THE WEAK OPE AND DIMENSION-EIGHT OPERATORS

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We discuss recent work which identifies a potential flaw in standard treatments of weak decay amplitudes, including that of $\varepsilon'/\varepsilon$. The point is that (contrary to conventional wisdom) dimension-eight operators contribute to weak amplitudes at order $G_F\alpha_s$ and without $1/M_W^2$ suppression. The effect of dimension-eight operators is estimated to be at the 100% level in a sum rule determination of the operator $Q_i^{(6)}$ for $\mu = 1.5$ GeV, suggesting that presently available values of $\mu$ are too low to justify the neglect of these effects.

1 Motivation

1.1 Calculating Kaon Weak Amplitudes

The modern approach to calculating a kaon weak nonleptonic amplitude $\mathcal{M}$ involves use of the operator product expansion,

$$\mathcal{M} = \sum_{d} \sum_{i} C_i^{(d)}(\mu) \langle Q_i^{(d)} \rangle_\mu,$$

in which the nonleptonic weak hamiltonian $\mathcal{H}_W$ is expressed as a linear combination of local operators $Q_i^{(d)}$. There is a sum over the dimensions (starting here at $d = 6$) of the local operators and a sum over all operators of a common dimension. In practice, the following hybrid methodology is employed:

1. The Wilson coefficients $C_i^{(d)}(\mu)$ are calculated in $\overline{MS}$ renormalization.

2. The operator matrix elements $\langle Q_i^{(d)} \rangle_\mu$ are calculated in cutoff renormalization at the energy scale $\mu$. The term ‘cutoff’ means specifically that $\mu$ serves as a ‘separation scale’ which distinguishes between short-distance and long-distance physics. Three different approaches falling into this category are quark models, $1/N_c$ expansion methods, and lattice-QCD evaluations.

The reason for this hybrid approach is that it is not practical to carry out the (low energy) kaon matrix element evaluations with $\overline{MS}$ renormalization. Typical choices for the scale $\mu$ fall in the range $0.5 \leq \mu(\text{GeV}) \leq 3$, the lower part used in quark-model and $1/N_c$ evaluations and the upper part in lattice simulations.

The purpose of this talk is to describe some recent results:

1. In a pure cutoff scheme, dimension-eight operators occur in the weak hamiltonian at order $G_F\alpha_s/\mu^2$, $\mu$ being the separation scale. This can be explicitly demonstrated (see Sect. 2) in a calculation involving a LR weak hamiltonian.

2. In dimensional regularization (DR), the $d = 8$ operators do not appear explicitly in the hamiltonian at order $G_F\alpha_s$. However, the use of a cutoff scheme for the calculation of the matrix elements of dimension-six operators requires a careful matching onto DR for which dimension-eight operators do play an important role.

These findings mean that hybrid evaluations, in the sense described above, of kaon matrix elements at low $\mu$ will contain (unwanted) contributions from dimension-eight operators. At the very least, this will introduce an uncertainty of unknown magnitude into the evaluation.
2.1 $\epsilon'/\epsilon$ in the Chiral Limit

The determination of $\epsilon'/\epsilon$ can be shown to depend upon the matrix elements $\langle (\pi\pi)|Q^{(6)}|K\rangle$ and $\langle (\pi\pi)|Q^{(6)}|K\rangle$. In the chiral limit of vanishing light-quark mass, the latter matrix element (as well as that of operator $Q^{(6)}_1$) can be inferred from certain vacuum expectation values, $\langle 0|Q^{(6)}_1|0\rangle \equiv \langle O^{(6)}_{1,8}\rangle$, where $O^{(6)}_8$ are dimension-six four-quark operators. The use of soft-meson techniques to relate physical amplitudes to those in the world of zero light-quark mass is a well-known procedure of chiral dynamics.

2.2 Sum Rules for $\langle O^{(6)}_{1,8}\rangle$

Numerical values for $\langle O^{(6)}_{1,8}\rangle$ in cutoff renormalization can be obtained from the following sum rules:

$$
\frac{16\pi^2}{3}(O^{(6)}_{1,8})_{(c.o.)} = \int_0^\infty ds \ln \frac{s + \mu^2}{s} \Delta \rho \\
2\pi(\alpha_s O^{(6)}_8)_{(c.o.)} = \int_0^\infty ds \frac{s^2}{s + \mu^2} \Delta \rho ,
$$

(2)

where $\Delta \rho(s)$ is the difference of vector and axialvector spectral functions, and $\Delta \Pi(Q^2)$ is the corresponding difference of isospin polarization functions ($Im \Delta \Pi = \pi \Delta \rho$).

2.3 Physics of a LR Operator

One can probe the influence of $d = 8$ operators by considering the K-to-$\pi$ matrix element $M(p)$,

$$
M(p) = \langle \pi^-|\mathcal{H}_{LR}|K^-\rangle ,
$$

(3)

where $\mathcal{H}_{LR}$ is a LR hamiltonian obtained by flipping the chirality of one of the quark pairs in the usual LL hamiltonian $\mathcal{H}_W$. The reason for defining such a LR operator is that, in leading chiral order, its K-to-$\pi$ matrix element is nonzero and yields information on $\langle O^{(6)}_1\rangle$ and $\langle O^{(6)}_8\rangle$.

To demonstrate this, we proceed to the chiral limit to find

$$
M \equiv M(0) = \lim_{p \to 0} M(p) = \frac{3G_F M^2}{32\sqrt{2}\pi^2 F^2} \int_0^\infty dQ^2 \frac{Q^4}{Q^2 + M^2} \Delta \Pi .
$$

(4)

This result is exact — it is not a consequence of any model. Information about $\langle O^{(6)}_1\rangle$ and $\langle O^{(6)}_8\rangle$ is obtained by performing an operator product expansion on $\Delta \Pi(Q^2)$. Working to first order in $\alpha_s$ we have

$$
M = \frac{G_F}{2\sqrt{2} F^2} \left[ (O^{(6)}_{1,8})_{(c.o.)} + \frac{3}{8\pi} \ln \frac{M^2}{\mu^2} (\alpha_s O^{(6)}_8)_{(c.o.)} + \frac{3}{16\pi^2} \frac{\xi^{(8)}_1}{\mu^2} + \cdots \right] ,
$$

(5)

The three additive terms in Eq. (5) are proportional respectively to the quantities $\langle O^{(6)}_1\rangle$, $\langle O^{(6)}_8\rangle$ and $\xi^{(8)}$. The last of these $\xi^{(8)}$ contains the effect of the $d = 8$ contributions. For dimensional reasons, $\xi^{(8)}$ must be accompanied by an inverse squared energy.

This turns out to be the factor $\mu^{-2}$.

In Table 1 we display the numerical values (in units of $10^{-7}$ GeV$^2$) of the three terms of Eq. (5) for various choices of $\mu$. Observe for the lowest values that the dimension-eight term dominates the contribution from $\langle O^{(6)}_1\rangle$. Only when one proceeds to a sufficiently large value like $\mu = 4$ GeV is the $d = 8$ influence suppressed.

3 Dimensional Regularization

Suppose one wishes to express the entire analysis in terms of $\overline{MS}$ quantities. To do so requires converting matrix elements in cutoff renormalization to those in $\overline{MS}$ renormalization. Recall, in dimensional regularization

$^6$Although the $d = 8$ operators arising from $Q^{(6)}_s$ have been determined$^5$, to our knowledge the individual $d = 8$ LR operators comprising $\xi^{(8)}$ have not.
one calculates in $d$ dimensions and for dimensional consistency introduces a scale $\mu_{d.r.}$.

The dimensionally regularized matrix element for $\langle O_1^{(6)} \rangle$ is found from the $d$-dimensional integral
\[
\langle O_1^{(6)} \rangle_{\mu}^{(d.r.)} = \langle O_1^{(6)} \rangle_{\mu}^{(c.o.)} + \frac{d-1}{(4\pi)^{d/2}} \int_{\mu^2}^{\infty} dQ^2 Q^d \Delta \Pi(Q^2) .
\] (6)

The term in Eq. (6) containing the integral is proof that the dimensionally regularized matrix element $\langle O_1^{(6)} \rangle_{\mu}^{(d.r.)}$ will contain short-distance contributions. As written, this term becomes divergent for four dimensions and also is scheme-dependent. In the $\overline{\text{MS}}$ approach, the divergent factor $2/\epsilon - \gamma + \ln(4\pi)$ is removed. The NDR scheme involves a certain procedure for treating chirality in $d$-dimensions. The final result is a relation (given here to $\mathcal{O}(\alpha_s)$) between the cutoff and $\overline{\text{MS}}$-NDR matrix elements,
\[
\langle O_1^{(6)} \rangle_{\mu}^{(\overline{\text{MS}}-\text{NDR})} = \langle O_1^{(6)} \rangle_{\mu}^{(c.o.)} + \frac{3}{8\pi} \left[ \ln \frac{\mu_{d.r.}^2}{\mu^2} - \frac{1}{6} \right] \langle \alpha_s \mathcal{O}_8^{(6)} \rangle_{\mu}^{(c.o.)} + \frac{3}{16\pi^2} \frac{\mathcal{E}_{\mu}^{(8)}}{\mu^2} + \ldots
\] (7)

The effect of the $d = 8$ contribution to the weak OPE now appears in the $d = 6$ $\overline{\text{MS}}$-NDR operator matrix element. Note also that the parameter $\mu_{d.r.}$ is distinct from the separation scale $\mu$.

4 Evaluation of $B_{7,8}^{(3/2)}$

To suppress the effect of dimension-eight operators on the determinations of Eq. (3), one should evaluate the two sum rules for $\langle O_{1,8} \rangle^{(c.o.)}_{\mu}$ at a large value of $\mu$ (e.g. $\mu \geq 4$ GeV) and then use renormalization group equations to run the matrix elements down to lower values of $\mu$ (e.g. $\mu = 2$ GeV). Alternative approaches might involve the finite energy sum rule framework or QCD-lattice simulations at sufficiently large $\mu$.

5 Concluding Remarks

This talk has dealt with an important aspect of calculating kaon weak matrix elements, the role of dimension-eight operators. In this regard, Eq. (7) is of special interest. It reveals that the relation between $\overline{\text{MS}}$-NDR and cutoff matrix elements will involve not only mixing between operators of a given dimension but also mixing between operators of differing dimensions. The net result of our work is that existing work on $\epsilon'/\epsilon$ will be affected, especially for methods which take $\mu \leq 2$ GeV.

Acknowledgments

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References

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\[\begin{array}{|c|c|c|c|}
\hline
\mu \text{ (GeV)} & \text{Term 1} & \text{Term 2} & \text{Term 3} \\
\hline
1.0 & -0.12 & -3.84 & 0.64 \\
1.5 & -0.28 & -3.49 & 0.30 \\
2.0 & -0.44 & -3.24 & 0.17 \\
4.0 & -0.89 & -2.63 & 0.04 \\
\hline
\end{array}\]