FLOW OF ELECTRORHEOLOGICAL FLUIDS UNDER THE CONDITIONS OF SLIP ON THE BOUNDARY.

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Abstract. We derive general conditions of slip of a fluid on the boundary. Under these conditions the velocity of the fluid on the immovable boundary is a function of the normal and tangential components of the force acting on the surface of the fluid. A problem on stationary flow of an electrorheological fluid in which the terms of slip are specified on one part of the boundary and surface forces are given on the other is formulated and studied. Existence of a solution of this problem is proved by using the methods of penalty functions, monotonicity and compactness. It is shown that the method of penalty functions and the Galerkin approximations can be used for the approximate solution of the problem under consideration.

1. Introduction

Electrorheological fluids are smart materials which are concentrated suspensions of polarizable particles in a nonconductive dielectric liquid. In moderately large electric fields, the particles form chains along the field lines, and these chains then aggregate to form columns [9]. These chainlike and columnar structures cause dramatic changes in the rheological properties of the suspensions. The fluids become anisotropic, the apparent viscosity (the resistance to flow) in the direction orthogonal to the direction of electric field abruptly increases, while the apparent viscosity in the direction of the electric field changes not so drastically.

The chainlike and columnar structures are destroyed under the action of large stresses, and then the apparent viscosity of the fluid decreases and the fluid becomes less anisotropic.

On the basis of experimental results, the following constitutive equation was developed in [3]:

$$\sigma_{ij}(p, u, E) = -p\delta_{ij} + 2\varphi(I(u), |E|, \mu(u, E))\varepsilon_{ij}(u), \quad i, j = 1, \ldots, n, \quad n = 2 \text{ or } 3.$$ (1.1)

Here, $\sigma_{ij}(p, u, E)$ are the components of the stress tensor which depend on the pressure $p$, the velocity vector $u = (u_1, \ldots, u_n)$ and the electric field strength $E = (E_1, \ldots, E_n)$, $\delta_{ij}$ are the components of the unit tensor (the Kronecker delta), and $\varepsilon_{ij}(u)$ are the components of the rate of strain tensor

$$\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$ (1.2)

Moreover, $I(u)$ is the second invariant of the rate of strain tensor

$$I(u) = \sum_{i,j=1}^{n} (\varepsilon_{ij}(u))^2.$$ (1.3)
shown in [3] that it can be represented as follows:

\[
(\mu(u,E))(x) = \left(\frac{u(x)}{|u(x)|}, \frac{E(x)}{|E(x)|}\right)^2 \Omega = \frac{\sum_{i=1}^{n} u_i(x) E_i(x)^2}{(\sum_{i=1}^{n} u_i(x))^2(\sum_{i=1}^{n} E_i(x))^2}.
\]  

(1.4)

So \(\mu(u,E)\) is the square of the scalar product of the unit vectors \(\frac{u}{|u|}\) and \(\frac{E}{|E|}\). The function \(\mu\) is defined by (1.4) in the case of an immovable frame of reference.

As the scalar product of two vectors is independent of the frame of reference, the constitutive equation (1.1) is invariant with respect to the group of Galilei transformations of the frame of reference that are represented as a product of time-independent translations, rotations and uniform motions.

The presence of the function \(\mu\) in the constitutive equation (1.1) is connected with the anisotropy of the electrorheological fluid under which the viscosity of the fluid depends on the angle between the vectors of velocity and the vector of electric field.

The function \(\mu\) defined by (1.4), (1.5) is not specified at \(E = 0\) and at \(u = 0\), and there does not exist an extension by continuity to the values of \(u = 0\) and \(E = 0\). However, at \(E = 0\) there is no influence of the electric field, and the function \(\mu(u,E)\) need not to be specified at \(E = 0\). Likewise, in case that the measure of the set of points \(x\) at which \(u(x) = 0\) is zero, the function \(\mu\) need not also be specified at \(u = 0\). But in the general the function \(\mu\) can be defined as follows:

\[
\mu(u,E)(x) = \left(\frac{\alpha \bar{I} + u(x) + \bar{u}}{\alpha \sqrt{n + |u(x)| + \bar{u}}}, \frac{E(x)}{|E(x)|}\right)^2 \Omega, \tag{1.6}
\]

where \(\bar{I}\) denotes a vector with components equal to one, and \(\alpha\) is a small positive constant. If \(u(x) \neq 0\) almost everywhere in \(\Omega\), one can choose \(\alpha = 0\).

The viscosity function \(\varphi\) is identified by approximation of flow curves, see [3], and it was shown in [3] that it can be represented as follows:

\[
\varphi(I(u),|E|,\mu(u,E)) = b(|E|,\mu(u,E))(\lambda + I(u))^{-\frac{1}{2}} + \psi(I(u),|E|,\mu(u,E)), \tag{1.7}
\]

where \(\lambda\) is a small parameter, \(\lambda \geq 0\).

The equations for the functions \(E\) and \((p,v)\) are separated, (see [3]). Because of this, we assume here and thereafter that the function of electric field \(E\) is known.

Various problems on stationary flow of electrorheological fluids under mixed boundary conditions such that velocities and surface forces are prescribed on different parts of the boundary are investigated in [3]. This formulation assumes that the fluid adheres to a hard boundary, that is the velocity of the fluid on the hard boundary is equal to the velocity of the boundary.

But at some conditions, wall effects appear, the velocity of a fluid on the hard boundary can be different from the velocity of the hard boundary. In particular, hard particles of electrorheological suspensions may slip along the hard boundary.

It was shown experimentally that magnetic suspensions, whose conduct is similar to the conduct of electrorheological fluids, exhibit wall effect, see [5]. This effect depends both on the surface roughness of the wall and on the force pressing the particles against the surface of the wall.
In Section 2, we derive the boundary conditions of slip. In Sections 3 and 4, we formulate a boundary value problem on stationary flow of the electrorheological fluid under the condition of slip on the boundary and present a theorem on the existence of a solution of this problem. Sections 5 and 6 are devoted to the proof of the existence result and construction of approximate solutions by using the method of penalty functions. In Section 7, we show that Galerkin approximations can be used for approximate solution of our problem.

Since the constitutive equations of nonlinear viscous and viscous fluids are partial cases of the equation (1.1), the results presented in this paper can be applied to nonlinear viscous and viscous fluids.

2. Frictional force and the velocity of slip on a hard boundary.

Let $\Omega \subset \mathbb{R}^n$ be a domain in which a fluid flows. Let $S$ be the boundary of $\Omega$ and $S_1$ be a part of $S$ which corresponds to a hard immovable wall. We assume that the fluid slips on $S_1$. Let $F(s) = \sum_{i=1}^{n} F_i(s)\zeta_i$ be an external surface force acting on the fluid. Here $\zeta_i$ are unit vectors directed along the coordinate axes $x_i$, $F_i$ scalar functions of points $s$ of $S_1$.

We represent the function $F$ in the form
\[ F(s) = F^\nu(s) + F^\tau(s), \quad s \in S_1, \]  
where $F^\nu$ and $F^\tau$ are the normal and the tangential surface forces.

\[ F^\nu(s) = F_\nu(s)\nu(s), \quad F_\nu(s) = \sum_{i=1}^{n} F_i(s)\nu_i(s), \]  
\[ F^\tau(s) = F(s) - F^\nu(s) = \sum_{i=1}^{n} F_{\tau i}(s)\zeta_i, \quad F_{\tau i}(s) = F_i(s) - F_\nu(s)\nu_i(s), \]

where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outward normal to $S_1$.

Analogously, the velocity vector $u$ on the boundary is represented in the form
\[ u(s) = \sum_{i=1}^{n} u_i(s)\zeta_i = u^\nu(s) + u^\tau(s), \]
\[ u^\nu(s) = u_\nu(s)\nu(s), \quad u_\nu(s) = \sum_{i=1}^{n} u_i(s)\nu_i(s), \]
\[ u^\tau(s) = u(s) - u^\nu(s) = \sum_{i=1}^{n} u_{\tau i}(s)\zeta_i, \]

where
\[ u_{\tau i}(s) = u_i(s) - u_\nu(s)\nu_i(s). \]

We consider the following boundary conditions on $S_1$:
\[ u_\nu(s) = 0, \quad s \in S_1, \]  
\[ F^\tau(s) = -\chi(F_\nu(s), |u^\tau(s)|^2)u^\tau(s), \quad s \in S_1. \]

Here $\chi$ is the function of slip that depends on the normal component of the surface force $F_\nu$ and on the square of the module of the tangential velocity $u^\tau$.

Formula (2.7) is a generalization of Navier’s condition of slip in which $\chi$ is a positive constant, the nonlinear modification of Navier’s condition of slip in which $\chi$ is a function
of \(|u|^\), and Coulomb’s law of friction in which \(\chi = \infty \) at \(|F^r| < c_1|F^\nu|\) and \(\chi = c\) at \(|F^r| = c_1|F^\nu|, c, c_1\) are positive constants.

We note that problems on flow of nonlinear viscous fluids in which \(\chi\) is a function of \(|u^r|\) were investigated in [8].

The function \(\chi\) accepts positive values, \(\chi\) does not depend of \(F^\nu\) at \(F^\nu > 0\), and it rises as \(F^\nu\) decreases. The sign minus in (2.7) designates that the velocity of slip of the fluid is in opposition to the tangential surface force, i.e. the frictional force is in opposition to the direction of motion, and the module of the slip velocity is equal to \(|F^r(s)|/(\chi(F^\nu(s), |u^r(s)|^2))^{-1}\).

In the special case that \(\chi(y_1, y_2) = \infty\) for an arbitrary \((y_1, y_2) \in \mathbb{R} \times \mathbb{R}_+,\) formulas (2.6), (2.7) imply \(u\big|_{S_1} = 0\), i.e. the fluid adheres to the hard boundary, and in the case of \(\chi(y_1, y_2) = 0\) for an arbitrary \((y_1, y_2) \in \mathbb{R} \times \mathbb{R}_+,\) (2.7) yields \(F^r\big|_{S_1} = 0\), i.e. the frictional force is equal to zero. The relation (2.6) designates that the fluid does not flow through the hard wall \(S_1\).

For the constitutive equation (1.1) the components \(F_i\) of the surface force \(F = (F_1,\ldots,F_n)\) are defined by

\[
F_i = [-p\delta_{ij} + 2\varphi(I(u), |E|, \mu(u, E))\epsilon_{ij}(u)]\nu_j\big|_{S_1}, \quad i = 1,\ldots,n, \tag{2.8}
\]

and the normal component \(F^\nu\) of the surface force is determined by (2.2). In (2.8) and below the Einstein convention on summation over repeated index is applied.

Let \(P\) be an operator of regularization given by

\[
P\nu(x) = \int_{\mathbb{R}^n} \omega(|x - x'|)\nu(x') \, dx', \quad x \in \Gamma, \tag{2.9}
\]

where

\[
\omega \in C^\infty(\mathbb{R}_+), \quad \text{supp}\omega \subseteq [0, a], \quad \omega(z) \geq 0, \quad z \in \mathbb{R}_+,
\]

\[
\int_{\mathbb{R}^n} \omega(|x|) \, dx = 1, \quad a \text{ is a small positive constant.} \tag{2.10}
\]

Here, we assume that the function \(\nu\) is extended to \(\mathbb{R}^n\).

We denote by \(F_{\nu}(p, u)\) the normal component of the surface force calculated by the regularized functions of pressure \(p\) and velocity \(u\). According to (2.2) and (2.8), the function \(F_{\nu}(p, u)\) is defined as follows:

\[
F_{\nu}(p, u) = [-p + 2\varphi(I(Pu), |E|, \mu(Pu, E))\epsilon_{ij}(Pu)\nu_i\nu_j]\big|_{S_1}. \tag{2.11}
\]

We change the function \(F^\nu\) in (2.7) for the function \(F_{\nu}(p, u)\). Then, we obtain the following boundary condition:

\[
F^r(s) = -\chi(F_{\nu}(p, u)(s), |u^r(s)|^2)u^r(s), \quad s \in S_1. \tag{2.12}
\]

From the physical point of view, (2.12) denotes that the model is not local, the velocity of slip at a point \(s \in S_1\) depends on the averaged normal surface force \(F_{\nu}(p, u)\) which in its turn is defined by the values of pressure and the derivatives of the velocity at points belonging to some small vicinity of the point \(s\). This is natural from the physical view-point.

Such nonlocal approach is also connected with the fact that the velocity of slip depends on the surface roughness which is not a local characteristic.
Taking (2.2), (2.3) and (2.4) into account, we represent (2.12) in the following form:

\[ F_i - \left( \sum_{k=1}^{n} F_k r_k \right) u_i = -\chi \left( F_{rv}(p, u), \sum_{k=1}^{n} u_{r_k}^2 \right) u_{ri} \quad \text{on } S_1, \quad i = 1, \ldots, n, \]  

(3.5)

Finally, we obtain by (2.8) and (2.13) the following boundary condition of slip:

\[ 2\varphi(I(u), |E|, \mu(u, E)) [\varepsilon_{ij}(u) u_{ij} - \varepsilon_{kj}(u) u_{jk} u_{k} u_{i}] = -\chi(F_{rv}(p, u), \sum_{k=1}^{n} u_{r_k}^2) u_{ri} \quad \text{on } S_1, \]

\[ i = 1, \ldots, n, \]  

(2.14)

where \( F_{rv}(p, u) \) is defined by (2.11).

3. GOVERNING EQUATIONS AND ASSUMPTIONS.

We consider stationary flow problem under the Stokes approximation, i.e. we ignore inertial forces which are assumed to be small as compared with the internal forces caused by the viscous stresses. Then, the motion equations take the following form:

\[ \frac{\partial p}{\partial x_i} - 2 \frac{\partial}{\partial x_j} [\varphi(I(u), |E|, \mu(u, E)) \varepsilon_{ij}(u)] = K_i \quad \text{in } \Omega, \quad i = 1, \ldots, n, \]  

(3.1)

where \( K_i \) are the components of the volume force vector \( K \).

The velocity function \( u \) meets the incompressibility condition

\[ \text{div } u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad \text{in } \Omega. \]  

(3.2)

We assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n = 2 \) or \( 3 \) with a Lipschitz continuous boundary \( S \). Suppose that \( S_1 \) and \( S_2 \) are open non-empty subsets of \( S \) such that \( S_1 \cap S_2 = \emptyset \), and \( \overline{S}_1 \cup \overline{S}_2 = S \).

We consider mixed boundary conditions for the functions \( u, p \). Wherein, the terms of slip (2.6), (2.14) are specified on \( S_1 \) and surface forces are given on \( S_2 \), i.e.

\[ [-p \delta_{ij} + 2\varphi(I(u), |E|, \mu(u, E)) \varepsilon_{ij}(u)] u_j \bigg|_{S_2} = F_i, \quad i = 1, \ldots, n. \]  

(3.3)

We assume that

**(C1):** \( \varphi : (y_1, y_2, y_3) \rightarrow \varphi(y_1, y_2, y_3) \) is a function continuous in \( \mathbb{R}^2_+ \times [0, 1] \), and for an arbitrarily fixed \( (y_2, y_3) \in \mathbb{R}^2_+ \times [0, 1] \) the function \( \varphi(., y_2, y_3) : y_1 \rightarrow \varphi(y_1, y_2, y_3) \) is continuously differentiable in \( \mathbb{R}^2_+ \), and the following inequalities hold:

\[ a_2 \geq \varphi(y_1, y_2, y_3) \geq a_1 \]  

(3.4)

\[ \varphi(y_1, y_2, y_3) + 2 \frac{\partial}{\partial y_1} (y_1, y_2, y_3)y_1 \geq a_3 \]  

(3.5)

\[ \left| \frac{\partial}{\partial y_1} (y_1, y_2, y_3) \right| y_1 \leq a_4, \]  

(3.6)

where \( a_i, 1 \leq i \leq 4 \), are positive numbers.

Inequality (3.4) indicates that the viscosity is bounded from below and from above by positive constants. The inequality (3.5) implies that for fixed values of \( |E| \) and \( \mu(u, E) \) the derivative of the function \( I(v) \rightarrow G(v) \) is positive, where \( G(v) \) is the second invariant of the stress deviator

\[ G(v) = 4[\varphi(I(v), |E|, \mu(u, E))]^2 I(v). \]
This means that in the case of simple shear flow the shear stress increases with increasing shear rate. (3.6) is a restriction on \( \partial y \) for large values of \( y_1 \). These inequalities are natural from the physical point of view.

Relative to the function of slip \( \chi \), we assume that the following conditions are satisfied:

(C2): \( \chi : (y_1, y_2) \rightarrow \chi(y_1, y_2) \) is a function continuous in \( \mathbb{R} \times \mathbb{R}_+ \), and for an arbitrarily fixed \( y_1 \in \mathbb{R} \), the function \( \chi(y_1, \cdot) : y_2 \rightarrow \chi(y_1, y_2) \) is continuously differentiable in \( \mathbb{R}_+ \), and the following inequalities hold:

\[
\begin{align*}
  b_2 &\geq \chi(y_1, y_2) \geq b_1, \\
  \chi(y_1, y_2) + 2 \frac{\partial \chi}{\partial y_2}(y_1, y_2)y_2 &\geq b_3, \\
  \left| \frac{\partial \chi}{\partial y_2}(y_1, y_2) \right| y_2 &\leq b_4,
\end{align*}
\]

where \( (y_1, y_2) \in \mathbb{R} \times \mathbb{R}_+ \), and \( b_i, 1 \leq i \leq 4 \), are positive numbers. Inequalities (3.7)–(3.9) are analogous to the ones of (3.4)–(3.6). Inequality (3.7) means that the function of slip is bounded from below and from above by positive constants. (3.8) implies that for fixed value of \( y_1 \), i.e. the value of \( F_{\nu_1}(p, u)(s), s \in S_1 \), the derivative of the function \( |u^r| \rightarrow |F^r| = \chi(F_{\nu_1}(p, u), |u^r|^2)|u^r| \) is not less than \( b_3 \). Indeed, denoting \( z = \frac{y_2}{2} \), we obtain

\[
\frac{\partial (\chi(y_1, z^2)z)}{\partial z} = \chi(y_1, z^2) + 2 \frac{\partial \chi}{\partial y_2}(y_1, z^2)z \geq b_3,
\]

that is the frictional force increases as the velocity of slip increases.

The inequality (3.9) is a restriction on \( \frac{\partial \chi}{\partial y_2} \) for large values of \( y_2 \). The inequalities (3.7)–(3.9) are natural from the physical viewpoint.

We suppose also that

\[
K = (K_1, \ldots, K_n) \in L_2(\Omega)^n, \quad F = (F_1, \ldots, F_n) \in L_2(S_2)^n.
\]

4. Boundary value problem.

We study a problem of searching for a pair of functions \((u, p)\) which satisfy the motion equation (3.1), the condition of incompressibility (3.2) and the boundary conditions (2.6), (2.14) and (3.3).

Consider the following spaces

\[
Z = \{ v | v \in H^1(\Omega)^n, \quad v_\nu|_{S_1} = 0 \}, \quad W = \{ v | v \in Z, \quad \text{div} \; v = 0 \}.
\]

Lemma 4.1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n = 2 \) or \( 3 \) with a Lipschitz continuous boundary \( S \), and let \( S_1 \) be an open nonempty subset of \( S \). Then the expression

\[
\| v \|_Z = \left( \int_\Omega I(v) \, dx + \int_{S_1} \sum_{k=1}^n v_{\nu_k}^2 \, ds \right)^{1/2}
\]

defines a norm in \( Z \) and \( W \) being equivalent to the norm of \( H^1(\Omega)^n \).

For a proof see in [7], Section 1.7.

Everywhere below we use the following notations: If \( Y \) is a normed space, we denote by \( Y^* \) the dual of \( Y \), and by \((f, h)\) the duality between \( Y^* \) and \( Y \), where \( f \in Y^* \), \( h \in Y \). In particular, if \( f \in L_2(\Omega) \) or \( f \in L_2(\Omega)^n \), then \((f, h)\) is the scalar product in \( L_2(\Omega) \) or in \( L_2(\Omega)^n \), respectively. The sign \( \rightharpoonup \) denotes weak convergence in a Banach space.
Denote by $B$ the operator of divergence, i.e.

$$Bu = \text{div} u. \quad (4.4)$$

It is obvious that $B$ is a linear continuous mapping of $Z$ into $L_2(\Omega)$, i.e. $B \in \mathcal{L}(Z, L_2(\Omega))$. We denote by $B^*$ the adjoint to $B$ operator.

We introduce operators $M : Z \rightarrow Z^*$ and $A : Z \times L_2(\Omega) \rightarrow Z^*$ as follows:

$$(M(u), h) = 2\int_\Omega \varphi(I(u), |E|, \mu(u, E))\varepsilon_{ij}(u)\varepsilon_{ij}(h) \, dx, \quad u, h \in Z, \quad (4.5)$$

$$(A(u, p), h) = \int_{S_1} \chi(F_{\nu\nu}(p, u), \sum_{k=1}^n u^2_{\tau_k}) u_{\tau_i} h_{\tau_i} \, ds, \quad (u, p) \in Z \times L_2(\Omega), \quad h \in Z. \quad (4.6)$$

Consider the problem: find a pair $(u, p)$ such that

$$(M(u), h) + (A(u, p), h) - (B^* p, h) = (K + F, h), \quad h \in Z, \quad (4.7)$$

$$(Bu, q) = 0, \quad q \in L_2(\Omega). \quad (4.8)$$

Here we use the notations

$$(K, h) = \int_\Omega K_i h_i \, dx, \quad (F, h) = \int_{S_2} F_i h_i \, ds. \quad (4.9)$$

A solution of the problem (4.7)–(4.9) will be called a generalized solution of the problem (3.1), (3.2), (3.3), (2.6) and (2.14). Indeed, by use of Green’s formula it can be seen that, if $(u, p)$ is a solution of the problem (3.1), (3.2), (3.3), (2.6) and (2.14), then $(u, p)$ is a solution of the problem (4.7)–(4.9). On the contrary, if $(u, p)$ is a solution of the problem (4.7)–(4.9), then $(u, p)$ is a solution of the problem (3.1), (3.2), (3.3), (2.6) and (2.14) in the sense of distributions.

**Theorem 4.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz continuous boundary $S$, and suppose that the conditions (C1), (C2) and (3.10) are satisfied. Then there exists a solution of the problem (4.7)–(4.9).

5. Auxiliary results.

We consider four functions $v_1, v_2, v_3, v_4$ such that

$$v_1 \in L_2(\Omega), \quad v_1(x) \geq 0 \text{ a.e. in } \Omega, \quad v_2 \in L_\infty(\Omega), \quad v_2(x) \in [0, 1] \text{ a.e. in } \Omega, \quad v_3 \in L_2(\Omega), \quad v_4 \in H^1(\Omega)^n. \quad (5.1)$$

We set $v = (v_1, v_2, v_3, v_4)$ and define the operator $M_v : Z \rightarrow Z^*$ as follows:

$$(M_v(u), e) = 2\int_\Omega \varphi(I(u), v_1, v_2)\varepsilon_{ij}(u)\varepsilon_{ij}(e) \, dx + \int_{S_1} \chi(F_{\nu\nu}(v_3, v_4), \sum_{k=1}^n u^2_{\tau_k}) u_{\tau_i} e_{\tau_i} \, ds, \quad u, e \in Z. \quad (5.2)$$

**Lemma 5.1.** Suppose that the conditions (C1), (C2) and (5.1) are satisfied. Then

$$(M_v(u) - M_v(w), u-w) \geq \mu_1 \|u-w\|_Z^2, \quad u, w \in Z, \quad (5.3)$$

$$\|M_v(u) - M_v(w)\|_{Z^*} \leq \mu_2 \|u-w\|_Z, \quad u, w \in Z, \quad (5.4)$$
where

\[ \mu_1 = \min(2a_1, 2a_3, b_1, b_3), \]
\[ \mu_2 = 2a_2 + 4a_4 + b_2 + 2b_4. \]  

**Proof.** We present the operator \( M_v \) in the form

\[ M_v = M_1 + M_2, \]  

(5.6)

\[ (M_1(u), e) = 2 \int_{\Omega} \varphi(I(u), v_1, v_2)\varepsilon_{ij}(u)\varepsilon_{ij}(e) \, dx, \]  

(5.7)

\[ (M_2(u), e) = \int_{S_1} \chi(F_{ri}(v_3, v_4), \sum_{k=1}^{n} u_{r_k}^2) u_{ri} \, ds, \quad u, e \in Z. \]  

(5.8)

Let \( u, w \) be arbitrary functions in \( Z \) and \( h = u - w \).

We introduce the function \( \gamma \) as follows:

\[ \gamma(t) = \int_{\Omega} \varphi(I(w + th), v_1, v_2)\varepsilon_{ij}(w + th)\varepsilon_{ij}(e) \, dx, \quad t \in [0, 1], \quad e \in Z. \]  

(5.10)

It is obvious that

\[ \gamma(1) - \gamma(0) = \frac{1}{2}(M_1(u) - M_1(w), e). \]  

(5.11)

By using the theorem on the differentiability of a function represented as an integral, we conclude that \( \gamma \) is differentiable at any point \( t \in (0, 1) \). Therefore

\[ \gamma(1) = \gamma(0) + \frac{d\gamma}{dt}(\xi), \quad \xi \in (0, 1), \]  

(5.12)

where

\[
\frac{d\gamma}{dt}(\xi) = \int_{\Omega} \left[ \varphi(I(w + \xi h), v_1, v_2)\varepsilon_{ij}(h)\varepsilon_{ij}(e)
+ 2 \frac{\partial \varphi}{\partial y_1} (I(w + \xi h), v_1, v_2)\varepsilon_{km}(w + \xi h)\varepsilon_{km}(h)\varepsilon_{ij}(w + \xi h)\varepsilon_{ij}(e) \right] \, dx.
\]

(5.13)

Taking note of the inequality

\[ |\varepsilon_{km}(w + \xi h)\varepsilon_{km}(h)| \leq I(w + \xi h)^{\frac{1}{2}} I(h)^{\frac{1}{2}}, \]  

(5.14)

and (3.4), (3.6), (5.11)–(5.13), we obtain

\[ \|M_1(u) - M_1(w)\|_{Z^*} \leq (2a_2 + 4a_4) \left( \int_{\Omega} I(u - w) \, dx \right)^{\frac{1}{2}} \leq (2a_2 + 4a_4) \|u - w\|_Z. \]  

(5.15)

Define the function \( g \) as follows:

\[ g(\alpha, x) = \begin{cases} 
\frac{\partial \varphi}{\partial y_1}(\alpha, v_1(x), v_2(x)), & \text{if } \frac{\partial \varphi}{\partial y_1}(\alpha, v_1(x), v_2(x)) < 0, \\
0, & \text{if } \frac{\partial \varphi}{\partial y_1}(\alpha, v_1(x), v_2(x)) \geq 0,
\end{cases} \]  

where \( \alpha \in \mathbb{R}_+, \ x \in \Omega. \)
Then, taking $e = h$ in (5.13) and applying (3.4), (3.5), (5.9) and (5.14), we obtain
\[
\frac{d\gamma}{dt}(\xi) = \int_{\Omega} \left[ \varphi(I(w + \xi h), v_1, v_2)I(h) + 2 \frac{\partial \varphi}{\partial y_1}(I(w + \xi h), v_1, v_2)(\varepsilon_{ij}(w + \xi h)\varepsilon_{ij}(h))^2 \right]dx \geq \min(a_1, a_3) \int_{\Omega} I(u - w) \, dx
\]
(5.16)
(5.11), (5.12) and (5.16) imply
\[
(M_1(u) - M_1(w), u - w) \geq 2 \min(a_1, a_3) \int_{\Omega} I(u - w) \, dx.
\]
(5.17)
We introduce the function $\gamma_1$ as follows:
\[
\gamma_1(t) = \int_{S_1} \chi \left( F_{rv}(v_3, v_4), \sum_{k=1}^{n} (w_{rk} + \xi \eta_{rk})^2 \right) (w_{ri} + \xi \eta_{ri}) \, ds, \quad t \in [0, 1], \quad e \in Z,
\]
where $h$ is defined by (5.9).

By analogy with the foregoing, we obtain
\[
(M_2(u) - M_2(w), e) = \gamma_1(1) - \gamma_1(0) = \frac{d\gamma_1}{dt}(\xi_1)
\]
\[
= \int_{S_1} \left[ \chi \left( F_{rv}(v_3, v_4), \sum_{k=1}^{n} (w_{rk} + \xi \eta_{rk})^2 \right) h_{ri} e_{ri} \right. + 2 \frac{\partial \chi}{\partial y_2} \left( F_{rv}(v_3, v_4), \sum_{k=1}^{n} (w_{rk} + \xi \eta_{rk})^2 \right) \nabla \cdot (w_{rk} + \xi \eta_{rk}) \eta_{ri} \, ds, \quad \xi_1 \in (0, 1),
\]
(5.18)
and (3.7)–(3.9) imply
\[
\|M_2(u) - M_2(w)\|_{Z^*} \leq (b_2 + 2b_4) \left( \int_{S_1} \sum_{i=1}^{n} (u_{ri} - w_{ri})^2 \, ds \right)^{\frac{1}{2}} \leq (b_2 + 2b_4) \|u - w\|_Z,
\]
(5.19)
\[
(M_2(u) - M_2(w), u - w) \geq \min(b_1, b_3) \int_{S_1} \sum_{k=1}^{n} (u_{rk} - w_{rk})^2 \, ds.
\]
(5.20)
Taking (5.15), (5.17), (5.19) and (5.20) into account, we obtain (5.3)–(5.5).

Let $\alpha$ be a positive number. Define the operator $A_\alpha : Z \to Z^*$ as follows:
\[
(A_\alpha(u), h) = \int_{S_1} \chi \left( F_{rv}(\frac{1}{\alpha} B u, u), \sum_{k=1}^{n} u_{rk}^2 \right) u_{ri} h_{ri} \, ds \quad u, h \in Z.
\]
(5.21)
Consider the problem: find a function $u_\alpha$ satisfying
\[
u_\alpha \in Z,
\]
(5.22)
\[
(A_\alpha(u_\alpha), h) + \frac{1}{\alpha}(Bu_\alpha, Bh) = (K + F, h), \quad h \in Z.
\]
(5.23)
The problem (5.22), (5.23) is an approximation of the problem (4.7)–(4.9) in which the function of pressure $p$ is replaced by the function $-\alpha^{-1} \text{div} u$; in this case we do not assume that $\text{div} u_\alpha = 0$.

**Theorem 5.1.** Let $\Omega$ be a bounded in $\mathbb{R}^n$, $n = 2$ or 3, with a Lipschitz continuous boundary $S$. Suppose that the conditions (C1), (C2) and (3.10) are satisfied. Then for an arbitrary $\alpha > 0$, there exists a solution of the problem (5.22), (5.23).
**Proof.** Let \( \{Z_m\}_{m=1}^\infty \) be a sequence of finite dimensional subspaces in \( Z \) such that
\[
\lim_{m \to \infty} \inf_{h \in Z_m} \|v - h\|_Z = 0, \quad v \in Z, \tag{5.24}
\]
\[
Z_m \subset Z_{m+1}, \quad m \in \mathbb{N}. \tag{5.25}
\]
We seek an approximate solution of the problem (5.22), (5.23) in the form
\[
u_{\alpha m} \in Z_m, \quad (M(u_{\alpha m}), h) + (A_\alpha(u_{\alpha m}), h) + \frac{1}{\alpha}(Bu_{\alpha m}, Bh) = (K + F, h), \quad h \in Z_m. \tag{5.26}
\]
By (3.4), (3.7), (3.10) and (4.3), we obtain
\[
y(h) = (M(h), h) + (A_\alpha(h), h) + \frac{1}{\alpha}(Bh, Bh) - (K + F, h) \\
\geq 2a_1 \int_\Omega I(h) \, dx + b_1 \int_{S_1} n \sum_{i=1}^n h_{\tau_i}^2 \, ds - \|K + F\|_{Z^*} \|h\|_Z \\
\geq \mu_1 \|h\|_Z^2 - \|K + F\|_{Z^*} \|h\|_Z, \tag{5.27}
\]
where \( \mu_1 = \min(2a_1, b_1) \).

Therefore, \( y(h) \geq 0 \) for \( \|h\|_Z \geq r = \|K + F\|_{Z^*} \mu_1^{-1} \).

From the corollary of Brouwer's fixed point theorem (cf. [2]), it follows that there exists a solution of (5.26) with
\[
\|u_{\alpha m}\|_Z \leq r, \quad \|M(u_{\alpha m}) + A_\alpha(u_{\alpha m})\|_{Z^*} \leq c, \quad m \in \mathbb{N},
\]
where the second inequality follows from (3.4) and (3.7). Therefore, we can extract a subsequence \( \{u_{\alpha \eta}\}_{\eta=1}^\infty \) such that
\[
u_{\alpha \eta} \to \nu_\alpha \quad \text{in } Z, \tag{5.28}
\]
\[
u_{\alpha \eta} \to \nu_\alpha \quad \text{in } L_2(\Omega)^n \text{ and a.e. in } \Omega, \tag{5.29}
\]
\[
(M(u_{\alpha \eta}) + A_\alpha(u_{\alpha \eta})) \to \theta \quad \text{in } Z^*. \tag{5.30}
\]
Let \( \eta_0 \) be a fixed positive integer and \( h \in Z_{\eta_0} \). Observing (5.29), (5.30), we pass to the limit in (5.26) with \( m \) replaced by \( \eta \) and obtain
\[
(\theta + \frac{1}{\alpha} B^* Bu_\alpha, h) = (K + F, h), \quad h \in Z_{\eta_0}. \tag{5.31}
\]
Since \( \eta_0 \) is an arbitrary positive integer, by (5.24), we obtain
\[
\theta + \frac{1}{\alpha} B^* Bu_\alpha = K + F \quad \text{in } Z^*. \tag{5.32}
\]
We present the operators \( M \) and \( A_\alpha \) in the form
\[
M(u) = \tilde{M}(u, u), \quad A_\alpha(u) = \tilde{A}_\alpha(u, u), \tag{5.33}
\]
where the operators \( (u, v) \to \tilde{M}(u, v) \) and \( (u, v) \to \tilde{A}_\alpha(u, v) \) are mappings of \( Z \times Z \) into \( Z^* \) according to
\[
(\tilde{M}(u, v), h) = 2 \int_\Omega \varphi(I(v), |E|, \mu(u, E)) \varepsilon_{ij}(v) \varepsilon_{ij}(h) \, dx, \tag{5.34}
\]
\[
(\tilde{A}_\alpha(u, v), h) = \int_{S_1} \chi(F_\varphi(\frac{1}{\alpha} Bu, u, \sum_{k=1}^n v_{\tau_k}^2) v_{\tau_i} h_{\tau_i} \, ds. \tag{5.35}
\]
Denote
\[ X_\eta(v) = (\tilde{M}(u_\alpha, u_\alpha) + \tilde{A}_\alpha(u_\alpha, u_\alpha) + \frac{1}{\alpha} B^* B(u_\alpha - v) - \tilde{M}(u_\alpha, v) - \tilde{A}_\alpha(u_\alpha, v), u_\alpha - v), \quad v \in Z. \] (5.36)

By Lemma 5.1, see (5.3), we obtain
\[ X_\eta(v) \geq 0, \quad \eta \in \mathbb{N}, \quad v \in Z. \] (5.37)

We have
\[ \|\tilde{M}(u_\alpha, v) - \tilde{M}(u_\alpha, v)\|_{Z^*} \]
\[ \leq 2\left\{ \int_{\Omega} [\varphi(I(v), |E|, \mu(u_\alpha, E)) - \varphi(I(v), |E|, \mu(u_\alpha, E))]^2 I(v) dx \right\}^{\frac{1}{2}}. \] (5.38)

(C1), (5.29), (5.38), and the Lebesgue theorem give
\[ \tilde{M}(u_\alpha, v) \to \tilde{M}(u_\alpha, v) \quad \text{in} \quad Z^*. \] (5.39)

Likewise, we obtain
\[ \tilde{A}_\alpha(u_\alpha, v) \to \tilde{A}_\alpha(u_\alpha, v) \quad \text{in} \quad Z^*. \] (5.40)

(5.26), (5.28) and (5.33) yield
\[ (\tilde{M}(u_\alpha, u_\alpha) + \tilde{A}_\alpha(u_\alpha, u_\alpha), u_\alpha) + \frac{1}{\alpha}(Bu_\alpha, Bu_\alpha) = (K + F, u_\alpha) \to (K + F, u_\alpha). \] (5.41)

By (5.30) and (5.32), we obtain
\[ \lim [(\tilde{M}(u_\alpha, u_\alpha) + \tilde{A}_\alpha(u_\alpha, u_\alpha), v)] + \frac{1}{\alpha}(Bu_\alpha, Bv) = (K + F, v), \quad v \in Z. \] (5.42)

Observing (5.39)–(5.42), we pass to the limit in (5.36). Then by (5.37), we find
\[ (K + F - \tilde{M}(u_\alpha, v) - \tilde{A}_\alpha(u_\alpha, v) - \frac{1}{\alpha} B^* B v, u_\alpha - v) \geq 0, \quad v \in Z. \] (5.43)

We choose \( v = u_\alpha - \gamma h, \gamma > 0, h \in Z, \) and consider \( \gamma \to 0. \) Then, Lemma 5.1, see (5.2), (5.4), (5.34) and (5.35), implies
\[ (K + F - M(u_\alpha) - A_\alpha(u_\alpha) - \frac{1}{\alpha} B^* B u_\alpha, h) \geq 0. \] (5.44)

This inequality holds for any \( h \in Z. \) Replacing \( h \) by \( -h \) shows that equality holds true in (5.44). Therefore, \( u_\alpha \) is a solution of the problem (5.22), (5.23). \( \blacksquare \)

We will use also the following lemma:

**Lemma 5.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n, n = 2 \) or 3, with a Lipschitz continuous boundary \( S, \) and let the operator \( B \in \mathcal{L}(Z, L_2(\Omega)) \) be defined by (4.4). Then, the inf-sup condition
\[ \inf_{\mu \in L_2(\Omega)} \sup_{v \in Z} \frac{(Bu, \mu)}{{\|v\|}_Z \|\mu\|_{L_2(\Omega)}} \geq \beta_1 > 0 \] (5.45)
holds true. The operator \( B \) is an isomorphism from \( W^\perp \) onto \( L_2(\Omega), \) where \( W^\perp \) is orthogonal complement of \( W \) in \( Z, \) and the operator \( B^* \) that is adjoint to \( B, \) is an isomorphism from \( L_2(\Omega) \) onto the polar set
\[ W^0 = \{ f \in Z^*, (f, u) = 0, \quad u \in W \}. \] (5.46)
Moreover,
\[
\|B^{-1}\|_{L(L_2(\Omega),W^{'})} \leq \frac{1}{\beta_1},
\]
(5.47)
\[
\|(B^*)^{-1}\|_{L(W_0,L_2(\Omega))} \leq \frac{1}{\beta_1}.
\]
(5.48)

For a proof see in [7], Section 6.1.2.

6. Proof of Theorem 4.1.

Let \(\{\alpha_i\}\) be a sequence of positive numbers such that \(\lim \alpha_i = 0\). Consider the problem: given \(\alpha_i\), find \(u_{\alpha_i}\) satisfying
\[
u_{\alpha_i} \in Z,
\]
(6.1)
\[
(M(u_{\alpha_i}), h) + (A_{\alpha_i}(u_{\alpha_i}), h) + \alpha_i^{-1} (Bu_{\alpha_i}, Bh) = (K + F, h), \quad h \in Z.
\]
(6.2)
The existence of a solution of the problem (6.1), (6.2) follows from the Theorem 5.1.

Taking \(h = u_{\alpha_i}\) in (6.2), we obtain
\[
(M(u_{\alpha_i}), u_{\alpha_i}) + (A_{\alpha_i}(u_{\alpha_i}), u_{\alpha_i}) + \alpha_i^{-1} \|Bu_{\alpha_i}\|_{L_2(\Omega)}^2 \leq \|K + F\|_Z \|u_{\alpha_i}\|_Z.
\]
(6.3)
(3.4), (3.7) and (6.3) imply
\[
2\alpha_1 \int_\Omega I(u_{\alpha_i})dx + b_1 \int_{S_1} \sum_{k=1}^n u_{\alpha_i,\tau k}^2 ds + \alpha_i^{-1} \|Bu_{\alpha_i}\|_{L_2(\Omega)}^2 \leq \|K + F\|_Z \|u_{\alpha_i}\|_Z.
\]
(6.4)

It follows from here and (4.3) that
\[
\|u_{\alpha_i}\|_Z \leq c_1,
\]
(6.5)
\[
\alpha_i^{-\frac{1}{2}} \|Bu_{\alpha_i}\|_{L_2(\Omega)} \leq c_2.
\]
(6.6)

Therefore, a subsequence \(\{u_{\alpha_m}\}\) can be extracted from the sequence \(\{u_{\alpha_i}\}\) such that
\[
u_{\alpha_m} \rightharpoonup u \quad \text{in } Z,
\]
(6.7)
\[
u_{\alpha_m} \rightarrow u \quad \text{in } L_2(\Omega) \quad \text{and a.e. in } \Omega,
\]
(6.8)
\[
u_{\alpha_m}|_{S_1} \rightarrow u|_{S_1} \quad \text{in } L_2(S_1) \quad \text{and a.e. in } S_1,
\]
(6.9)
\[
Bu_{\alpha_m} \rightarrow 0 \quad \text{in } L_2(\Omega).
\]
(6.10)

(6.2) yields
\[
\alpha_m^{-1} B^* Bu_{\alpha_m} = K + F - M(u_{\alpha_m}) - A_{\alpha_m}(u_{\alpha_m}) \quad \text{in } Z^*.
\]
(6.11)

By virtue of (3.4), (3.7), (6.5) and (6.9) the right-hand side of (6.11) is bounded in \(Z^*\). Therefore, Lemma 5.2, yields
\[
\alpha_m^{-1} \|Bu_{\alpha_m}\|_{L_2(\Omega)} \leq c_3,
\]
(6.12)
and we can consider that
\[
\alpha_m^{-1} Bu_{\alpha_m} \rightarrow p \quad \text{in } L_2(\Omega).
\]
(6.13)

By analogy with the proof of Theorem 5.1, we pass to the limit in (6.2) using (6.7)–(6.10) and (6.13). As a result, we obtain that the pair \((u, p)\) is a solution of the problem (4.7)–(4.9).
Remark. Suppose that the condition of slip has the following form:
\[
2\varphi(I(u), |E|, \mu(u, E))\left[\varepsilon_{ij}(u)\nu_j - \varepsilon_{kj}(u)\nu_j\nu_k\nu_i\right]
\]
\[
= -\chi(F\nu(p, u), \sum_{k=1}^{n}(Pu_{\nu k})^2)u_{ri} \quad \text{on } S_1, \quad i = 1, \ldots, n,
\]
(6.14)

compare with (2.14). Here \( P \) is the operator of regularization defined by (2.9), (2.10).

In this case, the function \( \chi \) in the operator \( A \) in (4.6) is defined by just the same expression as in (6.14).

The condition of slip (6.14) is reasonable from the physical point of view and under such condition the Theorem 5.1 remains true without the restrictions (3.8) and (3.9).

7. Galerkin method for the problem (4.7)–(4.9).

Let \( \{N_m\} \) be a sequence of finite-dimensional subspaces in \( L_2(\Omega) \) such that:
\[
\lim_{m\to\infty} \inf_{y \in N_m} \|w - y\|_{L_2(\Omega)} = 0, \quad w \in L_2(\Omega).
\]
(7.1)

\[
N_m \subset N_{m+1}, \quad m \in \mathbb{N},
\]
(7.2)

\[
\inf_{\mu \in N_m} \sup_{v \in Z_m} \frac{(B_m v, \mu)}{\|v\|_{L_2(\Omega)} \|\mu\|_{L_2(\Omega)}} \geq \beta > 0, \quad m \in \mathbb{N},
\]
(7.3)

where the operators \( B_m \in \mathcal{L}(Z_m, N_m^*) \) are defined as follows:
\[
(B_m v, \mu) = \int_{\Omega} \mu \, dv \, dx \quad v \in Z_m, \quad \mu \in N_m.
\]
(7.4)

Let \( B_m^* \in \mathcal{L}(N_m, Z_m^*) \) be the adjoint operator of \( B_m \) with \( (B_m v, \mu) = (v, B_m^* \mu) \) for all \( v \in Z_m \) and all \( \mu \in N_m \).

We introduce the spaces \( W_m \) and \( W_m^0 \) by:
\[
W_m = \{v \in Z_m, \quad (B_m v, \mu) = 0, \quad \mu \in N_m\},
\]
(7.5)

\[
W_m^0 = \{q \in Z_m^*, \quad (q, v) = 0, \quad v \in W_m\}.
\]
(7.6)

Lemma 7.1. Let \( \{Z_m\}_{m=1}^{\infty}, \{N_m\}_{m=1}^{\infty} \) be sequences of finite-dimensional subspaces in \( Z \) and \( L_2(\Omega) \) such that (7.3) holds true. Then the operator \( B_m^* \) is an isomorphism from \( N_m \) onto \( W_m^0 \), and the operator \( B_m \) is an isomorphism from \( W_m^\bot \) onto \( N_m^* \), where \( W_m^\bot \) is an orthogonal complement of \( W_m \) in \( Z_m \). Moreover,
\[
\|(B_m^*)^{-1}\|_{\mathcal{L}(W_m^0, N_m)} \leq \frac{1}{\beta}, \quad \|B_m^{-1}\|_{\mathcal{L}(N_m^*, W_m^\bot)} \leq \frac{1}{\beta}, \quad m \in \mathbb{N}.
\]
(7.7)

For a proof see in [7], Section 6.1.2.

We seek an approximate solution of the problem (4.7)–(4.9) of the form:
\[
(u_m, p_m) \in Z_m \times N_m,
\]
(7.8)

\[
(M(u_m), h) + (A(u_m, p_m), h) - (B_m^* p_m, h) = (K + F, h) \quad h \in Z_m,
\]
(7.9)

\[
(B_m u_m, q) = 0, \quad q \in N_m,
\]
(7.10)

Theorem 7.1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n, n = 2 \) or 3, with a Lipschitz continuous boundary \( S \), and suppose that the conditions (C1), (C2) and (3.10) are satisfied. Let also \( \{Z_m\}_{m=1}^{\infty} \) and \( \{N_m\}_{m=1}^{\infty} \) be sequences of finite-dimensional subspaces in \( Z \) and \( L_2(\Omega) \) which satisfy the conditions (5.24), (5.25), (7.1)–(7.3). Then for an arbitrary \( m \in \mathbb{N} \), there exists
a solution of the problem (7.8)–(7.10), and a subsequence \( \{u_k, p_k\} \) can be extracted from the sequence \( \{u_m, p_m\} \) such that:

\[
\begin{align*}
  u_k &\to u \quad \text{in } Z, \\
p_k &\to p \quad \text{in } L_2(\Omega),
\end{align*}
\]

where \((u, p)\) is a solution of the problem (4.7)–(4.9).

**Proof.** 1) We determine a mapping \( A_1 : L_2(\Omega) \times Z \times Z \to Z^* \) as follows:

\[
(A_1(\mu, v, w), h) = \int_{S_1} \chi(F_{rv}(\mu, v), \sum_{k=1}^n w^2_{r_k} w_{r_k} d\tau, ds,
\]

\[
\mu \in L_2(\Omega), \quad (v, w, h) \in Z^3.
\]

Consider the following the problem: given a pair \((v_m, \mu_m) \in Z_m \times N_m\), find \((\hat{v}_m, \hat{\mu}_m)\) satisfying

\[
(\hat{v}_m, \hat{\mu}_m) \in Z_m \times N_m,
\]

\[
(\hat{M}(v_m, \hat{v}_m), h) + (A_1(\mu_m, v_m, \hat{v}_m), h) - (B^*_m \hat{\mu}_m, h) = (K + F, h), \quad h \in Z_m,
\]

\[
(B_m, \hat{v}_m, q) = 0, \quad q \in N_m.
\]

It follows from (7.5) and (7.14)–(7.16) that \(\hat{v}_m\) is a solution of the problem:

\[
\hat{v}_m \in W_m,
\]

\[
(\hat{M}(v_m, \hat{v}_m), h) + (A_1(\mu_m, v_m, \hat{v}_m), h) = (K + F, h) \quad h \in W_m.
\]

By Lemma 5.1, the operator

\[
\hat{M}(v_m, .) + A_1(\mu_m, v_m, .) : v \to \hat{M}(v_m, v) + A_1(\mu_m, v_m, v)
\]

is a mapping of \(Z\) into \(Z^*\) that is strictly monotone, coercive and continuous. Therefore, there exists a unique solution of the problem (7.17), (7.18), and by Lemma 7.1, there exists a unique function \(\hat{\mu}_m \in N_m\) such that:

\[
B^*_m \hat{\mu}_m = \hat{M}(v_m, \hat{v}_m) + A_1(\mu_m, v_m, \hat{v}_m) - K - F \quad \text{in } Z_m^*.
\]

In this case the pair \((\hat{v}_m, \hat{\mu}_m)\) is a unique solution of the problem (7.14)–(7.16).

We take \(h = \hat{v}_m\) in (7.18). Then by (3.4) and (3.7), we obtain

\[
\|\hat{v}_m\|_Z \leq c_1, \quad (v_m, \mu_m) \in Z_m \times N_m, \quad m \in \mathbb{N}.
\]

Lemma 7.1 and (7.19) imply

\[
\|\hat{\mu}_m\|_{L_2(\Omega)} \leq c_2, \quad (v_m, \mu_m) \in Z_m \times N_m, \quad m \in \mathbb{N}.
\]

For \(m \in \mathbb{N}\), we introduce a mapping \(B_m : Z_m \times N_m \to Z_m \times N_m\) as follows: \((v_m, \mu_m) \in Z_m \times N_m, B_m(v_m, \mu_m) = (\hat{v}_m, \hat{\mu}_m)\), where \((\hat{v}_m, \hat{\mu}_m)\) is the solution of the problem (7.14)–(7.16).

Let \(\{g_k, \alpha_k\} \in Z_m \times N_m\) and \(g_k \to g, \alpha_k \to \alpha\). By using (C1), (C2), (7.20), and (7.21), one can verify that \(B_m(g_k, \alpha_k) \to B_m(g, \alpha)\). Hence, \(B_m\) is a continuous mapping of \(Z_m \times N_m\) into itself.

Moreover, (7.20) and (7.21) yield that the mapping \(B_m\) maps a compact convex set

\[
d_m = \{(v, \mu) \in Z_m \times N_m, \quad \|v\|_Z \leq c_1, \quad \|\mu\|_{L_2(\Omega)} \leq c_2\}
\]
into itself. Therefore, the Schauder principle implies that there exists a pair \((u_m, p_m) \in Z_m \times N_m\) such that:

\[
B_m(u_m, p_m) = (u_m, p_m).
\] (7.22)

In addition, the pair \((u_m, p_m)\) is a solution of the problem (7.8)--(7.10) for any \(m\), and we have

\[
\|u_m\|_Z \leq c_1, \quad \|p_m\|_{L_2(\Omega)} \leq c_2, \quad m \in \mathbb{N}.
\] (7.23)

Hence, a subsequence \(\{u_k, p_k\}\) can be extracted from the sequence \(\{u_m, p_m\}\) such that:

\[
u_k \rightharpoonup u_0 \text{ in } Z, \quad u_k \to u_0 \text{ in } L_2(\Omega) \text{ and a.e. in } \Omega,
\] (7.24)

\[
p_k \to p_0 \text{ in } L_2(\Omega), \quad F_{rv}(p_k, u_k) \to F_{rv}(p_0, u_0) \text{ in } L_\infty(S_1), \quad M(u_k) + A(u_k, p_k) \to \Theta \text{ in } Z^*. \quad (7.25)
\]

Let \(k_0\) be a fixed positive integer and let \(h \in Z_{k_0}, q \in N_{k_0}\). By (7.24), (7.26), (7.28), we pass to the limit in (7.9), (7.10) with \(m\) changed by \(k\), which gives

\[
(\Theta - B^* p_0, h) = (K + F, h), \quad h \in Z_{k_0}, \quad \int_\Omega q \text{ div } u_0 \, dx = 0, \quad q \in N_{k_0}.
\] (7.29)

(7.30)

Since \(k_0\) is an arbitrary positive integer, we obtain by (5.24), (7.1), (7.29) and (7.30) that

\[
\Theta - B^* p_0 = K + F, \quad \text{in } Z^*
\] (7.31)

\[
\text{div } u_0 = 0.
\] (7.32)

We determine a mapping \(J_k : Z \to Z^*\) as follows:

\[
(J_k(v), h) = (\tilde{M}(u_k, v) + A_1(p_k, u_k, v), h), \quad k = 0, 1, 2, \ldots.
\] (7.33)

It follows from (5.34) and (7.13) that

\[
J_k(u_k) = M(u_k) + A(u_k, p_k),
\] (7.34)

and Lemma 5.1 gives

\[
(J_k(u_k) - J_k(v), u_k - v) \geq 0, \quad v \in Z, \quad k = 0, 1, 2, \ldots.
\] (7.35)

(7.25), (7.27) and the Lebesgue theorem imply

\[
\lim(J_k(v), u_k) = (J_0(v), u_0), \quad \lim(J_k(v), v) = J_0(v), v,
\] (7.36)

By (7.28), (7.31) and (7.34), we have

\[
\lim(J_k(u_k), v) - (B^* p_0, v) = (K + F, v).
\] (7.37)

Taking into account that

\[
(B_k^* p_k, u_k) = (p_k, B u_k) = 0,
\]

we get from (7.9), (7.24) and (7.34) that

\[
(J_k(u_k), u_k) = (K + F, u_k) \to (K + F, u_0).
\] (7.38)
Upon (7.36)–(7.38), we pass to the limit in (7.35), which gives
\[(K + F - J_0(v) + B^* p_0, u_0 - v) \geq 0, \quad v \in Z.\] (7.39)

Take here \(v = u_0 - \xi h, \xi > 0, h \in Z,\) and let \(\xi\) tend to zero. By Lemma 5.1, see (5.4), we get
\[(K + F - J_0(u_0) + B^* p_0, h) \geq 0, \quad h \in Z.\] (7.40)

Therefore, the pair \(u = u_0, p = p_0\) is a solution of the problem (4.7)–(4.9).

2) We will show that the solution of the problem (7.8)–(7.10) converge to the solution of (4.7)–(4.9) strongly.

Let
\[Y_k = (J_k(u_k) - J_0(u_0), u_k - u_0).\] (7.41)

Obviously
\[Y_k = (J_k(u_k) - J_k(u_0), u_k - u_0) + (J_k(u_0) - J_0(u_0), u_k - u_0).\] (7.42)

Upon (7.24), (7.36)–(7.38) and (7.41) \(\lim_{k \to \infty} Y_k = 0.\) By (7.24), (7.36) the second addend in (7.42) tends also to zero, and so
\[\lim (J_k(u_k) - J_k(u_0), u_k - u_0) = 0.\] (7.43)

Observing (7.33), (7.43) and Lemma 5.1 see (5.3), we obtain (7.11).

We take \(h \in Z_k\) in (4.8) and (7.9) and subtract (4.8) from (7.9). Then, we obtain
\[(B^*(p_k - \mu), h) = (M(u_k) + A(u_k, p_k) - M(u_0) - A(u_0, p_0), h) + (B^*(p_0 - \mu), h),\]
\(h \in Z_k, \quad \mu \in N_k.\) (7.44)

This equality together with (7.3) yields
\[\|p_k - \mu\|_{L^2(\Omega)} \leq \sup_{h \in Z_k} \frac{(B^*(p_k - \mu), h)}{\beta^* h_{\|Z\|}} \leq \beta^{-1} A_k + c\|p_0 - \mu\|_{L^2(\Omega)}, \quad \mu \in N_k,\] (7.45)

where
\[A_k = \|M(u_k) + A(u_k, p_k) - M(u_0) - A(u_0, p_0)\|_{Z^*}.\] (7.46)

Hence
\[\|p_0 - p_k\|_{L^2(\Omega)} \leq \inf_{\mu \in N_k} (\|p_0 - \mu\|_{L^2(\Omega)} + \|p_k - \mu\|_{L^2(\Omega)}) \leq \beta^{-1} A_k + (c + 1) \inf_{\mu \in N_k} \|p_0 - \mu\|_{L^2(\Omega)}.\] (7.47)

By (7.11) and (7.26), we obtain \(\lim_{k \to \infty} A_k = 0,\) and (7.1) implies
\[\lim_{k \to \infty} \inf_{\mu \in N_k} \|p_0 - \mu\|_{L^2(\Omega)} = 0,\]
so that (7.12) follows from (7.47).
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