DOMINIONS IN VARIETIES OF NILPOTENT GROUPS

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Abstract. We investigate the concept of dominion (in the sense of Isbell) in several varieties of nilpotent groups. We obtain a complete description of dominions in the variety of nilpotent groups of class at most two. Then we look at the behavior of dominions of subgroups of groups in $N_2$ when taken in the context of $N_c$ for $c > 2$. Finally, we establish the existence of nontrivial dominions in the category of all nilpotent groups.

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Section 1. Introduction

Suppose a group $G$ and a subgroup $H$ of $G$ are given. Are there any elements $g \in G \setminus H$ such that any two group morphisms which agree on $H$ must also agree on $g$? What if we require all the groups involved to be nilpotent?

To put this question in a more general context, let $C$ be a full subcategory of the category of all algebras (in the sense of Universal Algebra) of a fixed type, which is closed under passing to subalgebras. Let $A \in C$, and let $B$ be a subalgebra of $A$. Recall that, in this situation, Isbell [5] defines the dominion of $B$ in $A$ (in the category $C$) to be the intersection of all equalizer subalgebras of $A$ containing $B$. Explicitly,

$$ \text{dom}_C^A(B) = \left\{ a \in A \mid \forall C \in C \forall f, g: A \to C, \text{ if } f|_B = g|_B \text{ then } f(a) = g(a) \right\}. $$
Therefore, the question with which we opened this discussion may be restated in terms of the dominion of $H$ in $G$ in the category of context.

If $\text{dom}_A^C(B) = B$ we will say that the dominion of $B$ in $A$ is \textit{trivial}, and we will say it is \textit{nontrivial} otherwise.

In the case where the category $C$ is actually a \textit{variety} (or more generally, a right closed category; see [5]), we can look at the amalgamated coproduct of two copies of $A$, amalgamated over the subalgebra $B$. This is the pushout of the diagram

$$
\begin{array}{ccc}
B & \xrightarrow{i} & A \\
\downarrow{i} & & \\
A
\end{array}
$$

and is denoted by $A \amalg_B^C A$. If we write $(\lambda, \rho)$ for the universal pair of maps from $A$ to $A \amalg_B^C A$, then $\text{dom}_A^C(B)$ is the equalizer of $\lambda$ and $\rho$; one can in fact verify that $\text{dom}_A^C(B) = \lambda(A) \cap \rho(A)$. By a classical theorem of Schreier, if $C = \text{Group}$, then for all $A \in C$ and for every subgroup $B$ of $A$, $\text{dom}_A^C(B) = B$.

Dominions are related to group amalgams, and particularly to \textit{special amalgams}. Recall that an amalgam of $A$ and $C$ with core $B$ consists of groups $A$, $C$, and $B$, equipped with one-to-one morphisms $\Phi_A: B \to A$, $\Phi_C: B \to C$. We will denote this situation by writing $[A, C; B]$. We say that the amalgam is \textit{weakly embeddable} in $C$ if there exists a group $M \in C$ and one-to-one mappings

$$
\lambda_A: A \to M, \quad \lambda_C: C \to M, \quad \lambda_B: B \to M
$$

such that

$$
\lambda_A \circ \Phi_A = \lambda_B, \quad \lambda_C \circ \Phi_C = \lambda_B.
$$

We usually identify $B$ with its images in $A$ and $C$.

We say the amalgam is \textit{strongly embeddable} if, furthermore, there is no identification between elements of $A \setminus B$ and $C \setminus B$. Finally, by a \textit{special amalgam} we mean an amalgam $[A, C; B]$, where there is an isomorphism $\alpha: A \to C$ such that $\alpha \circ \Phi_A = \Phi_C$. In this case, we usually write $[A, A; B]$, with $\alpha = \text{id}_A$ being understood.

It is not hard to see that $B$ equals its own dominion in $A$ (in the category $C$) if and only if the special amalgam $[A, A; B]$ is strongly embeddable, and that the dominion of $B$ is in general the least subgroup $D$ of $A$, such that $D$ contains $B$, and $[A, A; D]$ is strongly embeddable. We refer the reader to the survey article by Higgins [4] for the details.

Dominions are also related to epimorphisms, and in fact were introduced by Isbell to study them. Recall that given a category of algebras $C$, a map
$f: A \to C$ is an epimorphism if and only if it is right cancellable in $C$. That is, if for all pairs of maps $g, h: C \to K$ in $C$, $g \circ f = h \circ f$ implies that $g = h$. Clearly, $f: A \to C$ is an epimorphism in $C$ if and only if $\text{dom}_C^K(f(A)) = C$.

We note that in categories of algebras, all surjective maps are epimorphisms; in some categories, such as the category of all groups, the converse also holds. But the converse does not hold in general: for example, the embedding $Z \hookrightarrow Q$ is an epimorphism in the category of all rings.

Unfortunately, the connection to epimorphisms is not relevant in the context of the present work. Peter Neumann has proven [15] that in all “reasonable” categories of solvable groups, all epimorphisms are necessarily surjective; specifically, he showed that in any full category which consists of solvable groups and is closed under taking quotients, all epimorphisms are surjective. This was later extended substantially by S. McKay [9].

Despite these negative results, one should not set aside dominions as worthless in the context of varieties of nilpotent groups. Their relation to amalgams can be useful; for example, the work on dominions in $N_2$ done here is used elsewhere [11] to characterize the special amalgamation bases in that variety. Also, from a very general point of view, finding nontrivial dominions should be enough to be of interest. Any result that says that we can predict the behavior of a function at some point based on partial information (in this case, the value of the morphism at points in the dominion of $H$ but not in $H$, based on the value on $H$) has the potential of being useful. Aside from this, the dominion construction determines a special class of subalgebras, those which equal their own dominion; equivalently, those which are “closed” under the closure operation induced by the construction. When not all subalgebras are closed, that is when not every subalgebra equals its own dominion, it is possible for this class to have interesting properties of their own. For example, Bergman has shown [1] that in the category of Orderable Groups (groups in which an order can be defined which is compatible with the operations), the class of dominion-closed subgroups of an orderable group $G$ are precisely the subgroups $T$ for which the amalgamated coproduct $G \amalg^\text{Group}_TG$ is also orderable.

In Section 2 we prove some of the basic properties of dominions, and recall the basic definitions associated to nilpotent groups. In Section 3 we will study the variety of nilpotent groups of class at most two. Then, in Section 4 we will generalize the arguments in Section 3 to the variety of 2-Engel groups. In Section 5 we will study how dominions of subgroups of $N_2$-groups behave when we change the variety of context from $N_2$ to $A^2 \cap N_c$, for $c > 2$; and in Section 6 we will look at the category of all metabelian nilpotent groups. In that section, we will have some things to say about dominions in the variety of all metabelian groups as well. Then, in Section 7 and Section 8 we will
expand these investigations to cover the case where the category of context is \( \mathcal{N}_c \), and \( \mathcal{N}_\tilde{c} \), respectively. Finally, in Section 9 we will mention some related results in other varieties of nilpotent groups.

The contents of this work are part of the author’s doctoral dissertation, which was conducted under the direction of Prof. George M. Bergman, at the University of California at Berkeley. It is my very great pleasure to express my deep gratitude and indebtedness to Prof. Bergman for his advice and encouragement; his many suggestions have improved this work in ways too numerous to list explicitly; he also caught and helped correct many mistakes. Any errors that remain, however, are entirely my own responsibility.

Section 2. Preliminary definitions and results

The group operation will be written multiplicatively unless otherwise stated; given a group \( G \), the identity element of \( G \) will be denoted by \( e_G \), with the subscript omitted if it is understood from context. Given two elements \( x \) and \( y \) in \( G \), we write \( x^y = y^{-1}xy \), and we will denote their commutator by \( [x, y] = x^{-1}y^{-1}xy \). Given two subsets \( A, B \) of \( G \) (not necessarily subgroups), we denote by \( [A, B] \) the subgroup of \( G \) generated by all elements \( [a, b] \) with \( a \in A \) and \( b \in B \). We also define inductively the left-normed commutators of weight \( c + 1 \):

\[
[x_1, \ldots, x_c, x_{c+1}] = [[x_1, \ldots, x_c], x_{c+1}]; \quad c \geq 2.
\]

The centralizer in \( G \) of a subgroup \( H \) will be denoted by \( C_G(H) \). We will denote the center of \( G \) by \( Z(G) \).

A variety of groups is a full subcategory of \( \text{Group} \) which is closed under taking subgroups, quotients, and arbitrary direct products. We will first establish some basic properties of dominions in varieties (and more general categories) of groups.

Lemma 2.1. Let \( C \) be a category of groups.

(i) If \( A \in C \), then \( \text{dom}^C_A(\_\_\_\_) \) is a closure operator on the lattice of subgroups of \( A \).

(ii) Given a homomorphism \( h: A \to A' \), with \( A, A' \in C \), if \( B \) is a subgroup of \( A \), then \( h(\text{dom}^C_A(B)) \subseteq \text{dom}^C_{A'}(h(B)) \).

(iii) If \( A, C \in C \) and \( B, C \) are subgroups of \( A \), with \( B \subseteq C \), then

\[
\text{dom}^C_C(B) \subseteq \text{dom}^C_A(B).
\]

Proof: (i) and (ii) are immediate from the definition of dominion; (iii) follows from (ii) by considering the inclusion map \( i: C \to A \).
Lemma 2.2. Let $C \subseteq D$ be two full subcategories of groups, and let $G \in C$. If $H$ is a subgroup of $G$, then $\text{dom}_G^D(H) \subseteq \text{dom}_G^C(H)$.

Proof: The set of pairs of maps from $G$ to groups in $C$ is a subset of the set of pairs of maps from $G$ to groups in $D$. The inclusion now follows from the definition of the dominion as the intersection of equalizer subgroups of $G$.

Lemma 2.3. Let $C$ be a full subcategory of Group, and let $G \in C$ be a group. If $H$ and $K$ are subgroups of $G$, then

$$\langle \text{dom}_G^C(H), \text{dom}_G^C(K) \rangle \subseteq \text{dom}_G^C(\langle H, K \rangle).$$

Proof: Let $L = \langle H, K \rangle$. Recall that $\text{dom}_G^C(\cdot)$ is a closure operator on the subgroups of $G$. Now, $H \subseteq L$ implies that the dominion of $H$ is contained in the dominion of $L$; the same is true of the dominion of $K$, and therefore it is also true of the subgroup that the dominion of $H$ and the dominion of $K$ generate, establishing the result.

Theorem 2.4. Let $G_1$ and $G_2$ be groups, and let $H_1$ be a subgroup of $G_1$, $H_2$ a subgroup of $G_2$. If $C$ is any full subcategory of Group such that $G_1$, $G_2$, and $G_1 \times G_2$ are in $C$, then

$$\text{dom}_G^C_{G_1 \times G_2}(H_1 \times H_2) = \text{dom}_G^C_{G_1}(H_1) \times \text{dom}_G^C_{G_2}(H_2).$$

Proof: We can identify $G_1$ with the subgroup $G_1 \times \{e\}$ of $G_1 \times G_2$, and similarly $G_2$ with the subgroup $\{e\} \times G_2$. It follows, from Lemma 2.3, that $\text{dom}_G^C_{G_1}(H_1)$ and $\text{dom}_G^C_{G_2}(H_2)$ are contained in $\text{dom}_G^C_{G_1 \times G_2}(H_1 \times H_2)$. Therefore

$$\text{dom}_G^C_{G_1 \times G_2}(H_1 \times H_2) \supseteq \langle \text{dom}_G^C_{G_1}(H_1), \text{dom}_G^C_{G_2}(H_2) \rangle = \text{dom}_G^C_{G_1}(H_1) \times \text{dom}_G^C_{G_2}(H_2).$$

Now let $(g_1, g_2) \notin \text{dom}_G^C_{G_1}(H_1) \times \text{dom}_G^C_{G_2}(H_2)$. Assume, without loss of generality, that $g_1 \notin \text{dom}_G^C_{G_1}(H_1)$. Therefore there is a group $K \in C$, and a pair of maps $\psi, \phi: G_1 \to K$ such that $\psi|_{H_1} = \phi|_{H_1}$, but $\psi(g_1) \neq \phi(g_1)$. Let $\pi: G_1 \times G_2 \to G_1$ be the canonical projection, and compare the maps $\psi \circ \pi$ with $\phi \circ \pi$. By construction, they agree on $H_1 \times H_2$, but disagree on $(g_1, g_2)$. Therefore, $(g_1, g_2) \notin \text{dom}_G^C_{G_1 \times G_2}(H_1 \times H_2)$. 

\qed
Remark 2.5. We add a caution, however. Theorem 2.4 implies that the same result holds for a finite number of direct factors, and that the analogous result holds for the direct sum of an arbitrary number of factors. However, in the case of an infinite direct product, equality may no longer hold. An example of this will be given below, in Example 3.35.

Lemma 2.6. If $C$ is a full subcategory of groups which is closed under quotients, then normal subgroups are dominion-closed. That is, if $N$ is a normal subgroup of $G$, with $G \in C$, then $\text{dom}^C_G(N) = N$.

Proof: Compare the maps $\pi, \zeta: G \to G/N$, where $\pi$ is the canonical epimorphism onto the quotient and $\zeta$ is the zero map. They both agree on $N$, and disagree on any element not in $N$. \qed

Corollary 2.7. Let $C$ be a full subcategory of groups which is closed under quotients. If $G$ is a group in $C$ and $H$ is a subgroup of $G$, then

$$\text{dom}^C_G(H) \subseteq H[H,G].$$

Proof: The subgroup $H[H,G]$ is the normal closure of the subgroup $H$; hence it contains $H$, and is normal in $G$. By Lemma 2.6,

$$\text{dom}^C_G(H) \subseteq \text{dom}^C_G(H[H,G]) = H[H,G].$$ \qed

Remark 2.8. It follows from Lemma 2.6 that dominions in any variety $V$ of abelian groups are trivial (that is, $\text{dom}^V_G(H) = H$ for all groups $G$ and subgroups $H$ in the variety), since any subgroup is normal. In fact, we see that for any variety $V$, if $G \in V$ is abelian, then $\text{dom}^V_G(H) = H$ for any subgroup $H$ of $G$. We express this situation by saying that dominions of subgroups of abelian groups are trivial (in any variety $V$).

Recall that a class of groups $P$ is a pseudovariety if it is closed under quotients, subgroups, and finite direct products.

Proposition 2.9. Let $\{P_i\}_{i \in I}$ be a nonempty collection of pseudovarieties. Let $G \in \bigcup P_i$, $H$ a subgroup of $G$, and $N \triangleleft G$ such that $N \subseteq H$. Then

$$\text{dom}^{\bigcup P_i}_{G/N}(H/N) = \text{dom}^{\bigcup P_i}_G(H)/N.$$

Proof: We have that $\text{dom}^{\bigcup P_i}_{G/N}(H/N) \subseteq \text{dom}^{\bigcup P_i}_G(H/N)$, using Lemma 2.1(ii) and setting $h: G \to G/N$ to be the canonical surjection.

For the reverse inclusion, assume $x \notin \text{dom}^{\bigcup P_i}_G(H)$. Then there exists a group $K \in \bigcup P_i$ and a pair of maps $f, g: G \to K$ such that $f|_H = g|_H$ and $f(x) \neq g(x)$.\end{document}
Consider the induced homomorphisms \((f \times f), (f \times g) : G \to K \times K\), and let \(L\) be the subgroup of \(K \times K\) generated by the images of \(G\) under these two morphisms. Since \(N\) is normal in \(G\), and contained in \(H\), the common image of \(N\) under these two maps will be normal in \(L\). This image may be written as the set

\[
(f \times f)(N) = \left\{ (f(n), f(n)) \in K \times K \mid n \in N \right\}.
\]

We claim that \((f \times f)(x)\) and \((f \times g)(x)\) are not in the same coset of \((f \times f)(N)\) in \(L\). This will prove the claim, since we can then mod out by \((f \times f)(N)\) to obtain an induced pair of maps \(G/N \to L/(f \times f)(N)\) which agree on \(H/N\) and disagree on \(xN\).

Indeed, \(\left((f \times f)(x)\right)\left((f \times g)(x)\right)^{-1} = (e, f(x)g(x)^{-1})\), and we know that the second coordinate is not trivial, because \(f\) and \(g\) disagree on \(x\). Hence,

\[
\left((f \times f)(x)\right)\left((f \times g)(x)\right)^{-1}
\]

is not a diagonal element, and in particular cannot lie in \((f \times f)(N)\). This proves the proposition.

**Remark 2.10.** Note that a collection of groups is a union of pseudovarieties if and only if it contains the pseudovariety generated by each of its members. In particular, a collection of groups is a union of pseudovarieties if and only if it is closed under subgroups, quotients, and squares, where the square of a group \(G\) is the group \(G \times G\).

**Proposition 2.11.** (Cf. Corollary 2.4 in [5]) Let \(\mathcal{C}\) be a full subcategory of Group, and let \(G \in \mathcal{C}\). If \(H\) is a subgroup of \(G\), then

\[
C_G(H) = C_G(\text{dom}^\mathcal{C}_G(H)).
\]

**Proof:** Since \(H \subseteq \text{dom}^\mathcal{C}_G(H)\), we automatically have

\[
C_G(\text{dom}^\mathcal{C}_G(H)) \subseteq C_G(H).
\]

To prove the reverse inclusion, let \(g \in C_G(H)\) and consider the inner automorphism \(\phi_g\) of \(G\) given by conjugation by \(g\). Since \(g \in C_G(H)\), it follows that \(\phi_g\) fixes \(H\) pointwise. Therefore \(\phi_g|_H = \text{id}_G|_H\), so it follows that \(\phi_g\) also fixes \(\text{dom}^\mathcal{C}_G(H)\). That is, for all \(d\) in \(\text{dom}^\mathcal{C}_G(H)\), \(d = \phi_g(d) = g^{-1}dg\), hence \(gd = dg\). So \(g\) lies in \(C_G(\text{dom}^\mathcal{C}_G(H))\), as claimed.
Corollary 2.12. (Cf. Cor 2.5 in [5]) Let $C$ be a full subcategory of $\text{Group}$, and let $G$ be a group in $C$. If $H$ is an abelian subgroup of $G$, then $\text{dom}_C^G(H)$ is also abelian.

Proof: By Proposition 2.11, since all elements of $H$ centralize $H$, they also centralize $\text{dom}_C^G(H)$. Therefore, every element of the dominion of $H$ commutes with every element of $H$; hence every element of the dominion centralizes $H$, and therefore also centralizes the dominion of $H$. \hfill \square

Next we recall the basic definitions and terminology associated to nilpotent groups. We refer the reader to [14] and [17] for the proofs.

Definition 2.13. For a group $G$ we define the lower central series of $G$ recursively as follows: $G_1 = G$, and $G_{c+1} = [G_c, G]$ for $c \geq 1$. We call $G_c$ the $c$-th term of the lower central series of $G$; $G_c$ is generated by elements of the form $[x_1, \ldots, x_c]$ for $c \geq 2$, and $x_i$ ranging over the elements of $G$. We sometimes also write $G' = G_2 = [G, G]$ for the commutator subgroup of $G$.

The factor groups $G_{i−1}/G_i$ are called the lower central factors of $G$.

A group $G$ is nilpotent of class $c$ if and only if $G_{c+1} = \{e\}$.

We will denote by $\mathcal{A}$ the variety of all abelian groups; by $\mathcal{A}^2$ the variety of all metabelian groups (that is, solvable groups of solvability length at most two); by $\mathcal{N}_c$ the variety of all nilpotent groups of class at most $c$; by $\mathcal{A}^2 \cap \mathcal{N}_c$ the variety of all metabelian nilpotent groups of class at most $c$; by $\mathcal{B}_n$ the Burnside variety of exponent $n$, consisting of all groups that satisfy the identity $x^n = e$; by $\mathcal{Nil}$ the category of all nilpotent groups, and by $\mathcal{A}^2 \cap \mathcal{Nil}$ the category of all metabelian nilpotent groups. Note that the last two classes are not varieties, since they are not closed under arbitrary direct products.

The following lemma, which is easily established by direct computation, and the definitions that follow it, will be useful in subsequent considerations.

Lemma 2.14. The following hold for any elements $x$, $y$, $z$, and $w$ of an arbitrary group $G$:

(a) $xy = yx[x, y]$; $x^y = x[x, y]$.
(b) $[x, y]^{-1} = [y, x]$.
(c) $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$.
(d) $[x, zw] = [x, w][x, z]^w = [x, w][x, z][x, z, w]$.

\hfill \square

Definition 2.15. Let $G$ be a group, generated by elements $x_1, \ldots, x_n$.

We define the basic commutators and the ordering among them recursively, as follows:
(i) The letters \(x_1,\ldots,x_n\) are basic commutators of weight one, ordered by setting \(x_i < x_j\) if \(i < j\).

(ii) If the basic commutators \(c\) of weight less than \(k\) have been defined and ordered, define the basic commutators of weight \(k\) by the rules: 
\[
[c_1, c_2] \text{ is a basic commutator of weight } k \text{ if } \\
\text{(a) weight}(c_1) + \text{weight}(c_2) = k, \\
\text{(b) } c_1 > c_2, \\
\text{(c) If } c_1 = [c_3, c_4], \text{ then } c_2 \geq c_4.
\]

(iii) Then continue the order by setting \(c > c'\) if weight\((c) > \text{weight}(c')\), and ordering those of order \(k\) lexicographically; explicitly, we set 
\[
[c_1, c_2] < [c'_1, c'_2] \text{ if } c_1 < c'_1, \text{ or if } c_1 = c'_1 \text{ and } c_2 < c'_2.
\]

**Theorem 2.16.** (Propositions 31.52 and 31.53 in [14]) If \(G = \langle g_1,\ldots,g_n \rangle\) is nilpotent of class \(c\), and \(\alpha: F(x_1,\ldots,x_n) \to G\) is the map from the relatively free nilpotent group of class \(c\) and rank \(n\) sending \(x_i\) to \(g_i\), then every element \(g \in G\), \(g \neq e\) is a product \(g = \alpha(c_1^{m_1} \cdots c_\ell^{m_\ell})\), where the \(c_\lambda\) are basic commutators of weight less than or equal to \(c\), \(c_1 < \cdots < c_\ell\), and the \(m_\lambda\) are nonzero integers. If \(G\) is freely generated in \(N_c\) by the \(g_i\), then the \(m_\lambda\) take independently all nonzero integral values, and the representation is unique.

**Definition 2.17.** All elements of a group \(G\) are said to be **commutators of weight** 1. The commutators of weight \(n\) are defined recursively as the elements \([x, y]\) such that \(x\) is a commutator of weight \(k\), \(y\) is a commutator of weight \(m\), and \(k + m = n\).

Note that an element \(g \in G\) may be considered to be a commutator of several different weights.

**Lemma 2.18.** (Lemma 33.35 in [14]) Every commutator of weight \(n\) of a group \(G\) is a product of left-normed commutators of weight \(n\) and their inverses, and it belongs to \(G_n\), the \(n\)-th term of the lower central series of \(G\).

**Section 3. Dominions in \(N_2\)**

A group \(G\) is nilpotent of class at most two if and only if \(G' \subseteq Z(G)\); that is, if and only if commutators are central. From this, it is easy to show the following two results:

**Proposition 3.19.** Let \(G \in N_2\). Then \(\forall x, y, z \in G\)
\[
[x, y, z] = [x, z][y, z], \quad [x, yz] = [x, y][x, z].
\]

**Corollary 3.20.** Let \(G \in N_2\). Then for all \(x, y \in G\) and for all \(n \in \mathbb{Z}\),
\[
[x^n, y] = [x, y]^n = [x, y^n].
\]
It is not hard to verify that the group $F(x, y)$ presented by

$$F(x, y) = \langle x, y \mid [x, y, x] = [x, y, y] = e \rangle.$$ 

is the relatively free $N_2$-group of rank two, freely generated by $x$ and $y$.

Let $F = F(x_1, \ldots, x_n)$ be the relatively free $N_2$-group in $n$ generators. In $F$ there is a normal form for the elements. Namely, every element can be written uniquely in the form $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} u$ where $u$ is of the form

$$u = \prod_{1 \leq i < j \leq n} [x_j, x_i]^{m_{ij}} \quad m_{ij} \in \mathbb{Z};$$

see [2] for the details.

Furthermore, if $g = x_1^{m_1} \cdots x_n^{m_n} u$ and $g' = x_1^{p_1} \cdots x_n^{p_n} u'$, then

$$gg' = x_1^{m_1+p_1} \cdots x_n^{m_n+p_n} v \quad \text{where}$$

$$v = uu' \prod_{1 \leq i < j \leq n} [x_j, x_i]^{p_{ji}} = uu' \prod_{1 \leq i < j \leq n} [x_j, x_i]^{m_{ji}p_i}$$

and

$$g^{-1} = x_1^{-m_1} \cdots x_n^{-m_n} u^{-1} \prod_{1 \leq i < j \leq n} [x_j, x_i]^{m_{ij}m_i}.$$

**Lemma 3.23.** Let $F(x_1, \ldots, x_n)$ be the relatively free group on $n$ generators in $N_2$. Let $H = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle$, and $g \in G$ where

$$g = x_1^{m_1} \cdots x_n^{m_n} \prod_{1 \leq i < j \leq n} [x_j, x_i]^{m_{ij}}.$$ 

Then $g \in H$ if and only if $\forall i, j \in \{1, \ldots, n\}$, $a_i|m_i$ and $a_ia_j|m_{ji}$.

**Proof:** Suppose that $m_{ji} = a_ja_i k_{ji}$, with $k_{ji} \in \mathbb{Z}$ for each $i$ and $j$. Then

$$g = (x_1^{a_1})^{k_1} \cdots (x_n^{a_n})^{k_n} \prod_{1 \leq i < j \leq n} [x_j^{a_j}, x_i^{a_i}]^{k_{ji}},$$

which clearly lies in $H$.

The converse follows from (3.21) and (3.22), after noting that the generators of $H$ are of the prescribed form.
Theorem 3.24. Let $F(x, y)$ be the relatively free $N_2$-group on two generators, and let $n > 1$. Let $H$ be the subgroup generated by $x^n$ and $y^n$. Then $[x, y]^n \in \text{dom}_{F(x, y)}^N(H)$, and $[x, y]^n \notin H$.

Proof: That $[x, y]^n \notin H$ follows from Lemma 3.23. Let $G \in N_2$, and let $f, g : F(x, y) \to G$ be two morphisms such that $f|_H = g|_H$. Then:

$$f([x, y]^n) = f([x^n, y]) = [f(x^n), f(y)] = [g(x^n), f(y)] = [g(x), f(y)]^n$$

and by a symmetric argument, this term equals $g([x, y]^n)$. In particular, $[x, y]^n$ lies in $\text{dom}_{F(x, y)}^N(H)$.

A similar argument, noting that commutators are central and the image of a commutator is a commutator, yields:

Corollary 3.25. Let $G \in N_2$, and let $H$ be a subgroup of $G$. Let $x$ and $y$ be elements of $G$, $q > 0$, and $x', y' \in G'$ such that $x^q x', y^q y' \in H$. Then $[x, y]^q$ lies in $\text{dom}_{G^q}(H)$.

We can get more information thanks to a theorem of Maier, which we now state:

Lemma 3.26. (B. Maier, Satz 3 in [12]) Let $G, K \in N_2$, and suppose that $[G, K; H]$ is an amalgam. The amalgam is strongly embeddable in a group $M \in N_2$ if and only if the following conditions hold:

(3.27) $G' \cap H \subseteq Z(K)$ and $K' \cap H \subseteq Z(G)$.

(3.28) $\forall q \geq 0$, $g \in G$, $g' \in G'$, $k \in K$, $k' \in K'$, if $g^q g', k^q k' \in H$ then both the element $[g, k^q k']$ of $G$ and the element $[g^q g', k]$ of $K$ belong to $H$ and are equal.

In fact, condition (3.27) is a consequence of (3.28), as noted by Maier (see Remark 3 in [12]). Set $q = 0$, and take $g = k' = e$ and $g' \in G' \cap H$. Taking any $k \in K$, the hypothesis of (3.28) is satisfied, so we conclude that $[g', k] = [e, e] = e$; so $g'$ commutes with $k$. If we let $k$ range over all of $K$, we have that $g'$ lies in the center of $K$, giving the first inclusion in (3.27). The second inclusion is proved in the same way.

Corollary 3.29. Let $G \in N_2$ and let $H$ be a subgroup of $G$. Then the special amalgam $[G, G; H]$ is strongly embeddable in a group $M \in N_2$ if and only if

(3.30) $\forall q \geq 0$, $\forall x, y \in G$, $\forall x', y' \in G'$, if $x^q x', y^q y' \in H$ then $[x, y]^q \in H$.

Proof: We will show that if a subgroup $H$ of $G$ satisfies (3.30), then $G$ and $H$ satisfies (3.28), with $K = G$. To establish this, we apply bilinearity
Let \( q > 0 \) of \( H \) own dominion, and it already contains \( x, y \) want to show that \([x, y]\) to get that both \([x^q x', y]\) and \([x, y^q y']\) are equal to \([x, y]^q\); so condition (3.28) becomes just the condition that the latter element lie in \( H \), and this is condition (3.30); condition (3.27) now follows from (3.28), as noted above. \( \square \)

**Theorem 3.31.** Let \( G \in \mathcal{N}_2 \), and let \( H \) a subgroup of \( G \). Let \( D \) be the subgroup of \( G \) generated by all elements of \( H \) and all elements of the form \([x, y]^{q} \), where \( x, y \in G \), \( q \geq 0 \), and there exists \( x', y' \in [G, G] \) such that \( x^q x', y^q y' \in H \). Then \( D = \text{dom}_{G}^{N_2}(H) \).

**Proof:** It suffices to show that \( D \) and \( G \) satisfy (3.30), as then \( D \) will be its own dominion, and it already contains \( H \) and is contained in the dominion of \( H \).

Let \( q > 0 \), and let \( x, y \in G \), \( x', y' \in [G, G] \) such that \( x^q x', y^q y' \in D \). We want to show that \([x, y]^q \in D \). By Corollary 3.25, \( D \subseteq \text{dom}_{G}^{N_2}(H) \), and it follows from Corollary 2.7 that we can perturb \( x^q x', y^q y' \in D \) by elements of \([H, G]\) to get elements of \( H \); i.e., we have \( x^q x'', y^q y'' \in H \) for appropriate \( x'', y'' \in [G, G] \). Hence \([x, y]^q \in D \) by construction. \( \square \)

We also note the following property:

**Corollary 3.32.** Let \( G \in \mathcal{N}_2 \) and let \( H \) be a subgroup of \( G \). If \( d \) lies in \( \text{dom}_{G}^{N_2}(H) \), then there exists \( n > 0 \) such that \( d^n \in H \).

**Proof:** Let \( d \in \text{dom}_{G}^{N_2}(H) \). By Theorem 3.31 there exists \( h \in H \) and \( x_1, \ldots, x_m, y_1, \ldots, y_m \in G \), \( x_1', \ldots, x_m', y_1', \ldots, y_m' \in [G, G] \), \( q_1, \ldots, q_m > 0 \) such that \( x_i^q x_i', y_i^q y_i' \in H \) for each \( i \), and

\[
d = h[x_1, y_1]^{q_1} [x_2, y_2]^{q_2} \cdots [x_m, y_m]^{q_m}.
\]

Note that \([x_i, y_i]^{q_i} \in H \) for each \( i \). Let \( n = \prod q_i \). Since commutators are central, we have

\[
d^n = (h[x_1, y_1]^{q_1} [x_2, y_2]^{q_2} \cdots [x_m, y_m]^{q_m})^n
= h^n [x_1, y_1]^{q_1 n} [x_2, y_2]^{q_2 n} \cdots [x_m, y_m]^{q_m n}.
\]

Since \( q_i^2 | q_i n \) for each \( i \), it follows that every term on the right hand side lies in \( H \), and therefore, \( d^n \in H \), as claimed. \( \square \)

Recall that a subgroup of a finitely generated nilpotent group is finitely generated, and that a finitely generated nilpotent group in which every element is torsion is finite.

**Corollary 3.33.** Let \( G \in \mathcal{N}_2 \) be any group, and let \( H \) be any subgroup of \( G \). Then \( H \) is normal in \( \text{dom}_{G}^{N_2}(H) \). If \( G \) is finitely generated, then \( H \) is of finite index in \( \text{dom}_{G}^{N_2}(H) \).
Proof: Since \( \text{dom}^{N_2}_G(H) \) is generated by \( H \) and central elements of \( G \), it follows that it normalizes \( H \). This proves the first assertion. To establish the second assertion, note that if \( G \) is finitely generated, then so is the dominion of \( H \), and that \( \text{dom}^{N_2}_G(H)/H \) is a torsion group by Corollary 3.32. Therefore, it is a finitely generated torsion nilpotent group, hence finite. 

**Example 3.34.** Our ubiquitous example. Let \( F = F(x, y) \) be the relatively free \( N_2 \)-group of rank 2, and let \( n \) be an integer greater than 1. Let \( H \) be the subgroup of \( F \) generated by \( x^n \) and \( y^n \). From Lemma 3.23 it follows that \( H \) consists exactly of all elements of the form

\[
x^a y^b [x, y]^c \quad a, b, c \in \mathbb{Z}.
\]

Note that this is the situation we had in Theorem 3.24; there we proved that the dominion contained the element \([x, y]^n\), which is not in \( H \). Using Theorem 3.31 we can now see that the dominion is actually *generated* by \( H \) and \([x, y]^n\), which is a subgroup of \( G \) strictly larger than \( H \). We will use variations of this example below.

In Theorem 2.4 we noted that the dominion construction respects finite direct products, and added a caution that the analogous result does not hold for an infinite number of direct factors. We will now provide the example we promised in Remark 2.5.

**Example 3.35.** Let \( F \) be the relatively free \( N_2 \) group of rank two generated by \( x \) and \( y \).

Let \( G = F^{\mathbb{Z}^+} \) be the direct product of countably many copies of \( F \), indexed by the positive integers. For each \( i > 0 \), let \( H_i \) be the subgroup of \( F \) generated by \( x^i \) and \( y^i \). Let \( H \) be the subgroup of \( G \) given by \( \prod H_i \).

Example 3.34 shows that \( \text{dom}^{N_2}_F(H_i) = \langle x^i, y^i, [x, y]^i \rangle \), that is all elements of \( F \) which can be written in the form \( x^{ai} y^{bi} [x, y]^{ci} \) with \( a, b, \) and \( c \) in \( \mathbb{Z} \).

We claim that

\[
\text{dom}^{N_2}_G(H) \subsetneq \bigoplus_{i>0} \text{dom}^{N_2}_F(H_i).
\]

Consider the element \( d = ([x, y]^i)_{i>0} \). This is an element of \( \prod \text{dom}^{N_2}_F(H_i) \) by the discussion in the preceding paragraph. To reach a contradiction, assume that \( d \in \text{dom}^{N_2}_G(H) \). From Corollary 3.32 it follows that there exist some \( n > 0 \) with \( d^n \in H \). Since \( d^n = ([x, y]^{in})_{i>0} \), if \( d^n \in H \), it follows that \( i^2 | in \) for all \( i > 0 \), which is clearly impossible. \( \square \)
Section 4. Dominions in the variety of 2-Engel groups

The proof that there are instances of nontrivial dominions in the variety $\mathcal{N}_2$ only relies on the fact that for a group $G \in \mathcal{N}_2$,

$$(4.36) \quad \forall x, y \in G \forall n \in \mathbb{Z} \quad [x^n, y] = [x, y] = [x, y^n];$$

so we might ask which are the groups that satisfy (4.36).

**Lemma 4.37.** Let $G$ be a group that satisfies

$$\forall x, y \in G \quad [x^{-1}, y] = [x, y]^{-1} = [x, y^{-1}].$$

Then $G$ satisfies the identity $[[x, y], y] = e$. In particular, if $G$ satisfies (4.36), it also satisfies the identity $[[x, y], y] = e$.

**Proof:** In any group we have that $[x, y] = [y, x]^{-1}$, and by Lemma 2.14(b) and our hypothesis we have $[x, y] = [y, x]^{-1} = [y^{-1}, x]$. Therefore,

$$[x, y]^y = y^{-1}[x, y]y = y^{-1}[y^{-1}, x]y = y^{-1}yx^{-1}y^{-1}xy = x^{-1}y^{-1}xy = [x, y].$$

Hence $[x, y]$ commutes with $y$, so $[[x, y], y] = e$ as claimed.

**Definition 4.38.** A group $G$ is said to be 2-Engel if it satisfies the law

$$[[x, y], y] = e.$$  

In general, the $k$-th Engel Law is given by $[x, y, \ldots, y] = e$, where the $y$ occurs $k$ times.

In fact, we have a converse to Lemma 4.37:

**Lemma 4.39.** (Lemma 5.42 in [17]) Let $G$ be a group in which $[x, y]$ commutes with $y$. Then $\forall n \in \mathbb{Z}, \ [x, y^n] = [x, y]^n$.

**Corollary 4.40.** A group $G$ is 2-Engel if and only if for all $x, y \in G$, and for all $n \in \mathbb{Z}$, $[x, y^n] = [x, y]^n = [x^n, y]$.

The reader may find it amusing to verify the following result:

**Lemma 4.41.** For a group $G$ the following are equivalent:

(i) $G$ is a 2-Engel group.

(ii) $G$ satisfies (4.36).

(iii) For all $x, y \in G$, $[x, y^{-1}] = [x, y]^{-1} = [x^{-1}, y]$.

(iv) For all $x, y \in G$, $[x, y^2] = [x, y]^2 = [x^2, y]$. 

(v) There exists an integer n such that for all \( x, y \in G \), and \( i = 0, 1, 2 \),

\[
[x, y^{n+i}] = [x, y]^{n+i} = [x^{n+i}, y].
\]

(Cf. Exercise 2.3.4, pp. 31 in [3])

(vi) For all \( x, y \in G \), \( [x^{-1}, y] = [x, y^{-1}] \).

(vii) For all \( x, y \in G \), \( [x^2, y] = [x, y^2] \).

(viii) Every subgroup of \( G \) which is generated by at most two elements is nilpotent of class at most 2.

From the comments above, it follows that:

**Theorem 4.42.** The variety of 2-Engel groups has instances of nontrivial dominions.

**Section 5. Dominions of subgroups of \( N_2 \)-groups in \( A^2 \cap N_c \)**

We turn our attention to the variety of metabelian nilpotent groups of class at most \( c \), with \( c > 1 \). In this section we will prove that these varieties also contain instances of nontrivial dominions. We will in fact prove a bit more: that there is a finitely generated group \( G \), nilpotent of class 2, such that for each given \( c > 1 \), there exists a subgroup \( H_c \) of \( G \) such that

\[ H_c \subseteq \text{dom}_{N_2}^G(H_c) = \text{dom}_{A^2 \cap N_c}^G(H_c). \]

In fact, we will prove the corresponding result with the variety \( A^2 \cap N_c \) replaced by the variety \( N_c \) itself in Section 7. This will of course imply it for the case we are now contemplating. However, the technique and calculations are more transparent in the metabelian case. Thus we include it first as an introduction to the method, rather than deducing it from the more general result later.

**Definition 5.43.** Let \( G \) be a group, and \( x \) and \( y \) elements of \( G \). We write

\[
[x, n y] = [x, y, \ldots, y].
\]

In a metabelian group, of all the basic commutators only the left normed may be nontrivial. The left normed basic commutators on \( x_1, x_2, \ldots, x_n \) will look as follows:

\[
[x_{i_1}, x_{i_2}, \ldots, x_{i_m}] 
\]

where \( i_1 > i_2 \), and \( i_2 \leq i_3 \leq i_4 \leq \cdots \leq i_m \).

The following lemma is easy to establish using Lemma 2.14:
Lemma 5.44. Let $G$ be a group satisfying $[G_2,G_3] = \{e\}$, and let $x$, $y$, $z$, $w$, and $t$ be elements of $G$. Then
\[
[x,y][z,w],t] = [x,y,t][z,w],t],
[t,[x,y][z,w]] = [t,[x,y][t,[z,w]].
\]
In particular, the above identities hold in any metabelian group.

Lemma 5.45. Let $G$ be a group satisfying $[G_2,G_3] = \{e\}$, and $x_1, \ldots, x_m$ be elements of $G$, with $m \geq 3$. Then for any $n > 0$ the following identities hold:
\[
[\ldots[[x_1,x_2],x_3],\ldots],x_m]^n = [[\ldots[[x_1,x_2],x_3],\ldots]^n,x_m]
\]
\[
\vdots
\]
\[
= [[\ldots[[x_1,x_2]^n,x_3],\ldots],x_m].
\]
The following result is easily proven using induction:

Lemma 5.46. Let $G$ be a group such that $[G_2,G_3] = \{e\}$, and let $x$ and $y$ be elements of $G$. Then for any $n > 0$, the following identities hold:
\[
y^n,x] = [y,x][]^n[y,x,y][]^n[y,x,2y][]^n[x_1,x,(n-1)y][]^n;
\]
\[
y,x^n = [y,x][]^n[y,x,x][]^n[y,x,3x][]^n[x_1,nx][]^n.
\]
In particular, (5.47) and (5.48) hold in any metabelian group.

Note that (5.47) and (5.48), in a group satisfying $[G_2,G_3] = \{e\}$ may be rewritten
\[
y^n,x] = [y^n,x][y,x,y]^{-n}[]^n[y,x,2y]^{-n}[]^n[x_1,x,(n-1)y]^{-n};
\]
\[
y,x^n = [y,x^n][y,x,x]^{-n}[]^n[y,x,3x]^{-n}[]^n[x_1,nx]^{-n}.
\]
The variety $A^2$ mentioned in all further results in this section could be replaced with the larger variety of all groups $G$ in which $[G_2,G_3] = \{e\}$. We state the results for $A^2$ for simplicity.

Lemma 5.51. Let $c > 1$ and let $G \in A^2 \cap N_c$. Let $y,z \in G$, $p$ a prime such that $p \geq c$. If
\[
[z,y^p,z] = [z^p,y,y] = e
\]
then $(y,z)_3$ has exponent $p$. 

As noted following Definition 5.43, commutators of weight greater than $c > 0$ (basic commutators of weight $m$) with respect to the property that there exists a basic commutator $c_0$ in $y$ and $z$ of weight $m_0$ which is not of exponent $p$.

As noted following Definition 5.43, $c_0 = [z, n y, k z]$; since weight$(c_0) = m_0$, we must have $n + k = m_0 - 1 \geq 2$. Therefore, if $n > 1$ we have

$$c_0^p = [z, n y, k z]^p = [[z, y]^p, (n - 1) y, k z] \quad \text{(by Lemma 5.45)}$$

$$= [[z^p, y][z, y, z]^{-\frac{p}{2}} \cdots [z, y, (c - 2)z]^{-\frac{p}{c - 1}}, (n - 1) y, k z] \quad \text{(by (5.50))}$$

$$= [[z^p, y], (n - 1) y, k z] \cdots [[z, y, (c - 2)z]^{-\frac{p}{c - 1}}, (n - 1) y, k z]$$

$$= [z^p, n y, k z][z, y, z, (n - 1) y, k z]^{-\frac{p}{2}} \cdots [z, y, (c - 2)z, (n - 1) y, k z]^{-\frac{p}{c - 1}} \quad \text{(by Lemma 5.45)}.$$  

The first factor is trivial, since $[z^p, y, y] = e$. All the subsequent factors are commutators in $y$ and $z$ of weight $> m_0$. In particular, they are all of exponent $p$ by choice of $m_0$. Since $p \geq c$, the powers $-\frac{p}{i}$ to which they are raised are multiples of $p$, hence all the factors on the right hand side are trivial. Thus, $c_0^p = e$. If, on the other hand, $n = 1$, then we simply use (5.49) instead of (5.50) to “pull in” the exponent, and proceed as above, to conclude that $c_0^p = e$, a contradiction to the choice of $c_0$.

\[ \square \]

**Remark 5.53.** It is worth noting that Lemma 5.51 also holds if (5.52) is replaced by any one of:

$$[z, y^p, z] = [z, y^p, y] = e; [z^p, y, z] = [z^p, y, y] = e; [z^p, y, z] = [z, y^p, z] = e.$$

The only difference in the proof would be that under the first condition, one would handle both the $n > 1$ and the $n = 1$ case as the $n > 1$ case is handled in Lemma 5.51; under the second one, we would handle them both like the $n = 1$ case; and under the last condition, one would interchange the methods used for $n > 1$ and $n = 1$. However, only (5.52) is relevant to our study of dominions in $A^2 \cap N_c$. In any case, all four conditions are equivalent, which will follow from Lemma 5.51 and Lemma 7.69.

**Corollary 5.54.** Let $c > 1$ and let $G \in A^2 \cap N_c$. Let $y, z \in G$, $p$ a prime such that $p \geq c$. If (5.52) holds, then $[z, y^p] = [z, y]^p = [z^p, y]$.

**Proof:** From (5.47) and (5.48) we have

$$[z^p, y] = [z, y]^p[z, y, z]^{\frac{p}{2}} \cdots [z, y, (c - 2)z]^{\frac{p}{c - 1}}$$

$$[z, y^p] = [z, y]^p[z, y, y]^{\frac{p}{2}} \cdots [z, y, (c - 2)y]^{\frac{p}{c - 1}}.$$
Every factor on the right hand side of both expressions is raised to a multiple of \( p \), and that all but the first term lie in \( \langle z, y \rangle_3 \). Since the latter subgroup is of exponent \( p \) by Lemma 5.51, it follows that \([z, y^p] = [z, y]^p = [z^p, y]\) as claimed.

**Theorem 5.55.** Fix \( c > 1 \). Let \( p \) be a prime, \( p \geq c \), and let \( G = F^{(2)}(x, y) \) be the relatively free nilpotent group of class 2 on two generators. Let \( H \) be the subgroup of \( G \) generated by \( x^p \) and \( y^p \). Then \( \text{dom}_{G}^{A^2 \cap N_c}(H) \) is generated by \( x^p, y^p, \) and \([x, y]^p\). In particular, it equals \( \text{dom}_{G}^{N_2}(H) \) and properly contains \( H \).

**Proof:** Let \( V = A^2 \cap N_c \). We look at the amalgamated coproduct of two copies of \( G \) over the subgroup \( H \) in the variety \( V \), denoted by \( F = G \amalg_{H} V G \).

This is the group with presentation

\[
\langle x, y, z, w \mid x^p = z^p; \ y^p = w^p; \ (\langle x, y \rangle)_2 = \{e\}; \ (\langle z, w \rangle)_2 = \{e\} \rangle
\]

in the variety determined by the identities

\[
[[x_1, x_2], [x_3, x_4]] = e \quad \text{and} \quad [x_1, \ldots, x_c, x_{c+1}] = e;
\]

where by \((\langle x, y \rangle)_2 = \{e\}\) we mean that the subgroup generated by \( x \) and \( y \) is nilpotent of class two, etc.

Then \([x, y]^p \in \text{dom}_{G}^{V}(H)\) if and only if \([x, y]^p = [z, w]^p \) in \( F \). We note that, since the subgroup generated by \( x \) and \( y \), and that generated by \( z \) and \( w \), are both nilpotent of class two, if follows that

\[
[x, y]^p = [x^p, y] = [z^p, y], \quad [z, w]^p = [z, w^p] = [z, y^p]
\]

so we want to see whether \([z^p, y] = [z, y^p]\). Note that

\[
[z, y^p, z] = [z, w^p, z] = e \quad \text{and} \quad [z^p, y, y] = [x^p, y, y] = e.
\]

In particular, by Lemma 5.51, \( \langle z, y \rangle_3 \) is of exponent \( p \), and by Corollary 5.54, \([z^p, y] = [z, y^p]\). Hence \([x, y]^p \in \text{dom}_{G}^{A^2 \cap N_c}(H)\).

Therefore, by Example 3.34, \( \text{dom}_{G}^{N_2}(H) \subseteq \text{dom}_{G}^{A^2 \cap N_c}(H) \); the reverse inclusion now follows from Lemma 2.2.

Note that, if we take \( G = F^{(2)}(x, y) \) and \( H = \langle x^p, y^p \rangle \), and we let \( c > p \), then the argument in Lemma 5.51 breaks down. This raises the question of whether for any \( G \in N_2 \), any subgroup \( H \) of \( G \), and any given \( g \) in \( G \setminus H \), there exists a \( k \) (depending on \( g \)) such that \( g \) is not in the dominion of \( H \) in \( G \), in the variety \( A^2 \cap N_k \). Put another way, we ask whether dominions of subgroups of \( N_2 \) groups are trivial in the category of all meta-abelian nilpotent groups (note that this category is not a variety). We will answer this question in Section 8.
Section 6. Dominions in $A^2 \cap \text{Nil}$

In this section we will investigate dominions in the category of all metabelian nilpotent groups. We take the opportunity to obtain some results about dominions in the variety of all metabelian groups as well. We begin by making some general observations on $A^2$:

**Lemma 6.56.** (Lemma 34.51 in [14]) Let $G$ be metabelian, and let $x \in G'$. Then for all elements $y_1, \ldots, y_n \in G$ and every permutation of $n$ elements $\sigma \in S_n$, 
\[
[x, y_1, \ldots, y_n] = [x, y_{\sigma(1)}, \ldots, y_{\sigma(n)}].
\]

**Lemma 6.57.** (Theorem 36.33 in [14]) For a relatively free group $F$ in the variety $A^2$, the left normed basic commutators of weight $\geq 2$ freely generate a free abelian subgroup of the derived group $F'$.

We add a caution:

**Lemma 6.58.** (Proposition 36.24 in [14]) The basic commutators of weight at least two in $F$, the free group of rank two in $A^2$, do not generate the derived group $F'$, even though they generate $F'$ modulo every term of the lower central series.

We will first obtain results about dominions in $A^2$. We will then derive the results we want by restricting our attention to groups within $A^2 \cap \text{Nil}$.

**Lemma 6.59.** Let $G \in A^2$, $y \in G'$, $x, z \in G$, and let $H$ be a subgroup of $G$. If $y, [y, x]$ and $[y, z]$ lie in $H$, then $[y, x, z] \in \text{dom}_{G^2}^2(H)$.

**Proof:** Let $K \in A^2$, and let $f, g: G \to K$ be two group morphisms such that $f|_H = g|_H$. Then 
\[
f([y, x, z]) = f([y, x], f(z)] = g([y, x]), f(z)] \quad \text{(since } [y, x] \in H)\]
\[
= [g(y), g(x), f(z)] = [f(y), f(z), g(x)] \quad \text{(using Lemma 6.56 and the fact that } y \in H)\]
\[
= [f([y, z]), g(x)] = g([y, z]), g(x)] \quad \text{(since } [y, z] \in H)\]
\[
= g([y, z, x]) = g([y, x, z]) \quad \text{(by Lemma 6.56)}
\]
so $[y, x, z] \in \text{dom}_{G^2}^2(H)$, as claimed.

**Theorem 6.60.** Let $G \in A^2$, $x \in G'$, and $y_1, \ldots, y_n \in G$. Let 
\[
H = \langle x, [x, y_1], \ldots, [x, y_n] \rangle.
\]

Then $[x, y_1, \ldots, y_n] \in \text{dom}_{G^2}^2(H)$.

**Proof:** Follows from Lemma 6.59 by induction on $n$. 

\[\square\]
The next two results express essentially the same thing as Lemma 6.59 and Theorem 6.60, but in terms of conjugation rather than commutation.

**Lemma 6.61.** Let \( G \in A^2 \), and let \( H \) be a subgroup of \( G \). Let \( x \in G' \), and let \( y \) and \( z \) be elements of \( G \). If \( x, x^y, \) and \( x^z \) lie in \( H \), then \( x^{yz} \) lies in \( \text{dom}^A_G(H) \).

**Proof:** Recall that \( x^y = y^{-1}xy = x[y, y] \). Applying Lemma 2.14(d), we obtain
\[
x^{yz} = x[x, z][x, y][x, y, z].
\]
Since \( x \) and \( x^y \) both lie in \( H \), it follows that \( [x, y] \) also lies in \( H \). Similarly, \( [x, z] \) lies in \( H \). By Lemma 6.59, \( [x, y, z] \in \text{dom}^A_G(H) \), so \( x^{yz} \in \text{dom}^A_G(H) \), as claimed. \( \square \)

**Theorem 6.62.** Let \( G \in A^2 \) and let \( H \) be a subgroup of \( G \). Let \( x \in G' \), and \( y_1, \ldots, y_n \) be elements of \( G \). If \( x \in H \) and \( x^{y_i} \in H \) for \( i = 1, \ldots, n \), then \( x^{y_1 \cdots y_n} \) also lies in \( \text{dom}^A_G(H) \).

**Theorem 6.63.** Let \( G = F(x, y) \) be the free metabelian group on two generators. Let \( H \) be the subgroup of \( G \) generated by \([y, x] \), \([y, x, y]\) and \([y, x, x]\). Then \( H \subseteq \text{dom}^A_G(H) \). Moreover, \( \text{dom}^A_G(H) = G' \), so \( \text{dom}^A_G(H)/H \) is an abelian group of infinite rank.

**Proof:** By Lemma 6.57, \( H \) is a free abelian group on the generators \([y, x]\), \([y, x, y]\) and \([y, x, x]\). Applying Theorem 6.60 to \( H \) and these generators, we see that \( D = \text{dom}^A_G(H) \) contains all basic commutators of weight \( k \geq 2 \).

We claim that \( G' \) is generated by all elements of the form \([y, x]^g\), where \( g \) ranges over all elements of \( G \). Indeed, using Lemma 2.14(c) and noting that \([y, x]^{-1} = [x, y]\), we can decompose any commutator \([z, w]\) into a product of conjugates of \([y, x], [y, x^{-1}], [y^{-1}, x], [y^{-1}, x^{-1}]\) and their inverses. But \([y, x^{-1}] = ([y, x]^{-1})^{-1}\), and similarly for \([y^{-1}, x]\) and \([y^{-1}, x^{-1}]\), so that \( G' \) is indeed generated by all elements of the form \([y, x]^g\).

Since \( D \) contains all basic commutators, it certainly contains \([y, x]^x\) and \([y, x]^y\). It will suffice to show that it also contains \([y, x]^{y^{-1}}\) and \([y, x]^{x^{-1}}\).

Let \( u = [y, x]^y \). Then \( u \in D \cap G' \), and \( u^{y^{-1}} = [y, x] \in D \). By Lemma 6.61, the dominion of \( D \) must also contain \( u^{y^{-1}}y^{-1} = u^{y^{-2}} = [y, x]y^{-1} \). Since \( D \) is its own dominion, this elements lies in \( D \). A similar calculation yields \([y, x]^{x^{-1}} \in D \). This proves that \( G' \subseteq D \). Since \( G' \) is normal in \( G \), \( \text{dom}^A_G(G') = G' \), so \( D = G' \) as claimed.

By Lemma 6.57, it follows that \( D \) contains a free abelian group on countably many generators, and \( H \) is the subgroup generated by three of these generators. Therefore, \( D/H \) is abelian of infinite rank, as claimed. \( \square \)
Corollary 6.64. There are nontrivial dominions in \( A^2 \cap \text{Nil} \). Specifically, if \( F \) is the relatively free \( N_4 \) group of rank 2, with generators \( x \) and \( y \), then the dominion of the subgroup generated by \([x,y], [x,y,x], \) and \([x,y,y]\) is the entire derived subgroup of \( F \).

Proof: \( F \) is the quotient of the relatively free \( A^2 \) group on two generators, modulo the fifth term of the lower central series. By Proposition 2.9, we get the result. \( \square \)

Remark 6.65. Note that this example also establishes that the finite index clause of Corollary 3.33 does not hold for general varieties of nilpotent groups, by looking at \( A^2 \cap N_4 \).

Section 7. Dominions of subgroups of \( N_2 \)-groups in \( N_c \)

As promised at the beginning of Section 5, we will now prove the analog of Theorem 5.55 for the varieties \( N_c \). The proof is indeed very similar to that of Theorem 5.55, with the added complications that arise from the nonabelian nature of the commutator subgroup in the more general case.

We first require two technical lemmas, which we now state:

Lemma 7.66. (Struik, Theorem H3 in [18]) Let \( x_1, \ldots, x_s \) be any elements of a nilpotent group \( G \), and let \( u_1 < u_2 < \ldots \) be a system of basic commutators on the \( x_i \). Let \( \sigma \) be a fixed permutation of \( \{1,2,\ldots,s\} \), and let \( n > 0 \). Then

\[
(x_1 x_2 \cdots x_s)^n = x_{\sigma(1)}^n x_{\sigma(2)}^n \cdots x_{\sigma(s)}^n u_1^{f_1(n)} u_2^{f_2(n)} \cdots
\]

where

\[
f_i(n) = a_1 \binom{n}{1} + a_2 \binom{n}{2} + \cdots + a_{w_i} \binom{n}{w_i}
\]

with \( a_k \) an integer, and \( w_i \) the weight of \( u_i \) in the \( x_j \). \( \square \)

Lemma 7.69. (Struik, Lemma H2 in [18]) Let \( n \) be a fixed integer and let \( G \) be a nilpotent group of class \( c \). If \( x_j \in G \), then

\[
[x_1, \ldots, x_{i-1}, x_i^n, x_{i+1}, \ldots, x_r] = [x_1, \ldots, x_r]^n v_1^{f_1(n)} v_2^{f_2(n)} \cdots
\]

where the \( v_k \) are basic commutators in \( x_1, \ldots, x_r \) of weight \( > r \), and every \( x_j \) appears in each commutator \( v_k \) for \( 1 \leq j \leq r \). The \( f_i \) are of the form (7.68) where \( w_i \) is the weight of \( v_i \) minus \( (r-1) \). \( \square \)
We will find (7.70) useful in situations when we have commutators in some terms, some of which are shown as powers, and we want to “pull the exponent out.” But at other times we will want to reverse this process and pull exponents “into” a commutator. In such situations, we will use (7.70) to express \([b_1, \ldots, b_r]^n\) in terms of other commutators. Let us call the resulting identity (7.70’); that is,

\[(7.70') \quad [b_1, \ldots, b_r]^n = [b_1, \ldots, b_{i-1}, b_i^n, b_{i+1}, \ldots, b_r] \cdots v_2^{-f_2(n)} v_1^{-f_1(n)}.\]

There is a slight refinement to the above formulas for the case \(r = 2\) and \(n\) a prime. Namely,

**Lemma 7.71.** (Struik; see equations labeled (57) and (58) in [19]) Let \(F\) be the absolutely free group with generators \(x\) and \(y\), and let \(p\) be a prime. Then

\[(7.72) \quad [y^p, x] \equiv [y, x]^p \left( \prod_{i=4}^{q(p+1)} c_i^{p\beta_i} \right) [y, x, p-1y] \pmod{F_{p+2}}\]

\[(7.73) \quad [y, x^p] \equiv [y, x]^p \left( \prod_{i=4}^{q(p+1)} c_i^{p\alpha_i} \right) [y, px] \pmod{F_{p+2}}\]

where \(q(n)\) is the number of basic commutators of weight less than or equal to \(n\) in two symbols \(x\) and \(y\), \(c_4 < c_5 < \ldots < c_{q(p+1)}\) are the basic commutators of weight \(> 2\) and weight \(\leq p + 1\), and \(\alpha_i\) and \(\beta_i\) are integers.

To finish our preparatory lemmas, we need a description of the basic commutators on two generators. This description is easily established by induction on the weight:

**Lemma 7.74.** Let \(F(x, y)\) be the free group on two generators. Then every basic commutator of weight \(\geq 3\) is of the form

\[(7.75) \quad [y, x, y, c_4, \ldots, c_r] \quad \text{or} \quad [y, x, x, c_4, \ldots, c_r]\]

where \(c_4, \ldots, c_r\) are basic commutators in \(x\) and \(y\).

We now proceed as in Section 5:

**Lemma 7.76.** Let \(G \in N_c\) and let \(p\) be a prime, with \(p \geq c\). If \(z, y \in G\) are such that \([z, y^p, z] = [z^p, y, y] = e\), then \((z, y)_3\) is of exponent \(p\).

**Proof:** Assume the lemma is false. Since every commutator of weight \(> c\) has exponent \(p\), any counterexample would have smaller weight. Let \(m_0\)
be maximal with respect to the property that there exist an element $c_0$ in $\langle z, y \rangle_{m_0}$ which is not of exponent $p$.

First we prove that a basic commutator $c_0$ of weight exactly $m_0$ is of exponent $p$, by writing it as in (7.75) and then using (7.70') to “pull in” the exponent, and express $c_0^p$ as a product of a commutator which is trivial by hypothesis, and commutators of higher weight each raised to a power which is a multiple of $p$, to reach a contradiction.

Next we assume that $c_0$ is a product of basic commutators of weight at least $m_0$, and apply (7.67) to reach a contradiction.

**Corollary 7.77.** Let $c > 0$, and let $G \in N_c$, $p$ a prime with $p \geq c$. If $z, y \in G$ are such that

$$\left[ z, y^p, z \right] = \left[ z^p, y, y \right] = e,$$

then $[z, y^p] = [z, y]^p = [z^p, y]$.

**Proof:** By Lemma 7.76, $\langle z, y \rangle_3$ is of exponent $p$. We now apply Struik’s formulas from Lemma 7.71, to get

$$[z^p, y] = [z, y]^p \left( \prod_{i=4}^{q(c)} c_i^{p \beta_i} \right), \quad [z, y^p] = [z, y]^p \left( \prod_{i=4}^{q(c)} c_i^{p \alpha_i} \right).$$

Each $c_i$ is of weight at least three, hence of exponent $p$. Therefore, we conclude that $[z, y^p] = [z, y]^p = [z^p, y]$ as claimed.

**Theorem 7.78.** Fix $c > 1$. Let $p$ be a prime, $p \geq c$, and let $G = F^{(2)}(x, y)$ be the relatively free nilpotent group of class two on two generators. Let $H$ be the subgroup of $G$ generated by $x^p$ and $y^p$. Then $\text{dom}^{N_c}_G(H)$ is generated by $x^p$, $y^p$, and $[x, y]^p$. In particular, it equals $\text{dom}^{N_2}_G(H)$, and properly contains $H$.

**Proof:** Once again, we look at the amalgamated coproduct of two copies of $G$ over the subgroup $H$ in the variety $N_c$, denoted by $F = G \amalg^{N_c}_H G$. This is the group with presentation

$$\langle x, y, z, w \mid x^p = z^p; y^p = w^p; \ (\langle x, y \rangle)_2 = \{e\}; \ (\langle z, w \rangle)_2 = \{e\} \rangle$$

in the variety determined by the identity $[x_1, \ldots, x_c, x_{c+1}] = e$; where by $(\langle x, y \rangle)_2 = \{e\}$ we again mean that the subgroup generated by $x$ and $y$ is nilpotent of class 2, etc.

Then $[x, y]^p \in \text{dom}^{N_c}_G(H)$ if and only if $[x, y]^p = [z, w]^p$ in $F$. The rest of the proof now proceeds like the proof of Theorem 5.55.
Corollary 7.79. For every $c > 1$, any category $\mathcal{V}$ with $\mathcal{N}_2 \subseteq \mathcal{V} \subseteq \mathcal{N}_c$ has instances of nontrivial dominion. Namely, if $G = F^{(2)}(x, y)$ is the relatively free $\mathcal{N}_2$ group on two generators, and $p$ is a prime with $p \geq c$, then

$$\langle x^p, y^p \rangle \not\subseteq \text{dom}^{\mathcal{N}_2}_G (\langle x^p, y^p \rangle) = \text{dom}^{\mathcal{V}_G}_G (\langle x^p, y^p \rangle).$$

□

Remark 7.80. Corollary 7.79 includes, among others, the varieties $A_k \cap \mathcal{N}_c$ of all groups which are nilpotent of class at most $c$ and solvable of length at most $k$.

The same comments as in the closing of Section 5 apply here. So we ask whether dominions of subgroups of $\mathcal{N}_2$ groups are trivial in the category of all nilpotent groups. We will answer this question in Section 8.

Section 8. Dominions of subgroups of $\mathcal{N}_2$-groups in $\mathcal{N} \mathcal{I} \mathcal{I}$

In this section we will prove results similar to Theorem 7.78 without the assumption that $p \geq c$. The arguments are very similar to those used above, and so we will only sketch the proofs.

Lemma 8.81. Let $c > 1$, $p$ a prime, and let $a > 0$. Let $G \in \mathcal{N}_c$, and $y, z \in G$. Suppose that for every $i \geq a$,

$$\langle z, y^i \rangle = \langle z, y \rangle = e.$$ (8.82)

Let $m_0 \geq 3$. If $\langle z, y \rangle_{m_0+1}$ is of exponent $p^k$, and $k \geq a$, then $\langle z, y \rangle_{m_0}$ is of exponent $p^{\text{ord}_p(c!)+k}$.

Proof: The first step is to show that every basic commutator of weight exactly $m_0$ has exponent $p^{\text{ord}_p(c!)+k}$. We do this by applying (7.70′) in the same manner as we did before, using the term $\text{ord}_p(c!)$ to make sure we can factor out at least a $p^k$ from the exponent $f_i(n)$. Then, to prove that an arbitrary element of $\langle z, y \rangle_{m_0}$ is also of exponent $p^{\text{ord}_p(c!)+k}$, we write it as a product of basic commutators of weight at least $m_0$ and apply (7.67) and the fact that the result holds for basic commutators of weight at least $m_0$. □

Theorem 8.83. Let $c > 1$, $G \in \mathcal{N}_c$, $z, y \in G$, $a > 0$ and $p$ a prime. Suppose that for each $i \geq a$, $G$ satisfies (8.82). Then $\langle z, y \rangle_3$ is of exponent $p^{(c-3)\text{ord}_p(c!)+a}$.

Proof: Since $G \in \mathcal{N}_c$, we have $\langle z, y \rangle_c = \{e\}$, so it is of exponent 1. However, in order to apply Lemma 8.81 we note that it is also of exponent $p^a$.

By Lemma 8.81, $\langle z, y \rangle_{c-1}$ is of exponent $p^{\text{ord}_p(c!)+a}$; $\langle z, y \rangle_{c-2}$ is of exponent $p^{2\text{ord}_p(c!)+2}$; etc. Continuing this until the third term, we obtain that $\langle z, y \rangle_3$ is of exponent $p^{(c-3)\text{ord}_p(c!)+a}$ as claimed. □
Corollary 8.84. Let $c > 1$, $G ∈ N_c$, $z, y ∈ G$, $a > 0$ and $p$ a prime. Suppose that for each $i ≥ a$, $G$ satisfies (8.82). If $N ≥ (c − 2)ord_p(c!) + a$, then

$$[z^p N, y] = [z, y]^p N = [z, z^p N].$$

Proof: By Theorem 8.83, $⟨z, y⟩_3$ is of exponent $p^{(c − 3)ord_p(c!) + a}$. Let

$$N ≥ (c − 2)ord_p(c!) + a.$$

Apply Lemma 7.69 to $[z^p N, y]$ and $[z, y]^p N$, and use the fact that $⟨z, y⟩_3$ is of exponent $p^{(c − 3)ord_p(c!) + a}$ to obtain the result.

Remark 8.85. Note that (8.82) is a stronger hypothesis than (5.52); but that if we set $p > c$ and $a = 1$, Corollary 8.84 yields the same conclusion as Corollary 7.77.

Theorem 8.86. Let $c > 1$, $p$ a prime. Let $G ∈ N_2$, and let $H$ be a subgroup of $G$. Let $a ≥ 0$ and let $N ≥ (c − 2)ord_p(c!) + a$. If $x^p N$, $y^p N ∈ H$, then $[x, y]^{p 2N − a} ∈ dom_{N_c}^G (H)$.

Proof: Let $F = G ⊔ H$, and let $(λ, ρ)$ be the universal pair of maps of $G$ into $F$. For simplicity, write $λ(x) = x$, $λ(y) = y$, $ρ(x) = z$ and $ρ(y) = w$.

By hypothesis, $x^p N = z^p N$ and $y^p N = w^p N$. Note that if $i ≥ N$, then

$$[z, y^p, z] = [z, (y^p N)^{p^{i−N}}, z] = [z, (w^p N)^{p^{i−N}}, z] = [z, w, z] = e,$$

since $ρ(G)$ is a nilpotent group of class at most two. Analogously, for every $i ≥ N$, we have $[z^p, y, y] = e$.

We also have that

$$[x, y]^{p 2N − a} = [x^{p 2N − a}, y] = [z^{p 2N − a}, y]$$

and

$$[z, w]^{p 2N − a} = [z, w^{p 2N − a}] = [z, y^{p 2N − a}],$$

so it suffices to show that $[z^{p 2N − a}, y] = [z, y^{p 2N − a}]$ in $F$.

Also, we have that

$$2N − a = N + N − a ≥ ((c − 2)ord_p(c!) + a) + N − a = (c − 2)ord_p(c!) + N.$$

By Corollary 8.84, it follows that

$$[z^{2N − a}, y] = [z, y]^{2N − a} = [z, y^{2N − a}]$$

in $F$; therefore $[x, y]^{p 2N − a} ∈ dom_{N_c}^G (H)$, as claimed. □
To finish this section we present an example, suggested by George Bergman, of a nontrivial dominion of a subgroup of an $N_2$-group in $\text{Nil}$.

**Example 8.87.** Let $\mathbb{Z}\left[\frac{1}{p}\right]$ be the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{p}$. Let $G$ be the group with underlying set

$$\mathbb{Z}\left[\frac{1}{p}\right] \oplus \mathbb{Z}\left[\frac{1}{p}\right] \oplus \mathbb{Z}\left[\frac{1}{p}\right]$$

and multiplication $(a, b, c) \cdot (x, y, z) = (a + x, b + y, c + z + bx)$. This group is isomorphic to the multiplicative group of upper triangular special $3 \times 3$ matrices over $\mathbb{Z}\left[\frac{1}{p}\right]$, via the identification

$$(x, y, z) \leftrightarrow \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

Let $H$ be the subgroup generated by $(1, 0, 0)$ and $(0, 1, 0)$. It is not hard to verify that $H$ is the subgroup with underlying set $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and that $\text{dom}_{N_2}^G(H)$ is the subgroup with underlying set $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\left[\frac{1}{p}\right]$.

We claim that the dominion of $H$ in $G$ in the category $\text{Nil}$ is also equal to the subgroup with underlying set $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}\left[\frac{1}{p}\right]$. For this, it suffices to show that for all $c > 1$ and all $i > 0$,

$$\left(0, 0, \frac{1}{p}\right) \in \text{dom}_{N_c}^G(H).$$

Fix $c > 1$ and $i > 0$. Let $N = (c - 2)\text{ord}_p(c!) + i$ and let $x = (\frac{1}{p^N}, 0, 0)$, $y = (0, -\frac{1}{p^N}, 0)$.

Then $x^{p^N} = (1, 0, 0) \in H$, and $y^{p^N} = (0, -1, 0) \in H$. By Theorem 8.86,

$$[x, y]^{p^{2N-i}} \in \text{dom}_{N_c}^G(H).$$

Calculating directly we have that $[x, y] = (0, 0, \frac{1}{p^N})$, and hence

$$[x, y]^{p^{2N-i}} = \left(0, 0, \frac{1}{p^i}\right) \in \text{dom}_{N_c}^G(H),$$

as claimed.

Therefore, $H \not\subseteq \text{dom}_{N_2}^G(H) = \text{dom}_{N_c}^G(H)$ for all $c > 1$, so

$$H \not\subseteq \text{dom}_{G_2}^N(H) = \text{dom}_{G_2}^N(H).$$

In particular, there are instances of nontrivial dominions in $\text{Nil}$. 

$\square$
Note that in Example 8.87, \( G \) lies in \( N_2 \); since the dominions of \( H \) in \( N_2 \) and in \( \mathcal{N}il \) are equal and properly contain \( H \), it follows that the dominion of \( H \) in \( G \) in the category \( A^2 \cap \mathcal{N}il \) is also nontrivial. Hence, there are nontrivial dominions of subgroups of \( N_2 \)-groups in the category \( A^2 \cap \mathcal{N}il \).

Also worthy of note is the fact that even though \( H \) itself is finitely generated, \( \text{dom}_{G}^{N_2}(H) \) is not finitely generated. Note as well that \( G \) is not finitely generated. In fact, one can prove that dominions of subgroups of finitely generated nilpotent groups (of any class) are trivial in \( \mathcal{N}il \), and that dominions of subgroups of finitely generated \( N_2 \)-groups are trivial in \( A^2 \cap \mathcal{N}il \). For the proofs of these assertions, see [10].

Section 9. Dominions in other varieties of nilpotent groups

In this section, we will mention briefly some results which give information on dominions in other varieties of nilpotent groups.

First, it is known that any nonabelian torsion free locally nilpotent variety \( \mathcal{V} \) (that is, the relatively free groups in \( \mathcal{V} \) are torsion free, and every finitely generated group in \( \mathcal{V} \) is nilpotent) contains \( N_2 \). See for example [8]. Therefore, we have:

**Theorem 9.88.** Every nonabelian torsion free locally nilpotent variety \( \mathcal{V} \) has instances of nontrivial dominions. Namely, if \( p \) is a prime greater than the nilpotency class of the relatively free \( \mathcal{V} \) group on four generators, then

\[
\langle x^p, y^p \rangle \subset \text{dom}_{\mathcal{V}}^{F(2)(x,y)}(\langle x^p, y^p \rangle) = \langle x^p, y^p, [x, y]^p \rangle.
\]

**Proof:** Let \( c \) be the nilpotency class of the relatively free \( \mathcal{V} \) group on four generators. As we noted above, \( \mathcal{V} \) will contain \( F(2)(x,y) \).

For every subgroup \( H \) of \( F(2)(x,y) \), the amalgamated coproduct

\[
F(2)(x,y) \amalg_{H} \mathcal{V} \cong F(2)(x,y)
\]

is generated by four elements, and hence is a quotient of the relatively free \( \mathcal{V} \)-group of rank four. In particular, it is nilpotent of class at most \( c \).

Let \( p \) be a prime, \( p \geq c \). If \( \mathcal{V}' \) is the (nilpotent) variety generated by the relatively free \( \mathcal{V} \) group on four generators, then

\[
F(2)(x,y) \amalg_{H} \mathcal{V}' \cong F(2)(x,y) \amalg_{H} \mathcal{V}' \cong F(2)(x,y);
\]

hence

\[
\text{dom}_{F(2)(x,y)}^{\mathcal{V}}(\langle x^p, y^p \rangle) = \text{dom}_{F(2)(x,y)}^{\mathcal{V}'}(\langle x^p, y^p \rangle)
\]

which by Corollary 7.79 equals \( \langle x^p, y^p, [x, y]^p \rangle \), as claimed.
In the case of the varieties $\mathcal{B}_p \cap \mathcal{N}_c$, that is, nilpotent groups of class at most $c$ and exponent $p$, with $p$ a prime greater than $c$, (the latter condition is sometimes called of small class), the answer is provided by a theorem of Maier [13], which implies that in these varieties dominions are trivial. Specifically, given two groups $G$ and $K$ in the variety in question, and an amalgam $[G,K;H]$, Maier obtains necessary and sufficient conditions for the weak embeddability of the amalgam into a group $M$ in the variety. He then proves:

**Theorem 9.89.** (Maier, Corollary 1.3 in [13]) Suppose that $c < p$, and let $G, K \in \mathcal{B}_p \cap \mathcal{N}_c$. If $[G,K;H]$ is an amalgam, and is weakly embeddable in a group $M \in \mathcal{B}_p \cap \mathcal{N}_c$, then it is strongly embeddable in a group $N$ in $\mathcal{B}_p \cap \mathcal{N}_c$.

Since a special amalgam $[G,G;H]$ is always weakly embeddable, we conclude that:

**Corollary 9.90.** Suppose that $c < p$. Let $G \in \mathcal{B}_p \cap \mathcal{N}_c$, and let $H$ be a subgroup of $G$. Then $\text{dom}_{G}^{\mathcal{B}_p \cap \mathcal{N}_c}(H) = H$.

Finally, we look at the nonabelian varieties of nilpotent groups of class at most two. First we state the classification of subvarieties of $\mathcal{N}_2$. It follows as a corollary from the classification of subvarieties of $\mathcal{N}_3$, due to Jónsson and Remeslennikov (see [7] and [16]).

**Theorem 9.91.** Every variety of nilpotent groups of class at most 2 may be defined by the identities

$$x^m = [x_1, x_2]^n = [x_1, x_2, x_3] = e$$

for unique nonnegative integers $m$ and $n$ satisfying $n|m/\gcd(2,m)$, yielding a bijection between pairs of nonnegative integers $(m,n)$ satisfying this condition, and varieties of nilpotent groups of class at most two.

**Theorem 9.92.** Let $\mathcal{V}$ be a variety of nilpotent groups of class 2 corresponding to the pair $(m,n)$, and suppose that $n$ is not square free. Then there are nontrivial dominions in $\mathcal{V}$.

**Proof:** If $m = n = 0$, then $\mathcal{V} = \mathcal{N}_2$ and there is nothing to prove. Otherwise, let $G$ be the relatively free group of rank 2 in $\mathcal{V}$, generated by $x$ and $y$. Let $p$ be a prime number such that $p^2|n$. Let $H = \langle x^p, y^p \rangle$. If $m > 0$, it is not hard to verify that

$$H = \left\{ x^{ap}y^{bp}[x,y]^{cp^2} \mid 0 \leq pa, pb < m, \quad 0 \leq cp^2 < n \right\}.$$ 

However, $[x,y]^p \in \text{dom}_{G}^{\mathcal{V}}(H)$, so $H \subseteq \text{dom}_{G}^{\mathcal{V}}(H)$, as claimed. If $m = 0$, we proceed as above noting that $a$ and $b$ may now be any integers, instead of being bounded by $m/p$. This proves the theorem.
What about other varieties of nilpotent groups of class two? We can settle the matter for a few of the remaining varieties.

For $m > 0$, we may use the facts that a finite nilpotent group is the direct product of its Sylow subgroups, that a finitely generated torsion nilpotent group is finite, and that dominions respect finite direct products, to reduce to the case where $m$ is a prime power. If $m = p$ a prime, then we are in the situation of Corollary 9.90, so dominions are trivial. This leaves the varieties $(p^n, p)$ with $n > 1$, and $(0, n)$ with $n$ square free still open. We ask:

**Question 9.93.** Are dominions trivial in the subvarieties of $N_2$ corresponding to pairs of positive integers $(p^a, p)$ with $p$ a prime, $a > 1$? Are dominions trivial in the subvarieties corresponding to pairs $(0, n)$ with $n > 1$ square free?

My guess is that dominions will be trivial in those varieties, but I am at present unable to prove this guess.

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