Optimal scheduling of entropy regulariser for continuous-time linear-quadratic reinforcement learning

Lukasz Szpruch∗ Tanut Treetanthiploet† Yufei Zhang‡

Abstract. This work uses the entropy-regularised relaxed stochastic control perspective as a principled framework for designing reinforcement learning (RL) algorithms. Herein agent interacts with the environment by generating noisy controls distributed according to the optimal relaxed policy. The noisy policies on the one hand, explore the space and hence facilitate learning but, on the other hand, introduce bias by assigning a positive probability to non-optimal actions. This exploration-exploitation trade-off is determined by the strength of entropy regularisation. We study algorithms resulting from two entropy regularisation formulations: the exploratory control approach, where entropy is added to the cost objective, and the proximal policy update approach, where entropy penalises policy divergence between consecutive episodes. We focus on the finite horizon continuous-time linear-quadratic (LQ) RL problem, where a linear dynamics with unknown drift coefficients is controlled subject to quadratic costs. In this setting, both algorithms yield a Gaussian relaxed policy. We quantify the precise difference between the value functions of a Gaussian policy and its noisy evaluation and show that the execution noise must be independent across time. By tuning the frequency of sampling from relaxed policies and the parameter governing the strength of entropy regularisation, we prove that the regret, for both learning algorithms, is of the order $O(\sqrt{N})$ (up to a logarithmic factor) over $N$ episodes, matching the best known result from the literature.

Key words. Continuous-time reinforcement learning, linear-quadratic, entropy regularisation, exploratory control, proximal policy update, regret analysis

AMS subject classifications. 62L05, 49N10, 93E35, 94A17

1 Introduction

Reinforcement learning (RL) is concerned with sequential decision-making in an uncertain environment and is a core topic in machine learning. Two key concepts in RL are exploration, which corresponds to learning via interactions with the environment, and exploitation, which corresponds to optimising the objective function given accumulated information (see [30]). They are at odds with each other, as learning the environment often requires acting suboptimally with respect to existing knowledge [21]. Thus it is crucial to develop effective exploration strategies and to optimally balance exploration and exploitation.

Randomisation and entropy-regularised relaxed control. A common exploration technique in RL is to adopt randomised actions. Randomisation generates stochastic policies that explore the environment, but simultaneously introduces additional biases by assigning positive probability to, potentially, non-optimal actions. The precise impact of randomisation has been studied extensively for discrete-time RL problems, including finite state-action models [10, 4] and linear-quadratic (LQ) models [23, 7, 28]. This is often quantified by the regret of learning, which measures the difference between the value function of designed policies and the optimal value function that could have been achieved had the agent known the environment. It is shown that to achieve the optimal regret, the level of randomisation has to decrease at a proper rate over the learning process.
For RL problems in continuous time and space, [33] proposes an entropy-regularised relaxed control framework to systematically design stochastic policies. Relaxed control formulation has been introduced in classical control theory for the existence of optimal controls (see e.g., [36, 24]), but its entropy regularised variants have recently attracted much attention for their wide applications in learning. Entropy regularisation ensures the optimal stochastic policy has a nondegenerate distribution over actions and hence explores the environment (see e.g., [33, 34, 6, 19, 12, 17, 18]). It also leads to a policy that is Lipschitz stable with respect to perturbations of the underlying model [25]. Such a stability property is crucial for the analysis of algorithm regret [16, 31]. These advantages make entropy-regularised relaxed stochastic control formulation a good choice for the principled design of efficient learning algorithms.

However, despite the recent influx of interest on entropy-regularised control problems, to the best of our knowledge there is no work on how to control the entropy regularising weight to achieve the best possible algorithm regret. A large degree of entropy regularisation often signifies a high level of exploration, which facilitates more efficient learning but also creates a larger exploration cost. Furthermore relaxed control formulation does not constitute a complete learning algorithm as the set of admissible policies for the agent are functions mapping time and space into actions and not into probability distributions over all actions.

Therefore, finding the optimal scheduling of entropy regularisation requires addressing the following two questions: A) How to follow a relaxed policy to interact with a continuous-time system? B) How do the policy execution and regularising weight affect the exploration-exploitation trade-off over the entire learning process?

**Main Contributions.** This work addresses the above questions for finite horizon continuous-time episodic LQ-RL problems. In this problem, the environment is modelled by a linear stochastic differential equation with unknown drift parameter and known deterministic diffusion coefficient, and the agent is minimising known quadratic costs over a finite horizon.

- We prove that a Gaussian relaxed policy can be approximated with a single episode by piecewise constant independent Gaussian noises on a time grid. The type of execution noise corresponds to a properly scaled discrete-time white noise process (see Section 2.5 for details). The error of this noisy execution is of half-order in both the mesh size of the grid and the variance of the relaxed policy in the high-probability sense, and is of first-order in both the mesh size and the variance under expectation (Theorem 2.2). This result should be contrasted with the heuristic implementation of a relaxed policy suggested in [33, 12], which involves independent agents interacting over a large number of episodes (see Example 2.1).

- We use entropy-regularised relaxed control problem formulation to design algorithms that achieve the best-known regret for episodic LQ-RL problems. At each episode, the algorithm executes a Gaussian relaxed policy on a time grid, estimates the unknown parameters by a Bayesian approach based on observed trajectories, and then designs a Gaussian policy by solving an entropy-regularised control problem. Two widely used entropy regularisation formulations are analysed: the exploratory control approach (i.e., Algorithm 1) where entropy serves as an exploration reward [33, 34, 12, 17, 18], and the proximal policy update approach (i.e., Algorithm 2) where entropy penalises the divergence of policies between two consecutive episodes [27, 15, 20]. By optimising the execution mesh size and regularising weight, both algorithms achieve an expected regret of the order $\mathcal{O}(\sqrt{N})$ (up to a logarithmic factor) over $N$ episodes, matching the best possible results from the literature (see Theorems 2.3 and 2.4).

**Related works.** To the best of our knowledge, this is the first theoretical work on the execution of a relaxed policy, on the optimal scheduling of entropy regulariser for RL problems, and on regret bounds of learning algorithms with proximal policy updates.

Optimal control of stochastic systems with parametric uncertainty has been studied in the classical adaptive control literature, where the goal is to construct a stationary policy that minimises the long-term average cost [26]. For LQ adaptive control problems, the convergence of adaptive control algorithms has been shown in [9] as the time goes to infinity, and non-asymptotic regret bounds have been established in [1, 5, 23, 7, 28, 11]. The problem studied here is different. Our main objective is to construct optimal  

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1 The terminologies “relaxed policy” and “stochastic policy” will be used interchangeably throughout the paper.
time-dependent policies for finite-horizon problems, and to establish a principled framework for designing RL algorithms using the perspective of entropy-regularised stochastic control. Compared to the adaptive control literature, the regret analysis in this work requires novel techniques, including precise performance estimation of a randomised policy and quantification of the impact of the regularising weight on the regret bound for continuous-time systems.

Available theoretical results on regret bounds for continuous-time RL are very limited, including [13, 14] for continuous-time Markov decision processes (with finite states and actions), [11] for infinite-horizon LQ-RL problems with stationary policies, [16, 2, 31] for finite-horizon problems with time dependent policies. Specifically, for RL problems with continuous-time LQ models, [2] shows that when the agent knows a priori that the optimal control of the true model exploits the parameter space, a pure exploitation algorithm based on optimal policies of the current estimated models leads to a logarithmic regret. In the more general setting where the optimal control of the true model does not guarantee exploration, [31] proposes a phased-based algorithm that explicitly separates exploitation and exploration phases and achieves a square-root regret. The algorithm therein uses a fixed deterministic exploration policy for exploration throughout the learning process and (deterministic) optimal policies of the current estimated models for exploitation.

The regret analysis of randomised policies considered in this paper is more challenging than those for deterministic policies. Since the randomised policies explore and exploit simultaneously, one needs to carefully disentangle the impacts of the injected noises on exploration and exploitation in order to recover the square-root regret bounds. Moreover, the proximal policy update approach (i.e., Algorithm 2) results in randomised policies depending on all previously estimated models. Such a memory dependence has to be carefully controlled via the regularising weights to optimise the regret.

Organisation of the paper: Section 2 formulates the LQ-RL problem with stochastic policies and presents two learning algorithms with their regret bounds. In particular, Section 2.2 introduces two entropy regularisation formulations for designing learning algorithms. It is proven that both formulations yield Gaussian policies, providing a principled approach to algorithm design. Section 2.3 analyses sampling procedures to execute a Gaussian policy, and establishes bounds on the execution error in terms of the sampling frequency. Section 2.4 optimises the policy sampling frequency and entropy regularisation weight to achieve square-root regrets for both algorithms. The proofs of the execution error bounds and algorithm regrets are presented in Sections 3 and 4, respectively.

Notation: For each $n \in \mathbb{N}$, we denote by $\mathbb{S}^n$ (resp. $\mathbb{S}_+^n$) the space of $n \times n$ symmetric, (resp. symmetric positive definite) matrices, and by $\lambda_{\text{min}}(A)$ the smallest eigenvalues of a given matrix $A \in \mathbb{S}^n$. For each $T > 0$, complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and Euclidean space $(E, | \cdot |)$, we denote by $L^\infty([0, T]; E)$ (resp. $C([0, T]; E)$) the space of measurable (resp. continuous) functions $\phi : [0, T] \to E$ satisfying $\|\phi\|_\infty = \text{ess sup}_{t \in [0, T]} |\phi_t| < \infty$, and by $L^\infty(\Omega \times [0, T]; E)$ the space of measurable functions $\phi : \Omega \times [0, T] \to E$ satisfying $\|\phi\|_{L^\infty} = \text{ess sup}_{(\omega, t) \in \Omega \times [0, T]} |\phi_t(\omega)| < \infty$. For notational simplicity, we write $\mathbb{R}_{\geq 0} = [0, \infty)$, and denote by $\mathcal{N}(m, s^2)$, $m \in \mathbb{R}$ and $s \geq 0$, the Gaussian measure on $\mathbb{R}$ with mean $m$ and standard deviation $s$.

2 Problem formulation and main results

This section studies a LQ-RL problem, where the state dynamics involves unknown drift coefficients. We prove that executing optimal relaxed policies of suitable entropy-regularised control problems leads to learning algorithms with optimal regrets. For the clarity of presentation, we assume all variables are one-dimensional, but the analysis and results can be naturally extended to a multidimensional setting.

2.1 LQ-RL with stochastic policies

LQ control. Let us first recall the standard LQ control problem over feedback controls. Let $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a Brownian motion $W = (W_t)_{t \in [0, T]}$ is defined, and $\mathcal{F}_0 \subset \mathcal{F}$ be a $\sigma$-algebra independent of $W$. For fixed $\theta^* = (A^*, B^*) \in \mathbb{R}^{1 \times 2}$, consider minimising the
The agent then constructs a new stochastic policy due to the sampling process. We use \( \theta \) of an upper bound on Episodic LQ-RL and stochastic policy. Now assume that the parameter sequentially updating and executing (\( \nu \) in RL is first constructing a stochastic policy \( \phi \) of an upper bound on the system optimally while simultaneously learning the parameter \( \theta \). Then the LQ control problem becomes an LQ-RL problem: the agent’s objective is to control the environment by sampling noisy actions according to the policy \( \phi \). Sampling actions from a stochastic policy is referred to as its controlled dynamics (see Section 2.4). The mechanism for sequentially updating and executing (\( \nu \) in RL is first constructing a stochastic policy \( \phi \)) by sampling from a stochastic policy is referred to as its controlled policy. The cost functions \( f \) and \( g \) satisfy for all \( (t, x, a) \in [0, T] \times \mathbb{R} \times \mathbb{R} \),

\[
 f(t, x, a) = Q_t x^2 + 2S_t x a + R_t a^2 + 2p_t x + 2q_t a, \quad g(x) = M x^2 + 2m x. \quad (2.3)
\]

We impose the following conditions on the the coefficients.

**H.1.** \( T > 0, x_0, A^*, B^* \in \mathbb{R}, \bar{\sigma}, Q, S, R, p, q \in C([0, T]; \mathbb{R}), M \geq 0, m \in \mathbb{R}, \) and for all \( t \in [0, T], \bar{\sigma}_t, R_t > 0 \) and \( Q_t - S_t^2 R_t^{-1} \geq 0 \).

Given \( \theta^* \), under (H.1), standard LQ control theory (see e.g., [35]) shows that (2.1)-(2.2) admits the optimal policy \( \phi^{\theta^*} \), and for all \( \theta = (A, B) \in \mathbb{R}^{1 \times 2} \) and \( (t, x) \in [0, T] \times \mathbb{R} \)

\[
 \phi^\theta(t, x) = k^\theta_t + K^\theta_t x, \quad \text{with} \quad K^\theta_t := -(B P^\theta_t + S_t) R_t^{-1}, \quad k^\theta_t := -(B \eta^\theta_t + q_t) R_t^{-1}, \quad (2.4)
\]

where \( P^\theta \in C([0, T]; \mathbb{R}_{\geq 0}) \) and \( \eta^\theta \in C([0, T]; \mathbb{R}) \) satisfy for all \( t \in [0, T] \),

\[
 \frac{d}{dt} P^\theta_t + 2 A P^\theta_t - (B P^\theta_t + S_t) R_t^{-1} + Q_t = 0, \quad P^\theta_0 = M; \quad (2.5a)
\]

\[
 \frac{d}{dt} \eta^\theta_t + (A - (B P^\theta_t + S_t) R_t^{-1} B) \eta^\theta_t + p_t - (B P^\theta_t + S_t) R_t^{-1} q_t = 0, \quad \eta^\theta_T = m. \quad (2.5b)
\]

**Episodic LQ-RL and stochastic policy.** Now assume that the parameter \( \theta^* = (A^*, B^*) \) is unknown to the agent. Then the LQ control problem becomes an LQ-RL problem: the agent’s objective is to control the system optimally while simultaneously learning the parameter \( \theta^* \). In an episodic RL framework, agent learns about \( \theta^* \) by executing a sequence of policies and observing realisations of the corresponding controlled dynamics (see Section 2.4 for details). For simplicity, we assume that the agent knows the form of the dynamics (2.2) (except for the coefficient \( \theta^* \)), the diffusion coefficient \( \bar{\sigma} \), the cost functions \( f \) and \( g \) in (2.3) and the time horizon. The agent is unaware of the exact value of \( \theta^* \) but possesses prior knowledge of an upper bound on \( \theta^* \) as stated in (H.2).

Note that learning the dynamics often requires explicit exploration. Indeed, as shown in [31, 2], a greedy policy \( \phi^\theta \), as defined in (2.4), based on the present estimate \( \theta \) of \( \theta^* \) in general does not guarantee exploration and consequently fails to converge to the optimal solution. As alluded to in Section 1, a common strategy in RL is first constructing a stochastic policy \( \nu : [0, T] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \), which maps the current state to a probability measure over the action space, based on the current estimated model, and then interacting with the environment by sampling noisy actions according to the policy \( \nu \).

Here we outline the episodic learning procedure with stochastic policies. At the \( m \)-th episode, the agent designs a stochastic policy \( \nu_m \), interacts with (2.2) by sampling from \( \nu_m \), and observes a trajectory of the corresponding state dynamics. Sampling actions from a stochastic policy is referred to as its execution, and the resulting randomised policy is denoted by \( \phi^m \). The agent then constructs a new stochastic policy \( \nu_{m+1} \) based on all previous observations, and proceeds to the next learning episode. The mechanism for sequentially updating and executing \( \nu_m \) is referred to as a learning algorithm. A agent aims to design a learning algorithm that minimizes the growth rate of cumulative loss as the number of interactions tends to infinity. The precise formulation of this framework, accounting for different sources of randomness in state observation, policy construction, execution, and evaluation, will be discussed in detail in Section 2.4.

\[\text{Note that a random policy maps the current state and time to the original action space, but involves additional randomness due to the sampling process. We use } \varphi \text{ to distinguish it with the deterministic policy } \phi.\]
2.2 Policy construction via regularised relaxed control

A principle approach to construct stochastic policies is to solve entropy-regularised relaxed control problems (see e.g., [33, 34, 19, 12, 17]). Here we fix \( \theta = (A, B) \) as the current estimate of \( \theta^* \) after the \( m \)-th episode, and present two relaxed control formulations that are widely used in the literature. Let \( \varrho > 0 \) be a regularising weight, and for each measure \( \bar{\nu} \) on \( \mathbb{R} \), let \( \mathcal{H}(\cdot \| \bar{\nu}) : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\} \) be the relative entropy with respect to \( \bar{\nu} \) such that for all \( \nu \in \mathcal{P}(\mathbb{R}) \),

\[
\mathcal{H}(\nu \| \bar{\nu}) = \begin{cases} 
\int_{\mathbb{R}} \ln \left( \frac{\nu(da)}{\bar{\nu}(da)} \right) \nu(da), & \nu \text{ is absolutely continuous with respect to } \bar{\nu}, \\
\infty, & \text{otherwise}.
\end{cases}
\]

**(Approach 1)** (Exploration reward). A stochastic policy can be chosen as the optimal policy of an exploratory control problem (see e.g., [33, 34, 25, 12, 17, 18]): consider minimising

\[
\tilde{J}_{\varrho}^\theta(\nu) = \mathbb{E} \left[ \int_0^T \left( \int_{\mathbb{R}} f(t, X_t^{\theta, \nu}, a) \nu(t, X_t^{\theta, \nu}; da) + \varrho \mathcal{H}(\nu(t, X_t^{\theta, \nu}) \| \nu_{\text{Leb}}) \right) \, dt + g(X_T^{\theta, \nu}) \right]
\]

over all \( \nu \in \mathcal{M} \), where \( \mathcal{M} \) consists of all measurable functions \( \nu : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) such that the following controlled dynamics

\[
dX_t = \int_{\mathbb{R}} (AX_t + Ba) \nu(t, X_t; da) \, dt + \sigma_t \, dW_t, \quad t \in [0, T]; \quad X_0 = x_0
\]

admits a unique square integrable solution \( X^{\theta, \nu} \) satisfying \( \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}} |a|^2 \nu(t, X_t^{\theta, \nu}; da) \, dt \right] < \infty \), \( \bar{\sigma} \), \( f \) and \( g \) are the same functions as in (2.1)-(2.2), and \( \nu_{\text{Leb}} \) is the Lebesgue measure on \( \mathbb{R} \).

The term \( \varrho \mathcal{H}(\cdot \| \nu_{\text{Leb}}) \) is an additional reward to encourage exploration. Under (H.1), standard verification arguments (see e.g., [33, 34]) show that the optimal policy of (2.7)-(2.8) is Gaussian:

\[
\nu_{\text{opt}}^\theta(t, x) = \mathcal{N} \left( k^\theta_t + K^\theta_t x, \frac{\theta}{2R^2} \right), \quad (t, x) \in [0, T] \times \mathbb{R},
\]

where \( K^\theta \) and \( k^\theta \) are defined as in (2.4). Note that \( \varrho > 0 \) ensures that \( \nu_{\text{opt}}^\theta(t, x) \) executes each action with positive probability and hence explores the parameter space.

**(Approach 2)** (Proximal policy update). A stochastic policy can also be determined by penalising the divergence of policies between two consecutive episodes. Recall that the greedy policy \( \phi^\theta \) in (2.4) satisfies the following pointwise minimisation condition (see e.g., [35, p. 317]):

\[
\phi^\theta(t, x) = \arg \min_{a \in \mathbb{R}} H^\theta(t, x, a, 2(P^\theta_t x + \eta^\theta_t)), \quad (t, x) \in [0, T] \times \mathbb{R},
\]

where \( P^\theta \) and \( \eta^\theta \) are defined as in (2.5), and \( H^\theta : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is the Hamiltonian of (2.1)-(2.2) with \( \theta^* = \theta \):

\[
H^\theta(t, x, a, y) := (Ax + Ba)y + f(t, x, a), \quad (t, x, a, y) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.
\]

Instead of taking the greedy update (2.10), the agent determines a stochastic policy (for the \( (m + 1) \)-th episode) by minimising the following entropy-regularised Hamiltonian: for each \( (t, x) \in \mathcal{P}(\mathbb{R}) \),

\[
\nu_{\text{opt}}^\theta(t, x) = \arg \min_{\nu \in \mathcal{P}(\mathbb{R})} \left( \int_{\mathbb{R}} H^\theta(t, x, a, 2(P^\theta_t x + \eta^\theta_t)) \nu(da) + \varrho \mathcal{H}(\nu \| \nu^m(t, x)) \right),
\]

where \( \nu^m \) is the relaxed policy for the \( m \)-th episode.

The relative entropy in (2.11) enforces the updated policy \( \nu_{\text{opt}}^\theta(t, x) \) to stay close to \( \nu^m(t, x) \). This prevents an excessively large policy update, and ensures that the updated policy \( \nu_{\text{opt}}^\theta \) explores the space if \( \nu^m \) is a relaxed policy. Similar ideas have been used to design proximal policy gradient methods in [27, 20] and
The policy \( \nu^\theta \) in (2.11) is Gaussian if the previous policy \( \nu^m \) is Gaussian. Indeed, let \( \nu^m(t, x) = \mathcal{N}(k_t^m + K_t^m x, (\lambda_t^m)^2) \) for some \((t, x) \in [0, T] \times \mathbb{R}\), then one can directly verify that

\[
\nu^\theta(t, x) = \mathcal{N}(k_t^{m+1} + K_t^{m+1} x, (\lambda_t^{m+1})^2), \quad (t, x) \in [0, T] \times \mathbb{R},
\]

where

\[
k_t^{m+1} = h_t^m k_t^\theta + (1 - h_t^m) k_t^m, \quad K_t^{m+1} = h_t^m K_t^\theta + (1 - h_t^m) K_t^m,
\]

\[
(\lambda_t^{m+1})^{-2} = \frac{2R_t}{\theta} + (\lambda_t^m)^{-2}, \quad h_t^m = \frac{2R_t}{2R_t + \theta(\lambda_t^m)^{-2}},
\]

with \( K_t^\theta \) and \( k_t^\theta \) defined in (2.4).

Observe that in the present LQ setting, solving the relaxed control problems in Approaches 1 and 2 leads to Gaussian-measure-valued policies. The mean of these Gaussian measures is affine in the state and depends on the estimated parameter \( \theta \), while the variance of these Gaussian measures depends on the regularising weight \( \theta \). In the following, we demonstrate how to follow these Gaussian policies to interact with the dynamics (2.2), and how to choose the regularising weight for an efficient learning algorithm.

### 2.3 Execution of Gaussian relaxed policy

This section studies the execution of a Gaussian relaxed policy via randomisation. The precise definition of Gaussian policies is given as follows.

**Definition 2.1.** For each \( k, K \in L^\infty([0, T]; \mathbb{R}) \) and \( \lambda \in L^\infty([0, T]; \mathbb{R}_\geq 0) \), we define a Gaussian relaxed policy \( \nu : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \), denoted by \( \nu \sim \mathcal{G}(k, K, \lambda) \), such that for all \((t, x) \in [0, T] \times \mathbb{R}, \nu(t, x) = \mathcal{N}(k_t + K_t x, \lambda_t^2) \).

In the following, we introduce sampling procedures to execute a fixed Gaussian policy \( \nu \sim \mathcal{G}(k, K, \lambda) \), where the coefficients \((k, K, \lambda)\) are known to the agent. This is the task encountered in each episode of the previously mentioned episodic learning problem, where a Gaussian policy \( \nu \) has been constructed based on estimated parameters. We allow the coefficients of the Gaussian policy to be arbitrary bounded functions, which include the policies (2.9) and (2.12) as special cases. Note that executing a Gaussian policy does not require knowing the true parameter \( \theta^m \) in (2.1), as the agent can simply take feedback controls based on observed system states.

To execute a Gaussian relaxed policy \( \nu \), at each given time and state, the agent will sample a control action based on the distribution \( \nu(t, x) \sim \mathcal{N}(k_t + K_t x, \lambda_t^2) \). This suggests interacting with the dynamics (2.2) using a randomised policy \( \varphi(t, x) = k_t + K_t x + \lambda_t \xi_t \), where \( \xi_t \sim \mathcal{N}(0, 1) \) is an injected Gaussian noise independent of the Brownian motion \( W \) in (2.2). The precise definition of the class of randomised policies is given below.

**Definition 2.2.** For each \( k, K \in L^\infty([0, T]; \mathbb{R}) \), \( \lambda \in L^\infty([0, T]; \mathbb{R}_\geq 0) \), and measurable process \( \xi : \Omega \times [0, T] \to \mathbb{R} \) satisfying for all \( t \in [0, T] \), \( \xi_t \) is an \( \mathcal{F}_t \)-measurable standard normal random variable, we define a randomised policy with Gaussian noise \( \varphi : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R} \), denoted by \( \varphi \sim \mathcal{R}(k, K, \lambda, \xi) \), such that for all \((\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}, \varphi(\omega, t, x) = k_t + K_t x + \lambda_t \xi_t(\omega) \).

Note that the process \( \xi \) in Definition 2.2 models additional Gaussian noises used for executing a relaxed policy, and is assumed, without loss of generality, sampled at \( t = 0 \) (and hence independent of the Brownian motion \( W \)). However, Definition 2.2 only imposes a Gaussianity of \( \xi_t \) for each \( t \in [0, T] \), without requiring an independence of \((\xi_t, \xi_s)\) for different \( t, s \in [0, T] \). In the sequel, we simply refer to \( \varphi \sim \mathcal{R}(k, K, \lambda, \xi) \) as a randomised policy if no confusion occurs.

To compare the performances of the Gaussian relaxed policy and a randomised policy, we introduce the following cost of a relaxed policy \( \nu \sim \mathcal{G}(k, K, \lambda) \) as in [33] (cf. (2.7)):

\[
\mathcal{J}_0^\nu(\nu) := \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}} f(t, X_t^{\nu, x}; a) \nu(t, X_t^{\nu, x}; da) \, dt + g(X_T^{\nu, x}) \right],
\]

where \( f \) and \( g \) are given as in (2.7).
where $f$ and $g$ are defined in (2.3), and $X^{\theta^*, \nu}$ is the solution to the following relaxed dynamics:

$$\begin{align*}
\text{d}X_t &= \int_{\mathbb{R}} \left( A^* X_t + B^* a \right) \nu(t, X_t; da) \, dt + \sigma_t \, dW_t, \quad t \in [0, T]; \quad X_0 = x_0.
\end{align*}$$

(2.15)

Note that if $\lambda \equiv 0$, then $\nu(t, x) = \delta_{\phi(t, x)}$ and $\tilde{J}^{\theta^*}_0(\nu) = J^{\theta^*}(\phi)$, where $\phi(t, x) = k_t + K_t x$, $\delta_a$ is the Dirac measure supported at $a \in \mathbb{R}$, and $J^{\theta^*}(\phi)$ is defined in (2.1). In other words, (2.14)-(2.15) incorporates (2.1)-(2.2) as a special case. We also define the expected cost of a randomised policy $\varphi \sim \mathcal{R}(k, K, \lambda, \xi)$ (with fixed realisation of $\xi$):

$$J^{\theta^*}(\varphi) := \mathbb{E} \left[ \int_0^T f \left( t, X^\varphi_t, \varphi(\cdot, t, X^\varphi_t) \right) \, dt + g(X^\varphi_T) \bigg| \mathcal{F}_0 \right],$$

(2.16)

where $X^\varphi$ satisfies the following controlled dynamics (cf. (2.2)):

$$\begin{align*}
\text{d}X_t &= (A^* X_t + B^* \varphi(\cdot, t, X_t)) \, dt + \sigma_t \, dW_t, \quad t \in [0, T]; \quad X_0 = x_0.
\end{align*}$$

(2.17)

Note that the expectation in (2.16) is only taken over the Brownian motion $W$ in (2.17). Hence, $J^{\theta^*}(\varphi)$ is a random variable, as its value depends on the realisations of the injected noise $\xi$.

The following lemma quantifies the difference between $\tilde{J}(\nu)$ with $\nu \sim \mathcal{G}(k, K, \lambda)$ and $J(\varphi)$ with $\varphi \sim \mathcal{R}(k, K, \lambda, \xi)$.

**Lemma 2.1.** Suppose (H.1) holds. Then for all $\nu \sim \mathcal{G}(k, K, \lambda)$ and $\varphi \sim \mathcal{R}(k, K, \lambda, \xi)$,

$$J^{\theta^*}(\varphi) - \tilde{J}^{\theta^*}_0(\nu) = \int_0^T h_{1,t} \lambda_t \xi_t \, dt + \int_0^T h_{2,t} \lambda_t^2 (\xi_t^2 - 1) \, dt + \int_0^T \int_0^T h_{3,t,r} \lambda_t \lambda_r \xi_t \xi_r \, dr \, dt,$$

(2.18)

where $h_1, h_2 : [0, T] \to \mathbb{R}$ and $h_3 : [0, T]^2 \to \mathbb{R}$ are some bounded functions whose sup-norms depend only on the sup-norms of $k, K$ and the coefficients in (H.1).

The proof of Lemma 2.1 is given in Section 3.

Note that the first two terms on the right-hand side of (2.18) vanish under expectation, while the third term has a non-zero expectation if the injected noise $\xi$ is correlated across time. Indeed, the following illustrative example shows that repeated sampling of the relaxed policy across multiple episodes may not yield the relaxed cost $\tilde{J}^{\theta^*}_0(\nu)$, and therefore is not the correct execution of a relaxed policy (cf. the heuristic implementation of a relaxed policy suggested in [12, Section 3]). The proof follows from a direct computation and the strong law of large numbers (omitted for brevity).

**Example 2.1.** Let $\theta^* = (0, 1), \mu, \sigma \in \mathbb{R}$ and $\lambda \geq 0$. Consider the relaxed policy $\nu : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ such that $\nu(t, x) = \mathcal{N}(\mu x, \lambda^2)$ for all $(t, x) \in [0, 1] \times \mathbb{R}$, and the cost (cf. (2.14)):

$$\tilde{J}^{\theta^*}_0(\nu) = \mathbb{E} \left[ \int_0^1 \int_{\mathbb{R}} |a|^2 \nu(t, X^\nu_t; da) \, dt \right], \quad \text{with} \quad X^\nu_t = \int_0^t \int_{\mathbb{R}} a \nu(s, X^\nu_s; da) \, ds + \sigma W_t, \quad \forall t \in [0, 1].$$

Let $(\xi_i)_{i \in \mathbb{N}}$ be $\mathcal{F}_0$-measurable independent standard normal random variables, and for each $i \in \mathbb{N}$, let $\varphi^i : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ be such that $\varphi^i(\cdot, t, x) = \mu x + \lambda \xi^i_t$ for all $(t, x) \in [0, 1] \times \mathbb{R}$, and the cost (cf. (2.16)):

$$J^{\theta^*}(\varphi^i) := \mathbb{E} \left[ \int_0^1 (\varphi^i(\cdot, t, X^i_t))^2 \, dt \bigg| \mathcal{F}_0 \right], \quad \text{with} \quad X^i_t = \int_0^t \varphi^i(\cdot, s, X^i_s) \, ds + \sigma W_t, \quad \forall t \in [0, 1].$$

Then $\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N J^{\theta^*}(\varphi^i) = \tilde{J}^{\theta^*}_0(\nu) + \frac{\lambda^2}{2\mu} \left( e^{2\mu} - 2\mu - 1 \right)$ a.s.

Instead, we show that the relaxed cost (2.14) can be realised by sampling the target policy with sufficient independence across different time points within a single episode. Specifically, we consider executing a relaxed policy on a time grid using piecewise constant independent Gaussian noises, defined as follows.
Definition 2.3. For each $\nu \sim \mathcal{G}(k, K, \lambda)$ and grid $\pi = \{0 = t_0 < \cdots < t_N = T\}$ with mesh size $|\pi| = \max_{i=0,\ldots,N-1}(t_{i+1} - t_i)$, we say $\varphi : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a piecewise constant independent execution of $\nu$ on $\pi$, denoted by $\varphi \sim \mathcal{R}(k, K, \lambda, \xi)$, if there exist $\mathcal{F}_\nu$-measurable independent standard normal random variables $(\zeta_i)_{i=0}^{N-1}$ such that for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}$, $\varphi(\omega, t, x) = k_t + K_t x + \lambda_t \zeta_i(\omega)\cdot 1_{t_i \leq t < t_{i+1}}$.

Remark 2.1. At first glance, it seems natural to consider injecting an exploration noise $\xi = (\xi_t)_{t \in [0, T]}$ that consists of (uncountably many) pairwise independent standard Gaussian random variables. However, imposing pairwise independence over the index set $[0, T]$ leads to highly irregular sample paths of $\xi$, which introduces measure-theoretical issues. For instance, [29, Corollary 4.3] shows that, if $\xi$ is a process of (essentially) pairwise-independent non-random constant random variables, then almost all sample paths of $t \mapsto \xi_t(\omega)$ are not Lebesgue measurable\(^3\). This implies that the time integrals of $\xi$ in Lemma 2.1 become ill-defined, which prevents us from using $\xi$ as an exploration noise.

Due to the technical challenges associated with the use of a “continuum” of i.i.d. random variables, we have opted for a relaxed approach by ensuring sufficient independence on a grid, and adjusting the sampling frequency through the mesh size. This allows us to bypass these measure-theoretical difficulties while still achieving the desired level of independence in our exploration noises.

It is worth noting that one can also consider changing the coefficients of randomised policies only at grid points. This will introduce an additional time discretisation error, which can be controlled based on the time regularity of $(k, K, \lambda)$. Discrete-time policies of this nature have been analysed in [2, Section 2.3] for LQ-RL problems with deterministic policies. The analysis therein shows that discrete-time policies achieve similar regret as continuous-time policies, with an additional term that explicitly depends on the time step sizes used in the algorithm. We anticipate that a similar regret analysis can be performed in the present setting with stochastic policies, whose detailed analysis is left for future work.

The following theorem establishes the convergence of $J^\nu(\varphi) - \tilde{J}^\nu_0(\nu)$ with respect to the mesh size $|\pi|$ and the standard deviation $\lambda$, where $\nu \sim \mathcal{G}(k, K, \lambda)$ and $\varphi \sim \mathcal{R}(k, K, \lambda, \xi)$ are given in Definitions 2.1 and 2.3, respectively. This consequently proves that $\varphi$ is a proper execution of $\nu$. The explicit dependence of the convergence rate on $\lambda$ is crucial for determining the appropriate choice of entropy regularisation weight and for analysing the algorithm performance in Section 2.4.

Theorem 2.2. Suppose (H.1) holds and let $\nu \sim \mathcal{G}(k, K, \lambda)$. Then there exists a constant $C > 0$, depending only on the sup-norms of $k, K$ and the coefficients in (H.1), such that

1. for all grids $\pi$ of $[0, T]$, all $\varphi \sim \mathcal{R}(k, K, \lambda, \xi)$ and all $\delta \geq 0$,
   \[
   \mathbb{P}\left(\left|J^\nu(\varphi) - \tilde{J}^\nu_0(\nu)\right| \geq C\sqrt{|\pi||\lambda||\xi|}(1 + \|\lambda\|_{\infty})(1 + \ln \left(\frac{1}{\delta}\right))\right) \leq \delta,
   \]

2. if $\pi_n = \{i\delta\}_{i=0}^{n}$ and $\varphi^n \sim \mathcal{R}(k, K, \lambda, \xi^n)$ for all $n \in \mathbb{N}$, then it holds a.s. that
   \[
   \left|J^{\varphi^n}(\varphi^n) - \tilde{J}^{\varphi^n}_0(\nu)\right| \leq Cn^{-\frac{1}{2}}(\ln n)||\lambda||_{\infty}(1 + ||\lambda||_{\infty}), \quad \text{for all sufficiently large } n \in \mathbb{N},
   \]

3. for all grids $\pi$ of $[0, T]$ and $\varphi \sim \mathcal{R}(k, K, \lambda, \xi^n)$, $|\mathbb{E}[J^{\varphi^n}(\varphi) - \tilde{J}^{\varphi^n}_0(\nu)]| \leq C||\pi||||\lambda||^2_{\infty}$.

The proof of Theorem 2.2 is given in Section 3.

Remark 2.2. Roughly speaking, Theorem 2.2 shows that if a randomised policy $\varphi \sim \mathcal{R}(k, K, \lambda, \xi)$ is driven by an external noise $\xi$ with independent marginals across time, then its cost $J^\nu(\varphi)$ well represents the relaxed cost $\tilde{J}^\nu_0(\nu)$. Such a process $\xi$ is closely related to the white noise in the distribution theory, and we refer the reader to Section 2.5 for a detailed discussion.

\(^3\)A measurable process $\xi$ of (essentially) pairwise independent random variables does not exist on a usual product probability space, but can be constructed on an extended product space, as shown in [29]. In particular, in [29, Proposition 5.6], given the index set $I = [0, 1]$ and a probability space $(\Lambda, \mathcal{F}, P)$, the author constructs an extended product probability space $(I \times \Lambda, \mathcal{I} \otimes \mathcal{F}, \Lambda \otimes P)$ (known as the rich Fubini extension), on which a continuum of (essentially) i.i.d. random variables is defined. Here $\mathcal{I}$ and $\Lambda$ are $\sigma$-algebra and probability measure on $I$, respectively, whose specific choices are part of the construction. The implication of [29, Corollary 4.3] is that $\lambda$ cannot be chosen as the Lebesgue measure in this context.
2.4 Tuning entropy regularising weight for optimal regret

Based on Theorem 2.2, this section analyses the episodic learning algorithms outlined in Sections 2.1 and 2.2. In particular, it optimises the policy sampling frequency and entropy regularisation weight to achieve the best possible algorithm regret.

Probabilistic learning framework. We start by recalling the generic learning procedure for solving (2.1)-(2.2) with unknown \( \theta^* \): for each episode \( m \in \mathbb{N} \), let \( \nu^m = \mathcal{N}(k^m_t + K^m_t x, (\lambda^m)^2) \) be a given Gaussian relaxed policy. Here \( k^m, K^m, \lambda^m : \Omega \times [0, T] \to \mathbb{R} \) are designed based on observations from previous episodes. The agent then chooses a grid \( \pi_m \) of \([0, T]\) and executes \( \nu^m \) on \( \pi_m \) via a policy \( \varphi^m \): for all \((\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R} \),

\[
\varphi^m(\omega, t, x) = k^m_t(\omega) + K^m_t(\omega)x + \lambda^m_t(\omega)\xi^m_t(\omega),
\]

where \( \xi^m \) is a piecewise constant process on \( \pi_m \) generated by independent standard normal variables \((\xi^n_i)_{i \in \mathbb{N}} \) (cf. Definition 2.3). The agent observes a trajectory of the corresponding state process \( X^m \) governed by the following dynamics (cf. (2.2)):

\[
dX_t^m = \theta^* Z_t^m dt + \sigma_t dW_t^m, \quad X_0^m = x_0, \quad \text{with } Z_t^m = \left( X_t^m, \varphi^{(\cdot, t, X_t^m)} \right),
\]

where \( W^m \) is an independent Brownian motion corresponding to the \( m \)-th episode.\(^4\) The agent then constructs an estimate of \( \theta^* \), denoted by \( \tilde{\theta}_m \), based on all observations \((X^n, \xi^n)_{n=1}^m \), and solves a regularised relaxed control problem (see e.g., Approaches 1 and 2) based on \( \theta_m \) and a properly chosen weight \( \theta_m > 0 \). This determines the relaxed policy \( \nu^{m+1} \) for the next episode.

Bayesian inference. To derive a concrete learning algorithm, we infer the parameter \( \theta^* \) by a Bayesian approach as in [31]. Let \( \mathcal{F}^{\text{ob}}_m \) be the observation \( \sigma \)-algebra generated by the state processes and the injected noises after the \( m \)-th episode:

\[
\mathcal{F}^{\text{ob}}_m = \sigma \{ X^n_t, \xi^n_t \mid t \in [0, T], n = 1, \ldots, m \} \vee \mathcal{N}_0, \quad m \in \mathbb{N} \cup \{0\},
\]

where \( \mathcal{N}_0 \) is the \( \sigma \)-algebra generated by \( \mathbb{P} \)-null sets. By [22, Section 7.6.4], for each \( m \in \mathbb{N} \), the likelihood function of \( \theta^* \) with observations \((X^n, \xi^n)\) is given by

\[
\ell(\theta^* \mid X^m, \xi^m) \propto \exp \left( -\frac{1}{2} \int_0^T \theta^* Z^n_s dX^n_s - \frac{1}{2} \int_0^T \frac{1}{\sigma^2_s} (\theta^* Z^n_s)^2 \right),
\]

where \( \propto \) stands for proportionality up to a constant independent of \( \theta^* \). Hence, given the initial belief that \( \theta^* \) follows the prior distribution \( \pi(\theta^* \mid \mathcal{F}^{\text{ob}}_0) = \mathcal{N}(\theta_0, V_0) \) for some \( \theta \in \mathbb{R}^{1 \times 2} \) and \( V_0 \in S^2_+ \), the posterior distribution of \( \pi(\theta^* \mid \mathcal{F}^{\text{ob}}_m) \) after the \( m \)-th episode is given by

\[
\pi(\theta^* \mid \mathcal{F}^{\text{ob}}_m) \propto \pi(\theta^* \mid \mathcal{F}^{\text{ob}}_0) \prod_{n=1}^m \ell(\theta^* \mid X^n, \xi^n) \propto \exp \left( -\frac{(\theta^* - \tilde{\theta}_m)V^{-1}_m(\theta^* - \tilde{\theta}_m)^\top}{2} \right),
\]

where \( \tilde{\theta}_m \) and \( V_m \) are given by:

\[
V_m := \left( V_0^{-1} + \sum_{n=1}^m \int_0^T \frac{1}{\sigma^2_s} Z^n_s (Z^n_s)^\top ds \right)^{-1} \in S^2_+,
\]

\[
\tilde{\theta}_m := \left( \theta_0 V_0^{-1} + \sum_{n=1}^m \left( \int_0^T \frac{1}{\sigma^2_s} Z^n_s dX^n_s \right)^\top \right) V_m \in \mathbb{R}^{1 \times 2}.
\]

For simplicity, we assume that the agent has prior knowledge of the magnitude of \( \theta^* \):

\(^4\)Without loss of generality, we assume that \((\xi^n_i, W^n)_{n \in \mathbb{N}}\) are defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), \((\xi^n)_{n \in \mathbb{N}}\) are measurable with respect to the initial \( \sigma \)-algebra \( \mathcal{F}_0 \), and all Brownian motions \((W^n)_{n \in \mathbb{N}}\) are independent.
H.2. \( \Theta \) is a compact subset of \( \mathbb{R}^{1 \times 2} \) such that \( \theta^* \in \text{int}(\Theta) \).

Condition (H.2) suggests a truncated maximum a posteriori estimate \( \theta_m \) as follows:

\[
\theta_m := \Pi_\Theta(\theta_m, V_m) \in \mathbb{R}^{1 \times 2},
\]

where \( \Pi_\Theta : \mathbb{R}^{1 \times 2} \times S_+^2 \to \Theta \) is a measurable function such that \( \Pi_\Theta(\theta, V) = \theta \) for all \( \theta \in \Theta \) and \( V \in S_+^2 \). Possible choices of \( \Pi_\Theta \) include the orthogonal projection for closed and convex \( \Theta \), or a function that maximises the posterior distribution (2.23) over all \( \theta^* \in \Theta \). We emphasise that the a-priori bound of \( \theta^* \) is only imposed to simplify the analysis, which along with the projection step \( \Pi_\Theta \) ensures the estimates \((\theta_m)_{m \in \mathbb{N}}\) are a-priori bounded. In the case without prior knowledge on \( \theta^* \), one can remove the projection step, and instead adopt a sufficiently long initial exploration step to guarantee a uniform bound of \((\hat{\theta}_m)_{m \in \mathbb{N}}\) with high probability (see e.g., [28, 2]).

**Performance measure.** In the above learning procedure, the agent solves suitable regularised relaxed control problems and executes the relaxed policies through randomisation. To measure the performance of a learning algorithm in this setting, we consider the regret of learning defined as follows (see e.g., [16, 31, 2]):

for each \( N \in \mathbb{N} \),

\[
\text{Reg}(N) = \sum_{m=1}^N \left( J^{\theta^*}(\varphi^m) - J^{\theta^*}(\varphi^{\theta^*}) \right),
\]

where \( J^{\theta^*}(\varphi^{\theta^*}) \) is the optimal cost that agent can achieve knowing the parameter \( \theta^* \) (see (2.1)), and \( J^{\theta^*}(\varphi^m) \) is the expected cost of the randomised policy \( \varphi^m \) for the \( m \)-th episode:

\[
J^{\theta^*}(\varphi^m) := \mathbb{E} \left[ \int_0^T f(t, X_t^m, \varphi^m(\cdot, t, X_t^m)) \, dt + g(X_T^m) \left| \mathcal{F}_{m-1}^{\text{ob}} \vee \mathcal{Z}_m \right. \right],
\]

where \( \mathcal{F}_{m-1}^{\text{ob}} \) is defined as in (2.21), and \( \mathcal{Z}_m = \sigma\{\xi_t^m \mid t \in [0, T]\} \). The regret \( \text{Reg}(N) \) characterises the cumulative loss from taking sub-optimal policies up to the \( N \)-th episode. Agent’s aim is to construct policies whose regret grows sublinearly with respect to \( N \).

Note that the expectation in (2.27) is only taken with respect to the Brownian motion \( W^m \), and consequently, for each \( m \in \mathbb{N} \), \( J^{\theta^*}(\varphi^m) \) is a random variable depending on the realisations of the Brownian motions \((W^n)_{n=1}^{m-1}\) and the injected noises \((\xi^{n^m})_{n=1}^{m}\).

**Regret analysis.** Given the parameter inference scheme (2.24)-(2.25), the performance of a learning algorithm now depends on the construction and execution of relaxed policies. In the sequel, we propose two algorithms based on regularised control problems in Approaches 1 and 2, and choose the regularising weights \((\varrho_m)_{m \in \mathbb{N}}\) and execution grids \((\pi_m)_{m \in \mathbb{N}}\) to minimise the algorithm regrets.

The first algorithm computes relaxed policies by solving the relaxed control problem (2.7) with exploration rewards. We summarise the algorithm as follows.

**Algorithm 1:** Learning with exploration reward

1. **Input:** \( \theta_0 \in \mathbb{R}^{1 \times 2}, V_0 \in S_+^2 \), a truncation function \( \Pi_\Theta \), and a Gaussian relaxed policy \( \nu^1 \).
2. for \( m = 1, 2, \ldots \) do
3. Determine a grid \( \pi_m \) of \([0, T]\) and exercise \( \nu^m \) on \( \pi_m \) via \( \varphi^m \) as in (2.19).
4. Obtain the updated estimate \( \theta_m \) via (2.25).
5. Determine \( \varrho_m > 0 \) and compute \( \nu^{m+1} \) by (2.9) with \( \theta = \hat{\theta}_m \) and \( \varrho = \varrho_m \).
6. end

The following theorem chooses \((\varrho_m)_{m \in \mathbb{N}}\) and \((\pi_m)_{m \in \mathbb{N}}\) such that Algorithm 1 achieves a regret of the order \( \mathcal{O}(\sqrt{N} \ln N) \) in expectation. This recovers the regret order of the phased-based algorithm with deterministic policies in [31]. Note that the additional exploration reward vanishes as the number of episodes tends to infinity.
Theorem 2.3. Suppose (H.1) and (H.2) hold. Let \( \theta_0 \in \mathbb{R}^{1 \times 2} \), \( V_0 \in \mathbb{S}^2_+ \), \( \varrho_0 > 0 \) and \( \nu^1 \sim \mathcal{G}(k_0^\theta, K^\theta, \sqrt{\theta_0/(2R)}) \). Then there exists \( c_0 > 0 \) such that if one sets \( \varrho_m = \varrho_0 m^{-\frac{2}{7}} \ln(m+1) \) and \( |\pi_m| \leq c_0 \) for all \( m \in \mathbb{N} \), then there exists a constant \( C \geq 0 \) such that the regret of Algorithm 1 satisfies
\[
\mathbb{E}[\text{Reg}(N)] \leq C\sqrt{N} \ln N, \quad \forall N \in \mathbb{N} \cap [2, \infty).
\]

Remark 2.3. The proof of Theorem 2.3 is given in Section 4.2, where the key steps are outlined at the end of this section for the reader’s convenience. Note that for simplicity we choose \( \nu^1 \) based on \( \theta_0 \), but the same regret order can be achieved with \( \nu^1 \sim \mathcal{G}(k^1, K^1, \lambda^1) \) for any deterministic \( k^1, K^1 \) and \( \lambda^1 \). Moreover, we focus on optimising the regret order in expectation, and obtain a simple deterministic scheduling of the sampling frequency \( (|\pi_m|)_{m \in \mathbb{N}} \) and regularising weights \( (\varrho_m)_{m \in \mathbb{N}} \). The analysis can be extended to ensure that a similar regret bound holds with probability at least \( 1 - \delta \), for any given \( \delta \in (0, 1) \), but \((\pi_m)_{m \in \mathbb{N}} \) and \((\varrho_m)_{m \in \mathbb{N}} \) have to be chosen depending explicitly on \( \delta \) (see e.g., [28, 16, 2]). We leave a rigorous analysis of such a high-probability regret for future research.

The second algorithm computes relaxed policies with the proximal policy update (2.11). We summarise the algorithm as follows.

**Algorithm 2:** Learning with proximal policy update

1. **Input:** \( \theta_0 \in \mathbb{R}^{1 \times 2} \), \( V_0 \in \mathbb{S}^2_+ \), a truncation function \( \Pi_{\theta_0} \), and a Gaussian relaxed policy \( \nu^1 \).
2. **for** \( m = 1, 2, \ldots \) **do**
   1. Determine a grid \( \pi_m \) of \([0, T]\) and exercise \( \nu^m \) on \( \pi_m \) via \( \varphi^m \) as in (2.19).
   2. Obtain the updated estimate \( \theta_m \) via (2.25).
   3. Determine \( \varrho_m > 0 \) and compute \( \nu^{m+1} \) by (2.13) with \( \theta = \theta_m \) and \( \varphi = \varrho_m \).
3. **end**

The following theorem shows that Algorithm 2 achieves a similar sublinear regret as Algorithm 1. Note that unlike Algorithm 1, the regularising weight \( \varrho_m \) tends to infinity as the learning proceeds.

Theorem 2.4. Suppose (H.1) and (H.2) hold. Let \( \theta_0 \in \mathbb{R}^{1 \times 2} \), \( V_0 \in \mathbb{S}^2_+ \), \( \varrho_0 > 0 \) and \( \nu^1 \sim \mathcal{G}(k_0^\theta, K^\theta, \theta_0) \). Then there exists \( c_0 > 0 \) such that if one sets \( \varrho_m = \varrho_0 m^{-\frac{2}{7}} \ln(m+1) \) and \( |\pi_m| \leq c_0 \) for all \( m \in \mathbb{N} \), then there exists a constant \( C \geq 0 \) such that the regret of Algorithm 2 satisfies
\[
\mathbb{E}[\text{Reg}(N)] \leq C\sqrt{N} \ln(N \ln(N)), \quad \forall N \in \mathbb{N} \cap [3, \infty).
\]

Remark 2.4. The proof of Theorem 2.4 is given in Section 4.3, whose key steps are outlined below. The argument is more involved than that of Theorem 2.3. This is due to the fact that \( \nu^{m+1}, m \in \mathbb{N} \), is a convex combination of all previous policies (\( (\nu^m)_{m=0}^\infty \) (cf. (2.13)), and hence the cost \( J^{\theta^*}(\nu^{m+1}) \) depends explicitly on \( (\theta_n)_{m=0}^\infty \). This memory dependence requires quantifying the precise dependence of the performance gap \( J^{\theta^*}(\nu^{m+1}) - J^{\theta^*}(\theta^*) \) on the accuracy of \( (\theta_n)_{m=0}^\infty \).

Sketched proofs of Theorems 2.3 and 2.4. The key idea of the regret analysis is to balance the exploration–exploitation trade-off for Algorithms 1 and 2. To this end, we decompose the regret into
\[
\text{Reg}(N) = \sum_{m=1}^{N} \left( J^{\theta^*}(\varphi^m) - J^{\theta^*}(\nu^m) \right) + \sum_{m=1}^{N} \left( J^{\theta^*}(\varphi^m) - J^{\theta^*}(\theta^*) \right), \quad (2.28)
\]
where \( \varphi^m(t, x) = k^m + K^m x \) with \( (k^m, K^m) \) is defined in (2.19). The first term induces the exploration cost due to randomisation, and the second term describes the exploitation cost caused by the sub-optimal policies \( \varphi^N_{n=1} \). In particular, we prove that \( \mathbb{E}[J^{\theta^*}(\varphi^m) - J^{\theta^*}(\nu^m) | \mathcal{F}_{m-1}] = O(\|\lambda^m\|_\infty^2 + \sqrt{|\pi_m|\|\lambda^m\|_\infty^2}) \) and \( J^{\theta^*}(\varphi^m) - J^{\theta^*}(\theta^*) = O(\|K^m - K^{\theta^*}\|_\infty^2 + \|k^m - k^{\theta^*}\|_\infty^2) \). We then reduce the exploitation loss to the parameter estimate error by using Step 4 of Algorithms 1 and 2, and prove that \( J^{\theta^*}(\varphi^m) - J^{\theta^*}(\theta^*) = O(|\theta_m - \theta^*|^2) \) for Algorithm 1 and \( J^{\theta^*}(\varphi^m) - J^{\theta^*}(\theta^*) = O \left( \sum_{n=1}^{m} \frac{1}{n} \|\lambda^n\|_\infty^2 \|\theta_n - \theta^*\|^2 \right) \) for Algorithm 2 (see Proposition 4.10). We further quantify the estimation accuracy by \( |\theta_m - \theta^*|^2 = O \left( \sum_{n=1}^{m} \min_{t \in [0, T]} \|\lambda^n\|_\infty^2 \right) \), provided that the mesh size \( |\pi_m| \) is sufficiently small (see Section 4.1). These analyses show that the exploration cost in (2.28) increases in \( (\lambda^n)_{n \in \mathbb{N}} \) and the exploitation loss in (2.28) decreases in \( (\lambda^n)_{n \in \mathbb{N}} \).
Observe that for both Algorithms 1 and 2, the magnitude of $\lambda^{n+1}$ is determined by the weights $(\theta_n)_{n \in \mathbb{N}}$ (cf. (2.9) and (2.13)). Hence, by choosing the weights $(\theta_n)_{n \in \mathbb{N}}$ as in Theorems 2.3 and 2.4, we can balance the exploration and exploitation and obtain algorithms with optimal regret orders (up to a logarithmic factor).

2.5 Discussions

Random execution and scaled white noise. The execution noise in Definition 2.3 is a properly scaled discrete-time white noise. Discretised white noise has been used in [8] for exploration. More precisely, let $\phi$ be a deterministic policy that is affine in the state, let $\bar{W}$ be an independent Brownian motion, and for each $n \in \mathbb{N}$, let $\pi_n$ be a uniform grid with mesh size $\delta_n = T/n$. Given $\varepsilon > 0$, [8] considers the randomised policy $\varphi_{n,\varepsilon} = \phi + \varepsilon \sum_{i=0}^{n-1} \frac{W_{i+1} - W_i}{\delta_n} 1_{[\delta_n, (i+1)\delta_n]}$ for all large $n \in \mathbb{N}$, and the normalised cost $J^{\theta^*}(\varphi_{n,\varepsilon}) - Cn\varepsilon^2$ for some constant $C > 0$. As $\varphi_{n,\varepsilon}$ behaves similar to $\phi + \varepsilon \bar{W}$ with $\bar{W}$ being a white noise, the term $n\varepsilon^2$ is necessary for the finiteness of the normalised cost. Indeed, for a suitable choice of $C$, the normalised cost converges to $J^{\theta^*}(\phi)$ as $n \to \infty$.

Instead of fixing $\varepsilon$ as in [8], the piecewise constant randomisation in Definition 2.3 adjusts $\varepsilon$ according to the mesh size. To see it, let $\lambda > 0$ and for each $n \in \mathbb{N}$, let $\varepsilon_n = \lambda \sqrt{\delta_n}$. By Theorem 2.2, $J^{\theta^*}(\varphi_{n,\varepsilon_n})$ approximates the relaxed cost $J^{\theta^*}_0(\nu)$ with $\nu \sim \mathcal{N}(\phi, \lambda)$. The normalisation term $n\varepsilon_n^2 = O(\lambda^2)$ corresponds to the exploration cost $J^{\theta^*}_0(\nu) - J^{\theta^*}(\phi)$ in (4.15) (see also [33]).

Random execution and chattering lemma. In the control theory, the chattering lemma (see e.g., [24]) ensures that there exists a sequence of strict control processes converging to a relaxed control process. However, the proof is nonconstructive and typically requires a compact action space. Also there is no explicit convergence rate.

Our work focuses on the LQ setting and provides an explicit approximation of Gaussian relaxed policies. We also quantify the precise approximation error in terms of the mesh size and the variance of the Gaussian relaxed policies.

LQ-RL with controlled diffusion. For general LQ-RL problems whose state dynamics involves controlled diffusions, the state process is no longer Gaussian. Moreover, as the noise of the state dynamics can degenerate, the Bayesian estimation (2.24) may not be well-defined. In this case, one has to derive learning algorithms in a problem dependent way.

For instance, let $\phi$ be a given policy and consider the state process $X^\phi$ satisfying the following one-dimensional dynamics:

$$dX_t = (AX_t + B\phi(t, X_t)) dt + (CX_t + D\phi(t, X_t) + \bar{\sigma}) dW_t, \quad t \in [0, T]; \quad X_0 = x_0. \quad (2.29)$$

Note that by observing $X^\phi$ continuously, the agent can infer the constants $(C, D, \bar{\sigma})$ using finite episodes. Indeed, the agent can compute the process $Y^\phi_t = CX^\phi_t + D\phi(t, X^\phi_t) + \bar{\sigma}$, $t \in [0, T]$, from the rate of increase of the quadratic variation of $X^\phi$. Then the agent can recover $D$ from the random variables $Y^\phi_0, i = 1, 2, 3$, with constant policies $\phi^{(1)} \equiv -1, \phi^{(2)} \equiv 0$, and $\phi^{(3)} \equiv 1$. The constants $C$ and $\bar{\sigma}$ can further be inferred from observed paths of $Y^\phi$, by executing a policy $\phi$ such that paths of $X^\phi$ are non-constant almost surely.

Given $(C, D, \bar{\sigma})$, the agent may eliminate the observation noise and simplify the learning process. To see it, suppose that $D, \bar{\sigma} \neq 0$. Then the agent will estimate $(A, B)$ by executing the policy $\phi(t, x) = -D^{-1}(Cx + \bar{\sigma})$. This leads to a deterministic state dynamics: $dX^\phi_t = \theta Z^\phi_t dt$, with $\theta = (A, B)$ and $Z^\phi_t := (X^\phi_t, \phi(t, X^\phi_t))$. As $\bar{\sigma}$ is non-zero and $X^\phi$ is in general non-constant, $X^\phi$ and $\phi(\cdot, X^\phi)$ are linearly independent, and $0^T Z^\phi (Z^\phi_t)^T dt$ is invertible. This allows to identify $\theta$ using one episode. A complete study of LQ-RL with controlled diffusion is left to future research.
3 Proofs of Lemma 2.1 and Theorem 2.2

Proof of Lemma 2.1. Let \( X^\nu \) be the solution to (2.15) associated with the policy \( \nu \sim \mathcal{G}(k, K, \lambda) \), and let \( X^\varphi \) be the solution to (2.17) associated with the policy \( \varphi \sim \mathcal{R}(k, K, \lambda, \xi) \). Recall that for all \( t \in [0, T] \),

\[
dX^\nu_t = (A^* X^\nu_t + B^* (k_t + K_t X^\nu_t)) dt + \tilde{\sigma}_t dW_t, \quad X_0^\nu = x_0,
\]

\[
dX^\varphi_t = (A^* X^\varphi_t + B^* (k_t + K_t X^\varphi_t + \lambda_t \xi_t)) dt + \tilde{\sigma}_t dW_t, \quad X_0^\varphi = x_0.
\]

Then by Duhamel’s principle, for all \( t \in [0, T] \),

\[
X^\nu_t = \Phi_t x_0 + \Phi_t \int_0^t \Phi_s^{-1} (B^* k_s + \tilde{\sigma}_s W_s) ds, \quad X^\varphi_t = \Phi_t x_0 + \Phi_t \int_0^t \Phi_s^{-1} B^* \lambda_s \xi_s ds,
\]

where \( (\Phi_t)_{t \in [0, T]} \) is the solution to \( d\Phi_t = (A^* + B^* K_t) \Phi_t dt \) with \( \Phi_0 = I \). Therefore, for all \( t \in [0, T] \),

\[
\mathbb{E}[X^\varphi_t - X^\nu_t | \mathcal{F}_0] = \Phi_t \int_0^t \Phi_s^{-1} B^* \lambda_s \xi_s ds = \Phi_t \int_0^T \Phi_s^{-1} B^* \lambda_s \xi_s ds,
\]

\[
\mathbb{E}[(X_t^\varphi)^2 - (X_t^\nu)^2 | \mathcal{F}_0] = \left( \Phi_t \int_0^t \Phi_s^{-1} B^* \lambda_s \xi_s ds \right)^2 + 2 \mathbb{E}[X_t^\nu | \mathcal{F}_0] \Phi_t \int_0^t \Phi_s^{-1} B^* \lambda_s \xi_s ds
\]

\[
= \Phi_t^2 (B^*)^2 \int_0^T \int_0^t 1_{s \leq t} \Phi_u^{-1} \lambda_u \xi_u ds du + 2 \left( \Phi_t x_0 + \Phi_t \int_0^t \Phi_s^{-1} B^* k_s ds \right) \Phi_t \int_0^T 1_{s \leq t} \Phi_s^{-1} B^* \lambda_s \xi_s ds.
\]

Substituting the expressions of \( \nu \sim \mathcal{G}(k, K, \lambda) \) and \( \varphi \sim \mathcal{R}(k, K, \lambda, \xi) \) into (2.14) and (2.16), respectively, and applying Fubini’s theorem, we have

\[
J^\varphi(\varphi) - J_0^\varphi(\nu) = \int_0^T \left( Q_t + 2 S_t K_t + R_t (K_t)^2 \right) \mathbb{E}[(X_t^\varphi)^2 - (X_t^\nu)^2 | \mathcal{F}_0] dt
\]

\[
+ \int_0^T 2 \left( S_t k_t + R_t K_t k_t + p_t + q_t K_t \right) \mathbb{E}[X_t^\varphi - X_t^\nu | \mathcal{F}_0] dt
\]

\[
+ \int_0^T R_t \lambda_t (\xi_t^2 - 1) dt + ME[(X_t^\varphi)^2 - (X_t^\nu)^2 | \mathcal{F}_0] + 2m \mathbb{E}[X_t^\nu - X_t^\varphi | \mathcal{F}_0]
\]

\[
+ \int_0^T 2 (S_t + R_t K_t) \mathbb{E}[X_t^\varphi - X_t^\nu | \mathcal{F}_0] \lambda_t \xi_t dt + \int_0^T 2 (R_t k_t + q_t) \lambda_t \xi_t dt.
\]

Substituting \( \mathbb{E}[X_t^\varphi - X_t^\nu | \mathcal{F}_0] \) and \( \mathbb{E}[(X_t^\varphi)^2 - (X_t^\nu)^2 | \mathcal{F}_0] \) into (3.1) and applying Fubini’s theorem to reorder the integrations yield the desired conclusion.

To prove Theorem 2.2, the following concentration inequalities of independent normal random variables will be used, whose proof is given in Appendix A.

Lemma 3.1. There exists a constant \( C \geq 0 \) such that for all independent standard normal random variables \( (\xi_i)_{i \in \mathbb{N}} \), for all \( N \in \mathbb{N} \), \( (\rho_i)_{i=1}^N \subset \mathbb{R} \), \( (\beta_{ij})_{i,j=1}^N \subset \mathbb{R} \) and \( \delta > 0 \),

(1) \( \mathbb{P} \left( \left| \sum_{i=1}^N \rho_i \xi_i \right| \geq C \| \rho \|_2 (1 + \sqrt{\ln (\frac{1}{\delta})}) \right) \leq \delta \), with \( \| \rho \|_2 = \sqrt{\sum_{i=1}^N \rho_i^2} \).

(2) \( \mathbb{P} \left( \left| \sum_{i=1}^N \rho_i (\xi_i^2 - 1) \right| \geq C \| \rho \|_2 (1 + \ln (\frac{1}{\delta})) \right) \leq \delta \), with \( \| \rho \|_2 = \sqrt{\sum_{i=1}^N \rho_i^2} \).

(3) \( \mathbb{P} \left( \sum_{i,j=1,i \neq j}^N \beta_{ij} \xi_i \xi_j \right) \geq C \| \beta \|_2 (1 + \ln (\frac{1}{\delta})) \right) \leq \delta \), with \( \| \beta \|_2 = \sqrt{\sum_{i,j=1,i \neq j}^N \beta_{ij}^2} \).
Proof of Theorem 2.2. Throughout this proof, let $C$ be a generic constant independent of $\lambda, \xi, \delta$ and $\pi$, but may possibly depend on the sup-norms of the coefficients in (H.1) and the functions $k$ and $K$. Observe that the functions $h_i$, $i = 1, \ldots, 3$, in Lemma 2.1 are uniformly bounded by some constant $C \geq 0$. Let $\pi = \{0 = t_0 < \cdots < t_N = T\}$ be a partition of $[0,T]$ with mesh size $|\pi|$, and let $\varphi \sim \mathcal{R}(\mu, \lambda, \xi^\pi)$ with $\xi^\pi = \sum_{i=0}^{N-1} \xi_i 1_{t_i \leq \xi_{i+1}}$. We omit the superscript $\pi$ in $\xi^\pi$ for notational simplicity.

To prove Theorem 2.2 Item (1), it suffices to upper bound all terms of (2.18) in high probability. The definition of $\xi$ implies that $\int_0^T h_{1,t} \lambda_t \xi_t dt = \sum_{i=0}^{N-1} \zeta_i \left( \int_{t_i}^{t_{i+1}} h_{1,t} \lambda_t dt \right)$. By Lemma 3.1 Item (1), it holds with probability at least $1 - \delta$ that

$$\left| \int_0^T h_{1,t} \lambda_t \xi_t dt \right| \leq C \left( 1 + \sqrt{\ln \left( \frac{1}{\delta} \right)} \right) \sqrt{\sum_{i=0}^{N-1} \left( \int_{t_i}^{t_{i+1}} h_{1,t} \lambda_t dt \right)^2} \leq C \sqrt{T} \sqrt{|\lambda|_{\infty}} \left( 1 + \sqrt{\ln \left( \frac{1}{\delta} \right)} \right).$$

(3.2)

Also applying Lemma 3.1 Item (2) to $\int_0^T h_{2,t} \lambda_t^2 (\xi_t^2 - 1) dt$ implies that with probability at least $1 - \delta$,

$$\left| \int_0^T h_{2,t} \lambda_t (\xi_t^2 - 1) dt \right| \leq C \sqrt{T} \sqrt{|\lambda|_{\infty}} \left( 1 + \ln \left( \frac{1}{\delta} \right) \right).$$

(3.3)

It now remains to quantify the last term of (2.18). Observe that

$$\int_0^T \int_0^T h_{3,t,r} \lambda_t \lambda_r \xi_t \xi_r dt dr dt = \sum_{i,j=0,i \neq j}^{N-1} \zeta_i \zeta_j \left( \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} h_{3,t,r} \lambda_t \lambda_r dr dt \right) + \sum_{i=0}^{N-1} \zeta_i^2 \left( \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} h_{3,t,r} \lambda_t \lambda_r dr dt \right).$$

By Lemma 3.1 Item (3), it holds with probability at least $1 - \delta$,

$$\left| \sum_{i,j=0,i \neq j}^{N-1} \zeta_i \zeta_j \left( \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} h_{3,t,r} \lambda_t \lambda_r dr dt \right) \right| \leq C |\lambda|_{\infty}^2 \left( 1 + \ln \left( \frac{1}{\delta} \right) \right) \sqrt{\sum_{i,j=0,i \neq j}^{N-1} (t_{i+1} - t_i)^2 (t_{j+1} - t_j)^2} \leq C |\lambda|_{\infty}^2 \left( 1 + \ln \left( \frac{1}{\delta} \right) \right) T |\pi|,$$

while by Lemma 3.1 Item (2), it holds with probability at least $1 - \delta$,

$$\left| \sum_{i=0}^{N-1} \zeta_i^2 \left( \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} h_{3,t,r} \lambda_t \lambda_r dr dt \right) \right| \leq \sum_{i=0}^{N-1} \left( \zeta_i^2 - 1 \right) \left( \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} h_{3,t,r} \lambda_t \lambda_r dr dt \right) \leq C |\lambda|_{\infty}^2 \left( 1 + \ln \left( \frac{1}{\delta} \right) \right) \sum_{i=0}^{N-1} (t_{i+1} - t_i)^4 + C |\lambda|_{\infty}^2 T |\pi| \leq C |\lambda|_{\infty}^2 \left( 1 + \ln \left( \frac{1}{\delta} \right) \right) |\pi|.$$

Summarising these inequalities shows that with probability at least $1 - \delta$,

$$\left| \int_0^T \int_0^T h_{3,t,r} \lambda_t \lambda_r \xi_t \xi_r 1_{r \leq t} dt dr dt \right| \leq C |\pi| |\lambda|_{\infty}^2 \left( 1 + \ln \left( \frac{1}{\delta} \right) \right).$$

(3.4)

Combining (3.2), (3.3) and (3.4) with the inequality that $\sqrt{\ln \left( \frac{1}{\delta} \right) \leq 1 + \ln \left( \frac{1}{\delta} \right)}$ for all $\delta \geq 0$ proves Theorem 2.2 Item (1).

We proceed to prove Theorem 2.2 Item (2). Let $C$ be the constant in Theorem 2.2 Item (1), and for each $n \in \mathbb{N}$, let $\delta_n = \frac{1}{n^2}$, and $A_n = \left| J(\varphi^n) - \tilde{J}(\nu) \right| \geq C \sqrt{|\pi_n|} |\lambda|_{\infty} \left( 1 + |\lambda|_{\infty} \right) \left( 1 + \ln \left( \frac{1}{\delta_n} \right) \right)$. By
Theorem 2.2 Item (1), $\sum_{n=1}^{\infty} P(A_n) \leq \sum_{n=1}^{\infty} \delta_n < \infty$, which along with the Borel–Cantelli lemma shows that $P(\limsup_{n \to \infty} A_n) = 0$. Consequently, for $\mathbb{P}$-a.s. $\omega \in \Omega$, there exists $N_\omega \in \mathbb{N}$, such that for all $n \geq N_\omega$,

$$|J(\varphi^n) - J(\nu)| \leq C \sqrt{\pi n} \|\lambda\|_\infty (1 + \|\lambda\|_\infty) (1 + \ln (\frac{1}{n})) \leq C n^{-\frac{1}{2}} (\ln n) \|\lambda\|_\infty (1 + \|\lambda\|_\infty).$$

It remains to prove Theorem 2.2 Item (3). By Lemma 2.1 and the fact that $\xi_t \sim \mathcal{N}(0, 1)$ for all $t \in [0, T]$ and $|\mathbb{E}[\xi_t \xi_s]| \leq 1_{|t-s| \leq |\pi|}$, it follows from (2.18) that

$$|\mathbb{E}[J^{\theta}(\varphi) - J^{\theta}_0(\nu)]| = |E \left[ \int_0^T \int_0^T h_{3,t,r} \lambda_t \lambda_t \xi_t \xi_t \, dt \right] | \leq C \|\lambda\|_\infty^2 \int_0^T \int_0^T |\mathbb{E}[\xi_t \xi_s]| \, dr \, dt \leq C \|\lambda\|_\infty^2 \int_0^T 2|\pi| \, dt \leq C |\pi| \|\lambda\|_\infty^2.

This proves the desired estimate in Theorem 2.2 Item (3).

4 Regret analysis for episodic learning

4.1 Analysis of piecewise constant randomised policies

This section analyses a sequence of piecewise constant randomised policies that is adapted to the observation filtration. The precise definition of these policies are given in Setting 4.1, which includes as special the policies randomised generated by Algorithms 1 and 2.

Setting 4.1. Let $\theta_0 \in \mathbb{R}^{1 \times 2}$, $V_0 \in \mathbb{S}_2^2$, and for each $m \in \mathbb{N}$, let $K^m, k^m \in L^\infty(\Omega \times [0, T]; \mathbb{R})$, let $\lambda^m \in L^\infty([0, T]; \mathbb{R}_{\geq 0})$, let $\pi_m = (t^m)_{m=1}^\infty$ be a grid of $[0, T]$, let $\varphi^m : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be defined as in (2.19), let the processes $(X^m, Z^m)$ be defined as in (2.20), and let $(\bar{\theta}_m, V_m)$ be defined as in (2.24). Assume that $L := \sup_{m \in \mathbb{N}}(\|K^m\|_{L^\infty} + \|k^m\|_{L^\infty} + \|\lambda^m\|_{L^\infty}) < \infty$, and for all $m \in \mathbb{N}$, $K^m$ and $k^m$ are $\mathcal{F}_m \otimes \mathcal{B}([0, T])$-measurable, with $\mathcal{F}_m$ defined by (2.21).

The following proposition establishes a subexponential concentration behaviour of $(V^{-1}_m)_{m \in \mathbb{N}}$.

Proposition 4.1. Suppose (H.1) and Setting 4.1 hold. Then there exists $C > 0$, depending only on the coefficients in (H.1) and $L$ in Setting 4.1, such that for all $m \in \mathbb{N}$ and $\delta > 0$,

$$\mathbb{P} \left( \left| V^{-1}_m - V^{-1}_0 \right| - \sum_{n=1}^{m} \mathbb{E} \left[ \int_0^T \frac{1}{d^\top_i Z^\top_i (Z^m)^\top \mathrm{d}t} \right] \right) \geq C \max \left( \ln (\frac{\delta}{\pi}), \sqrt{\frac{m \ln (\frac{\pi}{\delta})}{} \right) \leq \delta,$n

Proof. Throughout this proof, let $C$ be a generic constant depending only on the coefficients in (H.1) and $L$ in Setting 4.1. For each $q \in [1, 2]$ and $\sigma$-algebra $\mathcal{G}$, let $||| \cdot |||_{q, \mathcal{G}} : \Omega \to [0, \infty]$ be the conditional Orlicz norm defined in [31, Definition 4.1], such that for all random variables $X : \Omega \to \mathbb{R}$,

$$|||X|||_{q, \mathcal{G}} := \mathbb{E}_{\text{ess inf}} \{ Y \in L^0(\mathcal{G}; [0, \infty)) : \mathbb{E} \left[ \exp(|X|^q |Y|^q) |\mathcal{G} \right] \leq 2 \},$$

where $L^0(\mathcal{G}; [0, \infty))$ is the set of all $\mathcal{G}$/$\mathcal{B}([0, \infty))$-measurable functions $Y : \Omega \to [0, \infty)$.

We first prove that there exists a constant $C > 0$ such that $||(\int_0^T |X^m|^2 \mathrm{d}t)^{1/2}||_{2, \mathcal{F}_m} \leq C$ for all $m \in \mathbb{N}$. Observe from (2.20) that, for each $m \in \mathbb{N}$, $X^m = \bar{X}^m + \bar{Y}^m$, where

$$dX^m_t = (A^\top X^m_t + B^\top (K^m_t X^m_t + k^m_t)) \, dt + \partial_t \, dW^m_t, \quad t \in [0, T]; \quad X^m_0 = x_0,$n

$$dY^m_t = (A^\top Y^m_t + B^\top (K^m_t Y^m_t + \lambda^m_t \xi^m_t)) \, dt, \quad t \in [0, T]; \quad Y^m_0 = 0. \quad (4.1)$$

By [31, Proposition 4.7], $||(\int_0^T |X^m_t|^2 \mathrm{d}t)^{1/2}||_{2, \mathcal{F}_m} \leq C$ for all $m \in \mathbb{N}$. Moreover, let $\Phi^m : \Omega \times [0, T] \to \mathbb{R}$ be such that for all $\omega \in \Omega \times [0, T]$, $d\Phi^m_t(\omega) = (A^\top + B^\top K^m_t (\omega)) \Phi^m_t(\omega) \, dt$ and $\Phi^m_0(\omega) = 1$, then $Y^m_t = \Phi^m_t \int_0^t (\Phi^m_s)^{-1} B^\top \lambda^m_s \xi^m_s \, ds$ for all $t \in [0, T]$. Observe that $\Phi^m$ is invertible $d\mathbb{P} \otimes dt$ a.e., and by
the uniform boundedness of \((K^m)_{m\in\mathbb{N}}\), \(||\Phi^m||_{L^\infty} + ||(\Phi^m)^{-1}||_{L^\infty} \leq C\) for all \(m \in \mathbb{N}\). Hence, by Hölder’s inequality,
\[
\left(\int_0^T |\mathcal{Y}_m|^2 \, dt\right)^{\frac{1}{2}} \leq C \sup_{t \in [0,T]} |\mathcal{Y}_m| \leq C \sup_{t \in [0,T]} \int_0^t |\lambda^m_t \xi^m_s| \, ds \leq C||\lambda^m||_{L^\infty}\left(\int_0^T |\xi^m_s|^2 \, ds\right)^{\frac{1}{2}},
\]
Assume without loss of generality that \(||\lambda^m||_{L^\infty} > 0\). By the convexity of \(x \mapsto e^x\) and Jensen’s inequality, for a sufficiently large \(\tilde{C} > 0\),
\[
\mathbb{E}\left[\exp\left(\frac{1}{C_2^2||\lambda^m||_{L^\infty}^2} \int_0^T |\mathcal{Y}_m|^2 \, dt\right)|\mathcal{F}^\text{ob}_{m-1}\right] \leq \mathbb{E}\left[\exp\left(\frac{1}{C_2^2||\lambda^m||_{L^\infty}^2} \int_0^T |\xi^m_s|^2 \, ds\right)|\mathcal{F}^\text{ob}_{m-1}\right] \leq 2,
\]
where we used that \((\xi^m_t)_{t \in [0,T]}\) are standard normal random variables independent of \(\mathcal{F}^\text{ob}_{m-1}\). Consequently, \(||\int_0^T |\mathcal{Y}_m|^2 \, dt||_{L^2|\mathcal{F}^\text{ob}_{m-1}} \leq C\), which implies that \(||\int_0^T |X^m_t|^2 \, dt||_{L^2|\mathcal{F}^\text{ob}_{m-1}} \leq C\) for all \(m \in \mathbb{N}\).

Observe that for all \(m \in \mathbb{N}\) and \(t \in [0,T]\), \(\varphi^m(\cdot,t,X^m_t) = K^m_t X^m_t + k^m_t + \lambda^m_t \xi^m_t\). By the uniform boundedness of \(K^m\) and \(k^m\), \(||\int_0^T |\varphi^m(\cdot,t,X^m_t)|^2 \, dt||_{L^2|\mathcal{F}^\text{ob}_{m-1}} \leq C\) for all \(m \in \mathbb{N}\), which along with \(\tilde{\sigma}^{-1} \in L^\infty([0,T];\mathbb{R})\) and [31, Proposition 4.3] implies that
\[
\left\|\int_0^T \frac{1}{\tilde{\sigma}^2_s} Z^m_s (Z^m_s)^\top \, ds\right\|_{L^1|\mathcal{F}^\text{ob}_{m-1}} \leq C \left\|\int_0^T \left(\varphi^m(\cdot,s,X^m_s)\varphi^m(\cdot,s,X^m_s)^\top\right) \, ds\right\|_{L^1|\mathcal{F}^\text{ob}_{m-1}}
\leq C \left(1 + \left\|\int_0^T |X^m_t|^2 \, dt\right\|_{L^2|\mathcal{F}^\text{ob}_{m-1}} + \left\|\int_0^T |\varphi^m(\cdot,t,X^m_t)|^2 \, dt\right\|_{L^2|\mathcal{F}^\text{ob}_{m-1}}\right) \leq C.
\]
As \(V^{-1} = V^{-1} - \sum_{n=1}^m E\left[\int_0^T \frac{1}{\tilde{\sigma}^2_s} Z^m_s (Z^m_s)^\top \, ds|\mathcal{F}^\text{ob}_{n-1}\right]\) is \(\mathcal{F}^\text{ob}_{n-1}\)-measurable, by the Bernstein inequality for martingale difference sequence in [31, Proposition 4.4], for all \(m \in \mathbb{N}\) and \(\varepsilon > 0\),
\[
\mathbb{P}\left(\left|\sum_{n=1}^m E\left[\int_0^T \frac{1}{\tilde{\sigma}^2_s} Z^m_s (Z^m_s)^\top \, ds|\mathcal{F}^\text{ob}_{n-1}\right]\right| \geq m\varepsilon\right) \leq 2 \exp\left(-mC \min\left(\varepsilon^2,\varepsilon\right)\right).
\]
The desired result follows by substituting \(\varepsilon\) such that \(2 \exp\left(-mC \min\left(\varepsilon^2,\varepsilon\right)\right) = \delta\).

We now establish upper and lower bounds of \(\sum_{n=1}^m E\left[\int_0^T \frac{1}{\tilde{\sigma}^2_s} Z^m_s (Z^m_s)^\top \, ds|\mathcal{F}^\text{ob}_{n-1}\right]\) in terms of \(m\).

**Proposition 4.2.** Suppose (H.1) and Setting 4.1 hold. Then there exists a constant \(C > 0\), depending only on the coefficients in (H.1) and \(L\) in Setting 4.1, such that

1. for all \(m \in \mathbb{N}\), \(\sum_{n=1}^m \left\|\mathbb{E}\left[\int_0^T \frac{1}{\tilde{\sigma}^2_s} Z^m_s (Z^m_s)^\top \, ds|\mathcal{F}^\text{ob}_{n-1}\right]\right\| \leq Cm\).

2. if \(||m||\lambda^m||_{L^\infty}^2 \leq \frac{1}{C_1} \min_{t \in [0,T]} (\lambda^m_t)^2(1)\) for all \(m \in \mathbb{N}\), then
\[
\sum_{n=1}^m \lambda \min \left(\mathbb{E}\left[\int_0^T \frac{1}{\tilde{\sigma}^2_s} Z^m_s (Z^m_s)^\top \, ds|\mathcal{F}^\text{ob}_{n-1}\right]\right) \geq \frac{1}{C} \sum_{n=1}^m \min_{t \in [0,T]} (\lambda^m_t)^2(1), \quad \forall m \in \mathbb{N},
\]
where \(a \land 1 = \min(a,1)\) for all \(a \in \mathbb{R}\).

**Proof.** For all \(m \in \mathbb{N}\), by the definition of \(Z^m\) in (2.20), the measurability conditions of \(K^m\) and \(k^m\) and
the fact that $\mathbb{E}[-\eta^m_t | \mathcal{F}^m_{t-1}] = 0$ for all $t \in [0,T]$ implies that for $d\mathbb{P}$-a.s.,

$$
\mathbb{E} \left[ \int_0^T \frac{1}{\sigma^2} Z_t^m (Z_t^m)^\top dt | \mathcal{F}^m_{t-1} \right] (\omega) = \mathbb{E} \left[ \int_0^T \frac{1}{\sigma^2} \left( K^m_t X^m_t + k^m_t + \lambda^m_t \xi^m_t \right) \left( K^m_t X^m_t + k^m_t + \lambda^m_t \xi^m_t \right)^\top dt | \mathcal{F}^m_{t-1} \right] (\omega) \\
= \mathbb{E} \left[ \int_0^T \frac{1}{\sigma^2} \left( \left( \tilde{X}^m_t \omega, \lambda^m_t \xi^m_t \right)(\omega) \tilde{X}^m_t \omega, \lambda^m_t \xi^m_t) \right) \right] dt \\
+ \mathbb{E} \left[ \int_0^T \frac{1}{\sigma^2} \left( \left( K^m_t(\omega) \tilde{X}^m_t \omega, k^m_t(\omega) \right) \left( K^m_t(\omega) \tilde{X}^m_t \omega, k^m_t(\omega) \right)^\top \right) \right] dt 
$$

(4.2)

(4.3)

where $\tilde{X}^m_t \omega$ is the state process $X^m$ conditioned on $\mathcal{F}^m_{t-1}$, such that $\mathbb{P}$-a.s. $\omega$ and for all $t \in [0,T]$

$$
\tilde{X}^m_t(\omega) = \Phi^m_t(\omega)x_0 + \Phi^m_t(\omega) \int_0^t (\Phi^m_s(\omega))^{-1} \left( B^m(k^m(\omega) + \lambda^m_t \xi^m_t) \right) ds + \tilde{\sigma}_t dW_t^m(\cdot),
$$

(4.4)

with $d\Phi^m_t(\omega) = (A^* + B^T K^m_t(\omega)) \Phi^m_t(\omega) dt$ and $\Phi^m_0(\omega) = 1$. By the uniform boundedness of $K^m$, $k^m$ and $\lambda^m$, $\mathbb{E} \left[ \sup_{t \in [0,T]} |\tilde{X}^m_t(\omega)|^2 \right] \leq C$ for all $m \in \mathbb{N}$. Hence, applying the Cauchy–Schwarz inequality to (4.2)-(4.3) and using the Gaussianity of $(\xi^m_t)_{t \in [0,T]}$ yield Item (1).

To prove Item (2), we first establish a lower bound of (4.2). By (4.4) and the independence between $\xi^m_t$ and $W^m_t$, for all $t \in [0,T]$, the

$$
\mathbb{E} \left[ \left| \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t \right| \right] = \left| \lambda^m_t \Phi^m_t(\omega) \int_0^t (\Phi^m_s(\omega))^{-1} \lambda^m_s \mathbb{E} \left[ \xi^m_s \xi^m_t \right] ds \right|.
$$

$$
\mathbb{E} \left[ |k^m_t(\omega) \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t| \right] = \left| k^m_t(\omega) \lambda^m_t \Phi^m_t(\omega) \int_0^t (\Phi^m_s(\omega))^{-1} \lambda^m_s \mathbb{E} \left[ \xi^m_s \xi^m_t \right] ds \right|.
$$

Note that $\mathbb{E} \left[ \xi^m_s \xi^m_t \right]$ is 1 if there exists $i_0$ such that $s, t \in [t_{i_0}, t_{i_0+1}]$, and is zero otherwise. By the uniform boundedness of $K^m$, there exists $C \geq 0$ such that for all $m \in \mathbb{N}$,

$$
\Lambda_{\text{min}} \left( \int_0^T \frac{1}{\sigma^2} \left( \left( \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t \right)(\omega) \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t) \right) \right) \geq -C|\pi_m||\lambda^m|^2_{\text{HS}}.
$$

We then prove that the minimum eigenvalue of (4.3) is bounded away from zero. Observe that for each $K \in \mathbb{R}$, a direct computation shows that all eigenvalues of the matrix $\Psi(K) := (\frac{1}{K})(\frac{1}{K})^\top + (0 0)^\top$ are positive, which along with the continuity of the map $S^2 \ni S \mapsto \Lambda_{\text{min}}(S) \in \mathbb{R}$ implies that $c_0 := \inf_{|K| \leq L} \Lambda_{\text{min}}(\Psi(K)) > 0$, which the constant $L$ in Setting 4.1. Then for all $m \in \mathbb{N}$ and $t \in [0,T]$, the

$$
\Lambda_{\text{min}} \left( \int_0^T \frac{1}{\sigma^2} \left( \left( \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t \right)(\omega) \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t) \right) \right) \geq 0.
$$

(4.5)

We then prove that the minimum eigenvalue of (4.3) is bounded away from zero. Observe that for each $K \in \mathbb{R}$, a direct computation shows that all eigenvalues of the matrix $\Psi(K) := (\frac{1}{K})(\frac{1}{K})^\top + (0 0)^\top$ are positive, which along with the continuity of the map $S^2 \ni S \mapsto \Lambda_{\text{min}}(S) \in \mathbb{R}$ implies that $c_0 := \inf_{|K| \leq L} \Lambda_{\text{min}}(\Psi(K)) > 0$, which the constant $L$ in Setting 4.1. Then for all $m \in \mathbb{N}$ and $t \in [0,T]$, the

$$
\Lambda_{\text{min}} \left( \int_0^T \frac{1}{\sigma^2} \left( \left( \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t \right)(\omega) \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t) \right) \right) \geq 0.
$$

(4.6)

We then prove that the minimum eigenvalue of (4.3) is bounded away from zero. Observe that for each $K \in \mathbb{R}$, a direct computation shows that all eigenvalues of the matrix $\Psi(K) := (\frac{1}{K})(\frac{1}{K})^\top + (0 0)^\top$ are positive, which along with the continuity of the map $S^2 \ni S \mapsto \Lambda_{\text{min}}(S) \in \mathbb{R}$ implies that $c_0 := \inf_{|K| \leq L} \Lambda_{\text{min}}(\Psi(K)) > 0$, which the constant $L$ in Setting 4.1. Then for all $m \in \mathbb{N}$ and $t \in [0,T]$, the

$$
\Lambda_{\text{min}} \left( \int_0^T \frac{1}{\sigma^2} \left( \left( \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t \right)(\omega) \tilde{X}^m_t(\omega) \lambda^m_t \xi^m_t) \right) \right) \geq 0.
$$

(4.7)
Consequently, by using Fubini’s theorem, the concavity of \( S^2 \Rightarrow S \mapsto \Lambda_{\min}(S) \in \mathbb{R} \) and Jensen’s inequality,
\[
\Lambda_{\min}\left( \mathbb{E} \left[ \frac{1}{\sigma_t^2} \left( \begin{pmatrix} K_m^*(\omega) \tilde{X}^m_\omega + k_m^*(\omega) & K_m^*(\omega) \tilde{X}^m_\omega + k_m^*(\omega) \end{pmatrix}^\top + \begin{pmatrix} 0 & 0 \end{pmatrix} \frac{(\lambda^m)^2}{\lambda^m} \right) dt \right] \right)
\geq \Lambda_{\min}\left( \mathbb{E} \left[ \frac{1}{\sigma_t^2} \left( \begin{pmatrix} K_m^*(\omega) \tilde{X}^m_\omega + k_m^*(\omega) & K_m^*(\omega) \tilde{X}^m_\omega + k_m^*(\omega) \end{pmatrix}^\top + \begin{pmatrix} 0 & 0 \end{pmatrix} \frac{(\lambda^m)^2}{\lambda^m} \right) dt \right] \right) \geq \hat{c}_0 \min_{t \in [0,T]} \left( 1 \wedge (\lambda^m)^2 \right),
\]

for some \( \hat{c}_0 > 0 \), where the final inequality follows from (4.6) and (4.7). Hence setting \( \pi_m \|\lambda^m\|_{L^\infty}^2 \leq \min_{t \in [0,T]} ((\lambda^m)^2 \wedge 1) \) with \( C \) given in (4.5) and using (4.2) (4.3), (4.5) and (4.8) prove Item (2). □

Based on Propositions 4.1 and 4.2, the following lemma estimates the accuracy of \( (\hat{\theta}_m)_{m \in \mathbb{N}} \) in terms of \( (V_m^{-1})_{m \in \mathbb{N}} \), which extends [31, Proposition 4.6] to randomised policies.

**Lemma 4.3.** Suppose (H.1) and Setting 4.1 hold. Then there exists a constant \( C \geq 0 \), depending only on the coefficients in (H.1) and \( \theta_0, \lambda, L \) in Setting 4.1, such that for all \( m \in \mathbb{N} \),
\[
\mathbb{P} \left( \|\hat{\theta}_m - \theta^*\|^2 \leq C \left( \Lambda_{\min}(V_m^{-1}) \right)^{-1} (1 + \ln m) \right) \geq 1 - 1/m. \tag{4.9}
\]
Assume further that (H.2) holds. Then for all \( m \in \mathbb{N} \),
\[
\mathbb{E} \left[ \|\theta_m - \theta^*\|^2 \right] \leq C \left( \mathbb{E} \left[ \Lambda_{\min}(V_m^{-1}) \right]^{-1} (1 + \ln m) + \frac{1}{m} + \mathbb{P}(\hat{\theta}_m \notin \Theta) \right), \tag{4.10}
\]
with \( \theta_m \) defined by (2.25).

**Proof.** Throughout this proof, let \( C > 0 \) be a generic constant which is independent of \( \delta \in (0,1) \) and \( m \in \mathbb{N} \) and may take different values at each occurrence.

By the same arguments for Lemma 4.5 and Proposition 4.6 in [31] and the fact that \( \inf_{t \in [0,T]} \bar{\sigma}_t > 0 \), it follows that for all \( \delta \in (0,1) \) and \( m \in \mathbb{N} \),
\[
\mathbb{P} \left( \Lambda_{\min}(V_m^{-1}) \|\hat{\theta}_m - \theta^*\|^2 \leq C (1 + \ln(\det V_m^{-1}) + \ln(\frac{1}{\delta})) \right) \geq 1 - \delta. \tag{4.11}
\]

By Proposition 4.1 and Proposition 4.2 Item (1), it follows that for all \( \delta \in (0,1) \) and \( m \in \mathbb{N} \), with probability at least \( 1 - \delta \),
\[
\ln(\det V_m^{-1}) \leq C \ln |V_m^{-1}| \leq C \ln \left( \sum_{n=1}^m \mathbb{E} \left[ \int_0^T \frac{1}{\sigma_t^2} \left( Z_t^n (Z_t^n)^\top \right) dt \right] |F_{n-1}^{\sigma b} \right) + C \max \left( \ln \left( \frac{2}{\delta} \right), \sqrt{m \ln \left( \frac{2}{\delta} \right)} \right) \leq C \left( m + C \max \left( \ln \left( \frac{2}{\delta} \right), \sqrt{m \ln \left( \frac{2}{\delta} \right)} \right) \right) \leq C \left( 1 + \ln m + \ln \left( \frac{1}{3} \right) \right). \tag{4.12}
\]
Substituting \( \delta = 1/(2m) \) in (4.11) and (4.12) implies that
\[
\mathbb{P} \left( \|\hat{\theta}_m - \theta^*\|^2 \leq C \left( \Lambda_{\min}(V_m^{-1}) \right)^{-1} (1 + \ln m) \right) \geq 1 - 1/m.
\]
Let \( A \) be the event such that the estimate \( \|\hat{\theta}_m - \theta^*\|^2 \leq C \Lambda_{\min}(V_m^{-1})^{-1} (1 + \ln m) \) holds. Under (H.2), \( \theta_m \) and \( \theta^* \) are uniformly bounded (cf. (2.25)). Hence, the first inequality follows from the fact that \( \theta_m = \hat{\theta}_m \) on the event \( \{\hat{\theta}_m \in \Theta\} \). □
The remaining part of this section is devoted to analyse the expected costs of \((\varphi_m)_{m \in \mathbb{N}}\). The following lemma proves stability of \((2.5)\) and regularity of the cost function \(J^{\theta^*}\) in \((2.1)\). The proof adapts the arguments of Lemma 3.1 and Proposition 3.7 in [2] to the present setting, and is given in Appendix A.

**Lemma 4.4.** Suppose \((H.1)\) holds.

1. The map \(\mathbb{R}^{1 \times 2} \ni \theta \to P^\theta, \eta^\theta \in C^1([0,T];\mathbb{R})\) is continuously differentiable, where \(P^\theta \in C([0,T];\mathbb{R}_{\geq 0})\) and \(\eta^\theta \in C([0,T];\mathbb{R})\) are the solution to \((2.5)\).

2. For each \(L \geq 0\), there exists a constant \(C \geq 0\), depending only on \(L\) and the coefficients in \((H.1)\), such that for all \(K, k \in C([0,T];\mathbb{R})\) with \(\|K\|_\infty, \|k\|_\infty \in [0,L]\),

\[
J^{\theta^*}(\phi) - J^{\theta^*}(\phi^{\theta^*}) \leq C(\|K - K^{\theta^*}\|_\infty^2 + \|k - k^{\theta^*}\|_\infty^2),
\]

where \(\phi : [0,T] \times \mathbb{R} \to \mathbb{R}\) satisfies \(\phi(t,x) = Ktx + kt\) for all \((t,x) \in [0,T] \times \mathbb{R}\), and \(\phi^{\theta^*}\) is the optimal policy of \((2.1)\).

Based on Lemma 4.4, the next proposition quantifies the performance gaps between piecewise constant randomised policies \((\varphi_m)_{m \in \mathbb{N}}\) and the optimal policy \(\varphi^{\theta^*}\).

**Proposition 4.5.** Suppose \((H.1)\) and Setting 4.1 hold. Then there exists a constant \(C \geq 0\), depending only on the coefficients in \((H.1)\) and \(L\) in Setting 4.1, such that for all \(m \in \mathbb{N}\),

\[
\mathbb{E}[J^{\theta^*}(\varphi^m) - J^{\theta^*}(\varphi^{\theta^*})] \leq C \left((1 + |\pi_m|)\|\lambda_m\|_\infty^2 + \mathbb{E}[\|K^m - K^{\theta^*}\|_\infty^2] + \mathbb{E}[\|k^m - k^{\theta^*}\|_\infty^2]\right).
\]

**Proof.** We first prove that there exists a constant \(C \geq 0\) such that for all \(K, k \in L^\infty([0,T];\mathbb{R})\) with \(\|K\|_\infty, \|k\|_\infty \in [0,L]\) and \(\lambda \in L^\infty([0,T];\mathbb{R}_{\geq 0})\), it holds with \(\nu(t,x) := N(Ktx + kt, \lambda_t^2)\) for all \((t,x) \in [0,T] \times \mathbb{R}\),

\[
|\bar{J}^{\theta^*}_0(\nu) - J^{\theta^*}(\varphi^{\theta^*})| \leq C \left(\|\lambda\|_\infty^2 + \|K - K^{\theta^*}\|_\infty^2 + \|k - k^{\theta^*}\|_\infty^2\right),
\]

(4.13)

with \(\bar{J}^{\theta^*}_0(\nu)\) defined as in \((2.14)\). Indeed, let \(\phi(t,x) := Ktx + kt\) for all \((t,x) \in [0,T] \times \mathbb{R}\). Then

\[
\bar{J}^{\theta^*}_0(\nu) - J^{\theta^*}(\varphi^{\theta^*}) = \left(\bar{J}^{\theta^*}_0(\nu) - J^{\theta^*}(\varphi)\right) + \left(J^{\theta^*}(\varphi) - J^{\theta^*}(\phi^{\theta^*})\right).
\]

(4.14)

For the first term of \((4.14)\), observe that \(X^{\theta^*,\varphi}\) and \(X^{\theta^*,\nu}\) satisfy the same dynamics:

\[
dX_t = (A^*X_t + B^*(K_tX_t + k_t)) \, dt + \sigma_t \, dW_t, \quad t \in [0,T]; \quad X_0 = x_0,
\]

and hence \(X^{\theta^*,\varphi} = X^{\theta^*,\nu}\) due to the uniqueness of solutions. Thus, by \((2.14)\),

\[
\bar{J}^{\theta^*}_0(\nu) = \mathbb{E} \left[ \int_0^T \left( f(t, X_t^{\theta^*,\varphi}, K_tX_t^{\theta^*,\varphi} + k_t) + R_t\lambda_t^2 \right) \, dt + g(X_T^{\theta^*,\varphi}) \right] = J^{\theta^*}(\varphi) + \int_0^T R_t\lambda_t^2 \, dt,
\]

(4.15)

which along with \(R \in L^\infty([0,T];\mathbb{R})\) shows \(\bar{J}^{\theta^*}_0(\nu) - J^{\theta^*}(\varphi) \leq C\|\lambda\|_\infty^2\). By Lemma 4.4, the second term of \((4.14)\) can be estimated by \(J^{\theta^*}(\varphi) - J^{\theta^*}(\phi^{\theta^*}) \leq C(\|K - K^{\theta^*}\|_\infty^2 + \|k - k^{\theta^*}\|_\infty^2)\), which subsequently leads to \((4.13)\).

Now for each \(m \in \mathbb{N}\), consider

\[
\bar{J}^{\theta^*}_0(\nu^m) = \mathbb{E} \left[ \int_0^T \left( \int_\mathbb{R} f(t, X_t^{\nu^m}, a)\nu^m(t, X_t^{\nu^m}; da) \right) \, dt + g(X_T^{\nu^m}) \big| \mathcal{F}_{m-1}^\nu \right],
\]

where \(\nu^m\) is defined in Setting 4.1, and \(X^{\nu^m}\) satisfies for all \(t \in [0,T]\),

\[
dX_t^{\nu^m} = \int_\mathbb{R} (A^*X_t^{\nu^m} + B^*a)\nu^m(t, X_t^{\nu^m}; da) \, dt + \sigma_t \, dW_t^{\nu^m}, \quad X_0^{\nu^m} = x_0.
\]
The assumptions in Setting 4.1 ensures that \((K^m, k^m)_{m \in \mathbb{N} \cup \{0\}}\) are uniformly bounded. Then by (4.13), there exists \(C \geq 0\) such that for all \(m \in \mathbb{N},
\end{equation}
Moreover, Theorem 2.2 and the uniform boundedness of \(K^m, k^m, \Lambda^m\) imply that
\[
\mathbb{E} \left[ J^\varphi(\varphi^m) - J^\varphi_0(\nu^m) \right] \leq C|\pi_m||\lambda^m||\lambda^\varphi_0|.
\]
which along with the tower property of conditional expectations and (4.16) leads to the desired estimate. \(\Box\)

### 4.2 Regret analysis of Algorithm 1

This section establishes the regret of Algorithm 1 (i.e., Theorem 2.3). For notational simplicity, we denote by \(C > 0\) a generic constant which is independent of the episode number \(m\) and may take different values at each occurrence. The following lemma proves that the policies generated by Algorithm 1 satisfies Setting 4.1, and hence the results in Section 4.1 apply to these policies. The proof is given in Appendix A.

**Lemma 4.6.** Suppose (H.1) and (H.2) hold and \(\sup_{m \in \mathbb{N}} \vartheta_m < \infty\). Let \(k^1, K^1 \in C([0, T]; \mathbb{R}), \lambda^1 \in C([0, T]; \mathbb{R}_{\geq 0})\) and \(\nu^1 \sim \mathcal{G}(k^1, K^1, \lambda^1)\). Then the policies \((\varphi^m)_{m \in \mathbb{N}}\) from Algorithm 1 satisfy Setting 4.1.

We first establish the convergence rate of \((\vartheta_m)_{m \in \mathbb{N}}\) for suitable choices of \((\vartheta_m)_{m \in \mathbb{N}}\) and \((\pi_m)_{m \in \mathbb{N}}\).

**Proposition 4.7.** Suppose (H.1) and (H.2) hold. Let \(\vartheta_0 \in \mathbb{R}^{1 \times 2}, V_0 \in \mathbb{S}^2_\varphi, \vartheta_0 > 0\) and \(\nu^1 \sim \mathcal{G}(k^{b_0}, K^{b_0}, \sqrt{\alpha/2n})\). Then there exist constants \(c_0, C > 0\) such that if one sets
\[
\vartheta_m = \vartheta_0 m^{-1/2} \ln (m + 1) \quad \text{and} \quad |\pi_m|\vartheta_{m-1} \leq c_0(\vartheta_{m-1} \wedge 1), \quad \forall m \in \mathbb{N},
\]
then \(\mathbb{E}[(\vartheta_m - \vartheta^*)^2] \leq Cm^{-1/2} \text{ for all } m \in \mathbb{N}\).

**Proof.** We first establish a lower and upper bound of \((V^{-1}_m)_{m \in \mathbb{N}}\) in terms of \(m\). Note that by (2.9) and the choice of \(\nu^1\), Algorithm 1 defines \(K^m = K^{b_0} - m, k^m = K^{b_0} - n^m\) and \(\lambda^m = \frac{n^m}{2R}\) for all \(m \in \mathbb{N}\). As \(0 < \inf_{t \in [0, T]} R_t \leq \sup_{t \in [0, T]} R_t < \infty\), by choosing a sufficiently small \(c_0 > 0\) and setting \(|\pi_m|\vartheta_{m-1} \leq c_0(\vartheta_{m-1} \wedge 1)\) for all \(m \in \mathbb{N}\), Proposition 4.2 Item (2) implies that for all \(m \in \mathbb{N},
\end{equation}
with some constant \(r > 0\) independent of \(m\). By the definition of \((\vartheta_m)_{m \in \mathbb{N}}\), there exists \(\tilde{N}_0 \in \mathbb{N}\) such that \(\vartheta_m \leq 1\) for all \(m \geq \tilde{N}_0\). In particular, for \(m \geq 4\tilde{N}_0\), it follows from (4.17) that
\[
\sum_{n=1}^{m} \Lambda_m \left( \mathbb{E} \left[ \int_0^T \frac{1}{\sigma^2_s} Z^n_s(Z^n_s)^\top ds | F^n_{n-1} \right] \right) \geq r \sum_{n=1}^{m} \vartheta_0(n-1)^{-1/2} \ln n \geq r \int_{N_0}^{m} (x-1)^{-1/2} \ln x dx
\]
\[
\geq r \int_{m/4}^{m} (x-1)^{-1/2} \ln x dx \geq r \ln \left( \frac{m}{4} \right) \int_{m/4}^{m} (x-1)^{-1/2} dx \geq rm^{1/2} \ln (m + 1).
\]
By applying Proposition 4.1 with \(\delta = 1/m\), for each \(m \in \mathbb{N}\), with probability at least \(1 - 1/m,\)
\[
\left| V^{-1}_m - V^{-1}_0 - \sum_{n=1}^{m} \mathbb{E} \left[ \int_0^T \frac{1}{\sigma^2_t} Z^n_t(Z^n_t)^\top dt | F^{ob}_{n-1} \right] \right| \leq C \sqrt{m \ln (m + 1)}.
\]
Let \(A_m\) be the event that (4.19) holds. Without loss of generality, assume that \(\tilde{N}_0\) is sufficiently large such that \(C \ln^{-1/2} (m + 1) \leq r/2\) for all \(m \geq 4\tilde{N}_0\), with the constant \(C\) in (4.19). Then, for any \(m \geq 4\tilde{N}_0\), by (4.18) and (4.19), under the event \(A_m,\)
\[
\Lambda_m \left( V^{-1}_m \right) \geq \sum_{n=1}^{m} \Lambda_m \left( V^{-1}_0 + \mathbb{E} \left[ \int_0^T \frac{1}{\sigma^2_t} Z^n_t(Z^n_t)^\top dt | F^{ob}_{n-1} \right] \right) \geq \sum_{n=1}^{m} \mathbb{E} \left[ \int_0^T \frac{1}{\sigma^2_t} Z^n_t(Z^n_t)^\top dt | F^{ob}_{n-1} \right] \geq rm^{1/2} \ln (m + 1) - \frac{1}{2} rm^{1/2} \ln (m + 1),
\]
\[
\geq \frac{1}{2} rm^{1/2} \ln (m + 1),
\]
which along with $(\Lambda_{\min}(V^{-1})^{-1})^{-1} \leq (\Lambda_{\min}(V^{-1})^{-1})^{-1}$ and $\mathbb{P}(A_m^c) \leq 1/m$ shows that for all $m \geq 4\bar{N}_0$,

$$
\mathbb{E}\left[(\Lambda_{\min}(V^{-1})^{-1})^{-1}\right] = \mathbb{E}\left[(\Lambda_{\min}(V^{-1})^{-1})^{-1}1_{A_m}\right] + \mathbb{E}\left[(\Lambda_{\min}(V^{-1})^{-1})^{-1}1_{A_m^c}\right]
\leq C m^{-1/2}(\ln(m + 1))^{-1} + \frac{1}{m}(\Lambda_{\min}(V^{-1})^{-1})^{-1} \leq C m^{-1/2}(\ln(m + 1))^{-1}.
$$

(4.20)

Now consider the event such that the estimate (4.9) in Lemma 4.3 and (4.19) hold simultaneously. Then for all $m \geq 4\bar{N}_0$, with probability at least $1 - 2/m$,

$$
|\hat{\theta}_m - \theta^*|^2 \leq C m^{-1/2}(\ln(m + 1))^{-1}(1 + \ln m).
$$

(4.21)

As $\theta^* \in \text{int}(\Theta)$ by (H.2), one can choose a sufficiently large $\bar{N}_0 \in \mathbb{N}$ such that for all $m \geq 4\bar{N}_0$, $C m^{-1/2}(\ln(m + 1))^{-1}(1 + \ln m) \leq \inf_{\theta \notin \Theta} |\theta - \theta^*|^2$, with the constant $C$ in (4.21). Hence, $\hat{\theta}_m \in \Theta$ on the event that (4.21) holds, i.e. $\mathbb{P}(\hat{\theta}_m \notin \Theta) \leq 2/m$ for all $m \geq 4\bar{N}_0$. Combining this with the estimate (4.10) in Lemma 4.3 and (4.20), for all $m \geq 4\bar{N}_0$,

$$
\mathbb{E}[|\theta_m - \theta^*|^2] \leq C \left(C m^{-1/2}(\ln(m + 1))^{-1}(1 + \ln m) + \frac{1}{m} + \frac{2}{m} \right) \leq C m^{-1/2}.
$$

By the uniform boundedness of $(\theta_m)_{m \in \mathbb{N}} \subset \Theta$, there exists a sufficiently large $C$ such that the estimate holds for all $m \in \mathbb{N}$. □

**Proof of Theorem 2.2.** By Lemma 4.4 Item (1), $\mathbb{R}^{1^2} \ni \theta \mapsto (K^\theta, k^\theta) \in C([0, T]; \mathbb{R})^2$ is continuously differentiable and hence locally Lipschitz continuous. Then the fact that $(\theta_m)_{m \in \mathbb{N}} \subset \Theta$ and the boundedness of $\Theta$ (see (H.2)) imply that for all $m \in \mathbb{N}$ and $\mathbb{P}$-a.s.,

$$
\|K^m - K^{\theta^*}\|^2 + \|k^m - k^{\theta^*}\|^2 \leq \|K^{\theta_{m-1}} - K^{\theta^*}\|^2 + \|k^{\theta_{m-1}} - k^{\theta^*}\|^2 \leq C|\theta_{m-1} - \theta^*|^2.
$$

(4.22)

Hence, by Proposition 4.5 and the fact that $\|\Lambda^m\|^2_{\infty} \leq C\varrho_{m-1}$ for all $m \in \mathbb{N}$,

$$
\mathbb{E}[|J^{\theta^*}(\varphi^m) - J^{\theta^*}(\varphi^{\theta^*})|] \leq C\left((1 + |\pi_m|)\varrho_{m-1} + \mathbb{E}[|\theta_{m-1} - \theta^*|^2]\right), \quad \forall m \in \mathbb{N}.
$$

Combining this with Proposition 4.7 shows that there exists $c_0 > 0$ such that if one sets

$$
\varrho_m = \varrho_{0m^{-1/2}} \ln(m + 1), \quad |\pi_m| \varrho_{m-1} \leq c_0(\varrho_{m-1} + 1), \quad \forall m \in \mathbb{N},
$$

(4.23)

then for all $N \in \mathbb{N}$,

$$
\mathbb{E}\left[\text{Reg}(N)\right] = \sum_{m=1}^{N} \mathbb{E}[|J^{\theta^*}(\varphi^m) - J^{\theta^*}(\varphi^{\theta^*})|] \leq C \sum_{m=1}^{N} \left(|\pi_m| \varrho_{m-1} + \varrho_{m-1} + \mathbb{E}[|\theta_{m-1} - \theta^*|^2]\right)
\leq C \sum_{m=1}^{N} \left(|\pi_m| \varrho_{m-1} + \varrho_{m-1} + m^{-1/2}\right).
$$

In particular, by choosing $|\pi_m| \leq \frac{c_0}{\max(\sup_{m \in \mathbb{N}} \varrho_{m-1}, 1)}$, the condition (4.23) holds and

$$
\mathbb{E}\left[\text{Reg}(N)\right] \leq C \sum_{m=1}^{N} \left(\varrho_{m-1} + m^{-1/2}\right) \leq CN^{1/2} \ln(N + 1).
$$

□

### 4.3 Regret analysis of Algorithm 2

This section establishes the regret of Algorithm 2 (i.e., Theorem 2.4). As in Section 4.2, $C > 0$ denotes a generic constant which is independent of the episode number $m$ and may take different values at each occurrence. The following lemma proves an analogue of Lemma 4.6 for Algorithm 2. The proof is given in Appendix A.

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Lemma 4.8. Suppose (H.1) and (H.2) hold. Let $k^1, K^1 \in C([0,T];\mathbb{R})$, $\lambda^1 \in C([0,T];\mathbb{R}_{>0})$ and $\nu^1 \sim \mathcal{G}(k^1, K^1, \lambda^1)$. Then the policies $(\varphi^m)_{m \in \mathbb{N}}$ from Algorithm 2 satisfy Setting 4.1.

The following lemma will be used to quantify the dependence of $\|K^m - K^\theta\|_\infty$ and $\|k^m - k^\theta\|_\infty$ on $(\theta_m - \theta_m^*)_{m=0}^\infty$.

Lemma 4.9. Suppose (H.1) holds and for each $m \in \mathbb{N}$, let $h^m$ be defined by (2.13). Then $h^m = 1 - \frac{(\lambda^m)^2}{(\lambda^m)^2}$ for all $m \in \mathbb{N}$ and $t \in [0,T]$. Assume further that $(\varphi_m)_{m \in \mathbb{N}}$ is increasing, i.e., $\varphi_m \leq \varphi_{m+1}$ for all $m \in \mathbb{N}$. Then $h^m \leq \frac{1}{m}$ for all $m \in \mathbb{N}$ and $t \in [0,T]$.

Proof. By (2.13), for all $m \in \mathbb{N}$,

$$1 - h^m = \frac{\varphi_m(\lambda^m)^2 - 2R_t}{2R_t + \varphi_m(\lambda^m)^2} = \frac{\varphi_m(\lambda^m)^2 - 2R_t}{\varphi_m(\lambda^m)^2} = \frac{(\lambda^m)^2}{(\lambda^m)^2}.$$

Assume further that $(\varphi_m)_{m \in \mathbb{N}}$ is increasing, for (2.13), for all $m \in \mathbb{N}$,

$$\frac{\varphi_m(\lambda^m)^2 - 2R_t}{2R_t + \varphi_m(\lambda^m)^2} = 1 + \frac{\varphi_m(\lambda^m)^2 - 2R_t}{\varphi_m(\lambda^m)^2} = \frac{(\lambda^m)^2}{(\lambda^m)^2} \geq m - 1,$$

which along with $h^m = (1 + \frac{\varphi_m(\lambda^m)^2 - 2R_t}{\varphi_m(\lambda^m)^2})^{-1}$ leads to the desired estimate. \hfill \Box

Based on Lemma 4.9, we quantify the accuracy of $K^m$ and $k^m$ in terms of $(\theta_n)_{n=0}^\infty$.

Proposition 4.10. Suppose (H.1) and (H.2) hold. Let $\nu^1 \sim \mathcal{G}(k^1, K^1, \lambda_1)$ for some $k^1, K^1 \in C([0,T];\mathbb{R})$ and $\lambda^1 \in C([0,T];\mathbb{R}_{>0})$. Then there exists a constant $C \geq 0$ such that for all $\theta_0 \in \mathbb{R}^1 \times \mathbb{R}$, $V_0 \in \mathbb{S}_{+}^2$, $m \in \mathbb{N}$ and increasing sequences $(\varphi_m)_{m \in \mathbb{N}} \subset (0,\infty)$,

$$\|K^m - K^\theta\|_\infty + \|k^m - k^\theta\|_\infty^2 \leq C \sum_{n=1}^m \frac{(\lambda_n)^2}{\lambda_n + 1} \|\theta_n - \theta\|_\infty^2 + \left(\|k_1 - k^\theta\|_\infty^2 + \|k_1 - k^\theta\|_\infty^2\right),$$

where for each $m \in \mathbb{N}$, $K^m$ and $k^m$ are defined by (2.13).

Proof. For all $m \in \mathbb{N}$ and $t \in [0,T]$, by (2.13), $h^m \in [0,1]$ and

$$|K^m - K^\theta| = \|K^m - K^\theta\|_\infty + (1 - h^m)|K^m - K^\theta|.$$

Then by using the convexity of $\mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$,

$$|K^m - K^\theta| \leq h^m|K^m - K^\theta| + (1 - h^m)|K^m - K^\theta|.$$

Similar to (4.22), one has $\|K^m - K^\theta\|_\infty^2 \leq C\|\theta_m - \theta^\star\|_\infty^2$, which subsequently leads to

$$|K^m - K^\theta|^2 \leq C\|\theta_m - \theta^\star\|_\infty^2 + (1 - h^m)|K^m - K^\theta|.$$

Applying the same argument to $|k^m - k^\theta|$ leads to the estimate:

$$|k^m - k^\theta| \leq |k^m - k^\theta| + |k^m - k^\theta| \leq C\|\theta_m - \theta^\star\|_\infty^2 + (1 - h^m)(|K^m - K^\theta|^2 + |k^m - k^\theta|_\infty^2).$$

For each $m \in \mathbb{N}$ and $t \in [0,T]$, let $\kappa_{m,t} := |K^m - K^\theta| + |k^m - k^\theta|_\infty^2$, $\Delta_m := |\theta_m - \theta^\star|_\infty^2$ and $\eta_{m,t} := \prod_{k=1}^m (1 - h^k) = (\lambda^m)^2$ by Lemma 4.9. Then for all $m \in \mathbb{N}$ and $t \in [0,T]$, (4.24) and the fact that $h^m \leq \frac{1}{m}$ imply that

$$\eta_{m,t}^{\kappa_{m,t} \Delta_m + \eta_{m-1,t}^{k_{m-1,t}}} \leq C \sum_{n=1}^m \frac{(\lambda_n)^2}{\lambda_n + 1} \Delta_n + \eta_{m-1,t}^{k_{m-1,t}} \leq C \sum_{n=1}^m \frac{(\lambda_n)^2}{\lambda_n + 1} \Delta_n + \eta_{m-1,t}^{k_{m-1,t}}.$$
Multiplying both sides by $\eta_m$ and using $\frac{2m+1}{\eta_m} = \frac{(\lambda_m^{m+1})^2}{(\lambda_1^{m+1})^2}$ yield

$$k_{m+1,t} \leq C \sum_{n=1}^{m} \frac{1}{n} \left( \frac{\lambda_{n+1}}{\lambda_1^{n+1}} \right)^2 \Delta_n + \left( \frac{\lambda_{m+1}}{\lambda_1^2} \right)^2 \kappa_{1,t}. $$

Taking the supremum over $t$ leads to the desired inequality. 

To simplify the estimate in Proposition 4.10, we quantify the behavior of $(\lambda^m)_{m \in \mathbb{N}}$ in terms of $m$.

**Lemma 4.11.** Suppose (H.1) holds. Let $\theta_0 > 0$ and $\inf_{t \in [0,T]} \lambda_{t} > 0$. Then there exist constants $C \geq \epsilon > 0$ such that if one sets $\varrho_m = \varrho_0 m^{1/2} \ln(m+1)$ for all $m \in \mathbb{N}$, then

$$\varrho m^{-1/2} \ln(m+1) \leq (\lambda^m)^2 \leq \varrho m^{-1/2} \ln(m+1), \quad \forall t \in [0,T], m \in \mathbb{N} \cap [2, \infty).$$

where $\lambda^m$, for $m \in \mathbb{N} \cap [2, \infty)$, is defined by (2.13).

**Proof.** Observe from (H.1) that $\inf_{t \in [0,T]} R_t > 0$. Then by (2.13) and the choice of $(\varrho_m)_{m \in \mathbb{N}}$, for all $m \geq 2$ and $t \in [0,T]$,

$$(\lambda_t^m)^{-2} = 2R_t \varrho_{m-1}^{-1} + (\lambda_t^{m-1})^{-2} = 2R_t \sum_{k=1}^{m-1} \varrho_k^{-1} + (\lambda_t^1)^{-2} \geq C \sum_{k=1}^{m-1} k^{-1/2} (\ln(k+1))^{-1}^{-1} \geq C (\ln(m+1))^{-1} \sum_{k=1}^{m-1} k^{-1/2} \geq C (\ln(m+1))^{-1} \int_1^m x^{-1/2} dx \geq C m^{1/2} (\ln(m+1))^{-1}. $$

Rearranging the terms leads to the desired upper bound.

On the other hand, observe that there exists $C > 0$ such that for all $x \geq 1$,

$$\frac{1}{\sqrt{(\ln(x+2))}} \geq \frac{\ln(x+2)}{2} \geq C m^{-1/2} (\ln(x+1))^{-1}. $$

Then by the facts that $\sup_{t \in [0,T]} R_t < \infty$ and $(\lambda_t)^{-2} \leq C$, for all $m \geq 2$ and $t \in [0,T]$,

$$(\lambda_t^m)^{-2} = 2R_t \sum_{k=1}^{m-1} \varrho_k^{-1} + (\lambda_t^1)^{-2} \leq C \sum_{k=1}^{m-1} k^{-1/2} (\ln(k+1))^{-1} \leq C \sum_{k=1}^{m-1} \frac{(k+2) \ln(k+2) - 2k}{2k^{1/2}(k+2) \ln^2(k+2)} \leq C \int_1^m \frac{(x+2) \ln(x+2) - 2x}{2x^{1/2}(x+2) \ln^2(x+2)} dx \leq C m^{1/2} (\ln(m+2))^{-1} \leq C m^{1/2} (\ln(m+1))^{-1}. $$

Rearranging the terms yields to the desired lower bound.

Based on Lemma 4.11, the next proposition proves the convergence rate of $(\theta_m)_{m \in \mathbb{N}}$ in terms of $m$.

**Proposition 4.12.** Suppose (H.1) and (H.2) hold. Let $\theta_0 \in \mathbb{R}^{1 \times 2}$, $V_0 \in S^2_{++}$ and $\varrho_0 > 0$. Then there exist constants $c_0, C > 0$ such that if one sets $\nu^0 \sim \mathcal{G}(k^0, K^0, \varrho_0)$, $\varrho_m = \varrho_0 m^{1/2} \ln(m+1)$ and $|\pi_m| \leq c_0$ for all $m \in \mathbb{N}$, then $E[|\theta_m - \theta^*|^2] \leq C m^{-1/2}$ for all $m \in \mathbb{N}$.

**Proof.** By Lemma 4.11, for all $m \geq 2$, $||\lambda^m||^2_{\infty} \leq (\bar{C}/C) \sup_{t \in [0,T]} (\lambda_t^m)^2$ and $\sup_{m \in \mathbb{N}} ||\lambda^m||^2_{\infty} < \infty$. Let $c_0 := \frac{1}{2} \min \left( \frac{1}{\varrho_0}, \frac{1}{(\sup_{m \in \mathbb{N}} ||\lambda^m||^2_{\infty})} \right) > 0$, with $C$ being the constant in Proposition 4.2 Item (2). Hence, if $|\pi_m| \leq c_0$ for all $m \in \mathbb{N}$, then $|\pi_m| ||\lambda^m||^2 \leq \frac{1}{2} \min_{t \in [0,T]} (\lambda_t^m)^2 \wedge 1$ for all $m \in \mathbb{N}$, and

$$\sum_{n=1}^{m} A_{\min} \left( E \left[ \int_0^T \frac{1}{\sigma_s^2} Z^n_s(Z^n_s)^\top ds | F_{n-1}^{\text{ob}} \right] \right) \geq r \sum_{n=1}^{m} \min_{t \in [0,T]} ((\lambda_t^n)^2 \wedge 1), \quad \text{(4.25)}$$

with some constant $r > 0$ independent of $m$. Moreover, Lemma 4.11 implies that there exists $N_0 \in \mathbb{N}$ such that for all $t \in [0,T]$ and $m \geq N_0$, $(\lambda_t^m)^2 \leq 1$ for all $t \in [0,T]$. Hence, using (4.25) and following the same argument as in Proposition 4.7 yield (cf. (4.18)):

$$\sum_{n=1}^{m} A_{\min} \left( E \left[ \int_0^T \frac{1}{\sigma_s^2} Z^n_s(Z^n_s)^\top ds | F_{n-1}^{\text{ob}} \right] \right) \geq rm^{1/2} \ln(m+1), \quad \forall m \geq 4\bar{N}_0.$$
Proof of Theorem 2.4. Note that the choice of \((\varrho_m)_{m \in \mathbb{N}}\) ensures that \((\varrho_m)_{m \in \mathbb{N}}\) is increasing. By Propositions 4.10 and 4.12, for all \(m \in \mathbb{N}\),

\[
\mathbb{E}\left[\left\|K^{m+1} - K^{\vartheta^*}\right\|_\infty^2 + \mathbb{E}\left[\left\|k^{m+1} - k^{\vartheta^*}\right\|_\infty^2\right]\right] \\
\leq C \sum_{n=1}^{m} \frac{1}{n} \left|\lambda_{n+1}^{m+1}\right|^2 \mathbb{E}\left[\|\theta_n - \vartheta^*\|^2\right] + \left|\frac{\lambda_{n+1}^{m+1}}{\lambda^2}\right|^2 \left(\left\|K^{\theta_0} - K^{\vartheta^*}\right\|_\infty^2 + \left\|k^{\theta_0} - k^{\vartheta^*}\right\|_\infty^2\right)
\]

\[
\leq C \sum_{n=1}^{m} \frac{1}{n^{3/2}} \left|\lambda_{n+1}^{m+1}\right|^2 + C \left|\frac{\lambda_{n+1}^{m+1}}{\lambda^2}\right|^2
\]

By Lemma 4.11, for all \(m \in \mathbb{N}\),

\[
\mathbb{E}\left[\left\|K^{m+1} - K^{\vartheta^*}\right\|_\infty^2 + \mathbb{E}\left[\left\|k^{m+1} - k^{\vartheta^*}\right\|_\infty^2\right]\right] \leq C m^{-\frac{1}{\delta}} \ln(m + 1) \left(\sum_{n=1}^{m} \frac{1}{n \ln(n + 1)} + 1\right)
\]

\[
\leq C m^{-\frac{1}{\delta}} \ln(m + 1) \left(\ln \left(\ln(m + 1)\right) + 1\right).
\]

Summing over the index \(m\) then yields that for all \(N \in \mathbb{N}\),

\[
\sum_{m=1}^{N} \left(\mathbb{E}\left[\left\|K^{m+1} - K^{\vartheta^*}\right\|_\infty^2 + \mathbb{E}\left[\left\|k^{m+1} - k^{\vartheta^*}\right\|_\infty^2\right]\right]\right) \leq C N^{-\frac{1}{\delta}} \ln(N + 1) \left(\ln \left(\ln(N + 1)\right) + 1\right).
\]

Note that the choices of \(\varrho_m\) and \(\pi_m\) and Lemma 4.11 imply that \(\left|\lambda^{m}\right|_\infty^2 + |\pi_m||\lambda^{m}\|_\infty^2 \leq C m^{-\frac{1}{\delta}} \ln(m + 1)\), for all \(m \in \mathbb{N} \cap [2, \infty)\), which along with (4.26) and Proposition 4.5 shows that for all \(N \in \mathbb{N}\),

\[
\mathbb{E}\left[\text{Reg}(N)\right] = \mathbb{E}\left[J^{\vartheta^*}(\varphi^1) - J^{\vartheta^*}(\varphi^{\vartheta^*})\right] + \sum_{m=2}^{N} \mathbb{E}\left[J^{\vartheta^*}(\varphi^{m}) - J^{\vartheta^*}(\varphi^{\vartheta^*})\right]
\]

\[
\leq C \left(\sum_{m=1}^{N} m^{-\frac{1}{\delta}} \ln(m + 1) + N^{-\frac{1}{\delta}} \ln(N + 1) \left(\ln \left(\ln(N + 1)\right) + 1\right)\right)
\]

\[
\leq C N^{-\frac{1}{\delta}} \ln(N + 1) \left(\ln \left(\ln(N + 1)\right) + 1\right).
\]

This proves the desired estimate. \(\square\)

Appendix A Proofs of technical lemmas

Proof of Lemma 3.1. Throughout this proof, let \(C\) be a generic constant independent of \(N\), \((\varrho_i)_{i=1}^{N}\), \((\beta_{ij})_{i,j=1}^{N}\) and \(\delta\). Item (1) follows directly from the general Hoeffding inequality [32, Theorem 2.6.3]. To prove Item (2), observe that \((\zeta_i^2 - 1)_{i=1}^{N}\) are mean-zero sub-exponential random variables. Hence by Bernstein’s inequality in [32, Theorem 2.8.2], there exists \(C \geq 0\) such that for all \(\delta \geq 0\), it holds with probability at least \(1 - \delta\) that

\[
\left|\sum_{i=1}^{N} \nu_i(\zeta_i^2 - 1)\right| \leq C \max \left(\left\|\nu\right\|_2 \sqrt{\ln \left(\frac{2}{\delta}\right)} \left\|\nu\right\|_\infty \sqrt{\ln \left(\frac{2}{\delta}\right)}\right) \leq C \left\|\nu\right\|_2 \left(1 + \ln \left(\frac{1}{\delta}\right)\right),
\]

where the last inequality used the fact that \(\|\nu\|_\infty \leq \|\nu\|_2\) and the Cauchy-Schwarz inequality.

It remains to prove Item (3). The independence of \((\zeta_i)_{i \in \mathbb{N}}\) implies that \(\mathbb{E}\left[\sum_{i,j=1,i \neq j}^{N} \beta_{ij} \zeta_i \zeta_j\right] = 0\). Then, by considering \(A \in \mathbb{R}^{N \times N}\) with diagonal entries \(a_{ii} = 0\) for all \(i = 1, \ldots, N\), and off diagonal entries \(a_{ij} = \rho_{ij}^N\) for all \(i \neq j\), and by applying the Hanson–Wright inequality (see [32, Theorem 6.2.1]), there exists a constant \(C \geq 0\) such that for all \(s \geq 0\),

\[
\mathbb{P}\left(\left|\sum_{i,j=1,i \neq j}^{N} \beta_{ij} \zeta_i \zeta_j\right| \geq s\right) \leq 2 \exp \left(-C \min \left(\frac{s^2}{\|A\|_F^2}, \frac{s}{\|A\|_F}\right)\right),
\]
where \(\|A\|_F\) is the Frobenius norm of \(A\). Consequently for all \(\delta > 0\), it holds with probability at least \(1 - \delta\) that
\[
\left| \sum_{i,j=1, i \neq j}^N \beta_{ij} \zeta_i \zeta_j \right| \leq C \|A\|_F \max \left( \sqrt{\ln \left( \frac{2}{\delta} \right)}, \ln \left( \frac{2}{\delta} \right) \right) \leq C \|A\|_F \left( 1 + \ln \left( \frac{1}{\delta} \right) \right),
\]
where the last inequality used the Cauchy-Schwarz inequality.

**Proof of Lemma 4.4.** Item (1) has been proved in [2, Lemma 3.1] for \(S = p = q = M = m = 0\). The argument there is based on the implicit function theorem and extends straightforwardly to the present setting with general convex quadratic cost functions (details omitted).

For Item (2), let \(C\) be a generic constant depending only on \(\Lambda\) and the coefficients in (H.1), and define \(\|X\|_{S^2} = E[\sup_{t \in [0, T]} |X_t|^2]^{1/2}\) for any process \(X : \Omega \times [0, T] \rightarrow \mathbb{R}\). Consider the control processes \(U^{\theta^*} = \phi^\theta(\cdot, X^{\theta^*, \phi^*})\) and \(U^\phi = \phi(\cdot, X^{\theta^*, \phi})\) \(\text{d}P \otimes \text{d}t\) a.e. As \(\phi^{\theta^*}\) is an optimal policy, a quadratic expansion of (2.1) at \(U^{\theta^*}\) (see [2, Proposition 3.7]) implies that
\[
J^\theta(\phi) - J^{\theta^*}(\phi) = 2E \left[ \int_0^T \left( X^{\theta^*, \phi}_t - X^{\theta^*, \phi^*}_t \right)^\top \left( Q_t S_t R_t \right)^\top \left( X^{\theta^*, \phi}_t - X^{\theta^*, \phi^*}_t \right) \text{d}t \right] + 2E \left[ M(X^{\theta^*, \phi}_T - X^{\theta^*, \phi^*}_T)^2 \right] \leq C \left( \|X^{\theta^*, \phi} - X^{\theta^*, \phi^*}\|_{S^2} + \|\phi^\theta(\cdot, X^{\theta^*, \phi}) - \phi^{\theta^*}(\cdot, X^{\theta^*, \phi^*})\|_{S^2} \right). \tag{A.1}
\]

Recall that \(X^{\theta^*, \phi}\) satisfies
\[
dx_t^{\theta^*, \phi} = (A^* X_t^{\theta^*, \phi} + B^*(K_t X_t^{\theta^*, \phi} + k_t)) \text{d}t + \sigma_t \text{d}W_t, \quad t \in [0, T]; \quad X_0^{\theta^*, \phi} = x_0. \tag{A.2}
\]

The analytical solution to (A.2) and the assumption that \(\|K\|_\infty, \|k\|_\infty \leq L\) yield \(\|X^{\theta^*, \phi}\|_{S^2} \leq C\). Observe that \(X_0^{\theta^*, \phi} = X_0^{\theta^*, \phi^*}\) and for all \(t \in [0, T],\)
\[
d(X_t^{\theta^*, \phi} - X_t^{\theta^*, \phi^*}) = \left( A^* (X_t^{\theta^*, \phi} - X_t^{\theta^*, \phi^*}) + B^* (K_t X_t^{\theta^*, \phi} + k_t - K_t^{\theta^*} X_t^{\theta^*, \phi^*} - k_t) \right) \text{d}t \leq \left( (A^* + B^* K_t^{\theta^*})(X_t^{\theta^*, \phi} - X_t^{\theta^*, \phi^*}) + B^* (K_t - K_t^{\theta^*}) X_t^{\theta^*, \phi} + B^* (k_t - k_t^{\theta^*}) \right) \text{d}t.
\]

Let \(\Phi_t \in C([0, T]; \mathbb{R})\) be the fundamental solution of \(d\Phi_t^\theta = (A^* + B^* K_t^{\theta^*}) \Phi_t^\theta \text{d}t\). Then
\[
X_t^{\theta^*, \phi} - X_t^{\theta^*, \phi^*} = \Phi_t^\theta \int_0^t (\Phi_s^\theta)^{-1} (B^* (K_t - K_t^{\theta^*}) X_s^{\theta^*, \phi} + B^* (k_t - k_t^{\theta^*})) \text{d}s, \quad t \in [0, T],
\]
which along with \(\|K^\theta\|_\infty < \infty\) and \(\|X^{\theta^*, \phi}\|_{S^2} \leq C\) shows \(\|X^{\theta^*, \phi} - X^{\theta^*, \phi^*}\|_{S^2} \leq C(\|K - K^\theta\|_\infty + \|k - k^\theta\|_\infty)\). Furthermore, for all \(t \in [0, T],\)
\[
\phi^\theta(t, X_t^{\theta^*, \phi}) - \phi^{\theta^*}(t, X_t^{\theta^*, \phi^*}) = K_t X_t^{\theta^*, \phi} + k_t - K_t^{\theta^*} X_t^{\theta^*, \phi^*} - k_t^{\theta^*}
= K_t^{\theta^*} (X_t^{\theta^*, \phi} - X_t^{\theta^*, \phi^*}) + (K_t - K_t^{\theta^*}) X_t^{\theta^*, \phi} + k_t - k_t^{\theta^*}.
\]

Hence by \(\|K^\theta\|_\infty < \infty\) and \(\|X^{\theta^*, \phi}\|_{S^2} \leq C\), \(\|\phi^\theta(\cdot, X^{\theta^*, \phi}) - \phi^{\theta^*}(\cdot, X^{\theta^*, \phi^*})\|_{S^2} \leq C(\|K - K^\theta\|_\infty + \|k - k^\theta\|_\infty)\), which along with (A.1) yields the desired estimate.

**Proof of Lemma 4.6.** Recall that for all \(m \in \mathbb{N}\), Algorithm 1 defines \(K^{m+1} = K^\theta_m\) and \(k^{m+1} = k^\theta_m\). Lemma 4.4 Item (1) and (H.2) imply that \(\sup_{\theta \in \Theta} (\|K^\theta\|_\infty + \|k^\theta\|_\infty) < \infty\). By the definition of (2.25), \((\theta_m)_{m \in \mathbb{N}}\) takes values in a bounded set \(\Theta\), and hence \(\sup_{m \in \mathbb{N}} (\|K^m\|_{L^\infty} + \|k^m\|_{L^\infty}) \leq L\). The uniform boundedness of \(\|X^m\|_\infty\) follows from (2.9), \(\inf_{t \in [0, T]} R_t > 0\), and the condition that \(\sup_{m \in \mathbb{N}} \theta_m < \infty\). Finally, for each \(m \in \mathbb{N}\), by (2.24) and (2.25), \(\theta_m\) is \(F_m^\text{ob}\) measurable, which along with the continuity of \(\mathbb{R}^2 \times [0, T] \ni (\theta, t) \mapsto (K^\theta_t, k^\theta_t) \in \mathbb{R}^2\) leads to the desired \(F_m^\text{ob} \otimes \mathcal{B}([0, T])\)-measurability of \(K^{m+1}\) and \(k^{m+1}\).
Proof of Lemma 4.8. By similar arguments as those for Lemma 4.6, \( L := \sup_{m \in \mathbb{N}}(\|K^m\|_{L^\infty} + \|k^m\|_{L^\infty}) < \infty \), and for all \( m \in \mathbb{N} \), \( K^m \) and \( k^m \) are measurable. Observe that for each \( m \in \mathbb{N} \) and \( t \in [0, T] \), \( K_t^{m+1} \) (resp. \( k_t^{m+1} \)) is a convex combination of \( (K_t^0)_{n=0}^m \) (resp. \( (k_t^0)_{n=0}^m \)) according to the weights \( (h_t^n)_{n=0}^m \). Hence \( K^m \) and \( k^m \) are measurable, and \( \sup_{m \in \mathbb{N}}(\|K_t^{m+1}\|_{L^\infty} + \|k_t^{m+1}\|_{L^\infty}) \leq L \), independent of \( \theta_0, V_t, (\varphi_n)_{m \in \mathbb{N}} \) and \( (\pi_m)_{m \in \mathbb{N}} \). Finally, by (2.13), \( \varphi_m > 0 \) and \( R_k \geq 0 \) for all \( t \in [0, T] \), we have \( \lambda_t^{m+1} \leq \lambda_t^m \) for all \( t \in [0, T] \), which along with the fact that \( \|\lambda_0^m\|_{L^\infty} < \infty \) implies the uniform boundedness of \( (\lambda^m)_{m \in \mathbb{N}} \).

\[ \square \]

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