THE AUTOMORPHISM GROUPS
OF COMPLEX HOMOGENEOUS SPACES

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Abstract. If $G$ is a (connected) complex Lie Group and $Z$ is a generalized flag manifold for $G$, then the open orbits $D$ of a (connected) real form $G_0$ of $G$ form an interesting class of complex homogeneous spaces, which play an important role in the representation theory of $G_0$. We find that the group of automorphisms, i.e., the holomorphic diffeomorphisms, is a finite-dimensional Lie group, except for a small number of open orbits, where it is infinite dimensional. In the finite-dimensional case, we determine its structure. Our results have some consequences in representation theory.

§1. We determine the automorphism groups for a certain interesting class of complex homogeneous spaces. Denote by $Z$ a generalized flag manifold for a connected complex semisimple Lie group $G$. A real form $G_0$ (which we assume to be connected) of $G$ acts on $Z$ with a finite number of orbits, thus there are always open orbits (cf. [22]). These open orbits play a key role in the representation theory of $G_0$. An open $G_0$-orbit $D$ in $Z$ has a $G_0$-invariant complex structure. The identity component of the group of holomorphic diffeomorphisms of $D$ will be denoted by $\text{Hol}(D)$. In the main theorem below we determine $\text{Hol}(D)$ for each measurable open orbit (see Definition 2.1). In the case where $D$ is measurable, $D$ carries a $G_0$-invariant (usually) indefinite hermitian metric and we determine its group of hermitian isometries. Generally the open orbits $D$ are non-compact, however, our results include the cases where $G_0$ is a compact real form and $D$ is compact, so $D = Z$. The compact case is contained in [12], [21], [3] and [2], from various points of view.

In general, for a complex manifold $X$, $\text{Hol}(X)$ is a (finite-dimensional) Lie group if $X$ is compact and may or may not be a Lie group if $X$ is non-compact. For example, $\text{Hol}(\mathbb{C}^n)$ is infinite dimensional. Our main interest is when $G_0$ (so $D$) is non-compact. We give a precise condition for $\text{Hol}(D)$ to be a Lie group.

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Our main results are contained in the following theorem.

**Main Theorem.** Suppose \( G \) is a connected simple complex Lie group and \( D \) is an open measurable \( G_0 \)-orbit in a generalized complex flag manifold for \( G \).

1. If there is a \( G_0 \)-equivariant holomorphic fibration of \( D \) over the hermitian symmetric space for \( G_0 \) (and \( D \) is not equal to the hermitian symmetric space itself), then \( \text{Hol}(D) \) is not a finite-dimensional Lie group.
2. If no such fibration exists, then \( \text{Hol}(D) \) is a Lie group and, except for the cases listed in Table 1.1 below, we have:
   a. \( \text{Hol}(D) = G_0 \) if \( G_0 \) is non-compact and,
   b. \( \text{Hol}(D) = G \) if \( G_0 \) is compact.

\[
\begin{array}{|c|c|c|}
\hline
G_0 & Z & \text{Hol}(D) \\
\hline
\text{SO}_e(2p, 2q + 1), p \neq 0 & \{\text{pure spinors in } \mathbb{C}^{2p+2q+1}\} & \text{SO}_e(2p, 2q + 2)/\mathbb{Z}_2 \\
\hline
\text{SO}(2n + 1) & \mathbb{C}P^{2n-1} & \text{SO}(2n + 2, \mathbb{C})/\mathbb{Z}_2 \\
\hline
\text{Sp}(n, \mathbb{R}) & \mathbb{C}P^{2n-1} & \text{SU}(n, n)/\mathbb{Z}_{2n} \\
\hline
\text{Sp}(p, q), pq \neq 0 & \mathbb{C}P^{2n-1} & \text{SU}(2p, 2q)/\mathbb{Z}_{2p+2q} \\
\hline
\text{Sp}(n) & \mathbb{C}P^{2n-1} & \text{SL}(2n, \mathbb{C})/\mathbb{Z}_{2n} \\
\hline
G_2, \text{split} & \text{quadric in } \mathbb{C}P^6 & \text{SO}_e(3, 4) \\
\hline
G_2, \text{compact} & \mathbb{C}P^{2n-1} & \text{SO}(7, \mathbb{C}) \\
\hline
\end{array}
\]

\( D \) is any open orbit in \( Z \).

Table 1.1

3. The group of hermitian isometries (i.e., the group of holomorphic diffeomorphisms preserving the hermitian metric) is
   a. \( \text{Hol}(D) \) if \( G_0 \) is non-compact and,
   b. a compact real form of \( \text{Hol}(D) \) if \( G_0 \) is compact.

In Proposition 3.11 we will see how the case of a semisimple group reduces to the case of simple groups.

The method of proof is to study the Lie algebra of global holomorphic vector fields on \( D \) using some standard techniques from representation theory. In most cases, this Lie algebra is just \( g \). However, in other cases, it is a bigger finite-dimensional Lie algebra \( g^1 \). In each of these cases we find a group \( G_0^1 \) which has (complexified) Lie algebra \( g^1 \) and has an effective action on \( D \).

Our results have several consequences for the representations associated to the open orbits. In the cases listed in Table 1.1 we view \( G_0^1 \) as acting on \( D \). The irreducible representations occurring in Dolbeault cohomology spaces on \( D \) extend
to (irreducible) representations of $G_0^1$. Also, the results have implications for a space of maximal compact subvarieties of $D$, which in turn plays a role in certain realizations of these representations. This will be discussed in the final section.

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§2. Some detailed information is obtained on the Lie algebra of global holomorphic vector fields on $D$. It will follow that $\text{Hol}(D)$ is usually finite dimensional and its structure will be narrowed down to a few possibilities. We start with some notation.

Let $G$ be a connected simple complex Lie group. As mentioned in the introduction, the semisimple case can be reduced to the simple case. Fix a generalized complex flag manifold $Z$ for $G$. Then, $Z$ is (biholomorphic to) $G/Q$ where $Q$ is a parabolic subgroup of $G$. We follow the common practice of denoting the Lie algebra of a Lie group by the corresponding gothic letter. Thus, the Lie algebras of $G$ and $Q$ will be denoted by $\mathfrak{g}$ and $\mathfrak{q}$, respectively. A connected real form of $G$ will be denoted by $G_0$, with Lie algebra $\mathfrak{g}_0$. By Theorem 2.6 of [22], $G_0$ acts on $Z$ with a finite number of orbits. Hence we know that open orbits always exist. Fix a Cartan involution $\theta$ and let $K$ (respectively, $K_0$) denote the fixed-point group of $\theta$ in $G$ (respectively, $G_0$). Fix an arbitrary open orbit $D$ in $Z$. For $z_0 \in D$, $\text{Stab}_G(z_0) = L_0$, a real form of $L$. We assume from now on that $D$ is a measurable open $G_0$-orbit in $Z$.

**Definition 2.1.** The orbit $D$ is measurable if and only if $D$ carries a $G_0$-invariant measure.

By Theorem 6.3 of [22], $D$ is measurable if and only if $\mathfrak{q} \cap \overline{\mathfrak{q}} = \mathfrak{l}$. This is equivalent to the statement that $\text{Stab}_{G_0}(z_0) = L_0$, a real form of $L$. We describe this in terms of the Lie algebras as follows. Let $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$ be the roots of $\mathfrak{h}$ in $\mathfrak{g}$. Then, there exists $\lambda_0 \in i\mathfrak{t}_0^*$ such that $\Delta(\mathfrak{h}, \mathfrak{l}) = \{\alpha \in \Delta \mid \langle \lambda_0, \alpha \rangle = 0\}$ and $\Delta(\mathfrak{h}, \overline{\mathfrak{q}}) = \{\alpha \in \Delta \mid \langle \lambda_0, \alpha \rangle < 0\}$. Also, $\overline{\mathfrak{q}}$ is the complex conjugate (with respect to the real form $\mathfrak{g}_0$) of a subalgebra $\mathfrak{u}$ and $\overline{\mathfrak{q}} = \mathfrak{l} + \mathfrak{u}$ is the parabolic opposite to $\mathfrak{q}$. We assume from now on that $D$ is a measurable open $G_0$-orbit in $Z$.

**Remark.** It is the measurable open orbits that play an important role in representation theory. If $G_0$ contains a compact Cartan subgroup or if $Z$ is the full flag manifold (so $Q$ is a Borel subgroup), then all open orbits are measurable. Also, if one open $G_0$-orbit in $Z$ is measurable, then all open orbits are measurable. A simple example of a non-measurable open orbit is the (unique) open orbit of $\text{SL}(3, \mathbb{R})$ on $\mathbb{CP}^2$.

The hermitian symmetric spaces $G_0/K_0$ are examples of measurable open orbits.
In this case, writing the Cartan decomposition as \( g = \mathfrak{t} \oplus p, \) \( p \) splits into \( p_+ \oplus p_- \) as representations of \( K \). One sees that \( G_0/K_0 \) is an open orbit in \( G/KP_- \). It is well known that \( G_0/K_0 \) is biholomorphic to a bounded domain in some \( \mathbb{C}^N \). Thus the Lie algebra of global holomorphic vector fields is infinite dimensional, so our arguments in this case are a little different from the general case. We will treat this case first.

**Proposition 2.2.** If \( B = G_0/K_0 \) is a hermitian symmetric space, then \( Hol(B) \) is finite-dimensional.

*Proof.* This is standard. It is contained in Chapter VIII of [7].

We now give a condition for \( Hol(D) \) to be infinite dimensional.

**Proposition 2.3.** If \( B = G_0/K_0 \) is of hermitian symmetric type and \( D \) is an open orbit with \( p_- \not\subset \mathfrak{u} \) then the natural fibration \( \pi : D \to B \) is holomorphically trivial, in particular \( D \cong B \times K_0 \cdot z_0 \). Furthermore, the group of holomorphic diffeomorphisms of \( D \) is not a finite-dimensional Lie group.

*Proof.* Recall the following decomposition of \( G_0 \). In the complex group \( G \), \( P_+KP_- \) is a dense open set and \( G_0 \subset P_+KP_- \). Furthermore, the decomposition of \( g \in G_0 \) as \( g = p_+kp_- \) is unique. Then the Harish-Chandra embedding maps \( gK_0 \in B \) to \( \xi \in p_+ \), where \( p_+ = \exp(\xi) \). Denote the image of the Harish-Chandra embedding by \( B \). Then \( B \) is a bounded domain biholomorphic to \( B = G_0/K_0 \).

Now suppose that \( D \) has \( p_- \subset \mathfrak{u} \). Then \( L_0 \subset K_0 \) and \( L\mathfrak{u} \subset KP_- \). It is clear that \( D = G_0 \cdot z_0 \subset P_+K \cdot z_0 \). The latter is biholomorphic to \( p_+ \times K_0/(K_0 \cap L_0) \). To see this, note that the map \( \Phi : p_+ \times K_0/(K_0 \cap L_0) \to P_+K \cdot z_0 \) given by \( \Phi(\xi,k) = \exp(\xi)k \cdot z_0 \) is complex analytic. It is clearly onto. It is one-to-one: if \( \exp(\xi)k \cdot z_0 = \exp(\xi')k' \cdot z_0 \), then by the uniqueness of the Harish-Chandra decomposition, \( \exp(\xi) = \exp(\xi') \); so \( \xi = \xi' \). Then \( k \cdot z_0 = k' \cdot z_0 \). We must determine the inverse image of \( D \) under \( \Phi \). Note that elements of \( G_0 \) can be written as \( g = \exp(\xi)kp_- \) with \( \xi \in B \). Thus, \( g \cdot z_0 = \exp(\xi)k \cdot z_0 \). As \( \Phi \) is bijective, \( \Phi^{-1}(D) = B \times K \cdot z_0 \). But \( K \cdot z_0 = K_0 \cdot z_0 \cong K_0/(K_0 \cap L_0) \). This proves the first part of the proposition.

As \( B \) is a bounded domain in some \( \mathbb{C}^N \), the space of holomorphic functions \( f : B \to \mathfrak{t} \) is infinite dimensional. Now, \( K \cdot z_0 = K_0 \cdot z_0 \), so \( K \) is contained in the automorphism group of \( K_0 \cdot z_0 \). Thus, for each holomorphic \( f : B \to \mathfrak{t} \), there is an automorphism of \( B \times K_0 \cdot z_0 \) defined by \( \varphi(\xi,z) = (\xi,(\exp(f(\xi)))z) \). Note that \( \mathfrak{t} \neq 0 \) since \( p_- \neq \mathfrak{u} \). This provides an infinite-dimensional family in \( Hol(D) \). \( \square \)

Proposition 2.3 is (1) of the main theorem, since there exists a holomorphic fibration \( D \to B \) as in (1) if and only if \( \mathfrak{u} \) properly contains \( p_+ \) or \( p_- \). Also note that \( B \) is simply connected, whence \( D \to B \) cannot have discrete fibers.
Assume from now on that \( \mathcal{U} \) does not contain \( p_- \) or \( p_+ \). In these cases, as we will see, \( Hol(D) \) is a (finite-dimensional) Lie group. Now consider the Lie algebra of holomorphic vector fields on \( D \), that is, the global sections \( H^0(D, \mathcal{T}) \) of the vector bundle \( \mathcal{T} \) of holomorphic vector fields on \( D \). Note that \( \mathcal{T} \) is the holomorphic homogeneous vector bundle for the representation \( g / q \) of \( Q \). Since \( G_0 \) acts on \( D \) and on \( \mathcal{T} \), it is clear that \( H^0(D, \mathcal{T}) \) is a representation of \( G_0 \). There is considerable machinery available to study representations of this type. See, for example, [17] and [25]. We will very briefly describe the ingredients that we will need. For any finite-dimensional representation \( F \) of \( Q \), let \( \mathcal{F} \rightarrow Z \) be the corresponding holomorphic, homogeneous vector bundle on \( Z \). By restriction, we obtain a homogeneous vector bundle on \( D \) denoted by \( \mathcal{F} \rightarrow D \). By [25], the Dolbeault cohomology spaces \( H^p(D, \mathcal{F}) \) are continuous, admissible representations of \( G_0 \). The Harish-Chandra module (i.e., the subspace of \( K_0 \)-finite vectors in \( H^p(D, \mathcal{F}) \)) is a cohomologically parabolically induced \((g, K_0)\)-module \( \mathcal{R}^p_q(F) \). The definition of \( \mathcal{R}^p_q(F) \) is given in Definition 1, page 432 of [25] (and in §6.3 of [16], with slightly different conventions). It follows immediately that \( \mathcal{R}^0_q(F) \subset \text{Hom}_q(\mathcal{U}(g), F)_{K_0-\text{finite}} \). Here, \( \mathcal{U}(g) \) is the universal enveloping algebra of \( g \) and the subscript ‘\( K_0 \)-finite’ indicates the subspace of \( K_0 \)-finite vectors.

**Proposition 2.4.** If \( E \) is a finite-dimensional representation of \( G_0 \), then

\[
\text{Hom}_{G_0}(E, H^0(D, \mathcal{T})) \neq 0 \implies \text{Hom}_q(E, g / q) \neq 0.
\]

**Proof.** If \( \phi : E \rightarrow H^0(D, \mathcal{T}) \) is a nonzero \( G_0 \)-homomorphism, then the image of \( \phi \) lies in \( H^0(D, \mathcal{T}) \). But \( H^0(D, \mathcal{T}) \subset \text{Hom}_q(\mathcal{U}(g), g / q) \). So \( \phi \) defines a nonzero element of \( \text{Hom}_q(E, \text{Hom}_q(\mathcal{U}(g), g / q)) \cong \text{Hom}_q(E, g / q) \).

**Proposition 2.5.** \( H^0(D, \mathcal{T}) \) is finite dimensional and \( Hol(D) \) is a Lie group unless \( G_0 / K_0 \) is hermitian symmetric and \( p_+ \subset \mathcal{U} \) or \( p_- \subset \mathcal{U} \).

**Proof.** Fix a positive root system \( \Delta^+ \subset \Delta \) so that \( \Delta(u) \subset \Delta^+ \). (We may choose \( \Delta^+ = \Delta^+ \cup \Delta^-(u) \) with \( \Delta^+(l) \) an arbitrary positive system for \( \Delta(l) \).) Suppose \( F \) is a finite dimensional \( Q = L \mathcal{U} \) representation with trivial \( \mathcal{U} \) action. Then, viewing \( F \) as a \( \mathcal{Q} = L \mathcal{U} \) representation with trivial \( \mathcal{U} \) action, by Lemma 5.15 of [17], \( \text{Hom}_q(\mathcal{U}(g), F)_{L_0 \cap K_0-\text{finite}} = \mathcal{U}(g) \otimes \mathcal{F} \). Furthermore, \( (\mathcal{U}(g) \otimes \mathcal{F})_{K_0-\text{finite}} \) is a highest-weight Harish-Chandra module (with respect to \( \Delta^+ \)). By a result of Harish-Chandra (cf. [5], Theorem 1 and Corollary 2, page 761), the only case for which there exist infinite-dimensional highest-weight \((g, K)\)-modules is when \( \Delta^+ \) contains \( \Delta(p_+) \) or \( \Delta(p_-) \).

Now consider the holomorphic tangent bundle \( \mathcal{T} \). The action of \( \mathcal{U} \) on \( g / q \) is not in general trivial, so the above does not apply directly. Instead, we form a filtration \( g / q = F_1 \supset F_2 \supset \cdots \supset F_N \supset 0 \) so that \( F_i / F_{i+1} \) does have a trivial \( \mathcal{U} \) action. (For instance, we could take a composition series for \( g / q \), so that
each $F_i/F_{i+1}$ is irreducible as $q$-module, in which case the action of $\mathfrak{p}$ is necessarily trivial.) As $\text{Hom}_q(\mathcal{U}(\mathfrak{g}), \cdot)_{(L_0 \cap K_0)}$-finite is exact, there is a filtration of $\text{Hom}_q(\mathcal{U}(\mathfrak{g}), \mathfrak{g}/q)_{(L_0 \cap K_0)}$-finite with quotients $\text{Hom}_q(\mathcal{U}(\mathfrak{g}), (F_i/F_{i+1}))_{(L_0 \cap K_0)} \cong (\mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{p}} (F_i/F_{i+1})))$. Now we may conclude that, unless $\Delta^+$ contains one of $\Delta(p_{\pm})$, $\text{Hom}_q(\mathcal{U}(\mathfrak{g}), \mathfrak{g}/q)_{K_0}$-finite is finite dimensional.

Now note that as we were free to choose $\Delta^+(l) \subset \Delta$, it follows that $\Delta(p_{\pm}) \cap \Delta(l) = 0$. Thus, $\Delta^+$ contains one of $\Delta(p_{\pm})$ if and only if $\Delta(u)$ contains one of $\Delta(p_{\pm})$, that is, if and only if $p_+ \subset \mathfrak{u}$ or $p_- \subset \mathfrak{p}$.

That $\text{Hol}(D)$ is a Lie group now follows from Theorem 3.1 of [9]. □

By Proposition 2.5 we may use Proposition 2.4 to restrict the possibilities for $\text{Hol}(D)$. Note, it is clear that

\begin{equation}
\text{Hom}_q(E, \mathfrak{g}/q)
\end{equation}

is non-zero for $E = \mathfrak{g}$ and, of course, that $G_0 \subset \text{Hol}(D)$.

**Corollary 2.7.** If (2.6) is zero for all irreducible representations $E \not\cong \mathfrak{g}$, then $\text{Hol}(D) = G_0$ or $G$.

**Proof.** The Lie algebra of $\text{Hol}(D)$ lies between $\mathfrak{g}_0$ and $\mathfrak{g}$. As $\mathfrak{g}$ is simple the only possibilities for $\text{Hol}(D)$ are $G_0$ and $G$. □

We look for finite-dimensional representations $E$ other than $E \cong \mathfrak{g}$ such that (2.6) is non-zero. We start off with a fact about the structure of simple Lie algebras.

**Lemma 2.8.** If $\mathfrak{g}$ is a simple Lie algebra, then the Weyl group is transitive on roots of a given length. Therefore, there are two possibilities:

1. There is just one root length and the highest root, which we denote by $\gamma_{\ell}$, is the unique $\Delta^+$-dominant root.
2. There are two root lengths and there is a unique dominant long root $\gamma_{\ell}$ and a unique dominant short root $\gamma_s$.

**Proof.** This is standard. See, for example, [8], page 53.

**Theorem 2.9.** If $\mathfrak{g}$ is simple and has just one root length and $\mathfrak{p}$ does not contain $p_+$ or $p_-$, then

$$\text{Hol}(D) = \begin{cases} 
G_0, & \text{if } G_0 \text{ is non-compact} \\
G, & \text{if } G_0 \text{ is compact}
\end{cases}$$

**Proof.** Suppose $E$ is an irreducible finite-dimensional representation of $G_0$. Then, by Corollary 2.7, it is enough to check that (2.6) is zero unless $E \cong \mathfrak{g}$. Since $\Delta^+$ contains $\Delta(u)$, the highest-weight space in $E$ is cyclic for $q = l + \mathfrak{p}$. So, a nonzero $q$
homomorphism $\phi : E \to g/\mathfrak{q}$ maps the highest-weight vector to a (nonzero) weight vector in $g/\mathfrak{q}$ (of the same weight). This weight must be dominant. As there is just one root length, Lemma 2.8 says that this highest weight of $E$ must be $\gamma_\ell$. Now Corollary 2.7 applies.

In case $G_0$ is non-compact, $D \neq Z$ and $G$ does not act on $D$. Thus, $\text{Hol}(D) = G_0$. When $G_0$ is compact, $D = Z$ and $G$ always acts on $D$. □

When there are two root lengths in $g$, then by Lemma 2.8 we have the possibility that the Lie algebra of holomorphic vector fields is $g$ or $g \oplus E_{\gamma_s}$, where $E_{\gamma_s}$ is the irreducible finite-dimensional representation of $g$ with highest weight $\gamma_s$. In connection with this case we have the following lemma.

**Lemma 2.10.** Let $g$ be a simple Lie algebra with two root lengths. Then there is a simple Lie algebra $g^1$ containing $g$ as a subalgebra so that, as $g$-modules, $g^1 \cong g \oplus E_{\gamma_s}$. The following table lists the possibilities:

| $g$  | $g^1$   |
|------|---------|
| $B_n$ | $D_{n+1}$ |
| $C_n$ | $A_{2n-1}$ |
| $F_4$ | $E_6$   |
| $G_2$ | $B_3$   |

**Table 2.11**

*Proof.* This is easily checked by direct calculation.

§3. We now consider the simple Lie algebras $g$ having two different root lengths. We use Lemma 3.1 below to restrict the possibilities for $Z = G/Q$ for which $\text{Hol}(D) \neq G_0$ or $G$. We then treat each of the possible flag manifolds $Z$ separately.

The following lemma will help us determine when (2.6) is zero for $E = E_{\gamma_s}$.

**Lemma 3.1.** Suppose there are roots $\beta_1, \ldots, \beta_m \in \Delta^+$, not necessarily distinct, so that $\gamma_s - \sum_{j=1}^k \beta_j$ is a root for each $k = 1, \ldots, m$ and $\gamma_s - \sum_{j=1}^m \beta_j$ is a long root in $u$. Then $\text{Hom}_q(E_{\gamma_s}, g/\mathfrak{q}) = 0$.

*Proof.* Suppose $\phi \in \text{Hom}_q(E_{\gamma_s}, g/\mathfrak{q})$ is nonzero. If $v_+$ is a highest weight vector in $E_{\gamma_s}$ then $\phi(v_+)$ is a root vector $X_{\gamma_s}$ of weight $\gamma_s$. Since weights of $E_{\gamma_s}$ all have norm less than or equal to $\|\gamma_s\|$, the long root $\gamma_s - \sum_{j=1}^m \beta_j$ is not a weight. Thus $X_{-\beta_1} \cdots X_{-\beta_m} \cdot v_+ = 0$. On the other hand, $\text{ad}(X_{-\beta_1}) \cdots \text{ad}(X_{-\beta_m}) \cdot X_{\gamma_s} \neq 0$ in $g$ since $\gamma_s - \sum_{j=1}^k \beta_j$ is a root for each $k = 1, \ldots, m$. Since the root $\gamma_s - \sum_{j=1}^m \beta_j$ is in $u$, $\text{ad}(X_{-\beta_1}) \cdots \text{ad}(X_{-\beta_m}) \cdot X_{\gamma_s} \neq 0$ in $g/\mathfrak{q}$. But this is a contradiction; $0 = \phi(X_{-\beta_1} \cdots X_{-\beta_m} \cdot v_+) = \text{ad}(X_{-\beta_1}) \cdots \text{ad}(X_{-\beta_m}) \cdot X_{\gamma_s} \neq 0$. □
For the simple Lie algebras with two root lengths, we number the simple roots as follows:

The fundamental weight corresponding to $\alpha_j$ is denoted by $\lambda_j$. Recall that a weight $\lambda$ determines a parabolic subgroup $Q$ with Lie algebra $q = q(\lambda)$ by $\Delta(h, q) = \{\alpha \in \Delta | \langle \lambda, \alpha \rangle \leq 0\}$. Each parabolic subalgebra is conjugate to some $q(\lambda)$, in fact, is conjugate to one with $\lambda = \sum_{j \in \Phi} \lambda_j$ for some $\Phi \subset \{1, 2, \ldots, \text{rank } g\}$.

**Theorem 3.2.** $Hol(D)$ is $G$ if $G_0$ is compact and is $G_0$ if $G_0$ is non-compact, except possibly in the following cases:

- Type $B_n$: $Q$ is determined by $\lambda_n$.
- Type $C_n$: $Q$ is determined by $\lambda_1$.
- Type $G_2$: $Q$ is determined by $\lambda_1$.

**Proof.** To apply Lemma 3.1 we find, for each simple Lie algebra having more than one root length, roots $\beta_j$ as in the lemma so that $\langle \lambda_j, \gamma_s - \sum_{j=1}^{m} \beta_j \rangle > 0$ for all $\lambda_j$ except those listed in the theorem.

For type $B_n$, $\gamma_s = \sum_{j=1}^{n} \alpha_j$ and $\gamma_s - \alpha_n$ is a long root. (So $m = 1, \beta_1 = \alpha_n$ in Lemma 3.1.) Clearly $\langle \lambda_j, \gamma_s - \alpha_n \rangle > 0$ except for $j = n$.

For type $C_n$, $\gamma_s = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n$ and $\gamma_s - \alpha_1$ is a long root. (So $m = 1, \beta_1 = \alpha_1$ in Lemma 3.1.) Clearly $\langle \lambda_j, \gamma_s - \alpha_1 \rangle > 0$ except for $j = 1$.

For type $G_2$, $\gamma_s = 2\alpha_1 + \alpha_2$ and $\gamma_s - 2\alpha_1 (= \alpha_2)$ is a long root. (So $m = 2, \beta_1 = \beta_2 = \alpha_2$ in Lemma 3.1.) Clearly $\langle \lambda_j, \gamma_s - 2\alpha_1 \rangle > 0$.

For type $F_4$, $\gamma_s = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$ and $\gamma_s - \alpha_2$ is a long root. (So $m = 1, \beta_1 = \alpha_2$ in Lemma 3.1.) Clearly $\langle \lambda_j, \gamma_s - \alpha_2 \rangle > 0$ for all $j$. Thus, every parabolic subalgebra of $F_4$ satisfies the condition of Lemma 3.1. □

We now prove our main theorem by looking at each flag manifold listed in Theorem 3.2.
**Type B}_n.** Let \( p, q \) be positive integers such that \( p + q = n \). The complex group \( G = \text{SO}(2n+1, \mathbb{C}) \) is defined by the symmetric form on \( \mathbb{C}^{2n+1} \):

\[
(w, z) = \sum_{j=1}^{2p} w_jz_j - \sum_{j=2p+1}^{2p+2q+1} w_jz_j.
\]

The relevant real form \( G_0 \) of \( B}_n \) is the connected component of the isometry group of \( (, ) \) restricted to \( \mathbb{R}^{2n+1} \). That is, \( G_0 \cong \text{SO}_e(2p, 2q+1) \). The complex flag manifold \( Z \) is the space of all maximal isotropic subspaces of \( \mathbb{C}^{2n+1} \), also known as the space of pure spinors. The set of \( (p + q) \)-planes \( \zeta \in Z \) such that the hermitian form \( \langle w, z \rangle := (w, \bar{z}) \) restricted to \( \zeta \) has signature \( (p, q) \) contains two open \( G_0 \)-orbits. They are \( D_\pm = G_0, z_0 \), where \( z_0 = \text{span}_{\mathbb{C}}\{e_1, e_3, e_4, \ldots, e_{2n-1} \pm i \epsilon_{2n}\} \). These are the only open orbits of \( G_0 \) in \( Z \).

Let \( G^1 \) be the isometry group of \( (w, z)^1 = \sum_{j=1}^{2p} w_jz_j - \sum_{j=2p+1}^{2p+2q+2} w_jz_j \) on \( \mathbb{C}^{2n+2} \). Let \( G_0^1 \) be the connected component of the isometry group of \( (, )^1 \) restricted to \( \mathbb{R}^{2n+2} \). Then, \( G_0^1 \cong \text{SO}_e(2p, 2q+2) \). The space of maximal isotropic subspaces in \( \mathbb{C}^{2n+2} \) has two connected components (since the dimension is even). Take \( Z^1 \) to be the component containing \( z_0^1 = \text{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_{2n+1} + i e_{2n+2}\} \). The other component contains \( \text{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_{2n-1} + i e_{2n}, e_{2n+1} - i e_{2n+2}\} \). Then \( Z^1 \) is a flag manifold for \( G^1 \). Again, there are two open orbits \( D_{\pm}^1 \) consisting of planes of signature \( (p, q + 1) \) with respect to the hermitian form \( \langle w, z \rangle^1 := (w, \bar{z})^1 \).

Define a map \( \pi : Z^1 \to Z \) by \( \pi(\zeta) = \zeta \cap (\mathbb{C}^{2n+1} \times \{0\}) \). Note that \( \pi(\zeta) \in Z \) as it is clearly isotropic and \( \text{dim}(\pi(\zeta)) = n \). It is also clear that \( \pi \) is \( G_0 \)-equivariant. By equivariance, \( \pi \) is onto. To see that \( \pi \) is one-to-one it is enough to check that \( \pi^{-1}(z_0) = \{z_0^1\} \). Suppose \( \pi(\zeta) = \zeta \cap (\mathbb{C}^{2n+1} \times \{0\}) = z_0 \). Then \( z_0 \subset \zeta \) and there is \( v \in \zeta \) so that \( v \in z_0^{1 \perp} \cap z_0^{\perp} = (z_0 + \mathbb{C} \cdot v)^\perp \). But \( (z_0 + \mathbb{C} \cdot v)^\perp = \text{span}_{\mathbb{C}}\{e_{2n+1}, e_{2n+2}\} \). As \( v \) is isotropic it is either \( e_{2n+1} + i e_{2n+2} \) or \( e_{2n+1} - i e_{2n+2} \). Only \( v = e_{2n+1} + i e_{2n+2} \) gives \( \zeta \in Z^1 \), so \( \zeta = z_0^1 \).

We may conclude that \( G \) acts transitively on \( Z^1 \cong Z \) and \( G_0 \) acts transitively on \( D_{\pm}^1 \cong D_{\pm} \). Equivalently, \( G^1 \) and \( G_0^1 \) act on \( Z \) and \( D_{\pm} \), respectively. It also follows that the compact real form acts on the \( G_0 \)-orbit \( Z \).

**Type C}_n.** Let \( \omega(w, z) = \sum_{j=1}^{n} (w_jz_{n+j} - w_{n+j}z_j) \) be the standard symplectic form on \( \mathbb{C}^{2n} \). Then \( G = \text{Sp}(n, \mathbb{C}) \) is the complex group preserving \( \omega \). The complex flag manifold under consideration is \( Z = \{\omega\text{-isotropic lines in } \mathbb{C}^{2n}\} \), which is just \( \mathbb{CP}^{2n-1} \), since any line is automatically isotropic. There are two families of real forms: \( \text{Sp}(n, \mathbb{R}) \) and \( \text{Sp}(p, q) \).

Define \( G_0 = U(n, n) \cap G \), where \( U(n, n) \) is the isometry group of \( \langle w, z \rangle = \sum_{j=1}^{n} (w_jz_{n+j} - w_{n+j}z_j) \). Then \( G_0 \cong \text{Sp}(n, \mathbb{R}) \). There are two open \( G_0 \)-orbits, \( D_{\pm} : \) the positive lines and the negative lines. But, it is clear from Witt’s theorem that \( G_0^1 = \text{SU}(n, n) \) acts transitively on \( D_{\pm} \).
For the other real forms, take $G_0 = U(2p, 2q) \cap G$, where $U(2p, 2q)$ is the isometry group of $\langle w, z \rangle = \sum_{j=1}^{p} (w_j \bar{z}_j + w_{n+j} \bar{z}_{n+j}) - \sum_{j=p+1}^{p+q} (w_j \bar{z}_j + w_{n+j} \bar{z}_{n+j})$, where $p + q = n$. Then $G_0 \cong \text{Sp}(p, q)$, (cf. [7], p. 445). For $pq \neq 0$, $G_0$ has two open orbits consisting of positive and negative lines. Also, $G_0^1 = SU(2p, 2q)$ acts transitively on each of these orbits. If $pq = 0$, then $G_0$ is compact and acts transitively on $Z = \mathbb{CP}^{2n-1}$. The unitary group $SU(2n)$ also acts transitively on $Z$.

**Type G$_2$.** Let $G$ be the complex group of type G$_2$ and let $G^1$ be the complex group of type B$_3$. Let $G_0$ and $G_0^1$ be the split real forms of $G$ and $G^1$, respectively. We first recall how $G_0 \subset G^1_0$ and $G \subset G^1$. The split real form $G_0$ of type G$_2$ is the automorphism group of the split octonions (i.e., the Cayley numbers), $\tilde{O}$. There is a natural inner product of signature $(4, 4)$ defined on $\tilde{O}$. It is easy to see that any automorphism must preserve the inner product. Since an automorphism must fix $1 \in \tilde{O}$, it must preserve $\text{Im} \tilde{O} := (\mathbb{R} \cdot 1)^\perp$, the pure imaginary octonions. Moreover, an automorphism is completely determined by its restriction to $\text{Im} \tilde{O}$. We see, then, that $G_0 \subset \text{SO}_e(3, 4) \cong G^1_0$. Complexifying, we obtain a complex symmetric form $(\cdot, \cdot)_C$ on $(\text{Im} \tilde{O})_C$. Similarly, $G$ sits inside the isometry group $G^1$ of $(\cdot, \cdot)_C$. There is a corresponding hermitian form defined by $\langle w, z \rangle := (w, \bar{z})_C$.

Consider the flag manifold for $G^1$ defined by $Z = \{\text{isotropic lines in } (\text{Im} \tilde{O})_C \cong \mathbb{C}^7\}$. This is a flag manifold for $G$ under the action of $G$ as a subgroup of $G^1$. The proof of this is similar to the proof of Claim 3.5 below and is sketched at the end of this section. We will see that $G^1_0 = \text{SO}_e(3, 4)$ acts on $Z$ with two open orbits $D_\pm$ and that $G_0$ acts transitively on both.

The action of $G^1_0 \cong \text{SO}_e(3, 4)$ has two open orbits, $D_\pm$, respectively, the positive and negative lines (with respect to $\langle \cdot, \cdot \rangle$) in $Z$. We verify this as follows. Write $z = x + iy \in (\text{Im} \tilde{O})_C = \text{Im} \tilde{O} + i \text{Im} \tilde{O}$ with $x, y \in \text{Im} \tilde{O}$. Then $z$ is positive and isotropic if and only if:

$$\langle x, x \rangle = \langle y, y \rangle > 0 \text{ and } \langle x, y \rangle = 0 \ .$$

Suppose $z' = x' + iy'$ is another positive isotropic vector. By scaling $z$, we may assume $(x, x) = (x', x')$ and $(y, y) = (y', y')$. Now, Witt’s theorem says that there is an isometry of $\text{Im} \tilde{O} \cong \mathbb{R}^{3, 4}$ taking $x$ to $x'$ and $y$ to $y'$, i.e., taking $z$ to $z'$. One can check that this isometry can be chosen to lie in $\text{SO}_e(3, 4)$.

**Claim 3.5.** $G_0$ acts transitively on $D_\pm$.

We use the following lemma, which follows easily from the development in Chapter 6 of [6].

**Lemma 3.6.** Suppose $A$, $A' \subset \tilde{O}$ are normed subalgebras both isomorphic to either the quaternions $\mathbb{H}$ or the $2 \times 2$ real matrices $M_2(\mathbb{R})$, then any isomorphism $A \rightarrow A'$ extends to an automorphism of $\tilde{O}$. 

Again, by Witt’s theorem, there is an isometry $h$ as we now demonstrate.

Now, by Lemma 3.6, $h$ by elements of $\text{SL}(2, \mathbb{R})$. As homogeneous spaces, the two open orbits are isomorphic to $A = A' \oplus A''$. Moreover, multiplication in $\bar{\Omega}$ is given by

$$
(a + be)(c + de) = (ac + \bar{d}b) + (da + bc)e,
$$

where the bar denotes conjugation in $A$: the usual conjugation for $\mathbb{H}$ and

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} = \begin{pmatrix}
\delta & -\beta \\
-\gamma & \alpha
\end{pmatrix}
$$

for $M_2(\mathbb{R})$. Similarly for $\bar{\Omega} = A' \oplus A''$. Now suppose $f : A \to A'$ is an isomorphism. Then, as one can readily check, $F(a + be) := f(a) + f(b)e'$ is an automorphism of $\bar{\Omega}$. □

Remark 3.8. In case $A = A'$, $F_{\epsilon}(a + be) := f(a) + f(b)e'$ extends $f$ to an automorphism of $\bar{\Omega}$ for any $e' \in A^\perp$ with $(\epsilon', \epsilon') = -1$. In fact, any automorphism of $\bar{\Omega}$ that preserves $A$ must send $\epsilon$ to such an $\epsilon'$. Therefore, all extensions of $f$ are of this form.

Proof of Claim 3.5. If $z$ and $z'$ are two elements in $D_+$ with $z = x + iy$ and $z' = x' + iy'$, then both decompositions must satisfy (3.4). By Proposition 6.40 of [6], the subalgebras $A$ and $A'$ of $\bar{\Omega}$ generated by $\{x, y\}$ and $\{x', y'\}$ are isomorphic to $\mathbb{H}$, say, $\phi : A \to \mathbb{H}$ and $\phi' : A' \to \mathbb{H}$. Then, $f = \phi^{-1} \circ \phi' : A' \to A$ is an isomorphism from $A'$ to $A$.

Now, both $x$, $y$ and $f(x')$, $f(y')$ satisfy (3.4). By rescaling $z'$, if necessary, we may assume $(x, x) = (f(x'), f(x'))$. Also, $\text{Aut}(\mathbb{H})$ consists solely of the conjugations by unit quaternions, which is the map $\text{SU}(2) \to \text{SO}(3)$, acting on $\text{Im}(\mathbb{H}) \cong \mathbb{R}^3$. By Witt’s theorem, there is an orthogonal map $h$ (i.e. an automorphism of $\mathbb{H}$) sending $f(x')$ to $x$ and $f(y')$ to $y$. Then, $h \circ f$ extends to an automorphism $g$ of $\bar{\Omega}$, by Lemma 3.6. Thus, we have an element $g \in G_0$ with $g(z') = z$.

For $z, z' \in D_-$, the argument is similar. Now, however, the subalgebras $A$ and $A'$ generated by $\{x, y\}$ and $\{x', y'\}$ are isomorphic to $M_2(\mathbb{R})$. Again, there exists an isomorphism $f : A' \to A$. The group $\text{Aut}(M_2(\mathbb{R}))$ consists solely of conjugations by elements of $\text{SL}(2, \mathbb{R})$, which is the action of $\text{SO}(1, 2)$ on $\mathbb{R}^{1,2} \cong M_2(\mathbb{R}) \cap \text{Im} \bar{\Omega}$. Again, by Witt’s theorem, there is an isometry $h$ of $\mathbb{R}^{1,2}$ such that $h(f(x')) = x$ and $h(f(y')) = y$. One can check that $h$ may be taken to lie in $\text{SO}(1, 2) \cong \text{SL}(2, \mathbb{R})$. Now, by Lemma 3.6, $h \circ f$ extends to $g \in \text{Aut}(\bar{\Omega})$. Thus, we have an element $g \in G_0$ with $g \cdot z' = z$. This completes the proof of Claim 3.5. □

Remark 3.9. As homogeneous spaces, the two open orbits are $D_+ \cong G_0/\text{U}(2)$ and $D_- \cong G_0/(\text{SL}(2, \mathbb{R}) \times \text{U}(1))$. This is a consequence of Lemma 3.6 and Remark 3.8, as we now demonstrate.
For $D_+$, write $A = H = \bigoplus_{j=1}^{4} \text{Re} e_j$ with $e_1$ = identity and $e_2, e_3, e_4$ units (i.e. $(e_j, e_j) = 1$) with $e_2 e_3 = e_4$. Suppose $g \in G_0$ fixes $e_2 + ie_3 \in D_+$. Then $g$ fixes both $e_2$ and $e_3$. As $g$ is an automorphism, it must also fix $G_4$, hence all of $A$. By Remark 3.8, the automorphisms fixing $A$ are of the form $F_{a\alpha}$, where $\alpha$ is an element of $A$ of length one. But, since $A \cong H$, the length-one elements form a group isomorphic to $SU(2)$. The stabilizer of the line $C \cdot (e_2 + ie_3)$ also includes the scalars. Therefore, $\text{Stab}_{G_0}(e_2 + ie_3) \cong U(2)$.

For the orbit $D_-$, take $A = M_2(\mathbb{R})$. We pick our base point to be the isotropic vector $x + iy$ with $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Again, if $g \in G_0$ fixes $x + iy$, then $g$ fixes $A$. By Remark 3.8, the automorphisms of $\tilde{O}$ fixing $A$ are all of the form $F_{\beta e}$ where $\beta \in A$ and $(\beta, \beta) = 1$, i.e., $\det(\beta) = 1$. Thus, the group of automorphisms fixing the vector $x + iy$ is $\text{SL}(2, \mathbb{R})$. As scalars also fix the line $C \cdot (x + iy)$, we have $\text{Stab}_{G_0}(x + iy) \cong \text{SL}(2, \mathbb{R}) \times U(1)$.

We conclude the type $G_2$ case by sketching a proof that $Z$ is a flag manifold for $G$. For this it is enough to show that a compact real form of $G_2$ acts transitively on $Z$. To do this we use a slightly different realization for $G$. A compact real form of $G_2$ is the automorphism group of $O$, where $O$ is the nonsplit octonions. There is a positive definite symmetric form on $O$ preserved by the automorphism group, $\text{Aut}(O)$. As above $\text{Aut}(O)$ preserves $\text{Im} O = (\mathbb{R} \cdot 1)^\perp$, thus $\text{Aut}(O) \subseteq \text{SO}(7)$. Now complexify the form to obtain a symmetric form on $(\text{Im} O)_C$. The isometry group is conjugate to $G^1$ (as the form is equivalent to the complex form arising from the split octonions). The corresponding flag manifold is the space of isotropic lines in $\text{Im} O$.

Now let $z = x + iy$ and $z' = x' + iy'$ be isotropic vectors in $(\text{Im} O)_C$. Then (3.4) holds for both $z$ and $z'$. The subalgebras $A, A'$ generated by $\{x, y\}$ and $\{x', y'\}$ are both isomorphic to $H$. Now argue as in the case $D_+$ using the fact, analogous to Lemma 3.6, that an isomorphism $A \to A'$ extends to an automorphism of $O$.

Part (3) of the main theorem now follows immediately since the groups $G_0$ or $G_0^1$ preserve the metric and the corresponding complex groups do not. This completes the proof of the main theorem.

Now suppose that $G$ is semisimple. Write $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ with $\mathfrak{g}_1, \ldots, \mathfrak{g}_N$ simple and let $G_i$ be the subgroups of $G$ corresponding to the $\mathfrak{g}_i$. A flag manifold for $G$ is the product of flag manifolds $Z_i \cong G_i/Q_i$ for the $G_i$. A measurable open $G_0$-orbit $D$ is the product of measurable open $G_{i,0}$-orbits $D_i$ in the $Z_i$. Recall that by Proposition 2.3, writing $\mathfrak{q}_i = \mathfrak{l}_i + \mathfrak{u}_i$, if

\begin{equation}
\tag{3.10}
\mathfrak{p}_{i, -} \subset \mathfrak{u}_i \text{ or } \mathfrak{p}_{i, +} \subset \mathfrak{u}_i
\end{equation}

then $\text{Hol}(D_i)$ is not a (finite-dimensional) Lie group. Thus, if at least one of the factors $D_i$ satisfies (3.10) then $\text{Hol}(D)$ is not a Lie group.
Proposition 3.11. If no factor $D_i$ satisfies (3.10) or is a hermitian symmetric space then $\text{Hol}(D) \cong \text{Hol}(D_1) \times \cdots \times \text{Hol}(D_N)$.

Proof. By the same argument as in 2.5, $\text{Hol}(D)$ is finite dimensional. We look for representations $E$ such that $\text{Hom}_q(E, g/q) \neq 0$. The only such irreducible representations $E$ are $E \cong g_\beta$ and $E \cong g_1^1$, the $g_i^1$ occurring when $Z_i$ appears in Table 1.1. It follows that $\text{Hol}(D) = \text{Hol}(D_1) \times \cdots \times \text{Hol}(D_n)$.

§4. The main theorem has several consequences in representation theory. As mentioned earlier, interesting representations are those occurring as Dolbeault cohomology of a line bundle over an open orbit $D$. These are the representations associated to elliptic co-adjoint orbits by the orbit method. See for example [14], [15], [16] and [18] for some of the important properties of these representations. Suppose we have a holomorphic homogeneous line bundle $L_\chi \to G_0/L_0$ corresponding to the character of $L_0$ whose differential is $\chi \in i t_0^\ast$. If

\begin{equation}
\langle \chi + \rho, \beta \rangle < 0, \text{ for all } \beta \in \Delta(u)
\end{equation}

(where $\rho$ is half the sum of the positive roots) then the Dolbeault cohomology space $H^p(G_0/L_0, L_\chi) = 0$ for $p \neq s = \dim C(K_0 \cdot z_0)$ and $H^s(G_0/L_0, L_\chi)$ is an irreducible admissible representation. When $G_0$ is non-compact the main theorem gives the cases where $\text{Hol}(D) = G_0^1 \nsubseteq G_0$. From the case-by-case descriptions given in Section 3, it is clear that $D \cong G_0/L_0 \cong G_0^1/L_0^1$ is an open orbit in a flag manifold for $G^1$. The character defining $L_\chi$ extends uniquely to a character of $L_0^1$ whose differential we will call $\chi^1$. Thus the bundles $L_\chi \to G_0/L_0$ and $L_{\chi^1} \to G_0^1/L_0^1$ are holomorphically and $G_0$-equivariantly equivalent, implying the Dolbeault cohomology spaces are $G_0$-isomorphic. Thus, in the range where (4.1) holds, the representations $H^s(G_0^1/L_0^1, L_\chi)$ are irreducible representations of $G_0^1$ which remain irreducible when restricted to $G_0$. Put differently, the irreducible representations $H^s(G_0/L_0, L_\chi)$ of $G_0$ extend to representations of $G_0^1$. This is a somewhat rare phenomenon. Wolf [20] obtained some general results on the irreducibility of a representation when restricted to a subgroup, which have some overlap with this application of our main theorem. Also, see [10] for results on restricting representations of this type. More recently, A. Dvorsky has a number of new results along these lines.

A well-known and important example of the phenomenon discussed above occurs in quantum electrodynamics. A family of “massless” representations of the de Sitter group $(SO(2, 3))$ extend to representations of the conformal group $(SO(2, 4))$, cf. [1]. As these representations can be realized naturally in Dolbeault cohomology (for the open orbits in the first entry of Table 1.1 for $p = 1$ and $q = 1$), the geometric explanation for this extension is an instance of our main theorem.

When $G_0$ is compact, for arbitrary $\chi$, $H^p(G_0/L_0, L_\chi) \cong H^p(G_0^1/L_0^1, L_{\chi^1})$ as $G_0$ representations for all $p$. By the Bott-Borel-Weil theorem we see that certain
representations of $G_0$ remain irreducible when restricted to $G_0$. The following table gives the highest weights $\chi$ and $\chi^1$ of these representations.

| $\mathfrak{g}^1$ | $\chi^1$ | $\mathfrak{g}$ | $\chi$ |
|-----------------|---------|---------------|--------|
| $D_{n+1}$       | $a\lambda_{n+1}$ | $B_n$         | $a\lambda_n$ |
| $A_{2n-1}$      | $a\lambda_1$    | $C_n$         | $a\lambda_1$ |
| $B_3$           | $a\lambda_1$    | $G_2$         | $a\lambda_1$ |

$a \in \mathbb{Z}_+$

Table 4.2

Our results also have implications for the study of deformations of maximal compact subvarieties in $D$, as in [4], [19], [24]. In particular, let $V_0$ be the compact subvariety $K_0 \cdot z_0$ in $D = G_0/L_0$, where $G_0$ is non-compact and $D$ is, as usual, an open orbit in a generalized flag manifold for $G$. The cycle space $M_D$ studied in [19] is the space of translations of $V_0$ by elements of $G$ which remain in $D$: $M_D = \{ g \cdot V \subset D \mid g \in G \}$. The space $M_D$ plays a key role in constructing a transform, often called a Penrose transform, for the representations $H^s(D, L_\chi)$. In most cases, $M_D$ has the dimension that one computes using Kodaira-Spencer theory (cf. [11]). However, in some cases, the Kodaira-Spencer theory predicts a larger space of deformations. For some of these cases (but not all), our results explain the discrepancy by demonstrating that there is a larger group of automorphisms $G_1$ acting on $D$. Compare with [13].

For example, when $\mathfrak{g}$ is of type $C_n$ and $G_0 = \text{Sp}(n, \mathbb{R})$ we have seen that the space of positive lines in $\mathbb{C}P^{2n-1}$ is an open orbit. Then $V_0 \cong \mathbb{C}P^{n-1}$. Kodaira-Spencer theory predicts that the (infinitesimal) deformations of $V_0$ in $D$ come from $H^0(V_0, \mathcal{N})$, where $\mathcal{N}$ is the holomorphic conormal bundle of $V_0$ in $D$. In this example, $H^0(V_0, \mathcal{N}) \cong \{ \text{symmetric } n \times n \text{ matrices} \} \oplus \{ \text{skew-symmetric } n \times n \text{ matrices} \}$ as representations of $K_0$. Let’s denote the deformation space of $[19]$ for the groups $G_0 = \text{Sp}(n, \mathbb{R})$ and $G_1 \cong \text{SU}(n, n)$ acting on $D$ by $M_D$ and $M_D^1$, respectively. One can check that in this case $M_D \cong G_0/K_0$ and $M_D^1 \cong G_1/K_0^1$ (cf. [24]). As $\dim_{\mathbb{C}}(G_0/K_0) = \dim(G_0/K_0) + \frac{n(n-1)}{2}$, the deformation space of Kodaira-Spencer is strictly larger than $M_D$ in dimension. The extra $\frac{n(n-1)}{2}$ dimensions come from the skew matrices $B$ by the action of

$$\exp \left( \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right) \in U(n, n) \ltimes \text{Sp}(n, \mathbb{R})$$
on $V_0$. Note that the infinitesimal deformations predicted by Kodaira-Spencer theory all come from $M_D^1$.

As a final remark, note that the list in [2] of nilpotent co-adjoint orbits ‘sharing’ an orbit with a bigger group has a lot in common with our list. The open orbits we are considering here are $G_0$-equivariantly biholomorphic to the elliptic co-adjoint
orbits. It would be interesting to understand the connection between our list and that of [2] in terms of nilpotent orbits as limits of elliptic orbits.

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