Asymptotic Solutions of Compaction in Porous Media

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Abstract

Compaction in reactive porous media is modelled as a reaction-diffusion process with a moving boundary. Asymptotic analysis is used to find solutions for the coupled nonlinear compaction equations, and a traveling wave solution is obtained above the reaction zone.

Keywords: Reaction-diffusion, Darcy flow, asymptotic analysis, porous media.

Citation detail: X. S. Yang, Asymptotic solutions of compaction in porous media, Applied Mathematics Letters, 14, 765-768 (2001).

1 Introduction

The accurate modelling of compaction and reactive flow in porous media such as sand and shale sediments is very important in civil engineering and oil industry [1]. The general mathematical model of compaction and reactive mineral flow considers the fluid-sediment system as a porous medium consisting of multiple mineral species. The interstitial volume of the porous solid phase is saturated with pore fluids. Due to the action of gravitational overburden loading and the density difference between the two phases, the solid phase compacts by reducing the porosity (volume fraction of the pores), thus leading to the expulsion of the pore fluid out of porous media. During compaction and continuous burial, the mineral species such as water-rich clay smectite react with pore fluids and are then transformed into a more stable mineral species such as illite, releasing free water into the porous environment[1,2]. In this paper, a reaction-diffusion model together with some asymptotic analysis is presented.

2 Mathematical Model

Let the volume fractions of solid reactant species (smectite) and water be $\psi$, $\phi$, respectively. By proper non-dimensionalization and appropriate scalings [3,4,5] in a 1-D basin $0 < z < h(t)$, where $h(t)$ is the ocean floor and $z = 0$ is the basement rock, we can write down the non-dimensional compaction equations as

$$\psi_t = -e^{\beta(h-z-z^*)}\psi - \frac{\lambda}{1-\phi_0}[\psi(\frac{\phi}{\phi_0})^m(\phi_z - \phi)]_z, \quad m \geq 7. \quad (1)$$

$$\phi_t = \lambda[(\frac{\phi}{\phi_0})^m(\phi_z - \phi)]_z + \frac{\alpha_0}{\beta}e^{\beta(h-z-z^*)}\psi, \quad m \geq 7. \quad (2)$$

The boundary conditions are

$$\phi_z = \phi = 0, \quad \text{at} \quad z = 0, \quad (3)$$

$$\phi = \phi_0, \quad \psi = \psi_0, \quad (4)$$

$$\dot{h}(t) = \dot{s} + \frac{\lambda}{1-\phi_0}[(\frac{\phi}{\phi_0})^m(\phi_z - \phi)]_z \quad \text{at} \quad z = h(t). \quad (4)$$
where $\lambda = O(1)$ and $\beta \gg 1$ are compaction constant and non-dimensional activation energy of the one-step dehydration mineral reaction with a critical temperature $z^*$ and the amount $a_0 = O(1)$ of free water released from the reaction. $\dot{s}$ is the rate of new sediment accumulation at the basin top, and thus can be taken as a prescribed constant ($\dot{s} = 1$). $\phi_0$ and $\psi_0$ are the initial values of $\phi$ and $\psi$ at the ocean floor $z = h(t)$, respectively. In addition, all the variables ($\phi$, $\psi$, $z$, $t$) and the parameters ($\lambda$, $m$, $\phi_0$, $\psi_0$, $\dot{s}$) are non-negative. The reaction term $\exp[\beta(h-z-z^*)]$ is only dominant within a thin region of a width of $O(1/\beta)$ near $h-z-z^* \approx 0$, in other words, the reaction region is located at $z \approx \theta^*$ defined as
\[
\theta^* = h - z^*. \tag{5}
\]
Clearly, the present problem is a non-linear diffusion problem with a boundary $h(t)$ moving at a speed of $\dot{h}(t)$, which can be solved numerically by using the predictor/corrector implicit finite-difference method.

## 3 Travelling Wave Solution

Numerical simulations [3,6] and real data observations [7] imply that the moving boundary $h(t)$ moving essentially at a nearly constant speed $\dot{h} = c$, although the specific value $c$ depends on the boundary conditions, and is yet to be determined. The constant moving boundary implies there exist travelling wave solutions. Thus, we define a new variable by
\[
\zeta = z - \theta^* = z - h(t) + z^*, \quad -\theta^* \leq \zeta \leq z^*, \tag{6}
\]
so that the model equations (1) and (2) become
\[
-c\psi_\zeta = -e^{-\beta\zeta}\psi - \frac{\lambda}{1 - \phi_0}\psi_\zeta \cdot \left[\phi_\zeta + (\phi)\right]_\zeta, \tag{7}
\]
\[
-c\phi_\zeta = \lambda\left[\phi_\zeta + (\phi_0)\right]_\zeta + \frac{a_0}{\beta} e^{-\beta\zeta}\psi, \tag{8}
\]
The fact that $\beta \sim m \gg 1$ allows us to seek asymptotic solutions in different regions with these distinguished limits. $\beta \gg 1$ implies that the reaction is essentially restraint in a very narrow zone with a width of $O(1/\beta)$ at $z = \theta^*$. Above this region ($z > \theta^*$), we have $\zeta > 0$ so that $\exp(-\beta\zeta) \to 0$. Below this reaction region ($\zeta < 0$), the reaction is essentially completed (i.e., the volume fraction of smectite $\psi \to 0$) and consequently $\psi \exp(-\beta\zeta) \to 0$. We shall see that this is true below in equation (24) because $\psi \to 0$ as $\zeta \to -\infty$ (or $\eta \to -\infty$).

### Outer Solutions

In the outer region above the reaction zone ($\zeta > 0$), the reaction terms are negligible, then we have
\[
c\psi_\zeta = \frac{\lambda}{1 - \phi_0}\left[\psi_\zeta \cdot (\phi_\zeta + (\phi_0)\right]_\zeta, \tag{9}
\]
\[
-c\phi_\zeta = \lambda\left[\phi_\zeta + (\phi_0)\right]_\zeta. \tag{10}
\]
The integrations together with top boundary conditions (4) give
\[
\psi = \frac{\dot{s}\psi_0}{c - \frac{\lambda}{1 - \phi_0}(\phi_\zeta + (\phi_0))}, \tag{11}
\]
and
\[
c\phi + \lambda(\phi_\zeta + (\phi_0)) = c\phi_0 + (c - \dot{s})(1 - \phi_0), \tag{12}
\]
whose solution can be further written as a quadrature. This solution is only valid in the region above the reaction zone ($z = \theta^*$). On the other hand, the travelling solution will not be appropriate
in the region below the reaction zone \((\zeta < 0)\) because the exponential term \((\phi/\phi_0)^m \ll 1\) due to \(\phi < \phi_0\) and \(m \gg 1\). However, we can define a typical value of \(\phi^*\) by

\[
\phi^* = \phi_0 e^{-\frac{\ln m}{m}},
\]

so that \(\phi \sim \phi^*\) in the region below the reaction zone \((\zeta < 0)\). The reaction term is still negligible. Thus we rewrite equations (1) and (2) in terms of \(\Phi\) defined by

\[
\phi = \phi^* e^{\Phi} = \phi_0 e^{\frac{\Phi - \ln m}{m}},
\]

then we have

\[
\psi_t = -\frac{\phi^* \lambda}{m(1 - \phi_0)} [\psi e^{\Phi} (\frac{1}{m} \Phi_z - 1)]_z.
\]

\[
\phi^* \Phi_t = \lambda \phi^* e^{\Phi} [e^{\frac{1}{m} \Phi_z - 1}]_z.
\]

By using \(1/m \ll 1\), the above equations becomes

\[
\psi_t \approx 0,
\]

\[
\Phi_t + \lambda e^{\Phi} \Phi_z = 0.
\]

It is straightforward to write down the solution together with the boundary condition \(\Phi_z = m\) at \(z = 0\). We have

\[
\Phi = \ln(\frac{1 + mz}{1 + m\lambda t}).
\]

As \(z = h(t)\), we have \(\Phi_\infty = \ln((1 + mh)/(1 + m\lambda t)) \to \ln(h/\lambda t) = \ln(c/\lambda)\) as \(t \to \infty\) or \(mh(t) \to \infty\) due to \(dh/dt = c = \text{const}\) and \(h = 0\) at \(t = 0\). However, \(z = h(t)\) is not usually reached since solution (22) is below the reaction region. When \(z \gg 1\), the above solution shall match the inner solutions as \(\eta \to -\infty\).

**Inner Solutions**

In the reaction zone, we use the stretched variables defined by

\[
\eta = \beta \zeta + \ln \beta, \quad \phi = \phi^* e^{\frac{\Phi}{m}} = \phi_0 e^{\frac{\Phi - \ln m}{m}},
\]

so that equations (7) and (8) become

\[
-c\psi_{\eta} = -e^{-\eta}\psi - \frac{A\lambda \phi^*}{\beta(1 - \phi_0)} [\psi e^{\Phi} (A\Phi_{\eta} - 1)]_{\eta},
\]

\[
-c\phi^* \Phi_{\eta} = \lambda \phi^* [e^{\Phi} (A\Phi_{\eta} - 1)]_{\eta} + \frac{a_0}{A} e^{-\eta}\psi,
\]

where \(A = \frac{\beta}{m} = O(1)\). By using \(1/\beta \ll 1\), equation (21) becomes

\[
c\psi_{\eta} = e^{-\eta}\psi,
\]

whose solution is

\[
\psi = C \exp\left[-\frac{1}{c} e^{-\eta}\right], \quad C = \psi_0 \exp\left[-\frac{1}{c} e^{(\beta \zeta - \ln \beta)}\right],
\]

where \(C\) depends on \(c\), and \(c\) will be determined later in (29) by matching. Substituting this solution into (22), integrating once from \(-\infty\) to \(\eta\) and using \(\Phi \to \Phi_\infty\) as \(\eta \to -\infty\), we get

\[
c\phi^* \Phi + \lambda \phi^* e^{\Phi} (A\Phi_{\eta} - 1) - B = -\frac{ca_0}{A} C,
\]

where \(B = c\phi^* \Phi_\infty - \lambda \phi^* \exp(\Phi_\infty)\). As \(\eta \to \infty\), we have

\[
[c\phi^* \Phi + \lambda \phi^* e^{\Phi} (A\Phi_{\eta} - 1)]_{+\infty} = -\frac{ca_0}{A} C,
\]

which implies a jump through the reaction region.
Matching

By rewriting solution (25) in terms of $\phi$, we have approximately

$$c\phi + \lambda(\frac{\phi}{\phi_0})^n(\phi_0 - \phi) \approx c\phi^*\Phi_\infty - \lambda\phi^*e^{\Phi_\infty} - \frac{C_{ca} A}{A}, \quad (27)$$

and matching this to solution (12), we have

$$c\phi_0 + (c - \dot{s})(1 - \phi_0) = c\phi^*\Phi_\infty - \lambda\phi^*e^{\Phi_\infty} - \frac{C_{ca} A}{A}, \quad (28)$$

which determines $c$, leading to

$$c = \frac{\dot{s}(1 - \phi_0)}{1 - \phi^*\Phi_\infty + \frac{AaC}{A}} - \frac{\lambda\phi^*e^{\Phi_\infty}}{(1 - \phi^*\Phi_\infty + \frac{AaC}{A})}. \quad (29)$$

Since $\Phi_\infty$ is a function of $c$, we now have an implicit equation for $c$, which depends essentially on the initial values of $\phi_0$ and $\psi_0$.

In summary, although it is very difficult to seek directly solutions for the couple nonlinear reaction-diffusion equations, we use a hybrid method to get the asymptotic solutions and travelling wave solutions in different regions. The matching of these solutions can thus determine the boundary moving velocity $\dot{h}(t) = c$, which shows how the reaction inside the porous media affect the evolution of its top boundary.

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