Tight embedding of modular lattices into partition lattices:
progress and program

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ABSTRACT: A famous Theorem of Pudlak and Tuma states that each finite lattice $L$ occurs as sublattice of a finite partition lattice. Here we derive, for modular lattices $L$, necessary and sufficient conditions for cover-preserving embeddability, i.e. as tight as it gets. Some of the remaining open questions are purely combinatorial, such as deciding whether certain binary matroids are in fact graphic.

1 Introduction

Every concept not explained is standard and can e.g. be explored in either [G] or [O]. By definition a tight embedding $f: L \to L'$ between finite lattices is a cover-preserving lattice homomorphism. Let $\text{Part}(n)$ be the semimodular lattice of all partitions of the set $[n] := \{1, 2, \ldots, n\}$. When $L$ wants to tightly embed into $\text{Part}(n)$, then $L$ must be semimodular itself. About twenty five years ago I made strides towards finding necessary and sufficient conditions for a modular lattice $L$ to be tightly embeddable into $\text{Part}(n)$. This is because modular lattices enjoy a much richer structure theory than merely semimodular ones. One modular lattice $L_0$ together with a tight embedding into $\text{Part}(5)$ is given in Figure 1(a). For instance, $13, 25, 4$ is shorthand for the partition $\{\{1, 3\}, \{2, 5\}, \{4\}\}$. The two elements larger lattice $L_1$ in Figure 1(b) does not admit a tight embedding into $\text{Part}(5)$.

Figure 1

(a) A tight partition embedding of $L_0$
(b) $L_1$ is not tightly partition embeddable

1Without further mention, all structures considered in this paper, which is a large-scale expansion of [W3], will be finite. It is easy to see that cover-preserving implies injective.
Generally the necessary conditions for tight embeddability into $\text{Part}(n)$ turned out [W2] to be quite close to the sufficient ones. To close the gap roughly speaking half of the remaining work is lattice-theoretic (strengthening the necessary conditions), whereas the other half (softening the sufficient conditions) is purely combinatorial and the main topic of this article. Along the way many results of [W2] will be presented in crisper ways.

That each lattice $L$ is embeddable at all into some $\text{Part}(n)$ (although $n$ being super-exponential in $|L|$) was established in a celebrated Theorem of Pudlak-Tuma from 1980. The techniques of the present article are completely different from the ones in [PT], and rather hark back to [HW]. The latter in turn was heavily influenced by the groundbreaking paper of Jónsson and Nation [JN], which was the first to exploit 2-distributivity to investigate modular lattices.

Here comes a brief Section break up. In Section 2 we point out two straightforward sufficient conditions for tight partition embeddability of a modular lattice $L$. One is the tight embeddability of the subdirectly irreducible factors of $L$. The other condition (a consequence of the first or easily shown directly) is the distributivity (= 1-distributivity) of $L$. Lesser known 2-distributivity turns out to be a necessary condition. The next two Sections omit modular lattices altogether. Section 3 reviews the definition of a partial linear space and familiar concepts such as connected components and cycles. Related further concepts, apparently first introduced in [W2], are reviewed and partly trimmed. Section 4, which constitutes a good third of the article, links graphs (and matroids) to partial linear spaces. Some intriguing interplay arises between cycles in partial linear spaces on the one hand and circuits in graphs on the other. How Sections 3 and 4 relate to modular lattices only dawns in Section 5, and becomes apparent in Section 6. The latter also looks into the future.

I am grateful to Manoel Lemos both for Lemma 5 itself and the creative momentum it triggered within me.

2 First steps

After handling easy-going distributive lattices (2.1) we turn to lesser known 2-distributive lattices and show that this condition is necessary for tight partition embeddability (2.2). As to sufficiency, it is enough to embed subdirectly irreducible modular lattices (2.3). Finally we state a numerical inequality that holds in all modular lattices. Its sharpness is sufficient for tight partition embeddability (2.4).

2.1 An order ideal in a poset $(P, \leq)$ is a subset $X \subseteq P$ such that from $y \leq x \in X$ follows $y \in X$. The set $D(P, \leq)$ of all order ideals is closed under $\cap$ and $\cup$, whence it is a (necessarily distributive) sublattice of the powerset lattice $\mathcal{P}(P)$. We denote by $\mathcal{J}(L)$ the set of nonzero join-irreducibles of a lattice $L$, for each $a \in L$ put $J(a) := \{ p \in \mathcal{J}(L) : p \leq a \}$, and write $(J, \leq)$ for the poset arising from restricting the lattice ordering to the subset $J = J(L)$. By Birkhoff’s Theorem each distributive lattice $L = D$ is isomorphic to $D(J, \leq)$ via $a \rightarrow J(a)$. Since therefore $D$ is isomorphic to sublattice of $\mathcal{P}(J)$, and $\mathcal{P}(J)$ is tightly embedded in $\text{Part}(|J|)$ in obvious

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Footnote: Four out of seven open questions throughout the article do not involve modular lattices at all. Two of them (Questions 5 and 6) ask whether certain types of binary matroids must in fact be graphic. We thus sollicite the input of combinatorists.
2.2 As usual we denote by $M_n$ the length two modular lattice with $n$ join irreducibles, and write $D_2$ for the 2-element lattice. It is easily seen that each interval of $\text{Part}(n)$ is isomorphic to a direct product of partition lattices. In particular a length 2 interval is isomorphic to $\text{Part}(3) = M_3$ or to $\text{Part}(2) \times \text{Part}(2) = D_2 \times D_2$. Thus $M_4$ cannot be a covering sublattice of $\text{Part}(n)$. A length 3 interval of $\text{Part}(n)$ is isomorphic to either $\text{Part}(4)$ or $\text{Part}(3) \times \text{Part}(2)$ or $\text{Part}(2) \times \text{Part}(2) \times \text{Part}(2)$. Since none of these lattices has more than 6 atoms, none of them is isomorphic to the subspace lattice of a nondegenerate projective plane.

2.3 A lattice $L$ is called 2-distributive if

$$a \land (b \lor c \lor d) = (a \land (b \lor c)) \lor (a \land (b \lor d)) \lor (a \land (c \lor d))$$

for all $a, b, c, d \in L$. When $L$ is modular, which is always the case for us, the dual identity holds as well. Evidently (1) is a (wide-ranging) generalization of the distributive law. One can prove [W1] that $L$ is 2-distributive iff it doesn’t contain a length 3 interval isomorphic to the ‘thick’ subspace lattice of a nondegenerate projective plane. It will be handy to call a modular lattice thin if it is 2-distributive and doesn’t contain a covering sublattice $M_4$. In particular every modular lattice that is tightly partition embeddable must be thin. The lattice in Figure 1(b) shows that the converse fails.

2.4 Let $s = s(N)$ be the number of subdirectly irreducible[4] congruences $\theta_i$ of $N$. Putting $N_i := N/\theta_i$ it follows that $a \mapsto (a\theta_1, \ldots, a\theta_s)$ is an injective lattice homomorphism from $N$ into $N_1 \times \ldots \times N_s$. This is illustrated for $N_1 = M_3$ and $N_2 = D_2$ in Figure 2(a),(b),(c).

Theorem 1: If the subdirectly irreducible factors of the modular lattice $N$ are tightly partition embeddable, then so is $N$.

Proof. Putting $N_i := N/\theta_i$ let $f_i : N_i \to \text{Part}(n_i)$ be tight embeddings ($1 \leq i \leq s$). In order to get a tight embedding $f : N \to \text{Part}(n)$ let

$$T : N \to N_1 \times \cdots \times N_s, \quad a \mapsto (T_1(a), \ldots, T_s(a))$$

be a subdirect embedding of $N$. Thus $T$ is injective and all component maps $T_i : N \to N_i$ are surjective. In order to see that the homomorphism $f(a) := (f_1(T_1(a)), \ldots, f_s(T_s(a)))$ is a tight embedding of $N$ into $\text{Part}(n_1) \times \ldots \times \text{Part}(n_s)$, take any covering pair $a \prec b$ in $N$. By the injectivity of $T$ there is at least one index, say $i = 1$, such that $T_1(a) < T_1(b)$. From $a \prec b$ and the surjectivity of $T_1$ follows that in fact $T_1(a) < T_1(b)$. By footnote 5 there is no other $j$ with $T_j(a) < T_j(b)$, thus $T_j(a) = T_j(b)$ for all $j > 1$. Since $f_1$ is cover-preserving, one has

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3For $A \subseteq J$ put $f(A) := \{ x : x \in J \setminus A \} \cup \{ A \}$. Then $A \mapsto f(A)$ is a tight lattice embedding $\mathcal{P}(J) \to \text{Part}(\mathcal{P}(J))$.

4The smallest nondegenerate projective plane is the Fano-plane in Figure 6 which has 7 points. Its subspace lattice is isomorphic to $L(GF(2)^3)$, where $GF(2)$ is the 2-element field.

5The $\theta_i$’s constitute all co-atoms of the Boolean [G,p.316] congruence lattice $\text{Con}(N)$, and hence satisfy $\theta_1 \land \ldots \land \theta_s = \Delta$ where $\Delta = \{ (x,x) : x \in N \}$ is the identity-congruence. More specifically, the $\theta_i$’s bijectively match the perspectivity classes of prime quotients. Thus for each covering pair $a \prec b$ in $N$ there is exactly one $\theta_i$ with $(a,b) \not\in \theta_i$. 

3
$f_1(T_1(a)) \prec f_1(T_1(b))$, and so $f(b) = (f_1(T_1(b)), f_2(T_2(a)), \ldots, f_s(T_s(a)))$ is indeed an upper cover of $f(a)$. Finally, putting $n := n_1 + \cdots + n_s$ observe that $\text{Part}(n_1) \times \cdots \times \text{Part}(n_s)$ is isomorphic to the sublattice of $\text{Part}(n)$ that consists of all partitions refining $\{\{1, \ldots, n_1\}, \{n_1 + 1, \ldots, n_1 + n_2\}, \ldots, \{\ldots, n - 1, n\}\}$. QED

In (d) to (f) of Figure 2 all of this is illustrated for $N_1 = M_3$ and $N_2 = D_2$. Speaking of $M_3$ and $D_2$, if the modular lattice $FM(P, \leq)$ freely generated by the poset $(P, \leq)$ happens to be finite, then $FM(P, \leq)$ is a subdirect product of $M_3$’s and $D_2$’s [Wi], and so $FM(P, \leq)$ tightly embeds into $\text{Part}(n)$ for $n$ large enough.

2.5 We write $d(N)$ for the height of a modular lattice $N$. One can show [HW, Thm.6.4] that always

\[(2) \quad |J(N)| \geq 2d(N) - s(N).\]

For instance for $N = L_1$ in Figure 1 this becomes $9 \geq 2 \cdot 4 - 1$. For distributive lattices $D$ equality takes place in (2) because in fact $|J(D)| = d(D) = s(D)$. All $s(D)$ many factor lattices $D/\theta_i$ are isomorphic to the 2-element lattice $D_2$. Equality in (2) also takes place for $M_3$, namely $3 = 2 \cdot 2 - 1$. A complete characterization of the modular lattices for which (2) is sharp, follows in Section 6. All of them are tightly partition embeddable.

\[6\]Thus, apart from 2.1, Theorem 1 provides another reason for the tight partition embeddabiliy of distributive lattices.
3 Partial linear spaces on their own

A partial linear space (PLS) is an ordered pair \((J, \Lambda)\) consisting of a set \(J\) of points and a set \(\Lambda\) of 3-element\(^7\) subsets \(l \subseteq \Lambda\) called lines such that

\[
(3) \quad |l \cap l'| \leq 1 \text{ for all distinct } l, l' \in \Lambda.
\]

We first introduce paths (3.1) and cycles (3.2) in a PLS in unsurprising ways. Lesser known will be these concepts from [W2]: quasi-isolated midpoints (3.3), unique midpoints (3.4), non-decreasingly constructible PLSes (3.5), and the rank of a PLS (3.6).

3.1 Because of (3) any distinct points \(p, q\) of a PLS \((J, \Lambda)\) lie on at most one common line which we then denote by \([p, q]\). For \(n \geq 2\) a tuplet \(P = [p_1, p_2, \ldots, p_n]\) is called a path in \((J, \Lambda)\) if all lines \([p_i, p_{i+1}]\) \((1 \leq i < n)\) exist, are distinct, and for all \(1 \leq i < j < n\) it holds that \([p_i, p_{i+1}] \cap [p_j, p_{j+1}] \neq \emptyset\) iff \(j = i + 1\). Consequently the underlying point set \(P^* := [p_1, p_2] \cup \cdots \cup [p_{n-1}, p_n]\) has cardinality \(2n - 1\). Two points \(p, q\) are connected if \(p = q\) or there is a path \([p, \ldots, q]\). This yields an equivalence relation whose \(c(J, \Lambda)\) many classes are the connected components of \((J, \Lambda)\). Call \(p\) isolated when \(\{p\}\) is a connected component.

3.2 A cycle\(^8\) \(C = (p_1, p_2, \ldots, p_n)\) is a path \([p_1, p_2, \ldots, p_n]\) such that the line \([p_n, p_1]\) exists and features a new point, i.e. \(C^* := [p_1, \cdots, p_n]^* \cup [p_n, p_1]\) has cardinality \(2n\). Thus \((1, 2, 5)\) is a cycle in the PLS depicted in Figure 3(a). In contrast \([1, 2, 3, 4]\) is a path which is no cycle; albeit \([4, 1]\) = \([4, 1, 5]\) exists, one has \(5 \in [1, 2, 3, 4]^*\).

**Figure 3**

(a) The path \([1, 2, 3, 4]\) is no cycle
(b) An acyclic PLS
(c) Another drawing

Necessarily each cycle \(C = (p_1, p_2, \ldots, p_n)\) has \(n \geq 3\) and we call the points \(p_i\) the \(C\)-junctions. For each \(C\)-line \(l = [p_i, p_{i+1}]\) \((\text{where } n+1 := 1)\) the unique point in \(l \setminus [p_i, p_{i+1}]\) is called the \(C\)-midpoint of \(l\). Thus \(C\)-lines can only intersect in a \(C\)-junction, never in a \(C\)-midpoint.

For instance, consider \((J_1, \Lambda_1)\) depicted in Figure 4 (i). It features the cycle \(C = (1, 3, 5, 7)\). The \(C\)-midpoints of \([1, 3]\), \([3, 5]\), \([5, 7]\), \([7, 1]\) are \(2, 4, 6, 8\) respectively and they ‘physically’ appear in the middle of the drawn lines. This cannot always be achieved simultaneously for all cycles.

\(^7\)Usually also lines of cardinality > 3 are considered, but for us only cardinality 3 matters. This relates to the fact that \(M_3\) is tightly embeddable into \(Part(n)\) but \(M_4\) is not.

\(^8\)Albeit ‘cycle’ is used in graph theory, for us ‘cycle’ always refers to PLSes. We shall soon be concerned with the corresponding structure in graphs (which we name ‘circuits’).
For instance $C' = (2, 3, 5)$ is a cycle of $(J_1, \Lambda_1)$ whose line $\{2, 3\}$ has the $C'$-midpoint 1. Similarly $C'' = (1, 2, 5)$ is a cycle of the PLS in Figure 3(a) whose lines $\{2, 5\}$ and $\{5, 1\}$ have $C''$-midpoints 3 and 4 respectively.

![Figure 4](image)

A PLS without cycles is called acyclic, an example being shown in Figure 3(b).

**3.3** We say that a PLS $(J, \Lambda)$ is a QIMP, if each $\ell \in \Lambda$ contains at least one quasi-isolated point, i.e. one which is on no other line. In particular, any $\ell \in \Lambda$ that happens to occur in cycles $C$ and $C'$ satisfies:

\[(4) \ (C\text{-midpoint of } \ell) = (C'\text{-midpoint of } \ell) = (\text{unique quasi-isolated point of } \ell)\]

This explains the acronym QIMP (= quasi-isolated midpoints). Evidently each acyclic PLS, no matter how we draw it, is a QIMP. Thus Figures 3(b) and 3(c) represent the same acyclic QIMP with quasi-isolated points 2, 3, 4, 6, 7, 8, 9.

Starting with any graph $G$ and ‘plotting’ one new point on each edge obviously yields a QIMP $(J, \Lambda(G))$, see Figure 5. Conversely, let $(J, \Lambda)$ be any QIMP. Fix any $\ell = \{p, q, r\}$ in $\Lambda$ and let $q$ only be incident with $\ell$. If $(J_0, \Lambda_0)$ is defined by $J_0 := J \setminus \{q\}$ and $\Lambda_0 := \Lambda \setminus \{\ell\}$ then clearly $(J_0, \Lambda_0)$ remains a QIMP. By induction $(J_0, \Lambda_0) = (J, \Lambda(G_0))$ for some graph $G_0$. If $G$ arises from $G_0$ by addition of the edge $\{p, r\}$ then $(J, \Lambda) = (J, \Lambda(G))$ by the isolation property of $q$.

**3.4** We say the PLS $(J, \Lambda)$ is a UMP (unique midpoints) if for any two cycles $C$, $C'$ and each $\ell \in \Lambda$ that simultaneously is a $C$-line and a $C'$-line, the $C$-midpoint coincides with the $C'$-midpoint. In other words, the first ‘=’ in (4) takes place. For instance $(J_1, \Lambda_1)$ in Figure 4(i) is not a UMP, and neither is the PLS in Figure 3(a). (Why?)

![Figure 5](image)

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\(^9\) All graphs appearing in this paper are assumed to be simple, i.e without multiple edges and loops.
Let us generalize the fact that each QIMP is a UMP. Let $(J^1, \Lambda^1)$ to $(J^t, \Lambda^t)$ be PLSes such that $J^2 \cap J^1 = \{p_2\}$, $J^3 \cap (J^1 \cup J^2) = \{p_3\}$, and so on until $J^t \cap (J^1 \cup \cdots \cup J^{t-1}) = \{p_t\}$. Here the points $p_i$ need not be distinct. Putting $J := \bigcup_{i=1}^t J^i$ and $\Lambda := \bigcup_{i=1}^t \Lambda^i$ it is clear that $(J, \Lambda)$ is again a PLS. We call it a tree of the PLSes $(J^i, \Lambda^i)$. Using induction on $i$ one sees that each cycle $C$ of $(J, \Lambda)$ must be such that for some fixed element $j \in [t]$ all $C$-lines are contained in $\Lambda^j$. In particular it follows that not just each QIMP, but each tree of QIMPes has unique midpoints. Up to connectedness, the converse holds as well:

Lemma 2: The PLS $(J, \Lambda)$ is a UMP iff each connected component of $(J, \Lambda)$ is a tree of QIMPes.

3.5 We call a PLS $(J, \Lambda)$ nondecreasingly constructible (ndc) if there is an ordering $(\ell_1, \ldots, \ell_t)$ of $\Lambda$ such that $\ell_{i+1} \not\subseteq \ell_1 \cup \ldots \cup \ell_i$ for all $1 \leq i < t$. Obviously 'QIMP $\Rightarrow$ ndc' since any ordering of $\Lambda$ will do. In view of Lemma 2, more generally 'UMP $\Rightarrow$ ndc'. Although $(J_1, \Lambda_1)$ in Figure 4(i) is not a UMP, it is ndc since

$$\ell_1 = \{1, 2, 3\}, \ \ell_2 = \{3, 4, 5\}, \ \ell_3 = \{5, 6, 7\}, \ \ell_4 = \{7, 8, 1\}, \ \ell_5 = \{2, 9, 5\}$$

is a nondecreasing ordering of $\Lambda_1$. In contrast, one checks that $(J_2, \Lambda_2)$ in Figure 6(a) is not ndc.

3.6 Define the rank (more precisely: PLS-rank) of $(J, \Lambda)$ as

$$rk(J, \Lambda) := |J| - |\Lambda|.$$  

For instance $rk(J_1, \Lambda_1) = 9 - 5 = 4$ and $rk(J_2, \Lambda_2) = 7 - 7 = 0$.

Let $(J'_1, \Lambda'_1)$ to $(J'_c, \Lambda'_c)$ be the connected components of the PLS $(J', \Lambda')$. Because of

$$rk(J', \Lambda') = \sum_{i=1}^c |J'_i| - \sum_{i=1}^c |\Lambda'_i| = \sum_{i=1}^c (|J'_i| - |\Lambda'_i|) = \sum_{i=1}^c rk(J'_i, \Lambda'_i)$$

Up to terminology this is shown in [W2, Lemma 13]: QIMP is called mpi, and UMP is called regular. In [W2,p.216] a general but clumsier definition of the rank of a PLS is given (in terms of 'point splittings'). As pointed out by Jim Geelen, the general definition boils down to (5) in our scenario where all lines have cardinality 3.
we will usually restrict ourselves to connected PLSes \((J, \Lambda)\). Then \(\Lambda\) (if nonempty) can be ordered in such a way \((\ell_1, \ldots, \ell_t)\) that \((\ell_1 \cup \ldots \cup \ell_i) \cap \ell_{i+1} \neq \emptyset\) for all \(1 \leq i < t\). Upon adding \(\ell_{i+1}\) the rank \(rk\) of the PLS so far by (5) changes in one of three ways:

\((7+)\) If \(|(\ell_1 \cup \ldots \cup \ell_i) \cap \ell_{i+1}| = 1\) then \(rk\) increases to \(rk + (2 - 1) = rk + 1\).

\((7)\) If \(|(\ell_1 \cup \ldots \cup \ell_i) \cap \ell_{i+1}| = 2\) then \(rk\) remains \(rk + (1 - 1) = rk\).

\((7-)\) If \(|(\ell_1 \cup \ldots \cup \ell_i) \cap \ell_{i+1}| = 3\) then \(rk\) decreases to \(rk + (0 - 1) = rk - 1\).

Observe that the non-decreasing orderings of \(\Lambda\) in 3.5 are exactly the orderings that avoid the rank-decreasing case \((7-)\).

In Section 6 we shall look at PLSes \((J, \Lambda)\) whose universe \(J\) happens to be the set \(J(L)\) of join-irreducibles of a modular lattice \(L\). For instance we will match the points of \(J_1\) in Figure 4 with the 9 join-irreducibles of \(L_1\) in Figure 1. Unsurprisingly, such PLSes give rise to various new concepts. Novel features also arise when matroids enter the scene, and that happens now\(^{12}\).

4 Matroids and partial linear spaces

In 4.1 we define how a matroid models a PLS. Our main focus will be on binary matroids, in particular graphic ones. Subsection 4.2 shows in detail how each UMP can be modeled by a graph (i.e. its associated graphic matroid). Most of the remainder of Section 4 is dedicated to finding a necessary condition for a PLS to be modeled by a graph. Cycles are boring types of PLSes, but not so when it comes to being modeled by a graph. After investigating both their standard and non-standard modeling graphs (4.3, 4.4) we are in a position to state Lemma 5. Under the overall assumption that the modeling is ‘cycle-friendly’ (4.5) it restricts the shape of PLSes that want to be modeled by a graph. If all cycles in \((J, \Lambda)\) have cardinality at most 4 then cycle-friendliness is for free (Corollary 6). A dual kind of property is ‘circuit-friendliness’ in 4.6.

4.1 Let \(M = M(E)\) be any simple matroid with universe \(E\). We say that \(M(E)\) weakly models the PLS \((J, \Lambda)\) if there is a bijection \(\psi : J \rightarrow E\) such that

\((8)\) \(\psi(\ell)\) is dependent in \(M\) for all \(\ell \in \Lambda\) (dependence condition of the first\(^{13}\) kind).

For instance the PLS in Figure 6(a) is weakly modeled by a binary matroid as shown in Figure 6(b). (To unclutter notation often the explicite mention of \(\psi\) will be omitted.)

In contrast we claim that there is no weak binary matroid modeling the PLS \((J_3, \Lambda_3)\) in Figure 7. By way of contradiction, we may assume the two top horizontal lines in Figure 7 are \(\psi\)-labeled as they are, where \(a, a', b, b'\) belong to some vector space \(GF(2)^n\). In view of (8) the three vertical lines force the labelling of the bottom line. Applying (8) to the diagonal line yields

\(^{12}\) In a nutshell, Section 4 links matroids and PLSes, Section 5 links matroids and modular lattices, and Section 6 puts the pieces together.

\(^{13}\) A second kind will be introduced in Section 6.
\[ b' = (a + a') + (a + b) = a' + b, \] and so \[ a + a' + b + b' = a + (a' + b) + b' = a' + b + b' = a \] which contradicts the injectivity of \( \psi \).

Figure 7: No binary matroid weakly models the PLS \((J_3, \Lambda_3)\)

In the sequel we often focus on the most natural subclass of binary matroids \( M(E) \), i.e. the class of graphic matroids. Thus by definition \( E \) is the edge set of a graph \( G = (V, E) \) with vertex set \( V \), and a subset of \( E \) is dependent iff it contains the edge set of a circuit\(^{14}\). Consequently if the graphic matroid \( M(E) \) weakly models \((J, \Lambda)\) then each line \( \ell \) maps to a triangle of \( G \). Instead of saying '\( M(E) \) weakly models \((J, \Lambda)\)' we usually say the graph \( G \) weakly models \((J, \Lambda)\).

4.1.1 For any matroid \( M(E) \) weakly modeling \((J, \Lambda)\) we may compare the PLS-rank \( \text{rk}(J, \Lambda) \) with the matroid-rank \( \text{mrk}(E) \). For instance Figure 6(b) shows a binary matroid \( M(E) \) with \( E = GF(2)^3 \) that weakly models \((J_2, \Lambda_2)\). One calculates \( \text{rk}(J_2, \Lambda_2) = 0 < 3 = \text{mrk}(E) \). Actually the converse inequality will occur more often:

\[ (9) \quad \text{If } (J, \Lambda) \text{ is ndc and weakly modeled by a matroid } M(E) \text{ then } \text{mrk}(E) \leq \text{rk}(J, \Lambda). \]

Proof of (9). Let \( (\ell_1, \ldots, \ell_t) \) be a nondecreasing listing of \( \Lambda \). For \( t = 1 \) the inequality in (9) becomes \( 2 \leq 2 \). If generally \( \Lambda' = \{\ell_1, \ldots, \ell_i\} \) and \( E' \subseteq E \) matches \( J' := \ell_1 \cup \ldots \cup \ell_i \) via \( \psi \) then \( \text{mrk}(E') \leq \text{rk}(J', \Lambda') \) by induction. Upon adding \( \ell_{i+1} = \{a, b, c\} \) only cases (7+) or (7) occur by the ndc assumption. In case (7+) the PLS-rank increases by 1. As to the matroid, say \( a \in E' \) and \( b, c \notin E' \). By submodularity \( \text{mrk}(E' \cup \{b, c\}) - \text{mrk}(E') \leq \text{mrk}(\{a, b, c\}) - \text{mrk}(\{a\}) = 1 \), and so the mrk-rank increases by at most 1. In case (7), the PLS-rank remains the same. As to the matroid, say \( a, b \in E' \) and \( c \notin E' \). Since \( c \) is in the closure of \( E' \) by (8), also the matroid-rank remains the same. Thus the inequality gets perpetuated. QED

4.1.2 We say a matroid \( M(E) \) (or graph \( G = (V, E) \)) models the PLS \((J, \Lambda)\) if additionally to (8) one has

\[ \begin{align*}
\text{Proof of (9). Let } (\ell_1, \ldots, \ell_t) \text{ be a nondecreasing listing of } \Lambda. \text{ For } t = 1 \text{ the inequality in (9) becomes } 2 \leq 2. \text{ If generally } \Lambda' = \{\ell_1, \ldots, \ell_i\} \text{ and } E' \subseteq E \text{ matches } J' := \ell_1 \cup \ldots \cup \ell_i \text{ via } \psi \text{ then } \text{mrk}(E') \leq \text{rk}(J', \Lambda') \text{ by induction. Upon adding } \ell_{i+1} = \{a, b, c\} \text{ only cases (7+) or (7) occur by the ndc assumption. In case (7+) the PLS-rank increases by 1. As to the matroid, say } a \in E' \text{ and } b, c \notin E'. \text{ By submodularity } \text{mrk}(E' \cup \{b, c\}) - \text{mrk}(E') \leq \text{mrk}(\{a, b, c\}) - \text{mrk}(\{a\}) = 1, \text{ and so the mrk-rank increases by at most 1. In case (7), the PLS-rank remains the same. As to the matroid, say } a, b \in E' \text{ and } c \notin E'. \text{ Since } c \text{ is in the closure of } E' \text{ by (8), also the matroid-rank remains the same. Thus the inequality gets perpetuated. QED}
\end{align*} \]

\[ \text{4.1.2 We say a matroid } M(E) \text{ (or graph } G = (V, E) \text{) models the PLS } (J, \Lambda) \text{ if additionally to (8) one has}\]

\(^{14}\)By definition our circuits need not be simple, i.e. we allow repetition of vertices but not edges. Thus there is no harm identifying a cycle with its underlying edge set.
For instance, letting \( a, b, c, d \) be any independent vectors of \( GF(2)^n \) (thus \( n \geq 4 \)), the 9-element binary matroid \( M(E) \) defined by Figure 4(ii) satisfies (8). Since \( rk(J_1, \Lambda_1) = 4 = mrk(E) \), it satisfies (10) as well, i.e. \( M(E) \) models \( (J_1, \Lambda_1) \).

Question 1: What are necessary or sufficient conditions for a PLS to possess a graphic (or at least binary) modeling matroid?

4.2 Let \((J^1, \Lambda^1)\) be a connected QIMP. Dropping from each line \( \ell \in \Lambda^1 \) one quasi-isolated point results in a set \( \{p_1, \ldots, p_n\} \) of sticky-points. Put \( V^1 = \{0, 1, \ldots, n\} \). Guided by [W2, p.218] we define a graph \( G^1 = (V^1, E^1) \) that will weakly model \((J^1, \Lambda^1)\). As visualized in the first column of Figure 8 (ignore that one line is dashed), each sticky-point \( p_i \) is mapped (by \( \psi \)) to the edge \( \{0, i\} \). Thus the edges assigned to the sticky-points 'stick together' in the common vertex 0. Furthermore, if the line \([p_i, p_j]\) exists then its midpoint \( q \) is mapped to \( \{i, j\} \). In this way (8) is satisfied. Ditto the second column in Figure 8 shows that \((J^2, \Lambda^2)\) is weakly modeled by \( G^2 = (V^2, E^2) \).

Let us check that actually proper modeling takes place. By (5) one has \( rk(J^1, \Lambda^1) = 3 = |V^1| - 1 = mrk(E^1) \) and \( rk(J^2, \Lambda^2) = 3 = mrk(E^2) \). Thus (10) holds twice, and so each \( G^i \) models \((J^i, \Lambda^i)\) \((i = 1, 2)\). The construction of \( G^1, G^2 \) generalizes to arbitrary QIMPes. The obtained graph will be called the \textit{standard} modeling graph of the QIMP.

4.2.1 Let the PLSees \((J^i, \Lambda^i)\) be modeled by graphs \( G^i \), and let \((J, \Lambda)\) be a tree of these PLSees in the sense of 3.4. Then there is a natural graph that models \((J, \Lambda)\) itself [W2, Lemma 12]. We merely illustrate this for the two PLSees (=QIMPes) in Figure 8. Merging \( q \) of \((J^1, \Lambda^1)\) with \( p_1' \) of \((J^2, \Lambda^2)\) yields the tree top right in Figure 8. Continuing our indexing scheme let's denote this UMP by \((J_4, \Lambda_4)\) (recall Lemma 2). For later use note that

\[
(11) \quad rk(J_4, \Lambda_4) = |J'_4| - |\Lambda_4| = |J^1| + |J^2| - 1 - |\Lambda^1| - |\Lambda^2| = rk(J^1, \Lambda^1) + rk(J^2, \Lambda^2) - 1.
\]
4.2.2 If by mimicking the merging of the points $q$ and $p_1'$, we merge the corresponding (dashed) edges $\psi(q) = \{3, 2\}$ of $G^1$ and $\psi(p_1') = \{1', 0'\}$ of $G^2$ (rendered dashed in Figure 8), then we get a graph $G_4 = (V_4, E_4)$. It evidently weakly models $(J_4, \Lambda_4)$. In fact (10) carries over as well:

$$mrk(E_4) = |V_4| - 1 = |V^1| + |V^2| - 2 - 1 = mrk(E^1) + mrk(E^2) - 1$$

$$\Rightarrow rk(J_1, \Lambda^1) + rk(J^2, \Lambda^2) - 1 \equiv rk(J_4, \Lambda_4)$$

In view of Lemma 2 we have thus sketched a proof of [W2, Lemma 14(a)], which we restate as follows.

**Lemma 3:** Each UMP $(J, \Lambda)$ can be modeled by a graph.

Having grasped how the standard graph $G$ of a UMP arises, it will henceforth be more convenient to label the edges rather than the vertices of $G$. In this way we can use the same labels as for the points of the UMP. Thus the third column of Figure 8 gives way to Figure 9. While each line in $\Lambda_4$ gives rise to a triangle in $G_4$, it's no harm that the converse fails: The triangle $\{2, 4, 6\}$ in $G_4$ doesn't match a line of $\Lambda_4$ (but see 4.6).
Figure 9: The UMP \((J_4, \Lambda_4)\) and its standard graph \(G_4\)

4.2.3 In Figure 10(a) we blow up \((J_4, \Lambda_4)\) of Figure 9 to \((J_5, \Lambda_5)\) with \(J_5 = J_4 \cup \{11\}\) and \(\Lambda_5 = \Lambda_4 \cup \{5, 10, 11\}\). Unfortunately the modeling graph \(G_4\) doesn’t adapt accordingly since the edges 5 and 10 cannot be completed to a triangle \(\{5, 10, 11\}\). Nevertheless, Figure 10(b) proves that \((J_5, \Lambda_5)\) has another modeling graph \(G_5 = (V_5, E_5)\); one checks that indeed all lines map to triangles and that \(rk(J_5, \Lambda_5) = 11 - 6 = |V_5| - 1 = mrk(E_5)\).

Figure 10: Also non-UMPes can have modeling graphs

4.3 In this Subsection we consider cycles \(C = (p_1, \ldots, p_n)\) on their own, i.e. not embedded in a larger PLS. Thus consider a PLS of type \((C^*, \Lambda^*)\) with \(\Lambda^* = \{[p_i, p_{i+1}] : 1 \leq i \leq n\}\), and \(C^*\) being the underlying set of \(C\)-junctions \(p_i\) and \(C\)-midpoints \(q_i \in [p_i, p_{i+1}]\). See Figure 11(a) where \(n = 6\). Any cycle \(C\) is a QIMP (see 2.2) whose modeling standard graph is a wheel \(W = (V, E)\) as in Figure 11(b). Thus these transformations occur:

- junctions of \(C\) → spokes of \(W\)
- midpoints of \(C\) → rims of \(W\)
4.3.1 Can cycles be modeled by other graphs as well? Since cycles are ndc it follows from (9) that in any case

\[ \text{mrk}(E) \leq \text{rk}(C^*, \Lambda^*) \]

for each weakly modeling graph \( G = (V, E) \).

That \( < \) can take place in (13) is witnessed by the cycle \( C_6 \) in Figure 12, which has a weakly modeling non-wheel graph \( G_6 = (V_6, E_6) \). Indeed one checks that \( \text{mrk}(E_6) = 4 < 5 = \text{rk}(C_6^*, \Lambda_6^*) \).

The good news is, when rank-consistency (10) is postulated (i.e. 'weakly' is dropped in (13)), then Lemma 4 provides what we want.

Figure 12: Cycles can also be modeled by non-wheels

Lemma 4: If the PLS \((J, \Lambda)\) is a cycle \((C^*, \Lambda^*)\), and has a a modeling graph \( G = (V, E) \), then \( G \) must be a wheel.

Proof. Suppose \( C = (p_1, \ldots, p_n) \). For each \( r \in C^* \) let \( r' \in E \) be the associated edge in the modeling graph \( G = (V, E) \). We claim it suffices to show that \( \{p'_1, \ldots, p'_n\} \) is of type

\[ \{p'_1, \ldots, p'_n\} = \text{star}(v), \]

i.e. all edges incident with some vertex \( v \) of \( G \). Indeed, since each \( \{p_i, q_i, p_{i+1}\} \in \Lambda^* \) yields a triangle \( \{p'_i, q'_i, p'_{i+1}\} \) of \( G \), it will follow from (14) that \( G \) is a wheel \( W \), thus proving Lemma 4.

(15) If (14) fails then the edge set \( \{p'_1, \ldots, p'_n\} \) contains the edge set of a circuit of \( G \).
Proof of (15). Because for each line \( \{p_i, q_i, p_{i+1}\} \) the edge set \( \{p'_i, q'_i, p'_{i+1}\} \) is a triangle of \( G \), each edge \( p'_{i+1} \) is incident with edge \( p'_i \) (modulo \( n \)). Say \( p'_2 \) is incident with \( p'_1 = \{v_1, v_2\} \) in \( v_2 \). Since (14) fails there is \( i \geq 2 \) such that \( p'_i \) is incident with \( v_2 \) but \( p'_{i+1} \) is not, see Figure 13. If \( p'_{i+1} \) is incident with \( v_1 \) then \( \{p'_1, p'_i, p'_{i+1}\} \) is a triangle (whence circuit) of \( G \). Otherwise consider \( p'_{i+2} \). As shown in Figure 13 there are two options for \( p'_{i+2} \). Whichever option takes place, if \( p'_{i+2} \) is incident with \( p'_1 \) then we get again a circuit of \( G \), if not continue with \( p'_{i+3} \), and so on. Because at the latest \( p'_n \) is incident with \( p'_1 \), there must be a cycle in \( G \). This proves (15).

Figure 13: Visualizing the proof of Lemma 4

Now by way of contradiction assume that (14) fails. On the one hand \( \{p'_1, \ldots, p'_n\} \) spans the universe \( E \) of the graphic matroid since each \( q'_i \) is in a circuit \( \{p'_i, q'_i, p'_{i+1}\} \). On the other hand \( \{p'_1, \ldots, p'_n\} \) is dependent since it contains a circuit by (15). Hence \( mrk(E) = mrk(\{p'_1, \ldots, p'_n\}) \leq n - 1 \). Yet \( rk(C^*, \Lambda^*) = |C^*| - |\Lambda^*| = 2n - n = n \), and so \( rk(C^*, \Lambda^*) \neq mrk(E) \). This contradicts rank-consistency, and thus proves (14).

4.3.2 Figure 12 shows that the rank-consistency in Lemma 4 cannot be dropped. However, it can be dropped (thus the word ‘weak’ below) for small cycles:

(16) Let the cycle \( (C^*, \Lambda^*) \) be small in the sense that \( |\Lambda^*| \in \{3, 4\} \). If \( G = (V, E) \) weakly models \( (C^*, \Lambda^*) \) then \( G \) must be a wheel.

Proof of (16). Suppose first (case 1) that strict inequality \( < \) takes place in (13). Then \( |V| - 1 = mrk(E) < rk(C^*, \Lambda^*) = |\Lambda^*| \). If \( |\Lambda^*| = 3 \) then \( |V| \leq 3 \), whence \( |E| \leq \binom{3}{2} < 6 = |C^*| \). If \( |\Lambda^*| = 4 \) then \( |V| \leq 4 \), whence \( |E| \leq \binom{4}{2} < 8 = |C^*| \). In both subcases this contradicts the bijectivity of \( C^* \to E \). Now suppose (case 2) that equality takes place in (13). Then the claim follows from Lemma 4. QED

4.4 Unfortunately, Lemma 4 only applies to isolated cycles and not to embedded cycles. Specifically, we claim that although \( G_7 \) models \( (J_7, \Lambda_7) \) in Figure 14, the former contains a cycle \( C \) of rank higher than \( rk(J_7, \Lambda_7) \) that maps to a non-wheel in \( G_7 \). Indeed, one checks that the lines of \( \Lambda_7 \) map to triangles and that \( rk(J_7, \Lambda_7) = 13 - 8 = |V_7| - 1 = mrk(E_7) \). The cycle \( C = (1, 2, 3, 4, 5, 6) \) of \( (J_7, \Lambda_7) \) maps to edges in \( G_7 \) that are not the spokes of a wheel. The PLS-rank of \( (C^*, \Lambda^*) \) is 6, thus higher than the PLS-rank 5 of its host \( (J_7, \Lambda_7) \). (It is noteworthy that \( (J_7, \Lambda_7) \) is benign enough to be ndc.)

The cardinality argument breaks down when \( |\Lambda^*| = 5 \) since then \( |E| \leq \binom{5}{2} = 10 = |C^*| \). And indeed things can go wrong, as witnessed by Figure 12.
4.5 Let \((J, \Lambda)\) be a PLS containing a cycle \(C = (p_1, \ldots, p_n)\), thus with \(C\)-junctions \(p_i\) and \(C\)-midpoints \(q_i \in [p_i, p_{i+1}]\). As will be seen, a path between two junctions \(p_i\) and \(p_j\) is benign, whereas paths between \(q_i\), \(q_j\) or between \(q_i\), \(p_j\) pose problems. Specifically, a \textit{type 1 midpoint-link} (of \(C\)) is a path \(P = (q_i, \ldots, p_j)\) with \(P^* \cap C^* = \{q_i, p_j\}\), and a \textit{type 2 midpoint-link} is a path \(P\) between two midpoints, i.e. \(P = (q_i, \ldots, q_j)\) with \(P^* \cap C^* = \{q_i, q_j\}\).

A type 1 (resp. type 2) midpoint-link of \(C\) is \textit{close} if \(p_j \in \{p_i, p_{i+1}\}\) (resp. \(q_j = q_{i+1}\)). For instance, the cycle \((1, 3, 5)\) in \((J_5, \Lambda_5)\) has a (type 1) close midpoint-link \((4, 8, 10, 5)\). And the cycle \((1, 3, 5)\) in \((J_2, \Lambda_2)\) has the (type 2) close midpoint-links \((2, 4)\) and \((2, 6)\) and \((4, 6)\). In contrast, the cycle \((1, 3, 5, 7)\) of \((J_1, \Lambda_1)\) in Figure 4(i) has a non-close midpoint-link \((2, 5)\).

A cycle \textit{loves} close midpoint-links if all its midpoint links are close. The PLS \((J, \Lambda)\) as a whole is said to \textit{love close midpoint-links} if all its cycles love close midpoint-links. Trivially QIMPs lack midpoint-links altogether. Since each UMP is a tree of QIMPs, each cycle of a UMP is contained in one of its QIMP components We thus conclude:

\[(17)\] Each UMP loves close midpoint-links (because it has no midpoint-links).

The PLSes in Figure 3(a) and \((J_5, \Lambda_5)\) in Figure 10 are 'proper' examples of PLSes loving close midpoint-links. Neither \((J_1, \Lambda_1)\) nor \((J_7, \Lambda_7)\) in Figure 14 loves close midpoint-links; thus they shun close midpoint-links.

\[4.5.1\] A weak modeling graph \(G\) of a PLS \((J, \Lambda)\) is \textit{cycle-friendly} if the midoints of each cycle \(C\) of \((J, \Lambda)\) map to a circuit \(\Gamma\) of \(G\). This forces \(\Gamma\) to be of a very specific shape. Namely, since each \(\ell \in \Lambda\) maps to a triangle in \(G\), a quick sketch confirms that the edges in \(\Gamma\) are the rimes of a \textit{wheel} whose spokes bijectively correspond to the junctions of \(C\). The most obvious example of
a cycle-friendly modeling graph is the standard graph of a QIMP. More generally, akin to (17) one concludes at once from the fact that UMPes are trees of QIMPes:

(18) The standard graph of a UMP is cycle-friendly.

Notice that for a UMP that happens to be a mere cycle the proviso 'standard' in (18) is not necessary (Lemma 4). Modeling graphs of PLSes that are not UMPes, may (Fig. 10), or may not (Fig. 14), be cycle-friendly. Speaking of Figure 10, recall that this PLS also loves close midpoint-links. This is no coincidence:

Lemma 5: If \((J, \Lambda)\) admits a cycle-friendly weak modeling graph \(G\), then \((J, \Lambda)\) loves close midpoint-links.

Proof \(^{16}\). Consider a cycle \(C\) of \((J, \Lambda)\), without much loss of generality let’s take the one in Figure 11(a). By assumption it is mapped onto a subwheel of \(G\) (see Figure 11(b)). By way of contradiction assume \(C\) had non-close midpoint-links.

Case 1: Suppose there is a non-close type 1 midpoint-link, such as \((q_2, p_4)\) in Figure 15(a). As shown in Figure 15(b) this yields a cycle \(C'\) in \((J, \Lambda)\) with junctions \(p_4, q_2, q_2, p_3\). By assumption \(C'\) maps to a subwheel of \(G\), in such a way that \(p_4, q_2\) are mapped to incident edges (being spokes). But this contradicts Figure 11(b) where these edges are not incident. (Generally in each wheel \(p_i\) is incident with \(q_j\) only when \(j \in \{i - 1, i\}\).)

Case 2: Suppose there is a non-close type 2 midpoint-link, such as \((q_2, q_5)\) in Figure 15(c). Then, as illustrated in Fig. 15(d), there is a cycle \(C''\) in \((J, \Lambda)\) with junctions \((q_2, q_5)\) in \((J, \Lambda)\) with junctions (among others) \(q_2, q_5\). As in case 1 they are mapped to incident edges of \(G\). This contradicts Figure 11(b) where \(q_2, q_5\) do not touch. QED

\(^{16}\)The overall proof idea is due to Manoel Lemos but the author cautioned cycle-friendliness.
Corollary 6: If all cycles of \((J, \Lambda)\) are small, and if \((J, \Lambda)\) shuns close midpoint-links, then \((J, \Lambda)\) doesn’t admit modeling graphs.

Proof. By the smallness of cycles and by (16) each modeling graph is cycle-friendly. The claim thus follows from Lemma 5. QED

It follows that \((J_1, \Lambda_1)\) in Figure 4(i) doesn’t admit modeling graphs. This sideshow continues in footnote 25.

4.6 We say that a weakly modeling graph \(G\) of \((J, \Lambda)\) is circuit-friendly if each chordless circuit of \(G\) either corresponds to a line, or to the midpoints of a cycle of \((J, \Lambda)\). Notice the similarities but also differences between 'circuit-friendly' and 'cycle-friendly'. In particular, circuit-friendly has got nothing to do with wheels. Akin to (17), but by quite different arguments [W2, Lemma 14] it also holds that

\[
\text{(19) } \text{the standard graph of a UMP is circuit-friendly.}
\]

For instance the chordless cycle \(\{2, 4, 6\}\) of \(G_4\) in Figure 9 corresponds to the midpoints of a cycle of the UMP \((J_4, \Lambda_4)\). In contrast the cycle \(\{4, 7, 9, 10\}\) of \(G_4\) doesn’t behave that way (since it has edge 8 as a chord). Some non-UMPes qualify as well. Thus one checks [17] that \(G_5\) in Figure 10 circuit-friendly models \((J_5, \Lambda_5)\). For instance the chordless circuit \(\{3, 7, 9, 11\}\) of \(G_5\) maps to the midpoints of a cycle in \((J_5, \Lambda_5)\). Again circuits with chords, such as \(\{1, 2, 4, 5\}\), do

\[\text{Figure 15}\]

\[\text{(a) Midpoint link of type 1}\]

\[\text{(b) The 'new' cycle induced by it}\]

\[\text{(c) Midpoint link of type 2}\]

\[\text{(d) The 'new' cycle induced by it}\]

\[\text{Footnote 25:}\]

\[\text{Footnote 14:}\]

\[\text{Footnote 17:}\]

\[\text{The PLS \((J_5, \Lambda_5)\) belongs to a general, albeit clumsy, class of non-UMPes (called 'quasiregular' in [W2, Lemma 15]) that can be modeled circuit-friendly, and that love close midpoint-links. See Question 6 for a more elegant approach to generalize UMPes.}\]

17
not behave that way.

**Question 2:** What are the relations between cycle-friendly and circuit-friendly modeling graphs, and (apart from Lemma 5) how do PLSes loving close midpoint-links tie in?

Be it as it may, Theorem 10 will show that circuit-friendly beats cycle-friendly. But then again, better one sparrow (=cycle friendly) in the hand than two sparrows (=circuit friendly) in the bush.

## 5 Matroids and modular lattices

Both tight $k$-linear representations of modular lattices, and tight embeddings of them into partition lattices (Sections 1, 2), fit the common hat of tight embeddings into flat lattices of matroids. Accordingly we investigate various kinds of matroids *modeling modular lattices*. This is akin to Section 4 where we considered matroids *modeling PLSes*. But PLSes are absent in Section 5 and only return in Section 6.

### 5.1 For any field $k$ let $LM(k^n)$ be the (modular) subspace lattice of $k^n$. A *$k$-linear representation* of $L$ is any homomorphism $\Phi : L \to LM(k^n)$. Article [HW] classifies up to isomorphism\textsuperscript{18} all $k$-linear representations of certain ‘acyclic’ modular lattices. (More about them in Section 6.) For general modular lattices it is already nontrivial establishing the mere existence of injective, let alone tight $k$-linear representations.

Existence is all we care about in the present article, and instead of $LM(k^n)$ we more generally look at $LM(K)$, which by definition is the lattice\textsuperscript{19} of flats (=closed subsets) of the matroid $M(K)$. Whenever the closure operator $P(K) \to P(K) : X \mapsto \overline{X}$ is essential we write $M(K, \overline{\cdot})$ rather than $M(K)$. In the special case where $K = k^n$ and $\overline{X}$ is the subspace generated by $X \subseteq k^n$, it can be notationally better to write $\langle X \rangle$ instead of $\overline{X}$. The Lemma below is Lemma 5 in [W2].

**Lemma 7:** Let $L$ be a modular lattice with $J = J(L)$, and let $M(K, \overline{\cdot})$ be a matroid. There is a tight embedding $\Phi : L \to LM(K)$ iff the following holds. There is an injection $\varphi : J \to K$ such that the induced submatroid $M(\varphi(J), \neg)$ is simple and such that (20) and (21) hold:

\begin{align*}
(20) & \quad \varphi(J(a)) = \varphi(J(a)) \text{ for all } a \in J \text{ (dependency\textsuperscript{20} condition of the second kind).} \\
(21) & \quad \text{mrk}(\varphi(J)) = d(L) \text{ (rank consistency of the second kind).}
\end{align*}

This is good and well, but how does $\Phi$ arise from $\varphi$, and vice versa? Given $\varphi$ with (20) and (21), one can put $\Phi(a) := \varphi(J(a))$. Conversely, given any tight homomorphism $\Phi$, for each $p \in J$ pick any $p' \in \Phi(p) \setminus \Phi(p_*)$ and define $\varphi : J \to K$ by $\varphi(p) := p'$. Here $p_* < p$ is the unique lower cover of $p$ in $L$. It is crucial to distinguish the closure operators $\overline{\cdot}$ and $\neg$. For instance

\textsuperscript{18}The $k$-linear representations $\Phi$ and $\Phi'$ are *isomorphic* if there is a vector space isomorphism $f : k^n \to k^n$ such that $\Phi'(a) = f(\Phi(a))$ for all $a \in L$.

\textsuperscript{19}Such lattices are also known as *geometric* lattices.

\textsuperscript{20}The term ‘closure condition’ would perhaps fit better but we opted for ‘dependency condition’ in order to match the terminology in (8).
In view of the above we say the simple matroid $M(E, -)$ models the modular lattice $L$ if there is a bijection $\varphi : J(L) \to E$ satisfying (20) and (21). As will be further investigated later, condition (20) is the sibling of (8), and (21) the sibling of (10). Consider the three increasingly special cases where $M(E, -)$ incorporates linear dependency (thus $E \subseteq k^m$ for some field $k$), or where particularly $k = GF(2)$, or where $M(E, -)$ is graphic. We then speak of $k$-linear, binary and graphic matroids modeling $L$. In the graphic case we usually speak (akin to Section 4) of the graph $G = (V, E)$ modeling $L$.

5.2 We first trim Lemma 7 to the $k$-linear case (Corollary 8) and then to graphs (Theorem 9).

Corollary 8: Suppose the modular lattice $L$ has height $n = d(L)$ and is modeled by the $k$-linear matroid $M(E, -)$ via $\varphi : J(L) \to E$. Then $\Phi(a) := \langle \varphi(J(a)) \rangle$ ($a \in L$) provides a tight embedding $\Phi : L \to LM(k^m)$. Conversely each tight embedding $\Phi' : L \to LM(k^m)$ comes from such a $\varphi$-induced tight embedding $\Phi : L \to LM(k^m)$.

Proof. Since $L$ is modeled by the $k$-linear $M(E, -)$ one has $mrk(E) = d(L) = n$, and so $E$ can be viewed as a subset of $k^m$ for some $m \geq n$. Lemma 7 hence yields a $k$-linear representation $\Phi : L \to LM(k^m)$. Since the subspace $\Phi(L)$ of $k^m$ is isomorphic to $k^n$, we may view $\Phi$ as a map $L \to LM(k^m)$. Similarly the claim about $\Phi'$ follows from Lemma 7. QED

5.3 Recall that $Part(m)$ is the lattice of all set partitions of $[m]$. The complete graph $CG(m) = ([m], K)$ has vertex set $[m]$ and $|K| = \binom{m}{2}$ many edges. It is well known that the graphic matroid $M(K, -)$ has rank $m - 1$, and its height $m - 1$ flat lattice $LM(K)$ is isomorphic to $Part(m)$ via $B \mapsto comp(B)$. Here for any edge set $B \subseteq K$ we define $comp(B)$ as the partition of $V$ whose blocks are the vertex sets of the connected components of the subgraph $([m], B)$ of $([m], K)$.

Theorem 9: Let $L$ be a modular lattice of height $n = d(L)$. If the graph $G = ([m + 1], E)$ models $L$ via $\varphi : J(L) \to E$ then $\Phi(a) := comp(\varphi(J(a)))$ ($a \in L$) provides a tight embedding $\Phi : L \to Part(n + 1)$. Conversely, each tight embedding $\Phi' : L \to Part(m)$ comes from such a $\varphi$-induced tight embedding $\Phi : L \to Part(n + 1)$.

Proof. That $\varphi$ as described induces a tight embedding $\Phi : L \to Part(n + 1)$ follows from Lemma 7 and the remarks above. The converse claim is slightly more subtle than in Corollary 8. Thus let $\Phi' : L \to Part(m)$ be any tight embedding. By Lemma 7 (and the remarks above) $\Phi'$ induces a graph $G' = ([m], E)$ that models $L$. By gluing together potential disconnected components of $G'$ one gets a connected graph $G$. A moment’s thought shows that $G$ still models $L$. By connectedness $G$ now has $mrk(E) + 1 \geq d(L) + 1 = n + 1$ vertices. Applying Lemma 7 in the other direction yields a tight embedding $\Phi : L \to Part(n + 1)$. It is fair to say that $\Phi'$ ‘comes from’ $\Phi$. QED

5.3.1 To fix ideas, consider $\Phi'_2 : L_2 \to Part(5)$ as defined in Figure 2 (f). Note that $5 > d(L_2) + 1$. By Lemma 7 the tight embedding $\Phi'_2$ must be induced by a modeling graph $G'_2$. One checks that $G'_2$ has two connected components, a triangle and a single edge. The obtained connected modeling graph $G_2$ is shown in Figure 16(b). Since it now has 4 vertices, it induces a tight embedding $\Phi_2 : L_2 \to Part(4)$. 

\[ \varphi(J(a)) = \varphi(J(a)) \] by (20) whereas generally $\varphi(J(a)) \neq \varphi(J(a))! $
One verifies that $G_2$ in Figure 16(c) is another modeling graph of $L_2$. Indeed, (21) is trivial and (20) will be verified as the variant (20') in a moment. By Theorem 9 $G_2$ yields a tight embedding $L_2 \rightarrow \text{Part}(4)$ that e.g. maps $b \in L_2$ to the partition $\text{comp}(J(b)) = \text{comp}([p,q]) = ([1,3,4],[2])$. Notice that $G_2$ is 2-connected, whereas $G_2$ is not.

5.3.2 The criterion (20') below helps to handle the dependency condition (20) in the graph case. Namely, let $G = (V,E)$ be a graph, $L$ a modular lattice, and $\varphi : J(L) \rightarrow E$ a bijection. Then (20) is by [W2, Lemma 10] equivalent to this condition:

(20') For each chordless circuit $\varphi(X) \subseteq E$ of $G$ it holds that $q \leq \bigvee (X \setminus \{q\})$ for all $q \in X$.

For instance, the only chordless circuit $\{p,q,r,s\}$ of $G_2$ in Figure 16(c) satisfies (20'):

\[
p \leq q \lor r \lor s, \quad q \leq p \lor r \lor s, \quad r \leq p \lor q \lor s, \quad s \leq p \lor q \lor r
\]

5.4 A variation of Lemma 7 shows [W2, Theorem 4] that even each semimodular lattice $L$ is tightly embeddable in a geometric lattice $LM(K)$, but possibly not in a 'nice' one like $LM(k^n)$ or $\text{Part}(n+1)$. That’s because the matroid modeling $L$ is constructed as a submatroid of a certain matroid $(K,\overline{\varnothing})$ triggered by the height function $d : L \rightarrow \mathbb{N}$. This makes $LM(K)$ quite unpredictable. The proof of [W2, Theorem 4] cuts short previous ones by Dilworth 1973 and Grätzer-Kiss 1986.

6 Putting the pieces together and looking ahead

Theorem 9 is good and well, but how can one get graphs modeling modular lattices, and what is the role of PLSes in all of this? A brief answer is as follows. Our work in Section 4 enables us to construct a graph $G$ modeling a given PLS $(J,\Lambda)$. But since $G$ is supposed to model a modular lattice $L$, our $(J,\Lambda)$ better be linked to $L$ somehow. In Section 6 we add the missing link (essentially we force $J = J(L)$) and put the pieces together.

Specifically, the Fundamental Theorem (FT) of Projective Geometry links complemented modular lattices $L$ with well-known types of PLSes, i.e. projective planes $(J,\Lambda)$. Here $J$ can be

\footnote{This also shows that not each tight embedding of a subdirectly reducible height a modular lattice $L$ into $\text{Part}(n+1)$ arises from tight embeddings of its subdirect factors $L_i$ (as it does in Theorem 1).}
identified with the set of atoms of $L$. However a complemented $L$ (except for trivial cases) isn’t tightly partition embeddable, and so projective planes need to be adapted appropriately. In brief (6.1), the adapted PLSes $(J, \Lambda)$ on the one hand replace an unordered set of atoms by a partially ordered set $J = J(L)$ of join irreducibles. On the other they obey the crucial equality $rk(J, \Lambda) = d(L)$. Since from now on always $J$ equals the set $J(L)$ of join irreducibles, this opens the possibility that a graph (or matroid) simultaneously models the PLS $(J, \Lambda)$ (in the sense of Section 4) and the lattice $L$ (in the sense of Section 5). Combining Theorem 9 with the new material leads to Theorem 11 in 6.2 which states the tight partition embeddability for a large class of modular lattices (related to UMPes). Having sketched (in 6.3) an unpleasant state of affairs in [W2] we outline (in 6.4) a program that aims to fix that. In brief, it works when certain binary matroids are actually graphic. Thus we attempt to shift the burden from lattice theory to matroid theory.

6.1 Let $L$ be a modular lattice. If $\ell \subseteq J(L)$ is maximal w.r.t. the property that any distinct $p, q \in \ell$ yield the same join $p \lor q$ (which we denote by $\bar{\ell}$), and if $|\ell| = 3$ then $\ell$ is called a line (of $L$). Two lines $\ell$ and $\ell_0$ are equivalent if $\bar{\ell} = \bar{\ell_0}$. Any maximal family $\Lambda$ of mutually inequivalent lines yields a partial linear space $(J, \Lambda)$ (since (3) is satisfied), which we call a MoPLS. For instance, take $L = L_1$ from Figure 1 which is rendered again in Figure 17(a); one has $J(L_1) = \{9\}$. For instance $\ell = \{1, 2, 4\}$ is a line with $\bar{\ell} = \top$. The line $\ell_0 = \{3, 8, 9\}$, and six more, are equivalent to $\ell$. One verifies that the five lines in Figure 17(b) yield a MoPLS $(J(L_1), \Lambda)$.

![Figure 17](image)

6.1.1 MoPLSes are important because they extend Birkhoff’s Theorem (2.1) in elegant ways to modular lattices $L$. Specifically, an order ideal $X$ of $(J, \leq)$ is called $\Lambda$-closed if for all $\ell \in \Lambda$ it follows from $|\ell \cap X| \geq 2$ that $\ell \subseteq X$. If $L(J, \leq, \Lambda)$ denotes the closure system of all $\Lambda$-closed order ideals then $a \mapsto J(a)$ turns out [HW, Thm. 2.5] to be a lattice isomorphism from $L$ onto $L(J, \leq, \Lambda)$. Each MoPLS of $L$ has $s(L)$ many connected components, they being in bijection with the subdirectly irreducible factors of $L$. If $L = D$ is distributive then $s(D) = |J(D)|$, and so $L(J, \leq, \Lambda)$ boils down to $D(J, \leq)$ in 2.1.

6.1.2 A modular lattice is acyclic if all (equivalently: one) its MoPLSes are acyclic. Inequality

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22 While $|\ell| > 3$ occurs we will only be interested in lattices $L$ all of whose lines have cardinality 3. Recall the corresponding remark about PLSes.

23 The acronym Mo emphasizes that this PLS arises from a modular lattice. In [HW] the terminology ‘base of lines of $L$’ was used.
(2) is sharp exactly for acyclic modular lattices. A weaker form of local acyclicity of MoPLSes (to be glimpsed in footnote 23) characterizes the 2-distributivity of a lattice. For instance $L_1$ is 2-distributive but not acyclic.

6.1.3 Speaking of 2-distributivity, recall from 2.2 how 'thin' enhances this property. According to [W2, Lemma 19] each MoPLS $(J, \Lambda)$ of a thin lattice $L$ satisfies

$$(22) \quad rk(J, \Lambda) = d(L).$$

For instance $L_0$, $L_1$ from Figure 1 are thin and one checks that $rk(J_i, \Lambda_i) = d(L_i) = 4$ for $i = 0, 1$. For modular lattices which are not 2-distributive equality (22) probably fails; e.g. $LM(GF(2)^3)$ has $(J_2, \Lambda_2)$ as unique MoPLS and (22) fails since $rk(J_2, \Lambda_2) = 0$.

6.2 One may be led to say that Theorem 9 settles the tight embeddability of a thin height $n$ lattice $L$ into Part$(n + 1)$: It works iff there is a connected graph $G = ([n + 1], E)$ and a bijection $\varphi : J(L) \to E$ that obeys (20') and (21). But how can the existence of such a $\varphi$ be decided for concrete lattices? That’s why in Section 4 we learned about graphs modeling PLSes. Nevertheless, graphs modeling PLSes merely constitute a crutch to the actually relevant, but more enigmatic graphs modeling thin lattices.

Let us look at this crutch more closely. The good news is that in view of link (22) rank consistency for (Mo)PLSes (i.e. (10)) is equivalent to rank consistency for lattices (i.e. (21)). Bad news is that the dependency condition for lattices (i.e. (20) or (20')) is not equivalent to the dependency condition for (Mo)PLSes (i.e. (8)). For instance $\overline{G_2}$ in Figure 16 satisfies (20') but not (8). However, the following remains an open question:

**Question 3**: Is there a 'freak' graph $G$ and a MoPLS $(J, \Lambda)$ of a thin lattice $L$ such that $G$ models $(J, \Lambda)$ yet not $L$?

The hoped for answer is 'no'. Putting it another way, while the existence of a graph modeling a MoPLS of $L$ may not be necessary for tight partition embeddability of $L$, it may well be sufficient. Adding the extra ingredient of circuit-friendliness, it actually is sufficient:

**Theorem 10**: If the thin lattice $L$ has a MoPLS that is circuit-friendly modeled by a graph then $L$ is tightly partition embeddable.

**Proof.** Putting $J = J(L)$ let $G = (\varphi(J), E)$ circuit-friendly model the MoPLS $(J, \Lambda)$ of $L$. We need to verify (20') and (21). As to rank consistency (21), recall that this is automatic whenever $G$ (normally) models $(J, \Lambda)$. As to (20'), let $\varphi(X) \subseteq E$ be a chordless circuit of $G$. By definition of circuit-friendliness in 4.6 the set $X$ is either a line $X = \{p, q, r\}$ or the set of $C$-midpoints of a cycle $C$ in $(J, \Lambda)$. For $X = \{p, q, r\}$ condition (20') holds in view of $p \lor q = p \lor r = q \lor r$. If $X$ is a set of $C$-midpoints then (20') holds by [W2, Lemma 20]. QED

From Theorem 10 and (19) follows:

**Theorem 11**: If the height $n$ thin lattice $L$ has a MoUMP then there is a tight embedding $\Phi : L \to Part(n + 1)$.

In line with Question 3 (and also Question 1) we further ask:
Question 4: What kind of PLS can occur at all as MoPLS of a thin lattice?

Recall from 2.2 that 2-distributive modular lattices are characterized by the avoidance of length three interval sublattices isomorphic to the subspace lattice of a nondegenerate projective plane. In particular $LM(GF(2)^3)$ or any other $LM(k^3)$ are avoided. It thus is plausible\footnote{Nevertheless it is subtle to see how exactly such a configuration forces the existence of a certain length three interval sublattice.} that no seven lines of a 2-distributive modular lattice relate to each other as in Figure 6(a). Actually one can show [HPR, p.388] that even a sparser, so-called triangle configuration (Figure 6(c)) cannot occur as part of a MoPLS of a 2-distributive lattice. Whether e.g. the somewhat similar configuration in Figure 3(c) can occur in a 2-distributive modular lattice remains an open question.

6.3 Apart from cycles in MoPLSes of modular lattices $L$ let us glimpse at cycles of 'essential' elements in $L$ which play a crucial role in [W2]. Each line $\ell$ of a lattice $L$ belongs to a line-interval. This is the tight $M_3$-sublattice of $L$ whose top is $\ell$, and whose bottom is the meet of the three lower covers of $\ell$. Conversely, each interval sublattice $[b,a] \simeq M_3$, whose top has only three lower covers in $L$, occurs this way (often hosting more than one line). The lattice in Figure 17(a) has five line-intervals. One of them is $[5,a_4]$ which hosts the two lines $\{1,6,8\}$ and $\{1,7,8\}$. Generally the top $a$ of a line-interval $[b,a]$ is called $M_3$-element (or essential) in [W2,p.211]. There are subtle connections between cycles in MoPLSes (in the sense of 3.2) and 'cycles of $M_3$-elements' in the sense of [W2,p.225]. Each cycle of essential elements in $L$ induces a cycle in any given MoPLS of $L$, but the converse fails. For instance the cycle $(2,3,5)$ in the MoPLS of Figure 17(b) doesn’t yield a cycle of the corresponding three essential elements $a_1$, $a_2$, $a_3$. Various types of $M_3$-elements are distinguished in [W2]. For instance a thin lattice with a $M_3$-element of type (3.3s), such as $L_1$ in Figure 17(a), is not tightly embeddable\footnote{By Theorem 9 therefore no graph models $L_1$. Nevertheless $L_1$ doesn’t trigger ‘yes’ for Question 3 because by Corollary 6 also no graph models $(J_1,\Lambda_1)$ (nor any other MoPLS of $L_1$).} into a partition lattice by [W2, Theorem 6]. Due to Subsection 6.4 there is hope to steer future research away from messy cycles of $M_3$-elements, and more towards matroid theory.

6.4 Here we look into the future. After preliminaries from matroid theory (6.4.1) we introduce a novel type of PLSes (6.4.2). In 6.4.3 we briefly hark back to the Pudlak-Tuma Theorem mentioned in the introduction.

6.4.1 Let $M(E)$ be a matroid such that $E = E_1 \cup E_2$ and $E = E_1 \cap E_2 = \{z\}$. Suppose each circuit $C$ of $M(E)$ is a circuit in either one of the submatroids $M(E_1)$ and $M(E_2)$, or it is of type

\begin{equation}
C = (C_1 \cup C_2) \setminus \{z\} \text{ where } C_i \text{ is a circuit of } M(E_i) \text{ and } C_1 \cap C_2 = \{z\}.
\end{equation}

In this situation $M(E)$ is isomorphic to the so-called parallel connection [O,p.240] of $M(E_1)$ and $M(E_2)$. Furthermore, if both $M(E_i)$ are graphic, then so is $M(E)$.

6.4.2 Call the PLS $(J,\Lambda)$ a graph-trigger if each binary matroid that models $(J,\Lambda)$ must be graphic. For instance:

\begin{equation}
\text{(24) Each acyclic PLS is a graph-trigger.}
\end{equation}

Proof of (24). It suffices to prove the claim for connected PLSes, and this we do by induction
on the number of lines. Thus let the connected \((J, \Lambda)\) be modeled by the binary matroid \(M(E)\) and put \(n = rk(J, \Lambda) = mrk(E)\). We write \(p'\) for the image of \(p \in J\) in \(E\). If \(|\Lambda| = 1\), then \(|E| = |J| = 3\) and \(mrk(E) = rk(J, \Lambda) = 2\) by (8). Hence \(M(E)\) is isomorphic to the graphic matroid induced by a triangle.

Suppose now that \(|\Lambda| > 1\). By acyclicity there is a \(\ell \in \Lambda\) that intersects each other line in at most one point \(z\) (and intersection \(\{z\}\) does occur by connectedness). Putting \(\ell = \{x, y, z\}\), \(J_1 = J \setminus \{x, y\}\), \(\Lambda_1 = \Lambda \setminus \{\ell\}\), \(J_2 = \ell\), \(\Lambda_2 = \{\ell\}\), consider the PLSes \((J_1, \Lambda_1)\) and \((J_2, \Lambda_2)\), as well as their corresponding submatroids \(M(E_1)\) and \(M(E_2)\) of \(M(E)\). As above it follows that \(M(E_2)\) is induced by a triangle with edges \(x', y', z'\). To unravel the structure of \(M(E_1)\) first observe that \(M(E_1)\) keeps on weakly modeling \((J_1, \Lambda_1)\). Since each acyclic PLS being ndc it follows from (9) that \(mrk(E_1) \leq rk(J_1, \Lambda_1) = n - 1\). Yet the inequality cannot be strict because of \(n = rk(J, \Lambda) = mrk(E)\) and \(mrk(\{x', y', z'\}) = 1\) and the submodularity of \(mrk\). Hence \(mrk(E_1) = rk(J_1, \Lambda_1)\), and so \(M(E_1)\) models \((J_1, \Lambda_1)\). By induction \(M(E_1)\) must be graphic. By the remarks above it now suffices to show that each circuit \(C\) of \(M(E)\) which neither lies in \(E_1\) or \(E_2\) is of type (23). Evidently the case distinction below covers all cases.

Case 1: \(x' \in C\) but \(y' \notin C\) (or dually with \(x', y'\) switched). Then \(C \setminus \{x'\} \subseteq E_1\), and so \(x'\) is in the closure of \(E_1\). This leads to the contradiction \(mrk(E_1) = mrk(E_1 \cup \{x'\}) = n\). Hence Case 1 is impossible.

Case 2: \(x', y' \in C\). If \(z'\) was in \(C\), then \(\{x', y', z'\}\) was a dependent subset of \(C\), by definition of a circuit implies \(C = \{x', y', z'\}\). This is impossible because \(C \notin E_2\). To fix ideas, say \(C = \{x', y', a', b', c'\}\) (and \(z' \notin C\)). Since \(M(E)\) is binary we may think of these elements as lying in some \(GF(2)\)-vector space, it holds that \(x' + \cdots + c' = 0\) but no proper subset of \(C\) sums to 0. Hence \(z' + a' + b' + c' = (x' + y') + a' + b' + c' = 0\) but no proper subset of \(\{z', a', b', c'\}\) sums to 0. It follows that \(C_1 = \{z', a', b', c'\}\) and \(C_2 = \{x', y', z'\}\) are circuits as required in (23). This proves (24). QED

The relevance of graph-triggers derives from this result:

**Theorem 12:** Let \(L\) be a thin lattice. If \(L\) has a MoPLS which is a graph-trigger, then \(L\) is tightly embeddable into a partition lattice.

**Proof.** Let \((J, \Lambda)\) be a MoPLS of \(L\) which is a graph-trigger. As for any PLS of \(L\), according to [W1, Theorem 16] (which improves [HW, Thm. 5.1]) there is a binary matroid \(M(E)\) which models both\(^{26}\) the PLS \((J, \Lambda)\) and the lattice \(L\). Because \((J, \Lambda)\) is a graph-trigger, \(M(E)\) is in fact graphic. Therefore, because \(M(E)\) models \(L\), Theorem 9 yields a tight embedding into \(\text{Part}(n + 1)\) for \(n = d(L)\). QED

Fact (24) in conjunction with Theorem 12 shows that the graph-trigger concept makes sense. Each acyclic PLS is a QIMP and perhaps the proof of (24) can be fine-tuned to handle QIMPes. If so, in view of Lemma 2, we may even dare to ask:

**Question 5:** Is each UMP a graph-trigger?

\(^{26}\)The construction of this ‘bi-modeling’ binary (or generally \(k\)-linear) matroid \(M(E)\) proceeds by induction on the height of \(L\). In doing so, the MoPLS \((J, \Lambda)\) yields some ‘local’ PLS which, loosely speaking, is the difference between \((J, \Lambda)\) and the MoPLS induced by the interval sublattice \([0, a]\). Here \(a\) is any co-atom of \(L\). It is a crucial consequence of 2-distributivity that this PLS is acyclic, even when \((J, \Lambda)\) itself is not.
If yes, we would have an elegant proof of Theorem 11 which, recall, currently relies on the ugly $M_3$-cycles in Lemma 20 of [W2]. In view of (17) and Corollary 6 we generalize Question 5 to

**Question 6:** Is each PLS that loves close midpoint-links a graph-trigger?

When tackling Question 6 the recent criterion [M, Thm.1] for establishing the graphicness of certain binary matroids $M(E)$ may come in handy. What makes the Mighton-test appealing is that any one suitable base of $M(E)$ can be targeted. That fits us since when $M(E)$ models a Question 6 type PLS then some bases are better than others (depending on which point set in the PLS they match).

In case ‘yes’ is even provable for Question 6, then combinatorists can hand the flag to modular latticians. They will attempt to adapt the nonembeddability argument for $L_1$ (thus [W2, Thm.6] mentioned in 6.3) to arbitrary lattices that shun close midpoint-links. If successful, the tight partition embeddability problem would finally be solved!

**6.4.3** Finally, recall the remarks in Section 1 about the super-exponentiality occuring in the Pudlak-Tuma Theorem. Hence our last question.

**Question 7:** Can parts of the machinery developed in this article be adapted to handle non-tight but injective embeddings of modular lattices $L$ into moderate-size partition lattices?

A natural place to start would be to look at $L = M_4, M_5,$ and so forth. What are the smallest values of $n$ for which these lattices are embeddable into $\text{Part}(n)$?

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