Exact thermalization dynamics in the “Rule 54” Quantum Cellular Automaton

Katja Klobas,† Bruno Bertini,† and Lorenzo Piori‡,†

†Rudolf Peierls Centre for Theoretical Physics, Clarendon Laboratory, Oxford University, Parks Road, Oxford OX1 3PU, United Kingdom
‡Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, 85748 Garching, Germany
§Munich Center for Quantum Science and Technology, Schellingstraße 4, 80799 München, Germany

We study the out-of-equilibrium dynamics of the quantum cellular automaton known as “Rule 54”. For a class of low-entangled initial states, we provide an analytic description of the effect of the global evolution on finite subsystems in terms of simple quantum channels, which gives access to the full thermalization dynamics at the microscopic level. As an example, we provide analytic formulae for the evolution of local observables and Rényi entropies. We show that, in contrast to other known examples of exactly solvable quantum circuits, Rule 54 does not behave as a simple Markovian bath on its own parts, and displays typical non-equilibrium features of interacting (integrable) many-body quantum systems such as finite relaxation rate and interaction-induced dressing effects. Our study provides a rare example where the full thermalization dynamics can be solved exactly at the microscopic level.

Recently, a useful arena to construct tractable models of many-body physics out of equilibrium has been identified in quantum circuits. In the simplest setting, one considers a one-dimensional (1D) set of qudits evolved by a “brickwork” circuit, where a given initial state is updated by sequences of local unitary gates (cf. Fig. 1). While in general quantum circuits offer no simplification with respect to local-Hamiltonian dynamics [8], the discreteness of the evolution makes it possible to construct non-trivial solvable models, with notable examples given by random [9–12] and dual unitary circuits [13]. These systems proved to be useful minimal models for the quantum chaotic dynamics, enabling the analysis of aspects that are notoriously hard to tackle in traditional systems [9–37]. However, when seen as models for thermalization, they are not typical: for instance, in dual-unitary circuits the action of the global evolution on any subsystem is purely Markovian, even in the absence of noise [14, 19, 38].

In this letter, we present an exact solution for the thermalization dynamics in a quantum circuit that provably exhibits typical features of interacting many-body systems: the quantum version of the “Rule 54” cellular automaton [39]. The latter can be regarded as the simplest interacting integrable system and over the last years has been shown to provide an ideal ground for studying interacting many-body dynamics [40–51].

Our approach is based on a general tensor network (TN) algorithm introduced in Ref. [52, 53] (see also [54]) to describe the evolution of any subsystem in the thermodynamic limit. Specifically, we identify a set of algebraic relations obeyed by the tensors of Rule 54 that enable us to follow such an algorithm analytically. We use this to derive exact formulae for the evolution of local observables, two-point correlation functions and Rényi entropies, providing a rare example where the full microscopic dynamics can be solved exactly, beyond non-interacting models and the perfectly Markovian regime.

Rule 54 is defined by a 1D lattice of qubits where the time evolution is discrete and generated by the unitary operator $U = U_o U_s$, with $U_o = \prod_j U_{2j-1,2j+1}$, $U_s = \prod_j U_{2j,2j+1,2j+2}$. In the local computational basis $\{|s_j\rangle\}_{s_j=0,1}$, the matrix elements of the three-site unitary gate $U_{j-1,j,j+1}$ read

$$U^{s_1',s_2',s_3'}_{s_1,s_2,s_3} = \delta_{s_1,s_1'} \delta_{s_2,s_2'} \delta_{s_3,s_3'}, \quad (1)$$

where $\chi(s_1,s_2,s_3) \equiv (s_1 + s_2 + s_3 + s_1 s_3) \mod 2$. The operators $U_{j-1,j,j+1}$ and $U_{j+1,j+2,j+3}$ commute, which allows us to write $U$ in the form of a brickwork quantum circuit (Fig. 1). We consider periodic chains of length $2L$, eventually taking $L \to \infty$.

![Quantum circuit representation of the Rule 54 QCA. The dynamics can be equivalently given in terms of three-site gates (left) and as an MPO (right).](image)
We study a quench protocol \cite{55, 56} where the system is initialised in a low-entangled state, which we take to be a matrix product state (MPS) \cite{57, 58} \( |\Psi_0\rangle = \sum_{i_1,\ldots,i_{2L}=0,1} \text{tr} (A_{11}^{i_1} \cdots A_{2L}^{i_{2L}}) |i_1,\ldots,i_{2L}\rangle \), where \( A_j \) are \( \chi \)-dimensional matrices (\( \chi \) is called the bond dimension).

It is straightforward to see that \( U_c, U_o \) can be represented as two-site shift invariant matrix product operators (MPO) with \( \chi = 2 \) \cite{59}, so that the evolution can be computed by applying a sequence of MPOs to \( |\Psi(0)\rangle \).

Note that this representation is completely general, since any quantum cellular automata can be expressed exactly in terms of MPOs acting on a computational basis states which are mapped onto one another so that no entanglement is generated \cite{67}. In contrast, typical initial states approach a volume-law entanglement state described by a Generalized Gibbs Ensemble \cite{51}. Our goal is to identify initial MPSs that thermalize, i.e., whose local properties at large times approach those of an infinite-temperature state. One therefore expects the fixed points of the transfer matrix \( \bar{W}_\infty \), corresponding to the infinite temperature state [Note that the correlation functions on the infinite temperature state are given by the TN analogous to the one in Fig. 2, with the initial state replaced by the (appropriately normalised) identity operator]. Thus, we begin by analysing the latter.

As our first main result, we show that the fixed points of \( \bar{W}_\infty \) can be computed analytically: this gives direct access to all infinite-temperature multi-point correlation functions, generalising the findings of Ref. \cite{46} to the quantum case. Specifically, we find that the leading left and right eigenvectors can be expressed in terms of the \( 3 \times 3 \) local tensors \( A_{sr} \), \( B_{sr} \) and the boundary vector \( |b\rangle \) (the explicit expressions are reported in the Supplemental Material)

\[
\begin{align*}
\begin{bmatrix} A_{sr} & B_{sr} & \downarrow \end{bmatrix}
\end{align*}
\]

that fulfil the following set of local relations, which we
we consider initial MPSs with this symmetry. Second, we observe that Eqs. (6) define a fixed point of $\bar{W}$ (with the graphical representation given in Fig. 2), provided that the following boundary identities are fulfilled

$$
\begin{align*}
\langle \bar{L} \mid \bar{W} = \langle L \rangle. 
\end{align*}
$$

Eqs. (9) should be seen as a consistency equation for the bulk tensors defining the initial MPS, and the boundary vectors $|v_\nu\rangle$, $|w_\nu\rangle$. A priori, it is not obvious that a solution exists, but we find that this is indeed the case. In particular, considering the simplest case of initial product states ($\chi = 1$), given by a pair of one-site states, $|\Psi_0\rangle = \otimes_{j=1}^L (|\psi_1\rangle \otimes |\psi_2\rangle)$, $\bar{U} = |\psi_1\rangle |\psi_1\rangle^*$, $\bar{V} = |\psi_2\rangle |\psi_2\rangle^*$, the solution of Eq. (9) exists for the one-parameter family of states

$$
|\psi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\varphi} |1\rangle), \quad |\psi_2\rangle = |0\rangle. 
$$

The tensors $|v_1\rangle$, $|w_1\rangle$ are then uniquely determined [59]. This is our second main result: we have written the fixed points corresponding to the initial states (11) as MPSs with bond dimension three. Note that the fact that $\langle L \rangle$ and $\langle R \rangle$ are not product states implies that the dynamics is not purely Markovian, in contrast to the case of dual-unitary circuits [14, 19]. Accordingly, the evolution of a given subsystem does not only depend on its state, and the quench protocol displays typical features of interacting many-body quantum systems.

In order to illustrate the power of our analytic solution, let us consider $\lim_{L \to \infty} \langle \Psi(t) | O_{1:2x} | \Psi(t) \rangle$, where $O_{1:2x}$ is a local operator spanning $2x$ sites. Making use of (4) and representing the r.h.s. diagrammatically (cf. (10)), we find that the expectation value can be expressed in terms of a time-independent map (matrix) $C_{2x}$

$$
\langle \Psi(t) | O_{1:2x} | \Psi(t) \rangle = \langle \Phi | O_{1:2x} | \Phi \rangle = \langle \Phi | O_{1:2x} | 2x \rangle, 
$$

where the matrix $C_{2x}$ and vectors $\langle \Phi | O_{1:2x} | 2x \rangle$, $|\Phi_{2x}\rangle$ are defined graphically as

$$
C_{2x} = \begin{array}{c}
\begin{array}{c}
2x \\
\vdots \\
2x
\end{array}
\end{array}, \quad |\Phi_{2x}\rangle = \begin{array}{c}
\begin{array}{c}
2x \\
\vdots \\
2x
\end{array}
\end{array}. 
$$

The prefactor $1/4$ in the diagrammatic expression of $\bar{W}_\infty$ comes from the normalisation of the infinite temperature state. To prove the fixed-point condition (8), we first apply the first relation of (6a), which replaces the bottom-most tensor $A$ with $B$, and introduces the two-site tensor $C$ in the second and third leg. The tensor $C$ is then repeatedly moved up using the third relation in (6b), until it is absorbed at the top by applying the left-most relation in (6b). This gives the MPS with the exchanged roles of $A$ and $B$. The procedure is then repeated, by using the right-most relation in (6a) and the second identity in (6b), to complete the proof of (8). The form of the right leading vector $|R_\infty\rangle$ is analogous, with the only difference that the roles of $A$ and $B$ are exchanged and the diagram is reflected (flipped from left to right).

Besides giving access to all infinite temperature correlations, the expressions of $\langle L_\infty \rangle$ and $|R_\infty\rangle$ provide a natural basis to infer the structure of the fixed points $\langle L \rangle$ and $\langle R \rangle$ corresponding to thermalizing initial states. Specifically, we search for fixed points taking the same form up to the boundary vector $|b\rangle$ (cf. (5)). This is because, at large times after the quench, the action of the system on its own parts has to be indistinguishable from an infinite-temperature reservoir.

To complete the ansatz, we then just have to specify a boundary vector for $\langle L \rangle$. This is done as follows. First, given the two-site shift invariance of the problem,
From Eqs. (12)–(13), we see that the time evolution of operators supported on a small number of sites can be computed either analytically or in a numerically exact fashion for arbitrarily large times, by diagonalising the matrix C0. However, since the matrix has dimensions $9\times2^{2x}$, this becomes increasingly hard as $x$ grows. Nevertheless, Eq. (12) can be used to show that the late-time behaviour of any operator with finite support is exponentially decaying, and to compute exactly the corresponding characteristic time $\tau$. To see this, we use the identity

$$\langle \Psi(t) | \mathcal{O}_{1:2x} | \Psi(t) \rangle - \text{tr}[\mathcal{O}_{1:2x} \rho_{\infty,2x}] \sim e^{-t/\tau}$$  \hspace{1cm} (15)$$

which can be proven graphically using Eqs. (6) and (9). Eq. (14) implies that for any fixed $x$ the asymptotics of the expectation value is governed by the matrix $C_0$ (cf. (13)), independent of $x$. It is easy to show that the spectrum of $C_0$ is $\{1, \lambda, \lambda^*, 0\}$, with $\lambda = (-3 + i\sqrt{7})/8$, while the only eigenvector associated with eigenvalue one is $|b\rangle \otimes |b\rangle$. This is the state that one would find at the bottom of the diagram on the l.h.s. of (14) when computing the expectation value of $\mathcal{O}_{1,2x}$ in the infinite temperature state. This means that the local observable approaches exponentially its infinite-temperature stationary value

$$\langle \mathcal{O}_{1:2x} | \rho_{\infty,2x} | \mathcal{O}_{1:2x} \rangle \sim e^{-t/\tau}$$  \hspace{1cm} (16)$$

where $\rho_{\infty,2x} = 1/2^x$ is the infinite temperature state on $x$ sites and $\tau^{-1} = -2 \log |\lambda| = 2 \log 2$. Thus, as anticipated before, the relaxation rate is (in general) finite.

Eq. (12) can also be used to study (equal-time) two-point functions of generic local observables $a_x$ and $b_y$ (in this case one has to choose $O_{x,y} = a_x b_y$) and explicitly identify a maximal and a minimal velocity of correlation spreading. Indeed, the brickwork structure imposes a maximal speed $v_{\text{max}} = 2$ (in our units), therefore for $t < x/2$ connected correlations are strictly zero. Additionally, Eq. (14) implies that, for $t > 3x/2$, correlations are exponentially suppressed. Namely, there exist a minimal speed $v_{\text{min}} = 2/3$.

As a last physical application, we show that our solution allows us to compute explicitly the full time evolution of the Rényi entropies after the quench. This is especially relevant in light of their accessibility in recent state-of-the-art quantum simulation experiments [68–72]. By repeating the reasoning that led to Eq. (4) in the case of $2n$ copies (or “replicas”) of the time-evolution operator ($n$ copies of $U^t$ and $n$ copies of $(U^t)^*$) [59] we find that $\langle L \rangle$ and $\langle R \rangle$ give access to the full time evolution of the entanglement entropies of a semi-infinite interval in the thermodynamic limit [73]. This is formalised by

$$S_n(t) = \frac{\log[\text{tr}[\rho_H^n(t)]]}{1 - n} = \frac{\log[\langle (L)^\otimes (R)^* \rangle^n_{2n}]}{1 - n},$$  \hspace{1cm} (16)$$

where $\rho_H(t)$ is the density matrix reduced to one of the halves, $\mathcal{S}_{2n}$ is the operator performing a periodic shift by one copy in the $2n$-replica space [59], and $(\cdot)^*$ denotes complex conjugation.

Eq. (16) can again be written as the matrix element of the $t$-th power of a finite dimensional matrix. In particular, in this case the matrix is $3^n \times 3^n$ and we denote it by $T_n$ [59]. This means that for small $n$ we can compute $S_n(t)$ for arbitrarily large times: see, e.g., Fig. 3, where we report numerical data for $n = 2, 3, 4$. As expected for interacting integrable systems [74–78], after an initial transient, we see a clear ballistic growth with asymptotic rate given by $s_n \equiv \lim_{t \to \infty} S_n(t)/t = (\log \lambda_n)/(1 - n)$, where $\lambda_n$ denotes the leading eigenvalue of $T_n$. Remarkably, $\lambda_n$ can be determined exactly for arbitrary $n$. Indeed, by studying the structure of $T_n$ we are able to establish the following property, whose proof will be reported elsewhere [79].

Property 1. $\lambda_n$ is the only real solution of

$$4^n x^3 - (x + 1)^2 = 0.$$  \hspace{1cm} (16)$$

To the best of our knowledge, Property 1 provides the first exact predictions for the asymptotic growth of Rényi entropies in an interacting (non-dual-unitary) system. Furthermore, it also allows us to perform an analytic continuation and obtain predictions for arbitrary (non-integer) values of $n$. In particular, taking the limit $n \to 1$, we obtain the asymptotic growth for the von Neumann entanglement entropy $s_1 = \log 2$. This result is particularly significant, since it represents the first exact confirmation of the Alba-Calabrese formula [74] for the entanglement growth in the presence of interactions [80]. It is also interesting to consider the limit $n \to \infty$. In this...
case we obtain $s_\infty = 2/3 \log 2$, which reveals that the min-entropy spreads with the minimal velocity of correlations (or, equivalently, of quasi-particles).

In summary, in this letter we presented an exact description of the finite-time non-equilibrium dynamics generated by a quantum quench in an interacting integrable model: the quantum cellular automaton Rule 54. The fundamental ingredient for our derivations are a set of tensor network identities, the zipping conditions (5), that enabled us to characterise exactly the action of the system on its own parts. It would be interesting to understand whether this approach can be extended to families of initial states relaxing to non-trivial GGEs, rather than to the infinite temperature state, which would enable the investigation of richer relaxation dynamics and more general dressing effects. For instance, this would provide new exact results for the spreading of entanglement beyond the dual-unitary case.

Acknowledgments: We thank Tomaz Prosen and Pasquale Calabrese for valuable comments on the manuscript. LP acknowledges interesting discussions on related topics with Ignacio Cirac, Dmitry Abanin, Alessio Lerose, and Michael Sonner. This work was supported by EPSRC under grant EP/S020527/1 (KK), the Royal Society through the University Research Fellowship No. 201101 (BB) and the Alexander von Humboldt foundation (LP).

[1] M. Rigol, V. Dunjko, and M. Olshanii, Nature 452, 854 (2008).
[2] M. Cazalilla and M. Rigol, New J. Phys. 12, 055006 (2010).
[3] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. 83, 863 (2011).
[4] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Adv. Phys. 65, 239 (2016).
[5] F. H. L. Essler and M. Fagotti, J. Stat. Mech. 2016, 064002 (2016).
[6] P. Calabrese, F. H. Essler, and G. Mussardo, J. Stat. Mech. 2016, 064001 (2016).
[7] J.-S. Caux, J. Stat. Mech. 2016, 064006 (2016).
[8] In fact, any continuous evolution driven by a local-Hamiltonian can be approximated arbitrarily well by a quantum circuit [83].
[9] A. Chan, A. De Luca, and J. Chalker, Phys. Rev. X 8 (2018).
[10] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Phys. Rev. X 7, 031016 (2017).
[11] A. Nahum, S. Vijay, and J. Haah, Phys. Rev. X 8, 021014 (2018).
[12] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Phys. Rev. X 8, 021013 (2018).
[13] B. Bertini, P. Kos, and T. Prosen, Phys. Rev. Lett. 123, 210601 (2019).
[14] B. Bertini, P. Kos, and T. Prosen, Phys. Rev. X 9 (2019).
[15] S. Gopalakrishnan and A. Lamacraft, Phys. Rev. B 100, 064309 (2019).
[16] B. Bertini, P. Kos, and T. Prosen, SciPost Phys. 8, 067 (2020).
[17] B. Bertini, P. Kos, and T. Prosen, SciPost Phys. 8, 068 (2020).
[18] B. Bertini and L. Piroli, Phys. Rev. B 102, 064305 (2020).
[19] L. Piroli, B. Bertini, J. I. Cirac, and T. Prosen, Phys. Rev. B 101, 094304 (2020).
[20] P. W. Claeys and A. Lamacraft, Phys. Rev. Research 2, 033032 (2020).
[21] B. Gutkin, P. Braun, M. Akilia, D. Waltner, and T. Guhr, arXiv:2001.01298 (2020).
[22] S. A. Rather, S. Aravinda, and A. Lakshminarayan, Phys. Rev. Lett. 125, 070501 (2020).
[23] P. Kos, B. Bertini, and T. Prosen, arXiv:2006.07304 (2020).
[24] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Phys. Rev. X 8, 031058 (2018).
[25] V. Khemani, A. Vishwanath, and D. A. Huse, Phys. Rev. X 8, 031057 (2018).
[26] A. Chan, A. De Luca, and J. T. Chalker, Phys. Rev. Lett. 121, 060601 (2018).
[27] C. Sündenbergh, D. Pérez-García, D. A. Huse, N. Schuch, and J. I. Cirac, Phys. Rev. B 98, 134204 (2018).
[28] A. J. Friedman, A. Chan, A. De Luca, and J. T. Chalker, Phys. Rev. Lett. 123, 210603 (2019).
[29] Y. Li, X. Chen, and M. P. A. Fisher, Phys. Rev. B 100, 134306 (2019).
[30] B. Skinner, J. Ruhman, and A. Nahum, Phys. Rev. X 9, 031009 (2019).
[31] A. Zabalo, M. J. Gullans, J. H. Wilson, S. Gopalakrishnan, D. A. Huse, and J. H. Pixley, Phys. Rev. B 101, 060301 (2020).
[32] Y. Bao, S. Choi, and E. Altman, Phys. Rev. B 101, 104301 (2020).
[33] S. Choi, Y. Bao, X.-L. Qi, and E. Altman, Phys. Rev. Lett. 125, 030505 (2020).
[34] A. Lavasani, Y. Alavirad, and M. Barkeshli, arXiv:2004.07243 (2020).
[35] P. Kos, B. Bertini, and T. Prosen, arXiv:2010.12494 (2020).
[36] P. W. Claeys and A. Lamacraft, arXiv:2009.03791 (2020).
[37] T. Rakovszky, F. Pollmann, and C. von Keyserlingk, Phys. Rev. Lett. 122 (2019).
[38] A. Lerose, M. Sonner, and D. A. Abanin, arXiv:2009.10105 (2020).
[39] A. Bobenko, M. Bordemann, C. Gunn, and U. Pinkall, Commun. Math. Phys. 158, 127 (1993).
Here we report some useful information complementing the main text. In particular

- In Section A we derive the tensor-network representation of Rule 54;

- In Section B we study the spectrum of the transfer matrix in space;

- In Section C we provide the explicit form of matrices and boundary vectors defining the MPS representation of the fixed points

- In Section D we provide additional detail on the calculation of Rényi entropies.

Appendix A: Tensor network representation of Rule 54

Time evolution in the model is obtained by consecutive application of the three-site local unitary gate $U$ given by (1). Defining the following two tensors,

$$s_1 s_2 s_3 = \delta_{\chi(s_1, s_2, s_3), s_4},$$
$$s_k = \prod_{j=1}^{k-1} \delta_{s_j, s_{j+1}},$$

one realizes that the three-site operator can be equivalently expressed as

$$U = \begin{array}{c}
\vdots \\
\arrow{2}
\end{array} = \begin{array}{c}
\vdots \\
\arrow{1}
\end{array} = \begin{array}{c}
\vdots \\
\arrow{2}
\end{array} \ (A1)$$

The equivalence between the two sides in Fig. 1 then follows from a simple identity satisfied by the second tensor,

$$\begin{array}{c}
\vdots \\
\arrow{2}
\end{array} = \begin{array}{c}
\vdots \\
\arrow{1}
\end{array} = \begin{array}{c}
\vdots \\
\arrow{2}
\end{array} \ (A2)$$

Appendix B: Spectrum of the transverse transfer matrix

The unitarity of time-evolution implies

$$1 = \langle \Psi(t) | \Psi(t) \rangle = \text{tr} \left( \tilde{W}^L \right) = \sum_j \lambda_j^L, \quad (B1)$$

where the second equality follows directly from the definition of the transverse transfer matrix (3) and $\lambda_j$ are eigenvalues of $\tilde{W}$. This equality holds for any $L$, which means that $\lambda_j \in \{0, 1\}$ and both the geometric and algebraic multiplicity of the eigenvalue 1 has to be 1.
Appendix C: MPS representation of the leading eigenvectors

Here we report an explicit representation for the $3 \times 3$ matrices $A_{sr}$, $B_{sr}$, and $C_{s_1r_1s_2r_2}$ fulfilling relations (6a)–(6b) and (9). The non identically vanishing matrices in the set are given by

\[
A_{00} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad A_{01} = A_{10} = \frac{1}{2} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
B_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
C_{0001} = C_{0010} = \frac{1}{4} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}, \quad C_{0101} = C_{1010} = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_{0110} = C_{1001} = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad C_{0011} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_{1100} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_{1111} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

while the appropriate boundary vectors $|b\rangle$, $|v_1\rangle$, $|w_1\rangle$ are

\[
|b\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad |v_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad |w_1\rangle = \sqrt{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

Note that this choice of magnitude of boundary vectors implies the normalisation of leading eigenvectors of the transverse transfer matrix,

\[
\langle L_\infty | R_\infty \rangle = \langle b | b \rangle^2 = 1, \quad \langle L | R \rangle = \langle b | v \rangle \langle b | w \rangle = 1.
\]

The latter equations follow from Eq. (13) and $\langle b \otimes b | C_0 = \langle b \otimes b \rangle$.

Appendix D: Tensor network formulation of Rényi entanglement entropies

We start by considering the density matrix at time $t$ (at time 0 the system has been initialised in an MPS fulfilling (9)) reduced to one of the halves

\[
\rho_H(t) = \frac{1}{N_L}
\]

Here we took open boundary conditions and introduced the constant $N_L$ to ensure $\text{tr}[\rho_H(t)] = 1$. Note that $\lim_{L\to\infty} N_L = (\langle B_L | \otimes \langle B_L | \rangle) | R \rangle \langle L | (|B_R\rangle \otimes |B_R\rangle)^{\ast}$ where $\langle B_L \rangle, |B_R\rangle \in \mathbb{C}^{2^M}$ are the vectors in the $t$-direction
encoding the (left and right) boundary conditions, the tensor product acts on different copies, and \((\cdot)^*\) denotes complex conjugation. Note that \(\langle B_L | \) and \(| B_R \rangle\) are defined in one copy of the time-sheet, while \(\langle L | , | R \rangle \in \mathbb{C}^{2^t}\) live in the tensor product of two copies.

Considering now the trace of, e.g., the third power of the reduced density matrix we have

\[
\text{tr}[\rho^3_H(t)] = \frac{1}{N^3_L}, \quad (D2)
\]

where green and red areas represent the TNs associated with \(U\) and \(U^\dagger\) respectively, cf. Eq. (D1). Using the t-channel language of Eq. (3), we see that this object can be written as

\[
\text{tr}[\rho^3_H(t)] = \frac{1}{N^3_L} ((B_L | \otimes (B_L|^*)^\otimes 3 (\tilde{W}^\otimes 3) L S_6((\tilde{W}^*)^\otimes 3) L ((| B_R \rangle^* \otimes | B_R \rangle)), (D3)
\]

where \(S_6\) is the periodic shift by one in the space of the 6 replicas. Namely it acts on the Hilbert space

\[
\mathcal{H}_6 = (\mathbb{C}^{2^t})^\otimes 6 \quad (D4)
\]

as follows

\[
S_6 | i_1 \rangle \otimes | i_2 \rangle \otimes | i_3 \rangle \otimes | i_4 \rangle \otimes | i_5 \rangle \otimes | i_6 \rangle = | i_2 \rangle \otimes | i_3 \rangle \otimes | i_4 \rangle \otimes | i_5 \rangle \otimes | i_6 \rangle \otimes | i_1 \rangle , \quad i_j = 0, 1, \ldots 2^t - 1, \quad (D5)
\]

where \(
\{| i \rangle \}_{i=0}^{2^t-1} \) is a basis of \(\mathbb{C}^{2^t}\).

In the thermodynamic limit we can make the replacement

\[
\tilde{W} \mapsto | R \rangle \langle L | , \quad \tilde{W}^* \mapsto | R \rangle^* \langle L | , \quad N_L \mapsto ((B_L | \otimes (B_L|^*) | R \rangle \langle L | (| B_R \rangle \otimes | B_R \rangle^*), (D6)
\]

and use

\[
((B_L | \otimes (B_L|^*) | R \rangle = (B_L|^* \otimes (B_L |) | R \rangle^* , \quad \langle L | (| B_R \rangle \otimes | B_R \rangle^* = \langle L |^* (| B_R \rangle^* \otimes | B_R \rangle), (D7)
\]

to find

\[
\lim_{L \to \infty} \text{tr}[\rho^3_H(t)] = \langle L |^\otimes 3 S_6(| R \rangle^*)^\otimes 3. \quad (D8)
\]

An analogous reasoning considering a generic \(n\)-th power leads to (16). Note that for periodic boundary conditions the r.h.s. of (D8) is replaced by its square.

Finally, we note that (D8) can be represented in terms of the matrix element of the \(t\)-th power of a time-independent matrix. Indeed, “unfolding” the tensors (5), i.e.

\[
\begin{align*}
s_1 s_2 & \rightarrow s_1 \quad (D9) \\
s_1 s_2 & \rightarrow s_1 \quad (D10)
\end{align*}
\]

we have

\[
\langle L |^\otimes 3 S_6(| R \rangle^*)^\otimes 3 = \langle U_3 | T_n^t | D_n \rangle, \quad (D10)
\]

with

\[
T_n = \begin{array}{cc}
2n \\
\hline
2n \\
\end{array}, \quad | D_n \rangle = \begin{array}{cc}
2n \\
\hline
2n \\
\end{array}, \quad \langle U_3 | = \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}. \quad (D11)
\]