NONCOMMUTATIVE SPACE-TIME FROM QUANTIZED
TWISTORS

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Abstract

We consider the relativistic phase space coordinates \((x_\mu, p_\mu)\) as composite, described by functions of the primary pair of twistor coordinates. It appears that if twistor coordinates are canonically quantized the composite space-time coordinates are becoming noncommutative. We obtain deformed Heisenberg algebra which in order to be closed should be enlarged by the Pauli-Lubanski four-vector components. We further comment on star-product quantization of derived algebraic structures which permit to introduce spin-extended deformed Heisenberg algebra.

1 Introduction

Space-time description of relativistic point particles does not provide a natural geometrization of spin degrees of freedom. It is well acknowledged however that spin degrees of freedom play essential role in the description of space-time as dynamical system, what is illustrated e.g. by spin foam approaches to quantum gravity (see e.g. [1],[2]) or the use of spin networks in loop quantum gravity (see e.g. [3],[4]). Well known geometrization of the spin degrees of freedom is provided by superspace extensions of space-time (see e.g. [5],[6]), with finite-dimensional Grassmann algebra attached to each space-time point. In this paper we shall introduce geometric spin degrees in different way by considering as primary the twistor geometry (see e.g. [7],[8]) with basic spinorial coordinates, and consider space-time coordinates as their composites.

The twistors in \(D = 4\) are the fundamental conformal \(SU(2,2)\) spinors and introduce primary conformal geometry, with single twistors well suited to the description of massless elementary objects [7]-[9]. In fact single twistor space has exactly the structure of phase space for massless particle, with all possible choices of helicity [11]. Massive particles with arbitrary spin can be described if we introduce two-twistor space, which contains the phase space for massive particles with spin [9],[12]-[15]. The pair of twistors is needed as well if we wish to introduce the space-time coordinates given as composites of twistor components [7]. In this report we shall consider canonically quantized pairs of twistors which provide particular choice of noncommutative composite space-time coordinates and leads to the generalization of the standard QM phase space structure.
In order to introduce the coordinates describing point in complex Minkowski spacetime one should employ two nonparallel twistors \( t_{A,i} \), where \( A = 1, 2, 3, 4 \) are the \( SU(2, 2) \) \( \simeq O(4, 2) \) indices, and \( i = 1, 2 \) is the internal \( U(2) \) index. In two-twistor complex space \( T^{(2)} = T \otimes T \in t_{A,i} \) \( (t_{A,1} \in T \otimes 1, t_{A,2} \in 1 \otimes T) \) one can introduce the following canonical twistorial Poisson brackets (PB) \[7\]

\[
\{\bar{t}^{A,i}, t_{B,j}\} = \delta^A_B \delta_i^j \\
\{\bar{t}^{A,i}, \bar{t}^{B,j}\} = \{t_{A,i}, t_{B,j}\} = 0
\]

where \( \bar{t}^{A,i} = g^{AB} \bar{t}_B^i \) and \( g^{AB} \) describes the Hermitian \( SU(2, 2) \) metric. The standard choice of twistor coordinates, described by the pairs of 2-component Weyl spinors \( (t_{A,i} = (\pi_{a,i}, \omega^{\hat{a} i})) \), corresponds to the metric \( g^{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

One can introduce the bilinear Hermitian products of twistors \( t^{A,i}, \bar{t}_{B,i} \) which after the use of (1) describe the twistorial realization of conformal algebra \( O(4, 2) \) \[7\]. In particular for the Poincare algebra generators \( P_\mu = P_{\dot{\alpha}\beta}, \quad M_{\mu\nu} = (M_{\alpha\beta}, M_{\dot{\alpha}\dot{\beta}}) \) we get the formulae

\[
P_{\dot{\alpha}\beta} = \pi^{\dot{\alpha}}_\beta \\
M_{\alpha\beta} = \omega^{\dot{\alpha}}_\beta \\
M_{\dot{\alpha}\dot{\beta}} = \bar{\omega}_{\dot{\alpha}}^\beta
\]

The complex Minkowski space-time coordinates parametrize two-planes in twistor space, which are determined by the pair of incidence equation for two twistors \( t_A^i = (\pi_{a,i}, \omega^{\hat{a} i}) \)

\[
\omega^{\dot{a}\dot{b}} = \pi^{\dot{a}}_\alpha \pi^{\dot{b}}_\beta \\
z^{\dot{a}\dot{b}} = (\sigma_\mu)^{\dot{a}\dot{b}} z^\mu
\]

which provides the known composite formula for \( z^{\dot{a}\dot{b}} \) \[7\]-\[9\]

\[
z^{\dot{a}\dot{b}} = \frac{i}{\pi^{1\alpha}_\alpha \pi^{2\alpha}_\alpha} (\pi^{1\alpha}_\alpha \omega^{2\beta}_\beta - \pi^{2\alpha}_\alpha \omega^{1\beta}_\beta) = x^{\dot{a}\dot{b}} + iy^{\dot{a}\dot{b}}.
\]

One chooses that real part \( x^{\dot{a}\dot{b}} = \Re z^{\dot{a}\dot{b}} \) describes the composite real physical Minkowski space. Twistorial PB (1) induce further on the composite space-time coordinates the noncommutative structure.

We mention that the relations (4)-(5) has been already used as defining the coordinates of quantum free fields (see e.g. \[10\]). Our aim here is to incorporate the nonvanishing PB of composite Minkowski space-time coordinates \( x_\mu = \frac{1}{2}(\sigma_\mu)_{\alpha\beta} x^{\alpha\beta} \) into enlarged deformed relativistic Heisenberg algebra. It appears that in order to get the closure of PB algebra it is necessary to add to space-time coordinates \( x_\mu = \Re z_\mu \) (see (5)) and fourmomenta \( p_\mu \) (see (2)) the Pauli-Lubanski fourvector components \( (x_\mu, P_\mu, W_\mu) \).

\[
W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\tau} P^\nu M^\rho\tau.
\]

After insertion in (6) of composite formulae (2)-(3) we obtain Pauli-Lubansky fourvector as expressed fourlinearly in twistor coordinates \( t_{A,i}, \bar{t}_A^i \). In such a way we obtain spin-extended deformed Heisenberg (SEDH) algebra with the basis described by the generators \( (x_\mu, P_\mu, W_\mu) \).
2 PB structure of spin-extended deformed Heisenberg algebra

Let us introduce the Hermitean $2 \times 2$ matrix described by the $U(2, 2)$--invariant scalar products ($r = 1, 2, 3$)

$$K^i_j = \bar{t}^A i t_{A,j} = (K^j_i)^\dagger = (\tau_r)^j_i k_r \delta^i_j k_0$$

where $\tau_r$ are Pauli matrices and $\bar{k}_r = k_r$, $\bar{k}_0 = -k_0$. The scalar products $K^i_j$ introduce the internal $U(2)$ PB algebra induced by the canonical twistorial PB (1):

$$\{ k_r, k_s \} = \epsilon^{rst} \bar{k}_r (\tau^\rho)_{ij}$$

In order to describe the fourlinear twistor formula for $W_\mu$ in compact way one can introduce four composite vierbeins by means of the formula (see also [16])

$$e^{(r)}_{\mu} = \frac{1}{2} (\sigma_\mu)^{\alpha\beta} \bar{\pi}_{\alpha \beta} (\tau^\rho)^j_i \pi_{\alpha j}$$

where $\tau^{(\rho)} = (1, 2, \tau_r)$. One obtains from the comparison with (2) that $e^{(0)}_{\mu} = p_\mu$ and one gets the orthogonality relations

$$e^{(\rho)}_{\mu} e^{(\tau)}_{\nu} = |f|^2 \eta^{\rho\tau}, \quad f = \bar{\pi}_{\alpha} \pi^{\alpha 2} |f|^2 = p^2.$$  

Due to (10) the set of composite frame fields $e^{(\rho)}_{\mu}$ depends on seven independent degrees of freedom which describe eight degrees ($\pi_{\alpha i}, \bar{\pi}_{\alpha i}$) factorized by $U(1)$ phase $\pi_{\alpha i} \rightarrow e^{i\gamma} \pi_{\alpha i}$. Further one can derive the formulae (see also [6])

$$W_\mu = k_r e^{(r)}_{\mu} \quad k_r = -\frac{1}{p^2} e^{(r)}_{\mu} W_\mu$$

or more explicitly ($W_{\alpha\bar{\beta}} = (\sigma^\mu)_{\alpha\bar{\beta}} W_\mu$)

$$W_{\alpha\bar{\beta}} = k_3 (\pi^{\alpha}_1 \bar{\pi}^{\bar{\beta}}_1 - \pi^{\alpha}_1 \bar{\pi}^{\bar{\beta}}_1) + k_2 \pi^{2}_\alpha \bar{\pi}^{\bar{2}}_\beta + k_1 \pi^{1}_\alpha \bar{\pi}^{\bar{1}}_\beta$$

where $k_\pm = k_1 \pm ik_2$.

The relations (11) provide the covariant formulae for the generators $k_r$ of internal $SU(2)$ algebra (see [7]) which describe the Lorentz-invariant three spin projections. From (10)-(11) follows that

$$W_\mu W^\mu = p^2 t^2 \quad t^2 = k_1^2 + k_2^2 + k_3^2$$

and after quantization of PB (8) we obtain the well-known relativistic spin square spectrum with $t^2$ replaced by quantum spin square $s(s + 1)$ ($s = 0, \frac{1}{2}, 1, \ldots$).

We see from (11)-(12) that the component $k_0$ does not enter into the definition of composite Pauli-Lubansky fourvector, but one can show that it contributes to the imaginary
part $y_\mu = \Im z_\mu$ of composite complex Minkowski space coordinates (5). One can derive the following general formula

$$y_\mu = -\frac{1}{p^2} k^\rho e_\mu^{(\rho)} = -\frac{1}{p^2}(t^0 p_\mu - W_\mu).$$

(14)

The choice simplifying the formulae (see (12)) for the spin fourvector $W_\mu$ and complex Minkowski space are obtained if we choose $t_3 = t \neq 0$ and $t_0 = t_1 = t_2 = 0$. We obtain that

$$W_{\alpha\beta} = t(\pi_{\alpha\bar{\beta}}^1 - \pi_{\alpha\beta}^2), \quad z_\mu = x_\mu + \frac{i}{p^2} W_\mu.$$ (15)

Using explicit formulae (2), (5), (8) and (12) one can derive the following PB algebra (see also [13]; we denote further the twistor functions (2) and (12) by small letters)

$$\{x_\mu, p_\nu\} = \eta_{\mu\nu} \quad \{p_\mu, p_\nu\} = 0 \quad (16)$$

$$\{x_\mu, x_\nu\} = -\frac{1}{(p^2)^2}\epsilon_{\mu\nu\rho\sigma}w^\rho p^\sigma \quad (17)$$

$$\{w_\mu, x_\nu\} = -\frac{1}{p^2} w_{[\mu} p_{\nu]} \quad \{w_\mu, p_\nu\} = 0 \quad (18)$$

$$\{w_\mu, w_\nu\} = \epsilon_{\mu\nu\rho\sigma}w^\rho p^\sigma, \quad (19)$$

which are consistent with the relation $p_\mu W^\mu = 0$.

The nonpolynomial PB algebra (16)-(19) is consistent with dimensionalities $[x_\mu] = m^{-1}$, $[p_\mu] = [w_\mu] = m$. We mention that the PB subalgebra with generators $(p_\mu, w_\mu)$ was e.g. studied in [17] (see Appendix I) as relativistic spin algebra. We add that the nonpolynomial factor $\lambda^2 = p^{-2}$ can not be replaced by constant inverse mass square because of the following nonvanishing PB

$$\{x_\mu, \frac{1}{p^2} p_\mu\} = -2\frac{1}{(p^2)^2}p_\mu \Rightarrow \{x_\mu, \lambda^2\} = -2\lambda^4 p_\mu. \quad (20)$$

### 3 Quantization of spin-extended deformed Heisenberg algebra

The general Poisson brackets can be quantized if we use Kontsevich quantization method [18]-[19] which solved the problem of existence of associative $\star$-product representing multiplication on quantized Poisson manifold. By this method naive quantization of phase space functions via replacement $\{\cdot, \cdot\} \rightarrow \frac{i}{\hbar}[\cdot, \cdot]$ has been modified in a way which leads to the validity of Jacobi identities. The star product representation of the algebra describing quantized Poisson brackets is obtained after performing the Weyl map of elements of SEDH algebra

$$f(\hat{Y}_a) \overset{W}{\rightarrow} f(Y_a) \quad Y_a = (x_\mu, p_\mu, w_\mu, \Lambda)$$ (21)
and introducing the following homomorphic map of the products

$$f(\hat{Y}_a) \cdot g(\hat{Y}_b) \xrightarrow{W} f(Y_a) \ast g(Y_b)$$

(22)

where $\Pi_j$ are bidifferential operators maximally of $2j$ order and

\begin{align*}
\Pi_0(f, g) &= f \cdot g \\
\Pi_1(f, g) - \Pi_1(g, f) &= \{f, g\} \\
f(Y_a) \ast (g(Y_a) \ast h(Y_a)) + \text{cycl} &= 0.
\end{align*}

(23) \quad (24) \quad (25)

If the algebraic manifold with coordinates $Y_a$ is not a flat one in the bidifferential operators $\Pi_j$ one should introduce suitably covariantized derivatives [21].

The SEDH PB algebra \((16)-(19)\) \(\{Y_a, Y_b\} = \omega_{ab}(Y)\) can be introduced as dual to the 2-form $\Omega_2$

\(\{Y_a, Y_b\} = \omega_{ab}(Y) \xrightarrow{\text{dual}} \Omega_2 = \omega_{ab}(Y) dY_a \wedge dY_b\)

(26)

where $\omega_{ab} \omega_{bc} = \delta^c_a$.

One can show that $\Omega_2$ in (26) has the form (see (13))

$$\Omega_2 = dp^\mu \wedge dx_\mu + \Omega_2^{\text{Sour}},$$

(27)

where

$$\Omega_2^{\text{Sour}} = \frac{1}{2(r^2)^{1/2}} \epsilon_{\mu\nu\rho\sigma} w^\rho p^\sigma \left( \frac{1}{p^2} dp^\mu \wedge dp^\nu - \frac{1}{t^2} dw^\mu \wedge dw^\nu \right)$$

(28)

The two-form (27) can be obtained from the canonical Liouville one-form $\theta_1$ on $T \times T$

$$\theta_1 = i(\tau^A_i \wedge dt_{A,i}) = \frac{i}{2} (\omega_\mu^a d\pi_{a,i} + \pi_{a,i} d\omega^\mu_i - \text{c.c.})$$

(29)

after introducing the coordinates $Y_a$ as functions (see (2), (5) and (8)) of the pair of twistor coordinates $t_i \in T \times T$. It should be added that the one-form (29) pulled back on one-dimensional trajectories ($\int \theta = \int dt L$) in generalized phase space $Y_a = (x_\mu, p_\mu, w_\mu)$ defines the action of massive particle with spin characterized by the fourvector $w_\mu$ (see [15]).

4 Outlook

The basic PB structure \((16)-(19)\) of our extended deformed phase space requires for its application consistent quantization. In standard QM there is well-known Wigner formulation (see e.g. [23]) realizing Weyl correspondence between quantum-mechanical operators and phase-space classical functions. In the geometric scheme presented in this paper the basic PB are more complicated, but fortunately one can quantize them via Kontsevich star-product [18]-[20]. An important property of the $\ast$-quantization formula given by
is its dependence only on the Poisson structure function $\omega_{ab}(Y)$ (see (26)) and its derivatives to all orders. In such a way one gets the modification of naive correspondence rule between classical PB and quantum mechanical phase space commutators in order to achieve the validity of Jacobi identity (from naive quantization prescription one obtains nonvanishing Jacobiator with leading $\hbar^2$ term). We obtain the quantization rules for SEDH algebra by calculating the perturbative formula for $\star$-product, which takes for Poisson structure (26) the following perturbative form in the space of functions $f(Y_a), g(Y_a)$ depending on spin-deformed extended phase-space coordinates $Y_a$ (compare with (22))

\begin{equation}
\begin{aligned}
f \star g &= fg + \hbar \omega_{ab} f_{,a} g_{,b} + \frac{\hbar^2}{2} \omega_{ab} \omega_{cd} f_{,ac} g_{,bd} \\
&+ \frac{\hbar^2}{3} \omega_{ab} \omega_{cd,b} (f_{,ac} g_{,bd} - f_{,bc} g_{,ad}) + O(\hbar^3)
\end{aligned}
\end{equation}

where $f \equiv f(Y_a), f_{,a} \equiv \frac{\partial}{\partial Y_a} f(Y_a)$ etc. One can check that using (30) the Jacobi relation (25) is satisfied up to the $\hbar^3$ terms.

Concluding, using Kontsevich $\hbar$-expansion of general $\star$-product formula, the perturbative quantization of our PB (16)-(19) can be achieved. In such a way one gets the spin-extended deformed QM and one can further test possible applications. In particular we conjecture that the presence of additional spin coordinate $w_{\mu}$ can help to provide new ways of describing the relativistic (stringy?) spin dynamics with infinite set of spin values.

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