\( \eta - \eta' \)-Glueball mixing from photon-meson transition form factors and decay ratio

\[ D_s \rightarrow \eta l\nu / \eta' l\nu \]

V.V. Anisovich*
St. Petersburg Nuclear Physics Institute, Gatchina, 188350, Russia

D.V. Bugg
Queen Mary and Westfield College, Mile End Rd., London E1 4NS, UK

D.I. Melikhov†
Nuclear Physics Institute, Moscow State University, 119899, Moscow, Russia

V.A. Nikonov‡
St. Petersburg Nuclear Physics Institute, Gatchina, 188350, Russia

We have determined the \( \eta/\eta' \) mixing angle and the regions of the allowed admixture of the glueball component with \( J^{PC}=0^{++} \) in \( \eta \) and \( \eta' \), based on transition form factors \( \eta \rightarrow \gamma \gamma^* \) and \( \eta' \rightarrow \gamma \gamma^* \) at \( 0 \leq Q^2 \leq 20 \text{ GeV}^2/c^2 \), and the branching ratio \( D_s \rightarrow \eta l\nu / \eta' l\nu \). For \( \eta \) and \( \eta' \) wave functions, \( \eta = \cos \alpha (\cos \theta \bar{n} n - \sin \theta \bar{s} s) + \sin \alpha G \) and \( \eta' = \cos \alpha' (\sin \theta' \bar{n} n + \cos \theta' \bar{s} s) + \sin \alpha' G \) where \( \bar{n} n = (\bar{u} u + \bar{d} d) / \sqrt{2} \) and \( G \) means a glueball, a shape of the allowed region in the \( (\theta, \alpha, \alpha') \)-space is determined which is located inside the borders \( 0.56 \leq \sin \theta \leq 0.66 \), \( \sin^2 \alpha \leq 0.08 \), \( \sin^2 \alpha' \leq 0.12 \). This corresponds to \( \theta_F = -17.0^\circ \pm 2.6^\circ \).

The problem of a precise determination of the \( \eta \) and \( \eta' \) contents lies in the fact that these states have been considered for a long time as candidates for states with a significant admixture of glueball components. The actual presence of such mixing can be determined by analyzing extensive experimental information. The conventional way of determining the \( \eta/\eta' \) mixing angle from mass formulae suggests values in the range \(-22^\circ < \theta_F < -10^\circ \) (\( \theta_F \) is the \( \eta/\eta' \) mixing angle \( \theta \approx 54.74^\circ + \theta_F \), depending on whether linear or quadratic mass formulae are used). Experiments on meson transitions which involve \( \eta \) and \( \eta' \) open a possibility of an alternative determination of the pseudoscalar mixing angle from form factor data.

We consider two different processes for a determination of the mixing angles: (i) transition form factors \( \eta, \eta' \rightarrow \gamma \gamma^* \) at \( Q^2 \leq 20 \text{ GeV}^2/c^2 \), including \( \eta \) and \( \eta' \) partial decay widths into \( \gamma \gamma \) which correspond to \( Q^2 = 0 \), and (ii) semileptonic electroweak transitions \( D_s \rightarrow \eta l\nu / \eta' l\nu \).

I. TRANSITIONS \( \eta \rightarrow \gamma \gamma^* \) AND \( \eta' \rightarrow \gamma \gamma^* \)

The approach developed in Refs. [3] to the description of elastic and transition form factors lies in taking into account a truly Strong-QCD part and the \( O(\alpha_s) \) corrections to the form factors; thus we can expect to give a reasonable description in the region of intermediate momentum transfers. Our strategy is as follows. The accurate structure of the soft pion wave function is determined by fitting the data on elastic pion form factor [2]. Then, having determined the pion wave function, we extract the soft photon wave function from data on the \( \pi^0 \rightarrow \gamma \gamma^* \) transition form factor; we find it to be pretty similar to the pion soft wave function [3], quite in line with vector meson dominance. Furthermore, assuming universality of the wave functions of the ground-state pseudoscalar meson nonet, we have at hand the nonstrange component of the \( \eta \) and \( \eta' \) wave function. The main SU(3) breaking effect in the strange and nonstrange components of the meson wave functions comes from the mass difference of the nonstrange and strange quarks; hence we can also determine the wave function of the strange component. We thus calculate the \( \eta, \eta' \rightarrow \gamma \gamma^* \) transition form factors [3] and can analyze them by comparison with the data. All lengthy technical details for the calculation of the elastic and transition form factors have been given in Refs. [3] and will not be presented here. We only briefly outline the main steps.

We split the meson wave function into soft and hard components, \( \Psi_S^\pi \) and \( \Psi_H^\pi \) such that \( \Psi_S \) is large at \( s = (m^2 + k_F^2)/(x(1-x)) < s_0 \) where \( s \) is the \( q\bar{q} \) invariant energy squared and \( m \) is quark mass; \( \Psi_H \) prevails at \( s > s_0 \). The parameter \( s_0 \) is a boundary of the soft and hard regions and is expected to have the value of several \( \text{GeV}^2 \). We perform the splitting of the wave function into the soft and the hard components using a simple step-function ansatz

\[
\Psi_\pi = \Psi_S^\pi \theta(s_0 - s) + \Psi_H^\pi \theta(s - s_0) .
\]
The hard component $\Psi^H_\pi$ is represented as a convolution of the one-gluon exchange kernel $V^{\alpha_s}$ with $\Psi^S_\pi$:

$$\Psi^H_\pi = V^{\alpha_s} \otimes \Psi^S_\pi ,$$

thus one comes to the following expansion of the elastic pion form factor as a series in $\alpha_s$:

$$F_\pi = F_\pi^{SS} + 2 F_\pi^{SH} + O(\alpha^2_s) ,$$

where $F_\pi^{SS}$ is a truly Strong-QCD part of the form factor, and $F_\pi^{SH}$ is an $O(\alpha_s)$ term with one-gluon exchange. The first term dominates the pion form factor at small $Q^2$. The second term gives a minor contribution at small $Q^2$ but provides the leading $\alpha_s(Q^2)/Q^2$ behaviour of the elastic form factor at large $Q^2$.

The soft wave function $\Psi^S_\pi$ is responsible for the meson elastic form factor behaviour both at small and moderately large $Q^2$. The value of $s_0$ and the soft wave function are variational parameters of this approach.

Applying this strategy to the pion elastic form factor numerical analysis we have found $s_0 = 9$ GeV$^2$ provides the best description of the data. This value corresponds to an extended soft region and thus we relate a large portion of the pion form factor to the soft contribution. Although particular values of the soft and hard contributions to the form factor are model-dependent quantities we find that a good description of the form factor at small $Q^2$ yields a substantial soft contribution to the form factor at $Q^2 \simeq 10 - 20$ GeV$^2$. It is convenient to represent $\Psi^S_\pi$ in terms of the relative momentum $k^2 = (s - 4m^2)/4m$ GeV as follows:

$$\Psi^S_\pi = \psi_\pi(k^2) = \frac{g_\pi(k^2)}{k^2 + \kappa_0^2} ,$$

where $\kappa_0^2 = 0.1176$ GeV$^2$. The reconstructed wave function $\psi_\pi(k^2)$ is shown in Fig. 1a, while Figs. 1a,b give the elastic pion form factor calculated with this wave function (experimental data from Ref. [5]).

![FIG. 1. The reconstructed wave functions of pion (a) and photon (b).](image)

FIG. 1. The reconstructed wave functions of pion (a) and photon (b).

Similarly, for the description of the photon-pion transition form factor we introduced the photon $q\bar{q}$ wave function and split it into soft and hard components as follows

$$\Psi_\gamma = \Psi^S_\gamma \theta(s_0 - s) + \Psi^H_\gamma .$$

The corresponding expansion of the photon-meson transition form factor has the form (see Fig. 3):

$$F_{\gamma\pi} = F_{SS} + F_{SPt} + F_{SH(1)} + F_{SH(2)} ,$$

where $F_{SH(1)} + F_{SH(2)}$ are $O(\alpha_s)$ terms.

![FIG. 3. Diagrams relevant to the description of the meson-photon transition form factor at low and moderately high $Q^2$: $F_{SS}$-term (a), $F_{SPt}$-term (b), $F_{SH(1)}$, $F_{SH(2)}$-terms (c,d).](image)

FIG. 3. Diagrams relevant to the description of the meson-photon transition form factor at low and moderately high $Q^2$: $F_{SS}$-term (a), $F_{SPt}$-term (b), $F_{SH(1)}$, $F_{SH(2)}$-terms (c,d).
and the soft-hard terms dominate in the large-$Q^2$ region. Representing

$$\Psi^S_\gamma = \psi_\gamma(\vec{k}^2) = \frac{g_\gamma(\vec{k}^2)}{\vec{k}^2 + m^2}, \quad (8)$$

we determine the $\psi_\gamma(\vec{k}^2)$ (Fig. 1b) from data on the \(\pi^0 \to \gamma \gamma^*\) transition form factor \(\gamma\). Fig. 1b demonstrates the description of the data for the $\pi^0 \to \gamma \gamma^*$ transition; the partial width for the decay $\pi^0 \to \gamma \gamma$, $\Gamma_{\gamma \gamma} = 7.23$ eV, provides the normalization of $\gamma \gamma^*$ transition form factor and, as result, normalises the photon wave function $\psi_\gamma(\vec{k}^2)$.

To calculate $\eta \to \gamma \gamma^*$ and $\eta' \to \gamma \gamma^*$ transition form factors, we should take into account the mixing of non-strange quark component, $n\bar{n} = (\bar{u} u + d \bar{d})/\sqrt{2}$, the strange one, $s\bar{s}$, and the glueball component, $G$:

$$\Psi_\eta = \cos \alpha \left[ \cos \theta \psi_{n\bar{n}}(\vec{k}^2) - \sin \theta \psi_{s\bar{s}}(\vec{k}^2) \right] + \sin \alpha \psi_G,$$

$$\Psi_\eta' = \cos \alpha' \left[ \sin \theta' \psi_{n\bar{n}}(\vec{k}^2) + \cos \theta' \psi_{s\bar{s}}(\vec{k}^2) \right] + \sin \alpha' \psi_G. \quad (9)$$

The orthogonality condition reads

$$\cos \alpha \cos \alpha' \sin(\theta' - \theta) + \sin \alpha \sin \alpha' = 0, \quad (10)$$

determining the mixing angle $\theta'$. In the spirit of the quark model, the universality of soft wave functions of the $0^-$-neton is assumed; this implies $\psi_{n\bar{n}}(\vec{k}^2) = \psi_\pi(\vec{k}^2)$. For the $s\bar{s}$-component we take into account the SU(3)$_{\text{flavour}}$ breaking effect due to the strange/nonstrange quark mass difference

$$\psi_{s\bar{s}}(\vec{k}^2) = \frac{g_{s\bar{s}}(\vec{k}^2)}{\vec{k}^2 + m_s^2 + \Delta^2}, \quad (11)$$

where $\Delta^2 = m_s^2 - m^2$ with $m_s - m = 150$ MeV. The factor $N$ corresponds to the renormalization of $\psi_{s\bar{s}}(\vec{k}^2)$ after introducing $\Delta$.

Similarly, in the spirit of the quark model, we find for the nonstrange and strange components of the soft photon wave function:

$$\psi_{\gamma \to n\bar{n}}(\vec{k}^2) = \frac{g_{\gamma}(\vec{k}^2)}{\vec{k}^2 + m^2}, \quad \psi_{\gamma \to s\bar{s}}(\vec{k}^2) = \frac{g_{\gamma}(\vec{k}^2)}{\vec{k}^2 + m_s^2}, \quad (12)$$

Having fixed the $n\bar{n}$ and $s\bar{s}$ components of the meson and soft photon wave function, we calculate the $\eta \to \gamma \gamma^*$ and $\eta' \to \gamma \gamma^*$ transition form factors:

$$F_{\eta \to \gamma \gamma^*}(Q^2) = \cos \alpha \left[ \cos \theta \ F_{n\bar{n}}(Q^2) - \sin \theta \ F_{s\bar{s}}(Q^2) \right], \quad (13)$$

$$F_{\eta' \to \gamma \gamma^*}(Q^2) = \cos \alpha' \left[ \sin \theta' \ F_{n\bar{n}}(Q^2) + \cos \theta' \ F_{s\bar{s}}(Q^2) \right].$$

The partial width for the transition $\text{meson} \to \gamma \gamma^*$ is equal to

$$\Gamma_{\gamma \text{meson}}(Q^2) = \frac{\pi^2 m_{\text{meson}}^3}{4} F_{\gamma \text{meson} \to \gamma \gamma^*}(Q^2). \quad (14)$$

Fig. 4 presents the data for the partial widths $\Gamma_{\gamma \eta}(Q^2)$ and $\Gamma_{\gamma \eta'}(Q^2)$, and their fit with $\theta, \alpha$ and $\alpha'$ being parameters. The region of the allowed $(\theta, \alpha, \alpha')$ values on the 90% confidence level is shown in Fig. 5 (the region I).

\[ Q^2 \text{dependence for the ratios (a) } \Gamma_{\gamma \eta}(Q^2)/\Gamma_{\gamma \text{calc}}(Q^2) \text{ and (b) } \Gamma_{\gamma \eta'}(Q^2)/\Gamma_{\gamma \text{calc}}(Q^2) \text{ where } \Gamma_{\gamma \text{calc}}(Q^2) \text{ is the calculated quantity shown in Fig. 2. The fitted curves correspond to } \sin^2 \alpha = 0.02, \sin^2 \alpha' = 0.08, \text{ and } \sin \theta = 0.62. \]

\[ \text{II. SEMILEPTONIC DECAYS } D_s \to n\ell\nu/n'\ell'\nu \]

Exclusive semileptonic decays $D_s \to n\ell\nu/n'\ell'\nu$ probe the $s\bar{s}$ component of $\eta$ and $\eta'$ and thus provide a test of the mixing angle.

The decay rates are expressed through the mixing angles $\theta, \alpha, \alpha'$ and the form factor $f_+$. The semileptonic transition $D_s \to n\ell\nu/n'\ell'\nu$. The kinematically accessible $q^2$-regions in $\eta$ and $\eta'$ decays are $0 \leq q^2 \leq (M_{D_s} - M_n)^2$ and $0 \leq q^2 \leq (M_{D_s} - M_{n'})^2$.

The form factors for the semileptonic transition between pseudoscalar mesons $P(M_1) \to P(M_2)$ induced by the quark weak transition $c \to s$ are defined as follows:

\[ \text{FIG. 4. } Q^2 \text{ dependence for the ratios (a) } \Gamma_{\gamma \eta}(Q^2)/\Gamma_{\gamma \text{calc}}(Q^2) \text{ and (b) } \Gamma_{\gamma \eta'}(Q^2)/\Gamma_{\gamma \text{calc}}(Q^2) \text{ where } \Gamma_{\gamma \text{calc}}(Q^2) \text{ is the calculated quantity shown in Fig. 2. The fitted curves correspond to } \sin^2 \alpha = 0.02, \sin^2 \alpha' = 0.08, \text{ and } \sin \theta = 0.62. \]
\begin{equation}
\langle P(M_2, p_2) | \bar{s} \gamma_\mu c(0) | P(M_1, p_1) \rangle = f_+(q^2) P_\mu + f_-(q^2) q_\mu .
\end{equation}

(15)

FIG. 5. Allowed $(\theta, \alpha, \alpha')$-region: slices at different $R = \sin^2 \alpha/\sin^2 \alpha'$. The regions I and II are due to the $\eta/\eta' \rightarrow \gamma \gamma^*$ and $D_s \rightarrow \eta \nu/\eta' \nu\bar{\nu}$ constraints, correspondingly.

In considering the form factor of interest we use the dispersion relation formulation of the light-cone quark model \cite{10}: the form factors of the light-cone quark model of Ref. \cite{11} at spacelike momentum transfers are represented as double spectral representations over the invariant masses of the initial and final $q\bar{q}$ pairs, and form factors in the timelike region are obtained by analytical continuation in $q^2$. This procedure represents the form factors at $q^2 > 0$ through the light cone wave functions of the initial and final mesons and allows direct calculation of the decay form factors in the timelike region. It should be emphasized that we derive the analytical continuation in the region $q^2 \leq (m_2 - m_1)^2$. For the constituent quark masses used in the Isgur-Wise model \cite{12} $m_u = 0.33$ GeV, $m_s = 0.55$ GeV, and $m_c = 1.82$ GeV we adopt for considering the decay process, this allows a direct calculation of the form factor $D_s \rightarrow \eta'$ transitions in the whole kinematical decay region $0 \leq q^2 \leq (M_{D_s} - M_{\eta'})^2$, as $M_{D_s} - M_{\eta'} < m_c - m_s$. For the $D_s \rightarrow \eta$ transition this is not the case as $M_{D_s} - M_{\eta} > m_c - m_s$. For this mode we directly calculate the form factors in the region $0 \leq q^2 \leq (m_2 - m_1)^2$ and perform numerical extrapolation in $(m_c - m_s)^2 \leq q^2 \leq M_{D_s} - M_{\eta}$. All relevant technical details can be found in \cite{11}.

The transition form factors are expressed through the light-cone wave functions of the initial and final mesons. For $\eta$ and $\eta'$ we know such functions from the previous consideration. For the $D_s$ meson we assume the $D_s$ wave function to be approximated by a simple one-parameter exponential function $w(k) = \exp(-k^2/2\beta^2)$ and adopt the value $\beta = 0.56$ GeV from the Isgur-Wise model \cite{12}.

The results of calculating the form factors in the region $q^2 < (m_c - m_s)^2$ are fitted by the function

\begin{equation}
f_+(q^2) = f_+(0)/[1 - \alpha_1 q^2 + \alpha_2 q^4]
\end{equation}

with better than $0.5\%$ accuracy. We found $f_+(0) = 0.8$, $\alpha_1 = 0.192$ GeV$^{-2}$, and $\alpha_2 = 0.008$ GeV$^{-2}$.

For the form factor $f_+$ in the region $(m_c - m_s)^2 \leq q^2 \leq M_{D_s} - M_{\eta}$ this formula is used for numerical extrapolation. Numerical analysis shows the accuracy of this extrapolation procedure to be very high. The decay rates are calculated from the form factors via the formulae from \cite{13}. The calculated decay rates depend on the content of $\eta$ and $\eta'$ mesons, and their ratio with $|V_{cs}| = 0.975$ reads

\begin{equation}
\frac{\Gamma(D_s \rightarrow \eta' \nu\bar{\nu})}{\Gamma(D_s \rightarrow \eta \nu\bar{\nu})} = 0.28 \frac{\cos^2 \theta' \cos^2 \alpha'}{\sin^2 \theta \cos^2 \alpha}
\end{equation}

(17)

It should be pointed out that the obtained decay rates are calculated neglecting a nontrivial structure of the constituent quark transition form factor. This is a conventional but rather crude approximation: in particular, the quark transition form factor should contain a pole at $q^2 = M_{\text{res}}^2$ with $M_{\text{res}}$ the mass of a resonance with appropriate quantum numbers. It is not clear yet whether the transition form factor differs significantly from unity or not in the kinematical decay region. Anyway, the transition form factor is a rising function at $q^2 > 0$. This property yields an important consequence for the ratio of the decay rates $\Gamma(D_s \rightarrow \eta')/\Gamma(D_s \rightarrow \eta)$: as the phase space of the decay $D_s \rightarrow \eta$ is larger than that of the decay $D_s \rightarrow \eta'$, taking into account the (rising) constituent quark form factor will decrease the theoretical value of the ratio. We can try to estimate the effect caused by a nontrivial form factor taking a simple monopole $q^2$ dependence:

\begin{equation}
f_c(q^2) = \frac{1}{1 - q^2/M_{\text{res}}^2}
\end{equation}

(18)

with $M_{\text{res}} = M_{D_s} = 2.1$ GeV. This yields the following shift in the predicted ratio

\begin{equation}
\frac{\Gamma(D_s \rightarrow \eta' \nu\bar{\nu})}{\Gamma(D_s \rightarrow \eta \nu\bar{\nu})} = 0.23 \frac{\cos^2 \theta' \cos^2 \alpha'}{\sin^2 \theta \cos^2 \alpha}
\end{equation}

(19)

CLEO gives $\Gamma(D_s \rightarrow \eta')/\Gamma(D_s \rightarrow \eta) = 0.35 \pm 0.16$ \cite{14} implying a new restriction on the region of the allowed $(\theta, \alpha, \alpha')$: the region II on Fig. 5.

III. CONCLUSION

Based on the data for transitions $\eta/\eta' \rightarrow \gamma \gamma^*$ \cite{14} and $D_s \rightarrow \eta \nu/\eta' \nu\bar{\nu}$ \cite{14}, we have determined the region...
of the allowed mixing angles for $\eta_1, \eta_8$ and glueball component in $\eta$ and $\eta'$. The determined $\eta_1/\eta_8$ mixing angle $\theta_P = -17.0^\circ \pm 2.6^\circ$ is in between the values given by the linear and quadratic mass formulae. It looks reasonable to suppose the glueball component in $\eta'$ meson prevails that in $\eta$ meson, $R = \sin^2 \alpha/\sin^2 \alpha' < 1$: in this case the glueball component probabilities are in the borders $\sin^2 \alpha \leq 0.08$, $\sin^2 \alpha' \leq 0.12$. The significant values of the glueball components in $\eta$ and $\eta'$ are in an agreement with enlarged production of these mesons in the radiative $J/\psi$ decay \[1\]. We think that the decays $J/\psi \rightarrow \gamma \eta/\gamma \eta'$ can provide an additional important information about glueball components in $\eta$ and $\eta'$. However these decays need a special consideration that is beyond our present analysis.

ACKNOWLEDGEMENT

We are indebted to F.E. Close, S.S. Gershtein, and B.S. Zou for valuable remarks and discussions. VVA thanks F.E. Close for hospitality at Rutherford–Appleton Laboratory when carrying out this work. This investigation is supported by INTAS-RFBR grant 95-0267.

[1] L. Montanet et al. (PDG), Phys. Rev. D 50 (1994) 1443.
[2] V.V. Anisovich, D.I. Melikhov, and V.A. Nikonov, Phys. Rev. D 52 (1995) 5295.
[3] V.V. Anisovich, D.I. Melikhov, and V.A. Nikonov, "Photon-meson transition form factors $\gamma\pi^0$, $\gamma\eta$, and $\gamma\eta'$ at low and moderately high $Q^2$, \texttt{hep-ph/9607217} (1996), to be published in Phys. Rev. D 55 (1997).
[4] F.E. Close, private communication.
[5] C. Bebek et al., Phys. Rev. D 13 (1976) 25; Phys. Rev. D 17 (1978) 1693; S.R. Amendolia et al., Nucl. Phys. B 277 (1986) 168.
[6] CLEO Collaboration, V. Savinov, "A measurement of the form factors of light pseudoscalar mesons at large momentum transfers", \texttt{hep-ex/9507003} (1995), unpublished; CELLO Collaboration, H.J. Behrend et al., Z. Phys. C 49 (1991) 401; TCP/2$\gamma$ Collaboration, H. Aihara et al., Phys. Rev. Lett. 64 (1990) 172.
[7] Crystall Ball Collaboration, D. Williams et al., Phys. Rev. D 38 (1988) 1365.
[8] P. Butler et al., Phys. Rev. D 42 (1990) 1368.
[9] N. Isgur and M. Wise, Phys. Lett. B 232 (1989) 113.
[10] D.I. Melikhov, Phys. Rev. D 53 (1996) 2460.
[11] W. Jaus, Phys. Rev. D 41 (1990) 3394; Phys. Rev. D 53 (1996) 1349.
[12] D. Scora and N. Isgur, Phys. Rev. D 52 (1995) 2783.
[13] F. Gilman and R. Singleton, Phys. Rev. D 41 (1990) 142.
[14] CLEO Collaboration, G.Brandenburg et al., Phys. Rev. Lett. 75 (1995) 3804.