ALMOST-PRIMES REPRESENTED BY $p + a^m$

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Abstract. Let $a \geq 2$ be a fixed integer in this paper. By using the method of Goldston, Pintz and Yıldırım, we will prove that there are infinitely many almost-primes which can be represented as $p + a^m$ in at least two different ways.

1. Introduction

In 1934, Romanoff [9] proved that the integers of the form $p + 2^m$ have a positive density. Thereafter, many works have been done involving the so-called Romanoff’s constant:

$$c = \lim_{x \to \infty} \inf \frac{\# \{n \leq x : n = p + 2^m \}}{x}.$$  

For example, Chen and Sun [1] proved that $c > 0.0868$, this result is improved by Habsieger and Roblot [6] to 0.0933 and by Pintz [7] to 0.09368. Their works mainly based on studying the mean values involving $r(n)$, the number of different representations of $n$ in the form $p + 2^m$.

Prachar [8] studied a more generalized problem. He proved that if $a > 1$ and $(m_j)$ is a strictly increasing sequence of non-negative integers, then the number of distinct integers $\leq x$ which can be expressed in the form $p + a^{m_j}$ is

$$\gg \frac{x}{\log x} \# \{m_j : a^{m_j} \leq x \}.$$  

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In this paper, we take interest in almost-primes with \( r(n) \geq 2 \). It is early in 1950 that Erdős [2] proved that there are infinitely many integers satisfying
\[
    r(n) \gg \log \log n,
\]
but his method can not be applied to attack the problem on almost-primes. The main result of this paper is the following theorem:

**Theorem 1.1.** Let \( a \geq 2 \) be an fixed integer. Then there exists a positive integer \( R \), such that there are infinitely many integers \( n \) satisfying:

1. \( n \) has at most \( R \) distinct prime divisors;
2. \( n \) can be represented as \( p + a^m \) in at least two different ways.

We should mention to Friedlander and Iwaniec [4] who claimed: “We believe (although we did not check all details) that the method presented here can, when combined with the Fundamental Lemma, produce infinitely many almost-prime integers which have two different representations in the form \( p + a^m \).” Therefore, what we do in this paper is just to “check the details”.

Throughout the paper, we denote \( \varepsilon \) to be a sufficiently small positive real number, and write
\[
    \Lambda^\flat(n) = \begin{cases} 
    \log n, & \text{if } n \text{ is a prime,} \\
    0, & \text{otherwise.} 
    \end{cases}
\]
As usual, \( \tau_k(n) \) is the divisor function and \( \varphi(n) \) is the Euler’s function.

**2. Basic Considerations**

The proof of Theorem 1.1 is based on the lower-bound sieve and the method of Goldston, Pintz and Yıldırım (see eg. [4], [5] and [10]).

Let \( N \) be a sufficiently large integer, we write
\[
    \mathcal{M} = \left\{ a^m : 1 \leq m \leq \frac{\log N}{2 \log a} \right\}
\]
and \( \mathcal{H} = \{ a^m : 1 \leq m \leq k \} \) a subset of \( \mathcal{M} \). Let
\[
    Q(X) = \prod_{1 \leq j \leq k} (X - a^j),
\]
and \( \omega(d) \) denote the number of solutions \( n \pmod{d} \) of \( Q(n) \equiv 0 \pmod{d} \). Note that if \( p \mid a \), then \( \omega(p) = 1 \); if \( p \nmid a \), then \( \omega(p) < p \) since \( Q(0) \neq 0 \).
Almost-primes represented by $p + a^m$. Therefore, $\omega(p) < p$ for every prime $p$, in another word, $\mathcal{H}$ is “admissible”.

We write
\[
\det \mathcal{H} = \sum_{1 < i < j < k} (a^j - a^i)^2 = a^{k(k-1)} \prod_{1 < j < k} (a^j - 1)^{2(k-j)},
\]
and let $\Delta$ be the product of all prime divisors of $a$ and all primes $p$ for which $a^j \equiv 1 \pmod{p}$ with some $1 \leq j \leq k$. Then we can easily check the following three things:

(i) Since $\mathcal{H}$ is admissible, $\Delta$ is divisible by all primes $p \leq k + 1$. In practice, we shall choose $k$ to be an even integer, therefore $k + 2$ is not a prime.

(ii) If $p \nmid \Delta$, then $\omega(p) = k$.

(iii) For any $a^m \in \mathcal{M}$, we have $\Delta \mid Q(a^m)$ since
\[
Q(a^m) = \prod_{1 < j < k} (a^m - a^j) = a^{k(k+1)/2} \prod_{m-k < j < m-1} (a^j - 1).
\]

Now we consider the sequence $(a_n)$ supported on the dyadic segment $(\frac{N}{2}, N]$ as well as $(Q(n), \Delta) = 1$ with
\[
a_n = \left( \sum_{a^m \in \mathcal{M}} \Lambda'(n - a^m) - \log N \right) \left( \sum_{\nu | Q(n)} \lambda_\nu \right),
\]
where $(\lambda_\nu)$ is an upper-bound sieve supported on squarefree numbers $\nu < D = N^{1-2\epsilon}$, $(\nu, \Delta) = 1$, whence the summation over $\nu$ is non-negative. Here we choose $(\lambda_\nu)$ to be the Selberg’s $\Lambda^2$-sieve, that is
\[
\sum_{\nu | n} \lambda_\nu = \left( \sum_{d \mid n} \rho_d \right)^2
\]
where $(\rho_d)$ is a sequence of real numbers supported on squarefree numbers $d$ with $d < \sqrt{D}$, $(d, \Delta) = 1$ which satisfies $\rho_1 = 1$ and
\[
|\rho_d| \leq 1
\]
for all $d$ (see Lemma 6.1). Thus
\[
\lambda_\nu = \sum_{[d_1, d_2] = \nu} \rho_{d_1} \rho_{d_2}
\]
and $|\lambda_\nu| \leq \tau_3(\nu)$ for all $\nu$. If we can give a proper lower bound for the number of almost-primes $n$ such that $a_n > 0$, we will prove Theorem 1.1. Therefore, we need to apply a lower-bound sieve to $n$.

Let

$$T = \{ N/2 < n \leq N : (Q(n), \Delta) = 1 \},$$

$$T_1 = T \cap \left\{ n : \sum_{a^m \in \mathcal{M}} \Lambda^b(n - a^m) - \log N > 0 \right\},$$

$$T_2 = T \cap \left\{ n : \sum_{a^m \in \mathcal{M}} \Lambda^b(n - a^m) - \log N \leq 0 \right\},$$

and $\mathcal{A} = (a_n)_{n \in T_1}$. We choose the sifting set $\mathcal{P} = \{ p \geq k + 2 : p \nmid a \}$ since it is easy to deduce $(n, a) = 1$ from $(Q(n), \Delta) = 1$, and as usual, denote

$$P(z) = \prod_{\substack{p < z \atop p \in \mathcal{P}}} p.$$

Let $(\lambda'_d)$ be a lower-bound sieve of level $D' = N^\epsilon$, then the sifting function

$$S(\mathcal{A}, \mathcal{P}, z) = \sum_{n \in T_1} a_n \sum_{n \in T_1} a_n \sum_{d \mid (n, P(z))} \lambda'_d$$

(2.4)

$$= \sum_{n \in T_1} a_n \sum_{d \mid (n, P(z))} \lambda'_d - \sum_{n \in T_2} a_n \sum_{d \mid (n, P(z))} \lambda'_d = S_1 - S_2$$

say. If we can produce a positive lower bound of $S(\mathcal{A}, \mathcal{P}, z)$ for $z = D'^{1/2}$, we will deduce that there are infinitely many integers $n$ which have at most $s\epsilon^{-1} + k + 2$ distinct prime factors and satisfy $a_n > 0$.

Now we give a careful look at $S_2$, we write

$$T_{21} = \{ n \in T_2 : n - a^m \text{ is not a prime for any } a^m \in \mathcal{M} \},$$

$$T_{22} = T_2 \setminus T_{21} = \{ n \in T_2 : \exists a^m \in \mathcal{M} \text{ such that } n - a^m \text{ is a prime} \}.$$
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Then,

$$- \sum_{n \in T_{21}} a_n \sum_{d \mid (n, P(z))} \lambda_d = (\log N) \sum_{n \in T_{21}} \left( \sum_{\nu \mid Q(n)} \lambda_{\nu} \right) \left( \sum_{d \mid (n, P(z))} \lambda_d' \right)$$

$$= (\log N) \sum_{N/2 < n \leq N} \left( \sum_{\nu \mid Q(n)} \lambda_{\nu} \right) \left( \sum_{d \mid (n, P(z))} \lambda_d' \right)$$

$$- (\log N) \sum_{N/2 < n \leq N} \left( \sum_{\nu \mid Q(n)} \lambda_{\nu} \right) \left( \sum_{d \mid (n, P(z))} \lambda_d' \right) \exists a^m \in \mathcal{M}, \text{ s.t. } n - a^m \text{ is prime}$$

$$\geq (\log N) \sum_{N/2 < n \leq N} \left( \sum_{\nu \mid Q(n)} \lambda_{\nu} \right) \left( \sum_{d \mid (n, P(z))} \lambda_d' \right) - (\log N) \sum_{N/2 < n \leq N} \left( \sum_{\nu \mid Q(n)} \lambda_{\nu} \right).$$

Noticing that

$$S_1 = \sum_{N/2 < n \leq N} \sum_{a^m \in \mathcal{M}} \Lambda^b(n - a^m) \left( \sum_{\nu \mid Q(n)} \lambda_{\nu} \right) \left( \sum_{d \mid (n, P(z))} \lambda_d' \right)$$

$$- (\log N) \sum_{N/2 < n \leq N} \left( \sum_{\nu \mid Q(n)} \lambda_{\nu} \right) \left( \sum_{d \mid (n, P(z))} \lambda_d' \right),$$

we finally get from (2.4) that

(2.5)

$$S(A, P, z) \geq \sum_{N/2 < n \leq N} \sum_{a^m \in \mathcal{M}} \Lambda^b(n - a^m) \left( \sum_{\nu \mid Q(n)} \lambda_{\nu} \right) \left( \sum_{d \mid (n, P(z))} \lambda_d' \right)$$

$$- (\log N) \sum_{N/2 < n \leq N} \left( \sum_{\nu \mid Q(n)} \lambda_{\nu} \right) \sum_{n \in T_{22}} \sum_{d \mid (n, P(z))} \lambda_d'$$

$$= S_3 - S_4 - S_5.$$
Before doing further calculations, we should study the reduced composition of sieve-twisted sums.

3. Reduced Composition of Sieves

Let \((\lambda_d)\) be a finite sequence supported on squarefree numbers and write

\[ \theta_n = \sum_{d|n} \lambda_d. \]

For \(g(d)\) a multiplicative function supported on finite set of squarefree numbers with \(0 \leq g(p) < 1\), we denote \(h(d)\) the multiplicative function supported on squarefree numbers with

\[ h(p) = \frac{g(p)}{1 - g(p)}. \]

We call \(g\) a density function and \(h\) the relative density function of \(g\). Now we consider the sieve-twisted sum

\[ G = \sum_d \lambda_d g(d). \]

**Lemma 3.1.** *It holds that*

\[ G = VG^*, \]

where

\[ V = \prod_p (1 - g(p)) \quad \text{and} \quad G^* = \sum_d \theta_d h(d). \]

**Proof.** This is Lemma A.1 of [3].

Next, we consider the reduced composition of two sieve-twisted sums of the following type:

\[ G' \ast G'' = \sum_{(d_1, d_2) = 1} \lambda_{d_1} \lambda_{d_2} g'(d_1) g''(d_2). \]

We have
Lemma 3.2.

\[ G' \ast G'' = \sum_{(b_1, b_2) = 1} \sum_{p | b_1 b_2} \theta'_{b_1} \theta''_{b_2} g'(b_1) g''(b_2) \prod_{p | b_1 b_2} (1 - g'(p) - g''(p)). \]

Proof. This is Lemma A.2 of [3]. □

Now assume that \((\lambda')\) is an upper-bound sieve (either from the beta-sieve or from the Selberg’s sieve), \((\lambda'')\) is a beta-sieve of level \(D''\), while \(g''\) is supported on the divisors of \(P(z'') = \prod_{p \leq z''} p\) for some \(z'' \leq D''\) and satisfying

\[ \prod_{w \leq p < w'} (1 - g(p))^{-1} \leq \left( \frac{\log w'}{\log w} \right)^\kappa \left( 1 + O\left( \frac{1}{\log w} \right) \right) \]

for some \(\kappa > 0\) and any \(0 < w < w'\). If we denote by \(h^{(1)}(d)\) and \(h^{(2)}(d)\) the multiplicative functions supported on squarefree numbers with

\[ h^{(1)}(p) = \frac{g'(p)}{1 - g'(p) - g''(p)} \quad \text{and} \quad h^{(2)}(p) = \frac{g''(p)}{1 - g'(p) - g''(p)}, \]

then we get (at primes)

\[ g^{(1)} = \frac{h^{(1)}}{1 + h^{(1)}} = \frac{g'}{1 - g''} \quad \text{and} \quad g^{(2)} = \frac{h^{(2)}}{1 + h^{(2)}} = \frac{g''}{1 - g'} \]

respectively. Thus Lemma 3.2 indicates

\[ G' \ast G'' = \prod_{p} (1 - g'(p) - g''(p)) \sum_{(b_1, b_2) = 1} \theta'_{b_1} \theta''_{b_2} h^{(1)}(b_1) h^{(2)}(b_2) \]

\[ = \prod_{p} (1 - g'(p) - g''(p)) \sum_{b_1} \theta'_{b_1} h^{(1)}(b_1) \sum_{(b_2, b_1) = 1} \theta''_{b_2} h^{(2)}(b_2). \]

From Lemma 3.1 and the Fundamental Lemma of the sieve we know that

\[ \sum_{(b_2, b_1) = 1} \theta''_{b_2} h^{(2)}(b_2) = \prod_{p | b_1} (1 - g^{(2)}(p))^{-1} \sum_{(d, b_1) = 1} \lambda_d g^{(2)}(d) = 1 + O(e^{-s''}), \]

where \(s''\) is the critical strip.
provided that \( s'' = \log D'' / \log z'' \) is sufficiently large. Inserting this into (3.7) and noticing that \( \theta'_{b_1} \geq 0 \), we obtain

\[
G' \ast G'' = (1 + O(e^{-s''})) \prod_p (1 - g'(p) - g''(p)) \left( \sum_{b_1} \theta'_{b_1} h^{(1)}(b_1) \right)
\]

\[
= (1 + O(e^{-s''})) \prod_p (1 - g'(p) - g''(p))(1 - g^{(1)})^{-1} \left( \sum_d \lambda'(d) g^{(1)}(d) \right)
\]

\[
= (1 + O(e^{-s''})) \prod_p (1 - g''(p)) \left( \sum_d \lambda'(d) g^{(1)}(d) \right).
\]

Therefore, we conclude:

**Proposition 3.3.** Suppose that \((\lambda')\) is an upper-bound sieve, \((\lambda'')\) is a beta-sieve of level \(D''\). Let \(g''\) be a density function supported on the divisors of \(P(z'')\) for some \(z'' \leq D''\). Then

\[
(3.8) \quad G' \ast G'' = (1 + O(e^{-s''}))V''G^{(1)}
\]

provided that \( s'' = \log D'' / \log z'' \) is sufficiently large, where

\[
V'' = \prod_p (1 - g''(p)), \quad G^{(1)} = \sum_d \lambda'(d) g^{(1)}(d)
\]

with \(g^{(1)}\) defined in (3.6).
4. Estimation of $S_5$

From (2.5) we know that

$$|S_5| = \left| \sum_{n \in T_{22}} \left( \sum_{a^m \in M} \Lambda^b(n - a^m) - \log N \right) \left( \sum_{\nu | Q(n)} \lambda_{\nu} \right) \left( \sum_{d | (n,P(z))} \chi_d' \right) \right|$$

$$\leq \sum_{n \in T_{22}} \log \frac{N}{T} - \sqrt{N} \left| \sum_{\nu | Q(n)} \lambda_{\nu} \right| \left| \sum_{d | (n,P(z))} \chi_d' \right|$$

$$\leq \left( \log 2 + O\left( \frac{1}{\sqrt{N}} \right) \right) \sum_{N/2 < n \leq N} \left( \sum_{\nu | Q(n)} \lambda_{\nu} \right) \left| \sum_{d | (n,P(z))} \chi_d' \right|$$

$$= \left( \log 2 + O\left( \frac{1}{\sqrt{N}} \right) \right) \left[ \sum_{N/2 < n \leq N} \sum_{\nu | Q(n)} \lambda_{\nu} \right.$$

$$- \sum_{N/2 < n \leq N} \left( \sum_{\nu | Q(n)} \lambda_{\nu} \right) \left( \sum_{d | (n,P(z))} \chi_d' \right) \left. \right]$$

$$= \left( \log 2 + O\left( \frac{1}{\sqrt{N}} \right) \right) \left[ 2 \sum_{N/2 < n \leq N} \sum_{\nu | Q(n)} \lambda_{\nu} \right.$$

$$- \sum_{N/2 < n \leq N} \left( \sum_{\nu | Q(n)} \lambda_{\nu} \right) \left( \sum_{d | (n,P(z))} \chi_d' \right) \left. \right]$$

$$= \left( \log 2 + O\left( \frac{1}{\sqrt{N}} \right) \right) (2S_{51} - S_{52})$$

say. In order to estimate $S_{51}$, we introduce an upper-bound beta-sieve ($\lambda''$) of level $D'$. Then
\[ S_{51} \leq \sum_{\substack{N/2 < n \leq N \\ (Q(n), \Delta) = 1}} \left( \sum_{\nu | Q(n)} \lambda_{\nu} \right) \left( \sum_{d | (n, P(z))} \lambda_d' \right) \]

\[ = \sum_{d | P(z)} \lambda_d'' \sum_{(\nu, \Delta d) = 1} \lambda_{\nu} \sum_{\substack{N/2 < n \leq N \\ (Q(n), \Delta) = 1}} \sum_{\substack{Q(n) \equiv 0 \pmod{\nu} \\ n \equiv 0 \pmod{d}}} 1. \]

Notice that the condition \((\nu, d) = 1\) is automatic since \(d \mid n, \nu \mid Q(n)\) and \((d, a) = 1\). The innermost sum can be represented as

\[ \sum_{\substack{\alpha \pmod{\Delta} \\ (Q(\alpha), \Delta) = 1 \quad \beta \pmod{\nu} \\ (Q(\beta), \Delta) = 1 \quad \delta \pmod{\Delta d} = 1 \quad \mu(\delta) = 1 \quad \alpha \pmod{\Delta} \quad \beta \pmod{\nu} \quad \delta \pmod{\Delta d} = 1 \quad \mu(\delta) = 1 \quad 1}} \left( \frac{N/2}{\nu \mid [\Delta, d]} + O(1) \right), \]

where

\[ \sum_{\substack{\alpha \pmod{\Delta} \\ (Q(\alpha), \Delta) = 1 \quad \mu(\delta) = 1 \quad \alpha \pmod{\Delta} \quad \delta \pmod{\Delta d} = 1 \quad \mu(\delta) = 1 \quad 1}} \]

\[ = \sum_{\substack{\alpha \pmod{\Delta} \\ (Q(\alpha), \Delta) = 1 \quad \delta \pmod{(Q(\alpha), \Delta)} \quad \delta \pmod{\Delta d} = 1 \quad \mu(\delta) = 1 \quad \alpha \pmod{\Delta} \quad \mu(\delta) = 1 \quad 1}} \]

\[ = \sum_{\substack{\delta \pmod{\Delta d} \quad \mu(\delta) \cdot \frac{\Delta}{(\Delta, d) \delta \omega(\delta)} = \frac{\Delta}{(\Delta, d)} \prod_{p \mid \Delta} \left( 1 - \frac{\omega(p)}{p} \right) \prod_{p \mid (\Delta, d)} \left( 1 - \frac{\omega(p)}{p} \right)^{-1} \]

with

\[ \gamma(H) = \prod_{p \mid \Delta} \left( 1 - \frac{\omega(p)}{p} \right). \]
Moreover, from \((\nu, \Delta) = 1\) we know that
\[
\sum_{\beta \pmod{\nu}} 1 = \tau_k(\nu).
\]

Therefore, summing up the above four formulae we get
\[
S_{51} \leq \gamma(H) \sum_{d \mid P(z)} \sum_{(\nu, \Delta d) = 1} \lambda_{\nu} \lambda'_{\Delta d} \frac{\Delta}{(\Delta, d)} \tau_k(\nu)
\[
\times \left( \frac{N/2}{\nu[\Delta, d]} + O(1) \right) \prod_{p \mid (\Delta, d)} \left( 1 - \frac{\omega(p)}{p} \right)^{-1}.
\]

Since \(|\lambda_{\nu}(d)| \leq \tau_3(d)\) and \(|\lambda'_{d}| \leq 1\), we have
\[
S_{51} \leq \frac{N}{2} \gamma(H) \sum_{d \mid P(z)} \sum_{(\nu, \Delta d) = 1} \lambda_{\nu} \lambda'_{d} \frac{\tau_k(\nu)}{d\nu} \prod_{p \mid (\Delta, d)} \left( 1 - \frac{\omega(p)}{p} \right)^{-1}
\[
+ O \left( \gamma(H) \sum_{d < D'} \sum_{\nu < D} \tau_3(\nu) \tau_k(\nu) \frac{\Delta}{(\Delta, d)} \prod_{p \mid (\Delta, d)} \left( 1 - \frac{\omega(p)}{p} \right)^{-1} \right)
\[
= \frac{N}{2} \gamma(H) \sum_{d \mid P(z)} \sum_{(\nu, \Delta d) = 1} \lambda_{\nu} \lambda'_{d} \frac{\tau_k(\nu)}{d\nu} \prod_{p \mid (\Delta, d)} \frac{p}{p - \omega(p)} + O(\Delta DD'(\log D)^{3k-1}).
\]

From Proposition 3.3 we get for sufficiently large \(s\) that
\[
S_{51} \leq (1 + O(e^{-s})) \frac{N}{2} \gamma(H) G_1 V(z) + O(\Delta DD'(\log D)^{3k-1}),
\]
where
\[
V(z) = \prod_{p < z \atop p \mid \Delta} \left( 1 - \frac{1}{p} \right) \prod_{k+2 \leq p < z \atop p \mid (\Delta, p) \atop p \mid a} \left( 1 - \frac{1}{p - \omega(p)} \right)
\]
and
\[
G_1 = \sum_{\nu < D \atop (\nu, \Delta) = 1} \lambda_{\nu} \frac{\tau_k(\nu)}{\varphi(\nu)}.
\]

Analogously,
\[
S_{52} = (1 + O(e^{-s})) \frac{N}{2} \gamma(H) G_1 V(z) + O(\Delta DD'(\log D)^{3k-1}).
\]
Therefore,

\[(4.5) \quad S_5 \ll N\gamma(\mathcal{H})G_1V(z) + N^{1-\frac{c}{2}}.\]

5. Evaluation of $S_3$ and $S_4$

First, we mention that $S_4 = S_{51}\log N$, where $S_{51}$ is defined in the previous section. Thus (4.2) implies that

\[(5.1) \quad S_4 \leq (1 + O(c^{-s})) \frac{N\log N}{2}\gamma(\mathcal{H})G_1V(z) + O(\Delta DD'(\log N)^{3k}).\]

In order to calculate $S_3$, we change the order of summation to get

\[(5.2) \quad S_3 = \sum_{d \mid P(z)} \lambda'_d \sum_{a^m \in \mathcal{M}} U_d^{(m)},\]

where

\[(5.3) \quad U_d^{(m)} = \sum_{\substack{N/2 < n \leq N \\ (Q(n),\Delta) = 1 \\ n \equiv 0 \pmod{d}}} \Lambda^\flat(n - a^m) \left( \sum_{\nu | Q(n)} \lambda_\nu \right).\]

Next we come to the evaluation of $U_d^{(m)}$.

\[
U_d^{(m)} = \sum_{\nu,\Delta \equiv 1} \lambda_\nu \sum_{\alpha \equiv 0 \pmod{\Delta}} \Lambda^\flat(n - a^m) \sum_{\beta \equiv 0 \pmod{\nu}} \sum_{\substack{N/2 < n \leq N \\ (Q(\alpha),\Delta) = 1 \\ (Q(\beta)\equiv 0 \pmod{\nu} \\ n \equiv \alpha \pmod{\Delta})}} \Lambda^\flat(n - a^m) \sum_{\substack{n \equiv \beta \pmod{\nu} \\ n \equiv 0 \pmod{d}}}.\]
Almost-primes represented by $p + a^m$

We write $R_1$ to be the summation with $(\beta - a^m, \nu) > 1$, then

$$
R_1 = \sum_{(\nu, \Delta d) = 1} \lambda_\nu \sum_{p \mid \nu} \log p \sum_{\alpha \equiv 0 (\text{mod } \Delta)} \sum_{(Q(\alpha), \Delta) = 1} \sum_{\beta \equiv 0 (\text{mod } \nu)} \sum_{(\beta - a^m, \nu) \equiv 1 \text{mod } p} 1 \quad \sum_{N/2 < n \leq N} \frac{1}{n} \approx \frac{\log D}{\varphi(d)} \sum_{p \equiv -a^m \text{ (mod } d)} \sum_{\nu \equiv -a^m \text{ (mod } d)} \frac{\log p}{p},
$$

where the implied constant depends only on $k$. If we denote by $R_2$ the summation with $(\beta - a^m, \nu) = 1$ and $(\alpha - a^m, \Delta) > 1$, then

$$
R_2 = \sum_{(\nu, \Delta d) = 1} \lambda_\nu \sum_{p \mid \Delta} \log p \sum_{\alpha \equiv 0 (\text{mod } \Delta)} \sum_{(Q(\alpha), \Delta) = 1} \sum_{\beta \equiv 0 (\text{mod } \nu)} \sum_{(\beta - a^m, \nu) \equiv 1 \text{mod } p} 1 \quad \sum_{N/2 < n \leq N} \frac{1}{n} \approx \frac{\log D}{\varphi(d)} \sum_{p \equiv -a^m \text{ (mod } d)} \sum_{\nu \equiv -a^m \text{ (mod } d)} \frac{\log p}{p},
$$

Therefore we conclude that

$$
U_d^{(m)} = \sum_{(\nu, \Delta d) = 1} \lambda_\nu \sum_{\alpha \equiv 0 (\text{mod } \Delta)} \sum_{((\alpha - a^m)Q(\alpha), \Delta) = 1} \sum_{\beta \equiv 0 (\text{mod } \nu)} \sum_{(\beta - a^m, \nu) \equiv 1 \text{mod } p} \Delta^2(n - a^m) \quad \sum_{N/2 < n \leq N} \frac{1}{n} \approx \Delta D(\log D)^2 \log \Delta. + O(D(\log D)^{3k}),
$$
where the implied constant depends only on \(k\).

For \((b, q) = 1\), we write

\[
E(x, q; b) = \sum_{n \equiv b \pmod{q} \atop n \leq x} \Lambda^k(n) - \frac{x}{\varphi(q)}
\]
as usual. Then

\[
U_d^{(m)} = \sum_{(\nu, \Delta d) = 1} \lambda_\nu \sum_{\alpha \equiv 0 \pmod{\Delta}} \frac{N}{2} - \frac{1}{\varphi(\nu[\Delta, d])} + R_d^{(m)}
\]

(5.5)

\[
+ O(D(\log D)^{3k}),
\]

where

\[
R_d^{(m)} = \sum_{(\nu, \Delta d) = 1} \lambda_\nu \sum_{\alpha \equiv 0 \pmod{\Delta}} \sum_{\beta \equiv 0 \pmod{\nu}} (E(N, \nu[\Delta, d]; b) - E(N/2, \nu[\Delta, d]; b))
\]

(5.6)

\[
\times \sum_{((\alpha - a^m)Q(\alpha), \Delta) = 1} \sum_{\beta \equiv 0 \pmod{\nu}} (E(N, \nu[\Delta, d]; b) - E(N/2, \nu[\Delta, d]; b))
\]

with \(b\) the residue class modulo \(\nu[\Delta, d]\) satisfying \(b \equiv \alpha - a^m \pmod{\Delta}\), \(b \equiv \beta - a^m \pmod{\nu}\) and \(b \equiv -a^m \pmod{d}\). Notice that we include in the error term a few terms for \(\Lambda^k(n)\) with \(n\) in the intervals \((N/2, N/2 + a^m]\) and \([N, N + a^m]\).

We can easily deduce that for every \(m\)

\[
\sum_d \lambda_d R_d^{(m)} \ll \Delta \sum_{q \leq D D'} \tau_{k+3}(q) \max_{(b, q) = 1} \left( |E(N, q; b)| + |E(N/2, q; b)| \right),
\]

while by Cauchy’s inequality and \(E(N, q; b) \ll N/\varphi(q)\)

\[
\sum_{q \leq D D'} \tau_{k+3}(q) \max_{(b, q) = 1} |E(N, q; b)| \ll \left( \sum_{q \leq D D'} \frac{\tau_{k+3}(q) N}{\varphi(q)} \right)^{1/2} \left( \sum_{q \leq D D'} \max_{(b, q) = 1} |E(N, q; b)| \right)^{1/2},
\]
and the Bombieri-Vinogradov theorem indicates
\[
\sum_{q \leq \Delta D'} \tau_{k+3}(q) \max_{(b,q)=1} |E(N, q; b)| \ll \frac{N}{(\log N)^{A+1}}
\]
for any positive real number \(A\). The same estimate holds for the sum involving \(E(N/2, q; b)\). Therefore,
\[
\sum_{d} \lambda_d' \tau_d^{(m)} \ll \frac{N}{(\log N)^{A+1}}.
\] (5.7)

In order to calculate the main term in (5.5), we need to evaluate the sum over \(\alpha\) and \(\beta\) respectively. Since \(\Delta \mid Q(a^m)\) for any \(m\), we have
\[
\sum_{\alpha \equiv 0 (\mod (\Delta, d))} 1 = \sum_{\alpha \equiv 0 (\mod (\Delta, d))} \frac{\Delta}{(\Delta, d)^2} \gamma(\mathcal{H}) \prod_{p \mid (\Delta, d)} \left(1 - \frac{\omega(p)}{p}\right)^{-1}
\]
by (4.1). For squarefree number \(\nu\) satisfying \((\nu, \Delta) = 1\), we write
\[
\tau_{k}^{(m)}(\nu) = \sum_{\beta \equiv 0 (\mod (\nu, \Delta))} 1,
\]
then \(\tau_{k}^{(m)}(\nu) = \tau_{k}(\nu_1)\tau_{k-1}(\nu_2)\) where \(\nu = \nu_1\nu_2\) with \((\nu_1, Q(a^m)) = 1\) and \(\nu_2 \mid Q(a^m)\). Therefore the main term in (5.5) is equal to
\[
= \frac{\Delta N \gamma(\mathcal{H})}{2\varphi([\Delta, d]) \cdot (\Delta, d)} \prod_{p \mid (\Delta, d)} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \sum_{(\nu, \Delta d) = 1} \lambda_\nu \frac{\tau_{k}^{(m)}(\nu)}{\varphi(\nu)}
\]
\[
= \frac{\Delta N \gamma(\mathcal{H})}{2\varphi(\Delta)} \cdot \frac{1}{\varphi(d)} \prod_{p \mid (\Delta, d)} \left(1 - \frac{1}{p}\right) \left(1 - \frac{\omega(p)}{p}\right)^{-1} \sum_{(\nu, \Delta d) = 1} \lambda_\nu \frac{\tau_{k}^{(m)}(\nu)}{\varphi(\nu)}.
\]
Summing over $d$, we get from Proposition 3.3 that

\[(5.8) \sum_d \lambda_d U_d^{(m)} = (1 + O(e^{-s})) \frac{\Delta N \gamma(\mathcal{H})}{2 \varphi(\Delta)} G^{(m)} \prod_{p < z \atop p \nmid \Delta} \left(1 - \frac{1}{p - 1}\right)\]

\[\times \prod_{p < z \atop p \nmid \Delta, p \nmid a} \left(1 - \frac{1}{p - \omega(p)}\right) + O\left(\frac{N}{(\log N)^{A+1}}\right),\]

\[\geq (1 + O(e^{-s})) \frac{\Delta N \gamma(\mathcal{H})}{2 \varphi(\Delta)} G^{(m)} V(z) + O\left(\frac{N}{(\log N)^{A+1}}\right),\]

where

\[G^{(m)} = \sum_{\nu < D \atop (\nu, \Delta) = 1} \lambda_{\nu} \frac{\tau_k^{(m)}(\nu)}{f(\nu)},\]

with $f(\nu)$ the multiplicative function satisfying $f(p) = p - 2$, and the error term mainly comes from (5.7).

Combining (5.1), (5.2) and (5.8) we finally arrive at

\[(5.9) S_3 - S_4 \geq (1 + O(e^{-s})) N \gamma(\mathcal{H}) V(z) \left(\frac{\Delta}{\varphi(\Delta)} \sum_{a m \in \mathcal{H}} G_2 \right) + \frac{\Delta}{\varphi(\Delta)} \sum_{a m \in \mathcal{M} \setminus \mathcal{H}} G_3 - G_1 \log N + O\left(\frac{N}{(\log N)^{A+1}}\right),\]

where $V(z)$ is given in (4.3) and

\[(5.10) G_2 = \sum_{\nu < D \atop (\nu, \Delta) = 1} \lambda_{\nu} \frac{\tau_k^{(m)}(\nu)}{f(\nu)}, \quad G_3 = \sum_{\nu < D \atop (\nu, \Delta) = 1} \lambda_{\nu} \frac{\tau_k^{(m)}(\nu)}{f(\nu)} (a m \in \mathcal{M} \setminus \mathcal{H}).\]

### 6. Choosing the Sifting Weights

In this section, we will choose the parameters $\lambda_{\nu}$ and give asymptotic formulae for $G_1$ and $G_2$. We follow the way given in [4].
Denote 
\[ g_1(\nu) = \frac{\tau_k(\nu)}{\varphi(\nu)}, \quad g_2(\nu) = \frac{\tau_{k-1}(\nu)}{f(\nu)}, \]
and 
\[ g_3(\nu) = \frac{\tau_k(m)(\nu)}{f(\nu)} \quad (a^m \in \mathcal{M} \setminus \mathcal{H}), \]
let \( h_i(\nu) \) be the relative density function of \( g_i(\nu) \). It is well-known from the Selberg’s \( \Lambda^2 \)-sieve theory that

\[ G_1 = \sum_{c<\sqrt{D} \atop (c,\Delta)=1} h_1(c)y_c^2, \]  

where

\[ y_c = \frac{\mu(c)}{h_1(c)} \sum_{m \equiv 0 \pmod{c}} \rho_m g_1(m). \]

Using the Möbius inversion formula on divisor-closed set we obtain

\[ \rho_m = \frac{\mu(m)}{g_1(m)} \sum_{c \equiv 0 \pmod{m}} h_1(c)y_c. \]

Therefore the initial condition \( \rho_1 = 1 \) is equivalent to

\[ \sum_{c<\sqrt{D} \atop (c,\Delta)=1} h_1(c)y_c = 1. \]

Now we choose

\[ y_c = \frac{1}{Y} \left( \log \frac{\sqrt{D}}{c} \right)^\ell, \]
for squarefree \( c \leq \sqrt{D} \), \( (c,\Delta) = 1 \) and \( y_c = 0 \) otherwise. Inserting this into (6.3) we find that

\[ Y = \sum_{c<\sqrt{D} \atop (c,\Delta)=1} h_1(c) \left( \log \frac{\sqrt{D}}{c} \right)^\ell, \]

where \( \sum^b \) means the summation goes through squarefree integers. Before going further, we give a result involving the sieve weight constituents which verifies (2.2).

**Lemma 6.1.** For any integer \( m \geq 1 \), we have \( |\rho_m| \leq 1 \).
Proof. From (6.2) and (6.4) we know that

$$
\rho_m = \frac{\mu(m)h_1(m)}{Yg_1(m)} \sum_{c<\sqrt{D/m}}^{b} \ h_1(c) \left( \log \frac{\sqrt{D}}{cm} \right)^{\ell}.
$$

Then the desired result follows from

$$
Y = \sum_{u|m}^{b} \sum_{c<\sqrt{D}}^{\ell} \ h_1(c) \left( \log \frac{\sqrt{D}}{c} \right)^{\ell} = \sum_{u|m}^{b} \sum_{c<\sqrt{D}/u}^{\ell} \ h_1(c) \left( \log \frac{\sqrt{D}}{cu} \right)^{\ell}
$$

\[
\geq \left( \sum_{u|m}^{b} h_1(u) \right) \sum_{c<\sqrt{D/m}}^{b} \ h_1(c) \left( \log \frac{\sqrt{D}}{cm} \right)^{\ell}
\]

\[
= \frac{h_1(m)}{g_1(m)} \sum_{c<\sqrt{D/m}}^{b} \ h_1(c) \left( \log \frac{\sqrt{D}}{cm} \right)^{\ell}.
\]

In order to calculate the sum in (6.5), we introduce the following lemma.

**Lemma 6.2.** Let $$\kappa$$ and $$\ell$$ be positive integers and assume $$g$$ is a multiplicative function supported on squarefree numbers such that

(6.6) $$g(p) = \frac{\kappa}{p} + O\left( \frac{1}{p^2} \right), \quad \kappa \geq 1.$$

Then, for $$x \geq 2$$,

(6.7) $$\sum_{m \leq x}^{b} \sum_{c<\sqrt{D/m}}^{\ell} g(m) \left( \log \frac{x}{m} \right)^{\ell} = \mathcal{G} \frac{\ell!}{(\ell + \kappa)!} \left( \log x \right)^{\ell + \kappa} \left( 1 + O\left( \frac{1}{\log x} \right) \right),$$

where

(6.8) $$\mathcal{G} = \prod_{p|\Delta} \left( 1 - \frac{1}{p} \right)^{\kappa} (1 + g(p)) \prod_{p|\Delta} \left( 1 - \frac{1}{p} \right)^{\kappa},$$

and the implied constant depends only on $$\kappa$$, $$\ell$$, $$\Delta$$ and on the one in (6.6).
Almost-primes represented by $p + a^n$

**Proof.** This is Corollary A.6 of [4].

Since $h_1(p) = k(p - k - 1)^{-1}$ for $p \nmid \Delta$, we get from Lemma 6.2 that

$$Y = \mathcal{S}(\Delta) \frac{(k + \ell)!}{(k + \ell)!} \left( \log \sqrt{D} \right)^{k + \ell} \left( 1 + O\left( \frac{1}{\log D} \right) \right),$$

where

$$\mathcal{S}(\Delta) = \prod_{p \nmid \Delta} \left( 1 - \frac{1}{p} \right)^k \left( 1 - \frac{k}{p - 1} \right)^{-1} \prod_{p \mid \Delta} \left( 1 - \frac{1}{p} \right)^k.$$

Analogously, (6.1) and (6.4) indicate that

$$Y^2 G_1 = \sum_{c < \sqrt{D}} \sum_{(c, \Delta) = 1} h_1(c) \left( \log \frac{\sqrt{D}}{c} \right)^{2\ell}$$

$$= \mathcal{S}(\Delta) \frac{(2\ell)!}{(k + 2\ell)!} \left( \log \sqrt{D} \right)^{k + 2\ell} \left( 1 + O\left( \frac{1}{\log D} \right) \right).$$

Applying (6.9) we obtain

$$G_1 = \mathcal{S}(\Delta)^{-1} \frac{(2\ell)!}{(k + 2\ell)!} \left( \log \sqrt{D} \right)^{k + 2\ell} \left( 1 + O\left( \frac{1}{\log D} \right) \right).$$

Next we calculate $G_2$. We have

$$G_2 = \sum_{c < \sqrt{D}} \sum_{(c, \Delta) = 1} \frac{1}{h_2(c)} \left( \sum_{m \equiv 0 \pmod{c}} \rho_m g_2(m) \right)^2.$$

Notice that $\rho_m$ is given in (6.2), whence

$$\sum_{m \equiv 0 \pmod{c}} \rho_m g_2(m) = \sum_{m \equiv 0 \pmod{c}} \frac{\mu(m) g_2(m)}{g_1(m)} \sum_{d \equiv 0 \pmod{m}} h_1(d) y_d$$

$$= \frac{\mu(c) g_2(c)}{g_1(c)} \sum_{d \equiv 0 \pmod{c}} h_1(d) y_d \sum_{u \equiv d \pmod{c}} \frac{\mu(u) g_2(u)}{g_1(u)}.$$

Since

$$\sum_{u \equiv d \pmod{c}} \frac{\mu(u) g_2(u)}{g_1(u)} = \prod_{p \nmid \Delta} \left( 1 - \frac{(k - 1)(p - 1)^2}{k(p - 2)} \right) = \prod_{p \nmid \Delta} \frac{p - k - 1}{k(p - 2)} = \frac{1}{h_1(d/c) f(d/c)},$$
we conclude that
\[ \sum_{m \equiv 0 \pmod{c}} \rho_m g_2(m) = \mu(c) \frac{\varphi(c) \tau_{k-1}(c)}{\tau_k(c)} \sum_{d \equiv 0 \pmod{c}} \frac{h_1(d)y_d}{h_1(d/c) f(d/c)} \]
\[ = \mu(c) \varphi(c) h_1(c) \frac{\tau_{k-1}(c)}{\tau_k(c)} \sum_{d \equiv 0 \pmod{c}} \frac{y_d}{f(d)}. \]

Inserting this into (6.12), we have
\[ G_2 = \sum_{c < \sqrt{D}} \frac{1}{h_2(c)} \left( \varphi(c) h_1(c) \frac{\tau_{k-1}(c)}{\tau_k(c)} \sum_{d \equiv 0 \pmod{c}} \frac{y_d}{f(d)} \right)^2 \]
\[ = \sum_{c < \sqrt{D}} \frac{h_2(c) \varphi(c)^2}{f(c)^2} \left( \sum_{d < \sqrt{D}/c} \frac{y_d}{f(d)} \right)^2 \]
\[ = \frac{1}{Y^2} \sum_{c < \sqrt{D}} \frac{h_2(c) \varphi(c)^2}{f(c)^2} \left( \sum_{d < \sqrt{D}/c} \frac{1}{f(d)} \left( \log \frac{\sqrt{D}}{cd} \right)^{\ell} \right)^2. \]

Applying Lemma 6.2 we have
\[ Y^2 G_2 = \mathcal{S}_1(\Delta)^2 \sum_{c < \sqrt{D}} \frac{1}{h_2(c)} \left( \log \frac{\sqrt{D}}{c} \right)^{2\ell+2} \left( 1 + O\left( \frac{1}{\log \sqrt{D}/c} \right) \right), \]
where
\[ \mathcal{S}_1(\Delta) = \prod_{p \nmid \Delta} \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p-2} \right) \prod_{p | \Delta} \left( 1 - \frac{1}{p} \right). \]

Applying Lemma 6.2 again we get
\[ Y^2 G_2 = \frac{\mathcal{S}_1(\Delta)^2}{(\ell + 1)^2} \cdot \mathcal{S}_2(\Delta) \frac{(2\ell + 2)!}{(k + 2\ell + 1)!} \left( \log \sqrt{D} \right)^{k+2\ell+1} \left( 1 + O\left( \frac{1}{\log D} \right) \right), \]
where
\[ \mathcal{S}_2(\Delta) = \prod_{p \nmid \Delta} \left( 1 - \frac{1}{p} \right)^{k-1} \left( 1 + \frac{k - 1}{p - k - 1} \right) \prod_{p | \Delta} \left( 1 - \frac{1}{p} \right)^{k-1}. \]
Combining with (6.9), we get
\begin{equation}
G_2 = \frac{\mathcal{G}_1(\Delta)^2 \mathcal{G}_2(\Delta)}{\mathcal{G}(\Delta)^2} \frac{(k + \ell)!^2(2\ell + 2)!}{(\ell + 1)!^2(k + 2\ell + 1)!} (\log \sqrt{D})^{1-k} \left(1 + O\left(\frac{1}{\log D}\right)\right).
\end{equation}
If we write
\begin{equation}
\mathcal{G}'(\Delta) = \prod_{p \nmid \Delta} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p + 2}\right),
\end{equation}
then a modicum of calculation shows that
\[ \frac{\mathcal{G}_1(\Delta)^2 \mathcal{G}_2(\Delta)}{\mathcal{G}(\Delta)} = \frac{\varphi(\Delta)}{\Delta} \mathcal{G}'(\Delta). \]
Therefore, comparing (6.13) with (6.11) we finally get
\begin{equation}
\frac{\Delta}{\varphi(\Delta)} G_2 = \frac{(2\ell + 1)\mathcal{G}'(\Delta)G_1 \log D}{(\ell + 1)(k + 2\ell + 1)} \left(1 + O\left(\frac{1}{\log D}\right)\right).
\end{equation}
The last task is to evaluate $G_3$, we will complete it in the next section.

7. Asymptotics of $G_3$ and Proof of the Theorem

First we will give a more precise form of the error term in Lemma 6.2.

**Lemma 7.1.** Under the assumption of Lemma 6.2, we have
\begin{equation}
\sum_{\substack{m \leq x \\ (m, \Delta) = 1}} g(m) \left(\log \frac{x}{m}\right)^\ell = \mathcal{G} \frac{\ell!}{(\ell + \kappa)!} (\log x)^{\ell + \kappa} + O\left( \frac{\log x}{\log(\Delta + 2)} \right),
\end{equation}
where $\mathcal{G}$ is given in (6.8) and the implied constant depends only on $\kappa, \ell$ and on the one in (6.6).

**Proof.** First we introduce the following asymptotic formula
\begin{equation}
\sum_{\substack{m \leq x \\ (m, \Delta) = 1}} g(m) = \frac{\mathcal{G}}{\kappa!} (\log x)^\kappa + O\left( \frac{\log x}{\log(\Delta + 2)} \right),
\end{equation}
the proof is analogous to the one given for Theorem A.5 in [4], the only difference is that the condition (A.15) appeared in [4] should be replaced
by
\[
\sum_{\substack{p \leq x \\ p \nmid \Delta}} g(p) \log p = \kappa \log x + O(\log \log (\Delta + 2))
\]
since
\[
\sum_{p \mid \Delta} \frac{\log p}{p} = \sum_{p \mid \Delta, p \leq \log (\Delta + 2)} \frac{\log p}{p} + \sum_{p \mid \Delta, p > \log (\Delta + 2)} \frac{\log p}{p}
\]
(7.3)
\[
\ll \log \log (\Delta + 2) + \frac{\log \log (\Delta + 2)}{\log (\Delta + 2)} \sum_{p \mid \Delta} 1
\]
\[
\ll \log \log (\Delta + 2).
\]

Then, using partial summation we can get (7.1) from (7.2).

**Lemma 7.2.** Under the assumption of Lemma 6.2, we have
\[
\sum_{\substack{m \leq x \\ (m, \Delta) = 1 \\ m \mid \Delta'}} g(m) \left( \log \frac{x}{m} \right)^{\kappa + \ell}
\]
\[
= \left( 1 + O \left( \frac{\log \log (\Delta' + 2)^{\kappa + 1}}{S \log x} \right) \right) \left( \log x \right)^{\kappa + \ell} \prod_{p \mid \Delta'} (1 + g(p)),
\]
where $S$ is given in (6.8), and the implied constant depends only on $\kappa$, $\ell$ and on the one in (6.6).

**Proof.** We have
\[
\sum_{\substack{m \leq x \\ (m, \Delta) = 1}} g(m) \left( \log \frac{x}{m} \right)^{\ell} = \sum_{\substack{m_1 \leq x \\ (m_1, \Delta') = 1}} g(m_1) \sum_{\substack{m_2 \leq x \\ (m_1 m_2, \Delta') = 1}} \left( \log \frac{x}{m_1 m_2} \right)^{\ell}
\]
\[
= \sum_{\substack{m_1 \leq x \\ (m_1, \Delta) = 1 \\ m_1 \mid \Delta'}} g(m_1) \sum_{\substack{m_2 \leq x/m_1 \\ (m_2, \Delta') = 1}} g(m_2) \left( \log \frac{x}{m_1 m_2} \right)^{\ell}
\]
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\[
(7.4) \quad = \sum_{m_1 \leq x \atop (m_1, \Delta) = 1 \atop m_1 | \Delta'} \sum_{b} g(m_1) \left[ \frac{\mathcal{G}'!}{(\kappa + \ell)!} \left( \log \frac{x}{m_1} \right)^{\kappa + \ell} + O\left( \left( \log \frac{x}{m_1} \right)^{\kappa + \ell - 1} \log \log(\Delta \Delta' + 2) \right) \right]
\]

where

\[
\mathcal{G}' = \prod_{p \nmid \Delta'} \left( 1 - \frac{1}{p} \right)^{\kappa} \left( 1 + g(p) \right) \prod_{p \mid \Delta'} \left( 1 - \frac{1}{p} \right)^{\kappa}.
\]

It is obvious that

\[
\sum_{m_1 \mid \Delta'} g(m_1) \leq \exp \left( \sum_{p \nmid \Delta'} g(p) \right) \ll \exp \left( \kappa \sum_{p \mid \Delta'} \frac{1}{p} \right) \ll \left( \log \log(\Delta' + 2) \right)^{\kappa},
\]

where the last step is analogous to (7.3). Therefore, the error term in (7.4) is

\[
O\left( (\log x)^{\kappa + \ell - 1} \log \log(\Delta \Delta' + 2))^{\kappa + 1} \right).
\]

Now, using Lemma 7.1 to calculate the left hand side of (7.4), we have

\[
\sum_{b} g(m_1) \left( \log \frac{x}{m_1} \right)^{\kappa + \ell} = \frac{\mathcal{G}}{\mathcal{G}'} (\log x)^{\kappa + \ell} \left( 1 + O\left( \frac{\left( \log \log(\Delta \Delta' + 2) \right)^{\kappa + 1}}{\mathcal{G} \log x} \right) \right),
\]

where \( \mathcal{G} \) is given in (6.8). Since

\[
\frac{\mathcal{G}}{\mathcal{G}'} = \prod_{p \nmid \Delta} \left( 1 + g(p) \right),
\]

the desired result is obtained. \( \square \)

Now we begin to calculate \( G_3 \). As in section 6, we have

\[
(7.5) \quad G_3 = \sum_{c < \sqrt{D} \atop (c, \Delta) = 1} \frac{1}{h_3(c)} \left( \sum_{m \equiv 0 \pmod{c}} \rho_m g_3(m) \right)^2.
\]
From (6.2) we know that

\[
\sum_{m \equiv 0 \pmod{c}} \rho_m g_3(m) = \sum_{m \equiv 0 \pmod{c}} g_3(m) \mu(m) \frac{\varphi(m)}{\tau_k(m)} \sum_{d \equiv 0 \pmod{m}} h_1(d) y_d
\]

\[
= \frac{\mu(c) \varphi(c) g_3(c)}{\tau_k(c)} \sum_{d \equiv 0 \pmod{m}} h_1(d) y_d \sum_{u \mid \frac{d}{c}} \frac{\mu(u) \varphi(u) g_3(u)}{\tau_k(u)}
\]

\[
= \frac{\mu(c) \varphi(c) g_3(c)}{\tau_k(c)} \sum_{d \equiv 0 \pmod{m}} h_1(d) y_d \prod_{p \mid \frac{d}{c}} \left(1 - \frac{(p-1) \tau_k(m)}{k(p-2)} \right)
\]

\[
= \frac{\mu(c) \varphi(c) g_3(c)}{\tau_k(c)} \sum_{d \equiv 0 \pmod{m}} h_1(d) y_d \frac{f_1(b/c)}{\tau_k(b/c) f(b/c)},
\]

where \( f_1 \) is the multiplicative function with

\[
f_1(p) = k(p-2) - (p-1) \tau_k^{(m)}(p).
\]

Therefore,

\[
(7.6) \sum_{m \equiv 0 \pmod{c}} \rho_m g_3(m) = \frac{\mu(c) \varphi(c) g_3(c) h_1(c)}{\tau_k(c)} \sum_d h_1(d) f_1(d) \frac{\tau_k(d)}{\tau_k(b/c) f(b/c)} y_{dc}.
\]

Recalling the definition of \( y_{dc} \), we can express the summation over \( d \) as

\[
\frac{1}{Y} \sum_{d < \sqrt{D/c} \atop (d, \Delta c) = 1} h_1(d) f_1(d) \frac{\tau_k(d)}{\tau_k(b/c) f(b/c)} \left( \log \frac{\sqrt{D}}{dc} \right)^\ell.
\]

Since

\[
\frac{h_1(p) f_1(p)}{\tau_k(p) f(p)} = \begin{cases} 
\frac{1}{f(p)} & \text{if } p \mid Q(a^m), \\
\frac{\mu(p) h_1(p)}{f(p)} & \text{if } p \nmid Q(a^m),
\end{cases}
\]
we have (note that $\Delta \mid Q(a^m)$)

\[(7.7)\]

\[
\frac{h_1(d) f_1(d)}{e(d) f(d)} y_{dc} = \frac{1}{Y} \sum_{d} \frac{1}{f(u)} \mu(v) h_1(v) \left( \log \frac{\sqrt{D}}{uv} \right)^{\ell}
\]

\[
= \frac{1}{Y} \sum_{u < \sqrt{D}/c} \sum_{v < \sqrt{D}/uc} \sum_{(uv, \Delta c) = 1} \sum_{u|Q(a^m), (v, Q(a^m)) = 1} \mu(v) h_1(v) \left( \log \frac{\sqrt{D}}{uv} \right)^{\ell}
\]

It is obvious that $\mu(v) h_1(v) / f(v) \ll v^{\epsilon - 2}$, therefore writing

\[
\left( \log \frac{\sqrt{D}}{uv} \right)^{\ell} = \sum_{j=0}^{\ell} \binom{\ell}{j} \left( \log \frac{\sqrt{D}}{uc} \right)^{\ell-j} \log^{j} v,
\]

we get

\[
\sum_v = \left( \log \frac{\sqrt{D}}{uc} \right)^{\ell} \sum_{v < \sqrt{D}/uc} \sum_{(v, Q(a^m)c) = 1} \mu(v) h_1(v) f(v) + O \left( \left( \log \frac{\sqrt{D}}{uc} \right)^{\ell-1} \right)
\]

\[
= \left( \log \frac{\sqrt{D}}{uc} \right)^{\ell} \prod_{p|Q(a^m)c} \left( 1 - \frac{k}{(p-k-1)(p-2)} \right) + O \left( \log \frac{\sqrt{D}}{uc} \right)^{\ell-1}.
\]

Inserting this into (7.7) and making use of Lemma 7.2 we have

\[
\sum_d h_1(d) f_1(d) \frac{y_{dc}}{e(d) f(d)} = \frac{1}{Y} \left( \log \frac{\sqrt{D}}{c} \right)^{\ell} \left( 1 + O \left( \frac{\phi(c)}{f(c)} \left( \log \frac{\sqrt{D}}{c} \right)^{-1} (\log \log N)^2 \right) \right)
\]

\[
\times \prod_{p|Q(a^m)c} \left( 1 - \frac{k}{(p-k-1)(p-2)} \right) \prod_{p|\Delta c} \left( 1 + \frac{1}{p-2} \right),
\]
combining with (7.5) and (7.6) we have
\[ Y^2G_3 = \sum_{c < \sqrt{D} \atop (c, \Delta) = 1} \frac{1}{h_3(c)} \frac{\varphi(c)^2 g_3(c)^2 h_1(c)^2}{\tau_k(c)^2} \left( \log \frac{\sqrt{D}}{c} \right)^{2g} \left( 1 + O \left( \frac{\varphi(c)^2 (\log \log N)^4}{f(c)^2 \log(\sqrt{D}/c)} \right) \right) \]
\[ \times \prod_{p \nmid Q(a^m)} \left( 1 - \frac{k}{(p - k - 1)(p - 2)} \right)^2 \prod_{p \nmid \Delta \atop p \nmid Q(a^m)} \left( 1 + \frac{1}{p - 2} \right)^2 \]
\[ = \sum_{c < \sqrt{D} \atop (c, \Delta) = 1} \sum_{b} \frac{\mathcal{S}_{1}^{(m)}}{h_3(c)} \frac{\varphi(c)^2 g_3(c)^2 h_1(c)^2}{\tau_k(c)^2} \left( \log \frac{\sqrt{D}}{c} \right)^{2g} \left( 1 + O \left( \frac{\varphi(c)^2 (\log \log N)^4}{f(c)^2 \log(\sqrt{D}/c)} \right) \right) \]
\[ \times \prod_{p \nmid Q(a^m)} \left( 1 - \frac{k}{(p - k - 1)(p - 2)} \right)^2 \prod_{p \nmid \Delta \atop p \nmid Q(a^m)} \left( 1 + \frac{1}{p - 2} \right)^2 \]
\[ \times \prod_{p \nmid u} \left( 1 - \frac{k}{(p - k - 1)(p - 2)} \right)^2 \left( 1 + O \left( \frac{\varphi(u)^2 (\log \log N)^4}{f(u)^2 \log(\sqrt{D}/u)} \right) \right), \]
where
\[ (7.8) \quad \mathcal{S}_{1}^{(m)} = \prod_{p \nmid Q(a^m)} \left( 1 - \frac{k}{(p - k - 1)(p - 2)} \right)^2 \prod_{p \nmid \Delta \atop p \nmid Q(a^m)} \left( 1 + \frac{1}{p - 2} \right)^2. \]

It is easy to verify that
\[ \frac{1}{h_3(p)} \frac{\varphi(p)^2 g_3(p)^2 h_1(p)^2}{\tau_k(p)^2} \left( 1 + \frac{1}{p - 2} \right)^{-2} = h_3(p) \]
for \( p \mid Q(a^m) \) and also
\[ \frac{1}{h_3(p)} \frac{\varphi(p)^2 g_3(p)^2 h_1(p)^2}{\tau_k(p)^2} \left( 1 - \frac{k}{(p - k - 1)(p - 2)} \right)^{-2} = h_3(p) \]
Almost-primes represented by $p + a^m$. Thus

$$Y^2 G_3 = \mathcal{G}_1^{(m)} \sum_{u < \sqrt{D}} h_3(u) \sum_{v < \sqrt{D/u}} h_3(v) \left( \log \frac{\sqrt{D}}{uv} \right)^{2\ell} \\
\times \left( 1 + O \left( \frac{\varphi(uv)^2 (\log \log N)^4}{f(uv)^2 \log(\sqrt{D}/uv)} \right) \right).$$

Making use of Lemma 7.1 we obtain

$$Y^2 G_3 = \mathcal{G}_1^{(m)} \sum_{u < \sqrt{D}} h_3(u) \left[ \mathcal{G}_2^{(m)} \frac{(2\ell)!}{(k + 2\ell)!} \left( \log \frac{\sqrt{D}}{u} \right)^{k+2\ell} \\
+ O \left( \frac{\varphi(u)^2}{f(u)^2} \log \frac{\sqrt{D}}{u} \right)^{k+2\ell-1} (\log \log N)^4 \right],$$

where

$$(7.9) \quad \mathcal{G}_2^{(m)} = \prod_{p \nmid Q(a^m)} \left( 1 - \frac{1}{p} \right)^k \left( 1 + \frac{k}{p - k - 2} \right) \prod_{p | Q(a^m)} \left( 1 - \frac{1}{p} \right)^k.$$

Therefore, Lemma 7.2 implies

$$(7.10) \quad Y^2 G_3 = \frac{(2\ell)! \mathcal{G}_1^{(m)} \mathcal{G}_2^{(m)}}{(k + 2\ell)!} \prod_{p \nmid \Delta} \left( 1 + \frac{k - 1}{p - k - 1} \right) \cdot (\log \sqrt{D})^{k+2\ell}$$

$$\times \left( 1 + O \left( \frac{(\log \log N)^k}{\log D} \right) \right)$$

$$+ O \left( \mathcal{G}_1^{(m)} \prod_{p \nmid \Delta} \left( 1 + \frac{k - 1}{p - k - 1} \frac{(p - 1)^2}{(p - 2)^2} \right) \cdot (\log \sqrt{D})^{k+2\ell-1} \right).$$
The error term can be disposed in the following way:

$$S_1^{(m)} \prod_{p\nmid \Delta \atop p
mid Q(a^m)} \left(1 + \frac{k-1}{p-k-1} \frac{(p-1)^2}{(p-2)^2} \right) \ll \prod_{p\nmid \Delta \atop p
mid Q(a^m)} \left(\frac{p-1}{p-2}\right)^2 \left(1 + \frac{k-1}{p-k-1} \frac{(p-1)^2}{(p-2)^2} \right),$$

the product over primes $p \leq \log N$ is $O((\log \log N)^{k+1})$, while for $p > \log N$ we have

$$\left(\frac{p-1}{p-2}\right)^2 \left(1 + \frac{k-1}{p-k-1} \frac{(p-1)^2}{(p-2)^2} \right) \leq 1 + \frac{2k}{\log N},$$

therefore the corresponding product is

$$\ll \left(1 + \frac{2k}{\log N}\right)^{\omega(Q(a^m))} \ll \left(1 + \frac{2k}{\log N}\right)^{\frac{k \log N}{\log \log N}} = 1 + O\left(\frac{1}{\log \log N}\right),$$

since $Q(a^m) \leq N^{\frac{3}{2}}$. Whence the last $O$-term in (7.10) is $O((\log D)^{k+2\ell-1}(\log \log N)^{k+1})$. Analogously, If we denote by

$$S^{(m)} = \frac{S_1^{(m)} S_2^{(m)}}{S(\Delta)} \prod_{p\nmid \Delta \atop p
mid Q(a^m)} \left(1 + \frac{k-1}{p-k-1} \right),$$

then it is easy to show that

$$S^{(m)} = \prod_{p\nmid \Delta} \left(1 - \frac{k}{(p-k-1)(p-2)}\right) \prod_{p\nmid \Delta} \left(1 + \frac{1}{p-k-2}\right).$$

Therefore,

$$S_1^{(m)} S_2^{(m)} \prod_{p\nmid \Delta \atop p
mid Q(a^m)} \left(1 + \frac{k-1}{p-k-1}\right) \ll \log \log N.$$

Hence the total error in (7.10) is $O((\log D)^{k+2\ell-1}(\log \log N)^{k+1})$. 
Almost-primes represented by \( p + a^m \)

Recalling the asymptotic formula of \( Y \) and \( G_1 \) in (6.9) and (6.11) respectively, we can deduce from (7.10) that

\[
G_3 = G^{(m)}_1 + O\left( \frac{G_1 (\log \log N)^{k+1}}{\log D} \right).
\]  

Now we come to the proof of Theorem 1.1. Inserting (6.15) and (7.12) into (5.9), we have

\[
S_3 - S_4 \geq (1 + O(e^{-s})) \frac{N \gamma(H) V(z) G_1}{2} \left[ k(2\ell + 1) \mathcal{G}'(\Delta) \log D \right] \left( 1 + O\left( \frac{1}{\log D} \right) \right)
\]
\[
+ \frac{\Delta}{\varphi(\Delta)} \sum_{\substack{a \in M \setminus \mathcal{H}}} G^{(m)}_1 - \log N + O\left( (\log \log N)^{k+1} \right) + O\left( \frac{N}{(\log N)^{A+1}} \right)
\]
\[
\geq (1 + O(e^{-s})) \frac{N \gamma(H) V(z) G_1}{2} \.log N \left[ \frac{k(2\ell + 1) \mathcal{G}'(\Delta)}{2(\ell + 1)(k + 2\ell + 1)} (1 - 4\varepsilon) \right]
\]
\[
+ \frac{1}{2 \log a} \prod_{\substack{p|\Delta}} \left( 1 - \frac{k}{(p - k - 1)(p - 2)} \right) - 1 + O\left( \frac{(\log \log N)^{k+1}}{\log N} \right)
\]
\[
+ O\left( \frac{N}{(\log N)^{A+1}} \right).
\]

Combining with (2.5) and (4.5) we get

\[
S(\mathcal{A}, \mathcal{P}, z) \geq (1 + O(e^{-s})) \frac{N \gamma(H) V(z) G_1}{2} \log N \left[ \frac{k(2\ell + 1) \mathcal{G}'(\Delta)}{2(\ell + 1)(k + 2\ell + 1)} (1 - 4\varepsilon) \right]
\]
\[
+ \frac{1}{2 \log a} \prod_{\substack{p|\Delta}} \left( 1 - \frac{k}{(p - k - 1)(p - 2)} \right) - 1 + O\left( \frac{(\log \log N)^{k+1}}{\log N} \right)
\]
\[
+ O\left( \frac{N}{(\log N)^{A+1}} \right).
\]
Therefore, \( S(\mathcal{A}, \mathcal{P}, z) \) has a positive lower bound provided that

\[
\frac{k(2\ell+1)\mathcal{G}'(\Delta)}{2(\ell+1)(k+2\ell+1)}(1-4\varepsilon) + \frac{1}{2\log a} \prod_{p|\Delta} \left(1 - \frac{k}{(p-k-1)(p-2)}\right)^{-1} > 0,
\]

we verify this in the following way.

Firstly, (6.14) implies that

\[
\mathcal{G}'(\Delta) = \prod_{p|\Delta} \left(1 - \frac{3}{p(p+2)}\right) \geq \prod_{n>k+1} \left(1 - \frac{3}{n(n+2)}\right) = \frac{k}{k+3}.
\]

Secondly, we have

\[
\prod_{p|\Delta} \left(1 - \frac{k}{(p-k-1)(p-2)}\right) \geq \gamma_k,
\]

where

\[
\gamma_k = \prod_{p>k+2} \left(1 - \frac{k}{(p-k-1)(p-2)}\right) = \prod_{p>k+2} \left(1 + \frac{k}{(p-k-2)(p-1)}\right)^{-1}.
\]

We can prove that there are infinitely many \( k \) such that \( \gamma_k \) has absolute lower-bound by studying the mean value.

**Lemma 7.3.** It holds for any \( K \geq 1 \) that

\[
\frac{1}{K} \sum_{K<k<2K} \frac{\log 1}{\gamma_k} \ll 1,
\]

where the implied constant is absolute.

**Proof.** It is sufficient to prove that

\[
\frac{1}{K} \sum_{K<k<2K} \sum_{k+2<p<4K} \frac{k}{(p-k-2)(p-1)} \ll 1.
\]
Almost-primes represented by \( p + a^m \)

The left hand side is equal to

\[
\frac{1}{K} \sum_{K+2 < p \leq 4K} \frac{1}{p-1} \sum_{K < k \leq \min(2K, p-3)} \frac{k}{p-k-2}
\]

\[
= \frac{1}{K} \left( \sum_{K+2 < p \leq 2K+3} \frac{1}{p-1} \sum_{K < k \leq p-3} \frac{k}{p-k-2} \right.
\]

\[
+ \sum_{2K+3 < p \leq 4K} \frac{1}{p-1} \sum_{K < k \leq 2K} \frac{k}{p-k-2} \left. \right) \]

\[
= \frac{1}{K} (\mathcal{K}_1 + \mathcal{K}_2)
\]

say, where

\[
\mathcal{K}_1 \ll \sum_{K+2 < p \leq 2K+3} \sum_{K < k \leq p-3} \frac{1}{p-k-2} = \sum_{K+2 < p \leq 2K+3} \sum_{k < p-K-2} \frac{1}{k}
\]

\[
\ll \sum_{K+2 < p \leq 2K+3} \log p \ll K.
\]

On the other hand,

\[
\mathcal{K}_2 \ll \sum_{2K+3 < p \leq 4K} \sum_{K < k \leq 2K} \frac{1}{p-k-2} = \sum_{2K+3 < p \leq 4K} \left( \log \frac{p-K-2}{p-2K-2} + O(1) \right)
\]

\[
= - \sum_{2K+3 < p \leq 4K} \log \left( 1 - \frac{K}{p-K-2} \right) + O(K)
\]

\[
\ll \sum_{2K+3 < p \leq 4K} \frac{K}{p-K-2} + K \ll K.
\]

The desired result is obtained.

It follows from Lemma 7.3 that \( \gamma_k \) is bounded below by a positive absolute constant for some even number \( k \) in any dyadic segment. Choosing such a \( k \), sufficiently large in terms of \( \varepsilon \) and \( a \), and choosing \( \ell = [\sqrt{k}/2] \), we find that the left hand side of (7.14) is positive. This completes the proof of Theorem 1.1.

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