Multiplicativity of Maximal $p$–Norms in Werner–Holevo Channels for $1 < p \leq 2$

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Abstract

Recently, King and Ruskai [1] conjectured that the maximal $p$–norm of the Werner–Holevo channel is multiplicative for all $1 \leq p \leq 2$. In this paper we prove this conjecture. Our proof relies on certain convexity and monotonicity properties of the $p$–norm.

1 Introduction

A quantum channel $\Phi$ is described by a completely positive trace–preserving map [3, 4], which acts on an input density matrix $\rho$ to yield the output $\Phi(\rho)$. Under the effect of noise present in the channel, pure input states are typically transformed into mixed output states. The amount of noise present in the channel can be estimated by determining how close the output $\Phi(\rho)$ is to a pure state when the input $\rho$ is pure. In other words, the output purity provides a yardstick for the level of noise in the channel. There are various measures of output purity, one of them being the maximal $p$–norm of the channel. It is defined as follows

$$\nu_p(\Phi) := \sup_\rho \{ ||\Phi(\rho)||_p \},$$

where $||\Phi(\rho)||_p = \left[ \text{Tr} \left( (\Phi(\rho))^p \right) \right]^{1/p}$ is the $p$–norm of $\Phi(\rho)$ and $1 \leq p \leq \infty$. The case $p = \infty$ corresponds to the operator norm. In [1] the supremum is taken over all input density matrices. However, due to convexity of the map $\rho \mapsto ||\Phi(\rho)||_p$, it suffices to restrict this supremum to pure states. It is clear that $||\Phi(\rho)||_p \leq 1$, since $\Phi(\rho)$ is a density matrix. The equality holds
if and only if the latter is a pure state. Hence $\nu_p(\Phi) \leq 1$ with equality if and only if there is a pure state $\rho$ for which the output $\Phi(\rho)$ is also pure. Thus $\nu_p(\Phi)$ provides a measure of the maximal purity of outputs from a quantum channel $\Phi$.

The maximal $p$–norms of two quantum channels $\Phi$ and $\Psi$ are said to be multiplicative if

$$\nu_p(\Phi \otimes \Psi) = \nu_p(\Phi) \cdot \nu_p(\Psi).$$

(2)

This multiplicativity was conjectured by Amosov, Holevo and Werner in [5]. In the limit $p \to 1$, (2) implies the additivity of another natural measure of the output purity, namely the von Neumann entropy. The multiplicativity (2) has been proved explicitly for various cases. For example, it is valid for all integer values of $p$, when $\Phi$ and $\Psi$ are tensor products of depolarizing channels [6]. It also holds for all $p \geq 1$ when $\Psi$ is an arbitrary quantum channel and $\Phi$ is any one of the following: (i) an entanglement breaking channel [12], (ii) a unital qubit channel [10] or (iii) a depolarizing channel in any dimension [11]. However, it is now known that the conjecture is not true in general. A counterexample to the conjecture was given in [7], for $p > 4.79$ in the case in which $\Phi$ and $\Psi$ are Werner–Holevo channels, defined by (3).

The Werner–Holevo channel $\Phi_d$ of dimension $d < \infty$ is defined by its action on any complex $d \times d$ matrix $\mu$ as follows

$$\Phi_d(\mu) = \frac{1}{d - 1} (I \text{Tr}(\mu) - \mu^T).$$

(3)

Here $\mu^T$ denotes the transpose of $\mu$, and $I$ is the $d \times d$ unit matrix. In particular, the action of the channel on any density matrix $\rho$ is given by

$$\Phi_d(\rho) = \frac{1}{d - 1} (I - \rho^T).$$

(4)

Werner and Holevo [7] proved that the conjecture (2) was false for $p > 4.79$ when $\Phi = \Psi = \Phi_d$ with $d = 3$. The validity of the conjecture for smaller values of $p$ for this channel was an open question. Recently it was proved [1, 2] that (2) is true for $p = 2$ for the above channel $\Phi_d$ with $d \geq 2$. In fact, in [1] the multiplicativity (2) was proved in a more general setting, namely one in which $\Phi$ is a Werner–Holevo channel but $\Psi$ is any arbitrary channel. Moreover, in [1] the multiplicativity (2) was conjectured to hold for all $1 \leq p \leq 2$ for Werner–Holevo channels. This paper provides a proof of this conjecture.

The precise statement of our result is given in Theorem 1 of Section 2. We would like to note that while writing our results, we were made aware of an almost simultaneous but independent and alternative proof of the conjecture put forth in [1]. This is contained in the recently posted body of work in [2]. However, not only do we present an alternative approach to the same conjecture, but this paper also provides the result encapsulated in Lemma 3, which would be of independent interest.

2 Main result

Following the discussion in the Introduction, we write the maximal $p$–norms for a single Werner–Holevo channel $\Phi_d$ and the product channel $\Phi_{d_1} \otimes \Phi_{d_2}$ as
\[ \nu_p(\Phi_d) = \max_{|\psi\rangle \in \mathcal{H}, ||\psi|| = 1} \{||\Phi_d(|\psi\rangle \langle \psi|)||_p\} , \]
and
\[ \nu_p(\Phi_{d1} \otimes \Phi_{d2}) = \max_{|\psi_{12}\rangle \in \mathcal{H}_{12}, ||\psi_{12}|| = 1} \{||\Phi_1 \otimes \Phi_2(|\psi_{12}\rangle \langle \psi_{12}|)||_p\} , \tag{5} \]
respectively. In the above, \( \mathcal{H} \simeq \mathbb{C}^d \) and \( \mathcal{H}_i \simeq \mathbb{C}^{d_i} \) for \( i = 1, 2 \). Our main result is stated in the following theorem.

**Theorem 1** Let \( \Phi_d \) denote a Werner–Holevo channel of dimension \( d \). Then the multiplicativity of the maximal \( p \)-norms
\[ \nu_p(\Phi_{d1} \otimes \Phi_{d2}) = \nu_p(\Phi_{d1}) \nu_p(\Phi_{d2}) , \tag{6} \]
holds for all \( 1 \leq p \leq 2 \), for arbitrary dimensions \( d_1, d_2 \geq 2 \).

To prove Theorem 1 we will make use of the method developed in [9] and of certain results proved in it. It is useful to consider the Schmidt decomposition of \( |\psi_{12}\rangle \)
\[ |\psi_{12}\rangle = \sum_{\alpha=1}^d \sqrt{\lambda_\alpha} |\alpha; 1\rangle \otimes |\alpha; 2\rangle . \tag{7} \]
Here \( d = \min[d_1, d_2] \) and \( \{|\alpha; j\rangle\} \) is an orthonormal basis in \( \mathcal{H}_i, \, i = 1, 2 \). The Schmidt coefficients \( \lambda_\alpha, \alpha = 1, 2, \ldots, d \), and hence also the vector of Schmidt coefficients \( \Lambda := (\lambda_1, \ldots, \lambda_d) \), vary in the \( (d-1) \)-dimensional simplex \( \Sigma_d \), defined by the constraints
\[ \lambda_\alpha \geq 0 \quad ; \quad \sum_{\alpha=1}^d \lambda_\alpha = 1 . \tag{8} \]
Note that the vertices of \( \Sigma_d \) correspond to unentangled vectors \( |\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \). To prove Theorem 1 it is sufficient to show that the maximum on the RHS of (6) is achieved for unentangled vectors. In other words, we need to prove that this maximum occurs at the vertices of the simplex \( \Sigma_d \).

Using the Schmidt decomposition (7), the input to the product channel can be expressed as
\[ |\psi_{12}\rangle \langle \psi_{12}| = \sum_{\alpha, \beta=1}^d \sqrt{\lambda_\alpha \lambda_\beta} |\alpha; 1\rangle \langle \beta; 1| \otimes |\alpha; 2\rangle \langle \beta; 2| . \tag{9} \]
The output of the channel is the density matrix given by
\[ \sigma_{12}(\Lambda) := (\Phi_1 \otimes \Phi_2)(|\psi_{12}\rangle \langle \psi_{12}|) = \sum_{\alpha, \beta=1}^d \sqrt{\lambda_\alpha \lambda_\beta} \Phi_1(|\alpha; 1\rangle \langle \beta; 1|) \otimes \Phi_2(|\alpha; 2\rangle \langle \beta; 2|) . \tag{10} \]
We prove Theorem 1 by showing that
\[ [\nu_p(\Phi_{d1} \otimes \Phi_{d2})]^p = [\nu_p(\Phi_{d1})]^p [\nu_p(\Phi_{d2})]^p \quad \text{for any } d_1, d_2 \geq 2 \quad \text{and } 1 < p \leq 2 . \]
The multiplicativity (6) holds trivially for $p = 1$ since $\text{Tr} \tilde{\rho} = 1$ for any density matrix $\tilde{\rho}$. Note that

$$\left[ \nu_p(\Phi_{d_1} \otimes \Phi_{d_2}) \right]^p = \max_{|\psi_{12} \rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2} \left\{ \| (\Phi_1 \otimes \Phi_2)(|\psi_{12}\rangle \langle \psi_{12}|) \|^p \right\}$$

$$= \max_{\lambda \in \Sigma_{d_1 d_2}} \sum_{i=1}^{d_1 d_2} (E_i(\lambda))^p,$$

where $\{E_i(\lambda), i = 1, 2, \ldots, d_1 d_2\}$ denotes the set of eigenvalues of the channel output $\sigma_{12}(\lambda)$.

These eigenvalues were studied in detail in [9] and were found to be divided into the three classes given below. Here we assume for definiteness that $d_1 \leq d_2$, so that $d = d_1$.

1. There are $d(d - 1)$ eigenvalues given by

$$e_{\alpha\beta} := \frac{1}{(d_1 - 1)(d_2 - 1)}(1 - \lambda_\alpha - \lambda_\beta), \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2, \ldots, d.$$

2. There are $d$ eigenvalues given by

$$h_\alpha := \frac{(1 - \lambda_\alpha)}{(d_1 - 1)(d_2 - 1)}; \quad \alpha = 1, 2, \ldots, d,$$

each of multiplicity $d_2 - d_1$.

3. There are $d$ eigenvalues of the form

$$g_\alpha := \frac{\gamma_\alpha}{(d_1 - 1)(d_2 - 1)}, \quad \alpha = 1, 2, \ldots, d,$$

where $\gamma_\alpha$ are the roots of the equation

$$\frac{d}{\prod_{\alpha=1}^{d}(1 - 2\lambda_\alpha - \gamma)} \left\{ 1 + \sum_{\alpha' = 1}^{d} \frac{\lambda_{\alpha'}}{(1 - 2\lambda_{\alpha'} - \gamma)} \right\} = 0.$$

Using the constraint (8) we find that

$$\sum_{1 \leq \alpha, \beta \leq d, \alpha \neq \beta} e_{\alpha\beta} = \frac{d_1 - 2}{d_2 - 1}; \quad \sum_{\alpha=1}^{d} h_\alpha = \frac{1}{d_2 - 1};$$

and using the fact that the sum of all eigenvalues of $\sigma_{12}(\lambda)$ is equal to 1 we get

$$\sum_{\alpha=1}^{d} g_\alpha = \frac{d_2 - d_1}{d_2 - 1}. $$
These relations allow us to define the following sets of non-negative variables

\[ e_{\alpha\beta} := \left( \frac{d_2 - 1}{d_1 - 2} \right) e_{\alpha\beta} \quad \alpha, \beta = 1, 2, \ldots, d; \alpha \neq \beta, \quad (14) \]

\[ h_{\alpha} := (d_2 - 1) h_{\alpha} \quad \alpha = 1, 2, \ldots, d, \quad (15) \]

\[ g_{\alpha} := \left( \frac{d_2 - 1}{d_2 - d_1} \right) g_{\alpha} \quad \alpha = 1, 2, \ldots, d, \quad (16) \]

such that the sum of each of these sets of variables is equal to unity, i.e.

\[ \sum_{1 \leq \alpha, \beta \leq d \atop \alpha \neq \beta} e_{\alpha\beta} = 1; \quad \sum_{\alpha = 1}^{d} h_{\alpha} = 1; \quad \sum_{\alpha = 1}^{d} g_{\alpha} = 1. \]

Hence, we can write

\[ ||\sigma_{12}(\lambda)||_p^p = (E_{i}(\lambda))^p \]

\[ = c_1 \sum_{1 \leq \alpha, \beta \leq d \atop \alpha \neq \beta} e_{\alpha\beta}^p + c_2 \sum_{\alpha = 1}^{d} h_{\alpha}^p + c_3 \sum_{\alpha = 1}^{d} g_{\alpha}^p \]

\[ := T_1(\lambda) + T_2(\lambda) + T_3(\lambda), \quad (17) \]

where \( c_1, c_2 \) and \( c_3 \) are constants depending on the dimensions \( d_1 \) and \( d_2 \).

The function \( f(x) := x^p \), where \( 1 < p \leq 2 \), is convex for \( x \geq 0 \). Hence, \( T_1(\lambda) \) is a convex function of the variables \( e_{\alpha\beta} \). These variables are affine functions of the Schmidt coefficients \( \lambda_1, \ldots, \lambda_d \). Hence, \( T_1(\lambda) \) is a convex function of \( \lambda \) and attains its global maximum at the vertices of the simplex \( \Sigma_d \). The same argument applies to \( T_2(\lambda) \) since the variables \( h_{\alpha} \) are also affine functions of the Schmidt coefficients. The function \( T_3(\lambda) \) is however not necessarily a convex function of \( \lambda \). In spite of this, it too achieves its maximum value at the vertices of \( \Sigma_d \). This follows from the following theorem.

**Theorem 2** The function \( T_3(\lambda) \) is Schur-convex in \( \lambda \in \Sigma_d \) i.e., \( \lambda \prec \lambda' \implies T_3(\lambda) \leq T_3(\lambda') \), where \( \prec \) denotes the stochastic majorization (see the Appendix).

Since every \( \lambda \in \Sigma_d \) is majorized by the vertices of \( \Sigma_d \), Theorem 2 implies that \( T_3(\lambda) \) also attains its maximum at the vertices. Thus \( ||\sigma_{12}(\lambda)||_p^p = T_1(\lambda) + T_2(\lambda) + T_3(\lambda) \) is maximized at the vertices of \( \Sigma_d \). As was observed, this implies the multiplicativity \( \Box \).

To prove Theorem 2 we use the following lemma, which is proved in Section 3.

**Lemma 3** Let \( f(x) := \sum_{i=1}^{n} x_i^p \), where \( x = (x_1, x_2, \ldots, x_n) \) with each \( x_i \geq 0 \) and \( \sum_{i=1}^{n} x_i = 1 \). For \( 1 < p < 2 \), \( f(x) \) is a monotonically non-increasing function of the elementary symmetric polynomials \( s_q(x_1, x_2, \ldots, x_n) \) for \( 2 \leq q \leq n \), where

\[ s_k(x_1, x_2, \ldots, x_n) := \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k} \quad \text{for} \quad k = 1, 2, 3, \ldots, n. \quad (18) \]
Note that $T_3(\lambda) := c_3 \sum_{\alpha=1}^{d} \tilde{g}_\alpha^p$, where
\[ \tilde{g}_\alpha \geq 0 \quad \text{and} \quad \sum_{\alpha=1}^{d} \tilde{g}_\alpha = 1. \]
The variables $\tilde{g}_\alpha$ are proportional to the roots $\gamma_\alpha$ of eq. (13) (see (12) and (16)), which are obviously functions of the Schmidt vector $\lambda$. Hence, by Lemma 3, $T_3(\lambda)$ is a monotonically non-increasing function of the elementary symmetric polynomials
\[ \tilde{s}_k(\lambda) := s_k(\gamma_1, \gamma_2, \ldots, \gamma_d), \quad k = 0, \ldots, d. \]
Therefore, to prove Theorem 2 it suffices to show that the functions $\tilde{s}_k(\lambda)$ are Schur concave in $\lambda \in \Sigma_d$. This Schur–concavity property of $\tilde{s}_k(\lambda)$ was proved explicitly in [9]. The proof of Lemma 3 therefore allows us to establish Theorem 2 and hence Theorem 1, our main result. This is given in Section 3. Our proof is analogous to that of Theorem 1 of [14].

3 Proof of Lemma 3

The variables $x_1, x_2, \ldots, x_n$, defined in Lemma 3, can be viewed as the eigenvalues of an $n \times n$ density matrix $\rho_n$ (say), and hence as the roots of the characteristic equation $\det(\rho_n - x I) = 0$. Since the roots are the zeros of the product $\prod_{i=1}^{n}(x - x_i)$, the characteristic equation can be expressed in terms of these roots as follows:
\[ \sum_{k=0}^{n} x^k (-1)^{n-k} s_{n-k}(x_1, x_2, \ldots, x_n) = 0. \]
Here the coefficient $s_{n-k}(x_1, x_2, \ldots, x_n)$ denotes the $(n-k)^{th}$ elementary symmetric polynomial of the variables $x_1, x_2, \ldots, x_n$ (defined by (18)). We consider equation (20) to implicitly define the variables $x_j \equiv x_j(s_1, s_2, \ldots, s_n)$ as functions of the elementary symmetric polynomials. This can be done unambiguously as long as there are no multiple roots, i.e. $x_i \neq x_j$ for $i \neq j$, $i,j = 1,2,\ldots,n$. We restrict our attention to this case at first, and prove that in the absence of multiple roots, $\partial f/\partial s_i < 0$ for each $i \geq 2$. This will enable us, by continuity arguments, to conclude that $f$ is indeed a monotonically non-increasing function of the elementary symmetric polynomials $s_2, s_3, \ldots, s_n$ everywhere.

Let us first prove that $\partial f/\partial s_i < 0$ when the roots $x_1, x_2, \ldots, x_n$ are all different. In this case we can view the variables $x_j$ to be implicitly defined by (20). Then differentiating with respect to $s_m$, for $2 \leq m \leq n$, we get
\[ \frac{\partial x_j}{\partial s_m} = \frac{(-1)^{m+1}x_j^{n-m}}{\prod_{i \neq j}(x_j - x_i)}. \]
Using the chain rule and the definition of the function \( f(x) \) we get

\[
\frac{\partial f}{\partial s_m} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial s_m} = \sum_{j=1}^{n} \left( -1 \right)^{m+1} x_j^{n-m} p x_j^{p-1} \prod_{i \neq j} (x_j - x_i). \tag{22}
\]

To prove that \( f(x) \) is a monotonically non-increasing function of \( s_m \) for each \( m = 2, 3, \ldots, n \), we use some standard results from Numerical Analysis [15]. It is known that there is a unique polynomial of degree \( (n-1) \) which interpolates a given function \( g(x) \) at the points \( x_1, x_2, \ldots, x_n \). The coefficient of \( x^{n-1} \) of this polynomial is given by

\[
a_{n-1} = \sum_{j=1}^{n} \frac{g(x_i)}{\prod_{i \neq j} (x_j - x_i)},
\]

called the Newton Divided Difference [15] of the function \( g(x) \). The expression on the RHS of (22) implies that \( \partial f/\partial s_m \) is the Newton Divided difference of the following function

\[
g(x) \equiv g_m(x) = (-1)^{m+1} p x^{n-m+p-1}. \tag{23}
\]

By the Hermite Gennochi theorem [15], the Newton Divided Difference is also given by the integral of \( g^{(n-1)}(p_1 x_1 + \ldots + p_n x_n) \) over the probability simplex \( \{(p_1, p_2, \ldots, p_n), p_i \geq 0, \sum_{i=1}^{n} p_i = 1\} \), where \( g^{(n-1)}(x) \) denotes the \((n-1)^{th}\) derivative of \( g(x) \).

From (23) we obtain

\[
g_m^{(n-1)}(x) = (-1)^{m+1} p(p - m + 1)(p - m + 2) \ldots (p - m + n - 1) x^{p-m+1}. \tag{24}
\]

It is easy to see that for all \( 2 \leq m \leq n \), \( g_m^{(n-1)}(x) < 0 \) for all \( x \), since \( 1 < p < 2 \). Hence the integral over the probability simplex is negative and we get \( \partial f/\partial s_m < 0 \) as required.

Therefore, \( \partial f/\partial s_m < 0 \) everywhere except on the manifolds on which two or more of the roots \( x_1, x_2, \ldots, x_n \) coincide. By continuity we deduce that \( f \) is a monotonically non-increasing function of the elementary symmetric polynomials \( s_2, s_3, \ldots, s_n \) everywhere. \( \blacksquare \)

Note that in Lemma we considered \( 1 < p < 2 \). For the case \( p = 2 \), proceeding analogously to the proof of the above lemma, we find the following: in the absence of multiple roots

\[
\frac{\partial f}{\partial s_2} < 0 \text{ whereas } \frac{\partial f}{\partial s_m} = 0 \text{ for all } m = 3, 4, \ldots, n. \tag{25}
\]

Hence by continuity, \( f \) is a monotonically non-increasing function of the elementary symmetric polynomial \( s_2 \) everywhere. The latter is however a Schur–concave function of the Schmidt vector \( \lambda \) [9]. Hence for the case \( p = 2 \) as well, Theorem applies and the multiplicativity stated in Theorem holds.
Appendix

A real-valued function $\Phi$ on $\mathbb{R}^n$ is said to be Schur convex (see [17]) if

$$x < y \implies \Phi(x) \leq \Phi(y).$$

Here the symbol $x < y$ means that $x = (x_1, x_2, \ldots, x_n)$ is majorized by $y = (y_1, y_2, \ldots, y_n)$ in the following sense: Let $x^\downarrow$ be the vector obtained by rearranging the coordinates of $x$ in decreasing order

$$x^\downarrow = (x_1^\downarrow, x_2^\downarrow, \ldots, x_n^\downarrow)$$

means $x_1^\downarrow \geq x_2^\downarrow \geq \ldots \geq x_n^\downarrow$.

For $x, y \in \mathbb{R}^n$, we say that $x$ is majorized by $y$ and write $x < y$ if

$$\sum_{j=1}^{k} x_j^\downarrow \leq \sum_{j=1}^{k} y_j^\downarrow, \quad 1 \leq k \leq n,$$

and

$$\sum_{j=1}^{n} x_j^\downarrow = \sum_{j=1}^{n} y_j^\downarrow.$$

In the simplex $\Sigma_d$, defined by the constraints [3], the minimal point is $(1/d, \ldots, 1/d)$ (the baricenter of $\Sigma_d$), and the maximal points are the permutations of $(1, 0, \ldots, 0)$ (the vertices).

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