REDUCTION BY SYMMETRIES IN SINGULAR QUANTUM-MECHANICAL PROBLEMS: GENERAL SCHEME AND APPLICATION TO AHARONOV-BOHM MODEL

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Abstract. We develop a general technique for finding self-adjoint extensions of a symmetric operator that respect a given set of its symmetries. Problems of this type naturally arise when considering two- and three-dimensional Schrödinger operators with singular potentials. The approach is based on constructing a unitary transformation diagonalizing the symmetries and reducing the initial operator to the direct integral of a suitable family of partial operators. We prove that symmetry preserving self-adjoint extensions of the initial operator are in a one-to-one correspondence with measurable families of self-adjoint extensions of partial operators obtained by reduction. The general scheme is applied to the three-dimensional Aharonov-Bohm Hamiltonian describing the electron in the magnetic field of an infinitely thin solenoid. We construct all self-adjoint extensions of this Hamiltonian, invariant under translations along the solenoid and rotations around it, and explicitly find their eigenfunction expansions.

1. Introduction

The spectral analysis of Schrödinger operators can be often facilitated if some symmetries of the problem are known. This happens if the spectral problem for the operators representing symmetries can be explicitly solved. In this case, the initial operator can be represented as a direct sum (or, more generally, direct integral) of partial operators corresponding to fixed eigenvalues of symmetries. In this way, two- and three-dimensional problems can be often reduced to solving certain one-dimensional Schrödinger equations (this is the case, for example, for spherically symmetric problems).

The present paper is concerned with such a reduction by symmetries in the case of singular quantum-mechanical problems. It is well known [4, 17] that strong singularities in the potential may lead to the lack of self-adjointness of the corresponding Schrödinger operator on its natural domain. As a result, the quantum model is no longer fixed uniquely by the potential and different quantum dynamics described by various self-adjoint extensions of the initial Schrödinger operator are possible. In this paper, we are interested in the extensions that respect some given set of symmetries of the initial operator. We propose a general technique for the reduction of such symmetry preserving extensions and apply it to the spectral analysis of the three-dimensional Aharonov-Bohm Hamiltonian describing the electron in the magnetic field of an infinitely thin solenoid.

In a general form, the problem can be posed as follows. Suppose $H$ is a closed operator in a separable Hilbert space $\mathcal{H}$ and $\mathcal{X}$ is a subset of the algebra $L(\mathcal{H})$ of linear bounded everywhere defined operators in $\mathcal{H}$. We assume that $\mathcal{X}$ is involutive
(i.e., the adjoint $T^*$ of every $T \in \mathfrak{X}$ belongs to $\mathfrak{X}$) and consists of pairwise commuting operators. Further, we assume $\mathfrak{X}$ is a set of symmetries of $H$, by which we mean that $H$ commutes with elements of $\mathfrak{X}$ in the sense of the next definition.

**Definition 1.1.** A linear operator $H$ in $\mathfrak{H}$ with the domain $D_H$ is said to commute with $T \in L(\mathfrak{H})$ if $T\Psi \in D_H$ and $HT\Psi = TH\Psi$ for any $\Psi \in D_H$.

Our aim is to show how spectral decompositions of symmetries can be used to represent closed (and, in particular, self-adjoint) symmetry preserving extensions of $H$ as direct integrals of suitable partial operators. To clarify main ideas, we assume for a while that $\mathfrak{H}$ falls into an orthogonal direct sum of eigenspaces of symmetries. That is, we suppose that there is a countable family $\{\mathfrak{G}(s)\}_{s \in \mathcal{S}}$ of nontrivial closed pairwise orthogonal subspaces of $\mathfrak{H}$ such that

$$\mathfrak{H} = \bigoplus_{s \in \mathcal{S}} \mathfrak{G}(s)$$

and

$$T\Psi = g_T(s)\Psi, \quad \Psi \in \mathfrak{G}(s),$$

for every $s \in \mathcal{S}$ and $T \in \mathfrak{X}$, where $g_T(s)$ are some complex numbers (since $\mathfrak{G}(s)$ are nontrivial, the eigenvalues $g_T(s)$ are defined uniquely). Let $P_s$ denote the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{G}(s)$. We would like to have a decomposition of the form

$$H = \bigoplus_{s \in \mathcal{S}} \mathcal{H}(s),$$

where $\mathcal{H}(s)$ are closed operators in $\mathfrak{G}(s)$ for all $s \in \mathcal{S}$ and $\bigoplus_{s \in \mathcal{S}} \mathcal{H}(s)$ is, by definition, the operator in $\mathfrak{H}$ whose graph consists of all $(\Psi, \tilde{\Psi}) \in \mathfrak{H} \oplus \mathfrak{H}$ such that $(P_s\Psi, P_s\tilde{\Psi})$ is in the graph of $\mathcal{H}(s)$ for every $s \in \mathcal{S}$. As the operator $\bigoplus_{s \in \mathcal{S}} \mathcal{H}(s)$ obviously commutes with all projections $P_s$, equality (2) can hold only if $H$ commutes with $P_s$ for every $s \in \mathcal{S}$. In fact, the latter condition is also sufficient for the existence of decomposition (2) (see Theorems 2 and 3 in Sec. 45 of [3]). Given a decomposition of form (2), it can be easily shown that $\mathcal{H}(s)$ is actually the restriction of $H$ to the domain $D_H \cap \mathfrak{G}(s)$ for every $s \in \mathcal{S}$. Thus, the partial operators $\mathcal{H}(s)$ are uniquely determined by $H$.

In general, the condition that $\mathfrak{G}(s)$ are eigenspaces of symmetries is insufficient to guarantee that $H$ commutes with $P_s$ for all $s \in \mathcal{S}$ (otherwise we could choose $\mathfrak{X}$ to be the one-element set containing the identity operator in $\mathfrak{H}$ and consider the decomposition $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}'^\perp$, where $\mathfrak{H}'$ is an arbitrary closed subspace of $\mathfrak{H}$; it would follow that every closed operator in $\mathfrak{H}$ commutes with the orthogonal projection onto $\mathfrak{H}'$). The commutation between $H$ and $P_s$ can be ensured, however, if we require that

$$P_s \in \mathcal{A}(\mathfrak{X})$$

for every $s \in \mathcal{S}$, where $\mathcal{A}(\mathfrak{X})$ is the smallest strongly closed subalgebra of $L(\mathfrak{H})$ containing the set $\mathfrak{X}$ and the identity operator in $\mathfrak{H}$ (in other words, $\mathcal{A}(\mathfrak{X})$ is the strong closure of the algebra generated by $\mathfrak{X}$ and the identity operator). Indeed, let $\mathcal{M}$ be the subset of $L(\mathfrak{H})$ consisting of all operators commuting with $H$. Since the sum and product of any two operators in $L(\mathfrak{H})$ commuting with $H$ also commute with $H$, we conclude that $\mathcal{M}$ is an algebra. Moreover, it easily follows from the closedness of $H$ that $\mathcal{M}$ is closed in the strong operator topology (see Lemma 1
in $[27]$) and, hence, $A(X) \subset M$. Condition (3) therefore implies that $P_s \in M$ or, in other words, that $H$ commutes with $P_s$ for all $s \in \mathcal{S}$.

We shall say that an eigenspace decomposition of form (1) is exact for $X$ if condition (3) is fulfilled for all $s \in \mathcal{S}$. Given an exact decomposition for $X$, it is easy to describe symmetry preserving closed extensions of $H$. Indeed, suppose (E) holds. Since $H$ is the sum of its partial operators (adjoint) extensions of $H_L$, it follows from (3) that $H(s)$ is a closed extension of $H$ commuting with all elements of $X$. Conversely, if $\tilde{H}$ is a closed extension of $H$ commuting with all elements of $X$, then the above considerations applied to $\tilde{H}$ instead of $H$ show that $\tilde{H}$ falls into the direct sum of its partial operators $\tilde{H}(s)$, which are obviously closed extensions of $H(s)$. Moreover, $\tilde{H}$ is densely defined if and only if all $\tilde{H}(s)$ are densely defined, in which case we have $\tilde{H} = \bigoplus_{s \in \mathcal{S}} \tilde{H}(s)$. Hence, $\tilde{H}$ is self-adjoint if and only if $\tilde{H}(s)$ is self-adjoint for every $s \in \mathcal{S}$.

Thus, given an exact decomposition for $X$, the operator $H$ falls into the direct sum of its partial operators $H(s)$, and the symmetry preserving closed (resp., self-adjoint) extensions of $H$ are precisely the direct sums of closed (resp., self-adjoint) extensions of $H(s)$.

Exactness condition (3) can be given another, equivalent, formulation that is better suited for verifying in concrete applications. More specifically, we claim that (E) holds for all $s \in \mathcal{S}$ if and only if the following condition is fulfilled:

(E) For every $s, s' \in \mathcal{S}$ such that $s \neq s'$, there is $T \in X$ such that $g_T(s) \neq g_T(s')$.

Indeed, suppose (E) holds. Since $X$ is involutive, the elements of $A(X)$ can be characterized using von Neumann’s bicommutant theorem (see Sec. 4 for details): an operator in $L(\mathcal{S})$ belongs to $A(X)$ if and only if it commutes with every element of commutant of $X$, i.e., with every operator in $L(\mathcal{S})$ that commutes with all elements of $X$. To prove (3), we have to verify this condition for $P_s$. For every $s, s' \in \mathcal{S}$ such that $s \neq s'$, we use (E) to choose an operator $T_{s,s'} \in X$ such that $g_{T_{s,s'}}(s) \neq g_{T_{s,s'}}(s')$.

Given $s \in \mathcal{S}$ and a finite set $K \subset \mathcal{S}$, we define the operator $T^K_s \in L(\mathcal{S})$ by the equality

$$T^K_s = \prod_{s' \in K, s' \neq s} \frac{T_{s,s'} - g_{T_{s,s'}}(s')}{g_{T_{s,s'}}(s) - g_{T_{s,s'}}(s')}.$$  

We then have $T^K_s P_{s'} = 0$ for $s' \in K \setminus \{s\}$ and $T^K_s P_s = P_s$, whence

$$P_s P^K = T^K_s P^K,$$

where $P^K = \sum_{s' \in K} P_{s'}$ is the orthogonal projection of $\mathcal{S}$ onto $\bigoplus_{s' \in K} \mathcal{S}(s')$. Let $R$ be an operator in $L(\mathcal{S})$ commuting with all elements of $X$. Then $R$ commutes with $T^K_s$, and it follows from (3) that

$$P_s R P^K = T^K_s P^K R P^K = P^K R P^K T^K_s = P^K R P^K P_s$$

(note that the operators $P_s$, $P^K$, and $T^K_s$ pairwise commute). Thus, $P_s$ commutes with the operator $P^K R P^K$ for every $s \in \mathcal{S}$ and finite set $K \subset \mathcal{S}$. We now choose finite sets $K_1 \subset K_2 \subset \ldots$ such that $\mathcal{S} = \bigcup_{n=1}^{\infty} K_n$. Since $R$ is the strong limit of $P^K R P^K$ as $n \to \infty$, we conclude that $P_s$ commutes with $R$ for every $s \in \mathcal{S}$, and (3) is proved. We now show that, conversely, (3) implies (E). Suppose, to the contrary, that (3) holds and there exist $s_1, s_2 \in \mathcal{S}$ such that $s_1 \neq s_2$ and...
$g_T(s_1) = g_T(s_2)$ for every $T \in \mathfrak{X}$. We then have $\hat{P}T = T\hat{P} = g_T(s_1)\hat{P}$ for every $T \in \mathfrak{X}$, where $\hat{P} = P_{s_1} + P_{s_2}$ is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{S}(s_1) \oplus \mathfrak{S}(s_2)$. It follows that every element of $\mathfrak{X}$ commutes with $\hat{P}R\hat{P}$ for every $R \in L(\mathfrak{H})$. In view of (3) and the characterization of elements of $\mathcal{A}(\mathfrak{X})$ given above, this implies that $P_{s_1}$ commutes with $\hat{P}R\hat{P}$ for every $R \in L(\mathfrak{H})$, which is obviously false. We thus arrive at a contradiction and our claim is proved.

The above discussion is based on the assumption that the Hilbert space $\mathfrak{H}$ falls into a direct sum of eigenspaces of symmetries. In general, this assumption does not hold and one has to use direct integral decompositions instead. More specifically, we can try to find a positive measure $\nu$, a $\nu$-measurable family $\mathfrak{S}$ of Hilbert spaces, and a unitary operator $V : \mathfrak{H} \rightarrow \int^\oplus \mathfrak{S}(s) \, d\nu(s)$ such that every $T \in \mathfrak{X}$ is representable in the form $V^{-1}T_g\mathfrak{S}V$ for some $\nu$-measurable complex function $g$, where $T_g\mathfrak{S}$ is the operator of multiplication by $g$ in $\int^\oplus \mathfrak{S}(s) \, d\nu(s)$ (we refer the reader to Appendix B for the notions related to direct integrals of Hilbert spaces). A triple $(\nu, \mathfrak{S}, V)$ satisfying this condition is said to be a diagonalization for $\mathfrak{X}$. Given a diagonalization $(\nu, \mathfrak{S}, V)$, we would like to have a decomposition of $H$ of the form

$$H = V^{-1} \int^\oplus \mathcal{H}(s) \, d\nu(s) \, V,$$

where $\mathcal{H}$ is a $\nu$-measurable family of closed operators in $\mathfrak{S}$. This decomposition is an analogue of (2) in the direct integral setting. As we have seen, such a decomposition, in general, does not exist even in the direct sum case. It turns out, however, that this problem does not arise if we confine ourselves to exact diagonalizations that are singled out by the requirement that

$$V^{-1}T_g\mathfrak{S}V \in \mathcal{A}(\mathfrak{X}),$$

for every $\nu$-measurable $\nu$-essentially bounded function $g$. It will be shown that, in contrast to direct sum decompositions, exact diagonalizations exist for every involutive set $\mathfrak{X} \subset L(\mathfrak{H})$ of pairwise commuting operators. Exactness condition (6) (which is an analogue of (3)) implies that $VHV^{-1}$ commutes with $T_g\mathfrak{S}$ for every $\nu$-measurable $\nu$-essentially bounded $g$ since, as shown above, $H$ commutes with all elements of $\mathcal{A}(\mathfrak{X})$. This allows us to apply the von Neumann’s reduction theory [32] (or, more precisely, its generalization [24, 25] for the case of unbounded operators) to $VHV^{-1}$ and obtain decomposition (4) as a result. Moreover, given an exact diagonalization $(\nu, \mathfrak{S}, V)$ for $\mathfrak{X}$ and a decomposition of form (5), the symmetry preserving closed (resp., self-adjoint) extensions of $H$ are precisely the operators of the form

$$V^{-1} \int^\oplus \mathcal{H}(s) \, d\nu(s) \, V,$$

where $\mathcal{H}$ is a $\nu$-measurable family of operators in $\mathfrak{S}$ such that $\mathcal{H}(s)$ is a closed (resp., self-adjoint) extension of the partial operator $\mathcal{H}(s)$ for $\nu$-a.e. $s$. Condition (4) is, as a rule, inconvenient for concrete applications because it involves the algebra $\mathcal{A}(\mathfrak{X})$ rather than the set $\mathfrak{X}$ itself. We shall see, however, that a simple exactness criterion similar to condition (E) can be obtained under very mild restrictions on the measure $\nu$ (Theorem 1.2). This generalized condition (E) can usually be easily verified for concrete examples.

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1Throughout the paper, a.e. means either 'almost every' or 'almost everywhere'.
The results formulated above suggest that, in general, one can find the symmetry preserving self-adjoint extensions of $H$ by doing the following steps:

(I) Find an exact diagonalization $(\nu, \mathcal{G}, V)$ for $X$.

(II) Compute partial operators $\mathcal{H}(s)$ satisfying (5).

(III) Find self-adjoint extensions of the partial operators $\mathcal{H}(s)$.

In practice, the operator $H$ often comes as the closure of some non-closed operator $\hat{H}$. While $\hat{H}$ is usually given by some explicit formula, finding an explicit description of $H$ may be a difficult task. We shall see, however, that every symmetry of $\hat{H}$ is also a symmetry of $H$ (Lemma 3.1). Moreover, finding partial operators of $H$ can be effectively reduced to some computations involving $\hat{H}$ (Proposition 5.3). For this reason, the knowledge of $\hat{H}$ is actually sufficient for doing steps (I)-(III) for $H$.

We apply the general construction described above to the three-dimensional model of an electron in the magnetic field of an infinitely thin solenoid. The Hamiltonian for this model is formally given by the differential expression

$$H = \frac{\hbar^2}{2m_e} \sum_{j=1}^{3} \left( i\partial_{x_j} + \frac{e}{\hbar c} a_j(x) \right)^2,$$

where $e$ and $m_e$ are the electron charge and mass respectively, $c$ is the velocity of light, and the vector potential $a = (a_1, a_2, a_3)$ has the form

$$a_1(x) = \frac{\hbar c}{e} \frac{\phi x_2}{x_1^2 + x_2^2}, \quad a_2(x) = -\frac{\hbar c}{e} \frac{\phi x_1}{x_1^2 + x_2^2}, \quad a_3(x) = 0,$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Here, $\phi$ is the flux of the magnetic field through the solenoid measured in the units of the flux quantum $2\pi\hbar c/e$.

This model was originally considered by Aharonov and Bohm in [2]. Self-adjoint extensions in the two-dimensional variant of this model, as well as their eigenfunction expansions and corresponding scattering amplitudes, were analysed in [31, 1, 9].

The vector potential $a$ is smooth outside the $x_3$-axis $Z = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Hence, (7) naturally determines an operator $\mathcal{H}_\phi$ in the space $C_0^\infty(\mathbb{R}^3 \setminus Z)$ of smooth functions on $\mathbb{R}^3$ with compact support contained in $\mathbb{R}^3 \setminus Z$,

$$(8) \quad (\mathcal{H}_\phi \Phi)(x) = \sum_{j=1}^{3} \left( i\partial_{x_j} + \frac{e}{\hbar c} a_j(x) \right)^2 \Phi(x) =$$

$$= \left( -\Delta + \frac{2i\phi}{x_1^2 + x_2^2} (x_2 \partial x_1 - x_1 \partial x_2) + \frac{\phi^2}{x_1^2 + x_2^2} \right) \Phi(x), \quad \Phi \in C_0^\infty(\mathbb{R}^3 \setminus Z)$$

(to simplify notation, we have dropped the factor $\hbar^2/2m_e$ in (7)). Passing to $\Lambda$-equivalence classes, where $\Lambda$ is the Lebesgue measure on $\mathbb{R}^3$, we obtain a (non-closed) densely defined symmetric operator $\hat{H}_\phi$ in $L_2(\mathbb{R}^3)$:

$$D_{\hat{H}_\phi} = \{ [\Phi]_\Lambda : \Phi \in C_0^\infty(\mathbb{R}^3 \setminus Z) \},$$

$$\hat{H}_\phi [\Phi]_\Lambda = [\mathcal{H}_\phi \Phi]_\Lambda, \quad \Phi \in C_0^\infty(\mathbb{R}^3 \setminus Z),$$

where $[\Phi]_\Lambda$ denotes the $\Lambda$-equivalence class corresponding to $\Phi$. We define the operator $\check{H}_\phi$ in $L_2(\mathbb{R}^3)$ as the closure of $\hat{H}_\phi$,

$$(9) \quad H^\phi = \overline{\hat{H}_\phi},$$
Let \( G \) be the Abelian group of linear operators in \( \mathbb{R}^3 \) generated by translations along the \( x_3 \)-axis and rotations around the \( x_3 \)-axis. Given \( G \in G \), we denote by \( T_G \) the unitary operator in \( L_2(\mathbb{R}^3) \) taking \( \Psi \) to \( \Psi \circ G^{-1} \) for any \( \Psi \in L_2(\mathbb{R}^3) \). Clearly, \( G \to T_G \) is a linear representation of \( G \) in \( L_2(\mathbb{R}^3) \). It is straightforward to check that \( \tilde{H}^\phi \) (and, hence, \( H^\phi \)) commutes with \( T_G \) for any \( G \in G \). The above abstract scheme can therefore be applied to \( H = H^\phi \) and the set \( X \) consisting of all \( T_G \) with \( G \in G \). As a result, we shall describe all self-adjoint extensions of \( H^\phi \) commuting with \( T_G \) for any \( G \in G \) and find eigenfunction expansions for such extensions.

The paper is organized as follows. In Sec. 2 we formulate the results concerning self-adjoint extensions and eigenfunction expansions for the Aharonov-Bohm model. In Secs. 3–5, we elaborate on the general scheme of constructing symmetry preserving extensions outlined above. Sec. 3 is devoted to preliminaries concerning the commutation properties of operators in Hilbert space. In Sec. 4 we establish the existence of exact diagonalizations and prove an exactness criterion, analogous to condition (E) discussed above. In Sec. 5 we consider direct integral decompositions for closed operators possessing a given set of symmetries and describe their symmetry preserving extensions. The aim of Secs. 6–10 is to derive eigenfunction expansions for the Aharonov-Bohm model. Sec. 6 is concerned with self-adjoint extensions and eigenfunction expansions for one-dimensional Schrödinger operators. In Sec. 7 we treat the measurability questions for families of such operators. In Appendices A and B we give the necessary background material concerning measure theory and direct integral decompositions of operators in Hilbert space respectively.

2. Formulation of results for the Aharonov-Bohm model

Let \( \rho \) be the counting measure on \( \mathbb{Z} \), which assigns to each finite set of integers the number of points in this set. We define the positive Borel measure \( \nu_0 \) on \( S = \mathbb{Z} \times \mathbb{R} \) by setting \( \nu_0 = \rho \times \lambda \), where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \). For any \( \nu_0 \)-integrable \( f \), the function \( p \to f(m, p) \) is integrable for every \( m \in \mathbb{Z} \) and we have

\[
\int_S f(m, p) \, d\nu_0(m, p) = \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(m, p) \, dp.
\]

For \( \kappa \in \mathbb{R} \), let the positive Borel measure \( \nu_{\kappa} \) on \( \mathbb{R} \) be given by

\[
d\nu_{\kappa}(E) = \frac{1}{2} \Theta(\kappa|E|) |E| \, dE,
\]

where \( \Theta \) is the Heaviside function, i.e., \( \Theta(E) = 1 \) for \( E \geq 0 \) and \( \Theta(E) = 0 \) for \( E < 0 \). Given \(-1 < \kappa < 1\) and \( \vartheta \in \mathbb{R} \), we define the positive Borel measure \( \nu_{\kappa, \vartheta} \) on

\[\text{Appendices A and B give the necessary background material concerning measure theory and direct integral decompositions of operators in Hilbert space respectively.}\]
\[ \mathcal{V}_{\kappa, \vartheta} = \int_{-\alpha}^{\alpha} \frac{\tilde{V}_{\kappa, \vartheta}}{2 \sin \vartheta} \delta_{E_{\kappa, \vartheta}} + \tilde{V}_{\kappa, \vartheta}, \quad \vartheta \in \mathbb{R}, \]  

where

\[ \vartheta_{\kappa} = \frac{\pi \kappa}{2}, \]

the positive Borel measure \( \tilde{V}_{\kappa, \vartheta} \) on \( \mathbb{R} \) is given by

\[ d\tilde{V}_{\kappa, \vartheta}(E) = \frac{1}{2} (E - \kappa \sin^2(\vartheta + \vartheta_\kappa)) dE, \]

and \( \delta_{E_{\kappa, \vartheta}} \) is the Dirac measure at the point

\[ E_{\kappa, \vartheta} = -\frac{(\sin(\vartheta + \vartheta_\kappa))^{1/\kappa}}{(\sin(\vartheta_\kappa))}. \]

For \( \kappa = 0 \), the measure \( \mathcal{V}_{\kappa, \vartheta} \) is defined by taking the limit \( \kappa \to 0 \) in formulas (11), (13), and (14). This yields

\[ \mathcal{V}_{0, \vartheta} = \begin{cases} 2 \pi |E_{0, \vartheta}| \delta_{E_{0, \vartheta}} + \tilde{V}_{0, \vartheta}, & \vartheta \neq \pi, \\ \pi^2 |E_{0, \vartheta}| \delta_{E_{0, \vartheta}} + \tilde{V}_{0, \vartheta}, & \vartheta = \pi, \end{cases} \]

where

\[ E_{0, \vartheta} = -e^{\pi \cot \vartheta} \]

and the positive Borel measure \( \tilde{V}_{0, \vartheta} \) on \( \mathbb{R} \) is given by

\[ d\tilde{V}_{0, \vartheta}(E) = \frac{1}{2} (\cos \vartheta - \ln E \sin \vartheta/\pi)^2 + \sin^2 \vartheta dE. \]

For every \( \phi \in \mathbb{R} \), let the subset \( A^\phi \) of \( S \) be given by

\[ A^\phi = \{ (m, p) \in S : |m - \phi| < 1 \}. \]

Given a Borel real function \( \theta \) on \( A^\phi \), let \( \mu_{\theta}^\phi \) be the measure-valued map on \( S \) such that

\[ \mu_{\theta}^\phi(s) = \begin{cases} \mathcal{V}_{|m - \phi|} & s \in S \setminus A^\phi, \\ \tilde{V}_{m - \phi, \theta(s)} & s \in A^\phi, \end{cases} \]

for every \( s = (m, p) \in S \).

**Lemma 2.1.** Let \( F \) be a bounded Borel function on \( \mathbb{R} \) with compact support. Then \( (\kappa, \vartheta) \to \int F(E) d\mathcal{V}_{\kappa, \vartheta}(E) \) is a Borel function on \( (-1, 1) \times \mathbb{R} \) that is bounded on \( [-\alpha, \alpha] \times \mathbb{R} \) for every \( 0 \leq \alpha < 1 \). If, in addition, \( F \) is continuous, then \( (\kappa, \vartheta) \to \int F(E) d\mathcal{V}_{\kappa, \vartheta}(E) \) is continuous on \( (-1, 1) \times \mathbb{R} \).

**Corollary 2.2.** Let \( \phi \in \mathbb{R} \) and \( \theta \) be a Borel real function on \( A^\phi \). For any compact set \( K \subset \mathbb{R} \), \( s \to \mu_{\theta}^\phi(s|K) \) is a Borel function on \( S \) that is bounded on every compact subset of \( S \).

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3For brevity, we denote by \( \mu_{\theta}^\phi(s|K) \) the measure \( \mu_{\theta}^\phi(s) \) of the set \( K \): \( \mu_{\theta}^\phi(s|K) = (\mu_{\theta}^\phi(s))(K) \). A similar notation will be used for any maps whose values are also maps.
Proposition 2.3. Let $\phi \in \mathbb{R}$ and $\theta$ be a Borel real function on $A^\phi$. Then there is a unique positive Borel measure $M$ on $S \times \mathbb{R}$ such that

$$M(K' \times K) = \int_{K'} \mu^\phi_\theta(s|K) \, dv_0(s)$$

for every compact sets $K' \subset S$ and $K \subset \mathbb{R}$. If $f$ is an $M$-integrable complex function, then the function $E \to f(s, E)$ is $\mu_\theta^\phi(s)$-integrable for $v_0$-a.e. $s$, the function $s \to \int f(s, E) \, d\mu_\theta^\phi(s|E)$ is $v_0$-integrable, and

$$\int f(s, E) \, dM(s, E) = \int d\nu_0(s) \int f(s, E) \, d\mu_\theta^\phi(s|E).$$

Note that the right-hand side of (20) is well-defined in view of Corollary 2.2. Given $\phi \in \mathbb{R}$ and a Borel real function $\theta$ on $A^\phi$, the measure $M$ satisfying the conditions of Proposition 2.3 will be denoted by $M^\phi_\theta$.

For any $z, \kappa \in \mathbb{C}$, we define the function $u^\kappa(z)$ on $\mathbb{R}_+$ by setting

$$u^\kappa(z|r) = \nu^{1/2 + \kappa} \gamma(z^2), \quad r \in \mathbb{R}_+,$$

where the entire function $\gamma(z)$ is given by

$$\gamma(z) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{\Gamma(\kappa + n + 1)n!2^{2n}}, \quad z \in \mathbb{C}.$$

The function $\gamma(z)$ is closely related to Bessel functions: for $z \neq 0$, we have $\gamma(z) = z^{-\kappa/2} J_\kappa(z^{1/2})$, where $J_\kappa$ is the Bessel function of the first kind of order $\kappa$.

For $-1 < \kappa < 1$, $\vartheta \in \mathbb{R}$, and $z \in \mathbb{C}$, we define the function $u^\kappa(\vartheta)$ on $\mathbb{R}_+$ by setting

$$u^\kappa(\vartheta)(z) = \frac{u^\kappa(z) \sin(\vartheta + \vartheta_\kappa) - u^{-\kappa}(z) \sin(\vartheta - \vartheta_\kappa)}{\sin \pi \kappa}, \quad 0 < |\kappa| < 1,$$

and

$$u^\kappa(\vartheta)(z|r) = \lim_{\kappa \to 0} u^\kappa(\vartheta)(z|r) = u^0(z|r) \cos \vartheta + \frac{2}{\pi} \left[ (\ln \frac{r}{2} + \gamma) u^0(z|r) - \sqrt{r} Y(zr^2) \right] \sin \vartheta, \quad r \in \mathbb{R}_+,$$

where the entire function $Y(z)$ is given by

$$Y(z) = \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{(n!)^2 2^{2n}} z^n, \quad c_n = \sum_{i=1}^{n} \frac{1}{i},$$

and $\gamma = \lim_{n \to \infty} (c_n - \ln n) = 0, 577 \ldots$ is the Euler constant.

Given $\phi \in \mathbb{R}$ and a Borel real function $\theta$ on $A^\phi$, let $W^\phi_\theta$ be the complex function on $S \times \mathbb{R} \times (\mathbb{R}^3 \setminus Z)$ such that

$$W^\phi_\theta(s, E, x) = \frac{e^{ipx_3}}{2\pi \sqrt{F_x}} \frac{(x_1 + ix_2)}{r_x} \left( \frac{x_1 + ix_2}{r_x} \right)^{m} \times \left\{ \begin{array}{ll} u^{m-\phi}(E|r_x), & s \in S \setminus A^\phi, \\ u^{m-\phi}_\theta(E|r_x), & s \in A^\phi, \end{array} \right.$$}

for all $s = (m, p) \in S$, $E \in \mathbb{R}$, and $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus Z$, where $r_x = \sqrt{x_1^2 + x_2^2}$.

In view of (22), (24), and (25), $x \to W^\phi_\theta(s, E, x)$ is a locally square-integrable

---

4To compute the limit of $u^\kappa(\vartheta)(z|r)$ as $\kappa \to 0$, one has to apply L'Hôpital's rule and use the equality $\Gamma(1 + n)/\Gamma(1 + n) = c_n - \gamma$ (see (12), Sec. 1.7.1, formula (9)).
function on \( \mathbb{R}^3 \) for all \( s \in S \) and \( E \in \mathbb{R} \). Let \( L_2^s(\mathbb{R}^3) \) denote the subspace of \( L_2(\mathbb{R}^3) \) consisting of all its elements vanishing \( \Lambda \)-a.e. outside some compact subset of \( \mathbb{R}^3 \).

**Proposition 2.4.** Let \( \phi \in \mathbb{R} \) and \( \theta \) be a Borel real function on \( A^\phi \). Then there is a unique unitary operator \( W : L_2(\mathbb{R}^3) \to L_2(S \times \mathbb{R}, M^\phi_0) \) such that

\[
(W \Psi)(s, E) = \int W^\phi_\theta(s, E, x) \Psi(x) \, dx, \quad \Psi \in L_2^s(\mathbb{R}^3),
\]

for \( M^\phi_0 \)-a.e. \( (s, E) \).

Given \( \phi \in \mathbb{R} \) and a Borel real function \( \theta \) on \( A^\phi \), the operator \( W \) satisfying the conditions of Proposition 2.4 will be denoted by \( W^\phi_\theta \). Let \( f \) be the function on \( S \times \mathbb{R} \) defined by the relation

\[
f(m, p; E) = p^2 + E.
\]

We define the self-adjoint operator \( H^\phi_\theta \) in \( L_2(\mathbb{R}^3) \) by setting

\[
H^\phi_\theta = (W^\phi_\theta)^* T^M_\theta W^\phi_\theta,
\]

where \( M = M^\phi_0 \) and \( T^M_\theta \) is the operator of multiplication by \( f \) in \( L_2(S \times \mathbb{R}, M) \) (see Sec. A.7 for its precise definition).

**Theorem 2.5.** Let \( \phi \in \mathbb{R} \). For every Borel real function \( \theta \) on \( A^\phi \), the operator \( H^\phi_\theta \) is a self-adjoint extension of \( H^\phi \) commuting with \( T_G \) for any \( G \in \mathcal{G} \). Conversely, every self-adjoint extension of \( H^\phi \) commuting with \( T_G \) for any \( G \in \mathcal{G} \) is equal to \( H^\phi_\theta \) for some Borel real function \( \theta \) on \( A^\phi \). Given Borel real functions \( \theta \) and \( \tilde{\theta} \) on \( A^\phi \), we have \( H^\phi_\theta = H^\phi_{\tilde{\theta}} \) if and only if \( \theta(s) - \tilde{\theta}(s) \in \pi \mathbb{Z} \) for \( \nu_0 \)-a.e. \( s \in A^\phi \).

## 3. Commutation of Operators and von Neumann Algebras

In this section, we give some background material for the treatment of diagonalizations in the next section. It should be noted that, as far as diagonalizations are concerned, the conditions imposed on \( X \) in Introduction are excessively restrictive. In fact, it suffices to assume (and we do so in this and the next sections) that \( X \) is an arbitrary set of closed densely defined operators rather than an involutive subset of \( L(\mathcal{H}) \).

**Lemma 3.1.** Let \( \mathcal{H} \) be a Hilbert space, \( T \in L(\mathcal{H}) \), and \( R \) be an operator in \( \mathcal{H} \) commuting with \( T \). If \( R \) is densely defined, then \( R^* \) commutes with \( T^* \). If \( R \) is closable, then the closure \( \bar{R} \) of \( R \) commutes with \( T \).

**Proof.** Suppose \( R \) is densely defined. Let \( \Psi \in D_R \) and \( \Phi = R^* \Psi \). Then we have

\[
\langle R \Psi', \Psi \rangle = \langle \Psi', \Phi \rangle \quad \text{for any } \Psi' \in D_R.
\]

Hence, we obtain

\[
\langle R \Psi', T^* \Psi \rangle = \langle R T \Psi', \Psi \rangle = \langle T \Psi', \Phi \rangle = \langle \Psi', T^* \Phi \rangle, \quad \Psi' \in D_R.
\]

This means that \( T^* \Psi \in D_{R^*} \) and \( R^* T^* \Psi = T^* R^* \Psi \), i.e., \( R^* \) commutes with \( T^* \).

Suppose now that \( R \) is closable. Let \( \Psi \in D_{\bar{R}} \). Then there is a sequence \( \Psi_n \in D_R \) such that \( \Psi_n \to \Psi \) and \( R \Psi_n \to \bar{R} \Psi \) in \( \mathcal{H} \). Since \( R \) commutes with \( T \), we have \( T \Psi_n \in D_R \) for all \( n \). The continuity of \( T \) implies that \( T \Psi_n \to T \Psi \) and \( R T \Psi_n = T R \Psi_n \to T \bar{R} \Psi \). This means that \( T \Psi \in D_{\bar{R}} \) and \( \bar{R} T \Psi = T \bar{R} \Psi \). \( \square \)
Given a set $\mathcal{X}$ of closed densely defined operators in a Hilbert space $\mathcal{H}$, let $\mathcal{X}'$ denote its commutant, i.e., the subalgebra of $L(\mathcal{H})$ consisting of all operators commuting with every element of $\mathcal{X}$. Let $\mathcal{X}^*$ be the set consisting of the adjoints of the elements of $\mathcal{X}$. By Lemma 3.1, we have

$$\tag{31} (\mathcal{X}')^* = (\mathcal{X}^*)'.$$

The set $\mathcal{X}$ is called involutive if $\mathcal{X}^* = \mathcal{X}$.

Recall [10] that a subalgebra $\mathcal{M}$ of $L(\mathcal{H})$ is called a von Neumann algebra if it is involutive and coincides with its bicommutant $\mathcal{M}''$. By the well-known von Neumann’s bicommutant theorem (see, e.g., [10], Sec. I.3.4, Corollaire 2), an involutive subalgebra $\mathcal{M}$ of $L(\mathcal{H})$ is a von Neumann algebra if and only if it contains the identity operator and is closed in the strong operator topology. It follows from (31) that $\mathcal{X}'$ is an involutive subalgebra of $L(\mathcal{H})$ for any involutive set $\mathcal{X}$ of closed densely defined operators in $\mathcal{H}$. Moreover, it is easy to show (see Lemma 1 in [27]) that $\mathcal{X}'$ is always strongly closed and, therefore, is a von Neumann algebra for involutive $\mathcal{X}$ by the bicommutant theorem.

A closed densely defined operator $T$ in $\mathcal{H}$ is called affiliated with a von Neumann algebra $\mathcal{M}$ if $T$ commutes with every element of $\mathcal{M}'$. If $\mathcal{X}$ is a set of closed densely defined operators in $\mathcal{H}$, then every element of $\mathcal{X}$ is obviously affiliated with the algebra $\mathcal{A}(\mathcal{X}) = (\mathcal{X} \cup \mathcal{X}^*)''$. As shown by the next lemma, $\mathcal{A}(\mathcal{X})$ is actually the smallest von Neumann algebra with this property.

**Lemma 3.2.** Let $\mathcal{X}$ be a set of closed densely defined operators in a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ be a von Neumann algebra in $\mathcal{H}$. Then $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{M}$ if and only if every operator in $\mathcal{A}(\mathcal{X})$ is affiliated with $\mathcal{M}$.

**Proof.** If every element of $\mathcal{A}(\mathcal{X})$ is affiliated with $\mathcal{M}$, then $\mathcal{M}' \subseteq \mathcal{X}'$, whence $\mathcal{M}' \subseteq (\mathcal{X}^*)'$ by (31). It follows that $\mathcal{M}' \subseteq \mathcal{X}' \cap (\mathcal{X}^*)' = (\mathcal{X} \cup \mathcal{X}^*)'$ and, hence, $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{M}'' = \mathcal{M}$. Conversely, if $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{M}$, then $\mathcal{M}' \subseteq (\mathcal{X} \cup \mathcal{X}^*)' \subseteq \mathcal{X}'$ and, hence, every element of $\mathcal{A}(\mathcal{X})$ is affiliated with $\mathcal{M}$. \hfill $\square$

The algebra $\mathcal{A}(\mathcal{X})$ will be called the von Neumann algebra generated by $\mathcal{X}$, and $\mathcal{X}$ will be referred to as a set of generators of $\mathcal{A}(\mathcal{X})$. If $\mathcal{X} \subseteq L(\mathcal{H})$, then $\mathcal{A}(\mathcal{X})$ is just the smallest von Neumann algebra containing $\mathcal{X}$. By Lemma 3.2, a closed densely defined operator $T$ is affiliated with a von Neumann algebra $\mathcal{M}$ if and only if $\mathcal{A}(T) \subseteq \mathcal{M}$ (here and subsequently, we write $\mathcal{A}(T)$ instead of $\mathcal{A}([T])$, where $\{T\}$ is the one-point set containing $T$).

Let $\mathcal{X} \subseteq L(\mathcal{H})$ be an involutive set of pairwise commuting operators. Then $\mathcal{X} \subseteq \mathcal{X}'$ and, hence, $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{X}'$ because $\mathcal{X}'$ is a von Neumann algebra. As $\mathcal{X}' = \mathcal{A}(\mathcal{X})'$, it follows that $\mathcal{A}(\mathcal{X})$ is Abelian.

We say that two sets $\mathcal{X}$ and $\mathcal{Y}$ of closed densely defined operators in $\mathcal{H}$ are equivalent if $\mathcal{A}(\mathcal{X}) = \mathcal{A}(\mathcal{Y})$. We say that $\mathcal{X}$ is equivalent to a closed densely defined operator $T$ if $\mathcal{X}$ is equivalent to the one-element set $\{T\}$.

**Remark 3.3.** If $\mathcal{X} \subseteq L(\mathcal{H})$ is an involutive set, then the smallest strongly closed algebra containing $\mathcal{X}$ is obviously a von Neumann algebra and, hence, coincides with $\mathcal{A}(\mathcal{X})$. The above definition of $\mathcal{A}(\mathcal{X})$ therefore complies with that used in Introduction.

Given a spectral measure $\mathcal{E}$ (see Sec. [A.9]), we denote by $\mathcal{P}_\mathcal{E}$ the set of all operators $\mathcal{E}(A)$, where $A$ is an $\mathcal{E}$-measurable set. As the elements of $\mathcal{P}_\mathcal{E}$ pairwise commute, the algebra $\mathcal{A}(\mathcal{P}_\mathcal{E})$ is Abelian.
Lemma 3.4. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{E}$ be a spectral measure in $\mathcal{H}$. Then the following statements hold:

1. The algebra $\mathcal{A}(\mathcal{P}_\mathcal{E})$ coincides with the set of all $J_\mathcal{E}^f$, where $f$ is an $\mathcal{E}$-measurable $\mathcal{E}$-essentially bounded complex function.

2. A closed densely defined operator $T$ in $\mathcal{H}$ is affiliated with $\mathcal{A}(\mathcal{P}_\mathcal{E})$ if and only if $T = J_\mathcal{E}^g$ for an $\mathcal{E}$-measurable complex function $g$.

Proof. See Lemma 10 in [27].

A family of maps $\{g_\iota\}_{\iota \in I}$ is said to separate points of a set $S$ if $S \subset D_{g_\iota}$ for all $\iota \in I$ and for any two distinct elements $s_1$ and $s_2$ of $S$, there is $\iota \in I$ such that $g_\iota(s_1) \neq g_\iota(s_2)$.

To cover both positive and spectral measures, the next definition is formulated in terms of a general $\mathcal{A}$-valued measure (see Sec. A.2).

Definition 3.5. Let $\mathcal{A}$ be a topological Abelian group and $\nu$ be a $\sigma$-finite $\mathcal{A}$-valued measure. A family $\{g_\iota\}_{\iota \in I}$ of maps is said to be $\nu$-separating if $I$ is countable and $\{g_\iota\}_{\iota \in I}$ separates points of $S_\nu \setminus N$ for some $\nu$-null set $N$.

The next result gives a complete description of systems of generators for $\mathcal{A}(\mathcal{P}_\mathcal{E})$.

Proposition 3.6. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{E}$ be a standard spectral measure in $\mathcal{H}$, and $\mathcal{X}$ be a set of closed densely defined operators in $\mathcal{H}$. Then $\mathcal{A}(\mathcal{X}) = \mathcal{A}(\mathcal{P}_\mathcal{E})$ if and only if the following conditions hold

1. $\mathcal{A}(\mathcal{X}) \subset \mathcal{A}(\mathcal{P}_\mathcal{E})$.

2. There is an $\mathcal{E}$-separating family $\{g_\iota\}_{\iota \in I}$ of $\mathcal{E}$-measurable complex functions such that $J_{g_\iota}^\mathcal{E} \in \mathcal{X}$ for all $\iota \in I$.

Proof. By Lemma 3.2 and statement 2 of Lemma 3.4, condition 1 holds if and only if every element of $\mathcal{X}$ is equal to $J_{g_\iota}^\mathcal{E}$ for some $\mathcal{E}$-measurable complex function $g$. Hence the proposition follows from Theorem 3 in [27].

Example 3.7. Let $T$ be a normal\footnote{See Sec. A.3} operator in a separable Hilbert space and $\mathcal{E}_T$ be its spectral measure (see Sec. A.9). Let $g$ be the identical function on $\mathbb{C}$: $g(z) = z$, $z \in \mathbb{C}$. Then the family containing the single function $g$ separates points of $\mathbb{C}$, and Proposition 3.6 implies that the operator $T$ is equivalent to the set $\mathcal{P}_{\mathcal{E}_T}$ of its spectral projections. Let $\zeta \in \mathbb{C}$ and $h_\zeta$ be the function on $\mathbb{C} \setminus \{\zeta\}$ defined by the relation $h_\zeta(z) = (z - \zeta)^{-1}$. If $\zeta$ is not an eigenvalue of $T$, then $\mathcal{E}_T(\{\zeta\}) = 0$ and the family containing the single function $h_\zeta$ is $\mathcal{E}_T$-separating. It follows from Proposition 3.6 that the operator $(T - \zeta)^{-1}$ is equivalent to $\mathcal{P}_{\mathcal{E}_T}$ (and, hence, to $T$). Let $A \subset \mathbb{C}$ be a set having an accumulation point in $\mathbb{C}$ and let $f_\zeta(z) = e^{\zeta z}$ for $\zeta \in A$ and $z \in \mathbb{C}$. It is easy to show (see example 7 in [27] for details) that the family $\{f_\zeta\}_{\zeta \in A}$ contains a countable subfamily separating the points of $\mathbb{C}$. By Proposition 3.6 we conclude that the set of all operators $e^{\zeta T}$ with $\zeta \in A$ is equivalent to $\mathcal{P}_{\mathcal{E}_T}$.

\footnote{Recall that a closed densely defined linear operator $T$ in a Hilbert space is called normal if the operators $TT^*$ and $T^*T$ have the same domain of definition and coincide thereon. In particular, self-adjoint and unitary operators are normal.}
4. Diagonalizations

Given a positive $\sigma$-finite measure $\nu$, we say that a $\nu$-a.e. defined family $\mathcal{S}$ of Hilbert spaces is $\nu$-nondegenerate if $\mathcal{S}(s) \neq \{0\}$ for $\nu$-a.e. $s$.

**Definition 4.1.** Let $\mathfrak{X}$ be a set of closed densely defined operators in a Hilbert space $\mathcal{H}$. A triple $(\nu, \mathcal{S}, V)$, where $\nu$ is a positive $\sigma$-finite measure, $\mathcal{S}$ is a $\nu$-nondegenerate $\nu$-measurable family of Hilbert spaces, and $V$ is a unitary operator from $\mathcal{H}$ to $\int^\mathcal{S} \mathcal{S}(s) d\nu(s)$, is called a diagonalization for $\mathfrak{X}$ if every $T \in \mathfrak{X}$ is equal to $V^{-1}T\nu,\mathcal{S}V$ for some complex $\nu$-measurable function $g$. A diagonalization $(\nu, \mathcal{S}, V)$ for $\mathfrak{X}$ is called exact if condition (6) holds for any complex $\nu$-measurable $\nu$-essentially bounded function $g$.

The next theorem gives an exactness criterion for diagonalizations, similar to condition (E) discussed in Introduction in the context of direct sum decompositions.

**Theorem 4.2.** Let $\mathcal{H}$ be a separable Hilbert space and $\mathfrak{X}$ be a set of closed densely defined operators in $\mathcal{H}$. A diagonalization $(\nu, \mathcal{S}, V)$ for $\mathfrak{X}$, where $\nu$ is standard, is exact if and only if there is a $\nu$-separating family $\{g_\iota\}_{\iota \in I}$ of $\nu$-measurable complex functions such that

$$V^{-1}T\nu,\mathcal{S}V \in \mathfrak{X}, \quad \iota \in I. \quad (32)$$

Sometimes the analysis of diagonalizations for $\mathfrak{X}$ simplifies if we replace $\mathfrak{X}$ with an equivalent set $\mathfrak{Y}$. The next result shows that passing to equivalent sets is always possible.

**Proposition 4.3.** Let $\mathcal{H}$ be a separable Hilbert space and $\mathfrak{X}$ and $\mathfrak{Y}$ be equivalent sets of closed densely defined operators in $\mathcal{H}$. Then every (exact) diagonalization for $\mathfrak{X}$ is an (exact) diagonalization for $\mathfrak{Y}$.

We say that normal operators $T_1$ and $T_2$ in $\mathcal{H}$ commute if their spectral projections commute (if $T_2 \in L(\mathcal{H})$, then this definition agrees with Definition 1.1, see [13]). If a set $\mathfrak{X}$ of closed densely defined operators admits a diagonalization, then $\mathfrak{X}$ consists of normal pairwise commuting operators because operators of multiplication by functions are normal and commute with each other. The converse statement is provided by the next proposition.

**Proposition 4.4.** For every set of normal pairwise commuting operators in a separable Hilbert space, there exists a diagonalization $(\nu, \mathcal{S}, V)$, where $\nu$ is standard.

Before proceeding with the proofs of the above results, we give some simple examples of diagonalizations.

**Example 4.5.** Let $Z$ be the set of all absolutely continuous square-integrable complex functions on $\mathbb{R}$ having square-integrable derivatives. Let $P$ be the one-dimensional operator of momentum, i.e., the operator in $L_2(\mathbb{R})$ with the domain $D_P = \{|f|_\lambda : f \in Z\}$ satisfying the relation

$$\langle P|f|_\lambda(x)\rangle = -i\overline{f'(x)}, \quad f \in Z,$$

for $\lambda$-a.e. $x$ (as in Sec. 2, $\lambda$ is the Lebesgue measure on $\mathbb{R}$). Let $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be the operator of the Fourier transformation: $$(\mathcal{F}f)(p) = (2\pi)^{-1/2} \int f(x)e^{ipx} \, dx.$$ Then we have $FPFP^{-1} = T_{g^\lambda}$, where $g$ is the identical function on $\mathbb{R}$: $g(p) = p$, $p \in \mathbb{R}$. By Theorem 4.2, we conclude that $(\lambda, \mathcal{L}_{C,\lambda}, \mathcal{F})$ is an exact diagonalization.
for $P$ (here, $I_{C,\lambda}$ is a constant family of Hilbert spaces, see Sec. 4.3 by (119)), we have $L_2(\mathbb{R}) = \int\oplus I_{C,\lambda}(x) \, dx$. In view of Proposition 4.3, $(\lambda, I_{C,\lambda}, F)$ is also an exact diagonalization for the set $\{ e^{iP} \}_{a \in \mathbb{R}}$ of translations in $L_2(\mathbb{R})$.

**Example 4.6.** Let $P$, $F$, and $g$ be as in Example 4.3 and let $H = P^2$ be the Hamiltonian of the one-dimensional free particle. Then $\mathcal{F}H\mathcal{F}^{-1} = T^2_\lambda$ and, hence, $(\lambda, I_{C,\lambda}, F)$ is a diagonalization for $H$. Since the family containing the single function $g^2$ is not $\lambda$-separating, this diagonalization is not exact. Let $\mathbb{R}_+ = (0, \infty)$ and $V: L_2(\mathbb{R}) \to L_2(\mathbb{R}_+, \mathbb{C}^2, \lambda)$ be the unitary operator such that $(Vf)(p) = ((\mathcal{F}f)(p), (\mathcal{F}f)(-p))$, $f \in L_2(\mathbb{R})$, for $\lambda$-a.e. $p \in \mathbb{R}_+$. Then $VHV^{-1}$ is the operator of multiplication by $g^2$ in $L_2(\mathbb{R}_+, \mathbb{C}^2, \lambda)$. In view of (119), it follows that $VHV^{-1} = T^2_{g^2, \lambda}$, where $T_{g^2, \lambda}$ is a constant family of Hilbert spaces, see Sec. B.3; by (119), we have $\mathcal{F}T_{g^2, \lambda} = \mathcal{F}H\mathcal{F}^{-1}$.

We now turn to the proofs of Theorem 4.2 and Propositions 4.3 and 4.4. In the rest of this section, we assume that $\mathcal{H}$ is a separable Hilbert space. For brevity, we say that $(\nu, \mathcal{S}, V)$ is an $\mathfrak{H}$-triple if $\nu$ is a positive $\sigma$-finite measure, $\mathcal{S}$ is a $\nu$-nondegenerate $\nu$-measurable family of Hilbert spaces, and $V$ is a unitary operator from $\mathcal{S}$ to $\int \mathcal{S}(s) \, d\nu(s)$.

Given an $\mathfrak{H}$-triple $t = (\nu, \mathcal{S}, V)$, we define the map $\mathcal{E}^t$ on $\sigma(D_\nu)$ by the relation

$$\mathcal{E}^t(A) = V^{-1} T^t_{\nu, \lambda} V, \quad A \in \sigma(D_\nu),$$

where $\chi_A$ is equal to unity on $A$ and vanishes on $S_\nu \setminus A$.

**Lemma 4.7.** Let $t = (\nu, \mathcal{S}, V)$ be an $\mathfrak{H}$-triple. Then $\mathcal{E}^t$ is a spectral measure such that $\mathcal{E}^t$-measurable and $\mathcal{E}^t$-null sets coincide with $\nu$-measurable and $\nu$-null sets respectively. For any $\nu$-measurable complex function $g$, we have $J^{\mathcal{E}^t}_g = V^{-1} T^t_{\nu, \lambda} V$.

**Proof.** Clearly, $\mathcal{E}^t$ is an $L(\mathfrak{H})$-valued $\sigma$-additive function satisfying condition (a) of Sec. 4.2. Clearly, $\mathcal{N}_t \subseteq \mathcal{N}_{\mathcal{E}^t}$. If $N \in \mathcal{N}_{\mathcal{E}^t}$, then $T^t_{\nu, \lambda} = 0$ and, hence, $\nu(N) = 0$ because $\mathcal{S}(s) \neq 0$ for $\nu$-a.e. $s$. It follows that $N_\nu = N_{\mathcal{E}^t}$ and, therefore, condition (b) of Sec. 4.2 is also fulfilled. Thus, $\mathcal{E}^t$ is a spectral measure having the same null sets as $\nu$. Since $D_{\mathcal{E}^t} = \sigma(D_\nu)$, we have $\sigma(D_{\mathcal{E}^t}) = \sigma(D_\nu)$, i.e., $\nu$-measurable sets coincide with $\mathcal{E}^t$-measurable sets. For any $A \in D_{\mathcal{E}^t}$, and $\Psi \in \mathfrak{H}$, we have $\mathcal{E}^t_\Psi(A) = \int_A ||(V\Psi)(s)||^2 \, d\nu(s)$ (see Sec. A.9). Hence, a $\nu$-measurable function $g$ is $\mathcal{E}^t_\Psi$-integrable if and only if $s \to ||(V\Psi)(s)||^2 f(s)$ is a $\nu$-integrable function, in which case we have

$$\int g(s) \, d\mathcal{E}^t_{\Psi}(s) = \int ||(V\Psi)(s)||^2 g(s) \, d\nu(s).$$

Let $g$ be a $\nu$-measurable function and $\Psi \in \mathfrak{H}$. By (117), we have

$$\Psi \in D_{J^t_g} \iff |g|^2 \text{ is } \mathcal{E}^t_{\Psi}\text{-integrable} \iff$$

$$\iff s \to ||g(s)(V\Psi)(s)||^2 \text{ is } \nu\text{-integrable} \iff V\Psi \in D_{J^t_{|g|^2}}$$

and, therefore, the domains of $J^t_g$ and $V^{-1} T^t_{\nu, \lambda} V$ coincide. Now (113) and (33) imply that

$$\langle \Psi, J^t_g \Psi \rangle = \int g(s) \, d\mathcal{E}^t_{\Psi}(s) = \langle V\Psi, T^t_{\nu, \lambda} V\Psi \rangle$$

for any $\Psi \in D_{J^t_g}$. Hence, $J^t_g = V^{-1} T^t_{\nu, \lambda} V$. \qed
Proposition 4.3 follows immediately from the next lemma.

**Lemma 4.8.** Let \( \mathfrak{X} \) be a set of closed densely defined operators in \( \mathfrak{H} \). An \( \mathfrak{H} \)-triple \( t \) is an (exact) diagonalization for \( \mathfrak{X} \) if and only if \( A(\mathfrak{X}) \subset A(\mathcal{P}_t) \) (resp., \( A(\mathfrak{X}) \subset A(\mathcal{P}_t) \)).

**Proof.** Let \( t \) be an \( \mathfrak{H} \)-triple. Lemma 4.7, statement 2 of Lemma 3.4 and Lemma 3.2 imply that

\[
\text{\( t \) is a diagonalization for } \mathfrak{X} \iff \\
\text{every element of } \mathfrak{X} \text{ is equal to } J^g_t \text{ for some } \mathcal{E}_t\text{-measurable } g \iff \\
\text{every element of } \mathfrak{X} \text{ is affiliated with } A(\mathcal{P}_t) \iff A(\mathfrak{X}) \subset A(\mathcal{P}_t).\]

In view of statement (1) of Lemma 3.4 an \( \mathfrak{H} \)-triple \( t \) is an exact diagonalization for \( \mathfrak{X} \) if and only if it is a diagonalization for \( \mathfrak{X} \) and \( A(\mathfrak{X}) \subset A(\mathcal{P}_t) \). By the above, these two conditions are equivalent to the equality \( A(\mathfrak{X}) = A(\mathcal{P}_t) \). \( \square \)

**Proof of Theorem 5.1.** Let \( t = (\nu, \mathfrak{G}, V) \) be a diagonalization for \( \mathfrak{X} \), where \( \nu \) is standard. By Lemma 4.8 and Proposition 3.6 \( t \) is exact if and only if there is an \( \mathcal{E}_t \)-separating family \( \{g_i\}_{i \in I} \) of \( \mathcal{E}_t \)-measurable complex functions such that \( J^g_t \in \mathfrak{X} \) for all \( i \in I \). Hence, the required statement follows from Lemma 4.7. \( \square \)

**Proof of Proposition 4.4.** Let \( \mathfrak{X} \) be a set of pairwise commuting normal operators in \( \mathfrak{H} \) and \( \mathfrak{H} = \bigcup_{T \in \mathcal{X}} \mathcal{P}_T \), where \( \mathcal{E}_T \) is the spectral measure of \( T \) (see Sec. A.9). Then \( \mathfrak{H} \) is an involutive subset of \( L(\mathfrak{H}) \) whose elements pairwise commute and, therefore, \( \mathfrak{M} = A(\mathfrak{H}) \) is an Abelian von Neumann algebra. Since \( A(T) = A(\mathcal{P}_T) \subset \mathfrak{M} \) for any \( T \in \mathfrak{X} \) (see Example 3.7), every element of \( \mathfrak{X} \) is affiliated with \( \mathfrak{M} \). It follows from Lemma 3.2 that \( A(\mathfrak{X}) \subset \mathfrak{M} \). Hence, the algebra \( A(\mathfrak{X}) \) is Abelian. By Théorème 2 of Sec. II.6.2 in [10], there are a finite Borel measure \( \nu \) on a compact metrizable space, a \( \nu \)-measurable family \( \mathfrak{G} \) of Hilbert spaces, and a unitary operator \( V : \mathfrak{H} \to \int^\oplus \mathfrak{G}(s) \, d\nu(s) \) such that \( A(\mathfrak{X}) \) coincides with the set of all operators \( V^{-1}\mathcal{H}_g V \), where \( g \) is a \( \nu \)-measurable \( \nu \)-essentially bounded complex function. This means that \( (\nu, \mathfrak{G}, V) \) is an exact diagonalization for \( A(\mathfrak{X}) \). Since \( \mathfrak{X} \) is equivalent to \( A(\mathfrak{X}) \), Proposition 4.3 implies that \( (\nu, \mathfrak{G}, V) \) is also an exact diagonalization for \( \mathfrak{X} \). \( \square \)

5. Reduction by symmetries

In this section, we assume that

(A) \( \mathfrak{H} \) is a Hilbert space, \( \mathfrak{X} \) is an involutive subset of \( L(\mathfrak{H}) \), and \( (\nu, \mathfrak{G}, V) \) is an exact diagonalization for \( \mathfrak{X} \).

Given a \( \nu \)-measurable family \( \mathcal{H} \) of closed operators in \( \mathfrak{G} \), we define the operator \( Q_{\mathcal{H}} \) in \( \mathfrak{H} \) by setting

\[
Q_{\mathcal{H}} = V^{-1} \int^\oplus \mathcal{H}(s) \, d\nu(s) \, V.
\]

The structure of closed operators in \( \mathfrak{H} \) commuting with operators in \( \mathfrak{X} \) and of their closed (in particular, self-adjoint) symmetry preserving extensions is described by the next theorem.

**Theorem 5.1.** Let (A) be satisfied. Then the following statements hold:

1. \( \mathcal{H} \) is a closed operator in \( \mathfrak{H} \) commuting with all elements of \( \mathfrak{X} \) if and only if \( \mathcal{H} = Q_{\mathcal{H}} \) for some \( \nu \)-measurable family \( \mathcal{H} \) of closed operators in \( \mathfrak{G} \).
2. Let $\mathcal{H}$ and $\hat{\mathcal{H}}$ be $\nu$-measurable families of closed operators in $\mathfrak{S}$. Then $Q_{\hat{\mathcal{H}}}$ is an extension of $Q_{\mathcal{H}}$ if and only if $\hat{\mathcal{H}}(s)$ is an extension of $\mathcal{H}(s)$ for $\nu$-a.e. $s$.

In particular, $Q_{\hat{\mathcal{H}}} = Q_{\mathcal{H}}$ if and only if $\hat{\mathcal{H}}(s) = \mathcal{H}(s)$ for $\nu$-a.e. $s$.

3. Let $\mathcal{H}$ be a $\nu$-measurable family of closed operators in $\mathfrak{S}$. Then $Q_{\mathcal{H}}$ is self-adjoint if and only if $\hat{\mathcal{H}}(s)$ is self-adjoint for $\nu$-a.e. $s$.

Proof. Let $\mathcal{H}$ be a $\nu$-measurable family of closed operators in $\mathfrak{S}$. Since $\mathfrak{X} \subset L(\mathfrak{S})$, every element of $\mathfrak{X}$ is representable in the form $V^{-1}T^{\nu,\mathfrak{S}}V$, where $g$ is $\nu$-essentially bounded. By Proposition 3.21, $\int q_{\mathcal{H}}(s) \, dv(s)$ is a closed operator commuting with all $T^{\nu,\mathfrak{S}}$ and, therefore, $Q_{\mathcal{H}}$ is a closed operator commuting with all elements of $\mathfrak{X}$. Conversely, let $H$ be a closed operator in $\mathfrak{H}$ commuting with all elements of $\mathfrak{X}$. Let $\mathcal{M}$ denote the subalgebra of $L(\mathfrak{S})$ consisting of all operators commuting with $H$. By Lemma 1 in [27], $\mathcal{M}$ is strongly closed. Since $\mathfrak{X}$ is involutive, $A(\mathfrak{X})$ coincides with the smallest strongly closed subalgebra of $L(\mathfrak{S})$ containing $\mathfrak{X}$. We hence have $A(\mathfrak{X}) \subset \mathcal{M}$, i.e., $H$ commutes with all operators in $A(\mathfrak{X})$. As the diagonalization $(\nu, \mathfrak{S}, V)$ is exact, it follows from (6) that $VHV^{-1}$ commutes with all operators $T^{\nu,\mathfrak{S}}$, where $g$ is a $\nu$-measurable $\nu$-essentially bounded function. By Proposition 3.21, we have $VHV^{-1} = \int q_{\mathcal{H}}(s) \, dv(s)$ for some $\nu$-measurable family $\mathcal{H}$ of closed operators in $\mathfrak{S}$. This means that $H = Q_{\mathcal{H}}$ and statement 1 is proved. Statements 2 and 3 follow immediately from Proposition 3.20 and Corollary 3.23 respectively.

In particular, Statements 1 and 2 of Theorem 5.1 imply the existence and uniqueness (up to $\nu$-equivalence) of decomposition (5) for any closed operator $H$ commuting with all elements of $\mathfrak{X}$.

Corollary 5.2. Let (A) be satisfied and $\mathcal{H}$ be a $\nu$-measurable family of closed operators in $\mathfrak{S}$. Then the closed (resp., self-adjoint) extensions of $Q_{\mathcal{H}}$ commuting with all elements of $\mathfrak{X}$ are precisely the operators $Q_{\hat{\mathcal{H}}}$, where $\hat{\mathcal{H}}$ is a $\nu$-measurable family of operators in $\mathfrak{S}$ such that $\hat{\mathcal{H}}(s)$ is a closed (resp., self-adjoint) extension of $\mathcal{H}(s)$ for $\nu$-a.e. $s$.

In concrete examples, $H$ usually comes as the closure of some non-closed operator $\hat{\mathcal{H}}$ and $\mathcal{H}$ are a $\nu$-a.e. defined family $\mathcal{H}$ of closed operators. In this case, the next proposition may be used to prove that actually equality (5) holds.

Proposition 5.3. Let (A) be satisfied, $\hat{\mathcal{H}}$ be an operator in $\mathfrak{H}$, and $\mathcal{H}$ be a $\nu$-a.e. defined family of closed operators in $\mathfrak{S}$ such that $Q_{\mathcal{H}}$ is an extension of $\hat{\mathcal{H}}$. Suppose $D_{\hat{\mathcal{H}}}$ is taken to itself by all operators in $\mathfrak{X}$ and there is a sequence $\xi_1, \xi_2, \ldots$ of elements of $V(D_{\hat{\mathcal{H}}})$ such that the linear span of $(\xi_j(s), \mathcal{H}(s)\xi_j(s))$ is dense in the graph $G_{\mathcal{H}(s)}$ of $\mathcal{H}(s)$ for $\nu$-a.e. $s$. Then $\mathcal{H}$ is $\nu$-measurable, $\hat{\mathcal{H}}$ is closable, and $Q_{\mathcal{H}} = Q_{\mathcal{H}}$.

Proof. Since $\xi_j \in D_{\mathcal{H}V^{-1}}$ for all $j = 1, 2, \ldots$, we have $\mathcal{H}(s)\xi_j(s) = (V\hat{H}V^{-1})\xi_j(s)$ for $\nu$-a.e. $s$ and, hence, $s \rightarrow \mathcal{H}(s)\xi_j(s)$ are $\nu$-measurable sections of $\mathfrak{S}$. It follows that the family $\mathcal{H}$ is $\nu$-measurable. Statement 1 of Theorem 5.1 implies that $Q_{\mathcal{H}}$ is a closed operator commuting with all elements of $\mathfrak{X}$. Since $Q_{\mathcal{H}}$ is an extension of $\mathcal{H}$ and $D_{\hat{\mathcal{H}}}$ taken to itself by all operators in $\mathfrak{X}$, it follows that $\hat{\mathcal{H}}$ is closable and commutes with all elements of $\mathfrak{X}$. By Lemma 3.1, we conclude that $H = \hat{\mathcal{H}}$. 

also commutes with all operators in \( \mathcal{X} \). By statement 1 of Theorem 5.1 there is a \( \nu \)-measurable family \( \mathcal{H}_0 \) of closed operators in \( \mathcal{S} \) such that \( H = Q_{\mathcal{H}_0} \). As \( Q_{\mathcal{H}} \) is a closed extension of \( \hat{H} \) it is also an extension of \( H \). By statement 2 of Theorem 5.1 it follows that \( \mathcal{H}(s) \) is an extension of \( \mathcal{H}_0(s) \) for \( \nu\text{-a.e. } s \). Since \( V(D_H) \subset V(D_H) \) and \( V(D_H) \) coincides with the domain of \( \int^\dagger \mathcal{H}_0(s)\,d\nu(s) \), we have \( \xi_j(s) \in D_{\mathcal{H}_0(s)} \) and \( \mathcal{H}_0(s)\xi_j(s) = \mathcal{H}(s)\xi_j(s) \) for all \( j \) and \( \nu\text{-a.e. } s \). Thus, the linear span of \( (\xi_j(s), \mathcal{H}(s)\xi_j(s)) \) is contained in \( G_{\mathcal{H}_0(s)} \) and, hence, \( G_{\mathcal{H}_0(s)} \) is dense in \( G_{\mathcal{H}(s)} \) for \( \nu\text{-a.e. } s \). In view of the closedness of \( \mathcal{H}_0(s) \), this implies that \( \mathcal{H}_0(s) = \mathcal{H}(s) \) for \( \nu\text{-a.e. } s \). □

6. ONE-DIMENSIONAL SCHRODINGER OPERATORS

In this section, we consider self-adjoint realizations of one-dimensional Schrödinger operators and their eigenfunction expansions. In Sec. 3 we shall use the results of Sec. 2 to represent self-adjoint extensions of the Aharonov-Bohm Hamiltonian as direct integrals of suitable one-dimensional Schrödinger operators. The proof of Theorem 2.5 given in Sec. 10 is based on combining such a representation with the analysis of the one-dimensional problem given in this section. For the most part, our treatment of self-adjoint extensions is standard and proofs are omitted. We refer the reader to [23, 29, 33] for a detailed exposition of this material. At the same time, our analysis of eigenfunction expansions is based on recently developed (or rather rediscovered, see remark 6.8) approach [15, 16, 19] and is presented in somewhat more detail.

Let \( -\infty \leq a < b \leq \infty \) and \( \lambda_{a,b} \) be the restriction to \((a,b)\) of the Lebesgue measure \( \lambda \) on \( \mathbb{R} \). We denote by \( \mathcal{D} \) the space of all continuously differentiable functions on \((a,b)\) whose derivative is absolutely continuous on \((a,b)\) (i.e., absolutely continuous on every segment \([c,d]\) with \( a < c \leq d < b \)). Given a locally integrable complex function \( q \) on \((a,b)\), we denote by \( l_q \) the linear operator from \( \mathcal{D} \) to the space of complex \( \lambda_{a,b}\)-equivalence classes such that

\[
(34) \quad l_q f(r) = -f''(r) + q(r)f(r)
\]

for \( \lambda\text{-a.e. } r \in (a,b) \). For every \( c \in (a,b) \) and complex numbers \( z_1 \) and \( z_2 \), there is a unique solution \( f \) of the equation \( l_q f = 0 \) such that \( f(c) = z_1 \) and \( f'(c) = z_2 \). This implies that solutions of \( l_q f = 0 \) constitute a two-dimensional subspace of \( \mathcal{D} \). For any functions \( f, g \in \mathcal{D} \), their Wronskian \( W_r(f,g) \) at point \( r \in (a,b) \) is defined by the relation

\[
(35) \quad W_r(f,g) = f(r)g'(r) - f'(r)g(r).
\]

Clearly, \( r \to W_r(f,g) \) is an absolutely continuous function on \((a,b)\). If \( f \) and \( g \) are such that \( r \to W_r(f,g) \) is a constant function on \((a,b)\) (this is the case, in particular, when \( f \) and \( g \) are solutions of \( l_q f = l_q g = 0 \) for some locally integrable complex \( q \) on \((a,b)\)), its value will be denoted by \( W(f,g) \). It follows immediately from (35) that the following identities hold for any \( f_1, f_2, f_3, f_4 \in \mathcal{D} \) and \( r \in (a,b) \):

\[
(36) \quad W_r(f_1, f_2)W_r(f_3, f_4) + W_r(f_1, f_3)W_r(f_4, f_2) + W_r(f_2, f_3)W_r(f_1, f_4) = 0,
\]

\[
(37) \quad W_r(f_1 f_2, f_3 f_4) = f_1(r)f_2(r)W_r(f_3, f_4) + f_3(r)f_4(r)W_r(f_1, f_2) + W_r(f_1, f_3)W_r(f_2, f_4) + W_r(f_1, f_2)W_r(f_3, f_4).
\]

From now on, we assume that \( q \) is a locally integrable real function on \((a,b)\). Let

\[
(38) \quad \mathcal{D}_q = \{ f \in \mathcal{D} : f \text{ and } l_q f \text{ are both square-integrable on } (a,b) \}.
\]
A $\lambda_{a,b}$-measurable complex function $f$ is said to be left (right) square-integrable on $(a, b)$ if $\int_a^b |f(r)|^2 dr < \infty$ (resp., $\int_a^b |f(r)|^2 dx < \infty$) for any $c \in (a, b)$. The subspace of $\mathcal{D}$ consisting of left (right) square-integrable on $(a, b)$ functions $f$ such that $l_q f$ is also left (resp., right) square-integrable on $(a, b)$ will be denoted by $\mathcal{D}_q^l$ (resp., $\mathcal{D}_q^r$). We obviously have $\mathcal{D}_q = \mathcal{D}_q^l \cap \mathcal{D}_q^r$. It follows from (34) by integrating by parts that

$$\int_c^d (l_q f)(r)g(r) - f(r)(l_q g)(r) \, dr = W_d(f, g) - W_e(f, g)$$

for any $f, g \in \mathcal{D}$ such that $l_q f$ and $l_q g$ are locally square-integrable on $(a, b)$ and every $c, d \in (a, b)$. This implies the existence of limits $W_a(f, g) = \lim_{r \downarrow a} W_r(f, g)$ and $W_b(f, g) = \lim_{r \uparrow b} W_r(f, g)$ for $f, g \in \mathcal{D}_q^l$ and $f, g \in \mathcal{D}_q^r$ respectively. Moreover, it follows that

$$\langle l_q f, [g] \rangle - \langle [f], l_q g \rangle = W_b(\bar{f}, g) - W_a(f, g)$$

for any $f, g \in \mathcal{D}_q$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(a, b)$.

For any linear subspace $Z$ of $\mathcal{D}_q$, let $L_q(Z)$ be the linear operator in $L_2(a, b)$ defined by the relations

$$\mathcal{D}_{L_q}(Z) = \{ [f] : f \in Z \},$$

$$L_q(Z)[f] = l_q f, \quad f \in Z.$$

We define the minimal operator $L_q$ by setting

$$L_q = L_q(\mathcal{D}_q^0),$$

where

$$\mathcal{D}_q^0 = \{ f \in \mathcal{D}_q : W_a(f, g) = W_b(f, g) = 0 \text{ for any } g \in \mathcal{D}_q \}.$$  

By (39), the operator $L_q(Z)$ is symmetric if and only if $W_a(\bar{f}, g) = W_b(\bar{f}, g)$ for any $f, g \in Z$. In particular, $L_q$ is a symmetric operator. Moreover, $L_q$ is closed and densely defined and its adjoint $L_q^*$ is given by

$$L_q^* = L_q(\mathcal{D}_q).$$

If $T$ is a symmetric extension of $L_q$, then $L_q^*$ is an extension of $T^*$ and, hence, of $T$. In view of (42), this implies that $T$ is of the form $L_q(Z)$ for some subspace $Z$ of $\mathcal{D}_q$.

**Remark 6.1.** Self-adjoint operators of the form $L_q(Z)$ can be naturally viewed as self-adjoint realizations of the differential expression $-d^2/dr^2 + q$. If $L_q(Z)$ is self-adjoint, then equality (42) and the closedness of $L_q$ imply that $L_q(Z)$ is an extension of $L_q$ because $L_q(\mathcal{D}_q)$ is an extension of $L_q(Z)$. Thus, the self-adjoint realizations of the expression $-d^2/dr^2 + q$ are precisely the self-adjoint extensions of the minimal operator $L_q$.

We say that a linear subspace $X$ of $\mathcal{D}_q^l$ is a left boundary space if

1. $W_a(\bar{f}, g) = 0$ for any $f, g \in X$,
2. for any $g \in \mathcal{D}_q^l$ such that $W_a(f, g) = 0$ for all $f \in X$, we have $g \in X$.

Throughout this and the next section, all equivalence classes are taken with respect to $\lambda_{a,b}$. We shall drop the subscript and write $[f]$ instead of $[f]_{\lambda_{a,b}}$.
Replacing $D_q^l$ with $D_q^r$ and $a$ with $b$, we obtain the definition of a right boundary space.

Given a left boundary space $X$ and a right boundary space $Y$, the operator $L_q(X \cap Y)$ is a self-adjoint extension of $L_q$. The operators of this form are called self-adjoint extensions of $L_q$ with separated boundary conditions.

If $W_a(f,g) = 0$ for any $f,g \in D_q^l$, then $q$ is said to be in the limit point case (l.p.c.) at $a$. Otherwise $q$ is said to be in the limit circle case (l.c.c.) at $a$. Similarly, $q$ is said to be in the l.p.c. at $b$ if $W_b(f,g) = 0$ for any $f,g \in D_q^r$ and to be in l.c.c. at $b$ otherwise. Clearly, $q$ is in l.p.c. at $a$ (at $b$) if and only if $D_q^r$ is a left (resp., $D_q^r$ is a right) boundary space. According to the well-known Weyl alternative, $q$ is in l.c.c. at $a$ if and only if there is $z \in \mathbb{C}$ such that all solutions of $l_{q-z}f = 0$ are left square-integrable on $(a,b)$. Moreover, if all solutions of $l_{q-z}f = 0$ are left square-integrable for some $z \in \mathbb{C}$, then the same is true for all $z \in \mathbb{C}$.

**Remark 6.2.** The concept of left and right boundary spaces introduced above is a formalization of the notion of a real boundary condition. It is useful for formulating and proving some results in general terms, without explicitly mentioning limit point and limit circle cases. An example of such a formulation is provided by Proposition 6.3 below.

**Remark 6.3.** Let $f$ and $g$ be linear independent solutions of $l_{q-z}f = l_{q-z}g = 0$, where $\text{Im } z \neq 0$. Suppose $f$ satisfies a real boundary condition at $a$ (i.e., belongs to some left boundary space). Let $A$ denote the set of all $\zeta \in \mathbb{C}$ such that $g + \zeta f$ belongs to some right boundary space. Then $A$ is either a one-point set or a circle depending on whether $q$ is in l.p.c. or l.c.c. at $b$. Moreover, $A$ is the limit of the circles $A_\epsilon$ obtained by replacing $b$ with a regular endpoint $c \in (a,b)$ in the definition of $A$. Such a limiting procedure was originally used by Weyl [34] to distinguish between l.p.c. and l.c.c.

If $q$ is in l.p.c. at both $a$ and $b$, then (42) implies that $L_q^*$ is symmetric and, therefore, $L_q$ is self-adjoint.

Suppose now that $q$ is in l.c.c. at $a$. Then

$$D^l_{q,f} = \{g \in D_q^l : W_a(f,g) = 0\}$$

is a left boundary space for any nontrivial real solution $f$ of $l_qf = 0$. Moreover, every left boundary space coincides with $D^l_{q,f}$ for some nontrivial real $f$ satisfying $l_qf = 0$. Let $f_1, f_2 \in \mathbb{D}$ be linearly independent functions such that $l_qf_1 = l_qf_2 = 0$. As $q$ is in l.c.c. at $a$, $f_1$ and $f_2$ are left square-integrable on $(a,b)$ and, therefore, the function

$$\delta_q^g(r) = \frac{1}{W(f_1,f_2)} \left[ f_1(r) \int_a^r (l_qg)(\rho)f_2(\rho)\,d\rho - f_2(r) \int_a^r (l_qg)(\rho)f_1(\rho)\,d\rho \right]$$

is well-defined and belongs to $D$ for any $g \in D_q$. It is straightforward to check that $\delta_q^g$ does not depend on the choice of the solutions $f_1$ and $f_2$ and $l_q\delta_q^g = l_qg$. Hence, the function

$$\gamma_q^g = g - \delta_q^g$$

satisfies the equation

$$l_q\gamma_q^g = 0.$$
If \( q \) is in l.c.c. at \( a \) and l.p.c. at \( b \), then every self-adjoint extension of \( L_q \) has separated boundary conditions and, hence, coincides with the operator
\[
(47) \quad L^f_q = L_q(D^f_q \cap D^r_q)
\]
for some nontrivial real solution \( f \) of \( l_q f = 0 \). Note that \( L^f_q \) determines \( f \) uniquely up to a nonzero real coefficient. In this case, \( L_q \) has deficiency indices \((1, 1)\). This implies, in particular, that the orthogonal complement \( G_{L^f_q} \cap G_{L_q} \) of the graph \( G_{L_q} \) of \( L_q \) in the graph \( G_{L^f_q} \) of \( L^f_q \) is one-dimensional.

**Lemma 6.4.** Suppose \( q \) is in l.c.c. at \( a \) and in l.p.c. at \( b \). Let \( T \) be a self-adjoint extension of \( L_q \) and \( g \) be a real function in \( L_{q} \). Then \( \gamma_q^g \) is a real nontrivial solution of \((46)\) and \( T = L_{q}^{\gamma_q^g} \).

**Proof.** The reality of \( \gamma_q^g \) follows from \((44)\) and \((45)\) because we can choose \( f_1 \) and \( f_2 \) in \((44)\) to be real. It follows easily from \((44)\) that
\[
W_r(\delta_q^g(h \gamma_q), h) = \frac{1}{W(f_1, f_2)} \left[ W_r(f_1, h) \int_a^r (l_q g)(\rho)f_2(\rho)\, d\rho - W_r(f_2, h) \int_a^r (l_q g)(\rho)f_1(\rho)\, d\rho \right], \quad r \in (a, b),
\]
for any \( h \in D \), where \( f_1, f_2 \) are linearly independent solutions of \( l_q f_1, f_2 = 0 \). This implies that
\[
(48) \quad W_a(\delta_q^g, h) = 0
\]
for any \( h \in D^a_q \) and, therefore, \( W_a(g, h) = W_a(\gamma_q^g, h) \). If \( \gamma_q^g \) were trivial, we would have \( W_a(g, h) = W_b(g, h) = 0 \) for any \( h \in D_q \) (recall that \( q \) is in l.p.c. at \( b \)) and, hence, \([g] \in D_{L_q}^l \) by \((44)\) and \((40)\). Thus, \( \gamma_q^g \) is nontrivial. Let \( f \) be a nontrivial real solution of \( l_q f = 0 \) such that \( T = L_q^f \). Then we have \( W_a(f, g) = 0 \), and it follows from \((45)\) and \((48)\) that \( W(f, \gamma_q^g) = 0 \). This means that \( \gamma_q^g = Cf \) for some real \( C \neq 0 \) and, therefore, \( T = L_q^{\gamma_q^g} \). \( \square \)

If \( q \) is locally square-integrable on \((a, b)\), then \( D^a_q \supset C_0^\infty(a, b) \), where \( C_0^\infty(a, b) \) is the space of smooth functions on \((a, b)\) with compact support. In view of \((40)\), this implies that \( ([f], l_q f) \in G_{L_q} \) for any \( f \in C_0^\infty(a, b) \).

**Lemma 6.5.** There exists a countable set \( A \subset C_0^\infty(a, b) \) such that the elements \( ([f], l_q f) \) with \( f \in A \) are dense in the graph \( G_{L_q} \) of \( L_q \) for any locally square-integrable real function \( q \) on \((a, b)\).

**Proof.** Given \( a < \alpha \leq \beta < b \), we denote by \( Z_{\alpha, \beta} \) the linear subspace of the space \( C_0^\infty(a, b) \) consisting of all its elements vanishing outside of \([\alpha, \beta]\). We make \( Z_{\alpha, \beta} \) a normed space by setting
\[
\|f\| = \sup_{\alpha \leq r \leq \beta} (|f(r)| + |f''(r)|), \quad f \in Z_{\alpha, \beta}.
\]
Let \( C[\alpha, \beta] \) be the space of all continuous functions on the segment \([\alpha, \beta]\). Then \( f \to (f, f'') \) is an isometric embedding of \( Z_{\alpha, \beta} \) into \( C[\alpha, \beta]^2 \) if the latter space is endowed with the norm
\[
\|(f, g)\| = \sup_{\alpha \leq r \leq \beta} (|f(r)| + |g(r)|), \quad f, g \in C[\alpha, \beta].
\]
Note that the space $C[\alpha, \beta]^2$ with this norm is separable because $C[\alpha, \beta]$ endowed with its ordinary supremum norm is separable. Since every subspace of a separable metric space is separable, it follows that $Z_{\alpha, \beta}$ is separable. We now pick sequences $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ such that $a < \alpha_j \leq \beta_j < b$ for all $j$ and $\alpha_j \to a$ and $\beta_j \to b$ as $j \to \infty$. For each $j = 1, 2, \ldots$, we choose a countable dense subset $A_j$ of $Z_{\alpha_j, \beta_j}$, and define the set $A$ by the relation $A = \bigcup_{j=1}^{\infty} A_j$. Let $q$ be a locally square-integrable real function on $(a, b)$. Then $L_q$ is a closed extension of the operator $L_{q,0} = L_q(C^\infty_0(a, b))$. It is easy to verify by a direct computation that $L_{q,0} = L_q(D_q)$. In view of (42), it follows that $L_q$ is the closure of $L_{q,0}$. This means that elements of the form $(|f|, l_q f)$ with $f \in C^\infty_0(a, b)$ are dense in the graph of $L_q$, and it suffices to prove that every such element can be approximated by $(|g|, l_q g)$ with $g \in A$. For this, we choose a $j$ such that $f \in Z_{\alpha_j, \beta_j}$, and find a sequence $g_1, g_2, \ldots$ of elements of $A_j$ converging to $f$ in $Z_{\alpha_j, \beta_j}$. As $l_q$ obviously induces a continuous map from $Z_{\alpha_j, \beta_j}$ to $L_2(a, b)$ and $f \to [f]$ is a continuous embedding of $Z_{\alpha_j, \beta_j}$ into $L_2(a, b)$, we conclude that $|g_k| \to |f|$ and $l_q g_k \to l_q f$ in $L_2(a, b)$ as $k \to \infty$. \hfill \square

We now consider the eigenfunction expansions associated with $L_q$. Let $O \subset \mathbb{C}$ be an open set. We say that a map $u : O \to D$ is analytic if the maps $z \to u(z)r$ and $z \to \partial_r u(z)r$ are analytic in $O$ for any $r \in (a, b)$. Given a left or right boundary space $X$, there always exists an analytic map $v : \mathbb{C}_+ \to D$ such that $v(z)$ is a nontrivial solution of $l_q - z v(z) = 0$ belonging to $X$ for all $z \in \mathbb{C}_+$ (see, e.g., [29], Lemma 9.8). An analytic map $u : \mathbb{C} \to D$ is said to be real-entire if $u(E)$ is real for real $E$. Let $L^\alpha_2(a, b)$ denote the subspace of $L_2(a, b)$ consisting of all its elements vanishing $\lambda$-a.e. in some neighborhoods of $a$ and $b$. The next proposition gives a way of constructing eigenfunction expansions for self-adjoint extensions of $L_q$ with separated boundary conditions.

**Proposition 6.6.** Let $X$ and $Y$ be, respectively, left and right boundary spaces, $T = L_q(X \cap Y)$, and $u : \mathbb{C} \to D$ be a real-entire map such that $u(z)$ is a nontrivial solution of $l_q - z u(z) = 0$ belonging to $X$ for all $z \in \mathbb{C}$. Then there exists a unique pair $(\sigma, U)$, where $\sigma$ is a positive Borel measure on $\mathbb{R}$ and $U$ is a unitary operator from $L_2(a, b)$ to $L_2(\mathbb{R}, \sigma)$, such that

$$ (U \psi)(E) = \int_a^b u(E|r)\psi(r) \, dr, \quad \psi \in L^\alpha_2(a, b), $$

for $\sigma$-a.e. $E$ and

$$ UF(T)U^{-1} = \mathcal{T}^\sigma $$

for any Borel complex function $F$ on $\mathbb{R}$. The domain $D_\sigma$ of $\sigma$ contains all compact subsets of $\mathbb{R}$. Let $O \subset \mathbb{C}$ be an open set and $\tilde{u} : O \to D$ be an analytic map such that $l_q - z \tilde{u}(z) = 0$ and $W(u(z), \tilde{u}(z)) \neq 0$ for all $z \in O$ and $\tilde{u}(E)$ is real for $E \in O \cap \mathbb{R}$. Then for any continuous function $\varphi$ on $O \cap \mathbb{R}$ with compact support, we have

$$ \int \varphi(E) \, d\sigma(E) = \lim_{\eta \downarrow 0} \int \Phi_\eta(E + i\eta) \varphi(E) \, dE, $$

where $\Phi_\eta$ is the analytic function in $O \cap \mathbb{C}_+$ defined by the relation

$$ \Phi_\eta(z) = \frac{1}{\pi W(v(z), \tilde{u}(z))} \frac{W(v(z), u(z))}{W(u(z), \tilde{u}(z))}. $$

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8We denote by $\mathbb{C}_+$ the open upper half-plane of the complex plane: $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im} z > 0\}$. 

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and \( v: \mathbb{C}_+ \to \mathcal{D} \) is an analytic map such that \( v(z) \) is a nontrivial solution of \( l_{a-z}v(z) = 0 \) belonging to \( Y \) for all \( z \in \mathbb{C}_+ \) (clearly, \( \Phi_{\phi} \) does not depend on the choice of \( v \)).

**Proof.** Given \( \psi \in L^2_0(a,b) \), let the entire function \( \hat{\psi} \) be defined by the relation
\[
\hat{\psi}(z) = \int_a^b u(z|\tau)\psi(\tau)\,d\tau, \quad z \in \mathbb{C}.
\]

Let \( Q_0 \) be the set of all Borel sets \( A \subset \mathbb{R} \) with the following property: there is \( \psi_A \in L^2_0(a,b) \) such that \( \hat{\psi}_A \) is nonzero on \( A \) and \( 1/\hat{\psi}_A \) is bounded on \( A \). Let \( Q \) be the set of all finite unions of elements of \( Q_0 \). Every \( E \in \mathbb{R} \) has a neighborhood belonging to \( Q_0 \) (otherwise we would have \( \hat{\psi}(E) = 0 \) for all \( \psi \in L^2_0(a,b) \) and, hence, \( u(E) = 0 \)). This implies that every bounded Borel subset of \( \mathbb{R} \) belongs to \( Q \) (note that a Borel subset of an element of \( Q_0 \) is also an element of \( Q_0 \)). Given \( A \in Q_0 \), we define the bounded Borel function \( F_A \) on \( \mathbb{R} \) by setting \( F_A(E) = |\hat{\psi}(E)|^2 \) for \( E \in A \) and \( F_A(E) = 0 \) for \( E \in \mathbb{R} \setminus A \).

Suppose \( \sigma \) and \( U \) satisfy the conditions of the proposition. If \( A \in Q_0 \), then \( |\hat{\psi}_A|^2 F_A \) is a \( \sigma \)-integrable function that is equal to unity on \( A \) and vanishes on \( \mathbb{R} \setminus A \). It follows that \( Q_0 \subset D_\sigma \) and, hence, \( Q \subset D_\sigma \). This implies, in particular, that all compact subsets of \( \mathbb{R} \) belong to \( D_\sigma \). In view of (49), (50), and the unitarity of \( U \), we have
\[
\sigma(A) = \int |\hat{\psi}_A(E)|^2 F_A(E)\,d\sigma(E) = \langle U\psi_A, T_{F_A} U\psi_A \rangle = \langle \psi_A, F_A(T)\psi_A \rangle, \quad A \in Q_0.
\]

We thus see that \( \sigma(A) \) for \( A \in Q_0 \) is uniquely determined by (49) and (50). This implies that \( \sigma \) is also uniquely determined on \( Q \) because every set in \( Q \) is a finite disjoint union of elements of \( Q_0 \). Since \( Q \) is a \( \sigma \)-ring, the uniqueness of \( \sigma \) follows from Lemma 6.3. As \( L^2_0(a,b) \) is dense in \( L^2(a,b) \), \( U \) is uniquely determined by (49).

The existence of \( \sigma \) and \( U \) is guaranteed by Theorem 3.4 in [19]. Moreover, as shown in [19], there exists a real-entire \( \theta: \mathbb{C} \to \mathcal{D} \) such that \( \theta(z) \) is a solution of \( l_{a-z}\theta(z) = 0 \) satisfying \( W(\theta(z), u(z)) = 1 \) for all \( z \in \mathbb{C} \) and (51) holds for \( \hat{u} \) replaced with \( \theta \). Substituting \( f_1 = u(z) \), \( f_2 = v(z) \), \( f_3 = \hat{u}(z) \), and \( f_4 = \theta(z) \) in (38) and dividing the result by \( \pi W(u(z), v(z))W(u(z), \theta(z))W(u(z), \hat{u}(z)) \) yields
\[
\Phi_{\phi}(z) = \Phi_{\theta}(z) + \frac{W(\hat{u}(z), \theta(z))}{\pi W(u(z), \theta(z))W(u(z), \hat{u}(z))} = \Phi_{\theta}(z) + \frac{1}{\pi W(u(z), u(z))} W(\hat{u}(z), \theta(z)),
\]
for any \( z \in O \cap \mathbb{C}_+ \). Hence (51) follows because the last term in the right-hand side is analytic in \( O \) and real on \( O \cap \mathbb{R} \).

**Corollary 6.7.** In the notation of Proposition 6.6, for every \( f \in L^2_0(\mathbb{R}, \sigma) \), we have
\[
(U^{-1} f)(r) = \int u(E|\tau) f(E)\,d\sigma(E)
\]
for \( \lambda \cdot \text{a.e. } r \in (a,b) \). If \( \sigma(\{E\}) \neq 0 \) for some \( E \in \mathbb{R} \), then \( [u(E)] \) is an eigenfunction of \( T \).

**Proof.** Given \( f \in L^2_0(\mathbb{R}, \sigma) \) and \( r \in (a,b) \), let \( \tilde{f}(r) \) denote the right-hand side of (53). In view of unitarity of \( U \), we have
\[
\langle \psi, U^{-1} f \rangle = \langle U\psi, f \rangle = \int d\sigma(E) f(E) \int_a^b \overline{\psi(r)} u(E|\tau) u(E)\,d\tau = \int_a^b \overline{\psi(r)} \tilde{f}(r)\,dr
\]
for any \( \psi \in L^2(\mathbb{R}, m) \), whence (53) follows. In particular, we have \( U^{-1}[\chi_E] \sigma = \sigma([E]) [u(E)] \), where \( \chi_E \) is the characteristic function of the one-point set \( \{ E \} \). By (50), this implies that \([u(E)]\) is an eigenfunction of \( T \) if \( \sigma([E]) \neq 0 \). □

The measure \( \sigma \) satisfying the conditions of the theorem will be called the spectral density corresponding to the map \( u \). Being a Radon measure, \( \sigma \) is uniquely determined by the values of \( \int \varphi(E) d\sigma(E) \), where \( \varphi \) is a continuous function on \( \mathbb{R} \) with compact support (see Sec. A.8). Hence, formulas (51) and (52) are, in principle, sufficient for computing the spectral density.

Remark 6.8. Proposition 6.6 is only a slight modification of Theorem 3.4 in [19]. The main difference is that we allow \( \tilde{u} \) to be defined on an open subset of \( \mathbb{C} \) rather than on the entire complex plane. This possibility will be used in the proof of Proposition 6.9 below. For \( \tilde{u} \) globally defined, Proposition 6.6 can also be easily derived using Kodaira’s general approach [18] based on matrix-valued measures. Indeed, if we set \( s_1(z) = \tilde{u}(z)/W(u(z), \tilde{u}(z)) \) and \( s_2(z) = u(z) \) for \( z \in \mathbb{C} \), then the only nonreal entry \( M_{22}(z) \) of the characteristic matrix \( M \) defined by formula (1.13) in [18] is equal to \( \pi \Phi(z) \) and, therefore, Proposition 6.6 essentially coincides with Theorem 1.3 of [18] in this case. A simple direct proof given in [19] follows the lines of the Titchmarsh \( m \)-coefficient theory [90] and does not involve matrix-valued measures. It essentially relies on the technique developed in [15], where the case of l.p.c. at both endpoints was considered (a treatment in the same spirit for the case of l.c.c. at one of the endpoints can be found in [6]). A similar approach to finding the spectral density was also proposed in [10].

We now apply the above general theory to the potential

\[
q_\kappa(r) = \frac{\kappa^2 - 1/4}{r^2}
\]

on the positive semi-axis \( \mathbb{R}_+ = (0, \infty) \), where \( \kappa \) is a real parameter. We set

\[
h_\kappa = L_{q_\kappa}, \quad \kappa \in \mathbb{R}.
\]

Our aim is to obtain explicit description of all self-adjoint extensions of \( h_\kappa \) and their eigenfunction expansions.

The equation \( l_{q_\kappa} f = 0 \) has linearly independent solutions \( r^{1/2 \pm \kappa} \) for \( \kappa \neq 0 \) and \( r^{1/2} \) and \( r^{1/2} \ln r \) for \( \kappa = 0 \). Hence, \( q_\kappa \) is in l.p.c. at both 0 and \( \infty \) for \( |\kappa| \geq 1 \), while for \( |\kappa| < 1 \), \( q_\kappa \) is in l.p.c. at \( \infty \) and in l.c.c. at 0.

For any \( \kappa \in \mathbb{R} \), let the real-entire function \( X_\kappa \) and the real-entire map \( u_\kappa : \mathbb{C} \rightarrow \mathcal{D} \) be defined by (23) and (22) respectively. For \( z \neq 0 \), we have \( X_\kappa(z) = z^{-\kappa/2} J_\kappa(z^{1/2}) \), where \( J_\kappa \) is the Bessel function of the first kind of order \( \kappa \). Hence, \( u_\kappa(z|r) = z^{-\kappa/2} r^{1/2} J_\kappa(rz^{1/2}) \) for \( z \neq 0 \) and every \( r \in \mathbb{R}_+ \), and the equalities

\[
l_{q_\kappa - z} u_\kappa(z) = 0
\]

hold for \( z \neq 0 \) because \( J_\kappa \) is a solution of the Bessel equation. By continuity, these equalities also hold for \( z = 0 \).

Further, for any \( \kappa \in \mathbb{R} \), we define the analytic map \( v_\kappa : \mathbb{C}_+ \rightarrow \mathcal{D} \) by the relation

\[
v_\kappa(z|r) = r^{1/2} H_\kappa^{(1)}(rz^{1/2}), \quad r \in \mathbb{R}_+, \quad z \in \mathbb{C}_+,
\]

where \( H_\kappa^{(1)} \) is the first Hankel function of order \( \kappa \). Since \( H_\kappa^{(1)} \) is a solution of the Bessel equation, we have \( l_{q_\kappa - z} v_\kappa(z) = 0 \) for any \( z \in \mathbb{C}_+ \). It follows from the
relation $H^{(1)}_\kappa = e^{i\pi\kappa} H^{(1)}_{\kappa/2}$ (12), Sec. 7.2.1, formula (9)) that

$$v^{-\kappa}(z) = e^{i\pi\kappa}u^\kappa(z), \quad \kappa \in \mathbb{R}, \ z \in \mathbb{C}_+.$$  \hspace{1cm} (58)

It is known (12), Sec. 7.13.1, formula (1)) that

$$H^{(1)}_{\kappa/2}(z) \sim \left(\frac{\pi z}{2}\right)^{-1/2} e^{i(z-\pi\kappa/2-\pi/4)}, \quad -\pi < \arg z < 2\pi,$$

for $z \to \infty$. It follows that $v^\kappa(z)$ is right square-integrable for all $\kappa \in \mathbb{R}$ and $z \in \mathbb{C}_+$. Using the expressions (12), Sec. 7.11, formulas (27) and (29)) for the Wronskians of Bessel functions,

$$W_z(J_{\kappa}, J_{-\kappa}) = -\frac{2}{\pi z} \sin \pi\kappa, \quad W_z(J_{\kappa}, H^{(1)}_{\kappa/2}) = \frac{2i}{\pi z}$$

and taking (58) into account, we find that

$$W(u^\kappa(z), u^{-\kappa}(z)) = -\frac{2}{\pi} \sin \pi\kappa$$

for all $\kappa \in \mathbb{R}$ and $z \in \mathbb{C}$ and

$$W(v^\kappa(z), u^\kappa(z)) = -\frac{2i}{\pi} z^{-\kappa/2}, \quad W(v^\kappa(z), u^{-\kappa}(z)) = -\frac{2i}{\pi} z^{\kappa/2} e^{-i\pi\kappa}$$

for any $\kappa \in \mathbb{R}$ and $z \in \mathbb{C}_+$.

As shown above, $q_\kappa$ is in l.p.c. at both 0 and $\infty$ for $|\kappa| \geq 1$. Hence, $h_\kappa$ is self-adjoint. The next proposition gives the corresponding eigenfunction expansion.

**Proposition 6.9.** Let $|\kappa| \geq 1$ and the positive measure $\mathcal{V}_\kappa$ on $\mathbb{R}$ be defined by (10). Then there is a unique unitary operator $U_\kappa : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}, \mathcal{V}_\kappa)$ such that

$$\langle U_\kappa \psi \rangle(E) = \int_0^\infty w^{\kappa}(E|r)\psi(r) dr, \quad \psi \in L_2^0(\mathbb{R}_+),$$

for $\mathcal{V}_\kappa$-a.e. $E$. For any Borel complex function $F$ on $\mathbb{R}$, we have

$$U_\kappa F(h_\kappa U_\kappa^{-1} = T^\mathcal{V}_\kappa F.$$  \hspace{1cm} (62)

**Proof.** As $h_\kappa = h_{-\kappa}$ and $\mathcal{V}_\kappa = \mathcal{V}_{-\kappa}$ for any $\kappa \in \mathbb{R}$, it suffices to consider the case $\kappa \geq 1$. Since $L_2^0(\mathbb{R}_+)$ is dense in $L_2(\mathbb{R}_+)$, $U_\kappa$ is uniquely determined by (12). As $q_\kappa$ is in l.p.c. at both ends, $X = D^l_{q_\kappa}$ and $Y = D^r_{q_\kappa}$ are left and right boundary spaces respectively and $h_\kappa = L_{q_\kappa}(X \cap Y)$. By (22), $u^\kappa(z)$ is left square-integrable and, hence, $u^\kappa(z) \in X$ for all $z \in \mathbb{C}$. The statement will therefore follow from Proposition 6.6 if we show that $\mathcal{V}_\kappa$ coincides with the spectral density $\sigma_\kappa$ corresponding to $u^\kappa$. Let the analytic map $\tilde{u}_1 : \mathbb{C} \setminus [0, \infty) \to \mathcal{D}$ satisfying $l_{q_\kappa} - z \tilde{u}_1(z) = 0$ be defined by the relation $\tilde{u}_1(z|r) = i\pi e^{i\pi\kappa/2} r^{1/2} H^{(1)}_{\kappa/2}(r z^{1/2})$ for any $r \in \mathbb{R}_+$. For $E < 0$, we have $\tilde{u}_1(E) = 2r^{1/2} K_\kappa(r \sqrt{|E|})$, where $K_\kappa$ is the modified Bessel function of the second kind (12), Sec. 7.2.2, formula (15)). Hence, $\tilde{u}_1(E)$ is real for $E < 0$. Since $\tilde{u}_1(z) = i\pi e^{i\pi\kappa/2} v^\kappa(z)$ for $z \in \mathbb{C}_+$, it follows from (61) and the uniqueness of analytic continuation that $W(\tilde{u}_1(z), u^\kappa(z)) = 2e^{i\pi\kappa/2} z^{-\kappa/2} \neq 0$ for all $z \in \mathbb{C} \setminus [0, \infty)$. As $W(\tilde{u}_1(z), v^\kappa(z)) = 0$ for $z \in \mathbb{C}_+$, formulas (61) and (62) applied to $u = u^\kappa$, $v = v^\kappa$, and $\tilde{u} = \tilde{u}_1$ ensure that $\sigma_\kappa$ is identically zero on $(0, \infty)$. Let the analytic map $\tilde{u}_2 : \mathbb{C} \setminus (-\infty, 0] \to \mathcal{D}$ be given by $\tilde{u}_2(z|r) = r^{1/2} Y_\kappa(r z^{1/2})$, where $Y_\kappa$ is the Bessel function of the second kind. We have $l_{q_\kappa} - z \tilde{u}_2(z) = 0$ for any $z \in \mathbb{C} \setminus (-\infty, 0]$.
because $Y_\alpha$ satisfies the Bessel equation. Since $Y_\alpha$ is real for positive real arguments, $\tilde{u}_2(E)$ is real for $E > 0$. As $H^{(1)}_\alpha = J_\alpha + iY_\alpha$, it follows from (59) that

$$W_z(H^{(1)}_\alpha, Y_\alpha) = W_z(H^{(1)}_\alpha, Y_\alpha) = \frac{2}{\pi z},$$

whence $W(u^\kappa(z), \tilde{u}_2(z)) = 2z^{-\kappa/2}/\pi \neq 0$ for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $W(v^\kappa(z), \tilde{u}_2(z)) = 2/\pi$ for $z \in \mathbb{C}_+$. In view of (61), this implies that

$$\frac{1}{\pi} W(u^\kappa(z), \tilde{u}_2(z)) W(v^\kappa(z), \tilde{u}_2(z)) = \frac{iz^\kappa}{2}, \quad z \in \mathbb{C}_+.$$

Hence, formulas (51) and (52) applied to $u = u^\kappa$, $v = v^\kappa$, and $\tilde{u} = \tilde{u}_2$ ensure that $\sigma_\alpha$ coincides with $Y_\alpha$ on $(0, \infty)$. It remains to note that $\sigma_\alpha(\{0\}) = 0$ because otherwise $[u^\kappa(0)]$, which is not square-integrable, would be an eigenfunction of $h_\alpha$ by Corollary 6.4.

**Remark 6.10.** Up to a change of variables, the operator $U_\alpha$ coincides with the well-known Hankel transformation. Its treatment given above is similar to that in [15, 16, 19]. It should be noted that the second solution $\tilde{u}$ used for calculating the spectral density is required to be globally defined in [15, 19]. This makes it necessary to distinguish between integer and noninteger values of $\kappa$. Using locally defined $\tilde{u}$ allows us to treat all values of $\kappa$ in a uniform way.

If $-1 < \kappa < 1$, then $q_\alpha$ is in l.c.c. at $0$ and in l.p.c. at $\infty$ and, hence, $h_\alpha$ has different self-adjoint extensions. Let the real-entire map $w^\kappa : \mathbb{C} \to D$ be defined by the relation

$$w^\kappa(z) = u^\kappa_{\pi/2+\vartheta}(z), \quad z \in \mathbb{C},$$

where $u^\kappa_\vartheta(z)$ is given by (24) and (25) and $\vartheta_\kappa$ is defined by (12). It follows immediately from (24) and (25) that

$$u^\kappa_\vartheta(z) = w^\kappa(z) \cos(\vartheta - \vartheta_\kappa) + w^\kappa(z) \sin(\vartheta - \vartheta_\kappa)$$

for all $z \in \mathbb{C}$, $\vartheta \in \mathbb{R}$, and $-1 < \kappa < 1$. Using the equality (12, Sec. 7.2.4, formula (33))

$$\pi Y_0(z) = 2 \left( \gamma + \ln \frac{z}{2} \right) J_0(z) - 2 Y(z^2), \quad z \neq 0,$$

where $Y_0$ is the Bessel function of the second kind of order 0, we derive from (25) and (63) that

$$w^0(z|r) = \sqrt{r} Y_0(rz^{1/2}) - \frac{1}{r} u^0(z|r) \ln z$$

for any $r \in \mathbb{R}_+$ and $z \neq 0$. As $Y_0$ is a solution of the Bessel equation, it follows that $l_{q_\alpha - z} w^0(z) = 0$ for $z \neq 0$. By continuity, this equality also holds for $z = 0$. In view of (24) and (56), it follows that

$$l_{q_\alpha - z} w^\kappa(z) = 0$$

for every $z \in \mathbb{C}$ and $-1 < \kappa < 1$. By (24), (60), and (63), $u^\kappa(0)$ and $w^\kappa(0)$ are linearly independent for $0 < |\kappa| < 1$. Since $w^\kappa(0|r) = \sqrt{r}$ and $w^0(0|r) = 2(\ln r/2 + \gamma)\sqrt{r}/\pi$ for all $r > 0$, the functions $u^\kappa(0)$ and $w^\kappa(0)$ are also linearly independent. In view of (64), we conclude that $u^\kappa_\vartheta(0)$ is a nontrivial real element of $D$ satisfying $l_{q_\alpha} u^\kappa_\vartheta(0) = 0$ for any $-1 < \kappa < 1$ and $\vartheta \in \mathbb{R}$, and every real solution
of \( l_\rho f = 0 \) coincides, up to a real nonzero coefficient, with \( u_\vartheta' (0) \) for some \( \vartheta \in \mathbb{R} \). This implies that every self-adjoint extension of \( h_\kappa \) is equal to the operator

\[
h_{\kappa, \vartheta} = L_{a_\kappa}^{\vartheta}
\]

for some \( \vartheta \in \mathbb{R} \). Given \( \vartheta \in \mathbb{R} \), let the positive measure \( \nu_{\kappa, \vartheta} \) on \( \mathbb{R} \) be defined by \( (11) \) and \( (15) \) for \( 0 < |\kappa| < 1 \) and \( \kappa = 0 \) respectively. The next proposition gives eigenfunction expansions for \( h_{\kappa, \vartheta} \).

**Proposition 6.11.** For every \(-1 < \kappa < 1 \) and \( \vartheta \in \mathbb{R} \), there is a unique unitary operator \( U_{\kappa, \vartheta} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}, \nu_{\kappa, \vartheta}) \) such that

\[
(U_{\kappa, \vartheta} \psi)(E) = \int_0^\infty u_\vartheta^r (E | r) \psi(r) \, dr, \quad \psi \in L_2^0(\mathbb{R}_+),
\]

for \( \nu_{\kappa, \vartheta} \)-a.e. \( E \). For any Borel complex function \( F \) on \( \mathbb{R} \), we have

\[
U_{\kappa, \vartheta} F(h_{\kappa, \vartheta}) U_{\kappa, \vartheta}^{-1} = T^\nu_{\kappa, \vartheta} F.
\]

For the proof of Proposition 6.11 we need the next auxiliary result.

**Lemma 6.12.** For every \( z, z' \in \mathbb{C} \) and \(-1 < \kappa < 1 \), we have

\[
W_0(u^\kappa(z), u^\kappa(z')) = W_0(u^\kappa(z), u^\kappa(z')) = 0, \quad W_0(u^\kappa(z), u^{-\kappa}(z')) = -\frac{2}{\pi} \sin \pi \kappa.
\]

**Proof.** Given \( z \in \mathbb{C} \) and \(-1 < \kappa < 1 \), we define the smooth function \( a_\kappa^z \) on \( \mathbb{R} \) by setting \( a_\kappa^z(r) = X_\kappa(z r^2) \). For \( r \in \mathbb{R}_+ \), we have \( u^\kappa(z | r) = r^{1/2 + \kappa} a_\kappa^z(r) \). In view of \( (67) \), it follows that

\[
W_r(u^\kappa(z), u^\kappa(z')) = r^{1+2 \kappa} W_r(a_\kappa^z, a_\kappa^{z'}),
\]

\[
W_r(u^\kappa(z), u^{-\kappa}(z')) = r W_r(a_\kappa^z, a_\kappa^{-z'}) - 2 \kappa a_\kappa^z(r) a_\kappa^{-z'}(r)
\]

for every \( r \in \mathbb{R}_+ \) and \( z, z' \in \mathbb{C} \). Since \( a_\kappa^z(0) = 2^{-\kappa}/\Gamma(1+\kappa) \) for any \( z \in \mathbb{C} \), we obtain

\[
W_0(u^\kappa(z), u^\kappa(z')) = 0, \quad W_0(u^\kappa(z), u^{-\kappa}(z')) = -\frac{2}{\pi} \sin \pi \kappa.
\]

The statement of the lemma for \( 0 < |\kappa| < 1 \) now follows immediately from \( (24) \) and \( (63) \). Given \( z \in \mathbb{C} \), we define the smooth function \( b_z \) on \( \mathbb{R} \) by setting \( b_z(r) = (\gamma - \ln 2) X_\kappa(z r^2) - Y(z r^2) \). For \( r \in \mathbb{R}_+ \), we have \( \pi u^{\kappa}(z | r)/2 = r^{1/2} \ln r a_\kappa^0(r) + r^{1/2} b_z(r) \). In view of \( (37) \), it follows that

\[
\frac{\pi}{2} W_r(u^{0}(z), u^{0}(z')) = r W_r(a_\kappa^0, b_z) + r \ln r W_r(a_\kappa^0, a_\kappa^0) + a_\kappa^0(r) a_\kappa^0(r),
\]

\[
\frac{\pi^2}{4} W_r(u^{0}(z), u^{0}(z')) = r \ln^2 r W_r(a_\kappa^0, a_\kappa^0) + r \ln r W_r(a_\kappa^0, b_z) + W_r(b_z, a_\kappa^0) + r W_r(b_z, b_z) + b_z(r) a_\kappa^0(r) - a_\kappa^0(r) b_z(r)
\]

for every \( r \in \mathbb{R}_+ \) and \( z, z' \in \mathbb{C} \). Since \( a_\kappa^0(0) = 1 \) and \( b_z(0) = \gamma - \ln 2 \) for any \( z \in \mathbb{C} \), these equalities and the left relation in \( (69) \) imply the required statement for \( \kappa = 0 \).

**Proof of Proposition 6.11** Since \( L_\nu^0(\mathbb{R}_+) \) is dense in \( L_2(\mathbb{R}_+) \), \( U_{\kappa, \vartheta} \) is uniquely determined by \( (68) \). As \( g_\kappa \) is in l.c.c. at 0 and in l.p.c. at \( \infty \), \( X = D_{g_\kappa}^u \) and \( Y = D_{g_\kappa}^r \) are left and right boundary spaces respectively, and in view of \( (47) \) and \( (67) \), we have \( h_{\kappa, \vartheta} = L_{g_\kappa}(X \cap Y) \). By \( (64) \) and Lemma 6.12 we have \( W_0(u_\vartheta^0(z), u_\vartheta^0(0)) = 0 \), i.e., \( u_\vartheta^0(z) \in X \) for all \( z \in \mathbb{C} \). The statement will therefore
follow from Proposition 6.6 if we show that $\mathcal{V}_{\kappa, \vartheta}$ coincides with the spectral density $\sigma_{\kappa, \vartheta}$ corresponding to $u_0^\kappa$. Let $\tilde{u}_0^\kappa : \mathbb{C} \to \mathcal{D}$ be the real-entire map defined by the relation

$$ (70) \quad \tilde{u}_0^\kappa(z) = u_0^\kappa(z) = u^\kappa(z) \sin(\vartheta - \vartheta_\kappa) - w^\kappa(z) \cos(\vartheta - \vartheta_\kappa), \quad z \in \mathbb{C}. $$

By (61), (66), and Lemma 6.12 $\tilde{u}_0^\kappa(z)$ is a solution of $l_{\eta_{-z}} \tilde{u}_0^\kappa(z) = 0$ linearly independent of $u_0^\kappa(z)$ for any $z \in \mathbb{C}$. The proof of the equality $\mathcal{V}_{\kappa, \vartheta} = \sigma_{\kappa, \vartheta}$ depends on whether $0 < |\vartheta| < 1$ or $\kappa = 0$. We consider both cases separately.

1. $0 < |\vartheta| < 1$. In view of (24), (64), (61), and Lemma 6.12 we have

$$ \frac{1}{\pi} \frac{W(u^\kappa(z), \tilde{u}_0^\kappa(z))}{W(u_0^\kappa(z), \tilde{u}_0^\kappa(z))} = \Phi_{\kappa, \vartheta}(z), \quad z \in \mathbb{C}_+, $$

where $\Phi_{\kappa, \vartheta}$ is the meromorphic function in $\mathbb{C} \setminus [0, -i\infty)$ that is given by

$$ \Phi_{\kappa, \vartheta}(z) = \frac{1}{2} \cos(\vartheta + \vartheta_\kappa) - e^{-i\pi \kappa} z^\kappa \cos(\vartheta - \vartheta_\kappa) + \frac{1}{2} \sin(\vartheta + \vartheta_\kappa) - e^{-i\pi \kappa} z^\kappa \sin(\vartheta - \vartheta_\kappa). $$

It is easy to see that $\Phi_{\kappa, \vartheta}$ has no singularities on $(0, \infty)$ and

$$ \text{Im} \Phi_{\kappa, \vartheta}(E) = \frac{1}{2} \frac{\Theta(E) \sin^2 \pi \kappa}{E^\kappa - \vartheta_\kappa + \sin(\vartheta + \vartheta_\kappa) - \frac{1}{2} \cos(\vartheta + \vartheta_\kappa)} = E > 0, $$

where $\vartheta_\pm = \vartheta \pm \vartheta_\kappa$. Hence, Proposition 6.6 applied to $u = u_0^\kappa$, $\vartheta = \vartheta_\kappa$, and $\tilde{u} = \tilde{u}_0^\kappa$ ensures that $\sigma_{\kappa, \vartheta}$ coincides with $\mathcal{V}_{\kappa, \vartheta}$ on $(0, \infty)$. For $\vartheta \in [-|\vartheta_\kappa|, |\vartheta_\kappa|] + \pi \mathbb{Z}$, $\Phi_{\kappa, \vartheta}$ is real on $(-\infty, 0)$ and has no singularities on this set. Proposition 6.6 therefore implies that $\sigma_{\kappa, \vartheta}$ is zero on $(-\infty, 0)$ for such $\vartheta$. If $\vartheta \in (|\vartheta_\kappa|, |\vartheta_\kappa|) + \pi \mathbb{Z}$, then $\Phi_{\kappa, \vartheta}$ has a simple pole at the point $E_{\kappa, \vartheta}$ given by (14) and, hence, is representable in the form

$$ \Phi_{\kappa, \vartheta}(z) = g(z) + \frac{A}{E_{\kappa, \vartheta} - z}, $$

where $g$ is a function, analytic in $\mathbb{C} \setminus [0, -i\infty)$ and real on $(-\infty, 0)$ and

$$ A = \lim_{z \to E_{\kappa, \vartheta}} (E_{\kappa, \vartheta} - z) \Phi_{\kappa, \vartheta}(z) = \frac{\sin \pi \kappa |E_{\kappa, \vartheta}|}{2 \kappa \sin(\vartheta + \vartheta_\kappa) \sin(\vartheta - \vartheta_\kappa)}. $$

It therefore follows from Proposition 6.5 that $\sigma_{\kappa, \vartheta}$ is equal to $\pi A \delta_{E_{\kappa, \vartheta}}$ on $(-\infty, 0)$. Thus, $\sigma_{\kappa, \vartheta}$ coincides with $\mathcal{V}_{\kappa, \vartheta}$ for all $\vartheta$. It remains to note that $\sigma_{\kappa, \vartheta} \{0\}$ is not square-integrable, which would be an eigenfunction of $h_{\kappa, \vartheta}$ by Corollary 6.7.

2. $\kappa = 0$. In view of the equality $H_0^{(1)} = J_0 + i Y_0$, formulas (57) and (63) yield

$$ w^0(z) = -iw^0(z) + (i - \pi^{-1} \ln z)u^0(z), \quad z \in \mathbb{C}_+. $$

By (64), it follows that

$$ W(v^0(z), u^0(z)) = -\frac{2i}{\pi} W(v^0(z), w^0(z)) = \frac{2}{\pi} \left(1 + \frac{i}{\pi} \ln z\right), \quad z \in \mathbb{C}_+. $$

In view of (64), (70), (71), and Lemma 6.12 we have

$$ \frac{1}{\pi} \frac{W(u^0(z), \tilde{u}_0^0(z))}{W(v^0(z), u_0^0(z))W(u_0^0(z), \tilde{u}_0^0(z))} = \Phi_{0, \vartheta}(z), \quad z \in \mathbb{C}_+. $$
Lemma 7.1. Let $\Phi_{0,\vartheta}$ be the meromorphic function in $\mathbb{C} \setminus [0, -i\infty)$ that is given by

$$\Phi_{0,\vartheta}(z) = \frac{1}{2} (i - \pi^{-1} \ln z) \cos \vartheta - \sin \vartheta$$

It is easy to see that $\Phi_{0,\vartheta}$ has no singularities on $(0, \infty)$ and

$$\text{Im} \, \Phi_{0,\vartheta}(E) = \frac{1}{2} \frac{1}{(\cos \vartheta - \ln E \sin \vartheta/\pi)^2 + \sin^2 \vartheta}, \quad E > 0.$$  

Hence, Proposition 6.6 applied to $\Phi_{0,\vartheta}$ coincides with $\lambda_A \delta_{E_{0,\vartheta}}$ on $(-\infty, 0)$. Thus, $\sigma_{0,\vartheta}$ coincides with $V_{0,\vartheta}$ on $(0, \infty)$. For $\vartheta \in \pi \mathbb{Z}$, $\Phi_{0,\vartheta}$ is real on $(-\infty, 0)$ and has no singularities on this set. Proposition 6.6 therefore implies that $\sigma_{0,\vartheta}$ is zero on $(\infty, 0)$ for such $\vartheta$. If $\vartheta \not\in \pi \mathbb{Z}$, then $\Phi_{0,\vartheta}$ has a simple pole at the point $E_{0,\vartheta}$ given by (16) and, hence, is representable in the form

$$\Phi_{0,\vartheta}(z) = g(z) + \frac{A}{E_{0,\vartheta} - z},$$

where $g$ is a function, analytic in $\mathbb{C} \setminus [0, -i\infty)$ and real on $(-\infty, 0)$ and

$$A = \lim_{z \to E_{0,\vartheta}} (E_{0,\vartheta} - z) \Phi_{0,\vartheta}(z) = \frac{\pi |E_{0,\vartheta}|}{2 \sin^2 \vartheta}.$$ 

It therefore follows from Proposition 6.6 that $\sigma_{0,\vartheta}$ is equal to $\pi A \delta_{E_{0,\vartheta}}$ on $(-\infty, 0)$.

Remark 6.13. Eigenfunction expansions similar to those described by Proposition 6.11 can be found in [39], where they were obtained by a different method. A treatment of such expansions based on the theory of operators in Hilbert space was given in [16]. Our consideration differs from that in [16] by a choice of parametrization of self-adjoint extensions. The advantage of our choice is that the eigenfunctions $u_{0,\vartheta}(E|r)$ and the spectral densities $V_{\kappa,\vartheta}$ are continuous in $\kappa$ at $\kappa = 0$.

Remark 6.14. Note that the function $q_{\kappa}$ given by (34) is real not only for real $\kappa$ but also for purely imaginary $\kappa$. The eigenfunction expansions for this case can be found in [39]. For the analysis of the Aharonov-Bohm model, we need not to consider such values of $\kappa$.

7. Measurable families of one-dimensional Schrödinger operators

In what follows, we fix $-\infty \leq a < b \leq \infty$ and set $\mathfrak{h} = L^2((a,b))$.

Let $\nu$ be a $\sigma$-finite positive measure. A $\nu$-a.e. defined map $\xi$ is said to be a $\nu$-measurable family of functions on $(a,b)$ if $\xi(s)$ is a locally integrable complex function on $(a,b)$ for $\nu$-a.e. $s$ and $s \rightarrow \int_a^b |\xi(s)|^2 \, dr$ is a $\nu$-measurable complex function for any $a < \alpha \leq \beta < b$.

Lemma 7.1. Let $\nu$ be a $\sigma$-finite positive measure. Then the following statements hold:

1. Let $\xi$ and $\eta$ be $\nu$-measurable families of locally square-integrable functions on $(a,b)$. Then $s \rightarrow \xi(s)\eta(s)$ is a $\nu$-measurable family of functions on $(a,b)$.
2. $\xi$ is an $\mathfrak{h}$-valued $\nu$-measurable map if and only if $\xi$ is a $\nu$-measurable family of functions on $(a,b)$ such that $\xi(s) \in \mathfrak{h}$ for $\nu$-a.e. $s$. 


(3) Let \( \xi \) be a \( \nu \)-a.e. defined map such that \( \xi(s) \) is a continuous function on \((a, b)\) for \( \nu \)-a.e. \( s \). Then \( \xi \) is a \( \nu \)-measurable family of functions on \((a, b)\) if and only if \( s \rightarrow \xi(s|r) \) is a \( \nu \)-measurable complex function for any \( r \in (a, b) \).

(4) Let \( r_0 \in (a, b) \), \( \xi \) be a \( \nu \)-measurable family of functions on \((a, b)\), and \( \eta \) be a \( \nu \)-a.e. defined map such that, for \( \nu \)-a.e. \( s \), \( \eta(s) \) is a complex function on \((a, b)\) satisfying the equality \( \eta(s|r) = \int_{a}^{r} \xi(s|r) \, dr \) for all \( r \in (a, b) \). Then \( \eta \) is a \( \nu \)-measurable family of functions on \((a, b)\).

(5) Let \( \xi \) be a \( \nu \)-measurable family of functions on \((a, b)\) such that \( \xi(s) \) is absolutely continuous on \((a, b)\) for \( \nu \)-a.e. \( s \). Then \( s \rightarrow \xi(s) \) is a \( \nu \)-measurable family of functions on \((a, b)\).

**Proof.** 1. For any \( a < \gamma \leq \delta < b \), let \( \chi_{\gamma, \delta} \) be the function on \((a, b)\) that is equal to unity on \([\gamma, \delta]\) and vanishes outside this segment. Then the linear span of all \([\chi_{\gamma, \delta}]\) with rational \( \gamma \) and \( \delta \) is dense in \( \mathfrak{h} \). Let \( e_1, e_2, \ldots \) be a basis in \( \mathfrak{h} \) obtained by orthogonalization of this system. Given \( a < \alpha \leq \beta < b \), we have

\[
\int_{\alpha}^{\beta} \xi(s|r)\eta(s|r) \, dr = \langle [\chi_{\alpha, \beta} \xi(s)], [\chi_{\alpha, \beta} \eta(s)] \rangle = \sum_{i=1}^{\infty} \langle [\chi_{\alpha, \beta} \xi(s)], e_i \rangle \langle e_i, [\chi_{\alpha, \beta} \eta(s)] \rangle = \sum_{i=1}^{\infty} \int_{\alpha}^{\beta} \xi(s|r)e_i(r) \, dr \int_{\alpha}^{\beta} e_i(r)\eta(s|r) \, dr
\]

for \( \nu \)-a.e. \( s \), where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathfrak{h} \). Since each \( e_i \) is \( \lambda_{a,b} \)-equivalent to a linear combination of \( \chi_{\gamma, \delta} \), the right-hand side is a \( \nu \)-measurable complex function of \( s \).

2. Let \( \chi_{\alpha, \beta} \) and \( e_i \) be as in the proof of (1). If \( \xi \) is an \( \mathfrak{h} \)-valued \( \nu \)-measurable map, then \( s \rightarrow \langle \psi, \xi(s) \rangle \) is a \( \nu \)-measurable function for any \( \psi \in \mathfrak{h} \). In particular, \( s \rightarrow \langle [\chi_{\alpha, \beta}], \xi(s) \rangle = \int_{\alpha}^{\beta} \xi(s|r) \, dr \) is a \( \nu \)-measurable function for any \( a < \alpha \leq \beta < b \), i.e., \( \xi \) is a \( \nu \)-measurable family of functions on \((a, b)\). Conversely, if \( \xi \) is a \( \nu \)-measurable family of functions on \((a, b)\), then \( s \rightarrow \int_{\alpha}^{\beta} e_i(r)\xi(s|r) \, dr \) is a \( \nu \)-measurable function for every \( i = 1, 2, \ldots \). If, in addition, \( \xi(s) \in \mathfrak{h} \) for \( \nu \)-a.e. \( s \), then \( \int_{\alpha}^{\beta} e_i(r)\xi(s|r) \, dr = \langle e_i, \xi(s) \rangle \) and, therefore, \( \xi \) is an \( \mathfrak{h} \)-valued \( \nu \)-measurable map.

3. Let \( \xi \) be a \( \nu \)-measurable family of functions on \((a, b)\) and \( r \in (a, b) \). As \( \xi(s) \) is continuous at \( r \) for \( \nu \)-a.e. \( s \), we have \( \xi(s|r) = \lim_{n \to \infty} n \int_{r-1/2n}^{r+1/2n} \xi(s) \, ds \) for \( \nu \)-a.e. \( s \). This implies that \( s \rightarrow \xi(s|r) \) is a \( \nu \)-measurable function because \( s \rightarrow n \int_{r-1/2n}^{r+1/2n} \xi(s) \, ds \) are \( \nu \)-measurable functions. Conversely, let \( s \rightarrow \xi(s|r) \) be a \( \nu \)-measurable function for every \( r \in (a, b) \) and let \( a < \alpha < \beta < b \). Given \( n = 1, 2, \ldots \) and a function \( f \) on \((a, b)\), we denote by \( S_n(f) \) the Riemann sum \( \frac{2 \alpha}{n} \sum_{k=1}^{n} f(r_k^n) \) for \( f \) on \([\alpha, \beta]\), where \( r_k^n = \alpha + (\beta - \alpha)k/n \). As \( \xi(s) \) is continuous for \( \nu \)-a.e. \( s \), \( \int_{\alpha}^{\beta} \xi(s|r) \, dr \) is the limit of \( S_n(\xi(s)) \) for \( \nu \)-a.e. \( s \). This means that \( s \rightarrow \int_{\alpha}^{\beta} \xi(s|r) \, dr \) is a \( \nu \)-measurable function because \( s \rightarrow S_n(\xi(s)) \) is a \( \nu \)-measurable function for every \( n \).

4. Clearly, \( \eta(s) \) is a continuous function on \((a, b)\) for \( \nu \)-a.e. \( s \) and \( s \rightarrow \eta(s|r) \) is a \( \nu \)-measurable function for any \( r \in (a, b) \). Hence, the statement follows from 3.

---

9As in the preceding section, we write \([f] \) in place of \([f]_{\lambda_{a,b}} \), where \( \lambda_{a,b} \) is the restriction to \((a, b)\) of the Lebesgue measure \( \lambda \) on \( \mathbb{R} \).
5. For \( \nu \)-a.e. \( s \), the function \( \xi(s)' \) is locally integrable and \( \int_a^b \xi'(s|r) \, dr = \xi(s|\beta) - \xi(s|\alpha) \). Hence, the statement follows from 3. \( \square \)

**Lemma 7.2.** Let \( \nu \) be a \( \sigma \)-finite positive measure and \( v \) be a \( \nu \)-measurable family of real locally square-integrable functions on \( (a,b) \). Then \( s \rightarrow L_v(s) \) is a \( \nu \)-measurable family of operators in \( \mathfrak{h} \).

**Proof.** By Lemma 6.3 there is a sequence \( \varphi_1, \varphi_2, \ldots \) of functions in \( C_0^\infty(a,b) \) such that the vectors \( ([\varphi_j], l_{\nu} \varphi_j) \) are dense in \( G_{L_v(s)} \) for any locally square-integrable real function \( q \) on \( (a,b) \). For each \( j = 1, 2, \ldots \), let \( \xi_j \) and \( \eta_j \) be \( \nu \)-a.e. defined maps such that \( \xi_j(s) = [\varphi_j] \) and \( \eta_j(s) = l_{\nu(s)} \varphi_j \) for \( \nu \)-a.e. \( s \). Since \( v(s) \) is locally square integrable for \( \nu \)-a.e. \( s \), the vectors \( (\xi_j(s), \eta_j(s)) \) are dense in \( G_{L_v(s)}(\nu) \) for \( \nu \)-a.e. \( s \).

By statements 1 and 5 of Lemma 7.1 \( \eta_j \) is a \( \nu \)-measurable family of functions on \( (a,b) \), and statement 2 of Lemma 7.1 implies that \( \eta_j \) is an \( \mathfrak{h} \)-valued \( \nu \)-measurable map for any \( j = 1, 2, \ldots \). As \( \xi_j \) are obviously \( \mathfrak{h} \)-valued \( \nu \)-measurable maps for all \( j \), we conclude that \( s \rightarrow L_v(s) \) is a \( \nu \)-measurable family of operators in \( \mathfrak{h} \). \( \square \)

**Lemma 7.3.** Let \( \nu \) and \( v \) be as in Lemma 7.2 and \( \xi \) be a \( \mathcal{D} \)-valued \( \nu \)-a.e. defined map such that

\[(72) \quad l_{\nu(s)} \xi(s) = 0 \]

for \( \nu \)-a.e. \( s \). Suppose there is \( r_0 \in (a,b) \) such that the functions \( s \rightarrow \xi(s|r_0) \) and \( s \rightarrow \xi'(s|r_0) \) are \( \nu \)-measurable. Then \( \xi \) is a \( \nu \)-measurable family of functions on \( (a,b) \).

**Proof.** Clearly, it suffices to show that \( \xi \) is a \( \nu \)-measurable family of functions on \( (\alpha, \beta) \) for any real numbers \( \alpha \) and \( \beta \) such that \( a < \alpha < r_0 < \beta < b \). We fix such \( \alpha \) and \( \beta \) and set

\[ A_N = \left\{ s \in S_v : \int_\alpha^\beta v(s|r)^2 \, dr < N^2/\beta - \alpha \right\} \]

for each \( N = 1, 2, \ldots \). By statement 1 of Lemma 7.1, \( A_N \) is a \( \nu \)-measurable set. By the Cauchy–Bunyakovsky inequality, we have

\[(73) \quad \int_\alpha^\beta |v(s|r)| \, dr < N, \quad s \in A_N. \]

Let \( \nu_N = \nu|_{A_N} \). To prove our statement, it suffices to show that the set

\[ Q_N = \{ r \in (\alpha, \beta) : s \rightarrow \xi(s|r) \text{ and } s \rightarrow \xi'(s|r) \text{ are } \nu_N\text{-measurable functions} \} \]

coincides with \( (\alpha, \beta) \) for all \( N = 1, 2, \ldots \). Indeed, this condition implies that \( s \rightarrow \xi(s|r) \) is a \( \nu_N \)-measurable function for all \( N \) and \( r \in (\alpha, \beta) \). Because \( S_v \setminus \bigcup_{N=1}^\infty A_N \) is a \( \nu \)-null set, this means that \( s \rightarrow \xi(s|r) \) is a \( \nu \)-measurable function for every \( r \in (\alpha, \beta) \), and statement 3 of Lemma 7.1 ensures that \( \xi \) is a \( \nu \)-measurable family of functions on \( (\alpha, \beta) \). Note that \( r_0 \in Q_N \) for all \( N \). Hence, to prove the equality \( Q_N = (\alpha, \beta) \), it suffices to verify that the set \( R_{N,r} = \{ r \in (\alpha, \beta) : |r - \rho| < 1/2N \} \)

\(^{10}\)We denote by \( \xi'(s|r) \) the derivative of \( \xi(s) \) at point \( r \): \( \xi'(s|r) = (\xi(s))'(r) \).
is contained in \( Q_N \) for every \( N = 1, 2, \ldots \) and \( r \in Q_N \). Fix \( N \) and \( r \in Q_N \) and let \( \xi_0, \xi_1, \ldots \) be \( \mathcal{D} \)-valued \( \nu \)-a.e. defined maps such that, for \( \nu \)-a.e. \( s \), the relations

\[
\xi_0(s|\rho) = \xi(s|\rho) + \xi'(s|\rho)(\rho - r),
\]

(74)

\[
\xi_n(s|\rho) = \xi_0(s|\rho) + \int_r^\rho dp' \int_r^{p'} v(s|t)\xi_{n-1}(s|t) \, dt, \quad n = 1, 2, \ldots,
\]

hold for all \( \rho \in (a, b) \). As \( r \in Q_N \), \( \xi_0 \) is a \( \nu_N \)-measurable family of functions on \((a, b)\), and it follows from statements 1 and 4 of Lemma 7.1 that \( \xi_n \) is a \( \nu_N \)-measurable family of functions on \((a, b)\) for every \( n = 0, 1, \ldots \). Since \( \xi(s) \) satisfies (72) for \( \nu \)-a.e. \( s \), it follows that, for \( \nu \)-a.e. \( s \), the equality

\[
\xi(s|\rho) = \xi_0(s|\rho) + \int_r^\rho dp' \int_r^{p'} v(s|t)\xi(s|t) \, dt
\]

(75)

holds for any \( \rho \in (a, b) \). Let \( \rho \in R_{N,r} \). It follows from (72), (73) and (75) that

\[
|\xi(s|\rho) - \xi_n(s|\rho)| \leq |\xi(s|\rho) - \xi_{n-1}(s|\rho)|/2^n
\]

for \( \nu_N \)-a.e. \( s \) and all \( n = 0, 1, \ldots \). We hence have

\[
|\xi(s|\rho) - \xi_n(s|\rho)| \leq |\xi(s|\rho) - \xi_0(s|\rho)|/2^n
\]

for \( \nu_N \)-a.e. \( s \). This means that, for any \( \rho \in R_{N,r} \), the sequence \( \xi_n(s|\rho) \) converges to \( \xi(s|\rho) \) for \( \nu_N \)-a.e. \( s \) and, therefore, \( s \to \xi(s|\rho) \) is a \( \nu_N \)-measurable function for every \( \rho \in R_{N,r} \). As \( \xi'(s|\rho) \) is the limit of \( k(\xi(s|\rho + 1/k) - \xi(s|\rho)) \) as \( k \to \infty \) for \( \nu \)-a.e. \( s \), it also follows that \( s \to \xi'(s|\rho) \) is a \( \nu_N \)-measurable function for every \( \rho \in R_{N,r} \). Hence \( R_{N,r} \subset Q_N \) and the lemma is proved.

**Corollary 7.4.** Let \( \nu \) and \( v \) be as in Lemma 7.2. Then there are \( \nu \)-measurable families \( \xi_1 \) and \( \xi_2 \) of functions on \((a, b)\) such that \( \xi_1(s) \) and \( \xi_2(s) \) are linearly independent real elements of \( \mathcal{D} \) satisfying equation (72) for \( \nu \)-a.e. \( s \).

**Proof.** Choose \( r_0 \in (a, b) \). Let \( \xi_1 \) and \( \xi_2 \) be \( \nu \)-a.e. defined \( \mathcal{D} \)-valued maps such that \( \xi_1(s) \) and \( \xi_2(s) \) are solutions of (72) satisfying the conditions

\[
\xi_1(s|r_0) = \xi_2(s|r_0) = 1, \quad \xi_1'(s|r_0) = \xi_2(s|r_0) = 0
\]

for \( \nu \)-a.e. \( s \) (such solutions always exist, see, e.g., [23], Chapter V, Sec. 16, Theorem 2). Obviously, \( \xi_1(s) \) and \( \xi_2(s) \) are linearly independent for \( \nu \)-a.e. \( s \), and Lemma 7.3 implies that \( \xi_1 \) and \( \xi_2 \) are \( \nu \)-measurable families of functions on \((a, b)\).

**Proposition 7.5.** Let \( \nu \) and \( v \) be as in Lemma 7.2. Suppose \( v(s) \) is in l.c.c. at \( a \) and in l.p.c. at \( b \) for \( \nu \)-a.e. \( s \). If \( \xi \) is a \( \nu \)-measurable family of elements of \( \mathcal{D} \) such that \( \xi(s) \) is a nontrivial real solution of (72) for \( \nu \)-a.e. \( s \), then \( s \to L_{v(s)}^\xi(s) \) is a \( \nu \)-measurable family of self-adjoint operators in \( \mathfrak{h} \). If \( \mathcal{R} \) is a \( \nu \)-measurable family of operators in \( \mathfrak{h} \) such that \( \mathcal{R}(s) \) is a self-adjoint extension of \( L_{v(s)} \) for \( \nu \)-a.e. \( s \), then there exists a \( \nu \)-measurable family \( \xi \) of elements of \( \mathcal{D} \) such that \( \xi(s) \) is a nontrivial real solution of (72) and \( \mathcal{R}(s) = L_{v(s)}^\xi(s) \) for \( \nu \)-a.e. \( s \).

**Proof.** By Lemma 7.2, \( s \to L_{v(s)} \) is a \( \nu \)-measurable family of operators in \( \mathfrak{h} \). This means that there is a sequence \( \zeta_1, \zeta_2, \ldots \) of \( \mathfrak{h} \oplus \mathfrak{h} \)-valued \( \nu \)-measurable maps such that the linear span of \( \zeta_1(s), \zeta_2(s), \ldots \) is dense in the graph \( G_{L_{v(s)}} \) of \( L_{v(s)} \) for \( \nu \)-a.e. \( s \).
Let $\xi$ be a $\nu$-measurable family of elements of $\mathcal{D}$ such that $\xi(s)$ is a nontrivial real solution of (72) for $\nu$-a.e. $s$ and let $\tau$ be a smooth function on $(a,b)$ that is equal to unity in a neighborhood of $a$ and vanishes in a neighborhood of $b$. Let $\nu$-a.e. defined maps $g$ and $h$ be such that

$$
g(s) = [\tau \xi(s)], \quad h(s) = -[\tau'' \xi(s) + 2\tau' \xi'(s)]$$

for $\nu$-a.e. $s$. We obviously have $g(s) \in \mathfrak{h}$ and $h(s) \in \mathfrak{h}$ for $\nu$-a.e. $s$ and, therefore, statements 1, 2, and 5 of Lemma 6.4 imply that $g$ and $h$ are $\mathfrak{h}$-valued $\nu$-measurable maps. Let $\zeta$ be an $\mathfrak{h} \oplus \mathfrak{h}$-valued $\nu$-measurable map such that $\zeta(s) = (g(s), h(s))$ for $\nu$-a.e. $s$. Note that

$$
l_{\nu(s)}(\tau \xi(s)) = h(s)
$$

for $\nu$-a.e. $s$ and, therefore, $\tau \xi(s) \in D_{\nu(s)}$ for $\nu$-a.e. $s$ (see (38)). Since $\xi(s)$ is a nontrivial real solution of (72) and $\tau \xi(s)$ coincides with $\xi(s)$ in a neighborhood of $a$, it follows from (41) and (43) that $\tau \xi(s) \in D_{\nu(s)}(\xi(s))$ and $\tau \xi(s) \notin D_{\nu(s)}^\nu$ for $\nu$-a.e. $s$. In view of (40), (47), and (76), we conclude that $\zeta(s) \in L_{\nu(s)}^\nu \setminus G_{L_{\nu(s)}}$ for $\nu$-a.e. $s$. As $G_{L_{\nu(s)}} \oplus G_{L_{\nu(s)}}$ is one-dimensional, this implies that the linear span of the sequence $\zeta(s), \zeta_1(s), \zeta_2(s), \ldots$ is dense in the graph of $L_{\nu(s)}^\nu$ for $\nu$-a.e. $s$. This means that $s \rightarrow L_{\nu(s)}^\nu$ is a $\nu$-measurable family of operators in $\mathfrak{h}$.

Conversely, let $\mathcal{R}$ be a $\nu$-measurable family of operators in $\mathfrak{h}$ such that $\mathcal{R}(s)$ is a self-adjoint extension of $L_{\nu(s)}$ for $\nu$-a.e. $s$. Then both $s \rightarrow G_{\mathcal{R}(s)}$ and $s \rightarrow G_{L_{\nu(s)}}$ are $\nu$-measurable families of subspaces of $\mathfrak{h} \oplus \mathfrak{h}$. By statement 1 of Lemma B.6, $s \rightarrow G_{\mathcal{R}(s)} \oplus G_{L_{\nu(s)}}$ is also a $\nu$-measurable family of subspaces of $\mathfrak{h} \oplus \mathfrak{h}$. Since $\mathcal{R}(s) \oplus G_{L_{\nu(s)}}$ is nontrivial for $\nu$-a.e. $s$, there is an $\mathfrak{h} \oplus \mathfrak{h}$-valued $\nu$-measurable map $\eta$ such that $\eta(s) \in G_{\mathcal{R}(s)} \oplus G_{L_{\nu(s)}}$ for $\nu$-a.e. $s$. Let $Q = \{s \in S_\nu : \eta(s) = -\overline{\eta(s)}\}$ and $\zeta$ be a $\nu$-a.e. defined map such that $\zeta(s) = i\eta(s)$ for $\nu$-a.e. $s \in Q$ and $\zeta(s) = \eta(s) + \overline{\eta(s)}$ for $\nu$-a.e. $s \in S_\nu \setminus Q$. Then $\zeta$ is an $\mathfrak{h} \oplus \mathfrak{h}$-valued $\nu$-measurable map such that

$$
\zeta(s) = \overline{\eta(s)}
$$

and $\zeta(s) \neq 0$ for $\nu$-a.e. $s$. Moreover, since $G_{\mathcal{R}(s)}$ and $G_{L_{\nu(s)}}$ are both invariant under complex conjugation, we have $\zeta(s) \in G_{\mathcal{R}(s)} \oplus G_{L_{\nu(s)}}$ for $\nu$-a.e. $s$. Let $g$ and $h$ be $\mathfrak{h}$-valued $\nu$-measurable maps such that $\zeta(s) = (g(s), h(s))$ for $\nu$-a.e. $s$. For $\nu$-a.e. $s$, we have

$$
g(s) \in D_{\mathcal{R}(s)} \setminus D_{L_{\nu(s)}}.
$$

As $g(s) \in D_{\mathcal{R}(s)}$ for $\nu$-a.e. $s$, there exists a $\nu$-a.e. defined map $\tilde{g}$ such that $\tilde{g}(s) \in D_{\nu(s)}$ and $g(s) = \tilde{g}(s)$ for $\nu$-a.e. $s$. It follows from (77) that $\tilde{g}(s)$ is real for $\nu$-a.e. $s$. By statement 2 of Lemma 6.4, $\tilde{g}$ and $h$ are $\nu$-measurable families of functions on $(a,b)$. Let $\xi$ be a $\nu$-a.e. defined map such that $\xi(s) = \overline{\tilde{g}(s)}$ for $\nu$-a.e. $s$. In view of (75), it follows from Lemma 6.4 that $\xi(s)$ is a nontrivial real solution of (72) and $\mathcal{R}(s) = L_{\nu(s)}^\nu$ for $\nu$-a.e. $s$. Let $\xi_1$ and $\xi_2$ be as in Corollary 7.4. Since

---

\footnote{Given $\psi = (\psi_1, \psi_2) \in \mathfrak{h} \oplus \mathfrak{h}$, we set $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2)$, where $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are elements of $\mathfrak{h}$ such that $\tilde{\psi}_{1,2}(r) = \psi_{1,2}(r)$ for $\lambda$-a.e. $r \in (a,b)$.}
h(s) = R(s)g(s), we have \( h(s) = I_{\nu}(s) \tilde{g}(s) \) for \( \nu \)-a.e. \( s \). Equations (44) and (45) imply that, for \( \nu \)-a.e. \( s \), equality

\[
(79) \quad \xi(s|r) = \tilde{g}(s|r) - \frac{1}{W(\xi_1(s), \xi_2(s))} \left[ \xi_1(s|r) \int_{a}^{r} h(s|\rho) \xi_2(s|\rho) d\rho - \xi_2(s|r) \int_{a}^{r} h(s|\rho) \xi_1(s|\rho) d\rho \right]
\]

holds for all \( r \in (a, b) \). By statements 3 and 5 of Lemma 7.1, \( s \rightarrow W(\xi_1(s), \xi_2(s)) \) is a \( \nu \)-measurable function. It therefore follows from statements 1 and 4 of Lemma 7.1 that \( \xi \) is a \( \nu \)-measurable family of functions on \( (a, b) \).

8. Self-adjoint extensions of the three-dimensional Aharonov–Bohm Hamiltonian

In this section, we use the results of Secs. 5 and 7 to represent the self-adjoint extensions of \( H^0 \) commuting with all operators \( T_G, G \in \mathcal{G} \), as direct integrals of one-dimensional Schrödinger operators. Let \( \mathfrak{h} = L^2(\mathbb{R}_+) \) and the positive Borel measure \( \nu_0 \) on \( S = \mathbb{Z} \times \mathbb{R} \) be as in Sec. 2. We begin by constructing a suitable unitary operator \( V: L_2(\mathbb{R}^3) \rightarrow L_2(S, \mathfrak{h}, \nu_0) \) and then prove that \( (\nu_0, \mathfrak{h}, \nu_0, V) \) is actually an exact diagonalization for the set of the operators \( T_G \). After that, we use Proposition 5.3 to obtain a representation of form (3) for \( H^0 \). Finally, we combine Corollary 5.2 and Proposition 7.5 to get an explicit description of the self-adjoint extensions of \( H^0 \) commuting with \( T_G \).

We denote by \( \lambda_+ \) the restriction to \( \mathbb{R}_+ \) of the Lebesgue measure \( \lambda \) on \( \mathbb{R} \). For \( \Phi \in C_0^\infty(\mathbb{R}^3 \setminus Z) \), let the map \( \Phi \) from \( S \) to \( C_0^\infty(\mathbb{R}_+) \) be defined by the relation

\[
(80) \quad \tilde{\Phi}(s|r) = \sqrt{\frac{\nu}{2\pi}} \int_{-\infty}^{\infty} dx_3 \int_{0}^{2\pi} d\varphi \Phi(r \cos \varphi, r \sin \varphi, x_3) e^{ipx_3 + im\varphi}
\]

for any \( s = (m, p) \in S \) and \( r > 0 \).

**Lemma 8.1.** There is a unique unitary operator \( V: L_2(\mathbb{R}^3) \rightarrow L_2(S, \mathfrak{h}, \nu_0) \) such that the equality

\[
(81) \quad (V[\Phi]_\lambda)(s) = [\tilde{\Phi}(s)]_{\lambda_+}
\]

holds for \( \nu_0 \)-a.e. \( s \) for any \( \Phi \in C_0^\infty(\mathbb{R}^3 \setminus Z) \).

**Proof.** Let \( \Phi \in C_0^\infty(\mathbb{R}^3 \setminus Z) \). Then \( s \rightarrow \int_{0}^{r} \tilde{\Phi}(s|r) dr \) is a continuous and, hence, \( \nu_0 \)-measurable function for any \( c, d \in \mathbb{R}_+ \). This means that \( \tilde{\Phi} \) is a \( \nu_0 \)-measurable family of functions on \( (0, \infty) \), and statement 2 of Lemma 7.1 implies that \( s \rightarrow [\tilde{\Phi}(s)]_{\lambda_+} \) is an \( \mathfrak{h} \)-valued \( \nu_0 \)-measurable map. Let \( f \) be the function on \( \mathbb{R}^3 \) defined by the formula

\[
f(x_1, x_2, p) = \int_{-\infty}^{\infty} \Phi(x_1, x_2, x_3) e^{ipx_3} dx_3.
\]

Clearly, \( f \) is continuous and, hence, Lebesgue measurable. It follows from the Fubini theorem and the Parseval identity for the one-dimensional Fourier transformation that \( f \) is square-integrable and

\[
(82) \quad \int |f(x_1, x_2, p)|^2 dx_1 dx_2 dp = 2\pi \int |\Phi(x_1, x_2, x_3)|^2 dx_1 dx_2 dx_3.
\]
Let $g$ be the continuous function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ defined by the relation $g(\varphi, p, r) = f(r \cos \varphi, r \sin \varphi, p)$. By the Fubini theorem, we have

$$\tilde{\Phi}(s|r) = \sqrt{r} \frac{2\pi}{2\pi} \int_0^{2\pi} g(\varphi, p, r)e^{im\varphi} \, d\varphi$$

for any $s = (m, p) \in S$ and $r \in \mathbb{R}^+$. Further, by the Fubini theorem and Parseval identity for the Fourier series expansion, we have

$$\int dv_0(s) \int_0^\infty |\tilde{\Phi}(s|r)|^2 \, dr = \frac{1}{2\pi} \int_0^\infty r \, dr \int_{-\infty}^\infty dp \sum_{m \in \mathbb{Z}} |\tilde{\Phi}(m, p|r)|^2 = \frac{1}{2\pi} \int |f(x_1, x_2, p)|^2 \, dx_1 dx_2 dp$$

and in view of (82), we conclude that $s \to [\tilde{\Phi}(s)]_{\lambda_+}$ is a square-integrable map and

$$\int \|\tilde{\Phi}(s)|_{\lambda_+}^2 dv_0(s) = \int \tilde{\Phi}(s) \int_0^\infty |\tilde{\Phi}(s|r)|^2 \, dr = \int |\Phi(x_1, x_2, x_3)|^2 \, dx_1 dx_2 dx_3.$$

Since $[\tilde{\Phi}]_{\lambda}$ with $\Phi \in C_0^\infty(\mathbb{R}^3 \setminus Z)$ are dense in $L_2(\mathbb{R}^3)$, it follows that there exists a unique isometric operator $V: L_2(\mathbb{R}^3) \to L_2(S, \eta, v_0)$ satisfying (81). It remains to check that the image of $V$ is the entire space $L_2(S, \eta, v_0)$. For any $\xi \in L_2(S, \eta, v_0)$ and $h \in \eta$, $s \to \langle \xi(s), h \rangle$ is a $\nu_0$-square integrable function because $|\langle \xi(s), h \rangle| \leq \|\xi(s)\|\|h\|$ for $\nu_0$-a.e. $s$. Given $\psi \in C_0^\infty(\mathbb{R}^3)$, $\xi \in L_2(S, \eta, v_0)$, and $m \in \mathbb{Z}$, we denote by $F_{\xi, \psi, m}$ the element of $L_2(\mathbb{R})$ such that $F_{\xi, \psi, m}(p) = \langle \xi(m, p), [\psi]_{\lambda_+} \rangle$ for $\lambda$-a.e. $p$. For $\psi \in C_0^\infty(\mathbb{R}^3)$, $\chi \in C_0^\infty(\mathbb{R})$, and $m \in \mathbb{Z}$, let $\Phi_{\psi, \chi, m} \in C_0^\infty(\mathbb{R}^3 \setminus Z)$ be such that

$$\Phi_{\psi, \chi, m}(r \cos \varphi, r \sin \varphi, x_3) = \frac{1}{\sqrt{r}} \psi(r) \chi(x_3)e^{-im\varphi}$$

for any $r > 0$ and $x_3, \varphi \in \mathbb{R}$. Then we have

$$\Phi_{\psi, \chi, m}(k, p|r) = \delta_{km}\hat{\psi}(p)\hat{\chi}(r),$$

where $\delta_{km} = 0$ for $k \neq m$ and $\delta_{km} = 1$ for $k = m$ and $\hat{\chi}(r) = \int \chi(x_3)e^{ipx_3} \, dx_3$ is the Fourier transform of $\chi$. It follows that

$$\langle \xi, V[\Phi_{\psi, \chi, m}] \rangle_{L_2(S, \eta, v_0)} = \int \hat{\chi}(p)\langle \xi(m, p), [\psi]_{\lambda_+} \rangle_{\eta} dp = \langle F_{\xi, \psi, m}, [\chi]_{\lambda} \rangle_{L_2(\mathbb{R})}$$

for every $\xi \in L_2(S, \eta, v_0)$. Suppose now that $\xi$ is orthogonal to every element of $\text{Im} \, V$. Since $[\chi]_{\lambda}$ with $\chi \in C_0^\infty(\mathbb{R})$ are dense in $L_2(\mathbb{R})$, equation (85) implies that $F_{\xi, \psi, m} = 0$, i.e., $\langle \xi(s), [\psi]_{\lambda_+} \rangle = 0$ for $\nu_0$-a.e. $s$ for any $\psi \in C_0^\infty(\mathbb{R}^3)$. As $[\psi]_{\lambda_+}$ with $\psi \in C_0^\infty(\mathbb{R}^3)$ are dense in $\eta$, it follows that $\xi = 0$ and, therefore, $\text{Im} \, V = L_2(S, \eta, v_0)$. The lemma is proved.

Lemma 8.2. Let $V$ be as in Lemma 8.1. Then $(v_0, \mathcal{I}_{\eta, v_0}, V)$ is an exact diagonalization for the set of the operators $T_G$ with $G \in \mathcal{G}$. Proof. For $\alpha, \beta \in \mathbb{R}$, let $G_{\alpha\beta} \in \mathcal{G}$ be defined by the relation

$$G_{\alpha\beta} x = (x_1 \cos \alpha - x_2 \sin \alpha, x_1 \sin \alpha + x_2 \cos \alpha, x_3 + \beta),$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Clearly, each element of $\mathcal{G}$ is equal to $G_{\alpha\beta}$ for some $\alpha, \beta \in \mathbb{R}$. We have

$$\Phi \circ G_{\alpha\beta}^{-1}(m, p) = e^{im\alpha + i\beta} \Phi(m, p), \quad (m, p) \in S.$$
for any $\Phi \in C^\infty_0(\mathbb{R}^3 \setminus Z)$ and, therefore,

$$VT_{G_{\alpha\beta}} V^{-1} = T_{G_{\alpha\beta}}^\nu, \quad \mathcal{S} = \mathcal{I}_{h,\nu_0}$$

where $\mathcal{S} = \mathcal{I}_{h,\nu_0}$ and the function $g_{\alpha,\beta}$ on $S$ is given by

$$g_{\alpha,\beta}(m,p) = e^{i\alpha m + i\beta p}.$$

Suppose $(m,p)$ and $(m',p')$ are such that $g_{\alpha,\beta}(m,p) = g_{\alpha,\beta}(m',p')$ for all $(\alpha, \beta) \in \mathbb{Q}^2$, where $\mathbb{Q}$ is the set of rational numbers. Then we have

$$e^{i\alpha(m - m')} + i\beta(p - p') = 1$$

for all $(\alpha, \beta) \in \mathbb{Q}^2$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, it follows that $m = m'$ and $p = p'$. This means that the family $\{g_{\alpha,\beta}\}_{(\alpha, \beta) \in \mathbb{Q}^2}$ separates the points of $S$. The desired statement now follows from Theorem [4.2].

The next lemma gives a representation of form (8) for $H^\phi$.

**Lemma 8.3.** Let $\phi \in \mathbb{R}$, $V$ be as in Lemma [8.1] and $H^\phi$ be the map on $S$ such that

$$(89) \quad H^\phi(m,p) = h_{m-\phi} + p^21_h, \quad (m,p) \in S,$$

where $1_h$ is the identity operator in $h$ and the operator $h_\kappa$, $\kappa \in \mathbb{R}$, is given by (55). Then $H^\phi$ is a $\nu_0$-measurable family of operators in $h$ and we have

$$H^\phi = V^{-1} \int_{\mathbb{R}} H^\phi(s) d\nu_0(s) V.$$

**Proof.** It easily follows from (8) that

$$(90) \quad (\mathcal{H}^\phi \Phi)(r \cos \varphi, r \sin \varphi, x_3) =$$

$$= \left(-\partial^2_{x_3} - \partial^2_r - \frac{1}{r} \partial_r - \frac{1}{r^2} \left(\partial^2_{\varphi} + 2i \phi \partial_{\varphi} - \phi^2\right)\right) F_\phi(r, \varphi, x_3)$$

for any $\Phi \in C^\infty_0(\mathbb{R}^3 \setminus Z)$, where the operator $\mathcal{H}^\phi$ in $C^\infty_0(\mathbb{R}^3 \setminus Z)$ is given by (8) and $F_\phi$ is the smooth function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ which represents $\Phi$ in the cylindrical coordinates,

$$F_\phi(r, \varphi, x_3) = \check{\Phi}(r \cos \varphi, r \sin \varphi, x_3).$$

Substituting (90) in (80) and integrating by parts yields

$$(91) \quad \mathcal{H}^\phi \Phi(s) = -\check{\Phi}(s)'' + q_{m-\phi} \check{\Phi}(s) + p^2 \check{\Phi}(s)$$

for every $s = (m,p) \in S$, where $q_m$, $\kappa \in \mathbb{R}$, is given by (54). Since $\check{\Phi}(s) \in C^\infty_0(\mathbb{R}^3 \setminus Z)$ and $F_\phi$ is the smooth function on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ which represents $\Phi$ in the cylindrical coordinates,

$$(92) \quad H^\phi(s)(V[\check{\Phi}]_{\lambda})(s) = I_{q_{m-\phi}} \check{\Phi}(s) + p^2 \check{\Phi}(s)_{\lambda_s}$$

for $\nu_0$-a.e. $s = (m,p)$. Since every element of $D_{\tilde{H}^\phi}$ is equal to $[\check{\Phi}]_{\lambda}$ for some $\check{\Phi} \in C^\infty_0(\mathbb{R}^3 \setminus Z)$, this implies that $V^{-1} \int_{\mathbb{R}} H^\phi(s) d\nu_0(s) V$ is an extension of $\tilde{H}^\phi$. In view of (55), Lemma [6.3] implies the existence of a sequence $f_1, f_2, \ldots$ of elements of $C^\infty_0(\mathbb{R}^3 \setminus Z)$ such that $(f_j)_{\lambda_s}$ are dense in the graph of $h_\kappa$ for any $\kappa \in \mathbb{R}$. Since $p^21_h$ is a bounded everywhere defined operator in $h$, it follows that the elements $((f_j)_{\lambda_s}, H^\phi(s)(f_j)_{\lambda_s})$ are dense in the graph of $H^\phi(s)$ for all $s \in S$. Let $\chi$ be a nonzero element of $C^\infty_0(\mathbb{R})$ and $\hat{\chi}(p) = \int_{-\infty}^{\infty} \chi(x)e^{ipx} dx$ be its Fourier transform.
For every $j = 1, 2 \ldots$ and $m \in \mathbb{Z}$, we define $\xi_{jm}$ to be the $\nu_0$-equivalence class such that

$$\xi_{jm}(k,p) = \delta_{km} \hat{\chi}(p)[f_j]_{\lambda}$$

for $\nu_0$-a.e. $(k,p)$, where $\delta_{km} = 0$ for $k \neq m$ and $\delta_{km} = 1$ for $k = m$. Let $\Phi_{jm} \in C^\infty_0 (\mathbb{R}^3 \setminus Z)$ be defined by the relation $\Phi_{jm} = \Phi_{f_j, \chi, m}$, where $\Phi_{\phi, \chi, m}$ for $\phi \in C^\infty_0 (\mathbb{R}^+)$ is given by (83). It follows from (81) and (83) that $(V[\Phi_{jm}]\chi)(s) = \xi_{jm}(s)$ for $\nu_0$-a.e. $s$ and, hence, $V[\Phi_{jm}]A = \xi_{jm}$. Thus, $\xi_{jm} \in V(D_{H_0})$ for all $j = 1, 2 \ldots$ and $m \in \mathbb{Z}$. Since $\hat{\chi}$ is the restriction to $\mathbb{R}$ of a nontrivial entire function, the set of its zeros is at most countable. This implies that the elements $(\xi_{jm}(s), \mathcal{H}^0(s)\xi_{jm}(s))$ are dense in the graph of $\mathcal{H}^0(s)$ for $\nu_0$-a.e. $s$. The statement of the lemma therefore follows from (9) and Proposition 5.3.

Let $\xi \in \mathbb{R}$ and $\theta$ be a Borel real function on $A^\phi$, where $A^\phi$ is given by (18). We denote by $\mathcal{H}_\phi$ the operator-valued map on $S$ such that

$$\mathcal{H}_\phi(s) = \begin{cases} 
\mathcal{H}^\phi(s), & s \in S \setminus A^\phi, \\
h_{m,0,\xi}(s) + p^2 1_B, & s \in A^\phi,
\end{cases}$$

for every $s = (m,p) \in S$, where $\mathcal{H}^\phi$ is defined by (89) and the operator $h_{m,0,\xi}$ for $-1 < \kappa < 1$ and $\phi \in \mathbb{R}$ is given by (67).

**Lemma 8.4.** Let $\phi \in \mathbb{R}$. For any Borel real function $\theta$ on $A^\phi$, $\mathcal{H}_\phi$ is a $\nu_0$-measurable family of self-adjoint operators in $\mathfrak{h}$ such that $\mathcal{H}_\phi(s)$ is an extension of $\mathcal{H}(s)$ for every $s \in S$. If $\mathcal{H}$ is a $\nu_0$-measurable family of operators in $\mathfrak{h}$ such that $\mathcal{H}(s)$ is a self-adjoint extension of $\mathcal{H}(s)$ for $\nu_0$-a.e. $s$, then $\mathcal{H}$ is $\nu_0$-equivalent to $\mathcal{H}_\phi$ for some Borel real function $\phi$ on $A^\phi$.

**Proof.** Let $v$ be the map on $S$ defined by by the relation

$$v(m,p) = q_{m,0}, \quad (m,p) \in S,$$

where $q_{m,0, \kappa} \in \mathbb{R}$, is given by (54). Then $v$ is a $\nu_0$-measurable family of functions on $\mathbb{R}_+$, and (55) implies that

$$L_{v(s)} = h_{m,0,\phi}$$

for any $s = (m,p) \in S$. Clearly, $v(s)$ is in l.c.c. at 0 if and only if $s \in A^\phi$. Given a Borel real function $\theta$ on $A^\phi$, we define the map $\xi_{\theta}$ on $A^\phi$ by setting

$$\xi_{\theta}(s) = w^{m,0}(\cos (\theta(s) - \vartheta_{m,0} + w^{m,0}(\sin (\theta(s) - \vartheta_{m,0})))$$

for every $s = (m,p) \in A^\phi$, where $w^\kappa$, $v^\kappa$, and $\vartheta_{m,0}$ are given by (22), (63), and (12) respectively. By (64), (67), and (61), we have $l_{v(s)}\xi_{\theta}(s) = 0$ for every $s \in A^\phi$. It follows from (64), (67), (61), and (60) that

$$h_{m,0,\xi}(s) = L_{v(s)}\xi_{\theta}(s), \quad s = (m,p) \in A^\phi.$$

Since $\xi_{\theta}$ is obviously a $\nu_0|_{A^\phi}$-measurable family of functions on $\mathbb{R}_+$, it follows from Proposition 5.5 that $s \rightarrow h_{m,0,\xi}(s)$ is a $\nu_0$-measurable family of operators in $\mathfrak{h}$ on $A^\phi$. Since $(m,p) \rightarrow p^2 1_B$ is a $\nu_0$-measurable family of operators in $\mathfrak{h}$ by Lemma 3.3, the $\nu_0$-measurability of $\mathcal{H}_\phi$ on $A^\phi$ follows from (93) and Lemma 5.3. As $\mathcal{H}_\phi$ is $\nu_0$-measurable on $S \setminus A^\phi$ by (94) and Lemma 5.3, we see that $\mathcal{H}_\phi$ is a $\nu_0$-measurable family of operators in $\mathfrak{h}$. It follows immediately from (89) and (93) that $\mathcal{H}_\phi(s)$ is a self-adjoint extension of $\mathcal{H}(s)$ for every $s \in S$. 


Let $\mathcal{H}$ be a $\nu_0$-measurable family of operators in $\mathfrak{h}$ such that $\mathcal{H}(s)$ is a self-adjoint extension of $\mathcal{H}^0(s)$ for $\nu_0$-a.e. $s$. Let $\mathcal{R}$ be a $\nu_0$-a.e. defined family of operators in $\mathfrak{h}$ such that $\mathcal{R}(s) = \mathcal{H}(s) - p^2 1_{\mathfrak{h}}$ for $\nu_0$-a.e. $s = (m, p)$. By Lemma 3.3 $\mathcal{R}$ is $\nu_0$-measurable. In view of (89) and (95) $\mathcal{R}(s)$ is a self-adjoint extension of $L_\nu(s)$ for $\nu_0$-a.e. $s$. By Proposition 7.3 there exists a $\nu_0$-a.e. defined map $\xi$ on $A^0$ such that $\xi$ is a $\nu_0|A^0$-measurable family of functions on $\mathbb{R}_+$, $\xi(s)$ is a nontrivial real element of $\mathcal{D}$ satisfying $l_{\nu(s)}\xi(s) = 0$ for $\nu_0$-a.e. $s \in A^0$, and $\mathcal{R}(s) = L_\nu^{\xi(s)}$ for $\nu_0$-a.e. $s \in A^0$. By [50], (95), and Lemma 6.12 $u^{\nu_0}(0)$ and $w^{\nu_0}(0)$ are real linearly independent solutions of $l_\nu u^{\nu_0}(0) = l_\nu w^{\nu_0}(0) = 0$ for every $s = (m, p) \in A^0$. Hence, there exist $\nu_0$-a.e. defined real functions $C_1$ and $C_2$ on $A^0$ such that

$$\xi(s) = C_1(s)u^{\nu_0}(0) + C_2(s)w^{\nu_0}(0)$$

for $\nu_0$-a.e. $s = (m, p) \in A^0$. In view of Lemma 6.12 we have

$$C_1(s) = \frac{\pi}{2} W(\xi(s), w^{\nu_0}(0)), \quad C_2(s) = -\frac{\pi}{2} W(\xi(s), u^{\nu_0}(0))$$

for $\nu_0$-a.e. $s = (m, p) \in A^0$. It follows from statements 3 and 5 of Lemma 7.1 that $C_1$ and $C_2$ are $\nu_0$-measurable functions on $A^0$. Since $\xi(s) \neq 0$, we have $C_1(s)^2 + C_2(s)^2 \neq 0$ for $\nu_0$-a.e. $s \in A^0$. Let $\theta$ be a Borel function on $A^0$ such that

$$\theta(s) = \vartheta_{m^0} + \tau(C_2(s)) \arccos \frac{C_1(s)}{C_1(s)^2 + C_2(s)^2}$$

for $\nu_0$-a.e. $s = (m, p) \in A^0$, where $\tau(y) = 1$ for $y \geq 0$ and $\tau(y) = -1$ for $y < 0$. We then have

$$\cos(\theta(s) - \vartheta_{m^0}) = \frac{C_1(s)}{C_1(s)^2 + C_2(s)^2}, \quad \sin(\theta(s) - \vartheta_{m^0}) = \frac{C_2(s)}{C_1(s)^2 + C_2(s)^2}$$

for $\nu_0$-a.e. $s = (m, p) \in A^0$. This means that $\xi(s)$ is proportional to the function $\xi_\theta(s)$ given by (94) and it follows from (97) that $\mathcal{R}(s) = h_{\nu_0, \theta(s)}$ for $\nu_0$-a.e. $s = (m, p) \in A^0$. Hence, $\mathcal{H}(s) = \mathcal{H}^\theta_\nu(s)$ for $\nu_0$-a.e. $s \in A^0$. As $\mathcal{H}^\theta(s)$ is self-adjoint for all $s \in S \setminus A^0$, we have $\mathcal{H}(s) = \mathcal{H}^\theta(s) = \mathcal{H}^\theta_0(s)$ for $\nu_0$-a.e. $s \in S \setminus A^0$. We therefore have $\mathcal{H}(s) = \mathcal{H}^\theta_0(s)$ for $\nu_0$-a.e. $s$.

Given $\phi \in \mathbb{R}$ and a Borel real function $\theta$ on $A^0$, we define the operator $R^\theta_\phi$ in $L_2(\mathbb{R}^3)$ by the relation

$$(98) \quad R^\theta_\phi = V^{-1} \int_\mathbb{R}^3 \mathcal{H}^\theta_\nu(s) d\nu_0(s) V,$$

where the unitary operator $V \colon L_2(\mathbb{R}^3) \to L_2(S, \mathfrak{h}, \nu_0)$ is as in Lemma 8.1.

**Proposition 8.5.** Let $\phi \in \mathbb{R}$. For any Borel real function $\theta$ on $A^0$, $R^\theta_\phi$ is a self-adjoint extension of $H^\phi$ commuting with $T_G$ for all $G \in \mathcal{G}$. Every self-adjoint extension of $H^\phi$ commuting with $T_G$ for all $G \in \mathcal{G}$ is equal to $R^\phi_\theta$ for some Borel real function $\theta$ on $A^0$. Given Borel real functions $\theta$ and $\theta$ on $A^0$, we have $R^\phi_\theta = R^\phi_\theta$ if and only if $\theta(s) - \theta(s) \in \pi \mathbb{Z}$ for $\nu_0$-a.e. $s \in A^0$.

**Proof.** If $\theta$ is a Borel real function on $A^0$, then it follows immediately from Corollary 8.2 and Lemmas 8.2, 8.3, and 8.4 that $R^\phi_\theta$ is a self-adjoint extension of $H^\phi$ commuting with $T_G$ for all $G \in \mathcal{G}$.
Conversely, let \( \hat{H} \) be a self-adjoint extension of \( H^\phi \) commuting with \( T_G \) for all \( G \in \mathcal{G} \). By Corollary 5.2 and Lemmas 8.2 and 8.3 there is a \( \nu_0 \)-measurable family \( \mathcal{H} \) of operators in \( \mathfrak{h} \) such that \( \mathcal{H}(s) \) is a self-adjoint extension of \( H^\phi(s) \) for \( \nu_0 \)-a.e. \( s \) and

\[
(99) \quad \hat{H} = V^{-1} \int_{\mathbb{R}} \mathcal{H}(s) \, d\nu_0(s) \, V.
\]

By Lemma 8.4 there is a Borel real function \( \vartheta \) on \( A^\phi \) such that \( \mathcal{H}(s) = H^\phi_{\vartheta}(s) \) for \( \nu_0 \)-a.e. \( s \). In view of (98) and (99), it follows that \( \hat{H} = R^\phi_{\vartheta} \).

Let \( \vartheta \) and \( \tilde{\vartheta} \) be Borel real functions on \( A^\phi \). By Proposition 13.20 the equality \( R^\phi_{\vartheta} = R^\phi_{\tilde{\vartheta}} \) holds if and only if \( \mathcal{H}^\vartheta_{\vartheta}(s) = \mathcal{H}^\vartheta_{\tilde{\vartheta}}(s) \) for \( \nu_0 \)-a.e. \( s \). By (67) and (68), the latter condition is fulfilled if and only if \( u^m_{\vartheta(s)}(0) \) is proportional to \( u^m_{\tilde{\vartheta}(s)}(0) \) for \( \nu_0 \)-a.e. \( s = (m,p) \in A^\phi \). In view of (64), this is true if and only if \( \vartheta(s) - \tilde{\vartheta}(s) \in \pi \mathbb{Z} \) for \( \nu_0 \)-a.e. \( s \in A^\phi \).

\[ \Box \]

9. Direct integrals of measures

**Definition 9.1.** Let \( \nu \) be a \( \sigma \)-finite positive measure, \( \Sigma \) be a \( \sigma \)-algebra, and a map \( \mu \) be such that \( \mu(s) \) is a \( \sigma \)-finite positive measure for \( \nu \)-a.e. \( s \). Let the \( \delta \)-ring \( Q^\Sigma_{\mu,\nu} \) be defined by the relation

\[
Q^\Sigma_{\mu,\nu} = \{ A \in \Sigma : A \in D^\mu_{\mu(s)} \text{ for } \nu \text{-a.e. } s \}
\]

We say that \( \mu \) is a \( (\nu,\Sigma) \)-measurable family of measures if \( \Sigma = \sigma(Q^\Sigma_{\mu,\nu}) \) and \( s \mapsto \mu(s|A) \) is a \( \nu \)-measurable function for any \( A \in Q^\Sigma_{\mu,\nu} \).

**Lemma 9.2.** Let \( \nu, \Sigma, \mu, \) and \( Q^\Sigma_{\mu,\nu} \) be as in Definition 9.1. Let \( K \subset Q^\Sigma_{\mu,\nu} \) be closed under finite intersections and satisfy \( \sigma(K) = \Sigma \). Let a map \( \xi \) be such that, for every \( A \in Q^\Sigma_{\mu,\nu} \), \( \xi(s) \) is a \( \mu(s) \)-integrable complex function on \( A \) for \( \nu \)-a.e. \( s \). If \( s \mapsto \int_A \xi(s|E) \, d\mu(s|E) \) is a \( \nu \)-measurable function for any \( A \in K \), then the same is true for every \( A \in Q^\Sigma_{\mu,\nu} \).

If \( \xi(s) \) is \( \mu(s) \)-equivalent to unity for \( \nu \)-a.e. \( s \), this lemma reduces to the next result.

**Corollary 9.3.** Let \( \nu, \Sigma, \mu, \) and \( K \) be as in Lemma 9.2. If \( s \mapsto \mu(s|A) \) is a \( \nu \)-measurable function for any \( A \in K \), then \( \mu \) is a \( (\nu,\Sigma) \)-measurable family of measures.

For the proof of Lemma 9.2 we need the next definition.

**Definition 9.4.** We say that a nonempty set of sets \( Q \) is an \( \alpha \)-class if it satisfies the following conditions

1. If \( A, B \in Q \) and \( A \cap B = \emptyset \), then \( A \cup B \in Q \).
2. If \( A, B \in Q \) and \( B \subset A \), then \( A \setminus B \in Q \).
3. If \( A_1 \supset A_2 \supset \ldots \) is a nonincreasing sequence of elements of \( Q \), then \( \bigcap_{i=1}^{\infty} A_i \in Q \).

**Remark 9.5.** In the context of \( \delta \)-rings, the role of \( \alpha \)-classes is essentially the same as that of Dynkin systems (see, e.g., Sec. 2 in [5]) with respect to \( \sigma \)-algebras. In particular, Lemma 9.6 below is an analogue of Theorem 2.4 in [5].
Lemma 9.6. Let $Q$ be an $\alpha$-class. If there exists a set $K \subset Q$ that is closed under finite intersections and satisfies $Q \subset \sigma(K)$, then $Q$ is a $\delta$-ring.

Proof of Lemma 9.2. It is straightforward to check that the set

$$L = \{ A \in Q_\mu^\Sigma : s \to \int_A \xi(s|E) d\mu(s|E) \text{ is a } \nu\text{-measurable function} \}.$$ 

is an $\alpha$-class containing $K$. As $\sigma(K) = \Sigma$, we have $L \subset \sigma(K)$, and Lemma 9.3 implies that $L$ is a $\delta$-ring. As $\sigma(L) = \Sigma$, every $A \in Q_\mu^\Sigma$ is representable as the union of a sequence $A_1 \subset A_2 \subset \ldots$ of elements of $L$. Hence,

$$\int_A \xi(s|E) d\mu(s|E) = \lim_{k \to \infty} \int_{A_k} \xi(s|E) d\mu(s|E)$$

for $\nu$-a.e. $s$ and, therefore, $s \to \int_A \xi(s|E) d\mu(s|E)$ is $\nu$-measurable. Thus, $Q_\mu^\Sigma = L$ and the lemma is proved.

Proof of Lemma 9.6. Let $L$ be the smallest $\alpha$-class containing $K$. We first prove that $L$ is a $\delta$-ring. To this end, it suffices to show that $L$ is closed under finite intersections because every $\alpha$-class with this property is a $\delta$-ring. For $A \in L$, we set

$$L^A = \{ B \in L : A \cap B \in L \}.$$ 

Let $A \in L$ and $B, C \in L^A$ be such that $B \cap C = \emptyset$. As $(A \cap B) \cap (A \cap C) = \emptyset$ and both sets $A \cap B$ and $A \cap C$ belong to $L$, the set $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ also belongs to $L$ by condition 1 of Definition 9.4. This means that $B \cup C \in L^A$.

If $B, C \in L^A$ are such that $C \subset B$, then $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$. Since both $A \cap B$ and $A \cap C$ belong to $L$ and $A \cap C \subset A \cap B$, we have $A \cap (B \setminus C) \in L$ by condition 2 of Definition 9.4 and, hence, $B \setminus C \in L^A$.

Now let $B_1 \supset B_2 \supset \ldots$ be a nonincreasing sequence of elements of $L^A$ and let $B = \bigcap_{i=1}^\infty B_i$. Since $A \cap B = \bigcap_{i=1}^\infty (A \cap B_i)$ and the sets $A \cap B_1, A \cap B_2, \ldots$ constitute a nonincreasing sequence of elements of $L$, we have $A \cap B \in L$ by condition 3 of Definition 9.4 and, therefore, $B \in L^A$.

It follows from the above that $L^A$ is an $\alpha$-class for any $A \in L$. If $B \in K$, then we obviously have $K \subset L^B$ and, hence, $L^B = L$. It follows that $A \cap B \in L$ for arbitrary $A \in L$ and $B \in K$ and, therefore, $B \in L^A$. This means that $K \subset L^A$ for every $A \in L$. Hence, $L^A = L$ for every $A \in L$ and, consequently, $A \cap B \in L$ for every $A, B \in L$. Thus, $L$ is a $\delta$-ring.

We now show that $Q$ is a $\delta$-ring. As for $L$, it suffices to show that $A \cap B \in Q$ for any $A, B \in Q$. As $L$ is a $\delta$-ring, every element of $\sigma(K) = \sigma(L)$ is a countable union of elements of $L$. Let $A_1 \subset A_2 \subset \ldots$ and $B_1 \subset B_2 \subset \ldots$ be nondecreasing sequences of elements of $L$ such that $A = \bigcup_{i=1}^\infty A_i$ and $B = \bigcup_{i=1}^\infty B_i$. As $L$ is a ring, $C_i = A_i \cap B_i$ belongs to $L$ (and, hence, to $Q$) for every $i = 1, 2, \ldots$. It follows from condition 2 of Definition 9.4 that $C_i = A \setminus C_i$ belongs to $Q$ for every $i = 1, 2, \ldots$. As $C_1 \supset C_2 \supset \ldots$, condition 3 of Definition 9.4 implies that $A \setminus B = \bigcap_{i=1}^\infty C_i$ belongs to $Q$. Applying condition 2 of Definition 9.4 again, we conclude that $A \cap B = A \setminus (A \setminus B)$ belongs to $Q$. 

\footnote{It is easy to see that the intersection of any set of $\alpha$-classes is again an $\alpha$-class (note that such an intersection is always nonempty because $\emptyset$ is an element of every $\alpha$-class). Hence, given a set of sets $K$, there exists the smallest $\alpha$-class containing $K$.}
Given sets of sets $Q_1$ and $Q_2$, we denote by $Q_1 \boxtimes Q_2$ the set of all sets $A_1 \times A_2$, where $A_1 \in Q_1$ and $A_2 \in Q_2$. Let $A$ be a set. For any $s$, we define its section $A_s$ by the relation $A_s = \{ E \colon (s, E) \in A \}$.

Let $\nu$ be a $\sigma$-finite positive measure, $\Sigma$ be a $\sigma$-algebra, and $\mu$ be a $\nu$-measurable family of measures. We set $\Xi_{\mu, \nu} = \sigma(D_\nu \boxtimes \Sigma)$ and denote by $\Delta_{\mu, \nu}$ the set of all $A \in \Xi_{\mu, \nu}$ such that $A_s \subseteq D_{\mu(s)}$ for $\nu$-a.e. $s$ and $s \to \mu(s|A_s)$ is a $\nu$-integrable function.

**Proposition 9.7.** Let $\nu$ be a $\sigma$-finite positive measure, $\Sigma$ be a $\sigma$-algebra, and $\mu$ be a $\nu$-measurable family of measures. Then there is a unique $\Xi_{\mu, \nu}$-compatible measure $M$ such that $\Delta_{\mu, \nu} \subset D_M$ and

$$M(A) = \int \mu(s|A_s) \, d\nu(s)$$

for any $A \in \Delta_{\mu, \nu}$. Moreover, the equality $\Xi_{\mu, \nu} \cap D_M = \Delta_{\mu, \nu}$ is fulfilled. If $\nu$ is $\Sigma$-compatible for some $\sigma$-algebra $\Sigma$, then $M$ is $\sigma(\Sigma \boxtimes \Sigma)$-compatible.

**Proof.** Let $Q = Q_{\mu, \nu}^\Sigma$, $\Xi = \Xi_{\mu, \nu}^\Sigma$, $\Delta = \Delta_{\mu, \nu}^\Sigma$, and $K = \Delta \cap (D_\nu \boxtimes Q)$. Since $Q$ is a $\delta$-ring, $K$ is closed under finite intersections. Given $B \in D_\nu$, $C \in Q$, and $n = 1, 2, \ldots$, we set $B_n = \{ s \in B : \mu(s|C) \leq n \}$. Clearly, $B = N \cup \bigcup_{n} B_n$, where $N$ is a $\nu$-null set. Since $B_n \times C \in K$ for all $n$ and $N \times C \in K$, we conclude that $B \times C \in \sigma(K)$. Thus, $D_\nu \boxtimes Q \subset \sigma(K)$. As $\sigma(Q) = \Sigma$, it follows that $\sigma(K) = \Xi$. As $\Delta$ is obviously an $\alpha$-class, Lemma 9.6 implies that $\Delta$ is a $\delta$-ring. Let the function $m \colon \Delta \to \mathbb{R}$ be such that $m(A)$ is equal to the right-hand side of (100) for any $A \in \Delta$. It follows from the monotone convergence theorem that $m$ is a $\sigma$-additive function satisfying the condition (a) of Sec. A.2. Hence, the existence and uniqueness of the measure $M$, as well as the equality $\Xi \cap D_M = \Delta$, are ensured by Lemma A.3. Now suppose $\nu$ is $\Sigma$-compatible and let $L$ denote the set of all $A \in \Xi$ such that there exist $B_1, B_2 \in \sigma(\Sigma \boxtimes \Sigma)$ satisfying the conditions $B_1 \subset A \subset B_2$ and $M(B_2 \setminus B_1) = 0$. Then $L$ is a $\sigma$-algebra that contains $D_\nu \boxtimes \Sigma$ and, hence, coincides with $\Xi$. In view of the $\Xi$-compatibility of $M$, this implies that $M$ is $\sigma(\Sigma \boxtimes \Sigma)$-compatible. \hfill $\Box$

**Definition 9.8.** Under the conditions of Proposition 9.7 we call the measure $M$ the direct integral of $\mu$ with respect to $(\nu, \Sigma)$ and denote it by $(\Sigma)\int \mu(s) \, d\nu(s)$.

**Lemma 9.9.** Let $\nu$ be a $\sigma$-finite positive measure, $\Sigma$ be a $\sigma$-algebra, $\mu$ be a $\nu$-measurable family of measures, and $M = (\Sigma)\int \mu(s) \, d\nu(s)$. An $M$-measurable set $N$ is an $M$-null set if and only if $N_s$ is a $\mu(s)$-null set for $\nu$-a.e. $s$.

**Proof.** Let $\Xi = \Xi_{\mu, \nu}^\Sigma$ and $N$ be an $M$-measurable set. By the $\Xi$-compatibility of $M$, there is a set $N' \in \Xi$ such that $N' \supseteq N$ and $N' \setminus N$ is an $M$-null set. If $N$ is an $M$-null set, then $N'$ is also an $M$-null set, and it follows from Proposition 9.7 that $N'_s \in D_{\mu(s)}$ for $\nu$-a.e. $s$ and $\int \mu(s|N'_s) \, d\nu(s) = 0$, i.e., $N'_s$ is a $\mu(s)$-null set for $\nu$-a.e. $s$. Since $N_s \subseteq N'_s$, this implies that $N_s$ is a $\mu(s)$-null set for $\nu$-a.e. $s$. Conversely, suppose $N_s$ is a $\mu(s)$-null set for $\nu$-a.e. $s$. By the above $(N' \setminus N)_s$ is a $\mu(s)$-null set and, hence, $N'_s$ is a $\mu(s)$-null set for $\nu$-a.e. $s$. Proposition 9.7 now implies that $M(N') = 0$ and, therefore, $N$ is an $M$-null set. \hfill $\Box$

**Proposition 9.10.** Let $\nu$, $\Sigma$, $\mu$, and $M$ be as in Lemma 9.9. For any $M$-integrable complex function $f$, the function $E \to f(s, E)$ is $\mu(s)$-integrable for $\nu$-a.e. $s$ and
we have

\[ (101) \quad \int f(s, E) \, dM(s, E) = \int d\nu(s) \int f(s, E) \, d\mu(s|E). \]

Proof. Let \( \Xi = \Xi^\Sigma_{\mu, \nu} \). As \( M \) is \( \Xi \)-compatible, every \( A \in D_M \) is representable in the form \( A = B \cup N \), where \( B \in D_M \cap \Xi \) and \( N \) is an \( M \)-null set. By Proposition 9.7 we have \( B \in \Delta^\Sigma_{\mu, \nu} \). Hence, \( B \in D_{\mu(s)} \) for \( \nu \)-a.e. \( s \), the function \( s \rightarrow \mu(s|B_s) \) is \( \nu \)-integrable, and \( M(B) = \int \mu(s|B_s) \, d\nu(s) \). Since \( A_s = B_s \cup N_s \), it follows from Lemma 9.9 that \( A_s \in D_{\mu(s)} \) and \( \mu(s|A_s) = \mu(s|B_s) \) for \( \nu \)-a.e. \( s \). Hence, the function \( s \rightarrow \mu(s|A_s) \) is \( \nu \)-integrable, and \( M(A) = \int \mu(s|A_s) \, d\nu(s) \). This proves the proposition for \( f = X_A \), where \( X_A \) is the function on \( S_M \) that is equal to unity on \( A \) and vanishes on \( S_M \setminus A \). We say that a function \( f \) on \( S_M \) is simple if it can be represented in the form \( f = \sum_{i=1}^n c_i X_{A_i} \), with \( c_i \in \mathbb{C} \) and \( A_i \in D_M \). Clearly, the proposition is true if \( f \) is a simple function. If \( f \) is a nonnegative \( M \)-integrable function, then there is a nondecreasing sequence \( f_k \) of nonnegative simple functions such that \( f_k(s, E) \leq f(s, E) \) and \( f_k(s, E) \rightarrow f(s, E) \) as \( k \rightarrow \infty \) for \( M \)-a.e. \( (s, E) \). Let \( g_k \) be a \( \nu \)-integrable function such that \( g_k(s) = \int f_k(s, E) \, d\mu(s|E) \) for \( \nu \)-a.e. \( s \). The dominated convergence theorem implies that

\[ \int f(s, E) \, dM(s, E) = \lim_{k \rightarrow \infty} \int f_k(s, E) \, dM(s, E) = \lim_{k \rightarrow \infty} \int g_k(s) \, d\nu(s). \]

By the monotone convergence theorem, there exists a \( \nu \)-integrable function \( g \) such that \( g(s) = \lim_{k \rightarrow \infty} g_k(s) \) for \( \nu \)-a.e. \( s \), and we have

\[ (102) \quad \int f(s, E) \, dM(s, E) = \int g(s) \, d\nu(s). \]

By Lemma 9.9 for \( \nu \)-a.e. \( s \), the relations \( f_k(s, E) \leq f(s, E) \) and \( f_k(s, E) \rightarrow f(s, E) \) hold for \( \mu(s|E) \) a.e. \( E \). In view of the existence of \( \lim_{k \rightarrow \infty} g_k(s) \), the monotone convergence theorem implies that the function \( E \rightarrow f(s, E) \) is \( \mu(s|E) \)-integrable and \( \int f(s, E) \, d\mu(s|E) = g(s) \) for \( \nu \)-a.e. \( s \). Substituting this equality in (102) yields (101). To complete the proof, it remains to note that every \( M \)-integrable function is a linear combination of nonnegative \( M \)-integrable functions. \( \square \)

Corollary 9.11. Let \( \nu \), \( \Sigma \), \( \mu \), and \( M \) be as in Lemma 9.9 and \( f \) be an \( M \)-measurable complex function. Then \( E \rightarrow f(s, E) \) is a \( \mu(s|E) \)-measurable complex function for \( \nu \)-a.e. \( s \). If \( E \rightarrow f(s, E) \) is \( \mu(s|E) \)-integrable for \( \nu \)-a.e. \( s \), then \( s \rightarrow \int f(s, E) \, d\mu(s|E) \) is a \( \nu \)-measurable function. If, in addition, \( f \) is nonnegative and \( s \rightarrow \int f(s, E) \, d\mu(s|E) \) is a \( \nu \)-integrable function, then \( f \) is \( M \)-integrable.

Proof. Let \( A_1 \subset A_2 \subset \ldots \) be a sequence of elements of \( D_M \) such that \( S_M = \bigcup_{k=1}^\infty A_k \). Let \( B_k = A_k \cap \{(s, E) \in S_M : |f(s, E)| \leq k\} \) and \( f_k = X_{B_k} f \), where \( X_{B_k} \) is as in the proof of Proposition 9.10. Then \( S_M \) coincides with \( \bigcup_{k=1}^\infty B_k \) up to an \( M \)-null set and, therefore, \( f_k(s, E) \rightarrow f(s, E) \) as \( k \rightarrow \infty \) for \( M \)-a.e. \( (s, E) \). Since \( f_k \) are \( M \)-integrable for all \( k \), Lemma 9.9 and Proposition 9.10 imply that, for \( \nu \)-a.e. \( s \), the functions \( E \rightarrow f_k(s, E) \) are \( \mu(s|E) \)-integrable and converge \( \mu(s|E) \)-a.e. to the function \( E \rightarrow f(s, E) \). Hence, the latter is \( \mu(s|E) \)-measurable for \( \nu \)-a.e. \( s \). If it is also \( \mu(s|E) \)-integrable for \( \nu \)-a.e. \( s \), then it follows from the dominated convergence theorem that \( \int f(s, E) \, d\mu(s|E) = \lim_{k \rightarrow \infty} \int f_k(s, E) \, d\mu(s|E) \) for \( \nu \)-a.e. \( s \). Since \( s \rightarrow \int f_k(s, E) \, d\mu(s|E) \) are \( \nu \)-measurable functions by Proposition 9.10 we...
conclude that \( s \to \int f(s, E) d\mu(s|E) \) is \( \nu \)-measurable. If \( f \) is nonnegative and the function \( s \to \int f(s, E) d\mu(s|E) \) is \( \nu \)-integrable, then Proposition 9.10 implies that
\[
\int f_k(s, E) dM(s, E) \leq \int d\nu(s) \int f(s, E) d\mu(s|E)
\]
for all \( k \). The \( M \)-integrability of \( f \) therefore follows from the monotone convergence theorem. \( \square \)

**Lemma 9.12.** Let \( K \) be a set of sets closed under finite intersections and \( \nu \) be a positive \( \sigma \)-finite \( \sigma(K) \)-compatible measure. Suppose \( f \) is a \( \nu \)-measurable complex function that is \( \nu \)-integrable on every set in \( K \). If \( \int_A f(s) d\nu(s) = 0 \) for every \( A \in K \), then \( f(s) = 0 \) for \( \nu \)-a.e. \( s \).

**Proof.** If \( f \) satisfies the conditions of the lemma, then the same is true for its real and imaginary parts. So we can assume that \( f \) is real. Let
\[
Q = \{ A \in \sigma(K) : f \text{ is } \nu \text{-integrable on } A \text{ and } \int_A f(s) d\nu(s) = 0 \}.
\]
Clearly, \( Q \) is an \( \alpha \)-class satisfying \( K \subset Q \subset \sigma(K) \) and, therefore, is a \( \delta \)-ring by Lemma 9.6. As \( \nu \) is \( \sigma(K) \)-compatible and \( f \) is real, every \( A \in Q \) can be represented in the form \( A = A_+ \cup A_- \), where \( A_+ \in \sigma(K) \) are such that \( f(s) \geq 0 \) for \( \nu \)-a.e. \( s \in A_+ \) and \( f(s) \leq 0 \) for \( \nu \)-a.e. \( s \in A_- \). Since \( A_{\pm} \) are elements of \( \sigma(Q) \) contained in an element of \( Q \), we conclude that \( A_{\pm} \in Q \) and, therefore, \( \int_{A_{\pm}} f(s) d\nu(s) = 0 \). This implies that \( f(s) = 0 \) for \( \nu \)-a.e. \( s \in A \). Since \( S_\nu \) is, up to a \( \nu \)-null set, a countable union of elements of \( Q \), we have \( f(s) = 0 \) for \( \nu \)-a.e. \( s \). \( \square \)

**Proposition 9.13.** Let \( \nu \) be a \( \sigma \)-finite positive measure, \( \Sigma \) be a countably generated \( \sigma \)-algebra, and \( \mu \) be a \((\nu,\Sigma)\)-measurable family of measures such that \( \mu(s) \) is \( \Sigma \)-compatible for \( \nu \)-a.e. \( s \). Then there is a unique (up to \( \nu \)-identity) \( \nu \)-measurable family \( \mathcal{G} \) of Hilbert spaces satisfying the conditions:

1. \( \mathcal{G}(s) = L_2(G, \mu(s)) \) for \( \nu \)-a.e. \( s \), where \( G \) is the largest set in \( \Sigma \).
2. \( A \) \( \nu \)-a.e. defined section \( \xi \) of \( \mathcal{G} \) is \( \nu \)-measurable if and only if \( s \to \int_A \xi(s|E) d\mu(s|E) \) is a \( \nu \)-measurable function for any \( A \in \mathcal{Q}_\mu(\nu) \).

Let \( K \subset \mathcal{Q}_\mu(\nu) \) be closed under finite intersections and satisfy \( \sigma(K) = \Sigma \). If \( \xi \) is a \( \nu \)-a.e. defined section of \( \mathcal{G} \) such that \( s \to \int_A \xi(s|E) d\mu(s|E) \) is a \( \nu \)-measurable function for any \( A \in K \), then \( \xi \) is \( \nu \)-measurable.

**Proof.** Let \( L_0 \) be a countable set generating \( \Sigma \). Without loss of generality, we can assume that \( L_0 \) is closed under finite intersections. Let \( B_1 \subset B_2 \subset \ldots \) be a sequence of elements of \( \mathcal{Q}_\mu(\nu) \) such that \( G = \bigcup_{i=1}^{\infty} B_i \) and let \( \mathcal{L} \) be the set of all sets of the form \( A \cap B_i \) for some \( A \in L_0 \) and \( i = 1, 2, \ldots \). Then \( \mathcal{L} \) is a countable subset of \( \mathcal{Q}_\mu(\nu) \) that is closed under finite intersections and satisfies \( \Sigma = \sigma(\mathcal{L}) \). Let \( \mathcal{G} \) be \( \nu \)-a.e. defined family of Hilbert spaces satisfying (1). We choose a numbering \( A_1, A_2, \ldots \) of the set \( \mathcal{L} \) and endow \( \mathcal{G} \) with a \( \nu \)-measurable structure by setting
\[
\xi_i^\mathcal{G}(s) = [\chi_{A_i}]_\mu(s), \quad i = 1, 2, \ldots,
\]
for \( \nu \)-a.e. \( s \), where the function \( \chi_{A_i} \) is equal to unity on \( A_i \) and vanishes on \( G \setminus A_i \). Since \( (\xi_i^\mathcal{G}(s), \xi_j^\mathcal{G}(s)) = (\mu(s|A_i \cap A_j)) \) and \( \mathcal{Q}_\mu(\nu) \) is a ring, \( s \to (\xi_i^\mathcal{G}(s), \xi_j^\mathcal{G}(s)) \) is a \( \nu \)-measurable function for any \( i, j = 1, 2, \ldots \). If \( \psi \in \mathcal{G}(s) \) is orthogonal to \( \xi_i^\mathcal{G}(s) \) for all \( i \), then Lemma 9.12 implies that \( \langle \psi \rangle _{\mu(s)} = 0 \). This means that the linear span of the sequence \( \xi_1^\mathcal{G}(s), \xi_2^\mathcal{G}(s), \ldots \) is dense in \( \mathcal{G}(s) \) for \( \nu \)-a.e. \( s \). Thus, \( \mathcal{G} \) endowed
with the sequence $\xi_1, \xi_2, \ldots$ is indeed a $\nu$-measurable family of Hilbert spaces. If $\xi$ is a $\nu$-measurable section of $\mathcal{G}$, then $s \to \langle \xi(s), [\chi_A]_{\mu(s)} \rangle = \int_A \xi(s|E) \, d\mu(s|E)$ is a $\nu$-measurable function for any $A \in \mathcal{L}$, and Lemma 9.2 implies that this is also true for every $A \in Q^\Sigma_{\mu,\nu}$. Thus, $\mathcal{G}$ satisfies (2). Suppose now that $K \subset Q^\Sigma_{\mu,\nu}$ is closed under finite intersections and satisfies $\sigma(K) = \Sigma$. By Lemma 9.2 if $\xi$ is a $\nu$-a.e. defined section of $\mathcal{G}$ such that $s \to \int_A \xi(s|E) \, d\mu(s|E)$ is a $\nu$-measurable function for any $A \in K$, then this is also true for any $A \in Q^\Sigma_{\mu,\nu}$ and it follows from (2) that $\xi$ is $\nu$-measurable.

**Proposition 9.14.** Let $\nu, \mu, \Sigma$, and $\mathcal{G}$ be as in Proposition 9.13. Let $\mathcal{H} = \int^\oplus \mathcal{G}(s) \, d\nu(s)$ and $M = \langle \Sigma \rangle = \int^\oplus \mu(s) \, d\nu(s)$. Given an $M$-measurable function $f$, let $\hat{f}$ denote the $\nu$-equivalence class such that $\hat{f}(s)$ is the $\mu(s)$-equivalence class of the map $E \to f(s, E)$ for $\nu$-a.e. $s$. Then the following statements hold:

1. $M$-measurable functions $f_1$ and $f_2$ are $M$-equivalent if and only if $\hat{f}_1 = \hat{f}_2$.
2. If $f \in L^2(S_M, M)$, then $\hat{f} \in \mathcal{H}$ and the operator $Q : L^2(S_M, M) \to \mathcal{H}$ taking $f$ to $\hat{f}$ is unitary.
3. If $g$ is an $M$-measurable complex function, then $\hat{g}(s)$ is $\mu(s)$-measurable for $\nu$-a.e. $s$, $s \to \mathcal{T}^\mu(s)$ is a $\nu$-measurable family of operators in $\mathcal{G}$, and

\[
Q \mathcal{T}^\mu(s) Q^{-1} = \int^\oplus \mathcal{T}^\mu(s) \, d\nu(s).
\]

**Proof.** 1. Since $f_1$ and $f_2$ are $M$-measurable, the set $N = \{ (s, E) \in S_M : f_1(s, E) \neq f_2(s, E) \}$ is $M$-measurable. By Lemma 9.12, $f_1$ and $f_2$ are $M$-equivalent if and only if $N_s$ is a $\mu(s)$-null set for $\nu$-a.e. $s$. Clearly, this condition holds if and only if $f_1 = f_2(s)$ for $\nu$-a.e. $s$ and, hence, $f_1 = f_2$.

2. Let $f \in L^2(S_M, M)$. By Corollary 9.11, $\hat{f}(s)$ is a $\mu(s)$-measurable function for $\nu$-a.e. $s$. Since $|f|^2$ is $M$-integrable, Proposition 9.11 implies that $|\hat{f}(s)|^2$ is $\mu(s)$-integrable and, therefore, $\hat{f}(s) \in \mathcal{G}(s)$ for $\nu$-a.e. $s$. For every $A \in Q^\Sigma_{\mu,\nu}$, the function $\chi_A \hat{f}(s)$, where $\chi_A$ is as in the proof of Proposition 9.13, is a product of two $\mu(s)$-square-integrable functions and, hence, is $\mu(s)$-integrable for $\nu$-a.e. $s$. Applying Corollary 9.11 to the function $s \to \chi_A(E) |f(s, E)|$, we conclude that $s \to \int_A \hat{f}(s|E) \, d\mu(s|E)$ is a $\nu$-measurable function for any $A \in Q^\Sigma_{\mu,\nu}$. In view of Proposition 9.13, this means that $\hat{f}$ is a $\nu$-measurable section of $\mathcal{G}$. By Proposition 9.10, $s \to \|\hat{f}(s)\|^2$ is a $\nu$-integrable function and we have $\|f\|^2 = \int \|\hat{f}(s)\|^2 \, d\nu(s)$. This means that $\hat{f}$ behaves like $f$ and the operator $Q$ is isometric. We now prove that $Q$ is unitary. For this, it suffices to show that its image is dense in $\mathcal{H}$. In other words, we have to show that every $\xi \in \mathcal{H}$ that is orthogonal to $\hat{f}$ for every $f \in L^2(S_M, M)$ is equal to zero. Given $A \in Q^\Sigma_{\mu,\nu}$, let $K_A$ denote the set of all $B \in D_\nu$ such that $B \times A \in D_M$. It is clear that $K_A$ is closed under finite intersections and $\sigma(K_A) = \sigma(D_\nu)$. For a set $C \subset S_M$, let $X_C$ denote the function on $S_M$ that is equal to unity on $C$ and vanishes on $S_M \setminus C$. For $A \in Q^\Sigma_{\mu,\nu}$ and $B \in K_A$, the function $X_{B \times A}$ is obviously $M$-square-integrable and, therefore, we have

\[
\int_B \langle [\chi_A]_{\mu(s)}, \xi(s) \rangle \, d\nu(s) = \int \langle X_{B \times A}(s), \xi(s) \rangle \, d\nu(s) = \langle X_{B \times A}, \xi \rangle = 0.
\]

Applying Lemma 9.12 to $K = K_A$, we conclude that $\langle [\chi_A]_{\mu(s)}, \xi(s) \rangle = 0$ for $\nu$-a.e. $s$ for every $A \in Q^\Sigma_{\mu,\nu}$. Acting as in the proof of Proposition 9.13 we choose a
sequence $A_1, A_2, \ldots$ of elements of $Q^{\infty}_{\mu, \nu}$ such that the linear span of $[\chi_{A_i}]_{\mu(s)}$ is dense in $\mathcal{S}(s)$ for $\nu$-a.e. $s$. Then, for $\nu$-a.e. $s$, $\xi(s)$ is orthogonal to all vectors $[\chi_{A_i}]_{\mu(s)}$ and, hence, is equal to zero. Thus, $\xi = 0$ and the unitarity of $Q$ is proved.

3. By Corollary 5.11, $\hat{g}(s)$ is $\mu(s)$-measurable for $\nu$-a.e. $s$. Let $C_1 \subset C_2 \subset \ldots$ be a sequence of elements of $D_M$ such that $S_M = \bigcup_{j=1}^{\infty} C_j$ and $g$ is $M$-essentially bounded on $C_j$ for every $j = 1, 2, \ldots$. For $i, j = 1, 2, \ldots$, we define the function $h_{ij}$ on $S_M$ by setting $h_{ij}(s, E) = \chi_{A_i}(E)X_{C_j}(s, E)$, where $A_i$ are as in the proof of (2). It is clear that $h_{ij}$ is $M$-square-integrable and, therefore, $\hat{h}_{ij}$ is a $\nu$-measurable section of $\mathcal{S}$ for all $i, j$. To prove the $\nu$-measurability of the family $s \to T_{\hat{g}(s)}^{\mu(s)}$, it suffices to show that $\hat{h}_{ij}(s)$ belongs to the domain of $T_{\hat{g}(s)}^{\mu(s)}$ and the linear span of the vectors $(\hat{h}_{ij}(s), T_{\hat{g}(s)}^{\mu(s)}\hat{h}_{ij}(s))$ is dense in the graph of $T_{\hat{g}(s)}^{\mu(s)}$ for $\nu$-a.e. $s$. For $j = 1, 2, \ldots$, let $P_j$ be a $\nu$-a.e. defined map such that $P_j(s) = T_{\chi_{C_j}}^{\mu(s)}$ for $\nu$-a.e. $s$, where $C_{j,s} = \{E : (s, E) \in C_j\}$. Then, for $\nu$-a.e. $s$, $P_j(s)$ is an orthogonal projection commuting with $T_{\hat{g}(s)}^{\mu(s)}$ and satisfying the equality

$$(103) \quad \hat{h}_{ij}(s) = P_j(s)[\chi_{A_i}(s)]_{\mu(s)}$$

for all $i, j$. In view of Lemma 5.9, $\hat{g}(s)$ is $\mu(s)$-essentially bounded on $C_{j,s}$ for $\nu$-a.e. $s$. Hence $\text{Im} P_j(s)$ is contained in the domain of $T_{\hat{g}(s)}^{\mu(s)}$ and $T_{\hat{g}(s)}^{\mu(s)}P_j(s)$ is a bounded operator for $\nu$-a.e. $s$. In particular, it follows from (103) that $\hat{h}_{ij}(s)$ is in the domain of $T_{\hat{g}(s)}^{\mu(s)}$ for $\nu$-a.e. $s$. Let $\mathcal{G}_s$ be the subset of the graph of $T_{\hat{g}(s)}^{\mu(s)}$ consisting of all its elements $(\psi, \tilde{\psi})$ such that $\psi, \tilde{\psi} \in \text{Im} P_j(s)$ for some $j = 1, 2, \ldots$. For $\nu$-a.e. $s$, we have $\lim_{j \to \infty} P_j(s)\psi = \psi$ for every $\psi \in \mathcal{S}(s)$. As $P_j(s)$ commute with $T_{\hat{g}(s)}^{\mu(s)}$, this implies that $\mathcal{G}_s$ is dense in the graph of $T_{\hat{g}(s)}^{\mu(s)}$ for $\nu$-a.e. $s$. Given $(\psi, \tilde{\psi}) \in \mathcal{G}_s$ and $\varepsilon > 0$, we can find a finite linear combination $\tau$ of vectors $[\chi_{A_i}]_{\mu(s)}$ such that $\|\psi - \tau\| < \varepsilon$. Then we have

$$\|\psi - P_j(s)\tau\| = \|P_j(s)(\psi - \tau)\| < \varepsilon,$$

$$\|\tilde{\psi} - T_{\hat{g}(s)}^{\mu(s)}P_j(s)\tau\| = \|T_{\hat{g}(s)}^{\mu(s)}P_j(s)(\psi - \tau)\| < \|T_{\hat{g}(s)}^{\mu(s)}P_j(s)\| \varepsilon,$$

where $j$ is such that $\psi, \tilde{\psi} \in \text{Im} P_j(s)$. In view of (103), $P_j(s)\tau$ is a linear combination of $\hat{h}_{ij}(s)$, and it follows from the above inequalities that the linear span of the vectors $(\hat{h}_{ij}(s), T_{\hat{g}(s)}^{\mu(s)}\hat{h}_{ij}(s))$ is dense in $\mathcal{G}_s$ and, hence, in the graph of $T_{\hat{g}(s)}^{\mu(s)}$ for $\nu$-a.e. $s$. The $\nu$-measurability of the family $s \to T_{\hat{g}(s)}^{\mu(s)}$ is thus proved. Set $T = \int_{\mathcal{S}(s)} T_{\hat{g}(s)}^{\mu(s)} d\nu(s)$.

Let $f$ belong to the domain of $T_g^{M}$ and $F = T_g^{M}f$. In view of Lemma 5.10, $\hat{F}(s)$ is $\mu(s)$-equivalent to $\hat{g}(s)f(s)$ for $\nu$-a.e. $s$. This means that $Qf = \hat{f} \in DT$ and $QT_g^{M}f = TQf$. It follows that $T$ is an extension of $Q^2T_g^{M}Q^{-1}$. To finish the proof, we have to show that $Q^{-1}G$ belongs to the domain of $T_g^{M}$ for any $G \in DT$. Let $\psi = gQ^{-1}G$. For $\nu$-a.e. $s$, we have $(T(G))(s,E) = \hat{g}(s,E)\xi(s,E) = \varphi(s,E)$ for $\mu(s)$-a.e. $E$. As $T \xi \in \mathcal{D}$, it follows that the function $E \to \varphi(s,E)$ is $\mu(s)$-square-integrable for $\nu$-a.e. $s$ and the function $s \to \int |\varphi(s,E)|^2 d\mu(s|E)$ is $\nu$-integrable. In view of Corollary 5.11, this implies that $\varphi$ is $M$-square-integrable, i.e., $Q^{-1}G$ is in the domain of $T_g^{M}$.

\[\square\]
10. Eigenfunction expansions

In this section, we prove the results formulated in Sec. 2.

Proof of Lemma 2.1. Let the entire function \( f \) be such that \( f(z) = \sin z/z \) for \( z \neq 0 \) and \( f(0) = 1 \). We define the continuous function \( g \) on \((-1, 1) \times \mathbb{R}_+\) by setting

\[
g(\kappa, E) = -\frac{\ln E}{\pi f(\pi \kappa)} f \left( \frac{i\kappa}{2} \ln E \right).\]  

(104)

It follows that

\[
g(\kappa, E) = \begin{cases} \
\frac{-\ln E}{\pi}, & \kappa = 0, \\
\frac{\ln E}{\pi E^{\kappa/2} - E^{\kappa/2}}, & 0 < \kappa < 1.
\end{cases}\]  

(105)

For every \( \vartheta \in \mathbb{R} \) and \(-1 < \kappa < 1\), we define the function \( t_{\kappa, \vartheta} \) on \( \mathbb{R}_+ \) by the relation

\[
t_{\kappa, \vartheta}(E) = 2 + g(\kappa, E)^2 + 4 g(\kappa, E)(E^{-\kappa/2} + E^{\kappa/2}) \sin 2\vartheta.
\]  

(106)

It follows from (13), (17), and (105) by a straightforward calculation that

\[
d\tilde{\mathcal{V}}_{\kappa, \vartheta}(E) = t_{\kappa, \vartheta}(E)^{-1} \Theta(E) \, dE\]  

(107)

for all \( \vartheta \in \mathbb{R} \) and \(-1 < \kappa < 1\). By the Cauchy–Bunyakovsky inequality, we have

\[
| - c \cos 2\vartheta + d \sin 2\vartheta | \leq \sqrt{c^2 + d^2}
\]

for any \( c, d \in \mathbb{R} \). Applying this bound to \( c = g(\kappa, E)^2 \cos \pi \kappa \) and

\[
d = g(\kappa, E)(E^{-\kappa/2} + E^{\kappa/2}) = g(\kappa, E)\sqrt{g(\kappa, E)^2 \sin^2 \pi \kappa + 4},
\]

we deduce from (106) that \( t_{\kappa, \vartheta}(E) \geq \tilde{f}(g(\kappa, E)^2) \), where \( \tilde{f}(y) = 2 + y - \sqrt{y^2 + 4y} \), \( y \geq 0 \). Since

\[
\tilde{f}(y) = \frac{4}{2 + y + \sqrt{y^2 + 4y}} \geq \frac{2}{2 + y}, \quad y \geq 0,
\]

we conclude that \( t_{\kappa, \vartheta}(E)^{-1} \geq 1 + \frac{1}{2}g(\alpha, E)^2 \leq \frac{1}{2 \sin^2 \pi \alpha}(E^\alpha + E^{-\alpha}) \)

(108)

for all \( E > 0 \), \( \vartheta \in \mathbb{R} \), and \(-\alpha \leq \kappa \leq \alpha\). Let \( F \) be a bounded Borel function on \( \mathbb{R} \) with compact support and \( |B| = \mathcal{L}^2(\mathbb{R}) \). Since the function \( (\kappa, \vartheta) \rightarrow t_{\kappa, \vartheta}(E)^{-1} F(E) \) is continuous on \( B \) for every \( E > 0 \), relations (107) and (108) and the dominated convergence theorem imply that \( (\kappa, \vartheta) \rightarrow \int F(E) \, d\tilde{\mathcal{V}}_{\kappa, \vartheta}(E) \) is a continuous function on \( B \) that is bounded on \([-\alpha, \alpha] \times \mathbb{R} \) for every \( 0 < \alpha < 1 \).

Let

\[
B' = \{(\kappa, \vartheta) \in B : \vartheta \in (|\vartheta_\kappa|, |\vartheta_\kappa| + \pi \mathbb{Z})\}.
\]

It follows from (11) and (15) that

\[
\int F(E) \, d\tilde{\mathcal{V}}_{\kappa, \vartheta}(E) = \int F(E) \, d\tilde{\mathcal{V}}_{\kappa, \vartheta}(E) + b_F(\kappa, \vartheta),
\]

where the function \( b_F \) on \( B \) is defined by the relation

\[
b_F(\kappa, \vartheta) = \begin{cases} \
\tilde{g}(\kappa, |E_{\kappa, \vartheta}|) F(E_{\kappa, \vartheta}), & (\kappa, \vartheta) \in B', \\
0, & (\kappa, \vartheta) \in B \setminus B'.
\end{cases}\]  

(109)
and the continuous function \( \tilde{g} \) on \((-1, 1) \times \mathbb{R}_+ \) is given by

\[
\tilde{g}(\kappa, E) = \frac{1}{2} E \pi^2 f(\pi \kappa) \left( g(\kappa, E) + \frac{1}{\cos^2 \vartheta} \right).
\]

For every \((\kappa, \vartheta) \in B'\), we have \(|\cot \vartheta \tan \vartheta| < 1\), and it follows from (14) and (16) that

\[
E_{\kappa, \vartheta} = - \exp \left[ \frac{\pi \cot \vartheta}{2 \cos \vartheta} f(\vartheta) f_1(\cot \vartheta \tan \vartheta) \right],
\]

where \(f_1\) is the continuous function on \((-1, 1)\) such that \(f_1(y) = y^{-1} \ln((1 + y)(1 - y))^{-1}\) for \(y \neq 0\) and \(f_1(0) = 2\). Thus, \((\kappa, \vartheta) \rightarrow E_{\kappa, \vartheta}\) is a continuous function on \(B'\) and, hence, \(b_F\) is a Borel function on \(B\). Estimating \(g(\kappa, E)\) as above, we obtain

\[
(110) \quad \tilde{g}(\kappa, E) \leq \frac{\pi^2 E}{2 \sin^2 \pi \alpha}(E^\alpha + E^{-\alpha} + 2), \quad (\kappa, E) \in [-\alpha, \alpha] \times \mathbb{R}_+,
\]

for every \(0 < \alpha < 1\). In view of (109), this implies that \(b_F\) is bounded on \([-\alpha, \alpha] \times \mathbb{R}\) for every \(0 \leq \alpha < 1\). To complete the proof, it remains to show that \(b_F\) is continuous on \(B\) if \(F\) is continuous. Let \(-1 < \kappa < 1\). It follows from (14) and (16) that \(|E_{\kappa, \vartheta}|\) strictly decreases from \(\infty\) to \(0\) as \(\vartheta\) varies from \(|\vartheta|\) to \(\pi - |\vartheta|\). Hence, for every \(E > 0\), there is a unique \(\tau_E(\kappa) \in (|\vartheta|, \pi - |\vartheta|)\) such that \(|E_{\kappa, \tau_E(\kappa)}| = E\). The continuity of \(E_{\kappa, \vartheta}\) in \((\kappa, \vartheta)\) implies that \(\tau_E\) is a continuous function on \((-1, 1)\) for every \(E > 0\). Let \(\beta > 0\) be such that \(F(E) = 0\) for every \(E \leq -\beta\). Given \(0 < \alpha < 1\) and \(0 < \delta\), we define the open subset \(B_{\alpha, \delta}\) of \(B\) by setting

\[
B_{\alpha, \delta} = \{(\kappa, \vartheta) \in (-\alpha, \alpha) \times \mathbb{R} : \vartheta \in (\tau_\delta(\kappa) - \pi, \tau_\delta(\kappa) + \pi \mathbb{Z})\}.
\]

If \(\delta < 1\), then it follows from (109) and (110) that

\[
|b_F(\kappa, \vartheta)| \leq \frac{2 \pi^2 \delta^{1-\alpha}}{\sin^2 \pi \alpha} \sup_{E \in \mathbb{R}} |F(E)|, \quad (\kappa, \vartheta) \in B_{\alpha, \delta}.
\]

Given \((\kappa, \vartheta) \in B \setminus B'\) and \(\varepsilon > 0\), we pick an arbitrary \(\alpha \in (|\vartheta|, 1)\) and choose \(\delta > 0\) so small that the right-hand side of the last inequality is less than \(\varepsilon\). Then \(B_{\alpha, \delta}\) is a neighborhood of \((\kappa, \vartheta)\), where the absolute value of \(b_F\) is less than \(\varepsilon\). This proves that \(b_F\) is continuous at every point of \(B \setminus B'\). Since \(b_F\) is obviously continuous on \(B'\), the lemma is proved. \(\Box\)

Proposition 2.3 follows immediately from the next lemma.

**Lemma 10.1.** Let \(\phi \in \mathbb{R}\) and \(\theta\) be a Borel real function on \(A^\phi\). Then \(\mu^\phi_{\theta}\) is a \((\nu_0, B_{\mathbb{R}})\)-measurable family of measures, where \(B_{\mathbb{R}}\) is the Borel \(\sigma\)-algebra on \(\mathbb{R}\), and \(M = (B_{\mathbb{R}}) \Delta \mu^\phi_{\theta}(s) d\nu_0(s)\) is the unique Borel measure on \(S \times \mathbb{R}\) satisfying the conditions of Proposition 2.3.

**Proof.** Let \(K\) be the set of compact subsets of \(\mathbb{R}\). Clearly, \(K\) is contained in the domain of \(\mu^\phi_{\theta}(s)\) for all \(s \in S\). Hence, the \((\nu_0, B_{\mathbb{R}})\)-measurability of \(\mu^\phi_{\theta}\) is ensured by Corollary 2.2 and Corollary 9.3 (note that \(\sigma(K) = B_{\mathbb{R}}\)).

Let \(B_S\) and \(B_{S \times \mathbb{R}}\) denote the Borel \(\sigma\)-algebras of \(S\) and \(S \times \mathbb{R}\) respectively. By Definition 9.8 \(M\) satisfies the conditions of Proposition 9.7 for \(\mu = \mu^\phi_{\theta}\), \(\nu = \nu_0\), \(\Sigma = B_{\mathbb{R}}\), and \(\bar{\Sigma} = B_S\). By Corollary 2.2 we have \(K' \times K \in \Delta^{+}_{\nu_0}\) for any compact sets \(K' \subset S\) and \(K \subset \mathbb{R}\). Since \(B_{S \times \mathbb{R}} = \sigma(B_S \otimes B_{\mathbb{R}})\), Proposition 9.7 implies that \(M\) is a Borel measure on \(S \times \mathbb{R}\) satisfying (20). If \(f\) is an \(M\)-measurable function, then equality (21) is ensured by Proposition 9.10. Thus, \(M\) satisfies the conditions of Proposition 2.3.
Corollary 10.3. We conclude that $L_\Phi$ from (112)

Proof. The statement follows immediately from Lemma 10.2 and Lemma B.3.

\[
\Phi = \left\{ \begin{array}{ll}
U_{m-\phi}, & s \in S \setminus A^\phi, \\
U_{m-\phi,\theta(s)}, & s \in A^\phi,
\end{array} \right.
\]

for all $s = (m, p) \in S$, where the operators $U_{m}$ and $U_{m,\theta}$ satisfy the conditions of Proposition 6.9 and Proposition 6.11 respectively. By Lemma 10.1, the assumptions of Proposition 9.13 are fulfilled for $\Sigma = B_{S \times \mathbb{R}}$, $\mu = \tilde{\nu}_\phi$, and $\nu = \nu_0$. A $\nu_0$-measurable family $\mathcal{S}$ of Hilbert spaces satisfying the conditions of Proposition 9.13 for such $\Sigma$, $\mu$, and $\nu$ will be denoted by $\mathcal{S}_\phi$.

Lemma 10.2. Let $\phi \in \mathbb{R}$, $\theta$ be a Borel real function on $A^\phi$, and $\psi \in L^2_2(\mathbb{R}_+)$. Then $s \rightarrow \Phi_\phi^\phi(s) \psi$ is a $\nu_0$-measurable section of $\mathcal{S}_\phi$.

Proof. As follows from Proposition 9.13 applied to the set $K$ of all compact subsets of $\mathbb{R}$, it suffices to show that

\[
I_K(s) = \int_K \Phi(s) \psi(E) \, d\mu(s) 
\]

is a Borel function on $S$ for any compact set $K \subset \mathbb{R}$. Given $-1 < \kappa < 1$, let the function $f_K^\phi$ on $\mathbb{R}$ be defined by the relation

\[
f_K^\phi(\theta) = \cos(\theta - \vartheta_\kappa) \int_K F_\kappa^{(1)}(E) \, dV_{\kappa,\theta}(E) + \sin(\theta - \vartheta_\kappa) \int_K F_\kappa^{(2)}(E) \, dV_{\kappa,\theta}(E),
\]

where $\vartheta_\kappa$ is defined by (12) and the continuous functions $F_\kappa^{(1)}$ and $F_\kappa^{(2)}$ on $\mathbb{R}$ are given by

\[
F_\kappa^{(1)}(E) = \int_0^\infty u^\kappa(E|r) \psi(r) \, dr, \quad F_\kappa^{(2)}(E) = \int_0^\infty w^\kappa(E|r) \psi(r).
\]

It follows from (19), (64), (68), and (111) that $I_K(s) = f_K^{m-\phi}(\theta(s))$ for every $s = (m, p) \in A^\phi$. On the other hand, $p \rightarrow I_K(m, p)$ is a constant function on $\mathbb{R}$ for every $m$ satisfying $|m - \phi| \geq 1$. Since $f_K^\phi$ is a Borel function on $\mathbb{R}$ by Lemma 2.1, we conclude that $I_K$ is a Borel function on $S$. □

In what follows, we set $\mathfrak{h} = L_2(\mathbb{R}_+)$. Corollary 10.3. Let $\phi \in \mathbb{R}$ and $\theta$ be a Borel real function on $A^\phi$. Then $\Phi_\phi^\phi$ is a $\nu_0$-measurable family of operators from $\mathcal{I}_{\mathfrak{h}, \nu_0}$ to $\mathcal{S}_\phi$.

Proof. The statement follows immediately from Lemma 10.2 and Lemma 10.3. □

Given $\phi \in \mathbb{R}$ and a Borel real function $\theta$ on $A^\phi$, we define the linear operator $U_\theta^\phi$ from $L_2(S, \mathfrak{h}, \nu_0)$ to $\int_\Phi \mathcal{S}_\phi(s) \, d\nu_0(s)$ by setting

\[
U_\theta^\phi = \int_\Phi \mathcal{S}_\phi(s) \, d\nu_0(s).
\]
Since $U^\phi(s)$ is a unitary operator from $\mathfrak{h}$ to $\mathcal{S}^\phi_0(s)$ for $\nu_0$-a.e. $s$, it follows from Corollary 10.3 and Proposition 12.22 that $U^\phi_0$ is a unitary operator. If $f$ is an $M_\phi^0$-measurable complex function, we denote by $f|_\theta^\phi$ the $\nu_0$-equivalence class such that $f|_\theta^\phi(s)$ is the $\mu^\phi_\theta(s)$-equivalence class of the map $E \to f(s, E)$ for $\nu_0$-a.e. $s$. In view of Lemma 10.4, all conditions of Proposition 9.14 are fulfilled for $\Sigma = B_\mathbb{R}$, $\nu = \nu_0$, $\mu = \mu^\phi_\theta$, $M = M_\phi^0$, $\mathcal{G} = \mathcal{S}^\phi_0$, and $\bar{f} = f|_\theta^\phi$. It follows from statement 2 of Proposition 9.14 that the correspondence $f \to f|_\theta^\phi$ induces a unitary operator $Q_\theta^\phi$ from $L_2(S \times \mathbb{R}, M_\phi^0)$ to $\int_0^\infty \mathcal{S}^\phi_0(s) d\nu_0(s)$.

Note that the uniqueness of $W$ in Proposition 2.4 is ensured by the density of $L_2^0(\mathbb{R}^3)$ in $L_2(\mathbb{R}^3)$. Hence, Proposition 2.4 follows from the next lemma.

**Lemma 10.4.** Let $\phi \in \mathbb{R}$, $\theta$ be a Borel real function on $A^\phi$, and $V$ be as in Lemma 5.1. Then the operator $W = (Q_\theta^\phi)^{-1}U^\phi_0 V$ satisfies the conditions of Proposition 2.4.

**Proof.** Let $\Phi \in C^\infty_0(\mathbb{R}^3 \setminus Z)$ and $\Psi = [\Phi]_\Lambda$. For $s \in S$ and $E \in \mathbb{R}$, let $h_\Phi(s, E)$ denote the right-hand side of (28). For $-1 < \kappa < 1$ and $m \in \mathbb{Z}$, we define the continuous function $f_{m,\Psi}^\kappa$ on $\mathbb{R} \times \mathbb{R} \times (\mathbb{R}^3 \setminus Z)$ by the relation

$$f_{m,\Psi}^\kappa(E, \theta, x) = \frac{e^{ipx_3}}{2\pi \sqrt{r_x}} \left( \frac{x_1 + ix_3}{r_x} \right)^m u_0^\phi(E|x_x)$$

for all $E, \theta \in \mathbb{R}$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus Z$, where $r_x = \sqrt{x_1^2 + x_2^2}$. Set $f_{m,\Psi}^\kappa(E, \theta) = \int f_{m,\Psi}^\kappa(E, \theta, x) \Psi(x) dx$. It follows from (27) and (28) that $h_\Phi(s, E) = f_{m,\Psi}^\kappa(E, \theta(s))$ for all $s = (m, p) \in A^\phi$ and $E \in \mathbb{R}$. If $|m - \phi| > 1$, then (27) and (28) imply that $(p, E) \to h_\Phi(m, p; E)$ is a continuous function on $\mathbb{R}^2$. Since $f_{m,\Psi}^\kappa$ is a continuous function on $\mathbb{R}^2$ for all $m \in \mathbb{Z}$ and $-1 < \kappa < 1$, it follows that $(p, E) \to h_\Phi(m, p; E)$ is a Borel function on $\mathbb{R}^2$ for every $m \in \mathbb{Z}$. Thus, $h_\Phi$ is a Borel (and, hence, $M_\phi^0$-measurable) function on $S \times \mathbb{R}$.

Let $J$ be the map on $S \times \mathbb{R}$ such that

$$J(s, E) = \begin{cases} u_{m-\phi}^\kappa(E), & s \in S \setminus A^\phi, \\ u_{m-\phi}^\kappa(E), & s \in A^\phi, \end{cases}$$

for all $s = (m, p) \in S$ and $E \in \mathbb{R}$. Passing to the polar coordinates in the $(x_1, x_2)$-plane in the integral in (28), we obtain

$$h_\Phi(s, E) = \int_0^\infty J(s, E|x|) \hat{\Phi}(s|x|) dx$$

for all $s \in S$ and $E \in \mathbb{R}$, where the $C^\infty_0(\mathbb{R}_+)$-valued map $\hat{\Phi}$ on $S$ is given by (80). It follows from (19), (111), Proposition 6.9 and Proposition 6.11 that, for every $s \in S$ and $\psi \in L_2(\mathbb{R}_+)$, the equality

$$\langle U^\phi_0(s) \psi(E) = \int_0^\infty J(s, E|x|) \psi(r|x|) dx$$

holds for $\mu^\phi_\theta(s)$-a.e. $E$. In view of (81), equalities (112), (114) and (115) imply that the function $E \to h_\Phi(s, E)$ is $\mu^\phi_\theta(s)$-equivalent to $(U^\phi_0 V \Psi)(s)$ for $\nu_0$-a.e. $s$, whence $U^\phi_0 V \Psi = h_\Phi|_\theta^\phi$. Since $U^\phi_0 V \Psi = Q_\theta^\phi W \Psi = (W \Psi)|_\theta^\phi$, it follows from statement 1 of Proposition 9.14 that $W \Psi$ is $M_\phi^0$-equivalent to $h_\Phi$. Thus, (28) holds for $\Psi = [\Phi]_\Lambda$ with $\Phi \in C^\infty_0(\mathbb{R}^3 \setminus Z)$. 
For a general $\Psi \in L^2(\mathbb{R}^3)$, we choose a sequence $\Phi_n \in C^\infty(\mathbb{R}^3 \setminus Z)$ such that $\Psi_n = [\Phi_n]_\Lambda$ converge to $\Psi$ in $L_2(\mathbb{R}^3)$ and the supports of all $\Phi_n$ are contained in a fixed compact set for all $n$. Since $x \to W_\theta^\phi(s, E, x)$ is locally square-integrable, we have $h_{\Psi_n}(s, E) \to h_{\Psi}(s, E)$ for all $s \in S$ and $E \in \mathbb{R}$. As $W\Psi_n \to W\Psi$ in $L_2(S \times \mathbb{R}, M_\theta^\phi)$ by continuity of $W$, it follows that $W\Psi$ is $M_\theta^\phi$-equivalent to $h_{\Psi}$. \hfill \Box

Lemma 10.4 and the uniqueness statement of Proposition 2.4 imply that

$$W_\theta^\phi = (Q_\theta^\phi)^{-1}U_\theta^\phi V.$$  

Theorem 2.5 follows from Proposition 8.3 and the next statement.

**Proposition 10.5.** Let $\phi \in \mathbb{R}$, be a Borel real function on $\Lambda^\phi$, and $R_\theta^\phi$ be defined by (98). Then we have $H_\theta^\phi = R_\theta^\phi$.

Proof. For brevity, we set $\mu = \mu_\theta^\phi$, $U = U_\theta^\phi$, $\mathfrak{g} = \mathfrak{g}_\theta^\phi$, and $h = |\phi|_\theta^\phi$. It follows from (19), (29), (39), (93), (111), and Propositions 6.9 and 6.11 that

$$U(s)\mathcal{H}_\theta^\phi(s)U(s)^{-1} = T^\mu(s)$$

for $\nu_0$-a.e. $s$. By Lemma 8.4, $\mathcal{H}_\theta^\phi$ is a $\nu_0$-measurable family of operators in $\mathfrak{g}$. Hence, it follows from (98), (112), Corollary 10.3 statement 1 of Proposition 11.22, and Proposition 10.24 that $s \to T^\mu(s)$ is a $\nu_0$-measurable family of operators in $\mathfrak{g}_\theta^\phi$ and

$$UV R_\theta^\phi(\mathfrak{g})^{-1} = \int T^\mu(s) \, dB_0(s).$$

By statement 3 of Proposition 11.14, the operator in the right-hand side is equal to $QT^M Q^{-1}$. Equalities (30) and (116) therefore imply that $H_\theta^\phi = R_\theta^\phi$. \hfill \Box

**Remark 10.6.** Once Proposition 2.4 is established, the part of Theorem 2.5 stating that $H_\theta^\phi$ is a self-adjoint extension of $H_\theta^\phi$ commuting with $T_G$ for all $G \in \mathcal{G}$ can be proved by a direct computation that does not involve direct integrals of operators. Indeed, let $\Phi \in C^\infty(\mathbb{R}^3 \setminus Z)$, $\Psi = [\Phi]_\Lambda$, $h_{\Psi}$ be as in the proof of Lemma 10.4. $G_{\alpha\beta} \in \mathcal{G}$ be given by (86), and $\psi_{\alpha\beta} = T_{G_{\alpha\beta}} \Psi$. Since $\psi_{\alpha\beta} = [\Phi \circ G_{\alpha\beta}^{-1}]_\Lambda$, formulas (87) and (114) imply that $h_{\psi_{\alpha\beta}}(s, E) = g_{\alpha\beta}(s)h_{\Psi}(s, E)$ for all $s \in S$ and $E \in \mathbb{R}$, where $g_{\alpha\beta}$ is given by (88). By Proposition 2.4, we have $WT_{G_{\alpha\beta}} \Psi = T_{g_{\alpha\beta}} W\Psi$, where $M = M_\theta^\phi$ and $W = W_\theta^\phi$, whence $T_{G_{\alpha\beta}} = W^{-1} T^M_{g_{\alpha\beta}} W$. Since every element of $\mathcal{G}$ is equal to $G_{\alpha\beta}$ for some $\alpha, \beta \in \mathbb{R}$, it follows from (30) that $H_\theta^\phi$ commutes with all $T_G$ with $G \in \mathcal{G}$. Further, it follows from (64), (56), and (113) that

$$-J(s, E)'' + q_{m-\phi} J(s, E) = E J(s, E)$$

for every $s \in S$ and $E \in \mathbb{R}$, where $q_\kappa, \kappa \in \mathbb{R}$, is given by (34). Substituting (11) in (114) and integrating by parts therefore yields $h_{H_\theta^\phi}(s, E) = (E + p^2)h_{\Psi}(s, E)$ for all $s \in S$ and $E \in \mathbb{R}$. By Proposition 2.4, we conclude that $W\Psi$ belongs to the domain of $T^M_I$ and $WH_\theta^\phi \Psi = T^M_I W\Psi$. In view of (79), this means that $H_\theta^\phi$ is an extension of $H_\theta^\phi$. We thus see that Proposition 8.3 (and, hence, all the direct integral machinery developed in Secs. 4, 5, and 7) is really needed only for the proof of the converse statement, namely, that every self-adjoint extension of $H_\theta^\phi$ commuting with all $T_G$ is equal to $H_\theta^\phi$ for a suitable $\theta$. 

$\Box$
measures as functions on particular, in contrast to the standard approach (see, e.g., [5, 8]), we define positive in such a way as to provide a unified treatment of positive and spectral measures. In paper. Our aim here is not only to fix the notation but also to formulat e definitions A.1.

A.1. Rings and algebras of sets. Recall that a nonempty set of sets $Q$ is called a ring of sets if $A \cup B \in Q$ and $A \setminus B \in Q$ for any $A, B \in Q$. As $A \cap B = A \setminus (A \setminus B)$, every ring is closed under finite intersections. A ring $Q$ is called a $\sigma$-ring (a $\delta$-ring) if it is closed under countable unions (resp., under countable intersections). For a set of sets $Q$, we denote by $\sigma(Q)$ (resp., $\delta(Q)$) the $\sigma$-ring (resp., $\delta$-ring) generated by $Q$, i.e., the smallest $\sigma$-ring (resp., $\delta$-ring) containing $Q$. If $Q$ is a $\delta$-ring, then $\sigma(Q)$ is just the set of all countable unions of elements of $Q$. A $\sigma$-ring $Q$ is said to be countably generated if there is a countable set $\nu = \{A_i\}$ such that $Q = \sigma(\nu)$. A $\sigma$-ring $Q$ is called a $\sigma$-algebra if it has the largest element with respect to inclusion.

The $\sigma$-ring generated by all open sets of a topological space $X$ is obviously a $\sigma$-algebra. It is called the Borel $\sigma$-algebra of $X$ and its elements are called Borel subsets of $X$. A map $f$ is called a Borel map from $X$ to a topological space $Y$ if $X$ is contained in the domain $D_f$ of $f$ and $f^{-1}(A) \cap X$ is a Borel subset of $X$ for any Borel subset $A$ of $Y$.

A.2. Measures. Let $A$ be a topological Abelian group and $\nu$ be an $A$-valued map. Let $\mathcal{N}_\nu$ denote the set of all $N \in D_\nu$ with the property: if $N' \subset N$ and $N' \in D_\nu$, then $\nu(N') = 0$. We say that $\nu$ is an $A$-valued $\sigma$-additive function if $\nu(A) = \sum_{i \in I} \nu(A_i)$ for any $A \in D_\nu$ and any countable partition $A = \bigcup_{i \in I} A_i$ with $A_i \in D_\nu$. An $A$-valued $\sigma$-additive function is called an $A$-valued measure if its domain $D_\nu$ is a $\delta$-ring and the following completeness conditions are satisfied:

(a) If $A \in \sigma(D_\nu)$ and the family $\{\nu(A_i)\}_{i \in I}$ is summable in $A$ for any countable partition $A = \bigcup_{i \in I} A_i$ with $A_i \in D_\nu$, then $A \in D_\nu$.

(b) If $N \in \mathcal{N}_\nu$ and $N' \subset N$, then $N' \in D_\nu$ (and, hence, $N' \in \mathcal{N}_\nu$).

Elements of $\mathcal{N}_\nu$ are called $\nu$-null sets and elements of $\sigma(D_\nu)$ are called $\nu$-measurable sets. A measure $\nu$ is called $\sigma$-finite if $\sigma(D_\nu)$ is a $\sigma$-algebra. In this case, we denote by $S_\nu$ the largest element of $\sigma(D_\nu)$ with respect to inclusion. A measure $\nu$ is called finite if it is $\sigma$-finite and $S_\nu \in D_\nu$. A measure $\nu$ is called positive if it is $\mathbb{R}$-valued and $\nu(A) \geq 0$ for any $A \in D_\nu$.

Remark A.1. In most expositions of measure theory (see, e.g., [3, 8]), a positive measure is defined as a $\sigma$-additive function on a $\sigma$-algebra taking values in the extended real semi-axis $\mathbb{R}_+ = [0, \infty]$. Such an $\mathbb{R}_+$-valued measure $\nu$ is called $\sigma$-finite if the largest set in $D_\nu$ is a countable union of elements of $D_\nu$ with a finite measure. It is called complete if every subset of an element of $D_\nu$ with zero measure also belongs to $D_\nu$. Complete $\sigma$-finite $\mathbb{R}_+$-valued measures can be naturally identified with positive $\sigma$-finite measures in our sense (note that a complete $\sigma$-finite $\mathbb{R}_+$-valued measure $\nu$ restricted to the set of all elements of $D_\nu$ with a finite measure

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is a positive \( \sigma \)-finite measure in our sense). The advantage of our definition is that it does not involve the extended real axis and, therefore, makes it possible to treat positive measures in the same way as vector-valued measures.

Given a \( \sigma \)-ring \( Q \), a measure \( \nu \) is called \( Q \)-compatible if \( Q \subset \sigma(D_\nu) \) and for every \( A \in D_\nu \), there is an \( N \in N_\nu \) such that \( A \cup N \in Q \). In this case, \( D_\nu \) consists of all sets of the form \( B \cup N \), where \( B \in Q \cap D_\nu \) and \( N \in N_\nu \). If \( X \) is a topological space and \( B \) is its Borel \( \sigma \)-algebra, then a \( B \)-compatible measure is called a Borel measure on \( X \).

**Remark A.2.** Positive Borel measures on a topological space \( X \) are usually defined as \( \mathbb{R}_+ \)-valued \( \sigma \)-additive functions on the Borel \( \sigma \)-algebra of \( X \). If such a measure is \( \sigma \)-finite, then its completion corresponds (as described in Remark [A.1]) to a positive Borel measure on \( X \) in our sense. Since we consider only complete measures (see condition (b) above), we introduce the notion of \( Q \)-compatibility to relate measure and topology.

**Lemma A.3.** Let \( \mathfrak{A} \) be a sequentially complete topological Abelian group and \( \nu \) be an \( \mathfrak{A} \)-valued \( \sigma \)-additive function such that \( D_\nu \) is a \( \delta \)-ring. Then there is a unique \( \sigma(D_\nu) \)-compatible \( \mathfrak{A} \)-valued measure \( \hat{\nu} \) such that \( D_\nu \subset D_{\hat{\nu}} \) and \( \hat{\nu} \) coincides with \( \nu \) on \( D_\nu \). If \( \nu \) satisfies (a), then \( D_\nu = D_{\hat{\nu}} \cap \sigma(D_\nu) \).

The measure \( \hat{\nu} \) is called the completion of the \( \sigma \)-additive function \( \nu \).

**Proof.** We give the proof only for positive \( \nu \) because the lemma is used in this paper only in this case. The existence and uniqueness of \( \nu \) is then guaranteed by the well-known results on the extension of positive measures. Let \( \nu \) satisfy (a) and \( A \in D_\nu \cap \sigma(D_\nu) \). Let \( \{A_i\}_{i \in I} \) be a countable partition of \( A \) such that \( A_i \in D_\nu \) for all \( i \in I \). Since \( \hat{\nu}(A_i) = \nu(A_i) \) for all \( i \in I \), the \( \sigma \)-additivity of \( \hat{\nu} \) implies that the family \( \{\nu(A_i)\}_{i \in I} \) is summable. Condition (a) hence ensures that \( A \in D_\nu \) and, therefore, \( D_\nu \supset D_{\hat{\nu}} \cap \sigma(D_\nu) \). As the opposite inclusion also obviously holds, the lemma is proved. \( \square \)

Let \( \nu \) be an \( \mathfrak{A} \)-valued measure and \( A \) be a \( \nu \)-measurable set. The restriction \( \nu|_A \) of \( \nu \) to \( A \) is, by definition, the restriction of the map \( \nu \) to the domain \( D_{\nu|_A} \) consisting of all \( \nu \)-measurable sets contained in \( A \). Clearly, \( \nu|_A \) is a \( \sigma \)-finite measure for any \( \nu \)-measurable set \( A \) and \( S_{\nu|_A} = A \).

Let \( \nu_1 \) and \( \nu_2 \) be positive measures and \( Q \) be the \( \sigma \)-ring generated by all sets of the form \( A_1 \times A_2 \), where \( A_1 \in D_{\nu_1} \) and \( A_2 \in D_{\nu_2} \). Then there is a unique positive \( Q \)-compatible measure \( \nu \) such that \( \nu(A_1 \times A_2) = \nu_1(A_1)\nu_2(A_2) \) for every \( A_1 \in D_{\nu_1} \) and \( A_2 \in D_{\nu_2} \). This measure is called the product of \( \nu_1 \) and \( \nu_2 \) and is denoted by \( \nu_1 \times \nu_2 \).

**A.3. Standard measures.** Let \( \mathfrak{A} \) be a topological Abelian group. An \( \mathfrak{A} \)-valued measure \( \nu \) is called standard if it is \( \sigma \)-finite and there exists a complete separable metric space \( X \) such that \( X \subset S_\nu \), \( S_\nu \setminus X \) is a \( \nu \)-null set, and \( \nu|_X \) is a Borel measure on \( X \). If \( \nu \) is a standard measure and \( A \) is a \( \nu \)-measurable set, then \( \nu|_A \) is also a standard measure.

**Remark A.4.** Standard measures were first introduced in [22]. Unlike [22], we consider only complete measures, and standard measures in our sense are actually the completions of those in the sense of [22]. In the probabilistic context, such complete measures were introduced in [26] under the name of Lebesgue measure.
spaces. The class of standard measures is broad enough to include most measures encountered in applications and, at the same time, is narrow enough to exclude measures with a pathological behavior.

A.4. Measurable maps. Let \( \nu \) be a \( \sigma \)-finite \( \mathcal{A} \)-valued measure. A map \( f \) is said to be defined \( \nu \text{-a.e.} \) if \( S_\nu \setminus D_f \) is a \( \nu \)-null set. Given a set \( X \), a map \( f \) is said to be an \( X \)-valued \( \nu \text{-a.e.} \) defined map if \( S_\nu \setminus f^{-1}(X) \) is a \( \nu \)-null set. If \( X \) is a topological space, then a map \( f \) is called an \( X \)-valued \( \nu \)-measurable map if \( f \) is an \( X \)-valued \( \nu \text{-a.e.} \) defined map and \( f^{-1}(B) \cap S_\nu \) is a \( \nu \)-measurable set for any Borel subset \( B \) of \( X \). We say that some property \( P(s) \) holds for \( \nu \text{-a.e.} \) \( s \), if there is a \( \nu \)-null set \( N \) such that \( P(s) \) holds for all \( s \in S_\nu \setminus N \).

Given a \( \nu \)-measurable set \( A \), we say that \( f \) is an \( X \)-valued \( \nu \)-measurable map on \( A \) if \( f \) is an \( X \)-valued \( \nu|_A \)-measurable map. A property \( P(s) \) is said to hold for \( \nu \text{-a.e.} \) \( s \in A \) if it holds for \( \nu|_A \text{-a.e.} \) \( s \).

A complex \( \nu \text{-a.e.} \) defined function \( f \) is said to be \( \nu \)-essentially bounded if there is \( C > 0 \) such that \( |f(s)| \leq C \) for \( \nu \text{-a.e.} \) \( s \).

Remark A.5. Note that the domain \( D_f \) of a \( \nu \text{-a.e.} \) defined or \( \nu \)-measurable map \( f \) is not assumed to be contained in \( S_\nu \). Moreover, \( X \)-valued \( \nu \text{-a.e.} \) defined (and, in particular, \( \nu \)-measurable) maps are allowed to take values outside \( X \) on some \( \nu \)-null set contained in \( S_\nu \setminus D_f \). This implies in particular, that using “the set of all \( X \)-valued \( \nu \)-measurable maps” would lead to the same kind of set-theoretic problems as the use of “the set of all sets”.

Let \( X \) be a complete separable metric space, \( f \) be an \( X \)-valued \( \nu \text{-a.e.} \) defined map, and \( f_1, f_2, \ldots \) be a sequence of \( X \)-valued \( \nu \)-measurable maps such that \( f_n(s) \) converge to \( f(s) \) in \( X \) as \( n \to \infty \) for \( \nu \text{-a.e.} \) \( s \). Then \( f \) is \( \nu \)-measurable.

A.5. Equivalence classes. Let \( \nu \) be a \( \sigma \)-finite \( \mathcal{A} \)-valued measure. Two \( \nu \text{-a.e.} \) defined maps \( f \) and \( g \) are called \( \nu \)-equivalent if \( f(s) = g(s) \) for \( \nu \text{-a.e.} \) \( s \). All \( \nu \text{-a.e.} \) defined maps fall into disjoint classes of \( \nu \)-equivalent maps, which are called \( \nu \)-equivalence classes. In every \( \nu \)-equivalence class, we choose an arbitrary fixed element. Given a \( \nu \text{-a.e.} \) defined map \( f \), we denote by \([f]_\nu\) such a chosen element belonging to the \( \nu \)-equivalence class containing \( f \). Thus, the map \([f]_\nu\) is \( \nu \)-equivalent to \( f \) for any \( \nu \text{-a.e.} \) defined map \( f \), and we have \([f]_\nu = [g]_\nu\) for every pair of \( \nu \)-equivalent maps \( f \) and \( g \). If \( X \) is a topological space, then \( f \) is an \( X \)-valued \( \nu \)-measurable map if and only if so is \([f]_\nu\).

In this paper, \( \nu \)-equivalence classes \textit{per se} are not used. Whenever we speak of \( \nu \)-equivalence classes, we always refer to representatives of the form \([f]_\nu\), where \( f \) is a \( \nu \text{-a.e.} \) defined map. There are two reasons for such a redefinition of the notion of a \( \nu \)-equivalence class. First, the same arguments as in Remark A.5 show that the \( \nu \)-equivalence classes in the true sense cannot be considered as well-defined sets. Hence, using them as elements of sets is not sound from the viewpoint of foundations of mathematics. Choosing a fixed representative in each class allows us to circumvent this difficulty. Second, since \( \nu \)-equivalence classes in our sense are just some maps, all definitions and notations introduced for maps become directly applicable to \( \nu \)-equivalence classes.
A.6. **Integrable functions.** Let \( \nu \) be a \( \sigma \)-finite positive measure. A \( \nu \)-measurable complex function \( f \) is called \( \nu \)-integrable if

\[
I_\nu(f) = \sup_{A_i \in D_\nu, c_i \in \mathbb{R}} \sum_{i=1}^n c_i \nu(A_i) < \infty,
\]

where the supremum is taken over all finite sets \( A_1, \ldots, A_n \) of disjoint elements of \( D_\nu \) and \( c_1, \ldots, c_n \in \mathbb{R} \) such that \( c_i \leq |f(s)| \) for \( \nu \)-a.e. \( s \in A_i \). The integral \( \int f(s) \, d\nu(s) \) of a \( \nu \)-integrable function \( f \) is a complex number that is uniquely determined by the conditions that \( \int f(s) \, d\nu(s) \) be linear in \( f \) and coincide with \( I_\nu(f) \) if \( f(s) \geq 0 \) for \( \nu \)-a.e. \( s \).

Clearly, passing to a \( \nu \)-equivalent function does not affect its \( \nu \)-integrability. In particular, a \( \nu \)-a.e. defined function \( f \) is \( \nu \)-integrable if and only if \( [f]_\nu \) is \( \nu \)-integrable. If \( A \) is a \( \nu \)-measurable set and \( f \) is a \( \nu|_A \)-integrable function, then we say that \( f \) is a \( \nu \)-integrable function on \( A \). In this case, we write \( \int_A f(s) \, d\nu(s) \) in place of \( \int f(s) \, d\nu|_A(s) \).

**Remark A.6.** As for \( \nu \)-measurable functions, the domain of a \( \nu \)-integrable function is not assumed to be contained in \( S_\nu \).

A.7. **\( L_2 \)-spaces.** Let \( \nu \) be a \( \sigma \)-finite positive measure and \( \mathfrak{H} \) be a separable Hilbert space.\(^{13}\) We denote by \( \mathcal{M}(\mathfrak{H}, \nu) \) the set of all elements \( ([f]_\nu) \), where \( f \) is an \( \mathfrak{H} \)-valued \( \nu \)-measurable map. The set \( \mathcal{M}(\mathfrak{H}, \nu) \) has a natural structure of a vector space over \( \mathbb{C} \) (for any \( \mathfrak{H} \)-valued \( \nu \)-a.e. defined maps \( f \) and \( g \) and any \( k \in \mathbb{C} \), we set \( [f]_\nu + [g]_\nu = [f + g]_\nu \) and \( k[f]_\nu = [kf]_\nu \)). Given a \( \nu \)-measurable set \( A \), we denote by \( L_2(A, \mathfrak{H}, \nu) \) the linear subspace of \( \mathcal{M}(\mathfrak{H}, \nu|_A) \) consisting of all its elements \( f \) such that \( s \to \|f(s)\|^2 \) is a \( \nu \)-integrable function. Defining the scalar product of \( f, g \in L_2(A, \mathfrak{H}, \nu) \) by the formula

\[
(f, g) = \int_A (f(s), g(s)) \, d\nu(s),
\]

we make \( L_2(A, \mathfrak{H}, \nu) \) into a Hilbert space.

For \( \mathfrak{H} = \mathbb{C} \), we denote the space \( L_2(A, \mathfrak{H}, \nu) \) by \( L_2(A, \nu) \). If \( A \subset \mathbb{R}^n \) and \( \nu \) is the Lebesgue measure on \( \mathbb{R}^n \), the space \( L_2(A, \nu) \) is denoted by \( L_2(A) \).

Given a \( \nu \)-measurable complex function \( g \), we denote by \( \mathcal{T}_g^\nu \) the operator of multiplication by \( g \) in \( L_2(S_\nu, \nu) \). By definition, the graph of \( \mathcal{T}_g^\nu \) consists of all pairs \((f_1, f_2)\) of elements of \( L_2(S_\nu, \nu) \) such that \( f_2(s) = g(s)f_1(s) \) for \( \nu \)-a.e. \( s \). The operator \( \mathcal{T}_g^\nu \) is closed and densely defined and its adjoint is equal to \( \mathcal{T}_{\bar{g}}^\nu \), where \( \bar{g} \) is the complex conjugate function of \( g \). In particular, if \( g \) is real, then \( \mathcal{T}_g^\nu \) is self-adjoint.

A.8. **Radon measures.** Let \( X \) be a separable metrizable locally compact space. A positive Borel measure \( \nu \) on \( X \) is said to be a Radon measure on \( X \) if every compact subset of \( X \) belongs to \( D_\nu \). For any Radon measure \( \nu \) on \( X \), \( f \to \int f(s) \, d\nu(s) \) is obviously a positive linear functional on the space \( C_0(X) \) of continuous functions on \( X \) with compact support. The well-known Riesz representation theorem (see, e.g., Sec. 29 in \([5]\)) states that the converse is also true: if \( l \) is a positive linear

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\(^{13}\)Note that a Hilbert space is separable if it is either finite-dimensional or infinite-dimensional with a countable orthonormal basis. In particular, we consider \( \mathbb{C} \) as a one-dimensional Hilbert space with the scalar product \( \langle \alpha, \beta \rangle = \alpha \bar{\beta}, \alpha, \beta \in \mathbb{C} \).
functional on $C_0(X)$, then there is a unique Radon measure $\nu$ on $X$ such that $l(f) = \int f(s) \, d\nu(s)$ for any $f \in C_0(X)$.

A.9. Spectral measures. We refer the reader to Chapter 5 of [7] for a detailed exposition of the theory of spectral measures. Here, we only give a brief summary of the facts needed in this paper. Let $H$ be a separable Hilbert space and $L(H)$ be the space of bounded everywhere defined linear operators in $H$ endowed with the strong operator topology. A finite $L(H)$-valued measure $E$ is called a spectral measure in $H$ if $E(A)$ is an orthogonal projection in $H$ for any $A \in \mathcal{D}_E$ and $E(S_E)$ is the identity operator in $H$. For any $\Psi \in H$, we define the positive measure $E\Psi$ as the completion of the positive $\sigma$-additive function $A \rightarrow \langle E(A)\Psi, \Psi \rangle$ on $\mathcal{D}_E$, where $\langle \cdot, \cdot \rangle$ is the scalar product on $H$.

Let $E$ be a spectral measure. Given an $E$-measurable complex function $f$, the integral $J_E^f$ of $f$ with respect to $E$ is defined as the unique linear operator in $H$ such that

$$D_{J_E^f} = \left\{ \Psi \in H : \int |f(s)|^2 \, dE\Psi(s) < \infty \right\}$$

$$(117) \quad \langle \Psi, J_E^f \Psi \rangle = \int f(s) \, dE\Psi(s), \quad \Psi \in D_{J_E^f}.$$

For any $E$-measurable complex function $f$, the operator $J_E^f$ is closed and densely defined and its adjoint is equal to $J_{E^*}^f$.

For every normal operator $T$, there is a unique Borel spectral measure $E_T$ on $\mathbb{C}$ such that $J_{E_T}^g = T$, where $g$ is the identity function on $\mathbb{C}$. The operators $E_T(A)$, where $A$ is a Borel subset of $\mathbb{C}$, are called the spectral projections of $T$. If $f$ is an $E_T$-measurable complex function, then the operator $J_{E_T}^f$ is also denoted as $f(T)$.

APPENDIX B. DIRECT INTEGRAL DECOMPOSITIONS OF OPERATORS IN HILBERT SPACE

One of the main mathematical tools used in this paper are the direct integral decompositions of operators in Hilbert space. In its original form [32], the theory of such decompositions (also known as von Neumann’s reduction theory) is applicable only to bounded operators and, therefore, is insufficient for quantum-mechanical applications, where unbounded operators play a prominent role. An extension of the reduction theory to unbounded operators was given in [24] (most results of [24] were earlier formulated without proofs in [25]). The approach proposed in [24] is based on the observation [28] that the properties of a closed operator in a Hilbert space can be encoded in a set of four bounded operators forming its so called characteristic matrix. This allows one to reformulate problems concerning the direct integrals of arbitrary closed operators in terms of direct integrals of bounded operators. Although this approach makes it possible to extend the results of the original von Neumann’s theory to unbounded operators, it seems to be somewhat inconvenient for applications to concrete problems, where the computation of the characteristic matrices may turn out to be a difficult task. In particular, the definition of the measurability of families of operators given in [24] may be difficult to verify for concrete examples. From the viewpoint of applications, the definition

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14A detailed account of the reduction theory for bounded operators can be found in the book [10].
of measurability proposed in [25] and formulated directly in terms of the graphs of operators seems to be more suitable (both definitions are actually equivalent, see Remark B.12 below).

In this appendix, we give a self-contained exposition of the theory of direct integral decompositions of (generally, unbounded) operators in Hilbert space. Our treatment is based on the formulation of measurability proposed in [25] and does not involve characteristic matrices. Such an approach allows us to derive all results in a straightforward manner: unlike [24], we do not need the theory for bounded operators as a prerequisite. Moreover, it becomes possible to give a concise treatment of direct integral decompositions of sums and products of operators (see Propositions [B.24 and B.25 below), whose original analysis in [24] involves lengthy computations of characteristic matrices.

In what follows, we use the measure-theoretic framework described in Appendix A. Throughout this appendix, \( \nu \) denotes a \( \sigma \)-finite positive measure.

B.1. **Measurable families of Hilbert spaces.** A \( \nu \)-a.e. defined map \( \mathcal{S} \) is called a \( \nu \)-a.e. defined family of Hilbert spaces if \( \mathcal{S}(s) \) is a separable Hilbert space for \( \nu \)-a.e. \( s \). A map \( \xi \) is called a \( \nu \)-a.e. defined section of \( \mathcal{S} \) if \( \xi(s) \in \mathcal{S}(s) \) for \( \nu \)-a.e. \( s \).

A \( \nu \)-a.e. defined family of Hilbert spaces \( \mathcal{S} \) is called \( \nu \)-measurable if it is equipped with a sequence \( \{ (\xi^\nu_i)_{i=1}^\infty \} \) of \( \nu \)-a.e. defined sections of \( \mathcal{S} \) such that the linear span of the vectors \( \xi^\nu_i(s), \xi^\nu_j(s), \ldots \) is dense in \( \mathcal{S}(s) \) for \( \nu \)-a.e. \( s \), and \( s \to \langle \xi^\nu_i(s), \xi^\nu_j(s) \rangle \) is a \( \nu \)-measurable complex function for any \( i, j = 1, 2, \ldots \). A \( \nu \)-a.e. defined section \( \xi \) of \( \mathcal{S} \) is called \( \nu \)-measurable if \( \xi \to \langle \xi^\nu(s), \xi(s) \rangle \) is a \( \nu \)-measurable complex function for any \( i = 1, 2, \ldots \).

We say that \( \nu \)-measurable families \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) of Hilbert spaces are \( \nu \)-identical if they are \( \nu \)-equivalent and have the same \( \nu \)-measurable sections.

A sequence \( e_1, e_2, \ldots \) of elements of a Hilbert space \( \mathcal{S} \) is said to be a generalized orthonormal system in \( \mathcal{S} \) if \( \langle e_i, e_j \rangle = 0 \) for \( i \neq j \) and \( \|e_i\| \) is equal to either 1 or 0 for any \( i = 1, 2, \ldots \).

Let \( \mathcal{S} \) be a \( \nu \)-a.e. defined family of Hilbert spaces and \( \xi_1, \xi_2, \ldots \) be a sequence of \( \nu \)-a.e. defined sections of \( \mathcal{S} \). We say that \( \nu \)-a.e. defined sections \( \xi_1, \xi_2, \ldots \) of \( \mathcal{S} \) constitute an orthonormal sequence associated with \( \xi_1, \xi_2, \ldots \) if the following conditions hold

(a) For \( \nu \)-a.e. \( s \), the sequence \( \xi_1(s), \xi_2(s), \ldots \) is a generalized orthonormal system in \( \mathcal{S}(s) \) whose linear span coincides with that of \( \xi_1(s), \xi_2(s), \ldots \).

(b) There exist \( \nu \)-measurable complex functions \( f_{ij} \) defined for \( i \leq j \) such that

\[
\xi_j(s) = \sum_{i=1}^j f_{ij}(s) \xi_i(s) \quad \text{for all } j = 1, 2, \ldots \text{ and } \nu \text{-a.e. } s.
\]

**Lemma B.1.** Let \( \mathcal{S} \) be a \( \nu \)-a.e. defined family of Hilbert spaces and \( \xi_1, \xi_2, \ldots \) be a sequence of \( \nu \)-a.e. defined sections of \( \mathcal{S} \) such that \( s \to \langle \xi_i(s), \xi_j(s) \rangle \) is a \( \nu \)-measurable complex function for any \( i, j = 1, 2, \ldots \). Then there exists an orthonormal sequence associated with \( \xi_1, \xi_2, \ldots \).

**Proof.** The required sequence is constructed by applying the standard orthogonalization procedure to \( \xi_1(s), \xi_2(s), \ldots \), see the proof of Lemma 1 in Sec. II.1.2 of [10] for details.

If \( \xi \) and \( \eta \) are \( \nu \)-measurable sections of a \( \nu \)-measurable family \( \mathcal{S} \) of Hilbert spaces, then \( s \to \langle \xi(s), \eta(s) \rangle \) is a \( \nu \)-measurable complex function. Indeed, by Lemma [B.1] there exists an orthonormal sequence \( \tilde{\xi}_1, \tilde{\xi}_2, \ldots \) associated with \( \tilde{\xi}_1^\nu, \tilde{\xi}_2^\nu, \ldots \). By (a),
the linear span of \( \tilde{\xi}_1(s), \tilde{\xi}_2(s), \ldots \) is dense in \( \mathcal{G}(s) \) for \( \nu \)-a.e. \( s \). This implies that 
\[
\langle \xi(s), \eta(s) \rangle = \sum_{i=1}^{\infty} \langle \xi(s), \xi_i(s) \rangle \langle \xi_i(s), \eta(s) \rangle
\]
for \( \nu \)-a.e. \( s \). Hence the required statement follows because \( s \to \langle \xi(s), \tilde{\xi}_i(s) \rangle \) and \( s \to \langle \tilde{\xi}_i(s), \eta(s) \rangle \) are \( \nu \)-measurable complex functions by (b) and the \( \nu \)-measurability of \( \xi \) and \( \eta \).

**Lemma B.2.** Let \( \mathcal{G} \) be a \( \nu \)-measurable family of Hilbert spaces and \( \xi_1, \xi_2, \ldots \) be \( \nu \)-measurable sections of \( \mathcal{G} \). Let \( \xi \) be a \( \nu \)-a.e. defined map such that \( \xi(s) \) belongs to the closed linear span of \( \xi_1(s), \xi_2(s), \ldots \) for \( \nu \)-a.e. \( s \). If \( s \to \langle \xi_j(s), \xi(s) \rangle \) is a \( \nu \)-measurable function for all \( j = 1, 2, \ldots \), then \( \xi \) is a \( \nu \)-measurable section of \( \mathcal{G} \).

**Proof.** By Lemma B.1 there exists an orthonormal sequence \( \tilde{\xi}_1, \tilde{\xi}_2, \ldots \) associated with \( \xi_1, \xi_2, \ldots \). By condition (b), \( \tilde{\xi}_j \) is a \( \nu \)-measurable section of \( \mathcal{G} \) for every \( j = 1, 2, \ldots \). By condition (a), \( \tilde{\xi}_1(s), \tilde{\xi}_2(s), \ldots \) is a generalized orthonormal system whose closed linear span coincides with that of \( \xi_1(s), \xi_2(s), \ldots \) and, hence, contains \( \xi(s) \) for \( \nu \)-a.e. \( s \). This implies that \( \xi(s) = \sum_{j=1}^{\infty} \langle \tilde{\xi}_j(s), \xi(s) \rangle \tilde{\xi}_j(s) \) for \( \nu \)-a.e. \( s \) and, therefore, \( \xi \) is a \( \nu \)-measurable section of \( \mathcal{G} \). \( \square \)

**B.2. Direct sums of measurable families.** Let \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be \( \nu \)-measurable families of Hilbert spaces. Then there is a unique (up to \( \nu \)-identity) \( \nu \)-measurable family \( \mathcal{G} \) of Hilbert spaces satisfying the conditions

1. \( \mathcal{G}(s) = \mathcal{G}_1(s) \oplus \mathcal{G}_2(s) \) for \( \nu \)-a.e. \( s \).
2. A \( \nu \)-a.e. defined map \( \zeta \) is a \( \nu \)-measurable section of \( \mathcal{G} \) if and only if there are \( \nu \)-measurable sections \( \xi \) and \( \eta \) of \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) respectively such that \( \zeta(s) = \langle \xi(s), \eta(s) \rangle \) for \( \nu \)-a.e. \( s \).

Indeed, every \( \nu \)-a.e. defined map \( \mathcal{G} \) satisfying (1) becomes a \( \nu \)-measurable family of Hilbert spaces satisfying (2) if we choose \( \xi^0_1, \xi^0_2, \ldots \) in such a way that \( \xi^0_k(s) = (\xi^0_k(s), 0) \) and \( \xi^0_{k-1}(s) = (0, \xi^0_k(s)) \) for \( \nu \)-a.e. \( s \) and every \( k = 1, 2, \ldots \). The \( \nu \)-measurable family \( \mathcal{G} \) of Hilbert spaces satisfying (1) and (2) is called the direct sum of \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) and is denoted by \( \mathcal{G}_1 \oplus \mathcal{G}_2 \).

**B.3. Constant families.** Let \( \mathcal{H} \) be a separable Hilbert space. Then there exists a unique (up to \( \nu \)-identity) \( \nu \)-measurable family \( \mathcal{G} \) of Hilbert spaces satisfying the conditions

1. \( \mathcal{G}(s) = \mathcal{H} \) for \( \nu \)-a.e. \( s \).
2. A \( \nu \)-a.e. defined map \( \xi \) is a \( \nu \)-measurable section of \( \mathcal{G} \) if and only if \( \xi \) is an \( \mathcal{H} \)-valued \( \nu \)-measurable map.

Indeed, let \( e_1, e_2, \ldots \) be an orthonormal basis in \( \mathcal{H} \) and \( \mathcal{G} \) be a \( \nu \)-a.e. defined map satisfying (1). To make \( \mathcal{G} \) into a \( \nu \)-measurable family, we require \( \xi^0_1, \xi^0_2, \ldots \) to be \( \nu \)-a.e. defined maps such that \( \xi^0_i(s) = e_i \) for \( \nu \)-a.e. \( s \). If \( \xi \) is a \( \nu \)-measurable section of \( \mathcal{G} \), then \( \xi \) is an \( \mathcal{H} \)-valued \( \nu \)-measurable map because \( \xi(s) = \sum_i \langle \xi^0_i(s), \xi(s) \rangle e_i \) for \( \nu \)-a.e. \( s \) and, therefore, \( \mathcal{G} \) satisfies (2). The \( \nu \)-measurable family \( \mathcal{G} \) of Hilbert spaces satisfying (1) and (2) is denoted by \( \mathcal{I}_{\mathcal{H}, \nu} \).

**B.4. Measurable families of operators.** Let \( \mathcal{G} \) be a \( \nu \)-measurable family of Hilbert spaces. A \( \nu \)-a.e. defined map \( \mathcal{G}' \) is said to be a \( \nu \)-a.e. defined family of subspaces of \( \mathcal{G} \) if \( \mathcal{G}'(s) \) is a linear (not necessarily closed) subspace of \( \mathcal{G}(s) \) for \( \nu \)-a.e. \( s \). A \( \nu \)-a.e. defined family \( \mathcal{G}' \) of subspaces of \( \mathcal{G} \) is called \( \nu \)-measurable if there is a sequence \( \xi_1, \xi_2, \ldots \) of \( \nu \)-measurable sections of \( \mathcal{G} \) such that the linear span of \( \xi_1(s), \xi_2(s), \ldots \) is dense in \( \mathcal{G}'(s) \) for \( \nu \)-a.e. \( s \) (such a sequence will be called a \( \nu \)-measurable basis in \( \mathcal{G}' \)).
Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be $\nu$-measurable families of Hilbert spaces. A $\nu$-a.e. defined map $\mathcal{R}$ is called a $\nu$-a.e. defined family of operators from $\mathcal{G}_1$ to $\mathcal{G}_2$ if $\mathcal{R}(s)$ is an operator (possibly not everywhere defined and unbounded) from $\mathcal{G}_1(s)$ to $\mathcal{G}_2(s)$ for $\nu$-a.e. $s$. A $\nu$-a.e. defined family $\mathcal{R}$ of operators from $\mathcal{G}_1$ to $\mathcal{G}_2$ is called $\nu$-measurable if there are sequences $\xi_1, \xi_2, \ldots$ and $\eta_1, \eta_2, \ldots$ of $\nu$-measurable sections of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively such that the linear span of the vectors $(\xi_j(s), \eta_j(s))$ is dense in the graph $G_{\mathcal{R}(s)}$ of the operator $\mathcal{R}(s)$ for $\nu$-a.e. $s$. In other words, $\mathcal{R}$ is $\nu$-measurable if $s \to G_{\mathcal{R}(s)}$ is a $\nu$-measurable family of subspaces of $\mathcal{G}_1 \oplus \nu \mathcal{G}_2$.

Given a $\nu$-measurable family $\mathcal{G}$ of Hilbert spaces, we say that $\mathcal{R}$ is a $\nu$-a.e. defined (or measurable) family of operators in $\mathcal{G}$ if it is a $\nu$-a.e. defined (resp., $\nu$-measurable) family of operators from $\mathcal{G}_1$ to $\mathcal{G}_2$. If $\mathcal{H}$ is a separable Hilbert space, then $\nu$-a.e. defined ($\nu$-measurable) families of operators in $\mathcal{I}_{\mathcal{H}, \nu}$ are called $\nu$-a.e. defined (resp., $\nu$-measurable) families of operators in $\mathcal{H}$.

**Lemma B.3.** Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be $\nu$-measurable families of Hilbert spaces and $\mathcal{R}$ be a $\nu$-a.e. defined family of operators from $\mathcal{G}_1$ to $\mathcal{G}_2$ such that $\mathcal{R}(s)$ is an everywhere defined bounded operator for $\nu$-a.e. $s$. Then the following statements hold

1. If $\mathcal{R}$ is $\nu$-measurable then $s \to \mathcal{R}(s)\xi(s)$ is a $\nu$-measurable section of $\mathcal{G}_2$ for any $\nu$-measurable section $\xi$ of $\mathcal{G}_1$.

2. Suppose there exists a $\nu$-measurable basis $\xi_1, \xi_2, \ldots$ in $\mathcal{G}_1$ such that $s \to \mathcal{R}(s)\xi_j(s)$ is a $\nu$-measurable section of $\mathcal{G}_2$ for all $j = 1, 2, \ldots$. Then $\mathcal{R}$ is $\nu$-measurable.

**Proof.**

1. Let $\xi$ be a $\nu$-measurable section of $\mathcal{G}_1$ and $\xi_1, \xi_2, \ldots$ and $\eta_1, \eta_2, \ldots$ be $\nu$-measurable sections of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively such that the linear span of the vectors $(\xi_j(s), \eta_j(s))$ is dense in $G_{\mathcal{R}(s)}$ for $\nu$-a.e. $s$. By Lemma B.1 there exists an orthonormal sequence $\tilde{\xi}_1, \tilde{\xi}_2, \ldots$ associated with $\xi_1, \xi_2, \ldots$. By condition (b) of Sec. B.1 we have $\mathcal{R}(s)\tilde{\xi}_j(s) = \sum_{i=1}^{\infty} f_{ij}(s)\eta_i(s)$ with some $\nu$-measurable functions $f_{ij}$ for any $j = 1, 2, \ldots$ and $\nu$-a.e. $s$. Hence, $s \to \mathcal{R}(s)\tilde{\xi}_j(s)$ is a $\nu$-measurable section of $\mathcal{G}_2$ for any $j = 1, 2, \ldots$. By condition (a) of Sec. B.1 the linear span of $\tilde{\xi}_1(s), \tilde{\xi}_2(s), \ldots$ coincides with that of $\xi_1(s), \xi_2(s), \ldots$ and, therefore, is dense in $\mathcal{G}_1(s)$ for $\nu$-a.e. $s$. This implies that $\xi(s) = \sum_{j=1}^{\infty} (\xi_j(s), \xi(s))\tilde{\xi}_j(s)$ for $\nu$-a.e. $s$. By the continuity of $\mathcal{R}(s)$, it follows that $\mathcal{R}(s)\xi(s) = \sum_{j=1}^{\infty} (\xi_j(s), \xi(s))\mathcal{R}(s)\tilde{\xi}_j(s)$ for $\nu$-a.e. $s$ and, therefore, $s \to \mathcal{R}(s)\xi(s)$ is a $\nu$-measurable section of $\mathcal{G}_2$.

2. Let $\nu$-measurable sections $\eta_1, \eta_2, \ldots$ of $\mathcal{G}_2$ be such that $\eta_j(s) = \mathcal{R}(s)\xi_j(s)$ for all $j = 1, 2, \ldots$ and $\nu$-a.e. $s$. As the linear span of $\xi_1(s), \xi_2(s), \ldots$ is dense in $\mathcal{G}_1(s)$ and $\mathcal{R}(s)$ is everywhere defined and continuous, the linear span of $(\xi_j(s), \eta_j(s))$ is dense in $G_{\mathcal{R}(s)}$ for $\nu$-a.e. $s$. This means that $\mathcal{R}$ is $\nu$-measurable. \qed

**Lemma B.4.** Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be $\nu$-measurable families of Hilbert spaces and $\mathcal{R}$ be a $\nu$-measurable family of operators from $\mathcal{G}_1$ to $\mathcal{G}_2$ such that $\mathcal{R}(s)$ is an everywhere defined bounded operator for $\nu$-a.e. $s$. Let $\mathcal{G}'_1$ be a $\nu$-measurable family of subspaces of $\mathcal{G}_1$. Then the images of $\mathcal{G}'_1(s)$ under $\mathcal{R}(s)$ constitute a $\nu$-measurable family of subspaces of $\mathcal{G}_2$.

**Proof.** Let $\xi_1, \xi_2, \ldots$ be a $\nu$-measurable basis in $\mathcal{G}_1$. By the continuity of $\mathcal{R}(s)$, the linear span of $\mathcal{R}(s)\xi_1(s), \mathcal{R}(s)\xi_2(s), \ldots$ is dense in the image of $\mathcal{G}'_1(s)$ under $\mathcal{R}(s)$ for $\nu$-a.e. $s$. Hence the statement follows because $s \to \mathcal{R}(s)\xi_j(s)$ is a $\nu$-measurable section of $\mathcal{G}_2$ for any $j = 1, 2, \ldots$ by Lemma B.3. \qed
Let $\mathcal{S}$ be a $\nu$-measurable family of Hilbert spaces and $\mathcal{S}'$ and $\mathcal{P}$ be $\nu$-a.e. defined maps such that $\mathcal{S}'(s)$ is a closed subspace of $\mathcal{S}(s)$ and $\mathcal{P}(s)$ is the orthogonal projection of $\mathcal{S}(s)$ onto $\mathcal{S}'(s)$ for $\nu$-a.e. $s$. Then $\mathcal{P}$ is a $\nu$-measurable family of operators in $\mathcal{S}$ if and only if $\mathcal{S}'$ is a $\nu$-measurable family of subspaces of $\mathcal{S}$.

Proof. If $\mathcal{P}$ is $\nu$-measurable, then so is $\mathcal{S}'$ by Lemma B.4. Let $\mathcal{S}'$ be $\nu$-measurable and $\xi_1, \xi_2, \ldots$ be a $\nu$-measurable basis in $\mathcal{S}'$. Let $\xi$ be a $\nu$-measurable section of $\mathcal{S}$. For $\nu$-a.e. $s$, we have $(\mathcal{P}(s)\xi(s), \xi_j(s)) = (\xi(s), \xi_j(s))$. Hence, $s \to (\mathcal{P}(s)\xi(s), \xi_j(s))$ is a $\nu$-measurable section of $\mathcal{S}$ for all $j = 1, 2, \ldots$. In view of Lemma B.2, this implies that $s \to \mathcal{P}(s)\xi(s)$ is a $\nu$-measurable section of $\mathcal{S}$ and, therefore, $\mathcal{P}$ is $\nu$-measurable by Lemma B.3.

Let $\mathcal{S}'$ and $\mathcal{S}''$ be closed subspaces of a Hilbert space $\mathcal{H}$ and $\mathcal{P}'$ and $\mathcal{P}''$ be the orthogonal projections of $\mathcal{S}$ onto $\mathcal{S}'$ and $\mathcal{S}''$ respectively. Then we have $\mathcal{S}' \cap \mathcal{S}'' = \ker \mathcal{R}$, where $\mathcal{R} = 1_\mathcal{S} - (\mathcal{P}' + \mathcal{P}'')/2$ and $1_\mathcal{S}$ denotes the identity operator in $\mathcal{S}$. Indeed, suppose $\psi \in \ker \mathcal{R}$ and $\psi \notin \mathcal{S}' \cap \mathcal{S}''$. Assume, for definiteness, that $\psi \notin \mathcal{S}'$. Then $\|\mathcal{P}'\psi\| < \|\psi\|$. Since $\psi = (\mathcal{P}'\psi + \mathcal{P}''\psi)/2$ and $\|\mathcal{P}'\psi\| \leq \|\psi\|$, this implies that $\|\psi\| < \|\psi\|$. We thus obtain a contradiction and the statement is proved.

Lemma B.5. Let $\mathcal{S}$ be a $\nu$-measurable family of Hilbert spaces and $\mathcal{S}'$ and $\mathcal{S}''$ be $\nu$-measurable families of closed subspaces of $\mathcal{S}$. Then

1. If $\mathcal{S}'(s) \subset \mathcal{S}'(s)$ for $\nu$-a.e. $s$, then $s \to \mathcal{S}'(s) \cap \mathcal{S}''(s)$, where $\mathcal{S}'(s) \cap \mathcal{S}''(s)$ is the orthogonal complement of $\mathcal{S}'(s)$ in $\mathcal{S}(s)$, is a $\nu$-measurable family of subspaces of $\mathcal{S}$.
2. $s \to \mathcal{S}'(s) \cap \mathcal{S}''(s)$ is a $\nu$-measurable family of subspaces of $\mathcal{S}$.

Proof. 1. Let $\mathcal{P}'$ and $\mathcal{P}''$ be $\nu$-a.e. defined maps such that $\mathcal{P}'(s)$ and $\mathcal{P}''(s)$ are orthogonal projections of $\mathcal{S}(s)$ onto $\mathcal{S}'(s)$ and $\mathcal{S}''(s)$ respectively for $\nu$-a.e. $s$. By Lemma B.5, $\mathcal{P}'$ and $\mathcal{P}''$ are $\nu$-measurable families of operators in $\mathcal{S}$. It follows from Lemma B.3 that $s \to \mathcal{P}'(s) - \mathcal{P}''(s)$ is also a $\nu$-measurable family of operators in $\mathcal{S}$. As $\mathcal{P}'(s) - \mathcal{P}''(s)$ is the orthogonal projection of $\mathcal{S}(s)$ onto $\mathcal{S}'(s) \cap \mathcal{S}''(s)$ for $\nu$-a.e. $s$, the desired statement follows from Lemma B.3.

2. Let $\nu$-measurable families $\mathcal{P}'$ and $\mathcal{P}''$ of operators in $\mathcal{S}$ be as in the proof of (1) and $\mathcal{R}$ be a $\nu$-a.e. defined map such that $\mathcal{R}(s) = 1_\mathcal{S}(s) - (\mathcal{P}'(s) + \mathcal{P}''(s))/2$ for $\nu$-a.e. $s$. It follows from Lemma B.3 that $\mathcal{R}$ is a $\nu$-measurable family of operators in $\mathcal{S}$. Since $\mathcal{S}'(s) \cap \mathcal{S}''(s) = \ker \mathcal{R}(s) = (\im \mathcal{R}(s))^\perp$ for $\nu$-a.e. $s$, the statement follows from Lemma B.3 and (1).

Lemma B.7. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be $\nu$-measurable families of Hilbert spaces and $\mathcal{R}$ be a $\nu$-measurable family of operators from $\mathcal{S}_1$ to $\mathcal{S}_2$. Then

1. If $\mathcal{R}(s)$ is invertible for $\nu$-a.e. $s$, then $s \to \mathcal{R}(s)^{-1}$ is a $\nu$-measurable family of operators from $\mathcal{S}_2$ to $\mathcal{S}_1$.
2. If $\mathcal{R}(s)$ is closed for $\nu$-a.e. $s$, then $s \to \ker \mathcal{R}(s)$ is a $\nu$-measurable family of subspaces of $\mathcal{S}_1$.

Proof. For $\nu$-a.e. $s$, let $\mathcal{S}(s)$ be every defined bounded operator from $\mathcal{S}_1(s) \oplus \mathcal{S}_2(s)$ to $\mathcal{S}_2(s) \oplus \mathcal{S}_1(s)$ taking $(\psi_1, \psi_2)$ to $(\psi_2, \psi_1)$. Obviously, $\mathcal{S}$ is a $\nu$-measurable family of operators from $\mathcal{S}_1 \oplus_{\nu} \mathcal{S}_2$ to $\mathcal{S}_2 \oplus_{\nu} \mathcal{S}_1$. If $\mathcal{R}(s)$ is invertible for $\nu$-a.e. $s$, then $G_{\mathcal{R}(s)^{-1}}$ coincides with the image of $G_{\mathcal{R}(s)}$ under $\mathcal{S}(s)$ for $\nu$-a.e. $s$ and, therefore, (1) follows from Lemma B.3. Suppose now that $\mathcal{R}(s)$ is closed for $\nu$-a.e.
s. Since \( s \to \mathcal{S}_1(s) \times \{0\} \) is a \( \nu \)-measurable family of closed subspaces of \( \mathcal{S}_1 \oplus_\nu \mathcal{S}_2 \) and \( G_{\mathcal{R}_1(s)} \cap (\mathcal{S}_1(s) \times \{0\}) = \text{Ker} \mathcal{R}(s) \times \{0\} \) for \( \nu \)-a.e. \( s \), statement 2 of Lemma B.10 implies that \( s \to \text{Ker} \mathcal{R}(s) \times \{0\} \) is a \( \nu \)-measurable family of closed subspaces of \( \mathcal{S}_1 \oplus_\nu \mathcal{S}_2 \), whence (2) obviously follows.

\[ \square \]

**Lemma B.8.** Let \( \mathcal{S}_1, \mathcal{S}_2, \) and \( \mathcal{S}_3 \) be \( \nu \)-measurable families of Hilbert spaces and \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be \( \nu \)-measurable families of closed operators from \( \mathcal{S}_1 \) to \( \mathcal{S}_2 \) and from \( \mathcal{S}_2 \) to \( \mathcal{S}_3 \) respectively. Then \( s \to \mathcal{R}_2(s) \mathcal{R}_1(s) \) is a \( \nu \)-measurable family of operators from \( \mathcal{S}_1 \) to \( \mathcal{S}_3 \).

**Proof.** Let \( \mathcal{S}' \) and \( \mathcal{S}'' \) be \( \nu \)-a.e. defined maps such that, for \( \nu \)-a.e. \( s \), \( \mathcal{S}'(s) \) and \( \mathcal{S}''(s) \) are subspaces of \( \mathcal{S}_1(s) \oplus_\nu \mathcal{S}_2(s) \oplus_\nu \mathcal{S}_3(s) \) consisting of all \( (\psi_1, \psi_2, \psi_3) \) with \( (\psi_1, \psi_2) \in G_{\mathcal{R}_1(s)} \) and \( (\psi_2, \psi_3) \in G_{\mathcal{R}_2(s)} \) respectively. Clearly, both \( \mathcal{S}' \) and \( \mathcal{S}'' \) are \( \nu \)-measurable families of closed subspaces of \( \mathcal{S}_1 \oplus_\nu \mathcal{S}_2 \oplus_\nu \mathcal{S}_3 \). Let \( \mathcal{S} \) be a \( \nu \)-a.e. defined map such that, for \( \nu \)-a.e. \( s \), \( \mathcal{S}(s) \) is the everywhere defined bounded operator from \( \mathcal{S}_1(s) \oplus_\nu \mathcal{S}_2(s) \oplus_\nu \mathcal{S}_3(s) \) to \( \mathcal{S}_1(s) \oplus_\nu \mathcal{S}_3(s) \) taking \( (\psi_1, \psi_2, \psi_3) \) to \( (\psi_1, \psi_3) \). Then \( G_{\mathcal{R}_2(s) \mathcal{R}_1(s)} \) is the image of \( \mathcal{S}'(s) \cap \mathcal{S}''(s) \) under \( \mathcal{S}(s) \) for \( \nu \)-a.e. \( s \). Hence, the result follows from Lemma B.3 and statement 2 of Lemma B.6.

\[ \square \]

**Lemma B.9.** Let \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) be \( \nu \)-measurable families of Hilbert spaces and \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be \( \nu \)-measurable families of closed operators from \( \mathcal{S}_1 \) to \( \mathcal{S}_2 \). Then \( s \to G_{\mathcal{R}_2(s) + \mathcal{R}_1(s)} \) is a \( \nu \)-measurable family of operators from \( \mathcal{S}_1 \) to \( \mathcal{S}_2 \).

**Proof.** Let \( \mathcal{S}' \) and \( \mathcal{S}'' \) be \( \nu \)-a.e. defined maps such that, for \( \nu \)-a.e. \( s \), \( \mathcal{S}'(s) \) and \( \mathcal{S}''(s) \) are subspaces of \( \mathcal{S}_1(s) \oplus_\nu \mathcal{S}_2(s) \oplus_\nu \mathcal{S}_2(s) \) consisting of all \( (\psi_1, \psi_2, \psi_3) \) with \( (\psi_1, \psi_2) \in G_{\mathcal{R}_1(s)} \) and \( (\psi_1, \psi_3) \in G_{\mathcal{R}_2(s)} \) respectively. Clearly, both \( \mathcal{S}' \) and \( \mathcal{S}'' \) are \( \nu \)-measurable families of closed subspaces of \( \mathcal{S}_1 \oplus_\nu \mathcal{S}_2 \oplus_\nu \mathcal{S}_2 \). Let \( \mathcal{S} \) be a \( \nu \)-a.e. defined map such that, for \( \nu \)-a.e. \( s \), \( \mathcal{S}(s) \) is the everywhere defined bounded operator from \( \mathcal{S}_1(s) \oplus_\nu \mathcal{S}_2(s) \oplus_\nu \mathcal{S}_2(s) \) to \( \mathcal{S}_1(s) \oplus_\nu \mathcal{S}_2(s) \) taking \( (\psi_1, \psi_2, \psi_3) \) to \( (\psi_1, \psi_2 + \psi_3) \). Then \( G_{\mathcal{R}_2(s) + \mathcal{R}_1(s)} \) is the image of \( \mathcal{S}'(s) \cap \mathcal{S}''(s) \) under \( \mathcal{S}(s) \) for \( \nu \)-a.e. \( s \). Hence, the result follows from Lemma B.3 and statement 2 of Lemma B.6.

\[ \square \]

**Lemma B.10.** Let \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) be \( \nu \)-measurable families of Hilbert spaces and \( \mathcal{R} \) be a \( \nu \)-measurable family of closed operators from \( \mathcal{S}_1 \) to \( \mathcal{S}_2 \). Let \( \xi \) be a \( \nu \)-measurable section of \( \mathcal{S}_1 \) such that \( \xi(s) \in D_{\mathcal{R}(s)} \) for \( \nu \)-a.e. \( s \). Then \( s \to \mathcal{R}(s)\xi(s) \) is a \( \nu \)-measurable section of \( \mathcal{S}_2 \).

**Proof.** Let \( \xi \) be a \( \nu \)-a.e. defined section of \( \mathcal{S}_1 \oplus_\nu \mathcal{S}_2 \) such that \( \xi(s) = (\xi(s), \mathcal{R}(s)\xi(s)) \) for \( \nu \)-a.e. \( s \). It suffices to show that \( \xi \) is \( \nu \)-measurable. Let \( \xi_1, \xi_2, \ldots \) be a \( \nu \)-measurable basis in \( \mathcal{S}_1 \). Let \( \mathcal{P} \) be a \( \nu \)-a.e. defined map such that \( \mathcal{P}(s) \) is the orthogonal projection of \( \mathcal{S}_1(s) \oplus_\nu \mathcal{S}_2(s) \) onto \( G_{\mathcal{R}(s)} \) for \( \nu \)-a.e. \( s \). For \( j = 1, 2, \ldots \), let \( \xi_j \) be a \( \nu \)-a.e. defined map such that \( \xi_j(s) = \mathcal{P}(s)(\xi_j(s), 0) \) for \( \nu \)-a.e. \( s \). By Lemma B.3, \( \mathcal{P} \) is a \( \nu \)-measurable family of operators in \( \mathcal{S}_1 \oplus_\nu \mathcal{S}_2 \), and Lemma B.3 implies that \( \xi_j \) is a \( \nu \)-measurable section of \( \mathcal{S}_1 \oplus_\nu \mathcal{S}_2 \) for all \( j = 1, 2, \ldots \). Moreover, the linear span of \( \xi_1(s), \xi_2(s), \ldots \) is dense in \( G_{\mathcal{R}(s)} \) for \( \nu \)-a.e. \( s \). As \( \langle \xi_j(s), \xi(s) \rangle = \langle \xi_j(s), \xi(s) \rangle \) for \( \nu \)-a.e. \( s \), we conclude that \( s \to (\xi_j(s), \xi(s)) \) is a \( \nu \)-measurable function for all \( j = 1, 2, \ldots \). Hence, \( \xi \) is \( \nu \)-measurable by Lemma B.2.

\[ \square \]

\(^{15}\)Indeed, let \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) be Hilbert spaces, \( R \) be a closed operator from \( \mathcal{N}_1 \) to \( \mathcal{N}_2 \), \( \mathcal{P} \) be the orthogonal projection of \( \mathcal{N}_1 \oplus_\nu \mathcal{N}_2 \) onto \( G_R \), and \( \psi_1, \psi_2, \ldots \) be elements of \( \mathcal{N}_1 \) whose linear span is dense in \( \mathcal{N}_1 \). If \( (\psi, \psi') \in G_R \) is orthogonal to all vectors \( P(\psi_1, 0) \), then \( \langle \psi, \psi_j \rangle = \langle (\psi, \psi'), P(\psi_j, 0) \rangle = 0 \) for all \( j \) and, hence, \( \psi = 0 \). As \( G_R \) is a graph of an operator, it follows that \( \psi' = 0 \). Thus, the linear span of \( P(\psi_j, 0) \) is dense in \( G_R \).
Remark B.11. The definition of measurability of a family of operators given in this section is mainly the same as in [25]. The only difference is that, in contrast to [25], we do not require operators to be closed and densely defined. This is essential for Propositions B.3 and B.5 because the properties of being closed and densely defined are not inherited by sums and products of operators.

Remark B.12. Let $R$ be a closed operator in a Hilbert space $H$, $P(R)$ be the orthogonal projection of $H\oplus H$ onto $G_R$, and $\pi_{1,2}(H): H\oplus H \to H$ and $j_{1,2}(H): H \to H\oplus H$ be the canonical projections and embeddings respectively. For $i, k = 1, 2$, we set $P_k(R) = \pi_i(H)P(R)j_k(H)$. The $2 \times 2$-matrix composed of bounded operators $P_k(R)$ is called the characteristic matrix of (generally, unbounded) operator $R$. In [24], a $\nu$-a.e. defined family $R$ of closed operators in a $\nu$-measurable family $\mathcal{S}$ of Hilbert spaces was called $\nu$-measurable if $s \to P_k(\mathcal{R}(s))\xi(s)$ is a $\nu$-measurable section of $\mathcal{S}$ for all $i, k = 1, 2$ and every $\nu$-measurable section $\xi$ of $\mathcal{S}$. It follows from Lemmas B.3 and B.5 that this definition is equivalent to that given in this section because

$$P(\mathcal{R}(s)) = \sum_{i, k=1}^{2} j_i(\mathcal{S}(s))P_k(\mathcal{R}(s))\pi_k(\mathcal{S}(s))$$

for $\nu$-a.e. $s$ and $s \to \pi_{1,2}(\mathcal{S}(s))$ and $s \to j_{1,2}(\mathcal{S}(s))$ are obviously $\nu$-measurable families (in our sense) of operators from $\mathcal{S} \oplus_\nu \mathcal{S}$ to $\mathcal{S}$ and from $\mathcal{S}$ to $\mathcal{S} \oplus_\nu \mathcal{S}$ respectively.

Remark B.13. A $\nu$-a.e. defined family $\mathcal{R}$ of operators in a $\nu$-measurable family $\mathcal{S}$ of Hilbert spaces is said to be weakly $\nu$-measurable [24] if $s \to \mathcal{R}(s)\xi(s)$ is a $\nu$-measurable section of $\mathcal{S}$ for every $\nu$-measurable section $\xi$ of $\mathcal{S}$ satisfying $\xi(s) \in D_{\mathcal{R}(s)}$ for $\nu$-a.e. $s$. Lemma B.10 states that every $\nu$-measurable family of closed operators is also weakly $\nu$-measurable. In [24], where Lemma B.10 was originally proved, a question was posed whether this statement can be reverted, i.e., whether every weakly $\nu$-measurable family of closed operators is $\nu$-measurable. In [14], it was shown by constructing a counterexample that the answer is negative.

B.5. Direct integrals of Hilbert spaces. Given a $\nu$-measurable family $\mathcal{S}$ of Hilbert spaces, we denote by $\mathcal{M}(\mathcal{S}, \nu)$ the set of all $\nu$-equivalence classes $[f]_\nu$, where $f$ is a $\nu$-measurable section of $\mathcal{S}$. Clearly, $\mathcal{M}(\mathcal{S}, \nu)$ has a natural structure of a complex vector space (for any $\nu$-measurable sections $\xi$ and $\eta$ and any $k \in \mathbb{C}$, we set $[\xi + \eta]_\nu = [\xi]_\nu + [\eta]_\nu$ and $k[\xi]_\nu = [k\xi]_\nu$). Now suppose $\mathcal{S}'$ is a $\nu$-a.e. defined family of subspaces of $\mathcal{S}$. We denote by $\langle \mathcal{S}' \rangle \int_\nu \mathcal{S}'(s)\,d\nu(s)$ the linear subspace of the space $\mathcal{M}(\mathcal{S}, \nu)$ consisting of all its elements $\xi$ such that $\xi(s) \in \mathcal{S}'(s)$ for $\nu$-a.e. $s$ and $s \to ||\xi(s)||^2$ is a $\nu$-integrable function. The space $\langle \mathcal{S}' \rangle \int_\nu \mathcal{S}'(s)\,d\nu(s)$ is endowed with the scalar product defined by the relation

$$\langle \xi, \eta \rangle = \int \langle \xi(s), \eta(s) \rangle\,d\nu(s).$$

For any $\nu$-measurable family $\mathcal{S}$, the space $\langle \mathcal{S}' \rangle \int_\nu \mathcal{S}'(s)\,d\nu(s)$ is complete (the proof is essentially the same as that of completeness of ordinary $L_2$-spaces) and, hence, is a Hilbert space.

As a rule, the $\nu$-measurable family $\mathcal{S}$ can be easily deduced from the context. So we usually omit the prefix $\langle \mathcal{S}' \rangle$ and write $\int_\nu \mathcal{S}'(s)\,d\nu(s)$ in place of
(\mathcal{S})-\int_\mathcal{S} G'(s) \, d\nu(s)$. For any separable Hilbert space $h$, we obviously have
\begin{equation}
L_2(S\nu, h, \nu) = (I_{h\nu})-\int_\mathcal{S} h \, d\nu(s).
\end{equation}

Given a $\nu$-a.e. defined section $\xi$ of $\mathcal{S}$ and a $\nu$-measurable set $A$, we denote by $\xi^A$ the $\nu$-equivalence class such that $\xi^A(s) = \xi(s)$ for $\nu$-a.e. $s \in A$ and $\xi^A(s) = 0$ for $\nu$-a.e. $s \in S\nu \setminus A$.

**Lemma B.14.** Let $\mathcal{S}$ be a $\nu$-measurable family of Hilbert spaces and $\mathcal{S}'$ be a $\nu$-measurable family of subspaces of $\mathcal{S}$. Then there is a $\nu$-measurable basis in $\mathcal{S}'$ consisting of elements of $\int_\mathcal{S} \mathcal{S}'(s) \, d\nu(s)$.

**Proof.** Let $\xi_1, \xi_2, \ldots$ be a $\nu$-measurable basis in $\mathcal{S}'$. Multiplying $\xi_j$ by suitable $\nu$-measurable functions, we can ensure that $\|\xi_j(s)\| \leq 1$ for all $j = 1, 2, \ldots$ and $\nu$-a.e. $s$. Let $A_1, A_2, \ldots$ be elements of $D\nu$ such that $S\nu = \bigcup_{k=1}^{\infty} A_k$. Then $\xi^A_k$ with $j, k = 1, 2, \ldots$ obviously constitute the required $\nu$-measurable basis in $\mathcal{S}'$. $\square$

**Lemma B.15.** Let $\mathcal{S}$ and $\mathcal{S}'$ be as in Lemma B.14 and $\mathcal{S}' = \int_\mathcal{S} \mathcal{S}'(s) \, d\nu(s)$. Then $\mathcal{S}' = \int_\mathcal{S} \mathcal{S}'(s) \, d\nu(s)$ and $\overline{\mathcal{S}'}(s) = \int_\mathcal{S} \overline{\mathcal{S}'(s)} \, d\nu(s)$.

**Proof.** Let $\eta \in \mathcal{S}'^\perp$ and $\xi \in \mathcal{S}'$. As $\xi^A \in \mathcal{S}'$ for any $A \in D\nu$, we obtain $\int A (\xi(s) \nu, \eta(s)) \, d\nu(s) = \langle \xi, \eta \rangle = 0$ and, hence, $\langle \xi(s), \eta(s) \rangle = 0$ for $\nu$-a.e. $s$. By Lemma B.14 there is a $\nu$-measurable basis $\xi_1, \xi_2, \ldots$ in $\mathcal{S}'$ such that $\xi_j \in \mathcal{S}'$ for all $j = 1, 2, \ldots$. Since $\langle \xi_j(s), \eta(s) \rangle = 0$ for $\nu$-a.e. $s$ and all $j = 1, 2, \ldots$, we have $\eta(s) \in \mathcal{S}'^\perp$ for $\nu$-a.e. $s$. This implies that $\mathcal{S}' = \int_\mathcal{S} \mathcal{S}'(s)^\perp \, d\nu(s)$. The equality for the closure of $\mathcal{S}'$ now follows from the relations $\overline{\mathcal{S}'} = (\mathcal{S}'^\perp)^\perp$ and $\overline{\mathcal{S}'}(s) = (\mathcal{S}'(s)^\perp)^\perp$. $\square$

Let $\mathcal{S}$ be a $\nu$-measurable family of Hilbert spaces and $\mathcal{H} = \int_\mathcal{S} \mathcal{S}(s) \, d\nu(s)$. Given a $\nu$-measurable complex function $g$, we denote by $T_{g\mathcal{E}}$ the operator of multiplication by $g$ in $\mathcal{H}$. By definition, the graph of $T_{g\mathcal{E}}$ consists of all pairs $(\xi_1, \xi_2) \in \mathcal{H} \oplus \mathcal{H}$ such that $\xi_2(s) = g(s)\xi_1(s)$ for $\nu$-a.e. $s$. The operator $T_{g\mathcal{E}}$ is closed and densely defined and its adjoint is equal to $T_{g^*\mathcal{E}}$, where $g^*$ is the complex conjugate function of $g$. In particular, if $g$ is real, then $T_{g\mathcal{E}}$ is self-adjoint. If $g$ is $\nu$-essentially bounded, then $T_{g\mathcal{E}}$ is everywhere defined and bounded.

**Lemma B.16.** Let $\mathcal{S}$ be a $\nu$-measurable family of Hilbert spaces, $\mathcal{H} = \int_\mathcal{S} \mathcal{S}(s) \, d\nu(s)$, and $\mathcal{S}'$ be a closed linear subspace of $\mathcal{H}$ such that $T_{g\mathcal{E}} \xi \in \mathcal{S}'$ for every $\xi \in \mathcal{S}'$ and every $\nu$-essentially bounded complex function $g$. Then there is a $\nu$-measurable family $\mathcal{S}'$ of closed subspaces of $\mathcal{S}$ such that $\mathcal{S}' = \int_\mathcal{S} \mathcal{S}'(s) \, d\nu(s)$.

**Proof.** By the hypothesis, the projection $P$ of $\mathcal{H}$ onto $\mathcal{S}'$ commutes with $T_{g\mathcal{E}}$ for every $\nu$-essentially bounded function $g$. By Lemma B.14 there is a $\nu$-measurable basis $\xi_1, \xi_2, \ldots$ in $\mathcal{S}$ consisting of elements of $\mathcal{S}$. For each $j = 1, 2, \ldots$, we set $\eta_j = P\xi_j$. Let $\mathcal{S}'$ be a $\nu$-measurable family of subspaces of $\mathcal{S}$ such that $\mathcal{S}'(s)$ is the closed linear span of $\eta_1(s), \eta_2(s), \ldots$ for $\nu$-a.e. $s$. Let $\mathcal{H} = \int_\mathcal{S} \mathcal{S}'(s) \, d\nu(s)$. If $\xi \in \mathcal{H}$ is such that $\langle \xi^A, \xi \rangle = \int A (\xi(s), \xi(s)) \, d\nu(s) = 0$ for every $\nu$-measurable set $A$ and $j = 1, 2, \ldots$, then $\langle \xi_j(s), \xi(s) \rangle = 0$ for all $j$ and $\nu$-a.e. $s$ and, therefore, $\xi = 0$. This means that the linear span of all $\xi^A$ is dense in $\mathcal{H}$. Note that $\eta_j^A = T_{\chi_A\mathcal{E}} P\xi_j = P\xi_j^A$, where $\chi_A(s) = 1$ for $s \in A$ and $\chi_A(s) = 0$ for $s \in S\nu \setminus A$. Hence, the linear span
of \( \eta_j^1 \) is dense in \( \mathcal{H}' \). Since \( \tilde{\mathcal{H}} \) is closed and contains all \( \eta_j^1 \), we conclude that 
\( \mathcal{H}' \subset \tilde{\mathcal{H}} \). Let \( \eta \in \mathcal{H}' \). Then we have 
\( \langle \eta_j^1, \eta \rangle = \int_A \langle \eta_j(s), \eta(s) \rangle \, d\nu(s) = 0 \) and, hence, 
\( \langle \eta_j(s), \eta(s) \rangle = 0 \) for all \( j \) and \( \nu \)-a.e. \( s \). This implies that \( \eta(s) \in \mathcal{G}'(s)^\perp \) for \( \nu \)-a.e. \( s \), i.e., \( \eta \in \tilde{\mathcal{H}}^\perp \). We therefore obtain \( \mathcal{H}' \subset \tilde{\mathcal{H}}^\perp \). As \( \mathcal{H}' \) is closed, it follows that \( \tilde{\mathcal{H}} \subset \mathcal{H}' \) and, hence, \( \tilde{\mathcal{H}} = \mathcal{H}' \).

\( \square \)

B.6. Direct integrals of operators. Let \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be \( \nu \)-measurable families of Hilbert spaces. Let \( \mathcal{H}_{1,2} = \int^\oplus \mathcal{G}_1,2(s) \, d\nu(s) \) and \( \mathcal{R} \) be a \( \nu \)-a.e. defined family of operators from \( \mathcal{G}_1 \) to \( \mathcal{G}_2 \). The direct integral \( \int^\oplus \mathcal{R}(s) \, d\nu(s) \) of the family \( \mathcal{R} \) is defined as the linear operator from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) whose graph consists of all pairs 
\( (\xi, \eta) \in \mathcal{H}_1 \oplus \mathcal{H}_2 \) such that 
\( \langle \eta(s), \xi \rangle = \langle \mathcal{R}(s)\xi, \eta \rangle \) for \( \nu \)-a.e. \( s \).

Let \( \mathcal{H}_1 = \int^\oplus \mathcal{G}_1(s) \, d\nu(s) \) and \( \mathcal{H}_2 = \int^\oplus \mathcal{G}_2(s) \, d\nu(s) \). Then there is a unique unitary operator 
\( U_{\mathcal{G}_1,\mathcal{G}_2} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \) such that 
\( \langle \mathcal{U}_{\mathcal{G}_1,\mathcal{G}_2}(\xi, \eta), (\xi, \eta) \rangle = \langle (\xi, \eta), (\xi, \eta) \rangle \) for any \( (\xi, \eta) \in \mathcal{H}_1 \oplus \mathcal{H}_2 \) and \( \nu \)-a.e. \( s \). We call \( U_{\mathcal{G}_1,\mathcal{G}_2} \) the natural isomorphism between \( \mathcal{H}_1 \oplus \mathcal{H}_2 \) and \( \mathcal{H}_1 \).

The next statement follows immediately from the definition of \( U_{\mathcal{G}_1,\mathcal{G}_2} \).

**Lemma B.17.** Let \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be \( \nu \)-measurable families of Hilbert spaces, \( \mathcal{R} \) be a \( \nu \)-a.e. defined family of operators from \( \mathcal{G}_1 \) to \( \mathcal{G}_2 \), and \( R = \int^\oplus \mathcal{R}(s) \, d\nu(s) \). Then we have 
\( U_{\mathcal{G}_1,\mathcal{G}_2}(G_R) = (\mathcal{G}_1 \oplus_{\nu} \mathcal{G}_2) \, \int^\oplus \mathcal{G}_R(s) \, d\nu(s) \).

**Lemma B.18.** Let \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be \( \nu \)-measurable families of Hilbert spaces, \( \mathcal{H}_{1,2} = \int^\oplus \mathcal{G}_1,2(s) \, d\nu(s) \), and \( G \) be a closed linear subspace of \( \mathcal{H}_1 \oplus \mathcal{H}_2 \). Suppose \( \mathcal{G} \) is a \( \nu \)-measurable family of closed subspaces of \( \mathcal{G}_1 \oplus_{\nu} \mathcal{G}_2 \) such that 
\( U_{\mathcal{G}_1,\mathcal{G}_2}(G) = \int^\oplus \mathcal{G}(s) \, d\nu(s) \). Then \( G \) is a graph of an operator from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) if and only if \( G(s) \) is a graph of an operator from \( \mathcal{G}_1(s) \) to \( \mathcal{G}_2(s) \) for \( \nu \)-a.e. \( s \).

**Proof.** Let \( U = U_{\mathcal{G}_1,\mathcal{G}_2} \) and \( \mathcal{G}' \) be a \( \nu \)-a.e. defined map such that 
\( \mathcal{G}'(s) = \{0\} \times \mathcal{G}_2(s) \) for \( \nu \)-a.e. \( s \). Clearly, \( \mathcal{G}' \) is a \( \nu \)-measurable family of closed subspaces of \( \mathcal{G}_1 \oplus_{\nu} \mathcal{G}_2 \). Let \( \mathcal{H}' = \int^\oplus \mathcal{G}'(s) \, d\nu(s) \) and \( \mathcal{H}'' = \int^\oplus \mathcal{G}'(s) \cap \mathcal{G}(s) \, d\nu(s) \). Since \( \mathcal{H}' = \mathcal{H}' \cap U(G) \) and \( U^{-1}(\mathcal{H}) = \{0\} \times \mathcal{H}_2 \), we have 
\( G \cap (\{0\} \times \mathcal{H}_2) = U^{-1}(\mathcal{H}'') \). Hence, \( G \) is a graph of an operator if and only if \( \mathcal{H}'' \) is trivial. By statement 2 of Lemma B.6, \( s \to \mathcal{G}'(s) \cap \mathcal{G}(s) \) is a \( \nu \)-measurable family of subspaces of \( \mathcal{G}_1 \oplus_{\nu} \mathcal{G}_2 \) and, therefore, \( \mathcal{H}'' \) is trivial if and only if \( \mathcal{G}'(s) \cap \mathcal{G}(s) \) is trivial for \( \nu \)-a.e. \( s \), i.e., if \( \mathcal{G}(s) \) is a graph of an operator for \( \nu \)-a.e. \( s \). \( \square \)

**Lemma B.19.** Let \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be \( \nu \)-measurable families of Hilbert spaces, \( \mathcal{R} \) be a \( \nu \)-measurable family of operators from \( \mathcal{G}_1 \) to \( \mathcal{G}_2 \), and \( R = \int^\oplus \mathcal{R}(s) \, d\nu(s) \). Then \( R \) is closable if and only if \( \mathcal{R}(s) \) is closable for \( \nu \)-a.e. \( s \), in which case 
\( R = \int^\oplus \mathcal{R}(s) \, d\nu(s) \).

**Proof.** Let \( U = U_{\mathcal{G}_1,\mathcal{G}_2} \). As \( U(G_R) = \int^\oplus G_{\mathcal{R}(s)} \, d\nu(s) \), Lemma B.15 yields 
\( U(G_R) = \int^\oplus G_{\mathcal{R}(s)} \, d\nu(s) \).

The operator \( R \) is closable if and only if \( \mathcal{R}_R \) is a graph of an operator. By Lemma B.18, the latter condition holds if and only if \( \mathcal{G}_R(s) \) is a graph of an operator for \( \nu \)-a.e. \( s \), i.e., if \( \mathcal{R}(s) \) is closable for \( \nu \)-a.e. \( s \). Suppose now that \( R \) is closable and \( R' = \int^\oplus \mathcal{R}(s) \, d\nu(s) \). As \( \mathcal{R}_R = \mathcal{R}_R \) and \( \mathcal{G}_R(s) = \mathcal{G}_R(s) \) for \( \nu \)-a.e. \( s \), equality (120) and Lemma B.17 imply that 
\( U(G_{\mathcal{R}_R}) = U(G_{\mathcal{R}'}) \) and, hence, \( R = R' \). \( \square \)
Proposition B.20. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be $\nu$-measurable families of Hilbert spaces and $\mathcal{R}$ and $\mathcal{R}'$ be $\nu$-measurable families of closed operators from $\mathcal{G}_1$ to $\mathcal{G}_2$. Let $R = \int^\oplus \mathcal{R}(s) \, d\nu(s)$, and $R' = \int^\oplus \mathcal{R}'(s) \, d\nu(s)$. If $R$ is an extension of $R'$, then $\mathcal{R}(s)$ is an extension of $\mathcal{R}'(s)$ for $\nu$-a.e. $s$. If $R = R'$, then $\mathcal{R}(s) = \mathcal{R}'(s)$ for $\nu$-a.e. $s$.

Proof. Let $U = U_{\mathcal{G}_1, \mathcal{G}_2}$. Since $U(G_R) = \int^\oplus G_{\mathcal{R}(s)}(s) \, d\nu(s)$, Lemma B.14 implies that there exists a sequence $\zeta_1, \zeta_2, \ldots$ of elements of $U(G_R)$ such that the linear span of $\zeta_1(s), \zeta_2(s), \ldots$ is dense in $G_{\mathcal{R}(s)}(s)$ for $\nu$-a.e. $s$. If $R$ is an extension of $R'$, then $U(G_R) \subseteq U(G_{R'})$ and, hence, $\zeta_j \in U(G_R)$ for all $j = 1, 2, \ldots$. Since $U(G_R) = \int^\oplus G_{\mathcal{R}(s)}(s) \, d\nu(s)$, it follows that $\zeta_j(s) \in G_{\mathcal{R}(s)}(s)$ for $\nu$-a.e. $s$ and all $j = 1, 2, \ldots$. The linear span of $\zeta_1(s), \zeta_2(s), \ldots$ is therefore contained in $G_{\mathcal{R}(s)}(s)$ for $\nu$-a.e. $s$. As $\mathcal{R}(s)$ is closed, it follows that $G_{\mathcal{R}(s)}(s) \subseteq G_{\mathcal{R}(s)}$, i.e., $\mathcal{R}(s)$ is an extension of $\mathcal{R}'(s)$ for $\nu$-a.e. $s$. If $R = R'$, then, by the above, $\mathcal{R}(s)$ and $\mathcal{R}'(s)$ are extensions of each other and, therefore, are equal for $\nu$-a.e. $s$. \hfill $\square$

Proposition B.21. Let $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{H}_1$, and $\mathcal{H}_2$ be as in Lemma B.18. Let $\mathcal{R}$ be a $\nu$-measurable family of closed operators from $\mathcal{G}_1$ to $\mathcal{G}_2$. Then $R = \int^\oplus \mathcal{R}(s) \, d\nu(s)$ is a closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ satisfying the condition:

(M) If $\xi \in D_R$ and $g$ is a $\nu$-measurable $\nu$-essentially bounded function, then $T_g^\nu \mathcal{R}(s) \xi \in D_R$ and $RT_g^\nu \mathcal{R}(s) \xi = T_g^\nu \mathcal{R}(s) \xi$.

Conversely, if $R$ is a closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ satisfying (M), then there is a unique (up to $\nu$-equivalence) $\nu$-measurable family $\mathcal{R}$ of closed operators from $\mathcal{G}_1$ to $\mathcal{G}_2$ such that $R = \int^\oplus \mathcal{R}(s) \, d\nu(s)$.

Proof. Let $U = U_{\mathcal{G}_1, \mathcal{G}_2}$, $\mathcal{R}$ be a $\nu$-measurable family of closed operators from $\mathcal{G}_1$ to $\mathcal{G}_2$, and $R = \int^\oplus \mathcal{R}(s) \, d\nu(s)$. By Lemma B.19 $R$ is closed. If $\xi \in D_R$ and $g$ is a $\nu$-measurable $\nu$-essentially bounded function, then $(T_g^\nu \mathcal{R}(s) \xi, T_g^\nu \mathcal{R}(s) \xi)$ belongs to $G_R$ by the very definition of the direct integral of operators. This means that (M) is fulfilled. Conversely, let $R$ be a closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ satisfying (M). If $(\xi, \eta) \in G_R$ and $g$ is a $\nu$-measurable $\nu$-essentially bounded function, then $(T_g^\nu \mathcal{R}(s) \xi, T_g^\nu \mathcal{R}(s) \xi)$ belongs to $G_R$ by (M) and, hence, $T_g^\nu \mathcal{R}(s) U(\xi, \eta) = U(T_g^\nu \mathcal{R}(s) \xi, T_g^\nu \mathcal{R}(s) \eta)$ belongs to $U(G_R)$. This means that $U(G_R)$ is invariant under $T_g^\nu \mathcal{R}(s)$. By Lemma B.16 there is a $\nu$-measurable family $\mathcal{G}$ of closed subspaces of $\mathcal{G}_1 \oplus \mathcal{G}_2$ such that $U(G_R) = \int^\oplus \mathcal{G}(s) \, d\nu(s)$. By Lemma B.18 there is a $\nu$-measurable family $\mathcal{R}$ of closed operators from $\mathcal{G}_1$ to $\mathcal{G}_2$ such that $\mathcal{G}(s) = G_{\mathcal{R}(s)}(s)$ for $\nu$-a.e. $s$. If $R' = \int^\oplus \mathcal{R}(s) \, d\nu(s)$, then it follows from Lemma B.17 that $U(G_{R'}) = U(G_R)$ and, hence, $R = R'$. The uniqueness of $\mathcal{R}$ is ensured by Proposition B.20. \hfill $\square$

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $R$ be an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. We define the linear subspaces $G_{R^*}$ and $G_R$ of $\mathcal{H}_2 \oplus \mathcal{H}_1$ by setting

$$G_{R^*} = \{(-\psi_2, \psi_1) : (\psi_1, \psi_2) \in G_R\}, \quad G_R = (G_{R^*})^\perp.$$ 

The operator $R$ is densely defined if and only if $G_{R^*}$ is the graph of an operator, in which case $G_{R^*} = G_R$.

Proposition B.22. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be $\nu$-measurable families of Hilbert spaces, $\mathcal{R}$ be a $\nu$-measurable family of operators from $\mathcal{G}_1$ to $\mathcal{G}_2$, and $R = \int^\oplus \mathcal{R}(s) \, d\nu(s)$. Then the following statements hold:
1. Suppose \( \mathcal{R}(s) \) is closed for \( \nu \)-a.e. \( s \). The operator \( R \) is invertible if and only if \( \mathcal{R}(s) \) is invertible for \( \nu \)-a.e. \( s \), in which case \( s \rightarrow \mathcal{R}(s)^{-1} \) is a \( \nu \)-measurable family of operators from \( \mathcal{S}_2 \) to \( \mathcal{S}_1 \) and \( R^{-1} = \int_{\mathbb{R}} \mathcal{R}(s)^{-1} \, d\nu(s) \).

2. The operator \( R \) is densely defined if and only if \( \mathcal{R}(s) \) is densely defined for \( \nu \)-a.e. \( s \), in which case \( s \rightarrow \mathcal{R}(s)^* \) is a \( \nu \)-measurable family of operators from \( \mathcal{S}_2 \) to \( \mathcal{S}_1 \) and \( R^* = \int_{\mathbb{R}} \mathcal{R}(s)^* \, d\nu(s) \).

**Proof.** Let \( \mathcal{S}_{1,2} = \int_{\mathbb{R}} \mathcal{S}_{1,2}(s) \, d\nu(s) \).

1. By statement 2 of Lemma \([B.7]\) \( s \rightarrow \text{Ker}(\mathcal{R}(s)) \) is a \( \nu \)-measurable family of subspaces of \( \mathcal{S}_1 \). As \( \text{Ker} R = \int_{\mathbb{R}} \text{Ker}(\mathcal{R}(s)) \, d\nu(s) \), we conclude that \( R \) is invertible if and only if \( \text{Ker} \mathcal{R}(s) = \{0\} \) for \( \nu \)-a.e. \( s \), i.e., if \( \mathcal{R}(s) \) is invertible for \( \nu \)-a.e. \( s \). Let \( R \) be invertible and \( R' = \int_{\mathbb{R}} \mathcal{R}(s)^{-1} \, d\nu(s) \). For any \( (\eta, \xi) \in \mathcal{S}_2 \oplus \mathcal{S}_1 \), we have

\[
(\eta, \xi) \in \mathcal{S}_R^{-1} \iff (\xi, \eta) \in \mathcal{S}_R \iff (\xi(s), \eta(s)) \in \mathcal{S}_{\mathcal{R}(s)} \text{ for } \nu\text{-a.e. } s \iff (\eta(s), \xi(s)) \in \mathcal{S}_{\mathcal{R}(s)^{-1}} \text{ for } \nu\text{-a.e. } s \iff (\eta, \xi) \in \mathcal{S}_{R'}
\]

We thus have \( \mathcal{S}_{R^{-1}} = \mathcal{S}_{R'} \) and, hence, \( R = R' \). By statement 1 of Lemma \([B.7]\) \( s \rightarrow \mathcal{R}(s)^{-1} \) is a \( \nu \)-measurable family of operators from \( \mathcal{S}_2 \) to \( \mathcal{S}_1 \).

2. Let \( U_{12} = U_{\mathcal{S}_1, \mathcal{S}_2} \) and \( U_{21} = U_{\mathcal{S}_2, \mathcal{S}_1} \). For \( \nu \)-a.e. \( s \), let \( \mathcal{S}(s) \) be everywhere defined bounded operator from \( \mathcal{S}_1(s) \oplus \mathcal{S}_2(s) \) to \( \mathcal{S}_2(s) \oplus \mathcal{S}_1(s) \) taking \( (\psi_1, \psi_2) \) to \( (-\psi_2, \psi_1) \). Clearly, \( \mathcal{S} \) is a \( \nu \)-measurable family of unitary operators from \( \mathcal{S}_1 \oplus \nu \mathcal{S}_2 \) to \( \mathcal{S}_2 \oplus \nu \mathcal{S}_1 \). Note that \( G_{\mathcal{R}(s)}(\nu) \) is the image of \( G_{\mathcal{R}(s)}(\nu) \) under \( \mathcal{S}(s) \) for \( \nu \)-a.e. \( s \). Hence, \( s \rightarrow G_{\mathcal{R}(s)}^{\circ} \) is a \( \nu \)-measurable family of subspaces of \( \mathcal{S}_2 \oplus \nu \mathcal{S}_1 \) by Lemma \([B.3]\). Let \( S = \int_{\mathbb{R}} \mathcal{S}(s) \, d\nu(s) \) and \( \tilde{S} \) be the unitary operator from \( \mathcal{S}_1 \oplus \mathcal{S}_2 \) to \( \mathcal{S}_2 \oplus \mathcal{S}_1 \) defined by the relation \( \tilde{S}(\xi, \eta) = (\eta, \xi) \). It is straightforward to check that \( U_{21} \tilde{S} = SU_{12} \). As \( G_{\mathcal{R}}^{\circ} = \tilde{S}(G_R) \), we have \( U_{21}(G_{\mathcal{R}}^*) = S(U_{12}(G_R)) \). Since \( U_{12}(G_R) = \int_{\mathbb{R}} G_{\mathcal{R}(s)}(\nu) \, d\nu(s) \) by Lemma \([B.7]\) and the image of \( \int_{\mathbb{R}} G_{\mathcal{R}(s)}(\nu) \, d\nu(s) \) under \( S \) is equal to \( \int_{\mathbb{R}} G_{\mathcal{R}(s)}^{\circ}(\nu) \, d\nu(s) \), we conclude that \( U_{21}(G_{\mathcal{R}}^*) = \int_{\mathbb{R}} G_{\mathcal{R}(s)}^{\circ}(\nu) \, d\nu(s) \). It now follows from Lemmas \([B.6]\) and \([B.15]\) and the unitarity of \( U_{21} \) that \( s \rightarrow G_{\mathcal{R}(s)}^\ast \) is a \( \nu \)-measurable family of subspaces of \( \mathcal{S}_2 \oplus \nu \mathcal{S}_1 \) and

\[(121) \quad U_{21}(G_{\mathcal{R}}^*) = \int_{\mathbb{R}} G_{\mathcal{R}(s)}^\ast(\nu) \, d\nu(s)\]

In view of (121), Lemma \([B.18]\) implies that \( R \) is densely defined if and only if \( G_{\mathcal{R}(s)}^\ast(\nu) \) is the graph of an operator for \( \nu \)-a.e. \( s \), i.e., if \( \mathcal{R}(s) \) is densely defined for \( \nu \)-a.e. \( s \). Suppose \( R \) is densely defined and \( R' = \int_{\mathbb{R}} \mathcal{R}(s)^\ast(\nu) \, d\nu(s) \). As \( G_{\mathcal{R}(s)}^\ast(\nu) = \mathcal{R}(s)^\ast(\nu) \) for \( \nu \)-a.e. \( s \), it follows that \( s \rightarrow \mathcal{R}(s)^\ast \) is a \( \nu \)-measurable family of operators from \( \mathcal{S}_2 \) to \( \mathcal{S}_1 \). Finally, since \( G_{\mathcal{R}}^\ast = G_{R^\ast} \), equality (121) and Lemma \([B.7]\) imply that \( U_{21}(G_{R^\ast}) = U_{21}(G_{R^\ast}) \) and, hence, \( R^* = R' \). \( \square \)

**Corollary B.23.** Let \( \mathcal{S} \) be a \( \nu \)-measurable family of Hilbert spaces, and \( \mathcal{R} \) be a \( \nu \)-measurable family of closed operators in \( \mathcal{S} \). The operator \( \int_{\mathbb{R}} \mathcal{R}(s) \, d\nu(s) \) is self-adjoint if and only if \( \mathcal{R}(s) \) is self-adjoint for \( \nu \)-a.e. \( s \).

**Proof.** The statement follows from Propositions \([B.22]\) and \([B.20] \) \( \square \)

**Proposition B.24.** Let \( \mathcal{S}_1, \mathcal{S}_2, \) and \( \mathcal{S}_3 \) be \( \nu \)-measurable families of Hilbert spaces and \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be \( \nu \)-measurable families of closed operators from \( \mathcal{S}_1 \) to \( \mathcal{S}_2 \) and from \( \mathcal{S}_2 \) to \( \mathcal{S}_3 \) respectively. Let \( R_{1,2} = \int_{\mathbb{R}} \mathcal{R}_{1,2}(s) \, d\nu(s) \). The operator \( R_2R_1 \) is
densely defined if and only if the operator $R_2(s)R_1(s)$ is densely defined for $\nu$-a.e. $s$. The operator $R_2R_1$ is closable if and only if the operator $R_2(s)R_1(s)$ is closable for $\nu$-a.e. $s$, in which case $s \to \overline{R_2(s)R_1(s)}$ is a $\nu$-measurable family of operators from $\mathcal{S}_1$ to $\mathcal{S}_3$ and $\overline{R_2R_1} = \int_0^\infty \overline{R_2(s)R_1(s)} \, d\nu(s)$.

Proof. Let $\mathcal{S}_i = \int_0^\infty \mathcal{S}_i(s) \, d\nu(s)$, $i = 1, 2, 3$, and $R = \int_0^\infty R_2(s)R_1(s) \, d\nu(s)$. Clearly, $R$ is an extension of $R_2R_1$. Let $(\xi, \eta) \in G_R$. Then $\xi(s) \in D_{R_2(s)}$ and, hence, $\xi(s) \in D_{R_1(s)}$ for $\nu$-a.e. $s$. Let $\tilde{\eta}$ be a $\nu$-a.e. defined section of $\mathcal{S}_2$ such that $\tilde{\eta}(s) = R_1(s)\xi(s)$ for $\nu$-a.e. $s$. By Lemma B.10 $\tilde{\eta}$ is $\nu$-measurable. Let $A_1 \subset A_2 \subset \ldots$ be a sequence of elements of $D_{\nu}$ such that $S_{\nu} = \bigcup_{k=1}^\infty A_k$ and $\|\tilde{\eta}(s)\| \leq k$ for $\nu$-a.e. $s \in A_k$. Set $\xi_k = \xi^{Ak}$, $\eta_k = \eta^{Ak}$, and $\tilde{\eta}_k = \tilde{\eta}^{Ak}$. We obviously have $\xi_k \in \mathcal{S}_1$, $\eta_k \in \mathcal{S}_2$, and $\tilde{\eta}_k \in \mathcal{S}_3$ for all $k$. As $\tilde{\eta}(s) \in D_{R_2(s)}$ and $\eta(s) = R_2(s)\tilde{\eta}(s)$ for $\nu$-a.e. $s$, we have $(\xi_k, \eta_k) \in G_R$, and $(\xi_k, \eta_k) \in G_{R_2}$. Since $(\xi_k, \eta_k) \to (\xi, \eta)$ in $\mathcal{S}_1 \oplus \mathcal{S}_3$ as $k \to \infty$, we conclude that $G_{R_2R_1}$ is dense in $G_R$ and, hence, $D_{R_2R_1}$ is dense in $D_{R}$. In view of statement 2 of Proposition B.22 it follows that $R_2R_1$ is densely defined if and only if $R_2(s)R_1(s)$ is densely defined for $\nu$-a.e. $s$. Since $\overline{G_R} = \overline{G_{R_2R_1}}$, the operator $R_2R_1$ is closable if and only if $R$ is closable. In view of Lemma B.19 the latter condition holds if and only if $R_2(s)R_1(s)$ is closable for $\nu$-a.e. $s$ (note that $s \to R_2(s)R_1(s)$ is a $\nu$-measurable family of operators from $\mathcal{S}_1$ to $\mathcal{S}_3$ by Lemma B.8). If $R_2R_1$ is closable, then Lemma B.19 implies that $\overline{R_2R_1} = \overline{R} = \int_0^\infty \overline{R_2(s)R_1(s)} \, d\nu(s)$.

Proposition B.25. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be $\nu$-measurable families of Hilbert spaces, $R_1$ and $R_2$ be $\nu$-measurable families of closed operators from $\mathcal{S}_1$ to $\mathcal{S}_2$, and $R_{1,2} = \int_0^\infty R_{1,2}(s) \, d\nu(s)$. The operator $R_1 + R_2$ is densely defined if and only if the operator $R_1(s) + R_2(s)$ is densely defined for $\nu$-a.e. $s$. The operator $R_1 + R_2$ is closable if and only if the operator $R_1(s) + R_2(s)$ is closable for $\nu$-a.e. $s$, in which case $R_{1,2} = \int_0^\infty R_{1,2}(s) \, d\nu(s)$.

Proof. Let $\mathcal{S}_{1,2} = \int_0^\infty \mathcal{S}_{1,2}(s) \, d\nu(s)$ and $R = \int_0^\infty (R_1(s) + R_2(s)) \, d\nu(s)$. Clearly, $R$ is an extension of $R_1 + R_2$. Let $(\xi, \eta) \in G_R$. Then $\xi(s) \in D_{R_1(s)} \cap D_{R_2(s)}$ for $\nu$-a.e. $s$. Let $\eta^{(1)}$ and $\eta^{(2)}$ be $\nu$-a.e. defined sections of $\mathcal{S}_2$ such that $\eta^{(1,2)}(s) = R_{1,2}(s)\xi(s)$ for $\nu$-a.e. $s$. By Lemma B.10 $\eta^{(1)}$ and $\eta^{(2)}$ are $\nu$-measurable. Let $A_1 \subset A_2 \subset \ldots$ be a sequence of elements of $D_{\nu}$ such that $S_{\nu} = \bigcup_{k=1}^\infty A_k$ and $\|\eta^{(1)}(s)\| + \|\eta^{(2)}(s)\| \leq k$ for $\nu$-a.e. $s \in A_k$. Set $\xi_k = \xi^{Ak}$, $\eta_k = \eta^{Ak}$, and $\tilde{\eta}_k^{(1,2)} = (\eta^{(1,2)})^{Ak}$. We obviously have $\xi_k \in \mathcal{S}_1$ and $\eta_k \in \mathcal{S}_2$ for all $k$. As $(\xi_k, \eta_k) \in G_{R_{1,2}}$, we have $(\xi_k, \eta_k) \in G_{R_1+R_2}$ for all $k$. Since $(\xi_k, \eta_k) \to (\xi, \eta)$, we conclude that $G_{R_1+R_2}$ is dense in $G_R$ and, hence, $D_{R_1+R_2}$ is dense in $D_R$. In view of statement 2 of Proposition B.22 it follows that $R_1 + R_2$ is densely defined if and only if $R_1(s) + R_2(s)$ is densely defined for $\nu$-a.e. $s$. Since $\overline{G_R} = \overline{G_{R_1+R_2}}$, the operator $R_1 + R_2$ is closable if and only if $R$ is closable. In view of Lemma B.19 the latter condition holds if and only if $R_1(s) + R_2(s)$ is closable for $\nu$-a.e. $s$ (note that $s \to R_1(s) + R_2(s)$ is a $\nu$-measurable family of operators from $\mathcal{S}_1$ to $\mathcal{S}_2$ by Lemma B.9). If $R_1 + R_2$ is closable, then Lemma B.19 implies that $\overline{R_1 + R_2} = \overline{R} = \int_0^\infty \overline{R_1(s) + R_2(s)} \, d\nu(s)$.

Remark B.26. Propositions B.20, B.21 and B.22 were proved in [23] (see also [20]). They are generalizations of corresponding statements proved in [32] for bounded
operators. Propositions B.24 and B.25 were obtained in [11] under additional assumption that $\nu$ is a finite Borel measure on a metrizable compact space and were proved in [21] in their general form.

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