = \frac{c}{12\pi \phi_t} \sigma_s$ and $\tilde{b} = \frac{c}{12\pi \phi_t} b$, we can rewrite

$$\tilde{\sigma}_s = \frac{1}{2} \left( 1 + \tilde{b} + \sqrt{1 + 6\tilde{b} + \tilde{b}^2} \right).$$

(4.30)

For $b = 0$, $\tilde{\sigma}_s = 1$. In other words, in $b \to 0$ limit we obtain

$$\sigma_s = \frac{12\pi \phi_t}{c}.$$

(4.31)

Entanglement entropy becomes

$$S_{\text{gen}}(\sigma_s) = S_0 + \frac{c}{6} \log \frac{12\pi \phi_t}{\epsilon_B c}.$$

(4.32)

4.2. Entanglement entropy for regions without gravity

Next we calculate entropy of $[0, d - 0]$, i.e. entanglement entropy between bath and the two JT + matter systems. In this case, there are at least two candidate of the quantum extremal surfaces. The first one is entanglement entropy of the interval $[0, d]$ on a circle of the total length $\pi\ell + d$. This is usual entanglement entropy for a single interval and is evaluated as

$$S_{\text{no-island}} = \frac{c}{3} \log \left[ \frac{\pi\ell + d}{\pi \epsilon_B} \sin \left( \frac{d}{\pi \ell + d} \right) \right].$$

(4.33)

The second one is a quantum extremal surface where we have the island. Assuming the matter fields are holographic and use the Ryu–Takayanagi formula, this is the twice of the entropy between single JT gravity + matter system we calculated in $(4.25)$:

$$S_{\text{island}} = 2S_{\text{gen}}(\sigma_s).$$

(4.34)
Entanglement entropy of the region $[0_+, d - 0]$ is given by the minimal value of these candidates:

$$S_{\text{ent}} = \min\{S_{\text{no-island}}, S_{\text{Island}}\}.$$  \hspace{1cm} (4.35)

Assuming that $S_0$ is sufficiently large, for small $d$ the no-island saddle dominates. This grows when we increase the length $d$. When $d$ is extremely large, we expect that the island saddle dominates. We can estimate the point where the dominant saddle exchanges. For large $d$, the length of the bath and the wormhole are almost the same. Therefore, we can evaluate the no-island saddle as

$$S_{\text{no-island}} \approx \frac{c}{3} \log \frac{d}{\epsilon_B}. \hspace{1cm} (4.36)$$

On the other hand, the island saddle is almost constant when $d \approx \pi \ell$ is large:

$$S_{\text{Island}} = 2S_{\text{gen}}(\sigma_s) = 2S_0 + \frac{c}{3} + \frac{c}{3} \log \frac{12\pi \phi_t}{\epsilon_B}. \hspace{1cm} (4.37)$$

When these two are equal,

$$S_{\text{Island}} = S_{\text{no-island}} \rightarrow \frac{c}{3} \log \frac{d}{\epsilon_B} = 2S_0 + \frac{c}{3} + \frac{c}{3} \log \frac{12\pi \phi_t}{c \epsilon_B}. \hspace{1cm} (4.38)$$

This happens when

$$\frac{c}{12\pi \phi_t} d = \exp \left( \frac{6S_0}{c} + \frac{3}{4} \right). \hspace{1cm} (4.39)$$

or

$$d = \frac{12\pi \phi_t}{c} e^{\frac{6S_0}{c} + \frac{3}{4}} = \frac{3\pi}{4} \ell_0 e^{\frac{6S_0}{c} + \frac{3}{4}}. \hspace{1cm} (4.40)$$

Here $\ell_0 = \frac{16g}{\sqrt{c}}$ is the wormhole length when the bath length is 0, which is the minimal wormhole length of this traversable wormhole setup. Therefore, when the wormhole is exponentially large with respect to the ratio $\frac{6S_0}{c}$, the island saddle dominates. For 4D traversable wormholes, these parameters become

$$S_0 = \frac{\pi r_e^2}{G_N} = \frac{\pi q^2}{g^2}, \hspace{1cm} c = q. \hspace{1cm} (4.41)$$

Therefore, the ratio

$$\frac{S_0}{c} = \frac{\pi q}{g^2}, \hspace{1cm} (4.42)$$

is huge when $q$ is large. This means that it becomes hard to develop the island for large $q$ i.e. the magnetic charge and the mass of the original black hole are large. Semiclassical limit is good only when $q$ is large, so it seems to be hard to develop islands. On the other hand, if we use the SYK values, these parameters becomes

$$S_0 = NS_0, \hspace{1cm} c = \frac{N}{2}. \hspace{1cm} (4.43)$$

\textsuperscript{9}Precisely speaking, when we define bulk fermions mass from the SYK fermion operator dimensions through the usual AdS/CFT dictionary, the bulk fermions are not massless.
In the SYK model $s_0$ is given by $s_0 = \frac{1}{2} \log 2 - \frac{\pi^2}{96} \log q$ in the large $q$ expansion \cite{2} and $s_0 \approx 0.2324 \approx \frac{1}{4} \log(1.592)$ for $q = 4$ \cite{22}. Therefore the ratio is

$$\frac{s_0}{c} = 2s_0,$$

(4.44)

which is not big and does not depend on $N$. In this type of theory it is easy to develop the island because $e^{\frac{s_0}{c}}$ is order one quantity (w.r.t. order $N$ counting) and $\ell_0 e^{\frac{6s_0}{c} + \frac{1}{4}}$ is not so big.

4.3. JT + several 2D holographic matters

In the 4D traversable wormhole/magnetically charged blackholes setup \cite{1,20}, we obtain large number of free 2D fermions because of the Landau degeneracy. Similarly, we can consider to couple 2D several holographic matters where the bulk fields have their own 3D gravity dual. In other words, there are $N_{hi}$ holographic matters that are coupled to JT gravity. We label these holographic matters by $\lambda$ and denote their Newton constants by $G^{(3)}_{N_{hi}}$, AdS radius by $l^{(AdS3)}_{hi}$ and the metric by $\gamma^{(3)}_{\lambda\mu\nu}$. The Brown–Henneaux central charge is given by $c^\lambda = \frac{\rho^{(AdS3)}_{hi}}{8\pi G^{(3)}_{N_{hi}}}$. For holographic CFTs, the ‘quantum’ stress tensor expectation value is \cite{9}

$$T^\text{mu}_\nu = \lim_{\epsilon \to 0} \frac{1}{8\pi G^{(3)}_{N_{hi}}(\epsilon)} \left[ K_{0\mu\nu} - K_{\lambda\mu} h^{(2)}_{\lambda\nu} - \frac{1}{l^{(AdS3)}_{hi}} h^{(2)}_{\lambda\mu\nu} \right] \bigg|_{z=\epsilon}. \quad (4.45)$$

Here, $h^{(2)}_{\lambda\mu\nu}$ is the induced metric on the UV cutoff surface $z = \epsilon$ and is expanded as

$$h^{(2)}_{\lambda\mu\nu} = \frac{g^{(2)}_{\epsilon\mu\nu}}{c^2} + \cdots$$

(4.46)

where the 2D metric $g^{(2)}_{\epsilon\mu\nu}$ is the metric in JT gravity, which will be set to be that of AdS$_3$, after imposing the equation of motion for JT gravity. Note that we impose the same asymptotic boundary conditions for all the $N_{hi}$ holographic matters because all the holographic CFTs are coupled to the same 2D gravity. Now, the equation of motion for JT gravity becomes

$$- \frac{1}{8\pi G^{(2)}_N} (\nabla_\mu \nabla_\nu \phi - g^{(2)}_{\mu\nu} \nabla^2 \phi + g^{(2)}_{\mu\nu} \phi) = \sum_{i=1}^{N_{hi}} \frac{1}{8\pi G^{(3)}_{N_{hi}}} \left[ K_{0\mu\nu} - K_{\lambda\mu} h^{(2)}_{\lambda\nu} - \frac{1}{l^{(AdS3)}_{hi}} h^{(2)}_{\lambda\mu\nu} \right] \bigg|_{z=\epsilon}. \quad (4.47)$$

Note that all the holographic matters shares the same boundary condition $h^{(2)}_{\lambda\mu\nu} = \frac{g^{(2)}_{\epsilon\mu\nu}}{c^2} + \cdots$, which represent the coupling to the same 2D JT gravity. $K_{0\mu\nu}$ is the extrinsic curvature of the UV cutoff surface in the $i$\textsuperscript{th} holographic CFT.

These can be thought of as ‘Randall Sundrum junctions’, since several 3D geometries are connected through the same JT gravity. In particular, we can consider the special case where all the holographic matters are the same theory. In this case, they have the same central charges and we obtain

$$- \frac{1}{8\pi G^{(2)}_N} (\nabla_\mu \nabla_\nu \phi - g^{(2)}_{\mu\nu} \nabla^2 \phi + g^{(2)}_{\mu\nu} \phi) = \frac{1}{8\pi G^{(2)}_N} \sum_{i=1}^{N_{hi}} \left[ K_{0\mu\nu} - K_{\lambda\mu} h^{(2)}_{\lambda\nu} - \frac{1}{l^{(AdS3)}_{hi}} h^{(2)}_{\lambda\mu\nu} \right] \bigg|_{z=\epsilon}. \quad (4.48)$$
When we have two holographic matters, the setup is similar to the defect CFT model [67, 68]. In this case, JT brane exists between two holographic matters and looks like a defect between them. We can consider the parameter regime where $N_H$ is of order $1/G_N^{(2)}$ so that we can balance the JT classical fields and 'highly quantum' holographic matter effects. In other words, each $c_i$ is $O(1)$ in the $1/G_N^{(2)}$ expansion but is still sufficiently large $c_i \gg 1$ so that we can use large $c$ limit for these holographic CFTs.

This JT + several holographic matters model is convenient when we try to change the boundary of holographic CFT$_2$ gradually since we can associate different boundary conditions for different holographic matters. Similar situation can arise from the 4D holographic CFT with $U(1)$ symmetry and strong magnetic fields [51, 69]. In this situation the dual 5D geometry flows from AdS$_5$ to AdS$_3 \times \mathbb{R}^2$. The magnetic flux $q$ which is vertical to $\mathbb{R}^2$ plays the role of $N_H$ there.

This JT gravity + several holographic matters model is convenient for our purpose in latter sections.

4.4. Traversable wormholes with partial couplings

Until now we assume that we introduce the periodic all the matter fields through the CFT in the outside on which we do not have gravity. Here we introduce the periodic boundary condition only for $c_j < c$ matter fields and we introduce a boundary condition in the no gravity region for $c_s = c - c_j$ fields that keeps a half of conformal symmetry [70]. These introductions of boundaries cut the direct coupling [71, 72]. In the SYK model, we can still have similar traversable wormholes because of the all to all coupling nature [73] and we expect the same thing for JT gravity. We assume that the boundary conditions are the same in both side of the boundary so that the CFT achieve the minimal energy on a finite strip. We deal with $c_j$ as a continuous parameter. This is valid when the CFTs are collections of free CFTs. In a model with holographic matters, we consider the model in section 4.3 and then we can treat $c_j$ as a continuous parameter. For simplicity, we put these boundary conditions at $x = d/2$ on the strip $[0, d]$. In this case, the ground state energy for $c_j$ fields on a flat finite strip is

$$T_{++} = T_{--} = -\frac{c_j}{48\pi} \frac{1}{(1 + a)^2}. \tag{4.49}$$

Therefore, if we couple the matter to AdS$_2$ for $x \in [\pi a/2, \pi(1 + a/2)]$ region, the energy in the AdS$_2$ region is

$$\langle T^{\text{AdS}_2} \rangle_{y^+, y^+} = c_s \frac{1}{48\pi} - \frac{c_j}{48\pi (1 + a)^2} = c_s \frac{a(a + 2)}{48\pi (a + 1)^2}. \tag{4.50}$$

Therefore, if we decouple the fields, we only introduce the positive energy in the AdS$_2$ region. The negative energy we introduce is

$$\langle T^{\text{AdS}_2} \rangle_{y^+, y^+} = -\frac{c_j}{48\pi} \frac{(1 - a)(3 + a)}{(a + 1)^2} + \frac{c_s}{48\pi} \frac{a(a + 2)}{(a + 1)^2}$$

$$= -\frac{1}{48\pi} \frac{c_j(1 - a)(3 + a) - c_s a(a + 2)}{(a + 1)^2}$$

$$= -\frac{1}{48\pi} \frac{-ca^2 - 2ca + 3cj}{(a + 1)^2} = -\frac{c_{\text{eff}} a}{16\pi}, \tag{4.51}$$

10 Here $j$ of $c_j$ stands for ‘join’, which means that we are connecting in the no gravity region. $s$ of $c_s$ stands for ‘split’, which means that we split the direct coupling by introducing boundaries.
where \( c_{\text{eff}}(a) = \frac{-a^{3} - 2ac + 3c_{j}}{3a + 1} \) plays a role of ‘effective central charges’ for making traversable wormholes. Therefore the effective central charge decreases and the wormhole length becomes longer and longer. In particular for \( d \ll \ell \), \( c_{s} \) do not give any contribution and the wormhole length is determined as 
\[
\ell = \frac{2 \hat{\phi}_{t}}{c_{\text{eff}}(a) G_{N}} = \frac{2 \hat{\phi}_{t}}{G_{N} 3c_{j} - 2ca - ca^{2}}
\]  
(4.52)

and the length of the bath (region without gravity) in terms of parameter \( a \) is
\[
d = \pi \ell a = \frac{2 \hat{\phi}_{t}}{G_{N} 3c_{j} - 2ca - ca^{2}}. \quad (4.53)
\]

This fix the parameter \( a \) and the wormhole length \( \ell \) in terms of \( d \). For \( \frac{G_{N}}{2c_{s}} d \gg 1 \), the parameter \( a \) becomes
\[
a = \frac{d}{\pi \ell} \approx \sqrt{1 + \frac{3c_{j}}{c} - 1}. \quad (4.54)
\]

Since this is smaller than 1, the wormhole length \( \pi \ell \) is larger than \( d \), which is expected from the achronal ANEC. We should also notice that the traversable wormhole solution does not exist when \( c_{j} = c \) (or equivalently \( c_{j} = 0 \)) because we do not introduce negative null energy in the bulk any more and it is not possible to construct the traversable wormhole solutions. Actually, even when \( c_{a} \) is the order one and not the order of \( 1/G_{N}^{2} \), the wormhole length \( \ell \) becomes very small and we cannot ignore the quantum gravity effect so that we cannot trust the classical treatment of the wormholes.

Entanglement entropy of the no gravity region is also calculated in the same manner. Here we assume that CFTs are given by holographic CFTs, and BCFTs are modeled by the AdS/BCFT model [74, 75] with tensionless Cardy brane. Here we consider entanglement entropy for the interval \([0 + d, d - 0]\), which means entanglement between CFT on no gravity region and quantum mechanics QM_{L} and QM_{R} in the full quantum mechanical description. In the configuration without islands, entanglement entropy of quantum fields are given by\(^{11}\)
\[
S_{\text{no-island}} = \frac{c_{j}}{3} \log \left( \frac{\pi \ell + d}{\epsilon_{B} \pi} \sin \frac{\pi d}{\pi \ell + d} \right) + \frac{c_{s}}{3} \log \left( \frac{2(\pi \ell + d)}{\epsilon_{B} \pi} \sin \frac{\pi d}{(\pi \ell + d)} \right) + 2 \log g_{s}
\]
\[\approx \frac{c_{j}}{3} \log \frac{d}{\epsilon_{B}} + \frac{c_{s}}{3} \log \frac{2d}{\epsilon_{B}} + 2 \log g_{s},
\]
\[= \frac{c}{3} \log \frac{d}{\epsilon_{B}} + \frac{c_{s}}{3} \log 2 + 2 \log g_{s}, \quad \text{(for \( \pi \ell \gg d \))}. \quad (4.56)
\]
\(^{11}\) Entanglement entropy of the region \( A = [0, l] \) on a strip \([0, l] \) is given by [76]
\[
S_{A} = \frac{c}{6} \log \left( \frac{2L}{t} \sin \frac{l}{t} \right) + \log g + c_{l},
\]  
(4.55)

where \( \log g \) is the boundary entropy [77] and \( c_{l} \) is the non universal constant without the presence of the boundaries.
Figure 16. We describe the three descriptions for the quantum extremal surface which does not contain islands. (Left) The JT gravity plus matter CFT description. We are computing entanglement entropy of matter fields. (Middle) We use the 3D gravity description for matter CFT assuming that the matters are holographic. We are computing the lengths of the RT/HRT surfaces. The shaded region is the entanglement wedge, which is surrounded by the entangling surface and the RT/HRT surfaces. (Right) The full quantum mechanical description. We calculate entanglement entropy of CFT using the large $N$ expansion. This is a candidate saddle of the full answer.

On the other hand, in the island configuration the generalized entropy functional is

$$S(\sigma_a, \sigma_b) = 2S_0 + 2\pi \phi(\sigma_a) + 2\pi \phi(\sigma_b) + \frac{c_j}{6} \log \left( \frac{\pi \ell + d}{\pi} \right) \frac{\sin^2 \frac{\sigma_a}{\pi \ell + d}}{\epsilon_B \ell \sin \frac{\pi}{4}} + \frac{c_j}{6} \log \left( \frac{2(\pi \ell + d)}{\pi} \right) \frac{\sin^2 \frac{\sigma_b}{2(\pi \ell + d)}}{\epsilon_B \ell \sin \frac{\pi}{4}} + \frac{c_s}{6} \log \left( \frac{2(\pi \ell + d)}{\pi} \right) \frac{\sin^2 \frac{\sigma_a}{2(\pi \ell + d)}}{\epsilon_B \ell \sin \frac{\pi}{4}}.$$  

Assuming $\pi \ell \gg d, \sigma_a, \sigma_b$, this entropy functional reduces to the extremal black hole one\(^{12}\)

$$S(\sigma_a, \sigma_b) \approx 2S_0 + 2\pi \phi(\sigma_a) + 2\pi \phi(\sigma_b) + \frac{c_j}{6} \log \frac{\sigma_a}{\epsilon_B} + \frac{c_j}{6} \log \frac{\sigma_b}{\epsilon_B} + \frac{c_s}{6} \log \frac{\sigma_a}{\epsilon_B} + \frac{c_s}{6} \log \frac{\sigma_b}{\epsilon_B} = 2S_0 + 2\pi \phi(\sigma_a) + 2\pi \phi(\sigma_b) + \frac{c}{3} \log \frac{\sigma_a}{\epsilon_B} + \frac{c}{3} \log \frac{\sigma_b}{\epsilon_B}. \quad (4.58)$$

So the entanglement entropy only get the boundary entropy contribution in the no island configuration, but except that the structure is basically the same with the former case without boundaries (figures 16–18).

\(^{12}\)Without the assumption that the BCFT is replaced by the AdS/BCFT model, we will get entanglement entropy for an interval with the presence of the boundary, which is not universal.
5. Four coupled JT gravity

In this section, we consider the four coupled JT gravity coupled to conformal matters. The setup is the generalization of the two coupled JT gravity and similar to the four coupled SYK models. First, we assume that the matter fields consists of at least two decoupled CFTs: CFT_{LR} with central charge \( c_{LR} \) and CFT_{12} with central charge \( c_{12} \). In other words, the JT gravity + matter theory is given by the action

\[
S = I_{grav}[g^{(2)}_{ij}, \phi] + I_{CFT_{LR}}[g^{(2)}_{ij}, \chi_{LR}] + I_{CFT_{12}}[g^{(2)}_{ij}, \chi_{12}],
\]

(5.1)

where \( I_{grav}[g^{(2)}_{ij}, \phi] \) is the JT gravity (2.11), the \( \chi_{LR} \) abstractly represents the fields for CFT_{LR} and the \( \chi_{12} \) abstractly represents the fields for CFT_{12}. Then, we also prepare two copies of CFT_{LR} and CFT_{12} on lengths \( d_{LR} \) and \( d_{12} \) without gravity. We couple JT + matter system through these CFTs on no gravity region in a way that is shown in figure 19.

5.1. CFT on two traversable wormholes

To construct the wormhole solutions in four coupled JT gravities, first we consider CFT on a cylinder of a length \( L = 2\pi(1 + b) \). Then, the expectation value of the stress tensor is

\[
\langle T^{(1)}_{tt} \rangle = \langle T^{(1)}_{\sigma \sigma} \rangle = \frac{1}{24\pi} \frac{1}{(1 + b)^2}.
\]

(5.2)

In terms of null energy, they become

\[
\langle T^{(1)}_{y^+ y^+} \rangle = \langle T^{(1)}_{y^- y^-} \rangle = -\frac{1}{48\pi} \frac{1}{(1 + b)^2}.
\]

(5.3)

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Figure 18. The configuration for CFT with the central charge $c_s$ with boundary conditions. We suppress the $c_j$ CFT without boundaries that are treated in the same manner with the former case. (Left) The JT gravity + matter CFT description coupled to BCFT in the no gravity region. (Middle) We take the 3D gravity description assuming that the matter CFT is holographic. The 3D spacetime now has a Cardy brane, which is dual of boundary condition in AdS/BCFT setup. (Right) The full quantum mechanical description. We couple BCFT to quantum mechanics. Again note that we suppress the $c_j$ CFT in this picture, which directly connects two quantum mechanics.

Figure 19. The descriptions of the theory we consider in the four coupled JT gravities.

We couple the region $\sigma \in [0, \pi]$ to AdS$_2$ metric. Furthermore, we put another JT gravity on the region $\sigma \in [\pi(1 + b), \pi(2 + b)]$. Therefore, this CFT is sitting on two different wormholes! Then, the null energy in each AdS$_2$ region is

$$\langle T^\text{AdS}_2 \rangle = \langle T^\text{AdS}_2 \rangle = \frac{c}{48\pi} + \frac{c}{48\pi} \left( \frac{1}{(1 + b)^2} \right) = \frac{c}{48\pi} \left( \frac{b(b + 2)}{(1 + b)^2} \right).$$

First of all, this coupling between two side actually do not introduce the negative energy in the bulk. Rather, it is introducing a positive energy in the 2D bulk. This contribution takes the same form with (4.50), in which we introduce the boundary conditions for CFT in the no gravity region.
Let us understand this when this CFT on two traversable wormholes is holographic. One interesting thing is that now two traversable wormholes are directly connected through 3D spacetime which is dual to the entangled state of bulk matter. From 3D gravity point of view, this configuration is actually a traversable wormhole in 3D since two asymptotic boundaries are connected in the interior as depicted in the figure 20. The positive energy is interpreted as a cost to make this 3D wormhole traversable. This positive energy in the bulk makes wormholes longer. This is also similar to the SYK interaction $\mu_{12}(H_{12}^{LL} + H_{12}^{RR})$ in section 3.3, which causes the decrease of the energy gap in L–R wormhole phase rather than increasing the energy gap.

Taking $b \to \infty$, we obtain the CFT on infinite line with coupling to AdS$_2$ on the region $\sigma \in [0, \pi]$:

$$\langle T_{\text{AdS}_2}^{\pm} \rangle = \frac{c}{48\pi}.$$  \hfill (5.5)

This induces a positive null energy on AdS$_2$ region.

The length in flat space region on which CFT lives is given by

$$d_{12} = \pi \ell b.$$  \hfill (5.6)

When we take $d_{12} = 0$, then the 3D gravity geometry looks like that in wedge holography [78].

5.2. **Wormhole solutions in four coupled JT gravities**

5.2.1. **Variational ansatz.** Here we first consider the variational approximation. We approximate the geometry by the $t = 0$ slice of the eternal black holes (= thermo field double states). The wormhole length is $\ell = \frac{1}{2\pi T_H}$. The energy par a black hole is

$$M_{\text{BH}} = \frac{\pi \vartheta_T}{4G_N} T_H^3 = \frac{\vartheta_T}{16\pi G_N} \frac{1}{\ell^2}.$$  \hfill (5.7)
The quantum matter contribution is

\[ E_{\text{mat}} = 2\frac{c_{\text{LR}} + c_{12}}{24\ell} - 2\frac{c_{\text{LR}}\pi}{6(\pi\ell + d_{\text{LR}})} - \frac{c_{12}\pi}{6 \cdot 2(\pi\ell + d_{12})}. \]  

(5.8)

The total variational energy is

\[ E(\ell) = 4M_{\text{BH}} + E_{\text{mat}} = \frac{\dot{\phi}\dot{r}}{4\pi G_N} \frac{1}{\ell^3} + \frac{c_{\text{LR}} + c_{12}}{12\ell} - \frac{c_{\text{LR}}\pi}{3(\pi\ell + d_{\text{LR}})} - \frac{c_{12}\pi}{12(\pi\ell + d_{12})}. \]  

(5.9)

We minimize this energy

\[ \frac{\partial E}{\partial \ell} = -\frac{\dot{\phi}\dot{r}}{2\pi G_N} \frac{1}{\ell^3} - \frac{c_{\text{LR}} + c_{12}}{12\ell^2} + \frac{c_{\text{LR}}\pi^2}{3(\pi\ell + d_{\text{LR}})^2} + \frac{c_{12}\pi^2}{12(\pi\ell + d_{12})^2} = 0. \]  

(5.10)

Before we analyze the length of the wormhole, we first consider some special cases. First we consider the case of \( d_{12} = 0 \). In this case, the energy from the \( c_{12} \) CFT completely cancels. Therefore, the length of the wormhole becomes the same with the case without \( c_{12} \) CFT.

Second, we consider the case of \( d_{\text{LR}} = 0 \) and \( d_{12} \rightarrow \infty^{13} \). In this case, the matter energy becomes

\[ E_{\text{mat}} = -\frac{3c_{\text{LR}} - c_{12}}{12\ell}. \]  

(5.11)

In this regime, the wormhole length becomes that of single traversable wormhole with effective central charge \( c_{\text{LR}} - \frac{c_{12}}{3} \) for \( c_{12} < 3c_{\text{LR}} \) cases. On the other hand when \( c_{12} > 3c_{\text{LR}} \) the matter energy no more introduce the negative energy in AdS \(_2\) region and we cannot make traversable wormholes. This means that we cannot take \( d_{12} \) to be large while keeping the length \( \ell \) to be finite. There is a cost to entangle the \( c_{12} \) fields through bulk and there is a limit to support such entanglement.

### 5.2.2. Solving the JT gravity equation of motion directly.

We solve the equation of motion of the JT gravity (2.15) directly. Finally we found that the variational method gives the exact answer. We assume the form

\[ ds^2 = -\frac{dt^2}{f^2} + \frac{d\sigma^2}{f^2 \sin^2 \frac{\sigma}{f}}, \]

\[ \phi(\sigma) = \frac{\dot{\phi}}{\pi\ell} \left[ \frac{\pi\ell}{\ell} - \frac{2\sigma}{\ell \tan \frac{\sigma}{f}} + 1 \right] + y. \]  

(5.12)

Then, first we obtain

\[ y = \frac{c_{\text{LR}} + c_{12}}{24\pi}. \]  

(5.13)

The null energy becomes

\[ \langle T^{x+}x_+^{\text{AdS}_2} \rangle = \frac{c_{\text{LR}}}{48\pi} - \frac{c_{\text{LR}}}{12\pi(1 + a)^2} + \frac{c_{12}}{48\pi} - \frac{c_{12}}{48\pi(1 + b)^2} = -\frac{c_{\text{eff}}(a, b)}{16\pi}. \]  

(5.14)

\(^{13}\)Here we mean \( d_{12} \gg 1 \). In this regime, the solution with \( \pi \ell \approx d_{12} \) may not be trusted since the quantum gravity effect becomes important.
here we defined the effective central charge through

\[ c_{\text{eff}}(a,b) = \frac{c_{\text{LR}}}{3} \left( \frac{4}{(1+a)^2} - 1 \right) - \frac{c_{12}}{3} \left( 1 - \frac{1}{(1+b)^2} \right). \]  
\[ (5.15) \]

Then, we can use the same equation (4.52) in a single traversable wormhole case. The (inverse) length is determined as

\[ \ell' = \frac{c_{\text{eff}}(a,b) \pi G_N}{2 \phi_t}. \]  
\[ (5.16) \]

Using \( \ell' = \ell^{-1} \), the definition of \( c_{\text{eff}}(a,b) \), \( d_{\text{LR}} = \pi \ell a \) and \( d_{12} = \pi \ell b \), we obtain

\[ \hat{\phi}_t = \frac{c_{\text{LR}} + c_{12}}{8 \pi G_N \ell^3} = \frac{c_{\text{LR}} \pi^2}{48 \ell^2} - \frac{c_{12} \pi^2}{12 (\pi \ell + d_{\text{LR}})^2} = \frac{c_{12} \pi^2}{12 (\pi \ell + d_{12})^2}. \]  
\[ (5.17) \]

This is the same equation with (5.10) that we obtained from the variational approximation.

5.2.3. Entanglement entropy of wormhole solutions. We consider entanglement entropy in the wormhole solutions in four coupled JT gravities. Because of symmetry, we can focus on the L–R wormhole solution without loss of generality. We first consider entanglement entropy between L and R. Similarly to the single traversable wormhole case, there only be island configurations because only specifying the entangling surface on the region without gravity does not divide the system to two parts. The generalized entropy functional becomes

\[ S_{\text{gen}}(\sigma_1, \sigma_2) = 2 S_0 + 2 \pi \phi(\sigma_1) + 2 \pi \phi(\sigma_2) + \frac{c_{\text{LR}}}{6} \log \left( \frac{\pi \ell + d_{\text{LR}}}{\pi} \right) \left( \frac{2 \sin^2 \frac{\sigma_1 + \sigma_2}{2}}{\epsilon_B \ell \sin \frac{\sigma_1}{2} \sin \frac{\sigma_2}{2}} \right) \]
\[ + \frac{c_{\text{LR}}}{6} \log \left( \frac{\pi \ell + d_{\text{LR}}}{\pi} \right) \left( \frac{2 \sin^2 \frac{\sigma_1 + \sigma_2 + d_{\text{LR}}}{2}}{\epsilon_B \ell \sin \frac{\sigma_1}{2} \sin \frac{\sigma_2}{2}} \right) \]
\[ + \frac{c_{12}}{6} \log \left( \frac{2(\pi \ell + d_{12})}{\pi} \right) \left( \frac{2 \sin^2 \frac{\sigma_1 + \sigma_2 + d_{12}}{2}}{\epsilon_B \ell \sin \frac{\sigma_1}{2} \sin \frac{\sigma_2}{2}} \right). \]  
\[ (5.18) \]

Entanglement entropy is obtained by taking the extremal of the generalized entropy. Generically the saddle point of (5.18) can be found numerically. For a simple case of \( b_1 = b_2 = d_{\text{LR}}/2 \), the extremal point is given by \( \sigma_1 = \sigma_2 = \frac{\pi}{2} \). In this case, entanglement entropy between L and R system is

\[ S_{\text{LR}} = 2 S_0 + 4 \frac{\hat{\phi}_t}{\ell} + \frac{c_{\text{LR}}}{3} \log \left( \frac{\ell}{\epsilon_B} \left( \frac{\pi \ell + d_{\text{LR}}}{\pi \ell} \right)^2 \right) + \frac{c_{12}}{3} \log \left( \frac{2(\pi \ell + d_{12})}{\pi \ell} \right), \]  
\[ (5.19) \]

where \( b_1, b_2 \) are the distance from the AdS_2 asymptotic boundary in no gravity region and \( \sigma_1, \sigma_2 \) are the positions of quantum extremal surfaces.

Next we consider entanglement entropy between 1 and 2 system. In this case, we can consider the configuration without islands. Entanglement entropy is simply given by the matter entanglement entropy

\[ S_{12} = \frac{c_{12}}{3} \log \left( \frac{2(\pi \ell + d_{12})}{\pi \epsilon_B} \sin \frac{\pi (b_L + b_R + \pi \ell)}{2(\pi \ell + d_{12})} \right). \]  
\[ (5.20) \]
Therefore, in L–R wormhole solution entanglement entropy between L and R is much larger than that between 1 and 2 because of $2S_0$ contribution. We can think of entanglement entropy as an order parameter, which is an analog of the spin operators in the four coupled SYK models.

5.3. Symmetric solution

We can find a solution which becomes symmetric under the change of 1–2 and L–R direction when $c_{LR} = c_{12}$. We call this solution symmetric solution. The solution is simply given by the four extremal black holes

$$\phi(z) = \phi_0 + \frac{\bar{\phi}_r}{z}$$

and all the CFTs are in the vacuum state on infinite lines. There energy stress tensor vanishes $\langle T^{\mu\nu}_{\text{mat}} \rangle = 0$ everywhere, so the set of the dilaton profile and the quantum stress tensor satisfies the equation of motion. This solution exists for any $c_{LR}$ and $c_{12}$ they are not symmetric under the exchange of 1–2 and L–R direction when $c_{LR} \neq c_{12}$. The energy of this solution is 0, which is bigger than the wormhole states.

We consider entanglement entropy in this solution. In this solution, the island always dominate since the entropy in configurations without islands diverges. The entropy functional is

$$S(\sigma_1, \sigma_2) = 2S_0 + \frac{2\pi\bar{\phi}_r}{\sigma_1} + \frac{2\pi\bar{\phi}_r}{\sigma_2} + \frac{c_{LR}}{6} \log \left( \frac{(\sigma_1 + b)^2}{\epsilon_B \sigma_1} \right) + \frac{c_{12}}{6} \log \left( \frac{(d_{12} + \sigma_1 + \sigma_2)^2}{\sigma_1 \sigma_2} \right).$$

Assuming $d_{12}$ is small ($d_{12} \ll \frac{\bar{\phi}_r}{c}$) and also $\sigma_1 = \sigma_2$, the entropy functional becomes

$$S(\sigma, \sigma) = 2 \left[ S_0 + \frac{4\pi\bar{\phi}_r}{\sigma} + \frac{c_{LR}}{6} \log \left( \frac{(\sigma + b)^2}{\epsilon_B \sigma} \right) \right] + \frac{c_{12}}{3} \log 2.$$
Note that the matter contribution in the L–R coupling only gives a constant in this parameter regime. Thus, we obtain the same solution with the zero temperature case for small $b$

$$\sigma_s = \frac{12\pi \phi}{c_{LR}}. \quad (5.24)$$

Entanglement entropy is given by

$$S(\sigma_s, \sigma_s) = 2S_0 + \frac{c_{LR}}{3} + \frac{c_{LR}}{3} \log \frac{12\pi \phi}{\epsilon \phi_{CLR}} + \frac{c_{12}}{3} \log 2. \quad (5.25)$$

In particular in doubly holographic models, a portion of 3D spacetime is contained in the entanglement wedge of the right side. This is a 3D island in the state. In a doubly holographic setup, the JT branes are connected to other branes through the 3D direction, though the branes themselves are disconnected. This is still enough to allow the state to have order $2S_0$ entropy for both L–R direction and 1–2 direction.

We consider entanglement entropy in this solution. In this solution, the island always dominates since the entropy in configurations without islands diverges. The entropy functional is (figure 21)

$$S(\sigma_L, \sigma_R) = 2S_0 + \frac{2\pi \phi}{\sigma_L} + \frac{2\pi \phi}{\sigma_R} + \frac{c_{12}}{6} \log \frac{(\sigma_L + b_L)^2}{\epsilon \sigma_L} + \frac{c_{12}}{6} \log \frac{(\sigma_R + b_R)^2}{\epsilon \sigma_R} + \frac{c_{LR}}{6} \log \frac{(d_{LR} + \sigma_L + \sigma_R)^2}{\sigma_L \sigma_R}. \quad (5.26)$$

where $b_L, b_R$ are the distances from the points on which asymptotic AdS$_2$ boundaries are located. $\sigma_L, \sigma_R$ are the positions of quantum extremal surfaces. Assuming $d_{LR}$ is small ($d_{LR} \ll \frac{\phi}{c}$), $b_L = b_R \equiv b$ and $\sigma_L = \sigma_R \equiv \sigma$, the entropy functional becomes

$$S(\sigma, \sigma) = 2 \left[ S_0 + \frac{4\pi \phi}{\sigma} + \frac{c_{LR}}{6} \log \frac{(\sigma + b)^2}{\epsilon \sigma} \right] + \frac{c_{12}}{3} \log 2. \quad (5.27)$$

Note that the matter contribution in the L–R coupling only gives a constant in this parameter regime. Thus, we obtain the same solution with the zero temperature case for small $b$

$$\sigma_s = \frac{12\pi \phi}{c_{LR}}. \quad (5.28)$$

Entanglement entropy is given by

$$S(\sigma_s, \sigma_s) = 2S_0 + \frac{c_{LR}}{3} + \frac{c_{LR}}{3} \log \frac{12\pi \phi}{\epsilon \phi_{CLR}} + \frac{c_{12}}{3} \log 2. \quad (5.29)$$

In particular in doubly holographic models, a portion of 3D spacetime is contained in the entanglement wedge of the right side. This is a 3D island in this state. In a doubly holographic setup, the JT branes are directly connected to other branes through the 3D direction, though the branes themselves are disconnected. This is still enough to allow the state to have order $2S_0$ entropy for both L–R direction and 1–2 direction.
Figure 22. The configurations for RT surface when the matters are holographic. (Left) The symmetric solution. The dashed circle at the center represent the ‘infinity’ of this geometry. (Right) Island phase in the L–R wormhole solution.

Let us compare with the possible island phase in entanglement entropy between system 1 and 2. We can consider the island phase in figure 22. This involves the computation of two interval entanglement entropy, which is not universal. For simplicity, we consider the case that the bulk matter fields are holographic. Then, the generalized entropy functional is given by

$$S(\sigma_L, \sigma_R) = 2S_0 + \frac{\phi(\sigma_L)}{4G_N} + \frac{\phi(\sigma_R)}{4G_N}$$

$$+ \frac{c_{LR}}{6} \log \left( \frac{\pi \ell + d_{LR}}{\ell^2 \sin \frac{\sigma_L}{2} \sin \frac{\sigma_R}{2} \sin \frac{\sigma_{LR}}{2} \sin \frac{\sigma_{LR} + \phi_{LR}}{2}} \right)$$

$$+ \frac{c_{12}}{6} \log \left( \frac{2(\pi \ell + d_{12})}{\ell^2 \sin \frac{\sigma_L}{2} \sin \frac{\sigma_R}{2} \sin \frac{\sigma_{12}}{2} \sin \frac{\sigma_{12} + \phi_{12}}{2}} \right).$$

(5.30)

For $\ell \gg d_{12}, d_{LR}$ and assuming $\sigma_L, \sigma_R \ll \ell$, the generalized entropy functional (5.30) reduces to that for symmetric solution (5.26). Therefore, entanglement entropy for symmetric solution is basically the same with the entanglement entropy for long wormhole.

5.4. $\mathbb{Z}_2$ symmetry breaking at special point

Suppose we have the same fields in L–R and 1–2 direction i.e. the same matter contents and in particular the same central charge $c_{LR} = c_{12}$ and the same distance $d_{LR} = d_{12}$. Then the system is $\mathbb{Z}_2$ invariant, which is similar to the $\mathbb{Z}_2$ symmetry (3.16) in four coupled SYK model, because the asymptotic boundary condition in 2D/3D gravity prescription is symmetric. The configurations we found are shown in figure 23 in 2D gravity description and in figure 24 when the matter fields are holographic. We considered the wormhole solution which connect left and right systems in section 5.2. We can exchange the role of L–R direction and 1–2 direction and obtain the 1–2 wormhole solution. Because the energy in the wormhole solution is negative whereas the energy in symmetric solution is 0, the wormhole solutions are dominant saddles. Since each wormhole state is not invariant under $\mathbb{Z}_2$ symmetry that exchange L–R ↔ 1–2 direction, this symmetry is broken by each wormhole configuration.

We can use entanglement entropy as an order parameter. The difference of entanglement entropy in L–R wormhole phase is

$$S_{LR} - S_{12} = 2S_0 + 4\frac{\phi_1}{\ell} + \frac{2c}{3} \log \frac{\pi \ell + d}{\pi \ell}. $$

(5.31)
On the other hand, entanglement entropy in the symmetric solution is

$$S_{LR} - S_{12} = 0. \quad (5.32)$$

Therefore, the difference of entanglement entropy quantitatively characterizes the pattern of symmetry breaking.

### 5.5. Partial coupling in four coupled JT gravities and phase transition

Here we study what happens when we decrease the L–R coupling starting from the L–R wormhole saddle. To reduce the coupling between L and R, we introduce boundaries for L–R CFTs as we did in section 4.4. Here we separate the central charge as $c_{LR} = c_{LR}^{1} + c_{LR}^{2}$ and introduce the boundary conditions for $c_{LR}^{1}$ CFTs whereas we continue to impose the transparent boundary conditions for $c_{LR}^{2}$ CFT in the no gravity region. We imagine that the boundary conditions are described by BCFT and moreover the boundary conditions are the same (or more precisely the CPT conjugate) between 1 and 2 systems.
We already calculated these BCFT contribution in (4.50). Using this, the energy which we should minimize in L–R wormhole phase is

$$E(\ell) = \frac{\bar{\phi}}{4\pi G_N \ell^2} + \frac{c_{LR} + c_{12}}{12\ell} - \frac{c_{LR}^{'}}{3(\pi \ell + d_{LR})} - \frac{c_{12}^{'}}{12(\pi \ell + d_{LR})} - \frac{c_{LR}}{12(\pi \ell + d_{LR})}. \quad (5.33)$$

Note that when $d_{12} = d_{LR}$, the split $c_{LR}^'$ CFT gives the same contribution with $c_{12}^'$ CFT that sits on two traversable wormholes as we observed in (5.4). In this case, introducing boundaries have the same effect with replacing $(c_{LR}, c_{12}) \rightarrow (c_{LR}^{'}, c_{12} + c_{LR}^{'})$.

On the other hand, in 1–2 wormhole phase the energy which we should minimize is

$$E(\ell) = \frac{\bar{\phi}}{4\pi G_N \ell^2} + \frac{c_{LR} + c_{12}}{12\ell} - \frac{c_{12}^{'}}{3(\pi \ell + d_{12})} - \frac{c_{LR}^{'}}{12(\pi \ell + d_{12})} - \frac{c_{12}}{12(\pi \ell + d_{12})}. \quad (5.34)$$

Therefore, in this phase the energy function is not changed.

We can also have symmetric solutions. This is because BCFT on a half line with single boundary has vanishing stress tensor $\langle T_{\mu\nu} \rangle = 0$. This solution is still given by the dilaton profile (5.21). The ADM energy for symmetric solutions is 0.

For generic parameters $d_{12}, d_{LR}$, we show a plot in figure 25. The phase diagram is very similar with that in the four coupled SYK in figure 4, though the energy for 1–2 wormhole and the symmetric solution do not meet.

We can also have symmetric solutions. This is because BCFT on a half line with single boundary has vanishing stress tensor $\langle T_{\mu\nu} \rangle = 0$. This solution is still given by the dilaton profile (5.21). The ADM energy for symmetric solutions is 0.

For generic parameters $d_{12}, d_{LR}$, we show a plot in figure 25. The phase diagram is very similar with that in the four coupled SYK in figure 4, though the energy for 1–2 wormhole and the symmetric solution do not meet.
5.6. Changing central charge $c_{LR}$ and $c_{12}$ keeping $c_{LR} + c_{12}$ fixed

Here we consider to change the central charge $c_{LR}$ and $c_{12}$ while we are keeping $c_{LR} + c_{12} \equiv c_{\text{tot}}$ to be fixed. What we are imagining is to cut some part of $c_{12}$ CFT and reconnect in L–R direction. We can also do the opposite i.e. cutting some part of $c_{LR}$ and reconnect in 12 direction. We can gradually change these central charges assuming that $c_{LR}$ consists from decoupled many CFTs like free fermions or several holographic matters introduced in section 4.3 so that we can gradually change these central charges.

Let us consider the simple case of $d_{LR} = d_{12} \ll \ell$. The energy in the L–R wormhole saddle is

$$E = 2(M_L + M_R) = -2\frac{\phi_t \ell^2}{16\pi G_N} = -4\frac{\phi_t}{16\pi G_N} \left( \frac{c_{LR} \pi G_N}{2\phi_t} \right)^2 = -\frac{c_{LR}^2 \pi G_N}{16\phi_t}. \quad (5.36)$$

The energy in the 1–2 wormhole saddle is

$$E = 2(M_1 + M_2) = -2\frac{\phi_t \ell^2}{16\pi G_N} = -\frac{c_{12}^2 \pi G_N}{16\phi_t} = -\frac{(c_{\text{tot}} - c_{LR})^2 \pi G_N}{16\phi_t}. \quad (5.37)$$

Therefore for $d_{LR}, d_{12} \ll \ell$ regime, the energy as a function of $c_{LR}$ on each solutions is

$$E = \begin{cases} 
-\frac{c_{LR}^2 \pi G_N}{16\phi_t} & \text{for L–R wormhole solution} \\
-\frac{(c_{\text{tot}} - c_{LR})^2 \pi G_N}{16\phi_t} & \text{for 1–2 wormhole solution} \\
0 & \text{for symmetric solution.} 
\end{cases} \quad (5.38)$$

When we increase the central charge $c_{12}$ from 0 to some value greater than $c_{LR}$, there is a phase transition. The symmetric saddle have zero energy and the wormhole saddles always have smaller energy.

For generic parameters $d_{LR}, d_{12}$, we plot an example in figure 26. The plot is very similar to that in the four coupled SYK model in figure 8.

5.7. JT gravity with both of holographic matters and free CFTs

Since our construction of the solutions only depends on the central charges of matters, we can consider several pattern in our models if we do not care about the $Z_2$ symmetry at the $c_{LR} = c_{12}$ point. In particular we can consider (figure 27)

(a) LR CFT: free CFT, 12 CFT: free CFT
(b) LR CFT: holographic, 12 CFT: holographic
(c) LR CFT: holographic, 12 CFT: free CFT

The case (a) is basically the two dimensional picture of our setup whereas the case (b) is the three dimensional picture in which both of L–R and 1–2 CFTs have holographic dual. We can also consider where the L–R CFT has holographic dual whereas the 1–2 CFT is a bunch of free fields. One can interpret that the free CFTs are end-of-the-world brane degrees of freedoms [23, 41, 79]. From the perspective of CFTs on flat space, JT gravity + Holographic matter + free CFT theory gives an interface between holographic CFTs and free CFTs. If these interfaces
Figure 26. The plot of the energy in L–R wormhole, in 1–2 wormhole and in the symmetric solution. The parameters are taken to be \( \phi_8 = 1, \, d_{LR} = 1, \, d_{12} = 1, \) and \( c_{tot} = 5. \)

Figure 27. Configuration of the four coupled JT gravities with L–R holographic matters and 1–2 free CFTs. (Left) Traversable wormholes in L–R direction. The L–R CFT is capped off. In the middle there are ‘nothing’ but the matter fields on the branes are entangled. (Right) Traversable wormholes in 1–2 direction. In this case L–R CFT makes a 3D traversable wormhole.

exist, they give a way to embed the holographic states into the free CFT Hilbert spaces. Actually the four coupled model serves a way for such an embedding as we will see later.

Note that the JT gravity coupled to order \( 1/G_N \) fermions can arise from the 4D magnetically charged near extremal black holes with a single 4D fermion \([9]\). Similarly, the JT gravity coupled to ‘order \( 1/G_N \)’ number of holographic CFTs can arise from the 4D black holes \([51]\) in the Randall–Sundrum II model \([80]\). A model with a dark photon with couplings to both of 4D fermions and holographic CFT will give rise to a JT gravity both with 2D free fermions and 2D holographic CFTs. In this case, originally we have 4D gravity + 4D fermions + 4D holographic CFT with UV cutoff, which is equivalent to the brane world model where the 4D fermions and 4D gravity live on the Planck brane of 5D gravity and in particular the 4D fermions are EOW brane degrees of freedom. From these 4D theory we obtain 2D nearly AdS_2 gravity + 2D free fermions + 2D holographic CFTs near horizons in near extremal magnetically charged black holes.
5.8. Wick rotation and bra-ket wormholes

5.8.1. State preparation interpretation of the four coupled JT gravities. It is interesting to consider the Wick rotation in the fully quantum mechanical description. Here we first work in the Euclidean signature so the Wick rotation do nothing but we change the interpretation of the Euclidean geometry by exchanging the role of the space and the Euclidean time.

Since these configurations are time reflection symmetric, these geometries can be understood as Euclidean path integrals for state preparations, see figure 28. For example, we can take the time slice in the middle of the 1–2 CFT path integrals. Then, these Euclidean path integrals give states \(|\Psi\rangle\) for the copies of Hilbert space of CFTs with central charge \(c_{12}\):

\[|\Psi_{12}\rangle \in \mathcal{H}_{\text{CFT}12} \otimes \mathcal{H}_{\text{CFT}12}.\]

The CFT LR and the domain walls play the role to characterize the entanglement structure between two copies, but their Hilbert spaces do not appear finally in this interpretation. In this interpretation, we can think that the label 1 is for ket and the label 2 for bra.

Now, we consider the wormhole configurations in this state preparation interpretation. The wormholes are now connecting the ‘bra’ and ‘ket’ path integral. Therefore, the wormholes are interpreted as the bra-ket wormhole [52], which is a kind of Euclidean wormhole.

As we observed, we have several saddles so we take the saddle point that minimizes the partition function. The configurations are given by exchanging the role of Euclidean time and the spacial direction in wormhole/symmetric saddles: the 2D gravity prescriptions are given by figure 23 and the 3D gravity prescriptions are given by figure 24. The evaluation is completely the same of the partition function, which is given by \(Z \approx e^{-LE_0}\) where \(E_0\) is the energy evaluated in the wormhole solutions and \(L\) is the length of the spacial direction\(^{14}\), which we take to be very large. Then, the dominant saddle point is given by the wormhole configuration that has minimum energy. The symmetric saddle cannot be dominant since this have larger energy than

\(^{14}\) In the traversable wormhole picture, \(L\) corresponds to the inverse temperature \(\beta\).
wormhole saddles. Therefore, for \(d_{12} \ll d_{\text{LR}}\) case, we obtain

\[
c_{12} > c_{\text{LR}} : \text{bra-ket wormhole phase,}
\]
\[
c_{12} < c_{\text{LR}} : \text{no bra-ket wormhole phase}
\]

and the phase transition happens at \(c_{12} = c_{\text{LR}}\).

5.8.2. Projections on bra-ket wormhole states. It is interesting to consider the transition in sections 5.6 and 5.5 in the state preparation interpretation. First let us consider the partial couplings in section 5.5. In that context, we introduced boundary conditions for \(c_{12}^j\) fields. After changing the role of time and space direction, these boundary conditions become spacelike. These spacelike boundaries are called boundary states [81–83]. Since the boundary conditions we introduce are local, the boundary states \(|B\rangle\) are schematically represented as

\[
|B\rangle = \prod_x |\psi_x\rangle.
\]

In our context, we fix the future boundary condition for \(c_{12}^j\) CFTs. These are interpreted as local projection of quantum states [84] onto boundary states. The projection operator \(P = |B\rangle\langle B|\) is then schematically written as

\[
P = \prod_x \langle\psi_x|\langle\psi_x|.
\]

The introduction of the spacelike boundaries for \(c_{12}^j\) CFTs is interpreted as

\[
|\Psi_{12}\rangle \rightarrow (P_{c_{12}^j} \otimes \mathbb{I}_{c_{12}^j})|\Psi_{12}\rangle = |P_{c_{12}^j}^L, B_{c_{12}^j}^L, B_{c_{12}^j}^R |\Psi_{12}\rangle.
\]

Here \(P_{c_{12}^j} = |B_{c_{12}^j}\rangle\langle B_{c_{12}^j}|\) is the projection operator onto the boundary state \(|B_{c_{12}^j}\rangle\) and \(|B_{c_{12}^j}^L, B_{c_{12}^j}^R |\Psi_{12}\rangle \in \mathcal{H}_{\text{CFT}_{12}^j} \otimes \mathcal{H}_{\text{CFT}_{12}^j}\) where \(\mathcal{H}_{\text{CFT}_{12}^j} \otimes \mathcal{H}_{\text{CFT}_{12}^j}\) is the Hilbert space for remaining CFT with the central charge \(c_{12}^j\) after projections. In this context, the partition function for traversable wormholes with partial couplings evaluates the (unnormalized) norm \(|\langle B_{c_{12}^j}^L, B_{c_{12}^j}^R |\Psi_{12}\rangle|^2\). The interpretation of the phase transition in section 5.5 becomes as follows. Initially we do not specify the final state and we have bra-ket wormholes. Then, we select the final state for \(c_{12}^j\) CFT by introducing the boundaries. Finally, at \(c_{\text{LR}} = c_{12}^j\) there is a phase transition and beyond that point bra-ket wormholes disappear!

Dividing by the normalization \(\langle\Psi_{12}|\Psi_{12}\rangle\), we can compute the two point function of projection operators:

\[
\langle P_{c_{12}^j}^L P_{c_{12}^j}^R \rangle = \frac{\langle\Psi_{12}|P_{c_{12}^j}^L P_{c_{12}^j}^R |\Psi_{12}\rangle}{\langle\Psi_{12}|\Psi_{12}\rangle}.
\]

This corresponds to the ratio of the partition functions:

\[
\langle P_{c_{12}^j}^L P_{c_{12}^j}^R \rangle = e^{-L\left(E_{0}(c_{12}^j) - E_0(c_{12})\right)},
\]

where \(E_{0}(c_{12}^j)\) is the energy as a function of \(c_{12}^j\), an example of which is given in figure 25. The behavior of the two point function (5.44) is shown the left panel of figure 30.
5.8.3. Entangling operations on bra-ket wormhole states. Next, we consider the case in section 5.6 where we consider to first split some of CFT\textsubscript{12}, say CFT\textsubscript{12}' with central charge \( c_{12}' \), and then reconnect them in L–R direction. This amounts to changing \((c_{LR},c_{12}) \rightarrow (c_{LR} + c_{12}',c_{12} - c_{12}')\). Here we attaching the spacelike perfectly transparent condition for \( c_{12}' \) CFTs. These gluings are interpreted as attaching local entanglement \cite{84} between two sides. In other words, we partially project the state onto a maximally entangled state, which is schematically written as

\[
|\psi\rangle_{LR} = \prod_{t} \left( \sum_{n_t} |n_t\rangle_{L}|n_t\rangle_{R} \right).
\]  

\[
(5.45)
\]
The projection operator onto maximally entangled state is then schematically written as
\[
E = |I\rangle_{LR} \langle I|_{LR} = \prod_x \left( \sum_{n_x} |n_x\rangle_L \langle n_x|_R \right) \left( \sum_{m_x} \langle m_x|_L \langle m_x|_R \right).
\] (5.46)

After this entangling procedure, the state becomes
\[
|\Psi_{12}\rangle \rightarrow E_{c_{12}} |\Psi_{12}\rangle = |I_{c_{12}}\rangle \langle I_{c_{12}}|_{\Psi_{12}}.
\] (5.47)

Here \(E_{c_{12}} = |I_{c_{12}}\rangle \langle I_{c_{12}}|\) is the projection operator onto the maximally entangled state \(|I_{c_{12}}\rangle\in H_{CFT_{12}} \otimes H_{CFT_{12}}\) and \(\langle I_{c_{12}}|_{\Psi_{12}}\rangle \in H_{CFT_{12}} \otimes H_{CFT_{12}}\) where \(H_{CFT_{12}} \otimes H_{CFT_{12}}\) is the Hilbert space for remaining CFT with the central charge \(c_{12} = c - c_{s_{12}}\) after projections. In this context, the partition function for traversable wormholes with partial couplings evaluates the (unnormalized) norm \(|\langle I_{c_{12}}|_{\Psi_{12}}\rangle|^2\). The interpretation of the phase transition in section 5.6 becomes as follows, which is similar to the boundary state cases. Initially we do not specify the final state and we have bra-ket wormholes. Then, we select the final state for \(c_{12}\ CFT\) by introducing entanglement between two sides. Finally, at \(c_{LR} = c_{12}\) there is a phase transition and beyond that point bra-ket wormholes disappear\(^{15}\).

Therefore, both of projections and entangling operation induce the disappearance of the bracket wormholes. In other words, two pairs of cosmological spacetime annihilate in Euclidean regime.

Dividing by the normalization \(\langle \Psi_{12}|_{\Psi_{12}}\rangle\), we can compute the one point function of entangling operators:
\[
\langle E_{c_{12}} \rangle = \frac{\langle \Psi_{12}|_{E_{c_{12}}}| \Psi_{12}\rangle}{\langle \Psi_{12}|_{\Psi_{12}}\rangle}.
\] (5.48)

This again corresponds to the ratio of the partition functions:
\[
\langle E_{c_{12}} \rangle = e^{-L E_0(c_{12}) + L E_0(0)},
\] (5.49)

where \(E_0(c_{12})\) is the energy as a function of \(c_{12}\), an example of which is given in figure 26. Initially at \(c_{12} = 0\), the entangling operator is the identity operator and the expectation value is 1. The behavior of the two point function (5.49) is shown in the right panel of figure 30. Here we focus on the case where initially \(c_{LR} = 0\). In this case, the initial state are two decoupled bra-ket wormholes, which is expected to be a product state \(|\Psi_{12}\rangle = |B\rangle_L|B\rangle_R\). Then at \(c_{12} = c_{12}\) we do entangling operation for all the CFT\(\)\(_{12}\). The property of the maximally entangled state leads to \(|I|_{LR}(|B\rangle_L|B\rangle_R) = 1\). The expectation value of the entangling operator becomes
\[
\langle E_{c_{12}} \rangle = \frac{(|B\rangle_L|B\rangle_R)(|I|_{LR}(|B\rangle_L|B\rangle_R))}{(|B\rangle_L|B\rangle_R)(|B\rangle_L|B\rangle_R)} = 1.
\] (5.50)

Therefore, when we increase \(c_{12}\) from 0 to \(c_{12}\), entangling operator expectation value decreases but finally should come back to 1 at \(c_{12} = c_{12}\). The inclusion of the phase transition in (30) correctly reproduce this behavior. On the other hand, if we always use the bra-ket wormhole solution, the entangling operator expectation becomes much smaller than we expect. We interpret this as a kind of information loss and the exchange of saddle is needed to reproduce the correct behavior for factorized state \(|B\rangle_L|B\rangle_R\).

\(^{15}\) Using central charges as variables is similar to the toy model of black hole evaporation in [85].
5.8.4. Lorentzian continuation. It is interesting to consider Lorentzian continuation of the state preparation interpretation. The 2D metric and the dilaton profile is given by

\[ ds^2 = -\frac{dt^2 + d\sigma^2}{\ell^2 \cos^2 \frac{\sigma}{\ell}}. \]

\[ \phi(\sigma) = \frac{\ell}{\pi} \left( 1 + \frac{\sigma}{\ell} \tan \frac{\sigma}{\ell} \right). \] (5.51)

We analytically continue to \( \sigma \to i\tau, t \to ix \). Then, the metric and the dilaton profile become

\[ ds^2 = -\frac{dt^2 + dx^2}{\ell^2 \cosh^2 \frac{\tau}{\ell}}, \]

\[ \phi(t) = \frac{\ell}{\pi} \left( 1 - \frac{t}{\ell} \tanh \frac{t}{\ell} \right). \] (5.52)

These are interpreted as closed universes \([52]\). The Lorentzian geometry (5.52) corresponds to the Friedmann–Lemaître–Robertson–Walker (FLRW) universe that emerges from a singularity at \( \eta = -\infty \), reach the maximum size at \( \eta = 0 \) and recollapses at \( \eta = \infty \). If the conformal matter is holographic, the original metric is given by that of the global coordinate:

\[ ds^2_{(3)} = \frac{i^2}{\ell^2} \text{AdS}_3(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\sigma^2). \] (5.53)

After the analytic continuation \( \sigma \to i\tau \) and \( t \to ix \), the metric becomes

\[ ds^2_{(3)} = \frac{i^2}{\ell^2} \text{AdS}_3(-\sinh^2 \rho \, dt^2 + d\rho^2 + \cosh^2 \rho \, dx^2). \] (5.54)

This is the metric for BTZ black holes \([87]\). Let us consider the bra-ket wormhole phase. In 2D gravity description, there are two closed universes with CFT_{LR} and CFT_{12}. CFT_{12} are entangled with the same CFT_{12} on a Minkowski spacetime whereas CFT_{LR} are entangled with the other closed universe. In 3D gravity description when matter fields are holographic, matter quantum entanglement is geometrized in a way depicted in 32. The length of the circle which the CFT is living on gives the temperature of the CFT, or entanglement temperature of the thermo field double state in 2D gravity description. Each temperature is given by

\[ \begin{align*}
\frac{1}{\beta_{12}} &= \frac{\pi \ell + d_{12}}{2} \\
\frac{1}{\beta_{LR}} &= \frac{2(\pi \ell + d_{LR})}{\pi \ell + d_{LR}} \Rightarrow \begin{cases} 
T_{12} = \frac{1}{\pi \ell + d_{12}} \\
T_{LR} = \frac{1}{2(\pi \ell + d_{LR})}
\end{cases}
\end{align*} \] (5.55)

where \( \beta_{12}(\beta_{LR}) \) is the inverse temperature for CFT_{12} (CFT_{LR}) and \( T_{12}(T_{LR}) \) is the temperature.

In 3D gravity description, this is consistent that we obtain a black hole geometry (figure 31).

Now we consider entanglement entropy between two copies of CFT_{12}. First it is easy to consider the 3D gravity description. In this case, we can use the holographic entanglement entropy formula \([88–90]\) for bulk matter CFTs. We have three horizons in figure 32, which area gives candidate saddles. The area of the horizon, or equivalently the thermal entropy for CFT is given by \( S = \frac{c}{2} L [91] \) where \( c \) is the central charge, \( L \) is the length of the spacial circle

\[ \text{We shift } \sigma \to \sigma + \frac{\pi}{\ell} \text{ so that the } t = 0 \text{ slice become } \sigma = 0. \]

\[ \text{In three dimensions with negative cosmological constant, we can construct Euclidean wormhole solutions } [35, 86], \text{ which becomes FLRW universe after analytic continuation to Lorentzian signature. The geometry here is the 2D analog of the relation between Euclidean wormholes and the closed universes.} \]
and $\beta$ is the inverse temperature. Therefore, holographic entanglement entropy between two sides is given by

$$S_{\text{ent}} = \min \left\{ \frac{\pi c_{LR}}{3\beta_{LR}} L, \frac{\pi c_{12}}{3\beta_{12}} L \right\},$$

(5.56)

where the first one is the horizon area of L–R black hole and the second one is that of 1–2 black holes. Because the bra-ket wormhole is dominated when $c_{LR} < c_{12}$. Moreover, we found that $T_{LR} < T_{12}$ (here we focus on the case $d_{12} = d_{LR}$) in (5.55). Therefore entanglement entropy between two side is

$$S_{\text{ent}} = \frac{\pi c_{LR}}{3\beta_{LR}} L.$$ 

(5.57)

In particular, when $c_{LR} = 0$ entanglement entropy becomes 0. This is consistent with the observation in [52]. In this case, we have the product state $|\Psi_{12}\rangle = |B\rangle |B\rangle$ and each of closed universe is contained in the entanglement wedge of CFT on a flat space. Though we only work
in holographic bulk matters cases, we expect this will be justified by replica wormhole argument [23, 24] even for non holographic bulk matters. We will not fully analyze this problem but discuss later in the discussion section.

It is interesting when CFT$^{12}$ is a collection of free CFTs and CFT$_{LR}$ is holographic. Even in this case (5.57) will be correct. Then, we use holographic entanglement entropy formula for states of free CFTs! This is natural from the perspective of quantum error correction [92]. Entanglement is a nature of state, not of the theory, and for the holographic code in free CFT we expect that we can use the holographic entanglement entropy formula.

In the no bra-ket wormhole phase, closed universes and CFT$_{LR}$ do not appear anymore in Lorentzian signature. The only role of the JT gravity is to determine the entanglement temperature between two copies of CFT$_{12}$:

$$\beta_{12} = 2(\pi \ell + d_{12}) \quad \rightarrow \quad T_{12} = \frac{1}{2(\pi \ell + d_{12})}. \quad (5.58)$$

Here \(\ell\) is the wormhole length in the Euclidean signature and it depends on \(\tilde{\phi}_r\). Thought we have two JT gravity in Euclidean signature, they annihilate in the Euclidean regime and disappears in the Lorentzian signature.

6. Discussion

6.1. Summary

Here we describe the summary of this paper. In this paper, we construct models of four coupled SYK models and nearly AdS$_2$ gravities and study them. In both of SYK models and JT gravities, these coupling can be thought of as coupling two traversable wormholes. This coupling introduces entanglement between two traversable wormholes. These theories show the first order phase transitions at the symmetric point. Both of them exhibit the $\mathbb{Z}_2$ symmetry breaking that exchanges the wormhole configurations. These models also have symmetric solutions under $\mathbb{Z}_2$ symmetry, which have larger energy than wormhole configurations. The direct interactions at the boundaries act in a state dependent way. In one wormhole phase the wormhole length is shortened when we introduce the interactions. On the other hand, in the other wormhole phase the direct boundary interaction lengthens the wormhole length.

In the four coupled SYK models, we study the solutions and their properties numerically at large $N$ limit beyond the low energy description by the Schwartzian action. Our tool is the Schwinger–Dyson equation. Each of the wormhole configurations disappears in some parameter regimes. An order parameter is given by the (difference of) spin operators that consist from the fundamental fermions. The first order phase transition only exists when the coupling of the four coupled interaction is small. For larger couplings (mass terms), the phase transition disappears and the theory exhibits the crossover. The decreasing energy gap, or the lengthening the wormhole length still exists even in the cross over regime. The theory has a duality, which becomes a symmetry at a special point. Based on the action for collective fields $G$ and $\Sigma$, we can evaluate the effective potential for the four coupled theory. By explicitly studying this potential, the symmetric solution contains unstable direction.

We also studied the four coupled nearly AdS$_2$ gravities, or four coupled JT gravities. Here we meant that the theory have four boundaries. We assume that the JT gravities are coupled to at least two different CFTs. Then, we couple these JT + CFTs through CFTs on flat space without dynamical gravity. The quantum mechanical dual description looks like four 2D CFTs that are connected through domain walls where each domain wall supports quantum mechanics that is dual to JT gravity + CFTs. We construct two wormhole solutions and one symmetric
solution in this setup. When we coupled two traversable wormholes, the matter fields on different wormholes are entangled. If these matter fields are holographic, the dual 3D geometry directly connect two wormholes, which is the realization of the \( \text{ER} = \text{EPR} \). Furthermore, this 3D geometry becomes a traversable wormhole from the perspective of CFTs on flat space without dynamical gravity. When we increase entanglement between wormholes there is a phase transition where the wormhole configuration changes. One of the interesting feature of this gravity model is that we can compute entanglement entropy. We study entanglement entropy which plays a role of order parameter. In some configuration, we need to take into account island contribution, which leads to the wormhole entropy.

We also consider the wick rotation of four coupled JT gravities. In this case the wormholes are interpreted as bra-ket wormholes. By considering the JT gravity coupled to both of free CFTs and holographic CFTs, these bra-ket wormhole configuration serves a way to embed holographic states into free CFTs’ Hilbert spaces. The change of the coupling is interpreted as either of partial projection onto local product states or partial entangling operation. The first order phase transition happens when we increase the number of projection or entangling operation. The first order phase transition is now interpreted as a bra-ket wormhole transition, which means a transition of entanglement structure in family of states prepared by Euclidean path integrals.

6.2. Similarity with symmetry breaking in holographic QCD

In the four coupled JT gravities with holographic matter fields, the symmetry breaking is realized as a brane connection in 3D space time. Actually, there is already a well known example of symmetry breaking by brane connections. That is the holographic QCD model known as Sakai–Sugimoto–Witten model [93, 94]. In that model, the connection of D8 branes and anti D8 branes (denoted by \( \text{D8} \) branes), represent the breaking or restoration of the chiral symmetry. At zero temperature, the D8 branes and \( \text{D8} \) branes are connected in the bulk, which means the chiral symmetry is broken. On the other hand, at finite temperature and when the gluon sector is deconfined, there are two configurations [95]. One configuration is the same with that in the zero temperature, which means the chiral symmetry breaking. In the second configuration branes end on the horizon in the black hole geometry. In Euclidean signature, these branes are actually connected in the shrinking thermal circle in the bulk. When we go to the Lorentzian signature, the branes of two sides in the thermo field double state is actually connected at the Einstein–Rosen (ER) bridge.

A difference is that in the holographic QCD case there are chiral symmetries that act on each defect in the dual picture and single brane connection breaks these symmetries whereas our four coupled model do not have such an symmetry acting on each defects. The situation in the holographic QCD is more similar to the complex SYK symmetry breaking [96, 97] where each complex SYK cluster has a \( U(1) \) symmetry [98] and the symmetry is spontaneously broken in coupled models [99]. Another comment is that the symmetry breaking also plays an important role in quantum chaos in the SYK or JT gravity from semiclassical analysis [100].

6.3. The connection to the bubble of wormholes

The story of Sakai–Sugimoto–Witten model is related to the bubble of wormholes [101, 102]. In type IIA string theory, when we put the theory on a \( \mathbb{R}^{1,8} \times I \) and introduce the 8 D8-branes with an O8-plane on each boundary of an interval \( I \), we obtain the type I’ theory [103]. In the strong coupling limit we obtain M-theory ending on the end of the world M9 brane that carries the 10d \( E_8 \) super Yang–Mills (SYM) [103, 104], which is an ‘end of the world brane degrees of freedom’.
The configurations above are supersymmetric and stable. However, when we flip a chirality of one of the boundaries, the supersymmetry is completely broken [101]. A possible scenario when two branes are separated is that the spacetime hole is nucleated and spacetime completely disappears. These are described by the $\mathbb{Z}_2$ orbifold of the bubble of nothing bounce solution [96, 105]. An interesting thing is that these bubbles connect two boundaries and plays a role of wormhole from the end of the world M9 brane perspective. After this bubble is nucleated, the $E_8$ SYM degrees of freedom is connected and we obtain a single $E_8$ degrees of freedom, which is similar to the D8 brane connections and the Higgs mechanism in the holographic QCD. This is more similar with the situation in section 5.7 where L–R CFT is holographic and 1–2 CFT is free CFTs that are interpreted as the EOWs brane degrees of freedom. The first similarities are the point that the L–R CFT is separated to two pieces under the wormhole transition by nucleating the ‘bubble of nothing’ in the middle. Second, we obtain a single free fields that are sit on two traversable wormholes, which is similar to obtaining a single $E_8$ SYM. When the separation of the two M9 branes are smaller than the string scale, we expect that the theory is tachyonic and perturbatively unstable. Actually, there is a tachyonic string theory with single $E_8$ gauge degrees of freedom [102], which is the counterpart of the chiral symmetry breaking in the holographic QCD. This is analogous to the SYK symmetric unstable solution in this paper and it is interesting to study more how they are parallel.

6.4. Interpretations of results in 4D traversable wormhole setup

In sections 5.5 and 5.6, we study the change of boundary conditions in no gravity region and study their effects. It is interesting to consider the interpretation of these changes in 4D traversable wormhole setup in [9, 51].

First, we consider the change of the distance between two traversable wormholes. Then, this causes the change of magnetic field configurations. Since each 2D free fermion propagates along the magnetic lines, this will cause the change of number of 2D fermions that are propagating along from L to R, or 1 to 2. This changes the central charge $c_{LR}$ and $c_{12}$. On the other hand, the total central charge $c_{tot} = c_{LR} + c_{12}$ will not be changed because these are total number of 2D free fermions propagating in the wormhole, which is determined by the magnetic flux that penetrates the sphere $S^2$. Therefore, we can interpret that the change of boundary conditions in section 5.6 is modeling this change of magnetic field configurations and its effects.

Next, we consider to insert monopole anti-monopole pairs to cut the magnetic line outside the wormholes. Then some of magnetic lines terminate on (anti-) monopoles. The number of magnetic lines that are cut depends on the total monopole charges. Therefore, fermions on such magnetic lines essentially live on intervals with boundaries on the location of (anti-) monopoles. 4D fermions around monopoles or dyons and their connection to boundary CFT are discussed in [106–108]. We can interpret that the partial couplings by the introduction of boundaries outside the wormholes in section 5.5 are modeling this insertion of monopole anti-monopole pairs.

In more realistic 4D setup, the things outside the wormholes are also described by the 4D physics whereas we consider the situation where we can do what we want to do outside the wormholes. For example, the mouses of wormholes can attract with each other because they have opposite charges. To avoid this, we can introduce a rotation to avoid attraction but it leads to a radiation and wormholes still have finite life times [9]. Because the transitions between wormholes involve topology changes, which are non perturbative effects and it may takes longer time than a life time.
6.5. Generalization to many coupled SYK/JT gravities

It is interesting to consider to couple more SYK/JT gravities. In the SYK side, there are several works on the SYK chains [109–113]. In a similar manner, we can consider $n$-coupled SYK models. It is interesting to study this generalization in detail.

We can also consider more general couplings of JT gravities based on our construction of the four coupled JT gravities. One generalization is to introduce many domain walls in the full quantum mechanical description. Note that many D5 branes on black brane backgrounds and their connection to dimer states are discussed in [114, 115]. Based on the relation between brane connections and wormhole configuration in section 6.2, it is natural to expect that the states with many domain walls will have similar structure (figure 33).

Another generalization is to consider the quantum mechanical domain walls with JT gravity dual to junctions of CFTs, and then couple different junctions. For example, we can consider the junction of five CFTs as depicted in figure 34. Actually, JT gravity + $N$ free CFTs can be thought of as a junction of $N$ free CFTs. In 4D traversable wormholes, free fermions on flat space region without dynamical gravity effectively live on different lengths of intervals [9]. Therefore, it is more natural to think of them as a junction CFT. In 3D gravity description, we also already encounter this type in section 4.3. The $N_H = 5$ setup there can be thought of as the junction of five holographic CFTs in figure 34. Based on these junctions, we can couple them. It is interesting study this type of couplings and their entanglement structures.

6.6. Relation to replica wormholes

It is interesting to consider a relation to replica wormholes [23, 24]. Let us consider the replica method justification of the calculation of entanglement entropy in (5.56) in a partially doubly holographic model. In other words, CFT$_{LR}$ is holographic and CFT$_{12}$ is a collection of free fields. We mainly consider the bra-ket wormhole phase.

We consider the $n$th Rényi entropy. In this case, we encounter the situation with $4 \times n = 4n$ domain walls in the Euclidean path integral. This is an example of many coupled JT gravities that we discussed in section 6.5. Then we can evaluate the replica partition function using the JT gravity description. We describe replica symmetric configurations in figure 35. The first configuration in figure 35 will be relevant to compute entanglement entropy between two closed universes, which gives the minimal one in the bra-ket wormhole phase, see (5.56).
Figure 34. Three description of a junction of five CFTs. (Left) 2D gravity description. The red line denotes the JT gravity + five CFTs. There are also five CFTs on regions without dynamical gravity. They are connected by the transparent boundary conditions at junction point. (Middle) 3D gravity description when the matter CFTs have holographic duals. Five AdS3 geometries are joined at an end of the world brane. (Right) Full quantum mechanical description. Five CFTs on half lines are joined at center on which a quantum mechanical degrees of freedom with gravity dual. This quantum mechanical degrees of freedom is not conformal and we can think of the theory as non conformal junction CFTs.

Figure 35. (Left) Quantum mechanical description of the Rényi entropy for the density matrix $\rho_L = Tr_0 |\Psi_{12}\rangle \langle \Psi_{12}|$ with replica number 3. (Middle) Replica wormhole configuration with replica symmetry in 3D gravity description for CFTBR. (Right) Disconnected configuration with replica symmetry in 3D gravity description for CFTBR.

configurations are replica wormholes for 3D gravities that are holographic duals of bulk matters. The replicas of bra-ket wormholes are connected in this configuration. If we view these configurations as traversable wormholes, the 3D gravity parts are traversable wormholes with $2n$ boundaries. Because CFT$\text{BR}$ is entering on all 2D traversable wormholes, the matter stress tensor from CFT$\text{BR}$ on each AdS$_2$ region is evaluated

$$\langle T^{\text{AdS}_2}_{++}\rangle = \frac{c_{\text{BR}}}{48\pi\ell^2} \frac{\pi}{n^2(\pi\ell + d_{\text{BR}})^2}. \tag{6.1}$$

Therefore as we increase the replica number $n$, the energy from CFT$\text{BR}$ increases and it approaches to $\frac{c_{\text{BR}}}{48\pi\ell^2}$. However as we observed around (5.11), even in this limit the finite $\ell$ solution exists because in the bra-ket wormhole phase the central charges satisfy $c_{\text{BR}} < c_{12}$ whereas...
the condition to have finite $\ell$ solution is $c_{LR} < 3c_{12}$\(^{18}\). Therefore we expect that the replica wormhole solutions always exists for any Rényi index $n$. These configurations are interpreted as both of replica wormholes and traversable wormholes, which gives a connection between them.

On the other hand, the second configuration in figure 35 will be relevant to compute entanglement entropy between a closed universe and CFT on a flat space without dynamical gravity. Though we expect that there are solutions with finite wormhole length $\ell$ for $n \approx 1$, it is not apparent whether there are saddles with these configurations because CFT\(_{12}\) no more introduces negative energy for generic replica number $n$.

We also expect that there are also replica non symmetric solutions generically. We skip the full analysis of other saddles as a future problem.

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**Data availability statement**

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

**Appendix A. Some formulas for 2D CFT**

In the Lorentzian signature, the metric is

$$d\mathcal{s}^2 = -dt^2 + d\sigma^2 = -dy^+dy^-,$$

(A.1)

where $y^\pm = t \pm \sigma$ is a null coordinate. The stress energy tensor is

$$T_{\mu\nu} = \begin{pmatrix} T_{tt} & T_{t\sigma} \\ T_{\sigma t} & T_{\sigma\sigma} \end{pmatrix}.$$

(A.2)

The energy measured in time $t$ is

$$E = \int T_{tt} \, dx.$$

(A.3)

In the light cone coordinate $\omega^\pm = t \pm \sigma$, this becomes

$$\begin{pmatrix} T_{++} & T_{+-} \\ T_{+-} & T_{--} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} T_{tt} + 2T_{t\sigma} + T_{\sigma\sigma} & T_{tt} - T_{\sigma\sigma} \\ T_{tt} - T_{\sigma\sigma} & T_{tt} - 2T_{t\sigma} + T_{\sigma\sigma} \end{pmatrix}.$$

(A.4)

\(^{18}\)Here the role of $c_{LR}$ and $c_{12}$ are exchanged compared to the notation around (5.11).
Figure 36. (Left) The conformal transformation $\omega^\pm(y^\pm)$ from the cylinder to the Minkowski space. The patch $-\frac{L}{2} \leq y^\pm \leq \frac{L}{2}$ is mapped to the entire Minkowski space, which is the diamond surrounded by the red solid line. (Right) The same conformal diagram with boundaries. The boundaries are located on $\sigma = -\frac{L}{2}$ and $\sigma = 0$.

Or inversely,

$$
\begin{pmatrix}
T_{tt} & T_{t\sigma} \\
T_{t\sigma} & T_{\sigma\sigma}
\end{pmatrix} =
\begin{pmatrix}
T_{++} + 2T_{+-} + T_- & T_{++} - T_- \\
T_{++} - T_- & T_{++} - 2T_{+-} + T_-
\end{pmatrix}.
$$

When we perform a conformal transformation $\omega^\pm(y^\pm)$, the stress tensor is transformed as (figure 36)

$$
\left(\frac{d\omega^+}{dy^+}\right)^2 T_{++}^{(\omega)} = T_{++}^{(\nu)} + \frac{c}{24\pi} (\omega^+, y^+),
$$

where $T_{++}^{(\omega)}$ is the stress tensor in the coordinate system $ds^2 = -d\omega^+ d\omega^-$ whereas $T_{++}^{(\nu)}$ is the stress tensor in the coordinate system $ds^2 = -dy^+ dy^-$. Let us think of $\omega^\pm$ is a coordinate of Minkowski space and $y^\pm$ is that of a cylinder with the length $L$.

Then, the conformal transformation $\omega^\pm(y^\pm)$ from the cylinder to the Minkowski space is

$$
\omega^\pm(y^\pm) = \frac{L}{2 \pi} \tan \left(\frac{\pi}{L} y^\pm\right).
$$

The coordinate transformation becomes

$$
\frac{dy^+ dy^-}{\cos^2(\frac{\pi}{L} y^+) \cos^2(\frac{\pi}{L} y^-)}
$$

and the null boundary $y^\pm = \pm \frac{L}{2}$ is located at infinity. After the Weyl transformation $ds^2 \rightarrow \tilde{ds}^2 = \cos^2(\frac{\pi}{L} y^+) \cos^2(\frac{\pi}{L} y^-)$, the null boundary $y^\pm = \pm \frac{L}{2}$ is located at finite distance from the interior and we can continue the geometry beyond them and obtain the cylinder with the metric $ds^2 = -dy^+ dy^-$. Because on the Minkowski vacuum we do not have the stress tensor
expectation value, we obtain the stress energy tensor expectation value on the cylinder with the length $L$:

$$\langle T^{(y)}_{++} \rangle = -\frac{c}{24\pi} \left\{ \omega^+, y^+ \right\} = -\frac{c}{24\pi} \frac{2\pi^2}{L^2}. \quad (A.9)$$

Then, we obtain

$$T_{\mu\nu} = T_{\sigma\sigma} = T^{(y)}_{++} + T^{(y)}_{--} = -\frac{c}{12} \frac{2\pi^2}{L^2}. \quad (A.10)$$

In particular, the energy density with $L = 2\pi$ is $T_{\mu\nu} = -\frac{c}{2\pi}$. The ground state energy of the system is

$$E = \int_0^L T_{\mu\nu} d\sigma = -\frac{c}{12} \frac{2\pi^2}{L}. \quad (A.11)$$

When the period is $2\pi$, we obtain $E = -\frac{c}{12\pi}$.

We can treat boundary CFT in the same manner. We consider the boundary CFT on a strip with the length $L_{\text{bdy}} = L/2$. The geometry is obtained by just cutting the cylinder along $\sigma = 0, L/2$ and imposing a boundary condition. We introduce the ‘same’ boundary condition on two boundaries. The same means that they are CPT conjugate with each others. The formulas are the same that we obtained in the cylinder case. The null energy on the strip is

$$\langle T^{(y)}_{++} \rangle = -\frac{c}{24\pi} \left\{ \omega^+, y^+ \right\} = -\frac{c}{24\pi} \frac{2\pi^2}{L^2} = -\frac{c}{24\pi} \frac{\pi^2}{2L_{\text{bdy}}} \quad (A.12)$$

The energy density and the pressure is

$$T_{\mu\nu} = T_{\sigma\sigma} = T^{(y)}_{++} + T^{(y)}_{--} = -\frac{c}{12} \frac{2\pi^2}{L^2} = -\frac{c}{24} \frac{\pi}{L_{\text{bdy}}}. \quad (A.13)$$

In particular, the energy density for $L_{\text{bdy}} = \pi$ is $T_{\mu\nu} = -\frac{c}{24\pi}$. The ground state energy of the system is

$$E = \int_0^{L_{\text{bdy}}} T_{\mu\nu} d\sigma = -\frac{c}{24} \frac{\pi}{L_{\text{bdy}}}. \quad (A.14)$$

A.1. CFT on AdS$_2$

We now consider to put the boundary CFT on AdS$_2$. To compute the stress tensor, we should care about the Weyl anomaly. The stress tensor is given by

$$\langle T^{(y)}_{\mu\nu} \rangle = i \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \log Z[g]. \quad (A.15)$$

For example, the free scaler case in classical limit, the partition function is

$$Z[g] = e^{S[g]} = \exp \left( -i \frac{1}{2} \int d^2x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right). \quad (A.16)$$
The energy stress tensor is
\[
T^g_{\mu\nu} = i \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \log Z[g] = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi.
\] (A.17)

The classical action is invariant under the Weyl transformation \( g_{\mu\nu} \to e^{2\omega} g_{\mu\nu} \) since the factor \( \sqrt{-g} \to e^{2\omega} \sqrt{-g} \) and \( g^{\mu\nu} \to e^{-2\omega} g^{\mu\nu} \) cancel. However the partition function transforms anomalously as
\[
Z[g = e^{2\omega} \hat{g}] = \exp \left\{ i \frac{c}{24\pi} \int d^2 x \sqrt{-\hat{g}} \left[ \hat{R} + (\hat{\nabla} \omega)^2 \right] \right\} Z[\hat{g}] .
\] (A.18)

Therefore, two stress tensors are related by\(^19\)
\[
\langle T^g_{\mu\nu} \rangle = i \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \log Z[g] \\
= i \frac{2}{e^{2\omega} \sqrt{-\hat{g}}} \frac{\delta}{\delta \left( e^{-2\omega} g^{\mu\nu} \right)} \log \left[ \exp \left\{ i \frac{c}{24\pi} \int d^2 x \sqrt{-\hat{g}} \left[ \hat{R} + (\hat{\nabla} \omega)^2 \right] \right\} Z[\hat{g}] \right] \\
= \langle T^\hat{g}_{\mu\nu} \rangle - \frac{c}{12\pi} \left[ \partial_\mu \omega \partial_\nu \omega - \frac{1}{2} \hat{g}_{\mu\nu} (\hat{\nabla} \omega)^2 - \hat{\nabla}_\mu \hat{\nabla}_\nu \omega + \hat{g}_{\mu\nu} \hat{\nabla}^2 \omega \right].
\] (A.19)

In AdS\(_2\) case, the quantum stress tensor is related to the stress tensor on a strip by
\[
T^\text{AdS}_{\mu\nu} = \hat{T}^\text{strip}_{\mu\nu} + \frac{c}{24\pi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\mu\nu} - \frac{c}{24\pi} g_{\mu\nu}.
\] (A.20)

Here the second term means \( T_{++} = T_{--} = \frac{c}{2\pi} \) and \( T_{+-} = T_{-+} = 0 \).

\(\text{A.2. Holographic CFT on AdS}_2\)

We consider holographic CFTs partially on AdS\(_2\) for the region \([0, \pi]\) of \([0, \pi(1 + a)]\). To put the theory on the length \(\pi(1 + a)\), it is convenient to rescale the global AdS\(_3\) metric
\[
d\ell^2 = l^2_{\text{AdS}_3} \left[ -\frac{4}{(1 + a)^2} \cosh^2 \rho \, d\tau^2 + d\rho^2 + \frac{4}{(1 + a)^2} \sinh^2 \rho \, d\phi^2 \right].
\] (A.21)

Then, for \(0 < \phi < \pi\), the UV cutoff surface is given by the condition
\[
e^{-\rho(t,\phi)} = \frac{\epsilon}{1 + a} \sin \phi.
\] (A.22)

\(^{19}\)In the anomalous term, the first two terms take the form of scalar energy stress tensor and the last two terms is the stress energy tensor for dilaton in JT gravity.
The induced metric on the cutoff surface is

\[
\begin{align*}
\text{d}s^2_{\text{ind}} &= h_{\mu\nu} \text{d}x^\mu \text{d}x^\nu \\
&= -\frac{l_{\text{AdS}3}^2}{\epsilon^2} \left( \frac{1}{\sin \phi} + \frac{\epsilon^2}{(1+a)^2} \frac{\sin \phi}{\sin \phi} \right)^2 \text{d}r^2 \\
&\quad + \frac{l_{\text{AdS}3}^2}{\epsilon^2} \left[ \frac{1}{\tan^2 \phi} + \frac{1}{\epsilon^2} \left( \frac{1}{\sin \phi} - \frac{\epsilon^2}{(1+a)^2} \frac{\sin \phi}{\sin \phi} \right)^2 \right] \text{d}\phi^2 \\
&= \frac{l_{\text{AdS}3}^2}{\epsilon^2} \left( -\text{d}t^2 + \frac{\text{d}\phi^2}{\sin^2 \phi} \right) + \cdots = \frac{l_{\text{AdS}3}^2}{\epsilon^2} \text{d}s^2_{\text{AdS}2} + \cdots. \\
\end{align*}
\] (A.23)

The normal vector \( n_A \) is given by

\[
\begin{align*}
n_t &= 0, \\
n_\rho &= \frac{2l_{\text{AdS}3}}{\sqrt{4 + \frac{(1+a)^2}{\sin^2 \phi \cosh^2 \rho}}}, \\
n_\phi &= \frac{2l_{\text{AdS}3}}{\tan \phi \sqrt{4 + \frac{(1+a)^2}{\sin^2 \phi \cosh^2 \rho}}}. \\
\end{align*}
\] (A.24)

Then, we can calculate \( K_{\mu\nu} = P_{\mu}^A P_{\nu}^B \nabla_A n_B \) where \( P_{\mu}^A \) is the projection on to the cutoff surface. \( K_{\mu\nu} \) in \((t, \phi)\) coordinate is

\[
\begin{align*}
K_{tt} &= -l_{\text{AdS}3} \left[ \frac{1}{\epsilon^2 \sin^2 \phi} - \frac{1}{2 \tan^2 \phi} \right] + \mathcal{O}(\epsilon), \\
K_{\phi\phi} &= l_{\text{AdS}3} \left[ \frac{1}{\epsilon^2 \sin^2 \phi} - 1 + \frac{1}{2 \tan^2 \phi} \right] + \mathcal{O}(\epsilon) \\
K_{t\phi} &= K_{\phi t} = 0. \\
\end{align*}
\] (A.25)

and

\[
\begin{align*}
K &= K_{\mu\nu} h^{\mu\nu} = \frac{1}{l_{\text{AdS}3}} (2 - \epsilon^2) + \mathcal{O}(\epsilon^3). \\
\end{align*}
\] (A.26)

The induced metric is expanded as

\[
\begin{align*}
h_{tt} &= l_{\text{AdS}3}^2 \left[ -\frac{1}{\epsilon^2 \sin^2 \phi} - \frac{2}{(1+a)^2} \right] + \mathcal{O}(\epsilon) \\
h_{\phi\phi} &= l_{\text{AdS}3}^2 \left[ \frac{1}{\epsilon^2 \sin^2 \phi} - \frac{2}{(1+a)^2} + \frac{1}{\tan^2 \phi} \right] + \mathcal{O}(\epsilon) \\
h_{t\phi} &= h_{\phi t} = 0. \\
\end{align*}
\] (A.27)
Therefore, the holographic stress energy tensor [66] is

\[
T_{\mu\nu} dx^\mu \, dx^\nu = -\frac{1}{8\pi G_N} \left( K_{\mu\nu} - K h_{\mu\nu} + \frac{1}{l_{\text{AdS}}} h_{\mu\nu} \right) dx^\mu \, dx^\nu \\
= -\frac{l_{\text{AdS}}}{8\pi G_N} \left[ \left( \frac{1}{2 \sin^2 \phi} + \frac{1}{2} - \frac{2}{(1 + a)^2} \right) dr^2 \\
+ \left( \frac{1}{2 \sin^2 \phi} + \frac{1}{2} - \frac{2}{(1 + a)^2} \right) d\phi^2 \right] \\
= -\frac{c}{24\pi} \frac{(1 - a)(a + 3)}{(1 + a)^2} (dr^2 + d\phi^2) - \frac{c}{24\pi} \frac{1}{\sin^2 \phi} (-dr^2 + d\phi^2) \\
= -\frac{c}{48\pi} \frac{(1 - a)(a + 3)}{(1 + a)^2} ((dx^+)^2 + (dx^-)^2) - \frac{c}{24\pi} g_{\mu\nu}^\text{AdS} dx^\mu \, dx^\nu. \quad (A.28)
\]

Here we used the central charges for holographic CFTs \( c = \frac{3l_{\text{AdS}}}{2\pi G_N} \). This reproduce the stress energy tensor from conformal anomaly argument \((A.20)\). Note that when \( a = 1 \), the stress tensor vanishes.

A.3. Some formula for entanglement entropy

First we consider the system on an infinite line. Entanglement entropy for single interval \([0, x]\) is

\[
S_A = \frac{c}{3} \log \frac{x}{\epsilon} + c', \quad (A.29)
\]

where \( \epsilon \) is a UV cutoff and \( c' \) is a non universal constant.

Entanglement entropy for single interval \([0, x]\) on a circle of the length \( L \) is

\[
S_A = \frac{c}{3} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi x}{L} \right) + c'. \quad (A.30)
\]

In free Dirac fermion case, entanglement entropy for two intervals of lengths \( l_1, l_2 \) with the distance \( D \) (i.e. intervals \([-l_1, 0] \cup [D, D + l_2]\) between them \([72, 84]\) is\(^{20}\)

\[
S_A = \frac{c}{3} \log \left( \frac{l_1 l_2}{\epsilon} \frac{D(l_1 + l_2 + D)}{(l_1 + D)(l_2 + D)} \right) + 2c'. \quad (A.32)
\]

If the system is in a finite cylinder of the length \( L \), entanglement entropy becomes

\[
S_A = \frac{c}{3} \log \left[ \left( \frac{L}{\pi \epsilon} \right)^2 \sin \frac{\pi l_1}{L} \sin \frac{\pi l_2}{L} \sin \frac{\pi l_1 + l_2 + D}{L} \right] + 2c'. \quad (A.33)
\]

Note that in \( L \to \infty \) limit entanglement entropy on a circle \((A.33)\) recovers the that on an infinite line \((A.32)\).

\(^{20}\) When we write the interval as \([x_1, x_2] \cup [x_1, x_4]\), entanglement entropy becomes

\[
S_A = \frac{c}{3} \log \left( \frac{|x_2| |x_4|}{e^2} \frac{|x_2| |x_4|}{|x_1| |x_3|} \right) + 2c'. \quad (A.31)
\]
In CFT with holographic dual, entanglement entropy for single interval is given by

\[ S_A = \min \{ S_1, S_2 \}, \]  

\[ \text{(A.34)} \]

where \( S_1 \) and \( S_2 \) are given by

\[ S_1 = \frac{c}{3} \log \frac{l_1}{\epsilon} + \frac{c}{3} \log \frac{l_2}{\epsilon}, \]

\[ S_2 = \frac{c}{3} \log \frac{D}{\epsilon} + \frac{c}{3} \log \frac{D + l_1 + l_2}{\epsilon}. \]  

\[ \text{(A.35)} \]

For \( S_1 \), each geodesic connects the endpoints of each interval whereas in \( S_2 \) two geodesics connects the end points of different intervals. On a circle, entanglement entropy for two intervals is given by

\[ S_A = \min \{ S_1, S_2 \}. \]  

\[ \text{(A.36)} \]

with

\[ S_1 = \frac{c}{3} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi l_1}{L} \right) + \frac{c}{3} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi l_2}{L} \right), \]

\[ S_2 = \frac{c}{3} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi D}{L} \right) + \frac{c}{3} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi (D + l_1 + l_2)}{L} \right). \]  

\[ \text{(A.37)} \]

In general CFT, we can express entanglement entropy for two intervals as

\[ S_A = \frac{c}{3} \log \left( \frac{l_1 l_2 \eta}{\epsilon^2} \right) + \log F(\eta) + 2c_1', \]

\[ \eta = \frac{D(l_1 + l_2 + D)}{(l_1 + D)(l_2 + D)}. \]  

\[ \text{(A.38)} \]

with theory dependent function \( F(\eta) \) which always satisfies \( F(1) = 1 \) and \( F(0) = 1 \). On a circle, entanglement entropy becomes

\[ S_A = \frac{c}{3} \log \left( \frac{L}{\pi \epsilon} \sin \frac{\pi l_1}{L} \sin \frac{\pi l_2}{L} \right) + \log F(\eta) + 2c_1', \]

\[ \eta = \frac{\sin \frac{\pi D}{L} \sin \frac{\pi (l_1 + l_2 + D)}{L}}{\sin \frac{\pi D}{L} \sin \frac{\pi l_1}{L} \sin \frac{\pi (l_2 + D)}{L}}. \]  

\[ \text{(A.39)} \]

Next we consider CFT on a manifold with boundaries on which we impose a conformal boundary condition. When system is put on a semi infinite line \([0, \infty), \) entanglement entropy for an interval \([0, x)\) is

\[ S_A = \frac{c}{6} \log \frac{2x}{\epsilon} + \log g + \frac{1}{2} c_1', \]  

\[ \text{(A.40)} \]

where \( c_1' \) is the same non universal constant in \( \text{(A.29)} \) and \( \log g \) is the boundary entropy that depends on the choice of boundary conditions. Entanglement entropy between the interval \([0, x)\) on an interval of the length \( L \) with boundaries at 0, \( L \) is

\[ S_A = \frac{c}{6} \log \left( \frac{2L}{\pi \epsilon} \sin \frac{\pi x}{L} \right) + \frac{1}{2} c_1' + \log g. \]  

\[ \text{(A.41)} \]
In free Dirac fermions, entanglement entropy for single interval of length $l$ with the distance $D$ from the boundary at $x = 0$ (i.e. intervals $[D, l + D]$ in a half line $[0, \infty)$) becomes

$$S_A = \frac{c}{6} \log \left( \frac{\ell^2}{\epsilon^2} \frac{4D(l + D)}{(l + 2D)^2} \right) + c'_1. \quad (A.42)$$

On a finite interval $[0, L]$ with the same boundary condition on each boundary, entanglement entropy becomes

$$S_A = \frac{c}{6} \log \left( \frac{L}{\pi \epsilon} \right)^2 \left( \sin \frac{2\ell}{\pi} \sin \frac{2\pi D}{\pi} \sin \frac{2\pi(l + D)}{\pi} \right)^2 + c'_1. \quad (A.43)$$

In AdS/BCFT models [74, 75], entanglement entropy for single interval is given by

$$S_A = \min \{ S_{\text{con}}, S_{\text{dis}} \}, \quad (A.44)$$

where $S_{\text{con}}$ and $S_{\text{dis}}$ are given by

$$S_{\text{con}} = \frac{c}{3} \log \frac{\ell}{\epsilon}, \quad S_{\text{dis}} = \frac{c}{6} \log \frac{2D}{\epsilon} + \frac{c}{6} \log \frac{2\ell}{\epsilon} + 2 \log g. \quad (A.45)$$

$S_{\text{con}}$ is the length of the geodesics that connects two endpoints, whereas $S_{\text{dis}}$ is the sum of two disconnected geodesics that end on the end of the world brane. On a finite interval, holographic entanglement entropy for single interval becomes

$$S_A = \min \{ S_{\text{con}}, S_{\text{dis}} \}. \quad (A.46)$$

with

$$S_{\text{con}} = \frac{c}{3} \log \left( \frac{2L}{\pi \epsilon} \sin \frac{\pi \ell}{2L} \right), \quad S_{\text{dis}} = \frac{c}{6} \log \left( \frac{2L}{\pi \epsilon} \sin \frac{\pi D}{L} \right) + \frac{c}{6} \log \left( \frac{2L}{\pi \epsilon} \sin \frac{\pi(l + D)}{L} \right) + 2 \log g. \quad (A.47)$$

For general BCFT, we can express entanglement entropy for two intervals as

$$S_A = \frac{c}{3} \log \left( \frac{\ell_1 \eta}{\epsilon^2} \right) + \log G(\eta) + c'_1, \quad \eta = \frac{4D(l + D)}{(l + 2D)^2}. \quad (A.48)$$

with theory dependent function $G(\eta)$ which always satisfies $G(1) = 1$ and $G(0) = g^2$. On a circle, entanglement entropy becomes

$$S_A = \frac{c}{6} \log \left( \frac{2L}{\pi \epsilon} \right)^2 \sin^2 \left( \frac{\pi \ell}{2L} \right) \eta + \log G(\eta) + 2c'_1, \quad \eta = \frac{\sin \frac{\pi D}{L} \sin \frac{\pi(l + D)}{L}}{\sin^2 \frac{\pi(l + D)}{2L}}. \quad (A.49)$$
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