We study the exact solution of $N = 2$ supersymmetric $SU(N)$ Yang-Mills theory in the framework of the Whitham-Toda hierarchy. We show that it is in fact obtainable by modulating the solution of the (generalized) Toda lattice associated with moduli of curves. The relation between the holomorphic pre-potential of the low energy effective action and the $\tau$-function of the (generalized) Toda lattice is also clarified.

1

Recently Seiberg and Witten obtained exact expressions for the metric on moduli space and the dyon spectrum of $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory by using a version of the Montonen-Olive duality and holomorphy of 4d supersymmetric theories. Their approach has been generalized to the case of other Lie group. Especially surprising in these results is unexpected emergence of elliptic (or hyperelliptic) curves and their periods. Although these objects appear in the course of determining the holomorphic pre-potential $\mathcal{F}$ of the exact low energy effective actions, physical significance of the curves themselves is unclear yet. It will be important to clarify their physical role. An interesting step in this direction has been taken from the view of integrable systems, in which the correspondence between the Seiberg-Witten solution and a Gurevich-Pitaevsky solution to the elliptic Whitham equations is pointed out.

The exact solution (Seiberg-Witten type) of $N=2$ supersymmetric $SU(N)$ Yang-Mills theory is described by the following data:

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Data 1 "family of the hyperelliptic curves"

\[ C : y^2 = P(x)^2 - \Lambda^{2N} , \quad P(x) = x^N + \sum_{k=0}^{N-2} u_{N-k} x^k, \]  \hspace{1cm} (1)

where \( u = (u_2, \ldots, u_N) \) are the order parameters and \( \Lambda \) is the \( \lambda \)-parameter of this theory.

Data 2 "the meromorphic differential on \( C \)"

\[ dS = \frac{x \frac{dP(x)}{dx}}{y} \, dx. \]  \hspace{1cm} (2)

Using these two data the holomorphic pre-potential \( F = F(a) \) is prescribed from the relation,

\[ \frac{\partial F}{\partial a_i} = \oint_{\beta_i} dS, \quad a_i = \oint_{\alpha_i} dS \quad (1 \leq i \leq N - 1) \]  \hspace{1cm} (3)

where \( \alpha_i, \beta_i \) are the standard homology cycles of the curve \( C \).

In this talk we discuss the exact solution of \( N = 2 \) SUSY \( SU(N) \) Yang-Mills theory in the framework of the Whitham equations \( 8 \) \ and clarify the relation with Toda lattice. Especially we would like to explain (i) the integrable structure of \( SU(N) \) Seiberg-Witten solution has its origin in (generalized) \( N \)-periodic Toda lattice (or chain), and (ii) the holomorphic pre-potential \( F(a) \) is obtainable from the \( \tau \)-function of this Toda lattice (or chain) by modulation (Whitham’s averaging method).

2

2.1

Let us begin by describing the integrable structure which already appears implicitly in these data. For the convenience we introduce the meromorphic function \( h \) by

\[ h = y + P(x), \]  \hspace{1cm} (4)

and consider the partial-derivation of \( dS \) by the moduli parameter \( a_i \) with fixing this function \( h \). It satisfies

\[ \left. \frac{\partial}{\partial a_i} dS \right|_{fix \, h} = dz_i, \]  \hspace{1cm} (5)
where $dz_i$ is the normalized holomorphic differential, i.e.
\[ \oint \alpha_j dz_i = \delta_{i,j} \]
Therefore these holomorphic differentials satisfy
\[ \frac{\partial}{\partial a_i} dz_j = \frac{\partial}{\partial a_j} dz_i \quad (1 \leq i, j \leq N-1), \]
where the derivations by the moduli parameters $a = (a_1, \ldots, a_{N-1})$ are performed by fixing $h$. These equations are the compatibility conditions for the system of holomorphic differentials under their evolutions by the moduli parameters $a$. So these compatibility conditions define a integrable system. $dS$ gives a solution for this system.

Nextly let us explain the moduli curve $C$ in the data can be understood as the spectral curve of $(N$-periodic) Toda chain. $(N$-periodic) Toda chain is a 1d integrable system defined by the equations,
\[ \partial_t^2 \phi_n = e^{-(\phi_{n+1}-\phi_n)} - e^{-(\phi_{n-1}-\phi_n)}, \quad \phi_{n+N} = \phi_n \quad (n \in \mathbb{Z}). \]
(7)
The Lax representation of $(N$-periodic) Toda chain is given by
\[ \partial_t \tilde{\Psi} = B \tilde{\Psi}, \quad C \tilde{\Psi} = x \tilde{\Psi}, \]
(8)
where $\tilde{\Psi} = (\Psi(1), \ldots, \Psi(N))$. $B = (B_{i,j})$ and $C = (C_{i,j})$ are $N \times N$ matrices of the following forms,
\[ B_{i,j} = d_i \delta_{i+1,j} - d_j \delta_{i,j+1} + h d_N \delta_i N \delta_j - \frac{d_N}{h} \delta_i 1 \delta_j N, \]
\[ C_{i,j} = d_i \delta_{i+1,j} + d_j \delta_{i,j+1} + b_i \delta_i j + h d_N \delta_i N \delta_j 1 + \frac{d_N}{h} \delta_i 1 \delta_j N. \]
(9)
$x$ and $h$ in (8) and (9) are the scalar variables independent of $t$. By identifying $b_n$ and $d_n$ with $\partial_t \phi_n / 2$ and $\exp\{-(\phi_{n+1}-\phi_n)/2\} / 2$ respectively, the compatibility condition of equations (8), $\partial_t C = [B, C]$, gives Toda chain equation (6).

The spectral curve of linear system (8) can be determined from the condition,
\[ \det | x - C | = 0, \]
(10)
which we can solve with respect to $h$:
\[ h = \pm \frac{y + P(x)}{\Lambda^N}, \]
(11)
where \( P(x) \) is the monic polynomial of degree \( N \) and \( \Lambda^N = 2d_1 \cdots d_N \). \( y \) in (11) is given by
\[
y^2 = P(x)^2 - \Lambda^2 N,
\]
which defines the spectral curve. It coincides with the moduli curve \( C \).

2.3

The solution of linear system (8), which becomes a meromorphic function on the spectral curve (except at the divisor of \( h \)), has been constructed by Krichever. It is shown that there exist \( N - 1 \) additional charges and that the solution \( \Psi(n) \) is allowed to depend on the corresponding new parameters \( \theta = (\theta_1, \ldots, \theta_{N-1}) \). It has the following form,
\[
\Psi(n, t, \theta)(p) = \exp\{-n \int P dB_0 + t \int P dB_1 + \sqrt{-1} \sum_{i=1}^{N-1} \theta_i \int P dz_i\} \times \text{theta functions},
\]
where \( dB_0, 1 \) are the normalized meromorphic differentials specified by their behaviors around the divisor of \( h \).

Now we will describe the modulation of wave function (13). Suppose there exists an observer whose length scales \( T_0, T_1 \) and \( a_i (1 \leq i \leq N-1) \) are very slow relative to \( n, t \) and \( \theta_i \) :
\[
T_0 = -\epsilon n, \quad T_1 = \epsilon t, \quad a_i = \sqrt{-1} \epsilon \theta_i, \quad (\epsilon \ll 1)
\]
and then the spectral curve is "slowly" varying on these observer’s scales. On this circumstance the wave function will have the following WKB form,
\[
\Psi(n, t, \theta) = A(T_0, T_1, a, \epsilon) e^{\frac{1}{\epsilon} S(T_0, T_1, a)}.
\]
Taking up the 0-th order approximation leads to the existence of a meromorphic differential, \( dS \), which satisfies
\[
\frac{\partial}{\partial a_i} dS = dz_i, \quad \frac{\partial}{\partial T_0} dS = dB_0, \quad \frac{\partial}{\partial T_1} dS = dB_1.
\]
By setting \( T_0 = 0 \) and \( T_1 = 1 \) these equations reduce to those in (5). The meromorphic differential \( dS \) in data (2) can be obtainable from the modulation of wave function (13).
2.4

According to the prescription the \( \tau \)-function can be also introduced by studying the behaviors of wave function around the divisor of \( h \). Let us consider the modulation of this \( \tau \)-function. With the same spirit as for the case of wave function we can derive the expansion,

\[
\ln \tau(n, t, \theta) = \epsilon^{-2} (F(T_0, T_1, a) + \mathcal{O}(\epsilon)) .
\]

(17)

The quantity, \( F \), which is the 0-th order approximation, turns out to satisfy

\[
F(T_0 = 0, T_1 = 1, a) = \mathcal{F}(a),
\]

where \( \mathcal{F} \) is the pre-potential of \( N = 2 \) SUSY \( SU(N) \) Yang Mills theory.

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