On the Universality of Nonperturbative effects in Stabilized 2D Quantum Gravity

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Abstract

In this letter I study the universality of the nonperturbative effects and the vacua structure of the stochastic stabilization of the matrix models which defines Pure 2D Quantum Gravity. I show also that there is not tunneling, in the continuum limit, between the one-arc and three-arc solutions of the simplest matrix model which defines the flow between Pure Gravity and the Lee-Yang model.
1. Pure Quantum Gravity in two dimensions is an ill defined theory because the topological expansion is given by a non Borel summable series\[1, 2, 3\]. In the matrix models approach\[1\], the topological expansion is defined by the perturbative series of a matrix model with an unbounded potential, and the non Borel summability of the topological expansion is related to tunneling from the local minimum of the matrix potential to the unbounded region\[3\].

There are several proposals for stabilize an unbounded matrix model\[4\]. In the stochastic stabilization\[5\] the matrix model is mapped to a one dimensional matrix model, where the unbounded region of the potential is related to the local minimum of the stabilized potential. In fact, the perturbative ground state of the one dimensional stabilized model is the original matrix model.

In a previous paper\[6\] the analytical ground state of the stochastic stabilization of the fourth matrix potential has been studied, taking into account nonperturbative corrections. The nonperturbative effects reproduce the asymptotic behaviour of the topological expansion, and is related to tunneling from the perturbative vacuum, which defines the topological expansion, to a nonperturbative vacuum, which break down the symmetries of the original matrix model.

The stabilized matrix model is given by a one dimensional Fermi gas. In the planar limit the Fermi energy is placed at the local minimum of the stabilized potential\[5\]. The perturbative corrections decrease the Fermi energy, and it become below the local minimum\[6, 7, 8\]. This is very important
because the nonperturbative ground state is given by a combination

\[ \Psi = \Psi_{\text{pert}} + \Psi_{\text{nonpert}} \]  \hspace{1cm} (1)

where \( \Psi_{\text{pert}} \) is a perturbative state around the global minimum of the stabilized potential which defines the topological expansion. Around the local minimum \( \Psi_{\text{nonpert}} \) is the state with energy below the potential, hence it cannot be a perturbative eigenstate around the local minimum because the WKB wave functions around the local minimum have energy above it. Henceforth \( \Psi_{\text{nonpert}} \) can be interpreted as a nonperturbative vacuum and (1) defines a ground state where the nonperturbative effects are given by tunneling between a perturbative vacuum and a nonperturbative one\(^6\).

In this short letter I study the universality of the vacua structure and the nonperturbative effects in stabilized matrix models. The cubic, fourth and sixth matrix models are considered. I show that there is not tunneling between one-arc and three-arc solution in the sixth matrix model.

2. The cubic potential is the simplest matrix model which defines Pure Quantum Gravity:

\[ W = \text{Tr} \Phi^2 - \frac{2}{3} g \text{Tr} \phi^3 \]  \hspace{1cm} (2)

and the stabilized potential is\(^9\)

\[ V = \frac{1}{2} \left\{ g^2 \lambda^4 - 2g \lambda^3 + \lambda^2 + 2g \lambda - 1 \right\}. \]  \hspace{1cm} (3)

The perturbative Fermi energy is given by the condition:

\[ \frac{N}{\pi} \int d\lambda \sqrt{2(E_F - V)} = N - \frac{1}{2} \]  \hspace{1cm} (4)

where the minus sign in \( N - 1/2 \) is because the Fermi gas is given by \( N \) free fermions which must fill the first \( N \) eigenvalues of the WKB quantization
The perturbative Fermi energy is given now by:

\[
\frac{N}{\pi} \int_{-a}^{a} d\lambda \sqrt{2(E_F - V)} - \frac{1}{\pi^2} g \int_{-a}^{b} d\lambda \frac{\lambda^2}{\sqrt{2(E_F - V)}} + O(1/N) = N - \frac{1}{2}.
\]  

\[
\Delta_1 = -bg \sqrt{b^2 - a^2}
\]  

1. The stabilized potential of the fourth and next order matrix models has interaction terms, but the Hartree-Fock approach is exact to all orders in $1/N$.  

The fourth potential is

\[
W = Tr \Phi^2 - \frac{2}{4} g Tr \phi^4
\]  

and its stabilized potential in the Hartree-Fock approach\footnote{The stabilized potential of the fourth and next order matrix models has interaction terms, but the Hartree-Fock approach is exact to all orders in $1/N$.} is\footnote{The stabilized potential of the fourth and next order matrix models has interaction terms, but the Hartree-Fock approach is exact to all orders in $1/N$.}

\[
V = \frac{1}{2} \left\{ g^2 \lambda^6 - 2g \lambda^4 + \lambda^2 + 2g \lambda^2 - 1 \right\} + \frac{1}{N^2} g \lambda^2 + \text{Fock}.
\]  

\[
\Delta_1 = -bg \sqrt{b^2 - a^2}
\]
where $a$ is the cut of the eigenvalue density and $b$ is the local minimum of the stabilized potential\[10\].

3. The sixth potential:

$$V = Tr\Phi^2 - \frac{2}{4}gTr\phi^4 + \frac{2}{6}\alpha Tr\Phi^6$$

(13)

is the simplest potential which defines the flow between Pure Gravity and
the Lee-Yang model\[1, 11\]. If $\alpha$ is positive there are three kind of vacua in
the planar limit\[12\]: the one-arc vacuum, where the eigenvalue density $\rho(\lambda)$
is defined on one interval; the two-arc vacuum, where $\rho(\lambda)$ is defined on two
intervals and an infinite family of three-arc vacua where $\rho(\lambda)$ is defined on
three intervals.

Pure Quantum Gravity is the continuum limit of the one-arc vacuum.
In this model there is a line of critical points which defines Pure Quantum
Gravity. The line ends at the tricritical point which defines the Lee-Yang
model. But at the critical line there are three-arc and two-arc solutions. In
Refs. \[11, 12\] it has been argued that the tunneling between one-arc solution
and three-arc solutions at the critical line is the origin of the nonperturbative
instability of the KdV flow between Lee-Yang an Pure Gravity. I will show
that the tunneling between one-arc and two or three arc solutions does not
survive in the continuum limit.

The stabilized potential in the Hartree-Fock approach is\[13\]

$$V = V_0 + \frac{1}{N}V_1 + \text{Fock}$$

$$V_0 = \frac{1}{2} \left\{ \alpha^2 \lambda^{10} - 2g\alpha\lambda^8 + (g^2 + 2\alpha)\lambda^6 - 2(g + \alpha)\lambda^4 \right\}$$

$$+ \frac{1}{2} \left\{ (1 + 2g - 2\alpha\omega)\lambda^2 - 1 \right\}$$
\[ V_1 = \frac{1}{N} \left\{ \frac{1}{2} g \lambda^2 - \frac{3}{2} \alpha \lambda^4 \right\} \]  

(14)

where \( \omega \) is a self-consistent parameter

\[ \omega = \frac{1}{N} \langle Tr \Phi^2 \rangle \]

\[ = \frac{1}{\pi} \int d\lambda \lambda^2 \sqrt{2(E_F - V) + O(1/N^2)}. \]  

(15)

In the planar limit there is an infinite set of ground states of the above Hamiltonian which are the multi-cut solutions of the original matrix model[13].

The one-arc and three-arc solutions are represented in the Figure 1. The one-arc solution has four local degenerate minima \( b, -b, c \) and \( -c \), and the Fermi energy is placed at these local minima. Hence, the eigenvalues are restricted to the interval \((-a, a)\) where \( a \) is the cut of the eigenvalue density of the original matrix model.

The three-arc solutions have four nondegenerate local minima and the Fermi energy is placed at the minimum \( b \). There are eigenvalues in the three intervals \((-d, -c), (-a, a)\) and \((c, d)\) of figure 1. These intervals are the support of the eigenvalue density of the three-arc solutions of the original matrix model.

In the one-arc solution the Fermi energy is given by the condition

\[ \frac{N}{\pi} \int_{-a}^{a} d\lambda \sqrt{2(E_F - V)} - \frac{1}{\pi} \int_{a}^{b} d\lambda \frac{V_1}{\sqrt{2(E_F - V)}} + O(1/N) = N - \frac{1}{2} \]  

(16)

which must be supplemented by a condition to the self-consistent parameter \( \omega \) at leading order

\[ \omega = \frac{1}{\pi} \int_{-a}^{a} d\lambda \lambda^2 \sqrt{2(E_F - V)}. \]  

(17)

In the one-arc solution (figure 1.a) the difference between the Fermi energy and the value of the potential at the two local minima are zero at leading
order and at subleading order are:

\[
\Delta_1(b) = \left[ -2\alpha(c^2 - b^2) + \frac{1}{c\sqrt{c^2 - a^2}} \{V_1(b) - V_1(c)\} \right] \\
\times \left[ \frac{1}{b\sqrt{b^2 - a^2}} - \frac{1}{c\sqrt{c^2 - a^2}} \right]^{-1} \\
\Delta_1(c) = \left[ -2\alpha(c^2 - b^2) + \frac{1}{b\sqrt{b^2 - a^2}} \{V_1(b) - V_1(c)\} \right] \\
\times \left[ \frac{1}{b\sqrt{b^2 - a^2}} - \frac{1}{c\sqrt{c^2 - a^2}} \right]^{-1}. \tag{18}
\]

Numerically one can check that \(\Delta_1(b)\) is negative and \(\Delta_1(c)\) is positive. Hence, at subleading order in \(1/N\) the Fermi energy is below the local minimum \(b\) and above the local minimum \(c\).

This result is very natural because the local minimum \(c\) is related to the nonzero minimum of the original potential. Well defined configurations of the original potential must be related to perturbative ground states of the stochastic stabilization[14].
Henceforth there exist tunneling between the one-arc solution, where all the eigenvalues are restricted to the central well of the stabilized potential and the perturbative vacuum with an eigenvalue around the local minimum $c$. This perturbative vacuum is a three-arc vacuum. Hence the tunneling from the main well to the local minimum $c$ can be interpreted as tunneling between the one-arc vacuum and the three-arc vacuum. From standard WKB calculation the tunneling is proportional to:

$$\exp \{ -N(\Gamma_1 + \Gamma_2) \} \quad (19)$$

where

$$\Gamma_1 = \int_a^b d\lambda \sqrt{2(V-E_F)}$$

$$\Gamma_2 = \int_b^c d\lambda \sqrt{2(V-E_F)} \quad (20)$$

The critical point $g_c$ which defines perturbative Pure Quantum Gravity, is given when $a = b$ in the one-arc solution (figure 1.a). The double scaling limit is the limit:

$$N \rightarrow \infty \quad \quad g \rightarrow g_c \quad \quad N(g - g_c)^{5/4} = z \quad (21)$$

where $z$ is finite. In the double scaling limit $\Gamma_1$ goes to zero and

$$\exp \{ -N\Gamma_1 \} \rightarrow \exp \{ -Cz \} \quad (22)$$

where $C$ is a universal constant for all matrix models. $\Gamma_2$ remains finite at $g = g_c$, and, in the double scaling limit

$$\exp \{ -N\Gamma_2 \} \rightarrow 0 \quad (23)$$
hence, in the double scaling limit the tunneling between the perturbative one-arc solution and the perturbative three-arc solution does not survive. In the double scaling limit the nonperturbative effects are given only by $\Gamma_1$ which can be interpreted as tunneling between the main well of the stabilized potential and the local minimum $b$. Because the Fermi energy is below the value of the potential at the local minimum $b$, the stabilized potential of the sixth matrix model has the same nonperturbative behaviour that the model studied in Ref. [6], and following the same reasoning that in this reference one can construct the ground state as a linear combination of a perturbative ground state around the global minimum and a wave function around the local minimum $b$, which is nonperturbative.

Henceforth, the origin of the instability of the flow between Lee-Yang and Pure Gravity is the tunneling between the perturbative vacuum which defines Pure Gravity and a nonperturbative vacuum which break down the symmetries of the matrix model. This is also the origin of the instability of the loop equation[6, 8].

In the two-arc vacuum the distribution of eigenvalues differs macroscopically from the one-arc vacuum, but only tunneling of one eigenvalue may survive in the double scaling limit[16]. Hence there is not tunneling between one-arc and two-arcs vacuum in the double scaling limit.
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