Unit vectors for similar oblate spheroidal coordinates and vector transformation

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Abstract. The unit vectors transformation between the Cartesian and the novel similar oblate spheroidal coordinates, and vice versa, is derived. It facilitates transformation of vector fields between these two types of orthogonal coordinates and can advantageously simplify solutions of problems exhibiting oblate spheroidal geometry. Several examples demonstrate the use of the derived relations. Generalized sine and cosine applicable in the similar oblate spheroidal coordinate system are introduced as well.

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1. Introduction

Recently finalized analytical solution of the similar oblate spheroidal (SOS) orthogonal coordinate system [1,2] can be a powerful tool for a description of physical processes inside or in the vicinity of the bodies with geometry of an oblate spheroid. Such bodies range (but are not limited to) from planets with a small oblateness (like the Earth with the ratio of the semi-axes difference to the major semi-axis length $\approx 1:300$), through elliptical galaxies up to significantly flattened objects like disk galaxies.

In the field of atmospheric physics, it was stated [1] that the SOS coordinates could be of help for better modeling of geopotential surfaces, allowing for a better description of the spatial variation of the apparent gravity. Still more advantageously, the SOS coordinates could be used for modeling of the potential and atmosphere of significantly more oblate celestial bodies, e.g., the gas giants. Further, similar oblate spheroids are frequently used for modeling of iso-density levels inside galaxies [3,4]. Therefore, the SOS coordinates can be helpful also in this field.

Similar oblate spheroidal coordinates are distinct from the well-known confocal oblate spheroidal coordinates [5], which do not possess similarity between the individual spheroids within the family of spheroidal coordinate surfaces. Moreover, they are distinct from all the standard orthogonal coordinate systems [5], as one coordinate surfaces family is not of the second degree (and even not of the fourth degree), but it is created by general power functions (i.e., with a real number exponent) rotated around an axis.

Although the coordinate transformation from the SOS coordinates to the Cartesian system as well as the metric scale factors and the Jacobian determinant was already reported [2], a description of the transformation of a vector field $\mathbf{A}$ between the SOS system and the Cartesian coordinates is still missing. It would be of advantage to have a possibility to transform between vectors (e.g., force vector, fluid velocity vector field) expressed in the SOS and in the Cartesian coordinates. Therefore, a clear physical motivation exists for the derivation of the formulas enabling to transform vectors or vector fields.

The transformation of a vector field $\mathbf{A}$ between two orthogonal coordinate systems requires unit vectors to be determined. In case of the vector transformation from the similar oblate spheroidal orthogonal coordinate system (coordinates denoted $R$, $\nu$, $\lambda$) to the Cartesian coordinate system (coordinates $x_{3D}$, $y_{3D}$, $z_{3D}$), the expressions for the unit vectors in the SOS system $\hat{R}$, $\hat{\nu}$, $\hat{\lambda}$ are to be found.
The vector field \( \mathbf{A} \) can be then written in terms of the unit vectors as
\[
\mathbf{A} = A_x \mathbf{\hat{x}} + A_y \mathbf{\hat{y}} + A_z \mathbf{\hat{z}} = A_R \mathbf{\hat{R}} + A_\nu \mathbf{\hat{\nu}} + A_\lambda \mathbf{\hat{\lambda}},
\]
(1)
where \( \mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{\hat{z}} \) are the unit vectors in the Cartesian coordinates. \( A_x, A_y, A_z \) are the components of the vector \( \mathbf{A} \) in the Cartesian coordinates, while \( A_R, A_\nu, A_\lambda \) are the components of the vector \( \mathbf{A} \) in the SOS coordinates. When the unit vectors \( \mathbf{\hat{x}}, \mathbf{\hat{y}}, \mathbf{\hat{z}} \) are selected to be equal to the basis vectors of the Cartesian coordinate system, as is done in this article, then the above relation (1) represents a way of the vector \( \mathbf{A} \) transformation from the SOS coordinates to the Cartesian coordinates.

The following text deals with the derivation of the unit vectors \( \mathbf{\hat{R}}, \mathbf{\hat{\nu}}, \mathbf{\hat{\lambda}} \) of the SOS coordinate system, which then enables transformation of vectors and vector fields from the SOS coordinates to the Cartesian coordinates. The inverse transformation is dealt with as well: as both coordinate systems are orthogonal, the derivation of the inverse transformation relations does not pose further tedious work.

The organization of the text is as follows: First, the relevant formulas for the SOS coordinates and their transformations to the Cartesian coordinates [2] are summarized (section 2); then, also the combinatorial identities relevant for the derivations in this article are reminded (section 3). The main parts of the article are sections 4 and 5, in which the unit vectors are determined in two distinct regions of the space. A matrix form of the unit vector transformations is reported in section 6. Finally, several examples of the use of the derived unit vector transformation are shown (section 7).

2. SOS coordinates and their derivatives, metric scale factors, and Jacobian

2.1. SOS coordinates

For SOS coordinates [1], the basic coordinate surfaces of the \( R \) coordinate are similar oblate spheroids. They are generated by rotating similar ellipses, given in Cartesian coordinates by the formula
\[
x^2 + (1+\mu)z^2 = R^2,
\]
around the minor axis. The \( R \) coordinate value is equal to the major semi-axis length (i.e., to the equatorial radius) of the particular spheroid from the family. The parameter \( \mu \) characterizes the whole family of similar ellipses—as well as the similar oblate spheroids generated by rotating the ellipses—having the same eccentricity \( e \):
\[
e = \sqrt{\frac{\mu}{1 + \mu}}.
\]
(3)
The parameter \( \mu > 0 \) for oblate spheroids, and the minor and major semi-axes of each member of the spheroid family have the ratio \( (1+\mu)^{-1/2} \). As a limit (when \( \mu = 0 \)), a sphere is determined by Eq. (2) with corresponding spherical coordinate surfaces of the \( R \) coordinate.

A special—called “reference” in what follows—spheroid is introduced with the equatorial radius \( R_0 \). Usually, this reference spheroid of the SOS coordinate system with the equatorial radius \( R_0 \) would coincide with the surface of the body under investigation for which the SOS coordinate system is applied.

The second set of the coordinate surfaces, orthogonal to the similar oblate spheroids defined above, are power functions \( z \sim x^{1+\mu} \) rotated around the minor axis of the above spheroids [1,2]. The labeling, i.e., the coordinate \( \nu \) corresponding to these surfaces, is equivalent to the so called parametric latitude [2]. The coordinate \( \nu \) is also equivalent to the parameter used for the standard parametric equation of the ellipse with a special \( R_0 \) major semi-axis, i.e., \( x = R_0 \cos \nu \) and \( z = \frac{R_0}{\sqrt{1+\mu}} \sin \nu \). This reference ellipse generates the reference spheroid with the equatorial radius \( R_0 \) mentioned above. An example of the SOS coordinate system section with \( x-z \) plane is displayed in Fig. 1.

Finally, the third set of the coordinate surfaces, orthogonal to the previous two, are then semi-infinite planes containing the rotation axis. The associated coordinate is the longitude angle \( \lambda \), which is the same as its equivalent coordinate in the spherical coordinate system.
Fig. 1. One quadrant in $x$–$z$ plane composed of the SOS coordinate system (with $\mu=2$) lines around the reference spheroid with the equatorial radius $R_0=1$. Both constant-$R$ coordinate surfaces (spheroids, here represented by their sections by $x$–$z$ plane) and orthogonal trajectories to them (power functions) having a constant value of $\nu$ are displayed. The angular spacing of the parametric angle $\nu$ lines is $3^\circ$. The boundary (a straight line in 2D section of 3D cone surface) between the small-$\nu$ and the large-$\nu$ regions is displayed as well.

A key role in the derivation of the SOS coordinate transformations plays the dimensionless parameter $W$ defined as

$$W = \left(\frac{R}{R_0}\right)^\mu \frac{\sin \nu}{\cos^{1+\mu} \nu},$$

which will be used frequently in this article. In what follows, the calculation is restricted to the parametric latitude $\nu$ positioned only in the first quadrant of the $x$–$z$ plane of the Cartesian coordinate system, thus the generating ellipse is calculated only for $\nu \in (0, \frac{\pi}{2})$. Due to the symmetry, the solution in the other quadrants can be easily obtained. With this restriction, the parameter $W$, eq. (4), is always non-negative, which simplifies further derivations.

The SOS coordinates can be transformed to the Cartesian coordinates with expressions including infinite power series with generalized binomial coefficients [2]. The solution is based on the Lagrange’s inversion theorem (see the solution of trinomial equation in [6,7]; nevertheless, the solution of the problem using infinite series with generalized binomial coefficients was first reported in [8]). The expressions cannot be written in a closed form; nevertheless, they can still be denoted as “analytical.”

It appeared [2] that the expressions have to be derived separately in so called ”the small-$\nu$ region” and separately in ”the large-$\nu$ region” of the space, and that the border between the two regions is defined by the formula

$$W_{\text{border}} (R, \nu) = \sqrt{\frac{\mu^\nu}{(1+\mu)^{1+\mu}}} = \text{constant.}$$

In Fig. 1, the border (being a line in 2D section) between the two regions is displayed as well.

In order to fulfill the aim of this article, the already derived relations for the SOS coordinates [2] are needed. Therefore, they are listed in the following sub-sections. The formulae are listed separately for the small-$\nu$ region and for the large-$\nu$ region.
2.2. Formulae for the small-\(\nu\) region

The SOS coordinates \(R, \nu, \lambda\) transformation to the 3D Cartesian coordinates are as follows [2] in the small-\(\nu\) region:

\[
x_{3D} = \cos \lambda \, x(\nu, R) = \cos \lambda \, R \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k, \tag{6}
\]

\[
y_{3D} = \sin \lambda \, x(\nu, R) = \sin \lambda \, R \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k, \tag{7}
\]

\[
z_{3D} = z(\nu, R) = W \frac{\sqrt{1 + \mu}}{1 + \mu} \sum_{k=0}^{\infty} \left( -\frac{1+\mu}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k \tag{8}
\]

where the parameter \(W\) was defined in (4). The convergence limit for the series in (6)–(8) is \(W < W_{\text{border}}\), where \(W_{\text{border}}\) is given by (5). The power series contain generalized binomial coefficients [9] of the form \(\left( a - \frac{\mu k}{k} \right)\) where the real number \(a\) can depend on \(\mu\).

Although the transformation using infinite power series may seem to be complicated, it has an advantage of a relatively simple differentiability and integrability, which operations result again in infinite power series. The partial derivatives of \(x_{3D}, y_{3D}, z_{3D}\) with respect to \(R, \nu\) and \(\lambda\) can be thus calculated (see [2]):

\[
\frac{\partial x_{3D}(\nu, R, \lambda)}{\partial R} = \cos \lambda \frac{\partial x(\nu, R)}{\partial R} = \cos \lambda \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k, \tag{9}
\]

\[
\frac{\partial y_{3D}(\nu, R, \lambda)}{\partial R} = \sin \lambda \frac{\partial x(\nu, R)}{\partial R} = \sin \lambda \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k, \tag{10}
\]

\[
\frac{\partial z_{3D}(\nu, R)}{\partial R} = \frac{\partial z(\nu, R)}{\partial R} = W \frac{\sqrt{1 + \mu}}{1 + \mu} \sum_{k=0}^{\infty} \left( \frac{\mu+1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k, \tag{11}
\]

\[
\frac{\partial x_{3D}(\nu, R, \lambda)}{\partial \nu} = \cos \lambda \frac{\partial x(\nu, R)}{\partial \nu} = \cos \lambda \frac{1}{\mu W} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k - \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( \frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k \right\}, \tag{12}
\]

\[
\frac{\partial y_{3D}(\nu, R, \lambda)}{\partial \nu} = \frac{1}{\mu W} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k - \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( \frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k \right\}, \tag{13}
\]

\[
\frac{\partial z_{3D}(\nu, R, \lambda)}{\partial \nu} = \frac{\partial z(\nu, R)}{\partial \nu} = \frac{1}{\sqrt{1 + \mu}} \frac{1}{\mu W} \frac{\partial W}{\partial \nu} \left\{ (1 + \mu) \sum_{k=0}^{\infty} \left( -\frac{1+\mu}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k - \sum_{k=0}^{\infty} \left( -\frac{1+\mu}{2} - \frac{\mu k}{k} \right) \left( \frac{1+\mu}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k \right\}, \tag{14}
\]

\[
\frac{\partial x_{3D}(\nu, R, \lambda)}{\partial \lambda} = \frac{\partial x(\nu, R)}{\partial \lambda} = -\sin \lambda \, R \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( \frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k, \tag{15}
\]

\[
\frac{\partial y_{3D}(\nu, R, \lambda)}{\partial \lambda} = \frac{\partial y(\nu, R)}{\partial \lambda} = \cos \lambda \, R \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \left( \frac{1}{2} - \frac{\mu k}{k} \right) \left( W^2 \right)^k, \tag{16}
\]

\[
\frac{\partial z_{3D}(\nu, R, \lambda)}{\partial \lambda} = 0. \tag{17}
\]
and they can be further used for the derivation of the metric scale factors \([2]\),

\[
h_R = \sqrt{\sum_{k=0}^{\infty} \left( -\frac{\mu k}{k} \right) (W^2)^k},
\]

\[
h_\nu = \frac{R}{\sqrt{1+\mu}} \frac{\partial W}{\partial \nu} \sqrt{\sum_{k=0}^{\infty} \left( -\frac{(\mu+2) - \mu k}{k} \right) (W^2)^k},
\]

\[
h_\lambda = x(\nu, R) = R \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \frac{-1}{-\frac{1}{2} - \mu k} (W^2)^k.
\]

Here,

\[
\frac{\partial W}{\partial \nu} = \left( \frac{R}{R_0} \right)^\mu \frac{1+\mu \sin^2 \nu}{\cos^{2+\mu} \nu}.
\]

To complete the list of the already derived relations \([2]\) for the SOS coordinate system, also the Jacobian determinant in the small-\(\nu\) region is reported

\[
J = h_R h_\nu h_\lambda = R^2 \sqrt{1+\mu} \frac{\partial W}{\partial \nu} \sum_{k=0}^{\infty} \left( -\frac{\mu+3}{2} - \frac{\mu k}{k} \right) (W^2)^k,
\]

as well as the Jacobian divided by the square of the \(h_R\) scale factor

\[
\frac{J}{h_R^2} = R^2 \sqrt{1+\mu} \frac{\partial W}{\partial \nu} \sum_{k=0}^{\infty} \left( -\frac{\mu+3}{2} - \frac{\mu k}{k} \right) (W^2)^k.
\]

### 2.3. Formulae for the large-\(\nu\) region

3D SOS coordinates transformation to the Cartesian coordinates in the large-\(\nu\) region is as follows \([2]\):

\[
x_{3D} = x(\nu, R) \cos \lambda = \cos \lambda W^{-\frac{1}{1+\mu}} R \sum_{k=0}^{\infty} \left( -\frac{1}{2(1+\mu)} + \frac{\mu}{1+\mu} k \right) \frac{-1}{2(1+\mu)} \left( W^{-\frac{2}{1+\mu}} \right)^k,
\]

\[
y_{3D} = y(\nu, R) \sin \lambda = \sin \lambda W^{-\frac{1}{1+\mu}} R \sum_{k=0}^{\infty} \left( -\frac{1}{2(1+\mu)} + \frac{\mu}{1+\mu} k \right) \frac{-1}{2(1+\mu)} \left( W^{-\frac{2}{1+\mu}} \right)^k,
\]

\[
z_{3D} = z(\nu, R) = \frac{1}{\sqrt{1+\mu}} R \sum_{k=0}^{\infty} \left( -\frac{\mu + 1}{2(1+\mu)} k \right) \frac{-1}{\frac{3}{2} + \frac{\mu}{1+\mu}} \left( W^{-\frac{2}{1+\mu}} \right)^k,
\]

where, again, the parameter \(W\) equals to \((4)\). The convergence limit for the series in \((24)-(26)\) is \(W > W_{\text{border}}\), where \(W_{\text{border}}\) is given by \((5)\).

From \((6)\) and \((8)\), as well as from \((24)\) and \((26)\), it follows that the ratio of \(x\) and \(z\) coordinates in the \(x-z\) plane is dependent solely on the parameter \(W\) (i.e., not separately on \(R\)). It means that—for \(W=\text{constant}\) while changing \(R\)—the points \((x, z)\) lie on a straight line starting at zero, regardless the particular values of \(R\) and \(\nu\). In 3D, it forms a cone instead. This is valid for both the small-\(\nu\) region and the large-\(\nu\) region, and even in any vicinity of the convergence limit \(W=W_{\text{border}}\). As a consequence of this, also the border between the small- and the large-\(\nu\) regions (characterized by \(W_{\text{border}}\) being constant according to \((5)\)) is a straight half-line starting at zero, regardless if the series in \((6)-(8)\) and \((24)-(26)\) converge or not at this border line.
The partial derivatives in the large-$\nu$ region are:

\[
\frac{\partial x_{3D}(\nu, R, \lambda)}{\partial R} = \cos \lambda \frac{\partial x(\nu, R)}{\partial R} = \cos \lambda R \frac{1}{\mu W^{\frac{1}{1+\mu}}} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{\frac{2}{1+\mu}} \right)^{k-\frac{1}{2}} \right\},
\]

(27)

\[
\frac{\partial y_{3D}(\nu, R, \lambda)}{\partial R} = \sin \lambda \frac{\partial x(\nu, R)}{\partial R} = \sin \lambda R \frac{1}{\mu W^{\frac{1}{1+\mu}}} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{\frac{2}{1+\mu}} \right)^{k-\frac{1}{2}} \right\},
\]

(28)

\[
\frac{\partial z_{3D}(\nu, R)}{\partial R} = \frac{\partial z(\nu, R)}{\partial R} = \frac{1}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{\frac{2}{1+\mu}} \right)^{k},
\]

(29)

\[
\frac{\partial x_{3D}(\nu, R, \lambda)}{\partial \nu} = \cos \lambda \frac{\partial x(\nu, R)}{\partial \nu} = \cos \lambda \frac{1}{\mu W^{\frac{1}{1+\mu}}} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{\frac{2}{1+\mu}} \right)^{k-\frac{1}{2}} \right\},
\]

(30)

\[
\frac{\partial y_{3D}(\nu, R, \lambda)}{\partial \nu} = \sin \lambda \frac{\partial x(\nu, R)}{\partial \nu} = \sin \lambda \frac{1}{\mu W^{\frac{1}{1+\mu}}} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{\frac{2}{1+\mu}} \right)^{k-\frac{1}{2}} \right\},
\]

(31)

\[
\frac{\partial z_{3D}(\nu, R)}{\partial \nu} = \frac{\partial z(\nu, R)}{\partial \nu} = \frac{1}{\mu W} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{\frac{2}{1+\mu}} \right)^{k} \right\},
\]

(32)

\[
\frac{\partial x_{3D}(\nu, R, \lambda)}{\partial \lambda} = \frac{\partial x(\nu, R)}{\partial \lambda} = -\sin \lambda W^{\frac{1}{1+\mu}} R \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \frac{1}{\mu W} \frac{\partial W}{\partial \nu} \left( \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{\frac{2}{1+\mu}} \right)^{k} \right),
\]

(33)

\[
\frac{\partial y_{3D}(\nu, R, \lambda)}{\partial \lambda} = \frac{\partial y(\nu, R)}{\partial \lambda} = \cos \lambda W^{\frac{1}{1+\mu}} R \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \frac{1}{\mu W} \frac{\partial W}{\partial \nu} \left( \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{\frac{2}{1+\mu}} \right)^{k} \right),
\]

(34)

\[
\frac{\partial z_{3D}(\nu, R)}{\partial \lambda} = 0.
\]

(35)

The metric scale factors can be summarized as follows [2]:

\[
h_{R} = \frac{1}{\sqrt{1+\mu}} \left( \sum_{k=0}^{\infty} \left( \frac{1}{1+\mu} k \right) \left( W^{-\frac{1}{1+\mu}} \right)^{k} \right),
\]

(36)

\[
h_{\nu} = \frac{R}{1+\mu} W^{-\frac{2+\mu}{1+\mu}} \frac{\partial W}{\partial \nu} \left( \sum_{k=0}^{\infty} \left( \frac{2+\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^{k} \right),
\]

(37)

\[
h_{\lambda} = \frac{1}{\sqrt{1+\mu}} \frac{1}{\mu W} \frac{\partial W}{\partial \nu} \left( \sum_{k=0}^{\infty} \left( \frac{1+\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^{k} \right).
\]

(38)

Finally, also the Jacobian determinant for the large-$\nu$ region is reported

\[
\mathcal{J} = h_{R} h_{\nu} h_{\lambda} = \frac{R^{2} W^{-\frac{3+\mu}{1+\mu}}}{(1+\mu)^{3}} \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^{k},
\]

(39)
as well as the Jacobian divided by the square of the $h_R$ scale factor

$$\frac{J}{h_R^2} = \frac{R^2}{\sqrt{1 + \mu}} \frac{\partial W}{\partial \nu} \sum_{k=0}^{\infty} \left( -\frac{1}{2} \frac{\mu + 3}{\mu + 1} + \frac{\mu}{2} \frac{1 + \mu}{1 + \mu} \right) \left( -\frac{1}{2} \frac{\mu + 3}{\mu + 1} + \frac{\mu}{2} \frac{1 + \mu}{1 + \mu} \right)^k \left( W^{-\frac{3}{1 + \mu}} \right)^k. \quad (40)$$

3. Useful combinatorial identities

Operations with the above reported series are not as straightforward as a simple multiplication or summation of two expressions. Nevertheless, it is possible to deal with them with a help of known combinatorial identities [8–11]. The important ones, used in the derivations in this article, are listed in this section. They involve generalized binomial coefficients, which can be expressed in several ways as follows:

$$\binom{\beta}{k} = \frac{\beta!}{k! (\beta - k)!} = \frac{\Gamma (\beta + 1)}{k! \Gamma (\beta - k + 1)} = \frac{1}{k!} \prod_{j=1}^{k} (\beta - j + 1). \quad (41)$$

In the expressions, $\beta$ is a real number and $k$ is a non-negative integer. The symbol $\Gamma$ denotes Euler’s Gamma function. The last expression on the right side, containing the product called the falling factorial, is of advantage for numerical calculations of the generalized binomial coefficients. The value of the falling factorial is taken to be 1, i.e., an empty product, when $k=0$. Note also that the generalized binomial coefficient is in fact a $k$-th degree polynomial in $\beta$.

The following set of identities, useful for the present derivations, was found in the past for the generalized binomial coefficients. The crucial importance have the Pólya and Szegö identities (see [8], or [9] – expressions No. 1.120 and 1.121 – or [12])

$$\sum_{m=0}^{k} \frac{a + bm}{a + bm} \binom{a + bm}{m} \binom{c + b (k - m)}{k - m} = \binom{a + c + bk}{k}, \quad (42)$$

$$\sum_{m=0}^{k} \frac{a}{a + bk} \binom{a + bk}{k} r^k = \binom{a + t + bm}{m} \binom{c - t + b (k - m)}{k - m}. \quad (43)$$

for which the following equation for $p$ is fulfilled:

$$-rp^b + p = 1. \quad (44)$$

Both series in (42) and in (43) have the convergence limit

$$|r| < \left| \frac{(b-1)^{k-1}}{b^k} \right| \quad (45)$$

(for the discussion on the convergence limit, see [2,13]). The variables $p$, $a$, $b$, $r$ in the above relations are real numbers.

Other important identities for further derivations involving generalized binomial coefficients are the Hagen–Rothe identity [10],

$$\sum_{m=0}^{k} \frac{a + bm}{a + bm} \binom{a + bm}{m} \binom{c + b (k - m)}{k - m} = \binom{a + c + bk}{k}, \quad (46)$$

and the Jensen identity (see [9], expression No. 3.143)

$$\sum_{m=0}^{k} \frac{a + bm}{m} \binom{a + bm}{m} \binom{c + b (k - m)}{k - m} = \sum_{m=0}^{k} \frac{a + t + bm}{m} \binom{c - t + b (k - m)}{k - m}. \quad (47)$$

Also the $t$ variable is real.
The well-known Cauchy product applied on for the series with generalized binomial coefficients is
\[
\left[ \sum_{k=0}^{\infty} \frac{a}{a + bk} \binom{a + bk}{k} r^k \right] \left[ \sum_{k=0}^{\infty} \binom{c + bk}{k} r^k \right] = \sum_{k=0}^{\infty} r^k \sum_{m=0}^{k} \frac{a}{a + bm} \binom{a + bm}{m} \binom{c + b(k - m)}{k - m}, \tag{48}
\]
or similarly
\[
\left[ \sum_{k=0}^{\infty} \frac{a}{a + bk} \binom{a + bk}{k} r^k \right] \left[ \sum_{k=0}^{\infty} \binom{c + bk}{k} r^k \right] = \sum_{k=0}^{\infty} r^k \sum_{m=0}^{k} \binom{a}{m} \binom{c + b(k - m)}{k - m}. \tag{49}
\]

By combining the Hagen–Rothe identity (46) with the first Cauchy product (48), the following identity is obtained:
\[
\left[ \sum_{k=0}^{\infty} \frac{a}{a + bk} \binom{a + bk}{k} r^k \right] \left[ \sum_{k=0}^{\infty} \binom{c + bk}{k} r^k \right] = \sum_{k=0}^{\infty} \binom{a + c + bk}{k} r^k. \tag{50}
\]

From this relation, it also follows (for \(c = 0\)) that
\[
\sum_{k=0}^{\infty} \frac{a}{a + bk} \binom{a + bk}{k} r^k = \frac{\sum_{k=0}^{\infty} \binom{a + bk}{k} r^k}{\sum_{k=0}^{\infty} \binom{bk}{k} r^k}. \tag{51}
\]

By combining the Jensen identity (47) with the second type Cauchy product (49), the following identity is obtained:
\[
\left[ \sum_{k=0}^{\infty} \binom{a + bk}{k} r^k \right] \left[ \sum_{k=0}^{\infty} \binom{c + bk}{k} r^k \right] = \left[ \sum_{k=0}^{\infty} \binom{a + t + bk}{k} r^k \right] \left[ \sum_{k=0}^{\infty} \binom{c - t + bk}{k} r^k \right], \tag{52}
\]

For the generalized binomial coefficients, also the following identities hold \[9\]:
\[
\binom{\beta + k}{k} \frac{1}{\beta + k} = \binom{\beta + k - 1}{k} \frac{1}{\beta} = (-1)^{k+1} \binom{-\beta}{k} \frac{1}{-\beta}. \tag{53}
\]

Thus, there are three equivalent forms for the quantity expressed by the generalized binomial coefficients. In order to complete the set of the useful binomial identities, the following are also listed \[9\]:
\[
\binom{\alpha}{k} = \binom{\alpha}{k - 1} \frac{\alpha - k + 1}{k}, \tag{54}
\]
\[
\binom{\alpha + 1}{k} - \binom{\alpha}{k} = \binom{\alpha}{k - 1}, \tag{55}
\]

where \(\alpha\) is a real number. Further, the binomial theorem
\[
(1+r)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} r^k \tag{56}
\]
will be later used. Finally, the generating function of the central binomial coefficients, which has a closed form
\[
\sum_{k=0}^{\infty} \binom{2k}{k} r^k = \frac{1}{\sqrt{1 - 4r}} \quad \text{where} \quad |r| < \frac{1}{4} \tag{57}
\]
is to be mentioned \[11\], together with its more general counterpart
\[
\sum_{k=0}^{\infty} \binom{a + 2k}{k} r^k = \frac{1}{\sqrt{1 - 4r}} \left( \frac{1 - \sqrt{1 - 4r}}{2r} \right)^a \quad \text{where} \quad |r| < \frac{1}{4}. \tag{58}
\]
4. Unit vectors in the SOS coordinate system for the small-\( \nu \) region

In this section, the unit vectors are derived in the small-\( \nu \) region with the help of the relations reported in the previous sections.

4.1. Unit vector in \( R \) direction

The norm in the case of \( R \)-direction is (see (9)–(11))

\[
\sqrt{\left( \frac{\partial x_{3D}(\nu, R, \lambda)}{\partial R} \right)^2 + \left( \frac{\partial y_{3D}(\nu, R, \lambda)}{\partial R} \right)^2 + \left( \frac{\partial z_{3D}(\nu, R)}{\partial R} \right)^2} = \sqrt{\cos^2 \lambda \left( \frac{\partial x(\nu, R)}{\partial R} \right)^2 + \sin^2 \lambda \left( \frac{\partial x(\nu, R)}{\partial R} \right)^2 + \left( \frac{\partial z(\nu, R)}{\partial R} \right)^2} = h_R, \tag{59}
\]

and thus, the unit vector is (again using (9)–(11))

\[
\hat{R} = \left( \begin{array}{c}
\frac{\partial x_{3D}(\nu, R, \lambda)}{\partial R}/h_R \\
\frac{\partial y_{3D}(\nu, R, \lambda)}{\partial R}/h_R \\
\frac{\partial z_{3D}(\nu, R)}{\partial R}/h_R
\end{array} \right) = \left( \begin{array}{c}
\cos \lambda \frac{\partial x(\nu, R)}{\partial R}/h_R \\
\sin \lambda \frac{\partial x(\nu, R)}{\partial R}/h_R \\
\frac{\partial z(\nu, R)}{\partial R}/h_R
\end{array} \right). \tag{60}
\]

First, the component containing derivative of \( x \) with respect to \( R \) is derived, see (9), (10) and (18):

\[
\frac{\partial x(\nu, R)}{\partial R}/h_\nu = \frac{\sum_{k=0}^{\infty} \left( -\frac{1}{2} - \mu k \right) (W^2)^k}{\sqrt{\sum_{k=0}^{\infty} \left( -\mu k \right) (W^2)^k}}. \tag{61}
\]

When Cauchy product combined with Jensen identity (52) is used for the numerator (with \( b=-\mu, a=c=-1/2, t=1/2 \)), then

\[
\sum_{k=0}^{\infty} \left( -\frac{1}{2} - \mu k \right) (W^2)^k = \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1}{2} - \mu k \right) (W^2)^k} \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \mu k \right) (W^2)^k = \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1}{2} - \mu k \right) (W^2)^k} \sum_{k=0}^{\infty} \left( -\mu k \right) (W^2)^k, \tag{62}
\]

and thus,

\[
\frac{\partial x(\nu, R)}{\partial \nu} / h_\nu = \frac{\sqrt{1 + \mu}}{\mu W} \left\{ \sum_{k=0}^{\infty} \left( \frac{\mu + 1}{k} - \mu k \right) (W^2)^k - \frac{\sum_{k=0}^{\infty} \left( -\frac{1}{2} - \mu k \right) - \frac{1}{2} - \mu k (W^2)^k}{\sqrt{\sum_{k=0}^{\infty} \left( -\mu k -(\mu + 2) \right) (W^2)^k}} \right\}. \tag{63}
\]

Similarly, we obtain for the \( z \) coordinate

\[
\frac{\partial z(\nu, R)}{\partial R} / h_R = \frac{W \sqrt{1 + \mu} \sum_{k=0}^{\infty} \left( \frac{\mu + 1}{2} \right) (W^2)^k}{\sqrt{\sum_{k=0}^{\infty} \left( -\mu k \right) (W^2)^k}}, \tag{64}
\]
and using (52) for the series in numerator (with \( b = -\mu, \ a = c = -(\mu+1)/2, \ t = (\mu+1)/2 \)), the following is received:

\[
\frac{\partial z(\nu,R)}{\partial R}/h_R = W \sqrt{1+\mu} \left[ \sum_{k=0}^{\infty} \left( -\frac{\mu k}{k} \right) (W^2)^k \right]^{1/2} \left[ \sum_{k=0}^{\infty} \left( -\frac{(1+\mu)\mu k}{k} \right) (W^2)^k \right]^{1/2} \left[ \sum_{k=0}^{\infty} \left( -\frac{\mu k}{k} \right) (W^2)^k \right]^{1/2}
\]

\[
= W \sqrt{1+\mu} \left[ \sum_{k=0}^{\infty} \left( -\frac{(1+\mu)\mu k}{k} \right) (W^2)^k \right].
\]

The unit vector \( \hat{R} (60) \) is then

\[
\hat{R} = \left( \frac{\cos \lambda}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1-\mu k}{k} \right) (W^2)^k, \right.
\]

\[
\sin \lambda \sum_{k=0}^{\infty} \left( -\frac{1-\mu k}{k} \right) (W^2)^k \left. \right) \right) \right) = \left( \frac{\cos \lambda}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1-\mu k}{k} \right) (W^2)^k, \right.
\]

\[
\sin \lambda \sum_{k=0}^{\infty} \left( -\frac{1-\mu k}{k} \right) (W^2)^k \right) \right) \right) \right).
\]

For the special case when \( \mu = 0 \) (i.e., the case equivalent to the spherical coordinates), the parameter \( W \), see (4), is simplified to

\[
W (\mu = 0) = \frac{\sin \nu}{\cos \nu},
\]

and the result is (thanks to the binomial theorem (56))

\[
\sqrt{\sum_{k=0}^{\infty} \left( -\frac{1-\mu k}{k} \right) (W^2)^k} = \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1}{k} \right) (W^2)^k} = \sqrt{(1+W^2)^{-1}} = \sqrt{\frac{1}{1+\sin^2 \nu}} = \cos \nu.
\]

The unit vector for the special case of the spherical coordinates \( (\mu=0) \) is thus

\[
\hat{R} (\mu = 0) = \left( \frac{\cos \lambda \cos \nu}{\sin \nu} \right),
\]

which is correct for the equivalent of the spherical coordinates (assuming that \( \nu \) is not a polar angle but increases when moving from the equator toward the pole).

Another partial confirmation that the above derivation of the unit vector is correct is that the unit vector length has to be equal to one. The length square

\[
\left( \frac{\cos \lambda}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1-\mu k}{k} \right) (W^2)^k \right)^2 + \left( \sin \lambda \sum_{k=0}^{\infty} \left( -\frac{1-\mu k}{k} \right) (W^2)^k \right)^2 + \left( W \sqrt{1+\mu} \sum_{k=0}^{\infty} \left( -\frac{(1+\mu)\mu k}{k} \right) (W^2)^k \right)^2
\]

\[
\text{can be clearly simplified to}
\]

\[
\sum_{k=0}^{\infty} \left( -\frac{1-\mu k}{k} \right) (W^2)^k + W^2 (1+\mu) \sum_{k=0}^{\infty} \left( -\frac{(1+\mu)\mu k}{k} \right) (W^2)^k.
\]
When (42) is used with \( r = W^2 \), \( b=\mu \), and \( a=-1 \) in the first term while \( a=-(\mu+1) \) in the second term, the relation can be rewritten to the form

\[
\frac{p^{-1+1}}{(1+\mu)p-\mu} + W^2 (1 + \mu) \frac{p^{-(1+\mu)+1}}{(1+\mu)p-\mu} = \frac{1}{(1+\mu)p-\mu} + W^2 (1 + \mu) \frac{p^{\mu-\mu}}{(1+\mu)p-\mu}.
\] (72)

According to (44)

\[
W^2 p^{-\mu} = p - 1,
\] (73)

and thus further simplification of the expression (72) follows:

\[
\frac{1}{(1+\mu)p-\mu} + (1 + \mu) \frac{p - 1}{(1+\mu)p-\mu} = \frac{1 + p - 1 + \mu p - \mu}{p + \mu p - \mu} = 1.
\] (74)

Indeed, the length of the unit vector is equal to one.

### 4.2. Unit vector in \( \nu \) direction

The norm in this case is (see (12) – (14))

\[
\sqrt{\left( \frac{\partial x_{3D}(\nu, R, \lambda)}{\partial \nu} \right)^2 + \left( \frac{\partial y_{3D}(\nu, R, \lambda)}{\partial \nu} \right)^2 + \left( \frac{\partial z_{3D}(\nu, R)}{\partial \nu} \right)^2}
\]

\[
= \sqrt{\cos^2 \lambda \left( \frac{\partial x(\nu, R)}{\partial \nu} \right)^2 + \sin^2 \lambda \left( \frac{\partial x(\nu, R)}{\partial \nu} \right)^2 + \left( \frac{\partial z(\nu, R)}{\partial \nu} \right)^2} = h_\nu.
\] (75)

Therefore, the unit vector is equal to

\[
\hat{\nu} = \left( \frac{\partial x(\nu, R, \lambda)}{\partial \nu}, \frac{\partial y_{3D}(\nu, R, \lambda)}{\partial \nu}, \frac{\partial z_{3D}(\nu, R)}{\partial \nu} \right) / h_\nu = \frac{\cos \lambda}{h_\nu} \left( \frac{\partial x(\nu, R)}{\partial \nu}, \frac{\partial y(\nu, R)}{\partial \nu}, \frac{\partial z(\nu, R)}{\partial \nu} \right).
\] (76)

The first component of the unit vector \( \hat{\nu} \) is thus proportional to (see (12), (13) and (19))

\[
\frac{\partial x(\nu, R)}{\partial \nu} / h_\nu = R \frac{1}{\mu W} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) (W^2)^k - \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \frac{-\frac{1}{2} - \mu k}{k} (W^2)^k \right\}
\]

\[
= \sqrt{1+\mu} \frac{1}{\mu W} \sqrt{\sum_{k=0}^{\infty} \left( -\frac{\mu k}{k} - (\mu+2) \right) (W^2)^k}
\]

\[
= \sqrt{1+\mu} \frac{1}{\mu W} \left\{ \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) (W^2)^k - \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) \frac{-\frac{1}{2} - \mu k}{k} (W^2)^k \right\}.
\] (77)

When the Cauchy product combined with the Jensen identity (52) is used for the first-term numerator (with \( b=\mu \), \( a=c=-1/2 \), \( t=(\mu+3/2) \)), the following identity is found (in a similar way as in the derivation of the unit \( R \) vector),

\[
\sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) (W^2)^k = \sum_{k=0}^{\infty} \left( \frac{1}{2} + \frac{\mu k}{k} \right) (W^2)^k \sum_{k=0}^{\infty} \left( -\frac{1}{2} - \frac{\mu k}{k} \right) (W^2)^k
\]

\[
= \sum_{k=0}^{\infty} \left( -\frac{(\mu+2)}{k} - \mu k \right) (W^2)^k \sum_{k=0}^{\infty} \left( \frac{1}{2} + \frac{\mu k}{k} \right) (W^2)^k,
\] (78)
and thus,

\[
\frac{\partial x(\nu, R)}{\partial \nu} / h_\nu = \frac{\sqrt{1 + \mu}}{\mu W} \left\{ \sum_{k=0}^{\infty} \binom{\mu + 1 - \mu k}{k} (W^2)^k - \frac{\sum_{k=0}^{\infty} \binom{\mu + 1 - \mu k}{k} (W^2)^k}{\sum_{k=0}^{\infty} \binom{-\mu - (\mu + 2)}{k} (W^2)^k} \right\}. \tag{79}
\]

As the identity (51) is valid, and as also the following identity holds thanks to the Cauchy product combined with Jensen identity (52) with \( b = -\mu, \ a = c = 0, \ t = (\mu + 1), \)

\[
\sqrt{\sum_{k=0}^{\infty} \binom{-\mu k}{k} (W^2)^k (W^2) = \sum_{k=0}^{\infty} \binom{-\mu - 1 - \mu k}{k} (W^2)^k \sum_{k=0}^{\infty} \binom{\mu + 1 - \mu k}{k} (W^2)^k \sum_{k=0}^{\infty} \binom{\mu + 1 - \mu k}{k} (W^2)^k ;}
\]

then, the second term in (79) is

\[
\frac{\sum_{k=0}^{\infty} \binom{-\frac{1}{2} - \mu k}{k} - \frac{1}{2} - \mu k (W^2)^k}{\sum_{k=0}^{\infty} \binom{-\mu - (\mu + 2)}{k} (W^2)^k} = \frac{\sum_{k=0}^{\infty} \binom{-\frac{1}{2} - \mu k}{k} (W^2)^k}{\sum_{k=0}^{\infty} \binom{-\mu - (\mu + 2)}{k} (W^2)^k}
\]

\[
= \frac{\sqrt{\sum_{k=0}^{\infty} \binom{-\mu - 1 - \mu k}{k} (W^2)^k \sum_{k=0}^{\infty} \binom{\mu + 1 - \mu k}{k} (W^2)^k \sum_{k=0}^{\infty} \binom{-\mu - (\mu + 2)}{k} (W^2)^k}}{\sum_{k=0}^{\infty} \binom{-\frac{1}{2} - \mu k}{k} (W^2)^k}
\]

\[
= \frac{1}{\sum_{k=0}^{\infty} \binom{-\mu - 1 - \mu k}{k} (W^2)^k}
\]. \tag{81}

In the final part of the above simplification, the relation (52) was again used with \( b = -\mu, \ a = (\mu + 1), \)

\( c = - (\mu + 2), \ t = - (\mu + 3/2) \) in the denominator. Therefore (by inserting (81) to (79)),

\[
\frac{\partial x(\nu, R)}{\partial \nu} / h_\nu = \frac{\sqrt{1 + \mu}}{\mu W} \left\{ \sum_{k=0}^{\infty} \binom{\mu + 1 - \mu k}{k} (W^2)^k - \frac{1}{\sum_{k=0}^{\infty} \binom{-\mu - 1 - \mu k}{k} (W^2)^k} \right\}. \tag{82}
\]
As Pólya and Szegö identity (42) holds with \( r = W^2, \ b = -\mu, \) and as \( a = \mu + 1 \) in the first term while it is \( a = -\mu - 1 \) in the denominator of the second term, then

\[
\frac{\partial x(\nu, R)}{\partial \nu} \bigg/ h_{\nu} = \frac{\sqrt{1 + \mu}}{\mu W} \left\{ \sqrt{\frac{p^{(\mu+1)} + 1}{(1 + \mu) p - \mu} - \frac{1}{\sqrt{1 + \mu p - \mu}}} \right\} = \frac{\sqrt{1 + \mu}}{\mu W} \left\{ \frac{\sqrt{p^{\mu+2}}}{(1 + \mu) p - \mu} - \frac{1}{\sqrt{1 + \mu p - \mu}} \right\}
\]

\[
= \frac{\sqrt{1 + \mu}}{\mu W} \left\{ \frac{p^{\sqrt{p^\mu}}}{(1 + \mu) p - \mu} - \sqrt{p^\mu} \sqrt{(1 + \mu) p - \mu} \right\} = \frac{\sqrt{1 + \mu}}{\mu W^{\sqrt{p^\mu}} \sqrt{p - (1 + \mu) p + \mu}} \sqrt{1 + \mu p - \mu}
\]

\[
= - \frac{\sqrt{1 + \mu}}{W} \sqrt{p - 1} \sqrt{(1 + \mu) p - \mu} .
\]  

(83)

As further (see (44), assuming \( r = W^2, \ b = -\mu \))

\[W^2 = \frac{p - 1}{p^\mu} = (p - 1) p^\mu \quad \Rightarrow \quad p^\mu = \frac{W^2}{p - 1}, \]  

(84)

then (using again (42), once with \( a = 0 \) while the second time with \( a = -1 \)) Eq. (83) changes to

\[
\frac{\partial x(\nu, R)}{\partial \nu} \bigg/ h_{\nu} = - \frac{\sqrt{1 + \mu}}{W} \sqrt{\frac{W^2}{p - 1} \frac{p - 1}{\sqrt{(1 + \mu) p - \mu}}} = - \frac{\sqrt{1 + \mu}}{\sqrt{\frac{p - 1}{(1 + \mu) p - \mu}}}
\]

\[
= - \frac{\sqrt{1 + \mu}}{\sqrt{1 + \mu p - \mu}} - \frac{1}{\sqrt{(1 + \mu) p - \mu}}
\]

\[
= - \sqrt{1 + \mu} \left\{ \sum_{k=0}^{\infty} \left( \frac{-\mu k}{k} \right) (W^2)^k - \sum_{k=0}^{\infty} \left( \frac{-1 - \mu k}{k} \right) (W^2)^k \right\}
\]

\[
= - \sqrt{1 + \mu} \left\{ \sum_{k=1}^{\infty} \left[ \left( \frac{-\mu k}{k} \right) - \left( \frac{-1 - \mu k}{k} \right) \right] (W^2)^k \right\}
\]

\[
= - \sqrt{1 + \mu} \left\{ \sum_{k=1}^{\infty} \left[ \left( \frac{-\mu k}{k} \right) - \left( \frac{-1 - \mu k}{k} \right) \right] (W^2)^k \right\} .
\]  

(85)

Note the change of the range of the \( k \) index in the last expression. Finally, due to the identity (55) with \( \alpha = -1 - \mu k \), it follows that

\[
\frac{\partial x(\nu, R)}{\partial \nu} \bigg/ h_{\nu} = - \sqrt{1 + \mu} \left\{ \sum_{k=1}^{\infty} \left( \frac{-1 - \mu k}{k - 1} \right) (W^2)^k \right\} = - \sqrt{1 + \mu} \left\{ \sum_{M=0}^{\infty} \left( \frac{-1 - \mu (M + 1)}{M} \right) (W^2)^{M+1} \right\}
\]

\[
= - W \sqrt{1 + \mu} \left\{ \sum_{M=0}^{\infty} \left( \frac{-1 - \mu M}{M} \right) (W^2)^M \right\} ,
\]  

(86)

where the index substitution \( M = k - 1 \) was applied.

The previous extensive calculation can be simplified with the direct use of the powerful relations (42) and (43). Equation (77) is then

\[
\frac{\partial x(\nu, R)}{\partial \nu} \bigg/ h_{\nu} = R \frac{1}{\mu W} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( \frac{-\mu k}{k} \right) (W^2)^k - \sum_{k=0}^{\infty} \left( \frac{-\mu k}{k} \right) \frac{2}{\mu - \mu k} (W^2)^k \right\}
\]

\[
= \frac{R}{\sqrt{1 + \mu}} \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( \frac{-\mu k - (\mu + 2)}{k} \right) (W^2)^k \right\}
\]
Then (with the help of (42))

$$\frac{\partial z(v, R)}{\partial \nu} \bigg/ h_\nu = \frac{1}{(1+\mu) p - \mu} \sqrt{p^{-1} + 1} + \sum_{k=0}^{\infty} \left(\frac{1}{W^2} \right)^k \sum_{k=0}^{\infty} \left(\frac{1}{W^2} \right)^k$$

(91)

Then (with the help of (42))

$$\frac{\partial z(v, R)}{\partial \nu} \bigg/ h_\nu = \sum_{k=0}^{\infty} \left(\frac{1}{W^2} \right)^k$$

(92)
The last expression can be again tested for \( \mu=0 \). Then, also \( W (\mu=0) = \frac{\sin \nu}{\cos \nu} \) and, when the binomial theorem (56) is used, we obtain

\[
\frac{\partial \varphi (\nu, R)}{\partial \nu}/h_\nu = \frac{\sqrt{\sum_{k=0}^{\infty} \left( -\frac{1}{k} \right) (W^2)^k}}{1 + \mu + \mu k} = \sqrt{(1+W^2)^{-1}} = \sqrt{\frac{1}{1+\sin^2 \nu}} = \cos \nu .
\] (93)

Generally, i.e., for any \( \mu \geq 0 \), it follows from (76) and from (89) and (92) that

\[
\hat{\nu} = \left( \frac{\partial x_{3D}(\nu, R)}{\partial \nu}/h_\nu, \frac{\partial x_{3D}(\nu, R)}{\partial \nu}/h_\nu, \frac{\partial x_{3D}(\nu, R)}{\partial \nu}/h_\nu \right) = \left( -\cos \lambda W \sqrt{1+\mu} \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1+\mu}{k} \right) (W^2)^k}, -\sin \lambda W \sqrt{1+\mu} \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1+\mu}{k} \right) (W^2)^k}, \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1}{k} \right) (W^2)^k} \right) .
\] (94)

For the special case of the spherical coordinates (\( \mu=0 \)), the unit vector is (see (90) and (93))

\[
\hat{\nu} (\mu=0) = \left( -\cos \lambda \sin \nu \right),
\] (95)

which is correct considering that \( \nu \) is not a polar angle, but it changes in the opposite direction than the usual polar angle of the spherical coordinates (i.e., \( \nu \) increases when moving from the equator toward the north pole), and the unit vector has thus opposite direction than the unit vector of the standard spherical polar coordinates.

Another partial confirmation that the above derivation of the unit vector is correct is that the unit vector length has to be equal to one. The length square

\[
\left( -\cos \lambda W \sqrt{1+\mu} \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1+\mu}{k} \right) (W^2)^k} \right)^2 + \left( -\sin \lambda W \sqrt{1+\mu} \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1+\mu}{k} \right) (W^2)^k} \right)^2 + \left( \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1}{k} \right) (W^2)^k} \right)^2
\] (96)

can be clearly simplified to

\[
W^2 (1+\mu) \sum_{k=0}^{\infty} \left( -\frac{1+\mu}{k} \right) (W^2)^k + \sum_{k=0}^{\infty} \left( -\frac{1}{k} \right) (W^2)^k .
\] (97)

When (42) is used with \( r = W^2 \), \( b=\mu \), and with \( a=-(\mu+1) \) in the first term while \( a=-1 \) in the second term, it results in

\[
W^2 (1+\mu) \frac{p^{-(1+\mu)+1}}{(1+\mu)p-\mu} + \frac{p^{-1}+1}{(1+\mu)p-\mu} = W^2 (1+\mu) \frac{p^{-\mu}}{(1+\mu)p-\mu} + \frac{1}{(1+\mu)p-\mu} .
\] (98)

According to (44)

\[
W^2 p^{-\mu} = p - 1 ,
\] (99)

and further simplification thus follows:

\[
(1+\mu) \frac{p - 1}{(1+\mu)p-\mu} + \frac{1}{(1+\mu)p-\mu} = \frac{p - 1 + \mu p - \mu + 1}{p + \mu p - \mu} = 1 .
\] (100)

It can be seen that the length of the unit vector is indeed equal to one.
4.3. Unit vector in $\lambda$ direction

The norm in this case is

$$\sqrt{\left(\frac{\partial x_{3D}(\nu, R, \lambda)}{\partial \lambda}\right)^2 + \left(\frac{\partial y_{3D}(\nu, R, \lambda)}{\partial \lambda}\right)^2 + \left(\frac{\partial z_{3D}(\nu, R)}{\partial \lambda}\right)^2} =$$

$$\sqrt{[x(\nu, R)]^2 \left(\frac{\partial \cos \lambda}{\partial \lambda}\right)^2 + [x(\nu, R)]^2 \left(\frac{\partial \sin \lambda}{\partial \lambda}\right)^2 + \left(\frac{\partial z(\nu, R)}{\partial \lambda}\right)^2} =$$

$$= \sqrt{[x(\nu, R)]^2 \sin^2 \lambda + [x(\nu, R)]^2 \cos^2 \lambda + x(\nu, R) = h_\lambda}, \quad (101)$$

and, therefore, the unit vector can be written as

$$\hat{\lambda} = \left(\frac{\partial x_{3D(\nu,R,\lambda)}}{\partial \lambda}/h_\lambda\right) \left(\frac{\partial y_{3D(\nu,R,\lambda)}}{\partial \lambda}/h_\lambda\right) \left(\frac{\partial z_{3D(\nu,R)}}{\partial \lambda}/h_\lambda\right) = \left(\begin{array}{c} -x(\nu, R) \sin \lambda / h_\lambda \\ x(\nu, R) \cos \lambda / h_\lambda \\ 0 \end{array}\right). \quad (102)$$

As $x(\nu, R) = h_\lambda$, the result is simply

$$\hat{\lambda} = \left(\begin{array}{c} -\sin \lambda \\ \cos \lambda \\ 0 \end{array}\right). \quad (103)$$

This result, valid for the SOS coordinates, is the same as for the spherical coordinates.

5. Unit vectors in SOS coordinate system for the large-$\nu$ region

In this section, the unit vectors are to be derived in the large-$\nu$ region. As the derivation is similar to the derivation for the small-$\nu$ region (only the particular parameters in the relations vary), the derivation for the unit vectors $\hat{R}$ and $\hat{\nu}$ is placed into Appendix A. The results of the derivations are then reported here.

5.1. Unit vector in $R$ direction

The unit vector $\hat{R}$ in the large-$\nu$ region is

$$\hat{R} = \left(\begin{array}{c} \cos \lambda \frac{W - \frac{\mu + 1}{\nu + 1} - \frac{\mu}{k} + \frac{\mu}{W - \frac{\mu + 1}{\nu + 1}}}{\sqrt{\frac{1 + \mu}{\nu + 1}}} \sqrt{\sum_{k=0}^{\infty} \left(\frac{-1 + \frac{\mu}{k}}{\nu + 1} k\right) \left(W - \frac{\mu + 1}{\nu + 1}\right)^k} \\ \sin \lambda \frac{W - \frac{\mu + 1}{\nu + 1} + \frac{\mu}{k} + \frac{\mu}{W - \frac{\mu + 1}{\nu + 1}}}{\sqrt{\frac{1 + \mu}{\nu + 1}}} \sqrt{\sum_{k=0}^{\infty} \left(\frac{-1 + \frac{\mu}{k}}{\nu + 1} k\right) \left(W - \frac{\mu + 1}{\nu + 1}\right)^k} \end{array}\right). \quad (104)$$

For the special case when $\mu = 0$ (i.e., equivalent to the spherical coordinates), the unit vector is

$$\hat{R}(\mu = 0) = \left(\begin{array}{c} \cos \lambda \cos \nu \\ \sin \lambda \cos \nu \\ \sin \nu \end{array}\right), \quad (105)$$

i.e., the same as in the small-$\nu$ region, as expected.
5.2. Unit vector in $\nu$ direction

The unit vector $\hat{\nu}$ in the large-$\nu$ region is (see Appendix A)

$$
\hat{\nu} = \left( \frac{\partial_{3D}(\nu, R)}{\partial \nu} / h_{\nu}, \frac{\partial_{3D}(\nu, R)}{\partial \nu} / h_{\nu}, \frac{\partial_{3D}(\nu, R)}{\partial \nu} / h_{\nu} \right) = \left( \begin{array}{c}
-\cos \lambda \sqrt{\sum_{k=0}^{\infty} \left( -1 + \frac{\mu}{1 + \mu} k \right) \left( W - \frac{2}{1 + \mu} \right) k}
-\sin \lambda \sqrt{\sum_{k=0}^{\infty} \left( -1 + \frac{\mu}{1 + \mu} k \right) \left( W - \frac{2}{1 + \mu} \right) k}
\frac{W - \frac{1}{1 + \mu}}{\sqrt{1 + \mu}} \sqrt{\sum_{k=0}^{\infty} \left( - \frac{1 + \mu}{1 + \mu} k \right) \left( W - \frac{2}{1 + \mu} \right) k}
\end{array} \right).
$$

(106)

The unit vector for the special case of spherical coordinates ($\mu=0$) is (see Appendix A)

$$
\hat{\nu} (\mu=0) = \left( \begin{array}{c}
-\cos \lambda \sin \nu
-\sin \lambda \sin \nu
0
\end{array} \right).
$$

(107)

This result, valid for the SOS coordinates, is the same as for the spherical coordinates.

In Appendix A, a proof that the unit vector length is equal to one in the large-$\nu$ region is derived as well, which partially confirms the correctness of the unit vector derivation.

5.3. Unit vector in $\lambda$ direction

In the large-$\nu$ region, the unit vector in $\lambda$-direction is simply (see (33) – (35) and (38))

$$
\hat{\lambda} = \left( \frac{\partial_{x3D}(\nu, R, \lambda)}{\partial \lambda} / h_{\lambda}, \frac{\partial_{y3D}(\nu, R, \lambda)}{\partial \lambda} / h_{\lambda}, \frac{\partial_{z3D}(\nu, R)}{\partial \lambda} / h_{\lambda} \right) = \left( \begin{array}{c}
-x (\nu, R) \sin \lambda / h_{\lambda}
-x (\nu, R) \cos \lambda / h_{\lambda}
0
\end{array} \right).
$$

(108)

This result, valid for the SOS coordinates, is the same as for the spherical coordinates.

6. Matrix expression of the transformation and the inverse transformation

Similarly as for the spherical-coordinate equivalent solution for $\mu=0$, for which the relation of the SOS unit vectors to the Cartesian unit vectors can be written in the matrix form

$$
\begin{pmatrix}
\hat{R} \\
\hat{\nu} \\
\hat{\lambda}
\end{pmatrix} =
\begin{pmatrix}
\cos \nu \cos \lambda & \cos \nu \sin \lambda & \sin \nu \\
-\sin \nu \cos \lambda & -\sin \nu \sin \lambda & \cos \nu \\
-\sin \lambda & \cos \lambda & 0
\end{pmatrix}
\begin{pmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{pmatrix},
$$

(109)

the same can be done for the SOS coordinates with $\mu>0$, as shown in the following paragraphs.

6.1. Unit vector transformation for SOS and Cartesian coordinates

Extensive matrix expressions, which would arise for the SOS unit vectors relation to the Cartesian unit vectors when directly using the found unit-vector relations (66), (94), (102) (in the small-$\nu$ region) and (104), (106), (108)) (in the large-$\nu$ region), can be significantly simplified by using the following notation:

$$
f_C = \sum_{k=0}^{\infty} \left( -1 - \mu k \right) (W^2)^k \quad \text{and} \quad f_S = W \sqrt{1 + \mu} \sum_{k=0}^{\infty} \left( - (1 + \mu) - \mu k \right) (W^2)^k
$$

(110)
in the small-$\nu$ region, and
\[
\begin{align*}
f_C &= \frac{W^{\frac{1}{1+\mu}}}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1+\mu}{1+\mu} k \right) \left( W^{-\frac{1}{1+\mu}} \right)^k \\
f_S &= \frac{W^{\frac{1}{1+\mu}}}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1+\mu}{1+\mu} k \right) \left( W^{-\frac{1}{1+\mu}} \right)^k
\end{align*}
\]

(111)
in the large-$\nu$ region. With the above notation, the unit vectors transformation can be written in both regions as
\[
\begin{pmatrix}
\hat{R} \\
\hat{\nu} \\
\hat{\lambda}
\end{pmatrix} = \begin{pmatrix}
f_C \cos \lambda & f_C \sin \lambda & f_S \\
-f_S \cos \lambda & -f_S \sin \lambda & f_C \\
-\sin \lambda & \cos \lambda & 0
\end{pmatrix} \begin{pmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{pmatrix},
\]

(112)
when taking the proper $f_C, f_S$ functions reported in (110) or (111) in the small- or in the large-$\nu$ region, respectively, depending on the value of $W$ (see (5)). This transformation remains still analytical (although generally not in a closed form) providing that the sums in (110) or (111) are convergent. The transformation is formally similar to the one valid for the spherical coordinates; only cosines and sines of the coordinate $\nu$ (i.e., the parametric latitude) are exchanged by $f_C, f_S$ functions for the SOS coordinates vector transformation.

### 6.2. Generalized sine and cosine

It does worth to note remarkable properties of $f_C, f_S$ functions. It can be shown with the help of (42) that
\[
f_C^2 + f_S^2 = 1. \tag{113}
\]
Further, with the help of (4), it can be seen that
\[
\begin{align*}
f_C (\nu = 0) &= 1 & \text{and} & f_S (\nu = 0) &= 0, \\
f_C (\nu = \frac{\pi}{2}) &= 0 & \text{and} & f_S (\nu = \frac{\pi}{2}) &= 1.
\end{align*}
\]
(114)
(115)
With the help of the binomial identities (53), (54), a relation between $f_S$ ((110) or (111)) and the $R$-coordinate metric scale factor ((18) or (36)) can be found:
\[
\frac{1 - h_R^2}{\mu} = f_S^2. \tag{116}
\]
Using (113), further relations between $f_C, f_S$ and $h_R$ are derived:
\[
h_R^2 = 1 - \frac{\mu}{1+\mu} f_S^2 = \frac{1}{1+\mu} + \frac{\mu}{1+\mu} f_C^2, \tag{117}
\]
and the inverse relations
\[
f_S^2 = \frac{1 + \mu}{\mu} (1 - h_R^2) \quad \text{and} \quad f_C^2 = \frac{(1+\mu) h_R^2 - 1}{\mu}. \tag{118}
\]
Specially, for $\mu=0$, we obtain (with the help of (4) and the binomial theorem)
\[
\lim_{\mu \to 0} \left( \frac{1 - h_R^2}{\mu} \right) = \lim_{\mu \to 0} \left( \frac{f_S^2}{1 + \mu} \right) = \lim_{\mu \to 0} (f_S^2) = \sin^2 \nu. \tag{119}
\]
From (113), it then follows that
\[
\lim_{\mu \to 0} (f_C^2) = \cos^2 \nu. \tag{120}
\]
In the extreme case of the spherical coordinates (i.e., when $\mu=0$), functions $f_C, f_S$ then became the standard trigonometric functions.
Using the relation (118), and the relation No. (91) from [2] for the closed form of the metric scale factor $h_R$ at the reference spheroid, i.e.,

$$h_{R0} = \frac{1}{\sqrt{1 + \mu \sin^2 \nu}},$$

the expressions for $f_C$ and $f_S$ on the reference spheroid (i.e., for $R=R_0$) can be written in a closed form:

$$f_{C0}^2 = \frac{(1 + \mu) h_{R0}^2 - 1}{\mu} = \frac{1}{1 + \mu \sin^2 \nu} \cos^2 \nu$$

$$f_{S0}^2 = \frac{1 + \mu}{\mu} (1 - h_{R0}^2) = \frac{1 + \mu}{1 + \mu \sin^2 \nu} \sin^2 \nu.$$  

(122)

(123)

The characteristics of the derivatives of $f_C$ and $f_S$ (not shown in this article) are also found to resemble the ones of the trigonometric functions.

The above reported properties of $f_C$ and $f_S$ lead to an introduction of their notation as the generalized cosine ($f_C$) and the generalized sine ($f_S$) in the frame of the SOS coordinate system. In Fig. 2, both functions are depicted for the parameter $\mu=2$ and for several values of the coordinate $R$.

With the help of (52) (Jensen’s identity) and the relations for $f_C$ and $f_S$, the following identity between the metric scale factors and $f_C, f_S$ can be derived as well:

$$h_\nu^2 = \frac{1 + \mu}{\left(\frac{\partial W}{\partial \nu}\right)^2 R^2} \frac{1}{1 + \mu}.$$  

(124)

This formula and the expressions (118) further help to derive the metric scale factor $h_\nu$ in terms of the powers of the metric scale factor $h_R$:

$$h_\nu^2 = \frac{1}{1 + \mu} \left(\frac{\partial W}{\partial \nu}\right)^2 R^2 \left\{ -1 + (2 + \mu) h_R^4 - (1 + \mu) h_R^4 \right\}.$$  

(125)

Note that $\left(\frac{\partial W}{\partial \nu}\right)/W = \frac{1 + \mu \sin^2 \nu}{\sin \nu \cos \nu}$ in this important relation is a function of only $\nu$.

### 6.3. Inverse transformation

As both the Cartesian and the SOS coordinates are orthogonal, the matrix (112) is an orthogonal matrix, which means that its inverse is simply its transpose. The Cartesian unit vectors are thus related to the SOS unit vectors by the relation

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} f_C \cos \lambda & -f_S \cos \lambda & -\sin \lambda \\ f_C \sin \lambda & -f_S \sin \lambda & \cos \lambda \\ f_S & f_C & 0 \end{pmatrix} \begin{pmatrix} \hat{R} \\ \hat{\nu} \\ \hat{\lambda} \end{pmatrix}.$$  

(126)

### 6.4. Vector transformation

With the above introduced notation ($f_C, f_S$), the unit vectors of the SOS coordinates are simplified (see (66), (94), (102), and (104), (106), (108)) to

$$\hat{R} = \begin{pmatrix} f_C \cos \lambda \\ f_C \sin \lambda \\ f_S \end{pmatrix}, \quad \hat{\nu} = \begin{pmatrix} -f_S \cos \lambda \\ -f_S \sin \lambda \\ f_C \end{pmatrix}, \quad \hat{\lambda} = \begin{pmatrix} -\sin \lambda \\ \cos \lambda \\ 0 \end{pmatrix}.$$  

(127)
Then, the vector $\mathbf{A}$, see (1), for which its components $A_R$, $A_\nu$, $A_\lambda$ in the SOS coordinates are known, can be expressed in the Cartesian coordinates components $A_x$, $A_y$, $A_z$ as

$$\mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} f_C \cos \lambda \\ f_C \sin \lambda \\ f_S \end{pmatrix} A_R + \begin{pmatrix} -f_S \cos \lambda \\ -f_S \sin \lambda \\ f_C \end{pmatrix} A_\nu + \begin{pmatrix} -\sin \lambda \\ \cos \lambda \\ 0 \end{pmatrix} A_\lambda$$

$$= \begin{pmatrix} A_R f_C \cos \lambda - A_\nu f_S \cos \lambda - A_\lambda \sin \lambda \\ A_R f_C \sin \lambda - A_\nu f_S \sin \lambda + A_\lambda \cos \lambda \\ A_R f_S + A_\nu f_C \end{pmatrix}.$$
7. Examples

7.1. Special case: $\mu=1$

One more example (along with $\mu=0$ case mentioned in the introduction to the section 6, and $\mu=2$ which is not shown here) when the transformation relations can be written in a closed form is the case when the oblateness parameter $\mu=1$. Then, in the small-$\nu$ region,

$$f_C = \sqrt{\sum_{k=0}^{\infty} \frac{(-1-k)}{k} (W^2)^k} \quad \text{and} \quad f_S = W\sqrt{2} \sqrt{\sum_{k=0}^{\infty} \frac{(-2-k)}{k} (W^2)^k}, \quad (129)$$

resulting in (using (53) with $\beta=k+1$ and therefore $\beta+k-1=2k$)

$$f_C = \sqrt{\sum_{k=0}^{\infty} \frac{2k}{k} (-W^2)^k} \quad \text{and} \quad f_S = W\sqrt{2} \sqrt{\sum_{k=0}^{\infty} \frac{1+2k}{k} (-W^2)^k}. \quad (130)$$

Using further the identities (57) and (58), the result is

$$f_C = \frac{1}{\sqrt{1-4(-W^2)}} = \frac{1}{\sqrt{1+4W^2}} \quad \text{and} \quad f_S = W\sqrt{2} \frac{1}{\sqrt{1-4(-W^2)}} = \frac{1}{\sqrt{1+4W^2}}. \quad (131)$$

Considering that, for $\mu=1$, the parameter $W$ (see (4)) is equal to

$$W = \left(\frac{R}{R_0}\right) \frac{\sin\nu}{\cos^2\nu}, \quad (132)$$

the following relation is obtained:

$$f_C = \sqrt{\frac{1}{\sqrt{1+4\left(\frac{R}{R_0}\right)^2 \sin^2\nu}} \frac{1}{\sqrt{1+4W^2}}} \quad \text{and} \quad f_S = \sqrt{1 - \frac{1}{\sqrt{1+4\left(\frac{R}{R_0}\right)^2 \sin^2\nu}}}. \quad (133)$$

Note that $f_C^2 + f_S^2 = 1$.

On the border line between the small-$\nu$ and the large-$\nu$ region, Eq. (5) is valid for $W$. Therefore, $W_{\text{border}} = \sqrt{1/(1+1)}^{1+1} = \frac{1}{2}$ on this line, and the value for $f_C$ and $f_S$ is there as follows: $f_C=\sqrt{1/\sqrt{2}}$, $f_S=\sqrt{1-1/\sqrt{2}}$.

In the large-$\nu$ region, the relations in the case when $\mu=1$ are

$$f_C = \frac{1}{\sqrt{2W}} \sum_{k=0}^{\infty} \left(\frac{-\frac{1}{2}+\frac{1}{2}k}{k}\right) (W^{-1})^k \quad \text{and} \quad f_S = \sqrt{\sum_{k=0}^{\infty} \left(\frac{-1+\frac{1}{2}k}{k}\right) (W^{-1})^k}. \quad (134)$$

For these infinite series, there is no simple well-known formula as for the series in the small-$\nu$ region. Therefore, the formula (42) has to be used to obtain a closed form. When used for the series contained in the $f_C$ expression, then

$$f_C = \frac{1}{\sqrt{2W}} \sum_{k=0}^{\infty} \left(\frac{-\frac{1}{2}+\frac{1}{2}k}{k}\right) (W^{-1})^k = \frac{1}{\sqrt{2W}} \sqrt{\frac{p^{-\frac{1}{2}+1}}{(1-\frac{1}{2})p+\frac{1}{2}}} \quad \text{where} \quad -W^{-1}p^{\frac{1}{2}}+p=1. \quad (135)$$
Therefore,

\[ f_C = \frac{1}{\sqrt{2W}} \sqrt{\frac{p^2}{\frac{1}{2}p + \frac{1}{2}}} \quad \text{where} \quad p^2 = W(p - 1), \quad (136) \]

and

\[ f_C = \frac{1}{\sqrt{2W}} \sqrt{2W \frac{p - 1}{p + 1}} = \sqrt{\frac{p - 1}{p + 1}} \quad \text{where} \quad W^2p^2 - (2W^2 + 1)p + W^2 = 0. \quad (137) \]

The solution of the quadratic equation on the right side is

\[ p_{1,2} = \frac{2W^2 + 1 \pm \sqrt{4W^2 + 1}}{2W^2}. \quad (138) \]

Finally,

\[ f_C = \sqrt{\frac{2W^2 + 1 + \sqrt{2W^2 + 1}}{2W^2 + 1}} = \sqrt{\frac{1 + \sqrt{4W^2 + 1}}{(4W^2 + 1) + \sqrt{4W^2 + 1}}} = \sqrt{\frac{1}{\sqrt{4W^2 + 1} + \sqrt{4W^2 + 1} \pm 1}} = \sqrt{\frac{1}{\sqrt{4W^2 + 1}}}, \quad (139) \]

where only the positive expression under the outer radical was assumed. The relation is the same as in the small-\( \nu \) region.

When the identity (42) is applied in the large-\( \nu \) region for the series contained in \( f_S \), the following is obtained for \( \mu = 1 \):

\[ f_S = \sum_{k=0}^{\infty} \left( -\frac{1 + \frac{1}{2}k}{k} \right) (W^{-1})^k = \sqrt{\frac{p^{-1} + 1}{(1 - \frac{1}{2})p + \frac{1}{2}}} \quad \text{where} \quad W^{-1} = \frac{p - 1}{p^2}. \quad (140) \]

As \( W^{-1} \) is non-negative (we are restricted to the first quadrant), \( p \) has to be equal or larger than one. Then,

\[ f_S = \sqrt{\frac{2}{p + 1}} \quad \text{where} \quad p_{1,2} = \frac{2W^2 + 1 \pm \sqrt{4W^2 + 1}}{2W^2}. \quad (141) \]

As \( W \) in the large-\( \nu \) region is larger than 1/2, only the solution with + is allowed on the right side of the previous relation in order to keep \( p \) equal or larger than one. Therefore,

\[ f_S = \sqrt{\frac{2}{2W^2 + 1 + \sqrt{4W^2 + 1}}} \quad \text{where} \quad p_{1,2} = \frac{2W^2 + 1 \pm \sqrt{4W^2 + 1}}{2W^2}. \quad (141) \]

As \( W \) in the large-\( \nu \) region is larger than 1/2, only the solution with + is allowed on the right side of the previous relation in order to keep \( p \) equal or larger than one. Therefore,

\[ f_S = \sqrt{\frac{2}{2W^2 + 1 + \sqrt{4W^2 + 1}}} \quad \text{where} \quad p_{1,2} = \frac{2W^2 + 1 \pm \sqrt{4W^2 + 1}}{2W^2}. \quad (141) \]

which is the same relation as in the small-\( \nu \) region. Therefore, the functions \( f_C, f_S \) according to (133) are to be used in the special case \( \mu = 1 \) in the relations (112), (126), (127), (128) for the vector transformations between the Cartesian and the SOS coordinate systems.

### 7.2. Points at the equator and at the pole for general \( \mu \)

The use of Eq. (128) can be demonstrated at a simple example. The vector \( \mathbf{A} \) at the equator (i.e., in the small-\( \nu \) region) of an oblate spheroid with the equatorial radius \( R_0 \) (i.e., \( \nu = 0 \), \( R = R_0 \)), pointing exactly in the direction from the center of the spheroid, and having thus components in the SOS system \((A_R, A_\nu, A_\lambda) = (A_R, 0, 0)\), can be written in the Cartesian coordinates using (128) as follows:

\[
\mathbf{A} = \begin{pmatrix}
A_x \\
A_y \\
A_z
\end{pmatrix} = \begin{pmatrix}
A_{RF_C} \cos \lambda - 0.5 f_S \cos \lambda - 0.5 \sin \lambda \\
A_{RF_C} \sin \lambda - 0.5 f_S \sin \lambda + 0.5 \cos \lambda \\
A_{RF} + 0.5 f_C
\end{pmatrix} = \begin{pmatrix}
A_{RF_C} \cos \lambda \\
A_{RF_C} \sin \lambda \\
A_{RF}\end{pmatrix}.
\quad (143)
\]
Assuming $R=R_0$ and $\nu=0$, then $W = 0$. Therefore (see (110)),
\[
    f_C = \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1+\mu k}{k} \right) (0)^k} = 1 \quad \text{and} \quad f_S = 0. \quad \sqrt{1+\mu} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-(1+\mu) - \mu k}{k} \right) (0)^k} = 0 \quad (144)
\]
and, further, the SOS coordinates of the point at which the vector is acting $(R, \nu, \lambda) = (R_0, 0, \lambda)$ are transformed to the Cartesian coordinates (see (6)–(8)) as $(R_0 \cos \lambda, R_0 \sin \lambda, 0)$. Then,
\[
    \mathbf{A}(R_0 \cos \lambda, R_0 \sin \lambda, 0) = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} A_R \cos \lambda \\ A_R \sin \lambda \\ 0 \end{pmatrix}, \quad (145)
\]
as expected for the vector at the equator pointing from the center.

Similar approach to the use of Eq. (128) in the large-$\nu$ region can be demonstrated at the pole. The vector $\mathbf{A}$ at the north pole of an oblate spheroid with the equatorial radius $R_0$ (i.e., $\nu=\pi/2$, $R=R_0$), pointing exactly in the direction from the center of the spheroid (i.e., along the rotation axis), and having thus components in the SOS system again $(A_R, A_\nu, A_\lambda) = (A_R, 0, 0)$, can be written in the Cartesian coordinates using (128) again as in (143). Assuming $R=R_0$ and $\nu=\pi/2$, $W^{-1} = 0$. Therefore (see (111)),
\[
    f_C = \frac{0}{\sqrt{1+\mu}} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1+\mu + \mu k}{k} \right) (0)^k} = 0 \quad \text{and} \quad f_S = \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1+\mu + \mu k}{k} \right) (0)^k} = 1. \quad (146)
\]
Further, the SOS coordinates of the point on the pole at which the vector is acting $(R, \nu, \lambda) = (R_0, \pi/2, \lambda)$ are transformed to the Cartesian coordinates (see (24)–(26)) as $(0, 0, R_0 \sqrt{1+\mu})$. Then, by using (143),
\[
    \mathbf{A} \left( 0, 0, \frac{R_0}{\sqrt{1+\mu}} \right) = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A_R \end{pmatrix}, \quad (147)
\]
as expected for the vector at the pole pointing from the center.

### 7.3. Special vector field transformation

As the last example, the vector field with a particular special dependence on the position in the SOS coordinates is transformed to the Cartesian coordinates. The particular vector field
\[
    \mathbf{A}(R, \nu, \lambda) = \begin{pmatrix} A_R \\ A_\nu \\ A_\lambda \end{pmatrix} = \begin{pmatrix} \frac{-C}{R} \\ 0 \\ 0 \end{pmatrix} \quad (148)
\]
is selected, where $C$ is a constant. This particular shape of the vector field means that only a component of the vector field in the SOS coordinates which is always perpendicular to the similar oblate spheroid surfaces exists, and, moreover, it has a special dependence on $R$ and $\nu$, particularly $-C \frac{R}{h_R}$, determined thus by (18) or by (36), depending on the particular region (the small-$\nu$ region or the large-$\nu$ region) of the space. In the Cartesian coordinates, the vector field is thus (see (128))
\[
    \mathbf{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} f_C \cos \lambda \\ f_C \sin \lambda \\ f_S \end{pmatrix} A_R = -C \begin{pmatrix} \frac{R f_C}{h_R} \cos \lambda \\ \frac{R f_C}{h_R} \sin \lambda \\ \frac{R f_C}{h_R} \lambda \end{pmatrix} \quad (149)
\]
where (8) was used in the last simplification.

\[
\frac{f_C}{h_R} = \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1 - \mu}{k} \right) (W^2)^k} = \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1}{k} \right) (W^2)^k},
\]

(150)

and

\[
\frac{f_S}{h_R} = W\sqrt{1 + \mu} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-(1+\mu) - \mu}{k} \right) (W^2)^k} = W\sqrt{1 + \mu} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1}{k} \right) (W^2)^k},
\]

(151)

where the relation (51) was used. With the further help of the identity (43), the square root can be removed:

\[
\frac{f_C}{h_R} = \sqrt{\sum_{k=0}^{\infty} \frac{-1}{1 - \mu} \left( \frac{-1 - \mu}{k} \right) (W^2)^k = \sqrt{p^{-1} - p^{2} - 1} = \sum_{k=0}^{\infty} \frac{-1}{2 - \mu} \left( \frac{-1 - \mu}{k} \right) (W^2)^k = \frac{x(\nu, R)}{R},
\]

(152)

where (6) was used in the last simplification, and

\[
\frac{f_S}{h_R} = W\sqrt{1 + \mu} \sqrt{\sum_{k=0}^{\infty} \frac{-1}{1 + \mu} \left( \frac{-1 - \mu}{k} \right) (W^2)^k = W\sqrt{1 + \mu} \sqrt{p^{-1 + \mu}}}
\]

\[= W\sqrt{1 + \mu} p^{-\frac{1 + \mu}{2}} = W\sqrt{1 + \mu} \sum_{k=0}^{\infty} \frac{-1}{2 - \mu} \left( \frac{-1 - \mu}{k} \right) (W^2)^k = \frac{(1 + \mu) z(\nu, R)}{R},
\]

(153)

where (8) was used in the last simplification.

Similarly, in the large-\(\nu\) region, with the help of (35)

\[
\frac{f_C}{h_R} = \frac{\sqrt{1 + \mu}}{\sqrt{1 + \mu}} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1 + \mu}{1 + \mu} \right) (W^{-\frac{2}{1 + \mu}})^k} = \frac{1}{\sqrt{1 + \mu}} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1 + \mu}{1 + \mu} \right) (W^{-\frac{2}{1 + \mu}})^k},
\]

(154)

and

\[
\frac{f_S}{h_R} = \frac{\sqrt{1 + \mu}}{\sqrt{1 + \mu}} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1 + \mu}{1 + \mu} \right) (W^{-\frac{2}{1 + \mu}})^k} = \sqrt{1 + \mu} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1 + \mu}{1 + \mu} \right) (W^{-\frac{2}{1 + \mu}})^k}.
\]

(155)

With the further help of the identity (43), the square root can be removed also here:

\[
\frac{f_C}{h_R} = W\frac{1 + \mu}{1 + \mu} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1 + \mu}{1 + \mu} \right) (W^{-\frac{2}{1 + \mu}})^k} = W\frac{1 + \mu}{1 + \mu} \sqrt{p^{-\frac{1}{1 + \mu}}}
\]

\[= W\frac{1 + \mu}{1 + \mu} \sum_{k=0}^{\infty} \left( \frac{-1 + \mu}{1 + \mu} \right) (W^{-\frac{2}{1 + \mu}})^k = \frac{x(\nu, R)}{R},
\]

(156)
where (24) was used in the last simplification, and

$$f_{S} = \frac{1}{hR} \sqrt{1 + \mu} \sum_{k=0}^{\infty} \frac{-1}{1 + \frac{\mu}{1 + \mu k}} \left( W \frac{z}{1 + \mu} \right)^{k} = \sqrt{1 + \mu} \sqrt{p} = \frac{(1 + \mu) z(\nu, R)}{R},$$

(157)

where (26) was used in the last simplification.

It can be seen that the relations looks the same in the small- and in the large-$\nu$ regions, only the proper expressions corresponding to the respective regions has to be used for $x$ and $z$. Then, (149) can be rewritten to the form

$$A = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = -C \begin{pmatrix} x(\nu, R) \cos \lambda \\ x(\nu, R) \sin \lambda \\ (1 + \mu) z(\nu, R) \end{pmatrix}. \quad (158)$$

Clearly, a linear dependence exists of the components of the special vector field $A$ on the Cartesian coordinates $x$ and $z$. Moreover, the $A_x$ (as well as $A_y$) component depends linearly only on $x$ coordinate (and not on $z$ coordinate), while the $A_z$ component depends linearly only on $z$ coordinate (and not on $x$ coordinate). This is a notable result, as the same magnitude of the vector $A$ projection to the $x$-$y$ plane exists, i.e., $\sqrt{A_x^2 + A_y^2} = -Cx(\nu, R)$, for the same distance $\sqrt{x_3^2 + y_3^2} = x(\nu, R)$ from the rotation axis, regardless of the height above the equatorial plane (i.e., regardless the $z$-coordinate value). The vector $A$ projection to the $x$-$y$ plane thus possesses cylindrical symmetry, whereas the vector $A$ $z$-component increases linearly when moving from the equator upwards (remember that the solution (158) is derived for the first quadrant only; nevertheless—due to the symmetry of the problem—the solution under the equatorial plane is a reflection of the solution above the equatorial plane).

In case when $A$ is a force vector field and the relation (148) holds for it, a stable trajectory of a body in such field would be circular when using top view projection while it would be a harmonic oscillator in $z$ direction (i.e., in the vertical direction), providing that the force in the horizontal direction is counteracted by the appropriate central centrifugal force, magnitude of which is constant regardless of the vertical position of the body. Such a trajectory is depicted in Fig. 3 which also shows the equidistant levels for the horizontal and vertical components of the force vector $A$. These are concentric cylinders (for the horizontal component) and planes parallel to $x$-$y$ plane (for the vertical component of the force vector).

Hypothetically, if an oblate spheroidal celestial object (e.g., elliptical or disc galaxy) would have the gravitational force inside the object of the shape given by (148), the trajectory depicted in Fig. 3 would be a stable orbit of—for example—a star. A potential corresponding to the gravitational force given by (148) and a mass distribution causing such potential are to be determined by a solution of corresponding differential equations. Nevertheless, this derivation is not in the scope of this article and will be published elsewhere.

8. Conclusions

The unit vectors in the SOS coordinates were found (see (127)). The expressions employ power series with generalized binomial coefficients which are reported in (110) and (111). The unit vector transformation between the Cartesian and the SOS coordinates, and vice versa, was derived, see (112) and (126). The obtained formulas can help to transform vector fields between the two types of orthogonal coordinates, Cartesian and SOS (see (128)). It would advantageously simplify the problems for a special geometry when, for example, the iso-density levels of an object are of similar oblate spheroidal shape, and a vector field is associated with such object. As a by-product, the generalized cosine and sine functions applicable in the similar oblate spheroidal coordinate system are introduced. They are important, e.g., for Laplace equation solution in the SOS system, which will be shown elsewhere. Several examples demonstrated
Fig. 3. A trajectory of a body in the field for which the force vector fulfills the relation (148). The equidistant levels of the horizontal and vertical components of the force vector are drawn as well the use of the derived transformations between the coordinate systems, including the case with a special dependence of the vector field on the position causing a stable trajectory.

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Author contributions PS derived all the reported relations, wrote the manuscript text, prepared all the figures, and reviewed the manuscript.

Data availability This paper deals with the derivation of theoretical relations of the SOS coordinate system. There are thus no data sets to be disclosed.

Declarations

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Appendix A. Unit vectors in the SOS coordinate system for the large-$\nu$ region

In appendix, the unit vectors $\hat{R}$ and $\hat{\nu}$ are derived in detail in the large-$\nu$ region.

A.1 Unit vector in $R$ direction

First, the component containing derivative of $x$ with respect to $R$, see (27) and (36), is derived:

$$\frac{\partial x (\nu, R)}{\partial R} / h_R = \frac{W^{-\frac{1}{1+\mu}}}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k . \quad (A.1)$$

When Cauchy product combined with Jensen identity (52) is used for the numerator (with $b=\mu/(1+\mu)$, $a=c=1/2/(1+\mu)$, $t=1/2/(1+\mu)$), then

$$\sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} \frac{k}{k} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k \sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} \frac{k}{k} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k \quad (A.2)$$

and thus,

$$\frac{\partial x (\nu, R)}{\partial R} / h_R = \frac{W^{-\frac{1}{1+\mu}}}{\sqrt{1+\mu}} \frac{\sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} \frac{k}{k} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k \sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} \frac{k}{k} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k}{\sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} \frac{k}{k} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k} . \quad (A.3)$$

Similarly for $z$ coordinate:

$$\frac{\partial z (\nu, R)}{\partial R} / h_R = \frac{1}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} \frac{k}{k} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k . \quad (A.4)$$

When—again—Cauchy product combined with Jensen identity (52) is used for the numerator (with $b=\mu/(1+\mu)$, $a=c=1/2$, $t=1/2$), the result is

$$\frac{\partial z (\nu, R)}{\partial R} / h_R = \frac{1}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} \frac{k}{k} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k \sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} \frac{k}{k} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k \quad (A.5)$$
The unit vector $\hat{R}$ is then

$$\hat{R} = \begin{pmatrix} \cos \lambda \frac{W^{\frac{1}{1+\mu}}}{\sqrt{1+\mu}} \left( \sum_{k=0}^{\infty} \left( -\frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^k \right) \\ \sin \lambda \frac{W^{\frac{1}{1+\mu}}}{\sqrt{1+\mu}} \left( \sum_{k=0}^{\infty} \left( -\frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^k \right) \\ \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^k} \end{pmatrix}.$$  \hspace{1cm} (A.6)

For the special case, when $\mu=0$ (i.e., equivalent to the spherical coordinates), the parameter $W$, see (4), is simplified, $W (\mu=0) = \sin \nu \cos \nu$. Thanks to the binomial theorem (56), the following is obtained:

$$\begin{align*}
\frac{W^{\frac{1}{1+\mu}}}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left( -\frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^k &= W^{-1} \sum_{k=0}^{\infty} \left( -\frac{1}{k} \right) (W^{-2})^k = W^{-1} \sqrt{(1+W^{-2})^{-1}} \\
&= W^{-1} \sqrt{\frac{1}{1+W^{-2}}} = \frac{\cos \nu}{\sin \nu} \sqrt{1 + \frac{\cos^2 \nu}{\sin^2 \nu}} = \frac{\cos \nu}{\sqrt{\cos^2 \nu + \sin^2 \nu}} = \cos \nu,
\end{align*}$$

and similarly,

$$\begin{align*}
\sqrt{\sum_{k=0}^{\infty} \left( -\frac{1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^k} &= \sqrt{\sum_{k=0}^{\infty} \left( -\frac{1}{k} \right) (W^{-2})^k} = \sin \nu.
\end{align*}$$

The unit vector for the special case of spherical-like coordinates ($\mu=0$) is thus

$$\hat{R} (\mu=0) = \begin{pmatrix} \cos \lambda \cos \nu \\ \sin \lambda \cos \nu \\ \sin \nu \end{pmatrix},$$

(A.9)

the same as in the small-$\nu$ region (see (69)), as expected.

### A.2 Unit vector in $\nu$ direction

The derivation in the large-$\nu$ region is very similar to the one for the small-$\nu$ region. The binomial identities listed in the section 3 can be used for the derivation. The shortest way is by the use of (42) and (43) identities. The first component of the unit vector is proportional to (see (30) and (37))

$$\begin{align*}
\frac{\partial x (\nu, R)}{\partial \nu} &= R \frac{1}{\mu W^{\frac{1}{1+\mu}}} \frac{\partial W}{\partial \nu} \left\{ \frac{1}{1+\mu} \sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^k - \sum_{k=0}^{\infty} \left( -\frac{1}{2} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+\mu}} \right)^k \right\} \\
&= \frac{R W^{-\frac{2+\mu}{1+\mu}}}{1+\mu} \frac{\partial W}{\partial \nu} \sqrt{\sum_{k=0}^{\infty} \left( -\frac{\mu}{k} \right) \left( W^{-\frac{2}{1+\mu}} \right)^k} \\
&= 1 + \frac{1}{\mu} \frac{1}{\left( \frac{2+\mu}{1+\mu} \right)^{p+\mu} - p^{\frac{1}{2}}} \left( \frac{1-\mu}{\frac{2+\mu}{1+\mu}} \right)^{p+\mu} \\
&= \frac{1 + \mu}{\mu} \sqrt{\frac{p^{\frac{2+\mu}{1+\mu}}}{(1-\frac{\mu}{1+\mu})^{p+\mu}}}.
\end{align*}$$

(A.10)
where the identities (42) and (43) were used. This can be further simplified to

\[
\frac{\partial x (\nu, R)}{\partial \nu} / h_\nu = \frac{1 + \mu}{\mu} \frac{p_{\frac{1}{1+\mu}} p_{\frac{1}{1+\mu}}^{\frac{2}{1+\mu}} + p_{\frac{1}{1+\mu}}^{\frac{1}{1+\mu}}}{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}} = 1 + \mu \frac{1 + \mu}{\mu} p_{\frac{1}{1+\mu}} \left[\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}\right] \sqrt{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}}
\]

\[
\frac{1}{\mu} p_{\frac{1}{1+\mu}} (1 - \mu - \mu) p - \mu \frac{1}{\mu} \sqrt{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}}
\]

and finally, according to (42), to

\[
\frac{\partial x (\nu, R)}{\partial \nu} / h_\nu = - \frac{1}{\mu} \sqrt{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}} = - \frac{1}{\mu} \sqrt{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}}
\]

(A.11)

Similar approach can be used also for z coordinate. The z-component of the unit vector is proportional to (see (32) and (37))

\[
\frac{\partial z (\nu, R)}{\partial \nu} / h_\nu = \frac{1}{\sqrt{1+\mu}} R \frac{1}{\mu} W \frac{\partial W}{\partial \nu} \left\{ \sum_{k=0}^{\infty} \left( - \frac{1}{2} + \frac{\mu}{1+\mu} k \right) \left( W - \frac{2}{1+\mu} \right)^k - \sum_{k=0}^{\infty} \left( - \frac{1}{2} + \frac{\mu}{1+\mu} k \right) - \frac{1}{2} \left( - \frac{2}{1+\mu} \right) \left( W - \frac{2}{1+\mu} \right)^k \right\}
\]

\[
= W \frac{1}{1+\mu} \sqrt{1+\mu} \frac{1}{\mu} \left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu} \frac{p - \frac{1}{2} + \frac{1}{1+\mu}}{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}}
\]

(A.12)

where the identities (42) and (43) were used which fulfill the relation (44), i.e.,

\[
-W - \frac{2}{1+\mu} \frac{p}{1+\mu} + p = 1
\]

(A.14)

Then,

\[
\frac{\partial z (\nu, R)}{\partial \nu} / h_\nu = W \frac{1}{1+\mu} \sqrt{1+\mu} \frac{1}{\mu} \left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu} \frac{p - \frac{1}{2} + \frac{1}{1+\mu}}{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}}
\]

\[
= W \frac{1}{1+\mu} \sqrt{1+\mu} \frac{1}{\mu} \left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu} \frac{p - \frac{1}{2} + \frac{1}{1+\mu}}{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}}
\]

\[
= W \frac{1}{1+\mu} \sqrt{1+\mu} \frac{1}{\mu} \left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu} \frac{p - \frac{1}{2} + \frac{1}{1+\mu}}{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}}
\]

\[
= W \frac{1}{1+\mu} \sqrt{1+\mu} \frac{1}{\mu} \left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu} \frac{p - \frac{1}{2} + \frac{1}{1+\mu}}{\left(1 - \frac{\mu}{1+\mu}\right) p + \frac{\mu}{1+\mu}}
\]

(A.15)
As, according to (A.14),

\[ p - 1 = W^{-\frac{2}{1+p}} p^{\frac{\mu}{1+p}}; \]  

(A.16)

then,

\[
\frac{\partial z (\nu, R)}{\partial \nu} / h_\nu = W^{\frac{1}{1+p}} \frac{1}{\sqrt{1 + \mu}} p^{-\frac{1}{1+p}} W^{-\frac{2}{1+p}} p^{\frac{\mu}{1+p}} \frac{1}{\sqrt{1 + \mu}} \frac{1}{\sqrt{1 + \mu}} \frac{p^{\frac{\mu}{1+p}}}{(1 - \frac{\mu}{1+p}) p + \frac{\mu}{1+p}} 
\]

(A.17)

According to the identity (42), the final result is

\[
\frac{\partial z (\nu, R)}{\partial \nu} / h_\nu = W^{\frac{1}{1+p}} \frac{1}{\sqrt{1 + \mu}} \left[ \sum_{k=0}^{\infty} \left( \frac{-1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+p}} \right)^k \right]. \]  

(A.18)

The unit vector \( \vec{\nu} \) is then

\[
\vec{\nu} = \left( \frac{\partial x_{3D}(\nu, R)}{\partial \nu} / h_\nu, \frac{\partial y_{3D}(\nu, R)}{\partial \nu} / h_\nu, \frac{\partial z_{3D}(\nu, R)}{\partial \nu} / h_\nu \right) = \begin{pmatrix} -\cos \lambda \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+p}} \right)^k} \\ -\sin \lambda \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+p}} \right)^k} \\ W^{\frac{1}{1+p}} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1}{1+\mu} + \frac{\mu}{1+\mu} k \right) \left( W^{-\frac{2}{1+p}} \right)^k} \end{pmatrix}. \]  

(A.19)

The components of the unit vector can be tested for \( \mu = 0 \), i.e., for spherical coordinates. Then, also \( W (\mu = 0) = \frac{\sin \nu}{\cos \nu} \), and

\[
-\sqrt{\sum_{k=0}^{\infty} \left( \frac{-1 + 0}{k} \right) \left(W^{-\frac{2}{1+p}}\right)^k} = -\sqrt{\sum_{k=0}^{\infty} \frac{-1}{k} (W^{-2})^k} = -\sqrt{(1+W^{-2})^{-1}} = -\sqrt{\frac{1}{1 + \frac{\cos^2 \nu}{\sin^2 \nu}}} = -\frac{\sin^2 \nu}{\sin^2 \nu + \cos^2 \nu} = -\sin \nu, \]  

(A.20)

where the binomial theorem (56) was used. Also the last component of the unit vector can be then simplified:

\[
W^{\frac{1}{1+p}} \sqrt{\sum_{k=0}^{\infty} \left( \frac{-1 + 0}{k} \right) \left(W^{-\frac{2}{1+p}}\right)^k} = \frac{1}{W} \sqrt{\sum_{k=0}^{\infty} \frac{-1}{k} (W^{-2})^k} = \frac{1}{W} \sqrt{(1+W^{-2})^{-1}} = \cos \nu \sqrt{\frac{1}{1 + \frac{\cos^2 \nu}{\sin^2 \nu}}} = \cos \nu, \]  

(A.21)

and the unit vector for the special case of the spherical coordinates \( (\mu = 0) \) is thus

\[
\vec{\nu} (\mu = 0) = \begin{pmatrix} -\cos \lambda \sin \nu \\ -\sin \lambda \sin \nu \\ \cos \nu \end{pmatrix}. \]  

(A.22)

This is the same result as for the small-\( \nu \) region, as expected for spherical-like coordinates.
Another proof, that the above derivation of the unit vector is correct, is that the unit vector length has to be equal to one also in the large-\( \nu \) region, and thus, its square has to be equal to one as well:

\[
\left(-\cos \lambda \sqrt{\sum_{k=0}^{\infty} \left(-1 + \frac{\mu}{k+1+\mu} \right) \left(W^{-\frac{2}{1+\mu}}\right)^k}\right)^2 + \left(-\sin \lambda \sqrt{\sum_{k=0}^{\infty} \left(-1 + \frac{\mu}{k+1+\mu} \right) \left(W^{-\frac{2}{1+\mu}}\right)^k}\right)^2 + \left(\frac{W^{-\frac{1}{1+\mu}}}{\sqrt{1+\mu}} \sum_{k=0}^{\infty} \left(-1 + \frac{\mu}{k+1+\mu} \right) \left(W^{-\frac{2}{1+\mu}}\right)^k\right)^2 = 1 .
\] (A.23)

The left side can be simplified to

\[
\sum_{k=0}^{\infty} \left(-1 + \frac{\mu}{k+1+\mu} \right) \left(W^{-\frac{2}{1+\mu}}\right)^k + \frac{W^{-\frac{2}{1+\mu}}}{1+\mu} \sum_{k=0}^{\infty} \left(-1 + \frac{\mu}{k+1+\mu} \right) \left(W^{-\frac{2}{1+\mu}}\right)^k = 1 ,
\] (A.24)

When (42) is used with \( r = W^{-\frac{2}{1+\mu}} , b=\mu/(\mu+1) \), and as \( a=-1 \) in the first term while \( a=-1/(1+\mu) \) in the second term, it can be rewritten to the form

\[
\frac{p^{-1+1}}{(1-\frac{\mu}{1+\mu})p + \frac{\mu}{1+\mu}} + \frac{W^{-\frac{2}{1+\mu}}}{1+\mu} \frac{p^{-1+1}}{(1-\frac{\mu}{1+\mu})p + \frac{\mu}{1+\mu}} = \frac{1}{1+\mu} \frac{1}{(1-\frac{\mu}{1+\mu})p + \frac{\mu}{1+\mu}} .
\] (A.25)

According to (44),

\[
p^{\frac{r}{1+\mu}} = (p-1) W^{\frac{2}{1+\mu}} ,
\] (A.26)

and thus further simplification follows:

\[
\frac{1}{(1-\frac{\mu}{1+\mu})p + \frac{\mu}{1+\mu}} + \frac{W^{-\frac{2}{1+\mu}}}{1+\mu} \frac{(p-1) W^{\frac{2}{1+\mu}}}{(1-\frac{\mu}{1+\mu})p + \frac{\mu}{1+\mu}} = \frac{1}{1+\mu} \frac{1}{(1-\frac{\mu}{1+\mu})p + \frac{\mu}{1+\mu}}
\]

\[
= \frac{1}{1+\mu} \left(\frac{1}{(1-\frac{\mu}{1+\mu})p + \frac{\mu}{1+\mu}}\right) + \frac{\mu + p}{\mu + p + \mu} = \frac{1}{1+\mu} \left(\frac{1}{(1-\frac{\mu}{1+\mu})p + \frac{\mu}{1+\mu}}\right)
\]

\[
\left(1+\mu\right) \left(\frac{1}{(1-\frac{\mu}{1+\mu})p + \frac{\mu}{1+\mu}}\right) + \frac{\mu + p}{\mu + p + \mu} = \frac{1}{1+\mu} .
\] (A.27)

Indeed, the length of the unit vector is equal to one.

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