A CLT for the third integrated moment of Brownian local time increments

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Abstract

Let \( \{L^x_t; (x, t) \in \mathbb{R}_1 \times \mathbb{R}_1^+\} \) denote the local time of Brownian motion. Our main result is to show that for each fixed \( t \)

\[
\frac{\int (L^{x+h}_t - L^x_t)^3 \, dx - 12h \int (L^{x+h}_t - L^x_t)L_t \, dx - 24h^2 t}{h^2} \overset{\mathcal{L}}{\Rightarrow} \sqrt{192} \left( \int (L^x_t)^3 \, dx \right)^{1/2} \eta
\]

as \( h \to 0 \), where \( \eta \) is a normal random variable with mean zero and variance one that is independent of \( L^x_t \). This generalizes our previous result for the second moment. We also explain why our approach will not work for higher moments.

1 Introduction

Let \( \{L^x_t; (x, t) \in \mathbb{R}_1 \times \mathbb{R}_1^+\} \) denote the local time of Brownian motion. Let

\[
\alpha_{p,t} = \int (L^x_t)^p \, dx
\]

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(an integral sign without limits is to be read as \(\int_{-\infty}^{\infty}\)) and let \(\eta = N(0, 1)\) be independent of \(\alpha_{p,t}\). The main result of [3] is the following weak limit theorem.

**Theorem 1.1** For each fixed \(t\)

\[
\frac{\int (L_i^{x+h} - L_i^x)^2 \, dx - 4ht}{h^{3/2}} \xrightarrow{\mathcal{L}} c\sqrt{\alpha_{2,t}} \eta \tag{1.2}
\]

as \(h \to 0\), where \(c = (64/3)^{1/2}\).

Equivalently

\[
\frac{\int (L_i^{x+1} - L_i^x)^2 \, dx - 4t}{t^{3/4}} \xrightarrow{\mathcal{L}} c\sqrt{\alpha_{2,1}} \eta \tag{1.3}
\]

as \(t \to \infty\).

In this paper we provide the analogous CLT for the third power.

**Theorem 1.2** For each fixed \(t\)

\[
\frac{\int (L_i^{x+h} - L_i^x)^3 \, dx - 12h \int (L_i^{x+h} - L_i^x)L_i^x \, dx - 24h^2 t}{h^2} \xrightarrow{\mathcal{L}} c\sqrt{\alpha_{3,t}} \eta \tag{1.4}
\]

as \(h \to 0\), where \(c = \sqrt{192}\).

Equivalently

\[
\frac{\int (L_i^{x+1} - L_i^x)^3 \, dx - 12 \int (L_i^{x+1} - L_i^x)L_i^x \, dx - 24t^2}{t} \xrightarrow{\mathcal{L}} c\sqrt{\alpha_{3,1}} \eta \tag{1.5}
\]

as \(t \to \infty\).

We explain below why the approach we use will not work for moments larger than three.

The equivalence of (1.4) and (1.5) follows from the scaling relationship

\[
\{L_{h^{-2}t}; (x, t) \in R^1 \times R^1_+\} \overset{\xrightarrow{\mathcal{L}}}{=} \{h^{-1}L_i^{hx}; (x, t) \in R^1 \times R^1_+\}, \tag{1.6}
\]

see e.g. [6] Lemma 10.5.2], which implies that

\[
\int (L_i^{x+h} - L_i^x)^3 \, dx \overset{\xrightarrow{\mathcal{L}}}{=} h^4 \int (L_i^{x+1} - L_i^x)^3 \, dx, \tag{1.7}
\]
and
\[ \int (L_t^{x+h} - L_t^x) L_t^x \, dx \xrightarrow{\ell} h^3 \int (L_{t/h}^{x+1} - L_{t/h}^x) L_{t/h}^x \, dx. \] (1.8)

Using this, and (1.4) with \( t = 1 \), and then setting \( h^2 = 1/t \) gives (1.5).

Theorem 1.2 is derived using the method of moments. Note that the right hand side of (1.4) is \( c \sqrt{\alpha_{p,t}} \eta \). Unfortunately, we can only show that \( \sqrt{\alpha_{p,t}} \eta \) is determined by its moments if \( p = 2 \) or \( 3 \), so we cannot use our approach to prove an analog of Theorem 1.2 for moments larger than three.

In Section 2 we give some estimates on the potential densities and transition densities of Brownian motion which are used throughout this paper. Their proof is deferred until Section 8. In Section 8 we show how Theorem 1.1 will follow from a result, Lemma 4.1, on the moments of an analogous expression where \( t \) is replaced by an independent exponential time. This Lemma is proven in Section 4. Other lemmas are that used in the proof of Theorem 1.1 are derived in Sections 5-7.

This paper extends the basic approach used in [3]. The main novelty in this paper is the need to subtract a non-random term in (1.4) in order to get a Central Limit Theorem. Dealing with this non-random subtraction term, and in particular the need to keep track of delicate cancellations, makes this paper considerably more difficult than [3]. Although, as mentioned, the approach of the present paper will not work for higher moments, Theorem 1.1 does suggest what a Central Limit Theorem for higher moments should look like. Here is our conjecture for the fourth integrated moment.

**Conjecture 1.1** For each fixed \( t \)

\[ \int (\Delta^h L_t^x)^4 \, dx - 24h \int (\Delta^h L_t^x)^2 L_t^x \, dx + 48h^2 \int (L_t^x)^2 - (\Delta^h L_t^x) L_t^x \, dx \]

\[ \xrightarrow{\ell} c_4 \sqrt{\alpha_{4,t}} \eta \] (1.9)

as \( h \to 0 \), where \( c_q = \sqrt{\frac{2q+1}{q+1}} \) and \( \Delta^h L_t^x = L_t^{x+h} - L_t^x \).
2 Estimates for the potential density of Brownian motion

Let \( p_t(x) \) denote the probability density of Brownian motion. The \( \alpha \)-potential density of Brownian motion

\[
\alpha (x) = \int_0^\infty e^{-\alpha t} p_t(x) \, dt = \frac{e^{-\sqrt{2\alpha} |x|}}{\sqrt{2\alpha}}. \tag{2.1}
\]

Let \( \lambda_\alpha \) be an independent exponential random variable with mean \( 1/\alpha \).

Kac’s moment formula, \([6, \text{Theorem 3.10.1}]\), states that

\[
E^{x_0} \left( \prod_{j=1}^n L_{\lambda_\alpha} x_j \right) = \sum \prod_{j=1}^n \alpha (x_{\pi(j)} - x_{\pi(j-1)}) \tag{2.2}
\]

where the sum runs over all permutations \( \pi \) of \( \{1, \ldots, n\} \) and \( \pi(0) = 0 \).

Let \( \Delta^h_x \) denote the finite difference operator on the variable \( x \), i.e.

\[
\Delta^h_x f(x) = f(x + h) - f(x). \tag{2.3}
\]

We write \( \Delta^h \) for \( \Delta^h_x \) when the variable \( x \) is clear.

The next lemma collects some facts about \( \alpha (x) \) that are used in this paper.

**Lemma 2.1** Fix \( \alpha > 0 \). For \( 0 < h \leq 1 \),

\[
\Delta^h_y \Delta^h_x \alpha (x - y) \bigg|_{y=x} = 2 \left( \frac{1 - e^{-\sqrt{2\alpha} h}}{\sqrt{2\alpha}} \right) = 2h + O(h^2), \tag{2.4}
\]

\[
\alpha (x) =: |\Delta^h \alpha (x)| \leq Ch \alpha (x), \tag{2.5}
\]

\[
w^\alpha (x) =: |\Delta^h \Delta^{-h} \alpha (x)| \leq \begin{cases} Ch \alpha (x), \\ Ch^2 \alpha (x), \end{cases} \quad \forall |x| \geq h. \tag{2.6}
\]

We have

\[
\int (w^\alpha (x))^q \, dx = O(h^{q+1}) \tag{2.7}
\]

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and
\[ \int_{|x| \geq h} (w^\alpha(x))^q \, dx = O(h^{2q}). \tag{2.8} \]

In addition, for any \( q \geq 2 \)
\[ \int \left( \Delta^h \Delta^{-h} u^\alpha(x) \right)^q \, dx = \left( 2^{q+1}/(q + 1) + O(h) \right) h^{q+1}. \tag{2.9} \]

In all these statements the constants \( C \) and the terms \( O(h^\cdot) \) may depend on \( \alpha \).

The proof is provided in Section 8.

We note that the same proof shows that for any \( \alpha_1, \ldots, \alpha_q > 0 \)
\[ \int \prod_{i=1}^{q} \left( \Delta^h \Delta^{-h} u^{\alpha_i}(x) \right) \, dx = \left( 2^{q+1}/(q + 1) + O(h) \right) h^{q+1}. \tag{2.10} \]

Remark 2.1 In Lemma 2.1 we have taken \( h \) positive. Using the fact that \( u^\alpha(x) \) is an even function of \( x \) it is easy to check that we obtain the analog of (2.5) for all \( |h| \leq 1 \) if on the right hand side we replace \( h \) by \( |h| \).

The following estimates, which will be used in the proof of Lemma 6.1 are also proven in Section 8.

Lemma 2.2 Let \( 0 < h \leq 1 \) and \( 0 < T < \infty \). Then for some \( C_T < \infty \)
\[ u_T(x) =: \int_0^T p_t(x) \, dt \leq C_T e^{-|x|}, \tag{2.11} \]
\[ v_T(x) =: \int_0^T |\Delta^h p_t(x)| \, dt \leq C_T h e^{-|x|}, \tag{2.12} \]
and
\[ w_T(x) =: \int_0^T |\Delta^h \Delta^{-h} p_t(x)| \, dt \leq C_T h^2 e^{-x^2/32T} \frac{e^{-x^2/32T}}{|x|}, \quad |x| \geq 2h. \tag{2.13} \]

Also
\[ \int w_T(x) \, dx \leq C_T h^2 |\log h|, \tag{2.14} \]
and for any \( q \geq 2 \)
\[ \int w_T^q(x) \, dx \leq C_T h^{q+1}, \tag{2.15} \]
and
\[ \int_{|x| \geq \sqrt{h}} w_T^q(x) \, dx \leq C_T h^{3q/2 + 1/2}. \tag{2.16} \]
Lemma 2.3 Let $0 < h \leq 1$ and $0 < \delta < T < \infty$. Then for some $C_{\delta,T} < \infty$

$$u_T(x) =: \sup_{\delta \leq t \leq T} p_t(x) \leq C_{\delta,T} e^{-x^2/2T}, \tag{2.17}$$

$$v_T(x) =: \sup_{\delta \leq t \leq T} |\Delta^h p_t(x)| \leq C_{\delta,T} h e^{-x^2/2T}. \tag{2.18}$$

and

$$w_T(x) =: \sup_{\delta \leq t \leq T} |\Delta^h \Delta^{-h} p_t(x)| \leq C_{\delta,T} h^2 e^{-x^2/2T}. \tag{2.19}$$

Lemma 2.4 Let $0 < h \leq 1$. For $q \geq 2$

$$\int \left( \int_0^\infty \Delta^h \Delta^{-h} p_t(x) \, dt \right)^q \, dx = (2^{q+1}/(q + 1) + O(h^{1/2})) h^{q+1}, \tag{2.20}$$

and

$$\int \left( \int_0^h \Delta^h \Delta^{-h} p_t(x) \, dt \right)^q \, dx = (2^{q+1}/(q + 1) + O(h^{1/2})) h^{q+1}. \tag{2.21}$$

3 Proof of Theorem 1.2

Theorem 1.2 will follow from the next lemma.

Lemma 3.1 For each integer $m \geq 0$ and $t \in R_+$

$$\lim_{h \to 0} \mathbb{E} \left( \left( \int \frac{(L_{t+h}^x - L_t^x)^3 \, dx - 12h \int L_t^x (L_{t+h}^x - L_t^x) \, dx - 24h^2 t}{h^2} \right)^m \right) \right)$$

$$= \begin{cases} 
\frac{(2n)!}{2^n n!} (192)^n \mathbb{E} \left( \left( \int (L_t^x)^3 \, dx \right)^n \right) & \text{if } m = 2n \\
0 & \text{otherwise}. 
\end{cases} \tag{3.1}
$$

Proof of Theorem 1.2 It follows from [2] (6.12) that for any $q$

$$\mathbb{E} \left\{ \left( \int (L_t^x)^q \, dx \right)^n \right\} \leq C_t^n (n!)^{(q-1)/2}. \tag{3.2}$$

Therefore, since $\sqrt{(2n)!} \leq 2^n n!$, the right hand side of (3.1), which is the $2n$–th moment of $\tilde{C} \sqrt{\int (L_t^x)^3 \, dx} \, \eta$ is bounded above by $\tilde{C}^{2n} C^n (2n)!$. 

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This implies that \( \tilde{C} \sqrt{\int (L_x^z)^3 \, dx} \, \eta \) is determined by its moments; (see \[4\] p. 227-228). Lemma \(3.1\) together with the method of moments, \[1\] Theorem 30.2, then gives us (1.4).

**Proof of Lemma 3.1** Let \( \lambda \zeta \) be an exponential random variable with mean \( 1/\zeta \). It follows from Lemma 4.1 below that for each integer \( m \geq 0 \),

\[
\lim_{h \to 0} E \left( \left( \frac{\int (\Delta \lambda \zeta) \, dx - 12h \int L_x^z (\Delta \lambda \zeta) \, dx - 24h^2 \lambda \zeta}{h^2} \right)^m \right) = \begin{cases} 
\frac{(2n)!}{2^n n!} (192)^n E \left( \left( \int (L_x^z)^3 \, dx \right)^n \right) & \text{if } m = 2n \\
0 & \text{otherwise.} \end{cases} \tag{3.3}
\]

We write (3.3) as

\[
\int_0^\infty e^{-\zeta s} E \left( \left( \frac{\int (\Delta \lambda \zeta) \, dx - 12h \int L_x^z (\Delta \lambda \zeta) \, dx - 24h^2 \lambda \zeta}{h^2} \right)^m \right) ds 
\longrightarrow 
\int_0^\infty e^{-\zeta s} E \left( \eta^m \left( 192 \int (L_x^z)^3 \, dx \right)^{m/2} \right) ds \tag{3.4}
\]
as \( h \to 0 \). For \( h > 0 \) let

\[
\hat{F}_{m,h}(s) := E \left( \left( \frac{\int (\Delta \lambda \zeta) \, dx - 12h \int L_x^z (\Delta \lambda \zeta) \, dx - 24h^2 \lambda \zeta}{h^2} \right)^m \right), \tag{3.5}
\]

and set

\[
\hat{F}_{m,0}(s) := E \left( \eta^m \left( 192 \int (L_x^z)^3 \, dx \right)^{m/2} \right). \tag{3.6}
\]

Then (3.4) can be written as

\[
\lim_{h \to 0} \int_0^\infty e^{-\zeta s} \hat{F}_{m,h}(s) \, ds = \int_0^\infty e^{-\zeta s} \hat{F}_{m,0}(s) \, ds. \tag{3.7}
\]

We consider first the case when \( m \) is even and write \( m = 2n \). In this case \( \hat{F}_{2n,h}(s) \geq 0 \) and the extended continuity theorem \[4\] XIII.1, Theorem 2a] applied to (3.7) implies that

\[
\lim_{h \to 0} \int_0^t \hat{F}_{2n,h}(s) \, ds = \int_0^t \hat{F}_{2n,0}(s) \, ds \tag{3.8}
\]
for all \( t \). In particular,
\[
\lim_{h \to 0} \int_t^{t+\delta} \hat{F}_{2n,h}(s) \, ds = \int_t^{t+\delta} \hat{F}_{2n,0}(s) \, ds.
\] 
(3.9)

It follows from the fact that \( L_s^x \) is almost surely continuous and increasing in \( s \) that \( \hat{F}_{2n,0}(s) \) is continuous in \( s \). (We saw in (3.2) that it is bounded.) Consequently,
\[
\lim \lim_{h \to 0} \frac{1}{h} \int_t^{t+\delta} \hat{F}_{2n,h}(s) \, ds = \hat{F}_{2n,0}(t).
\] 
(3.10)

When \( t = 0 \) we get
\[
\lim \lim_{h \to 0} \frac{1}{h} \int_0^{\delta} \hat{F}_{2n,h}(s) \, ds = 0.
\] 
(3.11)

To obtain (3.3) when \( m \) is even we must show that
\[
\lim_{h \to 0} \hat{F}_{2n,h}(t) = \hat{F}_{2n,0}(t).
\] 
(3.12)

This follows from (3.10) once we show that
\[
\lim \lim_{h \to 0} \frac{1}{h} \int_t^{t+\delta} \hat{F}_{2n,h}(s) \, ds = \lim_{h \to 0} \hat{F}_{2n,h}(t).
\] 
(3.13)

We proceed to obtain (3.13).

For \( s \geq t \) we write
\[
\int (\Delta_s^h L_s^x)^3 \, dx = \int \left( \Delta_s^h L_t^x + \Delta_s^h (L_s^x - L_t^x) \right)^3 \, dx
\] 
(3.14)

\[
= \int (\Delta_s^h L_t^x)^3 \, dx + 3 \int (\Delta_s^h L_t^x)^2 \Delta_s^h (L_s^x - L_t^x) \, dx + 3 \int \Delta_s^h L_t^x \left( \Delta_s^h (L_s^x - L_t^x) \right)^2 \, dx + \int \left( \Delta_s^h (L_s^x - L_t^x) \right)^3 \, dx
\]

and
\[
\int L_s^x (\Delta_s^h L_s^x) \, dx = \int \left( L_t^x + (L_s^x - L_t^x) \right) \left( \Delta_s^h L_t^x + \Delta_s^h (L_s^x - L_t^x) \right) \, dx
\]
\[
= \int L_t^x (\Delta_s^h L_t^x) \, dx + \int L_t^x \Delta_s^h (L_s^x - L_t^x) \, dx
\] 
(3.15)

\[
+ \int (L_s^x - L_t^x) (\Delta_s^h L_t^x) \, dx + \int (L_s^x - L_t^x) \Delta_s^h (L_s^x - L_t^x) \, dx
\]
so that
\[ \int (\Delta_x^h L_s^x)^3 \, dx - 12h \int L_s^x (\Delta_x^h L_s^x) \, dx - 24h^2 s = \int (\Delta_x^h L_t^x)^3 \, dx - 12h \int L_t^x (\Delta_x^h L_t^x) \, dx - 24h^2 t \]
\[ + 3 \int \{(\Delta_x^h L_t^x)^2 - 4hL_t^x\} \Delta_x^h (L_s^x - L_t^x) \, dx \]
\[ + 3 \int \Delta_x^h L_s^x \left\{ \left(\Delta_x^h (L_s^x - L_t^x)\right)^2 - 4h(L_s^x - L_t^x) \right\} \, dx \]
\[ + \int \left(\Delta_x^h (L_s^x - L_t^x)\right)^3 \, dx - 12h \int (L_s^x - L_t^x) \Delta_x^h (L_s^x - L_t^x) \, dx - 24h^2(t - s). \]

Note that, using \( \tilde{B}_t \) to denote an independent Brownian motion, and then using translation invariance
\[ \int \left(\Delta_x^h (L_s^x - L_t^x)\right)^3 \, dx - 12h \int (L_s^x - L_t^x) \Delta_x^h (L_s^x - L_t^x) \, dx - 24h^2(t - s) \]
\[ = \int \left(\Delta_x^h L_{s-t}^x\right)^3 \circ \theta_t \, dx - 12h \int \left\{ L_{s-t}^x \left( \Delta_x^h L_{s-t}^x \right) \right\} \circ \theta_t \, dx - 24h^2(t - s) \]
\[ \text{law} \int \left(\Delta_x^h L_{s-t}^x \circ \theta_t \right)^3 \, dx - 12h \int L_{s-t}^x \tilde{B}_t \left( \Delta_x^h L_{s-t}^x \circ \theta_t \right) \, dx - 24h^2(t - s) \]
\[ = \int \left(\Delta_x^h L_{s-t}^x \right)^3 \, dx - 12h \int L_{s-t}^x (\Delta_x^h L_{s-t}^x) \, dx - 24h^2(t - s). \] (3.17)

Also, using \( \tilde{L}_t^x \) to denote an independent copy of Brownian local time
\[ \int \left\{ \left(\Delta_x^h L_t^x\right)^2 - 4hL_t^x \right\} \Delta_x^h (L_s^x - L_t^x) \, dx \]
\[ = \int \left\{ \left(\Delta_x^h L_t^x\right)^2 - 4hL_t^x \right\} \left( \Delta_x^h L_{s-t}^x \circ \theta_t \right) \, dx \]
\[ \text{law} \int \left\{ \left(\Delta_x^h L_t^x\right)^2 - 4hL_t^x \right\} \Delta_x^h L_{s-t}^x B_t \, dx \]
\[ = \int \left\{ \left(\Delta_x^h L_t^x + B_t\right)^2 - 4hL_t^x + B_t \right\} \Delta_x^h L_{s-t}^x \, dx \]
\[ \text{law} \int \left\{ \left(\Delta_x^h L_t^x\right)^2 - 4hL_t^x \right\} \Delta_x^h L_{s-t}^x \, dx \]
where we have used the fact that \( \{L_t^x + B_t, x \in R^1\} \overset{\text{law}}{=} \{L_t^x, x \in R^1\} \)
which follows from time reversal. Similarly,
\[ \int \Delta_x^h L_t^x \left\{ \left(\Delta_x^h (L_s^x - L_t^x)\right)^2 - 4h(L_s^x - L_t^x) \right\} \, dx \]
\[ \text{law} \int \Delta_x^h \tilde{L}_t^x \left\{ \left(\Delta_x^h (L_{s-t}^x)\right)^2 - 4hL_{s-t}^x \right\} \, dx. \] (3.19)
Let
\[ \hat{G}_{m,h}(t,r) = \mathbb{E} \left( h^{-2} \int \left\{ (\Delta_h I_t)^2 - 4hI_t^2 \right\} \Delta_h \tilde{L}_r \, dx \right)^m \] (3.20)
and set
\[ \hat{G}_{m,0}(t,r) := \mathbb{E} \left\{ \eta^m \left( 64 \int (L_t^2 \tilde{L}_r \, dx)_{m/2} \right) \right\}. \] (3.21)

We then use the triangle inequality with respect to the norm \( \| \cdot \|_{2n} \) together with (3.16)-(3.19) to see that
\[ \hat{F}_{2n,h}^{1/(2n)}(s) \leq \hat{F}_{2n,h}^{1/(2n)}(t) + \hat{F}_{2n,h}^{1/(2n)}(s-t) + 3\hat{G}_{2n,h}^{1/(2n)}(t,s-t) + 3\hat{G}_{2n,h}^{1/(2n)}(s-t,t). \] (3.22)

Similarly we have
\[ \hat{F}_{2n,h}^{1/(2n)}(s) \geq \hat{F}_{2n,h}^{1/(2n)}(t) - \hat{F}_{2n,h}^{1/(2n)}(s-t) - 3\hat{G}_{2n,h}^{1/(2n)}(t,s-t) - 3\hat{G}_{2n,h}^{1/(2n)}(s-t,t). \] (3.23)

We now use the triangle inequality with respect to the norm in \( L^{2n}([t, t+\delta], \delta^{-1} ds) \) to see that
\[ \left\{ \frac{1}{\delta} \int_t^{t+\delta} \tilde{F}_{2n,h}(s) \, ds \right\}^{1/(2n)} \leq \hat{F}_{2n,h}^{1/(2n)}(t) + \left\{ \frac{1}{\delta} \int_t^{t+\delta} \tilde{F}_{2n,h}(s-t) \, ds \right\}^{1/(2n)} + 3 \left\{ \frac{1}{\delta} \int_t^{t+\delta} \tilde{G}_{2n,h}(s-t,t) \, ds \right\}^{1/(2n)} + 3 \left\{ \frac{1}{\delta} \int_t^{t+\delta} \tilde{G}_{2n,h}(s-t,t) \, ds \right\}^{1/(2n)} \] (3.24)
and
\[ \left\{ \frac{1}{\delta} \int_t^{t+\delta} \tilde{F}_{2n,h}(s) \, ds \right\}^{1/(2n)} \geq \hat{F}_{2n,h}^{1/(2n)}(t) - \left\{ \frac{1}{\delta} \int_t^{t+\delta} \tilde{F}_{2n,h}(s-t) \, ds \right\}^{1/(2n)} - 3 \left\{ \frac{1}{\delta} \int_t^{t+\delta} \tilde{G}_{2n,h}(s-t,t) \, ds \right\}^{1/(2n)} - 3 \left\{ \frac{1}{\delta} \int_t^{t+\delta} \tilde{G}_{2n,h}(s-t,t) \, ds \right\}^{1/(2n)} \] (3.25)
Hence, in light of (3.11), to prove (3.13) we need only show that for each $t$

$$\lim_{\delta \to 0} \lim_{h \to 0} \frac{1}{\delta} \int_0^\delta \hat{G}_{2n,h}(t, s) \, ds = 0 \quad (3.26)$$

and

$$\lim_{\delta \to 0} \lim_{h \to 0} \frac{1}{\delta} \int_0^\delta \hat{G}_{2n,h}(s, t) \, ds = 0. \quad (3.27)$$

We use $E^{y,z}(\cdot)$ to denote expectation with respect to the independent Brownian motions $B_t$ starting at $y$ and $\tilde{B}_t$ starting at $z$. Let $\lambda_\zeta, \lambda_\zeta'$ be independent exponential random variables with mean $1/\zeta, 1/\zeta'$. We show in Lemma 5.1 below that for each integer $n \geq 0$,

$$\lim_{h \to 0} E^{y,z} \left( \left( \int \left\{ \left( \frac{(\Delta^h L^x_\lambda)^2 - 4hL^x_\lambda} \right) \Delta^h \tilde{L}^x_\lambda \, dx \right) \right)^{2n} \right) = \frac{(2n)!}{2^n n!} \left( \int (L^x_\lambda)^{2} \tilde{L}^x_\lambda \, dx \right)^{n} \quad (3.28)$$

uniformly in $y, z$.

Just as (3.7) implied (3.8), it follows from (3.28) that

$$\lim_{h \to 0} \int_0^t \int_0^q \hat{G}_{2n,h}(s, r) \, dr \, ds = \int_0^t \int_0^q \hat{G}_{2n,0}(s, r) \, dr \, ds \quad (3.29)$$

for all $t$. In particular,

$$\lim_{h \to 0} \int_t^{t+\delta} \int_q^{q+\delta'} \hat{G}_{2n,h}(s, r) \, dr \, ds = \int_t^{t+\delta} \int_q^{q+\delta'} \hat{G}_{2n,0}(s, r) \, dr \, ds. \quad (3.30)$$

It follows as with $\hat{F}_{2n,0}(s)$ that $\hat{G}_{2n,0}(s, r)$ is continuous in $s, r$. Consequently,

$$\lim_{\delta, \delta' \to 0} \lim_{h \to 0} \frac{1}{\delta \delta'} \int_t^{t+\delta} \int_q^{q+\delta'} \hat{G}_{2n,h}(s, r) \, dr \, ds = \hat{G}_{2n,0}(t, q). \quad (3.31)$$

When $t = 0$ we get

$$\lim_{\delta, \delta' \to 0} \lim_{h \to 0} \frac{1}{\delta \delta'} \int_0^{\delta} \int_q^{q+\delta'} \hat{G}_{2n,h}(s, r) \, dr \, ds = 0. \quad (3.32)$$

Similarly we have

$$\lim_{\delta, \delta' \to 0} \lim_{h \to 0} \frac{1}{\delta \delta'} \int_t^{t+\delta} \int_0^{\delta'} \hat{G}_{2n,h}(s, r) \, dr \, ds = 0. \quad (3.33)$$
For \( s \geq t \) we write

\[
\int \left\{ \left( \Delta^h_{x} L^x_s \right)^2 - 4h L^x_s \right\} \Delta^h_{x} \bar{L}^x_r \, dx \tag{3.34}
\]

\[
= \int \left\{ \left( \Delta^h_{x} L^x_t + \Delta^h_{x} (L^x_s - L^x_t) \right)^2 - 4h \left( L^x_t + (L^x_s - L^x_t) \right) \right\} \Delta^h_{x} \bar{L}^x_r \, dx
\]

\[
= \int \left\{ \left( \Delta^h_{x} L^x_t \right)^2 - 4h L^x_t \right\} \Delta^h_{x} \bar{L}^x_r \, dx
\]

\[
+ \int \left\{ \left( \Delta^h_{x} (L^x_s - L^x_t) \right)^2 - 4h \left( L^x_s - L^x_t \right) \right\} \Delta^h_{x} \bar{L}^x_r \, dx \tag{3.35}
\]

\[
+ 2 \int \Delta^h_{x} L^x_t \Delta^h_{x} (L^x_s - L^x_t) \Delta^h_{x} \bar{L}^x_r \, dx
\]

Hence as before we obtain

\[
\left\{ \frac{1}{\delta_0} \int_t^{t+\delta} \int_0^{\tilde{\theta}} \tilde{G}_{2n,h}(s, r) \, dr \, ds \right\}^{1/(2n)} \geq \left\{ \frac{1}{\delta} \int_0^{\tilde{\theta}} \tilde{G}_{2n,h}(t, r) \, dr \right\}^{1/(2n)}
\]

\[ - \left\{ \frac{1}{\delta_0} \int_t^{t+\delta} \int_0^{\tilde{\theta}} \tilde{G}_{2n,h}(s - t, r) \, dr \, ds \right\}^{1/(2n)} \tag{3.36}
\]

\[ - \left\{ \frac{2}{\delta_0} \int_t^{t+\delta} \int_0^{\tilde{\theta}} \tilde{H}_{2n,h}(t, s - t, r) \, dr \, ds \right\}^{1/(2n)} \]

where

\[
\tilde{G}_{m,h}(s - t, r)
\]

\[
:: E \left( h^{-2} \int \left\{ \left( \Delta^h_{x} L^x_{s-t} \right)^2 - 4h L^x_{s-t} \right\} \circ \theta_t \Delta^h_{x} \bar{L}^x_r \, dx \right)^m
\]

\[
= \int E^{y,0} \left\{ \left( h^{-2} \int \left\{ \left( \Delta^h_{x} L^x_{s-t} \right)^2 - 4h L^x_{s-t} \right\} \Delta^h_{x} \bar{L}^x_r \, dx \right)^m \right\} p_t(y) \, dy
\]

and

\[
\tilde{H}_{m,h}(t, s, r) = E \left( h^{-2} \int \Delta^h_{x} L^x_t \left( \Delta^h_{x} L^x_s \circ \theta_t \right) \Delta^h_{x} \bar{L}^x_r \, dx \right)^m. \tag{3.38}
\]

We show in Lemma 6.1 below that for each integer \( n \geq 0 \),

\[
\lim_{h \to 0} E \left( \frac{\int \Delta^h_{x} L^x_t \left( \Delta^h_{x} L^x_s \circ \theta_t \right) \Delta^h_{x} \bar{L}^x_r \, dx}{h^2} \right)^{2n} \tag{3.39}
\]

\[
= \frac{(2n)!}{2^{n} n!} (64)^n E \left\{ \left( \int L^x_t \left( L^x_s \circ \theta_t \right) \bar{L}^x_r \, dx \right)^n \right\},
\]

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locally uniformly in $r, s, t$ on $t > 0$. \((3.26)\) then follows by arguing as we did to obtain \((3.33)\).

We can also write

\[
\int \left\{ (\Delta^h_x L^x_s)^2 - 4h L^x_s \right\} \Delta^h_x \tilde{L}^x_t \, dx
\]

\(= \int \left\{ (\Delta^h_x L^x_s)^2 - 4h L^x_s \right\} \Delta^h_x \tilde{L}^x_t \, dx
\]

\[\quad + \int \left\{ (\Delta^h_x L^x_s)^2 - 4h L^x_s \right\} \Delta^h_x (\tilde{L}^x_r - \tilde{L}^x_q) \, dx
\]

\[\quad \overset{(3.40)}{=} \int \left\{ (\Delta^h_x L^x_s)^2 - 4h L^x_s \right\} \Delta^h_x \tilde{L}^x_t \, dx
\]

\[\quad + \int \left\{ (\Delta^h_x L^x_s)^2 - 4h L^x_s \right\} \Delta^h_x (\tilde{L}^x_r - \tilde{L}^x_q) \circ \tilde{\theta}_q \, dx
\]

and hence as before this leads to

\[
\left\{ \frac{1}{\delta^2 \delta'} \int_0^\delta \int_q^{q+\delta'} \tilde{G}_{2n,h}(s, r) \, dr \, ds \right\}^{1/(2n)} \geq \left\{ \frac{1}{\delta} \int_0^\delta \tilde{G}_{2n,h}(s, q) \, ds \right\}^{1/(2n)}
\]

\[- \left\{ \frac{1}{\delta^2 \delta'} \int_0^\delta \int_q^{q+\delta'} \tilde{G}_{2n,h}(s, r - q) \, dr \, ds \right\}^{1/(2n)}
\]

\[\overset{(3.41)}{=} \quad \text{where}
\]

\[\tilde{G}_{m,h}(s, r - q) \overset{(3.42)}{=} E \left( h^{-2} \int \left\{ (\Delta^h_x L^x_s)^2 - 4h L^x_s \right\} \Delta^h_x \tilde{L}^x_{r-q} \circ \tilde{\theta}_q \, dx \right)^m
\]

\[\quad \overset{(3.43)}{=} \int E^{0,z} \left\{ \left( h^{-2} \int \left\{ (\Delta^h_x L^x_s)^2 - 4h L^x_s \right\} \Delta^h_x \tilde{L}^x_{r-q} \, dx \right)^m \right\} p_q(z) \, dz
\]

and then \((3.27)\) follows by arguing as we did to obtain \((3.32)\).

Thus we obtain \((3.12)\) and hence \((3.1)\) when $m$ is even.

In order to obtain \((3.12)\) when $m$ is odd we first show that

\[
\sup_{h > 0} \hat{F}_{2n,h}(t) \leq C t^{2n}.
\]

To this end, it clearly suffices to show that

\[
\sup_{h > 0} \hat{F}^{(0)}_{2n,h}(t) \leq C t^{2n}
\]
where
\[ \hat{F}_{m,h}(t) := E \left( \left( \frac{\int (\Delta_x L^x_t)^3 \, dx - 12h \int L^x_t (\Delta_x L^x_t) \, dx}{h^2} \right)^m \right). \] (3.45)

To see this we observe that by first changing variables and then using the scaling relationship (1.6) with \( h = \sqrt{t} \), we have
\[
\int (L^{x+h}_t - L^x_t)^3 \, dx = \sqrt{t} \int (L^{\sqrt{t}(x+ht^{-1/2})}_t - L^{\sqrt{t}x}_t)^3 \, dx \quad (3.46)
\]
and
\[
\int L^x_t (L^{x+h}_t - L^x_t) \, dx = \sqrt{t} \int L^{\sqrt{t}x}(L^{\sqrt{t}(x+ht^{-1/2})}_t - L^{\sqrt{t}x}_t) \, dx \quad (3.47)
\]
Therefore
\[
\frac{\int (L^{x+h}_t - L^x_t)^3 \, dx - 12h \int L^x_t (L^{x+h}_t - L^x_t) \, dx}{h^2} \leq t^2 \int (L^{x+ht^{-1/2}}_t - L^x_t)^3 \, dx - 12h^{3/2} \int L^x_t (L^{x+ht^{-1/2}}_t - L^x_t) \, dx \quad (3.48)
\]
so that for any integer \( m \)
\[
\hat{F}_{m,h}(t) = t^m \hat{F}_{m,ht^{-1/2}}(1). \quad (3.49)
\]
Therefore to prove (3.44) we need only show that
\[
\sup_t \sup_{h>0} \hat{F}_{2n,ht^{-1/2}}^{(0)}(1) \leq C. \quad (3.50)
\]
It follows from (3.12) that for some \( \delta > 0 \)
\[
\sup_{\{t,h\mid ht^{-1/2} \leq \delta\}} \hat{F}_{2n,ht^{-1/2}}^{(0)}(1) \leq C. \quad (3.51)
\]
On the other hand, for \(ht^{-1/2} \geq \delta\)

\[
\left| \frac{\left( \int(L_{1}^{x+ht^{-1/2}} - L_{1}^{x})^3 dx - 12(ht^{-1/2}) \int L_{1}^{x} (L_{1}^{x+ht^{-1/2}} - L_{1}^{x}) dx \right)}{(ht^{-1/2})^2} \right|
\leq \delta^{-2} \int(L_{1}^{x+ht^{-1/2}} - L_{1}^{x})^3 dx + 12\delta^{-1} \int L_{1}^{x} (L_{1}^{x+ht^{-1/2}} - L_{1}^{x}) dx
\leq 4\delta^{-2} \int(L_{1}^{x})^3 dx + 24\delta^{-1} \int(L_{1}^{x})^2 dx
\]

(3.52)

which has finite moments since \(\int(L_{1}^{x})^2 dx\) and \(\int(L_{1}^{x})^3 dx\) have finite moments, see (3.2). Using this and (3.51) we get (3.50) and hence (3.44). As already noted, this implies (3.43). It then follows from the Cauchy-Schwarz inequality that

\[
\sup_{h>0} |\hat{F}_{m,h}(t)| \leq Ct^{m}
\]

(3.53)

for all integers \(m\).

We next show that for any integer \(m\), the family of functions \(\{\hat{F}_{m,h}(t); h\}\) is equicontinuous in \(t\), that is, for each \(t\) and \(\epsilon > 0\) we can find a \(\delta > 0\) such that

\[
\sup_{\{s \mid |s-t| \leq \delta\}} \sup_{h>0} |\hat{F}_{m,h}(t) - \hat{F}_{m,h}(s)| \leq \epsilon.
\]

(3.54)

Let

\[
\Phi_{h}(t) := \frac{\int(L_{1}^{x+h} - L_{1}^{x})^3 dx - 12h \int L_{1}^{x} (L_{1}^{x+h} - L_{1}^{x}) dx - 24h^2 t}{h^2}.
\]

(3.55)

Applying the identity \(A^m - B^m = \sum_{j=0}^{m-1} A^j (A - B) B^{m-j-1}\) with \(A = \Phi_{h}(t), B = \Phi_{h}(s)\) gives

\[
\hat{F}_{m,h}(t) - \hat{F}_{m,h}(s) = \sum_{j=0}^{m-1} \Phi_{h}(t)^j (\Phi_{h}(t) - \Phi_{h}(s)) \Phi_{h}(s)^{m-j-1}
\]

(3.56)

Consequently by using the Cauchy–Schwarz inequality twice and (3.53), we see that

\[
\sup_{\{s \mid |s-t| \leq \delta\}} \sup_{h>0} |\hat{F}_{m,h}(t) - \hat{F}_{m,h}(s)| \leq C_{t,m} \sup_{\{s \mid |s-t| \leq \delta\}} \sup_{h>0} \|\Phi_{h}(t) - \Phi_{h}(s)\|_2.
\]

(3.57)
Using (3.16)-(3.19), we see that to obtain (3.54) it suffices to show that for any $\epsilon > 0$ we can find some $\delta > 0$ such that
\[
\sup_{\{s \mid s \leq \delta\}} \sup_{h > 0} \hat{F}_{2,h}(s) \leq \epsilon \tag{3.58}
\]
and for any $T < \infty$
\[
\sup_{\{t \leq T\}} \sup_{\{s \leq \delta\}} \sup_{h > 0} E \left[ \frac{1}{h^2} \int \left\{ (\Delta^h_{L_x} t)^2 - 4hL_x^2 \right\} \Delta^h_{L_x} s \, dx \right]^2 \leq \epsilon \tag{3.59}
\]
and
\[
\sup_{\{s \leq T\}} \sup_{\{t \leq \delta\}} \sup_{h > 0} E \left[ \frac{1}{h^2} \int \left\{ (\Delta^h_{L_x} t)^2 - 4hL_x^2 \right\} \Delta^h_{L_x} s \, dx \right]^2 \leq \epsilon. \tag{3.60}
\]
By (3.44)
\[
\sup_{h > 0} F_{2,h}(s) \leq Cs^2 \tag{3.61}
\]
which immediately gives (3.58), while (3.59) and (3.60) follow from Lemma 7.1 below. This establishes (3.54).

We now obtain (3.1) when $m$ is odd. By equicontinuity, for any sequence $h_n \to 0$, we can find a subsequence $h_{n_j} \to 0$, such that
\[
\lim_{j \to \infty} \hat{F}_{m,h_{n_j}}(t) \tag{3.62}
\]
converges to a continuous function which we denote by $\overline{F}_m(t)$. It remains to show that
\[
\overline{F}_m(t) \equiv 0. \tag{3.63}
\]
Let
\[
G_{m,h}(t) := e^{-t} \hat{F}_{m,h}(t) \quad \text{and} \quad \overline{G}_m(t) := e^{-t}\overline{F}_m(t). \tag{3.64}
\]
By (3.53)
\[
\sup_{h > 0} \sup_{t} |G_{m,h}(t)| \leq C \quad \text{and} \quad \lim_{t \to \infty} \sup_{h > 0} G_{m,h}(t) = 0. \tag{3.65}
\]
It then follows from (3.7) and the dominated convergence theorem that for all $\zeta > 0$
\[
\int_0^\infty e^{-\zeta s} \overline{G}_m(s) \, ds = 0. \tag{3.66}
\]
We obtain (3.63) by showing that $G_m(s) \equiv 0$.

It follows from (3.65) that $G_m(t)$ is a continuous bounded function on $R_+$ that goes to zero as $t \to \infty$. By the Stone–Weierstrass Theorem; (see [5, Lemma 5.4]), for any $\epsilon > 0$, we can find a finite linear combination of the form $\sum_{i=1}^{n} c_i e^{-\zeta_i s}$ such that

$$\sup_s |G_m(s) - \sum_{i=1}^{n} c_i e^{-\zeta_i s}| \leq \epsilon.$$  (3.67)

Therefore, by (3.66)

$$\int_0^{\infty} e^{-s} G_m^2(s) \, ds = \int_0^{\infty} e^{-s} \left( \sum_{i=1}^{n} c_i e^{-\zeta_i s} \right) G_m(s) \, ds$$  (3.68)

$$+ \int_0^{\infty} e^{-s} \left( G_m(s) - \sum_{i=1}^{n} c_i e^{-\zeta_i s} \right) G_m(s) \, ds$$

$$= \int_0^{\infty} e^{-s} \left( G_m(s) - \sum_{i=1}^{n} c_i e^{-\zeta_i s} \right) G_m(s) \, ds$$

$$\leq 2\epsilon \left( \int_0^{\infty} e^{-s} G_m^2(s) \, ds \right)^{1/2}$$  (3.69)

by the Cauchy–Schwarz inequality and (3.67). Thus $\int_0^{\infty} e^{-s} G_m^2(s) \, ds = 0$ which implies that $G_m(s) \equiv 0$.

\[ \square \]

## 4 Moments at exponential times

We often write $u_{h,-h}(0)$ for $\Delta_h \Delta^{-h} u^\zeta(0) = 2 \left( u^\zeta(0) - u^\zeta(h) \right)$.

**Lemma 4.1** For each $m$, as $h \to 0$

$$E \left( \left( \int_{\mathbb{R}} \frac{\left\{ (\Delta_x L^x_{\lambda})^3 - 6u_{h,-h}(0) L^x_{\lambda} \Delta_x L^x_{\lambda} - 6 \left( u^\zeta_{h,-h} \right)^2 L^x_{\lambda} \right\} \, dx}{h^2} \right)^m \right)$$

$$\Rightarrow \begin{cases} \left( \frac{2n}{2n} \right)^{m} \left( 192 \right)^n E \left\{ \int \left( L^x_{\lambda} \right)^3 \, dx \right\}^{n} \text{ if } m = 2n \\ 0 \text{ otherwise.} \end{cases}$$  (4.1)
We then use the product rule where the sum runs over all bijections of local time, Lemma 4.1 implies (3.3).

**Remark:** Of course, \( \int L_{\lambda_\zeta}^x \, dx = \lambda_\zeta \). Using (2.4) and the continuity of local time, Lemma 4.1 implies (3.3).

**Proof of Lemma 4.1** For any integer \( m \) we have

\[
E \left( \left( \int \left\{ \left( L_{\lambda_\zeta}^x \right)^3 - 6u_\zeta h h (0)L_{\lambda_\zeta}^x \Delta^h L_{\lambda_\zeta}^x - 6 \left( u_\zeta h h \right)^2 \Delta^h L_{\lambda_\zeta}^x \right\} \, dx \right)^m \right)
\]

\[
eq E \left( \prod_{i=1}^m \left( \int \left\{ \left( L_{\lambda_\zeta}^x \right)^3 - 6u_\zeta h h (0)L_{\lambda_\zeta}^x \Delta^h L_{\lambda_\zeta}^x - 6 \left( u_\zeta h h \right)^2 \Delta^h L_{\lambda_\zeta}^x \right\} \, dx_i \right) \right)
\]

\[
= \sum_{A,B,C} (-1)^{m-|B|-|C|} (6u_\zeta h h (0))^{|B|} \left( 6 \left( u_\zeta h h \right)^2 \right)^{|C|}
\]

\[
E \left( \prod_{i \in A} \left( \int (L_{\lambda_\zeta}^x)^3 \, dx_i \right) \left( \prod_{j \in B} \int L_{\lambda_\zeta}^x \Delta^h L_{\lambda_\zeta}^x \, dx_j \right) \left( \prod_{k \in C} \int L_{\lambda_\zeta}^x \, dx_k \right) \right),
\]

where the sum runs over all partitions of \([1, m]\) into three parts, \( A, B, C \).

We initially calculate

\[
E \left( \prod_{i \in A} \Delta^h L_{\lambda_\zeta}^x L_{\lambda_\zeta}^x L_{\lambda_\zeta}^x \Delta^h \Delta^h j \Delta^h j \Delta^h j \prod_{j \in B} L_{\lambda_\zeta}^x \Delta^h L_{\lambda_\zeta}^x \prod_{k \in C} L_{\lambda_\zeta}^x \right)
\]

and eventually we set \( y_i = x_i = z_i \) for all \( l \). Using (2.2) we have

\[
E \left( \prod_{i \in A} \Delta^h L_{\lambda_\zeta}^x L_{\lambda_\zeta}^x L_{\lambda_\zeta}^x \Delta^h \Delta^h j \Delta^h j \Delta^h j \prod_{j \in B} L_{\lambda_\zeta}^x \Delta^h L_{\lambda_\zeta}^x \prod_{k \in C} L_{\lambda_\zeta}^x \right)
\]

\[
= \left( \prod_{i \in A} \Delta^h L_{\lambda_\zeta}^x L_{\lambda_\zeta}^x L_{\lambda_\zeta}^x \Delta^h j \Delta^h j \Delta^h j \prod_{j \in B} L_{\lambda_\zeta}^x \Delta^h L_{\lambda_\zeta}^x \prod_{k \in C} L_{\lambda_\zeta}^x \right)
\]

\[
= \left( \prod_{i \in A} \Delta^h L_{\lambda_\zeta}^x L_{\lambda_\zeta}^x L_{\lambda_\zeta}^x \Delta^h j \Delta^h j \Delta^h j \prod_{j \in B} \prod_{j \in C} \prod_{j \in C} \right)
\]

where the sum runs over all bijections

\[
\sigma : \{1, 2, \ldots, m + 2|A| + |B|\} \mapsto \{x_i, y_i, z_i, i \in A\} \cup \{x_j, y_j, j \in B\} \cup \{x_k, k \in C\}.
\]

We then use the product rule

\[
\Delta^h_x \{f(x)g(x)\} = \{\Delta^h_x f(x)\} g(x + h) + f(x) \{\Delta^h_x g(x)\}
\]
to expand the right hand side of (4.4) into a sum of many terms, over all \( \sigma \) and all ways to allocate each \( \Delta^h_{x_1}, \Delta^h_{y_1}, \Delta^h_{z_1} \) or \( \Delta^h_{y_k} \) to a single \( u^{\zeta} \) factor.

Consider first the case where \( A = \{1, \ldots, m\} \). For a given term in the above expansion, we will say that \( x_i \) is 3-bound if \( x_i, y_i, z_i \) are adjacent, (in other words, for some \( j \) we have \((\sigma(j), \sigma(j + 1), \sigma(j + 2)) = (x_i, y_i, z_i) \) or one of its 6 possible permutations), and \( \Delta^h_{x_i}, \Delta^h_{y_i}, \Delta^h_{z_i} \) are all attached to the \( u^{\zeta} \) factors which connect \( x_i, y_i, z_i \). Thus if \((\sigma(j), \sigma(j + 1), \sigma(j + 2)) = (x_i, y_i, z_i) \), the \( \Delta^h_{x_i}, \Delta^h_{y_i}, \Delta^h_{z_i} \) are all attached to \( u^{\zeta}(y_i - x_i)u^{\zeta}(z_i - y_i) \). We return shortly to analyze this case.

If \( x_i \) is not 3-bound, we will say that it is 2-bound if any two of the three elements \( x_i, y_i, z_i \) are adjacent, for example if \( x_i, y_i \) are adjacent, (in other words either \((x_i, y_i) = (\sigma(j), \sigma(j + 1)) \) or \((y_i, x_i) = (\sigma(j), \sigma(j + 1)) \) for some \( j \)), and both \( \Delta^h_{x_i} \) and \( \Delta^h_{y_i} \) are attached to the factor \( u^{\zeta} \). In applying (4.5) we are free to choose which function plays the role of \( f \), and which the role of \( g \). In case \( x_i \) is 2-bound, taking our example with \((x_i, y_i) = (\sigma(j), \sigma(j + 1)) \), when using (4.5) we take \( g \) to be \( u^{\zeta}(x_i - y_i) \) and similarly when using (4.5) to expand \( \Delta^h_{y_i} \). In this way we guarantee that we have not added \( \pm h \) to the arguments of any other factors. By (2.4), setting \( x_i = y_i \) turns the factor \( \Delta^h_{x_i} \Delta^h_{y_i} u^{\zeta}(x_i - y_i) \) into \( \Delta^h \Delta^{-h} u^{\zeta}(0) \), and since for every such \( \sigma \) there is precisely one other bijection which differs from \( \sigma \) only in that it permutes \( x_i, y_i \), we obtain a factor of \( 2\Delta^h \Delta^{-h} u^{\zeta}(0) \). This is precisely what we would have obtained if instead of \( \Delta^h_{x_i} \Delta^h_{y_i} L^{x_i}_{\lambda_i} L^{y_i}_{\lambda_i} \) in (4.4) we had \( 2\Delta^h \Delta^{-h} u^{\zeta}(0) L^{x_i}_{\lambda_i} \). There are \( \binom{3}{2} = 3 \) ways to pick two letters from among \( \{x_i, y_i, z_i\} \). By considering all such cases, we obtain precisely what we would have obtained if instead of \( \Delta^h_{x_i} \Delta^h_{y_i} \Delta^h_{z_i} L^{x_i}_{\lambda_i} L^{y_i}_{\lambda_i} L^{z_i}_{\lambda_i} \) in (4.4) we had \( 6\Delta^h \Delta^{-h} u^{\zeta}(0) L^{x_i}_{\lambda_i} \Delta^h_{z_i} L^{z_i}_{\lambda_i} \). Consider then a term in the expansion of (4.4) with \( A = \{1, \ldots, m\} \) and \( J = \{i \mid x_i \text{ is 2-bound}\} \) non-empty, but \( \{i \mid x_i \text{ is 3-bound}\} = \emptyset \). By (4.4) there will be an identical contribution from the last line of (4.2) from any other \( A, B, C \) with \( B \subseteq J \) and \( C = \emptyset \). Since \( \sum_{B \subseteq J} (-1)^{|B|} = 0 \), we see that in the expansion of (4.2) there will not be any contributions from 2-bound \( x_i \)’s.

We emphasize that if \( x_i \) is 2-bound, and, for example, \((\sigma(j), \sigma(j + 1), \sigma(j + 2)) = (x_i, y_i, z_i) \), with both \( \Delta^h_{x_i}, \Delta^h_{y_i} \) attached to \( u^{\zeta}(y_i - x_i) \), then \( \Delta^h_{z_i} \) cannot be assigned to \( u^{\zeta}(z_i - y_i) \). Otherwise, \( x_i \) would be 3-bound.

We now return to analyze the case where \( x_i \) is 3-bound. Consider the case that \((\sigma(j), \sigma(j + 1), \sigma(j + 2)) = (x_i, y_i, z_i) \). We first apply the \( \Delta^h_{x_i} \) and
\[ \Delta^h z_i \] operators to \( u^\xi(y_i - x_i)u^\xi(z_i - y_i) \) to obtain \( \Delta^h x_i, u^\xi(y_i - x_i) \Delta^h z_i, u^\xi(z_i - y_i) \). Then by (4.5) we have
\[
\Delta^h y_i \left( \Delta^h x_i, u^\xi(y_i - x_i) \Delta^h z_i, u^\xi(z_i - y_i) \right) \\
= \left( \Delta^h y_i, \Delta^h x_i, u^\xi(y_i - x_i) \right) \Delta^h z_i, u^\xi(z_i - y_i - h) \\
+ \Delta^h x_i, u^\xi(y_i - x_i) \left( \Delta^h y_i, \Delta^h z_i, u^\xi(z_i - y_i) \right).
\] (4.6)

If we now set \( y_i = x_i = z_i \) we obtain
\[
\Delta^h \Delta^{-h} u^\xi(0) \left( u^\xi(0) - u^\xi(h) \right) \Delta^h \Delta^{-h} u^\xi(0) = 0. \] (4.7)

Thus, 3-bound variables such as \( x_i \) make no contribution to (4.4). However, there will be an analogous contribution from \( 6\Delta^h \Delta^{-h} u^\xi(0)L^x_{\lambda_i} \Delta^h L^x_{\lambda_i} \) which is not cancelled by 2-bound variables. This is the case where \( x_i, z_i \) are adjacent, say \((x_i, z_i) = (\sigma(j), \sigma(j + 1))\), and \( \Delta^h z_i \) is attached to the factor \( u^\xi(x_i - z_i) \). As before we may do this without adding an \( h \) to the arguments in any other factors. After setting \( x_i = z_i \) we obtain
\[
6\Delta^h \Delta^{-h} u^\xi(0) \left( u^\xi(h) - u^\xi(0) \right) = -3 \left( \Delta^h \Delta^{-h} u^\xi(0) \right)^2. \] Since we can also interchange \( x_i, z_i \), such \( x_i \) contribute \(-6 \left( \Delta^h \Delta^{-h} u^\xi(0) \right)^2\), which will be exactly canceled by the term \(-6 \left( \Delta^h \Delta^{-h} u^\xi(0) \right)^2 L^x_{\lambda_i}\). Furthermore, this completely exhausts the contribution to (4.2) of all \( A \neq \{1, \ldots, m\} \).

Thus, in estimating (4.4) we need only consider \( A = \{1, \ldots, m\} \) and those cases where each of the \( 3m \) \( \Delta^h \)'s are assigned either to unique \( u^\xi \) factors, or if two are assigned to the same \( u^\xi \) factor, it is not of the form \( u^\xi(x_i - y_i), u^\xi(x_i - z_i) \) or \( u^\xi(z_i - y_i) \).

For ease of exposition, in the following calculations we first replace the right hand side of (4.5) by \( \{\Delta^h f(x)\}g(x) + f(x)\{\Delta^h g(x)\} \), and return at the end of the proof to explain why this doesn’t affect the final result.

We use the notation
\[
E_* \left( \left( \int \left\{ \left( \Delta^h L^x_{\lambda_i} \right)^3 - 6u^\xi_{h,-h}(0)L^x_{\lambda_i} \Delta^h L^x_{\lambda_i} - 6 \left( u^\xi_{h,-h} \right)^2 L^x_{\lambda_i} \right \} dx \right) \right)^m \] (4.8)

to denote the expression obtained with this replacement. We can thus write
\[
E_* \left( \left( \int \left\{ \left( \Delta^h L^x_{\lambda_i} \right)^3 - 6u^\xi_{h,-h}(0)L^x_{\lambda_i} \Delta^h L^x_{\lambda_i} - 6 \left( u^\xi_{h,-h} \right)^2 L^x_{\lambda_i} \right \} dx \right) \right)^m \] \\
= 6^m \sum_{\pi, a} \int T_h(x; \pi, a) dx \] (4.9)
with
\[
T_h(x; \pi, a) = \prod_{j=1}^{3m} \left( \Delta_{x_{\pi(j)}}^{h} \right)^{a_1(j)} \left( \Delta_{x_{\pi(j-1)}}^{h} \right)^{a_2(j)} u^\xi(x_{\pi(j)} - x_{\pi(j-1)})
\] (4.10)

where the sum runs over all maps \( \pi : [1, \ldots, 3m] \mapsto [1, \ldots, m] \) with \( |\pi^{-1}(i)| = 3 \) for each \( i \), and all ‘assignments’ \( a = (a_1, a_2) : [1, \ldots, 2m] \mapsto \{0, 1\} \times \{0, 1\} \) with the property that for each \( i \) there will be exactly three operators of the form \( \Delta_{x_i}^{h} \) in (4.10), and if \( a(j) = (1, 1) \) for any \( j \), then \( x_{\pi(j)} \neq x_{\pi(j-1)} \). The factor \( 6^m = (3!)^m \) in (4.9) comes from the fact that \( |\pi^{-1}(i)| = 3 \) for each \( i \).

Let \( m = 2n \). Assume first that \( a = e \) where now \( e(2j) = (1, 1) \), \( e(2j-1) = (0, 0) \) for all \( j \).

### 4.1 \( a = e \) with \( \pi \) compatible with a pairing

When \( a = e \) we have
\[
T_h(x; \pi, e) = \prod_{j=1}^{3n} u^\xi(x_{\pi(2j-1)} - x_{\pi(2j-2)}) \Delta^h \Delta^{-h} u^\xi(x_{\pi(2j)} - x_{\pi(2j-1)}).
\] (4.11)

Let \( \mathcal{P} = \{(l_{2i-1}, l_{2i}), 1 \leq i \leq n\} \) be a pairing of the integers \([1, 2n]\). Let \( \pi \) as in (4.11) be such that for each \( 1 \leq j \leq 3n \), \( \{\pi(2j-1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\} \) for some, necessarily unique, \( 1 \leq i \leq n \). In this case we say that \( \pi \) is compatible with the pairing \( \mathcal{P} \) and write this as \( \pi \sim \mathcal{P} \). (Note that when we write \( \{\pi(2j-1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\} \) we mean as two sets, so, according to what \( \pi \) is, we may have \( \pi(2j-1) = l_{2i-1}, \pi(2j) = l_{2i} \) or \( \pi(2j-1) = l_{2i}, \pi(2j) = l_{2i-1} \).)

In this case we have
\[
T_h(x; \pi, e) = \prod_{i=1}^{n} \left( \Delta^h \Delta^{-h} u^\xi(x_{l_{2i}} - x_{l_{2i-1}}) \right)^3 \prod_{j=1}^{3n} u^\xi(x_{\pi(2j-1)} - x_{\pi(2j-2)}).
\] (4.12)

We now show that
\[
\int T_h(x; \pi, e) \prod_{j=1}^{2n} dx_j = \int T_{1,h}(x; \pi, a) \prod_{j=1}^{2n} dx_j + O(h^{4n+1})
\] (4.13)

where
\[
T_{1,h}(x; \pi, e) = \prod_{i=1}^{n} \left( 1_{|x_{l_{2i}} - x_{l_{2i-1}}| \leq h} \right) \left( \Delta^h \Delta^{-h} u^\xi(x_{l_{2i}} - x_{l_{2i-1}}) \right)^3.
\]
× \prod_{j=1}^{3n} u^\xi(x_{\pi(2j-1)} - x_{\pi(2j-2)}). \quad (4.14)

To prove (4.13) we write

\[ 1 = \prod_{i=1}^{n} \left( 1_{\{|x_{l_2i} - x_{l_2i-1}| \leq h\}} + 1_{\{|x_{l_2i} - x_{l_2i-1}| \geq h\}} \right), \quad (4.15) \]

insert this inside the integral on the left hand side of (4.13) and expand the product. It then suffices to show that

\[ \int \prod_{i \in A} 1_{\{|x_{l_2i} - x_{l_2i-1}| \leq h\}} \prod_{i \in A^c} 1_{\{|x_{l_2i} - x_{l_2i-1}| \geq h\}} \left( u^\xi(x_{l_2i} - x_{l_2i-1}) \right)^3 \]

\[ \times \prod_{j=1}^{3n} u^\xi(x_{\pi(2j-1)} - x_{\pi(2j-2)}) \prod_{j=1}^{2n} dx_j = O(h^{4n+1}) \quad (4.16) \]

whenever \(|A^c| \geq 1\). To see this we first choose \(j_k, k = 1, \ldots, n\) so that

\( \{x_{\pi(2j_k-1)} - x_{\pi(2j_k-2)}, k = 1, \ldots, n\} \cup \{x_{l_2i} - x_{l_2i-1}, i = 1, \ldots, n\} \)

generate \(\{x_1, \ldots, x_{2n}\}\). After changing variables, (4.16) follows easily from (2.7), (2.8) and the fact that \(u^\xi\) is bounded and integrable.

We then study

\[ \int T_{1,h}(x; \pi, e) \prod_{j=1}^{2n} dx_j. \quad (4.17) \]

Recall that for each \(1 \leq j \leq 3n\), \(\{\pi(2j - 1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\}\), for some \(1 \leq i \leq n\). We identify these relationships by setting \(i = \sigma(j)\) when \(\{\pi(2j - 1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\}\). In the present situation this means that \(\sigma : [1, 3n] \mapsto [1, n]\) with \(|\sigma^{-1}(i)| = 3\) for each \(1 \leq i \leq n\). (One for each occurrence of \(\{l_{2i-1}, l_{2i}\}\)). We write

\[ \prod_{j=1}^{3n} u^\xi(x_{\pi(2j-1)} - x_{\pi(2j-2)}) \]

\[ = \prod_{j=1}^{3n} \left( u^\xi(x_{l_2\sigma(j)-1} - x_{l_2\sigma(j-1)-1}) + \Delta h_j u^\xi(x_{l_2\sigma(j)-1} - x_{l_2\sigma(j-1)-1}) \right) , \quad (4.18) \]
where \( h_j = (x_{\pi(2j-1)} - x_{l_{2\sigma(j)-1}}) + (x_{l_{2\sigma(j)-1}} - x_{\pi(2j-2)}) \). Note that because of the presence of the term \( \prod_{i=1}^{n} \left( \mathbf{1}_{|x_{l_{2i}} - x_{l_{2i-1}}| \leq h} \right) \) in the integral in (4.17) we need only be concerned with values of \(|h_j| \leq 2h, 1 \leq j \leq 3n\).

We expand the product on the right hand side of (4.18) and obtain a sum of many terms. Using (2.5) and the fact that \(|h_j| \leq 2h, 1 \leq j \leq 3n\) we can see as above that

\[
\int T_{1,h}(x; \pi, e) \prod_{j=1}^{2n} dx_j 
= \int \prod_{i=1}^{n} \left( \mathbf{1}_{|x_{l_{2i}} - x_{l_{2i-1}}| \leq h} \right) \prod_{i=1}^{n} \left( \Delta^h \Delta^{-h} u^\zeta(x_{l_{2i}} - x_{l_{2i-1}}) \right)^3 \prod_{j=1}^{3n} u^\zeta(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}) \prod_{j=1}^{2n} dx_j + O(h^{4n+1})
\]

where \( x_{-1} = 0 \). Once again we can now see that

\[
\int T_{1,h}(x; \pi, e) \prod_{j=1}^{2n} dx_j 
= \int \prod_{i=1}^{n} \left( \Delta^h \Delta^{-h} u^\zeta(x_{l_{2i}} - x_{l_{2i-1}}) \right)^3 \prod_{j=1}^{3n} u^\zeta(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}) \prod_{j=1}^{2n} dx_j + O(h^{4n+1}).
\]

Using translation invariance and then (2.9) we have

\[
\int \prod_{i=1}^{n} \left( \Delta^h \Delta^{-h} u^\zeta(x_{l_{2i}} - x_{l_{2i-1}}) \right)^3 \prod_{j=1}^{3n} u^\zeta(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}) \prod_{j=1}^{2n} dx_j 
= \int \prod_{i=1}^{n} \left( \Delta^h \Delta^{-h} u^\zeta(x_{l_{2i}}) \right)^3 \prod_{j=1}^{3n} u^\zeta(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}) \prod_{k=1}^{2n} dx_{l_k} 
= (4 + O(h)) n h^{4n} \int \prod_{j=1}^{3n} u^\zeta(x_{l_{2\sigma(j)-1}} - x_{l_{2\sigma(j-1)-1}}) \prod_{k=1}^{n} dx_{l_{2k-1}}. \quad (4.21)
\]

Rewriting this and summarizing, we have shown that

\[
\int T_h(x; \pi, e) \prod_{j=1}^{2n} dx_j \quad (4.22)
\]
\[ \sum_{\pi} \int T_h(x; \pi, e) \prod_{j=1}^{2n} dx_j \]

\[ = \left(32h^4\right)^n \sum_{\sigma \in \mathcal{M}} \int \prod_{j=1}^{3n} u^\sigma (y_{\sigma(j)} - y_{\sigma(j-1)}) \prod_{k=1}^{n} dy_k + O(h^{4n+1}) \]

\[ = \left(\frac{16}{3} h^4\right)^n E \left\{ \left(\int (L_{\lambda, \epsilon})^3 dx\right)^n \right\} + O(h^{4n+1}) \]

where the last line follows from Kac’s moment formula, compare (4.9). To complete this subsection, let \( G \) denote the set of \( \pi \) which are compatible with some pairing \( P \). Since there are \( \frac{(2n)!}{2^n n!} \) pairings of \([1, \ldots, 2n]\), we have shown that

\[ \sum_{\pi \in G} \int T_h(x; \pi, e) \prod_{j=1}^{2n} dx_j \]

\[ = \frac{(2n)!}{2^n n!} \left(\frac{16}{3} h^4\right)^n E \left\{ \left(\int (L_{\lambda, \epsilon})^3 dx\right)^n \right\} + O(h^{4n+1}). \]

### 4.2 \( a = e \) but \( \pi \) not compatible with a pairing

In this subsection we show that

\[ \sum_{\pi \not\in G} \left| \int T_h(x; \pi, e) \prod_{j=1}^{2n} dx_j \right| = O(h^{4n+1}). \]

We return to (4.11) to obtain

\[ \left| \int T_h(x; \pi, e) \prod_{j=1}^{2n} dx_j \right| \]
\[
\leq \int \prod_{j=1}^{3n} u^\zeta(x_{\pi(2j-1)} - x_{\pi(2j-2)}) w^\zeta(x_{\pi(2j)} - x_{\pi(2j-1)}) \prod_{j=1}^{2n} dx_j.
\]

Let us show that when \(\pi\) not compatible with a pairing we can find \(n + 1\) linearly independent vectors from among the \(3n\) vectors

\[
x_{\pi(2j)} - x_{\pi(2j-1)}, \quad 1 \leq j \leq 3n.
\]  \hspace{5em} (4.27)

We will say that \(x\) and \(y\) are both ‘contained’ in \(x - y\). Since \(|\pi^{-1}(i)| = 3\) for each \(1 \leq i \leq 2n\), we can find \(j_1\) such that \(x_1\) is contained in \(x_{\pi(2j_1)} - x_{\pi(2j_1-1)}\). In addition, \(x_{\pi(2j_1)} - x_{\pi(2j_1-1)}\) will contain another \(x_{i_1}\). We then pick an integer from \([2, \ldots, 2n]\) – \(\{i_1\}\), say \(i_2\) and then find \(x_{\pi(2j_2)} - x_{\pi(2j_2-1)}\) which contains \(x_{i_2}\). \(x_{\pi(2j_2)} - x_{\pi(2j_2-1)}\) will contain another \(x_{i_3}\) where we may have \(i_3 = 1\) or \(i_3 = i_1\). In any event it is clear that in this manner we can obtain a sequence of vectors

\[
x_{\pi(2j_i)} - x_{\pi(2j_i-1)}, \quad 1 \leq i \leq n
\]  \hspace{5em} (4.28)

which are linearly independent, since for each \(i\), \(x_{\pi(2j_i)} - x_{\pi(2j_i-1)}\) contains some \(x_k\) not contained in any of the preceding \(x_{\pi(2j_i)} - x_{\pi(2j_i-1)}, 1 \leq l < i\). Then, if the \(n\) vectors in (4.28) contain all \(x_i, 1 \leq i \leq 2n\), it follows that the \(n\) vectors in (4.28) must contain disjoint pairs of \(x_i\)'s. As a consequence they cannot generate any vector of the form \(x_i - x_{i'}\) which is not among the \(n\) vectors in (4.28). Since by our assumption \(\pi\) is not compatible with a pairing, there are vectors of the form \(x_{\pi(2j)} - x_{\pi(2j-1)}\) which are different from the \(n\) vectors in (4.28). This proves our claim that we can find \(n + 1\) linearly independent vectors from among the \(3n\) vectors of (4.27) in case the \(n\) vectors in (4.28) contain all \(x_i, 1 \leq i \leq 2n\). But if they do not contain all \(x_i, 1 \leq i \leq 2n\), say they do not contain \(x_k\). There is some vector in (4.27) which contains \(x_k\), and it is clearly linearly independent of the vectors in (4.28).

Thus we have a sequence

\[
x_{\pi(2j_i)} - x_{\pi(2j_i-1)}, \quad 1 \leq i \leq n + 1
\]  \hspace{5em} (4.29)

of linearly independent vectors. Let \(J = \{j_i, 1 \leq i \leq n + 1\}\). We use (2.5) to bound (4.26) by

\[
\left| \int T_h(x; \pi, e) \prod_{j=1}^{2n} dx_j \right|
\]  \hspace{5em} (4.30)
\[ \leq Ch^{2n-1} \int \prod_{j=1}^{3n} u^\xi(x_{\pi(2j-1)} - x_{\pi(2j-2)}) \prod_{j \in J^c} u^\xi(x_{\pi(2j)} - x_{\pi(2j-1)}) \prod_{j \in J} u^\xi(x_{\pi(2j)} - x_{\pi(2j-1)}) \prod_{j=1}^{2n} dx_j. \]

We can complete the set of \( n + 1 \) vectors in (4.29) to a basis of \( x_i, 1 \leq i \leq 2n \) by choosing \( n - 1 \) vectors from among the vectors appearing as arguments of \( u^\xi \) in the second line of (4.30). We then bound the remaining \( u^\xi \) factors by a constant, change variables and use (2.7) with \( q = 1 \) to see that the integral on the right hand side of (4.30) is bounded by \( Ch^{2(n+1)} \). Combining this with (4.30) proves (4.25).

### 4.3 \( a \neq e \)

We now claim that

\[ \sum \sum_{a \neq e} \left| \int \mathcal{T}_h(x; \pi, a) \prod_{j=1}^{2n} dx_j \right| = O(h^{4n+1}). \quad (4.31) \]

If \( \mathcal{T}_h(x; \pi, a) \) contains \( k < 3n \) factors of the form \( w^\xi \), then we will have \( 2(3n - k) \) factors of the form \( v^\xi \). We use (2.5) then to bound

\[ \left| \int \mathcal{T}_h(x; \pi, a) \prod_{j=1}^{2n} dx_j \right| \leq Ch^{2(3n-k)} \int \mathcal{I}_h(x; \pi, a) \prod_{j=1}^{2n} dx_j. \quad (4.32) \]

where \( \mathcal{I}_h(x; \pi, a) \) is similar to \( \mathcal{T}_h(x; \pi, a) \) except that we have bounded the integrand by its absolute value and replaced each \( v^\xi \) by \( u^\xi \). We now show how to get a good bound on the integral of \( \mathcal{I}_h(x; \pi, a) \).

Choose \( j_1, j_2, \ldots, j_{2n} \) such that

\[ \text{span} \ \{x_{\pi(j_1)}, x_{\pi(j_2)} - x_{\pi(j_2-1)}, \ldots, x_{\pi(j_{2n})} - x_{\pi(j_{2n}-1)}\} = \text{span} \ \{x_1, \ldots, x_{2n}\}. \quad (4.33) \]

It is easy to see that this can be done. We now show that we can choose a permutation \( \sigma_1, \sigma_2, \ldots, \sigma_{2n} \) of \([1, 2n]\) such that for any \( 1 \leq k \leq 2n \)

\[ x_{\pi(j_k)}, x_{\pi(j_k-1)} \in \{x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_k}\}. \quad (4.34) \]

We take \( \sigma_1 = \pi(j_1) = \pi(1) \) and choose the \( \sigma_2, \ldots, \sigma_{2n} \) by induction so that (4.34) holds for each \( 1 \leq k \leq 2n \). This clearly holds for \( k = 1 \),
since by definition $x_{\pi(0)} = 0$. Assume we have chosen $\sigma_1, \sigma_2, \ldots, \sigma_l$ so that (4.34) holds for all $k \leq l$. Then among the remaining \{ $x_{\pi(j_{i+1})} - x_{\pi(j_i)}$ $\} \{ x_{\pi(j_{i+1})} - x_{\pi(j_i)}$ \} there will be at least one $i$ such that either $x_{\pi(j_i)}$ or $x_{\pi(j_i - 1)}$ is equal to one of $x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_l}$. This is because each element of \{ $x_{\pi(j_{i+1})} - x_{\pi(j_i)}$ $\} \{ x_{\pi(j_{i+1})} - x_{\pi(j_i)}$ \} is a difference of $x$’s so that by themselves we could never have

$$\text{span} \{ x_{\pi(j_{i+1})} - x_{\pi(j_i)} \} \cup \{ x_{\pi(j_{i+1})} - x_{\pi(j_i)} \} = \text{span} (\{ x_1, \ldots, x_{2n} \} - \{ x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_l} \}). \quad (4.35)$$

We then take such an $i$ and if $x_{\pi(j_{i+1})}$ is equal to one of $x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_l}$ set $\sigma_{i+1} = \pi(j_{i+1} - 1)$, while if $x_{\pi(j_i)}$ is equal to one of $x_{\sigma_1, \sigma_2, \ldots, x_{\sigma_l}}$ set $\sigma_{i+1} = \pi(j_i)$. Then (4.34) holds for $k = l + 1$ and completes our induction.

We will prove (4.25) by first bounding the $dx_{\sigma_{2n}}$ integral in (4.26) involving all factors containing $x_{\sigma_{2n}}$. We then bound the $dx_{\sigma_{2n-1}}$ integral involving all remaining factors containing $x_{\sigma_{2n-1}}$. We then iterate this procedure bounding in turn the $dx_{\sigma_{2n}}, dx_{\sigma_{2n-1}}, \ldots, dx_{\sigma_1}$ integrals. (4.34) guarantees that at each stage we are integrating a non-empty product of bounded integrable functions. Note that by (2.6) and (2.7) with $q = 1$

$$\sup_{a_1} \int \prod_{i=1}^p w^\xi(y + a_i) \, dy \leq Ch^{p-1} \sup_{a_1} \int w^\xi(y + a_1) \, dy = O(h^{p+1}) \quad (4.36)$$

for all $p \geq 1$.

Let $p_i$ denote the number of remaining $w^\xi$ factors containing $x_{\sigma_i}$ after we have bounded in turn the $dx_{\sigma_{2n}}, dx_{\sigma_{2n-1}}, \ldots, dx_{\sigma_{i+1}}$ integrals, that is, $p_i$ denotes the number of $w^\xi$ factors containing $x_{\sigma_i}$ but not any of $x_{\sigma_{2n}}, x_{\sigma_{2n-1}}, \ldots, x_{\sigma_{i+1}}$. Since there are a total of $k < 3n$ factors of the form $w^\xi$ in (4.32), we have that $\sum_{i=1}^{2n} p_i = k$. Let $k_0 = |\{ i \mid p_i \neq 0 \}|$.

If we apply our bounding procedure using (4.36) together with the fact that $w^\xi$ is bounded and integrable we see that

$$\int \mathcal{I}_h(x; \pi, a) \prod_{j=1}^{2n} dx_j = O(h^{\sum_{i=1}^{2n} (p_i + 1)(p_i \neq 0)}) = O(h^{k + k_0}). \quad (4.37)$$

It is easy to see that in (4.32) each $x_j$ appears in at most 3 factors of the form $w^\xi$. Thus each $p_i \leq 3$. Since $\sum_{i=1}^{2n} p_i = k$, we must have $k_0 \geq k/3$. 27
Combining (4.37) with (4.32) we have
\[
\left| \int T_h(x; \pi, a) \prod_{j=1}^{2n} dx_j \right| \leq Ch^{2(3n-k)+4k/3} = Ch^{4n+2(n-k/3)}
\] (4.38)
which proves (4.31) since \( k < 3n \).

Combining (4.31), (note the factor \( 6^m = 36^n \)) with the results of Subsections 4.1-4.2 we have thus shown that
\[
E_\pi \left( \left( \int \left\{ (\Delta_h \lambda_x^x)^3 - 6u_{h,-h}(0)\lambda_x^x \Delta_h \lambda_x^x - 6 \left( u_{h,-h}^x \right)^2 L_x^x \right\} dx \right)^{2n} \right)
\]
\[
= \frac{(2n)!}{2^nn!} (192h^4)^n E \left\{ \left( \int (L_x^x)^3 dx \right)^n \right\} + O(h^{4n+1}).
\] (4.39)

We now explain why we obtain the same expression on the right hand side when we have \( E \) instead of \( E_\pi \). In the paragraph following (4.8) we make use of precise cancellations to handle bound variables, which a priori might be affected by our modification of (4.5). Consider how (4.9) would look if we had used the product formula (4.5). Note that any estimates we used will still apply since these estimates involve integrating or bounding by the supremum, neither of which are affected by replacing any of the \( x \)'s are replaced by \( x + h \). (4.20) will be affected, but note from (4.5) that the only terms of the form \( u^x(x - y) \) that may have \( x \) replaced by \( x \pm h \) are those to which \( \Delta_h^x \) is not applied. Similarly \( y \) may be replaced by \( y \pm h \) only if \( \Delta_h^y \) is not applied to a term of the form \( u^x(x - y) \). Consequently, we still have all terms of the form \( \Delta^h \Delta^{-h} u^x \). Thus we obtain (4.22), except that some of the remaining \( u^x(x - y) \) may be replaced by \( u^x(x - y \pm h) \). Using (2.5) then leads to (4.22).

Thus, it only remains to show that for each \( n \)
\[
E \left( \left( \int \left\{ (\Delta_h \lambda_x^x)^3 - 6u_{h,-h}(0)\lambda_x^x \Delta_h \lambda_x^x - 6 \left( u_{h,-h}^x \right)^2 L_x^x \right\} dx \right)^{2n+1} \right)
\]
\[
= O(h^{2n+1}).
\] (4.40)
This follows from the fact that we cannot form any \( \pi \)'s in \( \mathcal{G} \).

5 Proof of Lemma 5.1

We use \( E^{y,z}(\cdot) \) to denote expectation with respect to the independent Brownian motions \( B_t \) starting at \( y \) and \( \tilde{B}_t \) starting at \( z \).
Lemma 5.1 Let \( \lambda_\zeta, \lambda_{\zeta'} \) be independent exponential random variables with mean \( 1/\zeta, 1/\zeta' \). For each integer \( m \geq 0 \),
\[
\lim_{h \to 0} E_{y,z}^{y,z} \left( \left( \int \left\{ (\Delta^h_x L^x\lambda_\zeta)^2 - 2 \Delta^h_x \Delta^{-h} u^{\zeta}(0) L^x\lambda_\zeta \right\} \Delta^h_x \tilde{L}^x_{\lambda_{\zeta'}} \ dx \right)^m \right) (5.1)
\]
\[
\begin{cases}
\frac{(2n)!}{2^n n!} (64)^n E_{y,z}^{y,z} \left\{ \left( \int (L^x\lambda_\zeta)^2 \tilde{L}^x_{\lambda_{\zeta'}} \ dx \right)^n \right\} & \text{if } m = 2n \\
0 & \text{otherwise}
\end{cases}
\]
uniformly in \( y, z \).

**Proof of Lemma 5.1** For any integer \( m \) we have
\[
E_{y,z}^{y,z} \left( \left( \int \left\{ \left( \Delta^h_x L^x\lambda_\zeta \right)^2 - 2 \Delta^h_x \Delta^{-h} u^{\zeta}(0) L^x\lambda_\zeta \right\} \Delta^h_x \tilde{L}^x_{\lambda_{\zeta'}} \ dx \right)^m \right) (5.2)
\]
\[
= E_{y,z}^{y,z} \left( \prod_{i=1}^m \left( \int \left\{ \left( \Delta^h_x L^x_{\lambda_\zeta} \right)^2 - 2 \Delta^h_x \Delta^{-h} u^{\zeta}(0) L^x_{\lambda_\zeta} \right\} \Delta^h_x \tilde{L}^x_{\lambda_{\zeta'}} \ dx \ dx_i \right) \right)
\]
\[
= \sum_{A \subseteq \{1, \ldots, m\}} (-1)^{m-|A|} (2\Delta^h x^{-h} u^{\zeta}(0))^{|A|} \left( \prod_{i \in A} \int (\Delta^h_{x_i} L^x_{\lambda_\zeta})^2 \Delta^h_{x_i} \tilde{L}^x_{\lambda_{\zeta'}} \ dx_i \right) \left( \prod_{k \in A^c} \int L^x_{\lambda_\zeta} \Delta^h_{x_k} \tilde{L}^x_{\lambda_{\zeta'}} \ dx_k \right)
\]
\[
= E_{y,z}^{y,z} \left( \prod_{i \in A} \Delta^h_{x_i} L^x_{\lambda_\zeta} \Delta^h_{y_i} L^y_{\lambda_{\zeta'}} \Delta^h_{z_i} \tilde{L}^z_{\lambda_{\zeta'}} \prod_{k \in A^c} L^u_{\lambda_\zeta} \Delta^h_{v_k} \tilde{L}^v_{\lambda_{\zeta'}} \right) (5.3)
\]
and eventually we set \( y_j = x_j = z_j \) and \( u_j = v_j \) for all \( j \). Using (2.2) we have
\[
E_{y,z}^{y,z} \left( \prod_{i \in A} \Delta^h_{x_i} L^x_{\lambda_\zeta} \Delta^h_{y_i} L^y_{\lambda_{\zeta'}} \Delta^h_{z_i} \tilde{L}^z_{\lambda_{\zeta'}} \prod_{k \in A^c} L^u_{\lambda_\zeta} \Delta^h_{v_k} \tilde{L}^v_{\lambda_{\zeta'}} \right) (5.4)
\]
\[
= \left( \prod_{i \in A} \Delta^h_{x_i} \Delta^h_{y_i} \Delta^h_{z_i} \prod_{k \in A^c} \Delta^h_{v_k} \right) E_{y,z}^{y,z} \left( \prod_{i \in A} L^x_{\lambda_\zeta} L^y_{\lambda_{\zeta'}} \tilde{L}^z_{\lambda_{\zeta'}} \prod_{k \in A^c} L^u_{\lambda_\zeta} \tilde{L}^v_{\lambda_{\zeta'}} \right)
\]
\[
= \left( \prod_{i \in A} \Delta^h_{x_i} \Delta^h_{y_i} \Delta^h_{z_i} \prod_{k \in A^c} \Delta^h_{v_k} \right)
\]
there will not be any contributions from bound cases where if two $\Delta^h$'s are assigned to the same $u^c$ factor, it is not of the form $u^c(x_i - y_i)$. 

Thus in estimating (5.4) we need only consider $A = \{1, \ldots, m\}$ and those cases where if two $\Delta^h$'s are assigned to the same $u^c$ factor, it is not of the form $u^c(x_i - y_i)$. 

with $\sigma(0) = y$ and the second sum runs over all bijections 

$\sigma': [1, \ldots, m] \mapsto \{z_i, i \in A\} \cup \{v_i, i \in A^c\}$ 

with $\sigma'(0) = z$.

We then use the product to expand the right hand side of (5.4) into a sum of many terms, over all $\sigma$ and all ways to allocate each $\Delta^h_{x_i}, \Delta^h_{y_i}$ or $\Delta^h_{z_i}$ to a single $v$ factor.

Consider first the case where $A = \{1, \ldots, m\}$. For a given term in the above expansion, we will say that $x_i$ is bound if $x_i, y_i$ are adjacent, (in other words either $(x_i, y_i) = (\sigma(j), \sigma(j + 1))$ or $(y_i, x_i) = (\sigma(j), \sigma(j + 1))$ for some $j$), and both $\Delta^h_{x_i}$ and $\Delta^h_{y_i}$ are attached to the factor $u^c(x_i - y_i)$. Setting $x_i = y_i$ turns the factor $\Delta^h_{x_i} \Delta^h_{y_i} u^c(x_i - y_i)$ into $\Delta^h \Delta^{-h} u^c(0)$, and as in the proof of Lemma 4.1 we can assume that our use of (4.5) for $\Delta^h_{x_i}$ and $\Delta^h_{y_i}$ does not introduce a $\pm h$ in the arguments of other factors. For every such $\sigma$ there is precisely one other $\sigma$ which agrees with $\sigma$ except that it permutes $x_i, y_i$, we obtain a factor of $2\Delta^h \Delta^{-h} u^c(0)$. This is precisely what we would have obtained if instead of $\Delta^h_{x_i} \Delta^h_{y_i} L_{\lambda^c}^u L_{\hat{\lambda}^c}^u$ in (5.4) we had $2\Delta^h \Delta^{-h} u^c(0) L_{\hat{\lambda}^c}^u$. Consider then any term in the expansion of (5.4) with $A = \{1, \ldots, m\}$ and $J = \{i \mid x_i \text{ is bound}\}$. By (4.4) there will be an identical contribution from the last line of (5.4) for any other $A$ with $A^c \subseteq J$. Since $\sum_{A^c \subseteq J} (-1)^{|A^c|} = 0$, we see that in the expansion of (5.2) there will not be any contributions from bound $x$'s. Furthermore, this completely exhausts the contribution to (5.2) of all $A \neq \{1, \ldots, m\}$.

Thus in estimating (5.4) we need only consider $A = \{1, \ldots, m\}$ and those cases where if two $\Delta^h$'s are assigned to the same $u^c$ factor, it is not of the form $u^c(x_i - y_i)$.
Again as in the proof of Lemma 4.1, in the following calculation we first replace the right hand side of (4.5) by \( \{ \Delta^h f(x) \} g(x) + f(x) \{ \Delta^h g(x) \} \), and return at the end of the proof to explain why this doesn’t affect the final result. We use the notation

\[
E_{y,z}^\pi \left( \left( \int \left\{ \left( \Delta^h x \right)^2 - 2 \Delta^h \Delta^{-h} u_{\lambda}(0) L^x_{\lambda} \right) \Delta^h L^x_{\lambda'} \ dx \right\} \right)^m \] (5.5)

\[
to denote the expression obtained with this replacement.

We can thus write

\[
E_{y,z}^\pi \left( \left( \int \left\{ \left( \Delta^h x \right)^2 - 2 \Delta^h \Delta^{-h} u_{\lambda}(0) L^x_{\lambda} \right) \Delta^h L^x_{\lambda'} \ dx \right\} \right)^m \] (5.6)

\[
= 2^m \sum_{\pi,a} \sum_{\pi',a'} \int T_h(x; \pi, \pi', a, a') \ dx
\]

with

\[
T_h(x; \pi, \pi', a, a')
\]

\[
= 2^m \prod_{j=1}^{2m} \left( \Delta^h_{x_{\pi(j)}} \right) a_{1(j)} \left( \Delta^h_{x_{\pi(j-1)}} \right) a_{2(j)} u_{\lambda}(x_{\pi(j)} - x_{\pi(j-1)})
\]

\[
\times \prod_{j=1}^{m} \left( \Delta^h_{x_{\pi(j)}} \right) a'_{1(j)} \left( \Delta^h_{x_{\pi(j-1)}} \right) a'_{2(j)} u_{\lambda'}(x_{\pi'(j)} - x_{\pi'(j-1)})
\]

where the first sum runs over all maps \( \pi : [1, \ldots, 2m] \mapsto [1, \ldots, m] \) with \( |\pi^{-1}(i)| = 2 \) for each \( i \), and all ‘assignments’ \( a = (a_1, a_2) : [1, \ldots, 2m] \mapsto \{0, 1\} \times \{0, 1\} \) with the property that for each \( i \) there will be exactly two factors of the form \( \Delta^h_{x_i} \) in the second line of (5.7), and if \( a(j) = (1, 1) \) for any \( j \), then \( x_{\pi(j)} \neq x_{\pi(j-1)} \). The factor \( 2^m \) in (5.6) comes from the fact that \( |\pi^{-1}(i)| = 2 \) for each \( i \). Similarly, the second sum runs over all permutations \( \pi' : [1, \ldots, m] \mapsto [1, \ldots, m] \), and all ‘assignments’ \( a' = (a'_1, a'_2) : [1, \ldots, 2m] \mapsto \{0, 1\} \times \{0, 1\} \) with the property that for each \( i \) there will be exactly one factor of the form \( \Delta^h_{x_i} \) in the last line of (5.7). Here we have set \( x_{\pi(0)} = y, x_{\pi'(0)} = z \).

From this point on the proof is very similar to that of Lemma 4.1. Let \( m = 2n \). Assume first that \( a = e \) where now \( e(2j) = (1, 1), e(2j - 1) = (0, 0) \) for all \( j \), and similarly for \( a' \).
5.1 \( a = a' = e \) with \( \pi, \pi' \) compatible with a pairing

When \( a = a' = e \) we have

\[
T_h(x; \pi, \pi', e, e) = \prod_{j=1}^{2n} u^\xi(x_{\pi(2j-1)} - x_{\pi(2j-2)}) \Delta^h \Delta^{-h} u^\xi(x_{\pi(2j)} - x_{\pi(2j-1)})
\]

\[
\times \prod_{j=1}^{n} u^\zeta(x_{\pi'(2j-1)} - x_{\pi'(2j-2)}) \Delta^h \Delta^{-h} u^\zeta(x_{\pi'(2j)} - x_{\pi'(2j-1)}).
\]  \( (5.8) \)

Let \( P = \{(l_{2i-1}, l_{2i}) \mid 1 \leq i \leq n\} \) be a pairing of the integers \([1, 2n] \). Let \( \pi \) as in \((5.5)\) be such that for each \( 1 \leq j \leq 2n \), \( \{\pi(2j-1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\} \) for some, necessarily unique, \( 1 \leq i \leq n \). In this case we say that \( \pi \) is compatible with the pairing \( P \) and write this as \( \pi \sim P \).

Similarly we say that \( \pi' \) is compatible with the pairing \( P \) if for each \( 1 \leq j \leq n \), \( \{\pi'(2j-1), \pi'(2j)\} = \{l_{2i-1}, l_{2i}\} \) for some, necessarily unique, \( 1 \leq i \leq n \), and write this as \( \pi' \sim P \). If \( \pi, \pi' \sim P \) we have

\[
T_h(x; \pi, \pi', e, e)
\]

\[
= \prod_{i=1}^{n} \left( \Delta^h \Delta^{-h} u^\xi(x_{l_{2i}} - x_{l_{2i-1}}) \right) \prod_{j=1}^{2n} u^\xi(x_{\pi(2j-1)} - x_{\pi(2j-2)})
\]

\[
\times \prod_{i=1}^{n} \Delta^h \Delta^{-h} u^\zeta(x_{l_{2i}} - x_{l_{2i-1}}) \prod_{j=1}^{n} u^\zeta(x_{\pi'(2j-1)} - x_{\pi'(2j-2)}).
\]  \( (5.9) \)

Set \( \sigma(j) = i \) when \( \{\pi(2j-1), \pi(2j)\} = \{l_{2i-1}, l_{2i}\} \), so that \( \sigma : [1, 2n] \mapsto [1, n] \) with \( \sigma^{-1}(i) \) = 2 for each \( 1 \leq i \leq n \). Similarly, set \( \sigma'(j) = i \) when \( \{\pi'(2j-1), \pi'(2j)\} = \{l_{2i-1}, l_{2i}\} \), so that that \( \sigma' \) is a permutation of \([1, n]\).

As in Sub-section \(4.1\) we can show that

\[
\int T_h(x; \pi, \pi', e, e) \prod_{j=1}^{2n} dx_j
\]

\[
= 4^nh^{4n} \prod_{j=1}^{2n} u^\xi(y_{\sigma(j)} - y_{\sigma(j-1)}) \prod_{j=1}^{n} u^\zeta(y_{\sigma'(j)} - y_{\sigma'(j-1)}) \prod_{k=1}^{n} dy_k + O(h^{4n+1})
\]

with \( y_{\sigma(0)} = y, y_{\sigma'(0)} = z \) and error term uniform in \( y, z \).

Let \( M_d \) denote the set of maps \( \sigma \) from \([1, \ldots, dn]\) to \([1, \ldots, n]\) such that \( |\sigma^{-1}(i)| = d \) for all \( i \). For each pairing \( P \), each map \( \pi \sim P \) gives rise as above to a map \( \sigma \in M_2 \). Also, any of the \( 2^n \) \( \pi \)'s obtained by
permuting the 2 elements in each of the 2n pairs, give rise to the same σ. In addition, for any \( \hat{\sigma} \in \mathcal{M}_2 \), we can reorder the 2n pairs of \( \pi \) to obtain a new \( \hat{\pi} \sim \mathcal{P} \) which gives rise to \( \hat{\sigma} \). A similar analysis applies to our \( \pi' \).

Thus we have shown that

\[
\sum_{\pi, \pi' \sim \mathcal{P}} \int \mathcal{T}_h(x; \pi, \pi', e, e) \prod_{j=1}^{2n} dx_j
\]

\[
= (32h^4)^n \sum_{\sigma \in \mathcal{M}_2, \sigma' \in \mathcal{M}_1} \int \prod_{j=1}^{2n} u^c(y_{\sigma(j)} - y_{\sigma(j-1)}) \prod_{k=1}^n dy_k + O(h^{4n+1})
\]

\[
= (16h^4)^n E^{y,z} \left\{ \left( \int (L_{\lambda_c}^x)^2 L_{\lambda_c'}^x dx \right)^n \right\} + O(h^{4n+1})
\]

where the last line follows from Kac’s moment formula. To complete this subsection, let \( \mathcal{G} \) denote the set of \( \pi, \pi' \) which are compatible with some pairing \( \mathcal{P} \). Since there are \( \frac{(2n)!}{2^n n!} \) pairings of \([1, \ldots, 2n]\), we have shown that

\[
\sum_{\pi, \pi' \in \mathcal{G}} \int \mathcal{T}_h(x; \pi, \pi', e, e) \prod_{j=1}^{2n} dx_j
\]

\[
= \frac{(2n)!}{2^n n!} (16h^4)^n E^{y,z} \left\{ \left( \int (L_{\lambda_c}^x)^2 L_{\lambda_c'}^x dx \right)^n \right\} + O(h^{4n+1}).
\]

The rest of the proof follows as in the proof of Lemma 4.1.

6 Proof of Lemma 6.1

Lemma 6.1 For each integer \( n \geq 0 \),

\[
\lim_{h \to 0} E \left( \left( \frac{\int \Delta_h^x L_{t_1}^x \left( \Delta_h^x L_{t_2}^x \circ \theta_{t_3} \right) \Delta_h^x \bar{L}_{t_3}^x dx}{h^2} \right)^{2n} \right)
\]

\[
= \frac{(2n)!}{2^n n!} (64)^n E \left\{ \left( \int L_{t_1}^x \left( L_{t_2}^x \circ \theta_{t_3} \right) \bar{L}_{t_3}^x dx \right)^n \right\},
\]

locally uniformly in \( t_1, t_2, t_3 \) on \( t_1 > 0 \).

Proof of Lemma 6.1 The proof of this Lemma is easier than that of Lemmas 4.1 and 5.1 since there are no subtraction terms. However, there
are complications due to the fact that we now work with non-random times \( t_1, t_2, t_3 \) rather than exponential times.

We begin by writing

\[
E \left( \left( \int \Delta_h x_{t_1} \left( \Delta_h x_{t_2} \circ \theta_{t_1} \right) \Delta_h \tilde{x}_{t_3} \, dx \right)^{2n} \right) \tag{6.2}
\]

\[
= E \left( \prod_{i=1}^{2n} \left( \int \Delta_h x_{t_1} \left( \Delta_h x_{t_2} \circ \theta_{t_1} \right) \Delta_h x_{t_3} \, dx_i \right) \right)
\]

\[
= \int E \left( \prod_{i=1}^{2n} \left( \Delta_h x_{t_1} \left( \Delta_h x_{t_2} \circ \theta_{t_1} \right) \Delta_h \tilde{x}_{t_3} \right) \right) 2n \, dx_i
\]

We first evaluate

\[
E \left( \prod_{i=1}^{2n} \left( \Delta_h x_{t_1} \left( \Delta_h y_{t_2} \circ \theta_{t_1} \right) \Delta_h \tilde{z}_{t_3} \right) \right) \tag{6.3}
\]

\[
= \prod_{i=1}^{2n} \left( \Delta_h x_{t_1} \Delta_h y_{t_2} \Delta_h \tilde{z}_{t_3} \right) \tag{6.4}
\]

\[
= \prod_{i=1}^{2n} \left( \Delta_h x_{t_1} \Delta_h y_{t_2} \Delta_h \tilde{z}_{t_3} \right)
\]

and then set all \( y_i = z_i = x_i \).

By Kac’s moment formula

\[
E \left( \prod_{j=1}^{2n} L_{t_1}^{x_i} \left( L_{t_2}^{y_i} \circ \theta_{t_1} \right) \right) \tag{6.5}
\]

\[
= \sum_{\pi_1, \pi_2} \int \left( \sum_{j=1}^{2n} r_{1,j} \leq t_1 \right) \prod_{j=1}^{2n} p_{r_{1,j}}(x_{\pi_1(j)} - x_{\pi_1(j-1)})
\]

\[
\int \left( \sum_{j=1}^{2n} r_{2,j} \leq t_2 \right) p_{r_{2,1} + \left( t_1 - \sum_{j=1}^{2n} r_{1,j} \right)}(y_{\pi_2(1)} - x_{\pi_1(2n)})
\]

\[
\prod_{j=2}^{2n} p_{r_{2,j}}(y_{\pi_2(j)} - y_{\pi_2(j-1)}) \prod_{j=1}^{2n} dr_{1,j} \, dr_{2,j}
\]

and

\[
E \left( \prod_{i=1}^{2n} \tilde{L}_{t_3}^{z_i} \right) \tag{6.5}
\]
\[
\sum_{\pi_3} \int_{\{\sum_{j=1}^{2n} r_{3,j} \leq t_3\}} \prod_{j=1}^{2n} p_{r_{3,j}}(z_{\pi_3(j)} - z_{\pi_3(j-1)}) \, dr_{3,j}
\]

where the sums run over all permutations \(\pi_j\) of \(\{1, \ldots, 2n\}\) and we also define \(\pi_j(0) = 0\) and \(x_0 = z_0 = 0\).

We then use the product rule (4.5) as before, to expand the right hand side of (6.3) into a sum of many terms, and then setting \(y_i = z_i = x_i\) we obtain:

\[
E \left( \prod_{i=1}^{2n} \left( \Delta_{x_i}^h L_{t_i}^{x_i} \left( \Delta_{x_i}^h L_{t_i}^{x_i} \circ \theta_{t_i} \right) \Delta_{x_i}^h \bar{L}_{t_i}^{x_i} \right) \right) = \sum_{\pi, a} \int T^*_h(x; \pi, a) \, dx
\]

where \(x = (x_1, \ldots, x_{2n}), \pi = (\pi_1, \pi_2, \pi_3), a = (a_1, a_2, a_d)\) and

\[
T^*_h(x; \pi, a) = \prod_{d=1}^{3} \int_{\mathcal{R}_d} \prod_{j=1}^{2n} \left( \left( \Delta_{x_{\pi_d(j)}}^h \right)^{a_{d,1}(j)} \left( \Delta_{\bar{x}_{\pi_d(j-1)}}^h \right)^{a_{d,2}(j)} \right) p_{r_{d,j}}(x_{\pi_d(j)} - x_{\pi_d(j-1)}) \prod_{j=1}^{m} dr_{d,j}.
\]

In (6.6) the sum runs over all triples of permutations \((\pi_1, \pi_2, \pi_3)\) and all \(a_d = (a_{d,1}, a_{d,2}) : [1, \ldots, 2n] \rightarrow \{0, 1\} \times \{0, 1\}\), with the restriction that for each \(d, i\) there is exactly one factor of the form \(\Delta_{x_{\pi_d(i)}}^h\). (Here we define \((\Delta_x^h)^0 = 1\) and \((\Delta_x^h)^1 = 1\). We have also set \(\pi_1(0) = \pi_3(0) = 0\) and \(\pi_2(0) = \pi_3(2n)\). In (6.7) we set \(\mathcal{R}_d = \{\sum_{j=1}^{2n} r_{d,j} \leq t_d\}\), \(p_r(x)\) may be either \(p_r(x), p_r(x + h)\) or \(p_r(x - h)\), and \(\bar{r}_{d,j} = r_{d,j}\) unless \(d = 2, j = 1\) in which case \(\bar{r}_{2,1} = r_{2,1} + (t_1 - \sum_{j=1}^{2n} r_{1,j})\). It is important to recognize that in the right hand side of (6.7) each difference operator is applied to only one of the terms \(p_r(\cdot)\).

Instead of (6.7) we first analyze

\[
T_h(x; \pi, a) = \prod_{d=1}^{3} \int_{\mathcal{R}_d} \prod_{j=1}^{2n} \left( \left( \Delta_{x_{\pi_d(j)}}^h \right)^{a_{d,1}(j)} \left( \Delta_{\bar{x}_{\pi_d(j-1)}}^h \right)^{a_{d,2}(j)} \right) p_{r_{d,j}}(x_{\pi_d(j)} - x_{\pi_d(j-1)}) \prod_{j=1}^{m} dr_{d,j}.
\]

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This differs from (6.7) in that we have replaced all \( p_i(x) \) by \( p_r(x) \). At
before it will be seen that this has no effect on the asymptotics.

As before, we first consider the case that \( a_1 = a_2 = a_3 = e \) where
\( e = (e(1), \ldots, e(2n)) \) and \( e(2j) = (1, 1), e(2j - 1) = (0, 0), j = 1 \ldots n \).
Let \( \mathcal{P} = \{(l_{2i-1}, l_{2i}), 1 \leq i \leq n\} \) be a pairing of the integers [1, 2n]. Let
\( \pi_1, \pi_2, \pi_3 \) be permutations of [1, 2n] such that for each \( 1 \leq d \leq 3, 1 \leq j \leq n \), \( \{\pi_d(2j - 1), \pi_d(2j)\} \) = \( \{l_{2i-1}, l_{2i}\} \) for some, necessarily unique,
\( 1 \leq i \leq n \). In this case we say that \( \pi \) is compatible with the pairing \( \mathcal{P} \).
( Note that \( \{\pi_d(2j - 1), \pi_d(2j)\} \) is not necessarily the same for each \( d \).)
Then by (6.8)

\[
\mathcal{T}_h(x; \pi, e) = \prod_{d=1}^{3} \int_{\sum_{j=1}^{n} r_{d,j} + s_{d,j} \leq t_{d}} \prod_{j=1}^{n} \left( \Delta^h \Delta^{-h} p_{r_{d,j}}(x_{\pi_d(2j)} - x_{\pi_d(2j-1)}) \right) dr_{d,j}
\]

\[
\times \prod_{j=1}^{n} p_{s_{d,j}}(x_{\pi_d(2j-1)} - x_{\pi_d(2j-2)}) ds_{d,j}.
\]

where again \( s_{d,j} = s_{d,j} \) unless \( (d, j) = (2, 1) \) in which case we have \( s_{2,1} = s_{2,1} + (t_1 - \sum_{j=1}^{n} r_{1,j} + s_{1,j}) \).

Set \( \sigma_d(j) = i \) when \( \{\pi_d(2j - 1), \pi_d(2j)\} = \{l_{2i-1}, l_{2i}\} \). Using the
approach of Sub-section 4.1 together with the estimates of Lemma 2.2 in
place of the estimates of Lemma 2.1 (in estimating error terms we take
absolute values of all integrands and extend the time integration of each
term to [0, T] with \( T = 2 \max(t_1, t_2, t_3) \)) we can show that

\[
\int \mathcal{T}_h(x; \pi, e) \prod_{j=1}^{2n} dx_j = \int \mathcal{T}_h(x; \pi, e) \prod_{j=1}^{2n} dx_j + O(h^{4n+1/2}) \tag{6.10}
\]

where

\[
\mathcal{T}_h(x; \pi, e) = \prod_{d=1}^{3} \int_{\sum_{j=1}^{n} r_{d,j} + s_{d,j} \leq t_{d}} \prod_{i=1}^{n} \Delta^h \Delta^{-h} p_{r_{d,i}}(x_{l_{2i}} - x_{l_{2i-1}}) dr_{d,i}
\]

\[
\times \prod_{j=1}^{n} p_{s_{d,j}}(x_{l_{2\sigma_d(j)-1}} - x_{l_{2\sigma_d(j-1)-1}}) ds_{d,j} \tag{6.11}
\]

The fact that the error term in (6.10) is \( O(h^{4n+1/2}) \) and not \( O(h^{4n+1}) \) is
due to the fact that we use (2.16) instead of (2.8).
Let $\tilde{A}_h(\pi, e)$ denote the integral on the right hand side of (6.10) so that

$$
\tilde{A}_h(\pi, e) = \int \prod_{d=1}^{3} \prod_{j=1}^{n} \left\{ \sum_{r_{d,j} + s_{d,j} \leq t_d} \Delta^h \Delta^{-h} p_{r_{d,i}}(x_{l_{2i}} - x_{l_{2i-1}}) dr_{d,i} \right\} \times \prod_{j=1}^{n} p_{s_{d,j}}(x_{l_{2\sigma_d(j)-1}} - x_{l_{2\sigma_d(j-1)-1}}) ds_{d,j} \prod_{i=1}^{2n} dx_i.
$$

We make the change of variables $x_{l_{2i}} \rightarrow x_{l_{2i}} + x_{l_{2i-1}}, \ i = 1, \ldots, n$ and write this as

$$
\tilde{A}_h(\pi, e) = \int \prod_{d=1}^{3} \prod_{j=1}^{n} \left\{ \sum_{r_{d,j} + s_{d,j} \leq t_d} \Delta^h \Delta^{-h} p_{r_{d,i}}(x_{l_{2i}}) dr_{d,i} \right\} \times \prod_{j=1}^{n} p_{s_{d,j}}(x_{l_{2\sigma_d(j)-1}} - x_{l_{2\sigma_d(j-1)-1}}) ds_{d,j} \prod_{i=1}^{2n} dx_i.
$$

We now rearrange the integrals with respect to $x_2, x_4, \ldots, x_{2n}$ and get

$$
\tilde{A}_h(\pi, e) = \int \left( \int \prod_{d=1}^{3} \prod_{j=1}^{n} \Delta^h \Delta^{-h} p_{r_{d,i}}(x) dx \right) \prod_{d=1}^{3} \prod_{j=1}^{n} p_{s_{d,j}}(x_{l_{2\sigma_d(j)-1}} - x_{l_{2\sigma_d(j-1)-1}}) ds_{d,j} dr_{d,i} \prod_{i=1}^{n} dx_{l_{2i-1}}.
$$

Let

$$
F(\sigma; s)
$$

$$
:= \int \prod_{d=1}^{3} \prod_{j=1}^{n} p_{s_{d,i}}(x_{l_{2\sigma_d(j)-1}} - x_{l_{2\sigma_d(j-1)-1}}) \prod_{i=1}^{n} dx_{l_{2i-1}}
$$

$$
= \int \prod_{d=1}^{3} \prod_{j=1}^{n} p_{s_{d,i}}(y_{\sigma_d(j)} - y_{\sigma_d(j-1)}) \prod_{i=1}^{n} dy_i,
$$

where we set $y_i = x_{l_{2i-1}}$. We can now write

$$
\tilde{A}_h(\pi, e)
$$
\[
\mathcal{A}_h(\pi, e) \leq \left( \int_{[0,\infty]^3} \left( \int \prod_{d=1}^{3} \left( \Delta^h \Delta^{-h} p_{r_d}(x) \right) \, dx \right) \, dr_d \right)^n \prod_{d=1}^{3} \prod_{i=1}^{n} ds_{d,i} \]

Using \(2 - e^{ih\lambda} - e^{-ih\lambda} = 2 - 2\cos(\lambda h) = 4\sin^2(\lambda h/2)\) we can write

\[
G_h(r) =: \prod_{d=1}^{3} \left( \Delta^h \Delta^{-h} p_{r_d}(x) \right) \, dx
\]

\[
= \int \left( \prod_{d=1}^{3} \left( \frac{1}{2\pi} \int e^{ix\lambda_{d,i}} \left( 2 - e^{ih\lambda_{d,i}} - e^{-ih\lambda_{d,i}} \right) e^{-r_d, i \lambda_{d,i}^2/2} \, d\lambda_{d,i} \right) \right) \, dx
\]

\[
= \left( \frac{4}{2\pi} \right)^{3} \int \left( \int e^{ix} \prod_{d=1}^{3} \lambda_{d,i} \prod_{d=1}^{3} \sin^2(\lambda_{d,i}h/2) e^{-r_d, i \lambda_{d,i}^2/2} \, d\lambda_{d,i} \right) \, dx
\]

\[
= \left( \frac{4}{2\pi} \right)^{3} \int \left( \int e^{ix} \prod_{d=2}^{3} \lambda_{d,i} \left( \int e^{ix\lambda_{d,i}} \sin^2(\lambda_{d,i}h/2) e^{-r_1, i \lambda_{d,i}^2/2} \, d\lambda_{d,i} \right) \prod_{d=2}^{3} \sin^2(\lambda_{d,i}h/2) e^{-r_d, i \lambda_{d,i}^2/2} \, d\lambda_{d,i} \right) \, dx
\]

with \(\lambda_{1,i} =: -\sum_{d=2}^{3} \lambda_{d,i}\) in the last equality. For the last equality we used Fourier inversion.

Since \(G_h, F \geq 0\), we have the following upper and lower bounds for \(\tilde{A}_h(\pi, e)\)

\[
\tilde{A}_h(\pi, e)
\]

and

\[
\tilde{A}_h(\pi, e)
\]

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\[
\geq \left( \int_{[0,h]^3} \left( \int \prod_{d=1}^3 \left( \Delta^h \Delta^{-h} p_{rd}(x) \right) \, dx \right) \, dr_d \right)^n
\]
\[
\times \int_{\{\sum_{i=1}^n s_{d,i} \leq t_d - nh, \forall d\}} F(\sigma; s) \prod_{d=1}^3 n \prod_{i=1}^n ds_{d,i}.
\]

We show that the two sides of the inequalities are asymptotically equivalent as \( h \to 0 \). The following Lemma is proven below.

**Lemma 6.2**

\[
\int_{\{\sum_{i=1}^n s_{d,i} \leq t_d, \forall d\}} F(\sigma; s) \prod_{d=1}^3 n \prod_{i=1}^n ds_{d,i}
\]
\[
- \int_{\{\sum_{i=1}^n s_{d,i} \leq t_d - nh, \forall d\}} F(\sigma; s) \prod_{d=1}^3 n \prod_{i=1}^n ds_{d,i} \leq C_T h.
\]

Referring to (6.10), using (2.20)-(2.21) we see that

\[
\int T_h(x; \pi, e) \prod_{j=1}^{2n} dx_j = \tilde{A}_h(\pi, e) + O(\theta^{4n+1/2})
\]
\[
= (8\theta)^n \int_{\{\sum_{i=1}^n s_{d,i} \leq t_d, \forall d\}} F(\sigma; s) \prod_{d=1}^3 n \prod_{i=1}^n ds_{d,i} + O(\theta^{4n+1/2})
\]
\[
= (8\theta)^n \int \left( \prod_{d=1}^3 \int_{\{\sum_{i=1}^n s_{d,i} \leq t_d\}} \prod_{i=1}^n p_{s_{d,i}}(y_{\sigma_d(i)} - y_{\sigma_d(i-1)}) \prod_{i=1}^n ds_{d,i} \right) \prod_{i=1}^n dy_i
\]
\[
\quad + O(\theta^{4n+1/2}).
\]

Recall that, in the paragraph containing (6.9), for a given pairing \( \mathcal{P} = \{(l_{2i-1}, l_{2i}) \mid 1 \leq i \leq n\} \) of the integers \([1, 2n]\), we define what it means for a collection of permutations \( \pi = (\pi_1, \pi_2, \pi_3) \) of \([1, 2n]\) to be compatible with \( \mathcal{P} \). We write this as \( (\pi_1, \pi_2, \pi_3) \sim \mathcal{P} \). Obviously, there are many such pairs. We can interchange the two elements of the pair \( \pi_d(2j-1), \pi_d(2j) \) without changing (6.18). There are \( 2^{3n} \) ways to do this. Furthermore, by permuting the pairs \( \{\pi_d(2j-1), \pi_d(2j)\} \) we give rise in (6.18) to all possible permutations \( \sigma_d \) of \([1, n]\). We thus obtain

\[
\sum_{(\pi_1,\pi_2,\pi_3) \sim \mathcal{P}} \int T_h(x; \pi, e) \prod_{j=1}^{2n} dx_j
\]
\[
(2^6 h^4)^n \sum_{\sigma} \int \left( \prod_{d=1}^{3} \int_{\{s_{d,i} \leq t_d\}} \prod_{i=1}^{n} p_{s_{d,i}} (y_{\sigma_d(i)} - y_{\sigma_d(i-1)}) \prod_{i=1}^{n} ds_{d,i} \right) \prod_{i=1}^{n} dy_{i} + O(h^{4n+1/2})
\]

\[
= (64 h^4)^n E \left\{ \left( \int L_{t_1}^{x} (L_{t_2}^{x} \circ \theta_{t_1}) \bar{L}_{t_3}^{x} dx \right)^n \right\} + O(h^{4n+1/2}).
\] (6.19)

Here the sum in the second line runs over all permutations \(\sigma = (\sigma_1, \ldots, \sigma_q)\) of \(\{1, \ldots, n\}\) and we set \(\sigma_d(0) = 0\). The fourth line follows from Kac’s moment formula.

Since there are \((2n)! / (2^n n!)\) pairings of the \(2n\) elements \(\{1, \ldots, m = 2n\}\) we see that

\[
\sum_{P} \sum_{(\pi_1, \pi_2, \pi_3) \sim P} \int \mathcal{T}_h(x; \pi, e) \prod_{j=1}^{2n} dx_j
\]

\[
= \frac{(2n)!}{2^n n!} (64 h^4)^n E \left\{ \left( \int L_{t_1}^{x} (L_{t_2}^{x} \circ \theta_{t_1}) \bar{L}_{t_3}^{x} dx \right)^n \right\} + O(h^{4n+1/2})
\] (6.20)

where the first sum runs over all pairings \(P\) of \(\{1, \ldots, 2n\}\).

Given the estimates of Lemma 2.2 we can show as in Section 4 that the contributions to (6.6) from \((a_1, a_2, a_3) = (e, e, e)\) for \(\pi\) not compatible with a pairing is \(O(h^{4n+1/2})\). The arguments of Section 4 will give a similar bound for \((a_1, a_2, a_3) \neq (e, e, e)\) with one possible exception. This will happen if \(\pi_2(1) = \pi_1(2n)\) so that the argument of the term \(p_{s_{2,1}}\) is zero, and two \(\Delta\) operators are applied to this \(p\). In that case, since \(\Delta^h \Delta^{-h} p_{s_{2,1}} (0) = 2 \Delta^h p_{s_{2,1}} (0)\), we seem to have lost one \(\Delta\) operator, all of which are used in Section 4 to obtain the required error estimate. The remedy will be found in the special nature of \(s_{2,1}\) as we now explain.

Instead of extending the time integration of each term to \([0, T]\), we first consider the region where \(\sum_{j=1}^{n} r_{1,j} + s_{1,j} \leq t_1/2\) so that \(\bar{s}_{2,1} \geq t_1/2\). Then

\[
|\Delta^h p_{s_{2,1}} (0)| = \frac{c |e^{-h^2/2 \bar{s}_{2,1} - 1}|}{\bar{s}_{2,1}^{1/2}} \leq \frac{ch^2}{\bar{s}_{2,1}^{3/2}} \leq c(t_1) h^2.
\] (6.21)

We then extend the time integration of each term to \([0, T]\) and proceed as before. On the other hand, if \(\sum_{j=1}^{n} r_{1,j} + s_{1,j} \geq t_1/2\), then for some \(j\) we have either \(r_{1,j} \geq \delta =: t_1/(4n)\) or \(s_{1,j} \geq \delta\). Say it is the latter. We
then use the $s_{1,j}$ integration for the bound, see (6.21),

$$
\int_0^T \int_0^T \left| \Delta^h p_{s_{2,1}+s_{1,j}}(0) \right| ds_{2,1} ds_{1,j} \leq ch^2 \int_0^T \int_0^T \frac{1}{(s_{2,1} + s_{1,j})^{3/2}} ds_{2,1} ds_{1,j} \leq ch^2
$$

and bound the other term involving $s_{1,j}$, be it $p_{s_{1,j}}(x), |\Delta^h p_{s_{1,j}}(x)|$ or $|\Delta^h \Delta^{-h} p_{s_{1,j}}(x)|$, by its supremum over $s_{1,j} \geq \delta$, using Lemma 2.3. 

Proof of Lemma 6.2.

Let $A \subseteq [0, t_1]^n \times [0, t_2]^n \times [0, t_3]^n$ and set

$$I(A) = \int_A F(\sigma; s) \prod_{d=1}^n \prod_{i=1}^n ds_{d,i}. \tag{6.23}$$

To prove Lemma 6.2 it suffices to show that

$$I(A) \leq C_T |A|^{1/2}. \tag{6.24}$$

We have

$$I(A) = \int_A \left( \int \prod_{d=1}^n \prod_{j=1}^n p_{s_{d,i}}(y_{\sigma_d(j)} - y_{\sigma_d(j-1)}) \prod_{i=1}^n dy_i \right) \prod_{d=1}^n \prod_{i=1}^n ds_{d,i}$$

$$= \int \left( \int_A \prod_{d=1}^n \prod_{j=1}^n p_{s_{d,i}}(y_{\sigma_d(j)} - y_{\sigma_d(j-1)}) ds_{d,i} \right) \prod_{i=1}^n dy_i.$$

Then by the Cauchy-Schwarz inequality

$$I(A) \leq |A|^{1/2} \int \left( \int_{[0,T]^n} \prod_{d=1}^n \prod_{j=1}^n p_{s_{d,i}}^2(y_{\sigma_d(j)} - y_{\sigma_d(j-1)}) ds_{d,i} \right)^{1/2} \prod_{i=1}^n dy_i$$

$$\leq |A|^{1/2} e^{3nT} \int \left( \prod_{d=1}^n \prod_{j=1}^n f(y_{\sigma_d(j)} - y_{\sigma_d(j-1)}) \right)^{1/2} \prod_{i=1}^n dy_i$$

where

$$f(y) = \int_0^\infty e^{-s} p_s^2(y) \, ds. \tag{6.27}$$
Since \( f(y) \) is the 1-potential density of planar Brownian motion evaluated at \((\sqrt{2} y, 0)\), we know that \( f(y) \) has a logarithmic singularity at \( y = 0 \) and has exponential falloff at \( \infty \) so that the last integral in (6.26) is finite.

7 Proof of Lemma 7.1

Lemma 7.1 Fix \( T < \infty \). For all \( s, t \leq T \)

\[
E \left[ \left( \int \left\{ \left( \Delta^h_t L_t^x \right)^2 - 4h L_t^x \right\} \Delta^h_s L_s^x \, dx \right)^2 \right] = 32h^4 E \left( \int (L_t^x)^2 L_s^x \, dx \right) + O \left( (s \land t)^{h^4+\epsilon} \right). \tag{7.1}
\]

Proof of Lemma 7.1: In order to prove (7.1) we must make use of the subtraction on the left hand side to eliminate all terms which are not \( O(\hbar^4) \), then isolate the main contribution which is the first term on the right hand side, and estimate all error terms. As we will see the terms which are not \( O(\hbar^4) \) come from ‘bound’ variables. Because we are not using exponential times, the subtractions do not exactly eliminate all bound variables, which makes the analysis more complicated than in previous sections.

We first write

\[
E \left[ \left( \int \left\{ \left( \Delta^h_t L_t^x \right)^2 - 4h L_t^x \right\} \Delta^h_s L_s^x \, dx \right)^2 \right] = I_1 - 8h I_2 + 16h^2 I_3 \tag{7.2}
\]

where

\[
I_1 = E \left[ \left( \int \left( \Delta^h_t L_t^x \right)^2 \Delta^h_s L_s^x \, dx \right)^2 \right]
\]

\[
= E \left[ \int \left( \Delta^h_t L_t^x \right)^2 \Delta^h_s L_s^x \, dx \int \left( \Delta^h_y L_y^x \right)^2 \Delta^h_y L_y^x \, dy \right]
\]

\[
= \int \int E \left( \left( \Delta^h_t L_t^x \right)^2 \left( \Delta^h_y L_y^x \right)^2 \right) E \left( \Delta^h_s L_s^x \Delta^h_y L_y^x \right) \, dx \, dy \tag{7.3}
\]

\[
I_2 = E \left[ \int L_t^x \Delta^h_s L_s^x \, dx \int \left( \Delta^h_y L_y^x \right)^2 \Delta^h_y L_y^x \, dy \right]
\]

\[
= \int \int E \left( L_t^x \left( \Delta^h_y L_y^x \right)^2 \right) E \left( \Delta^h_s L_s^x \Delta^h_y L_y^x \right) \, dx \, dy \tag{7.4}
\]

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and

\[ I_3 = E \left[ \left( \int L_t^x \Delta_t^h \tilde{L}_s^x \, dx \right)^2 \right] \]
\[ = E \left[ \int L_t^x \Delta_t^h \tilde{L}_s^x \, dx \int L_t^y \Delta_t^h \tilde{L}_s^y \, dy \right] \]
\[ = \int \int E \left( L_t^x L_t^y \right) E \left( \Delta_t^h \tilde{L}_s^x \Delta_t^h \tilde{L}_s^y \right) \, dx \, dy. \]  

(7.5)

By Kac’s moment formula and (4.5) we have

\[ G_s(x, y) = E \left( \Delta_t^h \tilde{L}_s^x \Delta_t^h \tilde{L}_s^y \right) = \int_{\{s_1 + s_2 \leq s\}} F_s(x, y) \, ds_1 \, ds_2 \]  

(7.6)

where

\[ F_s(x, y) \]
\[ = \Delta_t^h p_{s_1}(x) \Delta_t^h p_{s_2}(y - x - h) + p_{s_1}(x) \Delta_t^h \Delta_t^h p_{s_2}(y - x) 
\[ + \Delta_t^h p_{s_1}(y) \Delta_t^h p_{s_2}(x - y - h) + p_{s_1}(y) \Delta_t^h \Delta_t^h p_{s_2}(x - y). \]  

(7.7)

For any \( \epsilon > 0 \)

\[ |G_s(x, y)| \leq c_\epsilon s^{1/2} v_s(y - x - h) + c_\epsilon s^{1/2} u_s(y - x) 
\[ + c_\epsilon s^{1/2} v_s(x - y - h) + c_\epsilon s^{1/2} u_s(x - y). \]  

(7.8)

To see this we note the bounds

\[ \int_0^s p_r(x) \, dr \leq \int_0^s p_r(0) \, dr \leq c s^{1/2} \]  

(7.9)

and

\[ \int_0^s |\Delta_t^h p_r(x)| \, dr \leq 2 \int_0^s p_r(0) \, dr \leq c s^{1/2}. \]  

(7.10)

and interpolate to obtain

\[ u_s(x) \leq c_s^{1/2} u_s^{1-\epsilon}(x), \quad v_s(x) \leq c_s^{1/2} v_s^{1-\epsilon}(x). \]  

(7.11)

It follows from (7.8) and Lemma 2.2 that for any \( \epsilon > 0 \)

\[ \int |G_s(x, y)| \, dx \, dy \leq c s^{1/2} h^{2-\epsilon}. \]  

(7.12)

Clearly

\[ E \left( L_t^x L_t^y \right) = \int_{\{t_1 + t_2 \leq t\}} (A_{t_1, t_2}(x, y) + A_{t_1, t_2}(y, x)) \, dt_1 \, dt_2 \]  

(7.13)
where
\[ A_{t_1, t_2}(x, y) = p_{t_1}(x)p_{t_2}(y - x). \]  \hspace{1cm} (7.14)

By Kac’s moment formula and (4.5), compare (4.10),
\[
E \left( L_t^x \left( \Delta^h_y L_t^y \right)^2 \right) 
= 2 \sum_{\pi', t} \int \prod_{i=1}^3 \left( \Delta^h_{\pi'(i)} \right)^{a'_t(i)} \left( \Delta^h_{\pi'(i-1)} \right)^{a'_t(i)} p_t^x(\pi'(i) - \pi'(i - 1)) \, dt_i
\]
where the sum runs over all maps \( \pi' : [1, 2, 3] \mapsto \{x, y\} \) with \( |\pi'^{-1}(x)| = 1 \), \( |\pi'^{-1}(y)| = 2 \), and all ‘assignments’ \( a' = (a'_1, a'_2) : [1, 2, 3] \mapsto \{0, 1\} \times \{0, 1\} \) with the property that there will be exactly two factors of the form \( \Delta^h_y \) in (7.16) and none of the form \( \Delta^h_x \). The factor 2 comes from the fact that \( |\pi'^{-1}(x)| = 1 \), \( |\pi'^{-1}(y)| = 2 \). Recall that \( p_t^x(x) \) can be \( p_t(x), p_t(x + h) \) or \( p_t(x - h) \), but we always have \( \Delta^h \Delta^{-h} p_t^x(x) = \Delta^h \Delta^{-h} p_t(x) \). Also, we always take the \( p_t(\cdot) \) for a bound variable to be the \( g \) in (4.5).

Bound variables can come only from \( \pi'_1 = (x, y, y) \) and \( \pi'_2 = (y, y, x) \).

Setting
\[
f_t(h) = p_t(0) - p_t(h)
\]
we can write the contributions of \( \pi'_1 \) and \( \pi'_2 \) arising from a bound variable as
\[
\tilde{D}_{\pi'_1, t}(x, y) = p_{t_1}(x)p_{t_2}(y - x) \left( \Delta^h \Delta^{-h} p_{t_3}(0) \right)
= 2p_{t_1}(x)p_{t_2}(y - x) f_{t_3}(h)
= 2A_{t_1, t_2}(x, y) f_{t_3}(h)
\]
and
\[
\tilde{D}_{\pi'_2, t}(x, y) = p_{t_1}(y) \left( \Delta^h \Delta^{-h} p_{t_2}(0) \right) p_{t_3}(x - y)
= 2p_{t_1}(y) f_{t_2}(h) p_{t_3}(x - y)
= 2A_{t_1, t_3}(y, x) f_{t_2}(h).
\]

The non-bound contributions for \( \pi'_1 \) and \( \pi'_2 \) are
\[
\tilde{B}_{\pi'_1, t}(x, y) = p_{t_1}^x(x) \Delta^h p_{t_2}^x(y - x) \Delta^h p_{t_3}(0)
\]
\[
\tilde{B}_{\pi'_2, t}(x, y) = p_{t_1}^y(x) \Delta^h p_{t_2}^y(y - x) \Delta^h p_{t_3}(0)
\]
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and
\[ \bar{B}_{\pi',t}(x,y) = p_{t_1}(y) \Delta^{-h} p_{t_2}^* (0) \Delta^{-h} p_{t_3}^* (x-y) + \Delta^h p_{t_1}^* (y) p_{t_2}^* (0) \Delta^{-h} p_{t_3}^* (x-y) + \Delta^h p_{t_1}^* (y) \Delta^h p_{t_2}^* (0) p_{t_3}^* (x-y). \] (7.20)

and in addition there is a term from \( \pi' = (y, x, y) \) which is
\[ \bar{B}_{\pi',t}(x,y) = \Delta^h p_{t_1}^* (y) p_{t_2}^* (x-y) \Delta^h p_{t_3}^* (y-x) + \Delta^h p_{t_1}^* (y) \Delta^{-h} p_{t_2}^* (x-y) \Delta^h p_{t_3}^* (y-x). \] (7.21)

We observe that by (7.11) and Lemma 2.2, for any \( 1 \leq j \leq 3 \)
\[ \sup_{x,y} \int_{\{\sum_{i=1}^{3} t_i \leq t\}} |\bar{B}_{\pi',t}(x,y)| \prod_{i=1}^{3} dt_i \leq ct^j/2 h^{2-\epsilon}. \] (7.22)

Hence in view of (7.12) we see that for any \( \epsilon > 0 \) and \( 1 \leq j \leq 3 \)
\[ h \int \left( \int_{\{\sum_{i=1}^{3} t_i \leq t\}} |\bar{B}_{\pi',t}(x,y)| \prod_{i=1}^{3} dt_i \right) |G_s(x,y)| \, dx \, dy = O \left( (s \wedge t) h^{4+\epsilon} \right). \] (7.23)

Similarly
\[ E \left( \Delta^h L_{t_1}^x \right)^2 \left( \Delta^h L_{t_2}^y \right)^2 \] (7.24)
\[ = 4 \sum_{\pi,a} \int_{\{\sum_{i=1}^{4} t_i \leq t\}} \prod_{i=1}^{4} \left( \Delta^h \pi(i) \right)^{a_1(i)} \left( \Delta^h \pi(i-1) \right)^{a_2(i)} p_{t_1}^* (\pi(i) - \pi(i-1)) \, dt_i \]
where the sum runs over all maps \( \pi : [1, \ldots, 4] \mapsto \{x, y\} \) with \( |\pi^{-1}(x)| = |\pi^{-1}(y)| = 2 \), and all ‘assignments’ \( a = (a_1, a_2) : [1, \ldots, 4] \mapsto \{0, 1\} \times \{0, 1\} \) with the property that there will be exactly two factors of the form \( \Delta^h \) in (7.21) and similarly for \( \Delta^h \). The factor \( 4 = 2^2 \) comes from the fact that \( |\pi^{-1}(x)| = |\pi^{-1}(y)| = 2 \).

Writing \( \pi \) as a sequence \( (\pi(1), \pi(2), \pi(3), \pi(4)) \), we first consider \( \pi_1 = (x, x, y, y) \) and \( \pi_2 = (y, y, x, x) \). These are the only \( \pi \)'s which have two bound variables. We can write the contribution of \( \pi_1 \) arising from two bound variables as
\[ D_{\pi_1,t}(x,y) = p_{t_1}(x) \left( \Delta^h \Delta^{-h} p_{t_2} (0) \right) p_{t_3}(y-x) \left( \Delta^h \Delta^{-h} p_{t_4} (0) \right) \\
= 4 p_{t_1}(x) f_{t_2}(h) p_{t_3}(y-x) f_{t_4}(h) \\
= 4 f_{t_2}(h) f_{t_4}(h) A_{t_1,t_3}(x,y) \] (7.25)
and similarly
\[ D_{\pi_2,t}(x, y) = 4 f_{t_2}(h) f_{t_4}(h) A_{t_1, t_3}(y, x). \] (7.26)

The contribution of \( \pi_1 \) arising from one bound variable is
\[ B_{\pi_1,t}(x, y) = \pi_1(x) \left( \Delta^{-h}x p_{t_2}(0) \right) \Delta^{-h}p_{t_3}(y - x) \left( \Delta^{h} \Delta^{-h}p_{t_4}(0) \right) \]
\[ + \Delta^{h}p_{t_1}(x) p_{t_2}(0) \Delta^{-h}p_{t_3}(y - x) \left( \Delta^{h} \Delta^{-h}p_{t_4}(0) \right) \]
\[ + \Delta^{h}p_{t_1}(x) \Delta^{h}p_{t_2}(0) p_{t_3}(y - x) \left( \Delta^{h} \Delta^{-h}p_{t_4}(0) \right) \]
\[ + \pi_1(x) \left( \Delta^{h} \Delta^{-h}p_{t_2}(0) \right) \Delta^{h}p_{t_3}(y - x) \Delta^{h}p_{t_4}(0) \] (7.27)
and similar terms for \( \pi_2 \).

This is also a contribution of \( \pi_3 = (x, y, y, x) \) arising from one bound variable
\[ B_{\pi_3,t}(x, y) = \Delta^{h}p_{t_1}(x) p_{t_2}(y - x) \left( \Delta^{h} \Delta^{-h}p_{t_3}(0) \right) \Delta^{h}p_{t_4}(x - y) \]
\[ + \Delta^{h}p_{t_1}(x) \Delta^{-h}p_{t_2}(y - x) \left( \Delta^{h} \Delta^{-h}p_{t_3}(0) \right) \Delta^{h}p_{t_4}(x - y) \]
\[ + \Delta^{h}p_{t_1}(x) \Delta^{-h}p_{t_2}(y - x) \left( \Delta^{h} \Delta^{-h}p_{t_3}(0) \right) \Delta^{h}p_{t_4}(x - y) \]
and similar terms for \( \pi_4 = (y, x, x, y) \).

As before, we observe that by (7.11) and Lemma 2.2 for any \( 1 \leq j \leq 4 \)
\[ \sup_{x,y} \int_{0}^{4} |B_{\pi_j,t}(x, y)| \prod_{i=1}^{4} dt_i \leq c \epsilon^{j/2}h^{3-j}. \] (7.28)

Hence in view of (7.12) we see that for any \( \epsilon > 0 \) and \( 1 \leq j \leq 4 \)
\[ \int \left( \int_{0}^{4} |B_{\pi_j,t}(x, y)| \prod_{i=1}^{4} dt_i \right) |G_{s}(x, y)| dx dy = O \left( (s \wedge t)^{\epsilon}h^{4+\epsilon} \right). \] (7.29)

Taking note of the factor 4 in (7.24) and the factor 2 in (7.15) we now show that
\[ 4 \int \left( \int_{0}^{4} \left(D_{\pi_1,t}(x, y) + D_{\pi_2,t}(x, y) \right) \prod_{i=1}^{4} dt_i \right) G_{s}(x, y) dx dy \]
\[ -16h \int \left( \int_{0}^{4} \left( \bar{D}_{\pi_1,t}(x, y) + \bar{D}_{\pi_2,t}(x, y) \right) \prod_{i=1}^{3} dt_i \right) dx dy \] \[ +16h^2 \int \left( \int_{0}^{4} (A_{t_1,t_2}(x, y) + A_{t_1,t_2}(y, x)) dt_1 dt_2 \right) G_{s}(x, y) dx dy \]
\[ = O \left( (s \wedge t)^{\epsilon}h^{4+\epsilon} \right). \] (7.30)
We begin by rewriting (7.30). By symmetry it suffices to show that
\[
8 \int \left( \int_{\sum_{i=1}^{4} t_i \leq t} D_{\pi_1,t}(x,y) \prod_{i=1}^{4} dt_i \right) G_s(x,y) \, dx \, dy \tag{7.31}
\]
\[
-32h \int \left( \int_{\sum_{i=1}^{3} t_i \leq t} \tilde{D}_{\pi'_1,t}(x,y) \prod_{i=1}^{3} dt_i \right) G_s(x,y) \, dx \, dy
\]
\[
+32h^2 \int \left( \int_{t_1+t_2 \leq t} A_{t_1,t_2}(x,y) \, dt_1 \, dt_2 \right) G_s(x,y) \, dx \, dy = O \left( (s \wedge t)^{4+\epsilon} \right). \tag{7.32}
\]

Using the above expressions for $D_{\pi_1,t}(x,y)$, $\tilde{D}_{\pi'_1,t}(x,y)$ and relabeling the $t_i$'s this is equivalent to showing that
\[
32 \int \left( \int_{\sum_{i=1}^{4} t_i \leq t} A_{t_1,t_2}(x,y) f_{t_1}(h) f_{t_2}(h) \prod_{i=1}^{4} dt_i \right) G_s(x,y) \, dx \, dy \tag{7.32}
\]
\[
-64h \int \left( \int_{\sum_{i=1}^{3} t_i \leq t} A_{t_1,t_2}(x,y) f_{t_3}(h) \prod_{i=1}^{3} dt_i \right) G_s(x,y) \, dx \, dy
\]
\[
+32h^2 \int \left( \int_{t_1+t_2 \leq t} A_{t_1,t_2}(x,y) \, dt_1 \, dt_2 \right) G_s(x,y) \, dx \, dy = O \left( (s \wedge t)^{4+\epsilon} \right). \tag{7.32}
\]

This comes down to making precise the intuitive notion that $f_r(h)$ is $h$ times a delta-function in $r$, (in which case the left hand side would vanish).

To this end we note
\[
\int_{0}^{\infty} f_r(h) \, dr = \int_{0}^{\infty} (p_r(0) - p_r(h)) \, dr = h \tag{7.33}
\]
and for any $\delta > 0$
\[
\int_{\delta}^{\infty} f_r(h) \, dr = \int_{\delta}^{\infty} \frac{1 - e^{-h^2/2r}}{\sqrt{2\pi r}} \, dr \leq \int_{\delta}^{\infty} \frac{h^2/2r}{\sqrt{2\pi r}} \, dr = O(h^2/\sqrt{\delta}). \tag{7.34}
\]

We also note that
\[
\int_{t_1+t_2 \leq t} p_{t_1}(x) p_{t_2}(y-x) \, dt_1 \, dt_2 \tag{7.35}
\]
\[
\leq c \int_{t_2 \leq t_1} \frac{1}{\sqrt{t_1}} \frac{1}{\sqrt{t_2}} \, dt_1 \, dt_2 \leq C t^{2/3} h^{\epsilon/4}.
\]
We then write
\[
\int_{\sum_{i=1}^{4} t_i \leq t} A_{t_1, t_2}(x, y) f_{t_3}(h) f_{t_4}(h) \prod_{i=1}^{4} dt_i \quad (7.36)
\]
\[
= \left( \int_{\{t_1 + t_2 \leq t - 2h'\}} A_{t_1, t_2}(x, y) dt_1 dt_2 \right) \left( \int_0^{h'} f_r(h) dr \right)^2
\]
\[
+ \int_{C(t,h)} A_{t_1, t_2}(x, y) f_{t_3}(h) f_{t_4}(h) \prod_{i=1}^{4} dt_i
\]
where
\[
C(t, h) = \{ \sum_{i=1}^{4} t_i \leq t \} - \{ t_1 + t_2 \leq t - 2h' \} \times \{ t_3, t_4 \leq h' \} \quad (7.37)
\]
\[
\subseteq \left( [0, t]^4 \cap \{ t_3, t_4 \leq h' \} \right) \cup \{ t - 2h' \leq t_1 + t_2 \leq t \}.
\]
Using (7.33)-(7.35) we see that for \( \epsilon' \) small
\[
\left( \int_{\{t_1 + t_2 \leq t - 2h'\}} A_{t_1, t_2}(x, y) dt_1 dt_2 \right) \left( \int_0^{h'} f_r(h) dr \right)^2
\]
\[
= h^2 \int_{\{t_1 + t_2 \leq t\}} A_{t_1, t_2}(x, y) dt_1 dt_2 + O(t^{2/3} h^2 + \epsilon'/4)
\]
and
\[
\int_{C(t,h)} A_{t_1, t_2}(x, y) f_{t_3}(h) f_{t_4}(h) \prod_{i=1}^{4} dt_i = O(t^{2/3} h^2 + \epsilon'/4).
\]
A similar analysis applies to the second term in (7.32). Then taking \( \epsilon' = 8\epsilon \) and using (7.12) completes the proof of (7.32).

We have now dealt with all terms coming from \( I_2, I_3 \) and it only remains to consider the contribution of non-bound variables to \( I_1 \). We will show that this is
\[
32h^4 E \left( \int (L^x)^2 \tilde{L}^x dx \right) + O \left( (s \wedge t)^\epsilon h^{4+\epsilon} \right).
\]

The proof of (7.40) follows closely the proof of Lemma 6.1. The main contribution comes from \( \pi = (x, y, x, y) \) or \( (y, x, y, x) \) and \( a = e \). Taking
\[ \pi = (x, y, x, y) \text{ and } a = e \text{ we have} \]

\[
4 \int \left( \int_{\{ \sum_{i=1}^{4} t_i \leq t \}} p_{t_1}(x) \Delta_h \Delta^{-h} p_{t_2}(y-x) p_{t_3}(y-x) \Delta_h \Delta^{-h} p_{t_4}(y-x) \prod_{i=1}^{4} dt_i \right) G_s(x, y) \, dx \, dy
\]

Since as before

\[
| \int_{\{ \sum_{i=1}^{4} t_i \leq t \}} p_{t_1}(x) \Delta_h \Delta^{-h} p_{t_2}(y-x) p_{t_3}(y-x) \Delta_h \Delta^{-h} p_{t_4}(y-x) \prod_{i=1}^{4} dt_i | \leq c t^{\ell/2} u_t^{1-\epsilon} (x) u_t(y-x) w_t^2(y-x), \tag{7.41}
\]

we see that up to terms that are \( O((s \wedge t)^{\epsilon} h^{4+\epsilon}) \) we can replace \( G_s(x, y) \) in (7.41) by

\[
\int_{\{ s_1 + s_2 \leq s \}} \left( p_{s_1}(x) \Delta_h \Delta^{-h} p_{s_2}(y-x) + p_{s_1}(y) \Delta_h \Delta^{-h} p_{s_2}(x-y) \right) \, ds_1 \, ds_2. \tag{7.42}
\]

Thus consider

\[
4 \int \left( \int_{\{ \sum_{i=1}^{4} t_i \leq t \}} p_{t_1}(x) \Delta_h \Delta^{-h} p_{t_2}(y-x) p_{t_3}(y-x) \Delta_h \Delta^{-h} p_{t_4}(y-x) \prod_{i=1}^{4} dt_i \right) \left( \int_{\{ s_1 + s_2 \leq s \}} p_{s_1}(x) \Delta_h \Delta^{-h} p_{s_2}(y-x) \, ds_1 \, ds_2 \right) \, dx \, dy
\]

It now follows as in the proof of Lemma 6.1 that up to the error terms allowed in (7.40) this is equal to

\[
16h^4 \int \left( \int_{\{ t_1 + t_2 \leq t \}} p_{t_1}(x) p_{t_2}(0) \, dt_1 \, dt_2 \right) \left( \int_{\{ s \leq s \}} p_{s_1}(x) \, ds_1 \right) \, dx. \tag{7.43}
\]

The second term (7.42) gives the same contribution since up to another error term we can replace \( p_{s_1}(x) \) by \( p_{s_1}(y) \). There is a similar contribution from \( \pi = (y, x, y, x) \). Thus altogether we have

\[
64 \int \left( \int_{\{ t_1 + t_2 \leq t \}} p_{t_1}(x) p_{t_2}(0) \, dt_1 \, dt_2 \right) \left( \int_{\{ s \leq s \}} p_{s_1}(x) \, ds_1 \right) \, dx. \tag{7.44}
\]
Since by Kac’s moment formula
\[ E \left( \int (L_t^2 \tilde{L}_s^2) \, dx \right) \]
\[ = 2 \int \left( \int_{\{t_1 + t_2 \leq t\}} p_{t_1}(x)p_{t_2}(0) \, dt_1 \, dt_2 \right) \left( \int_{\{s_1 \leq s\}} p_{s_1}(x) \, ds_1 \right) \, dx \]
we obtain the main contribution to (7.40). The fact that all remaining \( \pi, a \) give error terms is now easy and left to the reader.

8 Proof of Lemmas 2.1–2.4

Proof of Lemma 2.1

Since
\[ \Delta^h_x \Delta^h_y u^\alpha(x - y) \]
\[ = \{ u^\alpha(x - y) - u^\alpha(x - y - h) \} - \{ u^\alpha(x - y + h) - u^\alpha(x - y) \} \] (8.1)
we have
\[ \Delta^h_x \Delta^h_y u^\alpha(x - y) \bigg|_{y=x} = \{ u^\alpha(0) - u^\alpha(-h) \} - \{ u^\alpha(h) - u^\alpha(0) \} \]
\[ = 2(u^\alpha(0) - u^\alpha(h)) = 2 \left( \frac{1 - e^{-\sqrt{2\alpha}h}}{\sqrt{2\alpha}} \right) \], (8.2)
which gives (2.4).

To obtain (2.5) we note that
\[ \Delta^h_x u^\alpha(x) = \left( \frac{e^{-\sqrt{2\alpha}|x+h|} - e^{-\sqrt{2\alpha}|x|}}{\sqrt{2\alpha}} \right) \]. (8.3)

Therefore
\[ |\Delta^h_x u^\alpha(x)| \leq \frac{e^{-\sqrt{2\alpha}|x|}}{\sqrt{2\alpha} \ (e^{\sqrt{2\alpha}(|x| - |x+h|)} - 1)} \]
\[ \leq e^{-\sqrt{2\alpha}|x|} \left( ||x|-|x+h|| + O(||x| - |x+h||^2) \right) \]
which gives (2.5), (since we allow \( C \) to depend on \( \alpha \).)
To obtain (2.6) we simply note that

\[ |\Delta^h \Delta^{-h} u^\alpha(x)| = |2u^\alpha(x) - u^\alpha(x + h) - u^\alpha(x - h)| \leq 2v^\alpha(x) \tag{8.5} \]

where we used the fact that \( u^\alpha(x) \) is an even function. The first part of (2.6) then follows from (2.5). When \(|x| \geq h\) we have

\[ \Delta^h \Delta^{-h} u^\alpha(x) = 2u^\alpha(x) - u^\alpha(x + h) - u^\alpha(x - h) \tag{8.6} \]

\[ = u^\alpha(x) \left( 2 - e^{-\sqrt{2\alpha} h} - e^{\sqrt{2\alpha} h} \right). \]

The statement in (2.8) follows trivially from (2.6).

For (2.9) we note that for \(|x| \leq h\)

\[ \Delta^h \Delta^{-h} u^\alpha(x) = 2u^\alpha(x) - u^\alpha(x + h) - u^\alpha(x - h) \tag{8.7} \]

\[ = (1 - u^\alpha(x + h)) + (1 - u^\alpha(x - h)) - 2(1 - u^\alpha(x)) \]

\[ = |x + h| + |x - h| - 2|x| + O(h^2). \]

When \(0 \leq x \leq h\) we therefore have

\[ \Delta^h \Delta^{-h} u^\alpha(x) = x + h + h - x - 2x + O(h^2) = (2 + O(h))(h - x). \tag{8.8} \]

Consequently

\[ \int_0^h \left( \Delta^h \Delta^{-h} u^\alpha(x) \right)^q dx = (2^q + O(h)) \int_0^h (h - x)^q dx \]

\[ = (2^q/(q + 1) + O(h)) h^{q+1}. \tag{8.9} \]

Similarly, when \(-h \leq x \leq 0\) it follows from (8.7) that

\[ \Delta^h \Delta^{-h} u^\alpha(x) = h - x + x + h + 2x + O(h^2) = (2 + O(h))(h + x). \tag{8.10} \]

Consequently

\[ \int_{-h}^0 \left( \Delta^h \Delta^{-h} u^\alpha(x) \right)^q dx = (2^q + O(h)) \int_{-h}^0 (h + x)^q dx \]

\[ = (2^q/(q + 1) + O(h)) h^{q+1}. \tag{8.11} \]

Using (8.9), (8.11) and (2.8) we get (2.9).
To obtain (2.7), we write
\[
\int |\Delta^h \Delta^{-h} u^\alpha(y)|^q \, dy = \int_{|y| \leq h} |\Delta^h \Delta^{-h} u^\alpha(y)|^q \, dy + \int_{|y| \geq h} |\Delta^h \Delta^{-h} u^\alpha(y)|^q \, dy \leq Ch^q \int_{|y| \leq h} 1 \, dy + Ch^{2q} \int_{|y| \geq h} u^\alpha(y) \, dy = O(h^{q+1}),
\]
where for the last line we use (2.6).

**Proof of Lemma 2.2** It follows from the fact that \(p_r(x) \leq p_r(y)\) for all \(r\) if \(|y| \leq |x|\), (2.1), and (2.5) that
\[
\int_0^T |\Delta^h p_t(x)| \, dt \leq e^{T/2} \int_0^\infty e^{-t/2} |\Delta^h p_t(x)| \, dt = e^{T/2} |\Delta^h \left( \int_0^\infty e^{-t/2} p_t(x) \, dt \right)| \leq C_T h e^{-|x|}.
\]
This gives (2.12).

For (2.13), we note that
\[
\left| \frac{d^2}{dx^2} p_t(x) \right| = \left| \frac{x^2/t - 1}{t \sqrt{2\pi t}} e^{-x^2/4t} \right| \leq \frac{C}{t^{3/2}} \left( \frac{x^2}{2t} + 1 \right) e^{-x^2/4t} \leq \frac{C}{t^{3/2}} e^{-x^2/4t},
\]
since \(\sup_{s>0} se^{-s} < \infty\). We use this and Taylor’s theorem to see that for some \(0 \leq h_t', h_t'' \leq h\),
\[
|\Delta^h \Delta^{-h} p_t(x)| = |2p_t(x) - p_t(x + h) - p_t(x - h)| \leq \frac{h^2}{2} \left| \frac{d^2}{dx^2} p_t(x + h_t') + \frac{d^2}{dx^2} p_t(x - h_t'') \right|
\leq \frac{Ch^2}{t^{3/2}} \left( e^{-(x+h_t')^2/4t} + e^{-(x+h_t'')^2/4t} \right).
\]
Therefore, when \(|x| \geq 2h\),
\[
|\Delta^h \Delta^{-h} p_t(x)| \leq \frac{Ch^2}{t^{3/2}} e^{-x^2/16t}.
\]
Consequently, when \( |x| \geq 2h \),

\[
\int_0^T |\Delta^h \Delta^{-h} p_t(x)| \, dt \leq C h^2 \int_0^T \frac{e^{-x^2/16t}}{t^{3/2}} \, dt \\
\leq C h^2 e^{-x^2/32T} \int_0^\infty \frac{e^{-x^2/32t}}{t^{3/2}} \, dt \\
= C h^2 e^{-x^2/32T} \int_0^\infty \frac{e^{-1/32t}}{t^{3/2}} \, dt \leq C_T h^2 \frac{e^{-x^2/32T}}{|x|},
\]

which proves (2.13).

Using (2.12) and (2.13) we see that

\[
\int w^q_T(x) \, dx = \int_{|x| \leq 2h} \left( \int_0^T |\Delta^h \Delta^{-h} p_t(x)| \, dt \right)^q \, dx \\
+ \int_{|x| \geq 2h} \left( \int_0^T |\Delta^h \Delta^{-h} p_t(x)| \, dt \right)^q \, dx \\
\leq 4 \int_{|x| \leq 2h} \left( \int_0^T |\Delta^h p_t(x)| \, dt \right)^q \, dx \\
+ \int_{|x| \geq 2h} \left( \int_0^T |\Delta^h \Delta^{-h} p_t(x)| \, dt \right)^q \, dx \\
\leq C_T \int_{|x| \leq 2h} h^q \, dx + C_T h^{2q} \int_{|x| \geq 2h} \frac{1}{|x|^q} \, dx \leq C_T h^{q+1},
\]

which gives us (2.15).

For (2.16) we note that when \( h \leq 1/4 \), \( \sqrt{h} \geq 2h \). Therefore, it follows from (2.13) that

\[
\int_{|x| \geq \sqrt{h}} w^q_T(x) \, dx \\
\leq C_T h^{2q} \int_{|x| \geq \sqrt{h}} \frac{1}{|x|^q} \, dx \leq C_T h^{3q/2 + 1/2}.
\]

Finally, to obtain (2.14) we use (2.12) and (2.13) to see that

\[
\int w_T(x) \, dx \leq 53
\]
\[ \int_{|x| \leq h} \int_{0}^{T} |\Delta^h \Delta^{-h} p_t(x)| \, dt \, dx + \int_{|x| \geq h} \int_{0}^{1} |\Delta^h \Delta^{-h} p_t(x)| \, dt \, dx \]
\[ \leq 2 \int_{|x| \leq h} \int_{0}^{T} |\Delta^h p_t(x)| \, dt \, dx + \int_{|x| \geq h} \int_{0}^{1} |\Delta^h \Delta^{-h} p_t(x)| \, dt \, dx \]
\[ \leq C_T \int_{|x| \leq h} h \, dx + C_T h^2 \int_{|x| \geq h} \frac{e^{-x^2/8}}{|x|} \, dx \leq C_T h^2 \log h. \]

\textbf{Remark 8.1} Using Remark 2.1 and (8.13) it is easy to check that we obtain the analog of (2.12) for all $|h| \leq 1$ if on the right hand side we replace $h$ by $|h|$.  

\textbf{Proof of Lemma 2.3} The proof of (2.17) is immediate. (2.19) follows from (8.15), and a similar application of the mean value theorem gives (2.18).  

\textbf{Proof of Lemma 2.4} Using \( 2 - e^{ihp} - e^{-ihp} = 2 - 2\cos(hp) = 4\sin^2(ph/2) \) we can write

\[ \int_{0}^{\infty} \Delta^h \Delta^{-h} p_t(x) \, dt \]
\[ = \frac{1}{2\pi} \int_{0}^{\infty} \int e^{ipx}(2 - e^{ipx} - e^{-ipx})e^{-tp^2/2} \, dp \, dt \]
\[ = \frac{4}{2\pi} \int_{0}^{\infty} \int e^{ipx} \sin^2(ph/2)e^{-tp^2/2} \, dp \, dt \]
\[ = \frac{8}{2\pi} \int e^{ipx} \frac{\sin^2(ph/2)}{p^2} \, dp. \]

Similarly

\[ \int_{0}^{h} \Delta^h \Delta^{-h} p_t(x) \, dt = \frac{8}{2\pi} \int e^{ipx} \frac{\sin^2(ph/2)}{p^2} \left( 1 - e^{-hp^2/2} \right) \, dp \]  
(8.22)

and

\[ \Delta^h \Delta^{-h} u^{1/2}(x) = \int_{0}^{\infty} e^{-t/2} \Delta^h \Delta^{-h} p_t(x) \, dt = \frac{8}{2\pi} \int e^{ipx} \frac{\sin^2(ph/2)}{1 + p^2} \, dp. \]  
(8.23)
Using (8.21) and the Fourier inversion formula we see that

\[
\int \left( \int_0^{\infty} \Delta^h \Delta^{-h} p_t(x) \, dt \right)^q \, dx \tag{8.24}
\]

\[
= \left( \frac{8}{2\pi} \right)^q \int \left( \int e^{ixp} \frac{\sin^2(p/2)}{p^2} \, dp \right)^q \, dx
\]

\[
= \left( \frac{8}{2\pi} \right)^q \int \left( \int e^{ix} \sum_{j=1}^{q} \prod_{j=1}^{q} \frac{\sin^2(p_jh/2)}{p_j^2} \, dp_j \right) \, dx
\]

\[
= \left( \frac{8}{2\pi} \right)^q \int \left( \int e^{ix} \sum_{j=2}^{q} \left( \int e^{ip_1} \frac{\sin^2(p_1h/2)}{p_1^2} \, dp_1 \right) \prod_{j=2}^{q} \frac{\sin^2(p_jh/2)}{p_j^2} \, dp_j \right) \, dx
\]

\[
\left. \right| \prod_{j=2}^{q} \frac{\sin^2(p_jh/2)}{p_j^2} \, dp_j
\]

where now \( p_1 = \sum_{j=2}^{q} p_j \). Scaling in \( h \) we then obtain

\[
\int \left( \int_0^{\infty} \Delta^h \Delta^{-h} p_t(x) \, dt \right)^q \, dx \tag{8.25}
\]

\[
= \frac{8^q h^{q+1}}{(2\pi)^q-1} \int \frac{\sin^2(p_1/2)}{p_1^2} \prod_{j=2}^{q} \frac{\sin^2(p_j/2)}{p_j^2} \, dp_j.
\]

Similarly we see that

\[
\int \left( \int_0^{h} \Delta^h \Delta^{-h} p_t(x) \, dt \right)^q \, dx \tag{8.26}
\]

\[
= \frac{8^q h^{q+1}}{(2\pi)^{q-1}} \int \frac{\sin^2(p_1/2)}{p_1^2} \left( 1 - e^{-p_1^2/2h} \right) \prod_{j=2}^{q} \frac{\sin^2(p_j/2)}{p_j^2} \left( 1 - e^{-p_j^2/2h} \right) \, dp_j
\]

and

\[
\int \left( \Delta^h \Delta^{-h} u^{1/2}(x) \right)^q \, dx \tag{8.27}
\]

\[
= \frac{8^q h^{q+1}}{(2\pi)^{q-1}} \int \frac{\sin^2(p_1/2)}{h^2 + p_1^2} \prod_{j=2}^{q} \frac{\sin^2(p_j/2)}{h^2 + p_j^2} \, dp_j.
\]

Using the fact that \( \frac{\sin^2(p/2)}{p^2} \) is bounded and

\[
\int e^{-p^2/2h} \, dp = Ch^{1/2} \tag{8.28}
\]
our Lemma follows from comparing (8.25)-(8.27) with (2.9). □

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