Mass problems and
intuitionistic higher-order logic

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First draft: February 7, 2014
This draft: August 13, 2014

Abstract
In this paper we study a model of intuitionistic higher-order logic which we call the Muchnik topos. The Muchnik topos may be defined briefly as the category of sheaves of sets over the topological space consisting of the Turing degrees, where the Turing cones form a base for the topology. We note that our Muchnik topos interpretation of intuitionistic mathematics is an extension of the well known Kolmogorov/Muchnik interpretation of intuitionistic propositional calculus via Muchnik degrees, i.e., mass problems under weak reducibility. We introduce a new sheaf representation of the intuitionistic real numbers, the Muchnik reals, which are different from the Cauchy reals and the Dedekind reals. Within the Muchnik topos we obtain a choice principle \( \forall x \exists y A(x, y) \Rightarrow \exists w \forall x A(x, wx) \) and a bounding principle \( \forall x \exists y A(x, y) \Rightarrow \exists z \forall x \exists y (y \leq_T (x, z) \wedge A(x, y)) \) where \( x, y, z \) range over Muchnik reals, \( w \) ranges over functions from Muchnik reals to Muchnik reals, and \( A(x, y) \) is a formula not containing \( w \) or \( z \).

For the convenience of the reader, we explain all of the essential background material on intuitionism, sheaf theory, intuitionistic higher-order logic, Turing degrees, mass problems, Muchnik degrees, and Kolmogorov’s calculus of problems. We also provide an English translation of Muchnik’s 1963 paper on Muchnik degrees.

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\(^1\)Simpson’s research was partially supported by the Eberly College of Science at the Pennsylvania State University, and by Simons Foundation Collaboration Grant 276282.
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1 Introduction

1.1 Intuitionism and the calculus of problems

1.1.1 Constructivism

In the early part of the 20th century, foundations of mathematics was dominated by Georg Cantor’s set theory and David Hilbert’s program of finitistic reductionism. The harshest critics of set theory and Hilbert’s program were the constructivists. Among the various constructivist schools were intuitionism, proposed by L. E. J. Brouwer in 1907; predicativism, proposed by Hermann Weyl in 1918; finitism, proposed by Thoralf Skolem in 1923; constructive recursive mathematics, proposed by Andrei Andrejevich Markov in 1950; and Bishop-style constructivism, proposed by Errett Bishop in 1967. Also among the constructivists were many other prominent mathematicians including Leopold Kronecker (1823–1891), who is sometimes regarded as “the first constructivist,” René Louis Baire (1874–1932), Emile Borel (1871–1956), Nikolai Nikolaevich Lusin (1883–1950) and Jules Henri Poincaré (1854–1913). For more about the various schools of constructivism and their history, see [33] and [34, Chapter 1].

1.1.2 Brouwer’s intuitionism

Intuitionism is a constructive approach to mathematics proposed by Brouwer. The philosophical basis of intuitionism was spelled out in Brouwer’s 1907 Ph.D. thesis, entitled “On the foundations of mathematics.” The mathematical consequences were developed in Brouwer’s subsequent papers, 1912–1928.

The following is quoted from [34, Chapter 1].

The basic tenets of Brouwer’s intuitionism are as follows.

1. Mathematics deals with mental constructions, which are immediately grasped by the mind; mathematics does not consist in the formal manipulation of symbols, and the use of mathematical language is a secondary phenomenon, induced by our limitations (when compared with an ideal mathematician with unlimited memory and perfect recall), and the wish to communicate our mathematical constructions with others.

2. It does not make sense to think of truth and falsity of a mathematical statement independently of our knowledge concerning the statement. A statement is true if we have a proof of it, and false if we can show that the assumption that there is a proof for the statement leads to a contradiction. For an arbitrary statement we can therefore not assert that it is either true or false.

3. Mathematics is a free creation: it is not a matter of mentally reconstructing, or grasping the truth about mathematical objects existing independently of us.
In Brouwer’s view, mathematics allows the construction of mathematical objects on the basis of intuition. Mathematical objects are mental constructs, and mathematics is independent of logic and cannot be founded upon the axiomatic method. In particular, Brouwer rejected Hilbert’s formalism and Cantor’s set theory.

An important feature of Brouwer’s work was weak counterexamples, introduced to show that certain statements of classical mathematics are not intuitionistically acceptable. A weak counterexample to a statement \( A \) is not a counterexample in the strict sense, but rather an argument to the effect that any intuitionistic proof of \( A \) would have to include a solution of a mathematical problem which is as yet unsolved.

In particular, the principle of the excluded middle (PEM), \( A \lor \neg A \), is valid in classical logic, but to accept it intuitionistically we would need a universal method for obtaining, for any \( A \), either a proof of \( A \) or a proof of \( \neg A \), i.e., a method for obtaining a contradiction from a hypothetical proof of \( A \). But if such a universal method were available, we would also have a method to decide the truth or falsity of statements \( A \) which have not yet been proved or refuted (e.g., \( A \equiv \text{“there are infinitely many twin primes”} \)), which is not the case. Thus we have a weak counterexample to PEM.

The above argument shows that PEM is not intuitionistically acceptable. However, intuitionists may accept certain special cases or consequences of PEM. In particular, since we cannot hope to find a proof of \( \neg (A \lor \neg A) \), it follows that \( \neg \neg (A \lor \neg A) \) is intuitionistically acceptable.

Excessive emphasis on weak counterexamples has sometimes created the impression that intuitionism is mainly concerned with refutation of principles of classical mathematics. However, Brouwer introduced a number of other innovations, such as choice sequences; see [5, Chapter 3] and [34, Chapters 4 and 12]. After 1912 Brouwer developed what has come to be known as Brouwer’s program, which provided an alternative perspective on foundations of mathematics, parallel to Hilbert’s program. For more on the history of intuitionism and Brouwer’s work, see [33] and [34, Chapter 1].

### 1.1.3 Kolmogorov’s calculus of problems

The great mathematician Andrei Nikolaevich Kolmogorov published two papers on intuitionism.

In Kolmogorov’s 1925 paper [15] he introduces minimal propositional calculus, which is strictly included in intuitionistic propositional calculus. Starting with minimal propositional calculus, one can add \( A \Rightarrow (\neg A \Rightarrow B) \) to get intuitionistic propositional calculus, and then one can add \( (\neg \neg A) \Rightarrow A \) to get classical propositional calculus. Furthermore, a propositional formula \( A \) is classically provable if and only if \( \neg \neg A \) is intuitionistically provable. This translation of classical to intuitionistic propositional calculus, due to Kolmogorov [15], predates the double-negation translations of Gödel and Gentzen.

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2On the contrary, logic is an application or part of mathematics (according to Brouwer).

3See also the English translation [16]
In Kolmogorov’s 1932 paper [14] he gives a natural but non-rigorous interpretation of intuitionistic propositional calculus, called the calculus of problems. Each proposition is regarded as a problem, and logically compound propositions are obtained by combining simpler problems. If $A$ and $B$ are problems, then:

1. $A \land B$ is the problem of solving both problem $A$ and problem $B$;
2. $A \lor B$ is the problem of solving either problem $A$ or problem $B$;
3. $A \rightarrow B$ is the problem of solving problem $B$ given a solution of problem $A$, i.e., of reducing problem $B$ to problem $A$; and
4. $\neg A$ is the problem of showing that problem $A$ has no solution;

but Kolmogorov does not give a rigorous definition of “problem.” For further discussion of these papers of Kolmogorov, see [3].

Arend Heyting was one of Brouwer’s principal students. His primary contribution to intuitionism was, ironical as it may sound, the formalization of intuitionistic logic and arithmetic. Heyting also proposed what is now called the proof interpretation for intuitionistic logic. In this interpretation, the meaning of a proposition $A$ is given by explaining what constitutes a proof of $A$, and proofs of a logically compound $A$ are explained in terms of proofs of its constituents. A version of this interpretation is described in [34, Chapter 1].

While Kolmogorov’s and Heyting’s work were independent of each other, they both acknowledged similarities between the calculus of problems and the proof interpretation. However, they regarded these respective interpretations as distinct. Later, in 1958, Heyting insisted that the two interpretations are practically the same and also extended them to predicate calculus. Since then, the two interpretations have been treated as the same and are widely known as the Brouwer/Heyting/Kolmogorov or BHK interpretation. However, as pointed out in [7], there are subtle differences between the two.

1.1.4 Other interpretations of intuitionism

Some other interpretations of intuitionistic propositional and predicate calculus are as follows:

- Algebraic semantics, widely known as Heyting algebra semantics, were probably first used by Stanislaw Jaśkowski in 1936.
- Topological semantics were implicit in Marshall Harvey Stone’s work published in 1937 and were introduced explicitly by Alfred Tarski in 1938.
- Beth models were introduced by Evert Willem Beth in 1956.
- Kripke models were introduced by Saul Aaron Kripke in 1965.

\footnote{See also the English translation [17].}
These interpretations provide a great many models of intuitionism with widely varying properties. Experts will recognize that our Muchnik topos may be viewed from various perspectives as a Kripke model, a topological model, and a Heyting algebra model.

1.2 Higher-order logic and sheaf semantics

1.2.1 Higher-order logic

Higher-order logic is a kind of logic where, in addition to quantifiers over objects, one has quantifiers over pairs of objects, sets of objects/pairs, sets of sets of objects/pairs, functions from objects to objects, functions from functions to functions, and so on. This augmentation of so-called first-order logic increases its expressive power and is a useful framework for certain foundational studies.

Higher-order logic calls for a many-sorted or typed language. In Subsection 2.2 below, we provide a detailed definition of the language of higher-order logic. This language together with appropriate axioms and rules of inference is sufficiently rich to permit the development of virtually all of intuitionistic mathematics.

1.2.2 Sheaf semantics for intuitionistic higher-order logic

Sheaf theory originated in the mid-20th century in a geometrical context. Subsequently it spread to many branches of mathematics including complex analysis, algebraic geometry, algebraic topology, differential equations, algebra, category theory, mathematical logic, and mathematical physics. Sheaf theory may be viewed as a general tool which facilitates passage from local properties to global properties. For more on the history of sheaf theory, see Gray [11].

The connection between sheaves and intuitionistic higher-order logic came from several sources. An important source was Dana Scott’s topological model of intuitionistic analysis [27, 28]. Another important source was category theory, an abstract approach to mathematics which was introduced by Samuel Eilenberg and Saunders Mac Lane in the context of algebraic topology. For an introduction to category theory, see [21]. Alexander Grothendieck and his coworkers gave a general definition of sheaves over sites (rather than merely over topological spaces) and were thus led to a class of categories known as Grothendieck topoi. Francis Lawvere realized that these categories provide enough structure to interpret intuitionistic higher-order logic. In collaboration with Myles Tierney, Lawvere developed the notion of elementary topos, a generalization of Grothendieck topoi. For more on sites and Grothendieck topoi, see [22] and [34, Chapters 14, 15]. For more on topos theory in general, see [12, 19].

In this paper we avoid the complications of category theory and topos theory. Instead we follow the sheaf-theoretic approach of Dana Scott, Michael Fourman, and Martin Hyland [8, 9, 29]. Subsection 2.1 below provides a definition of the category \( \text{Sh}(T) \) of sheaves over a fixed topological space \( T \). Subsection 2.3 explains how to interpret intuitionistic higher-order logic in \( \text{Sh}(T) \).
1.3 Recursive mathematics and degrees of unsolvability

1.3.1 Constructive recursive mathematics

Constructive recursive mathematics (mentioned above in Subsection 1.1) is a constructivist school that started in the 1930s. It is based on an informal concept of algorithm or effective procedure, with the following features.

- An algorithm is a set of instructions of finite size. The instructions themselves are finite strings of symbols from a finite alphabet.
- There is a computing agent (human or machine), which can react to the instructions and carry out the computations.
- The computing agent has unlimited facilities for making, storing, and retrieving steps in a computation.
- The computation is always carried out deterministically in a discrete step-wise fashion, without use of continuous methods or analog devices. In other words, the computing agent does not need to make intelligent decisions or enter into an infinite process at any step.

On this basis, a $k$-place partial function $f : \subseteq \mathbb{N}^k \rightarrow \mathbb{N}$ is said to be effectively calculable if there is an effective procedure with the following properties.

1. Given a $k$-tuple $(m_1, \ldots, m_k)$ in the domain of $f$, the procedure eventually halts and returns a correct value of $f(m_1, \ldots, m_k)$.
2. Given a $k$-tuple $(m_1, \ldots, m_k)$ not in the domain of $f$, the procedure does not halt and does not return a value.

Several formalizations of this informal idea of effectively calculable functions were developed. Kurt Friedrich Gödel used the primitive recursive functions in his famous incompleteness proof in 1931, and then later introduced general recursive functions in 1934 following a suggestion of Jacques Herbrand. Along completely different lines, Alonzo Church introduced the $\lambda$-calculus, a theory formulated in the language of $\lambda$-abstraction and application, and Haskell Brooks Curry developed his combinatory logic. The equivalence of $\lambda$-calculus with combinatory logic was proved by John Barkley Rosser, Sr. The equivalence of the Herbrand/Gödel recursive functions with the $\lambda$-definable functions was proved by Church and by Stephen Cole Kleene in 1936.

Alan Turing in 1936–1937 defined an interesting class of algorithms, now called Turing machines, and argued convincingly that the class of effectively calculable functions coincides with the class of functions computable by Turing machines. Independently of Turing, Emil Leon Post developed a mathematical model for computation in 1936. The Church/Turing thesis, also known as Church’s thesis, states that for each of the above formalisms, the class of functions generated by the formalism coincides with the informally defined class
of effectively calculable functions. This was proposed in 1936 and is now al-
most universally accepted, although no formal proof is possible, because of the
non-rigorous nature of the informal definition of effective calculability.

The study of constructive recursive mathematics was continued by Markov
and his students. Again, the functions computable by Markov algorithms were
shown to be the same as the Herbrand/Gödel recursive functions and the Tur-
ing computable functions. Markov’s approach to recursive mathematics was
constructive, but he explicitly accepted the following consequence of PEM:

“If it is impossible that an algorithmic computation does not termi-
nate, then it does terminate.”

This principle, known as Markov’s principle, was rejected by the intuitionists.
We comment further on Markov’s principle in Subsection 3.2 below.

As noted in [33], the discovery of precise definitions of effective calculability
and the Church/Turing thesis in the 1930’s had no effect on the philosophical
basis of intuitionism. Each of these definitions describes algorithms in terms
of a specific language, which is contrary to Brouwer’s view of mathematics as
the languageless activity of the ideal mathematician. Turing’s analysis is not
tied to a specific formalism, but his arguments are based on manipulation of
symbols and appeals to physical limitations on computing. Such arguments are
incompatible with Brouwer’s idea of mathematics as a free creation.

Our discussion above is based on [34, Chapter 1] and on [6, 26, 33].

1.3.2 Unsolvable problems and Turing degrees

A convincing example of a function which is not effectively calculable was given
by Turing in 1936 via the halting problem. Turing proved that there is no Turing
machine program which decides whether or not a given Turing machine program
will eventually halt. This was the first example of an unsolvable decision prob-
lem. Soon afterward, many other mathematical decision problems were shown
to be unsolvable, for instance Hilbert’s 10th problem (the problem of deciding
whether a given Diophantine equation has a solution in integers) and the word
problem for groups.

Eventually it became desirable to compare the amounts of unsolvability in-
herent in various unsolvable problems. Informally and vaguely, a problem A is
said to be solvable relative to a problem B if there exists a Turing algorithm
which provides a solution of A given a solution of B. If in addition B is not
solvable relative to A, then B is strictly more unsolvable than A, i.e., problem
B has a strictly greater degree of unsolvability than problem A.

The concept of oracle machines, described by Turing in 1939, gave a means
of comparing unsolvable problems. In 1944 Emil Post introduced the rigorous
notion of Turing reducibility and Turing degrees as a formalization of degrees of
unsolvability associated with decision problems. It was shown that the Turing
degrees form an upper semi-lattice, i.e., a partially ordered set in which any
finite set has a least upper bound. See [6, 26] and Subsection 5.1 below.
1.3.3 Mass problems

In order to formalize Kolmogorov’s calculus of problems, Yu. T. Medvedev [23] introduced mass problems. A mass problem is a subset of the Baire space $\mathbb{N}^\mathbb{N} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$. A mass problem is identified with its set of solutions. Informally, to “solve” a mass problem $P$ means to “find” or “construct” an element of the set $P \subseteq \mathbb{N}^\mathbb{N}$. Formally, if $P$ and $Q$ are mass problems, $P$ is said to be strongly reducible or Medvedev reducible to $Q$, written $P \leq_s Q$, if there exists an effectively calculable partial functional from the Baire space to itself which maps each element of $Q$ to some element of $P$. It can be shown that $\leq_s$ is a reflexive and transitive relation on the powerset of $\mathbb{N}^\mathbb{N}$. The strong degree or Medvedev degree of a mass problem $P$, denoted $\text{deg}_s(P)$, is the equivalence class consisting of all mass problems $Q$ which are strongly equivalent to $P$, i.e., $P \leq_s Q$ and $Q \leq_s P$. Following Kolmogorov’s ideas [14] concerning the calculus of problems, Medvedev proved rigorously that the collection of all strong degrees, denoted $D_s$, is a model of intuitionistic propositional calculus.

Later Albert Abramovich Muchnik [24]$^5$ introduced a variant notion of reducibility for mass problems, known as weak reducibility or Muchnik reducibility. A mass problem $P$ is said to be weakly reducible to a mass problem $Q$, written $P \leq_w Q$, if for each $g \in Q$ there exists an effectively calculable partial functional which maps $g$ to some $f \in P$. Again, $\leq_w$ is a reflexive and transitive relation on the powerset of $\mathbb{N}^\mathbb{N}$. The weak degree or Muchnik degree of a mass problem $P$, denoted $\text{deg}_w(P)$, is the equivalence class consisting of all mass problems $Q$ which are weakly equivalent to $P$, i.e., $P \leq_w Q$ and $Q \leq_w P$. Still following Kolmogorov’s ideas [14], Muchnik proved that the collection of all weak degrees, denoted $D_w$, is a model of intuitionistic propositional calculus.

Thus each of $D_w$ and $D_s$ provides a rigorous implementation of Kolmogorov’s non-rigorous calculus of problems. Muchnik [24] says that the difference between weak and strong reducibility of mass problems is analogous to the difference between proving the existence of a solution of a differential equation versus effectively finding such a solution.

Subsection 5.1 below provides further details on mass problems and Muchnik degrees. For a more extensive discussion, see [31]. For more on Muchnik degrees and their relationship to intuitionistic propositional calculus, see [18, 30, 32].

The purpose of this paper is to extend Muchnik’s interpretation of intuitionistic propositional calculus to intuitionistic higher-order logic. The extension is defined in terms of a sheaf model based on the Muchnik degrees. We call our sheaf model the Muchnik topos. We feel that our study of the Muchnik topos helps to strengthen the connection between two important subjects, intuitionism and degrees of unsolvability.

There is another line of research, known as realizability, which was initiated by Kleene in 1945 [13] and further developed by other researchers, especially Martin Hyland and Jaap van Oosten. Both the realizability interpretation and our Muchnik topos interpretation provide close connections between intuitionism and recursion theory. However, these two interpretations are quite different.

$^5$An English translation of Muchnik’s paper is included as an appendix to this paper.
One may draw the following analogy:

\[
\frac{\text{Medvedev reducibility}}{\text{realizability topos}} = \frac{\text{Muchnik reducibility}}{\text{Muchnik topos}}.
\]

For a historical account and survey of realizability, see [36]. For recent work on the realizability topos, see [20, 37].

### 1.4 Outline of this paper

The plan of this paper is as follows.

Sections 2 through 4 consist of background material concerning sheaf models and intuitionism. In Section 2 we describe sheaves (a.k.a., sheaves of sets) over topological spaces, and we explain how the sheaves over any fixed topological space form a model of intuitionistic higher-order logic. In Section 3 we discuss sheaf models over topological spaces of a particular kind, namely, poset spaces. We also discuss a choice principle which fails in some sheaf models but which holds in sheaf models over poset spaces and over the Baire space. In Section 4 we explain how the various number systems and the Baire space are standardly represented as sheaves within sheaf models of intuitionistic mathematics.

The heart of this paper is Section 5. In Subsection 5.1 we review the definitions of Turing degrees and Muchnik degrees, and we note that Muchnik degrees can be identified with upwardly closed sets of Turing degrees. We then define the Muchnik topos to be the sheaf model over the poset of Turing degrees. In Subsection 5.2 we introduce a new representation of the intuitionistic real number system, which we call the Muchnik reals. The idea is that a Muchnik real “comes into existence” only when we have enough Turing oracle power to compute it. In Subsection 5.3 we prove a choice principle and a bounding principle for the Muchnik reals. Thus it emerges that intuitionistic analysis based on the Muchnik reals bears some formal similarity to recursive analysis.

In an Appendix we provide an English translation of Muchnik’s paper [24]. This is the paper where Muchnik defined the Muchnik degrees and used them to interpret intuitionistic propositional calculus along the lines which had suggested by Kolmogorov. This paper [24] is important for us, because our Muchnik topos interpretation may be viewed as a natural extension of Muchnik’s interpretation, from intuitionistic propositional calculus [24, Section 1] to intuitionistic mathematics as a whole.

### 2 Sheaves and intuitionistic higher-order logic

In this section we provide background material on sheaf theory and intuitionistic higher-order logic. Our main references are [1] and [34, Chapter 14].

#### 2.1 Sheaves over a topological space

**Definition 2.1.** Let \( T \) be a topological space. Let \( \Omega = \{U \subseteq T \mid U \text{ is open}\} \). A sheaf over \( T \) is an ordered triple \( M = (M, E_M, |_M) \) (we omit the subscripts
on $E$ and $|$ when there is no chance of confusion), where $M$ is a set and $E$ and $|$ are functions, $E : M \to \Omega$ and $\cdot : M \times \Omega \to M$, with the following properties.

1. $a \upharpoonright E(a) = a$ for all $a \in M$.
2. $E(a \upharpoonright U) = E(a) \cap U$ for all $a \in M$ and all $U \in \Omega$.
3. $(a \upharpoonright U) \upharpoonright V = a \upharpoonright (U \cap V)$ for all $a \in M$ and all $U, V \in \Omega$.
4. $M$ is partially ordered by letting $a \leq b$ if and only if $a = b \upharpoonright E(a)$.
5. Say that $a, b \in M$ are compatible if $a \upharpoonright E(b) = b \upharpoonright E(a)$. Say that $C \subseteq M$ is compatible if the elements of $C$ are pairwise compatible. Then, any compatible set $C \subseteq M$ has a supremum or least upper bound with respect to $\leq$, denoted sup $C$. That is, for all $d \in M$ we have sup $C \leq d$ if and only if $a \leq d$ for all $a \in C$.

Elements of a sheaf $M$ are called sections of $M$. A global section is a section $a$ such that $E(a) = T$. The operations $E$ and $|$ are called extent and restriction respectively. Thus, for any $a \in M$ and $U \in \Omega$, $E(a) \in \Omega$ is the extent of $a$ and $a \upharpoonright U \in M$ is the restriction of $a$ to $U$.

**Example 2.2.** A good example of a sheaf over $T$ is

$$C_0(T, X) = \{a : \text{dom}(a) \to X \mid \text{dom}(a) \in \Omega, a \text{ is continuous}\}$$

where $X$ is any topological space, with $E$ and $|$ given by $E(a) = \text{dom}(a)$ is the domain\(^6\) of $a$, and $a \upharpoonright U = a \upharpoonright U$ is the restriction of $a$ to $U \cap \text{dom}(a) \in \Omega$.

**Example 2.3.** $\Omega$ itself is a sheaf over $T$, with $E(U) = U$ and $U \upharpoonright V = U \cap V$ for all $U, V \in \Omega$. Note that $\Omega \cong C_0(T, \{0\})$ where $\{0\}$ is the one-point space.

**Example 2.4.** Let $T$ and $\Omega$ be as in Definition 2.1. We define

$$\Omega_1 = \{(V, U) \mid V, U \in \Omega, V \subseteq U\}$$

with $E$ and $|$ given by $E((V, U)) = U$ and $(V, U) \upharpoonright W = (V \cap W, U \cap W)$ for all $(V, U) \in \Omega_1$ and all $W \in \Omega$. It can be shown that $\Omega_1$ is a sheaf over $T$. In fact, $\Omega_1 \cong C_0(T, S)$ where $S$ is the Sierpiński space, i.e., the topological space $\{0, 1\}$ with open sets $\emptyset, \{1\}, \{0, 1\}$. For details see [1, pages 16–17].

**Definition 2.5.** Let $(M, E_M, \upharpoonright_M)$ and $(N, E_N, \upharpoonright_N)$ be sheaves over $T$. We say that $(N, E_N, \upharpoonright_N)$ is a subsheaf of $(M, E_M, \upharpoonright_M)$ if $N \subseteq M$ and $E_N$ and $\upharpoonright_N$ are inherited from $M$, i.e., $E_N(a) = E_M(a)$ and $a \upharpoonright_N U = a \upharpoonright_M U$ for all $a \in N$ and all $U \in \Omega$.

**Example 2.6.** Let $X$ be a topological space. A function $a$ from a subset of $T$ into $X$ is said to be locally constant if for every $t \in \text{dom}(a)$ there exists an open

---

\(^6\)For any function $a$ we write $\text{dom}(a) = \text{the domain of } a$, and $\text{rng}(a) = \text{the range of } a$. 

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set $V \in \Omega$ such that $t \in V$ and $a$ is constant on $V \cap \text{dom}(a)$. Clearly locally constant functions are continuous. Let

$$
C^\text{lc}_\omega(T, X) = \{ a \in C_\omega(T, X) \mid a \text{ is locally constant} \}.
$$

Then $C^\text{lc}_\omega(T, X)$ is a subsheaf of $C_\omega(T, X)$. However,

$$
C^\text{c}_\omega(T, X) = \{ a \in C_\omega(T, X) \mid a \text{ is constant} \}
$$
is in general not a sheaf, hence not a subsheaf of $C^\text{lc}_\omega(T, X)$.

**Lemma 2.7.** Let $M$ be a sheaf over $T$.

1. For all $a, b \in M$, if $a \leq b$ then $E(a) \subseteq E(b)$.
2. If $C$ is a compatible subset of $M$, then $c = \sup C$ if and only if $E(c) = \bigcup_{a \in C} E(a)$ and $a \leq c$ for all $a \in C$.
3. If $\{U_i \mid i \in I\}$ is a family of open subsets of $T$, and if $c \in M$, then $\{c \upharpoonright U_i \mid i \in I\}$ is a compatible subset of $M$ and $\sup_{i \in I} (c \upharpoonright U_i) = c \upharpoonright \bigcup_{i \in I} U_i$.
4. Every bounded subset of $M$ is compatible and has a least upper bound.

**Proof.** The proof is straightforward. See [1, pages 13–16].

**Definition 2.8.** Let $M$ and $N$ be sheaves over $T$.

1. The **product sheaf** is

$$
M \times N = \{ (a, b) \mid a \in M, b \in N, E(a) = E(b) \}
$$

with $E$ and $\upharpoonright$ given by $E((a, b)) = E(a)$ and $(a, b) \upharpoonright U = (a \upharpoonright U, b \upharpoonright U)$.
2. For all $U \in \Omega$ the **restriction sheaf** is

$$
M \upharpoonright U = \{ a \upharpoonright U \mid a \in M \} = \{ a \in M \mid E(a) \subseteq U \},
$$

with $E$ and $\upharpoonright$ inherited from $M$.
3. A **sheaf morphism** $M \xrightarrow{\varphi} N$ is a mapping $\varphi : M \to N$ satisfying $E(\varphi(a)) = E(a)$ and $\varphi(a \upharpoonright U) = \varphi(a) \upharpoonright U$ for all $a \in M$ and all $U \in \Omega$.
4. The **function sheaf** is

$$
N^M = \{ (\varphi, U) \mid U \in \Omega, M \upharpoonright U \xrightarrow{\varphi} N \upharpoonright U \}
$$

with $E$ and $\upharpoonright$ defined by $E((\varphi, U)) = U$ and $(\varphi, U) \upharpoonright V = (\varphi \upharpoonright V, U \cap V)$ where $(\varphi \upharpoonright V)(a) = \varphi(a) \upharpoonright V$ for all $a \in M \upharpoonright (U \cap V)$ and all $V \in \Omega$.

**Remark 2.9.** It can be shown that the product sheaf, the restriction sheaf, and the function sheaf are indeed sheaves over $T$. For details see [1, pages 20–28].
Definition 2.10. For any sheaf $M$ we define the power sheaf $P(M)$ to be the function sheaf $\Omega^1_{\Omega_1}$ where $\Omega_1$ is as in Example 2.4.

Remark 2.11. An alternative definition of the power sheaf appears in [9, 34]. It can be shown that this alternative definition is equivalent to our Definition 2.10. For details see [1, pages 28–35].

Theorem 2.12. For any sheaf $M$ there is a natural one-to-one correspondence between the subsheaves of $M$ and the global sections of $P(M)$.

Proof. See [1, pages 35–39].

Remark 2.13. Given a topological space $T$, let $\text{Sh}(T)$ be the category whose objects are the sheaves over $T$ and whose morphisms are the sheaf morphisms over $T$. The category $\text{Sh}(T)$ is one of the most basic examples of a topos.

2.2 The language of higher-order logic

We now describe a many-sorted language $L$ for intuitionistic higher-order logic.

Definition 2.14. The language $L$ is defined as follows.

1. The sorts of $L$ are generated as follows.
   (a) There is a collection of ground sorts.$^7$
   (b) If $\sigma$ and $\tau$ are sorts, then so is $\sigma \times \tau$, the product sort of $\sigma$ and $\tau$.
   (c) If $\sigma$ and $\tau$ are sorts, then so is $\sigma \rightarrow \tau$, the function sort from $\sigma$ to $\tau$.
   (d) If $\sigma$ is a sort, then so is $P\sigma$, the power sort of $\sigma$.

2. The symbols of $L$ are:
   (a) for each sort $\sigma$, an infinite supply of variables $x^\sigma, y^\sigma, \ldots$;
   (b) for each sort $\sigma$, an existence predicate $E^\sigma$ of type $(\sigma)$, an equality predicate $=^\sigma$ of type $(\sigma, \sigma)$, and a membership predicate $\in^\sigma$ of type $(\sigma, P\sigma)$;
   (c) for all sorts $\sigma$ and $\tau$, a pairing operator $\pi^{\sigma, \tau}$ of type $(\sigma, \tau, \sigma \times \tau)$ and projection operators $\pi_1^{\sigma, \tau}$ and $\pi_2^{\sigma, \tau}$ of types $(\sigma \times \tau, \sigma)$ and $(\sigma \times \tau, \tau)$ respectively, and an application operator $\text{Ap}^{\sigma, \tau}$ of type $(\sigma \rightarrow \tau, \sigma, \tau)$;
   (d) propositional connectives $\neg, \land, \lor, \Rightarrow, \Leftrightarrow$;
   (e) quantifiers $\forall, \exists$.

When there is no danger of confusion, we may omit superscripts indicating sorts and types.

3. The terms of $L$ are generated as follows.

$^7$A.k.a., basic sorts or primitive sorts.
(a) Each variable of sort $\sigma$ is a term of sort $\sigma$.
(b) If $s$ and $t$ are terms of sort $\sigma$ and $\tau$ respectively, and if $\pi$ is of type $(\sigma, \tau, \sigma \times \tau)$, then $\pi st$ is a term of sort $\sigma \times \tau$.
(c) If $r$ is a term of sort $\sigma \times \tau$, and if $\pi_1, \pi_2$ are of type $(\sigma \times \tau, \sigma)$ and $(\sigma \times \tau, \tau)$ respectively, then $\pi_1 r$ and $\pi_2 r$ are terms of sort $\sigma$ and $\tau$ respectively.
(d) If $s$ and $t$ are terms of sort $\sigma$ and $\sigma \rightarrow \tau$ respectively, and if $A p$ is of type $(\sigma \rightarrow \tau, \sigma, \tau)$, then $A p ts$ is a term of sort $\tau$. We usually write $ts$ instead of $A p ts$.

4. The atomic formulas of $L$ are:
   (a) $r = s$, where $r$ and $s$ are terms of sort $\sigma$ and $=$ is of type $(\sigma, \sigma)$;
   (b) $s \in t$, where $s$ and $t$ are terms of sort $\sigma$ and $P \sigma$ respectively, and $\in$ is of type $(\sigma, P \sigma)$;
   (c) $Es$, where $s$ is a term of sort $\sigma$ and $E$ is of type $(\sigma)$.

5. The formulas of $L$ are generated as follows.
   (a) Each atomic formula is a formula.
   (b) If $A, B$ are formulas then so are $\neg A$, $A \land B$, $A \lor B$, $A \Rightarrow B$, $A \leftrightarrow B$.
   (c) If $A$ is a formula and $x$ is a variable, then $\forall x A$ and $\exists x A$ are formulas.

Definition 2.15. The complexity of a formula $A$ is the number of occurrences of propositional connectives $\neg, \land, \lor, \Rightarrow, \leftrightarrow$ and quantifiers $\forall, \exists$ in $A$. An occurrence of a variable $x^\sigma$ in $A$ is said to be bound in $A$ if it is within the scope of a quantifier $\forall x^\sigma$ or $\exists x^\sigma$ in $A$. A variable $x^\sigma$ is said to be free in $A$ if at least one occurrence of $x^\sigma$ in $A$ is not bound in $A$. A formula $A$ is called a sentence if no variables are free in $A$.

Remark 2.16. For a more extensive discussion, see [35]. We could include additional predicates and operators in $L$, but they are not needed for our purpose.

2.3 Sheaf models of intuitionistic mathematics

In this subsection we explain how sheaves over topological spaces provide models of intuitionistic higher-order logic and intuitionistic mathematics.

Definition 2.17. Let $T$ be a topological space. Let $\mu$ be a mapping which assigns to each ground sort $\sigma$ of $L$ a sheaf $M_{\sigma}$ over $T$. We inductively extend $\mu$ to the compound sorts of $L$ by letting $M_{\sigma \times \tau} = M_{\sigma} \times M_{\tau}$ (product sheaf), $M_{\sigma \rightarrow \tau} = M_{\tau}^{M_{\sigma}}$ (function sheaf), and $M_{P \sigma} = P(M_{\sigma})$ (power sheaf). For each sort $\sigma$ of $L$ and each section $a \in M_{\sigma}$, we extend $L$ by adding a constant symbol $a = a^\sigma$ of sort $\sigma$, which is now also a term of sort $\sigma$. The extended language is denoted $L(\mu)$. A term of $L(\mu)$ is said to be closed if it contains no variables.

Definition 2.18. To each $L(\mu)$-sentence $A$ we assign a truth value $[A] \in \Omega$. 

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1. To each closed $L(\mu)$-term $s$ of sort $\sigma$, we assign a value $\llbracket s \rrbracket \in M_\sigma$.
   (a) If $a \in M_\sigma$, let $\llbracket a \rrbracket = a$.
   (b) If $s$ and $t$ are closed terms of sorts $\sigma$ and $\tau$ respectively, let $\llbracket \pi s t \rrbracket = (\llbracket s \rrbracket \uparrow E(\llbracket t \rrbracket), \llbracket t \rrbracket \uparrow E(\llbracket s \rrbracket))$.
   (c) If $r$ is a closed term of sort $\sigma \times \tau$, then $\llbracket r \rrbracket = (a, b)$ for some $(a, b) \in M_\sigma \times M_\tau$ and we let $\llbracket \pi_1 r \rrbracket = a$ and $\llbracket \pi_2 r \rrbracket = b$.
   (d) Suppose $t$ is a closed term of sort $\sigma \rightarrow \tau$ with $\llbracket t \rrbracket = (\varphi, U) \in M_\tau M_\sigma$. If $s$ is a closed term of sort $\sigma$, let $\llbracket \text{Ap} t s \rrbracket = \llbracket t s \rrbracket = \varphi(\llbracket s \rrbracket \uparrow U)$.

2. For atomic $L(\mu)$-sentences $A$, we define $\llbracket A \rrbracket \in \Omega$ as follows.
   (a) If $r$ and $s$ are closed terms of sort $\sigma$, let
   $$\llbracket r = s \rrbracket = \bigcup \{ U \in \Omega \mid U \subseteq E(\llbracket r \rrbracket) \cap E(\llbracket s \rrbracket), \llbracket r \rrbracket \uparrow U = \llbracket s \rrbracket \uparrow U \}.$$ (b) If $s$ is a closed term of sort $\sigma$, let $\llbracket E^\sigma s \rrbracket = \llbracket s = s \rrbracket = E(\llbracket s \rrbracket)$.
   (c) If $s$ is a closed term of sort $\sigma$ and $t$ is a closed term of sort $P\sigma$ with $\llbracket t \rrbracket = (\varphi, U) \in P(M_\sigma) = \Omega_1^M_\sigma$, let $\llbracket s \uparrow t \rrbracket \in V$ where $\varphi(\llbracket s \rrbracket \uparrow U) = (V, E(\llbracket s \rrbracket) \cap U)$.

3. For non-atomic $L(\mu)$-sentences $A$, we define $\llbracket A \rrbracket \in \Omega$ by induction on the complexity of $A$, using the notation $S^\circ = \text{interior of } S$.
   (a) Propositional connectives:
   $$\llbracket \neg A \rrbracket = (T \setminus \llbracket A \rrbracket)^\circ,$$
   $$\llbracket A \wedge B \rrbracket = \llbracket A \rrbracket \cap \llbracket B \rrbracket, \quad \llbracket A \lor B \rrbracket = \llbracket A \rrbracket \cup \llbracket B \rrbracket, \quad \llbracket A \rightarrow B \rrbracket = (\llbracket A \rrbracket \rightarrow \llbracket B \rrbracket) \text{ where } (U \rightarrow V) = ((T \setminus U) \cup V)^\circ, \quad \llbracket A \leftrightarrow B \rrbracket = \llbracket A \rightarrow B \rrbracket \cap \llbracket B \rightarrow A \rrbracket.$$
   (b) Quantifiers:
   $$\llbracket \exists x^\sigma A(x^\sigma) \rrbracket = \bigcup_{a \in M_\sigma} \llbracket E a \wedge A(a) \rrbracket,$$
   $$\llbracket \forall x^\sigma A(x^\sigma) \rrbracket = \left( \bigcap_{a \in M_\sigma} \llbracket E a \rightarrow A(a) \rrbracket \right)^\circ.$$

**Definition 2.19.** For $L(\mu)$-sentences $A$ we write $\text{Sh}(T, \mu) \models A$ to mean that $\llbracket A \rrbracket = T$. An $L$-formula $A$ is said to be valid for sheaf models if for all topological spaces $T$ and all $\mu : \sigma \mapsto M_\sigma$ as above, $\text{Sh}(T, \mu) \models$ the universal closure of $A$.

The following theorem says that the axioms and rules of intuitionistic higher-order logic are valid for sheaf models. Let IHOL be the formal system of intuitionistic higher-order logic as formulated in [29] and [34, Chapter 14].
Theorem 2.20. The axioms and rules of IHOL are valid for sheaf models.

Proof. See [9, Theorem 7.3] and [34, Theorem 5.15].

For instance, substitution of equals is intuitionistically valid, hence provable in IHOL, so we have:

Theorem 2.21. Let \( x \) be a variable of sort \( \sigma \), let \( r \) and \( s \) be closed \( L(\mu) \)-terms of sort \( \sigma \), and let \( A(x) \) be an \( L(\mu) \)-formula with no free variables other than \( x \). Then \( \text{Sh}(T, \mu) \models r = s \Rightarrow (A(r) \iff A(s)) \), hence \([r = s] \cap [A(r)] \subseteq [A(s)]\).

Proof. For a much more detailed proof, see [1, pages 42–48].

Remark 2.22. By [9, 34] we know that intuitionistic mathematics is formalizable in IHOL. Thus Theorem 2.20 may be viewed as saying that, for any topological space \( T \), \( \text{Sh}(T) \) is a model of intuitionistic mathematics. Such models are known as sheaf models.

3 Poset spaces and choice principles

In this section we discuss sheaf models over a special class of topological spaces, the so-called poset spaces. We show that some special cases of the axiom of choice are valid for sheaf models over poset spaces and over the Baire space.

3.1 Poset spaces

Definition 3.1. A poset\(^8\) is a non-empty set \( K \) together with a binary relation \( \leq \) on \( K \) which is reflexive, antisymmetric, and transitive. A set \( U \subseteq K \) is said to be upwardly closed if for all \( \alpha \in U \) and \( \beta \in K \), \( \alpha \leq \beta \) implies \( \beta \in U \). The upwardly closed subsets of \( K \) are the open sets of a topology on \( K \), the Alexandrov topology. A poset space is a poset endowed with the Alexandrov topology. The category of sheaves over a poset space \( K \) is denoted \( \text{Sh}(K) \).

Lemma 3.2. Let \( K \) be a poset space.

1. For any \( \alpha \in K \) there is a smallest open set containing \( \alpha \), namely,
   \[ U_\alpha = \{ \beta \in K | \alpha \leq \beta \}. \]

2. \( K \) is locally connected, i.e., for any \( \alpha \in K \) and any neighborhood \( U \) of \( \alpha \), there is a connected neighborhood of \( \alpha \) included in \( U \).

3. For all families of subsets of \( K \) we have \( (\bigcap_{i \in I} S_i)^\circ = \bigcap_{i \in I} S_i^\circ \).

4. If \( U \) is an open subset of \( K \), and if \( X \) is a \( T_1 \) space\(^9\), then every continuous function \( f : U \to X \) is locally constant.

---

\(^8\)I.e., a partially ordered set.

\(^9\)A \( T_1 \) space is a topological space in which every point is a closed set. Examples of \( T_1 \) spaces are \( \mathbb{R}, \mathbb{N}, \mathbb{N}^n \), etc.
Proof. The proof is straightforward. See [1, pages 50–51].

Definition 3.3. A poset $K$ is said to be directed if for all $\alpha, \beta \in K$ there exists $\gamma \in K$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Lemma 3.4. Let $U$ be an upward closed subset in a directed poset $K$. Let $X$ be any set. Then, any locally constant function $f : U \to X$ is constant.

Proof. The proof is straightforward. See [1, pages 51–52].

3.2 Choice principles over poset spaces

In this subsection we show that sheaf models over poset spaces satisfy certain special cases of the axiom of choice.

Definition 3.5. Let $\sigma$ and $\tau$ be $L$-sorts. The axiom of choice for $\sigma \to \tau$, denoted $\text{AC}(\sigma, \tau)$, is the universal closure of

$$(\forall x \exists y A(x, y)) \Rightarrow \exists w \forall x A(x, wx)$$

where $x, y, w$ are variables of sort $\sigma, \tau, \sigma \to \tau$ respectively, and $A(x, y)$ is any $L$-formula in which $w$ does not occur.

Remark 3.6. A model of intuitionistic higher-order logic cannot satisfy $\text{AC}(\sigma, \tau)$ for all sorts $\sigma, \tau$ unless it is also a model of classical higher-order logic. This is because, as shown in [4], the full axiom of choice implies PEM. However, as we shall see, models such as $\text{Sh}(T, \mu)$ may satisfy $\text{AC}(\sigma, \tau)$ for some particular choices of $\sigma$ and $\tau$.

Definition 3.7. Let $T$ be a topological space, and let $X$ be a set. As in Example 2.6, let $C^{lc}_o(T, X)$ be the sheaf of locally constant functions from open subsets of $T$ into $X$. We define $\tilde{X}^{sh} = C^{lc}_o(T, X)$. Note that for each $x \in X$ there is a global section $\tilde{x}$ of $\tilde{X}^{sh}$ which maps $T$ into $\{x\}$. The sheaf $\tilde{X}^{sh}$ is called a simple sheaf. See [9] and [34, page 782].

Theorem 3.8. Let $K$ be a poset space. If $M_\sigma$ is a simple sheaf over $K$, then $\text{Sh}(K, \mu)$ satisfies $\text{AC}(\sigma, \tau)$.

Proof. We may safely assume that $A(x, y)$ has no free variables other than $x$ and $y$. Letting $U = [\forall x \exists y A(x, y)]$, it will suffice to show that $U \subseteq [\exists w \forall x A(x, wx)]$.

Let $X$ be a set such that $M_\sigma = \tilde{X}^{sh}$. For each $x \in X$ we have $\tilde{x} \in \tilde{X}^{sh}$ and $E(\tilde{x}) = K$, hence

$$U = \left( \bigcap_{a \in \tilde{X}^{sh}} (E(a) \Rightarrow [\exists y A(a, y)]) \right)^a$$

$$\subseteq \bigcap_{x \in X} (E(\tilde{x}) \Rightarrow [\exists y A(\tilde{x}, y)])$$

$$= \bigcap_{x \in X} [\exists y A(\tilde{x}, y)] \quad \text{(because } E(\tilde{x}) = K)$$

$$= \bigcap_{x \in X} \bigcup_{b \in M_\tau} (E(b) \cap [A(\tilde{x}, b)])$$.
Fix $\alpha \in U$. Using the axiom of choice externally, we choose for each $x \in X$ a $b_x \in M_\tau$ such that $\alpha \in E(b_x) \cap [A(\hat{x}, b_x)]$. Since $E(b_x) \cap [A(\hat{x}, b_x)]$ is open, it follows by Lemma 3.2 that $U_\alpha \subseteq E(b_x) \cap [A(\hat{x}, b_x)]$. We shall now define a sheaf morphism from $\hat{X}^{sh} \mid U_\alpha$ into $M_\tau$. Let $a \in X^{sh} \mid U_\alpha$ be given. For all $x, y \in X$ such that $x \neq y$ we have $a^{-1}(x) \cap a^{-1}(y) \neq \emptyset$, hence the set $\{b_x \mid a^{-1}(x) \mid x \in X\} \subseteq M_\tau$ is compatible, hence the least upper bound

$$\varphi(a) = \sup_{x \in X} (b_x \mid a^{-1}(x)) \in M_\tau$$

exists. We have $E(a) \subseteq U_\alpha \subseteq E(b_x)$, hence $E(\varphi(a)) = \bigcup_{x \in X} E(b_x \mid a^{-1}(x)) = \bigcup_{x \in X} a^{-1}(x) = E(a)$ by Lemma 2.7. Moreover, for any open set $V \subseteq K$ we have $\varphi(a \mid V) = \sup_{x \in X} (b_x \mid (a \mid V)^{-1}(x)) = \sup_{x \in X} (b_x \mid (a^{-1}(x) \cap V)) = (\sup_{x \in X} (b_x \mid a^{-1}(x))) \mid V = \varphi(a) \mid V$. Thus $\varphi$ preserves extent and restriction, so we have a sheaf morphism

$$\hat{X}^{sh} \mid U_\alpha \xrightarrow{\varphi} M_\tau \mid U_\alpha,$$

i.e., $(\varphi, U_\alpha) \in M_\tau^{X^{sh}}$.

We claim that $U_\alpha \cap E(a) \subseteq [A(a, (\varphi, U_\alpha)a)]$ for all $a \in \hat{X}^{sh}$. To see this, fix $\beta \in U_\alpha \cap E(a)$. For some $x \in X$ we have $a(\beta) = x$, hence $U_\beta \subseteq a^{-1}(x) \subseteq E(a)$ and $\hat{x} \mid U_\beta = a \mid U_\beta$, hence $U_\beta \subseteq [\hat{x} = a]$. Moreover $b_x \mid U_\beta = \varphi(a \mid U_\alpha) \mid U_\beta$ and $U_\beta \subseteq E(b_x)$, hence $U_\beta \subseteq [b_x = \varphi(a \mid U_\alpha)]$, and clearly $U_\beta \subseteq U_\alpha \cap E(a) = [\varphi(a \mid U_\alpha) = (\varphi, U_\alpha)a]$. Therefore, from $U_\beta \subseteq [A(\hat{x}, b_x)]$ it follows by Theorem 2.21 that $U_\beta \subseteq [A(a, (\varphi, U_\alpha)a)]$, and this proves the claim.

Our claim easily implies that

$$U_\alpha \subseteq \bigcap_{a \in \hat{X}^{sh}} ((K \setminus E(a)) \cup [A(a, (\varphi, U_\alpha)a)])^\circ$$

$$= [\forall x A(x, (\varphi, U_\alpha)x)].$$

But then, since $E((\varphi, U_\alpha)) = U_\alpha$, we have

$$U_\alpha \subseteq E((\varphi, U_\alpha)) \cap [\forall x A(x, (\varphi, U_\alpha)x)]$$

$$\subseteq \bigcup_{(\psi, V) \in M^{X^{sh}}} (E((\psi, V) \cap [\forall x A(x, (\psi, V)x)]))$$

$$= [\exists w \forall x A(x, w)].$$

Since $\alpha \in U$ was arbitrary, we conclude that $U \subseteq [\exists w \forall x A(x, w)]$. This completes the proof of Theorem 3.8. \hfill $\square$

**Remark 3.9.** Theorem 3.8 fails for sheaf models over arbitrary topological spaces. In particular, see [1, pages 77–79] and [34, page 788] for a proof that $\text{AC}(\sigma, \sigma)$ fails in $\text{Sh}(\mathbb{R}, \mu)$ for $M_\sigma = \mathbb{N}^{sh}$. See also Remark 4.18 below.
Remark 3.10. One might think that Theorem 3.8 should hold whenever $M_\sigma$ is a subsheaf of a simple sheaf over $K$. However, the following example shows otherwise. Let $K = \{-\infty\} \cup \{-i \mid i \in \mathbb{N}\}$ with the natural linear ordering, $-\infty < -j < -i$ for all $i, j \in \mathbb{N}$ with $i < j$. Let $M_\sigma = \mathcal{C}_0(K, \{0\}) \uparrow \{-i \mid i \in \mathbb{N}\}$.

For all $m, n \in \mathbb{N}$ let $b_{m,n} \in \mathcal{C}_0(K, \mathbb{N})$ be the constant function with domain $\{-i \mid i < m\}$ and value $n$. Let $M_\tau$ be the subsheaf of $\mathcal{C}_0(K, \mathbb{N})$ consisting of all $b_{m,n}$ such that $m < n$. Note that for each sheaf morphism $(\varphi, U) \in M_\sigma \rightarrow M_\tau$ we have $U \subseteq \{-i \mid i \in \mathbb{N}\}$. Let $x, y, w$ be variables of sort $\sigma, \tau, \sigma \rightarrow \tau$ respectively.

Easy calculations show that $\llbracket \forall x \exists y (y = y) \rrbracket = K$ and $\llbracket \exists w \forall x (wx = wx) \rrbracket = \{-i \mid i \in \mathbb{N}\}$. Thus AC$(\sigma, \tau)$ fails in Sh$(K)$ for the formula $A(x, y) \equiv (y = y)$.

In the vein of Theorem 3.8 and Remark 3.9, we now call attention to another principle which is valid for sheaf models over poset spaces but not over arbitrary topological spaces.

Definition 3.11. Let $GMP(\sigma)$ be the universal closure of $(\forall x (A(x) \lor \neg A(x)) \land \neg \exists x A(x)) \Rightarrow \exists x A(x)$

where $x$ is a variable of sort $\sigma$ and $A(x)$ is any $L$-formula. As will become clear in Subsection 4.1, $GMP(\sigma)$ for $M_\sigma = \hat{\mathbb{N}}^{\text{sh}}$ amounts to Markov’s principle as discussed in [34, page 203] and in Subsection 1.3.1 above. Thus $GMP(\sigma)$ may be viewed as a generalized Markov principle.

Theorem 3.12. Let $K$ be a poset space. If $M_\sigma$ is a simple sheaf over $K$, then Sh$(K, \mu)$ satisfies $GMP(\sigma)$.

Proof. See [1, pages 71–74].

Remark 3.13. Neither Markov’s principle nor its generalization in Theorem 3.12 holds for sheaf models over arbitrary topological spaces $T$. In fact, Markov’s principle fails over $T = \{0, 1\}^\mathbb{N} = \text{the Cantor space}$. See [1, pages 69–71].

3.3 Choice principles over the Baire space

Despite Remark 3.9, Theorem 3.8 is valid for sheaf models over some topological spaces other than poset spaces. We now show that the Baire space $\mathbb{N}^{\mathbb{N}}$ is one such topological space.

Lemma 3.14. Given a collection $\mathcal{U}$ of open sets in $\mathbb{N}^{\mathbb{N}}$, we can find a collection $\mathcal{V}$ of open sets in $\mathbb{N}^{\mathbb{N}}$ such that

1. $\bigcup \mathcal{V} = \bigcup \mathcal{U}$,
2. for all $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$, and
3. for all $V, V' \in \mathcal{V}$, if $V \neq V'$ then $V \cap V' = \emptyset$.

Proof. For each finite sequence $p$ of natural numbers, let $V_p = \{ f \in \mathbb{N}^{\mathbb{N}} \mid p \text{ is an initial segment of } f \}$. Given $\mathcal{U}$ as in the lemma, let $\mathcal{V} = \{ V_p \mid p \text{ minimal such that } \exists U (U \in \mathcal{U} \text{ and } V_p \subseteq U) \}$. Clearly $\mathcal{V}$ has the desired properties.
Theorem 3.15. If $M_\sigma$ is a simple sheaf over $\mathbb{N}^\mathbb{N}$, then $\text{Sh}(\mathbb{N}^\mathbb{N}, \mu) \models \text{AC}(\sigma, \tau)$.

Proof. Let $X$ be a set such that $M_\sigma = \hat{X}^\text{sh}$. As in the proof of Theorem 3.8, let $U = [\forall x \exists y A(x, y)]$ and note that

$$U \subseteq \bigcap_{x \in X} \bigcup_{b \in M_\tau} (E(b) \cap [A(\bar{x}, b)]).$$

For each $x \in X$ apply Lemma 3.14 to get a pairwise disjoint collection $V_x$ of open sets such that $U \subseteq \bigcup V_x$ and for all $V \in V_x$ choose such a $b$ and let $b_x, V = b \upharpoonright V$. Clearly $\{b_x, V \mid V \in V_x\} \subseteq M_\tau$ is compatible, so for all $a \in \hat{X}^\text{sh} \upharpoonright U$ define

$$\varphi(a) = \sup \sup_{x \in X} (b_x, V \upharpoonright a^{-1}(x)) \in M_\tau.$$ 

The verification that $\hat{X}^\text{sh} \upharpoonright U \xrightarrow{\varphi} M_\tau \upharpoonright U$ is a sheaf morphism, the proof that $U \cap E(a) \subseteq [A(a, \varphi, U)a]$ for all $a \in \hat{X}^\text{sh}$, and the final verification that $U \subseteq [\exists w \forall x A(x, wx)]$, are similar to the corresponding parts of the proof of Theorem 3.8. For further details, see [1, pages 80–83].

Remark 3.16. A different proof of Theorem 3.15 for the special case $M_\sigma = \hat{\mathbb{N}}^\text{sh}$ is given in [8, page 289] and [34, page 787].

Remark 3.17. Our proof of Theorem 3.15 uses only the property of the Baire space which is stated in Lemma 3.14. Therefore, Theorem 3.15 holds for sheaf models over all topological spaces with this property.

4 Sheaf representations of the number systems

Let $T$ be a topological space. In this section we discuss the representation of the number systems $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ and the Baire space $\mathbb{N}^\mathbb{N}$ within the sheaf model $\text{Sh}(T)$.

4.1 The natural numbers

Recall from Subsection 3.2 that $\hat{\mathbb{N}}^\text{sh} = C^b_o(T, \mathbb{N})$ where $\mathbb{N}$ is the set of natural numbers. In this subsection we argue that $\hat{\mathbb{N}}^\text{sh}$ is appropriately viewed as representing the natural number system within $\text{Sh}(T)$.

Definition 4.1. A system is an ordered triple $(X, c, f)$ where $X$ is a set, $c \in X$, and $f : X \to X$. A Peano system is a system which satisfies $\forall x (fx \neq c)$ and $\forall x \forall y (f(x) = f(y) \Rightarrow x = y)$ and

$$\forall Y ((Y \subseteq X \land c \in Y \land \forall x (x \in Y \Rightarrow f(x) \in Y)) \Rightarrow Y = X).$$

Theorem 4.2. The following familiar facts are intuitionistically valid.

1. Given a Peano system $(X, c, f)$ and a system $(X', c', f')$, there is a unique $h : X \to X'$ satisfying $h(c) = c'$ and $\forall x (h(f(x)) = f'(h(x)))$.  

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2. Any two Peano systems are isomorphic.

3. In any Peano system \((X, f, c)\) there are uniquely determined functions satisfying the primitive recursion equations, identifying \(c\) with 0 and \(f\) with the successor function \(S\). In particular, there are uniquely determined operations + and \(\cdot\) on \(X\) satisfying

\[
\begin{align*}
    x + c &= x, & x + f(y) &= f(x + y), \\
    x \cdot c &= c, & x \cdot f(y) &= (x \cdot y) + x
\end{align*}
\]

for all \(x, y \in X\).

**Proof.** See [34, Chapter 3].

**Remark 4.3.** We interpret Definition 4.1 and Theorem 4.2 in \(\text{Sh}(T)\) by letting \(X\) be a sheaf over \(T\), \(c\) a section of \(X\), \(f : X \to X\) a sheaf morphism, and \(Y\) a section of the power sheaf \(P(X)\). By Theorem 2.20 we know that Theorem 4.2 is valid in \(\text{Sh}(T)\). Therefore, the following theorem implies that \(\hat{\mathbb{N}}\) is the “correct” representation of the natural number system as a sheaf over \(T\).

**Theorem 4.4.** Let \(T\) be a topological space. Let \(\hat{0} \in \hat{\mathbb{N}}\) be the global section given by \(\hat{0}(t) = 0\) for all \(t \in T\). Let \(\hat{S} : \hat{\mathbb{N}} \to \hat{\mathbb{N}}\) be the sheaf morphism given by \((\hat{S}(a))(t) = a(t) + 1\) for all \(a \in \hat{\mathbb{N}}\) and all \(t \in E(a)\). Then \(\text{Sh}(T)\) satisfies that \((\hat{\mathbb{N}}, \hat{0}, \hat{S})\) is a Peano system.

**Proof.** A detailed proof is in [1, pages 58–61].

**Remark 4.5.** Similarly, the sheaves in \(\text{Sh}(T)\) corresponding to \(\mathbb{Z}\), the ring of integers, and \(\mathbb{Q}\), the field of rational numbers, are \(\hat{\mathbb{Z}}\) and \(\hat{\mathbb{Q}}\) respectively. See also [9, Chapter III] and [34, Chapter 15].

### 4.2 The Baire space

In this subsection we discuss the representation of the Baire space \(\mathbb{N}^\mathbb{N}\) within \(\text{Sh}(T)\). We begin by noting that, since the simple sheaf \(\hat{\mathbb{N}} = C^\mathbb{c}(T, \mathbb{N})\) represents \(\mathbb{N}\), the function sheaf \(\hat{\mathbb{N}}\) represents \(\mathbb{N}\).

**Theorem 4.6.** For any topological space \(T\), \(C^\mathbb{c}(T, \mathbb{N})\) and \(\hat{\mathbb{N}}\) are isomorphic as sheaves over \(T\). Hence \(C^\mathbb{c}(T, \mathbb{N})\) represents \(\mathbb{N}\) within \(\text{Sh}(T)\).

**Proof.** For a detailed proof, see [1, pages 62–64].

**Theorem 4.7.** If \(T\) is locally connected, then \(C^\mathbb{c}(T, \mathbb{N}) = C^\mathbb{c}(T, \mathbb{N})\), so the simple sheaf \(\hat{\mathbb{N}} = C^\mathbb{c}(T, \mathbb{N})\) represents \(\mathbb{N}\) within \(\text{Sh}(T)\).

**Proof.** Let \(U \subseteq T\) be open. Given a continuous function \(a : U \to \mathbb{N}\), for each \(i \in \mathbb{N}\) define a continuous function \(a_i : U \to \mathbb{N}\) by \(a_i(t) = (a(t))(i)\). If \(U\) is connected, then each \(a_i\) is constant on \(U\), hence \(a\) is constant on \(U\). Since \(T\) is locally connected, it follows that \(C^\mathbb{c}(T, \mathbb{N}) = C^\mathbb{c}(T, \mathbb{N})\). Therefore, by Definition 3.7 and Theorem 4.6, \(C^\mathbb{c}(T, \mathbb{N}) = \hat{\mathbb{N}}\) represents \(\mathbb{N}\) in \(\text{Sh}(T)\).
Corollary 4.8. For any poset space $K$, the simple sheaf $\hat{\mathbb{N}}^{sh} = C^k_0(K, \mathbb{N})$ represents $\mathbb{N}^\mathbb{N}$ within $\text{Sh}(K)$.

Proof. By Lemma 3.2 $K$ is locally connected, so Theorem 4.7 applies to $K$. □

Remark 4.9. Theorems 4.6 and 4.7 for $\mathbb{N}^\mathbb{N}$ hold more generally, for product spaces $X^Y$ where $X$ has the discrete topology. In other words, over any topological space $T$ the sheaves $\hat{X}^{sh}$ and $C_o(T, X^Y)$ are isomorphic, and if $T$ is locally connected then $C_o(T, X^Y) = C^k_o(T, X^Y) = \hat{X}^{sh}$.

4.3 The real numbers

In classical mathematics, the Cauchy reals (real numbers constructed as equivalence classes of Cauchy sequences of rational numbers) and the Dedekind reals (real numbers constructed as Dedekind cuts of rational numbers) are equivalent. Intuitionistically, they are not necessarily equivalent. In this subsection we discuss various sheaf models where they are and are not equivalent.

Definition 4.10. Classically, we use $\mathbb{R}$ to denote the real number system. Intuitionistically, we use $\mathbb{R}_C$ and $\mathbb{R}_D$ to denote the Cauchy reals and the Dedekind reals respectively. In particular, given a topological space $T$, we use $\mathbb{R}_C$ and $\mathbb{R}_D$ to denote the sheaves in $\text{Sh}(T)$ corresponding to the Cauchy reals and the Dedekind reals respectively. Recall from Subsection 2.1 that $C^k_o(T, X)$ (respectively $C^c_o(T, X)$) are the sheaves of continuous (respectively locally constant, constant) functions from open subsets of $T$ into $X$. If $M$ is any one of these sheaves over $T$, there is a natural isomorphism of $\hat{\mathbb{Q}}^{sh} = C_o(T, \mathbb{Q}) = C^c_o(T, \mathbb{Q})$ onto a subsheaf of $M$, corresponding to the natural embedding of $\mathbb{Q}$ into $\mathbb{R}$. If $M_1$ and $M_2$ are any two of these sheaves, we say that $M_1$ and $M_2$ are $\mathbb{Q}$-isomorphic, denoted $M_1 \cong_{\mathbb{Q}} M_2$, if there is an isomorphism of $M_1$ onto $M_2$ which commutes with the natural embeddings of $\hat{\mathbb{Q}}^{sh}$ into $M_1$ and $M_2$.

Theorem 4.11. Let $T$ be a topological space. Within $\text{Sh}(T)$ we have $\mathbb{R}_D \cong_{\mathbb{Q}} C_o(T, \mathbb{R})$. Moreover, if $T$ is locally connected then $\mathbb{R}_C \cong_{\mathbb{Q}} C^c_o(T, \mathbb{R})$.

Proof. See [8, pages 288–289], [9, pages 384–385], and [34, pages 784–789]. □

Corollary 4.12. In $\text{Sh}(\mathbb{R})$ we have $\mathbb{R}_C \not\cong_{\mathbb{Q}} \mathbb{R}_D$.

Proof. $\mathbb{R}$ is locally connected, so $\mathbb{R}_C \cong_{\mathbb{Q}} C^c_o(\mathbb{R}, \mathbb{R})$ and $\mathbb{R}_D \cong_{\mathbb{Q}} C_o(\mathbb{R}, \mathbb{R})$. On the other hand, there are continuous real-valued functions on $\mathbb{R}$ which are not locally constant, e.g., the identity function on $\mathbb{R}$. Thus $C^c_o(\mathbb{R}, \mathbb{R}) \not\cong C_o(\mathbb{R}, \mathbb{R})$, and from this it follows easily that $C^c_o(\mathbb{R}, \mathbb{R}) \not\cong_{\mathbb{Q}} C_o(\mathbb{R}, \mathbb{R})$. □

Corollary 4.13. Let $K$ be a poset space. In $\text{Sh}(K)$ we have $\mathbb{R}_C \cong_{\mathbb{Q}} \mathbb{R}_D \cong_{\mathbb{Q}} C^c_o(K, \mathbb{R}) = C_o(K, \mathbb{R})$. Moreover, $\text{Sh}(K, \mu)$ satisfies $\text{AC}(\sigma, \tau)$ for $M_\sigma = \mathbb{R}_C$. 

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Proof. By Lemma 3.2 $K$ is locally connected and all continuous functions from open subsets of $K$ into $\mathbb{R}$ are locally constant. Thus Theorem 4.11 implies that $\mathbb{R}_C \cong_{\mathbb{Q}} \mathbb{R}_D \cong_{\mathbb{Q}} C_0(K, \mathbb{R}) = C^{lc}_0(K, \mathbb{R}) = \mathbb{R}^{sh}$. Theorem 3.8 tells us that $\text{Sh}(K, \mu)$ satisfies $AC(\sigma, \tau)$ for $M_\sigma = \mathbb{R}^{sh}$, but since $\mathbb{R}_C \cong_{\mathbb{Q}} \mathbb{R}^{sh}$ we get the same conclusion for $M_\sigma = \mathbb{R}_C$. 

**Corollary 4.14.** Let $K$ be a directed poset space. In $\text{Sh}(K)$ we have $\mathbb{R}_C \cong_{\mathbb{Q}} \mathbb{R}_D \cong_{\mathbb{Q}} C^{lc}_0(K, \mathbb{R}) = C_0(K, \mathbb{R}) = C^{lc}_0(K, \mathbb{R}) = \mathbb{R}^{sh}$. Theorem 3.8 tells us that $\text{Sh}(K, \mu)$ satisfies $AC(\sigma, \tau)$ for $M_\sigma = \mathbb{R}^{sh}$, but since $\mathbb{R}_C \cong_{\mathbb{Q}} \mathbb{R}^{sh}$ we get the same conclusion for $M_\sigma = \mathbb{R}_C$. 

**Definition 4.15.** The axiom of countable choice is the special case $M_\sigma = \mathbb{N}^{sh}$ of $AC(\sigma, \tau)$ as formulated in Definition 3.5. More formally, for any topological space $T$ we say that $\text{Sh}(T)$ satisfies $AC_0$ if $\text{Sh}(T, \mu) \models AC(\sigma, \tau)$ for $M_\sigma = \mathbb{N}^{sh}$ and arbitrary $M_\tau$.

**Theorem 4.16.** Let $T$ be a topological space. If $\text{Sh}(T)$ satisfies $AC_0$, then $\text{Sh}(T)$ satisfies $\mathbb{R}_C \cong_{\mathbb{Q}} \mathbb{R}_D$.

**Proof.** It is known intuitionistically that the axiom of countable choice implies that the Cauchy reals and the Dedekind reals are isomorphic over $\mathbb{Q}$. Therefore, by Theorem 2.20, this implication holds in $\text{Sh}(T)$. See also [8, page 289] and [34, pages 274 and 788–789]. 

**Corollary 4.17.** Let $T$ be a locally connected topological space. If $\text{Sh}(T)$ satisfies $AC_0$, then $C_0(T, \mathbb{R}) = C^{lc}_0(T, \mathbb{R})$.

**Proof.** This is immediate from Theorems 4.11 and 4.16.

**Remark 4.18.** We noted in Remark 3.9 that $AC_0$ fails in $\text{Sh}(\mathbb{R})$. Now Corollaries 4.12 and 4.17 provide another proof of this fact.

**Theorem 4.19.** $AC_0$ and $\mathbb{R}_C \cong_{\mathbb{Q}} \mathbb{R}_D$ hold in $\text{Sh}(\mathbb{N}^{sh})$ and in $\text{Sh}(K)$ for any poset space $K$.

**Proof.** This is immediate from Theorems 3.8, 3.15, 4.16, and 4.19.

**Remark 4.20.** There are continuous functions from $\mathbb{N}^{sh}$ into $\mathbb{R}$ which are not locally constant. Thus $\mathbb{N}^{sh}$ is an example of a topological space $T$ such that in $\text{Sh}(T)$ we have $\mathbb{R}_C \cong_{\mathbb{Q}} \mathbb{R}_D \cong_{\mathbb{Q}} C_0(T, \mathbb{R}) \supset C^{lc}_0(T, \mathbb{R})$, hence $\mathbb{R}_C \not\cong_{\mathbb{Q}} C^{lc}_0(T, \mathbb{R})$.

## 5 The Muchnik topos and the Muchnik reals

In this section we discuss a particular sheaf model which we call the Muchnik topos. We show that the Muchnik topos provides a model of intuitionistic mathematics which is a natural extension of the well known Kolmogorov/Muchnik interpretation of intuitionistic propositional calculus via mass problems under weak reducibility, i.e., Muchnik degrees. Within the Muchnik topos we define a sheaf representation of the real number system which we call the Muchnik reals. We prove a choice principle and a bounding principle for the Muchnik reals.
5.1 The Muchnik topos

Definition 5.1. For \( f, g \in \mathbb{N}^\mathbb{N} \) we say that \( f \) is Turing reducible to \( g \), denoted \( f \leq_T g \), if \( f \) is computable using \( g \) as a Turing oracle. It can be shown that \( \leq_T \) is transitive and reflexive on \( \mathbb{N}^\mathbb{N} \). We say that \( f \) is Turing equivalent to \( g \), denoted \( f \equiv_T g \), if \( f \leq_T g \) and \( g \leq_T f \). Clearly \( \equiv_T \) is an equivalence relation on \( \mathbb{N}^\mathbb{N} \). The Turing degree of \( f \), denoted \( \text{deg}_T(f) \), is the equivalence class of \( f \) under \( \equiv_T \). The set of all Turing degrees is denoted \( \mathcal{D}_T \). We partially order \( \mathcal{D}_T \) by letting \( \text{deg}_T(f) \leq \text{deg}_T(g) \) if and only if \( f \leq_T g \).

Lemma 5.2. Some well known facts about the poset \( \mathcal{D}_T \) are as follows.

1. There is a bottom Turing degree \( 0 = \text{deg}_T(f) \) for computable \( f \in \mathbb{N}^\mathbb{N} \).
2. Any two Turing degrees have a supremum, i.e., a least upper bound, given by \( \sup(\text{deg}_T(f), \text{deg}_T(g)) = \text{deg}_T((f,g)) \) where \( (f,g) \in \mathbb{N}^\mathbb{N} \) is given by \( (f,g)(2i) = f(i) \) and \( (f,g)(2i+1) = g(i) \) for all \( i \in \mathbb{N} \).
3. However, two incomparable Turing degrees may or may not have an infimum, i.e., a greatest lower bound, in \( \mathcal{D}_T \).
4. Thus \( \mathcal{D}_T \) is an upper semi-lattice, hence a directed poset, but not a lattice.

Definition 5.3. A mass problem is a set \( P \subseteq \mathbb{N}^\mathbb{N} \). For \( P, Q \subseteq \mathbb{N}^\mathbb{N} \) we say that \( P \) is weakly reducible to \( Q \), denoted \( P \leq_w Q \), if for all \( g \in Q \) there exists \( f \in P \) such that \( f \leq_T g \). Clearly \( \leq_w \) is reflexive and transitive on the powerset of \( \mathbb{N}^\mathbb{N} \). We say that \( P \) is weakly equivalent to \( Q \), denoted \( P \equiv_w Q \), if \( P \leq_w Q \) and \( Q \leq_w P \). Clearly \( \equiv_w \) is an equivalence relation on the power set of \( \mathbb{N}^\mathbb{N} \). The weak degree or Muchnik degree of a mass problem \( P \), denoted \( \text{deg}_w(P) \), is the equivalence class of \( P \) under \( \equiv_w \). The set of all Muchnik degrees is denoted \( \mathcal{D}_w \). We partially order \( \mathcal{D}_w \) by letting \( \text{deg}_w(P) \leq \text{deg}_w(Q) \) if and only if \( P \leq_w Q \).

Remark 5.4. There is a natural embedding of the Turing degrees, \( \mathcal{D}_T \), into the Muchnik degrees, \( \mathcal{D}_w \), given by \( \text{deg}_T(f) \mapsto \text{deg}_w(\{f\}) \). This embedding is one-to-one and order-preserving, i.e., \( f \leq_T g \) if and only if \( \{f\} \leq_w \{g\} \). Moreover, this embedding preserves the bottom Turing degree and the supremum of any two Turing degrees. However, it does not preserve the infimum of two incomparable Turing degrees, even when the infimum exists.

Definition 5.5. A lattice is a poset such that for any two elements \( a \) and \( b \) there exists a supremum or least upper bound, \( \sup(a, b) \), and an infimum or greatest lower bound, \( \inf(a, b) \). A lattice is said to be complete if for every set of elements \( \{a_i\}_{i \in I} \) there exists a supremum or least upper bound, \( \sup_{i \in I} a_i \), and an infimum or greatest lower bound, \( \inf_{i \in I} a_i \). Note that every complete lattice has a top element and a bottom element. A complete lattice is said to be completely distributive if it satisfies \( \inf(\sup_{i \in I} a_i, b) = \sup_{i \in I} \inf(a_i, b) \) and \( \sup(\inf_{i \in I} a_i, b) = \inf_{i \in I} \sup(a_i, b) \) for all \( \{a_i\}_{i \in I} \) and all \( b \).

Remark 5.6. Our reference for lattice theory is Birkhoff, second edition [2]. Our reason for preferring the second edition to the third edition is explained in [30, Remark 1.5].
Definition 5.7. A set $U \subseteq \mathcal{D}_T$ is said to be upwardly closed if $\deg_T(f) \in U$ whenever $\deg_T(f) \subseteq U$ for some $f \leq_T g$. Let $\mathcal{U}(\mathcal{D}_T)$ be the set of upwardly closed subsets of $\mathcal{D}_T$. We partially order $\mathcal{U}(\mathcal{D}_T)$ by the subset relation: $U \subseteq V$ if and only if $U \subseteq V$. Clearly $\mathcal{U}(\mathcal{D}_T)$ is a complete and completely distributive lattice. To prove this, one uses only the fact that $\mathcal{D}_T$ is a poset.

Theorem 5.8. The posets $\mathcal{D}_w$ and $\mathcal{U}(\mathcal{D}_T)$ are dually isomorphic. That is, there is an order-reversing one-to-one correspondence between $\mathcal{D}_w$ and $\mathcal{U}(\mathcal{D}_T)$.

Proof. Define $\Psi : \mathcal{U}(\mathcal{D}_T) \rightarrow \mathcal{D}_w$ by letting $\Psi(U) = \deg_w(\{f \mid \deg_T(f) \in U\})$ for all $U \in \mathcal{U}(\mathcal{D}_T)$. It is straightforward to verify that $\Psi$ is one-to-one, onto, and order-reversing, i.e., $U \subseteq V$ if and only if $\Psi(U) \geq \Psi(V)$. In proving these properties, one uses only the fact that $\leq_T$ is reflexive and transitive on $\mathbb{N}^\mathbb{N}$. 

Corollary 5.9. $\mathcal{D}_w$ is a complete and completely distributive lattice. The lattice operations in $\mathcal{D}_w$ are given by

$$\sup(a, b) = \Psi(\Psi^{-1}(a) \cap \Psi^{-1}(b)), \quad \inf(a, b) = \Psi(\Psi^{-1}(a) \cup \Psi^{-1}(b)),$$

$$\sup_{i \in I} a_i = \Psi\left(\bigcap_{i \in I} \Psi^{-1}(a_i)\right), \quad \inf_{i \in I} a_i = \Psi\left(\bigcup_{i \in I} \Psi^{-1}(a_i)\right)$$

where $\Psi$ is as in the proof of Theorem 5.8. Moreover, the top degree in $\mathcal{D}_w$ is $\infty = \Psi(\emptyset) = \deg_w(\emptyset)$ and the bottom degree in $\mathcal{D}_w$ is $0 = \Psi(\mathcal{D}_T) = \deg_w(\{f \mid f \text{ is computable}\})$.

Definition 5.10. We define the Muchnik topos to be the sheaf model $\text{Sh}(\mathcal{D}_T)$. Here $\mathcal{D}_T$ is a poset space as usual, with the Alexandrov topology, where the open sets are the upward closed subsets of $\mathcal{D}_T$.

Remark 5.11. Our terminology “the Muchnik topos” is motivated by Theorem 5.8. Note that set $\Omega$ of truth values in $\text{Sh}(\mathcal{D}_T)$ is just $\mathcal{U}(\mathcal{D}_T)$. Moreover, the propositional connectives of Section 2 correspond via $\Psi$ to lattice operations in the Muchnik lattice $\mathcal{D}_w$. Namely, for all $L(\mu)$-sentences $A$ and $B$, letting $\Psi([A]) = a$ and $\Psi([B]) = b$ we have

$$\Psi([A \land B]) = \sup(a, b), \quad \Psi([A \lor B]) = \inf(a, b),$$

$$\Psi([A \implies B]) = \text{imp}(a, b) = \inf\{c \mid \sup(a, c) \geq b\},$$

$$\Psi([\neg A]) = \text{imp}(a, \infty).$$

Moreover, $\text{Sh}(\mathcal{D}_T, \mu) \models A \implies B$ if and only if $a \geq b$, i.e., $[A] \subseteq [B]$, and $\text{Sh}(\mathcal{D}_T, \mu) \models B$ if and only if $b = 0$, i.e., $[B] = \mathcal{D}_T$. Thus the Muchnik topos $\text{Sh}(\mathcal{D}_T)$ provides a natural extension of Muchnik’s $\mathcal{D}_w$ interpretation of intuitionistic propositional calculus [24, Section 1] to intuitionistic higher-order logic. See also our translation of [24] in the Appendix below.

Theorem 5.12. The Muchnik topos $\text{Sh}(\mathcal{D}_T)$ satisfies $AC(\sigma, \tau)$ whenever $M_\sigma$ is a simple sheaf. In particular, $\text{Sh}(\mathcal{D}_T)$ satisfies $AC(\sigma, \tau)$ for $M_\sigma = \mathbb{R}_C$. 

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Proof. This follows from Theorem 3.8 and Corollary 4.13 since \(D_T\) is a poset. \(\square\)

**Theorem 5.13.** In the Muchnik topos \(\text{Sh}(D_T)\), the Cauchy reals \(\mathbb{R}_C\) and the Dedekind reals \(\mathbb{R}_D\) are \(\mathbb{Q}\)-isomorphic to each other and to \(C_o(D_T, \mathbb{R}) = C_o(S(T, \mathbb{R}), \mathbb{R})\).

Proof. This follows from Corollary 4.14 because \(D_T\) is a directed poset. \(\square\)

### 5.2 The Muchnik reals

**Definition 5.14.** Let \(# : \mathbb{Q} \to \mathbb{N}\) be a standard Gödel numbering of the rational numbers. For instance, we could define \(#(q)\) for \(q \in \mathbb{Q}\) by

\[
#(q) = \begin{cases} 
1 & \text{if } q = 0, \\
2 \cdot 3^a \cdot 5^b & \text{if } q = a/b \text{ where } a, b \in \mathbb{N} \setminus \{0\} \text{ and gcd}(a, b) = 1, \\
4 \cdot 3^a \cdot 5^b & \text{if } q = -a/b \text{ where } a, b \in \mathbb{N} \setminus \{0\} \text{ and gcd}(a, b) = 1.
\end{cases}
\]

For real numbers \(x \in \mathbb{R}\) we define the Turing degree of \(x\) to be \(\text{deg}_T(x) = \text{deg}_T(f_x)\) where \(f_x \in \mathbb{N}^\mathbb{N}\) is given by \(f_x(i) = 1\) if \(i = #(q)\) for some \(q \in \mathbb{Q}\) such that \(q < x\), otherwise \(f_x(i) = 0\), for all \(i \in \mathbb{N}\).

**Definition 5.15.** In \(\text{Sh}(D_T)\), the Muchnik reals are the sections of the sheaf

\[\mathbb{R}_M = \{a \in C_o(D_T, \mathbb{R}) \mid \forall d \in \text{dom}(a) \Rightarrow \text{deg}_T(a(d)) \leq d\}.\]

For \(a \in \mathbb{R}_M\) such that \(a \neq \emptyset\), let \(\overline{a} \in \mathbb{R}\) be such that \(\text{rng}(a) = \{\overline{a}\}\), and let \(\widehat{a}\) be the maximal \(c \in \mathbb{R}_M\) such that \(a \leq c\), i.e., the unique \(\widehat{a} \in \mathbb{R}_M\) such that \(\text{rng}(\widehat{a}) = \{\overline{a}\}\) and \(\text{dom}(\widehat{a}) = \text{the Turing upward closure of } \{\text{deg}_T(\overline{a})\}\). For \(a = \emptyset\) let \(\widehat{a} = \emptyset\) and let \(\overline{a}\) be undefined. Note that \(\emptyset \neq a \leq b\) implies \(\overline{a} = \overline{b}\) and \(\widehat{a} = b\).

**Remark 5.16.** As we know from Theorem 5.13, the Cauchy reals \(\mathbb{R}_C\) and the Dedekind reals \(\mathbb{R}_D\) are represented in \(\text{Sh}(D_T)\) by \(C_o(D_T, \mathbb{R})\), the sheaf of constant functions from upward closed sets of Turing degrees into \(\mathbb{R}\). However, not all such constant functions are Muchnik reals. The Muchnik reals are those \(a \in C_o(D_T, \mathbb{R})\) such that either \(a = \emptyset\) or \(\text{dom}(a) \subseteq \text{the upward closure of } \{\text{deg}_T(\overline{a})\}\). Thus \(\mathbb{R}_M\) is a proper subsheaf of \(C_o(D_T, \mathbb{R})\), so \(\mathbb{R}_M \not\cong \mathbb{Q}\). \(\mathbb{R}_C \cong \mathbb{Q}\). Informally, a Muchnik real is a real number which “comes into existence” only when we have enough Turing oracle power to compute it.

### 5.3 A bounding principle for the Muchnik reals

By Theorem 5.12 the Muchnik topos \(\text{Sh}(D_T)\) satisfies a choice principle for \(\mathbb{R}_C\), the Cauchy reals. In this subsection we prove that for \(\mathbb{R}_M\), the Muchnik reals, \(\text{Sh}(D_T)\) satisfies not only a choice principle but also a bounding principle.

**Definition 5.17.** Let \(r, s, t\) be closed terms of sort \(\sigma\) where \(M_\sigma = \mathbb{R}_M\). Then \(a, b, c \in \mathbb{R}_M\) where \(a = [r]_T\), \(b = [s]_T\), \(c = [t]_T\). We define \([r \leq_T s]\) = \(E(a) \cap E(b)\) if \(a, b \neq \emptyset\) and \(\overline{r} \leq_T \overline{b}\), otherwise \([r \leq_T s] = \emptyset\). We define \([r \leq_T (s, t)] = E(a) \cap E(b) \cap E(c)\) if \(a, b, c \neq \emptyset\) and \(\overline{r} \leq_T (\overline{b}, \overline{t})\), otherwise \([r \leq_T (s, t)] = \emptyset\). Our bounding principle \(\text{BP}(\sigma, \sigma)\) for the Muchnik reals is
\[(\forall x \exists y A(x, y)) \Rightarrow \exists z \forall x (y \leq_T (x, z) \land A(x, y))\]

where \(x, y, z\) are variables of sort \(\sigma\) and \(A(x, y)\) is any \(L\)-formula which does not contain \(z\).

**Theorem 5.18.** The Muchnik topos \(\text{Sh}(\mathcal{D}_T)\) satisfies a combined *choice and bounding principle* \(\text{ACBP}(\sigma, \sigma)\) for the Muchnik reals,

\[(\forall x \exists y A(x, y)) \Rightarrow \exists w \exists z (wz \leq_T (x, z) \land A(x, wz))\]

where \(x, y, z\) are variables of sort \(\sigma\), \(w\) is a variable of sort \(\sigma \to \sigma\), \(A(x, y)\) is any \(L\)-formula which does not contain \(z\) or \(w\), and \(M_\sigma = \mathbb{R}_M\).

**Proof.** We may safely assume that \(A(x, y)\) has no free variables other than \(x\) and \(y\). Letting \(U = \exists x \exists y A(x, y)\) and \(V = \exists w \exists z \forall x (wz \leq_T (x, z) \land A(x, wz))\), it will suffice to show that \(U \subseteq V\). Fix \(c = \hat{c} \neq \emptyset\) in \(\mathbb{R}_M\) such that \(E(c) \subseteq U\). It will suffice to show that \(E(c) \subseteq V\).

For each \(a \neq 0\) in \(\mathbb{R}_M\) we have \(\deg_T((\overline{a}, \overline{a})) \in E(\overline{a}) \cap E(c) = E(\overline{a} \upharpoonright E(c)) \subseteq E(c) \subseteq U\), so choose \(b \in \mathbb{R}_M\) depending only on \(\overline{a}\) such that \(\deg_T((\overline{a}, \overline{b})) \in E(b) \cap [A(\overline{a} \upharpoonright E(c), b)]\). We then have \(\overline{b} \leq_T (\overline{a}, \overline{b})\) and \(E(b) \supseteq E(\overline{a}) \cap E(c)\), so by Theorem 2.21 it follows that \(E(a) \cap E(c) \subseteq \overline{b} \leq_T (a, c)\). Moreover, since \(b\) depends only on \(\overline{a}\), we have a sheaf morphism

\[\mathbb{R}_M \upharpoonright E(c) \xrightarrow{\cong} \mathbb{R}_M \upharpoonright E(c)\]

where \(\varphi(a \upharpoonright E(c)) = b \upharpoonright E(a) \cap E(c)\) for all \(a \in \mathbb{R}_M\). Thus \((\varphi, E(c)) \in \mathbb{R}_M^{\mathbb{N}}\) and \([[(\varphi, E(c))a] = b \upharpoonright E(a) \cap E(c)]\), so by Theorem 2.21 we have \(E(a) \cap E(c) \subseteq [[(\varphi, E(c))a \leq_T (a, c)]\cap [A(\overline{a} \upharpoonright E(c), b)]\). Since this holds for all \(a \in \mathbb{R}_M\), we now see that \(E(c) \subseteq V\), and the proof is complete.

**Corollary 5.19.** \(\text{Sh}(\mathcal{D}_T)\) satisfies \(\text{AC}(\sigma, \sigma)\) and \(\text{BP}(\sigma, \sigma)\) for \(M_\sigma = \mathbb{R}_M\).

**Proof.** Within the formal system IHOL, \(\text{AC}(\sigma, \sigma)\) and \(\text{BP}(\sigma, \sigma)\) are logical consequences of \(\text{ACBP}(\sigma, \sigma)\). Therefore, the corollary follows from Theorems 5.18 and 2.20. More details may be found in [1, pages 99–106].

**Remark 5.20.** In our proof of Theorem 5.18, one may avoid using the axiom of choice, as follows. First, given \(a \neq 0\) in \(\mathbb{R}_M\), let \(e_a\) be the smallest index \(e \in \mathbb{N}\) of a partial recursive functional \(\Phi_e\) such that \(\Phi_e((\overline{a}, \overline{b})) = \overline{b}\) for some \(b \in \mathbb{R}_M\) such that \(\deg_T((\overline{a}, \overline{b})) \in E(b) \cap [A(\overline{a} \upharpoonright E(c), b)]\). Then, choose \(b = \overline{b}\).

**Theorem 5.21.** \(\text{Sh}(\mathcal{D}_T)\) satisfies \(\text{AC}(\sigma, \tau)\) for \(M_\sigma = \mathbb{R}_M\) and \(M_\tau\) arbitrary.

**Proof.** Repeat the proof of Theorem 5.18 but skip the parts that involve \(\leq_T\).

**Remark 5.22.** Theorem 5.21 resembles Theorem 5.12. However, Theorem 5.12 applies only when \(M_\sigma\) is a simple sheaf, while in Theorem 5.21 we have \(M_\sigma = \mathbb{R}_M\) which is not a simple sheaf. See also Remark 3.10.


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Appendix: translation of Muchnik’s paper

In this appendix we offer a translation of Muchnik’s paper [24]. We started with a rough translation produced in 1964 by the United States Department of Commerce [25]. We have corrected some typographical and translation errors and updated some bibliographical references.

Siberian Mathematical Journal
Vol. IV, No. 6, November–December, 1963

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Strong and weak reducibility of algorithmic problems
Introduction

The abstract (arithmetical) analysis of algorithmic problems was initiated by S. Kleene and E. Post [1, 2]. E. Post introduced the concept of degree of unsolvability of a problem, while Kleene and Post investigated in [2] the class of degrees of unsolvability of arithmetical (in the sense of Gödel) sets. Papers along the same line were published subsequently.

The traditional algorithmic problems of algebra, number theory, topology, and mathematical logic were problems of solvability. This explains the predominant interest shown first in problems of solvability of arithmetic (i.e., problems of solvability of sets of natural numbers). Subsequently, however, in logic and its applications, problems arose connected to separability, enumerability, and isomorphism of sets [3, 4, 5, 8].

The definition of an algorithmic problem in abstract algorithm theory was formulated by Yu. T. Medvedev, in which all the previously known cases and many others were treated [3]. The problem of constructing an arithmetical function satisfying certain conditions is called a Medvedev problem (M-problem). To each M-problem \( P \) there corresponds a family of functions satisfying the conditions of the problem. Conversely, any family of functions \( A \) defines some M-problem \( P(A) \). The functions contained in the family corresponding to an M-problem \( P \) are called the solution functions of the M-problem \( P \).

To each M-problem there corresponds a certain degree of difficulty (an exact definition of degrees of difficulty is given below). It is possible to define in a natural fashion conjunction, disjunction, and other operations of propositional calculus on the degrees of difficulty. As was established by Yu. T. Medvedev, the calculus of M-problems is an interpretation of constructive propositional calculus. This is to be expected, since the calculus of M-problems is an elaboration of A. N. Kolmogorov’s calculus of problems (see [7]). The definition of reducibility of a family of functions (M-problems), which is basic in the calculus of M-problems, has a constructive character.

**Definition 1.** An M-problem \( P(A) \) (family \( A \)) is reducible to an M-problem \( P(B) \) (family \( B \)), if there exists a general method of transformation of any solution of the M-problem \( P(B) \) into a solution of the M-problem \( P(A) \), or more accurately, if there exists a partial recursive operator \( T \), which transforms each function \( f \) from the family \( B \) into some function \( g \) (which depends on \( f \)) from the family \( A \), \( g = T[f] \). The reducibility of the family \( A \) (M-problem \( P(A) \)) to \( B \) is denoted by \( A \leq B \) (\( P(A) \leq P(B) \)). The family of functions \( A \) (the M-problem \( P(A) \))\(^2\) is called solvable, if it contains at least one general recursive function. The M-problems (families) \( A \) and \( B \) are called equivalent (\( A \approx B \)) if they are reducible to each other.

The class of M-problems equivalent to an M-problem \( A \) is called the degree of difficulty of the M-problem \( A \) and is denoted by \( a = |A| \). The degrees of

---

1I.e., a function defined on the natural numbers \( \mathbb{N} \) and assuming values from \( \mathbb{N} \), which includes also 0.
2The definitions presented here apply equally well to families of functions and to the M-problems which they define.
difficulty form a partially ordered set $\Omega$: $|A| = a \leq b = |B|$ if the M-problem $A$ is reducible to $B$. $\Omega$ is a distributive lattice with implication and has a largest and a smallest element (see [3]). The investigation of M-problems, initiated by Yu. T. Medvedev, was continued by the author in [4].

Section 1

1. We describe here a second approach to the concept of reducibility of algorithmic problems, corresponding to classical, i.e., non-constructive, formulations.

Along with the problem of constructing an algorithm which solves a certain problem, it is possible to consider the problem of the existence of a required algorithm, without insisting on its concrete form. Then each condition imposed on the arithmetical functions (i.e., each family of functions) will be linked to two problems:

1. The problem of constructing one of the functions of this family: the M-problem.

2. The problem of proving the existence of a general recursive function in this family.\(^3\)

Problems of the second type will be called \textit{Ex-problems}. There is a pairwise one-to-one correspondence between the classes of families of functions, M-problems, and Ex-problems. The Ex-problem corresponding to the family of functions $A$ (M-problem $P(A)$) will be denoted by $P(A) = Q(A)$. The functions of the family defining the Ex-problem $Q$ will accordingly be called the \textit{solution functions} of the Ex-problem $Q$. An Ex-problem is called \textit{solvable} if its solution functions include a general recursive one.

An important method of establishing the solvability of an algorithmic problem $A$ is to reduce this problem to a different problem $B$, the solvability of which has already been established. Conversely, the unsolvability of a problem $A$ implies the unsolvability of any problem $B$ to which problem $A$ is reducible.

\textbf{Definition 2.} The Ex-problem $P(A)$ (family of functions $A$) is \textit{weakly reducible} to the Ex-problem $P(B)$ (family $B$) ($A \prec B$), if for any function $f$ of the family $B$ ($f \in B$) there exists a partial recursive operator $T$, which transforms the function $f$ into the function $g$ of the family $A$ ($g \in A$).

The choice of the function $f$ governs here not only $g$ but also the operator $T = T_f$. In this case we say that the problem $P(A)$ (family $A$) \textit{reduces weakly} to the problem $P(B)$ (family $B$) by means of the operators $\{T_f\}$.

The reducibility of families of functions (problems) in the sense of Medvedev’s definition will be called henceforth \textit{strong reducibility} (or simply \textit{reducibility}).

Inasmuch as each of the three objects: the family of functions, the M-problem, and the Ex-problem, defines uniquely the two others, we shall henceforth identify these objects and call them \textit{problems}.

\(^3\)It is easy to see here an analogy with the question of the existence of a solution of a differential equation and the problem of effectively finding a solution.
2. A natural question arises concerning the relation between these types of reducibility. It is clear that strong reducibility of a problem $A$ to a problem $B$ implies weak reducibility of $A$ to $B$. As shown by the example considered below, the converse is generally not true.

Let the problem $A$ be determined by a family consisting of one non-recursive function $f$, $A = K_f = \{f\}$, and let problem $B$ be determined by a family consisting of all the functions obtained from $f$ in the following manner: for each tuple of natural numbers $\mathbf{n} = \{n_1, \ldots, n_s\}$ we consider the function $f_{\mathbf{n}}(m)$:

$$
    f_{\mathbf{n}}(m) = \begin{cases}
        n_{i+1} & \text{for } 0 \leq i < s, \\
        f(i-s) & \text{for } i \geq s,
    \end{cases}
$$

i.e., we “place in front” of the sequence of values $\{f(i)\}$ the tuple $\mathbf{n}$:

$$
    B = K'_f = \{f_{\mathbf{n}}\}.
$$

It is easy to see that the problem $K_f$ reduces weakly to the problem $K'_f$:

$$
    K_f \vdash K'_f.
$$

For any function $f_{\mathbf{n}} \in B$ there exists a partial recursive operator (p.r.o.) $T$ which transforms $f_{\mathbf{n}}$ into $f$ (by “discarding” the first $s$ values of $f_{\mathbf{n}}$, where $\mathbf{n} = \{n_1, n_2, \ldots, n_s\}$).

However, the problem $A$ does not strongly reduce to the problem $B$.

Let us assume the opposite, i.e., that there exists a p.r.o. $T$ which transforms any function $f_{\mathbf{n}}$ into $f$. Any p.r.o. $T$ can be specified by means of a recursive sequence of pairs of tuples (see [4, 10])

$$
    \{(d_w, d'_w)\}, \ w = 0, 1, 2, \ldots.
$$

If the sequence of several first values of the function $h$ forms a tuple $d$, then we call $d$ a tuple of the function $h$. We shall also say that the function $h$ begins with the tuple $d$. If $h = T[e]$ and $d_w$ is a tuple of the function $e$, then $d'_w$ is a tuple of the function $h$. Inasmuch as $T[d_w] = f$ and $d_w$ is a tuple of the function $f_{d_w}$, then $d'_w$ is a tuple of the function $f$ (for each $w$). In view of the fact that $\{d'_w\}$ is a recursive sequence of tuples, the length of which is unlimited (in the aggregate), the function $f$ is recursive, yet we have assumed it to be non-recursive. This contradiction proves that the problem $A$ does not reduce strongly to $B$.

In the foregoing example, the problem $B$ was chosen somewhat artificially. For algorithmic problems which are usually considered in the theory of algorithms and its applications, the situation is different. If we confine ourselves to reducibility (strong and weak) by means of general recursive operators\(^4\) or even partial recursive operators applicable to each solution function of the problem to which we reduce another problem, then both types of reducibility are equivalent.

\(^4\)A general recursive operator is a p.r.o. which transforms functions which are everywhere defined (on $\mathbb{N}$) into functions which are everywhere defined.
for a broad class of problems. We shall return to this question in Section 2, and consider here in greater detail the calculus that results from the definition of weak reducibility of problems.

3. If problems $A$ and $B$ reduce weakly to each other, we shall call them *weakly equivalent*: $A \vdash B$. This relation is transitive, symmetrical, and reflexive. The class of all problems therefore breaks up into classes of weakly equivalent problems. The class of problems which are weakly equivalent to $A$ will be called the *weak degree of difficulty* of problem $A$. The weak degree of difficulty of the problem $A$ characterizes the problem of proving the existence (in the classical sense) of a computable solution function of the problem $A$.

A weak degree of difficulty $\tilde{a}$ exceeds $\tilde{b}$, $\tilde{b} \geq \tilde{a}$ or $\tilde{a} \leq \tilde{b}$, if the problem $A$ reduces weakly to the problem $B$ ($\tilde{a} = |A|$, $\tilde{b} = |B|$). We denote by $\Omega$ the partially ordered set of weak degrees of difficulty.

Between $\Omega$ and $\overline{\Omega}$ there is a one-sidedly univalent correspondence $\Omega \rightarrow \overline{\Omega}$; to each degree of difficulty $a \in \Omega$ there corresponds a weak degree of difficulty $\overline{a}$. The correspondence $a \rightarrow \overline{a}$ does not depend on the choice of the problem $A$, since equivalence of problems implies weak equivalence of problems. This relation is isotopic, since reducibility of problems implies weak reducibility. The solvable (smallest) degree 0 from $\Omega$ corresponds to the solvable weak degree 0 from $\overline{\Omega}$, and the improper (largest, i.e., defined by the empty class of functions) degree $\infty$ from $\Omega$ corresponds to an equal degree from $\overline{\Omega}$. We shall prove that $\Omega$ is a lattice and the indicated correspondence is a lattice homomorphism.

We note that $\overline{\Omega}$ admits a natural topological interpretation. Define a complete family of functions or points of Baire space (complete problem) to be any family (problem) $A$ having the following property: together with each function $f$ belonging to $A$, the family $A$ contains any function $g$ with respect to which the function $f$ is recursive.

We shall establish some properties of complete families. The union and intersection of any number of complete families (finite or infinite) are also complete families.

Let $B$ be some family of functions. The family consisting of all functions $\{g\}$, for each of which there exists a certain function $f$ from $B$, which is recursive with respect to this function $g$ will be called the completion $B'$ of the family $B$. It is obvious that $B'$ is the smallest complete family containing the family $B$, and the completion of a complete family $A$ coincides with $A$: $A' = A$. The family $B'$ is weakly equivalent to $B$. It is sufficient to establish that $B' \subseteq B$, since $B' \supseteq B$, from which follows $B \subseteq B'$. Indeed, for any function $g \in B'$ there exists a p.r.o. $T$ such that $T[g] = f \in B$.

Completions of two weakly equivalent families $A$ and $B$ coincide: $A \leftrightarrow B \rightarrow A' = B'$. Let $g \in A'$. Then there exists a function $f \in A$ and a p.r.o. $T_1$ such that $T_1[g] = f$. In view of $A \leftrightarrow B$, there exists a p.r.o. $T$ such that $T[f] = h \in B$. Then $T_2[g] = T[T_1[g]] = h \in B$. Therefore $g \in B'$. Conversely, if $g \in B'$, then $g \in A'$. Thus $A' = B'$.

In view of the foregoing, any weak degree $\tilde{a}$ defines uniquely a complete
family (problem) $A$, which we shall call the representative of $\pi$.

**Lemma.** Let $\pi$ and $\overline{\pi}$ be weak degrees of complete families $A$ and $B$ respectively. Then $\pi \geq \overline{\pi} \iff A \subset B \ ^5$, or using a different notation

$$\overline{A} \supseteq \overline{B} \iff A \subset B.$$ (1)

Let $g \in A$. Then there exists a p.r.o. $T$ such that $T[g] = f \in B$. In view of the completeness of the family $B$, $g \in B$. The relation $A \subset B \rightarrow A \supseteq B$ is obvious.

We now readily prove some theorems concerning the properties of $\Omega$.

**Theorem 1.** For any set of weak degrees $\{\pi_\xi\}$ there exist exact upper and lower bounds, denoted by $\bigvee \pi_\xi$ and $\bigwedge \pi_\xi$, respectively.

Let $A_\xi$ be a complete family with weak degree of difficulty $\pi_\xi$ and $A = \bigcup_\xi A_\xi$, $\overline{\pi} = \overline{|A|}$. We shall prove that $\pi = \inf \{\pi_\xi\}$. Obviously $A$ is a complete family and $\pi \leq \pi_\xi$ for any $\xi$. Further, let $\overline{\pi} \leq \pi_\xi$ for any $\xi$ and $B$ a complete family, $\overline{b} = |\overline{B}|$. Then $B \supseteq A_\xi$ and $B \supseteq A$, hence $\overline{b} \leq \pi$. We put $A^* = \bigcap_\xi A_\xi$. Obviously, $\pi^* \geq \pi_\xi$ for any $\xi$. In addition, if $\overline{b} \geq \pi_\xi$ for all $\xi$, and $B$ is a representative of $\overline{\pi}$, then $B \subset A_\xi$ and $B \subset \bigcap_\xi A_\xi = A^*$, i.e., $\overline{b} \geq \overline{\pi}$. Hence $\overline{\pi}^* = \sup \{\pi_\xi\}$.

From the proof of Theorem 1, we see that the operations of taking the exact upper and lower bounds in $\overline{\Omega}$ correspond to the operations of intersection and union of complete families of functions.

$\overline{\Omega}$ is a complete lattice, represented by subsets of the Baire space $J$. In the function space $J$ it is possible to introduce a topology by assigning as open sets the complete families of functions. This will be a $T_0$ space. (On this subject see G. D. Birkhoff, Lattice theory, Russian translation of the 2nd edition, IL 1952, Chapter IV, §§1 and 2.)

**Theorem 2.** Let $A$ and $B$ be arbitrary problems, $a = |A|, \pi = |\overline{B}|, b = |B|, \overline{b} = |\overline{B}|$. Then the problem $C = A \cup B$ with degree of difficulty $c = a \lor b$ has a weak degree $\overline{c} = \pi \lor \overline{b}$, and the problem $D$ with degree of difficulty $d = a \land b$ has a weak degree $\overline{d} = \pi \land \overline{b}$.

Following Yu. T. Medvedev [3], we choose problems $C$ and $D$ in the following fashion. We define the p.r.o.s $R_0$, $R_1$ and a two-place p.r.o. $R$:

$$f_i(n) = R_i[f],$$

$$f_i(n) = \begin{cases} i & \text{for } n = 0 \\ f(n-1) & \text{for } n > 0 \end{cases} \quad (i = 0, 1),$$

$$h(n) = R[f(m), g(m)],$$

$$h(n) = \begin{cases} f(m) & \text{for } n = 2m, \\ g(m) & \text{for } n = 2m + 1. \end{cases}$$

$\ ^5A \leftrightarrow B$ denotes that the statement $A$ is equivalent to statement $B$. 

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The problem $C$ consists of all the functions $f_0(n) = R_0[f]$, where $f \in A$, and all the functions $g_1(n) = R_1[g]$, where $g \in B$, and $|C| = c = a \lor b$. The problem $C_1$ consists of all the solution functions of problems $A$ and $B$,

$$|\tau_1| = \tau = \pi \lor \delta.$$

We shall prove that $C$ and $C_1$ are weakly equivalent, i.e., $|C_1| = |C|$. Indeed, each function $h \in C_1$ can be transformed with the aid of $R_0$ or $R_1$ into a function $h_i \in C$, and each function $h_i \in C$ can be reduced by means of an inverse transformation into $h \in C_1$ (i.e., $C_1$ reduces even strongly to $C$).

Further, the problem $D$ consists of all the functions $h = R[f, g]$, where $f$ runs through class $A$ and $g$ through class $B$. The problem $D_1$ consists of all the functions $e$ such that problems $A$ and $B$ reduce to the problem of computability $A_e = \{e\}$, i.e., for each function $e$ there exists p.r.o. $T_1$ and $T_2$ such that $T_1[e] \in A, T_2[e] \in B$. We shall prove that problems $D$ and $D_1$ are weakly equivalent:

1. $D_1 \sqsubseteq D$ (even $D_1 \leq D$). The relation $D_1 \leq D$ follows from the fact that class $D$ is contained in $D_1$, since any function $h \in D$ can be transformed with the aid of the p.r.o. $T_1$ ($T_2$) into the function $f(g)$, $f \in A$ ($g \in B$).

To this end it is sufficient to put

$$T_1[h] = f(m) = h(2m),$$
$$T_2[h] = g(m) = h(2m + 1).$$

2. $D \sqsubseteq D_1$. Let the function $e \in D_1$. Then there exist p.r.o. $T_1$ and $T_2$ such that

$$f = T_1[e] \in A, g = T_2[e] \in B, R[f, g] = h \in D$$

and

$$h = T[e] = R[T_1[e], T_2[e]] \in D.$$

The p.r.o. $T$ transforms the function $e$ into $h \in D$, from which it follows that $D \sqsubseteq D_1$.

The weak degree $\pi = \pi \lor \delta$ will be called the disjunction, and $\overline{\delta} = \pi \land \delta$ will be called the conjunction, of the weak degrees $\pi$ and $\delta$. Let us prove that the lattice $\Omega$ has an implication operator:

**Theorem 3.** For any weak degrees $\pi$ and $\delta$ there exists a smallest degree $\pi^+$ in the class of weak degrees $\pi$ such that $\pi \land \pi^+ \geq \delta$.

**Proof.** We consider the representatives of the weak degrees $\pi$ and $\delta$, i.e., the complete families (problems) $A$ and $B$, $|A| = \pi, |B| = \delta$. We denote by $C^*$ the family of all the functions $\{g\}$ such that for each pair of functions $[f, g]$, where $f \in A$ and $g \in C^*$, there exist a p.r.o. $T$ which transforms the pair $[f, g]$ into a function $e \in B$, $e = T[f, g]$. It is obvious that the family $C^*$ includes the family $B$ and that $\pi \land \pi^+ \geq \delta$ where $\pi^+ = |C^*|$. Let us prove that the problem $C^*$ reduces weakly to any problem $C$ such that $\pi \land \pi^+ \geq \delta$ where $\pi = |C|$. Let $C$ be such a problem and $g$ an arbitrary function from $C$. As follows from Theorem
The problem $D$, which consists of all of the functions $h = R[f, g]$ where $f$ runs through the family $A$ and $g$ through the family $C$, has the weak degree $d = \pi \land \overline{\pi}$. Inasmuch as $\overline{d} \geq b$, i.e., $|D| \leq |B|$, for any function $h \in D$, there exists a p.r.o. $T_1$ such that $e = T_1[h] \in B$. This means that for any pair of functions $[f, g]$ where $f \in A$ and $g \in C$, there exists a two-place p.r.o. $T = T_1R$ such that $e = T[f, g] = T_1[R[f, g]] \in B$. By definition of $C^*$, the function $g \in C^*$. It follows therefore that $C \subset C^*$ and $|C^*| \leq |C|$. This completes the proof.

We shall call $C^*$ the weak problem of reducibility of the problem $B$ to the problem $A$, and $\pi^*$ the implication, denoted by $\pi \supset \underline{\pi}$. Obviously $C^*$ is a complete family, i.e., the representative of $\pi^*$.

We note that implication, generally speaking, is not conserved in homomorphism of the lattices $\Omega \to \Pi$. Indeed, in the example discussed in Section 2, the problems $A$ and $B$ ($A = K_f, B = K'_f$) were related by $|A| > |B|$ and $|A| \vdash |B|$, or $a > b$ and $\pi = \underline{\pi}$. Therefore the implication $\underline{\pi} \supset \overline{\pi}$ is the solvable (trivial) weak degree (i.e., the degree of a solvable problem), and $b \supset a$ is an unsolvable degree and $\overline{b} \supset a \neq b \supset \overline{a}$.

Note that solvability of the weak degree $\pi \supset \underline{\pi}$ is equivalent to the relation $\pi \geq \overline{\pi}$. The proof of this is simple and will be omitted. Further consideration of this point is analogous to that of Yu. T. Medvedev with respect to the calculus of $\Omega$.

We consider an arbitrary segment $\Pi : 0 \leq x \leq d$. The weak degree $\neg x = x \supset d$ is called the negation of the weak degree $x$ (with respect to $d$). We introduce also the notation $\overline{\pi} \sim b$ for the degree $(\pi \supset \underline{b}) \cap (b \supset \overline{\pi})$.

The thought arises of the connection between the calculus of weak degrees $\Omega$ and the propositional calculus: elementary propositions can be interpreted as weak degrees, and the operations of propositional calculus correspond to like operations of the calculus of weak degrees. The truth of a formula corresponds to the solvability of a weak degree.

**Theorem 4.** All the axioms and rules of derivation of intuitionistic propositional calculus are satisfied for weak degrees of an arbitrary segment $0 \leq x \leq d$ in $\Omega$.

Theorem 4 follows from the existence of implication in the distributive lattice $\Omega$ (see Birkhoff, Lattice theory, Russian translation of the 2nd edition, Chapter XII, §7).

Let us discuss the consequences of this point. In spite of the fact that the definition of weak reducibility of problems has been chosen in accordance with classical premises, the calculus of weak degrees obtained thereby is an interpretation of constructive propositional calculus and does not include, for example, the law of the excluded third.

However, this should not surprise us, since the calculus of weak degrees $\Omega$ and the propositional calculus can be interpreted as weak degrees, and the operations of propositional calculus correspond to like operations of the calculus of weak degrees. The truth of a formula corresponds to the solvability of a weak degree.

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The question whether the weak degrees $\Omega$ are an exact interpretation of

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6 An interpretation of a logical calculus $K$ is called exact if all formulas true (solvable, realizable) in the interpretation are derivable in the calculus $K$. 

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constructive propositional calculus remains open. We note that the calculus of
degrees of difficulty $\Omega$, as shown recently by Yu. T. Medvedev, is an exact
interpretation of the constructive calculus.

Section 2

In this section we analyze the question of the relation between strong and weak
reducibility under certain limitations on the p.r.o.s by means of which the reducibility is realized, and on the problems themselves.

We need several new concepts. In the arguments that follow we shall find it
convenient to use the Baire space $J$.

Arithmetical functions can be interpreted as points in Baire space (considering the sequence of the values of these functions $[9, 10]$). To each problem $A$ in such an interpretation, there corresponds a certain set of points $M_A$ of the
Baire space, which defines it completely.

Let $\delta_\pi$ be a Baire interval, defined by a tuple $\pi = (n_1, n_2, \ldots, n_s)$. The problem which is defined by the set of points $M_A \cap \delta_\pi$ shall be called the interval $A_\pi$ of the problem $A$. In other words, $A_\pi$ is defined by the class of solution functions of the problem $A$ beginning with the tuple $\pi$. The interval $A_\pi$ is called non-empty if the set $M_A \cap \delta_\pi$ is non-empty. A problem $A$ is called uniform if any of its non-empty intervals is (strongly) reducible to it.

The problem of solvability $A_E$ of the set $E$ is defined by the class $K_A(E)$, consisting of one characteristic function of the set $E$. The problem of enumerability $C(E)$ is determined by the class $K_C(E) = \{f(n)\}$ of the functions that enumerate the set $E$, i.e., the set $E$ is the image of the function $f(n)$. The problem of separability $A_{E_0,E_1}$ of the sets $E_0$ and $E_1$ with empty intersection is determined by the class of functions $\{f(n)\}$ satisfying the condition

$$f(k) = \begin{cases} 0 & \text{for } k \in E_0, \\ 1 & \text{for } k \in E_1, \\ 0 \text{ or } 1 & \text{for } k \notin E_0 \cup E_1 \end{cases}$$

(1)

**Theorem 5.** The problem of enumerability of any non-empty set $E$ is uniform.

Indeed, let $C$ be the problem of enumerability of the set $E$ and let $C_\pi$ be a non-empty interval in it: $\pi = (n_1, n_2, \ldots, n_s)$. Thus $K_C_\pi$ consists of all the functions which enumerate the set $E$ and begin with the tuple $\pi$. The problem $C_\pi$ is (strongly) reducible to the problem $C$ by means of the p.r.o. $T$ which, being applied to any function $f$, shifts the sequence of its values by first adding the tuple $\pi$.

**Theorem 6.** The problem of separability $A_{E_0,E_1}$ is uniform for arbitrary $E_0, E_1$ ($E_0 \cap E_1 = \Lambda$, where $\Lambda$ is the empty set).

We denote the problem $A_{E_0,E_1}$ by $A$. Let $A_\pi$ be a non-empty interval of the problem $A$, $\pi = (n_1, n_2, \ldots, n_s)$. It is obvious that

$$n_k = \begin{cases} 0 & \text{for } k \in E_0, \\ 1 & \text{for } k \in E_1, \\ 0 \text{ or } 1 & \text{for } k \notin E_0 \cup E_1 \end{cases}, (k = 1, 2, \ldots, s)$$

(2)
The problem $A_{\pi}$ is (strongly) reducible to the problem $A$ by means of the p.r.o. $T$ which replaces the first $s$ values of any function by the tuple $\pi$. If $f(k)$ is a solution function of the problem $A$, then it satisfies the condition (1). But then the function $g = T[f]$ also satisfies the condition (1), as follows from (2) and from the definition of the p.r.o. $T$. In addition, the function $g$ begins with the tuple $\pi$ and hence is a solution function of the M-problem $A_{\pi}$, which was to be proved.

The problem of continuation of the partial function $f(m)$ is the problem $B_f$ defined by the class of functions (which are defined everywhere on $\mathbb{N}$) coinciding with the function $f(m)$ wherever the latter is defined. (We shall call such functions continuations of $f(m)$.) We note that a problem of separability is a particular case of a problem of continuation. Obviously we have:

**Theorem 7.** The problem of continuation of any partial function is uniform.

The proof of Theorem 7 is analogous to the proof of Theorem 6.

Inasmuch as the p.r.o.s used in the proofs of Theorems 1 and 2 are general recursive, each problem of enumerability or separability reduces to any of its non-empty intervals by means of a general recursive operator. Problems possessing this property will be called general recursively uniform. In addition to problems of enumerability and separability, problems of solvability are also general recursively uniform, since the operator of identical transformation reduces any function to itself.

An example of a non-uniform problem is the problem defined by the class $K = \{f, g\}$, where the degree of non-computability of the function $f$ is strictly greater than the degree of non-computability of the function $g$.

We shall call the problem $B$ closed if it corresponds to a closed set of points $M_B$ of the Baire space $J$. Obviously, solvability problems are closed.

**Theorem 8.** The continuation problem of any partial function is closed.

Let $\{g_k(m)\}$ be a convergent sequence of continuations of the function $f(m)$, and let $g(m) = \lim_{k \to \infty} g_k(m)$. We shall prove that $g(m)$ also continues $f(m)$. If the function $f(m)$ is defined for $m = m_0$, then $g_k(m_0) = f(m_0)$ for all $k$. Consequently $g(m_0) = f(m_0)$, as was to be proved.

**Corollary.** Any problem of separability is closed.

**Theorem 9.** The problem of enumerability $C(E)$ of any set $E$ containing more than one element is not closed.

Let $a \in E$ and let the function $f(m)$ enumerate the set $E$. We define the sequence of functions $\{f_k(m)\}$ enumerating the set $E$:

$$f_k(m) = \begin{cases} a & \text{for } m < k, \\ f(m - k) & \text{for } m \geq k. \end{cases}$$

I.e., perhaps not everywhere defined.
Obviously
\[ \lim_{k \to \infty} f_k(m) = g(m) \equiv a. \]

In view of the fact that the set \( E \setminus \{a\} \) is not empty, \( g(m) \) is not a solution function of the problem \( C(E) \), and consequently the problem \( C(E) \) is not closed.

However, it is possible to generalize the concept of closedness of a problem in such a way that enumerability problems as well as many other problems which are of interest for the recursive theory of sets are included. This concept is closely related with the theory of infinite games [11].

Let \( S \) be a set of points of the Baire space \( J \). We imagine two players I and II who move alternately, and their moves consist of choosing Baire intervals which intersect with the set \( S \). Player I chooses as his first move the Baire interval \( \delta_1 \), which intersects with \( S \). If player I chose in move \( m \) the Baire interval \( \delta_m \), then player II chooses in the \( m \)th move a sub-interval\(^8\) \( \delta_m^* \) of the interval \( \delta_m \) intersecting with \( S \). Then player I chooses in the \((m + 1)\)st move a sub-interval \( \delta_{m+1} \) of the interval \( \delta_m^* \) intersecting with \( S \). Let us assume that after the \( m \)th move of player I (II) the play is in the interval \( \delta_m (\delta_m^*) \). We agree that at the beginning the game is in the interval \( \delta_0 = J \). Player II wins if the sequence of intervals
\[ \delta_0 \supset \delta_1 \supset \delta_1^* \supset \delta_2 \supset \delta_2^* \supset \ldots \supset \delta_m \supset \delta_m^* \supset \delta_{m+1} \ldots \]
contracts to a point of the set \( S \). Otherwise, player I wins.

We fix once and for all some effective numbering of the Baire intervals by means of natural numbers. By a strategy of a player we mean a function \( r = \varphi(n) \) which indicates for each interval with number \( n \) the number \( r \) of a sub-interval of it. If the game is in the interval numbered \( n \), then the player making the next move chooses the interval with number \( \varphi(n) \). The strategy \( \varphi \) is called correct with respect to the set \( S \) if for any interval numbered \( n \) intersecting with \( S \), the interval numbered \( \varphi(n) \) also intersects with \( S \). We shall henceforth take strategy to mean a strategy which is correct with respect to the considered set. A strategy is called winning (for the set \( S \)) if player II, using this strategy, wins for any correct strategy of his opponent. A set \( S \) is called winning if there exists a winning strategy for this set. The M-problem \( A(S) \) and the class of functions \( K(S) \) defined by the set \( S \) will also be called winning in this case.

A set \( S \) (a problem \( A(S) \)) the complement of which is nowhere dense is called trivially winning. In order to win, it is sufficient for player II to choose as his first move an interval which is completely contained in \( S \), which is possible since the complement \( CS \) is nowhere dense.

If neither the set \( S \) nor its complement \( CS \) is trivially winning (or equivalently, neither \( S \) nor \( CS \) is nowhere dense), then they cannot be simultaneously winning. In fact, let \( S \) be a winning set and \( \varphi(n) \) its winning strategy. Let us consider the game with respect to \( CS \). Player I chooses as his first move an interval in which the set \( S \) is everywhere dense (such an interval exists, since \( S \) is not a set which is nowhere dense). Then player I applies strategy \( \varphi(n) \). For

\(^8\)We consider Baire sub-intervals which are proper parts of their intervals.
any strategy of player II, the sequence of intervals in which the game is situated will contract to a point belonging to $S$, i.e., player II loses.

An example of a winning set (problem) is a closed set $S$ (problem $A(S)$). In this case the sequence of intervals $\{\delta_m, \delta^*_m\}$ contracts always to a point of $S$. It follows therefore that problems of solvability and separability are winning. There exist also non-closed winning problems.

**Theorem 10.** Any problem of enumerability is a winning problem.

Let $GF$ be the problem of enumerability of a set $F$ of natural numbers, and let $\mathfrak{M}$ be the corresponding subset of $J$. Player II chooses a strategy $r = \varphi(n)$ in the following manner: let $n$ be the number of an interval $\delta_n = (n_1, n_2, \ldots, n_s)$ containing points in $\mathfrak{M}$, by virtue of which $n_1, n_2, \ldots, n_s \in F$. We denote by $n_{i+1}$ the smallest number belonging to the set $F$ which is not equal to $n_i$ for $i = 1, 2, \ldots, l$, and if there is no such number, then $n_{l+1} = n_l$. We put $\delta_{\varphi(n)} = (n_1, n_2, \ldots, n_l, n_{l+1})$. Obviously $\delta_{\varphi(n)}$ intersects with $\mathfrak{M}$. We consider the sequence of intervals in which the game occurs:

$$\delta_1, \delta^*_1, \ldots, \delta_m, \delta^*_m, \ldots$$

We note that: (1) for each $m$, all of the numbers of the tuples$^9$

$$\delta_m = (n_1, n_2, \ldots, n_p)$$

and $\delta^*_m = (n_1, n_2, \ldots, n_p)$

belong to $F$; (2) any number $q \in F$ will be sooner or later encountered in the tuples $\{\delta_m, \delta^*_m\}$, because going from $\delta_m$ to $\delta^*_m$ we add a still unchosen element of the set $F$ (if it exists). But then the sequence $\{\delta_m, \delta^*_m\}$ contracts to the point $b = (n_1, n_2, n_3, \ldots, n_l, \ldots)$ where the set $\{n_i\}$ coincides with $F$, i.e., $b \in \mathfrak{M}$. This proves the theorem.

A partial recursive operator $T$ is called fully applicable to a problem $B$ if it is defined on each solution function of the problem $B$. The class of all p.r.o.s which are fully applicable to the problem $B$ will be denoted by $Q_B$.

If a problem $A$ reduces strongly (weakly) to a problem $B$ by means of operators of a certain class $P$, then we say that $A$ is strongly (weakly) $P$-reducible to $B$.

Let $U$ be some class of p.r.o.s. A problem $B$ is called $U$-uniform if any of its non-empty intervals is strongly reducible to it by means of operators of the class $U$.

The class of p.r.o.s represented in the form of compositions$^{10}$ $RT[f]$ where $R \in P$ and $T \in U$ will be denoted by $PU$.

**Theorem 11.** Let $A$ be a closed problem, $B$ a $U$-uniform winning problem, and $P$ a subclass of $Q_B$. If the problem $A$ is weakly $P$-reducible to the problem $B$, then the problem $A$ is strongly $PU$-reducible to the problem $B$.

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$^9$As is well known, Baire intervals are identified with tuples of natural numbers.

$^{10}$RT[f] is the result of successive application of the p.r.o.s $R$ and $T$ to the function $f$. 

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Proof. Let us assume that no operator of class PU reduces the problem A to the problem $B$. We arrange all the p.r. operators of the class $P$ in some sequence

$$T_1, T_2, T_3, \ldots, T_s, \ldots$$

To each operator $T_s$ there corresponds a continuous function $\theta_s$ in the Baire space (see [10]) defined at each point of the set $M_B$, by virtue of $P \subset Q_B$. By virtue of our assumptions, including continuity of the functions $\theta_s$ and closedness of the set $M_A$, there exists for each $s$ an interval $\delta$ represented by the function $\theta_s$ in $CM_A$, the complement of $M_A$. Let $\varphi$ be the winning strategy for the set $M_B$. By the method indicated above, we obtain for $T_1$ an interval $\delta = \delta_1$. Let $n_1$ be the number of $\delta_1$; let $r_1 = \varphi(n_1)$; let $\delta^*_1$ be the interval numbered $r_1$; let $B_1$ be the problem defined by the set $M_B \cap \delta^*_1$, which is then a non-empty interval of the problem $B$.

Inasmuch as the problem $B$ is $U$-uniform, the problem $B_1$ is strongly $U$-reducible to $B$. But then the problem $A$ cannot be strongly $P$-reducible to $B_1$, since in accordance with our assumption the problem $A$ is not strongly $PU$-reducible to $B$. Consequently, there exists a non-empty interval $\delta_2$ intersecting with the set $M_B$ and transformed by the function $\theta_2$ into a subset of $CM_A$. Obviously it is possible to choose $\delta_2$ so as to make $\delta_2^* \subset \delta^*_1$. If $n_2$ is the number of $\delta_2$, then $r_2 = \varphi(n_2)$ is the number of a sub-interval $\delta^*_2$, $\delta^*_2 \subset \delta_2$, which also intersects with $M_B$. Let $B_2$ be the problem defined by the set $M_B \cap \delta^*_2$.

We define further in the same manner the intervals

$$\delta_3 \supset \delta^*_3 \supset \delta_4 \supset \delta^*_4 \supset \ldots$$

and the problems $B_3, B_4, \ldots$.

The problem $B$, by virtue of $U$-uniformity, is strongly $U$-reducible to any problem $B_s$, while the problem $A$ does not reduce strongly to $B_s$ by any $P$-operator. By virtue of the winning character of the problem $B$ and of the strategy $\varphi$, the sequence $\{\delta_s, \delta^*_s\}$ contracts to a point $f \in M_B$.

Inasmuch as for any $s$ the operator $T_s$ transforms the set $M_B$ into a subset of $CM_A$, we have $T_s[f] = g \in CM_A$ for any $s$, and this means that the problem $A$ does not reduce weakly to $B$ by means of operators in the class $P$, which contradicts the conditions of the theorem. Consequently, the assumption that the problem $A$ does not reduce strongly to $B$ by means of operators in the class $PU$ is incorrect. The theorem is proved.

Our desire to be as general as possible has made it necessary to formulate the theorem in a rather cumbersome manner. We present some simply formulated corollaries of Theorem 11.

**Corollary 1.** If a closed problem $A$ reduces weakly to a uniform winning problem $B$ by means of operators of the class $Q_B$, then $A$ reduces strongly to $B$.

A $U$-uniform problem is called *general recursively uniform* if $U$ is the class of general recursive operators [6, 10]. Problems of solvability, separability, and enumerability are general recursively uniform.
Corollary 2. If a closed problem $A$ reduces weakly to a general recursively uniform problem $B$ by means of general recursive operators, then problem $A$ reduces strongly to problem $B$ by means of a general recursive operator.

In conclusion, we formulate some unsolved problems.

1. Is it possible to strengthen the fundamental theorem in such a way that weak reducibility (by means of arbitrary partial recursive operators) of a closed problem $A$ to a uniform problem $B$ would imply strong reducibility of $A$ to $B$?

2. Under what “natural” conditions imposed on problems $A$ and $B$ does weak reducibility (by means of an arbitrary p.r.o.) imply strong reducibility?

3. What is the situation in the particular case when $A$ is a solvability problem and $B$ is a separability problem of enumerated recursively inseparable sets (we note that no non-trivial solvability problem $A$ can be reduced strongly to a separability problem $B$ [4, 12]).

Received 5 July 1962

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