SATAKE-FURSTENBERG COMPACTIFICATIONS AND GRADIENT MAP

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Abstract. Let $G$ be a real semisimple Lie group with finite center and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of its Lie algebra. Let $K$ be a maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$ and let $\tau$ be an irreducible representation of $G$ on a complex vector space $V$. Let $h$ be a Hermitian scalar product on $V$ such that $\tau(G)$ is compatible with respect to $\text{SU}(V, h)$. We denote by $\mu_p : \mathcal{P}(V) \to \mathfrak{p}$ the $G$-gradient map and by $O$ the unique closed orbit of $G$ in $\mathcal{P}(V)$, which is a $K$-orbit, contained in the unique closed orbit of the Zariski closure of $\tau(G)$ in $\text{SU}(V, h)$. We prove that up to equivalence the set of irreducible representations of parabolic subgroups of $G$ induced by $\tau$ are completely determined by the facial structure of the polar orbitope $E = \text{conv}(\mu_p(O))$. Moreover, any parabolic subgroup of $G$ admits a unique closed orbit in $O$ which is well-adapted to $\mu_p$. These results are new also in the complex reductive case. The connection between $E$ and $\tau$ provides a geometrical description of the Satake compactifications without root data. In this context the properties of the Bourguignon-Li-Yau map are also investigated. Given a measure $\gamma$ on $O$, we construct a map $\Psi_\gamma$ from the Satake compactification of $G/K$ associated to $\tau$ and $E$. If $\gamma$ is a $K$-invariant measure then $\Psi_\gamma$ is an homeomorphism of the Satake compactification and $E$. Finally, we prove that for a large class of measures the map $\Psi_\gamma$ is surjective.

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Let $G$ be a semisimple noncompact real Lie group with finite center and let $K$ be a maximal compact subgroup of $G$. Then $X = G/K$ is a symmetric space of noncompact type. Let $\tau : G \to \text{SL}(V)$ be an irreducible complex representation. There exists a Hermitian scalar product $h$ on $V$ such that $\tau(G) \subseteq \text{SU}(V, h)^C$ is compatible with respect to the Cartan decomposition of $\text{SU}(V, h)^C = \text{SU}(V, h) \exp(\text{su}(V, h))$, where $\text{su}(V, h) = \text{Lie}(\text{SU}(V, h))$ \cite[4.32 Proposition]{27}. In this paper we investigate the projective representation $\tau : G \to \text{PSL}(V)$. We will write $gv$ instead of $\tau(g)v$ for simplicity. There is a corresponding $G$-gradient map $\mu_p : \mathbb{P}(V) \to \mathbb{R}$, $\mu_p^\beta(x) = \langle \mu_p(x), \beta \rangle$ where $\langle \cdot, \cdot \rangle$ is an $\text{Ad}(K)$-invariant scalar product on $\mathfrak{p}$, is given by the fundamental vector field $\beta^\#$ induced by the $G$-action on $\mathbb{P}(V)$, i.e., $\beta^\#(p) := \frac{d}{dt} \bigg|_{t=0} \exp(t\beta)p$.

If $\mathfrak{a}$ is an Abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$ and $\pi_a$ is the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{a}$, then $\mu_a = \pi_a \circ \mu_p$ is the $A = \exp(\mathfrak{a})$-gradient map. The gradient map has been extensively studied in \cite{30, 31, 32} where the authors develop a geometrical invariant theory for actions of real Lie groups on real submanifolds of complex spaces.

The Zariski closure of $\tau(G)$ in $U(V, h)^C$ is a semisimple complex Lie group \cite[Proposition 3.4]{30} that we denote by $U^C$. It acts irreducibly on $V$. By Borel-Weil Theorem there exists a unique closed orbit of the $U^C$-action on $\mathbb{P}(V)$, that we denote by $O'$. By a Theorem of Wolf (Lemma \cite{45}) there exists a unique $G$ closed orbit $O$ contained in $O'$. $O$ is a $K$-orbit \cite{32} and it captures much of the informations of the projective representation $\tau$ and of the $G$-gradient map $\mu_p$.

**Proposition 1.** Let $A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subseteq \mathfrak{p}$ is an Abelian subalgebra. Then

$$\mu_a(\mathbb{P}(V)) = \mu_a(O).$$

If $\mathfrak{a}$ is a maximal Abelian subalgebra, then $\mu_a(\mathbb{P}(V))$ is the convex hull of a Weyl-group orbit.

As an application, the weights of $\tau$ are contained in the convex hull of the Weyl group orbit of $\mu_*$, where $\mu_*$ is the highest weight with respect to a positive Weyl chamber (see Proposition \cite{68}).

For $\beta \in \mathfrak{p}$, let $G^{\beta+} = \{g \in G : \lim_{t \to -\infty} \exp(t\beta)g\exp(-t\beta) \text{ exists} \}$. Then $G^{\beta+}$ is parabolic with Levi factor $G^\beta = \{g \in G : \text{Ad}(g)(\beta) = \beta \}$. It is well-known that any parabolic subgroup of $G$ arises as a $G^{\beta+}$ for some $\beta \in \mathfrak{p}$ \cite{10, 35}. In Section 3 we prove (see Theorem \cite{53}) the following result.

**Theorem 2.** Let $\beta \in \mathfrak{p}$ and let $W$ the eigenspace associated to the maximum eigenvalue of $\beta$. Then:

a) $\mathbb{P}(W) = \{z \in \mathbb{P}(V) : \mu_\beta^\#(z) = \max_{y \in \mathbb{P}(V)} \mu_\beta^\# \}$;

b) $G^{\beta+}$ preserves $W$ and acts irreducibly on $W$;

c) $W$ is the unique subspace of $V$ satisfying b);

d) $G^{\beta+}$ has a unique closed orbit in $O$ given by

$$\mathbb{P}(W) \cap O = \{z \in O : \mu_\beta^\#(z) = \max_{y \in O} \mu_\beta^\# \}.$$ 

This orbit is connected, it is a $(K^\beta)^o$-orbit, and it is full in $\mathbb{P}(W)$.

Hence the unique closed orbit of $G^{\beta+}$ contained in $O$ is well-adapted to $\tau$. Observe that $\mu_p(O)$ is a $K$-orbit but it is not true in general that $\mu_p$ defines a homeomorphism between $O$ and $\mu_p(O)$, as in the complex case,. Therefore, Theorem 1.2 in \cite{10} pag. 582 does not apply.
bodies have been largely studied in [11, 23, 37, 46] amongst many others.

Face of projective representations of convex body in \( P \) is called an orbitope in the sense of convex geometry. In Section 3, we prove (see Theorem 6.5) the following result.

There exists a bijection between the compactifications. We briefly recall the construction.

Let \( P = E \cap a \), where \( a \subseteq p \) is a maximal Abelian subalgebra. By a theorem of Kostant \( P = \pi_a(E) = \mu_a(O) \) and it is a polytope [40]. Hence any face of \( P \) is exposed [18] and so any face of \( E \) is exposed as well. The compact group \( K \) acts on the set of the faces of \( E \) and the Weyl group \( W = N_K(a)/K^a \) acts on the set of faces of \( P \), where \( N_K(a) = \{ k \in K : \text{Ad}(k)(a) = a \} \) is the normalizer of \( a \) and \( K^a = \{ k \in K : \text{Ad}(k)(z) = z, \text{for any } z \in a \} \) is the centralizer of \( a \) in \( K \).

Let \( F(E) \) and \( F(P) \) denote the sets of faces of \( E \) and \( P \), respectively. In [10] the authors proved \( F(P)/W \cong F(E)/K \). This means that \( P \) completely determines the boundary structure of \( E \) in the sense of convex geometry. In Section 3 we prove (see Theorem 6.5) the following result.

**Theorem 3.** There exists a bijection between \( F(P)/W \) and the set of irreducible representations of parabolic subgroups of \( G \) induced by \( \tau \), up to equivalence.

This correspondence between \( \tau \) and the gradient map provides a geometrical description of the Satake compactification associated to \( \tau \).

There are many different compactifications of symmetric spaces of noncompact type and they have different properties. For instance, Satake compactifications is used in the proof of the celebrated Mostow rigidity Theorem [14]. Other applications are given in harmonic analysis and global geometry [16, 27] and in ergodic theory and probability theory in the context of sets of subgroups of \( GL(n, \mathbb{R}) \) [21, 24]. The Satake compactifications of \( X = G/K \) are obtained by embedding \( X \) into some compact ambient space. These embeddings are given by faithful projective representations of \( G \). They are \( G \)-equivariant and so the \( G \)-action on \( X \) extends to the compactifications. We briefly recall the construction.

Let \( \tau : G \rightarrow PSL(V) \) be a faithful irreducible projective representation. Then the map

\[
i_\tau : G/K \rightarrow \mathbb{P}(\mathcal{H}(V)), \quad gK \mapsto [gg^*],
\]

where \( g^* \) denotes the adjoint of \( g \) with respect to the Hermitian scalar product \( h \) and \( \mathcal{H}(V) \) denotes the set of Hermitian endomorphism of \( V \), is well-defined and injective. \( G \) acts on \( \mathbb{P}(\mathcal{H}(V)) \) as follows \( g[A] = [gAg^*] \), so \( i_\tau \) is \( G \)-equivariant. The closure \( X_\tau^S = \overline{i_\tau(X)} \) in \( \mathbb{P}(\mathcal{H}(V)) \) is called the Satake compactification associated to \( \tau \). Satake [16, 17] gave a description of the boundary \( \partial X_\tau^S = X_\tau^S - i_\tau(X) \) in terms of \( \mu_\tau \)-connected subsets of simple roots [37]. We replace \( \mu_\tau \)-connected subsets of simple roots with \( \tau \)-connected subspaces of \( V \). A \( \tau \)-connected subspace \( W \) of \( V \) is well-adapted to \( O \) and it is described in terms of the facial structure of \( E \). Indeed (see Section 4), we prove that \( \mu_\beta(\mathbb{P}(W) \cap O) = \text{ext} F_W \), where \( F_W \) is a face of \( E \) and \( \text{ext} F_W \)
denotes the set of extreme points of $F_W$. We recall that $K$-action on a $K$-orbit in $\mathfrak{p}$ extends to a $G$-action \[33\]. Since $\text{ext} F_W$ is contained in the $K$-orbit $\mu_p(\mathcal{O})$, we define

$$Q(F_W) := \{ a \in G : a \text{ ext } F_W = \text{ ext } F_W \}.$$ 

We prove

$$Q(W) := \{ g \in G : gW = W \} = Q(F_W)$$

and $Q(W)$ is a parabolic subgroup of $G$ acting irreducibly on $W$. If $\beta \in C^H_F$, i.e., $F_W = F_\beta(\mathcal{E})$ is the exposed face of $\mathcal{E}$ defined by $\beta$, which is fixed by $H_F = K \cap Q(W)$ (see Subsection 2.1), then $Q(W) = G^{3+}$ and

$$\mathbb{P}(W) \cap \mathcal{O} = \{ z \in \mathcal{O} : \mu_\beta^\mathfrak{p}(z) = \max_{x \in \mathcal{O}} \mu_\beta^\mathfrak{p} \}$$

is the unique closed orbit of $Q(W)$ contained in $\mathcal{O}$. This means that the information captured by $W$ only depends on the face $F_W$.

In Section 5, following the strategy of Biliotti and Ghigi \[8\], the boundary components of Satake are described by $\tau$-connected subspaces (see Theorem 71). This allows the interpretation of $\overline{X}_\tau$ in terms of rational self maps of $\mathcal{O}$ (see Lemma 74 and Theorem 76). Roughly speaking, if $W$ is a $\tau$-connected subspace then the boundary component of $\overline{X}_\tau$ corresponding to $W$ corresponds to rational maps $\mathcal{O} \rightarrow \mathcal{O}(Q(W))$, where $\mathcal{O}(Q(W))$ denotes the unique closed orbit of $Q(W)$ contained in $\mathcal{O}$. These maps are the composition of an automorphism of $\mathcal{O}(Q(W))$ and the gradient flow of $\mu_\beta^\mathfrak{p}$ restricted to the unstable manifold relative to the maximum, where $\beta \in \mathfrak{p}$ satisfies $F_W = F_\beta(\mathcal{E})$. As a consequence, the Satake compactification associated to $\tau$ is completely described in terms of the facial structure of $\mathcal{E}$. This is new and the main difference between our paper and the papers \[8, 39, 47\]. In Section 6 we apply these results to the Bourguignon-Li-Yau map.

Given a probability measure $\gamma$ on $\mathcal{O}$, we define

$$\Psi_\gamma : G/K \rightarrow \mathfrak{p}, \quad gK \mapsto \int_{\mathcal{O}} \mu_\mathfrak{p}(\sqrt{gg^*}x)d\gamma(x),$$

which is called Bourguignon-Li-Yau map. The element $\rho(g) = \sqrt{gg^*}$ is the unique positive Hermitian endomorphism of $V$ such that $\rho(g)^{-1}g \in K$ and so $g = \rho(g)k$ is the polar decomposition of $g$. This map has been studied at different levels of generality by Hersch \[28\], Millson and Zombro \[12\], Bourguignon, Li and Yau \[17\] and Biliotti and Ghigi \[8\], to determine upper bounds for the first eigenvalue of the Laplacian acting on functions. The image of the Bourguignon-Li-Yau lies in $\mathcal{E}$. We say that $\gamma$ is $\tau$-admissible if for any hyperplane $H$ of $\mathbb{P}(V)$, we have $\gamma(\mathcal{O} \cap H) = 0$. The interpretation of the elements of $\overline{X}_\tau$ as rational maps gives a way to extend the Bourguignon-Li-Yau map to $\overline{X}_\tau$. In Section 6 we prove the following result (Theorem 88 and Theorem 89).

**Theorem 4.** Let $\gamma$ be a $\tau$-admissible measure. Then $\Psi_\gamma$ extends to $\overline{X}_\tau$ as a continuous map and $\Psi_\gamma(\partial \overline{X}_\tau) \subseteq \partial \mathcal{E}$. Moreover, $\Psi_\gamma(\overline{X}_\tau) = \mathcal{E}$ and $\Psi_\gamma(X) = \text{Int}(\mathcal{E})$.

Since $0 \in \text{Int} \mathcal{E}$, see Lemma 81 for any measure $\zeta$ defined by a Riemannian metric on $\mathcal{O}$, we can move the components of the gradient map with an automorphism of $\mathcal{O}$ in such a way these functions become functions of zero mean with respect to $\zeta$. Hence, one may apply the Rayleigh Theorem to get an estimate of the first eigenvalue. We discuss this estimation in Section 8. If $\nu$ is a $K$-invariant measure on $\mathcal{O}$, see Theorem 88 we get the following result.

**Theorem 5.** $\Psi_\nu$ defines an homeomorphism of $\overline{X}_\tau$ onto $\mathcal{E}$. 
Satake compactification of a symmetric space of noncompact type is homeomorphic to a polar orbitope. This result is not new since Korány [39] showed that $\mathcal{X}_\tau$ is homeomorphic to $\mathcal{E}$. On the other hand the map used by Korány is different from $\Psi$, and the techniques used by Korány is different from ours. The above Theorems generalize results proved in [8] for symmetric space of noncompact type of type IV. We point out that some parts of the paper [8] are extremely technical (see Section 2.4 p. 249 – 251 and Section 2.5) since when the paper [8] was written the authors were not aware of the facial structure of invariant convex compact subsets of polar representations [9, 10, 11]. The results of this paper generalize and better clarify some results proved in [8]. Finally, we give a short proof of a Theorem of Moore [43] (see Theorem 93), following the strategy of [8], stating that the Satake and Furstenberg compactifications are homeomorphic.

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2. Preliminaries

2.1. Convex geometry. It is useful to recall a few definitions and results regarding convex sets. The reader may refer for instance to [48] for more details.

Let $V$ be a real vector space with a scalar product $\langle \cdot, \cdot \rangle$ and let $E \subset V$ be a compact convex subset. The relative interior of $E$, denoted $\text{relint}E$, is the interior of $E$ in its affine hull. A face $F$ of $E$ is a convex subset $F \subset E$ with the following property: if $x, y \in E$ and $\text{relint}[x, y] \cap F \neq \emptyset$, then $[x, y] \subset F$. We say that a point $x \in E$ is an extreme point, and write $x \in \text{ext}E$, if $\{x\}$ is a face. By a Theorem of Minkowski, $E$ is the convex hull of its extreme points [48, p.19]. The faces of $E$ are closed [48, p. 62]. A face distinct from $E$ and $\emptyset$ will be called a proper face. The support function of $E$ is the function $h_E : V \to \mathbb{R}$, $h_E(u) = \max_{x \in E} \langle x, u \rangle$. If $u \neq 0$, the hyperplane $H(E, u) := \{x \in E : \langle x, u \rangle = h_E(u)\}$ is called the supporting hyperplane of $E$ for $u$. The set

$$F_u(E) := E \cap H(E, u)$$

is a face and it is called the exposed face of $E$ defined by $u$. In general not all faces of a convex subset are exposed.

Lemma 7 ([9, Lemma 3]). If $F$ is a face of a convex set $E$, then $\text{ext}F = F \cap \text{ext}E$.

Lemma 8 ([9, Lemma 8]). If $E$ is a compact convex set and $F \subset E$ is a face, then there is a chain of faces $F_0 = F \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E$ which is maximal, in the sense that for any $i$ there is no face of $E$ strictly contained between $F_{i-1}$ and $F_i$.

Lemma 9 ([Prop.5]). If $F \subset E$ is an exposed face, the set $C_F := \{u \in V : F = F_u(E)\}$ is a convex cone. If $K$ is a compact subgroup of $O(V)$ that preserves both $E$ and $F$, then $C_F$ contains a fixed point of $K$.

We denote by $C_F^K$ the elements of $C_F$ fixed by a compact group $K$.

The following result is well-known and a proof is given in [48, p. 62]

Theorem 10. If $E$ is a compact convex set and $F_1, F_2$ are distinct faces of $E$, then $\text{relint}F_1 \cap \text{relint}F_2 = \emptyset$. If $G$ is a nonempty convex subset of $E$ which is open in its affine hull, then $G \subset \text{relint}F$ for some face $F$ of $E$. Therefore $E$ is the disjoint union of the relative interiors of its faces.
The following result will be used to determine the image of the gradient map.

**Proposition 11.** Let $C_1 \subseteq C_2$ be two compact convex subsets of $V$. Assume that for any $\beta \in V$ we have

$$\max_{y \in C_1} \langle y, \beta \rangle = \max_{y \in C_2} \langle y, \beta \rangle.$$ 

Then $C_1 = C_2$.

**Proof.** We may assume without loss of generality that the affine hull of $C_2$ is $V$. Assume by contradiction that $C_1 \subsetneq C_2$. Since $C_1$ and $C_2$ are both compact, it follows that there exists $p \in \partial C_1$ such that $p \notin C_2$. Since every face of a compact convex set is contained in an exposed face [38], there exists $\beta \in V$ such that

$$\max_{y \in C_1} \langle y, \beta \rangle = \langle p, \beta \rangle.$$ 

This means the linear function $x \mapsto \langle x, \beta \rangle$ restricted on $C_2$ achieves its maximum at an interior point which is a contradiction. 

\[\square\]

### 2.2. Compatible subgroups and parabolic subgroups.

In the sequel we always refer to [16, 27, 32], see also [15, 36].

Let $U$ be compact connected Lie group. Let $U^C$ be its universal complexification which is a linear reductive complex algebraic group [11]. We denote by $\theta$ both the conjugation map $\theta : u^C \rightarrow u^C$ and the corresponding group isomorphism $\theta : U^C \rightarrow U^C$. Let $f : U \times iu \rightarrow U^C$ be the diffeomorphism $f(g, \xi) = g \exp \xi$. Let $G \subseteq U^C$ be a closed subgroup. Set $K := G \cap U$ and $p := g \cap iu$. We say that $G$ is compatible if $f(K \times p) = G$. The restriction of $f$ to $K \times p$ is then a diffeomorphism onto $G$. Hence $g = k \oplus p$ is the familiar Cartan decomposition and so $K$ is a maximal compact subgroup of $G$. Note that $G$ has finitely many connected components. Since $U$ can be embedded in $\text{GL}(N, \mathbb{C})$ for some $N$, and any such embedding induces a closed embedding of $U^C$, any compatible subgroup is a closed linear group. Moreover $g$ is a real reductive Lie algebra, hence $g = z(g) \oplus [g, g]$. Denote by $G_{ss}$ the analytic subgroup tangent to $[g, g]$. Then $G_{ss}$ is closed and $G^o = Z(G)^o \cdot G_{ss}$ [39, p. 442], where $G^o$, respectively $Z(G)^o$, denotes the connected component of $G$, respectively of $Z(G)$, containing $e$. The following lemma is well-known.

**Lemma 12.**

- a) If $G \subseteq U^C$ is a compatible subgroup, and $H \subseteq G$ is closed and $\theta$-invariant, then $H$ is compatible if and only if $H$ has only finitely many connected components.
- b) If $G \subseteq U^C$ is a connected compatible subgroup, then $G_{ss}$ is compatible.
- c) If $G \subseteq U^C$ is a compatible subgroup, and $E \subseteq p$ is any subset, then

$$G^E = \{g \in G : \text{Ad}(g)(z) = z, \forall z \in E\}$$

is compatible. Indeed, $G^E = K^E \exp(p^E)$, where

$$K^E = K \cap G^E = \{g \in K : \text{Ad}(g)(z) = z, \forall z \in E\}$$

and $p^E = \{v \in p : [v, E] = 0\}$.

A subalgebra $q \subseteq g$ is parabolic if $q^C$ is a parabolic subalgebra of $g^C$. One way to describe the parabolic subalgebras of $g$ is by means of restricted roots. If $a \subseteq p$ is a maximal subalgebra, let $\Delta(g, a)$ be the (restricted) roots of $g$ with respect to $a$, let $g_\lambda$ denote the root space corresponding to $\lambda$ and let $g_0 = m \oplus a$, where $m = z_e(a) = z(a) \cap \mathfrak{k}$. We denote by $z(a) = \{x \in g : [x, a] = 0\}$. Let
Π ⊂ Δ(g, a) be a base and let Δ⁺ be the set of positive roots. If I ⊂ Π, set Δ_I := span(I) ∩ Δ. Then

\[ \mathfrak{q}_I := \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Delta_I \cup \Delta_+} \mathfrak{g}_\lambda \]

is a parabolic subalgebra. Conversely, if \( \mathfrak{q} \subset \mathfrak{g} \) is a parabolic subalgebra, then there are a maximal subalgebra \( \mathfrak{a} \subset \mathfrak{p} \) contained in \( \mathfrak{q} \), a base \( \Pi \subset \Delta(\mathfrak{g}, \mathfrak{a}) \) and a subset \( I \subset \Pi \) such that \( \mathfrak{q} = \mathfrak{q}_I \). We can further introduce

\[ \mathfrak{a}_I := \bigcap_{\lambda \in I} \ker \lambda \quad \mathfrak{a}^I := \mathfrak{a}_I^\perp \]

\[ \mathfrak{n}_I = \bigoplus_{\lambda \in \Delta_+ - \Delta_I} \mathfrak{g}_\lambda \quad \mathfrak{m}_I := \mathfrak{m} \oplus \mathfrak{a}^I \oplus \bigoplus_{\lambda \in \Delta_I} \mathfrak{g}_\lambda. \]

Then \( \mathfrak{q}_I = \mathfrak{m}_I \oplus \mathfrak{a}_I \oplus \mathfrak{n}_I \). Since \( \theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda} \), it follows that \( \mathfrak{q}_I \cap \theta \mathfrak{q}_I = \mathfrak{a}_I \oplus \mathfrak{m}_I \). This latter Lie algebra coincides with the centralizer of \( \mathfrak{a}_I \) in \( \mathfrak{g} \). It is a Levi factor of \( \mathfrak{q}_I \) and

\[ \mathfrak{a}_I = \mathfrak{z}(\mathfrak{q}_I \cap \theta \mathfrak{q}_I) \cap \mathfrak{p}. \]

If we denote by \( \Delta_- \) the set of negative root, then \( \mathfrak{n}_I^- = \bigoplus_{\lambda \in \Delta_- - \Delta_I} \mathfrak{g}_\lambda \) is a subalgebra. It follows from standard commutation relations that \( \mathfrak{z}(\mathfrak{a}_I) \) normalizes \( \mathfrak{n}_I \) and \( \mathfrak{n}_I^- \) and the centralizer of \( \mathfrak{a}^I \) in either is reduced to zero. Then, keeping in mind \( \mathfrak{g} = \mathfrak{n}_I^- \oplus \mathfrak{q}_I, \mathfrak{q}_I \) is self-normalizing.

**Definition 16.** A subgroup \( Q \) of \( G \) is called parabolic if it is the normalizer of a parabolic subalgebra in \( \mathfrak{g} \).

The normalizer of \( \mathfrak{q}_I \) is the standard parabolic subalgebra \( Q_I \). Let \( R_I \) and let \( A_I \) be the unique connected Lie subgroups of \( G \) with Lie algebra equals to \( \mathfrak{n}_I \) and \( \mathfrak{a}_I \) respectively. \( R_I \) is the unipotent radical of \( Q_I \). The group \( Q_I \) is the semidirect product of \( R_I \) and of \( Z(A_I) \), i.e., the centralizer of \( A_I = \exp(a_I) \) in \( G \). Moreover, \( Z(A_I) = A_I \times M_I \), where \( M_I \) is a closed Lie group whose Lie algebra is \( \mathfrak{m}_I \). It is not connected in general but it is compatible. Since \( M_I \) is stable with respect to the Cartan involution, \( K_I = M_I \cap K \) is maximal compact in \( M_I \). It is also maximal compact in \( Q_I \) and the quotient

\[ X_I = M_I/K_I = Q_I/K_I A_I N_I \]

is a symmetric space of noncompact type for \( M_I \). Finally, as a consequence of the Iwasawa decomposition \( G = NAK \), where \( N = \exp(\mathfrak{n}) \), \( \mathfrak{n} = \mathfrak{n}_0 \), and \( NA \subset Q_I \), and so the following result holds

**Proposition 17.** \( G = KQ_I \).

Another way to describe parabolic subgroups of \( G \) is the following.

If \( \beta \in \mathfrak{p} \), the endomorphism \( \text{ad}(\beta) \) is diagonalizable over \( \mathbb{R} \). Denote by \( V_\lambda(\text{ad}(\beta)) \) the eigenspace of \( \text{ad}(\beta) \) corresponding to the eigenvalue \( \lambda \). Set

\[ \mathfrak{g}^{\beta+} := \bigoplus_{\lambda > 0} V_\lambda(\text{ad}(\beta)), \]

\[ \mathfrak{r}^{\beta+} := \bigoplus_{\lambda > 0} V_\lambda(\text{ad}(\beta)). \]
\[
G^{3+} := \{ g \in G : \lim_{t \to -\infty} \exp(t\beta)g \exp(-t\beta) \text{ exists} \}, \\
R^{3+} := \{ g \in G : \lim_{t \to -\infty} \exp(t\beta)g \exp(-t\beta) = e \}.
\]

The following result characterizes completely the parabolic subgroups of \( G \). The result is classical and a proof is given in [10].

**Lemma 18.** \( G^{3+} \) is a parabolic subgroup of \( G \) with Lie algebra \( \mathfrak{g}^{3+} \) and it is the semidirect product of \( G^3 \) with \( R^{3+} \). Moreover, \( G^3 \) is a Levi factor, \( R^{3+} \) is connected with Lie algebra \( \mathfrak{r}^{3+} \) and it is the unipotent radical of \( G^{3+} \). Every parabolic subgroup of \( G \) equals \( G^{3+} \) for some \( \beta \in \mathfrak{p} \).

**2.3. Basic properties of the gradient map.** Let \((Z, \omega)\) be a Kähler manifold. Assume that \( U^\mathfrak{c} \) acts holomorphically on \( Z \), that \( U \) preserves \( \omega \) and that there is a momentum map \( \mu : Z \to \mathfrak{u} \). If \( \xi \in \mathfrak{u} \) we denote by \( \xi^\# \) the induced vector field on \( Z \), i.e., \( \xi^\#(p) = \frac{d}{dt} \exp(t\xi)p \), and we let \( \mu^\xi \in C^\infty(Z) \) be the function \( \mu^\xi(z) := \langle \mu(z), \xi \rangle \), where \( \langle \cdot, \cdot \rangle \) is an \( \text{Ad}(U) \)-invariant scalar product on \( \mathfrak{u} \). That \( \mu \) is the momentum map means that it is \( U \)-equivariant and that \( d\mu^\xi = i_{\xi^\#}\omega \).

Let \( G \subset U^\mathfrak{c} \) be compatible. If \( z \in Z \), let \( \mu_p(z) \in \mathfrak{p} \) denote \( -i \) times the component of \( \mu(z) \) in the direction of \( ip \). In other words, if we also denote by \( \langle \cdot, \cdot \rangle \) the \( \text{Ad}(U) \)-invariant scalar product on \( iu \) requiring the multiplication by \( i \) is an orthogonal map from \( u \) onto \( iu \), then the formula \( \langle \mu_p(z), \beta \rangle = \langle i\mu(z), \beta \rangle = \langle \mu(z), -i\beta \rangle \) for any \( \beta \in \mathfrak{p} \) defines the gradient map
\[
\mu_p : Z \to \mathfrak{p}.
\]

Let \( \mu_\beta \in C^\infty(Z) \) be the function \( \mu_\beta(z) = \langle \mu_p(z), \beta \rangle = \mu^{-i\beta}(z) \). Let \( \langle \cdot, \cdot \rangle \) be the Kähler metric associated to \( \omega \), i.e. \( \langle v, w \rangle = \omega(v, Jw) \). Then \( \beta^\# \) is the gradient of \( \mu_\beta \). If \( M \subset Z \) is a locally closed \( G \)-invariant submanifold, then \( \beta^\# \) is the gradient of \( \mu_\beta|_M \) with respect to the induced Riemannian structure on \( M \). From now on we always assume that \( M \) is compact and connected.

**Theorem 19.** [Slice Theorem [32 Thm. 3.1]] If \( x \in M \) and \( \mu_p(x) = 0 \), there are a \( G_x \)-invariant decomposition \( T_xM = \mathfrak{g} \oplus x \oplus W \), open \( G_x \)-invariant subsets \( S \subset W \), \( \Omega \subset M \) and a \( G \)-equivariant diffeomorphism \( \Psi : G \times G_x S \to \Omega \), such that \( 0 \in S, x \in \Omega \) and \( \Psi([e, 0]) = x \).

Here \( G \times G_x S \) denotes the associated bundle with principal bundle \( G \to G/G_x \).

**Corollary 20.** If \( x \in M \) and \( \mu_p(x) = \beta \), there are a \( G^\beta \)-invariant decomposition \( T_xM = \mathfrak{g}^\beta \oplus x \oplus W \), open \( G^\beta \)-invariant subsets \( S \subset W \), \( \Omega \subset M \) and a \( G^\beta \)-equivariant diffeomorphism \( \Psi : G^\beta \times G_x S \to \Omega \), such that \( 0 \in S, x \in \Omega \) and \( \Psi([e, 0]) = x \).

This follows applying the previous theorem to the action of \( G^\beta \) with the gradient map \( \widehat{\mu_\omega}^\beta := \mu_{\omega^\beta} - i\beta \), where \( \mu_{\omega^\beta} \) denotes the projection of \( \mu \) onto \( \omega^\beta \). See [32] p.169 and [49] for more details.

**Corollary 21.** If \( \beta \in \mathfrak{p} \) and \( x \in M \) is a critical point of \( \mu_\beta \), then there are open invariant neighborhoods \( S \subset T_xM \) and \( \Omega \subset M \) and an \( \mathbb{R} \)-equivariant diffeomorphism \( \Psi : S \to \Omega \), such that \( 0 \in S, x \in \Omega \), \( \Psi(0) = x \). Here \( t \in \mathbb{R} \) acts as \( d\varphi_t(x) \) on \( S \) and as \( \varphi_t \) on \( \Omega \).

**Proof.** The subgroup \( H := \exp(\mathbb{R}\beta) \) is compatible. It is enough to apply the previous corollary to the \( H \)-action at \( x \). \( \square \)
Let \( x \in \text{Crit}(\mu_p^\beta) = \{ y \in M : \beta^#(y) = 0 \} \). Let \( D^2\mu_p^\beta(x) \) denote the Hessian, which is a symmetric operator on \( T_x M \). Denote by \( V_- \) (respectively \( V_+ \)) the sum of the eigenspaces of the Hessian of \( \mu_p^\beta \) corresponding to negative (resp. positive) eigenvalues. Denote by \( V_0 \) the kernel. Since the Hessian is symmetric we get an orthogonal decomposition
\[
T_x M = V_- \oplus V_0 \oplus V_+.
\]

Let \( \alpha : G \to M \) be the orbit map: \( \alpha(g) := gx \). The differential \( d\alpha_e \) is the map \( \xi \mapsto \xi^#(x) \). The following result is well-known. A proof is given in [10].

**Proposition 23.** If \( \beta \in p \) and \( x \in \text{Crit}(\mu_p^\beta) \) then
\[
D^2\mu_p^\beta(x) = d\beta^#(x).
\]
Moreover \( d\alpha_e(\mu^\beta \pm) \subset V_\pm \) and \( d\alpha_e(\mu^\beta) \subset V_0 \). If \( M \) is \( G \)-homogeneous these are equalities.

**Corollary 24.** For every \( \beta \in p \), \( \mu_p^\beta \) is a Morse-Bott function.

**Proof.** Corollary [21] implies that \( \text{Crit}(\mu_p^\beta) \) is a smooth submanifold. Since \( T_x \text{Crit}(\mu_p^\beta) = V_0 \) for \( x \in \text{Crit}(\mu_p^\beta) \), the first statement of Proposition [23] shows that the Hessian is nondegenerate in the normal directions. \( \square \)

Let \( g \in G \) and let \( \xi \in p \). It is easy check that
\[
(\text{dg})_p(\xi^#) = (\text{Ad}(g)(\xi))^#(gp).
\]
Therefore \( G^\beta \) preserves \( \text{Crit}(\mu_p^\beta) \).

**Corollary 25.** If \( M \) is \( G \)-homogeneous then \( G^\beta \)-orbits are open and closed in \( \text{Crit}(\mu_p^\beta) \).

**Proof.** Since \( T_x \text{Crit}(\mu_p^\beta) = V_0 = T_x G^\beta \cdot x \) for \( x \in \text{Crit}(\mu_p^\beta) \), the result follows. \( \square \)

Let \( c_1 > \cdots > c_r \) be the critical values of \( \mu_p^\beta \). The corresponding level sets of \( \mu_p^\beta \), \( C_i := (\mu_p^\beta)^{-1}(c_i) \) are submanifolds which are union of components of \( \text{Crit}(\mu_p^\beta) \). The function \( \mu_p^\beta \) defines a gradient flow generated by its gradient which is given by \( \beta^# \). By Theorem [19] it follows that for any \( x \in M \) the limit:
\[
\varphi_\infty(x) := \lim_{t \to +\infty} \exp(t\beta)x,
\]
exists. Let us denote by \( W_i^\beta \) the unstable manifold of the critical component \( C_i \) for the gradient flow of \( \mu_p^\beta \):
\[
W_i^\beta := \{ x \in M : \varphi_\infty(x) \in C_i \}.
\]
Applying Theorem [19] we have the following well-known decomposition of \( M \) into unstable manifolds with respect to \( \mu_p^\beta \).

**Theorem 27.** In the above assumption, we have
\[
M = \bigsqcup_{i=1}^r W_i^\beta,
\]
and for any \( i \) the map:
\[
(\varphi_\infty)|_{W_i} : W_i^\beta \to C_i,
\]
is a smooth fibration with fibres diffeomorphic to \( \mathbb{R}^{l_i} \) where \( l_i \) is the index (of negativity) of the critical submanifold \( C_i \).
Let $\beta \in \mathfrak{p}$. Proposition 23 implies that $\text{Max}(\beta) = \{x \in M : \mu^\beta_p(x) = \max_{y \in M} \mu^\beta_p\}$ is a smooth possibly disconnected submanifold of $M$.

**Lemma 29.** $\text{Max}(\beta)$ is $G^{\beta +}$-invariant.

*Proof.* $G^\beta$ preserves $\text{Crit}(\mu^\beta_p)$. By Proposition 23 $R^{\beta +}$ acts trivially on $\text{Max}(\beta)$. Therefore, it is enough to prove that $G^\beta$ preserves $\text{Max}(\beta)$.

Let $y \in \text{Max}(\beta)$. Since $\mu^\beta_p$ is $K$-equivariant, $K^\beta$ preserves $\text{Max}(\beta)$. Let $\xi \in \mathfrak{p}^\beta$ and let $\gamma(t) = \exp(t\xi)y$. Since $\beta_{\#}(\gamma(t)) = 0$ it follows that $\mu^\beta_p(\gamma(t))$ is constant and so $\exp(t\xi)y \in \text{Max}(\beta)$. Hence, keeping in mind that $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$, the result follows. $\square$

**Proposition 30.** $\text{Max}(\beta)$ contains a closed orbit of $G^{\beta +}$ which coincides with a $K^\beta$-orbit.

*Proof.* $(G^\beta)^o$ preserves any connected component of $\text{Max}(\beta)$. The restriction of $\mu^\beta_p$ on any connected component defines a $(G^\beta)^o$-gradient map [32]. By Corollary 6.11 in [32] p. 21, see also Proposition 31 $(G^\beta)^o$ has a closed orbit which coincides with a $(K^\beta)^o$-orbit. Since $G^\beta$ has a finite number of connected components and any connected component of $G^\beta$ intersects $K^\beta$, it follows that $G^\beta$ has a closed orbit which coincides with a $K^\beta$-orbit. This orbit is a closed orbit of $G^{\beta +}$ since $R^{\beta +}$ acts trivially on $\text{Max}(\beta)$, concluding the proof. $\square$

Using an $\text{Ad}(K)$-invariant inner product of $\mathfrak{p}$, we define $\nu^\beta_p(z) := \frac{1}{2} \parallel \mu^\beta_p(z) \parallel^2$. The function $\nu^\beta_p$ is $K$-invariant and it is called the norm square function. The following result is proved in [32] (see Corollary 6.11 and Corollary 6.12 p. 21).

**Proposition 31.** Let $x \in M$. Then:

- if $\nu^\beta_p$ restricted to $G \cdot x$ has a local maximum at $x$, then $G \cdot x = K \cdot x$
- if $G \cdot x$ is compact, then $G \cdot x = K \cdot x$

A strategy to analyzing the $G$-action on $M$ is to view $\nu^\beta_p$ as generalized Morse function. In [32] the authors proved the existence of a smooth $G$-invariant stratification of $M$ and they studied its properties.

### 2.4. Satake compactifications.

In this section we always refer to [16, 27, 47].

Let $G$ be a real noncompact connected semisimple Lie group $G$ with finite center and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of its Lie algebra. Let $K$ be a maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$ and let $\tau$ be an irreducible representation of $G$ with finite kernel on a complex vector space $V$. We also assume there exists a $K$-invariant Hermitian product $h$ on $V$ such that $\tau(G) \subset \text{U}(V,h)^C$ is compatible. With these data Satake [47] constructed a compactification $\overline{X}_\tau$ by the symmetric space $X = G/K$. A good references of symmetric spaces is [34]. We wish to recall the construction of the Satake compactifications and some of their relevant properties. Proofs can be found in the book [16] §I.1], see also [27], which we follow for most of the notation.

Put

$$\mathcal{H}(V) = \{A \in \text{End}(V) : A = A^*\},$$

where $A^*$ denotes the adjoint of $A$ with respect to $h$. Denote by $\pi : \mathcal{H}(V) - \{0\} \to \mathbb{P}(\mathcal{H}(V))$ the canonical projection and set

$$\mathcal{P}(V) = \pi(\{A \in \mathcal{H}(V) : A > 0\}) \subset \mathbb{P}(\mathcal{H}(V)).$$

$\mathcal{P}(V)$ consists of points $[A]$ such that $A$ is invertible and all its eigenvalues have the same sign. The following result is easy to check.


Lemma 32. (a) \( \overline{P(V)} = \pi(\{A \in H(V) : A \neq 0, A \geq 0\}) \). (b) The restriction of \( \pi \) to \( \{A \in H(V) : A > 0, \det A = 1\} \) is a homeomorphism onto \( \overline{P(V)} \). (c) The restriction of \( \pi \) to \( \{A \in H(V) : A \geq 0, \text{tr} A = 1\} \) is a homeomorphism onto \( \overline{P(V)} \).

Definition 33. For \( G, K, \tau, h \) as before, set

\[
i_\tau : X := G/K \to \overline{P(V)} \quad i_\tau(gK) = [\tau(g)\tau(g)^*].
\]

The Satake compactification of \( X \) associated to \( \tau \) and \( \langle \cdot, \cdot \rangle \) is the space \( \overline{X}^S_\tau := i_\tau(X) \). The closure is taken in \( \mathbb{P}(H(V)) \).

Since SL(V) and hence \( G \) acts on \( \mathbb{P}(H(V)) \) by

\[
g \cdot [A] := [gAg^*],
\]

\( \overline{X}^S_\tau \) is a \( G \)-compactification. We stress that \( \overline{X}^S_\tau \) depends only on \( G, K, \tau \) and \( h \). Satake gave a thorough description of the boundary \( \partial \overline{X}^S_\tau := \overline{X}^S_\tau - i_\tau(X) \) in terms of root data.

Let \( \Delta(g, a) \) be the (restricted) roots of \( g \) with respect to \( a \). Let \( \Delta(g, a) \) be the set of simple roots in \( \Delta(g, a) \) determined by the positive chamber \( a^+ \). Let \( \mu_\tau \) be the highest weight of \( \tau \) with respect to the partial ordering determined by \( \Delta \). If \( \lambda \) is another weight of \( \tau \), then it has the form

\[
\lambda = \mu_\tau - \sum_{\alpha \in \Delta(g, a)} c_{\alpha, \lambda} \alpha,
\]

where \( c_{\alpha, \lambda} \) are non negative integers. The support of \( \lambda \) is the set \( \text{supp}(\lambda) := \{\alpha \in \Pi : c_{\alpha, \lambda} > 0\} \).

Definition 35. A subset \( I \subset \Pi \) is \( \mu_\tau \)-connected if \( I \cup \{\mu_\tau\} \) is connected, i.e., it is not the union of subsets orthogonal with respect to the Killing form.

Connected components of \( I \) are defined as usual. The \( I \) is \( \mu_\tau \)-connected subset if and only if any connected component of \( I \) contains at least one element \( \alpha \) which is not orthogonal to \( \mu_\tau \).

Lemma 36 (\cite{7} Lemma 5 p. 87]). \( I \subset \Pi \) is \( \mu_\tau \)-connected if and only if \( I = \text{supp}(\lambda) \) for some weight \( \lambda \) of \( \tau \).

For example \( \emptyset = \text{supp}(\mu_\tau) \) and \( \Pi \) is \( \mu_\tau \)-connected since \( \tau \) is nontrivial on any simple factor of \( G \). Given \( \lambda \) a weight of \( \tau \), we denote by \( V_\lambda \) the corresponding eigenspace.

Let \( I \) be a \( \mu_\tau \)-connected subset of \( \Delta(g, a) \) and let

\[
V_I = \bigoplus_{\text{supp}(\lambda) \subset I} V_\lambda.
\]

Let \( Q_I \) be the normalizer of \( q_I \), see Section 2.2. \( Q_I \) is a parabolic subgroup of \( G \) given by

\[
Q_I = N_I A_I M_I.
\]

\( M_I \) is a Levi factor of \( Q_I \) that it is non connected in general.

Lemma 37 (\cite{7} Lemma 8 p. 89]). The subspace \( V_I \) is invariant under \( \tau(g), g \in Q_I \) and the induced representation of \( M_I \) on \( V_I \), denoted \( \tau_I : M_I \to GL(V_I) \), is a multiple of an irreducible faithful one.

Definition 38. If \( I \subset \Pi \) is \( \mu_\tau \)-connected, denote by \( I' \) the collection of all simple roots orthogonal to \( \{\mu_\tau\} \cup I \). The set \( J := I \cup I' \) is called the \( \mu_\tau \)-saturation of \( I \).

\( I \) is the largest \( \mu_\tau \)-connected subset contained in \( J \). Observe that union of \( \tau \)-connected subspaces is a \( \tau \)-connected subspace. Then \( I \) is the union of all \( \tau \)-connected subspaces contained in \( J \) and so the largest \( \mu_\tau \)-connected of \( J \) is unique.
Lemma 39 ([16] Prop. I.4.29 p. 70]). If $I$ is $\mu_\tau$-connected, then $Q_I = \{ g \in G : \tau(g)V_I = V_I \}$.

Fix a $\mu_\tau$-connected subset $I$. If $A \in \text{End}(V_I)$, let $A \oplus 0$ denote the extension of $A$ that is trivial on $V_I^\perp$. If $\pi_I : V \to V_I$ denotes orthogonal projection and $j_I : V_I \hookrightarrow V$ denotes the inclusion, then $A \oplus 0 = j_I \circ A \circ \pi_I$ and the map
\[
\psi_I : P(H(V_I)) \to P(H(V)), \quad \psi_I([A]) = ([A \oplus 0])
\]
embeds $P(V_I)$ in $P(V)$. Note that $K_I = M_I \cap K$ is a maximal compact subgroup of $M_I$ and $X_I = M_I/K_I$ is again a symmetric space of noncompact type. Hence we have a map $i_{\tau_I} : X_I \to P(V_I)$ defined as in ([34]). Finally define
\[
i_I = \psi_I \circ i_{\tau_I} : X_I \to P(V).
\]

Theorem 41 ([16] Cor. I.4.32]).
\[
X^S_\tau = \bigsqcup_{\mu_\tau\text{-connected }I} G.i_I(X_I).
\]

If $I = \Pi$ then $i_I(X_I) = X$. The sets $g.i_I(X_I)$ with $g \in G$ and $I \subseteq \Pi$ are called boundary components.

Lemma 42 ([16] Prop. I.4.29]). The boundary components are disjoint: if $I$ is $\mu_\tau$-connected and $g \in G$ then $g.i_I(X_I) \cap i_I(X_I) \neq \emptyset$ if and only if $g.i_I(X_I) = i_I(X_I)$ if and only if $g \in Q_I$.

3. Finite dimensional representations and gradient map

Let $G$ be real noncompact Lie group with finitely many connected components. $G$ admits maximal compact subgroups which are conjugate under the identity component $G^0$ ([34, 36]). In this section we always assume that $G$ is reductive, connected, linear and the center of $G$ is compact. If we denote by $K$ a maximal compact subgroup of $G$, then $G$ admits a Cartan involution $\theta$ with the fixed points set containing $K$. Since the center of $G$ is compact then it is contained in $K$ ([36]). We have the classical Cartan decomposition
\[ g = \mathfrak{k} \oplus \mathfrak{p}, \]
with $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The restriction of the Killing form of $g$ on $\mathfrak{p}$ defines a $G$-invariant metric on $G/K$ and with this metric $G/K$ is a simply connected complete symmetric space of noncompact type, i.e., of nonpositive curvature ([34]).

Let $V$ be a finite dimensional vector space over $\mathbb{C}$ and let $\tau : G \to \text{GL}(V)$ be a representation. We say that $\tau$ is irreducible if it has no non trivial invariant subspaces. The representation of $g$ associated by differentiation with a representation $\tau$ of $G$ is called the tangent representation associated with $\tau$ and it is also denoted by $\tau$. By ([27, 4.32 Proposition]), there exists a Hermitian scalar product $h$ on $V$ such that $\tau(G) \subset U(V, h)\mathbb{C}$ is compatible, where $U(V, h)$ is the unitary group of $V$ with respect to the Hermitian scalar product $h$. This means that $\tau(K) \subset U(V, h)$, $\tau(G) = \tau(K) \exp(\tau(\mathfrak{p}))$ with $\tau(\mathfrak{p}) \subset i\text{Lie}(U(V, h))$. From now on we identify $G$ with $\tau(G)$. The Zariski closure of $G$ in $U(V, h)\mathbb{C}$ is given by $U^C$, where $U$ is a compact connected subgroup of $U(V, h)$ and $G$ is compatible with respect to the Cartan decomposition of $U^C$ ([30] Lemma 1 p.3]. Moreover, keeping in mind that $G$ has a compact center, by Propositions 1 and 2 in ([30] p.4], there exist compact connected Lie subgroups $U_0$ and $U_1$ of $U$ which centralize each other, such that

a) $U^C = U_0^C \cdot U_1^C$ and the intersection $U_0 \cap U_1$ is a finite subgroup of the center of $U$;
b) $G = G_0 \cdot U_1^C$, where $G_0$ is a real form of $U_0^S$ which is compatible with the Cartan decomposition of $U_0^C$. Moreover, $g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, $u_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ and $G_0 = K_0 \exp(\mathfrak{p}_0)$.
Let \( \mu : \mathbb{P}(V) \to \mathbb{u} \) be the momentum map of \( U \) and let \( \mu_p : \mathbb{P}(V) \to \mathbb{p} \) be the \( G \)-gradient map associated to \( \mu \). Let \( \beta \in \mathfrak{p} \) and let

\[
\text{Max}(\beta) = \{ x \in \mathbb{P}(V) : \mu_p^\beta(x) = \max_{z \in \mathbb{P}(V)} \mu_p^\beta \}.
\]

Let \( \lambda_1 > \cdots > \lambda_k \) be the eigenvalues of \( \beta \). We denote by \( V_1, \ldots, V_k \) the corresponding eigenspaces. In view of the orthogonal decompositions \( V = V_1 \oplus \cdots \oplus V_k \), \( \mu_p^\beta \) is given by

\[
\mu_p^\beta([x_1 + \cdots + x_k]) = \frac{\lambda_1 \| x_1 \|^2 + \cdots + \lambda_k \| x_k \|^2}{\| x_1 \|^2 + \cdots + \| x_k \|^2}.
\]

Therefore \( \text{Max}(\beta) = \mathbb{P}(V_1) \). Since the gradient flow of \( \mu_p^\beta \) is given by

\[
\mathbb{R} \times \mathbb{P}(V) \to \mathbb{P}(V), \quad (t, [x_1 + \cdots + x_k]) \mapsto [e^{t\lambda_1}x_1 + \cdots + e^{t\lambda_k}x_k],
\]

the critical points of \( \mu_p^\beta \) are union of proper projective subspaces, \( \mathbb{P}(V_1) \cup \cdots \cup \mathbb{P}(V_k) \), of \( \mathbb{P}(V) \).

Set \( \Gamma = \exp(\mathbb{R}\beta) \). We recall that a submanifold \( M \subset \mathbb{P}(V) \) is called full if it is not contained in any proper projective linear subspace of \( \mathbb{P}(V) \).

**Lemma 43.** Let \( M \subset \mathbb{P}(V) \) be a full \( \Gamma \)-invariant closed subset. Then \( \text{Max}(\beta) \cap M \neq \emptyset \).

**Proof.** It is easy to check that the unstable manifolds of \( \mu_p^\beta \) are given by:

\[
W_1^\beta = \mathbb{P}(V) - \mathbb{P}(V_2 \oplus \cdots \oplus V_k),
\]

\[
W_2^\beta = \mathbb{P}(V_2 \oplus \cdots \oplus V_k) - \mathbb{P}(V_3 \oplus \cdots \oplus V_k),
\]

\[
\vdots
\]

\[
W_{k-1}^\beta = \mathbb{P}(V_{k-1} \oplus V_k) - \mathbb{P}(V_k),
\]

\[
W_k^\beta = \mathbb{P}(V_k).
\]

The unstable manifold of the global maximum is the complement of a proper projective subspace of \( \mathbb{P}(V) \) and so it is open and dense. In particular \( \mu_p^\beta \) has a unique local maximum which is the global maximum. Since \( M \) is full, it follows \( M \cap W_1^\beta \neq \emptyset \). Let \( y \in M \cap W_1^\beta \). Then, keeping in mind that \( M \) is closed and \( \Gamma \)-invariant, we have

\[
\lim_{t \to +\infty} \exp(t\beta)y \in \text{Max}(\beta) \cap M
\]

and so the result follows. \( \square \)

Let \( \mathfrak{a} \subset \mathfrak{p} \) be an Abelian subalgebra and let \( A = \exp(\mathfrak{a}) \). The \( A \)-gradient map is given by \( \mu_a = \pi_a \circ \mu_p \), where \( \pi_a \) is the orthogonal projection of \( \mathfrak{p} \) onto \( \mathfrak{a} \). It is well-known that \( \mu_a(\mathbb{P}(V)) \) is a polytope. Moreover \( \mu_a(\mathbb{P}(V)^A) \) is finite and \( \text{conv}(\mu_a(\mathbb{P}(V)^A)) = \mu_a(\mathbb{P}(V)) \) \([2]\ [26]\), where \( \mathbb{P}(V)^A = \{ x \in \mathbb{P}(V) : A \cdot x = x \} \) is the set of fixed point of \( A \).

**Proposition 44.** Let \( M \subset \mathbb{P}(V) \) be a full \( A \)-invariant closed subset such that \( \mu_a(M) \) is convex. Then \( \mu_a(\mathbb{P}(V)) = \mu_a(M) \).

**Proof.** Denote by \( C_1 = \mu_a(M) \) and by \( C_2 = \mu_a(\mathbb{P}(V)) \). By the previous Lemma, for any \( \beta \in \mathfrak{a} \), we have

\[
\max_{y \in C_2} \langle y, \beta \rangle = \max_{z \in \mathbb{P}(V)} \mu_a^\beta = \max_{z \in M} \mu_a^\beta = \max_{y \in C_1} \langle y, \beta \rangle.
\]

Since \( C_1 \subset C_2 \), applying Proposition \([11]\) we get \( C_1 = C_2 \). \( \square \)

**Lemma 45.** Let \( v \in \mathbb{P}(V) \) be such that \( U^C \cdot v \) is closed. Then there exists a unique closed orbit of \( G \) contained in \( U^C \cdot v \).
Proof. Assume that \( U^C \cdot v \) is closed. It is well-known that \( U^C \cdot v = U \cdot v \) is a flag manifold \([26]\). By \([25]\) p.197 there are irreducibly representations \( \psi_i : U_i^C \to GL(W_i) \) and \( \varphi_i : U_i^C \to GL(L_i) \), for \( i = 1, \ldots, k \) such that \( \mathbb{P}(V) = \bigoplus_{i=1}^k \mathbb{P}(W_i \otimes L_i) \). Moreover \( v = [v_i \otimes w_i] \in \mathbb{P}(V_i \otimes W_i) \) for some \( i \in \{1, \ldots, k\} \), where \( v_i \in W_i \), respectively \( w_i \in L_i \), is a highest weight vector of \( \psi_i \), respectively \( \varphi_i \), and \( U^C \cdot [v_i \otimes w_i] \) is the unique closed orbit of \( U^C \) on \( \mathbb{P}(W_i \otimes L_i) \) \([29]\). Then the compact orbit \( U^C \cdot [v_i \otimes w_i] \) fibers \( U^C \)-equivariantly over \( U_i^C \cdot [v_i] \) with fiber \( U_i^C \cdot [w_i] \). The result now follows directly by the Wolf’s result \([51]\). □

From now on, we always assume \( G \) is semisimple and \( \tau : G \to PGL(V) \) is irreducible. \( \tau \) induces a projective representation of \( U^C \) on \( \mathbb{P}(V) \) which is also irreducible. By Borel-Weil Theorem, the \( U^C \)-action on \( \mathbb{P}(V) \) has a unique closed orbit that we denote by \( O' \) \([29]\). By the above Lemma there exists a unique closed \( G \)-orbit contained in \( O' \) that we denote by \( O \). By Proposition \([31]\) \( O \) is a \( K \)-orbit. We claim there exists a link between the projective representation \( \tau : G \to PSL(V) \), the \( G \)-gradient map and \( O \). The next two results are useful.

Proposition 46. Let \( \mathfrak{a} \) be an Abelian subalgebra contained in \( \mathfrak{p} \) and let \( A = \exp(\mathfrak{a}) \). Let \( \beta \in \mathfrak{a} \) and let \( \mu_\beta : \mathbb{P}(V) \to \mathfrak{a} \) be the \( A \)-gradient map. The following items hold true:

(a) \( \text{Max}(\beta) = \mathbb{P}(W) \), where \( W \) is the eigenspace associated to the maximum of \( \beta \);
(b) \( \text{Max}(\beta) \cap O \) is not empty, and so \( \text{Max}(\beta) \cap O = \text{Max}_O(\beta) = \{ z \in O : \mu_\beta(z) = \max_{g \in O} \mu_\beta \} \);
(c) the unstable manifold relative to the maximum of \( \mu_\beta : O \to \mathbb{R} \), is given by \( O - \mathbb{P}(W^\perp) \);
(d) \( \mu_\mathfrak{a}(O) = \mu_\mathfrak{p}(\mathbb{P}(V)) \);
(e) if \( \mathfrak{a} \) is a maximal Abelian subalgebra contained in \( \mathfrak{p} \), then \( \mu_\mathfrak{p}(\mathbb{P}(V)) \) is the convex hull of a Weyl group orbit.

Proof. Denote by \( \pi : V - \{0\} \to \mathbb{P}(V) \) the natural projection. Let \( w \in V - \{0\} \) such that \( G \cdot \pi(w) = O \). Since the \( G \)-action on \( V \) is irreducible, it follows that \( G \cdot w \) is not contained in any subspace of \( V \). This means that \( O \) is full in \( \mathbb{P}(V) \). Applying Lemma \([43]\) and Proposition \([44]\) items (a), (b), (c), (d) hold. Assume that \( A = \exp(\mathfrak{a}) \), where \( \mathfrak{a} \) is a maximal Abelian subalgebra contained in \( \mathfrak{p} \). Since \( O \) is a \( K \)-orbit and \( \mu_\mathfrak{p} \) is \( K \)-invariant, keeping in mind that \( \mu_\mathfrak{a} = \pi_\mathfrak{a} \circ \mu_\mathfrak{p} \), applying a Theorem of Kostant \([10]\), we get \( \mu_\mathfrak{a}(O) \) is the convex hull of a Weyl group orbit \( \mathcal{W} \), concluding the proof. □

Corollary 47. \( \text{conv}(\mu_\mathfrak{p}(\mathbb{P}(V))) = \text{conv}(\mu_\mathfrak{p}(O)), \) and so it is a polar orbitope. In particular any face of \( \text{conv}(\mu_\mathfrak{p}(\mathbb{P}(V))) \) is exposed.

Proof. Let \( \mathfrak{a} \subset \mathfrak{p} \) be a maximal Abelian subalgebra contained in \( \mathfrak{p} \). The convex hull of \( \mu_\mathfrak{p}(\mathbb{P}(V)) \) is a \( K \)-invariant compact convex subset of \( \mathfrak{p} \) satisfying \( \pi_\mathfrak{a}(\text{conv}(\mu_\mathfrak{p}(\mathbb{P}(V)))) = \mu_\mathfrak{a}(\mathbb{P}(V)) \).

By Proposition \([46]\) we have \( \mu_\mathfrak{a}(\mathbb{P}(V)) = \mu_\mathfrak{a}(O) \). Applying \([11]\) Theorem 0.1 p. 424], we have \( \text{conv}(\mu_\mathfrak{p}(O)) = K \mu_\mathfrak{a}(O) = \text{conv}(\mu_\mathfrak{p}(\mathbb{P}(V))) \).

This implies that \( \text{conv}(\mu_\mathfrak{p}(\mathbb{P}(V))) \) is a polar orbitope. By \([11]\) Theorem 3.2 p. 597], every face of \( \text{conv}(\mu_\mathfrak{p}(\mathbb{P}(V))) \) is exposed, concluding the proof. □

In the sequel we denote by \( \mathcal{E} = \text{conv}(\mu_\mathfrak{a}(\mathbb{P}(V))) \). If \( \mathfrak{a} \subset \mathfrak{p} \) is a maximal Abelian subalgebra, then we denote by \( P = \mu_\mathfrak{a}(\mathbb{P}(V)) \).

We shall investigate the action of a parabolic subgroup of \( G \) on \( \mathbb{P}(V) \). We first briefly discuss the complex case.

Let \( Q \subset U^C \) be a parabolic subgroup. \( Q \) is connected. \([1]\).
Lemma 48. If $Q$ is a parabolic subgroup of $U^\mathbb{C}$, then $Q$ has a unique closed orbit which is contained in $O'$. This orbit is a complex $U \cap Q$-orbit, so a flag manifold, and it coincides with the maximum of a contraction of the momentum map.

Proof. Let $v \in \mathbb{P}(V)$ be such that $Q \cdot v$ is closed. By Proposition 17, $Q \cdot v$ is contained in $O'$, which is a flag manifold. Let $\xi \in u$ be such that $Q = (U^\mathbb{C})^{\xi}$. In [9] Proposition 3.9] the authors proved that
\[
\text{Max}_{O'}(\xi) := \{ x \in O' : \mu^\xi(x) = \max_{z \in O} \mu^\xi \}
\]
is the unique compact orbit of $Q$. It is a complex $U \cap Q$-orbit, and so it is connected, and a flag manifold. □

Now, we consider the real case. We start with the following Lemma.

Lemma 49. Let $\beta \in p$. Then any local maximum of $\mu^\beta_p$ restricted to $O$ is a global maximum of $\mu^\beta_p : \mathbb{P}(V) \longrightarrow \mathbb{R}$.

Proof. Let $x \in O$ be a local maximum of $\mu^\beta_p : O \longrightarrow \mathbb{R}$. The $G$-gradient map restricted to $O$
\[
\mu_p : K \cdot x \longrightarrow K \cdot \mu_p(x),
\]
is a smooth fibration. Hence $\mu_p(x)$ is a local maximum of the height function
\[
K \cdot \mu_p(x) \longrightarrow \mathbb{R}, \quad z \mapsto \langle z, \beta \rangle.
\]
By [10] proof of Proposition 3.1, p. 583, it follows that $\mu_p(x)$ is a global maximum. Since
\[
\text{Max}_{p \in O} \mu^\beta_p = \text{Max}_{z \in K \cdot \mu_p(x)} \langle \cdot, \beta \rangle,
\]
it follows that $x$ is a global maximum of $\mu^\beta_p : O \longrightarrow \mathbb{R}$. By Lemma 48, $x$ is a global maximum of $\mu^\beta_p : \mathbb{P}(V) \longrightarrow \mathbb{R}$ and the result is proved. □

Firstly, we assume that $G$ has a unique closed orbit $O$ in $\mathbb{P}(V)$. By Lemma 45, $O \subset O'$.

Theorem 50. Let $Q$ be a parabolic subgroup of $G$. Then $Q$ has a unique closed orbit in $\mathbb{P}(V)$. This orbit is connected and it is contained in $O$.

Proof. By Proposition 30, $Q$ has a closed orbit. By Proposition 17, it is contained in $O$. We prove this orbit is unique.

Let $\beta \in p$ such that $Q = G^{\beta+}$. The proof is given in three steps.

1. Any closed $G^{\beta+}$-orbit is contained in $\text{Max}_O(\beta)$.

Let $Q$ be a closed orbit of $G^{\beta+}$. We may assume that $Q = G^{\beta+} \cdot v$ and $v$ is a local maximum $\mu^\beta_p$ on $Q$. By Theorem 19, $R^{\beta+}$ acts trivially on $v$. Hence, by Proposition 23, keeping in mind that $O$ is $G$ homogeneous, $v$ is a local maximum of $\mu^\beta_p$ on $O$. By Lemma 49, $v$ is a global maximum on $O$. By Lemma 29, we have $Q \subset \text{Max}_O(\beta)$.

2. If $x \in \text{Max}_O(\beta)$, then $Q \cdot x$ is closed.

Since $R^{\beta+}$ acts trivially on $\text{Max}_O(\beta)$, it follows $Q \cdot x = G^{\beta} \cdot x \subset \text{Crit}(\mu^\beta_p)$. By Corollary 25, $Q \cdot x$ is closed. We thank the referee for pointing out this short argument.

3. $Q$ has a unique closed orbit and it is connected.
By definition of the gradient map, for any $\xi \in \mathfrak{p}$, we have
$$\langle \mu(z), -i\xi \rangle = \langle \mu_p(z), \xi \rangle.$$ 

By Lemma 43, we have
$$\text{Max}_{\mathcal{O}}(-i\beta) \cap \mathcal{O} = \text{Max}_{\mathcal{O}}(\beta).$$

By Lemma 43, $\text{Max}_{\mathcal{O}}(-i\beta)$ is connected and the unique closed orbit of $(U^C)^{\beta_+}$ on $\mathbb{P}(V)$. By Lemma 45, $(G^\beta)^o$ has a unique closed orbit on $\text{Max}_{\mathcal{O}}(-i\beta)$. By step 2, it follows that $G^{\beta_+}$ has a unique closed orbit. This orbit is connected and it coincides with $\text{Max}_{\mathcal{O}}(\beta)$. \hfill $\Box$

**Corollary 51.** Let $\beta \in \mathfrak{p}$. Then $\text{Max}_{\mathcal{O}}(\beta)$ is the unique closed orbit of $G^{\beta_+}$ and it is a $(K^\beta)^o$-orbit.

**Proof.** In the above Theorem, we prove that $\text{Max}_{\mathcal{O}}(\beta)$ is the unique compact orbit of $G^{\beta_+}$. It is connected and it is a $(G^\beta)^o$-orbit. By Proposition 31, it is a $(K^\beta)^o$-orbit. \hfill $\Box$

The unique closed orbit of a parabolic subgroup of $G$ is well-adapted to $\tau$.

**Theorem 52.** Let $\beta \in \mathfrak{p}$ and let $\lambda_1 > \cdots > \lambda_k$ be its eigenvalues. We denote by $V_1, \ldots, V_k$ the corresponding eigenspaces. Then

- $G^{\beta_+}$ preserves $V_1$. Moreover, $V_1$ is the unique subspace of $V$ on which $G^{\beta_+}$ acts irreducibly;
- $\text{Max}_{\mathcal{O}}(\beta) = \mathbb{P}(V_1) \cap \mathcal{O}$ is the unique closed orbit of $G^{\beta_+}$;

**Proof.** $\mathbb{P}(V_1) = \text{Max}_{x \in \mathbb{P}(V)} \mu_\beta^p$. By Proposition 29, we get $G^{\beta_+}$ preserves $\mathbb{P}(V_1)$ and so $V_1$. We claim that $G^{\beta_+}$ acts irreducibly on $V_1$. Since $R^{\beta_+}$ acts trivially on $\mathbb{P}(V_1)$, and $G^\beta$ is compatible, the representation of $G^{\beta_+}$ on $V_1$ splits. Assume that $V_1 = L \oplus Z$. By Proposition 31, $G^\beta$ has a closed orbit in both $\mathbb{P}(L)$ and $\mathbb{P}(Z)$ respectively. Therefore $G^{\beta_+}$ admits two closed orbits which is a contradiction. Now, we prove the uniqueness. Assume that $G^{\beta_+}$ acts irreducibly on $W$. Since $[\mathfrak{p}^{\beta_+}, \mathfrak{r}^{\beta_+}] \subset \mathfrak{r}^{\beta_+}$, by Engel Theorem 36, it follows $R^{\beta_+}$ acts irreducibly on $W$ and so it acts trivially on $\mathbb{P}(W)$. 

By Proposition 31, $\mathbb{P}(W)$ contains the unique closed orbit of $G^{\beta_+}$ which is full in $\mathbb{P}(V_1)$. This implies $\mathbb{P}(V_1) \subset \mathbb{P}(W)$ and so $V_1 = W$. Applying Corollary 51, we get that $\text{Max}_{\mathcal{O}}(\beta) = \mathbb{P}(V_1) \cap \mathcal{O}$ is the unique closed orbit of $G^{\beta_+}$. \hfill $\Box$

Now, we prove the general case.

Let $Q = G^{\beta_+}$ be a parabolic subgroup. By Lemma 45, there exists a unique closed $G$-orbit on $\mathcal{O}$, that we denote by $\mathcal{O}$. The proof of Theorem 40 shows that $Q$ has a unique closed orbit contained in $\mathcal{O}$. This orbit is connected and is given by
$$\mathbb{P}(V_1) \cap \mathcal{O} = \{ z \in \mathcal{O} : \mu_\beta^p(z) = \max_{y \in \mathcal{O}} \mu_\beta^y \},$$
where $V_1$ is the eigenspace associated to the maximum of $\beta$. Moreover,
$$\mathbb{P}(V_1) = \{ p \in \mathbb{P}(V) : \mu_p^\beta(p) = \max_{y \in \mathbb{P}(V)} \mu_\beta^y \} = \{ p \in \mathbb{P}(V) : \mu^{-i\beta}(p) = \max_{z \in \mathbb{P}(V)} \mu^{-i\beta} \}.$$ 

By Theorem 52, $V_1$ is the unique subspace of $V$ such that $(U^C)^{\beta_+}$ acts irreducibly on it. By the linearization Theorem, $R^{\beta_+}$ acts trivially on $\mathbb{P}(V_1)$. The Lie algebra of $G^\beta$ is a real form of the Lie algebra of $(U^C)^{\beta}$. Since $(U^C)^{\beta_+}$ is connected, keeping in mind that $V_1$ is a complex subspace, $G^\beta$, and so $G^{\beta_+}$, acts irreducibly on $V_1$.

Let $W \subset V$ be a subspace such that $G^{\beta_+}$ acts irreducibly on it. If $\beta = \beta_0 + i\beta_1 \in \mathfrak{p}_0 \oplus i\mathfrak{u}_1$, then $G^{\beta_+} = G_0^{\beta_0_+} \cdot (U^C_1)^{i\beta_1}$.

Let $Q$ be the parabolic subgroup of $U^C_0$ such that its Lie algebra is the complexification of the Lie algebra of $G_0^{\beta_0_+}$. Then $Q \cdot (U^C_1)^{i\beta_1}$ preserves $W$ and acts irreducibly on it. This implies that
the unipotent radical of \( Q \cdot (U^\mathbb{C})^{i\beta_1^+} \) acts trivially on \( \mathbb{P}(W) \). By Proposition 31, \( Q \) has a closed orbit on \( \mathbb{P}(W) \). By Lemma 18, this orbit is unique and it is contained in \( \mathcal{O}' \). Now, the unipotent radical of \( G^{\beta^+} \) acts trivially on the unique closed orbit of \( Q \cdot (U^\mathbb{C})^{i\beta_1^+} \) on \( \mathcal{O}' \). By Lemma 15, it is the unique closed orbit contained in the unique closed orbit of \( Q \cdot (U^\mathbb{C})^{i\beta_1^+} \) on \( \mathcal{O}' \). Hence, \( G^{\beta^+} \) has a unique closed orbit contained in \( \mathcal{O}' \), and so in \( \mathcal{O} \), and this orbit is connected. By Theorem 52, the unique compact orbit of the \( G^{\beta^+} \)-action on \( \mathcal{O} \) is given by \( \mathbb{P}(V_1) \cap \mathcal{O} \). Therefore, \( V_1 \subset W \) and so \( V_1 = W \). Summing up, we have proved the following result.

**Theorem 53.** Let \( \beta \in \mathfrak{p} \) and let \( \lambda_1 > \cdots > \lambda_k \) be its eigenvalues. We denote by \( V_1, \ldots, V_k \) the corresponding eigenspaces. Then

- \( G^{\beta^+} \) preserves \( V_1 \). Moreover, \( V_1 \) is the unique subspace of \( V \) on which \( G^{\beta^+} \) acts irreducibly;
- \( \mathbb{P}(V_1) \cap \mathcal{O} = \{ z \in \mathcal{O} : \mu_\beta^+(z) = \max_{r \in \mu_\beta(\mathcal{O})} \langle r, \beta \rangle \} \) is the unique closed orbit of \( G^{\beta^+} \) contained in \( \mathcal{O} \). Moreover, it is connected and a \( (K^{\beta^+})^0 \) orbit;

**Remark 54.** Let \( Q \) be a parabolic subgroup of \( G \) and let \( \mathcal{O} = K \cdot x \). Although \( \mu_p \) is not \( G \)-equivariant, if \( Q \cdot x \) is closed, then \( Q \cdot \mu_p(x) \subset K \cdot \mu_p(x) \) is closed and

\[
\mu_p(Q \cdot x) = Q \cdot \mu_p(x).
\]

Indeed, pick \( \beta \in \mathfrak{p} \) such that \( Q = G^{\beta^+} \). By Corollary 51, we have

\[
\mu_p(G^{\beta^+} \cdot x) = \{ z \in \mu_p(\mathcal{O}) : \langle z, \beta \rangle = \max_{r \in \mu_p(\mathcal{O})} \langle r, \beta \rangle \}.
\]

By Corollary 3.1 p. 593 and Proposition 3.9 p. 599, we have

\[
\{ z \in \mu(\mathcal{O}) : \langle z, \beta \rangle = \max_{r \in \mu_p(\mathcal{O})} \langle r, \beta \rangle \} = K^{\beta^+} \cdot \mu_p(x) = G^{\beta^+} \cdot \mu_p(x).
\]

We summarize the link between the parabolic subgroups of \( G \) and the \( G \)-gradient map.

Let \( \beta \in \mathfrak{p} \) and let \( Q = G^{\beta^+} \). Let \( \mathcal{O}(Q) \) and \( W(Q) \) denote the unique closed orbit of \( Q \) contained in \( \mathcal{O} \) and the unique irreducible submodule of \( Q \), respectively. \( W(Q) \) is the eigenspace associated to the maximum of \( \beta \in \mathfrak{p} \) and

\[
\mathbb{P}(W(Q)) \cap \mathcal{O} = \operatorname{Max}_\mathcal{O}(\beta) = \mathcal{O}(Q).
\]

By Remark 51 and Theorem 1.1 [10] p. 582, we have \( \mu_p(\mathcal{O}(Q)) = \operatorname{ext} F_\beta(\mathcal{E}) \). Since \( \mu_p^{-1}(F_\beta(\mathcal{E})) = \mathbb{P}(W(Q)) \), it follows

\[
\mathcal{O}(Q) = \mu_p^{-1}(F_\beta(\mathcal{E})) \cap \mathcal{O}.
\]

Let \( W \subset V \) be a complex subspace. We denote by

\[
\pi_W : V \rightarrow W,
\]

the orthogonal projection and by

\[
\hat{\pi}_W : \mathbb{P}(V) - \mathbb{P}(W) \rightarrow \mathbb{P}(V), \quad \hat{\pi}_W([v]) = [\pi_W(v)],
\]

its projectivization. \( \hat{\pi}_W \) is a meromorphic map. Since \( \mathbb{P}(W(Q)) = \operatorname{Max}(\beta) \), by Theorem 28 and Proposition 53, the domain of the map \( \pi_W(Q) \) is the unstable manifold of the maximum of \( \mu_\beta \). Moreover, \( \hat{\pi}_W \) coincides with \( \varphi_\infty \). Indeed, let \( \lambda_1 > \cdots > \lambda_k \) be the eigenvalues of \( \beta \). Let \( V_2, \ldots, V_k \) be the eigenspaces associated to \( \lambda_2, \ldots, \lambda_k \). Then

\[
\lim_{t \rightarrow +\infty} \exp(t\beta)[x_1 + x_2 + \cdots + x_k] = \lim_{t \rightarrow +\infty} [e^{t\lambda_1}x_1 + e^{t\lambda_2}x_2 + \cdots + e^{t\lambda_k}x_k] = [x_1] = \pi_W(x).
\]
By Theorem 27, an unstable manifold flows onto the corresponding critical set. Hence, keeping in mind that $O(Q) = \text{Max}_O(\beta)$, we get $\hat{\pi}_{W}(O) = O(Q)$. Summing up, we have proved the following result.

**Theorem 55.** Let $Q = G^{\beta^+}$ be a parabolic subgroup of $G$. Then

- a) $\mu_p(O(Q)) = \text{ext} F_\beta(\mathcal{E})$;
- b) $\mu_p^{-1}(F_\beta(\mathcal{E})) = \mathbb{P}(W(Q))$;
- c) $\mathbb{P}(W(Q)) \cap O = O(Q)$;
- d) $\hat{\pi}_{W}(O) = O(Q)$.

4. $\tau$-CONNECTED SUBSPACES, PARABOLIC SUBGROUPS OF $G$ AND GRADIENT MAP

In this section we explicitly determine, up to $K$-equivalence, the irreducible representations of parabolic subgroups of $G$ induced by $\tau$.

Given a $W \subseteq V$, $W \neq \{0\}$, set $Q(W) := \{g \in G : g(W) = W\}$.

**Definition 56.** $W$ is a $\tau$-connected subspace if $Q(W)$ is parabolic and acts irreducibly on $W$.

Let $O(Q(W))$ be the unique closed orbit of $Q(W)$ contained in $O$. By Theorem 55, we have

$$\mu_p(O(Q(W))) = \text{ext} F_W,$$

where $F_W \in \mathcal{F}(\mathcal{E})$. Now, $\mathcal{E} = \text{conv} (\mu_p(O))$ and $\mu_p(O)$ is a $K$-orbit. The $K$-action extends to a $G$ action on $\mu_p(O)$ [55]. The set of the extreme points of $F_W$ is contained in $O$. We define

$$Q_{F_W} = \{h \in G : h \text{ ext } F_W = \text{ext } F_W\}, \quad H_F = K \cap Q_{F_W}.$$

**Proposition 57.** $Q_{F_W} = Q(W)$. Moreover, if $\beta \in C_{F_W}^{HF}$, then $G^{\beta^+} = Q(W)$. Hence $Q(W)$ only depends on the face $F_W$.

**Proof.** Let $\beta \in C_{F_W}^{HF}$. By Theorem 55, $\mu_p^{-1}(F_\beta(\mathcal{E})) = \mathbb{P}(W)$ and $W(G^{\beta^+}) = W$. This implies $G^{\beta^+} \subseteq Q(W)$. Since $G^{\beta^+} = Q_{F_W}$ [10] Proposition 3.8, p.598], it follows that $Q_{F_W} \subseteq Q(W)$.

Let $g \in Q(W)$. By Proposition [17], $g = kp$ for some $k \in K$ and some $p \in Q_{F_W}$. Then $gW = W$ implies $kW = W$ and so, keeping in mind that $O(Q(W)) = \mathbb{P}(W) \cap O$, $k$ preserves $O(Q(W))$. By the $K$-equivariance of $\mu_p$, we get $k \text{ ext } F_W = \text{ext } F_W$ and so $k \in H_F \subseteq Q_{F_W}$. This proves $Q(W) \subseteq Q_{F_W}$, concluding the proof.

A $\tau$-connected subspace $W$ is completely determined by the closed orbit $O(Q(W))$.

**Proposition 58.** Let $W_1, W_2 \subseteq V$ be two $\tau$-connected subspaces. Then $W_1 \subseteq W_2$ if and only if $O(Q(W_1)) \subseteq O(Q(W_2))$.

**Proof.** The linear span of $O(Q(W_1))$ is $\mathbb{P}(W_1)$. Hence, if $O(Q(W_1)) \subseteq O(Q(W_2))$, then $\mathbb{P}(W_1) \subseteq \mathbb{P}(W_2)$.

If $W_1 \subseteq W_2$, then $\mathbb{P}(W_1) \cap O = O(Q(W_1)) \subseteq \mathbb{P}(W_2) \cap O = O(Q(W_2))$ and the result follows.

**Corollary 59.** Let $W_1, W_2 \subseteq V$ be $\tau$-connected subspaces. Then $W_1 \subseteq W_2$ if and only if $\text{relint } F_{W_1} \subseteq \text{relint } F_{W_2}$. Moreover $W_1 = W_2$ if and only if $\text{relint } F_{W_1} = \text{relint } F_{W_1}$.

**Proof.** If $W_1 \subseteq W_2$, then $O(Q(W_1)) = \mathbb{P}(W_1) \cap O \subseteq \mathbb{P}(W_2) \cap O = O(Q(W_2))$, and so

$$\text{ext } F_{W_1} = \mu_p(O(Q(W_1))) \subseteq \mu_p(O(Q(W_2))) = \text{ext } F_{W_2}.$$

Therefore $F_{W_1} = \text{conv} (\text{ext } F_{W_1}) \subseteq \text{conv} (\text{ext } F_{W_2}) = F_{W_2}$. Vice-versa, if $F_{W_1} \subseteq F_{W_2}$, then

$$O(Q(W_1)) = \mu_p^{-1}(F_{W_1}) \cap O \subseteq \mu_p^{-1}(F_{W_2}) \cap O = O(Q(W_2)).$$
By Proposition 62, we get $W_1 \subseteq W_2$. The last item follows from Theorem 10. Indeed, $F_{W_1} = F_{W_2}$ if and only if $\text{relint } F_{W_1} = \text{relint } F_{W_2}$.

**Remark 60.** Let $F \subseteq \mathcal{E}$ be a face. By Lemma 8 there exists a maximal chain

$$F = F_0 \subset F_1 \subset \cdots \subset F_k = \mathcal{E}$$

of faces. By Theorem 55, $\mu_\mathcal{F}^{-1}(F_i) = \mathbb{P}(W_i)$ and $W_i$ is a $\tau$-connected subspace of $\mathcal{E}$. By Corollary 59, we get a chain

$$\mathcal{O}(Q(W_0)) \subset \mathcal{O}(Q(W_1)) \subset \cdots \subset \mathcal{O}(Q(W_{k-1})) \subset \mathcal{O}(Q(W_k))$$

of homogeneous submanifolds of $\mathbb{P}(V)$.

Set $\mathcal{H}(\tau) = \{(W, Q(W)) : W \text{ is } \tau - \text{connected subspace of } \mathcal{E}\}$. The following Lemma is easy to check.

**Lemma 61.** Let $W$ be a $\tau$-connected subspace of $\mathcal{E}$ and let $g \in G$. Then

a) $gW$ is a $\tau$-connected subspace;

b) there exists $k \in K$ such that $gW = kW$;

c) $Q(gW) = Q(W)g^{-1} = kQ(W)k^{-1}$

$G$ acts on $\mathcal{H}(\tau)$ as follows:

$$g(W, Q(W)) := (gW, gQ(W)g^{-1}).$$

**Proposition 62.** The map

$$\mathcal{X} : \mathcal{H}(\tau) \rightarrow \mathcal{F}(\mathcal{E}), \quad (Q, Q(W)) \mapsto F_W,$$

is $K$-equivariant and bijective.

**Proof.** By Lemma 61 the map $\mathcal{X}$ is $K$-equivariant. By Theorem 55 and Corollary 59, the map $\mathcal{X}$ is bijective. □

Let $a \subseteq \mathfrak{p}$ be a maximal Abelian subalgebra and let $P = \mu_a(\mathcal{O}) = \mathcal{E} \cap a$. Fix a system of root of simple roots $\Pi \subseteq \Delta = \Delta(g, \mathfrak{a})$. We denote by $a_+$ the positive Weyl chamber associated to $\Pi$. Now, $\mu_\mathfrak{p}(\mathcal{O}) \cap a_+ = \{x\}$ and by Kostant convexity Theorem 10, $P = \text{conv}(W \cdot x)$.

A subset $B \subseteq a^*$ is connected if there is no pair of disjoint subsets $D, C \subseteq B$ such that $D \cup C = B$, and $D$ and $C$ are two subsets orthogonal with respect to the Killing Cartan form. Connected components are defined as usual. If $x$ is a nonzero vector of $a$, a subset $I \subseteq \Pi$ is called $x$-connected if any connected component of $I$ contains at least one root $\alpha$ such that $\alpha(x) \neq 0$. If $I \subseteq \Pi$ is $x$-connected, denote by $I'$ the collection of all simple roots orthogonal to $\{x\} \cup I$, i.e., if $\alpha \in I'$, then $\alpha$ is orthogonal to $I$ and satisfies $\alpha(x) = 0$. The set $J := I \cup I'$ is called the $x$-saturation of $I$. The largest $x$-connected subset contained in $J$ is $I$. So $J$ is determined by $I$ and $I$ is determined by $J$. Given a subset $I \subseteq \Pi$ we denote by $Q_I$ the parabolic subgroup with Lie algebra $\mathfrak{q}_I$ as defined in [13]. In [10], see also [19] [37], the following theorem is proved.

**Theorem 63.** Let $x \in a_+$ be a point such that $P = \text{conv}(W \cdot x)$. Then

a) Let $I \subseteq \Pi$ be a $x$-connected subset and let $J$ be its $x$-saturation. Then $Q_J \cdot x = Q_J \cdot x$ and $F := \text{conv}(Q_J \cdot x)$ is a face of $\mathcal{E}$. Moreover, $F = F_\beta(\mathcal{E}) = \text{conv}(K^\beta \cdot x)$, where $\beta \in a$ satisfies $Q_J = G^\beta$;

b) let $\beta' \in a$ be such that $Q_J = G^{\beta'}$. Then $F = F_{\beta'}(\mathcal{E}) = \text{conv}(K^{\beta'} \cdot x)$. Moreover $\{h \in G : h \text{ ext } F = \text{ ext } F\} = Q_J = G^{\beta'}$;

c) Any face of $\text{Conv}(\mathcal{E})$ is conjugate to one of the faces constructed in (a).
In the sequel, we denote by $F_I$ the face of $E$ such that $\text{ext } F_I = Q_I \cdot x$. Let $W_I = \mu_p^{-1}(F_I)$. By Proposition 57, $Q(W_I) = Q_J$. Applying Proposition 62 and Theorem 63, we have the following result.

**Corollary 64.** Let $W$ be a $\tau$-connected. Then there exists $k \in K$ and a $\{x\}$-connected subset $I$ of $\Pi$ such that $W = kW_I$. Moreover $Q(W) = kQ_J k^{-1}$, where $J$ is the saturation of $I$.

Now, we prove the following result.

**Theorem 65.** In the above setting, the map

$\mathcal{X} : I \mapsto (Q_I, W_I)$,

induces a bijection between $\{x\}$-connected subsets and the irreducible representations of parabolic subgroups of $G$ induced by $\tau$, up to the $K$-action.

**Proof.** Let $I_1, I_2 \subset \Pi$ be $x$-connected subsets. Let $J_1$ and $J_2$ be their $x$-saturations. Assume that there exists $k \in K$ such that $Q_{I_1} \cdot x = kQ_{I_2} \cdot x$. Then $Q_{J_1} = kQ_{J_2} k^{-1}$. By Lemma 1.2.1.11 in [50], see also [27] Proposition 2.18, pag.20], $J = J_1 = J_2$ and $k \in Q_J$. Since $I_1$, respectively, $I_2$ is the maximal $\tau$-connected subset of $J$, it follows that $I_1 = I_2$. This proves $\mathcal{X}$ is injective.

Let $Q$ be a parabolic subgroup of $G$ and let $W$ be the unique complex subspace of $V$ be such that $Q$ acts irreducibly on $W$. By Corollary 64 there exists $k \in K$ such that $W = kW_I$ and $Q \subseteq kQ_J k^{-1}$. By Theorem 53, $\mathcal{O}(Q) = \mathcal{O}(kQ_J k^{-1})$. Hence, keeping in mind that $\mathcal{O}(kQ_J k^{-1}) = k\mathcal{O}(Q_J)$, it follows that

$\text{ext } F_I = k^{-1}\text{ext } F_W$

By [10], Proposition 3.5, there exists $k_2 \in K$ such that $a \subset k_2 q k_2^{-1}$ and $(k_2 \text{ext } F_W) \cap a$ is a face of $P$. By the main result proved in [10], $\mathcal{P}(P)/\mathcal{W} \cong \mathcal{P}(E)/K$. Hence, there exists $\theta \in N_K(a)$ such that

$\text{ext } F_I \cap a = (\theta k_2 \text{ext } F_W) \cap a$.

By [10], Theorem 1.1, we have $\text{ext } F_I = \theta k_2 \text{ext } F_W$.

Set $k = \theta k_2$. Since $a \subset kqk^{-1}$, there exists a $\tilde{J}$ subset of $\Pi$ such that $Q_{J} = kQk^{-1}$. By Remark 54, we have

$Q_{\tilde{J}} \cdot x = Q_I \cdot x = \text{ext } F_I$.

Let $\tilde{I}$ denote the maximal $\{x\}$-connected subset contained in $\tilde{J}$. By Theorem 63, $Q_{\tilde{I}} \cdot x$ is a face and $Q_{E} \cdot x = Q_{\tilde{J}} \cdot x$, where $E$ is the saturation of $\tilde{I}$. Since $\tilde{I} \subseteq \tilde{J} \subseteq E$, it follows

$Q_{\tilde{I}} \cdot x = Q_J \cdot x = Q_{E} \cdot x$.

On the other hand $Q_I \cdot x = Q_J \cdot x$ and so, by Theorem 63, $Q_{E} = Q_J$. This implies $E = J$ and $\tilde{I} = I$ due to the fact that $I$, respectively $\tilde{I}$, is the maximal $\{x\}$-connected subspace of $J$. Moreover,

$Q_I \subset Q_J \subset Q_{\tilde{J}}$,

where $I \subset \tilde{J} \subset J$.

Let $I' = J - \{I\}$. The set $I'$ is perpendicular to $I$ and so the Langlands decomposition of $Q_J$ can be written as

$Q_J = N_I A_I M_I M_{I'}$,

see [15]. By [16] Lemma I.4.25, p. 69, $M_I$ acts trivially on $W_I$. Hence, the $Q_J$-action on $W_I$ is completely determined by the $Q_I$-action on $W_I$. 
\qed
Corollary 66. Let $Q$ be a parabolic subgroup and let $W$ be the unique complex subspace of $V$ such that $Q$ acts on irreducibly on $W$. Then there exists $k \in K$ and a $\{x\}$-connected subset $I$ of $\Pi$ such that $W = kW_I$ and $Q_I \subset kQk^{-1} \subset QJ$.

Example 67. Let $\text{SL}(3, \mathbb{R}) \cap \mathbb{C}^3$ as usual. We choice $\langle \xi, \nu \rangle = \text{Tr}(\xi \nu)$ as $\text{Ad}(\text{SO}(3))$-invariant scalar product on $\text{sym}_0(3)$. Let

$$a = \left\{ \begin{bmatrix} t & s \\ -t & -s \end{bmatrix}, t, s \in \mathbb{R} \right\}$$

be a maximal Abelian subalgebra of $\text{sym}_0(3)$. Then

$$A = \exp(a) = \left\{ \begin{bmatrix} e^t & e^s \\ e^{-t} & e^{-s} \end{bmatrix}, t, s \in \mathbb{R} \right\}$$

and the $A$-gradient map on $\mathbb{P}(\mathbb{C}^3)$ is given by

$$\mu_a([x, y, z]) = \left[ \begin{array}{c} \frac{\|x\|^2}{\|x\|^2 + \|y\|^2 + \|z\|^2} - \frac{1}{3} \\ \frac{\|y\|^2}{\|x\|^2 + \|y\|^2 + \|z\|^2} - \frac{1}{3} \\ \frac{\|z\|^2}{\|x\|^2 + \|y\|^2 + \|z\|^2} - \frac{1}{3} \end{array} \right]$$

By the Abelian convexity Theorem [2, 26], $\mu_a(\mathbb{P}(\mathbb{C}^3))$ is the convex hull of the image of the fixed point set of $A$. It is easy to check $\mathbb{P}(\mathbb{C}^3)^A = \{[e_1], [e_2], [e_3]\}$, where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{C}^3$. Therefore $\mu(\mathbb{P}(\mathbb{C}^3))$ is the convex hull of

$$x_1 = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}, x_2 = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}, x_3 = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

and so the image is an equilateral triangle.

The proper faces containing $x_2$ which are not Weyl equivalent are: $\{x_2\}$ and the side $x_2 - x_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. The segment $x_2 - x_1$ defines $\{x_2\}$ as exposed face and

$$\mu_a^{x_2-x_1}([x, y, z]) = \text{Tr}(\mu([x, y, z])(x_2 - x_1)) = -\frac{\|x\|^2 + \|y\|^2}{\|x\|^2 + \|y\|^2 + \|z\|^2}.$$
Therefore \( \text{Max}(x_2 - x_1) = \mathbb{P}(V_2) \), where \( V_2 = \text{Span}(e_2) \). Note that the vector \( x_2 - x_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \) defines as well \( \{x_2\} \) as an exposed face,

\[
\mu^\mathfrak{g}_{x_2-x_3}([x, y, z]) = \frac{\| y \|^2 - \| z \|^2}{\| x \|^2 + \| y \|^2 + \| z \|^2},
\]

and indeed \( \text{Max}(x_2 - x_3) = \mathbb{P}(V_2) \). Finally, the face corresponding of the segment \( x_2 - x_1 \) is defined by

\[
\beta = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},
\]

and

\[
\mu^\mathfrak{g}_\beta([x, y, z]) = \frac{\| x \|^2 + \| y \|^2 - 2 \| z \|^2}{\| x \|^2 + \| y \|^2 + \| z \|^2}.
\]

Therefore \( \text{Max}(\beta) = \mathbb{P}(V_1) \), where \( V_1 = \text{Span}(e_1, e_2) \).

5. Boundary components of Satake compactifications and gradient map

In this section we reformulate the Satake’s analysis in more geometrical terms. The elements of the Satake compactification associated to \( \tau \) are interpreted as rational maps of \( \mathcal{O} \). From now on we always assume that \( \tau : G \rightarrow \text{SL}(V) \) is irreducible and the kernel is finite. We also always refer to \([16, 27, 47]\) and we follow the notation introduced in Section 2.2.

5.1. \( \mu_\tau \)-connected subspaces, projections and rational maps. Let \( \mathfrak{a} \subset \mathfrak{p} \) be a maximal Abelian subalgebra. Then

\[
\mathfrak{z}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a},
\]

where \( \mathfrak{m} = \mathfrak{z}(\mathfrak{a}) \cap \mathfrak{k} \). If \( \mathfrak{a}' \subset \mathfrak{m} \) is a maximal Abelian subalgebra of \( \mathfrak{m} \), then \( \mathfrak{a}' + \mathfrak{i} \mathfrak{a} \subset \mathfrak{u} = \mathfrak{k} + \mathfrak{i} \mathfrak{p} \) is a maximal Abelian subalgebra of \( \mathfrak{u} \) and so \( \mathfrak{a}' + \mathfrak{i} \mathfrak{a} \subset \mathfrak{u}^\mathbb{C} = \mathfrak{u}^\mathbb{C} \) is a Cartan subalgebra. Given \( \mathfrak{a}, \mathfrak{a}' \), and \( \Pi \subset \Delta(\mathfrak{g}, \mathfrak{a}) \) be a basis one can choose a basis of \( (\mathfrak{a} + \mathfrak{i} \mathfrak{a}')^* \) adapted to \( \Pi \) and \( (\mathfrak{i} \mathfrak{a}')^* \). Indeed, it is possible to define a set of simple roots of \( \hat{\Delta} \) such that the projection of a subset of the simple roots of \( \hat{\Delta} \) onto \( \mathfrak{a}^* \) is equal to \( \Pi \) (see \([27, p.51 – 52]\) and \([34, p.272 – 273]\)). In particular the Borel subalgebra of \( \mathfrak{g}^\mathbb{C} \) is contained in \( q^\mathbb{C}_0 = (\mathfrak{m} + \mathfrak{a} + \mathfrak{n})^\mathbb{C} \).

Let \( \tilde{\mu}_\tau \) the highest weight of \( \mathfrak{g}^\mathbb{C} \) with respect to the partial ordering determined \( \hat{\Delta} \). Let \( x_\circ = [v_\circ] \), where \( v_\circ \) is any highest weight vector. It is well-known that

\[
\mu : \mathbb{P}(V) \longrightarrow \mathfrak{a}' \oplus \mathfrak{i} \mathfrak{a}, \quad \langle \mu(x_\circ), \xi \rangle = \tilde{\mu}_\tau(\xi),
\]

see \([8]\) that has opposite sign convection for \( \mu \), and \([5]\). By Proposition \([40]\) \( P = \mu^\mathfrak{g}_\mathfrak{a}(\mathbb{P}(V)) = \mu^\mathfrak{a}(\mathcal{O}) \). Now, \( G \cdot x_\circ \) is closed \([27, 4.29 \text{ Theorem p.59}] \), hence \( G \cdot x_\circ = \mathcal{O} \), and

\[
\langle \mu^\mathfrak{g}_\mathfrak{a}(x_\circ), \xi \rangle = i\tilde{\mu}_\tau(\xi).
\]

\( i\tilde{\mu}_\tau|_\mathfrak{a} \) is the highest weight of \( \mathfrak{g} \) with respect the induced order on \( \mathfrak{a}^* \). From now on, we denote by \( \mu_\tau = (i\tilde{\mu}_\tau|_\mathfrak{a}) \).

**Proposition 68.** The weights of \( \tau \) are contained in the convex hull of the Weyl group orbit \( \mathcal{W} \cdot \mu_\tau \).
Proof. Denote the weights of $\tau$ by $\mu_1, \ldots, \mu_p \in \mathfrak{a}^\ast$. It is easy to check that
$$\mu_\mathfrak{a}(\mathbb{P}(V)) = \text{conv}(v_1, \ldots, v_p),$$
where $v_1, \ldots, v_p \in \mathfrak{a}$ satisfy
$$\langle v_i, H \rangle = \mu_i(H),$$
for any $H \in \mathfrak{a}$ an for $i = 1, \ldots, n$. Since $\mu_\tau$ lies in the positive Weyl chamber, applying Proposition 46 we have
$$\mu_\mathfrak{a}(\mathbb{P}(V)) = \mu_\mathfrak{a}(O) = \text{conv}(W \cdot z_\tau),$$
where $z_\tau \in \mathfrak{a}$ satisfies
$$\langle z_\tau, H \rangle = \mu_\tau(H),$$
concluding the proof. \hfill \square

The boundary components of $\overline{X}_\tau^S$ has been described in terms of root data in the pioneering work of Satake 47. More precisely, given a $\mu_\tau$-connected subspace $I \subset \Pi$, Satake defined
$$V_I = \bigoplus_{\text{supp}(\lambda) \subset I} V_\lambda.$$
By Lemmata 37 and 39 we have $W_I = V_I$, $Q(W_I) = Q_J$ and $M_I$ acts irreducibly on $W_I$. By Theorem 41 the image of the map
$$i_{\tau_I} : M_I / K \cap M_I \rightarrow \mathbb{P}(V), \quad g(K \cap M_I) \mapsto [\tau_I(g(K \cap M_I)) \oplus 0],$$
lies in $\overline{X}_\tau^S$. By Theorem 41 and Theorem 53 boundary components of $\overline{X}_\tau^S$ arise from the faces of $P$ containing $z_\tau$ and which are not equivalent with respect to the Weyl group. Summing up, we have proved the following result.

**Theorem 69.** Let $\overline{X}_\tau^S$ be the Satake compactification associated to $\tau$. The $G$-action on $\overline{X}_\tau^S$ has a finite number of orbits. This number coincides with the number of the faces of $P = \mu_\mathfrak{a}(\mathbb{P}(V))$ containing $z_\tau$ which are not equivalent with respect to the Weyl group.

**Corollary 70.** The $G$-action on $\partial \overline{X}_\tau^S$ is transitive if and only if $G / K$ has rank one.

We describe the boundary components of $\overline{X}_\tau^S$ using the $G$-gradient map restricted on $O$.

Let $(\mathfrak{a}, \Pi)$ be a root datum and let $\mu_\tau$ the highest weight of $\tau$. If $I \subset \Pi$ is a $\mu_\tau$-connected subset, then $W_I$ is a $\tau$-admissible subspace and $Q(W_I) = Q_J$ where $J$ is the saturation of $I$.

Let $W$ be $\tau$-admissible. By Corollary 64 there exists $k \in K$ and a $\mu_\tau$-connected subset $I$ such that $W = kW_I$ and $Q(W) = kQ_Jk^{-1}$, where $J$ is a saturation of $I$. Set $M(W) = kM_Jk^{-1}$ and $K(W) = M(W) \cap K = kK_Jk^{-1}$. We claim that $M(W)$ does not depend on $k \in K$.

If $kW_I = \tilde{k}W_I$, then $\tilde{k}^{-1}kW_I = W_I$ and so $\tilde{k}^{-1}k \in K \cap Q_J$. Therefore
$$kM_Jk^{-1} = \tilde{k}M_J\tilde{k}^{-1}.$$
Since $M_J$ is compatible, it follows that $M(W)$ is compatible. By Theorem 53 the representation $\tau_W := (\tau_{|M(W)})_W$ is irreducible. Set $X_W = M(W) / K(W)$. $X(W)$ is a symmetric space of noncompact type 16. We split $V = W \oplus W_\perp$ and we define
$$i_{\tau_W} : X_W \rightarrow \mathbb{P}(H(W)), \quad i_{\tau_W}(gK(W)) \mapsto [\tau_W(g)\tau_W(g)^\ast],$$
which induces the following map
$$\psi_W : X_W \rightarrow \overline{\mathbb{P}(V)}, \quad gK(W) \mapsto [\tau_W(g)\tau_W(g)^\ast \oplus 0].$$
Assume that $W = W_I$. Since $J = I \cup I'$ and $I'$ is orthogonal to $I$, it follows that
$$M_J = M_I M_{I'},$$
and they commute. Since $M_I'$ acts trivially on $W_I$, it follows that $i_{W_I}$ depends only of $M_I/M_I \cap K$ and it is a boundary component of $X^S_\tau$. The same holds for any $\tau$-admissible subspace. Applying Theorem 71 and Lemma 42, we get the following result.

**Theorem 71.** The boundary of $X^S_\tau$ are exactly the subset $X^S_\tau$ of the form $i_W(X_W)$ for some $\tau$-connected subspace $W \subseteq V$, while $X = i_V(V)$. Hence

$$X^S_\tau = \bigcup_{W \tau\text{-connected}} i_W(X_W),$$

We shall think the elements of $X^S_\tau$ as rational maps of $O$.

Since $G \subset SU(V, h)\mathbb{C}$ is compatible,

$$\mathcal{M} = \{ g \in G : g = g^{-1} \},$$

is a submanifold and $\exp : p \to \mathcal{M}$ is a diffeomorphism with inverse $\log : \mathcal{M} \to p$. Then the map

$$g \mapsto \sqrt{g} = \exp \left( \frac{\log(g)}{2} \right),$$

is a diffeomorphism. Note that $\sqrt{q}$ is the Hermitian positive isomorphism of the polar decomposition of $g$ has isomorphism of $V$. We recall the following elementary fact. A proof is given \cite[Lemma 4.4, p.251]{notes}.

**Lemma 72.** Let $V$ be a Hermitian vector space and set $\mathcal{S} = \{ A \in \mathcal{H}(V) : A \geq 0 \}$. If $A \in \mathcal{S}$ there is a unique $B \in \mathcal{S}$ such that $B^2 = A$. Set $\sqrt{A} := B$. Then $\sqrt{\cdot}$ is a homeomorphism of $\mathcal{S}$ onto itself.

Set

$$\rho : G \to \mathcal{M}, \quad g \mapsto \sqrt{g} g^*.$$

Then $a = g \rho(g)^{-1} \in K$ and $g = a \rho(g)$ is the polar decomposition of $g$.

Let $W \subset V$ be a $\tau$-connected subspace. Let $g \in M(W)$. We define

$$R_{gW} : O \to O, \quad x \mapsto [\sqrt{\tau(g)\tau(g)^*} \pi_W(x)]$$

**Lemma 74.** In the above notation, the following hold:

- a. the locus of indeterminacy is $O - \mathbb{P}(W^\perp)$;
- b. $\text{Im} \ R_{gW} = O(Q(W))$;
- c. if $x \in O(Q(W))$, then $R_{gW}(x) = \rho(g)x$;
- d. $R_{gW} = L_{\rho(g)} \circ \pi_W$, where $L_{\rho(g)}$ is the automorphism of $O(Q(W))$ defined by $\rho(g)$;
- e. $\text{Im} \ R_{gW} = \text{Im} \ R_{g'W'}$ if and only if $W = W'$.

**Proof.** The domain of $R_{gW}$ is the domain of $\pi_W$ and so is given by $O - \mathbb{P}(W^\perp)$.

The Lie group $M(W)$ is compatible. By Proposition 57, it preserves $O(Q(W))$ and so $\rho(g)$ induces an automorphism of $O(Q(W))$. By Theorem 58, $\pi_W(O) = O(Q(W))$. Therefore $R_{gW}$ is given by the composition of the automorphism of $O(Q(W))$ defined by $\rho(g)$ and the rational map $\pi_W$. This proves item [a], [b], [c], [d]. By Proposition 58, $O(Q(W))$ determines $W$. Then $\text{Im} \ R_{gW} = \text{Im} \ R_{g'W'}$ if and only if $W = W'$, concluding the proof.

Given $W$ a $\tau$-connected subspace, we have a map

$$r_W : X_W \to \{ \text{Rational maps of } O \}, \quad gK(W) \mapsto R_{gW},$$

The following Lemma proves that the above map is injective.
Lemma 75. Let $\tau : G \to \mathbb{P}(V)$ be an irreducible representation. Let $O$ be a closed orbit of $G$ and let $g \in G$. If $g$ fixed pointwise $O$, then $g$ fixed pointwise any element of $\mathbb{P}(V)$.

Proof. Since the linear span of $O$ coincides with $\mathbb{P}(V)$, the automorphism $g$ fixed pointwise any element of $\mathbb{P}(V)$.

Let
$$ri_\tau : X^S_\tau \to \{\text{Rational maps of } O\},$$
denote the map defined as follow:
$$(ri_\tau)|_{X^W} = r_W.$$ By Lemmata 74 and 75 we have the following result.

Theorem 76. The map
$$ri_\tau : X^S_\tau \to \{\text{Rational maps of } O\}$$
is injective.

The next step is to understand convergence in the Satake compactification which kind of convergence we have of the corresponding rational maps.

Lemma 77. Let $p_n \to p \in X^S_\tau$ and let $\hat{p}_n$ and $\hat{p}$ be the corresponding rational map. Then $\hat{p}_n \to \hat{p}$ uniformly on compact subsets of an open dense connected subset of $O$.

Proof. By Lemma 32 we can find unique $A_n, A \in \mathcal{H}(V)$ such that $\text{Tr}(A_n) = \text{Tr}(A) = 1$, $p_n = [A_n], p = [A] \in X^S_\tau$ and $A_n \to A$ in $\text{End}(V)$. We denote by $W_n$, respectively $W$, the $\tau$-connected subspace of $V$ corresponding to $\hat{p}_n$, respectively $\hat{p}$. Then
$$N = \bigcup_{n \in \mathbb{N}} \mathbb{P}(W_n^\perp) \cup \mathbb{P}(W),$$
has no interior point and so $A_n \to A$ uniformly on the compact subset of $O - N$. Hence, by Lemma 72 we get the result. 

6. The Bourguignon Li-Yau map

In this section we introduce the Bourguignon-Li-Yau for the $G$-action on $\mathbb{P}(V)$. This map has been studied by different levels of generality by Hersch [28], Bourguignon, Li and Yau [17], Millson and Zombro [42], Biliotti and Ghigi [8, 12].

If $M$ is a compact manifold, denote by $\mathcal{M}(M)$ the vector space of finite signed Borel measures on $M$. These measures are automatically Radon [20, Thm. 7.8, p. 217]. Denote by $C(M)$ the space of real continuous function on $M$. It is a Banach space with the sup–norm. By the Riesz Representation Theorem [20, p.223] $\mathcal{M}(M)$ is the topological dual of $C(M)$. The induced norm on $\mathcal{M}(M)$ is the following one:

$$||\nu|| := \sup \left\{ \int_M f d\nu : f \in C(M), \sup_M |f| \leq 1 \right\}.$$  

We endow $\mathcal{M}(M)$ with the weak-∗ topology as dual of $C(M)$. Usually this is simply called the weak topology on measures. We use the symbol $\nu_\alpha \to \nu$ to denote the weak convergence of the net $\{\nu_\alpha\}$ to the measure $\nu$. Denote by $\mathcal{P}(M) \subset \mathcal{M}(M)$ the set of Borel probability measures on $M$. We claim $\mathcal{P}(M)$ is a compact convex subset of $\mathcal{M}(M)$. Indeed the cone of positive measures is closed and $\mathcal{P}(M)$ is the intersection of this cone with the closed affine hyperplane $\{\nu \in \mathcal{M}(M) : \nu(M) = 1\}$. Hence $\mathcal{P}(M)$ is closed. For a positive measure $|\nu| = \nu$, so $\mathcal{P}(M)$ is contained in the closed unit ball in $\mathcal{M}(M)$, which is compact in the weak topology by the
Banach-Alaoglu Theorem \cite{21} p. 425]. Since $C(M)$ is separable, the weak topology on $\mathcal{P}(M)$ is metrizable \cite{21} p. 426].

If $f : M \to N$ is a measurable map between measurable spaces and $\nu$ is a measure on $M$, the image measure $f_*\nu$ is defined by $f_*\nu(A) := \nu(f^{-1}(A))$. It satisfies the change of variables formula

\begin{equation}
\int_N u(y) d(f_*\nu)(y) = \int_M u(f(x)) d\nu(x).
\end{equation}

In the sequel for $g \in G$ and $\nu \in \mathcal{P}(M)$, we will use the notation $g \cdot \nu := g_*\nu$.

Let $\mathcal{O}$ be the unique closed orbit of $G$ on $\mathbb{P}(V)$ contained in the unique closed orbit of $U^C$. Define

$$F : \mathcal{P}(\mathcal{O}) \to \mathfrak{p}, \quad \gamma \mapsto \int_{\mathcal{O}} \mu_p(x) d\gamma(x).$$

By a standard formula of change of variable, we have

$$F(g \cdot \gamma) = \int_{\mathcal{O}} \mu_p(x) d(g \cdot \gamma)(x) = \int_{\mathcal{O}} \mu_p(gy) d\gamma(y).$$

Using the homomorphism $X = G/K \to G$, $gK \mapsto \sqrt{gg^*}$, we define the following map.

**Definition 80.** Given a probability measure $\gamma$ of $\mathcal{O}$, the Bourguignon-Li-Yau map $\Psi : X \to \mathfrak{p}$ is defined by

$$\Psi(\gamma) = \int_{\mathcal{O}} \mu_p(\rho(g)x) d\mu(x),$$

where $\rho(g) = \sqrt{gg^*}$ as in (73).

**Lemma 81.** $\Psi(\gamma)(X) \subseteq \mathcal{E}$ and $0 \in \text{Int}(\mathcal{E})$.

**Proof.** Since $\mathcal{E} = \text{conv}(\mu_p(\mathcal{O}))$ and $\gamma$ is a probability measure, it follows $\Psi(\gamma)(X) \subseteq \mathcal{E}$.

Let $\mathcal{O}'$ be the unique closed orbit of $U^C$. Since $U^C = U_0^C \cdot U_1^C$, $G = G_0 \cdot U_1^C$ and $G_0$ is a real form of $U_0$, then any simple factor of $U^C$ acts non-trivially on $V$. By Lemma 69 in \cite{8} p. 260, we have $0 \in \text{Int}(\text{conv}(i\mu(O')))$.

Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal Abelian subalgebra. By Proposition \cite{14} $\mu_{\mathfrak{a}}(\mathcal{O}') = \mu_{\mathfrak{a}}(\mathcal{O})$. Therefore

$$\pi_p(\text{conv}(i\mu(O'))) = K \mu_{\mathfrak{a}}(\mathcal{O}) = \mathcal{E},$$

where $\pi_p$ is the orthogonal projection of $i\mathfrak{u}$ onto $\mathfrak{p}$. Since $\pi_p$ is an open map, it follows that $0 \in \text{Int}(\mathcal{E})$. \hfill $\square$

**Definition 82.** We say that a probability measure $\gamma$ on $\mathcal{O}$ is $\tau$-admissible if for any hyperplane $H \subset \mathbb{P}(V)$, then $\gamma(\mathcal{O} \cap H) = 0$.

**Lemma 83.** Assume that $p_n \to p$ in $X^S_\tau$ and let $\hat{p}_n$ and $\hat{p}$ the corresponding rational maps. If $\gamma$ is $\tau$-admissible, then $\hat{p}_n \to \hat{p}$ $\gamma$-ae, i.e., $\gamma$ almost everywhere.

**Proof.** In Lemma \cite{71} we prove that $\hat{p}_n \to \hat{p}$ on a compact subset of $\mathbb{P}(V) - N$, where $N$ is a countably union of linear proper projective subspaces of $\mathbb{P}(V)$. Therefore $\gamma(N) = 0$ and so $p_n \to \hat{p}$, $\gamma$-ae. \hfill $\square$
Lemma 84. Let $\gamma$ be a $\tau$-admissible measure. Let $p \in X_W$ and let $\hat{p}$ the corresponding rational map. Then $\hat{p}_* \gamma$ is $\tau_W$-admissible measure. Moreover, if $\gamma$ is a $K$-invariant measure, then $\hat{p}_* \gamma$ is $K(W)$-invariant measure of $O(W)$.

Proof. Let $H \subset W$ be an hyperplane. It is easy to check that $\hat{p}_* \gamma(H) = \gamma(O - (W^\perp \cup H))$ and so it is $\tau_W$ admissible. The last statement follows from the fact that $\pi_W$ is $K(W)$-equivariant. □

Theorem 85. Let $\gamma \in \mathcal{P}(O)$ be a $\tau$-admissible measure. Then the Bourguignon-Li-Yau map $\Psi_\gamma$ admits a continuous extension to $\overline{X}_\tau^S$ that we still denote by $\Psi_\gamma$. This extension is unique and satisfies $\Psi_\gamma(\overline{X}_\tau^S) \subseteq \mathcal{E}$.

Proof. Let $p \in X_W$ and let $\hat{p}$ the corresponding rational map of $O$. Then $\hat{p} = \rho(g)\pi_{\hat{W}}$, where $g \in M(W)$. The function $\mu_p \circ \hat{p}$ is defined on $O - \{W^\perp\}$ and so $\gamma$-ae. It is also bounded on $O - \{W\}$ and so it $\gamma$ integrable. Therefore

$$\Psi_\gamma(p) = \int_O \mu_p(\hat{p}(x))d\gamma(x) = \int_{O - \{W^\perp\}} \mu_p(\hat{p}(x))d\gamma(x),$$

is well-posed. If $W = V$, then $p = \rho(g)$ for some $g \in G$, and so the definition agree with the definition of the Bourguignon-Li-Yau given in Definition 82 By Lemmata 77 and 83 the extension of $\Psi_\gamma$ is continuous. Finally, since $G/K$ is dense in $\overline{X}_\tau^S$, the extension is unique. □

Theorem 86. Let $W \subset V$ be a $\tau$-connected subspace. Then $\Psi_\gamma(X_W) \subseteq F_W$. Moreover, $(\Psi_\gamma)^{-1}(F_W) = \bigsqcup_{W' \subset W} X_{W'}^W = \overline{X}_W^\tau$.

Proof. Let $p \in X_W$ and let $\hat{p}$ the corresponding rational map. By Lemma 74 we have $\hat{p}(O) = O(Q(W))$. By Lemma 84 $\hat{p}_* \gamma$ is $\tau_W$-admissible. Since $O - W^\perp$ has full measure, we have

$$\Psi_\gamma(p) = \int_O \mu_p(\hat{p}(x))d\gamma(x) = \int_{O - W^\perp} \mu_p(\hat{p}(x))d\gamma(x) = \int_{O(Q(W))} \mu_p(y)d(\hat{p}_* \gamma)(y).$$

Since $\hat{p}(O) = O(Q(W))$ and $\mu_p(O(Q(W))) = \text{ext}(F_W)$, it follows $\Psi_\gamma(X_W) \subset F_W$. This also proves

$$(\Psi_\gamma)^{-1}(F_W) \supseteq \bigsqcup_{W' \subset W} X_{W'}^W.$$

Let $p \in X_{W'}$ be such that $\Psi_\gamma(p) \in F_W$. Let $\xi \in p$ be such that $F_W = F_\xi(\mathcal{E})$ and let $\hat{p} = L_{\rho(g)} \circ \pi_{\hat{W}'}$, the corresponding rational map of $p$. Then

$$\text{Max}_{z \in \mathcal{E}} \langle z, \xi \rangle = \langle \Psi_\gamma(p), \xi \rangle$$

$$= \int_O \mu_p(\rho(g)\pi_{\hat{W}'}x)d\gamma(x), \xi \rangle$$

$$= \int_{O(Q(W'))} \langle \mu_p(\rho(g)y), \xi \rangle d(\pi_{\hat{W}'})_* \gamma(y)$$

Since $\rho(g)$ is an automorphism of $O(Q(W'))$, it follows that

$$\langle \mu_p(x), \xi \rangle = \text{Max}_{z \in \mathcal{E}} \langle z, \xi \rangle,$$

$\pi_{\hat{W}'}, \gamma$-ae. By Lemma 84 $\pi_{\hat{W}'}, \gamma$ is a $\tau_W$-admissible.
Let $q \in X_{W'}$. Then
\[
\langle \Psi_{\gamma}(q), \xi \rangle = \left\langle \int_{\mathcal{O}(W')} \mu_p(\rho(g')\hat{\pi}_{W'}x) \, d\gamma(x), \xi \rightangle
= \left\langle \int_{\mathcal{O}(W')} \mu_p(\rho(g')y) \, d(\hat{\pi}_{W'}x), \xi \right\rangle
= \text{Max}_{x \in E(z)}(z, \xi),
\]
and so $\Psi_{\gamma}(X_{W'}) \subseteq F_W$. This implies $F_{W'} \subseteq F_W$ and by Proposition 59 we get $W' \subseteq W$ concluding the proof. \hfill \Box

**Corollary 87.** Let $\gamma$ be a $\tau$-admissible measure. Then $\Psi_{\gamma}(\partial X^S_{\tau}) \subseteq \partial E$.

Our aim is to prove that for any $\tau$-admissible measure, $\Psi_{\gamma}(X^S_{\tau}) = E$. We start, considering the smooth $K$-invariant measure $\nu$ which is, of course, $\tau$-admissible.

**Theorem 88.** The Bourguignon-Li-Yau map $\Psi_{\nu} : X^S_{\tau} \rightarrow E$ is an homeomorphism. Moreover, for any $W \subseteq V$ $\tau$-connected subspace
\[
\Psi_{\nu} : X_W \rightarrow \text{relint} F_W,
\]

is an homeomorphism.

**Proof.** By Corollary 5.3 p.153 in \cite{13}, the map
\[
\mathcal{F}_{\nu} : G \rightarrow p, \quad g \mapsto \int_{\mathcal{O}} \mu_p(x) \, d\nu(x),
\]
is a submersion onto Int($E$) which descents to a diffeomorphism $\mathcal{F}_{\nu} : G/K \rightarrow \text{Int}(E)$. Since $\Psi_{\nu}(gK) = \mathcal{F}_{\nu}(\rho(g)K)$, $\Psi_{\nu}$ is an homeomorphism.

Let $W \subseteq V$ be a $\tau$-connected subspace. By Lemma 84, $(\mathcal{F}_{\nu})_\ast$ is a $K(W)$ invariant measure on $\mathcal{O}(W)$ and so
\[
\Psi_{\nu} : X_W \rightarrow \text{relint} F_W,
\]
is an homeomorphism. By Theorems 10 and 86 we get that $\Psi_{\nu} : X^S_{\tau} \rightarrow E$ is an homeomorphism. \hfill \Box

**Theorem 89.** Let $\gamma$ be a $\tau$-admissible measure. Then the Bourguignon-Li-Yau map $\Psi_{\gamma} : X^S_{\tau} \rightarrow E$ is surjective.

**Proof.** Set $\gamma_t := t\gamma + (1 - t)\nu$. Define
\[
H : X^S_{\tau} \times [0, 1] \rightarrow E, \quad H(p, t) := \Psi_{\gamma_t}(p) = t\Psi_{\gamma}(p) + (1 - t)\nu(p).
\]
$\gamma_t$ is $\tau$-admissible measure on $\mathcal{O}$ for every $t \in [0, 1]$ and $H$ is continuous. By Theorem $86$, $\tau$-connected subspace $W \subseteq V$, we have $H(X_W \times [0, 1]) \subseteq F_W$ and so $H(\partial X^S_{\tau} \times [0, 1]) \subseteq \partial E$. Since $H(\cdot, 0) = \Psi_{\nu}(\cdot)$ is an homeomorphism, it has degree 1. Hence the same holds for $H(\cdot, 1) = \Psi_{\gamma}$. By a classical topological argument this yields the surjectivity of $H(\cdot, 1) = \Psi_{\gamma}$. \hfill \Box

Let $g = \mathfrak{k} \oplus p$. There is a splitting of algebra, see \cite{34},
\[
g = g_1 \oplus \cdots \oplus g_q,
\]
where $g_i = k_i \oplus p_i$ is an ideal of $g$, for $i = 1, \ldots, q$, and
\[
\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_q, \quad p = p_1 \oplus \cdots \oplus p_q.
Let \( G_1, \ldots, G_q, K_1, \ldots, K_q \) the corresponding analytic (connected) subgroups. Let \( x = x_1 + \cdots + x_p \in p \). Then
\[
K \cdot x = K_1 \cdot x_1 + \cdots + K \cdot x_p,
\]
and so one may easily check that
\[
\text{con}(K \cdot x) = \text{conv}(K_1 \cdot x_1) + \cdots + \text{conv}(K_p \cdot x_p).
\]
Let \( \tau : G \to \text{SL}(V) \) be an irreducible representation with finite kernel. Then any factor of \( G \) acts non trivially on \( V \). By [25, p.197], \( \mathcal{P}(V) = \mathcal{P}(V_1 \otimes \cdots \otimes V_q) \), \( \tau = \tau_1 \otimes \cdots \otimes \tau_q \) and the kernel of \( \tau_i \) is finite for any \( i = 1, \ldots, q \). Then
\[
\mu_\tau = \mu_{\tau_1} + \cdots + \mu_{\tau_q},
\]
and \( \mu_{\tau_i} \) is not zero for any \( i = 1, \ldots, q \). Therefore

\[
E = E_1 + \cdots + E_q,
\]
where \( E_i \) is the orbitope associated to the projective representation \( \tau_i : G_i \to \text{PSL}(V_i) \), for \( i = 1, \ldots, p \). By Theorem 88, we get the following result.

**Theorem 90.** If \( X \) is reducible, i.e., \( X = X_1 \times \cdots \times X_q \), then
\[
\mathcal{X}_{\mu_\tau} = \mathcal{X}_{\mu_{\tau_1}} \times \cdots \times \mathcal{X}_{\mu_{\tau_q}}.
\]

7. Furstenberg compactifications

Let \( G \) be a semisimple noncompact Lie group and \( K \) a maximal compact subgroup. Another way to compactly \( X = G/K \) was found by Furstenberg [22] in his search for an analogue of the Poisson formula for the unit disc. We recall very briefly the definition. In the sequel, we always refer to [16 §I.6]).

**Definition 91.** A compact homogeneous space \( O \) is called boundary of \( G \) or a \( G \)-boundary if for every probability measure \( \mu \) on \( O \), there exists a sequence \( g_j \in G \) such that \( g_j \cdot \mu \) converges to the delta measures \( \delta_x \) at some point of \( x \in O \). A \( G \)-boundary is called a boundary of \( X \).

Using Iwasawa structure theory, Moore [43, Thm. 1] proved that \( Y = G/P \) is a boundary if and only if \( P \) is parabolic. Let \( \nu \) be the \( K \)-invariant measure on \( O \). Then the map
\[
G \to \mathcal{P}(O) \quad g \mapsto g \cdot \nu
\]
descends to a continuous map \( i_O : X = G/K \to \mathcal{F}(O) \), which is injective if and only if \( P \) does not contain simple factors of \( G \) (see [43 Thm. 4] or [16 Prop. I.6.16]). In this case \( O \) is called a faithful Furstenberg boundary and the set
\[
\overline{X}_O^F := i_O(X)
\]
is called the Furstenberg compactification of \( X \) associated to the faithful boundary \( M \). Fix an irreducible complex representation \( \tau : G \to \text{GL}(V) \) such that \( P \) is the stabilizer of some \( x_0 \in \mathcal{P}(V) \) and ker \( \tau \) is finite. Such representations always exist and so \( O \) can be identified with the unique closed orbit of \( G \) in \( \mathcal{P}(V) \) contained in the unique closed orbit of \( U^C \).

**Theorem 93.** The map
\[
\Gamma : \overline{X}_\tau^S \to \overline{X}_O^F \quad \Gamma(p) := \hat{p}_* \nu
\]
is a \( G \)-equivariant homeomorphism of \( \overline{X}_\tau^S \) onto \( \overline{X}_O^F \) such that \( i_M = \Gamma \circ i_\tau \) (compare [51]).
Proof. Let \( p \in X^S_\tau \) and \( \hat{p} : \mathcal{O} \rightarrow \mathcal{O} \) be the corresponding rational map. \( \hat{p} \) is defined \( \mu \)-ae, since \( \nu \) is smooth and so \( \tau \)-admissible. Therefore \( \Gamma(p) = \hat{\mu} \cdot \mu \) is well-defined for any \( p \in X^S_\tau \). By Lemma 7.4 if \( p_n \rightarrow p \) in \( X^S_\tau \), then \( \hat{p}_n \rightarrow \hat{p} \) \( \mu \)-ae, and so \( \Gamma(p_n) \rightarrow \Gamma(p) \). This proves that \( \Gamma \) is continuous. Since \( i_M(X) \) is dense in \( X^S_\tau \) then the map is surjective. Now we prove it is injective.

Let \( p, q \in X^S_\tau \) and let \( \hat{p}, \hat{q} \) the corresponding rational map. If \( \hat{p} \cdot \mu = \hat{q} \cdot \mu \), then

\[
\int_{\mathcal{O}} \mu_p(x) d\hat{p} \cdot \mu(x) = \int_{\mathcal{O}} \mu_p(x) d\hat{q} \cdot \mu(x)
\]

and so

\[
\Psi_\nu(p) = \Psi_\nu(q),
\]

where \( \Psi_\nu \) is the Bourguignon-Li-Yau with respect to the \( K \)-invariant metric \( \nu \). Since \( \Psi_\nu \) is an homeomorphism, \( p = q \).

\( \square \)

8. A REMARK ON EIGENVALUE ESTIMATES

Let \((M, g)\) be a compact, connected orientable Riemannian manifold. It is well-known that the spectrum of the Laplacian \( \Delta_g = -d^*d \), acting on functions, form a discrete set. The first eigenvalue of the Laplacian operator, that we denote by \( \lambda_1(M, g) \), is one of the most natural and studied Riemannian invariants. There has been a considerable amount of work devoted to estimating the first eigenvalue in terms of other geometric quantities associated to \((M, g)\), see for instance [3, 4, 6, 7, 8, 11, 13, 11, 14, 15, 22]. More precisely one would like to study the quantity \( \lambda_1(M, g) \Vol(M, g)^{n/2} \), which is scale invariant. By the Rayleigh principle, upper bounds for the first eigenvalue are obtained by constructing functions with zero mean, sensitive to the geometry of the underlying manifold. Indeed, if \( f_1, \ldots, f_n \in C^\infty(M) \) have zero mean with respect to \((M, g)\), and so

\[
\int_M f_j(x) \vol_g(x) = 0,
\]

for \( j = 1, \ldots, n \), then

\[
\lambda_1(M, g) \leq \frac{\sum_{i=1}^n \int_M \| \nabla f_i(x) \|^2_g(x) \vol_g(x)}{\sum_{i=1}^n \int_M f_j^2(x) \vol_g(x)}.
\]

In the paper [28], Hersch studied the first eigenvalue on the unite sphere \( S^2 \subset \mathbb{R}^3 \).

Let \( g_0 \) be the restriction on \( S^2 \) of the canonical scalar product of \( \mathbb{R}^3 \), normalized to have volume \( 4\pi \). Then \((S^2, g_0)\) is an homogeneous Kähler manifold and its automorphism group is given by \( G = \text{PSL}(2, \mathbb{C}) \) acting by Möbius transformations. It is well-known that \( \lambda_1(S^2, g_0) = 2 \) and the three coordinate functions \( x, y, z \) are eigenfunctions of the Laplacian. These functions are the components of momentum map of \( S^2 \) with respect to \( \text{SO}(3) \)-action. Hersch showed that if \( g \) is an arbitrary Riemannian metric on \( S^2 \) (normalized to have volume \( 4\pi \)), then there is \( a \in G \) such that \( \int_{S^2} a^*x \vol_g(p) = \int_{S^2} a^*y \vol_g(p) = \int_{S^2} a^*z \vol_g(p) = 0 \). Moreover he showed that the right hand side in (94) is equal to 2 and so \( \lambda_1(S^2, g) \leq 2 \).

Bourguignon, Li and Yau [17] realized that this method applies also to estimate \( \lambda_1(\mathbb{P}^n(\mathbb{C}), g) \) if \( g \) is a Kähler metric. In [8] we recast the method of Hersch-Bourguignon-Li-Yau in terms of momentum map and applied it when \( M \) is an arbitrary Hermitian symmetric space, \( g_0 \) is the symmetric metric, \( G = \text{Aut}(M) \) and the functions are the components of the momentum map \( \mu : M \rightarrow \mathfrak{k} \) for \( K = \text{Isom}(M, g_0) \). The Bourguignon-Li-Yau map is the tool to deal with the first step for the unique closed orbit of \( G \) on \( \mathcal{O} \). We use the gradient map instead the momentum map. If \( g \) is a Riemannian metric on \( \mathcal{O} \), we denote by \( \nu := \vol_g / \Vol(\mathcal{O}, g) \) the corresponding Borel probability measure. Let \( e_1, \ldots, e_r \) be an orthonormal basis of \( \mathfrak{p} \) and set \( f_j := \langle \mu_p, e_j \rangle \). By
Theorem 89, there exists \( a \in G \) such that \( \int_{\mathcal{O}} \mu_p(ax) \vol_g(x) = 0 \) and so \( \int_{\mathcal{O}} a^* f_j(x) \vol_g(x) = 0 \) for \( j = 1, \ldots, r \). By Rayleigh’s Theorem we get the following result.

**Theorem 95.** In the above notation, we have

\[
\lambda_1(\mathcal{O}, g) \leq \frac{\sum_{j=1}^r \int_M |\nabla (a^* f_j)|^2 \vol_g}{\int_M a^* (|\mu_p|^2) \vol_g}
\]  

The second step is to actually compute the right hand side in (94). On the other hand, at the moment we are not able to compute the right hand side in (96) except for the Hermitian symmetric spaces \([8, 7]\). We believe that this computation can be carried out in much greater generality and that it would yield very interesting estimates. We leave this problem for future investigations.

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