Lift and drag in three-dimensional steady viscous and compressible flow

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In a recent paper, Liu, Zhu & Wu (2015, J. Fluid Mech. 784: 304; LZW for short) present a far-field theory for the aerodynamic force experienced by a body in a two-dimensional, viscous, compressible and steady flow. In this companion theoretical paper we do the same for three-dimensional flow. By a rigorous fundamental solution method of the linearized Navier-Stokes equations, we not only improve the far-field force formula for incompressible flow originally derived by Goldstein in 1931 and summarized by Milne-Thomson in 1968, both being far from complete, to its perfect final form, but also prove that this final form holds universally true in a wide range of compressible flow, from subsonic to supersonic flows. We call this result the unified force theorem (UF theorem for short) and state it as a theorem, which is exactly the counterpart of the two-dimensional compressible Joukowski-Filon theorem obtained by LZW. Thus, the steady lift and drag are always exactly determined by the values of vector circulation $\Gamma_{\phi}$ due to the longitudinal velocity and inflow $Q_{\psi}$ due to the transversal velocity, respectively, no matter how complicated the near-field viscous flow surrounding the body might be. However, velocity potentials are not directly observable either experimentally or computationally, and hence neither is the UF theorem. Thus, a testable version of it is also derived, which holds only in the linear far field and is exactly the counterpart of the testable compressible Joukowski-Filon formula in two dimensions. We call it the testable unified force formula (TUF formula for short). Due to its linear dependence on the vorticity, TUF formula is also valid for statistically stationary flow, including time-averaged turbulent flow.

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1. Introduction

In general, any aerodynamic force theory can be categorized into two groups. The first group is far-field force theory, by which a universal force formula can be rigorously deduced. Its central task is to identify the key physical quantities responsible for the forces, of which the first and most classic example is the Kutta-Joukowski lift theorem (K-J theorem for short, Joukowski 1906) for incompressible potential flow that focuses one’s attention to the circulation around an airfoil. The K-J theorem has motivated a series far-field theories (of which those that are relevant to our present study will be cited below in due course). But far-field approach alone is confined to steady flow only and

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cannot determine the relationships between the forces and the detailed flow processes and structures behind those universally identified key quantities at different specific flow conditions. This is the task of the second group, the near-field theories for both steady and unsteady flow, which is guided (for steady flow) by the results of far-field theories and always the main body of aerodynamic theories, as given in all monographs and textbooks of low- and high-speed aerodynamics.

Historically, however, various far-field theories had long been limited to incompressible and/or inviscid flow, and never reached their highest possible goal to be truly universal: to identify the key physical quantities responsible for aerodynamic forces within the general framework of the Navier-Stokes equation. The first breakthrough was made only very recently by Liu, Zhu & Wu (2015, LZW for short), who obtained a unified far-field aerodynamic force theory for two-dimensional viscous and compressible flow, valid from low-speed to supersonic regimes. In this theoretical paper, we present an exactly the same kind of theory but for three-dimensional flow.

To explain the motivation and orientation of our study, it is appropriate here to make a brief account of previous investigations about the far-field force theories, for both two- and three-dimensional flows due to their close relations. For both cases we rely crucially on the decomposition of a vector field with proper continuity and smoothness into a longitudinal field and a transversal field (not uniquely in general), namely the Helmholtz decomposition. In particular, let \( \mathbf{u} = \mathbf{U} + \mathbf{u}' \) where \( \mathbf{U} = U \mathbf{e}_x \) is the uniform incoming flow velocity, the disturbance velocity field \( \mathbf{u}' \) can always be written formally as

\[
\mathbf{u}' = \mathbf{u}_\phi + \mathbf{u}_\psi \equiv \nabla \phi + \nabla \times \psi, \quad \nabla \cdot \psi = 0,
\]

where \( \phi \) and \( \psi \) are called the velocity potential (scalar potential) of the longitudinal field and the vortical stream function (vector potential) of the transversal field, respectively. The latter should be distinguished from the full vector stream function of incompressible flow.

1.1. Far-field force theory in two dimensions

To make the notations for two-dimensional flow unified with three-dimensional flow in Cartesian coordinates \((x, y, z)\) with \( \mathbf{U} = U \mathbf{e}_x \) and \( \mathbf{e}_z \) being in vertical-up direction, a two-dimensional flow is assumed to occur on a \((x, z)\)-sectional plane, with vorticity \( \omega = \omega \mathbf{e}_y \) and \( \psi = (0, \psi, 0) \) so that \( \mathbf{u}_\psi = \nabla \psi \times \mathbf{e}_y \). A control surface \( S \) with unit outward normal \( \mathbf{n} \) in three dimensions is reduced to a closed loop, still denoted by \( S \) but has tangent unit vector \( \mathbf{t} \) so that \( \mathbf{n} \times \mathbf{t} = \mathbf{e}_y \). Then the Kutta-Joukowski lift formula obtained by Joukowski (1906) using far-field analysis is well known. In vector form, the lift force \( \mathbf{L} = L \mathbf{e}_z \) on a body of any shape in a two-dimensional incompressible, inviscid and steady flow reads

\[
\mathbf{L} = \rho_0 \mathbf{U} \times \mathbf{\Gamma}_\phi,
\]

where

\[
\mathbf{\Gamma}_\phi \equiv \mathcal{J}_S \mathbf{n} \times \nabla \phi dS = \mathbf{e}_y \{ \phi \}
\]

is the vector circulation \( \mathbf{\Gamma} = \mathbf{\Gamma} e_y \) of a bound vortex in the body, \( \rho \) is the fluid density with suffix 0 denoting the constant property at upstream infinity, \( \phi \) is the velocity potential and \( \{ \cdot \} \) denotes the jump as \( \mathbf{u}_\phi \) goes around the loop once. Note that (1.2) is completely independent of the size and geometry of \( S \). When the body is an airfoil with sharp trailing edge, the circulation \( \mathbf{\Gamma}_\phi \) can be determined by the Kutta condition (Kutta 1902). This well-known K-J formula has since served as the very basis of classic steady aerodynamics.
Lift and drag in three-dimensional steady viscous and compressible flow

Meanwhile, for this inviscid flow the drag force is zero, in consistent with the famous d’Alembert paradox.

In contrast to inviscid flow, in a steady viscous flow the vortical wake must extend to downstream unboundedly, and any contour $S$ surrounding the airfoil must cut through the wake, leaving some vorticity outside of $S$. Thus, one has to ask (i) whether $(1.2)$ is still effective, and (ii) if yes, whether the lift is still independent of the choice of $S$.

These questions were first studied experimentally by Bryant & Williams (1926). They found that the lift calculated by K-J formula $(1.2)$, with $\nabla \phi$ replaced by the measured total disturbance velocity $u'$, is a good approximation to that of the real viscous flow for typical aerodynamic applications:

$$L \approx \rho_0 U \times \Gamma,$$

where

$$\Gamma \equiv \int_S n \times u'dS = \int_V \omega dV, \quad \omega = \nabla \times u',$$

in which $V$ is the volume enclosed by $S$. Moreover, the experiment confirmed that $\Gamma$ may still be independent on $S$. In his theoretical explanation, Taylor (1926) points out that these positive answers require two conditions: (a) the intersect of $S$ and the wake has to be a vertical plane (“wake plane”, denoted by $W$) with normal $n = e_x$; and (b) the net vorticity flux through $W$ must vanish, which can be proven for steady viscous flow at large Reynolds number (for an improved proof of this issue see Wu, Ma & Zhou 2015). We call these conditions the first and second Taylor criteria (Liu, Zhu & Wu 2015), and $(1.4)$ the approximate Taylor lift formula.

Independent of the work of Taylor (1926), Filon (1926) makes a thorough analysis of the lift and drag problem for two-dimensional, viscous, incompressible and steady flow. He conforms that to the leading order the disturbance flow satisfies the Oseen equation (see $(1.8b)$ below), which is valid for an arbitrarily Reynolds number as long as the distance from the body is sufficiently large. After obtaining the complete solution of Oseen’s equation in the form of two series of typical solutions, Filon finds the complete solution for the transverse disturbance stream function $\psi$. Then he shows that the lift is the same as $(1.4)$ at infinity, while the drag is associated with a particular term in the solution, given by

$$D = \rho_0 U Q_\psi,$$

where

$$Q_\psi \equiv -\int_S (n \times \nabla) \cdot \psi dS = -[\psi]$$

represents an inflow at infinity at the tail.

Now, what LZW has achieved is to extend the above lift and drag formulas given by Joukowski, Taylor, and Filon for incompressible flow to fully viscous compressible and steady flow. In so doing, these authors also fully explain the appearance of two circulations, $\Gamma_\phi$ and $\Gamma$, and whether Filon’s result is valid only approximately at infinity. Starting from $(1.1)$ and to make the decomposition unique, the linearized compressible Navier-Stokes (N-S) equation has to be split into a longitudinal equation and a transversal equation as well, such that

$$\Pi + \rho_0 U \frac{\partial \phi}{\partial x} = 0, \quad (1.8a)$$

$$\left(\nabla^2 - 2k \frac{\partial}{\partial x}\right) u_\psi = 0. \quad (1.8b)$$
where $\Pi = p - \mu \nabla \cdot \mathbf{u}$ is the normal stress with $\mu$ being the longitudinal dynamic viscosity and $\nabla \cdot \mathbf{u}$ the dilatation. While (1.8a) is the linearized Bernoulli equation, (1.8b) is the transverse Oseen equation. Note that as shown by LZW, the two potentials $\phi$ and $\psi$ are inherently coupled in viscous flow at the body surface. Then, by linear far-field analysis and using the fundamental solution method, LZW have proven that the force $\mathbf{F}$ exerted on the body is

$$\mathbf{F} = \rho_0 \mathbf{U} \times \Gamma \phi + \rho_0 \mathbf{U} Q \psi,$$  \hspace{1cm} (1.9)

where the longitudinal circulation $\Gamma \phi$ and inflow $Q \psi$ are given by (1.3) and (1.7), respectively. Obviously, (1.9) is independent of the choice of $S$ since both (1.3) and (1.7) are independent of $S$ due to the generalized Stokes theorem. Thus, it is valid in the whole flow domain, not just at infinity. On the other hand, since there is no assumption of incompressibility in the whole derivation of (1.9), it is also true for viscous compressible flow in a wide range of Mach number, from subsonic to supersonic flows. Due to this universal validity, LZW calls (1.9) the Joukowski-Filon theorem, but we feel a better name would be compressible Joukowski-Filon theorem (CJ-F theorem for short), which states that the steady lift and drag are always exactly determined by the values of the circulation $\Gamma \phi$ and inflow $Q \psi$, no matter how complicated the near-field viscous flow surrounding the body might be.

Unfortunately, velocity potentials are not directly observable either experimentally or computationally, and hence neither are the integrands of the CJ-F theorem. This is why Filon's drag formula has seldom been noticed in aerodynamics community. But except providing universal and exact force formulas, the far-field theories have another task, namely to give asymptotic approximate formulas valid in linear far field only. Thus, LZW also derived a testable version of the CJ-F formula:

$$\mathbf{F} \cong \rho_0 \mathbf{U} \times \Gamma + \rho_0 \mathbf{U} W,$$  \hspace{1cm} (1.10)

where the circulation $\Gamma$ is given by (1.5) and

$$Q_W \equiv \int_W z \omega_y dS,$$  \hspace{1cm} (1.11)

where and after $W$ denotes a wake plane with $\mathbf{n} = e_x$, which is the downstream face of the outer boundary of $S$ and perpendicular to the incoming flow. Both $\Gamma$ and $Q_W$ depend only and linearly on the vorticity and thus holds in the linear far field of steady flow or time-averaged unsteady flow such as turbulence. By (1.10), the condition for the validity of (1.4) is obvious; but now it is also valid for viscous compressible flow. We call (1.10) the testable CJ-F formula, which has also been directly confirmed by a careful Reynolds-averaged Navier-Stokes (RANS) simulation of typical airfoil flow, and thereby enhanced our understanding of the CJ-F theorem.

Owing to this progress, the far-field force theory has for the first time been rigorously extended to viscous compressible flow. The CJ-F theorem is thus far the only force theory that has the same form in incompressible and compressible flows. Therefore, the far-field force theory in two dimensions has been completed.

### 1.2. Far-field force theory in three dimensions

The corresponding incompressible problem in three dimensions has been treated in two papers by Goldstein (1929, 1931), who follows Filon (1926) to apply the Oseen approximation at a great distance from the solid. In his first paper Goldstein discusses two series of solutions of the equations, which corresponds exactly to the longitudinal velocity $u_\phi$, and the transversal velocity $u_\psi$ given by (1.1). The first series yields a set of particular
integrals of (1.8a), in which the longitudinal velocities are associated with certain values of the pressure. In the second series, which is a set of particular integrals of (1.8b) and of the nature of a complementary function, the velocities are rotational, while the pressure does not appear. Thus, Filon’s drag formula (1.6) is shown to still hold for three-dimensional incompressible flow, where the transversal inflow \( Q_\psi \) is equivalent to the longitudinal outflow,

\[
Q_{\psi, in} = \int_S \mathbf{n} \cdot \nabla \phi dS, \quad (1.12)
\]

where subscript ‘in’ denotes incompressible. However, as pointed out by J. M. Burgers, the solution of Goldstein (1929) is valid only when the solid body is of revolution, and thus they are not sufficiently general.

In his 1931 paper, Goldstein investigates some more particular integrals, with special emphasis on singular solutions. He shows that for certain values of the pressure, the corresponding irrotational velocities \( \nabla \phi \) have singularities, which have to be cancelled by the suitable component of the transversal velocity \( u_\psi \). In particular, he divides \( u_\psi \) into three parts, \( v_1, v_2 \) and \( v_3 \), each of which satisfies (1.8b). Then, \( v_1 \) cancels out the singularities in \( \nabla \phi \), the sum \( v_1 + v_2 \) satisfies the condition of continuity \( \nabla \cdot (v_1 + v_2) = 0 \), and \( v_3 \) satisfies the condition of continuity and the condition at infinity separately. Based on these analyses, Goldstein (1931) obtains three major results:

(i) Filon’s drag (1.6) is found still true in three dimensions.

(ii) By comparing the orders of magnitude of the various terms in \( u' = \nabla \phi + v_1 + v_2 + v_3 \), a simple expression of integral form is obtained for the lift at infinity, which is exactly the same as Taylor’s two-dimensional lift formula (1.4).

(iii) Goldstein further shows that the force \( F \) can be expressed as

\[
F \equiv -\rho_0 U \int_W (v_2 + v_3) dS, \quad (1.13)
\]

so that \( D \) is the integral of \( U \cdot (v_2 + v_3) \) over \( W \). This is however just the linearized momentum theorem stating that the force exerted on the body is exactly the minus of the flux of the extra momentum.

Subsequently, Garstang (1936) obtains the complete solution of the equations discussed by Goldstein (1931), and thereby proves \( v_2 \cdot U = 0 \). On the other hand, in describing the results of Goldstein (1931) and Garstang (1936), Milne-Thomson (1968, pp. 702-706) finds that \( \Gamma \) defined by (1.5) can be further reduced to the circulation solely due to \( v_2 \). Namely, the results of Gastang and Milne-Thomson imply that in (1.4) and (1.13) one may set

\[
\Gamma \equiv \int_S \mathbf{n} \times v_2 dS, \quad D \equiv -\rho_0 \int_W U \cdot v_3 dS. \quad (1.14)
\]

Obviously, in contrast to two-dimensional incompressible flow, the above results have not yet been pursued to a mature stage. There is no universal force formula yet except Filon’s formula (1.6). Those force formulae in three dimensions, such as (1.4) and (1.13), are only valid approximately. The separate appearance of \( v_2 \) and \( v_3 \) in lift and drag of (1.14), respectively, is physically quite strange.

In our view, the main reason for this embarrassing situation is that these authors did not thoroughly utilize the process decoupling (1.1) and (1.8), as seen from their division of \( u' \) into four parts. Besides, nor did they find a simple method for solving far-field equations. Thus we may ask: if we persist the process decoupling and turn to the fundamental-solution method that has been proven for two-dimensional flow to be much neater and more straightforward than those classic techniques, is there any force
formula in three dimensions that is as neat and universal as the CJ-F theorem (1.9) in two dimensions? In this paper, we will give a positive answer to this question.

Naturally, the next relevant extension would be compressible flow. Toward this goal and within three dimensions, among others, Finn & Gilbarg (1957) have proven rigorously for subsonic nonlinear potential flow that the fluid exerts no net force to the body, which may be termed a “d’Almbert-like paradox” but is of course not the case for viscous flow. Then, based on some plausible assumptions, Lagerstrom (1964, pp. 34-38) has proposed that (1.4) should give the lift for viscous compressible flow.

Having reviewed these pieces of progress of three-dimensional far-field analyses, we may conclude that so far no profound and universal force theory is available. In other words, the far-field force theory in three dimensions is still far from complete.

1.3. Our work and this paper

In the rest of this paper we extend LZW’s two-dimensional theory to three dimensions in the same way. We show in §2 that, in terms of linearized far field, the velocity potential \( \phi \) and stream function \( \psi \) must have singularities (somewhat like the nonzero jumps \([ \phi ]\) and \([ \psi ]\) in two dimensions), for otherwise the body would be force-free. We then confirm the effectiveness (at least formally) of the far-field asymptotic lift formula (1.4), as well as a drag formula, for three-dimensional, steady, viscous and compressible flow over a wide range of Mach number and Reynolds number. In Section 3 we introduce the fundamental solutions of the linearized N-S equations and make a detailed analysis of the transversal far-field. The singularities in \( \phi \) and \( \psi \) are identified but proven to cancel each other to ensure finite velocity field. Then we arrive at a profound universal force formula, which we state as the unified force theorem (UF theorem for short). This neat theorem is however not yet a complete aerodynamic theory since it is not directly testable or measurable. Therefore, after finding the position where the linear far-field exists and discussing on the multiple circulations, we confirm that the far-field asymptotic lift formula (1.4), as well as the drag formula, do hold as a practical far-field force formula in Section 4, which we call testable unified force formula (TUF formula for short). In Section 5 we provide a simple physical explanation of the singularity in incompressible flow with a concrete model, which is the essence of the universality of the UF theorem and its existence is universal from incompressible flow to compressible flow. Conclusions and discussions are given in Section 6.

To be self-contained, the fundamental solution of three-dimensional steady linearized N-S equations is given in Appendix A. Appendix B gives some of the detailed algebra in proving the unified force theorem.

2. Far-field force formulae and their implications

For steady, viscous and compressible flow, the total force exerted on the body \( B \) can be expressed by a control-surface integral:

\[
F \equiv -\int_{\partial B} (-\Pi n + \tau) dS \tag{2.1a}
\]

\[
= -\int_S (\Pi n + \rho uu \cdot n - \tau) dS, \tag{2.1b}
\]

where \( \partial B \) is the boundary of body, \( S \) is an arbitrary control surface enclosing the body, and \( \tau = \mu \omega \times n \) is the shear stress.

Hereafter we assume \( S \) lies in sufficiently far field where the flow can be linearized and
Lift and drag in three-dimensional steady viscous and compressible flow is governed by (1.8). Then, using the exact continuity equation \( \nabla \cdot (\rho u) = 0 \) and omitting higher-order terms, there is
\[
\int_S \rho u \cdot n dS = \int_S \rho_0 (\nabla \phi + u_\psi) U \cdot n dS. \tag{2.2}
\]
Thus, the linearized version of (2.1b) is
\[
F = \rho_0 U \times \int_S n \times \nabla \phi dS - \rho_0 U \cdot \int_S n u_\psi dS + \mu \int_S \omega \times n dS, \tag{2.3}
\]
where the longitudinal equation (1.8a) has been used. Then, to transform the shear stress \( \mu \omega \times n \) we use the transverse equation (1.8b). Since
\[
\nabla \times (U \times u_\psi) = U \nabla \cdot u_\psi - U \nabla u_\psi = -U \cdot \nabla u_\psi,
\]
(1.8b) can be recast to
\[
\nabla \times (U \times u_\psi) = \nabla \times (\nu \omega),
\]
so that
\[
U \times u_\psi = \nu \omega + \nabla \eta \tag{2.4}
\]
for some scalar function \( \eta \), which satisfies the Poisson equation
\[
\nabla^2 \eta = -U \cdot \omega. \tag{2.5}
\]
Thus, from \( \nabla^2 \psi = -\omega \) follows \( \eta = U \cdot \psi \), and we have
\[
\nu \omega \times n = u_\psi n \cdot U - U n \cdot u_\psi + n \times \nabla (U \cdot \psi). \tag{2.6}
\]
Then, substituting (2.6) into (2.3) yields immediately
\[
F = \rho_0 U \times \Gamma_\phi + \rho_0 U Q_\psi + \rho_0 \int_S n \times \nabla (U \cdot \psi) dS, \tag{2.7}
\]
where the circulation \( \Gamma_\phi \) due to longitudinal field and inflow \( Q_\psi \) due to transverse field are defined by (1.3) and (1.7), respectively. Note that in two dimensions \( U \cdot \psi \equiv 0 \) and (2.7) recovers the compressible Joukowski-Filon formula as given by LZW. Remarkably, unlike two-dimensional flow where the lift is solely from \( \Gamma_\phi \), now the third term of (2.7) can also contribute to a lift via its \( z \)-component, which is directly associated with the vortical stream function \( \psi \).

Since \( \Gamma_\phi \) and \( Q_\psi \) are defined by the first equalities of (1.3) and (1.7), respectively, however, owing to the generalized Stokes theorem (e.g., Wu, Ma & Zhou 2006, p. 700), (2.7) would be identically zero unless \( \phi \) and \( \psi \) are either multi-valued or singular. This general “d’Almbert-like paradox” extends that observed by Finn & Gilbarg (1957) to not only viscous and rotational flow but also supersonic flow. For real viscous steady flow over a body, therefore, it is only the singularity or multi-valueness of velocity potentials (scalar and vector) that can ensure nonzero forces. Indeed, in two dimensions \( \phi \) and \( \psi \) must be multi-valued as discussed by LZW, while in a three-dimensional singly-connected domain \( \phi \) and \( \psi \) have to be singular, as first pointed out by Goldstein (1931) for incompressible flow. We shall see that this singularity does exist even for compressible flow. It should be stressed that since (2.7) depends only on the multi-valueness and singularity of \( \phi \) and \( \psi \), it must be valid for any choice of the control surface \( S \). Later in \( \S \) 5, after quantifying the singularity, we shall discuss its origin.

Of course, (2.7) is not yet the final form of force theory. But it can hardly be further pursued without knowing the specific singular property of \( \phi \) and \( \psi \). This will be done in the next section with the help of fundamental solutions, where the last term in (2.7) will
be proven to be exactly equal to the first term. Consequently, once again, the lift and
drag are totally determined by the longitudinal circulation $\Gamma_\phi$ and the transversal inflow $Q_\psi$, as in two-dimensional flow.

This being the case, here we turn to seeking the asymptotically approximate force
expression with observable physical quantities instead. For this purpose, we first rewrite
(2.3) as

$$
F = \rho_0 U \times \Gamma - \rho_0 U \cdot \int_S u_\psi n dS + \mu \int_S \omega \times n dS,
$$  \hspace{1cm} (2.8)

where the vector circulation $\Gamma$ is given by (1.5). Recalling the properties of the transversal
field or the general solution of (1.8b) (e.g. Goldstein 1931; Garstang 1936), the viscous
term in (2.8) can be omitted and the second term can be replaced by the integral on the
wake plane $W$:

$$
- \rho_0 U \cdot \int_S u_\psi n dS \cong - \rho_0 U \int_W u_{\psi x} dS.
$$  \hspace{1cm} (2.9)

To proceed, notice that in the Oseen approximation of unboundedly long steady wake, the
variation of flow properties in $x$-direction is much smaller than that in lateral directions,
namely $\partial/\partial y, \partial/\partial z \gg \partial/\partial x$ in the wake (similar to boundary layers and free shear layers,
but now the wake does not have to be thin). Then the second term of (2.9) is

$$
n = 2 : \int_W u_{\psi x} dy = - \int_W \frac{\partial u_{\psi x}}{\partial y} y dy \cong \int_W y \omega_z dS,
$$  \hspace{1cm} (2.10a)

$$
n = 3 : \int_W u_{\psi x} r dr d\theta = - \int_W \frac{\partial u_{\psi x}}{\partial r} r^2 dr d\theta \cong \frac{1}{2} \int_W \omega_\theta r dS.
$$  \hspace{1cm} (2.10b)

Since for $n = 3$ there is

$$
r \omega_y = y \omega_z - z \omega_y,
$$  \hspace{1cm} (2.11)

thus by (2.9), (2.8) is reduced to

$$
F \cong \rho_0 U \times \Gamma + \rho_0 U Q_W,
$$  \hspace{1cm} (2.12)

with $\Gamma$ being given by (1.5) and

$$
Q_W = \frac{\rho_0}{n-1} \int_W (z \omega_y - y \omega_z) dS,
$$  \hspace{1cm} (2.13)

with $n = 2, 3$ being the space dimensionality. This formula obviously includes and ex-
tends the two-dimensional testable CJ-F formula (1.10) and (1.11). Taylor’s formula (1.4)
and the lift proposed by Lagerstrom (1964, pp. 34-38) are both special cases of (2.12).
Namely, we arrive at a unified far-field asymptotic force formula for both two- and three-
dimensional flows. Thus, we name (2.12) the testable unified force formula. While thus
far (2.7) and its far-field asymptotic form, (2.12) and (2.13), were derived under the as-
sumed existence of linear far field, this assumption will be rigorously proved in the next
two sections from low-speed to supersonic flow.

Note that (2.13) has been derived by Wu, Ma & Zhou (2006, p. 630) as the far-field
linearized formula for the form drag, but in near-field analysis except form drag there are
also wave drag and induced drag. These two drags, however, no longer appear in linear far
field. Indeed, LZW has proven that the supersonic wave drag only leaves a signature in
far field as a modified vorticity distribution that is included in (1.11). On the other hand,
for incompressible flow, Wu, Ma & Zhou (2006, p. 629) has identified that the induced
drag comes from the $x$-component of the vortex force (volume integral of disturbance
Lamb-vector $\rho_0 \mathbf{u}' \times \mathbf{\omega}$ that, as is well known, can be cast to control-surface integral of $\mathbf{u}'$ quadratics; while Liu et al. (2014b) have further proven that for compressible flow the counterpart of this vortex force is

$$\int_V \nabla \cdot \left( \frac{1}{2} \rho |\mathbf{u}'|^2 \mathbf{n} - \rho \mathbf{u}' \mathbf{u}' \cdot \mathbf{n} \right) dV \equiv \rho_0 \int_S \left( \frac{1}{2} |\mathbf{u}'|^2 \mathbf{n} - \mathbf{u}' \mathbf{u}' \cdot \mathbf{n} \right) dS$$

as $S$ lies in linear far field and hence vanishes. Therefore, (2.13) is actually not the form drag alone but a synthetic result of all drag constituents identified in near field.

3. Unified force theorem

In this section we consider the viscous flow over a finite body, for which Lagerstrom (1964, p. 36) has pointed out that the linearization is feasible but without proof. The first proof has been given by LZW for two dimensions, by finding analytically the nontrivial solutions in the linear far field from subsonic to supersonic flows, as well as the detailed behavior of the longitudinal and transversal far-field velocities and vorticity. This method is now applied to three dimensions, which will lead to a force formula universally true for viscous, compressible and steady flow, for both two and three dimensions.

3.1. Fundamental solution method

For an observer in very far field, a body moving through a fluid appears as a singular point, and its action on the fluid appears as an impulse force. In this case the far-field disturbance flow is sufficiently weak and may well be governed by linearized N-S equations. Note that to calculate the impulse force there is no need to solve these equations under specified boundary conditions. Rather, it suffices to directly use the fundamental solution of the linearized steady N-S equations in free space. This is the basic idea in the study of linear differential equations, which is called fundamental solution method and has been successfully demonstrated by LZW for two dimensions.

Following LZW, we introduce (primed) disturbance quantities by

$$\mathbf{u} = U \mathbf{e}_x + \mathbf{u}', \quad \rho = \rho_0(1 + \rho'),$$

then the steady momentum and continuity equations are

\begin{align*}
(\nu_0 \mathbf{T}_\psi - \nu \mathbf{T}_\phi - U \partial_x \mathbf{I}) \cdot \mathbf{u}' - c^2 \nabla \rho' &= -f, \tag{3.2a} \\
\nabla \cdot \mathbf{u}' + U \partial_x \rho' &= 0, \tag{3.2b}
\end{align*}

where (hereafter the subscripts $\phi$ and $\psi$ denote longitudinal and transversal fields, respectively)

$$\mathbf{T}_\phi = \nabla \nabla, \quad \mathbf{T}_\psi = \nabla \nabla - \nabla^2 \mathbf{I}, \quad \mathbf{I} = \text{unit tensor}, \tag{3.2c}$$

and $\nu = \mu/\rho_0$ and $\nu_0 = \mu_0/\rho_0$ are the constant transversal and longitudinal kinematic viscosities, respectively, $c$ is the speed of sound, and $f$ represents an external body force, which in our case is the force exerted to the fluid by the body. In near-field formulation $f$ could have a compact distribution in $(x,t)$-space as used by Saffman (1992, p. 51), but below it will be idealized as a $\delta$-function of $x$, i.e.,

$$f = -\frac{\delta(x)}{\rho_0} \mathbf{F}, \tag{3.3}$$

where $\mathbf{F}$ is the total force exerted to the body, and the full-space integral of $\delta(x)$ is unit:

$$\int \delta(x) dx = 1. \tag{3.4}$$
Denote $G$ as the fundamental solution of (3.2) for $u'$, of which the derivation and expression are given in Appendix A, then the far-field $u'$ can be written as

$$u'(x) = \int G(x, x') \cdot f(x')dx'.$$

(3.5)

Since the total disturbance velocity $u'$ can be decomposed into a longitudinal part and a transversal part, see (1.1), it can be verified that $G$ can also be split into longitudinal and transversal parts,

$$G(x, x') = G_\phi(x, x') + G_\psi(x, x'),$$

(3.6)

so that the longitudinal and transversal velocities defined by (1.1) now read

$$u_\phi = \nabla \phi = \int G_\phi(x, x') \cdot f(x')dx',$$

(3.7a)

$$u_\psi = \nabla \times \psi = \int G_\psi(x, x') \cdot f(x')dx'.$$

(3.7b)

Here,

$$G_\phi(x, x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} T_\phi \left[ e^{i\xi x} \frac{i\xi U g}{\sqrt{\sigma^2 + \xi^2}} \right] d\xi,$$

(3.8a)

$$G_\psi(x, x') = -\frac{1}{2\pi} \int_{-\infty}^{\infty} T_\psi \left[ e^{i\xi x} \frac{i\xi U g}{\sqrt{\sigma^2 + \xi^2}} \right] d\xi$$

(3.8b)

are the fundamental solutions for the longitudinal and transversal processes, respectively, and

$$g_h = \frac{1}{2\pi} K_0(\sigma \sqrt{h^2 + \xi^2}), \quad \sigma = \sqrt{y^2 + z^2},$$

(3.8c)

with $K_0$ being the modified Bessel function of the second kind.

Compared to dealing with the disturbance velocity $u'$ directly, we find it sometimes more convenient to deal with the velocity potential $\phi$ and the vortical stream function $\psi$. By substituting (3.3) and (3.8) into (3.7), we can obtain

$$\phi = -\frac{1}{4\pi \rho_0 U} F \cdot \nabla \Phi,$$

(3.9a)

$$\psi = -\frac{1}{4\pi \rho_0 U} F \times \nabla \Psi,$$

(3.9b)

where

$$\Phi = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\xi x} K_0(\sigma \sqrt{\xi^2 + \frac{i\xi U}{\nu_0 + \frac{\xi^2}{i\xi U}}}) d\xi,$$

(3.10a)

$$\Psi = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\xi x} K_0(\sigma \sqrt{\xi^2 + \frac{i\xi U}{\nu}}) d\xi.$$  

(3.10b)

Here, since the integrals of (3.10a) and (5.7) are divergent in general, following Hadamard (1928) the symbol $\tilde{f}$ is used to denote the finite part of divergent integrals.
Lift and drag in three-dimensional steady viscous and compressible flow

3.2. The transversal far-field

By differentiating (5.7) with respect to \( x \), there is \((\text{Gradshteyn & Ryzhik 2007, p. 722})\)

\[
\frac{\partial \Psi}{\partial x} = e^{kx} \int_{-\infty}^{\infty} e^{i\xi x} K_0 \left( \sigma \sqrt{\xi^2 + k^2} \right) d\xi
\]

\[
= 2e^{kx} \int_{0}^{\infty} \cos(\xi x) K_0 \left( \sigma \sqrt{\xi^2 + k^2} \right) d\xi
\]

\[
= \frac{e^{-k(r-x)}}{r} \equiv \chi, \quad r = \sqrt{x^2 + y^2 + z^2},
\]

where

\[
k \equiv \frac{U}{2\nu},
\]

Then from (3.11c) we have

\[
\Psi = \int_{-\infty}^{x} \chi(t,y,z) dt = \int_{r-x}^{\infty} \frac{e^{-kt}}{t} dt,
\]

where we have used the upstream decaying condition \( \Psi(-\infty, y, z) = 0 \). Of course, (3.13) is independent of the Mach number and divergent at the positive \( x \)-axis \((r - x = 0)\), where the singular transverse velocity must be canceled by the longitudinal velocity.

By substituting (3.9b) into (3.7b), the transverse velocity is

\[
u_{\psi} = \frac{1}{4\pi \rho_0 U} \nabla \left( F \cdot \nabla \Psi \right).
\]

Since

\[
\nabla^2 \Psi = 2k \frac{\partial \Psi}{\partial x} = 2k \chi,
\]

(3.14) yields

\[
u_{\psi} = \frac{1}{4\pi \rho_0 U} \nabla \left( F \cdot \nabla \Psi \right) + v,
\]

where

\[
v \equiv -\frac{1}{4\pi \mu} F
\]

is the purely rotational velocity. Then, by (3.16) the vorticity is

\[
\omega = \nabla \times v = \frac{1}{4\pi \mu} F \times \nabla \chi.
\]

Because \( \nabla \chi \) is the only vorticity source term in (3.18), \( \chi \) is called the vorticity potential, which was first introduced by Lamb (1911) for the linearized far-field of steady axis-symmetrical flow. We now see it does exist for the linearized far-field of any steady three-dimensional flow. Furthermore, as a check of our algebra, we substitute (3.11c) into (3.18) to obtain

\[
\omega = \frac{k}{4\pi \mu} \frac{e^{-k(r-x)}}{r} \nabla (r-x) \times F + O \left( \frac{e^{-k(r-x)}}{r^2} \right),
\]

which agrees exactly the asymptotic vorticity expression obtained by Babenko & Vasiléyev (1973) for three-dimensional steady incompressible flow, see also Mizumachi (1984).
particular, in the $x$-axis there is
\[ \omega_x(x, 0, 0) = 0, \quad \omega_y(x, 0, 0) = - \frac{F_z}{4\pi \mu x^2}, \quad \omega_z(x, 0, 0) = - \frac{F_y}{4\pi \mu x^2}. \quad (3.20) \]

Thus the lift and side forces can also be written as
\[ F_y = 4\pi \mu x^2 \omega_z(x, 0, 0), \quad F_z = -4\pi \mu x^2 \omega_y(x, 0, 0). \quad (3.21) \]

Note that
\[ \int_W \chi dS = \pi \int_0^\infty \frac{e^{-k(\sqrt{x^2 + \sigma^2} - x)}}{\sqrt{x^2 + \sigma^2}} d\sigma^2 = 2\pi \int_x^\infty e^{-k(r-x)} dr = \frac{2\pi}{k}, \quad (3.22) \]
so the wake-plane integral of vorticity vanishes:
\[ \int_W \omega dS = \frac{1}{4\pi \mu} F \times e_x \frac{\partial}{\partial x} \int_W \chi dS = 0, \quad (3.23) \]
where the derivatives to $y$ and $z$ vanish due to the symmetry of $\chi$. Of course (3.23) holds independent of both Mach number and Reynolds number for both three- and two-dimensional flows (LZW).

On the other hand, from (3.17) and (3.22) we have
\[ F \approx -\rho_0 U \int_W \mathbf{v} dS, \quad (3.24) \]
which has exactly the same form as the force formula (1.13) for incompressible flow since $\mathbf{v} = \mathbf{v}_2 + \mathbf{v}_3$. But now it is also valid for compressible flow. Furthermore, since $\mathbf{v}$ is the purely rotational part of $\mathbf{u}'$, we may expect that $F$ can be solely expressed by the vorticity. This will be discussed in details in Section 4.

3.3. Unified force theorem

With the above preparations, we can now state the following innovative theorem:

**Unified force theorem.** For an $n$-dimensional steady flow of viscous and compressible fluid over a rigid body, $n = 2, 3$, the lift and drag exerted to the body are solely determined by the multi-valueness and singularities of the velocity potential $\phi$ in the circulation $\Gamma_{\phi}$ and the vortical stream function $\psi$ in the inflow $Q_{\psi}$, respectively:
\[ F = (n-1)\rho_0 U \times \Gamma_{\phi} + \rho_0 U Q_{\psi}, \quad (3.25) \]
where $\Gamma_{\phi}$ and $Q_{\psi}$ are given by (1.3) and (1.7), respectively, and are independent of the choice of control surface $S$.

**Proof.** First, as remarked previously, either $\Gamma_{\phi}$ or $Q_{\psi}$ vanishes due to the generalized Stokes theorem (e.g., Wu, Ma & Zhou 2006, p. 700), unless $\phi$ or $\psi$ are multi-valued or singular somewhere. This multi-valueness or singularity is independent of the integral surface $S$, and hence so is (3.25). In fact, $S$ can even be located in the nonlinear near field as long as the definition domain of $\phi$ and $\psi$ is properly extended; but the proof of the theorem can be made in the linearized far field where the formal solution (3.9) is valid.

Then, to prove (3.25), we only need to show that for $n = 3$ there is
\[ F = 2\rho_0 U \times \Gamma_{\phi} + \rho_0 U Q_{\psi}, \quad (3.26) \]
since then a comparison of (3.26) and (2.7) implies
\[ \rho_0 \int_S \mathbf{n} \times \nabla (U \cdot \psi) dS = \rho_0 U \times \Gamma_{\phi}, \quad (3.27) \]
and thus (3.25) follows at once. Note that after the existence of linear far field was proven in §§3.1 and 3.2, (2.7) has become a rigorous result and can be cited here.

To prove (3.26), observed that by substituting (3.16) and (3.9a) into (1.1), the total disturbance velocity can be written as

\[ u' = -\frac{1}{4\pi \rho_0 U} F \cdot \nabla (\Phi - \Psi) + v. \]  

(3.28)

Since \( u' \) and \( v \) must be regular, so must \( F \cdot \nabla (\Phi - \Psi) \). Then we can rewrite (3.9a) as

\[ \phi = -\frac{1}{4\pi \rho_0 U} [F \cdot \nabla (\Phi - \Psi) + F \cdot \nabla \Psi], \]

(3.29)

where the first term on the right hand side is regular, making no contribution to \( \Gamma_\phi \) due to the generalized Stokes theorem. Thus the longitudinal circulation \( \Gamma_\phi \) given by (1.3) reduces to (for details see Appendix B)

\[ \Gamma_\phi = -\frac{1}{4\pi \rho_0 U} \int_S (n \times \nabla)(F \cdot \nabla \Psi) dS = \frac{F \times e_x}{2\rho_0 U}, \]

(3.30)

Similarly, by directly substituting (3.9b) into the transversal inflow \( Q_\psi \) defined by (1.7), we find (for details see Appendix B)

\[ Q_\psi = \frac{1}{4\pi \rho_0 U} \int_S (n \times \nabla) \cdot (F \times \nabla \Psi) dS = \frac{F \cdot e_x}{\rho_0 U}. \]

(3.31)

Obviously, both \( \Gamma_\phi \) and \( Q_\psi \) are indeed independent of \( S \) and valid for arbitrary Mach number or Reynolds number. The proof is thus completed.

**Remark.** Although \( \phi, \psi, u_\phi \) and \( u_\psi \) are non-observable, once well established, the unified force theorem turns immediately \( \Gamma_\phi \) and \( Q_\psi \) with singular integrands to observable quantities in a generalized sense: they are just equivalent to the lift and drag (divided by \( (n - 1)\rho_0 U \) and \( \rho_0 U \), respectively) obtained by any experiments or computations!

In the above proof we only used the specific behavior of \( \Psi \). This approach makes the proof concise and general, but leaves the relevant physical mechanisms of lift and drag obscure. Thus, to explore the underlying physics we still have to directly analyse the disturbance velocity. We do this in the next section.

4. Testable unified force formula

In this section, we shall analyze the longitudinal far-field flow structures, estimate the distances of the linear far field from the body and clarify the far-field behavior of multiple circulations. After that, we further confirm that (2.12) and (2.13) are the far-field asymptotic approximation of (3.25), expressed solely in terms of observable vorticity field. Following LZW, we call this the testable unified force formula.

4.1. The longitudinal far-field

Like we did for \( \Psi \), to calculate the integral (3.10a) we also differentiate it with respect to \( x \),

\[ \frac{\partial \Phi}{\partial x} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\xi x} K_0 \left( \sigma \sqrt{\Lambda e^{i\theta}} \right) d\xi. \]

(4.1)

where

\[ \Lambda e^{i\theta} = \xi^2 + \frac{i\xi U}{\nu_0} + \frac{c^2}{\nu_0^2} = (1 - M^2)\xi^2 + i\frac{\nu_0 M^4}{U} \xi^3 + O(\nu_0^2), \quad M = \frac{U}{c}. \]

(4.2)
Obviously, the longitudinal part described by \( \Phi \) depends explicitly on the Mach number, \( M = U/c \), as seen from the key factors \( 1 - M^2 \) and \( M^4 \) in (4.2) in the Fourier space. As will be shown in the followings, (4.2) takes different leading-order forms for subsonic, transonic, and supersonic flows, corresponding to different structures of linear far field. In general, the velocity potential \( \phi \) can not be integrated exactly, but can be expressed by contour integrals, which is regular everywhere except at the positive \( x \)-axis. This fact indicates that the linearized far field indeed exists for compressible flow, which is even true for inviscid subsonic flow but only for viscous transonic and supersonic flows.

To dig as much information as possible from these integrals, we make a case-by-case analysis for different Mach-number regimes.

4.1.1. Subsonic flow

For subsonic flow, there is

\[
A \approx \beta^2 \xi^2, \quad \theta \approx 0, \quad \beta^2 = 1 - M^2 > 0.
\]

Then

\[
\frac{\partial \Phi}{\partial x} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\xi x} K_0 \left( \beta \sigma \sqrt{\xi^2} \right) d\xi = \frac{1}{r_\beta},
\]

where

\[
r_\beta^2 = x^2 + \beta^2 \sigma^2 = x^2 + \beta^2 (y^2 + z^2).
\]

Thus we have

\[
\Phi = \int_{r_\beta - x}^{\infty} \frac{1}{t} dt = -\ln(r_\beta - x).
\]

Then the longitudinal potential is

\[
\phi = \frac{1}{4\pi \rho_0 U} F \cdot \nabla \ln(r_\beta - x),
\]

which is singular at the positive \( x \)-axis and of which the singular longitudinal velocity can just cancel out that of the transversal field. The dilatation is

\[
\vartheta = \nabla^2 \phi = \frac{M^2}{4\pi \rho_0 U} F \cdot \nabla \left( \frac{x}{r_\beta^3} \right),
\]

which, of course, is always regular.

Note that the above results are also valid for incompressible flow by setting \( \beta = 1 \).

4.1.2. Supersonic flow

For supersonic flow, there is

\[
A \approx B^2 \xi^2, \quad \theta \approx \pi - 2\Lambda \xi,
\]

where

\[
B^2 = M^2 - 1 > 0, \quad \Lambda = \frac{\nu_0 M^4}{2B^2 U} \ll 1.
\]

Now, we need to find such a viscous solution that it is significant only near the Mach cone and decays exponentially elsewhere except near the positive \( x \)-axis (or wake region),
Lift and drag in three-dimensional steady viscous and compressible flow

where it must cancel out the singularity of the transversal field. Firstly, note that
\[
\frac{\partial \Phi}{\partial x} \approx \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\xi x} K_0 \left( i B \sigma \xi \sqrt{1 - 2i \Lambda \xi} \right) d\xi
\]
(4.11)
\[
\approx \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\xi x} K_0 \left( \Lambda B \sigma \xi^2 + i B \sigma \xi \right) d\xi,
\]
(4.12)
then the far-field decaying condition can be ensured since \(\Lambda B \sigma \xi^2 \geq 0\), which also verifies that the viscous effect has a vital role in supersonic flow. However, the contour integral in (4.12) is hard to integrate explicitly. Instead, we consider its approximation near the Mach cone by the asymptotic identity
\[
\lim_{z \to \infty} K_0(z) \approx \sqrt{\frac{\pi}{2}} z e^{-\frac{z}{2}}, \quad z \to \infty.
\]
(4.13)
Then (4.12) reduces to
\[
\frac{\partial \Phi}{\partial x} \approx \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\xi x} \sqrt{\frac{\pi}{2(i + \Lambda \xi) B \sigma \xi}} e^{-i(X + \Lambda \xi) B \sigma \xi} d\xi
\]
(4.14)
\[
\approx \frac{1}{\sqrt{2\pi B \sigma}} \int_{-\infty}^{\infty} e^{i\xi(x - B \sigma)} e^{-\Lambda B \sigma \xi^2} \frac{d\xi}{\sqrt{\xi}},
\]
(4.15)
\[
= \sqrt{\frac{2}{\pi B \sigma}} \int_{0}^{\infty} \cos \left[ (x - B \sigma) \xi - \frac{\pi}{4} \right] e^{-\Lambda B \sigma \xi^2} \frac{d\xi}{\sqrt{\xi}},
\]
(4.16)
where \(\sigma = \sqrt{y^2 + z^2} \gg 1\) and only the leading term is retained.

Although the integral of (4.16) can be worked out explicitly, here we left it out since its form is somewhat complicated and inconvenient to analyze. Instead, suppose \(F = De_x\) and substitute it into (3.9a), we have
\[
\phi = -\frac{D}{4\pi \rho_0 U} \frac{\partial \Phi}{\partial x}.
\]
(4.17)
Then from (4.17) and (4.16) the disturbance longitudinal velocity is
\[
\frac{1}{U} \frac{\partial \phi}{\partial x} \approx \frac{C_D}{8\pi} \sqrt{\frac{2}{\pi B \sigma}} \int_{0}^{\infty} \sin \left[ (x - B \sigma) \xi - \frac{\pi}{4} \right] e^{-\Lambda B \sigma \xi^2} \sqrt{\xi} d\xi,
\]
(4.18)
where \(C_D = 2D/\rho_0 U^2\). Evidently, there is
\[
\frac{\partial \phi}{\partial \sigma} \approx -B \frac{\partial \phi}{\partial x}.
\]
(4.19)
Since the disturbance must be largest along the Mach cone, there is
\[
\frac{1}{U} \frac{\partial \phi}{\partial x} \bigg|_{x=B\sigma} \approx -\frac{C_D}{8\pi} \sqrt{\frac{1}{\pi B \sigma}} \int_{0}^{\infty} e^{-\Lambda B \sigma \xi^2} \sqrt{\xi} d\xi = -\frac{G(3/4)C_D}{16\pi^{3/2} \Lambda^{3/4} B^{3/4} \sigma^{1/2}}.
\]
(4.20)
where \(G(\cdot)\) is the Gamma function and \(G(3/4) = 1.22542 \cdots\).

4.1.3. Sonic flow

The viscosity is also necessary for sonic flow, where there is
\[
\sqrt{A e^{i\theta}} = \sqrt{\frac{|\xi|^3}{Re_\theta}} e^{i\frac{3}{4} \text{sgn} \xi}, \quad Re_\theta = \frac{U}{\nu_\theta},
\]
(4.21)
and

\[ \frac{\partial \Phi}{\partial x} = \frac{1}{\pi} \int_{0}^{\infty} \left[ K_0 \left( \sigma \sqrt{\frac{\xi^3}{Re_\theta}} e^{-i\xi x} \right) + K_0 \left( \frac{\xi^3}{Re_\theta} e^{-i\xi x} \right) \right] d\xi, \quad (4.22) \]

which for \( \sigma \gg 1 \) reduces to

\[ \frac{\partial \Phi}{\partial x} \approx \sqrt{\frac{2}{\pi \sigma}} \int_{0}^{\infty} e^{-\sigma \sqrt{\frac{\xi^3}{2Re_\theta}}} \cos \left( \frac{\pi}{8} - \sigma \sqrt{\frac{\xi^3}{2Re_\theta}} + \frac{\xi^3}{2Re_\theta} \right) d\xi. \quad (4.23) \]

In particular, we have

\[ \frac{\partial^2 \Phi(0, \sigma)}{\partial x^2} \approx \sqrt{\frac{2}{\pi \sigma}} \int_{0}^{\infty} e^{-\sigma \sqrt{\frac{\xi^3}{2Re_\theta}}} \sin \left( \frac{3\pi}{8} + \sigma \sqrt{\frac{\xi^3}{2Re_\theta}} - \frac{\xi^3}{2Re_\theta} \right) d\xi. \quad (4.24) \]

and

\[ \frac{\partial^2 \Phi(0, \sigma)}{\partial x \partial \sigma} \approx -\sqrt{\frac{2}{\pi \sigma}} \int_{0}^{\infty} e^{-\sigma \sqrt{\frac{\xi^3}{2Re_\theta}}} \sin \left( \frac{3\pi}{8} + \sigma \sqrt{\frac{\xi^3}{2Re_\theta}} - \frac{\xi^3}{2Re_\theta} \right) d\xi. \quad (4.25) \]

Thus from (4.17) and (4.25), there is

\[ \frac{1}{U} \frac{\partial \phi}{\partial x} \bigg|_{x=0} \approx -\frac{C_D}{2\pi} \frac{\partial^2 \Phi(0, \sigma)}{\partial x^2} = -\frac{G(5/6)C_D Re_\theta^{\frac{2}{3}}}{4\sqrt{6\pi \frac{1}{2} \sigma^\frac{3}{2}}}, \quad (4.28) \]

where \( G(5/6) = 1.12879 \cdots \).

4.2. Distance of linear far-field from the body

We now use the preceding solutions of linear equations to predict how large the minimum distance \( r_m = \sqrt{x_m^2 + \sigma_m^2} \) from the body should be for them to become valid. LZW has shown that this estimate can be tested by numerical study. In this way, the existence of linear far field can be understood more concretely. For comparison, we also list the corresponding two-dimensional estimates given by LZW.

The estimate is based on a simple requirement that the order of magnitude of total disturbance velocity constituents, after being non-dimensionalized, is not larger than unity. As a familiar example, for small-\( Re \) incompressible flow over a sphere with drag coefficient \( C_D \sim Re^{-1} \) (e.g. Lagerstrom 1964, p. 85), there is \( r_m = O(1) \); then at far field with \( r > \text{const.} \), one should turn to the Oseen equation.

Now, let the characteristic length scale be unity so that \( Re = U/\nu \) and \( Re_\theta = U/\nu_\theta \), for the minimum streamwise distance \( x_m \) of the linear far field, by (3.28) it is straightforward to find

\[ x_m = O \left( \frac{C_D Re}{8\pi} \right), \quad (4.29) \]

which is independent of Mach number and fully determined by transversal process. This is comparable with the two-dimensional estimate \( r_m = O(C_D^2 Re/16\pi) \).
On the other hand, the lateral minimum distance $\sigma_m$ of linear far field is dominated by longitudinal process, which can be determined from (4.7), (4.20), and (4.28):

1. Subsonic far-field:
   \[
   \sigma_m = O\left( \sqrt{\frac{C_D}{8\pi\beta}} \right),
   \]
   comparable with the two-dimensional estimate $r_m = O(C_l/4\pi\beta)$.

2. Supersonic far-field:
   \[
   \sigma_m = O\left( \frac{G(3/4)^{\frac{1}{4}}B^{\frac{1}{4}}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}M^{\frac{1}{2}}C_D^4 R_\theta^2} \right),
   \]
   which is smaller than two-dimensional estimate $r_m \sim C_l^2 R_\theta$ or $C_l^2 R_\theta^3$.

3. Sonic far-field:
   \[
   \sigma_m = O\left( \frac{G(5/6)^{\frac{1}{3}}B^{\frac{1}{3}}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}C_D^4 R_\theta^2} \right),
   \]
   which is, remarkably, very much smaller than two-dimensional estimate $r_m \sim C_l^3 R_\theta^2$.

Clearly, in different Mach-number regimes and spatial directions, the dominant linearized far-field dynamic processes and flow structures are vastly distinct, with variance dependence on $C_D$, $Re$ or $Re_\theta$, and $M$. Of these distances $x_m$ is the farthest from the body. Fortunately, this large value is limited in the relatively narrow wake region and will not significantly affect the lift or side force, though it may have stronger effect on the drag. It should be stressed that, because $u_\phi$ and $u_\psi$ are infinite in the positive $x$-axis, one can not determine the location of the transversal and longitudinal fields separately as LZW did for two dimensions.

4.3. Multiple circulations

Similar to the longitudinal circulation $\Gamma_\phi$ given by (1.3) and the total circulation $\Gamma$ given by (1.5), we can define a transversal circulation (cf. LZW)

\[
\Gamma_\psi = \int_S n \times u_\psi dS = \Gamma - \Gamma_\phi.
\]

Let us now examine the far-field behavior of the three circulations. A substitution of the expression of $u'$, (3.28), into the first expression of (1.5) yields

\[
\Gamma = \int_S n \times v dS,
\]
where $v$ is given by (3.17). Let $S$ be a spherical surface with radius $r$. Then by substituting (3.17) into (4.32), we can obtain

\[
\Gamma = \frac{F \times e_x}{\rho_0 U} \left( 1 - \frac{1}{kr} + e^{-2kr} + \frac{1}{kr} e^{-2kr} \right),
\]
since $\Gamma_\phi$ is given by (3.30), we obtain

\[
\Gamma_\psi = \frac{F \times e_x}{\rho_0 U} \left( \frac{1}{2} - \frac{1}{kr} + e^{-2kr} + \frac{1}{kr} e^{-2kr} \right),
\]
which is dependent on $r$ or $S$ but independent of Mach number and Reynolds number. Note that, unlike two dimensions where the transversal circulation $\Gamma_\psi$ decays to zero as
$r \to \infty$, here it converges to the value of longitudinal circulation. This is the reason for the factor 2 in the unified force formula (3.25).

With these discussions, therefore, the generically non-observable $\Gamma_\phi$ becomes observable when it is used to measure the total vorticity in the total steady-flow region $V_{st}$:

$$\lim_{r \to \infty} \Gamma = 2 \Gamma_\phi = \int_{V_{st}} \omega dV. \quad (4.35)$$

### 4.4. Testable unified force formula

Although the remarks following the unified force theorem has turned $\Gamma_\phi$ and $Q_\psi$ to observable quantities, one is evidently still not satisfied if the integrands of $\Gamma_\phi$ and $Q_\psi$ are not observable in practice. We thus need to find the circumstances in which these integrands can be replaced by physically observable variables. Now the preceding analyses of the flow behaviour have revealed that the required circumstance is the linear far field, of which the existence has been confirmed by the estimates made in § 4.2. This permits us to give a testable version of the unified force formula (3.25), which we state first:

**Testable unified force formula.** For an $n$-dimensional steady viscous flow of compressible fluid over a rigid body, $n = 2, 3$, the force exerted on the body is given by (2.12) with $W$ being the downstream face of the outer boundary of $S$, which is perpendicular to the incoming flow and lies in the linear far field.

Here and below, we will call this result the TUF formula for short. Since the two-dimensional TUF has been addressed by LZW, we focus on the case $n = 3$.

**Remarks.**

1. One of the proofs of the TUF formula has been given in § 2, where use has been made of $\partial / \partial y, \partial / \partial z \gg \partial / \partial x$. In fact, this assumption can be removed. To see this, we first transform (2.12) to a form involving wake-plane integrals only, by using the identity (Wu, Ma & Zhou 2006, p. 700)

$$\int_V f dV = - \int_V x(\nabla \cdot f) dV + \int_{\partial V} x(n \cdot f) dS.$$

This casts (2.12) to

$$F \cong \rho_0 U \times \int_W x \omega_z dS - \frac{1}{2} \rho_0 U \int_W (y \omega_z - z \omega_y) dS. \quad (4.36)$$

Then, by substituting the expression of vorticity (3.18) into (4.36) we can directly confirm the validity of (2.12) free from the aforementioned assumption but only require that the steady linearized far-field is reached.

2. The TUF force formulas for $n = 2$ and $n = 3$ are never equivalent to each other, especially for the drag.

3. Due to its linear dependence on the vorticity, TUF formula (2.12) is supposed to be valid for statistically stationary flow, including time-averaged turbulent flow (with constant turbulent viscosity, see LZW).

### 5. Three-dimensional vorticity far field and singularity

Along with the result of LZW, we see that for both two and three dimensions the far-field asymptotic formulas of lift and drag are expressible solely in terms of vorticity integrals, from incompressible regime all the way to supersonic regime, no matter what complex processes and structures such as shocks, entropy gradient, and curved-shock generated vorticity field may occur. In other words, only vorticity leaves signature in far
Lift and drag in three-dimensional steady viscous and compressible flow

field since it decays in the wake most slowly and, what is more remarkably, because the vorticity is a transverse field, the TUF formula (4.36) is completely independent of the Mach number (the specific \( M \)-dependence of the vorticity field can only be identified by near-field flow behavior).

This being the case, we may well use the familiar difference in the physical behaviors of the incompressible vorticity field for \( n = 2 \) and \( 3 \) to interpret the distinction in (2.7), (2.12) or (4.36) for these two cases:

For \( n = 2 \), as discussed in details by LZW, vorticity lines are all straight and along the spanwise direction. The flow domain is doubly-connected, permitting multiple values of potentials \( \phi \) and \( \psi \). A steady wake must be cut by any boundary of the steady-flow sub-space \( V_{st} \subset V_{\infty} \), leaving the starting vortex system outside \( V_{st} \). This makes it inevitable that, mathematically, a body experiencing a force must have nonzero \( \left[ \phi \right] \) and/or \( \left[ \psi \right] \), which are responsible for the lift and drag, respectively, and surely independent of the choice of control surface.

In contrast, for \( n = 3 \), vorticity lines can be stretched and tilted, and eventually go to far field with \( \omega_x = -\nabla^2 \psi_x \) being the dominating component there, as indicated by the lift part of (4.36). Namely, the well-known trailing vortex couple is a universal phenomenon for any lifting body in three-dimensional viscous and compressible steady flow. This should explain why there is an extra term in (2.7) for \( n = 3 \) only, where \( U \cdot \psi = U \psi_x \) implies a contribution of \( \omega_x \) to both lift and drag. It is this extra term that makes a nonzero \( \Gamma_{\psi} \). But our finding that, at far field with \( r \to \infty \), each of \( \Gamma_{\psi} \) and \( \Gamma_{\phi} \) gives half of \( \Gamma \) (or total vorticity in \( V_{st} \)), is surprisingly interesting.

On the other hand, the flow domain is singly-connected, permitting no multi-valueness of \( \phi \) and \( \psi \). Mathematically, the only possible mechanism for providing nonzero force and being independent of the choice of control surface is the singularity of \( \phi \) and \( \psi \). The singularity has to disappear or be cancelled once we use \( \phi \) and \( \psi \) to construct observable flow quantities.

To better understand the above discussion, we write formally the longitudinal velocity potential \( \phi \) as

\[
\phi = \phi_f + \phi_r \cong \phi_f \quad \text{in the linearized far field},
\]

where \( \phi_f \) is the dominant term in the far-field, which may be multi-valued or singular, and \( \phi_r \) is the single-valued regular term, which decays faster than \( \phi_f \) at the far-field but may play a crucial role in the near-field. In particular, for two-dimensional incompressible flow, there is (Liu, Zhu & Wu 2015)

\[
\phi_f = \frac{L}{2\pi \rho_0 U} \arctan \left( \frac{z}{x} \right) + \frac{D}{2\pi \rho_0 U} \ln \sqrt{x^2 + z^2}.
\]

Similarly, suppose \( \mathbf{F} = (D, 0, L) \) for three-dimensions incompressible flow with \( \beta = 1 \), then (4.7) reduces to

\[
\phi_f = \frac{L}{4\pi \rho_0 U} \frac{z}{r(r - x)} - \frac{D}{4\pi \rho_0 U} \frac{1}{r}.
\]

This decomposition can also be directly applied to the transversal stream function \( \psi \). Thus we see clearly that, the lift and drag exerted to the body are solely determined by the multi-valueness and singularities of the velocity potential \( \phi \) and the vortical stream function \( \psi \).

While a complete analysis of the physical carrier of the singularity for compressible flow is too difficult to be done if not impossible, owing to the \( M \)-independence of (4.36) the interpretation first presented by Goldstein (1931) for incompressible is sufficient and worth recapitulating. We do this by a concrete line-vortex doublet model.
Suppose $F = Le_z$ and $\beta = 1$, then (4.7) reduces to

$$\phi = \frac{L}{4\pi \rho_0 U} \frac{z}{r(r-x)}, \quad (5.4)$$

which must be the dominate term in (1.1) for $y^2 \gg 1$, since the transversal part decays exponentially in this case. Then, in a footnote, Goldstein (1931) asserts that $z/r(r-x)$ gives the potential of a line doublet stretching from the origin along the $x$-axis to plus infinity. Here, a doublet has the same potential as a vortex filament bounding an infinitely small plane area if the strength of the doublet is equal to the product of the area and the strength of the vortex, and the doublet is along the normal to the area. Furthermore, Goldstein (1931) thought that a line doublet, stretching to infinity in one direction, has the same potential as a “horse-shoe” vortex (of the type encountered in approximate aerofoil theory), of infinitesimal breadth, if the strength of the doublet per unit length is equal to the product of the strength of the vortex and the breadth of the “horse-shoe”; and $z/r(r-x)$ is the potential of such a “horse-shoe” vortex, with the “trailing” vortices along the axis of $x$ from the origin to plus infinity, and the “bound” vortex of infinitesimal span, along the $y$-axis.

Obviously, Goldstein’s vortex doublet is precisely the far-field picture of the familiar trailing vortex couple. Compared to those methods used by Goldstein (1931) and Garstang (1936), this picture can be most intuitively visualized with our fundamental-solution method where the body is shrunk to a singular point. We now show that this vortex doublet is indeed the only possible source of singularity. Suppose that there is indeed such a “horse-shoe” with circulation $\Gamma$ and width $b$. Assume that $\Gamma b$ is fixed as $b \to 0$. Then the velocity induced by this vortex filament $C$ is (Wu, Ma & Zhou 2006, p. 81)

$$u' = \frac{\Gamma}{4\pi} \oint_C \frac{t \times r'}{r'^3} ds = \frac{\Gamma b e_z}{4\pi} \int_0^\infty \frac{dx'}{[(x-x')^2 + y^2 + z^2]^2} = \frac{\Gamma b e_z}{4\pi} \frac{1}{r(r-x)}, \quad (5.5)$$

Note that $u'$ has only one component $u'_z$, since fluids are entrained from the outside of the “horse-shoe” vortex, forming “upwash”, and pumped into it, forming “downwash”. But the downwash phenomenon can not be observed from (5.5) due to our assumption, i.e. $\Gamma \to \infty$ and $b \to 0$ but $\Gamma b$ remains fixed.

Now consider both $y^2 \gg 1$ and $z^2 \ll 1$, from (5.5) we have

$$\phi \approx \frac{\Gamma b}{4\pi r(r-x)} z, \quad (5.6)$$

Recall that in lifting-line theory there is $L \cong \rho_0 U \Gamma b$, we see (5.6) is identical to (5.4) in this special case. This simple argument seems to be the first confirmation of the assertion of Goldstein (1931).

Note that the above argument is not applicable to non-lifting flow $F = De_x$ where there is no singularity in longitudinal velocity potential $\phi$, as shown in the last term in (5.3). However, since the transversal stream function $\psi$ is still singular there is a drag. This can be seen more clearly by substituting $F = De_x$ into (3.96), which then reduces to

$$\psi = \frac{D \nabla \Psi \times e_x}{4\pi \rho_0 U} = \frac{D}{4\pi \rho_0 U} \frac{e^{-k(r-x)}}{r(r-x)} (0, -z, y), \quad (5.7)$$

This singularity comes from the fact that all vorticity is limited to the positive $x$-axis when viewed from far field.
6. Conclusions and discussions

6.1. Main findings of this paper

In this theoretical paper, we have studied the total lift and drag experienced by a body moving with constant velocity through a three-dimensional, externally unbounded, viscous and compressible fluid at rest at infinity. The major findings of this paper are summarized as follows.

1. A unified force theorem has been proven to hold universally for both two- and three-dimensional viscous, compressible and steady flow over a rigid body has been obtained. It states that the lift and drag exerted on the body are unified determined by the vector circulation $\Gamma_\phi$ due to the longitudinal velocity and scalar inflow $Q_\psi$ due to the transversal velocity, both being independent of the boundary of the domain used to calculate the circulation and inflow, as well as of the Reynolds number and Mach number.

2. The far-field asymptotic form of the exact unified force formula has also been obtained, solely expressed by vorticity integrals and valid if the domain boundary lies in linear far field. Its form is also independent of the Reynolds number and Mach number. This result is a reflection of the inherent flow physics: no matter how many interacting processes could appear in a nonlinear near-field flow, only the vorticity field has the farthest downstream extension and leaves signature in far field.

3. The unified force formula and its far-field asymptotics contains explicitly the spatial dimension $n$ ($n = 2, 3$), so the lift and drag are never the same for two and three dimensions. This fact is a result of the intrinsic difference of flow patterns in two and three dimensions. Unlike two dimensions where the lift and drag come solely from the multi-valueness of velocity potentials $\phi$ and $\psi$ in doubly-connected flow domain, now in three dimensions they come solely from the singularity of $\phi$ and $\psi$ in singly-connected flow domain, as first pointed out by Goldstein (1931). In the far field, as the body shrinks to a point the body-generated steady trailing vortex couple or “horse-shore” vortex degenerates to a line-vortex doublet of vanishingly small span, which is the only physical source of singularity in both incompressible flow and compressible flow.

4. Our progress in far-field aerodynamic theory is one more evidence of the fundamental importance of the concept and theory on process splitting and coupling, as systematically presented by Wu, Ma & Zhou (2015, Chapter 2), and a strong indication of the superiority of the fundamental solution method in resolving linearized far-field equations.

6.2. On the limitation of classic aerodynamic theory

Before the computer era, pioneers of aerodynamic theories could only solve the flow field analytically, and hence were confined mainly to the simplest inviscid and attached flow over streamlined bodies, using small-perturbation approaches. Consequently, in both low-speed aerodynamics (e.g. Kármán & Sears 1935) and high-speed aerodynamics (e.g. Liepmann & Roshko 1957) the viscosity has to be dropped (here and below we put aside the embedded boundary layer for calculating friction drag). Moreover, in the main body of classic theories one is focused to solving the linearized or weakly nonlinear equations for velocity potential $\phi$. Vorticity appears merely as an extra complexity, say behind curved shocks and in shock-boundary-layer interactions. Nevertheless, the theoretical achievements of classic aerodynamics with brilliant deep physical insight have remained the most valuable heritages, which should and can be fully inherited and developed in modern aerodynamics.

Then, our far-field theory for compressible aerodynamic force in both two and three dimensions represents a breakthrough of classic inviscid high-speed aerodynamics, in particular supersonic aerodynamics, as can be clearly seen in two aspects.
First, we have shown that in transonic and supersonic regimes, a linear far field consistent with uniform condition at infinity can exist only if viscous terms are retained in the linearized equations. For subsonic inviscid flow, such a far field can exist just because inviscid term takes leading order. This observation explains why in the past far-field theory was never available for transonic and supersonic flows.

Second, in classic supersonic aerodynamics, lift and drag are interpreted in terms of shock waves and expansion waves. While this interpretation is indeed true in near-field flow field (Liu, Zhu & Wu 2015, LZW for short), we have now shown that the unified force formula and its asymptotic form solely in terms of vorticity integrals, initiated for incompressible flow, are equally valid all the way to supersonic flow. Thus, it would certainly be a misconception to infer from the shock-expansion interpretation that the Kutta-Joukowski circulation theorem and the Prandtl (1918) vortex force theory (both established for incompressible flow) are no longer valid.

The physical root of the above shortage of classic aerodynamics can be clearly understood in terms of the universal coexistence of multiple fundamental dynamic and thermodynamic processes in real flows, as systematically analyzed by Wu, Ma & Zhou (2015, Chapter 2): a fluid motion always have a transverse or shearing process measured by vorticity, and a longitudinal process that can be subdivided into acoustic mode and entropy mode, measured by dilatation, pressure and certain other thermodynamic variables. In viscous flow these processes are coupled both inside the fluid and on boundary. In particular, the vorticity generation by tangent pressure gradient is a universal process-coupling mechanism. Therefore, as the incoming velocity changes from low-speed to high-speed flow, what happens is the switch of dominating process from transverse to longitudinal, but the real viscous flow can never be treated as having only a single process, either transverse or longitudinal. This artificial simplification just cuts down the inherent connection between low-speed and high-speed aerodynamics, both in physical concept and mathematical method.

6.3. On the development of modern aerodynamic theory

As mentioned in the beginning of this paper, the far-field approach of LZW and the present paper is just a minor aspect of complete aerodynamic theories and confined to steady flow. At the center of aerodynamics is always near-field theories. But our preceding findings based on far-field analysis have unambiguously indicated that modern aerodynamic theories need to be reformulated in the framework of the full Navier-Stokes equations, broader than that of classic aerodynamics. In particular, flow viscosity and multiple processes have to be included. Of course, in so doing the precious inheritance of classic theories have to be inherited and enriched.

Actually, the desired modernization of near-field low-speed aerodynamic theory has been advanced for decades. After the classic work of Kármán & Sears (1935), in 1950s one started to include the effects of boundary-layer separation on aerodynamic performance, e.g., Thwaites (1960), signifying that complex separated vortical flows have been within the concern of aerodynamicists. Later progresses have led to a few different formulations on the aerodynamic force and moment for both steady and unsteady viscous flows, see the review of Wu, Ma & Zhou (2006, Chapter 11). In contrast, despite a few isolated efforts reviewed in Liu et al. (2014a) and LZW, the modernization of high-speed near-field aerodynamic theory had long remained a nearly untouched subject till the longitudinal-transverse force theory of Liu et al. (2014a,b), also based on the process splitting and coupling. We believe that this theory is just one of the possible formulations, and some more could be developed with no principle difficulty, for example the compressible force element theory of Chang and coworkers (Chang, Su & Lei 1998).
Unlike its classic counterpart, modern aerodynamics no longer insists on finding analytical flow solutions (but any analytical advance will be a very valuable contribution). In addition to the invariant guidance and indispensable support of physical experiments, aerodynamic theory has to be developed hand in hand with the powerful computational fluid dynamics (CFD). But CFD can never replace theoretical studies. The demise of theoretical aerodynamics implies the demise of the whole aerodynamics. In this regards, one should be highly alert to a widely encountered bias as recently commented by Schmitz & Coder (2015): although most of the physical insight gained has arisen from classic aerodynamics, the progress in CFD has not propelled our understanding of aerodynamics much further forward.

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Appendix A. Fundamental solution of the linearized 3D steady compressible N-S equations

This appendix highlights the derivation of the fundamental solution of (3.2) for \( u' \) in three-dimensional steady flow.

Denote the Fourier transform and inverse transform of (3.2) in \( x \)-direction as

\[
\tilde{f}(\xi, y, z) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x, y, z) dx, \quad f(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \tilde{f}(\xi, y, z) d\xi.
\]  

(A 1)

Now equations in (3.2) are transformed to (upon eliminating \( \tilde{\rho}' \))

\[
(aM_1 - bM_2 - k^2I) \cdot \tilde{u}' = -\tilde{f},
\]  

(A 2)

where

\[
a = \nu \theta + \frac{c^2}{i\xi U}, \quad b = \nu, \quad k^2 = i\xi U,
\]  

(A 3a)

and

\[
M_1 = \vec{\nabla}\vec{\nabla}, \quad M_2 = \vec{\nabla}\vec{\nabla} - \vec{\nabla}^2 I, \quad \vec{\nabla} = (i\xi, \partial_y, \partial_z).
\]  

(A 3b)

To find the fundamental solution of (A 2) the following theorem is very useful (Lagerstrom, Cole & Trilling 1948, pp. 172–175):

**Theorem 1.** If \( M_1 \) and \( M_2 \) are two linear differential matrix operators such that

\[
M_1 \cdot M_2 = M_2 \cdot M_1 = 0, \quad M_1 - M_2 = LI,
\]  

(A 4)

where \( I \) is the unit matrix and \( L \) is a scalar linear differential operator, then the fundamental solution \( G(x, \xi) \) of (A 2) is given by

\[
G(x, x') = \frac{1}{k^2} \left( M_1 g \sqrt{\frac{\xi^2}{x^2}} - M_2 g \sqrt{\frac{x^2}{\xi^2}} \right),
\]  

(A 5)

where \( g_h(x, x') \) is the fundamental solution of the scalar operator \( L - h^2 \).

Now, since \( L = \partial_y^2 + \partial_z^2 - \xi^2 \), of which the fundamental solution \( g_h \) with far-field decaying condition is (Lagerstrom, Cole & Trilling 1948, p. 178)

\[
g_h = \frac{1}{2\pi} K_0(\sigma \sqrt{h^2 + \xi^2}), \quad \sigma = \sqrt{y^2 + z^2},
\]  

(A 6)
then the fundamental solution of (A2) comes from (A5) and (A6) directly, which we denote as \( \tilde{G} \),
\[
\tilde{G} = \frac{1}{2\pi k^2} \left[ M_1 K_0 \left( \sigma \sqrt{\frac{k^2}{a} + \xi^2} \right) - M_2 K_0 \left( \sigma \sqrt{\frac{k^2}{b} + \xi^2} \right) \right].
\] (A7)

Transforming back to the physical space, we obtain
\[
G = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{1}{2\pi k^2} \left[ M_1 K_0 \left( \sigma \sqrt{\frac{k^2}{a} + \xi^2} \right) - M_2 K_0 \left( \sigma \sqrt{\frac{k^2}{b} + \xi^2} \right) \right] d\xi,
\] (A8)

with \( a, b, k \) and \( M_1, M_2 \) being given by (A3a) and (A3b), respectively.

**Appendix B. The calculations of circulation and inflow**

Now, the circulation due to the longitudinal velocity and the inflow due to the transversal velocity are given by (3.30) and (3.31), respectively,
\[
\Gamma_\phi = -\frac{1}{4\pi \rho_0 U} \int_S (n \times \nabla)(F \cdot \nabla \Psi) dS, \quad (B1a)
\]
\[
Q_\psi = \frac{1}{4\pi \rho_0 U} \int_S (n \times \nabla) \cdot (F \times \nabla \Psi) dS, \quad (B1b)
\]
with
\[
\nabla \Psi = e^{-k(r-x)} \left( 1, -\frac{y}{r-x}, -\frac{z}{r-x} \right). \quad (B2)
\]

In fact, due to the exponential factor \( e^{-k(r-x)} \) in (B2), (B1b) can be reduced to a wake-plane integral with \( n = e_x \),
\[
Q_\psi \approx \frac{1}{4\pi \rho_0 U} \int_W \left( -\partial_x e_y + \partial_y e_z \right) \cdot (F \times \nabla \Psi) dS, \quad (B3)
\]
which can more or less simplify our analysis.

Since (B1) are linearly dependent on \( F \), we can estimate their results by assigning \( F \) with a specific value. Suppose \( F = De_z \), then \( \Gamma_\phi \equiv 0 \) since \( \partial \Psi / \partial x = \chi \) is regular. However, (B3) reduces to
\[
Q_\psi = -\frac{D}{4\pi \rho_0 U} \int_S \left[ \frac{\partial}{\partial y} \frac{ye^{-k(r-x)}}{r} + \frac{\partial}{\partial z} \frac{ze^{-k(r-x)}}{r} \right] dS
\]
\[
\approx \frac{D}{4\pi \rho_0 U} \int_W \frac{kr^2 + kr x + x}{r^3} e^{-k(r-x)} dS = \frac{D}{\rho_0 U}. \quad (B4)
\]

Due to the symmetry of \( y \) and \( z \) in (B2), for the lift or side force case we only need to consider \( F = Le_z \) in (B1a) or (B3). In this case, (B3) reduces to
\[
Q_\psi \approx \frac{L}{4\pi \rho_0 U} \int_W \frac{\partial}{\partial z} \frac{e^{-k(r-x)}}{r} dy dz = 0. \quad (B5)
\]

However, (B1a) needs more algebraic, which can be simplified by letting \( S \) be a sphere surface with \( n = e_r \). Thus, in this situation (B1a) reduces to
\[
\Gamma_\phi = (0, \Gamma_{\phi y}, \Gamma_{\phi z}), \quad (B6)
\]
where

\[ \Gamma_{\phi y} = \frac{L}{4\pi \rho_0 U} \int_S (e_y \times e_x) \cdot \nabla \left[ \frac{ze^{-k(r-x)}}{r(r - x)} \right] dS \]

\[ = \frac{L}{4\pi \rho_0 U} \int_S \left( \frac{z}{r} \frac{\partial}{\partial x} - \frac{x}{r} \frac{\partial}{\partial z} \right) \frac{ze^{-k(r-x)}}{r(r - x)} dS \]

\[ = \frac{L}{4\pi \rho_0 U} \int_S \left( \frac{z^2 + z^2 - rx + k^2(r-x)}{r^2(r - x)^2} \right) e^{-k(r-x)} dS \]

\[ = \frac{L}{4\pi \rho_0 U} \int_0^\pi \int_0^{2\pi} \frac{\cos^2 \theta + \sin^2 \theta \sin^2 \varphi - \cos \theta + kr \sin^2 \theta \sin^2 \varphi(1 - \cos \theta)}{(1 - \cos \theta)^2 e^{-kr(1-\cos \theta)}} \sin \theta d\theta d\varphi \]

\[ = \frac{L}{4\pi \rho_0 U} \int_{-1}^1 \left[ 1 + kr(1 + t) \right] e^{-kr(1-t)} dt = \frac{L}{2\rho_0 U}, \quad (B7) \]

and

\[ \Gamma_{\phi z} = \frac{L}{4\pi \rho_0 U} \int_S (e_z \times e_x) \cdot \nabla \left[ \frac{ze^{-k(r-x)}}{r(r - x)} \right] dS \]

\[ = \frac{L}{4\pi \rho_0 U} \int_S \left( \frac{y}{r} \frac{\partial}{\partial x} + \frac{x}{r} \frac{\partial}{\partial y} \right) \frac{ze^{-k(r-x)}}{r(r - x)} dS \]

\[ = -\frac{L}{4\pi \rho_0 U} \int_S \frac{1 + k(r-x)}{r^2(r - x)^2} yze^{-k(r-x)} dS = 0. \quad (B8) \]

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