Cosmological Perturbations Generated in the Colliding Bubble
Braneworld Universe

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Abstract

We compute the cosmological perturbations generated in the colliding bubble braneworld universe in which bubbles filled with five-dimensional anti-de Sitter space ($\text{AdS}^5$) expanding within a five dimensional de Sitter space ($\text{dS}^5$) or Minkowski space ($\text{M}^5$) collide to form a (3+1) dimensional local brane on which the cosmology is virtually identical to that of the Randall-Sundrum model. The perturbation calculation presented here is valid to linear order but treats the fluctuations of the expanding bubbles as (3+1) dimensional fields localized on the bubble wall. We find that for bubbles expanding in $\text{dS}^5$ the dominant contribution to the power spectrum is ‘red’, with $dP(k)/d[\log(k)] \approx 1/(m_4 \ell)^2 [1 + (R/\ell_{\text{ext}})^2] \ell^2 (k_c^2/k^2)$ where $R$ is the spatial curvature radius of the universe on the local brane at the moment of bubble collision, $\ell$ is the curvature radius of the bulk $\text{AdS}^5$ space within the bubbles, $\ell_{\text{ext}}$ is the curvature radius of the $\text{dS}^5$ (taken as infinite for the $\text{M}^5$ case), $k_c$ is the wavenumber of the spatial curvature scale, and $m_4$ is the four-dimensional Planck mass. The perturbations are minuscule and well below observational detection except when $(m_4 \ell)$ is not large or in certain cases where $R$ at the moment of collision exceeds $\bar{R} = \ell_{\text{ext}}(\ell_{\text{ext}}/\ell)$. Note: This paper supersedes a previous version titled “Exactly Scale-Invariant Cosmological Perturbations From a Colliding Bubble Braneworld Universe” in which we erroneously claimed that a scale-invariant spectrum results for the case of bubbles expanding in $\text{M}^5$. This paper corrects the errors of the previous version and extends the analysis to the more interesting and general case of bubbles expanding in $\text{dS}^5$.

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I. INTRODUCTION

One of us (MB) recently proposed [1] that the collision of two bubbles of a true anti-de Sitter space vacuum expanding within either de Sitter or Minkowski space can give rise to a braneworld universe consisting of a (3 + 1)-dimensional FLRW brane universe embedded in (4 + 1)-dimensional anti-de Sitter space. A $Z_2$ (or some larger) discrete symmetry unbroken outside the bubbles but broken in the AdS phase inside requires that after the collision a brane or domain wall form between the two bubbles when the choice of vacua of the colliding bubbles differs. This is the (3+1)-dimensional brane universe on which we live. The spacetime contained within the future lightcone of an observer situated on the (3 + 1)-dimensional brane universe resulting after the brane collision is virtually identical to that of Randall-Sundrum (RS) [2] cosmogony [3]. Consequently, gravity on the brane [4] and the results of any experiment performed there will be identical to those in the infinite, one-brane RS scenario. In the present scenario, however, the causal past of the local brane, and in particular how certain preferred initial conditions are established, is completely distinct.

The standard RS cosmology suffers from the bulk smoothness and horizon problems [5]. Although inflation on the (3 + 1)-dimensional brane proposed in some braneworld cosmologies can smooth out whatever inhomogeneities may have previously existed on the brane, it remains a complete puzzle why the (4 + 1)-dimensional bulk into which this brane expands should initially be homogeneous and isotropic. If the bulk is not homogeneous and isotropic at the outset, inhomogeneities in the bulk induce inhomogeneities on the brane at late times. The usual Randall-Sundrum scenario is also plagued with timelike boundaries at infinity and it is unclear what sort of boundary conditions should be imposed on these [8]. In the colliding bubble scenario these problems are avoided because a special initial state is singled out. In the de Sitter case, the five-dimensional Bunch-Davies (BD) [9] vacuum is singled out because this state is an attractor. For Minkowski space, there is a unique, well-defined vacuum to be used as an initial state. Consequently, our proposal resolves the five-dimensional bulk smoothness and horizon problems. A braneworld arising within the framework of the “no boundary” proposal suggests another possible way to get around this problem. [6,7]

In ref. [1] the colliding bubble braneworld universe was presented and a heuristic discussion of the physical mechanisms underlying the generation of cosmological density perturbations was given. The colliding bubble braneworld universe heavily relies of the dynamics of false vacuum decay, elucidated quite some time ago by a number of authors [10]. The idea of a cosmology in which entropy and matter-radiation is generated from brane collisions was proposed in the work of Dvali and Tye [11] on brane inflation and in the work of Khoury, Ovrut, Steinhardt, and Turok [12] on the ekpyrotic scenario. Perkins [13] considered a braneworld on an expanding bubble; however, unlike here, in his scenario bubble collisions are regarded as cataclysmic events to be avoided. Gorsky and Selivanov [14] considered some similar ideas involving a uniform external field and the Schwinger mechanism and Euclidean instantons. More recently, Gen et al. [15] have proposed an interesting variation on the colliding bubble braneworld scenario in which two bubbles collide, one nucleating within the other.

In this article we present a quantitative calculation of cosmological density perturbations generated in this model subject to two simplifying assumptions. Firstly, we assume that the
perturbations are linear. This assumption is well justified. Secondly, we ignore the self-
gravitation of the perturbations of the expanding bubbles, although gravity is completely
taken into account for the zeroth order, unperturbed solution. As we shall see, ignoring
the self-gravity of the perturbations is rather a questionable approximation. In all cases
of interest, dimensional arguments suggest that gravity may play an important role in the
dynamics of the expanding bubble perturbations at linear order. It is not clear whether
such corrections decouple for some miraculous reason, whether they merely result in an
$O(1)$ renormalization of the overall amplitude, or whether they spoil some of the appar-
ently delicate cancellations present in the calculation, generating perturbations of differing
magnitude and spectral character. Only by carrying out a fully five-dimensional calculation
of the density perturbations, taking into account the couplings of each mode of the bubble
wall to an infinite number of modes of five-dimensional gravity of the same total angular
momentum, will it be possible to answer this question definitively. Nevertheless, given the
technical difficulty of such a calculation, we still consider it worthwhile to calculate the lin-
ear perturbations within the framework of a $(3+1)$ dimensional theory. Interestingly, the
perturbations thus calculated are small, in most of parameter space well below the magni-
tude of those observed in our universe. Their spectral character, however, is of the wrong
type, suggesting the necessity of some other mechanism to generate the known cosmological
perturbations.

The organization of this paper is as follows. Section II describes how perturbations
prior to the bubble collision may be described as scalar fields on the two unperturbed
expanding bubbles and how these fluctuations of the bubble walls translate into density
perturbations at the moment of bubble collision. We also note the possible importance of
gravitational corrections to this picture. In section III we show how to describe the Bunch-
Davies vacuum for this scalar field in terms of the appropriate mode expansion. In section IV these results are combined to obtain the scalar power spectrum for bubbles expanding in five-dimensional Minkowski space \((M^5)\). Section V generalizes to the case of bubbles expanding in five-dimensional de Sitter space \((dS^5)\). Finally, in section VI we present some concluding remarks.

II. BUBBLE WALL VACUUM FLUCTUATIONS AND JUNCTION CONDITIONS AT THE COLLISION

We begin by discussing the perturbations on the expanding bubbles before collision. As explained in the work of Garriga and Vilenkin [16,17], if one ignores the coupling to gravity, there is but a single degree of freedom, that of translations of the bubble wall, completely described to the linear order of interest here by a free scalar field that lives on the bubble wall. The bubble wall, idealized here as infinitely thin, traces out a world volume trajectory of the shape of a hyperboloid (i.e., the locus of all points at a certain proper geodesic distance from the nucleation center). In the absence of perturbations, this hyperboloid is endowed with the internal geometry of de Sitter space. Perturbations are described by means of a scalar field \(\chi\) defined on the unperturbed hyperboloid \(\mathcal{H}\). At each point \(p\) on \(\mathcal{H}\), \(\chi\) is assigned a value equal to the distance to the perturbed brane worldvolume along the geodesic passing through \(p\) normal to \(\mathcal{H}\), with \(\chi\) taken positive for outward displacements. Of course, since each bubble nucleates as the result of quantum tunnelling, this hyperboloid does not really extend infinitely far into the past. The bubble materializes through quantum tunnelling within a compact (bounded) region. But it is not possible to determine in which rest frame the bubble has nucleated without seriously disturbing the outcome of the quantum tunnelling process. Consequently, the quantum state of the fluctuations must be \(SO(4,1)\) invariant. This requirement uniquely fixes the quantum state of the fluctuations. It follows that the scalar field \(\chi\) is in the Bunch-Davies vacuum. The bubble wall tension \(\tau\) appears in the action for \(\chi\) as an overall linear factor with units \((\text{mass})^4\), so that \(\phi = \tau^{1/2} \chi\) is the customarily normalized \((3 + 1)\)-dimensional scalar field with units of \((\text{energy})^1\). As pointed out by Garriga and Vilenkin, the mass of this scalar field is \(m^2 = -4H_b^2\) (where \(H_b\) is the Hubble constant on the bubble surface) is completely fixed by symmetry. Since the \(l = 1\) modes correspond to translation of the bubble, the change in action for these modes must vanish. This requirement completely fixes the mass.

Treating the perturbations of the expanding bubble wall as a scalar field in the manner just described would be exact if the fluctuations of the bubble did not gravitate. As pointed out in ref. [7] and demonstrated explicitly in appendix A of this paper, the relation \(m^2 = -4H_b^2\) continues to hold for weakly gravitating bubbles in \(AdS^5\) and \(dS^5\), and more generally remains exact for the \(l = 0\) and \(l = 1\) solutions corresponding to rigid spacetime translations of the nucleation centers of the bubble, no matter how strong the self-gravity. However, when the bubble self-gravity is taken into account, leading to a jump in the extrinsic curvature across the two sides as indicated by the Israel matching condition, this description ceases to be exact for the remaining modes. The inconsistency of the above solution in the presence of self-gravity of the wall can be seen by comparing the change in area \(\delta A/A\) as computed on the two sides having differing extrinsic curvature. These two quantities must be the same but are not. Physically, for the \(l = 2\) and higher modes, as the bubble wall oscillates, gravity
waves are emitted. To obtain a simple estimate to what extent it is justified to ignore self-gravity, we compare the order of magnitude of the Newtonian gravitational self-energy of the critical bubble $E_G$ to its characteristic non-gravitational energy $E_{NG}$. Let $\ell$ be the curvature radius of the $AdS^5$. The tension of the bubble wall is of the same order as the cosmological constant required on the local brane so that at late times its geometry is that of $M^4$, that is $\tau \approx \Lambda_4 = m_5^3/\ell$, where $m_5$ is the five dimensional Planck mass, related to the four-dimensional Planck mass $m_4$ through $m_4^2 = m_5^3/\ell$. It cannot be smaller, for otherwise the local brane would be unstable against bifurcations into two bubble wall branes. It follows that the critical bubble radius is of order $\Lambda_4/\Lambda_5 = \ell$. Here $\Lambda_5 = m_5^3/\ell^2$ is the negative cosmological constant in the $AdS^5$ bulk. It follows that the non-gravitational energy of the critical bubble is of order $E_{NG} = \Lambda_4 \ell^3$. Using the result for five-dimensional Newtonian gravity, we obtain $E_G = G_5 M^2/\ell^2 = m_5^3/\ell$ where $G_5 = m_5^{-3}$. If the corrections due to five-dimensional gravity may be characterized by the dimensionless parameter $\alpha = (E_G/E_{NG})$, this dimensionless parameter unfortunately is always of order unity.

The above order of magnitude calculation suggests that five-dimensional gravity plays an important role in determining the evolution of the bubbles. As the bubble walls fluctuate, they emit gravitational waves. The energy carried away by the gravity waves for particular mode is likely to comprise a substantial fraction of the energy originally in that mode. Moreover, gravity waves generated in the nucleation process may inject energy into a mode. Since each mode of the bubble of fixed total angular momentum mixes linearly with an infinite number of five-dimensional modes of the same total angular momentum, these corrections necessarily lack an effective (3+1) dimensional description. Moreover, unlike the case where a brane is surrounded by $AdS^5$ on both sides, for which there is a decoupling of long wavelengths, in this case because of the $M^5$ or $dS^5$ on the bubble exterior, we expect no such (3+1) dimensional description in the long wavelength limit. This is because $dS^5$ and $M^5$, unlike $AdS^5$, opens up rather than pinching off as one passes outward from the bubble wall. An understanding of the nature and importance of these corrections must await a full (4+1)-dimensional calculation.

Returning to the (3+1)-dimensional description, now consider how the fluctuations of the two expanding bubbles translate into cosmological perturbations on the local brane. This is completely determined by energy-momentum conservation \[\rho_L \bar{u}_L + \rho_R \bar{u}_R = \rho_F \bar{u}_F\] at the point of intersection, where $\rho_L$, $\rho_R$, and $\rho_F$ are the energy densities on the various branes in their respective brane rest frames. This may be explicitly demonstrated by sur-
FIG. 2. Spacetime Diagram of Bubble Collision Neighborhood. This diagram illustrates how energy-momentum conservation determines the outcome of the bubble collision, irrespective of the detailed microphysics taking place at the moment of collision. Here the three transverse dimensions generated by the $SO(3,1)$ symmetry are orthogonal to the plane of the page, the vertical direction indicating time and the horizontal one the ‘fifth’ spatial dimension. The two-dimensional vectors $\bar{u}_L$, $\bar{u}_R$, and $\bar{u}_F$ indicate the vectors tangent to the left, right, and final branes, respectively. $\rho_L$, $\rho_R$, and $\rho_F$, indicate the densities on these branes, respectively, that is the time-time component of the stress-energy as seen by an observer co-moving with the brane on the plane of the page.

rounding the collision surface by a small closed surface over which $T_{\mu\nu}$ is integrated. In the limit in which this surface is small, we can ignore the effect of curvature on parallel transport and the above conservation law follows. Because the bubble walls lack internal structure, $\rho_L = \rho_R = \tau$ where $\tau$ is the surface tension of the expanding bubble wall. However, rather than being constant, $\rho_F$ varies according to how much radiation-matter is deposited on the local brane during the collision.

Fluctuations in the fields $\chi_L$ and $\chi_R$ affect the outcome of the collision in several respects. Firstly, they displace the point of collision with respect to where the collision would have occurred in the absence of perturbations. Secondly, they alter the center-of-mass energy of the collision. Finally they impart a transverse velocity to the local brane. These effects are illustrated in Fig. 3.

Before calculating the above effects quantitatively, we first describe the degrees of freedom of the local brane. While the expanding bubble wall branes lack internal structure (their stress-energy is simply $T_{\mu\nu} = \tau g^{(4)}_{\mu\nu}$, where $g^{(4)}$ is the four-dimensional metric induced by the surrounding five-dimensional spacetime and $\tau$ is the constant brane tension), the local brane produced in the collision possesses an additional degree of freedom due to the radiation-matter deposited on the brane. This is simply the usual adiabatic mode of the conventional FLRW cosmological models. As illustrated in Fig. 4, in the absence of perturbations, the surfaces on which the universe on the local brane is at constant temperature are hyperboloids of constant cosmic time. Temperature decreases with increasing cosmic time. We assume that the excess energy deposited on the local brane after the collision takes the form of the
FIG. 3. Perturbations of the Bubble Collision. The dashed curves indicate the brane trajectories in the neighborhood of the collision as they would be in the absence of perturbations. The solid curves indicate the actual (perturbed) brane trajectories. In panels (a) and (b) the instant of brane collision is advanced or retarded in time. This has two effects. On the one hand, an earlier or later collision decreases or increases the velocities of the bubbles leading to an underdensity or overdensity at the moment of collision of the \((F)\) brane. On the other hand, this time delay warps the surface of collision, to correction to which acts in the same direction. Because the stress-energy content of our the local \((F)\) brane is cooling, this time delay leads to an underdensity or an overdensity in the respective cases. In (c) and (d) the spacetime position of the collision remains unaltered but the velocities of the incident branes are slightly decreased or increased, respectively, leading to an underdensity or overdensity, respectively. Not shown are the possible perturbations along the fifth dimension in position or in velocity. Because these modes couple to the matter on the brane starting only at quadratic order, they are not relevant to our study of linearized perturbations. A general perturbation is a linear superposition of all four modes.
FIG. 4. **Hyperplane of Local Universe (without perturbations).** The plane of the page corresponds to (1+1)-dimensional section of the (3+1)-dimensional hyperplane of points equidistant from the two nucleation centers. The point M, located at the vertex of the lightcone, is the midpoint of the geodesic connecting the two nucleation centers. The first hyperboloid represents a section of the spatially hyperbolic surface of bubble collision. This surface, and those of constant cosmic density or temperature in its future, has a hyperbolic spatial geometry, because it is generated by the $SO(3,1)$ residual symmetry of the two expanding bubble geometry. The vertex acts as a virtual “Big-Bang” of the universe on the local brane. Although the spatial geometry on the local brane is hyperbolic, parameters can be naturally chosen so that the spatial curvature radius today lies far beyond our present horizon, rendering the universe on the local brane effectively flat.

perfect fluid with fixed $w = p/\rho$ (taken in explicit calculations equal to $1/3$ corresponding to a radiation-dominated universe). In addition to this adiabatic mode, the local brane also has a bending mode corresponding to transverse displacements of this brane. These displacements in the “fifth” dimension are characterized by a scalar field $\chi$ on the brane, just as the displacements of the expanding bubble. The bending mode, however, does not affect the cosmological perturbations at linear order $\ddagger$. This is because the fields on the brane sense its bending through the perturbation of the induced metric, which to lowest order is quadratic. We therefore ignore excitations of this transverse mode.

Let $\mathbf{x}_C$ denote the two-dimensional vector indicating the displacement of the point of collision of the two bubbles as a result of the perturbations $\chi_L$ and $\chi_R$, as indicated in Fig. 2. Let $v_c = \tanh[2\beta]$ be the relative velocity of the two unperturbed bubbles before collision. The displacement $\mathbf{x}_C$ is computed by solving the two simultaneous equations

\[ \begin{align*}
\end{align*} \]

\[ \begin{align*}
\end{align*} \]

1This bending mode is absent in the Randall-Sundrum model, constructed using a $Z_2$ orbifold construction under which the bulk degrees of freedom on the two sides are but redundant copies of one another. Here, however, even though the zeroth order solution is $Z_2$ symmetric, the degrees of freedom on the two sides are distinct. Hence a bending mode is present.
FIG. 5. Hyperplane of Local Universe (with perturbations). In the presence of perturbations, the surfaces of constant cosmic temperature become warped by the processes illustrated in Fig. 3.

\[
\bar{x}_C = \chi_L \bar{n}_L + \lambda_L \bar{u}_L, \\
\bar{x}_C = \chi_R \bar{n}_R + \lambda_R \bar{u}_R
\]  

where \(\lambda_L\) and \(\lambda_R\) are undetermined multipliers. Here \(\bar{u}_L = (\cosh \beta, \sinh \beta)\) and \(\bar{u}_R = (\cosh \beta, -\sinh \beta)\) are the vectors tangent to the respective expanding bubble branes and \(\bar{n}_L = (\sinh \beta, \cosh \beta)\) and \(\bar{n}_R = (\sinh \beta, -\cosh \beta)\) are the respective outward normal vectors. Solving (2) yields the displacement

\[
\bar{x}_C = -\frac{1}{2} \frac{\chi_L + \chi_R}{\sinh \beta} \hat{t} + \frac{1}{2} \frac{\chi_L - \chi_R}{\cosh \beta} \hat{e}_5. 
\]  

The first term advances or retards the moment of bubble collision; the second excites the bending mode and therefore ignored for reasons already discussed.

The derivatives of \(\chi_L\) and \(\chi_R\) alter the velocity of the colliding branes. Without perturbations, the density at collision is \(\rho_{\text{coll}} = 2\tau \cosh[\beta]\). To linear order, it follows that

\[
\left( \frac{\delta \rho}{\rho} \right)_{\text{coll}} = \frac{1}{2} \tanh[\beta] \delta \beta 
\]  

where the perturbation in the boost parameter is

\[
\delta \beta = \frac{\partial \chi_L}{\partial t_L} + \frac{\partial \chi_R}{\partial t_R} - \coth[\beta](\chi_L + \chi_R). 
\]  

Here \(t_L\) and \(t_R\) are the forward time directions normal to the surface of collision and along the incident branes. The second term, calculated by setting the linear variations of the
quantity \((H_b^{-1} + \chi) \cosh \beta\) to vanish (and noting that \(H_b = 1\) in our units), reflects the fact that uniformly larger bubbles collide earlier and consequently less energetically.

Next we correct for the spatial warping of the surface of collision caused by the time delay \(\delta t_{pre} = -(1/\sinh \beta)(\chi_L + \chi_R)/2\) from eqn. (3). This is a time delay in a Milne universe, the spacetime prior to the bubble collision on the plane extending the unperturbed local brane backward in time. To calculate \(\delta \rho/\rho\) on a surface of constant mean spatial curvature, we apply the correction

\[
\left( \frac{\delta \rho}{\rho} \right)_{\text{dewarp}} = 3H_{pre}(1 + w)\delta t_{pre} = -\frac{3(1 + w)}{2 \sinh^2 \beta} (\chi_L + \chi_R).
\]

Here \(H_{pre}\) is the expansion rate of the Milne universe prior to the collision. Since \(H_{pre} = 1/\sinh[t]\) in units where \(H_b = 1\), the above result follows.

Combining the two contributions, we obtain

\[
\left( \frac{\delta \rho}{\rho} \right)_{\text{total}} = \frac{1}{2} \tanh[\beta] \left[ \left( \frac{\partial \chi_L}{\partial t_L} + \frac{\partial \chi_R}{\partial t_R} \right) - \coth[\beta](\chi_L + \chi_R) \right] - \frac{3(1 + w)}{2 \sinh^2[\beta]} (\chi_L + \chi_R).
\]

This formula indicates the density perturbations on a surface of constant mean spatial curvature. For bubble expanding in \(M^5\), \(\beta = t_{coll}\). For those expanding in \(dS^5\); however, \(\beta\) lags behind \(t_{coll}\), as shall be discussed in section V.

III. A HYPERBOLIC DESCRIPTION OF THE BUNCH-DAVIES VACUUM

To apply the matching formula above, it is necessary to expand the Bunch-Davies vacuum on each colliding bubble in terms of a mode expansion natural to the colliding bubble geometry and to that of the universe that arises on the local brane. Since eqn. (4) contains only the linear combination \((\chi_L + \chi_R)\), it is possible to adopt the fiction that there is only one bubble, with the perturbations at the surface of collision determined through the formula

\[
\frac{\delta \rho}{\rho} = \frac{1}{\sqrt{2}\tau} \left[ \tanh[\beta] \left( \frac{\partial \phi}{\partial t} - \coth[\beta] \phi \right) - \frac{3(1 + w)\phi}{\sinh^2[\beta]} \right]
\]

where the field \(\phi = \tau^{1/2}(\chi_1 + \chi_2)/\sqrt{2}\) has units of energy and the customary normalization.

We have adopted units in which \(H_b = 1\) and for bubbles expanding in Minkowski space \(\beta = t\). For the moment we further imagine that the bubble is past and forward eternal, extending

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the classical solution to the past to include an initially contracting phase and ignoring the bubble collision. Now the bubble surface has the geometry of maximally extended (3+1)-dimensional de Sitter space ($dS^4$), which can be covered by a series of hyperbolic coordinates, that make manifest the $SO(3, 1)$ subgroup of the full de Sitter symmetry group $SO(4, 1)$. The coordinates divide the spacetime into five patches labeled by roman numerals, as indicated in Fig. 6. For region I the line element is
\[
ds^2 = -dt^2 + \sinh^2[t] \cdot \left[ d\xi^2 + \sinh^2[\xi] d\Omega^2_{(2)} \right].
\] (9)

In region II the line element is
\[
ds^2 = d\sigma^2 + \sin^2[\sigma] \cdot \left[ -d\tau^2 + \cosh^2[\tau] d\Omega^2_{(2)} \right].
\] (10)

Here we use units of length with $H_b^{-1}$ set to unity, later restoring the correct physical units to the final results. With de Sitter space explicitly constructed as the embedding $W^2 + Z^2 + X^2 + Y^2 - T^2 = 1$ in (4+1)-dimensional Minkowski space, the embedding of region I is
\[
T = \sinh[t] \cosh[\xi],
W = \cosh[t],
Z = \sinh[t] \sinh[\xi] \cos[\theta],
X = \sinh[t] \sinh[\xi] \sin[\theta] \cos[\phi],
Y = \sinh[t] \sinh[\xi] \sin[\theta] \sin[\phi].
\] (11)

Regions III, IV, and V are essentially identical, being related by spatial and temporal reflections. The embedding for region II, on the other hand, is
\[ T = \sin[\sigma] \sinh[\tau], \]
\[ W = \cos[\sigma], \]
\[ Z = \sin[\sigma] \cosh[\tau] \cos[\theta], \]
\[ X = \sin[\sigma] \cosh[\tau] \sin[\theta] \cos[\phi], \]
\[ Y = \sin[\sigma] \cosh[\tau] \sin[\theta] \sin[\phi]. \]

(12)

Formally, many similarities are apparent between the dynamics of the vacuum bubble expanding in (4+1) dimensions considered here and those of linearized quantum fluctuations of the (3+1)-dimensional inflaton field in the single bubble open inflation \[21–23\]. In ref. \[22\] Bucher and Turok calculated how to describe the Bunch-Davies vacuum for a scalar field of arbitrary but uniform mass in terms of a hyperbolic mode expansion. In the case of interest here, \(m^2 = -4\) \[16,17\] (in the units with \(H_b = 1\) used here).

In the sequel we first treat the \(s\)-wave sector of the scalar field in isolation, using \(SO(3, 1)\) symmetry at the end of the calculation to generalize to all modes. In region II, we may expand the \(s\)-wave sector of the scalar field

\[ \hat{\phi}(\sigma, \tau) = \int_{-\infty}^{+\infty} d\zeta F_\zeta(\sigma) \frac{e^{-i|\zeta|\tau}}{\cosh[\tau]} \hat{a}_H(\zeta) + \text{h.c.} \]

(13)

where the spatial modes are given by

\[ F_\zeta(\sigma) = \frac{1}{4\pi \sqrt{\zeta}} S(\sigma; \zeta) = \frac{1}{4\pi \sqrt{|\zeta|}} \Gamma(1 - i\zeta) \frac{P_{2+\zeta}(\cos[\sigma])}{\sin[\sigma]} \]

(14)

and the annihilation and creation operators satisfy the usual commutation relations

\[ [\hat{a}_H(\zeta), \hat{a}^\dagger_H(\zeta')] = \delta(\zeta - \zeta') , \]
\[ [\hat{a}_H(\zeta), \hat{a}_H(\zeta')] = 0 , \]
\[ [\hat{a}^\dagger_H(\zeta), \hat{a}^\dagger_H(\zeta')] = 0 . \]

(15)

Here \(P_{2+\zeta}\) is a Legendre function. The special value in the subscript arises from the mass \(m^2 = -4H_b^2\). The hyperbolic vacuum \(|\text{Vac}_H\rangle\), a quantum state analogous to the well-known ‘Rindler’ vacuum, defined by the condition that

\[ \hat{a}_H(\zeta)|\text{Vac}_H\rangle = 0 \]

(16)

for all \(\zeta\) does not coincide with the \(SO(4, 1)\) de Sitter invariant Bunch-Davies (BD) vacuum, \(|\text{Vac}_{BD}\rangle\), defined by the conditions

\[ \hat{a}_{BD}(\zeta)|\text{Vac}_{BD}\rangle = 0 . \]

(17)

In ref. \[22\] it was demonstrated that the two sets of annihilation operators are related by the transformation

\[ \hat{a}_{BD}(\zeta) = \frac{e^{\pi|\zeta|/2}}{(e^{\pi|\zeta|} - e^{-\pi|\zeta|})^{1/2}} \hat{a}_H(\zeta) - \frac{e^{-\pi|\zeta|/2}}{(e^{\pi|\zeta|} - e^{-\pi|\zeta|})^{1/2}} \hat{a}^\dagger_H(\zeta). \]

(18)
Consequently, we may expand the scalar field $\hat{\phi}$ in terms of the more physical BD operators as follows

$$
\hat{\phi}(\sigma, \tau) = \int_{-\infty}^{+\infty} d\zeta \ F_\zeta(\sigma) \left( \frac{e^{\pi |\zeta|/2}e^{-i|\zeta|\tau} - e^{-\pi |\zeta|/2}e^{+i|\zeta|\tau}}{\cosh[\tau]} \right) \left( e^{\pi |\zeta|} - e^{-\pi |\zeta|} \right)^{1/2} \hat{a}_{BD}(\zeta) + h.c. \quad (19)
$$

Having obtained the mode expansion of $\hat{\phi}$ in region II, we now continue $\hat{\phi}$ into region I, obtaining the mode expansion there in terms of the operators $\hat{a}_{BD}(\zeta)$ and their conjugates.

This is in essence a classical field theory problem, technically complicated by the fact that the mode functions individually diverge near the lightcone separating regions I and II. In terms of the variable $u$, where $\tanh[u] = \cos[\sigma]$, we note that near this lightcone the following approximation holds

$$
S(u; \zeta) = \frac{\Gamma(1-i\zeta)P_2^{+i\zeta}(\tanh[u])}{\tanh[u]} \rightarrow \frac{e^{i\zeta u}}{\tanh[u]} \approx \sigma^{-1}(\sigma/2)^{-i\zeta} \quad (20)
$$

as $u \rightarrow +\infty$ ($\sigma \rightarrow 0+$) (approaching the lightcone from outside). For this special integral value of the scalar field mass, the power series expansion for the hypergeometric function appearing in the definition of the Legendre function has only a finite number of terms, and eqn. (20) can be cast into the following special form:

$$
S(\sigma; \zeta) = \left( \frac{\cot[\sigma/2]}{\sin[\sigma]} \right)^{i\zeta} \sin[\sigma] \times \left[ 1 - 6 \left( 1 - i\zeta \right) \sin^2[\sigma/2] + \frac{12}{(1-i\zeta)(2-i\zeta)} \sin^4[\sigma/2] \right]. \quad (21)
$$

In ref. [21] the following matching rules for the asymptotic behaviours from region II to region I were derived:

$$
e^{-i\zeta u} \frac{e^{+i\zeta \tau}}{\tanh[u] \cosh[\tau]} \rightarrow (+i) \frac{\sin[\zeta]}{\sinh[\zeta]} e^{(+i-1)\eta},
$$

$$
e^{-i\zeta u} \frac{e^{-i\zeta \tau}}{\tanh[u] \cosh[\tau]} \rightarrow 0. \quad (22)
$$

Two additional relations may be obtained by complex conjugation. Here $\eta$, defined by $e^n = \tanh[t/2]$, is the region I conformal time.

In region I the temporal mode functions, which satisfy the differential equation

$$
\frac{d^2T}{dt^2} + 3 \coth[t] \frac{dT}{dt} + \left[ (4) + \frac{(c^2 + 1)}{\sinh^2[t]} \right] T = 0, \quad (23)
$$

take the form

$$
T(t; \zeta) = \frac{e^{\pm \pi \zeta /2} \Gamma(1-i\zeta)P_2^{+i\zeta}(\cosh[t])}{\sinh[t]} = \left[ \frac{\coth(t/2)}{\sinh t} \right]^{i\zeta} \times \left[ 1 + \frac{6}{(1-i\zeta)} \sinh^2[t/2] + \frac{12}{(1-i\zeta)(2-i\zeta)} \sinh^4[t/2] \right]. \quad (24)
$$
The factor $e^{\pm \pi \zeta/2}$ reflects the ambiguity of whether the Legendre function is continued above or below the branch point at unit argument. Our normalization is the geometric mean of these two continuations.

Near the lightcone, in the $t \to 0$ or $\eta \to -\infty$ limit,

$$T(t; \zeta) \to t^{-1}(t/2)^{-i\zeta} \approx \frac{1}{2} e^{(-i\zeta-1)\eta}. \quad (25)$$

At large times, as $t \to +\infty$ (or $\eta \to 0$),

$$T(t; \zeta) \to \frac{(3/2)}{(2 - i\zeta)(1 - i\zeta)} e^t. \quad (26)$$

Using (20), we rewrite (19) in region II near the lightcone as

$$\hat{\phi}(u, \tau) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{d\zeta}{\sqrt{\zeta}} \frac{e^{i\zeta u}}{\cosh[\zeta]} \left( e^{i\pi\zeta/2} e^{-i|\zeta|\tau} - e^{-\pi/2} e^{i|\zeta|\tau} \right) \hat{a}_{BD}(\zeta) + \text{h.c.} \quad (27)$$

Applying the matching rules in eqn. (22), we obtain in the $\eta \to -\infty$ limit of region I the expansion

$$\hat{\phi}(\xi, \eta, \theta, \phi) = \frac{(-i)}{\sqrt{2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l+1} \int_{0}^{+\infty} \frac{d\zeta}{\sqrt{\zeta}} \frac{1}{R_l(\xi; \zeta)} R_l(\xi; \zeta) Y_{lm}(\theta, \phi) \times \left[ e^{+\pi\zeta/2} \frac{e^{(-i\zeta-1)\eta}}{2} \hat{a}_{BD}(+\zeta; l, m) - e^{-\pi\zeta/2} \frac{e^{+(i\zeta-1)\eta}}{2} \hat{a}_{BD}(-\zeta; l, m) \right] + \text{h.c.}, \quad (28)$$

where we have used the $SO(3, 1)$ symmetry to include the remaining non $s$-wave modes of the spherical harmonic expansion. Here the hyperbolic spherical functions are defined as

$$R_l(\xi; \zeta) = N_l(\zeta)(-)^{(l+1)} \sinh[l\xi] \frac{d^{l+1}}{d(\cosh \xi)^{(l+1)}} \cos[\xi] \quad (29)$$

where $N_l(\zeta) = [(\pi/2)^2(\zeta^2 + 1^2)(\zeta^2 + 2^2)\cdots(\zeta^2 + l^2)]^{-1/2}$. In particular,

$$R_0(\xi; \zeta) = \sqrt{\frac{2}{\pi}} \frac{\sin[\xi]}{\sinh[\xi]} \quad (30)$$

(See for example ref. [21].)

When matched onto the exact mode functions, eqn. (28) becomes

$$\hat{\phi}(\xi, t, \theta, \phi) = \frac{(-i)}{\sqrt{2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l+1} \int_{0}^{+\infty} \frac{d\zeta}{\sqrt{\zeta}} \frac{1}{R_l(\xi; \zeta)} R_l(\xi; \zeta) Y_{lm}(\theta, \phi) \times \left[ e^{+\pi\zeta/2} T(t; \zeta) \hat{a}_{BD}(+\zeta; l, m) - e^{-\pi\zeta/2} T^*(t; \zeta) \hat{a}_{BD}(-\zeta; l, m) \right] + \text{h.c.} \quad (31)$$

For a given wavenumber $\zeta$, the region I temporal dependence is as follows. First, for small $t$, the mode oscillates, suffering an infinite number of oscillations as $t \to 0 +$. Then, after
a certain time, dependent on $\zeta$, these oscillations freeze out. In the large $t$ limit, after the onset of this freeze out, the following asymptotic form becomes valid:

$$\dot{\phi}(\xi, t, \theta, \phi) = \frac{(-i)H_b}{\sqrt{2}} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{0}^{+\infty} \frac{d\zeta}{\sqrt{\zeta}} \frac{1}{(e^{\pi\zeta} - e^{-\pi\zeta})^{1/2}} R_l(\xi; \zeta) Y_{lm}(\theta, \phi)$$

$$\times \left[ e^{+\pi\zeta/2} \left( \frac{3}{2} \right)^{l} e^{t} (2 - i\zeta)(1 - i\zeta) \hat{a}_{BD}(+\zeta; l, m) - e^{-\pi\zeta/2} \left( \frac{3}{2} \right)^{l} e^{t} (2 + i\zeta)(1 + i\zeta) \hat{a}_{BD}(-\zeta; l, m) \right] + h.c. \quad (32)$$

Here $H_b = (\hbar c/\ell)$ has units of energy, restoring $\dot{\phi}$ to its correct physical dimension.

**IV. SCALAR POWER SPECTRUM I.: BUBBLES EXPANDING IN MINKOWSKI SPACE**

Subsequent to the bubble collision we assume a radiation-matter equation of state on our local brane. In other words, there is no inflation. After the bubble collision, the modes, initially frozen in by the rapid expansion, enter the horizon and become dynamical, first the modes of shortest wavelength followed by those of increasingly longer wavelengths. All modes of cosmological interest today lie far outside the horizon at the moment of bubble collision $t = t_{bc}$.

We expand the density contrast at the moment of bubble collision $t = t_{bc}$

$$\frac{\delta \rho}{\rho}(\xi, \theta, \phi) = \sqrt{2\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{0}^{+\infty} d\zeta \ A_{lm}(\zeta) \ \frac{R_l(\xi; \zeta)}{\zeta} Y_{lm}(\theta, \phi) \quad (33)$$

where the coefficients $A_{lm}(\zeta)$ are regarded as a classical Gaussian random field completely characterized by the correlators

$$\langle A_{lm}(\zeta) A_{l'm'}(\zeta') \rangle = \frac{1}{\zeta [\ln(\zeta)]} \ \frac{dP_{\delta \rho/\rho}(\zeta)}{d[\ln(\zeta)]} \ \delta(\zeta - \zeta') \ \delta_{l,l'} \ \delta_{m,m'}. \quad (34)$$

Here the conventions have been chosen so that $dP_{\delta \rho/\rho}/d[\ln(\zeta)]$ independent of $\zeta$ corresponds to a scale-free spectrum. The terms associated with the quantum operators $a_{BD}(\zeta; l, m)$ and $a_{BD}(-\zeta; l, m)$ both contribute in quadrature to $dP_{\delta \rho/\rho}/d[\ln(\zeta)]$. We apply eqn. (3) to the temporal mode functions given in eqn. (24), and squaring this in conjunction with (32) gives

$$\frac{dP_{\delta \rho/\rho}}{d[\ln(\zeta)]} = \frac{1}{8\pi^2} \left( \frac{H_b^4}{\tau} \right) \cdot \coth[\pi \zeta] \cdot \zeta^2 \cdot \left[ \frac{(\zeta - 2i)(\cosh[t] - i\zeta)}{(\zeta + i) \sinh[t] \cosh[t]} - \frac{18(1 + w)e^{-t}}{(2 - i\zeta)(1 - i\zeta)} \right]^2. \quad (35)$$

For the first term enclosed within the absolute value we have used the exact temporal dependence given in eqn. (24), whereas for the second dewarping term we have used the asymptotic form given in eqn. (26). For the first term the exact form was necessary, because of a delicate cancellation. Because of this cancellation, pointed out by Garriga and Tanaka [24] there is no ‘red’ contribution of the form $e^{-t}/\zeta^2$, and a ‘blue’ spectrum results. However, it should be noted that for all scales of cosmological interest, the magnitude of this ‘blue’ component is minuscule. The overall amplitude, by the dimensional considerations discussed in sect. II, is approximately $H_b^4/\tau \approx (1/\Lambda_4 \ell^4) \approx 1/(m_5 \ell)^3 \approx 1/(m_4 \ell)^2$ and thus tiny. Even
on the smallest scales just outside the apparent horizon at the moment of collision, with \( \zeta \sim e^t \), the fluctuations are suppressed by a factor of \( (m_4 \ell)^{-2} \). Therefore, the concerns of an excessive abundance of small black holes often associated with blue spectra do not apply in this case. On even smaller scales, we recover the even more singular vacuum divergences of Minkowski space associated with the quartic divergence of the operator \((\partial_t \phi)^2\). These divergences, however, do not have a cosmological significance.

We now re-express the power spectrum above in the flat coordinates, which are a reasonable approximation in the range \( 1 \ll \zeta \ll e^t \). We set \( \zeta = (k/k_c) \) where \( k_c \) is the wavenumber of the spatial curvature. In the flat case the expansion becomes

\[
\frac{\delta \rho}{\rho}(r, \theta, \phi) = \sqrt{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_{0}^{+\infty} dk \ A_{lm}(k) \ j_l(kr) \ Y_{lm}(\theta, \phi) \tag{36}
\]

with

\[
\langle A_{lm}(k) A_{lm}^{*}(k') \rangle = \frac{1}{k} \frac{dP_{\delta \rho/\rho}(k)}{d(ln k)} \ \delta(k - k') \ \delta_{l,l'} \ \delta_{m,m'}, \tag{37}
\]

the power spectrum for the density contrast becomes

\[
\frac{dP_{\delta \rho/\rho}(k)}{d(ln k)} = \frac{1}{2\pi^2} \frac{H_b^4}{\tau} \left( \frac{k}{k_c} \right)^2 e^{-2t} \left[ 1 + 9(1 + w) \left( \frac{k_c}{k} \right)^2 \right]^2. \tag{38}
\]

evaluated on superhorizon scales on a surface of unperturbed mean spatial curvature. In isolation the second term within the brackets arising from dewarping would have a red spectrum but theis term is dominated by and interferes with the first term. In the next section in which bubbles expanding in \( dS^5 \) are discussed, we shall see that the cancellation in the term corresponding to the energy on the surface of collision persists; however, in many cases the dewarping term becomes dominant on large scales and must be constrained by the CMB quadrupole.

V. SCALAR POWER SPECTRUM II.: BUBBLES EXPANDING IN DE SITTER SPACE

In the previous section we computed the scalar power spectrum for bubbles expanding in Minkowski space (\( M^5 \)). In this section we generalize to the case of a bubble expanding in de Sitter space (\( dS^5 \)). This calculation divides into two parts, one depending on the external geometry of the collision (developed in section II for the \( M^5 \) case) and another depending only on the internal geometry of the expanding bubbles (developed in section III). The latter part of the calculation, consisting of the identification of the Bunch-Davies vacuum and its expansion in terms of an appropriate set of modes once \( m^2/H_b^2 \) for the field \( \chi \) has correctly been identified, does not depend how the \( (3 + 1) \) dimensional spacetime of the expanding bubble has been embedded into \( (4 + 1) \) dimensions. Appendix A presents an explicit demonstration that \( m^2/H_b^2 \) is independent of the surrounding spacetime. On the other hand, those aspects dealing with the external geometry are substantially altered, as we now explain.
FIG. 7. \(M^6\) **Representation of Two Colliding Bubbles in dS\(^5\)**. Shown in the figure is the \(UV\)-plane with \(T = X = Y = Z = 0\) of six-dimensional Minkowski space \((M^6)\), in which five-dimensional de Sitter space \((dS^5)\) is represented as the invariant unit hyperboloid whose intersection with the \(UV\)-plane is the unit circle shown above. The two colliding bubble walls, with the internal geometry of \(dS^4\), are represented as the intersection of the two planes with the unit hyperboloid shown above as the oblique heavy lines at angles \(\pm \Theta\) with the \(V\) axis. The invariant separation of the nucleation centers of the two bubbles is \(\frac{1}{H_{dS^5}} \cdot (2\Theta)\) (where for convenience of calculation the \(dS^5\) geometry exterior to the bubble has been continued inside). In reality, the portion of \(dS^5\) to the right of the lines is replaced with \(AdS^5\) to represent the bubble interior. As \(\Theta \to \pi/2\), the bubbles collide at progressively later times, eventually never colliding when \(\Theta = \pi/2\) exactly. Larger separations of the nucleation centers may be contemplated, so that the bubbles never even come close to striking each other in the future. This is accomplished by setting \(\Theta = \pi/2\) and tilting the plane away from the positive \(T\) axis of \(M^6\), corresponding to spacelike separation in the flat coordinate slicing of \(dS^5\) so large that no spacelike geodesic exists that connects the two nucleation centers. The half opening angle \(\sigma\) represents the radius of the critical bubble, again in units of \(\frac{1}{H_{dS^5}}\) and adopting the fiction of extrapolating the exterior geometry inside the bubble. The surface of collision of the two bubbles is the represented as the branch of the intersection of the unit hyperboloid with the line \((U, V) = (U_c, 0) = (\cos \sigma / \cos \Theta, 0)\) formed by the intersection of the two planes representing the expanding bubbles lying in the forward time direction. In the \(UVT\)-subspace this branch consists of a single point, but when the \(XYZ\) dimensions are included, this single point opens up into a completely spatial hyperboloid, with the geometry of \(H^3\) and a intrinsic curvature radius of \(R = \sqrt{(\cos \Theta / \cos \sigma)^2 - 1}\). The Minkowski space limit is obtained by taking \(\sigma, \Theta \to 0\).
FIG. 8. **Colliding Bubbles in dS⁵.** Two colliding bubbles (with the geometry of dS⁴) expand and collide in dS⁵. The dS⁵ space surrounding the bubbles is represented as the unit hyperboloid in M⁶. The dS⁴ expanding bubble walls are represented as the intersection of vertical planes with this unit hyperboloid.

It is simplest to construct dS⁵ as the embedding of the unit hyperboloid in M⁶

\[ U^2 + V^2 + X^2 + Y^2 + Z^2 - T^2 = 1 \]  \hspace{1cm} (39)

where \( ds^2 = -dT^2 + dU^2 + dV^2 + dX^2 + dY^2 + dZ^2 \) is the usual M⁶ metric. The use of the redundant radial coordinate offers the technical advantage of rendering transformations between the various coordinate patches covering dS⁵ unnecessary. An expanding bubble with the geometry of dS⁴ and an expansion rate \( H_b \) in dimensionless units—that is, in comparison to the five-dimensional expansion rate of the surrounding de Sitter space, which has been set to one—is constructed as the intersection with the unit hyperboloid dS⁵ of a plane in M⁶ separated from the origin of M⁶ by an invariant distance

\[ \bar{U} = \frac{\sqrt{H_b^2 - 1}}{H_b} \]  \hspace{1cm} (40)

It is necessary that \( H_b > 1 \), for else there is not enough space for the bubble to fit inside dS⁵.

In particular, we shall consider collisions of bubbles nucleating at rest in the section defined by \( T = 0 \) in the M⁶ coordinates. These bubbles are described by the intersection of the unit hyperboloid with planes of the form \( \cos \Theta \, U - \sin \Theta \, V = \bar{V} \) where \( \Theta \) indicates the position of the bubble nucleation center on the unit circle \( U^2 + V^2 = 1, T = X = Y = Z = 0 \). In terms of the usual closed hyperbolic coordinates for dS⁴, the embedding is given by

\[ T = +\sqrt{1 - \bar{U}^2} \, \sinh[t], \]
We consider the collision of two such bubbles with nucleation centers separated by an invariant geodesic distance $0 < 2\Theta < \pi$. We choose two bubbles with nucleation centers situated at the angles $+\Theta$ and $-\Theta$ with the positive $U$ axis. Their surface of collision is easily calculated by noting that the two planes intersect on the line defined by

$$U = U_c = \frac{\bar{U}}{\cos \Theta}, \quad V = 0$$

(42)

where $\Theta$ must be large enough so that $\cos \Theta < \bar{U}$ to ensure that the bubbles do not initially overlap at $T = 0$. The surface of collision of the two bubbles is defined by the $T > 0$ branch of

$$T^2 - X^2 - Y^2 - Z^2 = U^2 - V^2 - 1 = \frac{\bar{U}^2}{\cos^2 \Theta} - 1 = R^2 > 0,$$

(43)

which is a completely spacelike hyperboloid with intrinsic curvature radius $R$. It is useful to make the transformation $\bar{U} = \cos \sigma$, so that $\sigma$ represents the half opening angle of the bubble. It follows that in terms of the internal bubble coordinates defined in eqn. (41)

$$T_{\text{coll}}^2 = R^2 = \frac{\cos^2 \sigma - \cos^2 \Theta}{\cos^2 \Theta} - 1 = \sin^2 \sigma \sinh^2 T_{\text{coll}}$$

(44)

so that

$$\sinh^2 T_{\text{coll}} = \frac{1}{\sin^2 \sigma} \left[ \frac{\cos^2 \sigma - \cos^2 \Theta}{\cos^2 \Theta} \right].$$

(45)

To compute the relative velocity of collision we set $\zeta = \pi$ for the $+\Theta$ bubble and $\zeta = 0$ for the $-\Theta$ bubble to single out the trajectories that collide at $T = R$, $X = Y = Z = 0$. The normalized tangent vectors to these trajectories expressed in terms of the $M^6$ coordinates are

$$T_{\pm} = \cosh t \frac{\partial}{\partial T} + \sinh t \left( \sin \Theta \frac{\partial}{\partial U} \mp \cos \Theta \frac{\partial}{\partial V} \right).$$

(46)

By taking their inner product, we determine that the relative velocity of the bubble walls at the collision $v_{\text{coll}}$, given by

$$3\text{In computing this invariant separation we adopt the fiction that the exterior } dS^5 \text{ geometry extrapolates inside the bubble.}$$

19
\[
\frac{1}{\sqrt{1 - v_{\text{coll}}^2}} = \cosh[2\beta_{\text{coll}}] = -\mathcal{T}_- \cdot \mathcal{T}_+ = \cosh^2[t_{\text{coll}}] + \sinh^2[t_{\text{coll}}] \cos[2\Theta] = 1 + 2 \cos^2[\Theta] \sinh^2[t_{\text{coll}}].
\]

The center-of-mass energy available in the collision is
\[
\rho_{\text{coll}} = 2\tau \cosh[\beta_{\text{coll}}] = 2\tau \sqrt{1 + \cos^2[\Theta] \sinh^2[t_{\text{coll}}]} = 2\tau \frac{\sin \Theta}{\sin \sigma}. \tag{48}
\]

Here \(\tau\) is the density (or equivalently tension) of the expanding bubble brane in its rest frame.

We compare the above result to that for Minkowski space, where \(\cos \Theta = 1\) and \(\rho_{\text{coll}} = 2\tau \cosh[\beta]\). In Minkowski space, when the expanding bubbles initially placed far enough apart, an arbitrarily large energy density is produced at the collision. In \(dS^5\), however, the most energetic possible \(\gamma\) factor is finite, attaining \(\gamma_\infty = \csc \sigma\) in the limit \(\Theta \to \pi/2\), in which the bubbles just barely remain within causal striking distance of each other. In Minkowski space \(\beta_{\text{coll}}\) and \(\tau\) are identical, whereas in de Sitter space \(\beta_{\text{coll}}\) falls behind \(\tau\) due to, one might say, the redshifting of the expansion of the bubble as a result of the expansion of the exterior \(dS^5\) space.

For bubbles expanding in de Sitter space, it is important to keep in mind the physical distinction between \(\beta_{\text{coll}}\) and \(t\). The latter quantity is globally defined. It is a coordinate of the internal geometry of the expanding \(dS^4\) bubble, acting as the argument of the mode functions and defining relative spatial relations on this surface. \(\beta_{\text{coll}}\), by contrast, is a locally defined quantity. It is a property of the bubble collision, expressing the relative orientations of the colliding bubbles. In the neighborhood of the bubble collision, for calculating what takes place it is justified to ignore the curvature of \(dS^5\) because displacements are small compared to \(H_{\text{dS}^5}^{-1}\). Therefore we may locally set up an approximately Minkowskian coordinate system in a small neighborhood of the collision. Thus we calculate the time delay
\[
\delta t_{\text{pre}} = -\frac{1}{\sinh[\beta_{\text{coll}}]} \frac{(\chi_L + \chi_R)}{2} = \sqrt{1 - \sin^2 \sigma} \sin \Theta \left(\frac{\hat{\chi}_L + \hat{\chi}_R}{2}\right). \tag{49}
\]

identical to the result for Minkowski space, given in eqn. (B).

The perturbation of the energy density on the surface of collision, as before, consists of two parts, one proportional to \(\dot{\chi}\) and another proportional to \(\dot{\chi}\). The coefficient of the former term is computed by taking the derivative of the logarithm of eqn. (48) with respect to \(\sigma\), which represents the radius of the bubble in units in which \(H_{\text{dS}^5}^{-1} = 1\), and dividing by \(\sin \sigma\) to convert to the units of \(\chi\) for which \(H_{\text{dS}^5}^{-1} = 1\), yielding
\[
\frac{\delta \rho_{\text{coll}}}{\rho} = -\cot[\sigma] \delta \sigma = -\cos \sigma \left(\frac{\dot{\chi}_L + \dot{\chi}_R}{2}\right). \tag{50}
\]

The perturbation due to the peculiar velocities of the bubbles is
\[
\left(\frac{\delta \rho_{\text{coll}}}{\rho}\right) = \tanh[\beta_{\text{coll}}] \left(\frac{\dot{\chi}_L + \dot{\chi}_R}{2}\right) = \sqrt{1 - \frac{\sin^2 \sigma}{\sin^2 \Theta}} \left(\frac{\hat{\chi}_L + \hat{\chi}_R}{2}\right). \tag{51}
\]
The derivation is exactly analogous to that for Minkowski space except that \( \tanh[t] \) is replaced with \( \tanh[\beta_{\text{coll}}] \) for the reasons discussed above.

Combining the two results, we obtain

\[
\left( \frac{\delta \rho_{\text{coll}}}{\rho} \right)_{\text{coll}} = \sqrt{1 - \sin^2 \sigma \left( \frac{\dot{\chi}_L + \dot{\chi}_R}{2} \right)} - \cos \sigma \left( \frac{\chi_L + \chi_R}{2} \right)
\]

\[
= \cos \sigma \left[ \tanh t \left( \frac{\dot{\chi}_L + \dot{\chi}_R}{2} \right) - \left( \frac{\chi_L + \chi_R}{2} \right) \right].
\]

(52)

We now calculate the correction due to the warping of the spatial geometry of the surface of bubble collision.

\[
\left( \frac{\delta \rho}{\rho} \right)_{\text{dewarp}} = 3(1 + w) H_{\text{pre}} \delta \tilde{t}_{\text{pre}} = -3(1 + w) \frac{H_{\text{pre}}}{\sinh[\beta_{\text{coll}}]} \chi \sin \sigma.
\]

(53)

The \( \sin \sigma \) factor converts from the natural units of \( \chi \), in which \( H_b = 1 \), to those of \( dS^5 \) in which its expansion rate is unity. This correction must be added to eqn. (52) to obtain the density contrast in a gauge in which the perturbation of the trace of the spatial curvature vanishes on surfaces of constant cosmic time. Here \( H_{\text{pre}} \), expressed as a sort of Hubble constant, is the extrinsic curvature of the surface of collision as an embedding in \( dS^5 \). We compute \( H_{\text{pre}} \) as follows. The portion \( U > 0 \) of the slice \( V = 0 \) may be covered by the following one-parameter family of spatial hyperboloids on \( dS^5 \), (with the same structure as the region I coordinates given in eqn. (11))

\[
T = \sinh[t_h] \cosh[\xi],
\]

\[
U = \cosh[t_h],
\]

\[
Z = \sinh[t_h] \sinh[\xi] \cos[\theta],
\]

\[
X = \sinh[t_h] \sinh[\xi] \sin[\theta] \cos[\phi],
\]

\[
Y = \sinh[t_h] \sinh[\xi] \sin[\theta] \sin[\phi],
\]

(54)

with the line element

\[
ds^2 = -dt_h^2 + \sinh^2[t_h] \cdot \left[ d\xi^2 + \sinh^2[\xi] d\Omega^2_2 \right].
\]

(55)

It follows that \( H_{\text{pre}} = \coth[t_h] \) where we set \( U = \cosh[t_h] = U_c = \cos \sigma / \cos \Theta \). Therefore,

\[
H_{\text{pre}} = \frac{\cos \sigma}{\sqrt{\cos^2 \sigma - \cos^2 \Theta}} = \sqrt{1 + \sin^2 \sigma \sinh^2 \tilde{t}}.
\]

(56)

We use the relation (from eqns. (15) and (18))

\[
\sinh[\beta_{\text{coll}}] = \frac{\cos \sigma \sinh[t_{\text{coll}}]}{\sqrt{1 + \sin^2 \sigma \sinh^2 t_{\text{coll}}}}.
\]

(57)

to obtain

\[
\left( \frac{\delta \rho}{\rho} \right)_{\text{dewarp}} = -3(1 + w) \frac{(1 + \sin^2 \sigma \sinh^2 \tilde{t})}{\cos \sigma \sinh^2 \tilde{t}} \chi.
\]

(58)
Combining eqns. (52) and (58), we obtain a total density contrast perturbation

\[
\left( \frac{\delta \rho}{\rho} \right)_{total} = \cos \sigma \left[ \tanh t \left( \frac{\dot{X}_L + \dot{X}_R}{2} \right) - \left( \frac{X_L + X_R}{2} \right) \right] - 3(1 + w) \frac{(1 + \sin^2 \sigma \sinh^2 t)}{\cos \sigma \sinh^2 t} \chi. \tag{59}
\]

expressed in a gauge with constant mean spatial curvature. In light of eqn. (59), the power spectrum for bubbles expanding in $M^5$ given in eqn. (53) is modified to the following expression more generally valid for bubbles expanding in $dS^5$.

\[
\frac{dP_{\delta \rho/\rho}}{d[\ln(\zeta)]} = \frac{1}{8\pi^2} \cdot \left( \frac{H^4_b}{\tau} \right) \cdot \coth[\pi \zeta] \cdot \zeta^2 
\times \left[ \frac{\cos[\sigma](\zeta - 2i)(\cosh[t] - i\zeta)}{(\zeta + i)\sinh[t]\cosh[t]} - \frac{18(1 + w)(1 + \sin^2[\sigma] \sinh^2[t])e^{-t}}{(2 - i\zeta)(1 - i\zeta)\cos[\sigma]} \right]^2. \tag{60}
\]

We now consider consider the various terms of eqn. (64). The first term, with the exception of the $\cos(\sigma)$ factor, which we ignore, is identical to the Minkowski space result. Because the mode functions for $\chi = (X_L + X_R)/2$ depend only on the internal geometry of the $dS^4$ expanding bubbles, these are identical to those for the bubbles expanding in $M^5$, namely those given in eqn. (24), with the same delicate cancellation resulting. The second term, however, differs markedly for the case of bubbles expanding in $dS^5$ when the colliding bubbles are of a size comparable to or larger than the $dS^5$ curvature radius. To make this more manifest, it is useful to change to the more physical variables $\ell, \ell_{ext}$, and $R$, where $\ell = H^{-1}_b$ is the critical bubble radius (as discussed in sect. II) comparable to the $AdS^5$ curvature radius), $\ell_{ext}$ is the curvature radius of the external $dS^5$ into which the bubble expands, and $R$ is the curvature radius of the surface of collision, which has the geometry of $H^3$. Since $R = \ell_{ext} \sin[\sigma] \sinh[t_{coll}]$ and $\sin[\sigma] = (\ell/\ell_{ext})$, we may rewrite the second term in the form

\[
\left( \frac{\delta \rho}{\rho} \right)_{dewarp} = -\frac{3(1 + w)}{\cos[\sigma]} \left( \frac{1 + (R/\ell_{ext})^2}{(R/\ell)^2} \right) \chi. \tag{61}
\]

When $R \ll \ell_{ext}$, the one in the numerator dominates, yielding the $M^5$ result of the previous section. However when $R \gg \ell_{ext}$, the second term dominates, leading to a larger amplitude for the second term with a red spectrum. We obtain the following power spectrum due to the dewarping, which is the dominant contribution on large scales for bubbles expanding in $dS^5$:

\[
\frac{dP_{\delta \rho/\rho}(k)}{d[\ln(k)\]} = \frac{O(1)}{(m_4\ell)^2} \left[ 1 + \frac{(R/\ell_{ext})^2}{(R/\ell)^2} \right]^2 \left( \frac{R}{\ell} \right)^2 \left( \frac{k_c}{k} \right)^2. \tag{62}
\]

Here we have used the fact that $H^4_b/\bar{\tau} \approx 1/(m_4\ell)^2$. In the $M^5$ limit, this expression is minuscule and well approximated as

\[
\frac{dP_{\delta \rho/\rho}(k)}{d[\ln(k)\]} = \frac{O(1)}{(m_4\ell)^2} \left( \frac{\ell}{R} \right)^2 \left( \frac{k_c}{k} \right)^2, \tag{63}
\]

reproducing the result of the previous section. By contrast, in the extreme $dS^5$ limit (where $R \gg \ell_{ext}$, or when the bubble at collision has expanded to a size well in excess of that of the $dS^5$ horizon), eqn. (62) becomes
\[
\frac{dP_{\delta\rho}(k)}{d(\ln(k))} = \frac{O(1)}{(m_4\ell)^2} \left( \frac{R^2}{(\ell_{\text{ext}}^2/\ell)^2} \right) \left( \frac{k_c}{k} \right)^2. \tag{64}
\]

The second term is small unless \( R \) exceeds \( \bar{R} \), where \( \ell : \ell_{\text{ext}} : \bar{R} \) forms a geometric progression.

Physically, one can easily identify why the dewarping term becomes large in the \( dS^5 \) case in comparison to the \( M^5 \) case for large \( R \). In the \( M^5 \) case, the geometry of the extension of the local brane to earlier times, prior to the bubble collision, has the geometry of a Milne universe, with \( H_{\text{pre}} \) rapidly falling off as \( R_{\text{coll}} \) increases. In the \( dS^5 \) case, however, the Milne geometry is replaced with a \( dS^4 \) geometry with the same curvature length as the exterior \( dS^5 \). In the notation employed in this section, this extension consists of the part of the intersection of \( V = 0 \) with the unit hyperboloid in \( M^6 \) below the surface of bubble collision. Consequently, after a certain point the decay of \( H_{\text{pre}} \) ceases, with \( H_{\text{pre}} \) approaching \( \ell_{\text{ext}}^{-1} \) rather than zero.

VI. CONCLUDING REMARKS

The main conclusion of this paper is that the cosmological perturbations generated in a colliding bubble braneworld are small on all observable scales, except when (1) \((m_4\ell)\) is not large (i.e., when the extra “fifth” dimension is not large, or when a substantial hierarchy between \( m_4 \) and \( m_5 \) is lacking), or when (2) the size of the bubbles at collision is large compared to \( \bar{R} \), where \( \ell : \ell_{\text{ext}} : \bar{R} \) form a geometric progression. Here \( \ell \) is the radius of the critical bubble, approximately equal to the curvature length of the \( AdS^5 \) space inside the bubble and \( \ell_{\text{ext}} \) is the curvature length (or apparent horizon size) of the exterior \( dS^5 \) into which the bubbles expand. Unfortunately, because the spectrum resulting from effect (2) is red, it is not possible to produce the observed nearly scale-invariant perturbations by adjusting \( R \) to lie near the edge of the allowed parameter space. Since the predicted spectrum dominates on large scales, the most stringent constraint is imposed by the observed CMB quadrupole. One must impose the condition

\[
\frac{1}{(m_4\ell)^2} \left( \frac{\ell}{R} \right)^2 \left[ 1 + \left( \frac{R}{\ell_{\text{ext}}} \right)^2 \right]^2 \left( \frac{k_c}{k_Q} \right)^2 \lesssim 10^{-10} \tag{65}
\]

where \( k_Q \) is the dominant wavenumber contributing to the CMB quadrupole. The perturbation calculations presented here, however, are subject to the caveats discussed in section II concerning the importance of gravitational corrections, which according to a naive order of magnitude analysis are expected to be important for all cases of interest.

In order to create a colliding bubble braneworld universe consistent with our universe, it is necessary that \( R \) be large enough so that the universe today is not dominated by curvature and thus not empty. Calculating the minimum \( R \) required so that the universe today is sufficiently flat involves many uncertainties, primarily due to our ignorance of the equation of state on the local brane immediately after the collision. Nevertheless, the crude calculation presented below is less sensitive to these uncertainties than might be expected. If we assume that all the energy liberated in the bubble collision is immediately converted into radiation and if we ignore the change as the universe cools in the effective number of relativistic spin degrees of freedom, it follows that \( \rho_{\text{rad}}R^4 \) remains constant as the universe
expands, where $R$ is the curvature radius of the negatively curved $H^3$ at constant cosmic time. Consequently, we require that

$$\rho_{\text{coll}} R_{\text{coll}}^4 \approx \rho_{0,\text{rad}} R_0^4 \approx \left(\frac{R_{0,\text{min}}}{\lambda_{\text{CMB}}}\right)^4 \approx 10^{118} \quad (66)$$

where $\lambda_{\text{CMB}} \approx 1$ mm is the mean wavelength of the CMB today and $R_{0,\text{min}}$ is the smallest plausible curvature length of the universe today consistent with present observation.

For bubbles expanding in $M^5$, $\rho_{\text{coll}} \approx \Lambda_4 (R/\ell) \approx m_4^2 \ell^{-2}(R/\ell)$. For bubbles expanding in $dS^5$, it follows from eqn. (48) that

$$\rho_{\text{coll}} \approx \Lambda_4 \frac{(R/\ell)}{\sqrt{1 + (R/\ell_{\text{ext}})^2}} \approx \frac{m_4^2}{\ell^2} \frac{(R/\ell)}{\sqrt{1 + (R/\ell_{\text{ext}})^2}}. \quad (67)$$

In the extreme de Sitter limit ($R \gg \ell_{\text{ext}}$) in which the bubbles cease to accelerate relative to each other, instead attaining a subluminal limiting relative velocity, this expression becomes

$$\rho_{\text{coll}} \approx \frac{m_4^2}{\ell^2} \left(\frac{\ell_{\text{ext}}}{\ell}\right). \quad (68)$$

For the case of bubbles expanding in $M^5$, we require that

$$(m_4 \ell)^2 \cdot \left(\frac{R}{\ell}\right)^5 \lesssim 10^{118}. \quad (69)$$

For bubbles expanding in $dS^5$, this becomes

$$(m_4 \ell)^2 \cdot \left(\frac{R}{\ell}\right)^5 \frac{1}{\sqrt{1 + (R/\ell_{\text{ext}})^2}} \lesssim 10^{118}. \quad (70)$$

We now examine the plausibility of obtaining a pair of colliding bubbles with $(R/\ell)$ this large in a typical two bubble collision. The nucleation rate for bubbles expanding in $M^5$ is approximately

$$\Gamma = \ell^{-5} \exp[-S_E] = \ell^{-5} \exp[-O(1) \cdot (m_4 \ell)^2]. \quad (71)$$

Here $S_E$ is the Euclidean action of the instanton. Setting $\Gamma \langle R \rangle^5 \approx 1$ where $\langle R \rangle$ is the mean separation of the colliding bubbles gives

$$\frac{\langle R \rangle}{\ell} = \exp\left[\frac{O(1)}{5} \cdot (m_4 \ell)^2\right], \quad (72)$$

which is enormous except for small $(m_4 \ell)$ in which the extra “fifth” dimension is not in any sense large (i.e., $(m_4 \ell) \lesssim 16$). For bubble nucleating in $dS^5$ corrections to this relation appear but we do not anticipate any difficulty in obtaining sufficiently large bubbles.

We parenthetically note that the large values of $(R/\ell)$ required, and easily obtained owing to the exponential suppression of bubble nucleation, raise the possibility of trans-Planckian
collisions and energy densities immediately after the bubble collision. Here the term ‘trans-
Planckian’ is taken in the five-dimensional sense. If $\rho_{\text{coll}} \gtrsim m_5^4$, one might worry that the
five-dimensional theory taken as a point of departure for these calculation may cease to be
valid. Too little is presently known about trans-Planckian physics to be able to determine
whether or not this poses an obstacle to the colliding bubble braneworld scenario in this
case. It is quite plausible that even if trans-Planckian densities are attained, the details of
the collision matter little in determining the perturbations on the local brane. It is likely
that the collision may be treated as a black box, slightly displaced in space and time as a
result of the fluctuations of the incoming branes. By calculating the relative positions of
these black boxes, it should be possible to predict the outcome at late times with a minimum
of assumptions regarding what takes place within. In any case the initial epoch is set up on
the surface of bubble collision by the colliding bubbles in a precise manner as opposed to
resulting from some random, acausal singularity.

The details of the bubble collision assumed here have been highly idealized. We have
assumed an instantaneous junction with no extent in the fifth dimension at which the two
colliding bubble branes join to form the local brane. Reality could be much more compi-
cated, with the junction extended both in time and the fifth dimension, and with a portion
of the available energy liberated in the collision escaping into the bulk. For example, when
the branes are formed through a scalar field, as in the case considered by Hawking et al. \[10\]
and with gravitational corrections taken into account by Wu \[10\], the ultrarelativistic branes
upon colliding undergo several bounces in the “fifth” dimension in which the $dS^5$ phase is
restored over a region of considerable thickness before coalescence to a single domain wall
would take place. However, since our junction conditions are based on no more than energy-
momentum conservation, these conditions remain equally valid when the pointlike junction
is replaced by a black box. If there is matter ejected into the bulk, if it is free falling, it
never falls back onto the local brane. To zeroth order, as a consequence of a generalized
Birkhoff’s theorem, the effect of this matter on the bulk geometry can be absorbed into a
single real parameter. There may be some interesting interaction between perturbations of
such matter escaping into the bulk and the perturbations of the matter on the brane.

One of the interesting features of the colliding bubble braneworld scenario is the paucity
of free parameters and unknown potentials on which the perturbations depend. The only
free parameters are the curvature scales $\ell$ and $\ell_{\text{ext}}$ of $AdS^5$ and $dS^5$, respectively, to some
extent the bubble wall tension $\tau$ (although the adjustability of this parameters is quite
limited), and to some extent the equation of state after the bubble collision.

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Note added: Shortly after the original version of this paper appeared, J. Garriga and
T. Tanaka announced to us that they had been working on calculating the perturbations in
the colliding bubble braneworld and had also obtained a scale-invariant spectrum but of a
different magnitude. In the resulting discussions it became apparent that our original paper
contained a number of errors. In particular we are grateful to J. Garriga and T. Tanaka for
pointing out to us the errors noted here in the footnote of section II. Their revised calculation
has already been placed on the archive. \[24\]
APPENDIX A: DEMONSTRATION THAT $M^2/H_B^2 = -4$ FOR BUBBLES EXPANDING IN $dS^5$

The result due to Garriga and Vilenkin [16,17] that $m^2 = -4H_b^2$ for bubbles expanding in five-dimensional Minkowski ($M^5$) space is shown here to apply equally well and without correction to the case of bubbles expanding in five-dimensional de Sitter space ($dS^5$). When $M^5$ is replaced by $dS^5$, an extra parameter appears, the ratio $H_{dS^5}/H_b$ (equal to zero in Minkowski space), which at first sight may appear to offer a possible corrections to $m^2 = -4H_b^2$. Using the translational symmetry of $dS^5$, we explicitly construct a nontrivial solution of the equations of motion for the displacement normal to the bubble represented by the field $\chi$, as defined in Section II. The form of this solution when expressed in terms of the coordinates internal to the $dS^4$ bubble wall geometry is independent of $H_{dS^5}/H_b$, thus proving the result.

It is most convenient to construct $dS^5$ as the embedding

$$U^2 + V^2 + X^2 + Y^2 + Z^2 - T^2 = 1$$

where $M^6$ has the usual line element $ds^2 = -dT^2 + dU^2 + dV^2 + dX^2 + dY^2 + dZ^2$. The section $U = \bar{U} = (\text{constant})$, where $0 < \bar{U} < 1$, in other words

$$V^2 + X^2 + Y^2 + Z^2 - T^2 = (1 - \bar{U}^2) = (H_b/H_{dS^5})^{-2}$$

describes the geometry of an expanding bubble with the internal geometry of $dS^4$ embedded in $dS^5$. For $U > \bar{U}$, the geometry actually should be that of the bubble interior, but here we concern ourselves only with the exterior $U < \bar{U}$ and the bubble wall itself $U = \bar{U}$. In terms of the closed coordinatization of the expanding bubble, we have the embedding

$$T = \sqrt{1 - \bar{U}^2} \sinh[\tau],$$
$$U = \bar{U},$$
$$V = \sqrt{1 - \bar{U}^2} \cosh[\tau] \cos[\zeta],$$
$$Z = \sqrt{1 - \bar{U}^2} \cosh[\tau] \sin[\zeta] \cos[\theta],$$
$$X = \sqrt{1 - \bar{U}^2} \cosh[\tau] \sin[\zeta] \sin[\theta] \cos[\phi],$$
$$Y = \sqrt{1 - \bar{U}^2} \cosh[\tau] \sin[\zeta] \sin[\theta] \sin[\phi].$$

It follows that $H_{dS^5}/H_b = \sqrt{1 - \bar{U}^2}$.

We form a bubble translation solution $\chi$ by taking the inner product

$$\chi = \bar{N} \cdot \bar{K}$$

where $\bar{K}$ is a Killing field of $dS^5$ and $\bar{N}$ is the outward unit normal vector field on the bubble wall. For $\bar{K}$ we arbitrarily choose the rotation

$$\bar{K} = U \frac{\partial}{\partial V} - V \frac{\partial}{\partial U}. $$

26
We construct $\overline{N}$ by noting that the vector field $-(\partial/\partial U)$ is everywhere normal to $dS^4$ and points out of the bubble. This field, however, is not entirely tangent to $dS^5$. It is therefore necessary to apply the projection operator, applying the transformation

$$\frac{\partial}{\partial U} \rightarrow (I - \overline{R} \otimes \overline{R}) \left(\frac{\partial}{\partial U}\right) = -\left(\frac{\partial}{\partial U} - \overline{U} \overline{R}\right)$$

$$= -(1 - \overline{U}^2) \frac{\partial}{\partial U} - \overline{U} \left(T \frac{\partial}{\partial T} + V \frac{\partial}{\partial V} + X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + Z \frac{\partial}{\partial Z}\right)$$

(A6)

where

$$\overline{R} = T \frac{\partial}{\partial T} + U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} + X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + Z \frac{\partial}{\partial Z}$$

(A7)

is the field in $M^6$ normal to $dS^5$. $\overline{R}$ is normalized on $dS^5$. Normalizing this projected field in eqn. (A6), we obtain

$$\overline{N} = -\frac{\sqrt{1 - \overline{U}^2}}{\sqrt{1 - \overline{U}^2}} \frac{\partial}{\partial U} - \frac{\overline{U}}{\sqrt{1 - \overline{U}^2}} \left(T \frac{\partial}{\partial T} + V \frac{\partial}{\partial V} + X \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Y} + Z \frac{\partial}{\partial Z}\right)$$

(A8)

It follows that

$$\chi = \overline{N} \cdot \overline{K} = \left(U \frac{\partial}{\partial V} - V \frac{\partial}{\partial U}, -\sqrt{1 - \overline{U}^2} \frac{\partial}{\partial U} + \frac{\overline{U}}{\sqrt{1 - \overline{U}^2}} V \frac{\partial}{\partial V}\right)$$

$$= -\frac{\overline{U}^3}{\sqrt{1 - \overline{U}^2}} V$$

(A9)

corresponds to a rigid translation of the bubble in $dS^5$. In terms of the internal coordinates

$$\chi = (\text{constant}) \cdot \cosh \tau \cos \zeta$$

(A10)

which is independent of $H_{dS^5}/H_b$, proving that $m^2/H_b^2$ is independent of this ratio and thus always equal to $-4$. 

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