Nonlinear boundary problem for Harmonic functions in higher dimensional Euclidean half-spaces

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Abstract

This paper concerns the existence of harmonic functions on the upper half-space $\mathbb{R}^n_+$ subject to linear and nonlinear Neumann boundary conditions with singular boundary data in Morrey-Lorentz spaces. The main tools developed were potential estimates for standard Riesz operator $I_\delta$ in Morrey-Lorentz spaces and BMO spaces as well as boundedness of certain Fourier multiplier $\sigma(D)$ operator in these spaces. As a byproduct of our potential estimates, we overcome the lack of strong-type trace theorems in Morrey spaces introducing a new functional space in order to show the existence and uniqueness of solutions from theorems developed in this functional setting. Moreover, we investigated qualitative properties of solutions such as self-similarity, positivity and radial-symmetry around the axis $\mathbb{R}^n_+$. Our results provide a new class for existence of harmonic functions in $\mathbb{R}^n_+$ with data $f$ in weak-Morrey space $w-M^\mu_p$ covering some previous ones, because $w-M^\mu_p$ is strictly larger than Morrey space $M^\mu_p$ which includes strictly Lebesgue $L^p$ space.

AMS MSC: 49J52, 42B37, 42B25, 35J05, 35J65, 42B15, 35J08, 35J67, 35J91

Keywords: Neumann boundary, Morrey-Lorentz spaces, harmonic functions, potential estimates, positive solutions, symmetry of solutions.

∗de Almeida, M.F. was supported by CNPq:409306/2016, FAPITEC:88887.157906/2017, Brazil (Corresponding author).
†Lima, L.S.M. was supported by CNPq:409306/2016, Brazil.
1 Introduction

In this paper we are interested in existence of harmonic functions on the upper half-space \( \mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\} \) subject to nonlinear Neumann boundary condition

\[
\begin{align*}
\Delta u & = 0 \quad \text{in} \ \mathbb{R}^n_+, \\
\frac{\partial u}{\partial \nu} & = V(x')u + b(x')|u|^\rho - 1u + f \quad \text{on} \ \partial \mathbb{R}^n_+,
\end{align*}
\]

(1.1)

where \( \rho > 1 \), \( \nu = (0, 0, \cdots, 0, -1) \) unit outward normal vector on \( \partial \mathbb{R}^n_+ \), \( b \) and \( V \) are certain singular potential on boundary \( \partial \mathbb{R}^n_+ \) and \( f \) is a suitable function. For \( b \equiv 0 \) the problem (1.1) becomes linear

\[-\Delta u = 0 \quad \text{in} \ \mathbb{R}^n_+ \quad \text{and} \quad \frac{\partial u}{\partial \nu} = V(x')u + f \quad \text{on} \ \partial \mathbb{R}^n_+.
\]

(1.2)

The problem of finding a harmonic function \( u \) on the upper half-space \( \mathbb{R}^n_+ \) with a weaker requirement in Neumann’s boundary data

\[
\frac{\partial u}{\partial \nu} = f \quad \text{on} \ \partial \mathbb{R}^n_+.
\]

(N)

or in Dirichlet’s boundary data

\[
u = g \quad \text{on} \ \partial \mathbb{R}^n_+,
\]

(D)

has been the focus of many papers, for instance [4, 14, 18, 24, 28, 33]. The most common condition to deal with \( f \) or \( g \) is linked to the continuous functions with higher order growth

\[
\int_{\partial \mathbb{R}^n_+} (1 + |x'|^2)^{-\frac{n-1}{2}} f(x')d\sigma < \infty \quad \text{and} \quad \int_{\partial \mathbb{R}^n_+} (1 + |x'|^2)^{-\frac{n}{2}} g(x')d\sigma < \infty
\]

(1.3)
to guarantee that layer potentials (see Section 4.4)

\[ N(f)(x) = \int_{\partial \mathbb{R}^n_+} G(x' - y', x_n) f(y') d\sigma \]

and

\[ D(g)(x) = \int_{\partial \mathbb{R}^n_+} \partial_n G(x' - y', x_n) g(y') d\sigma \]

are, respectively, harmonic functions in \( \mathbb{R}^n_+ \) satisfying (N) and (D), see Armitage [4, Theorem 1]. To weak the data \( f \) or \( g \) for a suitable class of functions space, look for estimate

\[
\int_{\partial \mathbb{R}^n_+} \frac{|g(x')|}{(1 + |x'|^2)^{\frac{1}{2}}} d\sigma \lesssim \int_{\partial \mathbb{R}^n_+} \frac{|g(x')|}{1 + |x' - x_0'|^n} d\sigma
\]

\[
\lesssim \int_{B(x_0',1)} |g(x')| d\sigma + \sum_{k=1}^{\infty} \int_{\{2^{k-1} < |x' - x_0'| < 2^k\}} \frac{|g(x')|}{1 + |x' - x_0'|^n} d\sigma
\]

\[
\lesssim \int_{B(x_0',1)} |g(x')| d\sigma + \sum_{k=1}^{\infty} 2^{-(k-1)n} 2^k (n-1) \int_{B(x_0',2^k)} |g(x')| d\sigma
\]

\[
\lesssim \sum_{k=0}^{\infty} 2^{-k} \int_{B(x_0',2^k)} |g(x')| d\sigma \lesssim (M_0g)(x_0'), \quad (1.4)
\]

and, similarly,

\[
\int_{\partial \mathbb{R}^n_+} \frac{|f(x')|}{(1 + |x'|^2)^{\frac{1}{2}}} d\sigma \lesssim \int_{B(x_0',1)} |f(x')| d\sigma + \sum_{k=1}^{\infty} 2^{(n-1) - k\alpha} 2^{k\alpha} \int_{B(x_0',2^k)} |f(x')| d\sigma
\]

\[
\lesssim (M_0f)(x_0') + (M_\alpha f)(x_0'), \quad (1.5)
\]

where \( M_\alpha \) stands for (tangential) fractional Hardy-Littlewood maximal function (see (3.1)) for \( 0 \leq \alpha < n - 1 \) and \( f, g \in L^1_{\text{loc}}(\mathbb{R}^{n-1}) \), we identify \( \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \). Therefore, via estimates (1.4) and (1.5), the data \( f \in \mathcal{X} \) or \( g \in \mathcal{X} \) can be more rough as soon as the maximal function \( M_\alpha \) is well defined there and maps continuously a function space \( \mathcal{X} \) to another function space \( \mathcal{Y} \), possibly \( \mathcal{X} = \mathcal{Y} \). Gardiner [18, Theorem 1] weakened the data \( f \) for locally integrable functions on \( \partial \mathbb{R}^n_+ \), in this case note that \( (M_\alpha f)(x_0') < \infty \) for \( 0 \leq \alpha < n - 1 \). Martell et. al [28] have shown, in particular, the equivalence between well-posedness of Dirichlet problem with data \( g \in \mathcal{X} \) and boundedness of \( M_\alpha \) in certain Köthe function space \( \mathcal{X} \) which recover a better family of linear functions spaces for initial data.

From Proposition 3.1 and Remark 4.3-A, we obtain that Hardy-Littlewood maximal operator \( M_0 \) and its fractional version \( M_\alpha \) maps continuously weak-Morrey space \( w-M_\lambda^p(\mathbb{R}^{n-1}) \) to \( w-M_\mu^p(\mathbb{R}^{n-1}) \), provided \( 0 < \alpha < (n - 1)/\lambda \) and \( r/\mu = p/\lambda \). Hence, in this scenario, is natural and we were able to show the well-posedness of linear equation (1.2) and its nonlinear version (1.1) with data in weak-Morrey space \( w-M_\lambda^p(\mathbb{R}^{n-1}) \) or Morrey-Lorentz space \( M_\mu^p(\mathbb{R}^{n-1}) \). The next lines we describe more precisely these results and their difficulties involved.
In our program, some key theorems related to Hardy-Littlewood maximal operator $M_\alpha$ and Riesz operator $I_\alpha$ must be proved in Morrey-Lorentz spaces. At first, employing a Fourier analysis easily we have

\[-\partial_n N = D \quad \text{and} \quad \partial_j N = DS_j = S_j D, \quad j = 1, 2, \cdots, n - 1, \tag{1.6}\]

where $S_j$ denotes the tangential $j$th Riesz transform defined on the upper half-space $\mathbb{R}^n_+$ or in $\mathbb{R}^{n-1}$. Hence, the double layer potential $N$ has certain smoothness in a function space, where $D$ and $S_j$ maps continuously. In Theorem 4.5 where we have shown that standard Mikhlin Fourier multiplier operators are bounded from $\mathcal{M}^\lambda_{p\kappa}(\mathbb{R}^n)$ to itself, in particular, we obtain that $S_j$ is bounded in $\mathcal{M}^\lambda_{p\kappa}(\mathbb{R}^{n-1})$ or $\mathcal{M}^\lambda_{p\kappa}(\mathbb{R}^n_+)$. Hence, from Theorem 4.8-(i) we obtain the following smoothness (see Corollary 4.9) for double layer potential

\[\|\partial_j N(f)\|_{\mathcal{M}^\mu_{\rho\kappa}(\mathbb{R}^n_+)} \leq C\|f\|_{\mathcal{M}^\lambda_{p\kappa}(\partial\mathbb{R}^n_+)}, \quad j = 1, 2, \cdots, n \tag{1.7}\]

for suitable parameters $p, r, \kappa, d, \lambda$ and $\mu$. Let us remark that Theorem 4.8 yields from our potential estimates, see Theorem 4.1 and Theorem 4.2, developed in Morrey-Lorentz spaces and in BMO spaces. These estimates are of independent interest and recovers some well-known theorems related to Riesz potentials and fractional maximal function $M_\alpha$, see details in Remark 4.3. In Section 3, as a consequence of [2, Lemma 4.1], we obtain the remarkable result

\[\|I_\alpha f\|_{\mathcal{M}^\lambda_{p\kappa}} \cong \|M_\alpha f\|_{\mathcal{M}^\lambda_{p\kappa}}.\]

Hence, to yield sharpness of restriction $r/\mu = p/\lambda$ (see Section 4.1) only needs work with fractional maximal function $M_\alpha$, as we did in Appendix (see Theorem 6.2 and Corollary 4.4).

The estimate (1.7) was fundamental to get well-posedness for linear boundary problem (1.2) as well as its nonlinear version. Indeed, the trace operator $(Tu)(x', x_n) = u(x', 0)$ is of weak-type bounded in Morrey spaces (see Adams [1, Theorem 5.1])

\[\|u(x', 0)\|_{\mathcal{M}^\mu_{\rho\kappa}(\partial\mathbb{R}^n_+)} \leq C\|\nabla u\|_{\mathcal{M}^\lambda_{\rho\kappa}(\mathbb{R}^n_+)}, \tag{1.8}\]

and from article of Ruiz and Vega [35], does not have any hope to show strong-type boundedness of trace operator in Morrey space $\mathcal{M}^\mu_{\rho\kappa}(\partial\mathbb{R}^n_+)$ via real interpolation technique. In a scenario of strong-type boundedness such as Lebesgue $L^d(\mathbb{R}^{n-1})$ space, we would proceed as in Ferreira, Everaldo and Montenegro [10], where they have shown the existence of solutions $u$ in functional space $D^1,p(\mathbb{R}^n_+) \cap L^q(\mathbb{R}^n_+)$ and taken data $f \in L^d(\mathbb{R}^{n-1})$ for $d = (n - 1)\frac{\rho - 1}{\rho}$. However, in Morrey spaces we can not proceed as [10] because we only have (1.8).

To overcome the lack of strong-type boundedness of trace operator in Morrey spaces, we follow the idea employed by Quittner and W. Reichel [32] in $L^1(\Omega) \times L^1(\partial\Omega)$ (see also [13] for a good description in $L^b(\Omega) \times L^q(\partial\Omega)$ and [5] for a study in $L^{p\infty}(\mathbb{R}^n_+) \times L^{q\infty}(\partial\mathbb{R}^n_+)$) and we employ a scaling analysis in order to yield a natural function space which will work as a substitute of (1.8), namely, the Banach space $A^{1,q\infty}_{1,\rho}$ endowed by norm

\[\|u\|_{A^{1,q\infty}_{1,\rho}} = \|\nabla u\|_{\mathcal{M}^\mu_{\rho\kappa}(\mathbb{R}^n_+)} + \|u\|_{\mathcal{M}^{\lambda\infty}_{\rho\kappa}(\partial\mathbb{R}^n_+)}.\]
Note that (1.2) has a scaling map of solutions
\[ u(x) \to u_\gamma(x) = \gamma^{\rho} u(\gamma x), \quad (1.9) \]
provided that \( u \) is a solution, \( V \) is homogeneous function of degree \(-1\) and \( f \) homogeneous of degree \(-(s + 1)\) on boundary \( \partial \mathbb{R}_+^n \). Hence, what makes \( A_{I_q}^{t, \infty} \) be invariant by scaling (1.9) is the same thing that makes (1.8) be invariant by such a scaling, namely, \( \frac{n-1}{\mu} = \frac{n}{\lambda} - 1 \). Therefore, from estimate (1.7) and Theorem 4.8-(ii), we were able to show the existence and uniqueness of mild solutions \( u \in A_{I_q}^{t, \infty} \) (see definition in Section 4.5.1) for linear problem (1.2), provided that \( V \in \mathcal{M}_{\rho, \infty}^{n-1} \) is small enough and \( f \in \mathcal{M}_{\rho, \infty}^\omega \) (see Theorem 4.10).

Employing a scaling analysis in (1.1), then the scaled function \( u_\gamma(x) = \gamma^{\rho} u(\gamma x) \), \( \gamma > 0 \), is a solution of (1.1) with data \( d(f)_\gamma = \gamma^{\rho} f(\gamma \cdot) \), provided \( u \) is a solution with data \( f \) and \( V \), \( b \), respectively, are homogeneous functions of degree \(-1\) and \( 0 \) on boundary \( \partial \mathbb{R}_+^n \). The scaling map
\[ u(x) \to u_\gamma(x) = \gamma^{\rho} u(\gamma x), \quad \gamma > 0 \quad (1.10) \]
is intrinsically linked to energy exponent \( \rho = \frac{n}{n-2} \) of (1.1). More precisely, for suitable functions \( u, V, b \) and \( f \), the problem (1.1) is Euler-Lagrange equation associated to energy functional \( E(u) \),
\[ E(u) = \frac{1}{2} \int_{\mathbb{R}_+^n} |\nabla u|^2 dx - \frac{1}{2} \int_{\partial \mathbb{R}_+^n} V(x') |u|^2 d\sigma \quad \text{and} \quad \frac{1}{2} \int_{\partial \mathbb{R}_+^n} b(x') |u|^{\rho+1} d\sigma + \int_{\partial \mathbb{R}_+^n} f(x') u d\sigma, \]
where \( d\sigma \) denotes the surface measure on \( \partial \mathbb{R}_+^n \). It is straightforward compute
\[ E(u_\gamma) = \gamma^{\rho} E(u) + 2 \gamma^{\rho} E(u) \quad (1.11) \]
provided \( V, b \) has the right homogeneity for scaling (1.10) and \( f = d(f)_\gamma \). Hence, the energy \( E(u_\gamma) = E(u) \) is preserved by scaling map exactly at \( \rho = \frac{n}{n-2} \). This exponent has an optimal behavior for existence or nonexistence of positive harmonic functions \( u \) in \( \mathbb{R}_+^n \) subject to Neumann nonlinear boundary conditions. More precisely, let \( c \in \mathbb{R} \) and consider the problem
\[ \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^n, \\ \frac{\partial u}{\partial \nu} = -cu^\rho & \text{on } \partial \mathbb{R}_+^n. \end{cases} \quad (1.12) \]

For \( \rho = \frac{n}{n-2} \), \( c < 0 \) and \( n \geq 3 \), a straightforward computation show us that (1.12) has a positive solution given explicitly by
\[ u(x', x_n) = \left( \frac{\varepsilon}{|x' - x_0'|^2 + |x_n - x_{0n}|^2} \right)^{n-1}, \quad \text{where} \quad \varepsilon = -\frac{(n-2)x_{0n}}{c}. \quad (1.13) \]
The case \( c > 0 \) and \( \rho = \frac{n}{n-2} \) is much more challenger. Employing Kelvin transform and moving sphere method, Li and Zhu [26] removed decaying \( u(x) = O(|x|^{2-n}) \) near \(+\infty\) in Escobar [9] and classified any positive classical solutions by \( u(x', x_n) = cA^{n/(n-2)}x_n + A \).
with $A > 0$. Let $1 \leq \rho < \frac{n}{n-2}$ and $c < 0$, by moving plane method Hu [22, Theorem 1.2] showed there’s no any positive classical solution of (1.12). However, for sub-critical $1 < \rho < \frac{n}{n-2}$ or supercritical $\frac{n}{n-2} \leq \rho < \infty$, Lou and Zhu [25, Theorem 1.1] have improved [26] and shown that all positive solutions of (1.12) are given by $u(x', x_n) = cA^p x_n + A$, provided $c > 0$. Hence, the sign of the constant in front the nonlinearity is fundamental in order to get existence of positive solutions, mainly for study of nonlinearities strictly supercritical $\frac{n}{n-2} < \rho < \infty$ as $c < 0$. Let us consider the boundary of (1.12) as follows

\[
\begin{align*}
\Delta u &= 0 & \text{in } \mathbb{R}^n_+,
\frac{\partial u}{\partial \nu} &= -c|u|^{\rho-1}u + V(x')u + f & \text{on } \partial \mathbb{R}^n_+.
\end{align*}
\] (1.14)

If $c = -1, f \equiv 0$ and $V(x') = -\alpha/|x'|^s$ for $s > 0$, Ferreira and Neves [12, Theorem 2.1] proved the Pohozaev identity

\[
\left( \frac{n-2}{2} - \frac{n-1}{\rho+1} \right) \int_{\mathbb{R}^n_+} |u|^{\rho+1} d\sigma - \frac{s-1}{2} \int_{\partial \mathbb{R}^n_+} \frac{\alpha}{|x'|^s} u^2 d\sigma = 0
\]

and have shown that (1.14) does not have positive solution, provided $\rho = \frac{n}{n-2}$ and $s \neq 1$ or $\rho \neq \frac{n}{n-2}$ and $s = 1$. However, if $s = 1$ and $\rho = \frac{n}{n-2}$, by a minimization and perturbation technique, Ferreira and Neves [12, Theorem 3.1] have shown existence of positive solutions. Moreover, by observing that energy functional $E(\cdot)$ over the Hilbert space $H = \{u \in D^{1,2}(\mathbb{R}^n_+): u|_0 \in L^2(\mathbb{R}^{n-1})\}$ satisfies Palais-Smale condition as $1 < \rho < \frac{n}{n-2}$ and $s = 0, \alpha = 1$, from Mountain-Pass theorem Abreu et al. [3, Proposition 2.13] have shown, among other things, the existence of ground state solution, that is, positive weak-solution. In other words, the presence of $Vu$ reverse nonexistence of positive solutions proved by Hu [22], if $V(x') = -\alpha$ is constant. If $f \neq 0$ and $f \in \mathcal{M}_{\rho_\infty}^\omega$ is small enough as $1 < p \leq \omega < n - 1$, from Theorem 4.11 we obtain that (1.14) has a unique mild solution $u \in A_{1q}^\infty$, provided $\frac{n-1}{n-2} < \rho < \infty, -c = b \in \mathcal{M}_{\omega}^{-1}$ and $V \in \mathcal{M}_{\ell_1}^{-1}$ is small enough. In particular, we can taken (see Remark 4.15)

\[
V(x') = \kappa_1 |x'|^{-1}, \quad b(x') = \kappa_2 |x'|^{-1} \quad \text{and} \quad f(x') = a \left( \frac{x_n}{|x'|} \right) |x'|^{-\frac{1}{p-1}}
\] (1.15)

and show the existence of self-similar symmetry solutions (see Theorem 4.13), that is, $u_\gamma(x) = u(x)$ and, moreover, these solutions are invariant by group of rotations around the axis $\overline{Ox_n}$ (see Theorem 4.14-B).

Let $-c = b$ and $V$ have to be non-negative constant sign, then the solution $u$ is positive in $\mathbb{R}^n_+$, provided $f$ is positive in certain subset $\mathcal{D} \subset \partial \mathbb{R}^n_+$ (see Theorem 4.14-A). Therefore, the nonlinear problem (1.14) with term $f \neq 0$ extend the range of $\rho$ for existence of positive solutions and includes, in particular, the supercritical case $\frac{n}{n-2} \leq \rho < \infty$ even for high singular data as in (1.15). But what happens in interval $\rho \in \left(1, \frac{n+1}{n-2}\right]$? We won’t be able to analysis this case, but we believe that employing Kelvin transformation in (1.14) and proceeding as [22, Theorem 1.1], we can prove nonexistence of positive solutions.

The organization of this paper is the following. In Section 2 we summarize basic definitions and properties of Lorentz spaces and Morrey-Lorentz spaces. In Section 3 we
prove boundedness of Hardy-Littlewood maximal function in Morrey-Lorentz spaces. In Section 4 we present the main result of this notes which were proved in Section 5.

2 Preliminaries

In this section we recall the main properties of the Lorentz and Morrey spaces, which would be necessary for the next sections.

2.1 The Lorentz spaces

Before get into details in Morrey spaces, let us recall some important properties of Lorentz space $L^{pd}(\Omega)$.

Let $\Omega \subseteq \mathbb{R}^n$ be a measure space endowed with Lebesgue measure $dm$, it is well known that Lebesgue space $L^p(\Omega)$ is completely determined by non-increasing rearrangement function $f^{*}(t)$ and distribution function $d_f(s)$ which are defined by

$$f^{*}(t) = \inf \{ s > 0 : d_f(s) \leq t \},$$

and $d_f(s) = |\{ x \in \Omega : |f(x)| > s \}|$, where $|E|$ stands for Lebesgue measure of a measurable set $E \subseteq \Omega$. Indeed, for $0 < p < \infty$ we have

$$\|f\|_{L^p(\Omega)} = \left( \int_0^{\|f\|_{L^p(\Omega)}} [t^{1/p}f^{*}(t)]^p \frac{dt}{t} \right)^{\frac{1}{p}} = \left( \int_0^{\|f\|_{L^p(\Omega)}} [d_f(s)^{1/p}ds] \right)^{\frac{1}{p}}. \quad (2.1)$$

This motives to define Lorentz space $L^{pd}(\Omega)$, by collection of all measurable function $f : \Omega \rightarrow \mathbb{R}$ such that $\|f\|_{pd} < \infty$, where

$$\|f\|_{pd} = \left( \frac{d}{p} \int_0^{\|f\|_{L^p(\Omega)}} [t^{1/p}f^{*}(t)]^p \frac{dt}{t} \right)^{\frac{1}{d}} = \left( p \int_0^{\|f\|_{L^p(\Omega)}} [d_f(s)^{1/p}ds] \right)^{\frac{1}{d}} \quad (2.2)$$

with $1 \leq p < \infty$ and $1 \leq d < \infty$. For $1 \leq p \leq \infty$ and $d = \infty$,

$$\|f\|_{pd} = \sup_{0 < t < \|f\|_{L^p(\Omega)}} t^{1/p}f^{*}(t) = \sup_{0 < s < \|f\|_{L^p(\Omega)}} [s^pd_f(s)]^{1/p}. \quad (2.3)$$

Note that $L^{\infty d}(\Omega) = \{ 0 \}$ for $1 \leq d < \infty$, $L^{\infty \infty}(\Omega) = L^\infty$, $L^{pp} \cong L^p$ for $1 \leq p < \infty$ and $L^{p\infty}(\Omega)$ denotes the weak-$L^p$ space. The Lorentz space $L^{pd}(\Omega)$ increase if the index $d$ increases. More precisely, the continuous inclusions

$$L^1(\Omega) \subset L^{pd_1}(\Omega) \subset L^p(\Omega) \subset L^{pd_2}(\Omega) \subset L^{p\infty}(\Omega) \quad (2.4)$$

holds for $1 < d_1 \leq p \leq d_2 < \infty$. Another important property, the scaling

$$\|f(\gamma \cdot)\|_{pd} = \gamma^{-\frac{p}{d}} \|f\|_{pd},$$

provided $\gamma x \in \Omega$ and $\gamma > 0$. 

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The quantities (2.2)-(2.3) does not satisfy the triangle inequality. However, if we consider the maximal function (see [20])

\[ f^\sharp(t) = \frac{1}{t} \int_0^t f^*(s) \, ds \quad (t > 0) \]

and define \( \|f\|_{pd} = \|f\|_{pd}^\sharp \), then \( f \mapsto \|f\|_{pd} \) defines a norm, when \( 1 < p < \infty \) and \( 1 \leq d \leq \infty \). Therefore, \( (L^pd, \|f\|_{pd}^\sharp) \) is a Banach space. Also, we have the inequality

\[ \|f\|_{pd}^\sharp \leq \frac{p}{p-1} \|f\|_{pd}^* \]

i.e., the functionals \( \|\cdot\|_{pd}^\sharp \) and \( \|\cdot\|_{pd}^* \) are equivalent.

Consider the interpolation functor \((\cdot, \cdot)_{\theta,d}\) constructed via the \( K_{\theta,d} \)-method and defined on the categories of quasi-normed or normed spaces. For \( 0 < p_1 < p < p_2 \leq \infty \) such that \( \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \) and \( 1 \leq d_1, d_2 \leq \infty \), we have (see [20] or [6, Theorems 5.3.1 and 5.3.2])

\[ (L^{p_1d_1}, L^{p_2d_2})_{\theta,d} = L^{pd}. \]

Moreover, \((\cdot, \cdot)_{\theta,d}\) is exact functor of exponent \( \theta \), where \( 0 < \theta < 1 \). The multiplication operator \( T_f(g) = fg \) works well in Lorentz spaces (see [30, Theorem 3.4, 3.5]).

**Lemma 2.1.** Let \( 1 \leq p_1, p_2 \leq \infty \), \( 1 < r \leq \infty \) and \( 1 \leq z_1, z_2 \leq \infty \) be such that

\[ \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{z_1} + \frac{1}{z_2} \geq 1, \]

where \( s \geq 1 \). If \( f \in L^{p_1z_1} \) and \( g \in L^{p_2z_2} \), then

\[ \|fg\|_{rs} \leq \frac{r}{r-1} \|f\|_{p_1z_1} \|g\|_{p_2z_2}. \] (2.7)

If \( f \in L^{p_1z_1} \) and \( g \in L^{p_2z_2} \), then

\[ \|fg\|_{L^1} \leq \|f\|_{p_1z_1} \|g\|_{p_2z_2}. \]

We finish this section with Minkowski type inequality in \( L^{pd} \)-spaces. More precisely, let \( K(\cdot, y) \in L^{rd}(\mathbb{R}^n) \) and \( f \) be an integrable function, then

\[ \left\| \int_{\mathbb{R}^n} K(x, y) f(y) \, dy \right\|_{L^{pd}(\mathbb{R}^n, dx)} \leq \int_{\mathbb{R}^n} \|K(x, y)\|_{L^{pd}(\mathbb{R}^n, dx)} |f(y)| \, dy \] (2.8)

for every \( 1 \leq p \leq \infty \) and \( 1 \leq d \leq \infty \).

### 2.2 The Morrey-Lorentz spaces

In this section we define and recall some properties of Morrey-Lorentz spaces, the reader can find more details in [1], [11] and [34].
Let $\Omega \subset \mathbb{R}^n$ and $B(x_0, \ell_Q)$ be a ball in $\mathbb{R}^n$, with $n \geq 1$. We say that $L^p$-functions $f$ supported in $Q := B(x_0, \ell_Q) \cap \Omega$, belongs to homogeneous Morrey space $\mathcal{M}^p_{\kappa\Omega}(\Omega)$, if the functional

$$\|f\|_{\mathcal{M}^p_{\kappa\Omega}} := \sup_{Q \subset \Omega} |Q|^\frac{1}{p} \|f\|_{L^\kappa(Q)}$$

is finite, where $1 \leq q \leq p < \infty$, $1 \leq \kappa \leq \infty$ and $|Q|$ denotes the Lebesgue measure of $Q$. Notice that $\mathcal{M}^p_{\infty}(\Omega)$ denotes the weak-Morrey space $w-M^p_{\kappa}(\Omega)$, the space $\mathcal{M}^p_{pk}(\Omega) = L^k(\Omega)$ denotes the Lorentz space for $p > 1$, $\mathcal{M}^p_{q\kappa}(\Omega) = \mathcal{M}^p_q(\Omega)$ denotes the Morrey space, for $1 \leq q \leq p < \infty$. The functions in $\mathcal{M}^p_{1,1}(\Omega) = \mathcal{M}^p_1(\Omega)$ can be identified by Radon measures whose total variation $|\mu|$ satisfies

$$\|\mu\|_{\mathcal{M}^p_1} = \sup_{Q \subset \Omega} \ell^n(\frac{1}{\kappa} - 1)|\mu|(Q) < \infty. \quad (2.9)$$

It is well known that Hardy-Littlewood maximal function $M_0$ is not bounded from $\mathcal{M}^p_1(\mathbb{R}^n)$ to itself (see for instance [15, Exemple]), however $M_0$ is bounded from $\mathcal{M}^p_1(\mathbb{R}^n)$ to $w-M^p_1(\mathbb{R}^n)$ (see Proposition 3.1). This motivated Gunawan et al. [16, Theorem 1.2] construct a function $g \in \mathcal{M}^p_{q\kappa}(\mathbb{R}^n)$ such that $g \notin \mathcal{M}^p_q(\mathbb{R}^n)$, therefore weak-Morrey spaces are strictly larger than Morrey spaces. Hence, the embeddings

$$L^p \hookrightarrow \mathcal{M}^p_q \hookrightarrow \mathcal{M}^p_{q\kappa} \quad (2.10)$$

must be understood in strict sense. We finish this section, recalling the Hölder inequality proved by Ferreira [11].

**Lemma 2.2.** Let $1 < q_i \leq \mu_i < \infty$ be such that $\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}, \frac{1}{\mu_1} = \frac{1}{\mu_2} + \frac{1}{\mu_3}$. Then there is a universal constant $C > 0$ such that

$$\|fg\|_{\mathcal{M}^p_{q_1\mu_1}} \leq C\|f\|_{\mathcal{M}^p_{q_2\mu_2}}\|g\|_{\mathcal{M}^p_{q_3\mu_3}}, \quad (2.11)$$

where $d_i \geq 1$ satisfies $\frac{1}{d_i} \leq \frac{1}{d_1} + \frac{1}{d_2}$.

## 3 Maximal functions

In what follows, the term $F \lesssim_p L$ will be written if there is $C = C(p) > 0$ such that $F(x) \leq CL(x)$, for all $x \in \mathbb{R}^n$. Also, $F \equiv_p L$ means that $F \lesssim_p L \lesssim_p F$.

The **uncentered fractional maximal function** $M_\alpha$ of a locally integrable function $f : \mathbb{R}^n \to [-\infty, \infty]$ is defined by

$$(M_\alpha f)(x) = \sup_{B(z, \ell) \ni x} |B(z, \ell)|^\frac{\alpha}{n} \int_{B(z, \ell)} |f(y)|dy, \quad (3.1)$$

where $0 \leq \alpha < n$, $B(z, \ell)$ is a ball centered at $z$ with radii $\ell > 0$ and $\int_{B(z, \ell)}$ denotes the average of $f$ in $B(z, \ell)$,

$$\int_{B(z, \ell)} |f(y)|dy = \int_{B(z, \ell)} |f(y)|dy \quad \text{if} \quad \int_{B(z, \ell)} |f(y)|dy = \left(\frac{1}{|B(z, \ell)|}\right) \int_{B(z, \ell)} |f(y)|dy.$$
Moreover, if easily we obtain for every \( \lambda < \infty \) and \( 1 \leq \kappa \leq \infty \) (\( p = \kappa = \infty \) is standard), again from Vitali type covering lemma there is constants such that (see [7, Theorem 3.8])

\[
(M_0 f)^* (t) \lesssim_n f^\varepsilon (t) \lesssim_n (M_0 f)^* (t), \quad (t > 0)
\]

for every \( f \in L^1_{loc} (\mathbb{R}^n) \). It follows from (2.5) that

\[
\| M_0 f \|_{L^p (\mathbb{R}^n)} = \left( \frac{\kappa}{p} \int_0^\infty [ (M_0 f)^* (t) ]^\kappa t^{\frac{\kappa}{p} - 1} dt \right)^\frac{1}{\kappa}
\]

\[
\lesssim_n \left( \frac{\kappa}{p} \int_0^\infty [ f^\varepsilon (t) ]^\kappa t^{\frac{\kappa}{p} - 1} dt \right)^\frac{1}{\kappa}
\]

\[
\lesssim_p \| f \|_{L^p (\mathbb{R}^n)},
\]

provided that \( f \in L^p (\mathbb{R}^n) \). Therefore,

\[
\| M_0 f \|_{M^\lambda \mathcal{P}_c} = \sup_Q |Q|^{\frac{1}{p} - \frac{1}{\kappa}} \| \chi_Q M_0 f \|_{p \mathcal{P}_c} \lesssim \sup_Q |Q|^{\frac{1}{p} - \frac{1}{\kappa}} \| \chi_Q f \|_{p \mathcal{P}_c} = \| f \|_{M^\lambda \mathcal{P}_c}
\]

for every \( 1 < p \leq \lambda < \infty \) with \( 1 \leq \kappa \leq \infty \) and \( f \in M^\lambda \mathcal{P}_c \). Moreover, by inequality (3.2) \( 0 \leq \alpha < n \) and we obtain

\[
\| M_0 f \|_{M^\lambda_1 \mathcal{P}_c} \leq 5^{-n} \| f \|_{M^\lambda_1 \mathcal{P}_c},
\]

for every \( 1 \leq \lambda < \infty \) and \( f \in M^\lambda_1 \mathcal{P}_c \). In resume, we obtain the following Proposition.

**Proposition 3.1.** Let \( 1 < p \leq \lambda < \infty \) and \( 1 \leq \kappa \leq \infty \). If \( f \in M^\lambda_\mathcal{P}_c \), the Hardy-Littlewood maximal function \( M_0 \) satisfies

\[
\| M_0 f \|_{M^\lambda_\mathcal{P}_c} \lesssim \| f \|_{M^\lambda_\mathcal{P}_c}.
\]

Moreover, if \( 1 \leq \lambda < \infty \) and \( f \in M^\lambda_1 \mathcal{P}_c \) then

\[
\| M_0 f \|_{M^\lambda_1 \mathcal{P}_c} \leq 5^{-n} \| f \|_{M^\lambda_1 \mathcal{P}_c}.
\]

**Remark 3.2.** Let \( 1 < \kappa = p < \infty \), then \( M^\lambda_\mathcal{P}_c = M^\lambda_\mathcal{P} \) and we obtain [8, Theorems 1,2] of Chiarenza and Frasca.

For \( 0 < \alpha < n \), the fractional maximal function and Riesz potential are very close, in view of

\[
(M_\alpha f) (x) \lesssim \sup_{E > 0} c^\alpha c^{n - \alpha} \int_{B(x, E)} \frac{|f (y)|}{|x - y|^{n - \alpha}} dy \lesssim I_\alpha(|f|) (x). \quad (3.4)
\]

The reverse inequality does not hold, however from lemma below Adams and Xiao have shown (see [2, Theorem 4.2])

\[
\| I_\alpha f \|_{M^\lambda_\mathcal{P}} \lesssim \| M_\alpha f \|_{M^\lambda_\mathcal{P}}, \quad (3.5)
\]

for all \( 1 < p \leq \lambda < \infty \), where \( I_\alpha f \) denotes the Riesz potential

\[
(I_\alpha f) (x) = c_{n\alpha} \int_{\mathbb{R}^n} \frac{f (y)}{|x - y|^{n - \alpha}} dy, \quad \text{where} \quad c_{n\alpha} = \frac{\Gamma \left( \frac{n - \alpha}{2} \right)}{2^n \pi^{n/2} \Gamma \left( \frac{n}{2} \right)}.
\]
Lemma 3.3 ([2]). Let $0 < \alpha < n$ and $I_\alpha f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then

(i) $(I_\alpha f)^\#(x) \cong M_\alpha f(x)$, for all $x \in \mathbb{R}^n$;

(ii) Given a cube $Q \subset \mathbb{R}^n$ and $\varepsilon, s > 0$, then

$$d_{I_\alpha f}(s) \leq |\{x \in Q : |(I_\alpha f)^\#(x)| > 2^{-1}\varepsilon s\}| + \varepsilon \{x \in Q : |(I_\alpha f)(x)| > 2^{-n-1}s\},$$

where $F^\#$ denotes the Fefferman-Stein sharp maximal function

$$F^\#(x) = \sup_{Q \ni x} \int_Q |F(y) - F_Q|dy. \quad (3.6)$$

The remarkable inequality (3.5) can be easily extended to $M^\lambda_p$. Indeed, let $1 \leq p \leq \lambda < \infty$ and $1 \leq d < \infty$ from Lemma 3.3(ii) we may infer

$$\|I_\alpha f\|_{L^p(\mathbb{R}^n)}^d = \int_0^{|Q|} \left[ (d_{I_\alpha f}(s))^{\frac{1}{p}} s \right]^d ds \leq \int_0^{\varepsilon |Q|} \left[ (d_{I_\alpha f}(2^{-1}\varepsilon s))^{\frac{1}{p}} s \right]^d ds + \int_0^{\varepsilon |Q|} \left[ \varepsilon d_{I_\alpha f}(2^{-n-1} s)\right]^{\frac{d}{p}} s^{d} ds \leq 2^d \varepsilon^{-d} \int_0^{\varepsilon |Q|} \left[ (d_{I_\alpha f}(s))^{\frac{1}{p}} s \right]^d ds \leq \varepsilon^{d} 2^{(n+1)d} \int_0^{\varepsilon |Q|} \left[ (d_{I_\alpha f}(s))^{\frac{1}{p}} s \right]^d ds. \quad (3.7)$$

Let $\varepsilon > 0$ be such that $\varepsilon^d 2^{(n+1)d} = 2^{-1}$, then

$$2^{-1} \int_0^{\varepsilon |Q|} \left[ (d_{I_\alpha f}(s))^{\frac{1}{p}} s \right]^d ds \lesssim 2^{d+p(n+1)d} \int_0^{\varepsilon |Q|} \left[ (d_{I_\alpha f}(s))^{\frac{1}{p}} s \right]^d ds. \quad (3.8)$$

The case $1 \leq p < \infty$ and $d = \infty$ can be proved similarly. These estimates jointly with Lemma 3.3(i) and inequality (3.4) gives us the following Proposition.

**Proposition 3.4.** Let $1 \leq p \leq \lambda < \infty$ and $1 \leq \kappa \leq \infty$. If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $0 < \alpha < n$, then

$$\|I_\alpha f\|_{M^\lambda_p} \cong \|M_\alpha f\|_{M^\lambda_p}. \quad (3.8)$$

We finish this section noting that, in weak sense, the partial derivative of $I_1$ satisfies

$$\partial_j (I_1 f)(x) = -R_j(f)(x),$$

where $R_j$ stands for $j$th Riesz transform. Since $R_j$ is bounded in Morrey space $M^\lambda_p$ for $1 < p < \lambda < \infty$, this leads to $\|\partial_j I_1 f\|_{M^\lambda_p} \leq C\|f\|_{M^\lambda_p}$. Hence, in view of (3.4), we ask if the partial derivative $\partial_j M_\alpha f$ belongs to certain Morrey spaces, for $f \in M^\lambda_p$. The argument of [21, Theorem 3.1] does not apply in this context, because the function $f(x) = |x|^{-\frac{n}{p}} \in M^\lambda_p(\mathbb{R}^n)$ can not be approximated by $C^\infty(\mathbb{R}^n)$-functions, see [38].
4 Main results

4.1 Riesz potentials

In this part we extend for Morrey-Lorentz space, some well known potential estimates for Riesz operator $I_\delta$. Before, let us recall that a measurable function $F$ is said to be in BMO space if,

$$\|F\|_{BMO} = \sup_{x \in \mathbb{R}^n} F^*(x) < \infty.$$ 

Our first potential estimate reads as follows.

**Theorem 4.1.** Let $n \geq 3$, $1 < p \leq \lambda < \infty$, $1 < r \leq \mu < \infty$ and $0 < \delta < n/\lambda$ be such that

$$(i) \, \frac{\delta}{n} = 1 - \frac{1}{\mu} \quad (ii) \, \frac{r}{\mu} = \frac{p}{\lambda}. \tag{4.1}$$

Then $I_\delta f \in \mathcal{M}_{p\mu}^\nu$ and there is a universal constant $C > 0$ such that

$$\|I_\delta f\|_{\mathcal{M}_{p\mu}^\nu} \lesssim_n \|f\|_{\mathcal{M}_{p\mu}^\nu}. \tag{4.2}$$

Moreover, if $\delta = n/\lambda$ we obtain

$$\|I_\delta f\|_{BMO} \lesssim_n \|f\|_{\mathcal{M}_{p\mu}^\nu}, \tag{4.3}$$

where $f \in \mathcal{M}_{p\mu}^\nu(\mathbb{R}^n)$ satisfies $\frac{k}{p} \leq \frac{\nu}{r}$ if $1 \leq \kappa, \nu < \infty$, or $\kappa = \nu = \infty$.

The Theorem 4.1 holds for $1 = p \leq \lambda < \infty$. In this case, we need of weak-type $(1, 1)$ boundedness of Hardy-Littlewood maximal function $M_0$ given in Proposition 3.1.

**Theorem 4.2.** Let $n \geq 3$, $1 \leq \lambda < \infty$, $1 < r \leq \mu < \infty$ and $0 < \delta < n/\lambda$ be such that

$$(i) \, \frac{\delta}{n} = 1 - \frac{1}{\mu} \quad (ii) \, \frac{r}{\mu} = \frac{1}{\lambda}. \tag{4.4}$$

Then, there is a universal constant $C > 0$ such that

$$\|I_\delta f\|_{\mathcal{M}_{r\infty}^\nu} \lesssim_n \|f\|_{\mathcal{M}_{r\infty}^\nu}. \tag{4.5}$$

Moreover, if $\delta = n/\lambda$ we obtain

$$\|I_\delta f\|_{BMO} \approx \|f\|_{\mathcal{M}_{r\infty}^\nu}, \quad \lambda > 1 \tag{4.6}$$

for all $f \in \mathcal{M}_{1}^\lambda(\mathbb{R}^n)$.

**Remark 4.3.**

(A) Under hypothesis of Theorem 4.1, the pointwise inequality (3.4) implies

$$\|M_\delta f\|_{\mathcal{M}_{p\mu}^\nu} \lesssim_n \|f\|_{\mathcal{M}_{p\mu}^\nu}, \tag{4.7}$$

for every $f \in \mathcal{M}_{p\mu}^\nu(\mathbb{R}^n)$. If $p = 1$, by Theorem 4.2 we obtain

$$\|M_\delta f\|_{\mathcal{M}_{r\infty}^\nu} \lesssim_n \|f\|_{\mathcal{M}_{r\infty}^\nu}. \tag{4.8}$$
(B) If $\lambda = 1$ in Theorem 4.2, then $0 < \delta < n$ and $r = \mu = \frac{n}{n-\delta}$ which leads to
\[
\|I_\delta f\|_{L^{\frac{n}{n-\delta}}(\mathbb{R}^n)} \leq C \|f\|_{L^1(\mathbb{R}^n)}.
\]

(C) Let $1 < \kappa = p < \infty$ and $1 < \nu = r < \infty$, we obtain Theorems 9 and 10 in [29].

(D) Let $\delta = n/p$ and $1 < p = \lambda = \kappa < \infty$, from Theorem 4.1 we obtain
\[
\|I_\delta f\|_{\text{BMO}} \lesssim n \|f\|_{L^p(\mathbb{R}^n)},
\]
note that $f \in L^\infty$ is not recovered.

(E) In (B), (C) and (D), the Riesz potential $I_\delta$ can be replaced by fractional maximal function $M_\delta$, in view the pointwise inequality (3.4).

4.2 The Sharpness

The next theorem show that hypothesis $r \mu = p \lambda$ is sharp. In view of norm equivalence between $I_\alpha$ and $M_\alpha$ gave in Proposition 3.4, only needs prove the sharpness for fractional maximal function as we did in Theorem 6.2, see details in Appendix.

**Corollary 4.4.** Let $1 < p < \lambda < \infty$, $1 < r < \mu < \infty$ and $n/\mu < \alpha < n/\lambda$ be such that
\[
\frac{\alpha}{n} = \frac{1}{\lambda} - \frac{1}{\mu} \quad \text{and} \quad \frac{r}{\mu} > \frac{p}{\lambda}.
\]
Then $I_\alpha$ does not maps continuously $\mathcal{M}^\lambda_{pr}$ into $\mathcal{M}^\mu_{r\nu}$.

Consequently, one has same sharpness in Theorem 4.8 and Corollary 4.9 below.

4.3 Fourier multipliers operators

Let $\sigma(D)$ be a Fourier multiplier operator
\[
\sigma(D)f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \sigma(\xi)\hat{f}(\xi) d\xi,
\]
where the symbol $\sigma \in C^\ell(\mathbb{R}^n \backslash \{0\})$ satisfies
\[
|D^\beta \sigma(\xi)| \leq C_\beta |\xi|^{-\delta - |\beta|}, \quad 0 \leq |\beta| \leq \ell,
\] (4.8)
for every $0 \leq \delta < \frac{n}{\lambda}$. By observing that $\sigma(D)f$ can be realized in terms of a convolution operator,
\[
\sigma(D)f(x) = (K \ast f)(x) = \int_{\mathbb{R}^n} K(x - y) f(y) dy,
\]
whenever $f \in L^2$ has compact support and $x \notin \text{supp} f$. Then, the proof of Theorem 4.1 works well in this context, since $K$ agrees in $\mathbb{R}^n \backslash \{0\}$ with a kernel $|\hat{K}(x)| \lesssim |x|^{\delta - n}$ as $\delta > 0$. This pointwise estimate can be obtained by means of Littlewood-Palay theory, in
Appendix (see Proposition 6.1) we sketch the prove (see [36, pg. 26] for \( \delta = 0 \)). More precisely, we outline that if

\[
|D_\xi^\beta \sigma(\xi)| \leq C_\beta |\xi|^{-|\beta|}, \quad 0 \leq |\beta| \leq \lfloor n/2 \rfloor,
\]

then \( K \) agrees with a smooth function \( \tilde{K} \) away from the origin and satisfies

\[
|D^\beta \tilde{K}(x)| \lesssim |x|^{-n-|\beta|}.
\]

Let \( \delta = 0 \) in (4.9), it follows from (4.10) that \( K(x) = \sigma^\vee(x) \) is a singular kernel, that is, \( \sigma(D) \) is a Calderon-Zygmund operator, because \( \|\hat{K}\|_\infty \leq C \) and (4.10) implies that \( |K(x)| \lesssim |x|^{-n} \) is locally integrable away from the origin and satisfies the standard Hörmander condition

\[
\int_{|x| \geq c|y|} |K(x - y) - K(x)| dx \leq A
\]

for every \( y \neq 0 \) and \( c > 1 \). By real interpolation in \( L^{p_k} \), we obtain the boundedness of \( \sigma(D) \) in \( L^{p_k} \), provided \( \sigma(\xi) \) satisfies the smoothness (4.12). Hence, we can prove the following theorem.

**Theorem 4.5.** Let \( 1 < p < \lambda < \infty \). If \( \sigma(\xi) \) is smooth away from the origin and satisfies

\[
|D_\xi^\beta \sigma(\xi)| \leq C_\beta |\xi|^{-|\beta|}, \quad 0 \leq |\beta| \leq \lfloor n/2 \rfloor,
\]

Then \( \sigma(D) \) maps continuously \( \mathcal{M}^{\lambda}_{p_\kappa} \) to itself, for every \( 1 \leq \kappa \leq \infty \).

Recall that the \( j \)th Riesz transform of \( f \in S'(\mathbb{R}^n) \) is defined by singular integral operator of convolution type

\[
R_j(f)(x) = \Gamma \left( \frac{n + 1}{2} \right) \pi^{-\frac{n+1}{2}} \lim_{\epsilon \to 0} \int_{|y| \geq \epsilon} \frac{y_j}{|y|^{n+1}} f(x - y) dy
\]

which can be characterized by a multiplier \( R_j : S(\mathbb{R}^n) \to S'(\mathbb{R}^n) \) with symbol

\[
P.V. \frac{x_j}{|x|^{n+1}}(\xi) = \sigma(R_j) = -\frac{i\xi_j}{|\xi|} \quad \text{in} \; S'(\mathbb{R}^n),
\]

where \( \xi \in \mathbb{R}^n \setminus \{0\} \). In particular, we obtain the following result (see [11, Lemma 2.3]).

**Corollary 4.6.** The \( j \)th Riesz transform \( R_j \) is bounded in \( \mathcal{M}^{\lambda}_{p_\kappa}(\mathbb{R}^n) \), as \( 1 < p < \lambda < \infty \) and \( 1 \leq \kappa \leq \infty \).

### 4.4 Harmonic functions on the upper half-space

Let \( g \in S(\mathbb{R}^{n-1}) \), the Poisson integral of \( g \) is defined by

\[
(Dg)(x) = \int_{\partial \mathbb{R}^n_+} \partial_n G(x' - y', x_n) g(y') d\sigma_{y'},
\]
where $\partial_n G(x', x_n) = P_{x_n}(x')$ denotes the harmonic Poisson kernel

$$P_{x_n}(x') = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \frac{x_n}{(x_n^2 + |x'|^2)^\frac{n}{2}}.$$ 

It is well known that

$$\lim_{x_n \to 0^+} (Dg)(x', x_n) = g(x')$$

and the Poisson integral $Dg$ is the unique $C^\infty(\mathbb{R}^n_+)$ solution of the Dirichlet problem

$$\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^n_+,
 u|_{\partial \mathbb{R}^n_+} = g & \text{on } \partial \mathbb{R}^n_+.
\end{cases}$$

where $u|_{\partial \mathbb{R}^n_+}$ should be understood in each point $x' \in \partial \mathbb{R}^n_+$ by

$$u|_{\partial \mathbb{R}^n_+}(x') = \lim_{x_n \to 0^+} (Dg)(x', x_n).$$

Passing the Fourier transform in $\Delta u$, we obtain the following ODE in $x_n$

$$(\partial^2_n - 4\pi|\xi'|^2)\hat{u} = 0,$$

whose solution is explicitly given by

$$\hat{u}(\xi', x_n) = e^{-2\pi|\xi'|x_n}\hat{g}(\xi), \text{ for } \hat{g}(\xi') = \hat{u}(\xi', 0).$$

It follows that $D$ can be defined by multiplier operator

$$(Dg)(x) = \int_{\mathbb{R}^{n-1}} e^{2\pi\mathrm{i}x' \cdot \xi'} e^{-2\pi|\xi'|x_n}\hat{g}(\xi')d\xi'.$$

Now, consider the normal derivative of $\hat{u}(\xi', x_n)$,

$$-\partial_n \hat{u}(\xi', x_n) = 2\pi|\xi'| e^{-2\pi|\xi'|x_n}\hat{g}(\xi')$$

and for $f \in \mathcal{S}(\mathbb{R}^{n-1})$ consider the Neumann problem

$$\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^n_+,
 \partial_n u = f(x') & \text{on } \partial \mathbb{R}^n_+.
\end{cases} \tag{N}$$

Then $\hat{f}(\xi') = -\lim_{x_n \to 0^+} \partial_n \hat{u}(\xi', x_n) = 2\pi|\xi'| \hat{g}(\xi')$ and (N) has a solution given by multiplier

$$(Nf)(x) = \int_{\mathbb{R}^{n-1}} e^{2\pi\mathrm{i}x' \cdot \xi'} e^{-2\pi|\xi'|x_n} \frac{i\xi}{|\xi|} \hat{f}(\xi')d\xi'. \tag{4.15}$$

Hence, from dominated converge theorem we obtain

$$(\partial_j Nf)(x) = \int_{\mathbb{R}^{n-1}} e^{2\pi\mathrm{i}x' \cdot \xi'} e^{-2\pi|\xi'|x_n} \frac{i\xi_j}{|\xi|} \hat{f}(\xi')d\xi' = (DS_j f)(x),$$

where $S_j$ denotes the tangential $j$th Riesz transform acting in $\mathbb{R}^n_+$ or $\mathbb{R}^{n-1}$ whose symbol is given by $\sigma(S_j) = \frac{i\xi_j}{|\xi|}$, $\xi' \in \mathbb{R}^{n-1}\setminus \{0\}$ and $j = 1, \ldots, n - 1$. 

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Proposition 4.7. Let \( n \geq 3 \) and \( f \in \mathcal{S}(\mathbb{R}^{n-1}) \), then

(i) \(-\partial_{n}Nf = Df\)

(ii) \( \partial_{j}Nf = DS_{j}f = S_{j}Df, \; j = 1, 2, \cdots, n - 1.\)

From Theorems 4.1 and 4.2 we will get the following estimates for layer potentials. But before, let \( \varphi(x)|_{0} = \varphi(x', 0) \) stands for restriction of \( \varphi \) to \( \partial \mathbb{R}^{n}_{+} = \mathbb{R}^{n-1}.\)

Theorem 4.8. Let \( n \geq 3, 1 < r \leq \mu < \infty \) and \( 1 \leq p \leq \lambda < n - 1 \) be such that \( \frac{n-1}{\lambda} = \frac{n-1}{\mu} + \frac{1}{r}.\)

Let \( f \in \mathcal{M}^{\lambda}_{p, \infty}(\partial \mathbb{R}^{n}_{+}) \) or \( f \in \mathcal{M}^{\lambda}_{1}(\partial \mathbb{R}^{n}_{+}).\)

(i) If \( \frac{n-1}{\lambda} = \frac{n-1}{\mu} + \frac{1}{r} \), there is a constant \( C > 0 \) such that

\[
\|D(f)\|_{\mathcal{M}^{p}_{d}(\mathbb{R}^{n}_{+})} \leq C\|f\|_{\mathcal{M}^{\lambda}_{p, \infty}(\mathbb{R}^{n-1})}
\]

(ii) If \( \frac{n-1}{\lambda} = \frac{n-1}{\mu} + 1 \), there is a positive constant \( C > 0 \) such that

\[
\|N(f)|_{0}\|_{\mathcal{M}^{p}_{d}(\partial \mathbb{R}^{n}_{+})} \leq C\|f\|_{\mathcal{M}^{\lambda}_{p, \infty}(\mathbb{R}^{n-1})},
\]

provided \( \frac{d}{p} \leq \frac{d}{r} \) and \( 1 \leq d, \kappa \leq \infty \), we shall take \( d = \infty \) and \( \kappa = 1 \), if \( p = 1 \). Moreover, we have

\[
\|N(f)|_{0}\|_{\text{BMO}} \lesssim \|f\|_{\mathcal{M}^{\lambda-1}_{p, \infty}(\mathbb{R}^{n-1})}.
\]

From Proposition 4.7, Theorem 4.8 and Corollary 4.6, we obtain the following regularity in \( \mathcal{M}^{\mu}_{r, d}(\mathbb{R}^{n}_{+}) \).

Corollary 4.9. Let \( n \geq 3, 1 < r \leq \mu < \infty \) and \( 1 \leq p \leq \lambda < n - 1 \) be such that \( \frac{n-1}{\lambda} = \frac{n-1}{\mu} + \frac{1}{r} \) and \( \frac{r}{\mu} = \frac{p}{\lambda} \), there is an universal constant \( C > 0 \) such that

\[
\|\partial_{l}N(f)\|_{\mathcal{M}^{p}_{d}(\mathbb{R}^{n}_{+})} \leq C\|f\|_{\mathcal{M}^{\lambda}_{p, \infty}(\mathbb{R}^{n-1})}, \; l = 1, 2, \cdots, n
\]

for every \( f \in \mathcal{M}^{\lambda}_{p, \infty}(\mathbb{R}^{n-1}) \) provided \( \frac{d}{p} \leq \frac{d}{r} \) and \( 1 \leq d, \kappa \leq \infty \). If \( p = 1 \), consider \( d = \infty \) and \( \kappa = 1 \).

4.5 Existence and Symmetries

Let us denote by \( A^{\infty}_{q} \) the space of locally integrable functions \( u: \mathbb{R}^{n} \rightarrow \mathbb{R} \) such that \( u|_{0} \in \mathcal{M}^{\lambda}_{q, \infty}(\mathbb{R}^{n-1}) \) and \( (\nabla u)|_{\partial \mathbb{R}^{n}_{+}} \in \mathcal{M}^{\mu}_{\infty}(\mathbb{R}^{n}_{+}) \) satisfies

\[
\|u\|_{A^{\infty}_{q}} = \|\nabla u\|_{\mathcal{M}^{\mu}_{\infty}} + \|u|_{0}\|_{\mathcal{M}^{\lambda}_{q, \infty}} < \infty.
\]

The space \( A^{\infty}_{q} \) is a Banach space, provided \( 1 \leq r \leq \mu < \infty \) and \( 1 \leq q \leq \lambda < \infty \). Moreover, thanks to \( \|f(\gamma \cdot)\|_{\mathcal{M}^{\mu}_{d, \infty}(\Omega)} = \gamma^{-\frac{d(\cdot)}{r}}\|f\|_{\mathcal{M}^{\mu}_{d, \infty}(\Omega)} \), the norm \( \| \cdot \|_{A^{\infty}_{q}} \) preserves the scaling map (1.9) if and only if \( (\gamma - 1)/\lambda = (\gamma - 1)/\mu - 1. \)
4.5.1 Well-posedness

Formally, the boundary nonlinear problem (1.1) is equivalent to the integral equation

\[ u = N(f) + \mathcal{T}_V(u) + \mathcal{B}(u), \]  

(4.16)

where \( \mathcal{T}_V(u) = N(Vu) \) and \( \mathcal{B}(u) = N(b|u|^{\rho-1}u) \). In this paper solutions of (1.1) will be understood as solutions of the integral equation (4.16) in sense of distributions, such solutions is called mild solutions for (1.1). Moreover, formally, the linear boundary problem

\[ \Delta u = 0 \text{ in } \mathbb{R}^n_+ \text{ and } \frac{\partial u}{\partial \nu} = f + V(x')u \text{ on } \partial \mathbb{R}^n_+ \]  

(4.17)

is equivalent to

\[ u(x) = N(f)(x) + \mathcal{T}_V(u)(x). \]

In the next theorem, we will consider \( 1 < r \leq \mu < \lambda \) and \( 1 < q \leq \lambda < \infty \) satisfying \( r/\mu = q/\lambda \), where \( \lambda = (n-1)\omega/(n-1-\omega) \) and \( 1 < \omega < n - 1 \).

**Theorem 4.10.** If \( 1 < p \leq \omega < n-1 \) for \( n \geq 3 \), then for every \( f \in \mathcal{M}_{p_\infty}^\omega(\partial \mathbb{R}^n_+) \) the linear problem (4.17) has a unique mild solution \( u \in A^{1\infty}_{rq} \) such that

\[ \|u\|_{A_{rq}^{1\infty}} \leq \frac{C}{1-L} \|f\|_{\mathcal{M}_{p_\infty}^\omega(\partial \mathbb{R}^n_+)}, \]  

(4.18)

provided \( V \in \mathcal{M}_{\ell_1\infty}^{n-1}(\partial \mathbb{R}^n_+) \) satisfies \( L = C\|V\|_{\mathcal{M}_{\ell_1\infty}^{n-1}} < 1 \) for all \( 1 < \ell_1 \leq n - 1 \).

As a byproduct of Theorem 4.10, we obtain the well-posedness for the problem (1.1). To this end, let \( 1 < r \leq \mu < \infty \) and \( 1 < q \leq \lambda < \infty \) be such that \( r/\mu = q/\lambda \). Also, note that

\[ \frac{n-1}{n-2} < \rho < \infty \text{ implies } 1 < \omega = (n-1)\frac{\rho-1}{\rho} < n - 1. \]

**Theorem 4.11.** Let \( \omega = (n-1)(\rho-1)/\rho \), where \( \frac{n-1}{n-2} < \rho < \infty \) for \( n \geq 3 \). Let \( 1 < p \leq \omega < n-1 \) and \( f \in \mathcal{M}_{p_\infty}^\omega \). Given \( 1 < \ell_1, \ell_2 \leq n - 1 \), consider \( V \in \mathcal{M}_{\ell_1\infty}^{n-1} \) and \( b \in \mathcal{M}_{\ell_2\infty}^{n-1} \).

**I** (Existence and uniqueness) There are \( \varepsilon > 0 \) and \( C > 0 \) such that if \( \|f\|_{\mathcal{M}_{p_\infty}^\omega} \leq \varepsilon/C \), the (BVP) (1.1) has a unique mild solution \( u \in A^{1\infty}_{rq} \) such that

\[ \|\nabla u\|_{\mathcal{M}_{\ell_1\infty}^\omega(\mathbb{R}^n_+)} + \|u\|_{\mathcal{M}_{\ell_2\infty}^\omega(\partial \mathbb{R}^n_+)} \leq \frac{2\varepsilon}{1-L}, \]  

(4.19)

provided \( L = C\|V\|_{\mathcal{M}_{\ell_1\infty}^{n-1}} < 1 \) and \( \mu < \lambda = (n-1)(\rho-1) \).

**II** (Stability of data) Moreover, the solution \( u \) depends continuously of the data \( f \) and potentials \( b \) and \( V \).

**Remark 4.12.**
(A) Theorem 4.11 works well in Morrey-Lorentz spaces, that is, if \( \|f\|_{\mathcal{M}_{p,d}} \) is small enough and \( C\|V\|_{\mathcal{M}_{q,d}^{-1}} < 1 \), the nonlinear boundary problem (1.1) has a unique mild solution \( u \in A_{rq}^{d_1 d_2} \) such that
\[
\|\nabla u\|_{\mathcal{M}_{r_1}^\mu (\mathbb{R}_+^n)} + \|u\|_{\mathcal{M}_{q,d}^\lambda (\partial \mathbb{R}_+^n)} \leq \frac{2\varepsilon}{1 - L}.
\]

(B) Item (A) works well for \( p = d, r = d_1 \) and \( q = d_2 \), that is, if \( f \in \mathcal{M}_{p,\infty} (\partial \mathbb{R}_+^n) \) is small enough, the unique mild solution \( u \) satisfy \( \nabla u \in \mathcal{M}_{p,\infty}^\mu \). However, we can not in general apply Morrey’s lemma in order to show that \( u \in C_0^{\alpha, \alpha} (\mathbb{R}_+^n) \) is Hölder continuous with exponent \( \alpha = 1 - \frac{n}{\mu} \), unless \( A_{rq}^{d_1 d_2} \) does not have scaling symmetry, since \( \mu = n(\rho - 1)/\rho \) if \( (n - 1)/\lambda = (n/\mu) - 1 \). See Theorem 4.13, for scaling symmetry.

Let us recall that \( u \in A_{rq}^{\infty} \) is a weak solution to (1.1) if
\[
\int_{\mathbb{R}_+^n} \nabla u \nabla \varphi dx = \int_{\partial \mathbb{R}_+^n} [Vu + b|u|^{p-1}u + f(y)] \varphi d\sigma,
\]
for every \( \varphi \in C^\infty_0 (\mathbb{R}_+^n) \). From prove of [11, Theorem 1.2(A)], the solution \( u \) of Theorem 4.11 is a weak-solution in the sense of (4.21). Then \( \langle u(x), \Delta \varphi (x) \rangle = 0 \) for all \( \varphi \in C^\infty_0 (\mathbb{R}_+^n) \) which yields \( u \in C^\infty (\mathbb{R}_+^n) \).

4.5.2 Symmetries

In the next theorems, we study self-similarity, symmetries and positivity of the solution obtained in Theorem 4.11. We point out that self-similarity is expected, since the space \( A_{rq}^{\infty} \) which we look for solutions can be invariant by scaling (1.10). In this case, we necessary have \( (n - 1)/\lambda = (n/\mu) - 1 \), then \( \lambda = (n - 1)(\rho - 1) \) give us \( \mu = n(\rho - 1)/\rho \). Note that \( \mu < \lambda \) if and only if \( \rho > n/(n - 1) \).

**Theorem 4.13.** Under the hypotheses of Theorem 4.11, let \( V \) and \( b \) be homogeneous function of degree \(-1\) and zero, respectively. Then the solution \( u \) corresponding to \( f \in \mathcal{M}_{p,\infty}^\mu \) is a self-similar solution, that is,
\[
u(x) = u_\gamma (x), \; \gamma > 0.
\]

Let \( A \) be the a subset of all rotations \( Ox_n \) around the axis \( \overrightarrow{Ox_n} = \{x_n = 0\} \). We recall that a function \( g \) is symmetric (resp. antisymmetric) under the action of \( A \) when \( g(T(x)) = g(x) \) (resp. \( g(T(x)) = -g(x) \)) for all \( T \in A \). Our result of radial symmetry and positivity of solutions read as follows:

**Theorem 4.14.** Under the hypotheses of Theorem 4.11. Let \( D \subset \mathbb{R}^{n-1} \) be a positive-measure space.

(A) If \( f, V \) and \( b \) are non-negative (resp. non-positive) in \( \partial \mathbb{R}_+^n \) and \( f \) is positive (resp. negative) in \( D \), then \( u \) is positive (resp. negative) in \( \mathbb{R}_+^n \).
Let $V$ and $b$ be radially symmetric in $\mathbb{R}^{n-1}$. If $f$ is symmetric (resp. antisymmetric) then the solution $u$ is symmetric (resp. antisymmetric) under the action of $A$.

**Remark 4.15.** We can combine Theorem 4.13 and Theorem 4.14-(B) to yield solutions self-similar and invariant by group of rotations around the axis $Ox_n$. For instance, let

$$V(x') = \kappa_1 |x'|^{-1}, \quad b(x') = \kappa_2 |x'|^{-1} \quad \text{and} \quad f(x') = a\left(\frac{x_n}{|x'|}\right) |x'|^{-\frac{1}{\rho-1}},$$

where $a(z) \in BC(\mathbb{R}_+)$ and $\kappa_1, \kappa_2$ are constants.

## 5 Proof of theorems

**Key estimates**

In this first part, we list important lemmas obtained from works of Adams [1] and Hedberg [19]. We put their proof just for reader convenience.

**Lemma 5.1** (Hedberg’s inequality). Let $0 < \delta < \alpha$, then there is a constant $C > 0$ such that

$$|(I_{\delta} f)(x)| \leq C \left[ M_\alpha f(x) \right]^\frac{\delta}{n} \left[ M_0 f(x) \right]^{1-\frac{\delta}{n}},$$

for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

**Proof.** Let $\rho > 0$ and $A_k(x) = \{ y \in \mathbb{R}^n : 2^k \rho < |y-x| < 2^{k+1} \rho \}$, to get a discretization of $I_{\delta} f$ as follows

$$\frac{1}{c_{n\delta}} |(I_{\delta} f)(x)| \leq \sum_{k \in \mathbb{Z}} \int_{A_k(x)} |x-y|^{\delta-n} |f(y)| \, dy
\leq \sum_{k \geq 0} (2^{k+1} \rho)^\delta (2^k \rho)^{-n} \int_{A_k(x)} |f(y)| \, dy + \sum_{k > 0} (2^{-k+1} \rho)^\delta (2^{-k} \rho)^{-n} \int_{A_{-k}(x)} |f(y)| \, dy.$$

It follows that

$$|(I_{\delta} f)(x)| \lesssim_n \rho^{\delta-\alpha} \left( \sum_{k=0}^{\infty} 2^{(k+1)(\delta-\alpha)} \right) \widetilde{M}_\alpha f(x) + \rho^\delta \left( \sum_{k=1}^{\infty} 2^{(k-1)\delta} \right) \widetilde{M}_0 f(x)
\cong \rho^{\delta-\alpha} M_\alpha f(x) + \rho^\delta M_0 f(x), \quad (5.1)$$

since the series above converges and by pointwise equivalence between the centered Hardy-Littlewood maximal function $\widetilde{M}_\alpha$ and $M_0$ with they uncentered version. Let $\rho = (M_\alpha f/M_0 f)^\frac{\alpha}{\delta}$ in (5.1), then

$$|(I_{\delta} f)(x)| \lesssim_{n\delta} \left[ M_\alpha f(x) \right]^\frac{\delta}{n} \left[ M_0 f(x) \right]^{1-\frac{\delta}{n}},$$

as we desired. \qed
An easily consequence of Hedberg's inequality (see [1]) reads as follows.

**Lemma 5.2.** Let $0 < \delta < \alpha$ and consider

$$\frac{1}{p_2} = \frac{b}{p_1} + \frac{a}{q}$$

and $b = 1 - \frac{\delta}{\alpha}$ and $a = \frac{\delta}{\alpha}$. Let $M_0 f \in L^{p_1 z_1}(\Omega)$ and $M_\alpha f \in L^{q z_3}(\Omega)$, then

$$\|I_\delta f\|_{p_2 z_2} \lesssim_a \|M_\alpha f\|^{\frac{\alpha}{q z_3}} \|M_0 f\|^{\frac{1 - \delta}{p_1 z_1}},$$

for every $1 \leq z_1, z_2, z_3 \leq \infty$.

**Proof.** Note that $\frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{q} + \frac{1}{a} - \frac{1}{\delta}$ and $\frac{1}{z_2} = \frac{1}{z_1} + \frac{1}{z_3}$. By Lemma 5.1 and Lemma 2.1, we have

$$\|I_\delta f\|_{p_2 z_2} \lesssim ||[M_\alpha f]^a||_{(q/a, z_3/a)} \|M_0 f\|^{b} ||[M_0 f]^b||_{(p_1/b, z_1/b)} = \|M_\alpha f\|^{a} \|M_0 f\|^{b}. \quad \Box$$

The fractional maximal function $M_{n/\lambda}$ maps continuously $\mathcal{M}^\lambda_{p\kappa}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. In fact, at first note that for Borel sets $A \subset \Omega$ with finite measure, we have

$$|A|^{\frac{1}{p} - 1} \int_A |f(x)| dx \leq \left( \frac{p}{p - 1} \right)^{\frac{1}{p}} \|f\|_{L^p(A)}^p, \quad (5.2)$$

provided $1 < p < \infty$ and $1 \leq k \leq \infty$ such that $\frac{1}{k} + \frac{1}{k'} = 1$. This inequality easily follows from Hölder inequality in $L^k(\mathbb{R}, dt/t)$,

$$\int_A |f(x)| dx = \int_0^{\int_A f^*(t) dt} \left( \int_0^A t^{1 - \frac{1}{p}} \left( \int_0^t f^*(s) ds \right) dt \right)^{\frac{1}{p'}} \|f\|_{L^p(A)}^p$$

$$\leq \left( \int_0^A \left( \int_0^t f^*(s) ds \right)^{\frac{1}{p}} \right)^{\frac{1}{p'}} \|f\|_{L^p(A)}^p \leq \left( \frac{p}{p - 1} \right)^{\frac{1}{p}} |A|^{\frac{1}{p} - 1} \|f\|_{L^p(A)}^p.$$

If $k = \infty$, it is well known that

$$\|f\|_{L^\infty(\Omega)} \approx \sup_{|A| < \infty} \|A|^{\frac{1}{p} - 1} \int_A |f(y)| dy. \quad (5.3)$$

Hence, we may infer

$$(M_{n/\lambda} f)(x) = \sup_{Q \ni x} \left\{ |Q|^{\frac{1}{p} - 1} \int_Q |f(y)| dy \right\}$$

$$\leq \left( \frac{p}{p - 1} \right)^{\frac{1}{p}} \sup_{Q \ni x} |Q|^{\frac{1}{p} - 1} |Q|^{\frac{1}{p} - \frac{1}{p'}} \|f\|_{L^p(Q)} \leq \left( \frac{p}{p - 1} \right)^{\frac{1}{p}} \|f\|_{\mathcal{M}^\lambda_{p\kappa}}.$$
Corollary 5.3. Let $1 < p \leq \lambda < \infty$ and $1 \leq k \leq \infty$, then $M_{n/\lambda}$ is a bounded map from $\mathcal{M}^{\lambda}_{pk}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Proof of Theorem 4.1. Setting $K(x) \equiv |x|^{\delta-n}$ we consider

$$I_\delta f(x) = T_1(f) + T_2(f)$$

$$:= \int_{|y-x|<\rho} K(x-y)f(y) dy + \int_{|y-x|\geq\rho} K(x-y)f(y) dy,$$  \hspace{1cm} (5.4)

for $\rho > 0$. If $x \in Q(x_0, \rho)$, we obtain $|y-x_0| < 2\rho$ when $y \in Q(x, \rho)$. Let us define

$$f'(y) = \begin{cases} f(y), & |y-x_0| < 2\rho \\ 0, & |y-x_0| \geq 2\rho \end{cases} \text{ and } f''(y) = f(y) - f'(y).$$

Hence, for $x \in Q(x_0, \rho)$ we can write

$$T_1(f) = (I_\delta f')(x) \text{ and } T_2(f) = (I_\delta f'')(x).$$

Let $\alpha = n/\lambda$ and $\delta = n/\lambda - n/\mu$, then $\frac{1}{\mu} = \frac{1}{p} \left( 1 - \frac{\delta}{\alpha} \right)$ is equivalent to $\frac{\mu}{\nu} = \frac{\lambda}{p}$. Then, by Lemma 5.2 with $(p_1, p_2, q) = (p, r, \infty)$ and $(\zeta_1, \zeta_2, \zeta_3) = (\kappa, \nu, \infty)$ satisfying $\frac{1}{\nu} \leq \frac{1}{\kappa}(1 - \frac{\delta}{\alpha})$, we estimate

$$\|I_\delta f'\|_{L^{r\nu}(Q(x_0, \rho))} \lesssim \|M_\alpha f'\|_{L^\infty(Q(x_0, \rho))}^{\frac{\mu}{\nu}}\|M_0 f'\|_{L^{p\nu}(Q(x_0, 2\rho))}^{1-\frac{\mu}{\nu}}.$$  \hspace{1cm} (5.5)

Noting that

$$\left( \frac{1}{\lambda} - \frac{1}{p} \right) \left( 1 - \frac{\delta}{\alpha} \right) = \frac{1}{\mu} - \frac{1}{r},$$

the inequality (5.5) give us

$$|Q(x_0, \rho)|^{\frac{\mu}{\nu} - \frac{1}{r}} \|T_1(f)\|_{L^{r\nu}(Q(x_0, \rho))} \lesssim \|M_\alpha f'\|_{L^\infty(Q(x_0, \rho))}^{\frac{\mu}{\nu}} \left( |Q(x_0, 2\rho)|^{\frac{\mu}{\nu} - \frac{1}{r}} \|M_0 f\|_{L^{p\nu}(Q(x_0, 2\rho))} \right)^{1-\frac{\mu}{\nu}}.$$ 

By Corollary 5.3, the fractional maximal operator $M_{n/\lambda}$ maps continuously $\mathcal{M}^{\lambda}_{pk}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. Since $M_0$ maps continuously $\mathcal{M}^{\lambda}_{pk}(\mathbb{R}^n)$ to itself (see Proposition 3.1), then

$$|Q(x_0, \rho)|^{\frac{\mu}{\nu} - \frac{1}{r}} \|T_1(f)\|_{L^{r\nu}(Q(x_0, \rho))} \lesssim \|f\|_{\mathcal{M}^{\lambda}_{pk}}.$$ 

In order to show the boundedness of $T_2(f)$, let $x \in Q(x_0, \rho)$ and note that

$$|T_2(f)(x)| \leq \int_{|y-x|>2\rho} |x-y|^{\delta-n} |f(y)| dy = \int_{2\rho}^{\infty} \ell^{\delta-n} d\pi(\ell),$$

where $\pi$ denotes the measure $\pi(\ell) = \int_{|y-x|<\ell} |f(y)| dy$. By inequality (5.2), we obtain

$$\pi(\ell) \lesssim |Q(x, \ell)|^{1-\frac{\mu}{\nu}} \|f\|_{L^{p\nu}(Q(x, \ell))} \lesssim |Q(x, \ell)|^{1-\frac{1}{r}} \|f\|_{\mathcal{M}^{\lambda}_{pk}}.$$
Hence, from polar coordinates and integration by parts, one has

\[
|T_2(f)(x)| \leq -\int_{2\rho}^\infty \frac{\partial}{\partial \ell} \ell^{\delta-n} \pi(\ell) d\ell \
\lesssim \left( \int_{2\rho}^\infty \ell^{\delta-\alpha-1} d\ell \right) \|f\|_{M^\lambda p_n} \
\cong \rho^{-\frac{n}{\alpha}} \|f\|_{M^\lambda p_n},
\]

because \(0 < \delta < \alpha\) and \(\delta - \alpha = -n/\mu\). It follows that

\[
\|T_2(f)\|_{L^r(Q(x_0,\rho))} \lesssim |Q|^{-\frac{1}{p}} \|f\|_{M^\lambda p_n} \left( \int_{0}^{|Q|} \ell^{\frac{\alpha}{\lambda}-1} d\ell \right)^\frac{1}{r} = |Q|^{-\frac{1}{p}} \|f\|_{M^\lambda p_n}.
\]

This finish the proof of (4.2). For \(1 < p \leq \lambda < \infty\) one has \(0 < \delta = n/\lambda < n\), then by Lemma 3.3(i) and Corollary 5.3 we obtain

\[
\|I_{n/\lambda} f\|_{\text{BMO}} = \sup_{x \in \mathbb{R}^n} (I_{n/\lambda} f)^\sharp(x) \cong \sup_{x \in \mathbb{R}^n} (M_{n/\lambda} f)(x) \leq \|f\|_{M^\lambda p_n}.
\]

Note that \(\kappa, \nu\) can be infinity. \(\square\)

**Proof of Theorem 4.2.** Noting that fractional maximal function \(M_{n/\lambda}\) satisfies

\[
(M_{n/\lambda} f)(x) = \sup_{B(z, \ell) \ni x} |B(z, \ell)|^{-\frac{1}{\lambda}} \int_{B(z, \ell)} |f(y)| dy = \|f\|_{M^\lambda 1},
\]

and noting that \(M_0\) is of weak-type \((1,1)\) (see Proposition 3.1), from Lemma 5.2 with \((p_2, p_1, q) = (r, 1, \infty)\) we obtain

\[
\|I_{\delta} f\|_{L^{r_\infty}(Q(x_0,\rho))} \lesssim \|\pi\|_{M^\lambda p_n} \|f\|_{L^1(Q(x_0,2\rho))}^{\frac{1}{r} - \frac{q}{r}}. \tag{5.6}
\]

Now, noting

\[
\pi(\ell) \leq \ell^{n-\frac{2}{\lambda}} (M_2 f)(x) \lesssim \ell^{n-\frac{2}{\lambda}} (M_2 f)(x) = \ell^{n-\frac{2}{\lambda}} \|f\|_{M^\lambda 1}
\]

and proceeding like before, we obtain

\[
|I_{\delta} f''(x)| \leq \int_{2\rho}^\infty \ell^{\delta-n} d\pi(\ell) \
\lesssim \left( \int_{2\rho}^\infty \ell^{\delta-\frac{2}{\lambda}-1} d\ell \right) \|f\|_{M^\lambda 1} \
\cong \rho^{-\frac{n}{\alpha}} \|f\|_{M^\lambda 1},
\]

for all \(x \in Q(x_0, \rho)\). It follows that

\[
\|I_{\delta} f''\|_{L^{r_\infty}(Q(x_0,\rho))} \lesssim \rho^{-\frac{n}{\alpha}} \rho^\frac{n}{\alpha} \|f\|_{M^\lambda 1}. \tag{5.7}
\]

The inequalities (5.6) and (5.7) will give us the estimate (4.4). Note that Lemma 3.3(i) easily yields

\[
\sup_{x \in \mathbb{R}^n} (I_{n/\lambda} f)^\sharp(x) \cong \sup_{x \in \mathbb{R}^n} (M_{n/\lambda} f)(x) = \|f\|_{M^\lambda 1}.
\]
provided $1 < \lambda < \infty$.

**Proof of Theorem 4.5.** Let us rewritten $\sigma(D)$ as follows $\sigma(D)f := T_1 + T_2$, where $T_1(f) = \sigma(D)f'$, $T_2(f) = \sigma(D)f''$ and

$$f'(y) = \begin{cases} f(y), & |y - x_0| < 2\rho \\ 0, & |y - x_0| \geq 2\rho \end{cases} \quad \text{and} \quad f''(y) = f(y) - f'(y),$$

for every $\rho > 0$. We observed that $\sigma(D)$ is a bounded operator in $L^{p_\kappa}(\mathbb{R}^n)$, provided that (4.12) holds. Then, we obtain

$$\|T_1(f)\|_{L^{p_\kappa}(Q(x_0,\rho))} \leq \|\sigma(D)f'\|_{L^{p_\kappa}} \lesssim \|f\|_{L^{p_\kappa}(Q(x_0,2\rho))}$$

which yields

$$|Q(x_0, \rho)|^{\frac{1}{\kappa} - \frac{1}{p}} \|T_1(f)\|_{L^{p_\kappa}(Q(x_0,\rho))} \lesssim |Q(x_0, \rho)|^{\frac{1}{\kappa} - \frac{1}{p}} \|f\|_{L^{p_\kappa}(Q(x_0,2\rho))} \lesssim \|f\|_{\mathcal{M}^\lambda_{p_\kappa}}.$$

In order to show boundedness of $T_2$, we starting remarking that

$$|\sigma(D)f''(x)| \lesssim \int_{2\rho}^{\infty} \ell^{-\frac{n}{2}} d\pi(\ell)$$

for every $x \in Q(x_0, \rho)$. Now we proceed like Theorem 4.1 to get

$$|T_2(f)(x)| \lesssim \left( \int_{2\rho}^{\infty} \ell^{-\frac{n}{2} - 1} d\ell \right) \|f\|_{\mathcal{M}^\lambda_{p_\kappa}} \lesssim \rho^{-\frac{n}{2}} \|f\|_{\mathcal{M}^\lambda_{p_\kappa}}.$$

It follows that $\|T_2(f)\|_{L^{p_\kappa}(Q(x_0,\rho))} \lesssim |Q|^{\frac{1}{\kappa} - \frac{1}{p}} \|f\|_{\mathcal{M}^\lambda_{p_\kappa}}$. Those inequalities will give us the theorem.\hfill $\Box$

**Proof of Theorem 4.8.** Part (i): We starting recalling

$$D(f)(x) = \int_{\partial \mathbb{R}^n_+} \partial_n G(x' - y, x_n) f(y) d\sigma_y = d_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{|x' - y|^2 + x_n^2} f(y) dy,$$

where $x \in \mathbb{R}^n_+$ and $y \in \partial \mathbb{R}^n_+$. Let $\Omega \subseteq \mathbb{R}^n_+$ be a cube and $\Omega'$ its projection on $\partial \mathbb{R}^n_+$. If $Q = Q(x_0, \ell_Q)$ is a subcube of $\Omega$ with sidelenth $\ell_Q$, then $Q' \subseteq \Omega'$ and $Q = Q' \times [0, \ell_Q] \subset \Omega' \times (0, \infty)$ (see Figure 1).

![Figure 1: The cube Q and its projection Q' on ∂R^n_+](image)

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It follows by Fubini’s theorem that
\[
\left| Q^{\frac{1}{p}} \right| \int |D(f)(x)| \, dx = \left| Q^{\frac{1}{p}} \right| \int |D(f)(x', x_n)| \, dx' \otimes dx_n
\]
\[
= \left| Q^{\frac{1}{p}} \right| \int |D(f)(x', \cdot)| \, L^1(I) \left\| L^1(Q') \right. .
\]
Hence, from (5.3) we can infer
\[
\| D(f) \|_{L^p(\Omega)} \leq C_k \left\| D(f)(x', \cdot) \right\|_{L^p((0, \infty))} \left\| L^p(\Omega) \right.
\]
\[
\text{Let } Z = \Omega' \times (0, \infty) \text{ and } 1 < p_1 < r < p_2 < \infty \text{ such that } \frac{1}{r} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, 0 < \theta < 1. \text{ Then, } D \text{ is continuous from } L^{p_1}(Z) \to L^{p_2}(\Omega). \text{ By interpolations}
\]
\[
\left( L^{p_1}(\Omega), L^{p_2}(\Omega) \right)_{\theta, d} = L^d(\Omega) \text{ and } \left( L^{p_1}(Z), L^{p_2}(Z) \right)_{\theta, d} = L^d(Z)
\]
we obtain
\[
\| D(f) \|_{L^d(\Omega)} \leq C^{1-\theta}_1 C^\theta_2 \left\| D(f)(x', \cdot) \right\|_{L^d((0, \infty))} \left\| L^d(\Omega) \right.
\]
Since \( \partial_n G(x' - y, \cdot) \in L^r(0, \infty) \) and \( L^r(0, \infty) \hookrightarrow L^d(0, \infty) \) holds, then by Minkowski inequality (2.8) one has
\[
\| D(f)(x', \cdot) \|_{L^d((0, \infty), dx_n)} \leq \int_{R^{n-1}} \| \partial_n G(x' - y, \cdot) \|_{L^d((0, \infty), dx_n)} |f(y)| \, dy
\]
\[
\leq \int_{R^{n-1}} \| \partial_n G(x' - y, \cdot) \|_{L^d((0, \infty), dx_n)} |f(y)| \, dy
\]
\[
\leq d_n \int_{R^{n-1}} \left( \int_0^\infty \frac{dx_n}{(1 + x_n^2 + y')^{2n/2}} \right)^{\frac{1}{1}} |f(y)| \, dy
\]
\[
\leq \int_{R^{n-1}} \frac{1}{|x' - y|} \left( \int_0^\infty \frac{dx_n}{(1 + x_n^2 + y')^{2n/2}} \right)^{\frac{1}{1}} |f(y)| \, dy.
\]
Inserting (5.9) into (5.8), by Theorem 4.1 and Theorem 4.2 in \( R^{n-1} \) with \( \delta = 1/r \) we have
\[
\| \Omega^{\frac{1}{p} - \frac{1}{2}} \| D(f) \|_{L^r(\Omega)} \leq \| \Omega^{\frac{1}{p} - \frac{1}{2}} \| \| f \|_{M^p_{\theta}(R^{n-1})} \| L^d(\Omega') \|
\]
\[
\leq \| f \|_{M^p_{\theta}(R^{n-1})} \| L^d(\Omega') \|
\]
For \( p = \kappa = 1 \), necessarily one has \( d = \infty \), because of Theorem 4.2. The case \( 1/r = \delta = (n - 1)/\lambda \) we won’t be able to do it.

**Part (ii):** Consider the double layer potential
\[
N(f)(x) = \int_{\partial B^+} G(x' - y', x_n) f(y') \, d\sigma y' = C_n \int_{R^{n-1}} \frac{1}{|x' - y'|^2 + x_n^2} f(y') \, dy'.
\]
Then $N(f)|_0$ can be seen as a Riesz potential $I_1$ on $\partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$, it follows from Theorems 4.1 and 4.2 the desired result.

**Proof of Theorem 4.10.** Let $1 < r \leq \mu < \infty$, $1 < q \leq \lambda < \infty$ and $1 < p \leq \omega < n - 1$ be such that
\[
\frac{1}{r} = \frac{n - 1}{\omega} - \frac{n - 1}{\mu} \quad \text{and} \quad \frac{n - 1}{\omega} = \frac{n - 1}{\lambda} + 1, \tag{5.10}
\]
which leads the restriction $\mu < \lambda = (n - 1)\omega/(n - 1 - \omega)$. From Theorem 4.8-(ii) and Corollary 4.9, we obtain
\[
\|N(f)\|_{A^1_{\lambda q}} = \|\nabla N(f)\|_{M^{\mu}_{\infty}} + \|N(f)\|_{M^{\lambda}_{\infty}} \leq C\|f\|_{M^{\omega}_{\infty}(\partial \mathbb{R}^n_+)}.
\]
Let $1 < r_0 \leq \lambda_0 < n - 1$ be such that (5.10) holds for $\omega = \lambda_0$ and $r = r_0$. Consider
\[
\frac{1}{\lambda_0} = \frac{1}{n - 1} + \frac{1}{\lambda} \quad \text{and} \quad \frac{1}{r_0} = \frac{1}{l_1} + \frac{1}{q},
\]
Then, by last estimate and Hölder inequality (2.11) we obtain
\[
\|T_V(u)\|_{A^1_{\lambda q}} = \|N(Vu)\|_{A^1_{\lambda q}} \leq C\|V\|_{M^{\lambda^+}_{\infty}} \|u\|_{M^{\lambda}_{\infty}} \leq C\|V\|_{M^{\lambda-1}_{\infty}} \|u\|_{A^1_{\lambda q}}.
\]
Let $L = C\|V\|_{M^{\lambda-1}_{\infty}} < 1$ and $B_\varepsilon$ be a closed ball in $A^1_{\lambda q}$, defined by
\[
B_\varepsilon = \left\{ u \in A^1_{\lambda q} : \|u\|_{A^1_{\lambda q}} \leq \frac{2\varepsilon}{1 - L} \right\}, \quad \varepsilon > 0.
\]
Consider the integral equation $\Phi(u) = N(f) + T_V(u)$, we will show that $\Phi$ has a unique fixed point in $B_\varepsilon$, for every $\varepsilon > 0$. Indeed, $\Phi : B_\varepsilon \to B_\varepsilon$ since
\[
\|\Phi(u)\|_{A^1_{\lambda q}} \leq C\|f\|_{M^{\omega}_{\infty}} + L\|u\|_{A^1_{\lambda q}} \leq \varepsilon + \frac{2\varepsilon L}{1 - L} < \frac{2\varepsilon}{1 - L}
\]
where $\varepsilon = C\|f\|_{M^{\omega}_{\infty}}$ and $0 < L < 1$. Also, we have
\[
\|\Phi(u) - \Phi(v)\|_{A^1_{\lambda q}} = \|T_V(u - v)\|_{A^1_{\lambda q}} \leq L\|u - v\|_{A^1_{\lambda q}},
\]
hence $\Phi$ is a contraction in $A^1_{\lambda q}$. Banach fixed theorem ensure us that $\Phi$ has a unique fixed point $u \in B_\varepsilon$, as we desired.

**Proof of Theorem 4.11.** Let us recall the inequality
\[
\|u\|_{\rho^{-1}} - |v|_{\rho^{-1}} \leq C\|u - v\|_{(\rho^{-1})}, \quad \text{for } \rho > 1.
\]
Hence, $\frac{\rho}{q} = \frac{1}{q} + \frac{\rho - 1}{q}$ and Hölder inequality (2.11) will give us
\[
\| |u|_{\rho^{-1}} - |v|_{\rho^{-1}} \|_{M^\lambda_{\rho(q)\infty}(\partial \mathbb{R}^n_+)} \leq C\|u - v\|_{M^\lambda_{\infty}} \| |u|_{\rho^{-1}} + |v|_{\rho^{-1}} \|_{M^\lambda_{\infty}} \leq C\|u - v\|_{M^\lambda_{\infty}} \left( \| |u|_{\rho^{-1}} \|_{M^\lambda_{\infty}} + \| |v|_{\rho^{-1}} \|_{M^\lambda_{\infty}} \right), \tag{5.13}
\]
Let $1 < r_1 \leq \nu < n - 1$ be such that
\[
\frac{1}{r} = \frac{n - 1}{\nu} - \frac{n - 1}{\mu} \quad \text{and} \quad \frac{n - 1}{\nu} = \frac{n - 1}{\lambda} + 1.
\]
From (5.11) we obtain
\[
\|B(u) - B(v)\|_{A^\infty_{1,q}} = \|N \left[ b \left( |u|^{\rho-1}u - |v|^{\rho-1}v \right) \right]\|_{A^\infty_{1,q}} \\
\leq C \left\| b \left( |u|^{\rho-1}u - |v|^{\rho-1}v \right) \right\|_{\mathcal{M}^\infty_{\infty}((\partial \mathbb{R}^n_+))}.
\]
Let \( \frac{1}{\ell_1} = \frac{1}{\ell_2} + \frac{q}{\rho} \) and \( \frac{1}{\nu} = \frac{1}{n-1} + \frac{1}{\lambda} \), then Hölder inequality (2.11) and (5.13) leads to
\[
\|B(u) - B(v)\|_{A^\infty_{1,q}} \leq C \|b\|_{\mathcal{M}^\infty_{\infty}} \left\| |u|^{\rho-1}u - |v|^{\rho-1}v \right\|_{\mathcal{M}^\infty_{\infty}((\partial \mathbb{R}^n_+))} \\
\leq C \|b\|_{\mathcal{M}^\infty_{\infty}} \left\| u - v \right\|_{\mathcal{M}^\infty_{\infty}} \left( \|u\|_{\mathcal{M}^\infty_{\infty}} + \||v|\|_{\mathcal{M}^\infty_{\infty}} \right).
\]
(14.14)

Now set \( \Psi(u) = N(f) + T_V(u) + B(u) \) and let \( u \in B_{\varepsilon} = \{ u \in A^\infty_{1,q} : \|u\|_{A^\infty_{1,q}} \leq \frac{2\varepsilon}{1-L} \} \), where \( L = C\|V\|_{\mathcal{M}^\infty_{\infty}} < 1 \) and \( \varepsilon > 0 \) will be chosen. The integral \( \Psi \) has a fixed point in \( B_{\varepsilon} \), for a suitable \( \varepsilon > 0 \). Indeed, by (5.12) and (5.14) one has
\[
\|\Psi(u) - \Psi(v)\|_{A^\infty_{1,q}} \leq \|B(u) - B(v)\|_{A^\infty_{1,q}} + \|T_V(u) - T_V(v)\|_{A^\infty_{1,q}} \\
\leq M \|u - v\|_{A^\infty_{1,q}} \left( \|u\|_{A^\infty_{1,q}} + \|v\|_{A^\infty_{1,q}} \right) + L \|u - v\|_{A^\infty_{1,q}} \\
\leq \|u - v\|_{A^\infty_{1,q}} \left( M \left( \frac{2\varepsilon}{1-L} \right) + L \right) \\
\leq \left( M2^\rho \left( \frac{\varepsilon}{1-L} \right) + L \right) \|u - v\|_{A^\infty_{1,q}},
\]
(15.15)
with \( M = C\|b\|_{\mathcal{M}^\infty_{\infty}} \). Now taking \( v = 0 \) in (15.15), the inequality (5.11) will give us
\[
\|\Psi(u)\|_{A^\infty_{1,q}} \leq \|N(f)\|_{A^\infty_{1,q}} + \|\Psi(u) - \Psi(0)\|_{A^\infty_{1,q}} \\
\leq C \|f\|_{\mathcal{M}^\infty_{\infty}} + \left( M2^\rho \left( \frac{\varepsilon}{1-L} \right) + L \right) \|u\|_{A^\infty_{1,q}} \\
\leq \varepsilon + \left( M2^\rho \left( \frac{\varepsilon}{1-L} \right) + L \right) \frac{2\varepsilon}{1-L} < \frac{2\varepsilon}{1-L},
\]
if we choose \( \varepsilon > 0 \) in such a way that
\[
L + M2^\rho \left( \frac{\varepsilon}{1-L} \right) < \frac{1}{2}.
\]
Therefore, \( \Psi \) is contraction in \( A^\infty_{1,q} \) and maps \( B_{\varepsilon} \) to itself, provided \( \varepsilon > 0 \) is small enough. The unique fixed point \( u \in B_{\varepsilon} \) of integral equation \( \Psi \), is the unique mild solution \( u \in A^\infty_{1,q} \) for nonlinear boundary problem (1.1), as we desired.

**Proof of Theorem 4.13.** If \( V(\gamma x') = \gamma^{-1}V(x') \) and \( b(\gamma x') = b(x') \), we need to prove that the rescaled function \( u_\gamma(x) = \gamma^{-(n-1)}u(\gamma x) \) satisfies
\[
[N(\cdot)]_\gamma + [T_V(\cdot)]_\gamma + [B(\cdot)]_\gamma = N(d(\cdot))_\gamma + B(u_\gamma) + T_V(u_\gamma),
\]
(16.15)
where \( d(\cdot)(x') = \gamma^{-(n-1)}f(\gamma x') \). Indeed, since \( G(\gamma x) = \gamma^{-(n-2)}G(x) \) and surface measure \( ds_\gamma \) on \( \partial \mathbb{R}^n_+ \) is homogeneous of degree \( n-1 \) one has
\[
N(\gamma x) = \int_{\partial \mathbb{R}^n_+} G(\gamma x' - y', \gamma x_n)f(y') ds_{\gamma y'} \\
= \int_{\partial \mathbb{R}^n_+} G(\gamma x' - y', \gamma x_n)f(\gamma y') ds_{\gamma y'} \\
= \gamma^{1-\frac{1}{n-1}} \int_{\partial \mathbb{R}^n_+} G(x' - y', x_n)d(\gamma y') ds_{\gamma y'} = \gamma^{-\frac{1}{n-1}} N(d(\gamma x))(x).
\]
For rescaled function \( u_\gamma(x) = \gamma^{\frac{1}{p-1}} u(\gamma x) \), let \( V \) be homogeneous of degree \(-1\) to get

\[
\mathcal{T}_V(u)(\gamma x) = \int_{\partial \mathbb{R}_+^n} G(\gamma x' - y', \gamma x_n) V(y') u(y') \, d\sigma_{y'}
= \int_{\partial \mathbb{R}_+^n} G(\gamma x' - \gamma y', \gamma x_n) V(\gamma y') u(\gamma y') \, d\sigma_{y'}
= \gamma^{-\frac{1}{p-1}} \int_{\partial \mathbb{R}_+^n} G(x' - y', x_n) V(y') u_\gamma(y') \, d\sigma_{y'}
= \gamma^{-\frac{1}{p-1}} \mathcal{T}_V(u_\gamma)(x)
\]

also, from homogeneity of \( b \), we have

\[
\mathcal{B}(u)(\gamma x) = \int_{\partial \mathbb{R}_+^n} G(\gamma x' - y', \gamma x_n) b(y') |u(y')|^{p-1} u(y') \, d\sigma_{y'}
= \int_{\partial \mathbb{R}_+^n} G(\gamma x' - \gamma y', \gamma x_n) b(\gamma y') |u(\gamma y')|^{p-1} u(\gamma y') \, d\sigma_{y'}
= \gamma^{(n-1)-(n-2)-\frac{1}{p-1}-\frac{1}{p}} \int_{\partial \mathbb{R}_+^n} G(x' - y', x_n) b(y') |u_\gamma(y')|^{p-1} u_\gamma(y') \, d\sigma_{y'}
= \gamma^{-\frac{1}{p-1}} \mathcal{B}(u_\gamma)(x).
\]

Hence, the equality (5.16) holds. Let \( u \in A_{1q}^{1\infty} \) be the unique \textit{mild solution} of (1.1), provided \( \|f\|_{\mathcal{M}^\omega_{p\infty}} \leq \varepsilon/C \) is small. It follows from (5.16) that \( u_\gamma \) is also a \textit{mild solution} with initial data \( d(f)_\gamma \)

\[
u_\gamma = N(d(f)_\gamma) + \mathcal{B}(u_\gamma) + \mathcal{T}_V(u_\gamma).
\]

Recall that \( \|u_\gamma\|_{A_{1q}^{1\infty}} = \|u\|_{A_{1q}^{1\infty}} \), moreover \( \mathcal{M}^\omega_{p\infty} \) satisfies \( \|f(\gamma \cdot)\|_{\mathcal{M}^\omega_{p\infty}} = \gamma^{-\frac{n-1}{p}} \|f\|_{\mathcal{M}^\omega_{p\infty}} \) which yields

\[
\|d(f)_\gamma\|_{\mathcal{M}^\omega_{p\infty}} = \gamma^{-\frac{n-1}{p}} \|f\|_{\mathcal{M}^\omega_{p\infty}} = \|f\|_{\mathcal{M}^\omega_{p\infty}}
\]

because \( \omega = \frac{(n-1)(p-1)}{p} \). Hence, by uniqueness in \( B_\varepsilon \), we obtain \( u(x) = u_\gamma(x) \) \( x \)-a.e. and \( f(x') = d(f)_\gamma(x') \), \( x' \)-a.e. \( \square \)

\textbf{Proof of Theorem 4.14.}

\textbf{Part A}: Note that the solution \( u \) obtained in the proof of Theorem 4.11 can be seen as limit in \( A_{1q}^{1\infty} \) of the following Picard sequence:

\[
u_1 = N(f), \quad u_{k+1} = u_1 + \mathcal{T}_V(u_k) + \mathcal{B}(u_k), \quad k \in \mathbb{N}.
\]

Since \( f \) is non-negative in \( \partial \mathbb{R}_+^n \) and positive in the measurable set \( D \) we have that

\[
u_1(x) = \int_{\partial \mathbb{R}_+^n} G(x' - y, x_n) f(y) d\sigma > 0 \quad \text{in} \quad \mathbb{R}_+^n.
\]

Moreover, using that \( V \) and \( b \) are non-negative functions in \( \partial \mathbb{R}_+^n \) one can prove that \( N(V u) + N(b|u|^{p-1} u) \) is non-negative in \( \partial \mathbb{R}_+^n \) provided that \( u \) restricted to \( \partial \mathbb{R}_+^n \) is non-negative. By an induction argument one can prove that all element of the sequence \( \{u_k\}_k \)
are positive. Note that convergence of the sequence \( \{u_k\}_k \) in \( A^{1,\infty}_{r q} \) implies the converge \( u_k \to u \) in \( M^1_{\infty}(\partial \mathbb{R}^n_+) \) and \( u_k \to u \) in \( I_1 M^\alpha_{r \infty}(\mathbb{R}^n_+) \) homogeneous Sobolev based in weak-Morrey spaces. The former implies convergence in measure on \( \partial \mathbb{R}^n_+ \) and the letter from Theorem 4.1 implies convergence in measure on the upper half-space \( \mathbb{R}^n_+ \). Therefore, there is a subsequence \( \{u_{k_j}\}_j \) which converges pointwise to \( u \) except a null set in \( (\partial \mathbb{R}^n_+, d\sigma) \) or in \((\mathbb{R}^n_+, dy)\). Since \( u_{k_j}(x) > 0 \), we conclude that \( u(x) \) is non-negative almost everywhere. Since \( u \) is a mild solution in \( A^{1,\infty}_{r q} \), then \( u = u_1 + TV(u) + B(u) \geq u_1 + 0 > 0 \).

**Part B:** Let \( T \in \mathcal{A} \) and \( x \in \mathbb{R}^n_+ \). Note that \( T \) fix the axis \( x_n > 0 \), that is, \( Tx = (T'x', x_n) \), where \( T' \in \mathcal{O}(n-1) \). If \( f \) is antisymmetric then

\[
N(f) ((T'x', x_n)) = C_n \int_{\mathbb{R}^{n-1}} \frac{1}{(|T'x' - y|^2 + x_n^2)^{\frac{n-2}{2}}} f(y) dy
= C_n \int_{\mathbb{R}^{n-1}} \frac{1}{(|T'(x' - T'y) - x_n^2 + x_n^2|^2)^{\frac{n-2}{2}}} f(y) dy
\]

Since \( T' \) is orthogonal in \( \mathbb{R}^{n-1} \), we obtain

\[
N(f) ((T'x', x_n)) = C_n \int_{\mathbb{R}^{n-1}} \frac{1}{(|x' - T'y|^2 + x_n^2)^{\frac{n-2}{2}}} f(y) dy
= C_n \int_{\mathbb{R}^{n-1}} \frac{1}{(|x' - z|^2 + x_n^2)^{\frac{n-2}{2}}} f(T'(z)) dz
= -C_n \int_{\mathbb{R}^{n-1}} \frac{1}{(|x' - z|^2 + x_n^2)^{\frac{n-2}{2}}} f(z) dz = -u_1(x)
\]  

where in (5.19) we making the change of variable \( z = T'y \). Now, suppose that \( u \) is antisymmetric. Since \( V \) is radially symmetric, easily we have

\[
T_V(u)((T(x)) = C_n \int_{\mathbb{R}^{n-1}} \frac{1}{(|T'x' - y|^2 + x_n^2)^{\frac{n-2}{2}}} V(y) u(y) dy
= -C_n \int_{\mathbb{R}^{n-1}} \frac{1}{(|x' - z|^2 + x_n^2)^{\frac{n-2}{2}}} V(z) u(z) dz = -T_V(u)(x).
\]

Similarly \( B(u)(Tx) = -B(u)(x) \), since \( b \) is radially symmetric on \( \mathbb{R}^{n-1} \) and \( u \) is antisymmetric under the action of \( \mathcal{A} \). So, an induction argument shows that Picard sequence (5.17) satisfies \( u_k(Tx) = -u_k(x) \), for all \( T \in \mathcal{A} \). Since we have convergence a.e. in \((\mathbb{R}^{n-1}, dx')\) and \((\mathbb{R}^n_+, dy)\) and these convergence preserve antisymmetry, it follows that the mild solution \( u \) is antisymmetric.

6 **Appendix**

**Proposition 6.1.** Let \( K \) be a distribution given by \( \hat{K} = \sigma \in C^{[n/2]}(\mathbb{R}^n \setminus \{0\}) \). If \( 0 \leq \delta < n \) and \( \sigma \) satisfies

\[
|D_x^\beta \sigma(\xi)| \leq C_\beta |\xi|^{-\delta - |\beta|}, \quad 0 \leq |\beta| \leq \lfloor n/2 \rfloor,
\]  

(6.1)
then $K$ agrees with a smooth function $\tilde{K}$ away from the origin and satisfies

$$|D^\beta \tilde{K}(x)| \lesssim_\beta |x|^{\delta-n-|\beta|}.$$ 

**Proof.** There is a sequence $\{\psi(2^{-j}\xi)\}_{j \in \mathbb{Z}}$ of $C^\infty$ functions of compact support in the frequency $\xi$-space $\mathbb{R}^n$ such that

$$\sum_{j=-l}^{l} \psi(2^{-j}\xi) \to 1 \quad \text{as} \quad l \to \infty, \quad \text{for every} \quad \xi \neq 0, \quad (6.2)$$

the function $\psi_j = \psi(2^{-j}\cdot)$ are supported in the shells $D_j = \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and satisfies, in particular,

$$\left|\partial_\xi^\beta \psi_j(\xi)\right| \lesssim_\beta 2^{-j|\beta|}, \quad \text{for all} \quad |\beta| \leq \lfloor n/2 \rfloor. \quad (6.3)$$

Let $\sigma_j(\xi) = \sigma(\xi)\psi_j(\xi)$, for each $j \in \mathbb{Z}$, and set $\tilde{K}_j = \sigma_j$. It follows from (6.2) that

$$\sum_{j=-l}^{l} K_j(x) \to K(x) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \quad \text{as} \quad l \to \infty,$$ 

modulus polynomials in $\mathbb{R}^n$. It suffices estimate $\sum_{j=-l}^{l} |\partial_x^\beta K_j(x)|$ for $x \neq 0$. Proceeding as in [36, p.246], only needs prove

$$\left|\partial_x^\beta K_j(x)\right| \lesssim_{\beta,M} |x|^{-M}2^{j(n-\delta+|\beta|-M)}, \quad (6.4)$$

for every $j \in \mathbb{Z}$ and $M \geq 0$. To this end, from Leibniz rule we obtain

$$\left|(-2\pi i)^\gamma [\partial_x^\beta K_j(x)]\right| = \left|\int_{D_j} \partial_\xi^\gamma \left[(2\pi i\xi)^\beta \sigma_j(x)\right] e^{2\pi i x \cdot \xi} d\xi\right| \lesssim_{\gamma} \int_{D_j} \left|\partial_\xi^\gamma (2\pi i\xi)^\beta \right| \left|\partial_\xi^{\gamma-\gamma'} \sigma_j(\xi)\right| d\xi \lesssim_{\beta,\gamma} \int_{D_j} 2^{|\beta-\gamma'|}2^{-j(\delta+|\gamma'|)} d\xi \lesssim_{\beta,\gamma} 2^{j(n-\delta+|\beta|-|\gamma|)}, \quad (6.5)$$

because in the shells $D_j$, one has $\left|\partial_\xi^\gamma (2\pi i\xi)^\beta\right| \lesssim_{\gamma,\beta} |\xi|^{\beta-\gamma'} \lesssim_{\beta,\gamma'} 2^{|\beta-\gamma'|}$ and

$$\left|\partial_\xi^k \sigma_j(\xi)\right| \lesssim_k \left|\partial_\xi^k \sigma_j(\xi)\right| \lesssim_k |x|^{-\delta-k|\beta'|} \lesssim_k 2^{-j(\delta+|k|)}$$

in view of inequalities (6.1) and (6.3). The inequality (6.5) easily implies the estimate (6.4). \qed

**Theorem 6.2.** Let $1 < p < \lambda < \infty$, $1 < r < \mu < \infty$ and $n/\mu < \alpha < n/\lambda$ be such that $\frac{\alpha}{n} = \frac{1}{x} - \frac{1}{\mu}$ and $\frac{\alpha}{\mu} > \frac{\mu}{x}$. Then $M_\alpha$ does not map continuously $\mathcal{M}^\lambda_{pc}$ into $\mathcal{M}^\mu_{r_\mu}$.
Proof. By assumption $1 < r < \mu < \infty$, the equation
\[
\left(\frac{2}{1-\delta}\right)^{\frac{1}{r}} (1-\delta)^{\frac{1}{\mu}} = 1
\] (6.6)
has a solution $\delta \in (0, 1)$. Fix a positive integer $N$ and consider the cube $E_0 = Q_{0,1} = \left[0, \left(\frac{2}{1-\delta}\right)^N\right]^n$. If $\frac{r}{\mu} > \frac{p}{\lambda}$, then we proceed as [37, Proposition 4.1] to get a family of sets $E_d = \bigcup_{j=1}^{2^n} Q_{d,j}$ and $F_{d-1} = \bigcup_{j=1}^{2^n} P_{d,j}$ for $0 \leq d \leq N$ such that $g(x) = |Q_{d,j}|^{-\frac{1}{r}} \chi_{E_d}(x) \in \mathcal{M}_p^\lambda$ and satisfies
\[
\|g\|_{\mathcal{M}_p^\lambda} \leq 1.
\] (6.7)
Indeed, in the first stage we partition $E_0$ into cubes and delete all, but let randomly an open middle cube $P_{0,1}$ of sidelength $\ell_{P_{0,1}} = \delta (\frac{2}{1-\delta})^{N-1}$ and $2^n$ closed cubes $Q_{1,j}$ of sidelengh $\ell_{Q_{1,j}} = (\frac{2}{1-\delta})^{N-1}$ for $j = 1, 2, \cdots, 2^n$. Now consider $F_0 = P_{0,1}$ and $E_1 = \bigcup_{j=1}^{2^n} Q_{1,j}$. In the second stage, we partition each $Q_{1,j}$ into subcubes, delete all except an open middle cube $P_{1,j}$ of sidelength $\ell_{P_{1,j}} = \delta (\frac{2}{1-\delta})^{N-1}$ and $2^n$ closed subcubes $Q_{2,j}$ of sidelength $\ell_{Q_{2,j}} = (\frac{2}{1-\delta})^{N-2}$, and set $F_1 = \bigcup_{j=1}^{2^n} P_{1,j}$ and $E_2 = \bigcup_{j=1}^{2^n} Q_{2,j}$ (see Figure 2). This process will stop in $d$-stage, where we randomly have obtained $2^{nd}$ cubes $Q_{d,j}$ of side $\ell_{Q_{d,j}} = (\frac{2}{1-\delta})^{N-d}$ and $2^{n(d-1)}$ open middle cubes $P_{d-1,j}$ of side $\ell_{P_{d-1,j}} = \delta (\frac{2}{1-\delta})^{N-d+1}$, hence
\[
F_{d-1} = \bigcup_{j=1}^{2^{n(d-1)}} P_{d-1,j} \quad \text{and} \quad E_d = \bigcup_{j=1}^{2^{nd}} Q_{d,j}.
\]

Let $l \leq d \leq N$ and note that
\[
\frac{|E_d \cap Q_{l,j}|}{|Q_{l,j}|} = (1-\delta)^{n(d-l)}.
\] (6.8)
In order to show that $\|g\|_{\mathcal{M}_p^\lambda} < 1$, first note that $d_{\chi_{E_d}}(s) = |\{x \in Q_{l,j} : |\chi_{E_d}(x)| > s\}| = |E_d \cap Q_{l,j}|$ if $s < 1$ and zero otherwise, then the decreasing rearrangement $\chi_{E_d}^*(t) = \int_t^1 d_{\chi_{E_d}}(s) \, ds$ and the property $\chi_{E_d}^{**}(t) = \chi_{E_d}(t)$ for all $t \leq 1$.

Figure 2: The cubes $Q_{d,j}$ and $P_{d-1,j}$ in $\mathbb{R}^2$
Therefore, the boundedness \( \|M_{\alpha} \chi_{E_N} \|_{M_{p_r}^n} \leq C \|\chi_{E_N} \|_{M_{p_n}^n} \) will never holds, for every \( 0 < \alpha < n/\lambda \) with \( (r/\mu) > (p/\lambda) \). Indeed, if \( M_{\alpha} \) is bounded from \( M_{p_n}^n \) to \( M_{p_r}^n \), let \( C = \|M_{\alpha} \|_{L(M_{p_n}^n, M_{p_r}^n)} \) which is finite. However, if \( (r/\mu) > (p/\lambda) \) then \( 1 > \|\chi_{E_N} \|_{M_{p_n}^n} \) which combined with (6.9) yields

\[
C > C\|\chi_{E_N} \|_{M_{p_n}^n} \geq \|M_{\alpha} \chi_{E_N} \|_{M_{p_r}^n} \geq \left( 1 + \delta^{\frac{2}{n-1}} \sum_{l=1}^{N-1} 2^{nl} (1 - \delta)^{n(N-l)} \right)
\]

taking \( N \) large enough, we obtain that \( C \) is arbitrarily large which is a contradiction. \( \square \)
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