Abstract. We study in a strip of $\mathbb{R}^2$ a combustion model of flame propagation with stepwise temperature kinetics and zero-order reaction, characterized by two free interfaces, respectively the ignition and the trailing fronts. The latter interface presents an additional difficulty because the non-degeneracy condition is not met. We turn the system to a fully nonlinear problem which is thoroughly investigated. When the width $\ell$ of the strip is sufficiently large, we prove the existence of a critical value $L_{e_c}$ of the Lewis number $L_e$ such that the one-dimensional, planar, solution is unstable for $0 < L_e < L_{e_c}$. Some numerical simulations confirm the analysis.

1. Introduction

This paper is devoted to the analysis of cellular instabilities of planar traveling fronts for a thermo-diffusive model of flame propagation with stepwise temperature kinetics and zero-order reaction. In non-dimensional form, the model reads:

\[
\begin{align*}
\Theta_t &= \Delta \Theta + W(\Phi, \Theta), \\
\Phi_t &= \frac{1}{L_e} \Delta \Phi - W(\Phi, \Theta),
\end{align*}
\]

(1.1)

where $\Theta$ and $\Phi$ are appropriately normalized temperature and concentration of deficient reactant, $L_e$ is the Lewis number and $W(\Phi, \Theta)$ is a reaction rate given by

\[
W(\Theta, \Phi) = \begin{cases} A, & \text{if } \Theta \geq \Theta_i \text{ and } \Phi > 0, \\ 0, & \text{if } \Theta < \Theta_i \text{ and/or } \Phi = 0. \end{cases}
\]

(1.2)

Here, $0 < \Theta_i < 1$ is the ignition temperature and $A > 0$ is a normalizing factor.

Combustion models involving discontinuous reaction terms, including the system (1.1)-(1.2), have been used by physicists and engineers since the very early stage of the development of the combustion science (see Mallard and Le Châtelier [32], primarily due to their relative simplicity and mathematical tractability (see, e.g., [4, 15, 16], and more recently [1, 3]). These models have drawn several mathematical studies on systems with discontinuous nonlinearities and related Free Boundary Problems which include, besides the pioneering work of K.-C. Chang [14], the references [12, 21, 22, 23, 24, 34, 35], to mention a few of them. In particular, models with ignition temperature were introduced in the mathematical description of the propagation of premixed flames to solve the so-called “cold-boundary difficulty” (see, e.g., [13, Section 2.2], [2]).

More specifically, in this paper we consider the free interface problem associated to the model (1.1)-(1.2). The domain is the strip $\mathbb{R} \times (-\ell/2, \ell/2)$, the spatial coordinates are denoted by $(x,y)$, $t > 0$ is the time. The free interfaces are respectively the ignition interface $x = F(t,y)$ and the trailing interface $x = G(t,y)$, $G(t,y) < F(t,y)$, defined by $\Theta(t,F(t,y),y) = \theta_i$, $\Phi(t,G(t,y),y) = 0$. 

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The system reads as follows, for $t > 0$ and $y \in (-\ell/2, \ell/2)$:

\[
\begin{align*}
\frac{\partial \Theta}{\partial t}(t, x, y) &= \Delta \Theta(t, x, y), & x < G(t, y), \\
\Phi(t, x, y) &= 0, & x < G(t, y), \\
\frac{\partial \Theta}{\partial t}(t, x, y) &= \Delta \Theta(t, x, y) + A, & G(t, y) < x < F(t, y), \\
\frac{\partial \Phi}{\partial t}(t, x, y) &= (\text{Le})^{-1} \Delta \Phi(t, x, y) - A, & G(t, y) < x < F(t, y), \\
\frac{\partial \Theta}{\partial x}(t, x, y) &= \Delta \Theta(t, x, y), & x > F(t, y), \\
\frac{\partial \Phi}{\partial x}(t, x, y) &= (\text{Le})^{-1} \Delta \Phi(t, x, y), & x > F(t, y),
\end{align*}
\]

(1.3)

where the normalizing factor $A$ will be fixed below. The functions $\Theta$ and $\Phi$ are continuous across the interfaces for $t > 0$, as well as their normal derivatives. As $x \to \pm \infty$, it holds

\[
\Theta(t, -\infty, y) = \Phi(t, +\infty, y) = 1, \quad \Theta(t, +\infty, y) = 0.
\]

(1.4)

Finally, periodic boundary conditions are assumed at $y = \pm \ell/2$.

As was noted in earlier studies (see [3, 5]), this system is very different from models arising in conventional thermo-diffusive combustion. Two are the principal differences. (i) The first one is that in the model considered here, the reaction zone is of order unity, whereas in the case of Arrhenius kinetics the reaction zone is infinitely thin. This fact suggests to refer to flame fronts for stepwise temperature kinetics as thick flames, in contrast to thin flames arising in Arrhenius kinetics. (ii) The second, even more important difference, is that, in the case of Arrhenius kinetics, there is a single interface separating burned and unburned gases. In contrast to that, in case of the stepwise temperature kinetics given by (1.3), there are two interfaces, namely the \textit{ignition interface} where $\Theta = \Theta_i$, located at $x = F(t, y)$, and \textit{trailing interface} at $x = G(t, y)$ being defined as a largest value of $x$ where the concentration is equal to zero. As a consequence of (i), the normal derivatives are continuous across both interfaces, in contrast to classical models with Arrhenius kinetics where jumps occur at the flame front (see e.g., [13] Section 11.8 and [33, 37] for the related Kuramoto-Sivashinsky equation). There have been a number of mathematical works in the latter case based on the method of [8] that we are going to extend below, see in particular [6, 7, 10, 11, 28, 29, 30] for the flame front, and the references therein. Finally, note that Free Boundary Problems with two interfaces have already been considered in the literature, especially in Stefan problems, see e.g., [19, 20, 39] (one-dimensional problem) and [18] (radial solutions).

The above system admits a one-dimensional traveling wave (planar) solution $(\Theta^{(0)}, \Phi^{(0)})$ which propagates with constant positive velocity $V$ (see [3 Section 4]). It is convenient to choose the normalizing factor $A = 1/R$ in such a way that $V = 1$, where the positive number $R = R(\theta_i)$ is given by:

\[
\theta_i R = 1 - e^{-R}, \quad 0 < \theta_i < 1.
\]

(1.5)

Thus, in the moving frame coordinate $x' = x - t$, the system for the travelling wave solution reads as follows:

\[
\begin{align*}
D_x \Theta^{(0)}(r') + D_\nu \nu \cdot \Theta^{(0)}(r') &= 0, & x < 0, \\
D_x \Theta^{(0)}(r') + D_\nu \nu \cdot \Theta^{(0)}(r') &= -R^{-1}, & (0, R), \\
\text{Le} D_x \phi^{(0)} + D_\nu \nu \cdot \phi^{(0)}(r') &= \text{Le} R^{-1}, & (0, R), \\
D_x \Theta^{(0)}(r') + D_\nu \nu \cdot \Theta^{(0)}(r') &= 0, & (R, +\infty), \\
\text{Le} D_x \phi^{(0)} + D_\nu \nu \cdot \phi^{(0)}(r') &= 0, & (R, +\infty),
\end{align*}
\]
whose solution is

$$\Theta^{(0)}(x') = \begin{cases} 
1, & x' \leq 0, \\
1 + \frac{1 - x' - e^{-x'}}{R}, & x' \in (0, R), \\
\theta_i e^{R-x'}, & x' \geq R,
\end{cases}$$

$$\Phi^{(0)}(x') = \begin{cases} 
0, & x' \leq 0, \\
\frac{e^{-LeR} - 1}{Le R} + \frac{x'}{R}, & x' \in (0, R), \\
1 + \frac{1 - e^{LeR}}{Le R e^{LeR}}, & x' \geq R.
\end{cases}$$

Figure 1. $\Theta^{(0)}$ (solid curve) and $\Phi^{(0)}$ (dashed curve) with $\theta_i = 0.75$, $Le = 0.75$ ($R = 0.60586$).

The existence of traveling fronts poses a natural question of one and multidimensional stability, or especially instabilities of such fronts. It is known (see [33, 37]) that diffusional-thermal instabilities of planar flame fronts, when the Lewis number is less than unity, generates cellular flames and pattern formation. In this paper, we focus our attention on instabilities of the traveling wave $(\Theta^{(0)}, \Phi^{(0)})$, and thus for the ignition and the trailing interfaces. Earlier studies have shown (see [5]) that instabilities depend on the Lewis number and occur only when the width of the strip $\ell$ is large enough (in [3], $\ell$ is taken to infinity), which motivates the present study.

The main result of the paper is the following:

Main Theorem. Let $0 < \theta_i < 1$ be fixed. There exist $\ell_0(\theta_i)$ sufficiently large, such that, whenever $\ell > \ell_0(\theta_i)$, there exists a critical value of the Lewis number, say $Le_c \in (0, 1)$ (see (6.2)). If $Le \in (0, Le_c)$, then the traveling wave solution to problem (1.3)-(1.4) is unstable with respect to smooth and sufficiently small two dimensional perturbation. Further, also the ignition and the trailing interfaces are pointwise unstable.

The paper is organized as follows. In Section 2, we introduce the main notation and the functional spaces. Section 3 is devoted to transforming problem (1.3)-(1.4) in a fully nonlinear problem for the perturbation of the traveling wave solution $(\Theta^{(0)}, \Phi^{(0)})$ in (1.6), set in a fixed domain (see (3.23)). We determine that the ignition interface meets the transversality (or non-degeneracy) condition of [8]. Unfortunately, this is not the case of the trailing interface which is of different nature. In short, the idea is to differentiate the mass fraction equation, taking advantage of the structure of the problems.

Then, in Section 4 we collect some tools that are needed to prove the main result. The theory of analytic semigroups plays a crucial role in all our analysis. For this reason, one of the main tools of this section is a generation result: we will show that a suitable realization of the linearized (at zero) elliptic operator associated with the fully nonlinear problem (3.23) generates an analytic semigroup and we characterize the interpolation spaces. This will allow us to prove an optimal regularity result for classical solutions to problem (3.23). Section 6 contains the proof of Theorem 1. Finally, Section 7 is devoted to a numerical method and computational results which show two-cell patterns (see [5] for further results).
2. Notation, functional spaces and preliminaries

In this section, we collect all the notation, the functional spaces and the preliminary results that we use throughout the paper.

2.1. Notation. We find it convenient to set, for each $\tau > 0$,

$$
S = \mathbb{R} \times (-\ell/2, \ell/2), \\
S = (\tau, +\infty) \times (-\ell/2, \ell/2), \\
H^1_\tau = (\tau, +\infty) \times \mathbb{R}, \\
H^1_\tau = (-\infty, \tau) \times (-\ell/2, \ell/2), \\
R_T = [0, T] \times [-\ell/2, \ell/2].
$$

Functions. Given a function $f : (a, b) \to \mathbb{R}$ and a point $x_0 \in (a, b)$, we denote by $\{f\}_{x_0}$ the jump of $f$ at $x_0$, i.e., the difference $f(x_0^+) - f(x_0^-)$ whenever defined. For each function $f : [-\ell/2, \ell/2] \to \mathbb{C}$ we denote by $f^\ast$ its $\ell$-periodic extension to $\mathbb{R}$. If $f$ depends also on $x$ running in some interval $I$, we still denote by $f^\ast$ its periodic (with respect to $y$) extension to $I \times \mathbb{R}$. For every $f \in L^2((-\ell/2, \ell/2))$ and $k \in \mathbb{Z}$, we denote by $\hat{f}_k$ the $k$-th Fourier coefficient of $f$, i.e.,

$$
\hat{f}_k = \frac{1}{\ell} \int_{-\ell/2}^{\ell/2} f e^{2\pi i k y} dy,
$$

where $e_h(y) = e^{2\pi hy}$ for each $h \in \mathbb{Z}$ and $y \in \mathbb{R}$. When $f$ depends also on the variable $x$ running in some interval $I$, $\hat{f}_k(x)$ stands for the $k$-th Fourier coefficient of the function $f(x, \cdot)$.

The time and the spatial derivatives of a given function $f$ are denoted by $D_t f = (f_1, D_x f = (f_2), D_y f = (f_3), D_{xx} f = (f_4), D_{xy} f = (f_5), D_{yy} f = (f_6))$, respectively. If $\beta = (\beta_1, \beta_2) \in \mathbb{N} \cup \{0\}$, then we set $D^\beta = D_{\beta_1} D_{\beta_2}^\ast$.

Finally, we denote by $\chi_A$ the characteristic function of the set $A \subset \mathbb{R}^d (d \geq 1)$.

Miscellanea. Throughout the paper, we denote by $c, \lambda$ a positive constant, possibly depending on $\lambda$ but being independent of $k$, $n$, $x$ and the functions that we will consider, which may vary from line to line. We simply write $c$ when the constant is independent also of $\lambda$.

The subscript “b” stands for bounded. For instance $C^b_b(\Omega; \mathbb{C})$ denotes the set of bounded and continuous function from $\Omega$ to $\mathbb{C}$. When we deal with spaces of real-valued functions we omit to write “\mathbb{C}”.

Vector-valued functions are displayed in bold.

2.2. Main function spaces. Here, we collect the main function spaces used in the paper pointing out the (sub)section where they are used for the first time.

The spaces $\mathcal{X}$ and $\mathcal{X}_{k+a}$ (Section 4). By $\mathcal{X}$ we denote the set of all pairs $f = (f_1, f_2)$, where $f_1 : \mathbb{S} \to \mathbb{C}$ and $f_2 : \mathbb{S}^\ast \to \mathbb{C}$ are bounded functions, $f_1 \in C(S_{\alpha_a}^\ast; \mathbb{C}) \cap C([0, R] \times [-\ell/2, \ell/2]; \mathbb{C}) \cap C(S_{\alpha_b}^\ast; \mathbb{C})$, $f_2 \in C([0, R] \times [-\ell/2, \ell/2]; \mathbb{C}) \cap C(S_{\alpha_a}^\ast; \mathbb{C})$ and $\lim_{x \to \pm \infty} f_1(x, y) = \lim_{x \to \pm \infty} f_2(x, y) = 0$ for each $y \in [-\ell/2, \ell/2]$. It is endowed with the sup-norm, i.e., $\|f\|_{\mathcal{X}} = \|f_1\|_{L^\infty(S_{\alpha_a}^\ast)} + \|f_2\|_{L^\infty(S_{\alpha_b}^\ast)}$.

For each $\alpha \in (0, 1]$, $\mathcal{X}_{\alpha}$ denotes the subset of $\mathcal{X}$ of all $f$ such that (i) $f_1 \in C^0_b(S_{\alpha_a}^\ast; \mathbb{C}) \cap C^0_b([0, R] \times [-\ell/2, \ell/2]; \mathbb{C}) \cap C^0_b(S_{\alpha_b}^\ast; \mathbb{C})$, (ii) $f_2 \in C^b_b([0, R] \times [-\ell/2, \ell/2]; \mathbb{C}) \cap C^b_b(S_{\alpha_b}^\ast; \mathbb{C})$, (iii) $f_1(\cdot, \ell/2) = f_1(\cdot, -\ell/2) = \nabla f_1(\cdot, \ell/2)$ if $\alpha = 1$ for $j = 1, 2$. It is endowed with the norm $\|f\|_{\mathcal{X}_{\alpha}} = \|f_1\|_{C^0_b(S_{\alpha_a}^\ast)} + \sum_{j=1}^{2}(\|f_2\|_{C^0_b([0, R] \times [-\ell/2, \ell/2]; \mathbb{C})) + \|f_2\|_{C^0_b(S_{\alpha_b}^\ast)})$.

For $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, $\mathcal{X}_{k+a}$ ($k \in \mathbb{N}$, $\alpha \in (0, 1)$) denotes the set of all $f \in \mathcal{X}$ such that $D^0 f = (D^0 f_1, D^0 f_2) \in \mathcal{X}$, $D^k f_1(\cdot, -\ell/2) = D^k f_1(\cdot, \ell/2)$ for each $|\alpha| \leq k$, $j = 1, 2$, and $D^k f \in \mathcal{X}_{\alpha}$ for $|\alpha| = k$. It is endowed with the norm $\|f\|_{k+a, \mathcal{X}} := \sum_{|\alpha| \leq k} \|D^k f\|_{\mathcal{X}_{\alpha}} + \sum_{|\alpha| = k} \|D^k f\|_{\mathcal{X}_{\alpha}}$.

The spaces $\mathcal{Y}_{\alpha}(a, b)$ and $\mathcal{Y}_{2+a}(a, b)$ (Section 5). For $\alpha \in (0, 1)$ and $0 \leq a < b$, we define by $\mathcal{Y}_{\alpha}(a, b)$ the space of all pairs $f = (f_1, f_2)$ such that $f_1 : [a, b] \times \mathbb{S} \to \mathbb{R}$, $f_2 : [a, b] \times \mathbb{S}^\ast \to \mathbb{R}$ and

$$
\|f\|_{\mathcal{Y}_{\alpha}(a, b)} = \sup_{0 < t < b} \|f(t, \cdot, \cdot)\|_{\alpha} + \sup_{(x, y) \in \mathbb{S}} \|f_1(\cdot, x, y)\|_{C^{1/2}(a, b ; \mathbb{C})} + \sup_{(x, y) \in \mathbb{S}^\ast} \|f_2(\cdot, x, y)\|_{C^{1/2}(a, b ; \mathbb{C})} < +\infty.
$$
Similarly, $\mathbf{y}_{2+\alpha}(a, b)$ denotes the space of all the pairs $\mathbf{u}$ such that $D_\gamma^{1+1}D_\tau^{1+2}D_\eta^{1+3}\mathbf{u}$ belongs to $\mathbf{y}_\alpha(a, b)$ for every $\gamma_1, \gamma_2, \gamma_3 \geq 0$ such that $2\gamma_1 + 2\gamma_2 + \gamma_3 \leq 2$. These are Banach spaces with the norms $\| \cdot \|_{\mathbf{y}_\alpha(a, b)}$ and $\| \mathbf{u} \|_{\mathbf{y}_{2+\alpha}(a, b)} = \sum_{2\gamma_1 + 2\gamma_2 + \gamma_3 \leq 2} \| D_\gamma^{1+1}D_\tau^{1+2}D_\eta^{1+3}\mathbf{u} \|_{\mathbf{y}_\alpha(a, b)}$.

If $a = 0$ and $b = T$ then we simply write $\mathbf{y}_\alpha$ and $\mathbf{y}_{2+\alpha}$ instead of $\mathbf{y}_\alpha(0, T)$ and $\mathbf{y}_{2+\alpha}(0, T)$, respectively.

3. Derivation of the Fully Nonlinear Problem

3.1. The system on a fixed domain. To begin with, we rewrite System (1.3) in the coordinates $t' = t$, $x' = x - t$, $y' = y$, $D_t = D_r - D_x$. Next, we look for the free interfaces respectively as:

$$G(t', y') = g(t', y') \quad \text{and} \quad F(t', y') = R + f(t', y'),$$

where $f$ and $g$ are small perturbations. In other words, the space variable $x'$ varies from $-\infty$ to $g(t', y')$, from $f(t', y')$ to $R + f(t', y')$, and eventually from $R + f(t', y')$ to $+\infty$. As usual, it is convenient to transform the problem on a variable domain to a problem on a fixed domain. To this end, we define a coordinate transformation in the spirit of [3] Section 2.1:

$$t' = \tau, \quad x' = \xi + \beta(\xi)g(\tau, \eta) + \beta(\xi - R)f(\tau, \eta), \quad y' = \eta,$$

where $\beta$ is a smooth mollifier, equal to unity in a small neighborhood of $\xi = 0$, say $[-\delta, \delta]$, and has compact support contained in $(-2\delta, 2\delta) \subset (-R, R)$. When $x' = \eta$, $\xi = 0$, and $\xi = R$ when $x' = R + f$. Then, the trailing front and the ignition front are fixed at $\xi = 0$ and $\xi = R$, respectively.

Thanks to the translation invariance, (1.6) holds with the variable $\xi$. For convenience, we introduce the notation

$$g(\tau, \xi, \eta) = \beta(\xi)g(\tau, \eta) + \beta(\xi - R)f(\tau, \eta)$$

and we expand $(1 + \varrho \xi)^{-1} = 1 - \varrho \xi + (\varrho \xi)^2(1 + \varrho \xi)^{-1}$. It turns out that

$$D_{\xi} = D_\xi - \varrho \xi D_\xi + (\varrho \xi)^2(1 + \varrho \xi)^{-1} D_\xi,$$

$$D_t = D_t - \varrho \xi D_t + \varrho \xi D_\xi - \varrho \xi(\varrho \xi)^2(1 + \varrho \xi)^{-1} D_\xi,$$

$$D_y = D_y - \varrho \xi D_\xi + \varrho \xi D_\xi - \varrho \xi(\varrho \xi)^2(1 + \varrho \xi)^{-1} D_\xi$$

and System (1.3) reads:

$$\begin{align*}
\Phi_r &= \Phi_{\xi} + \Delta \Phi + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho \xi)(1 + \varrho \xi)^{-2}\Phi_{\xi\xi} - 2\varrho \xi(1 + \varrho \xi)^{-1}\Phi_{\xi\eta}
\quad + \left[(\varrho \xi - \varrho \eta)(1 + \varrho \xi)^{-1} - 2\varrho \xi\varrho \eta(1 + \varrho \xi)^{-2} - \varrho \xi(1 + \varrho \xi)^{-3}\right]\Phi_{\xi}, \\
\Phi_r &= \Phi_{\xi} + \Delta \Phi + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho \xi)(1 + \varrho \xi)^{-2}\Phi_{\xi\xi} - 2\varrho \xi(1 + \varrho \xi)^{-1}\Phi_{\xi\eta}
\quad + \left[(\varrho \xi - \varrho \eta)(1 + \varrho \xi)^{-1} - 2\varrho \xi\varrho \eta(1 + \varrho \xi)^{-2} - \varrho \xi(1 + \varrho \xi)^{-3}\right]\Phi_{\xi}.
\end{align*}$$

(3.2)

where $\varrho \xi$ varies from $0$ to $T$, $\varrho \eta$ varies from $0$ to $T$, respectively.

$$\begin{align*}
\Phi_r &= \Phi_{\xi} + \Delta \Phi + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho \xi)(1 + \varrho \xi)^{-2}\Phi_{\xi\xi} - 2\varrho \xi(1 + \varrho \xi)^{-1}\Phi_{\xi\eta}
\quad + \left[(\varrho \xi - \varrho \eta)(1 + \varrho \xi)^{-1} - 2\varrho \xi\varrho \eta(1 + \varrho \xi)^{-2} - \varrho \xi(1 + \varrho \xi)^{-3}\right]\Phi_{\xi}, \\
\Phi_r &= \Phi_{\xi} + \Delta \Phi + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho \xi)(1 + \varrho \xi)^{-2}\Phi_{\xi\xi} - 2\varrho \xi(1 + \varrho \xi)^{-1}\Phi_{\xi\eta}
\quad + \left[(\varrho \xi - \varrho \eta)(1 + \varrho \xi)^{-1} - 2\varrho \xi\varrho \eta(1 + \varrho \xi)^{-2} - \varrho \xi(1 + \varrho \xi)^{-3}\right]\Phi_{\xi}.
\end{align*}$$

(3.3)

in $(0, +\infty) \times (-\infty, 0) \times (-\ell/2, \ell/2)$.

$$\begin{align*}
\Theta_r &= \Theta_{\xi} + \Delta \Theta + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho \xi)(1 + \varrho \xi)^{-2}\Theta_{\xi\xi} - 2\varrho \xi(1 + \varrho \xi)^{-1}\Theta_{\xi\eta} + R^{-1}
\quad + \left[(\varrho \xi - \varrho \eta)(1 + \varrho \xi)^{-1} - 2\varrho \xi\varrho \eta(1 + \varrho \xi)^{-2} - \varrho \xi(1 + \varrho \xi)^{-3}\right]\Theta_{\xi}, \\
\Phi_r &= \Phi_{\xi} + \Delta \Phi + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho \xi)(1 + \varrho \xi)^{-2}\Phi_{\xi\xi} - 2\varrho \xi(1 + \varrho \xi)^{-1}\Phi_{\xi\eta} + \left[(\varrho \xi - \varrho \eta)(1 + \varrho \xi)^{-1} - 2\varrho \xi\varrho \eta(1 + \varrho \xi)^{-2} - \varrho \xi(1 + \varrho \xi)^{-3}\right]\Phi_{\xi},
\end{align*}$$

(3.4)

in $(0, +\infty) \times (0, R) \times (-\ell/2, \ell/2)$ and

$$\begin{align*}
\Theta_r &= \Theta_{\xi} + \Delta \Theta + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho \xi)(1 + \varrho \xi)^{-2}\Theta_{\xi\xi} - 2\varrho \xi(1 + \varrho \xi)^{-1}\Theta_{\xi\eta}
\quad + \left[(\varrho \xi - \varrho \eta)(1 + \varrho \xi)^{-1} - 2\varrho \xi\varrho \eta(1 + \varrho \xi)^{-2} - \varrho \xi(1 + \varrho \xi)^{-3}\right]\Theta_{\xi}, \\
\Phi_r &= \Phi_{\xi} + \Delta \Phi + (\varrho_\eta^2 - \varrho_\xi^2 - 2\varrho \xi)(1 + \varrho \xi)^{-2}\Phi_{\xi\xi} - 2\varrho \xi(1 + \varrho \xi)^{-1}\Phi_{\xi\eta} + \left[(\varrho \xi - \varrho \eta)(1 + \varrho \xi)^{-1} - 2\varrho \xi\varrho \eta(1 + \varrho \xi)^{-2} - \varrho \xi(1 + \varrho \xi)^{-3}\right]\Phi_{\xi}.
\end{align*}$$

(3.3)
in \((0, +\infty) \times (R, +\infty) \times (-\ell/2, \ell/2)\). Moreover, \(\Theta\) and \(\Phi\) are continuous at the (fixed) interfaces \(\xi = 0\) and \(\xi = R\), and so are their first-order derivatives. Thus,
\[
[\Theta(\tau, \cdot, \cdot)]_0 = [\Phi_\xi(\tau, \cdot, \cdot)]_0 = 0
\]
and
\[
[\Phi(\tau, \cdot, \cdot)]_R = [\Phi_\xi(\tau, \cdot, \cdot)]_R = 0.
\]
Conditions \((1.4)\) hold at \(\xi = 0\) and \(\xi = R\) in \((0, +\infty) \times (R, +\infty) \times (-\ell/2, \ell/2)\).

### 3.2. Elimination of the interfaces.

From now on, with a slight abuse of notation, which does not cause confusion, we write \(t\) instead of \(\tau\) and \(x, y\) instead of \(\xi\) and \(\eta\).

In the spirit of \([8, 29]\), we introduce the splitting:
\[
\Theta(t, x, y) = \Theta^{(0)}(0) + \varrho(t, x, y) \Theta^{(0)}_x(x) + u(t, x, y),
\]
\[
\Phi(t, x, y) = \Phi^{(0)}(0) + \varrho(t, x, y) \Phi^{(0)}_x(x) + v(t, x, y),
\]
which is a sort of Taylor expansion of \((\Theta, \Phi)\) around the travelling wave solution \((\Theta^{(0)}, \Phi^{(0)})\). Thus, the pair \((u, v)\) plays the role of a remainder and, since we are interested in stability issues, we can assume that \(u\) and \(v\) are “sufficiently small” in a sense which will be made precise later on.

A long but straightforward computation reveals that the pair \((u, v)\) satisfies the differential equations
\[
u_t = \nabla u + \Delta u + \varrho(1 + \varrho_x) \varrho \Theta^{(0)}_x + \varrho \Theta^{(0)}_x - (1 + \varrho_x) \Theta^{(0)}_x = \varrho \Theta^{(0)}_x + \varrho \Theta^{(0)}_x
\]
\[
(3.7)
\]
in \((0, +\infty) \times (R \setminus \{0, R\}) \times (-\ell/2, \ell/2)\) and
\[
v_t = v_x + \frac{L e^{-1} \Delta v + \varrho(1 + \varrho_x) - (1 + \varrho_x) \varrho \Phi^{(0)}_x + v_x}{1 + \varrho_x} - (1 + \varrho_x) \Phi^{(0)}_x - (1 + \varrho_x) \Phi^{(0)}_x + v_x
\]
\[
- (1 + \varrho_x) \Phi^{(0)}_x - (1 + \varrho_x) \Phi^{(0)}_x + v_x
\]
\[
(3.8)
\]
in \((0, +\infty) \times ([0, R) \cup (R, +\infty)] \times (-\ell/2, \ell/2)\) and \(v = 0\) in \((0, +\infty) \times (-\ell/2, \ell/2)\).

Two steps are still needed:
(a) we have to determine the jump conditions satisfied by \(u\) and \(v\);
(b) again in the spirit of \([8, 29]\), we have to get rid of the function \(\varphi\) from the right-hand sides of \((3.7)\) and \((3.8)\). As we will see, some difficulties appear and, to overcome them, we will differentiate the differential equation \((3.8)\).

#### 3.2.1. The ignition interface \(x = R\).

Note that \(\Theta^{(0)}, \Phi^{(0)}\) belong to \(C^1(R)\). Thus, \([u]_R = [v]_R = 0\). Moreover, \(\Theta^{(0)}_x(R) = -\theta_i\) and \(\Phi^{(0)}_x(R) = (1 - \exp(-Le R))/R\), so that they do not vanish at the interface \(x = R\). The latter is a kind of transversality or non-degeneracy condition (see \([8]\)). Evaluating \((3.5)\) at \(x = R\), we get \(u(R) = \theta_i f\).

Next, differentiating \((3.5)\) and \((3.6)\) for \(x \neq R\), and taking the jumps across \(x = R\), it is not difficult to show that \([u_x]_R = -R^{-1} f\) and \([v_x]_R = R^{-1} Le f\). This is a key point since we are able to express \(f\) in terms of \(u\) and write
\[
f(t, y) = \theta_i^{-1} u(t, R, y).
\]
Summing up, the interface conditions at \(x = R\) are the following:
\[
[u]_R = [v]_R = 0, \quad u(R) + \theta_i R [u_x]_R = 0, \quad Le[u_x]_R + [v_x]_R = 0.
\]
3.2.2. The trailing interface $x = 0$. Taking the jump at $x = 0$ of both sides of (3.5) and (3.6), we get the conditions $u|_0 = v(0) = 0$ for $u$ and $v$. The trailing interface has a different nature with respect to the ignition interface. Indeed, since $\Theta_x(0)(0) = \Phi_x(0), (3.6)$, the non-degeneracy condition of $[8]$ is not verified and we are able to express $g$ in terms of neither of $u$ or $v$. On the other hand, $\Phi_x(0, 0) = -R^{-1}$ and $\Phi_{xx}(0, 0) = R^{-1}Le$, so that they do not vanish. Differentiating (3.5) and (3.6) for $x \neq 0$, and taking the jumps yields: $[u_x]_0 = R^{-1}g, v_x(0^+, \cdot) = -R^{-1}gLe$. Hence, we get the additional interface condition $Le[u_x]_0 + v_x(0^+, \cdot) = 0$, so that the interface conditions at $x = 0$ are

$$[u]_0 = v(0) = 0, \quad Le[u_x]_0 + v_x(0^+, \cdot) = 0.$$  \hfill (3.11)

We can also write

$$g(t, y) = -R Le^{-1}v_x(t, 0^+, y).$$  \hfill (3.12)

Although the front $g$ could be eliminated, the method of $[8]$ is not applicable since, in contrast to (3.9), $g$ is related to the derivative of $v$ in the equation (3.12).

3.3. Differentiation and new interface conditions. To overcome the difficulty pointed out above, the trick is to differentiate (3.8) with respect to $x$, taking advantage of the structure of the system and consider the problem satisfied by the pair $(u, v_x)$. From (3.10) and (3.11) we get the following interface conditions for $u$ and $w = v_x$:

$$[u]_0 = 0, \quad Le[u_x]_0 + w(0^+) = 0,$$

$$[w]_R = 0, \quad w(R) + \theta_1 R[u_x]_R = 0, \quad Le[u_x]_R + [w]_R = 0.$$  \hfill (3.13)

We missed two jump conditions: one at the trailing interface and the other one at the ignition interface. To obtain the additional condition at the trailing interface $x = 0$, we differentiate (3.6) twice in a neighborhood of $x = 0$ and take the trace at $x = 0^+$. Using (3.12), we get

$$\Phi_t + \Phi_x = Le R^{-1} + Le w(0^+, \cdot) + w_x(0^+, \cdot).$$  \hfill (3.14)

To get rid of $\Phi_{xx}(0^+, \cdot)$ from the left-hand side of (3.14), we observe that, for $x > 0$ sufficiently small, the second equation in (3.3) reduces to

$$\Phi_t + \Phi_x = Le R^{-1} \Delta \Phi + Le -1 g_y \Phi_{xx} - 2Le^{-1} g_y \Phi_{xy} - R^{-1} + (g_t - Le^{-1} g_{yy}) \Phi_x.$$  \hfill (3.15)

Taking the trace of (3.15) at $x = 0^+$ it is easy to check that $\Phi_{xx}(0^+, \cdot)(1 + g_y) = Le R^{-1}$. Hence, using (3.12) and (3.14) we get the additional interface condition

$$Le w(0^+, \cdot) + w_x(0^+, \cdot) = Le R^{-1} \{[1 + R^2 Le^{-2} (w_y(0^+, \cdot))^2]^{-1} - 1\}.$$  \hfill (3.16)

We likewise identify the additional interface condition at the ignition interface $x = R$. Differentiating twice (3.6) in a neighborhood of $x = R$, taking the jump at $x = R$ and using (3.9), (3.10) gives

$$[\Phi_{xx}]_R = -R^{-1}Le + Le [w]_R + [w_x]_R.$$  \hfill (3.17)

We need to compute $[\Phi_{xx}]_R$: in a neighborhood of $R^-$, the second equation in (3.3) yields

$$\Phi_t = \Phi_x + Le R^{-1} \Delta \Phi + Le^{-1} (f_y)^2 \Phi_{xx} - 2Le^{-1} f_y \Phi_{xy} - R^{-1} + (f_t - Le^{-1} f_{yy}) \Phi_x,$$

while in a neighborhood of $R^+$ from the second equation in (3.4) we get

$$\Phi_t = \Phi_x + Le R^{-1} \Delta \Phi + Le^{-1} (f_y)^2 \Phi_{xx} - 2Le^{-1} f_y \Phi_{xy} + (f_t - Le^{-1} f_{yy}) \Phi_x.$$  

Using the previous two equations it can be easily shown that $[\Phi_{xx}]_R (1 + (f_y)^2) = -R^{-1}Le$, which, together with (3.9) and (3.17), gives

$$Le[w]_R + [w_x]_R = -Le R^{-1} \{[1 + \theta_1^{-2} (w_y(0^+, \cdot))^2]^{-1} - 1\}.$$  \hfill (3.18)

This is the additional condition we were looking for.
3.4. Elimination of $g$ and its time and spatial derivatives. Formulae (3.9)-(3.12) enable the elimination of the fronts $f$ and $g$ from the differential equations satisfied by $u$ and $w$. First, they allow to write the following formula for $g$ (see (3.1)):

$$g(t, x, y) = \theta_t^{-1}\beta(x - R)u(t, R, y) - R \mathrm{Le}^{-1}\beta(x)w(t, 0^+, y).$$

Differentiation of (3.19) with respect to $x$ and $y$ is benign. The right-hand sides of (3.7) and (3.8) depend also on $g_t$. Hence, we need to compute such a derivative and express it in terms of (traces of) spatial derivatives of $u$ and $w$. Since

$$g_t(t, x, y) = \theta_t^{-1}\beta(x - R)u_t(t, R, y) - R \mathrm{Le}^{-1}\beta(x)w_t(t, 0^+, y),$$

we need to get rid of $u_t(t, R, y)$ and $w_t(t, 0^+, y)$. For simplicity, we forget the arguments $t$ and $y$. We evaluate (3.7) at $x = R^+$ (it would be equivalent at $x = R^-$). Recalling that all the derivatives of $g$ with respect of $x$ vanish and taking (3.19) into account, we get

$$u_t(R) = u_x(R^+) + \Delta u(R^+) + \theta_t^{-1}u_t(R)(u(R) + u_x(R^+)) - \theta_t^{-1}u_{yy}(R)u_x(R^+)
- 2\theta_t^{-1}u_y(R)u_{xy}(R^+) + \theta_t^{-2}(u_y(R))^2(u_{xx}(R^+) - u(R) - \theta_t) - \theta_t^{-1}u(R)u_{yy}(R).$$

Since $\theta_t$ is fixed, assuming that the perturbations are small we may invert and write

$$u_t(R) = [1 - \theta_t^{-1}(u(R) + u_x(R^+))]^{-1}[u_x(R^+) + \Delta u(R^+) - \theta_t^{-1}u_{yy}(R)u_x(R^+) - \theta_t^{-1}u(R)u_{yy}(R)
- 2\theta_t^{-1}u_y(R)u_{xy}(R^+) + \theta_t^{-2}(u_y(R))^2(u_{xx}(R^+) - u(R) - \theta_t)].$$

Similarly, differentiating and evaluating (3.8) at $x = 0^+$ we get

$$w_t(0^+) = w_x(0^+) + \Delta w(0^+) - R \mathrm{Le}^{-1}w_t(0^+)(\mathrm{Le}w(0^+) + w_x(0^+))
+ R \mathrm{Le}^{-2}[w_{yy}(0^+)(\mathrm{Le}w(0^+) + w_x(0^+)) + 2w_y(0^+)w_{xy}(0^+)]
+ R^2 \mathrm{Le}^{-3}(w_y(0^+))^2(-\mathrm{Le}^2w(0^+) + R^{-1} \mathrm{Le}^2 + w_{xx}(0^+)),
$$

so that

$$w_t(0^+) = [\mathrm{Le}w_x(0^+) + \Delta w(0^+) + R \mathrm{Le}^{-1}[w_{yy}(0^+)(\mathrm{Le}w(0^+) + w_x(0^+)) + 2w_y(0^+)w_{xy}(0^+)]
+ R^2 \mathrm{Le}^{-2}(w_y(0^+))^2(-\mathrm{Le}^2w(0^+) + R^{-1} \mathrm{Le}^2 + w_{xx}(0^+))]
\times [\mathrm{Le} + R(\mathrm{Le}w(0^+) + w_x(0^+))]^{-1}.$$  

A related remark is that Equation (3.20) for $g_t$, together with formulas (3.21)-(3.22), may be viewed as a second-order Stefan condition, see [2].

3.5. The final system. Using (3.7), (3.8), (3.13), (3.16), (3.18)-(3.22), we can write the final problem for $u = (u, w)$, which is fully nonlinear since the nonlinear part of the differential equations contains traces at $\xi = 0^+$ and $R$ of (first- and) second-order derivatives of the unknown $u$ itself.

Summing up, the pair $u = (u, w)$ solves the nonlinear system

$$\begin{cases}
D_t u(t, \cdot, \cdot) = \mathcal{L} u(t, \cdot, \cdot) + \mathcal{F}(u(t, \cdot, \cdot)), & t \geq 0, \\
\mathcal{B}(u(t, \cdot)) = \mathcal{H}(u(t, \cdot)), & t \geq 0,
\end{cases}$$

and satisfies periodic boundary conditions at $y = \pm \ell/2$, where

$$\mathcal{L} \mathbf{v} = (\Delta \zeta + \zeta_x, \mathrm{Le}^{-1}\Delta v + v_x),$$

$$\mathcal{B} \mathbf{v} = \begin{cases}
\zeta(0^+, \cdot) - \zeta(0^-, \cdot), \\
\zeta(R^+, \cdot) - \zeta(R^-, \cdot), \\
\mathrm{Le}[\zeta_x(0^+, \cdot) - \zeta_x(0^-, \cdot)] + v(0^+, \cdot), \\
\mathrm{Le} v(0^+, \cdot) + v_x(0^+, \cdot), \\
\zeta(R^+, \cdot) + \theta_t R[\zeta_x(R^+, \cdot) - \zeta_x(R^-, \cdot)], \\
\mathrm{Le} v(R^+, \cdot) - v(R^+, \cdot) - v(R^-, \cdot), \\
\mathrm{Le}[v(R^+, \cdot) - v(R^-, \cdot)] + v_x(R^+, \cdot) - v_x(R^-, \cdot)
\end{cases},$$

where $\zeta(0^+, \cdot) - \zeta(0^-, \cdot) = 1$, $\zeta(R^+, \cdot) - \zeta(R^-, \cdot) = 1$, $\zeta_x(0^+, \cdot) - \zeta_x(0^-, \cdot) = 1$, $v(0^+, \cdot) = 0$, $v_x(0^+, \cdot) = 0$, $\zeta(R^+, \cdot) = \zeta(R^-, \cdot) = 0$, $v(R^+, \cdot) = v(R^-, \cdot) = 0$, $v_x(R^+, \cdot) = v_x(R^-, \cdot) = 0$.
\[ \mathcal{F}(v) = (\Lambda(v) \mathcal{F}_1(v) - \mathcal{F}_2(v), \Lambda(v) D_\beta \mathcal{F}_1(v) + D_\beta \Lambda(v) \mathcal{F}_1(v) - Le^{-1} D_\beta \mathcal{F}_2(v)), \]

\[ \mathcal{F}_1(v) = \frac{\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Theta_{xx}^{(0)} v(0, \cdot)) + \zeta \cdot}{1 + \theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot)}, \]

\[ \mathcal{F}_2(v) = \{ (\theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot) + \theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)) \]

\[ \times (\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Theta_{xx}^{(0)} v(0, \cdot) + \zeta \cdot) \]

\[ + 2(\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot))\theta_i^{(0)} \]

\[ + 2(\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot))\beta_{yy} (0, \cdot) \}

\[ (1 + \theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot))^{-1} \]

\[ - \{ 2(\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)) (\theta_i^1 \beta^2 (- R) \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)) \]

\[ \times (\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Theta_{xx}^{(0)} v(0, \cdot) + \zeta \cdot) \]

\[ + [(\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)) \]^2 - (\theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot)) \}

\[ \times (\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Theta_{xx}^{(0)} v(0, \cdot) + \zeta \cdot)) \}

\[ - 2(\theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot)) \]

\[ \times [\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Theta_{xx}^{(0)} v(0, \cdot) + \zeta \cdot) \]

\[ - (\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)) \theta_i^{(0)} \]

\[ \times (1 + \theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot))^{-2} \]

\[ + \{(\theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot)) \}

\[ \times (\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Theta_{xx}^{(0)} v(0, \cdot) + \zeta \cdot)) \}

\[ \times (1 + \theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot))^{-3} \}

\[ \mathcal{G}_1(v) = \frac{\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Phi_{xx}^{(0)} v(0, \cdot) + \nu \cdot \nu}{\theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot)}, \]

\[ \quad \mathcal{G}_2(v) = \{ [Le \theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot) + \theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)] \]

\[ \times (\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Phi_{xx}^{(0)} v(0, \cdot) + \nu)] \]

\[ + 2(\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot))\Phi_{xx}^{(0)} \]

\[ + 2(\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot))\theta_{yy} (0, \cdot) \}

\[ (1 + \theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot))^{-1} \]

\[ - \{ 2(\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)) (\theta_i^1 \beta^2 (- R) \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)) \]

\[ \times (\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Phi_{xx}^{(0)} v(0, \cdot) + \zeta \cdot) \]

\[ + [(\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)) \]^2 - (\theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot)) \}

\[ \times (\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Phi_{xx}^{(0)} v(0, \cdot) + \zeta \cdot)) \]

\[ - 2(\theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot)) \]

\[ \times [\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Phi_{xx}^{(0)} v(0, \cdot) + \zeta \cdot) \]

\[ - (\theta_i^1 \beta R \zeta_{yy} (R^+, \cdot) - RLe^{-1} \beta v_{yy} (0, \cdot)) \Phi_{xx}^{(0)} \]

\[ \times (1 + \theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot))^{-2} \]

\[ + \{(\theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot)) \}

\[ \times (\theta_i^1 \beta R \theta_i^{(0)} (\zeta (R^+, \cdot) - RLe^{-1} \beta \Phi_{xx}^{(0)} v(0, \cdot) + \nu) \}

\[ \times (1 + \theta_i^1 \beta^2 (- R) \zeta (R^+, \cdot) - RLe^{-1} \beta^2 v(0, \cdot))^{-3},\]
\[ \Lambda(\nu) = (\theta_1 - \zeta(R^+, \cdot) - \zeta_x(R^+, \cdot))^{-1} \]
\[ \times [\zeta_x(R^+, \cdot) + \Delta \zeta(R^+, \cdot) - \theta_1^{-1} \zeta_y(R, \cdot) \zeta_x(R^+, \cdot) - 2 \theta_1^{-1} \zeta_y(R, \cdot) \zeta_{xy}(R^+, \cdot) \]
\[ + \theta_1^{-2} (\zeta_y(R, \cdot))^2 (\zeta_{xx}(R^+, \cdot) - \zeta(R^+) - \theta_1) - \theta_1^{-1} \zeta_y(R, \cdot) \zeta_y(R, \cdot) ] \]
\[ - [Le^2 + RLe(Le v(0^+, \cdot) + v_x(0^+, \cdot))]^{-1} \]
\[ \times \{ Le v_y(0^+, \cdot) + \Delta v(0^+, \cdot) \]
\[ + RLe^{-1} [v_y(0^+, \cdot)(Le v(0^+, \cdot) + v_x(0^+, \cdot)) + 2v_y(0^+, \cdot)v_{xy}(0^+, \cdot)] \]
\[ + R^2Le^{-2}(v_y(0^+, \cdot))^2 (-Le^2 v(0^+, \cdot) + R^{-1} Le^2 + v_{xx}(0^+, \cdot)) \}, \]
\[ \mathcal{H}_1(\nu) = 0 \text{ if } j \neq 4, 7, \quad \mathcal{H}_4(\nu) = -\frac{RLe(v_y(0^+, \cdot))^2}{Le^2 + R^2(v_y(0^+, \cdot))^2}, \quad \mathcal{H}_2(\nu) = \frac{Le(\zeta_y(R^+, \cdot))^2}{R(\theta_1^2 + (\zeta_y(R^+, \cdot))^2)}, \]
on smooth enough functions \( \nu = (\zeta, v) \), where \( \beta_R = \beta(-R) \).

**Remark 3.1.** Note that each smooth enough function \( \mathbf{u} \), which solves problem \([3.23]\), has its first component \( u_1 \) which is continuous on \( \{ R \} \times [-\ell/2, \ell/2] \). Therefore, the operator \( \mathcal{B} \) can be replaced with the operator \( \mathcal{\dot{B}} \) which is defined as \( \mathcal{B} \) with the fifth equation being replaced by the condition \( \frac{1}{2}(v_1(R^+, \cdot) + v_1(R^-, \cdot)) + \theta_1 R^2 D_xv_1(R^+, \cdot) - D_xv_1(R^+, \cdot) = 0 \).

We will use the above remark in Subsection 4.3

4. Tools

In this section we collect some technical results which are used in the next (sub)sections.

4.1. Preliminary results needed to prove Theorems 4.4 and Proposition 4.5. We find it convenient to set
\[ (\mathcal{F}_{k, \nu})(x) := \frac{1}{Z_k} \int_R e^{-\frac{s}{2}x} e^{-\frac{i}{2}Z_k|s|} f_k(x - s) ds, \quad x \in \mathbb{R}, \quad f \in C_b(S; \mathbb{C}), \quad k \in \mathbb{Z}, \]
where \( Z_k = Z_k(\lambda, \rho) = \sqrt{\rho^2 + 4\lambda^2 + 4k^2} \) for every \( k \in \mathbb{Z} \) and \( \lambda_k = (4\ell^2)^{-1/2} k^2 \pi^2 \). Moreover, we denote by \( \Sigma_0 \) the set of \( \lambda \in \mathbb{C} \) with positive real part.

**Lemma 4.1.** For every \( \rho \in (0, +\infty), \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > -\rho^{-1}(\text{Im}\lambda)^2 \) and \( f \in C_b(S; \mathbb{C}) \), the series \( \ell^{-1} \sum_{k \in \mathbb{Z}} (\mathcal{F}_{k, \nu})f \) defines a bounded and continuous function \( \mathcal{F}_{\lambda, \rho}f \) in \( \mathbb{R}^2 \) which, clearly, is periodic with respect to \( y \). Moreover,
\[ (i) \quad \mathcal{F}_{\lambda, \rho}f \in W^2_0(S; \mathbb{C}), \quad \lambda \mathcal{F}_{\lambda, \rho}f - \rho^{-1} \Delta \mathcal{F}_{\lambda, \rho}f - D_x \mathcal{F}_{\lambda, \rho}f = f \text{ in } S; \]
\[ (ii) \quad \nabla \mathcal{F}_{\lambda, \rho}f \in C_b(\mathbb{R}^2; \mathbb{C}) \times C_b(\mathbb{R}^2; \mathbb{C}) \text{ and} \]
\[ |\lambda| ||\mathcal{F}_{\lambda, \rho}f||_\infty + \sqrt{\lambda} ||\nabla \mathcal{F}_{\lambda, \rho}f||_\infty \leq c ||f||_\infty, \quad \lambda \in \Sigma_0; \quad (4.1) \]
\[ (iii) \quad \text{if further } \lim_{x \to -\infty} f(x, y) = 0 \text{ (resp. } \lim_{x \to +\infty} f(x, y) = 0) \text{ for every } y \in [-\ell/2, \ell/2], \text{ then} \]
\[ (\mathcal{F}_{\lambda, \rho}f)(x, y) \text{ vanishes as } x \to -\infty \text{ (resp. } x \to +\infty) \text{ for every } y \in \mathbb{R}; \]
\[ (iv) \quad \text{for every } f \in C^2_b(S; \mathbb{C}), \text{ such that } f(\cdot, -\ell/2) = f(\cdot, \ell/2), \text{ and } \lambda \in \mathbb{C} \text{ with } \text{Re}\lambda > -\rho^{-1}(\text{Im}\lambda)^2, \]
\[ \text{the function } \mathcal{F}_{\lambda, \rho}f \text{ admits classical derivatives up to the second-order which belong to } C^2_b(\mathbb{R}^2; \mathbb{C}). \]
Moreover,
\[ ||\mathcal{F}_{\lambda, \rho}f||_{C^{2, 0}_{x, r}(\mathbb{R}^2; \mathbb{C})} \leq c_\lambda ||f||_{C^2_b(S; \mathbb{C})}. \quad (4.2) \]

**Proof.** To begin with, we claim that, for each \( f \in C_b(S; \mathbb{C}) \) and \( \lambda \in \mathbb{C} \), such that \( \rho \text{Re}\lambda + (\text{Im}\lambda)^2 > 0 \), the series in the statement converges uniformly in \( \mathbb{R}^2 \). To prove the claim, we observe that \( \text{Re}(Z_k) > \rho \) and \( |Z_k| \geq \text{Re}(Z_k) \geq c_k(k + 1) \) for every \( k \in \mathbb{Z} \). Thus, we can estimate
\[ |(\mathcal{F}_{k, \nu})f(x)| \leq \sup_{x \in \mathbb{R}} |f_k(x)| \frac{1}{|Z_k|} \int_{\mathbb{R}} e^{-\frac{s}{2}x} e^{-\frac{i}{2}Z_k|s|} ds \leq \frac{c_\lambda}{k^2 + 1} ||f||_{C^1_b(S; \mathbb{C})}, \quad x \in \mathbb{R}, \quad k \in \mathbb{Z}, \quad (4.3) \]
and this is enough to infer that the series converges locally uniformly on \( \mathbb{R}^2 \) and, as a byproduct, that the operator \( \mathcal{F}_{\lambda, \rho} \) is bounded from \( C_b(S; \mathbb{C}) \) into \( C_b(\mathbb{R}^2; \mathbb{C}) \). Moreover, if \( f(\cdot, y) \) vanishes at
\(-\infty\) (resp. \(+\infty\)) for each \(y \in [-\ell/2, \ell/2]\), then by dominated convergence the function \(F_k f\) vanishes at \(-\infty\) (resp. \(+\infty\)) and, in view of the uniform convergence of the series which defines the function \(\mathcal{A}_{\lambda, \rho} f\), this is enough to conclude that this latter function tends to 0 as \(x \to -\infty\) (resp. \(x \to +\infty\)) for each \(x \in \mathbb{R}\).

Now, we prove properties (i), (ii) and (iv).

(i) Let us prove that the function \(\mathcal{A}_{\lambda, \rho} f\) is the unique solution to the equation \(\lambda u - \rho^{-1} \Delta u - D_x u = f^+\) in \(D = \{u \in C^1_b(\mathbb{R}^2; \mathbb{C}) \cap \bigcap_{\rho < \infty} W^{2, \rho}_{\text{loc}}(\mathbb{R}^2; \mathbb{C}) : \Delta u \in L^\infty(\mathbb{R}^2; \mathbb{C}) \cap C_0(\mathbb{S}; \mathbb{C}), u(\cdot, + \ell) = u\}\). For this purpose, for every \(n \in \mathbb{N}\) we introduce the functions \(u_n = \ell^{-1} \sum_{k=-n}^{n} (F_{k, \rho} f) e_k\) and \(f_n = \ell^{-1} \sum_{k=-n}^{n} f(k, e_k).\) Note that \(\lambda u_n - \rho^{-1} \Delta u_n - D_x u_n = f_n\) in \(\mathbb{R}^2\), for every \(n \in \mathbb{N}\), since the function \(F_{k, \rho} f(k \in \mathbb{Z})\) solves the differential equation \((\lambda + \rho^{-1} \lambda_k) u - \rho^{-1} u'' - u' = f_k\) in \(\mathbb{R}\). Thus,

\[
\langle u_n, \rho^{-1} \Delta \varphi - \varphi_x \rangle = \int_{\mathbb{R}^2} u_n(\rho^{-1} \Delta \varphi - \varphi_x) dxdy = \int_{\mathbb{R}^2} (\lambda u_n - f_n) \varphi dxdy =: \langle \lambda u_n - f_n, \varphi \rangle
\]

for every \(\varphi \in C^\infty_c(\mathbb{R}^2; \mathbb{C})\). Letting \(n\) tend to \(+\infty\) and applying the dominated convergence theorem, it follows that \(\mathcal{A}_{\lambda, \rho} f\) is a distributional solution to the equation \(\lambda \mathcal{A}_{\lambda, \rho} f - \rho^{-1} \Delta \mathcal{A}_{\lambda, \rho} f - D_x \mathcal{A}_{\lambda, \rho} f = f^+\).

By elliptic regularity (see e.g. [17]), we can infer that \(\mathcal{A}_{\lambda, \rho} f \in \bigcap_{\rho < \infty} W^{2, \rho}_{\text{loc}}(\mathbb{R}^2; \mathbb{C})\). Since \(\mathcal{A}_{\lambda, \rho} f\) is bounded and continuous in \(\mathbb{R}^2\) and \(\Delta \mathcal{A}_{\lambda, \rho} f + D_x \mathcal{A}_{\lambda, \rho} f = \lambda \mathcal{A}_{\lambda, \rho} f - f^+\) belongs to \(L^\infty(\mathbb{R}^2; \mathbb{C})\), again by classical results we can infer that \(\mathcal{A}_{\lambda, \rho} f \in C^1_b(\mathbb{R}^2; \mathbb{C})\) for each \(\gamma \in (0, 1)\) and, as a byproduct, that \(\Delta \mathcal{A}_{\lambda, \rho} f \in L^\infty(\mathbb{R}^2; \mathbb{C}) \cap C_0(\mathbb{S}; \mathbb{C})\). We can thus conclude that \(\mathcal{A}_{\lambda, \rho} f\) belongs to \(D\).

To prove uniqueness, we assume that \(v\) is another solution in \(D\) of the equation \(\lambda v - \rho^{-1} \Delta v - v_x = f^+\). The smoothness of \(v\) implies that, for each \(k \in \mathbb{Z}\), the function \(\hat{v}_k\) belongs to \(C^1_b(\mathbb{R}; \mathbb{C})\). Moreover, integrating by parts we obtain that

\[
\int_{\mathbb{R}} \hat{v}_k(\varphi'' dx = \frac{1}{\ell} \int_{\mathbb{S}} \varphi'' \psi_k dxdy = \frac{1}{\ell} \int_{\mathbb{S}} \varphi v_x \psi_k dxdy
\]

for each \(\varphi \in C^\infty_c(\mathbb{R})\). By Fubini theorem, \(\hat{v}_k\) belongs to \(W^{2, \rho}_{\text{loc}}(\mathbb{R}; \mathbb{C})\). Since \(\lambda v - \rho^{-1} \Delta v - v_x = f\) in \(S\), we can write

\[
\int_{\mathbb{S}} \varphi(x) v_x \psi_k dxdy = \lim_{n \to +\infty} \int_{\mathbb{S}} \varphi(x) v_{x}(x, y) \psi_n(y) e_k(y) dxdy
\]

\[= \lambda \rho \int_{\mathbb{R}} \varphi v_k dx - \lim_{n \to +\infty} \int_{\mathbb{S}} \varphi v_{yy} \psi_n \psi_k dxdy - \rho \int_{\mathbb{R}} \varphi \psi_k' dx - \rho \int_{\mathbb{R}} \varphi f_k dx, \quad (4.4)
\]

where \(\psi_n(y) = \psi(n|y|/\ell + 1 - n/2)\) for each \(y \in [-\ell/2, \ell/2]\), \(n \in \mathbb{N}\), and \(\psi\) is a smooth function such that \(\psi = 1\) in \((-\infty, 1/2]\) and \(\psi = 0\) outside \((-\infty, 3/4]\). Clearly, \(\psi_n\) converges to 1 in \(L^1((-\ell/2, \ell/2))\) as \(n\) tends to \(+\infty\). An integration by parts shows that

\[
\lim_{n \to +\infty} \int_{\mathbb{S}} \varphi(x) v_{yy} \psi_n(y) e_k(y) dxdy
\]

\[
= - \lim_{n \to +\infty} \frac{n}{\ell} \int_{\mathbb{R}^2} \varphi(x) \chi_A(x) \text{signum}(y) v_y(x, y) \psi' \left(\frac{n}{\ell} |y| + 1 - \frac{n}{2}\right) \chi_{B_n}(y) e_k(y) dxdy
\]

\[- \lambda \rho \int_{\mathbb{R}} \varphi(v_k(y) dx,
\]

\[
(4.5)
\]

where \(A = \text{supp}(\varphi)\) and \(B_n = \{ y \in \mathbb{R} : -\frac{\ell}{2} - \frac{n}{\ell} \leq |y| \leq \frac{\ell}{2} - \frac{n}{\ell} \}\) for each \(n \in \mathbb{N}\). We claim that the first term in the last side of (4.5) is zero. For this purpose, we split

\[
\frac{n}{\ell} \int_{\mathbb{R}^2} \varphi(x) \chi_A(x) \text{signum}(y) v_y(x, y) \psi' \left(\frac{n}{\ell} |y| + 1 - \frac{n}{2}\right) \chi_{B_n}(y) e_k(y) dxdy
\]

\[
= \frac{n}{\ell} \int_{\mathbb{R}^2} \varphi(x) \chi_A(x) \text{signum}(y)(v_y(x, y) - v_y(x, \ell/2)) \psi' \left(\frac{n}{\ell} |y| + 1 - \frac{n}{2}\right) \chi_{B_n}(y) e_k(y) dxdy
\]

\[+ \frac{n}{\ell} \int_{\mathbb{S}} \chi_{A}(x) \psi' \left(\frac{n}{\ell} |y| + 1 - \frac{n}{2}\right) \chi_{B_n}(y) e_k(y) dy
\]

\[= : \mathcal{K}_1 n(x, y) + \mathcal{K}_2 n(x, y).\]
for every \((x, y) \in \mathbb{R}^2\). Since \(v\) is \(\ell\)-periodic with respect to \(y\), \(D_y v\) is \(\ell\)-periodic with respect to \(y\) as well. Moreover, this latter function is 1/2-Hölder continuous in \(\mathbb{R}^2\) since it belongs to \(\mathcal{D}\). Hence, we can estimate

\[
|v_y(x, y) - v_y(x, \ell/2)| = |v_y(x, y) - v_y(x, -\ell/2)| \leq |v_y|_{C^{1/2}(\mathbb{R}^2)} \min\{|y - \ell/2|, |y + \ell/2|\}^{1/2}
\]

for every \(x \in A\) and \(y \in [-\ell/2, \ell/2]\). Thus,

\[
|\mathcal{K}_{1,n}| \leq c\|\varphi\|_\infty \|\psi\|_\infty n^{\frac{1}{2}} m(A \times B_n) \leq c\|\varphi\|_\infty \|\psi\|_\infty n^{-\frac{1}{2}}, \quad n \in \mathbb{N},
\]

where \(m(A \times B_n)\) denotes the Lebesgue measure of the set \(A \times B_n\), so that \(\mathcal{K}_{1,n}\) vanishes as \(n\) tends to \(+\infty\). As far as \(\mathcal{K}_{2,n}\) is concerned, we observe that

\[
\mathcal{K}_{2,n}(x, y) = -\frac{2in}{\ell} \int_A D_y v(x, \ell/2) \varphi(x) dx \int_\frac{-\ell/2}{\ell} \psi\left(\frac{n}{\ell}y + 1 - \frac{n}{\ell}\right) \sin\left(\frac{2k\pi}{\ell} y\right) dy
\]

\[
= \left[\frac{4k\pi i}{\ell} \int_\frac{-\ell/2}{\ell} \psi\left(\frac{n}{\ell}y + 1 - \frac{n}{\ell}\right) \sin\left(\frac{2k\pi}{\ell} y\right) dy - 2(-1)^k \sin\left(\frac{k\pi}{n}\right)\right].
\]

Hence, \(\mathcal{K}_{2,n}\) vanishes as \(n\) tends to \(+\infty\). The claim is so proved.

From (4.1) and (4.5), we can now infer that \(\rho^{-1} v^{\alpha}_\varphi = (\lambda + \rho^{-1}\lambda_k) v - \hat{v}_k\). Thus, \(v_k = \mathcal{K} f\) for every \(k \in \mathbb{Z}\) and, as a byproduct, we deduce that \(v = \mathcal{K} f\).

(ii) By classical results, the realization \(A_\varphi\) in \(L^\infty(\mathbb{R}^2; \mathcal{C})\) of the operator \(\rho^{-1} \Delta + D_\varphi\), with domain \(D(A_\varphi) = \{u \in C^1(\mathbb{R}^2; \mathcal{C}) \cap \bigcap_{\varepsilon > 0} \text{W}^{2,p}_\varepsilon(\mathbb{R}^2; \mathcal{C}) \colon \Delta u \in L^\infty(\mathbb{R}^2; \mathcal{C})\} \subseteq \mathcal{D}\), generates an analytic semigroup. Moreover, \(\|R(\lambda, A_\varphi)\|_{L(L^\infty(\mathbb{R}^2))} \leq c|\lambda|^{-1}\) for every \(\lambda \in \Sigma_0\) and \(\|\varphi\|_\infty \leq c\|\varphi\|_\infty \|\varphi\|_\infty^{1/2}\) for every \(\varphi \in D(A_\varphi)\). Note that the function \(R(\lambda, A_\varphi)f^g\) is \(\ell\)-periodic with respect to \(y\). Indeed, \(R(\lambda, A_\varphi)f^g\) and \(R(\lambda, A_\varphi)f^g(\cdot, \cdot + \ell)\) both solve (in \(D(A_\varphi)\)) the equation \(\lambda u - \rho^{-1} \Delta u = f^g\) and, by uniqueness, they coincide. Finally, since \(f^g\) is continuous in \(\mathcal{F}_s\), \(\Delta R(\lambda, A_\varphi)f^g\) belongs to \(C_{b}(\mathcal{C}; \mathcal{C})\). Hence, \(R(\lambda, A_\varphi)f^g\in \mathcal{D}\) and, by (i), it coincides with \(\mathcal{K} f\). Using the above estimates, inequality (4.1) follows immediately.

(iv) Since \(f \in C^0_b(S; \mathcal{C})\) and \(f(\cdot, -\ell/2) = f(\cdot, \ell/2)\), the function \(f^g\) belongs to \(C_b^0(\mathbb{R}^2; \mathcal{C})\). Hence, \(\Delta \mathcal{K} f = C(\lambda, A_\varphi f - f^g - D_\varphi)\) is an element of \(C^0_b(\mathbb{R}^2; \mathcal{C})\). Classical results (see e.g., [27]) yield (4.2).

Lemma 4.2. For \(g \in C_b(S^2_0; \mathcal{C})\) and \(\lambda \in \mathbb{C}\), such that \(\text{Re}\lambda > -\text{Le}^{-1}(\text{Im}\lambda)^2\), the function \(\mathcal{J}_g = \ell^{-1} \sum_{k \in \mathbb{Z}} (\mathcal{K}_k g) e_k\) is bounded and continuous in \(\mathbb{R}^2\). Here, \(\mathcal{K}_k : = \mathcal{K}_{L, \lambda}(\mathcal{C})\) for each \(k \in \mathbb{Z}\), and \(\mathcal{C}\) is the trivial extension of \(g\) to \(\mathbb{R} \times [-\ell/2, \ell/2]\). Moreover,

(i) \(\mathcal{J}_g\) belongs to \(\bigcap_{\varepsilon > 0} W^{2,p}_{\varepsilon}(\mathbb{R}^2; \mathcal{C})\) and \(\lambda \mathcal{J}_g - \ell^{-1} \Delta \mathcal{J}_g - D_\varphi \mathcal{J}_g = g\) in \(S^2_0\);

(ii) there exists a positive constant \(c_1\), independent of \(\lambda\), such that

\[
|\lambda| \|\mathcal{J}_g\|_\infty + \sqrt{|\lambda|} \|\nabla \mathcal{J}_g\|_\infty \leq c_1 \|g\|_\infty, \quad \lambda \in \Sigma_0;
\]

(iii) if further \(\lim_{x \to \pm \infty} g(x, y) = 0\) for each \(y \in [-\ell/2, \ell/2]\), then \(\mathcal{J}_g\) vanishes as \(x \to +\infty\) for each \(y \in \mathbb{R}\);

(iv) if \(g \in C^0_b(S^2_0; \mathcal{C})\) and \(g(\cdot, -\ell/2) = g(\cdot, \ell/2)\), then \(\mathcal{J}_g\) belongs to \(C^0_b(\mathbb{R}^2; \mathcal{C})\) for every \(\lambda \in \mathbb{C}\) such that \(\text{Re}\lambda > -\text{Le}^{-1}(\text{Im}\lambda)^2\). Moreover,

\[
\|\mathcal{J}_g\|_{C^2_b(\mathbb{R}^2; \mathcal{C})} \leq c_2 \|g\|_{C^0_b(S^2_0; \mathcal{C})}, \quad \lambda \in \Sigma_0;
\]

with the constant \(c_{2,\lambda}\) being independent of \(g\).

Proof. We split the proof into two steps: in the first one we prove properties (i), (ii) and (iii) and in the second step we prove property (iv).

Step 1. Let us fix \(g\) and \(\lambda\) as in the statement. Arguing as in the first part of the proof of Lemma [1] taking the continuity of the functions \(\mathcal{J}_g (k \in \mathbb{Z})\) into account, it can be checked that the series in the statement converges uniformly in \(\mathbb{R}^2\), so that the function \(\mathcal{J}_g\) is well defined and it vanishes as \(x \to +\infty\) for every \(y \in \mathbb{R}\), if \(\lim_{x \to \pm \infty} g(x, y) = 0\) for every \(y \in [-\ell/2, \ell/2]\).
To check properties (i) and (ii), for each \( n \in \mathbb{N} \) we set \( g_n := \psi_0 \ast \psi_n \), where \( \ast \) stands for convolution with respect to the variable \( x \) and \( (\psi_n) \) is a standard sequence of mollifiers. Clearly, \( \mathcal{F}_{\lambda} g_n = \mathcal{F}_{\lambda} \psi_0 \ast \psi_n \). The sequence \( (g_n) \) converges to \( \psi_0 \) pointwise in \( \mathbb{R}^2 \) as \( n \to +\infty \) and \( ||g_n||_\infty \leq ||g||_\infty \) for each \( n \in \mathbb{N} \). Thus, we can infer that \( \mathcal{F}_{\lambda} g_n \) converges to \( \mathcal{F}_{\lambda} \psi_0 \) pointwise in \( \mathbb{R} \) as \( n \) tends to \( +\infty \), for every \( k \in \mathbb{N} \) and, by dominated convergence, \( \mathcal{F}_{\lambda} g_n \) tends to \( \mathcal{F}_{\lambda} \psi_0 \) pointwise in \( \mathbb{R}^2 \).

Applying the classical interior \( L^p \)-estimates for the operator \( \mathcal{L}^{-1} \Delta + D_x \) and using (4.1), which allows us to write

\[
|\lambda||\nabla \mathcal{F}_{\lambda} g_n|_\infty + |\lambda|^{\frac{1}{2}}||\nabla \mathcal{F}_{\lambda} g_n|_\infty \leq c ||g_n|_\infty \leq c ||g||_\infty, \quad n \in \mathbb{N},
\]

we can estimate

\[
||\mathcal{F}_{\lambda} g_n||_{W^{2,p}(B(0,r);\mathbb{C})} \leq c_{p,r} \left(||\mathcal{F}_{\lambda} g_n||_{L^p(B(0,2r);\mathbb{C})} + ||\mathcal{L}^{-1} \Delta \mathcal{F}_{\lambda} g_n + D_x \mathcal{F}_{\lambda} g_n||_{L^p(B(0,2r);\mathbb{C})}\right)
\]

\[
\leq c_{p,r} \left(1 + |\lambda|\right)||\mathcal{F}_{\lambda} g_n||_{C(B(0,r);\mathbb{C})} + ||g_n||_{C(B(0,2r);\mathbb{C})}
\]

\[
\leq c_{p,r,\lambda} ||g_n||_\infty \leq c_{p,r,\lambda} ||g||_\infty
\]

for every \( p \in [1, +\infty) \) and \( r > 0 \). Hence, by compactness, we conclude that \( \mathcal{F}_{\lambda} g_n \) converges to \( \mathcal{F}_{\lambda} g \) in \( C^1(\overline{B(0,r)};\mathbb{C}) \) for every \( r > 0 \), \( \mathcal{F}_{\lambda} g \in W^{2,p}_0(\mathbb{R}^2;\mathbb{C}) \) for every \( p \in [1, +\infty) \) and \( \lambda \mathcal{F}_{\lambda} g - \mathcal{L}^{-1} \Delta \mathcal{F}_{\lambda} g - D_x \mathcal{F}_{\lambda} g = g \) in \( S^0 \). Finally, estimate (4.8) follows as once from (4.8).

Step 2. To complete the proof, here we check property (iv), which demands some additional effort. We begin by checking that the function \( \zeta = \mathcal{F}_{\lambda} g(0, \cdot) \) belongs to \( C_b^{2+\alpha}(\mathbb{R}; \mathbb{C}) \). For this purpose, we set \( \zeta_n = \ell^{-1} \sum_{k=-n}^{n} (\mathcal{F}_{\lambda} g)(0) e_k \) for each \( n \in \mathbb{N} \). Clearly, each function \( \zeta_n \) is smooth and

\[
\zeta_n = \frac{2\pi i}{\ell^2} \sum_{|k|<\ell_k} \frac{k}{|k|} e_k \int_0^\infty e^{\frac{k}{\ell} s} e^{-\frac{\pi}{2} \lambda_{\lambda} s} g_k(s) ds + \frac{2\pi i}{\ell^2} \sum_{|k|=\ell_k} \left( \frac{k}{|k|} \frac{\ell}{2\pi} \right) e_k \int_0^\infty e^{\frac{k}{\ell} s} e^{-\frac{\pi}{2} \lambda_{\lambda} s} g_k(s) ds
\]

\[
+ \frac{2i}{\ell} \sum_{|k|=\ell_k} e_k \int_0^\infty e^{\frac{k}{\ell} s} \left( e^{-\frac{\pi}{2} \lambda_{\lambda} s} - e^{-\frac{\pi}{2} \lambda_{\lambda} s} \right) g_k(s) ds + \frac{2i}{\ell} \sum_{|k|=\ell_k} e_k \int_0^\infty e^{\frac{k}{\ell} s} \frac{-\pi}{2\lambda_{\lambda} s} g_k(s) ds
\]

\[
=: J_0 + \sum_{h=1}^3 J_{n,h},
\]

where \( k_0 \in \mathbb{N} \) is chosen so that \( \pi k_0 > \ell \lambda \). As it is easily seen,

\[
\left| \frac{k}{|k|} \frac{\ell}{2\pi} \right| \leq \frac{c_{\lambda}}{k^2+1}, \quad k \in \mathbb{Z},
\]

so that \( J_{n,h} \) converges uniformly in \( \mathbb{R}^2 \) as \( n \to +\infty \). On the other hand,

\[
\left| e^{-\frac{\pi}{2} \lambda_{\lambda} s} - e^{-\frac{\pi}{2} \lambda_{\lambda} s} \right| = \int_0^1 \frac{d}{dr} e^{-\frac{\pi}{2} \lambda_{\lambda} s} dr \leq s \int_0^1 \left| \frac{\ell}{\ell_k} \right| e^{-\frac{\pi}{2} \lambda_{\lambda} r} dr \leq e^{-\frac{\pi}{2} \lambda_{\lambda} s} \left| \frac{\ell}{\ell_k} \right| \|

where we have set \( Z_k(r) = \left( \left( \ell^2 + 4\lambda \right) r + \frac{4\lambda^2}{\ell^2} \right)^{\frac{1}{2}} \). Note that \( \text{Re}(Z_k(r)) \geq c_{\lambda}|k| \) for \( |k| \geq k_0 \).

For such values of \( k \) and for \( \text{Re}(\lambda) > 0 \) (which implies that \( \text{Re}(Z_k(r)) > \ell \) for every \( r \in [0,1] \) and \( |k| \geq k_0 \)) we can estimate

\[
\left| \int_0^\infty e^{\frac{k}{\ell} s} \left( e^{-\frac{\pi}{2} \lambda_{\lambda} s} - e^{-\frac{\pi}{2} \lambda_{\lambda} s} \right) g_k(s) ds \right| \leq \left| \frac{\ell^2 + 4\lambda \ell}{4|k|} \right| ||g||_\infty \int_0^\infty r^{-\frac{\pi}{2\lambda_{\lambda}}} dr \int_0^\infty s e^{\frac{\pi}{2\lambda_{\lambda}} s} ||\text{Re}(Z_k(s))|| ds
\]

\[
\leq \frac{c_{\lambda}}{k^2+1} ||g||_\infty,
\]

so that the sequence \( (J_{2,n}) \) converges uniformly in \( \mathbb{R}^2 \). Next, we observe that

\[
J_{3,n}(y) = \frac{2i}{\ell} \int_0^\infty ds \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} K_n(s,y) g_0(s, y - \eta) d\eta, \quad y \in \mathbb{R}, \quad n \in \mathbb{N},
\]
where $K_n(x, y) = H_n(x, y) + H_n(x, -y)$ and
\[ H_n(x, y) = e^{\frac{\pi i}{2}} e^{-\frac{\pi}{e^{\pi i} - e^{\frac{\pi (k_0 - 1)}{2}}}} e^{\frac{\pi (k_0 - 1) n}{e^{\pi i} - e^{\frac{\pi (k_0 - 1)}{2}}}}, \quad g_n(x, y) = \ell^{-1} \sum_{k=-n}^{n} \hat{g}(k) e^{\frac{2\pi i k}{\ell}}, \]
for $x \geq 0, y \in \mathbb{R}^2$ and $n \in \mathbb{N}$. We set
\[ K(x, y) = e^{\left(\frac{\pi i}{2} - \frac{x (k_0 - 1)}{2}\right)^2} \left( e^{\frac{2\pi (k_0 - 1)}{e^{\pi i} - e^{\frac{\pi (k_0 - 1)}{2}}}} + e^{\frac{-\pi (k_0 - 1)}{e^{\pi i} - e^{\frac{\pi (k_0 - 1)}{2}}}}\right), \quad x \geq 0, y \in \mathbb{R}, \]
and prove that $J_{3,n}$ converges pointwise in $\mathbb{R}$ to the function $J_3$, defined by
\[ J_3(y) = \frac{i\ell}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ds \int_{-\frac{1}{2}}^{\frac{1}{2}} K(s, y) g^2(s, y - \eta) d\eta, \quad y \in \mathbb{R}. \]
For this purpose, we split
\[ J_{3,n} - J_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} ds \int_{-\frac{1}{2}}^{\frac{1}{2}} (g_n^2(s, \cdot - \eta) - g^2(s, \cdot - \eta)) K_n(s, \eta) d\eta 
+ \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \int_{-\frac{1}{2}}^{\frac{1}{2}} g^2(s, \cdot - \eta)(K_n(s, \eta) - K(s, \eta)) d\eta =: A_{1,n} + A_{2,n} \]
for every $n \in \mathbb{N}$ and observe that
\[ \|A_{1,n}\|_{L^\infty} \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \|g_n(s, \cdot) - g(s, \cdot)\|_{L^2((-\pi/2, \pi/2))} \|K_n(s, \cdot)\|_{L^2((-\pi/2, \pi/2))} ds, \quad n \in \mathbb{N}. \]
Since (i) $\|g_n(s, \cdot - \eta) - g(s, \cdot - \eta)\|_{L^2((-\pi/2, \pi/2), \mathbb{C})}$ vanishes as $n \to +\infty$, (ii) $\|g_n(s, \cdot) - g(s, \cdot)\|_{L^2((-\pi/2, \pi/2), \mathbb{C})} \leq 2\|g(s, \cdot)\|_{L^2((-\pi/2, \pi/2), \mathbb{C})} \leq 2\|g\|_{L^\infty}$ for $x \geq 0$ and $n \in \mathbb{N}$, and (iii) $|K_n| \leq 2|K|$ in $\mathbb{R}_+ \times \mathbb{R}$ for every $n \in \mathbb{N}$, the dominated convergence theorem shows that $A_{1,n}$ converges to zero pointwise in $\mathbb{R}$ as $n$ tends to $+\infty$. Moreover, $\|A_{1,n}\|_{L^\infty} \leq c\|g\|_{L^\infty}$. That theorem also shows that $A_{2,n}$ converges to zero pointwise in $\mathbb{R}$ as $n$ tends to $+\infty$; moreover, $\|A_{2,n}\|_{L^\infty} \leq c\|g\|_{L^\infty}$ for each $n \in \mathbb{N}$. Now, writing
\[ \zeta_n(y) = \zeta_n(0) + \int_{\mathbb{R}} \left(J_0(r) + \sum_{h=1}^{3} J_{h,n}(r)\right) dr, \quad y \in \mathbb{R}, \quad n \in \mathbb{N}, \]
and letting $n$ tend to $+\infty$, again by dominated convergence we conclude that
\[ \zeta' = \frac{2\pi i}{\ell^2} \sum_{|k| < k_0} k e_k \int_{0}^{+\infty} \frac{\ell e^{\pi i s}}{2\pi} e^{-\frac{\pi}{2} \hat{z}_s \hat{z}_k} \hat{g_k}(s) ds + \frac{2\pi i}{\ell^2} \sum_{|k| \geq k_0} \left( \frac{k}{2\pi} - \frac{\ell}{\pi} \right) e_k \int_{0}^{+\infty} e^{\frac{\pi i s}{2}} e^{-\frac{\pi}{2} \hat{z}_s \hat{z}_k} \hat{g_k}(s) ds 
+ \frac{2i}{\ell} \sum_{|k| \geq k_0} e_k \int_{0}^{+\infty} e^{\frac{\pi i s}{2}} \left( e^{-\frac{\pi}{2} \hat{z}_s \hat{z}_k} - e^{-\frac{\pi}{2} z_s \hat{z}_k} \right) \hat{g_k}(s) ds + \frac{i\ell}{\pi} \int_{0}^{+\infty} ds \int_{-\frac{1}{2}}^{\frac{1}{2}} K(s, \eta) g^2(s, \cdot - \eta) d\eta. \]
Denote by $\phi_1, \ldots, \phi_4$ the four terms in the right-hand side of the previous formula. Clearly, $\phi_1$ belongs to $C_{b}^{\theta}(\mathbb{R}; \mathbb{C})$. In particular, $\|\phi_1\|_{C_{b}^{\theta}(\mathbb{R}; \mathbb{C})} \leq c\|g\|_{L^\infty}$. As far as $\phi_2$ and $\phi_3$ are concerned, using (4.9), (4.10) and the arguments above, it can be easily shown that such functions belong to $C_{b}^{\theta}(\mathbb{R}; \mathbb{C})$ and $\|\phi_2\|_{C_{b}^{\theta}(\mathbb{R}; \mathbb{C})} + \|\phi_3\|_{C_{b}^{\theta}(\mathbb{R}; \mathbb{C})} \leq c\|g\|_{L^\infty}$. The function $\phi_4$ is the limit in $C_{b}(\mathbb{R}; \mathbb{C})$ of the sequence $(\phi_{4,n})$ defined by
\[ \phi_{4,n} = \frac{i\ell}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} ds \int_{-\frac{1}{2}}^{\frac{1}{2}} K(s, \eta) g^2(s, \cdot - \eta) d\eta, \quad n \in \mathbb{N}. \]
Clearly, each function $\phi_{4,n}$ is continuously differentiable in $\mathbb{R}$ and

$$
\phi_{4,n} = \frac{i\ell}{\pi} \int_{-\frac{\pi}{2}}^{+\infty} ds \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K_{y}(s,\eta)g^{\ast}(s,\cdot - \eta)d\eta,
$$

where

$$
K_{y}(x, y) = \frac{2\pi i}{\ell} \left( k_{0} - 1 \right) e^{\left( \frac{\pi}{2} - \frac{\pi k_{0}}{\pi} \right)x} \left( e^{\frac{2\pi i (k_{0} - 1)}{\pi y}} - e^{-\frac{2\pi i (k_{0} - 1)}{\pi y}} \right) + \frac{2\pi i}{\ell} e^{\left( \frac{\pi}{2} - \frac{\pi k_{0}}{\pi} \right)x} \left( e^{\frac{\pi i (x - 4iby)}{2}} - e^{\frac{\pi i (x + 4iby)}{2}} \right) =: L_{1}(x, y) + L_{2}(x, y)
$$

(4.11)

for $(x, y) \in \overline{H_{0}^{3}}$. Since $K$ is $\ell$-periodic with respect to $y$, it follows that $\int_{-\ell/2}^{\ell/2} K_{y}(s, \eta)d\eta = 0$. Hence, we can write

$$
\phi_{4,n}(y) = \frac{i\ell}{\pi} \int_{-\frac{\pi}{2}}^{+\infty} ds \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K_{y}(s, \eta)(g^{\ast}(s, y - \eta) - g(s, \eta))d\eta, \quad y \in \mathbb{R}, \ n \in \mathbb{N}
$$

By assumptions, $g \in C_{0}^{0}(S_{n}^{+}; \mathbb{C})$ and this allows us to estimate

$$
|K_{y}(s, \eta)(g^{\ast}(s, y - \eta) - g^{\ast}(s, \eta))| \leq c \min \{(s^{2} + \eta^{2})^{\frac{1}{2}} - 1, (e^{\pi/\ell} - 1)^{-1}\} ||g||_{C_{0}^{0}(0, +\infty) \times \mathbb{R} \times \mathbb{C}}
$$

for every $(s, \eta) \in \mathbb{R} \times \mathbb{R}^{+}$ and $y \geq 0$. Thus, we can let $n$ tend to $+\infty$ in (4.11) and conclude that $\phi_{4,n}$ converges uniformly in $\mathbb{R}^{2}$. As a byproduct, $\phi_{4}$ is continuously differentiable in $\mathbb{R}$,

$$
\phi_{4} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K_{y}(s, \eta)(g^{\ast}(s, \cdot - \eta) - g(s, \cdot))d\eta
$$

and $||\phi_{4}|| \leq c||g||_{C_{0}^{0}(S_{n}^{+}; \mathbb{C})}$.

To prove that $\phi_{4}$ belongs to $C_{0}^{0}(\mathbb{R}; \mathbb{C})$ we split

$$
\phi_{4} = \int_{0}^{+\infty} ds \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( L_{1}(s, \eta) + L_{2}(s, \eta)\chi_{(1, +\infty)}(s) \right)(g^{\ast}(s, \cdot - \eta) - g(s, \cdot))d\eta
$$

\[\] + \int_{0}^{1} ds \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_{2}(s, \eta)(g^{\ast}(s, \cdot - \eta) - g(s, \cdot))d\eta.

(4.12)

Since the function $L_{1} + L_{2}\chi_{(1, +\infty)} \in L^{1}(S_{n}^{+}; \mathbb{C})$, the first term in the right-hand side of (4.12) belongs to $C_{0}^{0}(\mathbb{R}; \mathbb{C})$ and its $C_{0}^{0}(\mathbb{R}; \mathbb{C})$-norm can be estimated from above by $c||g||_{C_{0}^{0}(S_{n}^{+}; \mathbb{C})}$. To estimate the other term, which we denote by $\Psi$, we observe that

$$
\frac{e^{\pi i (x - 4iby)}}{(e^{\pi i (x - 4iby)} - 1)^{2}} = \frac{\ell^{2} 4x^{2} + 32i\pi y - 64iy^{2}}{(x^{2} + 16y^{2})^{2}} + \psi(x, y), \quad (x, y) \in (0, 1) \times (-\ell/2, \ell/2),
$$

for some function $\psi \in L^{1}(0, 1) \times (-\ell/2, \ell/2); \mathbb{C})$. Thus,

$$
\Psi(y) = -\frac{128i\ell^{2}}{\pi^{2}} \int_{0}^{1} ds \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} s(y - \eta) \frac{g^{\ast}(s, \eta) - g(s, \eta)}{(s^{2} + 16(y - \eta)^{2})^{2}}d\eta
$$

\[\]

$$
-2 \int_{0}^{1} ds \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\psi(s, \eta) - \psi(s, -\eta))(g^{\ast}(s, y - \eta) - g(s, \cdot))d\eta =: \Psi_{1}(y) + \Psi_{2}(y)
$$

for every $y \in \mathbb{R}$. The function $\Psi_{2}$ is clearly $\alpha$-Hölder continuous in $[-\ell/2, \ell/2]$ since $\psi \in L^{1}(0, 1) \times (-\ell/2, \ell/2); \mathbb{C})$. Moreover, $||\Psi_{2}||_{C_{0}^{0}(\mathbb{R}; \mathbb{C})} \leq c||g||_{C_{0}^{0}(\mathbb{R}; \mathbb{C})}$. As far as the function $\Psi_{1}$ is concerned, we approximate it with the family of functions $\Psi_{1,h}$ defined by

$$
\Psi_{1,h}(y) = -\frac{128i\ell^{2}}{\pi^{2}} \int_{0}^{1} ds \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} s(y - \eta) \frac{g^{\ast}(s, \eta) - g(s, \cdot)}{(s^{2} + 16(y - \eta)^{2} + h^{2})^{2}}d\eta, \quad h > 0.
$$
Each of these functions is continuously differentiable in $\mathbb{R}$ with bounded derivative, so that, we can estimate

$$|\Psi_1(y_2) - \Psi_1(y_1)| \leq |\Psi_1(y_2) - \Psi_{1,h}(y_2)| + |\Psi_{1,h}(y_2) - \Psi_{1,h}(y_1)| + |\Psi_{1,h}(y_1) - \Psi_1(y_1)|$$

$$\leq 2\|\Psi_1 - \Psi_{1,h}\|_{\infty} + \|\Psi_{1,h}'\|_{\infty}|y_2 - y_1|$$  \hspace{1cm} (4.13)

for $y_1, y_2 \in [-\ell/2, \ell/2]$. Note that

$$\|\Psi_1 - \Psi_{1,h}\|_{\infty} \leq \frac{256\ell^2}{\pi^2}\|g\|_{C^2_b(S^+_0,\mathbb{C})} \int_{\mathbb{R}^2} \frac{|s\eta| h^2}{(s^2 + 16\eta^2 + h^2)(s^2 + 16\eta^2)^2} ds d\eta$$

$$\leq ch^2\|g\|_{C^2_b(S^+_0,\mathbb{C})} \int_0^{+\infty} \frac{\rho^{\alpha+1}}{(\rho^2 + h^2)^2} d\rho = ch^2\|g\|_{C^2_b(S^+_0,\mathbb{C})}$$

and

$$\|\Psi_{1,h}\|_{\infty} \leq \frac{128\ell^2}{\pi^2}\|g\|_{C^2_b(S^+_0,\mathbb{C})} \int_{\mathbb{R}^2} \frac{(s^2 + 48\eta^2 + h^2)|\eta|^{\alpha}}{(s^2 + 16\eta^2 + h^2)^2} ds d\eta$$

$$\leq c\|g\|_{C^2_b(S^+_0,\mathbb{C})} \int_0^{+\infty} \frac{\rho^{2+\alpha}}{(\rho^2 + h^2)^2} d\rho \leq c\|g\|_{C^2_b(S^+_0,\mathbb{C})} h^{\alpha-1}.$$  \hspace{1cm}

Replacing these inequalities into (4.13) and taking $h = |y_2 - y_1|$, we conclude that

$$|\Psi_1(y_2) - \Psi_1(y_1)| \leq c\|g\|_{C^2_b(S^+_0,\mathbb{C})}|y_2 - y_1|.$$  \hspace{1cm}

Therefore, $\phi'_4 \in C^2_b(\mathbb{R};\mathbb{C})$ and $\|\phi'_4\|_{C^2_b(\mathbb{R};\mathbb{C})} \leq c\|g\|_{C^2_b(S^+_0,\mathbb{C})}$. Putting everything together it follows that $\zeta \in C^{2+\alpha}_{p}(\mathbb{R};\mathbb{C})$ and $\|\zeta\|_{C^{2+\alpha}_{p}(\mathbb{R};\mathbb{C})} \leq c\|g\|_{C^2_b(S^+_0,\mathbb{C})}.$

Finally, we consider the function $\mathcal{A}g = \zeta = v \in C_b((0, +\infty) \times \mathbb{R};\mathbb{C}) \cap \bigcap_{p < +\infty} W^{2,p}_x((0, +\infty) \times \mathbb{R};\mathbb{C}).$ Since $L^{-1}_{\Delta} v + v_x \in C^0((0, +\infty) \times \mathbb{R};\mathbb{C})$ and by construction $v(0, \cdot) = 0$, by classical results (see, e.g., [27]) $v \in C^{2+\alpha}_{p}((0, +\infty) \times \mathbb{R})$ and

$$\|v\|_{C^{2+\alpha}_{p}(U_T^+;\mathbb{C})} \leq c(\|v\|_{\infty} + \|L^{-1}_{\Delta} v + v_x\|_{C^0(U_T^+;\mathbb{C})} + \|\zeta\|_{C^{2+\alpha}_{p}(\mathbb{R};\mathbb{C})})$$

$$\leq c(\|v\|_{\infty} + \lambda \mathcal{A}g - g\|_{C^0(U_T^+;\mathbb{C})} + \|\zeta\|_{C^{2+\alpha}_{p}(\mathbb{R};\mathbb{C})}) \leq c\|g\|_{C^2_b(S^+_0,\mathbb{C})}.$$  \hspace{1cm}

Formula (4.17) follows as once.

\textbf{Lemma 4.3.} For each $\lambda \in \mathbb{C}$ such that $\text{Re} \lambda > -(\text{Im} \lambda)^2$, $f \in C_b(\overline{S};\mathbb{C})$ and $g \in C_b(\overline{S}_0;\mathbb{C})$, we denote by $\mathcal{F}_f : H^2_\mathbb{R} \to \mathbb{C}$ and $\mathcal{A}g : H^2_\mathbb{R} \to \mathbb{C}$, respectively, the functions defined by

$$\mathcal{F}_f(x, y) = e^{-\frac{\lambda}{2}(x-R)} \sum_{k \in \mathbb{Z}} e^{-\frac{\lambda}{2} Y_k(x-R)} \mathcal{F}_{k,1} f(R)e_k(y), \quad (x, y) \in H^2_\mathbb{R},$$

$$\mathcal{A}g(x, y) = e^{-\frac{\lambda}{2} x} \sum_{k \in \mathbb{Z}} Y_k(x)(\mathcal{A}g)(0)e_k(y), \quad (x, y) \in H^2_\mathbb{R}.$$  \hspace{1cm}

Then, the following properties are satisfied.

\begin{itemize}
  \item[(i)] $\mathcal{F}_f$ belongs to $C^1_b(H^2_\mathbb{R};\mathbb{C}) \cap W^{2,p}_x(H^2_\mathbb{R};\mathbb{C})$, for each $p < +\infty$, solves the equation $\lambda u - L^{-1}_{\Delta} u - u_x = 0$ in $H^2_\mathbb{R}$ and, for each $M > 0$, there exists a constant $c_M > 0$ such that

$$|\lambda||\mathcal{F}_f|_{\infty} + \sqrt{|\lambda|}||\nabla \mathcal{F}_f|_{\infty} \leq c_M||f||_{\infty}, \quad |\lambda| \geq M;$$

\item[(ii)] $\lim_{x \to -\infty} \mathcal{F}_f(x, y) = \lim_{x \to -\infty} \mathcal{F}_f(x, y) = 0$ for every $y \in \mathbb{R}$ and $f \in C_b(\overline{S};\mathbb{C});$

\item[(iii)] if $f \in C^0_b(S;\mathbb{C})$ and $f(\cdot, -\ell/2) = f(\cdot, \ell/2)$ then the function $\mathcal{F}_f$ belongs to $C^{2+\alpha}_{p}(H^2_\mathbb{R};\mathbb{C})$ and $||\mathcal{F}_f||_{C^{2+\alpha}_{p}(H^2_\mathbb{R})} \leq c\|f\|_{C^0_b(S;\mathbb{C})}$ for each $\text{Re} \lambda > -(\text{Im} \lambda)^2;$

\item[(iv)] $\mathcal{A}g$ belongs to $C^1_b(H^2_\mathbb{R};\mathbb{C}) \cap W^{2,p}_x(H^2_\mathbb{R};\mathbb{C})$, for each $p < +\infty$, solves the equation $\lambda u - L^{-1}_{\Delta} u - D_x u = 0$ and, for every $M > 0$ there exists a positive constant $c_M'$ such that

$$|\lambda||\mathcal{A}g|_{\infty} + \sqrt{|\lambda|}||\nabla \mathcal{A}g|_{\infty} \leq c_M'||g||_{\infty}, \quad |\lambda| \geq M;$$

\end{itemize}
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that uniformly with respect to \( \bigcap \) where \( \bigcup \) for some \( \alpha \in (0, 1) \), then for each \( \alpha \in \rho(L) \) the function \( R(\lambda, L)f \) belongs to \( X_{2+\alpha} \) and \( \|R(\lambda, L)f\|_{2+\alpha} \leq c_{\lambda}\|f\|_{\alpha} \).
Proof. Since it is rather long, we split the proof into four steps. In Steps 1 and 2, we characterize \( \sigma(L) \), whereas in Step 3 we prove that \( L \) generates an analytic semigroup in \( \mathcal{X} \) as well as the estimate for the spatial gradient for the resolvent operator. Finally, in Step 4, we prove property (iii).

Step 1. Fix \( f \in \mathcal{X} \) and \( \lambda \in \mathbb{C} \) such that \( \text{Re} \lambda > -\text{Im} \lambda^2 \) and \( \lambda \not\in \Omega_k \) for each \( k \in \mathbb{N} \setminus \{0\} \), and assume that the equation \( Au = \mathcal{D}u = f \) admits a solution \( u = (u_1, u_2) \in D(L) \). The arguments in the proof of Lemma 4.1 show that for every \( k \in \mathbb{Z} \) the functions \( u_{1,k} \) and \( u_{2,k} \) (see Subsection 2.1), solve, respectively, the differential equations \( \lambda u_k - \hat{u}_k = \lambda \hat{u}_k = f_{1,k} \) and \( \lambda \hat{u}_k - \hat{v}_k - \hat{v}_k' + \lambda \hat{v}_k = f_{2,k} \in \mathbb{R}_+ \setminus \{R\} \). Moreover, they belong to \( C^1_\lambda(\mathbb{R}; \mathcal{C}) \) and \( C^1(\mathbb{R}_+, +\infty); \mathcal{C} \), respectively. Thus,

\[
\hat{u}_{1,k}(x) = (F_{k,1}f_1)(x) + c_{1,k}e^{\mu^+x}e^{\chi_{(-\infty,0)}(x)} + (c_{2,k}e^{\mu^+x} + c_{3,k}e^{\mu^-x})\chi_{(0,R)}(x) + c_{4,k}e^{\mu^+x}e^{\chi_{(R,\infty)}(x)}
\]

for every \( x \in \mathbb{R}, k \in \mathbb{Z} \) and

\[
\hat{u}_{2,k}(x) = (G_{k,f_2}) + (d_{1,k}e^{\mu^+x} + d_{2,k}e^{\mu^-x})\chi_{(0,R)}(x) + d_{3,k}e^{\mu^-x}\chi_{(R,\infty)}(x)
\]

in \( (0, R) \times \mathbb{R} \), where \( \mu_k^\pm = -\frac{1}{2} \pm \frac{1}{2} \mathbb{R}_k \) and \( c_{1,k}, c_{2,k}, c_{3,k}, c_{4,k}, d_{1,k}, d_{2,k}, d_{3,k} \) are complex constants determined imposing the conditions \( B(\hat{u}_{1,k}, \hat{u}_{2,k}) = 0 \) that the infinitely many functions \( \hat{u}_{1,k} \) and \( \hat{u}_{2,k} \) have to satisfy. It turns out that the above constants are uniquely determined if and only if \( \mathcal{D}(\lambda) \neq 0 \), as we are assuming, and as a byproduct,

\[
u = \frac{1}{\ell} \sum_{k \in \mathbb{Z}} p_{1,k}(F_{k,1}f_1)(R)e_k + \frac{2}{\ell} \sum_{k \in \mathbb{Z}} p_{2,k}(G_{k,f_2})(0)e_k, \quad \text{in} \quad H_0^\ell,
\]

\[
u = \frac{1}{\ell} \sum_{k \in \mathbb{Z}} p_{3,k}(F_{k,1}f_1)(R)e_k + \frac{2}{\ell} \sum_{k \in \mathbb{Z}} p_{4,k}(G_{k,f_2})(0)e_k,
\]

\[
u = \frac{1}{\ell} \sum_{k \in \mathbb{Z}} q_{1,k}(F_{k,1}f_1)(R)e_k + \frac{1}{\ell} \sum_{k \in \mathbb{Z}} q_{2,k}(G_{k,f_2})(0)e_k,
\]

\[
u = \frac{1}{\ell} \sum_{k \in \mathbb{Z}} q_{3,k}(F_{k,1}f_1)(R)e_k + \frac{1}{\ell} \sum_{k \in \mathbb{Z}} q_{4,k}(G_{k,f_2})(0)e_k,
\]

where \( \mathcal{A}_\ell f \) and \( \mathcal{A}_\ell g \) have been introduced in Lemmas 4.1 and 4.2.
and \( W_k = W_k(\lambda) = \theta_t R X_k - 1 + e^{(\nu^* - \mu^*_k)} R \).

In view of Lemmata 4.1 to 4.3, to prove that the pair \( \mathbf{u} \) defined by (4.17)–(4.21) belongs to \( D(L) \) and \( \lambda \mathbf{u} - \mathbf{Z} \mathbf{u} = \mathbf{f} \) we just need to consider the series in the above formulae, which we denote, respectively by \( \mathcal{P}_{\lambda, 2k+1} f_j \) \((j = 1, 2, k = 0, 1, 2) \), \( \mathcal{P}_{\lambda, 2k+2} f_j \) \((j = 1, 2, h = 0, 1) \). To begin with, we observe that, by \((i)\) in the proof of Lemma 4.1 we already know that \( \text{Re}(X_k) + \text{Re}(Y_k) \geq c_0 |k| \) and \(|X_k| + |Y_k| \leq c_1(1 + |k|) \) for each \( k \in \mathbb{Z} \). As a byproduct, taking also \((4.3)\) into account, we can infer that \([\mathcal{P}_{\lambda, 1} f_1(R)](R) \leq \lambda(1 + k^2)^{-1} \| f_1 \|_{\infty} \) and \([\mathcal{P}_{\lambda, 2} f_2(0)] \leq \lambda(1 + k^2)^{-1} \| f_2 \|_{\infty} \) for each \( k \in \mathbb{Z} \). Moreover, we can also estimate
\[
\|W_k\| \geq \theta_t R |X_k| - 1 - e^{(\nu^* - \mu^*_k)} R \geq \theta_t R |X_k| - 1 - e^{\frac{tR}{1-R}} \geq c_{\lambda} |k|, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{4.22}
\]

Putting everything together, we conclude that \( \| p_{1,k} \|_{C^k([-\infty, 0]; C)} \leq c_1 e^{-c_2 k} \) for each \( h \in \mathbb{N} \) and
\[
\| p_{2,k} \|_{C^k([-\infty, 0]; C)} + \sum_{i=1}^2 \| (p_{i+1+2,k}, q_{i,k}) \|_{C^i([0,R]; C^2)} + \| (p_{4+i+2,k}, q_{i+1,2,k}) \|_{C^i([R, +\infty); C^2)} \leq c_\lambda |k|^{-1}
\]
for each \( k \in \mathbb{Z} \setminus \{0\} \). Using these estimates, it is easy to check that \( \mathcal{P}_{\lambda, 1, f_1} \in C^3_0(\overline{H}_0; C) \) for \( \beta > 0 \) and \( \mathcal{P}_{\lambda, 2, f_2} \in C^\infty(\overline{H}_0; C) \cap C^2_0(\overline{H}_0; C) \). Moreover, they solve the equation \( \lambda w - \Delta_x w - D_x w = 0 \) in \( H_0^1 \). Since the series which define \( \mathcal{P}_{\lambda, 1, f_1} \) and \( \mathcal{P}_{\lambda, 2, f_2} \) converge uniformly in \( H_0^1 \) and each term vanishes as \( x \to -\infty \), uniformly with respect to \( y \in \mathbb{R} \), we immediately infer that \( \lim_{x \to -\infty}(\mathcal{P}_{\lambda, 1, f_1}(x,y) = \lim_{x \to -\infty}(\mathcal{P}_{\lambda, 2, f_2}(x,y), y = 0 \) for each \( y \in \mathbb{R} \). On the other hand, the functions \( \mathcal{P}_{\lambda, 1, f_1}, \mathcal{P}_{\lambda, 2, f_2} \) and \( \mathcal{P}_{\lambda, 1, f_1}, \mathcal{P}_{\lambda, 2, f_2} \) belong to \( C^\infty((0, R) \times \mathbb{R}; C) \cap C^3_0([0, R] \times \mathbb{R}; C) \) and solve, in \( (0, R) \times \mathbb{R} \), the equations \( \lambda w_t - \Delta_x w_{tt} - D_x w = 0 \) and \( \lambda w_t - \Delta_x w_{tt} - D_x w_{tt} = 0 \), respectively. Finally, the functions \( \mathcal{P}_{\lambda, 1, f_1}, \mathcal{P}_{\lambda, 2, f_2} \) belong to \( C^\infty(H_0^1; C) \cap C^3_0(\overline{H}_0^1; C) \), solves, in \( H_0^1 \), the equations \( \lambda u_t - \Delta_x u_{tt} - D_x u_{tt} = 0 \) and \( \lambda u_t - \Delta_x u_{tt} - D_x u_{tt} = 0 \), respectively, and vanish as \( x \) tends to \( +\infty \) for each \( y \in \mathbb{R} \). Therefore, the function \( \mathbf{u} \) defined by (4.17)–(4.21) belongs to \( \bigcap_{p<+\infty} W_{2,p}^{2,2}(\mathbb{R} \setminus \{0, R\}) \times \mathbb{R} \) and solves, in \( \mathbb{R} \), the equation \( \lambda \mathbf{u} - \mathbf{Z} \mathbf{u} = \mathbf{f} \) and \( \lim_{x \to +\infty} u_1(x,y) = \lim_{x \to -\infty} u_2(x,y) = 0 \) for each \( y \in \mathbb{R} \). Moreover, \( \| \nabla \mathbf{u} \|_{\infty} \leq c_\lambda \| \mathbf{f} \|_{\infty} \). To conclude that \( \mathbf{u} \in D(L) \), we have to check that \( \mathcal{B} \mathbf{u} = 0 \), but this is an easy task taking into account that all the series appearing in the definition of \( \mathbf{u} \) may be differentiated term by term and \( \mathcal{B}(\tilde{u}_{1,k}, \tilde{u}_{2,k}) = 0 \) for every \( k \in \mathbb{Z} \). We have so proved that \( \mathbf{u} \in D(L) \) and that
\[
\bigcup_{k \in \mathbb{N} \setminus \{0\}} \Omega_k \subset \sigma(L) \subset \{ \lambda \in \mathbb{C} : \text{Re} \lambda \leq -|\text{Im} \lambda|^2 \} \cup \bigcup_{k \in \mathbb{N} \setminus \{0\}} \Omega_k.
\]

**Step 2.** To complete the characterization of \( \sigma(L) \), let us check that \( \sigma(L) \supset \{ \lambda \in \mathbb{C} : \text{Re} \lambda \leq -|\text{Im} \lambda|^2 \} \cup \bigcup_{k \in \mathbb{N} \setminus \{0\}} \Omega_k \). Clearly, each \( \lambda \in \mathbb{C} \) such that \( \text{Re} \lambda \leq -|\text{Im} \lambda|^2 \) belongs to \( \sigma(L) \), since in this case \( \nu^*_k \) and \( \mu^*_k \) have nonpositive real parts so that the more general solution to the equation \( \lambda \mathbf{u} - \mathbf{Z} \mathbf{u} = \mathbf{0} \), which belongs to \( \mathbf{X} \) and is independent of \( y \), is determined up to 8 arbitrary complex constants and we have just 7 boundary condition. Thus, the previous equation admits infinitely many solutions in \( \mathbf{X} \). Similarly, if \( \lambda \in \Omega_k \) for some \( k \in \mathbb{N} \setminus \{0\} \), then the pair \( \mathbf{u} = (\tilde{u}_{1,k} e^{i \lambda}, \tilde{u}_{2,k} e^{i \lambda}) \), where
\[
\tilde{u}_{1,k}(x) = (\mathcal{F}_{\lambda, 1, f_1}(x) + (c_1,k, e^{i \nu^* \lambda} x + c_2, k, 0) \chi(0, R]|0, \infty)(x) + (c_3, k, e^{i \nu^* \lambda} x + c_4, k, 0) \chi(R, +\infty)(x), \quad x \in \mathbb{R},
\]
and \( \tilde{u}_{2,k} \) is still given by (4.10), is smooth, belongs to \( \mathbf{X} \) and solves the differential equation \( \lambda \mathbf{u} - \mathbf{Z} \mathbf{u} = \mathbf{f} \) for every choice of \( c_1, k, d_j, k, (i = 1, \ldots, 4, j = 1, 2, 3) \). Imposing the condition \( \mathcal{B} \mathbf{u} = 0 \), we get to a linear system of 7 equations in 7 unknowns whose determinant is \( \Delta \mathbf{k} \). Since \( \lambda \in \Omega_k \), the above equation admits infinitely many solutions in \( D(L) \).

**Step 3.** Since the roots of the dispersion relation have bounded from above real part (see also the forthcoming computations), Step 1 shows that the resolvent set \( \rho(L) \) contains a right-halfplane. Hence, to prove that \( L \) generates an analytic semigroup it remains to prove that \( \| \mathcal{B}(\lambda, R \lambda, \mathbf{f}) \|_{\infty} \leq c \| \mathbf{f} \|_{\infty} \) for each \( \lambda \in \Omega_k \) in a suitable right-halfplane. Again, in view of Lemmata 4.1 to 4.3, we can limit ourselves to dealing with the functions \( \mathcal{P}_{\lambda, 2k+i} f_j \) and \( \mathcal{P}_{\lambda, 2k+j} f_j \).
For each $\lambda \in \mathbb{C}$ with positive real part, we can refine the estimate for $\text{Re}(X_k)$ and $\text{Re}(Y_k)$; it turns out that
$$|X_k| \geq \text{Re}(X_k) = \sqrt{\frac{1+4\lambda+4\lambda k+\text{Re}(1+4\lambda+4\lambda k)}{2}} \geq \sqrt{2|\lambda| + 1} \sqrt{2\lambda_k} \geq \frac{\sqrt{3}}{3} \sqrt{|\lambda| + 1 + \lambda_k},$$
$$|X_k| \leq 2\sqrt{1+|\lambda| + \lambda_k}$$
and, similarly, $c_1 \sqrt{1+|\lambda| + \lambda_k} \leq \text{Re}(Y_k) \leq |Y_k| \leq c_2 \sqrt{1+|\lambda| + \lambda_k}$ for each $k \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ with positive real part. As a byproduct, we get $\{|(f_k, f_1)(R)| + |(S_k, f_2)(0)|\} \leq c_R(|\lambda| + 1 + k^2)^{-1}\|f\|_X$ for each $k$ and $\lambda$ as above. Moreover, using (4.22) we can also estimate $|W_k| \geq c_R \sqrt{1+|\lambda| + \lambda_k}$ for each $k \in \mathbb{Z}$ and $\lambda \in \Sigma_M := \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq M\}$ with $M$ large enough. Finally,
$$|Y_k - Le| = \frac{4Le\lambda + 4\lambda_k}{\sqrt{Le^2 + 4Le\lambda + \lambda_k + Le}} \geq \frac{Le|\lambda| + \lambda_k}{\sqrt{Le^2 + Le|\lambda| + \lambda_k}} \geq \frac{Le}{2} \sqrt{1+|\lambda| + \lambda_k}$$
for each $\lambda \in \Sigma_1$ and $k \in \mathbb{Z}$, since $Le \in (0,1)$. Hence, up to replacing $M$ with $M \vee 1$, if necessary, we can estimate
$$\sum_{j=1}^2 \sum_{k=1}^3 \sum_{i,k}^6 \left|p_{j,k}(x)\right|^2 + \sum_{j=1}^2 \left|q_{j,k}(x')\right|^2 + \sum_{j=1}^2 \left|q_{j,k}(x'')\right|^2 + \sum_{j=1}^2 \left|q_{j,k}(x''')\right|^2 \leq c_R(1+|\lambda| + \lambda_k)^{-\frac{1}{2}}$$
for each $k \geq 0$, $\lambda \in \Sigma_M$, $x < 0$, $x' \in (0, R)$ and $x'' > R$. We are almost done. Indeed, taking the above estimates and the fact that
$$\sum_{k \in \mathbb{Z}} \left|\|f\|_\infty \right|^2 (|\lambda| + 1 + r^2)^{-\frac{1}{2}} dr \leq \frac{c}{1+|\lambda|} \|f\|_\infty$$
into account, we easily conclude that
$$\sum_{k=1}^3 \left(\|P_{\lambda,2k-1}f_1\|_\infty + \|P_{\lambda,2k}f_2\|_\infty\right) + \sum_{k=1}^3 \left(\|\partial P_{\lambda,2k-1}f_1\|_\infty + \|\partial P_{\lambda,2k}f_2\|_\infty\right) \leq c_M|\lambda|^{-1}\|f\|_X$$
for every $\lambda \in \mathbb{C}$ with $\text{Re} \lambda \geq M$ and some positive constant $c_M$ independent of $\lambda$. Similarly,
$$k^{n-j} \left(|D_x^{(j)}p_{i,k}(x)| + |D_x^{(j)}q_{i,k}(x')| + |D_x^{(j)}q_{i,k}(x'')| + |D_x^{(j)}q_{i,k}(x''')| + |D_x^{(j)}p_{i+1,k}(x'')|\right) \leq c_M,$$
for each $x < 0$, $x' \in [0, R]$, $x'' \geq 0$, $i = 1, 2$, $j = 0, 1$ and
$$\sum_{k \in \mathbb{Z}} \left|\|f\|_\infty \right|^2 (|\lambda| + 1 + r^2)^{-1} dr \leq \frac{c}{\sqrt{1+|\lambda|}} \|f\|_\infty.$$
Thus, we deduce that
$$\sum_{k=1}^3 \left(\|\nabla P_{\lambda,2k-1}f_1\|_\infty + \|\nabla P_{\lambda,2k}f_2\|_\infty\right) + \sum_{k=1}^3 \left(\|\nabla \partial P_{\lambda,2k-1}f_1\|_\infty + \|\nabla \partial P_{\lambda,2k}f_2\|_\infty\right) \leq c_M|\lambda|^{-\frac{3}{2}}\|f\|_\infty.$$

**Step 4.** Finally, we show that if $f \in \mathcal{X}_\alpha$ then $u \in \mathcal{X}_{\alpha+\alpha}$. Again, in view of Lemmata 4.1-4.3 and the estimate $\|p_{i,k}\|_{C^\alpha_{0,+}((-\infty,0])} \leq c_1 e^{-c_2 k}$ (for every $h \in \mathbb{N}$) in Step 3, which shows that the function $P_{\lambda,1}f_1$ belongs to $C^\beta(H_0^{\alpha})$ for every $\beta > 0$ and $\|P_{\lambda,1}f_1\|_{C^\beta(H_0^{\alpha})} \leq c_\beta\|f\|_X$, it suffices to deal with the other functions $P_{\lambda,2k+1}f_j$ and $P_{\lambda,2k+2}f_j$. We adapt the arguments in Step 2 of the proof of Lemma 4.2. To begin with, we consider the function $P_{\lambda,2}f_2 \in C^2_{0}(H_0^{\alpha})$ which solves the equation $\lambda P_{\lambda,2}f_2 - \Delta P_{\lambda,2}f_2 - D_x P_{\lambda,2}f_2 = 0$ in $H_0^{\alpha}$. To prove that it belongs to $C^2_{0}(H_0^{\alpha})$, we check that $\langle P_{\lambda,2}f_2(0, \cdot) \rangle \in C^2_{0}(\mathbb{R})$. Note that $p_{2,k}(0) = \ell^k(\pi k)^{-1} + \tilde{p}_{2,k}$ for every $k \in \mathbb{Z} \setminus \{0\}$, where $\tilde{p}_{2,k} = O(k^{-2})$. Therefore, we can split
$$(P_{\lambda,2}f_2)(0, \cdot) = \frac{2}{Le} \sum_{k \in \mathbb{Z}} k^{-1} (S_k f_2)(0)e_k + \frac{2}{Le} \sum_{k \in \mathbb{Z}} \tilde{p}_{2,k}(S_k f_2)(0)e_k = \psi_1 + \psi_2.$$
Since $|\hat{p}_{2,k}(S_kf_2)(0)| \leq c|k|^{-4} \|f_2\|_4$ for every $k \in \mathbb{Z} \setminus \{0\}$, it follows immediately that $\psi_2 \in C_0^{2+\alpha}(\mathbb{R})$ and $\|\psi_2\|_{C_0^{2+\alpha}(\mathbb{R})} \leq c\|f_2\|_4$. As far as $\psi_1$ is concerned, a straightforward computation reveals that $\psi'_1 = c(\mathcal{A}_2f_2)(0,\cdot)$ so that, by Lemma 4.2, $\psi'_1 \in C_0^{2+\alpha}(\mathbb{R})$ and $\|\psi'_1\|_{C_0^{2+\alpha}(\mathbb{R})} \leq c\|f\|_4$. Thus, $(\mathcal{A}_2f_2)(0,\cdot)$ belongs to $C_0^{2+\alpha}(\mathbb{R})$ and $\|(\mathcal{A}_2f_2)(0,\cdot)\|_{C_0^{2+\alpha}(\mathbb{R})} \leq c_\lambda\|f\|_4$. By classical results for elliptic problems (see [27]), $\mathcal{A}_2f_2$ belongs to $C_0^{2+\alpha}(H_0^\alpha)$ and $\|(\mathcal{A}_2f_2)\|_{C_0^{2+\alpha}(H_0^\alpha)} \leq c_\lambda\|f\|_4$.

Next, we split $\mathcal{A}_2f_2$ into the sum of the functions

$$\mathcal{A}_2f_1 = \frac{1}{R} \sum_{k \in \mathbb{Z}} \frac{e^{R\alpha}}{W_k} (f_{k,1})(R)e_k, \quad \mathcal{A}_3f_1 = \frac{1}{R} \sum_{k \in \mathbb{Z}} \frac{e^{-R\alpha}}{W_k} (f_{k,1})(R)e_k.$$ 

The first function belongs to $C_0^{2}(H_0^\alpha;\mathbb{C})$ and $\mathcal{A}_3f_1 \to 0$ as $R \to \infty$. Since $\bar{\mathcal{G}}_{3,k} \leq c^{-1}$, the same arguments as above and Lemma 4.1 allow to show first that the function $((\mathcal{A}_3f_1)(R,\cdot))'$ belongs to $C_0^{2+\alpha}(\mathbb{R})$ and then to conclude that $\mathcal{A}_3f_1 \in C_0^{2+\alpha}(H_0^\alpha)$ and $\|\mathcal{A}_3f_1\|_{C_0^{2+\alpha}(H_0^\alpha)} \leq c_\lambda\|f\|_4$. The smoothness of the function $\mathcal{A}_2f_2$ is easier to prove, due to the uniform (in $[0, R]$) exponential decay to zero of the terms of the series. It turns out that $\|\mathcal{A}_2f_2\|_{C_0^{2+\alpha}(H_0^\alpha)} \leq c_\lambda\|f\|_4$.

Let us consider the function $\mathcal{A}_4f_2$, which we split it into the sum of the functions $\mathcal{A}_4f_1$ and $\mathcal{A}_4f_2$ defined, respectively, by

$$\mathcal{A}_4f_1(x) = \frac{\theta_jR}{W_k} e^{R\alpha}, \quad \mathcal{A}_4f_2(x) = \frac{\theta_jR}{W_k} e^{-R\alpha} x - \frac{Y_k}{X_kW_k}\frac{e^{R\alpha}}{W_k}(Y_k - Le) - \frac{Y_kR}{X_kW_k}(Y_k - Le).$$

Function $\mathcal{A}_4f_1$ is easier to analyze since it converges in $C_0^{2+\alpha}(H_0^\alpha)$ and $\|\mathcal{A}_4f_1\|_{C_0^{2+\alpha}(H_0^\alpha)} \leq c_\lambda\|f\|_4$. On the other hand, the series, which defines $\mathcal{A}_4f_2$ is easier to analyze since it converges in $C_0^{2+\alpha}(H_0^\alpha)$ and $\|\mathcal{A}_4f_2\|_{C_0^{2+\alpha}(H_0^\alpha)} \leq c_\lambda\|f\|_4$.

All the remaining functions $\mathcal{A}_{2k+j}f_j$ and $\mathcal{A}_{2k+j}f_j$ can be analyzed in the same way. The details are left to the reader. \hfill \Box

Now, we characterize the interpolation spaces $D_L(\alpha/2, \infty)$ and $D_1(1+\alpha/2, \infty)$. To simplify the notation, we introduce the operator $\mathcal{B}_0$, defined by

$$\mathcal{B}_0 u = (u_1(0^+, \cdot) - u_1(0^-), u_1(R^+, \cdot) - u_1(R^-, \cdot)), \quad u \in X, \tag{4.23}$$

and the sets $X_{\alpha, 0} = \{u \in X_{\alpha} : \mathcal{B}_0 u = 0\}$ (a.e $u \in (0, 1)$) and $X_{2+\alpha, 0} = \{u \in X_{2+\alpha} : \mathcal{B}_0 u = 0, \lim_{x \to \pm \infty}\|Lu\|_i(x,y) = 0\} (a.e. u \in (0, 1))$, equipped with the norm of $X_{\alpha}$ and $X_{2+\alpha}$, respectively.

**Proposition 4.5.** For each $\alpha \in (0, 1)$ the following characterizations hold:

(i) $D_L(\alpha/2, \infty) = X_{\alpha, 0}$, \hspace{1cm} (ii) $D_L(1+\alpha/2, \infty) = X_{2+\alpha, 0}$, \hspace{1cm} (4.24)

with equivalence of respective norms. Moreover,

$$X_{1, 0} \hookrightarrow D_L(1/2, \infty). \tag{4.25}$$

**Proof.** Throughout the proof, we assume that $\alpha$ is arbitrarily fixed in $(0, 1)$. \hfill \Box
Step 1: proof of (4.24)(i) and (4.25). Given \( f = (f_1, f_2) \in X_{\alpha, \mathbb{R}_0} \) and \( t > 0 \), we introduce the functions \( f_{1,t} \) and \( f_{2,t} \) defined by

\[
\begin{align*}
  f_{1,t}(x,y) &= \int_{H^+_0} f_1^t(x + tx', y + ty') \varphi(x', y') dx' dy', \quad (x, y) \in \mathbb{R}^2, \\
  f_{2,t}(x,y) &= \int_{H^+_0} f_2^t(x - tx', y + ty') \varphi(x', y') dx' dy', \quad (x, y) \in [0, R] \times \mathbb{R}, \\
  f_{1,t}(x,y) &= \int_{H^+_0} f_1^t(x + tx', y + ty') \varphi(x', y') dx' dy', \quad (x, y) \in H^+_R,
\end{align*}
\]

where \( \varphi \) is a positive smooth function with compact support in \((0, R) \times \mathbb{R}\), \( \| \varphi \|_{L^1(\mathbb{R}^2)} = 1 \) and \( f_2^t : H^+_R \to \mathbb{C} \) equals the function \( f_2^t \theta \) in \([0, R] \times \mathbb{R}\), whereas \( f_2^t(x, \cdot) = f_2^t(0, \cdot) \theta(x) \) if \( x < 0 \) and \( \theta \) is a smooth function compactly supported in \((-1, +\infty)\) and equal to 1 in \((-\frac{1}{2}, +\infty)\). Since \( \mathcal{B}_0 \mathcal{R} = \mathcal{R} \) and \( f_j(\cdot, -t/2) = f_j(\cdot, t/2) \) for \( j = 1, 2 \), the function \( f_1^t \) belongs to \( C^0_0(\mathbb{R}^2; \mathbb{C}) \). On the other hand, the functions \( f_2^t \) and \( f_2 \) belong to \( C^0(\mathcal{H}^+_R; \mathbb{C}) \) and to \( C^0_0(\mathcal{H}^+_R; \mathbb{C}) \), respectively. Moreover, \( f_1^t(\cdot, y) \) and \( f_j^t(\cdot, y) \) vanish at \( \pm \infty \) and at \( +\infty \), respectively, for every \( y \in \mathbb{R} \). Hence, if we set \( \tilde{f}_t = (f_{1,t}, f_{2,t}) \), then \( \tilde{f}_t \) belongs to \( X_\alpha \). \( \| \tilde{f}_t - f \|_{\alpha, \mathbb{R}_0} \leq c t \| \tilde{f}_t \|_{\alpha, \mathbb{R}_0} \) and \( \| \tilde{f}_t \|_{\alpha, \mathbb{R}_0} \leq c \| \tilde{f}_t \|_{\alpha, \mathbb{R}_0} \) for every \( t > 0 \). Similarly, since \( \int_{H^+_R} D^\gamma \varphi(x', y') dx' dy' = 0 \) for every multi-index \( \gamma \), we can write

\[
D^\gamma f_{1,t}(x,y) = t^{-|\gamma|} \int_{H^+_R} (f_1^t(x + tx', y + ty') - f_1^t(x,y)) D^\gamma \varphi(x', y') dx' dy', \quad (x, y) \in \mathbb{R}^2,
\]

so that \( |D^\gamma f_{1,t}(x,y)| \leq c t^{|\gamma|} \| \tilde{f}_t \|_{\alpha, \mathbb{R}_0} \) for every \( (x, y) \in \mathbb{R}^2 \). In the same way we can estimate the derivatives of the function \( f_{1,t} \), and conclude that \( t^{\alpha - |\gamma|} \| D^\gamma f_{1,t} \|_{\alpha, \mathbb{R}_0} \leq c t \| f \|_{\alpha, \mathbb{R}_0} \) for every \( t \in (0, T] \) and \( T > 0 \).

Now, we split \( \omega^{\alpha/2} LR(\omega, L) f = \omega^{\alpha/2} LR(\omega, L) f - \int_{\omega-1/2}^\omega \omega^{\alpha/2} LR(\omega, L) f \omega \omega \) for each \( \omega \in \rho(L) \cap \mathbb{R} \). By Theorem 4.1 for \( \omega \in \mathbb{R} \) sufficiently large (let us say \( \omega \geq \omega_0 > 0 \)), it holds that \( \| LR(\omega, L) \|_{L(\mathcal{X})} \leq M \) for some positive constant \( M \), independent of \( \omega \). Hence, using the above estimates, we can infer that \( \| \omega^{\alpha/2} LR(\omega, L) f \|_{\alpha, \mathbb{R}_0} \leq c \| f \|_{\alpha, \mathbb{R}_0} \). Next, consider the term \( v = LR(\omega, L) f \omega \omega^{-1/2} \), which belongs to \( D(\mathcal{L}) \) and solves the equation \( \omega v = \mathcal{L} f \omega \omega^{-1/2} \). If \( \omega \geq \omega_0 \), then we can estimate \( \| v \|_{\alpha, \mathbb{R}_0} \leq c \| \mathcal{L} f \omega \omega^{-1/2} \|_{\alpha, \mathbb{R}_0} \). Putting everything together, we conclude that \( \omega^{\alpha/2} \| LR(\omega, L) f \|_{\alpha, \mathbb{R}_0} \leq c \| f \|_{\alpha, \mathbb{R}_0} \) for each \( \omega \geq \omega_0 \) and this shows that \( f \in D_L(\alpha/2, \infty) \) and \( \| f \|_{D_L(\alpha/2, \infty)} \leq c \| f \|_{\alpha, \mathbb{R}_0} \).

Formula (4.25) can be proved just in the same way observing that, if \( f \) belongs to \( X_{\alpha, \mathbb{R}_0} \), then the functions \( f_1^t \), \( f_2^t \) and \( f_2 \) are bounded and Lipschitz continuous in \( \mathbb{R}^2 \) in \([0, R] \times \mathbb{R} \) and in \( H^+_R \), respectively.

The embedding “\( \hookrightarrow \)’” in (4.24)(i) is a straightforward consequence of two properties:

\[
(a) \quad \| \nabla u \|_{\mathbb{R}_0} \leq c \| u \|_{\alpha, \mathbb{R}_0}^{1/2} \| u \|_{D(L)}^{1/2}, \quad u \in D(L), \\
(b) \quad (X, X_{\alpha, \mathbb{R}_0})_{\alpha, \infty} \hookrightarrow (X, X_{\alpha, \mathbb{R}_0})_{\alpha, \infty}.
\]

Indeed, property (a) shows that \( X_{\alpha, \mathbb{R}_0} \) belongs to the class \( J_{1/2} \) between \( X \) and \( D(L) \), so that applying the reiteration theorem, we get \( D_L(\alpha/2, \infty) = (X, D(L))_{\alpha/2, \infty} \subset (X, X_{\alpha, \mathbb{R}_0})_{\alpha, \infty} \) and conclude using (b).

Proof of (a). It is an almost straightforward consequence of the estimate \( \| \nabla R(\lambda, M) u \|_{\mathbb{R}_0} \leq c |\lambda|^{-1/2} \| u \|_{\mathbb{R}_0} \) for each \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda \geq M \), contained in Theorem 4.1. Indeed, let \( M = L - M \). As it is easily seen, \( \rho(M) \supset (0, +\infty) \) and \( R(\lambda, M) = R(\lambda + M, L) \) for each \( \lambda > 0 \). It thus follows that \( \sqrt{\text{Re} \lambda} \| \nabla R(\lambda, M) u \|_{L(\mathcal{X} \times \mathcal{X})} \leq c \) for each \( \lambda > 0 \), so that we can estimate

\[
\| \nabla u \|_{\mathbb{R}_0} = \| \nabla R(\lambda, M)(\lambda u - L M u) \|_{\mathbb{R}_0} \leq c \| \lambda - \frac{1}{2} \| \lambda u - L M u \|_{\mathbb{R}_0} \leq c (\lambda^{\frac{1}{2}} \| u \|_{\mathbb{R}_0} + \lambda^{-\frac{1}{2}} \| L M u \|_{\mathbb{R}_0})
\]

for each \( u \in D(L) = D(M) \) and \( \lambda > 0 \). Minimizing with respect to \( \lambda > 0 \), we conclude the proof.

Proof of (b). Let us fix a nontrivial \( f \in (X, X_{\alpha, \mathbb{R}_0})_{\alpha, \infty} \). Since \( (X, X_{\alpha, \mathbb{R}_0})_{\alpha, \infty} \hookrightarrow (X, X_{\alpha, \mathbb{R}_0})_{\alpha/2, 1} \) and \( X_{\alpha, \mathbb{R}_0} \) is dense in \( (X, X_{\alpha, \mathbb{R}_0})_{\alpha/2, 1} \), we immediately deduce that \( f_3(\cdot, -t/2) = f_3(\cdot, t/2) \) for \( j = 1, 2 \).
and $\mathcal{B}_0 f = 0$. Next, we recall that
\[
\inf \{ \|g\|_\infty + t \|h\|_1 : f = g + h, g \in \mathbf{X}, h \in \mathbf{X}_{1,\mathcal{B}_0} \} \leq t^\alpha \|x_{x,1,\mathcal{B}_0}\|_{\infty}, \quad t > 0.
\]
Fix $(x_j, y_j) \in S_0^j$, $j = 1, 2$, and take $t = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$. Then, we can determine $g \in \mathbf{X}$ and $h \in \mathbf{X}_{1,\mathcal{B}_0}$ such that $f = g + h$ and
\[
\|g\|_\infty + \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \|h\|_1 \leq 2\|f\|_{(x_{x,1,\mathcal{B}_0})\|_{\infty}} (|x_2 - x_1|^2 + |y_2 - y_1|^2)^{\frac{\alpha}{2}}.
\]
From this estimate we can infer that
\[
|f_1(x_2, y_2) - f_1(x_1, y_1)| \\
\leq g_1(x_2, y_2) - g_1(x_1, y_1) + |h_1(x_2, y_2) - h_1(x_1, y_1)| \\
\leq 2\|g\|_\infty + \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \|h\|_1 \\
\leq 4\|f\|_{(x_{x,1,\mathcal{B}_0})\|_{\infty}} (|x_2 - x_1|^2 + |y_2 - y_1|^2)^{\frac{\alpha}{2}}.
\]
Hence, $f_1 \in C^0_b(S_0^\infty)$ and $\|f_1\|_{C^0_b(S_0^\infty)} \leq 5\|f\|_{(x_{x,1,\mathcal{B}_0})\|_{\infty}}$.

The same argument can be used to prove that $f_1, f_2 \in C^\alpha((0, R) \times (-\ell/2, \ell/2); \mathbb{C}) \cap C^0_b(S_R^\infty; \mathbb{C})$ and
\[
\|f_1\|_{C^0_b(S_R^\infty; \mathbb{C})} + \sum_{j=1}^J \|f_j\|_{C^\alpha((0, R) \times (-\ell/2, \ell/2); \mathbb{C})} + \|f_2\|_{C^0_b(S_R^\infty; \mathbb{C})} \leq c\|f\|_{(x_{x,1,\mathcal{B}_0})\|_{\infty}}.
\]

The proof of (b) is now complete.

Step 2: proof of (ii). The embedding "\(\rightarrow\)" easily follows from the first part of the proof. The other embedding follows from Theorem \[4.4\]. Indeed, fix $u \in D_L(1 + \alpha/2, \infty)$ and $\lambda \in \rho(L)$. Then, the function $f := \lambda u - Lu$ belongs to $D_L(\alpha/2, \infty) = \mathbf{X}_\alpha, \mathcal{B}_0$ and $\|f\|_{\lambda} \leq c\|u\|_{D_L(1 + \alpha/2, \infty)}$ for some positive constant $c$, independent of $u$. Since $u = R(\lambda, L)f$, Theorem \[4.4\] implies that $u \in \mathbf{X}_{2+\alpha}$ and $\|u\|_{2+\alpha} \leq c\|f\|_{\lambda} \leq c\|u\|_{D_L(1 + \alpha/2, \infty)}$. Clearly, $\mathcal{B}u = 0$, $\mathcal{B}0u = 0$ and $\lim_{x \to -\infty} (Lu)_1(x, y) = \lim_{x \to +\infty} (Lu)_2(x, y) = 0$, and this completes the proof.

Remark 4.6. From the classical theory of analytic semigroups (see e.g., [31]) and Proposition \[4.5\], it follows that the part $L_\alpha$ of $L$ in $\mathbf{X}_{\alpha, \mathcal{B}_0}$, i.e., the restriction of $L$ to $\mathbf{X}_{2+\alpha, \mathcal{B}_0}$, generates an analytic semigroup for each $\alpha \in (0, 1)$.

4.3. The lifting operators. In this subsection, we introduce some lifting operators which are used in the proof of the Main Theorem and Theorem \[5.1\].

To begin with, we consider the operator $\mathcal{M}$ defined by $\mathcal{M}\psi = (0, M\psi_1 + \frac{1}{2}(M\psi_2)(-\gamma, \cdots))$ on functions $\psi \in C([-\ell/2, \ell/2]; \mathbb{R}^2)$ such that $\psi(-\ell/2) = \psi(\ell/2)$, where
\[
(M\psi_1)(x, y) := |x|^2 \int_\mathbb{R} \varphi(\xi)\varphi_0(y + \xi x) d\xi, \quad (x, y) \in \mathbb{R}^2.
\]
Here, $\eta$ and $\varphi$ are smooth functions such that $\chi(-R/4, R/4) \leq \eta \leq \chi(-R/2, R/2)$, $\varphi$ is an even nonnegative function compactly supported in $(-1, 1)$ with $\|\varphi\|_{L^1(\mathbb{R})} = 1$. As it is easily seen, $\mathcal{M}\psi \in \mathbf{X}_{2+\alpha}$, $\mathcal{B}\mathcal{M}\psi = (0, 0, 0, \psi_1, 0, 0, \psi_2)$ for each $\psi$ as above.

Next, we introduce the operator $\mathcal{N}$ defined by
\[
(\mathcal{N}h)_1 = N_1 h_1 + (N_1 h_2) \circ \tau_R + \frac{N_2 h_3}{2L} + \frac{(N_2 h_5) \circ \tau_R}{2\theta_R} + N_3 \left(h_8 - \frac{1}{L^\alpha} h_3 - h_1^{\nu} \right)
\] + \left[ N_3 \left(h_9 - \frac{1}{L^\alpha} h_5 - h_2^{\nu} \right) \right] \circ \tau_R,
\]
\[
(\mathcal{N}h)_2 = \left[ N_1 \left(h_6 - \frac{L}{\theta_R} h_5 \right) \right] \circ \tau_R + N_2 h_4 + \frac{1}{2} \left( N_3 \left(h_7 - L^\alpha h_6 + \frac{1}{L^\alpha} h_3 \right) \right) \circ \tau_R
\]
on smooth enough functions $h : [-\ell/2, \ell/2] \to \mathbb{R}^9$, where $\tau_R(x, y) = (x - R, y)$,
\[
(\mathcal{N}\xi)(x, y) = \frac{1}{\sqrt{\eta(x)}}(\chi_{[0, +\infty)}(x) - \chi_{[-\infty, 0]}(x)) \int_\mathbb{R} \varphi(\sigma) \varphi_0(y + \sigma x^2) d\sigma,
\]
\[ (N_2\zeta)(x,y) = |x|\eta(x) \int_\mathbb{R} \varphi(\sigma)\zeta^2(y + \sigma|x|)d\sigma, \quad (N_3\zeta)(x,y) = \frac{x}{4}(N_2\zeta)(x,y), \]

for each \((x,y) \in \mathbb{R}^2\). Moreover, we set \(\mathcal{B}_*v = (\tilde{\mathcal{B}}v, \mathcal{B}_0\mathcal{L}v)\) for each \(v \in X_{2+\alpha}\), where \(\tilde{\mathcal{B}}\) is the operator in Remark 3.1 and the operator \(\mathcal{B}_0\) is defined in (4.23).

In the next lemma, we deal with real valued spaces. In particular, by \(\tilde{D}(L_\alpha)\) we denote the subset of \(D(L_\alpha)\) of real valued functions.

**Lemma 4.7.** The following properties are satisfied.

(i) The operator \(\mathcal{N}\) is bounded from the set \((X_{2+\alpha})^2 \times (X_{1+\alpha})^5 \times (X_\alpha)^2\) into \(X_{2+\alpha}\). Moreover, the operator \(P = I - \mathcal{N}\mathcal{B}_* : X_{2+\alpha} \to X_{2+\alpha}\) is a projection onto the kernel of \(B_*\), which coincides with \(\tilde{D}(L_\alpha)\).

(ii) Let \(\mathcal{J}\) denote the set of all functions \(u \in X_{2+\alpha}\) such that \(\mathcal{B}u = \mathcal{H}(u)\), \(\mathcal{B}_0(Lu + \mathcal{F}(u)) = 0\). Then, there exist \(r_0, r_1 > 0\) such that \(\mathcal{J} \cap B(0,r_0)\) is the graph of a smooth function \(\Upsilon : B(0,r_1) \to (I - P)(X_{2+\alpha})\) such that \(\Upsilon(0) = 0\).

**Proof.** (i) Showing that \(\mathcal{N}\) is a bounded operator is an easy task. Some long but straightforward computations reveal that \(\mathcal{B}_*\mathcal{N} = I\) on \((X_{2+\alpha})^2 \times (X_{1+\alpha})^5 \times (X_\alpha)^2\) and allow to prove that \(P\) is a projection onto \(\text{Ker}(B_*) = \tilde{D}(L_\alpha)\). In particular, we can split \(X_{2+\alpha} = \tilde{D}(L_\alpha) \oplus (I - P)(X_{2+\alpha})\).

The details are left to the reader.

(ii) Let \(\mathcal{N} : B(0,r_0) \subset \tilde{D}(L_\alpha) \to (I - P)(X_{2+\alpha})\) be the operator defined by \(\mathcal{N}(u,v) = (\mathcal{B}(u+v) - \mathcal{H}(u+v), \mathcal{B}_0(Lu + Lv + \mathcal{F}(u + v)))\) for each \((u,v) \in B(0,r_0)\), with \(r_0\) small enough to guarantee that \(\mathcal{N}\) is well defined. Since the functions \(\mathcal{F}\) and \(\mathcal{H}\) are quadratic at \(0\), it follows that \(\mathcal{N}(0,0) = 0\) and \(\mathcal{N}\) is Fréchet differentiable at \((0,0)\), with \(\mathcal{N}_0(0,0) = \mathcal{B}_*\). In view of (i), \(\mathcal{B}_*\) is an isomorphism from \((I - P)(X_{2+\alpha})\) to \((X_{2+\alpha})^2 \times (X_{1+\alpha})^5 \times (X_\alpha)^2\). Thus, we can invoke the implicit function theorem to complete the proof. \(\square\)

5. **Solving the nonlinear problem** (3.23)

Now we are able to solve the Cauchy problem

\[
\begin{aligned}
D_t u &= Lu + F(u), \\
\mathcal{B}u &= \mathcal{H}(u), \\
D_\gamma D_\gamma^\gamma u(\cdot,\cdot, -\ell/2) &= D_\gamma D_\gamma^\gamma u(\cdot,\cdot, \ell/2), \quad \gamma_1 + \gamma_2 \leq 2,
\end{aligned}
\]

for the unknown \(u = (u,w)\). Also in this section we assume that the function spaces that we deal with are real valued ones.

**Theorem 5.1.** Fix \(\alpha \in (0,1)\) and \(T > 0\). Then, there exists \(r_0 = r_0(T) > 0\) such that, for each \(u_0 \in B(0,r_0) \subset X_{2+\alpha}\) satisfying the compatibility conditions \(\mathcal{B}u_0 = \mathcal{H}(u_0), \mathcal{B}_0(Lu_0 + \mathcal{F}(u_0)) = 0\) and \(D^\gamma u_0(\cdot,\cdot, -\ell/2) = D^\gamma u_0(\cdot,\cdot, \ell/2)\) for each multi-index \(\gamma\) with length at most two, Problem (5.1) admits a unique solution \(u \in Y_{2+\alpha}\) with \(u(0,\cdot) = u_0\). Moreover, \(\|u\|_{Y_{2+\alpha}} \leq c\|u_0\|_{2+\alpha}\).

**Proof.** The proof can be obtained arguing as in the proof of [28, Theorem 4.1]. For this purpose, we just sketch the main points.

We first need to prove optimal regularity results for the linear version of Problem (5.1), i.e., with the problem

\[
\begin{aligned}
D_t u(t,\cdot,\cdot) &= Lu(t,\cdot,\cdot) + f(t,\cdot,\cdot), \quad t \in [0,T], \\
\mathcal{B}(u(t,\cdot,\cdot)) &= h(t,\cdot,\cdot), \quad t \in [0,T], \\
D_\gamma^2 D_\gamma^\gamma u(t,\cdot,-\ell/2) &= D_\gamma^2 D_\gamma^\gamma u(t,\cdot,\ell/2), \quad t \in [0,T], \quad \gamma_1 + \gamma_2 \leq 2, \\
u(0,\cdot) &= u_0,
\end{aligned}
\]

when \(f \in Y_{\gamma}, u_0 \in X_{2+\alpha}\), when \(h_j \equiv 0\) if \(j \neq 4,7, h_4 = \psi_1, h_7 = \psi_2, \psi = (\psi_1, \psi_2) \in C^{(1+\alpha)/2,1+\alpha}(0,T) \times (-\ell/2,\ell/2;\mathbb{R}^2)\) satisfy the compatibility conditions

- \(\mathcal{B}u_0 = h(0,\cdot), \mathcal{B}_0(Lu_0(0,\cdot) + f(0,\cdot)) = 0;\)

2see Remark 4.6
• \( f(0, t, -\ell/2) = f(0, t, \ell/2) \), \( D^\gamma u_0(\cdot, -\ell/2) = D^\gamma u_0(\cdot, \ell/2) \) and \( D_y^j \psi(\cdot, -\ell/2) = D_y^j \psi(\cdot, \ell/2) \)

for every multi-index \( \gamma \) with length at most two and \( j = 0, 1 \).

We also need to show the estimate

\[
\|u\|_{\mathcal{Y}_{2+\alpha}} \leq c_0(\|f\|_{\mathcal{Y}_\alpha} + \|u_0\|_{2+\alpha} + \|\psi\|_{C^{1+\alpha/2}, 1+\alpha}(0, T) \times (-\ell/2, \ell/2, \mathbb{R}^2))
\]

(5.3)

for its unique solution \( u \in \mathcal{Y}_{2+\alpha} \). This is the content of Steps 1 to 3.

Step 1. To begin with, we note that \( Mh \in C^{1+\alpha/2, 1+\alpha}(0, T) \times \mathbb{R}^n \) and \( C^{1+\alpha/2, 1+\alpha}(0, T) \times \mathbb{R}^n \)

for all \( h \in C^{1+\alpha/2, 1+\alpha}(0, T) \times (-\ell/2, \ell/2) \) such that \( D_y^j h(\cdot, -\ell/2) = D_y^j h(\cdot, \ell/2) \) for \( j = 0, 1 \), and

\[
\|Mh\|_{C^{1+\alpha/2, 1+\alpha}(0, T) \times \mathbb{R}^n} \leq c \|h\|_{C^{1+\alpha/2, 1+\alpha}(0, T) \times (-\ell/2, \ell/2, \mathbb{R}^2)}.
\]

Thus, the function \( f + \mathcal{L} \mathcal{M} \psi \) belongs to \( C^{\alpha/2}(0, T; \mathcal{X}) \). Moreover, by Proposition 4.3 and the compatibility conditions on \( u_0 \), it follows that \( u_0 - \mathcal{M}(\psi(0, \cdot)) \in D(L) \), \( \mathcal{L} u_0 + f(0, \cdot) \in D(L/\alpha, 2, \infty) \).

The theory of analytic semigroups (see e.g., [31, Chapter 4]), Theorem 1.3 and Proposition 4.5 show that there exists a unique function \( v \in C^{\alpha/2, \alpha}(0, T; \mathcal{X}) \cap C(0, T; D(L)) \) which solves the equation \( D_t v = \mathcal{L} v + f + \mathcal{L} \mathcal{M} \psi \) and satisfies the condition \( v(0, \cdot) = u_0 \). In addition, \( D_t v \) is bounded with values in \( \mathcal{X}_{\alpha, \alpha} \), (which implies that \( D_t v \in \mathcal{Y}_\alpha \) and \( \mathcal{L} v \in C^{\alpha, \alpha}(0, T; \mathcal{X}) \). By difference, \( \mathcal{L} v = L v \) is bounded in \( [0, T] \) with values in \( \mathcal{X}_\alpha \) and, in view of Theorem 4.4, \( \mathcal{L} v \) is bounded in \( [0, T] \) with values in \( \mathcal{X}_{\alpha, \alpha} \).

In particular, \( \mathcal{L} v = 0 \) and \( D_t^\alpha v(\cdot, -\ell/2) = D_t^\alpha v(\cdot, \ell/2) \) for every \( \alpha \leq 2 \).

Further, \( \|v(t, \cdot)\|_{2+\alpha} + \|D_t v(\cdot,\cdot)\|_{\mathcal{Y}_\alpha} \leq c(\|f\|_{\mathcal{Y}_\alpha} + \|u_0\|_{2+\alpha} + \|\psi\|_{C^{1+\alpha, \alpha}(0, T) \times (-\ell/2, \ell/2, \mathbb{R}^2)}) \) for every \( t \in [0, T] \).

Step 2. Let \( \tilde{w} \) be the function defined by \( \tilde{w}(t, \cdot, \cdot) = \int_0^1 e^{(t-s)L}(\mathcal{L} \mathcal{M} \psi(s, \cdot))ds \) for every \( t \in [0, T] \), where \( e^{(t-s)L} \) is the analytic semigroup generated by \( L \) in \( \mathcal{X} \). Taking 4.25 into account, it follows that \( \|\mathcal{L} \mathcal{M} \psi\|_{C^{1+\alpha/2, 1+\alpha}(0, T; D(\mathcal{L}))} \leq \|\psi\|_{C^{1+\alpha/2, 1+\alpha}(0, T) \times (-\ell/2, \ell/2, \mathbb{R}^2)} \). By the theory of analytic semigroups we infer that \( \tilde{w} \in C([0, T]; D(L)) \), \( D_t \tilde{w} \) is bounded in \( [0, T] \) with values in \( \mathcal{X}_{\alpha, \alpha} \), \( \tilde{w} \in C^{1+\alpha/2}(0, T; \mathcal{X}) \), \( D_t \tilde{w} = \mathcal{L} \mathcal{M} \psi - \mathcal{M}(\psi(0, \cdot)) \) and \( \|D_t \tilde{w}\|_{C^{1+\alpha/2}(0, T; D(L))} \leq c \|\psi\|_{C^{1+\alpha/2, 1+\alpha}(0, T) \times (-\ell/2, \ell/2, \mathbb{R}^2)} \) from these properties and using the same arguments as above, it can be easily checked that the function \( w = -L \tilde{w} + \mathcal{M}(\psi(0, \cdot)) \) is as smooth as \( v \) is. Moreover, \( D_t w = \mathcal{L} \tilde{w} = \mathcal{L} \mathcal{M} \psi \) and \( D_t^\alpha w(\cdot, -\ell/2) = D_t^\alpha w(\cdot, \ell/2) \) for every \( \alpha \leq 2 \).

Step 3. Clearly, the function \( u = v + w \in \mathcal{Y}_{2+\alpha} \) solves the Cauchy problem (5.2), satisfies (5.3), and it is the unique solution to the above problem in \( C^{1+\alpha/2}(0, T; \mathcal{X}) \cap C(0, T; D(L)) \). Moreover,

\[
\sup_{t \in [0, T]} \|u(t, \cdot)\|_{2+\alpha} + \|D_t u\|_{\mathcal{Y}_\alpha} \leq c(\|f\|_{\mathcal{Y}_\alpha} + \|u_0\|_{2+\alpha} + \|\psi\|_{C^{1+\alpha, \alpha}(0, T) \times (-\ell/2, \ell/2, \mathbb{R}^2)}).
\]

(5.4)

To conclude that \( u \in \mathcal{Y}_{2+\alpha} \) and it satisfies estimate (5.3), we use an interpolation argument. It is well known that \( \|\cdot\|_{C^{1/2}_y(H^1_t; \mathbb{R}^2)} \leq c \|\cdot\|_{C^{1/2}_y(H^1_t; \mathbb{R}^2)} \leq c \|\cdot\|_{C^{1/2}_y(H^1_t; \mathbb{R}^2)} \). Using this estimate and the formula

\[
u(t, x, y) - u(s, x, y) = \int_s^t D_t u(r, x, y) dr, \quad s, t \in [0, T], \quad (x, y) \in H^1_t \]

to show that \( \|u(t, \cdot) - u(s, \cdot)\|_{C^{1/2}_y(H^1_t; \mathbb{R}^2)} \leq \|D_t u\|_{\mathcal{Y}_\alpha} |t - s|^{1/2} \), we get \( \|u(t, \cdot) - u(s, \cdot)\|_{C^{1/2}_y(H^1_t; \mathbb{R}^2)} \leq c \|D_t u\|_{\mathcal{Y}_\alpha} \sup_{r \in [0, T]} \|u(r, \cdot)\|_{2+\alpha} |t - s|^{1/2} \).

In the same way, we can show that \( u_1 \in C^{3/2}(0, T; C^0_\alpha(\overline{H^1_t; \mathbb{R}^2})) \), \( u \in C^{3/2}(0, T; C^0_\alpha([0, R] \times [-\ell/2, \ell/2, \mathbb{R}^2])) \) and

\[
\|u\|_{C^{3/2}(0, T; C^0_\alpha([0, R] \times [-\ell/2, \ell/2, \mathbb{R}^2]))} + \|u_1\|_{C^{3/2}(0, T; C^0_\alpha(\overline{H^1_t; \mathbb{R}^2}))} \leq c \left( \|D_t u\|_{\mathcal{Y}_\alpha} + \sup_{r \in [0, T]} \|u(r, \cdot)\|_{2+\alpha} \right).
\]

Taking [5.4] into account we complete this step of the proof. In particular, from all the above results it follows that

\[
\nu(t, \cdot) = e^{LT} u_0 + \int_0^t e^{(t-s)L}[f(s, \cdot, \cdot) + \mathcal{L} \mathcal{M} \psi(s, \cdot)]ds - L \int_0^t e^{(t-s)L} \mathcal{M} \psi(s, \cdot) ds, \quad t \in [0, T].
\]

(5.5)
Step 4. Let \( r > 0 \) and \( \mathcal{C}_r \) be the space of all \( u \in Y_{2+a} \) such that \( D_x^2 D_y^2 u(\cdot, \cdot, \ell/2) = D_x^2 D_y^2 u(\cdot, \cdot, \ell/2) \), for every \( 0 \leq \gamma_1, \gamma_2 \) such that \( \gamma_1 + \gamma_2 \leq 2 \), \( \|u\|_{Y_{2+a}} \leq r \) and \( u(0, \cdot) = u_0 \).

In view of Steps 1-3, for every \( u_0 \in B(0, r_0) \subset Y_{2+\alpha} \) satisfying the compatibility conditions in the statement of the theorem, we can define the operator \( \Gamma \), which to every \( u \in \mathcal{C}_r \) (with \( r \) sufficiently small norm to guarantee that the nonlinear terms \( \mathcal{F}(u(t, \cdot)) \) and \( \mathcal{H}(u(t, \cdot, \cdot)) \) are well defined for every \( t \in [0, T] \)) assigns the unique solution \( v \) of the Cauchy problem (5.2) with \( f = \mathcal{F}(u) \) and \( \psi = \mathcal{H}(u) \). Since the maps \( u \mapsto \mathcal{F}(u) \), \( u \mapsto \mathcal{H}(u) \) are smooth in \( \mathcal{C}_r \) and quadratic at \( u = 0 \), we can estimate

\[
\|\mathcal{F}(u)\|_{Y_{2+a}} \leq c\|u\|_{Y_{2+a}}, \quad \|\mathcal{F}(u) - \mathcal{F}(v)\|_{Y_{2+a}} \leq cr\|u - v\|_{Y_{2+a}}, \\
\|\mathcal{H}(u)\|_{C^{(1+\alpha)/2,1+\alpha}(0, T) \times (\ell/2, \ell/2; \mathbb{R}^2)} \leq c\|u\|_{Y_{2+a}}, \\
\|\mathcal{H}(u) - \mathcal{H}(v)\|_{C^{(1+\alpha)/2,1+\alpha}(0, T) \times (\ell/2, \ell/2; \mathbb{R}^2)} \leq cr\|u - v\|_{Y_{2+a}}.
\]

These estimates combined with (5.3) show that \( r \) and \( r_0 \) can be determined small enough such that \( \Gamma \) is a contraction in \( \mathcal{C}_r \).

Uniqueness of the solution \( u \) to (5.1) follows from standard arguments, which we briefly sketch here. At first, for every \( t_0 \in [0, T] \), \( R, \delta > 0 \) and \( u_1 \in \mathcal{X}_{2+\alpha} \), which satisfies the compatibility conditions \( \mathcal{B}(u_1) = \mathcal{H}(u_1) \), \( \mathcal{B}(u_1 + \mathcal{F}(u_1)) = 0 \) and \( D^2 u_1(\cdot, \cdot, \ell/2) = D^2 u_1(\cdot, \cdot, \ell/2) \) for each multi-index \( \gamma \) with length at most two, we set

\[
\mathcal{Z}_{\delta, R}^R(u_1) = \{ u \in Y_{2+\alpha}(t_0, t_0 + \delta) : u(t_0, \cdot) = u_1, \|u - u_1\|_{Y_{2+\alpha}}(t_0, t_0 + \delta) \leq R \}.
\]

Given \( R > 0 \) we can determine \( r_1 > 0 \) and \( \delta > 0 \) (independent of \( t_0 \)) with \( \delta^{(3/2)}R \) sufficiently small such that, if \( u_1 \) belongs to \( B(0, r_1) \subset \mathcal{X}_{2+\alpha} \), then the Cauchy problem

\[
\begin{aligned}
D_t w & = \mathcal{L} w + \mathcal{F}(w), \\
\mathcal{B} w & = \mathcal{H}(w), \\
D_x^2 D_y^2 w(\cdot, \cdot, \ell/2) & = D_x^2 D_y^2 w(\cdot, \cdot, \ell/2), \\
w(t_0, \cdot) & = u_1,
\end{aligned}
\]

admits a unique solution \( w \in \mathcal{Z}_{\delta, R}^R(u_1) \). We are almost done. Indeed, let \( u \in \mathcal{C}_r \) be the unique fixed point of \( \Gamma \), and take \( r_0 \) small enough such that \( u \in B(0, r_1) \subset Y_{2+\alpha} \). Assume that \( v \in Y_{2+\alpha} \) is another solution to (5.1), and let \( t_0 > 0 \) denote the supremum of the set \( \{ \tau \in [0, T] : u(t, \cdot) = v(t, \cdot), t \in [0, \tau] \} \). Suppose by contradiction that \( t_0 < T \). Then, both \( u \) and \( v \) are solutions in \( Y_{2+\alpha}(t_0, t_0 + \delta) \) to the Cauchy problem (5.7), with \( u_1 := u(t_0, \cdot) = v(t_0, \cdot) \). Taking \( R \geq 2\max\{\|u\|_{Y_{2+a}}, \|v\|_{Y_{2+a}}\} \) large enough and \( \delta > 0 \) small enough, it follows that \( u \) and \( v \) both belong to \( \mathcal{Z}_{\delta, R}^R(u_1) \), so that they do coincide, leading us to a contradiction. \( \square \)

Remark 5.2. Since Problem (5.1) is autonomous, under the same assumptions as in Theorem 5.1 for each \( a > 0 \) and \( T > 0 \) there exists a unique solution \( u \in Y_{2+a}(a, a + T) \) such that \( u(a, \cdot) = u_0 \).

6. Proof of the main result

6.1. Study of the dispersion relation and the point spectrum. Since, we are interested in the instability of the travelling wave solution \((\Theta^{(0)}, \Phi^{(0)})\) to Problem 1.3-1.4, we need to determine a range of Lewis numbers \( Le \) which lead to eigenvalues of \( L_\alpha \) (see Remark 4.6) with positive real part. In view of Theorem 4.4, such eigenvalues will lie in \( \Omega_k \) (see (4.15)). For simplicity, we will look for positive real elements of \( \Omega_k \). Note that \( Le - Y_k \) vanishes for no \( \lambda \)'s with positive real part. To determine elements of \( \sigma(L_\alpha) \) with positive real part we need to analyze the reduced dispersion relation

\[
\mathcal{D}_{0,k}(\lambda, Le) = \exp \left( R \frac{R}{2}(Le - 1 - X_k(\lambda) - Y_k(\lambda, Le)) \right) - 1 + \theta_t R X_k(\lambda).
\]

We recall that \( X_k(\lambda) = \sqrt{1 + 4\lambda + 4\lambda^2}, \ Y_k(\lambda, Le) = Y_k(\lambda) = \sqrt{Le^2 + 4\lambda Le + 4\lambda^2} \) and \( \lambda_k = 4\pi^2 k^2 \ell^2 \) for each \( k \in \mathbb{N} \cup \{0\} \). Throughout this subsection we assume \( \theta_t \) is fixed in \((0, 1)\); so is \( R > 0 \) via (1.5).
Lemma 6.1. There exists \( \ell_0(\theta_i) \) such that, for all \( \ell > \ell_0(\theta_i) \), \( \mathcal{D}_{0,1}(0, \text{Le}) = 0 \) has a unique root \( \text{Le}_c = \text{Le}_c(1) \in (0, 1) \). Moreover, for each \( \ell \) fixed as above, there exists a maximal integer \( K \geq 1 \) such that, for \( k \in \{1, \ldots, K\} \), \( \mathcal{D}_{0,k}(0, \text{Le}) = 0 \) has a unique root \( \text{Le}_c(k) \in (0, 1) \). Finally, it holds:

\[
0 < \text{Le}_c(K) \leq \ldots \leq \text{Le}_c(2) \leq \text{Le}_c(1).
\]

**Proof.** An easy but formal computation shows that, if \( \text{Le}_c(k) \) is a root of \( \mathcal{D}_{0,k}(0, \cdot) \), then

\[
\text{Le}_c(k) = 1 + X_k(0) + Y_k(0, \text{Le}_c(k)) + 2R^{-1}\ln(1 - \theta_i R X_k(0)),
\]

or equivalently:

\[
\text{Le}_c(k) = \left(1 + X_k(0)\right)[R^2 + 2R\ln(1 - \theta_i R X_k(0))] + 2[\ln(1 - \theta_i R X_k(0))]^2 + 2R\ln(1 - \theta_i R X_k(0)).
\]

However, Formula \( 6.2 \) makes sense only if \( 1 - \theta_i R X_k(0) > 0 \). Hence, for each fixed \( \ell > 0 \) there exists \( K \) such that, for \( 1 - \theta_i R X_k(0) > 0 \) if and only if \( k \leq K \).

Further, \( \text{Le}_c(k) \) is required to meet the physical requirement that \( 0 < \text{Le}_c(k) < 1 \). In this respect, \( \ell \) should be large enough:

\[
\text{Le}_c(1) = \text{Le}_0 + \frac{16\pi^2 \theta_i e^{R(1 - \theta_i R)}}{2(2\theta_i e^{R} - 1)\ell^2} + o(\ell^{-2})
\]

as \( \ell \to +\infty \), where \( \text{Le}_0 = R(2\theta_i e^{R} - 1)^{-1} \) belongs to \( (0, 1) \) see [3, Formula (43), p. 2083]. Thus, there exists \( \ell_0(\theta_i) > 0 \) such that, if \( \ell > \ell_0(\theta_i) \), then \( \text{Le}_c(1) \in (0, 1) \).

Finally, it remains to prove property \( 6.1 \) for a fixed \( \ell > \ell_0(\theta_i) \) which in turn defines the integer \( K \geq 1 \). The latter property follows from the following estimate (see [5, Proposition 3.1]):

\[
\left(\sqrt{(\text{Le}_c(k))^2 + 4\lambda_k} - \text{Le}_c(k)\right) \frac{d\text{Le}_c(k)}{d\lambda_k} < 4\left(1 - \frac{\theta_i}{1 - \theta_i R}\right).
\]

Obviously, \( \frac{\theta_i}{1 - \theta_i R} = \frac{1 - e^{-R}}{R e^{-R}} = \frac{e^R - 1}{R} > 1 \), which implies that \( \frac{d\text{Le}_c(k)}{d\lambda_k} < 0 \). \( \square \)

In view of Lemma \( 6.1 \), our focus will be on the case when \( \text{Le} \in (0, \text{Le}_c(1)) \). Hereafter we will simply denote the critical value \( \text{Le}_c(1) \) by \( \text{Le}_c \), keeping in mind that \( \text{Le}_c \) at fixed \( 0 < \theta_i < 1 \) depends on \( \ell > \ell_0(\theta_i) \).

**Lemma 6.2.** The function \( \mathcal{D}_{0,1} \) is smooth in \( [0, \sqrt{\lambda_1}] \times [0, \text{Le}_c] \). Moreover, \( \frac{\partial \mathcal{D}_{0,1}}{\partial \lambda} \) is positive in \( [0, \sqrt{\lambda_1}] \times [0, \text{Le}_c] \) and vanishes on \( \sqrt{\lambda_1} \times [0, \text{Le}_c] \).

**Proof.** The proof of the positivity of \( \frac{\partial \mathcal{D}_{0,1}}{\partial \lambda} \) is straightforward and based on the observation that \( Y_1(\lambda, \text{Le}) - \lambda - 2\lambda > 0 \) for \( \lambda > \sqrt{\lambda_1} \) and \( Y_1(\sqrt{\lambda_1}, \text{Le}) = \text{Le} + 2\sqrt{\lambda_1} \). On the other hand, we observe that

\[
\frac{\partial \mathcal{D}_{0,1}}{\partial \lambda}(\lambda, \text{Le}) = -R \exp\left(\frac{R}{2}(\text{Le} - 1 - X_k(\lambda) - Y_k(\lambda, \text{Le}))\right)(X_k^{-1} + \text{Le} Y_k^{-1}) + 2(1 - e^{-R})X_k^{-1}.
\]

Since \( \text{Le} Y_k^{-1} < (X_k^{-1})^{-1} \) and \( \text{Le} - 1 - X_k(\lambda) - Y_k(\lambda, \text{Le}) < -2 \), we can estimate

\[
\frac{\partial \mathcal{D}_{0,1}}{\partial \lambda}(\lambda, \text{Le}) > 2(1 - (R + 1)e^{-R})X_k^{-1}, \quad (\lambda, \text{Le}) \in [0, \sqrt{\lambda_1}] \times [0, \text{Le}_c],
\]

and the positivity of \( \frac{\partial \mathcal{D}_{0,1}}{\partial \lambda} \) in \( [0, \sqrt{\lambda_1}] \times [0, \text{Le}_c] \) follows immediately. \( \square \)

We can now prove the following result.

**Proposition 6.3.** Under the assumptions of Lemma \( 6.1 \), there exist \( \lambda_* \in (0, \sqrt{\lambda_1}) \) and a decreasing, continuously differentiable function \( \tilde{\varphi} : (0, \text{Le}_c) \to (0, \lambda_*) \) such that \( \mathcal{D}_{0,1}(\tilde{\varphi}(\text{Le}), \text{Le}) = 0 \) for all \( \text{Le} \in (0, \text{Le}_c) \).
Proof. From Lemma 6.2 it follows that \( \frac{\partial D_{0,1}}{\partial \lambda}(0, \text{Le}_c) > 0 \). We can thus apply the implicit function theorem, which shows that there exist \( \delta_1, \delta_2 > 0 \) and a unique function \( \varphi \in C^1([\text{Le}_c - \delta_1, \text{Le}_c + \delta_1]) \) such that, if \((\lambda, \text{Le}) \in [\text{Le}_c - \delta_1, \text{Le}_c + \delta_1] \times [-\delta_2, \delta_2] \) is a root of \( D_{0,1} \), then \( \lambda = \varphi(\text{Le}) \). In view of the previous lemma, \( \varphi \) is a decreasing function. As a byproduct, we deduce that there exists \( k \) such that the real part of \( \sigma \) is decreasing, \( C \)-non-increasing, \( \partial \) \( \varphi \) non-increasing, \( \sigma \) \( \varphi \) real part, then it is an eigenvalue and it belongs to \( \Omega \). Therefore, its zeroes in \( \Pi \) are at most finitely many. Moreover, for each \( \lambda \), \( \varphi \) is holomorphic in the halfplane \( \Pi = \{ \lambda \in \mathbb{C} : \Re \lambda \geq 0 \} \) and the function \( x \mapsto f(x) = e^{-x(1+2\sqrt{x})} + 2\sqrt{x} - e^{-x(1+2\sqrt{x})} \) vanishes at \( x = 0 \) and its derivative is positive in \((0, +\infty)\).

\[ \varphi(\text{Le}) \] is a root of \( \Pi \) for \( \text{Le} \in (0, \text{Le}_c) \), extended beyond \( \text{Le}_c \). Here \( \theta_1 = 0.75, \ell = 100, \text{Le}_c \approx 0.5641, \lambda_* \approx 0.0315 \). Note that \( \sqrt{\lambda_1} = \pi/50 \approx 0.0628 \).

Corollary 6.4. The spectrum of the operator \( L \) contains elements with positive real parts. Moreover, the part of \( \sigma(L) \) in the right halfplane \( \{ \lambda \in \mathbb{C} : \Re \lambda \geq 0 \} \) consist of 0 and a finite number of eigenvalues.

Proof. By Theorem 4.4 we know that if \( \lambda \neq 0 \) is an element in the spectrum of \( L \) with nonnegative real part, then it is an eigenvalue and it belongs to \( \Omega_k \) for some \( k \in \mathbb{N} \cup \{0\} \). Hence, \( D_k(\lambda) = 0 \) or, equivalently, \( D_{0,k}(\lambda) = 0 \) for some \( k \in \mathbb{N} \cup \{0\} \). As it is immediately seen, each function \( \lambda \mapsto D_{0,k}(\lambda) \) is holomorphic in the halfplane \( \Pi = \{ \lambda \in \mathbb{C} : \Re \lambda \geq 0 \} \) and it does not identically vanish in it. Therefore, its zeroes in \( \Pi \) are at most finitely many. Moreover, for each \( \lambda \in \Pi \) and \( k \in \mathbb{N} \cup \{0\} \) we can estimate
\[
\Re X_k(\lambda) \geq \sqrt{\frac{1}{2} + 2\lambda_k}, \quad \Re Y_k(\lambda, \text{Le}) \geq \sqrt{\frac{\text{Le}^2}{2} + 2\lambda_k},
\]
so that the real part of \( D_{0,k}(\lambda, \text{Le}) \) diverges to \( +\infty \), as \( k \to +\infty \), uniformly with respect to \( \lambda \in \Pi \). As a byproduct, we deduce that there exists \( k_0 \in \mathbb{N} \) such that the nontrivial eigenvalues \( \lambda \in \Pi \) lie in \( \bigcup_{k=0}^{k_0} \Omega_k \) and this completes the proof. 

To prove the main result of this section, we also need the following result which is a variant of 25 Theorem 5.1.5] and 11 Theorem 4.3.

Figure 2. Numerical computation of the implicit curve \( \lambda = \varphi(\text{Le}) \) for \( \text{Le} \in (0, \text{Le}_c) \), extended beyond \( \text{Le}_c \). Here \( \theta_1 = 0.75, \ell = 100, \text{Le}_c \approx 0.5641, \lambda_* \approx 0.0315 \). Note that \( \sqrt{\lambda_1} = \pi/50 \approx 0.0628 \).
Lemma 6.5. Let $X$ be a complex Banach space, $r > 0$ and $T_n : B(0, r) \subset X \to X$ $(n \in \mathbb{N})$ be a bounded operator such that $T_n(x) = Mx + O(||x||^p)$ as $||x|| \to 0$, for some $p > 1$ and some bounded linear operator $M$ on $X$ with spectral radius $\rho > 1$. Further, assume that there exists an eigenvector $u$ of $M$ with eigenvalue $\lambda \in \mathbb{C}$ such that $|\lambda|^p > \rho$ and that there exists $x' \in X'$ such that $x'(u) \neq 0$. Then, there exist $c > 0$ and, for any $\delta > 0$, $x_0 \in B(0, \delta)$ and $n_0 \in \mathbb{N}$ (depending on $\delta$) such that the sequence $x_0, \ldots, x_{n_0}$, where $x_n = T_n(x_{n-1})$ for any $n = 1, \ldots, n_0$, is well defined and $|x'(x_{n_0})| \geq c|x'(u)|$.

Proof. Without loss of generality, we assume that $||u|| = 1$ and $||x'|| \leq 1$. Moreover, we choose $a, b > 0$ such that $||T_n(x) - Mx|| \leq b ||x||^p$ for each $x \in B(0, a) \subset X$ and $n \in \mathbb{N}$. Since $|\lambda|^p > \rho$, we can fix $\eta > 0$ such that $|\lambda|^p > \rho + \eta$ and, from the definition of the spectral radius of a bounded operator, we can also determine a positive constant $K$ such that $||M^n||_{L(X)} \leq K(\rho + \eta)^n$ for any $n \in \mathbb{N}$. Finally, we fix $\delta > 0$, choose $n_0 \in \mathbb{N}$ be such that $|\lambda|^{-n_0} < \delta$, and set $x_0 := \sigma u|\lambda|^{-n_0}$, where $\sigma \in (0, 1)$ is chosen so as to satisfy the conditions

$$
\sigma \leq \frac{a}{2}, \quad \frac{2^p b K}{|\lambda|^p - \rho - \eta} \sigma^{p-1} \leq \frac{1}{2} |x'(u)|.
$$

(6.3)

To begin with, we prove that the sequence $x_0, \ldots, x_{n_0}$ is well defined. For this purpose, in view of the condition in (6.3), it suffices to check that, if $x_k$ is well defined, then $||x_k|| \leq 2\sigma |\lambda|^{-n_k}$. We prove by recurrence. Clearly, $x_0$ satisfies this property. Suppose that the claim is true for $k = 0, \ldots, n-1$. Then, $x_n$ is well defined and it easy to check that

$$
x_n = M^n x_0 + \sum_{k=0}^{n-1} M^{n-1-k}(T_k + Mx_k) = M^n x_0 + \sum_{k=0}^{n-1} M^{n-1-k}(T_k + Mx_k).
$$

(6.4)

Thus, we can estimate

$$
||x_n|| \leq |\lambda|^n ||x_0|| + Kb \sum_{k=0}^{n-1} (\rho + \eta)^{n-1-k} ||x_k||^p.
$$

(6.5)

Let us consider the second term in the right-hand side of (6.5), which we denote by $S_n$. Since, we are assuming that $||x_k|| \leq 2\sigma |\lambda|^{-n_k}$ for each $k = 0, \ldots, n-1$, we can write

$$
S_n \leq 2^p Kb \sigma^p |\lambda|^p (n-n_0-1) \sum_{k=0}^{n-1} \left( \frac{\rho + \eta}{|\lambda|^p} \right)^{n-1-k} \leq \sigma |\lambda|^{-n_0} 2^p Kb \sigma^{p-1}
$$

(6.3)

and, using the second condition in (6.3) and the fact that $x'$ has norm which does not exceed 1, we conclude that

$$
S_n \leq \frac{1}{2} \sigma |\lambda|^{-n_0} |x'(u)| \leq \frac{1}{2} \sigma |\lambda|^{-n_0}.
$$

(6.6)

Since $|\lambda|^n \leq |\lambda|^{-n_0}$, from (6.5) and (6.6) the claim follows at once.

To conclude the proof, it suffices to use (6.4) with $n = n_0$, as well as (6.5) and (6.6) again, to estimate

$$
|x'(x_{n_0})| \geq |x'(M^n x_0)| - |x'(S_{n_0})| \geq \sigma |x'(u)| - \frac{1}{2} \sigma |x'(u)| = \frac{1}{2} \sigma |x'(u)|.
$$

The assertion follows with $c = \sigma/2$. \hfill \Box

Now, we can state and prove the following theorem.

Theorem 6.6. Let $0 < \theta_1 < 1$ be fixed, $\ell > \ell_0(\theta_1)$ as in Lemma 6.1, $L_{\infty} = L_{\infty}(1)$ as in Lemma 6.2. Then, for each $L \in (0, L_{\infty})$, the null solution $u$ of Problem 5.1 is poinwise unstable with respect to small perturbations in $X_{2+\alpha}$. More precisely, there exists a positive constant $C$ such that for each $y_0 \in \mathbb{R}$ and $\delta > 0$ there exist $u_0, u_1 \in B(0, \delta) \subset X_{2+\alpha}$ and $\tilde{u}, n_* \in \mathbb{N}$ depending on $\delta$ such that $\min\{|u_2(\tilde{u}, 0, y_0)|, |u_1(n_*, R, y_0)|\} \geq C$, where $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ and $u_* = (u_1^*, u_2^*)$ denote the solution to the Cauchy problem with initial datum $\tilde{u}_0$ and $u_0^*$, respectively.
Proof. We split the proof into two steps. The first one is devoted to prove an estimate which will allow us to apply Lemma 5.3. Then, in Step 2, we prove the pointwise instability.

Step 1. The smoothness of Υ implies that there exists $c > 0$ such that $||\Upsilon (v_0)||_{2+α} \leq c ||v_0||_{2+α}$, for each $v_0 \in D(L_0)$ with sufficiently small norm. Here, $\Upsilon$ is defined in Lemma 4.7(i). Fix $r$ so small that $||v_0 + \Upsilon (v_0)||_{2+α} \leq r_0$ for each $v_0 \in \overline{B(0, r)} \subset D(L_0)$, where $r_0 = r_0(1)$ is defined in the statement of Theorem 5.1.

For $n \in \mathbb{N}$, let $R_n : B(0, ρ) \subset D(L_0) \to \overline{D(L_0)}$ be the map defined by $R_n(v_0) = P(u_n(n, \cdot, \cdot))$ for each $v_0 \in B(0, ρ)$ (see Lemma 4.7), where $u_n$ is the solution to problem (5.1) with initial condition $u_0 = v_0 + \Upsilon (v_0)$ at $t = n - 1$. (Note that, by Lemma 4.7(i), $u_0$ satisfies the compatibility conditions in Theorem 5.1.) Further, by Remark 5.2, $u_n$ is well defined in the time domain $[n - 1, n]$. We claim that

$$||R_n(v_0) - e^{L_0 t} v_0||_{2+α} \leq c ||v_0||_{2+α}, \quad v_0 \in B(0, r).$$

(6.7) Estimate (6.7) follows from the integral representation of the solution of Problem (5.1) and estimates (6.6). Indeed, again by Remark 5.2, $u_n(n, \cdot, \cdot)$ is the value at $t = 1$ of the solution $u$ to Problem (5.1), with $u(0, \cdot) = u_0$ and, by the proof of Theorem 5.1 (see, in particular, formula (5.5)),

$$u(1, \cdot, \cdot) - e^{L_0 t} u_0 = \int_0^1 e^{(1-s)L}[\mathcal{F}(u(s, \cdot, \cdot)) + \mathcal{L}(\mathcal{H}(u(s, \cdot, \cdot)))] ds - \int_0^1 e^{(1-s)L} \mathcal{L}(\mathcal{H}(u(s, \cdot, \cdot))) ds.$$

Since $\mathcal{F}$ and $\mathcal{H}$ are quadratic at 0, it follows immediately that $||u(1, \cdot, \cdot) - e^{L_0 t} u_0||_{2+α} \leq c ||v_0||_{2+α}$. Noting that $P(u(1, \cdot, \cdot) - e^{L_0 t} u_0) = R_n(v_0) - e^{L_0 t} v_0$, formula (6.7) follows at once.

Step 2. Let us begin by proving that there exists $C > 0$ such that for any $y_0 \in \mathbb{R}$ and $δ > 0$ there exists $u_0 \in B(0, δ) \subset X_{2+α}$ and $n_0 \in \mathbb{N}$ depending on $δ$ such that $||u_2(n_0, 0^+, y_0)|| \geq C$, where $u = (u_1, u_2)$ is the solution to (5.1) with initial datum $u_0$ at time $t = 0$. For this purpose, we want to apply Lemma 6.5 with $X = D(L_0)$ endowed with the norm of $X_{2+α}$. To begin with, we observe that, by Corollary 6.4, there exists only a finite number of eigenvalues of $L$ (and hence of $L_0$) with positive real part. From the spectral mapping theorem for analytic semigroups it thus follows that the spectral radius $ρ$ of the operator $M = e^{L_0 γ}$ is larger than one and there exists an eigenvalue $λ$ such that $|λ| = ρ$. Let us fix $y_0 \in \mathbb{R}$ and $δ > 0$. It is not difficult to show that a corresponding eigenfunction is the function $w = (w_1 e_1(\cdot - 2πt^− y_0), w_2 e_1(\cdot - 2πt^− y_0))$, where

$$w_1(x) = e^{θ^− t} \chi_{(0, R)}(x) + (c_1 e^{θ^− t} + c_2 e^{θ^− t}) \chi_{(R, +∞)}(x),$$

$$w_2(x) = (d_1 e^{θ^+ t} + d_2 e^{θ^− t}) \chi_{(0, R)}(x) + d_3 e^{θ^− t} \chi_{(R, +∞)}(x),$$

for each $x \in \mathbb{R}$ and

$$c_1 = \frac{e^{θ^− t}}{e^{θ^− t} R - e^{θ^− t} R}, \quad c_2 = \frac{e^{θ^− t} R}{e^{θ^− t} R - e^{θ^− t} R}, \quad c_3 = \frac{θ_1 R e^{θ^+ t} R}{e^{θ^− t} R - e^{θ^− t} R} X_1,$$

$$d_1 = \frac{Le(Le + μ_1^±)}{(e^{θ^− t} R - e^{θ^− t} R) Y_1}, \quad d_2 = −(Le + μ_1^±) d_1, \quad d_3 = (1 - e^{θ^− t} R) d_1,$$

$$ν^± = −\frac{1}{2} ± \sqrt{1 + 4Λ + 4Λ}, \quad μ_1^± = −\frac{Le}{2} ± \sqrt{Le^2 + 4LeΛ + 4Λ}.$$

Note that

$$w_2(0^+, y_0) = (d_1 + d_2) = \frac{Le X_1 e^{θ^− t} R}{(e^{θ^− t} R - e^{θ^− t} R) Y_1} \neq 0.$$

Hence, if we set $x'(f) = f_2(0^+, y_0)$ for any $f \in D(L_0)$, then $x'(w) \neq 0$. As in Step 1, we fix $r > 0$ such that $||v_0 + \Upsilon (v_0)||_{2+α} \leq r_0$ for $j = 1, 2$ for each $v_0 = v_0 + iv_{0,2} \in B(0, r) \subset D(L_0)$. By Theorem 5.1 both $u_0(n, v_{0,1} + \Upsilon (v_{0,1}), n - 1)$ and $u_0(n, v_{0,2} + \Upsilon (v_{0,2}), n - 1)$ are well defined for each $n \in \mathbb{N}$. We can thus introduce the operator $T_n : B(0, r) \subset D(L_0) \to D(L_0) (n \in \mathbb{N})$ by setting $T_n(v_0) = P(u_0(n, v_{0,1} + \Upsilon (v_{0,1}), n - 1) + iP(u_0(n, v_{0,2} + \Upsilon (v_{0,2}), n - 1))$ for each $v_0 \in B(0, r)$, where $P$ is the projection in Lemma 4.7(iii). By the arguments in Step 1 we deduce that $||T_n(v_0) −$
\(e^{L_n}v_0\|_\mathcal{X} \leq C\|v_0\|_{\mathcal{X}}^2\) for some positive constant \(C\) and each \(v_0 \in B(0, r)\). We can thus apply Lemma 6.5 with \(M = e^{L_n}\), \(p = 2\) and conclude that there exist \(c > 0\) and, for each \(\delta > 0\), a function \(v_0 = v_{0,1} + iv_{0,2} \in B(0, \delta) \subset D(L_n)\) and \(n_0 \in \mathbb{N}\) such that \(v_n = T_n(v_{n-1})\) is well defined for each \(n \in \{1, \ldots, n_0\}\) and \(|x'(v_{n_0})| \geq c\). Since \(v_{n_0} = Pu(n_0, v_{0,1} + \Upsilon(v_{0,1}), 0) + iPu(n_0, v_{0,2} + \Upsilon(v_{0,2}), 0)\), where \(u(n_0, u_0, 0)\) denotes the value at \(n_0\) of the unique solution to problem (3.23) with initial datum \(u_0\) at time \(t = 0\), we have so proved that

\[
\left|(Pu(n_0, v_{0,1} + \Upsilon(v_{0,1}), 0))_2(0^+, y_0)\right|^2 + \left|(Pu(n_0, v_{0,2} + \Upsilon(v_{0,2}), 0))_2(0^+, y_0)\right|^2 \geq c^2. \tag{6.8}
\]

By definition, \(P = I - \mathcal{N}B_*\) (see Lemma 4.7(i)) and \((\mathcal{N}u)_2(0^+, \cdot) = 0\) for any function \(u\). Hence, 

\[
(Pu(n_0, v_{0,j} + \Upsilon(v_{0,j}), 0))_2(0^+, y_0) = (u(n_0, v_{0,j} + \Upsilon(v_{0,j}), 0))_2(0^+, y_0) \quad \text{for } j = 1, 2.
\]

From (6.8) it thus follows that there exists \(j\) such that \(|(u(n_0, v_{0,j} + \Upsilon(v_{0,j}), 0))_2(0^+, y_0)| \geq c/2\) and the thesis follows with \(C = c/2\), \(\bar{n} = n_0\) and \(\tilde{u} = u(n_0, v_{0,j} + \Upsilon(v_{0,j}), 0)\).

To prove the existence of \(u_*\) as in the statement of the theorem, it suffices to take as \(x'\) the functional defined by \(x'(f) = f_1(R, y_0)\) for each \(f \in D(L_n)\). The missing easy details are left to the reader. \(\square\)

**Remark 6.7.** Clearly, Theorem 6.6 implies the \(C^{2+\alpha}\)-instability of the null solution \(u\) to (5.1).

**Remark 6.8.** The \(C^{2+\alpha}\)-instability of the null solution \(u\) to (5.1) can be directly obtained by applying [25] Theorem 5.1.5, taking advantage of Step 1 of Theorem 6.6 and arguing as in [11] Corollary 4.5. Finally, it can also be proved in a slightly different way adapting the arguments in [30] Theorem 3.4.

From Theorem 6.6 we can now easily derive the proof of the main result of this paper.

**Proof of Main Theorem.** Taking the changes of variables and unknown in Subsections 3.1 and 3.2 into account, the result in Theorem 6.6 allows us to conclude easily that the normalized temperature \(\Theta\) and the normalized concentration of deficient reactant in problem (1.3)–(1.4) are unstable with respect to two dimensional \(C^{2+\alpha}\) perturbations. Similarly, using formulae (3.9) and (3.12) and again Theorem 6.6 we can infer that there exist initial data \((\Theta, \Phi)\) and \((\Theta_*, \Phi_*)\) with \(C^{2+\alpha}\)-norm, arbitrarily close to the travelling wave solution (4.6) such that the trailing front \(G\) (resp. the ignition front \(F\)) to problem (1.3)–(1.4) with initial datum \((\Theta(0, \cdot), \Phi(0, \cdot)) = (\Theta, \Phi)\) (resp. \((\Theta(0, \cdot), \Phi(0, \cdot)) = (\Theta_*, \Phi_*)) is not arbitrarily close to 0 (resp. \(R\)). \(\square\)

### 7. Numerical Simulation

In this section, we are going to use some high resolution numerical methods, including Chebyshev collocation and Fourier spectral method (see, e.g., [36, 38, 40]). We consider the problem (3.23) in the finite domain \(\Omega = \Omega_\alpha \cup \Omega_\beta \cup \Omega_\eta = \{[-A, 0] \cup [0, R] \cup [R, B]\} \times [-\ell/2, \ell/2]\), where \(A > 0\) and \(B > 0\) are large enough, see Figure 3. The independent variables are \(-A \leq \xi \leq B, -\ell/2 < \eta < \ell/2\).

![Figure 3. Computational domain.](image-url)
7.1. The linear system. The linearized system around the null solution of System (3.23) reads:

\[
\begin{aligned}
& u_\tau = u_\xi + u_{\xi\xi} + u_{\eta\eta}, \quad \text{in } \Omega, \\
& w = 0, \quad \text{in } \Omega_-, \\
& w_\tau = w_\xi + Le^{-1}(w_{\xi\xi} + w_{\eta\eta}), \quad \text{in } \Omega_0 \cup \Omega_+,
\end{aligned}
\]

with

\[
\mathcal{B}(u, w) = 0.
\]

We map \(\Omega_-, \Omega_0\) and \(\Omega_+\) to \(\mathbb{D} = [-1, 1] \times [0, 2\pi]\). Then, we consider in \(\mathbb{D}\) the system for the three pairs of unknowns \((u_1, u_1), (u_2, w_2)\) and \((u_3, w_3)\), corresponding respectively to \((u, w)\) in \(\Omega_-\), \(\Omega_0\) and \(\Omega_+\). The new independent variables are denoted by \(x \in [-1, 1]\) and \(y \in [0, 2\pi]\).

Therefore, System (7.1)-(7.2) is equivalent to:

\[
\begin{aligned}
D_\tau u_1 &= \frac{4}{B-R} D_x u_1 + \frac{4}{B-R} D_{xx} u_1 + \frac{4\pi^2}{L_1} D_{yy} u_1, \quad w_1 \equiv 0, \\
D_\tau u_2 &= \frac{4}{B-R} D_x u_2 + \frac{4}{B-R} D_{xx} u_2 + \frac{4\pi^2}{L_1} D_{yy} u_2, \\
D_\tau w_2 &= \frac{4}{B-R} D_x w_2 + \frac{4}{B-R} D_{xx} w_2 + \frac{4\pi^2}{L_1} D_{yy} w_2, \\
D_\tau u_3 &= \frac{4}{B-R} D_x u_3 + \frac{4}{B-R} D_{xx} u_3 + \frac{4\pi^2}{L_1} D_{yy} u_3, \\
D_\tau w_3 &= \frac{4}{B-R} D_x w_3 + \frac{4}{B-R} D_{xx} w_3 + \frac{4\pi^2}{L_1} D_{yy} w_3,
\end{aligned}
\]

together with the boundary conditions:

\[
\begin{aligned}
& u_1(-1) = u_3(1) = w_3(1) = 0, \quad u_1(1) = u_2(-1), \\
& w_2(-1) = \frac{4}{B-R} D_x u_2(-1) - \frac{4\pi^2}{L_1} D_{yy} w_2(-1), \quad D_x w_2(-1) = \frac{4}{B-R} w_2(-1), \\
& D_x w_2(1) = \frac{4}{B-R} D_x w_2(1) - \frac{4\pi^2}{L_1} D_{yy} w_2(1), \quad D_x w_3(-1) = \frac{4}{B-R} D_x w_3(-1) - \frac{4\pi^2}{L_1} w_2(1), \\
& D_x w_3(1) = 2 \frac{4}{B-R} D_x w_3(1) + 2 \frac{4\pi^2}{L_1} D_{yy} w_3(1), \quad u_3(1) = u_2(-1).
\end{aligned}
\]

Let us give a brief overview of the numerical method. Hereafter, we denote by \((u, w)\) any pair of unknowns \((u_i, w_i)\), \(1 \leq i \leq 3\). We discretize System (7.3)-(7.4) using a forward-Euler explicit scheme in time. Then, we use a discrete Fourier transform in the direction \(y \in (0, 2\pi)\), namely:

\[
\begin{aligned}
u(x, y) &= \sum_{k = -N_y/2}^{N_y/2} \hat{u}_k(x)e^{iky}, \\
w(x, y) &= \sum_{k = -N_y/2}^{N_y/2} \hat{w}_k(x)e^{iky},
\end{aligned}
\]

and

\[
\begin{aligned}
D_{yy} u(x, y) &= -\sum_{k = -N_y/2}^{N_y/2} k^2 \hat{u}_k(x)e^{iky}, \\
D_{yy} w(x, y) &= -\sum_{k = -N_y/2}^{N_y/2} k^2 \hat{w}_k(x)e^{iky}.
\end{aligned}
\]

Finally, we use a Chebyshev collocation method in \(x \in (-1, 1)\). Let \(\{l_j(x)\}_{j=0}^{N_x}\) be the Lagrange polynomials based on the Chebyshev–Gauss–Lobatto points \(\{x_j\}_{j=0}^{N_x} = \{\cos(j\pi/N_x)\}_{j=0}^{N_x}\). We set:

\[
\hat{u}_k(x) = \sum_{j=0}^{N_x} \hat{u}_{kj} l_j(x), \quad \hat{w}_k(x) = \sum_{j=0}^{N_x} \hat{w}_{kj} l_j(x).
\]

Denoting the differential matrix of order \(m\) associated to \(\{x_j\}_{j=0}^{N_x}\) by \(D^m = (d^{(m)}_{ij})_{i,j=0, \ldots, N_x}\), where \(d^{(m)}_{ij} = l_j^{(m)}(x_i)\) and \(l_j(x_i) = \delta_{ij}\), we eventually obtain

\[
\begin{aligned}
\hat{u}_k(x_1) &= \sum_{j=0}^{N_x} \hat{u}_{kj} d_{1j}, \\
D_x \hat{u}_k(x_1) &= \sum_{j=0}^{N_x} \hat{u}_{kj} d^{(1)}_{1j}, \\
D_{xx} \hat{u}_k(x_1) &= \sum_{j=0}^{N_x} \hat{u}_{kj} d^{(2)}_{1j},
\end{aligned}
\]

\[
\begin{aligned}
\hat{w}_k(x_1) &= \sum_{j=0}^{N_x} \hat{w}_{kj} d_{1j}, \\
D_x \hat{w}_k(x_1) &= \sum_{j=0}^{N_x} \hat{w}_{kj} d^{(1)}_{1j}, \\
D_{xx} \hat{w}_k(x_1) &= \sum_{j=0}^{N_x} \hat{w}_{kj} d^{(2)}_{1j}.
\end{aligned}
\]
As initial data, we take \( w_2(0, \cdot) = \varepsilon (1 + \sin^2(y)) \), \( y \in [0, 2\pi] \), which corresponds to \( \xi = R/2 \); the other unknowns are taken as 0. The following pictures are for \( \varepsilon = 10^{-2} \), \( A = B = 10 \), \( \ell = 100 \), \( \Delta t = 10^{-3} \). As expected, the two profiles blow up for Lewis number below critical.

![Figure 4. Evolution of \((u, w)\) solution of the linear problem (7.1)-(7.2) for \( \theta_i = 0.75 \), \( Le = 0.3 < Le_c \).](image)

7.2. The fully nonlinear system. By treating the nonlinearities explicitly, we can use the same algorithm as in the linear case. In the coordinates \((\xi, \eta)\), we approximate the mollifier \( \beta(\xi) \) by the following trapezoid, see Figure 5:

\[
\beta(\xi) = \begin{cases}
2 + \xi/\delta, & -2\delta < \xi < -\delta, \\
1, & -\delta \leq \xi \leq \delta, \\
2 - \xi/\delta, & \delta < \xi < 2\delta, \\
0, & \text{elsewhere},
\end{cases}
\]

Then, the fully nonlinear terms in System (3.23), namely \( F_1, F_2, G_1, G_2, G_3 \), as well as \( \varrho, \varrho \tau \), have to be computed separately in eight intervals, as they are zero elsewhere: \([-2\delta, -\delta] \cup \cdots \cup [R + \delta, R + 2\delta]\), see Figure 5. We refer to the Appendix for the formulas.

Hereafter, we present some typical numerical results for the fully nonlinear problem. Simulations were performed using a standard pseudo-spectral method with small time step \( \Delta t = 10^{-5} \) and small amplitude of initial perturbations (of order \( 10^{-4} \) to \( 10^{-3} \)), to ensure sufficient accuracy.

We consider the situation when ignition temperature is fixed at \( \theta_i = 0.75 \) and \( \ell = 100 \), in such a case \( Le_c \approx 0.5641 \). Three significant values of the Lewis number have been chosen in the interval \((0, Le_c)\), namely \( Le = 0.10 \), \( Le = 0.20 \) and \( Le = 0.50 \). Figures 6 and 7 represent the interface patterns and temperature levels. Numerically, we observe that, after a rapid transition period, a steady configuration consisting of “two-cell” patterns for the ignition and trailing interfaces is established. These simulations confirm the theoretical analysis, that is instability of the planar fronts for \( Le \in (0, Le_c) \).
**Figure 6.** Patterns of ignition (above) and trailing (below) interfaces. Here $\theta_i = 0.75$, $\ell = 100$, $Le_c \simeq 0.5641$.

**Figure 7.** Temperature levels around the ignition (above) and trailing (below) interfaces. Parameters are as in Figure 6.

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8. Appendix

In this Appendix, we write explicitly the formulas for the fully nonlinear terms in system (3.23), namely \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \). These have to be computed separately in 8 intervals, as they are zero elsewhere: \([-\delta, -\delta] \cup [-\delta, 0] \cup [0, \delta] \cup [\delta, 2\delta] \cup [R - 2\delta, R - \delta] \cup [R - \delta, R] \cup [R, R + \delta] \cup [R + \delta, R + 2\delta] \), see Figure 5. The formulas for \( \varphi_\varepsilon \) and \( \tau_{\varepsilon, \xi} \) can be easily derived.

**1.** For \( \xi \in [-2\delta, -\delta] \) and \( \xi = \frac{\delta}{2}(x - 1) \),
\[
\mathcal{F}_1(u, w) = \frac{2Le}{ALe - aRw(0^+)} u_x,
\]
\[
\mathcal{F}_2(u, w) = \frac{1}{Le - Rw(0^+)} \left\{ \frac{2Rw(0^+)}{A} u_x + \frac{8\pi^2R^2 \beta}{A^2} \left[ w_y(0^+) u_x + 2w_y(0^+) u_x \right] \right\}
+ \frac{Le}{(Le - Rw(0^+))^2} \left\{ \frac{16\pi^2 R^2 \beta w_y^2(0^+)}{A^2 \xi} \left( u_x + \frac{\beta}{A} u_x x \right) + \frac{4Rw(0^+)}{A^2} \left( 2 - \frac{Rw(0^+)}{Le} u_x x \right) \right\}.
\]

**2.** For \( \xi \in [-\delta, 0] \) and \( \xi = \frac{\delta}{2}(x - 1) \),
\[
\mathcal{F}_1(u) = \frac{2}{A} u_x, \quad \mathcal{F}_2(u, w) = 8\pi^2 \frac{R}{A^2 \xi} \left( w_y(0^+) u_{xx} + w_y(0^+) u_x y + \frac{2Rw(0^+)}{A^2} u_x y \right).
\]

**3.** For \( \xi \in [0, \delta] \) and \( \xi = \frac{\delta}{2}(x + 1) \),
\[
\mathcal{F}_1(u) = \frac{1}{Le} e^{-\frac{\xi}{2}(x + 1)} + \frac{2}{R^2} \frac{Le}{x}, \quad \mathcal{G}_1(w) = Le e^{-\frac{Le}{2}(x + 1) w(0^+)} + \frac{2}{R} w_x.
\]
\[
\mathcal{F}_2(u, w) = \frac{4\pi^2}{c^2 \xi} \left[ e^{-\frac{\xi}{2}(x + 1)} \left( -w_y(0^+) w(0^+) + w_y^2(0^+) \left( 1 - \frac{Rw(0^+)}{Le} \right) \right) \right]
+ \frac{8\pi^2}{c^2} \left[ w_y(0^+) u_x + 2w_y(0^+) \left( u_x y + \frac{w_y(0^+)}{Le} u_x x \right) \right].
\]
\[
\mathcal{G}_2(w) = \frac{4\pi^2}{c^2} \left[ \frac{Le}{x} e^{-\frac{\xi}{2}(x + 1)} \left( w_y(0^+) w(0^+) + w_y^2(0^+) \left( \frac{2\pi}{\xi} - w(0^+) - 1 \right) \right) \right]
+ \frac{8\pi^2}{c^2 \xi} \left[ w_y(0^+) w_x + 2w_y(0^+) w_x y + \frac{2Rw(0^+)}{Le} u_x x \right].
\]

**4.** For \( \xi \in [\delta, 2\delta] \) and \( \xi = \frac{\delta}{2}(x + 1) \),
\[
\mathcal{F}_1(u, w) = \frac{\beta w(0^+)}{Le + Rw(0^+)} e^{-\frac{\xi}{2}(x + 1)} \frac{R(R + 1)}{Le(R + Le + Rw(0^+))} u_x,
\]
\[
\mathcal{F}_2(u, w) = \frac{1}{Le + Rw(0^+)} \left\{ \frac{R^2 e^{-\frac{\xi}{2}(x + 1)}}{c^2 \xi} \left[ \frac{4\pi^2 \beta}{c^2} \left( 2w_y^2(0^+) - w(0^+) w_y(0^+) \right) - w^2(0^+) \right] \right\}
+ \frac{8\pi^2 \beta (2w_y(0^+) u_x - w_y(0^+) u_x) - 2w(0^+) u_x \right\}
+ \frac{Le}{(Le + Rw(0^+))^2} \left\{ e^{-\frac{\xi}{2}(x + 1)} \left[ -\frac{4\pi^2 R^2 \beta}{c^2 \xi} \left( \frac{4Rw(0^+)}{Le} w_y(0^+) u_x y + Rw_y(0^+) \left( \frac{\beta w(0^+)}{Le} + \frac{1}{R} \right) \right) \right] \right\}
+ \frac{R^2 w(0^+)}{Le} \left( \frac{\beta w(0^+)}{Le} + \frac{1}{R} \right) + 2Rw(0^+) u_x x \right\}.
\]
\[ \mathcal{G}_1(w) = \frac{L}{L + Rw(0^+)} \left[ e^{-\frac{L}{R}(x+1)} \beta w(0^+) \left( \frac{L}{L + Rw(0^+)} + \frac{2}{L} \right) - \beta w(0^+) \right], \]

\[ \mathcal{G}_2(w) = \frac{L}{L + Rw(0^+)} w - \beta w(0^+) \frac{L}{L + Rw(0^+)} e^{-\frac{L}{R}(x+1)}, \]

\[ \mathcal{G}_3(w) = \frac{1}{L + Rw(0^+)} \left\{ e^{-\frac{L}{R}(x+1)} \left[ 4\pi^2 R \beta w(0^+) \frac{L}{L + Rw(0^+)} \right] + \frac{4\pi^2 L w(0^+)}{R^2} \left( 1 + \beta \right) \right\} + \frac{L}{L + Rw(0^+)} \left\{ \frac{4\pi^2 R \beta w(0^+)}{R^2} \right\} \left( 1 + \beta \right) \]

\[ \mathcal{F}_1(u) = \frac{\beta_i}{\theta_i + u(R^+)} \left( \theta_i R^+ - \beta_i u(R^+) e^{-\frac{\beta_i}{R}(x+1)} \right), \]

\[ \mathcal{F}_2(u) = \frac{1}{\theta_i + u(R^+)} \left\{ e^{-\frac{\beta_i}{R}(x+1)} \left[ 4\pi^2 R \beta_i^2 \frac{w(0^+)}{R^2} \right] \left( \theta_i + u(R^+) + \frac{2}{R} \right) \right\} \]

\[ \mathcal{F}_3(u) = \frac{\beta_i}{\theta_i + u(R^+)} \left( \theta_i R^+ + w \right), \]

\[ \mathcal{F}_4(u) = \frac{\beta_i}{\theta_i + u(R^+)} \left( \theta_i R^+ \right). \]

\[ \mathcal{F}_5(u, w) = \frac{\theta_i}{\theta_i + u(R^+)} \left\{ e^{-\frac{\theta_i}{R}(x+1)} \left[ 4\pi^2 R \beta_i^2 \frac{w(0^+)}{R^2} \right] \right\} \left( \theta_i + u(R^+) + \frac{2}{R} \right) \]

\[ \mathcal{F}_6(u, w) = -\frac{u(R^+)}{\theta_i R} e^{-\frac{\beta_i}{R}(x+1)} + \frac{2}{R} u_{xx}, \quad \mathcal{F}_7(u, w) = -\frac{1}{\theta_i R} e^{-\frac{\theta_i}{R}(x+1)} + \frac{2}{R} w_x. \]
\[ \mathcal{F}_2(u) = e^{-\frac{B}{2}(x+1)} \frac{4\pi^2}{\theta_i^2} R \left[ u(R^+)^2 + u_y(R^+) \left( 1 + \frac{u(R^+)}{\theta_i} \right) \right] + \frac{8\pi^2}{\theta_i^2} \left( \frac{2u_y^2(R^+)}{\theta_i R} w_{xx} - w_x u_y(R^+) - 2u_y(R^+) w_{xy} \right), \]

\[ \mathcal{G}_3(u, w) = e^{-\frac{1}{2}(x+1)} \frac{4\pi^2}{\theta_i^2} R \left[ u(R^+) u_y(R^+) + Le u_y^2(R^+) \left( 1 + Le u(R^+) \right) \right] + \frac{8\pi^2}{Le \theta_i^2} \left( \frac{2u_y^2(R^+)}{\theta_i R} w_{xx} - w_x u_y(R^+) - 2u_y(R^+) w_{xy} \right), \]

(7) For \( \xi \in [R, R + \delta] \) and \( \xi = \frac{B-R}{2}(x+1) + R, \)

\[ \mathcal{F}_1(u) = u(R^+) e^{-\frac{B}{2}(x+1)} + \frac{2}{B-R} u_x, \]

\[ \mathcal{F}_2(u) = \frac{1}{\theta_i} \left[ e^{-\frac{B}{2}(x+1)} \frac{4\pi^2}{\theta_i^2} R \left( u(R^+) u_y(R^+) + \left( 1 + \frac{u(R^+)}{\theta_i} \right) u_y^2(R^+) \right) \right] \]

\[ - \frac{8\pi^2}{\theta_i^2} \left( \frac{u_y(R^+)}{\theta_i} \right) \left( w_{xx} - 2u_y(R^+) w_{xy} + \frac{2u_y^2(R^+)}{\theta_i R} w_{xx} \right), \]

(8) For \( \xi \in [R + \delta, R + 2\delta] \) and \( \xi = \frac{B-R}{2}(x+1) + R, \)

\[ \mathcal{F}_1(u) = \frac{\theta_i}{\theta_i - u(R^+)} \left( \beta_R u(R^+) e^{-\frac{B}{2}(x+1)} + \frac{2}{B-R} u_x \right), \]

\[ \mathcal{F}_2(u) = \frac{1}{\theta_i - u(R^+)} \left[ e^{-\frac{B}{2}(x+1)} \frac{4\pi^2}{\theta_i^2} R \left( u(R^+) u_y(R^+) + \left( 1 + \frac{u(R^+)}{\theta_i} \right) u_y^2(R^+) \right) \right] \]

\[ - \frac{8\pi^2}{\theta_i^2} \left( \frac{u_y(R^+)}{\theta_i} \right) \left( w_{xx} - 2u_y(R^+) w_{xy} + \frac{2u_y^2(R^+)}{\theta_i R} w_{xx} \right), \]

\[ \mathcal{G}_1(u, w) = \frac{\theta_i}{\theta_i - u(R^+)} \left[ e^{-\frac{B}{2}(x+1)} \frac{4\pi^2}{\theta_i^2} R \left( u(R^+) u_y(R^+) + \left( 1 + \frac{u(R^+)}{\theta_i} \right) u_y^2(R^+) \right) \right] \]

\[ - \frac{16\pi^2}{\theta_i^2} \left[ \frac{u_y^2(R^+)}{\theta_i R} \left( w_{xx} - 2u_y(R^+) w_{xy} + \frac{2u_y^2(R^+)}{\theta_i R} w_{xx} \right) \right], \]

\[ \mathcal{G}_2(u, w) = \frac{\theta_i}{\theta_i - u(R^+)} \left[ e^{-\frac{B}{2}(x+1)} \frac{4\pi^2}{\theta_i^2} R \left( u(R^+) u_y(R^+) + \left( 1 + \frac{u(R^+)}{\theta_i} \right) u_y^2(R^+) \right) \right] \]

\[ - \frac{4\pi^2}{\theta_i^2} \left( \frac{u_y(R^+)}{\theta_i} \right) \left( w_{xx} - 2u_y(R^+) w_{xy} + \frac{2u_y^2(R^+)}{\theta_i R} w_{xx} \right), \]

\[ \mathcal{G}_3(u, w) = \frac{\theta_i}{\theta_i - u(R^+)} \left[ e^{-\frac{B}{2}(x+1)} \frac{4\pi^2}{\theta_i^2} R \left( u(R^+) u_y(R^+) + \left( 1 + \frac{u(R^+)}{\theta_i} \right) u_y^2(R^+) \right) \right] \]

\[ - \frac{4\pi^2}{\theta_i^2} \left( \frac{u_y(R^+)}{\theta_i} \right) \left( w_{xx} - 2u_y(R^+) w_{xy} + \frac{2u_y^2(R^+)}{\theta_i R} w_{xx} \right). \]
\[
+ 2(1 + \beta_R \text{Le}) \text{Le} u^2(R^+)
\]
\[
+ \frac{8\pi^2 u^2(R^+)}{\text{Le} \theta_i \ell^2} \left( w - \frac{4\beta_R}{B - R} w_x + \frac{2(\beta_R^2 - 1)}{(B - R)^2} w_{xx} \right) + \frac{8u(R^+)}{\text{Le}(B - R)^2} w_{xx} \right).}

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