ON ASYMPTOTIC EXPANSION SOLVERS FOR HIGHLY OSCILLATORY SEMI-EXPLICIT DAES

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Abstract. The paper is concerned with the discretization and solution of DAEs of index 1 and subject to a highly oscillatory forcing term. Separate asymptotic expansions in inverse powers of the oscillatory parameter are constructed to approximate the differential and algebraic variables of the DAEs. The series are truncated to enable practical implementation. Numerical experiments are provided to illustrate the effectiveness of the method.

1. Introduction. Differential Algebraic Equations (DAEs) arise in numerous applications and in particular in circuit and device simulation [14, 15]. Accurate numerical solvers or discretization of such equations present many challenges [9, 10]. These challenges are compounded in the presence of highly oscillatory forcing terms and it is the purpose of this paper to address these. We wish to analyse the behaviour of the system on a time scale which is much larger than the period of the forcing term and obtain an efficient method that is not restricted by its high frequency. Considering the practical application in electronic engineering, the forcing terms on the right side are typically represented as a finite set of Fourier coefficients, such as sinusoidal AC analysis [7, 8, 11].

We are concerned with semi-explicit time-varying highly oscillatory DAEs of the form

\[ x'(t) = f(x, y) + \sum_{\eta = -M_1}^{M_1} a_{\eta}(t)e^{i\eta \omega t}, \quad t \geq 0, \quad x(0) = x_0, \quad (1) \]

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\[ 0 = g(x, y) + \frac{1}{\omega} \sum_{\eta = -M_2}^{M_2} b_\eta(t)e^{i\eta\omega t}. \]

where \( \eta \in \mathbb{Z} \), \( x(t) : \mathbb{R} \to \mathbb{C}^{d_1} \), \( y(t) : \mathbb{R} \to \mathbb{C}^{d_2} \), \( \omega > 1 \), while \( f(x, y) : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \to \mathbb{C}^{d_1} \) and \( g(x, y) : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \to \mathbb{C}^{d_2} \) are two analytic functions. We further assume that the Jacobian \( \partial g/\partial y \) is nonsingular which means that the DAEs are of index 1.

In Formula (1), without loss of the generality, it is convenient to set \( M_1 \leq M_2 \) and \( M_1, M_2 \in \mathbb{N} \). Note that the algebraic part possesses the factor \( \omega^{-1} \), unlike the ODE part: this is necessary for an asymptotic expansion while reflecting many problems of practical importance.

In this work, contrary to the received wisdom in the DAE theory, we convert the problem to an ODE system, since this allows us to express its solution as an asymptotic series. We introduce two asymptotic expansions of the solutions of the DAE (1) in inverse powers of \( \omega \),

\[ x(t) \sim p_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t)e^{im\omega t}, \quad (2) \]
\[ y(t) \sim q_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} q_{r,m}(t)e^{im\omega t}. \]

In these expansions, each term in this expansion can be obtained recursively using operations which, being independent of \( \omega \), do not involve high oscillation. Besides their intrinsic value as an analytic tool, our expansions can be employed as an exceedingly affordable and precise numerical method. Regardless of whether the DAE is linear or nonlinear, our asymptotic method only takes smaller computational expense. It is a crucial observation that our approach, unlike other computational methods for DAEs, requires virtually the same computational expense regardless of the frequency \( \omega \). Recall that the range of Fourier frequencies in the forcing terms of (1) is finite, restricted to \( \eta \in \{-M_j, \ldots, M_j\}, j = 1, 2 \). This will imply that the number of frequencies (i.e., the range of \( m \)) in each sum in (2–3) is also finite (and dependent upon \( r \), as well as \( M_j \) and \( M_k \)). This renders our task somewhat simpler.

The error of our asymptotic expansions is denoted by \( z_s(t, \omega) \) and \( \epsilon_s(t, \omega) \),

\[ z_s(t, \omega) = x(t) - p_{0,0}(t) - \sum_{r=1}^{s} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t)e^{im\omega t}, \]
\[ \epsilon_s(t, \omega) = y(t) - q_{0,0}(t) - \sum_{r=1}^{s} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} q_{r,m}(t)e^{im\omega t} \]

where \( s \) is a suitable truncation parameter and \( \omega \) is the frequency. Once \( \omega \) increases, we expect the errors \( z_s \) and \( \epsilon_s \) to decay for fixed \( s \): asymptotic convergence! Note that the expansion (2–3) is asymptotic and its convergence can be discussed only within this context. In other words, convergence means that for every \( \varepsilon > 0 \) there exists \( \omega_0 > 0 \) such that for \( \omega \geq \omega_0 \) and \( t \in [0, t^*] \) the expansions (2–3) are within \( \varepsilon \) from the exact solution, \( |z_s(t, \omega)| < \varepsilon \) and \( |\epsilon_s(t, \omega)| < \varepsilon \) for fixed \( s \).

However, a legitimate question is subject to which conditions is \( z_{s+1}(\omega) \), say, smaller than \( z_s(\omega) \) for some \( s \geq 1 \) and whether convergence (in a classical sense) takes place for fixed \( t \) and \( \omega \) as \( s \to \infty \). Like in the case with other asymptotic expansions, this is a hugely nontrivial problem and we hope to return to it in future papers.
In Section 2 we formulate the asymptotic expansion for linear DAEs with highly oscillatory terms. A numerical example is provided to illustrate the theoretical results. Linear equations, of course, are important on their own merit, but the purpose of the section is also to explore the range of phenomena and the difficulties of the asymptotic expansion (2–3) in a setting in which we can do virtually everything in an explicit and accessible manner. Section 3 is concerned with nonlinear DAEs and again it concludes with a numerical example. Finally, Section 4 explores the conclusions of our asymptotic expansion method.

2. Linear DAEs.

2.1. Theoretical analysis. Consider a set of linear DAEs

\[ x'(t) = A(t)x(t) + B(t)y(t) + \sum_{\eta=-M_1}^{M_1} a_{\eta}(t)e^{i\eta\omega t}, \quad t \geq 0, \quad x(0) = x_0, \]

\[ 0 = C(t)x(t) + D(t)y(t) + \frac{1}{\omega} \sum_{\eta=-M_2}^{M_2} b_{\eta}(t)e^{i\eta\omega t}. \]

(4)

where \( M_1 \leq M_2, M_1, M_2 \in \mathbb{N} \), \( x(t) : \mathbb{R} \rightarrow \mathbb{C}^{d_1}, y(t) : \mathbb{R} \rightarrow \mathbb{C}^{d_2}, \omega \gg 1 \), and the matrix functions are

\[ A(t) : \mathbb{R} \rightarrow \mathbb{C}^{d_1 \times d_1}, \quad B(t) : \mathbb{R} \rightarrow \mathbb{C}^{d_1 \times d_2}, \]

\[ C(t) : \mathbb{R} \rightarrow \mathbb{C}^{d_2 \times d_1}, \quad D(t) : \mathbb{R} \rightarrow \mathbb{C}^{d_2 \times d_2}. \]

The condition that \( D(t) \) is invertible is imposed so that the index of the DAE set is 1. The first equation of (4) is a differential equation. Had it been independent of \( y \), we could have expressed its solution \( x \) as an asymptotic series

\[ x(t) \sim p_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t)e^{i\eta\omega t}. \]

[3, 4]. Our contention is that, although the underlying framework is considerably more complicated, this is also the case for \( x \) in (4). Although our ansatz (2–3) predicts an infinite number of terms \( p_{r,m} \), the linearity of (4) implies that the only nonzero terms are \( p_{r,m} \) for \( |m| \leq M_2 \).

When \( D(t) \) is not singular, the solution of the algebraic equation in (4) is

\[ y(t) = -D^{-1}(t) \left[ C(t)x(t) + \frac{1}{\omega} \sum_{\eta=-M_2}^{M_2} b_{\eta}(t)e^{i\eta\omega t} \right]. \]

(5)

Thus, once we can write \( x \) in a form consistent with (2–3), we can do so also for \( y \). This motivates us to assume (and subsequently verify) the ansatz that both \( x \) and \( y \) possess an asymptotic expansion of this form.

Substituting the expression for \( y(t) \) in (5) into the differential equation in (4) yields

\[ x' = (A - BD^{-1}C)x + \sum_{\eta=-M_1}^{M_1} a_{\eta}(t)e^{i\eta\omega t} - \sum_{\eta=-M_2}^{M_2} \frac{1}{\omega} BD^{-1}b_{\eta}(t)e^{i\eta\omega t} \]

\[ = Ex - \frac{1}{\omega} \sum_{\eta=-M_2}^{M_2} c_{\eta}(t)e^{i\eta\omega t} + \sum_{\eta=-M_1}^{M_1} a_{\eta}(t)e^{i\eta\omega t}, \]
where
\[ E(t) = A(t) - B(t)D^{-1}(t)C(t), \quad c_\eta(t) = B(t)D^{-1}(t)b_\eta(t). \]

We now assume that the matrices \( A, B, C \) and \( D \) (hence also \( E \)) are constant. Prior to presenting a procedure for the determination of the coefficients of the asymptotic expansions, we recall from [6] that, given a smooth continuous function \( h(\tau) : \mathbb{R} \to \mathbb{C}^d \), it is true that
\[
\int_0^t h(\tau)e^{i\eta\omega\tau}d\tau \sim \sum_{k=1}^{\infty} \frac{1}{(1-i\eta\omega)^k} \left[ h^{(k-1)}(t)e^{i\eta\omega t} - h^{(k-1)}(0) \right]
\]
for \( \eta \neq 0 \) and \( \omega \gg 1 \).

Using variation of constants, the solution \( x \) has the form
\[
x(t) = e^{tE}x_0 - \frac{1}{\omega}e^{tE} \int_0^t e^{-\tau E}c_\eta(\tau)d\tau + e^{tE} \int_0^t e^{-\tau E}a_\eta(\tau)d\tau
\]
\[
= e^{tE}x_0 + e^{tE} \int_0^t e^{-\tau E}a_\eta(\tau)d\tau
\]
\[
+ e^{tE} \sum_{\eta=-M_1}^{M_1} \sum_{\eta \neq 0} \frac{1}{(-i\eta\omega)^r} \left[ \left( \frac{d^{r-1}}{d\tau^{r-1}}[e^{-\tau E}c_\eta(\tau)] \right) \bigg|_{\tau=t} - \left( \frac{d^{r-1}}{d\tau^{r-1}}[e^{-\tau E}a_\eta(\tau)] \right) \bigg|_{\tau=t} \right]
\]
\[
+ \sum_{r=2}^{\infty} \frac{1}{\omega^r} \left[ \sum_{\eta=-M_2}^{M_2} \frac{1}{(-i\eta\omega)^{r-2}} \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^{r-2-j} E^{r-2-j} c_\eta^{(j)}(t) \right]
\]
\[
+ \sum_{\eta=M_1+1}^{M_2} \frac{1}{(-i\eta\omega)^{r-1}} \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^{r-2-j} E^{r-2-j} c_\eta^{(j)}(t)
\]
\[ + \sum_{\eta=-M_1}^{M_1} \left[ \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^{r-2-j} e^{(r-2-j)E} c^{(j)}(t) \right] \]

\[ - \sum_{\eta=-M_1}^{M_1} \left[ \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} e^{(r-1-j)a^{(j)}(t)} \right] \]

\[ \sim p_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t)e^{im\omega t}, \]

where

\[ p_{0,0}(t) = e^{tE} x_0 + e^{tE} \int_0^t e^{-\tau E} a_0(\tau) d\tau, \]  

\[ p_{1,0}(t) = -e^{tE} \sum_{m=-M_1}^{M_1} \frac{1}{im} a_m(0) - e^{tE} \int_0^t e^{-\tau E} c_0(\tau) d\tau, \]

\[ p_{1,m}(t) = \frac{1}{im} a_m(t), \quad m = -M_1, \cdots, M_1, \quad m \neq 0, \]

\[ p_{r,0}(t) = -e^{tE} \sum_{m=-M_2}^{M_2} \frac{1}{(-im)^{r-1}} \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^{r-2-j} e^{(r-2-j)c_m^{(j)}(0)} \]

\[ + e^{tE} \sum_{m=-M_1}^{M_1} \frac{1}{(-im)^r} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} e^{(r-1-j)a_m^{(j)}(0)}, \]

\[ p_{r,m}(t) = \]

\[ \left\{ \begin{array}{ll}
\frac{1}{(-im)^{r-1}} \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^{r-2-j} e^{(r-2-j)c_m^{(j)}(t)}, & -M_2 \leq m \leq -(M_1 + 1), \\
\frac{1}{(-im)^{r-1}} \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^{r-2-j} e^{(r-2-j)c_m^{(j)}(t)}, & -M_1 \leq m \leq M_1, \\
-\frac{1}{(-im)^{r-1}} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} e^{(r-1-j)a_m^{(j)}(t)}, & |m| \leq M_1, \\
\frac{1}{(-im)^{r-1}} \sum_{j=0}^{r-2} \binom{r-2}{j} (-1)^{r-2-j} e^{(r-2-j)c_m^{(j)}(t)}, & M_1 + 1 \geq m M_2, 
\end{array} \right. \]

for \( r \in \mathbb{N} \) and \( r \geq 2 \).
We substitute the terms (6–10) into (5) for \( y \),
\[
y(t) \sim q_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-M_2}^{M_2} q_{r,m}(t)e^{im\omega t},
\]
where
\[
q_{0,0}(t) = -D^{-1}Cp_{0,0}(t), \tag{11}
\]
\[
q_{1,0}(t) = -D^{-1}Cp_{1,0}(t) - D^{-1}b_0(t), \tag{12}
\]
\[
q_{1,m}(t) = \begin{cases}
-D^{-1}b_m(t), & M_2 \leq m \leq -(M_1 + 1), \\
-D^{-1}Cp_{1,m}(t) - D^{-1}b_m(t), & |m| \leq M_1, \\
-D^{-1}b_m(t), & M_1 + 1 \geq m \leq M_2,
\end{cases} \tag{13}
\]
\[
q_{r,0}(t) = -D^{-1}Cp_{r,0}(t), \tag{14}
\]
\[
q_{r,m}(t) = -D^{-1}Cp_{r,m}(t), \quad |m| \leq M_2. \tag{15}
\]

We note again that the number of nonzero \( p_{r,m} \) and \( q_{r,m} \) is finite, this being an artefact of linearity. Indeed, Fourier modes at one level \( r \) feed exclusively to the same Fourier modes at the level \( r+1 \) (e.g., \( p_{r+1,m} \), \( m \neq 0 \), depends only upon \( p_{s,m} \) for \( s \leq r \), as well as the coefficients \( a_m, b_m \) and \( c_m \)). This makes the solution of linear systems considerably easier.

For ease of reference and compassion with the nonlinear setting of Section 3, we summarize the above expansion formally:

**Theorem 2.1.** Given the DAEs (4), if the numbers of the input oscillatory forcing terms are finite, which are from \(-M_1\) to \(M_1\) and \(-M_2\) to \(M_2\), respectively, where \(M_1 \leq M_2\), then its solution can be approximated by the asymptotic series (2–3). Furthermore, the finite number of the input terms determines the finiteness of the index \( m \) in the series of (2–3). In other words, the series (2–3) have the concrete form
\[
x(t) \sim p_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-M_2}^{M_2} p_{r,m}(t)e^{im\omega t}, \tag{16}
\]
\[
y(t) \sim q_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-M_2}^{M_2} q_{r,m}(t)e^{im\omega t}, \tag{17}
\]
with the coefficients \( p_{r,m} \) and \( q_{r,m} \) given by (6–15).

### 2.2. A numerical example
To illustrate the procedure just described, we consider a circuit as shown in Fig. 1, governed by linear DAEs
\[
C_{in} \frac{dv(t)}{dt} + \frac{v(t)}{R} + i(t) = h(t),
\]
\[
L \frac{di(t)}{dt} - v(t) = 0, \quad t \geq 0, \quad \begin{bmatrix} v(0) \\ i(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
\[
e(t) - Rh(t) - v(t) = 0,
\]
where the unknown functions \( v(t) \) and \( i(t) \) represent the voltage and current variables to be solved, respectively, while \( h(t) \) is an algebraic variable and \( e(t) \) is the (known) input function.
This can be rewritten in the standard form
\[
\frac{dx(t)}{dt} = Ax(t) + Bh(t)
\]
\[0 = Cx(t) + Dh(t) + e(t),\]
where
\[
A = \begin{bmatrix} -(C_{in}R)^{-1} & -C_{in}^{-1} \\ L^{-1} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} C_{in}^{-1} \\ 0 \end{bmatrix}, \quad x(t) = \begin{bmatrix} v(t) \\ i(t) \end{bmatrix},
\]
\[
C = [-1, 0], \quad D = -R.
\]
The forcing term is
\[
e(t) = A_{inj} \sin(2\pi f_\omega t)/(2\pi f_\omega) = \omega^{-1} A_{inj} \left(e^{i\omega t} - e^{-i\omega t}\right)/(2i),
\]
where \(\omega = 2\pi f_\omega\) and \(f_\omega\) is the oscillatory parameter. Thus, \(a_m(t) \equiv 0, b_1(t) = A_{inj}/(2i), b_0(t) = 0\) and \(b_{-1}(t) = -A_{inj}/(2i)\). That is, \(M_1 = 0\) and \(M_2 = 1\). The remaining values \(C_{in}, R, L\) are the circuit capacitance, resistance and inductance, while \(A_{inj}\) is a constant.

In line with Theorem 2.1, we expect our expansions to be of the form
\[
x(t) \sim p_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-M_2}^{M_2} p_{r,m}(t)e^{im\omega t},
\]
\[
h(t) \sim q_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-M_2}^{M_2} q_{r,m}(t)e^{im\omega t}.
\]
It is instructive to compare the absolute error at different values of \(\omega\) for the asymptotic method,
\[
z_s(t) = x(t) - p_{0,0}(t) - \sum_{r=1}^{s} \frac{1}{\omega^r} \sum_{m=-M_2}^{M_2} p_{r,m}(t)e^{im\omega t},
\]
\[
\epsilon_s(t) = h(t) - q_{0,0}(t) - \sum_{r=1}^{s} \frac{1}{\omega^r} \sum_{m=-M_2}^{M_2} q_{r,m}(t)e^{im\omega t},
\]
for different values of \(s \geq 1\). We compute the first few terms to show that the error tends to zero for \(\omega \to \infty\). In each case, we compare the pointwise error incurred by
a truncated expansion with the exact solution. Before calculating the coefficients $p_{r,m}$ and $q_{r,m}$, we let

\[
E = \begin{bmatrix}
-2/(C_{in}R) & -1/C_{in} \\
L^{-1} & 0
\end{bmatrix},
\quad c_m = BD^{-1}b_m,
\]

\[
c_1 = \begin{bmatrix}
-A_{inj}/(2iC_{in}R) \\
0
\end{bmatrix},
\quad c_{-1} = \begin{bmatrix}
A_{inj}/(2iC_{in}R) \\
0
\end{bmatrix}.
\]

Employing (6–10) and (11–15), we obtain the coefficients

\[
p_{0,0} = e^{tE}x_0,
\quad p_{1,0} = 0,
\]

\[
p_{r,m} = \frac{(-1)^r}{(im)^{r-1}} E^{r-2} c_m, \quad m = -1, 1,
\]

\[
p_{r,0} = \frac{1}{ir} e^{tE} E^{r-2} \left[ c_1(0) + (-1)^{r-1}c_{-1}(0) \right];
\]

\[
q_{0,0} = \begin{bmatrix}
-R^{-1} & 0
\end{bmatrix} e^{tE}x_0,
\quad q_{1,0} = 0,
\quad q_{1,m} = R^{-1}b_m(t),
\]

\[
q_{r,0} = \begin{bmatrix}
-R^{-1} & 0
\end{bmatrix} p_{r,0},
\quad q_{r,m} = \begin{bmatrix}
-R^{-1} & 0
\end{bmatrix} p_{r,m}, \quad m = -1, 1.
\]

For this example, the parameters are chosen as $L = 0.1$, $R = 10$, $C_{in} = 0.2533$ and $A_{inj} = 10$. Based on the calculations, the error of our method for the voltage $v(t)$ is the first component of $z_s(t)$. Figure 2 plots the real parts of the first components of these error functions $z_s(t)$, for $s = 0, 1, 2, 3$, in the interval $t \in [0, 2\pi]$ with $\omega = 200\pi$ and $\omega = 2000\pi$. These error functions are oscillatory and their amplitude (corresponding to pointwise error) is very small. With increasing $\omega$, the amplitudes decrease: this might be surprising to classical numerical analysts but par for the course once the logic underlying Taylor expansions is replaced by the rationale of asymptotic expansions. For example, once $\omega$ increases from $200\pi$ to $2000\pi$ with $s = 0$, the bound of the error function $z_0(t)$ changes from $0.000015$ to $1.5 \times 10^{-7}$. The second component functions of $z_s(t)$ are the error functions for $i(t)$ which are plotted in Fig. 3 in $t \in [0, 2\pi]$ for every $s$ with $\omega = 200\pi$ and $\omega = 2000\pi$. The error function for $h(t)$ is $\epsilon_s(t)$ which is shown in Fig. 4.

It is shown in Figs 2–4 that the error decreases significantly with increasing $s$ and $\omega$, as expected from our expansions. Note that $z_0$ and $z_1$ for $v(t)$ and $i(t)$ have the same behaviour since $p_{1,m} \equiv 0$ in this example.

For the linear constant coefficients DAEs, the MAPLE routine $dsolve$ without any numerical parameters can get the explicit solution. Thus, we use this routine to produce reference solutions of the DAE, to compare with our asymptotic method. With the oscillatory parameter $\omega$ increasing from $200\pi$ to $2000\pi$, the CPU time of the traditional adaptive method $dsolve$ for the original highly oscillatory DAE (16) requires from $0.86$ to $1.516$ seconds and the storage from $8.8 \times 10^5$ to $1.5 \times 10^7$, while the asymptotic method only takes from $0.203$ to $0.110$ seconds and from $1.1 \times 10^3$ to $1.2 \times 10^4$ kbytes to compute the solution. The traditional method, even though it computes the exact solution, needs more CPU time and storage with an increasing oscillatory parameter. On the contrary, the computational expense of the asymptotic method does not vary because we can deduce the explicit expression of the solution for the linear DAEs. This is fully in line with our theoretical analysis.

2.3. The analysis of the factor $\omega^{-1}$. In this subsection, we highlight the importance of the factor $\omega^{-1}$ in the algebraic part: in effect, it makes the entire expansion
Figure 2. The error functions for $v(t)$ in $t \in [0, 2\pi]$ are the first components of $z_s(t)$. The top row: $z_0$ (the left) and $z_1$ (the right) with $\omega = 200\pi$. The second row: $z_2$ (the left) and $z_3$ (the right) with $\omega = 200\pi$. The third row: $z_0$ (the left) and $z_1$ (the right) with $\omega = 2000\pi$. The bottom row: $z_2$ (the left) and $z_3$ (the right) with $\omega = 2000\pi$. 
Figure 3. The error functions for $i(t)$ in $t \in [0, 2\pi]$ are the second components of $z_{\delta}(t)$. The top row: $z_0$ (the left) and $z_1$ (the right) with $\omega = 200\pi$. The second row: $z_2$ (the left) and $z_3$ (the right) with $\omega = 200\pi$. The third row: $z_0$ (the left) and $z_1$ (the right) with $\omega = 2000\pi$. The bottom row: $z_2$ (the left) and $z_3$ (the right) with $\omega = 2000\pi$. 
Figure 4. The error functions for $h(t)$ in $t \in [0, 2\pi]$. The top row: $\epsilon_0$ (the left) and $\epsilon_1$ (the right) with $\omega = 200\pi$. The second row: $\epsilon_2$ (the left) and $\epsilon_3$ (the right) with $\omega = 200\pi$. The third row: $\epsilon_0$ (the left) and $\epsilon_1$ (the right) with $\omega = 2000\pi$. The bottom row: $\epsilon_2$ (the left) and $\epsilon_3$ (the right) with $\omega = 2000\pi$.

of the nonlinear case possible! To demonstrate this, let us get rid of the $\omega^{-1}$ factor and consider first the linear setting. Specifically, we consider the DAE

$$x'(t) = Ax(t) + By(t) + \sum_{\eta=-M_1}^{M_1} a_\eta(t)e^{i\eta \omega t}, \quad t \geq 0, \quad x(0) = x_0,$$
\begin{align*}
0 &= Cx(t) + Dy(t) + \sum_{\eta = -M_2}^{M_2} b_\eta(t) e^{i\eta \omega t}. \\
\text{Let us suppose that an asymptotic expansion } \textit{à la} \text{ Section 2 exists for } x, \text{ namely } \\
x(t) &\sim p_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t) e^{i m \omega t}. \\
\text{Substituting this into the algebraic part for } y(t), \text{ we thus have } \\
y(t) &= -D^{-1} C \left[ p_{0,0} + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m} e^{i m \omega t} \right] - \sum_{\eta = -M_2}^{M_2} D^{-1} b_\eta e^{i \eta \omega t} \\
&= (-D^{-1} C p_{0,0} - D^{-1} b_0) - \sum_{\eta = -M_2}^{M_2} D^{-1} b_\eta e^{i \eta \omega t} \\
&\quad - \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} D^{-1} C p_{r,m} e^{i m \omega t} \\
&\sim q_{0,0} + \sum_{\eta = -M_2}^{M_2} q_{0,\eta} e^{i \eta \omega t} + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} q_{r,m} e^{i m \omega t}. \\
\end{align*}

Note the presence of the \( \sum_{\eta = -M_2}^{M_2} q_{0,\eta} e^{i \eta \omega t} \) term at the \( \mathcal{O}(1) \) scale. This is fine for a linear equation but a similar state of affairs is far less manageable in a nonlinear setting. Specifically, we can no longer expand \( f(x, y) \) about the point \((p_{0,0}, q_{0,0})\) and our entire approach is no longer suitable which we will show in Section 3.

3. Nonlinear DAEs.

3.1. The general theory. We now proceed to the considerably more complicated case of nonlinear highly oscillatory DAEs.

\begin{align}
&x'(t) = f(x, y) + \sum_{\eta = -M_2}^{M_1} a_\eta(t) e^{i \eta \omega t}, \quad t \geq 0, \quad x(0) = x_0, \\
&0 = g(x, y) + \frac{1}{\omega} \sum_{\eta = -M_2}^{M_2} b_\eta(t) e^{i \eta \omega t}, \quad M_1 \leq M_2. \\
\end{align}

Insofar as the ordinary differential system

\begin{align*}
\dot{x}' = f(t, x) + \sum_{\eta = -M_1}^{M_1} a_\eta(t) e^{i \eta \omega t}, \quad t \geq 0, \quad x(0) = x_0,
\end{align*}

is concerned, Condon et al. have presented in [3, 4] the asymptotic expansion,

\begin{align*}
x(t) &\sim p_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t) e^{i m \omega t}.
\end{align*}
We similarly define:

\[
0 = \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial y} y' + \frac{1}{\omega} \sum_{\eta = -M_2}^{M_2} \left[ b'_\eta(t) + i\eta \omega b_\eta(t) \right] e^{i\eta \omega t}.
\]

Since the DAE index is one, \((\partial g / \partial y)^{-1}\) exists. Therefore we can rewrite (21) in the form:

\[
y' = -\left( \frac{\partial g}{\partial y} \right)^{-1} \frac{\partial q}{\partial x} x' - \frac{1}{\omega} \left( \frac{\partial g}{\partial y} \right)^{-1} \sum_{\eta = -M_2}^{M_2} b'_\eta(t) e^{i\eta \omega t}
\]

Multiplying both sides by \(\frac{\partial g}{\partial y}\), we get

\[
y' \frac{\partial g}{\partial y} = - \frac{\partial q}{\partial x} x' \frac{\partial g}{\partial y} - \frac{1}{\omega} \sum_{\eta = -M_2}^{M_2} \left[ b'_\eta(t) \right] e^{i\eta \omega t}
\]

This is similar to the differential equation. Therefore, the ansatz is that the variable \(y(t)\) has an asymptotic expansion of the form

\[
y(t) \sim q_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} q_{r,m}(t) e^{i\eta \omega t},
\]

which we proceed to confirm. Similarly to the linear case in Section 2, we want to determine the concrete relation between \(a_m, b_m\) and \(p_{r,m}, q_{r,m}\). Substituting (20) and (22) into the DAEs (19) and comparing the terms by scale and frequency, the coefficients in (20) and (22) can be obtained in a recursive manner. In the series (20) and (22), for fixed \(r\), the indexes \(m\) are finite because of the finiteness of \(M_1\) and \(M_2\). Before giving these explicitly, we introduce and explain notation that shall be employed in what follows.

The functions \(f(x,y)\) and \(g(x,y)\) are analytic and can be expanded in Taylor series about the point \((p_{0,0}, q_{0,0})\) (which itself depends on \(t\)). Thus, for any \(p \in \mathbb{C}^{d_1}\), \(q \in \mathbb{C}^{d_2}\), both sufficiently small in norm, we expand:

\[
f(p_{0,0} + p, q_{0,0} + q) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} f_{n,k}(p_{0,0}, q_{0,0}) [p, p, \ldots, p][q, q, \ldots, q],
\]

where

\[
f_{n,k} : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \mathbb{C}^{d_1} \times \cdots \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \cdots \times \mathbb{C}^{d_2} \to \mathbb{C}^{d_1},
\]

is linear in all its arguments except for \(p_{0,0}\) and \(q_{0,0}\) and symmetric in each of the two groups of arguments enclosed by square brackets: specifically, it is the derivative tensor

\[
f_{n,k}(p_{0,0}, q_{0,0}) = \left. \frac{\partial^n f(x,y)}{\partial x^k \partial y^{n-k}} \right|_{(x,y)=(p_{0,0}, q_{0,0})}
\]

We similarly define

\[
g_{n,k} : \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \mathbb{C}^{d_1} \times \cdots \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \cdots \times \mathbb{C}^{d_2} \to \mathbb{C}^{d_2}, \quad 0 \leq k \leq n,
\]

Matters become more complicated for the DAE (19) because we additionally need to consider the algebraic part. To this end, we differentiate the second algebraic equation formally,

\[
0 = \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial y} y' + \frac{1}{\omega} \sum_{\eta = -M_2}^{M_2} \left[ b'_\eta(t) + i\eta \omega b_\eta(t) \right] e^{i\eta \omega t}.
\]
so that
\[ g(p_{0,0} + p \cdot q_{0,0} + q) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} g_{n,k}(p_{0,0}, q_{0,0})[p, \ldots, p][q, q, \ldots, q]. \]

Substituting (20) and (22) into the nonlinear function on the right, we expand it in Taylor series at the point \((p_{0,0}, q_{0,0})\),

\[ f(x, y) \sim f(p_{0,0}, q_{0,0}) + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} f_{n,k}(p_{0,0}, q_{0,0}) \]

\[ \left[ \sum_{\ell_1=1}^{\infty} \frac{1}{\omega_{\ell_1}} \sum_{\sigma_1=-\infty}^{\infty} p_{\ell_1,\sigma_1} e^{i\sigma_1 \omega t}, \ldots, \sum_{\ell_n=1}^{\infty} \frac{1}{\omega_{\ell_n}} \sum_{\sigma_n=-\infty}^{\infty} p_{\ell_n,\sigma_n} e^{i\sigma_n \omega t} \right] \]

\[ \left[ \sum_{\ell_{k+1}=1}^{\infty} \frac{1}{\omega_{\ell_{k+1}}} \sum_{\sigma_{k+1}=\infty}^{\infty} q_{\ell_{k+1},\sigma_{k+1}} e^{i\sigma_{k+1} \omega t}, \ldots, \sum_{\ell_n=1}^{\infty} \frac{1}{\omega_{\ell_n}} \sum_{\sigma_n=-\infty}^{\infty} q_{\ell_n,\sigma_n} e^{i\sigma_n \omega t} \right], \]

where these indexes \(\sigma_1, \ldots, \sigma_k, \sigma_{k+1}, \ldots, \sigma_n\) are finite because of the finiteness of \(m\) in (20) and (22). Since \(f_{n,k}\) is linear in all its arguments exclusive of \((p_{0,0}, q_{0,0})\), extracting those linear arguments, we have

\[ f(x, y) \sim f(p_{0,0}, q_{0,0}) + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} f_{n,k}(p_{0,0}, q_{0,0}) \]

\[ \left[ p_{\ell_1,\sigma_1}, \ldots, p_{\ell_n,\sigma_n} \right] \left[ q_{\ell_{k+1},\sigma_{k+1}}, \ldots, q_{\ell_n,\sigma_n} \right] e^{i(\sigma_1 + \cdots + \sigma_n) \omega t}. \]

Let \(\ell_1 + \ell_2 + \cdots + \ell_n = r\) with \(\ell_1, \ldots, \ell_n \geq 1\) and \(\sigma_1 + \cdots + \sigma_n = m\) with \(\sigma_1, \ldots, \sigma_n \in \mathbb{Z}\). It follows that

\[ f(x, y) \sim f(p_{0,0}, q_{0,0}) + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \sum_{\ell_1+\cdots+\ell_n=r} \sum_{\sigma_1+\cdots+\sigma_n=m} f_{n,k}(p_{0,0}, q_{0,0}) \]

\[ \left[ p_{\ell_1,\sigma_1}, \ldots, p_{\ell_k,\sigma_k} \right] \left[ q_{\ell_{k+1},\sigma_{k+1}}, \ldots, q_{\ell_n,\sigma_n} \right] e^{i(\sigma_1 + \cdots + \sigma_n) \omega t}, \]

where the number of \(m\) is finite. To make this more transparent, we define the two index sets are

\[ \mathbb{I}^n_\circ = \{ \ell \in \mathbb{N}^n : 1^\top \ell = r \}, \quad \mathbb{I}^n_m = \{ \sigma \in \mathbb{Z}^n : 1^\top \sigma = m \}, \]

where \(\ell = (\ell_1, \ell_2, \ldots, \ell_n)^\top\) and \(\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)^\top\). To construct a numerical method for the every coefficient functions \(p_{r,m}\) and \(q_{r,m}\) in (20) and (22), we assume these two infinite sums \(n\) and \(r\) can be exchanged. Exchanging the index \(n\) and \(r\), we rewrite the Taylor expansion of \(f(x, y)\) in the form

\[ f(x, y) \sim f(p_{0,0}, q_{0,0}) + \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{\omega^r} \sum_{\ell \in \mathbb{I}^r_\circ, \sigma \in \mathbb{I}^r_m} \binom{n}{k} f_{n,k}(p_{0,0}, q_{0,0}) \]

\[ \left[ p_{\ell_1,\sigma_1}, \ldots, p_{\ell_k,\sigma_k} \right] \left[ q_{\ell_{k+1},\sigma_{k+1}}, \ldots, q_{\ell_n,\sigma_n} \right] e^{i\omega t} \]

\[ = f(p_{0,0}, q_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!(n-k)!} \sum_{\ell \in \mathbb{I}^r_\circ, \sigma \in \mathbb{I}^r_m} f_{n,k}(p_{0,0}, q_{0,0}) \]
\[
\begin{bmatrix}
p_{\ell_1, \sigma_1}, \ldots, p_{\ell_k, \sigma_k} \\
q_{\ell_{k+1}, \sigma_{k+1}}, \ldots, q_{\ell_n, \sigma_n}
\end{bmatrix} e^{\im \omega t}.
\]

Note that there is a measure of redundancy in the set \(I_{n,r}^o\). For example,
\[
I_{3,5}^o = \{(1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)\},
\]
consisting of six elements. Because of the built-in symmetry of \(f_{3,k}\), some of the elements above correspond to the same term and can be aggregated, but this depends on the value of \(k\). For example, for \(k = 1\) the second and third entry of each triplet can be permuted, hence we can associate a weight \(\theta_{n,k,r}(\ell)\) with each element and consider instead
\[
I_{3,1,5} = \{(1, 1, 3), (3, 1, 1), (1, 2, 2), (2, 1, 2)\},
\]
say, with the multiplicities
\[
\theta_{3,1,5}(3,1,1) = \theta_{3,1,5}(1,2,2) = 1, \quad \theta_{3,1,5}(1,1,3) = \theta_{3,1,5}(2,1,2) = 2.
\]

More formally, let
\[
I_{n,k,r} = \{\ell \in \mathbb{N}^n : \ell^\top \ell = r, \ell_1 \leq \ell_2 \leq \cdots \leq \ell_k \leq \ell_{k+1} \leq \cdots \leq \ell_n\}
\]
and let \(\theta_{n,k,r}(\ell)\) stand for the multiplicity of \(\ell \in I_{n,k,r}\), i.e. the number of terms of \(I_{n,r}^o\) that can be brought into the form \(\ell\) by permutations of the first \(k\) entries and of the last \(n-k\) entries. Note that \(k = 0\) and \(k = n\) make perfect sense: in each of these cases there is in \(I_{n,k,r}\) just a single monotone sequence.

For example, bearing in mind that \(n \leq r\),
\[
\begin{align*}
n = 1 : & \quad I_{1,1,r} = \{(r)\}, \quad \theta_{1,1,r}(r) = 1; \\
n = 2 : & \quad I_{2,0,r} = I_{2,2,r} = \{(i, r-i) : i = 1, \ldots, \lfloor r/2 \rfloor\}, \\
& \quad \theta_{2,0,r}(i, r-i) = \theta_{2,2,r}(i, r-i) = \\
& \quad \left\{ \begin{array}{ll}
1, & 2i = r, \\
2, & 2i < r;
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
n = 3 : & \quad I_{3,0,r} = I_{3,3,r} = \{\ell : \ell^\top \ell = r\}, \\
& \quad \theta_{3,0,r}(\ell) = \theta_{3,3,r}(\ell) = \\
& \quad \left\{ \begin{array}{ll}
1, & \ell_1 = \ell_2 = \ell_3, \\
2, & \ell_1 = \ell_2 \neq \ell_3, \\
2, & \ell_1 = \ell_3 \neq \ell_2, \\
2, & \ell_2 = \ell_3 \neq \ell_1, \\
3, & \text{otherwise};
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
n = 3, & \quad I_{3,1,r} = \{\ell : \ell_2 \leq \ell_3, \ell^\top \ell = r\}, \\
& \quad \theta_{3,1,r}(\ell) = \\
& \quad \left\{ \begin{array}{ll}
1, & \ell_2 = \ell_3, \\
2, & \ell_2 < \ell_3,
\end{array} \right.
\end{align*}
\]
\[
\begin{align*}
n = 3, & \quad I_{3,2,r} = \{\ell : \ell_1 \leq \ell_2, \ell^\top \ell = r\}, \\
& \quad \theta_{3,2,r}(\ell) = \\
& \quad \left\{ \begin{array}{ll}
1, & \ell_1 = \ell_2, \\
2, & \ell_1 < \ell_2.
\end{array} \right.
\end{align*}
\]
and so on.

We can now rewrite the Taylor expansion of \(f(x, y)\) in the form
\[
f(x, y) \sim f(p_{0,0}, q_{0,0})
\]
Likewise, 

\[ g(x, y) \sim g(p_{0,0}, q_{0,0}) \]

\[ + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m \in \mathbb{Z}} \sum_{n=1}^{\infty} \frac{1}{k!(n-k)!} \sum_{\ell \in I_{n,k,r}} \theta_{n,k,r}(\ell) \sum_{\sigma \in \mathcal{J}_{n,m}} g_{n,k}(p_{0,0}, q_{0,0}) [p_{\ell_1, \sigma_1}, \ldots, p_{\ell_k, \sigma_k}] [q_{\ell_{k+1}, \sigma_{k+1}}, \ldots, q_{\ell_n, \sigma_n}] e^{im\omega t}, \]

where

\[ g_{n,k}(p_{0,0}, q_{0,0}) = \frac{\partial^n g(x, y)}{\partial x^k \partial y^{n-k}} \bigg|_{(x,y) = (p_{0,0}, q_{0,0})} \]

Substitute (23) and (24) on both sides of the DAE (19),

\[ p'_{0,0} + \sum_{m \in \mathbb{Z}} im p_{1,m} e^{im\omega t} + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} (p'_{r,m} + im p_{r+1,m}) e^{im\omega t} \]

\[ = f(p_{0,0}, q_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=0}^{\infty} \frac{1}{k!(n-k)!} \sum_{m \in \mathbb{Z}} \sum_{\ell \in I_{n,k,r}} \theta_{n,k,r}(\ell) \sum_{\sigma \in \mathcal{J}_{n,m}} f_{n,k}(p_{0,0}, q_{0,0}) [p_{\ell_1, \sigma_1}, \ldots, p_{\ell_k, \sigma_k}] [q_{\ell_{k+1}, \sigma_{k+1}}, \ldots, q_{\ell_n, \sigma_n}] e^{im\omega t} \]

\[ + \sum_{\eta = -M_1}^{M_1} a_\eta e^{i\eta \omega t}, \]

\[ 0 = g(p_{0,0}, q_{0,0}) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=0}^{\infty} \frac{1}{k!(n-k)!} \sum_{m \in \mathbb{Z}} \sum_{\ell \in I_{n,k,r}} \theta_{n,k,r}(\ell) \sum_{\sigma \in \mathcal{J}_{n,m}} g_{n,k}(p_{0,0}, q_{0,0}) [p_{\ell_1, \sigma_1}, \ldots, p_{\ell_k, \sigma_k}] [q_{\ell_{k+1}, \sigma_{k+1}}, \ldots, q_{\ell_n, \sigma_n}] e^{im\omega t} \]

\[ + \sum_{\eta = -M_2}^{M_2} b_\eta e^{i\eta \omega t}. \]

We next separate the equations by different powers of \( \omega \). Thus, for \( r = 0 \),

\[ p'_{0,0} + \sum_{m \in \mathbb{Z}} im p_{1,m} e^{im\omega t} = f(p_{0,0}, q_{0,0}) + \sum_{\eta = -M_1}^{M_1} a_\eta e^{i\eta \omega t}, \]

\[ 0 = g(p_{0,0}, q_{0,0}) \]

while for \( r = 1 \) we have

\[ \sum_{m \in \mathbb{Z}} (p'_{1,m} + im p_{2,m}) e^{i\omega t} = \sum_{m \in \mathbb{Z}} f_{1,0}(p_{0,0}, q_{0,0}) q_{1,m} e^{i\omega t} \]

\[ + \sum_{m \in \mathbb{Z}} f_{1,1}(p_{0,0}, q_{0,0}) p_{1,m} e^{i\omega t}, \]
across a finite corresponding to nonzero terms depends upon both \((M_3, 2)\). The first few terms of

\[ f_{n,k}(p_{0,0}, q_{0,0})[p_{f_1, \sigma_1}, \cdots, p_{f_k, \sigma_k}, q_{\ell_{k+1}, \sigma_{k+1}}, \cdots, q_{\ell_n, \sigma_n}] e^{im\omega t}, \]

and for \(r \geq 2\)

\[ \sum_{m=-\infty}^{\infty} (p'_{r,m} + \im p_{r+1,m}) e^{im\omega t} = \sum_{n=1}^{r} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{m \in \mathbb{Z}} \sum_{\ell \in \mathbb{N}} \theta_{n,k,r}(\ell) \sum_{\sigma \in \mathcal{J}_{n,m}} f_{n,k}(p_{0,0}, q_{0,0}) \sum_{\ell' \in \mathbb{N}, k' \in \mathbb{N}} \theta_{n,k,r}(\ell') \sum_{\sigma' \in \mathcal{J}_{n,m}} g_{n,k}(p_{0,0}, q_{0,0}) \]

\[ 0 = \sum_{n=1}^{r} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{m \in \mathbb{Z}} \sum_{\ell \in \mathbb{N}, k \in \mathbb{N}} \theta_{n,k,r}(\ell) \sum_{\sigma \in \mathcal{J}_{n,m}} f_{n,k}(p_{0,0}, q_{0,0}) \sum_{\ell' \in \mathbb{N}, k' \in \mathbb{N}} \theta_{n,k,r}(\ell') \sum_{\sigma' \in \mathcal{J}_{n,m}} g_{n,k}(p_{0,0}, q_{0,0}) \]

Similarly to [3, 4], we have the initial conditions

\[ x(0) = p_{0,0}(0) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(0) e^{im\omega t} = x_0. \]

Separating powers of \(\omega\), we impose the initial conditions

\[ p_{0,0}(0) = x_0, \quad p_{r,0}(0) = -\sum_{m \neq 0} p_{r,m}(0), \quad r \in \mathbb{N}. \]

Note an important point: Each of the sums \(\sum_{m} p_{r,m} e^{im\omega t}\) and \(\sum_{m} q_{r,m} e^{im\omega t}\) ranges across a finite number of nonzero terms. In a nonlinear case the range of indices corresponding to nonzero terms depends upon both \((M_1, M_2)\) and \(r\) and it grows with \(r\). However, the crucial observation is that finiteness means that there is no issue of these Fourier series becoming divergent.

### 3.2. The first few terms of \(r\).

For \(r = 0\) (25) yields the non-oscillatory DAE

\[ p'_{0,0} = f(p_{0,0}, q_{0,0}) + a_0, \quad t \geq 0, \quad p_{0,0}(0) = x_0, \]

and the recursion

\[ p_{1,m} = \frac{a_m}{im}, \quad m = -M_1, \cdots, M_1, \quad m \neq 0. \]

In other words, \(p_{1,m} = 0\) provided \(m \notin [-M_1, M_1]\).

In the case of \(r = 1\), for \(m = 0\) we obtain from (26) a non-oscillatory DAE

\[ p'_{1,0} = f_{1,0}(p_{0,0}, q_{0,0}) q_{1,0} + f_{1,1}(p_{0,0}, q_{0,0}) p_{1,0}, \quad 0 = g_{1,0}(p_{0,0}, q_{0,0}) q_{1,0} + g_{1,1}(p_{0,0}, q_{0,0}) p_{1,0} + b_0, \]

as well as the recursions

\[ q_{1,m} = \frac{1}{im} \left[-p'_{1,m} + f_{1,0}(p_{0,0}, q_{0,0}) q_{1,m} + f_{1,1}(p_{0,0}, q_{0,0}) p_{1,m} \right], \]

\[ p_{2,m} = \frac{1}{im} \left[-p'_{2,m} + f_{2,0}(p_{0,0}, q_{0,0}) q_{2,m} + f_{2,1}(p_{0,0}, q_{0,0}) p_{2,m} \right]. \]

(The Jacobian matrix \(g_{1,0}(p_{0,0}, q_{0,0})\) is, by the index-1 assumption, invertible) for \(m = -M_2, \cdots, M_2, \ m \neq 0\). Note that the number of the nonzero terms \(q_{1,m}\) is
determined by nonzero $p_{1,m}$ and $b_{m}$, which are in turn determined by nonzero $a_{m}$ and $b_{m}$. In practical application, if the input data $a_{m}$, $m = -M_1, \ldots, M_1$, and $b_{m}$, $m = -M_2, \ldots, M_2$, are finite with $M_1 \leq M_2$, then $q_{1,m}$ and $p_{1,m}$ are finite for $m = -M_2, \ldots, M_2$. It is observed that the number of $p_{2,m}$ is associated with those of $q_{1,m}$ and $p_{1,m}$. The corresponding index $m$ of $p_{2,m}$ ranges from $-M_2$ to $M_2$.

The next scale is $r = 2$ whereby, after elementary algebra, we obtain for each $m \in \mathbb{Z}$

$$p_{2,m} + mp_{3,m} = f_{1.0}(p_{0,0}, q_{0,0})q_{2,m} + f_{1.1}(p_{0,0}, q_{0,0})p_{2,m} + \frac{1}{2} \sum_{\sigma_1 + \sigma_2 = m, \sigma_1, \sigma_2 \in \mathbb{Z}} f_{2.0}(p_{0,0}, q_{0,0})[q_{1,\sigma_1}, q_{1,\sigma_2}] + \sum_{\sigma_1 + \sigma_2 = m, \sigma_1, \sigma_2 \in \mathbb{Z}} f_{2.1}(p_{0,0}, q_{0,0})[p_{1,\sigma_1}, q_{1,\sigma_2}]$$

$$+ \frac{1}{2} \sum_{\sigma_1 + \sigma_2 = m, \sigma_1, \sigma_2 \in \mathbb{Z}} f_{2.2}(p_{0,0}, q_{0,0})[p_{1,\sigma_1}, p_{1,\sigma_2}]$$

$$0 = g_{1.0}(p_{0,0}, q_{0,0})q_{2,m} + g_{1.1}(p_{0,0}, q_{0,0})p_{2,m} + \frac{1}{2} \sum_{\sigma_1 + \sigma_2 = m, \sigma_1, \sigma_2 \in \mathbb{Z}} g_{2.0}(p_{0,0}, q_{0,0})[q_{1,\sigma_1}, q_{1,\sigma_2}]$$

$$+ \sum_{\sigma_1 + \sigma_2 = m, \sigma_1, \sigma_2 \in \mathbb{Z}} g_{2.1}(p_{0,0}, q_{0,0})[p_{1,\sigma_1}, q_{1,\sigma_2}] + \frac{1}{2} \sum_{\sigma_1 + \sigma_2 = m, \sigma_1, \sigma_2 \in \mathbb{Z}} g_{2.2}(p_{0,0}, q_{0,0})[p_{1,\sigma_1}, p_{1,\sigma_2}]$$

with the initial condition

$$p_{2,0}(0) = - \sum_{m \neq 0} p_{2,m}(0).$$

When $m \neq 0$, we derive the recursions

$$q_{2,m} = -g_{1.0}(p_{0,0}, q_{0,0})q_{2,m} - g_{1.1}(p_{0,0}, q_{0,0})p_{2,m} + \frac{1}{2} \sum_{\sigma_1 + \sigma_2 = m, \sigma_1, \sigma_2 \in \mathbb{Z}} g_{2.0}(p_{0,0}, q_{0,0})[q_{1,\sigma_1}, q_{1,\sigma_2}]$$

$$+ \sum_{\sigma_1 + \sigma_2 = m, \sigma_1, \sigma_2 \in \mathbb{Z}} g_{2.1}(p_{0,0}, q_{0,0})[p_{1,\sigma_1}, q_{1,\sigma_2}] + \frac{1}{2} \sum_{\sigma_1 + \sigma_2 = m, \sigma_1, \sigma_2 \in \mathbb{Z}} g_{2.2}(p_{0,0}, q_{0,0})[p_{1,\sigma_1}, p_{1,\sigma_2}]$$
Notice that the term $q_{2,m}$ is decided by

$$p_{3,m} = \frac{1}{im} \left\{ -p_{2,m} + f_{1,0}(p_{0,0}, q_{0,0})q_{2,m} + f_{1,1}(p_{0,0}, q_{0,0})p_{2,m} ight\} + \frac{1}{2} \sum_{\sigma_1 + \sigma_2 = m \atop \sigma_1, \sigma_2 \in \mathbb{Z}} f_{2,0}(p_{0,0}, q_{0,0})[q_{1,\sigma_1}, q_{1,\sigma_2}] + \sum_{\sigma_1 + \sigma_2 = m \atop \sigma_1, \sigma_2 \in \mathbb{Z}} f_{2,1}(p_{0,0}, q_{0,0})[p_{1,\sigma_1}, q_{1,\sigma_2}]
$$

$$+ \frac{1}{2} \sum_{\sigma_1 + \sigma_2 = m \atop \sigma_1, \sigma_2 \in \mathbb{Z}} f_{2,2}(p_{0,0}, q_{0,0})[p_{1,\sigma_1}, p_{1,\sigma_2}] \right\}.
$$

Notice that the term $q_{2,m}$ is decided by

$$p_{2,m} = [p_{1,\sigma_1}, q_{1,\sigma_2}]$$

$$p_{1,\sigma_1}, q_{1,\sigma_2}$$

$$p_{1,\sigma_1}, p_{1,\sigma_2}$$

where the maximum index set is $[-2M_2, 2M_2]$ of the couple $[q_{1,\sigma_1}, q_{1,\sigma_2}]$. Thus the index set of $m$ for $q_{2,m}$ is $[-2M_2, 2M_2]$. The same deduction for $p_{3,m}$ results in the index set of $m$ for $p_{3,m}$ is $[-2M_2, 2M_2]$.

### 3.3. The general expansion

Proceeding with full generality, we can easily convert the $r = 2$ example of the last subsection into a general rule. Thus, for any $r \geq 2$ in (27), we separate frequencies. For $m = 0$ we again obtain the non-oscillatory DAE

$$p_{r,0} = \sum_{n=1}^{r} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{\ell \in I_{n,k,r}} \theta_{n,k,r}(\ell) \sum_{\sigma \in J_{n,0}} f_{n,k}(p_{0,0}, q_{0,0})$$

$$\left[ p_{k_1,0}, \cdots, p_{k_r,0} \right] \left[ q_{k_{l_1} + 1,0}, \cdots, q_{k_l,0} \right],$$

$$0 = \sum_{n=1}^{r} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{\ell \in I_{n,k,r}} \theta_{n,k,r}(\ell) \sum_{\sigma \in J_{n,0}} g_{n,k}(p_{0,0}, q_{0,0})$$

$$\left[ p_{k_1,0}, \cdots, p_{k_r,0} \right] \left[ q_{k_{l_1} + 1,0}, \cdots, q_{k_l,0} \right],$$

accompanied by the initial conditions $p_{r,0}(0) = -\sum_{m \neq 0} p_{r,m}(0)$ to solve for $p_{r,0}$ and $q_{r,0}$. When $m \neq 0$ we have recursive formulae for $p_{r+1,m}$ and $q_{r,m}$:

$$q_{r,m} = -g_{1,0}^{-1}(p_{0,0}, q_{0,0}) \left\{ g_{1,1}(p_{0,0}, q_{0,0})p_{r,m} + \sum_{n=2}^{r} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{\ell \in I_{n,k,r}} \theta_{n,k,r}(\ell) \sum_{\sigma \in J_{n,m}} g_{n,k}(p_{0,0}, q_{0,0})$$

$$\left[ p_{k_1,0}, \cdots, p_{k_r,0} \right] \left[ q_{k_{l_1} + 1,0}, \cdots, q_{k_l,0} \right] \right\}$$

$$p_{r+1,m} = \frac{1}{im} \left\{ -p'_{r,m} + f_{1,0}(p_{0,0}, q_{0,0})q_{r,m} + f_{1,1}(p_{0,0}, q_{0,0})p_{r,m}$$

$$+ \sum_{n=2}^{r} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \sum_{\ell \in I_{n,k,r}} \theta_{n,k,r}(\ell) \sum_{\sigma \in J_{n,m}} f_{n,k}(p_{0,0}, q_{0,0})$$

$$\left[ p_{k_1,0}, \cdots, p_{k_r,0} \right] \left[ q_{k_{l_1} + 1,0}, \cdots, q_{k_l,0} \right] \right\}.$$
\[ \begin{bmatrix} p_{\ell_1, \sigma_1}, \cdots, p_{\ell_k, \sigma_k} \end{bmatrix} \begin{bmatrix} q_{\ell_{k+1}, \sigma_{k+1}}, \cdots, q_{\ell_n, \sigma_n} \end{bmatrix}. \]

Considering \( q_{r,m} \) is related to \( p_{r,m} \), \( [p_{\ell_1, \sigma_1}, \cdots, p_{\ell_k, \sigma_k}][q_{\ell_{k+1}, \sigma_{k+1}}, \cdots, q_{\ell_n, \sigma_n}] \), where the maximum index set if \([-r M_2, r M_2]\). Therefore the index of \( m \) for \( q_{r,m} \) is \([-r M_2, r M_2]\). also \( p_{r+1,m} \) is \([-r M_2, r M_2]\).

It is observed that the second sub-index of the terms \( q_{r,m} \) is associated with \( p_{\ell, \sigma}, \ell = 2, 3, \cdots, r; \sigma = -(r-1) M_2, \cdots, (r-1) M_2 \) and

\[ q_{\ell, \sigma}, \ell = 2, 3, \cdots, r-1; \sigma = -r M_2, \cdots, r M_2 \]

which have been calculated at the last \( r-1 \) steps.

In summary, we have proven an exact equivalent of Theorem 1 and a confirmation of our ansatzen.

**Theorem 3.1.** Provide the DAEs (19) with the finite term input \( a_\eta, \eta = -M_1, \cdots, M_1, b_\eta, \eta = -M_2, \cdots, M_2, \) and \( M_1 \leq M_2 \). Then the solutions of (19) can be approximated asymptotically by the ansatzen (20) and (22), also the indexes \( m \) of (20) and (22) are finite. In other words, the series (20) and (22) can be written as

\[ x(t) \sim q_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-r M_2}^{r M_2} q_{r,m}(t) e^{i m \omega t}, \]

(30)

\[ y(t) \sim p_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-(r-1) M_2}^{(r-1) M_2} p_{r,m}(t) e^{i m \omega t}. \]

(31)

3.4. A nonlinear example. To illustrate our approach, consider the nonlinear circuit in Fig. 5. Its behaviour is governed by the equations

\[\begin{align*}
C_{\text{in}} \frac{dv(t)}{dt} + \frac{v(t)}{R} + i(t) + S_{\text{inj}} \tanh \left( \frac{G_n}{S_{\text{inj}}} v(t) \right) &= h(t) + e_1(t), \\
L \frac{di(t)}{dt} - v(t) &= 0, \\
e(t) - Rh(t) - v(t) &= 0, \quad t \geq 0,
\end{align*}\]

(32)

where \( I_{\text{nl}} = S_{\text{inj}} \tanh (G_n v(t)/S_{\text{inj}}) \) (cf. Fig. 5).
The circuit DAEs may be rewritten in the form of (19)

\[
\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}, y(t)) + \frac{1}{C_{in}} e_1(t)
\]

\[
0 = g(\mathbf{x}, y(t)) + e(t),
\]

where \( \mathbf{x}(t) = [v(t) \ i(t)]^T, y(t) = h(t) \) and

\[
\mathbf{f}(\mathbf{x}, y) = \begin{bmatrix}
- \left[ \frac{v(t)}{R} + i(t) + S_{inj} \tanh \left( \frac{G_{inj} v(t)}{S_{inj}} \right) - h(t) \right] / C_{in} \\
\frac{v(t)}{L} 
\end{bmatrix},
\]

\[
g(\mathbf{x}, y) = - v(t) - R h(t), \quad e_1(t) = e^{i \omega t},
\]

\[
e(t) = \frac{A_{inj} \sin (2 \pi f_\omega t)}{2 \pi f_\omega} = \frac{1}{\omega} \frac{A_{inj}}{2i} (e^{i \omega t} - e^{-i \omega t}) ,
\]

with \( \omega = 2 \pi f_\omega \) and \( \mathbf{x}(0) = [1 \ 0]^T \). In addition, \( a_1(t) = [1/C_{in} \ 0]^T, b_1(t) = A_{inj}/(2i) \) and \( b_{-1}(t) = - A_{inj}/(2i) \). The source of nonlinearity is the term \( \tanh (G_{inj} v(t)/S_{inj}) \). The nonlinear DAE does not have a known analytical solution, hence we have used a reference solution, employing the MAPLE DAE routine procedure \textit{rkf45-dae} with the very high accuracy requirements \textit{AbsErr} = 10^{-10}, \textit{RelErr} = 10^{-10}. For the numerical simulations we have set \( L = 10^{-1}, R = 10, C_{in} = 0.2533, A_{inj} = 10, S_{inj} = 1/R \) and \( G_n = -1.1/R \). Our asymptotic expansion is of the form

\[
\mathbf{x}(t) \sim \begin{bmatrix} v(t) \\ i(t) \end{bmatrix} = \mathbf{p}_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \mathbf{p}_{r,m}(t) e^{im\omega t},
\]

\[
h(t) \sim q_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} q_{r,m}(t) e^{im\omega t}.
\]

The purpose of this example is to illustrate the computation of the first few terms in these expansions.

In the case \( r = 0 \), the non-oscillatory equations for \( \mathbf{p}_{0,0} \) and \( q_{0,0} \) are

\[
\mathbf{p}_{0,0}(t) = \mathbf{f}(\mathbf{p}_{0,0}, q_{0,0}) + \mathbf{a}_0(t), \quad t \geq 0, \quad \mathbf{p}_{0,0}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

\[
0 = g(\mathbf{p}_{0,0}, q_{0,0}),
\]

and the recurrence is \( \mathbf{p}_{1,m} = \mathbf{a}_m(t)/(im) \). Since \( \mathbf{a}_m = \mathbf{0} \) unless \( m = 1 \), the only nonzero term of this kind is \( \mathbf{p}_{1,1} = [-i/C_{in} \ 0]^T \).

Since

\[
\mathbf{f}_{1,0} = \begin{bmatrix} 1/C_{in} \\ 0 \end{bmatrix}, \quad \mathbf{g}_{1,0} = - R, \quad \mathbf{g}_{1,1} = [-1 \ 0],
\]

\[
\mathbf{f}_{1,1} = \begin{bmatrix} - \left\{ 1/R + G_n \left[ 1 - \tanh^2 \left( G_n \mathbf{p}_{1,0}(t)/S_{inj} \right) \right] \right\} / C_{in} \\ 1/L \end{bmatrix},
\]

where \( \mathbf{p}_{0,0} = [\mathbf{p}_{0,0}^1 \ \mathbf{p}_{0,0}^2]^T \), the leading terms \( \mathbf{p}_{1,0} \) and \( q_{1,0} \) based on our expansion are computed by the non-oscillatory DAEs

\[
\mathbf{p}_{1,0}(t) = \mathbf{f}_{1,0}(\mathbf{p}_{0,0}, q_{1,0}) + \mathbf{f}_{1,1}(\mathbf{p}_{0,0}, q_{0,0}) \mathbf{p}_{1,0}, \quad t \geq 0, \quad \mathbf{p}_{1,0}(0) = - \mathbf{p}_{1,1}(0),
\]

\[
0 = - R q_{1,0} + [\mathbf{a}_m]/b_m \mathbf{p}_{1,0}.
\]

For \( r = 1 \) and \( m \neq 0 \)

\[
q_{1,m} = \frac{-1}{R} \left\{ [1 \ 0] \mathbf{p}_{1,m} - b_m \right\},
\]
Figure 6. The error functions for \( v(t) \) are the first components of \( z_s(t) \) for \( \omega = 200\pi \) (top row) and \( \omega = 2000\pi \) (bottom row): from the left, the first components of \( z_0, z_1 \) and \( z_2 \) with \( t \in [0, 4] \).

Since \( f_{2,0} = f_{2,1} = 0, \ g_{2,0} = g_{2,1} = g_{2,2} = 0, \)
\[
f_{2,2} = \frac{(2G_n^2)/(C_{in}S_{inj}) \tanh \left( \frac{C_{in}}{S_{inj}} p_{1,0}(t) \right)}{0} \left[ 1 - \tanh^2 \left( \frac{C_{in}}{S_{inj}} p_{0,0}(t) \right) \right] \]
the DAE for \( r = 2 \) is
\[
p_{2,0}' = f_{1,0}\left(p_{0,0}, q_{0,0}\right) q_{2,0} + f_{1,1}\left(p_{0,0}, q_{0,0}\right) p_{2,0} + \frac{1}{2} f_{2,2}\left(p_{0,0}, q_{0,0}\right) p_{2,0}^2, \]
\[
0 = -Rq_{2,0} + \left[ -1 \quad 0 \right] p_{2,0}.
\]
The initial condition is
\[
p_{2,0}(0) = \left[ -\left\{ 2 - A_{inj}C_{in} + RG_n \left[ 1 - \tanh^2 \left( \frac{C_{in}}{S_{inj}} \right) \right] \right\} / (C_{in}^2 R) \right].
\]
Moreover,
\[
q_{2,m} = \frac{1}{R} \left[ -1 \quad 0 \right] p_{2,m}, \quad m \neq 0.
\]

There are three non-oscillatory DAEs arising from the asymptotic method. We also use the Maple routine \( \text{rkf45dae} \) to solve these DAE with the tolerances \( \text{AbsErr} = 10^{-10}, \text{RelErr} = 10^{-10} \).

Figures 6–8 display the errors \( z_s \) and \( \epsilon_s \), where
\[
z_0(t) = x(t) - p_{0,0},
\]
The errors for $i(t)$ are the second components of $z_s(t)$ in $t \in [0, 4]$ for $\omega = 200\pi$ (top row) and $\omega = 2000\pi$ (bottom row): from the left, the second component functions of $z_0$, $z_1$ and $z_2$.

\begin{align*}
    z_1(t) &= \mathbf{x}(t) - p_{0,0} - \frac{1}{\omega}(p_{1,0} + p_{1,1}e^{i\omega t}), \\
    z_2(t) &= \mathbf{x}(t) - p_{0,0} - \frac{1}{\omega}(p_{1,0} + p_{1,1}e^{i\omega t}) - \frac{1}{\omega^2}(p_{2,0} + p_{2,1}e^{i\omega t} + p_{2,-1}e^{-i\omega t}),
\end{align*}

for $\mathbf{x}(t) = [v(t) \ i(t)]^\top$ and

\begin{align*}
    \epsilon_0(t) &= h(t) - q_{0,0}, \\
    \epsilon_1(t) &= h(t) - q_{0,0} - \frac{1}{\omega}(q_{1,0} + q_{1,1}e^{i\omega t} + q_{1,-1}e^{-i\omega t}), \\
    \epsilon_2(t) &= h(t) - q_{0,0} - \frac{1}{\omega}(q_{1,0} + q_{1,1}e^{i\omega t} + q_{1,-1}e^{-i\omega t}) - \frac{1}{\omega^2}(q_{2,0} + q_{2,1}e^{i\omega t} + q_{2,-1}e^{-i\omega t})
\end{align*}

for $h(t)$ for the cases $\omega = 200\pi$ and $\omega = 2000\pi$. Consistently with our theory, the magnitude of the errors reduces as $\omega$ increases.

Concentrating on the CPU time, the asymptotic method is much cheaper than a Runge–Kutta method. The main computational expense of our approach for nonlinear DAEs lies in solving the unperturbed (i.e., non-oscillatory) DAEs using the Maple routine $rkf45dae$ with the tolerances $\text{AbsErr} = 10^{-10}$, $\text{RelErr} = 10^{-10}$. The conventional alternative is to use the $rkf45dae$ with the same tolerances to solve the original DAEs with highly oscillatory forcing terms. To retain similar error order $10^{-10}$ of the asymptotic method under the parameter $s = 3$ and $\omega = 2000\pi$, whose computational expense is 14.2 seconds and $4.9 \times 10^4$ of storage, the Maple routine $rkf45-dae$ for the original DAEs needs 69.688 seconds and $2.9 \times 10^5$ of storage.
4. Conclusions. Differential equations with highly oscillatory forcing terms are ubiquitous in many applications, since the forcing term corresponds to a high-frequency input into a dynamical system. Their solution by standard numerical methods is greatly restricted by the requirement that the step size should scale as the reciprocal of the largest frequency. This explains recent interest in computation which combines numerical and asymptotic insight, e.g. [1, 3, 4, 12]. This has been focussed on ordinary and partial differential equations [13], inclusive of the multi-frequency case [2], with a single paper on delay-differential equations [5]. However, many realistic problems occurring in circuit simulation require the solution of DAEs: this is among the first papers addressing the solution of DAE systems with high-frequency input using asymptotic-numerical techniques. We have demonstrated that the solution of such equations can be approximated to an exceedingly high precision using solely non-oscillatory computations – whether the solution of DAEs with no forcing terms or straightforward recursion. The oscillation is introduced into the computed solution only once the non-oscillatory ingredients are synthesised in a simple manner. Conversely (unless one is familiar with an asymptotic ‘frame of mind’), the precision for fixed computational cost increases with frequency.

This analysis emphasises the important point that this paper presents an initial foray into the novel and difficult area of DAEs with highly oscillatory input terms. In this paper we have presented a comprehensive analysis of one such DAE model, leading to a very effective numerical method. Yet, open questions abound, both in understanding further our approach (e.g., providing rigorous error bounds and convergence results for asymptotic expansions in a numerical setting) and extending it to other DAE models.

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