A Remark to the Theorem of
Le Calvez and Yoccoz

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Abstract

The theorem of Le Calvez and Yoccoz states that there are no minimal homeomorphism on the finite punctered 2-dimensional sphere $S^2$. We show that this does not hold for other surfaces. Moreover, we discuss why the fast-conjugation-method fails in the most cases to construct such homeomorphism.

1 Introduction

It is well-known which closed 2-dimensional manifolds admit minimal homeomorphism, but not for closed surfaces which are punctered. The case of the sphere was solved 1997:

**Theorem 1.1.** (Le Calvez and Yoccoz) Given a finite set $F \subset S^2$. There is no minimal homeomorphism on $S^2 - F$.

Proof: (see [CY])

One can ask further are there counterexamples for other surfaces. We show that for two surfaces the answer is yes:

**Theorem 1.2.** For any given non empty finite set $F$ of an orientable closed surface $X$ of genus equal 0 or 1, there is a minimal $C^\infty$ diffeomorphism on $X - F$.

2 Construction Of Quasi-Minimal Systems
By Minimal Flows

**Definition 2.1.** Let $(X,d)$ be a metric space and $G$ a topological group. A dynamical system $\Phi : G \times X \to X$ is called quasi-minimal if the union of its dense orbits is open. We call the non dense orbits the exceptional set.

**Theorem 2.2.** There exists a quasi-minimal flow on an orientable surface $X$ of genus $g = 1$ whose exceptional set is a point.
Proof: See section 14.4.b and section 14.6 in [KH] □

It is not clear if theorem [2.2] holds for surfaces of higher genus. The fixed point set for other quasi-minimal flows in [KH] corresponds the genus. For our examples we need an application of the following theorem:

**Theorem 2.3.** Given a real topologically transitive flow $\Phi : \mathbb{R} \times X \to X$ on a separable metric space and suppose there is no "isolated streamline". For all values of $t \in \mathbb{R}$, except a set of first category, the homeomorphisms $f = \Phi_t : X \to X$ are topologically transitive.

Proof: See [OU] □

**Corollary 2.4.** Let $X$ be a manifold of dimension $n > 1$ and $\Phi : \mathbb{R} \times X \to X$ a $C^r$ quasi-minimal flow where $r > 0$. If its induced vectorfield is bounded, then for all values of $t \in \mathbb{R}$, except a set of first category, the $C^1$ diffeomorphisms $f = \Phi_t : X \to X$ are quasi-minimal and their exceptional sets coincide with the exceptional set of the flow $\Phi$.

Proof: Take a generic $t$ such that $f = \Phi_t : X \to X$ is transitive. Choose a $x_0$ such that $O_{f, +}(x_0)$ and $O_{f, -}(x_0)$ is dense. This is a well-known fact and follows from Baire’s theorem. We show if any orbit $O_{\Phi, +}(x)$ is dense then $O_{f, +}(x)$ is dense too. Since the induced vectorfield is bounded, the orbit $O_{f, +}(x)$ must have an accumulation point on a segment of the orbit $O_{\Phi, +}(x_0)$. The accumulation point is dense, since $O_{f, +}(x_0)$ is dense and for any number $s$ we have $O_{f, +}(\Phi(s, x_0)) = \Phi(s, O_{f, +}(x_0))$, hence $O_{f, +}(x)$ is dense. The same argument works for the negative orbits. □

Proof of theorem [1.2]: Due to thereoem [2.2] we know that $X$ admits at least a quasi-minimal flow with only one exceptional point. Given a finite set $F \subset X$. We can assume that all elements of $F$ belongs to distinct orbits which are dense on both directions and the exceptional point belongs to $F$, otherwise we take a conjugation of the flow. We multiply the induced vector field $V$ of the flow with a function $0 \leq f$ that is exactly zero on $F$. The flow of $fV$ is quasi-minimal and the exceptional set is $X - F$. Now apply corollary [2.4] □

### 3 The Fast-Conjugation-Method

Fayad and Herman showed in [FH] that any compact manifold $M$ that admits a smooth free $S^1$ action must admit minimal diffeomorphisms. They used the fast-conjugation method and the theorem of Baires to proof that in the closure of the set $\{g\Phi_t g^{-1} | g \in \text{Diff}(M), t \in S^1\}$ with respect to the $C^r$ topology the minimal diffeomorphisms are generic. We can not extended this generic result to non compact manifolds or manifolds with a semi-free smooth $S^1$ action:

**Remark 3.1.** Given a continuous $S^1$ action on a manifold $M$. For any subset $C \subset S^1$ and $G \subset \text{Diff}(M)$ the closure of $F_{C, G} = \{g\Phi_t g^{-1} | g \in G, t \in C\}$ contain a generic subset $D_{C, G}$ such that for each element in $D_{C, G}$ every orbit is positive recurrent, hence any quasi-minimal homeomorphism in $D_{C, G}$ is minimal.
Proof: Take two bases \( \{U_i\} \) and \( \{V_i\} \) of the topology of \( M \) such that \( U_i \subset V_i \) and set \( R_i = \{f \in \text{Diff}(M) \mid \forall x \in U_i \exists n_x > 0 : f^{n_x} \in V_i\} \). Notice that since the \( S^1 \) action is continuous, we have \( F_{C,G} \subset R_i \). Since the space \( \text{Diff}(M) \) is a Baire space and the sets \( R_i \) are open, the set

\[
D := F_{C,G} \cap \bigcap_i R_i
\]

is generic in the closure of \( F_{C,G} \). We have that every point \( x \in M \) is positive recurrent for any \( f \in D \). Indeed, take a sequence of neighbourhoods \( V_{j(i)} \) such that \( V_{j(i)} \to x \). If \( x \) is not a periodic point, we can build at least a sequence \( n_i \to \infty \) such that \( f^{n_i}(x) \to x \), so every point is positive recurrent. If \( f \in D \) is quasi-minimal, then for every dense orbit we have \( \mathcal{O}_{f,i}(x) = \mathcal{O}_f(x) \), since each orbit is positive recurrent, thus \( f \) is a forward minimal homeomorphism on a dense open set. Due to theorem B of \( [G] \), there is no forward minimal homeomorphism on any noncompact locally compact space, thus the open set of dense orbits is compact, so the homeomorphism is forward minimal. \( \square \)

On the even-dimensional spheres \( S^{2n} \) we can find semi-free smooth \( S^1 \) actions that are free except on two fixed points. Because of the last remark and the fact that \( S^{2n} \) admits no minimal homeomorphism, we can not find a set \( F_{C,G} \) with a generic subset of quasi-minimal diffeomorphism.

References

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