Covariance within Random Integer Compositions

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Abstract. Fix a positive integer $N$. Select an additive composition $\xi$ of $N$ uniformly out of $2^{N-1}$ possibilities. The interplay between the number of parts in $\xi$ and the maximum part in $\xi$ is our focus. It is not surprising that correlations $\rho(N)$ between these quantities are negative; we earlier gave inconclusive evidence that $\lim_{N \to \infty} \rho(N)$ is strictly less than zero. A proof of this result would imply asymptotic dependence. We now retract our presumption in such an unforeseen outcome. Similar experimental findings apply when $\xi$ is a 1-free composition, i.e., possessing only parts $\geq 2$.

An unrestricted additive composition of $N$ is a sequence of positive integers, called parts, that sum to $N$. The number of unrestricted compositions is $2^{N-1}$. For example, the compositions of 5 are

{5}, {4, 1}, {3, 2}, {2, 3},
{1, 4}, {3, 1, 1}, {2, 2, 1}, {2, 1, 2},
{1, 3, 1}, {1, 2, 2}, {1, 1, 3}, {2, 1, 1, 1},
{1, 2, 1, 1}, {1, 1, 2, 1}, {1, 1, 1, 2}, {1, 1, 1, 1, 1}.

If a composition of $N$ is chosen uniformly at random from all possibilities, then

\[
1 + m_n = \mathbb{E}\text{(number of parts)} = 1 + \frac{n}{2},
\]

\[
s_n^2 = \mathbb{V}\text{(number of parts)} = \frac{n}{4}
\]

where $n = N - 1$, $\mathbb{E}$ denotes mean and $\mathbb{V}$ denotes variance (uncorrected for bias);

\[
1 + \mu_n = \mathbb{E}\text{(maximum part)} = 1 + \frac{1}{2^n} \left[ z^n \right] \sum_{k=1}^{\infty} \left( \frac{1}{1 - 2z} - \frac{1 - z^k}{1 - 2z + z^{k+1}} \right),
\]

\[
\sigma_n^2 = \mathbb{V}\text{(maximum part)} = \frac{1}{2^n} \left[ z^n \right] \sum_{k=1}^{\infty} (2k - 1) \left( \frac{1}{1 - 2z} - \frac{1 - z^k}{1 - 2z + z^{k+1}} \right) - \mu_n^2
\]

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where \([z^n]\) denotes the coefficient of \(z^n\) in the subsequent Taylor series expansion. For example, if \(N = 5\), then \(m_4 = 2\), \(s^2_4 = 1\), \(\mu_4 = 27/16\) and \(\sigma^2_4 = 247/256\). Up to small periodic fluctuations \([1, 5, 6]\), we have

\[
E(\text{maximum part}) \sim \frac{\ln(n)}{\ln(2)} + \left( \frac{\gamma}{\ln(2)} - \frac{1}{2} \right),
\]

\[
V(\text{maximum part}) \sim \frac{1}{12} + \frac{\pi^2}{6 \ln(2)^2}
\]
asymptotically as \(N \to \infty\).

A **restricted** additive composition of \(N\) obeys an extra condition that no parts are equal to 1. A more descriptive name is **1-free**. The number of restricted compositions is \(d_n\), the \(n\)th Fibonacci number, where \(d_0 = 0\), \(d_1 = 1\) and \(d_n = d_{n-1} + d_{n-2}\). For example, the 1-free compositions of 7 are

\[
\{7\}, \quad \{5, 2\}, \quad \{4, 3\}, \quad \{3, 4\}, \quad \{2, 5\}, \quad \{3, 2, 2\}, \quad \{2, 3, 2\}, \quad \{2, 2, 3\}.
\]

If a 1-free composition of \(N\) is chosen uniformly at random from all possibilities, then \([2, 3]\)

\[
1 + m_n = E(\text{number of parts}) = 1 + \frac{1}{d_n} [z^n] \frac{z^3}{(1 - z - z^2)^2},
\]

\[
s^2_n = V(\text{number of parts}) = \frac{1}{d_n} [z^n] \frac{z^3 (1 - z + z^2)}{(1 - z - z^2)^3} - m_n^2,
\]

\[
1 + \mu_n = E(\text{maximum part}) = 1 + \frac{1}{d_n} [z^n] \sum_{k=1}^{\infty} \left( \frac{1 - z^2}{1 - z - z^2} - \frac{1 - z^2 - z^k + z^{k+1}}{1 - z - z^2 + z^{k+1}} \right),
\]

\[
\sigma^2_n = V(\text{maximum part}) = \frac{1}{d_n} [z^n] \sum_{k=1}^{\infty} (2k-1) \left( \frac{1 - z^2}{1 - z - z^2} - \frac{1 - z^2 - z^k + z^{k+1}}{1 - z - z^2 + z^{k+1}} \right) - \mu_n^2.
\]

For example, if \(N = 7\), then \(m_6 = 5/4\), \(s^2_6 = 7/16\), \(\mu_6 = 13/4\) and \(\sigma^2_6 = 27/16\). Up to small periodic fluctuations, we conjecture that \([3]\)

\[
E(\text{maximum part}) \sim \frac{\ln(n)}{\ln(\varphi)} + \left( \frac{\gamma}{\ln(\varphi)} - 1 \right),
\]

\[
V(\text{maximum part}) \sim \frac{1}{12} + \frac{\pi^2}{6 \ln(\varphi)^2}
\]
as \(N \to \infty\), where \(\varphi = (1 + \sqrt{5})/2\) is the Golden mean.
What is missing among these results? We have not yet conveyed any sense of how the two quantities are interrelated. Define a correlation coefficient

\[ \rho(N) = \frac{\mathbb{E}[(\text{number of parts})(\text{maximum part})] - (1 + m_n)(1 + \mu_n)}{s_n \sigma_n} \]

\[ = \frac{\mathbb{E}[(\text{number of 1s})(\text{longest run of 0s})] - m_n \mu_n}{s_n \sigma_n} \]

where the first expression (involving integer compositions) is understandable but the second expression (involving bitstrings) needs clarification. Avoiding details for now, let us simply provide some numerical data in Table 1.

| \(n\) | \(\rho(N)\) for unrestricted case | \(\rho(N)\) for 1-free case |
|---|---|---|
| 100 | -0.441772 | -0.530911 |
| 200 | -0.361888 | -0.439875 |
| 300 | -0.319761 | -0.391011 |
| 400 | -0.292051 | -0.358533 |
| 500 | -0.271797 | -0.334641 |
| 600 | -0.256049 | -0.315973 |
| 700 | -0.243295 | -0.300791 |
| 800 | -0.232656 | -0.288084 |
| 900 | -0.223581 | -0.277216 |
| 1000 | -0.215704 | -0.267762 |
| 1100 | -0.208773 | -0.259428 |
| 1200 | -0.202606 | -0.251998 |
| 1300 | -0.197066 | -0.245313 |
| 1400 | -0.192050 | -0.239249 |

Table 1: Correlation between (number of parts) and (maximum part) within random integer compositions as a function of \(N = n + 1\).

Acceleration of convergence is possible for each sequence, suggesting (without proof [3]) that limits are nonzero as \(N \to \infty\). We now must retract such presumptive and unjustified thoughts. A more careful study leads to a revised conjecture:

\[ \rho(N) \sim C \ln(N)^{-5/2} \quad \text{for some } C < 0 \]

thus in particular \(\rho(N) \to 0^-\), consistent with asymptotic independence. This is plainly what intuition leads everyone to foresee. A rigorous treatment would be good to see someday.
1. **Unconstrained and Pinned Solus Bitstrings**

Given a random unconstrained bitstring of length \( n = N - 1 \), we have

\[
\mathbb{E}(\text{number of 1s}) = \frac{n}{2}, \quad \text{V}(\text{number of 1s}) = \frac{n}{4}
\]

because a sum of \( n \) independent Bernoulli(1/2) variables is Binomial(\( n, 1/2 \)). Expressed differently, the average density of 1s in a string is 1/2, with a corresponding variance 1/4. The word “unconstrained” offers that, in the sampling process, all \( 2^n \) strings are included and equally weighted.

If we append the string with a 1, calling this \( \eta \), then there is a natural way \cite{7} to associate \( \eta \) with an additive composition \( \xi \) of \( N \). For example, if \( N = 10 \),

\[
\eta = 0110100111 \longmapsto \xi = \{2, 1, 2, 3, 1, 1\}
\]

i.e., parts of \( \xi \) correspond to “waiting times” for each 1 in \( \eta \). The number of parts in \( \xi \) is equal to the number of 1s in \( \eta \) and the maximum part in \( \xi \) is equal to the duration of the longest run of 0s in \( \eta \), plus one.

In this paper, the word “constrained” refers to the logical conjunction of two requirements:

- A bitstring is **pinned** if its first bit is 0 and its last bit is 0.
- A bitstring is **solus** if all of its 1s are isolated.

The latter was discussed in \cite{2, 3}; additionally imposing the former is new. Given a random pinned solus bitstring of length \( n = N - 1 \), formulas for \( \mathbb{E}(\text{number of 1s}) \) and \( \text{V}(\text{number of 1s}) \) are best expressed using generating functions.

If we append the string with 1 to construct \( \eta \), then the associated \( \xi \) is a composition of \( N \) with all parts \( \geq 2 \). For example, if \( N = 15 \),

\[
\eta = 010001010010101 \longmapsto \xi = \{2, 4, 2, 3, 2, 2\}.
\]

It should now be clear why, starting with the original \( n \)-bitstring,

\[
1 + \mathbb{E}(\text{number of 1s}) = \mathbb{E}(\text{number of parts}),
\]

\[
1 + \mathbb{E}(\text{longest run of 0s}) = \mathbb{E}(\text{maximum part})
\]

for both scenarios, but the corresponding variances are always equal.

Nej & Satyanarayana Reddy \cite{8} gave an impressive recursion for the number \( F_n(x, y) \) of unconstrained bitstrings of length \( n \) containing exactly \( x \) 0s and a longest
run of exactly $y$ 0s:

$$F_n(x, y) = \begin{cases} 
\sum_{i=\kappa}^{y-1} F_{n-i-1}(x-i, y) + \sum_{j=0}^{y} F_{n-y-1}(x-y, j) & \text{if } 1 \leq x \leq n-2 \text{ and } \varepsilon_n(x, y) = 1, \\
\lambda_n(y) & \text{if } x = n-1 \text{ and } \varepsilon_n(x, y) = 1, \\
0 & \text{otherwise,}
\end{cases}$$

where $n \geq x \geq y$ (of course) and $\kappa = 0$,

$$\varepsilon_n(x, y) = \begin{cases} 
1 & \text{if } n \geq x \text{ and } \left\lfloor \frac{n}{n-x+1} \right\rfloor \leq y \leq x, \\
0 & \text{otherwise}
\end{cases}$$

and

$$\lambda_n(y) = \begin{cases} 
1 & \text{if } n \text{ is odd and } y = \frac{n-1}{2}, \\
2 & \text{otherwise.}
\end{cases}$$

Consequently, the numerator of $E[(\text{number of 1s})(\text{longest run of 0s})]$ for $n$-bitstrings is

$$\left\{ \sum_{x=0}^{n} \sum_{y=0}^{x} (n-x)y F_n(x, y) \right\}_{n=1}^{\infty} = \{0, 2, 11, 40, 122, 338, 881, 2202, 5337, 12634, \ldots\};$$

equivalently, the numerator of $E[(\text{number of parts})(\text{maximum part})]$ for $N$-compositions is

$$\left\{ \sum_{x=0}^{N} \sum_{y=0}^{x} (N-x)(y+1) F_n(x, y) \right\}_{N=1}^{\infty} = \{1, 4, 14, 42, 115, 296, 732, 1757, 4125, 9516, \ldots\}.$$

The denominator is $2^n$. Returning to the unrestricted example, the covariance for $N = 5$ is $\frac{40}{16} - (2) \left( \frac{27}{16} \right) = \frac{15}{16} - (1 + 2) \left( 1 + \frac{27}{16} \right)$. Correlations for selected small $N$ turn out to be

$$\{\rho(N) : N = 5, 11, 21, 51\} = \{-0.890799, -0.752444, -0.654958, -0.530128\}$$

and Table 1 exhibits values for larger $N = 101, 201, \ldots$. 
By a similar argument, we deduce the number \( G_n(x, y) \) of pinned solus bitstrings of length \( n \) containing exactly \( x \) 0s and a longest run of exactly \( y \) 0s. The recursion is identical to before (with \( F \) replaced by \( G \)). The initial conditions appear alike, but here we have \( \kappa = 1 \). Also, a different \( \varepsilon_n(x, y) \) applies:

\[
\varepsilon_n(x, y) = \begin{cases} 
1 & \text{if } n \geq x \text{ and } \left( \frac{n}{n - x + 1} \right) \leq y < x \text{ or } x = y = n \\
0 & \text{otherwise}
\end{cases}
\]

and a different \( \lambda_n(y) \):

\[
\lambda_n(y) = \begin{cases} 
1 & \text{if } n \text{ is odd and } y = \frac{n-1}{2}, \\
2 & \text{if } \left\lfloor \frac{n-1}{2} \right\rfloor < y < n - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Consequently, the numerator of \( E[(\text{number of 1s})(\text{longest run of 0s})] \) under constraints is

\[
\left\{ \sum_{x=0}^{n} \sum_{y=0}^{x} (n-x)y G_n(x, y) \right\}_{n=1}^{\infty} = \{0, 0, 1, 4, 10, 26, 54, 118, 230, 458, 864, 1632, \ldots\};
\]

equivalently, the numerator of \( E[(\text{number of parts})(\text{maximum part})] \) under restrictions is

\[
\left\{ \sum_{x=0}^{N} \sum_{y=0}^{x} (N-x)(y+1) G_n(x, y) \right\}_{n=1}^{\infty} = \{0, 2, 3, 8, 17, 34, 70, 131, 255, 466, 868, 1565, \ldots\}.
\]

The denominator is \( d_n \). Returning to the 1-free example, the covariance for \( N = 7 \) is \( \frac{26}{8} - \left( \frac{5}{4} \right) \left( \frac{13}{4} \right) = \frac{70}{8} - \left( 1 + \frac{5}{4} \right) \left( 1 + \frac{13}{4} \right) \). Correlations for selected small \( N \) turn out to be

\[
\{\rho(N) : N = 7, 11, 21, 51\} = \{-0.945611, -0.860467, -0.763395, -0.629068\},
\]

i.e., dependency is more significant than earlier. Table 1 exhibits values for larger \( N = 101, 201, \ldots \).

2. Sketches of Proofs I

Let \( \Omega \) be a set of finite bitstrings and \( \Omega_n^{x,y} \) be the subset of \( \Omega \) consisting of strings of length \( n \) containing exactly \( x \) 0s and a longest run of exactly \( y \) 0s. Let \( \Omega_n^{x,0} \) and \( \Omega_n^{x,1} \) be the subset of \( \Omega_n^{x,y} \) of strings starting with 0 and 1 respectively.
Assume that $\Omega$ consists of all unconstrained strings. If $\omega \in \Omega_{n,1}^x$, then $\omega$ is of the form $1\omega_1$ where $\omega_1 \in \Omega_{x-1,0}^x \cup \Omega_{x-1,1}^x$. If $\omega \in \Omega_{n,0}^x$, then $\omega$ is either of the form

$$00\ldots 0 \omega_2$$

where $\omega_2 \in \Omega_{n-i,1}^x$ and $1 \leq i \leq y - 1$

or

$$00\ldots 0 \omega_3$$

where $\omega_3 \in \Omega_{n-y,1}^x$ and $0 \leq j \leq y$.

We have

$$|\Omega_{n,0}^x| = \sum_{i=1}^{y-1} |\Omega_{n-i,1}^x| + \sum_{j=0}^{y-1} |\Omega_{n-y,1}^x| = \sum_{i=1}^{y-1} |\Omega_{n-i,1}^x|$$

(1)

hence

$$|\Omega_{n}^x| = \sum_{i=0}^{y-1} |\Omega_{n-i,1}^x| + \sum_{j=0}^{y-1} |\Omega_{n-y,1}^x|$$

upon addition. This proof of the recurrence for $F_n(x, y)$ appeared in [8].

Assume instead that $\Omega$ consists of all solus strings. If $\omega \in \Omega_{n,1}^x$, then $\omega$ is of the form $1\omega_1$ where $\omega_1 \in \Omega_{n-1,0}^x$. We have

$$|\Omega_{n,1}^x| = |\Omega_{n-1,0}^x| = |\Omega_{n-1}^x| - |\Omega_{n-1,1}^x|,$$

that is,

$$|\Omega_{n}^x| = |\Omega_{n,1}^x| + |\Omega_{n+1,1}^x| = |\Omega_{n-1,0}^x| + |\Omega_{n,0}^x|.$$

From formula (1) in the preceding,

$$|\Omega_{n,0}^x| = \sum_{i=1}^{y-1} |\Omega_{n-i,1}^x| + \sum_{j=0}^{y-1} |\Omega_{n-y,1}^x| = \sum_{i=1}^{y-1} |\Omega_{n-i,1}^x| + \sum_{j=0}^{y-1} |\Omega_{n-y,1}^x|$$

which gives a recurrence underlying what we called $\tilde{F}_n(x, y)$ in [9].

Let us turn attention to various boundary conditions. For either unconstrained or solus strings,

$$\Omega_{n-1,n-1}^{x,n-1} = \left\{ 00\ldots 0, \begin{array}{c} 00\ldots 1 \end{array} \right\};$$

if $n$ is odd, then

$$\Omega_{n-1,n-1}^{x,n-1} = \left\{ 00\ldots 0100\ldots 0 \begin{array}{c} (n-1)/2 \end{array} \right\};$$
if $n$ is even, then

$$\Omega^{n-1,n/2} = \begin{cases} 00\ldots0100\ldots0, & \text{if } (n-2)/2 \leq n/2 \leq (n-2)/2 \\ 00\ldots0100\ldots0, & \text{if } (n-2)/2 \leq n/2 \leq (n-2)/2 \end{cases}.$$ 

These imply the expression for $\lambda_n(y)$. For pinned strings, the latter two results hold, but the former becomes $\Omega^{n-1,n-1} = \emptyset$. The expression for $\varepsilon_n(x, y)$ comes from [8]:

$$F_n(x, y) > 0 \iff \begin{cases} x + \left\lfloor \frac{x}{y} \right\rfloor \leq n & \text{if } y > 0 \text{ and } y \nmid x, \\ x + \frac{x}{y} - 1 \leq n & \text{if } y > 0 \text{ and } y \mid x; \end{cases}$$

$$G_n(x, y) > 0 \iff \begin{cases} x + \left\lfloor \frac{x}{y} \right\rfloor \leq n & \text{if } y > 0 \text{ and } y \nmid x, \\ x + \frac{x}{y} - 1 \leq n & \text{if } (x > y > 0 \text{ or } x = y = n) \text{ and } y \mid x. \end{cases}$$

For completeness’ sake, we give the analog of Table 1 for pinned and solus strings.

| $n$  | $\rho$ for pinned case | $\rho$ for solus case |
|------|-------------------------|-----------------------|
| 100  | -0.445112               | -0.525562             |
| 200  | -0.363340               | -0.437637             |
| 300  | -0.320638               | -0.389680             |
| 400  | -0.292661               | -0.357617             |
| 500  | -0.272255               | -0.333956             |
| 600  | -0.256411               | -0.315434             |
| 700  | -0.243592               | -0.300351             |
| 800  | -0.232906               | -0.287715             |
| 900  | -0.223795               | -0.276900             |
| 1000 | -0.215891               | -0.267488             |
| 1100 | -0.208938               | -0.259187             |
| 1200 | -0.202753               | -0.251783             |
| 1300 | -0.197198               | -0.245119             |
| 1400 | -0.192170               | -0.239074             |

Table 2: Correlation between (number of 1s) and (longest run of 0s) within random bitstrings as a function of $n$. 
3. Sketches of Proofs II

Let $\Omega$ and $\Omega_{x,y}^n$ be as before. Assume that $\Omega$ consists of all pinned strings. If $\omega \in \Omega_{x,y}^n$, then $\omega$ is either of the form

$$\begin{align*}
00 \ldots 011 \ldots 10 \ldots, & \quad i = 0 \\
& \quad \geq 1 \\
00 \ldots 010 \ldots, & \quad 1 \leq i \leq y.
\end{align*}$$

(2)

or

(3)

On the one hand, the subset of strings $\omega$ satisfying (2) corresponds to $\Omega_{x,y}^{n-1}$ upon deleting the leftmost 1. On the other hand, the subset of strings $\omega$ satisfying (3) corresponds to

$$\Omega_{n-i-1}^{x-i,y} \quad \text{if} \quad i < y$$

and to

$$\Omega_{n-y-j}^{x-y,j} \quad \text{for some} \quad 0 \leq j \leq y \quad \text{if} \quad i = y$$

upon deleting the leftmost block of 0s and the leftmost 1. These observations give

$$|\Omega_{n}^{x,y}| = \sum_{i=0}^{y-1} |\Omega_{n-i-1}^{x-i,y}| + \sum_{j=0}^{y} |\Omega_{n-y-j}^{x-y,j}|$$

which interestingly is the same recurrence as that for $F_n(x, y)$. Clearly however

$$|\Omega_{n}^{0,0}| = 0, \quad |\Omega_{n}^{n,n}| = 1$$

are the initial conditions here.

Assume instead that $\Omega$ consists of all pinned solus strings. The condition described by (2) is no longer satisfied by any $\omega \in \Omega_{n}^{x,y}$ because 1s are now isolated. Hence the case $i = 0$ is removed from the summation, implying that $\kappa = 1$, which gives the recurrence for $G_n(x, y)$.

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