Comparing two approaches to the K-theory classification of D-branes

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Abstract

We consider the two main classification methods of D-brane charges via K-theory, in the case of vanishing $B$-field: the Gysin map approach and the one based on the Atiyah-Hirzebruch spectral sequence. Then, we find out an explicit link between these two approaches: the Gysin map provides a representative element of the equivalence class obtained via the spectral sequence. We also briefly discuss the case of rational coefficients, characterized by a complete equivalence between the two classification methods.

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1 Introduction

K-theory provides a good tool to classify D-brane charges (see [6], [21]). In the case of vanishing $B$-field, there are two main approaches in the literature. The first one consists of applying
the Gysin map to the gauge bundle of the D-brane, obtaining a K-theory class in space-time [18]. This approach is motivated by the Sen conjecture, stating that a generic configuration of branes and antibranes with gauge bundle is equivalent, via tachyon condensation, to a stack of coincident space-filling brane-antibranes equipped with an appropriate K-theory class [25]. The second approach consists of applying the Atiyah-Hirzebruch spectral sequence (AHSS, [3]) to the Poincaré dual of the homology cycle of the D-brane: such a sequence rules out some cycles affected by global world-sheet anomalies, e.g. Freed-Witten anomaly [9], and quotients out some cycles which are actually unstable, e.g. MMS-instantons [10]. We assume for simplicity that the space-time and the D-brane world-volumes are compact. For a given filtration of the space-time

\[ S^{23} = S^{10} \supset S^{9} \supset \cdots \supset S^{0}, \]

the second step of AHSS is the cohomology of \( S^{10} \), i.e.

\[ E_2^{q,0}(S) = H^q(S, \mathbb{Z}), \]

while the last step of AHSS is given by:

\[ E_{\infty}^{q,0}(S) = \ker \left( K^q(S) \rightarrow K^q(S^{q-1}) \right) / \ker \left( K^q(S) \rightarrow K^q(S^q) \right). \]

Hence, given a D-brane world-volume \( WY_p \) of codimension \( q = 10 - (p+1) \) with gauge bundle \( E \rightarrow WY_p \), if the Poincaré dual of \( WY_p \) in \( S \) survives until the last step of AHSS, it determines a class \( \{PD_S(WY_p)\} \in E_{\infty}^{q,0}(S) \) whose representatives belong to \( \ker(K^q(S) \rightarrow K^q(S^{q-1})) \).

These two approaches give different information, in particular AHSS does not take into account the gauge bundle: the aim of the present work is to relate them. We briefly anticipate the result. For a \( Dp \)-brane with world-volume \( WY_p \subset S \) and gauge bundle \( E \rightarrow WY_p \), let \( i : WY_p \rightarrow S \) be the embedding and \( i_! : K(WY_p) \rightarrow K(S) \) the Gysin map. We will show that \( i_!(E) \in \ker(K^q(S) \rightarrow K^q(S^{q-1})) \) for \( q = 10 - (p + 1) \), and that:

\[ \{PD_S(WY_p)\}_{E_{\infty}^{q,0}} = [i_!(E)]_{E_{\infty}^{q,0}}. \]

Thus, we must first use AHSS to detect possible anomalies, then we can use the Gysin map to get the charge of a non-anomalous brane: such a charge belongs to the equivalence class reached by AHSS, so that Gysin map gives more detailed information. For further remarks about this, we refer to the conclusions.

Moreover, we compare this picture with the case of rational coefficients. It is known that Chern character provides isomorphisms \( K(S) \otimes \mathbb{Q} \cong H^{ev}(S, \mathbb{Q}) \) and \( K^1(S) \otimes \mathbb{Q} \cong H^{odd}(S, \mathbb{Q}) \), and that AHSS with rational coefficients degenerates at the second step, i.e. at the level of cohomology. Therefore, we gain a complete equivalence between the two K-theoretical approaches, being both equivalent to the old cohomological classification.

The paper is organized as follows. In section 2 we discuss in detail the physical context underlying the K-theory classification of D-branes. In section 3 and 4 we introduce the topological tools needed to formulate our result, which is stated and proven in section 5. In section 6 we draw our conclusions.

## 2 Physical motivations

As we already said, for simplicity we assume the ten-dimensional space-time \( S \) to be compact, so that also D-brane world-volumes are compact. This seems not physically reasonable, but it has more meaning if we suppose to have performed the Wick rotation in space-time, so
that we work in a euclidean setting. In this setting we loose the physical interpretation of the D-brane world-volume as a volume moving in time and of the charge \( q \) (actually all the homology class \([q \cdot Y_{p,t}]\)) as a charge conserved in time. Thus, rather than considering the homology class of the D-brane volume at every instant of time, we prefer to consider the homology class of the entire world-volume in \( S \), using standard homology with compact support.

### 2.1 Classification

For a \( D_p \)-brane with \((p + 1)\)-dimensional world-volume \( WY_p \) and charge \( q \) we consider the corresponding homology cycle in \( S \): 

\[
q \cdot WY_p \in H_{p+1}(S, \mathbb{Z}) = \mathbb{Z}^{b_{p+1}} \oplus \mathbb{Z}_{p,n_i}.
\]

For what will follow, it is convenient to consider the cohomology of \( S \) rather than homology, defining the charge density:

\[
\text{PD}_S(q \cdot WY_p) \in H^{9-p}(S, \mathbb{Z}) = \mathbb{Z}^{b_{9-p}} \oplus \mathbb{Z}_{9-p,n_i}.
\]

This classification encounters some problems due to the presence of quantum anomalies. Two remarkable examples are the following:

- a brane wrapping a cycle \( WY_p \subset S \) is Freed-Witten anomalous if \( W_3(WY_p) \neq 0 \), hence not all the cycles are allowed ([9] and [6]);
- given a world-volume \( WY_p \) with \( W_3(WY_p) \neq 0 \), it can be interpreted as an MMS-instanton in the minkowskian setting ([16], [6]); in this case there are cycles intersecting \( WY_p \) in \( \text{PD}_{WY_p}(W_3(WY_p)) \) which, although homologically non-trivial in general, are actually unstable.

The two points above imply that:

- the numerator \( Z^{9-p}(S, \mathbb{Z}) \) is too large, since it contains anomalous cycles;
- the denominator \( B^{9-p}(S, \mathbb{Z}) \) is too small, since it does not cut all the unstable charges.

There are other possible anomalies, although not yet completely understood, some of which are probably related to cycles not representable by a smooth submanifold ([7], [4] and [6]).

We start by considering the case of world-volumes of even codimension in \( X \), i.e. we start with IIB superstring theory. To solve the problems mentioned above, one possible tool seems to be the Atiyah-Hirzebruch spectral sequence [3]. Given an appropriate filtration of the space-time manifold \( S = S_{10} \supset \cdots \supset S_0 \), such a spectral sequence starts from the even-dimensional simplicial cochains of \( S \) and, after a finite number of steps, it stabilizes to the graded group \( E^{\infty,0}_\infty(S) = \bigoplus_{2k} K_{2k}(S)/K_{2k+1}(S) \) where \( K_q(S) = \text{Ker}(K(S) \to K(S_q)) \). We can start from a representative of the Poincaré dual of the brane \( \text{PD}_S(q \cdot WY_p) \), which in our hypotheses is even-dimensional, and, if it survives until the last step, we arrive to a class \( \{\text{PD}_S(q \cdot WY_p)\} \in K_{9-p}(S)/K_{9-p+1}(S) \). The even boundaries \( d_2, d_4, \ldots \) of this sequence are 0, hence the important ones are the odd boundaries. In particular, one can prove that:
• $d_1$ coincides with the ordinary coboundary operator, hence the second step is the even cohomology of $S$ ($[24]$ and $[3]$);

• the cocycles not living in the kernel of $d_3$ are Freed-Witten anomalous, while the cocycles contained in its image are unstable because of the presence of MMS-instantons ($[6]$ and $[16]$).

As we will say in a while, there are good reasons to use K-theory to classify D-branes charges, hence, although the physical meaning of higher order boundaries is not completely clear, the behaviour of $d_3$ and the fact that the last step is directly related to K-theory suggest that the class $\{\text{PD}_S(q \cdot WY_p)\} \in E_{\infty}^{0-n,0}$ is a good candidate to be considered as the charge of the D-brane. Summarizing, we saw two ways to classify D-brane cycle and charge:

• the homological classification, i.e. $[q \cdot WY_p] \in H_{p+1}(S, \mathbb{Z})$;

• the classification via AHSS, i.e. $\{\text{PD}_S(q \cdot WY_p)\} \in E_{\infty}^{0-n,0}(S)$.

2.2 K-theory from Sen conjecture

2.2.1 Gauge and gravitational couplings

Up to now we only considered the cycle wrapped by the D-brane world-volume. There are other important features: the gauge bundle and the embedding in space-time, which enter the action via the two following couplings:

• the gauge coupling through the Chern character of the Chan-Paton bundle;

• the gravitational coupling through the $\hat{A}$-genus of the tangent and the normal bundle of the world-volume.

The unique non-anomalous form of these couplings, computed by Minasian and Moore in $[18]$, is:

$$S \supset \int_{WY_p} i^* C \wedge \text{ch}(E) \wedge e^\frac{A(T(WY_p))}{\sqrt{A(N(WY_p))}}$$ (1)

where $i : WY_p \to S$ is the embedding, $T(WY_p)$ and $N(WY_p)$ are the tangent bundle and the normal bundle of $WY_p$ in $S$, and $d \in H^2(WY_p, \mathbb{Z})$ is a class whose restriction mod 2 is $w_2(N(WY_p))$. The polyform that multiplies $i^* C$ has 0-term equal to $\text{ch}_0(E) = \text{rk}(E)$, leading to the previous action for $q = \text{rk}(E)$. Hence, (1) is an extension of the minimal coupling $q \int C_{p+1}$, and the charge is restored as the rank of the gauge bundle. In the case of anti-branes, we have to allow for negative charges, hence the gauge bundle is actually a K-theory class: a generic class $E - F$ can be interpreted as a stack of pairs of a brane $Y$ and an anti-brane $\overline{Y}$ with gauge bundle $E$ and $F$ respectively. For $i_\# : H^*(WY_p, \mathbb{Q}) \to H^*(S, \mathbb{Q})$ the Gysin map in cohomology, we define the charge density:

$$Q_{WY_p} = i_\# \left( \text{ch}(E) \wedge e^{\frac{\sqrt{A(T(WY_p))}}{\sqrt{A(N(WY_p))}}} \right).$$ (2)

Since new terms have appeared in the charge, we should discuss also their quantization, which is not immediate since the Chern character and the $\hat{A}$-genus are intrinsically rational
cohomology classes. To avoid the discussion of these problems \[19\], in the expression (1) we suppose $C$ to be globally defined, i.e. we suppose $G$ to be trivial in the de-Rham cohomology at any degree. For a general discussion see \[8\].

We put for notational convenience:

$$G(WY_p) = e^{\frac{i}{2}} \wedge \sqrt{\frac{\Lambda(T(WY_p))}{\Lambda(N(WY_p))}}.$$  

The action (1) is equal to:

$$S \supset \int_{PD(WY_p(ch(E))} i^* C \wedge G(WY_p).$$

Let $\{q_k \cdot WY_k\}$ be the set of branes appearing in the Poincaré dual of $ch(E)$ in $WY_p$: the first one is $PD(WY_p(ch_0(E)) = q \cdot WY_p$, so it gives rise to the action without gauge coupling. The other ones are lower dimensional branes. Let us consider the first one, i.e. $WY_1 = PD(WY_p(ch_1(E))$. Then the corresponding term in the action is $\int_{WY_1} i^* C \wedge G(WY_p)$, which can be written as $\int_{WY_1} i^* C \wedge G(WY_1) + \int_{WY_1} i^* C \wedge (G(WY_1) - G(WY_1))$. Since in the second term the sum $G(WY_1) - G(WY_1)$ has 0-term equal to 0, then $PD(WY_1) = G(WY_1) - G(WY_1)$ is made only by lower-dimensional subbranes. Let $WY_{1,1}$ be the first one: we get $\int_{WY_{1,1}} i^* C$, which is equal to $\int_{WY_{1,1}} i^* C \wedge G(WY_{1,1}) + \int_{WY_{1,1}} i^* C \wedge (1 - G(WY_{1,1}))$. The second term gives rise only to lower dimensional subbranes. Proceeding inductively until we arrive at D0-branes, whose $G$-term is 1, we can write:

$$\int_{WY_{1}} i^* C \wedge G(WY_p) = \sum_{h=0}^{m} \int_{WY_{1,h}} i^* C \wedge G(WY_{1,h})$$

where, for $h = 0$, it holds $WY_{1,0} = WY_1$. Proceeding in the same way for every $WY_{k}$, we obtain a set of subbranes $\{q_k \cdot WY_{k,h}\}$, which, using only one index, we still denote by $\{q_k \cdot WY_{k}\}$. Therefore we get:

$$S \supset \sum_{k} \int_{WY_{k}} i^* C \wedge G(WY_{k}).$$

From this expression we see that the brane $WY_p$ with gauge and gravitational couplings is equivalent to the set of sub-branes $WY_{k}$ with trivial gauge bundle. Moreover we now show that:

$$i_{\#}(ch(E) \wedge G(WY_p)) = \sum_{k} (i_{k})_{\#} G(WY_{k}) \quad (3)$$

i.e. the charge density of the two configurations are the same. To prove this, we recall the formulas:

$$i_{\#}(\alpha \wedge i^* \beta) = i_{\#}(\alpha) \wedge \beta$$

$$\int_{WY_p} \alpha = \int_{S} i_{\#}(\alpha) \quad (4)$$
for $\alpha \in H^*(W_Y p, \mathbb{Q})$ and $\beta \in H^*(S, \mathbb{Q})$. Thus:

$$\int_{W_Y p} i^* C \wedge \text{ch}(E) \wedge G(W_Y p) = \int_S i_# \left[ i^* C \wedge \text{ch}(E) \wedge G(W_Y p) \right]$$

$$= \int_S C \wedge i_# \left( \text{ch}(E) \wedge G(W_Y p) \right)$$

$$\sum_k \int_{W_Y p} i_k^* C \wedge G(W_Y (k)) = \sum_k \int_S (i_k)_# \left[ i_k^* C \wedge G(W_Y (k)) \right]$$

$$= \int_S C \wedge \sum_k (i_k)_# \left( G(W_Y (k)) \right).$$

Since the two terms are equal for every form $C$, we get formula (3). We thus get:

**Splitting principle:** a D-brane $W_Y p$ with gauge bundle is equivalent to a set of sub-branes $W_Y (k)$ with trivial gauge bundle, such that the total charge density of the two configurations is the same.

The physical interpretation of this conjecture is the phenomenon of tachyon condensation ([25], [27] and [6]): the quantization of strings extending from a brane to an antibrane leads to a tachyonic mode, which represents an instability and generates a process of annihilation of brane and antibrane world-volumes via an RG-flow [1], leaving lower dimensional branes. In particular, given a D-brane $W_Y p$ with gauge bundle $E \to W_Y p$, we can write $E = (E - \text{rk} E) + \text{rk} E$, so that $E - \text{rk} E \in \tilde{K}(W_Y p)$: thus we see this configuration as a triple made by a D-brane $W_Z p$ with gauge bundle $\text{rk} E$, a brane $W_Y p$ with gauge bundle $E$ and an antibrane $\overline{W_Z} p$ with gauge bundle $\text{rk} E$. Thus, by tachyon condensation, only $W_Z p$ remains (with trivial bundle, i.e., only with its own charge), while $W_Y p$ and $\overline{W_Z} p$ annihilate giving rise to lower dimensional branes with trivial bundle, as stated in the splitting principle. Moreover, if we consider a stack of pairs $(W_Y p, \overline{W_Y} p)$ with gauge bundles $E$ and $F$ respectively, this is equivalent to consider gauge bundles $E \oplus G$ and $F \oplus G$ respectively, since, viewing the factor $G$ as a stack of pairs $(W_Z p, \overline{W_Z} p)$ with the same gauge bundle, we have that by tachyon condensation $W_Z p$ and $\overline{W_Z} p$ disappear, leaving no other subbranes. This is exactly the physical interpretation of the stable equivalence relation of K-theory. This principle, as we will see, is an inverse of Sen conjecture, but we will actually use it to show Sen conjecture in this setting.

**Remark:** the splitting principle holds only at rational level, since it involves Chern characters and $\hat{A}$-genus. At integral level, we do not state such a principle.

### 2.2.2 K-theory

Since we are assuming the $H$-flux to vanish, in order not to be Freed-Witten anomalous the D-brane must be spin$^c$, hence, since the whole space-time is spin$^c$, also the normal bundle is. Thus we can consider the K-theory Gysin map $i_! : K(W_Y p) \to K(S)$. We recall the differentiable Riemann-Roch theorem ([12] and [21]):

$$\text{ch}(i_!(E)) \wedge \hat{A}(TS) = i_# \left( \text{ch}(E) \wedge e^2 \wedge \hat{A}(T(W_Y p)) \right).$$

(5)
Using (5) and (4) we obtain:

$$\int_{WY_p} i^* C \wedge ch(E) \wedge e^2 \wedge \frac{\sqrt{A(T(WY_p))}}{\sqrt{A(N(WY_p))}} = \int_S C \wedge ch(i(E)) \wedge \sqrt{\hat{A}(TS)}.$$

Thus we get:

$$S \supset \int_S C \wedge ch(i(E)) \wedge \sqrt{\hat{A}(TS)}$$

define:

$$Q_{WY_p} = ch(i(E)) \wedge \sqrt{\hat{A}(TS)}.$$  \hspace{1cm} (6)

In this way, (6) is another expression for $Q_{WY_p}$ with respect to (2), but with an important difference: the $\hat{A}$-factor does not depend on $WY_p$, hence all $Q_{WY_p}$ is a function only of $E$. Thus, we can consider $i(E)$ as the K-theory analogue of the charge density, considered as an integral K-theory class. The use of Chern characters, instead, obliges to consider rational classes, which, although they have to come from integral ones for Dirac quantization condition, they cannot contain information about the torsion part.

### 2.2.3 Sen conjecture

Let us consider the two expressions found for the rational charge density:

$$Q^{(1)}_{WY_p} = i_\#(ch(E) \wedge G(WY_p))$$

$$Q^{(2)}_{WY_p} = ch(i(E)) \wedge \sqrt{\hat{A}(TS)}.$$

$Q^{(2)}_{WY_p}$ is exactly the charge density of a D9-brane (whose world-volume coincides with $S$), whose gauge bundle is the K-theory class $i(E)$. Hence, expressing the charge in the form $Q^{(2)}_{WY_p}$ for each brane in our background is equivalent to think that there exists only one couple brane-antibrane of dimension 9 encoding all the dynamics. Hence we formulate the conjecture ([25] and [27]):

**Sen conjecture:** every configuration of branes and anti-branes with any gauge bundle is dynamically equivalent to a configuration with only a stack of coincident pairs brane-antibrane of dimension 9 with an appropriate K-theory class on it.

To see that the dynamics is actually equivalent, we use the splitting principle stated above: since $Q^{(1)}_{WY_p} = Q^{(2)}_{WY_p}$, the brane $WY_p$ with the charge $Q^{(1)}_{WY_p}$ and the D9-brane with charge $Q^{(2)}_{WY_p}$ split into the same set of subbranes (with trivial gauge bundle). We remark that to state Sen conjecture is necessary that the $H$-flux vanishes. Indeed, space-time is spin$^c$ (it is spin since space-time spinors exist, therefore also spin$^c$), hence Freed-Witten anomaly cancellation for D9-branes requires that $H = 0$. Actually, an appropriate stack of D9-branes can be consistent for $H$ a torsion class, but we do not consider this case in the present paper.
To formulate both the splitting principle and the Sen conjecture, we only considered the action, hence only rational classes given by Chern characters and $\hat{A}$-genus. Thus, we can classify the charge density in the two following ways:

- as a rational cohomology class $i_\#(\text{ch}(E) \wedge G(WY_p)) \in H^\text{ev}(S, \mathbb{Q})$;
- as a rational K-theory class $i_!E \in K_\mathbb{Q}(S) := K(S) \otimes \mathbb{Z} \mathbb{Q}$.

These two classification schemes are completely equivalent due to the fact that the Chern character:

$$\text{ch}(\cdot) \wedge \sqrt{\hat{A}}(TS) : K_\mathbb{Q}(S) \rightarrow H^\text{ev}(S, \mathbb{Q})$$

is an isomorphism. This equivalence is lost at the integral level, since the torsion part of $K(S)$ and $H^\text{ev}(S, \mathbb{Z})$ are in general different. Moreover, since at the integral level we do not apply the splitting principle, we do not really have Sen conjecture: the classification via Gysin map and cohomology are different, and the use of the Gysin map is just suggested by the equivalence at rational level, i.e., by the equivalene of the dynamics.

Moreover, for the integral case, we saw also the classification via AHSS. In the rational case, we can build the corresponding sequence AHSS$_\mathbb{Q}$ [3], ending at the groups $Q^\text{ev},0(S)$, but it stabilizes at the second step, i.e. at the level of cohomology. Hence, the class $\{i_\#(\text{ch}(E) \wedge G(WY_p))\} \in Q^\text{ev},0(S)$ is completely equivalent to the cohomology class $i_\#(\text{ch}(E) \wedge G(WY_p)) \in H^\text{ev}(S, \mathbb{Q})$.

2.3 Linking the classifications

To summarize, we are trying to classify the charges of D-branes in a compact euclidean space-time $S$. To achieve this, we can use cohomology or K-theory, with integer or rational coefficients, obtaining the possibilities showed in table 1.

|                | Integer                          | Rational                          |
|----------------|----------------------------------|-----------------------------------|
| Cohomology     | $\text{PD}_S(q \cdot WY_p) \in H^{9-p}(S, \mathbb{Z})$ | $i_\#(\text{ch}(E) \wedge G(WY_p)) \in H^\text{ev}(S, \mathbb{Q})$ |
| K-theory (Gysin map) | $i_!(E) \in K(S)$ | $i_!(E) \in K_\mathbb{Q}(S)$ |
| K-theory (AHSS) | $\{\text{PD}_S(q \cdot WY_p)\} \in E^\text{9},0(S)_{\infty}$ | $\{i_\#(\text{ch}(E) \wedge G(WY_p))\} \in Q^\text{ev},0(S)_{\infty}$ |

Table 1: Classifications

In the rational case, as we have seen, there is a complete equivalence of the three approaches, since the three groups we consider, i.e. $\bigoplus_{2k} H^{2k}(S, \mathbb{Q})$, $K_\mathbb{Q}(S)$ and $\bigoplus_{2k} Q_{2k}^{2k,0}(S)$ are all canonically isomorphic. Instead, in the integral case there are not such isomorphisms.
the three groups are all different), and there is a strong asymmetry due to the fact that in the homological and AHSS classifications the gauge bundle and the gravitational coupling are not considered at all, while they are of course taken into account in the Gysin map approach. Up to now we discussed the case of even-codimensional branes: that is because the Gysin map requires an even-dimensional normal bundle to be defined. We will discuss also the odd-dimensional case, considering the brane embedded in the suspension $S^1S$ of space-time, and the picture will be similar.

Since the integral approaches are not equivalent, we have to investigate the relations among them: it is clear how to link the cohomology class and the AHSS class, since the second step of AHSS is exactly the cohomology. Our aim is to link the Gysin map approach with the one based on AHSS.

### 3 Useful notions of K-theory

We briefly recall the main K-theoretical constructions, which will be used in the following.

#### 3.1 Products in K-theory

$K(X)$ has a natural ring structure given by tensor product: $[E] \otimes [F] := [E \otimes F]$. Such product restricts to $\tilde{K}(X)$. In general, we can define a product:

$$K(X) \otimes K(Y) \xrightarrow{\boxtimes} K(X \times Y)$$

where, if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projections, $E \boxtimes F = \pi_1^* E \otimes \pi_2^* F$. The fiber of $E \boxtimes F$ at $(x, y)$ is $E_x \otimes E_y$. We now prove that, fixing a marked point for $X$ and $Y$, this restricts to (v. [22]):

$$\tilde{K}(X) \otimes \tilde{K}(Y) \xrightarrow{\boxtimes} \tilde{K}(X \wedge Y).$$

For this, we first state that:

$$\tilde{K}(X \times Y) \simeq \tilde{K}(X \wedge Y) \oplus \tilde{K}(Y) \oplus \tilde{K}(X).$$

In fact:

- since $X$ is a retract of $X \times Y$ via the projection, we have that $\tilde{K}(X \times Y) = K(X \times Y, X) \oplus \tilde{K}(X) = \tilde{K}(X \times Y/X) \oplus \tilde{K}(X)$;
- since $Y$ is a retract $X \times Y/X$ via the projection, we also have $\tilde{K}(X \times Y/X) = K(X \times Y/X, Y) \oplus \tilde{K}(Y) = \tilde{K}(X \wedge Y) \oplus \tilde{K}(Y)$.

Combining, we obtain (9). The explicit isomorphism in (9) is given, for $\alpha = [E] - [F] \in \tilde{K}(X \times Y)$, by:

$$\alpha \mapsto (\alpha - \pi_1^* \alpha|_X - \pi_2^* \alpha|_Y) \oplus \pi_2^* \alpha|_Y \oplus \pi_1^* \alpha|_X.$$  

1 If $X = Y$ and $\Delta : X \to X \times X$ is the diagonal embedding, then $E \otimes F = \Delta^*(E \boxtimes F)$.
2 (9) is actually true for $\tilde{K}^{-n}(X \times Y)$ for any $n$, with the same proof.
Let \( \alpha \in \tilde{K}(X) \) and \( \beta \in \tilde{K}(Y) \): then \( \alpha \boxtimes \beta|_X = 0 \) and \( \alpha \boxtimes \beta|_Y = 0 \). In fact:

\[
\alpha \boxtimes \beta|_X = \alpha \otimes (\pi_2^* \beta)|_X = \alpha \otimes i_1^* \pi_2^* \beta = \alpha \otimes (\pi_2 i_1)^* \beta.
\]

But \( \pi_2 i_1 : X \to Y \) is the constant map with value \( y_0 \), and the pull-back of a bundle by a constant map is trivial. Hence \( (\pi_2 i_1)^* \beta = 0 \). Similarly for \( Y \). Hence, by \( (9) \), we obtain \( \alpha \boxtimes \beta \in \tilde{K}(X \wedge Y) \).

3.1.1 Non-compact case

For a generic (also non-compact) space \( X \), we use K-theory with compact support, i.e., \( K(X) = \tilde{K}(X^+) \). One can easily prove that \( X^+ \wedge Y^+ = (X \times Y)^+ \). Hence, product \( (8) \) exactly becomes:

\[
K(X) \otimes K(Y) \overset{\otimes}{\longrightarrow} K(X \times Y)
\]
also for the non-compact case.

3.2 Thom isomorphism

Let \( X \) be a compact topological space and \( E \overset{\pi}{\rightarrow} X \) a fiber bundle (not necessarily complex). We now show that \( K(E) \) has a natural structure of \( K(X) \)-module.

We do not have a natural pull-back \( \pi^* : K(X) \to K(E) \) since we consider the compactification \( E^+ \), and there are no possibilities to extend continuously \( \pi \) to \( E^+ \). Hence, we use the product \( (10) \): considering the embedding \( i : E \to X \times E \) given by \( i(e) = (\pi(e), e) \)\(^3\) which trivially extends to \( i : E^+ \to (X \times E)^+ \) by \( i(\infty) = \infty \), we can define a product:

\[
\alpha \otimes \beta \mapsto i^*(\alpha \boxtimes \beta).
\]

This product defines a structure of \( K(X) \)-module on \( K(E) \).

**Lemma 3.1** \( K(E) \) is unitary as \( K(X) \)-module.

**Proof:** Let us consider the following maps:

\[
\begin{align*}
\pi_1 & : X^+ \times E^+ \to X^+ \\
\pi_2 & : X^+ \times E^+ \to E^+ \\
i & : E^+ \to (X \times E)^+ \\
\tilde{\pi} & : X^+ \times E^+ \to X^+ \wedge E^+ = (X \times E)^+ \\
\tilde{\pi}_2 & : (X \times E)^+ \to E^+
\end{align*}
\]

where \( i(e) = (\pi(e), e) \) and the others are defined in the obvious way. Since the map:

\[
r : X^+ \times E^+ \to (X^+ \times \{\infty\}) \cup (\{\infty\} \times E^+)
\]

\(^3\)For such an embedding it is not necessary to have a marked point on \( X \).
given by \( r(x, e) = (x, \infty) \) and \( r(\infty, e) = (\infty, e) \).\(^4\) is a retraction, \( \tilde{\pi}^* : \tilde{K}((X \times E)^+) \to \tilde{K}(X^+ \times E^+) \) is injective (v. [2]). Then, by the definition of the module structure, for \( \alpha \in K(X) = \tilde{K}(X^+) \) and \( \beta \in K(E) = \tilde{K}(E^+) \) we reformulate (11) as:

\[\alpha \cdot \beta = i^*(\tilde{\pi}^*)^{-1}(\alpha \boxtimes \beta) = i^*(\tilde{\pi}^*)^{-1}(\alpha_1 \otimes \alpha_2 \beta).\]

For \( \alpha = 1 \), it is \( \alpha|_X = X \times \mathbb{C} \) and \( \alpha|_{(\infty)} = 0 \). Hence it is:

\[\begin{align*}
(1 \boxtimes \beta)|_{X \times E^+} &= \pi_2^* \beta|_{X \times E^+} \\
(1 \boxtimes \beta)|_{(\infty) \times E^+} &= 0.
\end{align*}\]

But:

- since \( \pi_2|_{X \times E^+} = (\tilde{\pi}_2 \circ \tilde{\pi})|_{X \times E^+} \), it is \( \pi_2|_{X \times E^+} = \tilde{\pi}^* \tilde{\pi}_2^* \beta|_{X \times E^+} \);

- since \( \tilde{\pi}_2 \circ \tilde{\pi}((\infty) \times E^+) = \{\infty\} \) and \( \beta \in \tilde{K}(E^+) \), it is \( \tilde{\pi}^* \tilde{\pi}_2^* \beta|_{(\infty) \times E^+} = 0.\)

Hence \( 1 \boxtimes \beta = \tilde{\pi}^* \tilde{\pi}_2^* \beta \), so that:

\[1 \cdot \beta = i^*(\tilde{\pi}^*)^{-1} \tilde{\pi}^* \tilde{\pi}_2^* \beta = i^* \tilde{\pi}_2^* \beta = (\tilde{\pi}_2 \circ i)^* \beta = \text{id}^* \beta = \beta.\]

\(\square\)

Let us consider a vector space \( \mathbb{R}^{2n} \) as a fiber bundle on a point \( \{x\} \). Then we have:

- \( K(x) = \mathbb{Z} \);

- \( K(\mathbb{R}^{2n}) = \tilde{K}(\mathbb{R}^{2n}^+) = \tilde{K}(S^{2n}) = \mathbb{Z}. \)

Hence \( K(x) \simeq K(\mathbb{R}^{2n}) \). The idea of Thom isomorphism is to extend this to a generic bundle \( E \to X \) with fiber \( \mathbb{R}^{2n} \). To achieve this, we try to write such isomorphism in a way that extends to a generic bundle. Actually, this generalization works for \( E \text{ Spin}^c \)-bundle of even dimension.

In fact, for the trivial bundle \( \{x\} \times \mathbb{R}^{2n} \), we have \( S^c_+(\mathbb{R}^{2n}) \simeq \mathbb{C}^{2n-1} \) as \( \text{Spin}^c(2n) \)-module, with \( \text{Spin}^c(2n) = \text{Spin}(2n) \otimes_\mathbb{R} \mathbb{C} \). Moreover, \( S^c_+(\mathbb{R}^{2n}) \) are submodules of \( S^c_+(\mathbb{R}^{2n}) = \mathbb{C}^{2n} \) as \( \text{Cl}(2n) \)-module, and, for \( v \in \mathbb{R}^{2n} \subset \text{Cl}(2n) \), we have \( v \cdot S^c_+(\mathbb{R}^{2n}) = S^c_-(\mathbb{R}^{2n}) \). We thus consider the following complex:

\[0 \to \mathbb{R}^{2n} \times S^c_+(\mathbb{R}^{2n}) \xrightarrow{c} \mathbb{R}^{2n} \times S^c_-(\mathbb{R}^{2n}) \to 0\]

where \( c \) is the Clifford multiplication by the first component: \( c(v, z) = (v, v \cdot z) \). Such sequence of trivial bundles on \( \mathbb{R}^{2n} \) is exact when restricted to \( \mathbb{R}^{2n} \setminus \{0\} \), hence the altered sum:

\[\lambda_{2n} = \left[\mathbb{R}^{2n} \times S^c_-(\mathbb{R}^{2n})\right] - \left[\mathbb{R}^{2n} \times S^c_+(\mathbb{R}^{2n})\right]\]

\(^4\)The map \( r \) is continuous because \( X \) is compact, so that its \( \infty \)-point is disjoint from it.

\(^5\)With respect to (11) we think \( \alpha \boxtimes \beta \in \tilde{K}(X^+ \times E^+) \) and we write explicitly \( (\tilde{\pi})^{-1} \).
naturally gives a class in $K(\mathbb{R}^{2n}, \mathbb{R}^{2n} \setminus \{0\})$ (v. [2]). The sequence is exact in particular in $\mathbb{R}^{2n} \setminus B_1(0)$, hence it defines a class:

$$\lambda_{\mathbb{R}^{2n}} \in K(\mathbb{R}^{2n}, \mathbb{R}^{2n} \setminus B_1(0)) = \tilde{K}(\mathbb{R}^{2n} / S^1(0)) = \tilde{K}(S^{2n}).$$

One can prove that, for $\eta = \mathcal{O}_{S^2}(1)$:

$$\lambda_{\mathbb{R}^{2n}} = (-1)^n \cdot (\eta - 1)^{\otimes n}$$

i.e., it is a generator of $\tilde{K}(S^{2n}) \cong \mathbb{Z}$.

For a generic Spin$^c$-bundle $\pi : E \to X$ of dimension $2n$, let $S^+_\mathbb{C}(E)$ be the bundle of complex spinors associated to $E$, i.e., we consider the spin$^c$-lift of the orthogonal frame bundle $SO(E)$ to Spin$^c(E)$, and we call $S^\mathbb{C}(E)$ the vector bundle with fiber $\mathbb{C}^{2n}$ associated to the representation Spin$^c(2n) \subset \text{Cl}(2n) \hookrightarrow \mathbb{C}^{2n}$: this bundle splits into $S^\mathbb{C}(E) = S^+_\mathbb{C}(E) \oplus S^-\mathbb{C}(E)$. Such bundle is naturally a $\mathbb{C}l(E)$-module.

We can lift $S^+_\mathbb{C}(E)$ to $E$ by $\pi^*$. Then we consider the complex:

$$0 \longrightarrow \pi^*S^+_\mathbb{C}(E) \overset{c}{\longrightarrow} \pi^*S^-\mathbb{C}(E) \longrightarrow 0$$

where $c$ is the Clifford multiplication given by the structure of $\mathbb{C}l(E)$-module: for $e \in E$ and $s_e \in (\pi^*S^+_\mathbb{C}(E))_e$, we define $c(s_e) = e \cdot s_e$. Such sequence is exact when restricted to $E \setminus B_1(E)$, where, for any fixed metric on $E$, $B_1(E)$ is the union of open balls of radius 1 on each fiber. Hence we naturally obtain:

$$\lambda_E = [\pi^*S^-\mathbb{C}(E)] - [\pi^*S^+_\mathbb{C}(E)]$$

as a class in $K(E, E \setminus B_1(E)) = \tilde{K}(\overline{B_1(E)} / S_1(E)) = \tilde{K}(E^+) = K(E)$. The following fundamental theorem holds (v. [15], [14] and, only for the complex case, [2] and [22]):

**Theorem 3.2 (Thom isomorphism)** Let $X$ be a compact topological space and $\pi : E \to X$ an even dimensional spin$^c$-bundle. For

$$\lambda_E = [\pi^*S^-\mathbb{C}(E)] - [\pi^*S^+_\mathbb{C}(E)] \in K(E)$$

the map:

$$T : K(X) \longrightarrow K(E)$$

$$\alpha \longrightarrow \alpha \cdot \lambda_E$$

is a group isomorphism.

We can now see that the construction for a generic $2n$-dimensional spin$^c$-bundle $E \to X$ is a generalization of the construction of $\mathbb{R}^{2n}$. In fact, for $x \in X$:

- $(\pi^*S^\pm\mathbb{C}(E))_x = E_x \times (S^\pm\mathbb{C}(E)) \cong \mathbb{R}^{2n} \times S^\pm\mathbb{C}(\mathbb{R}^{2n})$;

- Clifford multiplication restricts on each fiber $E_x$ to Clifford multiplication in $\mathbb{R}^{2n} \times S^\mathbb{C}(\mathbb{R}^{2n})$.

Hence:

$$\lambda_E|_{E_x} \cong \lambda_{\mathbb{R}^{2n}}.$$  \hspace{1cm} (13)

In particular, we see that, for $i : E^+_x \to E^+$, the restriction $i^* : K(E) \to K(E_x) \cong \mathbb{Z}$ is surjective.
3.3 Gysin map

Let $X$ be a compact smooth $n$-manifold and $Y \subset X$ a compact embedded $p$-submanifold such that $n - p$ is even and the normal bundle $\mathcal{N}(Y) = (TX|_Y)/TY$ is spin$^c$. Then, since $Y$ is compact, there exists a tubular neighborhood $U$ of $Y$ in $X$, i.e., there exists an homeomorphism $\varphi_U : U \rightarrow \mathcal{N}(Y)$.

If $i : Y \rightarrow X$ is the embedding, from this data we can naturally define an homomorphism, called **Gysin map**:

$$i_! : K(Y) \rightarrow \tilde{K}(X).$$

In fact:

- we first apply Thom isomorphism $T : K(Y) \rightarrow K(\mathcal{N}(Y)) = \tilde{K}(\mathcal{N}(Y)^+);$
- then we naturally extend $\varphi_U$ to $\varphi_U^+ : U^+ \rightarrow \mathcal{N}(Y)^+$ and apply $(\varphi_U^+)^* : K(\mathcal{N}(Y)) \rightarrow K(U);$
- there is a natural map $\psi : X \rightarrow U^+$ given by:

$$\psi(x) = \begin{cases} x & \text{if } x \in U \\ \infty & \text{if } x \in X \setminus U \end{cases}$$

hence we apply $\psi^* : K(U) \rightarrow \tilde{K}(X)$.

Summarizing:

$$i_!(\alpha) = \psi^* \circ (\varphi_U^+)^* \circ T(\alpha).$$

**Remark:** One could try to use the immersion $i : U^+ \rightarrow X^+$ and the retraction $r : X^+ \rightarrow U^+$ to have a splitting $K(X) = K(U) \oplus K(X,U) = K(Y) \oplus K(X,U)$. But this is false, since the immersion $i : U^+ \rightarrow X^+$ is not continuous: since $X$ is compact, $\{\infty\} \subset X^+$ is open, but $i^{-1}(\{\infty\}) = \{\infty\}$, and $\{\infty\}$ is not open in $U^+$ since $U$ is non-compact.

4 Atiyah-Hirzebruch spectral sequence

4.1 Spectral sequence for cohomological theories

As explained in [5], given the following assignements, for $p, q, r \in \mathbb{Z} \cup \{-\infty, +\infty\}$:

- for $-\infty \leq p \leq q \leq \infty$, an abelian group $H(p, q)$, such that $H(p, q) = H(0, q)$ for $p \leq 0$ and there exists $l \in \mathbb{N}$ such that $H(p, q) = H(p, +\infty)$ for $q > l$;
- for $p \leq q \leq r$, $a, b \geq 0$, two maps:

$$\Psi : H(p + a, q + b) \rightarrow H(p, q)$$

$$\Delta : H(p, q) \rightarrow H(q, r)$$
satisfying appropriate axioms (v. [5] chap. XV par. 7), we can define:

\[
E_p^r = \text{Im}(H(p, p + r) \xrightarrow{\Psi} H(p - r + 1, p + 1))
\]

\[
d_p^r = \Delta^{p-r+1,p+1+r+1} \big|_{\text{Im}(\Psi_{p,p+r}^{p+r+1})} : E_p^K \rightarrow E_{p+r}^r K
\]

\[
F^p H = \text{Im}(H(p, +\infty) \xrightarrow{\Psi} H(0, +\infty)).
\]

Then:

- the groups \(F^p H\) are a filtration of \(H(0, +\infty)\);
- \(E_{r+1}^p = H(E_r^p, d_r^p)\);
- the sequence \(\{E_r^p\}_{r \in \mathbb{N}}\) stabilizes to \(F^p H/F^{p+1} H\).

In particular, we have a commutative diagram:

![Commutative Diagram](image)

We have, as the reader can verify (v. [5] chap. XV):

- \(\text{Im}(\Psi_1) = E_p^p K\) and \(\text{Im}(\Psi_2) = E_r^{p+r} K\);
- \(d_p^r = \Delta_2 \big|_{\text{Im}(\Psi_1)} : E_p^K \rightarrow E_{p+r}^r K\).

In this language, the limit of the sequence is:

\[
E_0^p H(K) = E_\infty^K = \text{Im}(H(p, +\infty) \xrightarrow{\Psi} H(0, p+1)).
\]

Such limit is the associated graded group of the filtration of \(H(K) = H(0, +\infty)\) given by:

\[
F^p H(K) = \text{Im}(H(p, +\infty) \xrightarrow{\Psi} H(0, +\infty))
\]

i.e., \(E_0^p H(K) = F^p H(K)/F^{p+1} H(K)\).

Given a topological space \(X\) with a finite filtration:

\[
\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^m = X
\]

we can consider a generic cohomological theory \(H^\bullet\) and define:

- \(H(p, q) = \bigoplus_n H^n(X^{q-1}, X^{p-1})\);
- \(\Psi : H(p + a, q + b) \rightarrow H(p, q)\) is induced (by the axioms of cohomology) by the map of couples \(i : (X^{q-1}, X^{p-1}) \rightarrow (X^{q+b-1}, X^{p+a-1})\);
- \(\Delta : H(p, q) \rightarrow H(q, r)\) is the composition of the map \(\pi^* : H^\bullet(X^{q-1}, X^{p-1}) \rightarrow H^\bullet(X^{q-1})\) induced by \(\pi : (X^{q-1}, \emptyset) \rightarrow (X^{q-1}, X^{p-1})\), and the Bockstein map \(\beta : H^\bullet(X^{q-1}) \rightarrow H^{\bullet+1}(X^{q-1}, X^{q-1})\).
Remark: the shift by $-1$ in the definition of $H(p, q)$ is necessary to have $H(0, +\infty) = \bigoplus_n H^n(X)$. It would not be necessary if we declared $X^0 = \emptyset$, but this is not coherent with the case of finite simplicial complexes, since, in that case, $X^0$ denotes the 0-skeleton.

Since K-theory is a cohomological theory, it is natural to consider the spectral sequence associated to it for a given filtration $\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^n = X$: such sequence is called Atiyah-Hirzebruch spectral sequence (AHSS). In particular, groups and maps are defined in the following way (for $p \leq q \leq r; a, b \geq 0; p + a \leq q + b$):

- $H(p, q) = \bigoplus_n K^n(X^{q-1}, X^{p-1})$;
- $\Psi : K^\bullet(X^{q+b-1}, X^{p+a-1}) \to K^\bullet(X^{q-1}, X^{p-1})$ is induced by pull-back of the map $i : X^{q-1}/X^{p-1} \to X^{q+b-1}/X^{p+a-1}$;
- $\Delta : K^\bullet(X^{q-1}, X^{p-1}) \to K(X^{r-1}, X^{q-1})$ is the composition of the map $\pi^* : K^\bullet(X^{q-1}, X^{p-1}) \to K^\bullet(X^{q-1})$ induced by $\pi : X^{q-1} \to X^{q-1}/X^{p-1}$, and the K-theory Bockstein map $\delta : K^\bullet(X^{q-1}) \to K^{\bullet+1}(X^{r-1}, X^{q-1})$.

4.2 K-theory and simplicial cohomology

Lemma 4.1 For $k \in \mathbb{N}$ and $0 \leq i \leq k$, let:

$$X = \bigcup_{i=0,\ldots,k} X_i$$

be the one-point union of $k$ topological spaces. Then:

$$\tilde{K}^n(X) = \bigoplus_{i=0}^k \tilde{K}^n(X_i)$$

Proof: For $n = 0$, let us construct the isomorphism $\varphi : \tilde{K}(X) \to \bigoplus \tilde{K}(X_i)$: it is simply given by $\varphi(\alpha)_i = \alpha|_{X_i}$. To build $\varphi^{-1}$, let us consider $\{[E_i] - [n_i]\} \in \bigoplus \tilde{K}(X_i)$. By adding and subtracting a trivial bundle we can suppose $n_i = n_j$ for every $i, j$, so that we consider $\{[E_i] - [n]\}$. Since the intersection of the $X_i$ is a point and the bundles $E_i$ have the same rank, we can glue them to a bundle $E$ on $X$ (v. [2] pp. 20-21): then we declare $\varphi^{-1}(\{[E_i] - [n]\}) = ([E] - [n])$.

For $n = 1$, we first note that, for $\tilde{S}^1 X$ the unreduced suspension of $X \boxtimes \tilde{K}(\tilde{S}^1(\hat{X}_1 \cup X_2)) = \tilde{K}(\tilde{S}^1X_1 \cup \tilde{S}^1X_2)$, since quotienting by a contractible space (the linking between vertices of the cones and the joining point) we obtain the same space. Hence $\tilde{K}^1(X_1 \cup X_2) = \tilde{K}^1(X_1) \oplus \tilde{K}^1(X_2)$. Then, by induction, the thesis extends to finite families. \(\square\)

\(\text{\textsuperscript{6}}\)We remind that $K(\tilde{S}^1X) = K(S^1X)$.
Remark: we stress the fact that the previous lemma holds only for the one-point union of a finite number of spaces.

Theorem 4.2 Let $X$ be a $n$-dimensional simplicial complex, $X^p$ be the $p$-skeleton of $X$ for $0 \leq p \leq n$ and $C^p(X, \mathbb{Z})$ be the group of simplicial $p$-cochains. Then, for any $p$ such that $0 \leq 2p \leq n$ or $0 \leq 2p + 1 \leq n$, there are isomorphisms:

$$
\Phi^{2p} : C^{2p}(X, \mathbb{Z}) \rightarrow K(X^{2p}, X^{2p-1}) \\
\Phi^{2p+1} : C^{2p+1}(X, \mathbb{Z}) \rightarrow K^{1}(X^{2p+1}, X^{2p})
$$

which can be summarized by:

$$
\Psi^p : C^p(X, \mathbb{Z}) \rightarrow K^p(X^p, X^{p-1}).
$$

Moreover:

$$
K^1(X^{2p}, X^{2p-1}) = K(X^{2p+1}, X^{2p}) = 0.
$$

Proof: We denote the simplicial structure of $X$ by $\Delta = \{ \Delta^m_i \}_k$, where $m$ is the dimension of the simplex and $i$ enumerates the $m$-simplices, so that $X^{2p} = \bigcup_{i=0}^k \Delta^{2p}_i$. Then the quotient by $X^{2p-1}$ is given by $k$ spheres of dimension $2p$ attached to a point:

$$
X^{2p}/X^{2p-1} = \bigcup_i S^{2p}_i.
$$

By lemma (4.1) we obtain $\tilde{K}(X^{2p}/X^{2p-1}) = \bigoplus_i \tilde{K}(S^{2p})$, and, by Bott periodicity, $\tilde{K}(S^{2p}) = \tilde{K}(S^0) = \mathbb{Z}$. Hence:

$$
K(X^{2p}, X^{2p-1}) = \bigoplus_i \mathbb{Z} = C^{2p}(X, \mathbb{Z}).
$$

For the odd case, let $X^{2p+1} = \bigcup_{j=0}^h \Delta^{2p+1}_j$. We have by lemma (4.1)

$$
K^1(X^{2p+1}, X^{2p}) = \tilde{K}^1\left( \bigcup_j S^{2p+1}_j \right) = \bigoplus_j \tilde{K}^1(S^{2p+1}_j) \\
= \bigoplus_j \tilde{K}(S^{2p+2}_j) = \bigoplus_j \mathbb{Z} = C^{2p+1}(X, \mathbb{Z}).
$$

In the same way, $K^1(X^{2p}, X^{2p-1}) = \bigoplus_j \tilde{K}^1(S^{2p}_j) = \bigoplus_j \tilde{K}(S^{2p+1}_j) = 0$, and similarly for $K(X^{2p+1}, X^{2p})$. □
The explicit isomorphisms $\Phi^{2p}$ and $\Phi^{2p+1}$ are given by:

$$\Phi^{2p}(\Delta^{2p}_i) = \begin{cases} (-1)^p(\eta - 1)^{2p} \in \tilde{K}(S^{2p}_i) & \text{for } j \neq i \\ 0 \in \tilde{K}(S^{2p}_j) & \end{cases}$$

and:

$$\Phi^{2p+1}(\Delta^{2p+1}_i) = \begin{cases} (-1)^{p+1}(\eta - 1)^{3(p+1)} \in \tilde{K}(S^{2p+1}_i) & \\ 0 \in \tilde{K}(S^{2p+1}_j) & \text{for } j \neq i \end{cases}$$

where we put the overall factors $(-1)^p$ and $(-1)^{p+1}$ for coherence with (12).

**Remark:** such isomorphisms are canonical, since every simplex is supposed to be oriented and $\eta - 1$ is distinguishable from $1 - \eta$ also up to automorphisms of $X$ (in the first case the trivial bundle has negative coefficient, in the second case the non-trivial one, so that, for example, they have opposite first Chern class).

### 4.3 The spectral sequence

We now build the spectral sequence. We use the groups:

$$H(p, q) = \bigoplus_n K^n(X^{q-1}, X^{p-1}).$$

#### 4.3.1 The first step

The first step is:

$$E^p_1 = H(p, p + 1) = \bigoplus_n K^n(X^p, X^{p-1}).$$

We now consider the presence of the grading in the spectral sequence (v. [5]). Since $K^n$ is determined by the parity of $n$, we use the $\mathbb{Z}_2$-index $\sigma$:

$$E^p_1, \sigma = K^{p+\sigma}(X^p, X^{p-1}).$$

By theorem [12] we have isomorphisms:

$$E^{2p,0}_{1} \simeq C^{2p}(X, \mathbb{Z}) \quad E^{2p,1}_{1} = 0 \quad E^{2p+1,0}_{1} \simeq C^{2p+1}(X, \mathbb{Z}) \quad E^{2p+1,1}_{1} = 0.$$

Since $K(x_0) = \mathbb{Z}$ and $K^1(x_0) = 0$, we can write in a compact form:

$$E^p_1, \sigma \simeq C^p(X, K^\sigma(x_0)). \quad (18)$$

For $r = 1$, in the diagram (15) at page 13 it is $\Psi_1 = \Psi_2 = \text{id}$, hence $d^{p}_{1} = \Delta_2$, i.e., $d^{p}_{1} = \Delta^{p,p+1,p+2}$. In particular:

$$d^{p}_{1} : \bigoplus_n K^n(X^p, X^{p-1}) \longrightarrow \bigoplus_n K^n(X^{p+1}, X^p)$$
is the composition:

\[
\begin{array}{ccc}
\tilde{K}^{p+\sigma}(X^p/X^{p-1}) & \xrightarrow{d_1^{p,\sigma}} & \tilde{K}^{p+\sigma+1}(X^{p+1}/X^p) \\
\downarrow{\pi^*} & & \downarrow{\delta} \\
\tilde{K}^{p+\sigma}(X^p) & & 
\end{array}
\]

Another way to describe \(d_1^p\) can be obtained considering the exact sequence induced by the maps:

\[
\begin{array}{c}
i: X^p/X^{p-2} \to X^p/X^{p-1} \\
\pi: X^p/X^{p-2} \to X^p/X^{p-1} = \frac{X^p/X^{p-2}}{X^{p-1}/X^{p-2}} \\
j: X^p/X^{p-1} \to X^{p+1}/X^{p-1}
\end{array}
\]

These maps induce a commutative diagram:

\[
\begin{array}{ccc}
\tilde{K}^{p+\sigma}(X^p/X^{p-1}) & \xrightarrow{j^*} & \tilde{K}^{p+\sigma+1}(X^{p+1}/X^{p-1}) \\
\downarrow{\pi^*} & & \downarrow{i^*} \\
\tilde{K}^{p+\sigma}(X^p/X^{p-2}) & & \tilde{K}^{p+\sigma+1}(X^p/X^{p-2}) 
\end{array}
\]

where \(i^*, j^*, \pi^*\) are maps of the \(\Psi\)-type. We have that \(E_2^{p,\sigma} = \text{Im } i^*\) by (20).

We now prove that:

1. \(\text{Ker } d_1^{p,\sigma} = \text{Im } j^*\);
2. \(\text{Im } d_1^{p-1,\sigma} = \text{Ker } \pi^*\).
The first statement follows directly from (19) using the exact sequence:

$$\cdots \longrightarrow \tilde{K}^{p+\sigma}(X^{p+1}/X^{p-1}) \xrightarrow{j^*} \tilde{K}^{p+\sigma}(X^p/X^{p-1}) \xrightarrow{d_{p-1}^\sigma} \tilde{K}^{p+\sigma+1}(X^{p+1}/X^p) \longrightarrow \cdots$$

and the second by the exact sequence:

$$\cdots \longrightarrow \tilde{K}^{p+\sigma-1}(X^{p-1}/X^{p-2}) \xrightarrow{d_{p-1}^\sigma} \tilde{K}^{p+\sigma}(X^p/X^{p-1}) \xrightarrow{\pi^*} \tilde{K}^{p+\sigma}(X^p/X^{p-2}) \longrightarrow \cdots$$

Since \( \text{Im } i^* \simeq H^p(X,\mathbb{Z}) \) and \( d_{p,0}^p \) corresponds to the simplicial coboundary under this isomorphism, it follows that:

- cocycles in \( C^p(X,\mathbb{Z}) \) correspond to classes in \( \text{Im } j^* \), i.e., to classes in \( \tilde{K}^{p+\sigma}(X^p/X^{p-1}) \) that are restriction of classes in \( \tilde{K}^{p+\sigma}(X^{p+1}/X^{p-1}) \);
- coboundaries in \( C^p(X,\mathbb{Z}) \) corresponds to classes in \( \text{Ker } \pi^* \), i.e., to classes in \( \tilde{K}^{p+\sigma}(X^p/X^{p-1}) \) that are 0 when lifted to \( \tilde{K}^{p+\sigma}(X^p/X^{p-2}) \);
- \( \text{Im } \pi^* \) corresponds to cochains (not only cocycles) up to coboundaries and its subset \( \text{Im } i^* \) corresponds to cohomology classes;
- given \( \alpha \in \text{Im } i^* \), we can lift it to a class in \( \tilde{K}(X^p/X^{p-1}) \) choosing different trivializations on \( X^{p-1}/X^{p-2} \), and the different homotopy classes of such trivializations determine the different representative cocycles of the class.

### 4.3.3 The last step

**Notation:** we denote \( i_p : X^p \to X \) and \( \pi_p : X \to X/X^p \) for any \( p \).

We recall equation (17):

$$E^p_{\infty} = \text{Im}\left( H(p, +\infty) \xrightarrow{\Psi} H(0, p+1) \right)$$

which, in our case, becomes:

$$E^{p, \sigma}_{\infty} = \text{Im}\left( K^{p+\sigma}(X, X^{p-1}) \xrightarrow{\Psi} K^{p+\sigma}(X^p) \right)$$

where \( \Psi \) is obtained by the pull-back of \( i : X^p \to X/X^{p-1} \). Since \( i = \pi_{p-1} \circ i_p \), the following diagram commutes:

$$\begin{array}{ccc}
\tilde{K}^{p+\sigma}(X/X^{p-1}) & \xrightarrow{\Psi} & \tilde{K}^{p+\sigma}(X^p) \\
\downarrow{\pi_{p-1}^*} & & \downarrow{i_p^*} \\
\tilde{K}^{p+\sigma}(X) & & \\
\end{array}$$

**Remark:** in the previous triangle we cannot say that \( i_p^* \circ \pi_{p-1}^* = 0 \) by exactness, since by exactness \( i_p^* \circ \pi_p^* = 0 \) at the same level \( p \), as follows by \( X^p \to X \to X/X^p \).
By exactness of $K^{p+\sigma}(X, X^{p-1}) \xrightarrow{\pi_p} K^{p+\sigma}(X) \xrightarrow{i_p} K^{p+\sigma}(X^{p-1})$, we deduce that:

$$\text{Im } \pi_{p-1} = \text{Ker } i_{p-1}.$$  

Since trivially $\text{Ker } i_p \subset \text{Ker } i_{p-1}$, we obtain that $\text{Ker } i_p \subset \text{Im } \pi_{p-1}$. Moreover:

$$\text{Im } \Psi = \text{Im } (i_p \circ \pi^*) = \text{Im } \left( i_p \mid_{\text{Im } \pi_{p-1}} \right) \simeq \frac{\text{Im } \pi_{p-1}}{\text{Ker } i_p} = \frac{\text{Ker } i_{p-1}}{\text{Ker } i_p}$$

hence, finally:

$$E_{\infty}^{p,0} = \frac{\text{Ker } (K^p(X) \twoheadrightarrow K^p(X^{p-1}))}{\text{Ker } (K^p(X) \twoheadrightarrow K^p(X))}$$

$$E_{\infty}^{p,1} = 0$$  

(25)

i.e., $E_{\infty}^{p,0}$ is made by $p$-classes on $X$ which are 0 on $X^{p-1}$, up to classes which are 0 on $X^p$.

4.3.4 From the first to the last step

We now see how to link the first and the last step of the sequence. In general, as we have seen, it is:

$$E_1^p = H(p, p+1) \quad E_\infty^p = \text{Im } (H(p, +\infty) \xrightarrow{\Psi} H(0, p+1)).$$

There is a natural $\Psi$-map:

$$\iota : H(p, +\infty) \twoheadrightarrow H(p, p+1)$$

so that an element $\alpha \in E_1^p$ survives up to the last step if and only if $\alpha \in \text{Im } \iota$ and its class in $E_\infty^p$ is $\Psi \circ (\iota^{-1})(\alpha)$, which is well-defined since $\text{Ker } \iota \subset \text{Ker } \Psi$. We thus define, for $\alpha \in \text{Im } \iota \subset E_1^p$:

$$\{ \alpha \}_{E_\infty^p}^{(1)} := \Psi \circ (\iota^{-1})(\alpha).$$

where the upper 1 means that we are starting from the first step.

For AHSS, considering $p$ even and $\sigma = 0$, this becomes:

$$E_1^{p,0} = K(X^p, X^{p-1}) \quad E_\infty^{p,0} = \text{Im } (K(X, X^{p-1}) \xrightarrow{\Psi} K(X^p))$$

and:

$$\iota : K(X, X^{p-1}) \twoheadrightarrow K(X^p, X^{p-1}).$$

In this case, $\iota = i^*$ for $i : X^p/X^{p-1} \to X/X^{p-1}$. Thus, the classes in $E_1^{p,0}$ surviving up to the last step are the ones which are restrictions of a class defined on all $X/X^{p-1}$. Moreover, $\Psi = j^*$ for $j : X^p \to X/X^{p-1}$, and $j = i \circ \pi^p$ for $\pi^p : X^p \to X^p/X^{p-1}$. Hence $\Psi = (\pi^p)^* \circ \iota$, so that, for $\alpha \in \text{Im } \iota \subset E_1^{p,0}$:

$$\{ \alpha \}_{E_\infty^{p,0}}^{(1)} := (\pi^p)^*(\alpha).$$

(26)
Since in the following we’ll need to start from an element $\beta \in E_{2}^{p,0}$, which survives up to the last step, we also define:

$$\{\beta\}_{E_{2}^{p,0}}^{(2)}$$

as the class in $E_{2}^{p,0}$ corresponding to $\beta$.

### 4.4 Rational K-theory and cohomology

We now consider Atiyah-Hirzebruch spectral sequence in the rational case. In particular, we consider the groups:

$$H(p, q) = \bigoplus_{n} K_{Q}^{n}(X^{q-1}, X^{p-1})$$

where $K_{Q}^{n}(X, Y) := K^{n}(X, Y) \otimes \mathbb{Q}$. In this case the sequence is made by the groups $Q^{p, \sigma} = E_{r}^{p, \sigma} \otimes \mathbb{Q}$. In particular:

$$Q^{0,0}_{2} \simeq H^{0}(X, \mathbb{Q}) \quad Q^{1,0}_{2} = 0$$

$$Q^{p,0}_{\infty} = \frac{\text{Ker}(K_{Q}^{p}(X) \rightarrow K_{Q}^{p}(X^{p-1}))}{\text{Ker}(K_{Q}^{p}(X) \rightarrow K_{Q}^{p}(X^{p}))} \quad Q^{p,1}_{\infty} = 0.$$

Such sequence collapses at the second step (v. [3]), hence $Q^{p,0}_{\infty} \simeq Q^{p,0}_{2}$. Since:

- $\bigoplus_{p} Q^{p,0}_{\infty}$ is the graded group associated to the chosen filtration of $K(X) \oplus K^{1}(X)$;
- in particular, by (27), $\bigoplus_{2p} Q^{2p,0}_{\infty}$ is the graded group of $K(X)$ and $\bigoplus_{2p+1} Q^{2p+1,0}_{\infty}$ is the graded group of $K^{1}(X)$;
- $Q^{0,0}_{\infty} \simeq H^{p}(X, \mathbb{Q})$, thus has no torsion;

it follows that:

$$K_{Q}(X) = \bigoplus_{2p} Q^{2p,0}_{\infty} \quad K_{Q}^{1}(X) = \bigoplus_{2p+1} Q^{2p+1,0}_{\infty}$$

hence:

$$K_{Q}(X) \simeq H^{ev}(X, \mathbb{Q}) \quad K_{Q}^{1}(X) \simeq H^{odd}(X, \mathbb{Q}).$$

In particular, the isomorphisms of the last equation are given by Chern character:

$$\text{ch} : K_{Q}(X) \rightarrow H^{ev}(X, \mathbb{Q})$$

$$\text{ch} : K_{Q}^{1}(X) \rightarrow H^{ev}(S^{1}X, \mathbb{Q}) \simeq H^{odd}(X, \mathbb{Q})$$

are isomorphisms of rings.
5 Gysin map and AHSS

5.1 Even case

We choose a finite triangulation of $X$ which restricts to a triangulation of $Y$ (v. [20]). We use the following notation:

- we denote the triangulation of $X$ by $\Delta = \{\Delta^m_i\}$, where $m$ is the dimension of the simplex and $i$ enumerates the $m$-simplices;
- we denote by $X^p_\Delta$ the $p$-skeleton of $X$ with respect to $\Delta$.

**Theorem 5.1** Let $X$ be an $n$-dimensional compact manifold and $Y \subset X$ a $p$-dimensional embedded compact submanifold. Let:

- $\Delta = \{\Delta^m_i\}$ be a triangulation of $X$ which restricts to a triangulation $\Delta' = \{\Delta^m_i\}$ of $Y$;
- $D = \{D_j^{n-m}\}$ be the dual decomposition of $X$ with respect to $\Delta$;
- $\tilde{D} \subset D$ be subset of $D$ made by the duals of simplices in $\Delta'$.

Then:

- the interior of $|\tilde{D}|$ is a tubular neighborhood of $Y$ in $X$;
- the interior of $|\tilde{D}|$ does not intersect $X^{n-p-1}$, i.e.:

$$|\tilde{D}| \cap X^{n-p-1} \subset \partial|\tilde{D}|.$$

**Proof:** The $n$-simplices of $\tilde{D}$ are the dual of the vertices of $\Delta'$. Let $\tau = \{\tau^m_i\}$ be the first baricentric subdivision of $\Delta$. For each vertex $\Delta^0_i$ (thought as an element of $\Delta$), its dual is:

$$\tilde{D}^n_i = \bigcup_{\Delta^0_j \in \tau^m_i} \tau^n_j. \quad (28)$$

Moreover, if $\tau' = \{\tau^m_i\}$ is the first baricentric subdivision of $\Delta'$ and $D'$ is the dual of $\Delta'$ in $Y$, then:

$$D'^p_i = \bigcup_{\Delta^0_j \in \tau^m_i} \tau^n_j. \quad (29)$$

and:

$$\tilde{D}^n_i \cap Y = D'^p_i.$$

Moreover, let us consider the $(n-p)$-simplices in $\tilde{D}$ contained in $\partial \tilde{D}^n_i$ (for the fixed $i'$ of formula (28)), i.e. $\tilde{D}^{n-p} \cap \tilde{D}^n_i$: it intersects $Y$ transversally in the baricenters of each $p$-simplex of $\Delta'$ containing $\Delta^0_i$: we call such baricenters $\{b_1, \ldots, b_k\}$ and the intersecting $(n-p)$-cells $\tilde{D}^{n-p}_{j=1,\ldots,k}$. Since (for a fixed $i'$) $\tilde{D}^n_i$ retracts on $\Delta^0_i$, we can consider a local chart $(U_{i'}, \varphi_{i'})$, with $U_{i'} \subset \mathbb{R}^n$ neighborhood of 0, such that:

- $\varphi_{i'}^{-1}(U_{i'})$ is a neighborhood of $\tilde{D}^n_i$;
\[ \varphi_i(D_i'p) \subset U \cap \{0\} \times \mathbb{R}^p \text{, for } 0 \in \mathbb{R}^{n-p} \text{ (v. eq. 22)}; \]

\[ \varphi_i(\tilde{D}_j^{n-p}) \subset U \cap (\mathbb{R}^{n-p} \times \pi_p(\varphi(b_j))), \text{ for } \pi_p : \mathbb{R}^n \rightarrow \{0\} \times \mathbb{R}^p \text{ the projection.} \]

We now consider the natural foliation of \( U \) given by the intersection with the hyperplanes \( \mathbb{R}^{n-p} \times \{x\} \) and its image via \( \varphi^{-1} \): in this way, we obtain a foliation of \( \tilde{D}_i^{p} \) transversal to \( Y \). If we do this for any \( i' \), by construction the various foliations glue on the intersections, since such intersections are given by the \((n-p)\)-cells \( \{\tilde{D}_n^{n-p}\} \) and the interior gives a \( C^0 \)-tubular neighborhood of \( Y \).

Moreover, a \((n-p-r)\)-cell of \( \tilde{D} \), for \( r > 0 \), cannot intersect the brane since it is contained in the boundary of a \((n-p)\)-cell, and such cells intersect \( Y \), which is done by \( p \)-cells, only in their interior points \( b_j \).

We now consider triples \((X,Y,D)\) satisfying the following condition:

\((\#)\) \( X \) is an \( n \)-dimensional compact manifold and \( Y \subset X \) a \( p \)-dimensional embedded compact submanifold, such that \( n - p \) is even and \( N(Y) \) is \( \text{spin}^c \). Moreover, \( D \) and \( \tilde{D} \) are defined as in theorem 5.1.

**Lemma 5.2** Let \((X,Y,D)\) be a triple satisfying \((\#)\), \( U = \text{Int}|\tilde{D}| \) and \( \alpha \in K(Y) \). Then:

- there exists a neighborhood \( V \) of \( X \setminus U \) such that \( \iota(i)(\alpha) \big|_V = 0 \);
- in particular, \( \iota(i)(\alpha) \big|_{X^{n-p-1}} = 0 \).

**Proof:** By equation (14) at page 113

\[ \iota(i)(\alpha) = \psi^* \beta, \quad \beta = (\varphi_U^*)^* \circ T(\alpha) \in K(U^+). \]

Let \( \beta = [E] - [n] \), and let \( V_{\infty} \subset U^+ \) be a neighborhood of \( \infty \) which trivializes \( E \). Then \( (\psi^* E) \big|_{\psi^{-1}(V_{\infty})} \) is trivial. Hence, for \( V = \psi^{-1}(V_{\infty}) \):

\[ (\psi^* \beta) \big|_V = [(\psi^* E) \big|_V] - [n] = [n] - [n] = 0. \]

By theorem 5.1, \( X^{n-p-1}_D \) does not intersect the tubular neighborhood \( \text{Int}|\tilde{D}| \) of \( Y \), hence \( X^{n-p-1}_D \subset \psi^{-1}(V_{\infty}) = V \), so that \( (\psi^* \beta) \big|_{X^{n-p-1}_D} = 0 \).

\( \square \)

### 5.1.1 Trivial bundle

We start by considering the case of a trivial bundle.
Theorem 5.3 Let \((X, Y, D)\) be a triple satisfying (\#) and \(\Phi_{D}^{n-p} : C^{n-p}(X, \mathbb{Z}) \rightarrow K(X_{D}^{n-p}, X_{D}^{n-p-1})\) be the isomorphism stated in theorem 4.2. Let:

\[
\pi^{n-p} : X_{D}^{n-p} \rightarrow X_{D}^{n-p}/X_{D}^{n-p-1}
\]

be the projection and \(\hat{\text{PD}}(Y_{\Delta})\) be the representative of \(\text{PD}_{X}Y\) given by the sum of the cells dual to the \(p\)-cells of \(\Delta\) covering \(Y\). Then:

\[
i_{!} (Y \times \mathbb{C})_{|X_{D}^{n-p}} = (\pi^{n-p})^{*} (\Phi_{D}^{n-p} (\hat{\text{PD}}(Y_{\Delta}))).
\]

**Proof:** We define:

\[
(U^{+})_{D}^{n-p} = \dfrac{X_{D}^{n-p} \setminus U}{X_{D}^{n-p-1} \setminus \rho U}
\]

so that \((U^{+})_{D}^{n-p} \subset U^{+}\) sending the denominator to \(\infty\) (the numerator is exactly \(\hat{D}^{n-p}\) of theorem 5.1). We also define:

\[
\psi^{n-p} = \psi_{|X_{D}^{n-p}} : X_{D}^{n-p} \rightarrow (U^{+})_{D}^{n-p}.
\]

\(\psi^{n-p}\) is well-defined since the \((n-p)\)-simplices outside \(U\) and all the \((n-p-1)\)-simplices are sent to \(\infty\) by \(\psi\).

It is:

\[
\pi^{n-p} (X_{D}^{n-p}) \simeq \bigcup_{i \in I} S_{i}^{n-p}.
\]

We denote by \(\{S_{j}^{n-p}\}_{j \in J}\), with \(J \subset I\), the set of \((n-p)\)-spheres corresponding to \(\pi^{n-p} (X_{D}^{n-p} \setminus U)\). We define:

\[
\rho : \bigcup_{i \in I} S_{i}^{n-p} \rightarrow \bigcup_{j \in J} S_{j}^{n-p}
\]

as the projection, i.e., \(\rho\) is the identity of \(S_{j}^{n-p}\) for every \(j \in J\) and sends all the spheres in \(\{S_{i}^{n-p}\}_{i \in I \setminus J}\) to the attachment point. We have that:

\[
\psi^{n-p} = \rho \circ \pi^{n-p}.
\]

In fact, the boundary of the \((n-p)\)-cells intersecting \(U\) is contained in \(\partial U\), hence it is sent to \(\infty\) by \(\psi^{n-p}\), while all the \((n-p)\)-cells outside \(U\) are sent to \(\infty\): hence, the image of \(\psi^{n-p}\) is homeomorphic to \(\bigcup_{j \in J} S_{j}^{n-p}\) sending \(\infty\) to the attachment point. Thus:

\[
(\psi^{n-p})^{*} = (\pi^{n-p})^{*} \circ \rho^{*}.
\]

We put \(\mathcal{N} = \mathcal{N}(Y)\) and \(\tilde{\lambda}_{\mathcal{N}} = (\varphi_{U})^{*} (\lambda_{\mathcal{N}})\). By lemma 3.1 and equation (14) at page 13 it is \(i_{!} (Y \times \mathbb{C}) = \psi^{*} (\tilde{\lambda}_{\mathcal{N}})\). Then:

\[
i_{!} (Y \times \mathbb{C})_{|X_{D}^{n-p}} = \psi^{*} (\tilde{\lambda}_{\mathcal{N}})_{|X_{D}^{n-p}} = (\psi^{n-p})^{*} (\tilde{\lambda}_{\mathcal{N}} |_{(U^{+})_{D}^{n-p}})
\]

and

\[
\rho^{*} (\tilde{\lambda}_{\mathcal{N}} |_{(U^{+})_{D}^{n-p}}) = \Phi_{D}^{n-p} (\hat{\text{PD}}(Y_{\Delta}))
\]

since:
• $\widehat{PD}(Y_\Delta)$ is the sum of the $(n-p)$-cells intersecting $U$;

• hence $\Phi^{n-p}_D(\widehat{PD}(Y_\Delta))$ gives a $(-1)^{\frac{n-p}{2}}(\eta-1)^{\frac{n-p}{2}}$ factor to each sphere $S_j^{n-p}$ for $j \in J$ and 0 otherwise;

• but this is exactly $\rho^*(\tilde{\lambda}_N|_{(U^+)_D^{n-p}})$ since by equation (13) at page 12 it is, for $y \in Y$:

$$\lambda_N|_{\Lambda_\Delta^{n-p}} = \lambda_{R^{n-p}} = (-1)^{\frac{n-p}{2}}(\eta-1)^{\frac{n-p}{2}}$$

and for the spheres outside $U$, that $\rho$ sends to $\infty$, we have that:

$$\rho^*(\tilde{\lambda}_N|_{(U^+)_D^{n-p}})|_{\bigcup_{i \in \Gamma} S_i^{n-p}} = \rho^*(\tilde{\lambda}_N|_{\bigcup_{i \in \Gamma} S_i^{n-p}}) = \rho^*(\tilde{\lambda}_N|_{\{\infty\}}) = \rho^*(0) = 0.$$  

Hence:

$$i_!(Y \times \mathbb{C}) |_{X_D^{n-p}} = (\psi^{n-p})^*(\tilde{\lambda}_N|_{(U^+)_D^{n-p}}) = (\pi^{n-p})^* \circ \rho^*(\tilde{\lambda}_N|_{(U^+)_D^{n-p}}) = (\pi^{n-p})^* \Phi^{n-p}_D(\widehat{PD}(Y_\Delta)).$$



\[\Box\]

**Corollary 5.4** Let $(X,Y,D)$ be a triple satisfying $(\#)$ and $\Xi^{n-p} : H^{n-p}(X,\mathbb{Z}) \to \text{Im } \Psi \subset K(X_D^{n-p}, X_D^{n-p-2})$ be the isomorphism (21). Let:

$\bar{\pi}^{n-p} : X_D^{n-p} \to X_D^{n-p} / X_D^{n-p-2}$

be the projection. Then:

$$i_!(Y \times \mathbb{C}) |_{X_D^{n-p}} = (\bar{\pi}^{n-p})^*(\Xi^{n-p}(PD(Y))).$$

\[\textbf{Proof:}\] For $\tau \in Z^{n-p}(X,\mathbb{Z})$ and $\pi^*$ the map of the diagram (22), it is $\Xi^{n-p}_D(\tau) = \pi_* \Phi^{n-p}_D(\tau)$, and $(\bar{\pi}^{n-p})^* \circ \pi^* = (\pi^{n-p})^*$ since $\pi \circ \bar{\pi}^{n-p} = \pi^{n-p}$.

\[\Box\]

The following theorem encodes the link among Gysin map and AHSS: since the groups $E_r^{p,\sigma}$ for $r \geq 2$ and the filtration $\text{Ker}(K(X) \to K(X^{n-p}))$ of $K(X)$ do not depend on the particular simplicial structure chosen (v. 3), we can drop the dependence on $D$.

**Theorem 5.5** Let $X$ be an $n$-dimensional compact manifold and $Y \subset X$ a $p$-dimensional embedded compact submanifold, such that $n-p$ is even and $N(Y)$ is spin$^c$. Let $\{(E_r^{p,\sigma})\}$ be the Atiyah-Hirzebruch spectral sequence, and let $\Xi^{n-p} : H^{n-p}(X,\mathbb{Z}) \to E_2^{n-p,0}$ be the isomorphism induced by $\Phi^{n-p}$. Let us suppose that $\Xi^{n-p}PD(Y)$ is contained in the kernel of all the boundaries $d_r^{n-p,0}$ for $r \geq 2$.
With this data, we define a class:

\[
\{ \Xi^{n-p} \text{PD}(Y) \}_{E_{n-p,0}^{\infty}}^{(2)} = \frac{\text{Ker}(K(X) \to K(X^{n-p-1}))}{\text{Ker}(K(X) \to K(X^{n-p}))}.
\]

Then:

\[
\{ \Xi^{n-p} \text{PD}(Y) \}_{E_{n-p,0}^{\infty}}^{(2)} = [i_!(Y \times \mathbb{C})]_{E_{n-p,0}^{\infty}}.
\]

**Proof:** We use the cellular decomposition $D$ considered in the previous theorems. By equations (23) and (24) we have:

\[
E_{n-p,0}^{\infty} = \text{Im} \left( \tilde{K}(X/X_D^{n-p-1}) \xrightarrow{\Phi} \tilde{K}(X_D^{n-p}) \right)
\]

and, given a representative $\alpha \in \text{Ker}(K(X) \to K(X^{n-p-1})) = \text{Im} \pi_{n-p}^*$, we have that

\[
\left[ \alpha \right]_{E_{n-p,0}^{\infty}} = \pi_{n-p}^*(\alpha) = \alpha|_{X_D^{n-p}}.
\]

We have that:

- the class \( \{ \Xi^{n-p} \text{PD}(Y) \}_{E_{n-p,0}^{\infty}}^{(2)} \), which by construction is equal to \( \{ \Phi_{D_n}^{n-p} \text{PD}(Y) \}_{E_{n-p,0}^{\infty}}^{(1)} \), by formula (23) is given as an element of \( \tilde{K}(X_D^{n-p}) \) by \( (\pi^{n-p})^*(\Phi_{D_n}^{n-p} \text{PD}(Y)) \), for \( \pi^{n-p} : X_D^{n-p} \to X_D^{n-p}/X_D^{n-p-1} \);
- by lemma 5.3 it is \( i_!(Y \times \mathbb{C}) \in \text{Ker}(K(X) \to K(X_D^{n-p-1})) \), hence \( [i_!(Y \times \mathbb{C})]_{E_{n-p,0}^{\infty}} \) is well-defined, and, by exactness, \( i_!(Y \times \mathbb{C}) \in \text{Im} \pi_{n-p}^* \);
- by theorem 5.3 it is \( i_{n-p}^*(i_!(Y \times \mathbb{C})) = (\pi^{n-p})^*(\Phi_{D_n}^{n-p} \text{PD}(Y)) \);
- hence \( \{ \Phi_{D_n}^{n-p} \text{PD}(Y) \}_{E_{n-p,0}^{\infty}}^{(1)} = [i_!(Y \times \mathbb{C})]_{E_{n-p,0}^{\infty}} \).

\( \square \)

Let us consider a generic trivial bundle \( [r] = Y \times \mathbb{C}^r \). By lemma 3.1 at page 10 we have that \( [r] \cdot \lambda_N = \lambda_N^{r} \), hence theorem 5.3 becomes:

\[
i_!(Y \times \mathbb{C}^r)|_{X_D^{n-p}} = (\pi^{n-p})^*(\Phi_{D_n}^{n-p} \text{PD}(r \cdot Y_\Delta))
\]

and theorem 5.3 becomes:

\[
\{ \Xi^{n-p} \text{PD}(r \cdot Y) \}_{E_{n-p,0}^{\infty}}^{(2)} = [i_!(Y \times \mathbb{C}^r)]_{E_{n-p,0}^{\infty}}.
\]
5.1.2 Generic bundle

If we consider a generic bundle $E$ over $Y$ of rank $r$, we can prove that $i_!(E)$ and $i_!(Y \times \mathbb{C}^r)$ have the same restriction to $X_D^{n-r}$: in fact, the Thom isomorphism gives $T(E) = E \cdot \lambda_N$ and, if we restrict $E \cdot \lambda_N$ to a finite family of fibers, which are transversal to $Y$, the contribution of $E$ becomes trivial, so it has the same effect of the trivial bundle $Y \times \mathbb{C}^r$. We now prove this.

**Lemma 5.6** Let $(X,Y,D)$ be a triple satisfying $(\#)$ and $E \xrightarrow{\pi} Y$ a bundle of rank $r$. Then:

$$i_!(E)|_{X_D^{n-r}} = i_!(Y \times \mathbb{C}^r)|_{X_D^{n-r}}.$$

**Proof:** referring to the notations in the proof of lemma 3.1 at page 10, we have that:

$$E \cdot \lambda_N = i^*(\tilde{\pi}^*)^{-1}(E \boxtimes \lambda_N) = i^*(\tilde{\pi}^*)^{-1}(\pi_1^*E \otimes \pi_2^*\lambda_N).$$

Since $X_D^{n-r}$ intersects the tubular neighborhood in a finite number of cells corresponding under $\varphi_U$ to a finite number of fibers of $N$, it is sufficient to prove that, for any $y \in Y$, $(E \cdot \lambda_N)|_{N^+_y} = \lambda_N^y|_{N^+_y}$. First of all:

- $i(N^+_y) = (\{y\} \times N_y)^+ \subset (\{y\} \times N)^+$;
- $E \cdot \lambda_N|_{N^+_y} = (i|_{N^+_y})^*\{[(\tilde{\pi}^*)^{-1}(\pi_1^*E \otimes \pi_2^*\lambda_N)]|_{i(N^+_y)}\}$.

To obtain the bundle $[(\tilde{\pi}^*)^{-1}(\pi_1^*E \otimes \pi_2^*\lambda_N)]|_{i(N^+_y)}$, we can restrict $\tilde{\pi}$ to:

$$A = \tilde{\pi}^{-1}[i(N^+_y)] = \tilde{\pi}^{-1}[(\{y\} \times N_y)^+] = (\{y\} \times N_y^+) \cup (Y \times \{\infty\}) \cup (\{\infty\} \times N^+)$$

and consider $(\tilde{\pi}|_A)^{-1}[\tilde{\pi}^*(\pi_1^*E \otimes \pi_2^*\lambda_N)|_A]$. Moreover:

- $(\pi_1^*E \otimes \pi_2^*\lambda_N)|_{\{y\} \times N_y^+} = (\mathbb{C}^r \otimes \pi_2^*\lambda_N)|_{\{y\} \times N_y^+} \simeq \lambda_N^y|_{N_y^+}$;
- $(\pi_1^*E \otimes \pi_2^*\lambda_N)|_{Y \times \{\infty\}} = (\pi_1^*E \otimes 0)|_{Y \times \{\infty\}} = 0$;
- $(\pi_1^*E \otimes \pi_2^*\lambda_N)|_{\{\infty\} \times N^+} = (0 \otimes \pi_2^*\lambda_N)|_{\{\infty\} \times N^+} = 0$.

Hence, since the three components of $A$ intersect each other at most at one point, by lemma 4.1 at page 15 we obtain:

$$(\pi_1^*E \otimes \pi_2^*\lambda_N)|_A = (\pi_1^*(Y \times \mathbb{C}^r) \otimes \pi_2^*\lambda_N)|_A.$$
5.2 Odd case

We now consider the case of $n - p$ odd. We thus take into account the unreduced suspension $\hat{S}^1 X$ and the natural embedding $i^1 : Y \to \hat{S}^1 X$. Let $U$ be the tubular neighborhood of $Y$ in $X$, and let $U^1 \subset \hat{S}^1 X$ be the tubular neighborhood of $Y$ in $\hat{S}^1 X$ obtained by removing the vertices of the double cone to $\hat{S}^1 U$. Then, since $K^1(X) \simeq \hat{K}(\hat{S}^1 X)$, we consider the Gysin map:

$$i^1_! : K(Y) \to K^1(X).$$

With the neighborhood $U^1$ considered, we have that $\bar{S}^1(X^{n-p}_{p,D}|_{\partial U}) \subset \bar{U}^1$ and $\hat{S}^1(X^{n-p-1}_{p,D}|_{\partial U}) \subset \partial U^1$, where $\partial U^1$ contains also the vertices of the double cone. In this way we can reformulate the previous results in the odd case, considering $\hat{S}^1(X^{n-p})$ and $\hat{S}^1(X^{n-p-1})$ rather than $X^{n-p}_D$ and $X^{n-p-1}_D$.

We consider triples $(X,Y,D)$ satisfing the following condition:

($\#^1$) $X$ is an $n$-dimensional compact manifold and $Y \subset X$ a $p$-dimensional embedded compact submanifold, such that $n - p$ is odd and $N(Y)$ is spin$^c$. Moreover, $D$ is the dual decomposition of $\Delta$ as in theorem 5.1.

We now reformulate the same theorems stated for the even case, which can be proved in the same way. We remark that $\mathcal{N}_{\hat{S}_D} Y$ is spin$^c$ if and only $\mathcal{N}_X Y$ is, since $\mathcal{N}_{\hat{S}_D} Y = \mathcal{N}_X Y \oplus 1$ so that, by axioms of characteristic classes (v. [17]), $W_3$ must be the same.

**Lemma 5.7** Let $(X,Y,D)$ be a triple satisfying ($\#^1$) and $\alpha \in K(Y)$. Then:

- there exists a neighborhood $V$ of $X \setminus U^1$ such that $i^1_!(\alpha)|_V = 0$;
- in particular, $i^1_!(\alpha)|_{\hat{S}^1(X^{n-p-1})} = 0$.

\[ \square \]

**Theorem 5.8** Let $(X,Y,D)$ be a triple satisfying ($\#^1$) and $\Phi^{n-p}_D : C^{n-p}(X,\mathbb{Z}) \to K(\hat{S}^1(X^{n-p}_D),\hat{S}^1(X^{n-p-1}_D))$ be the isomorphism stated in theorem 4.2. Let:

$$\pi^{n-p} : \hat{S}^1(X^{n-p}_D) \to \hat{S}^1(X^{n-p}_D)/\hat{S}^1(X^{n-p-1}_D)$$

be the projection and $\hat{PD}(Y_\Delta)$ be the representative of $\hat{PD}_X Y$ given by the sum of the cells dual to the $p$-cells of $\Delta$ covering $Y$. Then:

$$i^1_!(Y \times \mathbb{C})|_{\hat{S}^1(X^{n-p}_D)} = (\pi^{n-p})^*\left(\Phi^{n-p}_D(\hat{PD}(Y_\Delta))\right).$$

\[ \square \]

**Corollary 5.9** Let $(X,Y,D)$ be a triple satisfying ($\#^1$) and $\Xi^{n-p}_D : H^{n-p}(X,\mathbb{Z}) \to \text{Im } \Psi \subset K(\hat{S}^1(X^{n-p}_D),\hat{S}^1(X^{n-p-2}_D))$ be the isomorphism (21). Let:

$$\hat{\pi}^{n-p} : \hat{S}^1(X^{n-p}_D) \to \hat{S}^1(X^{n-p}_D)/\hat{S}^1(X^{n-p-2}_D)$$

28
be the projection. Then:

$$i_1^*(Y \times \mathbb{C})|_{\hat{S}_1(X^{n-p})} = (\tilde{n}^{n-p})^*(\Xi^{n-p}(\text{PD}(Y))).$$

\[\square\]

**Theorem 5.10** Let $X$ be an $n$-dimensional compact manifold and $Y \subset X$ a $p$-dimensional embedded compact submanifold, such that $n - p$ is odd and $\mathcal{N}(Y)$ is spin$. Let $\{\{E_{r}^{p}, d_{r}^{p}\}\}$ be the Atiyah-Hirzebruch spectral sequence, and let $\Xi^{n-p} : H^{n-p}(X, \mathbb{Z}) \xrightarrow{\cong} E_2^{n-p,0}$ be the isomorphism induced by $\Phi^{n-p}$. Let us suppose that $\Xi^{n-p} \text{PD}(Y)$ is contained in the kernel of all the boundaries $d_{r}^{n-p,0}$ for $r \geq 2$.

With this data, we define a class:

$$\{\Xi^{n-p} \text{PD}(Y)\}^{(2)}_{E_\infty^{n-p,0}} \in E_\infty^{n-p,0} \simeq \frac{\text{Ker}(\tilde{K}(\hat{S}_1 X) \rightarrow K(\hat{S}_1 X^{n-p-1}))}{\text{Ker}(\tilde{K}(\hat{S}_1 X) \rightarrow K(\hat{S}_1 X^{n-p}))}.$$

Then:

$$\{\Xi^{n-p,0} \text{PD}(Y)\}^{(2)}_{E_\infty^{n-p,0}} = [i_1^*(Y \times \mathbb{C})]_{E_\infty^{n-p,0}}.$$

\[\square\]

### 5.3 The rational case

#### 5.3.1 Even case

We now analyze the case of rational coefficients. We define:

$$K_\mathbb{Q}(X) := K(X) \otimes \mathbb{Q}.$$

We can thus classify the D-brane charge density at rational level as $i_1(E) \otimes \mathbb{Q}$. Chern character provides an isomorphism $\text{ch} : K_\mathbb{Q}(X) \rightarrow H^{ev}(X, \mathbb{Q})$. Since the square root of $\hat{A}(TX)$ is a polyform starting with 1, it also defines an isomorphism, so that the composition:

$$\hat{\text{ch}} : K_\mathbb{Q}(X) \rightarrow H^{ev}(X, \mathbb{Q})$$

$$\hat{\text{ch}}(\alpha) = \text{ch}(\alpha) \wedge \sqrt{\hat{A}(TX)}$$

remains an isomorphism. Thus, the classifications with rational K-theory and rational cohomology are completely equivalent.

We can also define rational Atiyah-Hirzebruch spectral sequence $Q_{\mathbb{Q}}^{2k,\sigma}(X) := E_{\mathbb{Q}}^{2k,\sigma}(X) \otimes \mathbb{Q}$. Such sequence (v. [3]) collapses at the second step, i.e., at the cohomology: thus $Q_{\mathbb{Q}}^{2k,\sigma}(X) \simeq Q_2^{2k,\sigma}(X)$. An explicit isomorphism is given by the appropriate component of Chern character:

$$\frac{\text{ch}_{n-p} : \text{Ker}(K_\mathbb{Q}(X) \rightarrow K_\mathbb{Q}(X^{n-p-1}))}{\text{Ker}(K_\mathbb{Q}(X) \rightarrow K_\mathbb{Q}(X^{n-p}))} \rightarrow H^{n-p}(X, \mathbb{Q}).$$

29
For a bundle which is trivial on the \((n - p - 1)\)-skeleton, the lower components of \(ch\) are zero (v. [3]), hence \(ch_{n-p} = \hat{ch}_{n-p}\) but this is not in general true for the higher components. Moreover, since \(Q_{2k,0}^{\infty}\) has no torsion:

\[
K_{Q}(X) = \bigoplus_{2k} Q_{2k,0}^{\infty}
\]

and an isomorphism can be obtained splitting \(\alpha \in K_{Q}(X)\) as \(\alpha = \sum_{2k} \alpha_{2k}\) where \(ch(\alpha_{2k}) = ch_{k}(\alpha)\). We now link this isomorphism with the splitting principle stated at the beginning.

If we consider the brane \(Y\) with bundle \(E\) and the subbranes \(\{q_{k} \cdot Y_{k}\}\) verifying the splitting principle, we have that:

\[
i_{i}(E) \otimes_{Z} Q = \sum_{k} (i_{k})(Y_{k} \times C^{q_{k}}) \otimes_{Z} Q \tag{31}
\]

since the Chern characters of the two terms above are exactly the two terms of formula (3). Hence, if we look at the correspondence:

\[
K_{Q}(X) \longleftrightarrow \bigoplus_{2k} Q_{2k,0}^{\infty}
\]

\[
\alpha \longleftrightarrow \oplus_{2k} [\alpha_{2k}]_{Q_{2k,0}^{\infty}}
\]

for \(\alpha_{2k}\) such that \(ch(\alpha_{2k}) = ch_{k}(\alpha)\), we have in particular that \([\alpha_{2k}]_{Q_{2k,0}^{\infty}} = [(i_{k})(Y_{k} \times C^{q_{k}})]_{Q_{2k,0}^{\infty}}\).

However, we can also consider the subbranes \(i_{*}PD_{Y}(ch(E) \land G(Y))\), with trivial bundle. We call such subbranes \(\{q'_{k} \cdot Y'_{k}\}\). We have that:

\[
i_{i}(E) \otimes_{Z} Q \longleftrightarrow \oplus_{2k} [((i_{k})(Y'_{k} \times C^{q'_{k}}))]_{Q_{2k,0}^{\infty}}.
\]

In fact, we saw that \((i_{k})_{*}(q_{k} \cdot Y_{k}) = PD_{X} \hat{ch}_{k}(i_{i}(E))\). Hence:

\[
ch_{k}((i_{k})(Y_{k} \times C^{q_{k}})) = \hat{ch}_{k}((i_{k})(Y_{k} \times C^{q_{k}})) = (i_{k})_{#}(q_{k} \cdot 1)
\]

\[
= PD_{X}(i_{k})_{*}(q_{k} \cdot Y_{k}) = ch_{k}i(E).
\]

However, for the branes \(\{q'_{k} \cdot Y'_{k}\}\) formula (31) does not hold.

### 5.3.2 Odd case

In this case, we have the isomorphism \(ch : K_{Q}^{1}(X) \to H^{odd}(X, Q)\). Moreover, \(H^{odd}(X, Q) \cong H^{ev}(S^{1}X, Q)\). Hence we have the correspondence among:

- \(i_{i}^{1}(E) \in K_{Q}^{1}(X)\);
- \(\hat{ch}(i_{i}^{1}E) \in H^{ev}(S^{1}X, Q) \cong H^{odd}(X, Q)\);
- \(\oplus_{2k} [(i_{k}^{1})(Y_{k} \times C^{q_{k}})]_{Q_{2k+1,0}^{\infty}}\).

As before, for the splitting principle:

\[
i_{i}^{1}(E) \otimes_{Z} Q = \sum_{k} (i_{k}^{1})(Y_{k} \times C^{q_{k}}) \otimes_{Z} Q.
\]
6 Conclusions and future perspectives

To summarize, we considered the following classifications for charges of D-branes in a compact euclidean space-time $S$:

|                | Integer                                                                 | Rational                                                                 |
|----------------|------------------------------------------------------------------------|-------------------------------------------------------------------------|
| Cohomology     | $PD_S(q \cdot WY_p) \in H^{9-p}(S, \mathbb{Z})$                       | $i_\#(\text{ch}(E) \wedge G(WY_p)) \in H^{ev}(S, \mathbb{Q})$          |
| K-theory (Gysin map) | $i_t(E) \in K(S)$                                                 | $i_t(E) \in K_\mathbb{Q}(S)$                                           |
| K-theory (AHSS) | $\{PD_S(q \cdot WY_p)\} \in E^{9-p,0}_\infty(S)$                    | $\{i_\#(\text{ch}(E) \wedge G(WY_p))\} \in Q^{ev,0}_\infty(S)$       |

We can now explain the relations. We already saw the complete equivalence of the three rational classifications, due to the isomorphisms $H^*(S, \mathbb{Q}) \simeq K^*_\mathbb{Q}(S) \simeq \bigoplus_k Q_k^{k,0}$, which split into even and odd parts. For the integral classifications, the three approaches are not equivalent, and we have to clarify their relationships. The cohomological and AHSS approaches have a clear link as one can see in the table, but they do not take into account the gauge and gravitational couplings.

Since we have seen the link between Gysin map and Atiyah-Hirzebruch spectral sequence, we can also link these two approaches. We proved that $i_t(E) \in \text{Ker}(K^q(S) \rightarrow K^q(S^{q-1}))$ for $q = 10 - (p + 1)$, and that:

$$\{PD_S(WY_p)\}_{E^{q,0}} = [i_t(E)]_{E^{q,0}}.$$  

Thus, we can use AHSS to detect possible anomalies, then we can use the Gysin map to get the charge of a non-anomalous brane: such a charge belongs to the equivalence class reached by AHSS, so that Gysin map gives richer information. Some comments are in order. One could ask why the additional information provided by the Gysin map has to be considered: in fact, we have proven that it concerns the choice of a representative of the class, while, discussing AHSS in chapter 2, we have seen that one of its advantages is that it quotients out unstable configurations. It seems that such additional information keeps into account only instabilities. Actually, this is not the case. Let us consider a couple $(WY_p, i_t(E))$ made by a D-brane world-volume and its charge with respect to the Gysin map approach. The charge does not provide complete information about the world-volume, since $i_tE$ is a class in the whole space-time, exactly as the charge $q$ of an electron does not provide information about its trajectory. This is true also for the cohomological and AHSS classifications: two homologous world-volumes are not the same trajectory. If we consider two couples $(WY_p, i_t(E))$ and $(WY_p, i_t(F))$, we know that $[i_t(E) - i_t(F)]_{E^{q,0}} = 0$, which means that $i_t(E) - i_t(F)$ lies in the image of some boundaries of the AHSS. Let us suppose that it lies in the image of $d_3$. This means that there exists an unstable world-volume $WU_p$ with a
gauge bundle, e.g. the trivial one, such that \( i_!(WU_p \times \mathbb{C}) = i_!(E) - i_!(F) \), but the two terms of the latter equality concern different world-volumes with the same zero charge: in fact, \( WU_p \) has charge 0 because it lies in the image of \( d_3 \), and, since \( \text{rk}(E - F) = 0 \), \( i_!(E - F) \) is a representative of the class reached starting from \( 0 \cdot WY_p \). Actually the information contained in \( i_!(E - F) \) is partially contained in the charges of the sub-branes of \( WY_p \). Thus, we can apply the AHSS to the world-volume of the D-brane, then, if it corresponds to the trivial class we consider it as an unstable one, otherwise we can consider each representative of the class as an additional meaningful information.

Possible generalizations of this work are the following:

- admitting the presence of the \( B \)-field compatibly with Freed-Witten anomaly, considering also twisted K-theory and the corresponding twisted AHSS.
- considering the case of non-compact space-time and world-volumes, using the appropriate form of AHSS;
- studying branes with singularities, using the appropriate form of the Gysin map;

Acknowledgements

We would like to thank Loriano Bonora for the helpfulness he always showed since we started to work with him. We are also really grateful to Jarah Evslin for many suggestions and for having pointed out many subtleties. We also thank Giulio Bonelli and Ugo Bruzzo for useful discussions.

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33