The Quark–Hadron Transition in the Early Universe

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Abstract. We use recent lattice QCD outputs to work out the expansion law of the Universe during the cosmological quark–hadron transition. To do so, a suitable technique to exploit both pressure and energy density data, with the related error bars, is introduced. We also implement suitable techniques to relate the \(T\) range where lattice outputs are available with lower and higher \(T\)'s, for which we test suitable expressions. We finally compare the cosmological behavior found using lattice data with the one obtainable in the case the transition were first order, although not so far from the crossover transition we studied. Differences are small to be tested with cosmological data, but the coming of the era of precision cosmology might open a channel to inspect the QCD transition through them.

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1 Introduction

In the very early Universe, strongly interacting matter (SIM) was a quark–hadron plasma. Then cosmic expansion lowered the temperature \(T\) to values \(O(100–200\,\text{MeV})\) and, for a short period, SIM turned into a hadron gas made of \(\pi\) and mesonic resonances, plus rare nucleons carrying the cosmic baryon number \(B\). We shall refer to this process as Cosmic Quark–Hadron transition (CQHt). Soon after, when \(T\) shifts below \(m_\pi\) (pion mass), the only residual SIM shall be made of the baryons needed to carry \(B\).

CQHt might leave an imprint on today’s cosmic observables. This idea was widely explored in the mid-Eighties, when the option that CQHt had been a first order phase transition was considered likely. In that time lattice QCD outputs were already tentatively used to work out the cosmic expansion rate and the time dependence of the scale factor and several authors also obtained analytical determination of the time dependence of the scale factor, \(a(t)\), during a first order phase transition, occurring close to the critical temperature \(T_c\) [1]. One of the main ingredients to find a right solution of these problems was taking into account the presence of the lepton–photon component in the CQHt epoch.

Lattice outputs were considered to this very aim a few times and even quite recently (see, e.g., [2]). Often, a technique to exploit them was re-invented and a few errors occurred recursively, e.g. the neglect of the lepton–photon component.

A topic of this work will then be a general outline of the technique and the problems to exploit lattice data to cosmological aims.

A critical issue, concerning CQHt, was however risen by Witten [3] in the mid-Eighties, and this made then CQHt a hot subject. He showed that, in the case of a first order transition, \(B\) would tend to remain in the plasma. This led him to suggest that quark nuggets could still exist in today’s Universe, constituting the cosmic Cold Dark Matter (CDM) component. The apparent similarity between the present densities of the baryon and CDM components could then find a satisfactory explanation. This idea has been widely debated and the interest for it never completely faded (see, e.g., [4]).

Other researchers, although keeping to the idea that \(B\) kept in the plasma, until it existed, suggested that the transition would however reach its completion, leading however to an inhomogeneous \(B\) distribution. Neutron diffusion would then rapidly smooth out their inhomogeneities. Proton diffusion, instead, requires a coherent electron flow and the electron Thomson cross section is so large that proton inhomogeneities may last down to a temperature \(\sim 50\,\text{keV}\), being therefore able to affect Big–Bang Nucleosynthesis (BBN) [5]. This idea triggered a large deal of works [6], of whom we provide a (partial) list up to 1990.

As a matter of fact, a first order phase transition occurs in lattice QCD without dynamical quarks [7]. For dynamical quarks with vanishing mass, the transition is also first order. These outputs were given a great importance until physical mass quarks could not be treated on the lattice. Then, if \(u\) and \(d\) quarks are light and the strange quark is massive, according to phenomenological results, lattice outputs unequivocously yield no confinement phase transition, but a smooth crossover.
Some doubts can still exist on the nature of the transition leading to the breaking of chiral symmetry. Recent speculation related this transitions to the origin of cosmic magnetic fields [8], although a problem exists on their scale, which keeps related to the horizon scale at $t \simeq 10^{-5}\text{sec}$.

If CQHt is not a first order phase transition, its relevance for cosmology is no longer so great. It must also be outlined that the recent analysis of CMB, BAO and SNIa data [9], within the frame of a $\Lambda$CDM cosmology, has determined a reduced baryon density parameter $\omega_b = 0.0227 \pm 0.0006$, in fair agreement with what is predicted by homogeneous BBN, in accordance with the observational values of $^2H/H$ and $^4He/H$ ratios.

Some discrepancies however exist with observational $^3He$ and $^7Li$ abundances. They are likely to arise from marginal errors either in observations or in specific nuclear reaction rates. For instance, slight shifts in the rates of the $^7Be(d,p)^2He$ or $^4He(\alpha,\gamma)^7Be$ reactions [10] could put a remedy to the $^7Li$ discrepancy.

The possibility that the observational values of $^2H/H$ and $^4He/H$ ratios are an indication of inhomogeneous BBN, or derive from other peculiarities as particle decays during or after BBN, is still open, but is really second choice [11].

On the contrary, in the late Eighties the situation was not so strongly constrained: open cosmologies were then considered a valid option, while $\Lambda$CDM was mostly considered just as a counter–example in $n$–body simulations. The underlying idea, in the study of inhomogeneous BBN, was that is could relax the constraints on $\omega_b$, making it compatible with a present overall density parameter $\Omega_0 \sim 0.2$, so avoiding the need of non–baryonic DM.

The present situation is however such that, should fresh data reopen the option of first order CQHt, we should then take care that its features do not produce too strong proton inhomogeneities, radically perturbing the theory–observation agreement, as far as BBN is concerned. Only in this sense, perhaps, BBN remains a constraint to CQHt.

Let us then also remind that, although lattice QCD indicates that CQHt was crossover, this is still the outcome of a theoretical elaboration. No experimental confirm can be soon available and the most direct pattern to explore this physics is likely to be just cosmology. With the arrival of the era of precision cosmology, however, this might become a realistic option.

Besides of trying to provide a general setup on the use of lattice outputs in cosmology, this paper therefore aims at evaluating the impact that different CQHt patterns could have on cosmic observables. We shall then also consider the possibility that CQHt is a first order phase transition, keeping however quite close to recent lattice outputs, and will make a comparison between the cosmic evolutions in the two cases.

A further point we need to explore is the connection between the very high $T$ regime, where asymptotic freedom is approached, and the $T$–range where lattice outputs are available. We shall show that an analytical expression, which can be also seen as a generalization of the historical Bag Model [12], is able to fit high–$T$ lattice outputs, connecting them with the temperature range where the plasma behaves as an ideal gas. The procedure to provide this expression and to fit it to data is one of the outputs of this work.

Finally, we shall also need an expression to describe SIM in the very low $T$ regime, when it has turned into a hadron gas. Although SIM is then just a minor component of the Universe, its impact on the cosmic expansion is not yet fully negligible. In this paper we shall describe SIM, in the form of a hadron gas, by using a state equation inspired to the Hagedorn [13] model, whose parameters will be selected in order to connect it with the state equation resulting from lattice outputs.

Back in the Eighties, a large deal of work concerned also bubble nucleation, expansion, and coalescence, during CQHt (for a review see, e.g., [14]). In the absence of a phase transition they are hardly relevant.

Recent outputs on SIM bulk viscosity during ion collisions [15], however, could lead to conjecture some more intricate situation. Viscosity is an indication that particle reactions proceed too slowly to allow the system to settle in the equilibrium configurations of an ideal fluid. A tentative data interpretation could then be that reactions putting together 3 quarks have a low rate and, perhaps, require an intermediate di–quark state. In this case their rate would be controlled by the average di–quark concentration in high–$B$ quark gluon plasma.

Should this interpretation of bulk viscosity be correct, we expect viscosity to be negligible in the low–$B$ cosmological context.

One might however wonder whether the low rate for $3q \rightarrow B$ reactions could however mean that $B$ keeps being carried by quarks, almost until CQHt is (almost) complete. This would mean that the residual plasma component would gradually become richer in $B$, complicating the last stages of CQHt, when bulk viscosity could reappear and even the option of a first order transition might reopen.

In this work we shall not deepen these speculations, also because no significant experimental or lattice data can be exploited in such analysis. Accordingly, systems will be assumed to evolve through thermodynamical equilibrium states and no viscosity or heat conduction will be taken into account.

The plan of the paper is as follows: In the next Section we shall briefly outline some relations obtained from thermodynamical considerations in a cosmological context. Section 3 will just be devoted to show the lattice data we shall be using. In Section 4 we shall fit such data with a suitable analytical expression, able to reconnect them with the asymptotic freedom regime. In Section 5 instead, we shall focus on the low–$T$ region and discuss how to apply a Hagedorn–like expression to work out energy density and pressure. Then, in Section 6, the use of interpolating expressions, in the intermediate range, will be discussed. Cosmological considerations begin from Section 7, where lattice outputs are actually used to work out the cosmic expansion regime. For the sake of comparison,
In Section 8 we also deal with a first order transition, approaching lattice data. All that will enable us to plot the connections between a (the scale factor), t (time) and T (temperature); suitable combinations of such parameters and their evolution will be shown in Section 9. In Section 10 we shall gine some examples on the impact that the above detailed relations could have on cosmological observables, in the era of precision cosmology. Finally, Section 11 is devoted to drawing our conclusions.

2 Thermodynamics and cosmology

If we let apart the $3q \to B$ reaction, there can be little doubts that, around CQHt, relaxation times of particle interactions, including neutrinos, are well below the cosmic time. Similarly, the value of the baryon/entropy ratio, receiving no substantial contribution after CQHt, allows to neglect the chemical potential associated to $B$.

Within such context, starting from the thermodynamical identity $dU = -p dV + T dS$ and setting $\epsilon = \partial U/\partial V$, $\sigma = \partial S/\partial V$, we have that

$$\sigma T = \epsilon + p. \quad (1)$$

Let us then consider the free-energy $F = U - TS$, such that $dF = -p dV - S dT$. Comparing its second derivatives in respect to $V$ and $T$ when taken in different order, one has that

$$\sigma = \partial p/\partial T \quad (2)$$

so that the dependence on $T$ of

$$\epsilon = -p + T \partial p/\partial T \quad (3)$$

is fully determined, once $p(T)$ is known. If we then define

$$\Phi = p/T^4, \quad E = \epsilon/T^4, \quad (4)$$

the above equation yields

$$E(T) = 3\Phi(T) + T \frac{\partial \Phi}{\partial T}. \quad (5)$$

If $E$ and $\Phi$ depend only on $T$, this equation can be seen also as a differential equation with $\Phi(T)$ unknown. It is then equivalent to the relation $T^2 E = d(T^3 \phi)/dT$ and can be soon integrated yielding

$$\Phi(T) = \frac{T^3}{T^3} \Phi(T_r) + \frac{1}{T^3} \int_{T_r}^{T} d\tau \tau^2 E(\tau), \quad (6)$$

$T_r$ being a reference temperature. During CQHt, when $\Phi$ and $E$ actually depend just on $T$, eqs. (5) and (6) can be used to work out $\epsilon$ from $p$ and viceversa.

Should the transition be first order, however, we have two phases. The quark–gluon plasma would have a pressure $p_{qg}(T)$, while the hadron gas has a pressure $p_h(T)$ and the critical temperature $T_c$ is when $p_{qg}(T_c) = p_h(T_c)$. It must however also be $p_{qg}'(T_c) > p_h'(T_c)$ (here ’ indicates differentiation in respect to $T$) and therefore, according to eq. (3), $\epsilon_{qg}(T_c) > \epsilon_h(T_c)$.

When $T_c$ is reached, therefore, hadron bubbles must nucleate inside the plasma. If this requires no substantial supercooling, the Universe then undergoes an expansion at constant temperature, while the fraction of each horizon occupied by the plasma decreases and the fraction occupied by hadrons increases. Over large scales, the cosmic energy density then reads

$$\epsilon = \epsilon_{qg}(1 - \lambda) + \epsilon_h \lambda + 3\Phi(T_c)T_c^4. \quad (7)$$

Here $\lambda$ is the fraction of space occupied by the hadron gas, while the pressure of the cosmic lepton–photon component, obtainable from

$$\Phi(T_c) = \frac{\pi^2}{90} \left[ N_{bos} + \frac{7}{8} N_{fer} \right] \simeq 1.5627, \quad (8)$$

depends on the numbers $N_{bos, fer}$ of the boson, fermion relativistic spin states in the thermal soup. The above value arises from assuming 14.25 effective spin degrees of freedom, i.e. that $m$ particles are still fully relativistic. In Appendix A we show that, in the $T$ range considered, this is acceptable. When aiming at precise quantitative outputs, however, the fact that we are in a region where $m_\mu \sim T$ should not be disregarded.

If supercooling is quite small or infinitesimal, a first order phase transition implies quite a small or infinitesimal entropy input. In fact, let $S = a^3 \sigma = (a^3/T)(\epsilon + p)$ be the comoving entropy. If $T = T_c - \delta T$ (with infinitesimal $\delta T$), the average cosmic pressure will be

$$p = p_{qg}(1 - \lambda) + p_h \lambda + \Phi(T_c)T_c^4 \simeq$$

$$\simeq p(T_c) - [\epsilon_{qg}(T_c)(1 - \lambda) + \epsilon_h(T_c)\lambda] \frac{\delta T}{T_c}. \quad (9)$$

so that, if $\delta T$ does not depend on $\lambda$,

$$\frac{\partial p}{\partial \lambda} = [\epsilon_{qg}(T_c) - \epsilon_h(T_c)] \frac{\delta T}{T_c}. \quad (10)$$

Let us consider then the cosmological context, where the space–time metric reads

$$ds^2 = c^2 dt^2 - a^2(t)d\ell^2, \quad (11)$$

d$\ell$ being the infinitesimal comoving length element. Owing to the Friedman equation

$$d[a^3(\epsilon + p)] = a^3 dp \quad (12)$$

it is then

$$\dot{S} = (a^3/T)\dot{p} - S(\dot{T}/T) =$$

$$= (a^3/T) (\partial p/\partial \lambda) \dot{\lambda} + a^3(\partial p/\partial T)(\dot{T}/T) - S(\dot{T}/T) \quad (14)$$

(dots indicate here time differentiation) and the last two term at the r.h.s. cancel, owing to eq. (2), so that

$$\dot{S} = a^3 \epsilon_{qg}(T_c) - \epsilon_h(T_c) \frac{\delta T}{T_c} \simeq [S_{qg}(T_c) - S_h(T_c)] \frac{\delta T}{T_c} \dot{\lambda}. \quad (15)$$
fully vanishes as $\delta T/T_c \to 0$. This equation shows that only a tiny fraction of the entropy difference between the two phases may enrich cosmic entropy. The rest of the difference turns into lepton–photon entropy. This is why neglecting the latter component is strongly misleading.

When dealing with a supposed first order transition, we shall then consider a negligible supercooling and $S$ to be constant. Any other choice would require further assumptions.

3 Lattice QCD data

Lattice QCD deals with the non-perturbative regime for the QCD equation of state. Here we are referring to recent results obtained from [16]. They used a $(N_t = 6) \times 32^4$ size lattice to calculate the pressure $p$ and the trace anomaly $\epsilon - 3p$ for 20 values $T_i$ of temperature. The computation is performed using physical values for the masses of the up, down and strange quarks. No error is given for $p_i = p(T_i)$ estimates. The trace anomaly, instead, has a significant uncertainty which therefore affects their $\epsilon_i = \epsilon(T_i)$ estimates.

These estimates are reported in the Table herebelow ($T$’s in units of 100 MeV, the other quantities in units of $(100 \text{ MeV})^4$).

From this table the estimates of the coefficients $E(T)$ and $\Phi(T)$ can be deduced; they will be shown in a number of Figures hereafter.

Table I

| $T$  | $\epsilon$  | $p$  | $\Delta\epsilon$ |
|------|-------------|------|------------------|
| 7.19 | 39300       | 12600| 353              |
| 5.56 | 13800       | 4300 | 164.5            |
| 4.68 | 6800        | 2070 | 95.2             |
| 4.19 | 4350        | 1300 | 65.3             |
| 3.67 | 2520        | 714  | 48.02            |
| 3.24 | 1520        | 398  | 37.92            |
| 2.82 | 859         | 196  | 10.54            |
| 2.61 | 635         | 127  | 10.82            |
| 2.40 | 449         | 74.1 | 6.63             |
| 2.27 | 341         | 47.3 | 7.08             |
| 2.13 | 238         | 26.6 | 5.49             |
| 2.06 | 196         | 18.6 | 6.3              |
| 2.03 | 170         | 15.3 | 5.66             |
| 1.96 | 113         | 10.1 | 4.16             |
| 1.86 | 58.7        | 5.35 | 2.81             |
| 1.80 | 38.7        | 3.57 | 1.85             |
| 1.74 | 21.4        | 2.45 | 2.15             |
| 1.59 | 8.88        | 0.99 | 1.2              |
| 1.45 | 6.24        | 0.35 | 1.15             |
| 1.40 | 2.54        | 0.19 | 0.9              |

4 The high-$T$ regime

At high $T$, we expect $E = \epsilon/T^4$ to approach asymptotically a value

$$\tilde{E} = 3\tilde{\Phi} = (\pi^2/30)[16 + (7/2)N_{col}N_{fla}]$$  \hspace{1cm} (16)

$N_{col}$ and $N_{fla}$ being the number of colors and flavors. The above estimates of $\epsilon$ and $p$ show that, at $T = 719$ MeV, it is $E \approx 0.94 \times \tilde{E} = 15.627$, with $N_{col} = 3$ and $N_{fla} = 3$. In order to study the CQHt, however, we need an expression telling us how $E(T)$ and $\Phi(T)$ approach $\tilde{E}$ and $\tilde{\Phi}$ above 719 MeV.

The expression we shall propose, however, extends its validity down to $T \sim 200$ MeV, fitting lattice outputs all through this $T$ interval, being then suitable to deal also with a wide $T$-range where lattice outputs are available, and not only because it provides an easily treatable fit to them.

Let us then assume

$$\Phi = \tilde{\Phi} \left( 1 - \left[ \frac{T}{T_0} (1 + \delta_0) \right]^{s_0} \right)$$  \hspace{1cm} (17)

with

$$\delta_0 = \left[ \frac{T_1}{T} (1 + \delta_1) \right]^{s_1}, \quad \delta_1 = \left[ \frac{T_2}{T} (1 + \delta_2) \right]^{s_2}, \quad \ldots$$  \hspace{1cm} (18)

an expression which, a priori, depends on the successions $\{T_i\}$, $\{s_i\}$.

By using eq. (15), it is then possible to work out also the recursive expressions:

$$E = 3\Phi + s_0(\tilde{\Phi} - \Phi) \left( 1 - \frac{T\delta_0}{1 + \delta_0} \right)$$  \hspace{1cm} (19)

with

$$T\delta_r = -s_{r+1} \left( 1 - \frac{T\delta_{r+1}}{1 + \delta_{r+1}} \right) \delta_r \quad (r = 0, 1, \ldots)$$  \hspace{1cm} (20)

These expressions can be truncated at any order $\tilde{n}$, by assuming $\delta_{\tilde{n}} \equiv 0$.

If we take $\delta_0 = 0$ and $s_0 = 4$ we recover the MIT Bag–model expression with a bag constant $B = \tilde{\Phi}T_0^3$. Such expression is known not to fit lattice outputs; however, the values $s_0 \sim 1.48$ and $T_0 = 145$ MeV, are the best–fit to the highest 6 $\Phi$ points. The quality of this fit is shown in Figure 1.

A much better fit is however obtainable if we keep $\delta_0 \neq 0$ and set $\delta_1 = 0$. In Figure 1 we show some results obtained by freely selecting $s_0$ and working then out $s_1$, $T_0$ and $T_1$ from the highest 6 $\Phi$ points, as above. The detailed procedure followed in is described in Appendix B.

Clearly, the presence of the $\delta_1$ term has the result of increasing the bending in the middle–$T$ area: the expression (17) sets a $T$-point where the pressure turns negative. We do not expect such point to be reached. The hadron gas should replace the quark–gluon plasma before
Adding a $\delta_0$ ($>0$) correction means that, while $T$ decreases (and the quark–quark distance increases), the $T$–point gradually shifts to a higher value.

One could argue that subtracting from the ideal gas coefficient the power law term $(T_o/T)^{s_0}$ is a reasonable way to model the binding action due to gluon exchange, when the cosmic expansion tries to separate quarks beyond their confinement distance. The swiching on of the second power law correction $\delta_o = (T_1/T)^{s_1}$ could then be reminiscent of the expectation that, as the inter–quark distance increases furthermore, the exchanged confining gluons, being colored, may become source of further gluon emission, so causing a further strengthening of their binding action.

From Figure 1 we also see that reasonable $s_0$ values are between 1 and 1.1. Already at $s_0 = 1.1$ the $E(T)$ dependence exhibits a rebounce, which becomes more pronounced and shifts to higher $T$’s for greater $s_0$ values; $s_0 = 1.1$ appears then as an optimal choice to meet nine $E(T)$ error bars (for $T_i$ with $i = 12, \ldots, 20$), with no rebounce above $T_{12}$. If one takes a smaller $s_0$ (e.g., $s_0 = 1$), it is possible to meet also the $T_{11}$ error bar, at the expenses of shifting below $T_{12}$ and $T_{13}$. (Figure 1 evidently indicates that the fit with $\Phi(T_i)$ values is fair, even down to $i \sim 6–7$, well below 12. In the inner box we however magnify the low–$T$ area and show that, although the absolute values of $\Phi(T)$ are really well approached, the expression (17) misses the shift from positive to negative second derivative, essential to avoid the rebounce of $E(T)$ towards low $T$’s.)

As a matter of fact, a reliable analytical expression, meeting half lattice points and connecting the range of lattice data with asymptotic freedom, is to be implemented by a numerical fit to lattice data at lower $T$’s. It seems clear that the expressions (17) and (18), with $\delta_1 = 0$ and $s_0 \approx 1.1$ are suitable to this aim.

This choice is somehow corroborated by some curious numerical regularities. When $s_0 = 1.1$ is chosen, the data fit returns $s_1 \approx 2$, $T_o = 76.56\,\text{MeV}$ and $T_1 = 229.7\,\text{MeV}$; so that the $T_1/T_o$ ratio differs from 3 by $\sim 2:10^4$. Such regularities hold in a small interval around 1.1 (typically between $\sim 1.08$ and 1.12).

The expressions with $s_0 = 0.8$ will also be used, when aiming to mimic a first–order phase transition. In this case, the fitting procedure yields: $s_1 = 2.00$, $T_0 = 26.1\,\text{MeV}$ and $T_1 \sim 447\,\text{MeV}$.

We also tried to add an extra term to eqs. (17)–(18), allowing for $\delta_1 \neq 0$. The fitting procedure becomes then even more intricate. A fit with lattice data is obtained with the regular values $T_0 = 66\,\text{MeV}$, $T_1 = 2.15 T_0$, $T_2 = 3.3 T_0$, while $s_0 = 1$, $s_1 = 1.5$, $s_2 = 2$. This fit, however, is no improvement in respect to the one with $\delta_1 = 0$: it actually meets the $E(T)$ error bars more at their centers, but has a rebounce at greater $T$, making it useless already for $i = 12$; we avoid further graphics complications and we refrain from showing it in Figure 1.

5 The low–$T$ regime

Expressions of $E(T)$ and $\Phi(T)$, for $T \sim 100–200\,\text{MeV}$ may hardly keep any reference to an ideal relativistic gas. As a matter of fact, the lightest hadrons are pions and, if we tentatively assume for them relativistic ideal gas expressions when $T \sim m_{\pi}$, we work out that their mutual distance $D$, in average, is given by

$$D^{-3} \simeq \frac{\zeta(3)}{\pi^2} 3T^3 \simeq 0.37 T^3$$

(\(\zeta(3) \approx 1.202\)); this yields $D \sim 1.4/m_{\pi}$, a distance comparable with hadron hard core. Accordingly, hadron proper volumes cannot be neglected, let alone hadron–hadron interactions. Furthermore, as $T$ increases, we can hardly neglect heavier hadronic resonances.

There have been several approaches to the thermodynamics of the hadronic gas. In any case, however, because of their volume and of the temperature, hadron will never yield a major cosmic component. Leptons and photons will then have a density coefficient $E_{\text{el}} \sim (\pi^2/30) 15$, while

![Figure 1](image_url)
hadrons $E_s \ll (\pi^2/30)^3$. Accordingly, for our aims, taking the expressions inspired to the Hagedorn model will be fully reasonable. Such expressions contain 3 constant, that we shall tune to the low–$T$ lattice outputs.

The pressure of a Hagedorn gas with vanishing chemical potential reads then

$$p(T) = \frac{1}{6\pi^2} \int_{m_*}^{T_H} dm \, w(m) \int_0^\infty dk \, \frac{k^4 E^{-1}(k, m)}{\exp[E(k, m)/T] - 1}$$

(22)

with $E(k, m) = \sqrt{k^2 + m^2}$ and

$$w(m) = \left(\frac{\bar{m}}{m}\right) \gamma \exp\left(\frac{m}{T_H}\right).$$

(23)

Selecting $\gamma = 5/2$ allows a closed expression, using just the exponential integral function

$$E_i(x) = \int_x^\infty dt \, t e^{-t},$$

which reads

$$p = \Phi(T)T^4 \quad \text{with}$$

$$\Phi(T) = \alpha_0 \left(\frac{T}{T_H}\right)^{5/2} E_i \left[\frac{m_0(T^{-1} - T_H^{-1})}{T_H/T - 1}\right]$$

(24)

and depends on the Hagedorn temperature $T_H \simeq 200 \text{ MeV}$ and on the constants $\alpha_0$ and $m_0$, related to $\bar{m}$.

From this expression of $\Phi(T)$ and eq. (5) one easily works out that

$$E(T) = \frac{3}{2} \Phi(T) + \alpha_0 \left(\frac{T_H}{T}\right)^{5/2} \frac{m_0(T^{-1} - T_H^{-1})}{T_H/T - 1};$$

(25)

Figure 2 then shows the $\Phi(T)$ and $E(T)$ behaviors at low–$T$, for 5 reasonable choices of the $\alpha_0$ and $m_0$ constants. Although all parameter choices meet the general trend and the order of magnitude of lattice outputs, there is a specific feature that none of them approaches, the second $E(T)$ point, which is however high.

Accordingly, we shall use such expressions, in association with lattice outputs, just for $T < T_1$. Almost arbitrarily we chose then $\alpha_0 = 0.28$, $m_0 = 600 \text{ MeV}$. The same values will also be used when trying to mimic a first order transition, up to the temperature where the expression (24) of $\Phi(T)$ crosses the high–$T$ expression (17).

### 6 Fitting lattice outputs

In order to use lattice outputs in Friedmann equations, one needs interpolating them. In principle, mathematical and numerical libraries contain excellent interpolating routines, namely the `splint` and `spline` programs in *Numerical Recipes* [18], making use of a cubic polynomial.

However, if one uses such recipe to interpolate $\Phi(T)$ (let us also outline that interpolating $p(T)$ yields essentially identical results, as a counterproof of the outstanding efficiency of available routines), the critical difficulty

![Fig. 2. Lattice outputs compared with the low–$T$ Hagedorn–like expressions. The $E(T)$ points are reported with their 1–σ errors. The $\Phi(T)$ points are reported without errors (not provided in the lattice outputs used). Although the expressions meet the trend and the order of magnitude of lattice outputs, the high level of the second $E(T)$ point, if real, is clearly missed.](image)

![Fig. 3. Lattice outputs and their interpolation, as described in the text. The cyan solid lines represent the behavior of low–$T$ and high–$T$ extrapolating expressions, out of their validity range.](image)
arises when $E(T)$ is sought. The passage from $\Phi$ to $E$ implies differentiation, and this blows up small irregularities; the resulting $E(T)$ comprises several unmotivated oscillations, of clearly numerical origin.

In particular, there are fair reason to believe that $E(T)$ should be always increasing, as also lattice outputs indicate, towards its asymptotic freedom value. On the contrary, interpolation yields $T$ intervals, between contiguous $T_i$, where $E(T)$ decreases. This effect is particularly evident for $i \simeq 1$, 2, 3, ... and at $i > 11$–12.

We eliminate this problem with 3 actions: (i) At high $T$ we simply use the analytical expressions (17)–(18), starting from $i = 12$. (ii) We interpolate the central points of $E(T)$, instead of $\Phi(T)$, and make use of the integral expression (6) to obtain $\Phi(T)$. This will be done using $T_i$ as $T_*$, and taking the values of $E(T_i)$ and $\Phi(T_i)$ given by Hagedorn–like expressions with our selected parameters. (iii) We select the $E(T_i)$ values at $i = 2$ and 3, at hand within a 2–$\sigma$ error bar, at the minimal distance from the central points able to prevent the spline to yield any $E(T)$ decrease. Clearly other options are possible, but they appear more intricate.

The overall behaviors of $E(T)$ and $\Phi(T)$ is then shown in Figure 3 where the critical low–$T$ region is also magnified.

7 Use of lattice outputs in Friedmann equations

The $T$ dependence described in Figure 4 is then used to integrate the Friedmann equations.

It is then convenient to consider first entropy conservation, yielding that

$$E(T) + \Phi(T) + 4 \Phi_{\gamma}(a T)^3 = 4 (\Phi + \Phi_{\gamma})(a T)^3.$$ (26)

Here $a_i$, $T_i$ are scale factor and temperature at the “initial” conditions, where asymptotic freedom holds. In the computations here we took $T_i = 80$ MeV, so to avoid interferences with the electroweak transition. From eq. (26) it is easy to work out the $T(a)$ dependence and the deviations of the $a T$ product from constant.

The dynamical Friedmann equation then yields

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 \left[ E[T(a)] + 3 \Phi_{\gamma} \right] \left[ \frac{T(a)}{T_i} \right]^{-4};$$ (27)

by taking into account that the initial Hubble parameter $H_i \simeq 1/2t_i$ (in the radiative expansion regime), it is then

$$2 \int_{a_i}^a \frac{da}{a} \left( \frac{\dot{a} + \Phi + \Phi_{\gamma}}{E[T(a)]} \right)^{1/2} \left[ \frac{T_i}{T(a)} \right]^{-2} = t - t_i.$$ (28)

In this way one obtains $a(t)$ and thence $T(t)$.

Using this technique we work out the behaviors shown in Figures 5–8. Before discussing them, let us however outline how we proceed to build analogous curves holding in the case of a phase transition.

8 Strongly interacting matter in a first–order phase transition

In Figure 4 we show the $\Phi(T)$ curves we shall use to mimic a phase transition: at high–$T$ we shall assume that $\Phi(T)$ is given by an expression (17) with $l_1 = 0$ and $s_0 = 0.8$; at low–$T$, but also at $T > T_1$, we shall use the Hagedorn–like expression (24) with the same parameters as above. The two curves are taken above and below the temperature $T_c$ where they cross, respectively. Therefore, the overall $\Phi(T)$ behavior obtained in this way, although continuous, exhibits a (modest) shift of slope at $T_c$. Accordingly, the low– and high–$T$ curves for $E(T)$ do not intersect. In the Figure they are connected by a vertical line.

A cosmological phase transition, occurring close to the critical temperature, requires that, after $T_c$ is reached at a time $t_c$, the Universe stops cooling down. In any volume an increasing fraction of space will be then occupied by the hadron gas. When the high–$T$ plasma has completely vanished, at a time $t_i$, the cosmic temperature restarts decreasing.

During all this process $S$ is conserved, according to eq. (15).

Of course, one might also consider a transition occurring after a significant supercooling. In order to work out the cosmic expansion law, it would be necessary a supplement of information, yielding e.g., $\delta T/T$ as a function of $\lambda$.

The integration of Friedmann eqs. during a first order phase transition occurring without supercooling was first performed in [7]. Such integral was then rediscovered a few times by various authors.
Let \( p_c \) be the pressure at \( T_c \), including also the lepton–photon pressure. Also \( \epsilon \) and \( \sigma \) will include the lepton–photon contribution, all through this Section. They will be labeled with \( h \) \((\ell)\), to refer to the time when the transition begins (ends) and the same labels will be used for time and scale factor. Let us also remind that \( \lambda \) is the portion of space occupied by the low–\( T \) phase.

From the definition of comoving entropy, we have that

\[
\epsilon = T_c S/a^3 - p_c = \sigma_h T_c (a_h/a)^3 - p_c = \epsilon_h (1 - \lambda) + \epsilon_l \lambda \quad (29)
\]

with

\[
\sigma_h = S/a_h^3 = (\epsilon_h + p_c)/T_c.
\]

This eq. sets a link between \( a \) and \( \lambda \).

The dynamical Friedmann eq. reads then

\[
(\dot{a}/a)^2 = \left(8\pi G/3\right)[\sigma_h T_c (a_h/a)^3 - p_c] \quad (30)
\]

and, by taking

\[
u^2 = (a/a_h)^3/(\epsilon_h/p_c + 1),
\]

can be set into the form

\[
du \frac{1 - u^2}{-1/2} = (6\pi G p_c)^{1/2} dt \quad (31)
\]

whose integral reads

\[
u = \sin[(6\pi G p_c)^{1/2} t + A], \quad (32)
\]

\( A \) being determined by the conditions at \( t_h \).

In the case of the \( T \)–dependence of \( E \) and \( \Phi \) shown in Figure 4, we shall use the same equations as in the absence of a phase transition for \( t < t_h \) and \( > t_l \), and the above expressions during the transition.

### 9 Scale factor, temperature, time connections

When the cosmic temperature shifts from some thousands MeV to \( \sim 100 \text{MeV} \) and the CQHt occurs, scale factor and time increase by orders of magnitude.

In order to describe the cosmic evolution, it is then worth showing the behavior of precise combinations of \( a \), \( T \) and \( t \), starting from the initial values \( a_i \), \( T_i \) and \( t_i \).

Let us then recall first that, in a radiation dominated expansion, when the total number of spin states is constant, the product \( aT \) is also constant. Of course, during the CQHt, \( aT \) will have robust variations; entropy conservation prescribes that, at the end of the transition, \( a_f T_f = a_i T_i (g_i/g_f)^{1/3} \) with \( g_i/g_f \simeq (47.5 + 14.25)/14.25 \). Accordingly, the fact that \( aT \) shall increase by a factor \( \sim 1.6 \) can be soon predicted, and is independent from the detail of the transition.

Accordingly, in Figures 5–6 we show how \( aT \) varies, as a function of the scale factor or the temperature, from its initial to its final values, however known \( a \) \( p r i o r i \).

Let us then notice, in particular, the feature close to the transition end, when lattice outputs are used. It is due to the anomaly in \( E(T) \) at \( T_2 \) and similar features will be present in the next plots. Such features cannot be certainly predicted from conservation theorems.

In Figure 7 we then exhibit the time dependence of \( T \). The discrepancies between the two evolutions are evident,

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**Fig. 5.** Dependence of the \( aT \) product on the temperature. The black curve is derived by using lattice data; the cyan curve would hold in the case of a phase transition. Notice the vertical increase of \( aT \) while \( T \) remains constant during the phase transition.

**Fig. 6.** Dependence of the \( aT \) product on scale factor. Black and cyan colors as in the previous Figure.
during the transition. On the contrary, a slight offset between the two curves, after the transition is completed, is just noticeable.

As a matter of fact, also in this case we must recall that, during a radiative expansion when the number of spin degrees of freedom does not vary, \( tT^2 \) should be constant. This derives from the dynamical Friedmann equation, when it is written in the form

\[
\frac{(1/2)t^2}{(s/3G/90)} = \frac{(8\pi^3/90)g}{T^4},
\]

true if the relation \( a \propto t^{1/2} \) is strictly valid.

Even though we assume eq. (33) to be true before the transition, not only during the transition itself eq. (33) looses its validity, but also when \( g \) would be stabilized at a final value, we can only expect that \( a \propto (t + \delta t)^{1/2} \) with a suitable \( \delta t \) correction. This is what originates the minimal offset of the final curves in Figure 7.

In principle, therefore, the time and scale dependence of \( tT^2 \) are relevant both for the transition period and for the final settlement.

In Figure 8 we show then the explicit dependence of \( tT^2 \) on the scale factor \( a \). Here again the \( E(T_2) \) anomaly causes a clear feature. But the offset at the end of the transition, once again, is just barely noticeable.

This leads us to conclude that, even when we assume significantly different \( E(T) \) and \( \Phi(T) \) behaviors, the relations among \( T, a \) and \( t \) exhibit discrepancies mostly during the transition. We should select completely awkward – and unlikely – \( E(T) \) and \( \Phi(T) \) to cause a substantial shift in \( T(t) \) at the transition end.

We then conclude that, once the transition is completed, no significant dependence on the relations among \( T, a \) and \( t \) remains: a regime trend is recovered and, if no other quantity was altered during the transition, one can hardly expect signals from the relative values of such parameters.

### 10 Observable quantities

As an example of possible observables, let us consider non-relativistic Majorana spinors, mutually annihilating with a cross section \( \sigma_a \), such that the annihilation time \( \tau = (n\langle \sigma_a v_T \rangle^{-1} \) is \( O(10^{-6} - 10^{-5}) \) sec., occurring then across the quark–hadron transition. Here \( n \) is the particle number density, and \( v_T \) is their mutual velocity, in the temperature conditions considered.

Their comoving number density \( n_c = na^3 \) shall obey the equation

\[
\frac{dn_c}{dt} + n_c^2 \langle \sigma_a v_T \rangle/a^3 = 0
\]

and one must know the time dependences of \( \langle \sigma_a v_T \rangle \) and \( a^3 \), in order to integrate such equation. At large \( t \), such integral must yield \( n = \bar{n}(a_{dg}/a)^3, a_{dg} \) being the scale factor at a time \( t_{dg} \), the decoupling time. This means that the \( t \propto a^2 \) proportionality is normalized so that \( t/t_{dg} = (a/a_{dg})^2 \).

We shall however be cautious, before assuming that \( t_{dg} \) coincides with \( \tau \), so that \( \bar{n} = n(t_{dg}) \). For instance, assuming a purely radiative expansion during the decay
stages and $\langle a_v T \rangle = \text{const}$, one obtains that

$$n(t) = \frac{n(t_{dg})}{1 + 2(t_{dg}/\tau)(1 - (t_{dg}/t)^{1/2})} \left[ \frac{a_{dg}}{a} \right]^3.$$  \hspace{1cm} (35)

Therefore, at $t \gg t_{dg}$, it shall be $n(t) = \bar{n}(a_{dg}/a)^3$ with

$$\bar{n} = n(t_{dg})/(1 + 2t_{dg}/\tau).$$ \hspace{1cm} (36)

Assuming $\bar{n} = n(t_{dg})$ is therefore just a zero–th order approximation; on the contrary, taking $\tau = t_{dg}$, we see that the $\bar{n}/n(t_{dg})$ ratio is $\approx 0.33$. A different $T$ dependence of $\langle a_v T \rangle$, or a different $t$ dependence of $a$, would leave most of this treatment unchanged, just modifying the factor 2 in front of $(t_{dg}/\tau)$ in eq. (35). Accordingly, the detailed $t$ dependence of $a$ (or $\langle \sigma_T \rangle$) may change the residual amount of uninteracting Majorana spinors, possibly constituting CDM, by a factor $O(2)$.

The point is that cosmology is becoming a precision science and a variation by a fraction of percent, in the dependence of CDM, by a factor $O(2)$, might soon become appreciable.

If one aims then at connecting cosmological data with microphysical data, such as $\sigma_T$ and its $T$ dependence, a precise knowledge of $a(t)$ during the CQHT could be strictly required.

11 Conclusions

This paper was devoted to updating the treatment of the cosmological Quark–Hadron transition, taking into account recent lattice outputs.

Such transition had seemed to bear a major relevance, in the mid–Eighties, when most researchers seriously considered the option that it could be a first order phase transition. When lattice outputs suggested that it was a crossover, its interest faded.

With the coming of the era of precision cosmology, however, a detailed knowledge of expansion law and thermal history, when the time elapsed since the Big–Bang was $O(10^{-6} - 10^{-5} \text{ sec})$, could become vital to interconnect cosmological observables and microphysical parameters.

In this work we have seen that the most recent lattice data are still hard to be used to work out expansion law and thermal history. This is due, first of all, to the uncertainty on the $E(T)$ behavior, deduced from the trace of the stress–energy tensor, still given with wide error bars. Furthermore, even most recent data concern typically 20 temperature values $T_i$ and, perhaps, they are still not enough; we should need more $T_i$ points.

For instance, at the low–$T$ end, when strongly interacting matter can be considered a $\pi$+$\pi$-resonances gas, there is a clear anomaly in the $E(T)$ trend, however outlined by a single point $E(T_2)$. Such anomaly, e.g., vanifies the use of Hagedorn–like expressions in this $T$–range. In our treatment, we took such anomaly into account and saw that it causes evident features also in the cosmological expansion law. We however legitimately wonder whether such features, relying on a single estimate among 20, are to be considered safe.

In the attempt to overcome a part of such difficulties, however, we have proposed an expression able to fit almost half of the lattice points in the high–$T$ range, and to connect them with the very high $T$ regime. In our opinion this expression, which a priori depends on 4 parameters, but nicely fits more than 20 $c$ and $p$ values, could also allow some insight into the physics of the plasma; as a matter of fact, the best fit of the above parameters led to values unexpectedly regular, so that the expression might prove to be something more than a fitting algorithm.

The main results of this paper are however well summarized in the Figures and their captions, that can be useful to convey the overall message of this work.

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\[ \epsilon_{\mu}(T) = \frac{2}{\pi^2} \int_0^\infty dk \frac{k^2 W(k)}{\exp[W(k)/T] + 1}, \]

\[ 3p_{\mu}(T) = \frac{2}{\pi^2} \int_0^\infty dk \frac{k^4 W^{-1}(k)}{\exp[W(k)/T] + 1} \quad (A1) \]

where \( W(k) = \sqrt{k^2 + m_{\mu}^2} \), \( m_{\mu} \) being the mass of the particles. If one sets then \( \epsilon_{\mu} = ET^4 \), \( p_{\mu} = \Phi T^4 \), it is easy to see that

\[ E(T) = \frac{2}{\pi^2} \int_0^\infty dx x^3 Z(m_{\mu}/Tx) \frac{\exp(x)}{\exp[xZ(m_{\mu}/Tx)] + 1}, \]

\[ 3\Phi(T) = \frac{2}{\pi^2} \int_0^\infty dx x^3 Z^{-1}(m_{\mu}/Tx) \frac{\exp(x)}{\exp[xZ(m_{\mu}/Tx)] + 1} \quad (A2) \]

with \( Z = \sqrt{1 + (m_{\mu}/Tx)^2} \). These integrals can be easily performed numerically and yield the results shown in Figure [9].

Accordingly, around 100 MeV, energy density and pressure of the muon component have not yet substantially abandoned their ultrarelativistic values.

**Appendix B**

In order to fit the expression [17] with data, we follow these steps:

We select a value for \( s_o \) and use lattice outputs to obtain

\[ \psi_i = T_i \left[ 1 - \Phi(T_i)/\tilde{\Phi} \right]^{1/s_o}. \quad (B1) \]

We directly use just the six \( \psi_i \) values for \( i = 15, \ldots, 20 \). According to the expression [17], we expect that, in the \( T_{15} - T_{20} \) interval, at least, it is

\[ \psi(T) = T_o + \psi T^{-s_1} \quad (B2) \]

for suitable values of the parameters \( T_o, s_1 \) and \( \varphi = T^{s_1} \). This expression implies that

\[ \psi_i - \psi_j = \varphi(T_i^{-s_1} - T_j^{-s_1}) \quad (B3) \]

where, again, \( i, j, k, r \in 15 - 20 \). For any choice of \( i, j, k, r \), it is then possible to determine a value of \( s_1 \). There are however 15 possible combinations, yielding 15 values \( s_1 \), that we can average. Otherwise, we can directly treat the 15 combinations \( (T_i^{-s_1} - T_j^{-s_1})/(T_k^{-s_1} - T_r^{-s_1}) \) \( (i, j, k, r \in 15 - 20) \) as independent data, seeking the best fit \( s_1 \) value. For reasonable \( s_o \) values we expect the two procedures to yield the same \( s_1 \); we found consistent outputs (within \( 1 : 10^3 \)) for \( s_o \) between 0.8 and 1.25.

Using such \( s_1 \), we also average among \( \varphi \) values obtainable from eq. (B3), and then eq. (B2) yields a set of \( T_o \) values which are also suitably averaged.

Although using just the 6 top pressure values in this procedure, the agreement with many more pressure and energy density points eventually follows (as a matter of fact, only 4 points would be barely sufficient if we exclude \( T_{17} \)).
