Fano 3-folds of index 2

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Abstract

We study Fano 3-folds with Fano index 2: that is, 3-folds $X$ with rank $\text{Pic}(X) = 1$, $\mathbb{Q}$-factorial terminal singularities and $-K_X = 2A$ for an ample Weil divisor $A$. We give a first classification of all possible Hilbert series of such polarised varieties $X, A$ and deduce both the nonvanishing of $H^0(X, -K_X)$ and the sharp bound $(-K_X)^3 \geq 8/165$. We list families that can be realised in codimension up to 4.

1 Introduction

We work over the complex number field $\mathbb{C}$ throughout, and we denote the Picard number of $X$ by $\rho(X) = \text{rank Pic}(X)$.

Definition 1 A normal projective 3-fold $X$ is called a Fano 3-fold if and only if $X$ has $\mathbb{Q}$-factorial terminal singularities, $-K_X$ is ample, and $\rho(X) = 1$.

The Fano index $f = f(X)$ of a Fano 3-fold $X$ is

$$f(X) = \max\{m \in \mathbb{Z}_{>0} \mid -K_X = mA \text{ for some Weil divisor } A\},$$

where equality of divisors denotes linear equivalence of some multiple. A Weil divisor $A$ for which $-K_X = fA$ is called a primitive ample divisor.

Fano 3-folds are sometimes called $\mathbb{Q}$-Fano 3-folds to distinguish them from the classical nonsingular case. By \cite{Su}, Theorem 0.3, we know that $f \leq 19$. The case $f = 1$ is the main case with several hundred confirmed families and much work towards classification ongoing. In the case $f \geq 4$, \cite{Su1} contains a near classification of about 80 families (only a handful of which remain in doubt). We study the case $f = 2$ here. Typical classical examples include the cubic 3-fold in $\mathbb{P}^4$ and the intersection of two quadrics in $\mathbb{P}^5$; see \cite{IP}.
Table 12.2, for example, in which $r$ denotes the Fano index. Note that the anti-canonical divisor $-K_X$ is only expected to be a Weil divisor, although necessarily it will be $\mathbb{Q}$-Cartier.

A Fano 3-fold $X$ with primitive ample divisor $A$ has a graded ring

$$R(X, A) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nA)).$$

This graded ring is finitely generated. The Hilbert series of $X, A$ is defined to be that of the graded ring $R(X, A)$. A choice of minimal (homogeneous) generating set $x_0, \ldots, x_N \in R(X, A)$ determines an embedding

$$X \hookrightarrow \mathbb{P}^N = \mathbb{P}(a_0, \ldots, a_N)$$

for some weighted projective space (wps) $\mathbb{P}^N$, where $x_i \in H^0(X, \mathcal{O}_X(a_iA))$. With this embedding in mind, we say that $X, A$ has codimension $N - 3$.

A Fano 3-fold is Bogomolov–Kawamata stable, or simply stable, if

$$(-K_X)^3 \leq 3(-K_Xe_2(X)).$$

Fano 3-folds satisfying this condition fall into one half of Kawamata’s argument on the boundedness of Fanos [Ka2].

Our first result is a list of the possible Hilbert series of graded rings $R(X, A)$ for Fano 3-fold of Fano index 2 and their analysis in low codimension.

**Theorem 2** There are at most 1492 power series that are the Hilbert series of some $X, A$ with $X$ a Fano 3-fold of Fano index 2 and $A$ a primitive ample divisor, of which 1413 could correspond to stable Fano 3-folds. Of these power series, 36 can be realised by some stable $X, A$ of codimension $\leq 3$.

The proof is explained in section 3.3. Arguments of Kawamata [Ka1], [Ka2] and Suzuki [Su] impose conditions on geometrical data of $X, A$ which are then analysed by computer. The 36 cases in low codimension are listed in Tables 1–3 in section 4.2.

In the case $f = 1$, there are many Fano 3-folds with with empty anti-canonical linear system. This does not happen in index 2:

**Theorem 3** If $X, A$ is a Fano 3-fold of Fano index 2 then $H^0(X, -K_X) \neq 0$. 
We showed in [BS] that $-K_X$ also has sections whenever $f \geq 3$, and so this result shows that $-K_X$ has a section whenever $f \neq 1$.

When $f \geq 3$, the Riemann–Roch formula is a function of the singularities of $X$. Fano 3-folds with $f \leq 2$ have an extra parameter, their genus, $g = \dim H^0(X, A) - 2$. We have sharp bounds on the degree for various genera:

**Theorem 4** Let $X, A$ be a Fano 3-fold of Fano index 2 with $-K_X = 2A$. Let $g = \dim H^0(X, A) - 2$. Then $-2 \leq g \leq 9$, and the degree $A^3$ of $X$ is bounded below by $1/165$; according to small $g$ the sharp bounds are:

| genus $g$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|-----------|------|------|-----|-----|-----|
| lower bound for $A^3$ | $1/165$ | $1/35$ | $1/3$ | $1$ | $2$ |

Moreover, all these lower bounds are achieved by stable Fano 3-folds.

The graded ring approach to building classifications is well known; we describe it in this case in section 3.3. There are two main points to be aware of. First, although the the Hilbert series of any $R(X, A)$ will indeed appear in the list of Theorem 2, there may be power series in the list that do not correspond to such a graded ring: the list comprises candidates for the Hilbert series of Fano 3-folds, and an appearance on the list does not imply that a Fano 3-fold exists with that Hilbert series. Second, we cannot say exactly which rings occur—for there will be many degenerations, as in [B2], and classifying them will be difficult. But in many cases we can predict at least the smallest possible codimension of such a ring together with the weights of the corresponding wps. When the proposed ring has codimension $\leq 3$, then we are able to construct it. Gorenstein rings in codimension 4 are more subtle. In section 4.3 we describe a first classification into 35 cases.

In section 2 we discuss some typical examples. Then section 3 contains the proofs after first assembling the ingredients: the appropriate Riemann–Roch formula, bounds on the singularities and degree, and Kawamata’s bounded-ness result. In section 4 we study the converse question, that of constructing Fano 3-folds with given Hilbert series, listing results in Tables 1–4.

It is our pleasure to thank Miles Reid for his help and encouragement throughout this work.

2 Examples

We work out some examples, pointing out where each Ingredient 1–5 of section 3 fits in. We use our standard notation of Notation 6 below. To
run the graded ring method, we need a basket of singularities and an integer 
$g \geq -2$. In the first example, we choose $g = -2$.

Consider the basket of quotient singularities

$$
\mathcal{B} = \left\{ \frac{1}{3}(1, 2, 2), \frac{1}{5}(1, 4, 2), \frac{1}{11}(3, 8, 2) \right\},
$$

where $\frac{1}{r}(a, -a, 2)$ denotes the germ of the quotient singularity $\mathbb{C}^3 / (\mathbb{Z}/r)$ by the action

$$(x, y, z) \mapsto (\varepsilon^{a} x, \varepsilon^{-a} y, \varepsilon^2 z) \quad \text{where } \varepsilon = \exp(2\pi i / r).$$

(The third component $2$ is forced since such a singularity will be polarised locally by the canonical class.)

We want to make a Fano 3-fold $X, A$ of Fano index 2 such that $X$ has exactly the singularities of $\mathcal{B}$ and $\dim \mathcal{H}^0(X, A) = g + 2$. (It would be enough that $X$ has terminal singularities which contribute to Riemann–Roch as though they were the singularities of $\mathcal{B}$, but in practice a general element of any family we construct has quotient singularities.) According to Ingredient 1 in section 3 below, the basket $\mathcal{B}$ is a possibility since

$$
\sum_{\mathcal{B}} r - 1/r = (3 - 1/3) + (5 - 1/5) + (11 - 1/11) < 24.
$$

Applying the formula of Ingredient 2 with data $\mathcal{B}, g$ describes Hilbert series

$$
P(t) = \frac{1 - t^{38}}{(1-t^2)(1-t^3)(1-t^5)(1-t^{11})(1-t^{19})}.
$$

Certainly any hypersurface of the form

$$
X = X_{38} \subset \mathbb{P}(2, 3, 5, 11, 19)
$$

will have Hilbert series $P_X(t)$ equal to $P(t)$. One can check that a general member of this family is indeed a quasismooth index 2 Fano 3-fold with singularities exactly $\mathcal{B}$ and $\mathcal{H}^0(X, A) = 0$. This is one of eight hypersurfaces that are Fano 3-folds of Fano index 2: all eight are listed in Table 1 of section 4.2

Notice that any quasismooth Fano 3-fold of this form has a K3 elephant; that is, the section $S \in | -K_X |$ is a K3 surface, $S_{38} \subset \mathbb{P}(3, 5, 11, 19)$. 4
Examples from del Pezzo surfaces  A classical example is the Fano 3-fold \(X \subset \mathbb{P}^6\) whose hyperplane section is the del Pezzo surface of degree 5. The equations of a general such 3-fold are well known: they are the five maximal Pfaffians of a skew \(5 \times 5\) matrix of general linear forms on \(\mathbb{P}^6\). Extensions of other del Pezzo surfaces also give rise to index 2 Fano 3-folds. In \([RS]\), such series of families is called a ‘cascade’ and other examples using log del Pezzo surfaces are described there.

The main case of \([RS]\) is a series of families which, in order of increasing codimension (although, as with blowups of \(\mathbb{P}^2\), the opposite order is also natural) begins with the hypersurface \(X_{10} \subset \mathbb{P}(1^2,2,3,5)\), followed by \(X_{4,4} \subset \mathbb{P}(1^3,2^2,3)\) in codimension 2 and a family in codimension 3, \(X \subset \mathbb{P}(1^4,2^2,3)\).

These families arise using \(\mathcal{B} = \{\frac{1}{2}(1,2,2)\}\) and allowing \(g\) to vary, starting at \(g = 0\) and increasing. When \(g = 0\), the formula of Ingredient 2 with data \(\mathcal{B}, g\) describes Hilbert series

\[
P(t) = \frac{(1-t^{10})/(1-t)^2(1-t^2)(1-t^3)(1-t^5)},
\]

and, as before, this corresponds to a family of Fano hypersurfaces above of degree \(A^3 = 1/3\). When \(g = 1\), we get

\[
P(t) = \frac{(1-t^4)^2/(1-t)^3(1-t^2)^3(1-t^3)},
\]

corresponding to the codimension 2 complete intersection above of degree \(4/3 = 1/3 + 1\). Increasing \(g\) again to \(g = 2\), and thus \(A^3 = 7/3 = 2/3 + 2\) by Ingredient 3 below, describes Hilbert series

\[
P(t) = \frac{(t^4 + t^3 + 3t^2 + t + 1)/(1-t)^3(1-t^3)}.
\]

This description is not as revealing as previous ones, but it is easy to remedy. The numerator indicates that a further generator in degree 1 is needed, and with an extra factor of \(1 - t\), we see that

\[
P(t) = \frac{(-t^5 - 2t^3 + 2t^2 + 1)/(1-t)^4(1-t^3)}.
\]

Now it is clear that two new generators in degree 2 are required—these also serve to polarise the index 3 singularity in the basket. The final result is

\[
P(t) = \frac{(-t^9 + 2t^6 + 3t^5 - 3t^4 - 2t^3 + 1)/(1-t)^4(1-t^2)^2(1-t^3)}.
\]

This Hilbert series is realised by a graded ring with generators \(x_1, x_2, x_3, y_1, y_2, z\) in degrees 1, 1, 1, 2, 2, 3 and relations generated by the five maximal
Pfaffians of the following $5 \times 5$ skew matrix (where as usual we omit the leading diagonal of zeroes and leave the skew lower half implicit)

$$M = \begin{pmatrix}
  x_1 & x_2 & b_{14} & b_{15} \\
  x_3 & b_{24} & b_{25} \\
  b_{34} & b_{35} \\
  z
\end{pmatrix} \text{ of degrees } \begin{pmatrix}
  1 & 1 & 2 & 2 \\
  1 & 2 & 2 \\
  2 & 2 \\
  3
\end{pmatrix}.$$

The $b_{ij}$ are forms of degree 2 in the $x$ and $y$ variables. It is easy to check that for general $b_{ij}$, this defines the graded ring of a Fano 3-fold in $\mathbb{P}^6(1^4, 2^2, 3)$ as required, and we think of the $b_{ij}$ as being parameters defining a flat family of Fano 3-folds of index 2 with given basket $\mathcal{B}$ and genus $g$.

### 3 Hilbert series of Fano 3-folds

We describe in sections 3.1–3.2 the five ingredients that go into the raw list of power series that is the basis for our classification. Our raw list will include the Hilbert series of every index 2 Fano 3-fold, although it may contain power series not of this form. The derivation of the list is explained in section 3.3. Although it is not necessary for everything below, we suppose throughout that Fano 3-folds $X$ treated here will have Fano index $f(X) = 2$.

#### 3.1 Riemann-Roch formula

We explain the notion of a basket of singularities; see [R] or [B] section 2.1. Let $P \in X$ be a 3-dimensional terminal singularity of index $r > 1$ and $(U, P)$ a germ at $P$. (In particular, $rK_U$ is Cartier on $U$.)

If $(U, P)$ is a quotient singularity, then it isomorphic to the germ at the origin of some $\mathbb{C}^3/(\mathbb{Z}/r)$, with action

$$\varepsilon \cdot (x, y, z) \mapsto (\varepsilon^a x, \varepsilon^b y, \varepsilon^c y).$$

We denote this by $\frac{1}{r}(a, b, c)$ and recall that $\gcd(r, abc) = 1$ and $a + b = r$ up to permutations of $a, b, c$. When $X$ has Fano index 2, then we may suppose that $(a, b, c) = (a, r - a, 2)$. In particular, $r$ cannot be even.

If $(U, P)$ is not a quotient singularity, then $(U, P)$ can be deformed to a unique finite collection of terminal quotient singularities, say $\{Q_1, \ldots, Q_{n(P)}\}$ (where, as is usual, this is a set with possible repetitions) for some number...
\[ n(P) \geq 1. \] Each point \( Q_i \) is some quotient singularity \( \frac{1}{r}(a_i, -a_i, 2) \) where \( r_i \) and \( a_i \) are coprime, \( r_i \geq 2 \), and \( r = \text{lcm}\{r_1, \ldots, r_{n(P)}\} \). We call the set \( B(U, P) := \{Q_1, \ldots, Q_{n(P)}\} \) the basket of singularities of \( (U, P) \).

In the global case, we assemble all local baskets into one.

**Definition 5** Let \( X \) be a 3-fold with terminal singularities and \( \{P_1, \ldots, P_m\} \) the set of singular points of \( X \) of index \( \geq 2 \). Denoting germs \( P_i \in U_i(\subset X) \), we define the basket of singularities of \( X \) to be the disjoint union \( B(U_1, P_1) \cup \cdots \cup B(U_m, P_m) \) (a set with possible repetitions.)

**Notation 6** We denote a Fano 3-fold of index \( f = 2 \) for which \( -K_X = 2A \) by \( X, A \). The basket of singularities of \( X, A \) is denoted \( B \). A typical singularity of \( B \) is denoted \( \frac{1}{r}(a, -a, 2) \), and we use this notation whenever taking a sum over the elements of \( B \).

**Theorem 7** ([Ka1]) Let \( X \) be a Fano 3-fold with basket of singularities \( B \). Then \( -K_X c_2(X) > 0 \), the Euler characteristic \( \chi(O_X) = 1 \) and

\[
\chi(O_X) = \frac{-K_X c_2(X)}{24} + \sum \frac{r^2 - 1}{24r},
\]

the sum taken over \( B \) (see Notation 6).

This theorem gives the following bounds on the singularities in the basket:

**Ingredient 1 (Basket bound)** If \( B \) is the basket of a Fano 3-fold \( X \), then

\[
\sum \frac{r^2 - 1}{r} < 24 \quad \text{and} \quad -K_X c_2(X) = 24 - \sum \frac{r^2 - 1}{r}
\]

where each sum is taken over \( B \) (see Notation 6).

For the next theorem, recall that the plurigenera of a polarised variety \( X, A \) are denoted \( P_n(X, A) = \dim H^0(X, nA) \).

**Theorem 8** ([Su]) Let \( X \) be a Fano 3-fold of Fano index \( f \) and basket of singularities \( B \). Let \( A \) be a primitive Weil divisor with \( -K_X = fA \). Then

\[
\chi(O_X(nA)) = \chi(O_X) + \frac{n(n + f)(2n + f)}{12} A^3 + \frac{nAc_2(X)}{12} + \sum_B \left( -\frac{i_n r^2 - 1}{12r} + \sum_{j=1}^{i_n-1} \frac{bj(r - bj)}{2r} \right)
\]
where the sum is taken over points \( P = \frac{1}{r}(a, -a, 2) \) in \( \mathcal{B} \) (see Notation [6]) using notation: \( i_n \in [0, r - 1] \) is the local index of \( nA \) at \( P \) (see [R]), \( b \in [0, r - 1] \) satisfies \( ab \equiv 2 \, \text{mod} \, r \) and \( c \in [0, r - 1] \) is the residue of \( c \) mod \( r \).

By Kawamata–Viehweg vanishing, \( \chi(nA) = h^0(nA) \) for all \( n > -f \). So the Hilbert series \( P_{X,A}(t) \) of \( X \) is:

\[
P_{X,A}(t) = \sum_{n=0}^{\infty} P_n(X, A) t^n = \frac{1}{1 - t} + \frac{(f^2 + 3f + 2) t + (-2f^2 + 8)t^2 + (f^2 - 3f + 2)t^3}{12(1 - t)^4} A^3 + \frac{t}{(1 - t)^2} \frac{Ac_2(X)}{12} + \sum_{P \in \mathcal{B}} c_P(t)
\]

where, for a point \( P = \frac{1}{r}(a, -a, 2) \) in \( \mathcal{B} \),

\[
c_P(t) = \frac{1}{1 - t^r} \left( \sum_{k=1}^{r-1} \left( -i_k \frac{r^2 - 1}{12r} + \sum_{j=1}^{i_k-1} \frac{bj(r-bj)}{2r} \right) t^k \right).
\]

Setting \( f = 2 \) in the Hilbert series above gives the closed formula:

**Ingredient 2 (Hilbert series)** Using Notation [6] and the expression for \( c_P(t) \) from Theorem [8], the Hilbert series is

\[
P_{X,A}(t) = \frac{1}{1 - t} + \frac{t}{(1 - t)^2} A^3 + \frac{t}{(1 - t)^2} \frac{Ac_2(X)}{12} + \sum_{P \in \mathcal{B}} c_P(t)
\]

where the sum is taken over \( \mathcal{B} \).

Setting \( n = 1 \) in the Riemann–Roch formula, we compute the minimum possible value of the degree \( A^3 \).

**Ingredient 3 (Minimum degree)** Using Notation [6]

\[
0 < A^3 = -1 - \frac{Ac_2(X)}{12} - \sum \left( -i_1 \frac{r^2 - 1}{12r} + \sum_{j=1}^{i_1-1} \frac{bj(r-bj)}{2r} \right) + N
\]

for some integer \( N \geq 0 \), and where the sum is taken over \( \mathcal{B} \).
Since the Riemann–Roch formula also holds when $n = -1$, we have an additional constraint.

**Ingredient 4 (Polarisation condition)** Using Notation 7,

\[
1 + \sum \left( -(i-1) \frac{r^2 - 1}{12r} + \frac{(i-1)-1}{2r} \sum_{j=1}^{(i-1)-1} \frac{b_j(r-b_j)}{2r} \right) = \frac{Ac_2(X)}{12}.
\]

where the sum is taken over $\mathcal{B}$.

### 3.2 Kawamata Boundedness

Following [Ka2], let $X$ be a Fano 3-fold with Fano index 2 and $\mathcal{E} = (\Omega^1_X)^{**}$ the double dual of the sheaf of Kähler differentials of $X$. We do not define $\mu$-semistability (with respect to $-K_X$) here, since we do not use it further, but note the role it plays in strengthening the following bound on the degree.

**Theorem 9** ([Ka2]) Let $X$ be a Fano 3-fold with Fano index 2. Then $(-K_X)^3 \leq \frac{16}{5} (-K_Xc_2(X))$. If, furthermore, the sheaf $\mathcal{E}$ described above is $\mu$-semistable, then $(-K_X)^3 \leq 3(-K_Xc_2(X))$.

**Ingredient 5** Using Notation 7. $A^3 \leq \frac{4}{5} Ac_2(X)$. (In the stable case, the upper bound is $A^3 \leq \frac{3}{4} Ac_2(X)$.)

Applying the upper bound for $A c_2$ of Ingredient 1, this bound implies $A^3 \leq \frac{48}{5}$ (and $A^3 \leq 9$ in the stable case.) In fact, once all possible baskets are calculated, the sharp bound is $A^3 \leq 9$ achieved using the empty basket and $g = 9$; the next largest degree is $A^3 = \frac{25}{3}$ achieved by the basket $\{\frac{1}{3}(1,2,2)\}$ and $g = 8$ (which is not stable since $A c_2 = \frac{32}{3}$). In particular, for a fixed basket there are at most 9 different values for $g$ that give Fano Hilbert series. Comparing with [RS], we regard this as a bound on the number of blowups (or projections) that we can make from maximal $g$. A familiar instance of this bound is the maximum number of blowups of $\mathbb{P}^2$ that is a del Pezzo surface; again, 8 is the limit. In [RS], such a cascade of 8 log del Pezzo surfaces is constructed linking the hypersurface $S_{10} \subset \mathbb{P}(1,2,3,5)$ with a log del Pezzo of degree $25/3$. 
3.3 Proofs of Theorems 2–4

These proofs use some computer calculations which we describe rather than reproducing; we use the computer algebra system Magma \([\text{Mag}]\) for our calculations, and a short file with Magma code that can be run to generate these results is available at \([\text{BS}2]\).

**Listing the Hilbert series**  The first step is to construct the list of all power series according to the five ingredients assembled in section 3. This requires little comment. We simply compute all possible baskets satisfying Ingredients 1 and 4, together with all possible values for \(N\) using Ingredients 3 and 5, and then apply the Riemann–Roch formula of Ingredient 2. The result is all power series that could be the Hilbert series of a Fano 3-fold of index 2. There are 1492 such series in all. (If we impose the lower ‘stable’ bound of Ingredient 5, this number reduces to 1413.)

**Bounds on the degree**  It is easy to compute both upper and lower bounds on the degree (and also on \(A_2^c(X)\)) by computer check on the list. In Theorem 4 we list lower bounds for small \(g\), since our methods of construction reveal Fano 3-folds that realise these bounds. The following table includes all the degree bounds for each genus \(g\), even though we do not know whether all of them are sharp or not.

| genus \(g\) | \(-2\) | \(-1\) | 0 | 1 | 2 |
|-------------|------|------|---|---|---|
| lower bound for \(A_3^c\) | 1/165 | 1/35 | 1/3 | 1 | 2 |
| upper bound for \(A_3^c\) | 11/15 | 32/21 | 89/39 | 64/21 | 19/5 |
| total number of series (of which are unstable) | 337 | 470 | 303 | 174 | 97 |

| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 68/15 | 16/3 | 6 | 48/7 | 38/5 | 25/3 | 9 |
| 54 | 28 | 14 | 8 | 4 | 2 | 1 |
| 7 | 5 | 2 | 2 | 2 | 1 | 0 |

**Nonempty anticanonical system**  The proof of Theorem 3 is straightforward: since we do not discard candidate Hilbert series unless they are proved not to come from a Fano 3-fold, it is enough to calculate the coefficient of \(t^2\) in each one. Again, this is done by a computer on all 1492 Hilbert series.
Since \( H^0(X, -K_X) \neq 0 \), we can ask whether the linear system \(|-K_X|\) contains a K3 surface. Such a K3 section is sometimes impossible because its singular rank would be too big—see [B], Proposition 4. For example, \( \mathcal{B} = \{ \frac{1}{21}(10, 11, 2) \} \) with \( g = 0 \) satisfy all our numerical conditions (including stability)—they predict \( A^3 = 19/21 \) and \( (1/12)Ac_2(X) = 8/63 \). But a K3 section cannot exist because the corresponding surface singularity \( \frac{1}{21}(10, 11) \) has 20 exceptional curves in its resolution, pairwise orthogonal in the Picard group, which cannot happen in \( H^2(S) \) for a K3 surface \( S \). There are 171 such cases (of which 9 are unstable). (These examples all appear to be in high codimension, and we do not analyse them further.)

When the singular rank is \( \leq 19 \), we get an estimate of the degrees of generators of a model \( R(X, A) \) for a Fano 3-fold by comparing with K3 surfaces appearing in the K3 database [B]. This gives an idea of how to understand the result of Theorem 2 just as the estimates of [RS] 3.2.5 do in the case \( f \geq 3 \). But this list should not be taken as more than a guide. Using this K3 comparison, the number of Hilbert series per codimension is as follows.

| estimated codimension | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------------|---|---|---|---|---|---|---|---|
| total number of series| 8 | 26 | 2 | 35 | 13 | 59 | 25 | 99 |
| (of which are unstable)| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

We describe the 36 candidates in codimension \( \leq 3 \) in section 4.2 and the 35 candidates in codimension 4 in section 4.3 below.

**4 Constructing lists of Fano 3-folds**

The remaining claim is that 36 of the 1492 families can be realised in codimension \( \leq 3 \). This is achieved by the examples in Tables 1–3 below. We also explain the list in Table 4 of codimension 4 candidates and describe the role of unprojection methods.

**4.1 Models in low codimension**

**Finding the first generators** The following standard analysis of a Hilbert series bounds the minimum number of generators below. Since we are con-
centrating on codimension $\leq 3$, once we have confirmed that the ring must have at least eight generators if it exists, then we do not pursue it further.

Consider the data $B = \{ \frac{1}{3}(1,8,2) \}$, $g = 1$. Suppose there is a Fano $X, A$ with this data; we begin to describe $R(X, A)$. (Such $X$ would have $A^3 = 13/9$ and $AC_2 = 17/27$.) The Hilbert series expressed as a power series is

$$1 + 3t + 8t^2 + 17t^3 + 32t^4 + 54t^5 + 85t^6 + \cdots.$$ 

Any graded ring having this Hilbert series must have exactly three generators in degree 1. These span at most a 6-dimensional subspace in degree 2, so there are at least two new generators in degree 2. If this number is exactly two, then a similar argument shows that there must be at least one new generator in degree 3. The only alternative is to have three new generators in degree 2. Either way, we already have at least six generators in the ring.

Now we turn to the singularities. Certainly $X$ has a terminal singularity of index 9. Whether it is a quotient singularity or not, locally it must be a quotient by $\mathbb{Z}/9$ acting with at least one eigencordinate of weight 8—every case of index 9 in the classification of terminal singularities \[\text{M}\] and their baskets \[\text{R}\] has local eigencordinates $a, -a$, where $a = 1$ in this case. So $R(X, A)$ must have at least one generator of weight divisible by 9 and another of weight 8 modulo 9. So $R(X, A)$ is in codimension $\geq 4$. (One could also argue on the denominator of the Hilbert series: as a rational function, it has cyclotomic polynomial of degree 9 in its denominator, so the ring needs a generator in degree some multiple of 9 to cancel this.)

**Constructing varieties and additional generators** We carry out the analysis above systematically using a computer. When it suggests that a ring may be in codimension $\leq 3$, then we attempt to construct it as in the examples of section 2. The construction is straightforward, although there is one small twist: it happens frequently that we can find a graded ring with the right Hilbert series but that does not correspond to a Fano 3-fold.

For example, with $B = \{ \frac{1}{11}(2,9,2) \}$, $g = -1$, the Hilbert series is

$$P(t) = \frac{(1 - t^6)(1 - t^8)(1 - t^{10})}{\prod (1 - t^a)}$$

where the product is taken over $a \in \{1, 2, 2, 2, 3, 5, 11\}$. This is the Hilbert series of any $X_{6,9,10} \subset \mathbb{P}(1,2,2,3,5,11)$. But such a variety is not a Fano 3-fold: the equations cannot involve the variable of degree 11, so it will have
a cone singularity at the index 11 point. Moreover, there is no variable of degree 9 to polarise the singularity as we wanted. In this case, the solution is clear: adding a generator of degree 9 to the ring gives a codimension 4 model $X \subset \mathbb{P}(1, 2, 2, 3, 5, 9, 11)$. The family of complete intersections comprise a component of the Hilbert scheme that does not contain the desired varieties.

### 4.2 Classification in codimensions 1 to 3

The lists in Tables 1–3 contain families whose general element lies in codimension at most 3. (We denote the singularity $\frac{1}{3}(1, 2, 2)$ simply by $\frac{1}{3}$.) In addition to these, there will be degenerations of low codimension families that occur in higher codimension. For example, a Fano $X_{6,18} \subset \mathbb{P}(1, 2, 3, 5, 6, 9)$ for which the variable of degree 6 does not appear in the equation of degree 6 is not listed in the table of codimension 2 Fano 3-folds; its Hilbert series is that of the degree 1/15 hypersurface, and the degeneration occurs as the two singularities of index 3 in that hypersurface come together.

It is easy to confirm that the general element in each case of Tables 1–3 is a Fano 3-fold with the indicated properties.

| Fano hypersurface $X$ | Basket $\mathcal{B}$ | $A^3$ | $A\mathcal{C}/12$ |
|-----------------------|-----------------------|-------|------------------|
| $X_3 \subset \mathbb{P}(1, 1, 1, 1)$ | nonsingular | 3 | 1 |
| $X_4 \subset \mathbb{P}(1, 1, 1, 2)$ | nonsingular | 2 | 1 |
| $X_6 \subset \mathbb{P}(1, 1, 2, 3)$ | nonsingular | 1 | 1 |
| $X_{10} \subset \mathbb{P}(1, 1, 2, 3, 5)$ | $\frac{1}{3}$ (= $\frac{1}{3}(1, 2, 2)$) | 1/3 | 8/9 |
| $X_{18} \subset \mathbb{P}(1, 2, 3, 5, 9)$ | $2 \times \frac{1}{3}$, $\frac{1}{5}(1, 4, 2)$ | 1/15 | 26/45 |
| $X_{22} \subset \mathbb{P}(1, 2, 3, 7, 11)$ | $\frac{1}{3}$, $\frac{1}{3}(3, 4, 2)$ | 1/21 | 38/63 |
| $X_{26} \subset \mathbb{P}(1, 2, 5, 7, 13)$ | $\frac{1}{5}(2, 3, 2)$, $\frac{1}{7}(1, 6, 2)$ | 1/35 | 18/35 |
| $X_{38} \subset \mathbb{P}(2, 3, 5, 11, 19)$ | $\frac{1}{7}$, $\frac{1}{7}(1, 4, 2)$, $\frac{1}{11}(3, 8, 2)$ | 1/165 | 116/495 |

Table 1: Fano 3-folds in codimension 1

### 4.3 Classification in codimension 4

We continue our analysis of rings into higher codimension, although the method becomes more complicated and we do not check all possibilities rigorously. (Complete results are on the webpage [BS2].) We list in Table 4 the
35 Hilbert series with proposals for models in codimension 4—it is conceivable that there are other examples (other than degenerations) but we do not expect them. We list them according to the weights of the ambient wps $\mathbb{P}^7$, that is, the degrees of minimal generators of $R(X, A)$; for full details, see the webpage at [BS2]. The task is to construct these Fano 3-folds.

**Projection guided by a K3 section** If $|−K_X|$ contains a K3 surface $S$ we may compare our results with those of [B] as a guide to the constructions we might make. Consider, the following codimension 4 candidate:

$$X \subset \mathbb{P}(2, 2, 3, 5, 5, 7, 12, 17)$$

with $B = \{\frac{1}{17}(5, 12, 2)\}$. A section in degree 2 is a known K3 surface known with a Type 1 projection from a $\frac{1}{17}(5, 12)$ point:

$$(S \subset \mathbb{P}(2, 3, 5, 5, 7, 12, 17)) \rightarrow (T_{10,12,14,15,17} \subset \mathbb{P}(2, 3, 5, 5, 7, 12)).$$

The K3 surface $T$ can be constructed easily, and moreover it can be forced to contain a linear $\mathbb{P}(5, 12)$. This curve can be unprojected to construct $S$. This is how we proceed with cases in codimension 4 here, but there is a twist.

When we project from a quotient singularity—say the point $\frac{1}{17}(5, 12, 2)$ in the example above—the result will always contain a line of index 2 singularities. That is, projection automatically incurs canonical singularities that contribute to Riemann–Roch. (Recall from [ABR] that projection of 3-folds results in slightly worse singularities that we usually work with, but that typically these are Gorenstein and do not contribute in the Riemann–Roch formula.) And so the result of projection will not have Hilbert series already on our lists. This obstructs the inductive approach exemplified by [B].

There are two ways out. One would be to mimic classical methods and make a double projection. In the example above, after projecting from $\frac{1}{17}(5, 12, 2)$, we can project from the resulting $\frac{1}{12}(5, 7, 2)$ singularity. This second projection contracts the index 2 line, and we land in the family

$$Z_{10,12} \subset \mathbb{P}(2, 2, 3, 5, 5, 7).$$

To proceed, we would need an analysis of the exceptional locus of such double projections and a Type III-style unprojection result, following [B2] e.g. 9.16.

The approach we take is to consider also weak Fano 3-folds with canonical singularities. This avoids the bottleneck in codimension 3 for general
unprojection methods: by admitting some canonical singularities, in particular lines of index 2 with up to two non-isolated points of higher index on them, we see many more weak Fano 3-folds in codimension 3. We impose appropriate surfaces in these 3-folds and the apply general results of unprojection; this method is explained in detail in [BKR]. In short, we construct the example above by imposing the plane $\mathbb{P}(2, 5, 12)$ as a linear subspace in the codimension 3 weak Fano 3-fold

$$\mathbb{P}(2, 5, 12) \subset Y_{10, 12, 14, 15, 17} \subset \mathbb{P}(2, 5, 12).$$

Such $Y$ has a non-isolated singularity $\frac{1}{12}(2, 5, 7)$ on a line of index 2 singularities, which are all strictly canonical singularities.

**Codimension 4 candidates having no Type I projection**  The projection method outlined above constructs examples for 33 of the 35 candidates in codimension 4. The remaining two cases, which we do not construct, are:

$$X \subset \mathbb{P}^7(2, 3, 3, 4, 5, 6, 7)$$

with $\mathcal{B} = \{5 \times \frac{1}{3}(1, 2, 2), \frac{1}{5}(1, 4, 2)\}$, $A^3 = 1/15$, $\frac{1}{12}A_{C2}(X) = 11/45$ and Hilbert numerator $1 - t^8 - t^9 - 2t^{10} - t^{11} - 2t^{12} + \cdots + t^{33}$; and

$$X \subset \mathbb{P}^7(2, 3, 5, 6, 7, 7, 8, 9)$$

with $\mathcal{B} = \{3 \times \frac{1}{3}(1, 2, 2), \frac{1}{5}(2, 3, 2), \frac{1}{7}(1, 6, 2)\}$, $A^3 = 1/35$, $\frac{1}{12}A_{C2}(X) = 19/105$ and Hilbert numerator $1 - t^{12} - t^{13} - 2t^{14} - t^{15} - 2t^{16} - t^{17} - t^{18} + t^{19} + \cdots + t^{45}$.

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\[
\begin{array}{|c|c|c|c|}
\hline
\text{Fano 3-fold} & \text{Basket } \mathcal{B} & A^3 & Ac_2/12 \\
\hline
X_{2,2} \subset \mathbb{P}(1, 1, 1, 1, 1) & \text{nonsingular} & 3 & 1 \\
X_{4,4} \subset \mathbb{P}(1, 1, 1, 2, 2, 3) & \frac{1}{3} & 4/3 & 8/9 \\
X_{4,6} \subset \mathbb{P}(1, 1, 2, 2, 3, 3) & 2 \times \frac{1}{3} & 2/3 & 7/9 \\
X_{6,6} \subset \mathbb{P}(1, 1, 2, 2, 3, 5) & \frac{1}{3} (2, 3, 2) & 3/5 & 4/5 \\
X_{6,8} \subset \mathbb{P}(1, 1, 2, 3, 4, 5) & \frac{1}{3} (1, 4, 2) & 2/5 & 4/5 \\
X_{6,6} \subset \mathbb{P}(1, 2, 2, 3, 3, 3) & 4 \times \frac{1}{3} & 1/3 & 5/9 \\
X_{6,8} \subset \mathbb{P}(1, 2, 2, 3, 3, 5) & 2 \times \frac{1}{3}, \frac{1}{3} (2, 3, 2) & 4/15 & 26/45 \\
X_{6,10} \subset \mathbb{P}(1, 2, 2, 3, 5, 5) & 2 \times \frac{1}{3} (2, 3, 2) & 1/5 & 3/5 \\
X_{8,10} \subset \mathbb{P}(1, 2, 2, 3, 5, 7) & \frac{1}{3}, \frac{1}{5} (2, 5, 2) & 4/21 & 38/63 \\
X_{10,14} \subset \mathbb{P}(1, 2, 2, 5, 7, 9) & \frac{1}{5} (2, 7, 2) & 1/9 & 17/27 \\
X_{8,10} \subset \mathbb{P}(1, 2, 3, 4, 5, 5) & \frac{1}{3}, \frac{2}{3} (1, 4, 2) & 2/15 & 22/45 \\
X_{8,12} \subset \mathbb{P}(1, 2, 3, 4, 5, 7) & \frac{1}{3} (1, 4, 2), \frac{1}{3} (3, 4, 2) & 4/35 & 18/35 \\
X_{10,12} \subset \mathbb{P}(1, 2, 3, 4, 5, 9) & \frac{1}{3}, \frac{1}{9} (4, 5, 2) & 1/9 & 14/27 \\
X_{12,14} \subset \mathbb{P}(1, 2, 3, 4, 7, 11) & \frac{1}{11} (4, 7, 2) & 1/11 & 6/11 \\
X_{10,12} \subset \mathbb{P}(1, 2, 3, 5, 6, 7) & 2 \times \frac{1}{5}, \frac{1}{7} (1, 6, 2) & 2/21 & 31/63 \\
X_{14,16} \subset \mathbb{P}(1, 2, 5, 7, 8, 9) & \frac{1}{5} (2, 3, 2), \frac{1}{8} (1, 8, 2) & 2/45 & 58/135 \\
X_{10,12} \subset \mathbb{P}(2, 2, 3, 5, 5, 7) & 2 \times \frac{1}{5} (2, 3, 2), \frac{1}{7} (2, 5, 2) & 2/35 & 11/35 \\
X_{10,14} \subset \mathbb{P}(2, 2, 3, 5, 7, 7) & \frac{1}{7}, 2 \times \frac{1}{7} (2, 5, 2) & 1/21 & 20/63 \\
X_{12,14} \subset \mathbb{P}(2, 2, 3, 5, 7, 9) & \frac{1}{3}, \frac{1}{5} (2, 3, 2), \frac{1}{7} (2, 7, 2) & 2/45 & 43/135 \\
X_{14,18} \subset \mathbb{P}(2, 2, 3, 7, 9, 11) & 2 \times \frac{1}{7}, \frac{1}{11} (2, 7, 2) & 1/33 & 32/99 \\
X_{18,22} \subset \mathbb{P}(2, 2, 5, 9, 11, 13) & \frac{1}{5} (1, 2, 5), \frac{1}{15} (2, 11, 2) & 1/65 & 17/65 \\
X_{16,12} \subset \mathbb{P}(2, 3, 3, 4, 5, 7) & 4 \times \frac{1}{3}, \frac{1}{7} (3, 4, 2) & 1/21 & 17/63 \\
X_{12,14} \subset \mathbb{P}(2, 3, 4, 5, 7, 7) & \frac{1}{3}, (2, 3, 2), 2 \times \frac{1}{3} (3, 4, 2) & 1/35 & 8/35 \\
X_{14,16} \subset \mathbb{P}(2, 3, 4, 5, 7, 11) & \frac{1}{3}, \frac{1}{9} (2, 3, 2), \frac{1}{11} (4, 7, 2) & 4/165 & 116/495 \\
X_{18,20} \subset \mathbb{P}(2, 4, 5, 7, 9, 13) & \frac{1}{7} (2, 5, 2), \frac{1}{13} (4, 9, 2) & 1/91 & 16/91 \\
X_{18,20} \subset \mathbb{P}(2, 5, 6, 7, 9, 11) & \frac{1}{7} (2, 5, 2), \frac{1}{13} (5, 6, 2) & 2/231 & 103/693 \\
\hline
\end{array}
\]

Table 2: Fano 3-folds in codimension 2

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Fano 3-fold} & \text{Basket } \mathcal{B} & A^3 & Ac_2/12 \\
\hline
X_{2,2,2,2,2} \subset \mathbb{P}(1, 1, 1, 1, 1, 1) & \text{nonsingular} & 3 & 1 \\
X_{3,3,3,4,4,4} \subset \mathbb{P}(1, 1, 1, 1, 2, 2, 3) & \frac{1}{3} & 7/3 & 8/9 \\
\hline
\end{array}
\]

Table 3: Fano 3-folds in codimension 3
| Ambient $\mathbb{P}^7$ | Basket $\mathcal{B}$ | $A^3$ | $Ac_2/12$ |
|------------------------|---------------------|-------|-----------|
| $\mathbb{P}(1, 1, 1, 1, 1, 1)$ | nonsingular | 6 | 1 |
| $\mathbb{P}(1, 1, 1, 1, 1, 2, 3)$ | $\frac{1}{3}$ | 10/3 | 8/9 |
| $\mathbb{P}(1, 1, 1, 2, 2, 3, 3)$ | $2 \times \frac{1}{3}$ | 5/3 | 7/9 |
| $\mathbb{P}(1, 1, 2, 2, 2, 3, 5)$ | $\frac{1}{3}(2, 3, 2)$ | 8/5 | 4/5 |
| $\mathbb{P}(1, 1, 2, 2, 2, 3, 4, 5)$ | $\frac{1}{3}(1, 4, 2)$ | 7/5 | 4/5 |
| $\mathbb{P}(1, 1, 2, 2, 3, 3, 3)$ | $3 \times \frac{1}{3}$ | 1 | 2/3 |
| $\mathbb{P}(1, 1, 2, 2, 2, 3, 3, 5)$ | $\frac{1}{3}, \frac{1}{5}(2, 3, 2)$ | 14/15 | 31/45 |
| $\mathbb{P}(1, 1, 2, 2, 2, 3, 3, 4, 5)$ | $\frac{1}{3}, \frac{1}{5}(1, 4, 2)$ | 6/7 | 5/7 |
| $\mathbb{P}(1, 1, 2, 2, 3, 3, 4, 5)$ | $\frac{1}{3}, \frac{1}{5}(3, 4, 2)$ | 11/15 | 31/45 |
| $\mathbb{P}(1, 1, 2, 2, 3, 3, 5, 8, 11)$ | $\frac{1}{5}(3, 8, 2)$ | 5/7 | 5/7 |
| $\mathbb{P}(1, 2, 2, 3, 3, 5, 5, 7)$ | $\frac{1}{5}(1, 4, 2), \frac{1}{7}(2, 3, 2)$ | 3/7 | 5/7 |
| $\mathbb{P}(1, 2, 3, 3, 5, 8, 11)$ | $\frac{1}{5}(1, 4, 2), \frac{1}{7}(3, 4, 2)$ | 9/35 | 18/35 |
| $\mathbb{P}(1, 2, 3, 3, 4, 5, 9, 13)$ | $\frac{1}{5}, \frac{1}{7}(1, 1, 2)$ | 1/17 | 6/13 |
| $\mathbb{P}(1, 2, 3, 4, 5, 5, 6, 7)$ | $\frac{1}{7}(1, 6, 2), \frac{1}{7}(5, 6, 2)$ | 5/33 | 43/99 |
| $\mathbb{P}(1, 2, 3, 4, 5, 6, 7, 7)$ | $\frac{1}{7}(1, 6, 2), \frac{1}{7}(3, 4, 2)$ | 1/7 | 3/7 |
| $\mathbb{P}(1, 2, 3, 5, 6, 7, 8, 9)$ | $\frac{1}{7}(1, 10, 2)$ | 1/7 | 19/55 |
| $\mathbb{P}(1, 2, 5, 7, 8, 9, 10, 11)$ | $\frac{1}{7}(2, 3, 2), \frac{1}{7}(3, 10, 2)$ | 3/55 | 19/55 |
| $\mathbb{P}(2, 2, 3, 3, 4, 5, 5, 5)$ | $\frac{1}{3}, 3 \times \frac{1}{3}(2, 3, 2)$ | 2/15 | 13/45 |
| $\mathbb{P}(2, 2, 3, 3, 4, 5, 5, 7, 9)$ | $\frac{1}{3}, \frac{1}{7}(2, 3, 2), \frac{1}{7}(2, 3, 5, 2)$ | 13/105 | 92/315 |
| $\mathbb{P}(2, 2, 3, 3, 4, 5, 5, 7)$ | $3 \times \frac{1}{3}, \frac{1}{7}(2, 3, 2), \frac{1}{7}(2, 3, 5, 2)$ | 1/9 | 8/27 |
| $\mathbb{P}(2, 2, 3, 5, 7, 12, 17)$ | $\frac{1}{7}(5, 12, 2)$ | 1/17 | 5/17 |
| $\mathbb{P}(2, 3, 3, 4, 5, 5, 6, 7)$ | $\frac{1}{7}(1, 4, 2)$ | 1/15 | 11/45 |
| $\mathbb{P}(2, 3, 3, 4, 5, 7, 10, 13)$ | $\frac{1}{3}, \frac{1}{7}(3, 10, 2)$ | 2/39 | 28/117 |
| $\mathbb{P}(2, 3, 4, 5, 5, 6, 7, 7)$ | $\frac{1}{3}, \frac{1}{7}(1, 4, 2), \frac{1}{7}(2, 3, 2), \frac{1}{7}(3, 4, 2)$ | 1/21 | 64/315 |
| $\mathbb{P}(2, 3, 4, 5, 5, 6, 7, 9)$ | $\frac{1}{3}, \frac{1}{7}(2, 3, 2), \frac{1}{7}(4, 5, 2)$ | 2/45 | 28/135 |
| $\mathbb{P}(2, 3, 5, 6, 7, 7, 8, 9)$ | $\frac{1}{3}, \frac{1}{7}(2, 3, 2), \frac{1}{7}(1, 6, 2)$ | 1/35 | 19/105 |
| $\mathbb{P}(2, 5, 6, 7, 8, 9, 11)$ | $\frac{1}{3}, \frac{1}{7}(5, 6, 2)$ | 1/55 | 8/55 |

Table 4: Fano 3-folds in codimension 4