Leptonic mixing matrix in terms of Cabibbo angle.

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Abstract

We phenomenologically build a neutrino mass matrix which obeys the $\mu - \tau$ symmetry with only two free parameters, Cabibbo angle ($\lambda$) and a flavour twister parameter ($\eta$). For vanishing $\eta$, the model assumes TBM mixing. When $\eta = \lambda$, the model generates a solar angle and solar mass squared difference that coincide with the experimental best-fits. It motivates us to propose a mixing matrix in the neutrino sector, with $\theta_{12} < \sin^{-1}(1/\sqrt{3})$, $\theta_{13} = 0$ and $\theta_{23} = \pi/4$. The corrections in $\theta_{13}$ and $\theta_{23}$ are conducted following the breaking of $\mu - \tau$ symmetry, by choosing a proper unitary diagonalizing charged lepton matrix, and this ensures the fact that $\theta_{23}$ lies within the first octant. The Cabibbo angle ($\lambda = \sin \theta_c$) plays the role of a guiding parameter in both neutrino as well as leptonic sector. The effect of the Dirac CP violating phase $\delta_{cp}$ is also studied when it enters either through the charged leptonic mixing matrix or through neutrino mixing matrix.

1 Introduction

Tri-bimaximal mixing (TBM) \[1\] is one of the most popular kinds of mixing pattern. TBM is experienced in different flavour symmetry groups like $A_4$ \[2,11\], $S_4$ \[12,15\] and $\Delta(54)$ \[16,18\]. But recent experimental data questions the validity of TBM mixing as the first approximation.

To understand the unification of quarks and leptons is one of biggest goals in particle physics. In the “bottom-up” approach some phenomenological relations between quark and lepton mixing parameters are sought out. Starting from a simple parametrization of neutrino mass matrix following $\mu - \tau$ symmetry, we derive a new mixing scheme in

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the neutrino sector as

$$U_\nu(\lambda) = \begin{pmatrix} \sqrt{\frac{4+\lambda}{6+\lambda}} & 0 & \sqrt{\frac{2}{6+\lambda}} \sqrt{\frac{2}{2(6+\lambda)}} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6+\lambda}} & \sqrt{\frac{4+\lambda}{2(6+\lambda)}} \frac{1}{\sqrt{2}} & \sqrt{\frac{4+\lambda}{2(6+\lambda)}} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6+\lambda}} & \sqrt{\frac{2}{6+\lambda}} \sqrt{\frac{2}{2(6+\lambda)}} \frac{1}{\sqrt{2}} & \sqrt{\frac{2}{6+\lambda}} \sqrt{\frac{2}{2(6+\lambda)}} \frac{1}{\sqrt{2}} \end{pmatrix},$$  \hspace{1cm} (1)$$

where, \(\lambda\) is the Wolfenstein parameter \[19\]; \(\lambda \sim 0.2253 \pm 0.0007\) \[20\]. \(U_\nu(\lambda)\) predicts the atmospheric mixing angle \(\theta_{23}\) as maximal and reactor angle (\(\theta_{13}\)) as zero. But in contrast to TBM mixing, \(U_\nu(\lambda)\) predicts solar angle (\(\theta_{12}\)) rather with lesser value than \(\sin^{-1}(1/\sqrt{3})\),

$$\theta_{12} = \sin^{-1}\left(\sqrt{\frac{2}{6+\lambda}}\right).$$  \hspace{1cm} (2)$$

For vanishing \(\lambda\), \(U_\nu(\lambda)\) converges to general TBM mixing.

The present experimental data neither supports vanishing \(\theta_{13}\) \[21–25\] nor maximal \(\theta_{23}\). The mixing angle, \(\theta_{23}\) is expected to lie within the first octant \[26, 27\]. \(U_\nu\) is formulated in the basis where charged lepton mass matrix is diagonal and the charged lepton mixing matrix \(U_{eL}\) is an identity matrix. Under this assumption \(U_{PMNS}\) \[28\] is equated to \(U_\nu\). In order to evade the inadequacy appearing in atmospheric and reactor angle, we try to construct a charged lepton diagonalizing matrix \(U_{eL}\). We have the standard relation, \(U_{PMNS} = U_{eL}^\dagger U_\nu = U_{eL}^\dagger U_\nu^\dagger U_{TB}\). The charged lepton mixing matrix \(U_{eL}\) has to be a unitary matrix rather than an identity matrix. Following the work done by King \[29\], we try to formulate a diagonalizing charged lepton mixing matrix, \(U_{eL}\).

In this approach the Cabibbo angle \((\lambda \sim \sin \theta_c)\) seeds the whole parametrization in neutrino as well as charged lepton sector. Finally we shall construct the \(PMNS\) matrix, \(U_{PMNS} = U_{eL}^\dagger U_\nu\) and look into the mixing angles. We shall also try to investigate the same by introducing Dirac CP phase, \(\delta_D\) in the \(PMNS\) matrix. The CP phase can be introduced either from charged lepton or neutrino sector.

In the process of parametrization of the \(\mu - \tau\) symmetric mass matrix, we start with certain ansatz that the mass squared differences for both solar as well as atmospheric, can be expressed in terms of the Wolfenstein parameter \(\lambda\). This ansatz is phenomenological and is the outcome of the analysis of global neutrino data \[30\]. Thus we have

$$\Delta m^2_{sol} = \lambda^4 eV^2, \quad \Delta m^2_{atm} \sim 9\lambda^8 eV^2.$$

\(2\) Parametrization of \(\mu - \tau\) symmetric neutrino mass matrix

In ref. \[31\] the author proposed an exact symmetry of the neutrino mass matrix under the interchange of second and third generation of neutrinos. This symmetry is often called called \(\mu - \tau\) or \(2 - 3\) symmetry. Under this symmetry, matrix element of the neutrino mass matrix is invariant under the interchange of flavor basis vectors, \(|e\rangle \leftrightarrow \)
The appearance of minus sign ensures the positive mixing angles. With this symmetry we get $m_{e\mu} = -m_{e\tau}$ and $m_{\mu\mu} = m_{\tau\tau}$.

This symmetry is characterised by some significant features like predictions of maximal atmospheric, vanishing reactor and arbitrary solar mixing angles. The solar angle can be controlled with the proper choice of parameters present in the neutrino mass matrix,

$$M_{\mu\tau} = \begin{pmatrix} a & b & -b \\ b & c & d \\ -b & d & c \end{pmatrix} m_0, \quad (4)$$

$$\tan 2\theta_{12} = \frac{2\sqrt{2}b}{a - c + d}. \quad (5)$$

If the neutrinos follow normal hierarchy (NH) mass pattern, then out of the three absolute neutrino masses, one can be approximated to zero. We start with certain $\mu - \tau$ symmetric mass matrix that can give one mass eigenvalues as zero. We choose,

$$M_0 = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} m_0. \quad (6)$$

The mass eigenvalues are $m_{\nu 1} = 0$, $m_{\nu 2} = \xi m_0$ and $m_{\nu 3} = m_0$. But unfortunately $M_0$ predicts vanishing $\theta_{12}$. $M_0$ is mended to $M_{TB}$,

$$M_{TB} = \begin{pmatrix} \xi & \xi & -\xi \\ \frac{\xi}{2} + \xi & \frac{1}{2} - \xi \\ -\xi & \frac{1}{2} + \xi \end{pmatrix} m_0, \quad (7)$$

with mass eigenvalues as $m_{\nu 1} = 0$, $m_{\nu 2} = 3\xi m_0$ and $m_{\nu 3} = m_0$. $M_{TB}$ predicts $\theta_{12}$ as $\sin^{-1}(1/\sqrt{3})$ which is in accordance with TBM mixing. In order to account for the deviation from TBM mixing, we introduce flavour twister $\eta$ \cite{32, 33}. With $\eta$, we parametrize $M_{\mu\tau}(\xi, \eta, m_0),$

$$M_{\mu\tau} = \begin{pmatrix} \xi & \xi(1 + \frac{\eta}{4})^{1/2} & -\xi(1 + \frac{\eta}{4})^{1/2} \\ \xi(1 + \frac{\eta}{4})^{1/2} & \frac{1}{2}(1 + \xi\eta) + \xi & \frac{1}{2}(1 + \xi\eta) - \xi \\ -\xi(1 + \frac{\eta}{4})^{1/2} & \frac{1}{2}(1 - \xi\eta) - \xi & \frac{1}{2}(1 + \xi\eta) + \xi \end{pmatrix} m_0. \quad (8)$$

The above mass matrix predicts

$$\tan 2\theta_{12} = 2\sqrt{2} \frac{(4 + \eta)^{1/2}}{2 + \eta}. \quad (9)$$

We obtain the mass eigenvalues,

$$m_{\nu 1} = 0, \quad m_{\nu 2} = \frac{\xi}{2}(6 + \eta)m_0 \quad \text{and} \quad m_{\nu 3} = m_0. \quad (10)$$

So far what we have done from eq.(6) up to eq.(9), is simple algebraic rearrangement of the parameters. One significant outcome of this arrangement is that $\tan 2\theta_{12}$ is freed from other two parameters and is expressible in terms of $\eta$ only. But this parametrization will turn unnatural if we assume the parameters $\xi$, $\eta$ and $m_0$ as independent.
Figure 1: $m_{\nu^2}$ is assumed as $m_{\nu^3}(\lambda, \eta)$ with $3\lambda^4$ as the leading order. $\eta$ appears in the correction: $m_{\nu^2} = 3\lambda^4 + (a_0 + a_1\eta)\lambda^5$. The numerical values of $(a_0, a_1)$ is to be derived graphically. $\Delta m_{21}^2$ is plotted vs $\eta$, with different pairs of $(a_0, a_1)$. With respect to the 1σ range of $\eta$ derived with respect to 1σ range of solar angle data, this is preferable to choose the numerical value of $a_1$ as 1 and $a_0$ in the range $1 < a_0 < 2$.

ones. In principle, nature will assign certain numbers to the observational parameters. Hence such freedom is not expected. We then try to incorporate certain constraints in order to reduce the number of free parameters in $M_{\mu\tau}$ and also seek certain dominant parameter present in the same.

The two ansatz we made in eq.(3), can be recast as

$$m_{\nu^3} = \lambda^2 \text{eV},$$
$$m_{\nu^2} \sim 3\lambda^4 \text{eV} = 3\lambda^4 + \alpha \lambda^5 \text{eV},$$

(11)

where $\alpha$ is a correction parameter. With respect to 1σ range of $\Delta m_{21}^2$, we assign $\alpha$ a range $[1.533, 1.908]$. From eq.(10) and eq.(11), we can directly infer: $m_0 = \lambda^2$ and also the parameter $\xi$ depends on $\lambda$ and $\alpha$. We assume that $\alpha$ is not an independent parameter and it bears some correlation with $\eta$. We shall check the relevance of this assumption in the light of 1σ ranges of $\Delta m_{21}^2$ and $\eta$. 1σ range of solar angle confines $\eta$ within $[-0.0476, 0.606]$. We assume $\alpha$ to have a linear correlation with $\eta$: say $\alpha = a_0 + a_1\eta$. With different pairs of $(a_0, a_1)$ we plot $\Delta m_{21}^2$ vs. $\eta$ in fig.1. In the perspective of 1σ range, we find the suitable value of $a_1$ as 1 and that for $a_0$, $1 < a_0 < 2$. We choose the average value of $a_0$ as $3/2$. Thus eq.(11) becomes

$$m_{\nu^2} = 3\lambda^4 + (\frac{3}{2} + \eta)\lambda^5 \text{eV}.$$  
(12)

Following eq.(10), the above analysis leads to

$$\xi = \xi(\lambda, \eta) = \frac{6 + (\frac{3}{2} + \eta)\lambda}{(6 + \eta)}.$$  
(13)

Finally $M_{\mu\tau}$ in eq.(8), is expressible with only two free parameters $\lambda$ and $\eta$ where $\lambda = 0.2253 \pm 0.0007$ and $\eta$ is the flavour twister term which dominates in the expression of $M_{\mu\tau}(\lambda, \eta)$.

This is interesting to note that at $\eta = \lambda$, we obtain from eq.(9),

$$\sin^2 \theta_{12} = \frac{2}{6 + \lambda} \approx 0.321 \text{ (best-fit).}$$  
(14)

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The solar angle is solely dictated by the flavour twister \( \eta \). At \( \eta = \lambda \), \( \sin^2 \theta_{12} \) is found to be \( \sim 0.320 \) (best-fit). \( \sin^2 \theta_{12} \) coincides with TBM prediction at \( \eta = 0 \).

Choosing \( \lambda = 0.2253 \), \( \Delta m^2_{21} \) is plotted against \( \eta \). It is shown that at \( \eta = \lambda \), \( \Delta m^2_{21} \sim 7.62 \times 10^{-5} \text{eV}^2 \) (best fit) is obtained.

Again at \( \eta = \lambda \), we find

\[
\Delta m^2_{\text{sol}} = m_{\nu_2}^2 - m_{\nu_1}^2 \approx 7.6 \times 10^{-5} \text{eV}^2 \text{ (best-fit)}. \tag{15}
\]

TBM mixing is obtained at \( \eta = 0 \), and the corresponding mass parameter, \( \Delta m^2_{\text{sol}} = m_{\nu_2}^2 - m_{\nu_1}^2 \approx 7.4 \times 10^{-5} \text{eV}^2 \) which is one of the boundaries of 1\( \sigma \) range. The simultaneous close agreement of the solar angle and solar mass squared difference to the experimental best-fit at a single point thus hoists the preference for \( \eta = \lambda \). \( M_{\mu \tau} \) in eq. (3) with \( \eta = \lambda \) and \( \xi = \xi(\lambda) \) (eq.(13)) generates the diagonalizing matrix \( U_\nu(\lambda) \) in the exact form as shown in eq.(1),

\[
U_\nu(\lambda) = \begin{pmatrix} \frac{\sqrt{4+\lambda}}{\sqrt{6+\lambda}} & \frac{\sqrt{2}}{\sqrt{6+\lambda}} & 0 \\ -\frac{1}{\sqrt{6+\lambda}} & -\frac{\sqrt{2+\lambda}}{\sqrt{2(6+\lambda)}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6+\lambda}} & -\frac{\sqrt{2+\lambda}}{\sqrt{2(6+\lambda)}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{16}
\]
We approximate $U_\nu(\lambda)$ as

$$U_\nu(\lambda) \approx \begin{pmatrix} \sqrt{\frac{2}{3}} + \frac{\lambda}{12 \sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{\lambda}{12 \sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} - \frac{\lambda}{12 \sqrt{6}} & \frac{1}{\sqrt{3}} + \frac{\lambda}{24 \sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{\lambda}{12 \sqrt{6}} & -\frac{1}{\sqrt{3}} - \frac{\lambda}{24 \sqrt{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \quad (17)$$

We see that for vanishing $\lambda$, $U_\nu(\lambda)$ converges to TBM mixing matrix, $U_{TB}$.

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \sqrt{\frac{1}{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \quad (18)$$

This transition from $U_{TB}$ to $U_\nu(\lambda)$ can be interpreted in terms of an unitary matrix, $\tilde{U} = U_{TB} U^\dagger_\nu$ and is displayed below,

$$\tilde{U} = \begin{pmatrix} 1 & \lambda/24 & -\lambda/24 \\ -\lambda/24 & 1 & 0 \\ \lambda/24 & 0 & 1 \end{pmatrix}. \quad (19)$$

The unitary condition of $\tilde{U}$ is given as, $\tilde{U}^\dagger \tilde{U} = I + O(10^{-5})$. Up to this stage we are working on the the basis where charged lepton mass matrix is diagonal i.e., $U_{eL} = I$.

$M_{\mu\tau}(\lambda)$ which carries the information of absolute neutrino masses and mixing angles can be recast as

$$M_{\mu\tau}(\lambda) = \begin{pmatrix} \lambda(1 - \frac{\lambda}{24}) & \frac{\lambda}{2}(1 - \frac{\lambda}{24}) + \lambda & \frac{\lambda}{2}(1 - \frac{\lambda}{24}) - \lambda \\ \frac{\lambda}{2}(1 - \frac{\lambda}{24}) + \lambda & \frac{\lambda}{2}(1 - \frac{\lambda}{24}) - \lambda & \frac{\lambda}{2}(1 + \frac{5\lambda}{24}) + \lambda \\ \frac{\lambda}{2}(1 - \frac{\lambda}{24}) - \lambda & \frac{\lambda}{2}(1 + \frac{5\lambda}{24}) + \lambda & \lambda^2 + O(\lambda^5). \end{pmatrix} \quad (20)$$

In short, $M_{\mu\tau}$ predicts the solar angle, $\theta_{12} = 34.51^0$ (deviated from TBM prediction) atmospheric angle, $\theta_{23} = 45^0$ and reactor angle, $\theta_{13} = 0^0$. In order to account for the required deviations in these two mixing angles without disturbing the first, we try to correct $M_{\mu\tau}(\lambda)$ with a proper choice of a unitary charged lepton diagonalizing matrix, $U_{eL}$. The corrected neutrino mass matrix is $M_\nu = U_{eL}^\dagger M_{\mu\tau} U_{eL}$ is expected to give the complete picture of the mixing angles.

### 3 Charged lepton mixing matrix

In ref [29], a new idea of mixing was proposed by King. This was based on the relationship [35–38] consistent with the recent data.

$$\sin \theta_{13} = \frac{\sin \theta_c}{\sqrt{2}} = \frac{\lambda}{\sqrt{2}.} \quad (21)$$

The above ansatz implies $\theta_{13} \approx 9.2^0$. The author combined the above relation with TBM mixing leading to the proposal of the Tri-bimaximal-Cabibbo mixing:

$$\theta_{13} = \sin^{-1}(\lambda/\sqrt{2}), \quad \theta_{12} = \sin^{-1}(1/\sqrt{3}), \quad \theta_{23} = \pi/4. \quad (22)$$
With $\delta_{CP} = 0$, the mixing matrix assumes the following texture.

$$U_{TBC} = \begin{pmatrix}
\sqrt{\frac{2}{3}}(1 - \frac{1}{4}\lambda^2) & \frac{1}{\sqrt{3}}(1 - \frac{1}{4}\lambda^2) & \frac{1}{\sqrt{2}}\lambda \\
\frac{1}{\sqrt{6}}(1 + \lambda) & \frac{1}{\sqrt{3}}(1 - \frac{1}{4}\lambda) & \frac{1}{\sqrt{2}}(1 - \frac{1}{4}\lambda^2) \\
\frac{1}{\sqrt{6}}(1 - \lambda) & -\frac{1}{\sqrt{3}}(1 + \frac{2}{3}\lambda) & \frac{1}{\sqrt{2}}(1 - \frac{1}{4}\lambda^2).
\end{pmatrix} + O(\lambda^3).$$  \hspace{1cm} (23)

The above mixing scheme has its own limitations as it fails to lower the solar and atmospheric mixing angles from TBM predictions. Also it ignores the preference for $\theta_{23}$ to lie within the first octant. The significant feature of this mixing scheme lies in the prediction of a non-zero reactor angle.

We can think $U_{TBC}$ as one of the corrected version of $U_{TB}$. We hope that certain choice of the charged lepton mixing matrix, say $U^k_{eL}$ is responsible for the above texture. $U^{k*}_{eL} = U^{k*}_{TB}U_{TBC}$ where,

$$U^k_{eL} = \begin{pmatrix}
1 - \frac{1}{4}\lambda^2 & \frac{1}{2}\lambda & \frac{1}{8}\lambda^2 \\
\frac{1}{2}\lambda & 1 - \frac{1}{4}\lambda^2 & -\frac{1}{8}\lambda^2 \\
\frac{1}{2}\lambda & -\frac{1}{8}\lambda^2 & 1 - \frac{1}{4}\lambda^2.
\end{pmatrix}.$$

The charged lepton mixing matrix satisfies the unitary condition, $U_{eL}^{k*}U_{eL}^k = U_{eL}^{k*}U_{eL}^k = I + (O)(\lambda^4)$. We mend $U^{k*}_{eL}$ on associating it with $R_{23}(\theta) \hspace{1cm} (24)$ and construct the new charged lepton mixing matrix $U_{eL}$. Where, $U_{eL} = U^{k*}_{eL}R_{23}(\theta)$ and $R_{23}(\theta)$ is the rotational matrix in 2 - 3 plane.

$$R_{23} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - \frac{1}{2}\theta^2 & \theta \\
0 & -\theta & 1 - \frac{1}{2}\theta^2.
\end{pmatrix} \hspace{1cm} (25)$$

With $\theta = \lambda/3$, we obtain, the required diagonalizing charged lepton mixing matrix

$$U_{eL} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & -\frac{1}{2}\lambda\{1 - \frac{1}{6}(1 + \frac{2}{3})\} & -\frac{1}{2}\lambda\{1 + \frac{1}{6}(1 - \frac{2}{3})\} \\
\frac{1}{2}\lambda & 1 - \frac{\lambda^2}{2}(\frac{13}{24} - \lambda) & \frac{1}{2}\lambda - \frac{\lambda^2}{2}(1 + \frac{2}{3}) \\
\frac{1}{2}\lambda & -\frac{1}{2}\lambda - \frac{\lambda^2}{2}(1 - \frac{2}{3}) & 1 - \frac{\lambda^2}{24}(\frac{13}{3} + \lambda).
\end{pmatrix} + O(\lambda^4) \hspace{1cm} (26)$$

Upon incorporating $U_{eL}$ with the TBM mixing matrix, $U_{TB}$, $U' = U_{eL}^{\dagger}U_{TB}$.

$$U' = \begin{pmatrix}
\sqrt{\frac{7}{3}}(1 - \frac{\lambda^2}{2}) & \frac{1}{\sqrt{3}}(1 - \frac{\lambda^2}{2}) & \frac{1}{\sqrt{2}3}(1 - \frac{\lambda^2}{2}) \\
\frac{1}{\sqrt{6}}\{(1 + \frac{1}{3}\lambda) - \frac{7\lambda^2}{18}(1 + \frac{2}{3})\} & \frac{1}{\sqrt{3}}\{(1 - \frac{1}{3}\lambda) + \frac{7\lambda^2}{18}(1 + \frac{2}{3})\} & \frac{1}{\sqrt{2}3}(1 - \frac{1}{3}\lambda) - \Lambda_+ \\
\frac{1}{\sqrt{6}}\{(1 - \frac{1}{3}\lambda) - \frac{7\lambda^2}{18}(1 - \frac{2}{3})\} & \frac{1}{\sqrt{3}}\{(1 + \frac{1}{3}\lambda) + \frac{7\lambda^2}{18}(1 - \frac{2}{3})\} & \frac{1}{\sqrt{2}3}(1 + \frac{1}{3}\lambda) - \Lambda_+.
\end{pmatrix},$$

where, $\Lambda = \frac{11}{36}\sqrt{2}\lambda^2(1 \pm \frac{4}{13}\lambda)$.

Similar to $U_{TBC}$ in eq. \hspace{1cm} (24), $U'$ predicts same $\theta_{13}$ and $\theta_{12}$. But $U'$ generates a $\theta_{23}$ with lesser value than the maximal and of course $\theta_{23}$ lies within the first octant.

$$\sin^2\theta_{23} \approx \frac{1}{2} - \frac{\lambda}{3}(1 - \frac{\lambda^2}{18}) \approx 0.425.$$  \hspace{1cm} (28)

This result is very close to $\sin^2\theta_{23} = 0.427$ (best-fit).
4 Perturbing the $\mu - \tau$ symmetry of neutrino mass matrix

This is clear from the discussions in section (1) and section (2) that the solar angle can be suppressed within the neutrino sector whereas the lepton sector can control the atmospheric as well as the reactor angles. The corrected neutrino mass matrix with broken $\mu - \tau$ symmetry is,

$$M_\nu = U_{eL}^\dagger M_{\mu\tau} U_{eL} = M_\nu^{\mu\tau} + \Delta M_\nu,$$

(29)

where $\Delta M_\nu$ estimates the amount of perturbation to break the $\mu - \tau$ symmetry for producing the necessary deviations in the mixing angles. The diagonalizing matrix of $M_\nu(\lambda)$ has the following texture,

$$U_{PMNS} = U_{PMNS}(\lambda),$$

(30)

where, $\lambda_0 = \lambda^3/(108\sqrt{3})$.

The mixing angles are given by the three relations,

$$\sin^2 \theta_{13} = |U_{e3}|^2,$$
$$\sin^2 \theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2},$$
$$\sin^2 \theta_{23} = \frac{|U_{\mu3}|^2}{1 - |U_{e3}|^2}.\quad(31)$$

We obtain,

$$\sin^2 \theta_{13} = \frac{\lambda^2}{2},$$
$$\sin^2 \theta_{12} \approx \frac{1}{3} - \frac{\lambda}{18}(1 - \frac{\lambda}{24}),$$
$$\sin^2 \theta_{23} \approx \frac{1}{2} - \frac{\lambda}{3}(1 - \frac{\lambda^2}{18}).\quad(34)$$

The corresponding numerical values (with $\lambda = 0.2253$) of the mixing angle parameters are 0.0253 ($\sin^2 \theta_{13}$), 0.321 ($\sin^2 \theta_{12}$) and 0.425 ($\sin^2 \theta_{23}$).

5 The effect of Dirac CP phase, $\delta_{cp}$

The Dirac CP phase can enter $U_{PMNS}$, either through the charged lepton mixing matrix, $U_{eL}$ (lepton sector) or through $U_{\nu}$ (neutrino sector). $U_{\nu}$ comprises $\bar{U}$ and $U_{TB}$. Hence there are two possibilities by which $\delta_{cp}$ can enter the neutrino sector i.e., either through any of the two matrices.
5.1 $\delta_{cP}$ from $U_{eL}$ (leptonic sector)

We choose the texture of $U_{eL}$ with $\delta_{cP}$ in such a way that unitary condition is not violated,

$$
U_{eL}(\lambda, \delta_{cP}) = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & -\frac{1}{2} \lambda e^{-i\delta_{cP}} \{1 - \frac{1}{3} (1 + \frac{\lambda}{3})\} & -\frac{1}{2} \lambda e^{-i\delta_{cP}} \{1 + \frac{1}{3} (1 - \frac{2}{3})\} \\
\frac{1}{2} \lambda e^{i\delta_{cP}} & 1 - \frac{\lambda^2}{24} (1 - \frac{7}{3} - \lambda) & \frac{1}{3} \lambda - \frac{\lambda^2}{24} \{1 + \frac{2}{3}\} \\
-\frac{1}{3} \lambda - \frac{\lambda^2}{24} (1 - \frac{4}{3}) & 1 - \frac{\lambda^2}{24} (\frac{13}{3} + \lambda) \\
\end{pmatrix} + O(\lambda^4).
$$

(35)

The elements present in the $U_{PMNS} = U_{eL}^\dagger U_{\nu}$, except $U_{e1}$, $U_{e2}$, $U_{\mu2}$ and $U_{\tau3}$ are affected by $\delta_{cP}$. But the predictions of the mixing angles are not affected by the presence of CP phase and these coincide with the predictions in eqs.(32,33,34).

5.2 $\delta_{cP}$ from $U_{\nu}$ through $U_{TB}$ (neutrino sector)

We choose the TBM mixing matrix, $U_{TB}$,

$$
U_{TB}(\lambda, \delta_{cP}) = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \frac{1 + \lambda e^{-i\delta_{cP}}}{\sqrt{3}} & 0 \\
\frac{1 - \lambda e^{-i\delta_{cP}}}{\sqrt{3}} & 0 & 1 \\
0 & \sqrt{\frac{2}{3}} & 0 \\
\end{pmatrix}.R_{23}(\pi/4)
$$

(36)

and obtain $U_{\nu}(\lambda, \delta_{cP})$,

$$
U_{\nu}(\lambda, \delta_{cP}) = \begin{pmatrix}
\sqrt{\frac{2}{3}} + \frac{\lambda e^{i\delta_{cP}}}{12\sqrt{6}} & -\frac{\lambda}{12\sqrt{3}} + \frac{e^{-i\delta_{cP}}}{\sqrt{3}} & 0 \\
\frac{\lambda}{12\sqrt{6}} + \frac{e^{i\delta_{cP}}}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{\lambda e^{-i\delta_{cP}}}{24\sqrt{3}} \\
\frac{1}{\sqrt{3}} + \frac{e^{i\delta_{cP}}}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\end{pmatrix}
$$

(37)

Upon investigating the elements in PMNS matrix, $U_{PMNS} = U_{eL}^\dagger (\lambda).U_{\nu}$, we find only $|U_{e2}|$ to be affected by $\delta_{cP}$. The prediction of $\sin^2 \theta_{12}$ is affected while the rest preserves the same expression given in eqs.(32,33,34),

$$
\sin^2 \theta_{12} = \frac{1}{3} - \frac{\lambda}{18} (\cos \delta_{cP} - \frac{\lambda}{24}).
$$

(38)

5.3 $\delta_{cP}$ from $U_{\nu}$ through $\tilde{U}$

We choose,

$$
\tilde{U}(\lambda, \delta_{cP}) = \begin{pmatrix}
\frac{1}{27} \lambda e^{i\delta_{cP}} & \frac{1}{27} \lambda e^{-i\delta_{cP}} & 0 \\
\frac{1}{27} \lambda e^{i\delta_{cP}} & \frac{1}{27} \lambda e^{-i\delta_{cP}} & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

(39)
On introducing the Dirac CP phase $\delta_{cp}$ in the neutrino mixing matrix $U_\nu$ through $U_eL$, the prediction of $\sin^2 \theta_{12}$ is affected. In order to achieve the best-fit of solar angle, $\delta_{cp}$ must go to zero. TBM mixing is recovered at $\delta_{cp} = \pi/2$. Also it can be noted nonzero values of $\delta_{cp}$ is possible in the neighbourhood (right side only) of the best fit (0.32) of $\sin^2 \theta_{12}$.

The $U_\nu(\lambda)$ becomes,

$$U_\nu(\lambda, \delta_{cp}) \approx \begin{pmatrix} \sqrt{\frac{2}{3}} + \frac{\lambda e^{i\delta_{cp}}}{\sqrt{6}} & \frac{1}{\sqrt{3}} - \frac{\lambda e^{i\delta_{cp}}}{2\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} + \frac{\lambda e^{i\delta_{cp}}}{\sqrt{12}} & \frac{1}{\sqrt{3}} + \frac{\lambda e^{i\delta_{cp}}}{2\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} - \frac{\lambda e^{i\delta_{cp}}}{\sqrt{12}} & -\frac{1}{\sqrt{3}} - \frac{\lambda e^{i\delta_{cp}}}{2\sqrt{3}} & 0 \end{pmatrix}$$ (40)

Similarly, on investigating the PMNS matrix $U_{PMNS}$ on same footing as earlier, the effect of $\delta_{cp}$ is realized only in the solar angle prediction; other angles remain same as before[eqs.(32,34)],

$$\sin^2 \theta_{12} = \frac{1}{3} - \frac{\lambda}{18} (\cos \delta_{cp} - \frac{\lambda}{24}).$$ (41)

From the above analysis this is clear that $\delta_{cp}$ remains arbitrary if it resides in lepton sector. But the scenario is different if Dirac CP phase enters through neutrino sector in the sense that prediction of solar angle is affected. Comparing eqs.(33,38,41) it can be inferred that $\delta_{cp}$ must go to zero if we want $\sin^2 \theta_{12} = 0.321$. But interestingly, nonzero $\delta_{cp}$ with its value in the range $0 < \delta_{cp} < \pi/2$ are possible for $\sin^2 \theta_{12}$ in the range $0.32 < \sin^2 \theta_{12} < 0.33$ [Fig.5.3].

6 Discussion

Corrections to TBM mixing has been implemented from both charged leptonic sector as well as from neutrino sector. This is different from the works where corrections are conducted either from anyone of the two [35][44]. Our approach finds some similarity with the ref. [45]. But the correction matrix used by the author is different from what we derived in the eq.(19).

We derived the diagonalizing matrix $U_\nu$, in eq.(16) starting from parametrization of a $\mu-\tau$ symmetric neutrino mass matrix following NH mass pattern, $M_{\mu\tau}$ along with two
hypothesis that mass-squared differences can be expressed in terms of Wolfenstein parameter $\lambda \sim 0.2253$. In section 1, we have generated the neutrino mixing matrix $U_\nu(\lambda)$ starting from a $\mu - \tau$ symmetric neutrino mass matrix, giving rise to mass eigenvalues with normal spectrum. One question arises why out of the two mass hierarchies, we have preferred one than the others? In fact there is no strong reason behind that. But in ref. [31], the author showed that the $2 - 3$ symmetry present in the texture of the neutrino mass matrix in eq. (8) is possible only with NH hierarchy of the neutrino mass spectrum. Also in ref. [46], the analysis done by the author, based on ongoing CMB observations, including B mode polarization hints for NH pattern. The model within the $\mu - \tau$ symmetry regime, is expressible with two important parameters $\lambda$ and $\eta$ (flavour-twister). The model shows a strong preference towards the choice $\eta = \lambda$ which leads to the close agreement of solar mass squared difference, $\Delta m^2_{21}(\eta, \lambda)$ and solar mixing angle, $\theta_{12}(\eta)$ to the best-fits.

The charged lepton diagonalizing matrix, $U_{eL}$ we constructed in eq.(26), does not affect the solar angle further, lowers the atmospheric and produces nonzero reactor angles, when combined with $U_\nu$ in order to give $U_{PMNS}$. The results are summarised as : $\sin^2 \theta_{12} = 0.321$, $\sin^2 \theta_{23} = 0.425$ and $\sin^2 \theta_{13} = 0.025$. Our analyses shows that if Dirac CP phase resides in the leptonic sector, does not affect the predictions of mixing angles. On the contrary, if it emerges out of the neutrino sector, it affects only the prediction of $\theta_{12}$. Nonzero values of $\delta_{CP}$ within the range $0 - \pi/2$ are possible, and this corresponds to $\sin^2 \theta_{12}$ ranging from 0.321 - 0.333. This is to be emphasised that $\sin^2 \theta_{12} = 0.321$ is possible only when $\delta_{CP} = 0$.

With $\theta^\text{CKM}_{12} \approx 13.0^0$ and $\theta^\text{CKM}_{23} \approx 23.5^0$ [20], we try to investigate the quark-lepton complementarity (QLC) and selfcomplementarity (SLC) [47–50]. We take $\theta^\text{PMNS}_{12} = 34.51^0$, $\theta^\text{PMNS}_{13} = 9.2^0$ and $\theta^\text{PMNS}_{23} = 40.69^0$ as per the prediction of the model and find

\begin{align*}
\theta^\text{CKM}_{12} + \theta^\text{PMNS}_{12} &= 45^0 + \delta \theta, \quad \text{(QLC)} \\
\theta^\text{CKM}_{23} + \theta^\text{PMNS}_{23} &= 45^0 - \delta \theta, \quad \text{(QLC)} \\
\theta^\text{PMNS}_{12} + \theta^\text{PMNS}_{13} &= \theta^\text{PMNS}_{23} - \delta \theta. \quad \text{(SLC)}
\end{align*}

where $\delta \theta \approx 2^0$, represents the amount of deviation from the original QLC and SLC relations. This is interesting to note that all the three relations experience similar deviation $\delta \theta$ and also $\delta \theta \sim \theta^\text{CKM}_{23}$. One way to infer from the above results is either $\delta \theta$ is the error or the original three relations described above (with $\delta \theta = 0$), need little modification. This is due to the fact that $\delta \theta$ and $\theta^\text{CKM}_{23}$ are equivalent. We hope that these simultaneous similar deviations are not unnatural and by replacing $\delta \theta$ with $\theta^\text{CKM}_{23}$ we try to reformulate the original three phenomenological relations. Perhaps,

\begin{align*}
\theta^\text{CKM}_{12} - \theta^\text{CKM}_{23} + \theta^\text{PMNS}_{12} &= 45^0, \\
2\theta^\text{CKM}_{23} + \theta^\text{PMNS}_{23} &= 45^0, \\
\theta^\text{CKM}_{23} + \theta^\text{PMNS}_{12} + \theta^\text{PMNS}_{13} &= \theta^\text{PMNS}_{23},
\end{align*}

could be better QLC representations than the earlier. We hope that this little modification is permissible in the sense that $\theta^\text{CKM}_{23}$ is very small in comparison to the other members present in the above relations.
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