Integrating over the Coulomb branch in $\mathcal{N} = 2$ gauge theory

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We review the relation of certain integrals over the Coulomb phase of $d = 4$, $\mathcal{N} = 2$ $SO(3)$ supersymmetric Yang-Mills theory with Donaldson-Witten theory. We describe a new way to write an important contact term in the theory and show how the integrals generalize to higher rank gauge groups.

1. Introduction

In the past 15 years there has been a fruitful, sometimes dramatic, math/physics dialogue in the arena of 4-manifold theory and SYM. In the early 1980’s Donaldson defined diffeomorphism invariants of compact, oriented 4-folds $X$ using Yang-Mills instantons. These invariants are best organized in terms of a single function (or formal series), on the homology of $X$ called the the Donaldson-Witten function. Let $P \in H_0(X; \mathbb{Z})$, $S \in H_2(X; \mathbb{Z})$, then the Donaldson-Witten function can be written as:

$$Z_{DW}(pP + vS) = \sum d_{\ell,r} \frac{p^{\ell} v^r}{\ell! \, r!}$$

In 1988 E. Witten [28] interpreted $Z_{DW}$ as the generating function of correlation functions in topological $SO(3)$ SYM:

$$Z_{DW}(vS + pP) = \left\langle \exp[vI(S)+pO(P)] \right\rangle_{SO(3), \mathcal{N}=2}$$

where $I(S)$ and $O(P)$ are certain operators defined below. This lead to an interesting reformulation of the problem of computing Donaldson invariants and, in 1994, when the structure of the vacuum became apparent due to the work of Seiberg and Witten [24], Witten [22] gave a beautiful and simple expression for Donaldson invariants for $X$ of $b_2^+ > 1$ and “of simple type,” reproducing and extending the results of Kronheimer and Mrowka [17]. All known 4-manifolds of $b_2^+ > 1$ are of simple type.

With the publication of [31] the program of deriving 4-manifold invariants from supersymmetric Yang-Mills theory became a resounding success. Nevertheless, a few loose ends remained to be tied up. These chiefly concerned the derivation of Donaldson polynomials for 4-manifolds of $b_2^+ = 1$. In this case $Z_{DW}$ is not quite topologically invariant, and several subtle points must be addressed. The talk at the Strings 97 conference, entitled “Donaldson=Coulomb + Higgs,” and delivered by one of us reported on some work done in collaboration with E. Witten on the extension of the SYM approach to the derivation of the Donaldson polynomials for manifolds of $b_2^+ = 1$. In the course of this investigation a few new results on 4-manifolds were obtained. For example, a general formula relating Donaldson invariants to SW invariants, even for 4-manifolds not necessarily of simple type was derived (eqs. (35), (36) below). Using this one can show that all 4-manifolds of $b_1 = 0, b_2^+ > 1$ are of generalized simple type.

The results reported at Strings 97 have all been described in detail in [23]. Some interesting alternative derivations and new viewpoints on those results have recently been described by A. Losev, N. Nekrasov and S. Shatashvili in [20]. In this note, after reviewing some aspects of [23], we make a few additional comments on the “$u$-plane integrals.”

2. Topological field theory and the $u$-plane integral
2.1. Manifolds of $b_2^+ = 1$

Recall that for $X$ compact and oriented the intersection form on $H^2(X; \mathbb{R})$ has signature $((+1)^{b_2^+}, (-1)^{b_2^-})$, and hence is Lorentz for $b_2^+ = 1$. Examples of such manifolds are $S^2 \times S^2$, $CP^2$ and blowups thereof. This is also the signature of the operator $\ast$, and, given a metric one finds a unique solution up to sign, of the equation $\omega \cdot \omega = 1$. A choice of sign corresponds to a choice of orientation of instanton moduli space $\mathcal{M}$. Such an $\omega$ is called a period point. Using the period point we can define selfdual and anti-selfdual projections of 2-dimensional cohomology classes: $\lambda_+ \equiv (\lambda, \omega)\omega$, $\lambda_- \equiv \lambda - \lambda_+$.

2.2. Donaldson-Witten theory according to Donaldson

In Donaldson theory as formulated in [8], one starts with a principal $SO(3)$ bundle $E \to X$ over a compact, oriented, Riemannian four-manifold $X$, with fixed instanton number $c_2(E)$ and Stiefel-Whitney class $w_2(E)$ ($SO(3)$ bundles on a four-manifold are classified up to isomorphism by these topological data). The moduli space of ASD connections is then defined as

$$\mathcal{M}_{\text{ASD}} = \{ A : F_+(A) = 0 \}/\mathcal{G},$$

(3)

where $\mathcal{G}$ is the group of gauge transformations. To construct the Donaldson polynomials, one considers the universal bundle

$$P = (E \times \mathcal{A}(E))/(\mathcal{G} \times X),$$

(4)

which is a $G$-bundle over $(\mathcal{A}(E)/\mathcal{G}) \times X$ and as such has a classifying map

$$\Phi : P \to BG.$$  

(5)

The “observables” of Donaldson theory are then cohomology classes in $\mathcal{A}(E)$ obtained by the slant product pairing

$$\int_{\gamma} \Phi^*(\xi), \quad \gamma \in H_*(X), \quad \xi \in H^*(BG).$$

(6)

After restriction to $\mathcal{M}_{\text{ASD}} \subset \mathcal{A}(E)/\mathcal{G}$, we obtain cohomology classes in $\mathcal{M}_{\text{ASD}}$. In the case of simply-connected manifolds, we have two different types of observables

$$P \in H_0(X) \quad \to \quad \mathcal{O}(P) \in H^4(\mathcal{M}_{\text{ASD}}),$$

$$S \in H_2(X) \quad \to \quad I(S) \in H^2(\mathcal{M}_{\text{ASD}}).$$

(7)

The Donaldson invariants are defined by

$$d_{i,r} = \int_{\mathcal{M}_{\text{ASD}}} (\mathcal{O}(P))^i \wedge (I(S))^r.$$  

(8)

The main point is that $d_{i,r}$ are metric independent and hence diffeomorphism invariants of $X$ for $b_2^+ > 1$. When $b_2^+ = 1$ it turns out they are only piecewise constant as a function of $\omega$.

2.3. Donaldson-Witten theory according to Witten

In [28], Witten constructed a twisted version of $N=2$ SYM theory which has a nilpotent BRST charge (modulo gauge transformations)

$$\mathcal{Q} = e^{\hat{A}\hat{A}}Q_{\hat{a}\hat{A}},$$

(9)

where $Q_{\hat{a}\hat{A}}$ are the SUSY charges. Here $\hat{a}$ is a chiral spinor index and $A$ has its origin in the $SU(2)$ $\mathcal{R}$-symmetry. The field content of the theory is the standard twisted multiplet:

$$A, \psi_{\mu} = \psi_{\hat{a} \hat{\alpha}}, \phi; \quad D^+_{\mu \nu}, \chi^+_{\mu \nu} = \bar{\psi}_{\hat{a} \hat{\beta}};$$

$$\bar{\phi}, \eta = \bar{\psi}_{\hat{a}}.$$  

(10)

where \( 1/2D^+_{\mu \nu}dx^\mu dx^\nu \) is a self-dual 2-form derived from the auxiliary fields, etc. All fields are valued in the adjoint representation of the gauge group. After twisting, the theory is well defined on any Riemannian four-manifold, since the fields are naturally interpreted as differential forms and the $\mathcal{Q}$ charge is a scalar $\mathcal{Q}$.

The observables of the theory are $\mathcal{Q}$ cohomology classes of operators, and they can be constructed from zero-form observables $\mathcal{O}^{(0)}$ using the descent procedure. This amounts to solving the equations

$$d\mathcal{O}^{(i)} = \{ \mathcal{Q}, \mathcal{O}^{(i+1)} \}, \quad i = 0, \ldots, 3.$$  

(11)

The integration over $i$-cycles $\gamma^{(i)}$ in $X$ of the operators $\mathcal{O}^{(i)}$ is then an observable. These descent equations have a canonical solution: the one-form valued operator $K_{\hat{a}\hat{A}} = -i\delta^{\hat{A}}_{\hat{a}}Q_{\hat{a}\hat{A}}/4$ verifies

$$d = \{ \mathcal{Q}, K \},$$  

(12)

as a consequence of the supersymmetry algebra. The operators $\mathcal{O}^{(i)} = K^{i}\mathcal{O}^{(0)}$ solve the descent equations (11) and are canonical representatives.
One of the main results of [28] is that the generating functional \( \langle I(S) \rangle \) can be written as a correlation function of the twisted theory, involving the observables

\[
O(P) = \frac{1}{8\pi^2} \text{Tr} \phi^2, \\
I(S) = \frac{1}{4\pi^2} \int_S \text{Tr} \left[ \frac{1}{8} \psi \wedge \psi - \frac{1}{\sqrt{2}} \phi F \right].
\]

These operators correspond to the cohomology classes in (9). The relation between the above two formulations of Donaldson-Witten theory is described in detail in many reviews. See, for examples, [29,2,4,18].

2.4. Evaluation via low-energy effective field theory

One way of describing the main result of the work of Seiberg and Witten is that the moduli space of \( \mathcal{Q} \)-fixed points of the twisted \( SO(3) \) \( \mathcal{N} = 2 \) theory on a compact 4-fold has two branches, which we refer to as the Coulomb and Seiberg-Witten branches. On the Coulomb branch the expectation value: \( \langle \frac{1}{(4\pi^2)} \rangle = u \) breaks \( SO(3) \rightarrow SO(2) \) via the standard Higgs mechanism. The Coulomb branch is simply a copy of the complex \( u \)-plane. However, at two points, \( u = \pm 1 \), there is a singularity where the moduli space meets a second branch, the Seiberg-Witten branch, which is the moduli space of solutions of the SW equations modulo gauge equivalence \( \mathcal{M}_{SW} = \{ (A^D, M) : F_+(A^D) = MM, \Gamma \cdot DM = 0 \}/G \). Here \( A^D = \text{is an } SO(2) \text{ gauge field which is a magnetic dual to } A_u \).

Because of tunneling, the partition function on a compact space is a sum over all vacuum states. Hence:

\[
Z_{DW} = \langle e^{P O + v I(S)} \rangle = Z_u + Z_{SW}.
\]

In [31] Witten gave the expression for \( Z_{DW} \) in the case of manifolds of simple type with \( b_2^+ > 1 \). In this case, as we will see in a moment, \( Z_u = 0 \).

The simple type condition means that the only non-vanishing Seiberg-Witten invariants are associated to moduli spaces \( \mathcal{M}_{SW} \) of dimension zero. Below a generalization of Witten’s formula will be presented which holds for manifolds of \( b_2^+ > 0 \) and not necessarily of simple type.

3. Derivation of the \( u \)-plane integral.

We now sketch how to derive the contribution \( Z_u \). The full details are in [23]. The first step involves the identification of the low energy theory and action. Then we must map the operators \( I(S), O(P) \) to the low energy theory.

The untwisted low energy theory has been described in detail in [31,22]. It is an \( \mathcal{N} = 2 \) theory characterized by a prepotential \( F \) which depends on an \( \mathcal{N} = 2 \) vector multiplet. The effective gauge coupling is given by \( \tau(a) = F''(a) \), where \( a \) is the scalar component of the vector multiplet. The Euclidean Lagrange density for the \( u \)-plane theory can be obtained simply by twisting the physical theory. It can be written as

\[
\frac{i}{4\pi} F^aF_a + \frac{1}{16\pi^2} \{ \mathcal{Q}, F' \chi (D + F_+) \} \\
- \frac{1}{32\pi} \{ \mathcal{Q}, F' d \psi \} \\
- \frac{1}{32\pi} \{ \mathcal{Q}, F'' \chi \mu \lambda \chi \lambda \} \sqrt{g} d^4 x \\
+a(u) \text{Tr} R \wedge R + b(u) \text{Tr} R \wedge \tilde{R}
\]

where \( a(u), b(u) \) describe the coupling to gravity, and after integration of the corresponding differential forms we obtain terms proportional to the signature \( \sigma \) and Euler characteristic \( \chi \) of \( X \).

As for the operators, we have \( u = O(P) \) by definition. We may then obtain the 2-observables from the descent procedure. The result is that \( I(S) \rightarrow \tilde{I}(S) = \int_S K^2 u = \int_S \frac{1}{2\pi} (D_+ + F_-) + \cdots \) Here \( D_+ \) is the auxiliary field. It is important to work with off-shell supersymmetry because of contact terms. Even though we have the RG flow \( I(S) \rightarrow \tilde{I}(S) \) it does not necessarily follow that \( I(S_1)I(S_2) \rightarrow \tilde{I}(S_1)\tilde{I}(S_2) \) because there can be contact terms. If \( S_1 \) and \( S_2 \) intersect then in passing to the low energy theory we integrate out massive modes. This can induce delta function corrections to the operator product expansion modifying the mapping to the low energy theory to

\[
I(S_1)I(S_2) \rightarrow \tilde{I}(S_1)\tilde{I}(S_2) + \sum_{P \in S_1 \cap S_2} \epsilon_P T(P)
\]

where \( T(P) \) are local operators. Such contact terms were observed in [30] and were related to gluino condensates. The net effect is that the
mapping to the low energy theory is simply:

\[ \langle \exp[p\mathcal{O}(P) + I(S)] \rangle_{\text{Coulomb}} = \langle \exp[2pu + \tilde{I}_{\text{low}}(S) + S^2T(u)] \rangle_{U(1)} \]  

(17)

We will discuss the function \( T(u) \) in section 6 below. The explicit form is not needed for the remaining derivation of the integral. When evaluating this path integral standard arguments involving \( \Delta \) and \( \lambda \to \infty \), etc., must be applied with care because \( V \) has monodromy, and we must integrate by parts on the \( u \)-plane. Nevertheless, a simple scaling argument presented in \[2\] shows that the semiclassical evaluation of the partition function is exact, so the evaluation of the RHS of equation \[17\] above simply amounts to a semiclassical evaluation of an \( \mathcal{N} = 2 \) Maxwell partition function on a curved manifold. This boils down to several steps:

- Do the Gaussian integral on \( D \).
- Do the Fermion zeromode integral on \( \eta, \chi, \psi \).
- Do the photon path integral on \( U(1) \) gauge field \( A_\mu \).
- Evaluate the coupling of \( u \) to the background metric \( g_{\mu\nu} \).

The zeromode integral on \( a(u) \) finally gives the integral over the \( u \)-plane.

All of these steps are relatively straightforward. A few points which should be noted are:

First, and most importantly, since \( \eta, \chi \) always appear together and there is only one \( \eta \) zeromode (since it is a scalar) the Coulomb branch contributes only for manifolds of \( b_+^2 = 1 \).

A second point is the nature of the photon to gravity. This was derived in \[32\] for the case \( N_f = 0 \) and extended to theories with matter in \[23\]. In order to state the result one must remember that the \( u \)-plane integral describes a family of elliptic curves \( E_u \). The coupling to gravity is expressed in terms of quantities naturally associated to that family:

\[ A(u)^\chi B(u)^\sigma \sim \left( \frac{du}{da} \right)^{-\chi/2} \Delta^{\sigma/8}, \]

(20)

where \( \Delta \) is the discriminant of the elliptic curve and \( du/da \) is a period. The proof follows \[32\] and is based on R-charge and holomorphy.

### 3.1. The explicit expression for the \( u \)-plane integral

The final result of all the computations of the previous section is the expression:

\[ Z_u(p, S) = \int_C \frac{dudu}{y^{1/2}} \mu(\tau) e^{2pu + S^2\hat{T}(u)} \Psi \]

(21)

where \( \mu(\tau) = \frac{dx}{da}(da)^{-1/4} \chi \Delta^{\sigma/8} \) and

\[ \hat{T}(u) = T(u) + \frac{(du/da)^2}{8\pi \text{Im} \tau}. \]

(22)

Here, for simplicity, we have assumed that \( X \) is simply connected. The result is extended to non-simply connected manifolds in \[23\].

We would like to make a few remarks:

- This expression is also the answer for generalizations of Donaldson theory obtained by including hypermultiplets. In fact, this expression makes sense for any family of elliptic curves.

\footnote{We ignore some subtle overall phases described in \[23\].}
• The expression is not obviously well-defined because $\tau(u)$ has monodromy.

• The integrand has singularities at the cusps $u = u_i$ where $\Delta(u_i) = 0$ or $u = \infty$.

• Topological invariance is far from obvious because of the (A)SD projection: $\lambda = \lambda_+ + \lambda_-$, $\lambda_+ = (\lambda, \omega)\omega$.

• The expression for $Z_u$, while impressively complicated, only depends on the classical cohomology ring of $X$. It is only part of the answer to $Z_{DW}$.

3.2. The integral (21) makes sense

There are two points which must be checked before we can accept (21) as a sensible answer for $Z_u$. First, the integrand is expressed in terms of quantities which have monodromy. We must check that the integrand is in fact well-defined on the $u$-plane. Second, the integrand has singularities at the cusps $u = u_i$, and we must define the integral carefully by a regularization and limiting procedure.

The first point is easily checked since $\Psi$ is a modular form of weight $\left(\frac{1}{2}b^+_2 + 1, \frac{1}{2}b^-_2\right)$. (The modular invariance of $\hat{T}(u)$ is crucial for this point.) For single-valuedness in theories with fundamental matter it is sufficient to take $w_2(E) = w_2(X)$ and if we have adjoint matter we must have $w_2(X) = 0$, i.e., $X$ must be a spin manifold.

The second point is rather delicate. Near any cusp $u \sim u_i$ we can make a duality transformation to the local $\tau$-parameter: $q = e^{2\pi i \tau}$ such that $u \to u_i$ corresponds to $q \to 0$, i.e., $\text{Im} \tau \to +\infty$. If we set $\tau = x + iy$, we can regularize the integral by introducing a cutoff $\Lambda$ for $y$, and then taking the limit $\Lambda \to \infty$ at the end. The behaviour of the $u$-plane integral is given by:

$$Z_u \sim \lim_{\Lambda \to +\infty} \int^{\Lambda}_{-\Lambda} \frac{dy}{y^{1/2}} \int^{+1/2}_{-1/2} dx \cdot \sum_{\mu, \nu} q^\mu \bar{q}^\nu [1 + O(1/y)].$$

Then, one easily checks that $\nu \geq \lambda^2_2$, so $\lambda^2_2 > 0$ gives exponential convergence after $\int dx$. Curiously, this corresponds to the regularization used in evaluating string one-loop diagrams.

4. Wall-crossing and the relation of Donaldson and SW invariants.

4.1. Metric dependence

General results in topological field theory imply that $T = \{\mathbb{Q}, \Lambda\}$ and hence $\delta Z_u \sim \int d^2 u. \frac{\partial}{\partial S} \cdots$.

However, when field space is noncompact the total derivative can be nonzero. This is the case for Donaldson theory on manifolds of $b^+_2 = 1$. Indeed, one may derive the general variational formula

$$\frac{d}{dt} Z_u(\omega(t)) = \sum_{u_i} \oint_{u_i} \frac{da}{du} \left(1 - \frac{i}{2} \chi \Delta^8/8 \cdot e^{2pu + S^2T(u)} \cdot \text{Theta function} \right)$$

where an explicit expression for $\Theta$ was derived in [23].

Let us define a wall in the Kähler cone associated to $\lambda \in \Gamma$ by $\lambda_q = (\omega, \lambda) = 0$. For such a metric the line bundle $L$ with $c_1(L) = \lambda$ admits an abelian instanton and the variation of the integral diverges due to a new bosonic zero-mode. For a fixed correlator of order $\sim \sqrt{S}$ the metric variation of $Z_u$ vanishes, except when $\omega$ crosses a wall. Then $Z_u$ has a discontinuous change: the integral $Z_u(\omega)$ is not topologically invariant.

The essential problem is familiar from string theory one-loop amplitudes. It corresponds to the presence of “massless singularities.” Mathematically, at a cusp $y = \text{Im} \tau \to +\infty, q \to 0$, one considers the integral:

$$\lim_{\lambda \to +\infty} \int^{\Lambda}_{-\Lambda} \frac{dy}{y^{1/2}} \int^{+1/2}_{-1/2} dx. c(d) q^\mu (q \bar{q})^{1/2} q^{-\frac{i}{2} \lambda^2} \chi$$

$$= \frac{\mid \lambda \mid \cdot c(d) = \frac{i}{2} \lambda^2}{\lambda}$$

where we have taken $d = \frac{i}{2} \lambda^2$ (otherwise the integral $\frac{dx}{dx}$ gives zero). The contribution of a cusp $u = u_+$ to the discontinuity from a wall is thus given in terms of a residue formula:

$$\Delta Z_u \sim \oint_{u_+} \frac{da}{du} \left[1 - \frac{i}{2} \chi \Delta^8/8 \cdot e^{2pu + S^2T(u)} \cdot \text{exp}[2pu + S^2T - \frac{du}{da}(S, \lambda)] \right]$$

(26)
4.2. Donaldson and SW wall-crossing

There are two kinds of cusps: \( u = \infty \) and \( u = u_i \) on the complex plane. We refer to the finite cusps as SW cusps. Correspondingly there are two kinds of walls:

\[
\begin{align*}
    u = \infty : & \quad \lambda_+ = 0, \quad \lambda \in H^2(X, \mathbb{Z}) + \frac{1}{2} w_2(E) \\
    u = u_* : & \quad \lambda_+ = 0, \quad \lambda \in H^2(X, \mathbb{Z}) + \frac{1}{2} w_2(X)
\end{align*}
\]

(27)

The wall-crossing discontinuity of \( Z_u \) at \( u = \infty \) is:

\[
\Delta Z_u \sim \left[ q^{-\frac{1}{2}} \mu(\tau) e^{2p_0 + S^2 T(u) - i(\lambda, S) \frac{\pi}{\lambda}} \right] \partial \mu
\]

(28)

For the special case \( N_f = 0, u = \frac{1}{2} \vartheta_3 \), \( \mu(\tau) = \frac{\vartheta_1(\vartheta_3)}{\vartheta_2(\vartheta_3)^2} \), and one easily checks that this is identical to the famous formula of G"ottsche for the wall-crossing of \( Z_{DW} \).

However, at the finite cusps \( u = u_* \), \( Z_u \) changes but \( Z_{DW} \) does not! The change in \( Z_u \) is:

\[
\Delta Z_u \sim \int_{u_*}^\infty \frac{du}{(u - u_*) (u - u_*)^{\frac{1}{2} d(\lambda)}} \left[ 1 + O(u - u_*) \right]
\]

where \( d(\lambda) = \lambda^2 - \frac{2 \lambda + \lambda_0}{4} \) and \( \lambda \in H^2(X; \mathbb{Z}) + \frac{1}{2} w_2(X) \).

4.3. Mixing between the branches

At first equation (29) might appear to be a problem. In fact, it fits in quite beautifully with the general principle: Donaldson = Coulomb + Seiberg-Witten:

\[
Z_{DW} = \langle e^{p_0 + (1/S)} \rangle = Z_{Coulomb} + Z_{SW}
\]

(30)

Indeed, note that \( \lambda \in H^2(X; \mathbb{Z}) + \frac{1}{2} w_2(X) \) defines a Spin structure on \( X \) and then \( d(\lambda) = \lambda^2 - \frac{2 \lambda + \lambda_0}{4} \) is the dimension of SW moduli space:

\[
d(\lambda) = \dim \mathcal{M}_{SW}(\lambda).
\]

The Donaldson polynomials do not jump at SW walls so:

\[
0 = \delta Z_{DW} = \delta Z_{Coulomb} + \delta Z_{SW}
\]

(31)

The two terms on the RHS are nonvanishing. This is the mixing of Coulomb and “Higgs” branches.

4.4. Structure of the SW contributions

The cancellation between the changes of \( Z_u \) and \( Z_{SW} \) can in fact be turned to great advantage to derive the general relation between the Donaldson and SW invariants \([23]\).

We can compute \( \delta Z_{Coulomb} \) and therefore find \( \delta Z_{Higgs} \). Therefore, we can learn about the universal holomorphic functions in the effective Lagrangian \( L \) with the monopoles included. This Lagrangian comes from a prepotential \( \tilde{F}_M(a_D) \) for the \( N = 2 \) magnetic vector multiplet and also includes the coupling to the monopole hypermultiplet. The full action has the form:

\[
\{ \tilde{Q}, W \} + \frac{1}{16\pi} \tilde{q}_M \cdot F + p(u) Tr R \wedge R
\]

\[
+ \ell(u) Tr R \wedge R - \frac{1}{2 \pi} \left[ 4 \right]_D^w \psi \wedge \psi \wedge \psi \wedge \psi.
\]

(32)

This is due to the fact that the fourth descendant of the prepotential (which appears in \([13]\)) can be written as a \( \tilde{Q} \)-exact piece plus the terms involving the fields \( \psi, F \) that we have written in \([23]\). The part of the Lagrangian involving the monopole hypermultiplet can also be written as a \( \tilde{Q} \)-exact term after twisting, and is included in \( W \). The terms involving \( p(u), \ell(u) \) are again due to the coupling to gravity.

The action \([23]\) describes a TFT of a standard sort and can be evaluated using standard localization. The terms involving the fields \( \psi \) do not contribute on simply-connected manifolds, and we will drop them (their effect has been analyzed in \([23]\)). We obtain then for the generating functional of the SW contributions:

\[
Z_{SW} = \sum_{\lambda \in H^2(X; \mathbb{Z}) + \frac{1}{2} w_2(X)} \sum_{u_*} \langle e^{p_0 + (1/S)} \rangle_{\lambda, u_*}
\]

(33)

where

\[
\langle e^{p_0 + (1/S)} \rangle_{\lambda, u_*} = \int_{\mathcal{M}_\lambda} \exp \left( 2p_0 + \frac{i}{4\pi} \int_{\mathcal{M}_\lambda} F + S^2 T^*(u) \right)
\]

\[
\cdot \tilde{q}_M^{-\lambda^2/2} P(u)^{\pi/8} L(u)^{1/4}.
\]

(34)

In this equation, \( \tilde{q}_M = \exp(2\pi i \tilde{F}_M) \) and, by definition, the SW invariant is: \( SW(\lambda) \equiv \int_{\mathcal{M}_\lambda(a_D)} \frac{1}{2} d(\lambda) \). The integral \([34]\) is understood
in the sense that we must expand in $aD$ and isolate the correct power.

Using the fact $\Delta SW(\lambda) = \pm 1$ we can now derive the universal functions $\tilde{q}_M, P(u), L(u)$ and see that $T^*(u) = T(u)$.

Now, having obtained these universal functions we can drop the condition $b_2^+ = 1$ and give the Donaldson invariants for all manifolds of $b_2^+ > 0$:

$$Z_{DW} = Z_u + \sum_{\lambda} SW(\lambda) \Xi[\lambda]$$

where

$$\Xi[\lambda] = \sum_{u*} \int \frac{da}{(a-a_*)^{1+\frac{1}{4}d(\lambda)}} \cdot \exp[2pu - i\frac{d}{du}(S,\lambda) + S^2T(u)] \left(\frac{a-a_*}{q}\right)^{\frac{4}{3}} \left(\frac{d}{du}\chi/2\left(\frac{\Delta}{a-a_*}\right)\right)^{\sigma/8}. \tag{36}$$

This generalizes Witten’s famous formula $\tilde{Z}$ to manifolds not necessarily of simple type. One easily checks that $[\Delta(\frac{\partial}{\partial q})(N) Z = 0$ for $N$ large enough, and hence all 4-folds are of generalized simple type, for $b_1 = 0, b_2 > 1$. The result is also interesting for $X$ of simple type $b_2^+ > 1$, and $N_f > 0$. Here the answer can be expressed purely in terms of quantities associated to the elliptic curve at its points of degeneration:

$$Z_{DW} \sim \sum_{\lambda, u_*} SW(\lambda) \kappa_*^\delta \left(\frac{d}{du}\right)^{-(\delta + \sigma)} \cdot \exp\left[2pu + S^2T_u - i\left(\frac{d}{du}\right)(S,\lambda)\right] \tag{37}$$

where $y^2 = x^3 - \frac{c_4}{48}x^2 - \frac{c_6}{64}, \left(\frac{d}{du}\right)^2 = \frac{c_4(u_*)}{2c_6(u_*)}, \kappa_* = \frac{c_4(u_*)}{2\Xi(u_*)}$ and $\delta = (\chi + \sigma)/4$.

5. Evaluation of the $u$-plane integral and its basic properties

The question remains of computability of the $u$-plane integral. It can be computed in two ways: Indirectly, using basic properties of the integral and, at least at $N_f = 0$, directly using techniques developed in string perturbation theory. We first discuss the indirect method:

$Z_u$ is clearly very complicated. However, it is completely determined by four properties:

1. Wall-crossing: At order $p^2S^4$, $Z_u$ is piecewise polynomial with known discontinuities.

2. Vanishing theorems: $Z_u(X) = 0$ for special $X$’s and special gauge bundles, with $\omega$ in special chambers. An important example is $X = F_1$, the first Hirzebruch surface with $\omega \cdot f = 0$ for the fiber $f$ and $w_2(E) \cdot f \neq 0$.

3. The blowup formula: This relates the function $Z_{DW}$ on $X$ to $Z_{DW}$ on the blowup $Bl_p(X)$. Combined with 1, 2, this gives $Z_u$ on ruled surfaces.

4. Homotopy invariance: Since $Z_u$ only depends on the classical ring $H^*(X; Z)$ we can replace $X$ by an algebraic surface.

For simply connected 4-folds of $b_2^+ = 1$ $Z_{DW}$ satisfies exactly the same four properties, so in those cases $\int Z_{SW} = 0$ then we can immediately conclude that the above $u$-plane integral for $N_f = 0$ is an integral representation of the Donaldson invariants.

The reason these properties determine the integral is the following: at least for simply connected manifolds, we can use homotopy invariance to reduce to the case that $X$ is a rational surface. Any two rational surfaces, with any two given metrics, can be related to each other by blowups, blowdowns, and wall-crossing. Then we can reduce the computation to the case of $X = F_1$ in a chamber where $Z_u = Z_{SW} = Z_{DW} = 0$. We have already discussed wall-crossing. We will briefly review the vanishing theorems and blowup formula.

5.1. Vanishing theorems

On certain manifolds in special chambers and with special bundles the integral $Z_u$ vanishes. The intuitive principle behind the vanishing theorems is simple and physical: It costs a lot of action to confine nonzero flux in a small 2-cycle. Consider, for example a product manifold $b \times f$.\footnote{For example, if $X$ admits a metric of positive scalar curvature.}

\footnote{In fact, this result is logically independent of any use of physics or path integrals, and is completely rigorous from a mathematical viewpoint.}
For a metric product and connection the Maxwell action satisfies:

\[ S = \int_{b \times f} F \wedge *F = \frac{\text{vol}(b)}{\text{vol}(f)} \left( \int_F F \right)^2 + \frac{\text{vol}(f)}{\text{vol}(b)} \left( \int_b F \right)^2. \]  

(38)

If the gauge bundle is such that the flux \( \int_F F \) is nonzero for all electric line bundles (e.g. when \( w_2(E) \cdot f = 1 \)) then the action goes to infinity in the limit of small fibers: \( \text{vol}(f) \to 0 \). Hence the theta function decays exponentially fast:

\[ \Theta \sim \sum q^{\frac{1}{2} \lambda_2^2} q^{\frac{1}{2} \lambda_2^1} \sim e^{-y/\epsilon^2} \to 0. \]  

(39)

Thus the integrand vanishes pointwise.

This principle can be used to establish vanishing theorems. However, it must be applied with care since the integration region is noncompact. There are especially interesting subtleties at \( b_2 = 9 \), for elliptic surfaces. See [23] and references therein.

5.2. The blowup formula

Roughly speaking, the procedure of blowing up a manifold at a smooth point replaces the point \( P \) by a sphere - the exceptional divisor \( B \). It changes the intersection form by \( Q \to Q = Q \oplus (-1) \).

When the exceptional divisor is small: \( \omega \cdot B \sim 0 \) it can be replaced by a sum over local operators. This is quite analogous to the OPE in conformal field theory in which one replaces a disk (or even a handle) on a surface by an infinite sum of vertex operators. In Donaldson theory the only local BRST invariant operators are in the ring of polynomials generated by \( O \) and hence we expect a formula of the form:

\[ \langle \exp[I(S) + tI(B) + pO] \rangle_X = \sum_{k \geq 0} t^k \langle \exp[I(S) + pO] B_k(O) \rangle_X \]  

(40)

or more informally, \( \exp[tI(B)] = \sum_{k \geq 0} t^k B_k(O) \).

In fact, this equation, as well as explicit expressions for \( B_k \) can be derived quite straightforwardly from the \( u \)-plane integral [23]. The essential remark is that in the chamber \( B_+ = 0 \) the \( \Psi \)-function for \( \hat{X} \) factors as a product of the \( \Psi \)-function for \( X \) times a holomorphic function of \( u \), which can be interpreted as an insertion of 0-observables.

For \( N_f = 0 \) the blowup formulae of [23] agree with the results of [11] and [13]. In [20] the blowup formulae play a central logical role.

5.3. Direct evaluations

While the above basic properties indeed determine the \( u \)-plane integral completely, they do not lead to a very effective evaluation of these integrals. Any correlation function function \( \langle O(P)^{I(S)} \rangle_X \) is related to the chamber \( \omega \cdot f = 0 \) of \( F_1 \) by a finite number of blowups and wall-crossings. But the number of walls \( \not\sim \infty \) for \( \ell, r \not\sim \infty \). However, for \( N_f = 0 \) a direct evaluation is possible: \( \tau(u) \) maps the \( u \)-plane to the modular curve \( \mathcal{M} = \Gamma_0(4) \backslash \mathcal{H} \) and one can write the \( u \)-plane integral as:

\[ Z_u \sim \int_{\Gamma_0(4) \backslash \mathcal{H}} \frac{d\tau \wedge d\bar{\tau}}{y^2} \tilde{\mu}(\tau) \exp\left\{ 2pu + S^2 \tilde{T}(u) \right\} \Psi(41) \]

This is related to “theta lifts” in number theory or quantum corrections in 1-loop string amplitudes. The \( \Psi \) function is essentially the Narain theta function for signature \( (1, b_+) \), relevant, for example, to compactifications of heterotic string on \( K3 \times S^1 \).

The integral can be evaluated directly by calculations analogous to those described in, for examples, [13,14]. The explicit answers are given in [23]. The case of \( X = CP^2 \) turns out to be rather amusing. The unfolding technique which is used in the evaluations of [13,14] does not apply to this case. Instead, one must integrate by parts using a nonholomorphic modular form of weight \( 3/2, 0 \) for \( \Gamma_0(4) \) discovered by Zagier:

\[ G(\tau, y) = \sum_{n \geq 0} \mathcal{H}(n) q^n \]

\[ + \sum_{f = -\infty}^{\infty} q^{-f^2} \frac{1}{16\pi y^{1/2}} \int_1^\infty e^{-4\pi f^2 u} \frac{du}{u^{3/2}} \]  

(42)

where

\[ \mathcal{H}(\tau) = \sum_{n \geq 0} \mathcal{H}(n) q^n = -\frac{1}{12} + \frac{1}{3} q^3 + \frac{1}{2} q^4 + q^7 + \cdots \]  

(43)
is a generating function for Hurwitz class numbers. In [23] the $SU(2)$ invariants for $\mathbb{C}P^2$ were evaluated in terms of $\mathcal{H}(\tau)$. Comparing to previous results of Göttsche [12] leads to an interesting formula for class numbers:

$$
\mathcal{H}(4n) + \frac{1}{2}\mathcal{H}(16n) q^{2n} + \sum_{n \geq 0} \frac{1}{2} \mathcal{H}(16n + 8) q^{2n+1} - \sum_{n \geq 0} \left( \mathcal{H}(4n + 3) + \frac{1}{2}\mathcal{H}(16n + 12) \right) q^{(4n+3)/2} - \sum_{n \geq 0} \frac{1}{2} \mathcal{H}(16n + 4) q^{(4n+1)/2} = \sum_{n_1 \geq 0, n_2 \geq n_1} \frac{(-1)^{n_1+n_2}(2n_2 + 1)n_1}{\eta^3} \cdot \frac{\vartheta_2^4 + \vartheta_3^4}{8\theta_4}.
$$

(44)

6. Remarks on the contact term $T(u)$

We now return to the contact term $T(u)$. By working with off-shell supersymmetry in [16], $T(u)$ is guaranteed to be $\mathcal{Q}$-closed and is hence locally a holomorphic function of $u$. In [23] $T(u)$ was determined by the requirement that the function $\hat{T}(u)$ given in [23] is invariant under the SL(2, $\mathbb{Z}$) duality group and by some asymptotic constraints. It was rederived in [23] for a larger class of observables from a different point of view, but the argument is only simple for massless theories and $N_f < 4$. Here we present yet a third derivation. We restrict attention to the contact terms for 2-observables arising from the quadratic Casimir, but the method applies to arbitrary gauge group including matter with arbitrary masses. To find $T(u)$, one first notices that the prepotential of $SU(2)$, $N = 2$ supersymmetric gauge theories verifies

$$
\frac{\partial F}{\partial \tau_0} = \frac{1}{4} u,
$$

(45)

where in the asymptotically free theories $\tau_0$ is defined by $\Lambda_{N_f}^{4-N_f} = e^{i\pi \tau_0}$ and $\tau_0$ is the microscopic coupling for the $N_f = 4$ theory (the first definition is of course motivated by the RG equation). The relation (45) has been derived in many different contexts [22, 22, 23, 23, 5, 1] and holds for any matter content and bare masses for the hypermultiplets. We will denote derivatives of the prepotential with respect to the variables $a, \tau_0$ by the corresponding subindices. The duality transformation

$$
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})
$$

(46)

shifts the second term in (22) by

$$
- \frac{4i}{\pi} \frac{cF^2}{\pi \tau + d}.
$$

(47)

But this is precisely the structure of the shift for $F_{\tau_0 \tau_0}$:

$$
F_{\tau_0 \tau_0} \rightarrow F_{\tau_0 \tau_0} - \frac{cF^2}{\pi \tau + d}.
$$

(48)

as one can check using the duality transformation properties of the prepotential or following the approach in [1]. We then see that the contact term is given by

$$
T(u) = \frac{4}{\pi i} \frac{\partial^2 F}{\partial \tau_0^2}.
$$

(49)

An explicit expression for all the cases $0 \leq N_f \leq 4$ can be obtained by using the form of the elliptic curves and the Seiberg-Witten abelian differentials (for the asymptotically free theories with arbitrary masses, the expression has been obtained in [3] in a different context). It is given by

$$
T(u) = -\frac{1}{24} E_2(\tau) \left( \frac{du}{da} \right)^2 + \frac{1}{3} \left( u + \delta_{N_f, 3} \frac{\Lambda^2}{64} \right)
$$

(50)

in the case of the asymptotically free theories, and

$$
T(u) = -\frac{1}{24} E_2(\tau) \left( \frac{du}{da} \right)^2 - E_2(\tau_0) \frac{u}{3} + \frac{1}{9} R E_4(\tau_0)
$$

(51)

in the $N_f = 4$ case, where $R = \sum_n m_n^2 / 2$ and $E_2$, $E_4$ are the normalized Eisenstein series. These expressions are valid for the theories with arbitrary hypermultiplet masses. The same procedure can be applied to the higher rank theories for the contact term coming from the second quadratic Casimir, again with any hypermultiplet content and arbitrary masses.
Finally, we would like to notice that the parameter $\tau_0$ naturally arises in the context of Whitham hierarchies as a slow time variable. This should provide a link between this approach to the contact terms and the one in \cite{20}.

7. Extension to higher rank and other $u$-plane integrals

The $u$-plane integral can be also analyzed in the case of higher rank gauge groups \cite{21} as a tool to analyze the higher rank analogues of Donaldson invariants. We will consider for simplicity the case of $SU(N)$, although the formalism can be easily extended to other compact Lie groups. The integral is given by

$$Z(p, S; m_i, \tau_0) = \int_{\mathcal{M}_{\text{Coulomb}}} [d\tilde{a}d\tilde{\alpha}] A(u)^X B(u)^\sigma e^{U + S^2 T_V} \Psi.$$  \hspace{1cm} (52)

In this equation, $U = \sum_{I=2}^r b^I u_I$ is a linear combination of the Casimirs of the group. The 2-observable is derived from the quadratic Casimir $V = u_2$ and $T_V$ is the corresponding contact term, given by equation (49) above. A simple generalization of the argument of \cite{21} using holomorphy, modular properties and R-charge fixes the $A$, $B$ functions to be the natural generalizations of (20):

$$A^\chi = \alpha^\chi \left( \det \frac{\partial u_I}{\partial \alpha^J} \right)^{\chi/2}, \quad B^\sigma = \beta^\sigma \Delta^\sigma \Delta_{\lambda}^{I/8},$$ \hspace{1cm} (53)

where $\Delta_{\lambda}$ is the quantum discriminant associated to the genus $r$ hyperelliptic curve and $\alpha, \beta$ are constants on $\mathcal{M}_{\text{Coulomb}}$. (Equation (53) was independently derived in \cite{21}.) The lattice sum $\Psi$ is given in this case by the finite-dimensional integral

$$\Psi = \sum_{\lambda \in \Gamma} \int_{\mathbb{R}^r} d\eta^I d\chi^J \int_{-\infty}^{+\infty} \prod_I db^I \exp \left[ -i\pi \xi_{IJ}(\lambda^I_+, \lambda^I_-) - i\pi \tau_{IJ}(\lambda^I_+, \lambda^I_-) ight. \\
+ \frac{1}{2\pi} b^I (\text{Im} \tau)_{IJ} b^J - iV_I (S, \lambda^I_+) - \frac{1}{4\pi} V_I (S, \omega) b^I \\
- \frac{1}{4\pi} \tilde{\tau}_{IJ} \eta^I \chi^J (b^K + 4\pi \lambda^K_+ - i\pi (\tilde{\lambda} \cdot \tilde{\rho}, w_2(X))).$$ \hspace{1cm} (54)

Here, $\chi, \eta$ are Grassmann variables and $b^I$ are commuting variables (they have their origin in the zero modes of the $\chi, \eta$ fields, and in the auxiliary fields, respectively). In the lattice $\Gamma$ we have to consider non-abelian magnetic fluxes (or generalized Stiefel-Whitney classes) that are associated to the conjugacy classes of $\Lambda_{\text{weight}}/\Lambda_{\text{root}}$ \cite{20}. We sum then over root vectors of the form $\lambda^I = \lambda^I_0 + (C^{-1})^J J \pi^J$, \hspace{1cm} (55)

where $\lambda^I_0$, $\pi^I \in H^2(X, \mathbb{Z})$, $\alpha_I$, $I = 1, \ldots, r$, is a basis of simple roots, and $C$ is the Cartan matrix. The weight $\tilde{\rho}$ appearing in (34) is half the sum of the positive roots, and the term involving it is the appropriate generalization of the phase derived in \cite{22} for the rank one case (this term has been derived independently in \cite{20}).

Although the higher rank integral (52) is quite complicated, it can be analyzed in some detail. One can check single-valuedness of the integrand under the quantum monodromy group. The proper definition of the integral is rather subtle because of the nature of the singular loci of the moduli space. The superconformal loci are especially subtle. One can derive wall-crossing formulae, which are generically integrals of a residue. There is “wall-crossing for wall-crossing” arising from the contributions of codimension two submanifolds, and so forth. Using this one can generalize the above result for $Z_{DW}$ to higher rank gauge groups. A detailed presentation of these remarks will appear in \cite{21}.

8. Conclusion: Future directions

There are many interesting future directions in this subject. We mention just two here. First, it appears that an analogue of $Z_u$ can be written for any special Kähler geometry. The question remains as to the physical significance. Can we always find a physical system which is computing some invariants through $Z_u$? Examples of systems which could be especially interesting include the effective theory on the D3 probe in F-theory and the integrable system associated to variation of Hodge structures introduced by Donagi and Markman \cite{06}. Another direction, involving applications to
Gromov-Witten theory, has recently been proposed in [21].

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