Integrability of quotients in Poisson and Dirac geometry

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Abstract

We study the integrability of Poisson and Dirac structures that arise from quotient constructions. From our results we deduce several classical results as well as new applications. We study the integrability of quasi-Poisson quotients in full generality recovering, in particular, the integrability of quotients of Poisson manifolds by Poisson actions. We also give explicit constructions of Lie groupoids integrating two interesting families of geometric structures: (i) a special class of Poisson homogeneous spaces of symplectic groupoids integrating Poisson groups and (ii) Dirac homogeneous spaces.

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1 Introduction

Symplectic groupoids have become a key tool in the study of Poisson structures, being of central importance for the problem of quantization [17, 35] as well as other applications [20, 21, 24]. An important issue is that, unlike Lie algebras, which always admit integrations to Lie groups, not every Poisson manifold is “integrable” to a symplectic groupoid [72]. The precise conditions for the integrability of Poisson manifolds (and Lie algebroids in general) were found in [22, 23]. Although these results solve the abstract problem of integrability, it is still of fundamental importance to identify concrete classes of Poisson manifolds that are integrable and have explicit constructions for their integrating groupoids. Our work has been largely driven by this problem with special focus on Poisson structures (or more general Lie algebroids) that arise from quotient constructions. In this paper we extend most of the integrability results found in [31, 32, 67, 13] and we put them in a common framework.

The study of Poisson structures resulting from reduction by symmetries is intimately tied with more general geometric objects, known as Dirac structures [19, 18]. In simple terms, Dirac structures are to Poisson structures what closed 2-forms are to symplectic forms, and just as symplectic forms naturally arise as quotients of closed 2-forms, Poisson structures are often realized as quotients of Dirac structures. Extending Poisson to Dirac structures is also essential to make sense of their pullbacks [15]. It often happens that in order to understand the integrability of a Poisson structure obtained by reduction of a Dirac structure [15] it is easier to check whether the original Dirac structure is integrable and whether one of its integrations gives rise to an integration of the associated Poisson quotient, as e.g. in [13]. One of the main goals of this work is to understand how and when the following diagram closes:

\[
\begin{array}{ccc}
\{\text{Presymplectic groupoids}\} & \overset{\text{Reduction?}}{\longrightarrow} & \{\text{Symplectic groupoids}\} \\
\downarrow \text{Lie functor} & & \downarrow \text{Lie functor} \\
\{\text{Reducible Dirac structures}\} & \overset{\text{Reduction}}{\longrightarrow} & \{\text{Poisson structures}\}.
\end{array}
\]

It is not hard to find examples of integrable Dirac structures whose quotient Poisson manifold is nonintegrable, see Example 3.10. The central result in this paper characterizes the integrability of Poisson structures obtained as quotients of Dirac structures (see Theorem 3.12 below):

**Theorem 1.1.** Let \( q : S \to M \) be a surjective submersion and let \( \pi \) be a Poisson structure on \( M \). Then the following are equivalent:

1. \((M, \pi)\) is integrable;

2. the pullback Dirac structure \( L := q^!(T^*M) \) is integrable and the inclusion \( \ker Tq \subset L \) is integrable by a Lie groupoid morphism \( S \times_M S \to G \), where \( S \times_M S \rightrightarrows S \) is the submersion groupoid associated to \( q \) and \( G \) is a presymplectic groupoid.

Moreover, if \( \Phi : S \times_M S \to G \) is a Lie groupoid morphism as above, then \( M \) is integrable by \( G/\sim \), where \( \sim \) is the equivalence relation defined by

\[ g \sim \Phi(x)g\Phi(y) \]

for all compatible \( g \in G \) and \( x, y \in S \times_M S \).

In Section 3 we prove Theorem 1.1 along with its analogous version for Lie algebroids and their pullbacks, see Theorem 3.2. The main ingredient in the proof of Theorem 1.1 is the construction of a suitable double Lie groupoid [49] which integrates the multiplicative foliation determined by the kernel of the presymplectic form on \( G \). We can see that in this situation the orbit space of this double Lie groupoid provides the desired integration. In general, suppose
that $L$ is integrable and let $G(L)$ be its source-simply-connected integration. We can always integrate the kernel of the presymplectic form on $G(L)$ by a double Lie groupoid which should be seen as a desingularization of the null foliation on $G(L)$, see Remark 3.14. In fact, this double Lie groupoid is a higher version of the 0-symplectic groupoids of [37] which represents what we might call a “symplectic stack groupoid”. In light of this observation we can raise the question of whether we can find, for every Poisson manifold $M$, a surjective submersion $q : S \to M$ such that the pullback $q^*(T^*M)$ is integrable.

Theorem 1.1 turns out to have several interesting consequences and applications. For instance, Theorem 1.1 leads to an alternative proof of classical results concerning the existence of complete symplectic realizations and the Morita invariance of integrability, see Theorem 4.8 and Proposition 3.19. As new applications, we obtain:

- a criterion for the integrability of quotients of quasi-Poisson manifolds (Section 5);
- integration of certain types of homogeneous spaces in Poisson and Dirac geometry (Section 6).

**Integrability of quotients of quasi-Poisson manifolds.** The study of quasi-Poisson (q-Poisson) manifolds began with the finite-dimensional description of Poisson structures on representation varieties provided in [4]. Later on, it was realized that most of the known methods of Poisson reduction by symmetries could be described in terms of an even broader notion of q-Poisson manifold [61]. Roughly, a q-Poisson manifold (in the general sense of [61]) is a manifold endowed with a suitable (global or infinitesimal) action and a bilinear bracket on the space of smooth functions that fails to be Poisson in a way controlled by the action; an important feature is that the orbit space of such an action turns out to carry a genuine Poisson structure, provided it is smooth. Theorem 5.15 below gives a criterion for the integrability of the Poisson structure induced on the quotient while Proposition 5.22 gives an integration of this Poisson structure in terms of a symplectic groupoid obtained by reduction. The proofs of Theorem 5.15 and Proposition 5.22 are based on the work developed in Section 4 where we analyse a situation in which the leaf space of a vacant multiplicative foliation inherits a Lie groupoid structure.

We obtain some corollaries about Poisson reduction such as the following. Let $G$ be a Poisson group acting freely and properly by a Poisson action on a Poisson manifold $(S, \pi)$. Then the induced Poisson structure on $\overline{S} = S / G$ is integrable if $S$ is integrable [31, 32]. For the original q-Poisson g-manifolds of [1] we get a completely analogous result. If $(S, \pi)$ is a q-Poisson $G$-manifold, then there is a nonobvious but canonical Lie algebroid structure on $T^*S$ [44]. Theorem 5.15 implies that the Poisson structure induced on $\overline{S} = S / G$ is integrable if $T^*S$ is integrable. In both of these situations, the Lie algebroid $C$ that appears in Theorem 5.15 controlling the integrability of the quotient can be interpreted as the Lie algebroid of the level set corresponding to the unit of a Lie group valued moment map as in [17] in the former case and in the sense of [3] in the latter. This last observation is related to the integration of Poisson structures on moduli spaces of flat $G$-bundles that shall be studied in a companion paper, see [6].

**Integration of homogeneous spaces.** It is proven in [47] that Poisson groups are always integrable. More recently, Poisson homogeneous spaces of Poisson Lie groups [28] were also shown to be integrable [13]. Our tools allow us to extend these results and consider other types of homogeneous spaces; these applications are conducted in Section 6. In general, Poisson homogeneous spaces are Poisson manifolds endowed with a transitive Poisson action of a Poisson groupoid. We identify a natural class of integrable Poisson homogeneous spaces of symplectic groupoids which are in duality with respect to the classical Poisson homogeneous spaces of Poisson groups, see Theorem 6.34.
2 Preliminaries

2.1 Lie groupoids and Lie algebroids

We follow mainly the conventions of [30, 56]. A smooth groupoid is a groupoid object in the category of not necessarily Hausdorff smooth manifolds such that its source map is a submersion. The structure maps of a groupoid are its source, target, multiplication, unit map and inversion, denoted respectively \( s, t, m, u, i \). For the sake of simplifying the notation we also denote \( m(a, b) \) by \( ab \). By a Lie groupoid we mean a smooth groupoid such that its base and source-fibers are Hausdorff manifolds. In order to avoid ambiguity when dealing with several groupoids we use a subindex \( s = s_G, t_G, m_G \) to specify the groupoid under consideration.

A Lie algebroid \( A \) over a manifold \( M \) is a vector bundle \( A \to M \) and a bundle map \( a : A \to TM \) called the anchor that satisfies the following properties: \( \Gamma(A) \) has a Lie algebra structure \([\ , \ ]\) and the Leibniz rule holds:

\[
[u, fv] = f[u, v] + (\mathcal{L}_{a(u)}f)v,
\]

for all \( u, v \in \Gamma(A) \) and \( f \in C^\infty(M) \). See [36, 69] for the definition of Lie algebroid morphism.

Let \( G \rightrightarrows B \) be a Lie groupoid. Its tangent Lie algebroid \( A = A_G \) is the vector bundle \( A = \ker Ts|_B \) with the anchor given by the restriction of \( Tt \) and the bracket defined in terms of right invariant vector fields. The construction of the tangent Lie algebroid is functorial: a Lie groupoid morphism induces canonically a Lie algebroid morphism between the associated Lie algebroids. The functor thus induced is called the Lie functor and we denote it by \( \text{Lie} \). When a Lie algebroid is isomorphic to the tangent Lie algebroid of a Lie groupoid it is called integrable. Not every Lie algebroid is integrable and the general obstructions for integrability were found in [22]. If \( A \) is an integrable Lie algebroid, we denote by \( \mathcal{G}(A) \) its source-simply-connected integration (which is unique up to isomorphism).

An important result relating Lie groupoids and Lie algebroids, that will be frequently used in this paper, is the following. Let \( \phi : A \to B \) be a Lie algebroid morphism and suppose that \( A \) and \( B \) are integrable. Then, for every Lie groupoid \( K \) integrating \( B \), there is a unique Lie groupoid morphism \( \Phi : \mathcal{G}(A) \to K \) such that \( \text{Lie}(\Phi) = \phi \). This result is known as Lie’s second theorem [57, 56].

A case of interest in this paper is the following: let \( q : S \to M \) be a surjective submersion. Then the fiber product \( S \times_M S \) is a Lie groupoid over \( S \), where the source and target maps are the projections to \( S \) and the multiplication is defined by \( m((x, y), (y, z)) = (x, z) \). The Lie groupoid thus obtained is called the submersion groupoid associated to \( q \), its Lie algebroid is isomorphic to the distribution \( \ker Tq \to TS \).

If \( g \) is a Lie algebra acting on a manifold \( M \), we denote by \( u_M \) the vector field on \( M \) induced by the action of \( u \in g \) and by \( g_M \subset TM \) the distribution generated by all such vector fields. An infinitesimal action induces a Lie algebroid structure on the trivial bundle \( g \times M [50] \). More generally, a Lie algebroid \( A \) over \( M \) acts on a map \( J : S \to M \) if there is a Lie algebroid morphism \( \rho : \Gamma(A) \to \mathfrak{X}(S) \) such that \( X_S := \rho(X) \) is \( q \)-related to \( a(X) \) for all \( X \in \Gamma(A) \). In this situation, the pullback vector bundle \( J^*A \) inherits a Lie algebroid structure over \( S \) called the action Lie algebroid structure [51].

Let \( A \) be a Lie algebroid over \( M \) and let \( q : S \to M \) be a map such that \( Tq \) is transverse to the anchor of \( A \). The pullback Lie algebroid of \( A \) along \( q \), denoted \( q^*A [36] \) is the vector bundle \( q^*A = \{ X \oplus U \in TN \oplus f^*B : Tf(X) = a(U) \} \) endowed with the projection to \( TS \) as the anchor.
and with the Lie bracket given by \([X \oplus fU, Y \oplus gV] = [X, Y] \oplus \mathcal{L}_X gV - \mathcal{L}_Y fU + fg[U, V]\), where \(U, V\) are pullbacks of sections of \(A\) and \(f, g \in C^\infty(S)\). If \(q\) is a surjective submersion, we have that the natural inclusion \(\ker Tq \rightarrow q^*A\) is a Lie algebroid morphism.

### 2.2 Poisson structures and symplectic groupoids

A **Poisson structure** on a manifold \(M\) is a bivector field \(\pi \in \Gamma(\wedge^2 TM)\) such that \([\pi, \pi] = 0\), where \([\cdot, \cdot]\) is the Schouten bracket. A Poisson structure \(\pi\) turns \((M, \pi)\) into a **Poisson manifold**. Equivalently, a Poisson structure on \(M\) is determined by a Lie algebra structure on \(C^\infty(M)\) such that the Leibniz rule holds: \([fg, h] = f[g, h] + [f, h]g\) for all \(f, g, h \in C^\infty(M)\). In terms of a bivector field, the bracket \([\cdot, \cdot]\) is described as \([f, g] = \pi(df, dg)\) for all functions \(f, g\) on \(M\).

The cotangent bundle of a Poisson manifold is endowed with a Lie algebroid structure in the following way \([30]\). Let \(\pi^d : T^*M \rightarrow TM\) be the map defined by \(\alpha \mapsto i_\alpha \pi\). The Lie bracket on \(\Omega^1(M)\) is given by:

\[
[\alpha, \beta] = \mathcal{L}_{\pi^d(\alpha)}\beta - \mathcal{L}_{\pi^d(\beta)}\alpha - d\pi(\alpha, \beta),
\]

for all \(\alpha, \beta \in \Omega^1(M)\); the anchor of \(T^*M\) is the map \(\pi^d\). In this situation, we shall refer to \(T^*M\) as the **cotangent Lie algebroid** of \(M\). A Poisson morphism \(J : (P, \pi_P) \rightarrow (Q, \pi_Q)\) between Poisson manifolds is a smooth map that satisfies \(J^*: \{f, g\} = \{J^*f, J^*g\}\) for all \(f, g \in C^\infty(Q)\).

A **Lie groupoid** \(G \rightrightarrows B\) is a **symplectic groupoid** \([12, 41]\) if it is endowed with a symplectic form \(\omega\) such that the graph of the multiplication in \(G \times G \times G\) is Lagrangian with respect to the symplectic form \(pr_1^*\omega + pr_2^*\omega - pr_3^*\omega\). The base of a symplectic groupoid inherits a unique Poisson structure such that the target map is a Poisson morphism and the cotangent Lie algebroid of this Poisson structure is canonically isomorphic to the Lie algebroid of the symplectic groupoid \([51]\). If the cotangent Lie algebroid of a Poisson manifold \(M\) is integrable, we shall say that \(M\) is **integrable**. If \(M\) is an integrable Poisson manifold, we denote by \(\Sigma(M) \rightrightarrows M\) its source-simply-connected integration.

### 2.3 Dirac structures and presymplectic groupoids

Let \(M\) be a manifold. We denote by \(TM\) the direct sum \(TM \oplus T^*M\). The **Courant-Dorfman bracket** \([19, 18]\) on \(\Gamma(TM)\) is:

\[
[X + \alpha, Y + \beta] := [X, Y] + \mathcal{L}_X \beta - i_Y d\alpha,
\]

for all \(X, Y \in \Gamma(TM)\), \(\alpha, \beta \in \Gamma(T^*M)\). Define the bilinear pairing

\[
\langle X + \alpha, Y + \beta \rangle = \langle \alpha, Y \rangle + \langle \beta, X \rangle;
\]

a subbundle \(F \subset TM\) is called **isotropic** if \(\langle \cdot, \cdot \rangle|_F = 0\). A maximally isotropic subbundle of \(TM\) is called **Lagrangian**. A **Dirac structure** is a Lagrangian subbundle \(L \subset TM\) which is involutive with respect to the restricted Courant-Dorfman bracket. If \(L \subset TM\) is a Dirac structure, \((M, L)\) is called a **Dirac manifold**.

Take a Dirac structure \(L\) on \(M\). A function \(f\) on \(M\) is **admissible** if there exists a vector field \(X_f\) such that \(X_f + df \in \Gamma(L)\). Admissible functions form a Poisson algebra with the bracket \([f, g] = \mathcal{L}_{X_f}g\). If the distribution \(D = L \cap (TM \oplus 0)\) is of constant rank, its induced foliation is called the **null foliation of \(L\)**. The admissible functions are the functions constant along the leaves of \(D\), so they are identified with the smooth functions on the leaf space of \(D\) when this is smooth. A Dirac structure is called **reducible** if its null foliation is simple.

Take a smooth map \(f : (M, L_M) \rightarrow (N, L_N)\) between Dirac manifolds. We say that \(f\) is a **forward Dirac map** \([13, 1]\) if

\[
L_N = \mathfrak{F}_f(L_M) := \{Tf(X) \oplus \alpha : X \oplus f^*\alpha \in L_M\};
\]

5
We say that $f$ is a \textit{backward Dirac map} if
\[ L_M = \mathfrak{B}_f(L_N) := \{ X \oplus f^* \alpha : T f(X) \oplus \alpha \in L_M \}. \]

Generalizing the correspondence between Poisson manifolds and symplectic groupoids, presymplectic groupoids integrate Dirac structures \cite{13}. A 2-form $\omega$ on a Lie groupoid $G \rightrightarrows M$ is multiplicative if $\text{pr}_1^* \omega + \text{pr}_2^* \omega - \text{pr}_3^* \omega$ vanishes on the graph of the multiplication map in $G \times G \times G$. A \textit{presymplectic groupoid} is a Lie groupoid $G \rightrightarrows M$ endowed with a multiplicative 2-form $\omega$ such that $d \omega = 0$, $\dim G = 2 \dim M$ and $\ker \omega \cap \ker T \sigma \cap \ker T \tau = 0$. If $(G, \omega) \rightrightarrows M$ is a presymplectic groupoid, then there is a unique Dirac structure $L$ on $M$ such that $\tau : (G, \text{graph}(\omega)) \to (M, L)$ is a forward Dirac map and $L$ is canonically isomorphic to the Lie algebroid of $G$ \cite{15}.

We can see that a bivector field $\pi$ on $M$ is Poisson if and only if $\text{graph}(\pi) \to TM$ is a Dirac structure; in such a situation, if $q : S \to M$ is a submersion then $q^*(T^* M)$ is identified with a Dirac structure on $S$ such that $q$ is a backward Dirac map.

### 2.4 Double Lie groupoids

\textit{Definition 2.1} \cite{10, 49}. A double topological groupoid is a groupoid object in the category of topological groupoids and is represented by a diagram of the form

\[
\begin{array}{ccc}
G & \rightrightarrows & H \\
\downarrow & & \downarrow \\
K & \rightrightarrows & S.
\end{array}
\]

A \textit{double Lie groupoid} is a double topological groupoid as in the previous diagram such that:

\begin{enumerate}[(1)]
\item each of the side groupoids is a smooth groupoid,
\item $H$ and $K$ are Lie groupoids over $S$, and
\item the double source map $(s^H, s^K) : G \to H \times_S K$ is a submersion (the superindices $^H$, $^K$ denote the groupoid structures $G \rightrightarrows H$, $G \rightrightarrows K$ respectively).
\end{enumerate}

The infinitesimal counterpart of a double Lie groupoid is provided by the following concept.

\textit{Definition 2.2} \cite{49}. An \textit{LA-groupoid} is a Lie groupoid in the category of Lie algebroids, that is, a Lie groupoid $A_1 \rightrightarrows A_0$ where the structure maps are Lie algebroid morphisms over the structure maps of a base groupoid $G_1 \rightrightarrows G_0$ and the map $s'_{A_1} : A_1 \to s'_{G_1} A_0$ induced by $s_{A_1}$ is surjective. An LA-groupoid is \textit{vacant} if $s'_{A_1}$ is an isomorphism.

\textit{Example 2.3}. The tangent bundle of a Lie groupoid is an LA-groupoid.

A \textit{multiplicative foliation} is an LA-subgroupoid of the tangent groupoid of a Lie groupoid \cite{35, 59}.

### 3 Proof of the main result

#### 3.1 The case of Lie algebroids

We begin this section by proving the following version of Theorem \cite{31} which corresponds to Lie algebroids and their pullbacks. Later we specialize the argument for the case of Poisson and Dirac structures thus obtaining a slightly more refined integrability criterion in Theorem \cite{32} below. The following theorem deals with the integrability of Lie algebroids in terms of their pullbacks, answering the following question: \textit{given a surjective submersion $J : S \to M$ and a Lie algebroid $A$ over $M$, how is the integrability of $A$ related to the integrability of $q^! A$?} It is not enough for $q^! A$ to be integrable in order to conclude that so is $A$, see Example \cite{30}.

\textit{Definition 3.1}. Let $q : S \to M$ be a surjective submersion and let $A$ be a Lie algebroid on $M$. We say that $q^! A$ admits a \textit{q-admissible integration} if $q^! A$ is integrable and the inclusion $\ker T q \to q^! A$ is integrable by a Lie groupoid morphism $S \times_M S \to G$, where $S \times_M S \rightrightarrows S$ is the submersion groupoid.
The condition above generalizes the situation of principal bundles considered in [13, Definition 2.6]. Let us stress that the distribution $\ker Tq$ is always integrable by its monodromy groupoid $\mathcal{G}(\ker Tq)$ and Lie’s second theorem \cite{30} implies that, if $q^1A$ is integrable, then the inclusion $\ker Tq \to q^1A$ is integrable by a Lie groupoid morphism $\mathcal{G}(\ker Tq) \to \mathcal{G}(q^1A)$. In general, such a Lie groupoid morphism does not induce a morphism with source the submersion groupoid $S \times_M S \Rightarrow S$, see Lemma 3.15 and Proposition 3.16.

**Theorem 3.2.** Let $q : S \to M$ be a surjective submersion and let $A$ be a Lie algebroid on $M$. Then the following are equivalent:

1. $A$ is integrable.

2. the pullback Lie algebroid $q^1A$ admits a $q$-admissible integration $G$.

Moreover, if $\Phi : S \times_M S \to G$ is a Lie groupoid morphism such that $\text{Lie}(\Phi)$ is the inclusion $\ker Tq \to q^1A$, then $A$ is integrable by the quotient $G|_\sim$ of $G$ by the equivalence relation $g \sim \Phi(x)g\Phi(y)$ for all compatible $g \in G$ and $x, y \in S \times_M S$.

**Remark 3.3.** The previous result has been observed in the particular case in which $q : S \to M$ is a principal bundle and hence $S \times_M S \Rightarrow S$ is an action groupoid \cite{32, 6, 8, 33, 13}.

The fact that condition 1 implies condition 2 of the previous theorem is a consequence of the construction of pullback Lie groupoids.

**Definition 3.4** \cite{30}. Let $q : S \to M$ be a submersion and let $K \Rightarrow M$ be a Lie groupoid, then the pullback Lie groupoid $q^1K \Rightarrow S$ is the Lie groupoid

$$q^1K := \{(x, g, y) \in S \times K \times S : q(x) = t(g), q(y) = s(g)\},$$

where the source and target are the projections to $S$ and the multiplication is given by $m((x, a, y), (y, b, z)) = (x, m(a, b), z)$.

If $A$ is integrable by some Lie groupoid $K$, then $q^1A$ is integrable by the pullback groupoid $q^1K$, see \cite{30}, Corollary 1.9]. We have that $q^1K$ is a $q$-admissible integration of $q^1A$: we simply define the map $\Phi : S \times_M S \to q^1K$ by means of $\Phi(x, y) = (x, u(q(x)), y)$. The converse implication shall be proved by means of the following lemmas.

**Lemma 3.5.** Let $\phi : B \to A$ be a surjective Lie algebroid morphism over a submersion $q : S \to M$. Suppose that $a_B$ defines an isomorphism $\ker \phi \cong \ker Tq$. Then $B \cong q^1A$.

**Proof.** The map $\chi : B \to q^1A$ defined by $\chi(u) = (a(u), \phi(u))$ is a morphism of Lie algebroids thanks to the universal property of $q^1A$ \cite{30}, Proposition 1.8]. So $\chi$ is injective and since $B$ and $q^1A$ have the same rank, $\chi$ is an isomorphism.

In order to check that the equivalence relation $\sim$ defined in Theorem 3.2 induces a Lie groupoid structure on the quotient we shall see that this relation is induced by the orbits of a suitable double Lie groupoid.

Let $G$ and $H$ be Lie groupoids over $S$ and let $\Phi : H \to G$ be a morphism over the identity. Let $(H \times H) \times G \Rightarrow G$ be the action groupoid given by the action of $H \times H \times G \Rightarrow S \times S$ on the map $(t_G, s_G) : G \Rightarrow S \times S$ given by $(h, k) \cdot g = \Phi(h)g\Phi(k)^{-1}$. We denote an element of $(H \times H) \times G \Rightarrow G$ as $(a, u, v, b)$, where $a, b \in G$ and $u, v \in H$ are such that $\Phi(u)b = a\Phi(v)$ is defined.
Definition 3.6. The following maps make \((H \times H) \times G\) into a double Lie groupoid with sides \(G\) and \(H\) over \(S\): the groupoid structure over \(G\) is given by the action groupoid structure on \((H \times H) \times G \rightrightarrows G\) and the groupoid structure over \(H\) is given by the fiber product of the pair groupoid \(H \times H \rightrightarrows H\) and \(G \rightrightarrows S\) over the pair groupoid \(S \times S \rightrightarrows S\) along the maps \((s_H,s_H) : H \times H \to S \times S\) and \((t_G,s_G) : G \to S \times S\). Explicitly, the multiplications are given respectively by the formulae (1) and (2):

\[
m^{G}((a,u,v,b),(b,u',v',c)) = (a,uu',vv',c);
\]
\[
m^{H}((a,u,v,b),(a',v,w,b')) = (aa',u,w,bb').
\]

This double Lie groupoid is called a comma double Lie groupoid.

Example 3.7. Suppose that \(H\) is an action groupoid \(K \times S \rightrightarrows S\). The existence of a morphism from \(H\) to \(G\) is equivalent to the existence of a \(K \times K\)-group action on \(G\) which lifts the \(K\)-action on \(S\) and which is multiplicative in the sense of being a morphism \((K \times K) \times G \to G\) with respect to the pair groupoid structure on \(K \times K \rightrightarrows K\). In fact, if there is such a morphism \(\Phi\) we can define the action map

\[(x,y) \cdot g \mapsto \Phi(x,t_G(g))g\Phi(y,s_G(g))^{-1};\]

conversely, if there is a \(K \times K\)-action with such properties we can define a morphism from \(H\) to \(G\) as follows

\[(x,a) \mapsto (x,1) \cdot a,\]

for all \((x,a) \in K \times S\). This fact was first observed in [32]. In this situation, we have that \((H \times H) \times G\) is isomorphic to the action groupoid \((K \times K) \times G\) associated to the group action above.

Lemma 3.8. Suppose that \(\Phi : H \to G\) is a Lie groupoid morphism over the identity on \(S\). Consider the horizontal structure of the comma double groupoid \((H \times H) \times G \rightrightarrows G\). Then the following hold.

1. If \(H\) is a proper groupoid, then \((H \times H) \times G \rightrightarrows G\) is proper and the isotropy groups satisfy \(((H \times H) \times G)_g = H_{t_G(g)} \times H_{s_G(g)}\) for all \(g \in G\).

2. If \(\Phi(\bigcup_{p \in S} H_p)\) is a normal subgroupoid\(^1\), then \(G/(H \times H)\), the orbit space of \((H \times H) \times G \rightrightarrows G\), inherits a unique groupoid structure over \(M\), the orbit space of \(H\), in such a way that the quotient map \(G/(H \times H)\) is a morphism.

Proof. Suppose that \(H\) is proper. Take \(K \times K' \subset G \times G\) compact. Then \(F := (t,s) : (H \times H) \times G \rightrightarrows G \times G\) satisfies that

\[F^{-1}(K \times K') = ((H \times H)_M \times G) \cap (C \times C' \times K'),\]

where \(C = (F|_H)^{-1}(t(K) \times t(K'))\) and \(C' = (F|_H)^{-1}(s(K) \times s(K'))\) are compact since \(H\) is proper. Now \((H \times H)_M \times G \subset H \times H \times G\) is closed since it is the inverse image of the diagonal of \(S^2 \times S^2\) by a continuous map which is closed since \(S\) is Hausdorff; so \(F^{-1}(K \times K')\) is compact. This is enough to prove that \(F\) is a proper map. The second part is immediate.

The source, target, unit and inversion maps of \(G\) descend to corresponding structure maps between \(G/(H \times H)\) and \(M\). If \(\Phi(\bigcup_{p \in S} H_p)\) is a normal subgroupoid, then so does the multiplication on \(G\). Indeed, take \([g],[g'] \in (H \times H)/(H\times H)\) with \([s_G(g)],[t_G(g')],[s_G(g')],[t_G(g)]\). So there is a \(w \in H\) such that \(s_G(\Phi(w)) = t_G(g')\) and \(t_G(\Phi(w)) = s_G(g)\). Let us define \(m([g],[g']) := [g\Phi(w)g']\). We shall see that \(m\) is well defined. Suppose that \(\phi(\overline{a})\) is another element with the same source and target than \(\Phi(w)\). Let us denote \(g \sim g'\) if there exist \(h, h' \in H\) such that \(g' = \Phi(h)g\Phi(h')\).

\(^1\)A subgroupoid \(N \subset G\) is normal if \(axe^{-1} \in N\) for all \(a \in G\) and \(x \in N\) which are composable.
Since $g\Phi(w\tilde{w}^{-1})g^{-1} = \Phi(h)$ for some $h \in H$, we have that $g\Phi(w)g' = \Phi(h)g\Phi(\tilde{w})g'$ so $g\Phi(w)g' \sim g\Phi(\tilde{w})g'$. Now suppose that $g \sim k$ and $g' \sim k'$ with $w \in H$ as before, this means that there exist elements

$$P = (g, u, v, k), \quad R = (g', u', v', k') \in (H \times H) \times G.$$  

If we define $x := v^{-1}w'w \in H$ we have that $s_G(k) = t_G(\Phi(x))$ and $t_G(k') = s_G(\Phi(x))$, so

$$Q = (\Phi(w), v, u', \Phi(x)) \in (H \times H) \times G$$

allows us to define the multiplication $m^H(P, Q, R) \in (H \times H) \times G$ which relates $g\Phi(w)g'$ with $k\Phi(x)k'$. Thus the multiplication in $G/(H \times H)$ is well defined. The quotient map $q : G \to G/(H \times H)$ is a morphism by definition and the groupoid structure in $G/(H \times H)$ is the only one that makes this map into a Lie groupoid morphism.

Let us recall that the orbit space of a proper Lie groupoid with trivial isotropy groups inherits a manifold structure in a canonical way [56].

**Lemma 3.9.** Suppose that the Lie groupoid morphism $\Phi : H \to G$ over $S$ is an immersion and suppose that $H$ is proper with trivial isotropy groups. Let $M := S[H$ be the orbit space of $H$ and let $q : S \to M$ be the projection. Then the orbit groupoid $G/(H \times H) \to M$ is a Lie groupoid that satisfies $q^1\text{Lie}(G/H \times H) \cong \text{Lie}(G)$.

**Proof.** We have seen that the comma double Lie groupoid $(H \times H) \times G \rightrightarrows G$ associated to $\Phi$ is a proper Lie groupoid with trivial isotropy groups. Lemma 3.8 implies that there is a groupoid structure on the orbit space of $(H \times H) \times G \rightrightarrows G$ over $M$. The map

$$(t^G, s^G) : (H \times H) \times G \to G \times G$$

is an injective proper immersion whose image is an equivalence relation $R$. If $G$ is Hausdorff, then $(t^G, s^G)$ is a closed embedding and so Godement’s Theorem [65] implies that the orbit space $\overline{G} := G/(H \times H)$ is a Hausdorff smooth manifold.

If $G$ is not Hausdorff it is not clear that the image of $(t^G, s^G)$ is an embedded submanifold so we have to use instead the following argument. Essentially, we shall produce a slice for the $H \times H$-action at every point in $G$ and we shall see that the collection of these slices defines a smooth structure on the orbit space.

The quotient map $Q : G \to \overline{G}$ induces an isomorphism of topological groupoids $G \cong q^*\overline{G}$ given by the map $g \mapsto (t_G(g), q(g), s_G(g))$ (which is an open continuous bijection and hence a homeomorphism). Since $q$ is a submersion, we can find a chart for every point in $S$ such that $q$ looks like a projection; let $V_i = U_i \times F$ be such charts for $i = 1, 2$ and suppose that $q|_{V_i}$ is identified with the projection to $U_i$. Then we have that $W := s_G^{-1}(V_1) \cap t_G^{-1}(V_2)$ is homeomorphic to

$$\{(\overline{x}(k), x, k, \overline{s}(k), y) \in V_1 \times \overline{G} \times V_2 | x, y \in F, k \in \overline{s}^{-1}(U_1) \cap \overline{t}^{-1}(U_2)\} \subset q^*\overline{G},$$

where $\overline{s}, \overline{t}$ denote the source and target map induced on $\overline{G}$. This means that $\overline{s}^{-1}(U_1) \cap \overline{t}^{-1}(U_2)$ is defined as the inverse image of a point under the submersion

$$(p_1, p_2) \circ (t, s) : W \to F \times F,$$

where $p_i : V_i \to F$ is the projection. Then the open subset $\overline{s}^{-1}(U_1) \cap \overline{t}^{-1}(U_2)$ of $\overline{G}$ inherits a smooth structure; we shall see that this smooth structure does not depend on the identification that we used. First of all, a change of submersion charts for $q$ would change the smooth structure induced on $\overline{s}^{-1}(U_1) \cap \overline{t}^{-1}(U_2)$ by a diffeomorphism so it is independent of the choice of submersion charts. Let us see that the smooth structure on $\overline{s}^{-1}(U_1) \cap \overline{t}^{-1}(U_2)$ does not depend on the points of the $q$-fibers that we picked in order identify it with a submanifold of $G$. 

9
Suppose that on an open set \( V'_1 \subset S \) such that \( q(V'_1) = q(V'_2) \cong U_1 \) there is another submersion chart which allows us to identify it with \( U_1 \times F' \). For every \( (a, b) \in F \times F' \) define a map 
\[
\chi : s^{-1}(pr_2^{-1}(a)) \cap t^{-1}(V_2) \to s^{-1}(pr_2^{-1}(b)) \cap t^{-1}(V_2)
\]
by \( \chi(g) = m(g, \phi(s(g), y)) \), where \( (s(g), y) \in V_1 \times V_1' \) is defined by the conditions \( q(s(g)) = q(y) \) and \( pr_2(y) = b \) for \( pr_2 : V_1' \to F' \) the projection. We have that \( \chi \) is a diffeomorphism which restricts to a diffeomorphism between the smooth structures induced on \( \mathfrak{G}^{-1}(U_1) \cap \mathfrak{G}^{-1}(U_2) \) as above. This implies that they do not depend on the choice of a point on the \( q \)-fiber. We proceed analogously with a different choice of \( V_2 \). Since \( W \) as above is diffeomorphic to \( F \times \mathfrak{G}^{-1}(U_1) \cap \mathfrak{G}^{-1}(U_2) \times F \), the restricted quotient map \( Q|_{W} : W \to \mathfrak{G}^{-1}(U_1) \cap \mathfrak{G}^{-1}(U_2) \) is a submersion. As a consequence, \( G \) inherits a (unique) smooth structure such that \( Q \) has the universal property of a quotient, see [63, Chapter III].

Let us check that \( \mathfrak{G} \)-fibers in \( G \) are Hausdorff. Take \( C = s_G^{-1}(x) \times s_G^{-1}(x) \subset G \times G \); then \( f = (t_G, s^G) : (t_G, s^G)^{-1}(C) \cong H \times S (s_G^{-1}(x)) \to C \) is also an injective proper immersion since \( C \) is closed. Since \( s_G^{-1}(x) \) is Hausdorff, \( f \) is a closed embedding and so Godement’s Theorem implies that the quotient space \( \mathfrak{G}^{-1}(q(x)) \) is a Hausdorff manifold as well. The source, target, unit and inversion maps descend to smooth maps between \( G \) and \( M = S/H \) thanks to the universal property of the quotient. In particular, we get that the induced source map \( \mathfrak{G} : G \to M \) makes the following diagram commute:

\[
\begin{array}{ccc}
G & \overset{Q}{\longrightarrow} & G \\
\downarrow{s_G} & & \downarrow{=} \\
S & \overset{q}{\longrightarrow} & M.
\end{array}
\]

By taking derivatives we get that \( TQ \circ Ts_G = T\mathfrak{G} \circ TQ \), since the left hand side is fiberwise surjective so is \( T\mathfrak{G} \) and hence \( \mathfrak{G} \) is a submersion.

By counting dimensions we get that \((Q, Q)|_{G \times S G} \) is also a submersion and hence \( m \) induces a smooth multiplication map on \( G \):

\[
\begin{array}{ccc}
G \times S G & \overset{s_G}{\longrightarrow} & G \\
\downarrow{(Q, Q)} & & \downarrow{Q} \\
G \times_M G & \overset{\mathfrak{G}}{\longrightarrow} & G.
\end{array}
\]

Therefore, \( G \) is a Lie groupoid.

Finally, \( Q : G \to G \) induces a morphism of Lie algebroids \( \text{Lie}(G) \to \text{Lie}(G) \) to which it is associated a surjective vector bundle map \( \text{Lie}(G) \to q^* \text{Lie}(G) \) with kernel \( \text{Lie}(H) \). Then the isomorphism \( q^* \text{Lie}(G) \cong \text{Lie}(G) \) follows from Lemma 3.5.

**Proof of Theorem** 3.2. Suppose that condition 2 holds. Then the result follows from applying Lemma 3.9 to \((H \to S) = (S \times M S \to S)\) since the isotropy groups of \( H \) are trivial. Lemma 3.9 also implies that \( q^* A \cong q^* \text{Lie}(G/(H \times H)) \) but by definition of the bracket in \( q^* A \) this implies that \( A \cong \text{Lie}(G/(H \times H)) \).

### 3.2 The case of reducible Dirac structures

As motivation for Theorem 1.1 (Theorem 3.12 below), let us consider the following situation.

**Example** 3.10. Let \( M \) be \( S^2 \times \mathbb{R} \) with the Poisson structure \( \pi \) given by multiplying the canonical area form \( \omega_0 \) on \( S^2 \) by a positive function \( f \in C^\infty(\mathbb{R}) \). If \( q : S \to M \) is \( p \times \text{id}_\mathbb{R} \), where \( p : S^3 \to S^2 \) is the Hopf fibration, then we have that \( L = q^*(T^* M) \) is always integrable while \( M \) does not have to be, see Proposition 3.10.
In this subsection we shall consider the following refinement of Theorem 3.2 that takes the existence of (pre)symplectic forms into account.

**Definition 3.11.** Let $G \Rightarrow S$ be a $q$-admissible integration of a Dirac structure $L$ on $S$, where $q : S \to M$ is a surjective submersion, and let $\Phi : S \times_M S \to G$ be the associated Lie groupoid morphism. If $G$ is endowed with a presymplectic form $\omega$ whose infinitesimal counterpart is provided by $L$ and is such that $\Phi^*\omega = 0$, then we say that $G$ is a $q$-admissible presymplectic integration of $L$.

This concept generalizes [13, Definition 2.9]. If the $q$-fibers are connected, then the submersion groupoid $S \times_M S \Rightarrow S$ is source-connected and hence the condition $\Phi^*\omega = 0$ is automatic [11].

**Theorem 3.12.** Let $q : S \to M$ be a surjective submersion and let $(M, \pi)$ be a Poisson manifold. Then the following are equivalent.

1. $(M, \pi)$ is integrable.
2. the pullback $L = \mathcal{B}_q(\text{graph}(\pi))$ admits a $q$-admissible presymplectic integration $G$.

Moreover, if $\Phi : S \times_M S \to G$ is the Lie groupoid morphism associated to a $q$-admissible presymplectic integration, then the associated Lie groupoid $G/\sim$ integrating $T^*M$ inherits a natural symplectic structure, where $\sim$ is the equivalence relation

$$g \sim \Phi(x)g\Phi(y)$$

for all compatible $g \in G$ and $x, y \in S \times_M S$.

**Definition 3.13.** Let $\mathcal{G} \Rightarrow P$ be a Lie groupoid and let $\omega \in \Omega^2(P)$ be closed. We say that $\omega$ is $\mathcal{G}$-basic if (1) $\ker \omega$ coincides with the tangent distribution to the $\mathcal{G}$-orbits in $P$ and (2) $\tau^*\omega = s^*\omega$.

If $\omega$ is a $\mathcal{G}$-basic 2-form, it follows from considering the representation of $\mathcal{G}$ on the normal bundle of a $\mathcal{G}$-orbit that it induces a symplectic form on the orbit space of $\mathcal{G}$ provided it is a smooth manifold.

**Proof of Theorem 3.12.** If $\mathcal{G} \Rightarrow M$ is a symplectic groupoid, then $q^!\mathcal{G}$ is a $q$-admissible presymplectic integration of $L$: we just take the pullback of the symplectic form on $\mathcal{G}$ along the morphism $q^!\mathcal{G} \to \mathcal{G}$.

Let us call $H = S \times_M S$. Suppose that $(G, \omega)$ is a $q$-admissible integration of $L$ and let $\Phi : H \to G$ be the Lie groupoid morphism which integrates the inclusion of Lie algebroids $\ker Tq \to L$. Then the orbits of the comma double Lie groupoid $(H \times H) \times G \Rightarrow G$ associated to $\Phi$ are tangent to the kernel of $\omega$. Therefore, the action of $H \times H$ on the map $(\tau, s) : G \to S \times S$ induces vectors tangent to the $H \times H$-orbits of the following form, $u^l - v^l$, where $u, v \in \ker Tq$. But these tangent vectors generate the kernel of $\omega$, see [13, Proposition 2.2]. On the other hand, since $\Phi^*\omega = 0$ and $\omega$ is multiplicative, it follows that $(\tau^G)^*\omega = (s^G)^*\omega$, where $s^G, \tau^G$ are the source and target maps of $(H \times H) \times G \Rightarrow G$. Therefore, $\omega$ is basic with respect to $(H \times H) \times G \Rightarrow G$ and so the quotient Lie groupoid $G/(H \times H)$ inherits a symplectic structure.

**Remark 3.14.** Let $L$ be an integrable Dirac structure on $S$ whose kernel $D = L \cap (TS \oplus 0)$ is of constant rank. Then the induced presymplectic form on $\mathcal{G}(L)$ is basic with respect to the comma double Lie groupoid $\mathcal{K} := (\mathcal{G}(D) \times \mathcal{G}(D)) \times \mathcal{G}(L) \Rightarrow \mathcal{G}(L)$ corresponding to the canonical morphism $\mathcal{G}(D) \to \mathcal{G}(L)$. In other words, $\mathcal{K}$ is a higher version of a 0-symplectic groupoid [37]. So $\mathcal{K}$ represents what we may call a “symplectic stack groupoid".
3.3 More on admissible integrations

It is natural to ask when does a pullback Lie algebroid (or pullback Dirac structure) admit a q-admissible integration. In some special situations we can give a partial answer to this question. We shall apply the following observations to Example 3.10 in Proposition 3.16 below.

Let $A$ be a Lie algebroid on $M$ and let $q : S \to M$ be a surjective submersion with connected fibers. Suppose that $q^!A$ is integrable and let $\Psi : \mathcal{G}(\ker Tq) \to \mathcal{G}(q^!A)$ be the groupoid morphism induced by the inclusion $\ker Tq \to q^!A$. Consider the canonical morphism $\chi : \mathcal{G}(\ker Tq) \to S \times_M S$. Let us denote $\mathcal{M} = \Psi(\ker \chi)$.

Proposition 3.15. Suppose that the subgroupoid $\mathcal{M} \subset \mathcal{G}(q^!A)$ is normal. Then $q^!A$ admits a $q$-admissible integration if and only if $\mathcal{M}$ is an embedded submanifold.

Proof. We proceed as in [9, Proposition 4.4]. If there is a morphism $\Phi : S \times_M S \to G$ integrating the inclusion $\ker Tq \to q^!A$, then we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{G}(\ker Tq) & \xrightarrow{\Psi} & \mathcal{G}(q^!A) \\
\downarrow{\chi} & & \downarrow{p} \\
S \times_M S & \xrightarrow{\Phi} & G,
\end{array}
$$

where $p$ is the Lie groupoid morphism which integrates the identity on $q^!A$. Since $p$ is a local diffeomorphism and $\dim \mathcal{M} = p^{-1}(\mathcal{u}_G(S)) = \dim S$, we get that $\mathcal{M} \subset p^{-1}(\mathcal{u}_G(S))$ is an embedded submanifold. Indeed, $\mathcal{M}$ is open in $p^{-1}(\mathcal{u}_G(S))$ which is a submanifold by transversality of $p$ to $\mathcal{u}_G(S)$. Conversely, if $\mathcal{M}$ is a submanifold, then Godement Theorem implies that $G := G/\mathcal{M}$ is a Lie groupoid and, since $S \times_M S \cong S$ is source-connected, there is a Lie groupoid morphism $S \times_M S \to G$ integrating the inclusion $\ker Tq \to q^!A$.

\[\square\]

3.4 Some immediate applications

We present three straightforward applications of our main result.

(1) We study the integrability of Example 3.10 in the next proposition; the following result has been obtained with other techniques in [23, 17]. Compared to those treatments our approach seems more elementary although it uses some properties of the van Est map [73, 20].

Proposition 3.16 ([17]). Let $M$ be $S^2 \times \mathbb{R}$ with the Poisson structure $\pi$ given by multiplying the canonical form $\omega_0$ on $S^2$ by a positive function $f \in C^\infty(\mathbb{R})$. Then $\pi$ is integrable if and only if $f$ is constant or if it does not have any critical points.

Proof. Let $S$ be $S^2 \times \mathbb{R}$ and let $q : S \to M$ be the Hopf fibration $p : S^3 \to S^2$ times the identity on $\mathbb{R}$. The Dirac structure $L = q^!(\text{graph}(\pi))$ is always integrable and even explicitly so, see [15, §8]. Consider the monodromy groupoid $\mathcal{G}(\mathcal{F})$ of the foliation $\mathcal{F}$ induced by the projection of $L$ on $TS$. We have that $\mathcal{G}(\mathcal{F})$ is isomorphic to the pair groupoid $S^3 \times S^3 \to S^3$ times the unit groupoid on $\mathbb{R}$. There is an action of $\mathcal{G}(\mathcal{F})$ on the conormal bundle $\nu^*$ of $\mathcal{F}$ which is trivial in this case and we can see that $\mathcal{G}(L)$ is isomorphic to the twisted action groupoid $\mathcal{G}(\mathcal{F}) \rtimes_c \nu^*$, where $c$ is a differentiable 2-cocycle defined as follows. There is a natural transformation $\Phi$ between differentiable cohomology and Lie algebroid cohomology called the van Est map $\mathcal{F}$ which may be defined at the cochain level by an explicit morphism of chain complexes $[20]$:

$$
\Phi : (C^\bullet(\mathcal{G}, E), \delta) \to (\Omega^\bullet(\text{Lie}(\mathcal{G}), E), d),
$$

where $\mathcal{G}$ is any Lie groupoid and $E$ a representation of $\mathcal{G}$. In our case, the leafwise 2-form $f q^*\omega_0 \in \Omega^2(\mathcal{F})$ induces a 2-form $\Omega^2(\mathcal{F}, \nu^*)$ as $\omega := d_{dR}(f q^*\omega_0) = df \wedge q^*\omega_0$, where $d_{dR}$ is the
full de Rham differential on $S$. The 2-cocycle $c \in C^2(\mathcal{G}(\mathcal{F}), \nu^*)$ is induced by the 2-form $\omega \in \Omega^2(\mathcal{F}, \nu^*)$ as follows. Let $c_0 \in C^2(S^3 \times S^3, \mathbb{R})$ be such that $\Phi(c_0) = \rho^* \omega_0$. Since any pair groupoid is proper, its differentiable cohomology vanishes in positive degree [20] and so there is $s \in C^1(S^3 \times S^3, \mathbb{R})$ such that $ds = c_0$. Hence $\eta := \Phi(s) \in \Omega^1(S^3)$ is such that $\nu \circ \eta = q^* \omega_0$. By the explicit formula of $\Phi$ we can then see that $\Phi(df \cdot c_0) = df \wedge q^* \omega_0$ and so we can take $c = df \cdot c_0$.

The twisted multiplication on $\mathcal{G}(\mathcal{F}) \ltimes_v \nu^*$ is defined as

$$m((g, u), (h, v)) = (gh, u + v + c(g, h));$$

for every $g, h \in \mathcal{G}(\mathcal{F})$ for which the multiplication is defined. The morphism $\Psi : \mathbb{R} \times S \to \mathcal{G}(\mathcal{F}) \ltimes_v \nu^*$ induced by Lie’s second theorem applied to the inclusion $\ker Tq \to L$ is defined then by the formula

$$(t, x, u) \mapsto \left( e^{t \mathcal{X}} x, x, u \frac{df}{du}(u) \left( \int_0^t \eta(X_{S^3}) \right)_{\theta} + s(e^{t \mathcal{X}} x, x) \right) \wedge du;$$

for all $(t, x, u) \in \mathbb{R} \times S^3 \times \mathbb{R} = \mathbb{R} \times S$, where $X_{S^3}$ is the vector field induced by the canonical generator of Lie($\mathbb{R}$). Let us note that the integral in the previous formula does not vanish for $t = 2\pi$ otherwise $\omega_0$ would be exact as well. Let us denote $\mathcal{M} := \Psi(2\pi \mathbb{Z} \times S) \subset \mathcal{G}(\mathcal{F}) \ltimes_v \mathbb{R}$. If $f$ is constant, then $\mathcal{M} \cong u(S) \to \mathcal{G}(\mathcal{F}) \ltimes_v \mathbb{R}$. If $f$ is non constant and has no critical points, then $\mathcal{M} \cong S^3 \times \text{graph}(df/du) \times \mathbb{R}$ is a submanifold of $\mathcal{G}(\mathcal{F}) \ltimes_v \mathbb{R} \cong S^3 \times S^3 \times \mathbb{R} \times \mathbb{R}$. On the other hand, if $f$ is non constant and has some critical point, then $\mathcal{M}$ fails to be a manifold. Indeed, $\mathcal{M}$ is homeomorphic to $S^3 \times \mathcal{X}$, where $\mathcal{X} \cong \{(u, n \frac{df}{du}(u)) \in \mathbb{R}^2 | (u, n) \in \mathbb{R} \times \mathbb{R}\}$ and the graphs of the functions $u \mapsto n \frac{df}{du}(u)$ do not coincide but have a common point where $df/du$ vanishes. Since the $S^1$-action on $S$ is Hamiltonian with respect to $L$ with constant moment map $S \to \mathbb{R}$ [8], we have that $\mathcal{M} \subset \mathcal{G}(L)$ is a normal subgroupoid, see [3, Proposition 4.2]. So the result follows from Proposition 3.10.

$\square$

(2) Coisotropic reduction appears frequently in symplectic and Poisson reduction. The following result is concerned with the integrability of a quotient obtained by this method. A corollary of this result is [32, Thm. 3.6].

**Proposition 3.17.** Let $C$ be a coisotropic submanifold of a Poisson manifold $P$. Suppose that $\pi_{T^0 C}$ is injective and the coisotropic reduction $\overline{C}$ of $C$ is a smooth manifold. Then $\overline{C}$ is integrable if and only if the inclusion of Lie algebroids $T^0 C \to \pi^!(T^0 C)^0$ is integrable by a Lie groupoid morphism with source $C \times_{\overline{C}} C \rightrightarrows C$.

$\square$

(3) As the last application of this section we discuss the weak Morita invariance of integrability.

**Definition 3.18.** [24] Let $A_i$ be Lie algebroids over $M_i$, $i = 1, 2$. Then $A_1$ and $A_2$ are weakly Morita equivalent if there is a manifold $S$ and surjective submersions $q_i : S \to M_i$ with simply-connected fibers such that the pullback Lie algebroids $q_i^* A_i$ are isomorphic.

**Proposition 3.19.** If two Lie algebroids are weakly Morita equivalent, then one is integrable if and only if the other one is; moreover, for every Lie groupoid which integrates one of them we can find a Lie groupoid which is Morita equivalent to it and integrates the other Lie algebroid.

**Proof.** Let $G_i$ be an integration of $A_i$. Lie’s second theorem implies that a weak Morita equivalence between $A_i$ Lie algebroids over $M_i$, $i = 1, 2$ given by surjective submersions $q_i : S \to M_i$ induces a morphism from $(S \times_{M_i} S \rightrightarrows S) \cong \mathcal{G} (\ker Tq_i)$ to $q_i^*(G_i)$ integrating the inclusion of Lie algebroids so Theorem 3.12 implies the result.

$\square$

**Remark 3.20.** As we saw in the proof of Lemma 3.9, if $\Phi : H = S \times_M S \to G$ integrates the inclusion $\ker Tq \to q^! A$ corresponding to a surjective submersion $q : S \to M$, then the $s$-fibers of
the quotient groupoid \( \overline{G} = G/(H \times H) \) are diffeomorphic to the quotients of the \( s \)-fibers in \( G \) by the restricted \( H \)-action. As a consequence, if the \( q \)-fibers are 1-connected, then \( G \) is source-simply-connected if and only if so is \( \overline{G} \), see \[34\] Proposition 7.7. So the proof of Proposition \[3.19\] implies that a weak Morita equivalence between integrable Lie algebroids induces a Morita equivalence \[30\] between their source-simply-connected integrations, see also \[70\].

A weak Morita equivalence, as proved in \[23\], preserves many Lie algebroid invariants, among which there are the monodromy groups which control the integrability of a Lie algebroid. However, it is not quite explicit in \[23\] whether the uniform discreteness of these monodromy groups (which implies integrability) is preserved by such an equivalence.

4 Vacant double Lie groupoids and integrability

Now we shall consider two particular situations in which Theorem \[5.2\] can be applied and that have been of special interest in the literature concerning the problem of integrability.

1. **What is the relation between the integrability of Poisson manifolds and the existence of complete symplectic realizations, see \[14\] Ch. 6?**

2. **Let \((S, \pi)\) be a \( q \)-Poisson manifold for a Lie quasi-bialgebroid \((A, \delta, \chi)\) and suppose that the orbit space \( S/A \) is smooth. When is the reduced Poisson structure on the quotient \( S/A \) integrable?**

Question 1 was answered in \[23\] thanks to the path space model for the source-simply-connected integration of a Lie algebroid: a Poisson manifold admits a complete symplectic realization if and only if it is integrable. Question 2 has been answered in some particular cases, mainly those that deal with the integrability of Poisson manifolds resulting from reduction by symmetries \[55, 31, 32\]. In this section we are going to revisit Question 1 recovering its answer from a different viewpoint. Our approach to this problem will also allow us to give a complete answer to Question 2 that we discuss in Section 5.

4.1 Vacant double Lie groupoids and their orbit spaces

The global counterpart of a vacant LA-groupoid is given by the following notion.

**Definition 4.1** \[49\]. A double Lie groupoid in which the double source map is a diffeomorphism is called **vacant**.

Vacant double Lie groupoids are useful for us for the following reason. Let \( \mathcal{G} \) be a vacant double Lie groupoid with sides \( H, K \) over \( S \). There is a Lie groupoid structure \( \mathcal{G} \to S \) called **diagonal** that is given as follows. The source, target, unit and inversion map are the composition of the corresponding maps in \( H \) and \( K \):

\[
\begin{align*}
s^o &= s_H \circ s^K, & \quad t^o &= t_H \circ t^K = t_K \circ t^K, & \quad u^o &= u^H \circ u^K, & \quad i^o &= i^H \circ i^K = i^K \circ i^H. \\
\end{align*}
\]

Suppose that \( s_K \circ s^K(x) = t_K \circ t^K(y) \). The double source map condition implies that each element in \( \mathcal{G} \) is determined by any two consecutive sides of it, so there are unique \( u, v \in \mathcal{G} \) such that

\[
\begin{align*}
s^H(u) &= t^K(v), & \quad s^K(u) &= t^K(v), & \quad s^K(x) &= t^K(v), & \quad s^K(v) &= t^K(y), & & \quad s^K(v) &= t^K(y).
\end{align*}
\]

The multiplication is defined as \( \mathfrak{m}(x, y) = m^H(m^K(x, v), m^K(u, y)) \). With this structure, \( H, K \) are Lie subgroupoids of \( \mathcal{G} \) such that the multiplication map \( H \times_S K \to \mathcal{G} \) is a diffeomorphism. Notice that \((s^o)^{-1}(p)\) is Hausdorff for every \( p \in S \) since it is a subspace of \((s^K)^{-1}(p) \times (s^K)^{-1}(p)\).
Remark 4.2 ([49]). Equivalently, if $G$ is a Lie groupoid over $S$ and $H, K$ are Lie subgroupoids such that the multiplication map $H \times_S K \to G$ is a diffeomorphism, then $G$ is a vacant double Lie groupoid with sides $H$ and $K$. In fact, each element in $G$ can be written as $m(h, i) = m(t, \theta)$ for some $h, \theta \in H$, $t \in K$ so this defines action groupoids $H \times_S K \to K$ and $H \times_S K \to H$ for the actions $(h, i) \mapsto t$, $(h, i) \mapsto \theta$ respectively. These structures make $G$ with sides $H, K$ into a vacant double Lie groupoid over $S$.

**Proposition 4.3.** Let $B \rightrightarrows B$ be an LA-subgroupoid of $TK \rightrightarrows TS$ which is vacant and let $G$ and $H$ be the monodromy groupoids of $B$ and $B$ respectively. If the source map of $K$ restricts to a covering map between every compatible pair of $B$ and $B$-orbits, then $G$ is a vacant double Lie groupoid with sides $H$ and $K$.

**Proof.** Suppose that $s$ restricted to every $B$-orbit is a covering map. Take $[a], [b] \in G$ such that $s(a)$ and $t(b)$ are homotopic relative to their endpoints by means of $h$. Then we can lift $h$ to a homotopy from $a$ to $a'$ where $a'$ is pointwise composable with $b$, so we can define $m([a], [b])$ as the class of $t \mapsto m(a'(t), b(t))$. This multiplication over $H$ satisfies the interchange law with respect to $G \rightrightarrows K$. On the other hand, the homotopy lifting property of the leaves implies that the double source map $(s^H, s^K) : G \to H \times_S K$ is a diffeomorphism.

The next example, taken from [7], is a vacant multiplicative foliation in which the associated monodromy groupoids do not admit a double Lie groupoid structure. This shows that the covering map condition in Proposition 4.3 cannot be dropped.

**Example 4.4.** Take a free $\mathbb{R}^2$-action on a manifold $P$ such that there are at least three different orbits $O_i$. Let us choose $x_i \in O_i$ for $i = 1, 2, 3$ and take $v \in \mathbb{R}^2$ such that $|v| > 1$. Let us define the following manifold:

$$S = P - \{(v \cdot x_1) \cup \{z \cdot x_3 | z \in S^1\}).$$

We have a residual infinitesimal $\mathbb{R}^2$-action on $S$ which lifts to an $\mathbb{R}^2$-action by multiplicative vector fields on the pair groupoid $S \times S \rightrightarrows S$: $u_{S,S} = (u_S, u_S)$ for all $u \in \mathbb{R}^2$. These vector fields generate a distribution $B$ which is a subgroupoid of the tangent groupoid $T(S \times S) \rightrightarrows TS$ over $B := BS$ and hence it constitutes a vacant LA-groupoid.

The leaf of $B$ through $(x_1, x_2)$ is diffeomorphic to $\mathbb{R}^2$ minus a point. On the other hand, the leaves through $(x_1, x_3)$ and $(x_2, x_3)$ are diffeomorphic to a disk. Let us denote by $O_{ij}$ the leaf through the point $(x_i, x_j)$. Take a non trivial homotopy class $a \in \pi_1(O_{12}, (x_1, x_2)) \cong \mathbb{Z}$. Then $s(a) \in \pi_1(O_2) = 1$ is the trivial class. As a consequence, $a$ and the constant path based on $(x_2, x_3)$ should be composable. Since the isotropy group $\pi_1(O_{13}, (x_1, x_3)) = G(B)_{(x_1, x_3)} \rightrightarrows G(B)$ is trivial, there is no element in $G(B)_{(x_1, x_3)}$ which projects to $t(a) \in G(B)_{x_1} = \pi_1(O_1 - \{v \cdot x_1\})$ which is non trivial. It follows that $G(B)$ carries no groupoid structure over $G(B)$ which is compatible with the structure maps of $S \times S \rightrightarrows S$.

Now we shall see that, under certain conditions, the leaf space of a vacant multiplicative foliation which is integrable by a vacant double Lie groupoid inherits a (topological) groupoid structure.

**Proposition 4.5.** Let $G$ be a vacant double Lie groupoid with sides $H$ and $K$ over $S$. Then the following statements hold.

1. Consider the inclusions $H \to G$, $K \to G$ with respect to the diagonal structure $G \rightrightarrows S$ and the associated comma double Lie groupoids $(H \times H) \times G$, $(K \times K) \times G$. Then the orbit spaces of $(H \times H) \times G \rightrightarrows G$ and $(K \times K) \times G \rightrightarrows G$ are homeomorphic to the orbit spaces of $G \rightrightarrows K$ and $G \rightrightarrows H$ respectively.

2. Moreover, if the image of $\bigcup_{p \in S} H_p$ in $G \rightrightarrows M$ is a normal subgroupoid, then $K|G$ inherits a unique groupoid structure such that the quotient map $K \to K|G$ is a morphism and $K|G$ is isomorphic to $G/(H \times H)$ as a topological groupoid.
Proof. Let us denote the diagonal multiplication by \( m^s(x, y) = x \circ y \). Since \( G \) is a vacant, we can represent \( G \rightrightarrows K \) and \( G \rightrightarrows H \) as action groupoids \( K \times_S H \rightrightarrows K \) and \( H \times_S K \rightrightarrows H \), where the actions are given by the diffeomorphisms \( G \cong K \times_S H \cong H \times_S K \), i.e., if \((h, k) \in H \times_S K\) then \( h \circ k = h'' \circ h'' \) where \( h'' \in K \), \( h'' \in H \) are by definition the corresponding results of the action maps.

Define

\[
\psi : G/(H \times H) \to K/[G], \quad \psi([g]) = [\tau^K(g)], \quad \phi : K/G \to G/(H \times H), \quad \phi([k]) = [\mu^K(k)].
\]

Suppose that \( h, h' \in H \) and \( g \in G \) are such that \( h \circ g \circ h' \) is defined. Let us write \( g = a \circ b \) where \((a, b) \in K \times_S H\). Then \( g \circ h \circ h' \) are in the same \((H \times H) \times G\)-orbit but \( \tau^K(h \circ g \circ h') = h' a \) which is in the same \( G\)-orbit of \( \tau^K(g) = a \) so \( \psi \) is well defined.

Now \( \psi \phi([k]) = [k] \) and, on the other hand, \( \phi \psi([g]) = [\mu^K(\tau^K(g))] \). But \( s^H(g) \in H \) is such that \( u^K(\tau^K(g)) \circ s^H(g) = g \) and so \( \phi = \psi^{-1} \). The other statement is analogous.

If \( \bigcup_{p \in S} H_p \) in \( G \rightrightarrows S \) is a normal subgroupoid, then \( G/(H \times H) \) inherits a groupoid structure from \( G \) (Lemma \[33\]). Since \( u^K : K \to G \) is a groupoid morphism with respect to the diagonal structure and \( \phi \) is induced by \( u^K \), the bijection \( \phi : K/G \to G/(H \times H) \) also shows that if \([k], [k'] \in K/G\) are such that there is \( h \in H \) with \( s^H(h) = \tau^K(k') \) and \( \tau^K(h) = s^K(k) \), then \([u^K(k)] \) and \([u^K(k')] \) are composable in \( G/(H \times H) \) and so the multiplication in \( K/[G] \):

\[
m([k], [k']) := \phi^{-1}m([u^K(k)], [u^K(k')])
\]

is well defined, where \( m \) is the multiplication in \( G/(H \times H) \). And finally, if \( k \) and \( k' \) are composable in \( K \) then

\[
m_K(k, k') = \phi^{-1}m([u^K(k)], [u^K(k')])
\]

so the last statement also holds. \( \square \)

Remark 4.6. A vacant multiplicative foliation arises whenever a Lie group (or groupoid) acts on a Lie groupoid: \([56\), Lemma 5.9\] says that if we have a Lie groupoid action of \( G \rightrightarrows P \) on a map \( \mu := J \circ s = J \circ t : K \to P \) from another Lie groupoid \( K \rightrightarrows S \) which commutes with the structure maps, then the orbit space of this action is also a Lie groupoid if the action is principal on \( J : S \to P \). This result can be deduced from the previous lemma applied to the vacant double Lie groupoid

\[
\begin{array}{ccc}
G \times_P K & \longrightarrow & K \\
\uparrow & & \uparrow \\
G \times_P S & \longrightarrow & S,
\end{array}
\]

where the top and bottom structures are action groupoids and \( G \times_P K \rightrightarrows G \times_P S \) is the fiber product of the unit groupoid structure on \( G \) and \( K \rightrightarrows S \) along the maps \( s : G \to P \) and \( \mu : K \to P \).

In this situation, the diagonal groupoid structure integrates the semi-direct product structure on \( J^*\operatorname{Lie}(G) \times \operatorname{Lie}(K) \) \([57\]

Putting together Proposition \[43\] and Proposition \[45\] we obtain the following result, which shall be used in Section \[5\]. See also \[32\] for related results.

**Theorem 4.7.** Let \( B \rightrightarrows B \) be a vacant multiplicative foliation over a Lie groupoid \( K \rightrightarrows S \). Suppose that the source map \( s : K \to S \) induces a covering map between every \( B\)-orbit and the corresponding \( B\)-orbit. If the foliation induced by \( B \) is simple and its leaves are 1-connected, then the leaf space of \( B \) is a Lie groupoid.

**Proof.** Proposition \[43\] implies that the monodromy groupoids of \( B \rightrightarrows B \) form a vacant double Lie groupoid over \( K \rightrightarrows S \). Let \( G \) and \( H \) be the monodromy groupoids of \( B \) and \( B \) respectively. If the \( B\)-leaves are simply connected and the induced foliation is simple, we have that \( H \) is
isomorphic to a submersion groupoid. The unit embedding $u^V : \mathcal{H} \to \mathcal{G}$ is also a Lie groupoid morphism with respect to the diagonal groupoid structure. Then, by taking $\Phi = u^V$ in Theorem 3.2, we get that the orbit space of the comma double Lie groupoid $(\mathcal{H} \times \mathcal{H}) \ltimes \mathcal{G} \rightrightarrows \mathcal{G}$ is a Lie groupoid. On the other hand, Proposition 1.3 implies that the orbit space of $\mathcal{G}$ over $K$ is isomorphic as a topological groupoid to the orbit space of $(\mathcal{H} \times \mathcal{H}) \ltimes \mathcal{G} \rightrightarrows \mathcal{G}$ so the result holds.

4.2 Complete Lie algebroid actions and integrability

As an application of the arguments developed in this section we recover the following classical result which answers Question 1 above.

**Theorem 4.8 (23, 21).** A Lie algebroid $A \to M$ is integrable if it admits a complete action given by an injective vector bundle map $\rho : q^*A \to TS$, where $q : S \to M$ is a surjective submersion. Moreover, if the foliation induced by the $A$-action has no vanishing cycles, then $A$ is integrable by a Hausdorff Lie groupoid.

Let us stress that in the following proof we shall only use monodromy groupoids of foliations so we will not need to consider infinite-dimensional manifolds as in [22, 23]. Let us recall that a multiplicative vector field on a Lie groupoid is a vector field which is a Lie groupoid automorphism [50].

**Lemma 4.9.** Let $\mathcal{B} \rightrightarrows B$ be a vacant multiplicative foliation on a Lie groupoid $K \rightrightarrows S$. Suppose that there is a subspace $\mathcal{W} \subset \Gamma(\mathcal{B}) \subset \mathfrak{X}(K)$ consisting of complete multiplicative vector fields with the following property: for every $v \in \mathcal{B}_p$ and every $p \in S$, there exists $\mathcal{X} \in \mathcal{W}$ such that $\mathcal{X}_p = v$. Then the $\mathcal{B}$-orbits are coverings of the corresponding $B$-orbits by means of $\mathfrak{s} : K \to S$.

**Proof.** Take $p \in S$ and an ordered basis $\{e_i\}_{i=1...k} \subset B_p$. Let $\mathcal{X}_i \in \mathcal{W}$ be vector fields such that $\mathcal{X}_i(p) = e_i$ for all $i$. Define a map $\Phi : \mathbb{R}^k \to S$ as $\Phi(t_1, \ldots, t_k) = \Phi_{\mathcal{X}_i}^{t_i} \circ \cdots \circ \Phi_{\mathcal{X}_1}^{t_1}(p)$, where $\Phi_{\mathcal{X}_i}^{t_i}$ is the flow of $\mathcal{X}_i$. Then $\Phi$ is an injective immersion on a neighborhood $V \subset \mathbb{R}^k$ of the origin whose image $U = \Phi(V)$ is embedded in $S$ and is an open set in the $B$-orbit $\mathcal{O}$ that passes through $p$. For every $x \in K$ such that $\mathfrak{s}(x) = p$ we can define a neighborhood of $x$ in the $B$-orbit that passes through $x$ as follows. Take $U_x = \{\Phi_{\mathcal{X}_i}^{t_i} \circ \cdots \circ \Phi_{\mathcal{X}_1}^{t_1}(x) : (t_1, \ldots, t_k) \in V\}$. Then $\mathfrak{s} : U_x \to U$ is a diffeomorphism. Let $x, y \in K$ be such that $\mathfrak{s}(x) = \mathfrak{s}(y) = p$. Since each $\Phi_{\mathcal{X}_i}^{t_1}$ is an automorphism of $K$, we have that $U_x \cap \tilde{\mathcal{O}}_y = \emptyset$ if $x \neq y$. Then $\mathfrak{s}^{-1}(U) \cap \tilde{\mathcal{O}}_y \cong \bigsqcup_{x \mathfrak{s}^{-1}(p) \in \partial U[a]}$, where $\tilde{\mathcal{O}}$ is the $B$-orbit that passes through $x$. Therefore, $\mathfrak{s} : \tilde{\mathcal{O}} \to \mathcal{O}$ is a covering map.

**Proof of Theorem 4.8**. Take a complete action of $A$ on $q$ which induces an injective vector bundle map $\rho : q^*A \to TS$, where $q : S \to M$ is a surjective submersion. There is an induced action of $A$ on $q \circ \mathfrak{s}$, where $\mathfrak{s}$ is the source map of the submersion groupoid $K := S \times_M S \rightrightarrows S$. In fact, simply define the action map $\Gamma(A) \to \mathfrak{X}(K)$ as follows: $X \mapsto (\rho(X), \rho(X))$ for all $X \in \Gamma(A)$. This action induces a vacant multiplicative foliation $\mathcal{B} \rightrightarrows B$ over $S \times_M S \rightrightarrows S$, where $B := q^*A$ is the induced action Lie algebroid. Let $\mathcal{G}$ be the monodromy groupoid of $\mathcal{B}$ and let $H$ be the monodromy groupoid of $B$. In order to apply Proposition 1.3 we have to see that the $\mathcal{B}$-orbits cover the corresponding $B$-orbits by means of $\mathfrak{s}$. This follows from Lemma 1.9 by putting $\mathcal{W} = \{(\rho(X), \rho(X)) \in \mathfrak{X}(K) \mid \forall X \in \Gamma(A) \text{ of compact support}\}$.

Therefore, Proposition 1.3 implies that $\mathcal{G}$ is a vacant double Lie groupoid with sides $H$ and $K$. The diagonal structure $\mathcal{G} \rightrightarrows S$ integrates a Lie algebroid structure on the direct sum of $\ker Tq$ and $B$, denoted by $\ker Tq \ltimes B$, see [72]. But the canonical projection $\phi : \ker Tq \ltimes B \to A$ is a surjective Lie algebroid morphism with kernel $\ker Tq$. If we take a vector field $U$ tangent to
the $q$-fibers and $X \in \Gamma(A) \to \Gamma(B) \cong C^\infty(S) \otimes_{C^\infty(M)} \Gamma(A)$, we have that $[U, \rho(X)] \in \Gamma(\ker Tq)$. As a consequence, the right-invariant extensions $U^r, X^r$ in the diagonal groupoid $\mathcal{G} \Rightarrow S$ satisfy that $[U^r, X^r] = V^r$ for some $V \in \Gamma(\ker Tq)$. So the brackets of $\phi$-related sections in $\Gamma(\ker Tq \Rightarrow B)$ and in $\Gamma(A)$ are also $\phi$-related. Then [36, Proposition 1.5] implies that $\phi$ is a Lie algebra morphism. As a consequence, Lemma 3.5 implies that $\ker Tq \Rightarrow B \cong q^* A$. Since the inclusion $K = S \times_M S \rightarrow \mathcal{G}$ integrates the inclusion of Lie algebroids $\ker Tq \rightarrow q^* A$, Theorem 3.2 implies that $A$ is integrable.

If the foliation induced by the $A$-action has no vanishing cycles, then $H$ is Hausdorff. Since $\mathcal{G}$ is diffeomorphic to $H \times_S K$ by the vacant condition, it is Hausdorff as well. As we saw in the proof of Lemma 3.9, this implies that the orbit space of the comma double Lie groupoid $(K \times K) \ltimes \mathcal{G} \Rightarrow \mathcal{G}$ (which by Proposition 4.5 is isomorphic to the orbit space of $\mathcal{G} \Rightarrow H$) is then a Hausdorff integration of $A$.

**Remark 4.10.** A Poisson morphism $q : S \to M$ can be called complete if the $T^* M$-action it induces on $S$ is complete. If $S$ is a symplectic manifold and $q$ is a complete Poisson map which is a surjective submersion, then $q$ is called a complete symplectic realization. So the previous theorem implies that a Poisson manifold is integrable if it admits a complete symplectic realization, see [23] for the original proof of this fact.

## 5 Integrability of quotients of quasi-Poisson manifolds

The integrability of Poisson manifolds obtained by reduction has been studied in [55, 31, 67, 32]. Here we study the integrability of reduced Poisson structures in the more general framework provided by quasi-Poisson manifolds [61].

### 5.1 Quasi-Poisson manifolds

**Definition 5.1 ([63]).** A **Lie quasi-bialgebroid** is a Lie algebroid $A$ over $M$ endowed with a degree one derivation $\delta : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k+1} A)$ for all $k \in \mathbb{N}$ which is a derivation of the bracket on $A$,

$$\delta([u, v]) = [\delta(u), v] + (-1)^{p-1} [u, \delta(v)]$$

for all $u \in \Gamma(\wedge^p A), v \in \Gamma(\wedge^q A)$ and satisfies $\delta^2 = [\chi, \cdot]$, where $\chi \in \Gamma(\wedge^3 A)$ is such that $\delta(\chi) = 0$.

Since $\delta$ is a derivation, it is determined by its restriction to degree 0 and degree 1 where it is given respectively by a vector bundle map $a_\delta : A^* \to TM$ and a map $\Gamma(A) \to \Gamma(\wedge^2 A)$ called the **cobracket**.

**Example 5.2.** A **Lie bialgebroid** $(A, A^*)$ is a Lie quasi-bialgebroid $(A, \delta, \chi)$ in which $\chi = 0$ and hence the differential $\delta$ satisfies $\delta^2 = 0$ [52]. Since the dual of a differential which squares to zero is a Lie bracket, a Lie bialgebroid consists of a pair of Lie algebroid structures on $A$ and $A^*$ which are compatible in a suitable sense. A **Lie bialgebra** $(g, g^*)$ is a Lie bialgebroid over a point [27].

**Example 5.3.** Let $L$ be a Dirac structure on $M$. Then the choice of a Lagrangian subbundle $E \subset TM$ such that $TM = L \oplus E$ endows $L$ with a Lie quasi-bialgebroid structure.

**Definition 5.4 ([61]).** A **quasi-Poisson manifold** (or a **Hamiltonian space**) for a Lie quasi-bialgebroid $(A, \delta, \chi)$ on $M$ is given by an action $\rho : J^* A \to TS$ of $A$ on a smooth map $J : S \to M$ and a

---

2R. L. Fernandes pointed out to the author that this is an apparently stronger version of the original definition of complete Poisson map given in [13].
bivector field $\pi$ on $S$ such that:

\[ \frac{1}{2}[\pi, \pi] = \rho(\chi) \]
\[ \mathcal{L}_{\rho(U)}\pi = \rho(\delta(U)), \quad \forall U \in \Gamma(A), \]
\[ \pi^*J^* = \rho \circ a^*, \]

where $a_* : A^* \to TS$ is the component of $\delta$ in degree zero as before.

The main feature of this notion is the following: if $(S, \pi)$ is a quasi-Poisson manifold for $(A, \delta, \chi)$ and the $A$-action induces a simple foliation, then its leaf space $\overline{S}$ inherits a unique Poisson structure $\mathfrak{p}$ such that $\pi$ and $\mathfrak{p}$ are q-related, where $q : S \to \overline{S}$ is the projection.

**Example 5.5.** An infinitesimal Poisson action $\rho : g \to TS$ of the tangent Lie bialgebra $(g, g^*)$ of a Poisson group $[27, 47]$ can be expressed by saying that $(S, \pi, \rho)$ is a Hamiltonian space for $(g, \delta, 0)$, where $\delta$ is the differential dual to the bracket on $g^*$. If the foliation induced by the $g$-action is simple, then the quotient $S/g$ is a Poisson manifold. This is an infinitesimal version of the fact that the quotient of a free and proper Poisson action is again a Poisson manifold $[64]$.

**Example 5.6.** The original q-Poisson manifolds, which were introduced in $[4]$, are a special case of Definition 5.4. Consider a Lie algebra $g$ endowed with an $\text{Ad}$-invariant symmetric nondegenerate bilinear form $B$. Then there is a Lie quasi-bialgebra structure $(g, \delta, \chi)$ on $g$ which depends on $B$ and is determined by the splitting of $g \equiv g$ as the sum of the diagonal Lie subalgebra and the anti-diagonal $[2, 4]$. The q-Poisson manifolds corresponding to $(g, \delta, \chi)$ shall be called q-Poisson g-manifolds in accordance to $[4, 44]$.

### 5.2 Matched pairs of Lie algebroids and vacant multiplicative foliations

One of the observations that allowed us to reduce Question 1 of §3 to an application of Theorem 3.2 is the following: given an action of a Lie algebroid $A$ over $M$ on a surjective submersion $q : S \to M$, then the pullback Lie algebroid $q^*E$ splits as a direct sum of the action Lie algebroid $B := q^*A$ and the Lie subalgebroid $\ker Tq \to q^!A$. In order to answer Question 2 of §3 we shall use more general examples of these “split” Lie algebroids.

**Definition 5.7** ($[58]$). A pair of Lie algebroids $(B, C)$ over $S$ is a matched pair if their direct sum, denoted $B \ltimes C$, has a Lie algebroid structure for which $B$ and $C$ are Lie subalgebroids.

Equivalently, a pair of Lie algebroids $(B, C)$ constitutes a matched pair if there is a flat $C$-connection $D^C$ on $B$ and a flat $B$-connection $D^B$ on $C$ such that

\[ D^C_X[U, V] = [D^C_X U, V] + [U, D^C_X V] - D^C_{D^B_X U} V + D^C_{D^B_X V} U \]  
(3)
\[ D^B_U[X, Y] = [D^B_U X, Y] + [X, D^B_U Y] - D^B_{D^C_U X} Y + D^B_{D^C_U Y} X \]  
(4)
\[ a(D^C_X U) - a(D^B_X U) = [a(X), a(U)], \]  
(5)

for all $X, Y \in \Gamma(C)$ and for all $U, V \in \Gamma(B)$.

Matched pairs of Lie algebroids appear as the infinitesimal counterpart of vacant LA-groupoids $[49, 58]$. Let $(B, C)$ be a matched pair of Lie algebroids in which $B$ is a distribution on $S$. We have that $B$ and the flat $B$-connection $D^B$ constitute an infinitesimal multiplicative foliation (IM-foliation) $[10, 35]$ and so $\overline{\mathcal{S}}$. Thm. 7.2 implies that, if $C$ is integrable, then $B$ induces a multiplicative involutive distribution $B \subset TG(C)$ with the same rank of $B$, thus making $B \Rightarrow B$ into a vacant LA-groupoid, see also $[40]$.

We can give an infinitesimal condition that implies that the monodromy groupoids of $B \Rightarrow B$ fit into a vacant double Lie groupoid.
Definition 5.8. Let \((B,C)\) be a matched pair of Lie algebroids over \(S\). We say that \((B,C)\) is complete if there exists a subspace \(V \subset \Gamma(B)\) consisting of \(D^C\)-flat sections such that: (1) the vector field \(a(X)\) is complete for all \(X \in V\) and (2) for every \(p \in S\) and every \(v \in B_p\), there is \(X \in V\) with \(X_p = v\).

Proposition 5.9. Let \((B,C)\) be a complete matched pair of Lie algebroids on \(S\) in which \(B\) is a distribution on \(S\). Suppose that \(C\) is integrable and let \(B \rightleftharpoons B\) be the vacant multiplicative foliation induced by \((B,D^B)\) on \(G(C) \rightarrow S\). Then the monodromy groupoids of \(B \rightleftharpoons B\) constitute a vacant double Lie groupoid.

The proof of this Proposition is based on the following lemma.

Lemma 5.10. Let \((B,C)\) be a complete matched pair of Lie algebroids over \(S\) and let \(V \subset \Gamma(B)\) the associated subspace of \(C\)-flat sections. Suppose that \(C\) is integrable and \(B\) is a distribution on \(S\). Then we can lift the elements of \(V\) to complete multiplicative vector fields on \(G(C)\) which span the distribution \(B\) integrating the IM-foliation \((B,D^B)\).

Proof. Take \(X \in V\). Equations \(1\) and \(5\) imply that \((D^B_X, X)\) is a derivation of \(C\) and hence it lifts to a multiplicative vector field \(\bar{X}\) on \(G(C)\), see \([54, \text{Thm. 4.5}]\). So the distribution \(\mathcal{D} \subset T\bar{G}(C)\) that the lifted vector fields \(\bar{X}\) generate is multiplicative and has rank equal to the rank of \(B\). Now we shall see that every \(\bar{X}\) is complete. Consider the parallel transport \(P^C_x\) over the flow of \(X \in V\); \(P^C_x\) is a Lie algebroid automorphism for every \(s \in \mathbb{R}\). Let \(\Psi^s : G(C) \rightarrow G(C)\) be the groupoid automorphism induced by Lie’s second Theorem applied to \(P^C_x\). We have that the infinitesimal generator of the 1-parameter family of groupoid automorphisms \((\Psi^s)_{s \in \mathbb{R}}\) is a multiplicative vector field which induces the derivation \(D^B_X\) on \(C\), so by uniqueness of the lift we get that \(\left.\frac{d}{ds}\right|_{s=0} \Psi^s(x) = \bar{X}x\) for all \(x \in G(C)\). By construction, the flat connection induced by \(\mathcal{D}\) on \(C\) \([40]\) coincides with \(D^B\) and so \(\mathcal{D} = B\) by uniqueness of the integration of an IM-foliation.

Proof of Proposition \([5.9]\). Let \(B \rightleftharpoons B\) be the vacant multiplicative foliation on \(G(C) \rightarrow S\) associated to a complete matched pair of Lie algebroids \((B,C)\) over \(S\). Lemma \([5.10]\) together with Lemma \([4.9]\) imply that \(s : G(C) \rightarrow S\) induces a covering map between every pair of compatible \(B\) and \(B\)-orbits. Therefore, Proposition \([4.5]\) implies that the monodromy groupoid \(G(B)\) is a vacant double Lie groupoid with sides \(G(C)\) and \(G(B)\).

5.3 The integrability criterion

Given \((S, \pi)\) a Hamiltonian space for a Lie quasi-bialgebroid \((A,\delta,\chi)\) on \(M\) with moment map \(J : S \rightarrow M\) and action map \(\rho : J^*A \rightarrow TS\) where \(\rho\) is of constant rank, we can define a new Lie algebroid \(C \rightarrow T^*M\) as follows. Take \(C = \rho(J^*A)^0\), the annihilator of the distribution induced by \(A\); the anchor \(C \rightarrow TS\) is defined as \(\alpha \mapsto \pi^A(\alpha)\) and the Lie bracket is

\[
[\alpha,\beta] := \mathcal{L}_{\pi^A(\alpha)}(\beta) - d(i_{\pi^A(\beta)}\alpha),
\]

for all \(\alpha,\beta \in \Gamma(C)\). It turns out that if \(J\) is a surjective submersion, a q-Poisson manifold induces an action of \(A\) on \(C\) in the following sense.

Let \(J : S \rightarrow M\) be a surjective submersion, let \(A\) be a Lie algebroid over \(M\) and let \(C\) be a Lie algebroid over \(S\) such that \(Tq \circ a = 0\), where \(a\) is the anchor of \(C\).

Definition 5.11 \([33,57]\). An action of \(A\) on a Lie algebroid \(C\) over \(S\) is an \(A\)-action on \(J\) and a Lie algebra morphism \(\Gamma(A) \rightarrow \text{Der}(C)\) which is \(C^\infty(M)\)-linear, where \(\text{Der}(C)\) is the space of derivations of \(C\).

Lemma 5.12. Let \((S, \pi)\) be a Hamiltonian space for a Lie quasi-bialgebroid \((A,\delta,\chi)\) on \(M\) with moment map \(J : S \rightarrow M\). If \(J\) is a surjective submersion, then the map given by \(U \mapsto \mathcal{L}_{\rho(U)}\) for all \(U \in \Gamma(A)\) defines an infinitesimal action of \(A\) on \(C\).
Proof. The equation $\pi^4 J^* = \rho \circ a^*_x$ implies that $TJ \circ \pi^4 : C \to TM$ is the zero map. Take $U \in \Gamma(A)$, $\pi^4(\alpha) \oplus \alpha \in \Gamma(C)$ and let $\beta \in \Omega^1(S)$ be arbitrary. Then we have that

$$\langle \beta, \mathcal{L}_{\rho(U)}(\pi^4(\alpha)) \rangle = \mathcal{L}_{\rho(U)}(\langle \beta, \pi^4(\alpha) \rangle) - \langle \mathcal{L}_{\rho(U)}(\beta), \pi^4(\alpha) \rangle = \langle \rho(\delta(U)), \alpha \wedge \beta \rangle + \langle \beta, \pi^4(\mathcal{L}_{\rho(U)}(\alpha)) \rangle,$$

where we used the identity $\mathcal{L}_{\rho(U)} \pi = \rho(\delta(U))$ and the fact that

$$\mathcal{L}_{\rho(U)}(\pi(\alpha, \beta)) = (\mathcal{L}_{\rho(U)} \pi)(\alpha, \beta) + \pi(\mathcal{L}_{\rho(U)} \alpha, \beta) + \pi(\alpha, \mathcal{L}_{\rho(U)} \beta).$$

But $\langle \rho(\delta(U)), \alpha \wedge \beta \rangle = 0$ since $\alpha$ lies in the annihilator of $\rho(A)$. Then $\mathcal{L}_{\rho(U)} \pi^4(\alpha) = \pi^4(\mathcal{L}_{\rho(U)}(\alpha))$ and, as a consequence, $\mathcal{L}_{\rho(U)}(\pi^4(\alpha) \oplus \alpha) \in \Gamma(C)$. Since $\mathcal{L}_{\rho(U)}(\alpha) = f \mathcal{L}_{\rho(U)}(\alpha)$ by Cartan’s formula, the map $\psi$ is $C^\infty(M)$-linear and so we are done. \qed

The global counterpart of Definition 5.11 is given by the following concept. Let $G \to M$ be a Lie groupoid acting on a surjective submersion $J : S \to M$ and let $C$ be a Lie algebroid over $S$ such that $TJ \circ \mathfrak{a} = 0$. In this situation we have an action of $C$ on the projection $G \times_M S \to S$ given by $X \mapsto (0, \mathfrak{a}(X))$ for all $X \in \Gamma(C)$. Hence, there is an action Lie algebroid structure on $G \times_M C$ over $G \times_M S$. Let $p : C \to S$ be the vector bundle projection.

Definition 5.13. An action of $G \to M$ on $C$ is a Lie groupoid action of $G$ on $J \circ p : C \to S$ such that the structure maps of the action groupoid $G \times_M C \to C$ are Lie algebroid morphisms over the structure maps of the action groupoid $G \times_M S \to S$.

Remark 5.14. This definition is equivalent to the original one but has the advantage of clarifying the appearance of a vacant LA-groupoid structure on $G \times_M C \to C$ over $G \times_M S \to S$.

Theorem 5.15. Let $(S, \pi)$ be a Hamiltonian space for a Lie quasi-bialgebroid $(A, \delta, \chi)$ on $M$ with moment map $J : S \to M$ and action map $\rho : J^* A \to TS$ such that the $A$-action induced by $\rho$ is complete and $\rho$ is injective. Suppose that either one of the following two conditions holds:

1. the foliation induced by the $A$-action is simple and the $A$-orbits are 1-connected;

2. $J$ is a surjective submersion and there is a free and proper Lie groupoid action of $G \to M$ on $C$ along $J$ integrating the $A$-action on $C$ of Lemma 5.12 where $\text{Lie}(G) = A$.

Then the induced Poisson structure on the orbit space of the $A$ or $G$-action is integrable if and only if $C$ is integrable.

In the next subsection we will describe a symplectic groupoid that integrates the Poisson structure on the quotient of the $A$ or $G$-action in terms of an integration of $C$.

Remark 5.16. Part 2 of the previous result generalizes [67, Thm 3.4.4], which regards only Poisson groupoid actions.

Theorem 5.15 gives a general answer to Question 2 of Section 4. As we already mentioned, our proof is based on reducing this situation to a particular case of Theorem 1.7. The trick is simply to consider an auxiliary Dirac structure on $S$ in which $C$ sits inside as a Lie subalgebroid. Let us put

$$L = \{ \rho(X) + \pi^4(\alpha) \oplus \alpha : X \in A, \alpha \in \rho(A)^* \} \subset TS;$$

then we have that $L$ splits as the direct sum of $B = \{ \rho(X) : X \in A \}$ and $C$. Note that $L$ is indeed a Dirac structure because it can also be described as a pullback by means of a Manin pair morphism. In fact, there is a Manin pair morphism $R : (TS, TS) \to (A \oplus A^*, A)$ such that $L = A \circ R$ and $C = \text{ker} R$, see [14] and Section 6. Now that we have described $C$ as part of a matched pair of Lie algebroids $(B, C)$, we can check whether $(B, C)$ is complete in the sense of Definition 6.8.
Proof of Theorem 5.15. Statement 1. In the proof of Lemma 5.12 we saw that if \( U \in \Gamma(A) \), then \( \rho(U) \in \Gamma(B) \) is \( C \)-flat with respect to the \( C \)-connection on \( B \) induced by the matched pair decomposition \( L = B \bowtie C \). Hence the space \( \mathcal{V} \) generated by \( \{ \rho(U) \} \) where \( U \) is of compact support satisfies the conditions of Definition 5.8. Suppose that \( C \) is integrable and let \( \mathcal{G}(C) \) be its source-simply-connected integration. Proposition 5.13 implies that \( B \) induces a vacuum LA-subgroupoid \( \mathcal{B} \rightrightarrows B \) of \( T\mathcal{G}(C) \rightrightarrows TS \) which is integrable by a vacant double Lie groupoid consisting of monodromy groupoids: \( \mathcal{G}(B) \) with sides \( \mathcal{G}(B) \) and \( \mathcal{G}(C) \). Then Theorem 4.7 implies that the leaf space \( \mathcal{S} = S/A \) is integrable by the leaf space of \( \mathcal{B} \). The converse implication follows from Theorem 5.12 applied to \( L \), which is the pullback Lie algebroid of the cotangent Lie algebroid structure on \( T^*\mathcal{S} \) associated to the Poisson bracket. Since \( C \) is a Lie subalgebroid of \( L \), it is also integrable.

Statement 2. If we have a \( G \)-action on \( C \) which integrates the infinitesimal action of \( A \), then \( \Gamma(A) \) Thm. 3.6] implies that the \( G \)-action lifts to a \( G \)-action on \( \mathcal{G}(C) \) if \( C \) is integrable. If the \( G \)-action is principal on \( S \), then it is also principal on \( \mathcal{G}(C) \), see [30, Lemma 5.9] or Remark 4.6. Therefore, the Poisson structure on the quotient \( S/G \) is integrable. The converse implication is proved as in Statement 1.

Corollary 5.17 ([32]). Let \( G \) be a Poisson group acting freely and properly by a Poisson action on a Poisson manifold \((S, \pi)\). Then the induced Poisson structure on \( S/G \) is integrable if \( \pi \) is integrable.

Proof. It is clear from the definition that \( C \) is a Lie subalgebroid of \( T^*S \) with the Lie algebroid structure coming from \( \pi \). So \( C \) is integrable if \( \pi \) is. Since the \( g \)-action induces a free and proper \( G \)-action, the result follows from Theorem 5.15. \( \square \)

Corollary 5.17 implies, in particular, that the quotient of an integrable Poisson manifold by a free and proper action by automorphisms is also integrable, see [31].

Remark 5.18. In [32], Corollary 5.17 was actually proved under the assumption that \( G \) is a complete Poisson group so Theorem 5.15 already gives us a slightly sharper result in that situation. Moreover, it may also happen in principle that \( S/G \) is integrable even when \( S \) is not.

If \((S, \pi)\) is a q-Poisson \( g \)-manifold, then there is a nonobvious but canonical Lie algebroid structure on \( T^*S \), see [14, Thm. 1]; we shall denote it by \((T^*S)_g\). It is immediate that \( C \) as before is a Lie subalgebroid of \((T^*S)_g\) (provided the \( g \)-action is locally free). For the sake of making the analogy with the previous situation more evident, we shall also say in this case that \( S \) is integrable if \((T^*S)_g\) is.

Corollary 5.19. Let \((S, \pi)\) be a q-Poisson \( g \)-manifold in which the \( g \)-action integrates to a free and proper \( G \)-action. Then the Poisson structure induced on \( S/G \) is integrable if \( S \) is integrable. \( \square \)

5.4 Symplectic groupoids associated to q-Poisson manifolds

In this subsection we give a construction of a symplectic groupoid that integrates a quotient Poisson structure as in Theorem 5.15 in terms of an integration of the Lie algebroid \( C \).

Let \( L \) be a Dirac structure on \( S \) and consider its kernel \( B := \ker pr_2 = L \cap (TS \oplus 0) \). We have that \( B \) is a Lie subalgebroid as long as it has constant rank.

Definition 5.20. We shall call \( L \) a split Dirac structure if there is a Lie subalgebroid \( C \hookrightarrow L \) such that \( L \) is isomorphic as a vector bundle to the direct sum of the Lie subalgebroids \( B \) and \( C \).

If \( L \) is a split Dirac structure, then \( B \) and \( C \) constitute a matched pair of Lie algebroids; accordingly, we denote \( L \) as \( B \bowtie C \). We shall say that \( L \) is a split Dirac structure which is complete if \((B, C)\) is a complete matched pair of Lie algebroids. Such is the case of the Dirac structures induced by a q-Poisson manifold as in [9].
The projection $\text{pr}_2 : L \to T^* S$ together with the anchor constitutes an IM 2-form \[\mu \]. Notice that $C \to L$ inherits automatically an IM 2-form by restriction of $\text{pr}_2$. If $C$ is integrable, we have that the reduced groupoid given by the proof of Theorem 5.15 can also be obtained by reducing $G(L)$ by the kernel of the multiplicative 2-form integrating $\text{pr}_2|_C$.

**Lemma 5.21.** Let $L = B \rtimes C$ be a split Dirac structure on $S$ which is complete and suppose that $C$ is integrable. Then the vacant multiplicative foliation $B$ on $G(C) \rightrightarrows S$ induced by the matched pair $B \rtimes C$ coincides with the null distribution of the multiplicative 2-form $\omega$ on $G(C)$ whose infinitesimal counterpart is the inclusion $\mu : C \to T^* S$.

**Proof.** Let $\tilde{X} \in \mathfrak{x}(TG(C))$ be the lifted multiplicative vector field associated to the derivation $(D^B_X, X)$ for $X \in \mathcal{V}$, where $\mathcal{V} \subset \Gamma(B)$ is the space of complete $C$-flat sections associated to $(B, C)$. We have seen in Lemma 5.10 that $B$ is spanned by the vector fields of this form. Since $i_{\tilde{X}} \omega : TG(L) \to \mathbb{R}$ is a Lie groupoid morphism, in order to prove that $i_{\tilde{X}} \omega = 0$ we just have to check that its associated Lie algebroid morphism is zero.

Let us recall that the Lie algebroid morphism $\overline{\Lambda}_\mu : TC \oplus TC \to \mathbb{R}$ associated to $\omega : TG(C_L) \oplus TG(C) \to \mathbb{R}$ is defined by the linear 2-form $\Lambda_\mu = \mu^* \omega_{\text{can}}$, where $\omega_{\text{can}}$ is the canonical symplectic form on $T^* S$ \[\mu \]. On the other hand, we have that $\tilde{X}$ induces a Lie algebroid morphism $X' = C \to TC$ in the following way

$$X'(u) = T_p U(X_p) - \frac{1}{2} (D^B_X U)_p \quad \forall p \in S,$$

where $U \in \Gamma(C)$ is any section that satisfies $U_p = u \in C_p$ and $D^B_X U_p$ is the vertical tangent vector to $C_L$ at $u$ associated to $(D^B_X U)_p \mu$. So we have immediately that $i_{\tilde{X}} \Lambda_\mu = 0$ and hence $i_{\tilde{X}} \omega = 0$ which proves $B \subseteq \ker \omega$.

On the other hand, we have that $\ker \omega_{s(g)} \cap \ker T_{s(g)} \mathfrak{s} = 0$ since $\mu$ is injective and hence

$$\ker \omega_{g} \cap \ker T_g \mathfrak{s} \supseteq \ker \omega_{s(g)} \cap \ker T_{s(g)} \mathfrak{s} = 0 \quad \text{(7)}$$

for all $g \in G(C)$, [15, Lemma 3.1]. If $V \in \ker \omega_{g}$ and $W \in T_{s(g)} G$, then for (any) $X \in T_g G$ composable with $W$ we have that

$$\omega_{s(g)} (T s(V), W) = \omega_{g} (V, T \text{Im}(W, X)) - \omega_{g} (V, X) = 0$$

and so $T_g \mathfrak{s} (v) \in \ker \omega_{s(g)}$. But $T_g \mathfrak{s}$ restricted to $\ker \omega_{g}$ is injective by (7), so $\dim \ker \omega_{g} \leq \dim \ker \omega_{s(g)} = \dim B = \dim B$. Therefore, $B = \ker \omega$. \hfill $\Box$

The next Proposition is a refinement of Proposition 1.5 that takes the symplectic forms into account. It shows that the leaf space of the null foliation of the closed 2-form on $G(C) \rightrightarrows S$ described in the previous lemma is symplectomorphic to the leaf space of the kernel of the induced 2-form on a presymplectic groupoid which integrates $L$, provided the associated null foliations are simple.

**Proposition 5.22.** Let $L = B \rtimes C \subset TS$ be a split Dirac structure as in Lemma 5.21. Let $B \rightrightarrows B$ be the vacant multiplicative foliation induced by $B \rtimes C$ over $K = G(C) \rightrightarrows S$ and suppose that $B \rightrightarrows B$ is integrable by a vacant double Lie groupoid $G$ with sides $K$ and $H$ over $S$. Let $\omega$ be the 2-form on $K$ with infinitesimal counterpart the inclusion $C \to T^* S$. Then the following statements hold,

1. If $(\mathfrak{s}^K) \ast \mu = (\mathfrak{s}^K) \ast \omega$, then $(\mathfrak{s}^K) \ast \omega$ makes the diagonal groupoid $G \rightrightarrows S$ into a presymplectic groupoid integrating $L$.

2. If $H \rightrightarrows S$ is proper with trivial isotropy groups, then the orbit space of $G \rightrightarrows K$ is symplectomorphic to the orbit space $\mathcal{G}$ of the action groupoid $(H \times H) \times G \rightrightarrows G$ corresponding to the morphism $\phi = u^H : H \to \mathcal{G}$ as in Lemma 3.3.
\textbf{Proof.} Denote by $\varphi$ the tangent diagonal structure and by $\circ$ the horizontal and vertical compositions $T\mathfrak{m}^H, T\mathfrak{m}^V$ respectively. Take $(u, v), (u', v') \in TG_2$ (composable with respect to $\circ$). We have that 
\[ u \circ v = (u \circ \tilde{u}) \circ (\tilde{v} \circ v) = (u \bullet \tilde{v}) \circ (\tilde{v} \bullet v) \]
for unique $\tilde{u}$ and $\tilde{v}$. Let us note that 
\[ m^*_u \Omega(u, v, u', v') = \Omega((u \circ v, u' \circ v') = \omega(Tt^K((u \circ \tilde{v}) \circ (\tilde{v} \bullet v')), Tt^K((u' \bullet \tilde{v}) \circ (\tilde{v} \bullet v'))) = \omega(Tt^K(u \bullet \tilde{v}), Tt^K(u' \bullet \tilde{v})) = \omega(Tt^K(v), Tt^K(v')) \]
where we have used the fact that $t^H$ is a Lie groupoid morphism and the fact that $\omega$ is multiplicative. On the other hand, we have that 
\[ \omega(Tt^K(\tilde{v}), Tt^K(\tilde{v}')) = \omega(Tt^K(v), Tt^K(v')) \] 
so the infinitesimal multiplicative form corresponding to $\Omega$ is the one corresponding to $L$. By uniqueness of the corresponding multiplicative 2-form, we have that $(G, \Omega)$ is a presymplectic groupoid integrating $L$. Therefore, Part 1 of the Proposition holds.

Suppose that $H \rightrightarrows S$ is proper with trivial isotropy. Then Theorem 5.22 applied to the Lie groupoid $\mu^H : H \to G$ over $S$ implies that $\varpi$ is a Lie groupoid. On the other hand, Lemma 5.23 implies that we have an isomorphism of Lie groupoids $\varpi \cong K\!/G$. Lemma 5.24 implies that $\ker \omega = B$ and so $\omega$ is basic with respect to $\varpi \cong K$, i.e. $(t^K)^* \omega = (s^K)^* \omega$. As a consequence, $K\!/G$ inherits a symplectic form that we denote by $\varpi$.

In the proof of Proposition 5.23 we have seen that $t^K : G \to K$ induces the isomorphism of orbit spaces $\varpi = K\!/H \times H \cong K\!/G$. The projection to the orbit space of $G/$(H x H) factorizes as the composition $p \circ t^K : G \to K\!/G$, where $p : K \to K\!/G$ is the projection to the orbit space of $G \rightrightarrows K$. Since $p^* \varpi = \omega$, we get that $(p \circ t^K)^* \varpi = \varpi$ and then also $\varpi$ is basic and it induces the same symplectic form on $K\!/G$ which is induced by $\omega$. Therefore, Part 2 of the Lemma holds as well.

\textbf{Remark 5.23.} Suppose that we have a q-Poisson manifold $(S, \pi)$ for a Lie quasi-bialgebroid $(A, \delta, \chi)$ and moment map $J : S \to M$ such that the associated $A$-action on $C$ is integrable by a Lie groupoid action $a : G \times C \to C$ which preserves the inclusion $\mu : C \to T^*S$. If $C$ is integrable, then the lifted $G$-action on $G(C)$ preserves the multiplicative 2-form $\omega$ on $G(C)$ induced by $\mu$. In fact, the lifted $G$-action is obtained by applying Lie’s second theorem to $a$ and so it has to preserve $\omega$ as well.

The next example shows that if $G \rightrightarrows K$ is the monodromy groupoid of $B$ as in Proposition 5.22 we do not get in general a groupoid structure on $K\!/G$.

\textbf{Example 5.24.} Consider a free Hamiltonian action of the circle on a symplectic manifold $M$. For instance, take $M = \mathbb{C}^n - \{0\}$ for $n > 1$ with the $\mathbb{S}^1$-action given by rotations. Suppose there are three orbits $\mathbb{S}^1 p_i, i = 1, 2, 3$ within the same level of the moment map $J$. Consider $S = M - \{p_1, e^{i\pi/2} p_2\}$ and the infinitesimal action of $\mathbb{R}$ on $S$ induced by the $\mathbb{S}^1$-action, which still has the restriction of $J$ as moment map so this a hamiltonian action of $\mathbb{R}$ with the null Poisson tensor on $N$. Consider the Lie groupoid $K := S \times J S \rightrightarrows S$. Let us note that $K$ integrates $C$ as in the statement of Theorem 5.19. The inclusion $K \to S \times S$ endows $K$ with the multiplicative 2-form $\omega$ whose infinitesimal counterpart is the inclusion $C \to T^*S$. The pairs $P = (e^{i\pi/2} p_1, e^{-i\pi/2} p_2)$ and $P' = (e^{-i\pi/2} p_1, e^{i\pi/2} p_2)$ lie on the same leaf of $\ker \omega$ on $K$ and so
do $Q = (e^{i\pi/2}p_2, e^{i\pi/2}p_3)$ and $Q' = (e^{-i\pi/2}p_2, e^{-i\pi/2}p_3)$. However $m(P, Q)$ and $m(P', Q')$ lie on different leaves of $\ker \omega$ and as a result the leaf space of the null foliation on $K$ does not inherit a groupoid structure from $K$. On the other hand, the Poisson reduction of $S$ is isomorphic to $M/S^1$ which is integrable.

The following examples illustrate the integration of a quotient Poisson structure by applying the construction of Proposition 5.22.

**Example 5.25** ([22]). If $\Sigma(S) \rightarrow S$ is a source-simply-connected symplectic groupoid and the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ acts on $S$ by a Poisson action, then the dual of the action map $\phi : T^*S \rightarrow \mathfrak{g}^*$ is a Lie bialgebroid morphism. So $\phi$ induces a Poisson groupoid morphism $\Phi : \Sigma(S) \rightarrow G^*$ \cite{52, 51, 74, 67}. If the $g$-action is locally free, then $\Phi^{-1}(1)$ is a Lie groupoid that plays the role of $K$ as in Lemma 5.22, in fact, in our previous notation we have that $\text{Lie}(\Phi^{-1}(1)) = C$. Let $G$ be a Poisson group with tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. If the $g$-action induces a free and proper Poisson $G$-action on $M$, \cite{57, Thm. 3.6} implies that the $g$-action induced by $\Phi$ integrates to a $G$-action on the source-simply-connected integration $\Phi^{-1}(1)$ of $C$. The $G$-orbits on $\Phi^{-1}(1)$ are tangent to the kernel of $\omega$, the pullback of the symplectic form on $\Sigma(S)$ to $\Phi^{-1}(1)$, see Proposition 5.22. Since the $G$-action on $C$ is the restriction of the $G$-action on $T^*S$, Remark 5.23 implies that the $G$-action preserves $\omega$ so Proposition 5.22 implies that $\Phi^{-1}(1)/G$ is a symplectic groupoid that integrates $S/G$.

**Example 5.26.** Let $(S, \pi)$ be a $q$-Poisson $G$-manifold \cite{4}. Suppose that $G$ acts freely and properly on $S$. We have that \cite{44, Thm 1] states that the dual of the action map $T^*S \rightarrow \mathfrak{g}^*$ composed with the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by the bilinear form gives us a Lie algebroid morphism $(T^*S)_\mathfrak{g} \rightarrow \mathfrak{g}$. If $(T^*S)_\mathfrak{g}$ is integrable, then this morphism can be lifted to a moment map $\Phi : G(T^*S)_\mathfrak{g} \rightarrow G$ which makes $G(T^*S)_\mathfrak{g}$ into a $q$-Hamiltonian $\mathfrak{g}$-manifold (groupoid), see \cite{44, Thm 4}. If the $g$-action is locally free on $S$ (and hence on $G(T^*S)_\mathfrak{g}$), we have that $\Phi^{-1}(1)$ is a Lie groupoid. Indeed, \cite[Remark 3.3]{3} says that there is a $2$-form $\varpi \in \Omega^2(\mathfrak{g})$ such that, if we take a neighborhood $U$ of $1 \in G$ covered diffeomorphically by a neighborhood of $0 \in \mathfrak{g}$ using $\exp : \mathfrak{g} \rightarrow G$, then $\varpi := \omega - \Phi^* \log^* \varpi$ is symplectic on $\Phi^{-1}(1)$ and $\mu := \log \Phi$ is a classical moment map for the $g$-action. Since the $g$-action is locally free on $S$, it is also locally free on $G(T^*S)_\mathfrak{g}$. The usual property of moment maps $(\ker T\mu)^* = \mathfrak{g}_M$ implies that $\mu$ is a submersion on $\Phi^{-1}(1)$ and then so is $\Phi$.

As in the previous example, if we consider the source-simply-connected integration $\Phi^{-1}(1)$ of $C = \text{Lie}(\Phi^{-1}(1))$, we have a $G$-action by automorphisms on $\Phi^{-1}(1)$. Part 2 of Proposition 5.22 and Remark 5.23 imply that $\Phi^{-1}(1)/G$ is a symplectic groupoid which integrates the Poisson structure on $S/G$.

### 6 Integration of homogeneous spaces

The goal of this section is to offer some additional concrete applications of Theorem 5.22. In this respect, homogeneous spaces in Poisson and Dirac geometry are particularly relevant since their construction in terms of infinitesimal data typically involves some sort of reduction of a Dirac structure.

The concept of Poisson group \cite{27} can be generalized in two different directions:

- we can preserve the underlying group structure but generalizing the geometry over it in order to obtain the concept of “Dirac group”. When talking about Dirac structures and Dirac morphisms we can restrict ourselves to (1) the standard Courant algebroid and forward Dirac maps or (2) we can work with Manin pairs and Manin pair morphisms. The availability of these two approaches implies that there are two possible definitions of “Dirac group”, see \cite{60, 43};
**Definition** (6.3 ([53])). Analogously, the concepts of “Poisson action” and “Poisson homogeneous space” admit generalizations corresponding to each of the previous definitions and in all these cases there are classification results in terms of infinitesimal data. We shall study some particular examples of these constructions in which the associated Poisson or Dirac structures can be explicitly integrated thanks to Theorem 3.2, leaving the study of other of their features and applications for later work.

### 6.1 Dirac structures associated to Dirac-Lie group actions on homogeneous spaces

This subsection has two main goals: we give a general criterion for the quotient of an action Lie algebroid to be integrable based on Theorem 3.2 and the construction of [25]; then we apply this criterion to the Dirac homogeneous spaces of [62, 53] and obtain a result that generalizes the integration of Poisson homogeneous spaces in [13].

**Integrability of quotients of action Lie algebroids.** Let us consider an infinitesimal action of a Lie algebra \( l \) on a manifold \( M \) and the associated action Lie algebroid \( l \times M \). Suppose that there is a Lie subalgebra \( \mathfrak{k} \) of \( l \) whose restricted action is induced by the action of a Lie group \( K \). If \( K \) acts freely and properly on \( M \) and there is an \( K \)-action by automorphisms on \( l \) whose infinitesimal counterpart is the adjoint representation, then Lemma 3.3 implies that there is a unique Lie algebroid structure on the associated bundle \( A = ((l \times M)/K \) over \( M/K \) such that the quotient map \( q : M \to M/K \) satisfies that \( q^*A \) is isomorphic to the action Lie algebroid \( l \times M \). We shall see that the explicit integration of action Lie algebroids provided in [25] implies a simple criterion for the integrability of Lie algebroids of the form of \( A \); this construction generalizes the quotients of action groupoids studied in [8]. Then we shall see some applications to Dirac geometry of this fact.

**Definition 6.1 ([25]).** Let \( l \) be a Lie algebra acting on \( M \) and let \( L \) be a Lie group which integrates \( l \). The associated **Dazord model** is the quotient groupoid \( D_L := \text{hol}(\mathcal{F})/L \rightrightarrows M \) where \( \mathcal{F} \) is the foliation on \( M \times L \) given by the distribution \( D = \{(u_M, u'_L) \in TM \times TL : u \in l\} \) (here we assume that the Lie algebra structure on \( l \) is given by the left-invariant vector fields).

Since the action of \( L \) on \( P \times L \) given by \( a \cdot (p, b) = (p, ab) \) preserves \( D \), it induces an action by automorphisms of \( L \) on \( \mathcal{G}(\mathcal{F}) \), the monodromy groupoid. Since the \( L \)-action preserves holonomy, it descends to an action on the holonomy groupoid \( \text{hol}(\mathcal{F}) \); this action is free and proper and hence the quotient \( \text{hol}(\mathcal{F})/L \) is a Lie groupoid which integrates the action Lie algebroid \( l \times M \).

**Remark 6.2.** In the previous definition we can use the monodromy groupoid \( \text{mon}(\mathcal{F}) \) instead of \( \text{hol}(\mathcal{F}) \). The resulting quotient, that we shall also call Dazord model, \( D_L := \text{mon}(\mathcal{F})/L \) is then source-simply-connected. The advantage of \( \text{hol}(\mathcal{F}) \) is that it is better suited for our next application.

**Definition 6.3 ([53]).** Let \( l \) be a Lie algebra and let \( \mathfrak{k} \subset l \) be a Lie subalgebra. Let \( K \) be a Lie group with Lie algebra \( \mathfrak{k} \) and such that there is an action of \( K \) on \( l \) by automorphisms whose infinitesimal counterpart is the adjoint action of \( \mathfrak{k} \) on \( l \). In this situation \((l, K)\) is called a **Harish-Chandra pair**. A morphism of Harish-Chandra pairs \((l, K) \to (l', K')\) is a Lie group morphism \( F : K \to K' \) and a Lie algebra morphism \( f : l \to l' \) such that \( f \) commutes with the actions and \( T_l F = f|_l \).

**Proposition 6.4.** Let \((l, K)\) be a Harish-Chandra pair and let \( l \times M \) be an action Lie algebroid. Suppose that the \( l \)-action on \( M \) restricted to \( \mathfrak{k} \) coincides with the infinitesimal action corresponding to a free and proper action of \( K \) on \( M \). Let \( K_\circ \) be the connected component of the
unit. If the inclusion of Lie algebras \( \mathfrak{t} \rightarrow \mathfrak{l} \) is integrable by a Lie group morphism \( \psi : K_0 \rightarrow L \), then the associated quotient Lie algebroid \( (\mathfrak{t}/\mathfrak{k} \times \mathfrak{m})/\mathfrak{k} \) over \( M/\mathfrak{k} \) is integrable.

The proof is based on observing that, in this situation, we can promote \( \psi \) to a Lie groupoid morphism \( (M \times K_0) \rightarrow \mathcal{D}_L \) which integrates the inclusion of Lie algebroids \( \mathfrak{k} \times \mathfrak{m} \rightarrow \mathfrak{l} \times \mathfrak{m} \). Then (a particular case of) Theorem 6.2 implies the result.

Proof of Proposition 6.3. It is enough to prove the proposition under the assumption that \( K \) is connected. If \( K \) is not connected, the following discussion proves the statement for the principal bundle \( K_0 \rightarrow M \rightarrow M/K_0 \). But then the discrete group \( K/K_0 \) acts by automorphisms on \( (\mathfrak{l}/\mathfrak{k} \times \mathfrak{m})/\mathfrak{k} \) and hence the quotient \( (\mathfrak{l}/\mathfrak{k} \times \mathfrak{m})/\mathfrak{k} \) is also integrable.

Suppose that \( K \) is connected and let \( \psi : K \rightarrow L \) be a Lie group morphism such that \( \text{Lie}(\psi) \) is the inclusion \( \mathfrak{t} \rightarrow \mathfrak{l} \). Let us apply the Dazord construction to the restricted infinitesimal action of \( \mathfrak{t} \) on \( M \). We have a foliation \( \tilde{\mathcal{F}} \) on \( M \times \psi(K) \) given by the distribution \( \{u_M, u' \} : u \in \mathfrak{t} \) and then \( \text{hol}(\tilde{\mathcal{F}})/K \) integrates the Lie algebroid \( \mathfrak{t} \times \mathfrak{m} \). We shall see that there is a morphism of Lie groupoids \( \phi : \text{hol}(\tilde{\mathcal{F}})/K \rightarrow \text{hol}(\mathcal{F})/L \) which integrates the inclusion of Lie algebroids \( \mathfrak{t} \times \mathfrak{m} \rightarrow \mathfrak{l} \times \mathfrak{m} \) and then we will show that \( \text{hol}(\tilde{\mathcal{F}})/K \) is isomorphic to the action groupoid \( M \times K \rightarrow \mathcal{L} \).

Consider the subbundle \( D' \) of \( D = \{(u_M, u') \in TM \times TL : u \in \mathfrak{t} \} \) given by the infinitesimal action \( (u_M, u') \) for every \( u \in \mathfrak{t} \). The foliation \( \mathcal{F}' \) induced by \( D' \) restricts to \( \tilde{\mathcal{F}} \) on \( M \times \psi(K) \subset M \times L \) so this inclusion induces a Lie groupoid morphism \( \iota : \mathcal{G}(\mathcal{F}') \rightarrow \text{hol}(\mathcal{F}) \). We claim that \( \iota(a) \) is a unit for every loop \( a \) with trivial holonomy. Since the foliation induced by the left (or right) cosets of \( \psi(K) \) in \( L \) is transversely parallelizable \([56]\), a loop with trivial holonomy with respect to \( \tilde{\mathcal{F}} \) has also trivial holonomy with respect to \( \mathcal{F}' \). Moreover, a loop with trivial holonomy with respect to \( \mathcal{F}' \) has also trivial holonomy with respect to \( \mathcal{F} \), see the next lemma. From these observations we deduce that \( \iota \) induces a morphism \( \text{hol}(\tilde{\mathcal{F}}) \rightarrow \text{hol}(\mathcal{F}) \) which is clearly \( K \)-equivariant and hence it induces a Lie groupoid morphism \( \phi : \text{hol}(\tilde{\mathcal{F}})/K \rightarrow \text{hol}(\mathcal{F})/L \).

Now \( \text{hol}(\tilde{\mathcal{F}})/K \) is isomorphic to the action groupoid \( M \times K \rightarrow \mathcal{L} \). In fact, any path tangent to \( \tilde{\mathcal{F}} \) is of the form

\[
t \mapsto (pa(t), \psi(a(t)))
\]

where \( a \) is a path in \( K \). The foliation \( \mathcal{F}' \) is defined by the free action of a connected Lie group so its holonomy is trivial. Then \( \text{hol}(\mathcal{F}') \) is identified with the action groupoid \( (M \times \psi(K)) \times K \rightarrow M \times \psi(K) \) corresponding to the right action \( (p, a) \cdot b = (pb, a \psi(b)) \). As a consequence, the quotient of \( (M \times \psi(K)) \times K \) by \( \psi(K) \) defined by \( a \cdot (p, b, c) = (p, ab, c) \) is isomorphic to the action groupoid \( M \times K \rightarrow \mathcal{L} \).

Since we have a morphism \( \phi : M \times K \rightarrow \text{hol}(\mathcal{F})/L \) which integrates \( \mathfrak{t} \times \mathfrak{m} \rightarrow \mathfrak{l} \times \mathfrak{m} \), Theorem 6.2 implies the result.

In order to complete the previous proof, let us prove the following.

Lemma 6.5. A loop with trivial holonomy with respect to a subfoliation of \( \mathcal{F} \) has also trivial holonomy with respect to \( \mathcal{F} \).

Proof. The (usual) proof of Frobenius theorem as in \([65]\) shows that if \( \mathcal{F}' \) is a subfoliation of rank \( p \) of \( \mathcal{F} \) which has rank \( q \) on a manifold \( M \) of dimension \( p + q + r \), we can find local coordinates \( \{x^a\}_a \) around every point in \( M \) such that every leaf of \( \mathcal{F}' \) in that chart is given by \( x^a = \text{constant} \) for \( a > p \) and the leaves of \( \mathcal{F} \) are given by \( x^a = \text{constant} \) for \( a > p + q \). So in a chart like this, the holonomy germ along a path \( t \mapsto (a(t), x, y) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \) tangent to \( \mathcal{F} \) is given by the restriction to \( \{(a(0), x) \} \times \mathbb{R}^r \subset \{a(0)\} \times \mathbb{R}^{p+r} \) of the holonomy germ with respect to the same path. So choosing coordinates of this kind on each open set of some finite cover of a loop \( a \) tangent to \( \mathcal{F}' \) with trivial holonomy we see that the holonomy of \( a \) with respect to \( \mathcal{F} \) is trivial too.
Integration of Dirac homogeneous spaces. We start by recalling some concepts in order to define the Dirac homogeneous spaces.

Definition 6.6 ([46]). A Courant algebroid consists of a vector bundle $E \to M$, a tensor $\langle \cdot, \cdot \rangle \in \Gamma(E^* \otimes E^*)$ which induces a pointwise nondegenerate symmetric bilinear form that we call the metric, a bilinear bracket $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ called the Courant bracket and a vector bundle morphism $a : E \to TM$ called the anchor such that the following identities hold:

\[ a(X)(Y, Z) = \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle, \]
\[ [X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]], \]
\[ \frac{1}{2} a^*(d(X, X)) = \langle [X, X] \rangle; \]

for all $X, Y, Z \in \Gamma(E)$, where $b : E \to E^*$ is the isomorphism given by the metric.

Definition 6.7 ([46]). A subbundle $F \subset E$ of a vector bundle $E$ endowed with a metric $\langle \cdot, \cdot \rangle \in \Gamma(S^2(E^*))$ is called isotropic if $\langle \cdot, \cdot \rangle|_F = 0$; a subbundle of $E$ with the property $E = E^b$ is called Lagrangian. A Dirac structure in a Courant algebroid $E$ is a Lagrangian subbundle $L \subset E$ which is involutive with respect to the restricted Courant bracket. If $L$ is a Dirac structure inside the Courant algebroid $E$, then $(E, L)$ is called a Manin pair.

Remark 6.8 ([68]). It is a consequence of the axioms for a Courant algebroid $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], a)$ that the Leibniz rule holds and $a$ preserves brackets:

\[ [X, fY] = f[X, Y] + (\mathcal{L}_a(Y))(f)Y, \quad a([X, Y]) = [a(X), a(Y)]; \]

for all $X, Y \in \Gamma(E)$ and all $f \in C^\infty(M)$.

Example 6.9. A Courant algebroid over a point is a Lie algebra with an Ad-invariant metric.

Example 6.10. We have that $TM$ endowed with the Courant-Dorfman bracket and its canonical pairing is a Courant algebroid called the canonical Courant algebroid, see §2.3.

Example 6.11 ([30, 63]). If $(A, \delta, \chi)$ is a Lie quasi-bialgebroid, then the bundle $A \oplus A^*$ inherits a unique Courant algebroid structure such that $A$ sits inside it as a Dirac structure, the metric is the canonical pairing and the Courant bracket and anchor restricted to $A^*$ induce the differential $\delta$.

The general notion of morphism for Courant algebroids and Manin pairs is the following.

Let $E, F$ be Courant algebroids over $M, N$. We denote by $F$ the Courant algebroid $F$ with the opposite inner product. Given a smooth map $f : M \to N$, take $\Gamma_f \subset M \times N$, the graph of $f$.

Definition 6.12 ([5, 14]). A Courant morphism between $E$ and $F$ over $f$ is a Lagrangian subbundle $R \subset E \times F$ such that: (1) the anchors satisfy $a_E \times a_F(R) \subset T \Gamma_f$ and (2) if $u, v \in \Gamma(E \times F)$ restrict to sections of $R$, then so does their Courant bracket $[u, v]$. Composition of Courant morphisms $R, S$ is defined as the pointwise composition of relations $R \circ S$. A morphism of Manin pairs $(E, L), (F, K)$ over a smooth map $f : M \to N$ is a Courant morphism $R$ between $E$ and $F$ such that the composition satisfies $R \circ L = K$ and $\ker R \cap L = 0$.

Now we will review the definition of Dirac-Lie groups which generalizes the definition of Poisson groups and Lie groups endowed with the Cartan-Dirac structure [1].

Definition 6.13 ([43, 63]). A Dirac-Lie group is a Manin pair $(A, E)$ over a Lie group $H$ equipped with a Manin pair morphism $R_m : (A, E) \times (A, E) \to (A, E)$ over the multiplication map $m : H \times H \to H$ such that

\[ R_m \circ (R_m \times id_A) = R_m \circ (id_A \times R_m) \quad R_m \circ (\epsilon \times id_A) = R_m \circ (id_A \times \epsilon) \]

where $\epsilon$ is the inclusion of the trivial Manin pair over the unit of $H$ in $(A, E)$. A Dirac-Lie group is called exact if its underlying Courant algebroid is exact.
The main result of [43] is that Dirac-Lie groups are classified by $H$-equivariant Dirac-Manin triples, that we now recall.

**Definition 6.14** ([43]). Let $\mathfrak{d}$ be a Lie algebra and let $B \in S^2(\mathfrak{d})$ be $\mathfrak{d}$-invariant. Let $\mathfrak{g} \subset \mathfrak{d}$ be a coisotropic Lie subalgebra, i.e. $B^g(\mathfrak{g}^g) \subset \mathfrak{g}$, where $B^g : \mathfrak{d}^g \to \mathfrak{d}$ is defined by $B^g(\alpha) = B(\alpha, \cdot)$ and $\mathfrak{g}^g \subset \mathfrak{d}^g$ is the annihilator of $\mathfrak{g}$. Then $(\mathfrak{d}, \mathfrak{g})$ is called a Dirac-Manin pair.

In the previous definition, if $B$ is nondegenerate and $\mathfrak{g}$ is Lagrangian, we recover the definition of a Manin pair over a point.

**Definition 6.15.** Let $(\mathfrak{d}, \mathfrak{g})$ be a Dirac-Manin pair and let $\mathfrak{h} \subset \mathfrak{d}$ be a Lie subalgebra such that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ as a vector space. Then $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_B$ is called a Dirac-Manin triple. If there is an action by automorphisms of $H$ on $\mathfrak{d}$ whose infinitesimal counterpart is the action of $\mathfrak{h}$ on $\mathfrak{d}$ by inner derivations, $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_B$ is called an $H$-equivariant Dirac-Manin triple.

**Example 6.16.** An $H$-equivariant Dirac-Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_B$ in which $B$ is nondegenerate and both $\mathfrak{g}$ and $\mathfrak{h}$ are Lagrangian is called an $(H$-equivariant) Manin triple [27]. This structure induces a Poisson group structure on $H$. The corresponding Dirac-Lie group is $(H, \mathfrak{th}, \text{graph}(\pi))$, where $\pi$ is the Poisson structure that makes $H$ into a Poisson group, see [54].

**Example 6.17.** Let $\mathfrak{h}$ be a Lie algebra and let $B' \in S^2(\mathfrak{h})$ be $\mathfrak{h}$-invariant. Then the direct sum $\mathfrak{d} = \mathfrak{h} \oplus \mathfrak{h}$ with $B \in S^2(\mathfrak{d})$ the element which restricts to $B'$ on $\mathfrak{h} \oplus 0$ and $-B'$ on $0 \oplus \mathfrak{h}$ has the diagonal $\mathfrak{g} = \mathfrak{h}_\Delta$ as a coisotropic subalgebra, so $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_B$ is a Dirac-Manin triple, where we identify $\mathfrak{h}$ with its image in $\mathfrak{d}$ under the inclusion $u \mapsto (u, 0)$. When $B'$ is a nondegenerate quadratic form we are in the situation of the Cartan-Dirac structure [1]. For any integration $H$ of $\mathfrak{h}$, we have an $H$-action by automorphisms on $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_B$ which makes it into an $H$-equivariant Dirac-Manin triple. The corresponding Dirac-Lie group is isomorphic to $(H, T_\eta H, E_H)$, where $\eta$ is the Cartan 3-form on $H$ and $E_H$ is the Cartan-Dirac structure, see [1 [43].

**Definition 6.18** ([53]). A Dirac action of a Dirac-Lie group $\mathcal{H} = (H, \mathfrak{a}, E)$ on a Manin pair $\mathcal{M} = (M, B, F)$ is a Manin pair morphism $R_a : \mathcal{H} \times \mathcal{M} \to \mathcal{M}$ over an action map $a : H \times M \to M$ such that

\[ R_a \circ (R_a \times \text{id}_\mathfrak{g}) = R_a \circ (\text{id}_\mathfrak{h} \times R_a) \quad R_a \circ (\epsilon \times \text{id}_\mathfrak{g}) = \text{id}_\mathfrak{g} \]

where $\epsilon$ is the inclusion of the trivial Manin pair over the unit of $H$ in $(\mathfrak{a}, E)$.

Dirac-Lie group actions on homogeneous spaces were classified in [53] as follows. Let $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_B$ be an $H$-equivariant Dirac-Manin triple. Take the following data:

1. a closed Lie subgroup $K \hookrightarrow H$ and a Manin pair $(\mathfrak{n}, l)$ with bilinear form $\gamma \in S^2(\mathfrak{n})$ such that $\mathfrak{t} \subset \mathfrak{l}$;
2. a $K$-action by automorphisms on $(\mathfrak{n}, l)$ whose infinitesimal counterpart is the action by inner derivations of $\mathfrak{t}$ on $\mathfrak{n}$;
3. a morphism $(f, F) : (\mathfrak{n}, K) \to (\mathfrak{d}, H)$ of Harish-Chandra pairs such that $f(\gamma) = B$.

Define the map $\rho : \mathfrak{d} \to \mathfrak{X}(H)$ as follows: $X \in \mathfrak{d}$ goes to the vector field $X_H$ given by

\[ a \mapsto \text{Tr}_a (p_\mathfrak{h} \text{Ad}_a f(X)) \quad (10) \]

for all $a \in H$, where $p_\mathfrak{h} : \mathfrak{d} \to \mathfrak{h}$ is the projection along $\mathfrak{g}$. We have that $\rho$ is a Lie algebra morphism. The morphism $\rho \circ f$ defines an action of $\mathfrak{n}$ on $H$ with coisotropic stabilizers so it induces a Manin pair structure on $(H \times \mathfrak{n}, H \times l)$ [42]. Cosotropic reduction as in [42] of $(H \times \mathfrak{t}^+, H \times l)$ determines a Manin pair $(\mathcal{E}, A)$ over $H/K$ which admits a Dirac-Lie group action of the Dirac-Lie group associated to $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h})_B$. In particular, $A$ is isomorphic to a quotient Lie algebroid $(\mathfrak{l}/\mathfrak{t} \times H)/K$ as in the beginning of the section. One of the main results in [53] states that all the Manin pairs with such a property are of this form. So we can call the items in 1-3, the classifying data of the associated Dirac action. Let $(\mathcal{E}, A)$ be a Manin pair over $H/K$.
admitting a Dirac action of a Dirac-Lie group \((H,A,E)\). Suppose that in the classifying data 1-3 corresponding to \((\mathbb{E},A)\) we have that \(f\mid_1:1 \to \mathfrak{d}\) is injective and \(\mathfrak{t} = f(1) \cap \mathfrak{h}\). Then \((\mathbb{E},A)\) is a \textit{Dirac homogeneous space} in the sense of [62], see [53, §5.3].

**Theorem 6.19.** The Dirac homogeneous spaces are integrable.

**Proof.** It is enough to prove the result for Dirac homogeneous spaces over connected homogeneous spaces, see [13, Thm. 4.4] and its proof. Let \((\mathbb{E},A)\) be a Dirac homogeneous space over \(H/K\), where \((H,A,E)\) is a connected Dirac-Lie group. Let \((\mathfrak{d},\mathfrak{g},\mathfrak{h})_F\) be the \(H\)-equivariant triple associated to \((H,A,E)\). Since there is also a Dirac-Lie group structure over the 1-connected covering \(\tilde{H}\) of \(H\) and a canonical Dirac-Lie group morphism over the covering map \(\tilde{H} \to H\), we can assume, without loss of generality, that \(H\) is 1-connected. So there is a Lie group morphism \(j:H \to D\) integrating the inclusion \(\mathfrak{h} \hookrightarrow \mathfrak{d}\). Consider the classifying data 1-3 associated to \((\mathbb{E},A)\). Since \(f\mid_1:1 \to \mathfrak{d}\) is injective, there is an immersed connected subgroup \(L \subset D\) with Lie algebra \(\mathfrak{l}\). Then \(j(K_o) \subset L\), where \(K_o \subset K\) is the connected component of the unit. So the inclusion \(\mathfrak{t} \hookrightarrow \mathfrak{l}\) is integrable by \(\psi := j|_{K_o}:K_o \to L\). Therefore, Proposition [6.4] implies the result.

This is a generalization of the main result of [13] which concerns Poisson homogeneous spaces of Poisson groups. In fact, more generally, we have the following.

**Corollary 6.20.** The Dirac structures associated to Dirac actions of exact Dirac-Lie groups on exact Manin pairs over homogeneous spaces are integrable.

**Proof.** In this situation, the classifying data 1-3 of the Dirac action correspond to a Harish-Chandra pair morphism which is actually an isomorphism, see [53, Proposition 5.10], so the associated data reduces to a Lagrangian subalgebra of \(\mathfrak{d}\) and hence Theorem [6.19] implies the result.

Poisson groups and the Cartan-Dirac structure provide examples of exact Dirac-Lie groups so this result already covers the situations more commonly studied in the literature. Dirac homogeneous spaces [38] corresponding to Dirac-Lie groups in the sense of [60] are also integrable, see [6].

### 6.2 Poisson homogeneous spaces of symplectic groupoids and Poisson groups

**Definition 6.21 [71].** A \textit{Poisson groupoid} is a Lie groupoid \(\mathcal{G} \rightrightarrows M\) with a Poisson structure on \(\mathcal{G}\) such that the graph of the multiplication map is a coisotropic submanifold\(^3\) of \(\mathcal{G} \times \mathcal{G} \times \mathcal{G}\), where \(\mathcal{G}\) denotes \(\mathcal{G}\) with the opposite Poisson structure.

**Example 6.22.** A Poisson groupoid over a point is a \textit{Poisson group} [27].

**Example 6.23.** A Poisson groupoid whose bracket is nondegenerate is a symplectic groupoid.

The infinitesimal description of Poisson groupoids is provided by Lie bialgebroids [51] see [52]. If \((A, A^*)\) is a Lie bialgebroid over \(M\), we have that the map \(\pi^\#: a \circ a^*_\#: T^*M \to TM\) defines a Poisson structure \(\pi\) on \(M\), where \(a\) and \(a^*_\#\) are the anchors of \(A\) and \(A^*\) respectively; if \((A, A^*)\) is integrable by a Poisson groupoid \(\mathcal{G} \rightrightarrows M\), then \(\pi\) is the unique Poisson structure on \(M\) that makes \(\pi\) a Poisson morphism. If \((A, A^*)\) is the tangent Lie bialgebroid of a symplectic groupoid \(\mathcal{G} \rightrightarrows M\), then \(A\) is isomorphic to the cotangent Lie algebroid \(T^*M\) [52.2] and \(A^*\) is isomorphic to \(TM\).

**Definition 6.24 [45].** Let \(\mathcal{G} \rightrightarrows M\) be a Poisson groupoid. A \textit{Poisson action} of \(\mathcal{G}\) on a Poisson manifold \(P\) is a groupoid action \(a: \mathcal{G} \times M \to P\) of \(\mathcal{G}\) on \(J: P \to M\) such that its graph is coisotropic in \(\mathcal{G} \times P \times \overline{P}\). If there is a section \(s: M \to P\) of \(J\) such that \(P = \mathcal{G} \cdot s(M)\) for the induced action of \(\mathcal{G}\), \(P\) is called a \textit{homogeneous space} of \(\mathcal{G}\).

\(^3\)Let \((M, \pi)\) be a Poisson manifold. A submanifold \(C\) of \(M\) is coisotropic if \(\pi^\#(T^*C) \subset TC\), where \(T^*C\) is the annihilator of \(TC\).
Poisson groups are always integrable \[\text{[47]}\] so they allow us to consider two kinds of Poisson homogeneous spaces:

- Poisson manifolds endowed with a transitive Poisson action of a Poisson group as in the original treatment \[\text{[28]}\].
- Poisson manifolds endowed with a transitive Poisson action of some symplectic groupoid which integrates a Poisson group.

Let us explain the second situation in more familiar terms. The Lie functor establishes an equivalence of categories between the category of simply-connected Poisson groups and the category of Lie bialgebras \[\text{[27, 29]}\]. Hence, to every Poisson group \(G\) there is an associated 1-connected Poisson group \(\Sigma (G^*)\) with tangent Lie algebra \(g^*\). A Poisson action of a Poisson group \(G\) on a manifold \(M\) is Hamiltonian if it is encoded infinitesimally by a Poisson morphism \(J : M \to G^*\), see \[\text{[47]}\]. This concept generalizes the classical moment maps with target the dual of the Lie algebra \(g^*\). A Hamiltonian Poisson action of \(G\) induces an action of the source-simply-connected symplectic groupoid \(\Sigma (G^*) \rightrightarrows G^*\) which integrates \(G^*\); if such an action is transitive along the fibers of the moment map \(J\), then \(M\) is a Poisson homogeneous space of \(\Sigma (G^*) \rightrightarrows G^*\).

There are several explicit examples and applications of the original Poisson homogeneous spaces of Poisson groups, see \[\text{[28, 13]}\] and references therein, while there are few nontrivial examples of Hamiltonian Poisson actions and even fewer that come with a transitive symplectic groupoid action in the previous sense.

In this subsection we shall see the following topics: (i) as a corollary of Theorem 3.2 we shall produce a general criterion for the Poisson homogeneous space of a symplectic groupoid to be integrable; (ii) based on the general infinitesimal classification of Poisson homogeneous spaces provided in \[\text{[47]}\], for every Poisson homogeneous space of a (connected) Poisson group \(G\) we shall construct a Poisson homogeneous space of some symplectic groupoid \(\mathcal{G} \rightrightarrows G^*\) with the same underlying infinitesimal data; (iii) we shall prove that all the Poisson homogeneous spaces constructed in the previous way are explicitly integrable thanks to our general criterion.

**Integrability of Poisson homogeneous spaces of symplectic groupoids.** Let us recall the classification of Poisson homogeneous spaces for Poisson groupoids of \[\text{[42]}\]. For any Lie groupoid \(\mathcal{G} \rightrightarrows M\) with Lie algebroid \(A\), there is a canonical Lie groupoid structure on \(T^* \mathcal{G}\) over \(\mathcal{A}^*\) \[\text{[50]}\]. If \(\mathcal{G}\) is a Poisson groupoid, the target map \(\mathfrak{t} : T^* \mathcal{G} \to A^*\) is a Lie bialgebroid morphism \[\text{[50]}\] so it induces a Courant morphism \(R : \mathbb{T} \mathcal{G} \to A \oplus A^*\) \[\text{[3]}\]. Take a Dirac structure \(L \subset A \oplus A^*\). Then its pullback \(E := L \circ R\) is

\[
E = \{X^\tau + \pi_G^1(\alpha) \oplus \alpha : X \oplus \mathfrak{t}(\alpha) \in L\}. \tag{11}
\]

If \(L \cap (A \oplus 0)\) is of constant rank and there is a Lie subgroupoid \(\mathcal{K} \subset \mathcal{G}\) whose Lie algebroid is \(L \cap (A \oplus 0)\) and is suitably regular, then \(E\) is reducible and its Poisson quotient is identified with the quotient by left translations \(\mathcal{G}/\mathcal{K}\). The main result of \[\text{[42]}\] states that all the Poisson homogeneous spaces of \(\mathcal{G}\) are of this form.

**Example 6.25.** Taking \(L = A \oplus 0\), the Poisson homogeneous space associated to \(L\) is \(\mathcal{G}/\mathcal{G} = M\) itself. As a corollary of Theorem \[\text{[5, 13]}\] the Poisson structure on \(M\) is integrable if and only if the annihilator \(C \subset T^* G\) is integrable. In particular, \(M\) is integrable if so is \(\mathcal{G}\), see also \[\text{[67]}\].

If \(\mathcal{G} \rightrightarrows M\) is a symplectic groupoid, then \(E\) can be described in more familiar terms thanks to the fact that the Courant algebroid emerging as the double of its Lie bialgebroid is isomorphic to \(\mathbb{T} M\) by means of the map \(e^\pi : \mathbb{T} M \to \mathbb{T} M : X \oplus \alpha \mapsto X + \pi(\alpha) \oplus \alpha\), where \(\pi\) is the Poisson bracket on \(M\) \[\text{[46, 45]}\].

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Lemma 6.26. Let $L \subset TM$ be a Dirac structure on a Poisson manifold $(M, \pi)$. Suppose that $M$ is integrable by the symplectic groupoid $(\mathcal{G}, \omega)$. Then $E$ as in (11) is isomorphic as a Lie algebroid to the backward image $\mathcal{B}_t(L)$.

Proof. The map $\omega^\#: T\mathcal{G} \to T^*\mathcal{G}$ is an isomorphism of Lie groupoids and hence the target map of the cotangent groupoid, $\hat{t}: T^*\mathcal{G} \to TM$, is identified with $T\hat{t} \circ (\omega^#)^{-1}$. The Poisson groupoid structure on $\mathcal{G}$ is given by $\Pi := \omega^{-1}$ which satisfies $\Pi^\# = (\omega^#)^{-1}$. Since $\Pi^\#(t^*\theta) = \theta^r$ and $t$ is a Poisson morphism we have that

$$e^{-\omega}(E) = \{\Pi^\#(\alpha + t^*\theta) \oplus -t^*\theta : Tt\Pi^\#(\alpha) \oplus \theta \in e^\pi L\} = \mathcal{B}_t(L^{op});$$

where $L^{op} = \{X \oplus \alpha | X \oplus -\alpha \in L\}$ which is isomorphic to $L$. Therefore, we have the result. □

Proposition 6.27. Let $P = \mathcal{G}/\mathcal{K}$ be a Poisson homogeneous space of a symplectic groupoid $\mathcal{G} \rightrightarrows M$ and $L \subset TM$ its associated Dirac structure, where $\text{Lie}(\mathcal{K}) = L \cap T^*M$. Then $P$ is integrable if the inclusion of Lie algebroids $L \cap T^*M \rightrightarrows L$ is integrable by a Lie groupoid morphism with source $\mathcal{K}$.

Proof. Suppose that the inclusion of Lie algebroids $L \cap T^*M \rightrightarrows L$ is integrable by a Lie groupoid morphism $\psi: \mathcal{K} \to \mathcal{L}$. Lemma 6.26 implies that $E \cong t^\dagger_G L$ so $E$ is integrable by the pullback groupoid

$$t^\dagger_G \mathcal{L} = \{(x, k, y) \in \mathcal{G} \times \mathcal{L} \times \mathcal{G} : t_G(x) = t_L(k), t_G(y) = s_L(k)\}.$$

Now $P$ is the orbit space of the action groupoid $X := \mathcal{K} \times_M \mathcal{G} \rightrightarrows \mathcal{G}$ and we can define $\Psi: X \to t^\dagger_G \mathcal{L}$ as

$$(k, x) \mapsto (kx, \psi(k), x). \quad (12)$$

Since $\Psi$ integrates the inclusion $\text{Lie}(X) \rightrightarrows E$ and $X$ is isomorphic to the submersion groupoid $\mathcal{G} \times_P \mathcal{G} \rightrightarrows \mathcal{G}$, Theorem 3.2 implies that the Poisson structure on $P$ is integrable. □

Remark 6.28. In this situation, the pullback groupoid $t^\dagger_G \mathcal{L}$ which is Morita equivalent to $\mathcal{L}$. On the other hand, the Poisson structure $\pi$ on $P$, the Poisson homogeneous space associated to $L$, is integrable by a quotient of $t^\dagger_G \mathcal{L}$ as in Theorem 3.2 which is also Morita equivalent to $t^\dagger_G \mathcal{L}$. Therefore, $L$ and the cotangent Lie algebroid $T^*P$ associated to $\pi$ are integrable by Morita equivalent Lie groupoids.

Poisson homogeneous spaces of Poisson groups and their Hamiltonian counterparts.

Let $G$ be a Poisson group and let $\mathfrak{d}$ be the double of the tangent Lie bialgebra of $G$. We have an action Courant algebroid $\mathfrak{d} \times G$ given by the map $\rho: \mathfrak{d} \to \mathfrak{X}(G)$ defined by (10) (putting $H = G$), see [42]. The left-invariant trivialization of $TG$ defines an isomorphism with the action Courant algebroid $\mathfrak{d} \times G$. In what follows we shall identify $TG$ with $\mathfrak{d} \times G$ by means of this isomorphism. Let $I \times G \rightrightarrows \mathfrak{d} \times G$ be the Dirac structure associated to a Lagrangian subalgebra $I \subset \mathfrak{d}$. The Lie algebroid bracket $[,]$ and the Courant bracket $[[,]$ on $\mathfrak{d} \times G$ are related as follows:

$$[u, v] = [u, v] + a^*(du, v),$$

where $u, v \in C^\infty(G, \mathfrak{d})$, see [42]. Lemma 4.1. So if $u$ and $v$ take values in a Lagrangian subalgebra, then we have $[[u, v] = [u, v]$.

Definition 6.29 (13)). Let $G$ be a Poisson group. A Drinfeld double of $G$ is an integration $D$ of the double Lie algebra $\mathfrak{d}$ such that there is a Poisson group morphism $G \to D$ which integrates the inclusion of Lie algebras $\mathfrak{g} \to \mathfrak{d}$. 

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If $G$ is 1-connected, then it automatically admits a Drinfeld double given by the 1-connected integration of $\mathfrak{d}$. Throughout this section we shall use the following notation. If there are group morphisms $G \to D$, $G^* \to D$ which integrate the inclusions of Lie algebras $\mathfrak{g} \hookrightarrow \mathfrak{d}$, $\mathfrak{g}^* \hookrightarrow \mathfrak{d}$, we denote the image of an element $x \in G$ in $D$ as $\overline{x}$. Let $G$ be a Poisson group which admits a Drinfeld double $D$ and let us denote by $G^*$ the image in $D$ of the 1-connected integration of $\mathfrak{g}^*$.

Let us recall the integration of the Poisson structures on Poisson groups given in [43, 47]. There is a symplectic groupoid which integrates the Poisson structure on $G$. We take

$$
\mathcal{G} = \{(g, u, v, h) \in G \times G^* \times G^* \times G : \overline{gu} = v\overline{h}\},
$$

(13)

The source and target of this groupoid are the projections to $G$ and the multiplication is given by $\overline{1}$. In fact, $\mathcal{G}$ is also an integration for the Poisson structure on $G^*$: its source and target maps are the projections to $G^*$ and the multiplication is given by $\overline{2}$. See [47] for a description of the symplectic form on $\mathcal{G}$.

Take a Lagrangian subalgebra $I \subset \mathfrak{d}$. The general classification of [13] allows us to construct four associated (not necessarily smooth) homogeneous spaces in the following way. Let us denote $\mathfrak{k} = \mathfrak{g}^* \cap I$, $\mathfrak{k}' = \mathfrak{g} \cap I$ and let $K \subset G^*$, $K' \subset G$ be the connected subgroups integrating $\mathfrak{k}, \mathfrak{k}'$ respectively. In this situation we can integrate $I \times G$ with the following Lie groupoid which is an adaptation of [13] and was used in [13] to integrate the Poisson homogeneous spaces of Poisson groups:

$$
\mathcal{L} = \{(g, x, u, h) \in G \times L \times G^* \times G : \overline{gx} = u\overline{h}\},
$$

(14)

where $L \subset D$ is the connected subgroup which integrates $I \subset \mathfrak{d}$ and $a = x$ denotes again the inclusion of the corresponding element in $D$. The structure maps of $\mathcal{L}$ are analogous to those of $\mathcal{G}$, for example, the multiplication is $m((g, x, u, h), (h, y, v, k)) = (g, xy, uv, k)$. Similarly, we have an integration $\mathcal{L}'$ of the Dirac structure $I \times G^*$. The Lie algebroid $I \times G$ is integrated by

$$
\mathcal{K} = \{(g, x, u, h) \in G \times K \times G^* \times G : \overline{gx} = u\overline{h}\},
$$

(15)

with structure maps as those of $\mathcal{L}$. We have that $\mathcal{K}$ is a subgroupoid of both $\mathcal{G}$ and $\mathcal{L}$. Analogously, $\mathfrak{k}' \times G^*$ is integrated by a Lie subgroupoid $\mathcal{K}' \subset \mathcal{L}'$ defined in terms of $K'$. We can ask what are the conditions that ensure the smoothness of the following topological spaces (in which case they automatically become Poisson homogeneous spaces):

$$
G/K', \ G^*/K, \ G/\mathcal{K}, \ G/\mathcal{K}'.
$$

Notice that $\mathcal{G}/\mathcal{K}$ is a homogeneous space for $\mathcal{G} \rightrightarrows G$ while $\mathcal{G}/\mathcal{K}'$ is a homogeneous space for $\mathcal{G} \rightrightarrows G^*$.

**Proposition 6.30.** We have that $\mathcal{G}/\mathcal{K}$ is a manifold if and only if $\mathcal{K}$ is a closed subgroupoid of $\mathcal{G}$.

**Proof.** Suppose that $K$ is closed in $\overline{G^*}$. Then $\mathcal{K} \subset \mathcal{G}$ is a closed subgroupoid. The action groupoid $\mathcal{X} := K \times G \rightrightarrows \mathcal{G}$ which defines the orbit space $\mathcal{G}/\mathcal{K}$ is a proper Hausdorff Lie groupoid with trivial isotropy groups. In fact, the full action groupoid $S := \mathcal{G} \times G \rightrightarrows \mathcal{G}$ is proper. Take a product of compact subsets $X \times Y \subset \mathcal{G} \times \mathcal{G}$, then $Z = (X \times \mathcal{I}(Y)) \cap \mathcal{G} \times \mathcal{G}$ is also compact and so is $Z' = (t_\mathcal{S}, s_\mathcal{S}^{-1})(Z)$. But $Z'$ is homeomorphic to $(t_\mathcal{S}, s_\mathcal{S}^{-1})(X \times Y)$: if $(a, b) \in (t_\mathcal{S}, s_\mathcal{S}^{-1})(X \times Y)$, then $(a, b^{-1}) \in Z$ and hence $(a, b^{-1}) \in Z'$. Since $\mathcal{K} \subset \mathcal{G}$ is closed, $\mathcal{X} \rightrightarrows \mathcal{G}$ is also proper. As a consequence, the equivalence relation $R = (t_\mathcal{X}, s_\mathcal{X})(K \times G \rightrightarrows \mathcal{G})$ which defines the quotient $\mathcal{G}/\mathcal{K}$ is the image of an injective proper immersion and so it is a closed embedded submanifold such that $pr_3 : \mathcal{G} \times \mathcal{G} \rightrightarrows \mathcal{G}$ restricted to $R$ is a submersion. Therefore, Godement’s Theorem implies that $\mathcal{G}/\mathcal{K}$ is a Hausdorff manifold.

Conversely, suppose that $P = \mathcal{G}/\mathcal{K}$ is smooth. So the equivalence classes of $R$ are closed. Since $S = \{(1, x, x, 1) \in \mathcal{G} : \forall x \in K\}$ lies inside an equivalence class of $R$ it is also closed. Therefore, $\mathcal{K}$ is closed in $G^*$ as well.

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We know that $G^*/K$ is integrable provided it is smooth, see 13 or 6.1. We observe here that also $G/K$ is integrable.

**Corollary 6.31.** The Poisson manifolds $G/K$ of Proposition 6.30 are integrable by the quotient Lie groupoids

$$t_G^1L/(K \times_G G) \times (K \times_G G),$$

where $K \times_G G \supseteq G$ is the action groupoid given by the restriction of the multiplication on $G$.

**Proof.** The inclusion of Lie algebroids $\mathfrak{t} \times G \to \mathfrak{l} \times G$ is integrable by the inclusion $\mathfrak{K} \to \mathfrak{L}$ so Proposition 6.27 applied to the symplectic groupoid $G$ implies that $P = G/K$ is integrable. More concretely, the Poisson structure on $P$ is the reduction of a Dirac structure $E$ of the form $11$ which is isomorphic as a Lie algebroid to the pullback Dirac structure $t_G^1(1 \times G)$ on $G$, where $\tau_G : G \to G$ is the target map, see Lemma 6.26. Then $E \cong t_G^1(1 \times G)$ is integrable by the pullback Lie groupoid $t_G^1L$ and, by Theorem 6.22, the quotient Poisson structure on $P$ is integrable by the orbit space $t_G^1L/(K \times_G G) \times (K \times_G G)$ of the comma double Lie groupoid associated to the morphism $12$. □

**Remark 6.32.** We cannot deduce this result from the particular case of Theorem 3.2 which concerns submersions associated to quotients by free and proper Lie group actions since the dressing action on $G$ may be non complete and hence $K$ may not be an action groupoid.

Corollary 6.31 is based on the assumption that $G$ admits a Drinfeld double. In the next paragraph we shall adapt this integration result to connected Poisson groups at the expense of requiring an additional hypothesis on the source-simply-connected integration of the Poisson structure.

**Integration in the case of a connected Poisson group.** Suppose that $G$ is a connected Poisson group. Then there is a surjective Poisson group morphism $q : \tilde{G} \to G$ where $\tilde{G}$ is a connected Poisson group with the same Lie bialgebra. Since the Poisson tensor of $G$ vanishes on 1, the Poisson tensor of $\tilde{G}$ vanishes on $Z := \ker q$. As we have said, $\tilde{G}$ admits a Drinfeld double so we can define the corresponding Lie groupoids $\tilde{G}$, $K$, $L$.

**Lemma 6.33.** We have that $Z$ acts on $G$ and on $L$ by automorphisms and the map $\Phi : K \to L$ given by $\Phi(a, x, u, b) = (a, i(x), u, b)$ is $Z$-equivariant, where $i : K \to L$ is the inclusion.

**Proof.** The Poisson tensor $\pi$ on $\tilde{G}$ can be expressed in terms of the inclusions into $D$ and the projections $\text{pr}_1 : \mathfrak{d} \to \mathfrak{g}$, $\text{pr}_2 : \mathfrak{d} \to \mathfrak{g}^*$ as follows, see [47, Proposition 2.31]:

$$(r_{\alpha, \beta})(\alpha, \beta) = -\langle \text{pr}_1 \text{Ad}_{\mathfrak{d}}^{-1} \alpha, \text{pr}_2 \text{Ad}_{\mathfrak{d}}^{-1} \beta \rangle$$

for all $\alpha, \beta \in \mathfrak{g}^*$ and all $a \in G$. Using the explicit description of the Lie bracket on $\mathfrak{d}$ we get that $\text{pr}_2 \text{Ad}_{\mathfrak{d}}^{-1} \beta = \text{Ad}_{\mathfrak{a}}^\ast \beta$, where $\text{Ad}^\ast$ is the coadjoint action of $\tilde{G}$ on $\mathfrak{g}^*$. So we have that

$$\langle \text{pr}_1 \text{Ad}_{\mathfrak{d}}^{-1} \alpha, \text{Ad}_{\mathfrak{a}}^\ast \beta \rangle = 0$$

for all $\alpha, \beta \in \mathfrak{g}^*$ and all $a \in Z$. As a consequence, for all $\alpha$ we have that $\text{pr}_1 \text{Ad}_{\mathfrak{d}}^{-1} \alpha = 0$ and hence the adjoint action of $Z$ on $\mathfrak{d}$ fixes $\mathfrak{g}^*$. Therefore, for all $z \in Z$ and all $v \in G^*$ there exists $w \in G^*$ such that $zv = w\tau$, in such a situation we denote $w = zv$. Let us define an action of $Z$ on $G$ as follows:

$$z \cdot (a, u, v, b) = (za, u, zv, zb).$$

This is a free and proper action by automorphisms whose orbit space is a Lie groupoid which integrates the Poisson structure on $G$.

We can also define an action by automorphisms of $Z$ on $L$ as in (11) by means of the same formula: $z \cdot (a, x, u, b) = (za, x, z^2u, zb)$ for all $z \in Z$ and $(a, x, u, b) \in L$. So the map $\Phi$ as before is $Z$-equivariant. □
Theorem 6.34. Let $G$ be a connected Poisson group and let $P$ be a Poisson homogeneous space of the form $\Sigma(G)/K'$, where $K' \rightrightarrows G$ is source-connected. Suppose that the Dirac structure associated to $P$ is of the form $I \times G$ under the left trivialization $T G \cong g \times G$, where $I$ is a Lagrangian subalgebra of the double $\partial$. Then $P$ is integrable.

Proof. Let $P$ be a Poisson homogeneous space of the form $\Sigma(G)/K'$ whose associated Dirac structure is isomorphic to $I \times G \to \partial \times G$ via left trivialization. By Lie's second theorem there is a Lie groupoid morphism $\Psi : \Sigma(G) \to G/\mathbb{Z}$ which covers the identity on $T^* G \cong g^* \times G$. Since $K'$ is source-connected and $\text{Lie}(K') = \text{Lie}(K/\mathbb{Z}) \cong (I \cap g^*) \times G$, we have that $\Psi(K') \subset K/\mathbb{Z}$. So we can compose $\Psi|_{K'}$ with the morphism $\overline{\Phi} : K/\mathbb{Z} \to L/\mathbb{Z}$ induced by $\Phi$ as in the previous lemma and satisfy the condition of Proposition 6.27. As a consequence, $P$ is integrable.

To conclude we shall see a couple of simple families of Poisson homogeneous spaces of Poisson groups, their Hamiltonian counterparts and their integrations according to Corollary 6.31.

Example 6.35. Let $G$ be a 1-connected complete Poisson group and let $G^*$ be the 1-connected integration of $\mathfrak{g}^*$. In this situation, the dressing actions define a Lie group structure on $D = G \times G^*$ which integrates $\partial$ and so $D$ is a Drinfeld double for $G$ [47, Proposition 2.43]. As a consequence, $\Sigma(G) \rightrightarrows G$ is an action groupoid $G \times G^* \rightrightarrows G$ which is isomorphic to $G \rightrightarrows G$ as in (15). Let $I \subset \partial$ be a Lagrangian subalgebra which corresponds to a Poisson homogeneous space of $G^*$. Then $K$ as in (15) is isomorphic to an action groupoid as well $G \times K \rightrightarrows G$, where $K$ is $*$-is the connected integration of $\mathfrak{k}$. Therefore, we have the following: the Poisson homogeneous space $P = G/K$ is diffeomorphic to the quotient $(G \times G^*)/K$, where the $K$-action is defined as $x \cdot (a, u) = (a \cdot x^{-1}, xu)$ for all $(a, u) \in G \times G^*$ and all $x \in K$. So $P$ is a fiber bundle over $G^*/K$ with typical fiber $G$:

$$G \hookrightarrow P \to G^*/K.$$ 

Similarly, the integration of $P$ that we get by applying the argument in Proposition 6.27 is a fiber bundle construction. We have that $L$ as in (14) is an action groupoid $G \times L \rightrightarrows G$. The pullback groupoid $\Sigma L = \mathfrak{g}^* \times G^*$ is isomorphic to the product groupoid of $L$ and the pair groupoid $G^* \times G^* \rightrightarrows G^*$. Now $K \times K$ acts on $L$ by means of the action map $(x, y) \cdot (a, l) = (a \cdot x^{-1}, xly^{-1})$ for all $(x, y, a, l) \in K \times K \times G \times L$. On the other hand, we have a principal $K \times K$-action on $G^* \times G^*$ defined by $(x, y) \cdot (u, v) = (xu, yv)$ for all $(x, y, u, v) \in K \times K \times G^* \times G^*$. Both actions are multiplicative in the sense that they are groupoid morphisms:

$$(K \times K) \times L \to L,$$

$$(K \times K) \times (G^* \times G^*) \to G^* \times G^*,$$

with respect to the pair groupoid structure $K \times K \rightrightarrows K$. More precisely, they are Lie 2-group actions. So the quotient $(G^* \times G^*)/(K \times K)$ is isomorphic to the pair groupoid over $G^*/K$ and the Lie groupoid $P$ integrating $P$ that the proof of Proposition 6.27 gives us is a fibration of Lie groupoids:

$$L \to P \to (G^*/K) \times (G^*/K).$$

The following simple family of examples shows that there is not an obvious relationship such as (weak) Morita equivalence between the Poisson structure on $P = G/K$ and the Poisson structure on $G^*/K$.

Definition 6.36 [26]. Let $(G, \pi)$ be a Poisson group. An affine Poisson structure on $G$ is a Poisson structure $\Pi$ on $G$ such that the multiplication map $\mathfrak{m} : (G, \pi) \times (G, \Pi) \to (G, \Pi)$ is a Poisson map.

Affine Poisson structures are the same as Poisson homogeneous space structures on $G$. In particular, these structures are classified by Lagrangian subalgebras $l \subset \partial$ such that $l \cap \mathfrak{g} = \{0\}$.
Example 6.37. Take a Lagrangian subalgebra \( l \subset \mathfrak{d} \) such that \( l \cap \mathfrak{g}^* = \{0\} \). In this situation, the classification theorem of [45] implies that we can take \( \mathcal{G} := \Sigma(G) \) as the underlying manifold of a Poisson homogeneous space associated to \( l \times G \) (provided that it is Hausdorff). Since \( l \times G \) is isomorphic as a Lie algebroid to the cotangent Lie algebroid \( \mathfrak{g}^* \times G \cong T^*G \) [47, Ch. 5], we can integrate \( l \times G \) with the Lie groupoid \( \mathcal{G} \rightrightarrows G \) itself. But then the pullback groupoid\( \mathcal{G} \times \mathcal{G} \), which integrates \( \mathcal{G} \times \mathcal{G} \), is isomorphic to the pair groupoid \( \mathcal{G} \times \mathcal{G} \). Therefore, \( \mathcal{G} \times \mathcal{G} \) is already a symplectic groupoid which means that \( E \) is the graph of a symplectic structure. So we see that under the correspondence of Proposition 6.30 an affine Poisson group \( G^* \) corresponds to what might be called an “affine symplectic structure” on \( \mathcal{G} \), since it is determined by a symplectic structure on \( \mathcal{G} \) such that the action by right multiplication of \( G \) on itself is a symplectic action. See [12] for the definition of affine tensors on Lie groupoids.

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References

[1] A. Alekseev, H. Bursztyn, and E. Meinrenken. “Pure spinors on Lie groups”. In: Ásterisque (2009), pp. 131–199.
[2] A. Alekseev and Y. Kosmann-Schwarzbach. “Manin pairs and moment maps”. In: Journal of Differential geometry 56.1 (2000), pp. 133–165.
[3] A. Alekseev, A. Malkin, and E. Meinrenken. “Lie group valued moment maps”. In: Journal of Differential Geometry 48.3 (1998), pp. 445–495.
[4] A. Alekseev, E. Meinrenken, and Y. Kosmann-Schwarzbach. “Quasi-Poisson manifolds”. In: Canadian Journal of Mathematics 54 (2002), pp. 3–29.
[5] A. Alekseev and P. Xu. “Derived brackets and Courant algebroids”. Unfinished manuscript. 2002.
[6] D. Álvarez. “Integrability of quotients in Poisson and Dirac geometry”. PhD thesis. IMPA, 2019.
[7] D. Álvarez. On the symplectic leaves of Poisson groupoids. 2019. url: https://arxiv.org/abs/1909.08111.
[8] C. Blohmann and A. Weinstein. Hamiltonian Lie algebroids. 2018. url: https://arxiv.org/abs/1811.11143.
[9] O. Brahic and R. L. Fernandes. “Integrability and reduction of Hamiltonian actions on Dirac manifolds”. In: Indagationes Mathematicae 25.5 (2014), pp. 901–925.
[10] R. Brown and K. C. H. Mackenzie. “Determination of a double Lie groupoid by its core diagram”. In: Journal of pure and applied algebra 80.3 (1992), pp. 237–272.
[11] H. Bursztyn and A. Cabrera. “Multiplicative forms at the infinitesimal level”. In: Math. Annalen 353 (2012), pp. 663–705.
[12] H. Bursztyn and T. Drummond. Lie theory of multiplicative tensors. 2018. url: https://arxiv.org/abs/1905.11453.
[13] H. Bursztyn, D. Iglesias Ponte, and Jiang-Hua Lu. Dirac geometry and integration of Poisson homogeneous spaces. 2019. url: https://arxiv.org/abs/1905.11453.
[14] H. Bursztyn, D. Iglesias Ponte, and P. Severa. “Courant morphisms and moment maps”. In: Mathematical Research Letters 16.2 (2009), pp. 215–232.
[15] H. Bursztyn, M. Crainic, A. Weinstein, and C. Zhu. “Integration of twisted Dirac brackets”. In: Duke Mathematical Journal 123.3 (2004), pp. 549–607.
[16] A. Cannas da Silva and A. Weinstein. Geometric models for noncommutative algebras. Vol. 10. American Mathematical Soc., 1999.
[17] A. Cattaneo and G. Felder. “Poisson sigma models and symplectic groupoids”. In: Quantization of singular symplectic quotients. Springer, 2001, pp. 61–93.

[18] T. Courant. “Dirac manifolds”. In: Transactions of the American Mathematical Society 319.2 (1990), pp. 631–661.

[19] T. Courant and A. Weinstein. “Beyond poisson structures”. In: Séminaire sud-rhodanien de géométrie VIII. Travaux en Cours 27, Hermann, Paris 27 (1988), pp. 39–49.

[20] M. Crainic. “Differentiable and algebroid cohomology, Van Est isomorphisms, and characteristic classes”. In: Comm. Math. Helvetici (2003).

[21] M. Crainic and R. L. Fernandes. “A geometric approach to Conni’s linearization theorem”. In: Annals of Mathematics 173.2 (2011), pp. 1121–1139.

[22] M. Crainic and R. L. Fernandes. “Integrability of Lie brackets”. In: Annals of Mathematics (2003), pp. 575–620.

[23] M. Crainic and R. L. Fernandes. “Integrability of Poisson brackets”. In: Journal of Differential Geometry 66.1 (2004), pp. 71–137.

[24] M. Crainic, R. L. Fernandes, and D. Martínez. “Poisson manifolds of compact types (PMCT 1)”. In: Journal für die reine und angewandte Mathematik (Crelles Journal) (2017).

[25] P. Dazord. “Groupe d’holonomie et géométrie globale”. In: C. R. Acad. Sci. Paris 324.1 (1997), pp. 77–80.

[26] P. Dazord and D. Sondaz. “Groupes de Poisson affines”. In: Proceedings of the Seminar Sud-Rhodanien de Geometrie. Springer-Verlag, 1989.

[27] V. G. Drinfeld. “Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equations”. In: Soviet Math. Dokl. 27 (1983), pp. 68–71.

[28] V. G. Drinfeld. “On Poisson homogeneous spaces of Poisson-Lie groups”. In: Theoretical and Mathematical Physics 95.2 (1993), pp. 524–525.

[29] V. G. Drinfeld. “Quantum groups”. In: Proceedings of the International Congress of Mathematicians 1 (1987), pp. 798–820.

[30] J. P. Dufour and N. T. Zung. Poisson structures and their normal forms. Vol. 242. Birkhäuser, 2006.

[31] R. L. Fernandes, J. P. Ortega, and T. S. Ratiu. “The moment map in Poisson geometry”. In: American Journal of Mathematics 131.5 (2009), pp. 1261–1310.

[32] R. L. Fernandes and D. Iglesias Ponte. “Integrability of Poisson-Lie group actions”. In: Letters in Mathematical Physics 90.1 (2009), pp. 137–159.

[33] R. L. Fernandes and I. Struchiner. The global solutions to Cartan’s realization problem. 2019. url: https://arxiv.org/abs/1907.13614v1

[34] V. Ginzburg. “Grothendieck groups of Poisson vector bundles”. In: Journal of Symplectic Geometry 1.1 (2001), pp. 121–170.

[35] E. Hawkins. “A groupoid approach to quantization”. In: Journal of Symplectic Geometry 6.1 (2008), pp. 61–125.

[36] P. J. Higgins and K. C. H. Mackenzie. “Algebraic constructions in the category of Lie algebroids”. In: Journal of Algebra 129.1 (1990), pp. 194–230.

[37] B. Hoffman, R. Sjamaar, and C. Zhu. Stacky Hamiltonian actions and symplectic reduction. 2019. url: https://arxiv.org/abs/1808.01003v2

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[38] M. Jotz. “Dirac Lie groups, Dirac homogeneous spaces and the theorem of Drinfeld”. In: Indiana University Mathematics Journal (2011), pp. 319–366.

[39] M. Jotz. “The leaf space of a multiplicative foliation”. In: Journal of Geometric Mechanics 4.3 (2012), pp. 313–332.

[40] M. Jotz and C. Ortiz. “Foliated groupoids and infinitesimal ideal systems”. In: Indagationes Mathematicae 25 (2014), pp. 1019–1053.

[41] M. V. Karasev. “Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets”. In: Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya 50.3 (1986), pp. 508–538.

[42] D. Li-Bland and E. Meinrenken. “Courant algebroids and Poisson geometry”. In: International Mathematics Research Notices 2009.11 (2009), pp. 2106–2145.

[43] D. Li-Bland and E. Meinrenken. “Dirac Lie groups”. In: Asian Journal of Mathematics 18.5 (2014), pp. 779–816.

[44] D. Li-Bland and P. Severa. “Quasi-Hamiltonian groupoids and multiplicative Manin pairs”. In: International Mathematics Research Notices 10 (2011), pp. 2295–2350.

[45] Zhang-Ju Liu, A. Weinstein, and P. Xu. “Dirac structures and Poisson homogeneous spaces”. In: Communications in mathematical physics 192.1 (1998), pp. 121–144.

[46] Zhang-Ju Liu, A. Weinstein, and P. Xu. “Manin triples for Lie bialgebroids”. In: Journal of Differential Geometry 45.3 (1997), pp. 547–574.

[47] Jiang-Hua Lu. “Multiplicative and affine Poisson structures on Lie groups”. PhD thesis. University of California, Berkeley, 1990.

[48] Jiang-Hua Lu and A. Weinstein. “Groupoïdes symplectiques doubles des groupes de Lie-Poisson”. In: C. R. Acad. Sci. Paris 309.18 (1989), pp. 951–954.

[49] K. C. H. Mackenzie. “Double Lie algebroids and second-order geometry, I”. In: Advances in Mathematics 94.2 (1992), pp. 180–239.

[50] K. C. H. Mackenzie. General theory of Lie groupoids and Lie algebroids. Vol. 213. Cambridge University Press, 2005.

[51] K. C. H. Mackenzie and P. Xu. “Integration of Lie bialgebroids”. In: Topology 39.3 (2000), pp. 445–467.

[52] K. C. H. Mackenzie and P. Xu. “Lie bialgebroids and Poisson groupoids”. In: Duke Mathematical Journal 73.2 (1994), pp. 415–452.

[53] E. Meinrenken. “Dirac actions and Lu’s Lie algebroid”. In: Transformation Groups 22.4 (2016), pp. 1081–1124.

[54] E. Meinrenken. “Poisson geometry from a Dirac perspective”. In: Letters in Mathematical Physics 108.3 (2018), pp. 447–498.

[55] K. Mikami and A. Weinstein. “Moments and reduction for symplectic groupoids”. In: Publ. Res. Inst. Math. Sci. 24.1 (1988), pp. 121–140. issn: 0034-5318.

[56] I. Moerdijk and J. Mrčun. Introduction to foliations and Lie groupoids. Vol. 91. Cambridge University Press, 2003.

[57] I. Moerdijk and J. Mrčun. “On integrability of infinitesimal actions”. In: American Journal of Mathematics 124.3 (2000), pp. 567–593.

[58] T. Mokri. “Matched pairs of Lie algebroids”. In: Glasgow Mathematical Journal 39.2 (1997), pp. 167–181.

[59] C. Ortiz. “Multiplicative Dirac structures”. In: Pac. Journal of Mathematics 266.2 (2013), pp. 329–365.
C. Ortiz. “Multiplicative Dirac structures on Lie groups”. In: *Comptes Rendus Mathématique* 346.23-24 (2008), pp. 1279–1282.

D. Iglesias Ponte, C. Laurent-Gengoux, and P. Xu. “Universal lifting theorem and quasi-Poisson groupoids”. In: *Journal of the European Mathematical Society* 14.3 (2012), pp. 681–731.

P. Robinson. “The classification of Dirac homogeneous spaces”. PhD thesis. University of Toronto, 2014.

D. Roytenberg. “Courant algebroids, derived brackets and even symplectic supermanifolds”. PhD thesis. University of California, Berkeley, 1999.

M. A. Semenov-Tian-Shansky. “Dressing transformations and Poisson group actions”. In: *Publications of the Research Institute for Mathematical Sciences* 21.6 (1985), pp. 1237–1260.

J. P. Serre. *Lie algebras and Lie groups*. 2nd ed. Springer-Verlag, 1991.

M. Spivak. *A comprehensive introduction to differential geometry*. Vol. 1. Publish or Perish, 1970.

L. Stefani. “On morphic actions and integrability of LA-groupoids”. PhD thesis. ETH Zürich, 2008.

K. Uchino. “Remarks on the definition of a Courant algebroid”. In: *Letters in Mathematical Physics* 60.2 (2002), pp. 172–175.

A. Y. Vaintrob. “Lie algebroids and homological vector fields”. In: *Russian Mathematical Surveys* 52.2 (1997), pp. 428–429.

J. Villatoro. “Poisson manifolds and their associated stacks”. In: *Letters in Mathematical Physics* 108.3 (2018), pp. 897–926.

A. Weinstein. “Coisotropic calculus and Poisson groupoids”. In: *Journal of the Mathematical Society of Japan* 40.4 (1988), pp. 705–727.

A. Weinstein. “Symplectic groupoids and Poisson manifolds”. In: *Bulletin of the American mathematical Society* 16.1 (1987), pp. 101–104.

A. Weinstein and P. Xu. “Extensions of symplectic groupoids and quantization”. In: *J. reine angew. Math* 417 (1991), pp. 159–189.

P. Xu. “On Poisson groupoids”. In: *International Journal of Mathematics* 6.01 (1995), pp. 101–124.