COMPOSITION FORMULAS OF BESSEL-STRUVE KERNEL FUNCTION

K.S NISAR, S.R MONDAL, AND P.AGARWAL

Abstract. The generalized operators of fractional integration involving Appell’s function $F_3(\cdot)$ due to Marichev-Saigo-Maeda, is applied to the Bessel Struve kernel function $S_\alpha (\lambda z)$, $\lambda, z \in \mathbb{C}$ to obtain the results in terms of generalized Wright functions. The pathway integral representations Bessel Struve kernel function and its relation between many other functions also derived in this study.

1. Introduction

The fractional calculus is one of the most rapidly growing subject of applied mathematics that deals with derivatives and integrals of arbitrary orders. The applications of fractional integral operator involving various special functions has found in various sub fields such as statistical distribution theory, control theory, fluid dynamics, stochastic dynamical system, plasma, image processing, nonlinear biological systems, astrophysics, and in quantum mechanics (see [1–4]).

The influence of fractional integral operators involving various special functions in fractional calculus is very important as its significance and applications in various sub-fields of applied mathematical analysis. Many studies related to the fractional calculus found in the papers of Love [5], McBride [6], Kalla [7, 8], Kalla and Saxena [9, 10], Saigo [11–13], Saigo and Maeda [14], Kiryakova [21] gave various applications and extensions of hypergeometric operators of fractional integration. A comprehensive explanation of such operators can be found in the research monographs by Miller and Ross [20] and Kiryakova [21].

A useful generalization of the hypergeometric fractional integrals, including the Saigo operators ([11–13]), has been introduced by Marichev [22] (see details in Samko et al. [23], p. 194) and later extended and studied by Saigo and Maeda ([14], p.393) in term of any complex order with Appell function $F_3(\cdot)$ in the kernel, as follows:

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$, then the generalized fractional calculus operator involving the Appell functions, or Horn’s function are defined as follows:

\begin{align}
(1.1) & \quad \left( I^{\alpha,\alpha',\beta,\beta',\gamma}_{0,+} f \right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3 \left( \alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) dt, \\
(1.2) & \quad \left( I^{\alpha,\alpha',\beta,\beta',\gamma}_{0,-} f \right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3 \left( \alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) dt,
\end{align}

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Lemma 1. Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ be such that $\Re(\gamma) > 0$ and 

\[
\Re(\rho) > \max\{0, \Re(\alpha - \alpha' - \beta - \gamma), \Re(\alpha' - \beta')\}.
\]

Then there exists the relation

\[
\left( I_{\rho, +}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho - 1} \right)(x) = \Gamma \left[ \begin{array}{c} \rho, \\
\rho + \beta', \\
\rho + \gamma - \alpha - \alpha', \\
\rho + \beta' - \alpha'
\end{array} \right] x^{\rho - a - a' + \gamma - 1},
\]

where

\[
\Gamma \left[ \begin{array}{ccc} a, & b, & c \\
d, & e, & f \end{array} \right] = \frac{\Gamma(a) \Gamma(b) \Gamma(c) \Gamma(d) \Gamma(e) \Gamma(f)}{\Gamma(f)}.
\]

Lemma 2. Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ be such that $\Re(\gamma) > 0$ and 

\[
\Re(\rho) < 1 + \min\{\Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma)\}.
\]

Then there exists the relation

\[
\left( I_{\rho, -}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho - 1} \right)(x) = \Gamma \left[ \begin{array}{c} 1 - \rho - \gamma + \alpha + \alpha', \\
1 - \rho + \alpha + \beta', \\
1 - \rho - \alpha + \beta + \beta' - \gamma, \\
1 - \rho + \alpha - \beta
\end{array} \right] x^{\rho - a - a' + \gamma - 1}.
\]

2. Representations in term of generalized Wright functions

The generalized Wright hypergeometric function $p\psi_q(z)$ which is defined by the series

\[
p\psi_q(z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}.
\]

Here $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}$ ($i = 1, 2, \ldots, p; j = 1, 2, \ldots, q$). Asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ were studied in [26] and under the condition

\[
\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1
\]

in [27, 28]. Properties of this generalized Wright function were investigated in [29, 31]. In particular, it was proved [29] that $p\psi_q(z), z \in \mathbb{C}$ is an entire function under the condition (2.4). Interesting results related to generalized Wright functions are also given in [32]. For proceeding the coming subsections we need to recall the following definitions. The Bessel
and modified Bessel function of first kind, the Struve function $H_v(z)$ and modified Struve function $L_v(z)$ possess power series representation of the form \[33\]

\begin{align}
J_v(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+v}}{\Gamma(k+v+1) k!}, \\
I_v(z) &= \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+v}}{\Gamma(k+v+1) k!} \\
H_v(z) &= \left(\frac{z}{2}\right)^{v+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\frac{3}{2}) \Gamma(k+v+\frac{1}{2})} \left(\frac{z}{2}\right)^{2k} \\
L_v(z) &= \left(\frac{z}{2}\right)^{v+1} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\frac{3}{2}) \Gamma(k+v+\frac{1}{2})} \left(\frac{z}{2}\right)^{2k}
\end{align}

The purpose of this work is to investigate compositions of integral transforms \[1.1\] and \[1.2\] with the Bessel Struve kernel function $S_\alpha(\lambda z)$ defined for $z, \lambda \in \mathbb{C}$. The Bessel-Struve kernel $S_\alpha(\lambda z)$, $\lambda \in \mathbb{C}$, \[34\] which is unique solution of the initial value problem $l_\alpha u(z) = \lambda^2 u(z)$ with the initial conditions $u(0) = 1$ and $u'(0) = \lambda \Gamma(\alpha + 1)/(\sqrt{\pi} \Gamma(\alpha + 3/2))$ is given by $S_\alpha(\lambda z) = j_\alpha(i\lambda z) - ih_\alpha(i\lambda z)$, $\forall z \in \mathbb{C}$ where $j_\alpha$ and $h_\alpha$ are the normalized Bessel and Struve functions.

Moreover, the Bessel-Struve kernel is a holomorphic function on $\mathbb{C} \times \mathbb{C}$ and it can be expanded in a power series in the form

\begin{equation}
S_\alpha(\lambda z) = \sum_{n=0}^{\infty} \frac{(\lambda z)^n \Gamma(\alpha + 1) \Gamma(\frac{n+1}{2})}{\sqrt{\pi} n! \Gamma(\frac{z}{2} + \alpha + 1)}.
\end{equation}

Now, we derive the following theorems,

**Theorem 1.** Let $\alpha, \alpha', \beta, \beta', \gamma, \rho, p, b, c, \lambda, \nu \in \mathbb{C}$. Suppose that $\text{Re}(\gamma) > 0$ and $\text{Re}(\rho + n) > \max\{0, \text{Re}(\alpha + \alpha' + \beta - \gamma), \text{Re}(\alpha' - \beta')\}$. Then

\[
\left( I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S_\nu(\lambda t) \right) (x) = x^{\rho+\gamma-\alpha-\alpha'-1} \frac{\Gamma(\nu+1)}{\sqrt{\pi}}
\]

\[
\times \Psi_4 \left[ \frac{(1, \frac{1}{2}), (\rho, 1), (\rho + \gamma - \alpha - \alpha' - \beta, 1), (\rho + \beta' - \alpha', 1)}{\left(\nu + 1, \frac{1}{2}\right), (\rho + \beta', 1), (\rho + \gamma - \alpha - \alpha', 1), (\rho + \gamma - \alpha' - \beta, n)} \right] x_\lambda
\]

**Proof.** Using \[1.1\] and \[2.9\],

\[
\left( I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S_\nu(\lambda t) \right) (x) = \left( I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} \sum_{n=0}^{\infty} \frac{(\lambda)^n \Gamma(\nu+1) \Gamma((n+1)/2)\rho^{n+1}}{\sqrt{\pi} n! \Gamma(n/2 + \nu + 1)} \right) (x)
\]
By changing the order of integration and summation,
\[
\left( I_{0+}^{\alpha,\beta',\gamma} \lambda \rho^{-1} S_{\nu} (\lambda t) \right) (x)
= \sum_{n=0}^{\infty} \frac{\lambda^n \Gamma (\nu + 1) \Gamma ((n + 1)/2)}{\sqrt{\pi n! \Gamma (n/2 + \nu + 1)}} (I_{0+}^{\alpha,\beta',\gamma} \lambda \rho^{n-1})(x).
\]
Hence replacing \( \rho \) by \( \rho + n \) in Lemma [\ref{lem1}], we obtain
\[
\left( I_{0+}^{\alpha,\beta',\gamma} \lambda \rho^{-1} S_{\nu} (\lambda t) \right) (x)
= \sum_{n=0}^{\infty} \frac{\Gamma (\nu + 1) \Gamma (n + 1/2) \lambda^n}{\sqrt{\pi n! \Gamma (n/2 + \nu + 1)}}
\times \frac{\Gamma (\rho + n) \Gamma (\rho + n + \gamma - \alpha - \alpha' - \beta) \Gamma (\rho + n + \beta' - \alpha')}{\Gamma (\rho + n + \beta') \Gamma (\rho + n + \gamma - \alpha - \alpha') \Gamma (\rho + n + \gamma - \alpha' - \beta)} x^{\rho + n + \gamma - \alpha - \alpha' - 1}
\]
\[
= \sum_{n=0}^{\infty} \frac{\Gamma (\nu + 1) \Gamma (n + 1/2) \lambda^n}{\sqrt{\pi n! \Gamma (n/2 + \nu + 1)}}
\times \frac{\Gamma (\rho + n) \Gamma (\rho + n + \gamma - \alpha - \alpha' - \beta) \Gamma (\rho + n + \beta' - \alpha')}{\Gamma (\rho + n + \beta') \Gamma (\rho + n + \gamma - \alpha - \alpha') \Gamma (\rho + n + \gamma - \alpha' - \beta)} x^{\rho + n + \gamma - \alpha - \alpha' - 1}
\]
\[
= \frac{x^{\rho + \gamma - \alpha - \alpha' - 1}}{\sqrt{\pi}} \Gamma (\nu + 1) \sum_{n=0}^{\infty} \frac{\Gamma (n + 1/2)}{\Gamma (n/2 + \nu + 1)}
\times \frac{\Gamma (\rho + n) \Gamma (\rho + n + \gamma - \alpha - \alpha' - \beta) \Gamma (\rho + n + \beta' - \alpha')}{\Gamma (\rho + n + \beta') \Gamma (\rho + n + \gamma - \alpha - \alpha') \Gamma (\rho + n + \gamma - \alpha' - \beta)} \frac{(x\lambda)^n}{n!}
\]
\[
= \frac{x^{\rho + \gamma - \alpha - \alpha' - 1}}{\sqrt{\pi}} \Gamma (\nu + 1)
\times 4 \Psi_4 \left[ \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), (\rho, 1), (\rho + \gamma - \alpha - \alpha' - \beta, 1), (\rho + \beta' - \alpha', 1), (\nu + 1, \frac{1}{2}), (\rho + \beta', 1), (\rho + \gamma - \alpha - \alpha', 1), (\rho + \gamma - \alpha' - \beta, n) \right] \lambda x
\]
\[
Theorem 2. Let \( \alpha, \alpha', \beta, \beta', \gamma, \rho, p, b, c, \lambda, \nu \in \mathbb{C} \). Suppose that \( \operatorname{Re} (\gamma) > 0 \) and \( \operatorname{Re} (\rho - n) < 1 + \min \{ \operatorname{Re} (-\beta), \operatorname{Re} (\alpha + \alpha' - \gamma), \operatorname{Re} (\alpha + \beta' - \gamma) \} \). Then
\[
\left( I_{0-}^{\alpha,\beta',\gamma} \lambda \rho^{-1} S_{\nu} \left( \frac{\lambda}{t} \right) \right) (x)
= \frac{x^{\rho - \alpha - \alpha' + \gamma - 1}}{\sqrt{\pi}} \Gamma (\nu + 1)
\times 4 \Psi_4 \left[ \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right), (1 - \rho - \gamma + \alpha + \alpha', 1), (1 - \rho + \alpha + \beta', 1), (1 - \rho - \beta', 1), (\nu + 1, \frac{1}{2}), (1 - \rho, 1), (1 - \rho + \alpha + \alpha' + \beta' - \gamma, 1), (1 - \rho + n + \alpha + \beta, 1) \right] \lambda x
\]
\[
\text{Proof.} \ \text{Using \ref{lem2} and \ref{lem3}, and then changing the order of integration and summation,}
\]
\[
\left( I_{0-}^{\alpha,\beta',\gamma} \lambda \rho^{-1} S_{\nu} \left( \frac{\lambda}{t} \right) \right) (x) = \sum_{n=0}^{\infty} \frac{(\lambda)^n \Gamma (\nu + 1) \Gamma ((n + 1)/2)}{\sqrt{\pi n! \Gamma (n/2 + \nu + 1)}} (I_{0-}^{\alpha,\beta',\gamma} \lambda \rho^{n-1})(x).
\]
using Lemma 2 we obtain

\[
= \sum_{n=0}^{\infty} \frac{(\lambda)^n}{\sqrt{\pi n!}} \frac{\Gamma((n+1)/2)}{\Gamma(n/2 + \nu + 1)} \times \frac{\Gamma(1 - \rho - n - \gamma + \alpha + \alpha') \Gamma(1 - \rho - n + \alpha + \beta') \Gamma(1 - \rho - n - \beta) \Gamma(1 - \rho - n + \alpha + \beta)}{\Gamma(1 - \rho - n)}
\]

\[
x^{\rho - \alpha - \alpha' + \gamma - 1} \Gamma(\nu + 1) \Gamma((n + 1)/2)
\]

\[
= \frac{x^{\rho - \alpha - \alpha' + \gamma - 1} \Gamma(\nu + 1) \Gamma((n + 1)/2)}{\sqrt{\pi}} \frac{\Gamma(1 - \rho + n - \gamma + \alpha + \alpha') \Gamma(1 - \rho + n + \alpha + \beta') \Gamma(1 - \rho + n - \beta)}{\Gamma(1 - \rho + n + \alpha + \beta)} \frac{(x\lambda)^n}{\Gamma(n/2 + \nu + 1)}
\]

\[
= \frac{x^{\rho - \alpha - \alpha' + \gamma - 1}}{\sqrt{\pi}} \Gamma(\nu + 1) 
\times 4 \Psi_4 \left[ \left( \frac{\gamma}{2} - \frac{1}{2}, 1; (1 - \rho - \gamma + \alpha + \alpha', 1), (1 - \rho + \alpha + \beta', 1), (1 - \rho - \beta', 1) \right) \left( \nu + 1, \frac{1}{2} \right), (1 - \rho, 1), (1 - \rho + \alpha + \alpha' + \beta + \beta' - \gamma, 1), (1 - \rho + n + \alpha + \beta, 1) \right] |x\lambda|
\]

2.1. Representation of Bessel Struve kernel function in terms of exponential function. In this subsection we represent the Bessel Struve function in terms of exponential function. Also, we derive the Marichev Saigo Maeda operator representation of special cases. The representation Bessel Struve Kernel function in terms of exponential function as:

\[
(2.10) \quad S_{\frac{1}{2}}(x) = e^x,
\]

\[
(2.11) \quad S_{\frac{1}{2}}(x) = \frac{-1 + e^x}{x}.
\]

Now, we give the the following theorems

**Theorem 3**. Let \( \alpha, \alpha', \beta, \beta', \gamma, \rho, p, b, c, \lambda \in \mathbb{C} \). Suppose that \( \text{Re}(\gamma) > 0 \) and \( \text{Re}(\rho + n) > \max\{0, \text{Re}(\alpha + \alpha' + \beta - \gamma), \text{Re}(\alpha' - \beta')\} \). Then

\[
\left( I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} e^{\rho - 1} \right)(x) = x^{\rho - \alpha - \alpha' + \gamma - 1}
\]

\[
\times 3 \Psi_3 \left[ \left( \rho, 1 \right), \left( \rho + \gamma - \alpha - \alpha' - \beta, 1 \right), \left( \rho + \beta' - \alpha', 1 \right), \left( 1 - \rho - \beta', 1 \right) \right] \left( \nu + 1, \frac{1}{2} \right), (1 - \rho, 1), (1 - \rho + \alpha + \alpha' + \beta + \beta' - \gamma, 1), (1 - \rho + n + \alpha + \beta, 1) \right] |x|\]

**Proof.** From equation 1 and the definition of Bessel Struve function, we have

\[
\left( I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} e^{\rho - 1} \right)(x) = \left( I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} e^{\rho - 1} \sum_{n=0}^{\infty} \frac{\Gamma(-1/2 + 1) \Gamma(n + 1/2)}{\sqrt{\pi n!} \Gamma(n/2 - 1/2 + 1)} \rho^n \right)(x)
\]

\[
= \sum_{n=0}^{\infty} \frac{\Gamma(1/2) \Gamma(n + 1/2)}{\sqrt{\pi n!} \Gamma(n + 1/2)} \left( I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} e^{\rho + n - 1} \right)(x)
\]

This together with Lemma 3 yields

\[
\left( I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} e^{\rho - 1} \right)(x) = x^{\rho - \alpha - \alpha' + \gamma - 1} \sum_{n=0}^{\infty} \frac{x^n}{n!} \left[ \rho + n, \rho + n + \gamma - \alpha - \alpha' - \beta, \rho + n + \beta' - \alpha' \right] \left[ \rho + n + \beta', \rho + n + \gamma - \alpha - \alpha' - \beta \right]
\]
which is the desired result. □

**Theorem 4.** Let \( \alpha, \alpha', \beta, \beta', \gamma, \rho, \rho + \beta, \rho + \beta' \in \mathbb{C} \). Suppose that \( \text{Re} (\gamma) > 0 \) and \( \text{Re} (\rho + n) > \max \{0, \text{Re} (\alpha + \alpha' + \beta - \gamma), \text{Re} (\alpha' - \beta')\} \). Then

\[
\left( I_{0+, \alpha', \beta', \gamma, \rho} (0) \right) (x) = x^{\rho - \alpha - \alpha' + \gamma - 1} \frac{\pi}{\sqrt{\rho}} \times 4 \Psi_4 \left[ \begin{array}{c} (1, \frac{1}{2}) ; (\rho + \beta, \rho + \beta') ; (\rho + \gamma - \alpha - \alpha' - \beta, \rho + \beta - \alpha, \rho + \beta' - \alpha') \end{array} \right] \quad |x|
\]

\[
\times 4 \Psi_4 \left[ \begin{array}{c} (\frac{1}{2}, 1) ; (\rho + \gamma - \alpha - \alpha' - \beta, 1), (\rho + \beta - \alpha', 1) \end{array} \right] \quad |x|
\]

\[
\cdot \frac{\Gamma (3/2) \Gamma (n + 1)}{\sqrt{\pi n}! \Gamma (n + 3/2)} \left( I_{0+, \alpha', \beta', \gamma, \rho + n - 1} (x) \right)
\]

Proof. From equation 1 and the definition of Bessel Struve kernel function, we have

\[
\left( I_{0+, \alpha', \beta', \gamma, \rho} (0) \right) (x) = \left( I_{0+, \alpha', \beta', \gamma, \rho} (0) \right) (x) = \left( I_{0+, \alpha', \beta', \gamma, \rho} (0) \right) (x)
\]

Again using Lemma II, we can conclude that

\[
\left( I_{0+, \alpha', \beta', \gamma, \rho} (0) \right) (x) = \sum_{n=0}^{\infty} \frac{\Gamma (n + 1)}{\Gamma (n + 3/2)} \frac{\Gamma (n + 1/2)}{\sqrt{\pi n}! \Gamma (n + 3/2)} \Gamma (1/2 + 1) \Gamma (n + 1/2 + 1) t^n
\]

\[
= x^{\rho - \alpha - \alpha' + \gamma - 1} \frac{\pi}{\sqrt{\rho}} \times 4 \Psi_4 \left[ \begin{array}{c} (1, \frac{1}{2}) ; (\rho + \beta, \rho + \beta') ; (\rho + \gamma - \alpha - \alpha' - \beta, \rho + \beta - \alpha, \rho + \beta' - \alpha') \end{array} \right] \quad |x|
\]

\[
\times 4 \Psi_4 \left[ \begin{array}{c} (\frac{1}{2}, 1) ; (\rho + \gamma - \alpha - \alpha' - \beta, 1), (\rho + \beta - \alpha', 1) \end{array} \right] \quad |x|
\]

hence the conclusion. □

2.2. Relation between Bessel Struve kernel function and Bessel and Struve function of first kind. In this subsection we show the relation between \( S_n (x) \), and Bessel function \( I_0 (x) \) and Struve function \( L_0 (x) \) by choosing particular values of \( \alpha \)

\[
(2.12) \quad S_0 (x) = I_0 (x) + L_0 (x),
\]

\[
(2.13) \quad S_1 (x) = \frac{2I_1 (x) + L_1 (x)}{x},
\]

then we derive the Marichev Saigo Maeda operator representation of special cases

**Theorem 5.** Let \( \alpha, \alpha', \beta, \beta', \gamma, \rho, \rho + \beta, \rho + \beta' \in \mathbb{C} \). Suppose that \( \text{Re} (\gamma) > 0 \) and \( \text{Re} (\rho + n) > \max \{0, \text{Re} (\alpha + \alpha' + \beta - \gamma), \text{Re} (\alpha' - \beta')\} \). Then

\[
\left( I_{0+, \alpha', \beta', \gamma, \rho} (0) \right) (t) = x^{\rho - \alpha - \alpha' + \gamma - 1} \frac{\pi}{\sqrt{\rho}} \times 4 \Psi_4 \left[ \begin{array}{c} (1, \frac{1}{2}) ; (\rho + \beta, \rho + \beta') ; (\rho + \gamma - \alpha - \alpha' - \beta, \rho + \beta - \alpha, \rho + \beta' - \alpha') \end{array} \right] \quad |x|
\]

\[
\times 4 \Psi_4 \left[ \begin{array}{c} (\frac{1}{2}, 1) ; (\rho + \gamma - \alpha - \alpha' - \beta, 1), (\rho + \beta - \alpha', 1) \end{array} \right] \quad |x|
\]

\[
\cdot \frac{\Gamma (3/2) \Gamma (n + 1)}{\sqrt{\pi n}! \Gamma (n + 3/2)} \left( I_{0+, \alpha', \beta', \gamma, \rho} (x) \right)
\]
Proof. From equation (1.1) and the definition of Bessel Struve kernel function, we have

\[
\left( I_{0,+}^{\alpha', \beta', \gamma', \eta} \right) (x) = \left( \begin{array}{c}
I_{0,+}^{\alpha', \beta', \gamma', \eta} (I_0 (t) + L_0 (t)) \\
\end{array} \right) (x)
\]

Using Lemma 1, we obtain

\[
\sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{n+1}{2} \right)}{n! \Gamma \left( \frac{n}{2} + 1 \right)} \left( I_{0,+}^{\alpha', \beta', \gamma', \eta} \right) (x)
\]

which is the desired result.

Theorem 6. Let \( \alpha, \alpha', \beta, \beta', \gamma, \rho, p, b, c, \lambda \in \mathbb{C} \). Suppose that \( \Re (\gamma) > 0 \) and \( \Re (\rho + n) > \max \{0, \Re (\alpha + \alpha' + \beta + \gamma)\} \). Then

\[
\left( I_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma} \right) (x) \left( \begin{array}{c}
2I_1 (t) + L_1 (t) \\
t
\end{array} \right) = \frac{x^{\rho - \alpha - \alpha' + \gamma - 1}}{\sqrt{\pi}} \times \Psi_4 \left( \begin{array}{c}
(\frac{1}{2}, \frac{1}{2}), (\rho, 1), (\rho + \gamma - \alpha - \alpha' - \beta, 1), (\rho + \beta' - \alpha', 1); \\
(\frac{1}{2}, 1), (\rho + \beta', 1), (\rho + \gamma - \alpha - \alpha', 1), (\rho - \alpha - \alpha' - \beta, 1)
\end{array} \right) \]

Proof. From (1.1) and the definition of Bessel Struve kernel function, we have

\[
\left( I_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma} \right) (x) \left( \begin{array}{c}
2I_1 (t) + L_1 (t) \\
t
\end{array} \right) = \left( \begin{array}{c}
I_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma} \right) (x)
\]

Using Lemma 1, we obtain

\[
\frac{\Gamma (2)}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{n+1}{2} \right)}{n! \Gamma \left( \frac{n}{2} + 2 \right)} \left( I_{0,+}^{\alpha, \alpha', \beta, \beta', \gamma} \right) (x)
\]

which is the desired result.
3. Pathway Fractional Integration of Bessel Struve Kernel Function

By considering the idea of Mathai [36], Nair [35], introduced a pathway fractional integral operator and developed further by Mathai and Haubold [37], [38] is defined as follows:

Let \( f(x) \in L(a, b), \eta \in C, R(\eta) > 0, a > 0 \) and the pathway parameter \( \alpha < 1 \) as [39], then

\[
(3.14) \quad \left( F^{(\eta, \alpha)}_{0+} \right)(x) = x^\eta \int_0^1 \left[ 1 - \left( \frac{a(1-\alpha)}{x} \right)^{1-\alpha} \right] f(t) \, dt
\]

For a real scalar \( \alpha \), the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

\[
(3.15) \quad f(x) = c |x|^\gamma \left[ 1 - a (1-\alpha) |x|^\delta \right]^{\frac{\alpha}{1-\alpha}}
\]

provided that \(-\infty < x < \infty, \delta > 0, \beta \geq 0, \left[ 1 - a (1-\alpha) |x|^\delta \right] > 0, \) and \( \gamma > 0 \) where \( c \) is the normalizing constant and \( \alpha \) is called the pathway parameter. For real \( \alpha \) normalizing constant as follows:

\[
c = \begin{cases} \frac{\delta_{\alpha (1-\alpha)}}{2} \frac{\Gamma(\frac{\gamma}{\delta}) \Gamma(\frac{\delta}{\gamma}+1)}{\Gamma(\frac{\delta}{\gamma}) \Gamma(\frac{\gamma}{\delta}+1)}, & \text{for } \alpha < 1 \\ \frac{\delta_{\alpha (1-\alpha)}}{2} \frac{\Gamma(\frac{\gamma}{\delta}) \Gamma(\frac{\delta}{\gamma})}{\Gamma(\frac{\gamma}{\delta}) \Gamma(\frac{\delta}{\gamma})}, & \text{for } \frac{1}{1-\alpha} - \frac{\gamma}{\delta} > 0, \alpha > 1 \\ \frac{\delta_{1 (1-\alpha)}}{2} \frac{\Gamma(\frac{\gamma}{\delta})}{\Gamma(\frac{\gamma}{\delta})}, & \alpha \to 1 \end{cases}
\]

Note that for \( \alpha < 1 \) it is a finite range density with \( \left[ 1 - a (1-\alpha) |x|^\delta \right] > 0 \) and (3.15) remains in the extended generalized type-1 beta family. The pathway density in (3.15), for \( \alpha < 1 \), includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.'s. [39]. For instance, \( \alpha > 1 \) gives

\[
(3.16) \quad f(x) = c |x|^\gamma \left[ 1 + a (1-\alpha) |x|^\delta \right]^{\frac{\alpha}{1-\alpha}}
\]

provided that \(-\infty < x < \infty, \delta > 0, \beta \geq 0, \) and \( \alpha > 0 \) which is the extended generalized type-2 beta model for real \( x \). It includes the type-2 beta density, the F density, the Student-t density, the Cauchy density and many more. The composition of the integral transform operator (3.14) with the product of generalized Bessel function of the first kind is given in [39].

The purpose of this work is to investigate the composition formula of integral transform operator due to Nair, which is expressed in terms of the generalized Wright hypergeometric function, by inserting the generalized Struve function of the first kind \( S_\alpha (\lambda z) \) which is defined in the equation (2.9). The results given in this section are based on the preliminary assertions giving by composition formula of pathway fractional integral (3.14) with a power function.

Lemma 3. [35] Let \( \eta \in C, \text{Re} (\eta) > 0, \beta \in C \) and \( \alpha < 1. \) If \( \text{Re} (\beta) > 0, \) and \( \text{Re} \left( \frac{\eta}{1-\alpha} \right) > -1, \) then
The pathway fractional integration of the Bessel Struve Kernel function of the first kind is given by the following result.

**Theorem 7.** Let \( \eta, \sigma, p, b, c, \lambda \in \mathbb{C} \) and \( \alpha < 1 \) be such that \( \text{Re} (\eta) > 0, \text{Re} (\sigma) > 0 \), \( \text{Re} (\sigma + n) > 0 \) and \( \text{Re} \left( \frac{\pi}{2 - \alpha} \right) > -1 \) then the following formula hold

\[
(3.17) \quad \left\{ P_{0+}^{(\eta, \alpha)} \left[ t^{\beta - 1} \right] \right\} (x) = \frac{x^{\eta + \beta}}{[a (1 - \alpha)]^{\beta}} \left( \frac{\Gamma (\beta) \Gamma \left( 1 + \frac{n}{1 - \alpha} \right)}{\Gamma \left( 1 + \frac{n}{1 - \alpha} + \beta \right)} \right).
\]

**Proof.** Applying (2.9), (3.18) and changing the order of integration and summation, we get

\[
\left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma - 1} S_{\nu} (\lambda t) \right] \right) (x) = p^{(\eta, \alpha)} \left( t^{\sigma - 1} \sum_{n=0}^{\infty} \frac{\lambda^n \Gamma (\nu + 1) \Gamma ((n + 1) / 2) \Gamma ((n + 1) / 2)}{\sqrt{\pi n!} \Gamma (n/2 + \alpha + 1)} t^n \right) (x)
\]

\[
= \sum_{k=0}^{\infty} \frac{\lambda^n \Gamma (\nu + 1) \Gamma ((n + 1) / 2) \Gamma ((n + 1) / 2)}{\sqrt{\pi n!} \Gamma (n/2 + \nu + 1)} \left( P_{0+}^{(\eta, \alpha)} \left( t^{(n+\sigma)-1} \right) \right) (x)
\]

Using the conditions mentioned in the statement of the theorem and \( k \in K_0, R (p + n) > 0, \text{Re} \left( \frac{\pi}{2 - \alpha} \right) > -1 \). Applying Lemma 3 and using (3.17) with \( \beta \) replaced by \( \sigma + n \), we get

\[
\left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma - 1} S_{\nu} (\lambda t) \right] \right) (x) = \sum_{k=0}^{\infty} \frac{\lambda^n \Gamma (\nu + 1) \Gamma ((n + 1) / 2) \Gamma ((n + 1) / 2)}{\sqrt{\pi n!} \Gamma (n/2 + \nu + 1)} \frac{x^{\eta + \alpha}}{[a (1 - \alpha)]^{\sigma + n}}
\]

\[
\times \frac{\Gamma (\sigma + n) \Gamma \left( 1 + \frac{n}{1 - \alpha} \right)}{\Gamma \left( 1 + \frac{n}{1 - \alpha} + \sigma + n \right)}
\]

\[
= \frac{x^{\eta + \alpha} \Gamma (\nu + 1) \Gamma \left( 1 + \frac{n}{1 - \alpha} \right)}{\sqrt{\pi} [a (1 - \alpha)]^{\sigma}}
\]

\[
\times \sum_{k=0}^{\infty} \frac{\Gamma \left( \frac{n}{2} + \frac{1}{2} \right) \Gamma (\sigma + n)}{\Gamma \left( 1 + \frac{n}{2 - \alpha} + \sigma + n \right) n!} \frac{\lambda^k \Gamma (\sigma + n)}{[a (1 - \alpha)]^{\sigma + n}}
\]

which gives the desired result \( \square \)

By considering the relations given in (2.10) and (2.11), we obtain various new integral formulas for Bessel Struve functions involving in the Pathway fractional integration Operators:

**Theorem 8.** Let \( \eta, \sigma, p, b, c \in \mathbb{C} \) and \( \alpha < 1 \) such that \( \text{Re} (\eta) > 0, \text{Re} (\sigma + n) > 0 \) and \( \text{Re} \left( \frac{\pi}{2 - \alpha} \right) > -1 \) then the following formula hold

\[
(3.18) \quad \left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma - 1} \right] \right) (x) = x^{\eta + \sigma} \left( \frac{\Gamma (\sigma + 1)}{a (1 - \alpha)} \right) \times_1 \Psi_1 \left( \frac{(\sigma, 1)}{a (1 - \alpha)} \right) \frac{x}{[a (1 - \alpha)]^{\sigma}}
\]
Proof. Applying \((2.10)\) using \((3.14)\) with the help of Lemma 3 and changing the order of integration and summation, we get
\[
\left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma-1} \left( \frac{1}{e^t} \right) \right] \right) (x) = \left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma-1} S_{\frac{1}{2}}^{\pm} (\lambda t) \right] \right) (x)
\]
\[
= \left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma-1} \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi} n! \Gamma \left( \frac{n+3}{2} \right)} \right] \right) (x)
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left( P_{0+}^{(\eta, \alpha)} \left( t^{\sigma+n-1} \right) \right) (x)
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^{\eta+\sigma+n} \Gamma (\sigma + n) \Gamma \left( 1 + \frac{\eta}{1-\alpha} \right)}{\left( a (1-\alpha) \right)^{\sigma+n} \Gamma \left( 1 + \frac{\eta}{1-\alpha} + \sigma + n \right)}
\]
which completes the proof of the theorem. \[\square\]

**Theorem 9.** Let \(\eta, \sigma, p, b, c \in \mathbb{C}\) and \(\alpha < 1\) such that \(\text{Re} (\sigma + n) > 0\) and \(\text{Re} \left( \frac{n}{1-\alpha} \right) > -1\) then the following formula hold
\[
\left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma-1} \left( -1 + e^t \right) \right] \right) (x) = x^{\eta+\sigma} \Gamma \left( 1 + \frac{\eta}{1-\alpha} \right) \times_2 \Psi_2 \left[ \left( \frac{1}{2}, \frac{1}{2} \right); \left( 1 + \sigma + \frac{\eta}{1-\alpha}, 1 \right); \frac{x\lambda}{\left( a (1-\alpha) \right)} \right]
\]

Proof. Applying \((2.11)\) using \((2.9)\), \((3.14)\) and Lemma 1 and changing the order of integration and summation, we get
\[
\left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma-1} \left( \frac{1}{t} + e^t \right) \right] \right) (x) = \left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma-1} S_{\frac{1}{2}} (\lambda t) \right] \right) (x)
\]
\[
= \left( P_{0+}^{(\eta, \alpha)} \left[ t^{\sigma-1} \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi} n! \Gamma \left( \frac{n+3}{2} \right)} \right] \right) (x)
\]
\[
= \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{n+1}{2} \right) \lambda^n}{\sqrt{\pi} n! \Gamma \left( \frac{n+3}{2} \right)} \left( P_{0+}^{(\eta, \alpha)} \left( t^{\sigma+n-1} \right) \right) (x)
\]
\[
= \Gamma \left( \frac{3}{2} \right) \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{n+1}{2} \right) \lambda^n}{\sqrt{\pi} n! \Gamma \left( \frac{n+3}{2} \right)} \frac{x^{\eta+\sigma+n} \Gamma (\sigma + n) \Gamma \left( 1 + \frac{\eta}{1-\alpha} \right)}{\left( a (1-\alpha) \right)^{\sigma+n} \Gamma \left( 1 + \frac{\eta}{1-\alpha} + \sigma + n \right)}
\]
\[
= \Gamma \left( \frac{3}{2} \right) \frac{\lambda^n}{2 \left( a (1-\alpha) \right)^{\sigma+n}} \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{n+1}{2} \right) \Gamma (\sigma + n) \Gamma \left( 1 + \frac{\eta}{1-\alpha} + \sigma + n \right)}{\left( n! \Gamma \left( \frac{n+3}{2} \right) \right)^n \left( a (1-\alpha) \right)^n}
\]
Similarly one can derive the pathway integral representation of \((2.12)\), \((2.13)\). \[\square\]

**Conclusion**

In this present study, we consider Bessel Struve kernel function \(S_{\lambda} (\lambda z)\), \(\lambda, z \in \mathbb{C}\) to obtain the results in terms of generalized Wright functions by applying Marichev-Saigo-Maeda operators. Also, the pathway integral representations Bessel struve kernel function and its relation between many other functions also derived in this study.
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