Study of $\pi^0$ and $\eta$ decays containing dilepton

Chong-Chung Lih$^{1,2}$

$^1$Department of Optometry, Shu-Zen College of Medicine and Management, Kaohsiung Hsien 452, Taiwan

$^2$Department of Physics, National Tsing-Hua University, Hsinchu 300, Taiwan

(Dated: August 5, 2013)

Abstract

We calculate the momentum dependent form factors of $M \to \gamma^*\gamma^*(M = \pi^0, \eta)$ within the light-front quark model. Using the form factors, we examine the decays of $M \to l^+l^-$, $M \to l^+l^-\gamma$ and $M \to l^+l^-l^+l^-(l = e$ or $\mu)$ and compare our results with the experimental data and other theoretical predictions. In particular, for $\pi^0 \to e^+e^-$, we find that the decay branching ratio is $6.68 \times 10^{-8}$, which is closed to the recent measurement of $(7.48 \pm 0.29 \pm 0.25) \times 10^{-8}$ by E799 of KTeV/Fermilab.
I. INTRODUCTION

The neutral pseudoscalar meson decays of $M \to l^+l^-$, in particular $K_L \to \mu^+\mu^-$, have played very important roles to understand the Standard Model (SM). For the light pseudoscalar mesons of $\pi^0$ and $\eta$, the decays are dominated by the long distance (LD) contributions, described by the two photon intermediate state at the lowest order of QED. Since the short distance (SD) contributions in the SM are many orders of magnitude smaller, they can be neglected. Therefore, these decay modes are good processes to explore new physics beyond the SM.

The measurement on this process by the KTeV-E799 experiment at Fermilab has given

$$B(\pi^0 \to e^+e^-, x_D > 0.95) = (6.44 \pm 0.25 \pm 0.22) \times 10^{-8}$$

where $x_D \equiv (m_{2e}/m_{\pi})^2$ is the Dalitz variable with $m_{2e}$ being the $e^+e^-$ mass. By extrapolating the Dalitz branching ratio to the full range of $x_D$ with the overall radiative correction, one gets

$$B^{KTeV}_{\pi^0 \to e^+e^-} = (7.48 \pm 0.29 \pm 0.25) \times 10^{-8}.$$  

The decay of $\pi^0 \to e^+e^-$ has been well studied theoretically over the years. However, the KTeV result in Eq. (2) disagrees with the some theoretical predictions about $1.5 \sim 3.3$ standard deviations [2–7].

At the lowest order of QED, the decay branching ratio of $\pi^0 \to e^+e^-$ is found to be [8–10]:

$$B_{\pi^0 \to e^+e^-} = \frac{\Gamma(\pi^0 \to e^+e^-)}{\Gamma(\pi^0 \to 2\gamma)} = 2\beta \left( \frac{\alpha m_e}{\pi m_\pi} \right)^2 |A(m_\pi^2)|^2,$$

where $\beta \equiv \sqrt{1 - 4m_e^2/m_{\pi}^2}$ and $|A(m_\pi^2)|^2$ can be generally decomposed into $|\text{Im } A(m_\pi^2)|^2 + |\text{Re } A(m_\pi^2)|^2$. Here, $\text{Im } A$ denotes the absorptive contribution from the real photon in the intermediate state, which can be determined in a model-independent form [8–11]

$$|\text{Im } A|^2 = \frac{\pi^2}{4\beta^2} \left[ \ln \frac{1 - \beta}{1 + \beta} \right]^2,$$

leading to the unitary bound on the branching ratio as

$$B_{\pi^0 \to e^+e^-} > 2\beta \left( \frac{\alpha m_e}{\pi m_\pi} \right)^2 |\text{Im } A|^2 = 4.75 \times 10^{-8}.$$
The real part $\text{Re} A$ is given by the dispersive one, which can be written as the sum of SD and LD contributions,

$$\text{Re} \ A = \text{Re} \ A_{SD} + \text{Re} \ A_{LD}. \quad (6)$$

In the SM, the SD part is given by one-loop box and penguin diagrams\[12, 13\]. The LD one involves the form factor related to the $\pi^0\gamma\gamma$ vertex. Using the form factor, the LD amplitude one has

$$A_{LD} = \frac{2i}{\pi^2 m^2_e} \int d^4 q \frac{[P^2 q^2 - (P \cdot q)^2]}{q^2 (P - q)^2 [(q - p_e)^2 - m_e^2]} \frac{F(q^2, (P - q)^2)}{F(0, 0)}, \quad (7)$$

where $P$ and $p_e$ are the pion and electron momenta, respectively. The function $F(q^2, (P - q)^2)$ is the double form factor of $\pi^0 \rightarrow \gamma^*\gamma^*$. This form factor contains the nontrivial dynamics of the process and has been studied in various models\[3, 5–7, 14–16\]. In this paper, we calculate the form factor $F(q^2, (P - q)^2)$ within the light-front quark model (LFQM) and use this form factor to evaluate the decays of $\pi^0 \rightarrow e^+e^-$ and $e^+e^-\gamma$. We will also study $\eta$ decays, which contain a dilepton or dileptons.

This paper is organized as follows: In Sec. II, we present the relevant formulas for the matrix elements and form factors for $M \rightarrow \gamma^*\gamma^*$ ($M = \pi^0, \eta$). In Sec. III, we show our numerical results on the form factors and the branching ratios of meson $M$ decays with dilepton. We give our conclusions in Sec. IV.

II. THE FORM FACTORS

To calculate $M \rightarrow \gamma^*\gamma^*$ ($M = \pi^0, \eta$) transition from factors within the LFQM, we have to decompose the mesons into $Q\bar{Q}$ Fock states. Explicitly, $\pi^0$ may be described as $(u\bar{u} - d\bar{d})/\sqrt{2}$ and the valence state of $\eta$ can be written as\[17\]

$$|\eta\rangle = \Phi_8 \cos \theta_P |u\bar{u} + d\bar{d} - 2s\bar{s}\rangle/\sqrt{6} - \Phi_1 \sin \theta_P |u\bar{u} + d\bar{d} + s\bar{s}\rangle/\sqrt{3}, \quad (8)$$

where $\Phi_1, 8$ are the wave functions of the Fock states and $\theta_P \sim -20^o$ is the mixing angle. In the scheme of the $Q\bar{Q}$ state, the amplitude of $M \rightarrow \gamma^*\gamma^*$ with $CP$ conservation is given by:

$$A(Q\bar{Q}(P) \rightarrow \gamma^*(q_1, \epsilon_1) \gamma^*(q_2, \epsilon_2)) = ie^2 F_{Q\bar{Q}}(q_1^2, q_2^2) \epsilon_{\mu\nu\rho\sigma} \epsilon_1^\mu \epsilon_2^\nu q_1^\rho q_2^\sigma, \quad (9)$$

where $F_{Q\bar{Q}}(q_1^2, q_2^2)$ in Eq. \[9\] is a symmetric function under the interchange of $q_1^2$ and $q_2^2$. From the quark-meson diagram depicted in Fig. 1, we get
FIG. 1. Loop diagrams that contribute of $\pi^0 \to \gamma^* \gamma^*$.

$$A(Q\bar{Q} \to \gamma^*(q_1) \gamma^*(q_2)) = e_Q e_{\bar{Q}} N_c \int \frac{d^3 p_3}{(2\pi)^4} \Lambda_P \left\{ \text{Tr} \left[ \frac{i(-p_3 + m_{\bar{Q}})}{p_3^2 - m_{\bar{Q}}^2 + i\epsilon} \frac{i(p_2 + m_Q)}{p_2^2 - m_Q^2 + i\epsilon} \right] + (\epsilon_1 \leftrightarrow \epsilon_2, q_1 \leftrightarrow q_2) \right\}$$

$$+ (p_{1(3)} \leftrightarrow p_{3(1)}, m_Q \leftrightarrow m_{\bar{Q}}), \quad (10)$$

where $N_c$ is the number of colors and $\Lambda_P$ is a vertex function which related to the $Q\bar{Q}$ meson. In the light front (LF) approach, the LF meson wave function can be expressed by an anti-quark $\bar{Q}$ and a quark $Q$ with the total momentum $P$ as:

$$|M(P, S, S_z)\rangle = \sum_{\lambda_1 \lambda_2} \int [dp_1][dp_2] 2(2\pi)^3 \delta^3(P - p_1 - p_2)$$

$$\times \Phi_{M}^{SSz}(z, k_\perp) b_{\bar{Q}}^\dagger(p_1, \lambda_1) d_{Q}^\dagger(p_2, \lambda_2) |0\rangle, \quad (11)$$

and

$$[d^3p] = \frac{dp^+d^2p_\perp}{2(2\pi)^3}, \quad (12)$$

where $\Phi_{M}^{\lambda_1\lambda_2}$ is the amplitude of the corresponding $\bar{Q}(Q)$ and $p_{1(2)}$ is the on-mass shell LF momentum of the internal quark. In the momentum space, the wave function $\Phi_{M}^{SSz}$ is given by

$$\Phi_{M}^{SSz}(k_1, k_2, \lambda_1, \lambda_2) = R_{\lambda_1\lambda_2}^{SSz}(z, k_\perp) \phi(z, k_\perp), \quad (13)$$

where $\phi(z, k_\perp)$ describes the momentum distribution amplitude of the constituents in the bound state and $R_{\lambda_1\lambda_2}^{SSz}$ constructs a spin state $(S, S_z)$ out of light front helicity eigenstates $(\lambda_1 \lambda_2)[19]$. The LF relative momentum variables $(z, k_\perp)$ are defined by

$$p_1^+ = zP^+, \quad p_2^+ = (1 - z)P^+,$$

$$p_{1\perp} = zP_{\perp} - k_\perp, \quad p_{2\perp} = (1 - z)P_{\perp} + k_\perp. \quad (14)$$
The normalization condition of the meson state is given by
\[ \langle M(P', S', S'_z) | M(P, S, S_z) \rangle = 2(2\pi)^3 P^+ \delta^3(P' - P) \delta_{S'S} \delta_{S'_z S_z}, \] (15)
which leads the momentum distribution amplitude \( \phi(z, k_{\perp}) \) to
\[ N_c \int \frac{dz \, d^2k_{\perp}}{(2\pi)^3} |\phi(z, k_{\perp})|^2 = 1. \] (16)
We note that Eq. (13) can, in fact, be expressed as a covariant form\(^\text{[20–22]}\)
\[ \Phi^{SSz}(z, k_{\perp}) = \left( \frac{p_1^+ p_2^+}{2[M_0^2 - (m_Q - m_{\bar{Q}})^2]} \right)^{1/2} \mathcal{\Pi}(p_1, \lambda_1) \gamma^5 v(p_2, \lambda_2) \phi(z, k_{\perp}), \]
\[ M_0^2 = \frac{m_Q^2 + k_{\perp}^2}{z} + \frac{m_{\bar{Q}}^2 + k_{\perp}^2}{1 - z}. \] (17)
In principle, the momentum distribution amplitude \( \phi(z, k_{\perp}) \) can be obtained by solving the light-front QCD bound state equation\(^\text{[22]}\). However, before such first-principle solutions are available, we would have to be contented with phenomenological amplitudes. One example that has been used is the Gaussian type wave function\(^\text{[23–25]}\):
\[ \phi(z, k_{\perp}) = N \sqrt{N_c} \int dz \, d^2k_{\perp} \exp \left( -\frac{\vec{k}^2}{2\omega^2_M} \right) \phi(z, k_{\perp}), \] (18)
where \( N = 4(\pi/\omega^2_M)^{3/2}, \vec{k} = (k_{\perp}, k_z), \) and \( k_z \) defined through
\[ z = \frac{E_1 + k_z}{E_1 + E_2}, \quad 1 - z = \frac{E_2 - k_z}{E_1 + E_2}, \quad E_i = \sqrt{m_i^2 + \vec{k}^2} \] (19)
by
\[ k_z = \left( z - \frac{1}{2} \right) M_0 + \frac{m_Q^2 - m_{\bar{Q}}^2}{2M_0}, \quad M_0 = E_1 + E_2. \] (20)
and \( dk_z/dz = E_1 E_2 / z(1 - z) M_0. \) After integrating over \( p_{3-} \) in Eq. (10), we obtain
\[ A(Q\bar{Q} \to \gamma^*(q_1) \gamma^*(q_2)) = e_Q e_{Q'} N_c \int_{q_1^+}^{q_2^+} dp_{3-}^+ \int \frac{d^2p_{3\perp}}{(2\pi)^3} \prod_{i=1}^{3} p_i^+ \left[ \frac{\Lambda_P}{P^- - p_{3on}^- - p_{3on}^-} \right] \frac{1}{q_2 - p_{2on}^- - p_{3on}^-} + (\epsilon_1 \leftrightarrow \epsilon_2, q_1 \leftrightarrow q_2) + (p_{1(3)} \leftrightarrow p_{3(1)}) \right], \] (21)
and
\[ I = \text{Tr}[\gamma_5(- p_3 + m_Q) \phi_2(p_2 + m_Q) \phi_1(p_1 + m_Q)], \quad \vec{p}_{3on} = \frac{m_i^2 + p_{i\perp}^2}{p_i^+} \] (22)
where the subscript \( \{\text{on}\} \) represents the on-shell particles. One can extracted the vertex function \( \Lambda_P \) from Eqs. (10), (17) and (21), given by [18, 20, 21]:

\[
\frac{\Lambda_P}{P^- - p_{\text{ion}} - p_{\text{don}}} = \frac{\sqrt{p_1^+ p_3^+}}{\sqrt{2[M_0^2 - (m_Q - m_{\bar{Q}})^2]}} \phi(z, k_\perp),
\]

(23)

To calculated the trace \( I \), we have used the definitions of the LF momentum variables \( (z(x), k_\perp(k'_\perp)) \) and taken the frame with the transverse momentum \( (P - q_2)_\perp = 0 \) for the \( \bar{Q}Q \) state(\( P \)) and photon(\( q_2 \)) in Fig. 1a. Hence, the relevant quark variables are:

\[
p_1^+ = zP^+, \quad p_3^+ = (1 - z)P^+,
\]

\[
p_{1\perp} = zP_\perp - k_\perp, \quad p_{3\perp} = (1 - z)P_\perp + k_\perp.
\]

\[
p_2^+ = xq_2^+, \quad p_3^+ = (1 - x)q_2^+, \quad p_{2\perp} = xq_{2\perp} - k'_\perp, \quad p_{3\perp} = (1 - x)q_{2\perp} + k'_\perp.
\]

(24)

At the quark loop, it requires that

\[
k_\perp = (z - x)q_{2\perp} + k'_\perp.
\]

(25)

The trace \( I \) in Eq. (22) can be easily carried out. Thus, the form factor \( F(q_1^2, q_2^2) \) in Eq. (9) can be found to be:

\[
F_{\bar{Q}Q}(q_1^2, q_2^2) = -8 \sqrt{\frac{N_c}{3}} \int \frac{dx \ d^2k_\perp}{2(2\pi)^3} \Phi(z, k_\perp^2) \frac{c_Q}{1 - z} \frac{m_Q}{x(1 - x)q_2^2 - m_Q^2 - k_\perp^2} + (q_2 \leftrightarrow q_1)
\]

(26)

where \( c_Q \) is the quark electric charge factor and

\[
\Phi(z, k_\perp^2) = N \sqrt{\frac{z(1 - z)}{2M_0^2}} \sqrt{\frac{dk_z}{dz}} \exp \left( -\frac{k^2}{2\omega_M^2} \right),
\]

\[
\bar{k} = (\bar{k}_{\perp}, \bar{k}_z), \quad z = \frac{x}{r},
\]

\[
r = \frac{q_2^+}{P^+} = \frac{(m_{\bar{P}}^2 + q_2^2 - q_1^2) + \sqrt{(m_{\bar{P}}^2 + q_2^2 - q_1^2)^2 - 4q_2^2m_{\bar{P}}^2}}{2m_{\bar{P}}^2}.
\]

(27)

If \( q_1 \) and \( q_2 \) are on mass shell where \( r = 1 \), the form factors of \( \pi \rightarrow \gamma\gamma \) and \( \eta \rightarrow \gamma\gamma \) can be written as

\[
F_{\pi \rightarrow \gamma\gamma}(0, 0) = 8\sqrt{2} \sqrt{\frac{N_c}{3}} \int \frac{dx \ d^2k_\perp}{2(2\pi)^3} \Phi(x, k_\perp^2) \left\{ \frac{4}{9} \frac{m_u}{m_u^2 + k_\perp^2} - \frac{1}{9} \frac{m_d}{m_d^2 + k_\perp^2} \right\} ;
\]

\[
F_{\eta \rightarrow \gamma\gamma}(0, 0) = 16\sqrt{\frac{N_c}{3}} \int \frac{dx \ d^2k_\perp}{2(2\pi)^3} \left\{ \frac{\Phi^8(x, k_\perp^2) \cos \theta_P}{(1 - x)\sqrt{6}} \left( \frac{4}{9} \frac{m_u}{m_u^2 + k_\perp^2} + \frac{1}{9} \frac{m_d}{m_d^2 + k_\perp^2} - \frac{2}{9} \frac{m_s}{m_s^2 + k_\perp^2} \right) \right.
\]

\[
- \frac{\Phi^1(x, k_\perp^2) \sin \theta_P}{(1 - x)\sqrt{3}} \left( \frac{4}{9} \frac{m_u}{m_u^2 + k_\perp^2} + \frac{1}{9} \frac{m_d}{m_d^2 + k_\perp^2} + \frac{1}{9} \frac{m_s}{m_s^2 + k_\perp^2} \right) \right\}. \quad (28)
\]
III. NUMERICAL RESULT

To numerically calculate the transition form factors of $\pi^0$ and $\eta$ in Eq. (26) and (28), we need to specify the parameters appearing in $\phi(x,k_{\perp})$. To constrain the quark masses of $m_{u,d,s}$ and the meson scale parameters of $\omega_M$ in Eq. (26), we use the meson decay constants $f_M$ and its branching ratios of $M \rightarrow 2\gamma$, given by \[27\]

\[ f_{\pi^0} = 132 \text{ MeV}, \quad f_{\eta^8} = 169 \text{ MeV}, \quad f_{\eta^1} = 145 \text{ MeV}. \] (29)

and

\[ Br_{\pi^0 \rightarrow 2\gamma} = (98.832 \pm 0.034)\%, \quad Br_{\eta \rightarrow 2\gamma} = (39.30 \pm 0.2)\% \] (30)

respectively. Here, the explicit expression of $f_M$ is given by \[26\]

\[ f_M = 4 \frac{\sqrt{N_c}}{\sqrt{2}} \int \frac{dx d^2k_{\perp}}{2(2\pi)^3} \phi(x,k_{\perp}) \frac{m}{\sqrt{m^2 + k_{\perp}^2}}. \] (31)

From

\[ B_{M \rightarrow 2\gamma} = \frac{(4\pi\alpha)^2}{64\pi\Gamma_P} m_P^3 |F(0,0)_P \rightarrow 2\gamma|^2, \] (32)

we find that $|F(0,0)_{\pi^0(\eta) \rightarrow 2\gamma}| = 0.274(0.272)$ in $\text{GeV}^{-1}$. As an illustration, we extract $m_u = m_d = 0.24$, $m_s = 0.38$ and $\omega_\pi = 0.33$, $\omega_{\eta^1} = 0.42$, $\omega_{\eta^8} = 0.58$ in GeV, which will be used in our following numerical calculations.

A. $\pi^0(\eta) \rightarrow e^+e^-\gamma$

We now examine process of $\pi^0 \rightarrow e^+e^-\gamma$ with the form factor in Eq. (26). The interaction between the photon and leptons is given by the conventional QED \[14, 28\]. One easily obtains the differential decay rate

\[ \frac{d\Gamma(\pi^0 \rightarrow e^+e^-\gamma)}{\Gamma(\pi^0 \rightarrow \gamma\gamma) dq_1^2} = \frac{2\alpha}{3\pi q_1^2} \left( 1 - \frac{q_1^2}{m_\pi^2} \right)^3 \left( 1 - \frac{4m_e^2}{q_1^2} \right)^{1/2} \left( 1 + \frac{2m_e^2}{q_1^2} \right) |f(t)|^2, \] (33)

where $f(t) = F_\pi(q_1^2,0)/F_\pi(0,0)$ and $t = q_1^2/m_\pi^2$. Obviously, the branching ratio of $\pi^0 \rightarrow e^+e^-\gamma$ in the Eq. (33) depends on the factor of $1/q_1^2$. The function of $f(t)$ is an analytic function in the entire physics region of $4m_e^2 \leq q_1^2 \leq m_\pi^2$, related to

\[ F_\pi(q_1^2,0) = -4\sqrt{2} \int \frac{dx d^2k_{\perp}}{2(2\pi)^3} \tilde{\Phi}(z,k_{\perp}^2) \frac{1}{1-z} \left\{ \frac{4}{9} \left[ \frac{m_u}{x(1-x)q_1^2 - m_u^2 - k_{\perp}^2} + \frac{m_u}{m_u^2 + k_{\perp}^2} \right] \right\}^{\frac{1}{2}}. \] (34)
Integrating over $q_1^2$ in Eq. (33), we obtain the branching ratio

$$\frac{\Gamma(\pi^0 \rightarrow e^+e^-\gamma)}{\Gamma(\pi^0 \rightarrow \gamma\gamma)} = 1.18 \times 10^{-2},$$  \hspace{1cm} (35)$$

which agrees well with those by QED\[14, 15\] and vector meson dominance (VMD) model\[16\]. Our result is also close the experimental data: $B_{\pi^0 \rightarrow e^+e^-\gamma}^{exp} = (1.198 \pm 0.032) \times 10^{-2}$ \[27\].

Similarly, the branching ratios of $\eta \rightarrow e^+e^-\gamma$ and $\eta \rightarrow \mu^+\mu^-\gamma$ which normalized with $\eta$ total width are found to be

$$B_{\eta \rightarrow e^+e^-\gamma} = \frac{\Gamma(\eta \rightarrow e^+e^-\gamma)}{\Gamma_{\eta}} = 6.95 \times 10^{-3},$$  \hspace{1cm} (36)$$

$$B_{\eta \rightarrow \mu^+\mu^-\gamma} = \frac{\Gamma(\eta \rightarrow \mu^+\mu^-\gamma)}{\Gamma_{\eta}} = 2.94 \times 10^{-4}. \hspace{1cm} (36)$$

Ours result of $\eta \rightarrow e^+e^-\gamma$ is smaller than that in the CLEO data\[29\] but larger than the one in Ref.\[31\]. However, for the mode of $\eta \rightarrow \mu^+\mu^-\gamma$, our result agrees with Ref.\[31\] as well as that by the effective mass theory (EMT)\[32\]. Furthermore, our predictions in the two decay modes agree well with the experimental data in CELSIUS\[33\] and the PDG\[27\].

**B. $\pi^0 \rightarrow e^+e^-e^-e^-$ and $\eta \rightarrow \ell^+\ell^-\ell^+\ell^-$ ($\ell = e, \mu$)**

We examine the double lepton-pair decay of $\pi^0 \rightarrow e^+e^-e^-e^-$ with the form factors in Eq. (26). The decay matrix element is calculated by the conventional QED with the interaction of $\pi^0$ and two photons and the differential decay rate is given by

$$\frac{d\Gamma(\pi^0 \rightarrow e^+e^-e^-e^-)}{\Gamma(\pi^0 \rightarrow \gamma\gamma) dq_1^2 dq_2^2} = \frac{2}{q_1^2 q_2^2} \left( \frac{\alpha}{3\pi} \right)^2 \left| F_{\pi}(q_1^2, q_2^2) \right|^2 \lambda^{3/2} \left( \frac{1, q_1^2, q_2^2}{m^2_{\pi}, m^2_{\pi}} \right) G_1(q_1^2) G_2(q_2^2), \hspace{1cm} (37)$$

where

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ca),$$

$$G_1(q^2) = \left( 1 - \frac{4m^2_e}{q^2} \right)^{1/2} \left( 1 + \frac{2m^2_e}{q^2} \right). \hspace{1cm} (38)$$

After the integrations over $q_1^2$ and $q_2^2$, we obtain the branching ratio as follows:

$$B_{\pi^0 \rightarrow e^+e^-e^-e^-} \equiv \frac{\Gamma(\pi^0 \rightarrow e^+e^-e^-e^-)}{\Gamma(\pi^0 \rightarrow \gamma\gamma)} = 3.29 \times 10^{-5}, \hspace{1cm} (39)$$

which is smaller than that in Ref.\[14\], but larger than the one in Ref.\[15\] slightly. However, all results are consistent with the experimental data. We note that even if the form factor is replaced by an on-shell constant with $F(q_1^2, q_2^2) = F(0, 0)$, the branching ratio is found
to be very close to the result in Eq. (39). It might be a good approximation to neglect the momentum dependence of the form factor for the decay.

We can also perform the similar calculations for $\eta \rightarrow l^+l^-l^+l^-$ ($l = e$ or $\mu$) and we find

$$
\mathcal{B}_{\eta \rightarrow e^+e^-e^-} = 2.47 \times 10^{-5}, \\
\mathcal{B}_{\eta \rightarrow e^+\mu^-\mu^-} = 5.83 \times 10^{-7}, \\
\mathcal{B}_{\eta \rightarrow \mu^+\mu^-\mu^-} = 1.68 \times 10^{-9}.
$$

(40)

Our result on $\mathcal{B}_{\eta \rightarrow e^+e^-e^-}$ is in good agreement with the experimental data $\mathcal{B}_{\eta \rightarrow e^+e^-e^-}^{\text{exp}} = \left(2.7^{+2.1}_{-2.7}\text{stat} \pm 0.1\text{syst}\right) \times 10^{-5}$ [33] and Ref. [31]. For other modes, currently, our theoretical predictions are many orders of magnitude smaller than the experimental upper bounds [27, 33].

C. $\pi^0(\eta) \rightarrow \ell^+\ell^-$

We first calculate the real part of $\text{Re } A_{LD}$ in Eq. (7) at the pion momentum limit of $P^2 \rightarrow 0$. At this limit, the relevant form factor of Eq. (26), given by a triangular quark loop, would be simplify to

$$
F(q^2, q^2) = -8\sqrt{2} \int \frac{dx d^2 k_\perp}{(2\pi)^3} \Phi(z, k_\perp^2) \frac{1}{1 - z} \left\{ \frac{4}{9x(1 - x)q^2 - m_u^2 - k_\perp^2} - \frac{1}{9x(1 - x)q^2 - m_d^2 - k_\perp^2} \right\}.
$$

(41)

One could easily find

$$
\text{Re } A_{LD}(0) \simeq -20.74.
$$

(42)

The numerical result is in agreement with the most vector meson dominance(VMD) model at $P^2 \rightarrow 0$. This implies the equivalence between the VMD and LFQM descriptions on the form factors of hadrons with the relevant vector meson mass of $M_V \sim 2m_u$ in the VMD. To illustrate $\text{Re } A_{LD}(q^2)$ in the range $-m_\pi^2 \geq q^2 \geq m_\pi^2$, we use the dispersive framework proposed in Ref. [6]. The real part may be written by a once-subtracted dispersion relation [2, 6, 30]

$$
\text{Re } A_{LD}(q^2) = \text{Re } A(0) + \frac{q^2}{\pi} \int_0^\infty dq'^2 \frac{\text{Im } A(q'^2)}{(q'^2 - q^2)q^2}
$$

(43)
Extrapolating from \( q^2 = 0 \) to \( m_\pi^2 \), we find \( \text{Re} \ A_{LD}(m_\pi^2) = 11.18 \). Since the SD part of \( \text{Re} \ A_{SD} \) can be neglected, we get the branching ratio of the real part in Eq.(1) to be \( 1.93 \times 10^{-8} \). The total decay branching ratio is about \( 6.68 \times 10^{-8} \). Our prediction is smaller than the experimental value of \( B_{\pi^0 \rightarrow e^+e^-}^{\text{KTeV}} = (7.48 \pm 0.29 \pm 0.25) \times 10^{-8} \) measured by KTeV. We note that our result is larger than the values of \( (6.41 \pm 0.19) \times 10^{-8} \) and \( 6 \times 10^{-8} \) calculated in Ref.[5, 6] with the VMD and quark model(QM), respectively, but close to \( (7 \pm 1) \times 10^{-8} \) in the Chiral Perturbation Theory(ChPT)[3]. It is clear that we provide a method to calculate the form factor of \( \pi^0 \rightarrow \gamma^* \gamma^* \) and get a result in \( \pi^0 \rightarrow e^+e^- \) within the LFQM.

The \( \eta \rightarrow l^+l^- \) decay can be analyzed in a similar technique as \( \pi^0 \rightarrow e^+e^- \). In the momentum limit \( P^2 \rightarrow 0 \), we obtained

\[
\begin{align*}
\text{Re} \ A_{(2e)LD}(0) & \simeq -22.43, \\
\text{Re} \ A_{(2\mu)LD}(0) & \simeq -6.48 .
\end{align*}
\] (44)

Form the dispersive integral in Eq.(43) and Eq.(44), one obtains

\[
\begin{align*}
\text{Re} \ A_{(2e)LD}(m_\eta^2) & \simeq 27.11, \\
\text{Re} \ A_{(2\mu)LD}(m_\eta^2) & \simeq -2.81 .
\end{align*}
\] (45)

The SD contributions to the decays can be still ignored and the total branching ratios are given by

\[
\begin{align*}
B_{\eta \rightarrow e^+e^-} & = 4.47 \times 10^{-9}, \\
B_{\eta \rightarrow \mu^+\mu^-} & = 5.47 \times 10^{-6} .
\end{align*}
\] (46)

One notes that the value of \( B_{\eta \rightarrow e^+e^-} \) is larger than the CLEO result[2]. For the mode of \( \eta \rightarrow \mu^+\mu^- \), it is consistent with the CLEO[2] and VMD results[3]. It also agrees with the PDG data of \( 5.8 \pm 0.8 \times 10^{-5} \).

We summarized the related experimental and theoretical values of the decay branching ratios of \( \pi^0 \rightarrow e^+e^-\gamma \), \( \pi^0 \rightarrow e^+e^-e^-e^- \) and \( \pi^0 \rightarrow e^+e^- \) in Table I and \( \eta \rightarrow l^+l^-\gamma \), \( \eta \rightarrow l^+l^-l^+l^- \) and \( \eta \rightarrow l^+l^- \) in Table II.

**IV. CONCLUSIONS**
TABLE I. Summary of the decays of $\pi^0$ with lepton pair.

| Br     | Exp. data          | This work | Other models |
|--------|--------------------|-----------|--------------|
| $10^2 B_{e^+e^-}$ | $1.174 \pm 0.035[27]$ | 1.18      | $1.18[14][15][16]$ |
| $10^5 B_{e^+e^-e^+e^-}$ | $3.34 \pm 0.16[27]$ | 3.29      | $3.28[14], 3.46[15]$ |
| $10^8 B_{e^+e^-}$ | $7.48 \pm 0.29 \pm 0.25[1, 2]$ | 6.68      | $7 \pm 1[3], 8.3 \pm 0.4[4], 6.41 \pm 0.19[5], 6[6]$ |
|         | $6.46 \pm 0.33[27]$ |           | $< 4.7[7], 6.23 \pm 0.09[2, 30]$ |

TABLE II. Summary of the decays of $\eta$ with lepton pair.

| Br     | Exp. data               | This work | Other models |
|--------|-------------------------|-----------|--------------|
| $10^3 B_{e^+e^-}$ | $7.8 \pm 0.5_{stat} \pm 0.7_{syst}[33]$ | 6.95      | $9.4 \pm 0.7[29]$ |
|         | $7.0 \pm 0.7[27]$       |           | $6.31 - 6.46[31], 6.5[32]$ |
| $10^4 B_{\mu^+\mu^-}$ | $3.1 \pm 0.4[27]$ | 6.95      | $2.14 - 3.01[31], 3.0[32]$ |
| $10^5 B_{e^+e^-e^+e^-}$ | $2.7^{+2.1}_{-2.7_{stat}} \pm 0.1_{syst}[33]$ | 2.47      | $2.49 - 2.62[31]$ |
|         | $< 6.9[27]$             |           |              |
| $10^7 B_{\mu^+\mu^-e^+e^-}$ | $< 1.6 \times 10^3[27]$ | 5.83      | $1.57 - 2.21[31]$ |
| $10^9 B_{\mu^+\mu^-\mu^+\mu^-}$ | $< 3.6 \times 10^6[27]$ | 1.68      |              |
| $10^9 B_{e^+e^-}$ | $< 2.7 \times 10^4[27]$ | 4.47      | $13.7[5], 4.60 \pm 0.06[2, 30]$ |
| $10^6 B_{\mu^+\mu^-}$ | $5.8 \pm 0.8[27]$ | 5.47      | $5.8 \pm 0.2[3], 11.4[5]$ |
|         |                        |           | $5.11 \pm 0.20[2, 30], 5.2 \pm 1.2[34]$ |

We have calculated the form factors of $P \to \gamma^*\gamma^* (P = \pi^0, \eta)$ directly within the LFQM. In our calculations, we have adopted the Gaussian-type wave function and evaluated the form factors for the momentum dependences in the energy regions from $q^2 = 0$ to $m^2_P$. Using the form factors, we have examined $\pi^0 \to e^+e^-\gamma$ and $\pi^0 \to e^+e^-e^+e^-$ and shown that our results on the decay branching ratios agree well with the experimental data shown in Table I. Our predicted values are also close to those in the QED and VMD models [14–16]. For $\pi^0 \to e^+e^-$, we have found that $B_{\pi^0 \to e^+e^-}$ is $6.68 \times 10^{-8}$, which agrees with $(7 \pm 1) \times 10^{-8}$ in the ChPT [3] but larger than those in Refs. [5, 7]. We have demonstrated that the long-distance dispersive contribution in this model is possibly small. However, like other theoretical predictions, our result for $\pi^0 \to e^+e^-$ is also slightly smaller than the experimental data. Clearly, further theoretical studies as well as more precise experimental data such as those from the KTeV-
E799 experiment at Fermilab on the spectra of the decays with lepton pair are needed. About the $\eta$ decays, our results are all consistent with the experimental data. In particular, the branching ratios of $\eta \rightarrow 2e2\mu$, $\eta \rightarrow 4\mu$ and $\eta \rightarrow 2e$ are expected to be 4~5 orders of magnitude lower than the current experimental upper limits.

V. ACKNOWLEDGMENTS

This work is supported in part by the National Science Council of R.O.C. under Contract NSC-97-2112-M-471-002-MY3.

[1] E. Abouzaid, et al., Phys. Rev. D75, 012004 (2007).
[2] A.E. Dorokhov, Nucl. Phys. Proc. Suppl. (2008) 181-182: 37-41.
[3] Martin J. Savage, Michael Luke and Mark B. Wise, Phys. Lett. B291 (1992) 481.
[4] D. Gomez Dumm and A. Pich, Phys. Rev. Lett. 80 (1998) 4633.
[5] Ll. Ametller, A. Bramon and E. Masso, Phys. Rev. D48 (1993) 3388.
[6] L. Bergstrom, E. Masso, Ll. Ametller and A. Bramon, Phys. Lett. 126B (1983) 117.
[7] Mahendra Pratap and Jack Smith, Phys. Rev. D8 (1972) 5.
[8] S. D. Drell, Nuove Cimento 11 (1959) 693.
[9] S. M. Berman and D. A. Geffen, Nuove Cimento 18 (1960) 1192.
[10] L. Bergstrom, Z. Phys. C14 (1982) 129.
[11] C. Quigg and J. D. Jackson, UCRL Report No. 18487 (1968).
[12] T. Inami and C. S. Lim, Prog. Theor. Phys. 65 (1981) 297.
[13] C. Q. Geng and J. N. Ng, Phys. Rev. D41 (1990) 2351; G. Belanger and C. Q. Geng, Phys. Rev. D43 (1991) 140.
[14] Tadashi Miyazaki and Eiichi Takasugi, Phys. Rev. D8 (1973) 2051.
[15] N. M. Kroll and W. Wada, Phys. Rev. 98 (1955) 1355.
[16] K. S. Babu and E. Ma, Phys. Lett. B119 (1982) 449.
[17] Thorsten Feldmann and Peter Kroll, Eur. Phys. J. C58 (1998) 327.
[18] C. H. Chen, C. Q. Geng and C. C. Lih, Phys. Rev. D77, 014004 (2008); Int. J. Mod. Phys. A23, (2008) 3204.
[19] H. J. Melosh, Phys. Rev. D9 (1974) 1095.
[20] W. Jaus, Phys. Rev. D41 (1990) 3394; 44 (1991) 2851.
[21] Demchuk et la., Phys. Atom. Nucl 59 (1996) 2152.
[22] K. G. Wilson, T. S. Walhout, A. Harindranath, W. M. Zhang, R. J. Perry and S. D. Glazek
Phys. Rev. D49 (1994) 6720.
[23] C. Q. Geng, C. C. Lih and W. M. Zhang, Phys. Rev. D57 (1998) 5697; Phys. Rev. D62, 074017 (2000); Mod. Phys. Lett. A15 (2000) 2087.
[24] C. C. Lih, C. Q. Geng and W. M. Zhang, Phys. Rev. D59, 114002 (1999).
[25] C. Q. Geng, C. C. Lih and C. C. Liu, Phys. Rev. D62, 034019 (2000); C. H. Chen, C. Q. Geng, C. C. Lih and C. C. Liu, Phys. Rev. D75, 074010 (2007).
[26] H.Y. Cheng, C.Y. Cheung and C.W. Hwang, Phys. Rev. D55, 1559 (1997). C.W. Hwang,
Phys. Rev. D64, 034011 (2001).
[27] Particle Data Group, Phys. Lett. B667 (2008) 1.
[28] Tadashi Miyazaki, Nuove Cimento Lett. 5 (1972) 125.
[29] A. Lopez et la., Phys. Rev. Lett. 99, 122001 (2007).
[30] A.E. Dorokhov and M. A. Ivanov, Phys. Rev. D75 114007 (2007).
[31] Johan Bijnens, Fredrik Persson, Phys. Lett. A289 (2001) 301.
[32] Amand Faessler, C. Fuchs, M.I. Krivoruchenko, Phys. Rev. C61, 035206 (2000).
[33] M. Berlowski et la., hep-ex/0711.3531.
[34] Z. K. Silagadze, Phys. Rev. D74 054003 (2006).