Sparse time–frequency representation for signals with fast varying instantaneous frequency

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Abstract: Time–frequency (TF) distributions have been used for providing high-resolution representation in a large number of signal processing applications. However, high resolution and accurate instantaneous frequency (IF) estimation usually depends on the employed distribution and complexity of signal phase function. To ensure an efficient IF tracking for various types of signals, a class of complex-time distributions (CTD) has been developed. These distributions facilitate analysis in cases when standard distributions cannot provide satisfactory results (e.g. for highly non-stationary signal phase). In that sense, an ambiguity-based form of the fourth-order CTD is considered, in a new compressive sensing (CS) context. CS is an intensively growing approach in signal processing that allows efficient analysis and reconstruction of randomly under-sampled signals. In this study, randomly chosen ambiguity domain coefficients serve as CS measurements. By exploiting sparsity in the TF plane, it is possible to obtain highly concentrated IF using just small number of randomly chosen coefficients from the ambiguity domain. Moreover, in noisy signal case, this CS approach can be efficiently combined with the L-statistics producing robust TF representations. Noisy coefficients are first removed using the L-statistics and then reconstructed by using the CS algorithms. The theoretical considerations are illustrated using experimental results.

1 Introduction

Highly localised signal instantaneous frequency (IF) in the time–frequency (TF) plane has attracted attention of researchers and has been widely studied in the literature [1–6]. To deal with different types of signals, numerous TF distributions have been developed [7]. The quadratic distributions (Wigner–Ville distribution (WD) and distributions from the Cohen class) can perfectly localise linear frequency modulated signals. Therefore, in order to reduce or eliminate the cross-terms in the WD, distributions from the Cohen class are employed [5–8]. These are based on low-pass filtering of the ambiguity function (AF), which is two-dimensional (2D) Fourier transform of the WD. Having auto-terms around the origin and cross-terms dislocated from the origin in the ambiguity plane, resulting TF distribution has a reduced number of cross-terms, or is cross-terms free. However, in general there is a trade-off between localisation in the TF plane and cross-terms reduction, which is known as the TF uncertainty principle [4].

Signals such as radar signals or vibrating tones of musical instruments are characterised by the fast variations of the IF [9, 10]. These variations contain information about the related physical phenomena or the radar target, which can be efficiently analysed and extracted using the TF representations [7]. An interesting application of the TF representations assumes characterisation of the micro-Doppler effect that appears in radars [11–15]. It is produced by the fast moving parts of the target and can be used for target recognition [15]. When dealing with human walking, fast variations caused by micro movements can be hardly detected using standard quadratic TF representations (such as the WD, the S-method, or the Cohen class distributions). Furthermore, modifications of the S-method (adaptive and multi-window forms), improve the concentration in the TF plane, but still keep the limitations of the quadratic nature. Particularly, there will be a significant influence of higher-order-phase derivatives in the case of considered fast-varying signals (e.g. derivatives of sine/cosine signal modulations caused by rotating target parts). Therefore, the complex-time distributions (CTDs) [16–25] appear as an optimal choice for characterisation of fast-varying data, providing high concentration along IF, and consequently, the exact IF tracking.

The CTDs of different orders are applicable to various types of signals, depending on the phase function. The focus of this paper is on the fourth-order CTD, based on the ambiguity domain realisation. Most of the real signals exhibit sparsity property in a certain domain (time, frequency, TF etc.) [1, 2]. Sparsity means that, in certain domain, the signal could be represented with a small number of non-zero coefficients. Sparse signals are the subject of compressive sensing (CS) application, which is a new and intensively studied topic in signal processing [26–46]. CS uses signals sampled at the rates below Nyquist, and provides successful signal reconstruction using small set of randomly selected samples [27]. Random sampling is a necessary condition that should be satisfied, in order to reconstruct signal from incomplete set of samples. Most of the real-world signals are sparse in one domain, whereas they are dense in the other [28]. Here, we deal with signals that are sparse in the joint TF domain, that is, signals whose IF occupies only small part of the TF plane [1, 2, 34, 38]. By exploiting sparsity in the TF domain, the CS approach is applied to the signal in order to provide better IF localisation with reduced number of samples.

It was shown that the AF can be combined with CS to provide sparse TF representation, especially for linear IF [34, 38]. The observations are used from the ambiguity domain, usually from a priori defined area. The shape and size of the area is defined by the mask that should be large enough to collect auto-terms and small enough to avoid the cross-terms. Here, we deal with quite a complex signal structure, and thus we explore the use of ambiguity-based CTD in order to provide sparse and cross-terms free representation and to facilitate samples acquisition. In the cases of noisy signals, the robust form is provided by using the L-statistics.

This paper is organised as follows. The theoretical background on commonly used TF distributions is given in Section 2. Distributions with the complex-time argument are described in this section, as well. Section 3 presents the CS in the TF domain. A robust
approach for the IF estimation by using CTDs and CS is described in Section 4. Section 5 contains several examples which justify presented theory. Conclusions are given in Section 6.

2 Review of TF distributions

Real-world signals differ in their nature: stationarity, sparsity, number of components present in the signal etc. Therefore, different tools and methods have been used for signal processing and analysis. The target signals in this paper are non-stationary signals with fast oscillations of spectral content, which entail TF-based approaches [7]. The commonly used quadratic TF distribution is the WD defined as

\[
WD(t, \omega) = \int_{-\infty}^{\infty} x\left(t + \frac{T}{2}\right) x^*\left(t - \frac{T}{2}\right) e^{-j\omega t} dt. \tag{1}
\]

In the case of multicomponent signals the WD produces the cross-terms, whereas the spread factor will contain all odd phase derivatives in the cases of non-linear IF changes. The Cohen class distributions, the S-method, and the distributions with 'complex-time argument' are introduced with the aim to overcome drawbacks of the WD. We will focus on the ambiguity-domain-based distributions, that is, the Cohen class and class of 'complex-time' distributions. The AF is defined as a 2D Fourier transform of the WD

\[
A(\theta, \tau) = FT_{2D}(WD(t, \omega)) = \int_{-\infty}^{\infty} x\left(t + \frac{T}{2}\right) x^*\left(t - \frac{T}{2}\right) e^{-j\theta t} dt, \tag{2}
\]

where \( FT_{2D} \) denotes the 2D Fourier transform. Having in mind that signal terms are located around the origin in the ambiguity plane and that the cross-terms are dislocated, they can be easily reduced or completely removed by applying the low-pass kernel function. It leads to the distributions from the Cohen class defined as follows:

\[
CD(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(\theta, \tau) A(\theta, \tau) e^{-j\tau t} dt d\theta, \tag{3}
\]

where \( K(\theta, \tau) \) denotes a 2D kernel function.

However, in the cases of the signals with fast-varying IF, the Cohen class distributions cannot provide satisfactory results in tracking the IF changes. Therefore, higher-order distributions based on the complex-time argument should be used to provide high concentration in the TF plane, and consequently better IF estimation for signals with fast-varying IF.

The fourth-order CTD can be defined as

\[
CTD_4(t, \omega) = \int_{-\infty}^{\infty} x\left(t + \frac{T}{4}\right) x^*\left(t - \frac{T}{4}\right) e^{-j\omega t} dt, \tag{4}
\]

where \( a_i \) and \( b_j \) denote points on the unit circle. Let us consider the case where \( (a_1, b_1, a_2, b_2) = (1, 0, 0, 1) \). The corresponding fourth-order CTD is given in the form

\[
CTD_4(t, \omega) = \int_{-\infty}^{\infty} x\left(t + \frac{T}{4}\right) x^*\left(t - \frac{T}{4}\right) e^{-j\omega t} dt, \tag{5}
\]

and the corresponding spread factor is

\[
S(t, \tau) = \phi^{(5)}(t) \frac{x^5}{4!} + \phi^{(9)}(t) \frac{x^9}{9!} + \ldots \tag{6}
\]

Starting from the fourth-order moment function, it is possible to separate the real-time and complex-time AFs, as follows [6]:

\[
A_0(\theta, \tau) = \int_{-\infty}^{\infty} x\left(t + \frac{T}{4}\right) x^*\left(t - \frac{T}{4}\right) e^{-j\theta t} dt, \tag{7}
\]

\[
A_4(\theta, \tau) = \int_{-\infty}^{\infty} x^4\left(t + \frac{T}{4}\right) x^4\left(t - \frac{T}{4}\right) e^{-j\theta t} dt. \tag{8}
\]

Both of the AFs can be filtered using the kernel function

\[
A_{0k}(\theta, \tau) = K(\theta, \tau) A_0(\theta, \tau), \quad A_{4k}(\theta, \tau) = K(\theta, \tau) A_4(\theta, \tau). \tag{9}
\]

The resulting AF is obtained as [6]

\[
A(\theta, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(\epsilon) e^{-j\epsilon t} e^{j(\tau - \tau_1)} dt d\theta, \tag{10}
\]

where \( W(\epsilon) \) is a window function. Now, the CTD can be calculated as

\[
CTD(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\theta, \tau) e^{-j\pi-\epsilon\omega} dt d\theta. \tag{11}
\]

For signals with higher-order-phase non-stationarities, the distribution order can be increased to diminish the influence of higher-order-phase derivatives and to improve TF concentration.

In the sequel, we propose an approach to combine the CTD and the CS approach. This combination can be used to provide a sparse TF representation and an efficient IF estimation.

3 TF representations and CS

3.1 CS concept

CS approach [26] has been widely studied in the recent years. It provides successful signal reconstruction using an incomplete set of signal samples, that is, deals with signals sampled at the rate lower than Nyquist. Signal can be intentionally under-sampled. In the noisy signal cases, corrupted samples can be considered as missing ones, if we are able to locate these samples in the signal. To apply CS procedure, certain conditions should be satisfied, such as sparsity and incoherence between measurement and transform matrices [35, 36]. Sparsity refers to the property that the signal can be represented by a small number of non-zero coefficients in certain transform basis. Incoherent sensing provides successful signal reconstruction using a small set of signal samples. Many signals in real application satisfy these two conditions, allowing wide applications of CS.

The signal reconstruction from a small number of available samples is done using complex optimisation algorithms. Assume that we have a time-domain signal \( x \) of length \( N \). Let \( k \)-th be sparsifying basis for the signal \( x \). Therefore, signal can be
represented in the form of the basis matrix as follows:

\[ x = F_{N \times N}^{-1} S. \]  

(11)

If \( F_{N \times N}^{-1} \) denotes \( N \times N \) inverse Fourier transform matrix, then \( S \) denotes the vector of Fourier transform coefficients. In the matrix form, previous relation becomes

\[
\begin{bmatrix}
    x(0) \\
    x(1) \\
    \vdots \\
    x(N-1)
\end{bmatrix} = \frac{1}{N} \begin{bmatrix}
    1 & 1 & \cdots & 1 \\
    e^{j \frac{2\pi}{N}} & e^{j \frac{2(N-1)\pi}{N}} & \cdots & \cdots \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & e^{j \frac{2(N-1)\pi}{N}} & \cdots & e^{j \frac{2\pi}{N}}
\end{bmatrix}
\begin{bmatrix}
    S(0) \\
    S(1) \\
    \vdots \\
    S(N-1)
\end{bmatrix},
\]

(12)

where each element of the matrix \( F_{N \times N}^{-1} \) is an exponential term \( e^{j \frac{2\pi}{N} (kn/N)} \), \( k = 0, \ldots, N - 1; n = 0, \ldots, N - 1 \). Assume now that only \( M \) randomly distributed samples of the signal \( x \) are known, whereas the rest \( (N-M) \) samples are considered as missing. This means that the Fourier transform matrix is not a full \( N \times N \) matrix, but rather a randomly subsampled matrix that contains \( M \) out of \( N \) randomly selected rows of the original matrix. Random selection can be modelled as a matrix multiplication of the original matrix \( F_{N \times N}^{-1} \) with the incoherent measurement matrix \( \Phi_{M \times N} \). Matrix formed in such a way is called random partial Fourier matrix, and can be defined as

\[
F_p = \Phi_{M \times N} F_{N \times N}^{-1} = \frac{1}{N} \begin{bmatrix}
    1 & e^{j \frac{2\pi}{N}} & \cdots & e^{j \frac{2(N-1)\pi}{N}} \\
    1 & e^{j \frac{2\pi}{N}} & \cdots & e^{j \frac{2(N-1)\pi}{N}} \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & e^{j \frac{2\pi}{N}} & \cdots & e^{j \frac{2(N-1)\pi}{N}}
\end{bmatrix},
\]

(13)

where \((n_1, n_2, \ldots, n_M)\) correspond to the positions of the available samples. Randomly chosen \( M \) rows of the matrix \( F_{N \times N}^{-1} \) result in the measurement vector \( v \)

\[ v = F_p^{-1} S. \]  

(14)

The system of (14) is undetermined. To find the sparsest solution, among large number of possible solutions, different optimisation algorithms are used: greedy algorithms (matching pursuit (MP), orthogonal matching pursuit (OMP), stagewise orthogonal matching pursuit (StOMP), compressive sensing orthogonal matching pursuit (CoSaMP) etc.) [39–42], convex relaxation algorithms and the least absolute shrinkage and selection operator [43], non-iterative, and iterative solutions [29, 45]. Commonly used optimisation algorithms are based on \( l_1 \)-norm minimisation [39]

\[
\min_x \|S\|_{l_1} \quad \text{subject to} \quad v = F_p^{-1} S.
\]

(15)

### 3.2 Extended problem formulation – 2D partial Fourier transform matrix

The CS approach can be applied to the TF domain in order to provide sparse representation and better localisation of the signal in the TF plane [34, 38]. When TF domain is considered as a domain of sparsity, then the ambiguity domain can be used as a domain of observations. Hence, the measurements are randomly selected from the ambiguity domain.

The relationship between the AF and CTD can be written as

\[ A(\theta, \tau) = F^{2D} \cdot \text{CTD}(t, \omega), \]

(16)

where CTD denotes CTD and \( F^{2D} \) is \( 2D N^2 \times N^2 \) Fourier transform matrix. Therefore, we assume that the dense counterpart of the ambiguity domain is the fourth-order CTD. Matrix \( F^{2D} \) is produced as the Kronecker product of the identity matrix \( I \) and 1D Fourier transform matrix as follows:

\[
F_{N \times N}^{2D} = I_{N \times N} \otimes F_{1 \times N}^{1D},
\]

(17)

where \( \otimes \) denotes the Kronecker product. In the matrix form, we can write

\[
\begin{bmatrix}
    F_{N \times N}^{1D} & 0 & \cdots & 0 \\
    0 & F_{N \times N}^{1D} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & F_{N \times N}^{1D}
\end{bmatrix}
\begin{bmatrix}
    1_{N \times N} & 0 & \cdots & 0 \\
    0 & 1_{N \times N} & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1_{N \times N}
\end{bmatrix}
= \begin{bmatrix}
    1 & 1 & \cdots & 1 \\
    e^{-j \frac{2\pi}{N}} & e^{-j \frac{2(N-1)\pi}{N}} & \cdots & \cdots \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & e^{-j \frac{2\pi}{N}} & \cdots & e^{-j \frac{2(N-1)\pi}{N}}
\end{bmatrix}.
\]

(18)

Now we can define the CS problem in the ambiguity domain. For \( N \times N \) TF representation, at most \( K \times N \) non-zero components should exist, where \( K \) is the number of signal components [34, 38]. If we consider only the coefficients around the origin in the ambiguity plane, and use these coefficients as measurements for the CS procedure, the obtained AF is a measurement AF. It can be described as follows:

\[
A^M = \begin{bmatrix}
    0_{1 \times L} & 0_{L \times 1} & 0_{L \times L} \\
    0_{L \times 1} & 0_{L \times L} & 0_{L \times L} \\
    0_{1 \times L} & 0_{L \times 1} & 0_{L \times L}
\end{bmatrix} A.
\]

(19)

The matrix \( \Phi \) is of size \( J \times J \) (where \( J + 2L = N \)) and contains randomly positioned \( p\% \) of values ‘1’ and \((100-p)\%\) of values ‘0’. In the sequel, it will be referred to as the mask in the ambiguity domain, whereas \( A^M \) (vector of size \( J^2 \times 1 \)) will be referred as the masked AF. Note that the AF is first rearranged into vector \( A \) of size \( N^2 \times 1 \).

The sparse TF distribution can be obtained by minimising the following function:

\[
\min_{\sigma} F(\sigma) + G(\sigma),
\]

(20)

where \( F \) performs soft thresholding according to the relation: \( F(\sigma) = \max(0, 1 - \lambda / |\sigma|) \sigma \), \( \lambda \) is a regularisation parameter and \( G(\sigma) \) is defined as

\[
G(\sigma) = F_p(\sigma - A^M) \Phi.
\]

(21)

\( F_p \) denotes partial 2D Fourier transform matrix, \( F_p \) is its transpose, and \( \sigma \) is a row vector which is, after the optimisation problem is solved, rearranged into the matrix and forms a sparse TF representation. Matrix \( F_p \) is a row-subsampled matrix as only
those rows which correspond to the positions of measurements from the mask are kept, whereas the other rows are discarded.

4 Robust approach to the IF estimation

In the cases where the AF is corrupted by impulsive noise, the randomly selected coefficients used as measurements in the CS procedure will be noisy as well. The initial transform domain vector, used in CS optimisation problem, should be noise free. Therefore, the robust statistics can be applied to the coefficients of the AF in order to remove noisy peaks. This is achieved by using the L-statistics that performs well in the presence of impulsive and mixed noise [34, 38].

According to the L-estimation approach, the noisy measurement vector is sorted, and certain per cent of the smallest and the largest value coefficients is discarded. Therefore, the L-estimation-based minimisation problem can be defined as follows:

$$\arg \min_{S} \|S\|_{1} \text{ subject to } \psi = F_{P}^{-1}S.$$  (22)

A robust initial transform is obtained as

$$s_{0} = \sum_{i=0}^{Q} A_{\text{SORT}},$$  (23)

where $P$ and $Q$ denote the number of discarded coefficients, smallest and largest, respectively. The proposed algorithm for obtaining sparse TF representation is represented by the flowchart in Fig. 1.

Fig. 1  Flowchart of the proposed algorithm

Fig. 2  TF representations are of 90 × 90 size

a WD of the signal (24)
b, c Cohen class distributions, obtained by using Gaussian kernel with $\delta = 80, 20$
d IF estimated from the WD
e, f IFs estimated from the Cohen class distributions with Gaussian kernel $\delta = 80, 20$

Full line denotes the correct IF and dotted corresponds to the estimated IF
5 Experimental results

5.1 Example 1: simulated radar signal with fast-varying IF changes

Consider the multicomponent signal with the non-linear phase function in the form

\[
x(t) = e^{j(2 \cos (\theta) \cos (\theta) + 4.5 \theta^2)/2} + e^{j(\cos (\theta) + \cos (3\theta) + \cos (4\theta) - 3\theta^2)/2}.
\] (24)

To find a suitable TF representation for the observed signal, several TF representations are considered. The TF representations are of 90 × 90 size and they are displayed in Fig. 2a–c. First, the WD distribution is calculated.

We may observe that the WD cannot follow fast IF variations (see Fig. 2b and c), and, in addition, it introduces the cross-terms. Therefore, different Cohen class distributions are tested to avoid cross-terms. As a kernel function, the Gaussian low-pass filtering function is used for different values of parameter \( \delta \) (distributions obtained with two different values: \( \delta = 80 \) and \( \delta = 20 \) are shown in Fig. 2b and c). Mean square errors (MSEs) and relative MSEs (RMSEs) for different TF distributions and different number of CS measurements are given in Table 1. The distributions from the Cohen class enable controlling of the cross-terms amount, but again, fail to provide accurate IF tracking of the non-stationary signals, as it is shown in Fig. 2e and f.

The IFs estimated from the observed distributions are shown in Fig. 2 with full line, whereas the correct IF is shown with dotted line. CS-based TF distributions, calculated starting from the WD and the Cohen class distributions, are shown in Fig. 3. For the CS-based TF distribution, the TF mask is formed by using central region of 25 × 25 size in the ambiguity domain. This region contains 7.7% of the total number of samples. The amount of 50% of measurements is chosen randomly from the mask (3.8% of the total number of samples). Note that the resulting distributions are not sparse and do not follow accurately the IF changes. Therefore, the complex-time CS distribution is calculated. The standard form of the fourth-order CTD is shown in Fig. 4a. The ambiguity domain filtering with Gaussian kernel is used for the calculation of the CTD. The mask size is 25 × 25. The resulting sparse TF distribution (Fig. 4b) is provided by using 60% of the samples from the mask, which is about 5% of the total number of samples. Very similar results are obtained even for a lower number of measurements, for example, 40%. The IFs estimated from the sparse distributions, shown in Fig. 4a and b, are very close to the true IF of the signal (see Figs. 4c and d). Let us consider the influence of the mask size (\( J \times J \)) on the number of samples required to provide sparse TF representation. Fig. 5 shows that the masks of size 7 × 7 and 10 × 10 are too small to provide accurate IF representation, even if all samples in the masks are used as CS measurements.

### Table 1 MSEs of the IF estimation

| Distribution                        | Component 1 | Component 2 | Relative MSE–RMSE, % |
|-------------------------------------|-------------|-------------|----------------------|
|                                      | Component 1 | Component 2 |                      |
| Wigner distribution                 | 3.3192 × 10^3 | 79.7761     | 67.81                | 5.79                |
| Distribution from the Cohen class based on Gaussian kernel \( e^{-|\theta|^2} + |\theta|^2 \) with \( \delta = 120 \) | 1.1563 × 10^3 | 3.1624 × 10^2 | 28.3                | 77.43               |
| Distribution from the Cohen class based on Gaussian kernel \( e^{-|\theta|^2} + |\theta|^2 \) with \( \delta = 80 \) | 1.3880 × 10^3 | 1.5905e × 10^2 | 71.59               | 78.04               |
| Distribution from the Cohen class based on Gaussian kernel \( e^{-|\theta|^2} + |\theta|^2 \) with \( \delta = 20 \) | 1.6176 × 10^3 | 1.5958 × 10^2 | 61.02               | 54.82               |
| Fourth-order CTD                   |             |             |                      |
| 7 × 7 mask and all samples within the mask used | failed     | failed     | /                    | /                    |
| 10 × 10 mask and all samples within the mask used | failed     | failed     | /                    | /                    |
| 15 × 15 mask and 70% samples used   | 81.3099     | 57.1962     | 11.08                | 0.02                |
| 20 × 20 mask and 60% samples used   | 8.9314      | 20.5861     | 0.25                 | 3.43                |
| 25 × 25 mask                       | 17.7834     | 32.7570     | 1.23                 | 3.48                |
| 40% samples from the mask used      | 10.7177     | 9.2834      | 1.09                 | 0.36                |
| 50% samples from the mask used      | 7.3854      | 8.1211      | 0.30                 | 0.31                |
| 60% samples from the mask used      | 7.77% of the total number of samples | 50% samples from the mask used | 1.6176 × 10^3 | 1.5958 × 10^2 | 61.02               | 54.82               |

**Fig. 3** Sparse TF distributions obtained from

a WD
b Cohen class distribution based on Gaussian kernel

c Dotted line: the IF estimated from the distribution in Fig. 4a, full line – the true IF

d IF estimated from the distribution in Fig. 4b: full line is the true IF, dotted line – the estimated IF

**Fig. 4** Standard form of the fourth-order CTD

a Fourth-order CTD
b Sparse TF representation obtained by using 60% of randomly chosen measurements from 25 × 25 mask
c Dotted line: the IF estimated from the distribution in Fig. 4a, full line – the true IF
d IF estimated from the distribution in Fig. 4b: full line is the true IF, dotted line – the estimated IF
Larger masks (15 × 15 and 20 × 20) can provide an accurate IF estimation, but require a larger number of samples to be used as CS measurements (i.e. 70 and 60% of the measurements from the mask are used) in order to obtain an accurate IF estimation.

5.2 Example 2: noisy signal measurements

Let us now consider the simulated radar signal. Monocomponent signal is frequency modulated periodically and has fast IF changes. Non-uniform rotation of the reflecting point in radar systems could be described using this form of the signal [9]. The signal lasts for 96 s. It is sine modulated and sampled at frequency 48 Hz: $x(t) = e^{i(4\cos(\pi t) + (2/3)\cos(4\pi t) + (2/3)\cos(5\pi t))}$. The assumption is that the ambiguity domain measurements are corrupted by impulse

| SNR  | MSE  | RMSE, % |
|------|------|---------|
| -62  | 18.418 | 1.22    |
| -50  | 8.344  | 0.23    |
| -39  | 9.598  | 0.21    |
| -30  | 3.0854 | 0.18    |
| -23  | 4.9059 | 0.15    |

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noise. The TF distribution, obtained by using 50% of noisy ambiguity measurements from the mask of size 25 × 25, is shown in Fig. 6a.

The AF of the signal is corrupted by the impulse noise, which disables accurate IF estimation of the observed signal. The CS-based TF distribution (Fig. 6a), obtained by using noisy ambiguity measurements, is not sparse and does not provide accurate IF estimation. Therefore, the L-estimation approach is applied to the AF prior to the CS-based TF distribution calculation. Namely, 0.5% of the highest ambiguity coefficients are removed in the ambiguity domain (as a consequence of L-estimation). The estimation errors are given in Table 2 (the MSEs and the RMSEs). In all considered cases, the successful IF estimation is achieved.

### 5.3 Example 3: real radar signal

Finally, as an example of signal recorded in real noisy environment, let us observe a portion of radar data corresponding to the moving human body. The micro movements, appearing during the human walking, are hard to detect using the standard TF representations. In that sense, the high-resolution solutions provided by the CTDs allow detection of the smallest variations. The radar producing the signal operates at 2.4 GHz carrier frequency, and transmitted power level is 5 dBm. The sampling frequency is 1 kHz.

In that sense, the high-resolution solutions provided by the CTDs are shown in Figs. 8a and b. As it can be seen from Fig. 8, the IF estimated from the sparse TF corresponds to the IF estimated from the original CTD.

6 Conclusion

This paper deals with the IF estimation using CS-based sparse TF representation. The non-stationary signals with fast-varying phase functions are observed. Having in mind that the standard distributions fail to accurately estimate the IF of observed signals, we employ the CTDs. The CS observations are taken randomly from the masked AF calculated on a basis of the CTD. It has been shown that the sparse TF representation can be reconstructed from ~4% of the ambiguity domain measurements. Furthermore, the accuracy of IF estimation based on the resulting sparse TF distribution is proved by using the MSE.

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