A CESÀRO AVERAGE FOR AN ADDITIVE PROBLEM WITH AN ARBITRARY NUMBER OF PRIME POWERS AND SQUARES

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Abstract. In this paper we extend and improve all the previous results known in literature about weighted average, with Cesàro weight, of representations of an integer as sum of a positive arbitrary number of prime powers and a non-negative arbitrary number of squares. Our result includes all cases dealt with so far and allows us to obtain the best possible outcome using the chosen technique.

December 1, 2021

2010 Mathematics Subject Classification: Primary 11P32; Secondary 44A10, 33C10
Keywords and phrases: Goldbach-type theorems, Laplace transforms, Bessel functions, Cesàro average.

1. INTRODUCTION

The study of counting the number of the possible representation of a positive integer as sum of primes, prime powers and, in general, sum of elements that belong to some fixed subset of \( \mathbb{N} \) is classical in number theory and it has been highly developed in recent years. Probably, the most popular problem in this context is the ternary and binary Goldbach conjecture, which states that every odd number greater than 3 is sum of three primes and every even number greater than 2 is the sum of two primes, respectively. While the ternary conjecture has been partially solved by Vinogradov [27] and then definitively solved by Helfgott in a series of papers [13] [16] [17], the binary conjecture is still open; only partial results were obtained and, often, only conditional on the Riemann hypothesis (see, e.g., the historical account about the Goldbach binary problem [3]). Given the difficulty of the problem, it was decided to tackle simplified versions of it, such as, for example, considering the number of representations on average and with suitable weights. In this context are inserted the works related to the study of averages, with Cesàro weight, of functions that count the number representations of an integer as the sum of elements that are in some fixed subset of \( \mathbb{N} \). Similar averages of arithmetical functions are common in the literature: see, e.g., [2]. This approach was used in [21] for the binary Goldbach problem: see Langusco’s paper [18] for a thorough introduction. The presence of a smooth weight allowed to obtain an asymptotic formula with terms of decreasing orders of magnitude and depending on the non-trivial zeros of the Riemann Zeta function; furthermore, the weights allowed to obtain results independent of the Riemann hypothesis. Since the Cesàro weights depend on a non-negative real parameter \( k \) and are equal to 1 if \( k = 0 \) (that is, for \( k = 0 \) we have a simple average without weights) it is important to obtain results with the smallest possible \( k \). During the last few years there have been some improvements regarding the optimal \( k \) in the case of the Goldbach’s problem: in [21] results hold for \( k > 1 \), in [14] (assuming Riemann hypothesis) and in [8] (unconditionally) for \( k = 1 \) and in [4] for \( k > 0 \). Unfortunately, for case \( k = 0 \), that is, without the Cesàro weights, it is not yet possible to obtain in the same form or with the same quantity of terms as in the other cases (see, for example, [23], [5] and [25]). Given the flexibility of the technique introduced in [21], the latter has been applied to other types of additive problems (see [6], [7], [22], [19], [20]). In this paper, we prove a result which incorporates all the previous results in the case of Cesàro averages and we show how the technique, although very general and applicable to many problems, makes the lower bound of \( k \) worse as the number of primes and squares involved increases, so as to confirm what has already been suggested by the lower bound for \( k \) obtained in [6] and [7].

2. PRELIMINARY DEFINITIONS AND MAIN THEOREM

Let \( d, h, N \in \mathbb{N} \), \( d > 0 \), \( N \geq 2 \), \( \mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d \), \( \mathbf{r} := (r_1, \ldots, r_d) \in (\mathbb{N}^+)^d \), where \( 1 \leq r_1 \leq r_2 \leq \cdots \leq r_d \), \( \mathbf{t} := (t_1, \ldots, t_h) \in \mathbb{N}^h \) and, in general, with bold letters, for example \( \mathbf{f} \), we will indicate some vector that belongs to \( \mathbb{N}^\alpha \) or \( (\mathbb{N}^+)^\alpha \), for some positive integer \( \alpha \). With the symbol \( \| \cdot \| \) we will indicate the usual Euclidean norm, with the symbol \( \rho \), with or without subscripts, we will always indicate the non-trivial zeros of the Riemann zeta function and the series \( \sum \) will always indicates the sum over all non trivial zeros of \( \zeta(s) \), with or without subscripts. With \( \rho := (\rho_{s_1}, \ldots, \rho_{s_v}) \), where \( s_j, j = 1, \ldots, v \) belong to some subset of \( \mathbb{N}^+ \).
For every \( \mathfrak{J} \subseteq \mathfrak{G} \) we will define the scalar product
\[
\tau(\Psi, \mathfrak{J}) := \sum_{j \in \mathfrak{J}} \Psi_j
\]
where \( \Psi = (\Psi_1, \ldots, \Psi_d) \). In most cases along the paper we will use \( \Psi = \rho \); in addition, we will use the short definition \( \tau(1, \mathfrak{J}) := \sum_{j \in \mathfrak{J}} \frac{1}{r_j} \).

We will also indicate by \( \sum_{\mathfrak{J} \subseteq \mathfrak{G}} \) the sums over all the possible subsets of of \( \mathfrak{G} \). Taking \( n \in \mathbb{N} \), we set
\[
R_{d,h,r}(n) := \sum_{m_1^2 + \cdots + m_d^2 + \ell_1^2 + \cdots + \ell_h^2 = n} \Lambda(m_1) \cdots \Lambda(m_h)
\]
where \( \Lambda(m) \) is the usual von Mangoldt function. We want to find an asymptotic formula, as \( N \to +\infty \), for
\[
\sum_{n \leq N} R_{d,h,r}(n) \frac{(N-n)^k}{\Gamma(k+1)}
\]
where \( k > 0 \) is a real parameter and \( \Gamma(x) \) is the Euler Gamma function.

Let \( Z := \{ s \in \mathbb{C}, 0 \leq \Re(s) \leq 1 : \zeta(s) = 0 \} \) be the set of the non-trivial zeros of the Riemann Zeta function and let \( \mathfrak{J} \subseteq \mathfrak{G} \). We will use the symbols
\[
\sum_{\rho \in Z^{[\mathfrak{J}]}} = \sum_{\rho_{j_1}} \cdots \sum_{\rho_{j_{[\mathfrak{J}]}}}
\]
and
\[
\frac{1}{r} \Gamma\left(\frac{\rho}{r}\right) := \frac{1}{r_1} \Gamma\left(\frac{\rho_{j_1}}{r_{j_1}}\right) \cdots \frac{1}{r_d} \Gamma\left(\frac{\rho_{j_d}}{r_{j_d}}\right)
\]
where \( j_\alpha \in \mathfrak{J}, \alpha = 1, \ldots, [\mathfrak{J}] \), every \( \rho_{j_\alpha} \in Z \) and \( r_{j_\alpha} \) is the \( j_\alpha \)-th coordinate of the fixed vector \( r = (r_1, \ldots, r_d) \).

In analogy to the previous definition, we will use the following symbol
\[
\frac{1}{r} \Gamma\left(\frac{1}{r}\right) := \frac{1}{r_1} \Gamma\left(\frac{1}{r_1}\right) \cdots \frac{1}{r_d} \Gamma\left(\frac{1}{r_d}\right)
\].

We introduce the following abbreviation for the terms of the development:
\[
M_1(N,k,d,h,r) := \frac{1}{2^h} \sum_{\ell=0}^{h} \left(\begin{array}{c} h \\ \ell \end{array}\right) \pi^\frac{\ell}{2} (-1)^{\ell} N^{k+\tau(r,\mathfrak{G})+\frac{\tau}{2}} \frac{1}{r} \Gamma\left(\frac{1}{r}\right),
\]
\[
M_2(N,k,d,h,r) := \frac{N^{\frac{k+\tau}{2}(r,\mathfrak{G})}}{\pi^{k+\tau(r,\mathfrak{G})}} \sum_{\eta=0}^{h} \left(\begin{array}{c} \eta \\ 2\eta \end{array}\right) \sum_{\ell=0}^{\eta} \left(\begin{array}{c} \eta \\ \ell \end{array}\right) (-1)^{\eta-\ell} \mathcal{B}\left(\tau(r,\mathfrak{G})\right) \frac{1}{r} \Gamma\left(\frac{1}{r}\right),
\]
\[
M_3(N,k,d,h,r) := \frac{N^k (-1)^d}{\pi^h} \sum_{\ell=0}^{h} \left(\begin{array}{c} h \\ \ell \end{array}\right) (N\pi)^\frac{\ell}{2} (-1)^{h-\ell} \sum_{\rho \in Z^d} \frac{1}{r} \Gamma\left(\frac{\rho}{r}\right) \frac{N^{\tau(\rho,\mathfrak{G})}}{\pi^{k+\tau(\rho,\mathfrak{G})}} \mathcal{B}\left(\tau(r,\mathfrak{G})\right),
\]
\[
M_4(N,k,d,h,r) := \frac{N^{k/2} (-1)^d}{\pi^k} \sum_{\eta=0}^{h} \left(\begin{array}{c} \eta \\ 2\eta \end{array}\right) \sum_{\ell=0}^{\eta} \left(\begin{array}{c} \eta \\ \ell \end{array}\right) (-1)^{\eta-\ell} \sum_{\rho \in Z^d} \frac{1}{r} \Gamma\left(\frac{\rho}{r}\right) \frac{N^{\tau(\rho,\mathfrak{G})/2}}{\pi^{k+\tau(\rho,\mathfrak{G})/2}} \mathcal{B}\left(\tau(r,\mathfrak{G})\right),
\]
\[
M_5(N,k,d,h,r) := \frac{N^{k/2}}{\pi^k} \sum_{\mathfrak{I} \subseteq \mathfrak{G}} \frac{N^{\tau(\mathfrak{r},\mathfrak{I})}}{\pi^{\tau(\mathfrak{r},\mathfrak{G})}} (-1)^{|\mathfrak{G}\setminus\mathfrak{I}|} \sum_{\eta=0}^{h} \left(\begin{array}{c} \eta \\ 2\eta \end{array}\right) \sum_{\ell=0}^{\eta} \left(\begin{array}{c} \eta \\ \ell \end{array}\right) (-1)^{\eta-\ell} \sum_{\rho \in Z^{|\mathfrak{G}\setminus\mathfrak{I}|}} \frac{1}{r} \Gamma\left(\frac{\rho}{r}\right) \frac{N^{\tau(\rho,\mathfrak{G}\setminus\mathfrak{I})/2}}{\pi^{\tau(\rho,\mathfrak{G}\setminus\mathfrak{I})/2}} \mathcal{B}\left(\tau(r,\mathfrak{I}) + \tau(\rho,\mathfrak{G}\setminus\mathfrak{I})\right),
\]
where \( \rho_j \) runs over the non-trivial zeros of the Riemann Zeta function and \( J_v(u) \) are the Bessel \( J \) function of real argument \( u \) and complex order \( v \) and
\[
\mathcal{B}(x) = \mathcal{B}_{k,h,\eta,\ell,N}(x) = N^{\frac{h+\eta+\ell}{2}} \sum_{\mathbf{f} \in (N^\eta)^{h-\eta}} \frac{J_x+k+\frac{h-\eta+\ell}{2} \left(2\pi\sqrt{N||\mathbf{f}||}\right)}{||\mathbf{f}||^{x+k+\frac{h-\eta+\ell}{2}}},
\]
with
\[
\sum_{\mathbf{f} \in \mathcal{W}^*} := \sum_{f_1 \geq 1} \cdots \sum_{f_k \geq 1} .
\]
The convergence of the mentioned series will be proved in the section \[3\] The main result of this article is the following theorem:

**Theorem 1.** Let \( d, h \in \mathbb{N}, d > 0 \), let \( N \) be a sufficiently large integer. Let \( \mathcal{D} := \{1, \ldots, d\} \) and, for every \( \mathcal{J} \subseteq \mathcal{D} \) (or with the notation \( \mathcal{I} \subseteq \mathcal{D} \) let \( \tau (\mathcal{r}, \mathcal{J}) := \sum_{j \in \mathcal{J}} r_j, \) where \( 1 \leq r_1 \leq r_2 \leq \cdots \leq r_d \). Then, for \( k > \frac{d + h}{2} \), we have that
\[
\sum_{n \leq N} R_{d,h,r} (n) \frac{(N - n)^k}{\Gamma (k + 1)} = \sum_{j=1}^{5} M_j \left( N, k, d, h, r \right) + O_{r,d,h} \left( N^{k + h/2 + \tau (\mathcal{r}, \mathcal{D}) - 1/r_d} \right).
\]

It is important to underline that in some particular configurations of the parameters some terms of the asymptotic (but not the dominant term) could be incorporated in the error. Despite the apparently complicated form of the terms, it is not difficult to recognize the results obtained in the previous work on this topic, for example setting \( d = 2, h = 0 \) and \( \mathcal{r} = (1, 1) \) (the Goldbach numbers case [21]) or \( \mathcal{r} = (\ell_1, \ell_2) \), \( 1 \leq \ell_1 \leq \ell_2 \) integers (the generalized Goldbach numbers case [13]). Furthermore, it is quite natural to conjecture that at least the main term of this asymptotic is valid for \( k \geq 0 \) instead of \( k > \frac{d + h}{2} \) as suggested by similar studies but with other techniques (see, e.g., [10][11]). Now, let’s briefly explain the major ideas behind this theorem; one of the main tools of this technique is the formula, due to Laplace [24], namely

\[
(1) \quad \frac{1}{2\pi i} \int_{(a)} v^{-s} e^v \, dv = \frac{1}{\Gamma (s)}
\]
for \( \Re (s) > 0 \) and \( a > 0 \) (see formula 5.4(1) on page 238 of [12]), where
\[
\int_{(a)} := \int_{a-i\infty}^{a+i\infty}.
\]
From (1) and suitable hypotheses, which we will explain in detail in the next sections, we are able to write

\[
(2) \quad \sum_{n \leq N} R_{d,h,r} (n) \frac{(N - n)^k}{\Gamma (k + 1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}_{r_1} (z) \cdots \tilde{S}_{r_d} (z) \omega_2 (z)^h \, dz
\]
where \( z = a + iy, a > 0, y \in \mathbb{R} \), where

\[
(3) \quad \tilde{S}_{r} (z) := \sum_{m \geq 1} \Lambda (m) e^{-m^rz}, \quad \omega_2 (z) := \sum_{m \geq 1} e^{-m^2 z},
\]
are the series that embody the prime powers and the squares, respectively. Since, as we will see, it is possible to develop \( \tilde{S}_{r} (z) \) as an asymptotic formula, the idea is to substitute this formula for \( \tilde{S}_{r} (z) \), exchange the integral with all the terms which are obtained from the various products and finally calculate the error. Another important aspect to emphasize is that we work with squares, and so with \( \omega_2 (z) \), because this function is linked to the well-known Jacobi theta 3 function

\[
\theta_3 (z) := \sum_{m \in \mathbb{Z}} e^{-m^2 z} = 1 + 2\omega_2 (z)
\]
and \( \theta_3 (z) \) satisfies the functional equation

\[
\theta_3 (z) = \left( \frac{\pi}{iz} \right)^{1/2} \theta_3 \left( \frac{\pi^2}{z} \right), \, \Re (z) > 0
\]
(see, for example, Proposition VI.4.3, page 340, of [13]) which implies a functional equation for \( \omega_2 (z) \)

\[
(4) \quad \omega_2 (z) = \frac{1}{2} \left( \frac{\pi}{iz} \right)^{1/2} - \frac{1}{2} + \left( \frac{\pi}{iz} \right)^{1/2} \omega_2 \left( \frac{\pi^2}{z} \right).
\]
This is fundamental for the present technique, because this functional equation allows us to find the terms involving the Bessel \( J \) function and, since we do not have a functional equation of this type for other powers than squares, we can only deal with this particular case.
3. Settings

For our purposes, we need a general version of the formula (1), so we recall the following relations:

\[ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{iDu}}{(a + iu)} \, du = \begin{cases} \frac{D^{-1}e^{-aD}}{\Gamma(s)}, & D > 0 \\ 0, & D < 0 \end{cases} \]

with \( \Re(s) > 0, \Re(a) > 0 \) and

\[ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{(a + iu)} \, du = \begin{cases} 0, & \Re(s) > 1 \\ 1/2, & \Re(s) = 1 \end{cases} \]

with \( \Re(a) > 0 \) (see formulas (8) and (9) of [1]). We also need an integral representation of the Bessel \( J \) function with real argument \( u \) and complex order \( v \):

\[ J_v(u) := \frac{(u/2)^v}{2\pi i} \int_{(a)} s^{-v-1} e^{s-u^2/(4s)} \, ds \]

for \( a > 0, u, v \in \mathbb{C} \) with \( \Re(v) > -1 \) (see, e.g., equation (8) on page 177 of [28]). Assume that \( k > 0 \). From the definition of \( \tilde{S}_r(z) \) and \( \omega_2(z) \) (3), it is not difficult to note that

\[ \tilde{S}_{r_1}(z) \cdots \tilde{S}_{r_d}(z) \omega_2(z)^h = \sum_{n \geq 1} R_{d,h,r}(n) e^{-nz}. \]

Furthermore, from (5) and (6), we have that

\[ \sum_{n \leq N} R_{d,h,r}(n) (\frac{N-n}{\Gamma(k+1)})^k = \sum_{n \geq 1} R_{d,h,r}(n) \left( \frac{1}{2\pi i} \int_{(a)} e^{(N-n)z} z^{-k-1} \, dz \right). \]

Now we want to show that it is possible to exchange the integral with the series in the right side of (8). By the Prime Number Theorem, we have that

\[ \tilde{S}_{r_i}(a) \sim \frac{\Gamma(\frac{1}{r_i})}{r_j a^{1/r_j}} \]

as \( a \to 0^+ \) (see [20]) and

\[ |\omega_2(z)| \leq \omega_2(a) \leq \int_0^{+\infty} e^{-au^2} \, du \leq a^{-1/2} \int_0^{+\infty} e^{-v^2} \, dv \ll a^{-1/2} \]

and so

\[ \sum_{n \geq 1} |R_{d,h,r}(n) e^{-nz}| = \sum_{n \geq 1} R_{d,h,r}(n) e^{-na} = \tilde{S}_{r_1}(a) \cdots \tilde{S}_{r_d}(a) \omega_2(a)^h \ll r_d a^{-\tau(r,d)} - h/2. \]

From the trivial estimate

\[ |e^{Nz}| |z^{-k-1}| \ll e^{Na} \begin{cases} a^{-k-1}, & |y| \leq a \\ |y|^{-k-1}, & |y| > a \end{cases} \]

where \( f \asymp g \) means \( g \ll f \ll g \), we have

\[ \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}_{r_1}(z) \cdots \tilde{S}_{r_d}(z) \omega_2(z)^h \, dz \ll r_d a^{-\tau(r,d)} - h/2 \left( \int_{-a}^{a} a^{-k-1} \, dy + \int_{a}^{+\infty} y^{-k-1} \, dy \right) \ll r_d a^{-\tau(r,d)} - h/2 \]

for \( k > 0 \). Then, we can exchange the integral with the series and so we obtain the main formula (2).
4. Lemmas

In this section we present some technical lemmas that will be useful later and some basic facts in complex analysis. First, we recall that if \( z = a + iy, a > 0 \) and \( w \in \mathbb{C} \), we have that

\[
|z|^{-\text{Re}(w)-i\text{Im}(w)} \exp \left( (i\text{Re}(w) + \text{Im}(w)) \arctan \left( \frac{y}{a} \right) \right)
\]

and so

\[
|z|^{-w} = |z|^{-\text{Re}(w)} \exp \left( \text{Im}(w) \arctan \left( \frac{y}{a} \right) \right).
\]

We also recall the Stirling formula

\[
|\Gamma (x + iy)| \sim \sqrt{2\pi e^{-\pi}|y|/2} |y|^{x-1/2}
\]

which holds uniformly for \( x \in [x_1, x_2], x_1, x_2 \) fixed and \( |y| \to +\infty \) (see, e.g., [26], section 4.42).

Now we introduce the “explicit formula” of \( S_r (z), r \in \mathbb{N}^+ \).

**Lemma 2.** (Lemma 1 of [19]) Let \( r \geq 1 \) be an integer, let \( z = a + iy, a > 0, y \in \mathbb{R} \). Let

\[
T(z, r) := \frac{\Gamma (\frac{1}{r})}{\pi z^{1/r}} - \frac{1}{r} \sum_{\rho} z^{-\rho/r} \Gamma \left( \frac{\rho}{r} \right).
\]

Then

\[
\tilde{S}_r (z) = T(z, r) + E(a, y, r).
\]

where

\[
|E(a, y, r)| \ll_r 1 + |z|^{1/2} \begin{cases} 
1, & |y| \leq a \\
1 + \log^2 \left( \frac{2|y|}{a} \right), & |y| > a.
\end{cases}
\]

Note that in Lemma 1 of [19] \( T(z, r) \) is defined as

\[
T(z, r) := \frac{\Gamma (\frac{1}{r})}{\pi z^{1/r}} - \frac{1}{r} \sum_{\rho} z^{-\rho/r} \Gamma \left( \frac{\rho}{r} \right) - \log (2\pi)
\]

but in our context, to make the main term combinatorically more tractable, it is better to insert \( \log (2\pi) \) in the error term \( E(a, y, r) \). Furthermore, from [9] and [10] we immediately get the important estimate

\[
\left| \sum_{\rho} z^{-\rho/r} \Gamma \left( \frac{\rho}{r} \right) \right| \ll_r a^{-1/r} + 1 + |z|^{1/2} \begin{cases} 
1, & |y| \leq a \\
1 + \log^2 \left( \frac{2|y|}{a} \right), & |y| > a.
\end{cases}
\]

which can be rewritten, if \( 0 < a < 1 \) and \( r \geq 1 \), in the more compact form

\[
\left| \sum_{\rho} z^{-\rho/r} \Gamma \left( \frac{\rho}{r} \right) \right| \ll_r \begin{cases} 
a^{-1/r}, & |y| \leq a \\
a^{-1/r} + |z|^{1/2} \log^2 \left( \frac{2|y|}{a} \right), & |y| > a.
\end{cases}
\]

**Lemma 3.** Let \( \lambda \in \mathbb{N}^+, r_1, \ldots, r_\lambda \in \mathbb{N}^+ \) and \( r := (r_1, \ldots, r_\lambda) \in (\mathbb{N}^+)^\lambda \). Let \( \rho_j = \beta_j + i\gamma_j, j \in \{1, \ldots, \lambda\} \), run over the non trivial zeros of Riemann Zeta function and \( \alpha > 1 \) be a parameter. Then, for any fixed \( b > 1 \) and \( c \geq 0 \), the series

\[
\sum_{\rho_1: \gamma_1 > 0} \left( \frac{\gamma_1}{r_1} \right)^{\beta_1/r_1 - 1/2} \cdots \sum_{\rho_\lambda: \gamma_\lambda > 0} \left( \frac{\gamma_\lambda}{r_\lambda} \right)^{\beta_\lambda/r_\lambda - 1/2} \int_1^{+\infty} \log^c (bu) \exp \left( - \arctan \left( \frac{1}{u} \right) \right) \tau (\gamma, r, \lambda) \frac{du}{u^{\alpha + \tau (\beta, r, \lambda)}}
\]

converges if \( \alpha > \frac{\lambda}{2} + 1 \).

**Proof.** Following the proof of Lemma 2 of [19], we can see that

\[
\int_1^{+\infty} \exp \left( - \arctan \left( \frac{1}{u} \right) \right) \tau (\gamma, r, \lambda) \frac{du}{u^{\alpha + \tau (\beta, r, \lambda)}} \ll_{a,r} \tau (\gamma, r, \lambda)^{1 - \alpha - \tau (\beta, r, \lambda)} \int_0^{+\infty} e^{-w u^{\alpha + \tau (\beta, r, \lambda) - 2}} dw
\]

and the integral converges since \( 0 < \beta_j < 1, j = 1, \ldots, \lambda \) and \( \alpha > 1 \). Hence, we have to consider

\[
\sum_{\rho_2: \gamma_2 > 0} \cdots \sum_{\rho_\lambda: \gamma_\lambda > 0} \left( \frac{\gamma_1}{r_1} \right)^{\beta_1/r_1 - 1/2} \cdots \left( \frac{\gamma_\lambda}{r_\lambda} \right)^{\beta_\lambda/r_\lambda - 1/2} \tau (\gamma, r, \lambda)^{\alpha + \tau (\beta, r, \lambda) - 1}.
\]
Now, it is not difficult to note that
\[
\frac{\left(\frac{2\alpha}{r_1}\right)\beta_k/r_1 \cdots \left(\frac{2\alpha}{r_N}\right)\beta_k/r_N}{\tau(\gamma, r, 3\lambda)^{\tau(\beta r, 3\lambda)}} \leq 1
\]
so we analyze
\[
\sum_{\rho_1: \gamma_1 > 0} \cdots \sum_{\rho_N: \gamma_N > 0} \left(\frac{2\alpha}{r_1}\right)^{-1/2} \cdots \left(\frac{2\alpha}{r_N}\right)^{-1/2} \leq \sum_{\rho_1: \gamma_1 > 0} \left(\frac{\gamma_1}{r_1}\right)^{-\frac{1}{2}} \cdots \sum_{\rho_N: \gamma_N > 0} \left(\frac{\gamma_N}{r_N}\right)^{-\frac{1}{2}}
\]
by the inequality of arithmetic and geometric means. From the asymptotic formula of \( N(T) \), where \( N(T) \) is the number of non-trivial zeros of the Riemann zeta function with imaginary part \( 0 \leq \gamma \leq T \), it is not difficult to prove, putting \( \gamma(k) \) the imaginary part of the \( k \)-th non-trivial zeros of \( \zeta(s) \), that
\[
\gamma(k) \sim \frac{2\pi k}{\log(k)}
\]
as \( k \to +\infty \). So the series converges if \( \alpha > \frac{1}{2} + 1 \). The treatment is similar for the case \( c > 0 \). \( \square \)

**Lemma 4.** Let \( N, \lambda, \alpha \) be positive integers, let \( h \in \mathbb{Q}^+ \), let \( \rho_j = \beta_j + i\gamma_j \), \( j \in \{1, \ldots, \lambda\} \), run over the non-trivial zeros of the Riemann Zeta function, \(||\cdot||\) the Euclidean norm in \( \mathbb{R}^d \), \( d \in \mathbb{N}^+ \) and \( k > 0 \) a real number. For sake of simplicity we define \( \delta := \sum_{j=1}^{\lambda} \gamma_j \). Then, for every fixed integer \( b > 1 \) and \( c > 0 \),
\[
\sum_{\rho_1: \gamma_1 > 0} \cdots \sum_{\rho_N: \gamma_N > 0} \frac{\gamma_1^{\frac{1}{2}} \cdots \gamma_N^{\frac{1}{2}}}{\delta^{k+h+\alpha}} \sum_{\mathbf{f} \in (\mathbb{N}^+)^n} \int_0^\delta v^{k-1+h+\alpha+\tau(\beta r, 3\lambda)} e^{-||\mathbf{f}||^2 N v^2 / \delta^2 - v} \log^2 c \left( \frac{b d}{v} \right) dv
\]
converges if \( k > \frac{1}{2} - h \).

**Proof.** We consider the integral
\[
\sum_{\rho_1: \gamma_1 > 0} \cdots \sum_{\rho_N: \gamma_N > 0} \frac{\gamma_1^{\frac{1}{2}} \cdots \gamma_N^{\frac{1}{2}}}{\delta^{k+h+\alpha}} \sum_{\mathbf{f} \in (\mathbb{N}^+)^n} \int_0^\delta v^{k-1+h+\alpha+\tau(\beta r, 3\lambda)} e^{-||\mathbf{f}||^2 N v^2 / \delta^2 - v} \log^2 c \left( \frac{b d}{v} \right) dv
\]
Now we claim that we can exchange the integral with the multiple series \( \sum_{\mathbf{f} \in (\mathbb{N}^+)^n} \). To show this we consider
\[
\int_0^\delta v^{k-1+h+\alpha+\tau(\beta r, 3\lambda)} e^{-||\mathbf{f}||^2 N v^2 / \delta^2 - v} \log^2 c \left( \frac{b d}{v} \right) dv
\]
Now, since for every \( M \geq 1 \) we have
\[
\sum_{f_1 \leq M} e^{-f_1 N v^2 \delta^2 - \leq} \leq \sum_{f_1 \geq 1} e^{-f_1 N v^2 \delta^2 - \leq} = \omega_2(N v^2 \delta^2 - \leq) \ll_N \frac{\delta}{v}
\]
from (10) and so we have to deal with
\[
\int_0^\delta v^{k-2+h+\alpha+\tau(\beta r, 3\lambda)} \omega_2^{\alpha-1}(N v^2 \delta^2 - \leq) \exp(-v) dv \ll_{N, \alpha} \delta^{\alpha-1} \int_0^\delta v^{k+h+\alpha+\tau(\beta r, 3\lambda)} \exp(-v) dv \ll_{N, \alpha} \delta^{\alpha-1}
\]
which is convergent since \( k > 0 \), then we obtain
\[
\sum_{f_1 \geq 1} \int_0^\delta v^{k-1+h+\alpha+\frac{\beta_1+\ldots+\beta_N}{\delta}} e^{-f_1 N v^2 \delta^2 - \leq} \omega_2^{\alpha-1}(N v^2 \delta^2 - \leq) \exp(-v) dv
\]
by the Dominated Convergence Theorem. Clearly, we can repeat the same argument for every factor in the product \( \omega_2^{\alpha-1}(N v^2 \delta^2 - \leq) \) and so we can write (20) as
\[
\sum_{\rho_1: \gamma_1 > 0} \cdots \sum_{\rho_N: \gamma_N > 0} \frac{\gamma_1^{\frac{1}{2}} \cdots \gamma_N^{\frac{1}{2}}}{\delta^{k+h+\alpha}} \int_0^\delta v^{k-1+h+\alpha+\tau(\beta r, 3\lambda)} \omega_2^{\alpha}(N v^2 \delta^2 - \leq) \exp(-v) dv
\]
and again using (10) we have to deal with
\[
\sum_{\rho_1: \gamma_1 > 0} \cdots \sum_{\rho_N: \gamma_N > 0} \frac{\gamma_1^{\frac{1}{2}} \cdots \gamma_N^{\frac{1}{2}}}{\delta^{k+h+\alpha}} \int_0^\delta v^{k-1+h+\alpha+\tau(\beta r, 3\lambda)} \exp(-v) dv.
\]
Now, since \( k + h + \tau (\beta, r, \lambda) > 0 \), then
\[
\int_0^\delta v^{k-1+h+\tau(\beta, r, \lambda)} \exp (-v) \, dv \ll \int_0^{+\infty} v^{k-1+h+\tau(\beta, r, \lambda)} \exp (-v) \, dv < +\infty.
\]
Then, from arithmetic mean - geometric mean inequality, we get
\[
\sum_{\rho_1: \gamma_1 > 0} \cdots \sum_{\rho_\lambda: \gamma_\lambda > 0} \frac{1}{\delta^{k+h+\alpha}} \ll \sum_{\rho_1: \gamma_1 > 0} \gamma_1 \frac{1}{\delta^{k+h+\alpha}} \cdots \sum_{\rho_\lambda: \gamma_\lambda > 0} \gamma_\lambda \frac{1}{\delta^{k+h+\alpha}}
\]
and the series converges if \( k > \frac{1}{2} - h \). Clearly, if we have a log factor into the integral the bound for \( k \) is the same. Indeed, we note that
\[
\log^2 \left( \frac{\delta}{v} \right) \ll \log^2 (\delta) + \log^2 (bv)
\]
and
\[
\log^2 (\delta) \leq \log^2 \left( \max_{j=1, \ldots, \lambda} \gamma_j \right) := \log^2 (\lambda \gamma)
\]
so we have in (21) one series such that
\[
\sum_{\rho: \gamma > 0} \gamma \frac{1}{\delta^{k+h+\alpha}} \log^2 (\lambda \gamma)
\]
and clearly the log factor does not affect the bound for \( k \); if we have
\[
\int_0^{+\infty} v^{k-1+h+\tau(\beta, r, \lambda)} \exp (-v) \log^2 (bv) \, dv
\]
again, we have the same bounds for \( k \) and so the Lemma is proved. \( \square \)

5. PROOF OF THE MAIN THEOREM

In this section we prove the main theorem. We first show that the error bound in the main formula is “small”, then we prove that all the exchange of symbols is justified and finally we evaluate the integrals.

5.1. Error term. From (14), (15) and following the subdivision in (9), formula (2), we can write
\[
\tilde{S}_{r_1} (z) \cdots \tilde{S}_{r_d} (z) = T (z, r_1) \cdots T (z, r_d)
\]
\[
+ \sum_{j=1}^d E (a, y, r_j) \left( \prod_{i \neq j} \tilde{S}_{r_i} (z) \right) + \sum_{I \subseteq \mathbb{D}} c_d (I) \left( \prod_{i \in \mathbb{D} \setminus I} T (z, r_i) \right) \left( \prod_{i \in I} E (a, y, r_i) \right)
\]
for some suitable coefficients \( c_d (I) \), so we get
\[
\frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}_{r_1} (z) \cdots \tilde{S}_{r_d} (z) \omega_2 (z)^h \, dz = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} T (z, r_1) \cdots T (z, r_d) \omega_2 (z)^h \, dz
\]
\[
+ \frac{1}{2\pi i} \sum_{j=1}^d \int_{(a)} e^{Nz} z^{-k-1} E (a, y, r_j) \left( \prod_{i \neq j} \tilde{S}_{r_i} (z) \right) \omega_2 (z)^h \, dz
\]
\[
+ \sum_{I \subseteq \mathbb{D}} c_d (I) \int_{(a)} e^{Nz} z^{-k-1} \left( \prod_{i \in \mathbb{D} \setminus I} T (z, r_i) \right) \left( \prod_{i \in I} E (a, y, r_i) \right) \omega_2 (z)^h \, dz
\]
\[
=: A_1 + A_2 + A_3.
\]
Now we have to estimate the error term. From (9), (10) and (11) we obtain
\[
|A_2| \ll \sum_{j=1}^d \int_{(a)} |e^{Nz} | z^{-k-1} | E (a, y, r_j) | \prod_{i \neq j} | \tilde{S}_{r_i} (z) | \omega_2 (z)^h | \, dy
\]
\[
\ll_{r, d, h} e^{\hat{N}a} a^{-h/2} \sum_{j=1}^d a^{-\tau (r, \mathbb{D}) + 1/2} \left( \int_0^a a^{-k-1} \left( 1 + a^{1/2} \right) \, dy + \int_a^{+\infty} y^{-k-1} \left( 1 + y^{1/2} \left( 1 + \log^2 \left( \frac{N}{a} \right) \right) \right) \, dy \right)
\]
\begin{equation}
\lesssim_{r,d,h} e^{N a \alpha - k h/2} \sum_{j=1}^{d} a^{-\tau(r,\mathcal{D})+\frac{1}{r}}
\end{equation}
for \( k > 0 \).

For the estimation of \( A_3 \) we fix \( I \subset \mathcal{D} \) and we consider
\[
|A_{3.1}| := \int_{(a)} e^{N z} |z^{-k-1}| \prod_{i \in \mathcal{D} \setminus I} |T(z, r_i)| \prod_{j \in I} |E(a, y, r_j)| \omega_2(z^h) \ dy.
\]

We know from (9) and (15) that
\[
|T(z, r)| \lesssim_r a^{-1/r} + |E(a, y, r)|
\]

hence, using formula (10) it is enough to work with
\[
a^{-h/2} \int_{(a)} e^{N z} |z^{-k-1}| \prod_{i \in \mathcal{D} \setminus I} (a^{-1/r_i} + |E(a, y, r_i)|) \prod_{j \in I} |E(a, y, r_j)| \ dy.
\]

Now, observing that
\[
\prod_{i \in \mathcal{D} \setminus I} (a^{-1/r_i} + |E(a, y, r_i)|) = \sum_{J \subseteq \mathcal{D} \setminus I} a^{-\tau(r, J)} \prod_{i \in \mathcal{D} \setminus (I \cup J)} |E(a, y, r_i)|
\]

we have by (10),
\[
|A_{3.1}| \lesssim_{r,d,h} e^{N a \alpha - k h/2} \sum_{J \subseteq \mathcal{D} \setminus I} a^{-\tau(r, J)} \prod_{i \in \mathcal{D} \setminus (I \cup J)} |E(a, y, r_i)| |z^{-k-1}| \ dy
\]
\[
\lesssim_{r,d,h} e^{N a \alpha - k h/2} \sum_{J \subseteq \mathcal{D} \setminus I} a^{-\tau(r, J)} \int_0^{a} a^{-k-1} \left(1 + a^{1/2}\right)^{|\mathcal{D} \setminus J|} \ dy
\]
\[
+ e^{N a \alpha - k h/2} \sum_{J \subseteq \mathcal{D} \setminus I} a^{-\tau(r, J)} \int_a^{+\infty} y^{-k-1} \left(1 + y^{1/2} \left(1 + \log^2 \left(\frac{y}{a}\right)\right)\right)^{|\mathcal{D} \setminus J|} \ dy
\]
\[
\lesssim_{r,d,h} e^{N a \alpha - k h/2} \sum_{J \subseteq \mathcal{D} \setminus I} a^{-\tau(r, J)}
\]
for \( k > \frac{|\mathcal{D} \setminus I|}{2} \) and since this inequality must holds for all subsets \( J \subseteq \mathcal{D} \), we have to assume \( k > \frac{d}{2} \). Hence
\[
|A_3| \lesssim_{r,d,h} e^{N a \alpha - k h/2} \sum_{J \subseteq \mathcal{D}} \sum_{|I| \geq 2} a^{-\tau(r, J)}.
\]

Now we take \( a = 1/N \) and we observe that
\[
\sum_{|I| \geq 2} \sum_{J \subseteq \mathcal{D} \setminus I} N^{\tau(r, J)} \lesssim_d \max_{J \subseteq \mathcal{D}} \max_{|I| \geq 2} N^{\tau(r, J)} \lesssim_d N^{\tau(r, \mathcal{D}) - \frac{1}{r_1} - \frac{1}{r_2} \frac{1}{r_2} \frac{1}{r_3} \ldots} \lesssim_d N^{k + h/2 + \tau(r, \mathcal{D}) - 1/r_d}
\]

remembering that \( 1 \leq r_1 \leq r_2 \leq \ldots \leq r_d \). This error term is compatible with that of Theorem 1.

\section{Evaluation of the main term.}

According to (14) we rewrite \( A_1 \) in the following form
\[
A_1 = \frac{1}{2 \pi i} \int_{(1/N)} e^{N z} z^{-k-1} T(z, r_1) \ldots T(z, r_d) \omega_2(z^h) \ dz
\]
\[
= \frac{1}{2 \pi i} \frac{1}{r} \left(\frac{1}{r}\right) \int_{(1/N)} e^{N z} z^{-k-1-\tau(r, \mathcal{D})} \omega_2(z^h) \ dz
\]
\[
+ \frac{(-1)^d}{2 \pi i} \int_{(1/N)} e^{N z} z^{-k-1} \left(\sum_{\rho \in \mathbb{Z}^d} \frac{1}{r} \Gamma \left(\frac{\rho}{r}\right) z^{-\tau(r, \mathcal{D})}\right) \omega_2(z^h) \ dz
\]
\[
+ \frac{1}{2 \pi i} \sum_{\mathcal{D} \setminus J \supseteq I} (-1)^{|\mathcal{D} \setminus I|} \int_{(1/N)} e^{N z} z^{-k-1-\tau(r, I)} \left(\sum_{\rho \in \mathbb{Z}^d \setminus \mathcal{I}} \frac{1}{r} \Gamma \left(\frac{\rho}{r}\right) z^{-\tau(r, \mathcal{D} \setminus J)}\right) \omega_2(z^h) \ dz
\]
\[
=: I_1 + I_2 + I_3.
\]
$I_1$ corresponds to the terms $M_1$ and $M_2$ of Theorem 1. $I_2$ corresponds to the terms $M_3$ and $M_4$ and finally $I_3$ corresponds to $M_5$.

5.2.1. Evaluation of $I_1$. We study $I_1$. By (11) and the binomial theorem, we get

$$\omega_2(z)^h = \left(\frac{1}{2} \left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2} \omega_2\left(\frac{\pi^2}{z}\right)\right)^h$$

and so

$$I_1 = \frac{1}{2\pi i} \lim_{r \to 0} \left(\frac{1}{r}\right) \int_{(1/N)} e^{Nz} z^{-k-1-\tau(r, \mathcal{D})} \omega_2(z)^h \, dz$$

or

$$I_1 = \frac{1}{2\pi i} \lim_{r \to 0} \left(\frac{1}{r}\right) \int_{(1/N)} e^{Nz} z^{-k-1-\tau(r, \mathcal{D})} \omega_2(z)^h \, dz.$$

Our main goal is to show that, for a suitable $k$, we can exchange the integral with the involved series; in this case, with the series related to $\omega_2$. We consider two cases: if $\eta = h$ we get

$$I_{1,1} := \frac{1}{2\pi i} \sum_{\ell=0}^h \left(\frac{h}{\ell}\right) \pi^\ell \left(-1\right)^{h-\ell} \Gamma \left(\frac{1}{r}\right) \int_{(1/N)} e^{Nz} z^{-k-1-\tau(r, \mathcal{D})-\frac{\ell}{2}} \, dz$$

which corresponds to the term $M_1$ in Theorem 1 and, from the substitution $Nz = u$ and (11), we get

$$I_{1,1} = \frac{1}{2\pi i} \sum_{\ell=0}^h \left(\frac{h}{\ell}\right) \pi^\ell \left(-1\right)^{h-\ell} \Gamma \left(\frac{1}{r}\right) \int_{(1/N)} e^u u^{-k-1-\tau(r, \mathcal{D})-\frac{\ell}{2}} \, du$$

for $k + 1 + \tau(r, \mathcal{D}) + \frac{\ell}{2} > 0$, which is trivially true if $k > 0$. Now, fix $1 \leq \lambda \leq h - \eta$.

We consider the general case

(23) $I_{1,2,\lambda} := \sum_{f_1 \geq 1} \cdots \sum_{f_3 \geq 1} \int_{(1/N)} |e^{Nz}| |z|^{-k-1-\tau(r, \mathcal{D})-\frac{h-\eta+\ell}{2}} e^{-\pi^2 \text{Re}(1/z)(f_1^2 + \cdots + f_3^2)} \omega_2\left(\frac{\pi^2}{z}\right)^{h-\eta-\lambda} \, dz$.

Note that if $I_{1,2,\lambda}$ converges for all $\lambda$, the exchange between series and integral is justified. By the trivial estimate

$$\text{Re}\left(\frac{1}{z}\right) = \frac{N}{1 + y^2 N^2} \Rightarrow \begin{cases} N, & |y| \leq 1/N \\ 1/(Ny^2), & |y| > 1/N, \end{cases}$$

by (11) and by (10), we obtain

$$I_{1,2,\lambda} \ll_{h, \eta, \lambda} \sum_{f \in \mathbb{N}^+} \int_{(1/N)}^{1/N} N^{k+1+\tau(r, \mathcal{D})+\frac{\lambda+\ell}{2}} e^{-\pi^2 N \|f\|^2} \, dy$$

and

$$+ N^{h-\eta-\lambda} \sum_{f \in \mathbb{N}^+} \int_{1/N}^{1} y^{-k-1-\tau(r, \mathcal{D})+\frac{h-\eta+\ell}{2}} e^{-\pi^2 \|f\|^2} \, dy.$$
The first integral and the series trivially converge since $N$ is positive, then we can consider only the second integral. Making the substitution $v = \frac{x^2(\|f\|^2)}{Ny^2}$, we get

$$\sum_{f \in \mathbb{N}^d} \int_{1/N}^{\infty} y^{-k-1-\tau(r,\mathcal{D})} e^{-\frac{h-y-\ell}{2}\lambda} - \lambda e^{-\frac{x^2(\|f\|^2)}{Ny^2}} dy$$

$$\ll N, h, \eta, \lambda \sum_{f \in \mathbb{N}^d} ||f||^{-k+(r,\mathcal{D})-\frac{h-y-\ell}{2}+\lambda-1} - e^{-v} dv.$$ 

Now, the integral is convergent if $k + \tau(r,\mathcal{D}) - \frac{h-y-\ell}{2} + \lambda - 1 > 0$, which means $k > -\tau(r,\mathcal{D}) + \frac{h-y-\ell}{2} + \lambda + 1$ and the series is convergent if $k + \tau(r,\mathcal{D}) - \frac{h-y-\ell}{2} + \lambda + 1 > \lambda$, from the inequality of arithmetic and geometric means, and so $k > -\tau(r,\mathcal{D}) + \frac{h-y-\ell}{2}$. Since the inequalities must holds for all $1 \leq \lambda \leq h-\eta$, for all $0 \leq \ell \leq \eta$ and for all $0 \leq \eta \leq h-1$, we can conclude that we can exchange all the series with the integral if $k > -\tau(r,\mathcal{D}) + \frac{1}{2}$. Hence, using (7) we can finally write

$$I_{1,2} = \frac{1}{2\pi i} \sum_{h=0}^{N^k-r(r,\mathcal{D})} \sum_{\ell=0}^{(N-1)h-(1-e^{\eta})} \sum_{h=0}^{\eta} \sum_{\ell=0}^{\eta} \left( \frac{h}{\pi \eta} \right) \left( \frac{(N-1)h-(1-e^{\eta})}{\eta} \right) \sum_{f \in \mathbb{N}^d} ||f||^{-k+(r,\mathcal{D})-\frac{h-y-\ell}{2}+\lambda-1} - e^{-v} dv.$$ 

for $k > -\tau(r,\mathcal{D}) + \frac{1}{2}$. This term corresponds to $M_2$ in Theorem 1.

5.2.2. Evaluation of $I_2$. As in the previous case, we split the integral into two pieces

$$I_2 = \frac{(-1)^d}{2\pi i} \int_{1/N} e^{Nz-h-1} \left( \sum_{\rho \in \mathbb{Z}_d} \frac{1}{\Gamma \left( \frac{\rho}{r} \right) z^{-\tau(r,\mathcal{D})}} \omega_2(z)^h \right) dz$$

$$= \frac{(-1)^d}{2\pi i} \sum_{h=0}^{\eta} \sum_{\ell=0}^{\eta} \left( \frac{h}{\pi \eta} \right) \left( \frac{(N-1)h-(1-e^{\eta})}{\eta} \right) \sum_{f \in \mathbb{N}^d} ||f||^{-k+(r,\mathcal{D})+\frac{h-y-\ell}{2}+\lambda-1} - e^{-v} dv.$$ 

Let us consider $I_{2,1}$ which corresponds to $M_3$ in Theorem 1. We want to show that it is possible to exchange the integral with the product of the series involving the non-trivial zeros if the Riemann Zeta function. To prove this, we fix an arbitrary $1 \leq \lambda \leq d$ and we analyze

$$\sum_{\rho_1} \Gamma \left( \frac{\rho}{r_1} \right) \cdots \sum_{\rho_{\lambda}} \Gamma \left( \frac{\rho}{r_{\lambda}} \right) \int_{1/N} e^{Nz-h-1} \left( \prod_{s=\lambda+1}^{d} \sum_{\rho_s} \frac{1}{\Gamma \left( \frac{\rho}{r_s} \right) z^{-\tau(r,\mathcal{D})}} \right) dz.$$
with the convention that, if $\lambda = d$, then $\prod_{s=\lambda+1}^{d} \left| \sum_{p_s} \frac{\Gamma\left(\frac{\rho_s}{r_s}\right)}{r_s} z^{-\frac{\rho_s}{r_s}} \right| = 1$. From Stirling formula (13) and (17) we have that it is enough to study the convergence of

$$\sum_{\rho_1} |\gamma_1|^{\frac{2+\alpha}{\lambda}} \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{2+\alpha}{\lambda}} \int_{\mathbb{R}} |z|^{-k-1-\frac{\alpha}{2}-\tau(\beta, r, \lambda)}$$

$$\times \exp \left( \sum_{j=1}^{\lambda} \left( \frac{\gamma_j}{r_j} \arctan(Ny) - \frac{\pi |\gamma_j|}{2r_j} \right) \right) \prod_{s=\lambda+1}^{d} \left| \sum_{p_s} \frac{\Gamma\left(\frac{\rho_s}{r_s}\right)}{r_s} z^{-\frac{\rho_s}{r_s}} \right| dy.$$

We split the integral in $|y| \leq 1/N$ and $|y| > 1/N$. Assume that $|y| \leq 1/N$, then by (17), we have

$$\sum_{\rho_1} |\gamma_1|^{\frac{2+\alpha}{\lambda}} \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{2+\alpha}{\lambda}} \int_{-1/N}^{1/N} |z|^{-k-1-\frac{\alpha}{2}-\tau(\beta, r, \lambda)} \exp \left( \sum_{j=1}^{\lambda} \left( \frac{\gamma_j}{r_j} \arctan(Ny) - \frac{\pi |\gamma_j|}{2r_j} \right) \right) N^{d-\lambda} dy$$

$$\ll N^{\lambda \cdot d} \sum_{\rho_1} |\gamma_1|^{\frac{2+\alpha}{\lambda}} \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{2+\alpha}{\lambda}} \exp \left( -\frac{\pi |\gamma_1|}{4r_1} \right) \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{2+\alpha}{\lambda}} \exp \left( -\frac{\pi |\gamma_\lambda|}{4r_\lambda} \right)$$

and the series trivially converges, so assume that $|y| > 1/N$. It is enough considering the case

$$\sum_{\rho_1} |\gamma_1|^{\frac{2+\alpha}{\lambda}} \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{2+\alpha}{\lambda}} \int_{|y| > 1/N} |z|^{-k-1-\frac{\alpha}{2}-\tau(\beta, r, \lambda)+\frac{\alpha}{2}}$$

$$\times \exp \left( \sum_{j=1}^{\lambda} \left( \frac{\gamma_j}{r_j} \arctan(Ny) - \frac{\pi |\gamma_j|}{2r_j} \right) \right) \log^{2\alpha} (2N |y|) dy$$

for $1 \leq \alpha \leq d - \lambda$, since the powers of $N$ do not affect the study of the convergence and so can be omitted. Assume $y > 1/N$ and $\gamma_j > 0$, $j = 1, \ldots, \lambda$. Putting $Ny = u$ and using the well-known identity $\arctan(x) - \frac{x}{2} = -\arctan \left( \frac{1}{x} \right)$ we get

$$\sum_{\rho_1: \gamma_1 \geq 0} |\gamma_1|^{\frac{2+\alpha}{\lambda}} \cdots \sum_{\rho_\lambda: \gamma_\lambda \geq 0} |\gamma_\lambda|^{\frac{2+\alpha}{\lambda}} \int_{1}^{+\infty} u^{-k-1-\frac{\alpha}{2}-\tau(\beta, r, \lambda)+\frac{\alpha}{2}} \exp \left( -\arctan \left( \frac{1}{u} \right) \tau(\gamma, r, \lambda) \right) \log^{2\alpha} (2u) du$$

and, by Lemma 3, we have the convergence if $k > \frac{\lambda \cdot (\alpha - \ell)}{2}$, and since this inequality must be holds for all $0 \leq \ell \leq d$ and all $1 \leq \alpha \leq d - \lambda$ we can conclude that $k > \frac{d}{2}$. Now, fix $1 \leq \eta \leq \lambda$, assume that $\gamma_1, \ldots, \gamma_\eta > 0$ and $\gamma_{\eta+1}, \ldots, \gamma_\lambda < 0$. In this case, recalling that $y > 1/N$ and so $\frac{\gamma_j}{r_j} \arctan(Ny) - \frac{\pi |\gamma_j|}{2r_j} \leq -\frac{\pi |\gamma_j|}{2r_j}$ for $j > \eta$, we have to work with

$$\sum_{\rho_1: \gamma_1 \geq 0} |\gamma_1|^{\frac{2+\alpha}{\lambda}} \cdots \sum_{\rho_\eta: \gamma_\eta \geq 0} |\gamma_\eta|^{\frac{2+\alpha}{\lambda}} \sum_{\rho_{\eta+1}: \gamma_{\eta+1} < 0} |\gamma_{\eta+1}|^{\frac{2+\alpha}{\lambda}} \exp \left( -\frac{\pi |\gamma_{\eta+1}|}{2r_{\eta+1}} \right) \cdots \sum_{\rho_\lambda: \gamma_\lambda < 0} |\gamma_\lambda|^{\frac{2+\alpha}{\lambda}} \exp \left( -\frac{\pi |\gamma_\lambda|}{2r_\lambda} \right)$$

$$\times \int_{y > 1/N} y^{-k-1-\frac{\alpha}{2}-\tau(\beta, r, \lambda)+\frac{\alpha}{2}} \exp \left( \sum_{j=1}^{\eta} \left( \frac{\gamma_j}{r_j} \arctan(Ny) - \frac{\pi |\gamma_j|}{2r_j} \right) \right) \log^{2\alpha} (2Ny) dy$$

with $1 \leq \alpha \leq d - \lambda$. Letting $Ny = u$, we note that we have to deal with

$$\sum_{\rho_1: \gamma_1 \geq 0} |\gamma_1|^{\frac{2+\alpha}{\lambda}} \cdots \sum_{\rho_\eta: \gamma_\eta \geq 0} |\gamma_\eta|^{\frac{2+\alpha}{\lambda}} \sum_{\rho_{\eta+1}: \gamma_{\eta+1} < 0} |\gamma_{\eta+1}|^{\frac{2+\alpha}{\lambda}} \exp \left( -\frac{\pi |\gamma_{\eta+1}|}{2r_{\eta+1}} \right) \cdots \sum_{\rho_\lambda: \gamma_\lambda < 0} |\gamma_\lambda|^{\frac{2+\alpha}{\lambda}} \exp \left( -\frac{\pi |\gamma_\lambda|}{2r_\lambda} \right)$$

$$\times \int_{1}^{+\infty} u^{-k-1-\frac{\alpha}{2}-\tau(\beta, r, \lambda)+\frac{\alpha}{2}} \exp \left( -\arctan \left( \frac{1}{u} \right) \tau(\gamma, r, \lambda) \right) \log^{2\alpha} (2u) du$$

(also in this case we omit the powers of $N$ because they do not affect the convergence) and, since

$$\left( \frac{1}{u} \right)^{\frac{\alpha}{2}} < 1, \quad u > 1, \quad j = 1, \ldots, \lambda,$$

it is enough to consider

$$\sum_{\rho_1: \gamma_1 \geq 0} |\gamma_1|^{\frac{2+\alpha}{\lambda}} \cdots \sum_{\rho_\eta: \gamma_\eta \geq 0} |\gamma_\eta|^{\frac{2+\alpha}{\lambda}} \int_{1}^{+\infty} u^{-k-1-\frac{\alpha}{2}-\tau(\beta, r, \lambda)+\frac{\alpha}{2}} \exp \left( -\arctan \left( \frac{1}{u} \right) \tau(\gamma, r, \lambda) \right) \log^{2\alpha} (2u) du$$
and so, arguing as in the previous case, the convergence if \( k > \frac{n + d - \lambda}{2} \) and so, since \( \eta \leq \lambda \), a complete convergence in the case \( k > \frac{d}{2} \). If \( y < -1/N \) we get the same bounds for \( k \), by symmetry.

Now, since \( |D| = d \), we have

\[
I_{2,1} = \frac{(1)^d}{2^{n+1 \pi i}} \sum_{\ell = 0}^{h} \left( \frac{1}{\ell} \right) \pi^\frac{1}{2} (-1)^{h-\ell} \sum_{\rho \in \mathbb{Z}^d} \frac{1}{r} \Gamma \left( \frac{\rho}{r} \right) \int_{(1/N)} e^{Nz} z^{-k-1 - \frac{h}{2} - \tau(\rho, r, D)} \, dz
\]

\[
= \frac{N^k}{2^n} \sum_{\ell = 0}^{h} \left( \frac{1}{\ell} \right) (N\pi)^\frac{1}{2} (-1)^{h-\ell} \sum_{\rho \in \mathbb{Z}^d} \frac{1}{r} \Gamma \left( \frac{\rho}{r} \right) \frac{N^\tau(\rho, r, D)}{\Gamma \left( k + 1 + \frac{h}{2} + \tau(\rho, r, D) \right)}
\]

from (1).

Now, we analyze \( I_{2,2} \) (which corresponds to \( M_1 \) in Theorem 1) and we prove that we can switch the integral with the series involving the non-trivial zeros of the Riemann Zeta function and with the powers of \( \omega_2 \left( \frac{z^2}{z} \right) \). As the previous calculations, we fix \( 1 \leq \lambda \leq d \) and \( 1 \leq \alpha \leq h - \eta \). So we have to consider

\[
\sum_{\rho_1} \left| \frac{\Gamma \left( \frac{\rho_1}{r_1} \right)}{r_1} \right| \cdots \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| \prod_{s = \lambda + 1}^{d} \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| z^{-\rho_s} \omega_2 \left( \frac{z^2}{z} \right) \left| z^{-\tau(\rho, r, D)} \right| dz
\]

again with the convention \( \prod_{s = \lambda + 1}^{d} \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| z^{-\rho_s} = 1 \) if \( \lambda = d \). From (13) and recalling that the powers of \( N \) do not affect the convergence, it is enough to study the convergence of

\[
\sum_{\rho_1} \left| \frac{\Gamma \left( \frac{\rho_1}{r_1} \right)}{r_1} \right| \cdots \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| \prod_{s = \lambda + 1}^{d} \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| z^{-\rho_s} \omega_2 \left( \frac{z^2}{z} \right) \left| z^{-\tau(\rho, r, D)} \right| e^{-\frac{|\rho|^2 N}{1+N^2 z^2}}
\]

\[
\times \exp \left( \sum_{y = 1}^{\lambda} \left( \frac{\gamma_j}{r_j} \right) \arctan \left( Ny \right) - \frac{\pi |y|}{2r_j} \right) \prod_{s = \lambda + 1}^{d} \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| z^{-\rho_s} \left( 1 + y^2 N^2 \right)^{-\frac{h-\eta-\alpha}{2}} dy.
\]

If \( |y| \leq 1/N \) we have

\[
\sum_{\rho_1} \left| \frac{\Gamma \left( \frac{\rho_1}{r_1} \right)}{r_1} \right| \cdots \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| \prod_{s = \lambda + 1}^{d} \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| z^{-\rho_s} \omega_2 \left( \frac{z^2}{z} \right) \left| z^{-\tau(\rho, r, D)} \right| e^{-\frac{|\rho|^2 N}{1+N^2 z^2}}
\]

\[
\times \exp \left( \sum_{y = 1}^{\lambda} \left( \frac{\gamma_j}{r_j} \right) \arctan \left( Ny \right) - \frac{\pi |y|}{2r_j} \right) \prod_{s = \lambda + 1}^{d} \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| z^{-\rho_s} \left( 1 + y^2 N^2 \right)^{-\frac{h-\eta-\alpha}{2}} dy
\]

\[
\ll_{N,k,\alpha,\eta,\ell,r} \sum_{\rho_1} \left| \frac{\Gamma \left( \frac{\rho_1}{r_1} \right)}{r_1} \right| \cdots \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| \prod_{s = \lambda + 1}^{d} \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| z^{-\rho_s} \omega_2 \left( \frac{z^2}{z} \right) \left| z^{-\tau(\rho, r, D)} \right| e^{-\frac{|\rho|^2 N}{1+N^2 z^2}}
\]

and trivially the convergence, so we consider now \( |y| > 1/N \). If we fix \( 1 \leq \mu \leq d - \lambda \), from (17), it is enough to work with

\[
\sum_{\rho_1} \left| \frac{\Gamma \left( \frac{\rho_1}{r_1} \right)}{r_1} \right| \cdots \sum_{\rho_s} \left| \frac{\Gamma \left( \frac{\rho_s}{r_s} \right)}{r_s} \right| \prod_{s = \lambda + 1}^{d} \sum_{|y| > 1/N} \left| z^{-k-1 - \frac{h-\eta-\alpha}{2} + \frac{\mu}{2}} \right| z^{-\tau(\rho, r, D)} \right| e^{-\frac{|\rho|^2 N}{1+N^2 z^2}}
\]

\[
\times \exp \left( \sum_{y = 1}^{\lambda} \left( \frac{\gamma_j}{r_j} \right) \arctan \left( Ny \right) - \frac{\pi |y|}{2r_j} \right) \log^2 \left( 2N |y| \right) \left| y^{h-\eta-\alpha} \right| dy.
\]
Assume $y > 1/N$ and $\gamma_j > 0$, $j = 1, \ldots, \lambda$. Since $\arctan \left( \frac{1}{\sqrt{y}} \right) \gg \frac{1}{y}$, we have

$$
\sum_{\rho: \gamma_j > 0} \frac{\gamma_{\rho}}{1} \cdot \sum_{\rho: \gamma_j > 0} \gamma_{\rho} \cdot \frac{1}{y} \sum_{f \in (N^+)^n} \int_{1/N}^{+\infty} y^{-k-1} \left( b - b + \frac{h - h + \ell}{2} + \tau_y \right) \cdot e^{-\frac{4t_0^2}{y^2}}
\times \exp \left( -\frac{\tau \left( \gamma_y, (y), \lambda \right)}{N y} \right) \log^2 \left( 2N \right) y^{h - \eta - \alpha} dy.
$$

Putting $v = \sum_{\rho: \gamma_j > 0} \gamma_{\rho}$, we obtain

$$
\sum_{\rho: \gamma_j > 0} \frac{\gamma_{\rho}}{1} \cdot \sum_{\rho: \gamma_j > 0} \gamma_{\rho} \cdot \frac{1}{y} \sum_{f \in (N^+)^n} \int_{1/N}^{+\infty} y^{-k-1} \left( b - b + \frac{h - h + \ell}{2} + \tau_y \right) \cdot e^{-\frac{4t_0^2}{y^2}}
\times \exp \left( -\frac{\tau \left( \gamma_y, (y), \lambda \right)}{N y} \right) \log^2 \left( 2N \right) y^{h - \eta - \alpha} dv.
$$

Now, from (19) and by elementary manipulations we can study

$$
\sum_{\rho: \gamma_j > 0} \frac{\gamma_{\rho}}{1} \cdot \sum_{\rho: \gamma_j > 0} \gamma_{\rho} \cdot \frac{1}{y} \sum_{f \in (N^+)^n} \int_{1/N}^{+\infty} y^{-k-1} \left( b - b + \frac{h - h + \ell}{2} + \tau_y \right) \cdot e^{-\frac{4t_0^2}{y^2}}
\times \exp \left( -\frac{\tau \left( \gamma_y, (y), \lambda \right)}{N y} \right) \log^2 \left( 2N \right) y^{h - \eta - \alpha} dv.
$$

and so, by Lemma [19], we have the convergence if $k > \frac{1}{2} + \frac{h - h + \ell + \ell}{2}$. Since $\mu \leq d - \lambda$ and $0 \leq \ell \leq \eta$ we have the complete convergence for all possible cases if $k > \frac{d + h}{2}$. Now fix $1 \leq \xi \leq \lambda$ and assume that $\gamma_1, \ldots, \gamma_\xi > 0$ and $\gamma_{\xi+1}, \ldots, \gamma_\lambda < 0$. In this case, recalling that $y > 1/N$, we have to work with

$$
\sum_{\rho: \gamma_j > 0} \frac{\gamma_{\rho}}{1} \cdot \sum_{\rho: \gamma_j > 0} \gamma_{\rho} \cdot \frac{1}{y} \sum_{f \in (N^+)^n} \int_{1/N}^{+\infty} y^{-k-1} \left( b - b + \frac{h - h + \ell}{2} + \tau_y \right) \cdot e^{-\frac{4t_0^2}{y^2}}
\times \exp \left( -\frac{\tau \left( \gamma_y, (y), \lambda \right)}{N y} \right) \log^2 \left( 2N \right) y^{h - \eta - \alpha} dy.
$$

Now, since $y > 1/N$, if $\gamma_j < 0$, we observe that $\frac{\gamma_y}{r_j} \arctan (Ny) - \frac{\pi |\gamma_j|}{2r_j} \leq -\frac{\pi |\gamma_j|}{2r_j}$, $y^{-\beta_j/r_j} \leq N^{\beta_j/r_j} \leq N^{1/r_j}$ and

$$
\sum_{\rho: \gamma_j < 0} |\gamma_j| \cdot \frac{\gamma_{\rho}}{1} \cdot \sum_{\rho: \gamma_j < 0} \gamma_{\rho} \cdot \frac{1}{y} \sum_{f \in (N^+)^n} \int_{1/N}^{+\infty} y^{-k-1} \left( b - b + \frac{h - h + \ell}{2} + \tau_y \right) \cdot e^{-\frac{4t_0^2}{y^2}}
\times \exp \left( -\frac{\tau \left( \gamma_y, (y), \lambda \right)}{N y} \right) \log^2 \left( 2N \right) y^{h - \eta - \alpha} dy
$$

trivially converges, so it is enough to consider

$$
\sum_{\rho: \gamma_j > 0} \frac{\gamma_{\rho}}{1} \cdot \sum_{\rho: \gamma_j > 0} \gamma_{\rho} \cdot \frac{1}{y} \sum_{f \in (N^+)^n} \int_{1/N}^{+\infty} y^{-k-1} \left( b - b + \frac{h - h + \ell}{2} + \tau_y \right) \cdot e^{-\frac{4t_0^2}{y^2}}
\times \exp \left( -\frac{\tau \left( \gamma_y, (y), \lambda \right)}{N y} \right) \log^2 \left( 2N \right) y^{h - \eta - \alpha} dy
$$

and so, following the previous case, we have the convergence if $k > \frac{h + d - \lambda + \xi}{2}$ and since $\xi \leq \lambda$, we have the complete convergence if $k > \frac{d + h}{2}$. If $y < -1/N$ we have the same bounds, by the symmetry of the non-trivial zeros of the Riemann Zeta function, so we can finally exchange the integral with the series and get

$$
I_{2,2} = \frac{(-1)^d}{2\pi i} \sum_{n=0}^\eta \frac{\eta}{2\pi} \sum_{\ell=0}^\eta \frac{\eta}{2\pi} (-1)^{\eta - \ell} \sum_{\rho \in \mathbb{Z}^d} \frac{1}{r} \sum_{f \in (N^+)^{h - \eta}} e^{Nz - k - 1 - b + \frac{h - h + \ell}{2} + \tau_y} e^{-\frac{4t_0^2}{y^2}} dz
$$
which is, taking \( Nz = v \),

\[
\frac{(-1)^d}{2\pi i} \sum_{n=0}^h \frac{(h)}{2^n} \sum_{\ell=0}^\eta \frac{\eta}{\ell} \sum_{\rho \in \mathbb{Z}^d} (-1)^{\eta-\ell} \sum_{\rho \in \mathbb{Z}^d} \frac{1}{\Gamma} \left( \frac{\rho}{\mathbb{Z}^d} \right) N^{k+h-\eta+\ell+\tau(\rho, \mathbb{D})} \sum_{f \in \mathbb{N}^{h-\eta}} \int_{(1)} z^{-k-1-hz-\eta+\ell+\tau(\rho, \mathbb{D})} e^{-\frac{2N||f||^2}{z}} dv
\]

\[
= \frac{N^{k/2}}{\pi^k} (-1)^d \sum_{n=0}^h \frac{(h)}{2^n} \sum_{\ell=0}^\eta \frac{\eta}{\ell} \sum_{\rho \in \mathbb{Z}^d} \frac{1}{\Gamma} \left( \frac{\rho}{\mathbb{Z}^d} \right) N^{h-\eta+\ell+\tau(\rho, \mathbb{D})/2} \sum_{f \in \mathbb{N}^{h-\eta}} J_{k+h-\eta+\ell+\tau(\rho, \mathbb{D})} \left( \frac{2\pi \sqrt{N} ||f||}{z} \right)
\]

6. Evaluation of \( I_3 \)

In this section we evaluate

\[
I_3 = \frac{1}{2\pi i} \sum_{I \subseteq \mathbb{D}} \frac{-1}{|I|} \left( \frac{h}{\eta} \right) \sum_{\ell=0}^\eta \left( \frac{\eta}{\ell} \right) (-1)^{\eta-\ell} \frac{\rho}{\eta} \pi^{h-\eta}
\]

\[
\times \sum_{\rho \in \mathbb{Z}^d \setminus I} \frac{1}{\Gamma} \left( \frac{\rho}{\mathbb{Z}^d} \right) \int_{(1/N)} e^{Nz_{k-1-\eta+\tau(\rho, \mathbb{D})}} \left( \sum_{\rho \in \mathbb{Z}^d \setminus I} \frac{1}{\Gamma} \left( \frac{\rho}{\mathbb{Z}^d} \right) z^{-\eta+\tau(\rho, \mathbb{D})} \right) \omega_2 (z)^h dz
\]

which corresponds to \( M_3 \) in Theorem 1 and has a similar structure to \( I_2 \). If we fix \( I \subseteq \mathbb{D} \) we can repeat the previous argument to justify the exchange of the integral with the series only considering \( k + \tau(\rho, \mathbb{D}) \) instead of \( k \) and \( |\mathbb{D} \setminus I| \) instead of \( d = |\mathbb{D}| \). So, we can conclude that all exchanges are justified if \( k > \frac{|\mathbb{D} \setminus I|+h}{\eta} - \tau(\rho, \mathbb{D}) \), and so we have

\[
I_3 = \frac{1}{2\pi i} \sum_{I \subseteq \mathbb{D}} \frac{-1}{|I|} \left( \frac{h}{\eta} \right) \sum_{\ell=0}^\eta \left( \frac{\eta}{\ell} \right) (-1)^{\eta-\ell} \frac{\rho}{\eta} \pi^{h-\eta}
\]

\[
\times \sum_{\rho \in \mathbb{Z}^d \setminus I} \frac{1}{\Gamma} \left( \frac{\rho}{\mathbb{Z}^d} \right) \sum_{f \in \mathbb{N}^{h-\eta}} \int_{(1/N)} e^{Nz_{k-1-\eta+\tau(\rho, \mathbb{D})}} \left( \sum_{\rho \in \mathbb{Z}^d \setminus I} \frac{1}{\Gamma} \left( \frac{\rho}{\mathbb{Z}^d} \right) z^{-\eta+\tau(\rho, \mathbb{D})} \right) \omega_2 (z)^h dz
\]

and so, taking \( Nz = v \) and using (7), we have that

\[
I_3 = \frac{N^k}{2\pi i} \sum_{I \subseteq \mathbb{D}} N^{\tau(\rho, \mathbb{D})} \left( \frac{h}{\eta} \right) \sum_{\ell=0}^\eta \left( \frac{\eta}{\ell} \right) (-1)^{\eta-\ell} \frac{\rho}{\eta} \pi^{h-\eta}
\]

\[
\times \sum_{\rho \in \mathbb{Z}^d \setminus I} \frac{1}{\Gamma} \left( \frac{\rho}{\mathbb{Z}^d} \right) N^{h-\eta+\tau(\rho, \mathbb{D})} \sum_{f \in \mathbb{N}^{h-\eta}} \int_{(1)} e^{v_{k-1-\eta+\tau(\rho, \mathbb{D})}} \left( \sum_{\rho \in \mathbb{Z}^d \setminus I} \frac{1}{\Gamma} \left( \frac{\rho}{\mathbb{Z}^d} \right) z^{-\eta+\tau(\rho, \mathbb{D})} \right) \omega_2 (z)^h dv
\]

\[
= \frac{N^{k/2}}{\pi^k} \sum_{I \subseteq \mathbb{D}} N^{\tau(\rho, \mathbb{D})/2} \left( \frac{h}{\eta} \right) \sum_{\ell=0}^\eta \left( \frac{\eta}{\ell} \right) (-1)^{\eta-\ell}
\]

\[
\times \sum_{\rho \in \mathbb{Z}^d \setminus I} \frac{1}{\Gamma} \left( \frac{\rho}{\mathbb{Z}^d} \right) N^{h-\eta+\tau(\rho, \mathbb{D})/2} \sum_{f \in \mathbb{N}^{h-\eta}} J_{k+h-\eta+\tau(\rho, \mathbb{D})} \left( \frac{2\pi \sqrt{N} ||f||}{z} \right)
\]

and this completes the proof.
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