Bayesian and Non-Bayesian Parameters Estimation of Gamma distribution at Various Loss Functions

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Abstract. This paper deals with some Estimators of Bayes of the parameters of Gamma distribution (GD) under three different loss functions, represented via Precautionary loss function, Entropy loss function (ELF) and invarant of scale squared error (SE) loss function, assuming Gamma and Exponential priors for the shape and scale parameters respectively. Maximum likelihood estimator (MLE) and Lindley's approximation are used to obtain the Bayes estimates of the shape and scale parameters of GD. According to the Monte Carlo simulation method, those estimator\textsuperscript{s} have been compared based on the mean SE\textsuperscript{s} (MSE\textsuperscript{s}). The results show that the performance of the Estimator of Bayes\textsuperscript{s} under invarant of scale SE loss function produces the best estimates for the two parameters in all cases.

1. Introduction
The GD is a resilient one which widly used as a good fitting to each variable i.e., in meteorology, climatology, research of environmente, and other situations as physical [1]. The GD is extremely important in dependability analysis and testing of life. Hogg and et al. (2013), showed that, the GD is model as good for waiting periods and it is one for several non-negative random continuous type variables [2].

Assume that X is a random variable from GD with shape parameter (\( \alpha \)) and scale parameter (\( \beta \)), so, the possible density of GD function is definite as follows: [3]

\[
f(x; \alpha, \beta) = \frac{\beta \alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad ; \quad 0 < x < \infty \quad ; \quad \alpha, \beta > 0
\]

(1)

Since, the function of Gamma be

\[
\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1} e^{-x} dx , \quad \text{for } \alpha > 0
\]

Distribution of cumulative function (CDF) is:

\[
F(x; \alpha, \beta) = \frac{\int_{0}^{x} \frac{\beta^{u}}{\Gamma(\alpha)} u^{\alpha-1} e^{-u\beta} \, du}{\Gamma(\alpha)} = \frac{\gamma(\alpha, \beta X)}{\Gamma(\alpha)}
\]

where \( \gamma(\alpha, \beta X) \) is the lower incompSupposeee gamma function.
2. Non estimation of Bayesian

2.1. Moment Method. The moment method for estimating the two–parameter of GD can be derived as follows

\[ m_1 = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} \]
\[ m_2 = \frac{\sum_{i=1}^{n} x_i^2}{n} \]
\[ \mu'_1 = E(X) = \frac{\alpha}{\beta} \]
\[ \mu'_2 = E(X^2) = \frac{\alpha}{\beta^2} + \left( \frac{\alpha}{\beta} \right)^2 \]

From \( m_1 = \mu'_1 \) and \( m_2 = \mu'_2 \), yields

\[ \alpha^\text{MO} = \frac{n \bar{x}^2}{\sum_{i=1}^{n} x_i^2 - n \bar{x}^2} \]  
\[ \beta^\text{MO} = \frac{n \bar{x}}{\sum_{i=1}^{n} x_i^2 - n \bar{x}^2} \]  

2.2. MLE

Suppose \( x_1, x_2, ..., x_n \) is a sample being random from the GD with the pdf defined via equation (1). The MLE \( \hat{\theta} \) of the parameter as unknown \( \theta \) maximizes the function as likelihood \( L(x_1, x_2, ..., x_n; \theta) \), where,

\[ L(x_1, x_2, ..., x_n; \alpha, \beta) = \frac{\beta^n \alpha^n}{\Gamma(n)} \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\beta \sum_{i=1}^{n} x_i} \]

The likelihood function for two-parameter GD is

\[ \frac{\partial \ln L}{\partial \alpha} = -n \Psi'(\alpha) + n \ln \beta + \sum_{i=1}^{n} \ln x_i \]  
\[ \frac{\partial \ln L}{\partial \beta} = \frac{n \alpha}{\beta} - \sum_{i=1}^{n} x_i \]  

Observe that, it is difficult and complicated to solve (4), (5) analytically. Therefore, Newton-Raphson method using Hessian matrix can be used as a numerical solutions for the parameter estimates. Hessian matrix, is the second partial derivative of the log-likelihood function [5] which can construct as follows:

Suppose,
\[ g_1(\alpha) = -n \Psi'(\alpha) + n \ln \beta + \sum_{i=1}^{n} \ln x_i \]
\[ g_2(\beta) = \frac{n \alpha}{\beta} - n \bar{x} \]

The derivatives being partial of \( g_1(\alpha) \) regarding unknown parameters \( \alpha \), will be

\[ \frac{\partial g_1(\alpha)}{\partial \alpha} = -n \Psi''(\alpha) \]

Where, \( \Psi'(\alpha) \) is called a tri-gamma which is the derivative of \( \Psi(\alpha) \).
\[ \frac{dg_1(\alpha)}{d\beta} = \frac{n}{\beta} \]

The derivatives being partial of \( g_2(\beta) \) regarding unknown parameters \( \alpha, \beta \) will be

\[ \frac{dg_2(\beta)}{d\alpha} = \frac{n}{\beta} \]

\[ \frac{dg_2(\beta)}{d\beta} = -\frac{n\alpha}{\beta^2} \]

Hence,

\[ J_k = \begin{bmatrix} \frac{dg_1(\alpha)}{d\alpha} & \frac{dg_1(\alpha)}{d\beta} \\ \frac{dg_2(\beta)}{d\alpha} & \frac{dg_2(\beta)}{d\beta} \end{bmatrix} \left[ \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \]

where, \( J_k \) is the Jacobian matrix which must be a non-singular symmetric matrix so, its inverse can be found as

\[ J_k^{-1} = \frac{1}{|J_k|} \left[ \begin{array}{cc} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{array} \right] \]

\[ \left[ \begin{array}{c} \alpha_{k+1} \\ \beta_{k+1} \end{array} \right] = \left[ \begin{array}{c} \alpha_k \\ \beta_k \end{array} \right] - J_k^{-1} \begin{bmatrix} g_1(\alpha) \\ g_2(\beta) \end{bmatrix} \]

\[ = \left[ \begin{array}{c} \alpha_k \\ \beta_k \end{array} \right] - \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} -n\Psi(\alpha_k) + n\ln\beta_k + \sum_{i=1}^{n} \ln x_i \\ -na_{22}\Psi(\alpha_k) + na_{22}\ln\beta_k + a_{22}\sum_{i=1}^{n} \ln x_i - \frac{na_{12}\alpha_k}{\beta_k} + na_{12}\hat{x} \end{bmatrix} \]

\[ \left[ \begin{array}{c} \alpha_{k+1} \\ \beta_{k+1} \end{array} \right] = \left[ \begin{array}{c} \alpha_k \\ \beta_k \end{array} \right] - \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} -na_{22}\Psi(\alpha_k) + na_{22}\ln\beta_k + a_{22}\sum_{i=1}^{n} \ln x_i - \frac{na_{12}\alpha_k}{\beta_k} + na_{12}\hat{x} \\ na_{21}\Psi(\alpha_k) - na_{21}\ln\beta_k - a_{21}\sum_{i=1}^{n} \ln x_i + \frac{na_{11}\beta_k}{\beta_k} - na_{11}\hat{x} \end{bmatrix} \]  \hspace{1cm} (6)

The absolute value for the difference between the new value for \( \alpha \) and \( \beta \) in new iterative value with previous value for \( \alpha \) and \( \beta \) in last iterative represent the error term, its symbol is \( \varepsilon \), which a very small and assumed value is. Then, error term is formulated as

\[ \left[ \begin{array}{c} \varepsilon_{k+1}(\alpha) \\ \varepsilon_{k+1}(\beta) \end{array} \right] = \left[ \begin{array}{c} \alpha_{k+1} \\ \beta_{k+1} \end{array} \right] - \left[ \begin{array}{c} \alpha_k \\ \beta_k \end{array} \right] \]

where \( \alpha_k \) and \( \beta_k \) are the initial values for \( \alpha, \beta \) respectively, for which are assumed.

3. Estimation of bayesian

At the current part, few estimators of Bayesian for \( \alpha \) and \( \beta \) have been derived according to ML estimators as the initial values for \( \alpha \) and \( \beta \). Three different loss functions have been used as follows:

3.1. Posterior functions of density according to Exponential and Gamma priors

For estimating parameters of \( \alpha \) and \( \beta \) for GD, we adopt that \( \alpha \) of a \( \pi_1 \) prior (.), that follows Gamma \((a, b)\). We assume also, that \( \beta \) be of \( \pi_2 \) prior (.) and the function of density of \( \pi_2 \) (.) be Exponential and it is \( \pi_1 \) independent (.)
\[
\pi_1(\alpha) = \frac{(b)^{a(\alpha)}a^{-1}e^{-ba}}{\Gamma(\alpha)} ; \quad a > 0, \ b > 0, \alpha > 0 \tag{7}
\]
\[
\pi_2(\beta) = \begin{cases} \frac{c e^{-\beta c}}{\Gamma(\alpha)} & ; \quad c > 0, \beta \geq 0 \end{cases}
0. w \tag{8}
\]

The two formulas (7) and (8) are respectively the \(\alpha\) and \(\beta\) priors, since \(a, b, c\) are parameters being known. The pdf joint is given via

\[
J(x_1, x_2, ..., x_n; \alpha, \beta) = L(x_1, x_2, ..., x_n; \alpha, \beta)\pi_1(\alpha)\pi_2(\beta)
\]

\[
= \frac{\beta^{na}}{(\Gamma(\alpha))^n} \prod_{i=1}^{n} x_i^{a-1}e^{-\beta \sum_{i=1}^{n} x_i} \frac{(b)^{a(\alpha)}a^{-1}e^{-ba}}{\Gamma(\alpha)} c e^{-\beta c}
\]

the marginal \((x_1, x_2, ..., x_n)\) pdf is offered via

\[
J(x_1, x_2, ..., x_n) = \int_0^\infty \int_0^\infty L(x_1, x_2, ..., x_n; \alpha, \beta)\pi_1(\alpha)\pi_2(\beta) d\alpha d\beta
\]

The joint \(\alpha\) and \(\beta\) density functions are well-defined as following:

\[
h(\alpha, \beta | x_1, x_2, ..., x_n) = \frac{L(x_1, x_2, ..., x_n; \alpha, \beta)\pi_1(\alpha)\pi_2(\beta)}{\int_0^\infty \int_0^\infty L(x_1, x_2, ..., x_n; \alpha, \beta)\pi_1(\alpha)\pi_2(\beta) d\alpha d\beta}
\]

\[
= \frac{\beta^{na}}{(\Gamma(\alpha))^n} \prod_{i=1}^{n} x_i^{a-1}e^{-\beta \sum_{i=1}^{n} x_i} \frac{(b)^{a(\alpha)}a^{-1}e^{-ba}}{\Gamma(\alpha)} c e^{-\beta c} d\alpha d\beta
\]

3.2. Loss functions

In estimation of Bayesian, we regard 3 loss functions types.

3.2.1. Pre-cautionary loss function. Norstrom (1996) presented as an precautionary asymmetric loss function, that might be defined as follows[6]

\[
L_1(\hat{\theta}, \theta) = \frac{(\theta - \hat{\theta})^2}{\theta}
\]

The risk function \(R_1(\hat{\theta}, \theta)\) is the posterior expectation of the loss function \(L_1(\hat{\theta}, \theta)\) with respect to \(h(\theta | x_1, x_2, ..., x_n)\). That is \(R_1(\hat{\theta}, \theta) = \int_0^\infty L_1(\hat{\theta}, \theta) h(\theta | x_1, x_2, ..., x_n) d\theta\)

Thus, the value of \(\hat{\theta}\) that minimizes the posterior risk \(R_1(\hat{\theta}, \theta)\) is obtained via setting its first partial derivative regarding \(\hat{\theta}\) equal to 0. That is \(\hat{\theta}_1\) is the posterior mean.

\[
\hat{\theta}_1 = \sqrt{E(\theta^2 | x)}
\]

(9)

Generally,

\[
E[u(\alpha, \beta)] = \int_0^\infty \int_0^\infty u(\alpha, \beta) h(\alpha, \beta | x_1, ..., x_n) d\alpha d\beta
\]

Since \(u(\alpha, \beta)\) is whichever \(\alpha\) and \(\beta\) function. Thus,
\[ E[u(\alpha, \beta)] = \frac{\int_0^\infty \int_0^\infty u(\alpha, \beta) L(x_1, x_2, \ldots, x_n; \alpha, \beta) \pi_1(\alpha) \pi_2(\beta) \, d\alpha \, d\beta}{\int_0^\infty \int_0^\infty L(x_1, x_2, \ldots, x_n; \alpha, \beta) \pi_1(\alpha) \pi_2(\beta) \, d\alpha \, d\beta} \]

(i) estimation of Bayesian for \(\alpha\) under Precautionary loss function

To obtain estimation of Bayesian for \(\alpha\), assume that,

\[ u(\alpha, \beta) = \alpha^2 \]

Therefore, \[ E[\{\alpha^2\}|x] = \frac{\int_0^\infty \int_0^\infty \alpha^2 L(x_1, x_2, \ldots, x_n; \alpha, \beta) \pi_1(\alpha) \pi_2(\beta) \, d\alpha \, d\beta}{\int_0^\infty \int_0^\infty L(x_1, x_2, \ldots, x_n; \alpha, \beta) \pi_1(\alpha) \pi_2(\beta) \, d\alpha \, d\beta} \]

Note, it is not easy to get the 2 integrals ratio solution. Thus, approximate of Lindley utilized to find \[ E[\{\alpha^2\}|x] \] as following:

\[ u(\alpha, \beta) = \alpha^2 \]

\[ u_1 = \frac{\partial u(\alpha, \beta)}{\partial \alpha} = 2\alpha , \quad u_{11} = \frac{\partial^2 u(\alpha, \beta)}{\partial \alpha^2} = 2 \quad , \quad u_2 = \frac{\partial u(\alpha, \beta)}{\partial \beta} = 0 \quad , \quad u_{22} = \frac{\partial^2 u(\alpha, \beta)}{\partial \beta^2} = 0 \]

Recall independency of \(\alpha\) and \(\beta\). Therefore, the joint p.d.f of \(\alpha\) and \(\beta\) is offered via

\[ \pi(\alpha, \beta) = \frac{(b)^a(\alpha)^{a-1}e^{-b\alpha}}{\Gamma(a)} \cdot c e^{-c\beta} \]

\[ p = \ln r(\alpha, \beta) = (a - 1)\ln \alpha + a \ln b - b\alpha - \ln \Gamma(\alpha) + l_n c - c\beta \]

\[ p_1 = \frac{\partial p}{\partial \alpha} = \frac{a - 1}{a} - b , \quad p_2 = \frac{\partial p}{\partial \beta} = -c \]

\[ \ln L(x_1, \ldots, x_n; \alpha, \beta) = n \ln \beta - n \ln \Gamma(\alpha) - b \sum_{i=1}^{n} \ln x_i + (a - 1) \sum_{i=1}^{n} \ln \beta \]

\[ L_{21} = \frac{\partial^3 \ln l(\alpha, \beta)}{\partial \alpha \partial \beta^2} \]

\[ \frac{\partial \ln l(\alpha, \beta)}{\partial \alpha} = n \ln \beta - n \Psi(\alpha) + \sum_{i=1}^{n} \ln x_i \]

\[ \frac{\partial^2 \ln l(\alpha, \beta)}{\partial \alpha \partial \beta} = -\frac{n}{\beta} \]

\[ \frac{\partial^3 \ln l(\alpha, \beta)}{\partial \alpha \partial \beta^2} = -\frac{n}{\beta^2} \]

\[ L_{03} = \frac{\partial^3 \ln l(\alpha, \beta)}{\partial \beta^3} \]
\[ \frac{\partial \ln l(\alpha, \beta)}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} x_i \]

\[ \frac{\partial^2 \ln l(\alpha, \beta)}{\partial \beta^2} = -\frac{n\alpha}{\beta^2} \]

\[ \frac{\partial^3 \ln l(\alpha, \beta)}{\partial \beta^3} = \frac{2\alpha}{\beta^3} \]

\[ l_{30} = \frac{\partial^3 \ln l(\alpha, \beta)}{\partial \alpha^3} \]

\[ \frac{\partial \ln l(\alpha, \beta)}{\partial \alpha} = n \ln \beta - n \Psi(\alpha) + \sum_{i=1}^{n} \ln x_i \]

\[ \frac{\partial^2 \ln l(\alpha, \beta)}{\partial \alpha^2} = -n \Psi'(\alpha) \]

\[ \frac{\partial \ln l(\alpha, \beta)}{\partial \alpha^2} = -n \Psi''(\alpha) \]

\[ \sigma_{11} = \frac{1}{l_{20}} = \frac{1}{n \Psi'(\alpha)} \quad , \quad \sigma_{22} = -\frac{1}{l_{02}} = \frac{\beta^2}{n\alpha} \]

\[ E(\alpha^2) = \hat{\alpha}^2 + \frac{1}{2} (u_{11} \sigma_{11}) + p_1 u_1 \sigma_{11} + \frac{1}{2} (l_{30} u_1 \sigma_{11}^2) + \frac{1}{2} (l_{12} u_1 \sigma_{11} \sigma_{22}) \]

\[ = \hat{\alpha}^2 + \frac{1}{n \Psi'(\alpha)} + \frac{2\hat{\alpha}}{n \Psi'(\hat{\alpha})} \left( \frac{a-1}{a} - b \right) - \frac{n \Psi(\hat{\alpha}) \hat{\alpha}}{(n \Psi'(\hat{\alpha}))^2} - \frac{1}{\Psi'(\hat{\alpha})} \]

\[ = \hat{\alpha}^2 + \frac{2\hat{\alpha}}{n \Psi'(\hat{\alpha})} \left( \frac{a-1}{a} - b \right) - \frac{n \Psi(\hat{\alpha}) \hat{\alpha}}{(n \Psi'(\hat{\alpha}))^2} \]

Now, Substituting (10) into (9) yields,

\[ \hat{\alpha}_B = \sqrt{\hat{\alpha}^2 + \frac{2\hat{\alpha}}{n \Psi'(\hat{\alpha})} \left( \frac{a-1}{a} - b \right) - \frac{n \Psi(a) \hat{\alpha}}{(n \Psi'(\hat{\alpha}))^2}} \]

(ii) estimation of Bayesian for \(\beta\) under Precautionary loss function

Assume that,

\[ u(\alpha, \beta) = \beta^2 \]

then,

\[ u_1 = \frac{\partial u(\alpha, \beta)}{\partial \alpha} = 0 \quad , \quad u_{11} = \frac{\partial^2 u(\alpha, \beta)}{\partial \alpha^2} = 0 \quad , \quad u_2 = \frac{\partial u(\alpha, \beta)}{\partial \beta} = 2\beta \quad , \quad u_{22} = \frac{\partial^2 u(\alpha, \beta)}{\partial \beta^2} = 2 \]

Thus, \[ E(\beta^2) = \hat{\beta}^2 + \frac{1}{2} (u_{22} \sigma_{22}) + p_2 u_2 \sigma_{22} + \frac{1}{2} (l_{03} u_2 \sigma_{22}^2) \]

\[ = \hat{\beta}^2 + \frac{1}{2} \left( \frac{2\beta^2}{n \hat{\alpha}} \right) + \left( -c \frac{2\beta^3}{n \hat{\alpha}} + \frac{1}{2} \left( \frac{2n \hat{\alpha} \hat{\beta}^4}{\beta^3 n \hat{\alpha}} \right) \right) \]

\[ E(\beta^2) = \hat{\beta}^2 + \frac{3\hat{\beta}^2}{n \hat{\alpha}} - \frac{2c\beta^3}{n \hat{\alpha}} \]

(11)

After Substituting (11) into (9) yields,
\[ \hat{\beta}_B = \sqrt{\hat{\beta}^2 + \frac{3\hat{\beta}^2}{n\hat{\alpha}} - \frac{2c\hat{\beta}^3}{n\hat{\alpha}}} \]

Since \( \hat{\alpha}, \hat{\beta} \) are the MLEs.

### 3.2.2. Entropy function loss

In several practical cases, it seems to be additional accurate for expressing the loss in ratio terms. In such situation, Calabria and Pulcini (1994) emphasise that a beneficial asymmetric function loss is the normalized SE loss function

\[ L_2(\hat{\theta}, \theta) = \left( \frac{\hat{\theta}}{\theta} \right) - \ln \left( \frac{\hat{\theta}}{\theta} \right) - 1 \]

Therefore, the estimator of Bayes at ELF is offered via [7]:

\[ \hat{\theta}_2 = \left[ E(\theta^{-1}|X) \right]^{-1} \]  

(i) Estimator of Bayes for \( \alpha \) under ELF

Estimation of Bayesian for \( \alpha \) under ELF can be obtained as:

Assume that, \( u(\alpha, \beta) = \frac{1}{\alpha} \), so,

\[ u_1 = \frac{\partial u(\alpha, \beta)}{\partial \alpha} = -\alpha^{-2}, \quad u_{11} = \frac{\partial^2 u(\alpha, \beta)}{\partial \alpha^2} = 2\alpha^{-3}, \quad u_2 = \frac{\partial u(\alpha, \beta)}{\partial \beta} = 0, \quad u_{22} = \frac{\partial^2 u(\alpha, \beta)}{\partial \beta^2} = 0 \]

Thus,

\[ E \left( \frac{1}{\alpha} \right) \approx \frac{1}{\hat{\alpha}} + \frac{1}{2} \left( u_{11} \sigma_{11} \right) + \frac{1}{2} \left( L_{30} u_1 \sigma_{11}^2 \right) + \frac{1}{2} \left( L_{12} u_1 \sigma_{11} \sigma_{22} \right) \]

\[ E \left( \frac{1}{\alpha} \right) \approx \frac{1}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3 n \Psi'(\hat{\alpha})} - \frac{1}{\hat{\alpha}^2 n \Psi'(\hat{\alpha})} \left( \frac{a-1}{a} - b \right) + \frac{1}{2} \left( \frac{n \Psi''(\hat{\alpha})}{\hat{\alpha}^2 (n \Psi'(\hat{\alpha}))^2} \right) + \frac{1}{2} \left( \frac{1}{\hat{\alpha}^3 n \Psi'(\hat{\alpha})} \right) \]  

(13)

Now, Substituting (13) into (12) yields

\[ \hat{\alpha}_2 \approx \frac{1}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3 n \Psi'(\hat{\alpha})} - \frac{1}{\hat{\alpha}^2 n \Psi'(\hat{\alpha})} \left( \frac{a-1}{a} - b \right) + \frac{1}{2} \left( \frac{n \Psi''(\hat{\alpha})}{\hat{\alpha}^2 (n \Psi'(\hat{\alpha}))^2} \right) + \frac{1}{2} \left( \frac{1}{\hat{\alpha}^3 n \Psi'(\hat{\alpha})} \right) \]

(ii) Estimator of Bayes for \( \beta \) using ELF

Via the same way, the estimation of Bayes for parameter \( \beta \) scale under ELF has been derived as follows:

Suppose, \( u(\alpha, \beta) = \frac{1}{\beta} \), so,

\[ u_1 = \frac{\partial u(\alpha, \beta)}{\partial \alpha} = 0, \quad u_{11} = \frac{\partial^2 u(\alpha, \beta)}{\partial \alpha^2} = 0, \quad u_2 = \frac{\partial u(\alpha, \beta)}{\partial \beta} = -\beta^{-2}, \quad u_{22} = \frac{\partial^2 u(\alpha, \beta)}{\partial \beta^2} = 2\beta^{-3} \]

Thus, \( E \left( \frac{1}{\beta} \right) \approx \frac{1}{\hat{\beta}} + \frac{1}{2} \left( u_{22} \sigma_{22} \right) + \frac{1}{2} \left( L_{30} u_2 \sigma_{22} \right) + \frac{1}{2} \left( L_{31} u_2 \sigma_{11} \sigma_{22} \right) \]

\[ \approx \frac{1}{\hat{\beta}} + \frac{1}{2} \left( \frac{2\hat{\beta}^2}{\sqrt{n\hat{\alpha}}} \right) + \frac{c\hat{\beta}^2}{\hat{\beta}^3 n\hat{\alpha}} + \frac{1}{2} \left( \frac{-2n\hat{\alpha}}{\hat{\beta}^3} \right) \approx \frac{1}{\hat{\beta}} + \frac{c}{n\hat{\alpha}} \]

(14)

After Substituting (14) into (12) yields:

\[ \hat{\beta}_2 = \frac{1}{\hat{\beta}} + \frac{c}{n\hat{\alpha}} \]

Where \( \hat{\alpha}, \hat{\beta} \) are the MLEs.
3.2.3. invariant of scale SE loss function. The invariant of scale SE loss function has been discussed via De Groot (1970). It is symmetric as non-negative and loss function being continuous [8]. It is well-defined as
\[
L_3(\theta, \hat{\theta}) = \left( \frac{\theta - \hat{\theta}}{\theta} \right)^2
\]

Then, the Estimator of Bayes at invariant scale of SE loss function which minimizes the risk being posterior \(R_2(\hat{\theta}, \theta)\), where, \(R_2(\hat{\theta}, \theta) = \int_0^\infty L_2(\hat{\theta}, \theta)h(\theta|x_1, x_2, ..., x_n)\,d\theta\), is offered via[8]:
\[
\hat{\theta}_3 = \frac{E(\frac{1}{\alpha^2})}{E(\frac{1}{\alpha^2})}
\]

Assume that, \(u(\alpha, \beta)\) is whichever \(\alpha\) and \(\beta\) function, then,
\[
E[u(\alpha, \beta)] = \int_0^\infty \int_0^\infty u(\alpha, \beta) h(\alpha, \beta|x_1, ..., x_n)\,d\alpha\,d\beta
\]
\[
E[u(\alpha, \beta)] = \int_0^\infty \int_0^\infty u(\alpha, \beta) L(x_1, x_2, ..., x_n; \alpha, \beta)\pi_1(\alpha)\pi_2(\beta)\,d\alpha\,d\beta
\]

(i) estimation of Bayesian for \(\alpha\) under invariant scale of SE loss function
To obtain the estimation of Bayesian for the parameter \(\alpha\) shape under invariant scale of SE loss function, adopt such as
\[
u(\alpha, \beta) = \frac{1}{\alpha^2}
\]
\[
u_1 = \frac{\partial u(\alpha, \beta)}{\partial \alpha} = -2\alpha^{-3}, \quad \nu_{11} = \frac{\partial^2 u(\alpha, \beta)}{\partial \alpha^2} = 6\alpha^{-4}, \quad \nu_2 = \frac{\partial u(\alpha, \beta)}{\partial \beta} = 0, \quad \nu_{22} = \frac{\partial^2 u(\alpha, \beta)}{\partial \beta^2} = 0
\]
Thus, \(E\left(\frac{1}{\alpha^2}\right) \approx \frac{1}{\alpha^2} + \frac{1}{2}(\nu_{11} + \nu_{11}) + p_1\nu_1\nu_{11} + \frac{1}{2}(L_3\nu_1\nu_{11}) + \frac{1}{2}(L_1\nu_1\nu_{11}\nu_{22})\)
\[
\approx \frac{1}{\alpha^2} + \frac{1}{2}(6\alpha^{-4} + \frac{1}{\alpha^3}) + \frac{(a - 1)}{\alpha^3} - b) \left( -2\alpha^{-3} \right) + \frac{1}{2}(n\Psi^2(\hat{\alpha})\hat{\alpha}^{-3} \frac{1}{n\Psi^2(\hat{\alpha})})^2
\]
\[
+ \frac{1}{2} \left( n^{-2} \hat{\alpha}^{-3} n^{2} \Psi^2(\hat{\alpha}) + \nu_1 \right) \approx \frac{1}{\alpha^2} + \frac{1}{\alpha^4} - \frac{1}{\alpha^6} \left( \frac{a - 1}{\alpha^3} - b \right) + \frac{2}{\alpha^8} \left( \frac{n\Psi^2(\hat{\alpha})}{\alpha^3} \right) + \frac{1}{\alpha^{10} n\Psi^2(\hat{\alpha})} + \frac{1}{\alpha^6 n\Psi^2(\hat{\alpha})}
\]
Replacing (13) and (16) into (15) yields
\[
\hat{\alpha}_3 \approx \frac{1}{\alpha^2} + \frac{1}{\alpha^4 n\Psi^2(\hat{\alpha})} - \frac{1}{\alpha^6} \left( \frac{a - 1}{\alpha^3} - b \right) + \frac{2}{\alpha^8} \left( \frac{n\Psi^2(\hat{\alpha})}{\alpha^3} \right) + \frac{1}{\alpha^{10} n\Psi^2(\hat{\alpha})}
\]
\[
where \hat{\alpha} is the MLE.
(ii) estimation of Bayesian for \(\beta\) under invariant scale of SE loss function
To derive estimation of Bayesian for the parameter \(\beta\) scale under invariant scale of SE loss function, assume that
\[
u(\alpha, \beta) = \frac{1}{\beta^2}, \quad so,
\]
\[ u_1 = \frac{\partial u(a, \beta)}{\partial a} = 0, \quad u_{11} = \frac{\partial^2 u(a, \beta)}{\partial a^2} = 0, \quad u_2 = \frac{\partial u(a, \beta)}{\partial \beta} = -2\beta^{-3}, \quad u_{22} = \frac{\partial^2 u(a, \beta)}{\partial \beta^2} = 6\beta^{-4} \]

Therefore, \[ E \left( \frac{1}{\beta^2} \right) \approx \frac{1}{\beta^2} + \frac{1}{2} (u_{22} \sigma_{22}) + p_2 u_2 \sigma_{22} + \frac{1}{2} (L_{03} u_2 \sigma_{22}^2) + \frac{1}{2} (L_{21} u_2 \sigma_{11} \sigma_{22}) \]

\[ E \left( \frac{1}{\beta^2} \right) \approx \frac{1}{\beta^2} + \frac{1}{2} \left( \frac{6\beta^2}{\beta^3 n \alpha} \right) + \frac{2c}{\beta n \alpha} + \frac{1}{2} \left( \frac{2n \alpha - 2}{\beta^2} \right) \approx \frac{1}{\beta^2} + \frac{2c}{\beta n \alpha} + \frac{1}{\beta^2 n \alpha} \] \quad (17)

After Substituting (17) and (14) into (15) yields

\[ \hat{\beta}_3 \approx \frac{\frac{1}{\beta^2} + \frac{1}{2} \frac{c}{\beta n \alpha} + \frac{1}{2} \frac{c}{\beta^2 n \alpha}}{R} \] , since \( \hat{\alpha}, \hat{\beta} \) are the MLEs.

### 4. Simulation study

At the current part, simulation of Monte-Carlo is employed for comparison the 3 estimator’s performance (Maximum likelihood, Moments, and Estimator of Bayes s under different loss function) for two unknown parameters according to MSE’s where,

\[ \text{MSE} (\theta) = \frac{\sum_{i=1}^{R}(\hat{\theta}_i - \theta)^2}{R} \]

where \( R \) represents the number of replications.

In this paper, we have been generated \( R = 3000 \), samples of four diverse sizes of samples (\( n = 20, 30, 50, \) and \( 100 \)) with \( \alpha = 2,3 \) and \( \beta = 0.5,1 \). The parameters for the prior distribution of \( \alpha \) were elected as \( a=3, \) \( b=3 \) \( 50, \) and \( 100 \).

It is obvious that, the results for \( \alpha \) (expected values and MSE) at \( \beta = 0.5 \) being the same as the results corresponding if \( \beta = 1 \). The reason might be clarified simply as follows:

Based on method of moments, we get

\[ \hat{\alpha} \approx \frac{\sum_{i=1}^{n} x_i^2 - nx^2}{\sum_{i=1}^{n} x_i^2 - nx^2} = \frac{\beta}{n} \sum_{i=1}^{n} x_i \] \quad (18)

Notice that, \( x_1, x_2, ..., x_n \) is sample being random from a GD well-defined via (1), where every observation as \( x_i \) is created independently and identically via the equation as follow:

\[ x_i = \sum_{j=1}^{u_i} \frac{1}{\beta} \log (u_j), \quad i = 1, 2, ..., n \] \quad (15)

where \( u_i \) be a number being random followed distribution as uniform with (0,1), i.e., \( u_i \sim U(0,1) \)

After substituting (15) into (14) yields, \( \hat{\alpha} \approx \frac{\beta}{n} \sum_{i=1}^{n} \sum_{j=1}^{u_i} \frac{1}{\beta} \log (u_j) \)

Thus, \( \beta \) will be omitted from moments estimation for \( \alpha \). Recalling, the moments are the MLE initial value. As well, Bayesian estimators depends on MLE so that the results of expected MSE and values for \( \hat{\alpha} \) are similar as the corresponding value of \( \hat{\alpha} \) for different values of \( \beta \).

### 5. Discussion

Simulation of Monte-carlo results have been shortened in Tables (1 to 6) that represented via the estimated values and (MSE's) for estimating methods of \( \alpha \) and \( \beta \). The results show that:

1. Generally, the Estimator of Bayes s are the best in comparing to the each of MLE and the moments estimators.
2. The Bayes estimates performance under invariant of scale SE loss function for each of \( \alpha \) and \( \beta \) are the best, as it offers smallest MSE which was mentioned for all initial parameters values combination.
3. It is clear that the MSE values for estimation of unknown shape parameter are increasing when the shape parameter value increased. Values of MSE for whole estimates are amassed with the unknown scale parameter value upsurge for all cases.

4. The results for shape parameter $\alpha$ (estimated values and MSE’s) at any scale parameter of $\beta$ are the same. In other words, changing the scale parameter value will not affect the results of $\alpha$ due to cancelling $\beta$ through deriving the moment estimator which is regarded as the initial value of MLE.

| n  | $\hat{\alpha}_{MO}$ | $\hat{\alpha}_{ML}$ | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ |
|----|----------------------|----------------------|-------------------|-------------------|-------------------|
| 20 | 2.48639              | 2.33479              | 2.14737           | 2.11877           | 2.06815           |
| 30 | 2.29832              | 2.19465              | 2.08578           | 2.05860           | 2.01942           |
| 50 | 2.18314              | 2.11841              | 2.05839           | 2.03897           | 2.01247           |
| 100| 2.09072              | 2.05531              | 2.02736           | 2.01663           | 2.00228           |

| n  | $\hat{\alpha}_{MO}$ | $\hat{\alpha}_{ML}$ | $\hat{\alpha}_1$ | $\hat{\alpha}_2$ | $\hat{\alpha}_3$ |
|----|----------------------|----------------------|-------------------|-------------------|-------------------|
| 20 | 3.60049              | 3.44743              | 3.09156           | 3.0855            | 3.01686           |
| 30 | 3.40572              | 3.29932              | 3.08203           | 3.06486           | 3.00594           |
| 50 | 3.25553              | 3.18809              | 3.06634           | 3.04956           | 3.00829           |
| 100| 3.12605              | 3.08952              | 3.0322            | 3.02157           | 2.99869           |

| n  | $\hat{\beta}_{MO}$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ |
|----|---------------------|---------------------|-----------------|-----------------|-----------------|
| 20 | 0.63802             | 0.61058             | 0.52378         | 0.54285         | 0.51055         |
| 30 | 0.58915             | 0.38833             | 0.29354         | 0.29458         | 0.28012         |
| 50 | 0.29714             | 0.18510             | 0.15579         | 0.15440         | 0.14859         |
| 100| 0.13609             | 0.08313             | 0.07647         | 0.07594         | 0.07439         |

| n  | $\hat{\beta}_{MO}$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ |
|----|---------------------|---------------------|-----------------|-----------------|-----------------|
| 20 | 0.61598             | 0.44923             | 0.37063         | 0.37139         | 0.35835         |
| 30 | 0.61598             | 0.44923             | 0.37063         | 0.37139         | 0.35835         |
| 50 | 0.25924             | 0.18543             | 0.16827         | 0.16787         | 0.16457         |

| n  | $\hat{\beta}_{MO}$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ |
|----|---------------------|---------------------|-----------------|-----------------|-----------------|
| 20 | 0.63802             | 0.61058             | 0.52378         | 0.54285         | 0.51055         |
| 30 | 0.58915             | 0.38833             | 0.29354         | 0.29458         | 0.28012         |
Table 6: The estimated values of $\beta$ according to the estimation methods when $\beta = 1$

| $n$ | $\hat{\beta}_{MO}$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ |
|-----|--------------------|--------------------|----------------|----------------|----------------|
| 20  | 1.27605            | 1.22115            | 1.19739        | 1.16928        | 1.10595        |
|     | 1.11258            | 1.08359            | 1.09359        | 1.07467        | 1.08468        |
| 30  | 1.10256            | 1.14735            | 1.11661        | 1.11258        | 1.10595        |
|     | 1.06368            | 1.07708            | 1.04460        | 1.06298        | 1.03520        |
|     | 1.04133            | 1.04944            | 1.02830        | 1.04032        | 1.02099        |
|     | 1.03176            | 1.03176            | 1.01838        | 1.01135        | 1.01949        |
|     | 1.01135            | 1.00704            | 1.01646        |                |                |

Table 7: The MSE's of the estimators of $\beta$ when $\beta = 0.5$

| $n$ | $\hat{\beta}_{MO}$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ |
|-----|--------------------|--------------------|----------------|----------------|----------------|
| 20  | 0.09335            | 0.06094            | 0.06875        | 0.05533        | 0.05957        |
|     | 0.05055            | 0.05055            |                |                |                |
| 30  | 0.04509            | 0.03996            | 0.03111        | 0.02778        | 0.02835        |
|     | 0.02615            | 0.02615            |                |                |                |
|     | 0.02533            | 0.02533            |                |                |                |
| 50  | 0.02224            | 0.01936            | 0.01490        | 0.01412        | 0.01361        |
|     | 0.01338            | 0.01338            |                |                |                |
|     | 0.01386            | 0.01386            |                |                |                |
| 100 | 0.01023            | 0.00824            | 0.00681        | 0.00627        | 0.00662        |
|     | 0.00608            | 0.00608            |                |                |                |
|     | 0.00603            | 0.00603            |                |                |                |

Table 8: The MSE’s of the estimators of $\beta$ when $\beta = 1$

| $n$ | $\hat{\beta}_{MO}$ | $\hat{\beta}_{ML}$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ |
|-----|--------------------|--------------------|----------------|----------------|----------------|
| 20  | 0.10495            | 0.10495            | 0.10495        | 0.10495        | 0.10495        |
|     | 0.10495            | 0.10495            |                |                |                |
| 30  | 0.09135            | 0.09135            | 0.09135        | 0.09135        | 0.09135        |
|     | 0.09135            | 0.09135            |                |                |                |
|     | 0.09135            | 0.09135            |                |                |                |
| 50  | 0.08896            | 0.08896            | 0.08896        | 0.08896        | 0.08896        |
|     | 0.08896            | 0.08896            |                |                |                |
|     | 0.08896            | 0.08896            |                |                |                |
| 100 | 0.08675            | 0.08675            | 0.08675        | 0.08675        | 0.08675        |
|     | 0.08675            | 0.08675            |                |                |                |
|     | 0.08675            | 0.08675            |                |                |                |

References
[1] Apolloni, B, and Bassis, S 2009 Algorithmic inference of two-parameter GD, *Communications in Statistics - Simulation and Computation*, 38, 9, 1950 –1968.
[2] Hogg, R V, McKean, J W, and Craig A T 2014 *Introduction to Mathematical Statistics*. 7th ed. Pearson Education Publication.
[3] Douglas, C M, and George, C R 2003 *Applied Statistics and Probability for Engineering*, Third Edition, John Wiley and Sons, Inc.
[4] Montgomery, D C, and Runger, G C 2003 *Applied Statistics and Probability for Engineers*. 3rd Edition, John Wiley & Son, Inc., Hoboken.
[5] Al-Sultany, S A, and Mohammed, S A 2018 Comparison between Bayesian and Maximum Likelihood Methods for parameters and the Reliability function of Perks Distribution. *Iraqi Journal of Science*, 59(1B):369-376.
[6] Li, J, and Ren, H 2012 Estimation of One Parameter Exponential Family under a Precautionary Loss Function Based on Record Values. International *Journal Engineering and Manufacturing*, 3: 75-81.
[7] Dey, S 2010 estimation of Bayesian of the Shape Parameter of the Generalised Exponential Distribution under Different Loss Functions *Pak. J. Stat. Oper. Res.*, 6, 2, 163-174.
[8] Misra N, Vander Meulen E C, and Branden K V 2006a On estimating the scale parameter of the selected gamma population under the invariant of scale SE loss function. *Journal of Computational and Applied Mathematics* 186: 268–282.