The infinite Reynolds number limit and the quasi-dissipative anomaly

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December 11, 2020

Abstract

From a critical re-examination of Onsager’s (1949) pioneering paper, we find that his analysis was at odds with those of other workers because, unlike others, he did not actually take the limit of zero viscosity $\nu \to 0$. Instead, he simply set $\nu$ equal to zero, which is not the same thing. The final sentence of his paper, where he asserted that the detailed conservation symmetry does not hold globally because of the ‘infinite number of steps’ is contradicted by the analyses of Batchelor (1953) and Edwards (1965). Onsager’s work conflated the concepts of infinite Reynolds number limit (in the mathematical sense) with that of the breakdown of the continuum approximation. This was inconsistent, as the former is based on a continuum mechanics approach, which relies on the concept of an infinitely divisible fluid continuum; while the latter requires a physics approach, with recognition of the underlying molecular structure. We further show that his basic arguments are not in accord with experimental results (some of which were available at the time) which indicate that the onset of the zero-viscosity limit occurs at quite modest Reynolds numbers, where there is no possibility of the continuum approximation breaking down. This is the physical infinite Reynolds limit, which corresponds to the onset of scale-invariance. We conclude that attempts to make the inviscid Euler equation dissipative, by imposing constraints on its Fourier representation, amount at best to a quasi-dissipative interpretation of the increasing energy occupation of an infinite wavenumber space, and should be distinguished from the physics of viscous dissipation in real fluid turbulence.

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1 Introduction

In recent years the pioneering paper by Onsager [1] has exercised a growing influence on certain areas of fundamental research in turbulence. This is generally to do with the nature of dissipation and its relationship to the fluid viscosity in the limit of infinite Reynolds number (or, equivalently, of zero viscosity). The fact that dissipation appears to exist independently of viscosity in this limit, is often referred to as the dissipation anomaly, and basically this terminology seems to reflect an acceptance of Onsager’s view of the matter.

Of rather more concern, in this view of things the infinite Reynolds number limit is also seen as being equivalent to the breakdown of the continuum description which underpins the Navier-Stokes equation. Paradoxically, the Euler equation is regarded as surviving this catastrophe, despite depending on precisely the same underlying assumptions about the continuity of the fluid. It is argued that, in effect, this paradox is evaded by working with the Fourier representation of the Euler equation. As a result, the Euler equation is supposed to account for the dissipation by means of mechanisms which remain mysterious, but which are assumed to arise from constraints on its Fourier representation.

The purpose of the present paper is to make a critical examination of these ideas. In view of the potential for confusion, we begin by formally defining the dissipation rate and also establishing our notation.

1.1 The viscous dissipation

For a Newtonian fluid, the dissipation rate $\hat{\varepsilon}$ is formally defined in terms of the coefficient of kinematic viscosity $\nu$, thus:

$$\hat{\varepsilon} = \nu \sum_{\alpha,\beta} \left( \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right)^2; \quad (1.1)$$

see, for example, the book by Batchelor [2]. Note that we use Greek letters to denote the usual Cartesian tensor indices, where these take the values 1, 2 or 3, as appropriate for a three-dimensional space. There should be no confusion with the conventional use of Greek indices in Minkowski four-space for the same purpose.

For isotropic turbulence, we have that $u_\alpha(x,t) \equiv u(x,t)$ is a random variable with zero mean, and hence in this case $\hat{\varepsilon}$ is the instantaneous dissipation rate, and is also a random variable. For a turbulent flow we introduce the mean dissipation rate $\varepsilon$, as:

$$\varepsilon = \frac{\nu}{2} \sum_{\alpha,\beta} \left\langle \left( \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right)^2 \right\rangle, \quad (1.2)$$

where the angle brackets $\langle \ldots \rangle$ denote the operation of taking an average.

We will use the mean dissipation rate, along with the other averaged quantities, in our description of fluid turbulence; and we will use undecorated symbols such as $\varepsilon$ (and, for example, $E(k)$) for this purpose. It should be noted that this practice has long been usual (see for example the books by Batchelor [3] and by Landau and Lifshitz [4]). However, it is by no means universal and many people use $\bar{\varepsilon}$ or $\langle \varepsilon \rangle$ for the mean dissipation (or even $\epsilon$ rather than $\varepsilon$), so it is necessary to be quite specific about this. Note that hereafter, when we use the term ‘dissipation’, we shall always be referring to the mean dissipation rate. This is of course quite usual in the subject, but we should emphasise that we will never use the unqualified term to refer to the instantaneous dissipation.

3
2 Onsager (1949)

The paper by Onsager (herafter referred to as Onsager49) is in two parts. The first part deals with the statistical mechanics of two-dimensional vortices in an ideal (i.e. frictionless) fluid. Onsager concluded this part by wondering: “how soon will the vortices discover that there are three dimensions?”. He then commented:

The latter question is important because in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available.

In this paper we shall argue that this statement is misleading because it is untrue. By dissipation, in its usual sense, we mean the the conversion of macroscopic kinetic energy into microscopic random motion which is perceived as heat. This is not the same as the absorption of macroscopic kinetic energy in the infinitely large wavenumber space which exists if the viscosity is zero. In the present paper we consistently put the case that the limit of infinite Reynolds numbers must be explored by a standard limiting procedure. In this regard, we are concerned with the second part of Onsager49, which is entitled ‘Turbulence’.

Not surprisingly, in view of its date of publication, Onsager49 is rather old-fashioned in its notation and procedures. In making our commentary on this work, we will employ a fully modern notation, with the occasional qualification in order to indicate its equivalent in Onsager49. We will also divide our commentary into subsections, which we hope will be helpful to the reader. We note that equations (9) - (13) of Onsager49 constitute a real-space sub-section (in effect) and the Fourier wavenumber treatment begins with his equation (14). We shall discuss the real-space subsection first, with emphasis on his equation (11) which is nowadays often referred to as the Taylor dissipation surrogate.

2.1 The Taylor dissipation surrogate

Equation (11) of Onsager49 may be rewritten; first as:

\[ \varepsilon = -\frac{dU^2}{dt} = C_\varepsilon U^3 L, \]

where \( \varepsilon \) is the mean dissipation rate (Onsager calls it \( Q \)), the first term on the right hand side is the negative of the energy decay rate and the second term on the right is an intermediate step where \( C_\varepsilon \) is a coefficient which is a function of the Reynolds number; and secondly as:

\[ \lim_{R_\lambda \to \infty} \varepsilon = -\frac{dU^2}{dt} = C_{\varepsilon,\infty} \frac{U^3}{L}, \]

where \( C_{\varepsilon,\infty} \) is the asymptotic constant that \( C_\varepsilon \) becomes in the limit of large Reynolds numbers. Other symbols are: \( U \) is the root-mean-square velocity and \( L \) is the integral length scale.

Strictly speaking, it is the second of our two equations which is equivalent to Onsager’s equation (11); because he stated that ‘the Reynolds number must be sufficiently large’. We should also mention that we are considering the specific case of free decay. In 1949, the concept of stirring forces had not yet been introduced to the study of isotropic turbulence and so presumably Onsager saw no necessity to mention that it was force free.

At this point we begin our deconstruction of the turbulence section of Onsager49, by focusing on a specific quotation which encloses his equation (10) and goes as follows:
Experience indicates that for large Reynolds numbers the over-all rate of dissipation is completely determined by the intensity $U^2$ together with the “macroscale” $L$ of the motion, and the viscosity plays no primary role, except through the condition that the Reynolds number $\ldots$ must be sufficiently large.

This statement is misleading. The viscosity enters into the dissipation under all circumstances, and at all Reynolds numbers, through equation (1.2). In order to assess its behaviour at large Reynolds numbers it is necessary to apply a proper limiting procedure to this expression. That has to be done in wavenumber space. It was not done by Onsager, and we will return to how others have handled this in a subsequent section.

For the moment we confine our attention to the expression $C_\varepsilon U^3/L$. Although this has often been referred to as the ‘Taylor dissipation surrogate’, it is actually a surrogate for the rate of inertial transfer and becomes equal to the dissipation rate when the Reynolds number is large enough and $C_\varepsilon \to C_\varepsilon,\infty$. This corresponds to the onset of scale invariance of the inertial transfer through wavenumber $k$, and one can only discuss this further in the Fourier representation. Accordingly we defer such discussion to a later stage. For the moment we mention that these facts have been established by McComb et al. \[6, 7\], from both analysis and numerical simulation. We may also mention in passing that (as we shall see later) this asymptotic behaviour is observed for $R_\lambda \sim 100$, which is quite a large Reynolds number but is in no danger of imperilling the continuum nature of the fluid!

2.2 The Navier-Stokes equations (NSE) in wavenumber space

In Onsager, the NSE appear as equation (13). Then a Fourier-series representation is introduced through equations (14,14a), and equation (15) is the resulting discrete form of the NSE in wavenumber space. Between equations (13) and (14), another interesting comment arises, thus:

Before we can arrive at a completely self-contained theory we shall have to determine somehow, from the laws of dynamics, a statistical distribution in function space, and for the time being we do not know enough about how to describe such distributions.

This is less clear than it might be. One assumes that Onsager was referring to the probability distribution, but it is not clear whether he was concerned about the theoretical procedure for obtaining such a distribution, or about the mathematical aspects of representing it as a functional. In either case, both problems were solved by Edwards in 1964 [8], who obtained the turbulence probability distribution as an operator-product expansion about a Gaussian. This work was followed by others and a recent review (and extension of the Edwards analysis to the two-time case) can be found in the paper by McComb and Yoffe [9].

In introducing Fourier series, Onsager referred only to a finite volume $V$. Later on it became usual to refer to the fluid as occupying a box in the form of a cube with a side of length $L_{\text{box}}$, with periodic boundary conditions being imposed. If we take the limit $L_{\text{box}} \to \infty$, then we may replace sums over wave-vectors by integrals, according to

$$\lim_{L_{\text{box}} \to \infty} \left( \frac{2\pi}{L_{\text{box}}} \right)^3 \sum_k = \int d^3k.$$  

\[2.3\]

\[1\] Apparently this was first noted by Obukhov (see page 200 of the book by Lesieur [5]) and has since become an important concept in the theory of turbulence.
In the usual way we obtain \( u_\alpha(k, t) \), the Fourier transform of \( u_\alpha(x, t) \) from
\[
 u_\alpha(k, t) = \left( \frac{1}{2\pi} \right)^3 \int d^3x \ u_\alpha(x, t) \exp(-i k \cdot x); \tag{2.4}
\]
while the Fourier transform pair is completed by
\[
 u_\alpha(x, t) = \int d^3k \ u_\alpha(k, t) \exp(i k \cdot x). \tag{2.5}
\]
In the 1970s, it became usual to just go directly to the Fourier-transformed NSE, and the solenoidal form of this is now well known. For the velocity field \( u_\alpha(k, t) \) in wavenumber \( (k) \) space (see either [10] or [11]) we have the solenoidal NSE as:
\[
\left( \frac{\partial}{\partial t} + \nu k^2 \right) u_\alpha(k, t) = M_{\alpha\beta\gamma}(k) \int d^3j \ u_\beta(j, t) u_\gamma(k-j, t), \tag{2.6}
\]
where the inertial transfer operator \( M_{\alpha\beta\gamma}(k) \) is given by
\[
 M_{\alpha\beta\gamma}(k) = (2i)^{-1} [k_\beta P_{\alpha\gamma}(k) + k_\gamma P_{\alpha\beta}(k)], \tag{2.7}
\]
and \( i = \sqrt{-1} \), while the projector \( P_{\alpha\beta}(k) \) is expressed in terms of the Kronecker delta as
\[
 P_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}. \tag{2.8}
\]
Note that the use of the projector ensures that the velocity field remains solenoidal. Note also that our equation (2.6) is equivalent to equation (15) in Onsager49.

### 2.3 Moments and spectra

The covariance of the fluctuating velocity field may be introduced as
\[
 C_{\alpha\beta}(k, k'; t, t') = \langle u_\alpha(k, t) u_\beta(k', t') \rangle, \tag{2.9}
\]
and for isotropic, homogeneous turbulence we may write this as
\[
 \langle u_\alpha(k, t) u_\beta(-k, t') \rangle = P_{\alpha\beta}(k) \delta(k + k') C(k; t, t'), \tag{2.10}
\]
where we anticipate the effects of homogeneity in writing the left hand side. As is usual, the angle brackets \( \langle \ldots \rangle \) denote the operation of taking an average.

If we consider the case \( t = t' \), then we may introduce the spectral density function:
\[
 C(k, t) \equiv C(k; t, t), \tag{2.11}
\]
and the energy spectrum:
\[
 E(k, t) = 4\pi k^2 C(k, t). \tag{2.12}
\]
Lastly, we may also introduce the third-order moment, as:
\[
 \langle u_\alpha(k, t) u_\beta(k', t') u_\gamma(k'', t'') \rangle = \delta(k + k' + k'') C_{\alpha\beta\gamma}(k, k', k''; t, t', t''). \tag{2.13}
\]
This will be used to obtain the energy transfer spectrum when we consider the Lin equation.
2.4 The statistical equations

We form an equation for the covariance $C(k; t, t')$ in the usual way. Multiply each term in (2.6) by $u_{\alpha}(-k, t')$ and take the average, to obtain:

$$
\left( \frac{\partial}{\partial t} + \nu k^2 \right) C(k; t, t') = \frac{1}{2} M_{\alpha\beta\gamma}(k) \int d^3j \langle u_{\beta}(j, t)u_{\gamma}(k - j, t)u_{\alpha}(-k, t') \rangle,
$$

(2.14)

where we have used equations (2.8) and (2.10), cancelled the factor $\delta(k + k')$ across, and invoked isotropy, along with the property $Tr P_{\alpha\beta}(k) = 2$. We can also write this in the compact form:

$$
\left( \frac{\partial}{\partial t} + \nu k^2 \right) C(k; t, t') = J(k; t, t'),
$$

(2.15)

where $J(k; t, t')$ is just the right hand side of (2.14). The problem of expressing this in terms of the covariance is the well-known statistical closure problem.

Also using (2.6), we can obtain an equation describing the energy balance. On the time-diagonal, the covariance equation takes the form for isotropic turbulence:

$$
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) C(k; t, t) = Re \left[ M_{\alpha\beta\gamma}(k) \int d^3j \langle u_{\beta}(j, t)u_{\gamma}(k - j, t)u_{\alpha}(-k, t) \rangle \right] + F(k),
$$

(2.16)

where $F(k)$ is the energy injection spectral density, as defined in terms of the stirring forces, if present. Again, we can write this in a more compact form as:

$$
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) C(k; t, t) = H(k; t, t) + F(k),
$$

(2.17)

where $H(k; t, t)$ can be determined by comparison with the right hand side of (2.16).

The energy spectrum is related to the spectral density by equation (2.12). Accordingly, if we multiply (2.16) across by $4\pi k^2$, and rearrange terms, the governing equation of the energy spectrum takes the form:

$$
\frac{\partial E(k, t)}{\partial t} = W(k) + T(k, t) - 2\nu k^2 E(k, t),
$$

(2.18)

where the energy transfer spectrum is $T(k, t) = 4\pi k^2 H(k; t, t) \equiv 4\pi k^2 H(k, t)$. This equation for the energy spectrum is well known, and is nowadays increasingly referred to as the Lin equation [12] [11]. For later convenience, we may also rearrange it as:

$$
-T(k) = I(k) - 2\nu k^2 E(k),
$$

(2.19)

where $I(k) = \partial E(k, t)/\partial t$ or $W(k)$, according to whether we are studying free decay or forced, stationary isotropic turbulence. Obviously in the former case, one could put in the explicit time-dependences again.

More detailed discussions of these equations can be found in Chapter 3 of the book [11]. A schematic view of the energy transfer processes involved may be found in Fig. 1, where we have used the notation of equation (2.19).

2.5 Conservation properties of the inertial-transfer term

In order to demonstrate the conservation properties of the energy transfer spectrum $T(k)$, we make use of the triple-moment as defined by equation (2.13). Introducing the function $S(k, j; t)$ such that:

$$
T(k, t) = \int_0^\infty dj \, S(k, j; t),
$$

(2.20)
Figure 1: Schematic view of the terms in the Lin equation. The input spectrum \( I(k) \) represents either the work spectrum \( W(k) \) or \(-\partial E(k,t)/\partial t\). The other symbols are as defined in equation (2.18).

we have

\[
S(k,j;t) = 2\pi k^2 j^2 \int d\Omega_j M_{\alpha\beta\gamma}(k) \left[ C_{\beta\gamma\alpha}(j,k-j,-k;t) - C_{\beta\gamma\alpha}(-j,-k+j,k;t) \right], \quad (2.21)
\]

where the solid angle \( \Omega_j \) is defined from:

\[
\int d^3 j = \int_0^\infty j^2 \, dj \int d\Omega_j. \quad (2.22)
\]

Not that interchanging \( k \) and \( j \) on the right hand side of equation (2.21) maps the second term into the first, and vice versa, hence the sign of right hand side changes and we conclude:

\[
S(k,j;t) = -S(j,k;t). \quad (2.23)
\]

Therefore it follows from (2.20) that the integral of the transfer spectrum over all wavenumber space vanishes, and hence energy is conserved. A fuller treatment can be found in Chapter 3 of the book [11].

Now Onsager’s equivalent of our \( S(k,j) \) is \( Q(k,k') \) and his equation (17) is just our equation (2.23), written is a slightly different form. The point which we cannot emphasise strongly enough is that this antisymmetry guarantees the conservation of turbulence kinetic energy by nonlinear transfer for both the NSE and the Euler equations. We shall return to this point later. First, we turn to how others considered the actual limiting procedure which is involved in taking the infinite Reynolds number limit.
3 The infinite Reynolds number limit of Batchelor and others

The earliest work on turbulence in the first part of the twentieth century was on shear flows. Even then, systematic experiments with increasing values of the Reynolds number led to an appreciation of a limiting form of behaviour at large Reynolds numbers. In particular, the work of Nikuradse in 1932 on pipe flows showed the development of limiting behaviour of mean velocity profiles for the range \(4 \times 10^3 \leq Re \leq 3.2 \times 10^6\). For further details and references see the book by Goldstein [13]. A concise discussion can also be found in Section 1.6 of [10]. In addition, there was much interest in the eddy-viscosity concept, in which the randomizing motions of turbulent eddies were interpreted as being analogous to the randomizing motions of the molecules of the fluid. Such eddy viscosities depended on the type of flow, but were generally found to be something like two orders of magnitude larger than the molecular viscosity.

Of course this corresponded to a much larger dissipation rate than in laminar flow (usually interpreted as greater resistance to flow, in those days) but it should be emphasised that this dissipation was not a purely nonlinear effect. It was in fact a true dissipation of turbulent kinetic energy due to the effects of the fluid viscosity. This was where the cascade came in, as being a way of transferring turbulent energy to scales where it would be most effectively turned into heat. We shall come back to this point later, but for the moment we turn to the idea of an infinite Reynolds number limit.

In his book on turbulence, [3]: first published in 1953), Batchelor introduced a local (in wavenumber) Reynolds number for homogeneous turbulence and considered its behaviour as the viscosity tended to zero. If we denote this by \(R(k, t)\), then he defined it as:

\[
R(k, t) = \left( \frac{\langle E(k, t) \rangle}{\nu k^2} \right)^{\frac{1}{2}},
\]

where \(E(k, t)\) is the energy spectrum, and noted that the effect of decreasing \(\nu\) is to increase the dominance of the inertia forces over the viscous forces for the motion associated with that degree of freedom. He went on to say:

Consequently the region of wave-number space which is affected significantly by the action of viscous forces moves out from the origin towards \(k = \infty\) as the Reynolds number increases. In the limit of infinite Reynolds numbers the sink of energy is displaced to infinity and the influence of viscous forces is negligible for wave-numbers of finite magnitude.

Note that the emphasis is ours.

A more extensive discussion of the infinite Reynolds number limit can be found in Section 6.2 of Batchelor’s book (ibid), but the fragment given here is sufficient for our present purposes. This idea was extended by Edwards [14], who came to the same conclusion from a consideration of the Kolmogorov dissipation wavenumber in the limit of infinite Reynolds number, and represented the dissipation rate by a delta function at infinity. In order to study the stationary case, he balanced this with an input term in the form of a delta function at the origin.

Of course, if we interpret ‘infinity’ in the purely mathematical sense, then the above limiting cases are only valid for a picture of fluid motion which is part of continuum mechanics. In practice a real fluid has a molecular structure which limits the continuum approximation to length scales larger than inter-atomic spacings. To take account of this it is better to use the concept of the inertial flux of energy and this we will now do.
3.1 Scale-invariance of the inertial flux as the infinite Reynolds number limit

The concept of the inertial flux (and its scale-invariance) was already known to Obukhov in 1941 (see reference [5]), used by Onsager in 1945 [15], and discussed by Batchelor in 1953 [16]. But its first formal use was by Kraichnan [17] who used it to test the inertial-range behaviour of his direct-interaction approximation. In the process, he introduced a useful symbol for this flux, which he referred to as the transport power and defined (actually in the later form to be found in [18]) as

$$
\Pi(\kappa, t) = \int_{\kappa}^{\infty} dk T(k, t) = - \int_{\kappa}^{0} dk T(k, t),
$$

where \( \Pi(\kappa, t) \) is the flux of energy through wavenumber \( \kappa \) due to the nonlinear term.

It was understood at an early stage that the inertial flux would increase with increasing Reynolds number until it became equal to the dissipation rate \( \varepsilon \); and that this was the largest value that the flux could achieve. Increasing the Reynolds number further merely increased the extent of wavenumber space, as the Kolmogorov dissipation wavenumber was also increased. Accordingly, it was recognised that the condition:

$$
\Pi = \varepsilon,
$$

was the criterion for the onset of the inertial range of wavenumbers. This was stated in words in the original (1953) edition of Batchelor’s book [3] and in the lectures by Saffman [19]. It can be found in the books by Leslie [20], McComb [10,11], Davidson [21], Lesieur [5], and Sagaut and Cambon [12]. The criterion is used in direct numerical simulation (e.g. see references [22], [23] and [24]) and in theoretical work (e.g. see the papers by Chasnov [25], Bowman [26], Qian [27], Thacker [28], Falkovich [29], and Lundgren [30]), where the fact that the criterion holds over a range of wavenumbers is usually referred to as scale-invariance.

The onset of scale-invariance may be taken as equivalent to attaining the infinite Reynolds number limit. As the Reynolds number is increased further, there can be no resulting qualitative change. The inertial range of wavenumbers, with its constant flux of energy through it, merely becomes more extensive. In order to understand this better, it may be helpful to consider how the transfer spectrum \( T(k) \) behaves under these circumstances and this is the subject of the next sub-section.

3.2 Dependence of the transfer spectrum \( T(k) \) on Reynolds number

The first measurements of the transfer spectrum were obtained by Uberoi [31] in 1963 for freely decaying turbulence. Uberoi obtained the shape of \( T(k) \), as shown schematically in Fig. 1 by measuring the decay rate of the energy spectrum and the dissipation; and using the Lin equation (in the form given by equation (2.19)) to evaluate the transfer spectrum. A direct measurement of \( T(k) \) was first carried out in 1969 by Van Atta and Chen [32], and of course nowadays DNS and closure approximations produce this curve routinely.

As an aside, we should mention that Uberoi found his results rather surprising, in that he expected that the scale-invariance condition \( \Pi = \varepsilon \) would be paralleled by the condition \( T = 0 \), for the same range of wavenumbers. The fact that the transfer spectrum crossed the \( k \)-axis at a single point he attributed to the Reynolds number being too small to show the effect properly. This led Lumley to introduce an approximate criterion [33] to allow the inertial range to be identified using the transfer spectrum; and an example of its use can be found in [34].
more extensive investigations confirmed that \( T(k) \) possessed only a single zero-crossing [35], [36]. Ultimately, in 2008 this puzzle was formulated as a paradox and resolved by McComb [37], who introduced a decomposition of the transfer function, based on the antisymmetry of the exchange function \( S(k, j) \). More recently a modified Lin equation was introduced in order to avoid the paradox in the first place [38]. However, we mention these matters mainly for sake of completeness and we will not pursue them further here. What follows is based mainly on the more recent book by McComb [11], but we should also mention the comprehensive analysis of Tchoufag, Sagaut and Cambon [39], which deals with both real-space and spectral representations.

We can approach the idea of an infinite Reynolds number limit, by considering the maximum value of the flux which must correspond to the point where the transfer spectrum becomes zero. We will denote this value of the wavenumber by \( k^* \), and hence we may write:

\[
T(k^*) = 0 \quad \text{and hence} \quad \Pi_{max} = \Pi(k^*).
\] (3.4)

\( \Pi_{max} \) is the maximum value of the flux at any Reynolds number, but as the Reynolds number increases, it is bounded from above by the dissipation rate. Thus, introducing the symbol \( \varepsilon = \Pi_{max} \), we have the criterion for having reached the infinite Reynolds number limit:

Criterion for the onset of infinite Reynolds number behaviour: \( \frac{\Pi_{max}}{\varepsilon} = 1; \) or \( \frac{\varepsilon}{\varepsilon} = 1 \). (3.5)

In Fig. 2 we show the development of the infinite Reynolds number limit. In fact we plot the reciprocal of the above criterion, as this figure is taken from an investigation that was actually studying the behaviour of the dissipation rate [40]. Nevertheless, the qualitative behaviour should be perfectly clear. Those readers interested in seeing the details of how this behaviour develops in the spectral picture should consult reference [41].

It is of interest to reconcile the infinite Reynolds number limit of Edwards [14] with the present treatment. We may rewrite equation (2.19) for the stationary case, and then take the infinite Reynolds number limit, thus:

\[
-T(k) = W(k) - 2\nu k^2 E(k) = \varepsilon W \delta(k) - \varepsilon \delta(k - \infty),
\] (3.6)

where we note that energy conservation gives us \( \varepsilon W = \varepsilon \).

Of course, the Edwards result only applies (as does Batchelor’s) in continuum mechanics. But it can be useful if one is testing a statistical closure theory which is also applied to a fluid in the continuum mechanics description. This is where the transport power comes in. From equation (3.2) applied to (3.6), we have:

\[
\Pi(\kappa) = \varepsilon W = \varepsilon.
\] (3.7)

It may seem that (3.6) is in conflict with the experimental finding that the transfer spectrum has a single zero-crossing. It is of interest to note that Tchoufag et al. [39] found that, when the input and output regions were well separated, \( T(k) \) took the form of a quasi-plateau, being a linear region which was at a small angle to the axis. As the Reynolds number increases, this angle becomes smaller, and evidently the Edwards result would be appropriate for a true continuum fluid at where the Reynolds number is actually infinite. But of course the use of the criterion based on the flux brings it into conformity with the more physical picture.
Figure 2: Variation of the mean dissipation rate $\varepsilon$ for forced turbulence, divided by the peak inertial transfer rate $\varepsilon_T \equiv \Pi_{\text{max}}$, with increasing Taylor-Reynolds number, showing the onset of scale-invariance and hence the limit of infinite Reynolds numbers. Data are openly available online from the University of Strathclyde KnowledgeBase [http://dx.doi.org/10.15129/64a4a042-7d0d-48ce-8afa-21f9883d1e84].

4 The nature of viscous dissipation

In fluid dynamics, the term ‘dissipation’ invariably means ‘viscous dissipation’. Moreover, the term ‘viscosity’ is short for ‘coefficient of viscosity’, and assigns a magnitude to a process in which relative motion of a fluid is randomized and converted to molecular motion; which in turn corresponds to a rise in the temperature of the fluid. In other words, viscous dissipation is a thermodynamic process; although normally the thermodynamic aspects can be neglected for all practical purposes.

From this it follows that, contrary to Onsager49, when the viscosity is zero there is no viscous dissipation, and hence no dissipation at all. Evidently, with an infinitely large wavenumber space, energy can be absorbed and endlessly transferred to ever increasing wavenumbers. But this disappearance of turbulent kinetic energy can at best be described as quasi dissipation.

With reference to the last sentence in Onsager49, it is worth pointing out that Euler’s equation has been used to study absolute equilibrium ensembles in fluid dynamics. It is well known that these systems, when subject to an arbitrary initial random field, have equipartition solutions. Of course such systems have to be truncated to finite volumes of wavenumber space; but there is no reason to suppose that, as the size of the space is increased, there is any deviation from this behaviour. For a general treatment, see the book by Lesieur [5]; for some specific remarks relevant to our present discussion see the paper by Kraichnan [18]; and for some other applications see [42, 43].

When the viscous term is added back in (i.e. restoring the NSE), the effect is symmetry-breaking, because of the factor $k^2$. So the viscosity restores the uni-directional energy transfer process and also, through damping the higher-order modes, removes the need for truncation. This brings us back to the question of the supposed breakdown of the continuum description
at high Reynolds numbers. This was raised as a question by Batchelor (see page 5 of [3]), who answered his own question as follows:

However, the action of viscosity is to suppress strongly the small-scale components of the turbulence and we shall see that for all practical conditions the spectral distribution of energy dies away effectively to zero long before length scales comparable with the mean free path are reached. As a consequence we can ignore the molecular structure of the medium and regard it as a continuous fluid.

Leslie went further and presented an order-of-magnitude calculation in his book (see pages 3-4 of [20]), from which he concluded that, at a Reynolds number of $10^6$ in pipe flow, the smallest turbulence scales would be three orders of magnitude greater than the inter-molecular spacing of the fluid.

This calculation is unusual and perhaps unique. It is curious how often workers in the field express concern about possible singularities, or advocate some course of action to circumvent them, but never actually present even an estimate of the conditions under which they may be expected to occur.

5 Conclusion

We have concluded that, with the modern interpretation of the infinite Reynolds limit as being equivalent to the onset of scale-invariant inertial transfer, there is really no need to suppose that it amounts to the viscosity actually being equal to zero. Nor is there any reason to suppose that the simple symmetry $S(k, j) = -S(j, k)$, which guarantees both energy conservation in the NSE and equipartition in the Euler equation, will mysteriously disappear under any of the limiting processes available to us. Accordingly, we are unable to see what constitutes an anomaly. The Fourier representation of the Euler equation cannot be altered to destroy the above symmetry in any physically meaningful way. The Euler equation is inviscid and conservative and does not possess the ability to dissipate energy in the usual sense of real fluid mechanics.

The term anomaly is often used in connection with the observed limiting behaviour when the dimensionless dissipation is plotted against Reynolds number. But this process is fully understood at a spectral level, so there does not seem to be any justification of this usage. However, if the term is used to mean that the Euler equation can somehow be made to give the appearance of dissipating energy, then the most we would concede is that this could be described as a quasi-dissipation anomaly.

Lastly, it may be of interest to note that an informal introduction to the work of this paper is contained in the blog posts for 12 Nov, 19 Nov and 26 Nov 2020 at: blogs.ed.ac.uk/physics-of-turbulence/

Acknowledgement

SRY acknowledges support from the UK EPSRC (grant number EP/N028694/1) and from the STFC (grant number ST/G008248/1)

References

[1] L. Onsager. Statistical Hydrodynamics. *Nuovo Cim. Suppl.*, 6:279, 1949.
[2] G. K. Batchelor. *An introduction to fluid dynamics*. Cambridge University Press, Cambridge, 1967.

[3] G. K. Batchelor. *The theory of homogeneous turbulence*. Cambridge University Press, Cambridge, 2nd edition, 1971.

[4] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics*. Pergamon Press, London, English edition, 1959.

[5] Marcel Lesieur. *Turbulence in Fluids*. Springer, Dordrecht, 4th edition, 2008.

[6] W. David McComb, Arjun Berera, Matthew Salewski, and Sam R. Yoffe. Taylor’s (1935) dissipation surrogate reinterpreted. *Phys. Fluids*, 22:61704, 2010.

[7] W. D. McComb, A. Berera, S. R. Yoffe, and M. F. Linkmann. Energy transfer and dissipation in forced isotropic turbulence. *Phys. Rev. E*, 91:043013, 2015.

[8] S. F. Edwards. The statistical dynamics of homogeneous turbulence. *J. Fluid Mech.*, 18:239, 1964.

[9] W. D. McComb and S. R. Yoffe. A formal derivation of the local energy transfer (LET) theory of homogeneous turbulence. *J. Phys. A: Math. Theor.*, 50:375501, 2017.

[10] W. D. McComb. *The Physics of Fluid Turbulence*. Oxford University Press, 1990.

[11] W. David McComb. *Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures*. Oxford University Press, 2014.

[12] P. Sagaut and C. Cambon. *Homogeneous Turbulence Dynamics*. Cambridge University Press, Cambridge, 2008.

[13] S. Goldstein. *Modern developments in fluid dynamics*. Oxford University Press, 1938.

[14] S. F. Edwards. Turbulence in hydrodynamics and plasma physics. In *Proc. Int. Conf. on Plasma Physics, Trieste*, page 595. IAEA, 1965.

[15] L. Onsager. The Distribution of Energy in Turbulence. *Phys. Rev.*, 68:281, 1945.

[16] G. K. Batchelor. *The theory of homogeneous turbulence*. Cambridge University Press, Cambridge, 1st edition, 1953.

[17] R. H. Kraichnan. The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.*, 5:497–543, 1959.

[18] R. H. Kraichnan. Decay of isotropic turbulence in the Direct-Interaction Approximation. *Phys. Fluids*, 7(7):1030–1048, 1964.

[19] P. G. Saffman. Lectures on homogeneous turbulence. In N. Zabusky, editor, *Topics in nonlinear physics*, pages 485–614. Springer-Verlag, 1968.

[20] D. C. Leslie. *Developments in the theory of turbulence*. Clarendon Press, Oxford, 1973.

[21] P. A. Davidson. *Turbulence*. Oxford University Press, 2004.
[22] W. D. McComb and C. Johnston. Conditional mode elimination and scale-invariant dissipation in isotropic turbulence. *Physica A*, 292:346, 2001.

[23] T. Ishihara, T. Gotoh, and Y. Kaneda. Study of high-Reynolds number isotropic turbulence by direct numerical simulation. *Ann. Rev. Fluid Mech.*, 41:165, 2009.

[24] W. D. McComb, S. R. Yoffe, M. F. Linkmann, and A. Berera. Spectral analysis of structure functions and their scaling exponents in forced isotropic turbulence. *Phys. Rev. E*, 90:053010, 2014.

[25] J. R. Chasnov. Simulation of the Kolmogorov inertial subrange using an improved subgrid model. *Phys. Fluids A*, 3:188, 1991.

[26] John C. Bowman. On inertial-range scaling laws. *J. Fluid Mech.*, 306:167–181, 1996.

[27] J. Qian. Inertial range and the finite Reynolds number effect of turbulence. *Physical Review E*, 55:337, 1997.

[28] W. D. Thacker. A path integral for turbulence in incompressible fluids. *J. Math. Phys.*, 38:300, 1997.

[29] G. Falkovich. Introduction to developed turbulence. In M. Shats and H. Punzmann, editors, *Turbulence and coherent structures in Fluids, Plasmas and Nonlinear Media*, page 1. World Scientific, New Jersey, 2006.

[30] Thomas S. Lundgren. Turbulent scaling. *Phys. Fluids*, 20:31301, 2008.

[31] M. S. Uberoi. Energy transfer in isotropic turbulence. *Phys. Fluids*, 6:1048, 1963.

[32] C. W. van Atta and W. Y. Chen. Measurements of spectral energy transfer in grid turbulence. *J. Fluid Mech.*, 38:743–763, 1969.

[33] J. L. Lumley. The spectrum of nearly inertial turbulence in a stably stratified fluid. *J. Atmos. Sci.*, 21:99, 1964.

[34] W. D. McComb, M. J. Filipiak, and V. Shanmugasundaram. Rederivation and further assessment of the LET theory of isotropic turbulence, as applied to passive scalar convection. *J. Fluid Mech.*, 245:279–300, 1992.

[35] P. Bradshaw. Conditions for the existence of an inertial subrange in turbulent flow. Paper 1220, National Physical Laboratory, Aerodynamics Division, 1967.

[36] K. N. Helland, C. W. Van Atta, and G. R. Stegen. Spectral energy transfer in high Reynolds number turbulence. *J. Fluid Mech.*, 79:337–359, 1977.

[37] David McComb. Scale-invariance in three-dimensional turbulence: a paradox and its resolution. *J. Phys. A: Math. Theor.*, 41:75501, 2008.

[38] W. D. McComb. A modified Lin equation for the energy balance in isotropic turbulence. *Theoretical & Applied Mechanics Letters*, 10:0, 2020.

[39] J. Tchoufag, P. Sagaut, and C. Cambon. Spectral approach to finite Reynolds number effects on Kolmogorov’s 4/5 law in isotropic turbulence. *Phys. Fluids*, 24:015107, 2012.
[40] S. R. Yoffe. *Investigation of the transfer and dissipation of energy in isotropic turbulence*. PhD thesis, University of Edinburgh, 2012.

[41] V. Shanmugasundaram. Modal interactions and energy transfers in isotropic turbulence as predicted by local energy transfer theory. *Fluid. Dyn. Res*, 10:499, 1992.

[42] R. H. Kraichnan. Helical turbulence and absolute equilibrium. *Journal of Fluid Mechanics*, 59:745 – 752, 1973.

[43] R. H. Kraichnan and D. Montgomery. Two-dimensional turbulence. *Reports on Progress in Physics*, 43, 06 1980.