On homogeneous and isotropic universe

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18 June 2015

Abstract

We give a simple example of spacetime metric, illustrating that homogeneity and isotropy of space slices at all moments of time is not obligatory lifted to a full system of six Killing vector fields in spacetime, thus it cannot be interpreted as a symmetry of a four dimensional metric. The metric depends on two arbitrary and independent functions of time. One of these functions is the usual scale factor. The second function cannot be removed by coordinate transformations. We prove that it must be equal to zero, if the metric satisfies Einstein’s equations and the matter energy momentum tensor is homogeneous and isotropic. A new, equivalent, definition of homogeneous and isotropic spacetime is given.

Cosmological models constitute a classic part of general relativity and attract now great interest because of large amount of observational data. Most of the cosmological models are based on cosmological principle which requires our universe to be homogeneous and isotropic [1–5]. In particular, all space slices corresponding to constant time must be homogeneous and isotropic.

We consider spacetime $\mathcal{M}$ with coordinates $\{t, x^\mu\}$, $\mu = 1, 2, 3$, and metric of Lorentzian signature $(-+++)$. We assume that $t$ is the time coordinate and every section $t = \text{const}$ is spacelike. According to cosmological principle every section of constant time must be a three-dimensional space of constant curvature which is homogeneous and isotropic. Then the usual ansatz for the metric is

$$ds^2 = dt^2 + a^2 g_{\mu\nu}(x) dx^\mu dx^\nu. \quad (1)$$

where $g_{\mu\nu}(x)$ is a negative definite metric on three-dimensional space slices $\mathcal{S}$ of constant curvature which does not depend on time. This metric contains only one arbitrary function $a(t) > 0$ which is called the scale factor. This is the most general form of the homogeneous and isotropic metric in a suitable coordinate system (see, for example, [6]).

The form of the metric depends on the coordinates chosen on space slices. We choose stereographic coordinates. Then the metric takes the diagonal form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{a^2 \eta_{\mu\nu}}{(1 + b_0 x^2)^2} \end{pmatrix}, \quad (2)$$

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where \( b_0 = -1, 0, 1, \eta_{\mu\nu} := \text{diag}(- - -) \) is the negative definite Euclidean metric, and \( x^2 := \eta_{\mu\nu}x^\mu x^\nu \leq 0 \). Since we have chosen negative definite metric on space slices, the values \( b_0 = -1, 0, 1 \) describe constant curvature spaces of positive, zero, and negative curvature, respectively. For positive and zero curvature spaces, stereographic coordinates are defined on the whole Euclidean space \( x \in \mathbb{R}^3 \). For negative curvature spaces stereographic coordinates are defined inside the ball \(|x^2| < 1/b_0\).

Let us make the coordinate transformation \( x^\mu \mapsto x^\mu / a \). Then metric becomes non-diagonal, and the scale factor disappears:

\[
g = \begin{pmatrix}
1 + \frac{\dot{b}^2 x^2}{4b^2(1 + bx^2)^2} & \frac{\dot{b} x_\nu}{2b(1 + bx^2)^2} \\
\frac{\dot{b} x_\mu}{2b(1 + bx^2)^2} & \eta_{\mu\nu}
\end{pmatrix},
\]

(3)

where

\[
b(t) := \frac{b_0}{a^2(t)}
\]

(4)

and dot denotes derivative with respect to time.

We see that the metric of a homogeneous and isotropic universe can be non-diagonal, and may do not contain the scale factor. Moreover, the scalar curvature of space slices which is proportional to \( b(t) \) explicitly depends on time.

Let us now simply drop non-diagonal terms, add the scale factor, and put \( g_{00} = 1 \). Then the metric is

\[
g = \begin{pmatrix}
1 & 0 \\
0 & \frac{a^2 \eta_{\mu\nu}}{(1 + bx^2)^2}
\end{pmatrix},
\]

(5)

This metric contains two independent arbitrary functions of time: \( a(t) > 0 \) and \( b(t) \). It is nondegenerate for all values of \( b \) including zero. It is interesting because allows one to analyse, in a general case, solutions which go through zeroes \( b = 0 \). If such solutions exist, then the universe can change the sign of its space slices curvature during evolution.

Metric on space slices \( S \) of constant time \( t = \text{const} \) is

\[
h_{\mu\nu} = \frac{a^2 \eta_{\mu\nu}}{(1 + bx^2)^2},
\]

(6)

depending on time \( t \) as a parameter. Straightforward calculations show that three-dimensional curvature tensor for metric (6) has the form

\[
\hat{R}_{\mu\nu\rho\sigma} = -\frac{4b}{a^2} (h_{\mu\nu}h_{\rho\sigma} - h_{\nu\rho}h_{\mu\sigma}).
\]

Three-dimensional Ricci tensor and scalar curvature are

\[
\hat{R}_{\mu\nu} = -\frac{8b}{a^2} h_{\mu\nu}, \quad \hat{R} = -\frac{24b}{a^2}.
\]

We see that that all space slices are spaces of constant curvature. Therefore there are six independent Killing vector fields \( \hat{K}_i = \hat{K}_{i\mu} \partial_{\mu} \), \( i = 1, \ldots, 6 \), which act on space slices. Symmetry transformations on space slices must be prolonged on the whole spacetime. The usual assumption is that they act trivially on time coordinate \([6]\). That is Killing
vector fields on spacetime have zero time component: \( K_i := \{0, \hat{K}_i^\mu\} \). By construction, any linear combination of Killing vector fields \( K \) satisfies the Killing equation

\[
\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0, \quad \alpha, \beta = 0, 1, 2, 3,
\]

because they do not have time component and time enters space components as a parameter. The Killing equation can be rewritten for the contravariant components

\[
g_{\alpha\gamma} \partial_\beta K^\gamma + g_{\beta\gamma} \partial_\alpha K^\gamma + K^\gamma \partial_\gamma g_{\alpha\beta} = 0.
\]

This equation is fulfilled for all moments of time. The \((\alpha, \beta) = (0, 0)\) component of Killing equation (8) for diagonal metric (5) is identically satisfied. The \((\alpha, \beta) = (0, \mu)\) components reduce to equation

\[
\partial_0 \hat{K}^\mu = 0.
\]

The space components \((\alpha, \beta) = (\mu, \nu)\) of the Killing equation decouple

\[
h_{\mu\rho}(t, x) \partial_\nu \hat{K}^\rho + h_{\nu\rho}(t, x) \partial_\mu \hat{K}^\rho + \hat{K}^\rho \partial_\rho h_{\mu\nu}(t, x) = 0.
\]

These equations are identically satisfied for all moments of time by construction.

There is an interesting situation. On one hand, all space slices of metric (5) are homogeneous and isotropic. On the other hand, any homogeneous and isotropic metric must have form (1). The answer is the following. The whole four-dimensional metric (5) is not homogeneous and isotropic in a sense that Killing equations (7) are not fulfilled. Indeed, the six independent Killing vector fields on space slices are

\[
\hat{K}_{0\mu} = (1 + bx^2) \partial_\mu - \frac{2}{b} x_\nu x^\nu \partial_\nu,
\]

\[
\hat{K}_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad \mu, \nu = 1, 2, 3.
\]

The first three Killing vectors generate translations at the origin of the coordinate system \(x^2 = 0\), and the last three Killing vectors generate rotations. We see that the first three Killing vectors explicitly depend on time through \(b(t)\), and Eq. (9) is not fulfilled. This example shows that homogeneity and isotropy of space slices does not provide sufficient condition for the whole four-dimensional metric to be homogeneous and isotropic. The equivalent definition is the following.

The spacetime \(\mathbb{M}\) is called homogeneous and isotropic if:

1) All sections \(S\) of constant time \(t = \text{const}\) are three-dimensional spaces of constant curvature;

2) Extrinsic curvature of the embedding \(S \hookrightarrow \mathbb{M}\) is homogeneous and isotropic for all \(t\).

Indeed, the first requirement provides the existence of such coordinate system where metric is block diagonal [7]

\[
g = \begin{pmatrix} 1 & 0 \\ 0 & h_{\mu\nu}(t, x) \end{pmatrix},
\]

where \(h_{\mu\nu}(t, x)\) is a constant curvature metric for all \(t\). For this metric the extrinsic curvature is [8]

\[
K_{\mu\nu} = -\frac{1}{2} h_{\mu\nu}.
\]

If it is homogeneous and isotropic, then it must be proportional to the metric, and we get the differential equation

\[
\dot{h}_{\mu\nu} = fh_{\mu\nu},
\]
where \( f(t) \) is an arbitrary sufficiently smooth function of time.

If \( f = 0 \), then nothing should be proved, and the metric has form (1) for \( a = \text{const} \).

Let \( f \neq 0 \). Then we introduce new time coordinate \( t \mapsto t' \) defined by the differential equation

\[
dt' = f(t)dt.
\]

Afterwards equation (11) becomes

\[
\frac{dh_{\mu\nu}}{dt'} = h_{\mu\nu}.
\]

Its general solution is

\[
h_{\mu\nu}(t', x) = C e^{t' \overset{\circ}{g}_{\mu\nu}(x)}, \quad C = \text{const} \neq 0,
\]

where \( \overset{\circ}{g}_{\mu\nu}(x) \) is a constant curvature metric on \( S \) which do not depend on time. It implies Eq. (1).

In general relativity, we assume that metric satisfies Einstein’s equations

\[
R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \frac{1}{2}g_{\alpha\beta}\Lambda = -\frac{1}{2}T_{\alpha\beta},
\]

(12)

where \( R_{\alpha\beta} \) is the Ricci tensor, \( R \) is the scalar curvature, and \( \Lambda \) is the cosmological constant. The matter energy-momentum tensor is denoted by \( T_{\alpha\beta} \).

The cosmological principle requires not only the metric but also the energy-momentum tensor to be homogeneous and isotropic. The most general form of the homogeneous and isotropic energy-momentum tensor in the coordinate system defined by Eq.(5) is \( T_{\alpha\beta} \)

\[
T_{\alpha\beta} = \begin{pmatrix}
\mathcal{E} & 0 \\
0 & -\mathcal{P}h_{\mu\nu}
\end{pmatrix},
\]

(13)

where \( \mathcal{E}(t) \) and \( \mathcal{P}(t) \) are the matter energy density and pressure.

One can easily calculate the Einstein tensor for metric (5). The off-diagonal component is

\[
R_{0\mu} = -\frac{4\dot{b}x_\mu}{(1 + bx^2)^2}.
\]

To satisfy Einstein’s equations (12) we must put

\[
\dot{b} \Leftrightarrow b = \text{const}
\]

because all other terms are diagonal. Thus for homogeneous and isotropic matter we return to the original metric (2) on the equations of motion.

Metric (5) describes spacetime which has homogeneous and isotropic space slices. We have shown that this important property is not sufficient for describing homogeneous and isotropic universe. The reason is that three of the six Killing vectors on space slices do depend on time, and their lift to the whole spacetime does not satisfy the four-dimensional Killing equations. The sufficient condition for the metric to be homogeneous and isotropic is (i) all space slices must be spaces of constant curvature and (ii) time derivative of the spacial part of the metric must be homogeneous and isotropic (in the coordinate system described above). Fortunately, metric of type (5) seems to be excluded by Einstein’s equations, thus there is no reason to worry too much about this dilemma.

This work is supported by the Russian Science Foundation under grant 14-50-00005.
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