CONSTANTS OF THE MOTION AND THE QUANTUM MODULAR GROUP IN (2+1)-DIMENSIONAL GRAVITY

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Constants of motion are calculated for 2+1 dimensional gravity with topology \( \mathbb{R} \times T^2 \) and negative cosmological constant. Certain linear combinations of them satisfy the anti-de Sitter algebra \( so(2,2) \) in either ADM or holonomy variables. Quantisation is straightforward in terms of the holonomy parameters, and the modular group is generated by these conserved quantities. On inclusion of the Hamiltonian three new global constants are derived and the quantum algebra extends to the conformal algebra \( so(2,3) \).

1 Hamiltonian Dynamics

It is known that \((2+1)\)-dimensional gravity, with or without a cosmological constant \( \Lambda \), has (at least) two equivalent descriptions. For topology \( \mathbb{R} \times T^2 \) and \( \Lambda < 0 \) the \( \tau = \) extrinsic curvature development of the ADM canonical variables \( q^1, p_1, p_2 \) is generated by an effective Hamiltonian which is just the spatial volume

\[
H = \int_{T^2} d^2x \sqrt{(3)}g = \frac{1}{\sqrt{\tau^2 - 4\Lambda}} \bar{H}, \quad \bar{H} = \sqrt{p_1^2 + e^{-2q_1}p_2^2}.
\]

(1)

An alternative description is in terms of global, time-independent parameters \( r^\pm_1, r^\pm_2 \) which characterise the traces of holonomies (Wilson loops) and satisfy

\[
\{r^\pm_1, r^\pm_2\} = \frac{1}{\alpha}, \quad \{r^+, r^-\} = 0, \quad \text{and} \quad \alpha = \frac{1}{\sqrt{-\Lambda}} > 0.
\]

(2)

In Eq. 2 the subscripts 1, 2 refer to two intersecting paths \( \gamma_1, \gamma_2 \) on \( T^2 \) with intersection number +1, and the \( \pm \) refer to the two copies of \( SL(2,\mathbb{R}) \) in the decomposition of the spinor group of \( SO(2,2) \) as a tensor product \( SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R}) \).

The four real parameters \( r^\pm_1, r^\pm_2 \) of Eq. 2 are arbitrary, but are related, through a time-dependent canonical transformation, to the components of the complex moduli \( m = m_1 + im_2 \) and their momenta \( \pi = \pi^1 + i\pi^2 \) as follows

\[
m = \left( r^-_1 e^{it/\alpha} + r^+_1 e^{-it/\alpha} \right) \left( r^-_2 e^{it/\alpha} + r^+_2 e^{-it/\alpha} \right)^{-1},
\]

\[
\pi = -\frac{i\alpha}{2 \sin \frac{2\pi}{\alpha}} \left( r^+_2 e^{it/\alpha} + r^-_2 e^{-it/\alpha} \right)^2,
\]

(3)
where

\[ m_1 = q^2, m_2 = e^{-q^1}, \pi^1 = p_2, \pi^2 = -p_1e^{q^1} \quad \text{and} \quad \tau = \frac{2}{\alpha} \cot \frac{2t}{\alpha}, \quad (4) \]

with \( \tau \) monotonic in the range \( t \in (0, \frac{\alpha}{2}) \), and from Eqs. (3–4) the Hamiltonian (Eq. 1) is

\[ H = \alpha \frac{1}{\sqrt{\tau^2 - 4\Lambda}} (r_1^+ r_2^- - r_1^- r_2^+) \quad \text{and} \quad \bar{H} = \frac{\alpha}{2} (r_1^+ r_2^- - r_1^- r_2^+). \quad (5) \]

Since the \( r_\pm^1, r_\pm^2 \) are arbitrary the moduli and momenta of Eq. 3 can have arbitrary initial data \( m(t_0), p(t_0) \) at some initial time \( t_0 \).

### 2 Constants of the motion

#### 2.1 ADM Variables

Absolutely conserved quantities are obtained as in [6, 7], from the traces of the \( \text{SL}(2,\mathbb{R}) \) holonomies along the paths \( \gamma_1, \gamma_2, \gamma_1 \cdot \gamma_2 \) respectively,

\[ C_\pm^1 = C_1 \pm 2\sqrt{-\Lambda}C_4, \quad C_\pm^2 = C_2 \mp 2\sqrt{-\Lambda}C_5, \quad C_\pm^3 = C_3 \pm \sqrt{-\Lambda}C_6, \quad (6) \]

where

\[ C_1 = \frac{1}{2} e^{-q^1} \tau \left\{ \sqrt{1 - \frac{4\Lambda}{\tau^2}} \bar{H} - p_1 \right\}, \quad C_2 = \frac{1}{2} e^{q^1} \tau \left\{ \sqrt{1 - \frac{4\Lambda}{\tau^2}} \bar{H} - p_1 \right\}, \quad C_3 = \frac{1}{2} e^{q^1} \tau \left\{ q^2 \left( \sqrt{1 - \frac{4\Lambda}{\tau^2}} \bar{H} - p_1 \right) - p_2 e^{-2q^1} \right\}, \]

\[ C_4 = \frac{1}{2} \left\{ p_2 e^{-2q^1} + 2q^2 p_1 - p_2 (q^2)^2 \right\}, \quad C_5 = \frac{1}{2} p_2, \quad C_6 = p_1 - q^2 p_2. \quad (7) \]

Quantisation of these constants and the Hamiltonian (Eq. 3) in terms of these ADM variables has been discussed in [8, 9].

#### 2.2 Holonomy parameters

In terms of the time independent parameters \( r_\pm^1, r_\pm^2 \) the \( C_\pm^1 - C_\pm^3 \) are, from Eqs. (3–4)

\[ C_\pm^1 = (r_1^\mp)^2, \quad C_\pm^2 = (r_2^\pm)^2, \quad C_\pm^3 = r_1^\mp r_2^\pm, \quad (8) \]

and they are evidently time independent. Quantisation is straightforward in terms of these parameters. With the commutators (the quantisation of Eq. 4)

\[ [\hat{r}_1^\pm, \hat{r}_2^\pm] = \mp \frac{i\hbar}{\alpha}, \quad [\hat{r}^+, \hat{r}^-] = 0, \quad (9) \]
the combinations
\[
\hat{j}^\pm_0 = \mp \frac{\alpha}{2} (\hat{r}_1^\pm + \hat{r}_2^\pm), \quad \hat{j}^\pm_1 = \pm \frac{\alpha}{2} (\hat{r}_1^\pm - (\hat{r}_1^\pm)^2), \quad \hat{j}^\pm_2 = \pm \frac{\alpha}{2} (\hat{r}_1^\pm \hat{r}_2^\pm + \hat{r}_2^\pm \hat{r}_1^\pm),
\]
(10)
satisfy the two (±) Lie algebras of so(1, 2) \(\approx sl(2, \mathbb{R})\).

\[
[\hat{j}^\pm_a, \hat{j}^\pm_b] = 2i\hbar \epsilon_{abc} \hat{j}^c, \quad [\hat{j}^+_a, \hat{j}^-_b] = 0,
\]
(11)
where the \(\hat{j}^+_a\) depend only on the \(r^+\)'s and the \(\hat{j}^-_a\) only on the \(r^-\)'s.

The generators \(\hat{j}^\pm_\alpha\) and \(\hat{H}\) are not all independent. There are 3 Casimirs
\[
\hat{j} = \hat{j}^+_a \hat{j}^-_a = \frac{3\hbar^2}{4}, \quad \hat{j}^2 = \frac{1}{2} \hat{j}^+_a \hat{j}^-_a = \frac{\hbar^2}{2},
\]
(12)
This particular value of the Casimir \(\hat{j}\) (in Eq. [2]) corresponds to a particular discrete representation of \(SU(1, 1)\) which will be discussed elsewhere. Note that the only ordering ambiguity is in \(\hat{j}_2^\pm\) (Eq. [10]) but that any other ordering would only produce terms of \(O(\hbar^2)\) on the R.H.S. of Eqs. [1] and [12].

2.3 The Quantum Modular Group Generated

The modular group acts classically on the torus modulus and momentum and holonomy parameters as
\[
S : m \to -m^{-1}, \quad \pi \to \bar{\pi} \pi, \quad r_1^\pm \to r_2^\pm, \quad r_2^\pm \to -r_1^\pm,
\]
\[
T : m \to m + 1, \quad \pi \to \pi, \quad r_1^\pm \to r_1^\pm + r_2^\pm, \quad r_2^\pm \to r_2^\pm,
\]
(13)
and generates the entire group of large diffeomorphisms of \(\mathbb{R} \times T^2\).

With the ordering of Eq. [3] (the only ambiguity), the quantum action of the modular group is the same as the classical one, with no \(O(\hbar)\) corrections, and is generated by the \(SO(2, 2)\) anti-de Sitter subgroup by conjugation with the operators \(U_T\) and \(U_S\) where
\[
U_T = \exp \frac{i}{2\hbar}(j_0^+ + j_0^-) = \exp \frac{i\alpha}{2\hbar} C_2^- = \exp \frac{i\alpha}{2\hbar} (r_2^+)\],
\[
U_S = \exp \frac{i\pi}{2\hbar} j_0^+ = \exp \frac{i\pi\alpha}{4\hbar}(C_1^+ + C_2^-) = \exp \frac{i\pi\alpha}{4\hbar} ((r_1^+)^2 + (r_2^+)\].
(14)

The quantum algebra (Eq. [11]) and the identities (Eq. [12]) are invariant under the transformations of Eq. [13].

3 The Extended Algebra

Using Eq. [3] it can be checked that the Hamiltonian \(\hat{H}\) (Eq. [3]) does not commute with all the \(sl(2, \mathbb{R})\) generators (Eq. [10]), but instead defines a new globally constant three-vector \(\hat{v}_a\)
\[
[\hat{H}, \hat{j}^\pm_\alpha] = \pm i\hbar \hat{v}_a, \quad \text{where}
\]
(15)
\[ \hat{v}_0 = -\frac{\alpha}{2} (\hat{r}_1^+ \hat{r}_1^- + \hat{r}_2^+ \hat{r}_2^-), \quad \hat{v}_1 = \frac{\alpha}{2} (\hat{r}_1^+ \hat{r}_1^- - \hat{r}_2^+ \hat{r}_2^-), \quad \hat{v}_2 = -\frac{\alpha}{2} (\hat{r}_1^+ \hat{r}_2^- + \hat{r}_2^+ \hat{r}_1^-). \tag{16} \]

The extended algebra of the ten \( \hat{\mathcal{H}}, \hat{j}_a, \hat{v}_a \) (Eqs. 5, 10, 16), \( a = 0, 1, 2 \) then closes as follows

\[
[\hat{\mathcal{H}}, \hat{v}_a] = \frac{i\hbar}{2} (\hat{j}_a^+ - \hat{j}_a^-), \quad [\hat{v}_a, \hat{v}_b] = -\frac{i\hbar}{2} \epsilon_{abc} (\hat{j}_c^+ + \hat{j}_c^-), \quad [\hat{j}_a^\pm, \hat{v}_b] = i\hbar (\mp \eta_{ab} \hat{\mathcal{H}} + \epsilon_{abc} \hat{v}_c), \tag{17} \]

in addition to Eqs. 1 and 5, with the 3 additional identities (making, with Eq. 12, a total of 6 identities)

\[
\hat{v}^a \hat{j}_a^\pm = \hat{j}_a^\pm \hat{v}^a = \pm \frac{3i\hbar}{2} \hat{\mathcal{H}}, \quad \hat{v}_a \hat{v}^a = \hat{\mathcal{H}}^2 - \frac{\hbar^2}{2}. \tag{18} \]

The above 10-dimensional algebra is isomorphic to the Lie algebra of \( \text{so}(2, 3) \), whose corresponding group is the conformal group of 3-dimensional Minkowski space. The precise identifications with the dilatation \( D \) and the conformal accelerations are given in 4. Note that, in contrast to the generators \( \hat{j}_a^+ \) and \( \hat{j}_a^- \) (Eq. 11) of the two commuting \( \text{sl}(2, \mathbb{R}) \) subalgebras (Eq. 14), the vector \( \hat{v}_a \) (Eq. 14) and \( \hat{\mathcal{H}} \) (Eq. 5) require both the commuting \( \pm \) spinors (Eq. 9). Their action on the ADM and holonomy parameters is under study.

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