Odd Khovanov homology of principally unimodular bipartite graph-links

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Abstract

We define odd Khovanov homology over \( \mathbb{Z} \) for principally unimodular bipartite graph-links.

1 Introduction

This article is a sequel of our paper [6] and describes the integral version of odd Khovanov homology for graph-links. We refer the reader to papers [4, 5] for the definition of graph-links and to papers [1, 7] for the construction of odd Khovanov homology.

The definition of integer-valued odd Khovanov homology for graph-links faces the difficulty that the signs of the integer intersection matrix of an oriented chord diagram does not survive in general after mutations. But we can keep the signs if one is able to distinguish outer and inner chords. This reason confines us to bipartite graph-links. There is another restriction, which is necessary to retain the orientation after Reidemeister moves, — principal unimodularity [3]. So the definition of integer odd Khovanov homology is given for bipartite principally unimodular graph-links.

Bipartite principally unimodular graph-links can be considered as "classical" graph-links because the realizable graphs of this type are intersection graphs of chord diagrams that correspond to classical links. The question is whether this class of graph-links gives the "natural" definition of classical graph-links. Another problem is if the theory is meaningful, i.e. does there exist non-realizable bipartite principally unimodular graph-link.

2 Principally unimodular bipartite graph-links

In this section we define an orientable version of the Reidemeister on oriented bipartite graphs.

Let \( G \) be an oriented bipartite graph without loops and multiple edges and \( \mathcal{V} = \mathcal{V}(G) \) be the set of its vertices. We assume \( G \) be a labeled graph, i.e. every vertex in \( G \) is endowed with sign '+' or '-' . In other words, there is a map \( \text{sgn} : \mathcal{V} \to \{-1, 1\} \).
Fix an enumeration of vertices for $G$. We define the adjacency matrix $A(G) = (a_{ij})_{i,j=1,...,n}$ over $\mathbb{Z}_2$ as follows: $a_{ij} = 1$ and $a_{ji} = -1$ if and only if $v_i$ is the beginning and $v_j$ is the end of an edge in the graph $G$ (we shall denote this situation as $v_i \rightarrow v_j$), and $a_{ij} = 0$ if $v_i$ and $v_j$ are not adjacent. Besides we set $a_{ii} = 0$.

Any subset $s \subset V$ we shall call a state. Let’s define $G(s)$ to be the complete subgraph in $G$ with the set of vertices $s$ and denote $A(s) = A(G(s))$. Since $G$ is bipartite the set of vertices splits into a disjoint sum $V = V_0 \sqcup V_1$ and for every state $s$

$$A(s) = \begin{pmatrix} 0 & B(s) \\ -B(s)^\top & 0 \end{pmatrix}$$

(1)

where the rows of the matrix $B(s)$ correspond to the vertices in $s \cap V_0$ and the columns correspond to the vertices in $s \cap V_1$.

Let $v \in V$. The set of all vertices in $V$ adjacent to $v$ is called neighbourhood of the vertex $v$ and denoted $N(v)$.

Let us define the Reidemeister moves for labeled oriented bipartite graphs.

$\Omega_1$. The first Reidemeister move is an addition/removal of an isolated vertex labeled $+$ or $-$. $\Omega_2$. The second Reidemeister move is an addition/removal of two nonadjacent vertices $u$ and $v$ having the different signs and the same neighbourhoods so that the new graph remain bipartite. We require the orientations of the new edges to be compatible: for any vertex $w \in V(G)$ we change the direction of all the edges incident to $v$.

$\Omega_3$. The third Reidemeister move is defined as follows. Let $u,v,w$ be three vertices of $G$ with signs '-' and $u$ be adjacent only to $v$ and $w$ so that $u \rightarrow v$ and $u \rightarrow w$. Then we disconnect $u$ from $v$ and $w$. We set $u \rightarrow t$ (resp. $u \leftarrow t$) for all $t$ such that $v \rightarrow t$ (resp. $v \leftarrow t$) and set $u \rightarrow t$ (resp. $u \leftarrow t$) if $w \leftarrow t$ (resp. $w \rightarrow t$). In addition, we change the labels of $v$ and $w$ to '+' and '. The inverse operation is also called the third Reidemeister move.

$\Omega_4$. The fourth Reidemeister move is defined as follows. We take two adjacent vertices $u$ labeled $a$ and $v$ labeled $b$. Then we change the label of $u$ to $-b$ and the label of $v$ to $-a$ and change also the orientation of the edge $uv$. After that we change the adjacency for each pair $(t,w)$ of vertices where $t \in N(u)$ and $w \in N(v)$. We set the orientation of a new edge $tw$ so that the square $utwv$ be even, i.e. the number of codirectional edges in the round $utwv$ be even (see examples below).
Proposition 1. Let $G$ be an oriented bipartite labeled graph and $\tilde{G}$ differ from $G$ by orientation of edges. Then we can obtain $\tilde{G}$ by applying the moves $\Omega_2$ and $\Omega_4$ to the graph $G$.

Proof. Let $u$ and $v$ be two adjacent vertices in $G$. We can change the direction of the edge connecting $u$ and $v$ in the following way. We add two vertices $w, w'$ such that $N(w) = N(w') = \{v\}$ ($\Omega_2$ move). Then we add two vertices $t, t'$ such that $N(t) = N(t') = \{u, w\}$ and the square $uvtw$ is odd (another $\Omega_2$ move). Denote the obtained graph $G'$. Then we apply twice $\Omega_4$ move to the pair of vertices $w, t$. In the new graph $G''$ the adjacency of vertices are the same as in $G'$ and the directions of edges remain unchanged except the edge $uv$ which changes the direction because the square $uvtw$ in $G''$ is even. Then we remove vertices $w, w', t, t'$ to obtain the graph $G'''$ which differs from $G$ by the direction of the edge $uv$.

Repeating this operation we get to the graph $\tilde{G}$. \hfill \Box

This statement shows that the theory of oriented bipartite graphs with moves $R, \Omega_1, \ldots, \Omega_4$ is in fact the theory of undirected labeled bipartite graphs with the usual Reidemeister moves of graph-links which preserve the bipartite structure of the graphs. So we have to impose some additional constraints to make orientation of graphs significant.

Definition 1. Let $G$ be an oriented bipartite labeled graph. We call the orientation of $G$ principally unimodular if for each state $s \subset V$ we have $\det A(s)$ is equal either 0 or 1. The graph $G$ we call PU-oriented.

Any bipartite graph, which is realizable as the intersection graph of a chord diagram, is PU-oriented [2]. The inverse statement is not true, this graph has a PU-orientation but is not realizable:

The question whether exists a bipartite PU-oriented graph that can not be transformed by Reidemeister moves into a realizable graph is still open.

Lemma 1. Let $G$ be an oriented bipartite labeled graph. These statements are equivalent:

1. $G$ is PU-oriented;
2. any minor of the matrix $A(G)$ is equal to 0, $-1$ or 1;
3. any minor of the matrix $B(G)$ is equal to 0, $-1$ or 1.
Proof. 1 ⇒ 3. Let $B$ be a square submatrix of $B(G)$. Denote $s_0 \in V_0$ the set of vertices that correspond to the rows of $B$ and $s_1 \in V_1$ the set of vertices that correspond the columns of $B$. Then we have

$$A(s_0 \cup s_1) = \begin{pmatrix} 0 & B \\ -B^\top & 0 \end{pmatrix}$$

so $\det A(s_0 \cup s_1) = (\det B)^2 = 0$ or 1. Then $\det B = 0$, 1 or $-1$.

3 ⇒ 2. Let $C$ be a square matrix in $A(G)$. According to the splitting $V = V_0 \cup V_1$ the matrix $C$ can be written in the block form

$$C = \begin{pmatrix} 0 & B_1 \\ -B_2^\top & 0 \end{pmatrix}.$$

If the blocks $B_1$ and $B_2$ are not square then the rank of $C$ is less then the size of $C$ so $\det C = 0$. Otherwise, $\det C = \pm \det B_1 \det B_2 = 0$, $-1$ or 1 because $\det B_1, \det B_2 = 0, -1$ of 1.

The implication 2 ⇒ 1 is obvious.

**Proposition 2.** Let $G$ be PU-oriented. Then

1. any subgraph of $G$ is PU-oriented;
2. if $G'$ is obtained from $G$ by applying the moves $R, \Omega_1, \Omega_3, \Omega_4$ then $G'$ is PU-oriented.

Proof. The first statement of the proposition is evident.

Invariance under $R$ move. The matrix $A(G')$ is obtained from $A(G)$ by multiplication by $-1$ the row and the column corresponding to the vertex $v$ of the move $R$. Then for any state $s \in V$ we have $\det A(G'(s)) = \det A(G(s))$ if $v \notin s$ and $\det A(G'(s)) = (-1)^2 \det A(G(s)) = \det A(G(s))$ if $v \in s$. So all the determinants are equal to 0, $-1$ or 1, hence $G'$ is PU-oriented.

Invariance under $\Omega_1$ move. Assume we obtain $G'$ by adding an isolated vertex $v$. Then for any state $s \in V(G')$ we have $\det A(G'(s)) = 0$ if $v \in s$ and $\det A(G'(s)) = \det A(G(s))$ if $v \notin s$. Thus $G'$ is PU-oriented.

Invariance under $\Omega_3$ move. Without loss of generality we can renumber $V(G) = V(G')$ so that the vertices $u, v, w$ of the move have the indices 1, 2, 3. Then the adjacency matrices have the form

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & a \\ -1 & 0 & 0 & b \\ 0^\top & -a^\top & -b^\top & C \end{pmatrix}, \quad A(G) = \begin{pmatrix} 0 & 0 & 0 & a - b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ (b - a)^\top & -a^\top & -b^\top & C \end{pmatrix}.$$

We only need to check that the vector $a - b$ is the adjacency vector of the vertex $u$ in $G'$. Let us consider a vertex $t$ in $G$ and let $p$ be its index. Denote $A(G) = (a_{ij})_{i,j=1,\ldots,n}$ $A(G') = (a'_{ij})_{i,j=1,\ldots,n}$. If $t$ is not adjacent to $v$ and $w$ then $a_{2p} = a_{3p} = 0$ and $a'_{1p} = 0$ so $a'_{1p} = a_{2p} - a_{3p}$. If $t$ is adjacent to $v$ and not adjacent to $w$ then $a_{3p} = 0$ and $a'_{1p} = a_{2p} = a_{2p} - a_{3p}$. If $t$ is adjacent to $w$ and
not adjacent to \( v \) then \( a_{2p} = 0 \) and \( a'_{3p} = -a_{3p} = a_{2p} - a_{3p} \). If \( t \) is adjacent to \( v \) and \( w \) then \( a_{2p} = a_{3p} \) and \( a'_{3p} = 0 = a_{2p} - a_{3p} \). The case \( a_{2p} = 1, a_{3p} = -1 \) (or \( a_{2p} = -1, a_{3p} = 1 \)) is impossible because we would have

\[
A(G(\{u, v, w, t\})) = \begin{pmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1 \\
0 & -1 & 1 & 0
\end{pmatrix}
\]

with \( \det A(G(\{u, v, w, t\})) = 4 \) so \( G \) would not be PU-orientable.

For any state \( s \in \mathcal{V} \setminus \{u, v, w\} \) we have \( \det A(G(s)) = \det A(G'(s)) \), \( \det A(G(s \cup \{v\})) = \det A(G'(s \cup \{v\})) \), \( \det A(G(s \cup \{w\})) = \det A(G'(s \cup \{w\})) \), \( \det A(G(s \cup \{v, w\})) = \det A(G'(s \cup \{v, w\})) \). There are also equalities \( \det A(G(s \cup \{u\})) = 0 \), \( \det A(G(s \cup \{u, v\})) = \det A(G(s \cup \{u, w\})) = \det A(G(s \cup \{u, v, w\})) = \det A(G'(s \cup \{u\})) = \det A(G'(s \cup \{u, v\})) = \det A(G'(s \cup \{u, w\})) = \det A(G'(s \cup \{u, v, w\})) = 0 \).

We see that all the determinants \( \det A(G(s)), s \in \mathcal{V} \) are equal to some determinants of the graph \( G' \) and vice versa. Thus, \( G \) is PU-oriented iff \( G' \) is PU-oriented.

Invariance under \( \Omega_4 \) move. Without loss of generality we can suppose that \( u \in \mathcal{V}_0, v \in \mathcal{V}_1 \) and \( u \to v \) where \( u, v \) are the vertices of the move \( \Omega_4 \). We can also assume that \( u \) is the first vertex in \( \mathcal{V}_0 \) and \( v \) is the first in \( \mathcal{V}_1 \). Then the matrix \( B(G) \) looks like

\[
B(G) = \begin{pmatrix}
1 & c \\
d & B_1
\end{pmatrix}
\]

Adding/subtracting the first row of \( B(G) \) from others we obtain the matrix

\[
\tilde{B} = \begin{pmatrix}
1 & c \\
0 & \tilde{B}_1
\end{pmatrix}
\]

The elements of \( \tilde{B}_1 \) are \( 0, \pm 1 \) because otherwise we would have (up to sign change of rows and columns) a minor \( \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \) in \( B(G) \). Let \( D_1 \) be any square matrix in \( \tilde{B}_1 \) and \( D \) be the matrix in \( \tilde{B} \) obtained from \( D_1 \) by adding the first row and the first column. Then \( \det D_1 = \det D = 0, \pm 1 \) because \( D \) is equivalent by row transformation to a square submatrix in \( B(G) \). In particular, \( \det \tilde{B}_1 = \det B \). The matrix \( \tilde{B}_1 \) can be also obtained by adding/subtraction the column \( d \) to the columns of \( B_1 \).

The matrix \( B(G') \) is equal to \( \begin{pmatrix} -1 & c \\ d & \tilde{B}_1 \end{pmatrix} \). Let us consider any square submatrix \( D \) in \( B(G') \). If \( D \) contains neither the first row nor the first column then \( D \) is a submatrix of \( \tilde{B}_1 \) so \( \det D = 0, \pm 1 \). If \( D \) contains the first row but not the first column it is equivalent (by row transformations) to a submatrix in \( B(G) \). If \( D \) contains the first column but not the first row it is equivalent (by column transformations) to a submatrix in \( B(G) \). If \( D = \begin{pmatrix} -1 & c' \\ d' & C \end{pmatrix} \) then it
Let Proposition 3. \(C\) coincides with a submatrix in Definition 3. We call a cycle Definition 2. by requiring the result to be a PU-oriented bipartite graph.

Thus, any minor of the matrix \(B(G')\) is equal to 0, ±1 and by Lemma 1 the graph \(G'\) is PU-oriented.

The set of PU-oriented graphs is not stable under the second Reidemeister move. So we define the \textit{principally unimodular second Reidemeister move} \(\Omega^2_{PU}\) by requiring the result to be a PU-oriented bipartite graph.

**Definition 2.** A \textit{PU-oriented graph-link} is the class of equivalence of a PU-oriented bipartite labeled graph modulo moves \(R, \Omega_1, \Omega^2_{PU}, \Omega_3, \Omega_4\).

Let us consider several properties of PU-oriented graph.

**Proposition 3.** Let \(G\) be an oriented bipartite labeled graph. Then \(G\) is PU-oriented if and only if any graph \(G'\) obtained from \(G\) by a sequence of moves \(\Omega_4\) does not contain odd squares (4-cycles).

**Proof.** Necessity of the condition follows from Proposition 2.

Assume that applying \(\Omega_4\) to \(G\) doesn’t generate odd squares. We shall call such graphs stably even. Let us consider a state \(s\in \mathcal{V}\) and \(s_i = s\cap V_i, i = 0, 1, \) If \(#s_0 \neq #s_1\) then \(\det A(s) = 0\). So assume \(#s_0 \neq #s_1 = k\). We shall prove that \(\det A(s) = (\det B(s))^2 = 0\) or 1 by induction on \(k\).

For \(k = 1\) the statement is obvious. If \(k = 2\) then \(\det B(s) = 0, \pm 1\) because \(G\) has no odd squares.

Assume that for any stably even graph \(G'\) and \(s'_i \in V_i(G')\), \(i = 1, 2\) such that \(#s'_0 = #s'_1 < k\) we have \(\det B(G'(s'_0 \cup s'_1)) = 0, \pm 1\).

If the matrix \(B(s)\) contains only zeros then \(\det B(s) = 0\). Otherwise without loss of generality we can suppose that \(B(s) = \begin{pmatrix} 1 & c \\ d & B_1 \end{pmatrix}\). Then apply the move \(\Omega_4\) to the first vertex \(u\) in \(s_0\) and the first vertex \(v\) in \(s_1\). Denote the obtained graph as \(G'\). Denote also \(s' = s\setminus \{u, v\}\). Then we have \(B(G'(s)) = \begin{pmatrix} -1 & c \\ d & B_1 \end{pmatrix}\) where \(\det B_1 = \det B(s)\) (see the proof of \(\Omega_4\)-invariance in Proposition 2). But \(B_1 = B(G'(s'))\), \(G'\) is stably even and \(#s'_0 = #s'_1 < #s_0 = #s_1 = k\). Therefore, \(\det B(s) = \det B_1 = 0, \pm 1\).

Thus, for any state \(s\) of \(G\) we have \(\det B(s) = 0, \pm 1\) so \(G\) is PU-oriented by Lemma 1.

**Definition 3.** We call a cycle \(C\) in an oriented graph \textit{chordless} if \(C = G(\mathcal{V}(C))\), that is if any two vertices in \(C\), which are not neighbours in \(C\), are not adjacent in \(G\). A cycle \(C\) is \textit{even} if the number of codirectional edges of the cycle is even. Otherwise the cycle is \textit{odd}.

**Proposition 4.** Let \(G\) be a PU-oriented bipartite graph. Then any chordless cycle in \(G\) is even.
Proof. Let us consider a chordless cycle $C$. If the length $l$ of $C$ is 4 then $C$ is a square and, thus, is even. If $l > 4$ we choose any two neighbour vertices $u$ and $v$ of $C$ and apply the move $\Omega_4$. In the new graph the cycle $C$ splints into an even square that contains $u$ and $v$ and a chordless cycle $C'$ of length $l - 2$. The new cycle has the same parity as $C$. After repeating this procedure $l/2 - 2$ times, the rest of the cycle $C$ will be a square which is even due to Proposition 3. Therefore the original cycle $C$ is even.

**Corollary 1.** Any two PU-orientations of a bipartite graph $G$ coincide up to reversions $R$.

Proof. The difference between two principally unimodular orientations defines a cocycle $c$ in $H^1(G, \mathbb{Z}_2)$. By proposition the cocycle $c$ vanishes on any chordless cycle. But chordless cycles generate $H_1(G, \mathbb{Z}_2)$, therefore $c = 0$. Then $c = da$, $a \in C^0(G, \mathbb{Z}_2)$. This means that one of the considered PU-orientations can be obtained from the other by $R$ moves at the vertices $v$ such that $\alpha(v) \neq 0$.

Thus the theory of PU-oriented graph-links is in fact a theory of PU-orientable graph-links. We shall call graphs, which admit PU-orientation, *principally unimodular graphs* (or PU-graphs) and call the corresponding graph-links PU-graphs-links.

### 3 Odd Khovanov homology of PU-graph-links

Let $G$ be a PU-oriented bipartite labeled graph with $n$ vertices and $A = A(G)$ be its adjacency matrix.

Suppose $s \subset V = V(G)$. Consider the vector space

$$V(s) = \mathbb{Z} < x_1, \ldots, x_n | r^*_1, \ldots, r^*_n >$$

where relations $r^*_1, \ldots, r^*_n$ are given by the formula

$$r^*_i = \begin{cases} x_i - \sum_{\{j \mid v_j \in s\}} \text{sgn}(v_j)a_{ij}x_j, & \text{if } v_i \notin s, \\ -\sum_{\{j \mid v_j \in s\}} \text{sgn}(v_j)a_{ij}x_j, & \text{if } v_i \in s \end{cases} \quad (2)$$

We denote $\overline{A} = (\overline{a}_{ij})_{i,j=1,\ldots,n}$, $\overline{a}_{ij} = -\text{sgn}(v_j)a_{ij}$, the relation matrix. Let $A'$ be a submatrix in $A(G)$. We denote $\overline{A'}$ the corresponding submatrix (i.e. submatrix with the same sets of rows and columns as $A'$) in $\overline{A}$. We have $\text{rank}A' = \text{rank}\overline{A}$ and $\text{rank}A' = \text{rank}\overline{A}$.

The main technical consequence of the principal unimodularity is the following statement.

**Proposition 5.** For any state $s$ the $\mathbb{Z}$-module $V(s)$ is free.
Proof. The module $V(s)$ has no torsion if and only if for any $k$ the ideal $E_k(s) \subset \mathbb{Z}$ generated by all minors of corank $k$ in the relation matrix $\overline{A}(s)$ is equal to 0 or $\mathbb{Z}$. But by Lemma 1 all minors in $A$ are equal 0 or ±1, hence every minor in $\overline{A}$ is equal 0 or ±1.

The rank of $V(s)$ is equal to $\text{corank} \overline{A}(s) = \text{corank} A(s)$.

There is a natural bijection between states $s \subset \mathcal{V}$ and vertices of the hypercube $\{0, 1\}^n$. Every edge of the hypercube is of the type $s \to s \oplus i$ where $s \oplus i$ denotes $s \cup \{v_i\}$ if $v_i \not\in s$ and $s \setminus \{v_i\}$ if $v_i \in s$. We orient the arrow so that $v_i \not\in s$ if $\text{sgn}(v_i) = -1$ and $v_i \in s$ if $\text{sgn}(v_i) = 1$.

We assign to every edge $s \to s \oplus i$ the map $\partial^{s \oplus i}_{s} : \bigwedge^* V(s) \to \bigwedge^* V(s \oplus i)$ of exterior algebras defined by the formula

$$\partial^{s \oplus i}_{s}(u) = \begin{cases} x_i \wedge u & \text{if } x_i = 0 \in V(s), \\ u & \text{if } x_i \not= 0 \in V(s) \end{cases}$$

Lemma 2. $x_i = 0 \in V(s)$ iff $\text{corank} A(s \oplus i) = \text{corank} A(s) + 1$

Proof. Case 1. $\text{sgn}(v_i) = -1$. Then $v_i \not\in s$ and $s \oplus i = s \cup \{v_i\}$. The relation matrix of $s \oplus i$ up to numeration of vertices looks like

$$\overline{A}(s \oplus i) = \begin{pmatrix} \overline{A}(s) & -a^T \\ a & 0 \end{pmatrix},$$

and we have $x_i = \sum_{j: v_j \in s} \overline{a}_j x_j$. Equality $x_i = 0$ means that row $\overline{a}$ is linearly dependent on rows of the matrix $\overline{A}(s)$. This is equivalent to the equality $\text{rank}(\overline{A}(s)) = \text{rank} \left( \begin{pmatrix} \overline{A}(s) \\ \overline{a} \end{pmatrix} \right)$. So $\text{rank} \overline{A}(s) \leq \text{rank} \overline{A}(s \oplus i) \leq \text{rank} \overline{A}(s) + 1$.

But the ranks of $\overline{A}(s)$ and $\overline{A}(s \oplus i)$ are even because $\text{rank} A(s) = \text{rank} \overline{A}(s)$, $\text{rank} A(s \oplus i) = \text{rank} \overline{A}(s \oplus i)$ and the matrices $A(s)$ and $A(s \oplus i)$ are skew-symmetric. Then $\text{rank} \overline{A}(s \oplus i) = \text{rank} A(s)$ and $\text{rank} \overline{A}(s \oplus i) = \text{rank} A(s) + 1$.

Case 2. $\text{sgn}(v_i) = 1$. Then $v_i \in s$ and $s \oplus i = s \setminus \{v_i\}$. The intersection matrix of $s$ up to numeration of vertices has the form

$$\overline{A}(s) = \begin{pmatrix} \overline{A}(s \oplus i) & -a^T \\ a & 0 \end{pmatrix}.$$

Since $v_i \in S$, equality $x_i = 0$ means that the ranks of the matrices $\begin{pmatrix} \overline{A}(s \oplus i) & -a^T \\ a & 0 \end{pmatrix}$ and $\begin{pmatrix} \overline{A}(s \oplus i) & -a^T \\ 0 & 0 \end{pmatrix}$ coincide. But

$$\text{rank} \begin{pmatrix} \overline{A}(s \oplus i) & -a^T \\ a & 0 \end{pmatrix} = \text{rank} \begin{pmatrix} \overline{A}(s \oplus i) & 0 \\ 0 & 0 \end{pmatrix} \geq \text{rank} \overline{A}(s \oplus i) + 1.$$
So \( \text{rank} A(s) = \text{rank} A(s \oplus i) + 2 \) and \( \text{corank} A(s \oplus i) = \text{corank} A(s) + 1 \).

One can see that the reasoning can be reverted and the corank condition is equivalent to the equality \( x_i = 0 \).

**Corollary 2.** \( x_i = 0 \in V(s) \) iff \( x_i \neq 0 \in V(s \oplus i) \).

**Proposition 6 (Correctness of chain maps).** For any state \( s \) and index \( i \) the map \( \partial^s_{s \oplus i} : \Lambda^* V(s) \to \Lambda^* V(s \oplus i) \) is well defined.

**Proof.** We must check that for any element \( u \) and any index \( j \) there exist elements \( u_k \in V(s \oplus i) \) such that

\[
\partial^s_{s \oplus i}(r_j^s \wedge u) = \sum_k r_j^{s \oplus i} \wedge u_k \in V(s \oplus i).
\]

For any \( j \) we have \( r_j^s = r_j^{s \oplus i} + \alpha_j x_i \) for some \( \alpha_j \). If \( x_i = 0 \in V(s \oplus i) \) then

\[
\partial^s_{s \oplus i}(r_j^s \wedge u) = r_j^s \wedge u = r_j^{s \oplus i} \wedge u + \alpha_j x_i \wedge u = r_j^{s \oplus i} \wedge u
\]

in \( V(s \oplus i) \). If \( x_i \neq 0 \in V(s \oplus i) \) then

\[
\partial^s_{s \oplus i}(r_j^s \wedge u) = x_i \wedge r_j^s \wedge u = x_i \wedge r_j^{s \oplus i} \wedge u + \alpha_j x_i \wedge x_i \wedge u = \pm r_j^{s \oplus i} \wedge (x_i \wedge u).
\]

In any case the map \( \partial^s_{s \oplus i} \) is well defined.

**Remark 1.** For any \( s \) state and an index \( i \) there are two possibilities for the map \( \partial^s_{s \oplus i} : \Lambda^* V(s) \to \Lambda^* V(s \oplus i) \):

1. \( \text{rank} V(s \oplus i) = \text{rank} V(s) - 1 \). Then \( \partial^s_{s \oplus i} \) is an epimorphism with the kernel \( x_i \Lambda^* V(s) \);

2. \( \text{rank} V(s \oplus i) = \text{rank} V(s) + 1 \). Then \( \partial^s_{s \oplus i} \) is an isomorphism of \( \Lambda^* V(s) \) onto \( x_i \Lambda^* V(s \oplus i) \).

Every 2-face of the hypercube of states looks like

\[
\begin{array}{ccc}
\Lambda^* V(s) & \xrightarrow{\partial^s_{s \oplus i}} & \Lambda^* V(s \oplus i) \\
\partial^s_{s \oplus i-j} & \downarrow & \\
\Lambda^* V(s) & \xrightarrow{\partial^s_{s \oplus j}} & \Lambda^* V(s \oplus i)
\end{array}
\]

According to dimensions of spaces \( V(s') \), \( s' = s, s \oplus i, s \oplus j, s \oplus i \oplus j \) we have five types of diagrams:
Here the number at the place of state \( s' \) is equal to \( \text{rank}V(s') - \text{rank}V(s) = \text{corank}(A(s')) - \text{corank}(A(s)) \) and the label \( z = 1, x_i, x_j \) at the arrow for the map \( \partial^\prime \) means that \( \partial^\prime_{s'}(u) = z \wedge u. \)

**Proposition 7** (Commutativity of 2-faces). Any 2-face of the hypercube of states is commutative or anticommutative.

*Proof.* 2-faces of type 1 are anticommutative. 2-faces of types 2, 3 are commutative. A 2-face of type 4 is commutative because \( x_i = x_j = 0 \in V(s \oplus i \oplus j) \).

We need to look at type 5 more attentively. There are several possibilities.

1. \( \text{sgn}(v_i) = \text{sgn}(v_j) = -1 \). Then \( v_i, v_j \in s \oplus i \oplus j \). We can assume that \( v_i \) and \( v_j \) are the last vertices in \( s \oplus i \oplus j \).
   1.1. \( v_i, v_j \in V_0 \). The relation matrix \( \overline{A}(s \oplus i \oplus j) \) can be represented in the form

\[
\begin{pmatrix}
0 & \overline{B} \\
-\overline{B}^\top & -a^\top & -b^\top & 0
\end{pmatrix}
\]

(4)

where \( B = B(s) \). We have \( x_i = 0 \in V(s \oplus i) \). Then the rows of the matrix \( (\overline{B}^\top - a^\top - b^\top) \) generate the vector \( (0 1) \). Hence the rows of the matrix \( (-\overline{B}^\top - a^\top - b^\top) \) generate the vector \( (0 1 \alpha) \), \( \alpha \in \mathbb{Z} \). Analogously, the matrix \( (-\overline{B}^\top - a^\top - b^\top) \) generate the vector \( (0 \beta 1) \). These two vectors generate the vector \( (0 0 1 - \alpha\beta) \). If \( \alpha\beta \neq 1 \) then \( (1 - \alpha\beta)x_j = 0 \in V(s \oplus i \oplus j) \). Since \( V(s \oplus i \oplus j) \) is free we would have \( x_j = 0 \in V(s \oplus i \oplus j) \) but this is not the case. Thus, \( \alpha\beta = 1 \) and \( \alpha = \beta = \pm 1 \). Then \( x_i \pm x_j = 0 \in V(s \oplus i \oplus j) \) so the square is commutative or anticommutative.

1.2. \( v_i \in V_0, v_j \in V_1 \). The relation matrix \( \overline{A}(s \oplus i \oplus j) \) can be represented in the form

\[
\begin{pmatrix}
0 & \overline{B} & b^\top \\
0 & 0 & \overline{B}^\top \\
-\overline{B}^\top & -a^\top & 0 \\
-\overline{b} & -a & 0
\end{pmatrix}
\]

(5)

where \( B = B(s) \). We have \( x_i = 0 \in V(s \oplus i) \). Then the rows of the matrix \( (\overline{B}^\top - a^\top - b^\top) \) generate the vector \( (0 1) \). Hence the rows of the matrix \( (-\overline{B}^\top - a^\top - b^\top) \) generate the vector \( (0 1 - \alpha\beta) \). If \( \alpha\beta \neq 1 \) then \( (1 - \alpha \beta)x_j = 0 \in V(s \oplus i \oplus j) \). Since \( V(s \oplus i \oplus j) \) is free we would have \( x_j = 0 \in V(s \oplus i \oplus j) \) but this is not the case. Thus, \( \alpha\beta = 1 \) and \( \alpha = \beta = \pm 1 \). Then \( x_i \pm x_j = 0 \in V(s \oplus i \oplus j) \) so the square is commutative or anticommutative.
Since \( x_i = 0 \in V(s \oplus j) \) the vector \( (\overrightarrow{\pi} \overrightarrow{v}) \) depends on the rows of the matrix \( \text{rank} \begin{pmatrix} \overrightarrow{b} & \overrightarrow{b}^\top \\ \overrightarrow{a} & \overrightarrow{a}^\top \end{pmatrix} \). Then the vector \( \overrightarrow{\pi} \) depends on the rows of \( \overrightarrow{B} \) so \( x_i = 0 \in V(s) \) but this is not true. Thus, this case is impossible.

1.3. \( v_i, v_j \in V_1 \). This case can be considered analogously to the case 1.1.

2. \( \text{sgn}(v_i) = -1, \text{sgn}(v_j) = 1 \). Then \( v_i \in s \oplus i \oplus j \) and \( v_j \not\in s \oplus i \oplus j \).

2.1. \( v_i, v_j \in V_0 \). Without loss of generality we can assume that the relation matrix \( A(s \oplus i) \) has the form \((\mathbf{1})\) where \( B = B(s \oplus j) \). Since \( x_i = 0 \in V(s \oplus j) \) then \( \text{rank} \begin{pmatrix} \overrightarrow{b} \\ \overrightarrow{a} \end{pmatrix} = \text{rank} \overrightarrow{B} \). Since \( x_j = 0 \in V(s \oplus j) \) then \( \text{rank} \begin{pmatrix} \overrightarrow{b} \\ \overrightarrow{a} \end{pmatrix} = \text{rank} \overrightarrow{B} + 1 \) because \( x_i = 0 \in V(s \oplus i) \). Thus, this case is impossible.

2.2. \( v_i \in V_0, v_j \in V_1 \). The relation matrix \( A(s \oplus i) \) can be represented in the form \((\mathbf{1})\) with \( B = B(s \oplus j) \). Since \( x_j = 0 \in V(s \oplus j) \) we have \( \text{rank} \begin{pmatrix} -\overrightarrow{b}^\top \\ -\overrightarrow{a}^\top \end{pmatrix} = \text{rank} -\overrightarrow{B}^\top \). Then the vector \(-\overrightarrow{b}\) is generated by the rows of \(-\overrightarrow{B}^\top\) so the row of the matrix \( \left( -\overrightarrow{B}^\top -\overrightarrow{a}^\top \right) \) generate the vector \(-\overrightarrow{b}\). Hence the matrix \( \begin{pmatrix} -\overrightarrow{B}^\top & -\overrightarrow{a}^\top \\ 0 & -\delta \end{pmatrix} \) where \( \delta = -\overrightarrow{a}\). If \( \delta = 0 \) then \( \text{rank} \begin{pmatrix} -\overrightarrow{B}^\top & -\overrightarrow{a}^\top \\ -\overrightarrow{b} & -\delta \end{pmatrix} = \text{rank} \begin{pmatrix} -\overrightarrow{B}^\top & -\overrightarrow{a}^\top \\ 0 & -\delta \end{pmatrix} \) and \( x_j = 0 \in V(s \oplus i \oplus j) \) but this is not true.

If \( |\delta| > 1 \) then we must have \( B = 0 \) (otherwise we can find a minor in \( A(G) \) which is not equal to 0 and is a multiple of \( \delta \)). Hence, \( b = 0 \). If \( -\overrightarrow{a} = 0 \) then \( x_j = 0 \in V(\oplus_i \oplus j) \) that is not true. If \( -\overrightarrow{a} = \pm 1 \) then \( x_i \perp x_j = 0 \in V(\oplus_i \oplus j) \).

If \( \delta = \pm 1 \) then we have \( x_i \perp x_j = 0 \in V(\oplus_i \oplus j) \).

2.3. \( v_i \in V_1, v_j \in V_0 \). This case is considered analogously the case 2.2.

2.4. \( v_i, v_j \in V_1 \). This case is impossible by the same reason as the case 2.1.

3. \( \text{sgn}(v_i) = \text{sgn}(v_j) = -1 \). Then \( v_i, v_j \in s \oplus i \oplus j \).

3.1. \( v_i, v_j \in V_0 \). Then the matrix \( A(s) \) looks like \((\mathbf{1})\) where \( B = B(s \oplus i \oplus j) \).

Since \( x_i = 0 \in V(s \oplus i) \) we have \( \text{rank} \begin{pmatrix} \overrightarrow{b} \\ \overrightarrow{a} \end{pmatrix} = \text{rank} \begin{pmatrix} \overrightarrow{b} \\ \overrightarrow{a} \end{pmatrix} \). Then the vector \( \overrightarrow{\pi} \) is generated by the rows of the matrix \( \overrightarrow{B} \) and the vector \( \overrightarrow{B} \). It means that \( x_i = ax_j \in V(s \oplus i \oplus j) \). On the other hand, the same reasoning for \( x_j \) leads to the equality \( x_i = bx_j \in V(s \oplus i \oplus j) \) so \( x_i = \alpha \beta x_i \). If \( \alpha \beta \neq 1 \) then \( (1-\alpha \beta) x_i = 0 \) implies \( x_i = 0 \) since \( V(s \oplus i \oplus j) \) is free. But \( x_i \neq 0 \in V(s \oplus i \oplus j) \). Hence, \( \alpha \beta = 1 \), so \( \alpha = \beta = \pm 1 \) and \( x_i \perp x_j = 0 \in V(s \oplus i \oplus j) \).

3.2. \( v_i \in V_0, v_j \in V_1 \). The relation matrix \( A(s) \) can be represented in the form \((\mathbf{1})\) with \( B = B(s \oplus i \oplus j) \). Since \( x_i \neq 0 \in V(s \oplus i \oplus j) \) we have \( \text{rank} \begin{pmatrix} \overrightarrow{b} \\ \overrightarrow{a} \end{pmatrix} = \text{rank} \begin{pmatrix} \overrightarrow{b} \\ \overrightarrow{a} \end{pmatrix} \).
The equality $x_j \neq 0 \in V(s \oplus i \oplus j)$ implies $\text{rank} \begin{pmatrix} \overline{B} & b^\top \\ \alpha & \alpha \end{pmatrix} = \text{rank} \begin{pmatrix} \overline{B} \\ \alpha \end{pmatrix} + 1$. But $\text{rank} \begin{pmatrix} \overline{B} & b^\top \\ \alpha & \alpha \end{pmatrix} = \text{rank} \begin{pmatrix} \overline{B} \alpha & b^\top \alpha \end{pmatrix}$ since $x_j \notin V(s)$. Thus, this case is impossible.

3.3. The case $v_i, v_j \in \mathcal{V}_1$ is analogous to the case 3.1.

Thus, any diagram of type 5 is commutative or anticommutative. □

**Remark 2.** Following [7] we introduce another classification of two faces: anticommutative faces (type A), commutative faces (type C) and zero faces (types X and Y). 2-faces of type 1 have type A. 2-faces of types 2,3 have type C. Ass for diagrams of type 5 we assign to commutative diagrams the type C and to anticommutative ones the type A. A 2-face of type 4 is a zero face because $x_1 = x_j = 0 \in V(s \oplus i \oplus j)$, below we assign this face to type X or Y.

We call a vertex $v \in \mathcal{V}$ **inner** if $v \in \mathcal{V}_0$ and $\text{sgn}(v) = -1$ or $v \in \mathcal{V}_1$ and $\text{sgn}(v) = 1$. Otherwise $v$ is **outer**.

Diagrams of type 4 differ from diagrams of type 5 by a sign of the vertex $v_i$ or $v_j$. So the consideration of possible cases among cases 1.1-3.3 in Proposition 7 shows this statement is true.

**Lemma 3.** 1. In any diagram of type 5 the vertices $v_i, v_j$ are either both inner or both outer.
2. In any diagram of type 4 one the vertices $v_i, v_j$ is inner and the other is outer. □

Let us consider a 2-face of type 4 and let $v_i$ be the inner vertex of the face. We assign the face to the **type X** if $x_i = x_j \in V(s \oplus i)$ and assign to the **type Y** if $x_i = -x_j \in V(s \oplus i)$.

**Lemma 4.** Let us consider a 2-face of type 5. Then

- if $\text{sgn}(v_i) = \text{sgn}(v_j)$ we have $x_i \pm x_j \in \mathcal{V} \iff x_i \mp x_j \in \mathcal{V}(s \oplus i \oplus j)$;
- if $\text{sgn}(v_i) \neq \text{sgn}(v_j)$ we have $x_i \pm x_j \in \mathcal{V} \iff x_i \pm x_j \in \mathcal{V}(s \oplus i \oplus j)$.

For a 2-face of type 4 we have

- if $\text{sgn}(v_i) = \text{sgn}(v_j)$ then $x_i \pm x_j \in \mathcal{V}(s \oplus i) \iff x_i \pm x_j \in \mathcal{V}(s \oplus j)$;
- if $\text{sgn}(v_i) \neq \text{sgn}(v_j)$ then $x_i \pm x_j \in \mathcal{V}(s \oplus i) \iff x_i \mp x_j \in \mathcal{V}(s \oplus j)$.

**Proof.** Since any diagram of type 4 can be transformed in a diagram of type 5 by change of sign of the vertex $v_i$ we can prove this lemma only for diagrams of type 5.

Let us denote $s_\alpha = s \cap \mathcal{V}_\alpha$, $\alpha = 0, 1$.

1. Assume at first that $\text{sgn}(v_i) = \text{sgn}(v_j) = -1$ and $v_i, v_j \in \mathcal{V}_0$. Then $v_i, v_j \notin s$. We have $x_i + \alpha x_j = 0 \in V(s)$. This means there exist coefficients $\lambda_k$ where $v_k \in s_0$ such that

$$x_i + \alpha x_j = r_i^* + \alpha r_j^* + \sum_{k : v_k \in s_0} \lambda_k r_k^* \in \mathbb{Z}(x_i | v_k \in s_1) \oplus \mathbb{Z}(x_i, x_j).$$
In other words, we have equations \( \sum_k \lambda_k \text{sgn}(v_l) a_{kl} + \text{sgn}(v_l) a_{il} + \alpha \text{sgn}(v_l) a_{jl} = 0 \) for any \( l \in s_1 \). Then \( \sum_k \lambda_k a_{kl} = -a_{il} - \alpha \cdot a_{jl} \).

The equality \( x_i + \beta x_j \in V(s \oplus i \oplus j) \) means that there exist \( \mu_i, \, v_l \in (s \oplus i \oplus j)_1 = s_1 \) such that

\[
x_i + \beta x_j = \sum_{l : v_l \in (s \oplus i \oplus j)_1} \mu_i r^s_{l} \in \mathbb{Z}(x_k \mid v_k \in (s \oplus i \oplus j)_0).
\]

This is equivalent to the system of equations:

\[
\sum_l \mu_l a_{lk} = 0, \quad v_k \in s_0, \quad \sum_l \mu_l a_{li} = -\text{sgn}(v_l), \quad \sum_l \mu_l a_{lj} = -\beta \text{sgn}(v_j).
\]

Then

\[
\sum_{(k,l) : v_k \in s_0, v_l \in s_1} \lambda_l \mu_l a_{kl} = \sum_l \mu_l \sum_k \lambda_l a_{kl} = -\sum_l \mu_l (a_{il} + \alpha \cdot a_{jl}) = \sum_l \mu_l a_{li} + \alpha \sum_l \mu_l a_{lj} = -\text{sgn}(v_i) - \alpha \beta \text{sgn}(v_j).
\]

On the other hand, \( \sum_{k,l} \lambda_k \mu_l a_{kl} = -\sum_k \lambda_k \sum_j \mu_l a_{lk} = 0 \). Thus, \( \text{sgn}(v_i) \text{sgn}(v_j) + \alpha \beta = 0 \) so \( \alpha = -\beta \) that proves the statement of the lemma.

The case \( v_i, v_j \in V_1 \) is considered analogously.

2. Assume that \( \text{sgn}(v_i) = -1 \), \( \text{sgn}(v_j) = 1 \) and \( v_i \in V_0, v_j \in V_1 \). Then \( s_0 = s_0', s_1 = s_1' \cup \{v_j\} \) and \( (s \oplus i \oplus j)_0 = s_0' \cup \{v_i\}, \, (s \oplus i \oplus j)_1 = s_1' \).

The identity \( x_i + \alpha x_j = 0 \in V(s) \) means there exist \( \lambda_k, \, v_k \in s_0 \), such that

\[
x_i + \alpha x_j = r^s_{i} + \sum_{k : v_k \in s'_0} \lambda_k r^s_{k} \in \mathbb{Z}(x_k \mid v_k \in s_1) \oplus \mathbb{Z}(x_i).
\]

Then we have equations

\[
\sum_k \lambda_k a_{kl} + a_{il} = 0, \quad v_l \in s'_1, \quad \sum_k \lambda_k a_{kj} + a_{ij} = -\alpha \text{sgn}(v_j).
\]

The identity \( x_i + \beta x_j = 0 \in V(s \oplus i \oplus j) \) leads to equations

\[
\sum_{l : v_l \in s'_1} \mu_l a_{lk} = \beta a_{jk}, \quad v_k \in s'_0, \quad \sum_{l : v_l \in s'_1} \mu_l a_{li} + \beta a_{ji} = -\text{sgn}(v_i).
\]

Then

\[
\sum_{(k,l) : v_k \in s'_0, v_l \in s'_1} \lambda_k \mu_l a_{kl} = \sum_l \mu_l \sum_k \lambda_k a_{kl} = -\sum_l \mu_l a_{il} = \text{sgn}(v_i) + \beta a_{ji}.
\]

On the other hand,

\[
\sum_{(k,l) : v_k \in s'_0, v_l \in s'_1} \lambda_k \mu_l a_{kl} = -\sum_k \lambda_k \sum_l \mu_l a_{lk} = \beta \sum_k \lambda_l a_{kj} = -\beta a_{ij} - \alpha \beta \text{sgn}(v_j).
\]
Since \( a_{ji} = -a_{ij} \) we have \( \alpha \beta = -\text{sgn}(v_i)\text{sgn}(v_j) = 1 \). Thus \( x_i \pm x_j = 0 \in V(s) \) iff \( x_i \pm x_j = 0 \in V(s \oplus i \oplus j) \).

3. The case \( \text{sgn}(v_i) = \text{sgn}(v_j) = 1 \) is considered analogously the case 1.

Let us consider a 2-face of type 4. We assign the face to the type \( X \) if \( x_i = \text{sgn}(v_j)x_j \in V(s \oplus i) \) (by Lemma 4 this is equivalent to \( x_i = \text{sgn}(v_i)x_j \in V(s \oplus j) \)) and assign to the type \( Y \) if \( x_i = -\text{sgn}(v_j)x_j \in V(s \oplus i) \).

**Edge assignment.**

Let us denote the set of the edges in the hypercube as \( E \). We call edge assignment any map \( \epsilon : E \to \{\pm 1\} \) (see [7]). A 2-face is called even (resp. odd) if it contains even (resp. odd) number of edges \( e \) with \( \epsilon(e) = -1 \). A type \( X \) edge assignment is an edge assignment such that all faces of type \( A \) and \( X \) are even and all faces of type \( C \) and \( Y \) are odd. Similarly, type \( Y \) edge assignment is an edge assignment for which faces of type \( A \) and \( Y \) are even and faces of type \( C \) and \( X \) are odd.

**Lemma 5.** Each cube in the hypercube contains an even number of squares of type \( A \) and \( X \). Similarly, each cube contains an even number of squares of type \( A \) and \( Y \).

**Proof.** The proof is an analysis of all possible configurations of cubes using Lemmas 3, 4.

There are 18 possible cubes (up to symmetry of axes). Below the number at the place of state \( s' \) in the cube is equal to \( \text{rank}V(s') - \text{rank}V(s) \). These cases can be classified into the following groups.
Cases 1,6,13,18. The types of 2-faces are determined by ranks of states and their number can be counted explicitly. For example, in the case 6 the cube contains 4 faces of type C and 2 faces of type A.

Cases 2,5,7,10,14,15. These cases are not realizable because of Lemma 3. One can not attribute the vertices $v_i, v_j, v_k$ to inner and outer vertices so that the faces of type 4, incident to the state $s \oplus i \oplus j \oplus k$, have right configuration.

Cases 3,8,11,16. The faces of the cube include 4 commutative faces (cases 11, 16) or 2 A-faces and 2 C-faces (cases 3,8). The other two faces are opposite to each other and have the same type because there is a projection from one face to the other.

Cases 4,9,12,17. For example, let us consider the case 4. The cube has one anticommutative face and 2 commutative faces. The other three faces are incident to the state $s \oplus i$. Let us assume that $x_i = x_j = x_k \in V(s \oplus i)$ and $\text{sgn}(v_i) = \text{sgn}(v_j) = \text{sgn}(v_k) = -1$. Then these three faces have types A, Y and Y. We can see that if one changes the sign of any vertex $v_i, v_j, v_k$ or the sign of any variable $x_i, x_j, x_k$ in $V(s \oplus i)$ then two faces of the three change their type. So the number of A- and X-faces remains even.

**Lemma 6.** Any PU-oriented bipartite labeled graph $G$ has an edge assignment of type $X$ and one of type $Y$.

**Proof.** See [7, Lemma 1.2].

Given a type X or type Y edge assignment $\epsilon$ we define the chain complex

$$C(G) = \bigoplus_{s \subset V} \bigwedge^* V(s)$$

with differential

$$\partial_\epsilon(u) = \sum_{\{s, s' \subset V | s \rightarrow s' = \epsilon \in E\}} \epsilon(\epsilon) \partial_{\epsilon'}(u).$$

This lemmas were proved in [7].

**Lemma 7.** If $\epsilon$ and $\epsilon'$ are two edge assignment of the same type (X or Y) then the chain complex $(C(G), \partial_\epsilon)$ is isomorphic to $(C(G), \partial_{\epsilon'}).$

**Lemma 8.** If $\epsilon$ and $\epsilon'$ are two edge assignment of opposite types then there is isomorphism $(C(G), \partial_\epsilon) \cong (C(G), \partial_{\epsilon'}).$

**Definition 4.** Homology $\mathbb{KH}(G)$ of the complex $(C(G), \partial)$ is called reduced odd Khovanov homology of the labeled simple graph $G$.

The main theorem of the article states that odd Khovanov homology is in fact an invariant of PU-orientable graph-links.

**Theorem 1.** Khovanov homology $\mathbb{KH}(G)$ is invariant under $R, \Omega_1, \Omega_2^{PU}, \Omega_3, \Omega_4$ moves.
Proof. Let $G$ be a labeled graph and $\tilde{G}$ be a graph obtained from $G$ by some Reidemeister move $R, \Omega_1, \Omega_2^{P}, \Omega_3, \Omega_4$.

Invariance under $R$.

See [7, Lemma 2.3].

Invariance under $\Omega_1$.

Let $G$ be obtained from $G$ by addition an isolated labeled vertex $v$. Then its adjacency matrix $A(\tilde{G})$ looks like \[
\begin{pmatrix}
0 & 0 & a \\
0 & 0 & a \\
-a^\top & -a^\top & A(G)
\end{pmatrix}.
\]

The complex $C(\tilde{G})$ splits as a $\mathbb{Z}$-module into the sum $C \oplus C_v$ where $C$ corresponds to states $s \in V(\tilde{G})$ such that $v \not\in s$ and $C_v$ corresponds to states $s \in V(\tilde{G})$ such that $v \in s$. There is a natural bijection between states of $C(G)$, $C$ and $C_v$. Let $\epsilon$ be an edge assignment on $C(G)$. We define an edge assignment $\tilde{\epsilon}$ on $C(\tilde{G})$ as follows.

We set $\tilde{\epsilon} = \epsilon$ on $C_v$ and $\tilde{\epsilon}(e) = 1$ for all edges between $C$ and $C_v$. If $\text{sgn}(v) = 1$ we set $\tilde{\epsilon} = -\epsilon$ on $C$. If $\text{sgn}(v) = -1$ we set $\tilde{\epsilon} = \delta \cdot \epsilon$ on $C$ where $\delta(s \to s \oplus i) = 1$ if $\text{rank}(V(s \oplus i)) = \text{rank}(V(s)) + 1$ and $\delta(s \to s \oplus i) = -1$ otherwise.

Then $\tilde{\epsilon}$ is an edge assignment of the same type as $\epsilon$ because all squares with edges connecting $C$ and $C_v$ are of type $C$ or $A$ and the parity of other squares is the same as in $C(G)$.

The complex $(C(\tilde{G}), \partial_{\tilde{\epsilon}})$ is isomorphic to product of complexes $(C(G), \partial_{\epsilon}) \otimes C(v)$ where the complex $C(v)$ is equal to

\[
\mathbb{Z}_2 \xrightarrow{x} \wedge^* \mathbb{Z}_2(x)
\]
if $\text{sgn}(v) = -1$ and

\[
\wedge^* \mathbb{Z}_2(x) \xrightarrow{x=0} \mathbb{Z}_2
\]
if $\text{sgn}(v) = 1$. In any case $H_s(C(v)) = \mathbb{Z}_2 \cdot 1$, where $1 \in H_0(C(v))$ if $\text{sgn}(v) = 1$ and $1 \in H_1(C(v))$ if $\text{sgn}(v) = -1$. Thus, we have

\[
\text{Kh}(\tilde{G}) = \text{Kh}(G) \otimes \text{Kh}(v) \cong \text{Kh}(G).
\]

Invariance under $\Omega_2$.

Assume that we add vertices $v$ and $w$ to get the graph $\tilde{G}$ by $\Omega_2$ and $\text{sgn}(v) = 1$, $\text{sgn}(w) = -1$. We can write the adjacency matrix $A(\tilde{G})$ in the form

\[
\begin{pmatrix}
0 & 0 & a \\
0 & 0 & a \\
-a^\top & -a^\top & A(G)
\end{pmatrix}.
\]

For every state $s \in V(G)$ the following equations \[
\text{corank}(A(G(s))) = \text{corank}(A(\tilde{G}(s))),
\]
\[
\text{corank}(A(G(s \cup \{v\}))) = \text{corank}(A(G(s \cup \{w\}))) = \text{corank}(A(G(s \cup \{v, w\})) - 1.
\]
These equalities defines the type of the upper and left arrows of the complex.
The correspondent relation matrices are \[ \bigwedge^\ast \bigwedge^\ast V \] and the correspondent isomorphisms of the exterior algebras compatible with \[ \bar{C}, C_\sim \] of \( C_\sim \) and \( G \). Then the adjacency matrices of \( C \) and \( \bar{C} \) is well-defined because it vanishes any relation: \( f(v_1^\top) = \pi_{11} + \pi_{12} = 0 \) since \( a_{11} = a_{12} \) and \( \text{sgn}(v_1) = -\text{sgn}(v_2) \). Then \( \bigwedge^\ast V = \bigwedge^\ast \ker f \oplus x_2 \bigwedge^\ast V(s) \) and \( C_\sim = X \oplus x_2 C_\sim \). One can check that subcomplex \( X \to C_\sim \) is acyclic. Then homology of \( C(\bar{G}) \) coincides with the homology of the quotient complex

\[
\begin{array}{c}
C_{vw} \xrightarrow{x_2} C_v \\
\downarrow x_2 \\
C_v \xrightarrow{\partial} C.
\end{array}
\]

The quotient of this complex by subcomplex \( C \) appears to be acyclic too. Thus \( C(\bar{G}) \) has the same homology as \( C = C(G) \).

**Invariance under \( \Omega_3 \).**

Let the vertices \( u, v, w \) in the third Reidemeister move has the numbers 1, 2, 3 in the \( \mathcal{V}(G) = \mathcal{V}(\bar{G}) \). Then the adjacency matrices of \( G \) and \( \bar{G} \) looks like

\[
A(G) = \begin{pmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & a \\
-1 & 0 & 0 & b \\
\mathbf{0}^\top & -a^\top & -b^\top & D
\end{pmatrix}, \quad A(\bar{G}) = \begin{pmatrix}
0 & 0 & 0 & a - b \\
0 & 0 & 0 & a \\
0 & 0 & 0 & b \\
(b - a)^\top & -a^\top & -b^\top & D
\end{pmatrix}.
\]

The correspondent relation matrices are

\[
\begin{pmatrix}
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & \pi \\
-1 & 0 & 0 & b \\
\mathbf{0}^\top & -a^\top & -b^\top & D
\end{pmatrix} , \quad \begin{pmatrix}
0 & 0 & 0 & a - b \\
0 & 0 & 0 & \pi \\
0 & 0 & 0 & b \\
(b - a)^\top & a^\top & b^\top & D
\end{pmatrix}.
\]

Denote \( \bar{V}(s) = V(\bar{G}(s)) \). Then for any \( s \in \mathcal{V}(G) \setminus \{u, v, w\} \) we have \( V(s) \cong \bar{V}(s), V(s \oplus v) \cong \bar{V}(s \oplus v), V(s \oplus u) \cong \bar{V}(s \oplus u) \). Therefore, \( V(s \oplus v \oplus w) \cong \bar{V}(s \oplus v \oplus w) \) and the correspondent isomorphisms of the exterior algebras compatible with the differential.
Consider complexes $C(G)$ and $C(\tilde{G})$ in the form of cube:

For any state $s$ in $C_u$ define a linear function $f : V(s) \to \mathbb{Z}_2$ by the formula $f(\sum \lambda_i x_i) = \lambda_1$. The function is well-defined and there are decompositions $\bigwedge^* V(s) = \bigwedge^* \ker f \oplus \bigwedge^* V(s)$ and $C_u = X \oplus x_2 C_u$. Consider the following subcomplex

Factor complex $C \to x_2 C_u$ is acyclic so homology of $C(G)$ is isomorphic to homology of the subcomplex. This subcomplex contains acyclic subcomplex $X \to \partial(X)$. The maps $X \to C_{uv}$ and $X \to C_{uw}$ are isomorphisms, so after factorization we get the complex

Analogous reasonings reduce $C(\tilde{G})$ to the complex (in $\tilde{C}_{uw}$ we should define the function $f : V(s) \to \mathbb{Z}_2$ by the formula $f(\sum \lambda_i x_i) = \lambda_1 + \lambda_2 - \lambda_3$)

We show now that these complexes are isomorphic.
Let $\epsilon$ be an edge assignment of type $X$ on $C(G)$. We denote $\epsilon_0, \epsilon_u, \epsilon_{uv}$ etc. the restriction of $\epsilon$ to the edges of the spaces $C, C_u, C_{uv}$ etc. The restriction of the edge assignment $\epsilon$ to the edges between cubes $C_u$ and $C_{uv}$ will be denoted as $\epsilon_u(v)$, for edges between other cubes we use similar notation.

As in [7] we can assume that $\epsilon_{uv} = \epsilon_{uw}$.

We need explicit expression for the isomorphisms $V(s \oplus v \oplus w) \cong \bar{V}(s \oplus u \oplus v) \cong \bar{V}(s \oplus u \oplus w)$ etc. mentioned above. We define these isomorphisms on the generators as shown in the table

|        | $x_1$ | $x_2$ | $x_3$ |
|--------|-------|-------|-------|
| $V_{uv} \rightarrow V$ | $-x_2$ | 0 | $x_3 - x_2$ |
| $V_{uw} \rightarrow V$ | $-x_3$ | $x_2 - x_3$ | 0 |
| $V_{uwv} \rightarrow \bar{V}_u$ | $-x_2 = -x_3$ | $x_1$ | $-x_1$ |
| $V_{uw} \rightarrow \bar{V}_{uv}$ | $x_2$ | $x_1 - x_2$ | $-x_1$ |
| $V_{uw} \rightarrow \bar{V}_{uw}$ | $x_3$ | $x_1$ | $-x_1 - x_3$ |

Any other generator $x_i$, $i \geq 4$, goes to itself: $x_i \mapsto x_i$.

These maps induce isomorphisms $\phi_{uv}^\epsilon : C_{uv} \rightarrow C$, $\phi_{uw}^\epsilon : C_{uw} \rightarrow C$, $\phi_{uwv}^\epsilon : C_{uwv} \rightarrow C_u, \phi_{uw}^\epsilon : C_{uw} \rightarrow C_{uv}, \phi_{uwv}^\epsilon : C_{uwv} \cong C_{uw}$. Below for identification the cube $C_{uv} \rightarrow C$ we will use the modified isomorphism $\phi_{uv}^\epsilon = \delta \phi_{uv}^\epsilon$ where $\delta = -\epsilon_u(v)\epsilon_u(w)$. Then we choose the edge assignment in the image to extend the isomorphisms of spaces to chain maps of squares

The induced edge assignment $\bar{\epsilon}$ is indicated on the edges of the diagrams. The sign $\bar{\lambda}_i$, $i = 1, 2, 3$, is equal to $-1$ on edges $s \rightarrow s \oplus i$ in $C(G)$, such that $\text{rank} V(s \oplus i) = \text{rank} V(s) + 1$, and equal to 1 otherwise.

The maps $C_{uv} \rightarrow C_{uwv}$ and $C_{uw} \rightarrow C_{uwv}$ induce the same maps $C \rightarrow \bar{C}_u$. Indeed, the first square gives the map $\bar{\epsilon}_0(u)\delta_{\oplus 1} = \epsilon_{uv}(w)\phi_{uv}^\epsilon \delta_{\oplus 1} \delta_{\oplus 2} \delta_{\oplus 3}^{-1}$ whereas the second gives $\epsilon_{uw}(v)\phi_{uw}^\epsilon \delta_{\oplus 1} \delta_{\oplus 2} \delta_{\oplus 3} \delta_{\oplus 1} \delta_{\oplus 2} \delta_{\oplus 3}^{-1}$.

The identification map $(\phi_{uv}^\epsilon)^{-1} \phi_{uv}^\epsilon$ between spaces $C_{uv}$ and $C_{uw}$ is equal to $\delta \delta_{\oplus 1} \delta_{\oplus 1} \delta_{\oplus 2} \delta_{\oplus 3} \delta_{\oplus 1} \delta_{\oplus 2} \delta_{\oplus 3}^{-1}$. Then the two induced maps coincide if

$$
\epsilon_{uv}(w)\delta_{\oplus 1} \delta_{\oplus 2} \delta_{\oplus 3} \delta_{\oplus 1} \delta_{\oplus 2} \delta_{\oplus 3}^{-1} = \epsilon_{uw}(v)\delta_{\oplus 1} \delta_{\oplus 2} \delta_{\oplus 3} \delta_{\oplus 1} \delta_{\oplus 2} \delta_{\oplus 3}^{-1}.
$$
But $\delta = -\epsilon_u(v)\epsilon_u(w)$ and 

$$
\epsilon_{uv}(w)\epsilon_u(v)\phi^{uvw} \partial_s^{s_1\oplus 1\oplus 2} \partial_s^{s_2\oplus 3} \partial_s^{s_3\oplus 1} = -\epsilon_{uw}(v)\epsilon_u(w)\phi^{uvw} \partial_s^{s_1\oplus 1\oplus 2} \partial_s^{s_2\oplus 3} \partial_s^{s_3\oplus 1}
$$

because any 2-face in $C(G)$ equipped with an edge assignment anticommutes.

The first isomorphism of squares keeps the types of 2-faces and does not change the parity with respect to the edge assignments. Then the induced edge assignment has type $X$ on the square

\[
\begin{array}{c}
C \\
\downarrow \\
C_u
\end{array} \quad \begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow
\end{array} \quad \begin{array}{c}
\tilde{C}_u \\
\downarrow \\
\downarrow
\end{array}
\]

The second isomorphism keeps the parity of 2-faces of types 1,2,3, change the parity for types 4 and 5 because in this case there exist a unique edge of the square with $\tilde{\lambda}_1 = -1$. On the other hand, the isomorphism interchange squares A and $\tilde{C}$ of type 5 since the relation $x_1 = x_2 \in V_{uvw}$ is equivalent to $-x_3 = x_1 \in V_u$. X-squares become Y-squares and vice versa because $x_2 = \text{sgn}(v_1)x_1 = -x_1 \in V_{uv}$ turns into $x_1 = -x_3 = -\text{sgn}(v_3)x_3 \in V_{uw}$. Thus, $\tilde{\epsilon}$ again appears to be of type $X$ on the square

\[
\begin{array}{c}
C \\
\downarrow \\
C_u
\end{array} \quad \begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow
\end{array} \quad \begin{array}{c}
\tilde{C}_u \\
\downarrow \\
\downarrow
\end{array}
\]

We can extend $\tilde{\epsilon}$ to an edge assignment of type $X$ on the whole complex $C(\tilde{G})$ (it is possible because the quotient space obtained after one collapses the faces where $\tilde{\epsilon}$ is defined to a point has vanishing cohomology group $H^2$). Then contraction of the complex $C(\tilde{G})$ with the chosen edge assignment yields the complex isomorphic to the corresponding complex obtained from $C(G)$.

**Invariance under $\Omega_4$.**

Let the vertices $u$ and $v$ of the move $\Omega_4$ has numbers $p$ and $q$ in $V(G) = V(\tilde{G})$. The coefficients of adjacency matrices of $A(G) = (a_{ij})$ and $A(\tilde{G}) = (\tilde{a}_{ij})$ are connected by the formula

$$
\tilde{a}_{ij} = \begin{cases} 
 a_{ij} - a_{pq}a_{ip}a_{jq} + a_{pq}a_{iq}a_{jp}, & \{i, j\} \cap \{p, q\} = \emptyset, \\
 a_{ij}, & \{i, j\} \cap \{p, q\} \neq \emptyset, \{p, q\}, \\
 -a_{ij}, & \{i, j\} = \{p, q\}.
\end{cases}
$$

Consider the map $\phi$ acting on the states by the formula

$$
\phi(s) = \begin{cases} 
 s \cup \{u, v\}, & \{u, v\} \cap s = \emptyset, \\
 s \setminus \{u, v\}, & \{u, v\} \cap s = \{u, v\}, \\
 s, & \{u, v\} \cap s \neq \emptyset, \{u, v\}.
\end{cases}
$$

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with the linear maps \( \Phi : V(s) \to V(\phi(s)) \) defined by the formula

\[
\Phi(x_i) = \begin{cases} 
  x_i, & i \neq p, q, \\
  x_q, & i = p, \\
  x_p, & i = q.
\end{cases}
\]

Then the map \( \Phi \) is well-defined and after natural extension to homomorphisms of external algebras it determines map \( \Phi : C(G) \to C(\tilde{G}) \). The map \( \Phi \) does not change the types of 2-faces. If we choose the edge assignment \( \epsilon \) and \( \tilde{\epsilon} \) on the spaces \( C(G) \) and \( C(\tilde{G}) \) respectively, such that \( \epsilon = \phi^*(\tilde{\epsilon}) \) then \( \Phi \) appears to be a chain map. Thus the complexes \( (C(G), \partial_\epsilon) \) and \( (C(\tilde{G}), \partial_{\tilde{\epsilon}}) \) are isomorphic as well as their homology.

**Corollary 3.** Odd Khovanov homology \( \underline{\text{Kh}}(G) \) is an invariant of bipartite PU-graph-links.

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**References**

[1] J. Bloom, Odd khovanov homology is mutation invariant // Math. Res. Lett. 17(1) (2010), 1–10.

[2] A. Bouchet, Unimodularity and circle graphs // Discrete Math., 66 (1987), 203–208.

[3] A. Bouchet, W.H. Cunningham, J.F. Geelen, Principally unimodular skew-symmetric matrices // Combinatorica, 18(4) (1998), 461–486.

[4] D. Ilyutko, V. Manturov, Introduction to graph-link theory // Journal of Knot Theory and Its Ramifications, 18(6) (2009), 791–823.

[5] D. Ilyutko, V. Manturov, Graph links // Dokl. Akad. Nauk 428 (2009), no. 5, 591–594 (Russian).

[6] I. Nikonov, Khovanov homology of graph-links // arXiv:math.GT/

[7] P. Ozsváth, J. Rasmussen, Z. Szabó, Odd Khovanov homology // arXiv:math.QA/0710.4300.