Relativistic quantum protocols: “Bit Commitment” and “Coin Tossing”

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Abstract

The relativistic quantum protocols realizing the bit commitment and distant coin tossing schemes are proposed. The protocols are based on the fact that the non-stationary orthogonal extended quantum states cannot be reliably distinguished if they are not fully accessible for the measurement. As the states propagate from the domain controlled by one of the user to the domain accessible for the measurements performed by the other user, they become reliably distinguishable for the second user. Important for the protocol are both the quantum nature of the states and the existence of a finite maximum speed of the signal propagation imposed by the special relativity.

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1 Introduction

Many cryptographic problems reduce to a number of primitive cryptographic exchange protocols, such as the secret key distribution [1–3], bit commitment [4–6], and distant coin tossing [7] protocols. The bit commitment protocol is stronger than the distant coin tossing one in the sense that a distant coin tossing protocol can be formulated on the basis of a bit commitment protocol.

Informally, the bit commitment protocol is usually formulated in the following way. The protocol involves two participants (users) called A and B. At the commitment stage the user A chooses the value of a secret bit (0 or 1) and sends some information about his choice to user B in such a way that using the information provided by user A the user B cannot reliably determine the secret bit value chosen by A. To be more precise, in the ideal case the probability for user B to correctly identify the bit value chosen by user A is exactly 1/2 (i.e., it is equal to the probability of simply guessing the bit value) regardless of whether or not he uses the information supplied by user A. Then at the disclosure stage the user B can ask user A to provide him the rest information on the value of the chosen secret bit so that in the ideal case the user B reliably recovers the secret bit value. In addition, there should be no possibility for user A to change his mind and alter the chosen bit value after the commitment stage and before the disclosure state without being caught by user B.

The distant coin tossing protocol is formulated in the following way. The two distant users A and B, who do not trust each other and can employ any physically realizable opportunities to cheat, should exchange appropriate information so that at the end of the protocol (in the ideal case with the unit probability) they accept the arising bit as an honest lot. If the users have only access to the classical communication channel, the problem can even seem unsolvable.

Obviously, realization of any bit commitment protocol can be used to construct a distant coin tossing protocol. To achieve this purpose, user B can try and guess the bit chosen user A after the commitment stage but before the disclosure stage (remember that after the disclosure stage the secret bit chosen by user A is publicly known). The user B wins if he guesses the bit value chosen by user A and loses otherwise.

The following protocol is sometimes described as a simple example of the bit commitment protocol. User A writes down the chosen bit value on a paper sheet and places it into a safe which is then sent to user B (commitment stage) without the key which is only given to user B at the disclosure stage. In spite of the simplicity of the above example, it contains all the basic features of the protocols based on the classical information carriers. In this example, the user B obtains the complete rather than partial information on the secret bit at the commitment stage. Therefore, the laws of nature do not prohibit the user B to learn the secret bit value even before the disclosure stage if he has access to sufficient technical resources. A similar situation takes place in the protocols based on the computational complexity of some trap-functions (e.g., discrete logarithm) [8]. In the protocols of that type user A announces to
user B the value \( y \) (where \( y = a^b \mod p \); \( a, p \) are known in advance, and the parity of \( b \) is the secret bit value). In principle, the supplied information on bit \( b \) (i.e., the value of function \( y \)) is sufficient to learn the secret bit by calculating the discrete logarithm. However, all available classical numerical algorithms require exponentially large computational resources (although it was never proved that there exists no more efficient polynomial classical algorithm for this problem).

For the case where A and B can only exchange information through the classical communication channel, the problem was solved in Ref. \([8]\). Strictly speaking, the protocol proposed in Ref. \([8]\) is not secure against cheating by one of the users since it is based on the unproved computational complexity of the discrete logarithm problem \([8]\).

In these protocols the user B is given the complete rather than partial information on the secret bit already at the commitment stage. Therefore, in principle, user B can learn the secret bit even before the disclosure stage, for example, by using the quantum computer \([9,10]\) (which is currently, however, very far from the experimental realization).

Employing only the classical (non-relativistic) objects as the information carriers, it is impossible to construct an unconditionally secure bit commitment protocol (whose security is based on the fundamental laws of nature only rather than current technical limitations) where only “part” of a classical object (for example, a spatially extended signal which is only partly accessible to user B before the disclosure stage) is supplied to user B at the commitment stage. Since the part of the signal available to user B until the disclosure stage should have the same appearance to user B for both values 0 and 1 of the secret bit (otherwise the user B will have non-zero information about the secret bit before the beginning of the disclosure stage), the part of the classical object which is left with user A should be different for different secret bit values. The laws of classical non-relativistic physics do not prohibit an instantaneous modification of the part of the signal still controlled by user A converting 0 into 1 or vice versa before submitting it to user B immediately before the disclosure stage thus allowing cheating by user A. Therefore, no unconditionally secure bit commitment protocol can be realized within the framework of non-relativistic classical physics.

In the non-relativistic quantum protocols the information is carried by quantum systems. Schematically, the protocols can be described in the following way. First, the Hilbert state space \( \mathcal{H}_s \) is chosen to which the states of the information carriers belong. User A choses the states \( |\psi_{0,1}\rangle \in \mathcal{H}_s \), corresponding to 0 or 1 and send them to B. The states are usually chosen to be non-orthogonal. It is important that the state space \( \mathcal{H}_s \) is implicitly assumed to be fully accessible to both users A and B throughout the entire protocol. The requirement that the density matrices corresponding to both 0 and 1 look identical for user B until the disclosure stage begins results in the possibility of an undetectable cheating by user A employing an EPR-attack \([11,12]\). Roughly speaking, in this approach the protocol involves only the state space of the quantum system. However, this situation actually does not correspond to the real process of information transfer. To be more precise, we mean the following. The participants of the protocol cannot control the entire space. Instead, they control only certain domains (vicinities of their laboratories, measuring devices, etc). In addition, all the measurements occur in the real space and time (or space-time in the relativistic case). The non-relativistic quantum mechanics allows construction of entangled states of physically different systems (we are only interested in this case because it is impossible to perform a measurement which affects only one of the two identical systems). Therefore, if the users control only the non-overlapping domains, the entangled state from \( \mathcal{H}_s \otimes \mathcal{H}_a \) should automatically be non-local also in the coordinate space. The wave functions of both systems from \( \mathcal{H}_s \) and \( \mathcal{H}_a \) should be simultaneously different from zero in the domains controlled by users A and B (because otherwise the state will not be entangled for the users). The latter means that each user has access to both state spaces, \( \mathcal{H}_s \) and \( \mathcal{H}_a \), and can perform measurements and unitary transformations separately over both systems at his own discretion due to their physical distinguishability. Therefore, the locality of the transformations in the state space \( \mathcal{H}_s \otimes \mathcal{H}_a \) (in the sense of manipulation in only one of the state subspaces) does not imply the locality in the coordinate (position) space. In other words, in the non-relativistic quantum protocols of that kind (when the spatio-temporal structure of the information carrier states is not explicitly taken into account) the state space of the information carriers is accessible by both users. In this sense such protocols do not realize the idea of submitting only a part of information on the carrier of a secret bit.

In the non-relativistic case, explicit accounting for the effects of state propagation in the position space, when user B has access to only a part of a spatially extended state, can hardly introduce any new
Aspects to the indicated problems because of the absence of the maximum propagation speed.

Formulation of the problem where only the properties of the state space $\mathcal{H}$ are used does not correspond to the actual process of information transfer in the real space-time. It is more natural to consider a problem when the users are located in their respective laboratories and control some their spatial neighborhoods. It is natural to assume that neither A or B can control and have access simultaneously the entire space.

The state propagation effects (accounting for the spatio-temporal structure of the quantum states) were first explicitly used in quantum cryptography in Ref. [13] (which, in our opinion, was not assessed correctly [14,15]). Accounting for the restrictions imposed by the special relativity and quantum mechanics (quantum field theory) [16] substantially simplifies the proof of unconditional security of relativistic quantum cryptosystems [17]. Besides, the quantum field theory introduces additional fundamental restrictions, e.g. on the teleportation of quantum states [18].

Recently, the classical bit commitment and distant coin tossing protocols have been proposed which take into account the existence of finite maximum speed of signal (information) propagation [19]. The relativistic classical protocol [19] is unconditionally secure (i.e. its security is based on the fundamental laws of nature only) and in principle allows to delay the second stage of the protocol (disclosure of the secret bit value chosen by A) for arbitrarily long time. The implementation of this protocol requires that each of the users A and B control two spatially separated sites.

The idea of using orthogonal states in the bit commitment and coin tossing protocols was proposed earlier in Ref. [20]. The protocols suggested in Ref. [20] were based on two simple considerations. First, a pair of orthogonal (and, consequently, reliably distinguishable when completely accessible) states become efficiently non-orthogonal (only partly distinguishable) when restricted to a subspace. This is also true in non-relativistic quantum mechanics. Indeed, if we have a pair of spatially extended orthogonal states, $\psi_{0,1}(x) \in L^2(-\infty, \infty, dx)$

\[
(\psi_0, \psi_1) = \int_{-\infty}^{\infty} \psi_0^*(x)\psi_1(x)dx = 0,
\]

they become effectively non-orthogonal when restricted to a subspace (a finite domain $\Omega$ in the position space):

\[
(\psi_0, \psi_1)_\Omega = \int_{\Omega} \psi_0^*(x)\psi_1(x)dx \neq 0.
\]

The second important consideration is the existence of a finite maximum speed of both quantum states propagation and classical objects motion implied by the special relativity. This fact does not allow to instantaneously access the entire state (i.e., the domain where the state is present).

In contrast to Ref.[20], where the state had no “internal” degrees of freedom, taking into account the states with “internal” degrees of freedom (e.g. helicity for photons) allows to substantially simplify the protocols.

To be more precise, the states for $|\psi_{0,1}\rangle = \psi(x) \otimes |e_{0,1}\rangle$ ($\langle e_0|e_1\rangle = 0$) 0 and 1 are orthogonal (due to the internal degrees of freedom) even if they only partly accessible for the measurement (i.e. the measuring apparatus can access only a part of the entire spatial domain where $\psi(x) \neq 0$); however, in that case they are not reliably distinguishable: the probability of distinguishing between these two states (probability of obtaining a measurement result in one of the orthogonal channels) can be made arbitrarily small if only a part of the state is accessible since the probability of obtaining an outcome in a finite spatial domain is

\[
\langle \psi_i|\psi_i\rangle_\Omega = \int_{\Omega} |\psi(x)|^2dx < 1, \quad i = 0, 1
\]

so that by appropriately choosing the domain size and function $\psi(x)$ the probability can be made arbitrarily small (which actually follows from the normalization condition $\int_{-\infty}^{\infty} |\psi(x)|^2dx = 1$).

Schematically, our protocol can be described in the following way. User A controls a finite spatial domain and prepares a quantum state in it at the moment specified by the protocol. This state propagates into the quantum communication channel and becomes gradually accessible to the user B in the spatial domain which is not controlled by user B. Having access to only a part of the quantum state in the real position space, user B cannot reliably determine the secret bit (reliably distinguish between 0 and 1). Moreover, it is possible to choose the states in such a way that the probability of correct determination of the secret bit by user B is arbitrarily close to 1/2 (i.e. simple guessing probability) for arbitrarily
long (although agreed upon before the start of the protocol) time interval. There exist no fundamental restrictions on the length of this interval, although making it long enough may present a difficult technical problem. The existence of a finite maximum propagation speed allows to choose the states in such a way that the user A can no longer modify the chosen bit value after the prepared state has partly left the domain controlled by him. After the protocol duration time elapses and the states sent by user A become completely accessible to user B the latter acquires reliable information on the states with the probability arbitrarily close to 1. Important for this protocol are both the quantum nature of the states involved and the existence of a finite maximum speed of propagation imposed by the special relativity.

The states and measurements used in the protocols are described in Section 2 while the bit commitment and coin tossing protocols themselves for the states with finite supports are presented in Sections 3 and 4, respectively. The fundamental non-localizability of quantum states in the quantum field theory is accounted for in Section 5. The main results obtained in the paper are summarized in Conclusions.

2 States and measurements used in the protocol.

Since the protocol explicitly employs the spatio-temporal structure of the states, it cannot be formulated without specifying the system geometry. We shall consider a one-dimensional model containing all the important features dictated by the quantum field theory; a similar model is frequently used in quantum optics. We shall deal with a massless field whose states in the momentum representation are specified on the mass shell $k^2_0 - k^2 = 0$. Important for us are the states propagating in the positive direction of $x$-axis ($k > 0$). We assume that user A controls a neighbourhood of point $x_A$, while user B controls a neighborhood of point $x_B$ ($x_A < x_B$).

In the following all functions are assumed to depend on the difference $\tau = t - x$; the speed of light is assumed to be unit, $c = 1$. This representation reflects the intuitive picture of a packet moving with the speed of light. The momentum eigenstate $|k\rangle$, corresponding to the eigenvalue $k$ is a generalized eigenvector (to be more precise, a linear continuous functional on the elements from a dense subset in $\mathcal{H} = L^2(0, \infty, d\xi)$) and has the form

$$\langle \xi | k \rangle = \delta(k - \xi).$$

The states $|\psi\rangle$ from $\mathcal{H}$ can be expanded in the generalized states

$$|\psi\rangle = \int_0^{\infty} \langle k| \psi \rangle |k\rangle dk,$$

where the value taken by the functional $\langle k|$ on the element $|\psi\rangle$ is

$$\langle k| \psi \rangle = \int_0^{\infty} \psi(\xi)\delta(k - \xi)d\xi = \psi(k),$$

i.e. the amplitude of the state $|\psi\rangle$ in the $k$-representation. Accordingly, the amplitude of the state $|k\rangle$ in the $\tau$-representation is

$$\langle k|\tau \rangle = \frac{1}{\sqrt{2\pi}} e^{ik\tau}, \quad k \in (0, \infty), \quad \tau \in (-\infty, \infty), \quad \tau = t - x,$$

corresponding to the intuitive picture of a plane wave (a state with a definite momentum) moving with the speed of light.

It will be important for the protocol that the quantum states propagate with the maximum possible speed of light. In the $\tau$-representation the orthogonal states (packets) corresponding to 0 and 1 used in the protocol are written in the form

$$|\psi_{0,1}\rangle = \int_{-\infty}^{\infty} f(\tau)|\tau\rangle d\tau \otimes |e_{0,1}\rangle, \quad \langle e_0|e_1 \rangle = 0, \quad \langle e_{0,1}|e_{0,1} \rangle = 1,$$

where the states $|e_{0,1}\rangle$ describe the internal degrees of freedom (e.g. helicity for photons).

The state normalization condition takes the form

$$\langle \psi_{0,1}|\psi_{0,1} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)f(\tau')\langle \tau|\tau\rangle d\tau d\tau' = 1,$$
where
\[ \langle \tau | \tau' \rangle = \frac{1}{2\pi} \delta_+ (\tau - \tau') = \frac{1}{2\pi} \int_0^\infty e^{ik(\tau-\tau')} dk = \frac{1}{2} \frac{1}{\pi} \frac{1}{\tau - \tau'}, \]
(6)

We shall introduce the state amplitude in the \( k \)-representation defined as
\[ f(\tau) = \int_0^\infty f(k) e^{-ik\tau} dk. \]
(7)

Then taking into account Eqs. (5-7) the normalization condition becomes
\[ \langle \psi_{0,1} | \psi_{0,1} \rangle = \int_{-\infty}^\infty \int_{-\infty}^\infty f(\tau) f^*(\tau') \frac{1}{2} \delta(\tau - \tau') + \frac{i}{\pi} \frac{1}{\tau - \tau'} d\tau d\tau'. \]
(8)

Substitution of the state amplitude in the \( k \)-representation from Eq.(8) into Eq.(7) taking into account that [21]
\[ \int_{-\infty}^\infty e^{ik\tau} \frac{1}{\tau + a} d\tau = i\pi \cdot \text{sgn} k \cdot e^{-ia}, \]
(9)
yields
\[ \langle \psi_{0,1} | \psi_{0,1} \rangle = \int_{-\infty}^\infty |f(\tau)|^2 d\tau = 1. \]
(10)

The microcausality requirement [22] implies that the field operators generating the field states belonging to the Hilbert state space when acting on the vacuum vector should either commute or anticommute if they are related to two spatially-like domains. The commutator of two field operators is known to be a distribution (see details in Ref. [22]). If one wishes to talk about local properties of the distribution, the test functions should possess certain properties (actually, they should belong to the space \( J(\hat{x}) \) of infinitely smooth functions which vanish at infinity faster than any inverse polynomial). In other words, the states of a free field cannot have a finite support (i.e. to be zero outside a finite domain) which means that the states of a free field are fundamentally unlocalizable. However, one construct the states which are arbitrarily strongly localized in space and vanish at infinity with the rate arbitrarily close to the exponential one (e.g. see Refs. [23–27]). In addition, the functions from \( D(\hat{x}) \) with finite support form a dense set in \( J(\hat{x}) \) which means that any function from \( J(\hat{x}) \) can be approximated with functions from \( D(\hat{x}) \) with any desirable accuracy.

In the context of our one-dimensional model the non-localizability can be derived from the Wiener-Paley theorem [28] since the normalisation condition (10) together with Eq.(7) means the square integrability of the amplitude in the \( k \)-representation and imposes the restrictions on the asymptotic behaviour of the function \( f(\tau) \):
\[ f(\tau) = \int_0^\infty f(k) e^{-ik\tau} dk, \quad \int_{-\infty}^\infty \frac{\ln|f(\tau)|}{1 + \tau^2} d\tau < \infty. \]
(11)

Equation (11) implies that the function \( f(\tau) \) cannot have a finite support in \( \tau \) and cannot decay exponentially at infinity, although it can be arbitrarily strongly localized and possess decay rate arbitrarily close to the exponential law, for example
\[ f(\tau) \propto \exp \{-\alpha \tau/\ln(\ln(\ln(\ln|\ln|\tau|)))) \}, \]
(12)
where \( \alpha \) take any value.

For the reasons of convenience we shall first formulate the protocol for the states with finite support (since the functions from \( D(\tau) \) form a dense subset and any function \( f(\tau) \) can be approximated with functions from \( D(\tau) \) with any accuracy) and then introduce the necessary modification to account for the non-localizability of the states.

Let the state \( f(\tau) \) have a finite support, \( \text{supp} f(\tau) = (-\Delta \tau, \Delta \tau) \) (\( \Delta \tau \) can be chosen to be arbitrarily small). The states are formed by only the vectors \(|\tau\rangle \) belonging to the interval \((-\Delta \tau, \Delta \tau)\) on the light cone:
\[ |\psi_{0,1}\rangle = \int_{-\Delta \tau}^{\Delta \tau} f(\tau)|\tau\rangle d\tau \otimes |e_{0,1}\rangle. \]
(13)

In contrast to the non-relativistic quantum protocols which do not explicitly employ the spatio-temporal structure of the states and the state preparation effects are unimportant (to be more precise, the non-relativistic quantum mechanics allows an instantaneous preparation of any states from \( \mathcal{H} \), even those
which are non-local in the position space, at any moment of time), the situation is quite different in the field theory. Preparation of a state requires access to a finite spatio-temporal domain (even if the state support is assumed to be finite). In the one-dimensional model the state preparation requires either the access to the spatial domain of size $\Delta x = 2\Delta \tau$ if the state is prepared by a non-local source at a specified moment of time $t$ or a finite time interval $\Delta t = 2\Delta \tau/c$ if the state is generated by point-like source at point $x$. Therefore in the relativistic case the protocol can only be formulated after the system geometry is completely specified. The one-dimensional situation is the simplest one, since all the quantities here depend on a single variable $\tau = x - ct$. Bearing in mind that the actual experiments employ quasi-one-dimensional optical fiber systems, analysis of the one-dimensional model seems to be quite reasonable.

Consider now the “stretched” states used in the protocols. These states consist of the two halves separated by the interval $\tau_0$ on the light cone and can be written as

$$|\psi_{0,1}(\tau_0)\rangle = \frac{1}{\sqrt{2}} \int (f(\tau) + f(\tau - \tau_0))d\tau \otimes |e_{0,1}\rangle. \quad (14)$$

Here and below we adopt the normalization

$$\int_{-\infty}^{\infty} |f(\tau)|^2d\tau = \int_{-\infty}^{\infty} |f(\tau - \tau_0)|^2d\tau = 1, \quad \text{supp}f(\tau) \cap \text{supp}f(\tau - \tau_0) = \emptyset. \quad (15)$$

Since the initial state support belongs to the interval $(-\Delta \tau, \Delta \tau)$, the state preparation requires access to the domain $(-\Delta \tau, \Delta \tau + \tau_0)$ on the light cone (either domain $\Delta x = (-\Delta \tau, \Delta \tau + \tau_0)$ of the position space if the preparation is performed with a non-local apparatus at a specified moment of time or the time interval $\Delta t = (-\Delta \tau, \Delta \tau + \tau_0)$ if the state is produced by a local source at point $x$).

Consider now the individual measurements performed by user B over the quantum field states. The probability of obtaining an outcome by user B in the channel $\mathcal{P}_i$ for the input state $|\psi_{0,1}\rangle$ along the light cone branch $\tau = x - ct$:

$$U_{ch}\langle \tau_{ch}\rangle|\psi_{0,1}\rangle = |\psi_{0,1,\tau_{ch}}\rangle = \frac{1}{\sqrt{2}} \int (f(\tau - \tau_{ch}) + f(\tau - \tau_0 - \tau_{ch})))d\tau \otimes |e_{0,1}\rangle; \quad (17)$$

here $\tau_{ch}$ is the channel length. The state “extent” $(2\Delta \tau + \tau_0)$ and the communication channel length $\tau_{ch}$ should satisfy the inequality $\tau_{ch} < \tau_0 + 2\Delta \tau$, so that one can assume without loss of generality that $\tau_{ch} = 0$ (the channel length can be arbitrary until it does not exceed the state “extent”).

The probability of obtaining an outcome by user B in the channel $i$ ($\mathcal{P}_i$) in the interval $d\tau$ for the input state $|\psi_j(\tau_0)\rangle$ is

$$\Pr\{d\tau; i, j\} = \text{Tr}\{(\mathcal{M}(d\tau) \otimes \mathcal{P}_i)|\psi_j(\tau_0)\rangle\langle \psi_j(\tau_0)|\rangle = \delta_{ij} \frac{1}{2} \left\{ |f(\tau)|^2 + |f(\tau - \tau_0)|^2 \right\} d\tau. \quad (18)$$

This expression describes the probability density for obtaining an outcome in one of the orthogonal (distinguishable) channels for $0$ ($i = j = 0$) and $1$ ($i = j = 1$) in the interval $d\tau$. At the intuitive level such a measurement can be thought of as being realized with a very fast (formally with zero intrinsic time) photodetector operating in waiting mode. The measurement outcome is a random event occurring in the time interval $d\tau$ with the probability density given by Eq.(18).
The probability of detecting a state in the finite interval $\Delta(\tau)$ (for $i = j$) is

$$\Pr\{\Delta(\tau)\} = \int_{\Delta(\tau)} \Pr\{d\tau; i, i\} = \frac{1}{2} \left\{ \int_{\Delta(\tau)} |f(\tau)|^2 d\tau + \int_{\Delta(\tau)} |f(\tau - \tau_0)|^2 d\tau \right\}.$$

(19)

If the interval $\Delta(\tau)$ (accessible domain on the light cone) does not entirely cover the state support (for example, if only one half of the state is covered), the probability of obtaining an outcome is $1/2$. However, if an outcome is obtained the states are uniquely identified because of the orthogonality of the channels $\mathcal{P}_0$ and $\mathcal{P}_1$. Therefore, in the time interval $\Delta \tau \leq \tau \leq \tau_0 + \Delta \tau$ the probability of wrong state identification based on the measurement outcome is $1/4$. Accordingly, the probability of correct identification is $3/4$. On the other hand, for simple guessing the error probability is $1/2$.

It should be noted once again that the above measurement cannot be interpreted as a measurement which lasts for a finite time $\Delta(\tau)$: Every outcome occurs randomly at time $t$ with the probability density (18).

After the time $\tau_0 + 2\Delta \tau$ elapses and the entire state is found in the domain controlled by user B, the values 0 and 1 are uniquely identified because of the orthogonality of the corresponding states.

Hence, propagation of the states with the maximum possible speed allows an explicit and natural implementation of the idea of providing by user A of only a part of information (part of a quantum state) on the chosen secret bit. Quantum nature of the state is important for the protocol since for a classical signal whose shape is described by the function $f(\tau)$ with different polarizations $e_0$ or $e_1$ the probability of correct identification is 1 (rather than $3/4$) even if only a part of the signal is accessible. The correct identification probability of $3/4$ in the quantum case is actually a consequence of the state normalization requirement.

This result can also be derived in a somewhat different way allowing to clarify the peculiarity of the situations where only a part of the Hilbert state is accessible for measurements. Let us find the measurement minimizing the identification error in the problem of distinguishing between the two density matrices where the states are only partly accessible. The density matrices are written in the form

$$\rho_{0,1} = \left\{ \frac{1}{\sqrt{2}} \left( \int_{-\infty}^{\infty} |f(\tau) + f(\tau - \tau_0)|^2 d\tau \right) \left( \int_{-\infty}^{\infty} |f^*(\tau') + f^*(\tau' - \tau_0)|^2 d\tau' \right) \right\} \otimes |e_{0,1} \rangle \langle e_{0,1}| = (20)\rho(f) \otimes \rho(0, 1).$$

We shall now derive an expression for the identification error occurring when trying to distinguish between the states $\rho_{0,1}$ under the conditions where only a part of the entire space-time is accessible for measurements. Formally, the problem is reduced to the case where the domain $\Delta(\tau)$ is accessible for the measurements while the rest of the space-time (denoted as $\overline{\Delta}(\tau) = (-\infty, \infty) - \Delta(\tau)$) cannot be accessed by available measuring apparatus.

The measurement is described by the resolution of identity consisting of two terms. The first one is actually the identity operator in the subspace spanned by the basis vectors $|\tau\rangle$ belonging to the interval $\Delta(\tau)$, while the second term is the identity operator in the subspace spanned by the vectors form the inaccessible domain $\overline{\Delta}(\tau) = (-\infty, \infty) - \Delta(\tau)$:

$$I \otimes \mathbf{C}^2 = I(\Delta(\tau)) \otimes \mathbf{C}^2 + I(\overline{\Delta}(\tau)) \otimes \mathbf{C}^2 = \left( \int_{\Delta(\tau)} |\tau\rangle \langle \tau| d\tau \right) \otimes \mathbf{C}^2 + \left( \int_{\overline{\Delta}(\tau)} |\tau\rangle \langle \tau| d\tau \right) \otimes \mathbf{C}^2. \tag{21}$$

Suppose that the state $\rho_0$ is produced for the measurements with the a priori probability $\pi_0$, while the state $\rho_1$ with the probability $\pi_1$ ($\pi_0 + \pi_1 = 1$). In the following we shall assume that $\pi_0 = \pi_1 = 1/2$, i.e. 0 or 1 are chosen by the user A with equal probabilities.

Since only a part of the space-time is accessible for measurements (which automatically implies the restricted access to the Hilbert state space since the basis states are labeled by $\tau$), the total error contains two terms. The first one ($P_e(\overline{\Delta}(\tau))$) corresponds to the situation where the measuring apparatus (photodetector) operated by user B did not fire (the outcome occurred in the inaccessible domain). The second term ($P_e(\Delta(\tau))$) describes the error in the state identification arising in the case where the measurement outcome took place in the domain accessible for user B.
The probability of the event when the state was not detected by user B (his measuring apparatus did not fire) is

\[ P(\bar{\Delta}(\tau)) = \text{Tr}\{ (\pi_0 \rho_0 + \pi_1 \rho_1) \left(I(\bar{\Delta}(\tau)) \otimes C^2 \right) \} = \pi_0 p_0 + \pi_1 p_1, \quad p_0 = p_1 = p, \quad (22) \]

\[ p = \frac{1}{2} \int_{-\infty}^{\infty} [\mid f(\tau) \mid^2 + \mid f(\tau - \tau_0) \mid^2] d\tau. \]

The probability \( p_0 \) of the outcome in the inaccessible domain for the input state \( \rho_0 \) produced with the specified \textit{a priori} probability \( \pi_0 \) is

\[ p_0 = \frac{\pi_0 p}{\pi_0 p + \pi_1 p} = \pi_0, \quad (23) \]

and, similarly, for \( \rho_1 \) produced with the \textit{a priori} probability \( \pi_1 \),

\[ p_1 = \frac{\pi_1 p}{\pi_0 p + \pi_1 p} = \pi_1. \quad (24) \]

The probability of error, i.e. the probability of the event when the state \( \rho_1 \) is interpreted as state \( \rho_0 \), and \textit{vice versa}, is

\[ P_e(\bar{\Delta}(\tau)) = \pi_0 p_1 + \pi_1 p_0. \quad (25) \]

If only one half of the state is located in the accessible domain (the support of either \( f(\tau) \) or \( f(\tau - \tau_0) \) belongs to the inaccessible domain), the probability of wrong identification for the case of the outcome occurring in the inaccessible domain calculated according to Eqs. (22–25) is \( P_e(\bar{\Delta}(\tau)) = 1/2. \)

In the general case the measurement minimizing the wrong identification probability is given for a binary resolving function by the identity resolution

\[ \tilde{E}_0 + \tilde{E}_1 = I(\Delta(\tau)) \otimes I_{C^2} = I(\Delta(\tau)) \otimes (E_0 + E_1), \quad (26) \]

where, in contrast to Refs. [29,30], the resolution is specified in the subspace restricted to \( \Delta(\tau) \). The minimal error probability is found by the optimization with respect to all possible resolutions (see details in Refs. [29,30]):

\[ P_e(\Delta(\tau)) = \min_{\{E_0, E_1\}} \left( \pi_0 \text{Tr}\{ \rho_0 \tilde{E}_1 \} + \pi_1 \text{Tr}\{ \rho_1 \tilde{E}_0 \} \right), \quad (27) \]

where \( \pi_0 \) and \( \pi_1 \) are the probabilities of occurrence of density matrices \( \rho_0 \) and \( \rho_1 \), respectively (in our problem \( \pi_0 = \pi_1 = 1/2 \) are the probabilities of the preparation of 0 and 1 by user A).

Taking into account Eq.(26), the error probability can be reduced to the following form:

\[ P_e(\Delta(\tau)) = \pi_0 \text{Tr}\{ \rho(f) I(\Delta(\tau)) \} + \text{Tr}\{ \Gamma \tilde{E}_0 \}, \quad (28) \]

\[ \Gamma = \pi_1 \rho_1 - \pi_0 \rho_0. \]

Optimization of \( P_e(\Delta(\tau)) \) reduces to finding the minimum of \( \text{Tr}\{ \Gamma \tilde{E}_0 \} \) with respect to all possible operators \( \tilde{E}_0 \). Since the domain accessible for measurements is restricted to the interval \( \Delta(\tau) \), one has

\[ \text{Tr}\{ \Gamma \tilde{E}_0 \} = \text{Tr}_{\Delta(\tau)} \{ \Gamma \tilde{E}_0 \} = \text{Tr}_{\Delta(\tau)} \{ \rho(f) \} \otimes \text{Tr}\{ (\pi_1 \rho(1) - \pi_0 \rho(0)) E_0 \} = \quad (29) \]

\[ \left\{ \frac{1}{2} \int_{\Delta(\tau)} [\mid f(\tau) \mid^2 + \mid f(\tau - \tau_0) \mid^2] d\tau \right\} \otimes \text{Tr}\{ \Gamma E_0 \}. \]

Since \( 0 \leq E_0 \leq I(\Delta(\tau)) \otimes I_{C^2} \),

\[ \text{Tr}\{ \Gamma E_0 \} \geq \text{Tr}\{ \Gamma \} = \sum_i \gamma_i \quad (30) \]

The minimal possible error is determined by the negative eigenvalues \( \gamma_i \) of the operator \( \Gamma = \pi_1 \rho(1) - \pi_0 \rho(0) \) [30]. The operator \( \tilde{E}_0 \) should satisfy the conditions

\[ \langle \gamma_i | \tilde{E}_0 | \gamma_i \rangle = 1, \quad \gamma_i \leq 0, \quad (31) \]

\[ \langle \gamma_i | \tilde{E}_0 | \gamma_i \rangle = 0, \quad \gamma_i \geq 0, \]
where $|\gamma_i\rangle$ are the eigenvectors of the operator $\Gamma = \sum_i \gamma_i |\gamma_i\rangle \langle \gamma_i|$. In the basis $\{|e_0\rangle, |e_1\rangle\}$ the operator $\Gamma$ has the matrix
\[
\Gamma = \begin{pmatrix}
\pi_1 & 0 \\
0 & -\pi_0
\end{pmatrix}, \quad \text{negative}\quad \gamma_2 = -\pi_0 = -1/2.
\] (32)

Accordingly, the operator $\tilde{E}_0$ is
\[
\tilde{E}_0 = I(\Delta(\tau)) \otimes \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}, \quad \tilde{E}_1 = I(\Delta(\tau)) \otimes \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}.
\] (33)

For the minimal error one obtains
\[
P_e(\Delta(\tau)) = f^2(\Delta(\tau))(\pi_0 + \sum_{\gamma_i \leq 0} \gamma_i),
\] where the notation
\[
f^2(\Delta(\tau)) = \frac{1}{2} \int_{\Delta(\tau)} [|f(\tau)|^2 + |f(\tau - \tau_0)|^2] d\tau.
\] (35)

At the intuitive level this result can be interpreted in the following way. Suppose that one has to distinguish between a pair of single-photon extended states with different (orthogonal) helicities. The correct identification probability is only unit if the entire states are accessible for the measurements (accordingly, the error probability is zero). In spite of the orthogonality of the basis vectors describing different helicities, the states cannot be reliably identified due to their spatial extent if they (their spatial amplitudes) are not entirely accessible. Physically, this is related to the fact that there exist no helicity states beyond the spatial degrees of freedom. Because of the normalization condition with respect to the spatial degrees of freedom the probability of firing of any measuring device employed by user B does not exceed 1. Reliable distinguishing of the two states with even orthogonal helicities always requires a finite time since because of the restrictions imposed by special relativity the entire state cannot be accessed faster than the effective state “extent” divided by the speed of light.

For a large number of measurements the total error is the relative frequency of wrongly identified states. The fraction (probability) of outcomes in the accessible domain is
\[
N(\Delta(\tau)) = \text{Tr}\{(\pi_0 \rho_0 + \pi_1 \rho_1)(I(\Delta(\tau)) \otimes I_C^2)\},
\] (37)
and, similarly, the fraction of outcomes in the inaccessible domain is
\[
N(\overline{\Delta}(\tau)) = \text{Tr}\{(\pi_0 \rho_0 + \pi_1 \rho_1)(I(\overline{\Delta}(\tau)) \otimes I_C^2)\}.
\] (38)

The total error probability $P_e$ is the sum of the error occurring for the outcome taking place in the inaccessible domain multiplied by the probability of these outcomes and the product of the error occurring for the outcome taking place in the accessible domain and the probability of these outcomes:
\[
P_e = P_e(\overline{\Delta}(\tau)) \cdot N(\overline{\Delta}(\tau)) + P_e(\Delta(\tau)) \cdot N(\Delta(\tau)).
\] (39)

When only halves of the states are accessible, one has
\[
P_e = \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2},
\] (40)

\[
P_e(\overline{\Delta}(\tau)) = \frac{1}{2}, \quad N(\overline{\Delta}(\tau)) = \frac{1}{2}, \quad P_e(\Delta(\tau)) = 0, \quad N(\Delta(\tau)) = \frac{1}{2}.
Accordingly, the correct identification probability is 3/4.

The obtained result actually means the following. If user A prepares randomly and with equal probabilities either the state $\rho_0$ or $\rho_1$ and sends these states for measurements to user B, the probability of firing of the measuring device in one of the channels corresponding to 0 or 1 is 1/2. If the measuring apparatus employed by user B gave an outcome, the state is reliably identified. However, if the apparatus did not fire, the user B can only guess which state was sent to him by user A. The probability for the apparatus not to fire is 1/2; in that case the correct guess probability is also 1/2. Hence for these events the net probability of correct identification is $1/2 \cdot 1/2 = 1/4$. The total probability of correct identification of the state sent by user A is thus $1/2 + 1/4 = 3/4$.

In this case the probability $1 - P_e$ coincides with the probability of correct identification of the secret bit.

This probability of correct identification is too high (substantially exceeds 1/2) for development of a valid protocol. The situation is radically changed if the secret bit is constructed as parity bit of $N$ states. We shall see below that in this case the probability of correct identification of the parity bit exceeds 1/2 by only an exponentially small value, i.e. practically coincides with the probability of simple guessing which is the worst strategy for user B.

Let us now calculate the probability of the error of identification of the secret bit value when it is coded as parity bit of $N$ orthogonal states. For the case where the entire state space is accessible for the measurements, the problem of the parity bit of a string of $N$ bits each coded by one of the two non-orthogonal states was considered earlier in Ref. [32].

We shall first calculate the identification error for the outcomes occurring in the accessible domain. For a random string of $N$ bits associated with the density matrix $\rho_{0,1}$ described by $2^N$ possible combinations ($2N/2$ of which are even and $2N/2$ are odd) the problem is reduced to distinguishing between the two density matrices corresponding to even and odd strings:

$$\tilde{\rho}_0 = \frac{2}{2^N} \sum_{(i_1 \oplus i_2 \oplus \ldots \oplus i_N) = 0} \rho_{i_1} \otimes \rho_{i_2} \otimes \ldots \otimes \rho_{i_N} =$$

$$\frac{2}{2^N} (\rho(f) \otimes \rho(f) \otimes \ldots \otimes \rho(f)) \otimes \sum_{(i_1 \oplus i_2 \oplus \ldots \oplus i_N) = 0} \rho(i_1) \otimes \rho(i_2) \otimes \ldots \otimes \rho(i_N), \quad i_k = 0, 1, \quad k = 1, \ldots N;$$

$$\tilde{\rho}_1 = \frac{2}{2^N} \sum_{(i_1 \oplus i_2 \oplus \ldots \oplus i_N) = 1} \rho_{i_1} \otimes \rho_{i_2} \otimes \ldots \otimes \rho_{i_N} =$$

$$\frac{2}{2^N} (\rho(f) \otimes \rho(f) \otimes \ldots \otimes \rho(f)) \otimes \sum_{(i_1 \oplus i_2 \oplus \ldots \oplus i_N) = 1} \rho(i_1) \otimes \rho(i_2) \otimes \ldots \otimes \rho(i_N), \quad i_k = 0, 1 \quad k = 1, \ldots N.$$

The measurement minimizing the identification error for the density matrices $\tilde{\rho}_0$ and $\tilde{\rho}_1$ is given by the identity resolution

$$(I(\Delta(\tau)) \otimes I_{C^2})^{\otimes N} = \hat{E}_0 + \hat{E}_1 = I(\Delta(\tau))^\otimes \otimes \left(\hat{E}_0 + \hat{E}_1\right), \quad \hat{E}_0 + \hat{E}_1 = I_{C^2}^{\otimes N}. \quad (43)$$

In that case the error probability is

$$P_e(\Delta(\tau)) = \pi_0 \text{Tr} \left\{ \tilde{\rho}_0 \left( I(\Delta(\tau))^\otimes \otimes I_{C^2}^{\otimes N} \right) \right\} + \text{Tr} \left\{ \hat{\Gamma} \left( I(\Delta(\tau))^\otimes \otimes \hat{E}_0 \right) \right\}; \quad (44)$$

Accordingly, the minimal error is given by a formula similar to Eq.(28), and is determined by the negative eigenvalues $(\gamma_k)$ of the operator

$$\hat{\Gamma} = \left( f^2(\Delta) \right)^N \left( \pi_1 \hat{\rho}(1) - \pi_0 \hat{\rho}(0) \right) = \left( f^2(\Delta) \right)^N \Gamma. \quad (45)$$
In the basis of vectors ordered into the even and odd (with respect to the sum of subscripts) sets, the operator $\Gamma$ is written as

$$\Gamma = \frac{1}{2^N} \begin{pmatrix}
1 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & -1 & 0 \\
0 & 0 & \ldots & -1
\end{pmatrix},$$

(46)

Finally, the minimal error probability of the identification of the parity bit determined by $N$ orthogonal states which are only partly accessible, is ($\pi_0 = 1/2$)

$$P_e(\Delta(\tau)) = \left(\overline{f^2(\Delta)}\right)^N \left(\frac{1}{2} + \frac{1}{2^N} \sum_{n=0}^{2^N/2} (-1)\right).$$

(47)

If only one half of each state is accessible ($\overline{f^2(\Delta)} = 1/2$), one has

$$P_e(\Delta(\tau)) = \left(\frac{1}{2}\right)^N \left(\frac{1}{2} - \frac{1}{2^N} \cdot \frac{2^N}{2}\right) = 0.$$  

(48)

If the measurement outcome took place in the accessible domain, the identification error is zero because of the channel orthogonality. This formula should be understood in the following way. If all $N$ outcomes occurred in the accessible domain, the error probability is zero because of the channel orthogonality. The same is true if the outcomes in the accessible domain took place for $m$ states (in that case $N$ in Eqs.(47,48) should be replaced by $m$). In other words, the states which resulted in measurement outcomes occurring in the accessible domain become reliably known to user B.

However, the outcomes can also occur in the inaccessible domain.

As the states gradually propagate into the domain accessible for the measurements performed by user B, $\overline{f^2(\Delta)} \to 1$, the error probability $P_e \to 0$. Any two orthogonal states are reliably distinguishable when each of them is entirely accessible for the measuring apparatus.

Let us now calculate the probability of correct identification of the parity bit. The total number of binary strings of length $N$ is $2^N$. The outcomes can occur both in accessible and inaccessible domains. The total space of outcomes can be divided into two disjoint subsets. The first one corresponds to the event when all $N$ outcomes occurred in the accessible domain. In that case the probability of correct identification of the parity bit is $1$. However, the probability of this event for the case where only one half of each state is accessible $\overline{f^2(\Delta)}^N = 2^{-N}$.

The second subset corresponds to all other events when at least one outcome occurred in the inaccessible domain. The probability of all these events (when only one half of each state is accessible) is

$$\sum_{k=0}^{N-1} C_N^k \overline{f^2(\Delta)}^k (1 - \overline{f^2(\Delta)})^{N-k} = \sum_{k=0}^{N-1} C_N^k \frac{1}{2^k} \frac{1}{2^{(N-k)}} = 1 - \overline{f^2(\Delta)}^N = 1 - 2^{-N}.$$  

(49)

For these events the probability of error in the parity bit identification is $1/2$. Indeed, if user B has a string of length $k$ ($k \leq N - 1$) whose parity is reliably known to him. However, the parity of the rest part of the string of length $N - k$ corresponding to the outcomes in the inaccessible domain can be either odd or even with equal probabilities. Hence the parity of the full string consisting of $N$ bits is known with the probability of $1/2$, since the knowledge of the string of $k$ bits does not help in any way in finding the parity bit of the full string.

The total error in the parity bit determination is a sum of two contributions. The first one corresponds to the event when all outcomes took place in the accessible domain and the second one corresponds to all the rest events. Each contribution is a product of the probability of error in the parity bit identification and the probability of the event itself. One finally has

$$P_e(\text{parity}) = \frac{1}{2} \cdot \left(1 - 2^{-N}\right) + 0 \cdot 2^{-N} = \frac{1}{2} - \frac{1}{2} \cdot 2^{-N}.$$  

(50)
Accordingly, as long as only one half of each state is accessible to user B, the probability of correct parity bit identification by this user is

\[
P_c(\text{parity}) = 1 - P_c(\text{parity}) = \frac{1}{2} + \frac{1}{2} \cdot 2^{-N} \tag{51}
\]

and exceeds the simple guess probability by only an exponentially small value.

Thus, during the time interval \( \tau_0 (\Delta \tau < \tau < \Delta \tau + \tau_0) \) after the beginning of the protocol the user B has only exponentially small information on the secret bit.

However, this scheme where the secret bit is coded as a parity bit of a string of \( N \) bit is still insufficient for the development of a useful protocol since it allows the user A to cheat with unacceptably high probability (to delay his choice of the secret bit without being caught by user B).

To avoid this problem, each of \( N \) bits should be coded by a block of \( k \) identical bits (the number \( k \) will be specified below) which are randomly distributed over \( N \cdot k \) channels.

Finally, we shall give an expression for the probability of error when distinguishing between the two density matrices corresponding to 0 and 1 for the case where the parity bit is coded by the blocks of identical bits (all of them are either 0 or 1). In the protocol the secret parity bit is calculated over \( N \) bits each of which is represented by a block of length \( k \). This block-wise representation of each bit is necessary for the detection by user B of possible cheating by user A.

In that case the total number of binary string is \( 2^{N \cdot k} \). When each 0 and 1 is coded by blocks of length \( k \), the number of odd and even strings among them is

\[
S_{\text{odd,even}} = \frac{1}{2} \sum_{m=0}^{N-1} C_{N \cdot k}^{m \cdot k} = 2^{N \cdot k} \left( \frac{1}{2^k} \right) \sum_{l=1}^{k} \cos^{N \cdot k} \left( \frac{l \pi}{k} \right) \cos (Nl \pi) \approx 2^{N \cdot k}, \tag{52}
\]

i.e. is actually equal to the number of different ways of distributing \((N - l) \cdot k\) units and \( l \cdot k\) zeros (for \( 0 \leq l \leq N \)) among \( N \cdot k \) cells [30].

Note that if the position of each block was agreed upon in advance (i.e. the units and zeros from different blocks were not intermixed) the total number of possible odd and even strings would only be \( 2^N \) which is exponentially less than \( 2^{N \cdot k} \) (52) for large \( k \).

It will be important for the protocol that the rest \( N_{\text{rest}} = 2^{N \cdot k} - S_{\text{odd}} - S_{\text{even}} \ll 2^{N \cdot k} \) strings do not belong to either even or odd string sets coded by blocks of length \( k \).

In the complete basis ordered with respect to odd and even block-wise states and other states (for definiteness we assume \( k \) to be even) the operator similar to Eq. (46) is

\[
S_{\text{even}} \rightarrow \left\{ \begin{array}{c}
|e_0 \rangle \otimes |e_0 \rangle \ldots \ldots \ldots |e_0 \rangle \otimes |e_0 \rangle \\
|e_1 \rangle \otimes |e_1 \rangle \ldots \ldots \ldots |e_1 \rangle \otimes |e_1 \rangle \\
\text{other permutations of 0 and 1} \\
\end{array} \right\} \tag{53}
\]

\[
S_{\text{odd}} \rightarrow \left\{ \begin{array}{c}
|e_0 \rangle \otimes |e_0 \rangle \ldots \ldots \ldots |e_1 \rangle \otimes |e_1 \rangle \\
|e_1 \rangle \otimes |e_1 \rangle \ldots \ldots \ldots |e_0 \rangle \otimes |e_0 \rangle \\
\text{other permutations of 0 and 1} \\
\end{array} \right\} \tag{54}
\]

the operator similar to Eq. (46) takes the form

\[
\hat{\Gamma} = \begin{pmatrix}
\hat{I}_{S_{\text{odd}}} & 0 & 0 \\
0 & -\hat{I}_{S_{\text{even}}} & 0 \\
0 & 0 & \hat{0}
\end{pmatrix}, \tag{55}
\]

where \( \hat{I}_{S_{\text{odd}}} (\hat{I}_{S_{\text{even}}}) \) are unit \( S_{\text{odd}} \times S_{\text{odd}} (S_{\text{even}} \times S_{\text{even}}) \) matrices, and \( \hat{0} \) is the zero matrix of size \( N_{\text{rest}} \times N_{\text{rest}} \).
The measuring operators \( \hat{E}_0 \) and \( \hat{E}_1 \) in the same basis are written as
\[
\hat{E}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hat{1}_{\text{even}} & 0 \\ 0 & 0 & \hat{1} \end{pmatrix}, \quad \hat{E}_1 = \begin{pmatrix} \hat{I}_{\text{odd}} & 0 & 0 \\ 0 & \hat{0} & 0 \\ 0 & 0 & \hat{I} \end{pmatrix}.
\] (66)

The probability of error in identification of the block-wise parity bit under the conditions that \( N \cdot k \) outcomes took place in the accessible domain is
\[
P_e(\Delta(\tau)) = (\bar{f}^2(\Delta))^N \cdot k \left( \frac{1}{2} - \frac{1}{2^{(\Delta_{\text{even}} + \Delta_{\text{odd}})}} \sum_{\gamma_i \leq 0} (-1)^{\gamma_i} \right) = 0.
\] (77)

The error probability is zero for all the states which gave the outcomes in the accessible domain.

Let us now calculate the error in identification of the parity bit when coding with the blocks of length \( k \) is adopted. The outcomes can occur both in the accessible and in inaccessible domains. We shall first calculate the minimal number of outcomes which should occur in the accessible domain if the string parity is to be reliably identified. Since the direct calculation is rather difficult, we shall take advantage of the following approach (which is actually a straightforward modification of the Shannon typical sequence method [33,34]). For a moment we shall return to the situation where each block is represented by a block of unit length, \( k = 1 \). Since the set of all possible strings contains \( \Omega = 2^N \) elements, the information carried by each particular string is \( I = \log_2 |\Omega| \) and (to within the rounding error) coincides with the number of binary symbols required to specify each string. If each symbol (in our case, firing of the detector employed by user B in the accessible domain) occurs with the probability \( p \), the probability of the element identification is \( p^f \).

For block-wise coding the number of all possible strings is given by Eq. (52) and the number of binary symbols required to identify a particular string is
\[
I = \log_2 \left( \frac{2^{N \cdot k - 1}}{k} \sum_{l=1}^{k} \cos^{N \cdot k} \left( \frac{l\pi}{k} \right) \cos \left( Nl\pi \right) \right) = \alpha(N,k)(N \cdot k),
\] (58)

which yields the number of outcomes in the accessible domain required to identify the string parity.

Accordingly, the probability of this event is \((p = \bar{f}^2(\Delta) = 1/2)\)
\[
P_{\text{acc}} = p^\alpha(N,k)(N \cdot k) = 2^{-\alpha(N,k)(N \cdot k)}.
\] (59)

For these outcomes the error probability is zero. Accordingly, for the outcomes in the inaccessible domain we have
\[
P_{\text{unacc}} = 1 - P_{\text{acc}} = 1 - 2^{-\alpha(N,k)(N \cdot k)};
\] (60)
the parity bit identification error in that case is
\[
P_e(\text{parity}) = \frac{1}{2} \cdot \left( 1 - 2^{-\alpha(N,k)(N \cdot k)} \right) + 0 \cdot 2^{-\alpha(N,k)(N \cdot k)}.
\] (61)

Hence, the correct parity bit identification probability exceeds the simple guess probability by only an exponentially small value:
\[
P_e(\text{parity}) = 1 - P_e(\text{parity}) = \frac{1}{2} + 2^{-\alpha(N,k)(N \cdot k)}.
\] (62)

Note that in the order of magnitude the number of block-wise coded strings with the zeros and units of all blocks randomly distributed over the entire string is equal to the total number of strings \((\approx 2^{N \cdot k})\) and each block-wise coded string looks almost like if the block length were \( k = 1 \). Therefore, the parity bit identification requires the knowledge of almost the full string (to within the correction factor of \( \alpha(N,k) \) (52)).

If the block position were fixed, there would be allowed strings and \( N \) binary tests would be sufficient to identify the parity bit. However, the probability \( p \) of success in each such test is equal to the sum of probabilities of occurrence of 1 or 2 or ... \( k \) outcomes in the accessible domain
\[
p = \sum_{l=1}^{k} C_k^l \frac{1}{2^l} \frac{1}{2^{(k-l)}} = 1 - 2^{-k}.
\] (63)
Accordingly, the probability of having \( N \) outcomes (the probability of reliable identification of the parity bit by user B when he has only access to the state halves) \( P_{acc} = p^N = (1 - 2^{-k})^N \) would be high (for comparable \( N \) and \( k \)).

The only thing we should now to do is to demonstrate that after the time \( \tau_0 + \Delta \tau \approx \tau_0 \) elapses and the states become accessible to user B the probability of cheating by user A tends to zero. To be more precise, we should demonstrate that user A cannot change his mind (modifying the chosen secret bit) after the protocol was started without being detected by user B with sufficiently high probability.

In the protocol, \( N \cdot k \) states are sent simultaneously randomly distributed over \( N \cdot k \) channels. Possible cheating of user A is detected by user B with the help of a measurement described by the identity resolution of the form

\[
I^\otimes N \cdot k \otimes I_{C_2}^\otimes N \cdot k = (P_0(f) + P_1(f) + P_\perp)^\otimes N \cdot k ,
\]

where

\[
P_{0,1}(f) = \left( \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} [f(\tau) + f(\tau - \tau_0)] |\tau\rangle d\tau \right) \left( \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} [f^*(\tau') + f^*(\tau' - \tau_0)] \langle \tau' | d\tau' \right) \otimes |e_{0,1}\rangle \langle e_{0,1}| ,
\]

\[
P_\perp = I \otimes I_{C_2} - P_0(f) - P_1(f) .
\]

In each of \( N \cdot k \) quantum communication channels only three measurement outcomes are possible corresponding to \( P_0(f) \), \( P_1(f) \), and \( P_\perp(f) \). If user A sends the correct states, the different outcomes probabilities are

\[
Pr\{\rho_0, 0\} = Tr\{\rho_0 P_0(f)\} \equiv 1, \quad Pr\{\rho_1, 1\} = Tr\{\rho_1 P_1(f)\} \equiv 1 , \quad Pr\{\rho_0, 1\} = Tr\{\rho_0 P_1(f)\} \equiv 0, \quad Pr\{\rho_1, 0\} = Tr\{\rho_1 P_0(f)\} \equiv 0 , \quad Pr\{\rho_0, 1, \perp\} = Tr\{\rho_0 P_\perp(f)\} \equiv 0 ,
\]

which means that all the outcomes should only occur with the unit probability in the channels \( P_0(f) \) and \( P_1(f) \) if user A employs the correct states.

Any delay for a time longer than \( 2\Delta \tau \) introduced by user A means that he should employ the states which do not cover the front half of the correct extended state, i.e. user A begins the state preparation procedure after a time interval exceeding \( 2\Delta \tau \) has already elapsed after the protocol was initiated. For all such states \( \rho \) whose support does not cover the front half of the correct state the probability of the outcome in channels \( P_0(f) \) and \( P_1(f) \) does not exceed 1/2. Indeed,

\[
Tr\{\rho P_{0,1}(f)\} =
\]

\[
\frac{1}{2} \int_{-\Delta \tau}^{\Delta \tau} f(\tau)\rho(\tau, \tau')f^*(\tau')d\tau d\tau' + \frac{1}{2} \int_{\tau_0 - \Delta \tau}^{\tau_0 + \Delta \tau} \int_{\tau_0 - \Delta \tau}^{\tau_0 + \Delta \tau} f(\tau)\rho(\tau, \tau')f^*(\tau')d\tau d\tau' \leq
\]

\[
\frac{1}{2} \cdot \frac{1}{(2\Delta \tau)^2} \int_{-\Delta \tau}^{\Delta \tau} \int_{-\Delta \tau}^{\Delta \tau} |\rho(\tau, \tau')| d\tau d\tau' \leq \frac{1}{2}, \quad |\rho(\tau, \tau')| \leq 1 ,
\]

if the support of \( \rho \)
\[
\rho = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\tau, \tau') |\tau\rangle \langle \tau'| d\tau d\tau' , \quad Tr\{\rho\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(\tau - \tau')\rho(\tau, \tau') d\tau d\tau' = \int_{-\infty}^{\infty} \rho(\tau, \tau') d\tau = 1 ,
\]

does not cover the front half of the correct state,

\[
\text{supp}\rho(\tau, \tau') \cap \text{supp}\rho(\tau - \tau_0) = \emptyset .
\]

Hence, in the ideal communication channel any delay of the state by user A results in the outcome probabilities in the channels \( P_0(f) \) and \( P_1(f) \) not exceeding 1/2. To be more precise, in every individual experiment, even for the states delayed for more than \( 2\Delta \tau \), the measurement outcome can only occur in the channels \( P_0(f) \), \( P_1(f) \) and should never occur in the channel \( P_\perp(f) \). The probability of such an outcome is 1/2. However, the probability of the event where in \( k \) experiments all the measurement outcomes for delayed states occurred in the channels \( P_0(f) \) and \( P_1(f) \) only, and thus reproduce the correct statistics characteristic of the non-delayed states is as small as \( 2^{-k} \). This circumstance will later be used in the protocol.

Let us now formulate the protocol itself.
3 The Bit Commitment protocol for finite support states in the ideal communication channel.

- Before the protocol is started, the participant agree upon the states used (the localization interval $\Delta \tau$ and the state shape $f(\tau)$), as well as the duration $\tau_0$ of the protocol (the time during which user A retains the secret bit). In principle, time $\tau_0$ can be chosen arbitrarily long, although this may present a difficult technical problem. The users choose also $N$ and $k$.

- User A chooses a secret bit which is the parity bit of a string consisting of $N$ representatives $b = \sum_{i=1}^{N} a[i,j]$, where $a[i,j]$ is bit 0 or 1, a representative of $j$-th block ($j$ is the number of the block consisting of $k$ bits). All the bits in each block are identical.

- At the moment when the protocol is started (this moment is also agreed upon by the users in advance) user A begins to prepare $N \cdot k$ stretched states consisting of two peaks (halves) which are allowed to propagate into $N \cdot k$ quantum communication channels as they are being formed. The states could also be sent through a single communication channel in series, although this would substantially increase the time required for the protocol implementation. Bearing in mind the above remarks, we assume that the channel length is zero which actually means that user B controls only his laboratory (vicinity of the point $x_B$) and has no control over the rest space and the communication channel so that user A can in principle be located just at the threshold of the user B’ laboratory. Simultaneously, user A can control only the vicinity of point $x_A$ where the states are prepared.

The states from different blocks $a[i,j]$ are sent through different channels at random.

- User B can choose any moment of time from the interval $\Delta \tau < \tau < \tau_0 + \Delta \tau$ to start the disclosure stage when user A should announce through a classical communication channels which state were sent in each quantum channel and identify the quantum channels belonging to each block.

- User B performs the measurements described by the identity resolution (65–67). Although the states are orthogonal (and, consequently, reliably distinguishable), the non-local nature of the projection operators $P_{0,1}$ implies that the reliable distinguishability (67) can only be achieved with the correct states if one has access to the entire state which requires time $2\Delta \tau + \tau_0$. Then user B compares the results of his measurements in each quantum channel with the data supplied to him by user A through the classical channel. For ideal quantum channels, the outcomes obtained in all the channels belonging to the same block should yield the identical results (all 0 or all 1). User B abandons the protocol as soon as he finds a discrepancy between the results of his measurements and the data provided by user A for at least one of the quantum channel.

Note that if user A acts in an honest way (sends the correct stretched states at the beginning of the protocol, i.e. the parity of the string consisting of $N \cdot k$ bits is indeed chosen at the very beginning of the protocol), no redistribution of the quantum channels among the blocks can change the parity bit since otherwise the sets of odd and even strings would have common elements.

- If user A does not choose the bit value at the beginning of the protocol (to be more precise, if he chooses the secret bit value after the time $\Delta \tau$ but, of course, before the disclosure stage, $\Delta \tau < \tau < \tau_0 + \Delta \tau$) he will have to send the states different from $|\psi_{0,1}\rangle$. However, for any states different from the correct ones, the outcomes in each of the channels $P_{0,1}$ will occur with the probabilities not exceeding 1/2. To modify the secret bit, user A should delay his choice in at least one of the blocks as a whole, i.e. he should delay at least $k$ states. The probability of avoiding the detection of delay of $k$ states by user B is then $2^{-k}$ (see Eq.(68)).

- The probability for user B to have reliable information on the secret bit before he acquires access to the entire states does not exceed $1/2 + 2^{-\alpha(N,k)N-k}$ (see Eq.(62)).

Hence, the protocol allows to implement the original idea of bit commitment scheme when one of the participants provides only part of information analyzing which the second participant can only extract exponentially small information on the secret bit value before the disclosure stage. At the same time, user
A cannot change the chosen secret bit after the protocol is started (to be more precise, the probability of undetected modification of the chosen secret bit value after the protocol is started is exponentially small).

The outlined scheme allows to implement an honest protocol with the probability not worse than \(1 - 2^{-k}\) which is exponentially close to unit for large \(k\).

### 4 The Coin Tossing protocol for finite support states in the ideal communication channel.

Although the coin tossing protocol can be constructed on the basis of the bit commitment protocol, it is useful to formulate it explicitly.

- Just as in the outlined bit commitment protocol, the participants A and B agree upon the states used. When the protocol is started, each of them sends to the other \(N\) blocks containing \(k\) states randomly distributed among \(N\cdot k\) channels, the bits \(b_A\) and \(b_B\) chosen by users A and B, respectively, being coded as the parity bits of the strings consisting of \(N\) blocks. The users also agree in advance who is the winner if the final parity bit \(b = b_A \oplus b_B\) is 0 or 1.

- At an arbitrarily chosen moment of time \(\tau (-\Delta\tau < \tau < \tau_0 + \Delta\tau)\) one of the users, e.g. user A, announces for one half of all blocks through a classical channel which states were actually sent by him and identifies the blocks to which these states belong. After receiving these data, user B sends back to user A similar information for another half of his channels different from the channels disclosed by user A. Having obtained this information from B, user A discloses which channels belong to his still unrevealed blocks and the states that were actually sent through these channels. Then user B announces similar information about the rest of his channels. Since the channel length \(\tau_{ch} < \tau_0\), the exchange through the classical channel can be performed at the time when the users have access to only one half of each state.

- Just as in the above bit commitment protocol, the user cannot determine the parity bit chosen by the other participant with the probability exceeding 1/2 until the states become fully accessible to him.

- The users perform the measurements described by the identity resolution (65–67). Although the states are orthogonal (and, consequently, reliably distinguishable), the non-local nature of the projection operators \(P_{0,1}\) implies that the reliable distinguishability (67) can only be achieved with the correct states if one has access to the entire state which requires time \(2\Delta\tau + \tau_0\).

- After the time \(\tau_0 + \Delta\tau\) elapses and the states become fully accessible to both users, each of them compares the results of his measurements in each channel with the classical information provided by the other user. The protocol is abandoned if a discrepancy is found in at least one channel.

- Just as in the previous protocol, the probability for each user to obtain reliable information on the secret bit chosen by the other user before the states become entirely accessible, does not exceed the probability of simple guessing by an exponentially small value \(2^{-\alpha(N,k)Nk}\).

As a result, an honest parity bit (lot) \(b = b_A \oplus b_B\) arises with the probability exponentially close to unity \((1 - 2^{-k})\).

Obviously, even the correct states are sent by both users, one of them can abort the protocol claiming the discrepancy between his measurement results and the classical information provided by the other user if the arising parity bit does not suit him. However, this situation lies beyond the formulated problem and should be solved by different means.

Note that sending information through the classical channel is necessary to avoid the cheating strategy consisting in sending back the quantum states received from the other user (“send back” strategy). For example, one of the users can send no his own states at all and instead simply use a “mirror” to reflect back the states arriving from the other user. In that case the user (say, user A) which wins if the final
parity bit \( b = b_A \oplus b_B \) is zero can always cheat the other user since in this situation \( b_A \equiv b_B \) and, hence \( b = b_A \oplus b_B = b_B \oplus b_B \equiv 0 \).

Disclosure through the classical channel of the data on only one half of the quantum states sent by each user is also required to avoid the “send back” strategy. Had one user (say, user A) announced the information on all the quantum states, the second user (user B) could employ the “send back” strategy in quantum channels and simultaneously send through another classical channel the data just received from user A since \( \tau_{eh} < \tau_0 \). This strategy fails if only one half of all the quantum states are disclosed at first.

5 The Bit Commitment protocol for unlocalized states in the ideal communication channel.

So far we have considered the protocol employing the states with finite support (functions \( f(\tau) \in D(\tau) \)). The set of such functions forms a dense set in the space of functions describing the states of a free field (functions \( f(\tau) \in \mathcal{F}(\tau) \)). However, the field theory allows the states defined on the mass shell which are arbitrarily strongly localized and decaying with the rate arbitrarily close to the exponential law \( f(\tau) \propto e^{-\alpha \tau/\ln\ln...\ln\tau} \). Hence one can always choose the states in such a way that the measurements performed over them in a finite domain on the light cone \( \tau \) yield the net outcome probability arbitrarily close to unit, i.e. to make the contribution of the state tails at infinity arbitrarily small. To be more precise, the states (functions \( f(\tau) \)) and the measurement domain can be chosen in such a way that the probability of obtaining a result in the domain \( \Delta(\tau) \) be

\[
\Pr\{\Delta(\tau); i, i\} = \text{Tr}\left\{\left(\left(\int_{-\Delta \tau}^{\Delta \tau} \mathcal{M}(d\tau) \right) \otimes \mathcal{P}_i\right) |\psi_i\rangle \langle \psi_i|\right\} = \int_{-\Delta \tau}^{\Delta \tau} |f(\tau)|^2 d\tau = 1 - e^{-\xi} \to 1, \tag{69}
\]

where \( \xi \) can be arbitrarily large. Contribution from the state tails outside the domain \((-\Delta \tau, \Delta \tau)\)

\[
e^{-\xi} = \int_{|\tau| > \Delta \tau} |f(\tau)|^2 d\tau = \text{Tr}\left\{\left(\left(\int_{|\tau| > \Delta \tau} \mathcal{M}(d\tau) \right) \otimes \mathcal{P}_i\right) |\psi_i\rangle \langle \psi_i|\right\}. \tag{70}
\]

To preserve the analogy with the case of finite support states we shall write the stretched state with non-compact support in the form

\[
|\psi_{0,1}\rangle = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \left[f(\tau) + f(\tau - \tau_0)\right]|\tau\rangle d\tau \otimes |e_{0,1}\rangle, \quad f(\tau) \in \mathcal{F}(\tau),
\]

where function \( f(\tau) \) (\( f(\tau - \tau_0) \)), just as for the one-humped state (13) considered earlier, is strongly localized in the interval \((-\Delta \tau, \Delta \tau) \) \( (-\Delta \tau + \tau_0, \Delta \tau + \tau_0) \). The normalization condition yields

\[
\frac{1}{2} \int_{-\Delta \tau}^{\Delta \tau} |f(\tau)|^2 d\tau = \frac{1}{2} \int_{-\Delta \tau + \tau_0}^{\Delta \tau + \tau_0} |f(\tau - \tau_0)|^2 d\tau = \frac{1}{2} - \frac{1}{2}e^{-\xi} \to \frac{1}{2}, \tag{72}
\]

and

\[
\frac{1}{2} \int_{|\tau| > \Delta \tau} |f(\tau)|^2 d\tau + \frac{1}{2} \int_{|\tau + \tau_0| > \Delta \tau} |f(\tau - \tau_0)|^2 d\tau + \frac{1}{2} \int_{-\infty}^{\infty} [f^*(\tau)f(\tau - \tau_0) + f(\tau)f^*(\tau - \tau_0)]d\tau = e^{-\xi}. \tag{73}
\]

Measurements performed over the stretched state in a finite window \( \Delta(\tau_0) = (-\Delta \tau, \tau_0 + \Delta \tau) \) yields a result with the probability

\[
\Pr\{\Delta(\tau_0); i, i\} = \text{Tr}\left\{\left(\left(\int_{-\Delta \tau}^{\Delta \tau + \tau_0} \mathcal{M}(d\tau) \right) \otimes \mathcal{P}_{0,1}\right) |\psi_{0,1}\rangle \langle \psi_{0,1}|\right\} = 1 - \mathcal{O}(e^{-\xi}). \tag{74}
\]
The latter term arises due to the overlapping of the tails belonging to the two halves of the state centered at \( \tau = 0 \) and \( \tau = \tau_0 \) and does not exceed \( \mathcal{O}(e^{-\xi}) \).

Thus the statistics of measurements performed over the stretched states should yield the results in the interval \( (-\Delta \tau, \Delta \tau + \tau_0) \) with the probability \( 1 - \mathcal{O}(e^{-\xi}) \to 1 \) exponentially close to unit. The probability of obtaining a count beyond this interval does not exceed \( \mathcal{O}(e^{-\xi}) \) and can be made arbitrarily small by choosing appropriate \( f(\tau), \Delta \tau \), and \( \tau_0 \).

Preparation of a delocalized state with \( f(\tau) \in \mathcal{J}(\tau) \) formally requires an infinite time (if the state is generated by a point-like source) or access to the entire position space (if the state is prepared at a specified moment of time by a delocalized source). However, any realistic protocol should have a finite duration. To avoid the formal problems of that kind, it is convenient to argue in the following way as it is usually done in similar situations. User A controls the neighbourhood of point \( x_A \) and adiabatically turns on the source (at \( t \to -\infty \)) which generates the vector \( \psi_{0,1}(\tau_0) \) from the vacuum state. The source is described by the action of an \( \hat{S}(\tau, -\infty) \)-matrix on the vacuum state (we do not consider the problem of experimental realization of this source) and the produced state

\[
|\psi_{0,1}(\tau)\rangle = \hat{S}(\tau, -\infty)|0\rangle = \int_{-\infty}^{\tau} [f(\tau') + f(\tau' - \tau_0)]d\tau' \otimes |e_{0,1}\rangle,
\]

is sent into the communication channel as it is being formed.

At the intuitive level this source can be thought of as an atomic system (an atom) with a suitable spectrum excited by a classical field with the appropriately chosen amplitude shape which is turned on adiabatically and emits photons into the communication channel (preparation of unusual one- and two-photon states is discussed, e.g. in Ref. [35]).

User B performs measurements described by the identity resolution similar to Eq.(65):

\[
\mathcal{P}_0(\Delta) + \mathcal{P}_1(\Delta) + \mathcal{P}_\perp(\Delta) = I \otimes I_{C^2}, \quad I = \int_{-\infty}^{\infty} |\tau\rangle\langle \tau|d\tau,
\]

\[
\mathcal{P}_{0,1}(\Delta) = \left( \frac{1}{\sqrt{2}} \int_{-\Delta \tau}^{\Delta \tau + \tau_0} [f(\tau) + f(\tau - \tau_0)]|\tau\rangle d\tau \right) \left( \frac{1}{\sqrt{2}} \int_{-\Delta \tau}^{\Delta \tau + \tau_0} [f(\tau') + f(\tau' - \tau_0)]|\tau'| d\tau' \right) \otimes |e_{0,1}\rangle\langle e_{0,1}|,
\]

and then

\[
\mathcal{P}_\perp(\Delta) = I - \mathcal{P}_0(\Delta) - \mathcal{P}_1(\Delta).
\]

On the correct stretched states the measurement (76–78) yields the result with the probabilities

\[
\text{Tr}\{\rho_{0,1} \mathcal{P}_{0,1}(\Delta)\} = 1 - \mathcal{O}(e^{-\xi}),
\]

\[
\text{Tr}\{\rho_{0,1} \mathcal{P}_\perp(\Delta)\} = \mathcal{O}(e^{-\xi}).
\]

Similar to Eq.(68), for any states \( \rho \) which are not entirely concentrated in the sum of two intervals \( (-\Delta \tau, \Delta \tau) \) and \( (-\Delta \tau + \tau_0, \Delta \tau + \tau_0) \), the measurement (76–78) yields

\[
\int_{-\Delta \tau}^{\Delta \tau} \int_{-\Delta \tau}^{\Delta \tau} \delta_+(\tau - \tau') \rho(\tau, \tau')d\tau d\tau' = \frac{1}{2} - \frac{1}{2} e^{-\xi},
\]

\[
\int_{-\Delta \tau + \tau_0}^{\Delta \tau + \tau_0} \int_{-\Delta \tau + \tau_0}^{\Delta \tau + \tau_0} \delta_+(\tau - \tau') \rho(\tau, \tau')d\tau d\tau' = \frac{1}{2} - \frac{1}{2} e^{-\xi}.
\]

Hence, the delay of a state for more than \( 2\Delta \tau \) will result in the probability of obtaining a result on that state in the channel \( \mathcal{P}_{0,1} \) drops from almost 1 (79) to almost 1/2 (80),

\[
\text{Tr}\{\rho \mathcal{P}_{0,1}\} = \frac{1}{2} - \frac{1}{2} e^{-\xi}.
\]

Accordingly, the probability in the channel \( \mathcal{P}_\perp \) the probability rises from almost 0 (79) to almost 1/2 (80),

\[
\text{Tr}\{\rho \mathcal{P}_\perp\} = \frac{1}{2} - \frac{1}{2} e^{-\xi}.
\]
Just as in the previous case, user A prepares $N \cdot k$ states and sends them into the communication channels. As long as only the halves of the states are accessible ($\Delta \tau \leq \tau \leq \Delta \tau + \tau_0$), the probability for user B to obtain information on the secret parity bit chosen by user A does not exceed

$$P_c(\text{parity}) \approx \frac{1}{2} + \left(\frac{1}{2} - \mathcal{O}(e^{-\xi})\right)^{(N,k)Nk}.$$  

(83)

The probability of delaying the choice by user A for at least one of the blocks consisting of $k$ bits without being detected does not exceed

$$\left(\frac{1}{2} - \mathcal{O}(e^{-\xi})\right)^k.$$  

(84)

The probability of successfully completing the protocol (when all $N \cdot k$ states produce the results in channels $\mathcal{P}_{0,1}$) is

$$\left(1 - \mathcal{O}(e^{-\xi})\right)^{(N,k)Nk},$$

(85)

which can be made arbitrarily close to 1 by the appropriate choice of $N$, $k$, and $\xi$.

6 Conclusions.

Thus, the existence of maximum speed of quantum state propagation allows to develop the relativistic quantum bit and coin tossing protocols explicitly implementing the original idea of the protocol where one of the participants provides only part of information (part of quantum state) on the secret bit. However, the statistical nature of the measurement procedure in quantum mechanics does not allow (at least for the proposed protocol) to realize the honest protocol with the unit probability. Nevertheless, the honest protocol can be realized with the probability arbitrarily close to 1. In addition, the fundamental non-localizability of the quantum field states also imposes restrictions on the probability of the realization of the honest protocol in a finite time interval. Nevertheless, the possibility of construction of arbitrarily strongly localized states allows to develop an honest protocol with the success probability arbitrarily close to 1 for any time $\tau_0$ (time during which the bit secrecy is preserved).

In contrast to the non-relativistic protocols where only the structure of the states in the Hilbert space matters, the proposed relativistic protocols explicitly involve the stages of the state preparation and propagation in the space-time between the two distant users. Since the spin and helicity states do not exist separately from the spatial degrees of freedom of a quantum system, accounting for the spatial degrees of freedom extends the possibilities for construction of quantum cryptographic protocols.

It should be emphasized once again that the protocol is based on the orthogonal states. A non-zero error probability in distinguishing between the two orthogonal states arises due to the fact that a measurement can give no outcome at all (a photodetector will not fire, or the arrow of a classical device will not move) if the spatial domain accessible for the measurement does not “cover” the entire state. Therefore, the measurement can have three outcomes: the classical measuring device pointed to one of the channels $\mathcal{P}_0$ or $\mathcal{P}_1$, or did not change its state at all. If the device showed any particular result, the measured state is reliably identified. If the state is not entirely accessible, there is a non-zero probability for the device of not changing its state at all (showing no result at all), the larger the inaccessible part of the measured quantum state the larger this probability. In that case the observer can only guess what state he was dealing with (the state identification error in this situation being 1/2).

Since the spin and helicity do not exist separately from the spatial degrees of freedom, the restriction of access to the position space automatically restricts the access to the Hilbert state space. It is even possible to have the situation when the system state is completely inaccessible (the state amplitude is identical zero in the domain accessible for the measurement).

It should be noted that the considered situation is different from that discussed in Ref. [36] in connection with the analysis of Ref. [13] where a quantum cryptosystem based on orthogonal states was proposed. For a pair of orthogonal states of a composite system consisting of two subsystems $a$ and $b$ with the state space $\mathcal{H}_a \otimes \mathcal{H}_b$

$$|\psi_0\rangle = \alpha_0|\phi_0(a)\rangle \otimes |\phi_0(b)\rangle + \beta_0|\phi_1(a)\rangle \otimes |\phi_1(b)\rangle,$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are the spin states of the two particles, $\alpha_0$ and $\beta_0$ are real numbers, and $|\phi_0\rangle$ is an eigenstate of the spin operator $S_z$.
\[ |\psi_1 \rangle = \alpha_1 |\phi_0(a) \rangle \otimes |\phi_0(b) \rangle + \beta_1 |\phi_1(a) \rangle \otimes |\phi_1(b) \rangle, \]

where the states \( \alpha_{0,1} \) and \( \beta_{0,1} \) are such that the states \( |\psi_0 \rangle \) and \( |\psi_1 \rangle \) are orthogonal

\[ \langle \psi_0 | \psi_1 \rangle = 0. \]

If only one subsystem, e.g. \( \mathcal{H}_a \), is accessible, the states of the other subsystem \( b \) are non-orthogonal

\[ \rho_1 = \text{Tr}_{\mathcal{H}_a} \{|\psi_1 \rangle \langle \psi_1 |\}, \quad \rho_0 = \text{Tr}_{\mathcal{H}_a} \{|\psi_0 \rangle \langle \psi_0 |\}, \quad \text{Tr}_{\mathcal{H}_b} \{\rho_0 \cdot \rho_1\} \neq 0, \]

and, therefore, cannot be distinguished reliably. In our case the states remain orthogonal even after the restriction to a subspace and the absence of reliable distinguishability arises only due to the spatio-temporal structure of the states.

The protocol can also be extended to a noisy channel [38] since the initial orthogonality of the states employed allows to employ the classical codes [37].

In the proposed scheme the protocol duration time \( \approx \tau_0 \) is defined by the effective “extent” of the states which for photons can be estimated from the frequency spectrum width. The minimum attainable spectrum width in the visible range in the ring optical fiber resonators [39] is \( \Delta \omega \approx 10 \text{ kHz} \), the effective state length being \( L \approx c/\Delta \omega = 3 \times 10^{10}/10^4 = 3 \times 10^6 \text{ cm} \) (30 km). Accordingly, the time \( \tau_0 \approx 1/\Delta \omega \approx 10^{-3} \text{ s} \). Although there exist no fundamental restrictions on making time \( \tau_0 \) arbitrarily long (and, respectively, \( \Delta \omega \) arbitrarily small) this is very difficult technical problem. However, this circumstance does not matter for the Coin Tossing protocol, since the time \( \tau_0 \) for getting an honest lot can be arbitrary. On the contrary, for the Bit Commitment protocol the time \( \tau_0 \) is an important parameter since it determines the time interval during which the secrecy of the chosen bit is preserved. This is a rather general situation in the experimental realization of various systems for transfer and processing of “quantum information” where experimental realization of the possibilities formally allowed by the laws of quantum mechanics today requires solution of extremely difficult technical problems.

It should also be noted that the large-scale information transfer systems are based on optical fiber where the speed of signal propagation is somewhat lower than the speed of light in vacuum. However, this does not impose any restrictions since it is only necessary that the separation between the “halves” of the states used be larger than the channel length divided by the speed of light in the optical fiber.

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