Modeling few-body resonances in finite volume

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Under the assumption of separable interactions, we illustrate how the few-body quantization condition may be formulated in terms of phase shifts in general, which may be useful for describing and modeling of few-body resonances in finite volume.

I. INTRODUCTION

Few-hadron dynamics plays an important role in hadron and nuclear physics. There have been many good examples of physics processes that can only be understood through few-body interactions, such as, the u- and d-quark mass difference in \( \eta \to 3\pi \) \cite{1-7}, Efimov states \cite{8,9} and halo nuclei \cite{10,11}. The understanding of few-body interaction is also crucial in recent experimental efforts of exotic hadrons study, since most of exotic hadron states are expected to appear as few-hadron resonances.

On the theory side, lattice Quantum Chromodynamics (LQCD) provides an ab-initio method for the study of exotic hadron states. However, LQCD computation is usually performed in Euclidean space with certain periodic boundary condition, normally only discrete energy spectrum are measured in numerical simulation. Hence, mapping out few-hadron dynamics from discrete energy spectrum is a key step for the study of exotic hadron states.

In past few years, many progresses from different approaches \cite{23-53} have been made going beyond three-body threshold. Although few-body quantization conditions are formulated differently among these groups, it has been very clear \cite{52} that in few-body sectors, the few-body amplitudes are not directly extracted from lattice results. Particle interactions or its associated subprocess amplitudes are in fact essential ingredients in quantization condition. The infinite volume few-body amplitudes that are generated by particle interactions through coupled integral equations must be computed in a separate step once these dynamical ingredients are determined. In order to make predictions or fit lattice results, dynamical ingredients of quantization condition, such as interaction potentials or off-shell subprocess amplitudes must be modeled one way or another. In addition, number of partial waves involved in some physical processes may be large, which may add some extra complications on top of the uncertainty in modeling itself. Therefore, to have a reliable and controllable predictions, the modeling of dynamical ingredients must be constrained or guided by experimental data or effective theory. Nevertheless, there are two physical regions in which predictions and calculations may be made fairly reliable: (1) near threshold which is the region where the physical reaction can be described rather precisely by non-relativistic potential theory or relativistic effective perturbation theory; (2) near resonances region where resonance properties may be less affected by modeling and other partial waves.

In present work, we focus on near resonance region and aim to provide an approximate means for the modeling of few-body resonances in finite volume. Based on separable interaction potential assumption, we illustrate how the few-body quantization condition may be formulated in terms of subprocess phase shifts. Hence, the resonances may be modeled and inserted into quantization condition through phase shifts. Both two-body and three-body subprocess amplitudes appear Lüscher formula-like and solutions are given by algebra equations.

The paper is organized as follows. With separable interactions approximation, the technical details of formulating quantization conditions in terms of phase shifts are presented in Sec. II. A summary is given in Sec. III.

II. QUANTIZATION CONDITION UNDER SEPARABLE INTERACTIONS ASSUMPTION

Few-body quantization condition in finite volume can be formulated from homogeneous Faddeev type equations, see \cite{50-53}. As a simple example, we consider three non-relativistic identical bosons of mass \( m \) interacting with both pair-wise interaction and three-body force in follows. Due to exchange symmetry, only two independent Faddeev amplitudes are required: \( \mathcal{T}^{(2b)} \) and \( \mathcal{T}^{(3b)} \) that are associated with pair-wise two-body interaction, \( V^{(2b)} \), and three-body interaction, \( V^{(3b)} \), by

\[ \mathcal{T}^{(2b,3b)}(k_1, k_2) = -\langle k_1 | k_2 | m V^{(2b,3b)} | \Psi \rangle, \]

where \( \Psi \) stands for the three-body total wave function. The \( (k_1, k_2) \in \mathbb{R}^3 \), \( n \in \mathbb{Z}^3 \) refer to particle-1 and -2 momenta respectively, and third particle momentum is constrained by total momentum conservation,

\[ k_3 = -k_1 - k_2. \]

In follows, we also use symbols \( (k_{13}, k_{(13)2}) \) to describe two independent relative momenta of three particles,
where
\[
\mathbf{k}_{13} = \frac{\mathbf{k}_1 - \mathbf{k}_3}{2} = \frac{\mathbf{k}_1 + \mathbf{k}_2}{2}, \\
\mathbf{k}_{(13)2} = \sqrt{\frac{1}{3}} \left( \frac{\mathbf{k}_1 + \mathbf{k}_3}{2} - \mathbf{k}_2 \right) = -\sqrt{\frac{4}{3}} \mathbf{k}_2.
\]

The stationary states of three-body dynamics in finite volume is described by homogeneous Faddeev type equations, see [50–53],
\[
\mathcal{T}^{(2b)}(\mathbf{k}_1, \mathbf{k}_2) = \frac{1}{L^3} \sum_{\mathbf{p}_1} \frac{\tau^{(2b)}}{mE - \frac{p_1^2 + k_1^2 + (p_1 + k_2)^2}{2}} \cdot \left[ 2\mathcal{T}^{(2b)}(\mathbf{k}_2, \mathbf{p}_1) + \mathcal{T}^{(3b)}(\mathbf{p}_1, \mathbf{k}_2) \right],
\]
and
\[
\mathcal{T}^{(3b)}(\mathbf{k}_1, \mathbf{k}_2) = \frac{1}{L^6} \sum_{\mathbf{p}_1, \mathbf{p}_2} \frac{\tau^{(3b)}(\mathbf{K}; \mathbf{P})}{mE - \frac{p_1^2 + p_2^2 + (p_1 + p_2)^2}{2}} \mathcal{T}^{(2b)}(\mathbf{p}_1, \mathbf{p}_2),
\]
where
\[
(\mathbf{p}_1, \mathbf{p}_2) \in \frac{2\pi n}{L}, \quad n \in \mathbb{Z}^3.
\]
The symbol (\(\mathbf{K}, \mathbf{P}\)) stand for 6-dimensional vectors, they are related to relative momenta (\(\mathbf{k}_{13}, \mathbf{k}_{(13)2}\)) by
\[
\mathbf{K} = \{\mathbf{k}_{13}, \mathbf{k}_{(13)2}\} = \{\mathbf{k}_1 + \frac{\mathbf{k}_2}{2}, -\sqrt{\frac{4}{3}} \mathbf{k}_2\},
\]
\[
\mathbf{P} = \{\mathbf{p}_{13}, \mathbf{p}_{(13)2}\} = \{\mathbf{p}_1 + \frac{\mathbf{p}_2}{2}, -\sqrt{\frac{4}{3}} \mathbf{p}_2\}.
\]
The length of 6D vectors are given by
\[
K = \sqrt{k_{13}^2 + k_{(13)2}^2} = \sqrt{\frac{1}{2} \sum_{i=1}^3 k_i^2},
\]
\[
P = \sqrt{p_{13}^2 + p_{(13)2}^2} = \sqrt{\frac{1}{2} \sum_{i=1}^3 p_i^2}.
\]
Symbols \(\tau^{(2b)}\) and \(\tau^{(3b)}\) that are associated with twobody interaction \(V^{(2b)}\) and three-body interaction \(V^{(3b)}\) respectively are used to describe off-shell subprocess transition amplitudes between initial and final momenta states. For example, \(\tau^{(2b)}\) in (13) isobar channel with particle-2 carrying a momentum \(\mathbf{k}_2\) satisfies two-body inhomogeneous Lippmann-Schwinger equations,
\[
\tau^{(2b)}(\mathbf{k}_{13}; \mathbf{k}_{13}') = -m \mathcal{V}^{(2b)}(|\mathbf{k}_{13} - \mathbf{k}_{13}'|) + \frac{1}{L^3} \sum_{\mathbf{p}_1} \frac{m \mathcal{V}^{(2b)}(|\mathbf{k}_{13} - \mathbf{p}_1 - \mathbf{k}'_1|)}{mE - \frac{p_1^2 + k_1^2 + (p_1 + k_2)^2}{2}} \tau^{(2b)}(\mathbf{p}_1 + \frac{\mathbf{k}_2}{2}, \mathbf{k}_{13}'),
\]
and similarly \(\tau^{(3b)}\) satisfies a three-body equation,
\[
\tau^{(3b)}(\mathbf{K}; \mathbf{K}', \mathbf{k}_{13}') = -m \mathcal{V}^{(3b)}(|\mathbf{K} - \mathbf{K}'|) + \frac{1}{L^6} \sum_{\mathbf{p}_1, \mathbf{p}_2} \frac{m \mathcal{V}^{(3b)}(|\mathbf{K} - \mathbf{P}|)}{mE - \frac{p_1^2 + p_2^2 + (p_1 + p_2)^2}{2}} \tau^{(3b)}(\mathbf{P}, \mathbf{K}').
\]
\(\tau^{(2b)}\) and \(\tau^{(3b)}\) are dynamical input of finite volume Faddeev equations in Eq.(3) and Eq.(4), and must be solved first.

The quantization condition without cubic irreducible representation projection is given by
\[
0 = \det \left[ L^6 \delta_{\mathbf{k}_1, \mathbf{p}_1} \delta_{\mathbf{k}_2, \mathbf{p}_2} + \frac{L^3 \delta_{\mathbf{k}_2, \mathbf{p}_2} 2 \tau^{(2b)}(\mathbf{k}_{13}; \mathbf{p}_2 + \mathbf{k}_2)}{mE - \frac{3k_2^2}{4} - (\mathbf{p}_2 + \frac{\mathbf{k}_2}{2})^2} - \frac{3}{L^3} \sum_{\mathbf{p}} \frac{\tau^{(2b)}(\mathbf{k}_{13}; \mathbf{p} + \frac{\mathbf{k}_2}{2}) \tau^{(3b)}((\mathbf{p} + \frac{\mathbf{k}_2}{2}, -\sqrt{\frac{4}{3}} \mathbf{k}_2); \mathbf{P})}{mE - \frac{3k_2^2}{4} - (\mathbf{p} + \frac{\mathbf{k}_2}{2})^2} (mE - \mathbf{P}^2)
\]
where \(\tau^{(2b)}\) and \(\tau^{(3b)}\) in principle are given by the solutions of Eq.(7) and Eq.(8) respectively. In Sec.II.A, we will show that with separable interaction approximation Eq.(7) and Eq.(8) may be converted into algebra equations. Hence, the solutions of \(\tau^{(2b)}\) and \(\tau^{(3b)}\) are Lüscher formula-like, and can be formulated in terms of conventional two-body phase shifts in 3D and unconventional but mathematically convenient three-body phase shifts in 6D.

A. Separable interactions and algebra solutions of \(\tau^{(2b)}\) and \(\tau^{(3b)}\)

Under the assumption of separable short-range potentials for both \(V^{(2b)}\) and \(V^{(3b)}\), the partial wave expansion of potentials thus have the forms of
\[
\tilde{V}^{(2b)}(|\mathbf{k}_{13} - \mathbf{k}_{13}'|) = \sum_{LM} Y_{LM} (\mathbf{k}_{13}) g_L^{(2b)} (k_{13}) V_L^{(2b)} g_L^{(2b)} (k_{13}') Y_{LM}^* (\mathbf{k}_{13}'),
\]
and
\[
\tilde{V}^{(3b)}(|\mathbf{K} - \mathbf{K}'|) = \sum_{[J]} Y_{[J]} (\mathbf{k}) g_{[J]}^{(3b)} (K) V_{[J]}^{(3b)} g_{[J]}^{(3b)} (K') Y_{[J]}^* (\mathbf{k}'),
\]
where \(Y_{LM}(\mathbf{k}_{13})\) is 3D spherical harmonic function with quantum numbers \([LM]\) representing orbital angular momentum configurations between particle-1 and -3, while particle-2 acts as a spectator and is not involved in interaction. \(Y_{[J]}(\mathbf{k})\) stands for the 6D hyperspherical harmonic basis function, see Refs. [54, 55] and Appendix A, the quantum numbers \([J]\) represent a specific angular momentum configuration of three particles.
with a total angular momentum \( J \). Y_{\lfloor J \rfloor} (\hat{K}) may be constructed through two 3D spherical harmonic functions. For example, considering a configuration with angular momentum state \( |L_{13}M_{13} \rangle \) between particle-1 and -3 coupled with particle-2 in relative angular momentum state \( |J \rangle = |JM_{13}L_{13}(2) \rangle \), thus \( Y_{\lfloor J \rfloor} (\hat{K}) \) is given by

\[
Y_{\lfloor J \rfloor} (\hat{K}) = \sum_{M_{13}} \langle L_{13}M_{13}, L_{13}(2)M_{13} | JM \rangle Y_{\lfloor J \rfloor} (\hat{K}_{13}) Y_{L_{13}(2)} (\hat{k}_{13}) Y_{L_{13}(2)} (\phi),
\]

where

\[
\phi = \tan^{-1} \frac{k_{13}}{\hat{k}_{13(2)}}.
\]

The function \( P_{\lfloor J \rfloor L_{13}(2)} (\phi) \) is related to Jacobi polynomial by, also see [54, 55],

\[
P_{\lfloor J \rfloor L_{13}(2)} (\phi) = N_{\lfloor J \rfloor L_{13}(2)} (\sin \phi)L_{13} (\cos \phi)L_{13(2)}
\times P_{\lfloor J \rfloor L_{13}(2) + \frac{1}{2}} (\cos 2\phi),
\]

the normalization factor \( N_{\lfloor J \rfloor L_{13}(2)} \) is determined by orthonormal relation,

\[
\int_0^\pi d\phi \sin^2 \phi \cos^2 \phi P_{\lfloor J \rfloor L_{13}(2)} (\phi) P_{\lfloor J \rfloor L_{13}(2)} (\phi) = \delta_{J,J'}.
\]

The form factors, \( g_L^{(2b)} \) and \( g_J^{(3b)} \), and potential strengths, \( V_L^{(2b)} \) and \( V_J^{(3b)} \), may be considered as model parameters. Usually, the form factors, such as \( g_L^{(2b)} \), must show the correct threshold behavior,

\[
g_L^{(2b)} (k \to 0) \sim k^L.
\]

The potential strengths \( V_L^{(2b)} \) and \( V_J^{(3b)} \) may be used to model two-body and three-body resonances, for example, the two-particle resonance of mass \( m_R^{(2b)} \) in (13) isobar pair channel with particle-2 carrying momentum \( k_2 \) may be given by

\[
V_L^{(2b)} \propto \frac{1}{(E - \frac{3}{4}k_2^2 - m_R^{(2b)})}.
\]

A three-particle resonance of mass \( m_R^{(3b)} \) thus may be modeled similarly by

\[
V_L^{(3b)} \propto \frac{1}{E - m_R^{(3b)}}.
\]

Separable interactions suggest that \( \tau^{(2b)} \) in Eq.(7) and \( \tau^{(3b)} \) in Eq.(8) may be given by Lüscher formula-like algebra equations, see detailed discussion in Appendix A,

\[
\frac{\sqrt{mE - \frac{3}{4}k^2}}{16\pi^2} \tau^{(2b)} (k_{13}; K_{13}) = \sum_{LM,L'M'} Y_{LM}(k_{13}) g_L^{(2b)} (k_{13}) g_J^{(2b)} (k_{13}) Y_{L'M'} (k_{13})
\times \delta_{LM,L'M'} \frac{Y_J (\hat{K}_{13})}{\sqrt{mE - \frac{3}{4}k^2}} - \frac{(mE)^2}{128\pi^5} \tau^{(3b)} (k; K')
\times \frac{\sqrt{mE}}{\delta_{LM,L'M'}} \frac{Y_J (\hat{K}')}{\sqrt{mE}} - M_{LM,L'M'}^{(3b)} (\sqrt{mE})^{-1},
\]

and

\[
\frac{(mE)^2}{128\pi^5} \tau^{(3b)} (k; K') = \sum_{J,J'} Y_{\lfloor J \rfloor} (\hat{K}) g_J^{(3b)} (k) g_J^{(3b)} (k) Y_{\lfloor J' \rfloor} (\hat{K}') Y_{\lfloor J' \rfloor} (\hat{K}')
\times \frac{\delta_{J,J'} \cot \delta^{(3b)} (\sqrt{mE})}{\delta_{LM,L'M'}} (\sqrt{mE})^{-1}.
\]

Generalized Lüscher zeta functions, \( M_{LM,L'M'}^{(2b,k_2)} \) in 3D and \( M_{LM,L'M'}^{(3b)} \) in 6D, are given respectively by

\[
\frac{k}{16\pi^2} M_{LM,L'M'}^{(2b,k_2)} (k) = \frac{\delta_{LM,L'M'}}{16\pi^2}
+ \delta_{LM,L'M'} \int \frac{dp}{(2\pi)^3} \left( \frac{g_L^{(2b)} (p)}{g_L^{(2b)} (k)} \right)^2 \frac{1}{k^2 - p^2}
- \frac{1}{L^3} \sum_{p = \frac{2\pi n}{L}} \frac{g_L^{(2b)} (p) g_L^{(2b)} (k) Y_{LM} (\hat{p}) Y_{L'M'} (\hat{p})}{k^2 - p^2},
\]

and

\[
\frac{(mE)^2}{128\pi^5} M_{\lfloor J \rfloor, \lfloor J' \rfloor}^{(3b)} (\sqrt{mE}) = \delta_{\lfloor J \rfloor, \lfloor J' \rfloor} \frac{i(mE)^2}{128\pi^5}
+ \delta_{\lfloor J \rfloor, \lfloor J' \rfloor} \int \frac{dp}{(2\pi)^3} \left( \frac{g_J^{(3b)} (P)}{g_J^{(3b)} (\sqrt{mE})} \right)^2 \frac{1}{mE - P^2}
- \frac{1}{L^6} \sum_{P_1, P_2} \frac{g_J^{(3b)} (P) g_J^{(3b)} (P) Y_{\lfloor J \rfloor} (\hat{P}) Y_{\lfloor J' \rfloor} (\hat{P})}{mE - P^2}.
\]

The three-body phase shift \( \delta_J^{(3b)} \) is defined in a conventional way, which may be modeled and constrained by experimental data. The unconventional three-body phase shift \( \delta_J^{(3b)} \) may be interpreted as scattering of one particle off a short-range potential in 6D. It may only serve as a mathematically convenient tool for the modeling of three-body resonance of total spin-3.
B. Quantization condition with separable interactions approximation

Algebra solutions of \( \tau^{(2b)} \) in Eq.(18) and \( \tau^{(3b)} \) in Eq.(19) suggest that partial expansion of \( \tau^{(2b)}(k_1, k_2) \) may have the form of

\[
\tau^{(2b)}(k_1, k_2) = \sum_{LM} Y_{LM}(k_{13})g_L^{(2b)}(k_{13})T_{LM}^{(2b)}(k_2). \tag{22}
\]

The separable form of \( \tau^{(2b)}(k_1, k_2) \) thus allow one to further reduce Faddeev equations, Eq.(3) and Eq.(4), to

\[
T_{LM}^{(2b)}(k_2) = -\frac{1}{L^2} \sum_{p_2} \sum_{L'M'} \frac{2\tau^{(2b)}(p_2 + k_2/2)}{mE - \frac{1}{2}k_2^2 - (p_2 + k_2/2)^2} Y_{LM}(k_{13})g_L^{(2b)}(k_{13})T_{LM}^{(2b)}(p_2) \]

\[+ \frac{1}{L^3} \sum_{p_2} \sum_{L'M'} \left[ \frac{3}{L^3} \sum_{k_{1},p_{1}} \tau^{(3b)}(k_{13}; k_{13})g_L^{(2b)}(k_{13})T_{LM}^{(2b)}(k_2) \right] T_{LM}^{(2b)}(p_2), \tag{23}
\]

where \( \tau^{(2b)}(k_{13}') \) is defined by relation

\[
\tau^{(2b)}(k_{13}; k_{13}') = \sum_{LM} Y_{LM}(k_{13})g_L^{(2b)}(k_{13})T_{LM}^{(2b)}(k_{13}'). \tag{24}
\]

Therefore, a partially expanded quantization condition is given by

\[
\det \left[ \delta_{LM,L'M'} L^3 \delta_{k_2,p_2} + \frac{2\tau^{(2b)}(p_2 + k_2/2)}{mE - \frac{1}{2}k_2^2 - (p_2 + k_2/2)^2} \right] \]

\[+ \frac{1}{L^6} \sum_{k_{1},p_{1}} \tau^{(3b)}(k_{13}; k_{13})g_L^{(2b)}(k_{13})T_{LM}^{(2b)}(p_1) \]

\[= 0. \tag{25}
\]

As a specific example, let’s consider a simple case with only \( S \)-wave contributions in both two-body and three-body channels, that is to say, \( J = L_{13} = L_{(13)2} = 0 \). Thus \( \tau^{(2b)} \) and \( \tau^{(3b)} \) are given respectively by phase shifts \( \delta^{(2b)}_{L_{13}=0} \) and \( \delta^{(3b)}_{J=0} \) only,

\[
\tau^{(2b)}(k_{13}; k_{13}') = \tau^{(2b)}(k_{2})(\sqrt{mE - \frac{3}{4}k_2^2}) = \frac{4\pi}{\sqrt{mE - \frac{1}{2}k_2^2}} \times \frac{1}{\cot \delta^{(2b)}_0(\sqrt{mE - \frac{3}{4}k_2^2}) - M^{(2b,k_2)}_{00,00}(\sqrt{mE - \frac{3}{4}k_2^2})}. \tag{26}
\]

and

\[
\tau^{(3b)}(K; K') = \tau^{(3b)}(\sqrt{mE}) = \frac{128\pi^2}{(mE)^2} \cot \delta^{(3b)}_0(\sqrt{mE}) - M^{(3b)}_{00,00}(\sqrt{mE}). \tag{27}
\]

The quantization condition in this case is given by a simple form,

\[
\det \left[ L^3 \delta_{k_2,p_2} + \frac{2\tau^{(2b,k_2)}(\sqrt{mE - \frac{1}{2}k_2^2})}{mE - \frac{1}{2}k_2^2 - (p_2 + k_2/2)^2} \right] \]

\[+ \frac{1}{L^6} \sum_{k_{1},p_{1}} \tau^{(3b)}(k_{13}; k_{13})g_L^{(2b)}(k_{13})T_{LM}^{(2b)}(p_1) \]

\[= 0. \tag{28}
\]

The two-body and three-body resonances hence can be inserted through modeling of \( \delta^{(2b)}_{L_{13}=0} \) and \( \delta^{(3b)}_{J=0} \).

III. SUMMARY

In summary, with separable interaction approximation, we show that the subprocess transition amplitudes are Lüscher formula-like, and the quantization condition may be formulated in terms of both two-body and three-body phase shifts that may be useful for describing resonances in few-body interactions. Two-body phase shift
may be modeled and constrained by experimental data, and three-body phase shift may serve as a convenient tool for inserting three-body resonances with a specific spin into quantization condition.

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Appendix A: Lüscher formula in $D$-dimensional space

1. Scattering in $D$-dimensional space

Let's start with $N$-body Schrödinger equation in center of mass frame,

$$mE + \sum_{j=1}^{N-1} \nabla^2_{\xi_j} \psi(\xi; K) = mV(\xi)\psi(\xi; K).$$

(A1)

the relative coordinates of $N$-particle are given by

$$\xi_j = \sqrt{\frac{2j}{j+1}} \left( \frac{1}{j} \sum_{i=1}^{j} x_i - x_{j+1} \right),$$

$$q_j = \sqrt{\frac{j}{2(j+1)}} \left( \frac{1}{j} \sum_{i=1}^{j} k_i - k_{j+1} \right), \quad j = 1, \ldots, N-1,$$

(A2)

where $x_i$ and $k_i$ stand for the coordinate and momentum of $i$-th particle respectively. $D = 3(N-1)$ dimensional vector $(\xi, K)$ are defined by relative coordinates and momenta of particles,

$$\xi = \{\xi_1, \xi_2, \ldots, \xi_{N-1}\}, \quad \xi = |\xi| = \sqrt{\sum_{j=1}^{N-1} \xi_j^2},$$

$$K = \{q_1, q_2, \ldots, q_{N-1}\}, \quad K = |K| = \sqrt{\sum_{j=1}^{N-1} q_j^2}.$$

Where $D$-dimensional Laplace operator has a separable form between radial and orbital terms [54, 55],

$$\nabla^2_D = \sum_{j=1}^{N-1} \nabla^2_{\xi_j} = \frac{1}{\xi^{D-1}} \frac{\partial}{\partial \xi} \xi^{D-1} \frac{\partial}{\partial \xi} + \hat{L}^2(\Omega_D) \xi^2,$$

(A4)

where $\hat{L}^2(\Omega_D)$ is the grand orbital operator. The eigenstates of orbital equation

$$\hat{L}^2(\Omega_D)Y_{L}(\Omega_D) = L(L + D - 2)Y_{L}(\Omega_D)$$

(A5)

is given by hyperspherical harmonic $Y_{L}(\Omega_D)$ [54, 55], where $[L]$ is a set of $D-1$ quantum numbers, including total orbital angular momentum $L$. Hyperspherical harmonic $Y_{L}(\Omega_D)$ basis define a complete set of orthonormal angular function in $D$-dimensional space,

$$\int d\Omega_D Y_{L}^*(\Omega_D)Y_{L'}(\Omega_D) = \delta_{[L],[L']}. \quad (A6)$$

The scattering in $D$-dimensional space can also be described by Lippmann-Schwinger equation

$$\psi(\xi; K) = e^{iK\cdot\xi} + \int d\xi' G_D(\xi - \xi'; E)mV(\xi')\psi(\xi'; K),$$

$$G_D(\xi - \xi'; E) = \int \frac{dQ}{(2\pi)^D} e^{iQ\cdot(\xi - \xi')}, \quad (A7)$$

where Green’s function satisfies equation

$$[mE + \nabla^2_D] G_D(\xi - \xi'; E) = \delta(\xi - \xi'). \quad (A8)$$

The analytic expression of Green’s function and its partial wave expansion in terms of hyperspherical harmonic basis are given respectively by

$$G_D(\xi; E) = -i\frac{(mE)^{D-1}}{4(2\pi)^D} \frac{H_L^{(1)}}{\xi - \xi'}, \quad (A9)$$

and

$$G_D(\xi - \xi'; E) \xi \xi' = -i(mE)^{D-2} \sum_{[L]} Y_{L}(\Omega_D)\mathcal{H}^{(1)}_{L}(\sqrt{mE}\xi)\mathcal{J}_{L}(\sqrt{mE}\xi), \quad (A10)$$

where

$$\mathcal{J}_{L}(z) = \sqrt{\frac{\pi}{2}} \frac{J_{L+\frac{D-2}{2}}(z)}{z^{\frac{D-2}{2}}}, \quad N_{L}(z) = \sqrt{\frac{\pi}{2}} \frac{N_{L+\frac{D-2}{2}}(z)}{z^{\frac{D-2}{2}}}. \quad (A11)$$

Assuming potential $V(\xi)$ is spherical and short-range, and also using partial wave expansion of plane wave in $D$-dimensional space,

$$e^{iK\cdot\xi} = \sqrt{\frac{2}{\pi}}(2\pi)^{\frac{D}{2}} \sum_{[L]} i^L Y_{L}(\Omega_D)Y_{L}^*(\Omega_K)\mathcal{J}_{L}(\sqrt{mE}\xi), \quad (A13)$$

the asymptotic form of wave function is obtained,

$$\psi(\xi; K) \xrightarrow{\text{Large } \xi} \sqrt{\frac{2}{\pi}}(2\pi)^{\frac{D}{2}} \sum_{[L]} i^L Y_{L}(\Omega_D)Y_{L}^*(\Omega_K) \times \left[ \mathcal{J}_{L}(\sqrt{mE}\xi) + if_{L}^{(D)}(\sqrt{mE})\mathcal{H}^{(1)}_{L}(\sqrt{mE}\xi) \right], \quad (A14)$$
where $f_L^{(D)}$ is defined by
\[
\sqrt{\frac{2}{\pi}} \frac{(2\pi)^{\frac{D}{2}}}{(m\bar{E})^{D/2}} f_L^{(D)}(\sqrt{m\bar{E}})Y_{[L]}^*(\Omega_K) = -\int d\xi Y_{[L]}(\Omega_{\xi'}) J_L(\sqrt{m\bar{E}}\xi') mV(\xi')\psi(\xi',K).
\]
(A15)
Thus $f_L^{(D)}$ may be interpreted as partial wave scattering amplitude in $D$-dimensional space, and it can be parameterized in terms of $D$-dimensional phase shift $\delta_L^{(D)}(kE)$ [54, 55] by
\[
f_L^{(D)}(\sqrt{m\bar{E}}) = \frac{1}{\cot \delta_L^{(D)}(\sqrt{m\bar{E}}) - i}. \tag{A16}
\]

2. Lippmann-Schwinger equation in momentum space and separable potential approximation

The off-shell transition amplitude between initial and final momentum states $|K\rangle$ and $|K'\rangle$ may be introduced by
\[
t^{(D)}(K,K') = -\int d\xi e^{-iK\cdot \xi} mV(\xi)\psi(\xi,K), \tag{A17}
\]
thus Eq.(A7) can be converted into momentum space Lippmann-Schwinger equation,
\[
t^{(D)}(K,K') = -m\bar{V}(|K - K'|) + \int \frac{dQ}{(2\pi)^D} m\bar{V}_{L}(K,Q) t^{(D)}(Q,K'), \tag{A18}
\]
The partial wave expansion of above equation yields
\[
t^{(D)}_L(K,K') = -m\bar{V}_L(K,K') + \int \frac{Q^{D-1}dQ}{(2\pi)^D} m\bar{V}_{L}(K,Q) t^{(D)}_L(Q,K'), \tag{A19}
\]
where the expansion relations of potential and amplitude are given by
\[
\bar{V}(|K - K'|) = \sum_{[L]} Y_{[L]}^*(\hat{K}) \bar{V}_L(K,K') Y_{[L]}^*(\hat{K}'). \tag{A20}
\]
and
\[
t^{(D)}(K,K') = \sum_{[L]} Y_{[L]}^*(\hat{K}) t^{(D)}_L(K,K') Y_{[L]}^*(\hat{K}'). \tag{A21}
\]
Under assumption of separable potential,
\[
\bar{V}_L(K,K') = g_L^{(D)}(K) V_L g_L^{(D)}(K'), \tag{A22}
\]
where $g_L^{(D)}$ and $V_L$ stand for the form factor and interaction strength of potential, thus a closed algebra form of off-shell partial wave amplitude, $t^{(D)}_L(K,K')$, may be obtained, see [56],
\[
t^{(D)}_L(K,K') = -\frac{g_L^{(D)}(K) g_L^{(D)}(K')}{\frac{1}{mV_L} - \int \frac{Q^{D-1}dQ}{(2\pi)^D} \frac{\delta_L^{(D)}(Q)}{mE - Q^2}}. \tag{A23}
\]
Compared with on-shell scattering amplitude $f_L^{(D)}(\sqrt{m\bar{E}})$ in Eq.(A15), we find
\[
t^{(D)}_L(K,K') = \frac{g_L^{(D)}(K) g_L^{(D)}(K')}{\left(g_L^{(D)}(\sqrt{m\bar{E}})\right)^2} \times \frac{2}{(2\pi)^D} \frac{(2\pi)^D}{\cot \delta_L^{(D)}(\sqrt{m\bar{E}})}, \tag{A24}
\]
and also a useful relation
\[
\frac{1}{mV_L} = \int \frac{Q^{D-1}dQ}{(2\pi)^D} \frac{\left(g_L^{(D)}(Q)\right)^2}{mE - Q^2} + \left(g_L^{(D)}(\sqrt{m\bar{E}})\right)^2 \frac{2\pi (m\bar{E})^{D/2}}{(2\pi)^D} \left[1 - \cot \delta_L^{(D)}(\sqrt{m\bar{E}})\right]. \tag{A25}
\]
Therefore off-shell partial wave amplitude, $t^{(D)}_L(K,K')$ may be modeled of on-shell potential quantity: phase shifts $\delta_L^{(D)}(\sqrt{m\bar{E}})$.

3. Lüscher formula in $D$-dimensional space and separable potential approximation

Scattering solution in finite volume may be described by inhomogeneous Lippmann-Schwinger equation,
\[
\tau^{(D)}(K,K') = -m\bar{V}(|K - K'|) + \frac{1}{L^D} \sum_{p_1,\ldots,p_{N-1}} \frac{m\bar{V}(|K - Q|)}{mE - Q^2} \tau^{(D)}(Q,K'), \tag{A26}
\]
where $p_i \in \frac{2\pi n}{L}$, $n \in \mathbb{Z}^3$ and $Q^2 = \frac{1}{2} \sum_{l=1}^N p_l^2$. Considering partial wave expansion again,
\[
\tau^{(D)}(K,K') = \sum_{[L],[L']} Y_{[L]}(\hat{K}) \tau^{(D)}_{[L],[L']}(K,K') Y_{[L']}^*(\hat{K}'), \tag{A27}
\]
where
\[
\tau^{(D)}_{[L],[L']}(K,K') = \delta_{[L],[L']} m\bar{V}_L(K,K'), \tag{A28}
\]
\[
+ \sum_{[L]} \frac{1}{L^D} \sum_{p_1,\ldots,p_{N-1}} \frac{m\bar{V}_L(K,Q) Y_{[L]}(\hat{Q}) Y_{[L]}(\hat{Q})}{mE - Q^2} \times \tau^{(D)}_{[L],[L']}(Q,K'). \tag{A28}
\]
Again, the separable potential given in Eq. (A22) suggests that \( \tau^{(D)}_{[L],[L']} \) may have the separable form of

\[
\tau^{(D)}_{[L],[L']} (K, K') = g_L(K)C_{[L],[L']}(E)g_L'(K'), \tag{A29}
\]

where \( C_{[L],[L']}(E) \) satisfies a matrix equation,

\[
C_{[L],[L']}(E) = -\delta_{[L],[L']})mV_L + \frac{1}{L^D} \sum_{p_1, \ldots, p_{N-1}} \prod_{p} \sum_{|l|} \frac{g_L^{(D)}(Q)g_L^{(D)}(Q)}{mE - Q^2} C_{[l],[L']}(E). \tag{A30}
\]

Hence, a closed algebra form of solution of off-shell solution of finite volume amplitude, \( \tau^{(D)}_{[L],[L']} \), is obtained,

\[
\tau^{(D)}_{[L],[L']} (K, K') = \frac{g_L(K)g_L'(K')}{g_L(\sqrt{mE})g_L'(\sqrt{mE})} \left[ \mathcal{D}(\sqrt{mE}) \right]^{-1}_{[L],[L']}, \tag{A31}
\]

where

\[
\mathcal{D}_{[L],[L']}(\sqrt{mE}) = -\frac{\delta_{[L],[L']}}{g_L(\sqrt{mE})g_L'(\sqrt{mE})} + \frac{1}{L^D} \sum_{p_1, \ldots, p_{N-1}} \frac{g_L(Q)g_L'(Q)}{mE - Q^2} Y^*_L(Q)Y_{[L']}(Q). \tag{A32}
\]

Using relation given in Eq. (A25), thus \( \mathcal{D}_{[L],[L']} \) is linked to generalized Lüscher formula in \( D \)-dimensional space,

\[
\frac{2}{\pi} \left( \frac{2\pi}{mE} \right)^{D-2} l^{L-L'} \mathcal{D}_{[L],[L']}(\sqrt{mE}) = \delta_{[L],[L']} \cot \delta_L'(\sqrt{mE}) - M_{[L],[L']}(\sqrt{mE}), \tag{A33}
\]

where \( M_{[L],[L']} \) is generalized Lüscher’s zeta function in \( D \)-dimensional space,

\[
\frac{\pi}{2} \left( \frac{mE}{2\pi} \right)^{D-2} M_{[L],[L']}(\sqrt{mE}) = i\delta_{[L],[L']} \frac{\pi}{2} \left( \frac{mE}{2\pi} \right)^{D-2},
\]

\[
\frac{1}{L^D} \sum_{p_1, \ldots, p_{N-1}} \frac{i^{L-L'}g_L(Q)g_L'(Q)}{mE - Q^2} Y^*_L(Q)Y_{[L']}(Q) \left. \frac{\mathcal{D}(\sqrt{mE})}{\mathcal{D}(\sqrt{mE})} \right| \frac{1}{mE - Q^2} \tag{A34}
\]

Therefore, the inverse of \( \tau^{(D)}_{[L],[L']} \) is explicitly related to Lüscher formula by

\[
\frac{2}{\pi} \left( \frac{2\pi}{mE} \right)^{D-2} \left[ \frac{g_L(\sqrt{mE})g_L'(\sqrt{mE})}{g_L(K)g_L'(K')} \tau^{(D)}(K, K') \right]_{[L],[L']}^{-1} = i^{L-L'} \left[ \delta_{L_{[L],[L']}} \cot \delta_L'(\sqrt{mE}) - M_{[L],[L']}(\sqrt{mE}) \right]. \tag{A35}
\]

Generalized Lüscher zeta function can also be derived by considering hyperspherical harmonic basis function expansion of Green’s function. In infinite volume, the hyperspherical harmonic basis expansion of Green’s function is given by,

\[
\int \frac{dQ}{(2\pi)^D} \frac{e^{iQ(\xi-\xi')}}{mE - Q^2} \sum_{[L]} Y_{[L]}(\Omega) \mathcal{H}_{L}^{(1)}(\sqrt{mE}) J_L(\sqrt{mE}) \mathcal{J}^*_L(\Omega). \tag{A36}
\]

Similarly to expansion of infinite volume Green’s function, the expansion of finite volume Green’s function may be written as,

\[
\frac{1}{L^D} \sum_{p_1, \ldots, p_{N-1}} \frac{e^{iQ(\xi-\xi')}}{mE - Q^2} \sum_{[L]} Y_{[L]}(\Omega) \xi \mathcal{J}_{L}(\sqrt{mE}) \mathcal{J}^*_L(\Omega). \tag{A37}
\]

Combining Eq. (A36) and Eq. (A37), we obtain

\[
\frac{1}{L^D} \sum_{p_1, \ldots, p_{N-1}} \frac{e^{iQ(\xi-\xi')}}{mE - Q^2} \int \frac{dQ}{(2\pi)^D} \frac{e^{iQ(\xi-\xi')}}{mE - Q^2} \xi \mathcal{J}_{L}(\sqrt{mE}) \mathcal{J}^*_L(\Omega) \tag{A38}
\]

Next, using plane wave expansion formula given in Eq. (A13) and also replacing \( g^{(D)}_L(k) \) by \( k^L \), we thus find again Eq. (A34), which may also suggest that the form factor, \( g^{(D)}_L(k) \), may be chosen as \( g^{(D)}_L(k) \sim k^L \).

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