On the Value-Distribution of Hurwitz Zeta-Functions with Algebraic Parameter

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Abstract
We study the value-distribution of the Hurwitz zeta-function with algebraic irrational parameter \( \zeta(s; \alpha) = \sum_{n \geq 0} (n + \alpha)^{-s} \). In particular, we prove effective denseness results of the Hurwitz zeta-function and its derivatives in suitable strips containing the right boundary of the critical strip \( 1 + i \mathbb{R} \). This may be considered as a first "weak" manifestation of universality for those zeta-functions.

Keywords Zeta-functions · Universality · Approximation by algebraic numbers

Mathematics Subject Classification 11M35

1 Introduction and Statement of the Main Results

Let \( s = \sigma + it \) denote a complex variable. The Hurwitz zeta-function with a real parameter \( \alpha \in (0, 1] \) is for \( \sigma > 1 \) defined by the Dirichlet series expansion

\[
\zeta(s; \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s},
\]

and by analytic continuation elsewhere except for a simple pole at \( s = 1 \). This function was introduced by Hurwitz [12] in 1881/2 and generalizes the famous Riemann zeta-function which appears as \( \zeta(s) = \zeta(s; 1) \).
The Riemann zeta-function possesses a remarkable approximation property. In 1975, Voronin [22] proved that, roughly speaking, every non-vanishing analytic function \( f \) defined on a sufficiently small disk \( K \) centered at the origin can be approximated as good as we please by certain shifts of the Riemann zeta-function,

\[
\max_{s \in K} \left| \zeta \left( s + \frac{3}{4} + i \tau \right) - f(s) \right| < \varepsilon;
\]

furthermore, this approximation is a regular phenomenon: the set of shifts \( \tau \) satisfying the above inequality has positive lower density. Since a single function approximates elements of a huge class of target functions \( f \), this property is called universality.

Voronin’s celebrated universality theorem has been generalized and extended in various ways. It has been shown in 1979/81 by Gonek [8] and (independently) Bagchi [2] that the Hurwitz zeta-function \( \zeta(s; \alpha) \) satisfies the analogue of Voronin’s universality theorem whenever \( \alpha \) is rational or transcendental. It appears that for every such \( \alpha \neq \frac{1}{2}, 1 \), the target function \( f \) even may vanish in \( K \), for which those \( \zeta(s; \alpha) \) are said to be strongly universal. For this and more details, we refer besides the original works to [20].

Ever since the question whether the Hurwitz zeta-function with an algebraic irrational parameter is universal in this or another sense has been investigated, so far only with little success though. For instance, Garunkštis [6] showed by employing the continuity of \( \zeta(s; \alpha) \) with respect to \( \alpha \) the existence of zeros in the right half of the critical strip except for \( \alpha = \frac{1}{2}, 1 \) (which would also follow from universality). Also, Laurinčikas and the second author [15] obtained limit theorems for a Hurwitz zeta-function with an algebraic irrational parameter (unfortunately not sufficiently explicit for being used in a hypothetical proof of universality). Lastly, Mishou [17] considered the joint value distribution of the Riemann zeta-function and the Hurwitz zeta-function with algebraic parameter on the right of the line \( 1 + i \mathbb{R} \).

In this article, we study the behavior of the Hurwitz zeta-function \( \zeta(s; \alpha) \) on the left of \( 1 + i \mathbb{R} \), where the parameter \( \alpha \) is an algebraic irrational number. We incorporate ideas of Voronin [23] and Good [9] to obtain quantitative results; an additional feature in our reasoning is the use of the theory of approximation by algebraic numbers.

First, we have to introduce several notations which will be kept throughout the paper. The naive height of a complex polynomial \( P(X) \), denoted by \( H(P) \), is the maximum of the absolute values of its coefficients. If \( \alpha \) is an algebraic number, then its degree and height, which we denote by \( d(\alpha) \) and \( H(\alpha) \), are defined to be the degree and the height of its minimal polynomial over \( \mathbb{Z} \), respectively. All the constants appearing in the sequel are effectively computable. The numbers \( R, Q \) and \( M \) will always denote positive integers, while \( T, \sigma \) and \( d \) will be positive real numbers. We postpone the definitions for the number \( E = E(R, Q, \sigma) \), the set

\[
\mathcal{A}(Q, M) \subseteq \mathbb{A} := \{ \alpha \in [0, 1] : \alpha \text{ is algebraic irrational} \}
\]

and the number \( K = K(Q, M, \alpha) \) which appear in Theorem 1 until Sect. 3 (see (14), (37) and (38), respectively). Lastly, from here on the parameter \( \theta \) will denote the number occurring in Lemma 8. Our main result is the following:
Theorem 1 For every $\sigma \in (1/2, 1], N \in \mathbb{N}, A \in (0, 1], d > 0$ such that $d^3 - 2d^2 \geq 2\theta$ and $\epsilon \in (0, \theta/d^2)$, there exist positive numbers $c_0, c_1, c_2$ which depend on $\sigma$ and $N$, $c_3 = c_3(N, A)$, $c_4, c_5 = c_5(d, \epsilon, N)$ and $\nu = \nu(d, \epsilon, N)$, such that the following is true:

Let $\epsilon > 0$ and $a := (a_0, \ldots, a_N) \in \mathbb{C}^{N+1}$. Let also

$$R \geq c_0 \epsilon^{4/(1-2\sigma)}$$

and $Q_0 \geq c_1 R$ be positive integers satisfying the system of inequalities

$$c_2 \left( |a_k| + A^{-1/2} \right) \leq \mathbb{E} \left( \frac{\log \left( \frac{Q_0}{R+1} \right)}{2N \log Q_0} \right)^N k!(N-k)! (\log Q_0)^k, \quad k = 0, \ldots, N.$$

Then, for any $Q \geq c_3 (Q_0 + 1/e^8)$, $M \geq c_4 \exp(2Q^2)$, $\alpha \in A(Q, M) \cap [A, 1]$ of degree $d(\alpha) < d - 2\theta/d^2 + \epsilon$, where

$$d \leq \left( \frac{\theta}{1 - \sigma + \epsilon} \right)^{1/2}$$

(1)

and the right-hand side of the inequality is $+\infty$ for $\sigma = 1$, and any

$$T \geq c_5 \max \left\{ \left( K \exp \left( (M + 2) \exp \left( Q^2 \right) \right) \right)^{\frac{d^3}{d^2(d-d(\alpha)) - 2\theta + d^2}}, \epsilon^{-2\nu} \right\},$$

(2)

there is $\tau \in [T, 2T]$ with

$$|\zeta^{(k)}(\sigma + i\tau; \alpha) - a_k| < \epsilon, \quad k = 0, \ldots, N.$$  

(3)

Moreover, if $\mathcal{M}_T(\alpha, \sigma)$ is the set of those $\tau \in [T, 2T]$ for which

$$|\zeta^{(k)}(\sigma + i\tau; \alpha) - a_k| < \left( 2 \frac{Q^2 + 1}{Q^2 - 1} \right)^{1/2} \epsilon, \quad k = 0, \ldots, N,$$

then

$$\lim inf_{T \to \infty} \frac{1}{T} m (\mathcal{M}_T(\alpha, \sigma)) \geq \frac{1}{2} Q^{-2Q} \left( 1 - Q^{-2} \right),$$

where $m$ denotes the Lebesgue measure.

Observe that the theorem has meaning only when $\sigma > 1 - \xi$, where

$$\xi := \frac{\theta}{d^2} \approx 0.00186.$$
as follows from (1) for \( d \) satisfying \( d^3 - 2d^2 = 2\theta \). As a consequence, we obtain an effective but weak form of universality in the same manner as in [7]:

**Theorem 2** Let \( \epsilon \in (0, \xi) \), \( 1 - \xi + \epsilon \leq \sigma_0 \leq 1 \), \( s_0 = \sigma_0 + it_0 \) and \( f : \mathbb{C} \to \mathbb{C} \) be continuous and analytic in the interior of \( K = \{ s \in \mathbb{C} : |s - s_0| \leq r \} \), where \( r > 0 \). Let also \( 0 < A < 1 \) and \( \epsilon \in (0, |f(s_0)|) \). Then, for all but finitely many algebraic irrationals \( \alpha \) in \([A, 1]\) of degree at most \( d_0 = 2\theta/d_0^2 + \epsilon \), where

\[
d_0 \leq \left( \frac{\theta}{1 - \sigma_0 + \epsilon} \right)^{1/2}
\]

there exist real numbers \( \tau \in [T, 2T] \) and \( \delta = \delta(\epsilon, f, T) > 0 \) such that

\[
\max_{|s - s_0| \leq \delta r} |\zeta(s + i\tau; \alpha) - f(s)| < 3\epsilon,
\]

whenever \( T = T(\epsilon, f, \alpha) \) satisfies (2). The set of the exceptional \( \alpha \) can be described effectively, while the dependence of \( T \) on \( f \) arises from the first \( N \) Taylor coefficients of \( f \) for sufficiently large \( N \). Lastly, the number \( \delta \) is also effectively computable by choosing it to satisfy

\[
\max_{|s - s_0| = r} |\zeta(s + i\tau; \alpha)| \frac{\delta^N}{1 - \delta} \leq \epsilon(2 - \exp(\delta r)).
\]

The choice of \( \delta \) is possible since the left-hand side of (4) tends to zero as \( \delta \to 0 \), while the right-hand side tends to \( \epsilon > 0 \), resp. the left-hand side tends to infinity, but the right-hand side remains bounded as \( \delta \to 1 \). Now, one only needs to employ effective/ineffective bounds of \( \zeta(s; \alpha) \) in the \( t \)-aspect in order to compute effectively/ineffectively the number \( \delta \). We give such an order result in Lemma 7, which is the best known for \( \sigma \) close to 1 and is the case we are most interested. Observe that we have not imposed any restrictions on the range of \( r \). For instance, for \( r > 1 \) it follows from the functional equation of \( \zeta(s; \alpha) \) that the maximum in (4) is increasing with \( t > 0 \). In that case, \( \delta \) is tending to 0 and decreases the radius of the disc where \( \zeta(s; \alpha) \) can approximate the target function.

However, for the case \( 0 < r < \sigma_0 - 1/2 \), Yoonbok Lee (in private communication) has recently suggested us a refinement of the arguments used in the proof of Theorem 2 which results in dropping the dependence of \( \delta \) on \( T \). We are grateful to him, and we give a sketch of his idea after the proof of Theorem 2.

The restriction on the strip of universality reminds us of the case of the Dedekind zeta-function \( \zeta_K \), where \( K \) is an algebraic number field over \( \mathbb{Q} \). Reich [18], [19] proved that \( \zeta_K \) is universal in the sense of Voronin in the strip \( \max \{ 1/2, 1 - 1/d \} \), where \( d = [K : \mathbb{Q}] \) is the degree of the number field.

In the following section, we list several well-known results that will turn out useful for our proofs, in particular an approximate functional equation for \( \zeta(s; \alpha) \) and a Liouville-type inequality. In the succeeding two sections, we provide the proofs of the results mentioned above and we conclude with a few remarks which might be of interest with respect to further studies of this topic.
2 Preliminaries

Lemma 1 Let $x_1, \ldots, x_n$ be elements of a complex Hilbert space $H$ and let $a_1, \ldots, a_n$ be complex numbers with $|a_j| \leq 1$ for $1 \leq j \leq n$. Then, there exist complex numbers $b_1, \ldots, b_n$ with $|b_j| = 1$ for $1 \leq j \leq n$, satisfying the inequality

$$\left\| \sum_{j \leq n} a_j x_j - \sum_{j \leq n} b_j x_j \right\|^2 \leq 4 \sum_{j \leq n} \|x_j\|^2$$

Proof For a proof see [20, Lemma 5.2].

Lemma 2 Let $X$ be a locally convex vector space. Let $K \subseteq X$ be a closed convex set, and suppose that $z \in X \setminus K$. Then, there exists a continuous linear functional $\ell \in X^*$ and a constant $c \in \mathbb{R}$ such that $\ell(y) \leq c < \ell(z)$ for all $y \in K$.

Proof For a proof see [5, Theorem 8.73].

Lemma 3 If $x \neq y$ are positive real numbers, then

$$\left| \log \frac{x}{y} \right|^{-1} < \frac{\max\{x, y\}}{|x - y|}.$$

Proof We give shortly the proof. Assume without loss of generality that $x > y$. Then,

$$\log \frac{x}{y} = -\log \frac{x + y - x}{x} = -\log \left( 1 - \frac{x - y}{x} \right) = \sum_{n \geq 1} \frac{1}{n} \left( \frac{x - y}{x} \right)^n > \frac{x - y}{x}$$

and the assertion of the lemma follows.

Lemma 4 For $T > 0$ and $0 < \sigma \neq 1$

$$\sum_{n \leq T} \frac{1}{n} = \log T + \gamma + O \left( T^{-1} \right) \quad \text{and} \quad \sum_{n \leq T} \frac{1}{n^\sigma} = T^{1-\sigma} \frac{1}{1-\sigma} + \zeta(\sigma) + O \left( T^{-\sigma} \right),$$

here $\gamma$ is the Euler–Mascheroni constant.

Proof For a proof see [1, Theorem 3.2].

Lemma 5 For $0 < \alpha \leq 1$, $1/2 < \sigma \leq \sigma_0 < 1$ and $j = 0, 1$, we have

$$\sum_{1 \leq m \neq n \leq T} \frac{1}{(m+\alpha)^\sigma (n+\alpha)^\sigma} \left| \log \frac{n+\alpha}{m+\alpha} \right|^{-j} \ll_{\sigma_0} T^{2-2\sigma} (\log T)^j.$$
Proof If \( j = 0 \), then
\[
\sum_{1 \leq m \neq n \leq T} \frac{1}{(m + \alpha)^\sigma (n + \alpha)^\sigma} \lesssim \left( \sum_{n \leq T} \frac{1}{n^\sigma} \right)^2 \ll \sigma_0 T^{2-2\sigma}.
\]

If \( j = 1 \), then by Lemma 3 we obtain that
\[
\sum_{1 \leq m \neq n \leq T} \frac{1}{(m + \alpha)^\sigma (n + \alpha)^\sigma} \left\lvert \log \frac{n + \alpha}{m + \alpha} \right\rvert^{-1} < 4 \sum_{1 \leq m < n \leq T} \frac{1}{m^\sigma n^\sigma} \frac{n}{n-m}.
\]

We split the sum on the right-hand side of the latter inequality according to the cases \( m < n/2 \) and \( n/2 \leq m < n \). We use Lemma 4 to estimate the new sums. In the first case, we have that
\[
\sum_{1 < n \leq T} \sum_{m < \frac{n}{2}} \frac{1}{m^\sigma n^\sigma} \frac{n}{n-m} \leq 2 \sum_{1 < n \leq T} \sum_{m < \frac{n}{2}} \frac{1}{m^\sigma n^\sigma} < 2 \left( \sum_{n \leq T} \frac{1}{n^\sigma} \right)^2 \ll \sigma_0 T^{2-2\sigma},
\]

while in the second case, we set \( m = n - r \) and we get that
\[
\sum_{1 < n \leq T} \sum_{\frac{n}{2} \leq m < n} \frac{1}{m^\sigma n^\sigma} \frac{n}{n-m} < \sum_{1 < n \leq T} \sum_{r \leq \frac{n}{2}} \frac{1}{(n-r)^\sigma n^\sigma} \frac{n}{r} \leq 2 \sum_{n \leq T} \frac{1}{n^{2\sigma-1}} \sum_{r \leq T} \frac{1}{r} \ll \sigma_0 T^{2-2\sigma} \log T.
\]

\( \Box \)

We now present two lemmas regarding the order of the Hurwitz zeta-function in sufficiently narrow strips containing the vertical line \( 1 + i \mathbb{R} \).

Lemma 6 If \( 0 < \varepsilon < 1 \), then
\[
\zeta_1(s; \alpha):= \zeta(s; \alpha) - \alpha^{-s} \ll \varepsilon |t|^{\varepsilon}, \quad |t| \geq 2,
\]
uniformly for \( 1 - \varepsilon \leq \sigma \leq 3 \) and \( 0 < \alpha \leq 1 \).

Proof For a proof see [1, Theorem 12.23]. \( \Box \)

The latter lemma does not give us a sufficiently good result regarding the order of the Hurwitz zeta-function, that is the exponent \( \varepsilon \) on \( |t| \) decreases linearly as \( \varepsilon \) tends to 0 (and we approach the vertical line \( 1 + i \mathbb{R} \) from the left). This will be seen to be insufficient to prove Lemma 8. The following lemma is a generalization of a well-known result among many results of the same spirit regarding the Riemann zeta-function.

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Lemma 7 The following bound

\[ \zeta_1(s; \alpha) \ll |t|^{\eta(1-\sigma)^{3/2}} \log^{2/3} |t|, \quad |t| \geq 3, \]  

holds uniformly for \(1/2 \leq \sigma \leq 1\) and \(0 < \alpha \leq 1\), where \(\eta = 4.45\).

Proof For a proof see [11, Theorem 5]. \(\square\)

The next lemma provides a representation of the Hurwitz zeta-function in suitable strips which include the vertical line \(1 + i\mathbb{R}\). Recall that \([x]\) denotes the largest integer which is less than or equal to the real number \(x\).

Lemma 8 For every \(0 < \mu < 1\), there exists a positive number \(\nu = \nu(\mu, \sigma_0)\), such that

\[ \zeta(s; \alpha) = \sum_{0 \leq n \leq t^\mu} \frac{1}{(n + \alpha)^s} + O_{\mu, \sigma_0}(t^{-\nu}), \quad t \geq t_1 > 1, \]

uniformly in \(A(\mu) < \sigma_0 \leq \sigma \leq 2\) and \(0 < \alpha \leq 1\), where

\[ A(\mu) := 1 - \theta \mu^2, \]  

\(\theta = 4/(27\eta^2)\) and \(\eta\) is defined as in the previous lemma.

Proof If we set \(c = 1 + b\), where \(b = b(\mu) \in (0, 1]\) will be determined later on, and \(x = m + 1/2, m \in \mathbb{N}\), then the absolute convergence of \(\zeta_1(s; \alpha)\) in the half-plane \(\sigma > 1\) and Perron’s formula (see [13, Lemma 12.1]) imply that

\[ \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta_1(s + z; \alpha) \frac{(x + \alpha)^c}{z} \, dz = \sum_{n \leq m} \frac{1}{(n + \alpha)^s} \]

\[ + O_{\sigma_0} \left( \frac{1}{T} \sum_{n=1}^{\infty} \left( \frac{x + \alpha}{n + \alpha} \right)^c \left| \log \frac{x + \alpha}{n + \alpha} \right|^{-1} \right), \]  

uniformly in \(\sigma \geq \sigma_0 > 0\) and \(0 < \alpha \leq 1\). We estimate the sum in the error term:

\[ \left\{ \sum_{n < \frac{x}{2}} + \sum_{n > 2x} \right\} \left( \frac{x + \alpha}{n + \alpha} \right)^c \left| \log \frac{x + \alpha}{n + \alpha} \right|^{-1} \ll x^c \left\{ \sum_{n < \frac{x}{2}} + \sum_{n > 2x} \right\} \max\{x, n\} + \alpha \]

\[ \ll x^c \sum_{n \geq 1} \frac{1}{n^c \max\{x, n\}} \]

\[ \ll x^c b, \]  

(8)
while if we set \( q = m - n \) for \( x/2 \leq n < x \) and \( r = n - m \) for \( x < n \leq 2x \), we have

\[
\sum_{\frac{x}{2} \leq n \leq 2x} \left( \frac{x + \alpha}{n + \alpha} \right)^c \left| \log \frac{x + \alpha}{n + \alpha} \right|^{-1} \ll \sum_{\frac{x}{2} \leq n \leq 2x} \frac{x^c \max\{x, n\}}{n^c |x - n|} \ll x \left[ \sum_{0 \leq q \leq \frac{x-1}{2}} \frac{1}{q + \frac{1}{2}} + \sum_{r \leq \frac{2x+1}{2}} \frac{1}{r - \frac{1}{2}} \right] \]

(9)

Hence, we deduce from (7)–(9) that

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta_1(s + z; \alpha) \frac{(x + \alpha)^z}{z} \, dz = \sum_{n \leq m} \frac{1}{(n + \alpha)^s} + O_{\sigma_0} \left( \frac{x^c}{bT} + \frac{x \log x}{T} \right),
\]

(10)

uniformly in \( \sigma \geq \sigma_0 > 0 \) and \( 0 < \alpha \leq 1 \).

Let \( 1 - \kappa \leq \sigma \leq 2 \) be arbitrary, where \( \kappa = \kappa(\mu) \in [0, 1/2] \) will be determined later on. Let also \( T = 2t \) and consider the rectangle \( R \) with vertices \( 1 - 3\kappa - \sigma \pm iT \), \( c \pm iT \). By the calculus of residues, we get

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta_1(s + z; \alpha) \frac{(x + \alpha)^z}{z} \, dz = \zeta_1(s; \alpha) + \frac{(x + \alpha)^{1-s}}{1 - s} = \zeta_1(s; \alpha) + O \left( \frac{x^{1-\sigma}}{t} \right).
\]

(11)

Observe that Lemma 6 implies that

\[
\left\{\int_{1-3\kappa-\sigma-iT}^{c-iT} + \int_{c+iT}^{1-3\kappa-\sigma+iT} \right\} \zeta_1(s + z; \alpha) \frac{(x + \alpha)^z}{z} \, dz \ll x^c T^{3\kappa},
\]

(12)

while Lemma 7 yields

\[
\int_{1-3\kappa-\sigma-iT}^{1-3\kappa-\sigma+iT} \zeta_1(s + z; \alpha) \frac{(x + \alpha)^z}{z} \, dz \ll x^{-3\kappa-\sigma} \int_{-T}^{T} \frac{|\zeta_1(1 - 3\kappa + i(t + u))|}{|1 - 3\kappa + iu|} \, du \ll_\kappa x^{-2\kappa} T^{(3\kappa)^{3/2} \eta} (\log T)^2.
\]

(13)

From relations (10)–(13), we deduce

\[
\zeta_1(s; \alpha) = \sum_{n \leq m} \frac{1}{(n + \alpha)^s} + O_{\sigma_0} \left( \frac{x^c}{bT} + \frac{x \log x}{T} \right) + O \left( x^{1-\sigma} t^{-1} \right).
\]
\[ + O_\kappa \left( x^\kappa T^{-1+\kappa} + x^{-2\kappa} T^{(3\kappa)^{3/2}\eta} (\log T)^2 \right). \]

If we set \( m = \lfloor t^{\mu} \rfloor \), then the last three terms in the latter relation are bounded above by

\[
C(\sigma_0, \kappa, b) \left( t^{(1+b)\mu-1} + t^{\mu(1-\sigma)-1} + t^{(1+b)\mu+3\kappa-1} + t^{\kappa(-2\mu+3\kappa^{1/2}\eta)} (\log t)^2 \right),
\]

where \( C(\sigma_0, \kappa, b) > 0 \) is a constant. It is clear now that for \( \kappa = 4\mu^2/(27\eta^2) \) and \( 0 < b \ll \mu \) sufficiently small, the lemma follows.

**Lemma 9** For every \( d \geq 2 \), \( \epsilon \in (0, \theta/d^2) \) and \( k \in \mathbb{N}_0 \), there exists a positive number \( \nu = \nu(d, \epsilon, k) \) such that

\[
\zeta^{(k)}(s; \alpha) = \sum_{0 \leq n \leq t^{1/d}} \frac{(-\log(n + \alpha))^k}{(n + \alpha)^s} + O_d,\epsilon,k \left( t^{-\nu} \right), \quad t \geq t_1 > 0,
\]

uniformly in \( A \left( 1/d \right) + \epsilon \leq \sigma \leq 1 \) and \( 0 < \alpha \leq 1 \).

**Proof** For \( k = 0 \), the lemma is a special case of Lemma 8 with \( \mu = 1/d \) and \( \sigma_0 = A(\mu) + \epsilon \). For \( k \geq 1 \), the lemma will follow by applying Cauchy’s integral formula in the approximate functional equation of the case \( k = 0 \).

The next lemma originates from a work of Güting [10].

**Lemma 10** Let \( P(X) \) and \( Q(X) \) be non-constant integer polynomials of degree \( n \) and \( m \), respectively. Denote by \( \alpha \) a zero of \( Q(X) \) of order \( t \). Assuming that \( P(\alpha) \neq 0 \), we have

\[
|P(\alpha)| \geq (n + 1)^{1-m/t}(m + 1)^{-n/(2t)}H(P)^{1-m/t}H(Q)^{-n/t}(\max\{1, |\alpha|\})^n
\]

**Proof** For a proof see [4, Theorem A.1].

**Lemma 11** Let \( P(X) \) be a nonzero integer polynomial of degree \( n \) and \( \alpha \in (0, 1] \) be an algebraic number of degree \( d(\alpha) \) and height \( H(\alpha) \). Assuming that \( P(\alpha) \neq 0 \), we have

\[
|P(\alpha)| \geq (n + 1)^{1-d(\alpha)}(d(\alpha) + 1)^{-n/2}H(P)^{1-d(\alpha)}H(\alpha)^{-n}
\]

**Proof** Follows immediately from Lemma 10.
3 Two Auxiliary Theorems

In the sequel, we will use the abbreviation \( e(x) = \exp(2\pi ix) \) for \( x \in \mathbb{R} \). If \( Q \in \mathbb{N} \), we define the function

\[
(s, \theta, \alpha) \mapsto \zeta_Q(s, \theta, \alpha) := \sum_{0 \leq n \leq Q-1} \frac{e(\theta_n)}{(n + \alpha)^s},
\]

for every \((s, \theta, \alpha) \in \mathbb{C} \times \mathbb{R}^Q \times (0, 1] \).

We start with a modification of Good’s Lemma 9 in [9] on the effective approximation of vectors of complex numbers by suitable twisted Dirichlet polynomials. An alternative option would be to follow Voronin’s approach as in [14, Chapter 8, Section 2, Lemma 1]. Interestingly enough, we could not deduce a result for \( \sigma = 1 \) by the second way. And since also Good does not include the case of \( \sigma = 1 \), we shall add it in our proof.

Theorem 3 For every \( \sigma \in (1/2, 1] \) and \( N \in \mathbb{N} \), there exist positive numbers \( C_0, C_1 \) and \( C_2 \), depending on \( \sigma \) and \( N \), such that the following is true:

Let \( A \in (0, 1], \varepsilon > 0 \) and \( a = (a_0, \ldots, a_N) \in \mathbb{C}^{N+1} \). Let also

\[
R \geq C_0 \varepsilon^{4/(1-2\sigma)},
\]

and \( Q_0 \geq C_1 R \) be integers satisfying the system of inequalities

\[
C_2 \left( |a_k| + A^{-1/2} \right) \leq E(R, Q_0, \sigma) \left( \frac{\log Q_0}{2N \log Q_0} \right)^N k!(N-k)! (\log Q_0)^k,
\]

\( k = 0, \ldots, N \), where

\[
E(R, Q, \sigma) := \begin{cases} R^{1-\sigma} & \text{if } \sigma \neq 1, \\ \frac{2^{3+\sigma}(1-\sigma)}{\log \frac{Q}{R+1}} & \text{if } \sigma = 1. \end{cases}
\]

(14)

Then, for every \( Q \geq Q_0 \) and \( \alpha \in [A, 1] \), there exists \( \theta_0 \in [0, 1]^Q \) such that

\[
\left| \frac{\partial^k}{\partial s^k} \zeta_Q(s, \theta_0, \alpha) \right|_{s=\sigma} - a_k \left| < \varepsilon, \quad k = 0, \ldots, N. \right.
\]

Proof Let \( R = R(\varepsilon, \sigma, N) \) be a positive integer which will be specified later on. We consider for every integer \( Q > R \) the set of vectors

\[
\mathcal{D}_{RQ} := \{ z = (z_R, \ldots, z_{Q-1}) : |z_n| \leq 1, n = R, \ldots, Q-1 \}.
\]
and define the functions
\[ (z, \alpha) \mapsto g_k(z, \alpha) := \sum_{R \leq n \leq Q-1} z_n \frac{(-\log(n + \alpha))^k}{(n + \alpha)^\sigma}, \]
for every \((z, \alpha) \in \mathcal{D}_{RQ} \times (0, 1]\) and \(k = 0, \ldots, N\).

First, we will determine for a given vector of complex numbers \((A_0, \ldots, A_N)\) an integer \(Q\) such that, for every \(0 < \alpha \leq 1\), the system of equalities
\[ g_k(z, \alpha) = A_k, \quad k = 0, \ldots, N, \]
has a solution \(z_{\alpha} \in \mathcal{D}_{RQ}\), that is, \((A_0, \ldots, A_N)\) belongs to the set
\[ G := \{(g_0(z, \alpha), \ldots, g_N(z, \alpha)) : z \in \mathcal{D}_{RQ}\}. \]

Observe that \(G\) is a closed convex subset of the complex Hilbert space \(\mathbb{C}^{N+1}\) endowed with the inner product
\[ \langle (x_0, \ldots, x_N), (y_0, \ldots, y_N) \rangle := \sum_{0 \leq k \leq N} \Re(x_k \overline{y_k}). \]

Thus, in view of Lemma 2 it is sufficient to show that for sufficiently large \(Q\) and for arbitrary \(0 < \alpha \leq 1\) and nonzero \((\ell_0, \ldots, \ell_N) \in \mathbb{C}^{N+1}\), there is \(z \in \mathcal{D}_{RQ}\) such that
\[ \sum_{0 \leq k \leq N} \ell_k g_k(z, \alpha) = \sum_{0 \leq k \leq N} \ell_k A_k. \]

One can see that
\[ \sum_{0 \leq k \leq N} \ell_k g_k(\mathcal{D}_{RQ}, \alpha) = \left\{ z : |z| \leq V := \sum_{R \leq n \leq Q-1} \frac{1}{(n + \alpha)^\sigma} \left| \sum_{0 \leq k \leq N} \ell_k (-\log(n + \alpha))^k \right| \right\}. \]

Indeed, the inclusion of the set on the left-hand side in the set on the right-hand side is obvious, while if \(w = |w|e(\phi)\) belongs to the disc described in the right-hand side of (18), we can choose \(z \in \mathcal{D}_{RQ}\) with
\[ z_n = \frac{|w|}{V} e \left( \phi - \arg \left( \sum_{0 \leq m \leq N} \ell_m (-\log(n + \alpha))^m \right) \right) \]
such that
\[ \sum_{0 \leq k \leq N} \ell_k g_k(z, \alpha) = w. \]

Therefore, from (17) and (18) it is sufficient to show that, for sufficiently large \( Q \) and for arbitrary \( 0 < \alpha \leq 1 \) and nonzero \( (\ell_0, \ldots, \ell_N) \in \mathbb{C}^{N+1} \),
\[
\sum_{0 \leq k \leq N} |\ell_k| A_k | \leq \sum_{R \leq n \leq Q-1} \frac{1}{(n + \alpha)^\sigma} \left| \sum_{0 \leq k \leq N} \ell_k (-\log(n + \alpha))^k \right|. \tag{19}
\]

Now, consider the polynomial
\[
P(x) := \sum_{0 \leq k \leq N} (-1)^k \ell_k x^k, \quad x \in \mathbb{R}, \tag{20}
\]
and the following partition of the interval \([\log(R + \alpha), \log Q]\)
\[
x_k := \log(R + \alpha) + \frac{k}{N} \log\left(\frac{Q}{R + \alpha}\right), \quad k = 0, \ldots, N. \tag{21}
\]

If we set in addition
\[
G_k(x) := \prod_{0 \leq m \leq N} (x - x_m), \quad k = 0, \ldots, N,
\]

then it follows that
\[
|G_k^{(j)}(0)| \leq \sum_{0 \leq m_1 \leq N} \sum_{0 \leq m_2 \leq N} \cdots \sum_{0 \leq m_j \leq N} \left| -x_{m_j} \right| \leq \frac{N!}{(N - j)!} (\log Q)^{N-j} \tag{22}
\]
and
\[
|G_k(x_k)| = \prod_{0 \leq m \leq N} \left| \frac{k - m}{N} \log\left(\frac{Q}{R + \alpha}\right) \right| = \left( \frac{1}{N} \log\left(\frac{Q}{R + \alpha}\right) \right)^N k!(N-k)! \tag{23}
\]
for any \( j, k = 0, \ldots, N. \) In view of Lagrange’s interpolation theorem (see [3, Chapter 1, Section 1, E.6])
\[
P(x) = \sum_{0 \leq k \leq N} \frac{P(x_k)}{G_k(x_k)} G_k(x),
\]
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and relations (22) and (23), we obtain
\[
|j!|\ell_j| = \left| P^{(j)}(0) \right| = \left| \sum_{0 \leq k \leq N} \frac{P(x_k)G_k^{(j)}(0)}{G_k(x_k)} \right| \\
\leq \sum_{0 \leq k \leq N} \left| \frac{P(x_k)}{k!(N-k)!(N-j)!} \right| \left( \frac{N}{\log \frac{Q}{R+\alpha}} \right)^N
\]
for \( j = 0, \ldots, N \). Therefore,
\[
\frac{1}{N+1} \left( \frac{\log \frac{Q}{R+\alpha}}{2N \log Q} \right)^N \sum_{0 \leq j \leq N} j!|\ell_j|(N-j)!(\log Q)^j \leq \sum_{0 \leq k \leq N} |P(x_k)|. \tag{24}
\]
Let \( y_k, k = 1, \ldots, N \), be such that \( x_{k-1} \leq y_k \leq x_k \) and
\[
|P(y_k)| = \max_{x \in [x_{k-1}, x_k]} |P(x)| = \max_{x \in [-1, 1]} \left| P \left( x \frac{x_k - x_{k-1}}{2} + \frac{x_k + x_{k-1}}{2} \right) \right|
\]
for \( k = 1, \ldots, N \). Markov’s inequality (see [3, Chapter 5, Section 2, E.2]) states that
\[
\max_{x \in [-1, 1]} |\tilde{P}'(x)| \leq N^2 \max_{x \in [-1, 1]} |\tilde{P}(x)|
\]
for any \( \tilde{P} \in \mathbb{C}[X] \) of degree at most \( N \). Thus,
\[
\max_{x \in [x_{k-1}, x_k]} |P'(x)| = \max_{x \in [-1, 1]} \left| P' \left( x \frac{x_k - x_{k-1}}{2} + \frac{x_k + x_{k-1}}{2} \right) \right|
\]
\[
= \max_{x \in [-1, 1]} \left| \frac{2}{x_k - x_{k-1}} \frac{d}{dx} P \left( x \frac{x_k - x_{k-1}}{2} + \frac{x_k + x_{k-1}}{2} \right) \right|
\]
\[
\leq \frac{2N^2}{x_k - x_{k-1}} \max_{x \in [-1, 1]} \left| P \left( x \frac{x_k - x_{k-1}}{2} + \frac{x_k + x_{k-1}}{2} \right) \right|
\]
\[
= \frac{2N^2}{x_k - x_{k-1}} |P(y_k)|
\]
for \( k = 1, \ldots, N \). If we set now
\[
\mathcal{I}_k := \left\{ x \in [x_{k-1}, x_k] : |x - y_k| \leq S := \frac{\log \frac{Q}{R+\alpha}}{4N^3} \right\}, \quad k = 1, \ldots, N, \tag{26}
\]
then relations (21), (25), (26) and the mean-value theorem imply that for every \( x \in \mathcal{I}_k \) there is a \( \xi_x \) between the points \( x \) and \( y_k \) such that
\[
|P(x)| \geq |P(y_k)| - |P(y_k) - P(x)| = |P(y_k)| - |P'(\xi_x)(y_k - x)| \geq \frac{|P(y_k)|}{2}
\]
or
\[
\max_{x \in [x_{k-1}, x_k]} |P(x)| = |P(y_k)| \leq 2 \min_{x \in I_k} |P(x)| \tag{27}
\]
for \(k = 1, \ldots, N\). Since
\[
x_k - x_{k-1} = \frac{\log Q}{N} + \alpha, \quad k = 1, \ldots, N,
\]
at least one of the intervals \([y_k - S, y_k]\) and \([y_k, y_k + S]\) is contained in \(I_k\). We denote those intervals by
\[
J_k := [c_k - \sigma, c_k + S], \quad k = 1, \ldots, N.
\tag{28}
\]
Then, it follows from (20), (21), (27) and (28) that
\[
\sum_{R \leq n \leq Q-1} \frac{1}{(n + \alpha)^\sigma} \left| \sum_{0 \leq k \leq N} \ell_k (-\log(n + \alpha))^k \right| = \sum_{R \leq n \leq Q-1} \frac{|P(\log(n + \alpha))|}{(n + \alpha)^\sigma}
\geq \sum_{k \leq N} \frac{|P(\log(n + \alpha))|}{\sum_{\log(n + \alpha) \in J_k} (n + \alpha)^\sigma}
\geq \sum_{k \leq N} \frac{|P(y_k)|}{2} \sum_{e^{c_k} \leq n + \alpha \leq e^{c_k + S}} \frac{1}{(2n)^\sigma}.
\tag{29}
\]
Observe that
\[
\sum_{e^{c_k} \leq n + \alpha \leq e^{c_k + S}} \frac{1}{n^\sigma} \geq \left\{ \begin{array}{ll}
\frac{e^{c_k(1-\sigma)} (e^S(1-\sigma) - 1)}{1 - \sigma} + O(e^{-c_k}) , & \sigma < 1, \\
\log e^{c_k + S} + O(e^{-c_k}) , & \sigma = 1.
\end{array} \right.
\]
Since \(c_k \geq \log R\) for \(k = 1, \ldots, N\), the definition of \(S\) yields that
\[
\sum_{e^{c_k} \leq n + \alpha \leq e^{c_k + S}} \frac{1}{n^\sigma} \geq \left\{ \begin{array}{ll}
\frac{R^{1-\sigma}}{2(1 - \sigma)} \left[ \left( \frac{Q}{R + 1} \right)^{(1-\sigma)/(4N^3)} - 1 \right] , & \sigma < 1, \\
\log \frac{Q}{R + 1} \frac{1}{8N^3} , & \sigma = 1,
\end{array} \right.
\]
for sufficiently large \(R \gg 1\) and \(Q \geq C_1 R\), where \(C_1 = C_1(\sigma, N)\). Recall that the right-hand side part of the latter inequality is equal to \(2^{2+\sigma} \mathcal{E}(R, Q, \sigma)\). It follows now
from relations (27) and (29) that
\[
\sum_{R \leq n \leq Q-1} \frac{1}{(n + \alpha)^\sigma} \left| \sum_{0 \leq k \leq N} \ell_k (-\log(n + \alpha))^k \right| \geq 2E(R, Q, \sigma) \sum_{k \leq N} |P(y_k)| \\
\geq E(R, Q, \sigma) \sum_{0 \leq k \leq N} |P(x_k)|.
\]

(30)

Thus, in view of relations (24) and (29), if we choose \(Q \geq C_1 R\) large enough so that the system of inequalities
\[
|A_k| \leq E(R, Q, \sigma) \left( \frac{\log Q}{2N \log Q} \right)^N k!(N - k)! \left( \log Q \right)^k, \quad k = 0, \ldots, N,
\]

(31)
is satisfied, then relation (19) holds for arbitrary \(\alpha \in (0, 1]\) and any nonzero vector \((\ell_0, \ldots, \ell_N) \in \mathbb{C}^{N+1}\). Hence, for every \(\alpha \in (0, 1]\) the system (16) has a solution \(z_\alpha \in \mathcal{D}_{RQ}\) as long as \(Q \geq C_1 R\) satisfies (31).

If \(U_0 \gg \sigma, N\) is large enough so that the functions
\[
x \mapsto \frac{(\log x)^k}{x^\alpha}, \quad k = 0, \ldots, N,
\]

(32)
are decreasing in \([U_0, +\infty)\), then for every \(R > U_0, \alpha \in [A, 1]\) and \(k = 0, \ldots, N\), we have that
\[
\left| \sum_{0 \leq n \leq R-1} (-1)^n (-\log(n + \alpha))^k \right| \leq \frac{(-\log \alpha)^k}{\alpha} + \sum_{n \leq U} \frac{(-1)^n (\log(n + \alpha))^k}{(n + \alpha)^\sigma} \leq \frac{(-\log \alpha)^k}{\alpha^\sigma} + \max_{y \in [0, 1]} \sum_{n \leq U} \frac{(-1)^n (\log(n + y))^k}{(n + y)^\sigma} \leq C_2 A^{-1/2},
\]
where \(C_2 = C_2(\sigma, N) \geq 1\). Therefore, if we set
\[
A_k = A_k(\alpha) := a_k - \sum_{0 \leq n \leq R-1} \frac{(-1)^n (-\log(n + \alpha))^k}{(n + \alpha)^\sigma}, \quad k = 0, \ldots, N,
\]
it follows from (31) that for every \(\alpha \in [A, 1]\) the system of equalities (16) has a solution \(z_\alpha \in \mathcal{D}_{RQ}\) as long as \(Q \geq C_1 R\) satisfies the system of inequalities
\[
C_2 \left( |a_k| + A^{-1/2} \right) \leq E(R, Q, \sigma) \left( \frac{\log Q}{2N \log Q} \right)^N k!(N - k)! \left( \log Q \right)^k,
\]

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$k = 0, \ldots, N$. Since the right-hand side of these inequalities tends to infinity as $Q \to \infty$, the system is solvable for all sufficiently large $Q$.

Let $Q_0 \geq C_1 R$ be the smallest integer satisfying the aforementioned system, $Q \geq Q_0$ and $\alpha \in [A, 1]$. Let also $z_\alpha := (z_n)_{R \leq n \leq Q_0 - 1}$ be an element of $D_{RQ_0}$ such that
\[ g_k(z_\alpha, \alpha) = A_k(\alpha), \quad k = 0, \ldots, N. \tag{33} \]

From Lemma 1, there are real numbers $\theta_n$, $n = R, \ldots, Q - 1$, such that
\[
\left\| \sum_{R \leq n \leq Q_0 - 1} z_n \left( \frac{-\log(n + \alpha)^k}{(n + \alpha)^\sigma} - \frac{(-\log(n + \alpha)^k e(\theta_n)}{(n + \alpha)^\sigma} \right) \right\|_{C^{N+1}}^2 \\
\leq 4 \sum_{R \leq n \leq Q - 1} \left\| \frac{(-\log(n + \alpha)^k}{(n + \alpha)^\sigma} \right\|_{C^{N+1}}^2 \\
\leq 4 \sum_{0 \leq k \leq N} \sum_{R \leq n \leq Q - 1} \frac{(\log(n + 1)^2)^{2k}}{n^{2\sigma}} \\
\ll_{\sigma, N} R^{1 - 2\sigma}. \tag{34} \]

Let
\[
R \gg_{\sigma, N} \left( U_0 + \frac{1}{\varepsilon} \right)^{\frac{4}{\sigma-1}} \gg_{\sigma, N} \left( \frac{1}{\varepsilon} \right)^{\frac{4}{\sigma-1}}
\]
be sufficiently large and set $\theta_{0n} = (\theta_{0n})_{0 \leq n \leq Q - 1}$ to be
\[
\theta_{0n} := \begin{cases} n/2, & 0 \leq n \leq R - 1, \\ \theta_n, & R \leq n \leq Q - 1. \end{cases}
\]
Then, (15), (33) and (34) yield
\[
\left| \frac{\partial^k}{\partial s^k} Q(s, \theta_{0}, \alpha) \right|_{s = \sigma} - a_k = A_k(\alpha) - \sum_{R \leq n \leq Q - 1} \frac{(-\log(n + \alpha)^k e(\theta_n)}{(n + \alpha)^\sigma} \\
< \left| g_k(z_\alpha, \alpha) \sum_{R \leq n \leq Q_0 - 1} z_n \frac{(-\log(n + \alpha)^k}{(n + \alpha)^\sigma} \right| + \varepsilon \\
= \varepsilon
\]
for $k = 0, \ldots, N$. \hfill \Box

Before proving the next theorem, we need to introduce some notations. Let $\lambda : \mathbb{R} \to \mathbb{R}_+$ be an infinitely many times differentiable function with $\text{supp}(\lambda) \subseteq \mathbb{R}$.
We also assume that $\lambda$ is bounded above by 1. If $Q \geq 2$ is an integer, we set $\delta := Q^{-2}$ and define the function

$$
\theta \mapsto \Lambda_Q(\theta) := \prod_{0 \leq n \leq Q-1} \lambda\left(\frac{\theta_n}{\delta}\right),
$$

for any $\theta = (\theta_0, \ldots, \theta_{Q-1}) \in [-1, 1]^Q$. Then, supp $(\Lambda_Q) \subseteq [-1/2, 1/2]^Q$ and we can extend $\Lambda_Q$ onto all $\mathbb{R}^Q$ by periodicity with period 1 in each of the variables $\theta_n$, $n = 0, \ldots, Q - 1$. The function

$$
\theta \mapsto \lambda\left(\frac{\theta}{\delta}\right)
$$

extended to $\mathbb{R}$ by periodicity with period 1 has a Fourier expansion

$$
\lambda\left(\frac{\theta}{\delta}\right) := \sum_{n = -\infty}^{+\infty} c_n e(n \theta),
$$

where

$$
c_0 = \delta \quad \text{and} \quad c_n = \int_0^1 \lambda\left(\frac{\theta}{\delta}\right) e(-n \theta) d\theta \ll \frac{1}{n^2 \delta^2}, \quad n \in \mathbb{Z}\setminus\{0\}.
$$

The last relation follows from integrating twice by parts, and the implicit constant depends only on our choice of $\lambda$. Thus, the Fourier expansion of $\Lambda_Q$ is given by

$$
\Lambda_Q(\theta) := \sum_{m} d_m e(\langle m, \theta \rangle),
$$

where $m = (m_0, \ldots, m_{Q-1}) \in \mathbb{Z}^Q$ and

$$
d_m := \prod_{0 \leq n \leq Q-1} c_{m_n}.
$$

We define for every $m \in \mathbb{Z}^Q \setminus \{0\}$ and $x \in \mathbb{R}$ the polynomials

$$
Q^+_m(x) := \prod_{0 \leq n \leq Q-1, m_n > 0} (n + x)^{m_n} \quad \text{and} \quad Q^-_m(x) := \prod_{0 \leq n \leq Q-1, m_n < 0} (n + x)^{-m_n}.
$$

Let $\hat{M} := \mathbb{Z}^Q \cap [-M, M]^Q$ and

$$
\mathcal{P}(Q, M) := \left\{ P_m = Q^+_m - Q^-_m : m \in \hat{M} \setminus \{0\} \right\}.
$$
Observe that $P(Q, M)$ is a set of nonzero integer polynomials of degree at most $MQ$ and height bounded by a constant $H(Q, M)$. We also define the set

$$A(Q, M) = A_1 \cup A_2,$$

where

$$A_1 := \{ \alpha \in \mathbb{A} : d(\alpha) > MQ + 1 \},$$

$$A_{2m} := \left\{ \alpha \in \mathbb{A} \setminus A_1 : \forall x, y \in \mathbb{N} \cap \left[ 0, \exp\left(2Q^2\right) \right], \frac{Q_m^+(\alpha)}{Q_m^-(\alpha)} \neq \frac{x + \alpha}{y + \alpha} \right\},$$

for $m \in \hat{M} \setminus \{0\}$, and

$$A_2 := \bigcap_{m \in \hat{M} \setminus \{0\}} A_{2m}.$$

Observe that for any positive integers $Q$ and $M$, each one of the sets $A_{2m}, m \in \hat{M} \setminus \{0\}$, contains all algebraic irrational numbers $\alpha \in (0, 1]$ of degree $\leq MQ + 1$ with only finitely many exceptions that could possibly occur by the real solutions of the finitely many rational relations

$$\frac{Q_m^+(\alpha)}{Q_m^-(\alpha)} = \frac{x + \alpha}{y + \alpha}, \quad x, y \in \mathbb{N} \cap \left[ 0, \exp\left(2Q^2\right) \right], \quad m \in \hat{M} \setminus \{0\}.$$

This implies that, for any positive integers $Q$ and $M$, the set $A_2$ contains all algebraic irrational numbers $\alpha \in (0, 1]$ of degree $\leq MQ + 1$ except finitely many and, therefore, the set $\mathbb{A} \setminus A(Q, M)$ is finite.

Finally, we consider the curve

$$\mathbb{R} \times (0, 1] \ni (\tau, \alpha) \mapsto \gamma_Q(\tau, \alpha) := \left( \frac{\log (n + \alpha)}{2\pi} \right)_{0 \leq n < Q}.$$

**Theorem 4** For any $k \in \mathbb{N}_0$, $d$ satisfying $d^3 - 2d^2 > 2\theta$ and $\epsilon \in (0, \theta/d^2)$, there exist positive numbers $C_3 = C_3(k), C_4, C_5(d, \epsilon, k)$ and $v = v(d, \epsilon, k)$, such that the following is true:

Let $\epsilon > 0$, $Q \geq C_3/\epsilon^8$, $M \geq C_4 \exp\left(2Q^2\right)$, $\alpha \in A(Q, M)$ satisfying $d > d(\alpha) + 2\theta/d^2 - \epsilon$ and

$$T \geq C_5 \max \left\{ (K \exp\left((M + 2) \exp\left(Q^2\right)\right))^{d^3(d - d(\alpha)) - 2\theta + \epsilon d^2}, \epsilon^{-2v} \right\}.$$
where

\[
K = K(Q, M, \alpha) := [H(Q, M) (MQ + 2)]^{d(\alpha) - 1} [H(\alpha)(d(\alpha) + 1)^{1/2}]^{MQ+1}.
\]  

(38)

Then, we have that

\[
\left| \frac{1}{\delta Q T} \int_{T}^{2T} \Lambda(\gamma_Q(\tau, \alpha) - \theta_1) d\tau - 1 \right| < Q^{-2}
\]

and

\[
\int_{T}^{2T} \Lambda_Q(\gamma_Q(\tau, \alpha) - \theta_1) \left| \xi^{(k)}(\sigma + i\tau; \alpha) - \frac{\partial^k}{\partial s^k} \xi_Q(s + i\tau, Q, \alpha) \right|_{s=\sigma}^2 d\tau
\]

\[
< \varepsilon^2 \int_{T}^{2T} \Lambda_Q(\gamma_Q(\tau, \alpha) - \theta_1) d\tau
\]

for any \( \theta_1 \in \mathbb{R}^Q \) and \( A (1/d) + \epsilon \leq \sigma \leq 1 \).

**Proof** First, we will show that

\[
\left| \frac{1}{\delta Q T} \int_{T}^{2T} \Lambda(\gamma_Q(\tau, \alpha) - \theta_1) d\tau - 1 \right| < Q^{-2}
\]

for suitable \( Q, \alpha, T \) and any \( \theta_1 \in \mathbb{R}^Q \). The Fourier expansion of the function

\[
\theta \mapsto \Lambda_Q(\theta - \theta_1)
\]

is given by

\[
\Lambda_Q(\theta - \theta_1) := \sum_{m} h_m e(\langle m, \theta \rangle),
\]

where \( h_0 := \delta_Q \) and \( h_m := d_m e(\langle m, -\theta_1 \rangle) \), \( m \in \mathbb{Z}^Q \). For \( M \in \mathbb{N} \),

\[
\left| \sum_{m \notin \hat{M}} h_m e(\langle m, \theta \rangle) \right| \leq \sum_{m \notin \hat{M}} |h_m| \leq Q \left( \sum_{|n| > M} |c_n| \right) \left( \sum_{n = -\infty}^{+\infty} |c_n| \right)^{Q-1}.
\]  

(40)
From (35), we know that
\[
\sum_{|n| > M} |c_n| \ll \frac{1}{\delta^2 M} \quad \text{and} \quad \sum_{n = -\infty}^{+\infty} |c_n| \leq \left( \frac{A}{\delta} \right)^2,
\]
(41)

where \( A > 1 \) is an absolute constant. Therefore, from (40) we conclude that
\[
\Lambda_Q(\theta - \theta_1) = \sum_{m \in \hat{M}} h_m e(\langle m, \theta \rangle) + O\left( \frac{Q}{M} \left( \frac{A}{\delta} \right)^{2Q} \right).
\]
(42)

Observe that by \( \delta = Q^{-2} \), we have
\[
Q \left( \frac{A}{\delta} \right)^{2Q} \leq Q \delta^Q \left( \frac{A}{\delta} \right)^{3Q} \leq \delta^Q (AQ)^6 \ll \delta^Q \exp(Q^2).
\]
Hence, relation (42) can be written as
\[
\Lambda_Q(\theta - \theta_1) = \sum_{m \in \hat{M}} h_m e(\langle m, \theta \rangle) + O\left( \frac{\delta^Q \exp(Q^2)}{M} \right).
\]
(43)

In the sequel, we use the notations
\[
\ell(\tau) := \Lambda_Q(\gamma_Q(\tau, \alpha) - \theta_1) \quad \text{and} \quad \tilde{n} := n + \alpha
\]
in order to avoid extensive expressions. In view of (43), we have
\[
\int_{T}^{2T} \ell(\tau)d\tau = h_0 T + \sum_{m \in \hat{M}, \{0\}} h_m \int_{T}^{2T} e(\langle m, \gamma_Q(\tau, \alpha) \rangle) + O\left( \frac{T \delta^Q \exp(Q^2)}{M} \right),
\]
or
\[
\frac{1}{\delta^Q T} \int_{T}^{2T} \ell(\tau)d\tau = 1 + \frac{1}{\delta^Q T} \sum_{m \in \hat{M}, \{0\}} h_m \int_{T}^{2T} \left( \frac{Q^+_m(\alpha)}{Q^-_m(\alpha)} \right)^{i\tau} d\tau + O\left( \frac{\exp(Q^2)}{M} \right).
\]
(44)

It follows from the definition of \( h_m \) and (41) that
\[
\sum_{m} |h_m| \leq \left( \frac{A}{\delta} \right)^{2Q} \leq \delta^Q (AQ)^6 \ll \delta^Q \exp(Q^2).
\]
(45)
It also follows from \((36)\) and \((37)\) that if \(m \in \hat{M} \setminus \{0\}\) and \(\alpha \in \mathcal{A}(Q, M)\), then \(P_m(\alpha) = Q^+_m(\alpha) - Q^-_m(\alpha) \neq 0\). Thus, it follows from Lemma 3 that

\[
\int_T^{2T} \left( \frac{Q^+_m(\alpha)}{Q^-_m(\alpha)} \right)^{i\tau} d\tau \ll \log \frac{Q^+_m(\alpha)}{Q^-_m(\alpha)}\cdot \max\{Q^+_m(\alpha), Q^-_m(\alpha)\} < Q^{-1}. \tag{46}
\]

Now, Lemma 11 yields that, for every \(m \in \hat{M} \setminus \{0\}\) and \(\alpha \in \mathcal{A}(Q, M)\),

\[
\left| Q^+_m(\alpha) - Q^-_m(\alpha) \right| \geq [H(Q, M) (M Q + 1)]^{1-d(\alpha)} \left[ H(\alpha) (d(\alpha) + 1)^{1/2} \right]^{-M Q} > K^{-1}. \tag{47}
\]

Along with the estimate

\[
\max\{Q^+_m(\alpha), Q^-_m(\alpha)\} \ll \prod_{n=1}^Q n^M \ll \exp(M Q^2), \tag{48}
\]

we conclude from \((44)\)–\((48)\) that

\[
\frac{1}{\delta Q^2 T} \int_T^{2T} \ell(\tau) d\tau - 1 \ll \frac{\exp(Q^2)}{M} + \frac{K \exp((M + 1) Q^2)}{T}. 
\]

For \(Q \gg 1, M \gg \exp(2Q^2), \alpha \in \mathcal{A}(Q, M)\) and \(T \gg K \exp((M + 2) Q^2)\), with suitable constants in \(\gg\), we obtain

\[
\left| \frac{1}{\delta Q^2 T} \int_T^{2T} \Lambda(\gamma_Q(\tau, \alpha) - \theta_1) d\tau - 1 \right| < Q^{-2}. \tag{49}
\]

We proceed now with the proof of relation \((39)\). Let \(I\) denote the left-hand side of \((39)\). Let also \(\alpha \in \mathcal{A}(Q, M)\) and \(d > d(\alpha) + 2\theta/d^2 - \epsilon\). It follows from Lemma 9 that there exists a positive number \(\nu = \nu(d, \epsilon, k)\), such that

\[
\xi^{(k)}(s; \alpha) = \sum_{0 \leq n \leq t_1/d} \frac{(-\log(n + \alpha))^k}{(n + \alpha)^s} + O_{d, \epsilon, k}(t^{-\nu}), \quad t \geq t_1 > 0,
\]

uniformly in \(A(1/d) + \epsilon \leq \sigma \leq 1\) and \(0 < \alpha \leq 1\). By substituting this approximate functional equation in \(I\) for sufficiently large \(T \gg_d Q\), and by setting \(p(\tau) := \lfloor \tau^{1/d} \rfloor\),
it follows that $I \ll I_1 + I_2$, where

$$I_1 = \int_T^{2T} \ell(\tau) \left| \sum_{Q \leq n \leq p(\tau)} \frac{(-\log \tilde{n})^k}{\tilde{n}^{\sigma + i\tau}} \right|^2 d\tau \quad \text{and} \quad I_2 = \int_T^{2T} \ell(\tau) O_d, \epsilon, k \left( \tau^{-\nu} \right) d\tau.$$ 

Thus, it suffices to prove the theorem for $I_1$ and $I_2$. We start by estimating $I_1$:

$$I_1 \leq \left( \delta Q + O \left( \frac{\delta Q \exp(Q^2)}{M} \right) \right) \int_T^{2T} \left| \sum_{Q \leq n \leq p(\tau)} \frac{(-\log \tilde{n})^k}{\tilde{n}^{\sigma + i\tau}} \right|^2 d\tau + \left| \sum_{m \in M \setminus \{0\}} h_m \int_T^{2T} e(\langle m, \gamma_Q(\tau, \alpha) \rangle) \left| \sum_{Q \leq n \leq p(\tau)} \frac{(-\log \tilde{n})^k}{\tilde{n}^{\sigma + i\tau}} \right|^2 d\tau \right| \quad (50)$$

where $M_1 = \max \left\{ T, \tilde{n}^d \right\}$ and $M_2 = \max \left\{ T, \tilde{n}_1^d, \tilde{n}_2^d \right\}$. Since $\alpha \in (0, 1]$ and $\sigma \geq A(1/d) + \epsilon > 3/4$, we get that

$$\sum_{Q \leq n \leq p(2T)} \frac{(\log \tilde{n})^{2k}}{\tilde{n}^{2\sigma}} \ll_k \sum_{n \geq Q} \frac{(\log n)^{2k}}{n^{3/2}} \ll_k \frac{Q^{-1/2} \log Q)^{2k}}{Q^{-1/4}} \quad (51)$$

and

$$\sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{(\log \tilde{n}_1 \log \tilde{n}_2)^k}{\tilde{n}_1^{\sigma} \tilde{n}_2^{\sigma}} \ll_k \sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{(\log (2T))^k}{\tilde{n}_1 \tilde{n}_2^{\sigma}}. \quad (52)$$

Therefore,

$$S_1 \ll_k Q^{-1/4} T + \sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{(\log (2T))^k}{\tilde{n}_1 \tilde{n}_2^{\sigma}} \int_{T_2}^{2T} \left( \frac{\tilde{n}_2}{\tilde{n}_1} \right)^{i\tau} d\tau.$$

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\[ \ll_k Q^{-1/4} + (\log (2T))^2 \sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{1}{\bar{n}_1 \bar{n}_2} \left| \log \frac{\bar{n}_2}{\bar{n}_1} \right|^{-1} \]
\[ \ll_{k, \epsilon} Q^{-1/4} + p(2T)^{2 - 2\Lambda(1/d) - 2\epsilon} (\log p(2T))^{1 + 2k} \]
\[ \ll_{k, \epsilon} Q^{-1/4} + p(2T)^{2\theta/d^2 - \epsilon}. \] (53)

For the second sum, we have by interchanging integration and summation

\[ S_{2m} = \sum_{Q \leq n \leq p(2T)} \frac{(\log \bar{n})^{2k}}{\bar{n}^{2\sigma}} \int_{T_1}^{2T} \left( \frac{Q_m^+(\alpha)}{Q_m^-(\alpha)} \right)^{i\tau} d\tau \]
\[ + \sum_{Q \leq n_1 \neq n_2 \leq p(2T)} \frac{(\log \bar{n}_1 \log \bar{n}_2)^k}{\bar{n}_1 \bar{n}_2} \int_{T_2}^{2T} \left( \frac{Q_m^+(\alpha)\bar{n}_2}{Q_m^-(\alpha)\bar{n}_1} \right)^{i\tau} d\tau. \] (54)

Here, we consider two subcases, depending on whether \( \alpha \in A_1 \) or \( \alpha \in A_2 \). It follows from the definitions in (36) and (37) that, if \( m \in \mathcal{M} \setminus \{0\} \) and \( \alpha \in A_1 \), then

\[ Q_m^+(\alpha) - Q_m^-(\alpha) \neq 0 \quad \text{and} \quad Q_m^+(\alpha)\bar{n}_2 - Q_m^-(\alpha)\bar{n}_1 \neq 0. \] (55)

Thus, applying Lemma 11, it follows similar as in (46)–(48) that

\[ \int_{T_1}^{2T} \left( \frac{Q_m^+(\alpha)}{Q_m^-(\alpha)} \right)^{i\tau} d\tau \ll K \exp \left( MQ^2 \right) \] (56)

and

\[ \int_{T_1}^{2T} \left( \frac{Q_m^+(\alpha)\bar{n}_2}{Q_m^-(\alpha)\bar{n}_1} \right)^{i\tau} d\tau \ll \max \{Q_m^+(\alpha)\bar{n}_2, Q_m^-(\alpha)\bar{n}_1\} \quad \ll K p(2T)^{d(\alpha)} \exp \left( MQ^2 \right). \] (57)

From relations (51), (52), (54), (56) and (57), we obtain

\[ S_{2m} \ll_k \left( Q^{-1/4} + p(2T)^{2 - 2\sigma + d(\alpha)} (\log p(2T))^{2k} \right) K \exp \left( MQ^2 \right) \]
\[ \ll_{k, \epsilon} \left( Q^{-1/4} + p(2T)^{2\theta/d^2 - \epsilon + d(\alpha)} \right) K \exp \left( MQ^2 \right). \] (58)

If now \( \alpha \in A_2 \), then the second condition of relation (55) may not be satisfied. However, by the construction of the set \( A_2 \) this cannot happen too often. Indeed, for
every \( m \in \hat{M} \setminus \{0\} \), the equation
\[
\frac{Q_m^+(\alpha)}{Q_m(\alpha)} = \frac{x + \alpha}{y + \alpha}
\]
has at most one solution in the positive integers, \((x_m, y_m)\) say, with \(x_m \neq y_m\), as follows from the irrationality of \(\alpha\). In case, such a solution does not exist in the set \((\mathbb{N} \cap [Q, +\infty))^2\), the estimate for \(S_{2m}\) is the same as in (58). If it exists, then we have to add in (58) the term
\[
\frac{(\log(x_m + \alpha))^k (\log(y_m + \alpha))^\sigma}{(x_m + \alpha)^k (y_m + \alpha)^\sigma} T,
\]
where \(x_m\) and \(y_m\) are both greater than \(Q\) and at least one of them is greater than \(\exp\left(2Q^2\right)\). Therefore, for sufficiently large \(Q \gg_k 1\), the additional term is bounded by
\[
\frac{T}{\exp(Q^2) Q^{1/2}}.
\]
In view of the preceding and (45), (50), (53) and (58), we conclude that
\[
I_1 \ll_{k, \epsilon} \left[ \delta Q + O\left(\frac{\delta Q \exp(Q^2)}{M}\right) \right] \left(Q^{-1/4} + p(2T)^{20/d^2 - \epsilon}\right)
\]
\[+ \delta Q \exp(Q^2) \left[Q^{-1/4} + p(2T)^{20/d^2 - \epsilon + d(\alpha)}\right] K \exp(MQ^2) + \frac{T}{\exp(Q^2) Q^{1/2}}\]
or
\[
I_1 \ll_{k, \epsilon} Q^{-1/4} \left[2 + \frac{\exp(Q^2)}{M} + \frac{K \exp((M + 1)Q^2)}{T}\right] \delta Q T
\]
\[+ \left[1 + \frac{\exp(Q^2)}{M} + \frac{K \exp((M + 1)Q^2)}{T}\right] \delta Q p(2T)^{20/d^2 - \epsilon + d(\alpha)}.
\]
(59)

Observe that
\[
p(2T)^{20/d^2 - \epsilon + d(\alpha)} \ll_{d, \epsilon, k} T \frac{d^3 (d(\alpha) - d) + 20 - d^2}{d^3} T.
\]
Then, for \(Q \gg_k 1/\epsilon^8\), \(M \gg \exp(2Q^2)\), \(\alpha \in \mathcal{A}(Q, M)\) with \(d > d(\alpha) + 20/d^2 - \epsilon\) and
\[
T \gg_{d, \epsilon, k} \left(K \exp((M + 2)Q^2)\right) \frac{d^3 (d(\alpha) - d) + 20 + \epsilon d^2}{d^3},
\]
(60)
with suitable constants in $\gg$, we deduce from (49) and (59) that

$$I_1 \leq \frac{\varepsilon^2}{2} \int_{T}^{2T} \Lambda_Q \left( \gamma_Q(\tau, \alpha) - \theta_1 \right) d\tau$$

(61)

for every $A(1/d) + \varepsilon \leq \sigma \leq 1$ and $\theta_1 \in \mathbb{R}^Q$.

Finally, we estimate $I_2$ by

$$I_2 \ll d, \varepsilon, \kappa \frac{T}{\Lambda_1} Q \int_{T}^{2T} \Lambda_Q \left( \gamma_Q(\tau, \alpha) - \theta_1 \right) d\tau.$$  

(62)

The theorem now follows from (60)--(62).

4 Proofs of Theorem 1 and Theorem 2

**Proof of Theorem 1** Let $\sigma$, $N$, $A$, $\varepsilon$, $a$, $R$ and $Q_0$ be as in Theorem 3. Then, for every $Q \geq Q_0$ and $\alpha \in [A, 1]$, the system of inequalities

$$\left| \frac{\partial^k}{\partial \tau^k} \zeta_Q(s, \theta_0, \alpha) \left|_{s=\sigma} \right. \right| - a_k \right| < \frac{\varepsilon}{4}, \quad k = 0, \ldots, N,$$

has a solution $\theta_0 = \theta_0(\alpha)$. If we take $\delta = Q^{-2}$, then the inequality

$$|\theta_n - \theta_{0n}| \leq \delta$$

(63)

implies that

$$\left| \frac{\partial^k}{\partial \tau^k} \left( \zeta_Q(s, \theta_0, \alpha) - \zeta_Q(s, \theta_0, \alpha) \right) \left|_{s=\sigma} \right. \right| \leq \sum_{0 \leq n \leq Q-1} \frac{(\log(n + \alpha))^k |e(\theta_n) - e(\theta_{0n})|}{(n + \alpha)^\sigma} \ll \frac{1}{\alpha^\sigma} \delta Q \log^N Q \ll A^{-1/2} Q^{-1} \log^N Q \ll_{N, \Lambda} Q^{-1/2}$$

for $k = 0, \ldots, N$. Thus, the system of inequalities

$$\left| \frac{\partial^k}{\partial \tau^k} \zeta_Q(s, \theta_0, \alpha) \left|_{s=\sigma} \right. \right| - a_k \right| < \frac{\varepsilon}{2} < \left( \frac{2 Q^2 + 1}{Q^2 - 1} \right)^{1/2} \frac{\varepsilon}{2}, \quad k = 0, \ldots, N,$$

(64)

is satisfied whenever $\alpha \in [A, 1]$, $Q \gg_{N, \Lambda} Q_0 + 1/\varepsilon^4$ and (63) holds. On the other hand, Theorem 4 yields, for every $Q \geq C_3(N)/\varepsilon^8$, $M \geq C_4 \exp \left( 2Q^2 \right)$, $\alpha \in \mathcal{A}(Q, M) \cap \mathcal{R}$.
and \( d > d(\alpha) + 2\theta/d^2 - \epsilon \), the existence of a positive number \( v(d, \epsilon, N) \) such that, for every

\[
T \geq C_5(d, \epsilon, N, A) \max \left\{ \left( K \exp \left( (M + 2) \exp \left( Q^2 \right) \right) \right)^{d^2(d - d(\alpha) - 2\theta + \epsilon)d^2} \frac{1}{\epsilon^{2v}}, \epsilon^{-2v} \right\},
\]

we have

\[
\sum_{0 \leq k \leq N} \int_0^{2T} \Lambda_0 \left( \gamma_Q(\tau, \alpha) - \theta_0 \right) \left| \zeta^{(k)}(\sigma + i\tau; \alpha) - \frac{\partial}{\partial s^k} \zeta_Q(s + i\tau, 0, \alpha) \right|_{s = \sigma}^2 d\tau < \sum_{0 \leq k \leq N} \frac{\epsilon^2}{4(N + 1)} \int_0^{2T} \Lambda_0(\gamma_Q(\tau, \alpha) - \theta_0) d\tau
\]

\[
= \frac{\epsilon^2}{4} \int_0^{2T} \Lambda_0(\gamma_Q(\tau, \alpha) - \theta_0) d\tau < \frac{\epsilon^2}{4} \delta^Q(1 + Q^{-2}) T.
\]

(65)

for any \( A(1/d) + \epsilon \leq \sigma \leq 1 \) and

\[
\int_0^{2T} \Lambda_0(\gamma_Q(\tau, \alpha) - \theta_0) d\tau \geq \delta^Q \left( 1 - Q^{-2} \right) T.
\]

(66)

Now, let

\[
U_T(\alpha) := \left\{ \tau \in [T, 2T] : \Lambda_0(\gamma_Q(\tau, \alpha) - \theta_0) \neq 0 \right\}.
\]

(67)

By definition \( \Lambda_0 \) is bounded above by 1. This and (66) imply that

\[
m(U_T(\alpha)) \geq \delta^Q \left( 1 - Q^{-2} \right) T.
\]

(68)

We argue by contradiction to prove Theorem 1. To that end, we fix an integer \( Q \gg_{N, A} (Q_0 + C_3(N)/\epsilon^8) \), a real number \( A(1/d) + \epsilon \leq \sigma \leq 1 \) and assume that there is no solution \( \tau \) in \( U_T(\alpha) \) for the system of inequalities (3). In particular, for every \( \tau \in U_T(\alpha) \), we assume that there exists \( k_\tau \in \{0, \ldots, N\} \) such that

\[
\left| \zeta^{(k_\tau)}(\sigma + i\tau; \alpha) - a_{k_\tau} \right| \geq \epsilon.
\]
On the other hand for every such $\tau$ the curve $\gamma_Q(\tau, \alpha)$ is $\delta$-close to $\theta_0$ (63) as can be seen by (67). Hence, relation (64) is satisfied with $\theta = \gamma_Q(\tau, \alpha)$ and we can deduce for every $\tau \in U_T(\alpha)$ that

$$\sum_{0 \leq k \leq N} \left| \zeta^{(k)}(\sigma + i\tau; \alpha) - \frac{\partial^k}{\partial s^k} \zeta_Q(s + i\tau, 0, \alpha) \right|_{s=\sigma}^2 \geq \left| \zeta^{(k)}(\sigma + i\tau; \alpha) - \frac{\partial^k}{\partial s^k} \zeta_Q(s + i\tau, 0, \alpha) \right|_{s=\sigma}^2 \geq \frac{1}{2} \left| \zeta^{(k)}(\sigma + i\tau; \alpha) - a_k \right|^2 - \left| \frac{\partial^k}{\partial s^k} \zeta_Q(s + i\tau, 0, \alpha) \right|_{s=\sigma}^2 \geq \frac{\varepsilon^2}{4}.$$ 

But this contradicts (65) since the lower bound of the above inequalities holds uniformly in $\tau \in U_T(\alpha)$. Therefore, there exists $\tau \in U_T(\alpha)$ satisfying (3).

Moreover, we can provide a positive proportion of such solutions. Namely, if $M_T(\alpha, \sigma)$ is the set of those $\tau \in U_T(\alpha)$ for which the system of inequalities

$$\left| \zeta^{(k)}(\sigma + i\tau; \alpha) - a_k \right| < \left( 2 \frac{Q^2 + 1}{Q^2 - 1} \right)^{1/2} \varepsilon, \quad k = 0, \ldots, N,$$

is satisfied, then relations (64)–(68) yield that

$$m(M_T(\alpha, \sigma)) \geq \frac{1}{2} \delta_Q \left( 1 - Q^{-2} \right) T.$$ 

For if that was not true, we would have

$$\sum_{0 \leq k \leq N} \left| \zeta^{(k)}(\sigma + i\tau; \alpha) - \frac{\partial^k}{\partial s^k} \zeta_Q(s + i\tau, 0, \alpha) \right|_{s=\sigma}^2 \geq \frac{\varepsilon^2}{2} \frac{Q^2 + 1}{Q^2 - 1}$$

for every $\tau$ in the set of positive measure $U_T(\alpha) \setminus M_T(\alpha, \sigma)$. It would then follow from (64), (66) and (68) that

$$\int_T^{2T} \Lambda_Q \left( \gamma_Q(\tau, \alpha) - \theta_0 \right) \sum_{0 \leq k \leq N} \left| \zeta^{(k)}(\sigma + i\tau; \alpha) - \frac{\partial^k}{\partial s^k} \zeta_Q(s + i\tau, 0, \alpha) \right|_{s=\sigma}^2 d\tau \geq \frac{\varepsilon^2}{2} \frac{Q^2 + 1}{Q^2 - 1} \int_{U_T(\alpha) \setminus M_T(\alpha, \sigma)} \Lambda_Q \left( \gamma_Q(\tau, \alpha) - \theta_0 \right) d\tau$$

$$\geq \frac{\varepsilon^2}{2} \frac{Q^2 + 1}{Q^2 - 1} \left[ \int_{U_T(\alpha)} \Lambda_Q \left( \gamma_Q(\tau, \alpha) - \theta_0 \right) d\tau - m(M_T(\alpha, \sigma)) \right]$$

$$\geq \frac{\varepsilon^2}{4} \delta_Q \left( 1 + Q^{-2} \right) T,$$

which contradicts (65). \qed
Proof of Theorem 2  Beginning with the Taylor series of $f$,  

$$f(s) = \sum_{k \geq 0} \frac{f^{(k)}(s_0)}{k!} (s - s_0)^k,$$

valid for $s \in K$, we observe, by Cauchy’s formula

$$f^{(k)}(s_0) = \frac{k!}{2\pi i} \int_{|s-s_0|=r} \frac{f(s)}{(s-s_0)^k} ds,$$

that $|f^{(k)}(s_0)| \leq k!M r^{-k}$, where $M := \max_{|s-s_0|=r} |f(s)|$. Fixing a number $\delta_0 \in (0, 1)$, we get

$$|f^{(k)}(s_0)| \leq k! M \delta_0^k$$

for $|s - s_0| \leq \delta_0 r$. If $\varepsilon \in (0, |f(s_0)|)$, we can find $N = N(\delta_0, \varepsilon, M)$ such that

$$\Sigma_1 := \left| f(s) - \sum_{0 \leq k \leq N} \frac{f^{(k)}(s_0)}{k!} (s - s_0)^k \right| < \varepsilon,$$

for $|s - s_0| \leq \delta_0 r$.

Now, let $\delta \in (0, \delta_0)$. Then, of course, the latter inequality holds in particular for $s$ satisfying $|s - s_0| \leq \delta r$. Now, we apply Theorem 1 with $a_k = f^{(k)}(s_0), k = 0, \ldots, N$. Then, for $\alpha \in A(Q, M) \cap [A, 1]$ of degree at most $d_0 - 2\theta/d_0^2 + \epsilon$, and $T$ satisfying relation (2), there exists $t_1 \in [T, 2T]$ such that

$$|\zeta^{(k)}(\sigma_0 + it_1; \alpha) - f^{(k)}(s_0)| < \varepsilon, \quad k = 0, \ldots, N.$$  

Thus,

$$\Sigma_2 := \left| \sum_{0 \leq k \leq N} \frac{\zeta^{(k)}(\sigma_0 + it_1; \alpha)}{k!} (s - s_0)^k - \sum_{0 \leq k \leq N} \frac{f^{(k)}(s_0)}{k!} (s - s_0)^k \right| < \varepsilon \left( \delta r \right)^k \frac{k!}{k!} < \varepsilon \exp(\delta r),$$

for $|s - s_0| \leq \delta r$. Now, write $\tau = t_1 - t_0$, then $1 + it_1 = s_0 + i\tau$.

Next, we use the Taylor expansion for $\zeta(s; \alpha)$ on the shifted disk $K + i\tau$. For this purpose, we need to exclude the simple pole at $s = 1$; hence, we also request $T > r$.  

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Under this assumption, we have
\[
\zeta(s + i \tau; \alpha) = \sum_{k \geq 0} \frac{\zeta^{(k)}(s_0 + i \tau; \alpha)}{k!} (s - s_0)^k
\]
for \(s \in \mathcal{K}\). Let \(M(\tau) := \max_{|s - s_0| = r} |\zeta(s + i \tau; \alpha)|\). Then, again by Cauchy’s formula,
\[
\left| \frac{\zeta^{(k)}(s_0 + i \tau; \alpha)}{k!} (s - s_0)^k \right| \leq M(\tau) \delta^k
\]
for \(|s - s_0| \leq \delta r\). Hence,
\[
\Sigma_3 := \left| \zeta(s + i \tau; \alpha) - \sum_{0 \leq k \leq N} \frac{\zeta^{(k)}(s_0 + i \tau; \alpha)}{k!} (s - s_0)^k \right|
\leq M(\tau) \frac{\delta^N}{1 - \delta},
\]
for \(|s - s_0| \leq \delta r\). In combination with the above estimates, this yields
\[
|\zeta(s + i \tau; \alpha) - f(s)| \leq \Sigma_1 + \Sigma_2 + \Sigma_3 < \varepsilon + \varepsilon \exp(\delta r) + M(\tau) \frac{\delta^N}{1 - \delta}
\]
for \(|s - s_0| \leq \delta r\). Choosing now \(\delta > 0\) such that relation (4) is satisfied concludes the proof of the theorem. \(\square\)

If we assume now that \(r < \sigma_0 - 1/2\), then the disc where we approximate the target function lies in the half-plane \(\sigma > 1/2\). To drop the dependence of \(\delta\) on \(T\), we need to show that there is \(X > 0\) such that for any sufficiently large \(T > 0\), there is \(t_1 \in [T, 2T]\) satisfying the system (69) and \(M(\tau) \leq X\), where \(\tau = t_1 - t_0\). Indeed, from Theorem 1 we know that there exists \(b > 0\) such that at least \(bT\) numbers \(t_1 \in [T, 2T]\) satisfy (69). On the other hand, since
\[
\int_T^{2T} Y |\zeta(s + i \tau; \alpha)|^2 d\tau \leq YT, \quad Y = Y(\sigma_0, r, w, \alpha, \varepsilon) > 0,
\]
uniformly on \(|s - s_0| \leq r + w\), for a sufficiently small \(w > 0\), we can derive that
\[
\int_T^{2T} M(\tau)^2 d\tau \leq \int_T^{2T} Z \int_{|s - s_0| \leq r + w} |\zeta(s + i \tau; \alpha)|^2 d\sigma d\tau \leq ZYT, \quad Z = Z(\sigma_0, r, w) > 0.
\]
The first inequality above is an elementary relation from the theory of Bergman spaces (or spaces of square integrable analytic functions). Now, Chebyshev’s inequality implies that
\[
\# \{ \tau \in [T, 2T] : M(\tau) \leq X \} > T - X^{-2}YZT
\]
for any \( X > 0 \). Fixing \( X \geq (YZ/b)^{1/2} \) will ensure the existence of a suitable \( t_1 \). Moreover, \( X \) is independent of \( T \) and if we substitute it in (4) instead of \( M(\tau) \), we will be able to obtain \( \delta \) independently of \( T \) as well.

5 Concluding Remarks

We begin with a historical note. Hurwitz [12] himself treated only Hurwitz zeta-functions with a rational parameter. In his investigations on Dirichlet’s analytic class number formula, he studied Dirichlet series of the form
\[
\sum_{\substack{n \equiv a \mod m}} n^{-s}
\]
which can be rewritten as \( m^{-s} \zeta(s; \alpha m) \). It appears that switching from a rational to an irrational parameter does not affect analytic continuation, functional identities and the order of growth; however, the zero-distribution definitely depends and the general distribution of values might depend on the diophantine nature of the parameter. Further generalizations of the Riemann zeta-function and Dirichlet \( L \)-functions were in Hurwitz’s time also studied by Lerch and Lipschitz. For details, we refer to the monograph [16] by Laurinčikas and Garunkštis on the Lerch zeta-function (which covers the case of Hurwitz’s zeta).

Our next remark shall classify the various proofs of universality properties for the Hurwitz zeta-function \( \zeta(s; \alpha) \) in the literature so far, namely the results of Bagchi [2] and Gonek [8]. For rational \( \alpha \), there is a representation of \( \zeta(s; \alpha) \) in terms of Dirichlet \( L \)-functions with pairwise inequivalent characters which allows to apply a joint universality theorem for those due to Voronin [21] in order to deduce the desired approximation property; for \( \alpha \neq \frac{1}{2}, 1 \), the target function may even vanish (whereas for \( \alpha = \frac{1}{2} \) or 1 the Hurwitz zeta-function is essentially equal to a Dirichlet \( L \)-function and Riemann’s \( \zeta \), respectively, and too many zeros off the critical line would contradict classical density theorems). If the parameter \( \alpha \) is transcendental, one can mimic Voronin’s proof using the linear independence of the numbers \( \log(n + \alpha) \) for non-negative integers \( n \) (in place of the logarithms of the prime numbers in case of \( \zeta \)). In all these results, the approximating shifts form a set of positive lower density (as in the universality theorem for the Riemann zeta-function).

Improving the constant \( \eta \) in Lemma 7 would immediately increase the range in which we can prove universality. Since we pursued effective results, Lemma 7 is the best to our knowledge. However, if we drop effectiveness, then we can use the far superior result of Heath-Brown [11, Theorem 5] where \( \eta = 8\sqrt{5}/63 = 0.4918... \) is admissible.
Although our results are far from being satisfactory in comparison with the case of transcendental parameter, they shed light to another major topic in universality theory, that of effective lower bounds for the lower density of the set of approximating shifts as well as estimating explicitly $T$ such that an approximating shift $\tau$ lies in $[T, 2T]$. It is evident from the proof of Theorem 4 that if we have an estimate of the form

$$|P(\alpha)| \geq DH^C,$$  \hspace{1cm} (70)

for transcendental $\alpha$ and integral polynomial $P$, where $D$ and $C$ may depend on the degree of $P$ but not in its height $H$, then we can obtain the same lower bound for

$$\liminf_{T \to \infty} \frac{1}{T} M_T(\alpha, \sigma)$$

as in Theorem 1, whenever $\sigma$ is sufficiently close to 1. If in addition $D$ and $C$ are effectively computable, then we can estimate explicitly a $T$ such that $\tau \in [T, 2T]$. In direction of (70), we refer to the monograph of Bugeaud [4], where the classification of transcendental numbers into $S$-, $T$- and $U$-numbers is given in detail.

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