Local B-model Yukawa couplings from A-twisted correlators

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Using the exact formula for the A-twisted correlation functions of the 2D $\mathcal{N}=(2,2)$ gauged linear sigma model, we reconsider the computation of the B-model Yukawa couplings of the local toric Calabi–Yau varieties. Our analysis is based on an exact result that has been evaluated from the supersymmetric localization technique and careful treatment of its application. We provide a detailed description of a procedure to investigate the local B-model Yukawa couplings and also test our prescription by comparing the results with known expressions evaluated from the local mirror symmetry approach. In particular, we find that the ambiguities of classical intersection numbers of a certain class of local toric Calabi–Yau varieties discovered previously can be interpreted as degrees of freedom of the twisted mass deformations.

1. Introduction

The 2D $\mathcal{N}=(2,2)$ gauged linear sigma model (GLSM) \cite{1} has played significant roles in the study of topological quantum field theories and mirror symmetry. A physical proof of mirror symmetry in Ref. \cite{2} makes use of the duality properties of this model. Shortly after the work of Ref. \cite{1}, the topological A-twisted version of the GLSM was studied in Ref. \cite{3} and the genus-zero correlation functions were exactly computed for the A-models with target space a toric variety or a toric Calabi–Yau hypersurface. These developments led to a conjecture called the toric residue mirror conjecture \cite{4,5} (later proven in Refs. \cite{6–9}), which states the existence of an equality between an integration over the toric compactification\textsuperscript{1} of the moduli space of genus-zero maps in the A-model and the toric residue \cite{12} over the space of a holomorphic section in a toric variety in the B-model (see also Ref. \cite{13}). The toric residue mirror conjecture provides an efficient method to compute the B-model Yukawa couplings exactly for the mirror varieties of toric Calabi–Yau complete intersections.

On the other hand, by using the supersymmetric localization technique \cite{14}, the exact formula for the A-twisted $\mathcal{N}=(2,2)$ GLSM correlation functions on a supersymmetric two-sphere background has been clarified in Refs. \cite{15,16}.\textsuperscript{2} This approach does not depend on a particular choice of gauge

\textsuperscript{1}Toric compactifications can be generalized to stable toric quasimaps \cite{10} allowing some degeneracies of the moduli space of maps and further to stable quasimaps \cite{11}, which also include non-Abelian varieties.

\textsuperscript{2}It is worth noting that the supersymmetric localization of $\mathcal{N}=(2,2)$ gauge theories has also been performed on a different two-sphere background, which corresponds to a fusion of the $A$- and $\bar{A}$-twists on two hemispheres \cite{17,18}.
group and enables us to calculate the Yukawa couplings for GLSMs with non-Abelian gauge groups efficiently [15,19] (see also Ref. [20]). Meanwhile, the application for models with non-compact target spaces has not been thoroughly investigated. The aim of this work is to fill this gap and establish an explicit formalism to investigate the local B-model Yukawa couplings from the A-twisted GLSM correlation functions on local toric Calabi–Yau varieties.

In this paper, we will explicitly demonstrate how to apply the exact formula for A-twisted GLSM correlators in Ref. [15] for the explicit computation of the B-model Yukawa couplings of local toric Calabi–Yau varieties. We remind ourselves that the application of the formula to the local toric Calabi–Yau varieties requires careful treatment of the mass parameters of the chiral multiplets. Then we propose how to determine the proper mass deformations to conduct the exact calculation of the A-twisted GLSM correlators and check that the resulting A-twisted correlators coincide with known results for the B-model Yukawa couplings evaluated by the local mirror symmetry approach [22–24]. To the best of our knowledge, this relationship has not been thoroughly investigated before. Moreover, as a by-product, we also find that an ambiguity of the classical intersection numbers of a certain class of local toric Calabi–Yau varieties argued before in Ref. [23] can admit a new interpretation.

This paper is organized as follows. First we will take a brief look at the exact results for the GLSM correlation functions in Sect. 2. In Sect. 3, we explain the details of the topological properties of local toric Calabi–Yau varieties and propose the rules of mass assignment for applying the formula to non-compact backgrounds. Then in Sect. 4 we will evaluate the GLSM correlators for several examples of local toric Calabi–Yau threefolds and fourfolds, and demonstrate that the resulting expressions are completely consistent with known results obtained by the local mirror symmetry approach. Finally we will conclude and propose several future directions. In Appendix A, we collect the building blocks of the localization formula, which are useful for understanding the mirror transformation. Lists of local B-model Yukawa couplings for the local del Pezzo surface $K_{dP_2}$ and the local $A_2$ geometry are summarized in Appendixes B and C, respectively.

2. $\mathcal{N} = (2, 2)$ GLSM on the $\Omega$-deformed two-sphere

Here we will take a brief look at the 2D $\mathcal{N} = (2, 2)$ GLSM on the $\Omega$-deformed two-sphere $S^2_\hbar$, which is a one-parameter deformation of the A-twisted sphere. As shown in Ref. [15], the supersymmetric localization formula considerably simplifies the exact calculations of the A-twisted correlation functions and realizes the quantum cohomology relations appropriately.

2.1. Exact formula for GLSM correlation functions

Following the approach to curved space rigid supersymmetry advocated in Ref. [25], supersymmetric backgrounds in two dimensions were studied in detail in Ref. [26]. A remarkable result is that the topological A-twist on the two-sphere admits an interesting $U(1)$ equivariant deformation called the $\Omega$-deformation [27]. This deformation can be characterized by an equivariant parameter $\hbar$, which corresponds to a non-trivial expectation value for the graviphoton field. The ordinary topological A-twist is obtained by setting $\hbar = 0$.

Let $G$ be a rank $\text{rk}(G)$ gauge group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. In Ref. [15], an exact formula for the correlation functions of an $\mathcal{N} = (2, 2)$ GLSM with gauge group $G$ on the

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3 We refer the reader to Ref. [21] where several new aspects of twisted mass deformations have been clarified.
The parameters $d$ in the same direction as for the vector multiplet and GLSM. The hyperplanes can intersect at a point charges \[ 28 \].

for the chiral matter multiplets $\gamma_1$ gauge charges as are defined by the canonical pairing.

of $d$ in the vector multiplet inserted at the north and south poles, respectively. The order of the Weyl group of $G$ is denoted by $|W|$, and

$$
\prod_{i=1}^{r} z^{d_i} = \prod_{i=1}^{r} e^{2\pi \sqrt{-1} \left( \frac{\theta_i}{\pi} + \sqrt{-1} \xi_i \right) d_i} \quad (i = 1, \ldots, r),
$$

where $\theta_i$ and $\xi_i$ are theta angles and Fayet–Iliopoulos parameters for the central $U(1)' \subset G$. The parameters $d = (d_1, \ldots, d_r)$ represent magnetic charges for the central $U(1)'$, called GNO charges [28].

The factor $I_d(\Omega_1\Omega_2)$ is a differential form given by

$$
I_d(\Omega_1\Omega_2) = \Omega_1\left( u - \frac{d}{2} \hbar \right) \Omega_2\left( u + \frac{d}{2} \hbar \right) Z^{vec}_{\Phi_d}(u; \hbar) \prod_a Z^{0a}_{\Phi_d}(u; \hbar) du_1 \wedge \cdots \wedge du_{rk(G)},
$$

which consists of the one-loop determinants

$$
Z^{vec}_{\Phi_d}(u; \hbar) = (-1)^{\sum_{\alpha \in \Delta_+} (\alpha(\Phi^*) + 1)} \prod_{\alpha \in \Delta_+} \left( \alpha(u)^2 - \frac{\alpha(d)^2}{4} \hbar^2 \right)
$$

for the vector multiplet and

$$
Z^{0a}_{\Phi_d}(u; \hbar) = (-1)^{\delta_{a,2}} \prod_{\rho_{\alpha} \in R_a} h^{e_{\alpha} - \rho_{\alpha}(d) - 1} \frac{\Gamma \left( \frac{\rho_{\alpha}(u) + \lambda_{\alpha}}{\hbar} + \frac{r_{a} - \rho_{\alpha}(d)}{2} \right)}{\Gamma \left( \frac{\rho_{\alpha}(u) + \lambda_{\alpha}}{\hbar} - \frac{r_{a} - \rho_{\alpha}(d)}{2} + 1 \right)}
$$

for the chiral matter multiplets $\Phi_d$ in a representation $R_a$ with $R$-charge $r_{a}$ and the twisted mass $\lambda_{\alpha}$. Here $\Delta_+$ is the set of positive roots and $\rho_{\alpha}$ denote the weights of $R_a$. The products $\alpha(*)$ and $\rho_{\alpha}(*)$ are defined by the canonical pairing.

The one-loop determinant for chiral multiplets (2.5) has poles along hyperplanes defined from the $\Gamma$-function in the denominators. For later convenience, we will use a collective form for the gauge charges as $Q = \{ Q_i \in Z^{rk(G)} \subset \mathfrak{h}^* \}$ where $i$ labels all the components of the multiplets of the GLSM. The hyperplanes can intersect at a point $u = u_s = (u^*_{1}, \ldots, u^*_{rk(G)})$ and realize a codimension $rk(G)$ pole. Such intersecting hyperplanes simultaneously specify a subset of charge vectors with at least $rk(G)$ elements and we denote it by $Q_s \subset Q$.

A crucial ingredient of the exact formula (2.1), whose treatment will be described in the next section, is the Jeffrey–Kirwan residue operation [29] $^5$ JK-Res $[Q_s, \eta]$ at $u = u_s$ depending on a choice of a covector $\eta \in \mathfrak{h}^*$. As adopted in Ref. [15], we always choose $\eta$ to be parallel and pointing in the same direction as $\xi_i$ and therefore $\eta$ specifies the phase of the model.

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$^4$ Here we further assume that the rank of the gauge group $rk(G)$ is equal to the number of central $U(1)$ factors in $G$. This condition generically holds for the GLSMs with local toric Calabi-Yau backgrounds discussed in the following sections. 5

$^5$ See also Refs. [30,31] for the early developments.
2.2. The Jeffrey–Kirwan residue operation

In this section, we will briefly review the Jeffrey–Kirwan residue operation. For more details, we refer the reader to Refs. [7,32].

**Definition 2.1.** When \( Q_\ast \subset Q \) lies within an open half-space of \( \mathfrak{h}^\ast \), the associated intersection point \( u = u_\ast \) is called the projective point. Such an arrangement of hyperplanes is called a projective arrangement.

**Definition 2.2.** Consider an integrand \( \frac{du_1 \wedge \cdots \wedge du_{k(G)}}{Q_1(u) \cdots Q_{k(G)}(u)} \) with a projective point \( u_\ast = 0 \) and \( Q_\ast = \{Q_1, \ldots, Q_{k(G)}\} \). The cases with generic \( u_\ast \) can be realized by shifting the coordinates appropriately. Then the Jeffrey–Kirwan residue \( \text{JK-Res}_{u=0}[Q_\ast, \eta] \) is given by

\[
\text{JK-Res}_{u=0}[Q_\ast, \eta] = \frac{du_1 \wedge \cdots \wedge du_{k(G)}}{Q_1(u) \cdots Q_{k(G)}(u)} \begin{cases} 
\frac{1}{|\det(Q_1, \ldots, Q_{k(G)})|} & \text{if } \eta \in \text{Cone}(Q_1, \ldots, Q_{k(G)}), \\
0 & \text{if } \eta \notin \text{Cone}(Q_1, \ldots, Q_{k(G)}),
\end{cases}
\]

where \( \text{Cone}(Q_1, \ldots, Q_{k(G)}) \) is the closed cone spanned by \( Q_1, \ldots, Q_{k(G)} \).

A more constructive definition can be represented as follows. Let \( FL(Q_\ast) \) be a finite set of flags

\[
F = [F_0 = \{0\} \subset F_1 \subset \cdots \subset F_{k(G)} = \mathfrak{h}^\ast], \quad \dim F_j = j,
\]

such that \( Q_\ast \) contains a basis of each of the flags \( F_j, j = 1, \ldots, k(G) \). Then one can choose an ordered set \( Q_\ast^F = (Q_1, \ldots, Q_{k(G)}) \) such that the first \( j \) elements \( \{Q_{i_m}\}_{m=1}^j \) give a basis of \( F_j \).

**Definition 2.3.** For the above-defined ordered basis \( Q_\ast^F \) of a flag \( F \in FL(Q_\ast) \), the iterated residue \( \text{Res}_F \) of \( \omega = \omega_1 \wedge \cdots \wedge \omega_{k(G)} \) is defined by

\[
\text{Res}_F \omega = \text{Res}_{Q_{k(G)}(u)=0} \cdots \text{Res}_{Q_1(u)=0} (\omega_1 \wedge \cdots \wedge \omega_{k(G)}).
\]

Here in each step of the residue operations on the right-hand side, the higher variables remain as free parameters. Note that the iterated residue only depends on the flag \( F \), and does not depend on the choice of the ordered basis.

Let us define the closed simplicial cone for a flag \( F \in FL(Q_\ast) \) as

\[
s^+(F, Q_\ast) = \sum_{j=1}^{k(G)} R_{\geq 0} k_j^F, \quad k_j^F := \sum_{Q_i \in F_j} Q_i, \quad j = 1, \ldots, k(G), \tag{2.6}
\]

and denote by \( FL^+(Q_\ast, \eta) \) a set of flags such that the corresponding cone \( s^+(F, Q_\ast) \) contains \( \eta \).

**Definition 2.4.** By the partial sums of the elements of \( Q_\ast = \{Q_1, \ldots, Q_n\}, n \geq k(G) \), define a set

\[
\Sigma Q_\ast = \left\{ \sum_{i \in \pi} Q_i, \pi \subset \{1, \ldots, n\} \right\}.
\]

An element \( v \in \text{Cone}(Q_\ast) \) is called regular with respect to \( \Sigma Q_\ast \) if \( v \notin \text{Cone}_{\text{sing}}(\Sigma Q_\ast) \), where \( \text{Cone}_{\text{sing}}(\Sigma Q_\ast) \) is the union of the closed cones spanned by \( k(G) - 1 \) independent elements of \( \Sigma Q_\ast \). A regular \( v \notin \text{Cone}_{\text{sing}}(\Sigma Q_\ast) \) identifies one chamber in \( \mathfrak{h}^\ast \).
Then the following theorem can be proved [7].

**Theorem 2.5.** If \( \eta \in \text{Cone}(\mathbf{Q}_s) \) is regular with respect to \( \Sigma \mathbf{Q}_s \), the Jeffrey–Kirwan residue at a projective point \( \mathbf{u} = \mathbf{u}_s = 0 \) can be written in terms of the iterated residue as

\[
JK-\text{Res}_{\mathbf{u}=0}[\mathbf{Q}_s, \eta] = \sum_{F \in \mathcal{F}^c(\mathbf{Q}_s, \eta)} \nu(F) \text{ Res}_{\mathbf{u}}(F)
\]

(2.7)

where \( \nu(F) = 0 \) if \( \kappa^F_j, j = 1, \ldots, \text{rk}(G) \) are linearly independent, and \( \nu(F) = 1 \) (resp. \(-1\)) if \( \kappa^F_j \) are linearly dependent and the ordered basis \( \kappa^F := (\kappa^F_1, \ldots, \kappa^F_{\text{rk}(G)}) \in \mathfrak{h}^* \) is positively (resp. negatively) oriented, i.e., \( \text{sign}(|\det \kappa^F|) = 1 \) (resp. \(-1\)).

3. GLSM correlators and local B-model Yukawa couplings

It has been clarified in Ref. [15] that the A-twisted correlators given by Eq. (2.1) with vanishing \( \Omega \)-deformation precisely give the B-model Yukawa couplings of the mirror of compact Calabi–Yau manifolds and non-compact orbifolds. Here, we consider local toric Calabi–Yau varieties. In order to perform the Jeffrey–Kirwan residue operation for these backgrounds appropriately, it is necessary to introduce the twisted masses to matter fields and realize a projective hyperplane arrangement of Definition 2.1. The main purpose of this paper is to provide a detailed description of this procedure to investigate the local B-model Yukawa couplings.

3.1. Local toric Calabi–Yau varieties and correlation functions

Consider a local toric Calabi–Yau m-fold \( X \) described by a symplectic quotient \( X = \mathbb{C}^n/\mathbb{R}(\mathbb{C}^*)^r \) with \( m = n - r, \ m \geq 3 \). In terms of the GLSM, this background can be specified by \( n \) chiral matter multiplets \( \Phi_i, \ i = 1, \ldots, n \) with \( R \)-charge 0 and \( U(1)^r \) charge vectors \( \mathcal{Q}_i \in \mathbb{Z}^r \) satisfying the Calabi–Yau condition \( \sum_{i=1}^n \mathcal{Q}_i = 0 \). One can also assign the twisted masses \(-\lambda_i\) for the chiral multiplets \( \Phi_i \). Then a model can be specified by a set of charge vectors and twisted masses as

\[
\begin{pmatrix}
Q_1 & Q_2 & \cdots & Q_n \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n
\end{pmatrix}
\]

(3.1)

In the limit of the vanishing \( \Omega \)-deformation \( h = 0 \), the exact formula for correlation functions (2.1) in the geometric phase reduces to a simple expression,

\[
\langle \mathcal{O}(\mathbf{u}) \rangle_{h=0} = \sum_{\mathbf{d} \in (\mathbb{Z}_{\geq 0})^r} \mathbf{z}^\mathbf{d} \sum_{\mathbf{u}_s} JK-\text{Res}_{\mathbf{u}=\mathbf{u}_s}[\mathbf{Q}_s, \eta] \mathcal{O}(\mathbf{u}) \prod_{i=1}^n \mathcal{Z}_d^{\Phi_i}(\mathbf{u}) \ du_1 \wedge \cdots \wedge du_r,
\]

(3.2)

where the moduli parameters \( \mathbf{z} = (z_1, \ldots, z_r) \) are defined in Eq. (2.2) and the one-loop determinant for chiral multiplets in Eq. (2.5) or Eq. (A.5) takes a reduced form:

\[
\mathcal{Z}_d^{\Phi_i}(\mathbf{u}) := \mathcal{Z}_d^{\Phi_i}(\mathbf{u}; 0) = (\mathcal{Q}_i(\mathbf{u}) - \lambda_i)^{-\mathcal{Q}_i(\mathbf{d}) - 1}.
\]

(3.3)

Choosing the monomials of \( \mathbf{u} \) as the operators \( \mathcal{O}(\mathbf{u}) \), we focus on the following GLSM correlators:

\[
Y_{z_{i_1} \cdots z_{i_m}}(\mathbf{z}) := \langle h_{i_1} \cdots h_{i_m} \rangle_{h=0}, \quad 1 \leq i_1 \leq \cdots \leq i_m \leq r.
\]

(3.4)

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6 See also Refs. [19,20] for the treatment of manifolds with non-Abelian GLSM descriptions.

7 We refer the reader to Ref. [33] for a detailed introduction on this subject.
Remark 3.1. The phase structure of the Calabi–Yau variety $X$ is described by the secondary fan, which consists of a set of charge vectors $\{Q_1, \ldots, Q_n\}$ in $\mathfrak{h}^*$ (see, e.g., Ref. [33]). Although we will consider a particular choice of $\eta \in \mathfrak{h}^*$ inside the geometric phase of $X$ for actual computation of the GLSM correlators (3.4), the result does not depend on the choice of $\eta$ as long as the Jeffrey–Kirwan residue operation is carried out appropriately.

3.2. Criteria for the twisted mass deformations

Computations of the GLSM correlators (3.4) for local toric Calabi–Yau varieties require careful treatment. This is because generically the associated hyperplane arrangement without twisted mass parameters $\lambda_i$ becomes non-projective and one should not use the Jeffrey–Kirwan residue operation directly. Therefore it is necessary to introduce twisted masses for chiral multiplets in a proper way, such that the hyperplane arrangement of the model becomes projective. Although the necessity of this kind of prescription has been argued before, to the authors’ knowledge, there has been no satisfactory attempt to clarify the proper method for twisted mass insertions. Here we propose the following rules as a proper determination of the twisted mass deformations:

1. The twisted masses should not be inserted for chiral multiplets with $Q_i \neq 0$ describing a non-compact fiber coordinate, or a blow-up coordinate of a singularity.

2. Suppose a GLSM of a local toric Calabi–Yau variety of interest has a chiral multiplet $\Phi_0$, which is neutral under the $j$th $U(1)$ gauge symmetry and describes a non-compact fiber coordinate $X_0$ by its scalar component. Let $\lambda_i, i = 1, \ldots, s$ be twisted masses for other chiral multiplets $\Phi_i$ with non-zero charges with respect to the $j$th $U(1)$ gauge symmetry. If a divisor defined by $X_0 = 0$ contains a blow-up coordinate of a singularity, one needs to impose

$$\lambda_0 + \sum_{i=1}^{s} \lambda_i = 0$$

for the twisted mass $\lambda_0$ of $\Phi_0$ as a “Calabi–Yau condition on the divisor”.

3. As long as the above requirements are fulfilled, one can turn on generic twisted masses for the remaining chiral multiplets while respecting the symmetries of the model such as the permutation of the homogeneous coordinates of the toric variety.

In Sect. 4, by using the above prescription for various examples, we will explicitly check that the GLSM correlators (3.4) give the same results predicted by the mirror symmetry approach. Moreover, we find that the ambiguities of the intersection numbers for a certain class of varieties argued in Ref. [23] can be reinterpreted as degrees of freedom of the proper twisted mass deformations.

Remark 3.2. In this paper, we do not consider the so-called non-nef toric varieties such as $O(k) \oplus O(-2 - k) \rightarrow \mathbb{P}^1$ with $k \geq 1$ [34–36]. It would be interesting to investigate such varieties and try to extend our prescription.

\[\text{Note that the relevance of twisted mass deformations has also been clarified in Ref. [21] from the other perspective.}\]

\[\text{A typical example with this property will be shown in Eq. (4.5).}\]
3.3. Local Yukawa couplings and the mirror map

The topological B-model \(m\)-point Yukawa couplings of a Calabi–Yau \(m\)-fold \(X\) are defined by

\[
Y_{z_{i_1} \cdots z_{i_m}}(\mathbf{z}) = \int_{X^\vee} \Omega(\mathbf{z}) \wedge \nabla z_{i_1} a_{z_{i_1}} \cdots \nabla z_{i_m} a_{z_{i_m}} \Omega(\mathbf{z}) \in \text{Sym}^m(T^* M) \otimes \mathcal{L}^{-2},
\]

(3.5)

where \(\nabla\) is a flat connection called a Gauss–Manin connection (see, e.g., Ref. [33]) and \(T^* M\) is the cotangent bundle of the complex structure moduli space \(M\) of the Calabi–Yau \(m\)-fold \(X\). \(\mathcal{L}\) is a holomorphic line bundle over \(M\) whose section is given by the nowhere-vanishing holomorphic \(m\)-form \(\Omega(\mathbf{z})\) on \(X^\vee\). The topological A-model \(m\)-point Yukawa couplings for the mirror \(X\) of \(X^\vee\) can be obtained from Eq. (3.5) by changing the complex structure moduli parameters \(\{\log z_i\}\) into flat coordinates \(\{\log q_i\}\) parametrizing \(h_i \in H^{1,1}(X)\) as

\[
\widetilde{Y}_{h_1 \cdots h_m}(\mathbf{q}) = \frac{1}{X_0(\mathbf{z}(\mathbf{q}))^{\frac{1}{2}}} \sum_{j_1, \ldots, j_m=1}^m Y_{z_{j_1} \cdots z_{j_m}}(\mathbf{z}(\mathbf{q})) \frac{\partial \log z_{j_1}(\mathbf{q})}{\partial \log q_{j_1}} \cdots \frac{\partial \log z_{j_m}(\mathbf{q})}{\partial \log q_{j_m}}.
\]

(3.6)

This transformation map is called a mirror map. Note that the quantity \(X_0(\mathbf{z})^{\frac{1}{2}}\) is a square of the monodromy invariant fundamental period of \(X^\vee\) and is equal to 1 for local toric Calabi–Yau varieties, while for compact Calabi–Yau \(m\)-folds it is a non-trivial function of the moduli parameters \(\mathbf{z}\).

Generically the A-model three-point Yukawa coupling of a Calabi–Yau threefold, say \(X_3\), takes the following form [22,37–40]:

\[
\widetilde{Y}_{h_1 h_2 h_3}(\mathbf{q}) = \kappa_{h_1 h_2 h_3} + \sum_{d \in H_2(X_3, \mathbb{Z}) \setminus \{0\}} n_d \frac{d_i d_j d_k q^d}{1 - q^d}, \quad q^d = q_1^{d_1} q_2^{d_2} \cdots.
\]

(3.7)

This provides a generating function of integer invariants \(n_d\), which gives the Gromov–Witten invariants and enumerates the number of holomorphic maps \(\phi: \mathbb{P}^1 \rightarrow X_3\) of class \(d \in H_2(X_3, \mathbb{Z})\) intersecting with cycles dual to \(h_1, h_2,\) and \(h_3\). Here \(\kappa_{h_1 h_2 h_3}\) is called the classical triple intersection number and the summation with respect to \(d\) is taken only over non-negative elements.

For Calabi–Yau \(m\)-folds, the \(m\)-point Yukawa couplings can be factorized into \(m\)-point Yukawa couplings in accordance with the fusion rules of the underlying Frobenius algebra (see Refs. [41–43] for details). The so-obtained A-model three-point Yukawa couplings similarly provide generating functions of genus-zero Gromov–Witten invariants for \(m\)-folds. In the case of a Calabi–Yau fourfold \(X_4\), the A-model four-point Yukawa couplings are factorized into three-point Yukawa couplings as

\[
\widetilde{Y}_{h_1 h_2 h_3 h_4}(\mathbf{q}) = \eta^{\alpha \beta} \widetilde{Y}_{H_\alpha h_1 h_2}(\mathbf{q}) \widetilde{Y}_{H_\beta h_3 h_4}(\mathbf{q}),
\]

(3.8)

where \(H_\alpha, H_\beta \in H^{2,2}(X_4)\) represent the so-called primary elements generated by the wedge product of the elements in \(H^{1,1}(X_4)\) and \(\eta^{\alpha \beta}\) is the inverse matrix of the intersection matrix associated with \(H_\alpha\) and \(H_\beta\). The A-model three-point Yukawa couplings \(\widetilde{Y}_{h_1 h_2 H}(\mathbf{q})\) have a generic form [41–44]:

\[
\widetilde{Y}_{h_1 h_2 H}(\mathbf{q}) = \kappa_{h_1 h_2 H} + \sum_{d \in H_2(X_4, \mathbb{Z}) \setminus \{0\}} n_d(H) \frac{d_i d_j q^d}{1 - q^d}, \quad q^d = q_1^{d_1} q_2^{d_2} \cdots,
\]

(3.9)

where \(\kappa_{h_1 h_2 H}\) is the classical intersection number associated with cycles dual to \(h_1, h_2,\) and \(H\).

3.3.1. Mirror map from the localization formula

Starting from the factors (2.4) and (2.5) in the localization formula, one can find building blocks of the Givental \(I\)-function [45–47] from which the mirror map can be properly derived. Here we go straight to the point; some technical details are relegated to Appendix A.
For a local toric Calabi–Yau $m$-fold $X$ with GLSM description (3.1), one can construct the Givental $I$-function for $X$ with the following form:

$$I^{[\lambda]}_X(z; x; \hbar) = z^{x/h} \sum_{d \in (\mathbb{Z} \geq 0)^r} \frac{1}{n!} \left( \prod_{i=1}^n \phi_i(x, \lambda_i; \hbar) \right) z^d, \quad z^{x/h} = \prod_{j=1}^r z_j^{y_j/h}, \quad (3.10)$$

which consists of the building blocks defined in Eq. (A.8). To find the mirror map, it is sufficient to consider the case without twisted mass deformations. Expanding the $I$-function with $\lambda_i = 0$ around $\hbar = \infty$, we obtain

$$I^{[0]}_X(z; x; \hbar) = 1 + (I_{1,1}(z)x_1 + \cdots + I_{1,r}(z)x_r) \hbar^{-1} + O(\hbar^{-2}). \quad (3.11)$$

Then the inverse mirror map $\log q_i$, $i = 1, \ldots, r$ can be obtained by $\log q_i = I_{1,i}(z)$ [22,48] and the result is

$$\log q = \log z - \sum_{d \in (\mathbb{Z} \geq 0)^r \setminus \{0\}} \sum_I^\prime (-1)^{Q_I(d)} Q_I \frac{(-Q_I(d) - 1)!}{Q_I(d)!} z^d, \quad (3.12)$$

where we put $1/n! = 0$ for $n < 0$ and $\sum_I^\prime$ means that the summation is taken over all $I$ satisfying $Q_I(d) < 0$.

**Remark 3.3.** For a compact Calabi–Yau manifold, the $I$-function is generically annihilated by the Picard–Fuchs equation for the mirror. Then the higher-order terms of the expansion in Eq. (3.11) are expected to possess information on the genus-zero Gromov–Witten invariants (see, e.g., Ref. [49]). But for a local toric Calabi–Yau variety, say $X_{\text{local}}$, the information obtained from the $I$-function without twisted mass parameters (i.e., $\lambda_i = 0$ for all $i$) is sometimes not enough to get Gromov–Witten invariants. For instance, if $\dim H_4(X_{\text{local}}, \mathbb{Z}) = 0$, the higher-order terms $O(\hbar^{-2})$ in Eq. (3.11) vanish and cannot say anything about the Gromov–Witten invariants.

It is worth noting that by using the $I$-function with twisted mass parameters and the Birkhoff factorization [47], the equivariant local mirror symmetry for local toric Calabi–Yau threefolds was developed in Refs. [34–36] and the Gromov–Witten invariants have been computed.

### 4. Examples

Based on the criteria for the twisted mass deformations mentioned in Sect. 3.2, here we will compute the A-twisted GLSM correlators (3.4) for various local nef toric Calabi–Yau varieties in the geometric phase and demonstrate that the corresponding local B-model Yukawa couplings can be calculated appropriately. A novel aspect of the previously observed ambiguities for a certain class of varieties [23] will also be discussed.

#### 4.1. Local toric Calabi–Yau threefolds

First we will focus on the local nef toric Calabi–Yau threefolds whose exact properties have been studied by using the local mirror symmetry [22,23,34–36,50]. We confirm that the exact formula for the A-twisted GLSM correlation functions matches the previous results of the local B-model Yukawa couplings with the aid of our prescriptions. We also see that the ambiguities of the intersection numbers in Ref. [23] can be interpreted as degrees of freedom of the twisted mass deformations.

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10 See Ref. [21] for a recent development.
Fig. 1. 1) The upper left diagram describes the secondary fan of the resolved conifold and the right one represents the corresponding hyperplanes \( \{u = \lambda, u = 0\} \) corresponding to the charge vectors \( \{Q_{1,2} = 1, Q_{3,4} = -1\} \). The point \( u^* = \lambda \) is a projective point associated with \( Q^* = Q_{1,2} \). 2) The middle (resp. lower) left diagram describes the secondary fan of \( \mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1 \) (resp. \( \mathcal{O}(-3) \to \mathbb{P}^2 \)) and the right figure again describes the hyperplanes \( \{u = \lambda, u = 0\} \) associated with \( \{Q_{1,2} = 1, Q_4 = -2\} \) (resp. \( \{Q_{1,2,3} = 1, Q_4 = -3\} \)). The point \( u^* = \lambda \) is a projective point associated with \( Q^* = Q_{1,2} \) (resp. \( Q^* = Q_{1,2,3} \)). The orbifold phase of the middle (resp. lower) left diagram is given by the orbifold geometry \( \mathbb{C} \times \mathbb{C}^2 / \mathbb{Z}_2 \) (resp. \( \mathbb{C}^3 / \mathbb{Z}_3 \)).

4.1.1. Resolved conifold: \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 \)

Let us consider the resolved conifold described by a \( U(1) \) GLSM with the following charge vectors and the twisted masses:

\[
\begin{pmatrix}
Q_1 & Q_2 & Q_3 & Q_4 \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4
\end{pmatrix} = \begin{pmatrix}
1 & 1 & -1 & -1 \\
\lambda & \lambda & 0 & 0
\end{pmatrix},
\]

(4.1)

where the inclusion of the twisted mass parameters \( \lambda \neq 0 \) has been determined in accordance with the rules in Sect. 3.2, such that the pole \( u^* = \lambda \) of the exact formula associated with \( Q^* = \{Q_1, Q_2\} \) in Fig. 1 becomes a projective point. In other words, without turning on the twisted masses, the hyperplane arrangement of the model remains non-projective and the usage of the Jeffrey–Kirwan residue operation cannot be justified. By taking \( \eta \in \text{Cone}(Q_*) \), namely in the geometric phase, the projective point \( u^* = \lambda \) contributes to the Jeffrey–Kirwan residue in Eq. (3.4) and we obtain

\[
Y_{zzz}(z) = \sum_{d=0}^{\infty} z^d \text{Res}_{u=\lambda} \frac{u^3 (-u)^{2(d-1)}}{(u - \lambda)^{2(d+1)}} = \frac{1}{1 - z}.
\]

(4.2)

In this example the mirror map (3.12) becomes trivial, i.e., \( \log q = \log z \), and the A-model Yukawa coupling (3.6) is given by

\[
\tilde{Y}_{hhh}(q) = \frac{1}{1 - q}.
\]

(4.3)

Remark 4.1. Turning on the twisted masses in a symmetric manner for the base and fiber directions as \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda \) to maintain the flop symmetry \( z \leftrightarrow z^{-1} \) of the resolved conifold

---

\[11\] For the resolved conifold as well as the local \( \mathbb{P}^2 \), several aspects of twisted mass deformations have also been discussed in the context of correlator relations in Ref. [21].
interchanging the compact and the non-compact directions, we obtain

\[ Y'_{zzz}(z) = \sum_{d=0}^{\infty} z^d \text{Res}_{u=\lambda} \frac{u^3 (-u - \lambda)^{2(d-1)}}{(u - \lambda)^{2(d+1)}} = \frac{1}{2} + \frac{z}{1 - z}. \] (4.4)

Interestingly, this expression has a different classical triple intersection number and is consistent with the result argued in Ref. [23].

4.1.2. \( \mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1 \)

Next, we consider the local toric Calabi–Yau threefold \( \mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1 \) described by a \( U(1) \) GLSM with

\[ \left( \frac{Q_1}{\lambda_1}, \frac{Q_2}{\lambda_2}, \frac{Q_3}{\lambda_3}, \frac{Q_4}{\lambda_4} \right) = \left( \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{0}{-2\lambda}, \frac{0}{0} \right). \] (4.5)

Following the criteria in Sect. 3.2, a twisted mass parameter \( \lambda \neq 0 \) has been introduced such that \( u_* = \lambda \) associated with \( Q_\ast = \{Q_1, Q_2\} = 1 \) gives a projective point (see Fig. 1). In particular, according to rule 2, we have assigned a twisted mass \( \lambda_3 = -2\lambda \) for the neutral chiral multiplet \( \Phi_3 \) corresponding to \( \mathcal{O}(0) \) in order to satisfy the “Calabi–Yau condition on the divisor” \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \).

For \( \eta \in \text{Cone}(1) \) in the geometric phase, the projective point \( u_* = \lambda \) contributes to the Jeffrey–Kirwan residue in Eq. (3.4) and we obtain

\[ Y_{zzz}(z) = \sum_{d=0}^{\infty} z^d \text{Res}_{u=\lambda} \frac{u^3 (-2u)^{2d-1}}{2\lambda(u - \lambda)^{2(d+1)}} = -\frac{1}{2(1 - 4z)^2}. \] (4.6)

This agrees with the Yukawa coupling evaluated in Ref. [23]. The mirror map (3.12) becomes

\[ \log q = \log z + 2 \sum_{d=1}^{\infty} \frac{(2d - 1)!}{(d!)^2} z^d = \log z - 2 \log \frac{1 + \sqrt{1 - 4z}}{2}, \quad \leftrightarrow \quad z = \frac{q}{(1 + q)^2}, \] (4.7)

and the A-model Yukawa coupling (3.6) is given by

\[ \tilde{Y}_{hhh}(q) = -\frac{1}{2} + \frac{q}{1 + q}. \] (4.8)

4.1.3. Local \( \mathbb{P}^2 \): \( \mathcal{O}(-3) \to \mathbb{P}^2 \)

As an explicit example with \( \dim H_4(X, \mathbb{Z}) = 1 \), we consider the local Calabi–Yau threefold \( K_{\mathbb{P}^2} \) described by a \( U(1) \) GLSM with

\[ \left( \frac{Q_1}{\lambda_1}, \frac{Q_2}{\lambda_2}, \frac{Q_3}{\lambda_3}, \frac{Q_4}{\lambda_4} \right) = \left( \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{-3}{0} \right). \] (4.9)

Here we have introduced a twisted mass parameter \( \lambda \neq 0 \) for the base space \( \mathbb{P}^2 \) such that \( u_* = \lambda \) associated with \( Q_\ast = \{Q_1, Q_2, Q_3\} = 1 \) provides a projective point (see Fig. 1). For \( \eta \in \text{Cone}(1) \) in the geometric phase, the projective point \( u_* = \lambda \) gives a contribution to the Jeffrey–Kirwan residue in the GLSM correlator (3.4) and we obtain

\[ Y_{zzz}(z) = \sum_{d=0}^{\infty} z^d \text{Res}_{u=\lambda} \frac{u^3 (-3u)^{3d-1}}{(u - \lambda)^{3(d+1)}} = -\frac{1}{3(1 + 27z)}. \] (4.10)
As a second example with dim $H_4(X, \mathbb{Z}) = 1$, let us consider the local Hirzebruch surface $K_{F_0} = K_{\mathbb{P}^1 \times \mathbb{P}^1}$ described by a $U(1)^2$ GLSM with

$$
\begin{pmatrix}
Q_1 & Q_2 & Q_3 & Q_4 & Q_5 \\
\frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \frac{1}{\lambda_3} & \frac{1}{\lambda_4} & \frac{1}{\lambda_5}
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 0 & 0 & -2 \\
0 & 0 & 1 & 1 & -2 \\
\lambda & \lambda & \lambda & \lambda & 0
\end{pmatrix}.
$$

(4.12)

Here, based on the rules in Sect. 3.2, we have introduced a twisted mass parameter $\lambda \neq 0$ in a symmetric way for $z_1 \leftrightarrow z_2$ in the base space $\mathbb{P}^1 \times \mathbb{P}^1$, such that $u_* = (\lambda, \lambda)$ associated with $Q_* = \{Q_1, Q_2, Q_3, Q_4\}$ provides a projective point (see Fig. 2). For $\eta \in \text{Cone}(Q_1, Q_1 + Q_3)$ inside the geometric phase, the projective point $u_* = (\lambda, \lambda)$ gives a contribution to the Jeffrey–Kirwan residue in the GLSM correlation functions in Eq. (3.4) as

$$
Y_{z_1z_2}(z_1, z_2) = \sum_{d_1, d_2 = 0}^{\infty} z_1^{d_1} z_2^{d_2} \text{Res} \text{ Res}_{u_1 = \lambda, u_2 = \lambda} \frac{u_1 u_2 u_3}{(u_1 - \lambda)^{2(d_1 + 1)}(u_2 - \lambda)^{2(d_2 + 1)}} (-2u_1 - 2u_2)^{2d_1 + 2d_2 - 1},
$$

(4.13)

and each component has the following exact form:

$$
Y_{z_1z_1}(z_1, z_2) = \frac{(1 - 4z_2)^2 - 16z_1(1 + z_1)}{4\Delta_{F_0}(z_1, z_2)}, \quad Y_{z_1z_2}(z_1, z_2) = \frac{16z_1^2 - (1 - 4z_2)^2}{4\Delta_{F_0}(z_1, z_2)},
$$

$$
Y_{z_1z_2}(z_1, z_2) = \frac{16z_2^2 - (1 - 4z_2)^2}{4\Delta_{F_0}(z_1, z_2)}.
$$

(4.14)
Fig. 3. The left figure describes the secondary fan of the local $F_1$, and the right figure describes the hyperplanes \( (u_1 = p\lambda, u_1 - u_2 = 0, u_2 = \lambda, 2u_1 + u_2 = 0) \) corresponding to the charge vectors \( \{Q_1, Q_2, Q_3, Q_4, Q_5\} \). The points \( u_* = (p\lambda, \lambda), (\lambda, \lambda) \) are projective points associated with \( Q_* = \{Q_1, Q_3, Q_4\} \) and \( \{Q_2, Q_3, Q_4\} \).

where

\[
\Delta_{F_0}(z_1, z_2) = 1 - 8(z_1 + z_2) + 16(z_1 - z_2)^2. \tag{4.15}
\]

The above result is in agreement with the Yukawa couplings evaluated from the local mirror symmetry approach in Refs. [23,24]. The mirror map for $K_{F_0}$ is obtained by Eq. (3.12) as

\[
\log q_i = \log z_i + 2 \sum_{d_1, d_2 = 0}^{\infty} \frac{(2d_1 + 2d_2 - 1)!}{(d_1!)^2(d_2!)^2} z_1^{d_1} z_2^{d_2}, \quad i = 1, 2, \tag{4.16}
\]

and then the formula (3.7) correctly reproduces the Gromov–Witten invariants studied in Ref. [22].

4.1.5. Local $F_1$

As a third example with \( \text{dim } H_4(X, \mathbb{Z}) = 1 \), we consider the local Hirzebruch surface $K_{F_1}$ obtained by the one-point blow-up of $K_{\mathbb{P}^2}$, which can be described by a $U(1)^2$ GLSM with

\[
\begin{pmatrix}
Q_1 \\
\lambda_1 \\
Q_2 \\
\lambda_2 \\
Q_3 \\
\lambda_3 \\
Q_4 \\
\lambda_4 \\
Q_5 \\
\lambda_5 \\
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 0 & 0 & -2 \\
0 & -1 & 1 & 1 & -1 \\
p\lambda & 0 & \lambda & 0 \\
\end{pmatrix}. \tag{4.17}
\]

Here we have introduced a twisted mass parameter $\lambda \neq 0$ and a free parameter $p \neq -1/2$ for the base space $F_1$, such that $u_* = (p\lambda, \lambda)$ and $(\lambda, \lambda)$ associated with $Q_* = \{Q_1, Q_3, Q_4\}$ and $\{Q_2, Q_3, Q_4\}$ provide projective points respectively (see Fig. 3). Note that if we choose $p = -1/2$, the point $u_* = (p\lambda, \lambda)$ becomes non-projective. For $\eta \in \text{Cone}(Q_1, Q_1 + Q_3)$ in the geometric phase, the projective points $u_* = (p\lambda, \lambda), (\lambda, \lambda)$ contribute to the Jeffrey–Kirwan residue in Eq. (3.4) as

\[
Y_{z_1,z_2}(z_1, z_2) = \sum_{d_1,d_2=0}^{\infty} z_1^{d_1} z_2^{d_2} \left( \text{Res} \left. \frac{\text{Res} \left. \frac{\text{Res} \left. \frac{\text{Res} \left. u_1u_2u_k (-2u_1 - u_2)^{2d_1+d_2-1} \right.}{(u_1 - p\lambda)^{d_1+1}(u_2 - \lambda)^{2(d_2+1)}(u_1 - u_2)^{d_1-d_2+1}}} \right.}{u_2=\lambda, u_1=p\lambda, u_1=\lambda, u_2=\lambda} \right. \right) \tag{4.18}
\]
and we obtain
\[
Y_{z_1z_2}(z_1,z_2) = -6x + \frac{1}{3} + \frac{-1 - 4z_1^2 + z_2 - z_1(7 - 6z_2)}{3\Delta F_1(z_1,z_2)},
\]
\[
Y_{z_1z_2}(z_1,z_2) = 12x - \frac{2}{3} + \frac{-1 + 8z_1^2 + z_2 + z_1(2 - 3z_2)}{3\Delta F_1(z_1,z_2)},
\]
\[
Y_{z_2z_2}(z_1,z_2) = -24x + \frac{4}{3} + \frac{z_2(1 - 12z_1) - (1 - 4z_1)^2}{3\Delta F_1(z_1,z_2)},
\]
\[
Y_{z_2z_2}(z_1,z_2) = 48x - \frac{8}{3} + \frac{2(1 - 4z_1)^2 + z_2(1 + 60z_1)}{3\Delta F_1(z_1,z_2)},
\]
\[(4.19)\]
where
\[
\Delta F_1(z_1,z_2) = (1 - 4z_1)^2 - z_2(1 - 36z_1 + 27z_1z_2),
\]
\[(4.20)\]
and
\[
x = \frac{(p + 1)(3p + 1)}{18(2p + 1)^2}.
\]
\[(4.21)\]

The above expressions completely agree with the result in Ref. [23] (and Ref. [24] for\( p = 0 \) or equivalently \( x = 1/18 \)) and we also see that the free parameter \( x \) arising from a degree of freedom of the twisted mass of a chiral matter multiplet reproduces the “moduli parameter” discussed in Ref. [23].

**Remark 4.2.** In addition to the twisted masses represented in Eq. (4.17), we can also turn on a non-zero value for \( \lambda_2 \) as \( \lambda_1 = p\lambda, \lambda_2 = q\lambda, \lambda_3 = \lambda_4 = \lambda, \lambda_5 = 0 \) to make the poles associated with \( Q_a = \{Q_1, Q_3, Q_4\} \) and \( Q_a = \{Q_2, Q_3, Q_4\} \) projective. Then we obtain the same result with Eq. (4.19) while \( x \) is defined by
\[
x = \frac{(2q + 3)^2p^2 + (4q^2 + 15q + 12)p + (q + 1)(q + 3)}{6(2p + 1)^2(2q + 3)^2}.
\]
\[(4.22)\]
This means that additional mass deformation is possible in this case, but does not introduce an independent moduli parameter.

From Eq. (3.12), one can easily show that the mirror map for \( K_{F_1} \) is given by
\[
\log q_i = \log z_i + c_i \sum_{d_1, d_2=0}^{\infty} \frac{(-1)^{d_2}(2d_1 + 3d_2 - 1)!}{d_1!(d_2!)^2(d_1 + d_2)!} z_1^{d_1}z_2^{d_2}z_2^{-d_1}, \quad i = 1, 2,
\]
\[(4.23)\]
where \( c_1 = 2 \) and \( c_2 = 1 \). Combining this with the formula in Eq. (3.7), the Gromov–Witten invariants studied in Ref. [22] can be appropriately reproduced.

**4.1.6. Local \( F_2 \)**

As a fourth example with \( \dim H_4(X, \mathbb{Z}) = 1 \), let us study the local Hirzebruch surface \( K_{F_2} \) (a local \( A_1 \) geometry) described by a \( U(1)^2 \) GLSM with
\[
\begin{pmatrix}
Q_1 & Q_2 & Q_3 & Q_4 & Q_5 \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 0 & 0 & -2 \\
0 & -2 & 1 & 1 & 0 \\
\lambda & 0 & \epsilon & \epsilon & 0
\end{pmatrix}.
\]
\[(4.24)\]
The above expressions indeed agree with the result obtained in Ref. [23]. The mirror map (3.12) for $K_{Q_2}$ points $u$ projective points respectively (see Fig. 4). The mirror map (3.12) for $K_{Q_2}$ points $u$ are projective points associated with $Q_2 = \{Q_1, Q_3, Q_4\}$. The orbifold phase in the left figure is described by the orbifold geometry $\mathbb{C}^3/\mathbb{Z}_4$.

Here we have introduced twisted mass parameters $\epsilon, \lambda \neq 0$ with $\epsilon \neq \lambda$ for compact directions such that $u_\ast = (\lambda, \epsilon)$ and $(2\epsilon, \epsilon)$ associated with $Q_\ast = \{Q_1, Q_3, Q_4\}$ and $\{Q_2, Q_3, Q_4\}$ provide the projective points respectively (see Fig. 4).

In this model, the chiral multiplet $\Phi_5$, which is neutral under the second $U(1)$ gauge symmetry, describes the non-compact fiber coordinate $X_5$, and the compact divisor $\{X_5 = 0\}$ (i.e., $F_2$) contains the blow-up mode of the $\mathbb{Z}_2$ singularity. Therefore, following rule 2 in Sect. 3.2, we impose the “Calabi–Yau condition on the divisor” as $\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0$, which implies the need to take a limit $\epsilon \rightarrow 0$ in the final step. For $\eta \in \text{Cone}(Q_1, Q_1 + Q_3)$ in the geometric phase, the projective points $u_\ast = (\lambda, \epsilon)$ and $(2\epsilon, \epsilon)$ contribute to the Jeffrey–Kirwan residue in Eq. (3.4) as

$$
Y_{z_2}(z_1, z_2) = \lim_{\epsilon \rightarrow 0} \sum_{d_1, d_2 = 0}^\infty z_1^{d_1} z_2^{d_2} \left( \text{Res}_{u_2 = \epsilon} \text{Res}_{u_1 = \lambda} \text{Res}_{u_2 = \epsilon} \right) \frac{u_1 u_2 u_k (-2u_1)^{2d_1-1}}{(u_1 - \lambda)^{d_1+1}(u_2 - \epsilon)^{2(d_2+1)}(u_1 - 2u_2)^{d_1-2d_2+1}},
$$

and we obtain

$$
Y_{z_1}(z_1, z_2) = \frac{-1}{\Delta_{F_2}(z_1, z_2)}, \quad Y_{z_2}(z_1, z_2) = \frac{2z_1 - \frac{1}{2}}{\Delta_{F_2}(z_1, z_2)},
$$

$$
Y_{z_1 z_2}(z_1, z_2) = \frac{z_2(1 - 8z_1)}{(1 - 4z_2)\Delta_{F_2}(z_1, z_2)}, \quad Y_{z_2 z_2}(z_1, z_2) = \frac{z_2(24z_1 z_2 + 2z_1 - 2z_2 - \frac{1}{2})}{(1 - 4z_2)^2\Delta_{F_2}(z_1, z_2)},
$$

where

$$
\Delta_{F_2}(z_1, z_2) = (1 - 4z_1)^2 - 64z_1^2 z_2.
$$

The above expressions indeed agree with the result obtained in Ref. [23]. The mirror map (3.12) for $K_{Q_2}$ is given by

$$
\log q_i = \log z_i + c_i \log \frac{1 + \sqrt{1 - 4z_2}}{2} + 2\delta_i 1 \sum_{d_1, d_2 = 0}^{\infty} \frac{(2d_1 + 4d_2 - 1)!}{d_1!(d_2)!^2(d_1 + 2d_2)!} z_1^{d_1 + 2d_2} z_2^{d_2},
$$

Fig. 4. The left figure describes the secondary fan of the local $F_2$, and the right figure describes the hyperplanes $\{u_1 = \lambda, u_1 - 2u_2 = 0, u_2 = \epsilon, u_1 = 0\}$ corresponding to the charge vectors $\{Q_1, Q_2, Q_1, Q_2\}$. The points $u_\ast = (\lambda, \epsilon), (2\epsilon, \epsilon)$ are projective points associated with $Q_\ast = \{Q_1, Q_3, Q_4\}$, \{Q_2, Q_3, Q_4\}. The orbifold phase in the left figure is described by the orbifold geometry $\mathbb{C}^3/\mathbb{Z}_4$. The points $u_\ast = (\lambda, \epsilon), (2\epsilon, \epsilon)$ are projective points associated with $Q_\ast = \{Q_1, Q_3, Q_4\}$ and $\{Q_2, Q_3, Q_4\}$ provide the projective points respectively (see Fig. 4).
where \( i = 1, 2 \), and \((c_1, c_2) = (1, -2)\). The Gromov–Witten invariants computed in Ref. [22] from the local mirror symmetry approach can also be reproduced by using Eq. (3.7).

### 4.1.7. Local \( dP_2 \)

As a last example with \( \text{dim} \, H_4(X, \mathbb{Z}) = 1 \), we consider the local del Pezzo surface \( K_{dP_2} \), which can be obtained by a one-point blow-up of \( K_{F_0} \) or \( K_{F_1} \). This local toric variety can be described by a \( U(1)^3 \) GLSM with

\[
\begin{pmatrix}
    Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\
    \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6
\end{pmatrix} = \begin{pmatrix}
    1 & -1 & 1 & 0 & 0 & -1 \\
    -1 & 1 & 0 & 0 & 1 & -1 \\
    0 & 1 & -1 & 1 & 0 & -1 \\
\end{pmatrix}.
\]  

(4.29)

Here we have introduced a twisted mass parameter \( \lambda \neq 0 \) and two free parameters \( p \) and \( q \) respecting the interchanging symmetry \( z_2 \leftrightarrow z_3 \) for the base space \( dP_2 \), such that \( \mathbf{u}_* = ((p + 1)\lambda, p\lambda, p\lambda), (p + 1)\lambda, p\lambda, (q + 1)\lambda), (p + 1)\lambda, (q + 1)\lambda, p\lambda, (q + 2)\lambda, (q + 1)\lambda, (q + 1)\lambda) \) associated with \( Q_\pm = \{Q_1, Q_3, Q_4, Q_5\}, \{Q_1, Q_2, Q_5\}, \{Q_2, Q_3, Q_4\}, \) and \( \{Q_1, Q_2, Q_3\} \) provide the projective points, respectively. Note that the particular values for \( p \) and \( q \) satisfying \( 3p + 1 = 0, 2p + q + 2 = 0, 3q + 4 = 0 \) should be excluded in order to maintain the projective condition.

For \( \eta = (5, 3, 2) \) in the geometric phase, we find that the above projective points contribute to the Jeffrey–Kirwan residue in Eq. (3.4) as

\[
Y_{z_1z_2z_3}(z_1, z_2, z_3) = \sum_{d_1, d_2, d_3=0}^{\infty} z_1^{d_1} z_2^{d_2} z_3^{d_3} \left( \text{Res} \left. \frac{\text{Res} \text{Res} + \text{Res} \text{Res} \text{Res}}{u_2=(q+1)\lambda, u_3=p\lambda, u_1=(p+1)\lambda, u_2=p\lambda} \right|_{u_2=(q+1)\lambda, u_3=p\lambda, u_1=(p+1)\lambda, u_2=p\lambda} \right)
\]

\[
\times \frac{u_1 u_2 u_3}{(u_1 - u_2 - \lambda)d_1^{d_2} d_3^{d_3} (u_1 - u_2) - d_1 d_2 d_3} \times \frac{1}{(u_1 - u_3 - \lambda)d_1^{d_2} d_3^{d_3} (u_3 - p\lambda) d_3^{d_2} d_3^{d_3} (u_2 - p\lambda)^{d_2} d_3^{d_3} (u_3 - p\lambda)^{d_2} d_3^{d_3}}.
\]  

(4.30)

Note that the number of intersecting hyperplanes at the point \( \mathbf{u}_* = ((p + 1)\lambda, p\lambda, p\lambda) \) is larger than \( r = 3 \). To deal with this “degenerate point”, which requires careful treatment of the order of the iterated residue, we have applied Theorem 2.5 for a flag \( F \) with \( \kappa(F) = \{Q_5, Q_4 + Q_5, Q_1 + Q_3 + Q_4 + Q_5\} \) and \( \nu(F) = 1 \) as the only constituent in \( \mathcal{F} \mathcal{L}(Q_\pm, \eta) \) for \( Q_\pm = \{Q_1, Q_3, Q_4, Q_5\} \).

As a result, we finally obtain

\[
Y_{z_1z_2z_3}(z_1, z_2, z_3) = 6x + 2y - 1 + Y_{z_1z_2}(z_1, z_2, z_3)|_{p=q=0},
\]

\[
Y_{z_1z_2z_3}(z_1, z_2, z_3) = -3x - y + \frac{1}{2} + Y_{z_1z_2}(z_1, z_2, z_3)|_{p=q=0},
\]

\[
Y_{z_1z_2z_3}(z_1, z_2, z_3) = x + y + Y_{z_1z_2}(z_1, z_2, z_3)|_{p=q=0},
\]

\[
Y_{z_1z_2z_3}(z_1, z_2, z_3) = 2x - \frac{1}{2} + Y_{z_1z_2z_3}(z_1, z_2, z_3)|_{p=q=0},
\]

\[
Y_{z_2z_2z_3}(z_1, z_2, z_3) = -y - \frac{1}{4} + Y_{z_2z_2z_3}(z_1, z_2, z_3)|_{p=q=0},
\]

\[
Y_{z_2z_2z_3}(z_1, z_2, z_3) = -x + \frac{1}{4} + Y_{z_2z_2z_3}(z_1, z_2, z_3)|_{p=q=0}.
\]  

(4.31)
and

\[
Y_{z_1 z_2 z_3} (z_1, z_2, z_3) = Y_{z_1 z_2 z_3} (z_1, z_2, z_3), \quad Y_{z_1 z_2 z_3} (z_1, z_2, z_3) = Y_{z_1 z_2 z_3} (z_1, z_2, z_3), \quad Y_{z_1 z_2 z_3} (z_1, z_2, z_3) = Y_{z_1 z_2 z_3} (z_1, z_2, z_3),
\]

(4.32)

with \( z_2 \leftrightarrow z_3 \) symmetry. Here

\[
x = \frac{(12p^3 + 33p^2 + 19p + 3)q + 2(8p^3 + 14p^2 + 7p + 1) + (3pq + q)^2}{(3p + 1)^2(3q + 4)(2p + q + 2)},
\]

\[
y = \frac{5p^3(3q + 4) + 2p^2(3q + 4)^2 + 2p(6q^2 + 13q + 7) + 2(q + 1)^2}{(3p + 1)^2(3q + 4)(2p + q + 2)},
\]

(4.33)

and the analytic expressions of \( Y_{z_1 z_2 z_3} (z_1, z_2, z_3) \) \( p=q=0 \) are summarized in Appendix B. Note that \( p = q = 0 \) corresponds to \((x, y) = (1/4, -1/4)\).

The above results completely agree with the Yukawa couplings with the “moduli parameters” \( x \) and \( y \) given in Ref. [23]. The mirror map (3.12) for \( K_{dp_2} \) is given by

\[
\log q_i = \log z_i + \sum_{d_1, d_2, d_3 = 0}^{\infty} \sum_{(d_1, d_2, d_3) \neq (0,0,0)} \frac{(-1)^{d_1}(3d_1 + 2d_2 + 2d_3 - 1)!}{d_1!d_2!d_3!(d_1 + d_2)!(d_1 + d_3)!} \frac{x^{d_1+d_2+d_3}y^{d_1+d_2}z^{d_1}}{z_1^{d_1+d_2}z_2^{d_1}z_3^{d_3}},
\]

(4.34)

where \( i = 1, 2, 3 \), and the associated Gromov–Witten invariants studied in Ref. [22] are correctly reproduced by using Eq. (3.7).

4.1.8. Local \( A_2 \) geometry

Let us consider a local \( A_2 \) geometry with \( \dim H_4(X, \mathbb{Z}) = 2 \), which is a fibered \( A_2 \) geometry over \( \mathbb{P}^1 \) represented in Fig. 5. This geometry engineers the 4D pure \( SU(3) \) gauge theory and can be described by a \( U(1)^3 \) GLSM with [22,51,52]

\[
\begin{pmatrix}
Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 \\
\epsilon & \epsilon & 0 & 0 & 0 & \lambda
\end{pmatrix}.
\]

(4.35)
Here we have introduced the twisted mass parameters $\epsilon, \lambda \neq 0$ with $\epsilon \neq \lambda$ for a projective hyperplane arrangement in the geometric phase with $Q_n = \{Q_1, Q_2, Q_3, Q_6\}, \{Q_1, Q_2, Q_5, Q_6\},$ and $\{Q_1, Q_2, Q_3, Q_4\}$. We again confirm that the localization formula for the A-twisted GLSM correlation functions provides the exact expressions for local B-model Yukawa couplings appropriately, with the aid of our formalism.

Following rule 1 in Sect. 3.2, we have not included the mass parameters for the chiral multiplets $\Phi_3, \Phi_4,$ and $\Phi_5$ with charge vectors $Q_3, Q_4,$ and $Q_5,$ which blow up the singularities. Furthermore, the chiral multiplet $\Phi_4,$ which is neutral with respect to the first $U(1)$ gauge symmetry, describes a non-compact fiber coordinate $X_4$ and the compact divisor $\{X_4 = 0\}$ contains the blow-up mode of the $\mathbb{Z}_2$ singularity. Therefore, by following rule 2, we impose the “Calabi–Yau condition on the divisor” as $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, which implies the need to take a limit $\epsilon \to 0$ in the final step.

By taking, e.g., $\eta = (2, 1, 2)$ in the geometric phase, the projective points $u_\eta = (\epsilon, 2\epsilon, \lambda), (\epsilon, 2\lambda, \lambda),$ and $(\epsilon, 2\epsilon, 4\epsilon)$ contribute to the Jeffrey–Kirwan residue in the GLSM correlators (3.4) as

$$Y_{z_1 z_2 z_3}(z_1, z_2, z_3) = \lim_{\epsilon \to 0} \sum_{d_1, d_2, d_3 = 0}^{\infty} \epsilon^{d_1} \frac{z_1^{d_2} z_2^{d_3}}{z_3} \left( \frac{\text{Res}}{u_2 = 2\epsilon} \text{Res} + \text{Res} + \text{Res} + \text{Res} + \text{Res} \right)_{u_2 = 2\lambda, u_1 = \epsilon} \text{Res}_{u_3 = 4\epsilon, u_2 = 2\epsilon, u_1 = \epsilon},$$

where

$$M_1(z_1, z_2, z_3) = \sum_{d_1, d_2, d_3 = 0}^{\infty} \left( \frac{(-1)^d}{(d_1!)^2 d_2! d_3! (d_2 + 2d_3)!} \right) z_1^{d_1} z_2^{d_2} z_3^{d_3},$$

$$M_2(z_1, z_2, z_3) = \sum_{d_1, d_2, d_3 = 0}^{\infty} \left( \frac{(-1)^d}{(d_1!)^2 d_2! d_3! (4d_1 + 2d_2 + 2d_3)!} \right) z_1^{d_1} z_2^{d_2} z_3^{d_3}.$$
Fig. 6. The first (resp. second) figure in the left describes the secondary fan of $\mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathbb{P}^2$ (resp. $\mathcal{O}(-4) \to \mathbb{P}^3$), and the right figure describes the hyperplanes $\{u = \lambda, u = 0\}$ corresponding to the charge vectors $\{Q_{1,2,3}, Q_{4,5}\}$ (resp. $\{Q_{1,2,3,4,5}\}$). The point $u_\ast = \lambda$ is a projective point associated with $Q_\ast = Q_{1,2,3} = 1$ (resp. $Q_\ast = Q_{1,2,3,4,5} = 1$).

4.2.1. $\mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathbb{P}^2$

Let us consider the local toric Calabi–Yau fourfold $\mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathbb{P}^2$ described by a $U(1)$ GLSM with

$$
\left( \frac{Q_1}{\lambda_1}, \frac{Q_2}{\lambda_2}, \frac{Q_3}{\lambda_3}, \frac{Q_4}{\lambda_4}, \frac{Q_5}{\lambda_5} \right) = \left( \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{-1}{\lambda}, \frac{-2}{\lambda}, 0 \right),
$$

where, following the rules in Sect. 3.2, we have introduced a twisted mass parameter $\lambda \neq 0$ for the base space $\mathbb{P}^2$ such that $u_\ast = \lambda$ associated with $Q_\ast = \{Q_1, Q_2, Q_3\} = 1$ gives a projective point (see Fig. 6). For $\eta \in \text{Cone}(1)$ in the geometric phase, the projective point $u_\ast = \lambda$ contributes to the Jeffrey–Kirwan residue in the GLSM correlation function (3.4) as

$$
Y_{zzzz}(z) = \sum_{d=0}^{\infty} z^d \text{Res}_{u=\lambda} \frac{u^d (-u)^{d-1} (-2u)^{2d-1}}{(u - \lambda)^{3(d+1)}} = \frac{1}{2(1 + 4z)}.
$$

In this example, the mirror map (3.12) is trivial, i.e., $\log q = \log z$, and the A-model Yukawa coupling (3.6) takes a form

$$
\widetilde{Y}_{hhhh}(q) = \frac{1}{2(1 + 4q)}.
$$

One finds that the identity

$$
\frac{1}{1 - 4q} = \left( \sum_{d=0}^{\infty} \binom{2d}{d} q^d \right)^2
$$

is equivariant to the factorization (3.8) of the four-point Yukawa coupling $\widetilde{Y}_{hhhh}(q)$ into the three-point Yukawa coupling $\widetilde{Y}_{hhh2}(q)$ in Ref. [44]:

$$
\widetilde{Y}_{hhhh}(q) = 2\widetilde{Y}_{hhh2}(q)^2, \quad \widetilde{Y}_{hhh2}(q) = \sum_{d=0}^{\infty} \frac{1}{2} \binom{2d}{d} (-q)^d.
$$

4.2.2. Local $\mathbb{P}^3$: $\mathcal{O}(-4) \to \mathbb{P}^3$

Next, let us consider the local toric Calabi–Yau fourfold $K_{\mathbb{P}^3}$ described by a $U(1)$ GLSM with

$$
\left( \frac{Q_1}{\lambda_1}, \frac{Q_2}{\lambda_2}, \frac{Q_3}{\lambda_3}, \frac{Q_4}{\lambda_4}, \frac{Q_5}{\lambda_5} \right) = \left( \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{1}{\lambda}, \frac{-4}{\lambda} \right),
$$

where, following the rules in Sect. 3.2, we have introduced a twisted mass parameter $\lambda \neq 0$ for the base space $\mathbb{P}^3$ such that $u_\ast = \lambda$ associated with $Q_\ast = \{Q_1, Q_2, Q_3, Q_4, Q_5\} = 1$ gives a projective point (see Fig. 6). For $\eta \in \text{Cone}(1)$ in the geometric phase, the projective point $u_\ast = \lambda$ contributes to the Jeffrey–Kirwan residue in the GLSM correlation function (3.4) as

$$
Y_{zzzz}(z) = \sum_{d=0}^{\infty} z^d \text{Res}_{u=\lambda} \frac{u^d (-u)^{d-1} (-2u)^{2d-1}}{(u - \lambda)^{3(d+1)}} = \frac{1}{2(1 + 4z)}.
$$

In this example, the mirror map (3.12) is trivial, i.e., $\log q = \log z$, and the A-model Yukawa coupling (3.6) takes a form

$$
\widetilde{Y}_{hhhh}(q) = \frac{1}{2(1 + 4q)}.
$$

One finds that the identity

$$
\frac{1}{1 - 4q} = \left( \sum_{d=0}^{\infty} \binom{2d}{d} q^d \right)^2
$$

is equivariant to the factorization (3.8) of the four-point Yukawa coupling $\widetilde{Y}_{hhhh}(q)$ into the three-point Yukawa coupling $\widetilde{Y}_{hhh2}(q)$ in Ref. [44]:

$$
\widetilde{Y}_{hhhh}(q) = 2\widetilde{Y}_{hhh2}(q)^2, \quad \widetilde{Y}_{hhh2}(q) = \sum_{d=0}^{\infty} \frac{1}{2} \binom{2d}{d} (-q)^d.
$$
Fig. 7. The left figure describes the secondary fan of $\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$, and the right figure describes the hyperplanes $\{u_1 = \lambda, u_2 = \lambda, u_1 + u_2 = 0\}$ corresponding to the charge vectors $\{Q_{1,2}, Q_{3,4}, Q_{5,6}\}$. The point $u_* = (\lambda, \lambda)$ is a projective point associated with $Q_* = \{Q_{1,2}, Q_{3,4}\}$.

where a twisted mass parameter $\lambda \neq 0$ has been introduced such that $u_* = \lambda$ associated with $Q_* = \{Q_{1,2}, Q_{3,4}\} = 1$ provides a projective point (see Fig. 6). For $\eta \in \text{Cone}(1)$ in the geometric phase, the projective point $u_* = \lambda$ contributes to the Jeffrey–Kirwan residue in Eq. (3.4) and we obtain

$$
Y_{zzzz}(z) = \sum_{d=0}^{\infty} z^d \text{Res}_{u=\lambda} \frac{u^4 (4u)^{4d-1}}{(u - \lambda)^{4(d+1)}} = -\frac{1}{4 (1 - 4^4 z)}. \quad (4.45)
$$

The mirror map (3.12) for $K_{\mathbb{P}^3}$ is given by

$$
\log q = \log z + 4 \sum_{d=1}^{\infty} \frac{(4d - 1)!}{(dl)^4} z^d, \quad (4.46)
$$

and one can check that the factorization (3.8) into the three-point Yukawa couplings correctly reproduces the previous results for the Gromov–Witten invariants in Ref. [44].

4.2.3. $\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$

Finally, we consider the local toric Calabi–Yau fourfold $\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$ described by a $U(1)^2$ GLSM with

$$
\left( \begin{array}{cccccc}
Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\
\end{array} \right) = \left( \begin{array}{ccccccc}
1 & 1 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & 1 & -1 & -1 \\
\lambda & \lambda & \lambda & \lambda & 0 & 0 \\
\end{array} \right), \quad (4.47)
$$

where we have introduced a twisted mass parameter $\lambda \neq 0$ in a symmetric way for $z_1 \leftrightarrow z_2$ in the base space $\mathbb{P}^1 \times \mathbb{P}^1$, such that $u_* = (\lambda, \lambda)$ associated with $Q_* = \{Q_1, Q_2, Q_3, Q_4\}$ gives a projective point (see Fig. 7). For $\eta \in \text{Cone}(Q_1, Q_1 + Q_2)$ in the geometric phase, the projective point $u_* = (\lambda, \lambda)$ gives a contribution to the Jeffrey–Kirwan residue in the GLSM correlators (3.4) as

$$
Y_{zzz}(z_1, z_2) = \sum_{d_1, d_2=0}^{\infty} z_1^{d_1} z_2^{d_2} \text{Res}_{u_1 = \lambda, u_2 = \lambda} \frac{u_1 u_2 u_3 u_4 (u_1 - u_2)^{2(d_1 + d_2 - 1)}}{(u_1 - \lambda)^{2(d_1 + 1)}(u_2 - \lambda)^{2(d_2 + 1)}}. \quad (4.48)
$$
As a result, we obtain

\[ Y_{z_1z_1z_2}(z_1, z_2) = -\frac{5(1 - z_2)^2 + z_1(10 + 11z_1 + 10z_2)}{8\Delta'_F(z_1, z_2)}, \]

\[ Y_{z_1z_1z_2}(z_1, z_2) = \frac{(1 - z_2)^2 + z_1(6 - 7z_1 - 10z_2)}{8\Delta'_F(z_1, z_2)}, \]

\[ Y_{z_1z_2z_2}(z_1, z_2) = \frac{3 - 6(z_1 + z_2) + 3z_1^2 + 10z_1z_2 + 3z_2^2}{8\Delta'_F(z_1, z_2)}, \]

\[ Y_{z_2z_2z_2}(z_1, z_2) = \frac{(1 - z_1)^2 + z_2(6 - 10z_1 - 7z_2)}{8\Delta'_F(z_1, z_2)}, \]

\[ Y_{z_2z_2z_2}(z_1, z_2) = -\frac{5(1 - z_1)^2 + z_2(10 + 10z_1 + 11z_2)}{8\Delta'_F(z_1, z_2)}, \] (4.49)

where

\[ \Delta'_F(z_1, z_2) = 1 - 2(z_1 + z_2) + (z_1 - z_2)^2. \] (4.50)

In this example, the mirror map (3.12) becomes trivial, i.e., \( \log q_i = \log z_i, i = 1, 2 \), and we see that the factorization into the three-point Yukawa couplings (3.8) reproduces the previous results in Ref. [44] appropriately.

5. Conclusions

In this paper, we have thoroughly investigated the relationship between the topological B-model Yukawa couplings for backgrounds with non-compact directions and the exact localization formula for the A-twisted correlation functions of the 2D \( \mathcal{N} = (2, 2) \) gauged linear sigma models. Starting from the exact results for the A-twisted correlators of \( \mathcal{N} = (2, 2) \) gauged linear sigma models [15], we demonstrated how to extract appropriately the B-model Yukawa couplings for the local nef toric Calabi–Yau varieties. First we explained that the inclusion of the twisted masses for the chiral matter multiplets is indispensable in conducting explicit calculations for the Jeffrey–Kirwan residue formalism. Although it has been argued in the literature that a twisted mass deformation is required to deal with backgrounds with non-compact directions, a comprehensive study has not been conducted before. We addressed this important issue and proposed an algorithm to compute the GLSM correlation functions for local toric Calabi–Yau varieties appropriately.

We have also checked that our prescription for the twisted mass deformations of the GLSM correlation functions is totally consistent with known results for the Yukawa couplings evaluated from the local mirror symmetry approach. Moreover, we found that the ambiguities of classical intersection numbers of a certain class of local toric Calabi–Yau varieties argued previously are identified with the degrees of freedom of the assignment of the proper twisted mass parameters. In combination with the exact localization formula for the A-twisted GLSM correlation functions, our prescription would provide an alternative efficient formalism to compute the B-model Yukawa couplings for generic local nef toric Calabi–Yau varieties.

Finally we would like to comment on possible future research directions. Throughout this paper, we have not considered models whose target spaces are non-nef varieties [34–36]. It would be interesting to extend our analysis for the GLSM correlation functions and twisted mass deformations into such intriguing examples.
In our framework, the twisted masses are introduced only for the compact directions of the target space. It would be interesting to clarify the physical meaning of this requirement. One possible explanation is the following. From the viewpoint of the SUSY algebra, inclusion of the twisted masses modifies the central charges of the model and implies the existence of additional charged BPS particles in a specific vacuum. The appearance of such extra massless states can be naturally interpreted as D-branes wrapping on compact directions, as discussed in Refs. [53,54]; this is known to be indispensable to regularize appropriately a singularity of the model.

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Appendix A. Building blocks of $I$-functions from GLSM correlators

Let us consider a GLSM that flows in the IR to a 2D non-linear sigma model with a Fano or a Calabi–Yau variety $X$ as a target space. Following Ref. [19] (see also Ref. [55]), here we will show that the building blocks of the Givental $I$-function [45–47] for $X$ can be derived from the factors (2.4) and (2.5) in the localization formula.

First, let us reparametrize $u$ and $d$ as
\[
    u = x + \frac{d}{2} h - q' h, \quad d = q + q'.
\] (A.1)

Then the factor (2.4) can be decomposed as
\[
    Z^\text{vec}_d(u; h) = I_{\text{vec}}^\text{pert}(x) I_{q}^\text{vec}(x; h) I_{q'}^\text{vec}(x; -h),
\] (A.2)

where
\[
    I_{\text{vec}}^\text{pert}(x) = \prod_{\alpha \in \Delta_+} (-1)^{-\alpha(x)^2},
\] (A.3)
\[
    I_{q}^\text{vec}(x; h) = \prod_{\alpha \in \Delta_+} (-1)^{-\rho(q)\alpha(x) + \alpha(q)\frac{h}{2}}.
\] (A.4)

The factor (2.5) for $\Phi_a = \Phi$ with representation $R_a = R$, $R$-charge $r_a = 0$, and twisted mass $\lambda_a = \lambda$ can be decomposed as
\[
    Z^\Phi_d(u; h) = \begin{cases} 
    \prod_{\rho \in \mathcal{R}} \prod_{\rho = 0} \left( \rho(u) + \lambda + ph - \frac{\rho(d)}{2} h \right)^{-1}, & \text{if } \rho(d) \geq 0, \\
    \prod_{\rho \in \mathcal{R}} \prod_{\rho = 1} \left( \rho(u) + \lambda - ph - \frac{\rho(d)}{2} h \right), & \text{if } \rho(d) \leq -1,
\end{cases}
\] (A.5)

\[
    = I_{\text{pert}}^\Phi(x, \lambda) I_{q}^\Phi(x, \lambda; h) I_{q'}^\Phi(x, \lambda; -h),
\] (A.6)
where

\[ I^\Phi_{\text{pert}}(x, \lambda) = \prod_{\rho \in R} \frac{1}{\rho(x) + \lambda}, \quad (A.7) \]

\[ I^\Phi_q(x, \lambda; h) = \begin{cases} \prod_{\rho \in R} \prod_{p=0}^{\rho(q)} (\rho(x) + \lambda + ph)^{-1}, & \text{for } \rho(q) \geq 0, \\ \prod_{\rho \in R} \prod_{p=0}^{-\rho(q)} (\rho(x) + \lambda - ph)^{-1}, & \text{for } \rho(q) \leq -1. \end{cases} \quad (A.8) \]

Similarly, the factor (2.5) for \( \Phi_a = P \) with representation \( R_a = R, R \)-charge \( r_a = 2 \), and twisted mass \( \lambda_a = \lambda \) can be decomposed as

\[ Z^P_d(u; h) = \begin{cases} -\prod_{\rho \in R} \prod_{p=0}^{-\rho(d)} \left( \rho(u) + \lambda - ph - \frac{\rho(d)}{2} h \right), & \text{if } \rho(d) \leq 0, \\ -\prod_{\rho \in R} \prod_{p=0}^{\rho(d)} \left( \rho(u) + \lambda + ph - \frac{\rho(d)}{2} h \right)^{-1}, & \text{if } \rho(d) \geq 1, \end{cases} \quad (A.9) \]

\[ = I^P_{\text{pert}}(x, \lambda) I^P_q(x, \lambda; h) I^P_q(x, \lambda; -h) \prod_{\rho \in R} (-1)^{\rho(d)}, \quad (A.10) \]

where

\[ I^P_{\text{pert}}(x, \lambda) = -\prod_{\rho \in R} (\rho(x) + \lambda), \quad (A.11) \]

\[ I^P_q(x, \lambda; h) = \begin{cases} \prod_{\rho \in R} \prod_{p=0}^{\rho(q)} (-\rho(x) - \lambda + ph), & \text{for } \rho(q) \leq 0, \\ \prod_{\rho \in R} \prod_{p=0}^{-\rho(q)} (-\rho(x) - \lambda - ph)^{-1}, & \text{for } \rho(q) \geq 1. \end{cases} \quad (A.12) \]

After taking the above decomposition, the ingredients \( I^\Phi_{\text{loc}}(x; h) \) in Eq. (A.4), \( I^\Phi_q(x, \lambda; h) \) in Eq. (A.8) and \( I^P_q(x, \lambda; h) \) in Eq. (A.12) are known to provide the building blocks of the Givental \( I \)-function for the variety \( X \). There \( x \) and \( \lambda \) are identified with the equivariant cohomology elements or the Chern roots of \( X \) and the equivariant parameter acting on \( X \), respectively [19–21,55,56]. Starting from the \( I \)-function, one can find an associated quantum differential equation called the Picard–Fuchs equation, from which the mirror map and genus-zero Gromov–Witten invariants of \( X \) can be evaluated [45–47].

Appendix B. Local B-model Yukawa couplings of local \( dP_2 \)

The exact B-model Yukawa couplings \( Y^{(0)}_{z \bar{z} z}(z) = Y^{(0)}_{z \bar{z} z}(z, \zeta) \big|_{\rho=\bar{\rho}=0} \) for \( p = q = 0 \) of the local del Pezzo surface \( K_{dP_2} \) defined in Eq. (4.30) have the following expressions:

\[ Y^{(0)}_{z \bar{z} z}(z) = \frac{4 z^2 (z - \zeta)^2 (z + \zeta - 2) + z \left( z - 1 \right) \left( z - 1 \right) \left( 5 z - 2 \left( 9 z - 5 \right) - 1 \right)}{\Delta_{dP_2}(z)}, \]

\[ Y^{(0)}_{z \bar{z} z}(z) = \frac{-4 \left( 2 z^3 - 3 z^2 + 2 z^2 + z + 1 \right) \left( z^2 - 1 \right) \left( z - 1 \right)}{\Delta_{dP_2}(z)}, \]

\[ Y^{(0)}_{z \bar{z} z}(z) = \frac{4 \left( z - 1 \right) \left( z - 2 \right) \left( z - 1 \right) \left( z + 1 \right) z^2 + z \left( z - 1 \right) \left( 8 z - 2 \left( 9 z - 5 \right) - 4 \right)}{\Delta_{dP_2}(z)}, \]

\[ Y^{(0)}_{z \bar{z} z}(z) = \frac{4 \left( z^3 + z + 1 \right) z^2 + z \left( z + 1 \right) \left( 9 z^2 - 14 z + 4 \right) z^2 - 2 \left( 7 z^2 - 10 z + 2 \right) z^2}{\Delta_{dP_2}(z)}, \]

\[ Y^{(0)}_{z \bar{z} z}(z) = \frac{-16 \left( z^3 - z \right) z^2 - 8 \left( 4 z^3 - 4 z^2 - 2 z - 2 \right) z^2 + 2 z^2 + 2 z^2 + z \left( z - 1 \right) \left( z - 1 \right)}{\Delta_{dP_2}(z)}, \]

\[ + (45 z^3 - 52 z^2 + 9) z^2 + (20 z^3 - 38 z^2 + 20) z^2 - 8 z^2 + 8 z + 1 \left( z + 1 \right) \left( z - 1 \right) \left( z - 1 \right) \left( z - 1 \right) / \Delta_{dP_2}(z), \]
\[ Y^{(0)}_{2s2z3}(z) = (16(z_3^2 - z_3^3) z_1^4 + 8(2z_2^2 - 2z_2^3 - z_2^4z_2 - 2z_2^5 + z_2) z_1^2 + ((-9z_2^4 - 4z_2 + 8)z_3^2 \\
+ 2(4z_2^2 + z_2 - 4)(z_1 - 1)z_1 + (z_2 - 1)(z_3 - 1))/4\Delta_{dP2}(z), \]

and

\[
Y^{(0)}_{2s2z3}(z_1, z_2, z_3) = Y^{(0)}_{2s2z3}(z_1, z_3, z_2),
Y^{(0)}_{2s2z3}(z_1, z_2, z_3) = Y^{(0)}_{2s2z3}(z_1, z_3, z_2),
Y^{(0)}_{2s2z3}(z_1, z_2, z_3) = Y^{(0)}_{2s2z3}(z_1, z_3, z_2),
Y^{(0)}_{2s2z3}(z_1, z_2, z_3) = Y^{(0)}_{2s2z3}(z_1, z_3, z_2),
\]

where

\[
\Delta_{dP2}(z) = 16(z_2 - z_3)z_1^2 + 8((2z_2 + 2z_3 - 3)(z_2 - z_3) - (z_2 + z_3)(1 - z_2)(1 - z_3))z_1^2 \\
+ (4(9z_2z_3 - 2z_2 - 2z_3 - 2)(z_2 + z_3) + 1 - 30z_2z_3 - 27z_2^2z_3)z_1 - (1 - z_2)(1 - z_3). \]

These expressions indeed agree with the local B-model Yukawa couplings in Ref. [23] with \( x = 1/4 \) and \( y = -1/4 \).

**Appendix C. B-model Yukawa couplings of local \( A_2 \) geometry**

The exact B-model Yukawa couplings of the local \( A_2 \) geometry defined in Eq. (4.36) have the following expressions:

\[
Y_{2s2z3}(z) = 2z_1(-16z_2^2z_3^2(3z_2(2z_3(2z_3 - 1) + 4z_3 - 5) + 14) - 4) \\
+ 4z_1(32z_2(2z_3 - 2)z_2^2z_3 + 2(3z_2(32 - 9(z_2 - 3)z_2) - 8)z_1^4 + (8 - 9z_2z_3) + 11z_2 - 1) \\
+ z_2(6z_3 - 1 - 4z_3 + 1)(z_2(9z_2(2z_3z_3 - 2) - 4) + 4z_3 - 1)) /3(1 - 4z_1)^2\Delta_{dA}(z),
\]

\[
Y_{2s2z3}(z) = -2z_1(-54z_3^2z_2^3(4z_2(3z_2 + 1) - 3z_2 + 3) + z_2^3(9z_2z_2(4z_2(2z_2 + 5) - 2z_2 - 1) + 16) - 16) \\
- 2z_2z_3(21(4z_1 - 1)z_2 + 34) + 4z_2((4z_1 - 1)z_2 + 8z_3 - 1) /3(1 - 4z_1)\Delta_{dA}(z),
\]

\[
Y_{2s2z3}(z) = -2z_1(-54z_2^2z_3^3(3z_1(60z_2 + 8) - 15z_2 + 6) + z_2^3(9z_2z_2z_2(8z_2 + 11) - 8z_2 - 7) + 16) - 16) \\
- 4z_2z_3(33(4z_1 - 1)z_2 + 17) + 8z_2((8z_2 - 2)z_2 + 1) + 8z_3 - 1) /3(1 - 4z_1)\Delta_{dA}(z),
\]

\[
Y_{2s2z3}(z) = 2(-54z_2^2z_3^3(4z_2(3z_2 + 2) - 3z_2 + 2) + z_2^3(3z_2z_2z_2(2z_2 + 5) - 2z_2 - 1) + 32) - 32) \\
- 2z_2z_3(21(4z_1 - 1)z_2 + 22) + 4z_2((4z_1 - 1)z_2 + 8z_3 - 1) /3\Delta_{dA}(z),
\]

\[
Y_{2s2z3}(z) = (-27z_2^3z_3^3(4z_1(15z_2 + 4) - 15z_2 + 4) + z_2^3(3z_2z_2z_2(4z_2(2z_2 + 11) - 4z_2 - 9) + 44) - 16) \\
+ z_2(132(2z_2 - 2)z_2 - 5) + 8z_2((8z_2 - 2)z_2 + 1) + 8z_3 - 1) /3\Delta_{dA}(z),
\]

\[
Y_{2s2z3}(z) = 2(-54(4z_2 - 1)z_2^2z_3^2(3z_2 - 2) - z_2^3(4z_2(3z_2(5 - 5)z_2 + 32) - 16) \\
+ 3z_2(9z_2(z_3 - 3) + 32) + 16) + z_2(42z_2^2 - 44z_2 + 8) + z_2(5 - 4z_2 - 1)) /3\Delta_{dA}(z),
\]

\[
Y_{2s2z3}(z) = 4(4z_2 - 1)z_2^2(3z_2(3z_2 - 2)(12z_2 - 7) - 4) + z_2(4z_2(12z_2 - 5) + 2) - (1 - 4z_2) /3\Delta_{dA}(z),
\]

\[
Y_{2s2z3}(z) = 2(-4z_2 - 1)z_2^2(3z_2(9z_2(4z_2 - 11) + 44) - 16) \\
+ 2z_2(60z_2 - 31) + 4) - (1 - 4z_2) /3\Delta_{dA}(z),
\]

\[
Y_{2s2z3}(z) = 4(-4z_2 - 1)z_2^2(3z_2(9z_2 - 3z_3 - 5)z_2 + 32) - 16) \\
+ z_2(48z_2^2 - 44z_2 + 8) + 3z_2(5 - 4z_2 - 1) /3\Delta_{dA}(z),
\]

\[
Y_{2s2z3}(z) = (-4z_2 - 1)z_2^2(3z_2(9z_2 - 8z_2 + 80) - 64) \\
+ 4z_2(33z_2 - 52) + 16) - 8z_2 + 22z_2 - 8) /3\Delta_{dA}(z),
\]
where
\[ \Delta_{A_2}(z) = 729(1 - 4z_1)^3z_2^2z_3^4 + 108(4z_1 - 1)z_1^3(9z_2 - 2)z_3^2 + 2z_1^3(4z_1(9z_2(3z_3)(4z_3 - 7) + 8) - 8) \]
\[ + 9z_3(3z_3(4z_3 + 5) - 8) + 8) - 4z_1(4z_3 - 1)(9z_3 - 2) + (1 - 4z_2)^2. \]

**Appendix C.1. Derivation of \( \Delta_{A_2}(z) \)**

The factor \( \Delta_{A_2}(z) \) appearing in the above expressions can be regarded as a discriminant that describes the degenerate points of the mirror curve \[2,57\]. To find a mirror curve explicitly, let us first consider the mirror local Calabi–Yau threefold of the local \( A_2 \) geometry defined by

\[ X^\vee = \left( \omega_+, \omega_-, x_1, x_2, x_3, x_4, x_5, x_6 \right) \in \mathbb{C}^2 \times \left( \mathbb{C}^* \right)^6 \quad \text{subject to} \quad \omega_+ \omega_- = \sum_{i=1}^{6} x_i, \quad x_1x_2 = z_1x_3^2 \]
\[ x_3x_5 = z_2x_4^2, \quad x_4x_6 = z_3x_5^2 \quad \text{(C.1)} \]

where each \( x_i, i = 1, \ldots, 6 \) is the mirror coordinate corresponding to \( X_i \) in Fig. 5. Let us take a local coordinate \( x_1 = y, x_3 = 0, x_4 = x \), which corresponds to the mirror of a brane wrapping on a Lagrangian submanifold located at the external leg of the local atlas around \( X_1 = X_3 = X_4 = 0 \). Then \( x \) becomes the open string moduli parameter in the B-model. From Eq. (C.1), we obtain a mirror curve in \( X^\vee \) at \( \omega_+ = 0 \) or \( \omega_+ = 0 \) as

\[ y^2 + \left( 1 + x + z_2x_2^2 + z_2^2x_3x_3^3 \right) y + z_1 = 0. \quad \text{(C.2)} \]

The branch points can be obtained from

\[ \left( 1 + x + z_2x_2^2 + z_2^2x_3x_3^3 \right)^2 - 4z_1 = 0, \]

and the discriminant for \( x \) of the above equation given by

\[ 2^{12}z_1^2z_2^2z_3^6 \Delta_{A_2}(z) \]

describes the degenerate points of the mirror curve (C.2).

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