Reducing Urban Traffic Congestion Due To Localized Routing Decisions

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Balancing traffic flow by influencing drivers’ route choices to alleviate congestion is becoming increasingly more appealing in urban traffic planning. Here, we introduce a discrete dynamical model comprising users who make their own routing choices on the basis of local information and those who consider routing advice based on localized inducement. We identify the formation of traffic patterns, develop a scalable optimization method for identifying control values used for user guidance, and test the effectiveness of these measures on synthetic and real-world road networks.

Many of world’s major cities are increasingly grid-locked with a staggering estimated annual cost of $166B in the US alone [1]. The relentless urban population growth has created exorbitant traffic demands, which leads to recurring large-scale traffic jams [2–4]. Since it is expensive to satisfy the demand exclusively through further investment in infrastructure, there is a growing interest in optimizing transportation systems within the existing infrastructure [3, 7]. Modern information technologies can potentially offer effective solutions to these problems, such as ride-sharing through smart phones [8], congestion-aware routing schemes [9], and the use of autonomous vehicles [10, 11]. The deployment of smart devices is already impacting on the existing transportation networks and lead to a paradigm shift in traffic planning and management. However, not all these changes are for the better. Most navigation apps have been designed typically to minimize an individual driver’s travel time irrespective of street capacity along the route, safety or the route choices of other drivers; in many times this results in traffic chaos [12]. It is therefore important to understand the potential and limitations of these technologies and develop algorithmic tools for their effective use.

Detailed microscopic modeling of multi-agent systems characterizing the paths of individual users, such as cellular automata-based simulations [13], are available to model the traffic systems but usually require considerable computational power; it is also generally difficult to gain insight due to the overwhelming level of details. On the other hand, models based on traffic flow that coarse-grain the behaviors of individual users but maintain correlations at the network level, are simplistic but amenable to analysis. Link-based methods have been developed along this line, mainly for static assignment, selfish routing or centralized optimization [14, 15]. Such methods have also been extended to the more difficult dynamic traffic assignment problem [16, 17]. In reality, drivers do not have the full information of the traffic flow and unbounded computational capacity to determine the rational choices of routes [18, 19]. Instead, they typically adjust their route choice, especially in urban settings, en route according to the traffic conditions in downstream junctions, which has been investigated in some dynamic traffic assignment problems [20, 22].

In this study, we take into account such behavioral aspects and propose a dynamical model which includes both impulsive users who make their own decisions en route, and advice-susceptible users who follow the suggestions given by smart devices. The advice-susceptible users are incentivized to follow centrally-optimized routing suggestions that benefit traffic globally. Such strategy may be adopted in the future to alleviate traffic congestion. In fact, Electronic Road Pricing already operates successfully in Singapore [23], and recently the Israeli Ministry of Transportation has launched a pilot program to motivate drivers into driving in non-rush hours and carpooling through monetary incentives [24]. Our computational model offers complementary insights in support of such strategies. We focus on scenarios where commuters travel towards the city center at peak hours, during which they typically experience severe traffic congestion. We analyze the characteristics of emerging traffic patterns, develop an algorithm to determine the optimal incentive and investigate their impact on traffic congestion.

We model the urban road system as a network, where intersections are mapped to nodes and roads between them to edges (or links). We consider a scenario where drivers travel towards a universal destination $D$, which is relevant in the morning rush hour when a large number of people commute to the city center. The network is depicted as an undirected graph $G(V, E)$ of $N$ nodes, where each node $i \in V$ is connected to $k_i$ neighbors denoted by $\partial_i$, and each edge $(i, j) \in E$ represents two lanes $i \rightarrow j$ and $j \rightarrow i$, accommodating non-interacting traffic from $i$ to $j$ and $j$ to $i$, respectively. We denote the set of all lanes as $E$.

Assume that drivers can be classified into two groups according to whether they make their own route choices or follow the advice from navigation devices. In the former, a user makes routing decisions dynamically, based on her estimated time to destination $D$. Upon arriving to intersection $i$ at time $t$, the user faces a choice between $k_i$ possible roads $\{i \rightarrow j\}_{j=1}^{k_i}$. The user first estimates ($i$) the time it takes to travel through edge $i \rightarrow j$ as $g(\rho_{ij}^{t})$
where $\rho_{ij}^t$ is the number of users occupying edge $i \rightarrow j$ (i.e., traffic volume) at that time and $g(\rho_{ij}^t)$ is determined by the Greenshields model \cite{25}; see also the Supplemental Material \cite{26}; (ii) the remaining time $d_j$ needed to travel to $D$ from node $j$, which can be taken as shortest free traveling time or be based on past experience of congestion level. Afterwards, their route choices are made according to the probability

$$p_{ij}^{g,t}(\rho^t) = \frac{e^{-\beta g(\rho_{ij}^t)+d_j}}{\sum_{k \in \partial_i} e^{-\beta g(\rho_{ik}^t)+d_k}}, \quad (1)$$

where $\beta$ is a parameter determining the randomness of the decision making process. As shown in \cite{26}, the dependence on this parameter for $\beta > 1$ is relatively weak, and hence we choose $\beta = 1$ in what follows. Note that we do not limit the users from turning back. The awareness of congestion can be extended to road segments that are more distant, at the cost of higher computational complexity. Here, we focus on the one-step congestion-aware model.

In the latter group, the user follows the navigation advice aimed at improving the traffic efficiency. Their route choices at junction $i$ at time $t$ are determined by the localized probability

$$p_{ij}^{w,t}(w^t) = e^{-w_{ij}^t} / \sum_{k \in \partial_i} e^{-w_{ik}^t}, \quad (2)$$

where the weight variables $\{w_{ij}^t\}$ are optimized centrally. With the assumption that the fraction of users $n$ who are susceptible to the routing advice are distributed evenly in the network, on average the vehicle flow arriving at node $i$ at time $t$ will be diverted to the adjacent edges $\{i \rightarrow j\}_{j=1}^{k_i}$ according to the distribution

$$p_{ij}^t(\rho^t, w^t) = (1-n)p_{ij}^{g,t}(\rho^t) + np_{ij}^{w,t}(w^t). \quad (3)$$

At each time-step $t'$ a decision is made to enter edge $i \rightarrow j$, the user then spends some time $\tau_{ij}$ traveling on this edge with distribution $P(\tau_{ij})$, arriving to the end point at time $t = t' + \tau_{ij}$. The distribution of time spent can take several forms, including the typically used discrete Poisson distribution that we adopt here; see \cite{26} for details. The arrival probability depends on the traffic volume $\rho_{ij}^t$ at the time of entrance $t'$, i.e., $P(t-t'|\rho_{ij}^t)$, which is a realistic and important factor in traffic modeling. To express the dynamics we introduce the time-dependent flux $f_{ij}^t$ arriving to the end point $j$ of the edge $i \rightarrow j$ at time $t$. Assuming users enter the road system at time $t = 0$ with initial volume $\rho_i^0$, the dynamics of the traffic volume and flux on edge $i \rightarrow j$ ($i \neq D$) are governed by the discrete forward dynamics

$$\rho_{ij}^t = \rho_{ij}^{t-1} + \sum_{k \in \partial_i, k \neq D} f_{ki}^{t-1} + (\rho_{ij}^{t-1} - f_{ij}^{t-1}), \quad (4)$$

$$f_{ij}^t = \sum_{t'=1}^t [\rho_{ij}^{t'} - (\rho_{ij}^{t'-1} - f_{ij}^{t'-1})] P(t-t'|\rho_{ij}^0)$$

$$+ \rho_{ij}^0 P(t|\rho_{ij}^0). \quad (5)$$

Eq. (4) describes the traffic volume at edge $i \rightarrow j$ at time $t$; it is composed of the newly joined users who selected this junction at node $i$ at time $t-1$, and the users who were already traveling through this edge but have not yet reached the end point $j$. Eq. (5) states that the vehicles flux at the edge $i \rightarrow j$ end point at time $t$ comprises the fraction of traffic volume $\rho_{ij}^{t'} - (\rho_{ij}^{t'-1} - f_{ij}^{t'-1})$ entering the road segment at a different time $t'$, who have completed the trip on this road segment within a duration $t-t'$ as dictated by the probability $P(t-t'|\rho_{ij}^0)$, which is defined such that the mean traveling time follows the Greenshields models \cite{25,26}. The resulting model bears similarity to certain link-based models of dynamic traffic assignment \cite{17}. We assume that no vehicles leave the destination node, i.e., the destination $D$ is an absorbing boundary which satisfies $\rho_{Dj}^t = f_{Dj}^t = 0, \forall j \in \partial D$.

The model is simulated for a fixed time period $T$. To evaluate the efficiency of the system, we measure the average time of reaching the destination $D$ ahead of $T$

$$O = \frac{1}{\sum_{e \in E} \rho_e^T} \sum_{t=1}^T \sum_{j \in \partial D} f_{ij}^t \times (T-t), \quad (6)$$

and use it as the main measure of the transportation system’s performance. Other measures can be easily accommodated within the same framework but will not be considered here to keep the presentation simple.

We perform numerical experiments on both generated and realistic road networks. The generated networks are constructed by randomly rewiring a planar square lattice with shortcut edges, which is motivated by the recent observation that high-speed urban roads constitute effective long-range connections and render the system to exhibit small-world characteristics \cite{27}. The realistic road network used is extracted and adapted from the OpenStreetMap dataset \cite{28}, and converted to network formats by using the GIS F2E software \cite{29}. Two examples of the networks considered are shown in Fig. 1. Details of the network generation are described in \cite{26}.

**Model characterization without control**—The initial traffic volume is assigned independently and identically at random as $\rho_{ij}^0$ users departing from each node $i$, which can be proportional to the population at that node; users rest on the node’s neighboring edges $\{i \rightarrow j|j \in \partial i\}$ with equal probability $\rho_{ij}^0 = \rho_i^0/k_i$, constituting the initial traffic volume $\{\rho_{ij}^0\}$. After entering the system at time $t = 0$, all users drive towards the center node $D$. 
according to the instantaneous decision making rule of Eq. (4), i.e., $n = 0$ in Eq. (5). Clearly, the same framework can accommodate users entering the road network at any time, but here we will only consider the case of a $t = 0$ start. It leads to macroscopic dynamical traffic patterns governed by Eqs. (4) and (5). We define the traffic load level as

$$L = \frac{\sum_{e \in E} \rho_e^0}{\sum_{e \in E} \rho_e^{\text{lan}}}.$$  \hspace{1cm} (7)

The load level $L$ is similar to the demand-to-supply ratio introduced in [1], which is suggested to be a good predictor of the congestion level.

As the first consideration, we study the traffic patterns in the model in the absence of routing advice, $n = 0$. The movement of traffic mass can be visualized by contrasting the traffic volume $\rho_e$ to the distance-to-destination of each lane. To this end, we define the distance $\text{dist}(e, D)$ of lane $e = (i \rightarrow j)$ to destination $D$ as the shortest free traveling time from the midpoint of the lane to destination $\text{dist}(e, D) = d_{ij} + \text{free} / 2$. Fig. 2 demonstrates how the average traffic volumes $\langle\rho_e^t\rangle$ at specific distances change over time under different load levels. At the low load regime $L = 0.1$, shown in Fig. 2(a), the vehicles are able to move fairly quickly towards the destination $D$ from the initial positions at $t = 0$ to $t = 25$. The roads near the city center become congested, leading to a slow clearance of traffic from $t = 50$ to $t = 100$, which indicates that the limited connectivity of city center is a bottleneck of the traffic system. At high loads $L = 0.4$, shown in Fig. 2(b), the traffic volumes at large distance-to-destination decrease, while those at short distances increase over time, but at a much slower rate compared to the case of a small load $L = 0.1$. It indicates that the excessive demand creates congestion in the transportation network and leads to an increase in travel time. More details of system efficiency as a function of load are depicted in [20].

The simplest measure to reduce congestion is to improve the infrastructure, e.g., by building new roads or by increasing the capacity of existing ones. In particular, increasing the number of possible routes to the city center/destination node can significantly enhance the traffic clearance rate; yet it is rarely possible to do so due to the limited land availability especially near the city center. To examine the effect of network extension, we perform experiments by adding links from sites with the largest populations to the nearest neighbors of the destination node. From the relative frequency of the fractional change of objective function shown in Fig. 3, it is surprising to observe that the majority of link additions lead to a *decrease in system performance*. It suggests that the introduction of shortcuts being attractive to users, create congestion in the shortcut edges and nearby areas. The phenomenon is reminiscent of Braess’s paradox in static routing game which states that extension of network may lead to increase of selfish drivers’ traveling time [30]. In our model, drivers have limited knowledge and are unaware of long-distance traffic condition, so that the myopic decisions makes the system more prone to congestion. If users are aware of more global information like in routing game scenarios, it is possible that they may adapt, in a repeated game scenario, to avoid the already congested shortcuts, such that the probability of performance decrease becomes smaller.

Nevertheless, there is a small likelihood that adding a new link would lead to a significant improvement of the objective function $O$, which can be up to 10% for $L = 0.1$ and 20% for $L = 0.5$. Such improvement is more commonly observed in higher loads, but in the majority of cases the improvement is marginal. In either case, it is crucial to select the correct short-cut to invest in, which becomes a difficult task when the demand profile is fluctuating.

Model characterization with control—While extending the existing network is costly, it becomes increasingly more appealing and cost-effective to influence route-
choice of drivers in order to reduce congestion. In this section, we examine the particular type of instantaneous advice in the form of Eq. (2), which is adapted such that the objective function $O$ of Eq. (6) is maximized. The resulting highly nonlinear optimization problem is of a non-convex type, with a landscape characterized by multiple local maxima.

To solve this difficult optimization problem, we adopt an optimal control framework [31, 32], whereby the dynamics, Eqs. (4-5), is enforced as constraints in the Lagrangian formulation. The optimality conditions lead to a set of coupled non-linear equations solved by forward-backward iterations. To suppress divergent behavior due to radical changes of control parameters [31, 33], we employ gradient-ascent in the updates of control parameters [20]. The results shown in Fig. 3 demonstrate that the optimization algorithm successfully improves the system performance, as indicated by the fractional increase $\Delta O = O^{\text{optimal}} - O^{\text{original}}$ divided by the objective function before adding a link $O^{\text{original}}$. The results are aggregated from 10 networks generated from the small world network model of size $21 \times 21$ with rewiring probability $p_c = 0.05$. The parameters are $T = 100, n = 0$. One end of the new link is randomly chosen from the top 5 sites with the highest population, while the other end is randomly chosen from the nearest neighbors of the destination node. (a) Load level $L = 0.1$. (b) Load level $L = 0.5$.

Figure 3. Relative frequency of fractional change of the objective function $O$ after adding a link to the existing network, defined as the performance change $\Delta O = O^{\text{add-link}} - O^{\text{origin}}$ divided by the objective function before adding a link $O^{\text{origin}}$. The results are aggregated from 10 networks generated from the small world network model of size $21 \times 21$ with rewiring probability $p_c = 0.05$. The parameters are $T = 100, n = 0$. One end of the new link is randomly chosen from the top 5 sites with the highest population, while the other end is randomly chosen from the nearest neighbors of the destination node. (a) Load level $L = 0.1$. (b) Load level $L = 0.5$.

The Birmingham road network (BHM) and a small world network (SW) of size $21 \times 21$ are considered. (b) Time evolution of the fraction of traffic volume $\sum_e \rho^e / \sum_e \rho^e$ remaining on the Birmingham road network; at time $t = 0$, the system load is $L = 0.1$.

Figure 4. (a) Fractional change in objective function $O$ (defined as the performance change $\Delta O = O^{\text{optimal}} - O^{\text{original}}$) divided by the objective function without advice-susceptible users $O^{\text{original}}$) as a function of the fraction of advice-susceptible users $n$. The Birmingham road network (BHM) and a small world network (SW) of size $21 \times 21$ are considered. (b) Time evolution of the fraction of traffic volume $\sum_e \rho^e / \sum_e \rho^e$ remaining on the Birmingham road network; at time $t = 0$, the system load is $L = 0.1$.

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In Fig. 4, the evolution of the fraction of traffic $\sum_e \rho^e / \sum_e \rho^e$ remaining on the Birmingham road network at a given time as a function of the guided-users fraction $n$. One can observe a faster rate of traffic decrease when $n$ increases from 0 to 0.8, suggesting more users can reach the destination within the same time period with the increase in the number of advice-susceptible users.

Through modeling the dynamical evolution of transportation networks comprising users who make routing decisions based on local information, we unveil the traffic patterns and system performance under different levels of demand. A simple measure to meet excess demand is by extending the network, e.g., by adding new roads or increasing the capacity of existing ones. However, we demonstrate that the newly added shortcut links attract many users, resulting in increased congestion and a degradation in the system performance, which is reminiscent of the Braess’s paradox in static routing game. It highlights the importance of the selection of most beneficial roads to add, depending on the network structure and demand profile. This is a problem on its own right that could be addressed by our model in future studies, especially when taking into account the load volatility.

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On the other hand, balancing the traffic flow by influencing the route choices of users, offers a less costly and more flexible solution. This is possible through a current level of proliferation of routing devices. Our experiments on macroscopic traffic flow optimization by giving instantaneous and localized routing advice demonstrates its potential for improvement in system performance. The optimization process with first order gradient ascent update of control variables within optimal control is scalable, but may be trapped in local optima due to the problem non-linearity; it would be interesting to see if advanced methods from non-convex optimization could provide a trade-off between optimality and computational complexity. Moreover, the gradient based method may offer advantages in the online setting; when
there are sudden changes of local traffic conditions, it is sufficient to update a few steps based on the original result to quickly obtain an approximate solution. It may also become worthwhile to introduce measures in order to redistribute demand over time in order to reduce the traffic load at certain times. One way to achieve this is to shift the departure time of some users, which can be integrated into our optimization framework. These extensions can be tested at a low computational cost using our model and optimization method without the need for expensive large scale agent-based simulations.

Other possible generalizations include the introduction of spill-back mechanism, the integration of more non-local traffic condition information and cases of multiple destinations. Map data copyrighted OpenStreetMap contributors and available from https://www.openstreetmap.org. BL and DS acknowledge support from the Leverhulme Trust (RPG-2018-092), European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 835913. DS acknowledges support from the EPSRC programme grant TRANSNET (EP/R035342/1). AYL acknowledges support from the Laboratory Directed Research and Development program of Los Alamos National Laboratory under project numbers 20190059DR and 20200121ER.

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Supplemental Material

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I. THE MODEL

In this section, we provide additional details of the dynamical model proposed in the main text. Figure S1 summarizes the decision making rule Eq. (1) and dynamical rules Eq. (4)-Eq. (5) defined in the main text.

In the proposed dynamical model, the traffic volume \( \rho_{ij}^t \) is not explicitly limited by an upper bound to obey the road capacity constraint. Instead, a heavily-used road, say edge \( i \rightarrow j \), takes longer to travel through as reflected by the arrival probability \( P(t - t' | \rho_{ij}^t) \), resulting in a smaller probability of choosing it. This limits the traffic volume \( \rho_{ij}^t \) from growing indefinitely.

The form of the arrival probability \( P(t - t' | \rho_{ij}^t) \) reflects the vehicle flow on edge \( i \rightarrow j \), where a plausible choice is to match the corresponding mean traveling time \( E_{t \sim P}[t - t'] \) to the traveling time \( g(\rho_{ij}^t) \) predicted by a real-world traffic statistics. For a concrete example, we assume that the arrival time follows a Poisson distribution with the form of \( \text{Poi}(t - t' - t_{ij}^{\text{free}} | \lambda_{ij}^t (\rho_{ij}^t)) \), where \( t_{ij}^{\text{free}} \) is the free traveling time on edge \( i \rightarrow j \), and the parameter \( \lambda_{ij}^t \) of the Poisson distribution has the form \( \lambda_{ij}^t (\rho_{ij}^t) = g(\rho_{ij}^t) - t_{ij}^{\text{free}} \), such that the mean traveling time is \( E_{t \sim P}[t - t'] = g(\rho_{ij}^t) \). We further model the mean traveling time \( g(\rho) \) as a function of traffic volume by modifying the Greenshields model [1] (also see the section below for a brief description)

\[
g(\rho) = \begin{cases} 
  \frac{t_{ij}^{\text{free}}}{1 - \rho/\rho_{\text{jam}}}, & \rho < \rho_{\text{jam}} (1 - \epsilon), \\
  \frac{t_{ij}^{\text{free}}}{\epsilon}, & \rho \geq \rho_{\text{jam}} (1 - \epsilon).
\end{cases}
\]

where \( \rho_{\text{jam}} \) is the jam volume defined in Greenshields model, and \( \epsilon \) a cutoff parameter to allow for a slow-moving traffic flow even under the regime of severe traffic congestion.

Figure S1. (a) The decision making rule of the drivers who make their own route choice. When the drivers arrive to junction \( i \) at time \( t - 1 \), they assess the traffic volumes in nearest-neighbor down-stream roads \( \rho_{ij}^{t-1}, j \in \partial i \), based on which the corresponding travel are estimated as \( g(\rho_{ij}^{t-1}) \). Since they are not aware of the traffic conditions beyond the nearest neighbour roads, they estimate the remaining time needed to travel from node \( j \) to the destination \( D \) based on the free travel time, denoted as \( d_j \). Consequently, their probability for choosing route \( i \rightarrow j \) is \( p_{ij}^{t-1} = \exp(-\beta [g(\rho_{ij}^{t-1}) + d_j]) / Z_\beta \), where \( Z_\beta \) is a normalization factor. (b) Flow conservation in the discrete dynamics in Eq. (4). The traffic volume \( \rho_{ij}^t \) comprises (i) the existing users who have not left road \( i \rightarrow j \) at time \( t \) (black arrow); and (ii) newly joined users who selected this road at junction node \( i \) at time \( t - 1 \) (the orange arrow). Clearly, we also take account of users who left the road in the previous time step (green arrow). This illustrates the derivation of Eq. (4). (c) The existing traffic volume on edge \( i \rightarrow j \) can be decomposed into net traffic volume \( \Delta \rho_{ij}^t \) increase at different times \( t' \); the traffic condition at the time of their entrance (i.e. \( \rho_{ij}^{t'} \)) dictates the probability of travel time \( \tau \) needed to complete the journey on this road, denoted as \( P(\tau | \rho_{ij}^{t'}) \). This schematic helps explain the derivation of Eq. (5).
II. THE GREENSHIELDS MODEL

The traditional Greenshields model is characterized by a linear relation between traffic speed \( u \) and traffic density \( k \) (not to be confused with the connectivity \( k_i \) of node \( i \) in the main text) on a particular road section

\[
v = u_{\text{free}} \left( 1 - \frac{k}{k_{\text{jam}}} \right),
\]

(S1)

where \( u_{\text{free}} \) is the free flow speed and \( k_{\text{jam}} \) is the jam density. The traffic volume \( \rho \) is related to the traffic density \( k \) and road section length \( l \) through \( \rho = k \cdot l \), and similarly \( \rho_{\text{jam}} = k_{\text{jam}} \cdot l \). Therefore the traveling time on the road is

\[
t = \frac{l}{v} = \frac{l}{u_{\text{free}}} \left( 1 - \frac{\rho}{\rho_{\text{jam}}} \right),
\]

(S2)

where \( l/u_{\text{free}} \) is identified as the free traveling time \( t_{\text{free}} \).

III. SMALL-WORLD NETWORK

The small-world networks considered in the main text are generated according to the following procedure. We first generate a \( L_x \times L_y \) square lattice and identify the center node and its nearest neighbors as the city center, and define the city center cluster to be the destination node \( D \). We then randomly and independently disconnect each link with a fixed small probability \( p_r \) and re-connect one end of the link to a randomly chosen site of the network. For the \( 21 \times 21 \) square lattices considered in the main text, we used \( p_r = 0.05 \). Non-shortcut links were assumed to be of the same length and were assigned the same parameter values of \( t_{\text{free}} = 3 \) and \( \rho_{\text{jam}} = 16 \). For the shortcut links, we let \( t_{\text{free}} \) and \( \rho_{\text{jam}} \) be proportional to their lengths defined as the Euclidean distances between two sites; we also assume that users can drive on the shortcut edge with a higher speed than the non-shortcut edge since the shortcut edges represent high-speed urban roads in this model. These speed parameter choices do not seem to change the qualitative behavior we observe.

IV. BIRMINGHAM UK ROAD NETWORK

The Birmingham road network is extracted from the Open Street Map (OSM) datasets. For computational efficiency, we do not consider residential roads and keep only highways, primary roads and secondary roads and the corresponding ramps as classified by the OSM dataset type. They includes motorways, A roads and B roads in the UK road network. We neglect the M6 motorway in our analysis since it is rarely used in city commute within Birmingham, while its ramps can significantly complicate the network structure. The obtained OSM data comprises the polylines describing the geometric shapes of the roads and the corresponding attributes such as length \( l \), maximal speed \( v_{\text{max}} \) and number of lanes \( n_{\text{lane}} \).

We export these OSM data as in shapefile format and use the GIS F2E software to convert them into networks as described by edge lists and edge attributes. We then determine the free traveling time and jam volume of each edge as \( t_{\text{free}} = l/v_{\text{max}}, \rho_{\text{jam}} = l \cdot n_{\text{lane}} \). The resulting network has a large number of edges, many of which are of very short length. The nodes separating short-length edges are not considered decision-nodes of interest. Therefore, we apply some post-processing operations to the network, primarily by identifying the roundabouts and intersections of roads as decision-making nodes (DMNs), and combine the edges between two adjacent DMNs to a single edge.

When we combine two edges \((i,j)\) and \((j,k)\) to form a new edge \((i,k)\), the edge parameters are determined by \( t_{ik} = t_{ij} + t_{jk}, \rho_{ik} = \rho_{ij} + \rho_{jk} \). We define 20 seconds as one time step and round up the number of time steps needed to travel along edge \((i,j)\) to ensure \( t_{ij} \) is an integer.

V. OPTIMIZATION

The objective to be maximized is the system efficiency \( \mathcal{O} \) defined in Eq. (6) of the main text, while obeying the constraints representing the dynamics. In general it is difficult to deal with the non-linear equality constraints imposed by the forward dynamics using the non-linear programming framework. In this work, we adopt the optimal control
approach by introducing the Lagrangian

$$\mathcal{L} = \mathcal{O} + \sum_{t=1}^{T} \sum_{i \neq D} \sum_{j} A_{ij} \mu_{ij}^t \left[ - f_{ij}^t + \sum_{t' = 1}^{t} \left[ \rho_{ij}^{t'} - (\rho_{ij}^{t'-1} - f_{ij}^{t'-1}) \right] P(t-t' | \rho_{ij}^{t'}) + \rho_{ij}^0 P(t | \rho_{ij}^0) \right]$$

$$+ \sum_{t=1}^{T} \sum_{i \neq D} \sum_{j} A_{ij} \eta_{ij}^t \left[ - \rho_{ij}^t + \rho_{ij}^{t-1} (\rho_{ij}^{t-1} - f_{ij}^{t-1}) \sum_{k \in \partial_i, k \neq D} f_{ki}^{t-1} + (\rho_{ij}^{t-1} - f_{ij}^{t-1}) \right] \right], \quad (S3)$$

where $A_{ij} \in \{0, 1\}$ is the element of the adjacency matrix of the underlying graph, taking a value 1 if a link exists and 0 otherwise, and $\{\rho_{ij}^t, \eta_{ij}^t\}$ are the Lagrange multipliers enforcing the dynamics specified by Eqs. (4) and (5), respectively.

The optimal solution can be pursued by extremizing $\mathcal{L}$ with respect to the state variables $\{\rho_{ij}^t, f_{ij}^t\}$, the Lagrange multipliers $\{\rho_{ij}^t, \eta_{ij}^t\}$ and the control variables $\{w_{ij}^t\}$. The stationary point equations $\partial \mathcal{L} / \partial \eta_{ij}^t = 0$ and $\partial \mathcal{L} / \partial \mu_{ij}^t = 0$ return the equations of state, Eqs. (4) and (5) propagating forward in time, while $\partial \mathcal{L} / \partial \rho_{ij}^t = 0$ and $\partial \mathcal{L} / \partial f_{ij}^t = 0$ yield two backward propagation equations of the Lagrange multipliers, signaling the system’s deviation from optimum. Equation $\partial \mathcal{L} / \partial w_{ij}^t = 0$ dictates the optimal recommended weights $w_{ij}^t$ for the given objective function. The set of stationary point equations are solved by an iterative forward propagation of the state variables, followed by backward propagation of the Lagrange multipliers and an update of the control variables, which resembles the method of successive approximation in optimal control theory [2]. Similar techniques has been recently introduced for the impact maximization in spreading processes [3]. The control parameters $w_{ij}^t$ are gradually updated through gradient ascent as $w_{ij}^{t+1} = w_{ij}^t + s \partial \mathcal{L} / \partial w_{ij}^t$, with step size $s$ after a forward-backward propagation sweep, which will gradually drive the system towards a locally optimal state.

More specifically, the task is to maximize the average time ahead of $T$ reaching the destination with respect to $\{w_{ij}^t\}$

$$\max_{\mathbf{w}} \mathcal{O}(\mathbf{w}) = \frac{1}{\sum_{e \in \mathcal{E}} \rho_{ij}^D} \sum_{t=1}^{T} \sum_{j \in \partial D} f_{ij}^t(\mathbf{w}) \times (T - t), \quad (S4)$$

such that the following dynamical constraints are fulfilled,

$$\rho_{ij}^t = \rho_{ij}^{t-1} \sum_{k \in \partial_i, k \neq D} f_{ki}^{t-1} + (\rho_{ij}^{t-1} - f_{ij}^{t-1}), \quad \forall i \neq D \quad \quad (S5)$$

$$f_{ij}^t = \sum_{t'=1}^{t} \left[ \rho_{ij}^{t'} - (\rho_{ij}^{t'-1} - f_{ij}^{t'-1}) \right] Poi(t-t' | \rho_{ij}^{t'}) \mathbb{1}_{t \geq t_{free}^i}$$

$$+ \rho_{ij}^0 Poi(t-t_{free}^i | \rho_{ij}^0) \mathbb{1}_{t_{free}^i \geq t_{free}^i}, \quad \forall i \neq D \quad \quad (S6)$$

where $\mathbb{1}_{(\cdot)}$ is the indicator function. The dynamics satisfies the absorbing boundary conditions $\forall i j \in \mathcal{E}, i = D$

$$\rho_{ij}^t = 0, \quad f_{ij}^t = 0, \quad \forall t, \quad \forall i \neq D \quad \quad (S7)$$

and the initial conditions $\forall i j \in \mathcal{E}, i \neq D$

$$\rho_{ij}^0 = \tilde{\rho}_{ij}, \quad f_{ij}^0 = \rho_{ij}^0 Poi(0 | \lambda_{ij}^0 (\rho_{ij}^0)) \delta_{t_{free}^i, 0}, \quad \forall i \neq D \quad \quad (S8)$$

where $\tilde{\rho}_{ij}$ is the predefined initial traffic volume.

The Lagrangian to be extremized reads

$$\mathcal{L} = \frac{1}{\sum_{e \in \mathcal{E}} \rho_{ij}^D} \sum_{t=1}^{T} \sum_{j \in \partial D} f_{ij}^t \cdot (T - t) + \sum_{t=1}^{T} \sum_{i \neq D,j} A_{ij} \mu_{ij}^t \times \left[ - f_{ij}^t + \rho_{ij}^0 Poi(t-t_{free}^i | \rho_{ij}^0) \mathbb{1}_{t_{free}^i} + \sum_{t'=1}^{t_{free}^i} (\rho_{ij}^{t'} - \rho_{ij}^{t'-1} + f_{ij}^{t'-1}) Poi(t-t' | \rho_{ij}^{t'}) \right]$$

$$+ \sum_{t=1}^{T} \sum_{i \neq D,j} A_{ij} \eta_{ij}^t \left[ - \rho_{ij}^t + \rho_{ij}^{t-1} \sum_{k \in \partial_i, k \neq D} f_{ki}^{t-1} + (\rho_{ij}^{t-1} - f_{ij}^{t-1}) \right], \quad (S9)$$
The optimal solution satisfies the optimality condition $\partial L / \partial \{\mu_t^j, \eta_t^j, \rho_t^j, f_t^j, w_t^j\} = 0$. Variation of $L$ with respect to the Lagrange multipliers by differentiating $\partial L / \partial \mu_t^j = 0, \partial L / \partial \eta_t^j = 0$ returns the forward dynamical equations Eq. (S5) and Eq. (S6), while the variation with respect to the state variables by $\partial L / \partial f_{\tau mn} = 0$ and $\partial L / \partial \rho_{\tau mn} = 0$ yields two sets of dual equations $\forall 1 \leq \tau < T, mn \in \mathcal{E}, m \neq D$

$$\mu_{\tau mn} = \frac{1}{\sum_{e \in \mathcal{E}} \rho_e^\tau} (T - \tau) \mathbb{I}_{n=D}$$

$$+ \sum_{t=\tau+1}^{T} \mu_{t mn} \text{Poi}(t - \tau - 1 - t_{\text{free}}^\tau \mathbb{I}_{\tau < T})$$

$$+ \sum_{j \in \partial m} \eta_{\tau+1}^j p_{\tau mn}^j \mathbb{I}_{n \neq D} \mathbb{I}_{\tau < T} - \eta_{\tau mn}^+ \mathbb{I}_{\tau < T},$$

(S10)

$$\eta_{\tau mn} = \eta_{\tau mn}^- + \sum_{t=\tau+1}^{T} \mu_{t mn} \text{Poi}(t - \tau - 1 - t_{\text{free}}^\tau \mathbb{I}_{\tau < T})$$

$$- \sum_{t=\tau+1}^{T} \mu_{t mn} \text{Poi}(t - \tau - 1 - t_{\text{free}}^\tau \mathbb{I}_{\tau < T})$$

$$+ \sum_{t=\tau+1}^{T} \mu_{t mn}^+ (\rho_{ mn} - \rho_{mn}^\tau + f_{mn}^\tau)$$

$$\times \partial \text{Poi}(t - \tau - 1 - t_{\text{free}}^\tau \mathbb{I}_{\tau < T}) \frac{\partial \lambda_{\tau mn}^+}{\partial \rho_{mn}^\tau}$$

$$- (1 - n) \beta g'(p_{mn}) p_{mn}^\tau \eta_{\tau mn}^- + \sum_{j \in \partial m} p_{mn}^\tau \eta_{\tau mn}^j \sum_{k \in \partial m \backslash D} f_{km}^\tau.$$ 

Note that these are discrete dynamics that propagates backward in time, with the corresponding terminal condition

$$\mu_{\tau mn} = 0,$$  

(S12)

$$\eta_{\tau mn} = \mu_{\tau mn}^+ \text{Poi}(0) \lambda_{\tau mn}^T \delta_{t=0,0}$$

(S13)

Lastly, the variation with respect to the control parameter $w_{\tau mn}^\tau$ by $\partial L / \partial w_{\tau mn}^\tau = 0$ gives the optimality condition,

$$\frac{\partial L}{\partial w_{\tau mn}^\tau} = -n w_0 P_{\tau mn}^w w_{\tau mn}^+ \sum_{j \in \partial m} p_{mn}^\tau \eta_{\tau mn}^j \sum_{k \in \partial m \backslash D} f_{km}^\tau = 0.$$  

(S14)

In principle, the above optimal control problem can be solved by iterating the equations following the procedure outlined below, until a predefined accuracy or execution time is reached

1. Start from the initial condition of Eq. (S8), propagate the state variables forward according to Eq. (S8) and Eq. (S9).

2. Start from the terminal condition of Eq. (S12) and Eq. (S13), propagate the Lagrange multipliers dynamics backward according to Eq. (S10) and Eq. (S11).

3. Solve Eq. (S14) in order to update the control parameter $w_{\tau mn}^\tau$.

In practice, the update in Step 3 may result in radical changes of the control variables and lead to diverging behaviors of the algorithm [4]. We find that an adoption of a gradient ascent approach to update the control variable $w_{\tau mn}^\tau$ through

$$w_{\tau mn}^\tau \leftarrow w_{\tau mn}^\tau + s \frac{\partial L}{\partial w_{\tau mn}^\tau}$$

(S15)

is more efficient and provides a remedy to possible divergence issues. Together with Step 1 and Step 2, this update will gradually drive the system towards a locally optimal state. In practice, several different step sizes $s$ are used and different initial guesses of $\{w_{\tau mn}^\tau\}$ are given, among which the best solution is selected.
Figure S2. (a) Average arrival-time ahead of \( T \) reaching destination as a function of load \( L \). (b) Fraction of users reaching destination before \( T \) as a function of load. The experiments are performed on \( 21 \times 21 \) small world networks; parameters are \( T = 100, n = 0 \). For each data point with a specific network rewiring probability \( p_r \) and load \( L \), we obtained the average behavior of the quantities measured over 10 different network structures and 10 random initial traffic volumes.

VI. SYSTEM EFFICIENCY AS A FUNCTION OF LOAD

In Fig. S2, we quantify the level of congestion by contrasting the average arrival time to destination ahead of \( T \) defined in Eq. (6) and the fraction of users arriving at destination before \( T \), against the load level \( L \) on the same small world network ensembles. We observe that as load \( L \) increases, both quantities \( O \) and \( \frac{\sum_e (\rho^0_e - \rho^T_e)}{\sum_e \rho^0_e} \) decrease rapidly suggesting, unsurprisingly, that a higher usage of the traffic system is prone to lead to congestion in this model. The rapid decrease in objective function with increasing demands suggests an increase in travel time which grows faster than linear when more vehicles enter the system. Interestingly, as the rewiring probability \( p_r \) increases, both \( O \) and \( \frac{\sum_e (\rho^0_e - \rho^T_e)}{\sum_e \rho^0_e} \) drop, which indicates counter-intuitively, that additional shortcut edges contribute to a higher level of congestion. This can be understood by the fact that in the absence of routing advice, drivers are only aware of the local traffic conditions and tend to use the shortcut edges beyond the immediate neighboring junctions, which leads to high latency in the shortcut roads.

VII. EFFECT OF THE PARAMETER \( \beta \)

In this section, we examine the effect of the parameter \( \beta \) in the route choice model defined in the main text. We consider the scenario where the drivers make their own instantaneous decisions, i.e., \( n = 0 \). The relation between the objective function \( O \) and \( \beta \) is depicted in Fig. S3. For \( \beta \ll 1 \), the users behave like random walkers on the network; typically it takes a long time for them to hit the destination, as indicated by a small value of \( O \). The objective function \( O \) reaches the maximum at about \( \beta = 1 \) in both networks considered, and it deteriorates slightly with larger \( \beta \). As the system performance is not very sensitive to \( \beta \) when \( \beta \geq 1 \), we fix \( \beta = 1 \) in this study.

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Figure S3. $\mathcal{O}$ vs $\beta$. The other parameters are $T = 100$, $n = 0$, load level $L = 0.2$. The small world network is obtained by rewiring a $21 \times 21$ square lattice.