Renormalization Group Approach to the Normal State of Copper-Oxide Superconductors

J. González§, F. Guinea† and M.A.H. Vozmediano‡

§Instituto de Estructura de la Materia, CSIC
Serrano 123, 28006 Madrid, Spain
†Instituto de Ciencia de Materiales, CSIC
Cantoblanco, 28049 Madrid, Spain.
‡Departamento de Matemáticas, Universidad Carlos III de Madrid
Avda. Butarque 15, 28913 Leganés (Madrid), Spain.

Abstract

We study by means of renormalization group techniques the effect that on the two-dimensional electron liquid may have the van Hove singularities observed experimentally in the copper-oxide superconductors. We find significant deviations from Fermi liquid behavior, that lead to the appearance of an unstable fixed point in the renormalization group flow of the effective coupling constant. Besides the attenuation of electron quasiparticles already known on phenomenological grounds, our approach is able to explain the reduction in the dispersion of the band as well as the pinning of the Fermi level near the singularity, as observed in the photoemission experiments.

1e-mail: emgonzalez@iem.csic.es
2e-mail: paco@ccuam3.sdi.uam.es
1 Introduction

During recent years there has been great interest in finding theoretical models displaying the so-called “non-Fermi” liquid behavior\[1\]. This has not been a trivial search, mainly due to the fact that the Fermi liquid framework is quite robust in the context of interacting electron systems. The Fermi liquid concept based on Landau’s theory acts as the most valuable paradigm in the description of real three-dimensional metals\[2\]. The relevance of such paradigm is not constrained to situations in which the interaction strength is weak and goes beyond the range of applicability of perturbation theory. For some reasons, not quite well understood until recently, there are not many ways in which an electron system, in spatial dimension $D > 1$, may be destabilized out of the Fermi liquid picture. Some of the most outstanding real examples in which that happens are furnished by the copper-oxide compounds that become superconductors at high transition temperatures. In the metallic regime of these materials, some of the most prominent experimental properties (low-temperature resistivity, specific heat) have an abnormal behavior that cannot be explained by Fermi liquid theory\[3\]. Thus, for the copper-oxide compounds it is not only a question of explaining the unusual superconducting transition temperatures, but also of understanding the unconventional properties of the normal state —what may serve furthermore to uncover the superconducting mechanism.

The most modern approach to a general theory of interacting fermion systems has put forward the idea that the Fermi liquid describes a universality class in two and three spatial dimensions\[4, 5\], much in the same spirit that the Luttinger liquid is the class for a wide set of one-dimensional electron systems\[6\]. In order to apply the renormalization group methods to many-body fermion systems, the main achievement has been a careful analysis of the interactions on the grounds of the usual classification of the perturbations of a statistical theory\[4, 5\] —that is, relevant, irrelevant and marginal operators. Unlike for the one-dimensional fermion systems, this is not straightforward in higher dimensions, since the interaction involves in general excitations attached to different points of the Fermi surface. Thus, one has to talk properly of the existence of a manifold of coupling constants, not just of a finite number of them. Under very general assumptions, like the regularity of the interaction potential in that manifold of couplings, it turns out that there are essentially two different marginally relevant perturbations of the fermion system, corresponding to the superconductivity instability, which is always present, and to the charge density or spin density instability, which arises when the Fermi surface has the so-called nesting property. The rest of marginal interactions translates into the familiar $F$ function of Landau’s
Fermi liquid theory. Then, in this setup there is little room left to the description of additional deviations from Fermi liquid behavior, which may be conceived only at the expense of relaxing some of the initial assumptions.

Confronting such theoretical framework, there is clear experimental evidence that most of the copper-oxide superconductors have normal state properties incompatible with those of a Fermi liquid. The most notorious of them is the linear resistivity as a function of the temperature (at the optimal doping for the superconductivity), while a quadratic dependence is the only behavior that can be accounted for in Fermi liquid theory. On the other hand, the origin of the superconductivity in the cuprate oxides remains unexplained, although there is strong experimental evidence in favor of a mechanism based on electron-electron interactions. The electrons responsible for the conductivity in the normal state, and for the appearance of the superconducting condensate, are supposed to be localized on weakly coupled planes. The existence of important correlation effects has been used to propose, in the context of prototypes like the Hubbard or $t$-$J$ models, a novel two-dimensional electron liquid state, that should be responsible both for the unusual normal state properties as well as for the high-$T_c$ superconductivity. In the doped materials, however, there is nowadays clear evidence of the existence of a Fermi surface typical of a metallic band. Furthermore, the experiments based on angle-resolved photoemission spectroscopy have shown interesting features of the dispersion relation, like almost dispersionless bands and extended van Hove singularities (saddle points) near the Fermi level. Based on this experimental evidence, supplemented by phenomenological arguments, it has been proposed that the unusual properties of electrons near such singular points are the cause of the superconductivity in these systems. Although a variety of theoretical approaches have been attempted, a full treatment of interacting electrons near a saddle point in a 2D dispersion relation remains to be done.

The aim of this paper is to study the behavior of the two-dimensional electron liquid near a van Hove singularity by means of a renormalization group approach. Due to the presence of the saddle point in the dispersion relation, the usual description of the fermion system breaks down from the start. In the standard many-body treatment as well as in the modern renormalization group approach, the fact that one can linearize the dispersion relation at each point of the Fermi surface plays a crucial role. When that is possible, one may disregard the particular features of the Fermi surface. Thus, in the usual many-body approach one does not care about its fate under quantum corrections, since all its points are regarded as essentially equivalent. When there is a van Hove singularity at or very close to the Fermi surface that is no longer true. Physically this can be easily understood since at the point of
the singularity the density of states diverges logarithmically. From a technical point of view, the gradient of the dispersion relation at the saddle point vanishes, what makes unclear that the traditional many-body approach can be followed. Quite remarkably, however, the logarithmic divergences associated to a two-dimensional van Hove singularity make the problem specially suitable to renormalization group schemes, along the lines discussed in [4, 5].

In the following, we implement such a RG treatment of interacting electrons near a 2D van Hove singularity. The model to be studied is presented in the next section. Then, the renormalization of the coupling constant is worked out in section 3. In section 4 we discuss how the renormalized potential influences the quasiparticle properties. The renormalization group flow of the model is presented in section 5. A discussion of the physical consequences is given in section 6. The main conclusions are drawn in section 7.

2 Many-body theory of the 2D van Hove singularity

A van Hove singularity is a saddle point in the dispersion relation of the electron states $\varepsilon(k)$. In its vicinity, the density of states diverges logarithmically, in two dimensions, and shows cusps in three dimensions. The logarithmic divergence leads to a singular screening of the interactions, in the same way as for the 1D Luttinger liquid[17].

The copper-oxygen planes where the electrons near the Fermi level reside have an almost perfect tetragonal symmetry. Thus, the crystal wavevector $k$ is defined in a square Brillouin Zone, such that $-\pi < k_x, k_y < \pi$. Experiments show that the Fermi surface lies close to the high symmetry points (0, ±π) and (±π, 0). Thus, we expect two inequivalent van Hove “points” at the edges of the Brillouin Zone, as depicted in Fig. 1 (note that the points previously defined, are, by symmetry, extrema of the dispersion relation).

We can obtain a reasonable description of the low-energy properties of the system by expanding the dispersion relation around these points. Denoting them by $A$ and $B$, and shifting the origin of momenta to each of these points, we can write, in general,

$$\varepsilon_{A,B}(k) \approx (t \pm \Delta t)k_x^2 - (t \mp \Delta t)k_y^2$$

(1)

where $\Delta t$ is proportional to the deviation of the angle between the two separatrices from $\pi/2$.

Thus, the problem can be mapped onto a model with electrons with two flavors, which denote each of the two inequivalent singularities. The
full complexity of the interacting system is already present if we neglect this flavor degeneracy (which can be achieved, for instance, by means of an orthorhombic distortion which splits the two points). In the following, we will consider a system with electrons near a single singularity. Intersingularity (Umklapp) effects will be addressed in future work.

For a single singularity, a suitable scaling of lengths leads to the simpler dispersion relation

$$\varepsilon(k) = t(k_x^2 - k_y^2)$$  \hspace{1cm} (2)

We describe the situation in which the Fermi energy is right at the level of the van Hove singularity. The case in which the Fermi level deviates slightly from the position of the singularity can be treated by means of a small shift in the chemical potential of the system, which just introduces a small infrared cutoff in the renormalization group approach. There remains the question, though, about the physical significance of a situation in which the filling of the Fermi sea stops at the level of the singularity. According to our point of view, the shape of the Fermi line may suffer changes due to the interaction in the system and it is not clear, in principle, that the Fermi level may find a stable location near the singularity. We will see later on that this tuning of the Fermi level to the singularity is actually quite natural, from the point of view of the renormalization group approach.

We take then the chemical potential at $\mu = 0$ for the dispersion relation (2), so that the levels with $\varepsilon(k) < 0$ are filled and the levels with $\varepsilon(k) > 0$ are all empty. The Fermi line is given by the two straight asymptotes $k_x = \pm k_y$. Of course, in order to accomplish a proper description of the statistical system some bandwidth cutoff has to be introduced that regularizes the formally infinite Fermi sea of (2). We implement in what follows a cutoff in momentum space $(k_x, k_y)$ that preserves the SO(1,1) symmetry of the dispersion relation (2), by picking the states that lie in the region with $-\Lambda < \varepsilon(k) < \Lambda$. Thus, $\Lambda$ has the character of an energy (or frequency) cutoff. In fact, this possibility of controlling the energy scale of the processes in an invariant way is what allows us to carry out the renormalization program that we apply below. The behavior of the interactions as we reduce the cutoff $\Lambda$ gives us information about the form of the low-energy effective theory for the system. In this respect, the procedure is similar to that adopted in the usual description of the Luttinger liquid[17] or in the renormalization group approach to Fermi liquid theory[4, 5].

The first step in this approach to the low-energy effective theory of the van Hove singularity is to write down a classical interacting theory that is scale invariant. We take as starting point the system of electrons with the dispersion relation (2) and the typical four-fermion interaction of density-
times-density type. The action of the model reads, for the simplest case of a local interaction,

\[
S = \int dt' d^2r \sum_{\sigma} \left( i \Psi_\sigma^+(r,t') \partial_{t'} \Psi_\sigma(r,t') + t \, \Psi_\sigma^+(r,t') \left( \partial_t^2 - \partial_y^2 \right) \Psi_\sigma(r,t') \right) \\
- \frac{U}{2} \int dt' d^2r \sum_{\sigma} \Psi_\sigma^+(r,t') \Psi_\sigma(r,t') \sum_{\sigma'} \Psi_{\sigma'}^+(r,t') \Psi_{\sigma'}(r,t')
\]

(3)

\( \Psi_\sigma(r,t) \) is the electron field made out of all the modes that lie in the region determined by the cutoff, \( \sigma \) being the spin index \( \sigma = \uparrow, \downarrow \).

\[
\Psi_\sigma(r,t) = \frac{1}{(2\pi)^3} \int_S d\omega d^2k e^{-i\omega t + ikr} \, a_\sigma(k,\omega) \quad \mathcal{S} = \{ k \mid -\Lambda < \varepsilon(k) < \Lambda \}
\]

(4)

The scaling of the different terms in (3) is best analyzed by passing to the frequency-momentum \((\omega, k)\) representation, in which the action reads

\[
S = \int d\omega d^2k \sum_{\sigma} \left( \omega a_\sigma^+(k,\omega) a_\sigma(k,\omega) - t(k_x^2 - k_y^2) \, a_\sigma^+(k,\omega) a_\sigma(k,\omega) \right) \\
- \frac{U}{2} \int d\omega d^2k \left( \rho_\uparrow(k,\omega) + \rho_\downarrow(k,\omega) \right) V(k) \left( \rho_\uparrow(-k,-\omega) + \rho_\downarrow(-k,-\omega) \right)
\]

where \( \rho_\sigma(k,\omega) \) are the Fourier components of the density operator \( \Psi_\sigma^+(r,t) \Psi_\sigma(r,t) \)

\[
\rho_\sigma(k,\omega) = \frac{1}{(2\pi)^3} \int d\omega_d^2p \, a_\sigma^+(p-k,\omega) \rho_\sigma(p,\omega)
\]

(6)

The case of a purely local interaction corresponds to \( V(k) = 1 \). In what follows we will consider however the more realistic case in which the interaction has some finite range in real space. We will then suppose that the potential \( V(k) \) is constant but has a finite support in \( k \) space, so that the discussion can be carried out for a spin-independent interaction.

We determine the scaling dimension of the operator \( a_\sigma(k,\omega) \) in the usual way, by demanding the scale invariance of the free term in the action. Given the nonlinear character of the dispersion relation, it turns out that under a change of scale of the momenta

\[
k \rightarrow sk
\]

(7)

the frequency has to scale in the form

\[
\omega \rightarrow s^2 \omega
\]

(8)

This different scaling of the frequency and the momentum is something natural in the model, since to keep up with the approach to the Fermi level...
\( \varepsilon(k) = 0 \) the energy has to be reduced as the square of the reduction in the momentum scale. From (7) and (8) we determine the scaling of the electron operator \( a_\sigma(k, \omega) \) that renders invariant the first piece of (5)

\[
a_\sigma(k, \omega) \rightarrow s^{-3} a_\sigma(k, \omega)
\]  

(9)

A most important point is now to check whether by application of the scaling transformation (7), (8) and (9) the interaction strength \( U \) in the last term of (5) goes to zero, diverges or remains invariant in the infrared limit \( s \to 0 \). The first case would correspond to an irrelevant interaction, that should be neglected in the effective field theory at low energies, while in the second case the last term in (5) would play the role of a relevant perturbation of the noninteracting theory, spoiling the renormalization group approach. We see, however, that the actual case is that the interaction remains marginal at the classical level, i.e. the coupling strength \( U \) does not scale as \( s \to 0 \). Under these conditions, the effect of the quantum corrections in the model turn out to be crucial since, though small they may be, they are able to destabilize the naive scaling in one or the other direction.

The whole issue of the renormalization of the \( U \) parameter becomes of the utmost interest since, as follows from Shankar’s renormalization group approach to fermion systems, the interaction in Fermi liquid theory (constrained to the forward scattering channel) is not renormalized. It has to be pointed out, however, that the case of a dispersion relation like (2) becomes special, in the sense that the interaction is marginal at the classical level irrespective of the kinematics of the scattering process, as we have seen above. This enables us to pose the problem in terms of the renormalization of just one coupling constant \( U \), which affects evenly to all the scattering channels —apart from the BCS channel, which does require a particular kinematics.

In what follows we adopt a statistical field theory description of the model, sticking to a path integral formulation based on the action (5). The fundamental object in this approach is the partition function

\[
Z = \int \mathcal{D}\Psi \mathcal{D}\Psi^+ e^{iS(\Psi, \Psi^+)}
\]  

(10)

where, as stated above, we restrict the functional integration to the fermion modes within energies \(-\Lambda\) and \(\Lambda\) measured with respect to the level of the singularity. The Fermi sea that we are considering corresponds to a noncompact region in momentum space, though the number of states in the region becomes infinite only in the infinite volume limit. We will see, for instance, that the four-fermion interaction and the self-energy correction of the model
can be regularized by the use of just the bandwidth cutoff $\Lambda$. The dependence of the quantum corrections on $\Lambda$ gives us the information about how the different parameters of the model are renormalized when the cutoff is reduced from $\Lambda$ to $\Lambda - d\Lambda$, what in practice is equivalent to perform the partial integration of a slice of high-energy modes in the path integral. The recursive implementation of this renormalization group transformation, studied by the flow as $\Lambda \to 0$ of the scale dependent parameters, leads to the effective field theory governing the low-energy physics of the model.

3 Coupling constant renormalization

We study first the renormalization of the interaction potential $V(k)$, which gets the quantum corrections shown by the diagram in Fig. 2, to second order in perturbation theory. The figure represents the contribution to the corresponding vertex function $i\Gamma(k,\omega)$, i.e. the part of the interaction without the external legs. In terms of the fermion propagator $G^{(0)}(k,\omega)$ for each respective spin orientation, the vertex function to the given perturbative order $i\Gamma^{(2)}(k,\omega)$ reads

$$i\Gamma^{(2)} = -\frac{U^2}{(2\pi)^3} \int_{-\infty}^\infty d\omega_q \int d^2q \sum_\sigma G^{(0)}(q + k, \omega_q + \omega)G^{(0)}(q, \omega_q)$$

The fermion propagator to be used in our model is

$$G^{(0)}(q, \omega_q) = \frac{1}{\omega_q - t(q_x^2 - q_y^2) + i\epsilon \text{sgn} \omega_q}$$

which takes into account that the dispersion relation cannot be linearized at the singularity. The most important point in the computation of (11) is that the virtual states in the loop have to be kept within the band determined by the cutoff, i.e. only states with energies between $-\Lambda$ and $\Lambda$ are allowed. This bandwidth cutoff has the virtue of preserving the SO(1,1) symmetry of the model, that is nothing but the invariance under the continuous set of transformations

$$\begin{pmatrix} q'_x \\ q'_y \end{pmatrix} = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} q_x \\ q_y \end{pmatrix}$$

The change of variables (13) preserves the form of the dispersion relation (2) and it is an exact symmetry of the quantum theory. The implementation

\footnote{We do not mix up at this point the Cooper channel, which requires a particular kinematics, with the generic corrections to the four-fermion interaction, which is marginal in the model at the classical level irrespective of the values of the momentum exchange.}
of a cutoff that does not break the SO(1,1) invariance of the model is an important ingredient of the calculation since, as we will see, it enables us to reconstruct the complete expression of $i\Gamma^{(2)}(k,\omega)$ from just the dependence on, say, $k_x$ and $\omega$.

The loop integral in (11) is then most easily performed by making a change of variables that introduces the lines of constant energy. In the sector $R_1 \equiv \{ q\ | \ q_y \geq |q_x| \}$, for instance, we introduce the variables $r$ and $\phi$ by

$$
q_x = r \sinh \phi
$$

$$
q_y = r \cosh \phi
$$

(14)

The region of integration in that sector is mapped to $-\infty < \phi < +\infty$, $0 < r < \Lambda$. By doing first the integration in the $\omega_q$ complex plane, we get a partial contribution $i\Gamma^{(2)}_{R_1}$ to (11)

$$
i\Gamma^{(2)}_{R_1} = \frac{-i U^2}{2\pi^2} \int_0^{\sqrt{\Lambda/t}} dr \int_\alpha^\infty d\phi \frac{1}{\omega - tk_x^2 - 2tk_xr \sinh \phi + i\epsilon + i\epsilon \text{sgn} (\omega - \Lambda^2)}
$$

(15)

where $\alpha = \text{arcsh}((tr^2 - tk_x^2)/(2tk_xr))$ and we are computing setting $\omega > 0$, $k_x > 0$ and $k_y = 0$. As we have mentioned before, the complete dependence of the vertex function on $k$ and $\omega$ can be reconstructed, in general, by using the SO(1,1) symmetry of the model, as long as the vertex can only depend on invariants under that transformation group.

The angular integral in (15) can be easily done, while the remaining integral in the $r$ variable cannot be directly obtained. It is much simpler, however, to extract its $\Lambda$ dependence, which turns out to be

$$
i\Gamma^{(2)}_{R_1} \approx \frac{i U^2}{8\pi^2 \frac{t}{\Lambda}} \log \Lambda
$$

(16)

It can be checked that there is a similar contribution from the sector $q_y \leq -|q_x|$ and that the total contribution of the region with $q_x^2 - q_y^2 > 0$ equals that of the region $q_y^2 - q_x^2 > 0$. We have therefore

$$
i\Gamma^{(2)} \approx \frac{i U^2}{2\pi^2 \frac{t}{\Lambda}} \log \Lambda
$$

(17)

The presence of this dependence of the vertex function on the cutoff is a key result for the model, since it implies that the bare four-fermion interaction is renormalized as the bandwith cutoff is reduced towards the Fermi level. This property establishes a clear difference with the behavior of the forward scattering coupling constant in Fermi liquid theory, where it is shown that it
remains marginal to all orders in perturbation theory. In our case the cutoff dependence of (17) can be easily understood from the physical interpretation of the diagram in Fig. 2. The logarithm of Λ in (17) is nothing but a measure of the number of particle-hole excitations in which a particle below the Fermi level is promoted to an empty state above it. The logarithmic divergence can be alternatively seen as the divergence in the count of particle-hole transitions when the momentum transfer across the Fermi line is made arbitrarily small.

The complete dependence of the vertex function on $k_x$ and $\omega$ can be obtained starting from the knowledge of its imaginary part. This arises from the sum of four different contributions like that in (15). We get

$$\text{Im} \, \Gamma^{(2)} = \frac{1}{4\pi} \frac{U^2}{t} \left( \frac{|\omega + tk_x^2|}{tk_x^2} - \frac{|\omega - tk_x^2|}{tk_x^2} \right)$$

(18)

Given that this object can only depend on quantities invariant under the transformations (13), we may reconstruct the full expression

$$\text{Im} \, \Gamma^{(2)}(k, \omega) = \frac{1}{4\pi} \frac{U^2}{t} \left( \frac{|\omega + \epsilon(k)|}{\epsilon(k)} - \frac{|\omega - \epsilon(k)|}{\epsilon(k)} \right)$$

(19)

Moreover, we know that $\Gamma(k, \omega)$ must have an analytic continuation from the half-line $\omega > 0$ into the upper half-plane of the complex variable $\omega$ \cite{2}. This fact, together with the knowledge of the imaginary part (19), would suffice to determine the real part of the vertex function. Alternatively, one may introduce a different implementation of the cutoff, which makes the real part of $\Gamma^{(2)}$ directly available. The second version of the computation is sketched in Appendix I. There we apply the bandwidth cutoff to the frequency integral in (11), letting the remaining integrals in $q_x$ and $q_y$ from $-\infty$ to $+\infty$. Quite amazingly, these can be done analytically by passing to the complex plane, producing the same imaginary part as quoted in (19) and the real part

$$\text{Re} \, \Gamma^{(2)}(k, \omega) = \frac{1}{2\pi^2} \frac{U^2}{t} \log \frac{|\epsilon(k)| \Lambda}{|\omega^2 - \epsilon(k)^2|} + \frac{1}{2\pi^2} \frac{U^2}{t} \frac{\omega}{\epsilon(k)} \log \left| \frac{\omega - \epsilon(k)}{\omega + \epsilon(k)} \right|$$

(20)

It can be easily seen that the expression (19) corresponds for $\omega > 0$ to the imaginary part of the analytic function built from (20), with definite choices of the branch cuts for the logarithms. This shows the agreement between the two different implementations of the bandwidth cutoff and finishes up the complete determination of the vertex function to the one-loop order.

The renormalization of the vertex function under variations of the cutoff $\Lambda$ has important consequences from the physical point of view. Within the path integral formulation defined by (11), we may interpret $\Gamma^{(2)}$ as a quantum
correction to the four-fermion interaction term. We define therefore the four-fermion effective interaction by

\[ -iV_{\text{eff}} = -iU + i \frac{1}{2\pi^2} \frac{U^2}{t} \left( \log \frac{|\varepsilon(k)| \Lambda}{|\omega^2 - \varepsilon(k)^2|} + \frac{\omega}{\varepsilon(k)} \log \frac{\omega - \varepsilon(k)}{\omega + \varepsilon(k)} \right) \]  

(21)

The fundamental observation now is that, if \( V_{\text{eff}} \) is to play the role of effective interaction with which observables are computed in the quantum theory, it cannot depend on a particular choice of the cutoff \( \Lambda \). This means that, as we reduce for instance the value of \( \Lambda \), the value of the bare coupling constant \( U \) has to be adjusted so that the effective interaction of the theory remains cutoff independent. We recover in this way the idea underlying the original version of the renormalization group, in that the hamiltonian of the theory moves in the space of couplings as the length scale is varied\(^{[18]}\). By imposing the cutoff independence of \( V_{\text{eff}} \)

\[ \Lambda \frac{d}{d\Lambda} V_{\text{eff}} \equiv 0 \]  

(22)

we get from (21), to second order in perturbation theory,

\[ \Lambda \frac{d}{d\Lambda} U(\Lambda) = \frac{1}{2\pi^2} \frac{U^2}{t} \]  

(23)

This equation governs the flow of the coupling constant as the cutoff is lowered towards the Fermi level, leading to an effective theory for the low-energy processes of the model. We see that, in the case of a repulsive interaction we are interested in \( (U > 0) \), the bare four-fermion coupling constant decreases as \( \Lambda \to 0 \). It would seem, therefore, that perturbation theory should become more reliable at low energies if the starting value of \( U(\Lambda) \) was already small. One has to bear in mind, however, that the perturbative expansion in the model is given in powers of \( U/t \), so that one has to check if the “hopping” parameter \( t \) bears any dependence on \( \Lambda \) to ascertain whether the flow at low energies is to weak coupling or to strong coupling in terms of \( U/t \). We consider the renormalization of \( t \) in the next section, in the context of the quantum corrections to the two-point function.

### 4 Self-energy corrections

We undertake at this point the analysis of the self-energy function of the model, with the aim of studying the renormalization of the bare dispersion relation \( (2) \). The self-energy \( \Sigma(k, \omega) \) is given by the irreducible part of the
corrections to the two-point function and it is related to the full propagator by the equation:

\[ \frac{1}{G} = \frac{1}{G^{(0)}} - \Sigma \]  

(24)

The first perturbative contribution to \( \Sigma \) is shown diagrammatically in Fig. 3(a), where the dashed line represents the interaction potential \( V(k, \omega) \). We have started in our model with a constant interaction \( V(k) \), and for this reason it is clear that the mentioned diagram cannot give rise to a dependence on the external frequency or momentum. Under these conditions the one-loop diagram may only produce a renormalization of the two-point function by a constant, i.e. a renormalization of the chemical potential. We regard, however, this no-renormalization of the rest of parameters as a mere accident, since any slight modification of the potential (that still may keep it short-ranged) already introduces some nontrivial dependence \( V(k, \omega) \). We will recall this observation later on, when discussing the renormalization of \( \varepsilon(p) \).

The first nontrivial contributions to the self-energy in our model are given by the two-loop diagrams shown in Figs. 3(b) and 3(c). Since we are dealing with a constant interaction the dashed lines can be contracted to a point, making the two diagrams look the same. The only difference between diagram 3(b) and diagram 3(c) is that the first comes with a relative factor of \(-2\) with respect to the second due to the propagation of the two spin orientations around the fermion loop. The net effect is taken into account by computing diagram 3(b) for one of the spin orientations. We implement the same regularization adopted before, preserving the SO(1,1) invariance of the model. Thus we may take advantage of the one-loop particle-hole polarizability determined in the previous section. That is, at the two-loop level we have

\[ i\Sigma^{(2)}(p, \omega_p) = \frac{U^2}{(2\pi)^3} \int d\omega_k \, d^2k \, G^{(0)}(p - k, \omega_p - \omega_k) \Gamma^{(2)}(k, \omega_k) \]  

(25)

The additional loop integral is taken within the region \(-\Lambda < \varepsilon(k) < \Lambda\). By dimensional reasons, it is clear that for \( \omega_p, \varepsilon(p) << \Lambda \) the dominant behavior is given by the terms proportional to \( \omega_p \) and \( \varepsilon(p) \), as long as the cutoff is the only explicit scale in the theory and these quantities saturate the dimensions of the inverse propagator. Those are, on the other hand, the contributions in which we are interested in, since the term proportional to \( \omega_p \) in the self-energy renormalizes the scale of the electron field operator while the term proportional to \( \varepsilon(p) \) may give rise to a renormalization of the “hopping” parameter \( t \).

For the sake of a practical computation of \( \Sigma(p, \omega_p) \), we perform the loop integral by making a change of variables of the type (13) in the respective
sectors of positive and negative energy. In all cases, we first carry out the integral over the hyperbolic angle $\phi$, which is finite provided we keep the external momentum $p$ different from zero. To obtain the term linear in $\omega_p$, we take the derivative with respect to $\omega_p$ and set $\omega_p = 0$. In the region $k_y \geq |k_x|$, for instance, we get

$$\partial \frac{\partial \Re \Sigma^{(2)}_{\mathcal{R}_1}}{\partial \omega_p} \bigg|_{\omega_p=0} \approx \frac{1}{16\pi^4} \frac{U^2}{t^2} \int_0^\Lambda dr \int_0^\infty dx \sum_{n=\pm 1} \frac{x-n(1-\epsilon^2)}{\sqrt{(x-n(1-\epsilon^2))^2 + 4\epsilon^2}}$$

$$\times \left( -\log |x^2-1| + x \log \left| \frac{x-1}{x+1} \right| \right)$$

$$+ \frac{1}{8\pi^4} \frac{U^2}{t^2} \int_0^\Lambda dr \int_0^1 dx \sum_{n=\pm 1} \frac{1}{\sqrt{(x-n(1-\epsilon^2))^2 + 4\epsilon^2}}$$

$$\times \log \frac{2\epsilon}{x-n(1-\epsilon^2) + \sqrt{(x-n(1-\epsilon^2))^2 + 4\epsilon^2}} \right) (26)$$

where we use the parameter $\epsilon \equiv p_x/r$ as an infrared regulator (we have set for simplicity $p_y = 0$). The first term arises from the real part of $\Gamma^{(2)}$ times the imaginary part of $G^{(0)}$ while the second arises from the reverse combination. The terms containing the explicit dependence on the cutoff from $\Gamma^{(2)}$ cancel out in $\Sigma^{(2)}$, what is consistent with the fact that they are a local insertion in the one-loop diagram of Fig. 3(a). Another remarkable fact regarding equation (26) is that the infrared regulator $\epsilon$ cannot be sent to zero, if we want to maintain finite the above expression. This will have important consequences, as it turns out that the dependence of $\Sigma^{(2)}$ on the cutoff becomes entangled, in principle, with a dependence on the external momentum.

Putting together the pieces coming from the different integration regions, we arrive at the form of the linear dependence of $\Sigma^{(2)}$ on $\omega_p$

$$\Re \Sigma^{(2)}(p \to 0, \omega_p) \approx$$

$$\approx \omega_p \frac{1}{4\pi^4} \frac{U^2}{t^2} \int_0^\Lambda dr \frac{1}{r} \left( 2 \log 2 \log \left| \frac{\epsilon(p)}{tr^2} \right| + c_1 \right)$$

$$\approx \omega_p \frac{1}{4\pi^4} \frac{U^2}{t^2} \left( -2 \log 2 \log^2 \left( \Lambda/|p^2_x - p^2_y|^{1/2} \right) + c_1 \log \Lambda \right) \right) (27)$$

with $c_1 \approx -5.896$. We have used again the SO(1,1) symmetry to reconstruct the dispersion relation from the momentum dependence of the $\epsilon$ parameter. The above expression shows that there is a renormalization of the scale of the electron field as the cutoff is varied, though at this point this effect cannot be assessed in a way independent of the electron state. In other words, to the coefficient $c_1$ of the log $\Lambda$ term in (27) we should add the log $|p^2_x - p^2_y|^{1/2}$ coming...
from the squared logarithm contribution, leading to the result of a wavefunction renormalization depending on the electron momentum. This is what, in the field theory language, is called a nonlocal divergence, since it cannot be removed in the framework of quantum field theory by the subtraction of terms with power-law dependence on the frequency or the momentum. In the present situation it may also appear doubtful that the \( \log \left| p_x^2 - p_y^2 \right|^{1/2} \) cutoff dependence can be removed by a convenient redefinition of the couplings in the theory. However, the main difference of the condensed matter framework with respect to quantum field theory is that one may envisage the renormalization of the interaction potential and the Fermi surface, which are, in general, functions of the electron momentum. This approach to the problem of interacting fermions is not usually adopted, mainly because in Fermi liquid theory the interaction, taken as a manifold of couplings, is not irrelevant only in a very reduced number of channels. We have seen, though, that this type of restriction does not apply to our model, as the four-fermion interaction remains marginal no matter what is the kinematics of the momenta involved on it. The fact that the first nontrivial contribution in perturbation theory introduces a leading term of the type \( \log^2 \Lambda \), instead of a simple logarithm, points also at the solution of the problem. The absence of a \( \log \Lambda \) contribution to first order in perturbation theory is just a consequence of having fine-tuned the bare potential \( V(k) \) to unity, which now appears clearly unnatural since any slight (scale independent) deviation from that form introduces significant renormalization effects. We will follow this line of thought in order to apply with success the renormalization program in the next section.

We complete the discussion of the self-energy corrections by considering the contributions proportional to the dispersion relation \( \varepsilon(p) \). Their computation is carried out by differentiating now \( \Sigma^{(2)} \) with respect to \( p_x^2 \) or \( p_y^2 \), and then setting them to zero except where they act as an infrared regulator. Details of the calculation are given in Appendix II. We finally arrive at the result

\[
\text{Re} \Sigma^{(2)}(p \to 0, \omega_p = 0) \approx \varepsilon(p) \frac{1}{4\pi^4} \left\{ \frac{U^2}{t} \int^\Lambda dr \left( (-2 \log 2 + \frac{3}{2}) \log \left| \frac{\varepsilon(p)}{tr^2} \right| + c_2 \right) \right\} \approx \varepsilon(p) \frac{1}{4\pi^4} \frac{U^2}{t^2} \left( (2 \log 2 - \frac{3}{2}) \log^2 \left( \Lambda/ \left| p_x^2 - p_y^2 \right|^{1/2} \right) + c_2 \log \Lambda \right) \tag{28}
\]

with \( c_2 \approx 3.839 \). We can apply to this expression the same remarks made for the contributions linear in \( \omega_p \) to the self-energy. It is worth noting that the dependence on the cutoff in (28) is not the same as that renormalizing
the scale of the electron field in equation (27). This means that, besides the wavefunction renormalization also present in (28), there is a left-over that leads directly to the modification of the dispersion relation. However, the nonlocal character of the Λ-dependence prevents again to understand that correction, as it stands, as a pure renormalization of the hopping parameter \(t\). This issue will be conveniently clarified in the next section.

5 Renormalization

The goal of applying the renormalization group program to a statistical field theory of the type we are studying is to remove all the dependence on the cutoff from observable quantities by an appropriate redefinition of the couplings (in general sense) of the theory. This philosophy corresponds to the point of view of relating different cutoff-dependent bare theories to a unique cutoff-independent renormalized theory[19, 20]. In quantum field theory there are theorems that establish, in general, whether a given model is renormalizable, so that all the divergences as \(Λ \to \infty\) can be absorbed in the parameters of the theory. In the context of condensed matter systems, however, the criteria for interacting fermion systems are more unclear, as the renormalization group approach has been developed in recent years and limited mainly to known paradigms, like that of Fermi liquid theory, and to models with properties common to quantum field theories. It turns out, though, that the number of degrees of freedom that are available in the renormalization of a statistical fermion system is much larger than that in a quantum field theory. In the latter the number of marginal couplings is always finite (except in the case of spatial dimension equal to one) while in a fermion system with a nonvanishing charge density the Fermi surface (or line) and the interaction potential should be taken as dynamical variables influencing one to each other. In other words, there are two different functions, not just parameters, that can be set to enable the cutoff independence of the renormalized theory. The situation is in that respect reminiscent of string theory with massless background fields. Although the string interaction is clearly nonlocal, there is always a choice of the background condensates to each perturbative order that keeps the theory conformal invariant[21].

Coming back to our model, we should allow for redefinitions of the dispersion relation \(\varepsilon(k)\) and the interaction potential \(V(k)\) in order to get rid of the nonlocal cutoff dependences of the self-energy in (27) and (28). On the one hand, the saddle point in \(\varepsilon(k)\) has a topological character, in the sense that it cannot be removed by any smooth deformation of the dispersion relation. Therefore, we concentrate on the possibility of a modification of the
bare four-fermion interaction by the presence of the van Hove singularity. There are two different ways in which this can be understood, namely from a physical and a technical point of view. On physical grounds, the fact that the renormalization coefficients in (27) and (28) diverge as $p \to 0$ means that the renormalization process is out of control as one approaches the singularity. This instability can be interpreted as a signal that the physics is changing drastically at the flat region near the origin, in a way that our initial bare theory cannot encompass. On technical grounds, the instability can be traced back to the significant change that the quantum corrections bring about in the interaction potential at very low energies. This can be appreciated in the one-loop polarization tensor (20), which corrects the classical four-fermion interaction. The two effective interactions in the right-hand-side of (20) are the very source of the squared logarithm cutoff-dependence in the self-energy function, and they give the hint that similar terms have to be included in the bare interaction to absorb the nonlocal quantum corrections. From the point of view of a naive scaling, a logarithmic correction to the bare potential does not change the marginal character of the interaction, though it modifies in general the real space potential at long distances by a $1/r^2$ tail.

There is a sensible difference between the effective interactions that appear in (21), namely

$$V_1 \sim \log \left| \omega^2 - \epsilon(k)^2 \right| + iA_1$$
$$V_2 \sim \frac{\omega}{\epsilon(k)} \log \left| \frac{\omega - \epsilon(k)}{\omega + \epsilon(k)} \right| + iA_2$$

and those that can enter in a physical potential. A retarded interaction in real space gives rise to a potential in $(\omega, k)$ space that has an analytic extension from the real line to the upper half-plane of complex frequencies. As is well-known, though, the effective potentials that arise from the particle-hole polarizability $\Gamma^{(2)}$ are analytic up to $\omega = 0$ [2]. The imaginary parts of $V_1$ and $V_2$ are even functions of $\omega$, instead of odd functions as it should be to combine with the real part of respective analytic functions. For this reason, even if we renormalize the bare potential to absorb the nonlocal $\Lambda$-dependences that appear in the self-energy corrections, the effect of this renormalization can only cancel part of the effective interaction.

The other obvious condition that is required to any admissible modification of the bare potential is to have a real part that is an even function of $\omega$, leading to a real interaction in real space. Thus, we arrive at the conclusion that the real parts of the effective potentials $V_1$ and $V_2$ produced by the particle-hole polarizability are just the only dimensionless expressions that can be thought of. The second potential $V_2$ generates however too singular
log \Lambda contributions when inserted in the one-loop self-energy diagram, that cancel only for the specific imaginary part arising from \Gamma^{(2)} (and did not show up therefore in \Sigma^{(2)}). It seems then that we are only left with the real part of \Sigma_1 in our purpose of building up an analytic potential that may remove the nonlocal \Lambda-dependences of \Sigma^{(2)}. There are actually two possibilities, depending on whether the argument of the logarithm is scaled with \varepsilon(k)^2 or with \omega^2. We perform therefore a renormalization of the potential in the form

\[ U\tilde{V}(k, \omega) = UV + \frac{U^2}{t} V^2 \left( a_1 \log \left( \frac{\omega^2 - \varepsilon(k)^2}{\varepsilon(k)^2} \right) + a_2 \log \left( \frac{\omega^2 - \varepsilon(k)^2}{\omega^2} \right) \right) + O \left( \frac{U(U/t)^2}{t^2} \right) \]

and demand that \( a_1 \) and \( a_2 \) are such that, by insertion of \( \tilde{V}(k, \omega) \) in the one-loop self-energy diagram, the nonlocal \Lambda-dependences cancel up to the two-loop order. \( a_1 \) and \( a_2 \) are then uniquely determined, since as we already remarked there are independent quantum corrections to the scale of the electron field and to the dispersion relation \( \varepsilon(k) \).

When used at the one-loop level, the potential \( (30) \)

\[ \operatorname{Re} \Sigma^{(1)} \approx \omega_p \frac{1}{8\pi^4} \frac{U^2}{t^2} \left( -\frac{a_1}{2} \log^2 \left( \Lambda / |p_x^2 - p_y^2|^{1/2} \right) - 2.193 a_1 \log \Lambda \right) + \frac{3}{2} a_2 \log^2 \left( \Lambda / |p_x^2 - p_y^2|^{1/2} \right) - 2.193 a_2 \log \Lambda \]

\[ + \varepsilon(p) \frac{1}{8\pi^4} \frac{U^2}{t^2} \left( \frac{a_1}{4} \log^2 \left( \Lambda / |p_x^2 - p_y^2|^{1/2} \right) + 1.847 a_1 \log \Lambda \right) + \frac{a_2}{4} \log^2 \left( \Lambda / |p_x^2 - p_y^2|^{1/2} \right) - 2.153 a_2 \log \Lambda \]

(31)

If we now set \( a_1 = -14 \log 2 + 9 \) and \( a_2 = -2 \log 2 + 3 \), we see that all the nonlocal cutoff-dependences are removed from the self-energy function to order \( U^2/t^2 \)

\[ \Sigma = \Sigma^{(1)} + \Sigma^{(2)} + \ldots \]

\[ \approx \omega_p \frac{1}{4\pi^4} \frac{U^2}{t^2} \left( -6.893 \log \Lambda \right) - \varepsilon(p) \frac{1}{4\pi^4} \frac{U^2}{t^2} \left( -1.452 \log \Lambda \right) + O \left( \frac{(U/t^2)}{t^2} \right) \]

This amounts to accomplish half the renormalization program in the model, since there still remain the local \Lambda-dependences that have to be removed by the usual procedure of absorbing them in the scale of the bare electron field and in the bare hopping parameter.

We require the cutoff-independence of the full Green function given by

\[ \frac{1}{G} = \frac{1}{G^{(0)}} - \Sigma \]
\[
\approx Z_{\Psi}^{-1}(\Lambda) \left( \omega_p - t(\Lambda)(p_x^2 - p_y^2) \right) \\
+ Z_{\Psi}^{-1}(\Lambda) (\omega_p - \varepsilon(p)) \frac{1}{4\pi^4} \frac{U^2}{t^2} (6.893 \log \Lambda) \\
+ Z_{\Psi}^{-1}(\Lambda) \varepsilon(p) \frac{1}{4\pi^4} \frac{U^2}{t^2} (5.441 \log \Lambda) + O \left( \frac{(U/t)^3}{t} \right)
\]

(33)

since this object leads to observable quantities in the quantum theory. In (33) \(Z_{\Psi}^{1/2}(\Lambda)\) represents the scale of the bare electron field compared to that of the cutoff-independent electron field

\[
\Psi_{\text{bare}}(\Lambda) = Z_{\Psi}^{1/2}(\Lambda) \Psi
\]

(34)

From

\[
\Lambda \frac{d}{d\Lambda} G^{-1} \equiv 0
\]

(35)

we get the differential equations, to the two-loop order,

\[
\Lambda \frac{d}{d\Lambda} \log Z_{\Psi}(\Lambda) = 6.893 \frac{1}{4\pi^4} \frac{U^2}{t^2}
\]

(36)

\[
\Lambda \frac{d}{d\Lambda} t(\Lambda) = 5.441 \frac{1}{4\pi^4} \frac{U^2}{t}
\]

(37)

The quantities \(Z_{\Psi}(\Lambda)\) and \(t(\Lambda)\) play the role of effective parameters that reflect the behavior of the quantum theory as \(\Lambda \to 0\) and more states are integrated out from high-energy shells of the dispersion relation. The equations (36) and (37) have to be solved in conjunction with equation (23) for \(U(\Lambda)\), that completes the set of coupled renormalization group equations.

At last, we have accomplished successfully our purpose of absorbing all the dependences on the cutoff by renormalizing the interaction and other parameters of the theory. In the process we have been able to interpret the nonlocal \(\Lambda\)-dependent quantum corrections, that obstructed the knowledge about the physics close to the singularity, as a renormalization of the interaction potential. This may appear a little bit surprising, since it seems that we had gained a complete predictability about the behavior of the theory at arbitrarily small energies. However, this may not be necessarily the case.

The potentials in (30) that we have introduced to renormalize the interaction grow large at very small values of \(\omega^2\) or \(\varepsilon(k)^2\). Even if our renormalization group approach predicts that in certain regime the effective coupling constant \(U(\Lambda)/t(\Lambda)\) becomes small, so that perturbation theory may become reliable, there cannot be an absolute statement about the weakness of the interaction, since at very small frequencies or very large distances the interaction (30) enters necessarily the nonperturbative regime. It seems, therefore, that we have not obviated yet the existence of some limitation in the close approach
to the singularity, at least while we do not get information about the form of
the successive renormalizations of $V(k, \omega)$ to higher orders in perturbation
theory.

6 Physical properties

The solution of the coupled set of renormalization group equations (23), (36)
and (37) gives, in the limit $\Lambda \to 0$, the behavior of the statistical field theory
at low energies. In principle, the nontrivial renormalizations of the wavefunc-
tion of the electron, the hopping parameter and the four-fermion coupling
constant are quite relevant results, since none of them have a counterpart
in Fermi liquid theory\cite{4, 5}. The flow of the scale of the electron field in
(36), for instance, is such that it reflects the suppression of the weight of
fermion quasiparticles at the Fermi level. This agrees with the conclu-
sion obtained in the context of a purely phenomenological model of the copper-
oxide superconductors\cite{3}. The flow of the hopping parameter in (37)
gives information about how the dispersion relation is renormalized in the low-
energy limit. From that equation we may already draw the conclusion that
$\epsilon(k)$ has in any event a tendency to become flatter near the singularity. This
behavior is also in agreement with experimental observations\cite{10, 11}.

The main issue, however, is the determination of the coupling regime to
which the model flows at low energies. This is not just given by the flow of
the four-fermion coupling constant $U$, since the parameter that enters in the
power series expansion of perturbation theory is $U/t$. Thus, although $U(\Lambda)$
decreases as $\Lambda \to 0$, $t(\Lambda)$ also does and the consideration of the behavior of
$U/t$ requires further detail.

From (23) and (37) we obtain

$$
\Lambda \frac{d}{d\Lambda} \frac{U}{t} = \frac{1}{2\pi^2} \left( \frac{U}{t} \right)^2 - \frac{5.441}{4\pi^4} \left( \frac{U}{t} \right)^3
$$

We regard the expression at the right-hand-side of (38) as the beta function
of the effective coupling $U/t$, whose zeros correspond to fixed points of the
flow (see Fig. 4). Apart from the fixed point at the origin, another fixed
point shows up at a value of the coupling $(U/t)_{\text{critical}} \approx 2\pi^2/5.441$. One may
question whether this point is a sensible feature of the model, as it may not
lie in the region in which perturbation theory can be trusted. We think that
it is realistic to consider that the perturbative expansion is more precisely a

\footnote{It may seem that in (38) there is a $(U/t)^3$ term missing from coupling constant renormalization, but this contribution belongs to the RPA sum already encoded in the term at the right-hand-side of (23).}
power series in the parameter \( U/(2\pi^2 t) \), taking into account the phase space factors. Thus, it is very likely that the nontrivial fixed point that we obtain within perturbation theory is the evidence for a fixed point that remains at the nonperturbative level. A plot of the beta function \( \beta(U/t) \) in the right-hand-side of (38) is given in Fig. 4. The points with \( \beta > 0 \) have a flow that leads towards the origin as \( \Lambda \to 0 \) while those with \( \beta < 0 \) flow towards increasing values of \( U/t \), so that the nontrivial fixed point at \( (U/t)_{\text{critical}} \) is unstable in the infrared. The solution of (38) can be given in closed implicit form

\[
\frac{U}{t} = \frac{U_0/t_0}{1 - \frac{1}{2\pi^2} \frac{U_0}{t_0} \log \left( \frac{\Lambda}{\Lambda_0} \right) + \alpha \frac{U_0}{t_0} \log \left( \frac{U}{t U_0} \right) - \alpha \frac{U_0}{t_0} \log \left| \frac{1 - \alpha U/t}{1 - \alpha U_0/t_0} \right|} 
\]

(39)

where \( \alpha \approx 5.441/(2\pi^2) \). It seems, therefore, that the system has at least two different phases. One of them is within the weak coupling regime, in which the theory becomes asymptotically free in the low-energy limit. The other phase goes into the strong coupling regime, leading presumably to different physical properties that cannot be studied within perturbation theory.

Next, the behavior of \( t(\Lambda) \) can be easily understood in the following way. From equations (23) and (37) one may notice that the flow in \((t, U)\) space is given by straight lines

\[
\frac{dU}{dt} = \frac{2\pi^2}{5.441} \cdot \frac{U_0}{t_0} \cdot t \cdot \left( \frac{U_0}{t_0} \right)
\]

(40)

Then, it is clear that if we start with some initial value \( U_0/t_0 < 2\pi^2/5.441 \), the hopping parameter \( t(\Lambda) \) will decrease to some finite value as \( \Lambda \to 0 \). On the other hand, if \( U_0/t_0 > 2\pi^2/5.441 \) the running hopping \( t(\Lambda) \) will have no limit in its approach to zero, though our perturbative renormalization group methods become unreliable before reaching that value. We observe anyhow a definite trend of the hopping parameter towards decreasing values in the process of renormalization, which may persist in the strong coupling regime. This translates, in turn, into a tendency of the dispersion relation to become flatter around the singularity and, strictly speaking, a value of \( t(\Lambda) \approx 0 \) would mean that the dispersion relation \( \varepsilon(k) = t(k_x^2 - k_y^2) \) turns into a higher degree singularity. Quite remarkably, the experimental studies of copper-oxide superconductors by means of angle-resolved photoemission spectroscopy show very flat regions of the dispersion relation near the Fermi level[10, 11]. Similarly, an almost dispersionless region arises from numerical studies of the dynamics of a few holes in the half-filled two-dimensional square lattice[22]. All this evidence points in the direction that the correspondence with the physically relevant situation for the cuprates may take place from the strong coupling phase of our model, where the renormalization of the dispersion towards a flat band would persist at the nonperturbative level.
Regarding the renormalization of the electron wavefunction, we find again two different situations depending on whether we stay in the weak or in the strong coupling phase. In the weak coupling phase, we can integrate equation (36) in a straightforward way taking the initial data in the interval \((0, (U/t)_{\text{critical}})\)

\[
Z_{\Psi}(\Lambda) = e^{6.893/(4\pi^4) \int^\Lambda d\rho/\rho U(\rho)^2/\rho(\rho)^2}
\]  

(41)

As long as the running effective coupling \(U(\Lambda)/t(\Lambda)\) goes to zero when \(\Lambda \to 0\) according to (39), we can only expect a very weak reduction in the scale of the electron field at low energies. Thus, the attenuation of quasiparticles should not differ much in this case from that in Fermi liquid theory. The situation is completely different in the strong coupling phase, since there the scaling dimension of the electron field is actually changed from the noninteracting value. The anomalous dimension is most easily computed in the particular instance in which we sit at the critical coupling \((U/t)_{\text{critical}}\). Applying now (36) we obtain

\[
Z_{\Psi}(\Lambda) = \Lambda^{(6.893/(4\pi^4))(U/t)_{\text{critical}}^2}
\]  

(42)

By renormalizing then the model to the Fermi level \(\Lambda \to 0\), the attenuation of the electron quasiparticles becomes complete. This feature is similar to that proposed in Ref. [3] in a phenomenological approach to the copper-oxide superconductors. In our framework, however, it acquires a more precise meaning. The renormalization effects are so strong that they change at very low energies the form of the electron propagator. The relation between the unrenormalized and the renormalized (\(\Lambda\)-independent) electron Green function that follows from (42)

\[
G(k, \omega) = \Lambda^{(6.893/(4\pi^4))(U/t)_{\text{critical}}^2} G(k, \omega)_{\Lambda}
\]  

(43)

leads to the scaling behavior

\[
G(\rho k, \rho^2 \omega) = \rho^{-2+(6.893/(2\pi^4))(U/t)_{\text{critical}}^2} G(k, \omega)
\]  

(44)

This anomalous scaling is another way of looking at the breakdown of the quasiparticle picture. The equations (12) and (14) are only correct at the critical point \((U/t)_{\text{critical}}\), that is an unstable fixed point. Therefore it remains as an open question to what extent the breakdown of the Fermi liquid picture persists when leaving the critical point in the direction of the strong coupling phase.

We finish our discussion of the predictions that emerge from the renormalization group approach by addressing the question of the position of the Fermi level in the model. The results showing a clear deviation from Fermi liquid behavior depend on the coincidence of the Fermi level with the energy
level of the singularity, as it is the divergent density of states at that point the source of the unconventional behavior. Actually, the renormalization of the observables considered above still takes place to a great extent provided that the energy difference between the Fermi level and that of the singularity is much smaller than any other energy scale in the system. It arises the question, however, of whether this close approach of the two levels is something natural, or rather any slight perturbation may send the singularity away from the Fermi level. This problem has been studied in Ref. [23] from the point of view of a system with variable number of particles and fixed chemical potential. The conclusion reached there by a standard renormalization group analysis is that as \( \Lambda \rightarrow 0 \) the running Fermi energy is captured by the singularity. This feature is at odds with the unstable behavior of the Fermi energy in Fermi liquid theory, where only a repulsive fixed point is found for finite values of the chemical potential [4] —this fact does not have however physical relevance, as long as in closed systems the stability of the Fermi level is ensured by the Luttinger theorem.

Rather than reproducing the analysis of Ref. [23], we give here a simpler but more physical explanation of the pinning mechanism. The argument is suited for a system that may receive extra charge from a reservoir (as is the case of the copper-oxide compounds) and does not have therefore fixed number of particles. The number of particles in the reservoir is much larger than that in the system, so that the flow of particles into the latter is controlled by the chemical potential \( \mu \) of the reservoir. This quantity acts as a kind of external “pressure” of particles. However this external pressure is not seen in all its magnitude in the two-dimensional system, since it is partially counterbalanced there by the repulsion between the electrons. The competition between the two effects is what sets the value of the Fermi level \( \varepsilon_F \) in the system. By using a simple one-loop approximation we may express the correction to the chemical potential as the frequency and momentum-independent contribution of the Hartree diagram

\[
\varepsilon_F \approx \mu + i \frac{U}{8\pi^3} \int d\omega d^2kG^{(0)}(k, \omega)
\]

In terms of the density of states \( n(\omega) \) we may write this expression in the form

\[
\varepsilon_F = \mu - \frac{U}{4\pi} \int^{\varepsilon_F} d\omega n(\omega)
\]

We may obviate the influence of the particular shape of the dispersion relation by differentiating (46) with respect to \( \mu \), so that

\[
\frac{d\varepsilon_F}{d\mu} = 1 - \frac{U}{4\pi} n(\varepsilon_F) \frac{d\varepsilon_F}{d\mu}
\]
Then, we arrive at the differential equation
\[ \frac{d\varepsilon_F}{d\mu} = \frac{1}{1 + \frac{4\pi}{\varepsilon_F} n(\varepsilon_F)} \] (48)
which expresses how the filling level changes in the system under variations of the external chemical potential. Equation (48) shows that when the Fermi energy \( \varepsilon_F \) is very close to a level with a divergent density of states, as is the case of the van Hove singularity, the Fermi level is very weakly influenced by changes in the chemical potential of the reservoir. Of course, this does not prove that the level of filling has necessarily to fall close to the van Hove singularity, but the simple argument outlined gives a notion of the stability of the model with the particular filling considered in this paper. On the other hand, with the methods employed in Ref. [23] it is possible to show that the renormalization of the Fermi energy \( \varepsilon_F \) keeps up with the integration of the high-energy degrees of freedom. The result leads to the dynamical effect of pinning the Fermi level to the van Hove singularity as \( \Lambda \to 0 \). Quite remarkably, this feature is in correspondence with the experimental observation of many hole-doped copper-oxide compounds, for which the Fermi level is found very close to a pronounced peak of the photoemission spectra [10].

### 7 Conclusions

We have shown that a model of interacting electrons near a 2D van Hove singularity can be treated by means of renormalization group techniques. These have found recent application in the general description of interacting fermion systems [4, 5], which require specific treatment given the distinctive character of the many-body ground state, that is, the existence of an extended Fermi surface. One of the main accomplishments of Refs. [4, 5] has been the comprehension gained of Fermi liquid theory and its perturbations. In fact, under very general conditions like the isotropy of the Fermi surface and the short-range character of the interaction, it appears that the interacting fermion system falls necessarily into the Fermi liquid universality class. The problem that we have addressed, however, does not adhere to that well-established picture, as the anisotropy of the dispersion relation demands an appropriate treatment. As a result of that, we have found a nontrivial renormalization group flow of the couplings in the infrared. This flow is incompatible with conventional Fermi liquid theory, making the model one of the few systems beyond one dimension where the breakdown of a Fermi liquid is explicitly manifested.

We find that, for sufficiently weak coupling, the interaction strength is renormalized to zero, which is an attractive fixed-point. This behavior is
easy to understand, as it corresponds to the standard picture in which the repulsive interaction is screened at long distances by the particle-hole polarizability of the medium. On the other hand, our results suggest the existence of another unstable fixed-point, and of a strong coupling phase beyond it. This new phase arises as a consequence of the tendency of the dispersion relation to become flatter near the van Hove singularity. As long as the hopping parameter $t$ is renormalized towards zero in the infrared, the effective coupling constant $U/t$ flows to larger values in the new phase. The reduction of the dispersion of the band near the singularity is in remarkable correspondence with the recent experimental results from photoemission studies[10, 11]. Therefore, there are solid reasons to believe that the new phase predicted in the renormalization group approach should be a sensible feature of the system.

The precise knowledge of the nature of the strong coupling phase would require, though, different methods to that employed in this paper. Preliminary studies of finite clusters with next-to-nearest-neighbor interactions (excluding therefore a spin density wave instability) seem to indicate a trend towards ferromagnetic order[24], what is consistent with the development of a very flat band.

The attenuation of the electron quasiparticles that we have found in our model is in agreement with a phenomenological approach to the copper-oxide superconductors[3]. We have seen that this property arises from the electron wavefunction renormalization, which contains in general nonlocal contributions that can be studied rigorously within our renormalization group approach. The logarithmic corrections to the scale of the electron field are also consistent with another known property, that is, the abnormal behavior of the quasiparticle lifetime ($\sim \omega^{-1}$) in the model of the van Hove singularity[16]. A standard Kramers-Kronig relation applied to the expression (32) leads actually to an imaginary part of the electron Green function linear in $\omega$, in contrast to the conventional quadratic behavior in Fermi liquid theory.

Finally, we have seen that the chemical potential itself also shows a nontrivial scaling. At low energies, the Fermi level tends to be pinned near the van Hove singularity. This flow of the chemical potential, unusual in condensed matter systems, may explain the existence of a whole family of compounds where experiments show that the Fermi level is close to a van Hove singularity[10].

We have not explored here the role played by the intersingularity interactions, which may be important for the mechanism driving to the superconducting phase. In fact, for a sufficiently large coupling between the two singularities an attractive channel may open, leading to pair binding with $d$-wave order parameter[23]. It is worth noting that the Cooper channel shows a $\log^2 \Lambda$ dependence on the cutoff, in marked difference from the $\log \Lambda$ de-
pendence usual in the analysis of the Fermi liquid. In the context of our renormalization group approach to the model, that instability appears for a particular kinematics in the scattering process, corresponding to the channel of bound state formation, and it is therefore independent of the renormalization of the coupling constant accomplished above. It would be interesting then to undertake the analysis of the superconducting instability along the lines followed in the present paper, to elucidate whether a purely electronic mechanism may be responsible for the superconductivity of the copper-oxide compounds.
Appendix I

The one-loop vertex function $i\Gamma^{(2)}$ in (43) can be computed by letting the momentum integrals from $-\infty$ to $+\infty$, and carrying them out over the complex plane. In the region with $\omega_q > 0$ and for $\omega > 0$, for instance, we have

$$I_+^{(\omega, k_x)} = \int_{\omega_q > 0} d\omega_q \int dq_x dq_y \frac{1}{\omega + \omega_q - t ((q_x + k_x)^2 - q_y^2) + i\epsilon \text{sgn}(\omega + \omega_q)}$$

$$\times \frac{1}{\omega_q - t (q_x^2 - q_y^2) + i\epsilon \text{sgn} \omega_q}$$

$$= -\frac{i\pi}{t} \int_{\omega_q > 0} d\omega_q \int dq_y \frac{1}{\omega - tk_x^2 - 2tk_x \sqrt{\omega_q}/t + q_y^2 + i\epsilon}$$

$$-\frac{i\pi}{t} \int_{\omega_q > 0} d\omega_q \int dq_y \frac{1}{-\omega - tk_x^2 + 2tk_x \sqrt{(\omega + \omega_q)/t + q_y^2} + i\epsilon}$$

$$\times \frac{1}{\sqrt{(\omega + \omega_q)/t + q_y^2} + i\epsilon}$$

(49)

after picking up the residue of the two poles on the upper half-plane of the $q_x$ complex variable. The integrals over $q_y$ can be done by deforming the contour of integration in each case around a pole and a branch cut on the $q_y$ complex plane. The sum of all the contributions gives

$$I_+^{(\omega, k_x)} = -\frac{i\pi}{t} f(\omega, k_x) \int_{1}^{\Lambda/(tk_x^2 f^2)} dx \arcsin \frac{\sqrt{1 - 1/x} - \frac{\pi}{2}}{\sqrt{x - 1}}$$

$$+ \frac{i\pi}{t} g(\omega, k_x) \int_{1}^{\Lambda/(tk_x^2 g^2)} dx \arcsin \frac{\sqrt{1 - 1/x} - \frac{\pi}{2}}{\sqrt{x - 1}}$$

$$- \frac{i\pi}{t} g(\omega, k_x) \int_{0}^{\omega/(tk_x^2 g^2)} dx \arcsinh \frac{1}{x - 1}$$

$$- \frac{\pi^2}{t} |f(\omega, k_x)| + \frac{\pi^2}{t} |g(\omega, k_x)|$$

(50)

where $f(\omega, k_x) = \frac{\omega - tk_x^2}{2tk_x^2}$ and $g(\omega, k_x) = \frac{\omega + tk_x^2}{2tk_x^2}$

The part $I_-$ of the one-loop integral for $\omega_q < 0$ can be obtained in similar fashion, with the result that

$$I_-^{(\omega, k_x)} = \frac{i\pi}{t} f(\omega, k_x) \int_{-\Lambda/(tk_x^2 f^2)}^{0} dx \arcsinh \frac{-1/x}{\sqrt{1 - x}}$$

25
\begin{align*}
-\frac{i\pi}{t} g(\omega, k_x) \int_{-\Lambda/(tk_x^2g^2)}^{0} dx \frac{\text{arcsh}\sqrt{-1/x}}{\sqrt{1-x}} \\
+ \frac{i\pi}{t} g(\omega, k_x) \int_{0}^{\omega/(tk_x^2g^2)} dx \frac{\text{arcsh}\sqrt{1/x - 1}}{\sqrt{1-x}} \\
- \frac{\pi^2}{t} |f(\omega, k_x)| + \frac{\pi^2}{t} |g(\omega, k_x)|
\end{align*}

Both expressions (50) and (51) combine to produce a well-defined extension to the whole \((k_x, k_y)\) plane realizing the SO(1,1) invariance of the model, as quoted in the main text.
Appendix II

In order to compute the terms of $\text{Re} \Sigma^{(2)}(\mathbf{p}, \omega_p)$ linear in $\varepsilon(\mathbf{p})$, we divide the momentum integral in $(k_x, k_y)$ space in the four different sectors of positive and negative energy separated by the Fermi line $\varepsilon(\mathbf{k}) = 0$. In the region $k_y \geq |k_x|$, for instance, we differentiate with respect to $p^2_x$ and set the external momentum to zero (whenever it is not needed to regulate an infrared divergence). Thus we obtain

$$\frac{\partial}{\partial p^2_x} \text{Re} \Sigma^{(2)}_{R_1}(\mathbf{p} \to 0, 0) =$$

$$= - \frac{1}{16 \pi^4} \frac{U^2}{t} \int^\Lambda \frac{dr}{r} \frac{1}{r} \int_0^\infty dx \sum_{n=\pm 1} \frac{x - n(1 - \varepsilon^2) + 2n}{\sqrt{((x - n(1 - \varepsilon^2))^2 + 4\varepsilon^2)^3}}$$

$$\times \left( - \log |x^2 - 1| + x \log \left| \frac{x - 1}{x + 1} \right| \right)$$

$$- \frac{1}{4 \pi^3} \int^\Lambda \frac{dr}{r} \frac{1}{r} \int_0^\infty dx \sum_{n=\pm 1} \frac{x - n(1 - \varepsilon^2) + 2n}{\sqrt{((x - n(1 - \varepsilon^2))^2 + 4\varepsilon^2)^3}}$$

$$\times \log \left( \frac{2\varepsilon}{x - n(1 - \varepsilon^2) + \sqrt{(x - n(1 - \varepsilon^2))^2 + 4\varepsilon^2}} \right) \text{Im} \Gamma^{(2)} \big|_{\omega/|\varepsilon(\mathbf{k})|=x}$$

$$+ \frac{1}{8 \pi^3} \int^\Lambda \frac{dr}{r} \frac{1}{r} \int_0^\infty dx \sum_{n=\pm 1} \frac{n}{\varepsilon^2} \frac{1}{(x - n(1 - \varepsilon^2))^2 + 4\varepsilon^2} \text{Im} \Gamma^{(2)} \big|_{\omega/|\varepsilon(\mathbf{k})|=x}$$

$$- \frac{1}{4 \pi^3} \int^\Lambda \frac{dr}{r} \frac{1}{r} \int_0^\infty dx \sum_{n=\pm 1} \frac{x - n(1 - \varepsilon^2) + 2n}{(x - n(1 - \varepsilon^2))^2 + 4\varepsilon^2}$$

$$\times \frac{1}{x - n(1 - \varepsilon^2) + \sqrt{(x - n(1 - \varepsilon^2))^2 + 4\varepsilon^2}} \text{Im} \Gamma^{(2)} \big|_{\omega/|\varepsilon(\mathbf{k})|=x}$$

$$- \frac{1}{4 \pi^3} \int^\Lambda \frac{dr}{r} \frac{1}{r} \int_0^\infty dx \sum_{n=\pm 1} \frac{1}{\sqrt{((x - n(1 - \varepsilon^2))^2 + 4\varepsilon^2)}}$$

$$\times \frac{1}{x - n(1 - \varepsilon^2) + \sqrt{(x - n(1 - \varepsilon^2))^2 + 4\varepsilon^2}} \text{Im} \Gamma^{(2)} \big|_{\omega/|\varepsilon(\mathbf{k})|=x}$$

(52)

where $\varepsilon \equiv p_x/r$. It can be checked that, when the contributions from the four different sectors in $(k_x, k_y)$ space are put together, all the $1/\varepsilon^2$ terms cancel out. The leading contribution in $\log \varepsilon$ from the integrals in the $x$ variable may be evaluated analytically. The contribution independent of $\varepsilon$ is more difficult to extract, and it has been obtained by numerical computation of the above integrals, yielding the result quoted in the main text.
References

[1] J. Gan and E. Wong, *Phys. Rev. Lett.* **71** (1993) 4226. P.-A. Bares and X. G. Wen, *Phys. Rev. B* **48** (1993) 8636. C. Nayak and F. Wilczek, *Nucl. Phys. B* **417** (1994) 359. J. González, F. Guinea, and M. A. H. Vozmediano, *Nucl. Phys. B* **424** (1994) 595. S. Chakravarty, R. E. Norton, and O. F. Syljuasen, *Phys. Rev. Lett.* **74** (1995) 1423.

[2] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics*, Part 2 (Pergamon Press, Oxford, 1980).

[3] C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams and A. E. Ruckenstein, *Phys. Rev. Lett.* **63** (1989) 1996.

[4] R. Shankar, *Rev. Mod. Phys.* **66** (1994) 129.

[5] J. Polchinski, in *Proceedings of the 1992 TASI in Elementary Particle Physics*, J. Harvey and J. Polchinski eds. (World Scientific, Singapore, 1992).

[6] F. D. M. Haldane, *J. Phys. C* **14** (1981) 2585.

[7] J. González, M. A. Martín-Delgado, G. Sierra and M. A. H. Vozmediano, *Quantum Electron Liquids and High-T_c Superconductivity* (Springer-Verlag, Berlin, 1995).

[8] P. W. Anderson, *Science* **235** (1987) 1196. R. B. Laughlin, *Science* **242** (1988) 525.

[9] W. E. Pickett, H. Krakauer, R. E. Cohen and D. J. Singh, *Science* **255** (1992) 46.

[10] See Z.-X. Shen, W. E. Spicer, D. M. King, D. S. Dessau and B. O. Wells, *Science* **267** (1995) 343, and references therein.

[11] K. Gofron et al., *Phys. Rev. Lett.* **73** (1994) 3302.

[12] D. M. Newns, H. R. Krishnamurty, P. C. Pattnaik, C. C. Tsuei and C. L. Kane, *Phys. Rev. Lett.* **69** (1992) 1264, and references therein.

[13] J. Labbé and J. Bok, *Europhys. Lett.* **3** (1987) 1225. H. J. Schulz, *Europhys. Lett.* **4** (1987) 51. J. Friedel, *J. Phys.* (Paris) **48** (1987) 1787; **49** (1988) 1435. J. E. Dzyaloshinskii, *Pis’ma Zh. Eksp. Teor. Fiz.* **46** (1987) 97, [*JETP Lett.* **46** (1987) 118]. P. A. Lee and N. Read, *Phys. Rev. Lett.* **58** (1988) 2691.
[14] R. S. Markiewicz, *J. Phys. Condens. Matter* **2** (1990) 665. R. S. Markiewicz and B. G. Giessen, *Physica* (Amsterdam) **160C** (1989) 497.

[15] C. C. Tsuei, D. M. Newns, C. C. Chi and P. C. Pattnaik, *Phys. Rev. Lett.* **65** (1990) 2724.

[16] P. C. Pattnaik, C. L. Kane, D. M. Newns and C. C. Tsuei, *Phys. Rev. B* **45** (1992) 5714.

[17] J. Sólyom, *Adv. Phys.* **28** (1979) 201. V. J. Emery, in *Highly Conducting One-Dimensional Solids*, edited by J. T. Devreese, R. P. Evrard and V. E. Van Doren (Plenum, New York, 1979).

[18] L. P. Kadanoff, *Physics* **2** (1966) 263. K. G. Wilson, *Phys. Rev. B* **4** (1971) 3174.

[19] J. Zinn-Justin, Lectures delivered at the Cargèse Summer School 1973 (unpublished). K. Symanzik, in *Particles, Quantum Fields and Statistical Mechanics* (edited by M. Alexanian and A. Zepeda) (Springer Verlag, Berlin, 1975).

[20] D. J. Amit, *Field Theory, Renormalization Group and Critical Phenomena*, Chap. 8 (McGraw-Hill, New York, 1978).

[21] D. Friedan, *Phys. Rev. Lett.* **45** (1980) 1057. L. Alvarez-Gaumé, D. Z. Freedman and S. Mukhi, *Ann. Phys. (N.Y.)* **134** (1981) 85.

[22] E. Dagotto, A. Nazarenko and M. Boninsegni, *Phys. Rev. Lett.* **73** (1994) 728. E. Dagotto, A. Nazarenko and A. Moreo, *Phys. Rev. Lett.* **74** (1995) 310.

[23] J. González, F. Guinea and M. A. H. Vozmediano, report cond-mat/9502095.

[24] J. V. Alvarez, J. González, F. Guinea and S. Sorella, in preparation.
Figure Captions

Figure 1: Contour map of the dispersion relation for a two-dimensional square lattice with next-to-nearest-neighbor hopping.

Figure 2: One-loop order correction to the interaction potential.

Figure 3: Different self-energy diagrams to the two-loop order.

Figure 4: Plot of the beta function at the right-hand-side of (38).
Figure 2:
Figure 3:
