Efficient creation of multipartite entanglement in flux qubits

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Received 29 November 2009, in final form 23 February 2010
Published 22 June 2010
Online at stacks.iop.org/Nano/21/274015

Abstract

We investigate three superconducting flux qubits coupled in a loop. In this setup, tripartite entanglement can be created in a natural, controllable, and stable way. Both generic kinds of tripartite entanglement—the W type as well as the GHZ type entanglement—can be identified among the eigenstates. We also discuss the violation of Bell inequalities in this system and show the impact of a limited measurement fidelity on the detection of entanglement and quantum nonlocality.

1. Introduction

Entanglement is one of the most intriguing consequences of quantum mechanics [1] and has long been debated in a case against its completeness [2]. Entanglement manifests the nonlocality of quantum mechanics. Bipartite entanglement is well established in optical systems, see, e.g., [3–5], and has recently been verified in superconducting circuits [6, 7]. Quantum information processing uses entanglement as a resource [8, 9], i.e., the preparation of entangled states is essential for quantum computing.

In multipartite systems, different types of entanglement can be classified. Specifically, in a tripartite system, one can recover the same type of bipartite entanglement as in a two-particle system—the measurement of one of the particles reduces entanglement by taking out the particle being measured but leaves the entanglement between the remaining particles intact. On the contrary, there can also be tripartite entanglement where a single one-particle measurement completely destroys entanglement between all parties.

Quantum computing and control is very mature in liquid-state nuclear magnetic resonance and in atomic and optical physics. In the former, due to the use of pseudopure states, it is not obvious that strong entanglement can be created. Creation of entangled states of ions, neutral atoms, and photons [3–7] has been highly successful. The required experiments, though, become increasingly complex, caused by the very weak interaction of the objects to be entangled, which often is also only effective for a very short time.

On the other side, condensed matter systems typically have strong inter-particle interactions which can act effectively because the particles occupy fixed positions in space, e.g., in a lattice. Thus, entangled states are not that exotic in correlated condensed systems [10]. Going beyond the simplest system, namely the spin singlet, we consider a single triangle of spins with antiferromagnetic Ising coupling. Without external field, the ground state of the system is spanned by the three degenerate frustrated states in which the orientation of one of the spins differs from the other two. Creating a superposition of these states right away leads to maximally entangled Werner (W) states [11] which only contain bipartite entanglement. These superpositions can be created in the quantum version of this model, i.e., if the system is put into a transversal magnetic field. Spin Hamiltonians with ‘designed’ properties can be implemented using macroscopic devices such as superconducting qubits rather than elementary spins [12, 13]. In particular, superconducting flux qubits allow for a strong inductive qubit–qubit interaction.

The paper is organized as follows. In section 2 we propose a design for the coupling of three flux qubits. It provides a strong interaction whose strength can be designed over a
large range during fabrication. The form of the eigenstates is discussed in section 3; states in the proximity of Greenberger–Horne–Zeilinger (GHZ) states [14] as well as W states can be identified. Section 4 describes the characterization and detection of tripartite entanglement in this system. In section 5, we discuss the violation of Bell inequalities by the superconducting gap, the charging and Josephson energy of the junctions\[16\]. As the potential energy is dominated by the Josephson energy from the geometrical inductance of the qubits, \(\alpha\) is restricted by the mutual geometric inductances. However, a small self-inductance results in a very weak inductive coupling. This makes simple coupling schemes, like placing disconnected qubits close to each other, insufficient for the calculation of the coupling strength and neglect the inductive energy from the geometrical inductance of the triangle and the kinetic inductance of the shared lines. For convenience, we will consider the Josephson energy of the Qubit junctions (the ones on the outer edges of the triangle) and the large shared junctions separately.

The total Josephson energy of the qubit junctions is given by

\[ E_{\text{Jos}, Q} = -\frac{\hbar}{2e^2} \sum_{i=1}^{3} (\cos \phi_i^{(1)} + \cos \phi_i^{(2)} + \alpha \cos \phi_i^{(3)}). \]  

(1)

Combining the additional phases across the shared junctions into the phase \(\phi^{(i)}\) (e.g. \(\phi^{(1)} = \phi_s^{(1)} - \phi_s^{(2)}\) for qubit 1), the fluxoid quantization reads

\[ \phi_i^{(1)} + \phi_i^{(2)} + \phi_i^{(3)} + \frac{2\pi \Phi_i^{(i)}}{\Phi_0} = 0, \]  

(2)

where \(\Phi_0\) denotes the externally applied flux.

The coupling junctions are large compared to the qubit junctions and their critical currents are far above the persistent currents \(I_p\) in the qubits. Hence, their phases are small and we obtain for first order (for qubit 1)

\[ \phi^{(i)} \approx \frac{2I_p^{(i)} - I_p^{(2)} - I_p^{(3)}}{I_{C,S}}. \]  

(3)

where the critical current \(I_{C,S}\) is assumed to be equal for all three shared junctions.

In order to separate the effect of \(\phi^{(i)}\), we solve the fluxoid quantization for, e.g., \(\phi_s^{(i)}\)

\[ \alpha \cos \phi_s^{(i)} = \alpha \cos \left( \frac{2\pi \Phi_s^{(i)}}{\Phi_0} + \phi_1^{(i)} + \phi_2^{(i)} \right) + \frac{I_p^{(i)}}{I_{C,S}} \phi^{(i)}, \]  

(4)

where we again expanded to first order in \(\phi^{(i)}\) and used \(2\pi \Phi^{(i)}/\Phi_0 \approx \pi\) as well as the relation \(\phi_1^{(i)} = \phi_2^{(i)} = \pm \phi^*\) with \(\cos \phi^* = 1/2\alpha\) [17] for the minima of the potential landscape of a single qubit. Equation (1) then can be rewritten as

\[ E_{\text{Jos}, Q} = E_{\text{Jos, unc p}} - \frac{\hbar^2}{2e^2E_{C,S}} \sum_{i=1}^{3} I_p^{(i)} + \frac{\hbar^2}{2e^2E_{C,S}} \sum_{i=1}^{3} \sum_{j \neq i} I_p^{(i)} I_p^{(j)} \]  

(5)

with the Josephson energy of the uncoupled qubit system \(E_{\text{Jos, unc p}}\). This contribution to the coupling is thus found to be pairwise antiferromagnetic, with a larger shared junction resulting in a weaker coupling.

The second major contribution to the potential energy is the Josephson energy of the shared junctions\[5\],

\[ E_{\text{Jos, S}} = -E_{C,S} \sum_{i=1}^{3} \cos \phi_s^{(i)} \approx -\frac{E_{C,S}}{3 \sum_{i=1}^{3}} \left( 1 - \frac{\phi_s^{(i)}^2}{\Phi_0^2} \right). \]  

(6)

Using \(\phi_s^{(i)} = (I_p^{(i)} - I_p^{(3)})/I_{C,S}\) etc, it reads

\[ E_{\text{Jos, S}} = -3E_{C,S} + \frac{\hbar^2}{4e^2E_{C,S}} \sum_{i=1}^{3} I_p^{(i)^2} \]  

\[ - \frac{\hbar^2}{4e^2E_{C,S}} \sum_{i=1}^{3} \sum_{j \neq i} I_p^{(i)} I_p^{(j)} \]  

(7)

5 Note that we expand to second order in \(\phi_s^{(i)}\) here; this is justified by \(E_{I,S}/E_1 = I_{C,S}/I_C\) which makes (6) and (1) of the same order in \(I_C/I_{C,S}\).
yielding a ferromagnetic coupling with half the strength of the qubit term.

Hence the total coupling strength is given by

$$\Delta U^{(i)} = C^{(i)} = \frac{\hbar^2}{4e^2E_{1S}} I_{p}^{(i)} I_{p}^{(j)} + \mathcal{O}[(E_3/E_{1S})^2].$$

Expressing the persistent currents in the qubits in terms of the Josephson energy of the qubit junctions, \( I_p = I_c \sqrt{1 - 1/4\alpha^2} = 2eE_{1S}/\hbar \cdot \sqrt{1 - 1/4\alpha^2} \), the coupling is seen to depend directly on the size ratio \( r = E_{1S}/E_{1S} \) of the qubit’s and shared junctions,

$$C = rE_{1S}\left(1 - \frac{1}{4\alpha^2}\right).$$

In experiments with inductive coupling [16] or shared lines [18] between the qubits, the coupling is very small \((C/E_{1S} \approx 0.1\%-0.5\%)\), reflecting the small mutual inductance (geometric and/or bulk superconducting kinetic inductance) that can be achieved in such designs. In contrast, (9) allows for coupling strengths which are one order of magnitude stronger. The shared junctions are still large and linear: for a size ratio \( r \) of e.g. 1%-5%, the shared junctions basically behave like a linear inductor, \( L_{3S} = B_0/(2\pi I_{cS} \cos \phi_0) \approx \hbar^2/(4e^2E_{1S}) \). Thus, the shared junctions can be viewed as ultra-compact inductors making the qubit loops equivalent to the standard three-junction flux qubit with a moderate inductance [15, 19].

The currents in the qubits are quantum mechanically associated with \( \hat{\sigma}_z \) operators and the effective Hamiltonian reads in terms of the Pauli spin matrices

$$H = \sum_{i=1}^{3} \left( -\frac{1}{2} \epsilon_i \hat{\sigma}_z^{(i)} - \frac{1}{2} \Delta_i \hat{\sigma}_x^{(i)} + \sum_{j \neq i} J_{ij} \hat{\sigma}_z^{(i)} \hat{\sigma}_z^{(j)} \right).$$

The energy bias \( \epsilon \) can be tuned by the externally applied magnetic field. In this case, the antiferromagnetic coupling can be disregarded. The other excited states are accordingly the classical frustrated states with the three states \(|↑↑↑⟩, |↑↑↓⟩, \text{and } |↓↓↑⟩ \) being degenerate, as well as \(|↓↓↑⟩, |↑↑↑⟩, \text{and } |↑↑↓⟩ \).

Going to finite energy bias, we need to take the coupling into account \((C = 1.4\Delta \text{ in the following})\). The antiferromagnetic coupling energetically favors frustrated states. For \( \epsilon = 0 \), the ground state \(|E_1⟩\) therefore contains a larger contribution of frustrated states and a smaller contribution of polarized states,

$$|E_1⟩ = \frac{1}{\sqrt{6 + 2\delta^2}} \{|f_{\text{flu}}⟩ + \delta(|↑↑↑⟩ + |↓↓↓⟩)\},$$

where \(|f_{\text{flu}}⟩\) denotes the equal (non-normalized) superposition of all frustrated states

$$|f_{\text{flu}}⟩ = |↑↓↓⟩ + |↑↑↓⟩ + |↑↑↓⟩ + |↓↓↑⟩ + |↓↓↓⟩ + |↓↑↑⟩.$$ \hspace{1cm} (12)

and \( \delta \approx 0.2 \) is small, i.e. the aligned states \(|↑↑↑⟩ \) and \(|↓↓↓⟩ \) are suppressed.

The highest excited states \(|E_3⟩ \) and \(|E_4⟩ \) show the opposite behavior and consist for \( \epsilon = 0 \) mainly of a superposition of distinct polarized states,

$$|E_3⟩ = \frac{1}{\sqrt{2 + 6\delta_1^2}} \{|↓↓↓⟩ + |↑↑↑⟩ - \delta_1|f_{\text{flu}}⟩\}.$$ \hspace{1cm} (13)

with \( \delta_1 \approx 0.07 \), and

$$|E_4⟩ = \frac{1}{\sqrt{2 + 6\delta_2^2}} \{|↓↓↓⟩ - |↑↑↑⟩ + \delta_2|f_{\text{flu}}⟩\}.$$ \hspace{1cm} (14)

with \( \delta_2 \approx 0.1 \). Thus, we find these states to be in the proximity of GHZ states, commonly represented by \(|\text{GHZ}⟩ = (|↓↓↓⟩ \pm |↑↑↑⟩)/\sqrt{2} \).
Due to the large antiferromagnetic coupling, the eigenstates discussed above evolve into classical states already for small detuning of the energy bias. However, there are more regimes of tripartite entangled states among the spectrum of eigenstates: at finite positive and negative energy bias $\epsilon = \pm \epsilon * \approx \pm 2.6 \Delta$ two more anticrossings involving $|E_2\rangle$ and $|E_3\rangle$ are present. The explicit form of the state forming the lower branch at $\epsilon = -\epsilon *$ reads

$$|E_2\rangle_{-\epsilon} = \frac{1}{2\sqrt{1 + \delta^2}} \left( |\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\downarrow\downarrow\uparrow\rangle - |\uparrow\downarrow\downarrow\rangle \right) + \delta (|\downarrow\downarrow\downarrow\rangle + |\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle),$$

(15)

where $\delta \approx 0.09$. Therefore, this state is close to

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle),$$

(16)

and

$$|\text{W}\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$$

As $|\text{GHZ}\rangle$ and $|\text{W}\rangle$ can be transferred onto each other by purely local operations, they have identical entanglement properties.

The dash–dot lines in figure 2 indicate two two-fold degenerate subspaces. It is shown in the appendix that arbitrary states in these two-dimensional Hilbert spaces can be prepared by coupling these subspaces to the ground state via resonant driving. The subspaces are spanned by frustrated states.

Since any frustrated state $|f\rangle$ is an eigenstate of the coupling operator, $\hat{a}_c^{(1)}\hat{a}_c^{(2)} + \hat{a}_c^{(1)}\hat{a}_c^{(3)} + \hat{a}_c^{(2)}\hat{a}_c^{(3)}|f\rangle = -|f\rangle$, the form of an eigenstate prepared in these subspaces does not change with the coupling strength. Among the eigenstates, W type states can be found, with the maximally entangled state (with respect to the measure of global entanglement [23]) in the lower energy subspace for $\epsilon = 0$ taking the form

$$|\psi_{\text{max}}\rangle = \frac{1}{2\sqrt{6}} \left( 2(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle) - (1 - i\sqrt{3})(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle) - (1 + i\sqrt{3})(|\uparrow\uparrow\downarrow\rangle + |\downarrow\downarrow\downarrow\rangle) \right).$$

(17)

The local transformation $\hat{U}_c^{|\psi_{\text{max}}\rangle} = |W\rangle$ rotating $|\psi_{\text{max}}\rangle$ onto the common representation of a W state, $|W\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle)$, is given by

$$\hat{U}_c = \exp(-i\pi/3 \hat{R}_z(\pi/2)) \otimes (\hat{R}_z(2\pi/3)\hat{R}_z(\pi/2)) \otimes (\hat{R}_z(-2\pi/3)\hat{R}_z(\pi/2))^3,$$

(18)

where $\hat{R}_z(\theta)$ is a rotation around the z-axis (y-axis) by an angle $\theta$. Thus, we can see explicitly that the state $|\psi_{\text{max}}\rangle$ is indeed locally equivalent to a W state.

With respect to the occurrence of W states, the relative pairwise coupling strengths between the qubits are essential: equal mutual (antiferromagnetic) couplings like in the design proposed here make the system frustrated and facilitate W-like eigenstates. This is not given in, e.g. a linear chain of qubits

$$W = \alpha I - |\psi\rangle\langle\psi|,$$

(19)

where $\alpha$ is the maximal squared overlap of $|\psi\rangle$ with any biseparable or fully separable state. Determination of $\alpha$ is in general complicated [27–29], but we can use the aforementioned proximity of the states under investigation to W and GHZ states, respectively, to make use of known values for $\alpha$. We will discuss how to measure EWs at the end of this section.

Figure 3 shows the expectation values for two EWs $\langle E_2 | \psi_{\text{max}}^{(1)} \rangle | E_2 \rangle$ and $\langle E_2 | \psi_{\text{max}}^{(2)} \rangle | E_2 \rangle$ for varying energy bias.
The EW $\mathcal{V}_{\text{GW}}^{(1)}$ is constructed such as to detect states of the form given in (15), i.e., $|E_2\rangle$ at the left antiscattering. In order to construct the optimal EW, one would have to determine the maximal squared overlap of $|E_2\rangle$ in this point with any non-GHZ entangled state. Instead, we make use of the vicinity of $|E_2\rangle$ with $|\text{GHZ}\rangle$ defined in (16) and use $[30] \mathcal{V}_{\text{GW}}^{(2)} = \frac{1}{2} \mathcal{L} - |\text{GHZ}\rangle\langle\text{GHZ}|$. The optimal local decomposition of $\mathcal{V}_{\text{GW}}^{(2)}$ requires four experimental settings [31], see appendix B. Since the system is invariant under a combined flip of the spins and an inversion of the sign of $\epsilon$, $\mathcal{V}_{\text{GW}}^{(2)}$ (the optimal EW at the right antiscattering) is the totally flipped counterpart to $\mathcal{V}_{\text{GW}}^{(1)}$.

Figure 3 also shows the three-tangle $\tau$ [32], which allows for a reliable distinction between the two classes of entanglement, as it is zero for all W type states (and all separable states, of course), whereas it takes positive values for all states in the GHZ class. However, the three-tangle can only be measured by full state tomography which requires many more settings than the witness. Both quantities indicate a strong (limiting case: $\tau_{\text{max}} = \tau(|\text{GHZ}\rangle) = 1$, $\mathcal{V}_{\text{GW}}^{(1)}|_{\text{min}} = \mathcal{V}_{\text{GW}}^{(2)}|_{\text{min}} = -1/4$) tripartite entanglement of GHZ type over a large range of the energy bias.

In figure 4, the three-tangle and the expectation value of the W witness $\mathcal{W}_W$ for the state $|\psi_{\text{max}}^L\rangle$. The vanishing three-tangle excludes entanglement of the GHZ type, whereas the negative expectation value of the W witness indicates a W type entanglement.

So far, we made use of EWs as a tool for the detection of tripartite entanglement. Another common approach for the detection of entanglement are Bell inequalities. Multi-qubit states can contradict local realistic models in a new and stronger way than two-qubit states [34, 35], reflected by a stronger violation of Bell inequalities [36, 37]. In the case of three qubits, quantum mechanics predicts $|\langle \text{GHZ}| \mathcal{M}_{\text{GHZ}} |\psi\rangle < 2$ for any states $|\psi\rangle$. In comparison, the maximal expectation value of a Bell operator for a two-qubit state is $2\sqrt{2}$, in contrast to the local prediction of $\leq 2$, yielding a maximal violation only by a factor of $\sqrt{2}$. However, Bell inequalities are not distinctive to the type of entanglement; moreover, there are entangled states that do not violate any Bell inequality. Nevertheless, their violation as a sign for non-classical correlations is highly substantial as an ingredient to quantum information processing. Besides, it also allows for a comparison between the two-partite case and the tripartite case with respect to the robustness to limited measurement fidelity.

We again investigate the two states $|E_2\rangle$ and $|\psi_{\text{max}}^L\rangle$ for varying energy bias, see figures 5 and 6. For $|E_2\rangle$, we use the Bell operator [36] $\mathcal{M}_{\text{GHZ}} = \hat{\sigma}_x \hat{\sigma}_y + \hat{\sigma}_y \hat{\sigma}_z + \hat{\sigma}_z \hat{\sigma}_x + \hat{\sigma}_x \hat{\sigma}_y + \hat{\sigma}_y \hat{\sigma}_z + \hat{\sigma}_z \hat{\sigma}_x$. As for EWs, we can determine optimal Bell operators for given target states. In general, Bell inequalities for three qubits are constructed from the operator $\hat{m}(a, b, c) = (a \cdot \hat{\sigma}) \otimes (b \cdot \hat{\sigma}) \otimes (c \cdot \hat{\sigma})$, where $a$, $b$, and $c$ are real three-dimensional normalized vectors, which define a rotation of the Pauli matrices $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. A Bell operator is then obtained by

$$\hat{M} = \hat{m}(a, b, c') + \hat{m}(a, b', c) + \hat{m}(a', b, c) - \hat{m}(a', b', c').$$

(20)

In order to obtain an optimal Bell operator for a given state, one can optimize over the six unit vectors $a$, $a'$, $b$, $b'$, $c$, and $c'$ [23, 38]. The optimal values for $\hat{M}$ adapted to 6 The specific form of $\hat{M}_{\text{GHZ}}$ will be given later.
Figure 5. Three-tangle and expectation value of the Bell operator \( \hat{M}_{\text{GHZ}} \) for the state \(|E_2\rangle\). A significant violation of the corresponding Bell inequality can be observed.

Figure 6. Three-tangle and expectation value of the Bell operator \( \hat{M}_W \) for the state \(|\psi_{\text{max}}\rangle\). The maximal violation of the Bell inequality is not as high as for \(|E_2\rangle\) in figure 5; however, the violation persists over a larger range of \(\epsilon\).

The state \(|W\rangle\), as obtained by a numerical optimization, are listed in table 1. The Bell operator for the state \(|\psi_{\text{max}}^L\rangle\) then reads \(\hat{M}_W = \hat{U}^L \hat{M}_W \hat{U}^L\) with \(\hat{U}^L\) being the local propagator from (18).

Any experimental test of tripartite entanglement or the violation of Bell inequalities involving three qubits will be more fragile than a two-particle test and will be put in jeopardy by detector imperfections (as three-party correlations need to be measured in either case, the measurement fidelity enters with the third power) and fabrication uncertainties. However, for the Bell inequalities, the stronger violation might compensate for that. We consider the effect of a limited measurement fidelity \(f < 1\) on the expectation values of the EWS and Bell operators introduced above and compare the results to a representative two-particle case. We model a non-perfect measurement of a spin component \(\hat{\sigma}_i\) by the perfect measurement of a spin component \(\hat{\sigma}'_i\) which yields the correct measurement result with a probability \(f\) and ‘1’ otherwise,

\[
\hat{\sigma}'_i = f \hat{\sigma}_i + (1 - f) \mathbb{1}.
\] (21)

Figure 7 compares the decay of the Bell violation with decreasing measurement fidelity for the cases discussed above and for a representative two-qubit case, namely the violation of the Clauser–Horne–Shimony–Holt (CHSH) inequality [39] by a Bell pair \(|\psi_B\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\). We find that the larger initial violation for the tripartite cases compensates for the quicker decay of the violation and allows for a slightly lower minimal detector fidelity; table 2 summarizes the minimal detector fidelities for the detection of tripartite entanglement or violation of Bell inequalities. The requested measurement fidelity is already available for charge qubits, where significant progress has been achieved with dispersive readout inside a cavity, providing a visibility of more than 90% [40]. A similar design has been proposed for flux qubits [41]. Moreover, other experiments based on Josephson junction technology indicating similar fidelities have been performed [42–45, 6].

6. Related work

Related work has shown similar properties for a ring of exchange-coupled qubits [46] even in the ground state. Open/linear coupling topologies, albeit easier to prepare experimentally, require more complex pulse sequences [47–49]
because the eigenstates do not have tripartite entanglement; they become more efficient in connected networks [50]. Also, tripartite entanglement between two superconducting cavities and one qubit has been proposed [51]. Beyond tripartite entanglement, a circuit QED setup has been suggested for the fast preparation of an N-qubit GHZ state in superconducting flux or charge qubits [52].

7. Conclusions

In this paper, we introduced a system of three coupled flux qubits in a loop. We showed that it exhibits strong tripartite entanglement for a realistic and approachable set of parameters and that it is possible to detect and quantify this entanglement.

Acknowledgments

We acknowledge support by the Deutsche Forschungsgemeinschaft through SFB 631 and the NSERC discovery grants program and, in parts, by the National Science Foundation under Grant No. NSF PHY05-51164.

Appendix A. State preparation in a degenerate subspace

We describe here, how the highly entangled states (e.g., |ψL\rangle) in the degenerate subspaces introduced in section 3 can be prepared through driving by a resonant laser field. Let |g, n⟩ denote the ground state of the system dressed by |n⟩ photons and |e_{1/2}, n⟩ the excited states with one photon less. This constitutes a V-level scheme. Under resonant driving, these states are all degenerate but can be coupled using the effective transition Rabi frequencies ω_{1/2} (which depend on the coupling to the field) as

\[ \hat{H}_{\text{red}} = \hbar \begin{pmatrix} 0 & \omega_1 & \omega_2 e^{-i\varphi} \\ \omega_1 & 0 & 0 \\ \omega_2 e^{i\varphi} & 0 & 0 \end{pmatrix}, \]

where the phase ϕ can be introduced by a relative phase between the driving fields. Diagonalization of this Hamiltonian leads to eigenstates that are hybridized between the coupled qubit system and the driving photons. One finds that under the corresponding time-evolution operator the application of a pulse of length \( t = \pi/2(\omega_1^2 + \omega_2^2) \) to the ground state |g, n⟩ = (1, 0, 0) leads to the final state

\[ |\psi_f\rangle = \frac{1}{\sqrt{\omega_1^2 + \omega_2^2}} \begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 e^{i\varphi} \end{pmatrix}. \]

Using multiple coils, arbitrary combinations of ω_1 and ω_2 as well as relative phases can be produced and thus, arbitrary linear combinations of the degenerate states can be prepared. This type of preparation requires multiple, phase-locked microwave drives, ideally originating from the same source. This type of phase stability is routinely achieved in high-precision microwave control [53, 54]. As this is essentially a precise Rabi pulse, it is relatively robust against decoherence. However, after the preparation of the state, decoherence in the degenerate subspace essentially selects the eigenstates of the coupling to the environment, projected to the subspace, which will in general be not or much more weakly entangled. Hence, although the preparation of these states is as robust as any preparation of eigenstates, their maintenance is, in general, as fragile as that of a superposition.

Appendix B. Local decomposition of entanglement witnesses

Table B.1 lists the decompositions of the used entanglement witnesses into local projective measurements. The decompositions are optimal in that they require the minimal set of local measurements (e.g. four measurement settings for \( \mathcal{W}_{\text{MCH}} \) witness — i.e. \( \hat{\sigma}_z \) on all qubits —, \( \hat{\sigma}_z \otimes (\hat{\sigma}_z + \hat{\sigma}_x) \otimes \), and \( (\hat{\sigma}_z - \hat{\sigma}_x) \otimes \)).

| Operator | f_{min} | Operator | f_{min} |
|----------|---------|----------|---------|
| \( \mathcal{W}_{\text{GHZ}} \) | 84.3% | \( \mathcal{M}_{\text{GHZ}} \) | 81.4% |
| \( \mathcal{W}_{\text{MCH}} \) | 88.2% | \( \mathcal{M}_{\text{MCH}} \) | 78.4% |
| \( \mathcal{W}_{\text{F}} \) | 86.1% | \( \mathcal{M}_{\text{F}} \) | 81.2% |

| Table B.1. | Local decomposition |
|-------------|---------------------|
| \( \mathcal{W}_{\text{GHZ}}^{(1)} \) | \( \frac{1}{4} \left( 10 - 1^{\otimes 3} + 4 \hat{\sigma}_z^{\otimes 3} - 2 (\hat{\sigma}_z \hat{\sigma}_z \hat{1} + \hat{1} \hat{\sigma}_z + 1 \hat{\sigma}_z \hat{1} + 1 \hat{1} \hat{\sigma}_z) - (\hat{\sigma}_z + \hat{\sigma}_z)^{\otimes 3} - (\hat{\sigma}_z - \hat{\sigma}_z)^{\otimes 3} \right) \) |
| \( \mathcal{W}_{\text{MCH}}^{(1)} \) | \( \frac{1}{4} \left( 10 - 1^{\otimes 3} - 4 \hat{\sigma}_z^{\otimes 3} - 2 (\hat{\sigma}_z \hat{\sigma}_z \hat{1} + \hat{1} \hat{\sigma}_z + 1 \hat{\sigma}_z \hat{1} + 1 \hat{1} \hat{\sigma}_z) + (\hat{\sigma}_z + \hat{\sigma}_z)^{\otimes 3} + (\hat{\sigma}_z - \hat{\sigma}_z)^{\otimes 3} \right) \) |
| \( \mathcal{W}_{\text{F}}^{(1)} \) | \( \frac{1}{4} \left( 17 - 1^{\otimes 3} - 7 \hat{\sigma}_z^{\otimes 3} - 3 (\hat{\sigma}_z \hat{1} \hat{1} + \hat{1} \hat{1} \hat{\sigma}_z + 1 \hat{\sigma}_z \hat{1} + 1 \hat{1} \hat{\sigma}_z) + 5 (\hat{\sigma}_z \hat{\sigma}_z \hat{1} + \hat{\sigma}_z \hat{1} \hat{\sigma}_z + 1 \hat{\sigma}_z \hat{\sigma}_z) \right) \) |

\( -(\hat{1} - \hat{\sigma}_z + \hat{\sigma}_z) \otimes (\hat{1} - \hat{\sigma}_z - \sqrt{2} \hat{\sigma}_z - \frac{1}{\sqrt{2}} \hat{\sigma}_z) \otimes (\hat{1} - \hat{\sigma}_z + \sqrt{2} \hat{\sigma}_z + \frac{1}{\sqrt{2}} \hat{\sigma}_z) \otimes (\hat{1} - \hat{\sigma}_z - \frac{\sqrt{2} \hat{\sigma}_z}{\sqrt{2}} + \frac{1}{\sqrt{2}} \hat{\sigma}_z) \)
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