THEORY OF SQUARE-LIKE ABELIAN GROUPS IS DECIDABLE

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ABSTRACT. A group is called square-like if it is universally equivalent to its direct square. It is known that the class of all square-like groups admits an explicit first order axiomatization but its theory is undecidable. We prove that the theory of square-like abelian groups is decidable. This answers a question posed by D. Spellman.

INTRODUCTION

A group $G$ is called discriminating \cite{1} if every group separated by $G$ is discriminated by $G$. Here $G$ is said to separate (discriminate) a group $H$ if for any non-identity element (finite set of non-identity elements) of $H$ there is a homomorphism from $H$ to $G$ which does not map the element (any element of the set) to the identity. A group $G$ is discriminating iff $G$ discriminates $G^2$ \cite{1}. In particular, if $G$ embeds $G^2$ then $G$ is discriminating.

A group $G$ is called square-like \cite{5} if the groups $G^2$ and $G$ are universally equivalent. Any discriminating group is square-like \cite{4}. The notions of discriminating and square-like group were studied in \cite{1, 3, 4, 5, 6, 7, 8, 9}.

The class of square-like groups is first order axiomatizable \cite{5}, and the theory of the class is computably enumerable; an explicit first order axiom system was suggested in \cite{2, 3}, and also presented in \cite{5}. In \cite{5} square-like abelian groups were characterized in terms of Szmielew invariants.

The subclass of discriminating groups is not first order axiomatizable \cite{5}. Every square-like group is elementarily equivalent to a discriminating group \cite{3, 7}; so the class of square-like groups is the axiomatic closure of the class of discriminating groups.

The theory of square-like groups is undecidable \cite{3, 7}. The argument in \cite{7} is based on the obvious observation that any group embeds in a discriminating group, and so the universal theory of square-like groups coincide with the universal theory of all groups. The latter is undecidable because there exist finitely presented groups with unsolvable word problem. In \cite{3} a discriminating group that interprets the ring of integers is constructed; any theory that has the group as a model (and, in particular, the theory of square-like groups) is undecidable.

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The main result of the present paper is that the theory of square-like abelian groups is decidable. This answers a question posed by Dennis Spellman [12]. As a byproduct, we found characterizations of discriminating and square-like Szmielew groups.

1. Preliminaries

Here we collect some known definitions and facts we will use in the proofs.

Fact 1.1. [1] Proposition 1] A group $G$ is discriminating iff $G$ discriminates $G^2$. In particular, $G$ is discriminating if $G$ embeds $G^2$.

Fact 1.2. [1] Proposition 2] The direct product (restricted or not) of any family of discriminating groups is a discriminating group.

Fact 1.3. [1] Proposition 3] Any torsion-free abelian group is discriminating.

Fact 1.4. [4] Lemma 2.1] Any discriminating group is square-like.

Fact 1.5. [5] Theorem 3] The class of square-like groups is first order axiomatizable.

Fact 1.6. [3] Proposition 3.5] Any $\text{End}(G)$-invariant subgroup of a discriminating group $G$ is trivial or infinite.

Let $A$ be an abelian group. For a positive integer $n$ we denote

$$nA = \{na : a \in A\}, \quad A[n] = \{a \in A : na = 0\},$$

and write $\delta(A)$ for the largest divisible subgroup of $A$. We write $nA[k]$ for $(nA)[k]$. The subgroups $nA$, $A[n]$, $nA[k]$, and $\delta(A)$ are $\text{End}(A)$-invariant. We write $A^{(\kappa)}$ for the direct sum of $\kappa$ copies of $A$.

We write $\mathbb{Q}$ for the additive group of all rational numbers, and $\mathbb{Z}(p)$ for the additive group of rational numbers with denominator not divisible by a prime $p$. We write $\mathbb{Z}(n)$ for the cyclic group of order $n$, and $\mathbb{Z}(p^\infty)$ for the Prüfer $p$-group.

A Szmielew group is defined to be an abelian group of the form

$$(*) \quad \bigoplus_{p \text{ prime}} n \mathbb{Z}(p^n)^{\langle \kappa_p, n-1 \rangle} \oplus \mathbb{Z}(p^\infty)^{\langle \lambda_p \rangle} \oplus \mathbb{Z}(p)^{\langle \mu_p \rangle} \oplus \mathbb{Q}^{\langle \nu \rangle}$$

where $\kappa_{p,n-1}$, $\lambda_p$, $\mu_p$, $\nu$ are cardinals $\leq \omega$.

For a prime $p$, we call a Szmielew group of the form

$$\bigoplus_{n \geq 0} \mathbb{Z}(p^n)^{\langle \kappa_p, n-1 \rangle} \oplus \mathbb{Z}(p^\infty)^{\langle \lambda_p \rangle} \oplus \mathbb{Z}(p)^{\langle \mu_p \rangle} \oplus \mathbb{Q}^{\langle \nu \rangle}$$

a $p$-Szmielew group.

Fact 1.7. [11] Lemma A.2.3] Every abelian group is elementarily equivalent to a Szmielew group.
Let $p$ be a prime, and $n, k < \omega$. Let $\Phi_k(p, n)$ and $\Phi^k(p, n)$ be the sentences that say about an abelian group $B$ that
\[
\dim_p(p^nB[p]/p^{n+1}B[p]) = k \quad \text{and} \quad \dim_p(p^nB[p]/p^{n+1}B[p]) > k,
\]
$\Theta_k(p, n)$ and $\Theta^k(p, n)$ be the sentences that say that
\[
\dim_p(p^nB[p]) = k \quad \text{and} \quad \dim_p(p^nB[p]) > k,
\]
$\Gamma_k(p, n)$ and $\Gamma^k(p, n)$ be the sentences that say that
\[
\dim_p(p^nB/p^{n+1}B) = k \quad \text{and} \quad \dim_p(p^nB/p^{n+1}B) > k,
\]
$\Delta_k(p, n)$ and $\Delta^k(p, n)$ be the sentences that say that
\[
|p^nB| = k \quad \text{and} \quad |p^nB| > k.
\]
The sentences defined above are called the Szmielew invariant sentences.

Fact 1.8. [11, Section A.2] If $A$ is the Szmielew group $(\ast)$ then
- $A \models \Phi_k(p, n)$ iff $\kappa_{p, n} = k$,
- $A \models \Phi^k(p, n)$ iff $\kappa_{p, n} > k$,
- $A \models \Theta_k(p, n)$ iff $\lambda_p + \kappa_{p, n} + \kappa_{p, n+1} + \cdots = k$,
- $A \models \Theta^k(p, n)$ iff $\lambda_p + \kappa_{p, n} + \kappa_{p, n+1} + \cdots > k$,
- $A \models \Gamma_k(p, n)$ iff $\mu_p + \kappa_{p, n} + \kappa_{p, n+1} + \cdots = k$,
- $A \models \Gamma^k(p, n)$ iff $\mu_p + \kappa_{p, n} + \kappa_{p, n+1} + \cdots > k$.

Fact 1.9. [11, Theorem A.2.7] Every sentence of the first order language of abelian groups is equivalent, modulo the theory of abelian groups, to a positive Boolean combination of Szmielew invariant sentences.

Fact 1.10. [11, Theorem A.2.7] Two abelian groups are elementarily equivalent iff they satisfy the same Szmielew invariant sentences.

Abusing terminology, we call a sentence of the language of abelian groups consistent if it is true in some abelian group. By Fact 1.7, a sentence is consistent iff it holds in some Szmielew group.

Fact 1.11. [11, Theorem A.2.8] There is an algorithm that, given a finite conjunction of Szmielew invariant sentences, decides whether it holds in some Szmielew group.

Facts 1.9 and 1.11 are main ingredients of a proof of the Szmielew theorem on decidability of the theory of abelian groups; actually, they immediately imply the result. Indeed, given a sentence $\phi$, by Fact 1.9 and computable enumerability of the theory of abelian groups, we can effectively find a positive Boolean combination $\theta$ of Szmielew invariant sentences that is equivalent to $\neg \phi$, modulo the theory. A sentence $\phi$ is not in the theory iff $\theta$ is consistent; the latter can be effectively checked, by Fact 1.11.
We will use a similar method in our proof of decidability of the theory of square-like abelian groups.

2. Discriminating and square-like Szmielew groups

Let \( A \) be the Szmielew group \((*)\). For a prime \( p \), let \( I_p = \{ n : \kappa_{p,n-1} > 0 \} \).

In case when the set \( I_p \) is finite and nonempty, \( l_p \) denotes its maximal element; clearly, \( \kappa_{p,l_p-1} > 0 \).

**Proposition 2.1.** The following are equivalent:

1. \( A \) is discriminating;
2. for any prime \( p \) one of the following holds:
   i. \( \lambda_p = \omega \),
   ii. \( \lambda_p = 0 \), and if \( I_p \) is finite and nonempty then \( \kappa_{p,l_p-1} = \omega \).

**Proof.** (1)⇒(2). Suppose (1). Let \( p \) be a prime. The subgroup \( \delta(A) \cap A[p] \) is \( \text{End}(A) \)-invariant, and hence is trivial or infinite, by Fact 1.6. Then \( \lambda_p \) is 0 or \( \omega \). Suppose \( \lambda_p = 0 \), and \( I_p \) is finite and nonempty. Then the \( \text{End}(A) \)-invariant subgroup \( p^{l_p-1}A[p] \) is nontrivial and hence infinite, again by Fact 1.6. Then \( \kappa_{p,l_p-1} = \omega \).

(2)⇒(1). Suppose (2). Then for any prime \( p \) the group

\[
\bigoplus_{n>0} \mathbb{Z}(p^n)_{\kappa_{p,n-1}} \oplus \mathbb{Z}(p^\infty)_{\lambda_p}
\]

embeds it square. So \( A = B \oplus C \), where \( B \) embeds \( B^2 \), and \( C \) is torsion-free. By Facts 1.1, 1.3, and 1.2, \( A \) is discriminating. \( \square \)

**Proposition 2.2.** The following are equivalent:

1. \( A \) is square-like;
2. for any prime \( p \) one of the following holds:
   i. \( \lambda_p = \omega \),
   ii. \( \lambda_p = 0 \), and if \( I_p \) is finite and nonempty then \( \kappa_{p,l_p-1} = \omega \),
   iii. \( 0 < \lambda_p < \omega \), and \( I_p \) is infinite.

**Proof.** (1) ⇒ (2). Suppose (2) fails. Then, for some prime \( p \), (i), (ii), (iii) all fail. There are two possibilities:

(a) \( \lambda_p = 0 \), the set \( I_p \) is finite, nonempty, and \( \kappa_{p,l_p-1} < \omega \),
(b) \( 0 < \lambda_p < \omega \), and the set \( I_p \) is finite.

Suppose (a). Let \( \kappa = \kappa_{p,l_p-1} \). We have

\[
|p^{l_p-1}A[p]| = p^\kappa, \quad |p^{l_p-1}A^2[p]| = p^{2\kappa}.
\]

Suppose (b). Put \( l = l_p \) if \( I_p \neq \emptyset \), and \( l = 0 \) otherwise. We have

\[
|p^lA[p]| = p^{\lambda_p}, \quad |p^lA^2[p]| = p^{2\lambda_p}.
\]

For any positive integers \( s \) and \( t \) there is an existential sentence that says about an abelian group \( B \) that \( |sB[p]| \geq t \). Therefore in both cases (a) and (b) the groups \( A \) and \( A^2 \) are not universally equivalent, and so (1) fails.
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(2) ⇒ (1). Suppose (2). Let \( A' \) be the Szmielew group obtained from \( A \) by replacing
\[
\bigoplus_{n>0} \mathbb{Z}(p^n)^{(\kappa_{p,n-1})} \oplus \mathbb{Z}(p^\infty)^{(\lambda_p)}
\]
with
\[
\bigoplus_{n>0} \mathbb{Z}(p^n)^{(\kappa_{p,n-1})},
\]
for all \( p \) satisfying (3). Then \( A' \) is discriminating, by Proposition 2.1. Hence \( A' \) is square-like, by Fact 1.4. It is easy to check that \( A \) and \( A' \) satisfy the same Szmielew invariant sentences; therefore, by Fact 1.10, \( A \equiv A' \). Then, by Fact 1.5, the group \( A \) is square-like, too. □

Corollary 2.3. Any square-like abelian group is elementarily equivalent to a discriminating Szmielew group.

Proof. Let \( B \) be a square-like abelian group. By Fact 1.7, \( B \) is elementarily equivalent to a Szmielew group \( A \). By Fact 1.5, \( A \) is square-like. The argument at the end of the proof of Proposition 2.2 shows that \( A \) is elementarily equivalent to a discriminating Szmielew group \( A' \). □

3. Main result

Theorem 3.1. The theory of square-like abelian groups is decidable.

Proof. We need to find an algorithm which, given a sentence \( \phi \) of the language of abelian groups, decides whether \( \phi \) is true in some square-like abelian group, or, equivalently by Corollary 2.3, in some discriminating Szmielew group. By Fact 1.9, \( \phi \) is equivalent, modulo the theory of abelian groups, to a positive Boolean combination \( \theta \) of Szmielew invariant sentences. Since the theory of abelian groups is computably enumerable, \( \theta \) can be found effectively. We may assume that \( \theta = \bigvee_i \theta_i \), where each \( \theta_i \) is a conjunction of finitely many Szmielew invariant sentences. So it suffices to prove

Claim. There exists an algorithm that, given a consistent conjunction \( \psi \) of finitely many Szmielew invariant sentences, decides whether \( \psi \) holds in some discriminating Szmielew group.

For a prime \( p \), we call a conjunction of formulas of the forms
\[
\Phi_k(p,n), \Theta_k(p,n), \Gamma_k(p,n), \Delta_k(p,n), \\
\Phi^k(p,n), \Theta^k(p,n), \Gamma^k(p,n), \Delta^k(p,n)
\]
a \( p \)-conjunction. To prove the Claim, we show that
(A) there exists an algorithm that, given a prime \( p \) and a consistent \( p \)-conjunction \( \psi \), decides whether \( \psi \) holds in some discriminating \( p \)-Szmielew group, and
(B) the Claim follows from (A).
First we show (B): assuming (A), we prove the Claim.

Let $\psi$ be a conjunction of Szmielew invariant sentences, which holds in a Szmielew group $A$. We have $\psi = \bigwedge_p \psi_p$, where $p$ runs over a finite set of primes, and $\psi_p$ is a $p$-conjunction. There are three possibilities:

(a) $\psi$ has no conjuncts of the form $\Delta_k(p, n)$;

(b) $\psi$ has some conjuncts $\Delta_k(p, n)$ and $\Delta_l(q, m)$ with $p \neq q$;

(c) $\psi$ has a conjunct $\Delta_k(p, n)$, but has no conjuncts $\Delta_l(q, m)$ with $p \neq q$.

The following three lemmas prove (B).

**Lemma 3.2.** Assume (a). The following are equivalent:

(i) $\psi$ holds in some discriminating Szmielew group.

(ii) for all $p$ the sentence $\psi_p$ holds in some discriminating $p$-Szmielew group.

**Proof.** Suppose (i). We have $A = \bigoplus_p A(p)$, where $A(p)$ is a $p$-Szmielew group. Let $p$ be a prime. Then $A(p) \oplus \mathbb{Q}$ is a discriminating $p$-Szmielew group, by Proposition 2.1. Also, $A(p) \oplus \mathbb{Q} \models \psi_p$ because of (a). So (ii) holds.

Suppose (ii). For every prime $p$ choose a discriminating $p$-Szmielew group $A(p)$ in which $\psi_p$ holds. By Proposition 2.1, the Szmielew group $A = \bigoplus_p A(p)$ is discriminating. For every $p$ we have $A \models \psi_p$, because $A(p) \models \psi_p$ and $\psi$ satisfies (a). Therefore $A \models \psi$. So (i) holds. □

**Lemma 3.3.** Let $B$ be a discriminating abelian group.

(1) If $\Delta_k(p, n)$ or $-\Delta_k(p, n)$ holds in $B$ then $p^n B = 0$.

(2) Assume (b). If $B \models \psi$ then $B = 0$.

**Proof.** (1) The subgroup $p^n B$ is End($B$)-invariant and finite of order at most $k$. By Fact 1.6, the result follows.

(2) By (1), $p^n B = q^m B = 0$, and hence $B = 0$. □

Thus, for any $\psi$ with (b), in order to decide whether there is a discriminating Szmielew group that satisfies $\psi$, we need to decide whether $\psi$ holds in the trivial group, which can be done effectively.

**Lemma 3.4.** Assume (c). Then $\psi$ holds in some discriminating Szmielew group if and only if

(i) For any $q \neq p$ and $l > 0$, in $\psi$ there are no conjuncts of the forms $\Phi_l(q, m)$, $\Theta_l(q, m)$, $\Gamma_l(q, m)$, $\Phi_l(q, m)$, $\Theta_l(q, m)$, $\Gamma_l(q, m)$;

(ii) For any $q \neq p$, in $\psi$ there are no conjuncts of the forms $\Phi^0(q, m)$, $\Theta^0(q, m)$, $\Gamma^0(q, m)$;

(iii) the $p$-conjunction $\psi_p \land \bigwedge \{\Delta_s(p, 0) : s \in S\}$ holds in some discriminating $p$-Szmielew group, where $S$ is the set of all $s$ such that $\Delta_s(q, m)$ is a conjunct of $\psi$, for some $q \neq p$ and some $m$. 
Proof. First suppose that \( \psi \) holds in a discriminating Szmielew group \( A \). By (c) and Lemma 3.3(1), \( p^nA = 0 \), and so \( A \) is a \( p \)-Szmielew group. Therefore (i) and (ii) hold. Let \( s \in S \). Then for some \( m \) and \( q \neq p \) we have \( A \models \Delta^k(q, m) \), that is, \( |q^mA| > s \). As \( p^nA = 0 \), we have \( q^mA = A \); thus \( |A| > s \). Then \( A \models \Delta^k(p, 0) \). So (iii) holds.

Now suppose (i)–(iii) hold. By (iii) there is a discriminating \( p \)-Szmielew group \( A \) in which \( \psi_p \) and \( \{ \Delta^k(p, 0) : s \in S \} \) are true. We show that \( A \models \psi \). Since \( \Delta_k(p, n) \) is a conjunct of \( \psi \), we have \( p^nA = 0 \), by Lemma 3.3(1). As \( A \) is a \( p \)-Szmielew group, all the sentences \( \Phi_0(q, m), \Theta_0(q, m), \Gamma_0(q, m) \) with \( q \neq p \) hold in \( A \). Due to (i) and (ii), it remains to show that if \( \Delta^k(q, m) \) is a conjunct of \( \psi \), where \( q \neq p \), then it holds in \( A \). Suppose not. Then \( q^mA = 0 \), by Lemma 3.3(1). Therefore \( A = 0 \), contrary to \( A \models \Delta^k(p, 0) \). \(\square\)

Now we prove (A). From now on, let \( p \) be a fixed prime, and \( \psi \) be a \( p \)-conjunction which holds in some Szmielew group \( A \). We will show how to decide whether \( \psi \) holds in some discriminating \( p \)-Szmielew group.

There are four possibilities:

(a) \( \psi \) has a conjunct \( \Delta_k(p, n) \) with \( k \neq 1 \);
(b) \( \psi \) has a conjunct \( \Theta_k(p, n) \) with \( k > 0 \);
(c) \( \psi \) has no conjuncts of the forms \( \Delta_k(p, n) \) and \( \Theta_k(p, n) \);
(d) \( \psi \) has a conjunct \( \Delta_1(p, n) \) or \( \Theta_0(p, n) \), but (a) and (b) fail.

Lemma 3.5. If (a) then \( \psi \) fails in every discriminating abelian group.

Proof. Suppose \( \psi \) holds in an abelian group \( B \). Then \( |p^nB| = k \neq 1 \), and so \( p^nB \) is a nontrivial finite \( \text{End}(B) \)-invariant subgroup. Therefore \( B \) is not discriminating, by Fact 1.6. \(\square\)

Lemma 3.6. If (b) then \( \psi \) fails in every discriminating Szmielew group.

Proof. Suppose \( A \models \psi \), and \( A \) is a discriminating Szmielew group. Then

\[ \omega > k = \lambda_p + \kappa_{p,n} + \kappa_{p,n+1} + \ldots. \]

Hence \( \lambda_p < \omega \) and so, by Proposition 2.4, \( \lambda_p = 0 \). Then

\[ 0 < \kappa_{p,n} + \kappa_{p,n+1} + \ldots < \omega, \]

and so \( I_p \) is finite. Then we have \( n < l_p \), and \( \kappa_{p, l_p - 1} < \omega \). In this case \( A \) is not discriminating, by Proposition 2.4. A contradiction. \(\square\)

Lemma 3.7. If (c) then \( \psi \) holds in some discriminating \( p \)-Szmielew group.

Proof. We have \( A = \oplus_q A(q) \), where \( A(q) \) is a \( q \)-Szmielew group. Put

\[ A'(p) := A(p) \oplus \mathbb{Z}(p^\infty)^{(\omega)}. \]

By Proposition 2.4, \( A'(p) \) is a discriminating \( p \)-Szmielew group. Moreover, \( A'(p) \models \psi \). Indeed, for any sentence \( \theta \) of one of the forms

\[ \Phi_k(p, n), \Phi^k(p, n), \Theta^k(p, n), \Gamma_k(p, n), \Gamma^k(p, n), \Delta^k(p, n) \]

if \( A \models \theta \) then \( A'(p) \models \theta \). \(\square\)
It remains to consider case (d). We will need

**Lemma 3.8.** For any $n \geq k$ the sentence $\Gamma_l(p, k)$ is effectively equivalent in abelian groups to a positive Boolean combination of sentences of the forms $\Gamma_i(p, n)$ and $\Phi_j(p, s)$, where $k \leq s < n$ and $0 \leq i, j \leq l$.

**Proof.** It suffices to show that in abelian groups $\Gamma_l(p, k)$ is equivalent to

$$\Gamma_l'(p, k) := \bigvee_{i=0}^{l} (\Gamma_{l-i}(p, k+1) \land \Phi_i(p, k)).$$

A Szmielew group $A$ satisfies $\Gamma_l(p, k)$ if and only if

$$\mu_p + \kappa_{p,k} + \kappa_{p,k+1} + \cdots = l;$$

the latter holds if and only if, for some $i \in \{0, 1, \ldots, l\}$,

$$\mu_p + \kappa_{p,k+1} + \kappa_{p,k+2} + \cdots = l - i \quad \text{and} \quad \kappa_{p,k} = i,$$

which means that $\Gamma_l'(p, k)$ holds in $A$. \(\Box\)

Let $n < \omega$ be given. Replace in $\psi$ every conjunct $\Gamma_l(p, k)$, where $k < n$, with an equivalent positive Boolean combination of sentences of the forms $\Gamma_i(p, n)$ and $\Phi_j(p, s)$. The resulting formula is equivalent to a disjunction of $p$-conjunctions in each of which there is no conjunct $\Gamma_l(p, k)$ with $k < n$. Therefore it remains to prove the following statement, which allows to decide whether $\psi$ holds in some discriminating $p$-Szmielew group, in case (d).

**Lemma 3.9.** Suppose that $\psi$ has

(a) a conjunct $\Delta_1(p, n)$ or $\Theta_0(p, n)$;
(b) no conjuncts $\Delta_k(p, m)$ with $k \neq 1$ and $\Theta_k(p, m)$ with $k > 0$;
(c) no conjuncts $\Gamma_l(p, s)$ with $s < n$.

Then the following are equivalent:

1. $\psi$ fails in any discriminating $p$-Szmielew group;
2. there exist $m$ with $m < n$ and $i > 0$ such that
   (i) $\Phi_i(p, m)$ is a conjunct of $\psi$;
   (ii) for every $k$ with $m < k < n$ there is $j$ such that $\Phi_j(p, k)$ is a conjunct of $\psi$.

**Proof.** First we show that (b) implies that $\psi$ holds in some $p$-Szmielew group. If $\Delta_1(p, n)$ is in $\psi$ then $p^nA = 0$; therefore $A$ is a direct sum of cyclic $p$-groups and hence a $p$-Szmielew group. Suppose $\Delta_1(p, n)$ is not in $\psi$. Let $A = \oplus_q A(q)$, where each $A(q)$ is a $q$-Szmielew group. Since $\psi$ is a $p$-conjunction without conjuncts of the form $\Delta_k(p, n)$, the $p$-Szmielew group $A^{(p)} \oplus \mathbb{Q}$ satisfies $\psi$.

So we may assume that $A$ is a $p$-Szmielew group. By (a),

$$\lambda_p = \kappa_{p,n} = \kappa_{p,n+1} \cdots = 0.$$

Indeed, if $\Delta_1(p, n)$ is in $\psi$ then $p^nA = 0$; if $\Theta_0(p, n)$ is in $\psi$ then

$$0 = \lambda_p + \kappa_{p,n} + \kappa_{p,n+1} + \cdots.$$
In particular, the set $I_p$ is finite.

Suppose (2). Due to (i), we have $\kappa_{p,m} = i > 0$, and therefore $m < l_p \leq n$. Let $m < k < n$. By (ii) $\psi$ has a conjunct $\Phi_j(p,k)$; then $\kappa_{p,k} = j$. So $\kappa_{p,k} < \omega$ for all $k$ with $m \leq k < n$. In particular, $\kappa_{p,l_p-1} < \omega$. By Proposition 2.1 in this case $A$ cannot be discriminating, and (1) follows.

Assuming that (2) is not true, we show that (1) is not true, too. If $I_p = \emptyset$ then $A$ itself is discriminating, by Proposition 2.1.

Suppose $I_p \neq \emptyset$. First we show that there is $k < n$ such that $\kappa_{p,r} = 0$ for $r > k$, and for every $j$ the sentence $\Phi_j(p,k)$ is not a conjunct of $\psi$. Let $m = l_p - 1$ and $i = \kappa_{p,m}$. Then $m < n$ and $i > 0$. If (i) fails, put $k := m$. If (i) holds then (ii) fails, and therefore there is $k$ with $m < k < n$ such that for every $j$ the sentence $\Phi_j(p,k)$ is not a conjunct of $\psi$.

By Proposition 2.1 the $p$-Szmielew group $A \oplus \mathbb{Z}(p^{k+1})(\omega)$ is discriminating. Moreover,

$$A \oplus \mathbb{Z}(p^{k+1})(\omega) \models \psi.$$

Indeed, by (c) and the choice of $k$, a conjunct $\theta$ of $\psi$ can have only the forms

$$\Phi_j(p,r), \Theta_0(p,n), \Gamma_j(p,s), \Delta_1(p,n),$$

where $r \neq k$ and $s \geq n$, or the forms

$$\Phi^i(p,t), \Theta^i(p,t), \Gamma^i(p,t), \Delta^i(p,t).$$

Therefore $A \models \theta$ implies $A \oplus \mathbb{Z}(p^{k+1})(\omega) \models \theta$, for all such $\theta$. Here we use that $s \geq n > k$ when consider $\theta$ of the forms $\Theta_0(p,n)$ and $\Gamma_j(p,s)$. □

The proof of Theorem 3.1 is completed. □

4. Open questions

**Proposition 4.1.** The theory of square-like nilpotent groups is undecidable.

**Proof.** In fact, even the universal theory of square-like nilpotent groups is undecidable. Indeed, it coincides with the universal theory of nilpotent groups because any nilpotent group $G$ embeds in the discriminating nilpotent group $G^\omega$. As any finitely generated nilpotent group is residually finite, the universal theory of nilpotent groups coincides with the universal theory of finite nilpotent groups. The latter is undecidable [10]. □

**Question.** Is the theory of square-like 2-step nilpotent groups undecidable?

Note that the universal theory of square-like 2-step nilpotent groups is decidable. Indeed, as above, it coincides with the universal theory of 2-step nilpotent groups and with the universal theory of finite 2-step nilpotent groups. Obviously, the universal theory of 2-step nilpotent groups is computably enumerable, and the universal theory of finite 2-step nilpotent groups is co-computably-enumerable; so the result follows.

Thus, undecidability of the theory of square-like 2-step nilpotent groups cannot be shown like in the proof of Proposition 4.1. In [6, Theorem 5.1] we
proved undecidability of the theory of square-like groups by constructing a discriminating group which interprets the ring of integers.

**Question.** Is there a discriminating 2-step nilpotent group which interprets the ring of integers?

Existence of such a group would imply undecidability of the theory of square-like 2-step nilpotent groups.

**References**

[1] G. Baumslag, A. G. Myasnikov and V. N. Remeslennikov, Discriminating and co-discriminating groups, *J. Group Theory* 3 (2000), 467–479.

[2] O. Belegradek, Review of [5], *Math. Reviews*, MR1914831 (2003d: 20003).

[3] O. Belegradek, Discriminating and square-like groups, *J. Group Theory* 7 (2004), 521–532.

[4] B. Fine, A. M. Gaglione, A. G. Myasnikov and D. Spellman, Discriminating groups, *J. Group Theory* 4 (2001), 463–474.

[5] B. Fine, A. M. Gaglione, A. G. Myasnikov and D. Spellman, Groups whose universal theory is axiomatizable by quasi-identities, *J. Group Theory* 5 (2002), 365–381.

[6] B. Fine, A. M. Gaglione, D. Spellman, Every abelian group universally equivalent to a discriminating group is elementarily equivalent to a discriminating group, in *Combinatorial and geometric group theory*, Contemp. Math. 296 (Amer. Math. Soc., Providence, RI, 2002), 129–137.

[7] B. Fine, A. M. Gaglione, D. Spellman, The axiomatic closure of the class of discriminating groups, *Arch. Math.* 83 (2004), 106–112.

[8] B. Fine, A. M. Gaglione, D. Spellman, Discriminating and square-like groups. I. Axiomatics, in *Groups, statistics and cryptography*, Contemp. Math. 360 (Amer. Math. Soc., Providence, RI, 2004), 35–46.

[9] B. Fine, A. M. Gaglione, D. Spellman, Discriminating and square-like groups. II. Examples, *Houston J. Math.* 31 (2005), 649–674.

[10] O. G. Kharlampovich, Universal theory of the class of finite nilpotent groups is undecidable, *Math. Notes* 33 (1983), 254–263.

[11] W. Hodges, *Model theory*, Cambridge University Press, 1993.

[12] D. Spellman, Private communication, March 14, 2005.

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