Combined Paramagnetic and Diamagnetic Response of YBCO

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Abstract

It has been predicted that the zero frequency density of states of YBCO in the superconducting phase can display interesting anisotropy effects when a magnetic field is applied parallel to the copper-oxide planes, due to the diamagnetic response of the quasi-particles. In this paper we incorporate paramagnetism into the theory and show that it lessens the anisotropy and can even eliminate it altogether. At the same time paramagnetism also changes the scaling with the square root of the magnetic field first deduced by Volovik leading to an experimentally testable prediction. We also map out the analytic structure of the zero frequency density of states as a function of the diamagnetic and paramagnetic energies. At certain critical magnetic field values we predict kinks as we vary the magnetic field. However these probably lie beyond currently accessible field strengths.
I. INTRODUCTION

In 1993 Volovik [1] predicted that the density of states of a \textit{d}-wave superconductor is dominated by delocalised nodal quasi-particles and scales as the square root of the magnetic field. Since then there has been a lot of research on the role of nodal quasi-particles in determining the electronic density of states of the cuprate superconductors. Experimentally this is probed by looking at the specific heat as a function of magnetic field [2–5]. This is a bulk property and therefore complements STM and ARPES measurements which are surface probes. (On the other hand, it is a somewhat less direct probe of the electronic density of states.) Theoretically, groups have studied the effect of dirty superconductors [6], the role of the copper oxide chains in affecting the specific heat [7] and the anisotropic dependences on magnetic field direction [8]. In particular, the last paper predicted that the observed dependence of the density of states of an in plane magnetic field would depend on its orientation relative to the gap nodes. That paper only accounted for the diamagnetic response of the quasi-particles, as in Volovik’s original paper. In this paper we generalise this to include the paramagnetic response of the quasi-particles. The quasi-particle paramagnetism has been considered in different contexts in [9,10]. We focus on YBCO because only it has sufficient transport in the \textit{c} direction to allow the usual London theory to apply so that vortices to form out of the CuO planes [8]. Other relevant papers that treat different aspects of the problem include [11–13].

In the following section we present a general discussion of the theory and define the various terms. In section III we discuss the energy scales specific to YBCO. The two subsequent sections present the results for the physical case of relatively small magnetic field for the field perpendicular and parallel to the CuO planes, respectively. With small fields, a particularly simple approximation can be invoked leading to analytic results. In the parallel case, we predict a loss of anisotropy at sufficiently large magnetic fields. In section VI we present a more physical argument for the loss of anisotropy. We summarise our results in the conclusion. In the appendix, we allow for fields of arbitrary strength but only in the
situation where the field is perpendicular to the copper-oxide planes.

II. GENERAL THEORY

The approach we use is to consider a BCS-like formalism [14] with a \( d \)-wave gap. Since YBCO is a type-II superconductor, there are vortices above a critical field \( H_{c1} \) and we will focus on the regime \( H_{c1} \ll H \ll H_{c2} \). This condition ensures that \( \xi \ll R \ll \lambda \) where \( \xi \), \( R \) and \( \lambda \) are the coherence length, vortex radius and penetration depth, respectively. In fact, \( H_{c1} \) is so small (\( \sim 100 \) Gauss [15]) and \( H_{c2} \) so large (\( \sim 100 \) Tesla) that one is always in this experimental range. Around each vortex there is a rotating superfluid condensate. Quasiparticles are excited out of this condensate and are accordingly Doppler-shifted by the superfluid velocity. The effect of this Doppler shift is to give a finite density of states, even at zero frequency, and we refer to this as the diamagnetic effect. In addition, the individual quasi-particles have a paramagnetic response to the magnetic field due to their intrinsic spin and we refer to this as the paramagnetic effect. In this paper, we are interested in the competition between these two effects, there being two important geometries. The simpler one is when the field is parallel to the \( c \)-axis (\( H \parallel c \)) in which event we can study effects for all magnetic field values. The more physically interesting geometry, however, is that the field is parallel to the \( a - b \) plane (\( H \parallel ab \)). In this case the diamagnetism results in an anisotropic dependence of the density of states on the direction of the applied magnetic field, with a greater value when the field points along the gap antinodal direction and a lesser value when the field points along the nodal direction. We show that the paramagnetism lessens this effect and at a critical magnetic field makes it disappear altogether.

Due to the paramagnetic shift, the density of states is broken into two components which can be written semiclassically as

\[
N_0^{(\pm)} \propto \frac{1}{\pi R^2} \int \! dr \int \! dk |V| \delta \left( \left( \sqrt{\xi_k^2 + \Delta_k^2} \pm \mu H \right)^2 - V^2 \right).
\]

This is a simple generalisation of the formalism of [3] where we have included the paramagnetic term \( \pm \mu H \). We describe each factor in turn. The \( d \)-wave gap is given by
\[ \Delta_k = \Delta_0 \cos^2(2\phi) \] and \[ \xi_k = k^2 - \mu_c \] so that \[ \sqrt{\xi_k^2 + \Delta_k^2} \pm \mu H \] is the quasi-particle excitation energy. (Here we have defined the quasi-particle energy relative to the chemical potential \( \mu_c \).) The integral over \( k \) (with polar coordinates \( k \) and \( \phi \)) is a standard trace integral used to determine the density of states. We will use the delta function to do the radial \( k \) integral. The integral over \( r \) is an average over one unit cell of the vortex lattice; \( r \) is the distance from the vortex core and \( \beta \) is the vortex winding angle. It is convenient to nondimensionalise the radial integral to \( \rho = r/R \), where \( R \) is the inter-vortex distance

\[ R = a^{-1} \sqrt{\frac{\Phi_0}{\pi H}}, \]  

(2)
a is a geometrical constant accounting for the mismatch between the circular vortices and the regular lattice they fill out and \( \Phi_0 = \hbar c/2e \) is a flux quantum.

The factor \( V = v_s \cdot k_F \) is the Doppler shift which depends on the condensate velocity \( v_s \) (which in turn depends on \( r \)) and on the Fermi momentum \( k_F \). We first consider \( H \parallel c \); far from the vortex core (but still within the penetration depth scale) this is approximately

\[ V \approx \frac{E_H}{\rho} \sin(\beta - \phi) \quad H \parallel c. \]  

(3)

\( E_H = v_F a \sqrt{\pi H/4\Phi_0} \) is an energy scale associated with the diamagnetism (and \( v_F \) is the Fermi velocity.) This yields \( E_H = 2.4\sqrt{H} \) [8] where \( E_H \) is measured in meV and \( H \) in Tesla. In practise, when integrating over the winding angle \( \beta \), we change variables to \( \beta - \phi \) so the sinusoid is \( \sin \beta \). There are two differences when \( H \parallel ab \). Firstly, the differently aligned vortices implies

\[ V \approx \frac{E_H}{\rho} \sin \beta \sin(\phi - \alpha) \quad H \parallel ab. \]  

(4)

There is an additional parameter \( \alpha \) which describes the orientation of the field with respect to the axis directions in the \( ab \) plane. Specifically, when \( \alpha = 0 \), the field points along the antinodal direction of the gap function while when \( \alpha = \pi/4 \), the field points along the nodal direction. The second difference is that the energy scale \( E_H \) is smaller. This is due to the fact that the vortices must form out of the copper oxide planes. This leads to a rescaling of
the effective Fermi velocity by \( \sqrt{\lambda_{ab}/\lambda_c} \) where \( \lambda_i \) is the penetration depth in direction \( i \). \( E_H \) is accordingly reduced by a factor of about 2.5 relative to the \( H \parallel c \) case.

For now, we shall keep \( V \) arbitrary and thereby include both field alignments with the same analysis. To begin, we need to find the values of \( k \) for which the argument of the delta function is zero. For the \( +\mu H \) term this is given by

\[
\sqrt{\xi_k^2 + \Delta_k^2} + \mu H - |V| = 0.
\] (5)

A solution may or may not exist depending on the values of \( \phi, \beta \) and \( \rho \). (In principle we should also allow for the solution with \( +|V| \), but in practise there are no solutions to that equation.) We shall refer to the contribution arising from this term as \( N_0^{(+)} \) (the reason for this name is made clear below.) For the \( -\mu H \) term the analysis is a little more complicated. Specifically, we now must allow for the possibility of two distinct roots of the form

\[
\sqrt{\xi_k^2 + \Delta_k^2} - \mu H \pm |V| = 0.
\] (6)

Here we allow for both signs of \( |V| \). One should not confuse the \( \pm \) in (6) with the \( \pm \) in the choice of spin direction in (1). We shall call the respective contributions \( N_0^{(-+)} \) and \( N_0^{(--)} \) where the first sign refers to \( \mu H \) and the second to \( |V| \).

As stated, we use the \( \delta \) function to do the \( k \) integral. This leads to the following expressions for the three terms (where we normalise by recalling that for large fields, the total density of states should approach that of the normal state \( N \)):

\[
\frac{N_0^{(+)} + N_0^{(-+)}}{N} = \frac{1}{4\pi^2} \int_0^1 d\rho \int_0^{2\pi} d\beta \int_0^{2\pi} d\phi \ P \left( \frac{|V| - \mu H}{\sqrt{(|V| - \mu H)^2 - \Delta_k^2}} \right)
\]

\[
\frac{N_0^{(-)} + N_0^{(--)}}{N} = \frac{1}{4\pi^2} \int_0^1 d\rho \int_0^{2\pi} d\beta \int_0^{2\pi} d\phi \ P \left( \frac{|V| + \mu H}{\sqrt{(|V| + \mu H)^2 - \Delta_k^2}} \right)
\]

\[
\frac{N_0^{(-+)} + N_0^{(--)}}{N} = \frac{1}{4\pi^2} \int_0^1 d\rho \int_0^{2\pi} d\beta \int_0^{2\pi} d\phi \ P \left( \frac{\mu H - |V|}{\sqrt{(\mu H - |V|)^2 - \Delta_k^2}} \right).
\] (7)

The function \( P(x) \) equals \( x \) if \( x \) is real and positive, otherwise it is zero (this yields the ranges over which the respective solutions of Eqs.(5) and (6) exist.) Our goal is to do these three
integrals and thereby calculate the density of states and specific heat. It is convenient at this point to introduce the dimensionless quantities $\nu = E_H/\Delta_0$ and $\sigma = \mu H/\Delta_0$ describing the energies associated with the diamagnetism and paramagnetism respectively. Mathematically we can treat them as independent quantities. In reality they are dependent since $\nu \propto \sqrt{H}$ while $\sigma \propto H$ so that $\sigma \propto \nu^2$, but this can be imposed later.

We begin by assuming that the fields are small enough to use the “nodal approximation”, which amounts to replacing the gap function $\Delta_k$ by a local linear approximation in the neighbourhood of the gap nodes. This leads to a very simple $\phi$ integral. This approximation is valid in the limit of small $\nu$ and $\sigma$ since in that event the range of contributing $\phi$ values in (7) is very small and a linear approximation is valid throughout the allowed integration range. It is convenient to first take $H \parallel c$. This then sets up the small field $H \parallel ab$ calculation. The small field limit underlying the nodal approximation is physically relevant since it corresponds to currently accessible magnetic field strengths. However, it is also interesting to explore the structure for arbitrary fields; this is done in the appendix.

III. ENERGY SCALES

We now discuss the energy scales in the problem. It will be convenient to parameterise the diamagnetic and paramagnetic energies as follows. We define the coefficients $a$ and $b$ such that $\nu = a\sqrt{H}$ and $\sigma = bH$ and the functional relationship between $\nu$ and $\sigma$ is $\sigma = b\nu^2/a^2$. Knowing the coefficients $a$ and $b$ gives us the full information about the thermodynamic response of the system. For example, the cross-over field $H_c$ where $\nu = \sigma$ is given as $(a/b)^2$.

We begin by noting that for YBCO $\Delta_0$ is about 20 $meV$. The paramagnetic energy is simple since it is just $\mu H$ which equals 0.058$H$ where $H$ is measured in Tesla and the energy in $meV$ (henceforth these units will be understood.) Together these imply $b = 2.9\times10^{-3}$. The diamagnetic energy scale is trickier; to date it has been found directly from experimental measurements. This point is discussed in Refs. [13] where it is argued that $E_H$ is different for $H \parallel c$ and $H \parallel ab$. In the first case, $E_H \approx 2.4\sqrt{H}$ so that $a = 0.12$ and $\sigma = 0.20\nu^2$. 
The cross-over field is therefore \( H_e = 1.7 \times 10^3 \text{T} \). This is enormous and we can confidently state that for all experimentally accessible fields, the diamagnetism completely dominates the paramagnetism.

The \( H \parallel ab \) case is more ambiguous. Based on the ratio of the penetration depths in the \( c \) direction and the \( a \) and \( b \) directions, the authors of Refs. [16] estimated that \( E_H \approx 0.7 \sqrt{H} \) leading to \( a = 0.035 \) so that the cross-over field is \( H_e = 147 \text{T} \). On the other hand, experiments which compared the scaling of the specific heat with \( \sqrt{H} \) in the \( c \) and \( a \times b \) directions led to the conclusion that \( E_H < 0.4 \sqrt{H} \) [17]. Taking the upper limit of the inequality for concreteness, we would have \( a = 0.020 \) so that the cross-over field is \( H_e = 48 \text{T} \).

While the cross-over happens at rather large fields, we will show that the loss of anisotropy is apparent at smaller fields which are certainly experimentally accessible.

IV. \( H \parallel c \)

We first consider the field parallel to the \( c \)-axis. For this situation, all four gap nodes contribute identically. Also, we can use symmetry to restrict the range of \( \beta \) and thereby dispense with the absolute value signs around \( V \). Using the nodal approximation to do the \( \phi \) integral, we find

\[
\begin{align*}
\frac{N_0^{(+\mp)}}{N} & \approx \frac{2}{\pi} \int_0^1 \mathrm{d}\rho \int_0^{\pi/2} \mathrm{d}\beta \, P(\nu \sin \beta - \sigma \rho) \\
\frac{N_0^{(-\mp)}}{N} & \approx \frac{2}{\pi} \int_0^1 \mathrm{d}\rho \int_0^{\pi/2} \mathrm{d}\beta \, P(\nu \sin \beta + \sigma \rho) \\
\frac{N_0^{(+-)}}{N} & \approx \frac{2}{\pi} \int_0^1 \mathrm{d}\rho \int_0^{\pi/2} \mathrm{d}\beta \, P(\sigma \rho - \nu \sin \beta).
\end{align*}
\]

(Recall that \( P \) limits the integration ranges so that the corresponding zero solution of the original delta function argument exists.) It is interesting to remark that there is no limitation on the integration range of \( N_0^{(-\mp)} \), which contributes throughout the vortex unit cell for all values of \( \sigma \) and \( \nu \). The other two terms are restricted and in fact, their allowed ranges in the \( \rho - \beta \) space are complementary. There is a physical meaning to this. It says that in those regions of the vortex unit cell, it is energetically favourable for the quasi-particle to flip its
spin even though this costs it diamagnetic energy. In the limit where the diamagnetism is dominant, we have only $N_0^{(+,-)}$. In the limit where the paramagnetism is dominant, we have only $N_0^{(-,+)}$, that is all quasi-particles are spin down. In the regime where the diamagnetism and paramagnetism are competitive, the final result involves a nontrivial combination of both terms. These three terms can also be combined and expressed as

$$
\frac{N_0}{N} \approx \frac{4}{\pi} \int_0^1 d\rho \int_0^{\pi/2} d\beta \left( |\nu \sin \beta + \sigma \rho| + |\sigma \rho - \nu \sin \beta| \right). \tag{9}
$$

From here the analysis is simple and summing all three terms we conclude

$$
\frac{N_0}{N} \approx \sigma + \frac{\nu^2}{2\sigma} \quad \nu \leq \sigma
$$

$$
\approx \frac{2}{\pi} \left( \left( \sigma + \frac{\nu^2}{2\sigma} \right) \arcsin \left( \frac{\sigma}{\nu} \right) + \frac{3\nu}{2} \sqrt{1 - \frac{\sigma^2}{\nu^2}} \right) \quad \nu > \sigma. \tag{10}
$$

Clearly these two expressions agree when $\sigma = \nu$. They also give the correct behaviour for either $\sigma = 0$ or $\nu = 0$. For later purposes, it is be convenient to collectively express these as

$$
\frac{N_0}{N} \approx F(\sigma, \nu). \tag{11}
$$

The function $F$ has a nice scaling property. Specifically,

$$
F(\sigma, \nu) = \nu f(\sigma/\nu) \tag{12}
$$

where

$$
f(x) = x + \frac{1}{2x} \quad x > 1
$$

$$
= \frac{2}{\pi} \left[ \left( x + \frac{1}{2x} \right) \arcsin(x) + \frac{3}{2} \sqrt{1 - x^2} \right] \quad x < 1. \tag{13}
$$

Other than the prefactor in (12), the function $F$ is given by a simple universal form.

In Fig. 1 we show the function $f$ plotted as a function of $x$. To aid comparison, we plot the $x > 1$ expression as a dashed line in the range $x < 1$. The change in functional form at $x = 1$ represents a very mild nonanalyticity which is not observable by eye. Analysis of the function $f(x)$ shows that the first three derivatives are continuous at $x = 1$ and that the $x < 1$ branch suffers a $(1 - x)^{7/2}$ discontinuity as it approaches $x = 1$ from below.
FIG. 1. The function $f$ as a function of $x$. There is a change in functional form at $x = 1$; we have continued the $x > 1$ curve down to $x = 0$ as a dashed line for the sake of comparison.

V. $H \parallel ab$ AND ANISOTROPY

We now consider the more interesting situation where the magnetic field is parallel to the $ab$ planes. We continue to use the nodal approximation and again assume that the $\phi$ integral is dominated by the regions around the four gap nodes. We define $\phi_n = (2n - 1)\pi/4$ such that $\Delta(\phi_n) = 0$ and divide the $\phi$ integral into four domains centred on these values. In contrast to the previous case, there is a more complicated $\phi$ dependence since it appears in $V$ itself. However, to the same order of approximation, we ignore the variation of $\phi$ when evaluating $V$, and simply use the value of $\phi_n$ appropriate to that domain. The development then follows the previous case but where we must consider each of the four $\phi$ integrals (as indexed by $n$) and also keep track of the $\alpha$ dependence. We also restrict the $\beta$ range as before. The result is

$$\frac{N_0^{(+\, -)}}{N} \approx \sum_n \frac{1}{2\pi} \int_0^1 d\rho \int_0^{\pi/2} d\beta \, P(\nu | \sin(\phi_n - \alpha)| \sin \beta - \sigma \rho)$$

$$\frac{N_0^{(-\, -)}}{N} \approx \sum_n \frac{1}{2\pi} \int_0^1 d\rho \int_0^{\pi/2} d\beta \, (\nu | \sin(\phi_n - \alpha)| \sin \beta + \sigma \rho)$$
\[
\frac{N_0^{(-+)}-N_0}{N} \approx \sum_n \frac{1}{2\pi} \int_0^1 d\rho \int_0^{\pi/2} d\beta \ (-\nu|\sin(\phi_n - \alpha)| \sin \beta + \sigma \rho).
\]

By inspection each of these integrals has the same structure as Eq. (8) but with a rescaled value for \( \nu \). We then conclude
\[
\frac{N_0}{N} \approx \frac{1}{4} \sum_n F(\sigma, \nu|\sin(\phi_n - \alpha)|).
\]

By symmetry, the \( n = 1 \) term equals the \( n = 3 \) term and the \( n = 2 \) term equals the \( n = 4 \) term.

\[\text{FIG. 2. The electronic density of states as a function of } \alpha \text{ for } H = 10T \text{ and using the experimentally determined value } a = 0.02. \text{ The upper curve is for the physical value } b = 0.0029. \text{ The lower curve is for comparison and is without paramagnetism (i.e. } b = 0.)\]

As mentioned, one aspect of \( H \parallel ab \) is that there can be an interesting anisotropy as we vary the angle \( \alpha \). The maximum value is for \( \alpha = 0 \) while the minimum is for \( \alpha = \pi/4 \):
\[
\left. \frac{N_0}{N} \right|_{\alpha=0} \approx F(\sigma, \nu/\sqrt{2}) \\
\left. \frac{N_0}{N} \right|_{\alpha=\pi/4} \approx \frac{1}{2} (F(\sigma, 0) + F(\sigma, \nu))
\]

We define \( \Delta N \) to be the difference between these two values. There are two interesting limits that we can evaluate analytically:
\[
\Delta N \approx \frac{2}{\pi} (\sqrt{2} - 1) \nu - \frac{\sigma}{2} \quad \sigma \ll \nu
\]
When $\sigma = 0$, the first of these is the result found in [8]. We note that introducing the paramagnetism lessens the amount of anisotropy. This trend continues as $\sigma$ increases until, at the critical field where $\sigma = \nu$, the anisotropy completely vanishes. This last result can be found with more direct physical arguments which we present in Section [VI].

A relevant field scale is given by where $\Delta N$ is a maximum. Analysis of (17) shows that this happens at $H_m = 0.07H_e$. (Note that at this field, $\sigma = 0.26\nu$ so that the criterion $\sigma \ll \nu$ in (17) is roughly satisfied.) For the theoretical estimate of $a = 0.035$, we have $H_m = 10$ Tesla while for the experimental value (and the one we trust more) of $a = 0.02$, we have $H_m = 3.3$ Tesla. It is this field scale which effectively controls when the overall behaviour becomes significantly affected by the paramagnetism.

In Fig. 2 we show the angular dependence of the density of states for a fixed magnetic field as we increase the amount of paramagnetism. As we increase the paramagnetism, the density of states increases; this effect is particularly pronounced at $\alpha = \pi/4$, as we expect. Consequently, the amount of anisotropy decreases, again in agreement with our general discussion.
FIG. 3. The electronic density of states as a function of $\sqrt{H}$ (with $H$ measure in Tesla) for $a = 0.020$. The upper curve in each case is for $\alpha = 0$ while the lower curve is for $\alpha = \pi/4$; the difference between the curves is the magnitude of anisotropy. In the left figure we took $b = 0.0029$, the right figure is for comparison without paramagnetism (i.e. $b = 0$.)

Next we show the anisotropy as a function of magnetic field. In Fig. 3, we plot the field dependence of the density of states using the experimental estimates of $a = 0.02$ and $b = 0.0029$. The upper curve for each pair is for the field orientation $\alpha = 0$ while the lower one is for $\alpha = \pi/4$. Notice that they start with different slope but ultimately converge leading to a limited range of field strengths over which the anisotropy is present. For comparison we also show the results without anisotropy for which the two curves never converge. Also note that around the scale $E_m$ defined above, the two curves begin to converge. At around that scale there begins a noticeable nonlinear dependence of the density of states on $\sqrt{H}$, which is in principle measurable. The calculation presented here is incompatible with a density of states which is both isotropic and linear in $\sqrt{H}$, thus providing a clear experimental signature. The data of [17] is marginal in that the error bars are large enough to not be in conflict with theory. However, a more precise measurement or one that went to larger fields would presumably be able to settle this question. Preliminary data from another group [5] does seem to be consistent with a small anisotropy.
FIG. 4. The lower figure shows the absolute value of the anisotropy while the upper figure shows the relative value of the anisotropy ($\Delta N$ divided by the value at $\alpha = 0$). The solid line is from using $a = 0.020$ while the dashed line is from using $a = 0.035$.

In Fig. 4 we plot both the absolute anisotropy which initially grows with magnetic field as the overall scale of the diamagnetism increases with magnetic field. However at some field strength of order $H_m$ it stops growing as the paramagnetic effects begin taking hold and finally vanishes when the paramagnetic effect dominates. We also plot the relative anisotropy, \textit{i.e.} the absolute anisotropy divided by the value at $\alpha = 0$. This decreases abruptly with magnetic field (with a square root dependence for small field) so that even relatively small values of $\sigma$ can have a large affect on the anisotropy. We plot the results for two values of $a$ showing the strong dependence of this affect on the diamagnetic energy scale. Unfortunately this is not a well established number, however we have the greatest confidence in $a = 0.02$ which is shown as the solid curve in the figure.

VI. HEURISTIC ARGUMENT FOR LOSS OF ANISOTROPY

The anisotropy arises entirely from the diamagnetism, so it certainly makes sense that it is negligible in the limit that the paramagnetic effect is much stronger than the diamagnetic effect. However, it is not entirely intuitive that it should disappear completely when the effects are of comparable importance \textit{(i.e. when $\nu = \sigma$)}. In this section we present a heuristic explanation of the anisotropy. We begin by showing a picture of what happens when a small amount of paramagnetism is introduced to an otherwise purely diamagnetic problem. We then present the opposite limit of near perfect paramagnetism with a small amount of diamagnetism. We then bridge these two limits and in the process present a concrete explanation of why the anisotropy switches off precisely when $\nu = \sigma$. 

13
FIG. 5. Each “V” shows the dependence of the density of states on the argument $\omega$. It is locally linear but with a cusp at zero argument. We show the results for the field aligned along the gap antinode (top two) and aligned along the gap node (bottom two). In each row, the left hand figure is for the $\phi = 3\pi/4$ gap node and the right hand figure is for the $\phi = \pi/4$ gap node. The solid line in each case shows the result with no paramagnetism (note that there is no contribution from the bottom right diagram in this case.) The contribution from the two nodes for $\alpha = 0$ outweighs the one contribution for $\alpha = \pi/4$. We indicate the effect of adding paramagnetism with the dashed lines. Each solid line gets split into two contributions but the average of the two equals the result without paramagnetism, due to the linear behaviour, so there is no change to the net contribution. For the bottom right diagram this is not true and there is a net increase in the contribution. This lessens the anisotropy.

In Fig. 5 we show the dependence of the density of states as a function of frequency, in brief $N(\omega) \propto |\omega|$. Although all of our calculations involve the physical frequency being zero, due to the Doppler and paramagnetic shifts, our final answers involve evaluating this density of states at various arguments. For example, in Eq. (9), we can interpret the integrands $|\sigma \rho \pm \nu \sin(\phi - \alpha)|$ as being the density of states $N(\omega)$ evaluated at $\omega = \sigma \rho \pm \nu \sin(\phi - \alpha)$. All of the interplay between the paramagnetism and diamagnetism involves arguments being near zero so that the cusp in the function plays an important role. For concreteness, we
take $\beta = \pi/2$ and $\rho = 1$; we will discuss what happens elsewhere in the vortex unit cell in the subsequent discussion. In the purely diamagnetic limit, we set $\sigma = 0$. For $H \parallel c$, all nodes contribute identically and there is nothing interesting to say. Instead we will focus on $H \parallel ab$. In this case, the nodes at $\phi = \pi/4$ and $5\pi/4$ are identical as are the nodes at $\phi = 3\pi/4$ and $7\pi/4$, so we will only look at one of each of these pairs. Recall that the rescaled Doppler shift is $|\nu \sin(\phi - \alpha)|$. When $\alpha = 0$, all nodes contribute identically as $\nu/\sqrt{2}$; as indicated in the top two diagrams of the figure. When $\alpha = \pi/4$, one pair of nodes contributes as $\nu$ while the other does not. The dependence on the argument is linear, meaning a greater density of states for $\alpha = 0$ than for $\alpha = \pi/4$ (since $2\nu/\sqrt{2} > \nu$).

Imagine now that we add a small amount of paramagnetism, in other words set $\sigma$ to be a small but non-zero value. This induces a splitting in the argument at which we evaluate $N(\omega)$ to $\nu|\sin(\phi - \alpha)| \pm \sigma$. As long as $\nu \sin(\phi - \alpha)$ is larger in magnitude than $\sigma$, this splitting has no effect. This is because $N(\omega)$ is locally linear so that the extra amount from one sign of $\sigma$ is exactly compensated for by the other sign of $\sigma$. The only case that this does not apply to is the $\phi = \pi/4$ node when $\alpha = \pi/4$. In this event, both signs of $\sigma$ contribute equally and there is no cancellation. This implies an increase in the density of states for $\alpha = \pi/4$ and no change for $\alpha = 0$ so that the anisotropy is lessened, in accord with Eq. (17).

(We recall that this was all for the one point $\rho = 1$ and $\beta = \pi/2$. When we average over the vortex unit cell, there are regions of small $\beta$ where $\sigma \rho$ dominates the diamagnetic shift. This leads to corrections of order $\sigma^2$ to the final, average density of states. This does not affect our general conclusion that adding paramagnetism lessens the anisotropy.)

We now consider the opposite limit where the paramagnetism dominates. With no diamagnetism, we simply evaluate the density of states at $\pm \sigma$. This is clearly the same for all orientations and there is no anisotropy. We now add a small amount of diamagnetism, in other words set $\nu$ to be a small but non-zero value. Analogously to before, we have shifts in the argument by the amount $\pm \nu \sin(\phi - \alpha)$. As before, the local linear dependence of $N(\omega)$ means that the net effect cancels and there is no change to the densities of states and hence no introduction of anisotropy. Again, recall that this was for $\beta = \pi/2$ and $\rho = 1$. 

15
The argument just presented actually breaks down for small $\rho$ where the diamagnetism is greater than the paramagnetism and for which an altogether different mechanism preserves the lack of anisotropy. The diamagnetism is dominant in the region of the unit cell for which $\sigma \rho < \nu |\sin(\phi - \alpha)\sin \beta|$; the area of this region clearly scales as $\nu$. The value of the density of states in this region scales as $\nu$ (due to the local linear dependence of $N(\omega)$). Therefore, the total contribution scales as $\nu^2$. This is small when the paramagnetism dominates. Furthermore it leads to no anisotropy since the resultant sum over the nodes is proportional to $\sin^2(\pi/4 - \alpha) + \sin^2(3\pi/4 - \alpha) = 1$, which is clearly independent of $\alpha$.

The key points are the quadratic scaling in $\nu$ and the rescaling of $\nu$ by $\sin(\phi - \alpha)$. Taken together they lead to a completely isotropic $\alpha$ dependence.

For cases where $\sigma$ and $\nu$ are comparable, we continue to use the arguments of the above paragraph. As long as $\nu < \sigma$, the scaling with $\nu^2$ is perfect (as is also apparent in Eq. (10)) and there is no anisotropy. As soon as $\nu > \sigma$, the scaling breaks down. This is because the curve $\sigma \rho = |\nu \sin(\phi - \alpha) \sin \beta|$ exceeds the maximum value of $\rho = 1$ and the area of the vortex unit cell in which the diamagnetism dominates begins to saturate. The contribution of this part of the vortex unit cell then increases more slowly than $\nu^2$ and a nontrivial $\alpha$ dependence becomes possible. Nevertheless, the scaling is still approximately maintained even for $\nu > \sigma$ so that the anisotropy turns on very slowly. (This is related to the very smooth behaviour of the function $f(x)$ near $x = 1$ shown in Fig. 1.) Thus even for $\sigma$ quite a bit smaller than $\nu$, the anisotropy is strongly suppressed.

We remark that we made extensive use of the local linear behaviour of the density of states. This linear behaviour arises from using the nodal approximation for small fields. A more general discussion allowing for quadratic and higher powers will not affect the general conclusions but presumably would lead to a small amount of anisotropy for $\sigma > \nu$. 
VII. CONCLUSION

In this paper we have discussed the combined effect of paramagnetism and diamagnetism on the zero frequency quasiparticle density of states in the presence of a magnetic field. When the field is parallel to the $c$-axis, there is no interesting anisotropic effect. Furthermore, for the cuprates, the energy scales are such that the paramagnetism is negligible for any physically realisable magnetic field strength.

When the field is in the $ab$ planes, the diamagnetism is less important — this is related to the fact that it is more difficult to form vortices out of the copper-oxide planes. In fact, we expect that it is the paramagnetism which is dominant for all materials but YBCO. YBCO is special since the $c$–axis transport, necessary for the formation of vortices out of the plane, is small but not negligible. For this material, we expect that the paramagnetic and diamagnetic effects to be of comparable importance for magnetic fields of experimental interest.

One important issue is the anisotropy as we change the angle between the applied field direction and the gap nodes. Even a small amount of paramagnetism strongly suppresses the expected anisotropic response, providing a possible explanation of why this effect has not been observed in experiments. However, the same field scale where the paramagnetism begins to strongly suppress the anisotropy also controls where the linear dependence of $N$ on $\sqrt{H}$ begins to fail, as is apparent in Fig. 3. Therefore, if this is the explanation, one should also observe nonlinearities in such experimental curves. The experiments of [17] seem to indicate both no anisotropy and linear behaviour in $\sqrt{H}$, in contradiction with our conclusions. However, the error bars are large enough to still be consistent with our conclusions. More precise experiments or ones using larger fields are necessary to resolve this question. Preliminary data from [5] indicates that there is an observable anisotropy.

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APPENDIX A: GENERAL RESULTS FOR $H \parallel c$

In this appendix we generalise the results found using the nodal approximation by not considering either $\sigma$ or $\nu$ to be small but limiting ourselves to the somewhat simpler $H \parallel c$ case. Consider equations (7). For a given choice of $\rho$ and $\beta$, there is always some finite range of $\phi$ such that a solution to the $(- -)$ equation exists; this is not true for the $(+ -)$ and $(- +)$ equations. The resulting $\phi$ integrals correspond to the definition of the elliptic function and we conclude

$$
\begin{align*}
N_0^{(- -)} &= \frac{4}{\pi^2} \int d\beta d\rho \ G(a) \\
N_0^{(+ -)} &= \frac{4}{\pi^2} \int_{V<H} d\beta d\rho \ G(b) \\
N_0^{(- +)} &= \frac{4}{\pi^2} \int_{V>\mu H} d\beta d\rho \ G(c)
\end{align*}
$$

where we have defined the function

$$
G(x) = \begin{cases} 
K(x) & x < 1 \\
\frac{1}{x} K \left( \frac{1}{x} \right) & x \geq 1.
\end{cases}
$$

(A2)

$K(x)$ is the complete elliptic integral of the first kind and the parameters are

$$
a = \frac{\rho}{\nu \sin \beta + \sigma \rho} \quad b = \frac{\rho}{\sigma \rho - \nu \sin \beta} \quad c = \frac{\rho}{\nu \sin \beta - \sigma \rho}.
$$

(A3)

There is a logarithmic singularity when the argument of $G$ equals unity but this is integrable. The integration domain of the first equation in (A1) is $0 < \rho \leq 1$ and $0 < \beta \leq \pi/2$ so that $V = |V| > 0$. The domain is the same for the other two except for the added constraint on $V - \mu H$.

It is difficult to go further in an explicit analytic determination of the integrals. However, we can say a few things on a qualitative level. There are logarithmic singularities whenever one of the quantities $a$, $b$ or $c$ is unity. These possibilities correspond to curves of singularities in the $\rho - \beta$ space. Also, when $b$ (or, equivalently, $c$) equals infinity, there is a curve of zeros in the $\rho - \beta$ space. As $\nu$ and $\sigma$ are varied, singular things can happen to these special
curves. For example, in order to have a logarithmic singularity in $N_{0}^{(-)}$, we require that $a = 1$ which in turn requires

$$\rho = \frac{\nu}{1 - \sigma} \sin \beta. \quad (A4)$$

Clearly if $\sigma > 1$ there is no solution to this condition so that as $\sigma$ crosses through unity we will notice the abrupt disappearance of the curve of logarithmic singularities in the integrand of $N_{0}^{(-)}$. For $N_{0}^{(+)}$, it is precisely the converse. There are only logarithmic singularities if $\sigma > 1$ and none if $\sigma < 1$. Therefore, as $\sigma$ crosses through unity, we expect some nonanalyticity in the density of states.

Another possibility for $N_{0}^{(-)}$ is that the curve of singularities crosses through $\rho = 1$ and out of the allowed domain of the $\rho$ integral. By the previous equation, this will happen when $\nu = 1 - \sigma$. The analogous event happens for $N_{0}^{(+)}$ when $\nu = \sigma - 1$ and for $N_{0}^{(+)}$ when $\nu = 1 + \sigma$. The final possibility is that the curve of zero values for $N_{0}^{(+)}$ and $N_{0}^{(-)}$ crosses through $\rho = 1$ and out of the allowed domain of the $\rho$ integral. The curve of zero values (when $b$ and $c$ are infinite) requires $\rho = \nu \sin \beta/\sigma$ so this event occurs when $\nu = \sigma$. We already encountered the $\sigma = \nu$ criterion in our small argument analysis and is shown in Fig. 1. The others all require that either $\nu$ or $\sigma$ (or both) be of the order of unity and so were not captured in our small argument analysis.
FIG. 6. Top: A plot of the different lines in the $\nu - \sigma$ space along which we expect nonanalyticities in the density of states. The two parabolas represent possible traversals through this space as we increase the magnetic field. Bottom: the resultant dependence of the density of states for the two parabolas of the upper plot. The horizontal axis is in arbitrary units.

A plot of the different domains in the $\nu - \sigma$ space is shown in Fig. 6. Each of the solid lines represents some border along which one of the three terms experiences a nonanalyticity. As we change the magnetic field, we follow a parabola in this parameter space and thereby intersect various of these lines. The precise sequence of intersections depends on the energy scales associated with the paramagnetic and diamagnetic terms. There are five possible sequences depending on the sharpness of the parabola; we show two of them. For a very sharp parabola, the paramagnetic energy is dominant except at very small fields and the result is similar to a purely paramagnetic calculation. The major differences are 1) an initial $\sqrt{H}$ dependence for small enough fields and 2) the logarithmic singularity at $\sigma = 1$, which would be present for pure paramagnetism, is softened by the diamagnetism.

In a situation where the diamagnetism is dominant for low fields, the magnetic field dependence is rather similar to a purely diamagnetic case in which the density of states rises
smoothly and saturates at some critical value (at \( \nu = 1 \)). Again, this is softened, there is additional structure once the paramagnetism becomes comparable to the diamagnetism (\textit{i.e.} \( \sigma \sim \nu \)). For large enough fields the paramagnetism is always dominant. This means that \( N/N \) will always approach unity from above.

Unfortunately, most of this discussion is academic. As argued above, the cross-over field where \( \sigma = \nu \) for \( H \parallel c \) is about 1000T. The value where \( \sigma = 1 \) is about 350T while the value where \( \nu = 1 \) is about 70T. These are the scales where these affects would be visible, and they are probably beyond any experimental reach. For \( H \parallel ab \) it is reasonable to expect a similar structure to that shown in Fig. 6. The field where \( \sigma = \nu \) is about 100T. The value where \( \sigma = 1 \) is still 350T and the value where \( \nu = 1 \) is about 1000T (again, depending on the value chosen for \( a \)). Again, all of these fields are beyond experimental reach and so the results of this appendix could probably not be observed in experiment. Nevertheless it is of theoretical interest to understand the analytic structure of the density of states.
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