The Drinfel’d centres of String 2-groups

Christoph Weis

Let $G$ be a compact connected Lie group and $k \in H^4(BG, \mathbb{Z})$ a cohomology class. The String 2-group $G_k$ is the central extension of $G$ by the 2-group $[\ast/U(1)]$ classified by $k$. It has a close relationship to the level $k$ extension of the loop group $LG$. We compute the Drinfel’d centre of $G_k$ as a smooth 2-group. When $G$ is semisimple, we prove that the Drinfel’d centre is equal to the invertible part of the category of positive energy representations of $LG$ at level $k$ (as long as we exclude factors of $E_8$ at level 2).

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Introduction

Let $G$ be a compact connected Lie group and $k \in H^4(BG, \mathbb{Z})$ a cohomology class. The String 2-group $G_k$ is the central extension of $G$ by the 2-group $[\ast/U(1)]$ classified by $k$. It has a close relationship to the level $k$ extension of the loop group $LG$. In this note, we compute its Drinfel’d centre $\mathcal{Z} G_k$ in the context of smooth 2-groups. The result of this computation is interesting: we find that $\mathcal{Z} G_k$ recovers the invertible part of $\text{Rep}^k LG$ when $G$ is semisimple (as long as we exclude factors of $E_8$ at level 2).

Before stating the result in more detail, we introduce the model of the String 2-group and centre we use.

The String 2-group

A 2-group $[\text{BL04}]$ is a group object in the bicategory $\text{Cat}$ of categories. It is a monoidal groupoid $(C, 1, \otimes, \omega)$ all of whose objects are invertible: for all $x \in C$, there exists an object $x^{-1}$ such that
\(x \otimes x^{-1} \cong x^{-1} \otimes x \cong I\). The set of isomorphism classes of \(C\) forms a group \((\pi_0 C, \otimes)\), and the endomorphisms of \(I\) form an abelian group \((\pi_1 C, \circ = \otimes)\)\(^1\). Every object \(x \in \pi_0 C\) acts on \(\pi_1 C\) by conjugation:

\[f \mapsto \text{id}_x \otimes f \otimes \text{id}_{x^{-1}}\]

This assembles into an action \(\rho : \pi_0 C \to \text{Aut}(\pi_1 C)\). The associator of \(C\) satisfies the pentagon equation

\[\rho(g)(\omega(g', g'' , g''' )) \omega(g, g' , g'' , g''' ) \omega(g, g' , g'' , g''' ) = \omega(gg', g'' , g''' ) \omega(g, g' , g'' , g''' )\]

for all \(g, g', g'' , g''' \in \pi_0 C\). This is the equation of a group 3-cocycle in \(Z^3(\mathcal{B} \pi_0 C, \pi_1 C)\). Cohomologous 3-cocycles give rise to equivalent monoidal categories, and 2-groups are completely classified by the data \(\big(\pi_0 C, \pi_1 C, \rho, [\omega] \in H^3(\mathcal{B} \pi_0 C, \pi_1 C)\big)\)\(^2\). The data of a (1-)group \(G\) induces a 2-group (also denoted \(G\)) with objects \(G\), tensor product the multiplication on \(G\), and only identity morphisms. An abelian group \(A\) also defines a 2-group \([*/A]\) with a single object \(I\), whose endomorphisms form the group \(\pi_1 [*/A] = A\). Any 2-group \(C\) is an extension of 2-groups of this type:

\[\big[{*/\pi_1 C}\big] \to C \to \pi_0 C.\]

The data of this extension is encoded by the conjugation action \(\rho\) and the cohomology class \([\omega]\) of the associator. When \(\rho\) is trivial, one speaks of a central extension. Central extensions of \(G\) by \([*/A]\) are thus classified by \([\omega] \in H^3(\mathcal{B} G, A)\), where \(A\) carries the trivial \(G\)-module structure.

Smooth 2-groups are group objects in the bicategory of smooth groupoids (see Section 1). Just as for discrete 2-groups, there are smooth 2-groups \(G\) and \([*/A]\) associated to a Lie group \(G\) and an abelian Lie group \(A\). Central extensions of \(G\) by \([*/A]\) are also classified by \(H^3(\mathcal{B} G, A)\)\(^3\), though one has to work with Segal-Mitchison cohomology \([\text{Seg}70; \text{Bry}00]\) to make this precise. We recall Segal-Mitchison cohomology in Section 2. Let \(G\) be a compact connected Lie group. The short exact sequence of coefficients \(\mathbb{Z} \to \mathbb{R} \xrightarrow{\exp} U(1)\) gives rise to an isomorphism \(H^3(\mathcal{B} G, U(1)) \cong H^4(\mathcal{B} G, \mathbb{Z})\). By abuse of notation, we write \(k \in H^3(\mathcal{B} G, \mathbb{Z})\) to denote a \(U(1)\)-valued 3-cocycle on \(G\) representing \(k\). For simple \(G\), \(H^4(\mathcal{B} G, \mathbb{Z}) \cong \mathbb{Z}\), and \(k\) is often called a level.

The String 2-group \(G_k\) is the central extension of \(G\) by \([*/U(1)]\) classified by \(k\). A large body of work has been devoted to constructing and understanding the String 2-groups, with particular interest in the case where \(G\) is simple simply-connected and \(k\) is a generator of \(H^4(\mathcal{B} G, \mathbb{Z})\): The 2-group \(G_1\) is a model for the universal 3-connected cover of the Lie group \(G\). Such a 3-connected cover admits no incarnation as a finite-dimensional Lie group \([\text{Spi}11; \text{Footnote 2}]\), but it has been constructed variously as a topological group \([\text{Sto}96; \text{ST}04]\), an infinite-dimensional Lie (2-)group \([\text{BL}04; \text{BSC}07; \text{Hen}08; \text{NSW}13]\), a diffeological 2-group \([\text{Wal}12]\), and a smooth \(\infty\)-group \([\text{FSS}*12; \text{Bun}20]\). We use the model of \(G_k\) as a finite-dimensional smooth 2-group given in \([\text{Spi}11]\).

Let \(LG = C^\infty(S^1, G)\) denote the loop group of \(G\). Transgression \([\text{BM}94; \text{Wal}16]\) establishes a correspondence between String 2-groups \(\{G_k\}\) and central extensions \(\{LG_k\}\) of the loop group.\(^4\) The category \(\text{Rep}^G LG\) of positive energy representations of \(LG\) at level \(k\) is a linear braided monoidal category, defined when \(k\) satisfies a positivity condition. It is in fact a modular tensor category, and defines a 3-dimensional Topological Quantum Field Theory via the Reshetikhin-Turaev construction \([\text{RT}91]\). This is understood to be Chern-Simons theory with gauge group \(G\) at level \(k\) \([\text{Fre}04]\), see \([\text{Hen}17]\) for an argument in the simply-connected case.

\(^1\)There are two compatible group structures on \(\pi_1 C\) given by tensor product and composition. By the Eckmann-Hilton argument, these products agree and are commutative.

\(^2\)The notation indicates that \([*/A]\) is the quotient stack associated to the trivial \(A\)-action on the point \(*\).

\(^3\)The procedure requires a choice of connection on \(G_k\). The correspondence between \(\{G_k\}\) and \(\{LG_k\}\) is bijective when \(G\) is simply-connected.
The smooth centre

The centre of a monoid $M$ is the set of elements $z \in M$ such that $zm = mz$ for all $m \in M$. This concept admits a categorification to monoidal categories. The Drinfel’d centre $\mathcal{Z}C$ of a monoidal category $C$ is the monoidal category whose objects are pairs $(X, \gamma)$, where $X \in C$ and $\gamma : X \otimes - \to - \otimes X$ is a natural isomorphism satisfying the hexagon equation (recalled in Section 3). Such an isomorphism is called a half-braiding for $X$. Just as the centre of a monoid is a commutative monoid, the Drinfel’d centre of $C$ is a braided monoidal category. The centre $\mathcal{Z}C$ of a 2-group $C$ is again a 2-group (Lemma 3.4). The braiding $\beta$ makes $\mathcal{Z}C$ a braided categorical group [JS93]. The self-braidings $\beta_{x,x} \in \text{End}(x \otimes x) = \pi_1 \mathcal{Z}C$ of objects $x \in C$ assemble into a quadratic form [EGNO16, Ch 8.4]

$$q : \pi_0 \mathcal{Z}C \to \pi_1 \mathcal{Z}C$$

$$x \mapsto \beta_{x,x}.$$ 

The map $q$ encodes $\mathcal{Z}C$ up to braided monoidal equivalence [EM54].

If $C$ is a monoidal category with a smooth structure, the Drinfel’d centre of the underlying monoidal category has a distinguished subcategory on objects with smooth half-braidings. In this note, we compute this smooth Drinfel’d centre for the String 2-groups $G_k$. The result is another instance of the close relationship between $G_k$ and the associated central extension of the loop group, $LG_k$. We prove that for compact connected semisimple $G$ and positive-definite $k \in H^4(\mathbb{B}G, \mathbb{Z})$, the Drinfel’d centre of $G_k$ is

$$\mathcal{Z}G_k \simeq (\text{Rep}^k LG)^\times,$$

the 2-group of invertible objects and invertible morphisms in $\text{Rep}^k LG$, as long as we exclude factors of $E_8$ at level $k = 2$. The proof of the above statement is indirect: we compute the left hand side explicitly, and show the resulting braided categorical group agrees with that on the right hand side.

**Question.** Can all of $\text{Rep}^k LG$ be recovered as a generalised centre of the corresponding String 2-group $G_k$?

**Statement of Results**

Let $G$ be a compact simple simply-connected Lie group, and $\mathfrak{g}$ its complexified Lie algebra. There is a unique smallest positive-definite form $I : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ which is $\text{Ad}_G$-invariant and satisfies $I(X, X) \in 2\mathbb{Z}$ for all coroots of $\mathfrak{g}$ (a coroot is an element $X \in \mathfrak{g}$ satisfying $e^{2\pi i X} = 1$). We use $\text{exp} : \mathbb{C} \to \mathbb{C}^\times$ to denote the map $w \mapsto e^{2\pi i w}$. Its restriction $\exp : \mathbb{R} \to U(1)$ has kernel $\mathbb{Z} \hookrightarrow \mathbb{R}$.

**Theorem.** The Drinfel’d centre of the String 2-group $G_k$ is the braided categorical group with $\pi_0 \mathcal{Z}G_k = Z(G)$, $\pi_1 \mathcal{Z}G_k = U(1)$, and braided monoidal structure encoded by the quadratic form

$$q : Z(G) \to U(1)$$

$$z \mapsto \exp \frac{k}{2} I(\bar{z}, \bar{z}).$$

Here, $\bar{z} \in \mathfrak{g}$ denotes an arbitrary lift of $z \in Z(G)$ to the Lie algebra $\mathfrak{g}$ of $G$.

The inner product $\frac{1}{2} I(\bar{z}, \bar{z})$ is a real number. Its values on different lifts of $z \in Z(G)$ differ by integers. By composing with $\exp$, we get a well-defined map $Z(G) \to U(1)$.

\footnote{\text{Rep}^{k=2} LE_8 contains a non-trivial invertible object, while $\mathcal{Z}E_{8,2}$ does not. The invertibility of the non-identity object in $\text{Rep}^{k=2} LE_8$ may be viewed as an accident of low level. See for example the computation in [Fuc91].}

\footnote{For $SU(n)$, $I$ is the trace of the product of matrices: $I(X, Y) = \text{tr} XY$.}
After a choice of maximal torus of $G$, every element of the centre lifts to an element of the coweight lattice (see Section 5). Their norms under $I$ can be found e.g. in [Bou94]. We list the results for Lie groups with non-trivial centre in Table 1. The centreless Lie groups $E_8, F_4, G_2$ all have trivial Drinfel’d centre.

| Type | $G_k$ | $Z(G)$ | $q(z) = \exp \left( \frac{k}{2} I(z, z) \right)$ |
|------|-------|--------|----------------------------------|
| $A_{n-1}$ | SU$(n)_k$ | $\mathbb{Z}/n \langle \omega_1 \rangle$ | $q(\omega_1) = \exp \left( \frac{k(n-1)}{2n} \right)$ |
| $B_{n \geq 2}$ | Spin$(2n + 1)_k$ | $\mathbb{Z}/2 \langle \omega_1 \rangle$ | $q(\omega_1) = \exp \left( \frac{k}{4} \right) = (-1)^k$ |
| $C_{n \geq 2}$ | Sp$(n)_k$ | $\mathbb{Z}/2 \langle \omega_n \rangle$ | $q(\omega_n) = \exp \left( \frac{k(n-1)}{2n} \right) = (-1)^{k-n}$ |
| $D_{2n+1}$ | Spin$(4n + 2)_k$ | $\mathbb{Z}/4 \langle \omega_{2n+1} \rangle$ | $q(\omega_{2n+1}) = \exp \left( \frac{k(2n+1)}{8} \right)$ |
| $D_{2n \geq 4}$ | Spin$(4n)_k$ | $\mathbb{Z}/2 \langle \omega_{2n-1}, \omega_{2n} \rangle$ | $q(\omega_{2n-1}) = q(\omega_{2n}) = \exp \left( \frac{k(n)}{4} \right) = i^{k-n}$ |
| $E_6$ | $E_{6,k}$ | $\mathbb{Z}/3 \langle \omega_1 \rangle$ | $q(\omega_1) = \exp \left( \frac{2k}{3} \right)$ |
| $E_7$ | $E_{7,k}$ | $\mathbb{Z}/2 \langle \omega_7 \rangle$ | $q(\omega_1) = \exp \left( \frac{k}{7} \right) = i^k$ |

Table 1: The Drinfel’d centre of $G_k$ (displayed for groups with nontrivial centre). We denote by $\omega_i$ the $i$th coweight. The conventions for the numbering are taken from [Bou94].

A crucial step in our proof of the above theorem is the calculation of the centre of String 2-groups for $G = T$ a torus. They are also called categorical tori [Gan18]. Let $t = \text{Lie}(T)$ be the Lie algebra of a torus $T$, $\Lambda = \text{Hom}(T, U(1)) \subset t^*$ its character lattice and $\Pi = \text{Hom}(U(1), T) \subset t$ its cocharacter lattice. The group $H^4(\mathcal{B}T, \mathbb{Z})$ is the group of symmetric bilinear forms $\langle \cdot, \cdot \rangle : t \otimes t \to \mathbb{R}$, such that $\langle \pi, \pi \rangle \in 2\mathbb{Z}$ for all $\pi \in \Pi$. To describe the String 2-group associated to such a bilinear form, we pick a (not-necessarily-symmetric) bilinear form $J : t \otimes t \to \mathbb{R}$ which restricts to an integral form on $\Pi \otimes \Pi$ and satisfies $J(x, y) + J(y, x) = -\langle x, y \rangle$ for all $x, y \in t$.

The computation of the smooth Drinfel’d centre of the associated categorical torus $T_J$ was sketched in [FHLT10]. We provide a proof of the following statement in Section 4.

**Proposition.** The Drinfel’d centre of $T_J$ has Lie group of objects

$$\pi_0 \mathcal{Z} T_J = (\Lambda \oplus t)/\Pi,$$

where the inclusion $\Pi \hookrightarrow \Lambda \oplus t$ is $\pi \mapsto (-J(\cdot, \pi) - J(\pi, \cdot), \pi)$. Further, $\pi_1 \mathcal{Z} T_J = U(1)$, and the braided monoidal structure is encoded by the quadratic form

$$q : (\Lambda \oplus t)/\Pi \to U(1) \quad [\lambda, x] \mapsto \lambda(x) \exp(J(x, x)).$$

Every compact connected Lie group $G$ fits into a short exact sequence $Z \hookrightarrow \tilde{G} \to G$. Here $\tilde{G} = T \times \Pi_i G_i$ is a product of simple simply-connected groups $G_i$ with a torus $T$, and $Z \subset Z(\tilde{G})$ is a finite subgroup of its centre [MT94, Cor V.5.31]. A central extension of $G$ by $[\ast/\text{U}(1)]$ pulls back to a central extension of $\tilde{G}$ by $[\ast/\text{U}(1)]$, giving the String 2-group $\tilde{G}_k$. It is classified by the pullback degree 4 cohomology class in $H^4(\mathcal{B}G, \mathbb{Z}) \simeq H^4(\mathcal{B}T, \mathbb{Z}) \times \Pi_i H^4(\mathcal{B}G_i, \mathbb{Z})$ and we write $k = (J, \{ k_i \})$.

**Theorem.** The group $Z = \ker (\tilde{G} \to G)$ admits a unique lift to $\mathcal{Z} \tilde{G}_k$, and the quadratic form $q$ is trivial on $Z$. The Drinfel’d centre of $G_k$ is the subquotient

$$\pi_0 \mathcal{Z} G_k = Z^\perp/Z,$$
where $Z^\perp$ denotes the subgroup of $\pi_0\mathcal{Z}\tilde{G}_k$ on elements satisfying $q(x + z) = q(x)$ for all $z \in Z$. The quadratic form on $\pi_0\mathcal{Z}\tilde{G}_k$ descends to a quadratic form on $\pi_0\mathcal{Z}G_k$, and this quadratic form describes the braided monoidal structure of $\mathcal{Z}G_k$.

The Drinfel’d centre of the covering String 2-group $\tilde{G}_k = \times_i G_{i,k} \times T_J$ is given by

$$\pi_0\mathcal{Z}(J,\{k_i\}) = (\Lambda \oplus t)/\Pi \times \Pi G_{k_i},$$

and quadratic form the product of the quadratic forms $q_{k_i}$ and $q_J$ computed previously.

In the simply-connected case, every element of the centre admits a half-braiding, but in general the subset of the centre admitting a half-braiding varies with $k$. We present the result for the case $G = \text{SO}(4)$ here (the computation is sketched in more detail in Example 5.9). The relevant cohomology group $H^4(\mathbb{R}\text{SO}(4),\mathbb{Z})$ has two generators: the first Pontryagin class $p_1$ and the Euler class $\chi$. For the cohomology class $k = a \cdot p_1 + b \cdot \chi$, the Drinfel’d centre is given by

$$\mathcal{Z}\text{SO}(4)_k = \begin{cases} 
\text{Vec}_{\mathbb{Z}/2}^x & 2a + b \equiv 0 \mod 4 \\
\text{sVec}^x & 2a + b \equiv 2 \mod 4 \\
\text{Vec}^x & \text{else.}
\end{cases}$$

Here, $\text{Vec}_{\mathbb{Z}/2}^x$ denotes the trivially braided 2-group $[*/U(1)] \times \mathbb{Z}/2$, $\text{sVec}^x$ is super-$\mathbb{Z}/2$ with non-trivial self-braiding of $-1 \in \mathbb{Z}/2$, and $\text{Vec}^x = [*/U(1)]$ is the trivial Drinfel’d centre whose only object is the monoidal unit. As the notation indicates, they are the maximal braided sub-2-groups of the braided fusion categories $\text{Vec}[\mathbb{Z}/2]$, $\text{sVec}$ and $\text{Vec}$.

**Structure**

Section 1 is a quick introduction to groupoids with smooth structure. We review smooth 2-groups in Section 2, recalling in particular the model for the String 2-groups given in [SP11]. In Section 3 we discuss centres of smooth 2-groups as a special case of the notion of centre defined in [Str04]. We compute this centre for categorical tori in Section 4 and for the String 2-groups in Section 5. We end by showing that $\mathcal{Z}G_k = (\text{Rep}^k LG)^\times$ for $G$ compact connected semisimple.

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**1. Smooth groupoids**

We use $\text{Man}$ to denote the category of paracompact smooth manifolds and smooth maps. We equip it with the structure of a site by declaring covers to be surjective submersions $f : Y \to M$, and denote the $n$-fold fibre product of a cover along itself by $Y^{[n]} := Y \times_M Y \times_M \cdots Y$. 
Lie groupoids

A Lie groupoid $G_\bullet$ is a groupoid object in $\text{Man}$. Its data are two manifolds $G_1, G_0$ with a pair of surjective submersions $s, t : G_1 \rightarrow G_0$, and smooth maps

$$\circ : G_1 \times_{G_0} G_1 \rightarrow G_1 \quad \quad \text{id}_* : G_0 \rightarrow G_1 \quad \quad (-)^{-1} : G_1 \rightarrow G_1,$$

implementing composition, identities and inverses, respectively. A smooth functor $G_\bullet \rightarrow H_\bullet$ is a pair of smooth maps $G_0 \rightarrow H_0, G_1 \rightarrow H_1$ that preserve identities and composition. A smooth natural transformation between two smooth functors is a smooth map $G_0 \rightarrow H_1$ that makes the usual naturality diagrams commute — see [MP97] for an introduction.

**Example 1.1.** Every manifold $M$ defines a Lie groupoid $M \rightrightarrows M$, with only identity morphisms.

**Example 1.2.** A Lie group $G$ has an associated Lie groupoid $[*//G] := G \rightrightarrows *$.

A smooth functor $F_\bullet : G_\bullet \rightarrow H_\bullet$ is fully faithful if

$$
\begin{array}{ccc}
G_1 & \xrightarrow{F_1} & H_1 \\
\downarrow{(s,t)} & & \downarrow{(s,t)} \\
G_0 \times G_0 & \xrightarrow{F_0 \times F_0} & H_0 \times H_0
\end{array}
$$

is a pullback square, and it is essentially surjective if $s \circ p_2 : G_0 \times_{H_0} H_1 \rightarrow H_0$ is a surjective submersion (the pullback $G_0 \times_{H_0} H_1$ is formed along the maps $F_0 : G_0 \rightarrow H_0$ and $t : H_1 \rightarrow H_0$). A fully faithfully essentially surjective functor $F_\bullet$ is a smooth equivalence [MP97]. Given a submersion $f : Y \rightarrow G_0$ from a manifold to the space of objects of a Lie groupoid $G_\bullet$, the pullback groupoid $f^*G_\bullet$ is the groupoid $f^*G_1 \rightrightarrows Y$, with manifold of morphisms given by the pullback

$$
\begin{array}{ccc}
f^*G_1 & \xrightarrow{} & G_1 \\
\downarrow{(s,t)} & & \downarrow{(s,t)} \\
Y \times Y & \xrightarrow{f \times f} & G_0 \times G_0.
\end{array}
$$

It is constructed such that the natural functor $f^*G_\bullet \rightarrow G_\bullet$ is fully faithful. When $f$ is also surjective (and hence a cover), the functor $f^*G_\bullet \rightarrow G_\bullet$ is a smooth equivalence.

**Example 1.3.** The Čech groupoid $f^*M$ associated to a cover $f : Y \rightarrow M$ is $Y^{[2]} \rightrightarrows Y$ with composition given by the map $Y^{[2]} \times_Y Y^{[2]} = Y^{[3]} \rightarrow Y^{[2]}$ projecting out the middle factor. This Lie groupoid is equivalent to $M \rightrightarrows M$.

A crucial example for our computation is the Lie groupoid built from a smooth Čech 2-cocyle (Example [4]). We briefly recall how to compute smooth Čech cohomology $\check{H}^*(M, A)$ of a manifold $M$ with coefficients in an abelian Lie group $A$. The cochain complex computing Čech cohomology with respect to a cover $Y \rightarrow M$ is given by

$$C^p_c(M, A) := C^\infty(Y^{[p+1]}, A)$$

with differential $d_{\text{Čech}}$ the alternating sum over the pullbacks along the maps $\delta_i : Y^{[p]} \rightarrow Y^{[p-1]}$ that project out the $i$-th factor:

$$\delta_i^* : C^\infty(Y^{[p-1]}, A) \rightarrow C^\infty(Y^{[p]}, A).$$

Any two covers $Y, Z \rightarrow M$ admit a common refinement $Y \times_M Z \rightarrow M$. The assignment of cohomology groups assembles into a (contravariant) functor from the category of covers of $M$ into the category of graded abelian groups. Čech cohomology of $M$ is the colimit over this diagram. A cover is good if $Y^{[p]}$ is
The collection of good covers is a cofinal subset of the poset of covers of $M$. It is well-known that Čech cohomology with respect to any good cover computes sheaf cohomology for a paracompact manifold. Hence, the colimit we used to define Čech cohomology recovers sheaf cohomology, and can be computed using any good cover.

**Example 1.4.** A cocycle representing $[\lambda] \in \tilde{H}^1(M, A)$ is a map $\lambda : \gamma \to A$ for some cover $\gamma \to M$. The principal $[*/A]$-bundle classified by $\lambda$ is the Lie groupoid $E^\lambda : \gamma \times A \to \gamma$, with source and target given by the two projections $\gamma \times A \Rightarrow \gamma$, and composition given by

$$
((y_0, y_1, a), (y_1, y_2, b)) \mapsto (y_0, y_2, a + b + \lambda(y_0, y_1, y_2)).
$$

The cocycle condition ensures that this is associative (a condition checked on $\gamma^3$).

Lie groupoids, smooth functors and smooth natural transformations form a 2-category which we denote $\text{LieGpd}$. This 2-category of groupoids internal to $\text{Man}$ is not a good 2-category of smooth groupoids, because smooth functors are too strict to implement the principle of equivalence: A smooth equivalence $f^*G \to G$ usually does not admit an inverse. Any sufficiently generic cover $f : \coprod U_i \to M$ provides a counterexample.

The standard way to proceed is to invert these morphisms by force. We follow [Pro96] in writing $\text{LieGpd}[\text{W}^{-1}]$ to denote the localisation of $\text{LieGpd}$ at smooth equivalences. Every 1-morphism in $\text{LieGpd}[\text{W}^{-1}]$ can be represented by an anafunctor [Rob12]: a span

$$G_\bullet \xleftarrow{\sim} f^*G_\bullet \to H_\bullet,$$

whose left leg is a smooth equivalence. Morphisms between anafunctors are defined as natural transformations on a common refinement of the involved covers [Rob12]. In particular, if two anafunctors have the same left leg $G_\bullet \xleftarrow{\sim} f^*G_\bullet$, then all morphisms between them are represented by natural transformations between their right legs $f^*G_\bullet \to H_\bullet$. The introduction of anafunctors manifestly inverts smooth equivalences. They can be viewed as smooth functors defined on some cover of the source.

**Example 1.5.** The category of anafunctors from $M$ to $[*/G]$ is equivalent to the category of $G$-principal bundles over $M$. The subcategory of smooth functors $M \to [*/G]$ is that of topologically trivial bundles.

**Differentiable stacks**

Every Lie groupoid $G_\bullet$ defines a functor of bicategories $\text{Hom}(-, G_\bullet) : \text{Man}^{\text{op}} \to \text{Gpd}$, sending a manifold $M$ to the groupoid of smooth functors $(M \Rightarrow M) \to G_\bullet$. Given any functor $F : \text{Man}^{\text{op}} \to \text{Gpd}$, there is a natural way to evaluate it on the Čech groupoid (Example 1.3) associated to a cover $f : Y \to M$ (we suppress natural 2-cells in the diagram):

$$F(f^*M) := \text{lim} \left(F(Y) \Rightarrow F(Y^2) \Rightarrow F(Y^3)\right).$$

The limit on the right hand side is represented by the so-called groupoid of descent data (see e.g. [Vis04]) for $F$ with respect to $Y$. For $F = \text{Hom}(-, G_\bullet)$, this is the groupoid of smooth functors $F(f^*M) = \text{Hom}(f^*M, G_\bullet)$. The fact that smooth functors are too strict is reflected in the fact that the restriction functor $f^* := \text{Hom}(f, G_\bullet) : \text{Hom}(M, G_\bullet) \to \text{Hom}(f^*M, G_\bullet)$ is generally not an equivalence of categories.

**Definition 1.6.** A stack (of groupoids over the site of manifolds) is a functor of bicategories $F : \text{Man}^{\text{op}} \to \text{Gpd}$ satisfying descent: It sends coproducts to products $F(\coprod M_i) \xrightarrow{\sim} \prod_i F(M_i)$ and for every cover $f : Y \to M$, the restriction functor $F(f) : F(M) \to F(f^*M)$ is an equivalence of groupoids.
Stacks assemble into a bicategory, the full sub-bicategory $\text{SmSt} \subset [\text{Man}^{\text{op}}, \text{Gpd}]$ on functors which satisfy descent.

**Theorem 1.7.** [Girëd] The forgetful functor $\text{SmSt} \to [\text{Man}^{\text{op}}, \text{Gpd}]$ has a left adjoint, the stackification functor $L : [\text{Man}^{\text{op}}, \text{Gpd}] \to \text{SmSt}$.

**Definition 1.8.** The stack presented by $G_\bullet$ is the stackification of the associated 2-presheaf $\text{Hom}(-, G_\bullet)$.

The assignment $G_\bullet \mapsto \text{Hom}(-, G_\bullet)^\#$ defines a functor $\text{LieGpd} \to \text{SmSt}$.

**Definition 1.9.** The category $\text{DiffSt}$ of differentiable stacks is the full subcategory of $\text{SmSt}$ on the essential image of $\text{LieGpd} \to \text{SmSt}$.

**Theorem 1.10.** [Pro96] The functor $\text{LieGpd} \to \text{DiffSt}$, sending a Lie groupoid $G_\bullet$ to the stackification of $\text{Hom}(-, G_\bullet)$ induces an equivalence

$$\text{LieGpd}[W^{-1}] \cong \text{DiffSt}.$$

This tells us that inverting weak equivalences has the same effect as stackification. It allows us to use both the explicit description of $\text{LieGpd}[W^{-1}]$ and the nice formal properties of $\text{DiffSt} \subset \text{SmSt}$. We denote the stack presented by a Lie groupoid $G_\bullet$ by the same symbol, $G_\bullet : \text{Man}^{\text{op}} \to \text{Gpd}$. Its value $G_\bullet(M)$ on a manifold $M$ is the groupoid of morphisms $M \to G_\bullet$ in $\text{LieGpd}[W^{-1}]$, which can be computed as the groupoid of anafunctors $M \to G_\bullet$.

Every manifold $M$ defines a functor $\text{ev}_M : \text{DiffSt} \to \text{Gpd}$, given by evaluation on $M$. In particular, $\text{ev}_*$ sends each stack $X$ to its groupoid of points $X(*)$. We view the remaining data of a differentiable stack as equipping $X(*)$ with a smooth structure.

## 2. Smooth 2-groups

### From discrete to smooth 2-groups

A (discrete) 2-group is a monoidal category $(C, \otimes, 1)$ with the property that every morphism in $C$ has an inverse, and every object $x \in C$ has a weak inverse: an object $x^{-1} \in C$ satisfying $x^{-1} \otimes x \cong 1 \cong x \otimes x^{-1}$.

Up to equivalence, a 2-group $C$ is characterised by 4 invariants (see e.g. [BL04]):

1. $\pi_0 C$, the group of isomorphism classes of objects
2. $\pi_1 C$, the abelian group of endomorphisms of $1$
3. $\rho : \pi_0 C \to \text{Aut}(\pi_1 C)$, the conjugation action $x \mapsto \text{id}_x \otimes - \otimes \text{id}_{x^{-1}}$
4. $[\omega] \in H^3(\mathcal{B}\pi_0 C, \pi_1 C)$, the cohomology class of the associator.

We can build the 2-group associated to this data explicitly: The underlying groupoid is

$$\pi_0 C \rtimes_\rho \pi_1 C \Rightarrow \pi_0 C,$$

with both source and target morphisms given by projection to $\pi_0 C$.

The zero section $\pi_0 C \to \pi_0 C \rtimes_\rho \pi_1 C$ serves as the assignment of the identity morphism to each object. This canonically identifies the endomorphism group of each object $g \in \pi_0 C$ as $\text{End}(g) = \pi_1 C$.

There are no morphisms between objects corresponding to different elements $g \neq g' \in \pi_0 C$. (Note that in the model of smooth 2-groups we use, it will not be possible to realise the String 2-groups in such a way.) The composition of endomorphisms is group multiplication in $\pi_1 C$. The tensor product on objects and morphisms is given by the group multiplication in $\pi_0 C$ and $\pi_0 C \rtimes_\rho \pi_1 C$, respectively. The unitor isomorphisms are trivial. An associator for the tensor structure must pick out an endomorphism
\[ \omega(g, g', g'') \in \text{End}(gg'g'') = \pi_1 C \text{ for each triple } g, g', g'' \in \pi_0 C. \] Any cocycle \( \omega : \pi_0 C^3 \to \pi_1 C \) representing the cohomology class \([\omega]\) will do.

The notion of a smooth 2-group is the categorification of that of a Lie group (a smooth monoid with a smooth map sending each element to its inverse). One may avoid writing down conditions on the map \( g \mapsto g^{-1} \) by encoding the existence of the inverse indirectly:

**Definition 2.1.** A Lie group is a unital monoid \((G, \mu)\) in the 1-category of smooth manifolds, such that the map \((p_1, \mu) : G \times G \to G \times G\) is a diffeomorphism.

The smooth inverse appearing in the traditional definition of Lie groups can be recovered as

\[
G \xrightarrow{i_1} G \times G \xrightarrow{(p_1, \mu)^{-1}} G \times G \xrightarrow{p_2} G.
\]

This definition now categorifies verbatim.

**Definition 2.2.** A group object in a bicategory with finite products is a unital monoid \((G_\bullet, \otimes)\) such that

\[
(p_1, \otimes) : G_\bullet \times G_\bullet \to G_\bullet \times G_\bullet
\]

is an equivalence.

**Example 2.3.** A discrete 2-group is a group object in the bicategory of groupoids. That is a monoidal category in which each object and morphism is invertible.

**Definition 2.4.** ([SP11]) A smooth 2-group is a group object in \(\text{DiffSt}\).\(^7\)

**Example 2.5.** Every Lie group \(G\) defines a smooth 2-group. It has underlying Lie groupoid \(G \rightrightarrows G\), with tensor product given by multiplication.

**Example 2.6.** Let \(A\) be a Lie group. We denote by \([*/A]\) the Lie groupoid \(A \rightrightarrows *\). The multiplication on \(A\) equips it with a smooth 2-group structure iff \(A\) is abelian.

**Example 2.7.** The 2-category of discrete 2-groups is the full subcategory of the 2-category of smooth 2-groups on those whose underlying Lie groupoid is discrete.

**Example 2.8.** A strict smooth 2-group is a category object in the category of Lie groups. Up to equivalence, one may always bring the underlying Lie groupoid into the form \(G \times_\rho H \rightrightarrows G\). Here, \(G, H\) are Lie groups equipped with a \(G\)-action \(\rho : G \to \text{Aut}(H)\), and a target homomorphism \(t_0 : H \to G\). (This data \((G, H, t, \rho)\) satisfies further conditions and is known as a crossed module.) The source map can be chosen to be projection to \(G\), and the target map is given by \(t : (g, h) \mapsto t_0(h)g\). Composition is multiplication in \(H\) (i.e. \((t_0(h)g, h') \circ (g, h) = (t_0(h'h)g, h'h))\), tensor product is multiplication in the object and morphism groups, and all the other data is trivial. This example is spelt out in great detail in [Por08].

**String 2-groups and Segal-Mitchison cohomology**

String 2-groups are smooth 2-groups associated to a Lie group \(G\) and a cohomology class. The underlying discrete 2-group of a String 2-group is a 2-group with \(\pi_1 = U(1)\) and trivial action \(\rho : \pi_0 \to \text{Aut}(\pi_1)\).

These discrete String 2-groups are classified by their underlying “object group” \(\pi_0 = G_\delta\) and the class of the associator \(k \in H^3(A\text{G}_\delta, U(1)) \) — the latter is a class in discrete group cohomology. To understand String 2-groups as smooth 2-groups, we must talk about Lie group cohomology.

\(^7\)By a monoid object, we mean the maximally weak notion, often called a pseudomonoid.

\(^8\)In [SP11], smooth 2-groups are defined internal to a bicategory denoted \(\text{Bibun}_\ast \text{LieGpd}\) includes fully faithfully into \(\text{Bibun}_\ast\), and this induces an equivalence \(\text{DiffSt} \simeq \text{LieGpd}[W^{-1}] \simeq \text{Bibun}_\ast\).\[^8\]
Globally smooth group cohomology [Sta78; Bla85] \( H^\bullet_{\text{sm}}(BG, A) \) of a Lie group \( G \) with coefficients in a smooth \( G \)-module \( A \) is computed just as in the discrete case, except cocycles \( G^x_n \to A \) are required to be smooth maps. This cohomology theory is not very well-behaved. A short exact sequence of coefficients does not give rise to a long exact sequence in globally smooth group cohomology. Further, the group \( H^2_{\text{sm}}(BG, A) \), which ordinarily classifies extensions of \( G \) by \( A \), only detects those extensions \( A \to E \to G \) where \( E \) is topologically a trivial \( A \)-bundle over \( G \). One may fix this problem by only requiring smoothness in a neighbourhood of the identity of \( G \). The resulting cohomology theory is called locally smooth cohomology. More geometric is the cohomology theory introduced by Segal in [Seg70] and recalled below. It is equivalent to locally smooth cohomology by [WW15].

Lie group cohomology of \( G \) is the cohomology of its classifying space \( BG \). We use the model of \( BG \) as a simplicial manifold: the manifold of \( q \)-simplices is \( BG_q = G^q \) and the face maps \( d_i : G^q \to G^{q-1} \) are given by

\[
\begin{align*}
&d_i : (g_1, \ldots, g_q) \mapsto \begin{cases} 
(g_2, \ldots, g_q) & i = 0 \\
(\ldots, g_ig_{i+1}, \ldots) & 0 < i < q \\
(g_1, \ldots, g_{q-1}) & i = q.
\end{cases}
\end{align*}
\]

This simplicial manifold contains more homotopical information than the topological space \(|BG|\) obtained from \( BG \) by geometric realisation. The difference is detected by Segal-Mitchison cohomology unless the coefficient group is discrete (see Example 2.10).

A simplicial cover \( Y_\bullet \to BG \) is a collection of covers \( Y_q \to BG_q \), together with simplicial maps (which we will also denote by \( d_i \)) such that the corresponding squares in the diagram below commute.

\[
\begin{array}{ccccccc}
Y_0 & \leftarrow & Y_1 & \leftarrow & Y_2 & \leftarrow & Y_3 & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \cdots \\
* & \leftarrow & G & \leftarrow & G \times G & \leftarrow & G \times G \times G & \leftarrow & \cdots \\
\end{array}
\]

Associated to a simplicial cover is a Čech-simplicial double complex [Bry00] (recall that \( Y^{[p]} \) denotes the \( p \)-fold fibre product of \( Y \) over \( M \)):

\[
C^{p,q}(BG, A) = C^\infty(Y_q^{[p+1]}, A)
\]

\[
d = d^\text{Čech} + d^\text{simpl},
\]

where \( d^\text{Čech} \) denotes the Čech differential and \( d^\text{simpl} \) is the alternating sum of the pullback maps

\[
d_i^\ast : C^\infty(Y_q^{[p]}, A) \to C^\infty(Y_q^{[p+1]}, A).
\]

A simplical cover is good if \( Y_i \to BG_i \) is good for all \( i \). For good simplicial covers of \( BG \), we may always choose \( Y_0 \) to be a point.

**Definition 2.9.** Segal-Mitchison cohomology of \( G \) with coefficients in an abelian Lie group \( A \) is the cohomology of the Čech-simplicial complex associated to a good simplicial cover \( Y_\bullet \to BG \).

This is independent of the good cover chosen [Bry00]. Segal-Mitchison cohomology can be defined more generally with coefficients in any smooth \( G \)-module, but we will only need the case with trivial action.

**Example 2.10.** For certain coefficients \( A \), Segal-Mitchison cohomology reduces to other cohomology theories [Seg70, Bry00]:

- If \( A \) is a vector space, it agrees with globally smooth group cohomology.
- If \( A \) is discrete, it is the (singular) cohomology of the topological space \(|BG|\).
Segal-Mitchison cohomology associates to any short exact sequence of coefficients a long exact sequence of cohomology groups. The exponential exact sequence $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1)$ gives rise to a long exact sequence

$$\cdots \rightarrow H^p(\mathbb{R}G, \mathbb{R}) \rightarrow H^p(\mathbb{R}G, U(1)) \xrightarrow{\delta} H^{p+1}(\mathbb{R}G, \mathbb{Z}) \rightarrow H^{p+1}(\mathbb{R}G, \mathbb{R}) \rightarrow \cdots.$$  

Note that $H^*(\mathbb{R}G, \mathbb{R})$ is cohomology with respect to the sheaf of smooth $\mathbb{R}$-valued functions, not the sheaf of locally constant $\mathbb{R}$-valued functions. In particular, it is not the same as singular cohomology of $|\mathbb{R}G|$ with coefficients in discrete $\mathbb{R}$ and thus does not see the usual real characteristic classes. When $G$ is compact, $H^* (\mathbb{R}G, \mathbb{R})$ vanishes in positive degrees [Bla85] (see also [SP11, Cor 97]), so the map $\delta$ is an isomorphism for $p > 0$. In particular, $H^3 (\mathbb{R}G, U(1)) \simeq H^4 (\mathbb{R}G, \mathbb{Z}).$ If $G$ is also simply-connected and simple, then $G$ is in fact 2-connected and $\pi_3 G = \mathbb{Z}$ [MT91, Thm VI.4.17] [Noo09]. The group $H^4 (\mathbb{R}G, \mathbb{Z})$ can be computed as the simplicial cohomology of the geometric realisation $|\mathbb{R}G|$ (see Example 2.10). This topological space has homotopy groups $\pi_{i<4} |\mathbb{R}G| = 0$, $\pi_4 |\mathbb{R}G| = \mathbb{Z}$. The Hurewicz isomorphism and Universal Coefficient Theorem imply

$$H^3 (\mathbb{R}G, U(1)) \simeq H^4 (\mathbb{R}G, \mathbb{Z}) \cong \mathbb{Z}.$$  

From a Segal-Mitchison 3-cocycle, one may build a smooth 2-group [SP11]. We recall this construction now. Let $G$ be a Lie group, $A$ an abelian Lie group, and $k \in H^3 (\mathbb{R}G, A)$ a class in Segal-Mitchison cohomology. A Segal-Mitchison cocycle representing a cohomology class in $H^3 (\mathbb{R}G, A)$ is a simplicial cover $Y_\bullet \rightarrow \mathbb{R}G$ and a triple \(^8\)

$$(\lambda, \mu, \omega) \in C^\infty (Y_1, A) \times C^\infty (Y_2, A) \times C^\infty (Y_3, A).$$

Recall the Lie groupoid $E^\lambda : Y_1 \times A \rightrightarrows Y_2 \times A$ associated to the Čech 2-cocycle $\lambda$. Using the map $(d_0, d_2) : Y_2 \rightarrow Y_1 \times Y_1$, we can form the pullback groupoid $F^\lambda = (d_0, d_2)^*(E^\lambda \times E^\lambda) = Y_2 \times A \times A \rightrightarrows Y_2$. The tensor structure on $E^\lambda$ is given by the anafunctor

$$E^\lambda \times E^\lambda \overset{\cong}{\rightrightarrows} F^\lambda \rightarrow E^\lambda,$$

where the map $F^\lambda \rightarrow E^\lambda$ is given by $d_1 : Y_2 \rightarrow Y_1$ on objects, and by

$$Y_2 \times A \times A \rightarrow Y_1 \times A$$

$$(v_0, v_1, a, b) \mapsto (d_1(v_0), d_1(v_1), a + b + \mu(v_0, v_1)),$$

on morphisms. One may further pull back along $(d_0 d_0, d_2 d_0, d_1 d_0) : Y_3 \rightarrow Y_1 \times A$ to obtain another Lie groupoid $H^\lambda$. Then $(- \otimes -) \otimes -$ and $- \otimes (- \otimes -)$ are represented by spans $(E^\lambda)^3 \overset{\cong}{\rightarrow} H^\lambda \rightarrow E^\lambda$, and $\omega \in C^\infty (Y_3, A)$ defines a smooth natural transformation between them. The cocycle condition ensures that the above indeed defines a smooth 2-group.

There is a notion of central extension of smooth 2-groups [SP11, Defn 83] paralleling the definition for groups. Equivalence classes of central extensions of $G$ by $[*/A]$ are in bijective correspondence with elements of $H^3 (\mathbb{R}G, A)$, and a representative of this equivalence class may be built from a cocycle as above. The smooth String 2-group $G_k$ is the central extension of $G$ by $[*/U(1)]$ corresponding to $k \in H^3 (\mathbb{R}G, U(1))$ [SP11, Thm 100].

---

\(^8\)The Čech-simplicial complex has four groups in cohomological degree 3, but the cocycle condition implies that the component in $C^\infty (Y_0, A)$ is trivial.
3. The smooth centre

The Drinfel’d centre of a monoidal category \((C, \otimes, 1, \omega)\) is the category whose objects are pairs \((X, \gamma)\), where \(X \in C\) and \(\gamma : X \otimes - \cong - \otimes X\) is a \textit{half-braiding}: a natural isomorphism satisfying the \textit{hexagon equation}

\[
\omega(Y, Z, X) \circ \gamma(Y \otimes Z) \circ \omega(X, Y, Z) = (\gamma(Y) \otimes \text{id}_Z) \circ \omega(Y, X, Z) \circ (\text{id}_Y \otimes \gamma(Z)).
\]

It is the analogue of the centre of a monoid in the world of monoidal categories. We study the corresponding notion in the context of smooth 2-groups.

Drinfel’d centres of smooth 2-groups

In \cite{Str04}, the notion of the Drinfel’d centre was internalised to any braided monoidal bicategory (and thus in particular to the symmetric monoidal bicategory of stacks). Let \(\mathcal{B}\) be a symmetric monoidal bicategory with product \(\boxtimes\) and braiding \(b\), and \((C, \otimes)\) a monoid object in \(\mathcal{B}\).

**Definition 3.1.** Let \(U \in \mathcal{B}\). A \textit{U-family of centre pieces} for \((C, \otimes)\) is a pair \((u, \gamma)\) where \(u : U \to C\) is a morphism and

\[
\gamma : \otimes \circ (u \boxtimes \text{id}_C) \cong \otimes \circ (\text{id}_C \boxtimes u) \circ b_{U,C}
\]

an invertible 2-cell satisfying the hexagon equation (phrased internal to \(\mathcal{B}\)). We think of \(u\) as a \(U\)-point of \(C\) and call \(\gamma\) a \textit{half-braiding} for \(u\). A morphism of centre pieces \((u, \gamma) \to (u', \gamma')\) is a morphism \(u \to u'\) which commutes with the half-braidings. We denote by \(\mathcal{CP}(U, C)\) the category of \(U\)-families of centre pieces for \((C, \otimes)\).

Half-braidings may be pulled back along maps \(U' \to U\), and \(\mathcal{CP}(-, C)\) admits the structure of a 2-presheaf over \(\text{Man}\).

**Definition 3.2.** The centre \(\mathcal{Z}C\) of a monoid \(C\) in a braided monoidal bicategory \(\mathcal{B}\) is the representing object for the 2-presheaf \(\mathcal{CP}(\mathcal{Z}C, C) : \mathcal{B}^{op} \to \text{Cat}\).

In \cite{Pir21}, this centre was computed explicitly in the bicategory of crossed modules (see Example 2.3). We now specialise to \(\mathcal{B} = \text{SmSt}\), the bicategory of stacks. The 2-Yoneda Lemma says that for any stack \(\mathcal{F}\) and manifold \(M\), the category \(\mathcal{F}(M)\) is naturally equivalent to the category of 1-morphisms \(M \to \mathcal{F}\) \cite{Ler10}. Thus, a morphism \(u : M \to A\) is equivalently an object of the category \(A(M)\), which makes Definition 3.1 a parameterised version of the ordinary Drinfel’d centre in this case.

**Theorem 3.3.** \cite{Str04} The centre \(\mathcal{Z}C\) of a monoid \(C\) in \(\mathcal{B}\) exists if \(\mathcal{B}\) is finitely complete and closed. Denote the internal Hom of \(\mathcal{B}\) by \([\cdot, \cdot]\), then the centre is the limit

\[
\mathcal{Z}C = \text{lim} (C \Rightarrow [C, C] \Rightarrow [C \boxtimes C, C]).
\]

The centre is equipped with a monoidal product \(\otimes\), a monoidal morphism \(\mathcal{Z}C \to C\), and a braiding \(\beta : \otimes \Rightarrow \otimes \circ b_{\mathcal{Z}C,C}\).

The bicategory \(\text{SmSt}\) is finitely complete \cite{Str82}, so in particular fulfils the assumptions of the above theorem. Any smooth 2-group \(G \in \text{DiffSt} \subset \text{SmSt}\) thus has a centre \(\mathcal{Z}G \in \text{SmSt}\). The functor \(ev_U : \text{SmSt} \to \text{Gpd}\) which evaluates a stack on a manifold \(U\) induces a braided monoidal comparison functor \((\mathcal{Z}G)(U) \to \mathcal{Z}(G(U)))\). This functor is the inclusion of the smooth half-braidings and smooth maps of the associated centre pieces.

---

*The morphisms in the diagram are induced by the monoid structure of \(C\). We suppress the 2-cells.
The monoidal groupoid $G(U)$ is a 2-group, and hence every centre piece $(X, \gamma)$ has an inverse $(iX, i\gamma_i^{-1})$ [EGNO16, Ch 7.13], where $i : G(U) \to G(U)$ is a functor assigning inverses.

We would like to show that this inverse exists as a smooth half-braiding in $(\mathcal{Z}G)(U)$. One can pick a global inverse functor $i_{\text{glob}} : G \to G$ and directly show this, but the diagrams involved are unwieldy due to the presence of coherences. By uniqueness of inverses (up to isomorphism), it suffices to show the existence of inverses in $(\mathcal{Z}G)(U)$.

**Lemma 3.4.** The Drinfel’d centre of a group object in $\text{SmSt}$ is a group object in $\text{SmSt}$.

*Proof.* This is completely formal. The inclusion of group objects in $\text{SmSt}$ into monoid objects in $\text{SmSt}$ is a reflective localisation of $(2, 1)$-categories [SP11 Cor 59-Thm 61], and thus of $(\infty, 1)$-categories. Reflective localisations of $(\infty, 1)$-categories reflect limits [Wil13] so the limit exists in the bicategory of 2-groups, and agrees with the limit calculated in the bicategory of monoid objects. \qed

**Corollary 3.5.** The centre $\mathcal{Z}G$ of any 2-group in $\text{DiffSt}$ is a 2-group object in $\text{SmSt}$. Given a global inverse $i : G \to G$, the inverse of a centre piece $(X, \gamma) \in (\mathcal{Z}G)(U)$ is given by $(iU, i_X, i_{\text{glob}}^{-1})$.

*Proof.* Lemma 3.4 guarantees the existence of an inverse to each centre piece $(X, \gamma)$. Under the functor $\text{ev}_{U} : \text{SmSt} \to \text{Gpd}$, this inverse must be isomorphic to $(iX, i_X, i_{\text{glob}}^{-1})$ for any choice of inverse map $i$. \qed

**Lemma 3.6.** Let $C$ be a 2-group. Then $\pi_1 \mathcal{Z}C = (\pi_1 C)^{\pi_0 C}$, the invariants under the conjugation action $\rho$.

*Proof.* The trivial centre piece (the monoidal unit of $\mathcal{Z}C$) is the unit morphism $\mathcal{1} : * \to C$, with half-braiding $\gamma_1$ built as a composite of unitor morphisms $\mathcal{1} \otimes - \to - \to - \otimes \mathcal{1}$. The condition for a morphism $f \in \text{End}(\mathcal{1}) = \pi_1 C$ to be an endomorphism of $(\mathcal{1}, \gamma_1)$ is

$$\gamma_1(x) \circ (f \otimes \text{id}_x) = (\text{id}_x \otimes f) \circ \gamma_1(x)$$

for all $x \in C$. Tensoring with $\text{id}_{x^{-1}}$ shows that this is equivalent to $f$ being conjugation-invariant. \qed

The central extensions we are interested in have trivial conjugation action, and so $\pi_1 \mathcal{Z}C = \pi_1 C$ in this case.

**Braided 2-groups and quadratic forms**

Recall that a braiding $\beta$ on a monoid $(B, \otimes, \ldots)$ is a 2-morphism $\beta : - \otimes - \to - \otimes \text{op}$ which is compatible with units and satisfies the hexagon equation in both variables.

**Definition 3.7.** A braided smooth 2-group is a braided monoid $(B, \otimes, \omega, l, r, \beta) \in \text{DiffSt}$ whose underlying monoid $(B, \otimes, \omega, l, r)$ is a smooth 2-group.

A discrete braided 2-group is a braided monoidal groupoid whose underlying monoidal groupoid is a 2-group. They are also known as braided categorical groups [JS93]. The existence of a braiding on $B$ forces $\pi_0 B$ to be abelian and the action $\rho : \pi_0 B \to \text{Aut}(\pi_1 B)$ to be trivial.

**Theorem 3.8.** [EM54] Equivalence classes of discrete braided categorical groups $B$ with $\pi_0 B = A, \pi_1 B = B$ are in one-to-one correspondence with quadratic forms $q : A \to B$. Under this correspondence, $q(a)$ is the self-braiding $\beta_{a,a} \in \text{End}(a) \cong B$ of $a \in A$. 

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A quadratic form is a map \( q : A \to B \) such that \( q(n \cdot a) = n^2 \cdot q(a) \) for all \( n \in \mathbb{N}, a \in A \), and the associated form

\[
\sigma_q : A \times A \to B \\
(a, b) \mapsto q(a + b) / (q(a)q(b))
\]
is bilinear. We conjecture that Theorem 3.8 also holds true in the smooth case. We will only show one half of this statement, namely that braided 2-groups are captured by their quadratic form. Denote the bicategory of smooth braided 2-groups with \( \pi_0 = A, \pi_1 = B \) for two abelian Lie groups \( A, B \) by \( \mathbf{B2G}(A, B) \). It is easy to check that the quadratic form associated to any such braided 2-group is smooth, and that the Baer sum of braided 2-groups corresponds to pointwise product of the associated quadratic forms.

**Lemma 3.9.** The homomorphism

\[
\mathbf{B2G}(A, B) \to \text{Quad}^{\text{sm}}(A, B)
\]
that sends a braided 2-group to its quadratic form is injective. We assume the group of connected components of \( A \) is finitely generated.

**Proof.** We need to show that there is only one (braided) equivalence class of braided 2-groups that gives the trivial quadratic form \( q : A \to B \). The associator is detected by degree three Segal-Mitchison cohomology \( H^3(\mathcal{B}A, B) \) [SP11, Thm 99]. This cohomology group injects into the corresponding cohomology group of discrete groups [WW13]. By Theorem 3.8 the associator of the underlying discrete 2-group is trivial up to equivalence. As a result, we can assume that the associator of the smooth 2-group is trivial, so we are considering braiding on \( A \) instead of \( [*/B] \). The hexagon equations demand that the braiding be a bilinear map \( \beta : A \otimes A \to B \). As \( q(a) = \beta(a, a) \) is trivial, \( \beta \) is alternating. A braiding on \( A \otimes [*/B] \) is equivalent to the trivial braiding if there is a monoidal functor from any such braided 2-group is smooth, and that the Baer sum of braided 2-groups corresponds to pointwise product of the associated quadratic forms.

It remains to show that any alternating bilinear form \( \beta \) admits such a trivialisation \( \eta \). The group \( A \) splits as a product of a discrete group \( K \) and an abelian Lie group \( H \). Let \( \mathfrak{h} \) be the Lie algebra of \( H \), and \( x, x' \in \mathfrak{h}, k, k' \in K \). The value of a bilinear form on \( (k' \cdot \exp(\alpha x), k' \cdot \exp(\alpha' x')) \in A^{\times 2} = (K \times H)^{\times 2} \) is completely determined by its values on pairs of elements of \( k, k', x, x' \) (where evaluating on a Lie algebra element takes the derivative). Pick a set of generators \( I_K \) for \( K \) and \( I_\mathfrak{h} \) of the Lie algebra \( \mathfrak{h} \) of \( H \), as well as an ordering on \( I = I_K \cup I_\mathfrak{h} \). This allows encoding bilinear forms completely in an \( I \times I \) matrix whose entries are given by evaluating on the generators corresponding to the row and column. As \( \beta \) is alternating, it is represented by an antisymmetric matrix. We may simply pick \( \eta \) to be its upper triangular half.

Let \( \text{Quad}^{\text{sm}}(A, B) \to H^3(\mathcal{B}A, B) \) be the homomorphism extracting the monoidal structure of a braided categorical group from the quadratic form. There exist explicit formulae that recover an associator from the data of a quadratic form, see [Qui98, Bra20]. Taking monoidal equivalence classes gives the above map. Its kernel is the group of braiding for the 2-group with trivial associator. When the associator
is trivial, the hexagon equations reduce to character equations of the map \( \beta : A \times A \to B \) in each variable. Thus braidings are exactly bilinear maps \( A \otimes A \to B \). The image of the map \( \text{Quad}^{\text{sm}}(A, B) \to H^3(\mathcal{B}A, B) \) is called soft cohomology \( H^3_{\text{soft}}(\mathcal{B}A, B) \subset H^3(\mathcal{B}A, B) \) in [DS18]. Soft cohomology is the group of (equivalence classes of) associators that can be part of a braided monoidal structure. In summary, we get a sequence of groups, exact at \( \text{Quad}^{\text{sm}}(A, B) \):

\[
\text{Bilin}(A, B) \hookrightarrow \text{Quad}^{\text{sm}}(A, B) \longrightarrow H^3_{\text{soft}}(\mathcal{B}A, B) \to H^3(\mathcal{B}A, B).
\]

**Example 3.10.** We work out the case \( A = \mathbb{Z}/n, B = U(1) \). The relevant cohomology group is \( H^3(\mathcal{BZ}/n, U(1)) \simeq H^4(\mathcal{BZ}/n, \mathbb{Z}) = \mathbb{Z}/n \). The group of bilinear forms \( \mathbb{Z}/n \otimes \mathbb{Z}/n = \mathbb{Z}/n \to U(1) \) is \( \text{Bilin}(\mathbb{Z}/n, U(1)) = \mathbb{Z}/n \), generated by a primitive \( n \)-th root of unity. Quadratic forms \( q : \mathbb{Z}/n \to U(1) \) are determined by the value on a generator: \( q(k) = q(1)^k \). This defines a quadratic form iff \( q(1) \in \mathbb{Z}/(2n, n^2) \), so

\[
\text{Quad}(\mathbb{Z}/n, U(1)) = \begin{cases} \mathbb{Z}/n & n \text{ odd} \\ \mathbb{Z}/2n & n \text{ even.} \end{cases}
\]

The exact sequence of groups introduced above implies

\[
H^3_{\text{soft}}(\mathcal{BZ}/n, U(1)) = \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z}/2 & n \text{ even.} \end{cases}
\]

For odd \( n \), only the 2-group with trivial associator admits a braiding, while for even \( n \), the 2-group corresponding to \( [n/2] \in H^3(\mathcal{BZ}/n, U(1)) = \mathbb{Z}/n \) also does.

**Notation 3.11.** A quadratic form \( \mathbb{Z}/2 \to U(1) \) must send the generator of \( \mathbb{Z}/2 \) to a fourth root of unity, \( q(1) = i^k \). We borrow notation from the world of tensor categories to denote the corresponding braided 2-group \( (\mathbb{Z}/2, q) \) as below.

| \( (\mathbb{Z}/2, q) \) | 1 | i | -1 | -i |
|-----------------|---|---|----|----|
| \text{Vec}_{\mathbb{Z}/2}^\times | \text{Semi}^\times | \text{sVec}_{\mathbb{Z}/2}^\times | \text{Semi}^\times |

The braided tensor category we denote by \( \text{Semi} \) is known as the *semion category*.

### 4. The centres of categorical tori

A *categorical torus* \( \mathcal{T} \) is a central extension

\[
[\ast/U(1)] \to \mathcal{T} \to T
\]

of a (compact) torus \( T \) by \([\ast/U(1)]\). They are classified by \( H^3(\mathcal{B}T, U(1)) \simeq H^4(\mathcal{B}T, \mathbb{Z}) \). Let \( T \) be a torus, \( \Lambda = \text{Hom}(T, U(1)) \) its group of characters, \( \Pi = \Lambda^\vee = \pi_1 T \) its group of cocharacters and \( t = \text{Lie}(T) \) its Lie algebra, identified as the universal cover of \( T \) via \( \Pi \xrightarrow{\exp} T \). The topological space \( |\mathcal{B}T| \) is homotopy equivalent to an \( r \)-fold product of \( \mathbb{CP}^\infty \)'s, where \( r \) is the rank of the torus \( T \). The cohomology ring of \( \mathcal{B}T \) is naturally identified with \( H^*(\mathcal{B}T, \mathbb{Z}) = H^*(|\mathcal{B}T|, \mathbb{Z}) \simeq \text{Sym}^*(\Lambda) \), where \( \Lambda \) is placed in degree 2. We identify the group \( H^4(\mathcal{B}T, \mathbb{Z}) = \text{Sym}^2(\Lambda) \) with the group of symmetric bilinear forms \( I : t \times t \to \mathbb{R} \) such that for all \( \pi \in \Pi \), \( I(\pi, \pi) \in 2\mathbb{Z} \). An element \( \lambda \in \Lambda \) induces a map of Lie algebras \( D_e \lambda : t \to \mathbb{R} \). Then we send \( \lambda_1 \otimes \lambda_2 \in \Lambda \otimes \Lambda \) to the symmetric bilinear form \( I : (x, y) \mapsto D_e \lambda_1(x) \cdot D_e \lambda_2(y) + D_e \lambda_2(x) \cdot D_e \lambda_1(y) \).
Given a class $I \in \Pi^4(\mathcal{A}T, \mathbb{Z})$, we will use $\tau$ to denote the induced map $\Pi \to \Lambda = \Pi' \vee$, given by $\tau(\pi) = I(\pi, -)$. We further pick a (not necessarily symmetric) bilinear form $J$ on $t$ such that $J$ restricts to a $\mathbb{Z}$-valued form on $\Pi$, and $I = -(J + J')$ where $J'$ denotes the transpose of $J$. We recall a construction of the categorical torus $T_J$ given in [Gan18]. It is a strict smooth 2-group (Example [Z,8]). The underlying Lie groupoid is

$$\mathcal{T} \times (\Pi \times U(1)) \Rightarrow t,$$

with arrows $(x, \pi, w) = (x \xrightarrow{w} x + \pi)$. Composition and tensor product are given by

$$\left( x + \pi \xrightarrow{w'} x + \pi + \pi' \right) \circ \left( x \xrightarrow{w} x + \pi \right) = \left( x \xrightarrow{ww'} x + \pi + \pi' \right)$$

$$(x \xrightarrow{w} x + \pi) \otimes (x' \xrightarrow{w'} x + \pi') = \left( x + x' \xrightarrow{ww' \exp(J(\pi, x'))} x + x' + \pi + \pi' \right).$$

The associator and unitor cells are trivial. This categorical torus is classified up to equivalence by the symmetric bilinear form $I = -(J + J') \in \Pi^4(\mathcal{A}G, \mathbb{Z})$ [Gan18, Thm 4.1]. The paper [FHLT10] contains a sketch of calculation of the smooth Drinfel’d centre of $T_J$. We complement this with a proof based on the construction above.

**Proposition 4.1.** The Drinfel’d centre of $T_J$ has underlying Lie groupoid

$$\mathcal{Z}T_J = ((t \oplus \Lambda) \times (\Pi \times U(1)) \Rightarrow t \oplus \Lambda,$$

with arrows $(x, \lambda, \pi, w) : (x, \lambda) \to (x + \pi, \lambda + \tau(\pi))$. The braiding is given by

$$\beta_{[x,\lambda],[x',\lambda']} = \lambda(x') \exp(J(x', x)).$$

**Proof.** A half-braiding on an object $x \in t$ is a 2-cell $\gamma : x \otimes - \to - \otimes x$ (subject to the hexagon equation). Tensoring with $x$ on either side is a smooth functor $T_J \to T_J$. Hence each such 2-cell $\gamma$ is represented by a smooth natural transformation: a smooth map $\gamma : t \to t \times (\Pi \times U(1))$, which sends $y \in t$ to an endomorphism of $x + y = y + x$. In the absence of associators, the hexagon equation simplifies to the character equation $\gamma_{y + z} = \gamma_z \gamma_y$. The condition that $\gamma$ be natural gives it the form

$$\gamma(y) = \lambda(y) \cdot \exp(J(y, x)),$$

where $\lambda \in \Lambda$ is a smooth character of $T$. This allows a parameterisation of objects of $\mathcal{Z}T_J(*)$ by pairs $(x, \lambda) \in t \oplus \Lambda$.

Morphisms $(x, \lambda) \to (x', \lambda')$ are morphisms $x \to x'$ which are compatible with the braidings. The morphism $(x, \pi, w) \in t \times (\Pi \times U(1))$ is a morphism of centre pieces $(x, \lambda) \to (x + \pi, \lambda')$ precisely when

$$\lambda'(y) = \lambda(y) \cdot \exp(-J(y, \pi) - J(\pi, y)) = (\lambda + \tau(\pi))(y).$$

for all $y \in t$.

So far, we have calculated $\mathcal{Z}T_J(*)$. To deduce the smooth structure, note that any object $\tilde{x} \in T_J(V)$ can be represented by a smooth map $\tilde{x} : V \to t$, and the corresponding half-braidings are represented by smooth maps $\tilde{\gamma} : V \times t \to t \times (\Pi \times U(1))$. Naturality and the hexagon equation can now be checked pointwise.

The braiding on the centre can be computed in $\mathcal{Z}(T_J(*))$, where it takes the usual form $\beta_{(x,\gamma),(x',\gamma')} = \gamma(x')$ [ECNO16, Ch 8.5]. This completes the proof.
As mandated by Lemma $3.6$ $\pi_1 \mathcal{Z} T_J = U(1)$. The object group of the centre is

$$\pi_0 \mathcal{Z} T_J = \frac{t \oplus \Lambda}{\Pi},$$

with $\Pi \ni \pi \mapsto (\pi, \tau(\pi)) \in t \oplus \Lambda$. We end this section by computing the maximal compact subgroup of $\pi_0 \mathcal{Z} T_J$. The cohomology class corresponding to the symmetric bilinear form $I = -(J + J')$ induces the maps $\tau : \Pi \rightarrow \Lambda$ and $\tau_{\Pi} := t \otimes \tau : t \rightarrow t^\ast$. The case where $\tau_{\Pi}$ is an isomorphism (such levels are referred to as non-degenerate) was already analysed in [FLHT10]. The map $(t \oplus \Lambda)/\Pi \rightarrow \Lambda/\Pi$ is split by $s : \lambda \mapsto (\tau_{\Pi}^{-1}, \lambda)$. This furnishes an isomorphism $\pi_0 \mathcal{Z} T_J \cong t \oplus (\Lambda/\Pi)$. Note that $\Lambda/\Pi$ is finite, so $\pi_0 \mathcal{Z} T_J$ is a direct sum of $\mathbb{R}^{rk} \tau_{\Pi}$ with a finite abelian group.

For general level, we pick a splitting $\Pi \cong \Pi_{\ker} \oplus \Pi_{\coim}$, where $\Pi_{\ker} := ker \tau$. We tensor with $\mathbb{R}$ to obtain $t \cong t_{\ker} \oplus t_{\coim}$. Now $\tau_{\Pi}$ restricts to an isomorphism $\tau_{\Pi_{\ker}} : t_{\coim} \cong \im \tau_{\Pi_{\ker}} \subset t^\ast$. We denote the intersection $(\im \tau_{\Pi_{\ker}}) \cap \Lambda$ by $\Lambda_{\im_{\Pi_{\ker}}}$. Lastly, we pick a decomposition $\Lambda = \Lambda_{\im_{\Pi_{\ker}}} \oplus \Lambda_{\coim_{\Pi_{\ker}}}$. These isomorphisms assemble into

$$\pi_0 \mathcal{Z} T_J \cong \frac{\Lambda_{\coim_{\Pi_{\ker}}} \oplus t_{\ker}}{\Pi_{\ker}} \oplus \frac{t_{\coim} \oplus \Lambda_{\im_{\Pi_{\ker}}}}{\Pi_{\coim}} \cong \frac{\Lambda_{\coim_{\Pi_{\ker}}} \oplus T_{\ker} \oplus t_{\coim} \oplus \Lambda_{\im_{\Pi_{\ker}}} / \Pi_{\coim}}{\Pi_{\coim}},$$

where the second isomorphism uses that $\tau$ is non-degenerate when viewed as a map $\Pi_{\coim} \rightarrow \Lambda_{\im_{\Pi_{\ker}}}$.

As before, $\Lambda_{\im_{\Pi_{\ker}}} / \Pi_{\coim}$ is a finite group. Both $\Lambda_{\coim_{\Pi_{\ker}}}$ and $t_{\coim}$ are free, and $T_{\ker} := t_{\ker} / \Pi_{\ker}$ is a compact torus. One may now read off the maximal compact subgroup as $T_{\ker} \oplus \Lambda_{\im_{\Pi_{\ker}}} / \Pi_{\coim}$. We are justified in calling this the maximal compact subgroup: any element of the free subgroup $\Lambda_{\coim_{\Pi_{\ker}}} \oplus t_{\coim}$ generates a non-compact group, so all compact subgroups of $\pi_0 \mathcal{Z} T_J$ must intersect trivially with it — the maximal compact subgroup we computed above is the orthogonal complement of $\Lambda_{\coim_{\Pi_{\ker}}} \oplus t_{\coim}$. Using the explicit maps above, it is straightforward to compute the induced braiding on it. We call the resulting braided categorical group the maximal compact sub-2-group.

**Proposition 4.2.** The maximal compact sub-2-group of $\mathcal{Z} T_J$ is the braided 2-group with

$$\pi_0 \text{Cpct } \mathcal{Z} T_J = T_{\ker} \oplus \Lambda_{\im_{\Pi_{\coim}}},$$

$\pi_1 \mathcal{Z} T_J = U(1)$, and braiding encoded by the quadratic form

$$\hat{q}(t_{\ker}, \lambda) = \lambda(t_{\ker} + \tau_{\im_{\Pi_{\coim}}}^{-1} \lambda) \exp(J(\tau_{\im_{\Pi_{\coim}}}^{-1} \lambda, \tau_{\im_{\Pi_{\coim}}}^{-1} \lambda)).$$

**Proof.** This is a straightforward computation. The component $T_{\ker}$ does not contribute in the exponential because

$$J(t_{\ker}, t_{\ker}) = \frac{1}{2} \tau(t_{\ker})(t_{\ker}) = 0 \quad \text{and} \quad J(t_{\ker}, \tau_{\im_{\Pi_{\coim}}}^{-1} \lambda) + J(\tau_{\im_{\Pi_{\coim}}}^{-1} \lambda, t_{\ker}) = 0. \quad \square$$

We record one important feature of this subgroup.

**Lemma 4.3.** The map

$$u : \pi_0 \text{Cpct } \mathcal{Z} T_J \rightarrow T$$

that forgets the half-braiding is injective.

**Proof.** The decomposition of $t$ descends to a decomposition of the torus $T$. Then $u$ is the direct sum of injective maps

$$\pi_0 \text{Cpct } \mathcal{Z} T_J = T_{\ker} \oplus \Lambda_{\im_{\Pi_{\coim}}} \oplus \Lambda_{\coim_{\Pi_{\coim}}} \cong \frac{\Lambda_{\coim_{\Pi_{\coim}}} \oplus t_{\coim} / \Pi_{\coim}}{\Pi_{\coim}} = T_{\ker} \oplus T_{\coim}. \quad \square$$
5. The centres of String 2-groups

In this section, we compute the centre $\mathcal{Z}$ for the String groups $G_k$. We do this using obstruction theory. In particular, we establish an exact sequence of groups in Proposition 5.1 which relates the group $\pi_0\mathcal{Z}G_k$ to the ordinary centre $Z(G)$ and the group cohomology of $G$. When $G$ is simply-connected, this sequence allows us to deduce that every element of the ordinary centre $z \in Z(G)$ admits a unique lift to the Drinfel’d centre $\mathcal{Z}G_k$. This lift restricts to an element in $\mathcal{Z}T_k$, where $T_k$ is a maximal 2-torus for $G_k$. The computations of Section 4 allow us to deduce the resulting braided structure. We then treat the case of non-simply-connected $G$ by picking a simply-connected covering group $\tilde{G} \to G$ and checking which centre pieces in $\mathcal{Z}G_{\pi_1}$ descend to centre pieces of $\mathcal{Z}G_k$.

Finally, we show the Drinfel’d centre $\mathcal{Z}G_k$ agrees with the invertible part of $\text{Rep}^k LG$ (if $G$ is semisimple and there is no factor of $E_8$ at level 2).

Simply-connected Lie groups

Let $G$ be a compact simple simply-connected Lie group, pick a maximal torus $T \to G$, and let $t, t^*, \Lambda, \Pi$ be as in Section 4. The Lie algebra $t \subset g = \text{Lie}(G)$ is also called a Cartan algebra for $G$. The Weyl group $W = N(T)/T$, where $N(T)$ denotes the normaliser of $T$ in $G$, acts on $T$ by conjugation.

Recall from Section 4 that $H^1(\mathcal{B}T, Z)$ is identified with the group of $R$-valued symmetric bilinear forms $I$ on $t$ satisfying $I(\pi, \pi) \in 2\mathbb{Z}$ for all $\pi \in \Pi \subset t$. Borel [Bor53, Bor54] identified $H^1(\mathcal{B}G, Z)$ as the Weyl-invariant part $H^1(\mathcal{B}G, Z) = H^1(\mathcal{B}T, Z)^W$, see [Tod87] for a review. We identify $H^1(\mathcal{B}T, Z)^W$ with the group of those inner products $I$ as above that are invariant under the $W$-action on $t$. This identification holds true for general compact connected Lie groups [Hen17, Thm 6].

For $G$ compact simple and simply-connected, $H^1(\mathcal{B}G, Z) = H^1(\mathcal{B}T, Z)^W \simeq \mathbb{Z}$ has a distinguished generator: the basic positive-definite Weyl-invariant inner product $I : t \times t \to \mathbb{R}$, normalised such that short coroots have norm squared 2. We denote by $k \in \mathbb{Z}$ the cohomology class corresponding to $k \cdot I \in H^1(\mathcal{B}G, Z)$. As before, we denote the map induced by $k \cdot I$ on $\Pi$ by $\tau : \Pi \to \Lambda$ — recall it sends $\pi \mapsto k \cdot I(\pi, -)$. Every non-zero cohomology class $k$ induces an isomorphism $\tau_R = \tau \otimes \mathbb{R} : t \to t^*$.

One may pull back the extension of $G$ by $[\ast / U(1)]$ along the inclusion of a maximal torus $T \to G$ to obtain a maximal 2-torus. In [Gan18], Ganter shows that the maximal 2-torus of $G_k$ is the categorical torus $T_{Jk}$ associated to a (non-symmetric) bilinear form $J_k : t \times t \to \mathbb{R}$ such that $J_k + J_k^* = -k \cdot I$.

**Proposition 5.1.** Let $H_\omega$ denote the central extension of $H$ by $[\ast / A]$ (A an abelian Lie group) corresponding to $\omega \in H^3(\mathcal{B}H, A)$. Then there is an exact sequence

$$0 \to H^1(\mathcal{B}H, A) \to \pi_0\mathcal{Z}H_\omega \to Z(H) \to H^2(\mathcal{B}H, A).$$

**Proof.** Each centre piece $(g, \gamma) \in \mathcal{Z}H_\omega(*)$ must satisfy $g \in Z(H)$, otherwise $g x \neq x g$ for some $x \in H$. In the discrete case, one can work with a skeletal representative of $H_\omega$. The hexagon equation evaluated at $(x, y) \in G \times G$ is then

$$(d\gamma)(x, y) := \frac{\gamma(xy)}{\gamma(x)\gamma(y)} = \frac{\omega(x, g, y)}{\omega(g, x, y)\omega(x, y, g)} := \omega(x, y|g),$$

so $\gamma$ is a 1-cochain whose coboundary is $\omega(-, -|g)$. It is straightforward to check that $\omega(-, -|g)$ is indeed a 2-cocycle. The map $Z(H) \to H^2(\mathcal{B}H, A)$ assigns to each element the equivalence class of the corresponding cocycle: $g \mapsto [\omega(-, -|g)]$. The element $g$ admits a half-braiding precisely if $\omega(-, -|g)$ is a coboundary, which proves exactness at $Z(H)$.

The kernel of the map $\pi_0\mathcal{Z}H_\omega \to Z(H)$ is the group of half-braiding for the identity element of $H$. The associator $\omega$ can be chosen to be trivial whenever at least one of the entries is the identity, and the
The hexagon equation becomes the equation of an $A$-valued character on $H$. These are precisely the elements of $H^1(\mathcal{B}H, A)$.

We now port the above proof to the smooth case. Recall that the Lie groupoid modelling $H_\omega$ is of the form $L \rightrightarrows Y$, where $Y \to H$ is a surjective submersion and $L \to Y^{[2]}$ is an $A$-bundle. We pick a cover $\kappa : V \to Y$ such that the line bundle $L$ trivialises over $V^{[2]}$, and replace $H_\omega$ by the equivalent groupoid $\kappa^*H_\omega = V^{[2]} \times A \rightrightarrows V$. Then we pick a cover $\pi : W \to V \times V \to Y \times Y$ such that all six functors/vertices in the hexagon equation for $\gamma$ (ie. $(g \otimes \cdots, g \otimes (\cdots \otimes \cdots), (-\otimes -) \otimes g \ldots)$) are representable by smooth functors $\pi^*(H_\omega \times H_\omega) \to \kappa^*H_\omega$. Each 2-morphism/edge in the hexagon equation $\omega(z, -, -), \gamma(- \otimes -), \ldots$ is then represented by a smooth natural transformation. Each pair of functors $F_i, F_j$ gives a map $f_{ij} : W \to V^{[2]}$, and a smooth natural transformation $F_i \Rightarrow F_j$ is a section of the pullback bundle $f^*_{ij} (V^{[2]} \times A)$. Under the choices we made, these bundles are all trivial. The hexagon axiom reduces to the same equation as above, $d\gamma = \omega(-, -|g)$, except it is now an equation in Segal-Mitchison cohomology. The pair $(V \to H, W \to H \times H)$ forms the first two steps of a simplicial cover of $\mathcal{B}H$. The maps $\gamma : V \to A, \omega(-, -|g) : W \to A$ represent Segal-Mitchison cochains. \qed

**Corollary 5.2.** Each element $z \in Z(G)$ admits a unique half-braiding over $G_k$: \[ \pi_0 \mathcal{Z}G_k = Z(G). \]

**Proof.** Compactness of $G$ implies $H^* (\mathcal{B}G, U(1)) \simeq H^{*+1} (\mathcal{B}G, \mathbb{Z})$. The connectivity assumptions further imply $H^2(\mathcal{B}G, \mathbb{Z}) = H^3 (\mathcal{B}G, \mathbb{Z}) = 0$ (see Section [2]). The exact sequence of Proposition 5.1 shortens to an isomorphism. \qed

Every half-braiding for $z \in Z(G)$ over $G_k$ restricts to a half-braiding of $z$ over $T_{J_k}$. We thus get a restriction functor $r : \mathcal{Z}G_k \to \mathcal{Z}T_{J_k}$. To describe this restriction functor explicitly, we recall how the centre $Z(G)$ of a Lie group lifts to $t$ — see Chapter 13 of [Hal15] for proofs of the following facts. The centre $Z(G)$ includes into any maximal torus $T$ of $G$. It lifts to $t$ as the dual $\Phi^\vee$ of the root lattice $\Phi \subset t^*$ of $G$. For simply-connected $G$, the lattice $\Phi^\vee$ agrees with the coweight lattice (see e.g. [KK05]), and thus elements of $Z(G)$ lift to coweights in $\Phi^\vee \subset t$. The centre of a simply-connected compact Lie group may be computed from the coweights and cocharacter lattice as $Z(G) = \Phi^\vee / \Pi$.

**Theorem 5.3.** The Drinfel’d centre of $G_k$ is the braided categorical group specified by $\pi_0 \mathcal{Z}G_k = Z(G)$, $\pi_1 \mathcal{Z}G_k = U(1)$ and the quadratic form \[ q : Z(G) \to U(1) \quad z \mapsto \exp \left( \frac{1}{2} I(\bar{z}, \bar{z}) \right), \]

where $\bar{z}$ denotes any lift of $z \in Z(G)$ to $t$.

**Proof.** As $\pi_0 \mathcal{Z}G_k = Z(G)$ is finite, the functor $r : \mathcal{Z}G_k \to \mathcal{Z}T_{J_k}$ must land in the maximal compact subgroup $C_{\text{pct}} \mathcal{Z}T_{J_k}$, computed in Proposition 4.2. It fits into the commutative diagram

\[
\begin{array}{ccc}
\pi_0 \mathcal{Z}G_k & \xrightarrow{r} & \pi_0 C_{\text{pct}} \mathcal{Z}T_{J_k} \\
\downarrow{\bar{z}} & & \downarrow{u} \\
Z(G) & \xrightarrow{r_0} & T,
\end{array}
\]

where the left hand map is an isomorphism by Corollary 5.2, and the right hand map $u$ is injective by Lemma 4.3. The map $r_0 : Z(G) \to \pi_0 C_{\text{pct}} \mathcal{Z}T_{J_k}$ is uniquely fixed by the requirement that the bottom right triangle commute.
For $k = 0$, the map $u$ is the identity on $T$. The braiding is trivial on all of $T$, and restricts to the trivial braiding on $Z(G)$. For $k \neq 0$, the maximal compact subgroup is $\Lambda/\Pi$, $u$ is equal to $\tau^{-1}_R : \Lambda/\Pi \to t/\Pi = T$, and $r_0$ is the section

$$Z(G) = \Phi^\vee/\Pi \xleftarrow{\tau/\Pi} \Lambda/\Pi.$$  

Denote a lift of $z \in Z(G)$ to $\Phi^\vee$ by $\bar{z}$. The quadratic form $q$ computed in Proposition 4.2 pulls back to

$$q(z) = \bar{q}(\bar{z}) = \tau(\bar{z})\bar{z} \exp J_k(\bar{z}, \bar{z}) = \exp (k \cdot I(\bar{z}, \bar{z}) + J_k(\bar{z}, \bar{z})) = \exp \left( \frac{k}{2} I(\bar{z}, \bar{z}) \right).$$

Each lift $\bar{z}$ in this formula is a coweight. The norm of a coweight may be computed as the norm of the corresponding weight of the dual root datum. In the realm of compact simple simply-connected Lie groups, dualising root data simply exchanges the odd-dimensional Spin groups $B_n = \text{Spin}(2n+1)$ and the symplectic Lie groups $C_n = \text{Sp}(2n)$. All other groups are fixed by this operation. The norm is computed using the inner product on the dual of the Cartan of the dual root datum. It is normalised such that short roots have length squared 2. The values of the length squared of weights under this product can be read off from the explicit expansion for weights in terms of roots given in [Bou94]. In Table 1, we list the results of this computation.

**Example 5.4.** The Drinfel’d centre of $SU(2)_k$ is given by (see Notation 3.11)

$$\mathcal{Z}SU(2)_k = \begin{cases}
\text{Vec}_{\mathbb{Z}/2}^\times & k \equiv 0 \pmod{4} \\
\text{Semi}^\times & k \equiv 1 \pmod{4} \\
\text{sVec}^\times & k \equiv 2 \pmod{4} \\
\text{Semi}^\times & k \equiv 3 \pmod{4}.
\end{cases}$$

**Non-simply-connected Lie groups**

Any compact connected Lie group $G$ fits into a short exact sequence

$$0 \to Z \to \hat{G} \xrightarrow{\pi} G,$$

where the middle term is a product $\hat{G} = T \times \Pi_i G_i$ of a torus $T$ with simply-connected simple Lie groups $G_i$, and $Z \leftarrow Z(\hat{G})$ is a finite central subgroup [MT91, Cor V.5.31]. The degree 4 cohomology of $\hat{G}$ decomposes as

$$H^4(\mathcal{B}\hat{G}, \mathbb{Z}) = H^4(\mathcal{B}T, \mathbb{Z}) \oplus \prod_i H^4(\mathcal{B}G_i, \mathbb{Z}).$$

Hence any degree 4 cohomology class of $\hat{G}$ can be represented by a cocycle which is a product of cocycles pulled back from the individual factors.

**Lemma 5.5.** Let $H, H'$ be a pair of Lie groups and $\omega, \omega'$ cocycles representing associators. Denote by $\tilde{\omega} := p^+_H \omega + p^+_H \omega'$ the product cocycle on $H \times H'$. The centre $\mathcal{Z}(H \times H')_\omega$ is the braided abelian group $(\pi_0 \mathcal{Z} H_\omega \times \pi_0 \mathcal{Z} H'_{\omega'}, q)$, with quadratic form

$$\bar{q} : \pi_0 \mathcal{Z} H_\omega \times \pi_0 \mathcal{Z} H'_{\omega'} \to U(1) = \pi_1 \mathcal{Z}(H \times H')_\omega$$

given by the pointwise product of the quadratic forms on $\mathcal{Z} H_\omega$ and $\mathcal{Z} H'_{\omega'}$.

**Proof.** We prove this in the discrete setup. The argument is carried over to the smooth setting exactly as in the proof of Proposition 5.1. A centre piece for $(H \times H')_\omega$ is a tuple of elements $(h, h')$, equipped with a half-braiding $\gamma : H \times H' \to U(1)$ satisfying the hexagon equation

$$d\gamma = \tilde{\omega} (-, -)((h, h')).$$
We will now show that every such half-braiding is a product of half-braidings in $H_\omega$ and $H'_{\omega'}$ and vice versa. The cocycle $\tilde{\omega}$ splits as a product of $\omega$ and $\omega'$, and thus so does $\tilde{\omega}(-, -| (h, h'))$. The hexagon equation implies that the half-braiding splits as a product

$$\gamma((x, x')) = \gamma((x, e)) \gamma((e, x')) = \gamma_h(x) \gamma_{h'}(x').$$

The hexagon equation for $\gamma$ is now equivalent to the hexagon equations $d\gamma_h = \omega(-, -| h)$ and $d\gamma_{h'} = \omega'(-, -| h')$. Hence $\pi_0 \mathcal{Z}(H \times H')_\omega = \pi_0 \mathcal{Z}H_\omega \times \pi_0 \mathcal{Z}H'_{\omega'}$ is indeed the product. The quadratic form is given by the pointwise product because the half-braidings are.

The Weyl groups of $\tilde{\Pi}$ and $\Pi$ agree. Recall that we identified $\tilde{\Pi}$ as $\ker \pi$ lifts uniquely to $\pi_0 \tilde{\mathcal{G}}$: it must land in the maximal compact subgroup, which injects into $\tilde{G}$ as $\pi_0 \C_pct \mathcal{Z}T_{\mathcal{I}} \times \Pi_i Z(G_i) \to T \times \Pi_i Z(G_i)$ by Lemma 4.3.

Choose a maximal torus $T \hookrightarrow \tilde{G}$, then $T/Z \to G/Z = G$ is a maximal torus for $G$. The Lie algebras of these two tori may both be identified with their common universal cover, which we denote $\tilde{t}$. We write $\tilde{\Pi}$ for the fundamental group of $\tilde{T}$, embedded as a lattice in $\tilde{t}$. The fundamental group $\tilde{\Pi}_Z$ of $\tilde{T}/Z$ is an extension

$$\tilde{\Pi} \hookrightarrow \tilde{\Pi}_Z \to Z.$$

Considered as a lattice in $\tilde{t}$, it is the preimage of $Z \subset \tilde{T}$ in $\tilde{t}$:

$$\begin{array}{ccc}
\tilde{\Pi}_Z & \longrightarrow & \tilde{t} \\
\downarrow & & \downarrow \\
Z & \longrightarrow & T.
\end{array}$$

The Weyl groups of $\tilde{G}$ and $G$ under these choices of maximal tori are canonically isomorphic and the Weyl-actions on $t$ agree.

Recall that we identified $H^4(\mathcal{B}G, \mathbb{Z})$ with the group of symmetric bilinear forms $I$ on $t$ which are Weyl-invariant and satisfy $I(\pi, \tilde{\pi}) \in 2Z$ for all $\pi \in \tilde{\Pi}$. Under this identification with bilinear forms, the map $\pi^*: H^4(\mathcal{B}G, \mathbb{Z}) \to H^4(\tilde{\mathcal{G}}, \mathbb{Z})$ restricts the bilinear form along $\pi$ (see also [Hen17a]). The quotient map $\pi$ induces the identity on the Lie algebra $\tilde{t}$, and preserves the Weyl-action. Hence, the image of the map $\pi^*: H^4(\mathcal{B}G, \mathbb{Z}) \to H^4(\tilde{\mathcal{G}}, \mathbb{Z})$ consists precisely of those symmetric bilinear forms on $t$ that satisfy the even integrality condition not only for $\tilde{\Pi}$, but for $\tilde{\Pi}_Z$.

Now consider the unique lift of $Z$ to $\pi_0 \tilde{\mathcal{G}}_{\pi^* k}$. The quadratic form on $\pi_0 \tilde{\mathcal{G}}_{\pi^* k}$ restricts to

$$q: Z \to U(1)$$

$$z \mapsto \exp \frac{i}{2} I(\tilde{z}, \tilde{z}).$$

It vanishes on $z \in Z$ if and only if $I(\tilde{z}, \tilde{z}) \in 2Z$ for all lifts $\tilde{z} \in \tilde{t}$. As discussed above, the preimage of $Z$ under $\pi$ is precisely $\tilde{\Pi}_Z$, so this is equivalent to the integrality condition for $I \in H^4(\mathcal{B}G, \mathbb{Z})$ and we arrive at the following:

**Lemma 5.6.** Let $Z \hookrightarrow \tilde{G} \to G$ be as above. Then $\tilde{k} \in H^4(\mathcal{B}G, \mathbb{Z})$ is in the image of the map $H^4(\mathcal{B}G, \mathbb{Z}) \hookrightarrow H^4(\tilde{\mathcal{G}}, \mathbb{Z})$ if and only if the quadratic form on $\pi_0 \tilde{\mathcal{G}}_{\tilde{k}}$ vanishes when restricted along the unique lift $Z \hookrightarrow \pi_0 \tilde{\mathcal{G}}_{\tilde{k}}$.
A description of the maps $H^1(\mathcal{H}G, \mathbb{Z}) \hookrightarrow H^1(\mathcal{H}\tilde{G}, \mathbb{Z})$ for quotients $G = \tilde{G}/Z$ of simple simply-connected Lie groups may be found in [GW09, Table 1].

Consider the closed subgroup

$$Z^\perp := \left\{ x \in \pi_0 \mathcal{Z} \tilde{G} \mid q(x+z) = q(x), \forall z \in Z \right\}$$

of objects in $\mathcal{Z} \tilde{G}$ that braids trivially with every element of $Z$. By abuse of notation, we also denote by this the braided smooth sub-2-group of $\mathcal{Z} \tilde{G}$ obtained by pulling back along the inclusion $Z^\perp \hookrightarrow \pi_0 \mathcal{Z} \tilde{G}$. By Lemma 5.6, $q$ vanishes on $Z$, so in particular $Z \subset Z^\perp$. The quotient $Z^\perp/Z$ inherits a smooth structure, because $Z^\perp$ is an embedded Lie subgroup of $\pi_0 \mathcal{Z} \tilde{G}$, and $Z$ is a normal subgroup of $Z^\perp$.

**Theorem 5.7.** The Drinfel’d centre of $G_k$ is the smooth braided categorical group

$$\mathcal{Z} G_k = (Z^\perp/Z, q|_{Z^\perp}).$$

**Proof.** Let $(a, \gamma) \in \mathcal{Z} G_k$ be a centre piece for $G_k$. Pick a lift $\bar{a} \in Z(\tilde{G})$ of $a \in Z(G)$ against $\pi : \tilde{G} \to G$. Pulling back the half-braiding, we obtain a new centre piece $(\bar{a}, \pi^* \gamma) \in \mathcal{Z} \tilde{G}_{\pi^* k}$. Indeed, the hexagon equation $d\gamma = \omega(-,-, -|a)$ is preserved under pullback. The lifts of $a$ form a $\tilde{Z}$-torsor, and the induced quadratic form $q$ on the group of lifts is invariant under the $\tilde{Z}$-action: $q(\bar{a} + z) = \pi^* \gamma(\tilde{a} + z) = \gamma(a) = q(\bar{a})$ for all $z \in Z$. This construction extends to a smooth functor $\mathcal{Z} G_k \to Z^\perp/Z$, which preserves the quadratic form. We now build the inverse functor.

The pullback cocycle $\pi^* \omega$ representing the associator on $\mathcal{Z} G_{\pi^* k}$ can be chosen to be equivariant under translation by $Z$ in all variables. The hexagon equation implies that for any centre piece $(\bar{a}, \gamma) \in \mathcal{Z} \tilde{G}_{\pi^* k}$, $z \in Z$, and $\bar{x}, \bar{x}' \in \tilde{G}$,

$$\gamma(\bar{x} + z)/\gamma(\bar{x}) = \gamma(\bar{x}' + z)/\gamma(\bar{x}').$$

The projection $\pi : \tilde{G} \to G$ defines a simplicial cover $\pi_* : \mathcal{Z} \tilde{G}_* \to \mathcal{Z} G_*$, and $\gamma$ is a Segal-Mitchison cochain for $G$ defined with respect to this cover. One may check that $\gamma$ satisfies the cocycle condition precisely if it is $Z$-equivariant: $\gamma(\bar{x} + z) = \gamma(\bar{x})$ for all $\bar{x} \in \tilde{G}$, $z \in Z$. By the above equation, it is in fact enough to check it for a single $\bar{x} \in \tilde{G}$. The $Z$-equivariance of $\pi^* \omega$ implies that any $Z$-translate $\bar{a} + z$ of $\bar{a}$ admits $\gamma$ as a half-braiding. The quadratic form $q$ sends $(\bar{a}, \gamma) \mapsto \gamma(\bar{a})$ and $(\bar{a} + z, \gamma) \mapsto \gamma(\bar{a} + z)$. But $(\bar{a}, \gamma) \in Z^\perp$ implies $\gamma(\bar{a} + z) = \gamma(\bar{a})$. Hence, the functor $Z^\perp \to \mathcal{Z} G_k$ descends to a smooth functor $Z^\perp/Z \to \mathcal{Z} G_k$, which is manifestly inverse to the construction in the first half of the proof. □

**Remark 5.8.** The above calculation admits a more abstract description in the setting of additive monoidal categories. Denote by $C^{\oplus}$ the direct-sum completion of a (smooth) monoidal category $C$. Then $G_k^{\oplus}$ is a monoidal module category over $\mathcal{Z} G_{\pi^* k}^{\oplus}$, obtained by taking modules over the algebra corresponding to $Z \subset \tilde{G}$. The category of modules over an algebra receives a monoidal structure precisely when the algebra is commutative, which happens here if and only if the quadratic form vanishes on $Z$. The Drinfel’d centre of the category $C_A$ of modules over a commutative algebra $A \in \mathcal{Z} C$ was computed in [Sch01] (under completeness conditions which are satisfied here): it is the category $\mathcal{Z} \mathcal{C}_A = \text{loc}_A \mathcal{Z} C$ of local $A$-modules in $\mathcal{Z} C$. In the case at hand, $\mathcal{Z} G_k^{\oplus} = \text{loc}_Z \mathcal{Z} \mathcal{G}_{\pi^* k}^{\oplus} = (Z^\perp/Z, q|_{Z^\perp})^{\oplus}$. The statement about 2-groups can be recovered by restricting to simple objects.

**Example 5.9.** Let $G = \text{SO}(4)$. It is the quotient $\text{Spin}(4)/Z$, where $Z = Z/2 \hookrightarrow Z/2 \times Z/2 = Z(\text{Spin}(4))$ is a normal subgroup of $Z/2$ in the centre (under the decomposition $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$). We use Theorem 5.7 to compute the Drinfel’d centre of $G_k$, for $k \in H^1(\mathcal{B}\text{SO}(4), Z) \subset H^1(\mathcal{B}\text{Spin}(4), Z)$. The isomorphism $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$ allows us to identify the relevant cohomology group as freely generated by the second Chern class in each factor: $H^1(\mathcal{B}\text{Spin}(4), Z) = H^1(\mathcal{B}\text{SU}(2) \times \mathcal{B}\text{SU}(2), Z) =$
We write the cohomology class of the associator in this basis as $k = k_1 c_l + k_r c_r$. By Theorem 5.3, the Drinfel’d centre of Spin(4) at level $k$ is given by

$$\mathcal{Z} \text{Spin}(4)_k = \begin{pmatrix} \mathbb{Z}/2 \times \mathbb{Z}/2, q : & (1, 1) & \mapsto \exp \frac{k_1}{4} \\ (1, -1) & \mapsto \exp \frac{k_1 - k_r}{4} \\ (-1, 1) & \mapsto \exp \frac{k_1 + k_r}{4} \end{pmatrix}.$$  

The quadratic form $q$ is trivial on $Z$ if $k_1 + k_r \equiv 0 \mod 4$. This equation cuts out the subspace of cohomology classes in the image of $H^4(\mathcal{B}SO(4), \mathbb{Z}) \rightarrow H^4(\mathcal{B}\text{Spin}(4), \mathbb{Z})$. The Drinfel’d centre of $G_{k_1, k_r}$ is generated by $[(-1, 1)] = [(1, -1)] \in Z(\text{Spin}(4))/Z$ if $(-1, 1) \in Z^\perp$, otherwise it is trivial. This is equivalent to the condition that $k_l$ be even. In conclusion (see Notation 3.11),

$$\mathcal{Z} \text{SO}(4)_{k_1, k_r} \in 4\mathbb{Z} - k_1 = \begin{cases} \text{Vec}_{\mathbb{Z}/2}^{k_1} & k_l \equiv 0 \mod 4 \\ \text{sVec}^+ & k_l \equiv 2 \mod 4 \\ \text{Vec}^\times & \text{else.} \end{cases}$$

The computations in [CV98] relate the classes $c_l$ and $c_r$ to the more familiar first Pontryagin class $p_1$ and Euler class $\chi$, which generate $H^4(\mathcal{B}SO(4), \mathbb{Z})$. This allows the rephrasing of the above result that we gave in the introduction.

### Comparison to loop group representations

We end by comparing $\mathcal{Z} G_k$ (as computed by Theorems 5.3 and 5.7) to $(\text{Rep}^k LG)^\times$, the maximal sub-2-group of the category of positive energy representations of the loop group.

Let $G$ be a semisimple compact connected Lie group, i.e. a Lie group of the form $G = (\Pi_i G_i)/Z$, where all $G_i$ are compact simple simply-connected Lie groups and $Z$ is a finite central subgroup of the product. Let $k \in H^4(\mathcal{B} G, \mathbb{Z})$ be a cohomology class corresponding to a positive-definite bilinear form $I$ (under the identification made in Section 5). Below, we show that the braided 2-groups computed in the above subsections are equal to $(\text{Rep}^k LG)^\times$, as long as we explicitly exclude any factors of $E_8$ at level $k = 2$.

We use the model of $\text{Rep}^k LG$ as representations of a unitary vertex operator algebra: $\text{Rep}^k V_{G,k}$ — see [Hen17a, Eq (2)] for a definition of $V_{G,k}$ and its modules and [Hen17b, Sect 3] for a discussion of alternative models of $\text{Rep}^k LG$ and their relation. More than just tensor categories, these categories are unitary modular tensor categories [Gui19]. When we write $(\text{Rep}^k LG)^\times$, we mean the maximal unitary sub-2-group: we only retain $\otimes$-invertible objects and unitary morphisms. This way, we obtain a 2-group with

$$\pi_1(\text{Rep}^k LG)^\times = U(1)$$

(rather than $\mathbb{C}^\times$). By Lemma 3.6, the fundamental group $\pi_1 \mathcal{Z} G_k = U(1)$ agrees with that of $(\text{Rep}^k LG)^\times$. It remains to compare $\pi_0$ and the induced quadratic form. We will need the following result.

**Lemma 5.10.** Let $X$ be an invertible object in a unitary modular tensor category. Its self-braiding $\beta_{X,X}$ (considered as a complex number via the canonical isomorphism $\text{End}(X \otimes X) = \mathbb{C}$ sending $\text{id}_{X \otimes X} \mapsto 1$) is equal to its ribbon twist $\theta_X$.

**Proof.** This is best understood using string diagrams. The values of $\beta_{X,X}$ and $\theta_X$ are captured by the following equations (all strands are coloured with the object $X$).

$$\begin{align*}
\begin{array}{c}
\quad = \beta_{X,X} \cdot \\
\quad = \theta_X \cdot
\end{array}
\end{align*}$$
The left hand sides of these equations become equal upon taking the trace — they both form the figure eight. The traces of the right hand sides must thus also be equal:

\[ \beta_{X,X} \cdot \text{Tr}(\text{id}_X \otimes X) = \theta_X \cdot \text{Tr}(\text{id}_X). \]

In a unitary tensor category, \( \text{Tr}(\text{id}_Y) = 1 \) if \( Y \) is invertible. Invertibility of \( X \) and \( X \otimes X \) now implies \( \beta_{X,X} = \theta_X \).

Armed with this Lemma, we proceed as before: we first compute the simply-connected case, and then deal with quotients.

**Theorem 5.11.** For a compact simply-connected Lie group \( G \) and positive-definite level \( k \in H^4(\mathcal{B}G, \mathbb{Z}) \), there is a braided equivalence

\[ \mathcal{Z}^G_k \simeq (\text{Rep}^G_{V,k})^\times, \]

unless \( G \) contains a factor of \( E_8 \) to which \( k \) restricts as level 2.

**Proof.** The group of invertible objects of \( \text{Rep}^G_{V,k} \) was identified as

\[ \pi_0(\text{Rep}^G_{V,k})^\times = Z(G) \]

in \([Li01, \text{Prop 2.20}]\) (see also \([Hen17a, \text{Prop 7}]\)). This only fails when \( G \) contains a factor of \( E_8 \) and \( k \) restricts to level 2 on that factor. The category \( \text{Rep}^V_{E_8,2} \) contains an invertible object of order 2, despite \( Z(E_8) \) being trivial. Excluding this case, \( (\text{Rep}^G_{V,k})^\times \) is tensor equivalent to the underlying 2-group of \( \mathcal{Z}^G_k \), and it remains to compare the self-braidings.

By Lemma 5.10 the self-braiding on an invertible object is equal to its ribbon twist. The values of these are well known: On an invertible object corresponding to \( z \in Z(G) \),

\[ \beta_{z,z} = \theta_z = \exp \left( \frac{1}{2} I(\bar{z}, \bar{z}) \right), \]

where \( \bar{z} \) again denotes a lift of \( z \) to a Cartan of \( G \) (see \([Hen17a, \text{Prop 10}]\)). This formula is the same as that computed for simply-connected groups in Theorem 5.3 and Lemma 5.5 the quadratic form on \( \pi_0(\text{Rep}^k LG)^\times \) agrees with that on \( \pi_0 \mathcal{Z}^G_k \).

To show the above result also holds for non-simply-connected semisimple compact Lie groups \( G \), we use the relation between \( (\text{Rep}^G_{V,k})^\times \) and \( (\text{Rep}^G_{\tilde{G},\tilde{k}})^\times \), where \( \tilde{G} \) denotes the universal cover of \( G \) and \( \tilde{k} \) the pullback cohomology class. Recall that \( G \) is of the form \( \tilde{G}/Z \) where \( Z \hookrightarrow \tilde{G} \) is a finite central subgroup.

**Theorem 5.12.** Let \( G = \tilde{G}/Z \) be a semisimple compact connected Lie group and \( k \in H^4(\mathcal{B}G, \mathbb{Z}) \) positive-definite (such that \( \tilde{G} \) contains no factor of \( E_8 \) to which \( k \) pulls back as level 2). Then there is a braided equivalence

\[ \mathcal{Z}^G_k = (\text{Rep}^G_{V,k})^\times. \]

**Proof.** By \([HKL15, \text{Thm 3.4}]\) and \([CKM17]\), there is a braided equivalence

\[ \text{Rep}^G_{V,k} = \text{loc}_Z \text{Rep}^G_{\tilde{G},\tilde{k}}, \]

where the right-hand side is the category of local \( Z \)-modules in \( \text{Rep}^G_{\tilde{G},\tilde{k}} \). The maximal sub-2-group of the right-hand side may be computed by first restricting to the additive subcategory on invertible objects, taking local modules there, and then taking the maximal sub-2-group. (See also Remark 5.8 for a computation of \( \mathcal{Z}^G_k \) using this method.) The additive subcategory on invertibles is a pointed braided fusion category with simple objects \( Z(\tilde{G}) \) and braiding described by a quadratic form \( q \) (which agrees with that...
on $\mathcal{Z}\tilde{G}$ by Theorem [5.11]. The category of local $\mathcal{Z}$-modules is described in [DS18, Sect 3.1]: the local modules form a pointed braided fusion category with group of simple objects $\mathcal{Z}^\perp/\mathcal{Z}$ (where $\mathcal{Z}^\perp$ is defined as before Theorem [5.7]). The quadratic form on $\mathcal{Z}^\perp/\mathcal{Z}$ is given by the restriction of $q$ to $\mathcal{Z}^\perp$. Thus the maximal sub-2-group is

$$\left(\text{loc}_Z\text{Rep}\mathcal{V}_{\tilde{G},\tilde{k}}\right)^\times = \left(\mathcal{Z}^\perp/\mathcal{Z}, q|_{\mathcal{Z}^\perp}\right).$$

This agrees with the result of Theorem [5.7] completing the comparison. □

References

[BLa04] John C Baez and Aaron D Lauda. "Higher-dimensional algebra. V: 2-groups." In: Theory and Applications of Categories 12 (2004), pp. 423–491.

[Bla85] Philippe Blanc. "Cohomologie différentiable et changement de groupes". In: Astérisque 124.125 (1985), pp. 113–130.

[BM94] Jean-L Brylinski and Dennis A McLaughlin. "The geometry of degree-four characteristic classes and of line bundles on loop spaces I". In: Duke Mathematical Journal 75.3 (1994), pp. 603–638.

[Bor53] Armand Borel. "Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts". In: Annals of Mathematics (1953), pp. 115–207.

[Bor54] Armand Borel. "Sur l’homologie et la cohomologie des groupes de Lie compacts connexes". In: American Journal of Mathematics 76.2 (1954), pp. 273–342.

[Bou94] Nicolas Bourbaki. "Lie groups and Lie algebras". In: Elements of the History of Mathematics. Springer, 1994, pp. 247–267.

[Bra20] Oliver Braunling. "Quinn’s formula and abelian 3-cocycles for quadratic forms". In: Algebras and Representation Theory (2020), pp. 1–33.

[Bry00] Jean-L Brylinski. Differentiable cohomology of gauge groups. 2000. arXiv: math/0011069

[BSCS07] John C Baez, Danny Stevenson, Alissa S Crans, and Urs Schreiber. "From loop groups to 2-groups". In: Homology, Homotopy and Applications 9.2 (2007), pp. 101–135.

[Bun20] Severin Bunk. "Principal $\infty$-Bundles and Smooth String Group Models". In: Hamburger Beitraege Nr. 858, ZMP-HH/20-14 (Aug. 2020). arXiv: 2008.12263

[CKM17] Thomas Creutzig, Shashank Kanade, and Robert McRae. Tensor categories for vertex operator superalgebra extensions. 2017. arXiv: 1705.05017

[ČV98] Martin Čadek and Jiří Vanžura. "On 4-fields and 4-distributions in 8-dimensional vector bundles over 8-complexes". In: Colloquium Mathematicae. Vol. 76. 2. 1998, pp. 213–228.

[DS18] Alexei Davydov and Darren A Simmons. “Third cohomology and fusion categories”. In: Homology, Homotopy and Applications 20.1 (2018), pp. 275–302.

[EGNO16] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. Tensor categories. Vol. 205. American Mathematical Society, 2016.

[EM54] Samuel Eilenberg and Saunders MacLane. "On the groups $H (\Pi, n)$, II: methods of computation". In: Annals of Mathematics (1954), pp. 49–139.

[FHLT10] Dan Freed, Mike Hopkins, Jacob Lurie, and Constantin Teleman. “Topological quantum field theories from compact Lie groups. A celebration of the mathematical legacy of Raoul Bott, 367403”. In: CRM Proceedings Lecture Notes. Vol. 50. 2010.

[Fre09] Daniel Freed. "Remarks on Chern-Simons theory". In: Bulletin of the American Mathematical Society 46.2 (2009), pp. 221–254.

[FSS’12] Domenico Fiorenza, Urs Schreiber, Jim Stasheff, et al. “Čech cocycles for differential characteristic classes: an $\infty$-Lie theoretic construction”. In: Advances in Theoretical and Mathematical Physics 16.1 (2012), pp. 149–250.

[Fuc91] Jürgen Fuchs. "Simple WZW currents". In: Communications in mathematical physics 136.2 (1991), pp. 345–356.

[Gan18] Nora Ganter. “Categorical tori”. In: SIGMA. Symmetry, Integrability and Geometry: Methods and Applications 14 (2018).
[Gir66] Jean Giraud. Cohomologie non abélienne. 1966.

[Gui19] Bin Gui. "Unitarity of the modular tensor categories associated to unitary vertex operator algebras, I". In: Communications in Mathematical Physics 366.1 (2019), pp. 333–396.

[GW09] Krzysztof Gawedzki and Konrad Waldorf. "Polyakov-Wiegmann formula and multiplicative gerbes". In: Journal of High Energy Physics 2009.09 (2009), p. 073.

[Hal15] Brian Hall. Lie groups, Lie algebras, and representations: an elementary introduction. Vol. 222. Springer, 2015.

[Hen08] André G Henriques. "Integrating $L_{\infty}$-algebras". In: Compositio Mathematica 144.4 (2008), pp. 1017–1045.

[Hen17a] André G Henriques. "The classification of chiral WZW models by $H^1_+(BG,\mathbb{Z})$". In: Lie Algebras, Vertex Operator Algebras, and Related Topics 695 (2017), p. 99. arXiv:1602.02968.

[Hen17b] André G Henriques. "What Chern–Simons theory assigns to a point". In: Proceedings of the National Academy of Sciences 114.51 (2017), pp. 13418–13423.

[HKL15] Yi-Zhi Huang, Alexander Kirillov, and James Lepowsky. "Braided tensor categories and extensions of vertex operator algebras". In: Communications in Mathematical Physics 337.3 (2015), pp. 1143–1159.

[JS93] André Joyal and Ross Street. Braided tensor categories. In: Advances in Mathematics 102.1 (1993), pp. 20–78.

[KKJ05] Alexandre Kirillov and Alexander Kirillov Jr. Compact groups and their representations. 2005. arXiv:math/0506118.

[Ler10] Eugene Lerman. "Orbifolds as stacks?" In: L’Enseignement mathématique 56.3 (2010), pp. 315–363.

[Li01] Haisheng Li. "Certain Extensions of Vertex Operator Algebras of Affine Type". In: Communications in Mathematical Physics 217.3 (2001), pp. 653–696.

[MP97] Ieke Moerdijk and Dorette A Pronk. "Orbifolds, sheaves and groupoids". In: K-theory 12.1 (1997), pp. 3–21.

[MT91] Mamoru Mimura and Hiroshi Toda. Topology of Lie groups, I and II. Vol. 91. American Mathematical Society, 1991.

[Noo09] Matt Noonan. Homotopy groups of Lie groups. MathOverflow. 2009. URL: https://mathoverflow.net/q/8957.

[NSW13] Thomas Nikolaus, Christoph Sachse, and Christoph Wockel. "A smooth model for the string group". In: International Mathematics Research Notices 2013.16 (2013), pp. 3678–3721.

[Pir21] Mariam Pirashvili. On the centre of crossed modules of groups and Lie algebras. 2021. arXiv:2109.00981.

[Por08] Sven-S Porst. Strict 2-groups are crossed modules. 2008. arXiv:0812.1464.

[Pro96] Dorette A Pronk. "Etendues and stacks as bicategories of fractions". In: Compositio Mathematica 102.3 (1996), pp. 243–303.

[Qui98] Frank Quinn. "Group categories and their field theories". In: Geometry and Topology Monographs 2 (1998), pp. 407–453.

[Rob12] David M Roberts. "Internal categories, anafunctors and localisations". In: Theory and Applications of Categories 26.29 (2012), pp. 788–829.

[RT91] Nicolai Reshetikhin and Vladimir G Turaev. "Invariants of3-manifolds via link polynomials and quantum groups". In: Inventiones mathematicae 103.1 (1991), pp. 547–597.

[Sch01] Peter Schauenburg. "The monoidal center construction and bimodules". In: Journal of Pure and Applied Algebra 158.2-3 (2001), pp. 325–346.

[Seg70] Graeme B Segal. "Cohomology of topological groups". In: Symposia Mathematica. Vol. 4. Academic Press London. 1970, pp. 377–387.

[Sin75] Hoang Xuan Sinh. "Gr-catégories". PhD thesis. Université Paris, 1975.

[SP11] Christopher J Schommer-Pries. "Central extensions of smooth 2-groups and a finite-dimensional string 2-group". In: Geometry & Topology 15.2 (2011), pp. 609–676.

[ST04] Stephan Stolz and Peter Teichner. "What is an elliptic object?" In: London Mathematical Society Lecture Note Series 308 (2004), p. 247.

[Sta78] James D Stasheff. "Continuous cohomology of groups and classifying spaces". In: Bulletin of the American Mathematical Society 84.4 (1978), pp. 513–530.

[Sto96] Stephan Stolz. "A conjecture concerning positive Ricci curvature and the Witten genus". In: Mathematische Annalen 304.1 (1996), pp. 785–800.

[Str04] Ross Street. "The monoidal centre as a limit". In: Theory and Applications of Categories 13.13 (2004), pp. 184–190.
[Str82] Ross Street. "Characterization of bicategories of stacks". In: Category Theory. Springer. 1982, pp. 282–291.

[Tod87] Hiroshi Toda. "Cohomology of classifying spaces". In: Homotopy theory and related topics. Mathematical Society of Japan, 1987, pp. 75–108.

[Vis04] Angelo Vistoli. Notes on Grothendieck topologies, fibered categories and descent theory. 2004. arXiv:math/0412512

[Wal12] Konrad Waldorf. "A construction of string 2-group models using a transgression-regression technique". In: Analysis, geometry and quantum field theory 584 (2012), pp. 99–115.

[Wal16] Konrad Waldorf. "Transgressive loop group extensions". In: Mathematische Zeitschrift 286.1-2 (2016), 325–360. ISSN: 1432-1823. DOI: 10.1007/s00209-016-1764-0

[Wil13] Dylan Wilson. Are reflective subcategories of complete infinity categories complete? MathOverflow. 2013. URL: https://mathoverflow.net/q/132656

[WW15] Friedrich Wagemann and Christoph Wockel. “A cocycle model for topological and Lie group cohomology”. In: Transactions of the American Mathematical Society 367.3 (2015), pp. 1871–1909.