POSITIVE SCALAR CURVATURE ON MANIFOLDS WITH FIBERED SINGULARITIES

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Abstract. A (compact) manifold with fibered $P$-singularities is a (possibly) singular pseudomanifold $M_\Sigma$ with two strata: an open nonsingular stratum $M$ (a smooth open manifold) and a closed stratum $\beta M$ (a closed manifold of positive codimension), such that a tubular neighborhood of $\beta M$ is a fiber bundle with fibers each looking like the cone on a fixed closed manifold $P$. We discuss what it means for such an $M_\Sigma$ with fibered $P$-singularities to admit an appropriate Riemannian metric of positive scalar curvature, and we give necessary and sufficient conditions (the necessary conditions based on suitable versions of index theory, the sufficient conditions based on surgery methods and homotopy theory) for this to happen when the singularity type $P$ is either $\mathbb{Z}/k$ or $S^1$, and $M$ and the boundary of the tubular neighborhood of the singular stratum are simply connected and carry spin structures. Along the way, we prove some results of perhaps independent interest, concerning metrics on spin$^c$ manifolds with positive “twisted scalar curvature,” where the twisting comes from the curvature of the spin$^c$ line bundle.

1. Introduction

1.1. Manifolds with fibered singularities. In the paper [8], one of us studied the problem of when a spin manifold with Baas-type singularities admits a metric of positive scalar curvature. This was done when the singularities are a combination of the types $\mathbb{Z}/2$, $\eta = \text{a circle } S^1$ equipped with the non-bounding spin structure, and the Bott manifold of dimension 8 (a geometric generator of Bott periodicity in $KO$-homology). We will use the abbreviation $psc$-metric for metric of positive scalar curvature.

This paper considers a similar problem of existence of positive scalar curvature metrics for spin manifolds with fibered $P$-singularities, where $P$ is a closed manifold. We take $P$ to be a compact Lie group, either a cyclic group $\mathbb{Z}/k$ for some $k$, or $S^1$. In a sequel paper [10], we study the case where $P$ is a compact semisimple Lie group, such as $SO(3)$ or $SU(2)$. Another sequel [12] goes into more detail about the cases $P = \mathbb{Z}/2$ and $P = S^1$.

In more detail, let $P$ be a closed manifold, and $M$ be a compact manifold with boundary $\partial M$. We assume that the boundary $\partial M$ is the total space of a smooth fiber bundle $p: \partial M \rightarrow \beta M$ with the fiber $P$. We denote by $N(\beta M)$ the total space of the associated fiber bundle $\partial M \times_P C(P) \rightarrow \beta M$, where the manifold $P$ is replaced by the cone $C(P)$ fiberwise. The notation is a reminder that

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this will be a tubular neighborhood of the stratum $\beta M$. The associated singular space $M_\Sigma := M \cup_{\partial M} -N(\beta M)$ has a singular stratum $\beta M$ whose normal bundle has fibers homeomorphic to the cone on $P$, but where the normal bundle is not necessarily a trivial bundle. We call $M_\Sigma$ a manifold with fibered $P$-singularity. Clearly the original manifold $M$ with the given smooth fibration $p: \partial M \to \beta M$ uniquely determines the singular space $M_\Sigma$. It’s traditional to call $\beta M$ the Bockstein and $P$ the link.

The situation we study fits nicely into a more general framework. The compact metrizable space $M_\Sigma$ is taken to be a Thom-Mather stratified space (see [26]) of depth one. That means that $M_\Sigma$ is the union of a smooth closed manifold $\beta M$ (the singular stratum) and an open smooth manifold (the regular stratum) $M_\Sigma^{\text{reg}} = M_\Sigma \setminus \beta M$. There is an open neighborhood $N$ of $\beta M$ in $M_\Sigma$, with a continuous retraction $\pi: N \to \beta M$ and a continuous map $\rho: N \to [0, \infty)$ such that $\rho^{-1}(0) = \beta M$, where $\pi: N \to \beta M$ is a fiber bundle over $\beta M$ with fiber $C(P)$, the cone over $P$, as above. The original manifold $M$, called a resolution of $M_\Sigma$, can be identified with $M_\Sigma \setminus \rho^{-1}([0,1))$, so that the boundary $\partial M = \rho^{-1}(1)$ is the total space of a fibration over $\beta M$ with typical fiber $P$. Clearly the interior of $M$ can be identified with the regular stratum $M_\Sigma^{\text{reg}}$. Conversely, given a compact manifold $M$ with fibered boundary $P \to \partial M \to \beta M$, we obtain a Thom-Mather stratified space with two strata by collapsing the fibers.

1.2. Riemannian metrics on manifolds with fibered singularities. There are (at least) two natural definitions of a Riemannian metric on such a singular manifold $M_\Sigma$. The first possibility, which one can call a cylindrical metric, the definition used in [8], is a Riemannian metric in the usual sense on the nonsingular manifold $M$, which is a product metric $dr^2 + g\partial M$ on a collar neighborhood on the boundary $\partial M$, such that the compact Lie group $P$ acts freely by isometries of the boundary and the map $\partial M \xrightarrow{p} \beta M$ is a Riemannian submersion. The curvature of such a metric is defined as usual on the (nonsingular) manifold $M$. Such a metric also determines a Riemannian metric on the “Bockstein” manifold $\beta M$.

Assume $M_\Sigma$ has a psc-metric $g$ as above. Since the metric $g$ is assumed to be a product metric in a collar neighborhood of the boundary $\partial M$, the restriction $g|_{\partial M}$ is a $P$-invariant psc-metric. Thus the question of existence of a cylindrical psc-metric comes down to whether or not there exists a $P$-invariant psc-metric on $\partial M$ that extends to $M$. When $P = \text{SU}(2)$ or $\text{SO}(3)$, the boundary $\partial M$ always has a $P$-invariant psc-metric by [24], for which the quotient map is a Riemannian submersion, so the only question is whether or not this metric on $\partial M$ extends to $M$. By contrast, when $P = \mathbb{Z}/k$, the map $\partial M \to \beta M$ is a covering map, and $\partial M$ has a $P$-invariant psc-metric if and only if $\beta M$ has a psc-metric, for which there is a well-developed obstruction theory (e.g., [31, 32, 33, 11]). When $P = S^1$, the same is true; i.e., $\partial M$ has an $S^1$-invariant psc-metric if and only if $\beta M$ has a psc-metric, but this is now a hard theorem of Bérard-Bergery [5, Theorem C].

The problem with the notion of cylindrical metric is that it doesn’t really take into account the local structure near $\beta M$. A second possible definition is what one could call a conical metric on $M_\Sigma$. In this point of view, the primary object of study is the singular manifold $M_\Sigma$, not the
manifold \( M \) with non-empty boundary. A conical metric is again an ordinary Riemannian metric on the nonsingular part of \( M \) (which is diffeomorphic to the interior of \( M \)), but we require its local behavior near the singular stratum to look like \( dr^2 + r^2 g_P + p^* g_{\beta M} \), where \( g_P \) is a translation-invariant (standard) metric on \( P \), \( g_{\beta M} \) is a metric on the singular stratum \( \beta M \), and \( r \) is the distance to the singular stratum. Note that such a metric on the nonsingular stratum is necessarily incomplete, with \( M_{\Sigma} \) its metric completion. We assume that near the boundary of the tubular neighborhood \( N(\beta M) \), the metric transitions to a cylindrical psc-metric on \((M, \partial M)\), in a sense that will be made precise in Definition \[3.3\] and thus existence of a conical psc-metric is a stronger requirement than existence of a cylindrical psc-metric. When \( P = S^1 \) or \( SU(2) = S^3 \), the cone on \( P \) is homeomorphic to \( \mathbb{R}^2 \) or \( \mathbb{R}^4 \), and so a conical metric near the singular stratum \( \Sigma \) locally looks like an ordinary rotationally invariant Riemannian metric on a vector bundle over \( \beta M \), of fiber dimension 2, resp., 4, and actually extends to a smooth Riemannian metric on \( M_{\Sigma} \). Here is our main question:

**Question:** When does \( M_{\Sigma} \) admit a cylindrical or conical psc-metric?

**Remark 1.1.** In general, existence of a conical psc-metric is stronger than existence of a cylindrical psc-metric. However, they are the same provided \( P = \mathbb{Z}_k \), since then the term \( r^2 g_P \) drops out of the definition of conical metric. When \( P = S^1 \), there are some cases where existence of a cylindrical psc-metric implies existence of a conical psc-metric, but for the most part we will focus on the latter, which has greater geometric significance. See Remark \[3.13\] for further discussion.

### 1.3. Main results.

We consider two cases: when \( P = \mathbb{Z}_k \) or \( P = S^1 \).

1.3.1. *The case of \( P = \mathbb{Z}_k \).* We denote by \( \Omega_{n, \mathbb{Z}/k}^{\text{spin}}(\mathbb{Z}_k\text{-fb}) \) the bordism theory of spin manifolds with fibered \( \mathbb{Z}/k \)-singularities, and by \( \text{MSpin}\mathbb{Z}/k\text{-fb} \) the spectrum which represents this bordism theory.

In more detail, the group \( \Omega_{n, \mathbb{Z}/k}^{\text{spin}}(\mathbb{Z}_k\text{-fb}) \) consists of equivalence classes of maps \( f: M_{\Sigma} \to X \), where \( M_{\Sigma} \) is the singular space associated to an \( n \)-dimensional spin manifold \( M \) with fibered \( \mathbb{Z}/k \)-singularities (with given \( \mathbb{Z}/k \)-fold regular covering map \( p: \partial M \to \beta M \) preserving the spin structure on \( \partial M \) induced from the spin structure on \( M \)). Two such maps \( f: M_{\Sigma} \to X \) and \( f': M'_{\Sigma} \to X \) are said to be equivalent if there is a spin bordism \( M: M \leadsto M' \) between \( M \) and \( M' \) as spin manifolds with boundary, with a \( k \)-fold regular covering map \( \bar{p}: \partial \bar{M} \to \beta \bar{M} \) given by a free action of \( \mathbb{Z}/k \) on \( \partial \bar{M} \), such that the restrictions \( \bar{p}|_{\partial M} \) and \( \bar{p}|_{\partial M'} \) coincide with the corresponding maps \( p: \partial M \to \beta M \) and \( p': \partial M' \to \beta M' \), and there is a map \( \bar{f}: \bar{M} \to X \) restricting to \( f \) and \( f' \) on \( M \) and \( M' \). In particular, the manifold \( \beta \bar{M} \) gives a spin bordism of regular closed spin manifolds \( \beta \bar{M}: \beta M \leadsto \beta M' \).

The groups \( \Omega_{n, \mathbb{Z}/k}^{\text{spin}} \) are closely related to the regular spin bordism groups; indeed, there is an exact triangle

\[
\begin{array}{ccc}
\Omega_{n, \mathbb{Z}/k}^{\text{spin}} & \xrightarrow{i} & \Omega_{n, \mathbb{Z}/k}^{\text{spin}}(\mathbb{Z}_k/\mathbb{Z}) \\
\downarrow{\tau} & & \downarrow{\beta} \\
\Omega_{n, \mathbb{Z}/k}^{\text{spin}}(B\mathbb{Z}/k) & & \Omega_{n, \mathbb{Z}/k}^{\text{spin}}(\mathbb{Z}/k) \\
\end{array}
\]
Here \( i: \Omega^\text{spin}_* \rightarrow \Omega^\text{spin}_{*, Z/k-fb} \) is the homomorphism which considers a regular spin manifold as a manifold with empty singularity, the degree \(-1\) homomorphism \( \beta: \Omega^\text{spin}_{*, Z/k-fb} \rightarrow \Omega^\text{spin}_{*, 1}(BZ/k) \) takes a \( Z/k \)-fibered manifold \( M \) to the map \( \beta M \rightarrow BZ/k \) classifying the \( Z/k \)-fold regular covering \( p: \partial M \rightarrow \beta M \), and finally, \( \tau: \Omega^\text{spin}_{*, 1}(BZ/k) \rightarrow \Omega^\text{spin}_{*, 0} \) is a standard transfer.

We denote by \( KO \) the spectrum representing real \( K \)-theory, and by \( \alpha: MSpin \rightarrow KO \) the map of spectra corresponding to the index map \( \alpha: \Omega^\text{spin}_* \rightarrow KO_* \). The transfer map \( \tau \) in \( KO \) gives the map of spectra \( \tau^KO: KO \wedge (BZ/k)_+ \rightarrow KO \); we denote by \( KO^Z/k-fb \) the cofiber of the map \( \tau^KO \). Then there is a natural map \( \alpha^Z/k-fb: MSpin^Z/k-fb \rightarrow KO^Z/k-fb \) which makes the following diagram of spectra commute:

\[
\begin{array}{ccc}
MSpin \wedge (BZ/k)_+ & \xrightarrow{\tau} & MSpin \\
\downarrow{\alpha \wedge \text{Id}} & & \downarrow{\alpha} \\
KO \wedge (BZ/k)_+ & \xrightarrow{\tau^KO} & KO \\
\end{array}
\]

\( (2) \)

\[
\begin{array}{ccc}
KO & \xrightarrow{\alpha^Z/k-fb} & KO^Z/k-fb \\
\downarrow{\iota^KO} & & \\
KO & \xrightarrow{i^KO} & KO^Z/k-fb
\end{array}
\]

It turns out that the map \( \alpha^Z/k-fb \) is still not quite the right “index map”, since it can be nonzero on some psc-manifolds with fibered \( Z/k \)-singularities. For example, even-dimensional disks with a free \( Z/k \)-action on the boundary sphere will map nontrivially under \( \alpha^Z/k-fb \). However, by composing \( \alpha^Z/k-fb \) with the real assembly map \( KO_*-1(BZ/k) \rightarrow KO_*-1(\mathbb{R}[Z/k]) \) and its splitting we get the index homomorphism

\[
\text{ind}^Z/k-fb: \Omega^\text{spin}_{*, Z/k-fb} \rightarrow KO^Z/k-fb_*.
\]

(See Definition 2.5 and the comments just before it for more details.) The image contains the torsion (all of order 2) in the \( KO \)-theory of the real group ring of the cyclic group \( Z/k \). Here is our first main result on the existence of psc-metrics:

**Theorem A.** Let \( M \) be a spin manifold with fibered \( Z/k \)-singularities, of dimension \( n \geq 6 \). Assume that \( \partial M \) is non-empty, and both \( M \) and \( \partial M \) are connected and simply connected, and the action of \( Z/k \) on \( \partial M \) preserves the spin structure. Then \( M \) admits a metric of positive scalar curvature if and only if \( \text{ind}^Z/k-fb([M]) \) vanishes in the group \( KO^Z/k-fb_* \).

**Outline.** Here is an outline of the proof. In order to prove necessity of vanishing of \( \text{ind}^Z/k-fb([M]) \), we use \( C^* \)-algebraic index theory to show that \( \text{ind}^Z/k-fb([M]) \) is indeed an obstruction to the existence of a psc-metric on a spin manifold \( M \) with fibered \( Z/k \)-singularities. On the other hand, we show also that the existence of a psc-metric on \( M \) depends only on the corresponding bordism class \([M] \in \Omega^\text{spin}_{*, Z/k-fb} \). To prove that vanishing of the index \( \text{ind}^Z/k-fb([M]) \) is sufficient, we analyze the spectra \( MSpin^Z/k-fb \) and \( KO^Z/k-fb \). In particular, we construct the cofiber sequences

\[
\begin{align*}
\begin{cases}
MSpin(Z/k) & \rightarrow MSpin^Z/k-fb \\
KO(Z/k) & \rightarrow KO^Z/k-fb
\end{cases} \rightarrow \Sigma(MSpin \wedge BZ/k), \\
\begin{cases}
KO(Z/k) & \rightarrow KO^Z/k-fb \\
KO(Z/k) & \rightarrow KO^Z/k-fb
\end{cases} \rightarrow \Sigma(KO \wedge BZ/k),
\end{align*}
\]

(3)

where \( MSpin(Z/k) \) and \( KO(Z/k) \) denote \( MSpin \) and \( KO \) with \( Z/k \) coefficients respectively. Moreover, we show that the map \( \alpha^Z/k-fb \) is consistent with these decompositions.
To finish the proof, we use the transfer map \(T_\bullet: \Omega_{*+8}^{\text{spin}, \mathbb{Z}/k\text{-fb}}(BG) \to \Omega_{*+8}^{\text{spin}, \mathbb{Z}/k\text{-fb}}\) from \([35]\), where \(G = \text{PSp}(3)\). Recall that \(G\) is the isometry group of the standard metric on \(\mathbb{H}P^2\), and \(T_\bullet\) takes a map \(f: B \to BG\) to the total space \(E\) of the geometric \(\mathbb{H}P^2\)-bundle \(E \to B\) induced by \(f\).

The diagram (2) allows us to use (3) and the transfer map \(T_\bullet\) to show that all elements of the kernel \(\ker \text{ind}\mathbb{Z}/k\text{-fb} \subset \Omega_n^{\text{spin}, \mathbb{Z}/k\text{-fb}}\) of the index map can be represented by a manifold with fibered \(\mathbb{Z}/k\)-singularities carrying a psc-metric. □

1.3.2. The case \(P = S^1\). This case in some ways is similar to the case of \(\mathbb{Z}/k\)-singularities; however, it has new interesting features.

Let \(M\) be a manifold with fibered \(S^1\)-singularities, i.e., \(M\) comes with a free \(S^1\)-action on the boundary \(\partial M\). Let \(\partial M \to \beta M\) be the smooth \(S^1\)-fiber bundle given by this action. Since the cone on \(S^1\) is \(\mathbb{R}^2\), the corresponding pseudomanifold \(M_\Sigma\) is actually a smooth manifold, but with a distinguished codimension two submanifold \(\beta M\). There are two separate problems to consider: the existence of a cylindrical psc-metric, which is analogous to a psc-metric on a manifold with Baas-type singularities, or the existence of a conical psc-metric, which is about the pseudomanifold \(M_\Sigma\), but we focus primarily on the latter.

We assume that \(M\) is a spin manifold. However, it is worth pointing out that under our definition of fibered \(S^1\)-singularities, even if \(M\) is spin, \(\beta M\) or \(M_\Sigma\) may not be. For example, suppose \(M = D^{2n}\) is the unit disk in \(\mathbb{C}^n\), and we equip \(\partial M = S^{2n-1}\) with the usual free action of \(S^1\) by scalar multiplication by complex numbers of absolute value 1. Then \(\beta M = \partial M/S^1 = \mathbb{C}P^{n-1}\) is non-spin if \(n\) is odd, and \(M_\Sigma = \mathbb{C}P^n\) is non-spin if \(n\) is even. This phenomenon is related to the dichotomy between “even” and “odd” actions of \(S^1\) on a spin manifold, discussed in detail in \([41]\).

In general, the \(S^1\)-fiber bundle \(p: \partial M \to \beta M\) is given by some classifying map \(f: \beta M \to \mathbb{C}P^\infty\), which induces a complex line bundle \(L \to \beta M\). We split \(\partial M\) into the disjoint union of its path-components: \(\partial M = \bigsqcup \partial_i M\), and let \(\beta_i M = p(\partial_i M)\). Then for each component \(\partial_i M\) in \(\pi_0(\partial M)\), we have two possibilities:

(i) the action of \(S^1\) on \(\partial_i M\) is of even type, i.e., spin-structure preserving, or

(ii) the action of \(S^1\) on \(\partial_i M\) is of odd type, i.e., is not spin-structure preserving.

In the case (i), the spin structure on \(\partial_i M\) descends to a spin structure on \(\beta_i M\), but the vertical tangent bundle on \(L\) is not spin, since the \(S^1\)-invariant spin structure on \(S^1\) does not extend to a spin structure on \(D^2\) or on \(\mathbb{C}\). Hence \(M_\Sigma\) is spin\(^c\) but not spin. In the case (ii), \(w_2(\beta_i M)\) has to be in the kernel of the homomorphism \(p^*: H^2(\beta_i M; \mathbb{Z}/2) \to H^2(\partial_i M; \mathbb{Z}/2)\), since \(\partial M\) is spin. This means that \(c_1(L_{|\beta_i M}) \equiv w_2(\beta_i M) \mod 2\) and \(L_{|\beta_i M}\) determines a spin\(^c\)-structure on \(\beta_i M\). Thus the fiber bundle \(p: \partial M \to \beta M\) always splits into even and odd components:

\[
p^{\text{even}}: \partial^{\text{even}} M \to \beta M^{(\text{spin})}, \quad p^{\text{odd}}: \partial^{\text{odd}} M \to \beta M^{(\text{spin}^c)}.
\]

Thus the Bockstein operator \(\beta\) can be described as

\[
\beta: M \mapsto \{(\beta M^{(\text{spin})}, f|_{M^{(\text{spin})}}), (\beta M^{(\text{spin}^c)}, f|_{M^{(\text{spin}^c)}})\}.
\]
We denote by \( \Omega_{\text{spin}, S^1}^* \) the bordism theory of spin manifolds with fibered \( S^1 \)-singularities and by \( \text{MSpin}^{S^1_{\text{fb}}} \) the corresponding spectrum which represents this bordism theory; see section 3 for more details. The Bockstein operator \( \beta \) induces a homomorphism

\[
\beta: \Omega_{n-2}^{\text{spin}}(\mathbb{C}P^\infty) \oplus \Omega_{n-2}^{\text{spin}^c} \to \Omega_{n-1}^{\text{spin}}.
\]

Then we have a transfer map

\[
\tau: \Omega_{n-2}^{\text{spin}}(\mathbb{C}P^\infty) \oplus \Omega_{n-2}^{\text{spin}^c} \to \Omega_{n-1}^{\text{spin}}.
\]

The transfer \( \tau \) takes a pair \((N, L) \in \Omega_{n-2}^{\text{spin}}(\mathbb{C}P^\infty) \) or \((N, L) \in \Omega_{n-2}^{\text{spin}^c} \) (in second case \( L \) is a spin\(^c \)-structure on \( N \)) to the total space \( \widetilde{N} \) of the corresponding \( S^1 \)-fiber bundle \( \widetilde{N} \to N \). We show that there is an exact triangle

\[
(5) \quad \Omega_*^{\text{spin}} \xrightarrow{i} \Omega_*^{\text{spin}, S^1_{\text{fb}}} \xrightarrow{\beta} \Omega_*^{\text{spin}}(\mathbb{C}P^\infty) \oplus \Omega_*^{\text{spin}^c}
\]

where \( i: \Omega_*^{\text{spin}} \to \Omega_*^{\text{spin}, S^1_{\text{fb}}} \) takes a closed spin-manifold \( M \) to a manifold with empty fibered \( S^1 \)-singularity. At the level of spectra, we obtain the following cofibration:

\[
\Sigma(\text{MSpin} \wedge \mathbb{C}P^\infty) \vee \Sigma\text{MSpin}^c \to \Sigma\text{MSpin} \xrightarrow{i} \text{MSpin}^{S^1_{\text{fb}}}.
\]

Just as in the \( \mathbb{Z}/k \)-case, we consider corresponding \( K \)-theories and obtain the following commutative diagram of spectra:

\[
(6) \quad \Sigma(\text{MSpin} \wedge \mathbb{C}P^\infty) \vee \Sigma\text{MSpin}^c \xrightarrow{\tau} \text{MSpin} \xrightarrow{i} \text{MSpin}^{S^1_{\text{fb}}} \\
\Sigma K\text{O} \vee \Sigma KU \xrightarrow{\tau_{\text{KO}}} KO \xrightarrow{i_{\text{KO}}} K\text{O}^{S^1_{\text{fb}}}
\]

Here \( \alpha: \text{MSpin} \to K\text{O} \) and \( \alpha^c: \text{MSpin}^c \to K\text{U} \) are the corresponding index maps\(^1\). In particular, we have the index homomorphism \( \alpha^{S^1_{\text{fb}}}: \Omega_n^{\text{spin}, S^1_{\text{fb}}} \to KO_n^{S^1_{\text{fb}}} \). Here is the second main result, on the existence of a conical psc-metric on a spin manifold with fibered \( S^1 \)-singularities:

**Theorem B.** Let \( M \) be a spin manifold with fibered \( S^1 \)-singularities, of dimension \( n \geq 7 \). Assume that \( \partial M \) is non-empty and connected, and \( M \) and \( \beta M \) are simply connected. Then \( M \) admits a conical metric of positive scalar curvature if and only if \( \alpha^{S^1_{\text{fb}}}(\lbrack M \rbrack) \) vanishes in \( KO_n^{S^1_{\text{fb}}} \) under the above index map, and the relative index for extending the lift to \( \partial M \) of some metric of positive scalar curvature on \( \beta M \) to \( M \) vanishes.

**Outline.** Just as in the \( \mathbb{Z}/k \)-case, we show that the map \( \alpha^{S^1_{\text{fb}}} \) provides an obstruction to the existence of a psc-metric and that the existence of a psc-metric on \( M \) depends only on the bordism class \( \lbrack M \rbrack \in \Omega_n^{\text{spin}, S^1_{\text{fb}}} \). This guarantees the necessity in Theorem B. Again we need more information on the spectra \( KO^{S^1_{\text{fb}}} \) and \( \text{MSpin}^{S^1_{\text{fb}}} \). The final step consists of studying the kernels of the

\(^1\)For more about these maps, see [20].
index maps $\text{MSpin} \to \text{ko}$ and $\text{MSpin}^c \to \text{ku}$. In the case of $\text{MSpin}$, Stolz [35, 36] showed that $\ker \alpha : \Omega^\text{spin}_* \to \text{ko}_*$ is the image of a transfer map $T_\bullet : \Omega^\text{Spin}_*(BG) \to \Omega^{\text{spin}+8}_*$, where $G = \text{PSp}(3)$ and the transfer amounts to taking the total space of an $\mathbb{HP}^2$-bundle. One can do something similar in the (easier) case of $\ker \alpha^c : \Omega^\text{spin}^c_* \to \text{ku}_*$, where this time (since $\text{MSpin}^c$ splits 2-locally as a sum of $\Sigma^4\text{ku}$’s and some $\mathbb{Z}/2$ Eilenberg-Mac Lane spectra; see [29, §8] and [20, p. 184]), the transfer amounts to taking the total space of $\mathbb{CP}^2$-bundles instead of $\mathbb{HP}^2$ bundles. Details of this argument may be found in Section 5. Then (6) allows us to show that all elements of the kernel $\ker \alpha^{S^1-\text{fb}} \subset \Omega^{\text{spin},S^1-\text{fb}}_*$ of the index map on which the secondary (relative) index also vanishes are realized by $\mathbb{HP}^2$- or $\mathbb{CP}^2$-bundles and hence can be represented by manifolds with fibered $S^1$-singularities carrying a psc-metric.

The theorem also holds even when $\partial M$ is disconnected, assuming the appropriate indices vanish on each component of $\beta M$. □

1.4. Plan of the paper. The organization of the rest of the paper is quite straightforward. Section 2 deals in detail with the case of fibered $\mathbb{Z}/k$-singularities, and includes all the details of the proof of Theorem A. Section 3 explains the precise definition of conical psc-metrics in the case of fibered $S^1$-singularities, and includes all the details of the proof of Theorem B, except for two results of possibly independent interest which are needed for the proof of sufficiency, namely the spin$^c$ bordism theorem, which is proved in Section 4 and the theorem on $\mathbb{CP}^2$-bundles, which is proved in Section 5.

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2. The case of fibered $\mathbb{Z}/k$-singularities

This section will be about giving necessary and sufficient conditions for a manifold with fibered $\mathbb{Z}/k$-singularities to admit a psc-metric, under the additional conditions that $M$ and $\partial M$ are connected, simply connected, and spin. In particular, we prove Theorem A.

2.1. Some definitions. Recall that a compact closed manifold with fibered $\mathbb{Z}/k$-singularities means a compact smooth manifold $M$ with boundary, equipped with with a smooth free action of the group $P = \mathbb{Z}/k$ on $\partial M$. We denote by $\beta M$ the quotient space $\partial M/P$; we have a covering map $p : \partial M \to \beta M$. The associated singular space is $M_\Sigma = M/\sim$, where $\sim$ identifies any two points in $\partial M$ having the same image in $\beta M$. We use the notation $(M, \partial M \to \beta M)$ to emphasize the above

\[\text{There is an additional complication one has to be careful about. There are many spin}^c\text{-structures on } \mathbb{CP}^2, \text{ and we need one for which the image in } \text{ku}_4 \text{ vanishes. This is not the one given by the complex structure, for which the image under } \alpha^c \text{ in } \text{ku}_4 \cong \mathbb{Z} \text{ is the Todd genus, which is 1.} \]
\(Z/k\)-fibered structure. Note that a \(Z/k\)-manifold in the sense of Sullivan and Baas [38, 3] is a special case.

Just as in the case of Sullivan-Baas singularities, we say that a smooth manifold \(\overline{M}\) (with non-empty regular boundary \(\partial \overline{M}\)) is a \emph{manifold with fibered \(Z/k\)-singularities with boundary} \(\delta \overline{M} = M\) if the boundary \(\partial \overline{M}\) is given a splitting \(\partial \overline{M} = \delta \overline{M} \cup \partial_1 \overline{M}\) together with free \(Z/k\)-action on \(\partial_1 \overline{M}\). It is required that \(M \cap \partial_1 \overline{M} = \partial M\) so that the \(Z/k\)-action on \(\partial_1 \overline{M}\) restricts to a given free \(Z/k\)-action on \(\partial M\). In particular, we obtain that the \(Z/k\)-covering \(\partial_1 \overline{M} \to \beta \overline{M}\) restricted to \(\partial M\) coincides with the \(Z/k\)-covering \(\partial M \to \beta M\), where \(\partial(\beta \overline{M}) = \beta M\). We denote by \(\overline{M}_\Sigma\) the associated singular space where we identify those points in \(\partial_1 \overline{M}\) which belong to the same orbit under \(Z/k\)-action. Note that then \(\delta \overline{M}_\Sigma = M_\Sigma\) by definition.

**Remark 2.1.** We should emphasize that \(\overline{M}\) should be thought as a manifold with corners, where its \emph{corner} is the manifold \(\partial(\delta \overline{M}) = \delta \overline{M} \cap \partial_1 \overline{M} = -\partial(\partial_1 \overline{M})\); see Fig. 1.

![Figure 1. A manifold \(\overline{M}\) with fibered singularities and corners](image)

A \emph{metric of positive scalar curvature} on \(\overline{M}\) is a Riemannian psc-metric on \(\overline{M}\), which is a product metric in a collar neighborhood of the boundary \(\partial \overline{M}\), and with the metric on \(\partial_1 \overline{M}\) \(Z/k\)-invariant. If \(\delta \overline{M} = \emptyset\), it defines a psc-metric on a closed manifold with fibered \(Z/k\)-singularities.

2.2. **Bordism theory.** Here we set up the bordism theory and the variant of \(K\)-theory that will be needed in this section. We use a slight modification of the bordism theory of Baas [3]. Below we assume that all manifolds in this section are spin.

Let \(X\) be a topological space. Then the group \(\Omega_{\text{spin},Z/k-\text{fib}}^n(X)\) consists of equivalence classes of maps \(f: M_\Sigma \to X\), where \(M_\Sigma\) is the singular space associated to an \(n\)-dimensional spin manifold \((M, \partial M \to \beta M)\) with fibered \(Z/k\)-singularities, with \(Z/k\) preserving the spin structure on \(\partial M\) (this is only an issue if \(k\) is even). Two such maps \(f: M_\Sigma \to X\) and \(f': M'_\Sigma \to X\) are said to be equivalent if there exist a spin manifold \(\overline{M}\) with fibered \(Z/k\)-singularities with boundary \(\delta \overline{M} = M \sqcup -M'\) and a map \(\overline{f}: \overline{M} \to X\) restricting to \(f\) and \(f'\) on \(M\) and \(M'\) respectively.

We use notation \(\overline{M}: M \rightsquigarrow M'\). In particular, the manifold \(\beta \overline{M}\) gives a regular spin bordism between closed manifolds: \(\beta \overline{M}: \beta M \rightsquigarrow \beta M'\).
Proposition 2.2. The transfer \( \tau \) defines a map of bordism spectra \( \tau: \text{MSpin} \wedge (\mathbb{BZ}/k)_+ \to \text{MSpin} \), and \( \text{MSpin}^{Z/k-\text{fib}} \) is the cofiber of this map.\(^3\)

Proof. Clearly, this is just a restatement of the assertion above about the exact triangle (7). To explain the geometry involved, we write out the geometrical proof of the corresponding exact sequence:

\[
\cdots \to \Omega^\text{spin}_{n}(\mathbb{BZ}/k) \xrightarrow{\tau} \Omega^\text{spin}_{n} \xrightarrow{i} \Omega^\text{spin}_{n}^{Z/k-\text{fib}} \xrightarrow{\beta} \Omega^\text{spin}_{n-1}(\mathbb{BZ}/k) \xrightarrow{\tau} \cdots .
\]

That \( \beta \circ i = 0 \) is clear. To see that \( \tau \circ \beta = 0 \), observe that if \( M \) has fibered \( Z/k \)-singularities and \( \partial M = \tilde{N} \), \( \beta M = N \), then \( \tau \circ \beta([M]) = [\tilde{N}] \). And \( \tilde{N} \) bounds as a spin manifold since \( \partial M = \tilde{N} \). To see that \( i \circ \tau = 0 \), note that given \( N \to \mathbb{BZ}/k \) with corresponding covering \( \tilde{N} \), then \( i \circ \tau([N \to \mathbb{BZ}/k]) = [\tilde{N}] \). This is 0 since \( \tilde{N} \) bounds as a manifold with fibered \( Z/k \)-singularities; take \( W = \tilde{N} \times [0,1] \) with \( \partial_{1} W = \tilde{N} \times \{0\} \), \( \delta W = \tilde{N} \times \{1\} \), \( \beta W = N \times \{0\} \equiv N \).

To get \( \ker \beta \subseteq \text{image } i \), observe that if \( M \) has fibered \( Z/k \)-singularities and \( \beta M \) bounds as a spin manifold with mapping to \( \mathbb{BZ}/k \), say \( \beta M = \partial N \), then the associated \( k \)-fold covering \( \tilde{N} \) has boundary \( \partial M \), and \( M \) is bordant as a spin manifold with fibered \( Z/k \)-singularities to the closed manifold \( M \cup_{\partial M} -\tilde{N} \). To get \( \ker \tau \subseteq \text{image } \beta \), observe that if \( N \to \mathbb{BZ}/k \) is a spin manifold with specified \( k \)-fold covering \( \tilde{N} \), and if \( \tilde{N} \) is a spin boundary, say \( \partial M = \tilde{N} \), then \( M \) is a spin manifold with fibered \( Z/k \)-singularities and with \( \beta[M] = [N \to \mathbb{BZ}/k] \). Finally, to see that \( \ker i \subseteq \text{image } \tau \), suppose \( M \) is a closed manifold that bounds as a manifold with fibered \( Z/k \)-singularities. Then there is a \( W \) with \( \partial W \) decomposed into two pieces: one of them a copy of \( M \) and the other \( \tilde{M}' \) projecting down to some \( M' \); this shows \( M \) is bordant to \( \tilde{M}' \) in the image of \( \tau \).

\(^3\) Here, as usual, the subscript \( _{+} \) indicates the addition of a disjoint basepoint, and is needed to convert from reduced to unreduced homology theories.
2.3. Relevant $K$-theory. We denote by $KO$ the spectrum representing real $K$-theory. The transfer map $\tau: \text{MSpin} \wedge (B\mathbb{Z}/k)_+ \to \text{MSpin}$ has its analog in $K$-theories:

$$\tau^{KO}: KO \wedge (B\mathbb{Z}/k)_+ \to KO,$$

which is compatible with the transfer map for $k$-fold coverings of spin manifolds. Let $KO_{\ast}^{\mathbb{Z}/k}(\_)$ be the $K$-theory associated to the cofiber $KO_{\ast}^{\mathbb{Z}/k}$ of the transfer map [8].

From this definition it follows that the groups $KO_{\ast}^{\mathbb{Z}/k}$ fit into a long exact sequence

$$\cdots \to KO_n(B\mathbb{Z}/k) \xrightarrow{\tau^{KO}} KO_n^{\mathbb{Z}/k} \xrightarrow{\beta^{KO}} KO_{n-1}(B\mathbb{Z}/k) \xrightarrow{\tau^{KO}} KO_{n-1} \to \cdots,$$

where when we decompose $KO_n(B\mathbb{Z}/k)$ as $KO_n \oplus \widetilde{KO}_n(B\mathbb{Z}/k)$, $\tau^{KO}$ is multiplication by $k$ on the first summand and 0 on the second. A similar statement holds for spin bordism. Thus we have the following:

**Proposition 2.3.** We have exact sequences

$$0 \to \Omega^{\text{spin}}(\mathbb{Z}/k)_n \to \Omega^{\text{spin}, \mathbb{Z}/k} \xrightarrow{\beta} \Omega^{\text{spin}}_{n-1}(B\mathbb{Z}/k) \to 0$$

and

$$0 \to KO(\mathbb{Z}/k)_n \to KO_{\ast}^{\mathbb{Z}/k} \xrightarrow{\beta^{KO}} KO_{n-1}(B\mathbb{Z}/k) \to 0.$$

Here $\Omega^{\text{spin}}(\mathbb{Z}/k)$ and $KO_{\ast}(\mathbb{Z}/k)$ denote spin bordism and real $K$-theory with $\mathbb{Z}/k$ coefficients. If $k$ is odd, then the group $KO_{\ast}^{\mathbb{Z}/k}$ vanishes except when $n$ is even, when $KO_{\ast}^{\mathbb{Z}/k}$ is an extension of $\widetilde{KO}_{n-1}(B\mathbb{Z}/k)$ by $KO_n \otimes (\mathbb{Z}/k)$, and consists of odd torsion.

**Proof.** Most of this follows immediately from the exact sequences in Proposition 2.2 and (9).

When $k$ is odd, $KO_0(\text{pt}; \mathbb{Z}/k) = KO_0 \otimes (\mathbb{Z}/k)$, which is $\mathbb{Z}/k$ when $4 | n$, 0 otherwise. Since $\widetilde{H}_*(B\mathbb{Z}/k)$ is only nonzero in odd dimensions, the result follows. □

Finally, we need to consider the relationship between the spectra $\text{MSpin}^{\mathbb{Z}/k}$ and $KO^{\mathbb{Z}/k}$. Let $\alpha: \text{MSpin} \to KO$ be the usual Atiyah-Hitchin orientation for spin manifolds (corresponding to the $KO$-index of the Dirac operator). The following result is a consequence of the naturality of the transfer:

**Proposition 2.4.** There is a map of spectra $\alpha^{\mathbb{Z}/k}: \text{MSpin}^{\mathbb{Z}/k} \to KO^{\mathbb{Z}/k}$ making the following diagram commute:

$$\begin{array}{ccc}
\text{MSpin} \wedge (B\mathbb{Z}/k)_+ & \xrightarrow{\tau} & \text{MSpin} \\
\downarrow \alpha \wedge 1 & & \downarrow \alpha \\
KO \wedge (B\mathbb{Z}/k)_+ & \xrightarrow{\tau^{KO}} & KO \\
\end{array}$$

The last step is setting up the correct index map which will give the obstruction to a psc-metric on a compact manifold with fibered $\mathbb{Z}/k$-singularities. For this we need to recall the construction of
the assembly map (see for example [33]). This is much simpler in the case we need here of a finite group \( G \). There is a map of spectra

\[
\text{KO} \wedge BG_+ \xrightarrow{\text{asemb}} \text{KO}(\mathbb{R}[G]),
\]

where \( \text{KO}(\mathbb{R}[G]) \) denotes the \( K \)-theory spectrum of the real group ring, viewed as a Banach algebra, which can be defined in several ways. One method, developed by Loday [25] in a slightly different context, is to use the map on classifying spaces induced by the inclusion \( G \hookrightarrow GL(\mathbb{R}[G]) \). An alternative is to think of assembly as an index map. Given a class in \( KO_n(BG) \), represented, say, by the Dirac operator \( \partial_M \) on a closed spin manifold \( M^n \) with coefficients in some auxiliary bundle, together with a map \( M \to BG \) classifying a \( G \)-covering \( \tilde{M} \) of \( M \),

\[
\text{asemb}(\partial_M, M \to BG) = \text{ind}_{\mathbb{R}[G]}(\tilde{\partial_M}),
\]

where the right-hand side is the index of the lifted \( G \)-invariant operator on \( \tilde{M} \) with coefficients in \( \mathbb{R}[G] \). Since the assembly map for the trivial group is just the identity map, we can peel this off and consider also the reduced assembly map

\[
\widetilde{\text{asemb}} : \text{KO} \wedge BG \to \text{KO}(\mathbb{R}[G]),
\]

where \( \mathbb{R}[G] \) is the sum of the simple summands in the group ring corresponding to all irreducible (real) representations except for the trivial representation. When \( G \) is cyclic of odd order, all representations except for the trivial representation are of complex type, so the groups \( \widetilde{KO}_*(\mathbb{R}[G]) \) are torsion-free while \( \widetilde{KO}_*(BG) \) is torsion. Hence the map \( \widetilde{\text{asemb}} \) vanishes in homotopy. When \( G \) is cyclic of even order, it has exactly two irreducible representations of real type, the trivial representation and the sign representation. These are homomorphisms \( G \to O(1) = \{\pm 1\} \). The remaining representations are of complex type. In this case, it is shown in [33, Theorem 2.5] that \( \widetilde{\text{asemb}} \) is a split surjection onto the torsion of \( \widetilde{KO}_*(\mathbb{R}[G]) \), which consists of \( \mathbb{Z}/2 \) in each dimension \(* \equiv 1, 2 \pmod{8} \).

**Definition 2.5.** The index obstruction map (which appears in the statement of Theorem A) is the map \( \text{ind}_{\mathbb{Z}/k-\text{fb}} : \Omega_*^{\text{spin}, \mathbb{Z}/k-\text{fb}} \to \widetilde{KO}_*^{\mathbb{Z}/k-\text{fb}} \) defined by composing the map \( \alpha^{\mathbb{Z}/k-\text{fb}} \) of Proposition 2.4 with projection onto the inverse image of the torsion in \( \widetilde{KO}_{*-1}(\mathbb{Z}[k]) \). (Recall that assembly gives a split surjection from \( \widetilde{KO}_{*-1}(B\mathbb{Z}/k) \) onto the torsion in \( \widetilde{KO}_{*-1}(\mathbb{Z}[k]) \), and that we have the short exact sequence [11].)

### 2.4. Existence of a psc-metric

The next step is to prove a “bordism theorem” for psc-metrics on spin manifolds with fibered \( \mathbb{Z}/k \)-singularities, which will give a sufficient condition for such a manifold (under the conditions that \( M \) and \( \partial M \) are connected and simply connected) to admit a psc-metric.

---

4 The Baum-Connes assembly map (for a finite group) is a map \( KO^G_*(pt) \to KO_*(\mathbb{R}[G]) \), and is an isomorphism. The assembly map we use here is the composition of that map with the natural map \( KO_*(BG) = KO^G_*(EG) \to KO^n_*(pt) \).
Theorem 2.6 (Cf. [8, Theorem 7.4(1)]). Let $M$ be a spin manifold with non-empty fibered $\mathbb{Z}/k$-singularities, $\dim M = n \geq 6$. Assume that both $M$ and $\partial M$ are connected and simply connected, and $\mathbb{Z}/k$ preserves the spin structure on $\partial M$. If the bordism class $[M] \in \Omega_n^{\text{spin}, \mathbb{Z}/k-\text{fib}}$ contains a representative $M'$ admitting a psc-metric, then $M$ also admits a psc-metric.\footnote{Note that the representative $M' \in [M]$ could be a manifold with empty singularity, but that $M$ has to have $\partial M \neq \emptyset$.}

Proof. The proof is essentially the same as that of [8, Theorem 7.4(1)], using spin surgeries either in the interior of $M'$ or on $\beta M'$ (and then lifted to $\partial M'$).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\Omega_n^{\text{spin}, \text{psc}} & \xrightarrow{i} & \Omega_n^{\text{spin}, \mathbb{Z}/k-\text{fib}} \\
\downarrow & & \downarrow \\
\Omega_n^{\text{spin}} & \xrightarrow{i} & \Omega_n^{\text{spin}, \mathbb{Z}/k-\text{fib}} \\
\end{array}
\]

where the bottom row is exact by Proposition \[2.4\], and the top row consists of the subgroups of the groups on the bottom that are generated by manifolds admitting psc-metrics. It is easy to see that Theorem \[2.6\] implies the following

Corollary 2.7. Let $M$ satisfy the condition of Theorem \[2.6\] Then it admits a psc-metric if and only if its bordism class $[M] \in \Omega_n^{\text{spin}, \mathbb{Z}/k-\text{fib}}$ lies in the subgroup $\Omega_n^{\text{spin}, \mathbb{Z}/k-\text{fib}} \subseteq \Omega_n^{\text{spin}, \mathbb{Z}/k-\text{fib}}$.

We will now use the index obstruction map $\text{ind}_{\mathbb{Z}/k-\text{fib}} : \Omega_*^{\text{spin}, \mathbb{Z}/k-\text{fib}} \to KO_*^{\mathbb{Z}/k-\text{fib}}$ of Definition \[2.5\] in the main existence theorem for psc-metric on manifolds with fibered $\mathbb{Z}/k$-singularities:

Theorem 2.8. Let $M$ be a simply connected spin manifold with fibered $\mathbb{Z}/k$-singularities, of dimension $n \geq 6$. Assume that $\partial M$ is non-empty, connected, and simply connected, and that the action of $\mathbb{Z}/k$ preserves the spin structure on $\partial M$. If the element $\text{ind}_{\mathbb{Z}/k-\text{fib}}([M]) \in KO_n^{\mathbb{Z}/k-\text{fib}}$ vanishes, then $M$ admits a psc-metric.

Proof. First suppose $k$ is odd. By Proposition \[2.3\], $KO_n^{\mathbb{Z}/k-\text{fib}}$ vanishes when $n$ is odd, so in this case $\text{ind}_{\mathbb{Z}/k-\text{fib}}([M])$ automatically vanishes. But in this case $\Omega_n^{\text{spin}}(\mathbb{Z}/k)$ and $\tilde{\Omega}_n^{\text{spin}}(B\mathbb{Z}/k)$ also vanish, so the result follows from Corollary 2.7.

Now suppose $k$ is odd and $n$ is even. Since the homology groups $\tilde{H}_*(B\mathbb{Z}/k, \mathbb{Z})$ are concentrated in odd degrees, the Atiyah-Hirzebruch spectral sequence for $\tilde{\Omega}^n_*(B\mathbb{Z}/k)$ collapses and these groups are also concentrated in odd degrees, where all classes are linear combinations of generators of the form $L^{2m+1} \times N^{4p}$, where $L^{2m+1}$ denotes a lens space of dimension $\geq 3$. (There is a little trick here for getting rid of generators of the form $S^1 \times N^{4p}$ — see [32, proof of Theorem 1.3] and [9].) But $(D^{2m+2} \times N^{4p}, S^{2m+1} \times N^{4p} \to L^{2m+1} \times N^{4p})$ is a spin manifold with fibered $\mathbb{Z}/k$-singularities and positive scalar curvature. So subtracting off suitable positive scalar curvature classes from $[M]$, we can reduce to the case where $\beta([M])$ is trivial in bordism, and thus (by the exact sequence \[10\])
that $[M]$ lies in the image of $\Omega^\text{spin}_\ast(Z/k)_n$ under $i$. Now recall that $\Omega^\text{spin}_\ast$ is a direct sum of $ko_\ast$, detected by the Atiyah-Hitchin obstruction $\alpha$, and the image of the transfer map

$$T: \Omega^\text{spin}_\ast(B \text{PSp}(3)) \to \Omega^\text{spin}_{\ast+8},$$

representing total spaces of $\mathbb{H}P^2$-bundles with positive scalar curvature. Since $\text{ind}^{Z/k-\text{fb}}([M]) = 0$ and $\text{ind}^{Z/k-\text{fb}}$ is faithful on $ko_\ast \otimes (Z/k)$, the component of $[M]$ in $KO(Z/k)_n$ vanishes, and so $[M]$ in $\Omega^\text{spin}(Z/k)_n$ is represented by a manifold of positive scalar curvature. This completes the proof when $k$ is odd.

Now consider the case $k = 2^r \cdot s$, where $r \geq 1$ and $s$ is odd. The situation with the odd torsion is exactly the same as before, so it is no loss of generality to assume $k = 2^r$. Only the torsion behaves differently than when $k$ is odd, so we can localize at 2. After doing so, the spectrum $M\text{Spin}$ splits (additively, not multiplicatively) as a direct sum

$$M\text{Spin}_{(2)} = ko \vee M,$$

where $M$ consists of “higher summands”, i.e., suspensions of $ko$ and $ko(2)$ and Eilenberg-Mac Lane spectra $H\mathbb{Z}/2$, see [1]. Then as was shown by Stolz (see [33, 36]), the spectrum $M$ is in the image of the transfer map

$$T: M\text{Spin} \wedge \Sigma^8 B \text{PSp}(3)_+ \to M\text{Spin}.$$  

We proceed as before, assuming that $\text{ind}^{Z/k-\text{fb}}([M]) = 0$ and looking at the image of $[M]$ in $\widetilde{ko}_{n-1}(B\mathbb{Z}/k)$ in the exact sequence [10]. This is a sum of $\widetilde{ko}_{n-1}(B\mathbb{Z}/k)$ and a summand coming from $M$, represented by total spaces of $\mathbb{H}P^2$-bundles with positive scalar curvature. This second summand is the image under $\beta$ of $\mathbb{H}P^2$-bundles over manifolds with fibered $Z/k$-singularities, so these admit positive scalar curvature. Subtracting these off, we may assume that the image of $[M]$ lies in $\widetilde{ko}_{n-1}(B\mathbb{Z}/k)$. The periodization map $ko_{n-1}(\mathbb{Z}/k) \to KO_{n-1}(\mathbb{Z}/k)$ will turn out to be injective, so we can look at periodic $K$-homology instead.

Let’s look at $\widetilde{KO}_{n-1}(B\mathbb{Z}/2^n)$ in more detail. By [33 Theorem 2.5], this group is a direct sum of $2^{-n} - 1$ copies of the divisible group $Z/2\infty = Z/2^n$ for all even $n$, with one more copy of $Z/2\infty$ when $n$ is divisible by 4, and a copy of $Z/2$ when $n \equiv 2, 3 \pmod{8}$ [6]. The copies of $Z/2$ map nontrivially under the Baum-Connes assembly map, and lie in the image of corresponding summands in $\widetilde{ko}_{n-1}(B\mathbb{Z}/2^n)$, by [9 Lemma 2.8]. The analysis of the Atiyah-Hirzebruch spectral sequence in the proof of that lemma shows in fact that the “periodization” map

$$ko_{n-1}(B\mathbb{Z}/2^n) \to KO_{n-1}(B\mathbb{Z}/2^n)$$

is injective in all degrees. So if $[M]$ is in the kernel of $\text{ind}^{Z/2^n}$ we can assume $M$ represents a bordism class corresponding to one of the $Z/2\infty$ summands in $\widetilde{KO}_{n-1}(B\mathbb{Z}/2^n)$. By [9 §5], that means we can assume $\beta M$ is represented by a linear combination of lens spaces (when $n \equiv 0 \pmod{4}$) or lens spaces bundles over $S^2$ (when $n \equiv 2 \pmod{4}$).

---

[6] All of this comes from dualizing the Atiyah-Segal Theorem and looking at the decomposition of the group ring $R[Z/2^n] \equiv R^2 \oplus C^{2^{-n} - 1}$.  

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First suppose $\beta M$ is represented by a lens space. This case is very much like the corresponding step in the case with $k$ odd. We can take a disk $D^n$, equipped with the constant-curvature metric from the upper hemisphere in $S^n$, and $\beta M$ can then be identified with the quotient of $\partial D^n = S^{n-1}$ by an isometric $\mathbb{Z}/k$-action. So subtracting this off from the bordism class of $M$, we can reduce to the case where $M$ is a spin boundary and admits positive scalar curvature. This covers the case $n \equiv 0 \pmod{4}$. When $n \equiv 2 \pmod{4}$, the situation is only slightly more complicated. Because of [9], we proceed as in the case $n \equiv 0 \pmod{4}$ but with disks replaced by disk bundles over $S^2$. These still have positive scalar curvature so again we can subtract them off and assume the class $[M]$ lies in the image of $\Omega^\text{spin}(\mathbb{Z}/2^n)_n$. Again this group splits as a direct sum of $ko(\mathbb{Z}/2^n)_n$ and a piece coming from the spectrum $M$, represented by total spaces of $\mathbb{H}P^2$-bundles with positive scalar curvature. This latter piece is no problem, so finally we can assume $[M]$ is in the image of $ko(\mathbb{Z}/2^n)_n$ and has vanishing index obstruction. As before that means it represents a bordism class corresponding to one of the $\mathbb{Z}/2^\infty$ summands in $KO_n(B\mathbb{Z}/2^n)$, which come from lens spaces or lens spaces bundles over $S^2$. These have positive scalar curvature so we are done. \qed

2.5. Obstruction to the existence of a psc-metric. Let $M$ be a manifold with fibered $\mathbb{Z}/k$-singularities as above. Then the locally compact groupoid $C^*_\mathbb{R}(M; \mathbb{Z}/k)$ is defined exactly as in [34 §2]; it has unit space $N$, where $N = M \cup_{\partial M} \partial M \times [0, \infty)$ is $M$ with infinite cylindrical ends added to the boundary, and the groupoid structure is defined by the equivalence relation $\sim$ which is trivial on $M$ itself and given by the action of $P = \mathbb{Z}/k$ on $\partial M \times (0, \infty)$. Recall that $C^\mathbb{R}_*(M; \mathbb{Z}/k)$ is an extension

\begin{equation}
0 \to \Gamma_0((0, \infty) \times \beta M, \mathcal{A}) \to C^\mathbb{R}_*(M; \mathbb{Z}/k) \to C^\mathbb{R}(M) \to 0,
\end{equation}

where $\mathcal{A}$ is a bundle over $(0, \infty) \times \beta M$ with fibers isomorphic to $M_k(\mathbb{R})$ and with trivial Dixmier-Douady class, and there is also a “target” real $C^*$-algebra $C^\mathbb{R}_*(pt; \mathbb{Z}/k)$ which is an extension

\begin{equation}
0 \to C^\mathbb{R}_0((0, \infty), M_k(\mathbb{R})) \to C^\mathbb{R}_*(pt; \mathbb{Z}/k) \to \mathbb{R} \to 0.
\end{equation}

The long exact sequence of the extension [14] in $KO$-homology\footnote{We’re following a common abuse of terminology and calling $K$-homology or $KO$-homology the theory that goes by this name on spaces, even though it is contravariant on $C^*$-algebras.} has connecting map

\[
KO_{n-1}(pt) \cong KO_n((0, \infty)) \cong KO^{-n}(C^\mathbb{R}_0((0, \infty), M_k(\mathbb{R}))) \to KO^{1-n}(\mathbb{R}) \cong KO_{n-1}(pt)
\]

induced by the unital ring map $\mathbb{R} \to M_k(\mathbb{R})$, and hence given by multiplication by $k$. Thus $KO^{-n}(C^\mathbb{R}_*(pt; \mathbb{Z}/k)) \cong KO_n(pt; \mathbb{Z}/k)$. Here are our main results about this situation.

**Theorem 2.9** (Obstruction Theorem). Let $M$ be a spin manifold with fibered $\mathbb{Z}/k$-singularities. Assume that the action of $\mathbb{Z}/k$ on $\partial M$ preserves the induced spin structure. Then the Dirac operator on $M$ has a well-defined index

\[
\text{ind}_\mathbb{Z}/k(M) \in KO^{-n}(C^\mathbb{R}_*(pt; \mathbb{Z}/k)) \cong KO_n(pt; \mathbb{Z}/k)
\]
which is independent of the choice of Riemannian metric on \( M \). If \( M \) has psc-metric, then this index must vanish.

**Proof.** This is essentially [34, Definition 4.2] together with [34, Theorem 5.3], except for the fact that the covering map \( p: \partial M \to \beta M \) need not split. If one looks at the proof given there, the splitting is never used, so this is not a problem. \( \square \)

2.6. **Proof of Theorem A.** Now we can put all the pieces together. Recall the statement:

**Theorem 2.10** (Theorem A). Let \( M \) be a spin manifold with fibered \( \mathbb{Z}/k \)-singularities, \( \dim M = n \geq 6 \). Assume that \( M \) and \( \partial M \) are both non-empty and simply connected, and that \( \mathbb{Z}/k \) preserves the spin structure on \( \partial M \). Then the vanishing of the index obstruction \( \text{ind}^{\mathbb{Z}/k}(M) \) is necessary and sufficient for \( M \) to admit a psc-metric.

**Proof.** Sufficiency is in Theorem 2.8. For necessity, there are two pieces, due to the fact that \( \text{ind}^{\mathbb{Z}/k} \) has two components. The necessity of vanishing of the first component, in \( KO_n(\text{pt}; \mathbb{Z}/k) \), is covered by Theorem 2.9. The necessity of vanishing of the second component, in \( K\mathbb{O}(\mathbb{R}[\mathbb{Z}/k]) \), is simply [32, Theorem 3.3] or [33, Theorem 3.1], applied to \( \beta M \). \( \square \)

**Remark 2.11.** It is easy to modify the statement and proof to apply to the situation where \( \partial M \) is disconnected, but each component of \( \partial M \) is simply connected. We leave the details to the reader.

3. **The case \( P = S^1 \)**

3.1. **The goals.** In this section we modify the same program for manifolds with \( S^1 \)-fibered singularities. Since the cone on \( S^1 \) is \( \mathbb{R}^2 \), in this case the pseudomanifold \( M_\Sigma \) is actually a smooth manifold, but with a distinguished codimension-2 submanifold \( \beta M \). Our aim is prove Theorem B, thereby answering a version of the question in Section 1.2. Along the way we will prove some partial results about cylindrical metrics of positive scalar curvature.

3.2. **The setting.** The following is the exact analogue of the definitions in Section 2.1, but we want to restrict the metric to have a particular form near \( \beta M \).

**Definition 3.1.** A compact manifold with \( S^1 \)-fibered singularities will mean a compact smooth manifold \( M \) with boundary, equipped with a smooth free action of \( P = S^1 \) on \( \partial M \). We denote by \( \beta M \) the quotient space \( \partial M/S^1 \). We have a principal \( S^1 \)-bundle \( p: \partial M \to \beta M \) which we identify with the unit circle bundle of a corresponding complex line bundle \( \widetilde{p}: L \to \beta M \). The associated manifold is \( M_\Sigma = M/\sim \), where \( \sim \) identifies any two points in \( \partial M \) lying in the same \( S^1 \)-orbit.

We notice that \( M_\Sigma \) is indeed a manifold since the cone on \( S^1 \) is \( \mathbb{R}^2 \); in fact, we can identify \( M_\Sigma \) with the smooth manifold obtained by gluing the disk bundle \( T \) of \( L \) to \( M \) along their common boundary \( \partial M \cong \partial T \). (Of course we do this in such a way that the resulting manifold is oriented, so that one should think of \( \partial T = -\partial M \), and this, rather than \( \partial M \), is really the disk bundle of \( L \).)
Definition 3.2. A cylindrical metric of positive scalar curvature on $M$ is a Riemannian metric of positive scalar curvature on $M$, which is a product metric in a collar neighborhood of the boundary $\partial M$, and with the metric $S^1$-invariant on $\partial M$.

Clearly the disk bundle $T$ also serves as a tubular neighborhood of $\beta M \subset M_\Sigma$, and comes with the projection map $\tilde{p}: T \to \beta M$.

Definition 3.3. A conical metric of positive scalar curvature on $M$ is a metric of positive scalar curvature on $M_\Sigma$ which is $S^1$-invariant on the tubular neighborhood $T$ of $\beta M$, and which on this tubular neighborhood has the form $(\bar{p})^*g_{\beta M} + h$, where $g_{\beta M}$ is a metric of positive scalar curvature on $\beta M$ and $h$ is a hermitian metric on the line bundle $L$ over $\beta M$, smoothed out near the boundary $\partial T = -\partial M$ to match a metric of the form $dr^2 + g_{\partial M}$ on $M$. More precisely, we fix a connection on the line bundle $L$, giving a choice of horizontal tangent space at each point in the total space of $L$, and make the horizontal and vertical spaces perpendicular, with the metric $(\bar{p})^*g_{\beta M}$ on the horizontal tangent space and the metric $h$ on the vertical tangent space. In a neighborhood of $\partial M$, with $r$ the distance to $\beta M$, the metric will locally look like $dr^2 + f(r) g_{\partial M}$, where $f$ is $C^\infty$, $f(r) \equiv 1$ for $r \geq 1$, and $f(r)$ smoothly transitions to $r^2$ for $r < 1 - \varepsilon$ for some small $\varepsilon$.

This definition is consistent with the idea that a conical metric should have the approximate form $dr^2 + r^2 g_{S^1} + (\bar{p})^*g_{\beta M}$ near $\beta M$.

We notice that since the metric on a neighborhood of $\partial M$ is either exactly (in the cylindrical case) or approximately (in the conical case) of the form $g_{\partial M} + dr^2$, if $M$ admits a metric of positive scalar curvature (in either sense), so does $\partial M$. By [5] Theorem C], this implies that $\beta M$ has a metric of positive scalar curvature.

Remark 3.4. Since the paper [5] is published in a rather inaccessible place (not even indexed by MathSciNet or Zentralblatt), for the benefit of the reader, we repeat the essence of the argument here. Theorem C from [5] states that if one has a closed $n$-manifold $N$ with a free action of $S^1$, giving a principal $S^1$-bundle $p: N \to B$, then the manifold $N$ admits an $S^1$-invariant metric of positive scalar curvature if and only if $B$ admits positive scalar curvature. The “if” direction is relatively easy given [39] Theorem 3.5]: a metric $g_B$ of positive scalar curvature on $B$ lifts to an $S^1$-invariant metric $g_N$ on $N$ with totally geodesic fibers. Let $\kappa_N$ and $\kappa_B$ be the scalar curvatures of $N$ and $B$, respectively. By the O’Neill formulas [23], $\kappa_B = \kappa_N + ||A||^2$, where $A$ is the O’Neill tensor, and rescaling $g_N$ on the circle fibers, one can make $||A||^2$ small so that $\kappa_N > 0$.

For the other direction, assume we have an $S^1$-invariant metric $g_N$ of positive scalar curvature on $N$, and let $g_B$ be the induced metric on $B$ and let $\kappa_B$ and $\Delta_B$ be its scalar curvature and Laplacian. (We use the analysts’ sign convention in which $\Delta_B$ has nonpositive spectrum.) We need to have $n \geq 3$, since if $N$ is a closed 2-manifold fibering over $S^1$, then $N$ is a torus or Klein bottle and does not admit positive scalar curvature. Let $f$ be the function on $B$ which at $x \in B$ gives the length of the $S^1$-fiber over $x$. Bérard-Bergery computes from the O’Neill formulas that if $\kappa_N > 0$, then

$$\kappa_B + 2 \frac{\Delta_B f}{f} > 0.$$
From this it follows that after making the conformal change 
\[ \tilde{g}_B = f^{\frac{4}{n-2}}g_B \] the manifold \((B, \tilde{g}_B)\) has positive scalar curvature.

\[ \square \]

**Example 3.5.** (i) This example is adapted from [5, Example 9.2]. Let \(\beta M\) be a K3 surface (which is a simply connected spin 4-manifold with nonzero \(\tilde{A}\)-genus). Then \(\beta M\) does not admit a metric of positive scalar curvature, but there is a circle bundle \(\partial M\) over \(\beta M\) with simply connected total space \(\partial M\). To construct such a bundle, it is enough to choose a primitive element of 
\[ H^2(\beta M, \mathbb{Z}) \cong \mathbb{Z}^{22} \]
for the first Chern class \(c_1\). The manifold \(\partial M\) is necessarily spin, since \(T_{\partial M} \cong p^*T_{\beta M} \oplus V\), where \(V\) is the real tangent line bundle along the circle fibers, which is trivial, and thus \(w_2(T_{\partial M}) = p^*w_2(T_{\beta M}) = 0\). \(\partial M\) is a spin boundary, since \(\Omega^{\text{spin}}_{11} = 0\). So choosing \(M\) to have boundary \(\partial M\), we get a spin 6-manifold \(M\) with \(S^1\)-fibered singularities, which we can choose to be simply connected. By Van Kampen’s Theorem, the associated manifold \(M_\Sigma\) is also simply connected. As will be seen later (cf. Remark 3.6), \(M_\Sigma\) is spin\(^c\) but not spin, and it admits a metric of positive scalar curvature by [16, Theorem C], but \(\beta M\) does not. This gives a nontrivial example where the singular manifold \(M_\Sigma\) admits a metric of positive scalar curvature but \(M\) itself does not admit positive scalar curvature (either conical or cylindrical) in the sense of Definition 3.1.

(ii) We can also construct a “complementary” example where \(\beta M\) admits a metric of positive scalar curvature, and \(\partial M\) admits an \(S^1\)-invariant metric of positive scalar curvature, but \(M\) does not admit positive scalar curvature (either conical or cylindrical) in the sense of Definition 3.1. Take \(\Sigma^{10}\) to be an exotic 10-sphere with \(\alpha(\Sigma)\) nontrivial in \(KO_{10} \cong \mathbb{Z}/2\). Cut out a disk and let \(M = \Sigma \setminus \hat{D}^{10}\). Then \(\partial M = S^9\), which of course has a free \(S^1\) action with quotient space \(\beta M = \mathbb{C}P^4\). So one can glue in a standard tubular neighborhood of \(\mathbb{C}P^4\) action with quotient space \(\beta M = \mathbb{C}P^4\). Note that \(M_\Sigma\), \(M\), \(\partial M\), and \(\beta M\) are all simply connected, and that \(\partial M\) and \(\beta M\) admit standard metrics of positive curvature, but that this metric on \(\partial M\) cannot extend to a metric of positive scalar curvature on \(M\), since if it did, patching with a standard metric on \(T = D^{10}\) would give a metric of positive scalar curvature on \(\Sigma\), which is ruled out by the \(\alpha\)-invariant.

(iii) The following example is one where \(M_\Sigma\) admits positive scalar curvature but there is a torsion obstruction to positive scalar curvature in the sense of Definition 3.1 due to failure of \(\beta M\) to admit a metric of positive scalar curvature. Start with \(\beta M = \Sigma^{10} \# \mathbb{C}P^5\), where \(\Sigma^{10}\) is a homotopy 10-sphere with nonzero \(\alpha\)-invariant (i.e., representing the generator of \(KO_{10}\)). Since the \(\alpha\)-invariant is additive on connected sums, \(\beta M\) does not admit a metric of positive scalar curvature. However, the manifold \(\beta M\) is a fake complex projective space, so it admits a principal \(S^1\)-bundle for which the total space \(\partial M\) is a homotopy 11-sphere. There being no torsion in \(\Omega^{\text{spin}}_{11}\), the exotic sphere \(\partial M\) is a spin boundary and we can choose a spin 12-manifold \(M\) with \(S^1\)-fibered singularities having boundary this circle bundle over \(\beta M\). By [5, Theorem C], there is no \(S^1\)-invariant metric of positive scalar curvature on \(\partial M\). \[ \square \]
Remark 3.6. Let $M$ be a spin manifold with $S^1$-fibered singularities as in Definition 3.1. Assume for simplicity that $\partial M$ is connected. Since $M$ is a compact spin manifold with boundary, $\partial M$ carries a natural spin structure. The principal bundle $p: \partial M \to \beta M$ is the circle bundle associated to a complex line bundle $\tilde{p}: L \to \beta M$. The tangent bundle of $\partial M$ splits as the direct sum of $p^*(T(\beta M))$ and the real line bundle along the circle fibers, which is trivial. Thus $0 = w_2(\partial M) = p^*(w_2(\beta M))$. So $w_2(\beta M) \in \ker p^*$, which by the Gysin sequence is the $\mathbb{F}_2$-span of $c_1(L)$ reduced mod 2. Hence either $\beta M$ is spin, or else $L$ is nontrivial and $w_2(\beta M) = c_1(L) \mod 2$.

Note that the tangent bundle of the tubular neighborhood $N$ of $\beta M$ coincides with the direct sum $\tilde{p}^*T(\beta M) \oplus \tilde{p}^*L$, and that $N$ has a deformation retraction down to $\beta M$. By the additivity formula for Stiefel-Whitney classes, plus the fact that for a complex line bundle viewed as a real 2-plane bundle, $c_1$ reduces mod 2 to $w_2$, we see that $N$ (and hence also $M_\Sigma$) is spin exactly when $w_2(\beta M) = c_1(L) \mod 2$. Furthermore, if $\partial M$ and $\beta M$ are both simply connected, then from the long exact homotopy sequence

$$0 \to \pi_2(\partial M) \xrightarrow{p_*} \pi_2(\beta M) \xrightarrow{\partial} \pi_1(S^1) = \mathbb{Z} \to 0,$$

we see that $c_1(L) \in H^2(\beta M; \mathbb{Z}) \cong \text{Hom}(\pi_2(\beta M), \mathbb{Z})$ has to be non-zero mod 2, so $\beta M$ and $M_\Sigma$ cannot both be spin. The example given before of $\beta M = \mathbb{C}P^n$ and $M_\Sigma = \mathbb{C}P^n$ is instructive in this regard.

The fact that $\beta M$ and $M_\Sigma$ cannot both be spin in this case, whereas they will both be spin when the link $P$ is higher-dimensional, can be explained by the fact that a spin structure on a manifold is equivalent to a trivialization of the tangent bundle over the 2-skeleton (for a CW decomposition) (see, e.g., [23, Theorem II.2.10]). Since, in our case, $\beta M$ has codimension 2, we do not have room to push $\beta M$ away from this 2-skeleton. For that we would need the singular stratum to have codimension at least 3.

Example 3.7. The following is a more general version of the same example. Let $M_\Sigma$ be a simply connected spin manifold which is not spin, i.e., with $w_2(M) \neq 0$. ($\mathbb{C}P^n$ is an example if $n$ is even.) The spin condition means we have fixed a complex line bundle $\mathcal{L}$ on $M_\Sigma$ such that $c_1(\mathcal{L})$ reduces mod 2 to $w_2(M_\Sigma)$. Choose a submanifold $\beta M$ (which will have codimension 2 in $M_\Sigma$) dual to $\mathcal{L}$; by construction, $c_1(\mathcal{L})$, and thus also $w_2(M_\Sigma)$, is trivial on the complement $M$ of a tubular neighborhood $N$ of $\beta M$, so we get a spin manifold $M$ with boundary $\partial M$, and $\partial M$ is a circle bundle over $\beta M$. We get the line bundle $L$ associated to this circle bundle by pulling back $\mathcal{L}$ via the inclusion $\iota: \beta M \hookrightarrow M_\Sigma$, and $N$ can be identified with the disk bundle of $L$. Furthermore, $\beta M$ is a spin manifold, since

$$w_2(\beta M) + (c_1(L) \mod 2) = w_2(TM_\Sigma|_{\beta M}) = \iota^*w_2(M_\Sigma) = (c_1(L) \mod 2),$$

which says that $w_2(\beta M) = 0$. □

3.3. The case of $P = S^1$: geometry of the tubular neighborhood. For use later on, we want to study in more detail the geometry of the tubular neighborhood $T$ of $\beta M$ in a conical metric. Just
for this subsection, to simplify notation, we replace \( \beta M \) by \( B \) and \( \tilde{p} \) by \( p \). Consider the following general setting:

Let \( B \) be a Riemannian manifold, and let \( p: L \to B \) be a complex line bundle over \( B \), equipped with a connection \( \nabla^L \) and a hermitian metric \( h \). View the total space of \( L \) as a Riemannian manifold with the associated conical metric as in Definition 3.3. Then the projection \( p \) becomes a Riemannian submersion with totally geodesic flat fibers. We denote by \( A \) and \( T \) the associated O'Neill tensors (see [28]), and by \( F \) the curvature 2-form of the line bundle \( L \). [8]

**Proposition 3.8.** Let \( p: L \to B \) be a complex line bundle over \( B \) with connection \( \nabla^L \) and hermitian metric \( h \), as above. Denote by \( X \) and \( Y \) horizontal vector fields projecting to vector fields \( X_\ast \) and \( Y_\ast \) on \( B \), and by \( V \) and \( W \) vertical vector fields. Then

(i) \( A_X V = 0 \) and \( A_X Y = \frac{1}{2i} F(X_\ast, Y_\ast) \partial_z \), with \( \partial_z \) denoting differentiation by \( z \) in the fiber direction;

(ii) for \( v, w \) vertical unit tangent vectors and \( x, y \) horizontal unit tangent vectors, the sectional curvatures of \( L \) are

\[
K_{vw} = K_{vx} = 0, \quad K_{xy} = \tilde{K}_{x_\ast y_\ast} - \frac{3}{4} |F(x_\ast, y_\ast)|^2 \langle \partial_z, \partial_z \rangle_h;
\]

here \( x_\ast = dp(x) \), \( y_\ast = dp(y) \), and \( \tilde{K} \) is the sectional curvature on the base manifold \( B \).

**Proof.** By the construction of the conical metric, the derivative \( dp \) is an isometry on horizontal vectors, and so \( p \) is a Riemannian submersion. Similarly, the metric on the fibers is just the flat Euclidean metric defined by \( h \), and the fibers are totally geodesic. So the O'Neill tensor \( T \) vanishes identically. We have \( A_X V = 0 \) since if we lift a geodesic in \( B \) to a horizontal geodesic in \( L \), then locally, this geodesic and a vertical straight line span a totally geodesic flat submanifold in \( L \). Thus everything follows from [28] Corollary 1, p. 465] once we check that \( A_X Y = \frac{1}{2i} F(X_\ast, Y_\ast) \partial_z \). This in turn follows from the definition of the covariant derivative and curvature in terms of the connection. Let \( X_\ast \) and \( Y_\ast \) partial derivatives with respect to local coordinates in \( B \), lifted up to basic vector fields on \( L \). Then \([X_\ast, Y_\ast] = 0 \) on \( B \) and

\[
A_X Y = \frac{1}{2} [X, Y]^\ast = \frac{\partial_z}{2i} (p^* F)(X, Y) = \frac{1}{2i} F(X_\ast, Y_\ast) \partial_z
\]

(see for example [14 Ch. 1] for some of the relevant calculations). \( \square \)

**Corollary 3.9.** Let \( B \) be a closed Riemannian manifold, and let \( p: L \to B \) be a complex line bundle over \( B \), equipped with a connection and a hermitian metric \( h \). View the total space of \( L \) as a Riemannian manifold with the associated conical metric. If \( L \) has positive scalar curvature, then so does \( B \). Conversely, if \( B \) has positive scalar curvature, then by rescaling the metric \( h \) we can arrange for the scalar curvature of \( L \) to be bounded below by a positive constant.

**Proof.** Immediate from the sectional curvature formulas for \( L \) in Proposition 3.8 \( \square \)

---

[8] We normalize \( F \) to be purely imaginary; many authors normalize it to be real. This would introduce a factor of \( i \) but wouldn’t change the formula for the sectional curvature.
3.4. The case of \( P = S^1 \): obstruction theory. Now we want to set up an obstruction theory that will show that in some cases, a spin manifold \( M \) with \( S^1 \)-fibered singularities (in the sense of Definition 3.1) does not carry a metric of positive scalar curvature. For our first steps it will not actually matter whether we use the cylindrical metric definition (Definition 3.2) or conical metric definition (Definition 3.3), but we concentrate on the latter, which is in many respects more natural. We assume that \( M \) and \( \beta M \) are both simply connected. Then \( M_\Sigma \) will be as well (by Van Kampen’s Theorem). As we saw right after Definition 3.3 there is an obvious necessary condition: \( \beta M \) must admit a metric of positive scalar curvature (if \( M \) carries either a cylindrical or a conical psc-metric), and in addition, in the conical case \( M_\Sigma \) must admit a psc-metric (in the usual sense for closed manifolds).

Let \( M \) be a simply connected spin \( n \)-manifold with \( S^1 \)-fibered singularities and \( \partial M \rightarrow \beta M \) be the corresponding \( S^1 \)-bundle. Assume that \( \beta M \) is also simply connected. There are two cases to consider: when \( \beta M \) is spin or not spin.

If \( \beta M \) is spin, we assume that \( \alpha([\beta M]) = 0 \) in \( KO_{n-2} \), where \( \alpha: \Omega^{\text{spin}}x \rightarrow KOx \) is the index map. Then \( \beta M \) admits a metric of positive scalar curvature by [35]. If \( \beta M \) is not spin, then \( \beta M \) admits a metric of positive scalar curvature by [16]. In both cases, we fix a metric \( g_{\beta M} \) of positive scalar curvature on \( \beta M \). As pointed out in the proof of [16] Theorem C], once one has normalized the arclength of the \( S^1 \) fibers of the map \( p: \partial M \rightarrow \beta M \), this determines uniquely via [39] Theorem 3.5] an \( S^1 \)-invariant metric \( g_{\partial M} \) on \( \partial M \) with totally geodesic fibers, which will have positive scalar curvature if we make the fibers short enough. Attach an infinite cylinder \( \partial M \times [0, \infty) \) to \( M \) along \( \partial M \), give the end the product metric \( g_{\partial M} + dr^2 \), and extend to a metric \( g \) on \( M \cup_{\partial M} (\partial M \times [0, \infty)) \). Since \( g \) has positive scalar curvature except perhaps on a compact set (namely, \( M \)), the \( Cl_n \)-linear Dirac operator on this open manifold is Fredholm and has a well-defined index in \( KO_n \). We call it the relative index since it is essentially the same as the relative index invariant defined in [17] §4.

**Proposition 3.10.** Under these circumstances, the relative index in \( KO_n \) is an obstruction to \( g_{\partial M} \) extending to a metric of positive scalar curvature on \( M \) in either of the senses of Definition 3.1.

**Proof.** This is obvious, as existence of a metric of positive scalar curvature on \( M \) extending the given \( S^1 \)-invariant metric \( g_{\partial M} \) on \( \partial M \) implies that the Dirac operator has spectrum bounded away from 0, by the Lichnerowicz identity, and thus has vanishing index. Note by the way that the relative index can depend on the isotopy class of the positive scalar curvature metric on \( \beta M \), which was used to define the \( S^1 \)-invariant metric \( g_{\partial M} \) on \( \partial M \).

**Remark 3.11.** While computing the relative index of Proposition 3.10 is not always so easy, there is one case where one knows that it vanishes. Namely, if the free \( S^1 \)-action on \( \partial M \) extends to a (non-free) action on \( M \) with some component of the fixed set \( M^{S^1} \) having codimension 2, then \( M \) admits an \( S^1 \)-invariant psc-metric by [11] Theorem 2.4. (The theorem is stated for closed manifolds but the same argument applies to our case.)
Next we state and prove another obstruction theorem which appears in the “only if” part of Theorem B. This result is an adaption of [8, pp. 715–716], which corresponds to the special case where \( p \) is a trivial bundle and \( \partial M = \eta \times \beta M \), where \( \eta \) is \( S^1 \) with its non-bounding spin structure and \( \beta M \) is spin.

**Theorem 3.12.** Let \((M, \partial M)\) be a spin \( n \)-manifold with \( S^1 \)-fibered singularities, determined by an \( S^1 \)-bundle \( p: \partial M \to \beta M \). Assume that \( n \) is even, \( \beta M \) is spin, and \( M_\Sigma \) is spin\(^c\) but not spin, with spin\(^c\) line bundle \( L \) which is trivial over \( M \) and pulled back from \( L_{\beta M} \) over \( \beta M \) on the tubular neighborhood \(-D(L_{\beta M})\) of \( \beta M \). Then \( \text{ind}_{\partial / M_\Sigma}(L) \in K_n \) is an obstruction to existence of a conical metric of positive scalar curvature in the sense of Definition 3.1

**Proof.** For simplicity we will drop mention of the line bundle \( L \) as we are always working with the line bundle outlined in the statement of the theorem. Suppose \( M_\Sigma \) has a conical metric of positive scalar curvature. On \( M \) itself, \( \partial / M_\Sigma \) is just the usual Dirac operator on \( M \), and its square is just \( \nabla^* \nabla + \frac{1}{4} \kappa_M \), which is bounded away from 0. On \( T \) (a tubular neighborhood of \( \beta M \)), the spin\(^c\) line bundle enters also, and we get instead \( \nabla^* \nabla + \frac{1}{4} \kappa_T + R_L \), where the term \( R_L \) has the form

\[
R_L = \frac{1}{2} \sum_{j < k} F_L(e_j, e_k) \cdot e_j \cdot e_k,
\]

where one sums over an orthonormal frame and \( F_L \) is the curvature of the line bundle \( L \). Since the index is invariant under a homotopy of the metric, we can replace the original conical metric on \( T \) by the restriction of a metric on the \( \mathbb{C}P^1 \)-bundle compactification \( \mathbb{P}(L \oplus 1) \) of the line bundle \( L \) over \( \beta M \), smoothed out near \( \partial M \) to patch with a psc metric on \( M \), and by making the \( \mathbb{C}P^1 \) fibers very small, we can get the scalar curvature of \( T \) to dominate the \( R_L \) term. (This is exactly the same argument that was used in [8].) Hence the square of the spin\(^c\) Dirac operator can be bounded strictly away from 0 everywhere, and when \((M, \partial M)\) admits a conical metric of positive scalar curvature, the \( KU \)-index of \( \partial_{M_\Sigma} \) must vanish, which is what we wanted to show. \( \square \)

**Remark 3.13.** We believe that it should be possible to use Theorem 3.12 to construct a spin manifold with \( S^1 \)-fibered singularities that admits a cylindrical psc metric, but not a conical psc metric. This would show that the conditions of Definitions 3.2 and 3.3 are not equivalent. The idea would be to construct \( M^n \), a compact simply connected spin manifold, such that \( \partial M \) admits a free \( S^1 \)-action of *even* type (in the sense of [41]), so that \( \partial M/S^1 = \beta M \) is spin and simply connected. The action being of even type forces the Chern class \( c_1(L) \) classifying the line bundle \( L \) and the circle bundle \( p: \partial M \to \beta M \) to be *odd* (not to reduce to 0 mod 2). Then \( M_\Sigma \) would not be spin, and Theorem 3.12 is applicable. If \( n \geq 7 \) and \( \alpha(\beta M) = 0 \) in \( KO_{n-2} \), then \( \beta M \) admits a metric of positive scalar curvature by [35]. If one could choose this psc-metric so that its lift to an \( S^1 \)-invariant metric on \( \partial M \) extends over \( M \), then \((M, \partial M)\) would admit a cylindrical psc-metric. On the other hand, if \( \alpha^c(M_\Sigma) \neq 0 \), then \( M_\Sigma \) would not admit a conical psc-metric. Unfortunately we have not yet been able to construct a case where we can verify all these details simultaneously, so at the moment, that cylindrical and conical psc-metrics are not equivalent is only a conjecture.
3.5. The case of $P = S^1$: bordism theory. Here we want to consider the analogue of the definition from Section 2.2.

**Definition 3.14.** Let $X$ be a topological space. The group $\Omega^{\text{spin}, S^1-\text{fb}}(X)$ will consist of equivalence classes of maps $f_0: M_\Sigma \to X$, where $M_\Sigma$ is the closed manifold associated to an $n$-dimensional spin manifold $(M, \partial M)$ with $S^1$-fibered singularities in the sense of Definition 3.1. Two such maps $f_0: M_\Sigma \to X$ and $f_1: M'_\Sigma \to X$ are said to be equivalent if there is a spin bordism $\overline{M}$ between $M$ and $M'$ as spin manifolds with boundary, with a principal $S^1$-bundle $p: \partial \overline{M} \to \beta \overline{M}$ given by a free action of $S^1$ on $\overline{M}$, such that the restrictions $p|_{\partial M}$ and $p|_{\partial M'}$ coincide with the corresponding maps $p: \partial M \to \beta M$ and $p': \partial M' \to \beta M'$, and there is a map $\overline{f}: \overline{M} \to X$ restricting to $f_0$ and $f_1$ on $M$ and $M'$. In particular, the manifold $\beta \overline{M}$ gives a spin bordism of closed manifolds $\beta M \rightsquigarrow \beta M'$.

Exactly as in [3], it is easy to see that $\Omega^*_\text{spin}(S^1-\text{fb})(-)$ is a homology theory represented by a spectrum $\mathbb{M} \text{Spin}^{S^1-\text{fb}}$. We need the analogues of (7) and of Proposition 2.2, but this is somewhat more involved since, as we have seen in examples, $\beta M$ is not always spin. Thus the “Bockstein map” $\beta$ will take its values in a direct sum of two bordism groups, one spin and one spin$^c$. In fact (if $\partial M$ has multiple components), $\beta M$ can have components in both groups.

We have three natural transformations here. The first one

$$i: \Omega^*_\text{spin}(S^1-\text{fb})(-) \to \Omega^*_\text{spin}(\text{S}^1-\text{fb})(-)$$

is given by considering closed spin manifolds as manifolds with empty $S^1$-fibered singularity.

We notice that the Bockstein operator $M \mapsto \beta M$ comes together with a map $f: \beta M \to BS^1 = \mathbb{C}P^\infty$ classifying the $S^1$-bundle $\partial M \to \beta M$, or equivalently a line bundle $L$ over $\beta M$. There is a natural splitting of the Bockstein transformation into two pieces, $\beta_{\text{even}}$ and $\beta_{\text{odd}}$. This splitting can be traced back to [7, Propositions 2.2 and 2.3]. The two summands keep track of the components $\partial_i M$ of $\partial M$ where the $S^1$-action is of even type (preserves the spin structure) and those where it is of odd type (does not preserve the spin structure), respectively. In both cases, we get a pair $[N, L]$, where $N$ is one of the components $\beta_i M$ of $\beta M$ and $L$ is a line bundle on $N$. However, there is a difference:

$\beta_{\text{even}}$: In the even case, the spin structure on $\partial M$ descends to a spin structure on $N$, so that we can think of $[N, L]$ as living in $\Omega^*_\text{spin}(\mathbb{C}P^\infty)$.

$\beta_{\text{odd}}$: In the odd case, the manifold $N$ has a spin$^c$ structure determined by $L$, since in order to get a spin structure on $\partial_i M$, the class $w_2(\beta_i M)$ has to be in the kernel of $p^*$, which means $c_1(L_{|\beta_i M}) \equiv w_2(\beta_i M) \mod 2$ and $L_{|\beta_i M}$ determines a spin$^c$ structure on $\beta_i M$. Thus in the odd case, $[N, L]$ lives in $\Omega^*_\text{spin}^c$.

To sum up, we have a natural transformation

$$\Omega^*_\text{spin}(S^1-\text{fb})(-) \xrightarrow{\beta_{(\beta_{\text{even}}, \beta_{\text{odd}})}} \Omega^*_\text{spin}(\text{S}^1-\text{fb})(-) \oplus \Omega^*_\text{spin}^c(-)$$

of degree $-2$. Finally we have a transformation

$$\tau: \Omega^*_\text{spin}(\text{S}^1-\text{fb})(-) \oplus \Omega^*_\text{spin}^c(-) \to \Omega^*_\text{spin}(-)$$
of degree $+1$ which is a transfer: it sends a map $f : N \to X$ to $\tilde{f} : \tilde{N} \to X$, where $\tilde{N}$ is the total space of the $S^1$-bundle determined by the line bundle over $N$ given by either the map to $\mathbb{CP}^\infty$ or by the spin$^c$ structure.

**Theorem 3.15.** We have an exact triangle of (unreduced) bordism theories

$$
\Omega^\text{spin}_*(-) \xrightarrow{i} \Omega^\text{spin}_*, \beta = (\beta_{\text{even}}, \beta_{\text{odd}}) \xrightarrow{\tau} \Omega^\text{spin}, S^1\text{-fb}(\_ - ) \xrightarrow{\beta} \Omega^\text{spin}_*(\_ \times \mathbb{CP}^\infty) \oplus \Omega^\text{spin}^c_*(\_) \oplus \Omega^\text{spin}^c_*(\_ - )
$$

(15)

**Proof.** As this is all we really need, we will show that

$$
\ldots \xrightarrow{\beta} \Omega^\text{spin}^c_{n-1}(\mathbb{CP}^\infty) \oplus \Omega^\text{spin}^c_{n-2} \xrightarrow{\tau} \Omega^\text{spin}^c_n \xrightarrow{i} \Omega^\text{spin}^c_{n-1}, S^1\text{-fb} \xrightarrow{\beta} \Omega^\text{spin}^c_{n-2}(\mathbb{CP}^\infty) \oplus \Omega^\text{spin}^c_{n-2} \xrightarrow{\tau} \ldots
$$

(16)

is exact. First we note a few easy facts: $\beta \circ i = 0$, since the image of $i$ consists of (classes of) closed manifolds with empty boundary; $\tau \circ \beta = 0$, since $\tau \circ \beta([M, \partial M]) = [\partial M]$, and $\partial M$ bounds $M$; and $i \circ \tau = 0$, since if $M \xrightarrow{\beta} N$ is a principal $S^1$-bundle with associated complex line bundle $L$ over $N$ and $N$ spin$^c$ or spin and $M$ spin, then $i \circ \beta([N, L]) = [M]$. Let $W = M \times [0, 1]$ with $\beta W = N$. (See the comments in the parallel argument in the proof of Proposition 2.2)

Next, let’s show that $\ker \tau \subseteq \text{image } \beta$. Suppose we have a spin manifold $N_1$, with line bundle $L_1$, and a spin$^c$ manifold $N_2$ with line bundle $L_2$ associated to the spin$^c$ structure. Then $\tau([N_1, L_1] + [N_2, L_2]) = 0$ means that the disjoint union $M_1 \amalg M_2$ is a spin boundary, where $M_j$ is the circle bundle over $N_j$ defined by $L_j$. Let $P$ be a spin manifold with boundary $M_1 \amalg M_2$. Then $P$ is a spin manifold with $S^1$-fibered singularities and $\beta(P, \partial P) = [N_1, L_1] + [N_2, L_2]$

Let’s show that $\ker \beta \subseteq \text{image } i$. Suppose $\beta([M, \partial M]) = 0$. That means $(\beta M, L)$ bounds in the appropriate sense, i.e., if $\beta M = \beta_1 M \amalg \beta_2 M$ with $\beta_1 M$ spin and $\beta_2 M$ spin$^c$, then $[\beta_1 M, L_1] = 0$ in $\Omega^\text{spin}_{n-2}(\mathbb{CP}^\infty)$ and $[\beta_2 M] = 0$ in $\Omega^\text{spin}^c_{n-2}$. Then changing things up to bordism, we can assume $\beta M$ is actually empty, and then $\beta M$ lies in the image of $i$.

Finally, we show that $\ker i \subseteq \text{image } \tau$. Suppose $M$ is a closed spin $n$-manifold and $i([M]) = 0$. That means that $M$ bounds, not necessarily in the sense of spin manifolds, but in the sense of spin manifolds with $S^1$-fibered singularities. Choose $\overline{M}$ bounding $M$, with a free $S^1$-action on $\partial \overline{M}$. This means in particular that $M$ is the total space of a principal $S^1$-bundle over a spin or spin$^c$ manifold, so $[M]$ lies in the image of $\tau$.

**Lemma 3.16.** Split the domain of the transfer map $\tau$ of (16) into three summands:

$$
\Omega^\text{spin}_* \oplus \Omega^\text{spin}^c_* (\mathbb{CP}^\infty) \oplus \Omega^\text{spin}_*
$$

Then $\tau$ vanishes identically on $\Omega^\text{spin}_*$ and on $\Omega^\text{spin}^c_*(\mathbb{CP}^\infty)$, and its restriction to $\Omega^\text{spin}_*$ has finite image. There is a natural splitting of $\beta$ on the $\Omega^\text{spin}^c_{n-2}$ summand.

**Proof.** First suppose $N$ is a spin$^c$ manifold and $L$ is the line bundle defined by the spin$^c$ structure. Then the total space of $L$ is a spin manifold, as explained in Remark 3.6 and in Example 3.7. Hence the disk bundle $T(L)$ of $L$ is a spin manifold bounding the circle bundle of $L$, which is $\tau(N, L)$, and
so $\tau([N, L])$ is trivial in spin bordism. (Note that the $S^1$-action on $\tau([N, L])$ is free of odd type.) Mapping $N$ back to $(-T(L), -\partial T(L))$ gives us a splitting of $\beta$ from $\Omega_{n-2}^{\text{spin}}$ to $\Omega_n^{\text{spin}, S^1}$.

The other summand in the domain of $\tau$ is $\widetilde{\Omega}_{n-2}^{\text{spin}}(\mathbb{CP}^\infty)$. We can describe the transfer map $\tau$ on this summand with the aid of [6, §6]. (We are indebted to [35] for this reference.) This comes from a Thom map $\Sigma \mathbb{CP}^\infty \to S$, where $S$ is the sphere spectrum, which comes from stabilizing the map $\Sigma \mathbb{CP}^n \to S^{2n+1}$ collapsing all but the very top cell. Since this map is null-homotopic (in the limit as $n \to \infty$), the transfer map $\tau$ vanishes.

Note that it was asserted in [37, Proposition, p. 354] that $e \Omega^{\text{spin}}_{n-2}(\mathbb{CP}^\infty)$ is isomorphic to $\Omega_n^{\text{spin}, c}$, but this appears to be a misprint, since, for example, $\Omega_0^{\text{spin}, c} \cong \mathbb{Z}$ and $\widetilde{\Omega}_2^{\text{spin}}(\mathbb{CP}^\infty) = 0$. The correct statement may be found in [8, Remark 5.10], that there is an equivalence $\mathbb{MSpin}^c \cong \mathbb{MSpin} \wedge \Sigma^{-2}\mathbb{CP}^\infty$, or in other words $\Omega_n^{\text{spin}, c} \cong \widetilde{\Omega}_{n+2}^{\text{spin}}(\mathbb{CP}^\infty)$. (The geometric interpretation of this is that dualizing a complex line bundle over a closed spin manifold gives a codimension-2 spin $c$ submanifold.) So $\widetilde{\Omega}_{n-2}^{\text{spin}}(\mathbb{CP}^\infty) \cong \Omega_{n-4}^{\text{spin}}$.

Restating Theorem 3.15 and Lemma 3.16 in slightly different language, we have the following:

**Proposition 3.17.** The bordism spectrum $\mathbb{MSpin}^{S^1}$ sits in a cofiber sequence, which is rationally split, of the form

$$
\mathbb{MSpin} \to \mathbb{MSpin}^{S^1} \to \Sigma^2 \mathbb{MSpin} \vee \Sigma^2 \mathbb{MSpin}^c \vee \Sigma^4 \mathbb{MSpin}^c.
$$

**Proof.** This follows from the homotopy-theoretic description of the bordism groups and Lemma 3.16. \hfill \Box

**Definition 3.18.** The orientation maps $\alpha: \mathbb{MSpin} \to KO$ and $\alpha^c: \mathbb{MSpin}^c \to K$ for spin and spin $c$ manifolds respectively are defined by taking the index of the appropriate Dirac operator. Then the maps $\alpha$ and $\alpha^c$ give a map of spectra

$$
\text{ind}^{S^1}: \mathbb{MSpin}^{S^1} \to \Sigma^2 KO \vee \Sigma^2 K.
$$

In more detail, the $K$ component corresponds to the $\alpha^c$ invariant for $M_\Sigma$, and the $KO$ component corresponds to the $\alpha$ invariant for spin components of $\beta M$. In addition, the relative index of Proposition 3.10 is defined on the kernel of $\text{ind}^{S^1}$. Indeed, when $\text{ind}^{S^1}(M, \partial M) = 0$, $\beta M$ admits a metric of positive scalar curvature, and then by Corollary 3.9 there is a metric of positive scalar curvature on $\partial M$ extending over a tubular neighborhood of $\beta M$, and we can compute the index obstruction to extending this metric over $M$. The relative index might depend on the choice of a positive scalar curvature metric on $\beta M$. More specifically, fixing a path component of positive scalar curvature metrics on $\beta M$ is essentially equivalent (via the correspondence in [5]) to fixing a path component of the $S^1$-invariant psc-metrics on $\partial M$. Between any two of these, there is a relative index in $KO_n$, $n = \dim M$, and the relative index obstruction for extending an $S^1$-invariant psc-metric $g'_\partial$ on $\partial M$ over $M$ is the same as the relative index obstruction for another $S^1$-invariant psc-metric $g_\partial$ plus the relative index $i(g'_\partial, g_\partial)$, by [17, Theorem 4.41].
Theorem 3.19 (Cf. [8] Theorem 7.4(1))). Let $M$ be a simply connected spin manifold with $S^1$-fibered singularities, of dimension $n \geq 7$. Assume that each component of $\beta M$ is simply connected. If the class of $M$ in $\Omega_{n}^{\text{spin}},S^{1}\text{-fb}$ contains a representative $M'$ of positive scalar curvature, then $M$ admits a metric of positive scalar curvature.

Proof. As we saw in Lemma 3.16 and Proposition 3.17, $\Omega_{n}^{\text{spin}},S^{1}\text{-fb}$ is built out of spin and spin$^c$ bordism groups. For the spin bordism groups, the proof is essentially the same as that of [8, Theorem 7.4(1)], using spin surgeries either in the interior of $M'$ or on $\beta M'$ (and then lifted to $\partial M'$). The latter case requires a little bit of care, as we explain below.

Let $\tilde{M}: M \sim M'$ be a bordism in $\Omega_{n}^{\text{spin}},S^{1}\text{-fb}$. In particular, it means we have a bordism $(\beta \tilde{M}): (\beta M, f) \sim (\beta M', f')$, where $\bar{f}: \beta \tilde{M} \to \mathbb{C}P^{\infty}$ is a classifying map for the $S^1$ fiber bundle $\bar{p}: \partial \tilde{M} \to \beta \tilde{M}$, which restricts to corresponding classifying maps $f$ and $f'$ on the boundary $\partial(\beta M) = \beta M \sqcup -\beta M'$.

In order to “push” a psc metric from $M'$ to $M$, we have to show that the bordism $\tilde{M}: M \sim M'$ can be modified so that the embeddings $(M',\partial M') \hookrightarrow (\tilde{M},\partial \tilde{M})$ and $\beta M' \hookrightarrow \beta \tilde{M}$ are both 2-connected. The only non-standard part is a modification of the bordism $\beta \tilde{M}: \beta M \sim \beta M'$.

First we notice that the inclusion $i': \beta M' \hookrightarrow \beta \tilde{M}$ can be assumed to be 1-connected. Indeed, let $\iota: S^1 \hookrightarrow \beta \tilde{M} \setminus \beta M'$ be an embedding; then the composition $\bar{f} \circ \iota: S^1 \hookrightarrow \beta \tilde{M} \to \mathbb{C}P^{\infty}$ is null-homotopic. Hence the classifying map $\bar{f}$ can be extended to the manifold resulting from surgery along $S^1 \hookrightarrow \beta \tilde{M} \setminus \beta M'$. Moreover, the same argument shows that we can assume both manifolds $\beta M'$ and $\beta \tilde{M}$ are simply-connected.

The $S^1$-bundle $\bar{p}: \partial \tilde{M} \to \beta \tilde{M}$ restricts to the bundle $\bar{p}': \partial M' \to \beta M'$; together they give us the commutative diagram:

$$
\begin{array}{ccc}
\pi_2 \partial M' & \xrightarrow{\psi'} & \pi_2 \beta M' \\
\downarrow \bar{f} & & \downarrow \bar{f}' \\
\pi_2 \partial \tilde{M} & \xrightarrow{\bar{p}'} & \pi_2 \beta \tilde{M}
\end{array}
$$

(18)

Let $\phi: S^2 \to \beta M'$ be such that $\partial': [\phi] \to 1 \in \pi_1(S^1)$. Then $\bar{\partial}: [\bar{\phi}] \to 1 \in \pi_1(S^1)$, where $\bar{\phi} = \phi' \circ i': S^2 \to \beta \tilde{M}$.

Given these elements, we consider a map $\tilde{\psi}: S^2 \to \beta \tilde{M}$ such that $[\tilde{\psi}] = a \neq 0$ in the factor group $\pi_2 \beta M/\text{Im } i'_*$. If $\bar{\partial}([\tilde{\psi}]) = 0$, then we can do surgery along $S^2 \hookrightarrow \beta \tilde{M}$ and extend the map $\bar{f}$ to the resulting manifold. If $\bar{\partial}([\tilde{\psi}]) = k \in \pi_1 S^1$, we can adjust the element $[\tilde{\psi}]$ by adding $-k[\phi]$. This creates a representative $\tilde{\psi}: S^2 \to \beta \tilde{M}$ of the same element $a \in \pi_2 \beta M/\text{Im } i'_*$ such that $\bar{\partial}([\tilde{\psi}]) = 0$. This construction gives a bordism $\beta \tilde{M}: \beta M \sim \beta M'$, where the embedding $i': \beta M' \hookrightarrow \beta \tilde{M}$ is 2-connected. Finally, to obtain the desired modification of the bordism $\tilde{M}: M \sim M'$, we perform surgeries on the interior of $\tilde{M} \setminus \partial \tilde{M}$ using the standard method found, say, in [8, Theorem 7.4].

We have to use the restriction $n \geq 7$ since the Bockstein manifolds $\beta M$ and $\beta M'$ need to have dimension at least five. For the spin$^c$ bordism groups, the proof will be given in Theorem 4.3. □

Now we get to the main part of Theorem B. The following Theorem 3.20 relies on Corollary 5.2, which will be proved in Section 5.
Theorem 3.20. Let \((M, \partial M)\) be a spin \(n\)-manifold with \(S^1\)-fibered singularities, with \(n \geq 7\), determined by an \(S^1\)-bundle \(p: \partial M \to \beta M\). Assume that \(M\) and \(\beta M\) are connected and simply connected. Then \(M\) admits a conical metric of positive scalar curvature in the sense of Definition 3.1 if and only if the bordism class of \(M\) maps to 0 in \(K_{n-2} \oplus K_{n-2}\) under the map \(\text{ind}^{S^1-\text{fb}}\) from \(\Sigma_{n-2}\) and the relative index of Proposition 3.10 vanishes in \(K_n\) for some choice of a positive scalar curvature metric on \(\beta M\).

Remark 3.21. Note that the case of \(\eta\)-type singularities of [8, Theorem 1.1] is subsumed in Theorem 3.20. In the case considered there, \(p\) is a trivial bundle and \(\partial M = \eta \times \beta M\), where \(\eta\) is \(S^1\) with its non-bounding spin structure and \(\beta M\) is spin. In that case, \(M\_\chi\) is spin and the obstruction constructed in the proof on pages 715–716 of [8] is the component of \(\text{ind}^{S^1-\text{fb}}\) with values in \(K_n\).

Proof of Theorem 3.20. We begin with the necessity of the condition, which doesn’t involve the dimensional restriction \(n \geq 7\). Vanishing of the \(K_{n-2}\) term when \(\beta M\) is spin is needed for \(\beta M\) to admit a metric of positive scalar curvature, which as we have seen is a necessary condition. Similarly, vanishing of the \(KO_n\) term when \(M\_\chi\) is spin is needed for \(M\_\chi\) to admit a metric of positive scalar curvature, which is also a necessary condition. If \(\beta M\) is spin but not spin, then the \(K_{n-2}\) index automatically vanishes, and \(\beta M\) admits a metric of positive scalar curvature. Then by Corollary 3.9 with the right scaling of the metric on the fibers, there is a conical metric on the tubular neighborhood \(T\) of \(\beta M\) which also admits positive scalar curvature. Now the \(KU_n\) index of the spin Dirac operator \(\phi_{M\_\chi}\) on \(M\_\chi\) is also an obstruction to positive scalar curvature, by Theorem 3.12.

The argument for sufficiency uses the surgery method and the condition \(n \geq 7\), which guarantees that \(M\) and \(\beta M\) each have dimension at least 5, the range where the \(h\)-cobordism theorem applies. Suppose that \(\text{ind}^{S^1-\text{fb}}(M, \partial M) = 0\) and that \(M\) and \(\beta M\) are simply connected.

First assume that the \(S^1\)-action on \(\partial M\) is of odd type. In this case we view \((\beta M, L)\) as defining a class in \(\Omega^\text{spin}_{n-2}\). By Lemma 3.16, we have a splitting \(\Omega^\text{spin}_{n-2} \to \Omega^\text{spin}_{n, S^1-\text{fb}}\) obtained by taking the disk bundle \(D(L)\), which is spin since \(u_2(D(L)) = \overline{p^*}(w_2(\beta M)) = \overline{p^*}(c_1(L)) \) (mod 2) = 0 by the Gysin sequence. This admits a conical metric of positive scalar curvature, so subtracting this off, we can assume up to bordism that \(\beta M\) is empty. Then \(M\) is closed simply connected spin manifold which has vanishing \(\alpha\)-invariant, since for a closed manifold, the \(\alpha\)-invariant coincides with the relative index for extending a psc-metric from the empty boundary. Thus \(M\) admits positive scalar curvature by Stolz’s Theorem [35]. So we’re done with this case by Theorem 3.19.

Now assume that the \(S^1\)-action on \(\partial M\) is of even type and thus that \(\beta M\) is spin. Since \(\text{ind}^{S^1-\text{fb}}(M, \partial M) = 0\), we have the condition \(\alpha(\beta M) = 0\), and since \(\beta M\) is simply connected, it
admits a metric of positive scalar curvature by [35]. Lift such a metric to an $S^1$-invariant metric on $\partial M$. Suppose the relative index for extending this metric to a psc-metric over $M$ vanishes. Since the $S^1$-action on $\partial M$ is of even type, the disk bundle $D(L)$ of the line bundle $L$ over $\beta M$ is not spin, since $S^1$ with the $S^1$-invariant spin structure is $\eta$, which is not a spin boundary, and thus the disk bundle $D(L)$ is spin$^c$ but not spin. In this situation, the condition $\text{ind}_{S^1-fb}(M, \partial M) = 0$ gives vanishing of the $KU$-index of $\partial_M$. In this case, we need to use Corollary 5.2. By the method of proof, we can do spin$^c$ surgeries on $M$ away from the interior of $M$, possibly changing $\beta M$ and the line bundle $L$ over it, but preserving positive scalar curvature and the condition $\frac{1}{4}\kappa g + R_L > 0$. 

This has the effect of changing the class of $(M, \partial M)$ to a new $(M', \partial M')$ in the same bordism class in $\Omega_{n, spin, S^1-fb}$. The new $\beta M'$ will be in the same spin bordism class as $\beta M$, and we can arrange for it still to be simply connected. Since we can keep the spin$^c$ line bundle and the correction term $R_h$ trivial on $M'$, the new $M'$ will have a conical metric of positive scalar curvature. Again we conclude by Theorem 3.19.

3.6. Proof of Theorem B.

Remark 3.22. Theorem 3.20 can also be reformulated as the statement labeled Theorem B in Section 1, using the map $\alpha_{S^1-fb}$ obtained by naturality of the cofiber to get a commutative diagram of spectra

$$
\Sigma(M \text{Spin} \wedge \mathbb{CP}^\infty) \vee \Sigma \text{Spin}^c \xrightarrow{\tau} \text{MSpin} \xrightarrow{i} \text{MSpin}^{S^1-fb} \\
\Sigma(\text{KO} \wedge \mathbb{CP}^\infty) \vee \Sigma \text{KU} \xrightarrow{\gamma_{\text{KO}}} \text{KO} \xrightarrow{i_{\text{KO}}} \text{KO}^{S^1-fb}.
$$

Remark 3.23. Careful examination of the proof above shows that we don’t really need to assume that $\beta M$ is connected. The theorem still applies when $\beta M$ has more than one component, as long as each component is simply connected. In this situation, it is possible to have “mixing” of the various cases, as it is possible for some components of $\partial M$ to be of even type and for other components to be of odd type.

4. A SPINC SURGERY THEOREM

In this section we adapt the methods of Gromov and Lawson [16] and Hoelzel [19] to the situation needed in Section 3. Recall that to show that a manifold with $S^1$-fibered singularities admits a conical metric of positive scalar curvature, sometimes we need to deal with the effect of surgery on the Bochner term $\frac{1}{4}\kappa g + R_L$ in the square of the Dirac operator on a spin$^c$ manifold $M$ with the line bundle $L$ determined by the spin$^c$ structure. For the proof of Theorem 3.20 we need to know that under some circumstances, positivity of this term is preserved by spin$^c$ surgeries (which of course can change the line bundle $L$). This will provide a partial converse to [23 Corollary D.17], and show that under some circumstances, existence of a positive scalar curvature metric on $M$ together with a hermitian metric and connection on $L$ for which $\frac{1}{4}\kappa g + R_L > 0$ is equivalent to vanishing of $\alpha^c(M, L)$. 

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We need to consider coupling between the Riemannian curvature and the curvature of a line bundle $L$ (which is just given by an ordinary 2-form $\omega$, which after dividing by $2\pi i$, has integral de Rham class representing $c_1(L)$). Now recall [23, Lemma D.13], which says that any 2-form $\omega$ with $\frac{\omega}{2\pi i}$ in the de Rham class of $c_1(L)$ can be realized as the curvature of some unitary connection on $L$. To state the next theorem we need to recall the definition of spin$^c$ surgery.

**Definition 4.1.** Let $(M, L)$ be a closed spin$^c$ manifold (so $M$ is a closed oriented manifold and $L$ is a complex line bundle on $M$ with $c_2(L)$ reducing mod 2 to $w_2(M)$). We say that $(M', L')$ is obtained from $(M, L)$ by **spin$^c$ surgery in codimension $k$** if there is a sphere $S^{n-k}$ embedded in $M$ with trivial normal bundle, $M'$ is the result of gluing in $D^{n-k+1} \times S^{k-1}$ in place of $S^{n-k} \times D^k$, and there is a spin$^c$ line bundle $L$ on the trace of the surgery—a bordism from $M$ to $M'$—restricting to $L$ on $M$, and the bundle $L'$ on $M'$ is the restriction of $L$.

The following theorem deals with “twisted scalar curvature” on a spin$^c$ manifold, the twisting coming from the curvature of the spin$^c$ line bundle, as advertised in the Abstract of this paper.

**Theorem 4.2 (Spin$^c$ surgery theorem).** Let $M^n$ be a closed $n$-dimensional spin$^c$ manifold, with line bundle $L$ over $M$ defining the spin$^c$ structure. Assume that $M$ admits a Riemannian metric $g$ and $L$ admits a hermitian bundle metric $h$ and a unitary connection such that $\frac{1}{4} K_M + R_L > 0$ (in the notation of the proof of Theorem 3.20). Let $(M', L')$ be obtained from $(M, L)$ by spin$^c$ surgery in codimension $k \geq 3$, as in Definition 4.1. Then there is a metric $g'$ on $M'$, and $L'$ admits a hermitian bundle metric $h'$ and a unitary connection, such that $\frac{1}{4} K_{M'} + R_{L'} > 0$.

This leads to the following bordism theorem, which was needed in the proof of Theorem 3.20.

**Theorem 4.3 (Spin$^c$ bordism theorem).** Let $M^n$ be a connected closed $n$-dimensional spin$^c$ manifold **which is not spin**, with line bundle $L$ over $M$ defining the spin$^c$ structure. Assume that $M$ is simply connected and that $n \geq 5$. Also assume that there exists a pair $(M', L')$ in the same bordism class in $\Omega_n^{spin^c}$ with a metric $g'$ on $M'$ and a hermitian metric $h'$ and a unitary connection on $L'$ such that $\frac{1}{4} K_{M'} + R_{L'} > 0$. Then $M$ admits a Riemannian metric $g$ and $L$ admits a hermitian bundle metric $h$ and a unitary connection such that $\frac{1}{4} K_M + R_L > 0$.

**Proof of Theorem 4.3 assuming Theorem 4.2** The following argument is adapted from [16]. First of all, by Theorem 4.2 we can kill off $\pi_0$ and $\pi_1$ of $M'$ and assume that $M'$ is simply connected. Let $W$ be a spin$^c$ manifold with spin$^c$ line bundle $L$ and with $\partial W = M \amalg M'$, $L|_{M'} = L'$, $L|_M = L$. Doing spin$^c$ surgery on $W$ if necessary, we can also assume that $W$ is simply connected. In order for $M$ to be obtainable from $M'$ by spin$^c$ surgery in codimension $\geq 3$, we need the inclusion map $M \hookrightarrow W$ to be a 2-equivalence, i.e., to induce an isomorphism on $\pi_0$ and $\pi_1$ and a surjection on $\pi_2$. The $\pi_0$ and $\pi_1$ conditions certainly hold since $M$ and $W$ are simply connected. By the Hurewicz Theorem, we may identify $\pi_2(M)$ and $\pi_2(W)$ with $H_2(M)$ and $H_2(W)$, respectively, and thus we need to be able to kill $H_2(W, M)$ by surgery. Now since $n \geq 5$, $\pi_2(W)$ and $\pi_2(M)$ are represented by smoothly embedded 2-spheres. The normal bundles of these 2-spheres are determined by the second Stiefel-Whitney class $w_2$, which we can view by the Hurewicz Theorem and Universal Coefficient
Theorem as a map from \( \pi_2 \) to \( \mathbb{Z}/2 \), and the spin\(^c\) condition says that \( c_1(L) \) reduces mod 2 to \( w_2(W) \), and similarly \( c_1(L) \) reduces mod 2 to \( w_2(M) \neq 0 \). The kernel of \( w_2(W) \) in \( \pi_2(W) \) is represented by smoothly embedded 2-spheres with trivial normal bundles, and may be killed off by surgeries in the interior of \( W \). Doing this reduces \( H_2(W) \) to \( \mathbb{Z}/2 \) and kills \( H_2(W,M) \). Then by the proof of the \( h \)-Cobordism Theorem, using handle cancellations as in [27] or [21, Lemma 1], \( M \) can be obtained from \( M' \) by spin\(^c\) surgeries in codimension \( \geq 3 \), and we can apply Theorem 4.2 □

**Proof of Theorem 4.2** Let \( N^{n-k} \to M^n \) be the embedded sphere on which we are doing surgery. Recall the proof of the surgery theorem in [16] or in [19]: we “bend” the metric \( g \) on a neighborhood \( N^{n-k} \times D^k \) of \( N \), preserving the property of positive scalar curvature, so that close to \( N \) the metric looks like the standard metric on \( S^{n-k}(r_1) \times S^{k-1}(r_2) \times [0,1] \), for suitable \( r_1, r_2 > 0 \). Then it is clearly possible to glue in a handle isometrically, and since \( k-1 \geq 2 \) and thus the second factor has positive scalar curvature, we get positive scalar curvature on the new manifold \( M' \). In the course of this process, up to an error which can be taken arbitrarily small, the scalar curvature only goes up from \( \kappa_M \) to the scalar curvature of \( S^{n-k}(r_1) \times S^{k-1}(r_2) \), and thus during the bend, the property that \( \frac{1}{4} \kappa_M + R_L > 0 \) is preserved. (The estimates checking this are rather involved, but are explained in great detail in [40, §§2.3–2.6].) Now we can proceed as follows: once the bend is completed, in a still smaller neighborhood of \( N \), we can change the curvature 2-form \( \omega \) of \( L \) by any exact form, and we can choose this exact form to kill off \( R_L \) altogether in this small neighborhood, except in the case where \( n-k = 2 \) and \( \langle c_1(L), [N] \rangle \) is nonzero. Fortunately this case can never arise, because if \( N = S^2 \) and \( L|_N \) is topologically nontrivial, then since the trace of the surgery \( W^{n+1} \) is obtained by attaching a 3-handle to kill \( [N] \), the boundary map \( H^2(M) \to H^3(W,M) \) has to kill \( c_1(L) \), which shows that \( L \) cannot be the restriction of a line bundle over \( W \).

Thus we can assume \( R_L \) vanishes in a small neighborhood of the sphere on which we are doing the surgery, and then gluing in a standard handle preserves the curvature condition. □

**Remark 4.4.** Note the careful way in which Theorem 4.3 is written. One should not assume that if \( (M,L) \) is a spin\(^c\) and \( \alpha^c(M,L) = 0 \), then one can choose a metric on \( M \) and a hermitian metric and connection on \( L \) so that \( \frac{1}{4} \kappa_M + R_L > 0 \), for this is false. Indeed, suppose \( M \) is actually spin and \( L \) is the trivial bundle, so that \( \alpha^c(M,L) = 0 \) means that \( \hat{A}(M) = 0 \). This is not enough for \( M \) to admit positive scalar curvature, even with \( \dim M > 4 \) and \( M \) simply connected, because if \( M \) has dimension 1 or 2 mod 8, then \( \alpha(M) \) might be a non-zero 2-torsion class. Adding in the term \( R_L \) in this case only makes things worse, because in suitable coordinates, \( R_L \) has the form

\[
\begin{pmatrix}
\omega & 0 \\
0 & -\omega
\end{pmatrix},
\]

where the operator \( \omega \) is constructed from the curvature of \( L \), which can be any exact 2-form on \( M \), so \( \frac{1}{4} \kappa_M + R_L \) cannot be strictly positive in this case unless \( \kappa_M \) is strictly positive, which is impossible.

5. \( \mathbb{CP}^2 \) bundles

In this section we prove some results about spin\(^c\) bordism and \( \mathbb{CP}^2 \) bundles needed for the proof of Theorem 3.20. These might be of independent interest. (Indeed, they were hinted at in [22] p.
235], but the work of R. Jung alluded to there never appeared.) We will make extensive use of some calculations of Führing [15].

First we recall some basic facts about $\text{spin}^c$ bordism. The basic references are [37, Chapter XI], [29, §8], and [20]. There is an equivalence $\text{MSpin}^c \simeq \Sigma^{-2}\text{MSpin} \wedge \mathbb{C}P^\infty$, corresponding to the isomorphism of bordism groups $\Omega^\text{spin}^c_n \cong \Omega^{\text{Spin}}_{n+2}(\mathbb{C}P^\infty)$; see the proof of Lemma 3.16. Furthermore, classes in $\text{Spin}^c$ bordism are detected by their Stiefel Whitney numbers (which are constrained just by the Wu relations and the vanishing of $w_1$ and $w_3$) and integral cohomology characteristic numbers (where in addition to the Pontryagin classes, one can use powers of $c_1$ of the line bundle defining the $\text{spin}^c$ structure) [37, Theorem, p. 337]. *Note that the bordism class can change* depending on the choice of $\text{spin}^c$ structure. In fact, $\text{CP}^2$ plays the same role for $\text{spin}^c$ that $\mathbb{H}P^2$ plays for $\text{spin}$ according to the work of Stolz.

We now use $(\mathbb{C}P^2, \mathcal{O}(1))$ to construct a transfer map $T_{\text{MSpin}^c}: \Omega^\text{spin}^c_n(BG) \to \Omega^\text{spin}^c_{n+4}$, where $G$ is the Lie group $\text{SU}(3)$, as follows. The group $\text{SU}(3)$ acts transitively on $\mathbb{C}P^2 \cong G/H$, where $H = S(\text{U}(2) \times \text{U}(1))$, preserving the class of the bundle $\mathcal{O}(1)$. Thus given a $\text{spin}^c$ manifold $(M^n, L)$ and a map $f: M \to BG$, we can form the associated $\mathbb{C}P^2$ bundle $M \times f \mathbb{C}P^2$, by pulling back (under $f$) the $G/H$-bundle over $BG$ associated to the universal principal $G$-bundle. This manifold has dimension $n + 4$ and has a $\text{spin}^c$ structure inherited from the $\text{spin}^c$ structure on $M$ defined by $L$ and the $\text{spin}^c$ structure on $\mathbb{C}P^2$ defined by $\mathcal{O}(1)$.

We will also need another transfer map introduced in [35] and [36]. This is defined similarly, but with $G = \text{SU}(3)$ replaced by $\text{PSp}(3)$, $H = S(\text{U}(2) \times \text{U}(1))$ replaced by $P(\text{Sp}(2) \times \text{Sp}(1))$, and $\mathbb{C}P^2$ replaced by $\mathbb{H}P^2$. One obtains a transfer map $T_{\text{MSpin}^c}: \Omega^\text{spin}^c_n(\text{BPSp}(3)) \to \Omega^\text{spin}^c_{n+8}$, which can be extended to a similar map on any $\text{MSpin}$-module spectrum. We will apply it to $\text{MSpin}^c \simeq \text{MSpin} \wedge \Sigma^{-2}\mathbb{C}P^\infty$.

We’re now ready for one of the main theorems of this paper.

**Theorem 5.1.** The transfer maps $T_{\text{MSpin}^c}: \Omega^\text{spin}^c_n(BG) \to \Omega^\text{spin}^c_{n+4}$ and $T_{\text{MSpin}^c}: \Omega^\text{spin}^c_n(\text{PSp}(3)) \to \Omega^\text{spin}^c_{n+8}$ defined above have images lying in the kernel of $\alpha^c: \Omega^\text{spin}^c_+ \to KU_+$. The kernel of $\alpha^c$ is additively generated by the images of $T_{\text{MSpin}^c}$ and of $T_{\text{MSpin}^c}$.
Proof. First of all let’s observe that the image of the transfer $T_{\text{MSpin}^c}$ is contained in the kernel of $\alpha^c$. Consider a bundle $\mathbb{CP}^2 \to M \xrightarrow{p} N$ in the image of the transfer. Its tangent bundle is the direct sum of the tangent bundle along with fibers with $p^*TN$. By Atiyah-Singer, the value of $\alpha^c$ on $M$ is $\langle \hat{A}(M) \rangle (L_M)/2, [M] \rangle$. The line bundle $L_M$ is the tensor product of $O(1)$ on the fibers $\mathbb{CP}^2$ with $p^*L_N$, while $[M]$ is locally $[\mathbb{CP}^2] \times [N]$ and $\hat{A}(M)$ splits as a product of $\hat{A}(\mathbb{CP}^2)$ with $p^*\hat{A}(N)$. So the vanishing follows from the vanishing of the index for $\mathbb{CP}^2$, as computed above.

The argument for the image of the transfer $T_{\text{MSpin}}$ is similar. If a bundle $\mathbb{HP}^2 \to M \xrightarrow{p} N$ is in the image of the transfer, then since $\mathbb{HP}^2$ is 3-connected, the spin$^c$ line bundle $L_M$ on $M$ is just the pull-back $p^*L_N$ of the spin$^c$ line bundle $L_N$ on $N$. Since $\hat{A}(\mathbb{HP}^2) = 0$, we again see from Atiyah-Singer that $\alpha^c(M) = 0$.

Now we have to prove surjectivity. Since $T_{\text{MSpin}^c}$ is the induced map on homotopy groups of a map of spectra (which by abuse of notation we denote by the same symbol) $T: \text{MSpin}^c \wedge \Sigma^4BG_+ \to \text{MSpin}^c$ (see [15] for a very similar situation), we can prove the desired result by localizing separately at and away from the prime 2. Away from the prime 2, $\text{MSpin}^c$ is similar. If a bundle $\mathbb{HP}^2 \to M \xrightarrow{p} N$ is in the image of the transfer, then since $\mathbb{HP}^2$ is 3-connected, the spin$^c$ line bundle $L_M$ on $M$ is just the pull-back $p^*L_N$ of the spin$^c$ line bundle $L_N$ on $N$. Since $\hat{A}(\mathbb{HP}^2) = 0$, we again see from Atiyah-Singer that $\alpha^c(M) = 0$.

Now we have to prove surjectivity. Since $T_{\text{MSpin}^c}$ is the induced map on homotopy groups of a map of spectra (which by abuse of notation we denote by the same symbol) $T: \text{MSpin}^c \wedge \Sigma^4BG_+ \to \text{MSpin}^c$ (see [15] for a very similar situation), we can prove the desired result by localizing separately at and away from the prime 2. Away from the prime 2, $\text{MSpin}^c$ is equivalent to $\text{MSpin}^c \wedge \Sigma^4BG_+$ (see [37, p. 352]), so the result follows immediately from the analogous statement for the $\mathbb{CP}^2$ transfer on $\text{MSpin}$, which is [15, Theorem 1.4].

So we are reduced to a 2-local calculation. Since $\text{MSpin}^c$ is an $\text{MSpin}$-module spectrum, we can apply one of the main results (Theorem B) of [36]. This asserts that we have an additive splitting of $\text{MSpin}^c \simeq \text{MSpin} \wedge \Sigma^{-2}\mathbb{CP}^\infty$ into the image of $T_{\text{MSpin}}$ and $ko \wedge \Sigma^{-2}\mathbb{CP}^\infty$ after localizing at 2. Thus to prove our theorem we only need to show that the kernel of $\alpha^c$ on $ko \wedge \Sigma^{-2}\mathbb{CP}^\infty$ is in the image of $T_{\text{MSpin}^c}$.

Now by [2, Theorem 2.1],

\begin{equation}
ko \wedge \Sigma^{-2}\mathbb{CP}^\infty \simeq \bigvee_{k=0}^\infty \Sigma^{4k}ku.
\end{equation}

(The summands on the right are some of the suspensions $\Sigma^{4n(J)}ku$ of $ku$, indexed by partitions $J$ of size $n(J)$, that appear in the Anderson-Brown-Peterson additive splitting of $\text{MSpin}^c$ localized at 2. This splitting is implicit in [1] and written out in detail in [37, Ch. XI], [29, §8], and in [20]. The basic result is that, localized at 2, $\text{MSpin}^c$ splits additively as a direct sum of suspensions $\Sigma^{4n(J)}ku$ of $ku$, indexed by partitions $J$ of size $n(J)$, together with some suspensions of $HZ/2$ that start in fairly high dimension. The lowest-dimensional summand is $ku$ itself, coming from the empty partition $\emptyset$ with $n(\emptyset) = 0$, and $\alpha^c$ is just projection onto this bottom summand.)

For what we will do next, we need to explain where the splitting (19) comes from. It is well known (see [35, Corollary 5.5], [36, Proposition 2.3], [13, Theorem 1.6.1], and [30, Theorems 3.1.17 and 3.1.25]) that $H^\ast\text{MSpin}$, $H^\ast ko$, $H^\ast ku$, and $H^\ast$ of $\text{MSpin}$-module spectra which are bounded below and locally finite are all, as modules over the mod-2 Steenrod algebra, extended modules induced from the finite-dimensional subalgebra $A(1)$ generated by $1 = Sq^0$, $Sq^1$ and $Sq^2$. The Adams spectral sequences of all these spectra collapse, and so the splitting (19) comes from the splitting of $\Sigma^{-2}\mathbb{CP}^\infty$ as an $A(1)$-module into $\bigvee_{k=0}^\infty \Sigma^{4k}C$, where $C$ is the very simple 2-dimensional
A(1)-module symbolized by the diagram $\bullet_0 \overrightarrows \bullet_2$, where the curved arrow represents the action of $\text{Sq}^2$. And it is classical (due to Wood and Fujii) that $ko \wedge C \simeq ku$.

So in order to finish the proof, it suffices to consider the much simpler transfer map

$$T_{ku}: \bigvee_{k=0}^{\infty} \Sigma^{4k} ku \wedge \Sigma^4 BG_+ \to \bigvee_{k=0}^{\infty} \Sigma^{4k} ku. \tag{20}$$

In (20), the transfer $T_{ku}$ acts the same way on all the summands on the left: the action on each summand is just the fourfold suspension of the action on the previous one. The spin$^c$- $\alpha$-invariant map $\alpha^c$ just projects the sum on the right of (20) onto the bottom summand $ku$, so it will suffice to show that $T_{ku}$ sends $\Sigma^{4k} ku \wedge \Sigma^4 BG_+$ onto $\Sigma^{4k+4} ku$ on the right in (20), or by suspension invariance, that it sends $ku \wedge \Sigma^4 BG_+$ onto the summand $\Sigma^{4k} ku$ on the right in (20). But this is a simple characteristic class calculation. $T_{ku}$ sends $1 \in ku_0$, represented by a point, to the class of $(\mathbb{C}P^2, \mathcal{O}(1))$, which is in the kernel of $\alpha^c$. It thus projects to 0 in the bottom copy of $\pi_4(ku)$. We need to show that the class of $(\mathbb{C}P^2, \mathcal{O}(1))$ is, however, a generator of $\pi_4(\Sigma^4 ku)$. Recall that the right-hand side of (20) is identified with $ko \wedge \Sigma^{-2} \mathbb{C}P^\infty$. Since $M_{\text{Spin}}$ is identical to $ko$ up to dimension 7, $\pi_4(\Sigma^4 ku)$ is identified with $\bar{\Omega}_0^\text{spin} (\mathbb{C}P^\infty) \cong \bar{\Omega}_4^\text{spin}^c$, which has rank 2, with the two summands generated by $(\mathbb{C}P^1, \mathcal{O}(1))^2$ and by $(\mathbb{C}P^2, \mathcal{O}(1))$. (The latter is obtained by dualizing the line bundle $\mathcal{O}(1)$ on the spin manifold $\mathbb{C}P^3$, which generates the summand $H_6(\mathbb{C}P^\infty, \Omega_0^{\text{spin}})$ in $\bar{\Omega}_6^\text{spin} (\mathbb{C}P^\infty)$. Similarly, $ku_2$ is generated geometrically by the class $(\mathbb{C}P^1, \mathcal{O}(2))^n$ (which has Todd class 1), and this will go to the class of $(\mathbb{C}P^2, \mathcal{O}(1)) \times (\mathbb{C}P^1, \mathcal{O}(2))^n$, a generator of $\pi_4(\Sigma^4 ku)$. This completes the proof. \[\square\]

**Corollary 5.2.** Let $(M^n, L)$ be a simply connected spin$^c$ manifold, $L$ the line bundle over $M$ defining the spin$^c$ structure, with $\alpha^c([M, L]) = 0$. Then after changing $(M, L)$ up to spin$^c$ cobordism, we can assume that $M$ admits a Riemannian metric $g$ of positive scalar curvature and the line bundle $L$ over $M$ defining the spin$^c$ structure admits a hermitian metric $h$ with $\frac{1}{4} \kappa_g + \mathcal{R}_h > 0$.

**Proof.** As we have indicated, the idea of the proof is to reduce things to the case of $(\mathbb{C}P^2, \mathcal{O}(1))$, so the first step is to prove the theorem in this case. For this case, no cobordism is necessary; we use the Fubini-Study metric along with the usual connection on the dual of the tautological bundle. Then if $\omega$ is the Kähler form, this is also the curvature of $\mathcal{O}(1)$ and the Ricci tensor is 6 times the metric. So $\frac{1}{4} \kappa$ is 6 while the minimal eigenvalue of $\mathcal{R}$ is $-2$, so $\frac{1}{4} \kappa_g + \mathcal{R}_h \geq 6 - 2 = 4 > 0$.

Now we deal with the general case, using Theorem 5.1. Because of Theorem 4.3, we can deal separately with the images of the two different transfer maps that appear in Theorem 5.1. If $M$ is in the image of $T_{M_{\text{Spin}}}$ and is the total space of an $\mathbb{H}P^2$-bundle with structure group $\text{PSp}(3)$, we can rescale the $\mathbb{H}P^2$ fiber to have very small diameter and big curvature, and then the scalar curvature of the $\mathbb{H}P^2$ fiber will dominate everything else. So in that case the conclusion is obvious. Thus we can assume after making a spin$^c$ cobordism that $M^n$ is the total space of a $\mathbb{C}P^2$-bundle with $L_M$ restricting to $\mathcal{O}(1)$ on the fibers. Once again, by choosing the metric and connection so that on each fiber, we have a very small multiple of the Fubini-Study metric and the curvature of the line bundle is the Kähler form, the curvature of the fibers will dominate everything else, and the conclusion follows. \[\square\]
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