DC Conductivity in an $s$-Wave Superconducting Single Vortex System

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Abstract We study dynamics of a two-dimensional $s$-wave superconductor in the presence of a moving single vortex on the basis of the quasiclassical theory generalized by Kita (Phys Rev B 64:054503, 2001). We numerically calculate the linear response of a moving single vortex driven by a dc external current in a self-consistent way, in the sense that the gap equation, Maxwell equations and generalized quasiclassical equation with the impurity self-energy (self-consistent Born approximation) are solved simultaneously. We obtain Hall conductivity induced by vortex motion using the generalized quasiclassical equation, while we confirm that it vanishes in the conventional quasiclassical equation.

Keywords Superconductivity · Vortex · Quasiclassical theory · Impurity effect · Self-consistent Born approximation · Flux flow conductivity

1 Introduction

One of the long-standing and unsettled issues in vortex physics is microscopic calculation flux–flux Hall conductivity. Although there exist lots of references on microscopic calculation on Hall effect in vortex states [1,2], self-consistent calculations that cover both clean and dirty superconductors have not yet been reported. For example, in the pioneering work [3] on microscopic calculation of Hall conductivity of single vortex for clean $s$-wave superconductor in the Gor’kov formalism, only the contribution of
quasiparticles bounded near vortex cores has been taken into account. Self-consistent calculation for vortex system in the Gor’kov formalism is numerically prohibitive because this formalism contains high energy normal-state properties unnecessary to calculate low energy properties.

What we imply by ‘self-consistent calculation’ is the calculation that yields a set of Green function and electromagnetic fields satisfying the gap equation, Maxwell equations as well as equation of motion of Green function with the self-energy being a functional of Green functions. Importance of self-consistency in the calculation of vortex dynamics lies, as emphasized by Eschrig et al. [4], in the fact that the charge conservation is not necessarily guaranteed in non-self-consistent calculations.

The quasiclassical theory [1,5–7] of superconductivity describes low-energy properties and it can be derived from Gor’kov theory by integrating out high-energy and short-distance properties. This theory proved to be very useful for description of many properties in superconducting single vortex systems [1]. Self-consistent linear response of single vortex with respect to ac electric field has been obtained [4] within the Eilenberger–Eliashberg theory [5,6] (which we refer to as the ‘conventional’ quasiclassical theory). This theory is, however, unable to describe Hall effect.

To overcome this difficulty, several authors [8–10] generalized the quasiclassical theory such that the Hall effects are taken into account. While these generalized quasiclassical theories open a route to calculate Hall effect in vortex states, no reports have been done on self-consistent calculation on the basis of those theories.

In this paper, we present the results of self-consistent calculation of generalized quasiclassical equation derived by Kita [10,11] and discuss the linear response Hall conductivity of a moving single vortex driven by external current. We also compare our results with those obtained by the conventional quasiclassical equation.

2 Model

We consider two-dimensional $s$-wave superconductors with the isotropic dispersion (the Fermi velocity is $v_f$) in the normal state. The generalized quasiclassical theory [8,10] is formulated in terms of the quasiclassical propagator $\tilde{g}_\varepsilon(p_f, r, t)$, which is a function of energy $\varepsilon$, and momenta $p_f$ on the Fermi surface, position $r$ and time $t$. Hereafter, we sometimes drop the subscript $f$ in $p_f$ for convenience. Contrarily to the conventional quasiclassical theory, $\tilde{g}$ in the generalized quasiclassical theory [8,10] is not obtained directly by the integration of Nambu–Gor’kov–Kelydish Green function; instead, $\tilde{g}$ is an integration of the Green function prescribed by non-local gauge transformation [8,10]. We denote usual Nambu–Keldysh matrices: $\tilde{g} = \left( \begin{array}{cc} \hat{g}_R & \hat{g}_K \\ 0 & \hat{g}^A \end{array} \right)$ with $\hat{g}_{R,K} = \left( \begin{array}{cc} g_{R,K} & g^{R,K} \\ -g^{R,K} & g_{R,K} \end{array} \right) [1,10]$. The transport equation is given by [10]

$$\left[ \varepsilon \tilde{\tau}_3 + \tilde{\sigma}_{\text{imp}} + \tilde{\Delta}, \tilde{g} \right]_{\text{o}} + i\hbar v_f \cdot \partial_r \tilde{g} + \frac{i\hbar}{2} O_g \{ \tilde{\tau}_3, \tilde{g} \} = 0. \quad (1)$$

Here $\tilde{\sigma}_{\text{imp}}(r, t) = \frac{i\hbar}{2\tau_n} \langle \tilde{g}_\varepsilon(p, r, t) \rangle_p$ is the impurity self-energy in the self-consistent Born approximation. $\tau_n$ is the relaxation time in the normal state and the symbol
\[(\cdots)_p = \int \frac{d\theta_p}{2\pi} (\cdots)\] with \(p_r = (\cos \theta_p, \sin \theta_p)\) denotes the average on the two-dimensional Fermi surface.

The symbol \(\hat{\Delta}\) denotes the matrix \(\hat{\Delta} = \text{diag}(\Delta, \hat{\Delta})\) whose element is given by

\[
\hat{\Delta} = \begin{pmatrix}
0 & -\Delta \\
\Delta^* & 0
\end{pmatrix},
\]

\[
\Delta(r, t) = N_f V \int \left( f^K(p, r, t) \right) \frac{d\varepsilon}{p}.
\]

Here \(N_f\) denotes the density of state at the Fermi surface in the normal state; \(N_f = |p_f|/(2\pi \hbar^2|v_f|)\) for two-dimensional system of particles with the parabolic dispersion. The symbol \(V\) denotes the strength of attraction in the weak-coupling regime, \((N_f V)^{-1} = \ln \left( \frac{T}{T_c} \right) + \sum_{m=0}^{\epsilon_c/(2\pi k_BT)} \frac{1}{m+1/2}\), with the transition temperature \(T_c\) and a cut-off energy \(\epsilon_c\). In Eq. (1), \(\hat{\tau}_3 = \text{diag}(\hat{\tau}_3, \hat{\tau}_3)\) with \(\hat{\tau}_3 = \text{diag}(1, -1)\), \([A, B] = A \circ B - B \circ A\), \([A, B] = AB + BA\). The notation \(A \circ B\) for \(A_\varepsilon(p, r, t)\) and \(B_\varepsilon(p, r, t)\) is defined by

\[
A \circ B = \exp \left[ \frac{i\hbar}{2} \left( \partial_t \partial_{p'} - \partial_t \partial_{p'} + \partial_t \cdot \partial_{p'} - \partial_t \cdot \partial_{p'} \right) \right]
\times A_\varepsilon(p, r, t) B_\varepsilon(p', r', t') \bigg|_{\varepsilon', p', r', t' \to \varepsilon, p, r, t},
\]

which we approximate as

\[
A \circ B \sim AB + \frac{i\hbar}{2} \left( \frac{\partial A}{\partial \varepsilon} \frac{\partial B}{\partial t} + \frac{\partial A}{\partial t} \frac{\partial B}{\partial \varepsilon} + \frac{\partial A}{\partial p} \frac{\partial B}{\partial r} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial r} \right).
\] (2)

Here the gauge invariant derivatives are given by \([8, 10]\)

\[
\partial_t = \frac{\partial}{\partial t}, \quad \partial_r = \frac{\partial}{\partial r} \quad \text{on } g, \ g, \ E, \ B
\]

\[
\partial_t = \frac{\partial}{\partial t} + 2ie\Phi \frac{\partial}{\hbar r}, \quad \partial_r = \frac{\partial}{\partial r} - \frac{i2eA}{\hbar} \quad \text{on } f, \ \Delta
\]

\[
\partial_t = \frac{\partial}{\partial t} - 2ie\Phi \frac{\partial}{\hbar r}, \quad \partial_r = \frac{\partial}{\partial r} + \frac{2ieA}{\hbar} \quad \text{on } f^\dagger, \ \Delta^*.
\] (5)

Here the charge unit \(e\) is taken to be negative. We denote the vector potential \(A\) and the scalar potential \(\Phi\), which are related with electromagnetic fields as \(E = -\nabla \Phi - \frac{\partial A}{\partial t}\), \(B = \nabla \times A\). In the last term in the right-hand side of Eq. (1), which we call ‘the Hall term’ in the following part, we introduce the notation \([10]\)

\[
O_g = e(v_f \times B) \cdot \frac{\partial}{\partial p} + e v_f \cdot E \frac{\partial}{\partial \varepsilon}.
\] (6)

Normalization condition on the Green function is expressed in terms of the local density of state \(N(r, \varepsilon)\) as \(N(r, \varepsilon)/N_f \rightarrow 1\) at \(|\varepsilon/\Delta_\infty| \gg 1\), where

\[
N(r, \varepsilon) = \frac{N_f}{2} \left[ \left| g^{R}_\varepsilon(p, r) - g^{A}_\varepsilon(p, r) \right| \right]_p,
\] (7)
Here $\Delta_\infty$ denotes the modulus of the pair-potential in the bulk. We also solve the Maxwell equation $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$, $\nabla \times B = \mu_0 J$. The current and charge density around the vortex are given by

$$j(r, t) = -eN_f \int \left\{ v_f \left[ g^K_r(p, r, t) - \bar{g}^{K}_r(p, r, t) \right] \right\} \frac{d\varepsilon}{p^4} \quad (8)$$

$$\rho(r, t) = -eN_f \int \left\{ \left[ g^K_r(p, r, t) + \bar{g}_r^{K}(p, r, t) \right] \right\} \frac{d\varepsilon}{p^4}. \quad (9)$$

Here we remark that $\rho$ includes the screening effect [6,12]. Accordingly, we find the suppression of charge fluctuation, which can be ascribed to metallic screening effect$^1$. Note that $g$ in the present proceedings is not the conventional quasiclassical Green function (which we denote by $g^{c,K}$). In terms of $g^{c,K}$, Eq. (9) is rewritten as$^2$

$$\rho(r, t) = -2e^2N_f \Phi(r, t) - eN_f \int \left\{ \left[ g^{c,K}_r(p, r, t) + \bar{g}_r^{c,K}(p, r, t) \right] \right\} \frac{d\varepsilon}{p^4}, \quad (10)$$

which is a familiar expression in the quasiclassical theory of superconductivity and can be seen in literatures [6,7,1].

In the linear response, we split the physical quantities $O(=\tilde{g}, \tilde{\Delta}, \tilde{\sigma}^{imp}, A, \Phi)$ into the unperturbed part (denote by the subscript 0) moving with vortex velocity $v_v$ and a term of first order of $v_v$ [8,1]

$$O(r, t) = O_0(r - v_v t) + \Delta O(r). \quad (11)$$

We solve the generalized quasiclassical equation in a self-consistent way numerically. The numerical procedure to calculate the generalized quasiclassical equation is similar to that described in Ref. 4. The self-consistent solution of the quasiclassical equations ensures the conservation law of the charge density $\partial_t \rho + \nabla \cdot j(r) = 0$ within the linear response. We take the quasiclassical parameter $k_f \xi = 50$ with the Fermi wave-vector $k_f = |p_f|/\hbar$ and the coherent length $\xi = \hbar v_f/\pi \Delta_\infty$. We also take the impurity scattering rate in the normal state $\Gamma_n = 0.1 \Delta_\infty$, with $\Gamma_n = \hbar/2 \pi \tau_n$. In this paper, we choose a low temperature $T = 0.3 T_c$ and $v_v = |v_v| \hat{\chi}$ to see the Hall effect clearly.

### 3 Results

#### 3.1 Equilibrium Case

Figure 1a shows LDOS (7) in the equilibrium case for the generalized quasiclassical equation. For comparison, we show LDOS for the conventional quasiclassical equation in Fig. 1b.

$^1$ We numerically find in an equilibrium state that $\rho(r)$ around vortex core is the order of $|e| N_f \Delta_\infty/(k_f \xi)$ and confirm the charge neutrality condition holds; $2\pi \int_0^R drr \rho(r) \sim 0$ for $R$ being a few times larger than $\xi$.

$^2$ The discussion on the screening effect in terms of the gauge-invariant quasiclassical Green function is given in Chap.10.3 of Ref. [1].
Low energy excitations near a single vortex are exhausted by the Caroli-de Gennes-Matricon mode [13], or the Andreev bound states [1,14,15], the dispersion of which is given as a function of impact parameter $r$,

$$\varepsilon = E(r) = \frac{r}{C} \int_0^\infty \frac{|\Delta_0(s)|}{\sqrt{r^2 + s^2}} e^{-u(s)} ds, \quad C \equiv \int_0^\infty e^{-u(s)} ds,$$

(12)

with

$$u(s) \equiv \frac{2}{\hbar|v_f|} \int_0^{|s|} |\Delta_0(s')| ds'.$$

(13)

In Fig. 1a, b, the overall peak structures in the LDOS can be fitted by the dispersion Eq. (12). This observation confirms validity of our calculation.

By a close inspection of Fig. 1a, however, we can see that the LOS is asymmetric with respect to $\varepsilon$ and the main peak at the core is located at a positive energy, in contrast to (b). Similar asymmetry in LDOS and shift of the main peak have been observed in STS measurements [16,17].

We have checked validity of our result by examining the spatial dependence of the pair-potential and the circular current density near the vortex center. Figure 2 shows the
modulus of the pair-potential and the circular current density as functions of the radial coordinate from the vortex center. We see that the spatial dependence near the vortex center is steep. This shrinkage of vortex core was found and discussed by Kramer and Pesch [14]; we can obtain the initial slope of $|\Delta_0(r)|$ and that of the modulus of the current density, respectively, as

$$
|\Delta_0(r)| \rightarrow \frac{\pi \hbar |\mathbf{v}_f| N_f V}{8 C} \frac{r E'(r)_{r=0}}{2 k_B T}, \quad |j(r)| \rightarrow \frac{\pi \hbar N_f |e| |\mathbf{v}_f|^2}{8 C} \frac{r E'(r)_{r=0}}{k_B T} \quad (14)
$$

along a calculation similar to that in Ref. [14]. We see that the relations (14) hold in Fig. 2.

3.2 Linear Responses

In Fig. 3, we show electric field distributions for the generalized quasiclassical equation and radial profiles of the ‘average’ electric field, $\langle E_y \rangle_r = |\int E_y dS|/(|\mathbf{v}_y|)$. We can check that the Josephson relation in the flux flow state $\langle \mathbf{E} \rangle = (\mathbf{B}) \times \mathbf{v}_v$ [18] holds, where $\langle \quad \rangle$ denotes the spatial average. The Josephson relation in the present case reduces to $\lim_{\rho \rightarrow \infty} \langle E_y \rangle_r = \pi \hbar /|e|$. We see that this relation holds in Fig. 3b.

Figure 4 shows the spatial distribution of current density induced by vortex motion for the generalized quasiclassical equation (a) and the conventional quasiclassical equation (b). We see that there exists the Hall current in (a) while there does not in (b). We also confirm that the current density $j(r)$ approaches a constant vector $\mathbf{j}_{tr}$ far away from the vortex core ($\mathbf{j}_{tr}$ is identified as the transport current density). We can thus obtain the flux flow ohmic (longitudinal) $\sigma_O = j_{tr,y}/\langle E_y \rangle$ and Hall conductivities $\sigma_H = j_{tr,x}/\langle E_y \rangle$ using the generalized quasiclassical equation and obtain $\sigma_O \simeq 0.68 \sigma_B$ and $\sigma_H \simeq -0.32 \sigma_B$ with $\sigma_B = N_f p_f |e| / B$.

![Fig. 3](image-url) a Electric field distributions for the generalized quasiclassical equation. b Radial profiles of the ‘average’ electric field, $\langle E_y \rangle_r$.
Fig. 4 Current distributions for the generalized quasiclassical equation (a) and the conventional quasiclassical equation (b). The velocity of vortex is taken to be parallel to the x-direction.

4 Conclusion

On the self-consistent numerical calculation of the generalized quasiclassical equation, we have observed (i) asymmetric local density of states and the shift of the main peak at the core away from zero energy and (ii) the Hall effect. These results are not obtained by the conventional quasiclassical equation.

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