On an Extension of the Brownian Bridge with Applications in Finance

Mohamed Erraoui∗ Astrid Hilbert† and Mohammed Louriki‡

Abstract

The main purpose of this paper is to extend the information-based asset-pricing framework of Brody–Hughston–Macrina to a more general set-up. We include a wider class of models for market information and in contrast to the original paper, we consider a model in which a credit risky asset is modelled in the presence of a default time. Instead of using only a Brownian bridge as a noise, we consider another important type of noise. We model the flow of information about a default bond with given random repayments at a predetermined maturity date by the so-called market information process, this process is the sum of two terms, namely the cash flow induced by the repayment at maturity and a noise, a stochastic process set up by adding a Brownian bridge with length equal to the maturity date and a drift, linear in time, multiplied by a time changed Lévy process. In this model the information concerning the random cash-flow is modelled explicitly but the default time of the company is not since the payment is contractually set to take place at maturity only. We suggest a model, in which the cash flow and the time of bankruptcy are both modelled, which covers contracts, e.g. defaultable bonds, to be paid at hit. From a theoretical point of view, this paper deals with conditions, which allow to keep the Markov property, when we replace the pinning point in the Brownian bridge by a process. For this purpose, we first study the basic mathematical properties of a bridge between two Brownian motions.

Keywords: Lévy processes, Brownian bridges, Gaussian processes, Markov processes, Bayes theorem, enlargement of filtration, stochastic filtering theory, credit risk, information-based asset pricing, market filtration.

MSC 2010: 60G15, 60G40, 60G44, 60G51, 60J25.

1 Introduction

The authors’ wish to study information processes from a mathematical as well as modelling point of view led us to extend results by Brody et al. [4], Bedini et al. [2], Erraoui et al. [14], [15], and [13], and Louriki [24] in the field of random bridges by adding transformed Lévy processes and to apply the new results in an information-based framework of credit risk modelling or pricing credit derivatives. We choose to introduce the reader to the subject by supplying the main features of the fields of applications (given above) together with a few representative references. There are two main approaches to this type of problems: the structural models and the reduced-form ones.

∗Mathematics Department, Faculty of Sciences, Chouaib Doukkali University, Route Ben Maachou, 24000, El Jadida, Morocco. E-mail: erraoui@uca.ac.ma
†Linnaeus University, Vejdesplats 7, SE-351 95 Växjö, Sweden. E-mail: astrid.hilbert@lnu.se
‡Mathematics Department, Faculty of Sciences Semalalia, Cadi Ayyad University, Boulevard Prince Moulay Abdellah, P. O. Box 2390, Marrakesh 40000, Morocco. E-mail: louriki.1992@gmail.com
The *structural approach* makes explicit assumptions about a firm’s shareholders, assets, debts and capital structure. A firm is said to default if its assets are insufficient according to some measure. In this setting, a corporate liability may be characterized as an option on the firm’s assets. A corporate liability (bond) defaults if a payment agreed upon is missing or delayed.

In the *reduced-form approach* factors or reasons are not specified. Instead the dynamics of default is exogenously given in terms of a default rate or intensity. The value of the corresponding default-free asset is multiplied by a risk-adjusted discount, respectively growth factor.

For a deeper discussion of these approaches the reader is referred to Jeanblanc and Rutkowski [21], Hughston and Turnbull [20], Bielecki and Rutkowski [10], Duffie and Singleton [11], Schönbucher [30], Bielecki et al. [9], Lando [23], and Elizalde [12].

In the *information-based approach* introduced by Brody et al. [4] the authors take means to combine the economic intuition of an information flow generated by random market factors inherent to the structural approach with the tractability and empirical fit of the reduced-form approach. The conceptual tools are so-called market information processes realized by random bridges. The paper [4] is based on modelling the flow of information accessible to market participants concerning a cash flow corresponding to a future random payment $H_T$ at maturity $T$. The flow of information about $H_T$ is modelled with the natural, completed filtration generated by the process $\alpha_T = (\alpha_T^t, t \leq T)$, defined by

\begin{equation}
\alpha_T^t = \sigma t H_T + \beta_T^t.
\end{equation}

Here the first term on the right-hand side represents the true information revealed. It is disturbed by a noise term, the process $\beta_T^t$, that models the uncertainty about the true value of $H_T$ related to random market factors and incorporates rumors, speculation, misinterpretation, overreaction, and general disinformation in connection with financial activities. By remarking that at time $T$ investors have a perfect information about the values of $H_T$, the noise process must vanish at $T$. Brody et al. suggested that the natural choice for $\beta_T^t$ is a standard Brownian bridge of length $T$. This approach aims to bring the mathematical abstraction of financial modelling at the level of the specification of market filtration. In this way, the price process can be derived as an emergent phenomenon. For a deeper discussion of such an approach we refer the reader to [4], [5], [6], [8], [19] and [27]. This paper models default at predetermined times only, where the particularities of default were encoded in the a priory distribution of the random pay-off. Bedini et al. in [2] tackle the issue of giving an explicit description of the flow of information concerning the default time of a financial company by introducing Brownian bridges with random length. The time at which the default occurs is modelled by a strictly positive random time $\tau$. They consider the information process $\beta = (\beta_t, t \geq 0)$, a Brownian bridge with random length $\tau$ and pinning point 0:

\begin{equation}
\beta_t = W_{t \wedge \tau} - \frac{t \wedge \tau}{\tau} W_{\tau}, \quad t \geq 0,
\end{equation}

where $W$ is a Brownian motion independent of $\tau$. The idea is that this process leaks information concerning the default before it occurs. Inspired by [2], Erraoui et al. have introduced and mathematically analysed Gaussian bridges, gamma bridges and Lévy bridges with random length in [13], [14] and [15] respectively. In the paper [24] by Louriki, the uncertainty has been introduced in both pinning level and length level of the Brownian bridge to model the flow of information that motivates the holder of gas storage contract to act at a random time $\tau$ by injecting or withdrawing gas.

The aim of the current paper is twofold:
Firstly, we extend the information-based asset-pricing framework of Brody–Hughston–Macrina to include a wider class of models for market information by considering another important type of noise. Instead of using only a Brownian bridge as a noise, we add a product of time and a Lévy process. In the framework of financial modelling adding Lévy processes allows for jumps and provides stylized behaviour of information and price processes. A natural question arises: which conditions, allow to keep the properties when replacing the pinning point in the bridge by a process? To provide an answer to this question we first study the basic properties of bridges between two Brownian motions for instance the Markov property and their Doob-Meyer decomposition as semi-martingales in their own filtration. These processes do not inherit the Markov property from the Brownian bridge. Inspired by the construction of diffusions via a forward and backward Markov transition semi-groups examined in [25], we replace the pinning point in the Brownian bridge by a time reversed Brownian motion. As a result, the Markov property is preserved. In addition, sufficient and necessary conditions are given under which the process derived by replacing the pinning point in a Brownian bridge by a time changed Brownian motion is Markovian. The properties of Brownian bridges disturbed by transformed and time scaled Lévy process are investigated as well. This suggests that the flow of information concerning a random cash flow $H_T$ at some pre-established date $T > 0$ can be modelled with the natural, completed filtration generated by the process $\eta_T = (\eta_t^T, t \leq T)$, defined by

$$\eta_t^T = W_t - \frac{t}{T} W_T + \frac{t}{T} X_{T-t} + \sigma t H_T,$$

where $X$ is a Lévy process.

Secondly, in contrast to the model suggested by [4], we give a mathematical model, in which a credit risky asset is modelled in the presence of a default time. We propose a model, in which a non-defaultable cash flow with an agreed single payment $H_T$ at maturity $T$ and the time of bankruptcy $\tau$ of the writer of the associated asset are both modelled with the filtration generated by the information process $\kappa_T = (\kappa_t^T, t \leq T)$, defined by

$$\kappa_t^T = W_{t \land \tau} - \frac{t \land \tau}{\tau} W_{\tau} + \mu(t \land \tau) X_{t \land \tau} + \sigma t H_T.$$

The value of $H_T$ is revealed at the random time $\tau$. Precisely speaking the cash flow is measurable at time $T$ when the contract expires, but if the counterpart files a bankruptcy earlier, then you may find out the value of $H_T$. In this sense the completed natural filtration $\mathbb{P}_{\eta^T,c}$ generated by $\eta^T$ provides partial information on the default time before it occurs and the cash flow $H_T$. Getting information about $(\tau, H_T)$ relies on the theoretical properties of this process. The Markov property is proven, the random time modelling the default time is shown to be a stopping time with respect to the completed natural filtration of the respective information process. Explicit pricing formulas are derived for all cases, and the results of the simulations are displayed.

The rest of the paper is organized as follows. In section 2, we define two processes, the first one is derived by replacing the pinning point in a bridge by a Brownian motion while the second one is obtained by replacing the pinning point by a time reversed Brownian motion. We derive their canonical decompositions. The first one is not Markovian, however, we show that the second one is Markovian but it cannot be a bridge of a centred Gaussian Markov process. This requires proof techniques for the two results. We end section 2 with a sufficient and necessary condition for the Markov property of a bridge between Brownian motion and time-modified Brownian motion.
Section 3 deals with conditions, which allow to generalize the previous results when exchanging the time changed Brownian motion by a time-changed Lévy process. Section 4 presents an extension of the Brody-Hughston-Macrina approach to modelling defaultable bonds. We introduce the market model and prove the Markov property for the associated information. An explicit expression of the price of defaultable bond is also deduced. In section 5 we suggest a model in which the defaultable bond and the time of bankruptcy are both modelled. The Markov property of the information process is proven, the random time modelling the default time is shown to be stopping time. Explicit pricing formulas are derived for all cases, and some numerical illustrations are performed.

The following notations will be used throughout the paper: For a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), \(\mathcal{N}_0^\mathbb{P}\) denotes the collection of \(\mathbb{P}\)-null sets. If \(\theta\) is a random variable, then \(\mathbb{P}_\theta\) denotes the law of \(\theta\) under \(\mathbb{P}\). \(C(\mathbb{R}_+, \mathbb{R})\) denotes the canonical space, that is the space of continuous real-valued functions defined on \(\mathbb{R}_+\), \(\mathcal{C}\) the \(\sigma\)-algebra generated by the canonical process. If \(E\) is a topological space, then the Borel \(\sigma\)-algebra over \(E\) will be denoted by \(\mathcal{B}(E)\). The characteristic function of a set \(A\) is written \(I_A\).

\[ p(t, x, y), x, y \in \mathbb{R}, t \in \mathbb{R}_+, \] denotes the Gaussian density function with variance \(t\) and mean \(y\). If \(y = 0\), for simplicity of notation we write \(p(t, x)\) rather than \(p(t, x, 0)\).

Finally for any process \(Z = (Z_t, t \geq 0)\) on \((\Omega, \mathcal{F}, \mathbb{P})\), we define by:

(i) \(\mathbb{F}^Z = \left(\mathcal{F}^Z_t := \sigma(Z_s, s \leq t), \ t \geq 0\right)\) the natural filtration of the process \(Z\).

(ii) \(\mathbb{F}^{Z,c} = \left(\mathcal{F}^{Z,c}_t := \mathcal{F}^Z_t \vee \mathcal{N}_\mathbb{P}, \ t \geq 0\right)\) the completed natural filtration of the process \(Z\).

(iii) \(\mathbb{F}^{Z,c}_+ = \left(\mathcal{F}^{Z,c}_{t+} := \bigcap_{s>t} \mathcal{F}^{Z,c}_s = \mathcal{F}^Z_t \vee \mathcal{N}_\mathbb{P}, \ t \geq 0\right)\) the smallest filtration containing \(\mathbb{F}^Z\) and satisfying the usual hypotheses of right-continuity and completeness.

2 A bridge between two Brownian motions

It is known that the sum of a Brownian bridge and a product of time and a random variable is Markovian. A classical example is the Brownian bridge with random pinning point, see e.g. [6], [16], and [24]. The main objective of this section is to see if certain properties, in particular the Markov property, remain valid when we replace the random variable in the bridge by a time-changed Brownian motion. Let \(B\) and \(W\) be two independent Brownian motions. We proceed by considering the process \(\beta^T\) given by

\[ \beta^T_t = W_t - \frac{t}{T} W_T + \frac{t}{T} B_t, \ 0 \leq t \leq T, \ T > 0 \] (2.1)

and setting

\[ \beta^T = (\beta^T_t = W_t - \frac{t}{T} W_T, t \leq T). \] (2.2)

See [17] for another transformation. We start with the observation that \(\beta^T\) is a centred Gaussian process with covariance function given by

\[ \mathbb{E}[\beta^T_s \beta^T_t] = s \wedge t - \frac{st}{T} \left(1 - \frac{s \wedge t}{T}\right). \] (2.3)
Exploiting the properties of both the Brownian motion $B$ and the Brownian bridge $\beta^T$, direct calculation reveals that

$$E[\beta^T_t | \mathcal{F}^B_s \vee \mathcal{F}^\beta_T_s] = \frac{T-t}{T-s} \beta^T_s + \frac{t-s}{T-s} B_s, \ 0 \leq s \leq t < T, \quad (2.4)$$

which implies that $\beta^T$ is not a martingale with respect to $\mathbb{F}^B \vee \mathbb{F}^\beta_T$. However, we have the following result:

**Proposition 2.1.** The process $\beta^T$ is a quasi-martingale with respect to $\mathbb{F}^\beta_T \vee \mathbb{F}^B$.

**Proof.** We need to show that:

(i) For all $0 \leq t \leq T$, $E[|\beta^T_t|] < +\infty$.

(ii) $\sup_{\mathcal{P}} \sum_{i=1}^{n-1} E \left[ E \left[ \left| \beta^T_{t_{i+1}} - \beta^T_{t_i} \right| \mathcal{F}^B_{t_i} \vee \mathcal{F}^\beta_T_{t_i} \right] \right] < +\infty$, where $\mathcal{P}$ is the set of all finite partitions $0 = t_0 < t_1 < \ldots < t_n < t_n = T$ of $[0,T]$.

As regards statement (i), it is easy to see that $E[|\beta^T_t|] = \sqrt{\frac{2t}{\pi T}} \left( T - t + \frac{t^2}{T} \right) < +\infty$. Concerning the statement (ii), we have for $1 \leq i \leq n$

$$E \left[ \beta^T_{t_{i+1}} - \beta^T_{t_i} \right| \mathcal{F}^B_{t_i} \vee \mathcal{F}^\beta_T_{t_i} \right] = E \left[ \beta^T_{t_{i+1}} + \frac{t_{i+1} - t_i}{T} (B_{t_{i+1}} - B_{t_i}) \mathcal{F}^\beta_T_{t_i} \right] - \beta^T_{t_i} + \frac{t_{i+1} - t_i}{T} B_{t_i}.$$

Hence, we find

$$E \left[ \left| E \left[ \beta^T_{t_{i+1}} - \beta^T_{t_i} \right| \mathcal{F}^B_{t_i} \vee \mathcal{F}^\beta_T_{t_i} \right] \right] = \sqrt{\frac{2}{\pi}} \left( t_{i+1} - t_i \right) \sqrt{\frac{t_i (2T - t_i)}{T^2 (T - t_i)}} \leq \frac{2}{\sqrt{\pi}} \frac{t_{i+1} - t_i}{\sqrt{T - t_i}}. \quad (2.5)$$

It follows from (2.5) that

$$\sum_{i=1}^{n-1} E \left[ \left| E \left[ \beta^T_{t_{i+1}} - \beta^T_{t_i} \right| \mathcal{F}^B_{t_i} \vee \mathcal{F}^\beta_T_{t_i} \right] \right] \leq \frac{2}{\sqrt{\pi}} \sum_{i=1}^{n-1} \frac{t_{i+1} - t_i}{\sqrt{T - t_i}}. \quad (2.6)$$

The right hand side in (2.6) represents exactly the lower Darboux sum for the function $\sqrt{\frac{2}{\pi} (T - x)}$ on the interval $[0,T]$. Thus,

$$\sup_{\mathcal{P}} \sum_{i=1}^{n-1} E \left[ \left| E \left[ \beta^T_{t_{i+1}} - \beta^T_{t_i} \right| \mathcal{F}^B_{t_i} \vee \mathcal{F}^\beta_T_{t_i} \right] \right] \leq \frac{2}{\sqrt{\pi}} \sup_{\mathcal{P}} \sum_{i=1}^{n-1} \frac{t_{i+1} - t_i}{\sqrt{T - t_i}} \leq 4 \sqrt{\frac{T}{\pi}} < \infty, \quad (2.7)$$

which is the desired result. \qed
The process $\tilde{\beta}^T$ is not an $F^\tilde{\beta}$-Brownian motion, since its covariance function differs from $t \wedge s$. But it is a semi-martingale with respect to its natural filtration.

**Corollary 2.2.** The process $\tilde{\beta}^T$ is an $F^\tilde{\beta}$-semi-martingale.

**Proof.** Combining Theorem 18 of [26], Ch.III, and Proposition 2.1, we conclude that $\tilde{\beta}^T$ is an $F^{\beta} \vee F^B$-semi-martingale. Hence, the desired result is an immediate consequence of Stricker’s Theorem see, e.g., [26, Ch.II, Theorem 4].

Since $\tilde{\beta}^T$ is an $F^\tilde{\beta}$-semi-martingale, a natural question arises: what is the explicit form of its canonical decomposition? In the following proposition we provide an answer to this question.

**Proposition 2.3.** The canonical decomposition of $\tilde{\beta}^T$ in its natural filtration $F^\tilde{\beta}$ is given by

$$\tilde{\beta}^T_t = \int_0^t \sqrt{\frac{T^2 + s^2}{T^2}}dB_s - \int_0^t \frac{\tilde{\beta}^T_s}{T-s}ds + \int_0^t \int_0^s \frac{a(s,u)}{T-s}d\tilde{\beta}^T_u ds, \ t < T, \quad (2.8)$$

where, for $0 \leq u \leq s < T$,

$$a(s,u) = \frac{T-s}{T-s+s \tan^{-1}(\frac{s}{T})} \left( \frac{Tu}{T^2 + u^2} + \tan^{-1}\left(\frac{u}{T}\right) \right) + \frac{s \tan^{-1}(\frac{s}{T})}{T-s+s \tan^{-1}(\frac{s}{T})}.$$

**Proof.** Direct calculation reveals that the process, given by

$$\tilde{M}^T_t = \tilde{\beta}^T_t - \int_0^t \frac{B_s - \tilde{\beta}^T_s}{T-s}ds, \ t < T,$$

is a Gaussian martingale with respect to $F^{\beta} \vee F^B$. On the other hand for all $t > 0$, we have

$$E \left[ \int_0^t \left| \frac{\tilde{\beta}^T_s - B_s}{T-s} \right| ds \right] \leq \int_0^t \left( \frac{E[|\tilde{\beta}^T_s|]}{T-s} + \frac{E[|B_s|]}{T-s} \right) ds = \sqrt{\frac{2}{T\pi}} \int_0^t \left( \sqrt{\frac{s}{T-s}} + \sqrt{\frac{s}{T}} \right) ds < \infty.$$

Hence, by a well known result of filtering theory, see for instance Theorem 8.1.1 and Remark 8.1.1 in [22] or Proposition 2.30, p. 33 in [1], it follows that the process

$$\tilde{m}_t := \tilde{\beta}^T_t + \int_0^t \frac{\tilde{\beta}^T_s}{T-s}ds - \int_0^t \frac{E[B_s|F^\tilde{\beta}^T_t]}{T-s}ds, \ t < T \quad (2.9)$$

is an $F^{\tilde{\beta}}$-martingale, which is Gaussian by construction. Due to (2.9) and Theorem 9.4.1 in [22] we may assume that

$$E[B_t|F^\tilde{\beta}^T_t] = \int_0^t a(t,u)d\tilde{\beta}^T_u. \quad (2.10)$$

Therefore, we have only to derive the explicit form of the deterministic function $a$. Using the projection property of the conditional expectation, we have for all $0 \leq s \leq t < 1$,

$$E[\tilde{\beta}^T_s (B_t - E[B_t|F^\tilde{\beta}^T_t])] = 0.$$

Hence, we have

$$E[\tilde{\beta}^T_s B_t] - E[\tilde{\beta}^T_s \tilde{\beta}^T_t] a(t,t) + \int_0^t E[\tilde{\beta}^T_s \tilde{\beta}^T_u] \frac{\partial a(t,u)}{\partial u}du = 0.$$
Using (2.3), we get
\[
\frac{s^2}{T} - a(t,t) \left[ s - \frac{st}{T} + \frac{s^2 t}{T^2} \right] + \int_0^s \left[ u - \frac{us}{T} + \frac{u^2 s}{T^2} \right] \frac{\partial a(t,u)}{\partial u} du + \int_t^s \left[ s - \frac{su}{T} + \frac{s^2 u}{T^2} \right] \frac{\partial a(t,u)}{\partial u} du = 0.
\]
By taking the first derivatives with respect to \( s \) we obtain
\[
\frac{2s}{T} - a(t,t) \left[ 1 - \frac{t}{T} + 2 \frac{st}{T^2} \right] + \frac{\partial a(t,s)}{\partial s} + \int_0^s \left[ u - \frac{us}{T} + \frac{u^2 s}{T^2} \right] \frac{\partial a(t,u)}{\partial u} du + \frac{\partial a(t,s)}{\partial s} = 0. \tag{2.11}
\]
Taking further derivatives with respect to \( s \) we obtain
\[
(T^2 + s^2) \frac{\partial a(t,s)}{\partial s} + 2s a(t,s) + 2 \int_s^t a(t,u) du - 2T = 0, \tag{2.12}
\]
and
\[
4s \frac{\partial a(t,s)}{\partial s} + (T^2 + s^2) \frac{\partial^2 a(t,s)}{\partial^2 s} = 0. \tag{2.13}
\]
This implies that \( a(t,s) \) is of the form
\[
a(t,s) = c(t) \left[ \frac{s}{T^2(s^2 + T^2)} + \frac{\tan^{-1}(\frac{s}{T})}{T^3} \right] + d(t). \tag{2.14}
\]
Substituting (2.14) in (2.11) and (2.12), and taking \( s = t \) and \( s = 0 \), we obtain
\[
c(t) = T^3 \frac{T - t}{T - t + t \tan^{-1}(\frac{t}{T})},
\]
and
\[
d(t) = \frac{t \tan^{-1}(\frac{t}{T})}{T - t + t \tan^{-1}(\frac{t}{T})}.
\]
Finally, since \( \bar{m} \) is a Gaussian martingale of finite quadratic variation such that \( \langle \bar{m} \rangle_t = t + \frac{t^3}{3T^2} \), \( \bar{m} \) can be represented as
\[
\bar{m}_t = \int_0^t \sqrt{\frac{T^2 + s^2}{T^2}} d\bar{B}_s,
\]
where \( \bar{B} \) is an \( \mathbb{F}^{\bar{\beta}T} \)-Brownian motion. This completes the proof.

Remark 2.4. As an immediate consequence of (2.4) and (2.10), for \( s \leq t < T \), we can calculate the conditional expectation of \( \bar{\beta}_t \) given \( \mathcal{F}_s^{\beta T} \) by
\[
\mathbb{E}[\bar{\beta}_t | \mathcal{F}_s^{\beta T}] = \frac{T - t}{T - s} \bar{\beta}_s + \frac{t - s}{T - s} \int_0^s a(s,u) d\bar{\beta}_u^T,
\]
where the function \( a \) is defined above by the equation (2.3).
We observe that the drift diffusion coefficient in (2.8) depends on the trajectory of the process $\beta^T$ in a non-anticipative way. Consequently, $\beta^T$ is not an $F^{\beta^T}$-Markov process. Therefore, we come to the conclusion that we lose the Markov property, when we replace the pinning point in the bridge by a Brownian motion. Since the Markov property is one of the important tools in the information-based asset pricing framework, the process $\beta^T$ is not a good candidate for representing the noise in such a model. Inspired by the construction of diffusions via a forward and backward Markov transition semi-groups examined in [25], we consider the second process of this section, $\tilde{\beta}^T = (\tilde{\beta}^T_t, 0 \leq t \leq T)$ defined by

$$\tilde{\beta}^T_t = W_t - \frac{t}{T} W_T + \frac{t}{T} B_{T-t}, t \leq T. \quad (2.15)$$

**Proposition 2.5.** The process $\tilde{\beta}^T$ is an $F^{\tilde{\beta}^T}$-Markov process.

**Proof.** We observe that $\tilde{\beta}^T$ is a centred Gaussian process with covariance function given by

$$E[\tilde{\beta}^T_s \tilde{\beta}^T_t] = (s \wedge t) \frac{T^2 - (t \wedge s)^2}{T^2}. \quad (2.16)$$

By a direct calculation we verify that for all $0 \leq s \leq t \leq u \leq T$,

$$E[\tilde{\beta}^T_s \tilde{\beta}^T_t]E[\tilde{\beta}^T_u] = E[\tilde{\beta}^T_t \tilde{\beta}^T_u]E[\tilde{\beta}^T_s \tilde{\beta}^T_t].$$

By a well-known equivalent between this latter and the Markov property (see e.g. Ch.III p. 86 in [28]) we conclude that $\tilde{\beta}^T$ is a Markov process with respect to its natural filtration. \hfill \Box

By exploiting the Markov and Gaussian properties of the process $\tilde{\beta}^T$ we obtain:

**Proposition 2.6.** The canonical decomposition of $\tilde{\beta}^T$ in its natural filtration $F^{\tilde{\beta}^T}$ is given by

$$\tilde{\beta}^T_t = \int_0^t \frac{\sqrt{T^2 + s^2}}{T} dB^*_s - 2 \int_0^t \frac{s \tilde{\beta}^T_s}{T^2 - s^2} ds, \ 0 \leq t < T, \quad (2.17)$$

where $(\tilde{B}_t, 0 \leq t \leq T)$ is an $F^{\tilde{\beta}^T}$-Brownian motion.

**Proof.** Let us denote by $\tilde{M}^T$ the process given by

$$\tilde{M}^T_t = \tilde{\beta}^T_t + 2 \int_0^t \frac{s \tilde{\beta}^T_s}{T^2 - s^2} ds, \ t < T. \quad (2.18)$$

It is a matter of direct calculation to see that $\tilde{M}^T$ is an $F^{\tilde{\beta}^T}$-adapted integrable process. Furthermore, using the fact that $\tilde{\beta}^T$ is Markovian we have for any $s < t < T$,

$$E[\tilde{M}^T_t - \tilde{M}^T_s | F^T_s] = E[\tilde{\beta}^T_T - \tilde{\beta}^T_s | F^T_s] + 2 \int_s^t \frac{u}{T^2 - u^2} E[\tilde{\beta}^T_u | \tilde{\beta}^T_s] du = \frac{T^2 - t^2}{T^2 - s^2} \tilde{\beta}^T_s - \tilde{\beta}^T_t + 2 \frac{\tilde{\beta}^T_s}{T^2 - s^2} \int_s^t u du = 0,$$
where in the second equality we have used the fact that for any \( s < t < T \),
\[
\mathbb{E}[\beta_t^T | \beta_s^T] = \frac{\mathbb{E}[\beta_t^T \beta_s^T]}{\mathbb{E}[(\beta_t^T)^2]} \beta_s^T = \frac{T^2 - t^2}{T^2 - s^2} \beta_s^T.
\]
Consequently, \( \tilde{M}^T \) is a Gaussian martingale. Since \( \tilde{\beta}^T \) is of finite quadratic variation such that
\[
\langle \tilde{\beta}^T \rangle_t = t + \frac{t^3}{3T^2},
\]
the equality (2.18) can be rewritten as
\[
\tilde{\beta}^T_t = \int_0^t \frac{\sqrt{T^2 + s^2}}{T} d\tilde{B}_s - 2 \int_0^t \frac{s \tilde{\beta}_s^T}{T^2 - s^2} ds,
\]
where \( \tilde{B} \) is a Brownian motion with respect to \( \mathbb{F}^{\tilde{\beta}^T} \), which is the desired result. \( \square \)

**Remark 2.7.** The explicit solution of (2.17) is given by
\[
\tilde{\beta}^T_t = T^2 - t^2 \int_0^t \frac{\sqrt{T^2 + s^2}}{T} d\tilde{B}_s - 2 \int_0^t \frac{s \tilde{\beta}_s^T}{T^2 - s^2} ds, \quad t < T.
\] (2.19)

Since \( \tilde{\beta}_0^T = 0 \) and \( \tilde{\beta}_T^T = 0 \), the first thing we think about is that the process \( \tilde{\beta}^T \) may be a Gaussian-Markov bridge of length \( T \). However, in the following proposition, we show that this not the case.

**Proposition 2.8.** The process \( \tilde{\beta}^T \) is not a bridge of length \( T \) of a centred Gaussian Markov process. That is, there exists no centred Gaussian Markov process \( Y \) such as for every non-negative measurable function \( f \) and for any \( t \leq T \),
\[
\mathbb{E}[f(Y_t) | Y_T = 0] = \mathbb{E}[f(\tilde{\beta}_t^T)].
\] (2.20)

**Proof.** Assume that there exists a centred Gaussian Markov process \( Y \) such that (2.20) holds. Computing explicitly the left-hand side in (2.20), we obtain
\[
\mathbb{E}[f(Y_t) | Y_T = 0] = \int_{\mathbb{R}} f(y) \frac{p(\sigma_Y^2(T - t), y)p(\sigma_Y^2(t), y)}{p(\sigma_Y^2(T), 0)} dy
\]
\[
= \sqrt{\frac{\sigma_Y^2(T)}{\sigma_Y^2(T - t)}} \int_{\mathbb{R}} f(y) \exp \left( - \frac{y^2}{2\sigma_Y^2(T - t)} \right) p(\sigma_Y(t), y) dy,
\]
where \( \sigma_Y^2 \) is the variance of \( Y \). Hence, for \( f = 1 \) the equation (2.20) implies that
\[
\sigma_Y^2(T) = \sigma_Y^2(T - t) + \sigma_Y^2(t),
\] (2.21)
however, for \( f(x) = \exp \left( \frac{x^2}{2\sigma_Y^2(T - t)} \right) \) the equation (2.20) yields
\[
\frac{\sigma_Y^2(T - t) - \sigma_Y^2(t)}{\sigma_Y^2(T - t)} = \frac{\sigma_Y^2(T)}{\sigma_Y^2(T - t)},
\] (2.22)
where \( \sigma^2_{\tilde{\beta}^T} \) is the variance of \( \tilde{\beta}^T \). Combining (2.21) with (2.22) we obtain

\[
\sigma^2_{\tilde{\beta}^T}(t) = \frac{\sigma^2_Y(t)\sigma^2_Y(T-t)}{\sigma^2_Y(T)}. \tag{2.23}
\]

Thus,

\[
\sigma^2_{\tilde{\beta}^T}(t) = \sigma^2_{\tilde{\beta}^T}(T-t). \tag{2.24}
\]

Using that fact that \( \sigma^2_{\tilde{\beta}^T}(t) = tT^2 - t^2T^2 \), we conclude that this equality holds if and only if \( t = \frac{T}{2} \).

From the discussion above we see that the Markov property is not preserved by replacing the pinning point in the bridge with a Brownian motion. However, the process derived by replacing the pinning point by a time reversed Brownian motion is Markovian. The following theorem extends Proposition 2.5.

**Theorem 2.9.** Let \( B \) be a Brownian motion independent of a Brownian bridge \( \beta^T \), \( \psi \) be a non-constant function from \([0,T]\) into \( \mathbb{R}_+ \), and \( \sigma \in \mathbb{R} \). We introduce the process \( \hat{\beta}^T = (\hat{\beta}^T_t, 0 \leq t \leq T) \) defined by:

\[
\hat{\beta}^T_t = \beta^T_t + \sigma t B_{\psi(t)}, \quad t \in [0,T]. \tag{2.25}
\]

Then, we have

(i) If \( \psi \) is non-increasing on \([0,T]\), then \( \hat{\beta}^T \) is a Markov process with respect to its natural filtration.

(ii) If \( \psi \) is non-decreasing on \([0,T]\), then \( \hat{\beta}^T \) cannot be a Markov process with respect to its natural filtration.

**Proof.** Since \( \hat{\beta}^T \) is a centred Gaussian process, according to a well-known characterization of the Markov property in the Gaussian setting (see e.g. Ch.III p. 86 in [28]), \( \hat{\beta}^T \) is a Markov process if and only if for every \( s < t < u \leq T \), we have

\[
\mathbb{E}[\hat{\beta}^T_s \hat{\beta}^T_t] = \mathbb{E}[\hat{\beta}^T_s] \mathbb{E}[\hat{\beta}^T_t]. \tag{2.26}
\]

(i) Using the independence of \( \beta^T \) and \( B \), we have for all \( s,t \in [0,T] \)

\[
\mathbb{E}[\hat{\beta}^T_s \hat{\beta}^T_t] = (s \wedge t - \frac{st}{T}) + \sigma^2 st \psi(t \vee s). \tag{2.27}
\]

An easy computation shows that (2.26) holds for every \( s < t < u \leq T \), which implies that \( \hat{\beta}^T \) is an \( \mathbb{F}^{\hat{\beta}^T} \)-Markov process.

(ii) It is sufficient to prove that there exist \( s,t \) and \( u \) belonging to \([0,T]\) such that (2.26) does not hold. Let \( s,t \in (0,T) \) such that \( s < t \), \( \psi(s) < \psi(t) \), and \( u = T \). It is a simple matter to check that (2.26) holds if \( \psi(s) = \psi(t) \), this contradicts our assumption. Then, \( \hat{\beta}^T \) cannot be a Markov process with respect to its natural filtration.

The proof of the proposition is finished. \( \square \)
We have this following nice characterization in case the function $\psi$ is continuous on $[0, T]$.

**Corollary 2.10.** Suppose that $\psi$ is a continuous non-constant function from $[0, T]$ into $\mathbb{R}_+$. Then the process $\hat{\beta}^T$ is Markovian if and only if $\psi$ is non-increasing.

**Proof.** If the function $\psi$ is non-increasing, then by Theorem 2.9 the process $\hat{\beta}^T$ is a Markov process with respect to its natural filtration. Conversely, suppose that $\hat{\beta}^T$ is Markovian and let us show that $\psi$ is non-increasing. Assume that there exist $s < u$ such that $\psi(s) < \psi(u)$. Since, $\psi$ is continuous, there exist $t \in (s, u)$ such that $\psi(s) < \psi(t) < \psi(u)$. By an easy computation we show that (2.26) holds if and only if $\psi(s) = \psi(t)$, which contradicts the fact that $\psi(s) < \psi(t)$. Thus, $\psi$ is non-increasing.

**Remark 2.11.** From the proof of Corollary 2.10 it is clear that what is needed to have the equivalent is the fact that $\psi$ has the "intermediate value property". Therefore, the equivalent remains valid for discontinuous Darboux functions.

### 3 A bridge between a Brownian motion and a Lévy process

Now we are interested in finding non-Gaussian and not necessarily continuous processes for which if you multiply them by a time and you add a Brownian bridge you obtain a process that possesses the Markov property. The next result generalizes Theorem 2.9.

**Theorem 3.1.** Let $X$ be a right-continuous process having independent increments, $W$ be a Brownian motion independent of $X$, $\psi : [0, T] \rightarrow \mathbb{R}_+$ be a non-increasing function, and $\sigma \in \mathbb{R}$. Then the process $\xi^T = (\xi^T_t, 0 \leq t \leq T)$ defined by:

$$
\xi^T_t = W_t - \frac{t}{T} W_T + \sigma t X_{\psi(t)}, \quad t \in [0, T],
$$

(3.1)
is an $\mathbb{F}^{\xi^T}$-Markov process, that is, for all $0 \leq t < u \leq T$ and for every $x \in \mathbb{R}$, we have, $\mathbb{P}$-a.s.,

$$
\mathbb{P}[\xi^T_u \leq x | \mathcal{F}^\xi_t] = \mathbb{P}[\xi^T_t \leq x | \mathcal{F}^\xi_t].
$$

(3.2)

**Proof.** Since $\xi^T_0 = 0$ almost surely one directly checks that (3.2) holds for $t = 0$.

Now let us assume that $t > 0$. Using Theorem 1.3 in Blumenthal and Getoor [7], it is sufficient to prove that for each finite collection $0 < t_0 < t_1 \leq \cdots \leq t_n = t < u \leq T$ and for every $x \in \mathbb{R}$ one has

$$
\mathbb{P}[\xi^T_u \leq x | \xi^T_{t_0}, \ldots, \xi^T_{t_1}] = \mathbb{P}[\xi^T_t \leq x | \xi^T_{t_0}].
$$

(3.3)

First, we claim that for all $n \in \mathbb{N}$, all $(x_1, \cdots, x_n) \in \mathbb{R}^n$, and all $0 < t_1 < \cdots < t_n < u \leq T$,

$$
\mathbb{P}(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n | \xi^T_{t_0} = x_n, \ldots, \xi^T_{t_1} = x_1) = \mathbb{P}(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n | \xi^T_{t_0} = x_n).
$$

(3.4)

Indeed, setting

$$
A_k = \frac{W_{t_k} - W_{t_{k-1}}}{t_k - t_{k-1}}, \quad B_k = \sigma (X_{\psi(t_{k-1})} - X_{\psi(t_k)}), \quad y_k = \frac{x_k}{t_k} - \frac{x_{k-1}}{t_{k-1}}, \quad k = 2, \ldots, n,
$$
we have
\[ P(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n | \xi^T_{t_n} = x_n, \ldots, \xi^T_{t_1} = x_1) = P(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n | \xi^T_{t_n} = x_n, \xi^T_{t_{n-1}} = y_n, \ldots, \xi^T_{t_2} = y_2, \xi^T_{t_1} = y_1) = P(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n | \xi^T_{t_n} = x_n, A_n - B_n = y_n, \ldots, A_2 - B_2 = y_2). \] (3.5)

Since the vector \((A_2, \ldots, A_n, \beta^T_{t_n})\) is Gaussian, where \(\beta^T\) is the Brownian bridge of length \(T\) associated with \(W\), it is easy to check that \((A_2, \ldots, A_n)\) and \(\beta^T_{t_n}\) are independent. On the other hand, using that the process \(X\) has independent increments, it is easy to see that the vectors \((B_2, \ldots, B_n)\) and \((X_{\psi(t_n)}, X_{\psi(u)})\) are independent. From what has already been proved and the fact that the Brownian motion \(W\), and the process \(X\) are assumed to be independent we conclude that the vectors \((A_n - B^T_n, \ldots, A_2 - B_2)\) and \((\xi^T_{t_n}, X_{\psi(t_n)}, X_{\psi(u)})\) are independent. Thus, we have (3.4). On the other hand, we have
\[ P[\xi^T_u \leq x | \xi^T_{t_n} = x_n, \ldots, \xi^T_{t_1} = x_1] = \int_{\mathbb{R}^{n+1}} P[\xi^T_u \leq x | \xi^T_{t_n} = x_n, \ldots, \xi^T_{t_1} = x_1, X_{\psi(u)} = z, X_{\psi(t_n)} = z_n, \ldots, X_{\psi(t_1)} = z_1] \times P(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n, \ldots, X_{\psi(t_1)} \in dz_1 | \xi^T_{t_n} = x_n, \ldots, \xi^T_{t_1} = x_1). \] (3.6)

Using the fact that the process \(\beta^T = (\beta^T_t, 0 \leq t \leq T)\) is a Markov process with respect to its natural filtration and the fact that \(\alpha^T\) and \(X\) are assumed to be independent we obtain:
\[ P[\xi^T_u \leq x | \xi^T_{t_n} = x_n, \ldots, \xi^T_{t_1} = x_1, X_{\psi(u)} = z, X_{\psi(t_n)} = z_n, \ldots, X_{\psi(t_1)} = z_1] = P[\beta^T_u \leq x - \frac{u}{T} z | \beta^T_{t_n} = x_n - \frac{t_n}{T} z_n, \ldots, \beta^T_{t_1} = x_1 - \frac{t_1}{T} z_1, X_{\psi(u)} = z, X_{\psi(t_n)} = z_n, \ldots, X_{\psi(t_1)} = z_1] = P[\beta^T_u \leq x - \frac{u}{T} z | \beta^T_{t_n} = x_n - \frac{t_n}{T} z_n, \ldots, \beta^T_{t_1} = x_1 - \frac{t_1}{T} z_1] = P[\beta^T_u \leq x - \frac{u}{T} z | \beta^T_{t_n} = x_n - \frac{t_n}{T} z_n]. \] (3.7)

On the other hand, it follows from (3.4) that for every bounded measurable function \(g\) on \(\mathbb{R}^2\) we have
\[ \int_{\mathbb{R}^{n+1}} g(z, z_n) P(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n, \ldots, X_{\psi(t_1)} \in dz_1 | \xi^T_{t_n} = x_n, \ldots, \xi^T_{t_1} = x_1) = \int_{\mathbb{R}^2} g(z, z_n) P(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n | \xi^T_{t_n} = x_n, \ldots, \xi^T_{t_1} = x_1) = \int_{\mathbb{R}^2} g(z, z_n) P(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n | \xi^T_{t_n} = x_n). \] (3.8)

Inserting (3.7) and (3.8) into (3.6), we obtain
\[ P[\xi^T_u \leq x | \xi^T_{t_n} = x_n, \ldots, \xi^T_{t_1} = x_1] = \int_{\mathbb{R}^2} P[\beta^T_u \leq x - \frac{u}{T} z | \beta^T_{t_n} = x_n - \frac{t_n}{T} z_n] P(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n | \xi^T_{t_n} = x_n) = \int_{\mathbb{R}^2} P[\xi^T_u \leq x | \xi^T_{t_n} = x_n, X_{\psi(u)} = z, X_{\psi(t_n)} = z_n] P(X_{\psi(u)} \in dz, X_{\psi(t_n)} \in dz_n | \xi^T_{t_n} = x_n) = P[\xi^T_u \leq x | \xi^T_{t_n} = x_n]. \]
This ends the proof.

In the sequel we give an explicit expression of the transition density associated with the Markov process $X^T$ in case $X$ is a Lévy process and $\psi(t) = T - t$. For basic definitions and properties of Lévy properties we refer to [3] and [29].

**Theorem 3.2.** Let $T \in (0, \infty)$, $W$ a Brownian motion, $X$ a Lévy process with law $p_t^X(dx) = \mathbb{P}(X_t \in dx)$, for every $t > 0$. Assume that $W$ and $X$ are independent. The process $\zeta^T = (\zeta_t^T, 0 \leq t \leq T)$, defined by

$$\zeta_t^T = W_t - \frac{t}{T} W_T + \frac{t}{T} X_{T-t}, \quad t \in [0, T],$$

is an $\mathbb{R}^{c_T}$-Markov process. Moreover, its transition density is given by:

$$\mathbb{P}_{\zeta_{u}^T | \zeta_{t}^T=x}(dy) = \Psi_{t,u,T}(x,y)dy,$$

where,

$$\Psi_{t,u,T}(x,y) = \frac{\int \int p\left(\frac{t-u}{T-t} u - t, y - u \frac{y_2}{T} x - t \frac{y_2}{T} y_1 - t \frac{T}{2} y_2\right)p\left(\frac{t(T-t)}{T}, x, t \frac{T}{2} y_1 + t \frac{T}{2} y_2\right) p_{u-t}(dy_1)p_{T-u-t}(dy_2)}{\int p\left(\frac{t(T-t)}{T}, x, t \frac{T}{2} y_1\right)p_{T-u-t}(dy_1)}.$$  

(3.10)

**Proof.** Since $X$ has independent increments, it follows from Theorem 3.1 that the process $\zeta^T$ is a Markov process with respect to its natural filtration. It remains to determine its transition densities. For every bounded continuous function $g$ we have

$$\mathbb{E}[g(\zeta_{u}^T) | \zeta_{t}^T=x] = \int_{\mathbb{R}^2} \mathbb{E}[g(\zeta_{t}^T=x) | \zeta_{t}^T=x, X_{T-t} = y_1, X_{T-u} = y_2] \mathbb{P}(X_{T-t} \in dy_1, X_{T-u} \in dy_2 | \zeta_{t}^T=x)$$

$$= \int_{\mathbb{R}^2} \mathbb{E}[g(\beta_{u}^T + u \frac{y_2}{T} y_1) | \beta_{t}^T=x - \frac{t}{T} y_1] \mathbb{P}(X_{T-t} \in dy_1, X_{T-u} \in dy_2 | \zeta_{t}^T=x).$$  

(3.11)

On the other hand, using Bayes theorem, for every measurable function $g$ such that $g(X_{T-t}, X_{T-u})$ is integrable, we have

$$\mathbb{E}[g(X_{T-t}, X_{T-u}) | \zeta_{t}^T=x] = \int_{\mathbb{R}^2} g(y_1 + y_2, y_2)p\left(\frac{t(T-t)}{T}, x, t \frac{T}{2} y_1 + t \frac{T}{2} y_2\right) p_{u-t}(dy_1)p_{T-u-t}(dy_2)$$

$$\frac{\int p\left(\frac{t(T-t)}{T}, x, t \frac{T}{2} y_1\right)p_{T-u-t}(dy_1)}{\int p\left(\frac{t(T-t)}{T}, x, t \frac{T}{2} y_1\right)p_{T-u-t}(dy_1)}.$$  

Combining all this leads to the formula (3.10).
4 Extension of Brody-Hughston-Macrina Model

This section is based on a paper by Brody L.P. Hughston and A. Macrina [4]. The authors introduced a new reduced form approach to credit risk modelling that avoids the use of inaccessible stopping times. The new conceptual tool proposed is the so-called market information process. They model a single isolated cash flow represented by the random single dividend $H_T$ paid at a pre-described maturity time $T$ that constitutes the underlying default-free deterministic interest rate system and generalize to cash flows with several random payments, in case of a bond the coupon payments and the principle or face value. The current paper stays with the fundamental first case. Default is associated with the failure of the first agreed payment to be executed at the required time, which implies the abandoning of inaccessible stopping times. They assume that the à priori distribution of $H_T$ exists from the beginning and may represent “the best estimate for the distribution” given the data available initially. Brody et al. model the flow of partial information at an intermediate time $t \leq T$ to market participants about impending debt payments, that depends on random market factors and incorporates rumors, speculation, misinterpretation, overreaction, and general disinformation in connection with financial activities, with the process $\xi^T$, which is defined by the expression

$$\alpha^T_t = \sigma t H_T + \beta^T_t, \quad 0 \leq t \leq T.$$  

Here the first term on the right-hand side represents the true information revealed, disturbed by a noise term, the process $\beta^T$, that models the uncertainty about the true value of $H_T$ related to random market factors. By remarking that at time $T$ investors have a perfect information about the values of $H_T$, the noise process must vanish at $T$. Brody et al. suggested that the natural choice for $\beta^T$ is a standard Brownian bridge of length $T$. They assumed the absence of arbitrage, the existence of a pricing kernel and that the market filtration is generated by the information process. Under these conditions the price process $B^T = (B^T_t, 0 \leq t \leq T)$ for a default bond with payout $H_T$ is given by

$$B^T_t = P^T_t \mathbb{E}_P[H_T | \mathcal{F}^0_t],$$

where $\mathbb{P}$ is a risk-neutral measure and $P^T_t = (P^T_t, 0 \leq t \leq T)$ is the price of the associated default-free bond process which is assumed to be deterministic.
In contrast to the original paper, we consider the noise process $\zeta^T = (\zeta_t^T, 0 \leq t \leq T)$ studied in Theorem 3.2, i.e. the sum of a Brownian bridge and a transformed Lévy process of the form

$$\zeta_t^T = \beta_t^T + \frac{t}{T}X_{T-t}, \quad (4.3)$$

where a Lévy process $X$ has been time reversed, scaled and added to the Brownian bridge $\beta^T$ of length $T$. This means that the noise process $\zeta^T$ can be discontinuous. In addition, the terminal value $H_T$ of the singular cash flow could follow a discrete or a continuous distribution. The benefit of using Lévy processes is that they take into account jumps and the stylized characteristics of the market. The Markov property of $\zeta^T$ has been studied in Theorem 3.2. Following Brody et al. [4], our model is determined by the following postulates:

(i) A random variable $H_T$, a Brownian motion $W$, and a Lévy process $X$ are given on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and they are assumed to be independent.

(ii) The filtration $\mathcal{F} = \mathcal{F}_{T} = (\mathcal{F}_t^T)_{0 \leq t \leq T}$ is generated by the market information process $\eta^T$, given by the formula

$$\eta_t^T = \sigma t H_T + \zeta_t^T, \quad t \in [0, T], \quad (4.4)$$

where the process $\zeta^T$ is given by (4.3).

(iii) The short term interest rate $r$ is deterministic. Hence the value $P_t^T$ at time $t$ of a default-free bond maturing at time $T$ equals

$$P_t^T = \exp \left( - \int_t^T r(u)du \right), \quad t \in [0, T]. \quad (4.5)$$

(iv) The probability measure $\mathbb{P}$ is the risk-neutral probability, in the sense, that the price $B_t^T$ of a default bond with maturity $T$ equals

$$B_t^T = P_t^T \mathbb{E}[H_T | \mathcal{F}_t^T]. \quad (4.6)$$

4.1 Markov Property of the Market Information Process

The Markov property is a fundamental property in probability theory and statistics. It is often assumed in economic and financial modelling. In the following result we prove that the Market information process is Markovian even if the noise process $\zeta^T$ depends on a Lévy process. This property will help us to give the explicit expression of the price $B_t^T$ of a default bond with maturity $T$.

Lemma 4.1. The market information process $\eta^T$ in (4.4) is an $\mathcal{F}_t^T$-Markov process. That is, for every $0 \leq t \leq u \leq T$ and $x \in \mathbb{R}$, we have

$$\mathbb{P}(\eta_u^T \leq x | \mathcal{F}_t^T) = \mathbb{P}(\eta_u^T \leq x | \eta_t^T). \quad (4.7)$$
Proof. We will resume the same steps as of the proof of Theorem 3.1. Since $\eta_0^T = 0$ almost surely, it is easy to see that (4.7) is satisfied for $t = 0$.

Now let us assume that $t > 0$. Using Theorem 1.3 in Blumenthal and Getoor [7], it is sufficient to prove that for each finite collection $0 < t_0 < t_1 \leq \cdots \leq t_n = t < u$ and for every $x \in \mathbb{R}$ one has

$$
\mathbb{P}[\eta_u^T \leq x | \eta_{t_n}, \ldots, \eta_{t_0}] = \mathbb{P}[\eta_u^T \leq x | \eta_{t_n}^T],
$$

(4.8)
on the other hand,

$$
\mathbb{E}[\eta_u^T \leq x | \eta_{t_n}^T, \ldots, \eta_{t_0}^T] = \mathbb{E}[\eta_u^T \leq x | \eta_{t_n}^T, \frac{\eta_{t_{n-1}}^T - \eta_{t_{n-1}}^T}{t_n - t_{n-1}}, \ldots, \frac{\eta_{t_1}^T - \eta_{t_0}^T}{t_1 - t_0}]
$$

$$
= \mathbb{E}[\eta_u^T \leq x | \eta_{t_n}^T, A_n - B_n^T, \ldots, A_1 - B_1^T],
$$

(4.9)
where $(A_k)_{1 \leq k \leq n}$ and $(B_k^T)_{1 \leq k \leq n}$ are defined by

$$
A_k = \frac{W_{t_k} - W_{t_{k-1}}}{t_k - t_{k-1}}, \quad B_k = \frac{X_{T-t_{k-1}} - X_{T-t_k}}{T}.
$$

Since the vector $(A_1, \ldots, A_n, \beta_1^T, \beta_n^T)$ is Gaussian, where $\beta^T$ is the Brownian bridge associated with $W$ of length $T$, it is easy to check that the vectors $(A_1, \ldots, A_n)$ and $(\beta_1^T, \beta_n^T)$ are independent. Moreover, using that Lévy processes have independent increment, it is easy to see that the vectors $(B_1^T, \ldots, B_n^T)$ and $(X_{T-t}, X_{T-u})$ are independent. From what has already been proved and the fact that the random variable $H_T$, the Brownian motion $W$ and the Lévy process $X$ are assumed to be independent we conclude that the vectors $(A_n - B_n^T, \ldots, A_1 - B_1^T)$ and $(\eta_{t_n}^T, \eta_u^T)$ are independent, which proves (4.8).

Remark 4.2. (i) The Markov property of the market information process $\eta^T$ holds for random variables $H_T$ with arbitrary distribution. In the proof of Lemma 4.1 it was essential, however, to assume that $\beta^T$ is a Brownian bridge, as well as the fact that the random variable $H_T$ is multiplied by a linear function of $t$. Simple extensions of these assumptions can be done by the time change technique or by considering instead of a Brownian bridge a bridge associated with a Gaussian-Markov process.

(ii) The Markov property of the market information process $\eta^T$ remains valid in case the process $X$ has only independent increment.

### 4.2 Pricing Formula for a Default Bond

In view of the postulates (i) and (ii) of section 4 the price $B_t^T$ of a default bond with maturity $T$ equals

$$
B_t^T = P_t^T \mathbb{E}[H_T | \mathcal{F}_t^\eta^T].
$$

Since $\eta^T$ is a Markov process with respect to its natural filtration, see Lemma 4.1 and since $H_T$ is measurable with respect to $\sigma(\eta_t^T)$, it follows that

$$
B_t^T = P_t^T \mathbb{E}[H_T | \eta_t^T].
$$

In order to find the explicit pricing formula for a default bond, the explicit form of the conditional law of $H_T$ given $\eta_t^T$ is required. Bayes formula is the key to achieve this.
Lemma 4.3. The conditional expectation of $H_T$ given $\eta^T_t$ can be represented as

$$\hat{H}_t^T = \mathbb{E}[H_T|\eta^T_t] = H^T(\eta^T_t, t),$$

where $H^T : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ is a jointly continuous function given by:

$$H^T(x, t) = \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} p\left(\frac{t(T-t)}{T}, x - \sigma th, \frac{t}{T}y\right) p^X_{T-t}(dy) \mathbb{P}_{H_T}(dh)}{\int_{\mathbb{R}} \int_{\mathbb{R}} p\left(\frac{t(T-t)}{T}, x - \sigma th, \frac{t}{T}y\right) p^X_{T-t}(dy) \mathbb{P}_{H_T}(dh)}.$$  \hfill (4.11)

Proof. The proof of Lemma 4.3 is an immediate consequence of Bayes theorem and the fact that:

$$\mathbb{P}(\eta^T_t \in B|H_T = h) = \int_B q_t(h, x)dx,$$

for any $B \in \mathcal{B}(\mathbb{R})$, where $q_t$ is a non-negative function, measurable with respect to both variables jointly, and is given by:

$$q_t(h, x) = \int_{\mathbb{R}} p\left(\frac{t(T-t)}{T}, x - \sigma th, \frac{t}{T}y\right) p^X_{T-t}(dy).$$  \hfill (4.13)



Corollary 4.4. The price of a default bond equals

$$B_t^T = \exp\left(-\int_t^T r(u)du\right) \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} p\left(\frac{t(T-t)}{T}, \eta^T_t - \sigma th, \frac{t}{T}y\right) p^X_{T-t}(dy) \mathbb{P}_{H_T}(dh)}{\int_{\mathbb{R}} \int_{\mathbb{R}} p\left(\frac{t(T-t)}{T}, \eta^T_t - \sigma th, \frac{t}{T}y\right) p^X_{T-t}(dy) \mathbb{P}_{H_T}(dh)}.$$  \hfill (4.14)



4.3 Valuation of Bond Options

We consider the current value, i.e at time 0, of an option that is exercisable at a fixed time $t > 0$ on a credit-risky bond that matures at time $T > t$. Under the hypothesis of this section the value $C_0$ of a call option is

$$C_0^t = \exp\left(-\int_0^t r(u)du\right) \mathbb{E}[B_t^T - K]^+,$$

where $K$ is the strike price of the option.

Proposition 4.5. The value of the option is

$$C_0^t = P_0^t \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \left( P_t^T h - K \right) \int_{\mathbb{R}} p\left(\frac{t(T-t)}{T}, x - \sigma th, \frac{t}{T}y\right) p^X_{T-t}(dy) \mathbb{P}_{H_T}(dh) \right]^+ dx.$$  \hfill (4.16)

Proof. Using the formula of total probability, (4.11) and (4.14), we have

$$\mathbb{E}[(B_t^T - K)^+] = \int_{\mathbb{R}} [P_t^T H^T(x, t) - K]^+ \int_{\mathbb{R}} \int_{\mathbb{R}} p\left(\frac{t(T-t)}{T}, x - \sigma th, \frac{t}{T}y\right) p^X_{T-t}(dy) \mathbb{P}_{H_T}(dh) dx,$$

which establishes the formula (4.16).
4.4 Binary Bond

In this part we discuss the simplest non trivial case which corresponds to the case when the cash flow $H_t$ is a two-point random variable. Assume that $H_T \in \{h_0, h_1\}$ such that $\mathbb{P}(H_T = h_1) = p > 0$ and $\mathbb{P}(H_T = h_0) = 1 - p$. Then,

$$\mathbb{P}(H_T = h_0|\eta^T_t) = \left(1 + \int_{\mathbb{R}} p\left(\frac{t(T - t)}{T}, \eta^T_t - \sigma t h_1, \frac{t}{T} y\right) p_{T - t}^X(dy) \frac{p}{1 - p}\right)^{-1}$$

(4.17)

and

$$\mathbb{P}(H_T = h_1|\eta^T_t) = \left(1 + \int_{\mathbb{R}} p\left(\frac{t(T - t)}{T}, \eta^T_t - \sigma t h_0, \frac{t}{T} y\right) p_{T - t}^X(dy) \frac{1 - p}{p}\right)^{-1}$$

(4.18)

The bond price process $B_t^T$ associated with the given terminal cash flow is given by:

$$B_t^T = P_t^T[h_0\mathbb{P}(H_T = h_0|\eta^T_t) + h_1\mathbb{P}(H_T = h_1|\eta^T_t)].$$

(4.19)

Then, we find

$$B_t^T = \exp\left(-\int_t^T r(u)du\right)\left[1 + \int_{\mathbb{R}} p\left(\frac{t(T - t)}{T}, \eta^T_t - \sigma t h_1, \frac{t}{T} y\right) p_{T - t}^X(dy) \frac{p}{1 - p}\right] h_0 +$$

$$\left(1 + \int_{\mathbb{R}} p\left(\frac{t(T - t)}{T}, \eta^T_t - \sigma t h_0, \frac{t}{T} y\right) p_{T - t}^X(dy) \frac{1 - p}{p}\right)^{-1} h_1.\right.$$

(4.20)

4.4.1 Examples

**Gamma case:** By a standard gamma process $(\gamma_t, t \geq 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, we mean a subordinator without drift having the Lévy-Khintchine representation given by

$$\mathbb{E}(\exp(-\lambda \gamma_t)) = \exp\left(-t \int_0^\infty (1 - \exp(-\lambda x)) \frac{\exp(-x)}{x} dx\right),$$

(4.21)

where $\nu(dx) = \frac{\exp(-x)}{x} \mathbb{1}_{(0,\infty)}(x) dx$ is the so-called Lévy measure. On the other hand, for any $t > 0$, the random variable $\gamma_t$ follows a gamma distribution with density

$$f_{\gamma_t}(x) = \frac{x^{t-1} \exp(-x)}{\Gamma(t)} \mathbb{1}_{(0,\infty)}(x),$$

(4.22)

where $\Gamma$ is the gamma function, defined as usual for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) dt.$$
In this case, we choose the market information process as (4.4) to be
\[
\eta^T_t(t) = \sigma t H_T + W_t - \frac{t}{T} W_T + \frac{t}{T} \gamma_{T-t}, \quad t \in [0, T],
\]  
(4.24)
the binary bond price \(B_1^T\) derived in (4.20) becomes
\[
B_1^T(t) = P_t^T \left[ \left( 1 + \int \frac{p(T-t)}{T}, \eta^T_1(t) - \sigma th_1, \frac{t}{T} y \right) f_{\gamma_{T-t}}(y) dy \frac{p}{1-p} \right]^{-1} h_0 + \left( 1 + \int \frac{p(T-t)}{T}, \eta^T_1(t) - \sigma th_0, \frac{t}{T} y \right) f_{\gamma_{T-t}}(y) dy \frac{p}{1-p} \right]^{-1} h_1. 
\]  
(4.25)
For \(h \in \{h_1, h_2\}\) the integral expressions can be computed explicitly
\[
\int \frac{p(T-t)}{T}, \eta^T_1(t) - \sigma th, \frac{t}{T} y \right) f_{\gamma_{T-t}}(y) dy = \frac{\sqrt{T}}{\Gamma(T-t)\sqrt{2\pi t(T-t)}} \exp \left( - \frac{T(\eta^T_1(t) - \sigma th)^2}{2t(T-t)} \right) \times \int_0^\infty y^{\nu-1} \exp \left( - \beta y^2 - \alpha y \right) dy,
\]
where \(\nu = T-t, \beta = \frac{t}{2T(T-t)} \) and \(\alpha = T + \sigma th - \eta^T_1(t)T - t\). In order to compute the integral to the far right hand side, we apply the following known integral identity (see for instance [18])
\[
\int_0^\infty y^{\nu-1} \exp(-\beta y^2 - \alpha y) dy = (2\beta)^{-\nu/2} \Gamma(\nu) \exp \left( -\frac{\alpha^2}{8\beta} \right) D_{-\nu} \left( \frac{\alpha}{\sqrt{2\beta}} \right),
\]  
(4.26)
valid for \(\beta, \nu > 0 \) and \(\alpha \in \mathbb{R}\), where \(D\) is the parabolic cylinder function,
\[
\int_0^\infty x^{\nu-1} \exp \left( - \beta x^2 - \alpha x \right) dx = \frac{t}{T(T-t)} \Gamma(T-t) \exp \left( \frac{T(T-t + \sigma th - \eta^T_1(t))^2}{4t(T-t)} \right) \left( (T-t + \sigma th - \eta^T_1(t)) \sqrt{\frac{T}{t(T-t)}} \right).
\]  
(4.27)
This implies,
\[
B_1^T(t) = \exp \left( - \int_t^T r(u) du \right) \left[ \left( 1 + \frac{B_1A_1}{B_0A_0} \frac{p}{1-p} \right)^{-1} h_0 + \left( 1 + \frac{B_1A_1}{B_0A_0} \frac{1-p}{p} \right)^{-1} h_1 \right]. 
\]  
(4.28)
where,
\[
A_j = \exp \left( - \frac{T(\eta^T_1(t) - \sigma th_j)^2}{2t(T-t)} + \frac{T(T-t + \sigma th_j - \eta^T_1(t))^2}{4t(T-t)} \right), \quad j = 0, 1, 
\]  
(4.29)
\[
B_j = D_{t-T} \left( (T-t+\sigma th_j - \eta_1^T(t)) \sqrt{\frac{t}{T(T-t)}} \right), \quad j = 0, 1. \tag{4.30}
\]

Figure 2: The left hand side represents simulated paths of the information processes \(\eta_1^T\) in case the Lévy process \(X\) is gamma process, \(T = 1\), \(\sigma = 1\) and the random variable \(H_T\) follows a Bernoulli distribution with parameter \(p = 0.5\). Whereas, the right hand side represents simulated paths of the bond price processes \(B_1^T\) when the Lévy process \(X\) is a gamma process, \(T = 1\), \(\sigma = 1\), \(r(t) = 0\), and the default bond \(H_T\) follows a Bernoulli distribution with parameter \(p = 0.5\).

**Poisson case:** Let \(N = (N_t, t \geq 0)\) be a Poisson process with intensity \(\lambda\). Then \(N\) is a Lévy process, and for each \(n \in \mathbb{N}\), and for any \(t > 0\),

\[
\mathbb{P}(N_t = n) = Q_t(n) = \exp(-\lambda t) \frac{(\lambda t)^n}{n!}. \tag{4.31}
\]

The characteristic exponent of \(N\) is

\[
\psi(u) = \lambda(1 - \exp(iu)). \tag{4.32}
\]

That is, its Lévy-Khintchine representation is given by

\[
\mathbb{E}(\exp(iuN_t)) = \exp(-t\lambda(1 - \exp(iu))). \tag{4.33}
\]

In analogy to (4.4) we define the market information process by

\[
\eta_2^T(t) = \sigma th_T + W_t - \frac{t}{T}W_T + \frac{t}{T}N_{T-t}, \quad t \in [0, T]. \tag{4.34}
\]

Applying Corollary 4.4 the bond price is given by:

\[
B_2^T(t) = \exp \left( -\int_t^T r(u)du \right) \left[ \left( 1 + \frac{B_1A_1}{B_0A_0} \frac{p}{1-p} \right)^{-1} h_0 + \left( 1 + \frac{B_1A_1}{B_0A_0} \frac{1-p}{p} \right)^{-1} h_1 \right], \tag{4.35}
\]

where

\[
A_j = \sum_{i \geq 0} \exp \left( \eta_1^T(t) - \sigma th_j - t \frac{i}{2(T-t)} \right) \frac{(\lambda(T-t))^i}{i!}, \quad j = 0, 1. \tag{4.36}
\]
\[ B_j = \exp \left( - \frac{T(\eta_j^T(t) - \sigma t h_j)^2}{2t(T-t)} \right), \quad j = 0, 1. \] (4.37)

Figure 3: The left hand side represents simulated paths of the information processes \( \eta_j^T \) in case the Lévy process \( X \) is a Poisson process with rate parameter \( \lambda = 1, T = 1, \sigma = 1 \) and the random variable \( H_T \) follows a Bernoulli distribution with parameter \( p = 0.5 \). Whereas, the right hand side represents simulated paths of the bond price processes \( B_1^T \) when the Lévy process \( X \) is a gamma process, \( T = 1, r(t) = 0, \sigma = 1 \) and the default bond \( H_T \) follows a Bernoulli distribution with parameter \( p = 0.5 \).

Remark 4.6. The model can be extended to cash flows with a finite number of payments, e.g., coupon payments of bonds, see [4].

5 Information Based Pricing of Credit Risky Assets in the Presence of a Default Time

In the previous model for credit risk of a cash flow with a single random payment \( H_T \) at maturity \( T \) with a priori given distribution, the information about \( H_T \) made accessible to market participants before maturity is modelled explicitly but the default time is fixed to \( T \). However, information about definite default before maturity will influence the price immediately, most obvious in case of a default bond with payment at hit. Motivated by the problem of modelling the information concerning the default time of a financial company, Bedini et al [2] consider a new approach to credit risk in which the information about the time of bankruptcy \( \tau \) is modelled using the following information process

\[ \beta_t = W_{t \wedge \tau} - \frac{t \wedge \tau}{\tau} W_{\tau}, \quad t \geq 0, \] (5.1)

where \( W \) is a Brownian motion independent of \( \tau \). In their model, the information is carried by the process \( \beta \) through its completed natural filtration \( \mathbb{F}^{\beta,c} \). Similar to what has been done in Section 4, we can extend [2] to include a wider class of models by considering the following process:

\[ \zeta_t := W_{t \wedge \tau} - \frac{t \wedge \tau}{\tau} W_{\tau} + \mu (t \wedge \tau) X_{\tau - t \wedge \tau}, \quad t \geq 0, \] (5.2)
as an information process. But in this case the information about the cash flow and the information about the default time are modelled separately. In this section, we suggest a model, in which a non-defaultable cash flow with an agreed single payment $H_T$ at maturity $T$ and the time of bankruptcy of the writer of the associated asset are both modelled.

We fix a finite time horizon $[0, T]$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume a deterministic default-free discount bond system. The time $t$ price of a risk-free, zero-coupon bond with face value one maturing at time $T$ is

$$P^T_t = \exp \left( - \int_t^T r(s) ds \right).$$

For $t < T$, the time $t$ price of a cash flow occurring at time $T$, represented by a random variable $H_T$ is given by an expression of the form

$$B^T_t = P^T_t \mathbb{E}[H_T | \mathcal{F}_t],$$

where the $\sigma$-algebra $\mathcal{F}_t$ represents the information available to market participants at time $t$. We assume that the associated probability measure $\mathbb{P}$ is the risk-neutral measure in order for (5.3) to be consistent with the theory of no-arbitrage pricing. This way we model the price process $B^T = (B^T_t, 0 \leq t \leq T)$ of a limited-liability asset that pays $H_T$ at time $T$.

Now we present a simple model for the flow of market information. We assume the existence of an information process $\kappa^T = (\kappa^T_t, 0 \leq t \leq T)$ such that the market filtration is generated by $\kappa^T$:

$$\mathcal{F}_t = \mathcal{F}^{\kappa^T}_t = \sigma(\kappa^T_s, s \leq t).$$

This means that the information available in the market about the cash flow is assumed to be supplied by the information process $\kappa^T$. We define the information process $\kappa^T$ by

$$\kappa^T_t = \sigma t H_T + \mu(t \wedge \tau) X_{t \wedge \tau} - \frac{t \wedge \tau}{\tau} W_{t \wedge \tau},$$

where $\zeta$ was defined in (5.2), $W$ is a Brownian motion, $X$ is a Lévy process and $\tau$ is a random time taking value on $[0, T]$. The information process is composed of two parts, the term $\sigma t H_T$ that contains the true information, and grows in magnitude as $t$ increases, and a noise term, which decomposes into a Brownian bridge with random length $\tau$ and a time reversed Lévy process multiplied by a linear drift. Thus, $\zeta_0 = 0$, $\zeta_\tau = 0$ and $\zeta_{\tau} = 0$ which implies that $H_T$ is measurable with respect to $\mathcal{F}^{\kappa^T}_\tau$. This means that the cash flow is measurable at time $T$ when the contract expires, but if the counterpart files a bankruptcy earlier, then you may find out the value of $H_T$. Here the random time $\tau$ models the time of bankruptcy. It is well known that the stopping time property plays a key role in modelling the default times in mathematical finance. As a first step, we investigate this property for the unknown default time $\tau$ as well as basic properties of the process $\kappa^T$ with respect to the filtration $\mathbb{F}^{\kappa^T, \tau}$. Namely, the Bayes estimates, the structure of the a posteriori distribution of $\tau$ and the Markov property of the process $\kappa^T$.

We suppose that the Brownian motion $W$, the Lévy process $X$, the random time $\tau$, and the random variable $H_T$ are independent, and that the random variable $H_T$ follows a discrete distribution. From now on we denote by $\Delta = \{h_0, h_1, h_2, \ldots\} \subset \mathbb{R}$ its state space and by $p_i$ the probability that the random variable $H_T$ takes the value $h_i$. 

22
Moreover, for any $i \in \mathbb{N}$, the law of the random variable $H_T$, we obtain that $P_W$ follows a Bernoulli distribution with parameter $p = 0.5$, and the random time $\tau$ follows an exponential distribution over with rate parameter $0.1$.

**Proposition 5.1.** The random time $\tau$ is a stopping time with respect to $\mathbb{F}^{\kappa^T,c}$. Moreover, for any $0 < t \leq T$, the event $\{\tau \leq t\}$ is measurable with respect to the $\sigma$-algebra $\sigma(\kappa^T_\tau) \vee \mathcal{N}_P$.

**Proof.** Our proof starts with the observation that for any $t \in (0,T]$, $\kappa^T_t = \sigma t H_T$, while $\tau \leq t$. Using the formula of total probability, the fact that the Brownian motion $W$, the Lévy process $X$, the random time $\tau$, and the random variable $H_T$ are independent, and that for any $t \in (0,T)$ the law of the random variable $W_t - \frac{t}{T} W_T + \mu t X_{T-t}$ is absolutely continuous with respect to Lebesgue measure, we obtain that $P(t < \tau, \kappa^T_t = \sigma t H_T) = 0$. This implies that the process $(\mathbb{I}(\kappa^T_t = \sigma t H_T), 0 < t \leq T)$ is a modification of the process $(\mathbb{I}(\tau \leq t), 0 < t \leq T)$. On the other hand, since $H_T$ has a discrete distribution, for any $t > 0$, the event $\{\kappa^T_t = \sigma t H_T\}$ splits up disjointly into

$$
\mathbb{I}(\kappa^T_t = \sigma t H_T) = \sum_{i=0}^{\infty} \mathbb{I}(\kappa^T_t = \sigma t h_i) \mathbb{I}(H_T = h_i) = \sum_{i=0}^{\infty} \mathbb{I}(\kappa^T_t = \sigma t h_i) - \sum_{i=0}^{\infty} \mathbb{I}(\kappa^T_t = \sigma t h_i) \mathbb{I}(H_T \neq h_i). 
$$

Moreover, for any $i \in \mathbb{N}$,

$$
\mathbb{E}[\mathbb{I}(\kappa^T_t = \sigma t h_i) | H_T \neq h_i)] = \sum_{j=0}^{\infty} \int_{(0,T]} \mathbb{P}\left(\kappa^T_t = \sigma t h_i, H_T \neq h_i | \tau = r, H_T = h \right) \mathbb{P}_r(dr) p_j
$$

$$
= \sum_{j \neq i}^{\infty} \int_{(0,T]} \mathbb{P}\left(W_t - \frac{t \wedge r}{r} W_r + \mu (t \wedge r) X_{T-t} = \sigma t (h_i - h_j) \right) \mathbb{P}_r(dr) p_j
$$

$$
= \sum_{j \neq i}^{\infty} \int_{(t,T]} \mathbb{P}\left(W_t - \frac{t}{r} W_r + \mu t X_{T-t} = \sigma t (h_i - h_j) \right) \mathbb{P}_r(dr) p_j
$$

$$
= 0.
$$

(5.7)
Inserting (5.7) into the formula (5.6) and using the fact that the process \((\mathbb{I}_{\{\kappa^T_i = \sigma \cdot t \cdot n\}}, 0 < t \leq T)\) is a modification of the process \((\mathbb{I}_{\{\kappa^T_i = \sigma \cdot t \cdot n\}}, 0 < t \leq T)\) under the probability measure \(\mathbb{P}\) we obtain, \(\mathbb{P}\)-a.s.,

\[
\mathbb{I}(\tau \leq t) = \sum_{i=0}^{\infty} \mathbb{I}(\kappa^T_i = \sigma \cdot t \cdot n_i).
\] (5.8)

From what has already been proved we conclude that the event \(\{\tau \leq t\}\) is measurable with respect to the \(\sigma\)-algebra \(\sigma(\kappa^T) \vee \mathcal{N}_P\). This implies that the random time \(\tau\) is an \(\mathbb{F}^{\kappa^T}\)-\(\sigma\)-stopping time. \(\square\)

**Proposition 5.2.** Let \(t \in (0, T)\) and \(g\) be a measurable function on \((0, T] \times \mathbb{R}\) such that \(g(\tau, H_T)\) is integrable. Then, \(\mathbb{P}\)-a.s.,

\[
\mathbb{E}[g(\tau, H_T)|\kappa^T_i] = \sum_{i \geq 0} \int_{(0,t]} \frac{g(r, h_i)}{F_i(r)} \mathbb{P}_r(dr) \mathbb{I}(\kappa^T_i = \sigma \cdot t \cdot n_i) + \\
\sum_{i \geq 0} \sum_{j \geq 0} \int_{(t,T]} \int_{\mathbb{R}} \frac{g(r, h_i)}{F_i(r)} \mathbb{P}_r(dr) \mathbb{P}_j(r) \mathbb{I}(\kappa^T_i = \sigma \cdot t \cdot n_j) \mathbb{P}_r(dr) \mathbb{P}_j(r) \mathbb{I}(\kappa^T_i = \sigma \cdot t \cdot n_j),
\] (5.9)

Proof. As previously Bayes theorem is employed to derive the equality. It is not difficult to check that for all \(t > 0\),

\[
\mathbb{P}(\kappa^T_i \in dx|\tau = r, H_T = h) = q_t(x, r, h) \mu(dx).
\] (5.10)

where,

\[
q_t(x, r, h) = \sum_{i \geq 0} \mathbb{I}(x = h_i) \mathbb{I}(h = h_i) \mathbb{I}(r \leq t) + \int_{\mathbb{R}} p\left(\frac{t(r - t)}{r}, x - \mu y - \sigma th\right) p_{r-t}(dy) \mathbb{P}_r\mathbb{I}(x \neq h_i) \mathbb{I}(t < r),
\] (5.11)

and

\[
\mu(dx) = dx + \sum_{i \geq 0} \delta_{h_i}(dx).
\]

Since the function \(q_t\) is non-negative and jointly measurable and \(\mu\) is a \(\sigma\)-finite measure on \(\mathbb{R}\), we conclude from Bayes theorem that the conditional law of \((\tau, H_T)\) given \(\kappa^T_i\) is given by the formula (5.9). \(\square\)

**Theorem 5.3.** The information process \(\kappa^T = (\kappa^T_i, 0 \leq t \leq T)\) is an \(\mathbb{F}^{\kappa^T}\)-Markov process.

Proof. We need to show that for all \(t, h \geq 0\) and for every bounded measurable function \(f\) we have, \(\mathbb{P}\)-a.s.,

\[
\mathbb{E}[f(\kappa^T_{i+h}|\mathcal{F}^T_i)] = \mathbb{E}[f(\kappa^T_i)|\kappa^T_i].
\] (5.12)

For \(t = 0\) the statement is clear. Let us assume \(t > 0\). Since \(f(\kappa^T_{i+h}|\mathcal{F}_{\{\tau \leq t\}}) = f\left(\frac{t+h}{t} \kappa^T_i\right) \mathbb{I}_{\{\tau \leq t\}}\), which is measurable with respect to \(\sigma(\kappa^T) \vee \mathcal{N}_P\), it remains to prove (5.12) on the set \(\{t < \tau\}\), that is, \(\mathbb{P}\)-a.s.,

\[
\mathbb{E}[f(\kappa^T_{i+h}|\mathcal{F}^T_t)] = \mathbb{E}[f(\kappa^T_{i+h}|\kappa^T_{i+h})|\kappa^T_i].
\]
By the monotone class theorem it is sufficient to prove that for each finite collection \(0 < t_0 < t_1 < \cdots < t_n = t\),
\[
\mathbb{E}[f(T_{i+1}^T)\mathbb{1}_{\{t < \tau\}}|\kappa_t^T, \cdots, \kappa_t^n] = \mathbb{E}[f(T_{i+1}^T)\mathbb{1}_{\{t < \tau\}}|\kappa_t^n].
\]

Using the fact that
\[
\mathbb{E}[f(T_{i+1}^T)h(\kappa_t^n)\mathbb{1}_{\{t < \tau\}}|\kappa_t^n, \cdots, \kappa_t^0] = \mathbb{E}[f(T_{i+1}^T)h(\kappa_t^n)\mathbb{1}_{\{t < \tau\}}|\kappa_t^n, \frac{\kappa_t^n}{t_n}, \cdots, \frac{\kappa_t^1}{t_1} - \frac{\kappa_t^0}{t_0}],
\]
(5.13)
it remains to show that for any bounded measurable functions \(g\) and \(h\) on \(\mathbb{R}^n\) and \(\mathbb{R}\), respectively, we have
\[
\mathbb{E}[f(T_{i+1}^T)h(\kappa_t^n)\mathbb{1}_{\{t < \tau\}}g(A_1 - B_1(\tau), \cdots, A_n - B_n(\tau))]
= \mathbb{E}[\mathbb{E}[f(T_{i+1}^T)h(\kappa_t^n)\mathbb{1}_{\{t < \tau\}}g(A_1 - B_1(\tau), \cdots, A_n - B_n(\tau)), (5.14)
where \((A_k)_{1 \leq k \leq n}\) and \((B_k(\tau))_{1 \leq k \leq n}\) are given by
\[A_k = \frac{W_{t_k}}{t_k} - \frac{W_{t_{k-1}}}{t_{k-1}}, \quad B_k(\tau) = \mu(X_{\tau-t_{k-1}} - X_{\tau-t_k}).\]

Since the Brownian motion \(W\), the Lévy process \(X\), and the random time \(\tau\) are independent we have by the formula of total probability
\[
\mathbb{E}[f(T_{i+1}^T)h(\kappa_t^r)\mathbb{1}_{\{t < \tau\}}g(A_1 - B_1(\tau), \cdots, A_n - B_n(\tau))]
= \int_t^T \mathbb{E}[f(T_{i+1}^T)h(\kappa_t^r)g(A_1 - B_1(r), \cdots, A_n - B_n(r))]|\mathbb{P}_\tau(dr)
= \int_t^T \mathbb{E}[f(T_{i+1}^T)h(\kappa_t^r)]\mathbb{E}[g(A_1 - B_1(r), \cdots, A_n - B_n(r))]|\mathbb{P}_\tau(dr),
\]
(5.15)
where
\[\kappa_t^r = \sigma t H_T + \mu(t \wedge r)X_{\tau-t \wedge r} + W_{t \wedge r} - t \wedge \frac{r}{r}W_r, \quad t \geq 0.\]

On the other hand, with the understanding that \(\mathbb{P}(X_t \in dx) = p_t^X(dx)\) for \(t > 0\), we have
\[
\mathbb{E}[g(A_1 - B_1(r), \cdots, A_n - B_n(r))]
= \int_{\mathbb{R}^n} \mathbb{E}[g(A_1 - \mu x_1, \cdots, A_n - \mu x_n)]\mathbb{P}\left(\frac{B_1(r)}{\mu} \in dx_1, \cdots, \frac{B_n(r)}{\mu} \in dx_n\right)
= \int_{\mathbb{R}^n} \mathbb{E}[g(A_1 - \mu x_1, \cdots, A_n - \mu x_n)] \prod_{i=1}^n p_{t_i-t_{i-1}}^x(dx_i),
\]

25
which does not depend on \( r \). Thus, \( (5.15) \) implies that

\[
\mathbb{E}[f(\kappa_{i+h}^T)h(\kappa_i^T)1_{\{t<\tau\}}g(A_1 - B_1(\tau), \ldots, A_n - B_n(\tau))]
\]

\[
= \int_t^T \mathbb{E}[f(\kappa_{i+h}^T)h(\kappa_i^T)1_{\{t<\tau\}}] \mathbb{P}_r(\text{d}r) \mathbb{E}[g(A_1 - B_1(r), \ldots, A_n - B_n(r))]
\]

\[
= \mathbb{E}[f(\kappa_{i+h}^T)h(\kappa_i^T)1_{\{t<\tau\}}] \mathbb{E}[g(A_1 - B_1(r), \ldots, A_n - B_n(r))]
\]

\[
= \mathbb{E}[\mathbb{E}[f(\kappa_{i+h}^T)|\kappa_i^T]h(\kappa_i^T)1_{\{t<\tau\}}] \mathbb{E}[g(A_1 - B_1(r), \ldots, A_n - B_n(r))]
\]

\[
= \int_t^T \mathbb{E}[\mathbb{E}[f(\kappa_{i+h}^T)|\kappa_i^T]1_{\{t<\tau\}}h(\kappa_i^T)\mathbb{P}_r(\text{d}r) \mathbb{E}[g(A_1 - B_1(r), \ldots, A_n - B_n(r))]
\]

\[
= \mathbb{E}[\mathbb{E}[f(\kappa_{i+h}^T)|\kappa_i^T]1_{\{t<\tau\}}g(A_1 - B_1(\tau), \ldots, A_n - B_n(\tau))],
\]

which ends the proof.

\[\square\]

**Remark 5.4.** If the random variable \( H_T \) follows a law, which is absolutely continuous with respect to the Lebesgue measure, we can find a random time \( \nu \) with values on \((0, T]\) such that the process \( \mathcal{L}^T = (\mathcal{L}^T_t, 0 \leq t \leq T) \) where

\[
\mathcal{L}^T_t = \sigma t H_T + \mu(t \land \nu)X_{t \land \nu} + W_{t \land \nu} - \frac{t \land \nu}{\nu} W_{t},
\]

is not Markovian. See Theorem 3.9 in [24] for the proof of the case when \( \mu = 0 \).

**Proposition 5.5.** The price of a default bond equals

\[
B^T_t = \exp\left(-\int_t^T r(s)ds\right)\left[\frac{\kappa_i^T}{\sigma t}1_{\{t \leq \tau\}}\right]
\]

\[
+ \sum_{i \geq 0} \sum_{j \geq 0} \int_{(i,T]} \int_{\mathbb{R}} p\left(\frac{t-r}{r}, \kappa_i^T - \mu ty - \sigma th_j\right)p^{X}_{r-1}(dy)\mathbb{P}_r(\text{d}r)
\]

\[
\cdot \left[\int_{(i,T]} \int_{\mathbb{R}} p\left(\frac{t-r}{r}, \kappa_i^T - \mu ty - \sigma th_j\right)p^{X}_{r-1}(dy)\mathbb{P}_r(\text{d}r)\right]. \tag{5.16}
\]

**Proof.** The proof is an immediate consequence of \( (5.3), (5.9) \) and the fact that the market information process \( \kappa^T = (\kappa_i^T, 0 \leq t \leq T) \) is Markovian. \[\square\]
Corollary 5.6. The bond price process $B^T_t$ associated with a binary cash flow is given by

$$B^T_t = \exp \left( - \int_t^T r(u) du \right) \frac{\kappa_t^T}{\sigma_t} \mathbb{I}_{\{\tau \leq t\}} + \exp \left( - \int_t^T r(u) du \right) \times $$

$$\left[ \left( 1 + \int_{(t,T]} \int_{\mathbb{R}} \frac{p \left( \frac{t(r-t)}{r}, \kappa_t^T - \sigma th_1 - \mu ty \right)}{P_{r-t}^X(dy) \mathbb{P}_\tau(dr)} \right)^{-1} h_0 + \right. $$

$$\left. \int_{(t,T]} \int_{\mathbb{R}} \frac{p \left( \frac{t(r-t)}{r}, \kappa_t^T - \sigma th_0 - \mu ty \right)}{P_{r-t}^X(dy) \mathbb{P}_\tau(dr)} \left( \mathbb{P}(H_T = h_1) - \mathbb{P}(H_T = h_0) \right) \frac{1 - p}{p} \left( 1 + \int_{(t,T]} \int_{\mathbb{R}} \frac{p \left( \frac{t(r-t)}{r}, \kappa_t^T - \sigma th_0 - \mu ty \right)}{P_{r-t}^X(dy) \mathbb{P}_\tau(dr)} \right)^{-1} h_1 \right] \mathbb{I}_{\{t < \tau\}}, \quad (5.17)$$

where, $\mathbb{P}(H_T = h_1) = 1 - \mathbb{P}(H_T = h_0) = p > 0$.

Proposition 5.7. The value $C_0$ at time 0 of an option that is exercisable at a fixed time $t > 0$ on a credit-risky bond that matures at time $T > t$ is given by

$$C_0^t = \exp \left( - \int_0^t r(u) du \right) \mathbb{E}[(B^T_t - K)^+] = P_0^t F_\tau(t) \sum_{j \geq 0} (P^T_t h_j - K)^+ p_j$$

$$+ P_0^t \int_{\mathbb{R}} \left[ \sum_{j \geq 0} (P^T_t h_j - K) \int_{(t,T]} \int_{\mathbb{R}} p \left( \frac{t(r-t)}{r}, x - \sigma th - \mu ty \right) p_{r-t}^X(dy) \mathbb{P}_\tau(dr) p_j \right]^+ dx, \quad (5.18)$$

where $K$ is the strike price of the option.

Proof. First, we have

$$\mathbb{E}[(B^T_t - K)^+] = \mathbb{E}[(B^T_t - K)^+ \mathbb{I}_{\{\tau \leq t\}}] + \mathbb{E}[(B^T_t - K)^+ \mathbb{I}_{\{t < \tau\}}]. \quad (5.19)$$

Using (5.16) we obtain that

$$\mathbb{E}[(B^T_t - K)^+ \mathbb{I}_{\{\tau \leq t\}}] = \mathbb{E}[(P^T_t H_T - K)^+ \mathbb{I}_{\{\tau \leq t\}}] = F_\tau(t) \sum_{j \geq 0} (P^T_t h_j - K)^+ p_j. \quad (5.20)$$

On the other hand, by using (5.16) and the formula of total probability, we obtain

$$\mathbb{E}[(B^T_t - K)^+ \mathbb{I}_{\{t < \tau\}}] =$$

$$\int_{\mathbb{R}} \left[ \sum_{j \geq 0} (P^T_t h_j - K) \int_{(t,T]} \int_{\mathbb{R}} p \left( \frac{t(r-t)}{r}, x - \sigma th - \mu ty \right) p_{r-t}^X(dy) \mathbb{P}_\tau(dr) p_j \right]^+ dx. \quad (5.21)$$

We get (5.18) by inserting (5.20) and (5.21) into (5.19). \qed
Figure 5: The left hand side represents simulated paths of the bond price process $B^T$ in case where the Lévy process $X$ is a gamma process, $\mu = 0.5$ and the default time $\tau$ is uniformly distributed over $\{0.7, 0.8\}$. Whereas, the right hand side represents simulated paths of $B^T$ in case where $X$ is a Poisson process with rate parameter $\lambda = 1$, $\mu = 1$ and the default time $\tau$ is uniformly distributed over $\{0.7, 0.9\}$. In both pictures, $T = 1$, $r(t) = 0$, $\sigma = 1$ and the default bond $H_T$ follows a Bernoulli distribution with parameter $p = 0.5$.

Remark 5.8. The generalization to cash flows with a finite number of payments can be pursued along the lines detailed in Brody et al. [4].

Acknowledgements:
The authors would like to express particular thanks to Rainer Buckdahn, Dorje C. Brody and Paul Fischer for their comments.

References

[1] Bain, A.; Crisan, D. *Fundamentals of stochastic filtering*. Stochastic Modelling and Applied Probability, 60. Springer, New York, (2009).

[2] Bedini, M. L.; Buckdahn, R.; Engelbert, H. J. Brownian bridges on random intervals. Theory Probab. Appl. 61 (2017), no. 1, 15–39.

[3] Bertoin J. *Lévy Processes*. Cambridge Univeristy Press, Cambridge, (1996).

[4] Brody, D. C.; Hughston, L. P.; Macrina, A. Beyond hazard rates: a new framework to credit-risk modelling. In Advances in mathematical finance (eds M. Fu, R. Jarrow, J.-Y. J. Yen and R. Elliott), pp. 231–257, (2007).

[5] Brody, D. C.; Hughston, L. P.; Macrina, A. Information-based asset pricing. Int. J. Theor. Appl. Finance 11 (2008), no. 1, 107–142.

[6] Brody, D.C.; Hughston, L.P.; and Macrina, A. Dam rain and cumulative gain. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 464 (2008), no. 2095, 1801–1822.

[7] Blumenthal, R. M.; Getoor, R. K. *Markov processes and potential theory*. Academic Press, New York-London (1968).

[8] Brody, D. C.; Law, Y. T. Pricing of defaultable bonds with random information flow. Appl. Math. Finance 22 (2015), no. 5, 399–420.
[9] Bielecki, T.R; Jeanblanc, M.; Rutkowski, M. Modelling and valuation of credit risk. In Stochastic Methods in Finance, Bressanone Lectures 2003, eds. M. Fritelli and W. Runggaldier, LNM 1856, Springer, (2004).

[10] Bielecki, T.R; Rutkowski, M. Credit Risk: Modelling, Valuation and Hedging. Springer, (2002).

[11] Duffie, D.; Singleton, K.J. Credit Risk: Pricing, Measurement and Management. Princeton University Press, (2003).

[12] Elizalde, A. Credit risk models: I Default correlation in intensity models, II Structural models, III Reconciliation structural-reduced models, IV Understanding and pricing CDOs. www.abelelizalde.com, (2005).

[13] Erraoui, M.; Louriki, M. Bridges with random length: Gaussian-Markovian case. Markov Process. Related Fields 24 (2018), no. 4, 669–693.

[14] Erraoui, M.; Hilbert, A.; Louriki, M. Bridges with random length: gamma case. J. Theoret. Probab. 33 (2020), no. 2, 931–953.

[15] Erraoui, M.; Hilbert, A.; Louriki, On a Lévy process pinned at random time. Forum Math. 33 (2021), no. 2, 397–417.

[16] Ekström, E.; Vaicenavicius, J. Optimal stopping of a Brownian bridge with unknown pinning point. Stochastic Process. Appl. 130 (2020), no. 2, 806–823.

[17] Föllmer, H.; Wu, C.T.; Yor, M. Canonical decomposition of linear transformations of two independent Brownian motions motivated by models of insider trading. Stochastic Process. Appl. 84 (1999), no. 1, 137–164.

[18] Gradshteyn, I.S; Ryzhik, I.M. Table of integrals, series, and products. A P Academic Press (1996).

[19] Hoyle, E.; Hughston, L.P.; Macrina, A. Lévy random bridges and the modelling of financial information. Stochastic Process. Appl. 121 (2011), no. 4, 856–884.

[20] Hughston, L.P.; Turnbull, S. Credit risk: Constructing the basic building blocks. Economic Notes 30 (2001), 281–292.

[21] Jeanblanc, M.; Rutkowski, M. Modelling of default risk: An overview. In Mathematical Finance: Theory and Practice, Higher Education Press, (2000).

[22] Kallianpur, G.: Stochastic Filtering Theory. Applications of Mathematics, vol. 13. Springer, New York (1980).

[23] Lando, D. Credit Risk Modelling. Princeton University Press, (2004).

[24] Louriki, M. Brownian bridge with random length and pinning point for modelling of financial information. arXiv:1907.08047v2, (2019).
[25] Privault, N.; Zambrini, J. C. Markovian bridges and reversible diffusions with jumps. Ann. Inst. H. Poincaré Probab. Statist. 40 (2004), no. 5, 599–633.

[26] Protter, P. Stochastic Integration and Differential Equations. 2nd edn. Springer, Berlin, (2005).

[27] Rutkowski, M; Yu, N. An extension of the Brody-Hughston-Macrina approach to modelling of defaultable bonds. Int. J. Theor. Appl. Finance. 10 (2007), 557–589.

[28] Revuz D.; Yor M. Continuous Martingales and Brownian Motion. Springer-Verlag, Berlin, Third edition, (1999).

[29] Sato, K. I. Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge, (1999).

[30] Schönbucher, P.J. Credit Derivatives Pricing Models. John Wiley and Sons, (2003).