Wigner-Seitz cells in neutron star crust with finite range interactions

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The structure of Wigner-Seitz cells in the inner crust of neutron stars is investigated using a microscopic Hartree-Fock-Bogoliubov approach with finite range D1S and M3Y-P4 interactions. Large effects on the densities are found compared to previous predictions using Skyrme interactions. Pairing effects are found to be small, and they are attenuated by the use of finite range interactions in the mean field.

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I. INTRODUCTION

In recent years, the properties of the inner crust of neutron stars have been investigated within various frameworks, especially focusing on their microscopic structure and superfluid properties. The microscopic calculations of the inner crust matter are usually studied in the Wigner-Seitz (WS) approximation [1, 2]. Following the standard approaches, the inner crust consists of a lattice of WS cells, each cell containing a neutron-rich nucleus immersed in a sea of dilute gas of neutrons and relativistic electrons uniformly distributed inside the cell [2]. The first microscopic calculations of the properties of the inner crust matter were done by Negele and Vautherin in the 70s [1]. The calculations were performed assuming a set of non-interacting cells described in the Hartree-Fock (HF) approach with 11 representative cells. These cells are distributed in different zones of the inner crust with densities covering a range from 1.743 × 10^3ρ_0 to 0.5ρ_0, ρ_0 = 0.16 fm^−3 being the nuclear matter saturation density. The optimal number (N, Z) of neutrons and protons in each cell is obtained by searching for the lowest binding energy satisfying the β-stability condition of the cell.

More recently, the superfluid properties and their influence on the specific heat were investigated in the self-consistent Hartree-Fock-Bogoliubov (HFB) approach [3, 4]. The collective excitations and the cooling time of the inner crust of neutron stars were also studied in the framework of a spherically symmetric HFB + quasiparticle random phase approximation [5, 6] and of HFB approach at finite temperature [7].

All the above calculations were done within the non-relativistic framework with the SLy4 Skyrme interaction in the mean field channel and a density-dependent delta force for the pairing interaction. Later on, the properties of the WS cells have been studied in the non-relativistic work of Baldo et al. [9–11] using an energy functional method including the pairing correlations of protons and neutrons. The relativistic Hartree, or Relativistic Mean Field (RMF) approach [12] has also been applied to investigate the structure of WS cells and the influence of different boundary conditions on the structure of the cells.

However, there is so far no study of the inner crust in the HF approach using finite range interactions. The aim of this paper is to examine the behaviour of density distributions of protons and neutrons in the different WS cells obtained with a finite range interaction within a self-consistent HF-BCS calculation, where the HF equations are solved with the Dirichlet-Neumann mixed boundary conditions. For the numerical studies one obvious choice is the D1S interaction [13] which is widely employed in finite nuclei calculations. More recently, the M3Y-P4 interaction proposed by Nakada [14] seems to work satisfactorily well for describing nuclear ground states in the HFB framework. This finite-range effective interaction derives from the well-known M3Y interaction and it is interesting to study its predictions for neutron stars.

The structure of this paper is as follows. In Section II we present the HF and HF-BCS approaches using finite-range density-dependent interactions in both the mean field and pairing channels. The results of the calculations are presented in Section III. The evolution of the neutron and proton densities as well as the neutron pairing fields in the regions of the inner crust are discussed. The summary and perspectives of this study are given in Section IV.

II. HF-BCS CALCULATIONS WITH FINITE RANGE INTERACTIONS

A practical difficulty of HF calculations in WS cells is that the cell radius can be large (up to 40-50 fm). Solving directly the integro-differential equations in coordinate space can lead to inaccuracies and instabilities [15]. The method that we adopt consists in solving the HF equations in a basis well adapted to each WS cell under study. In subsection A we derive the HF equations in coordinate space, and in subsection B we give some details on their solution.
A. HF equations in coordinate space

We summarize here the analytical expressions needed for HF calculations with a finite-range interaction, taking as an illustrative example the Gogny interaction which contains a sum of two Gaussians, a zero-range density-dependent part and a zero-range spin-orbit part. Expressions for other types of finite-range interactions can be easily deduced from this case. For example, the M3Y-Pn interactions of Nakada have zero-range, density-dependent terms just like Gogny and Skyrme interactions and therefore, the corresponding contributions to the mean field potentials can be deduced from the expressions given here. In addition, the M3Y-Pn interactions also contain tensor terms, except for the M3Y-P4 interaction that we use in this work and therefore, we do not discuss the tensor contributions. Finally, the original M3Y-P4 interaction contains short-range spin-orbit components (with $\mu = 0.25 fm$ and $0.40 fm$, see Eq. 1) that we approximate for simplicity by a zero-range spin-orbit component as in the Gogny force. The corresponding spin-orbit strength $W_0 = 160 MeV fm^3$ is adjusted on the empirical $1p1/2 - 1p3/2$ proton splitting in $^4O$ as in [10]. Throughout this work we limit ourselves to spheroidal densities ranging from $1.410^{-3} \rho_0$ to about $0.5 \rho_0$, where $\rho_0 = 0.16 fm^{-3}$ is the nuclear matter saturation density.

In order to obtain the Hartree and Fock potentials, we use the multipole expansion of the Gaussian form factor [15]:

$$e^{-\frac{r_1 - r_2}{\rho_0^2}} = 4\pi \sum_{LM} (-)^M \psi_L^M(r_1, r_2) Y_L^M(r_1) Y_L^M(r_2).$$

We express the central part of the Gogny force in the form

$$V(|r_1 - r_2|) = 4\pi \sum_{S, l, j} \sum_{\nu=1}^{2} A_\nu(S)(-1)^{L+S+J+M} \psi_L^M(r_1, r_2)$$

$$\left( T_{(1)}^{(SL)J} \chi_{(2)}^{(SL)J} \right),$$

where

$$A_\nu(S=0) = W_\nu - H_\nu P^r + \frac{B_\nu - M_\nu P^r}{2},$$

$$A_\nu(S=1) = \frac{B_\nu - M_\nu P^r}{2}.$$ (3)

Here, $W_\nu$, $B_\nu$, $H_\nu$, $M_\nu$, $\mu_\nu$ are the parameters of the Gogny interaction or appropriate combinations of the parameters of the M3Y-Pn interactions, $P^r$ are the spin and isospin exchange operators, respectively. We have introduced the tensors $T_{(\mu)}^{(SL)J}$ which are tensorial products of a spherical harmonic $Y_L^M$ with a Pauli spin matrix

$$T_{(SL)J}^{(S')} = [\sigma_{ZZ} \otimes Y_L^M].$$ (4)

Because of the spherical symmetry assumption, the single-particle wave functions $\varphi_i(r, \sigma, q)$ can be factorized into a radial part $u_i(r)$, a spin-angular part $y_{ljm}(\hat{r}, \sigma)$, and an isospin part $\chi_m(\tau)$:

$$\varphi_i(r, \sigma, q) = \frac{u_{a}(r)}{r} y_{ljm}(\hat{r}, \sigma) \chi_{m}(\tau)$$ (5)

where

$$y_{ljm}(\hat{r}, \sigma) \equiv \sum_{m_s} <\frac{1}{2} m_s | j m > Y_{lm, i}(\hat{r}) \chi_{m_s}(\sigma),$$

$\chi_{m_s}(\sigma)$ being a spinor corresponding to a spin projection $m_s$, and the index $i$ stands for the following set of quantum numbers: the charge $q$, the principal quantum number $n$, the orbital angular momentum $l$, the total angular momentum $j$, and the magnetic quantum number $m$.

Using the tensors $T_{(\mu)}^{(SL)J}$ and the single-particle wave functions $\varphi_i(r, \sigma, q)$ we can calculate the direct (Hartree) and exchange (Fock) potentials in each $(l, j)$ partial wave. With the help of the results (A1) of Appendix A it is straightforward to obtain the radial integro-differential HF equations in coordinate space for the radial wave functions $u_i(r_1)$:

$$\frac{\hbar^2}{2m} [ -u''_{i}(r_1) + \frac{l_i (l_i + 1)}{r_1^2} u_i(r_1)] + U_i^D(r_1) u_i(r_1) - \int U_i^E(r_1, r_2) u_i(r_2) r_2^2 dr_2$$

$$+ \left[ j_i (j_i + 1) - l_i (l_i + 1) - \frac{3}{4} W_q^{LS0}(r_1) u_i(r_1) = \epsilon_i u_i(r_1)$$ (7)

where the local central potential $U_i^D$ contains the direct contributions of the finite range forces (nuclear and Coulomb) as well as the direct+exchange contributions of the zero-range density-dependent forces, while the direct+exchange contributions of the zero-range spin-orbit force are in both the $W_q^{LS1}$ component of $U_i^D(r)$ and in the one-body spin-orbit potential $W_q^{LS0}$:

$$U_i^D(r_1) = U_i^H(r_1) + U_q^{DD}(r_1) + V^{DC}(r_1) + W_q^{LS1}(r_1).$$ (8)
The non-local potential $U^E$ is composed of the exchange contributions of all finite range (nuclear and Coulomb) forces:

$$U^E(r_1,r_2) = U^F_i(r_1,r_2) + V^{EC}_i(r_1,r_2).$$  \hfill (9)

The corresponding expressions are given in Appendix A.

B. Solving HF equations in a basis representation

Solving the HF equations directly in coordinate space has several advantages: the results do not depend on the choice of a basis and its truncation, and the individual wave functions have a correct asymptotic behavior. However, in the regions of the inner crust where the density is below $10^{-2}\rho_0$ the radius of the WS cells can be larger than 40 fm and strong numerical instabilities may appear for single-particle wave functions with large orbital momentum ($l \sim 15 - 20$). The cause of the problem is that the HF wave functions behave like $r^{l+1}$ near the origin. Calculations requiring a certain number of nodes can fail if the HF potential is not accurate enough at a given iteration. To avoid this difficulty we have developed a basis expansion method [15] for solving the HF equations as a matrix diagonalization problem. The advantage of the basis expansion method is that it produces a high accuracy (measured by the orthogonality of the solutions) with a smaller number of points, while this may not be obtained with the coordinate space method.

The harmonic-oscillator basis functions is the most common choice for expanding the HF single-particle wave functions. Here, we choose instead, as a basis, the spherical Bessel functions normalized inside the Wigner-Seitz cell sphere.

The radial part $\frac{u_{nlj}(r)}{r}$ of the single-particle wave function in Eq. (3) can be expanded on the normalized spherical Bessel functions as

$$\frac{u_{nlj}(r)}{r} = \sum_{i=1}^{N} C_{nl,j} \tilde{j}_i(k_i^{(l)} r)$$  \hfill (10)

where $N$ is the dimension of the basis and $\tilde{j}_i(k_i^{(l)} r)$ the normalized spherical Bessel functions with boundary conditions that will be specified later. The single-particle wave function $\varphi_i(r,\sigma,q)$ of Eq. (5) can be expanded as

$$\varphi_{nljm}(r,\sigma,q) = \sum_{i=1}^{N} C_{nljm,i} \psi_{nljm,i}(r,\sigma,q),$$  \hfill (11)

where the orthonormal basis is

$$\psi_{nljm,i}(r,\sigma,q) = \tilde{j}_i(k_i^{(l)} r)\tilde{y}_{ljm}(\hat{r},\sigma)\chi_q(r).$$  \hfill (12)

Under the $l$ and $j$ conservation the HF Hamiltonian matrix is decomposed into $(l,j)$ blocks:

$$H^{(l,j)}_{ii'} = \langle \psi_{ljm,i} | H | \psi_{ljm,i'} \rangle ,$$  \hfill (13)

where $H$ stands for the HF Hamiltonian. The HF equations then become:

$$\sum_{i'=1}^{N} H^{(l,j)}_{ii'} C_{nlj,i'} = \epsilon_{nljm} C_{nlj,i}$$  \hfill (14)

where $H^{(l,j)}_{ii'}$ is a symmetric $N \times N$ matrix,

$$H^{(l,j)}_{ii'} = \frac{\hbar^2}{2m} (k_i^{(l)})^2 \delta_{ii'} + \iint V^{HF}_i(r_1,r_2) \tilde{y}_j(k_i^{(l)} r_1) \tilde{y}_j(k_i^{(l)} r_2) dr_1 dr_2 .$$  \hfill (15)

Here, the potentials $V^{HF}_i(r_1,r_2)$ can be obtained from the formulas (A1 - A6) of Appendix A. The HF equations are solved by an iterative procedure, starting from an initial Woods-Saxon potential. In the present calculation we use a basis size N=20. The convergence criteria is set on the single-particle energies $\epsilon_{nljm}$ to be 1 keV.

The HF calculations are done with finite-range interactions (the D1S [13] and M3Y-P4 [14] forces) imposing Dirichlet-Neumann boundary conditions at the edge of the cell as introduced in Ref. [3]. These boundary conditions for the single-particle wave functions are taken as follows: (i) the even parity wave functions vanish at the edge $r = R_{WS}$ of the box; (ii) the first derivative of the odd-parity wave functions vanish at $r = R_{WS}$. The purpose of these chosen boundary conditions is to obtain an approximately constant density at large distance from the center of the cell, thus simulating a lattice of nucleus-like systems embedded in a uniform neutron gas.

Of course, these boundary conditions are somewhat arbitrary and one could as well choose alternative boundary conditions. Thus, another kind of boundary conditions could be chosen in the following way: odd $l$ wave functions vanish at $r = R_{WS}$, the first derivatives of even $l$ wave functions vanish at $r = R_{WS}$. As shown in Ref. [11], the two kinds of boundary conditions can be used in the calculations of the neutron star inner crust. The difference of the binding energies per nucleon for each cell will increase with the increasing density (see Table 1 of Ref. [11]). However, the values of these uncertainties are smaller than the variations of the equilibrium configuration connected with the pairing effects [9, 11]. Therefore, in our study the boundary conditions and the spherical WS cell structure are kept the same as in Ref. [1]. We do not redetermine the number $(N,Z)$ of neutrons and protons in the considered cells. This study can be envisaged in the near future.

It should be noted that the Bloch boundary conditions can be used at the cost of more complex calculations as presented in Refs. [19, 21]. The validity in neutron star crust of the Wigner-Seitz approximation that we use here has been discussed in [19].
C. Pairing correlations in BCS approximation

The effects of pairing correlations are known to be substantial in the inner crust \[3, 4\] and it is necessary to include them in order to obtain a more realistic picture. In this first study of the inner crust with finite range interactions in the particle-hole (mean field) channel we adopt the simplified BCS picture for describing the neutron pairing fields, but the HF-BCS model remains completely self-consistent. Indeed, the BCS occupation numbers are recalculated at each HF iteration.

Various effective interactions can be used in the pairing channel. In this work the pairing correlations are treated in the BCS approximation with both the zero-range density-dependent and finite range density-dependent forces in the pairing channel.

The importance of the density dependence of the pairing interaction is well known in the theories of superfluidity in neutron stars. As shown in Ref. \[2\], it is impossible to deduce the magnitude of the pairing gaps in neutron stars with sufficient accuracy. The calculation of the \(^1S_0\) pairing gaps in pure neutron matter, or symmetric nuclear matter based on bare \(NN\) interaction depends strongly on the forces that are used.

In this work, we compare the results of the zero-range and finite-range interactions as pairing interactions. For the finite-range interactions, the effective Gogny D1S \[13\] or M3Y-P4 \[14\] interactions are used to calculate the pairing field.

For the zero-range force in the pairing channel, we use the form

\[ V(r_1 - r_2) = V_0 \left( 1 - \eta \frac{\rho(r)}{\rho_0} \right)^\alpha \delta(r_1 - r_2) \quad (16) \]

where \(V_0, \eta\), and \(\alpha\) are parameters that can be adjusted. We have studied the predictions of two types of pairing forces: a density-independent interaction (\(\eta = 0\)) that gives rise to volume pairing and a density-dependent delta force that can give rise to surface pairing. In Appendix B, we show the details of the calculations of pairing matrix elements.

Up to now, the magnitude of pairing correlations in neutron matter is still a subject of debate. The D1S Gogny interaction commonly used in finite nuclei calculations gives a maximum value of pairing gap in infinite neutron matter of about 2.4 MeV at a Fermi momentum \(k_F \approx 0.8 \text{ fm}^{-1}\) \[22\]. The maximum value of pairing gap in infinite matter using the M3Y-P4 interaction is about 3.2 MeV at the similar Fermi momentum \[14\]. On the other hand, the microscopic calculations of Refs. \[23, 24\] predict for the maximum gap a value of about 1 MeV. One observes that there are three different scenarios for pairing correlations in neutron matter. In our study, we use the density dependent delta force for the pairing interaction to simulate the third scenario. This is obtained by adopting the parameter values of Ref. \[3\]: \(V_0 = -330\) MeV fm\(^{-3}\), \(\eta = 0.7\), and \(\alpha = 0.45\). Thus, for each WS cell we perform three HF-BCS calculations with three different pairing forces. The BCS pairing window is chosen to be \(\pm 6\) MeV around the Fermi level.

### Table I: The Wigner-Seitz cells considered in this work. \(\rho, N, Z\) and \(R_{WS}\) are the baryonic densities, the numbers of neutrons and protons, and the WS cell radii, respectively. All values are taken from Ref. \[1\].

| \(N_{zone}\) | \(\rho/\rho_0\) | \(N\) | \(Z\) | \(R_{WS}[\text{fm}]\) |
|-----|-------------|-----|-----|----------------|
| 10  | 0.143 \times 10^{-2} | 140 | 40  | 53.6 |
| 9   | 0.250 \times 10^{-2} | 160 | 40  | 49.2 |
| 8   | 0.375 \times 10^{-2} | 210 | 40  | 46.3 |
| 7   | 0.549 \times 10^{-2} | 280 | 40  | 44.3 |
| 6   | 0.994 \times 10^{-2} | 460 | 40  | 42.2 |
| 5   | 0.233 \times 10^{-1} | 900 | 50  | 39.3 |
| 4   | 0.361 \times 10^{-1} | 1050| 50  | 35.7 |
| 3   | 0.557 \times 10^{-1} | 1300| 50  | 33.1 |
| 2   | 1.275 \times 10^{-1} | 1750| 50  | 27.6 |
| 1   | 2.968 \times 10^{-1} | 1460| 40  | 19.6 |
| 0   | 4.931 \times 10^{-1} | 950 | 32  | 14.4 |

III. RESULTS AND DISCUSSIONS

In the present study, we have performed the HF-BCS calculations for a set of 11 representative WS cells determined in Ref. \[1\]. The considered density range is from neutron drip density \(\rho_{min} = 1.743 \times 10^{-3} \rho_0\) to about \(\rho_{max} = 0.5\rho_0\). In this density range, the nuclear clusters are assumed to be spherical \[17\]. Above the density \(\rho_{max}\) the energy per baryon approaches the value of the uniform neutron system and the cells in the inner crust might deviate from the spherical shape \[27\]. Following Ref. \[1\] we denote the WS cells like a nucleus with \(Z\) protons and \(N\) neutrons. The eleven zones of the representative cells of the inner crust with mean densities and corresponding proton number \(Z\) and neutron number \(N\) in each cell are listed in Table I. The decreasing zone number \(N_{zone}\) are from 10 to 0, corresponding to the increasing density from the minimum density \(\rho_{min} = 2.79 \times 10^{-4} \text{ fm}^{-3}\) to the maximum density \(\rho_{max} = 7.89 \times 10^{-2} \text{ fm}^{-3}\). The WS cells are denoted like a nucleus as \(^{180}\text{Zr}, \; ^{200}\text{Zr}, \; ^{220}\text{Zr}, \; ^{320}\text{Zr}, \; ^{500}\text{Sn}, \; ^{950}\text{Sn}, \; ^{1180}\text{Sn}, \; ^{1800}\text{Sn}, \; ^{1350}\text{Sn}, \; ^{1500}\text{Zr}, \) and \(^{983}\text{Ge}\), as introduced in Ref. \[1\]. These \((N,Z)\) values must be considered as indicative since they were determined by using a different model for the HF mean field in Ref. \[1\]. The WS cell radii \(R_{WS}\) are calculated by the following relation

\[ <\rho> = \frac{A}{4\pi R_{WS}^3} \quad (17) \]

where \(\rho\) and \(A\) are the density and mass number of the considered cell, respectively. The values of the radii \(R_{WS}\) are also shown in the last column of Table I.

In isolated finite nuclei, for a given number of protons there is always a maximum number of bound neutrons.
This neutron stability limit defines the neutron drip line. Since the neutron-rich nuclei are quickly beta decaying, then the neutron drip line is usually drastically limited in the laboratory. This is not the case for the neutron-rich systems immersed in the inner crust of neutron stars. In the case of WS cells, the beta decay is blocked by the presence of the degenerate electron gas uniformly distributed inside the cell. Therefore, in this case the nuclei inside the inner crust of neutron stars can bind more neutrons than the nuclei in the vacuum. Furthermore, there are many delocalized neutrons forming a uniform neutron gas and filling the outer region of the WS cells.

A. HF and HF-BCS density distributions

First, we discuss the case of the WS cells calculated in the HF approach without pairing effects. Fig. 1 displays the HF proton and neutron density profiles of \(^{180}\)Zr, \(^{200}\)Zr, \(^{250}\)Zr, \(^{320}\)Zr and \(^{500}\)Zr systems obtained with D1S, M3Y-P4 and SLy4 interactions. One notes that the numerical HF calculations with SLy4 are the same as in Ref. 1. We observe that the HF calculations with the finite-range D1S and M3Y-P4 interactions give very similar results, while the results obtained with SLy4 interaction are different.

We will analyze the case of the cell \(^{180}\)Zr. One knows that the depopulation of the \(s\) state can lead to a depletion of the central density \(22, 23\). In the case of this cell, the \(5s_{1/2}\) neutron state is fully filled with SLy4 interaction and empty with D1S and M3Y-P4 interactions. Thus, the neutron densities in the center of the cell obtained with D1S and M3Y-P4 interactions are smaller by a factor of 2.3 than that obtained with SLy4 interaction. However, the neutron gas density obtained with D1S and M3Y-P4 interactions is more than 1.8 times greater than that of SLy4 interaction. We have checked that if one integrates the neutron density up to \(r = 10\) fm, where the neutron density profile becomes approximately constant, one finds about 80 neutrons and 90 neutrons with D1S (or M3Y-P4) and SLy4 interactions, respectively. Thus, it appears that large surface regions are observed in the case of D1S and M3Y-P4 interactions, and the neutron gas density of the cell \(^{180}\)Zr is much higher. Similar situations happen in two other cells \(^{200}\)Zr and \(^{250}\)Zr. In the case of cells \(^{320}\)Zr and \(^{500}\)Zr, where the \(5s_{1/2}\) neutron state is fully filled, one observes that the nuclear cluster region becomes larger with the SLy4 interaction, while its neutron gas density is smaller by a factor of 2 than those obtained with D1S and M3Y-P4. For the above cells one concludes that the neutron density in the center of the cell becomes smaller with D1S and M3Y-P4 interactions.

However, the situation is opposite at higher densities, such as in the cells \(^{95}\)Sn, \(^{110}\)Sn, \(^{180}\)Sn, \(^{135}\)Sn and \(^{150}\)Zr. These changes can be seen in Fig. 2 where the neutron densities of the nuclear clusters obtained with SLy4 interaction are always smaller than those obtained with D1S or M3Y-P4 interactions. One can also see that the surface thickness of the nuclear cluster with SLy4 interaction becomes larger by about 10% in the cells \(^{95}\)Sn, \(^{110}\)Sn, \(^{180}\)Sn and \(^{135}\)Sn. In the cell \(^{150}\)Zr, the nuclear cluster surface is similar with the three interactions. Since the neutron density in the center of this cell calculated with SLy4 interaction is smaller than that obtained with D1S or M3Y-P4 interactions, then its outer neutron gas density is a little higher. For the highest density corresponding to the cell \(^{98}\)Ge, there is still some trace of a central cluster and of an outer neutron gas in the case of SLy4 whereas this separation fades away with D1S and M3Y-P4 interactions.

As mentioned above, the first microscopic calculations of the WS cells in the inner crust matter were done in Ref. 1. The pairing effects were not taken into account in these calculations because it was assumed that their contributions would be small in comparison with the total binding energy of the considered system. However, calculations using an energy functional involving the neutron and proton pairing correlations by Baldo et al. 14 show that the mean density \(\rho\) of the equilibrium configuration \((Z, R_{WS})\) can be changed significantly due to the pairing effects. We therefore calculate also the WS cells in HF-BCS approximation. The D1S and M3Y-P4 interactions are used to construct the mean field and the pairing field. For comparison we have also considered a hybrid case where a density-dependent delta force is chosen for the pairing interaction and the D1S interaction is used in the mean field channel.

Fig. 3 shows the proton and neutron densities obtained in HF and HF-BCS approximations for 6 WS cells, in which the D1S Gogny interaction is used in both the mean field and pairing channels. One can see that the behaviour of the proton and neutron densities obtained in the two approximations are similar for \(^{95}\)Sn and \(^{110}\)Sn cells corresponding to the range of density \(\rho \sim 2.79 \times 10^{-4}\) fm\(^{-3}\) to \(5.77 \times 10^{-3}\) fm\(^{-3}\). One can conclude that the pairing effects are very small on these cells. For the higher density region \(\rho \sim 8.91 \times 10^{-3}\) fm\(^{-3}\) to \(4.75 \times 10^{-2}\) fm\(^{-3}\), corresponding to cells \(^{135}\)Sn, \(^{180}\)Sn and \(^{150}\)Zr, there are differences between the density profiles due to the pairing effects. This is because the occupancy of the 8\(s_{1/2}\) state is modified in HF-BCS calculations. Thus, the difference between HF and HF-BCS neutron densities at \(r = 0\) is around 10-11% for the three cells above. Indeed, the occupancies of the 8\(s_{1/2}\) neutron orbital are 0.91, 0.98 and 0.95 for cells \(^{135}\)Sn, \(^{180}\)Sn and \(^{150}\)Zr in HF-BCS, respectively, while this state is fully filled in HF calculations of these cells. In spite of the cell \(^{150}\)Zr, the HF-BCS neutron densities of the cells \(^{135}\)Sn, \(^{180}\)Sn have an extended “surface” before they reach the constant values corresponding to the neutron gas. The neutron gas densities of these cells are similar in both HF and HF-BCS approaches. For the highest-density cell \(^{98}\)Ge with \(\rho = 7.89 \times 10^{-2}\) fm\(^{-3}\), the behaviour of the neutron density is slightly changed due to the pairing effects. Although the D1S interaction is used in the pair-
FIG. 1: (Color online) The HF proton and neutron densities in zones 10 to 6 calculated with the Dirichlet-Neuman boundary conditions and using D1S [3], M3Y-P4 [4] and SLy4 [8] interactions.

FIG. 2: (Color online) Same as Fig. 1 but for higher density zones (from zone 5 to zone 0).
ing channel, it cannot produce a constant density around the outer edge of this cell. It seems that the cell 982 Ge most probably belongs to the deformed pasta phase.

Since it is not yet well established what are the pairing properties of neutron matter, we perform again the HF-BCS calculations for all WS cells of the inner crust matter using three different pairing interactions, namely the D1S, M3Y and delta interactions. The results of proton and neutron density distributions obtained with these three pairing interactions are shown in Fig. 3. Different combinations of mean fields and pairing interactions are used in this figure. One observes that the main features of the WS cells can be obtained with three kinds of pairing interactions. At the high density region, except for the cell 1350 Sn where the pairing effects on the neutron density are strong, the calculated proton and neutron densities are similar for the other cells such as 1100 Sn, 1500 Zr, 1800 Sn and 982 Ge. In the case of the 1350 Sn cell, the neutron density distribution of the nuclear cluster calculated with the zero-range pairing force is higher than those obtained with D1S or M3Y-P4 pairing interactions because the occupancy of the 8s1/2 neutron orbital corresponding to the delta-pairing interaction is larger than those of the D1S and M3Y-P4 pairing interactions. In all the cells we observe that the density distribution of the nuclear clusters obtained with the M3Y-P4 interaction is slightly less extended than those obtained with the D1S interaction. This effect may come from the differences of the ranges and the values of pairing gaps in infinite matter corresponding to these two effective interactions.

B. Pairing fields

Finally, we discuss briefly the BCS neutron pairing fields in the case of a delta-pairing interaction. Rewriting Eq. (16) as

$$V(r_1 - r_2) = V_{eff}(\rho(r))\delta(r_1 - r_2),$$  \hspace{1cm} (18)

the pairing field $\Delta(r)$ is a local function which can be expressed in terms of the local pairing density $\kappa(r)$ as

$$\Delta(r) = V_{eff}(\rho(r))\kappa(r),$$  \hspace{1cm} (19)

where the BCS pairing density $\kappa(r)$ is

$$\kappa(r) = \frac{1}{4\pi} \sum_i |u_i v_i | \varphi_i(r)|^2.$$  \hspace{1cm} (20)

Here, the factors $u_i$ and $v_i$ are the usual BCS amplitudes.

The results calculated for the WS cells from zone 10 to zone 1 (using D1S in the particle-hole mean field) are shown in Fig. 3 except for zone 0 which belongs to the deformed pasta phase. In the upper panel of Fig. 3 one can see that the pairing field is becoming very small almost everywhere in the cell. However, one can observe that in passing from the low density region of the neutron gas towards the higher density region of the cluster, the pairing field is increasing in the intermediate density region of the cluster surface in the cells 320 Zr and 500 Zr. This is a manifestation of the bell shape dependence of the pairing gap on density. For the cells corresponding to high baryonic densities shown in the lower panel of Fig. 3, the slope of the pairing field is changing very slowly when it is crossing the region between the nuclear cluster and the uniform neutron gas. Using the same density-dependent delta force in the pairing channel and the D1S force in the mean field channel, the pairing fields of the cells 950 Sn, 1500 Zr and 1800 Sn are about three times smaller (in absolute value) than those obtained by the HFB calculation using the SLy4 interaction in the mean field (see Figs. 5 and 6 in Ref. 3). We see that the pairing correlations are reduced strongly for all the cells in the present work, due to the use of finite range interactions in the mean field.

One can conclude that the behaviour of the pairing field in the inner crust matter is rather complex. On the other hand the magnitude of the pairing field inside the inner crust depends strongly on the scenario used for the pairing properties of infinite neutron matter as shown in Refs. 3, 4.

IV. CONCLUSIONS AND PERSPECTIVES

In this work, we have studied the properties of the WS cells in the inner crust matter of neutron stars using finite-range, density-dependent interactions in HF and HF-BCS approximations. Calculations are performed for 11 representative WS cells by imposing Dirichlet-Neumann boundary conditions at the edge of the cell. The study is done with the D1S and M3Y-P4 effective interactions in both mean field and pairing channels. For the pairing field we have also used a density-dependent, zero-range force whose parameters have been fixed to reproduce the pairing properties of infinite neutron matter as predicted by microscopic calculations which take into account polarization effects 3, 23. For the HF-BCS calculations using the zero-range pairing force, the HF mean field is calculated with the D1S interaction.

With the three different pairing interactions (D1S, M3Y-P4 and delta force), it is found that the behaviour of the proton and neutron density distributions are very similar in the low density region, $\rho \sim 2.79 \times 10^{-4}$ fm$^{-3}$ to $\rho \sim 5.77 \times 10^{-3}$ fm$^{-3}$. One can conclude that the pairing effects are very small in this region. In the higher density region, the pairing effects can make an extended “surface” for the neutron density distribution in the cell before they reach a constant density around the outer edge of the cell. It should be noted that pairing effects are weakened by the use of finite range interactions in the particle-hole (mean-field) channel, compared to the use of Skyrme interactions. However, one cannot obtain a constant neutron gas density in the outer part of the 982 Ge cell with finite-range density-dependent interactions. This cell corresponds to the highest density $\rho$.
= 7.89 × 10^{-2} \text{ fm}^{-3}, \text{ and it seems not to belong to the spherical case as assumed in our study.}

In all the cells we observe that the density distribution of the nuclear cluster obtained with the case of M3Y-P4 interaction is slightly smaller than those obtained with the D1S interaction. This effect may come from the difference of the ranges and the values of pairing gaps in infinite matter given by the two kinds of effective interactions. The largest effect on the densities comes from the use of finite range interactions in the “particle-hole” mean field, compared to the use of Skyrme interaction. This result shows that finite range interactions should be considered in order to describe microscopically Wigner-seitz cells.

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**Appendix A: Hartree-Fock potentials with finite range interactions**

We give here some general expressions of HF potentials corresponding to Gogny-type forces. They can be easily adapted to the case of the M3Y-P4 Nakada’s interaction \cite{nakada} if one adopts a zero-range approximation for the two-body spin-orbit component, as mentioned in subsec. II.A.

The contributions of the central force are separated into a Hartree (direct) and a Fock (exchange) potential:

\[
U^H_i (r_1) = \sum_{j} \sum_{\nu=1}^2 \delta_{j} \left( W_{\nu} + \frac{B_{\nu}}{2} - H_{\nu} \delta_{q_{q},q_{i}} \right) \left( \sum_{j} \frac{1}{2} \right) \int u_j^2(r_2)v_{\nu}^*(r_1,r_2)r_2^2dr_2 ,
\]

\[
U^F_i (r_1, r_2) = \sum_{j} \sum_{\nu=1}^2 \delta_{j} \left( W_{\nu} \delta_{q_{q},q_{i}} - H_{\nu} \delta_{q_{q},q_{i}} - M_{\nu} \right) \int u_j^2(r_2)v_{\nu}^*(r_1,r_2)r_2^2dr_2 ,
\]
The density-dependent component of the force is similar to the zero-range density-dependent term of Skyrme interactions \([3, 20]\), and we recall for completeness the corresponding contribution (direct plus exchange) to the mean field:

\[
U^{DD}_q(r) = \frac{t_3}{2}\{(2 + x_3)(2 + \alpha)\rho^{\alpha+1}(r) - (2x_3 + 1)
\times[2\rho^\alpha(r)\rho_q(r) + \alpha\rho^{\alpha-1}(r)(\rho_q^2(r) + \rho_n^2(r))]\} ,
\]

where \(t_3, x_3\) and \(\alpha\) are the parameters of the density-dependent force, \(\rho\) is the total nucleon density and \(q\) stands for protons or neutrons.

The direct plus exchange spin-orbit mean field is \([3]\):

\[
V_{qLS}^L(r) = W_0 \left\{ \frac{1}{r} \frac{d}{dr} (\rho(r) + \rho_q(r))L.s - \frac{1}{r} J(r) + J'(r) + \frac{1}{r} J_q(r) + J'_q(r) \right\} ,
\]

\[
= W_q^{L0}(r)L.s + W_q^{L1}(r) ,
\]

where \(J(r)\) is the spin density, and \(J' = \frac{dJ}{dr}\).

Finally, the direct and exchange Coulomb mean fields are:

\[
V^{DC}(r_1) = e^2 \sum_{j \in \text{protons}} \hat{L}_j^2 \int u_j^2(r_2)v_0^C(r_1, r_2)r_2^2 dr_2 ,
\]

\[
= e^2 \int \rho_p(r_2)v_0^C(r_1, r_2)r_2^2 dr_2 ,
\]

\[
V^{EC}(r_1, r_2) = e^2 \sum_{j_{iL}} \delta_{q_{i-1/2}q_{i+j_{iL}}}^2 \hat{L}_j^2 \hat{j}_j^2 u_j^* (r_1) u_j (r_2)
\times v_{C}^r(r_1, r_2) \left\{ \begin{array}{cc} l_i & l_j \\ 0 & 0 \end{array} \right\}^2 \left\{ \begin{array}{cc} j_i & j_j \\ 1/2 & 1/2 \end{array} \right\}^2 ,
\]

where \(v_{C}^r(r_1, r_2) = r_{1r_2}^L / r_{1r_2}^{L+1}\) are the multipoles of \(1/|r_1 - r_2|\).

**Appendix B: Pairing matrix elements**

Here, we give the main expressions for calculating the pairing fields in the BCS approximation. The gap equation is \([29]\):

\[
\Delta_b = -\frac{1}{2} \sum_b (-1)^{J_a + 1} j_a - 1 \zeta \hat{G}_0(aabb) \frac{\Delta_{\hat{b}}}{\sqrt{(\varepsilon_b - \lambda)^2 + \Delta_b^2}} ,
\]
FIG. 5: (Color online) The neutron pairing field $\Delta(r)$ of Eq. (19) in zones 10 to 1, calculated with the zero range pairing interaction and the D1S interaction in the particle-hole mean field.

where

$$ G_0(aabb) = \langle aa|V_p(1,2)|bb>_{00} \tag{B2} $$

is the $J = 0$ particle-particle matrix element of the $V_p(1,2)$ pairing interaction.

1. Particle-particle matrix elements with a finite-range interaction

For any interaction of the D1S or M3Y-P4 type the general particle-particle matrix element of the finite range part is:

$$ <ac|V_p(1,2)|bd>_{JM} = \int r_1^2 dr_1 r_2^2 dr_2 \sum_{\nu LSK}(−1)^\nu \times A_\nu(S)v_\nu^L(r_1, r_2)R_a(r_1)R_b(r_1)R_c(r_2)R_d(r_2) \times \left\{ \begin{array}{c} j_d & j_b & J \\ j_a & j_c & K \end{array} \right\} y_a\|T^{(SL)K}_{(1)}\|y_b > \times <y_c\|T^{(SL)K}_{(2)}\|y_d> \tag{B3} $$

Here, the total phase is $P = L + S + K + j_b + j_c$, $A_\nu(S)$ is given by Eq. (3), $R_i(r)$ is the radial part of the single-particle wave function of Eq. (4), and

$$ <y_i\|T^{(SL)K}_{(1)}\|y_j> \text{ is the reduced matrix element:} \tag{B4} $$

$$ \langle i_j i_j |T^{(SL)J}_{LJM}\|i_{j_i j_i} \rangle = (−1)^{j_i} \frac{\sqrt{2}}{\sqrt{4\pi}} \hat{S}_L \hat{J}_i \hat{J}_j \hat{J}_j \begin{pmatrix} l_i & l_i & L \\ j_i & j_i & J \end{pmatrix} \begin{pmatrix} l_i & l_i & L \end{pmatrix} \frac{1}{2} \frac{1}{2} S $$

For the pairing matrix elements $<aa|V_p(1,2)|bb>_{00}$ we thus obtain:

$$ <aa|V_p(1,2)|bb>_{00} = \hat{s}_a^{-1} \hat{s}_b^{-1} \int r_1^2 dr_1 r_2^2 dr_2 \sum_{\nu LSK} A_\nu(S) \times v_\nu^L(r_1, r_2)R_a(r_1)R_b(r_1)R_c(r_2)R_d(r_2) \times (−1)^{L+S} \times <y_a\|T^{(SL)K}_{(1)}\|y_b> \tag{B5} $$

The zero-range part of the Gogny or M3Y-P4 interactions does not contribute to the pairing matrix elements, while the small contribution of the zero-range spin-orbit component to the pairing field is neglected.

2. Particle-particle matrix elements with a zero-range interaction

The density-dependent delta interaction is taken of the form:

$$ V_p(1,2) = V_0 \left(1 - \eta \left(\frac{\rho(r_12)}{\rho_0}\right)^\alpha \right) \delta(r_1 - r_2) \tag{B6} $$
The delta function is expanded on the basis of spherical harmonics:

$$\delta(r_1 - r_2) = \frac{\delta(r_1 - r_2)}{r_1 r_2} \sum_{L \mu} (-1)^{\mu} Y_L^\mu(1)Y_L^{-\mu}(2)$$  \hspace{1cm} (B7)

The calculation of the coupled matrix element is similar to that of subsection B.1. The result is:

$$<aa|V_p(1,2)|bb>_{00} = \frac{1}{2} (-1)^{l_a + l_b} \hat{a}_a \hat{b}_b \int \frac{R_a^2(r)}{4\pi} \int \frac{R_b^2(r)}{4\pi} dr.$$  \hspace{1cm} (B8)

where

$$J_{aabb} = V_0 \int \left(1 - \eta \left(\frac{\rho(r)}{\rho_0}\right)^\alpha \right) R_a^2(r)R_b^2(r) dr.$$  \hspace{1cm} (B9)

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