On deformation theory of quantum vertex algebras

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Abstract

We study an algebraic deformation problem which captures the data of the general deformation problem for a quantum vertex algebra. We derive a system of coupled equations which is the counterpart of the Maurer-Cartan equation on the usual Hochschild complex of an associative algebra. We show that this system of equations results from an action principle. This might be the starting point for a perturbative treatment of the deformation problem of quantum vertex algebras. Our action generalizes the action of the Kodaira-Spencer theory of gravity and might therefore also be of relevance for applications in string theory.
1 Introduction

In [Bor] Borcherds has developed an abstract notion of vertex algebra as a certain singular commutative ring in a symmetric tensor category. He introduces a quantum vertex algebra as a singular braided commutative ring in a symmetric tensor category. He also remarks that, in addition, one could allow for the tensor category to be no longer symmetric but also braided. This is supposed to be of relevance for vertex operators with non-integer exponents in the formal series and should therefore be important for string theory with background fields (see [KO]). We will even drop the requirement of braidings and allow the inner ring, as well, as the tensor product of the outer category to become arbitrarily noncommutative. A generalization of vertex algebras where the inner ring has no longer to be braided has been developed under the name of field algebras in [BK]. We will drop most of the structure of a quantum vertex algebra and just consider the essential ingredients of the deformation problem, i.e we keep just those structures which can be deformed. In consequence, we consider a \( \mathbb{C} \)-linear, additive, monoidal category \( (\mathcal{C}, \cdot, \otimes) \) with composition \( \cdot \) and tensor product \( \otimes \) together with an inner associative ring \( R \). Finally, we make two additional generalizations:

- In addition to \( \otimes \) and the product on \( R \), we also allow the composition 
  \( \cdot \) of \( \mathcal{C} \) to become deformed.

- We pass from an inner associative ring \( R \) to an inner bialgebra object \( B \) in \( \mathcal{C} \) with an associative product and a coassociative coproduct. In [BD] chiral algebras are introduced as a coordinate independent generalization of vertex algebras. In the axiomatization used there, chiral algebras appear as a generalization of universal envelopes of Lie algebras in a certain monoidal category. One could consider in this case deformations of the outer category or of the inner algebra where - in the spirit of the theory of quantum groups - one could think of passing from the deformation theory of Lie algebras to the more general deformation theory of Hopf algebras. By passing from \( R \) to \( B \) as the inner object, we want to include the essential data of the deformation problem of this case, too.

We can therefore think of the deformation problem to be studied, here, as the most general deformation problem for the data of a quantum vertex
algebra (we will keep speaking of a quantum vertex algebra even in the most
general case where an inner bialgebra object is allowed and ⊗ is noncommu-
tative). We will state a conjecture below that even generalizations of this
deformation problem to higher category theory should not be possible. As
an additional requirement on the structure of $\mathbb{C}$, we will assume that there
exists a forgetful functor to the category of vector spaces over $\mathbb{C}$ (where we
allow vector spaces to become infinite dimensional).

For an associative algebra $A$ over $\mathbb{C}$, the deformations of the product $m$
to a new associative product are given by the solutions of the Maurer-Cartan
equation on the Hochschild complex. More precisely, let $\alpha$ be a bilinear map
on $A$ and $m + \alpha$ the deformed associative product. The Hochschild complex
of $A$ consists of all linear maps
$$\Phi : A^{\otimes n} \rightarrow A$$
Call $n$ the degree of $\Phi$. On the Hochschild complex there is a product $\circ$ given
for $\Phi_1$ of degree $n_1$ and $\Phi_2$ of degree $n_2$ by
$$\Phi_1 \circ \Phi_2 (a_1, \ldots, a_{n_1+n_2-1}) = \sum_{k=0}^{n_1-1} (-1)^{k(n_2-1)} \Phi_1 (a_1, \ldots, a_k, \Phi_2 (a_{k+1}, \ldots, a_{k+n_2}), a_{k+n_2+1}, \ldots, a_{n_1+n_2-1})$$
i.e. $\Phi_1 \circ \Phi_2$ is of degree $n_1 + n_2 - 1$. The Gerstenhaber bracket $[,]$ on the
Hochschild complex is the graded commutator of $\circ$, i.e.
$$[\Phi_1, \Phi_2] = \Phi_1 \circ \Phi_2 - (-1)^{(n_1-1)(n_2-1)} \Phi_2 \circ \Phi_1$$
Observe that $[,]$ lowers the degree by one, i.e. it is what is commonly called
an antibracket. We can now rewrite associativity of $m$ as
$$[m,m] = 0$$
because
$$[m,m] (a_1, a_2, a_3) = 2 (m \circ m) (a_1, a_2, a_3) = m (m(a_1, a_2), a_3) - m (a_1, m(a_2, a_3))$$
Now, the requirement that $m + \alpha$ should be associative, again, can be stated
as
$$0 = [m + \alpha, m + \alpha] = [m,m] + 2 [m, \alpha] + [\alpha, \alpha]$$
Using that \([m, m] = 0\), this gives
\[
[m, \alpha] + \frac{1}{2} [\alpha, \alpha] = 0
\]

Defining
\[
d_m \Phi = [m, \Phi]
\]
one checks that \(d_m\) defines a differential on the Hochschild complex, i.e. \((d_m)^2 = 0\). The cohomology with respect to \(d_m\) is the Hochschild cohomology of \(A\).

So, the requirement that \(\alpha\) gives a deformation of \(m\) which preserves associativity can be stated as
\[
d_m \alpha + \frac{1}{2} [\alpha, \alpha] = 0
\]
which is the *Maurer-Cartan equation* on the Hochschild complex of \(A\). We will show below that in the general deformation problem for a quantum vertex algebra, the Maurer-Cartan equation is generalized to a coupled system of differential equations.

The Maurer-Cartan equation is the starting point for a perturbative solution of the deformation problem. Let
\[
\alpha = \sum_{n=1}^{\infty} t^n \alpha_n
\]
where the \(\alpha_n\) are bilinear operators on \(A\), again, and \(t\) is the deformation parameter, i.e. in the perturbative treatment we consider \(\alpha\) as a formal power series in \(t\) and the deformation of \((A, m)\) as a deformation over \(\mathbb{C}[[t]]\). The first order deformations \(\alpha_1\) are parametrized by second Hochschild cohomology \(HH^2(A)\) of \(A\). Inserting (1) into the Maurer-Cartan equation, we get
\[
d_m \alpha_n = -\frac{1}{2} \sum_{k+l=n} [\alpha_k, \alpha_l]
\]
for all \(n\). Since \(n \geq 1\), we get \(k, l \leq n - 1\) on the right hand side and we can use (2) to solve the Maurer-Cartan equation perturbatively (in principle; the actual construction of the solution turns out to be a highly involved problem, see [Kon 1997] for a solution in the special case where \(A\) is the algebra of smooth functions on a manifold).
Now, let $M$ be a complex algebraic variety and suppose that $M$ is a Calabi-Yau variety. Consider the Hochschild complex of the holomorphic structure sheaf $\mathcal{O}_M$ of $M$. This has a similar structure as the Hochschild complex of an associative algebra. The Hochschild cohomology $HH^n(\mathcal{O}_M)$ of $\mathcal{O}_M$ decomposes as

$$HH^n(\mathcal{O}_M) \cong \bigoplus_{p+q=n} H^p \left( \bigwedge^q T_M \right) \quad (3)$$

where $H^p$ denotes usual sheaf cohomology and $T_M$ is the holomorphic tangent bundle of $M$. So, the Hochschild cohomology of $\mathcal{O}_M$ is given as the cohomology of the exterior powers of the tangent bundle. The first order deformations of the pointwise product structure of $\mathcal{O}_M$ are given by the component $HH^2(\mathcal{O}_M)$. This includes, especially, the component with $p = q = 1$ which parametrizes the first order Kodaira-Spencer deformations of the complex structure of $M$. The finite (i.e. noninfinitesimal) Kodaira-Spencer deformations are given by the solutions of the Kodaira-Spencer equation

$$\overline{\partial}_\gamma + \frac{1}{2} [\gamma, \gamma] = 0$$

where, in accordance with [3], $\gamma$ is a $(0,1)$-form valued vector field and $[,]$ denotes the commutator of vector fields and wedging. Observe that the Kodaira-Spencer equation is formally a Maurer-Cartan equation, again. In [BCOV] it was shown that the Kodaira-Spencer equation can be derived from an action principle and that the tree level expansion of the resulting field theory (Kodaira-Spencer theory of gravity) precisely solves the Kodaira-Spencer equation in a treatment completely analogous to [2]. The BV-quantization of the theory then shows that one has to include nonclassical ghost numbers which mathematically results in passing from the Kodaira-Spencer deformations to the total Hochschild complex of $\mathcal{O}_M$ with the cohomology given by [4] (in physics terminology passing from the purely marginal deformations given by Kodaira-Spencer theory to the extended moduli space of [Wit 1991], see [Kon 1994] for a mathematical approach to this space). The distinction between the complex and the cohomology on the mathematical side corresponds to the distinction between off-shell and on-shell states in physics (see e.g. [HM] for a detailed exposition of this point). The Maurer-Cartan equation on the Hochschild complex corresponds to the Master-equation of BV-quantization, the Gerstenhaber bracket corresponds to the BV-antibracket,
and the BV-action functional reduces to the classical action if one considers only classical fields because the Maurer-Cartan equation on (3) reduces to the Kodaira-Spencer equation if one considers only Kodaira-Spencer deformations. So, the essential requirements of BV-quantization (Master-equation and the correct reduction to the classical action on classical fields) are fulfilled just by the fact that the Kodaira-Spencer deformation can be embedded into the larger deformation theory on the Hochschild complex of $\mathcal{O}_M$ (with Maurer-Cartan type equations being preserved).

We show below that the coupled system of differential equations which we derive for the general deformation problem of quantum vertex algebras can likewise be derived from an action principle. This action can be understood as generalizing the action of the Kodaira-Spencer theory of gravity. We formulate a list of goals which would be interesting to reach in a perturbative (tree level, as well, as loop expansion) treatment of the resulting field theory. At present the perturbative study of the theory is limited by the fact that one would first have to find a suitable gauge fixing for the action (corresponding to the Tian gauge in the Kodaira-Spencer theory of gravity).

The paper is structured as follows: In section 2, we consider the deformation problem for the outer category $(\mathcal{C}, \bullet, \otimes)$, i.e. $\bullet$ and $\otimes$ receive deformations. In section 3, we include the inner bialgebra object $B$ into the consideration. We will see that one needs an additional assumption - which basically says that a reconstruction theorem between $B$ and its category of representations in $\mathcal{C}$ is supposed to hold - in order to be able to treat this case as an algebraic deformation problem. Under this assumption, the deformation problem can be reformulated as that for two monoidal categories $(\mathcal{M}, \bullet, \otimes)$ and $(\mathcal{C}, \bullet, \otimes)$ with a monoidal forgetful functor

$$\mathcal{F} : \mathcal{M} \rightarrow \mathcal{C}$$

where both compositions $\bullet$ and both tensor products $\otimes$ and the monoidal functor $\mathcal{F}$ become deformed. Here, and in the sequel, we will use the same symbol for the compositions in $\mathcal{M}$ and $\mathcal{C}$ (and likewise for the tensor products) if no confusion is possible. In section 4, we formulate the list of goals mentioned above. Section 5 contains some concluding remarks.

A disclaimer: We certainly do not claim that the structures which we study in this paper constitute a notion of quantum vertex algebras. Much more structure is needed for this. For part of this structure, one would expect
the existence of theorems saying that the deformation of these structural components canonically goes along with the deformations considered, here (much the same way as e.g. the counit in a Hopf algebra canonically goes along with the deformations of the product and coproduct). But the proof of such theorems is a nontrivial matter which needs further investigation. When we speak of the deformation problem of quantum vertex algebras in our context, we only want to point to the following fact: Vertex algebras can be considered as singular commutative rings in certain monoidal categories. In this sense, the deformation problem which we consider is the most general deformation problem for those part of structure which are the essential ones for the deformation of vertex algebras. Given the highly nontrivial nature of the question of deformations of quantum vertex algebras, we believe that such a boiled down approach is justified.

2 Deformation of the monoidal category

Let $\mathcal{A} := \text{Morph}(\mathcal{C})$ denote the class of morphisms of $\mathcal{C}$. Let $\boxtimes$ denote the tensor product of $\mathbb{C}$-linear tensor categories as used e.g. in [CF], i.e. we can understand a bilinear functor on $\mathcal{C}$ as a linear functor from $\mathcal{C} \boxtimes \mathcal{C}$ to $\mathcal{C}$. We also denote by $\boxtimes$ the corresponding tensor product on $\mathcal{A}$. Then let $\mathcal{T.A}$ denote the categorical tensor algebra of multilinear maps on $\mathcal{A}$, i.e.

$$\mathcal{T.A} = \bigoplus_{n=0}^{\infty} \{ \Phi : \boxtimes^n \mathcal{A} \to \mathcal{A} \text{ linear} \}$$

where the definition of the categorical direct sum is obvious.

In deforming the composition $\bullet$ to $\tilde{\bullet}$ we will assume that the domain and codomain maps on $\mathcal{C}$ remain undeformed, i.e. $a \bullet b$ is defined iff $a \tilde{\bullet} b$ is defined. For the deformation of $\otimes$ to $\tilde{\otimes}$, we make the following assumption: We assume that this deformation does not change the tensor product of objects on a $\mathbb{C}$-linear level, i.e the object maps of $\otimes$ and $\tilde{\otimes}$ should agree after applying the monoidal forgetful functors to the category $\mathcal{V}$ of (finite and infinite dimensional) vector spaces to both (remember that we suppose such forgetful functors always to exist, see the introduction). Especially, this means that after applying forgetful functors the domains and codomains
of $a \otimes b$ and $a\tilde{\otimes}b$ agree. In addition, we assume that we can fully study the deformation problem for $\otimes$ by studying the deformation problem for the morphism map of $\otimes$.

We write
\[
\tilde{\bullet} = \bullet + f
\]
and
\[
\tilde{\otimes} = \otimes + g
\]
where $f$ and $g$ are linear
\[
f,g : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}
\]
and addition $+$ can be understood as componentwise addition on the $\text{Hom}_C(\cdot,\cdot)$-sets after applying the forgetful functor to $\mathcal{V}$.

We start with the consideration of the deformation problem for $\bullet$. The requirement is that $\tilde{\bullet}$ should be associative, again. For $a,b,c$ in $\mathcal{A}$ and composable in the required order, this leads to
\[
\left(a\tilde{\bullet}b\right)\tilde{\bullet}c = a\tilde{\bullet}(b\tilde{\bullet}c)
\]
and therefore to
\[
\left[\bullet,f\right] + \frac{1}{2}[f,f] = \left[\bullet,f\right] + f \circ f = 0
\]
where $\circ$ and $[,]$ on $\mathcal{T}\mathcal{A}$ are defined in complete analogy to the case of the Hochschild complex of an associative algebra (see the introduction). Observe, here, that $f(a,b)$ is defined iff $a\bullet b$ is defined and that $\bullet$ is linear,
\[
\bullet : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}
\]
For $\Phi$ in $\mathcal{T}\mathcal{A}$ define $d_\bullet$ by
\[
d_\bullet \Phi = \left[\bullet,\Phi\right]
\]
We have $(d_\bullet)^2 = 0$ and $d_\bullet$ defines a differential on $\mathcal{T}\mathcal{A}$. Then associativity of $\tilde{\bullet}$ can be formulated as
\[
d_\bullet f + f \circ f = 0
\]
which is a Maurer-Cartan type equation.

Let us now pass to the deformation problem for $\tilde{\otimes}$. We assume that all the monoidal categories which we consider are strict monoidal. This is possible by a coherence theorem of Mac Lane (see [Mac]). Besides making calculations simpler, this assumption might also be justified on a deeper level: By the
very existence of the coherence theorem, one expects that when deriving the
deformation problem from a field theory action principle, the more general
case of a nontrivial associator should automatically arise by including BRST-
exact terms. Then the restriction imposed upon the deformation problem is,
again, associativity of \( \tilde{\otimes} \), leading to
\[
d \otimes g + g \circ g = 0 \tag{5}
\]
where \( d \otimes \) is defined by
\[
d \otimes \Phi = [\otimes, \Phi]
\]
for \( \Phi \) in \( \mathcal{T} \mathcal{A} \). Observe that by the restriction to the consideration of the
morphism map of \( \otimes \), we can, again, view \( \otimes \) as linear,
\[
\otimes : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}
\]
What remains to be considered is the compatibility constraint on \( \tilde{\bullet} \) and \( \tilde{\otimes} \) in
order that they define together a monoidal category, again. For \( a_1, a_2, a_3, a_4 \)
in \( \mathcal{A} \) such that
\[
(a_1 \otimes a_2) \bullet (a_3 \otimes a_4)
\]
exists this means that
\[
(a_1 \tilde{\otimes} a_2) \tilde{\bullet} (a_3 \tilde{\otimes} a_4) = (a_1 \tilde{\bullet} a_3) \tilde{\otimes} (a_2 \tilde{\bullet} a_4)
\]
i.e.
\[
(\bullet + f) ((\otimes + g) (a_1, a_2), (\otimes + g) (a_3, a_4)) = (\otimes + g) ((\bullet + f) (a_1, a_3), (\bullet + f) (a_2, a_4))
\]
which by using compatibility of \( \bullet \) and \( \otimes \) is equivalent to
\[
g (a_1, a_2) \bullet (a_3 \otimes a_4) + (a_1 \otimes a_2) \bullet g (a_3, a_4)
+ f (a_1 \otimes a_2, a_3 \otimes a_4) + f (a_1 \otimes a_2, g (a_3, a_4))
+ f (g (a_1, a_2), a_3 \otimes a_4) + f (g (a_1, a_2), g (a_3, a_4))
- f (a_1, a_3) \otimes (a_2 \bullet a_4) - (a_1 \bullet a_3) \otimes f (a_2, a_4)
- f (a_1, a_3) \otimes f (a_2, a_4)
- g (a_1 \bullet a_3, a_2 \bullet a_4) - g (a_1 \bullet a_3, f (a_2, a_4))
- g (f (a_1, a_3), a_2 \bullet a_4) - g (f (a_1, a_3), f (a_2, a_4)) = 0 \tag{6}
\]
We will now discuss the structural properties of the terms in \((6)\). Observe that \((6)\) contains terms of first, second, and third order in the fields \(f, g\) (especially, this means that these terms appear, upon expanding \(f\) and \(g\) according to \((1)\), for the first time in the respective order in the resulting expansion of \((6)\)). We start with considering the first order terms: There are six first order terms, three in each of the fields \(f\) and \(g\). Taking only the three first order terms in \(g\), we have

\[
(a_1 \otimes a_2) \cdot g(a_3, a_4) - g(a_1 \circ a_3, a_2 \cdot a_4) + g(a_1, a_2) \cdot (a_3 \otimes a_4) \tag{7}
\]

Remember that \(g\) is linear,

\[
g : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}
\]

But - as a consequence of our assumption that domain and codomain maps are preserved in the deformation - we can locally on the \(\text{Hom}\)-sets replace \(\boxtimes\) by \(\otimes\) after applying the forgetful functor to \(\mathcal{V}\) (alternatively, we can view this as evaluating an expression \(a \boxtimes b\) to \(\otimes (a \boxtimes b) = a \otimes b\)). We can then simply rewrite \((7)\) to

\[
(a_1 \otimes a_2) \cdot g(a_3 \otimes a_4) - g((a_1 \otimes a_3) \otimes (a_2 \cdot a_4)) + g(a_1 \otimes a_2) \cdot (a_3 \otimes a_4)
\]

where we now interpret \(g\) as a function of one variable on the tensor product. Defining

\[
d^{\otimes 2}_\circ g(a_1, a_2, a_3, a_4) = (a_1 \otimes a_2) \cdot g(a_3 \otimes a_4) - g((a_1 \otimes a_3) \otimes (a_2 \cdot a_4)) + g(a_1 \otimes a_2) \cdot (a_3 \otimes a_4)
\]

and similarly for a general \(n\)-variable function \(\Phi\) on the tensor product, we see that \(d^{\otimes 2}_\circ\) is just the Hochschild differential \(d_\circ\) with respect to \(\circ\), as introduced above, lifted to the twofold tensor product (hence the notation \(d^{\otimes 2}_\circ\)). It therefore immediately follows that

\[
(d^{\otimes 2}_\circ)^2 = 0
\]

Hence, the three terms in \((7)\) can be written as

\[
d^{\otimes 2}_\circ g(a_1, a_2, a_3, a_4)
\]
The three first order terms in $f$ are

$$-(a_1 \cdot a_3) \otimes f(a_2, a_4) + f(a_1 \otimes a_2, a_3 \otimes a_4) - f(a_1, a_3) \otimes (a_2 \cdot a_4) \quad (8)$$

By interchanging the roles of $\cdot$ and $\otimes$, we can - in complete analogy to the preceding argument - rewrite (8). This time, we imagine $a \boxtimes b$ as being evaluated to $\cdot (a \boxtimes b) = a \cdot b$ in order to interpret $f$ as a function of a single variable. With

$$d_{\otimes}^{\otimes^2} f(a_1, a_2, a_3, a_4)$$

$$= (a_1 \cdot a_3) \otimes f(a_2 \cdot a_4) - f((a_1 \otimes a_2) \cdot (a_3 \otimes a_4)) + f(a_1 \cdot a_3) \otimes (a_2 \cdot a_4)$$

we can rewrite the three first order terms in $f$ as

$$-d_{\otimes}^{\otimes^2} f(a_1, a_2, a_3, a_4)$$

Observe that we keep the upper index $\otimes^2$ for the differential lifted to the two fold tensor product since this is what occurs for the linear spaces before evaluating with $\cdot$. $d_{\otimes}^{\otimes^2}$ is the Hochschild differential $d_{\otimes}$ from above, lifted to the two fold tensor product. Especially, we have

$$\left(d_{\otimes}^{\otimes^2}\right)^2 = 0$$

Let us next turn to the consideration of the second order terms. There are, again, six terms: There is one term each which is quadratic in $f$, respectively $g$, and four terms which contain $f$ and $g$. From the mixed terms, two have $g$ left of $f$ and two vice versa. Let us start with the consideration of the latter ones: These are

$$f(g(a_1, a_2), a_3 \otimes a_4) + f(a_1 \otimes a_2, g(a_3, a_4))$$

Rewriting this, again as

$$f(g(a_1 \otimes a_2), a_3 \otimes a_4) + f(a_1 \otimes a_2, g(a_3 \otimes a_4))$$

we can view $g$ as a function of one variable and $f$ as a function of two variables on the twofold tensor product. Using the composition $\circ$ on $\mathcal{T}A$, we can write this as

$$(f \circ^2 g)(a_1, a_2, a_3, a_4)$$

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where the notation \( \circ^2 \)- which should in a more detailed form be written as \( \circ^{\otimes 2} \) - indicates the lift to the twofold tensor product, again. Similarly, we can rewrite the other two mixed terms as

\[- (g \circ^2 f) (a_1, a_2, a_3, a_4)\]

So, the second order terms can altogether be written as

\[\left( (f \circ^2 g) - (g \circ^2 f) + g \bullet g - f \otimes f \right) (a_1, a_2, a_3, a_4)\]

**Remark 1** Here and in the sequel, we neglect any switch functions for the arguments as they appear in terms like \( f \otimes f \) where the arguments appear in the order \( a_1, a_3, a_2, a_4 \) instead of \( a_1, a_2, a_3, a_4 \). These switches are obvious from the requirement that compositability in the undeformed category \( \mathcal{C} \) has to be satisfied (and consequently also holds then in the deformed setting) for all the expressions appearing.

Finally, we turn to the consideration of the third order terms. In usual Hochschild cohomology, one only has a differential (function of one variable) and a bracket (function of two variables) on the complex. Here, we encounter a new function of three variables which we introduce for

\[\Phi_1, \Phi_2, \Phi_3 : \mathcal{A}^{\otimes 2} \to \mathcal{A}\]

as

\[\text{Comp} (\Phi_1, \Phi_2, \Phi_3) (a_1, a_2, a_3, a_4) = \Phi_1 (\Phi_2 (a_1, a_2), \Phi_3 (a_3, a_4))\]

For general \( \Phi_1, \Phi_2, \Phi_3 \) of degrees \( n_1, n_2, n_3 \), respectively, \( \text{Comp} \) generalizes to an alternating sum of \( \Phi_2 \) and \( \Phi_3 \) inserted into \( \Phi_1 \) with different positions in the arguments of \( \Phi_1 \), again. Using this three variable function \( \text{Comp} \), we can rewrite (6) as

\[0 = d^{\otimes 2} f - d^\bullet g + f \otimes f - g \bullet g - (f \circ^2 g) + (g \circ^2 f)\]

- \( \text{Comp} (f, g, g) + \text{Comp} (g, f, f)\)

So, the deformation problem of a monoidal category leads to the coupled system of differential equations (4), (5), and (9) as replacing the Maurer-Cartan equation on the usual Hochschild complex.
Remark 2 By linearity in the three arguments, the third order terms involving $\text{Comp}$ are similar to the appearance of curvature terms. The only difference is that the curvature tensor is antisymmetric with respect to the last two indices while the two terms $\text{Comp}(f, g, g)$ and $\text{Comp}(g, f, f)$ appearing are symmetric in the last two variables. Can one understand the third order terms as true curvature contribution by taking into account the grading on the deformation complex (i.e. a kind of super-curvature)? If yes, does this imply that one no longer has a flat connection as in [Dub] on the moduli space of deformations? This would be similar to the situation one has for the appearance of superpotentials in string theory.

Let us assume the existence of a trace on $\mathcal{T}_A$ which we will write as $\int$. Then we claim that one can formally derive this coupled system of differential equations from the action

$$S \sim \int \{ f \circ d \cdot f + \frac{2}{3} f \circ f \circ f + g \circ d \circ g + \frac{2}{3} g \circ g \circ g$$

$$+ \lambda \circ^2 [d \circ^2 f - d \circ^2 g + f \otimes f - g \bullet g - (f \circ^2 g) + (g \circ^2 f)]$$

$$- \text{Comp}(f, g, g) + \text{Comp}(g, f, f) \}$$

with fields $f, g, \lambda$. Here, the field

$$\lambda : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

enters as a Lagrange multiplier into (10) in order to install the constraint (9).

Remark 3 For a bundle of quantum vertex algebras on a manifold $M$, we can view the integral $\int$ as including the integration over $M$ besides the trace.

Observe that variation of $S$ with respect to $f$ leads in $\mathcal{T}_A$ to an expression in degree three resulting from the first two terms and to an expression in degree four for the rest. Therefore these two expressions have to vanish separately and we get associativity of $\bullet + f$ from the first two terms of $S$. Similarly, we get associativity of $\otimes + g$. The action (10) describes a system of two Chern-Simons like theories, coupled by the constraint (9).
3 Inclusion of the inner bialgebra object

We now want to generalize (10) to include the deformation of the inner bialgebra $B$ object, too. Suppose, first, that $B$ is only an inner algebra object of $\mathcal{C}$ with associative product $\star$. Deforming $\star$ to $\tilde{\star}$ gives another Maurer-Cartan type equation from the requirement that $\tilde{\star}$ should be associative, again. The compatibility constraint with the deformation of $\mathcal{C}$ is that $\tilde{\star}$ should be a morphism in the deformed monoidal category $\left(\tilde{\mathcal{C}}, \tilde{\bullet}, \tilde{\otimes}\right)$. But the condition of being a morphism in $\tilde{\mathcal{C}}$ is not a purely algebraic condition but one of the type saying that $\tilde{\star}$ should behave well under all possible compositions with morphisms of $\tilde{\mathcal{C}}$. We therefore proceed in a slightly different way which we are now going to explain. It turns out to be useful to consider $B$ as a bialgebra object in $\mathcal{C}$, now, from the beginning.

Let $\text{Rep}_C(B)$ be the category of representations of $B$ on objects of $\mathcal{C}$. Since $B$ is a bialgebra object in $\mathcal{C}$ and therefore carries a coproduct, $\text{Rep}_C(B)$ is a monoidal category. We will now make an additional assumption: In the theory of usual Hopf algebras, there exist a number of so called reconstruction theorems which state that under certain conditions a Hopf algebra and its category of representations determine each other up to isomorphism (see e.g. [CP]). We will assume that in the same way the inner bialgebra object $B$ can be reconstructed from its category $\text{Rep}_C(B)$ of representations in $\mathcal{C}$. With this assumption, the deformation problem of the inner bialgebra object $B$ turns into the deformation problem for the monoidal category

$\mathcal{M} = \text{Rep}_C(B)$

where deformations of the product on $B$ correspond to deformations of the composition $\bullet$ in $\mathcal{M}$ while deformations of the coproduct of $B$ correspond to deformations of the tensor product $\otimes$ of $\mathcal{M}$. Here and in the sequel we use the same symbols for composition $\bullet$ and tensor product $\otimes$ of $\mathcal{C}$ and $\mathcal{M}$ if no confusion is possible. Hence, we double the deformation structure studied in the previous sections by studying the deformations of two monoidal categories $\mathcal{C}$ and $\mathcal{M}$. As the compatibility constraint - replacing the constraint that $\tilde{\star}$ should be a morphism of $\tilde{\mathcal{C}}$ - we use the requirement that the monoidal forgetful functor

$\mathcal{F} : \mathcal{M} \to \mathcal{C}$

should be deformed to a monoidal functor

$\tilde{\mathcal{F}} : \tilde{\mathcal{M}} \to \tilde{\mathcal{C}}$
With
\[ \tilde{\bullet} = \bullet + f \]
and
\[ \tilde{\otimes} = \otimes + g \]
for the deformation of \( \mathcal{C} \) as above and similarly
\[ \tilde{\bullet} = \bullet + \varphi \]
and
\[ \tilde{\otimes} = \otimes + \psi \]
for the deformation of \( \mathcal{M} \), plus
\[ \tilde{\mathcal{F}} = \mathcal{F} + \Omega \]
for the deformation of \( \mathcal{F} \), we have altogether five fields \( f, g, \varphi, \psi, \Omega \). Here, \( \Omega \) is viewed as a linear map between the morphism classes of \( \mathcal{M} \) and \( \mathcal{C} \) since the object classes remain undeformed after applying forgetful functors to \( \mathcal{V} \) (linearity of \( \Omega \) means, of course the local linearity on the \( \text{Hom} \)-sets). We will now study the constraints arising for \( \Omega \). Since \( \tilde{\mathcal{F}} \) has to be a monoidal functor, again, we have
\[ \tilde{\mathcal{F}} (a \bullet b) = \tilde{\mathcal{F}} (a) \bullet \tilde{\mathcal{F}} (b) \] (11)
and
\[ \tilde{\mathcal{F}} (a \otimes b) = \tilde{\mathcal{F}} (a) \otimes \tilde{\mathcal{F}} (b) \] (12)
for the morphisms of \( \tilde{\mathcal{M}} \). Let us study (11) in detail. (11) implies
\[ 0 = \mathcal{F} (\varphi (a, b)) + \Omega (a \bullet b) - \mathcal{F} (a) \bullet \Omega (b) \]
\[ -\Omega (a) \bullet \mathcal{F} (b) - f (\mathcal{F} (a), \mathcal{F} (b)) \]
\[ +\Omega (\varphi (a, b)) - \Omega (a) \bullet \Omega (b) - f (\mathcal{F} (a), \Omega (b)) - f (\Omega (a), \mathcal{F} (b)) \]
\[ -f (\Omega (a), \Omega (b)) \] (13)
Remember that, here, \( \mathcal{F} \) is not a field but the constant undeformed monoidal forgetful functor from \( \mathcal{M} \) to \( \mathcal{C} \). (13) contains five terms of first order in the fields, four terms of second order, and one term of third order. We start by considering the three first order terms in \( \Omega \). These are
\[ \mathcal{F} (a) \bullet \Omega (b) - \Omega (a \bullet b) + \Omega (a) \bullet \mathcal{F} (b) \]
Defining

\[ d_F^* \Phi (a_1, ..., a_{n+1}) = F(a_1) \cdot \Phi (a_2, ..., a_{n+1}) - \Phi (a_1 \cdot a_2, a_3, ..., a_{n+1}) \]
\[ \pm ... \pm \Phi (a_1, ..., a_{n-1}, a_n \cdot a_{n+1}) \pm \Phi (a_1, ..., a_n) \cdot F(a_{n+1}) \]

we have

\[ d_F^* \Omega (a, b) = F(a) \cdot \Omega (b) - \Omega (a \cdot b) + \Omega (a) \cdot F(b) \]

One proves by calculation - using the functoriality of \( F \) - that

\[ (d_F^*)^2 = 0 \]

We call \( d_F^* \) the \( F \)-twist of the differential \( d_* \).

We write \( F(\varphi)(a, b) \) for \( F(\varphi (a, b)) \) since we just apply \( F \) after \( \varphi \), here. Similarly, we write \( f \left( F^\otimes 2 \right) (a, b) \) for \( f (F(a), F(b)) \), indicating in the notation that \( F \) is first applied in both components. Then the first order terms of \( (13) \) can be summarized to

\[ F(\varphi) - f \left( F^\otimes 2 \right) - d_F^* \Omega \]  

(14)

Let us next consider the two second order terms in \( f \) and \( \Omega \). We have

\[ f (\Omega (a), F(b)) + f (F(a), \Omega (b)) \]

Compare this with the composition of \( f \) of degree two and \( \Omega \) of degree one to

\[ (f \circ \Omega) (a, b) = f (\Omega (a), b) + f (a, \Omega (b)) \]

In complete analogy to the \( F \)-twisted differential \( d_F^* \), we can introduce the \( F \)-twisted composition \( \circ_F \) and get

\[ (f \circ_F \Omega) (a, b) = f (\Omega (a), F(b)) + f (F(a), \Omega (b)) \]

Remark 4 Observe that the positions in which \( F \) appears in the \( F \)-twisted forms are determined by the requirement that the needed compositions are well defined.
Introducing $\Omega(\varphi)$ in complete analogy to the notation for the first order terms, we can summarize the four second order terms of (13) to

$$\Omega(\varphi) - \Omega \bullet \Omega - f \circ^F \Omega$$

(15)

Using, again, the notation introduced for the first order terms, we can rewrite the single third order term as $f\left(\Omega^\otimes 2\right)$. Together with (14) and (15), this leads to an equivalent reformulation of (13) as

$$0 = d^F \bullet \Omega + f\left(\mathcal{F} \otimes 2\right) - \mathcal{F}(\varphi) + f \circ^F \Omega$$

(16)

$$+ \Omega \bullet \Omega - \Omega(\varphi) + f\left(\Omega^\otimes 2\right)$$

In a completely analogous way, we can treat the constraint (12), leading to

$$0 = d_\otimes \bullet \Omega + g\left(\mathcal{F} \otimes 2\right) - \mathcal{F}(\psi) + g \circ^F \Omega$$

(17)

$$+ \Omega \otimes \Omega - \Omega(\psi) + g\left(\Omega^\otimes 2\right)$$

In summary, for the complete deformation problem of the two monoidal categories $\mathcal{C}$ and $\mathcal{M}$ together with the monoidal functor

$$\mathcal{F} : \mathcal{M} \to \mathcal{C}$$

we get a doubled set of the three coupled differential equations of the previous section plus the two constraints (16) and (17). So, in total, we get a coupled system of eight differential equations which replace for the general deformation problem of quantum vertex algebras the Maurer-Cartan equation of the usual Hochschild complex.

We claim that one can formally derive the complete system of eight coupled differential equations by variation from the action

$$S \sim \int \left\{ f \circ d_\bullet f + \frac{2}{3}f \circ f \circ f + g \circ d_\otimes g + \frac{2}{3}g \circ g \circ g \right\}$$

(18)

$$+ \varphi \circ d_\bullet \varphi + \frac{2}{3} \varphi \circ \varphi \circ \varphi + \psi \circ d_\otimes \psi + \frac{2}{3} \psi \circ \psi \circ \psi$$

$$+ \lambda_1 \varphi^2 \left[ d_\otimes^2 f - d_\bullet^2 g + f \otimes f - g \bullet g - (f \circ^2 g) + (g \circ^2 f) \right]$$

$$- \text{Comp}(f, g, g) + \text{Comp}(g, f, f)$$

$$+ \lambda_2 \varphi^2 \left[ d_\otimes^2 \psi + \varphi \otimes \varphi - \psi \bullet \psi - (\varphi \circ^2 \psi) + (\psi \circ^2 \varphi) \right]$$
\[-\text{Comp} (\varphi, \psi, \psi) + \text{Comp} (\psi, \varphi, \varphi)] \\
+ \lambda_3 \circ [d^F \Omega + f (F \otimes^2) - F (\varphi) + f \circ^F \Omega] \\
+ \Omega \cdot \Omega - \Omega (\varphi) + f \left( \Omega \otimes^2 \right) \right] \\
+ \lambda_4 \circ [d^F \Omega + g (F \otimes^2) - F (\psi) + g \circ^F \Omega] \\
+ \Omega \otimes \Omega - \Omega (\psi) + g \left( \Omega \otimes^2 \right) \right] \}

with fields \( f, g, \varphi, \psi, \Omega, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \). Here, the fields \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) appear as Lagrange multipliers for the constraints where \( \lambda_1, \lambda_2 \) are of the same type as \( f, g, \varphi, \psi \), i.e. they are linear two variable maps on the morphisms of \( \mathcal{C} \) and \( \mathcal{M} \), respectively, while \( \lambda_3, \lambda_4 \) are of the same type as \( \Omega \), i.e. one variable linear maps from the morphisms of \( \mathcal{M} \) to the morphisms of \( \mathcal{C} \).

The action \( (18) \) includes interesting special cases: The case

\( f = g = \psi = 0 \)

corresponds in the setting of \( [\text{Bor}] \) to the case where the outer category \( \mathcal{C} \) remains fixed and only the inner ring object \( R \) is deformed. This should correspond to a deformation theory for the field algebras of \( [\text{BK}] \). Imposing an additional commutativity constraint on the inner ring object \( R \) should lead to deformations of usual vertex algebras (see \( [\text{Tam}] \) for a study in the setting of the chiral algebras of \( [\text{BD}] \)) and - in the infinitesimal case - to the vertex algebra cohomology of \( [\text{KV}] \). It remains a task for future work to rigorously check the embedding of these cases into our framework.

Since we treated the deformation problem in an abstract setting, one is not restricted to apply it to the deformations of quantum vertex algebras. Another special case which is covered by \( (18) \) is the deformation problem for three dimensional topological quantum field theories (3d TQFTs for short). A 3d TQFT is a monoidal functor (Atiyah functor) from the category \( \text{Cobord} \) of three dimensional cobordisms with disjoint union as tensor product to the category \( \text{Vect} \) of finite dimensional vector spaces with the usual tensor product. The deformation problem for Atiyah functors is then the deformation problem for this monoidal functor with \( \text{Cobord} \) and \( \text{Vect} \) fixed, i.e it is included into the deformation theory given by \( (18) \) by making the special choice \( \mathcal{M} = \text{Cobord} \) and \( \mathcal{C} = \text{Vect} \) and

\( f = g = \varphi = \psi = 0 \)
The action \((18)\) generalizes the action of the Kodaira-Spencer theory of gravity of [BCOV] in the following sense: The BV-quantization of Kodaira-Spencer theory leads to the deformations described by the total Hochschild complex of the structure sheaf on a complex three dimensional Calabi-Yau manifold. Physically, this can be seen as the deformation theory of the BRST-complex of the topological open string. For the topological closed string a much more complicated deformation theory of the BRST-complex arises (see [HM]) where the \(A_\infty\)-deformations - leading to a Hochschild deformation complex as for the open string case - constitute only a special case which is very similar to the special case of quantum vertex algebra deformations where only the inner ring object \(R\) is deformed. It is not clear at present if the action \((18)\) leads to the deformations described in [HM] but it is interesting in any case that the deformation problem for quantum vertex algebras suggests a generalization of the action of Kodaira-Spencer theory.

### 4 A list of goals

We now give a list of goals one would like to reach in future work by studying the action \((18)\) in more detail.

- Find a gauge fixing for \((18)\) similar to the Tian gauge for the Kodaira-Spencer theory of gravity. Such a gauge fixing is the prerequisite for making use of \((18)\) in the study of the deformation problem of quantum vertex algebras.

- Study the tree level expansion of \((18)\). Can one derive a formality property for the deformation problem of quantum vertex algebras in this way as it is possible for Kodaira-Spencer theory on Calabi-Yau varieties or for the Hochschild complex in the case of [Kon 1997]?

- Study the loop corrections for \((18)\). Is there an analog of the holomorphic anomaly of Kodaira-Spencer theory? If yes, to what kind of structures does the extended moduli space - including non-classical ghost numbers - lead?

- In [Kon 1999] and [KS] it was conjectured that a quotient of the motivic Galois group which should be given by the Grothendieck-Teichmüller
group $GT$, as introduced in [Dri], acts as a symmetry on the extended moduli space of Kodaira-Spencer theory. The algebraic nature of the compatibility constraint appearing between deformations of $\otimes$ and $\boxtimes$ suggests that for the deformation theory of quantum vertex algebras the situation is very similar to the one studied in [Sch]. We therefore expect that the quantum analogue of $GT$, introduced in [Sch] in the form of a self-dual, noncommutative and noncocommutative Hopf algebra $\mathcal{H}_{GT}$, should appear as a symmetry on the moduli space related to (18). Can one verify this by studying the field theory given by (18)?

- In [Sch] we have given an argument that deformations on higher categorical analogs of the structures studied there have to be trivial. We expect that in a completely similar way the deformation problem of quantum vertex algebras can not be generalized in a nontrivial way to a similar deformation problem in the setting of monoidal 2-categories or higher monoidal $n$-categories. This should imply an unusual stability property for the field theory given by (18) against further deformations. We have seen that Maurer-Cartan type equations and their generalizations studied in this paper are strongly related to a perturbative treatment of the deformation problem (in the sense of deformations given by formal power series). Would such a stability property mean that the field theory given by (18) can be completely understood from a perturbative treatment? Does it mean that (18) is free of anomalies at any loop order and that the extended moduli space of (18) involves a more or less trivial extension?

- Study (18) in a concrete example. E.g., study the deformation problem for the chiral de Rham complex on the affine space (as introduced in [MSV]), i.e. the deformation problem for the tensor product of the Heisenberg and the Clifford vertex algebra. This should be of interest for studying the deformations of the models appearing in [Wit 2005].

5 Conclusion

We have studied the deformation problem of quantum vertex algebras in an abstract - or better boiled down - setting. We have derived a coupled system
of eight differential equations generalizing the Maurer-Cartan equation of the usual Hochschild complex. We have shown that this system of equations can formally be derived from an action principle which in a certain sense generalizes the action of Kodaira-Spencer theory.

One of the next steps in our work will be the study of (18) in a highly simplified example. For this, we will study the deformation problem for the case of topological minimal models where it does reduce to a deformation problem in usual Hochschild cohomology which can be formulated as a deformation problem for a prepotential on a noncommutative moduli space.

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