Remarks on Quantum Probability Backflow

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Abstract

It is known that for a non-relativistic quantum particle traveling freely on the $x$-axis, the positional probability can flow in the opposite direction to the particle’s velocity. The maximum possible amount of such backflow that can occur over any time interval has been determined previously as the largest positive eigenvalue of a certain hermitian observable, with the value $0.0384517\ldots$, or about 4% of the total probability on the line. The eigenvalue problem is now considered numerically in the more general case of states with momentum restricted to the range $p_0 < p < \infty$, for any given value $p_0$. It is found that the maximum possible backflow decreases monotonically, but never reaches 0, as $p_0$ increases through positive values; and it increases monotonically, but never reaches 1, as $p_0$ decreases through negative values. Both of these effects are non-classical. The results allow a simple interpretation of the classical limit, as an effective value of Planck’s constant goes to zero and probability backflow becomes impossible.

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1 Introduction

A consequence of wave-particle duality is that probability backflow can occur during the motion of a quantum ‘particle’, in particular one moving freely on a straight line, say the $x$-axis, and described by a wave packet composed of plane waves, each of which has positive wave number (momentum) [1–4]. Despite the fact that a particle in such a state would certainly be found on measurement of its momentum to be moving with constant velocity from left to right, the probability of finding it to the left of a given point on the axis, say $x = 0$, on measurement of its position, can increase over any given time interval. Thus it seems that the positional probability can flow in the opposite direction to the velocity of the particle.

It is important to see at the outset that the backflow phenomenon is not simply the familiar “spreading of a quantum wave packet”, such as a Gaussian packet with a constant average velocity, directed from left to right. Such a packet spreads in both directions as it travels, but that is not in itself surprising, because the packet is composed of left-moving as well as right-moving plane waves, so that there is a finite probability that the particle will be found on measurement of its momentum to be traveling in either direction. The backflow surprise is greatest when the probability flows in one direction even though the particle is — more precisely, would be found on measurement to be — certainly traveling in the opposite direction.

The reader may well consider that peculiar quantum effects involving incompatible observables such as position and momentum are to be expected. Furthermore, a quantum object has wave-like as well as particle-like properties and, from a purely mathematical point of view, probability backflow is one more example of retrogressive wave motion and is not particularly remarkable as such [5]. Nevertheless, in the context of quantum dynamics the effect is strikingly counterintuitive, with no classical analogue.

By what ‘surreal behavior’ can a ‘particle’ with velocity directed left-to-right, increase its chance of being found to the left of a given point? To make a connection with interpretations of other ‘surreal’ quantum effects in terms of Bohmian mechanics [6–10], it is enough here to note that while probability backflow is occurring, the Bohmian velocity [11,12], by definition parallel to the probability flux vector, is directed opposite to any velocity value obtained by measuring the momentum observable.

A quite different interpretation [2–4] follows Feynman’s suggested introduction of negative probabilities [13,14] as an aid to the interpretation of aspects of quantum behavior that defy a classical explanation, such as those associated with the double-slit experiment. Negative probabilities appear through the introduction of the Wigner function [15], leading in the present
context to the notion that while all probability flows in the same direction as the velocity for a quantum particle, just as for a classical particle, in the quantum case the flowing probability can be negative during some time intervals. The flow of negative probability in the direction of the velocity produces the same effect as a flow of positive probability in the opposite direction — probability backflow.

The possibility of probability backflow was first noticed over fifty years ago in the case of a free, non-relativistic particle [1], and the phenomenon was subsequently described in detail and quantified for that case [2]. It has been widely discussed and generalized in various directions since [3, 4, 16–24]. The maximum size of the effect is small [2, 20, 21]; in the case of a free particle, no more than 0.0384517... out of the total (unit) probability on the line, or about 4%, can flow backwards in any given time interval of length $T$, no matter how short or long.

The properties of this ‘quantum number’ $\lambda_{\text{max}} \approx 0.04$ provide another reason for interest in probability backflow. It is not only dimensionless and independent of the length of the time interval involved, but is also independent of the size of Planck’s constant $\hbar$ and of the mass $m$ of the particle [2]. Experimental measurement of its value might provide an unusual new test of the structure of quantum dynamics.

A possible experiment to observe probability backflow has been devised recently [25, 26], but to our knowledge no experiment has yet been proposed to go further and confirm the predicted maximum size $\lambda_{\text{max}}$ of the effect.

2 Purpose

The purpose of this note is to throw more light on the phenomenon of probability backflow in the case of a free, non-relativistic particle, by considering the mathematical question: How does the maximum backflow value $\lambda_{\text{max}}$ change as the least value of the momenta in the waves making up a packet is varied? In other words, what is the maximum backflow that can occur over a time interval of length $T$ if packets containing only momenta $p > p_0$ are considered, for given $-\infty < p_0 < \infty$?

When the choice $p_0 = 0$ is made, then $\lambda_{\text{max}} = 0.0384517...$ as indicated above. It is to be expected that $\lambda_{\text{max}}$ will decrease towards 0 as $p_0$ increases, because probability backflow surely becomes more and more unlikely as the minimum left-to-right velocity of the particle is increased. Conversely, as $p_0$ decreases below 0, the flow of some probability to the left is no longer counterintuitive, and it is to be expected that $\lambda_{\text{max}}$ will increase towards 1, representing the total probability on the line. But the shape of the graph of
\( \lambda_{\text{max}} \) \( v \). \( p_0 \) for \(-\infty < p_0 < \infty \) is quite unclear. At present the only known point on the graph is \((p_0 = 0, \lambda_{\text{max}} = 0.0384517 \ldots)\). The object here is to present and discuss a numerically-determined approximation to this graph.

3 The graph in question

For a non-relativistic particle with mass \( m \) moving freely on the \( x \)-axis, the value of \( \lambda_{\text{max}} \) is determined from the eigenvalue problem for the hermitian integral operator \( \hat{\Delta}_{(0,T)} \) that acts on momentum-space amplitudes \( \phi(p) \) as \(^2\)

\[
(\hat{\Delta}_{(0,T)} \phi)(p) = \frac{i}{2\pi} \int e^{i(p^2-q^2)T/2m\hbar - 1/p - q} \phi(q) dq.
\] (1)

This ‘probability-flow operator’ is constructed \(^2\) from the probability flux vector, and as a quantum observable has the meaning that its expectation value in a given quantum state of the particle at time \( t \), is the amount of probability that will flow across \( x = 0 \) from right to left in the time-interval \([t, t+T]\) during the free-particle evolution. Accordingly, the largest possible probability backflow during that time interval is given by the largest eigenvalue of \( \hat{\Delta}_{(0,T)} \). More precisely, since it is not certain that the spectrum of \( \hat{\Delta}_{(0,T)} \) is discrete \(^{21}\), \( \lambda_{\text{max}} \) is the least upper bound on that spectrum. Note that \( \hat{\Delta}_{(0,T)} \) is independent of \( t \) and hence the same is true of \( \lambda_{\text{max}} \).

The range of the variable \( p \) and of the integration over \( q \) in (1), both equal some chosen range of variation of momentum values. In the problem studied so far, as outlined in the Introduction, only amplitudes \( \phi(p) \) are considered where the range of integration and of \( p \) is \( 0 < p < \infty \). In states described by such amplitudes, the particle is certain to be found on measurement to have a positive momentum. In what follows, the more general range \( p_0 < p < \infty \) is considered, for various choices of \(-\infty < p_0 < \infty \). For each such choice, the momentum of the particle would certainly be found on measurement to lie in this more general range, between \( p_0 \) and \( \infty \).

The more general eigenvalue problem can be written in dimensionless form as

\[
-\frac{1}{\pi} \int_{u_0}^{\infty} \frac{\sin(u^2 - v^2)}{u - v} \varphi(v) dv = \lambda \varphi(u), \quad u_0 < u < \infty,
\] (2)

where

\[
\varphi(u) = e^{-ip^2T/4m\hbar} \phi(p), \quad \varphi(v) = e^{-iq^2T/4m\hbar} \phi(q),
\]

\[
u = \sqrt{T/4m\hbar} q, \quad \varphi(u_0) = \sqrt{T/4m\hbar} p_0.
\] (3)

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It is clear from (2) that the spectrum of \( \lambda \) values, and so the value of \( \lambda_{\text{max}} \), in this general case, is dependent on the values of \( p_0, T, \hbar \) and \( m \) only through the dimensionless parameter \( u_0 \) as in (3). When \( p_0 = 0 \), so that \( u_0 = 0 \), it is independent of \( T, \hbar \) and \( m \) as already noted above.

Fig. 1 shows the interesting part of the graph of \( \lambda_{\text{max}} \) vs. \( u_0 \), obtained numerically by discretizing a sufficiently large part of the range of \( u \) and \( v \) values in (3), and using the MATLAB \([27]\) eigenvalue solver \textit{eigs} repeatedly.

There are two surprises. The first is that no matter how large positive is \( u_0 \) as in (3), so no matter how large positive is the minimum cut-off momentum value \( p_0 \) for given \( T, \hbar \) and \( m \), some probability backflow can still occur, though \( \lambda_{\text{max}}(u_0) \), the maximum amount of backflow possible for given \( u_0 \), decreases steadily towards 0 as \( u_0 \) increases. This result, and the way in which \( \lambda_{\text{max}}(u_0) \) depends on the values of \( T \) and \( m \) through \( u_0 \) as in (3), may be useful in the design of experiments like that mentioned in the Introduction \([25,26]\).

The second surprise is that no matter how large negative is \( u_0 \), so no matter how large negative is the minimum cut-off momentum value \( p_0 \) for given \( T, \hbar \) and \( m \), the maximum amount \( \lambda_{\text{max}} \) of right-to-left probability flow is less than 1, though it increases steadily towards 1 as \( u_0 \) decreases.

Both these results are non-classical. For a classical particle traveling along the \( x \)-axis with an uncertain position and a constant but uncertain momentum \( p > p_0 \), the corresponding graph is discontinuous, and consists of two straight line segments, namely \( \lambda_{\text{max}} = 1 \) for \( p_0 < 0 \), and \( \lambda_{\text{max}} = 0 \) for \( p_0 > 0 \). The result for \( p_0 > 0 \) is obviously true; probability backflow cannot occur for a classical particle. To see that the result for \( p_0 < 0 \) is also correct, it suffices to consider the evolution of a classical probability density for momentum values that has compact support bounded below by \( p_0 \) and contained entirely on the negative \( p \)-axis, with in addition an initial probability density for position values with compact support entirely on the positive \( x \)-axis. Then the right-to-left probability flow across \( x = 0 \) in an appropriate time interval is clearly 1, the maximum possible.

Discussions of the ‘classical limit’ of probability backflow \([2,22]\) when \( p_0 = 0 \) have faced the obstacle that \( \lambda_{\text{max}} \) is independent of \( \hbar \) in that case, as noted in the Introduction. But the broadening of the treatment of the eigenvalue problem for \( \hat{\Delta}_{(0,T)} \) by considering each \(-\infty < p_0 < \infty \) leads to an appealing resolution of this difficulty. The discontinuous classical curve can be obtained in the ‘classical limit’ from the quantum curve by considering the graph of \( \lambda_{\text{max}}(u_0/\alpha) \) vs. \( u_0 \), effectively replacing \( \hbar \) by \( \hbar_{\text{eff}} = \alpha^2 \hbar \) in (3), and then allowing \( \alpha \to 0 \). Fig. 2 shows the graphs for \( \alpha = 1, 3/4, 1/2, 1/4 \) and \( 1/8 \). Note that all the curves pass through the same value \( \lambda_{\text{max}} \approx 0.04 \) at \( u_0 = 0 \); the maximum probability backflow in that previously-studied case
is indeed independent of the value of $\hbar$, and does not change as $\hbar_{\text{eff}} \to 0$, while the quantum curves nevertheless approach the discontinuous classical curve.

A final remark: A cursory inspection of the curve in Fig. 1 suggests that the function $\lambda_{\text{max}}(u_0) - 0.5$ may be odd about the point $u_0 = u_0^* \approx -1.16$ where $\lambda_{\text{max}} - 0.5 = 0$, that is to say, that

$$\lambda_{\text{max}}(u_0 - u_0^*) - 0.5 = -(\lambda_{\text{max}}(u_0^* - u_0) - 0.5),$$ 

but a more accurate numerical study is needed before this can be claimed true with confidence. Such a result would imply an important symmetry of the eigenvalue problem (2).

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Figure 1: Maximum R to L probability flow $\lambda_{\text{max}}$ plotted against $u_0$. Note $\lambda_{\text{max}} \approx 0.04$ when $u_0 = 0$. 
Figure 2: Maximum R to L probability flow plotted against $u_0$ for $\hbar_{\text{eff.}} = \alpha^2 \hbar$, with $\alpha = 1, 3/4, 1/2, 1/4, 1/8$ from left to right at the top, showing approach to the discontinuous ‘classical limit’ curve as $\hbar_{\text{eff.}} \to 0$. 