HILBERT FUNCTIONS OF
\(\mathcal{S}_n\)-STABLE ARTINIAN GORENSTEIN IDEALS

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Abstract. We describe the graded characters and Hilbert functions of certain graded artinian Gorenstein quotients of the polynomial ring which are also representations of the symmetric group. Specifically, we look at those algebras whose socles are trivial representations and whose principal apolar submodules are generated by the sum of the orbit of a power of a linear form.

1. Introduction

Let \( R = \mathbb{k}[x_1, \ldots, x_n] = \bigoplus_{k \geq 0} R_k \) be the standard graded polynomial ring in \( n \) variables over a field \( \mathbb{k} \) (which is one of \( \mathbb{Q}, \mathbb{R}, \) or \( \mathbb{C} \)) and let \( \mathcal{S}_n \) denote the symmetric group on \( n \) letters. We are interested in the Hilbert functions and graded characters of graded artinian Gorenstein algebras which are also representations of \( \mathcal{S}_n \). Specifically, we will examine quotients of \( R \) whose one-dimensional socles are spanned by a symmetric polynomial \( F \).

Every homogeneous polynomial \( F \) of degree \( d \) can be expressed as a linear combination of \( d \)-th powers of linear forms \( L_1, \ldots, L_m \) for some \( m \). If we symmetrize this expression for \( F \) by summing over all permutations and dividing by \( n! \), we can express \( F \) as a linear combination of \( F_1, \ldots, F_m \) where each \( F_i = \sum_{\sigma \in \mathcal{S}_n} \sigma L_i^d \) is the sum of the permutations of \( L_i^d \). In this paper, we will only consider polynomials \( F \) which are the sum of the orbit of the \( d \)-th power of a single linear form (i.e., \( m = 1 \)).

For example, let \( n = 4 \) and consider the linear form

\[ L = x_1 + x_2 + 2x_3 + 3x_4 \]

which has twelve distinct permutations \( \sigma_1 L, \ldots, \sigma_{12} L \) for \( \sigma_i \in \mathcal{S}_4 \), rather than twenty four. E.g., \((134)L = x_3 + x_2 + 2x_4 + 3x_1 \). The degree 7 symmetric polynomial

\[ F = \sigma_1 L^7 + \cdots + \sigma_{12} L^7 \]

spans the one-dimensional socle of the graded artinian Gorenstein algebra \( R/I_F \) where \( I_F \) consists of all \( f \in R \) with \( \partial F/\partial f = 0 \).
The dimensions of the homogeneous components of $R/I_F$ are recorded in its Hilbert function $HF_{R/I_F}(k) := \dim_k (R/I_F)_k$, although it is often convenient to present them in a generating function called a Hilbert series:

$$HS_{R/I_F}(t) := \sum_{k \geq 0} \dim_k (R/I_F)_k t^k.$$ 

Currently, there is no known description of all Hilbert functions that arise from graded Gorenstein algebras (except for $n = 2, 3$ [10, Theorem 1.44] [16]). In order to find the Hilbert function of an artinian graded algebra with a given socle, one typically computes the ranks of a collection of Catalecticant matrices — one for each degree [9].

In our example, however, $R/I_F$ has additional structure which will allow us to find that its Hilbert series is

$$HS_{R/I_F}(t) = 1 + 4t + 9t^2 + 12t^3 + 12t^4 + 9t^5 + 4t^6 + t^7.$$ 

Our point of view is illustrated as follows: since $F$ is a symmetric polynomial, $I_F$ is stable under the action of $S_n$. Thus, $I_F$ and its homogeneous components $(I_F)_k$ are representations of $S_n$. So, the quotient $R/I_F = \bigoplus_{k \geq 0} (R/I_F)_k$, where $(R/I_F)_k = R_k/(I_F)_k$, is a graded representation of $S_n$. If we denote the character of a finite dimensional representation $V$ of $S_n$ by $\chi_V : S_n \to k$, then the graded character of $R/I_F$ is defined by

$$\chi_{R/I_F}(t) := \sum_{k \geq 0} \chi_{(R/I_F)_k} t^k.$$ 

This encodes the algebra’s structure as a graded representation much like the Hilbert series does for its structure as a graded vector space.

Recall that since any representation of $S_n$ is a direct sum of irreducible representations, and the irreducible representations of $S_n$ are in one-to-one correspondence with partitions $\lambda \vdash n$, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\lambda_1 \geq \cdots \geq \lambda_r \geq 1$, we can write

$$\chi_{(R/I_F)_k} = \sum_{\lambda \vdash n} m_\lambda \chi^\lambda$$

where $m_\lambda \in \mathbb{N}$ and $\chi^\lambda$ is the character of the irreducible representation corresponding to $\lambda$. 
Writing $\chi^{(\lambda_1, \ldots, \lambda_r)}$ as $\chi^{\lambda_1, \ldots, \lambda_r}$, the graded character of $R/I_F$, for $F$ as above, is

$$
\chi_{R/I_F}(t) = \chi^4 + (\chi^4 + \chi^{31}) t \\
+ (\chi^4 + 2\chi^{31} + \chi^{22}) t^2 \\
+ (\chi^4 + 2\chi^{31} + \chi^{22} + \chi^{211}) t^3 \\
+ (\chi^4 + 2\chi^{31} + \chi^{22} + \chi^{211}) t^4 \\
+ (\chi^4 + 2\chi^{31} + \chi^{22}) t^5 \\
+ (\chi^4 + \chi^{31}) t^6 \\
+ \chi^4 t^7.
$$

In this paper, we describe the graded characters of Gorenstein algebras $R/I_F$ whose socles are spanned by a form $F$ which is the sum of the orbit of a power of a linear form $L$. (Our only additional requirements are that the coefficients of $L$ are real and that they do not sum to zero. The latter requirement is essentially equivalent to the embedding dimension of $R/I_F$ being $n$.) The graded characters of such algebras are palindromic, as in the example above, and the multiplicities of the irreducible characters in each degree can be computed directly from the Kostka-Foulkes polynomials. In fact, the graded character of $R/I_F$ depends on the degree of $F$ and the number of repeated coefficients in $L$, but not on the values of those coefficients. Furthermore, the Hilbert function of $R/I_F$ can be recovered from the graded character by replacing each $\chi^\lambda$ occurring in $\chi_{R/I_F}(t)$ with the numbers $f^\lambda$ which count the number of standard Young tableaux with shape $\lambda$ and which is equal to the dimension of the irreducible of type $\lambda$. For an introduction to the representation theory of the symmetric group we recommend [15] (and [1] to see how these representations are realized in the polynomial ring).

Since artinian Gorenstein algebras are characterized by having a one-dimensional socle, it follows that graded artinian Gorenstein algebras which admit an action of $\mathfrak{S}_n$ come in two types. These two types correspond to the only two one-dimensional representations of $\mathfrak{S}_n$. The socle of the algebra is either

(i) the trivial representation and spanned by a symmetric polynomial, or

(ii) the alternating representation and spanned by an alternating polynomial.
In [2], Bergeron, Garsia and Tesler described the graded character of artinian Gorenstein algebras of type (ii). The graded character of such an algebra is a multiple of the graded character of the coinvariant algebra $R_\mu$, where $\mu = (1, \ldots , 1) \vdash n$. Roth [14] extended their result to any artinian algebra whose socle is spanned by alternating forms.

Morita, Wachi and Watanabe [13] found the Hilbert function of each isotypic piece of the algebra $A(n, k) = \mathbb{k}[x_1, \ldots , x_n]/(x_1^k, \ldots , x_n^k)$ for each $n$ and $k$. The algebra $A(n, k)$ is of type (i) since its socle is spanned by the symmetric monomial $F = (x_1 \cdots x_n)^{k-1}$.

We will consider algebras $R/I_F$ of type (i) where we have chosen the socle of $R/I_F$ to be spanned by a symmetric polynomial $F$ of a specific type. The graded character $\chi_{R/I_F}(t)$ will be expressed in terms of the graded character of an algebra denoted $R_\mu$ where $\mu$ depends on $F$. In this situation, we will see that $\chi_{R/I_F}(t)$ is rather more complicated than a simple multiple of $\chi_{R_\mu}(t)$.

The remainder of Section 1 will cover: basic facts on Gorenstein algebras; the action of $S_n$ on $R$; and submodules of the apolarity module.

In Section 2, we describe the symmetric polynomials $F$ that we will consider and we examine the graded character of $R/I_F$ through its apolarity module $M = \bigoplus_{k \geq 0} M_k$ since their graded characters are equal. In fact, we produce a number of maps indexed by a degree $k$ — namely $\phi_k$, $\psi_k$, and $\nu_k$ — that have $M_k$ as the image of their composition (see Theorem 7). In this composition, $\nu_k$ can be dropped without changing the image and the maps $\phi_k$ and $\psi_k$ are closely related. It will follow that to compute the graded character of $R/I_F$, it suffices to determine the character of the image of $\phi_k$ for each $k \geq 0$.

In the final section, we relate $N = \bigoplus_{k \geq 0} \text{im}\phi_k$ to the homogeneous coordinate ring $\mathbb{k}[X]$ of the $S_n$-orbit of a projective point and further to an affine orbit $Y$ and the algebra $R_\mu$. The graded character of $R_\mu$ is intimately connected with the Kostka-Foulkes polynomials $K_{\lambda, \mu}(t)$ which we use to describe the graded characters of $N$ and $M$ (or equivalently, $\mathbb{k}[X]$ and $R/I_F$).

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1.1. Gorenstein Algebras. A standard graded algebra $A = R/I$ is Gorenstein if it contains a maximal $A$-regular sequence which generates
an irreducible ideal of $A$. If $A$ is artinian (i.e., finite dimensional as a $k$ vector space), then $A$ contains no regular elements $f$ with $A/(f) \neq 0$. Consequently, an artinian graded algebra $A$ is Gorenstein if $I$ is irreducible in $R$. That is, for any ideals $J_1, J_2 \subseteq R$ properly containing $I$, $J_1 \cap J_2 \neq I$.

The socle of $A$ is the ideal $\text{Soc } A = (0 : m) = \{ f \in A \mid fm = 0 \}$ where $m = \bigoplus_{k \geq 1} A_k$ is the homogeneous maximal ideal of $A$. A polynomial $f$ is in $\text{Soc } A$ if and only if $x_1^0 f = \cdots = x_n f = 0$. If $J \subseteq A$ is any nonzero ideal of the artinian algebra $A$, then $J$ contains an element of the socle of $A$. If $A$ is artinian and Gorenstein then its socle must be one-dimensional: if the socle is larger, it contains two linearly independent elements $f_1$ and $f_2$ for which $((f_1) + I) \cap ((f_2) + I) = I$.

1.2. The action of $S_n$ on $R$. We begin by setting our conventions for the action of $S_n$ on $R$. Let each $\sigma \in S_n$ act on a linear form $a_1x_1 + \cdots + a_nx_n \in R_1$ by

$$\sigma(a_1x_1 + \cdots + a_nx_n) = a_1x_{\sigma(1)} + \cdots + a_nx_{\sigma(n)} = a_{\sigma^{-1}(1)}x_1 + \cdots + a_{\sigma^{-1}(n)}x_n.$$ 

This action extends to an action on $R$ which is given by

$$(\sigma f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

In particular, if we write a monomial in $R$ as $x^b = x_1^{b_1}\cdots x_n^{b_n}$ for an exponent vector $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$, then

$$\sigma(x^b) = x_{\sigma(1)}^{b_1}\cdots x_{\sigma(n)}^{b_n} = x_1^{b_{\sigma^{-1}(1)}}\cdots x_n^{b_{\sigma^{-1}(n)}}.$$ 

Accordingly, we define $\sigma(b) = (b_{\sigma^{-1}(1)}, \ldots, b_{\sigma^{-1}(n)})$ for $\sigma \in S_n$ and for any $b \in \mathbb{N}^n$ or $\mathbb{k}^n$, so that $\sigma x^b = x^{\sigma b}$. With care, one can check that all of the above are left actions in that $(\tau \sigma)x^b = \tau(\sigma x^b)$ and $(\tau \sigma)b = \tau(\sigma b)$ for $\sigma, \tau \in S_n$. If $a \in \mathbb{k}^n$ then $f(\sigma a) = f(a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}) = (\sigma^{-1}f)(a)$. Furthermore, if $e_i$ is the $i$-th standard basis vector of $\mathbb{k}^n$, then $\sigma(e_i) = e_{\sigma(i)}$.

1.3. The apolarity module. Partial differentiation can be thought of as a $\mathbb{k}$-bilinear operator $\partial : R \times R \to R$ in the following way: The partial derivative of the monomial $x^e$ by the monomial $x^b$ is defined to
be

\[ \partial(x^b, x^c) = \frac{c_1! \cdots c_n!}{(c_1 - b_1)! \cdots (c_n - b_n)!} x^{c-b} \]

:= \frac{c!}{(c-b)!} x^{c-b}

when \( c \geq b \) (i.e., \( c_i \geq b_i \) for all \( i \)) and zero otherwise. We can extend this definition linearly in both components to define \( \partial(f, g) \), the partial derivative of a polynomial \( g \) by another polynomial \( f \). When \( f \) is a variable, \( \partial(f, g) = \partial g/\partial f \) is the usual partial derivative.

Partial differentiation endows \( R \) with an \( R \)-module structure which is different from that of a rank-one free module. To avoid confusion, we let \( S = R \) be the \( R \)-module with its scalar multiplication defined as: for \( f \in R \) and \( g \in S \), we set \( fg := \partial(f, g) \in S \). We call \( S \) the apolarity module of \( R \). So the bilinear operator \( \partial \) can be viewed as a map \( \partial : R \times S \to S \).

Our interest in the apolarity module derives from Macaulay’s correspondence between graded artinian Gorenstein quotients of \( R \) and principal submodules of the apolarity module of \( R \).

The polynomial ring \( R = \bigoplus_{k \geq 0} R_k \) is an \( \mathbb{N} \)-graded ring in the standard way: its homogenous components \( R_k \) are vector spaces consisting of the homogeneous forms of degree \( k \). The apolarity module \( S = \bigoplus_{k \geq 0} S_k \) decomposes similarly, where \( R_k = S_k \) as vector spaces. We will call \( S \) a graded \( R \)-module even though its graded components satisfy the slightly unconventional condition that \( R_k S_j \subseteq S_{j-k} \) for all \( j, k \in \mathbb{N} \) where \( S_{j-k} = 0 \) for \( k > j \). This convention allows a polynomial \( f \) to have the same degree regardless of whether it is in \( R \) or \( S \), though we need to remember that \( R \) acts by differentiation on \( S \) and hence decreases the degree of the elements of \( S \).

The bilinear operator \( \partial \) restricts to a bilinear form \( \partial_k : R_k \times S_k \to \mathbb{k} \). This bilinear form induces two maps \( R_k \to S_k^* \) and \( S_k \to R_k^* \). Let \( \overline{f} \) denote the polynomial obtained by taking the complex conjugate of the coefficients of the polynomial \( f \). Since \( \partial_k(f, \overline{f}) \neq 0 \) for any \( f \in R_k \setminus \{0\} \), the map \( R_k \to S_k^* \) is injective. Furthermore, \( R_k \) and \( S_k^* \) have the same dimension and hence the map \( R_k \to S_k^* \) is an isomorphism (and similarly for the map \( S_k \to R_k^* \)). Thus, the bilinear form \( \partial_k \) is a perfect pairing.

For any subspace \( M_k \) of \( S_k \), we define

\[ M_k^\perp = \{ f \in R_k \mid \forall g \in M_k, \partial_k(f, g) = 0 \} \]

One can check that \( \partial_k \) induces a well-defined bilinear form

\[ \partial_{k, M_k} : R_k/M_k^\perp \times M_k \to \mathbb{k} \]
which is also a perfect pairing. Thus, $M^*_k$ and $R_k/M_k^\perp$ are isomorphic vector spaces and hence $M_k$ and $R_k/M_k^\perp$ have the same dimension.

If we now consider a submodule $M \subseteq S$, its annihilator is the ideal

$$\text{Ann}(M) = \{ f \in R \mid \forall g \in M, \partial(f, g) = 0 \}.$$  

Furthermore, if $M$ is a graded submodule of $S$, then $\text{Ann}(M)$ is a homogeneous ideal and $R/\text{Ann}(M)$ is a graded algebra. One can check that the homogeneous components of $\text{Ann}(M)$ are $\text{Ann}(M)_k = M_k^\perp$ (or see [8, Proposition 2.5]). Thus, from the discussion above, the Hilbert function of $M$ (i.e., $\text{HF}_M(k) = \dim_k M_k$) is equal to the Hilbert function of $R/\text{Ann}(M)$.

Just as submodules $M$ of $S$ determine quotients $R/\text{Ann}(M)$ of $R$ by taking $\text{Ann}(M)_k = M_k^\perp$ for $k \geq 0$, homogeneous ideals of $R/\text{Ann}(M)$ determine graded quotients of $M$. In particular, the socle of $R/\text{Ann}(M)$ determines $M/\mathfrak{m}M$ (any basis of which represents a set of minimal generators of $M$) since the homogeneous components of $\text{Soc}(R/\text{Ann}(M))$ and $\mathfrak{m}M$ are orthogonal under $\partial_{k, M_k}$. This fact, proved by Macaulay [11, §60], shows that Gorenstein algebras are in one-to-one correspondence with principally generated submodules of the apolarity module. See [10, Lemma 2.12] for a modern treatment of these facts.

We would now like to consider submodules of $S$ which are also graded representations of $\mathfrak{S}_n$. We have already defined an action of $\mathfrak{S}_n$ on $R$ and, using the same action, $S$ is also a representation of $\mathfrak{S}_n$. Since the homogeneous components of $R$ and $S$ are stable under the action of $\mathfrak{S}_n$, $R$ and $S$ are graded representations. Partial differentiation is an invariant bilinear form in that $\partial_{k}(\sigma f, \sigma g) = \partial_{k}(f, g)$ for all $f \in R_k$, $g \in S_k$ and $\sigma \in \mathfrak{S}_n$. Therefore, the dual representation of $M_k$ is equivalent to $(R/\text{Ann}(M))_k$. (The action of $\mathfrak{S}_n$ on the dual $V^*$ of a representation $V$ is given by $(\sigma f)(v) = f(\sigma^{-1}v)$ for $\sigma \in \mathfrak{S}_n$, $f \in V^*$ and $v \in V$.) Since $\sigma$ and $\sigma^{-1}$ are conjugate, the symmetric group has the special property that all of its representations are self-dual. Thus, $R/\text{Ann}(M)$ and $M$ have the same graded characters.

Suppose $M = \langle g \rangle$ is the principal submodule of $S$ generated by $g$. As a vector space, $M$ consists of all partial derivatives of $g$. In this case, $R/\text{Ann}(M)$ is an artinian Gorenstein algebra which is isomorphic to $M$ as an $R$-module. If $g$ is a homogeneous polynomial of degree $d$, then $M$ is a graded $R$-module and its homogeneous components are given by $M_k = \text{im} \theta_{g,k}$ where $\theta_{g,k} : R_{d-k} \to S_k$ is the map

$$\theta_{g,k}(f) = \partial(f, g).$$
We set $\theta_g : R \to S$ to be the map $\theta_g = \bigoplus_{k \geq 0} \theta_{g,k}$ which can also be described by $\theta_g(f) = \partial(f, g)$. This map $\theta_g$ is called the \textit{Catalecticant map} of $g$ and its image is $M = \bigoplus_{k \geq 0} M_k = \langle g \rangle$.

If $M = \langle g \rangle$ is principally generated by a symmetric homogeneous polynomial $g$ of degree $d$ then $M$ is an $S_n$-stable subspace of $S$: if $f \in M$ then there is some $h \in R$ with $f = \partial(h, g)$ and hence, for any $\sigma \in S_n$,

$$\sigma f = \sigma \partial(h, g) = \partial(\sigma h, \sigma g) = \partial(\sigma h, g) \in M.$$ 

Using the same string of equalities, we can also see that $\theta_{g,k} : R^{d-k} \to S_k$ is equivariant. Thus, $M_k$ and $R^{d-k}/\ker \theta_{g,k} = (R/\Ann(M))_{d-k}$ are equivalent representations.

As $M_{d-k}$ is also equivalent to $(R/\Ann(M))_{d-k}$, the graded character of $M$ is palindromic in the sense that $\chi_{M_k} = \chi_{M_{d-k}}$ for all $k$.

2. \textbf{Principal Stable Submodules of the Apolarity Module.}

As mentioned above, we will be interested in artinan Gorenstein quotients of $R$ which are representations of $S_n$. Specifically we will look at $R/I_F$ where

$$I_F = \{ f \in R \mid \partial(f, F) = 0 \}$$

is the annihilator of $M = \langle F \rangle$, the principal submodule of $S$ generated by a homogeneous symmetric polynomial $F \in S$ of degree $d$.

If $F$ can be expressed as the sum of $d$-th powers of linear forms, say $\ell$ of them, then the Hilbert function $HF_{R/I_F}(k) = \dim_k(R/I_F)_k$ satisfies $HF_{R/I_F}(k) \leq \ell$ for all $k$. We will restrict our attention to homogeneous symmetric polynomials $F$ which are the sum of the $S_n$-orbit of a power of a linear form, not only because $R/I_F$ will have a bounded Hilbert function, but also because its associated principal apolar submodule will admit a useful “factored” presentation. We now construct such an $F$.

Suppose $a = (a_1, \ldots, a_n) \in \mathbb{K}^n$ and let

$$L = a_1 x_1 + \cdots + a_n x_n \in S_1.$$ 

Let $b_1, \ldots, b_r$ be all the distinct coordinates of $a$. Define $\mu_j = |\{ i \mid a_i = b_j \}|$ for $1 \leq j \leq r$. Reorder $b_1, \ldots, b_r$ so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 1$ and let $\mu = (\mu_1, \ldots, \mu_r)$. Thus the partition $\mu \vdash n$ associated to $a$ characterizes the number of repeated coordinates in $a$.

As mentioned in Section \ref{12}, $S_n$ acts on the left of tuples by $\sigma(a) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$. The left stabilizer of $a$ is

$$\langle S_n \rangle a = \{ \sigma \in S_n \mid \sigma a = a \}.$$
If we choose representatives $\sigma_1, \ldots, \sigma_\ell$ for the left cosets of $(\mathfrak{S}_n)_a$ then the elements in the orbit $Y = \{\sigma a \mid \sigma \in \mathfrak{S}_n\}$ of $a$ are in one-to-one correspondence with $\sigma_1, \ldots, \sigma_\ell$. Here $\ell = \binom{n}{\mu} := n!/(\mu_1! \cdots \mu_r!)$.

Using these representatives for the left-cosets, let $F \in S_d$ be the homogeneous symmetric polynomial

\[(*) \quad F = \sum_{i=1}^{\ell} \sigma_i L^d.\]

We now give an example of this construction.

**Example 1.** Let $R = \mathbb{k}[x_1, x_2, x_3]$ be the polynomial ring in $n = 3$ variables and let $S$ be its apolarity module. Let $a = (1, 1, 5)$ be a point in $\mathbb{k}^3$ and let

$L = x_1 + x_2 + 5x_3 \in S_1$

be the linear form which uses the entries of $a$ as its coefficients. Since $a$ has a coordinate that is repeated twice and another that occurs just once, its associated partition is $\mu = (2, 1)$.

The stabilizer of $a$ is $(\mathfrak{S}_3)_a = \{e, (12)\}$, where $e \in \mathfrak{S}_3$ is the identity permutation, and we may choose $\sigma_1 = e$, $\sigma_2 = (23)$ and $\sigma_3 = (13)$ as representatives for the left-cosets of $(\mathfrak{S}_3)_a$.

Finally, if we pick $d = 4$ as the degree of $F$, then

\[F = \sigma_1 L^4 + \sigma_2 L^4 + \sigma_3 L^4\]

\[= (x_1 + x_2 + 5x_3)^4 + (x_1 + 5x_2 + x_3)^4 + (5x_1 + x_2 + x_3)^4.\]

We will continue with this choice of $F$ in subsequent examples.

The polynomial $F$, defined by $(\star)$ above, is the sum of $\ell = \binom{n}{\mu}$ powers of linear forms since $\sigma_i L^d = (\sigma_i L)^d$. Thus, we have $HF_{R/I_F}(k) \leq \ell$ for all $k$. In order to obtain the Hilbert function of $R/I_F$ explicitly, we use the fact that $HF_{R/I_F} = HF_M$ and instead examine

\[M = \bigoplus_{k \geq 0} \text{im} \theta_{F,k}.\]

Recall that map $\theta_{F,k} : R_{d-k} \to S_k$ is given by $\theta_{F,k}(f) = \partial(f, F)$. 


Lemma 2. For \( F = \sum_{i=1}^{\ell} \sigma_i L^d \in S_d \) as described above and \( x^b \), a monomial of degree \( d - k \), we have

\[
\theta_{F,k}(x^b) = \frac{d!}{k!} \sum_{i=1}^{\ell} a_{\sigma_i^{-1}(b)} \sigma_i L^k.
\]

Proof. The proof is by induction on \( d - k \). If \( d - k = 0 \) then \( b = (0, \ldots, 0) \) and the formula obviously holds.

Assume that the formula above holds for every monomial \( x^b \) of some fixed degree \( d - k \). Differentiating our formula by \( x_j \)

\[
\partial(x^{b+e_j}, F) = \partial(x_j, \partial(x^b, F))
\]

\[
= \partial\left(x_j, \frac{d!}{k!} \sum_{i=1}^{\ell} a_{\sigma_i^{-1}(b)} \sigma_i L^k\right)
\]

\[
= \frac{d!}{k!} \sum_{i=1}^{\ell} a_{\sigma_i^{-1}(b)} \partial(x_j, \sigma_i L^k)
\]

\[
= \frac{d!}{k!} \sum_{i=1}^{\ell} a_{\sigma_i^{-1}(b)} \partial(x_j, \sigma_i L^{k-1})
\]

Thus,

\[
\partial(x^{b+e_j}, F) = \frac{d!}{k!} \sum_{i=1}^{\ell} a_{\sigma_i^{-1}(b)} \partial(x_j, \sigma_i L^{k-1})
\]

proving the formula holds for any monomial \( x^{b+e_j} \) of degree \( d - k + 1 \). \( \square \)

Example 3. Let \( L = x_1 + x_2 + 5x_3 \) and

\[
F = \sigma_1 L^4 + \sigma_2 L^4 + \sigma_3 L^4
\]

for \( \sigma_1 = e \), \( \sigma_2 = (23) \) and \( \sigma_3 = (13) \) as in Example II.

The coefficients of \( L \) give \( a = (1, 1, 5) \). The monomial \( x_1^2 = x^b \) has exponent \( b = (2, 0, 0) \). The formula from
Lemma 2 expresses $\theta_{F,2}(x_1^2)$ as
\[
\theta_{F,2}(x_1^2) = \frac{4!}{2!} \left( a^{\sigma_1^{-1}b} \sigma_1 L^2 + a^{\sigma_2^{-1}b} \sigma_2 L^2 + a^{\sigma_3^{-1}b} \sigma_3 L^2 \right)
\]
\[
= \frac{4!}{2!} \left( 1^{11} 0^5 0 \sigma_1 L^2 + 1^{2} 0^5 0 \sigma_2 L^2 + 1^0 1^{51} \sigma_3 L^2 \right)
\]
\[
= 12 \sigma_1 L^2 + 12 \sigma_2 L^2 + 300 \sigma_3 L^2
\]
\[
= 12(x_1 + x_2 + 5x_3)^2
\]
\[
+ 12(x_1 + 5x_2 + x_3)^2
\]
\[
+ 300(5x_1 + x_2 + x_3)^2.
\]

The above expression matches the definition of $\theta_{F,2}(x_1^2)$ as the second partial of $F$ with respect to $x_1$.

Similarly, the $x_1 x_2$-mixed partial of $F$ is
\[
\theta_{F,2}(x_1 x_2) = \frac{4!}{2!} \left( 1^{11} 1^5 0 \sigma_1 L^2 + 1^{11} 0^5 1 \sigma_2 L^2 + 1^0 1^{51} \sigma_3 L^2 \right)
\]
\[
= 12 \sigma_1 L^2 + 60 \sigma_2 L^2 + 60 \sigma_3 L^2
\]
\[
= 12(x_1 + x_2 + 5x_3)^2
\]
\[
+ 60(x_1 + 5x_2 + x_3)^2
\]
\[
+ 60(5x_1 + x_2 + x_3)^2.
\]

Recall that $\sigma_1, \ldots, \sigma_\ell$ are representatives for the left-cosets of $(\mathfrak{S}_n)_a$. We define the $\ell$-dimensional vector space $V$ as the span of these representatives of the left-cosets of the stabilizer of $a$. Then $V$ is a representation of $\mathfrak{S}_n$ where $\tau(\sigma_i) := \sigma_j$ if $\tau \sigma_i \in \sigma_j(\mathfrak{S}_n)_a$.

For any given degree $k$, we define $\phi_k : V \to S_k$ by setting
\[
\phi_k(\sigma_i) = \sigma_i L^k
\]
for $1 \leq i \leq \ell$ and extending linearly. The definition of $\phi_k$ does not depend on our choice of coset representatives since if $\tau \in \sigma_i(\mathfrak{S}_n)_a$, then $\tau = \sigma_i \gamma$ for some $\gamma \in (\mathfrak{S}_n)_a$ and hence $\tau L^k = \sigma_i \gamma L^k = \sigma_i L^k$. For similar reasons, $\phi_k$ is equivariant.

Let $\psi_k : R_k \to V$ be the linear map given by
\[
\psi_k(x^b) = \sum_{i=1}^{\ell} \binom{k}{b} a^{\sigma_i^{-1}b} \sigma_i.
\]

Example 4. Let $L = x_1 + x_2 + 5x_3$ and
\[
F = \sigma_1 L^4 + \sigma_2 L^4 + \sigma_3 L^4
\]
for $\sigma_1 = e$, $\sigma_2 = (23)$ and $\sigma_3 = (13)$ as in the previous two examples.
The map $\phi_2 : V \rightarrow S_2$ takes the following values on the basis $\{\sigma_1, \sigma_2, \sigma_3\}$ for $V$:

$$\phi_2(\sigma_1) = \sigma_1(x_1 + x_2 + 5x_3)^2$$

$$= x_1^2 + x_2^2 + 25x_3^2 + 2x_1x_2 + 10x_1x_3 + 10x_2x_3,$$

$$\phi_2(\sigma_2) = \sigma_2(x_1 + x_2 + 5x_3)^2$$

$$= x_1^2 + 25x_2^2 + x_3^2 + 10x_1x_2 + 2x_1x_3 + 10x_2x_3,$$

$$\phi_2(\sigma_3) = \sigma_3(x_1 + x_2 + 5x_3)^2$$

$$= 25x_1^2 + x_2^2 + 10x_1x_2 + 10x_1x_3 + 2x_2x_3.$$

The map $\psi_2 : R_2 \rightarrow V$ also takes the values,

$$\psi_2(x_1^2) = \frac{2!}{2010!} 1^{2} 0^{5} 5^{0} \sigma_1 + \frac{2!}{2010!} 1^{2} 0^{5} 5^{0} \sigma_2 + \frac{2!}{2010!} 1^{0} 1^{0} 5^{2} \sigma_3$$

$$= \sigma_1 + \sigma_2 + 25\sigma_3$$

and

$$\psi_2(x_1x_2) = \frac{2!}{11110!} 11115 \sigma_1 + \frac{2!}{11110!} 11105 \sigma_2 + \frac{2!}{11110!} 10115 \sigma_3$$

$$= 2\sigma_1 + 10\sigma_2 + 10\sigma_3.$$  

Continuing in this way, we get

$$\psi_2(x_1^2) = \sigma_1 + \sigma_2 + 25\sigma_3,$$

$$\psi_2(x_2^2) = \sigma_1 + 25\sigma_2 + \sigma_3,$$

$$\psi_2(x_3^2) = 25\sigma_1 + \sigma_2 + \sigma_3,$$

$$\psi_2(x_1x_2) = 2\sigma_1 + 10\sigma_2 + 10\sigma_3,$$

$$\psi_2(x_1x_3) = 10\sigma_1 + 2\sigma_2 + 10\sigma_3,$$

$$\psi_2(x_2x_3) = 10\sigma_1 + 10\sigma_2 + 2\sigma_3.$$

**Lemma 5.** For all $k \geq 0$, $\text{rank } \psi_k = \text{rank } \phi_k$.

**Proof.** The coefficient of $x^b \in S_k$ appearing in $\phi_k(\sigma_i)$ is equal to the coefficient of $\sigma_i^{-1}x^b = x^{\sigma_i^{-1}b}$ in $L^k$, which is $\binom{k}{b} a^{\sigma_i^{-1}b}$. If we order the monomial bases of $R_k$ and $S_k$ in the same manner, then the matrices of $\phi_k$ and $\psi_k$ are transposes of each other, and thus have the same rank.

**Lemma 6.** The map $\psi_k$ is equivariant.

**Proof.** For $\tau \in \mathfrak{S}_n$, $\tau \psi_k(x^b) = \sum_{i=1}^{\ell} \binom{k}{b} a^{\sigma_i^{-1}b} \tau \sigma_i$. In this expression, $\tau \sigma_i$ holds the place of the left coset $\tau \sigma_i(\mathfrak{S}_n)_a$. Let $\sigma_j$ be our chosen representative for this coset. Thus, $\tau \sigma_i(\mathfrak{S}_n)_a = \sigma_j(\mathfrak{S}_n)_a$ and hence $\tau \sigma_i = \sigma_j \gamma$ for some $\gamma \in (\mathfrak{S}_n)_a$. Rearranging we get $\sigma_i^{-1} = \gamma^{-1} \sigma_j^{-1} \tau$.
and therefore $a^{\sigma_i^{-1}b} = a^{\sigma_j^{-1}r_b}$, since $a^c = a^c$ for any $\gamma$ in the stabilizer of $a$ and any exponent vector $c$. Thus,

$$\tau \psi_k(x^b) = \sum_{i=1}^\ell \binom{k}{b} a^{\sigma_i^{-1}b} \tau \sigma_i = \sum_{j=1}^\ell a^{\sigma_j^{-1}r_b} \sigma_j = \psi_k(r_b)$$

□

Finally, let $\nu_k : R_k \to R_k$ be the non-singular equivariant linear scaling map defined by

$$\nu_k(x^b) = \binom{k}{b}^{-1} x^b = \frac{b_1! \cdots b_n!}{k!} x^b.$$

We thus have the following sequence of maps:

$$R_{d-k} \xrightarrow{\nu_{d-k}} R_{d-k} \xrightarrow{\psi_{d-k}} V \xrightarrow{\phi_k} S_k.$$

The following theorem gives the “factored” presentation of $\theta_{F,k}$ promised earlier.

**Theorem 7.** Let $F = \sum_{i=1}^\ell \sigma_i L^d \in S_d$ and let $\theta_{F,k} : R_{d-k} \to S_k$, $\nu_k$, $\phi_k$ and $\psi_k$ be as given above. For all $k$ with $0 \leq k \leq d$, we have

$$\theta_{F,k} = \frac{d!}{k!} \phi_k \circ \psi_{d-k} \circ \nu_{d-k}.$$

**Proof.** Applying the composition $\phi_k \circ \psi_{d-k} \circ \nu_{d-k}$ to a monomial $x^b \in R_{d-k}$ and using linearity gives

$$\phi_k \circ \psi_{d-k} \circ \nu_{d-k}(x^b) = \binom{d-k}{b} \phi_k \circ \psi_{d-k}(x^b) = \binom{d-k}{b} \sum_{i=1}^\ell \binom{d-k}{b} a^{\sigma_i^{-1}b} \phi_k(\sigma_i) = \sum_{i=1}^\ell a^{\sigma_i^{-1}b} \sigma_i L^k.$$

The result then follows from Lemma 2. □

**Example 8.** We continue our running example with

$$L = x_1 + x_2 + 5x_3,$$

$$F = \sigma_1 L^4 + \sigma_2 L^4 + \sigma_3 L^4$$

and with $\sigma_1 = e$, $\sigma_2 = (23)$ and $\sigma_3 = (13)$. The map $\nu_2$ takes the values $\nu_2(x_1^2) = \frac{2000}{2!} x_1^2 = x_1^2$ and $\nu_2(x_ix_j) = \frac{1110}{2!} = \frac{1}{2} x_i x_j$ for $i \neq j$. 
If we evaluate the composition $d_k \phi \circ \psi_{d-k} \circ \nu_{d-k}$, with $d = 4$ and $k = 2$, at $x_1^2$ and $x_1x_2$, we obtain

$$\frac{4!}{2!} \phi_2 \circ \psi_2 \circ \nu_2(x_1^2) = 12 \phi_2 \circ \psi_2(x_1^2)$$

$$= 12 \phi_2(\sigma_1 + \sigma_2 + 25\sigma_3)$$

$$= 12\sigma_1 L_1^2 + 12\sigma_2 L^2 + 300\sigma_3 L^2$$

$$= \theta_{F,2}(x_1^2)$$

and

$$\frac{4!}{2!} \phi_2 \circ \psi_2 \circ \nu_2(x_1x_2) = 12 \phi_2 \circ \psi_2(\frac{1}{2}x_1x_2)$$

$$= 12 \phi_2(\sigma_1 + 5\sigma_2 + 5\sigma_3)$$

$$= 12\sigma_1 L_1^2 + 60\sigma_2 L^2 + 60\sigma_3 L^2$$

$$= \theta_{F,2}(x_1x_2).$$

The Hilbert function of $R/I_F$ is equal to the Hilbert function of the image of $\theta_F$. Thus, to determine $HF_{R/I_F}(k)$ it suffices to find the rank of $\theta_{F,k} = \frac{d_k}{k!} \phi \circ \psi_{d-k} \circ \nu_{d-k}$. Since $\nu_{d-k}$ is non-singular, it suffices to determine the rank of the composition of $\phi_k$ and $\psi_{d-k}$. To that end, we need to examine the relationship between $\text{im} \psi_{d-k}$ and $\text{ker} \phi_k$.

Using the distinguished basis $\sigma_1, \ldots, \sigma_\ell$ of $V$, we can define the dot product of two vectors $v = c_1\sigma_1 + \cdots + c_\ell \sigma_\ell$ and $w = d_1\sigma_1 + \cdots + d_\ell \sigma_\ell$ to be $v \cdot w = c_1d_1 + \cdots + c_\ell d_\ell$.

In the following results, when we require $a$ to have real coordinates, we still allow $k$ to be one of $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.

**Proposition 9.** If the coordinates of $a$ are real then $\text{ker} \phi_k$ and $\text{im} \psi_k$ are orthogonal complements with respect to the dot product on $V$. In particular, their intersection is trivial.

**Proof.** Take $v \in \text{ker} \phi_k$ and $w \in \text{im} \psi_k$. If $v = \sum_{i=1}^\ell c_i \sigma_i$, then $\phi_k(v) = \sum_{i=1}^\ell c_i \sigma_i L^k = 0$. The coefficient of $x^b$ in $\sum_{i=1}^\ell c_i \sigma_i L^k$ is $\sum_{i=1}^\ell c_i \binom{k}{b} a^{\sigma_i^{-1} b}$ and this coefficient must be zero for each monomial $x^b$ of degree $k$.

Since $w \in \text{im} \psi_k$, there is a homogeneous polynomial $f = \sum_b r_b x^b$ of degree $k$ with $w = \psi_k(f) = \sum_b r_b \sum_{i=1}^\ell \binom{k}{b} a^{\sigma_i^{-1} b} \sigma_i$. Thus, $w = \psi_k(f)$...
\[ \sum_{i=1}^{\ell} d_i \sigma_i \] where \( d_i = \sum_b r_b \binom{k}{b} a^{\sigma^{-1}b} \). Since \( a \) is real, \( \overline{d_i} = d_i \). Therefore,

\[
v \cdot w = \sum_{i=1}^{\ell} c_i d_i \]

\[
= \sum_{i=1}^{\ell} c_i \sum_b r_b \binom{k}{b} a^{\sigma^{-1}b} \]

\[
= \sum_b r_b \sum_{i=1}^{\ell} c_i \binom{k}{b} a^{\sigma^{-1}b} \]

\[
= 0,
\]

since \( \sum_{i=1}^{\ell} c_i \binom{k}{b} a^{\sigma^{-1}b} = 0 \) for each \( x^b \) of degree \( k \). Furthermore, \( \phi_k \) and \( \psi_k \) have the same rank since their matrices are transposes of each other. Thus, \( \ker(\phi_k)^\perp = \im(\psi_k) \). \( \square \)

**Lemma 10.** If the coordinates of \( a \) are real and \( a_1 + \cdots + a_n \neq 0 \) then \( \ker(\phi_{k+1}) \subseteq \ker(\phi_k) \) and \( \im(\psi_k) \subseteq \im(\psi_{k+1}) \) for all \( k \geq 0 \).

**Proof.** Using Proposition 9, it suffices to show the containment of the kernels. If we suppose \( v = \sum_{i=1}^{\ell} c_i \sigma_i \in \ker(\phi_{k+1}) \), then \( \sum_{i=1}^{\ell} c_i \sigma_i L^{k+1} = 0 \). Therefore \( \partial(x_1 + \cdots + x_n, \phi_{k+1}(v)) \) is both zero and

\[
\partial(x_1 + \cdots + x_n, \phi_{k+1}(v)) = \sum_{i=1}^{\ell} c_i \partial(x_1 + \cdots + x_n, \sigma_i L^{k+1})
\]

\[
= \sum_{i=1}^{\ell} c_i (k+1)(a_1 + \cdots + a_n) \sigma_i L^k
\]

\[
= (a_1 + \cdots + a_n)(k+1) \phi_k(v).
\]

Since we have assumed that \( a_1 + \cdots + a_n \neq 0 \), we have \( v \in \ker(\phi_k) \). Thus, \( \ker(\phi_{k+1}) \subseteq \ker(\phi_k) \). \( \square \)

**Proposition 11.** If the coordinates of \( a \) are real and \( a_1 + \cdots + a_n \neq 0 \) then for any integer \( k \leq d/2 \), \( M_k, M_{d-k} \) and \( \im(\phi_k) \) are all equivalent representations.

**Proof.** Since \( \ker(\phi_j) \) and \( \im(\psi_j) \) are orthogonal complements, we have \( \ker(\phi_j) + \im(\psi_j) = V \) and \( \ker(\phi_j) \cap \im(\psi_j) = 0 \). If \( i \geq j \) then, by Lemma 10, \( \ker(\phi_i) \subseteq \ker(\phi_j) \) and hence \( \ker(\phi_i) \cap \im(\psi_j) = 0 \). If \( i < j \) then, by Lemma 10, \( \ker(\phi_j) \subseteq \ker(\phi_i) \) and hence \( \ker(\phi_i) + \im(\psi_j) = V \). \( \square \)

**Proposition 12.** If the coordinates of \( a \) are real and \( a_1 + \cdots + a_n \neq 0 \) then for any integer \( k \leq d/2 \), \( M_k, M_{d-k} \) and \( \im(\phi_k) \) are all equivalent representations.
Proof. Fix a non-negative integer $k$ for which $k \leq [d/2] \leq d - k$. As the kernels of the maps $(\phi_i)_{i \in \mathbb{N}}$ are decreasing in $i$ and the images of $(\psi_i)_{i \in \mathbb{N}}$ are increasing, we have $\ker \phi_{[d/2]} \subseteq \ker \phi_k$ and $\im \psi_{[d/2]} \subseteq \im \phi_{d-k}$. So, since $\ker \phi_{[d/2]} + \im \psi_{[d/2]} = V$, we have that $\ker \phi_k + \im \psi_{d-k} = V$.

Finally, $M_k = \im \theta_{F,k}$ and $\theta_{F,k} = \frac{d!}{k!} \phi_k \circ \psi_{d-k} \circ \nu_{d-k}$ by Theorem 7. Since $\nu_{d-k}$ is surjective, we see that $M_k$ can be expressed more simply as $M_k = \im (\phi_k \circ \psi_{d-k})$. Therefore we have the following chain of equivalences:

\[
M_k = \im (\phi_k \circ \psi_{d-k}) \\
\cong \im \psi_{d-k} / (\ker \phi_k \cap \im \psi_{d-k}) \\
\cong (\ker \phi_k + \im \psi_{d-k}) / \ker \phi_k \\
\cong V / \ker \phi_k \\
\cong \im \phi_k.
\]

As the graded character of $M$ is palindromic, $M_{d-k}$ and $M_k$ are also equivalent. \hfill \Box

Example 13. Let $L = x_1 + x_2 + 5x_3$ and $F = \sigma_1 L^4 + \sigma_2 L^4 + \sigma_3 L^4$

for $\sigma_1 = e, \sigma_2 = (23)$ and $\sigma_3 = (13)$ as in the previous examples.

Recall that $\chi^3, \chi^{21}$ and $\chi^{111}$ are the characters of the trivial, reduced-defining and alternating representations respectively. We will see in the next section that

\[
\chi_{\im \phi_k} = \begin{cases} 
\chi^3 & k = 0, \\
\chi^3 + \chi^{21} & k \geq 1.
\end{cases}
\]

Since the degree $d$ of $F$ is 4, Proposition 12 implies that the graded character of $M$ is

\[
\chi_M(t) = \chi^3 \\
+ (\chi^3 + \chi^{21})t \\
+ (\chi_3 + \chi^{21})t^2 \\
+ (\chi^3 + \chi^{21})t^3 \\
+ \chi^3 t^4.
\]

Proposition 14. Suppose that the coordinates of $a$ are real and $a_1 + \cdots + a_n \neq 0$. The Hilbert function of $M = \langle F \rangle$, the principal submodule
of $S$ generated by $F \in S_d$, is

$$HF_M(k) = \min(\text{rank}\, \phi_k, \text{rank}\, \phi_{d-k}).$$

**Proof.** The degree $k$ homogeneous component of $M$ is $M_k = \im \theta_k = \im(\phi_k \circ \psi_{d-k})$ by Theorem 7 and the fact that $\nu_{d-k}$ is surjective. Thus, $HF_M(k) = \dim_k M_k = \text{rank}(\phi_k \circ \psi_{d-k})$. Since $\ker \phi_k$ and $\im \psi_{d-k}$ are transverse, the rank of the composition of $\phi_k$ and $\psi_{d-k}$ is the minimum of their ranks. By Lemma 5, $\psi_{d-k}$ and $\phi_{d-k}$ have the same rank. □

**Example 15.** Let $L = x_1 + x_2 + 5x_3$ and

$$F = \sigma_1 L^4 + \sigma_2 L^4 + \sigma_3 L^4$$

for $\sigma_1 = e$, $\sigma_2 = (23)$ and $\sigma_3 = (13)$ as in the previous examples.

Since $X_{\im \phi_0} = X^3$, the map $\phi_0$ has rank $f^3 = 1$. Also, for $k \geq 1$, $X_{\im \phi_k} = X^3 + X^{21}$ and hence the map $\phi_k$ has rank $f^3 + f^{21} = 3$. By Proposition 14 with $d = 4$, the Hilbert series of $M$ (and also $R/I_F$) is

$$HS_M(t) = \sum_{k \geq 0} \dim_k M_k t^k = 1 + 3t + 3t^2 + 3t^3 + t^4.$$

For example, the coefficient of $t^4$ in the above series is $\dim_k M_4 = \min(\text{rank}\, \phi_4, \text{rank}\, \phi_0) = \min(3, 1) = 1$.

### 3. The Orbit of a Projective Point

From the results of the previous section, it suffices to know the character and dimension of each $\im \phi_k$ to determine the graded character and Hilbert function of $M$. In this section, we relate the image of the maps $\phi_k$ to the homogeneous coordinate ring of the $S_n$-orbit of a projective point. Through this connection, we will express the graded character of $M$ in terms of Kostka-Foulkes polynomials.

Recall that $L = a_1 x_1 + \cdots + a_n x_n$ is a fixed linear form. Let $p = [a_1 : \cdots : a_n] \in \mathbb{P}^{n-1}$ and $\sigma p = [a_{\sigma^{-1}(1)} : \cdots : a_{\sigma^{-1}(n)}]$ for $\sigma \in S_n$. The orbit of $p$ is the projective variety $X = \{\sigma p \mid \sigma \in S_n\}$. Its homogeneous coordinate ring is $k[X] = R/I_X$, a one-dimensional arithmetically Cohen-Macaulay ring, where $I_X$ is the ideal of homogeneous polynomials vanishing on $X$.

The following elementary lemma gives a sufficient condition for the projective orbit $X$ and the affine orbit $Y = \{\sigma a \mid \sigma \in S_n\}$ to have the same number of points.

**Lemma 16.** If $a_1 + \cdots + a_n \neq 0$ then $X$ and $Y$ contain the same number of points.
Proof. The set $X$ is obtained from $Y$ by identifying affine points which lie on the same line through the origin. Assume that $X$ and $Y$ do not have the same size, so there must be two distinct points $\sigma_i\mathbf{a}, \sigma_j\mathbf{a} \in Y$ which represent the same projective point. Thus, for $\tau = \sigma_j^{-1}\sigma_i$, $\mathbf{a}$ and $\tau\mathbf{a} = (a_{\tau^{-1}(1)}, \ldots, a_{\tau^{-1}(n)})$ are distinct points of $Y$, but are equal in $X$. So, there must be some non-zero $z \in \mathbb{C}$ with $a_i = za_{\tau^{-1}(i)}$ for all $1 \leq i \leq n$. If $i$ is contained in a cycle of $\tau^{-1}$ of length $m$ then $a_i = z a_{\tau^{-1}(i)} = z^2 a_{\tau^{-2}(i)} = \cdots = z^m a_i$. Thus, $z$ is an $m$-th root of unity. Also, since $\mathbf{a} \neq \tau\mathbf{a}$, we have $z \neq 1$. Therefore, the sum of the coordinates of $\mathbf{a}$ over the $m$-cycle of $\tau^{-1}$ containing $i$ is

$$a_i + a_{\tau^{-1}(i)} + \cdots + a_{\tau^{-(m-1)}(i)} = a_i + z^{-1} a_i + \cdots + z^{-(m-1)} a_i = a_i (1 + z^{-1} + \cdots + z^{-(m-1)}) = 0.$$  

Thus, by decomposing $\tau^{-1}$ into its cycles and summing over each cycle, we have expressed $a_1 + \cdots + a_n$ as a collection of disjoint sums which are all zero.  

Let $N = \bigoplus_{k \geq 0} N_k \subseteq S$ where $N_k = \text{im} \phi_k$. Since $\phi_k : V \rightarrow S_k$ is given by $\phi(\sigma_i) = \sigma_i L^k$, we have $N_k = \text{span}_k(\sigma_1 L^k, \ldots, \sigma_r L^k)$.

**Proposition 17.** The annihilator of $N$ is $I_X$ and, furthermore, $N$ and $k[X] = R/I_X$ are equivalent graded representations of $\mathfrak{S}_n$.

**Proof.** If $x^b$ is a monomial of degree $k$, then $\partial(x^b, L^k) = b! (\binom{k}{b} a^b = k! a^b)$ using the multinomial theorem. Thus, for an arbitrary polynomial $f \in R_k$, we have $\partial(f, L^k) = k! f(a)$ by linearity and hence

$$\partial(f, \sigma_i L^k) = \partial(\sigma_i^{-1} f, L^k) = k! (\sigma_i^{-1} f)(a) = k! f(\sigma_i a)$$

for all $1 \leq i \leq \ell$. By the definition of $I_X$, $f \in (I_X)_k$ if and only if $f(\sigma_i a) = 0$ for all $1 \leq i \leq \ell$. Thus, $f \in (I_X)_k$ if and only if $\partial(f, \sigma_i L^k) = 0$ for all $1 \leq i \leq \ell$ and hence $(I_X)_k = N^+_k = \text{Ann}(N)_k$ for all $k \geq 0$. Thus $I_X = \text{Ann}(N)$ as both are homogeneous ideals of $R$.

As mentioned in the introduction, the $\mathfrak{S}_n$-invariant perfect pairings $\partial_k : R_k \times S_k \rightarrow k$ induce equivalences between each $(R/\text{Ann}(N))_k$ and the dual representation of each $N_k$. Since the $N_k$ are self-dual, $R/I_X$ and $N$ are equivalent graded representations.  

**Proposition 18.** If $a_1 + \cdots + a_n \neq 0$ then $e_1 = \overline{a_1} + \cdots + \overline{a_n} \in k[X]$ is not a zero divisor. Consequently, $k[X]/(e_1) \cong \bigoplus_{i=0}^k (k[X]/(e_1))$, as representations for all $k \geq 0$.

**Proof.** If $e_1 f \in I_X$ for some $f \in R$, then $e_1(\sigma a) f(\sigma a) = 0$ for all $\sigma \in \mathfrak{S}_n$. As $e_1(\sigma a) = e_1(a) = a_1 + \cdots + a_n \neq 0$, we see that $f(\sigma a) = 0$
for all $\sigma \in \mathfrak{S}_n$. That is, $f \in I_X$. Consequently, $e_1$ is not a zero divisor of $k[X]$. Hence, $k[X]_k$ and $(e_1k[X]_{k-1}) \oplus (k[X]/(e_1))_k$ are equivalent representations. Also, $e_1k[X]_{k-1}$ and $k[X]_{k-1}$ are equivalent. Thus the result follows by induction. \hfill \Box

Again, let $Y = \{\sigma a \in A^n \mid \sigma \in \mathfrak{S}_n\}$ be the affine orbit of $a = (a_1, \ldots, a_n)$ and let $k[Y] = R/I_Y$ be its (inhomogeneous) coordinate ring. We define the associated graded algebra of $k[Y]$ to be

$$\text{gr}(k[Y]) = \bigoplus_{k \geq 0} k[Y]_{\leq k}/k[Y]_{<k-1}.$$  

Take a non-zero degree $k$ polynomial $f$ and express it as $f = f_k + \cdots + f_0$ where each $f_i$ is homogeneous of degree $i$ and $f_k \neq 0$. The leading form of $f$ is $\text{LF}(f) = f_k$. The associated graded algebra $\text{gr}(k[Y])$ is isomorphic to $R/\text{gr}(I_Y)$ where $\text{gr}(I_Y) = \{\text{LF}(f) \mid f \in I_Y\}$. One can see this as follows: two degree $k$ polynomials $f, g \in k[Y]_{\leq k}$ are equal modulo $k[Y]_{<k-1}$ if and only if the leading form of their difference is in $\text{gr}(I_Y)$. Thus, $k[Y]_{\leq k}/k[Y]_{<k-1}$ is isomorphic to $R_k/\text{gr}(I_Y)_k$. One can also check that this isomorphism is equivariant.

**Proposition 19.** If $a_1 + \cdots + a_n \neq 0$ then $k[X]/(e_1)$ and $\text{gr}(k[Y])$ are isomorphic graded algebras and equivalent representations of $\mathfrak{S}_n$.

**Proof.** It suffices to show that $I_X + (e_1) = \text{gr}(I_Y)$. Any non-zero element $f_k \in \text{gr}(I_Y)_k$ is the leading form of some polynomial $f = f_k + \cdots + f_0 \in I_Y$ where each $f_i \in R_i$ and $f_k \neq 0$. We homogenize $f$ with respect to $e_1(x)/e_1(a) = (x_1 + \cdots + x_n)/(a_1 + \cdots + a_n)$ to obtain

$$f' = f_k + \frac{e_1(x)}{e_1(a)} f_{k-1} + \cdots + \frac{e_1(x)^k}{e_1(a)^k} f_0.$$

As $f$ vanishes on $Y$, we see that the homogeneous polynomial $f'$ vanishes on $X$. Thus, $f' \in I_X$ and hence $f_k \in I_X + (e_1)$. This shows that $\text{gr}(I_Y) \subseteq I_X + (e_1)$.

Since $a_1 + \cdots + a_n \neq 0$, $X$ and $Y$ contain the same number of points. Thus, the artinian algebras $k[Y]$ and $k[X]/(e_1)$ must have the same vector space dimension and consequently $\text{gr}(I_Y) = I_X + (e_1)$. \hfill \Box

The content of the following proposition appears in other works (cf. [6, Theorem 4.5]), but we include it here for completeness.

**Proposition 20.** $k[Y]_{\leq k}$ and $\text{gr}(k[Y])_{\leq k}$ are equivalent representations.

**Proof.** Consider the following short exact sequence of representations:

$$0 \to k[Y]_{\leq k-1} \to k[Y]_{\leq k} \to k[Y]_{\leq k}/k[Y]_{<k-1} \to 0.$$
Since all short exact sequences of representations of finite groups split, $k[Y]_{\leq k}$ and $(k[Y]_{\leq k}/k[Y]_{\leq k-1}) \oplus k[Y]_{\leq k-1}$ are equivalent. So, by induction on $k$, $k[Y]_{\leq k}$ and $\text{gr}(k[Y])_{\leq k}$ are equivalent representations.

Connecting these equivalencies, we have shown that

$$N_k \cong k[X]_k \cong (k[X]/(e_1))_{\leq k} \cong \text{gr}(k[Y])_{\leq k} \cong k[Y]_{\leq k}.$$  

A remarkable fact, proved by Garsia and Procesi, is that $\text{gr}(k[Y])$ does not depend on the values of the coordinates of $a$, but simply on its associated partition $\mu$ [Remark 3.1]. In view of that, we will use

$$R_\mu = \text{gr}(k[Y])$$

to denote this algebra and $I_\mu = \text{gr}(I_Y)$ for the ideal appearing in its presentation as a quotient of $R$.

The algebra $R_\mu$ has a number of other descriptions. First, like any artinian algebra, $R_\mu$ is determined by its socle. The socle of $R_\mu$ is the unique irreducible representation of type $\mu$ which appears in the homogenous component $R_n(\mu)$ of degree

$$n(\mu) = \mu_2 + 2\mu_3 + \cdots + (r - 1)\mu_r.$$  

In fact, $R_n(\mu)$ is the lowest degree component of $R$ in which this irreducible representation occurs.

Originally, DeConcini and Procesi [4] defined the ring $R_\mu$ to be the cohomology ring of the variety of flags fixed by a unipotent matrix of shape $\mu = (\mu_1, \ldots, \mu_r)$. They showed that $R_\mu$ could be presented as a quotient of $Q[x_1, \ldots, x_n]$ by a homogeneous ideal $I_\mu$ and conjectured a set of generators for $I_\mu$. Tanisaki [17] conjectured a simpler set of generators for $I_\mu$ and, eventually, Weyman [18] proved these conjectures. Weyman also conjectured a minimal generating set for $I_\mu$, which Biagioli, Faridi and Rosas found to be minimal in some cases and redundant in others [3]. Garsia and Procesi used Tanisaki’s description of $I_\mu$ to show that $R_\mu = \text{gr}(k[Y])$ as previously mentioned.

We refer the reader to the introduction of [7] for the progression of papers that led to the graded character of $R_\mu$. As an ungraded representation, $R_\mu$ is equivalent to the representation afforded by the left cosets of the Young subgroup indexed by $\mu$ or, equivalently [11 §5.4], to the subrepresentation of $R_n(\mu)$ which is spanned by monomials of the form $\prod_{j=1}^{r} (x_{i_1, j} x_{i_2, j} \cdots x_{i_{j-1}, \mu_j})^{(j-1)}$ for distinct indices $i_{j,k} \in \{1, \ldots, n\}$.

The graded character of $R_\mu$ is given by

\begin{equation}
\chi_{R_\mu}(t) = \sum_{\lambda \in \mathcal{P}} K_{\lambda, \mu}(1/t)t^{n(\mu)}\chi^\lambda
\end{equation}
where \( K_{\lambda,\mu}(t) \in \mathbb{N}[t] \) are the Kostka-Foulkes polynomials [12 Chapter III.6]. If we apply the Frobenius character map \( \mathcal{F} \) to \( \chi_{R_\mu}(t) \) by replacing each \( \chi^\lambda \) with the Schur polynomial \( s_\lambda \), we get that
\[
\mathcal{F}(\chi_{R_\mu}(t)) = t^{n(\mu)}Q'_\mu(x_1, \ldots, x_n; t^{-1})
\]
where \( Q'_\mu(x_1, \ldots, x_n; t) \) is the modified Hall-Littlewood polynomial (see [5 §3]).

The twist in formula (\( \star \star \)) of having \( K_{\lambda,\mu}(1/t)t^{n(\mu)} \) where one might expect \( K_{\lambda,\mu}(t) \) makes the coefficient of \( K_{\lambda,\mu}(t) \) in degree \( k \) count the multiplicity of the irreducible representation of type \( \lambda \) that occurs in \( (R_\mu)_{n(\mu) - k} \). That is, exponents on \( t \) in \( K_{\lambda,\mu}(t) \) measure degrees down from the socle of \( R_\mu \), rather than up from the constants.

Since \( N_k \cong \text{gr}(\mathbb{k}[Y])_{\leq k} \cong (R_\mu)_{\leq k} \), the graded character of \( N \) is
\[
\chi_N(t) = \sum_{k \geq 0} \chi_{(R_\mu)_{\leq k}} t^k
\]
\[
= \frac{1}{1-t} \sum_{k \geq 0} \chi_{(R_\mu)_k} t^k
\]
\[
= \frac{1}{1-t} \chi_{R_\mu}(t).
\]

Finally, under the conditions of Proposition [12] we see that the graded character of \( M \),
\[
\chi_M(t) = \chi_{M_0} + \chi_{M_1} t + \chi_{M_2} t^2 + \cdots + \chi_{M_d} t^d,
\]
is determined by the values of \( \chi_{N_k} \) for \( k \leq d/2 \) since
\[
\chi_{M_k} = \chi_{N_k} = \chi_{M_{d-k}} \quad \text{for} \quad k \leq d/2.
\]
The graded character of \( R/I_F \) and \( M \) are equal.

**Example 21.** Our running example began with the form \( L = x_1 + x_2 + 5x_3 \) which has \( a = (1, 1, 5) \) for its coefficients and \( \mu = (2, 1) \) as its associated partition.

Using the table provided in [12 p.239], the Kostka-Foulkes polynomials \( K_{\lambda,\mu}(t) \) for \( \mu = (2, 1) \) are \( K_{3,\mu}(t) = t \) and \( K_{21,\mu}(t) = 1 \).

The socle of \( R_\mu \) is in degree \( n(\mu) = \mu_2 = 1 \). Thus,
\[
\chi_{R_\mu}(t) = K_{3,\mu}(1/t)t^{n(\mu)} X^3 + K_{21,\mu}(1/t)t^{n(\mu)} X^{211}
\]
\[
= X^3 + X^{211} t.
\]

Thus \( \text{im} \phi_0 = N_0 \cong (R_\mu)_0 \) has character \( \chi_{N_0} = X^3 \) and \( \text{im} \phi_k = N_k \cong (R_\mu)_{\leq k} \) has character \( \chi_{N_k} = X^3 + X^{211} \) for \( k \geq 1 \), as was asserted in Example [13].
We now work out the details of the example discussed in the introduction.

**Example 22.** Let \( F = \sum_{i=1}^{12} \sigma_i(x_1 + x_2 + 2x_3 + 3x_4)^7 \) as in the introduction. The coefficient vector \( a = (1, 1, 2, 3) \) of the linear form \( L = x_1 + x_2 + 2x_3 + 3x_4 \) has \( \mu = (2, 1, 1) \) as its associated partition. Using the table provided in [12, p.239], the Kostka-Foulkes polynomials \( K_{\lambda,\mu}(t) \) for \( \mu = (2, 1, 1) \) are

\[
\begin{align*}
K_{4,\mu}(t) &= t^3, \\
K_{31,\mu}(t) &= t + t^2, \\
K_{22,\mu}(t) &= t, \\
K_{211,\mu}(t) &= 1, \\
K_{1111,\mu}(t) &= 0.
\end{align*}
\]

The socle of \( R_{\mu} \) is in degree \( n(\mu) = 2\mu_2 + 2\mu_3 = 3 \). Thus, the graded character of \( R_{\mu} \) is

\[
\chi_{R_{\mu}}(t) = \sum_{\lambda \vdash n} K_{\lambda,\mu}(1/t)t^{n(\mu)}\chi^\lambda = \chi^4 + \chi^{31}t + (\chi^{31} + \chi^{22})t^2 + \chi^{211}t^3.
\]

The graded character of \( N \) is

\[
\chi_N(t) = \chi^4 + (\chi^4 + \chi^{31})t + (\chi^4 + 2\chi^{31} + \chi^{22})t^2 + (\chi^4 + 2\chi^{31} + \chi^{22} + \chi^{211})t^3(1 - t)^{-1}.
\]
By Proposition 12, $\chi_{M_k} = \chi_{M_{d-k}} = \chi_{N_k}$ for $0 \leq k \leq d/2$.

Thus, the graded character of $M$ (or $R/I_F$) is

$$
\chi_M(t) = \chi^4 + (\chi^4 + \chi^{31})t + (\chi^4 + 2\chi^{31} + \chi^{22})t^2 + (\chi^4 + 2\chi^{31} + \chi^{22} + \chi^{211})t^3 + (\chi^4 + 2\chi^{31} + \chi^{22} + \chi^{211})t^4 + (\chi^4 + 2\chi^{31} + \chi^{22})t^5 + (\chi^4 + \chi^{31})t^6 + \chi^4 t^7.
$$

The dimension $f^\lambda$ of the irreducible representation of type $\lambda$ can be found by counting the number of standard Young tableaux of shape $\lambda$. In particular, $f^4 = 1$, $f^{31} = 3$, $f^{22} = 2$, and $f^{211} = 3$. Thus, by substituting $f^\lambda$ for $\chi^\lambda$ in the graded character of $M$, we obtain its Hilbert series as

$$
\text{HS}_M(t) = 1 + 4t + 9t^2 + 12t^3 + 12t^4 + 9t^5 + 4t^6 + t^7.
$$

4. Further Work

The most broad question we propose is to determine the graded characters and Hilbert functions of level artinian quotients of $R$ by $\mathfrak{S}_n$-stable homogeneous ideals. A graded artinian algebra is level if its socle is contained in a single degree. The ring $R_\mu$ and the coinvariant algebra $R_{1^n}$, in particular, are the most well-studied algebras of this type. In the introduction, we mentioned the contributions of Bergeron, Garsia and Tesler [2], Roth [14], and Morita, Wachi and Watanabe [13] to this problem.

In this paper, we determined the graded characters of Gorenstein algebras whose socles were spanned by a single a symmetric polynomial $F$ that is the sum of the $\mathfrak{S}_n$-orbit of a power of a linear form (whose coefficients are real and do not sum to zero). It remains open to determine the graded character of $R/\text{Ann}(F)$ when $F$ is an arbitrary symmetric polynomial.

As mentioned in the introduction, every homogeneous symmetric polynomial $F \in S_d$ can be written as a linear combination of orbit sums of powers of linear forms $F_1, \ldots, F_m$ with $F_i = \sigma_1 L_{i1}^{d_i} + \cdots + \sigma_{\ell_i} L_{i\ell_i}^{d_i}$. We suggest that the graded character of $R/\text{Ann}(F)$ may depend on
the graded characters of $R/\text{Ann}(F_1), \ldots, R/\text{Ann}(F_m)$. If the linear forms determining each $F_i$ are chosen generically and $d$ is sufficiently large, we expect that the character of $R/\text{Ann}(F)$ will be the sum of the characters of $R/\text{Ann}(F_1), \ldots, R/\text{Ann}(F_m)$ in degrees where this is possible.

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