Pure Gaussian state generation via dissipation:
A quantum stochastic differential equation approach

Naoki Yamamoto
Department of Applied Physics and Physico-Informatics,
Keio University, Hiyoshi 3-1-14, Kohoku, Yokohama, Japan

Abstract

Recently the complete characterization of a general Gaussian dissipative system having a unique pure steady state was obtained in [Koga & Yamamoto 2012, Phys. Rev. A 85, 022103]. This result provides a clear guideline for engineering an environment such that the dissipative system has a desired pure steady state such as a cluster state. In this paper, we describe the system in terms of a quantum stochastic differential equation (QSDE) so that the environment channels can be explicitly dealt with. Then a physical meaning of that characterization, which cannot be seen without the QSDE representation, is clarified; more specifically, the nullifier dynamics of any Gaussian system generating a unique pure steady state is passive. In addition, again based on the QSDE framework, we provide a general and practical method to implement a desired dissipative Gaussian system, which has a structure of quantum state transfer.

*Electronic address: yamamoto@appi.keio.ac.jp
Towards quantum state preparation, which clearly plays a key part in quantum information processing, recently several dissipation-based approaches have been proposed. The basic idea of those approaches originates from the trivial fact that a thermal environment drives any state to the stable ground state. However, it has been shown that we can sometimes engineer a desired dissipative environment such that the corresponding stable state is a nontrivial and useful one, e.g., a highly entangled pure state \([1-14]\). More specifically, under some conditions we are allowed to synthesize an open quantum system described by the Markovian master equation

\[
\frac{d\hat{\rho}_t}{dt} = -i[\hat{H}, \hat{\rho}_t] + \sum_{k=1}^{m} \left( \hat{L}_k \hat{\rho}_t \hat{L}_k^\dagger - \frac{1}{2} \hat{L}_k^\dagger \hat{L}_k \hat{\rho}_t - \frac{1}{2} \hat{\rho}_t \hat{L}_k \hat{L}_k^\dagger \right),
\]  

such that \(\hat{\rho}_t\) must converge into a given desired pure state \(\hat{\rho}_\infty\); that is, the Hamiltonian \(\hat{H}\) and the dissipative channel \(\hat{L}_k\) \((k = 1, \ldots, m)\) are appropriately synthesized to achieve this goal. One of the main advantages of this approach is that the target state \(\hat{\rho}_\infty\) is clearly robust against any perturbation to the state \(\hat{\rho}_t\) during the dynamical process. In particular, it is independent on the initial state preparation.

In the finite dimensional case a necessary and sufficient condition for Eq. (1) to have a pure steady state was obtained in \([2, 4]\), and especially in \([4]\) the authors provided a sufficient condition for \(\hat{\rho}_\infty\) to be unique. The uniqueness characterization is of particular importance, because without such condition the desired convergence into the target state cannot be guaranteed. For infinite-dimensional systems, on the other hand, in \([13]\) the author particularly focused on a general Gaussian dissipative system and provided a complete parameterization of the system having a unique pure steady state. The merit of focusing on the class of Gaussian systems lies not only in its importance in quantum information technologies \([15, 16]\) but also in the fact that the parameterization is obtained in an easily-tractable manner in the phase space; actually the uniqueness of \(\hat{\rho}_\infty\) can be readily checked by simply calculating the rank of a specific matrix, while in the finite-dimensional case \([4]\) we are required to verify that there is no specific subspace in the Hilbert space.

In this paper, we study a Gaussian system having a unique pure steady state in terms of a quantum stochastic differential equation (QSDE) \([17-20]\). The use of a QSDE allows us to describe the dynamics of an open system in a form where the stochastic environment channels appear explicitly. The master equation (1) is obtained as a result of averaging out all such stochastic effects brought from the environment.

The contribution of this paper is twofold. The first one is that we clarify the physical meaning of the conditions for the Gaussian system to have a unique pure steady state, which cannot be clearly seen when dealing with only the master equation. This result is obtained through investigating the QSDE of the corresponding (complex) nullifier. In general, it is known that any pure Gaussian state can be characterized as the common zero eigenstate of the corresponding nullifier operators \([21]\); this is the reason why investigating the full behavior of the nullifier provides new information about the dynamic process towards the target pure state. Actually, we show that the nullifier dynamics of any Gaussian system
generating a unique pure steady state is *passive*. As a byproduct, the result is used to show a certain trade-off between the closeness of the steady state to a target *Gaussian cluster state* \((22, 25)\) and the convergence time into that steady state.

In the previous result \([13]\), although the mathematical characterization of the desired dissipative channel \(\hat{L}_k\) was obtained, its actual implementation was not discussed. Actually, the resulting desired dissipative channels usually have to non-locally act on the system, and no general method to effectively implement such dissipative channels is known. The second contribution of this paper is to give a partial answer to the question of how to practically construct a desired dissipative system. The proposed scheme has a structure of *quantum state transfer* from light to a matter \([26, 28]\); more specifically, a desired state of light is first generated and then that light field interacts with the oscillator system (memory), which as a result acquires the desired state by dissipation. This scheme is indeed practical, because, as shown in \([25]\), any pure Gaussian cluster state of light can be effectively generated using some beam splitters and OPOs. Note that the QSDE approach actually has to be taken in order to explicitly describe the input light field.

We use the following notations: for a matrix \(A = (a_{ij})\), the symbols \(A^\dagger\), \(A^\top\), and \(A^\#\) represent its Hermitian conjugate, transpose, and elementwise complex conjugate of \(A\), i.e., \(A^\dagger = (a^*_{ji})\), \(A^\top = (a_{ji})\), and \(A^\# = (a^*_{ij}) = (A^\dagger)^\top\), respectively. For a matrix of operators, \(\hat{A} = (\hat{a}_{ij})\), we use the same notation, in which case \(\hat{a}^*_{ij}\) denotes the adjoint to \(\hat{a}_{ij}\). \(I_n\) denotes the \(n \times n\) identity matrix. \(\Re\) and \(\Im\) denote the real and imaginary parts, respectively.

**II. PRELIMINARIES**

In this section, a brief introduction to a Gaussian system and its QSDE representation is given. Then we review the result of \([13]\).

**A. Gaussian dissipative systems**

A general \(n\)-mode bosonic system consists of \(n\) subsystems with canonical conjugate pairs \((\hat{q}_i, \hat{p}_i)\). Denote the vector of total system variables by \(\hat{x} := (\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n)^\top\). The canonical commutation relation \([\hat{q}_i, \hat{p}_j] = i\delta_{ij}\) then leads to

\[
\hat{x}^\dagger \hat{x} - (\hat{x}^\dagger \hat{x})^\top = i\Sigma, \quad \Sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.
\]

Now let \(\hat{\rho}\) be the density operator of this system and write the mean vector by \(\langle \hat{x} \rangle\) and the covariance matrix by \(V = \langle \Delta \hat{x} \Delta \hat{x}^\dagger + (\Delta \hat{x} \Delta \hat{x}^\dagger)^\top \rangle / 2\), \(\Delta \hat{x} = \hat{x} - \langle \hat{x} \rangle\), where the mean \(\langle \hat{X} \rangle = \text{Tr} (\hat{X} \hat{\rho})\) is taken elementwise. Note that the uncertainty relation \(V + i\Sigma/2 \geq 0\) holds. A Gaussian state can be characterized by only the mean vector and the covariance matrix. A particularly important fact is that the covariance matrix \(V\) corresponding to a pure Gaussian state always has the following general representation \([21, 29]\):

\[
V = \frac{1}{2} S S^\top, \quad S = \begin{pmatrix} Y^{-1/2} & 0 \\ XY^{-1/2} & Y^{1/2} \end{pmatrix},
\tag{2}
\]
where $X$ and $Y$ are $n \times n$ real symmetric and real positive definite matrices (i.e., $Y = Y^\top > 0$), respectively. In other words, a pure Gaussian state is completely parameterized by $X$ and $Y$. An important merit of this representation is that the complex graph matrix $Z := X + iY$ can be used for a graphical calculus for several Gaussian pure states \cite{21}. In particular, a pure Gaussian state $|\psi_Z\rangle$ having the covariance matrix (2) always satisfies

$$\hat{r}|\psi_Z\rangle = 0, \quad \hat{r} := (-Z, I_n)\hat{x} = \begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix} - Z \begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{pmatrix},$$

where the equation means that each entry of $\hat{r}$ acts on $|\psi_Z\rangle$. Conversely, if a pure Gaussian state $|\psi\rangle$ satisfies $(-Z, I_n)|\psi\rangle = 0$, then we have $|\psi\rangle = |\psi_Z\rangle$. The vector of operators $\hat{r}$ is called the nullifier for the pure Gaussian state $|\psi_Z\rangle$.

A linear system is such that the Hamiltonian $\hat{H}$ and $k$-th dissipative channel $\hat{L}_k$ in Eq. (I) are respectively characterized by

$$\hat{H} = \frac{1}{2} \hat{x}^\top G \hat{x}, \quad \hat{L}_k = c_k^\top \hat{x},$$

where $G = G^\top \in \mathbb{R}^{2n \times 2n}$ and $c_k \in \mathbb{C}^{2n}$. For this system, the time-evolution of $\langle \hat{x}_t \rangle$ and $V_t$ with the state $\hat{p}_t$ obeying Eq. (I) are given by $d\langle \hat{x}_t \rangle / dt = A\langle \hat{x}_t \rangle$ and $dV_t / dt = AV_t + V_t \hat{A}^\top + D$, respectively. Here, $A = \Sigma[G + \Im(C^\dagger C)]$ and $D = \Sigma \Re(C^\dagger C) \Sigma^\top$ with $C = (c_1, \ldots, c_m)^\top \in \mathbb{C}^{m \times 2n}$ (see \cite{20} for more detailed discussion). In this paper, we assume that the initial state of the dynamics is Gaussian; then, at any given time $t$ the state is also Gaussian with mean $\langle \hat{x}_t \rangle$ and covariance $V_t$, hence let us call such a linear system the Gaussian system. A steady state of the Gaussian system exists only when $A$ is a Hurwitz matrix, i.e., all the eigenvalues of $A$ have negative real parts. If it exists, the mean vector is $\langle \hat{x}_\infty \rangle = 0$ and the covariance matrix $V_\infty$ is given by the unique solution to the following matrix equation:

$$AV_\infty + V_\infty A^\top + D = 0. \tag{5}$$

**B. The QSDE framework**

The situation we have in mind is that the system interacts with countable set of environment channels. The time-evolution of an observable of this open system is described in terms of a QSDE. A most simple form of this equation is obtained when the environment channels are all independent vacuum fields with ideal Markovian approximation taken. Let $\hat{a}_i(t)$ be the annihilation operator of the $i$-th vacuum field; then the Markovian approximation means that $\hat{a}_i(t)$ instantaneously interacts with the system and satisfies the CCR $[\hat{a}_i(s), \hat{a}_i^\dagger(t)] = \delta_{ij}\delta(t-s)$. Define the field annihilation process operator by $\hat{A}_i(t) = \int_0^t \hat{a}_i(s)ds$, then this CCR leads to the following quantum Ito rule:

$$d\hat{A}_i d\hat{A}_j^\dagger = \delta_{ij} dt, \quad d\hat{A}_i d\hat{A}_j = d\hat{A}_i^\dagger d\hat{A}_j^\dagger = d\hat{A}_i d\hat{A}_j^\dagger = 0. \tag{6}$$

The system-field coupling in the time interval $[t, t + dt)$ is described by the unitary operation $\hat{U}(t + dt, t) = \exp[\sum_i (\hat{L}_i d\hat{A}_i^\dagger - \hat{L}_i^\dagger d\hat{A}_i)]$, where $\hat{L}_i$ is the system operator representing the
coupling with the \(i\)-th vacuum field. Then the system observable at time \(t\), \(j_t(\hat{X}) = \hat{U}_t^* \hat{X} \hat{U}_t\), obeys the Ito-type QSDE
\[
dj_t(\hat{X}) = j_t(i[\hat{H}, \hat{X}]) + \sum_{i=1}^m \left( \hat{L}_i^* \hat{X} \hat{L}_i - \frac{1}{2} \hat{L}_i^* \dot{\hat{L}}_i \hat{X} - \frac{1}{2} \dot{\hat{L}}_i^* \hat{L}_i \hat{X} \right) dt \\
+ \sum_{i=1}^m \left( j_t([\hat{X}, \hat{L}_i]) d\hat{A}_i^* - j_t([\hat{X}, \hat{L}_i]) d\hat{A}_i \right),
\]
(7)

where an additional system Hamiltonian \(\hat{H}\) has been added. Note that \(\hat{U}_{t+dt} = \hat{U}(t, t+dt)\hat{U}_t\).

The mean value \(\langle j_t(\hat{X}) \rangle\) is represented using the (unconditional) density operator \(\hat{\rho}_t\) by \(\langle j_t(\hat{X}) \rangle = \text{Tr}(\hat{X} \hat{\rho}_t)\), which leads to the master equation (1). The change of the field operator can also be dealt with explicitly; the output field \(\hat{A}_i := j_t(\hat{A}_i)\) after the interaction satisfies
\[
d\hat{A}_i = j_t(\hat{L}_i) dt + d\hat{A}_i.
\]
(8)

We are interested in the QSDE whose system Hamiltonian and dissipative channels are given by Eq. (1). Let us define \(\hat{A}_t = (\hat{A}_1, \ldots, \hat{A}_m)^\top\), then the vector of system quadratures \(\hat{x}_t = (j_t(\hat{q}_1), \ldots, j_t(\hat{q}_n), j_t(\hat{p}_1), \ldots, j_t(\hat{p}_n))^\top\) satisfies the linear QSDE [20, 30–35]:
\[
d\hat{x}_t = A\hat{x}_t dt - i\Sigma C^\dagger d\hat{A}_t + i\Sigma C^\top d\hat{A}_t^2,
\]
(9)

where the system matrices \(A\) and \(C\) were defined in Section 2 (a). It is easy to see that the linear QSDE (9) actually leads to the time-evolutions of the mean and the covariance matrix:
\[
d\langle \hat{x}_t \rangle / dt = A\langle \hat{x}_t \rangle \text{ and } d\Sigma / dt = AV + VA^\top + D. \]

Also the output field equation (8) of \(\hat{A}_t = (\hat{A}_1, \ldots, \hat{A}_m)^\top\) then becomes
\[
d\hat{A}_t = C\hat{x}_t dt + d\hat{A}_t.
\]
(10)

C. The dissipative Gaussian system generating a pure steady state

In [13], some conditions for a dissipative Gaussian system to have a unique pure steady state were obtained. A particularly useful result from the environment engineering viewpoint is the following (recall \(Z := X + iY\)):

**Theorem 1** [13]: Let \(V\) be a given covariance matrix of the form (2). Then, this is the unique solution of Eq. (5) if and only if the system matrices are represented by
\[
C = P^\top (-Z, I_n),
\]
\[
G = \begin{pmatrix}
XRX + YRY - \Gamma Y^{-1}X - XY^{-1}\Gamma^\top & -XR + \Gamma Y^{-1} \\
-RX + Y^{-1}\Gamma^\top & R
\end{pmatrix},
\]
(11)
(12)

where \(P\) is a complex \(n \times m\) matrix, \(R\) is a real \(n \times n\) symmetric matrix, and \(\Gamma\) is a real \(n \times n\) skew symmetric matrix (i.e., \(\Gamma + \Gamma^\top = 0\)), and moreover, \(P\) and \(Q := -iRY - Y^{-1}\Gamma^\top\) satisfy the following rank condition:
\[
\text{rank}(P, PQ, \ldots, Q^{n-1}P) = n.
\]
(13)
This theorem states that any dissipative linear system having a unique pure Gaussian steady state is completely parameterized by the three matrices $P, R$, and $\Gamma$, which further have to satisfy the rank condition (13). In [13], this result was obtained through a fully algebraic treatment of Eq. (5), and the physical meanings of the conditions were not discussed. As mentioned in Section 1, nevertheless, they will be clarified within the QSDE framework; for convenience of the later discussion, we note that $G$ satisfies

$$G \Sigma^\top \begin{pmatrix} -Z \\ I_n \end{pmatrix} = \begin{pmatrix} -Z \\ I_n \end{pmatrix} Q.$$  

(14)

III. DYNAMICS OF THE NULLIFIER

As seen in Eq. (3), the pure state $|\psi_Z\rangle$ is the common zero-eigenstate of the nullifier vector $\hat{r} = (Z, I_n)\hat{x}$. Hence it is worth to see the time-evolution of $\hat{r}_t$, when the conditions shown in Theorem 1 are satisfied. Noting Eqs. (11) and (14), we have

$$(-Z, I)A = (-Z, I)\Sigma G + \frac{1}{2i}(-Z, I)\Sigma(C^\dagger C - C^\top C^\sharp)
= Q^\top(-Z, I) + \frac{1}{2i}(-I, -Z)\left\{ \begin{pmatrix} -Z^\sharp \\ I \end{pmatrix} P^\sharp P^\top(-Z, I) - \begin{pmatrix} -Z \\ I \end{pmatrix} PP^\dagger(-Z^\sharp, I) \right\}
= Q^\top(-Z, I) + \frac{Z^\sharp - Z}{2i}P^\sharp P^\top(-Z, I) = (Q^\top - YP^\sharp P^\top)(-Z, I),$$

$$(-Z, I)(-i\Sigma C^\dagger) = -i(-Z, I)\Sigma \begin{pmatrix} -Z^\sharp \\ I \end{pmatrix} P^\sharp = i(Z - Z^\sharp)P^\sharp = -2YP^\sharp,$$

$$(-Z, I)(i\Sigma C^\top) = i(-Z, I)\Sigma \begin{pmatrix} -Z \\ I \end{pmatrix} P = 0.$$

Therefore, multiplying both sides of Eq. (9) by $(Z, I_n)$ from the left, we have

$$d\hat{r}_t = (Q^\top - YP^\sharp P^\top)\hat{r}_t dt - 2YP^\sharp d\hat{A}_t.$$  

(15)

Regarding the output process (10), as $C\hat{x}_t = P^\top(-Z, I_n)\hat{x}_t = P^\top \hat{r}_t$, it is written by

$$d\hat{A}'_t = P^\top \hat{r}_t dt + d\hat{A}_t.$$  

(16)

The coefficient matrix of the dynamics of $\hat{r}$ has the following property.

**Proposition 2:** The matrix $Q^\top - YP^\sharp P^\top$ is Hurwitz if and only if the rank condition (13) is satisfied.

**Proof:** Let $b \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ be the eigenvector and the eigenvalue of $Q^\top - YP^\sharp P^\top$, respectively; i.e., $(Q^\top - YP^\sharp P^\top)b = \lambda b$. Then, multiplying this equation by $b^\dagger Y^{-1}$ from the left, we have

$$b^\dagger(Y^{-1}Q^\top - P^\sharp P^\top)b = \lambda b\|Y^{-1/2}b\|^2,$$

where $\| \bullet \|$ denotes the standard Euclidean norm. This immediately yields $b^\dagger(Q^\top Y^{-1} - P^\top P^\top)b = \lambda^* \|Y^{-1/2}b\|^2$. Recall now that $Q = -iRY - Y^{-1}\Gamma^\top$, hence $QY^{-1} = \ldots$
\[ -iR - Y^{-1} \Gamma Y^{-1} \] is skew Hermitian. Therefore, adding the above two equations yields
\[-2\|P^T b\|^2 = (\lambda + \lambda^*)\|Y^{-1/2}b\|^2, \] and we have \( \Re(\lambda) = -\|P^T b\|^2/\|Y^{-1/2}b\|^2. \) Let us here assume that \( P^T b = 0. \) Then, we have \( Q^T b = \lambda b, \) and the matrix \( C := (P, Q P, \ldots, Q^{n-1} P) \) satisfies
\[ C^T b = \begin{pmatrix} P^T b \\ P^T Q^T b \\ \vdots \\ P^T (Q^T)^{n-1} b \end{pmatrix} = \begin{pmatrix} P^T b \\ \lambda P^T b \\ \vdots \\ \lambda^{n-1} P^T b \end{pmatrix} = 0. \]
But this is contradiction to the assumption \( [13], \) thus \( P^T b \neq 0. \) As a result, \( \Re(\lambda) \) is strictly negative, implying that the matrix \( Q^T - Y P^2 P^T \) is Hurwitz.

On the other hand, if \( C \) is not of rank \( n, \) there exists an eigenvector of \( Q^T - Y P^2 P^T, \) say \( b_0, \) that satisfies \( P^T b_0 = 0. \) Then, from the above discussion, the corresponding eigenvalue \( \lambda \) satisfies \( \Re(\lambda) = 0, \) hence \( Q^T - Y P^2 P^T \) is not Hurwitz.

Based on the above result, we can verify that the target pure Gaussian state is certainly generated. To see this, let us multiply all the entries of the nullifier dynamics \( (15) \) by the system-field composite state vector \( |\Psi\rangle = |\psi\rangle \otimes |0\rangle \) from the right (\( |0\rangle \) is the vacuum state). Then, due to the relation \( d\tilde{\tilde{A}}_t|0\rangle = 0, \) we have
\[ \frac{d}{dt} \tilde{\tilde{r}}_t|\Psi\rangle = (Q^T - Y P^2 P^T)\tilde{\tilde{r}}_t|\Psi\rangle. \]
The Hurwitz property of the matrix \( Q^T - Y P^2 P^T \) is equivalent to asymptotic stability of the dynamics, thus the nullifier vector \( \tilde{\tilde{r}}_t|\Psi\rangle = \tilde{U}_t^* \tilde{\tilde{r}}_t|\Psi\rangle \) converges to zero. Therefore, in the Schrödinger picture, \( \tilde{U}_\infty|\Psi\rangle \) is the common zero-eigenstate of \( \tilde{r}, \) meaning that the system state becomes the target pure Gaussian state \( |\psi_{2}\rangle \) as \( t \to \infty. \)

The Hurwitz property obtained above allows us to obtain a specific input-output relation from the incoming field \( \tilde{A}_t \) to the outgoing field \( \tilde{A}_t, \) through Eqs. \( (15) \) and \( (16). \) For this purpose, it is convenient to move into the frequency domain where both of these field operators as well as the internal system variable \( \tilde{x}_t \) are all Fourier transformed. The Fourier transformation of Eqs. \( (15) \) and \( (16) \) are given by
\[ i\omega \tilde{r}(\omega) = (Q^T - Y P^2 P^T)\tilde{r}(\omega) - 2Y P^2 \tilde{A}(\omega), \]
\[ \tilde{A}(\omega) = P^T \tilde{r}(\omega) + \tilde{\tilde{A}}(\omega), \]
where the tilde notation denotes the Fourier transformed operator. Note that for instance \( \tilde{A}_t(\omega) \) is not the Fourier transformation of \( \tilde{A}_t(t) \) but \( \tilde{\tilde{A}}_t(t). \) Also, more precisely, we should take Laplace transformation \( \tilde{\tilde{a}}_t(t) \to \tilde{A}_t(s) \) and set \( s = +0 + i\omega \) to obtain the Fourier transformation; for the rigorous treatment, see \( [30, 32]. \) As a result, we have the input-output map from \( \tilde{\tilde{A}}(\omega) \) to \( \tilde{\tilde{A}}(\omega): \)
\[ \tilde{\tilde{A}}(\omega) = F(\omega)\tilde{\tilde{A}}(\omega), \quad F(\omega) := I_m - 2P^T(i\omega - Q^T + Y P^2 P^T)^{-1}YP^2. \] (17)
Here we have used the Hurwitz property to justify that the initial contribution of the system was ignored in Eq. \( (17). \) The \( m \times m \) matrix \( F(\omega), \) called the transfer function matrix, has a striking property as shown below.
Proposition 3: The transfer function matrix $F(\omega)$ is unitary for all $\omega$.

Proof: The proof directly follows from Lemma 2 of [30], but here it is given for convenience of readers. First, to simplify the calculation, let us define $\bar{P} = Y^{1/2}P$ and $\bar{Q} = Y^{1/2}QY^{-1/2}$. As $Q = -iRY - Y^{-1}P^T$, $\bar{Q}$ is skew Hermitian; $\bar{Q} + \bar{Q}^\dagger = 0$. With this notation, the transfer function matrix is represented by $F(\omega) = I_m - 2\bar{P}^T(i\omega - \bar{Q}^\dagger + \bar{P}^\dagger \bar{P})^{-1}\bar{P}$.

Therefore, we have

$$F(\omega)^\dagger F(\omega) = I_m - 2\bar{P}^T(-i\omega - \bar{Q}^\dagger + \bar{P}^\dagger \bar{P})^{-1}\bar{P} - 2\bar{P}^T(i\omega - \bar{Q}^\dagger + \bar{P}^\dagger \bar{P})^{-1}\bar{P}^\dagger + 2\bar{P}^T(-i\omega - \bar{Q}^\dagger + \bar{P}^\dagger \bar{P})^{-1}(2\bar{P}^\dagger \bar{P})\bar{P}^\dagger - 2\bar{P}^T(i\omega - \bar{Q}^\dagger + \bar{P}^\dagger \bar{P})^{-1}\bar{P}^\dagger. \quad (18)$$

Here it follows from $\bar{Q} + \bar{Q}^\dagger = 0$ that

$$2\bar{P}^\dagger \bar{P}^T = (-i\omega - \bar{Q}^\dagger + \bar{P}^\dagger \bar{P}^\dagger) + (i\omega - \bar{Q}^\dagger + \bar{P}^\dagger \bar{P}^\dagger).$$

Substituting this expression for the last term of Eq. (18), we end up with the relation $F(\omega)^\dagger F(\omega) = I_m$, hence $F(\omega)$ is unitary for all $\omega$. \qed

This result states that the output power spectrum is flat in all the frequency domain, i.e., $(\tilde{A}(\omega)^* \tilde{A}(\omega)^\dagger) = F(\omega)^\dagger (\tilde{A}(\omega)^* \tilde{A}(\omega))^\dagger F(\omega)^\dagger = I_m$. This means that, at steady state, it is impossible to extract any information about the internal system, as long as the matrix $Q^\dagger - Y P^\dagger P$ is Hurwitz and the input fields are in vacuum or coherent states.

Now we arrive at the stage where the physical meanings of the conditions given in Theorem 1 can be clarified. First of all, the structure of Eqs. (15) and (16) as well as the above two propositions remind us that the dynamics of the nullifier is a generalization of that for the simple single-mode optical damped cavity whose QSDE is described by

$$d\hat{a}_t = (i\Delta - \frac{\kappa}{2})\hat{a}_t dt - \sqrt{\kappa}d\hat{A}_t, \quad d\hat{A}_t = \sqrt{\kappa}\hat{a}_t dt + d\hat{A}_t, \quad (19)$$

where $\hat{a}_t$ and $\hat{A}_t$ denote the intra-cavity mode and the incoming vacuum field mode, respectively. $\Delta$ and $\kappa$ denote the detuning and the damping rate, respectively. Clearly, the state evolves into the vacuum, and also we have $\langle \hat{A}^\dagger(\omega) \hat{A}(\omega)^\dagger \rangle = \langle \hat{A}(\omega)^* \hat{A}(\omega) \rangle = 1$ for all $\omega$, as in the nullifier case. These properties arise due to (i) that energy is not supplied through the Hamiltonian, (ii) that the field does not supply energy but simply brings about the damping of the system, and (iii) that the system is asymptotically stable. Mathematically, the first two statements mean that the dynamics does not contain the creation operators $\hat{a}^*_t$ and $\hat{A}^*_t$. Actually, regarding the first one, if the cavity contains a degenerate parametric amplifier, which is described by the Hamiltonian $\hat{H}_{DPA} = i(\hat{a}^* - \hat{a}^2)$, then the QSDE needs to be described in terms of both $\hat{a}$ and $\hat{a}^*$. The last condition (iii) guarantees that the state uniquely converges into the vacuum as well as that the output field does not contain any information about the system at steady state. Systems having the properties (i)-(iii) are in general called *passive systems* [30, 33, 35].

The above discussion implies that the nullifier dynamics is passive; more precisely, we obtain the physical meanings of the conditions (11), (12), and (13) as follows.

- The matrix $C$ has the form given in Eq. (11) so that the creation process $\hat{A}^*_t$ does not appear in the QSDE of $\hat{a}_t$; as mentioned above, this is equivalent to that there is no energy supply from the environment to the nullifier.
- The matrix $G$ has the form given in Eq. (12) so that the corresponding Hamiltonian $\hat{H} = \hat{x}^\top G \hat{x}/2$ does not supply energy for the nullifier.

- The rank condition (13) implies that the coefficient matrix $Q^\top - Y P^d P^\top$ is Hurwitz, or equivalently the asymptotic stability of the dynamics of the nullifier. This guarantees that the output power spectrum is flat in all frequencies, meaning that the output field does not contain any information about the system at steady state.

The last statement can be understood by studying the filtering equation [18, 36], which enables us to update the conditional state based on the measurement result of the output field. In general, when the state of the master equation reaches the steady state and it is pure, then the corresponding filtering equation is identical to the master equation, meaning that we do not obtain any new information through measuring the output field for updating our knowledge. Note that this does not mean that the system is not controllable.

IV. QUANTUM STATE TRANSFER FOR THE DISSIPATIVE SYSTEM ENGINEERING

We have seen in Theorem 1 how the dissipative channel $\hat{L}_k = c_k^\top \hat{x}$ with $C = (c_1^\top, \ldots, c_m^\top)$ should be chosen to engineer a desired Gaussian dissipative system. When we aim to generate a certain (useful) Gaussian state, however, it often turns out that the resulting $\hat{L}_k$ has to possess a specific structure which is hard to actually implement. For instance, a dissipative channel interacting with all the nodes, i.e., $\hat{L} = \ell_1 \hat{q}_1 + \ell_2 \hat{q}_2 + \ldots + \ell_n \hat{q}_n$, will be hard to implement. In this section, for the specific case where the system is subjected to $n$ independent input optical fields (i.e., $m = n$), we provide a practical procedure for implementing desired dissipative channels.

Let us first introduce the field quadratures $(k = 1, \ldots, n)$

$$\hat{Q}_k = (\hat{A}_k + \hat{A}_k^*)/\sqrt{2}, \quad \hat{P}_k = (\hat{A}_k - \hat{A}_k^*)/\sqrt{2}i,$$

which satisfy the canonical commutation relation $[d\hat{Q}_i, d\hat{P}_j] = \delta_{ij} dt$. Then, defining $\hat{Q} = (\hat{Q}_1, \ldots, \hat{Q}_n)^\top$ and $\hat{P} = (\hat{P}_1, \ldots, \hat{P}_n)^\top$, we find that the QSDE (9) is rewritten by

$$d\hat{x}_t = A\hat{x}_t dt + \sqrt{2\Sigma} (C_r^\top, C_i^\top) d\hat{W}_t, \quad \hat{W}_t = \begin{pmatrix} \hat{Q}_t \\ \hat{P}_t \end{pmatrix},$$

where $C_r = \Re(C)$ and $C_i = \Im(C)$.

The situation we have in mind is that a desired pure Gaussian state of light is first generated, and then, that state is transferred to the system through the system-field coupling; see Figure 1. This is the framework of the quantum state transfer [26–28]; in this case the system is called the memory and it should be independent on the input state we will transfer. More specifically, the target mode $\hat{W}^S_t$ is obtained from the vacuum mode $\hat{W}_t$ through the transformation $\hat{W}^S_t = S \hat{W}_t$ where the symplectic matrix $S$ is given in Eq. (2); then the quantum Ito rule (6) gives

$$d\hat{W}_t^S (d\hat{W}_t^S)^\top = S d\hat{W}_t d\hat{W}_t^\top S^\top = S (I_n + i\Sigma) S^\top dt/2 = (SS^\top/2 + i\Sigma/2) dt,$$
FIG. 1: Quantum state transfer from the input mode $\hat{W}_S$ to the memory mode $\hat{x}_t$.

implying that the covariance matrix of the input field $\hat{W}_S$ is certainly $V = SS^T/2$ (in the rigorous sense this statement should be given in terms of the power spectrum density; see [31]). Note that $S$ can contain a squeezing process, which was not included in the original QSDE framework of Hudson and Parthasarathy [17]; see [32] for a detailed discussion. Now the system (21) is written as

$$d\hat{x}_t = A\hat{x}_t dt + Bd\hat{W}_S, \quad B := \sqrt{2}\Sigma(C_r^T, C_i^T)S^{-1},$$

(22)

where, as shown above, the new input field $\hat{W}_S$ carries information of the target Gaussian state. The system, which serves as a memory, should satisfy the following two requirements:

(R1) The memory system (22) should not possess any information about the input state; that is, the system’s coefficient matrices $A$ and $B$ should be independent on $Z = X + iY$.

(R2) The state of the memory system (22) should converge to the target Gaussian state with covariance matrix $V_\infty = SS^T/2$. That is, $C$ and $G$ should be of the form (11) and (12) with $P$ and $Q$ satisfying the rank condition (13).

Below we give a characterization of the desired memory system:

Proposition 4: Assume that the system satisfies the requirements (R1) and (R2). Then, the system has to be of the form

$$d\hat{x}_t = -2\kappa^2\hat{x}_t dt - 2\kappa d\hat{W}_S,$$

(23)

where $\kappa$ is a scalar constant.

Proof. Note that $S^{-1} = \Sigma S^T \Sigma^T$. Then, substituting $C = P^T(-Z, I_n)$ for $B$ in Eq. (22), we have

$$B = \sqrt{2}\begin{pmatrix} -P_2Y^{1/2} - P_1Y^{-1/2}X & P_1Y^{-1/2} \\ -(YP_1 + XP_2)Y^{1/2} - (XP_1 - YP_2)Y^{-1/2}X & (XP_1 - YP_2)Y^{-1/2} \end{pmatrix},$$

where $P_1 = \Re(P)$ and $P_2 = \Im(P)$. First let us look at the (2,2) block matrix; since $X$ can take any symmetric matrix, here we set $X = 0$, implying $YP_2Y^{-1/2}$ is independent on $Y$. This readily implies that $P_2$ has to be of the form $P_2 = \sqrt{2}\kappa Y^{-1/2}$ with $\kappa$ a constant. Then, $XP_1Y^{-1/2}$ has to be independent on $X$ and $Y$. But as the (1,2) block matrix $\Theta := P_1Y^{-1/2}$ also has to be independent on $X$ and $Y$, thus this is the case for $X\Theta$ as well. Then, $\Theta = 0$
is only allowed, hence we obtain $P_1 = 0$. With these selection of $P_1$ and $P_2$, the (1,1) and (2,1) block matrices of $B$ take $-2\kappa I_n$ and zero, respectively. As a result, $B = -2\kappa I_{2n}$.

Next let us consider the matrix $A = \Sigma(G + \mathbb{3}(C^tC))$. From the above discussion, now we have $C = \sqrt{2}\kappa Y^{-1/2}(-Z, I_n)$, which leads to $A = \Sigma G - 2\kappa I_{2n}$. This means that the matrix $G$ given in Eq. (12) must be independent on $X$ and $Y$. Then, similar to the above discussion, by setting $X = 0$, we find that $G = (YR Y, \Gamma Y^{-1} ; Y^{-1} \Gamma^\top, R)$ has to be independent on $X$ and $Y$. But this requirement is only satisfied when $R = 0$ and $\Gamma = 0$; as a result, we have $G = 0$.

This proposition states that, in order to dissipatively generate a desired pure Gaussian state in the state transfer setup, we are required to prepare identical and independent oscillators as memories. Note that any pure Gaussian cluster state (see the next section) can be effectively generated from the vacuum fields by applying suitably combined two-mode squeezing Hamiltonians and beam splitters [25], hence the proposed scheme is practical.

V. EXAMPLES

A. Example 1: Gaussian cluster state generation

It was shown in [21] that the graph matrix $Z = X + iY$ can be used to capture several Gaussian graph states in a convenient graphical manner. In particular, the so-called canonical Gaussian cluster state [22–25], which plays an essential role in continuous-variable one-way quantum computation, corresponds to

$$Z = X + ie^{-2\alpha}I_n,$$

(24)

where $X$ is the symmetric adjacency matrix representing the graph structure of the cluster state; for instance, the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

represent the chain, T-shape, and square structures, respectively (see Figure 2). On the other hand, $Y = e^{-2\alpha}I$ corresponds to the approximation error of the state with covariance matrix (2) to the ideal cluster state; that is, bigger $\alpha$ means that the state well approximates the ideal cluster state having the graph structure assigned by $X$.

Now, the nullifier dynamics (15) is of the form

$$d\hat{r}_t = (Q^\top - e^{-2\alpha}P^\sharp P^\top)\hat{r}_tdt - 2e^{-2\alpha}P^\sharp d\hat{A}_t,$$

(25)

and the real part of the eigenvalue of the coefficient matrix $Q^\top - e^{-2\alpha}P^\sharp P^\top$ is $\Re(\lambda) = -e^{-2\alpha}\|P^\top b\|$, where $b$ is the corresponding eigenvector. This means that making $\alpha$ bigger, or equivalently making the state more close to the ideal cluster state, renders the stability of the nullifier dynamics worse. Another observation from a more practical viewpoint is as
follows; let us define the convergence time to the target by \( T = 1/\min|\Re(\lambda)| \) and denote the approximation error of the state to the ideal cluster one by \( \epsilon = e^{-2\alpha} \). Then, it is straightforward to find \( T\epsilon \geq c \) with \( c \) a constant. Therefore, in order to dissipatively generate a pure Gaussian state that is very close to a desired cluster state, the convergence time has to be long.

More generally, as shown in [21], the matrices \( X \) and \( Y \) respectively correspond to the ideal and realistic parts of a Gaussian graph state in the sense that the covariance matrix of \((-X, I_n)\hat{x}\) is given by \( Y/2 \). That is, \( Y \) can be regarded as the approximation error in approximating the ideal graph structure \( X \). Therefore, the above-mentioned trade-off holds for a general Gaussian graph state.

**B. Example 2: Two-mode squeezed state**

There exist a number of proposals to generate a steady two-mode squeezed state in for instance atomic ensembles or nano-mechanical oscillators. The system matrices describing the two-mode squeezed state are given by \( X = 0 \) and

\[
Y = \begin{pmatrix}
\cosh(2\alpha) & -\sinh(2\alpha) \\
-\sinh(2\alpha) & \cosh(2\alpha)
\end{pmatrix},
\]

where \( \alpha \) denotes the squeezing parameter representing the degree of entanglement. In [10, 11], the dissipative channels achieving this goal was shown to be

\[
\hat{L}_1 = \mu \hat{a}_1 + \nu \hat{a}_2^*, \quad \hat{L}_2 = \mu \hat{a}_2 + \nu \hat{a}_1^*,
\]

where \( \mu = \cosh(\alpha) \) and \( \nu = -\sinh(\alpha) \), while \( \hat{H} = 0 \). In our formulation, this corresponds to setting [13]:

\[
P = \begin{pmatrix}
i \cosh(\alpha) & i \sinh(\alpha) \\
i \sinh(\alpha) & i \cosh(\alpha)
\end{pmatrix}, \quad R = 0, \quad \Gamma = 0.
\]

However, the dissipative channels (26) are not easy to implement, since they are global and nontrivial coupling between the systems and the environment.

On the other hand, this dissipative system can be more easily implemented within the state transfer framework provided in Section 4, because we are only required to generate a two-mode squeezed state of optical fields and prepare two identical and independent oscillators. Note that a two-mode squeezed state of light can be effectively generated using a non-degenerate OPO.
VI. CONCLUSION

In this paper, the dissipation-based state preparation method for general Gaussian case, which was originally formulated in [13], was reconsidered in terms of the QSDE. This approach clarified that the nullifier dynamics of any Gaussian system generating a unique pure steady state is passive. As a byproduct, it was shown that there exists a trade-off between the closeness of the steady state to a given ideal graph state and the convergence time to that state. In addition, a convenient physical implementation method of a desired Gaussian dissipative system was provided; the scheme has the structure of quantum state transfer, which is a key ingredient in quantum information technologies.

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