Gauge invariants of linearized gravity with a general background metric

Deepen Garg\textsuperscript{1,*} and I Y Dodin\textsuperscript{1,2}

\textsuperscript{1} Department of Astrophysical Sciences, Princeton University, Princeton, NJ 08544, United States of America
\textsuperscript{2} Princeton Plasma Physics Laboratory, Princeton, NJ 08543, United States of America

E-mail: dgarg@princeton.edu

Received 13 August 2022; revised 21 October 2022
Accepted for publication 4 November 2022
Published 16 November 2022

Abstract

In linearized gravity with distributed matter, the background metric has no generic symmetries, and decomposition of the metric perturbation into global normal modes is generally impractical. This complicates the identification of the gauge-invariant part of the perturbation, which is a concern, for example, in the theory of dispersive gravitational waves (GWs) whose energy–momentum must be gauge-invariant. Here, we propose how to identify the gauge-invariant part of the metric perturbation and the six independent gauge invariants \textit{per se} for an arbitrary background metric. For the Minkowski background, the operator that projects the metric perturbation on the invariant subspace is proportional to the well-known dispersion operator of linear GWs in vacuum. For a general background, this operator is expressed in terms of the Green’s operator of the vacuum wave equation. If the background is smooth, it can be found asymptotically using the inverse scale of the background metric as a small parameter.

Keywords: general relativity, gravitational waves, gauge invariance, perturbative gravity, linear gravity

1. Introduction

In many problems related to gravity, the complicated structure of the Einstein field equations necessitates a perturbative approach within which the spacetime metric is split into a background metric and a small perturbation, and the equations are often linearized in the perturbation metric [1–3]. While this approach allows for a tractable answer for many interesting phenomena such as gravitational waves (GWs) and Jeans theory [2, chapter 7], it also introduces...
a gauge freedom that has to be dealt with. To be specific, let us consider the background metric to be $g_{\alpha \beta} = \mathcal{O}(1)$ and the perturbation metric to be $h_{\alpha \beta} = \mathcal{O}(a)$, where $a \ll 1$ is a small parameter. A coordinate transformation $x^\mu \to x'^\mu = x^\mu + \xi^\mu$, with $\xi^\mu = \mathcal{O}(a)$, induces a metric transformation $g_{\alpha \beta} \to g'_{\alpha \beta} = g_{\alpha \beta} - \xi^\mu g_{\alpha \mu} + \mathcal{O}(a^2)$, where $\xi^\mu$ is the Lie derivative along the vector field $\xi^\mu$ [1] and $g_{\alpha \beta}$ is the total spacetime metric. Within linearized gravity, where $\mathcal{O}(a^2)$ corrections are neglected and the background is $a$-independent by definition, this implies $g_{\alpha \beta} \to g'_{\alpha \beta} = g_{\alpha \beta}$ and $h_{\alpha \beta} \to h'_{\alpha \beta} = h_{\alpha \beta} - \xi^\mu g_{\alpha \mu}$. If $h_{\alpha \beta}$ is treated as a tensor field on the unperturbed spacetime, so its indices are manipulated using $g_{\alpha \beta}$ as the metric, one also has

$$h^{'\alpha \beta} = h^{\alpha \beta} + \xi^\gamma g^{\alpha \beta},$$

and as a reminder,

$$\xi^\alpha g^{\alpha \beta} = -\nabla^\alpha \xi^\beta - \nabla^\beta \xi^\alpha \equiv -2\nabla^{(\alpha} \xi^{\beta)}.$$  \hspace{1cm} (2)

The transformation (1) can be viewed as a gauge transformation (with $\xi^\mu$ being the gauge field) and, by general covariance, cannot have measurable effects. Thus, the physical, gauge-invariant, part of $h^{'\alpha \beta}$ is defined only up to the Lie derivative of $g^{\alpha \beta}$ along an arbitrary vector field, which is encoded by four functions (in a four-dimensional spacetime). Because the symmetric tensor $h^{\alpha \beta}$ is encoded by ten functions, this leaves room for six gauge-invariant degrees of freedom.

The nonphysical components of the perturbation metric obscure physical phenomena with coordinate artifacts, making it difficult to distinguish what is real and what is not. Thus, it is important to be able to identify the gauge-invariant degrees of freedom for a given metric perturbation and to represent the reduced equations of perturbation gravity in a gauge-invariant form. This problem has attracted considerable interest in many different contexts. In cosmological settings, the background can often be fixed to be Friedmann–Lemaître–Robertson–Walker metric, which then can be analyzed using Bardeen’s formalism [2, 4–7] or other methods, for example, using geodesic lightcone coordinates [8, 9]. Bardeen’s formalism has also been extended to second and higher-order perturbations [10–13] that are relevant for GWs produced in the early Universe, and also specifically for inflationary cosmologies [14, 15]. One can take the flat spacetime limit of Bardeen’s formalism to derive the gauge-invariant degrees of freedom for the Minkowski background [3, 16, 17]. Similarly, the gauge-invariant perturbations have been studied for the Schwarzschild and Kerr background as well [18–20].

While the analysis of isotropic backgrounds suffices for many settings, problems that involve GW–matter coupling [21–35] require a more general analysis. Usually, this coupling is studied by ignoring the backreaction of matter on metric oscillations [33, 36–40], because the interaction of GWs with cold collisionless matter is weak [29]. However, a more systematic theory is required to accommodate, for example, thermal effects [41], alternative GW polarizations [42], and fluid viscosity [43]. In particular, adequately describing linear transformations of GW modes in inhomogeneous matter (known as mode conversion for general waves [44]) requires that all gravitational perturbations be treated on the same footing and the GW polarization be derived rather than assumed a priori [45].

Ensuring the gauge invariance of the wave equation reported in [45] requires identification of the gauge-invariant variables for a general background metric. This has been studied, for example, using the Arnowitt–Deser–Misner (ADM) decomposition [46] for the background metric with a focus on application to the higher-order perturbations [47] (see also [48]). However, fundamental theory of general dispersive GWs must be covariant, and also it cannot rely on symmetry considerations [27] or normal-mode decomposition [46] that are commonly used for vacuum GWs. As known from plasma-wave theory [49], which deals with similar issues
for electromagnetic waves, these approaches become impractical once wave–matter coupling is introduced. (In particular, the wave polarization changes continuously as a function of the matter parameters [45], and the number of normal modes in the presence of matter is generally infinite [50, 51].) Thus, formulating GW theory in a covariant gauge-invariant form remains an important problem to solve.

Here, we explicitly identify the invariant part of a metric perturbation on a general background metric within linearized gravity. We start by showing that any metric perturbation \( h^{\alpha\beta} \) can be uniquely decomposed as

\[
h^{\alpha\beta} = \hat{\Pi}^{\alpha\beta}_{\text{inv}} h^{\gamma\delta} + \hat{\Pi}^{\alpha\beta}_{\text{g}} h^{\gamma\delta},
\]

where the operators \( \hat{\Pi}^{\alpha\beta}_{\text{inv}} \) and \( \hat{\Pi}^{\alpha\beta}_{\text{g}} \) are projectors that satisfy

\[
\begin{align*}
\hat{\Pi}^{\alpha\beta}_{\text{inv}} h^{\gamma\delta} + \hat{\Pi}^{\alpha\beta}_{\text{g}} h^{\gamma\delta} &= \delta^{\alpha}_\gamma \delta^{\beta}_\delta, \\
\hat{\Pi}^{\alpha\beta}_{\text{inv}} h^{\gamma\delta} + \hat{\Pi}^{\alpha\beta}_{\text{g}} h^{\gamma\delta} &= \hat{\Pi}^{\alpha\beta}_{\text{inv}} h^{\gamma\delta}, \\
\hat{\Pi}^{\alpha\beta}_{\text{g}} h^{\gamma\delta} + \hat{\Pi}^{\alpha\beta}_{\text{inv}} h^{\gamma\delta} &= \hat{\Pi}^{\alpha\beta}_{\text{g}} h^{\gamma\delta} = 0, \\
\hat{\Pi}^{\alpha\beta}_{\text{g}} h^{\gamma\delta} &= \hat{\Pi}^{\alpha\beta}_{\text{inv}} h^{\gamma\delta}.
\end{align*}
\]

(Parentheses in indices denote symmetrization, as usual, and \( u^\mu \) is any vector field.) In section 2, we present a method for how to calculate the operators \( \hat{\Pi}^{\alpha\beta}_{\text{inv}} \) and \( \hat{\Pi}^{\alpha\beta}_{\text{g}} \) for general \( g_{\alpha\beta} \). We assume the sign convention as in references [1, 52], so

\[
\{\nabla_\beta, \nabla_\alpha\} \xi^\gamma = R^\gamma_{\alpha\beta} \xi^\beta
\]

for any vector field \( \xi^\alpha \), where \( R^\gamma_{\alpha\beta} \) is the Ricci tensor. We assume that the matter is localized, so that at sufficiently large spatial distances (defined, say, in the center-of-mass frame), \( R^\gamma_{\alpha\beta} \)

---

3 For GW modes of certain types [45], the Minkowski-background model can also be relevant for studies of GW–matter coupling.
vanishes and they satisfy outgoing boundary conditions, i.e. no GWs are going in through a sufficiently large spatial sphere. Specifically, since GWs are vacuum tensor modes at infinity, we assume \( \partial_0 - n^i \partial_i = 0 \), where \( n^i \) is normal to the spatial sphere (the polarization does not have to be specified), and the Latin index \( i \) represents spatial coordinates. We assume that this applies to both \( h^{\alpha\beta} \) and \( h^{\prime\alpha\beta} \) and therefore to gauge fields as well. Then, one can proceed as follows.

### 2.1. Special case

To motivate the machinery that will be introduced in section 2.2, let us first discuss an auxiliary problem. Consider a vector field \( u^\alpha \) that transforms the gauge of a given metric perturbation \( h^{\alpha\beta} \) to the Lorenz gauge:

\[
h^{\prime\alpha\beta} = h^{\alpha\beta} - \mathcal{L}_{u} g^{\alpha\beta},
\]

(6)

\[
\nabla_\beta h^{\prime\alpha\beta} = 0,
\]

(7)

where the symbol \( \mathcal{L} \) denotes definitions. Let us assume for now that \( u^\alpha \) is divergence-free; i.e.

\[
\nabla_\alpha u^\alpha = 0.
\]

(8)

Then, equations (2)–(7) yield

\[
\tilde{Q}^\alpha_\beta u^\beta = \nabla_\beta h^{\prime\alpha\beta},
\]

(9a)

\[
\tilde{Q}^\alpha_\beta = -\delta^\alpha_\beta \nabla_\mu \nabla^\mu - R^\alpha_\beta,
\]

(9b)

where we have used equation (5). The hyperbolic operator \( \tilde{Q}^\alpha_\beta \) is similar to the one that appears in the driven Maxwell’s equation for the Lorenz-gauge electromagnetic vector potential in vacuum [53] except for the opposite sign in front of the Ricci tensor. In the presence of matter, GWs are dispersive (vacuum waves can be considered as a limit; see section 3), so \( \tilde{Q}^\alpha_\beta \) is generally invertible for fields of interest under the assumed boundary conditions. Then, one can introduce a unique Green’s operator of equation (9a) as

\[
\Xi^\alpha_\beta = (\tilde{Q}^{-1})^\alpha_\beta
\]

(10)

and express the solution of equation (9a) as follows:

\[
u^\alpha = \Xi^\alpha_\beta (\nabla_\beta \gamma) h^{\gamma\delta},
\]

(11)

where symmetrization with respect to the lower indices is added for convenience and does not affect the result. (As a side remark, the appearance of Green’s operator is not unexpected here; see [47], where Green’s functions of elliptic operators appear in a related problem for ADM-parameterized backgrounds.) Finding \( (\tilde{Q}^{-1})^\alpha_\beta \) is equivalent to finding waves generated by prescribed sources. Similar calculations for driven Maxwell’s equation in various covariant gauges can be found in [54]. See also [55].

Since, \( h^{\gamma\delta} \) is assumed to be such that the solution (11) satisfies the constraint (8), equation (7) is satisfied by

\[
h^{\prime\alpha\beta} = \tilde{\pi}^\alpha_\gamma \delta h^{\gamma\delta},
\]

(12)

where we defined

\[
\tilde{\pi}^\alpha_\gamma \delta \equiv \delta^\alpha_\gamma \delta^\beta_\delta + 2 \nabla^{(\alpha} (\Xi^{\beta)}_\gamma \nabla_\delta).
\]

(13)
In combination with equation (6), these results yield that
\[
    h^{\alpha\beta} = \tilde{\Pi}^{\alpha\beta}_{\gamma\delta} g^{\gamma\delta} + \mathcal{E}^\mu g^{\alpha\beta},
\]
(14a)
\[
    \mathcal{E}^\mu g^{\alpha\beta} = -2\nabla^{(\alpha} \Xi^{\beta)} \, \mu \nabla_\gamma g^{\delta} h^{\gamma\delta},
\]
(14b)
and a direct calculation shows that (appendix B)
\[
    \tilde{\Pi}^{\alpha\beta}_{\gamma\delta} \mathcal{E}^\mu g^{\gamma\delta} = 0.
\]
(15)
Equation (15) is similar to equation (4e) and makes the decomposition (14) close to equation (3), except it is constrained by equation (8). This can be taken as a hint that \( \tilde{\Pi}^{\alpha\beta}_{\gamma\delta} \) is close to the sought \( \tilde{\Pi}^{\alpha\beta}_{\gamma\delta} \). Hence, we approach the general case as follows.

2.2. General case

Now let us waive the Lorenz-gauge assumption (8) and consider applying \( \tilde{\Pi}^{\alpha\beta}_{\gamma\delta} \) to \( \mathcal{E}^\mu g^{\alpha\beta} \) with a general \( \mu^\alpha \). In this case, equation (7) is not necessarily satisfied, but a direct calculation shows that (appendix B)
\[
    \tilde{\Pi}^{\alpha\beta}_{\gamma\delta} \mathcal{E}^\mu g^{\gamma\delta} = \nabla^{(\alpha} \Xi^{\beta)} \, \mu \nabla_\gamma g^{\delta} \mathcal{E}^\mu g^{\gamma\delta}.
\]
(16)
Hence, the operator
\[
    \tilde{\Pi}^{\alpha\beta}_{\gamma\delta} \mathcal{E}^\mu g^{\gamma\delta} = -\nabla^{(\alpha} \Xi^{\beta)} \, \mu \nabla_\gamma g^{\delta} \mathcal{E}^\mu g^{\gamma\delta}
\]
automatically satisfies equation (4e). Let us substitute equation (13) and rewrite this operator as follows:
\[
    \tilde{\Pi}^{\alpha\beta}_{\gamma\delta} = \tilde{g}^{\alpha\beta}_{\gamma\delta} - \tilde{\Pi}^{\alpha\beta}_g \, \gamma\delta,
\]
(18a)
\[
    \tilde{\Pi}^{\alpha\beta}_g \, \gamma\delta = -2\nabla^{(\alpha} \Xi^{\beta)} \, (\gamma \nabla_\delta) + \nabla^{(\alpha} \Xi^{\beta)} \, \mu \nabla_\mu g^{\gamma\delta}.
\]
(18b)
This satisfies equations (4a), (4f), and (4g). (The latter ensures that \( \tilde{\Pi}^{\alpha\beta}_{\gamma\delta} \mu^\delta = 0 \) for all anti-symmetric \( \mu^\delta \), which is convenient.) The property (4c) is proven by a direct calculation (appendix C). Equation (4d) can be derived from equations (4a) and (4c), and the remaining property (4b) can then be obtained from equations (4a) and (4d).

Let us discuss how this intermediate result helps identify the invariant part of a metric perturbation. First, notice that
\[
    \tilde{\Pi}^{\alpha\beta}_g \, \gamma\delta h^{\gamma\delta} = -2\nabla^{(\alpha} \Xi^{\beta)} \, (\gamma \nabla_\delta) h^{\gamma\delta} + \nabla^{(\alpha} \Xi^{\beta)} \, \mu \nabla_\mu g^{\gamma\delta} h^{\gamma\delta}
\]
\[
    = -2\nabla^{(\alpha} \Xi^{\beta)} \, \mu \, \nabla_\mu g^{\gamma\delta} h^{\gamma\delta},
\]
(19)
where we introduced
\[
    \zeta^{\beta} \equiv \Xi^{\beta} (\gamma \nabla_\delta) h^{\gamma\delta} - \frac{1}{2} \Xi^{\beta} \, \mu \nabla_\mu g^{\gamma\delta} h^{\gamma\delta}.
\]
(20)
Hence, equation (3) can be rewritten as
\[
    h^{\alpha\beta} = \psi^{\alpha\beta} + \phi^{\alpha\beta},
\]
(21a)

\(^4\) Here and further, \( g_{\alpha\beta} \equiv \tilde{g}_{\alpha\beta} \) and \( R^\alpha_{\beta} \equiv \tilde{R}^\alpha_{\beta} \) serve as multiplication operators, and the assumed notation is \( \tilde{A} \tilde{B} = \tilde{A} (\tilde{B}) \) for any operators \( \tilde{A} \) and \( \tilde{B} \) and function \( f \) that they act upon. For example, \( \nabla_\mu \mathcal{E}^\mu g^{\gamma\delta} \equiv \nabla^\mu \mathcal{E}^\mu (\mathcal{E}^\mu g^{\gamma\delta}) \).
\[ \psi_{\alpha\beta} = b \Pi_{\alpha\beta} \gamma_{\gamma\delta} h_{\gamma\delta}, \]
\[ \phi_{\alpha\beta} = b \Pi_{\alpha\beta} g_{\gamma\delta} h_{\gamma\delta} = \mathcal{E}_{\gamma\delta} g_{\alpha\beta}, \]

Upon a gauge transformation (1), one obtains
\[ h'_{\alpha\beta} = \psi'_{\alpha\beta} + \phi'_{\alpha\beta}, \]
\[ \psi'_{\alpha\beta} = b \Pi_{\alpha\beta} \gamma_{\gamma\delta} h'_{\gamma\delta} = \psi_{\alpha\beta} + \frac{1}{4} \sum_{\gamma\delta} \xi_{\gamma\delta} g_{\alpha\beta}, \]
\[ \phi'_{\alpha\beta} = b \Pi_{\alpha\beta} g_{\gamma\delta} h'_{\gamma\delta} = \phi_{\alpha\beta} + \mathcal{E}_{\gamma\delta} g_{\alpha\beta} = \xi_{\gamma\delta} + \mathcal{E}_{\gamma\delta} g_{\alpha\beta}, \]

where we used equations (4d)–(4f). This means that \( \phi^{\alpha\beta} \), which is encoded by the four functions \( \zeta^\mu \), does not contain gauge-independent information. Hence, any solution that has nonzero \( \phi_{\alpha\beta} \) and zero \( \psi_{\alpha\beta} \) can be classified as a coordinate artifact. In contrast, \( \psi_{\alpha\beta} \) is gauge-invariant by equation (22b). By the argument presented in section 1, it is encoded by six independent functions, or gauge-invariant degrees of freedom. Also note that \( \psi_{\alpha\beta} \) does not necessarily satisfy the Lorenz-gauge condition \( \nabla_{\beta} \psi_{\alpha\beta} = 0 \).

Finding an explicit formula for the Green’s operator \( \bar{\Sigma}^{\alpha\beta} \) that determines \( \psi_{\alpha\beta} \) (equation (21b)) for a specific background geometry is beyond the scope of this paper, because our primary concern is the general framework needed for a dispersive-GW theory. (This is similar to the approach taken by others; for example, see references [47, 56, 57].) For a smooth background metric, \( \bar{\Sigma}^{\alpha\beta} \) can be found asymptotically within any predefined accuracy using methods of the Weyl symbol calculus (appendix A). The proof of existence of the exact operator and an asymptotic approximation of its Weyl symbol is, as usual [44, 55], sufficient within fundamental wave theory to properly define waves as dynamic objects. We elaborate on this subject in application to GWs in [58]. Alternatively, recursive construction of a parametrix can be used for small distances. This method can yield a converging expansion of the Green’s operator even for arbitrary globally hyperbolic spacetimes [59]. An example using Hadamard parametrices can be found in [54]. Also note that the very expression (21b) for the invariant part of the metric perturbation is not unique in general. In particular, any function of \( \psi_{\alpha\beta} \) is also an invariant.

### 2.3. Six gauge invariants

The six independent functions still need to be extracted from the sixteen gauge-invariant functions \( \psi^{\alpha\beta} \). To do so, let us consider \( h^{\alpha\beta} \) as a 16-dimensional (16D) field \( h^a \), or \( h \) in the index-free notation, of the form
\[ h = (h^{00}, h^{01}, h^{02}, h^{03}, h^{10}, \ldots, h^{32}, h^{33})^\top, \]
where \( ^\top \) denotes transpose. In other words,
\[ h^a = h^{\alpha\beta}, \quad h_b = h_{\gamma\delta}, \]
\[ \{ \alpha, \beta \} = \iota(a), \quad \{ \gamma, \delta \} = \iota(b), \]
where the index function \( \iota \) is defined via
\[ \iota(a) = \{ 1 + [(a - 1)/4 ], 1 + (a - 1) \mod 4 \}. \]

(Here and further, Latin indices from the beginning of the alphabet range from 1 to 16.) Let us define \( \mathcal{H} \) as a Hilbert space of one-component functions on the background spacetime with the usual inner product \( \langle \cdot, \cdot \rangle_1 \). Then, the 16D fields (equation (23)) can be considered as
vectors in the Hilbert space $\mathcal{H}_{16}$ that is the tensor product of 16 copies of $\mathcal{H}_1$, with the inner product

$$
\langle \xi, \varphi \rangle = \int d^4x \sqrt{-g} \xi^\alpha \varphi^\alpha = \sum_{a=1}^{16} (\xi_a, \varphi^a)_1,
$$

(27)

where $g \equiv \det g_{\alpha\beta}$. (Unlike in the rest of the paper, summation is shown explicitly here in order to emphasize the difference between $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_1$.) Then, $\Pi^{\alpha\beta}_{\text{inv}}$ induces an operator $\Pi^a_b$ on $\mathcal{H}_{16}$ defined via

$$
\Pi^a_b h^b \equiv \Pi^{\alpha\beta}_{\text{inv}} g^{\gamma\delta} h_{\gamma\delta},
$$

(28)

where we again assumed the notation as in equation (25). From equation (4), one finds that

$$
\Pi^a_b \Pi^b_c = \Pi^a_c, 
$$

(29a)

$$
\Pi^a_b \xi_a \delta^b_c = 0.
$$

(29b)

Equation (29a), which in the index-free notation can be written as $\Pi^2 = \hat{\Pi}$, means that $\hat{\Pi}$ is a projector. (Note that $\Pi^T \neq \Pi$, so the projector is not orthogonal but oblique.) Hence, each eigenvalue of $\Pi$ is either zero or unity and $\Pi$ is diagonalizable. This means that $\Pi$ can be represented as

$$
\Pi = \hat{V} \hat{J} \hat{V}^{-1},
$$

(30)

where $\hat{V}$ is a diagonalizing transformation and the operator $\hat{J}$ is such that each component of the vector $\hat{h}$ equals either zero or the corresponding component of $h$ for any $h$. Like $\Omega^{\alpha\beta}$, the diagonalizing transformation cannot be found exactly but can be found asymptotically using methods of the Weyl symbol calculus if the inhomogeneity of the background metric is weak. In this sense, our identification of the gauge invariants is intended as an algorithm rather than as an explicit answer.

Each linear operator in $\mathcal{H}_{16}$ is a $16 \times 16$ matrix of operators on $\mathcal{H}_1$. Then, $\hat{J}$ must be represented by a constant matrix $J$ of the form

$$
J = \text{diag} \left\{ \frac{1,1,\ldots,1,0,0,\ldots,0}{n}, \frac{16-n}{16-n} \right\},
$$

(31)

where, for clarity, we have ordered the basis such that the nonzero eigenvalues are grouped together and have indices $1,\ldots,n$. The gauge-invariant part of $\hat{h}$, which is given by equation (21b), can now be expressed as $\psi = \Pi h$. Using equation (30), one can also rewrite this as

$$
\psi = \hat{V} \Psi, \quad \Psi = J \hat{V}^{-1} h.
$$

(32)

Because $h$ is an arbitrary vector field parameterized by 16 functions and $\hat{V}$ is invertible, the field $\hat{V}^{-1} h$ is also parameterized by 16 functions. Then, $\Psi$ is parameterized by $n$ functions. But we know that $\psi$ is parameterized by six functions (section 1), and thus so is $\Psi$. Then, $n = 6$, and the nonzero elements of $\Psi$ are the sought invariants.

In summary, to find the gauge invariants, one needs to find the diagonalizing transformation $\hat{V}^a_b$ that brings $\Pi^a_b$ to the form given by equations (30) and (31). Then, the invariants can be found as

$$
\Psi^s = J^s_b (\hat{V}^{-1})^b_c h^c, \quad s = 1,2,\ldots,6.
$$

(33)
3. Example: Minkowski background

Except for toy models, problems involving wave propagation through inhomogeneous matter have no generic symmetries, so case studies are of little interest within the scope of this paper. What matter instead are the existence theorems, local analysis, and asymptotics. Hence, for an example, we will discuss only the simplest solvable case here, specifically, the case of the Minkowski background. Although interactions with matter generally curve the background, the Minkowski-background model can be a valid approximation for gravitational modes with a high refraction index. In addition, this example is instructive in that it allows direct benchmarking of our framework against known results.

3.1. Gauge invariants

In vacuum, when $R_{\alpha\beta} \to 0$, one has $\widehat{\Xi}^{\alpha\beta} \to -\delta^{\alpha\beta} \nabla^{-2}$. For the Minkowski background, $\widehat{\Xi}^{\alpha\beta}$ is further simplified to

$$\widehat{\Xi}^{\alpha\beta} \to -\delta^{\alpha\beta} \partial^{-2}. \quad (34)$$

Here, $\partial^{-2}$ is the operator inverse to $\partial^2$; i.e., $\varphi^{\alpha} = \partial^{-2} q^{\alpha}$ is the solution of $\partial^2 \varphi^{\alpha} = q^{\alpha}$ (appendix A). Formally, $\partial^{-2}$ is singular on free vacuum GWs, but the vacuum case can still be considered as a limit (section 3.2).

Using equation (34), one can rewrite equation (18a) as

$$\widehat{\Pi}^{\alpha\beta}_{\text{inv}} = \delta^{\alpha\beta} - 2 \partial^{-2} \delta^{\gamma\delta} \partial^{(\gamma} \partial^{\delta)} + \partial^{-2} \partial^{\alpha} \partial^{\beta} g^{\gamma\delta}. \quad (35)$$

Let us consider this operator in the Fourier representation, in which case it becomes a local matrix function of the wavevector $k_{\mu}$; namely,

$$\Pi^{\alpha\beta}_{\text{inv}} = \Pi^{\alpha\beta}_{\text{inv}} g^{\gamma\delta},$$

$$\Pi^{\alpha\beta}_{\text{inv}} g^{\gamma\delta} = \delta^{\alpha\beta} \delta^{\gamma\delta} - \frac{k^{(\alpha} \partial^{\beta)} g^{\gamma\delta}}{k^2} + g^{\gamma\delta} \frac{k^{\alpha} k^{\beta}}{k^2}. \quad (36)$$

Using that $\nabla_{\mu} \to \partial_{\mu} \to ik_{\mu}$ in the Fourier representation (and in particular, $\epsilon_{\gamma\delta} g^{\alpha\beta} = -2ik^{(\alpha} \epsilon^{\beta)}$), the properties (4) are easily verified. (At $k^2 = 0$, the usual rules of resonant-pole manipulation apply [54], but for the discussion below, which is restricted to the spectral representation, these details are not important.) One also finds by a direct calculation [60] that, as expected from equations (30) and (31),

$$\text{rank} \Pi = 6. \quad (37)$$

The invariant part of the metric perturbation (equation (21b)) is now given by

$$\psi^{\alpha\beta} = \Pi^{\alpha\beta}_{\text{inv}} h^{\gamma\delta},$$

or explicitly,

$$\psi^{\alpha\beta} = h^{\alpha\beta} - \frac{k^{\alpha} k^{\mu} h^{\beta\mu}}{k^2} - \frac{k^{\beta} k^{\mu} h^{\alpha\mu}}{k^2} + \frac{k^{\alpha} k^{\beta}}{k^2} h, \quad (38)$$

where $h \doteq \text{tr} h^{\alpha\beta}$. Without loss of generality, let us assume coordinates such that

$$k^{\alpha} = (\omega, 0, 0, k), \quad (39)$$
where \(k\) is the spatial wavenumber. Using this, the fact that \(k^2 = k^2 - \omega^2\), and also equation (25), the 16D vector \(\psi\) is found to be:

\[
\psi = \frac{1}{k^2} \begin{pmatrix}
  h^{00} k^2 - 2 h^{03} \omega k + \omega^2(h^{11} + h^{22} + h^{33}) \\
h^{01} k^2 - h^{13} \omega k \\
h^{02} k^2 - h^{23} \omega k \\
(h^{11} + h^{22}) k \omega \\
h^{01} k^2 - h^{13} \omega k \\
h^{11}(k^2 - \omega^2) \\
h^{12}(k^2 - \omega^2) \\
h^{01} \omega k - h^{13} \omega^2 \\
h^{02} k^2 - h^{23} \omega k \\
h^{12}(k^2 - \omega^2) \\
h^{22}(k^2 - \omega^2) \\
h^{02} \omega k - h^{23} \omega^2 \\
(h^{11} + h^{22}) k \omega \\
h^{01} \omega k - h^{13} \omega^2 \\
h^{02} \omega k - h^{23} \omega^2 \\
k^2(-h^{00} + h^{11} + h^{22}) + 2 h^{03} \omega k - h^{33} \omega^2
\end{pmatrix}.
\]

In order to extract the six gauge invariants from this \(\psi\), notice that the operator (28) is represented by a local function of \(k_\mu\), \(\Pi = \Pi\), and thus so is the diagonalizing transformation (30). Specifically, \(\tilde{V} = V\), and the columns of the matrix \(V\) are just the eigenvectors of \(\Pi\):

\[
V = (v_1, v_2, ..., v_{16}), \quad \Pi v_a = \lambda_a v_a,
\]

where \(\lambda_a \in \{0, 1\}\). The calculation of these eigenvectors and of the matrix \(V^{-1}\) can be automated [60], and the six gauge invariants (equation (33)) are readily found to be

\[
\Psi = \begin{pmatrix}
k^2(-h^{00} + h^{11} + h^{22}) + 2 \omega k h^{03} - \omega^2 h^{33} \\
k^2 - \omega^2 \\
\omega k h^{01} - \omega^2 h^{13} \\
\pi^2 - \omega^2 \\
\omega k h^{02} - \omega^2 h^{23} \\
\pi^2 - \omega^2 \\
\omega k(h^{11} + h^{22}) \\
\pi^2 - \omega^2 \\
h^{22} \\
h^{12}
\end{pmatrix}.
\]

The coordinate representation of these invariants is found by taking the inverse Fourier transform of equation (41).

Our result is in agreement with equations (2.45)–(2.47) in [3] (which operates with \(h_{\alpha\beta}\) instead of our \(h^{\alpha\beta}\)). This is seen from the fact that any linear combinations of our \(\Psi^s\) are gauge invariants too. In other words, instead of \(\Psi^s\), one can introduce the invariants as \(\tilde{\Psi}^s\) given by

\[
\tilde{\Psi}^s = C^n \Psi^n, \quad r, s = 1, 2, ..., 6,
\]

where \(C^n\) are the components of the 3D rotation matrix of the eigenframe.
or $\Psi = C\Psi$ in the index-free representation, where $C$ is an arbitrary matrix that may depend on $k_\mu$. This is particularly convenient at $k^2 \equiv k^2 - \omega^2 \to 0$, when $\Psi$ becomes singular. Specifically, by choosing

$$C = \text{diag} \{ k^2, k^2, k^2, k^2, 1, 1 \},$$

we obtain invariants that are well-behaved at all $k_\mu$:

$$\Psi = \begin{pmatrix}
k^2( - h^{00} + h^{11} + h^{22} ) + 2 \omega k h^{03} - \omega^2 h^{33} \\
\omega k h^{01} - \omega^2 h^{13} \\
\omega k h^{02} - \omega^2 h^{23} \\
\omega k( h^{11} + h^{22} ) \\
h^{22} \\
h^{12}
\end{pmatrix}.$$ (44)

As also mentioned in section 2.2, these invariants are not unique in that any function of them is an invariant too.

Let us also discuss why the original vectors $\psi$ and $\Psi$ are singular at $k^2 \to 0$. In this limit, the vectors $v_a$ (equation (40)) are well-behaved, and thus so is the matrix $V$. However, they cease to be linearly independent at $k^2 = 0$, so $V^{-1}$ becomes singular, and as a result, $\Pi$ becomes singular too. This means that no finite invariant projection of a generic $h^{\alpha \beta}$ can be defined in the Fourier space at $k^2 = 0$. The corresponding gauge-dependent part $\phi^{\alpha \beta}$ becomes singular as well in this limit, as seen from equations (20) and (21c), where $\Xi^{\alpha \beta}$ becomes singular (appendix A)\(^5\). Still, our general formulation correctly predicts the invariants (equation (44)) at zero $k^2$, and these invariants can be related to vacuum GWs as discussed in the next section.

### 3.2. Free GWs in the Minkowski space

By comparing equation (35) with, for example, equations (5.4) and (2.7) in [62], one finds that the equation for vacuum GWs in the Minkowski spacetime can be expressed as

$$\tilde{D}^{\alpha \beta} \gamma_\delta h^{\gamma \delta} = 0, \quad \tilde{D}^{\alpha \beta} \gamma_\delta = \partial^2 \tilde{\Pi}^{\alpha \beta \gamma \delta}.$$ (45)

In other words, in the special case of the Minkowski spacetime, the dispersion operator $\tilde{D}^{\alpha \beta} \gamma_\delta$ of vacuum GWs is exactly $\partial^2$ times the operator that projects a metric perturbation on the invariant subspace. Thus, as expected, using the operators introduced in this paper, the wave equation for the GWs in vacuum can be shown to directly specify the gauge invariants and naturally weed out the gauge artifacts.

Let us also briefly discuss monochromatic waves\(^6\), in which case, equation (45) becomes

$$k^2 \Pi^{\alpha \beta \gamma \delta} h^{\gamma \delta} = 0,$$ (46)

where the matrix $k^2 \Pi^{\alpha \beta \gamma \delta}$ is well-behaved for all $k_\mu$. Equation (46) can be written as the following six of equations, which determine the six gauge invariants (equation (44)):

$$k^2 h^{00} + \omega ( - 2 k h^{03} + \omega h^{33} ) = 0,$$ (47a)

$$k^2 h^{01} - \omega k h^{13} = 0.$$ (47b)

---

5 This is the same effect as the unlimited growth, at $x^\mu \to \infty$, of the gauge field that brings a generic $h^{\alpha \beta}$ to the Lorenz gauge. See appendix A in conjunction with equation (11), which is commonly known for the Minkowski background [61].

6 Cf a similar discussion in [63], except their equation (3.6) describes the trace-reversed metric perturbation.
\[ k^2 h^{02} - \omega k h^{23} = 0, \quad (47c) \]
\[ k \omega (h^{11} + h^{22}) = 0, \quad (47d) \]
\[ k^2 (h^{11} - h^{22}) = 0, \quad (47e) \]
\[ k^2 h^{12} = 0. \quad (47f) \]

For \( k^2 \neq 0 \), equation (47) indicate that all the six invariants (equation (44)) are zero, so only coordinate waves are possible in this case. For \( k^2 = 0 \), equations (47a)–(47d) yield

\[ \bar{\Psi}^1 = \bar{\Psi}^2 = \bar{\Psi}^3 = \bar{\Psi}^4 = 0, \quad (48) \]

and in particular, \( h^{11} + h^{22} = 0 \). However, equations (47e) and (47f) are satisfied identically at \( k^2 = 0 \), so the other two invariants,

\[ \bar{\Psi}^5 = h^{22} = -h^{11}, \quad \bar{\Psi}^6 = h^{12} = h^{21}, \quad (49) \]

can be arbitrary and represent the two tensor modes of the GWs in vacuum [3].

### 4. Conclusions

In summary, we propose a method for identifying the gauge-invariant part \( \psi^{\alpha\beta} \) of the metric perturbation \( h^{\alpha\beta} \) within linearized gravity for an arbitrary background metric \( g^{\alpha\beta} \) assuming that the inverse of a hyperbolic operator \( \tilde{Q}^{\alpha\beta} \) (equation (9b)). Specifically, we show that \( \psi^{\alpha\beta} = \bar{\Pi}^{\alpha\beta}_{\text{inv}} g^{\gamma\delta} \), where \( \bar{\Pi}^{\alpha\beta}_{\text{inv}} \) is a linear operator given by equation (18a). The six independent functions from the sixteen gauge-invariant functions \( \psi^{\alpha\beta} \) can be found using equation (33).

These results lead to a gauge-invariant quasilinear theory of dispersive GWs in an arbitrary background, as discussed in a companion paper [58] (see also [55]). For the Minkowski background, the well-known dispersion operator of linear GWs in vacuum is proportional to \( \bar{\Pi}^{\alpha\beta}_{\text{inv}} g^{\gamma\delta} \) (equation (45)), and thus specifies the gauge invariants directly streamlining the process of removing the gauge artifacts. We also show that this general formulation systematically yields the six known gauge invariants for the Minkowski background.

### Data availability statement

No new data were created or analyzed in this study.

### Acknowledgments

This material is based upon the work supported by National Science Foundation under the Grant No. PHY 1903130.

### Appendix A. Asymptotic representation of \( \tilde{\Xi}^{\alpha\beta} \)

An asymptotic approximation for the Green’s operator \( \tilde{\Xi}^{\alpha\beta} \) as the inverse of \( \tilde{Q}^{\alpha\beta} \) (equation (9b)) can be constructed using methods of the Weyl symbol calculus. These methods may not be particularly popular in general relativity, but they have become de facto standard in
For a homogeneous medium, the Weyl symbol of a given operator is obtained by replacing each $D$ with $\nabla^2$. In an inhomogeneous medium, the procedure is more elaborate. Assuming the adiabatic limit, we assume that the characteristic GW wavelength $\lambda$ is much smaller than the characteristic radius $L$ of the spacetime curvature, i.e., when $\epsilon = \lambda / L \ll 1$. Assuming the ordering $\lambda = O(1)$ and $L = O(\epsilon^{-1})$, one has $\nabla^2 = O(1)$ and $\hat{R} = O(\epsilon^2)$.

The operator $\hat{\mathcal{H}}_{\alpha\beta}$ defined in equation (10) can be written in the index-free representation as

$$\hat{\mathcal{H}} = - (\nabla^2 + \hat{R})^{-1}, \quad (A1)$$

where $\nabla^2 = \nabla^\mu \nabla^\mu$, $\hat{R}$ is the operator whose coordinate representation is the Ricci tensor $R^\alpha_{\beta}$, and $^{-1}$ denotes the operator inverse. In order for this inverse to exist (approximately), we assume the ordering $\lambda = O(1)$ and $L = O(\epsilon^{-1})$, one has $\nabla^2 = O(1)$ and $\hat{R} = O(\epsilon^2)$. Then,

$$\hat{\mathcal{H}} = - \nabla^{-2} - 2 \hat{R} \nabla^{-2} + O(\epsilon^4), \quad (A2)$$

where $\nabla^{-2}$ is the inverse of $\nabla^2$; i.e., $\varphi^\alpha = \nabla^2 q^\alpha$ is defined as the solution of $\nabla^2 \varphi^\alpha = q^\alpha$.

Because the operators in equations (A1) and (A2) are intended to act specifically on vector fields, one can also write them explicitly. For example, in normal coordinates, one has (appendix D)

$$\nabla^2 = \partial^2 - \frac{\hat{R}}{3}, \quad (A3)$$

and the corresponding inverse is

$$\nabla^{-2} = \partial^{-2} + \frac{1}{3} \partial^{-2} \hat{R} \partial^{-2} + O(\epsilon^4), \quad (A4)$$

so equation (A2) leads to

$$\hat{\mathcal{H}} = - \partial^{-2} - 2 \frac{2}{3} \partial^{-2} \hat{R} \partial^{-2} + O(\epsilon^4). \quad (A5)$$

The operator $\partial^{-2}$ that enters here is understood as the Green’s operator of the equation

$$\partial^2 \varphi^\alpha = q^\alpha. \quad (A6)$$

(This is the same equation that emerges in the well-known linear gravity in the Minkowski background [61]; see also equation (11).) Suppose that the right-hand side of equation (A6) is quasimonochromatic, i.e., $q^\alpha = Q^\alpha \exp(i\theta(x))$ with $\partial_\beta Q^\alpha = O(\epsilon)$ and $\partial_\beta k_\alpha = O(\epsilon)$, where $k_\alpha \equiv \partial_\alpha \theta$ is the local wavevector. Then,

$$\partial^{-2} = (k_\mu k^\mu)^{-1} + \hat{\Delta}, \quad (A7)$$

where $\hat{\Delta} = O(\epsilon)$ is a differential operator to act on the envelope $Q^\alpha$. If $k^2 \equiv k_\mu k^\mu$ approaches zero, as would be the case for GWs in the Minkowski vacuum, then $\varphi^\alpha$ grows indefinitely at

---

For a homogeneous medium, the Weyl symbol of a given operator is obtained by replacing each $\partial_\alpha$ with $-ik_\alpha$. In an inhomogeneous medium, the procedure is more elaborate [44, 64].

---

1. For example, see references [55, 66] for a modern reformulation of classic results. 
2. The operator $\hat{\mathcal{H}}_{\alpha\beta}$ is the local wavevector. Then,

$$\hat{\mathcal{H}}_{\alpha\beta} = -(\nabla^2 + \hat{R})^{-1},$$

where $\nabla^2 = \nabla_\mu \nabla^\mu$, $\hat{R}$ is the operator whose coordinate representation is the Ricci tensor $R^\alpha_{\beta}$, and $^{-1}$ denotes the operator inverse. In order for this inverse to exist (approximately), we assume the adiabatic limit. Specifically, we assume that the characteristic GW wavelength $\lambda$ is much smaller than the characteristic radius $L$ of the spacetime curvature, i.e., when $\epsilon = \lambda / L \ll 1$. Assuming the ordering $\lambda = O(1)$ and $L = O(\epsilon^{-1})$, one has $\nabla^2 = O(1)$ and $\hat{R} = O(\epsilon^2)$. Then,

$$\hat{\mathcal{H}} = - \nabla^{-2} - 2 \hat{R} \nabla^{-2} + O(\epsilon^4),$$

where $\nabla^{-2}$ is the inverse of $\nabla^2$; i.e., $\varphi^\alpha = \nabla^2 q^\alpha$ is defined as the solution of $\nabla^2 \varphi^\alpha = q^\alpha$.

Because the operators in equations (A1) and (A2) are intended to act specifically on vector fields, one can also write them explicitly. For example, in normal coordinates, one has (appendix D)

$$\nabla^2 = \partial^2 - \frac{\hat{R}}{3},$$

and the corresponding inverse is

$$\nabla^{-2} = \partial^{-2} + \frac{1}{3} \partial^{-2} \hat{R} \partial^{-2} + O(\epsilon^4),$$

so equation (A2) leads to

$$\hat{\mathcal{H}} = - \partial^{-2} - 2 \frac{2}{3} \partial^{-2} \hat{R} \partial^{-2} + O(\epsilon^4).$$

The operator $\partial^{-2}$ that enters here is understood as the Green’s operator of the equation

$$\partial^2 \varphi^\alpha = q^\alpha.$$
\(x^\mu \to \infty\). This is due to the fact that at \(k^2 \to 0\), \(q^\alpha\) acts as a resonant driving force for \(\varphi^\alpha\). No quasimonochromatic solution is possible in this case, and \(\varphi^\alpha\) necessarily diverges at infinity. In particular, this means that even if the Fourier spectrum of \(q^\alpha\) is analytic but includes harmonics with \(k^2 = 0\), the Fourier spectrum of the corresponding \(\varphi^\alpha\) is singular.

This indicates that the case \(k^2 = 0\) cannot be treated within the adiabatic approximation that we assume in this paper. However, it still can be considered as a limit, as discussed in section 3. Also, no such issues arise in problems that involve GW–matter coupling, because then \(k^2 \neq 0\). In this case, the term \(\Delta\) in equation (A7) can be calculated too, but there is no need to do this explicitly in the present paper. (The general approach to such calculations is described, for example, in [64] ) What matters instead is that \(\Xi\) is a well-defined object that, in principle, can be found within any predefined accuracy. As usual in fundamental wave theory [44], the zeroth-order or first-order approximation of the Green’s operator often suffices for practical applications [55, 64, 66, 67].

**Appendix B. Derivation of equations (15) and (16)**

Using equation (13) for \(\tilde{\pi}^{\alpha\beta}_{\gamma\delta}\) and equation (2) for \(\mathcal{L}_m\gamma^\delta\), one obtains

\[
\tilde{\pi}^{\alpha\beta}_{\gamma\delta}\mathcal{L}_m\gamma^\delta = -\left(\delta^\alpha_\gamma\delta^\beta_\delta + \nabla^\alpha\Xi^\beta_\gamma\nabla^\delta + \nabla^\beta\Xi^\alpha_\gamma\nabla^\delta\right)\left(\nabla^\gamma u^\delta + \nabla^\delta u^\gamma\right).
\]

(B1)

Then using equation (5) in the above equation yields

\[
\tilde{\pi}^{\alpha\beta}_{\gamma\delta}\mathcal{L}_m\gamma^\delta = -2\nabla^{(\alpha}u^{\beta)} - 2\nabla^{(\alpha}\Xi^{\beta)}_\gamma\left[\nabla^\gamma, \nabla\right]u^\delta - 2\nabla^{(\alpha}\Xi^{\beta)}_\gamma\nabla^\gamma\nabla^\delta u^\gamma - \nabla^{(\alpha}\Xi^{\beta)}_\gamma\nabla^2 u^\gamma
\]

\[
= -2\nabla^{(\alpha}u^{\beta)} - 2\nabla^{(\alpha}\Xi^{\beta)}_\gamma\left(\delta^\gamma_\alpha\nabla^2 + R^\gamma_\delta\right)u^\delta - 2\nabla^{(\alpha}\Xi^{\beta)}_\gamma\nabla^\gamma\nabla^\delta u^\delta.
\]

(B2)

Using equation (9b) for \(\tilde{Q}^{\alpha\beta}\) in combination with equation (10), one obtains

\[
\tilde{\pi}^{\alpha\beta}_{\gamma\delta}\mathcal{L}_m\gamma^\delta = -2\nabla^{(\alpha}u^{\beta)} + 2\nabla^{(\alpha}\Xi^{\beta)}_\mu\tilde{u}^{\mu} - 2\nabla^{(\alpha}\Xi^{\beta)}_\mu\nabla^\mu\nabla^\delta u^\delta
\]

\[
= -2\nabla^{(\alpha}u^{\beta)} + 2\nabla^{(\alpha}\Xi^{\beta)}_\delta u^\delta - 2\nabla^{(\alpha}\Xi^{\beta)}_\mu\nabla^\mu\nabla^\delta u^\delta
\]

\[
= -2\nabla^{(\alpha}\Xi^{\beta)}_\mu\nabla^\mu\nabla^\delta u^\delta.
\]

(B3)

For \(\nabla^\delta u^\delta = 0\), this leads to \(\tilde{\pi}^{\alpha\beta}_{\gamma\delta}\mathcal{L}_m\gamma^\delta = 0\), which is equation (15). Otherwise, notice that

\[
2\nabla^\delta u^\delta = 2g_{\delta\gamma}\nabla^\gamma u^\delta = 2g_{\gamma\delta}\nabla^\gamma u^\delta = -g_{\gamma\delta}\mathcal{L}_m\gamma^\delta.
\]

(B4)

Then, one can rewrite equation (B3) as

\[
\tilde{\pi}^{\alpha\beta}_{\gamma\delta}\mathcal{L}_m\gamma^\delta = \nabla^{(\alpha}\Xi^{\beta)}_\mu\nabla^\mu\nabla^\delta\mathcal{L}_m\gamma^\delta,
\]

(B5)

which is precisely equation (16).

**Appendix C. Derivation of equation (4c)**

Using equation (18b), we get
\[
\hat{\Omega}^{\alpha\beta}_{\gamma\delta} \hat{\Omega}^{\gamma\delta}_{\epsilon\lambda} = 4 \nabla^{(\alpha\beta)} (\gamma \nabla_{\delta} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) - 2 \nabla^{(\alpha\beta)} (\gamma \nabla_{\delta} \nabla^{(\gamma\delta)} \nu \nabla^{\nu} g_{\lambda\epsilon} - 2 \nabla^{(\alpha\beta)} \mu \nabla^{\mu} g_{\gamma\delta} \nabla^{(\gamma\delta)} \nu \nabla^{\nu} g_{\lambda\epsilon}). \quad (C1)
\]

Let us simplify the individual terms on the right-hand side separately. We start by expanding one pair of symmetrized indices to get

\[
4 \nabla^{(\alpha\beta)} (\gamma \nabla_{\delta} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon})
\]

\[
= 2 \nabla^{(\alpha\beta)} \gamma \nabla_{\delta} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) + 2 \nabla^{(\alpha\beta)} \gamma \nabla^{2} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon})
\]

\[
= 2 \nabla^{(\alpha\beta)} \gamma \nabla \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) + 2 \nabla^{(\alpha\beta)} \gamma \nabla^{2} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) + 2 \nabla^{(\alpha\beta)} \gamma \nabla^{(\gamma\delta)} (\nabla \nu) \nabla^{\nu} (\lambda \nabla_{\epsilon}). \quad (C2)
\]

Recognizing that the operator would act on a rank-2 tensor \(h^{\lambda\epsilon}\), we can use equation (5) for the commutator; hence,

\[
4 \nabla^{(\alpha\beta)} (\gamma \nabla_{\delta} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon})
\]

\[
= 2 \nabla^{(\alpha\beta)} \gamma \nabla \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) + 2 \nabla^{(\alpha\beta)} \gamma \left( R^{\gamma\delta} + \delta^{\gamma}_{\delta} \nabla^{2} \right) (\lambda \nabla_{\epsilon}). \quad (C3)
\]

The terms in the parenthesis on the right-hand side of the above equation can be expressed through \(\hat{Q}^{\alpha\beta}\) (equation (9b)), which is also the inverse of \(\hat{\Omega}^{\alpha\beta}\) (equation (10)); hence,

\[
4 \nabla^{(\alpha\beta)} (\gamma \nabla_{\delta} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon})
\]

\[
= 2 \nabla^{(\alpha\beta)} \gamma \nabla \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) - 2 \nabla^{(\alpha\beta)} \gamma \hat{Q}^{\delta}_{\epsilon} (\lambda \nabla_{\epsilon}) = 2 \nabla^{(\alpha\beta)} \gamma \nabla \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) - 2 \nabla^{(\alpha\beta)} \gamma \hat{Q}^{\delta}_{\epsilon} (\lambda \nabla_{\epsilon}). \quad (C4)
\]

Using a similar process, the second term is found to be

\[
2 \nabla^{(\alpha\beta)} \gamma \nabla \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) - 2 \nabla^{(\alpha\beta)} \gamma \hat{Q}^{\delta}_{\epsilon} (\lambda \nabla_{\epsilon})
\]

\[
= \nabla^{(\alpha\beta)} \gamma \nabla \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) + \nabla^{(\alpha\beta)} \gamma \hat{Q}^{\delta}_{\epsilon} (\lambda \nabla_{\epsilon}). \quad (C5)
\]

The third and the fourth terms are simply

\[
2 \nabla^{(\alpha\beta)} \mu \nabla^{\mu} g_{\gamma\delta} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) = 2 \nabla^{(\alpha\beta)} \mu \nabla^{\mu} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}),
\]

\[
\nabla^{(\alpha\beta)} \mu \nabla^{\mu} g_{\gamma\delta} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) = \nabla^{(\alpha\beta)} \mu \nabla^{\mu} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}).
\]

Combining all these expressions, we get

\[
\hat{\Omega}^{\alpha\beta}_{\gamma\delta} \hat{\Omega}^{\gamma\delta}_{\epsilon\lambda} = 2 \nabla^{(\alpha\beta)} \gamma \nabla \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) - 2 \nabla^{(\alpha\beta)} \gamma \nabla^{(\gamma\delta)} (\nabla \nu) \nabla^{\nu} g_{\lambda\epsilon} - \nabla^{(\alpha\beta)} \mu \nabla^{\mu} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}),
\]

\[
\nabla^{(\alpha\beta)} \mu \nabla^{\mu} g_{\gamma\delta} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) = \nabla^{(\alpha\beta)} \mu \nabla^{\mu} \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}). \quad (C6)
\]

Canceling the first term on the right-hand side with the fifth term, and the fourth term with the sixth term, we arrive at

\[
\hat{\Omega}^{\alpha\beta}_{\gamma\delta} \hat{\Omega}^{\gamma\delta}_{\epsilon\lambda} = -2 \nabla^{(\alpha\beta)} \gamma \nabla \nabla^{(\gamma\delta)} (\lambda \nabla_{\epsilon}) + \nabla^{(\alpha\beta)} \mu \nabla^{\mu} g_{\lambda\epsilon}. \quad (C7)
\]

Upon comparison with equation (18b), this leads to equation (4c).
Appendix D. Derivation of equation (A3)

For any vector field $u^\alpha$, one has
\[ \nabla^\beta \nabla_\beta u^\alpha = \nabla^\beta (\partial_\beta u^\alpha + \Gamma^\alpha_{\beta\lambda} u^\lambda) \]
\[ = \partial^\beta (\partial_\beta u^\alpha + \Gamma^\alpha_{\beta\lambda} u^\lambda) + g^\alpha_\gamma \Gamma^\gamma_{\beta\rho} (\partial_\beta u^\rho + \Gamma^\rho_{\beta\lambda} u^\lambda), \]
\[ (D1) \]
where $\Gamma^\alpha_{\beta\lambda}$ are the Christoffel symbols. In normal coordinates, the Christoffel symbols are zero, but their derivatives are not. This leads to
\[ 2u^\alpha = \partial^2 u^\alpha + u^\lambda \partial^\beta \Gamma^\alpha_{\beta\lambda}. \]
\[ (D2) \]
The derivatives of the Christoffel symbols can be expressed through the Riemann tensor $R^\rho_{\sigma\mu\nu}$ [68]:
\[ \partial_\nu \Gamma^\rho_{\mu\sigma} = -\frac{1}{3} (R^\rho_{\sigma\mu\nu} + R^\rho_{\mu\nu\sigma}). \]
\[ (D3) \]
Using the well-known symmetries of the Riemann tensor and of the Ricci tensor $R_{\sigma\nu} = R^\rho_{\sigma\rho\nu}$, one finds that
\[ \partial^\beta \Gamma^\alpha_{\beta\lambda} = -\frac{1}{3} (R^\alpha_{\lambda\beta\beta} + R^\alpha_{\beta\lambda\beta}) = -\frac{1}{3} R^\alpha_{\beta\lambda\beta}. \]
Hence, one can rewrite equation (D2) as
\[ \nabla^2 u^\alpha = \partial^2 u^\alpha - \frac{1}{3} R^\alpha_{\beta\lambda} u^\beta, \]
\[ (D4) \]
or equivalently, as
\[ (\nabla^2)^\alpha_{\beta} = \delta^\alpha_{\beta} \partial^2 - \frac{1}{3} R^\alpha_{\beta}. \]
\[ (D5) \]
In the index-free representation, this leads to equation (A3).

ORCID iD
Deepen Garg \(https://orcid.org/0000-0001-5226-1913\)

References

[1] Carroll S 2004 *Spacetime and Geometry: An Introduction to General Relativity* (San Francisco, CA: Addison-Wesley)
[2] Mukhanov V 2005 *Physical Foundations of Cosmology* (New York: Cambridge University Press)
[3] Flanagan E E and Hughes S A 2005 The basics of gravitational wave theory *New J. Phys.* 7 204
[4] Bardeen J M 1980 Gauge-invariant cosmological perturbations *Phys. Rev. D* 22 1882
[5] Malik K A and Wands D 2009 Cosmological perturbations *Phys. Rep.* 475 1
[6] Malik K A and Matravers D R 2013 Comments on gauge-invariance in cosmology *Gen. Relativ. Gravit.* 45 1989
[7] Fewster C J and Hunt D S 2013 Quantization of linearized gravity in cosmological vacuum spacetimes *Rev. Math. Phys.* 25 1330003
[8] Fantuzzi G, Marozzi G, Medeiros M and Schiaffino G 2021 The cosmological perturbation theory on the geodesic light-cone background *J. Cosmol. Astropart. Phys.* JCAP02(2021)014
[9] Fröb M B and Lima W C C 2022 Cosmological perturbations and invariant observables in geodesic lightcone coordinates J. Cosmol. Astropart. Phys. JCAP01(2022)034
[10] Nakamura K 2007 Second-order gauge-invariant cosmological perturbation theory: – Einstein equations in terms of gauge invariant variables – Prog. Theor. Phys. 117 17
[11] Nakamura K 2019 Second-order gauge-invariant cosmological perturbation theory: current status updated in 2019 (arXiv:1912.12805)
[12] Bruni M, Matarrese S, Mollerach S and Sonego S 1997 Perturbations of spacetime: gauge transformations and gauge invariance at second order and beyond Class. Quantum Grav. 14 2585
[13] De Luca V, Franciolini G, Kehagias A and Riotto A 2020 On the gauge invariance of cosmological gravitational waves J. Cosmol. Astropart. Phys. JCAP03(2020)014
[14] Fröb M B, Hack T-P and Higuchi A 2017 Compactly supported linearised observables in single-field inflation J. Cosmol. Astropart. Phys. JCAP07(2017)043
[15] Fröb M B, Hack T-P and Khavkine I 2018 Approaches to local gauge-invariant observables in inflationary cosmologies Class. Quantum Grav. 35 115002
[16] Moretti F, Bombacigno F and Montani G 2019 Gauge invariant formulation of metric f(R) gravity for gravitational waves Phys. Rev. D 100 084014
[17] Higuchi A 2012 Equivalence between the Weyl-tensor and gauge-invariant graviton two-point functions in Minkowski and de Sitter spaces (arXiv:1204.1684)
[18] Thompson J E, Chen H and Whiting B F 2017 Gauge invariant perturbations of the Schwarzschild spacetime Class. Quantum Grav. 34 174001
[19] Aksteiner S, Andersson L, Bäckdahl T, Khavkine I and Whiting B 2021 Compatibility complex for black hole spacetimes Commun. Math. Phys. 384 1585
[20] Nakamura K 2021 Gauge-invariant perturbation theory on the Schwarzschild background spacetime part I: – formulation and odd-mode perturbations (arXiv:2110.13508)
[21] Garg D and Dodin I Y 2020 Average nonlinear dynamics of particles in gravitational pulses: effective Hamiltonian, secular acceleration and gravitational susceptibility Phys. Rev. D 102 064012
[22] Baym G, Patil S P and Pethick C J 2017 Damping of gravitational waves by matter Phys. Rev. D 96 084033
[23] Bamba K, Nojiri S and Odintsov S D 2018 Propagation of gravitational waves in strong magnetic fields Phys. Rev. D 98 024002
[24] Asenjo F A and Mahajan S M 2020 Resonant interaction between dispersive gravitational waves and scalar massive particles Phys. Rev. D 101 063010
[25] Barta D and Vasúth M 2018 Dispersion of gravitational waves in cold spherical interstellar medium Int. J. Mod. Phys. D 27 1850040
[26] Chesters D 1973 Dispersion of gravitational waves by a collisionless gas Phys. Rev. D 7 2863
[27] Asseo E, Gerbal D, Heyvaerts J and Signore M 1976 General-relativistic kinetic theory of waves in a massive particle medium Phys. Rev. D 13 2724
[28] Macedo P G and Nelson A H 1983 Propagation of gravitational waves in a magnetized plasma Phys. Rev. D 28 2382
[29] Flauger R and Weinberg S 2018 Gravitational waves in cold dark matter Phys. Rev. D 97 123506
[30] Servin M, Brodin G and Marklund M 2001 Cyclotron damping and Faraday rotation of gravitational waves Phys. Rev. D 64 024013
[31] Moortgat J and Kuijpers J 2003 Gravitational and magnetosonic waves in gamma-ray bursts Astron. Astrophys. 402 905
[32] Forsberg M and Brodin G 2010 Linear theory of gravitational wave propagation in a magnetized, relativistic Vlasov plasma Phys. Rev. D 82 124029
[33] Isliker H, Sandberg I and Vlahos L 2006 Interaction of gravitational waves with strongly magnetized plasmas Phys. Rev. D 74 104009
[34] Duez M D, Liu Y T, Shapiro S L and Stephens B C 2005 Relativistic magnetohydrodynamics in dynamical spacetimes: numerical methods and tests Phys. Rev. D 72 024028
[35] Mendonça J T 2002 Gravitational waves in plasmas Plasma Phys. Control. Fusion 44 B225
[36] Brodin G, Marklund M and Dusby P K S 2000 Nonlinear gravitational wave interactions with plasmas Phys. Rev. D 62 104008
[37] Brodin G, Forsberg M, Marklund M and Eriksson D 2010 Interaction between gravitational waves and plasma waves in the Vlasov description J. Plasma Phys. 76 345
[38] Servin M, Brodin G, Bradley M and Marklund M 2000 Parametric excitation of Alfven waves by gravitational radiation Phys. Rev. E 62 8493
[39] Brodin G, Marklund M and Servin M 2001 Photon frequency conversion induced by gravitational radiation Phys. Rev. D 63 124003
[40] Brodin G, Marklund M and Shukla P K 2005 Generation of gravitational radiation in dusty plasmas and supernovae J. Exp. Theor. Phys. 81 135
[41] Kumar S, Nunes R C and Yadav S K 2019 Testing the warmness of dark matter Mon. Not. R. Astron. Soc. 490 1406
[42] Moretti F, Bombacigno F and Montani G 2020 Generation of gravitational radiation in dusty plasmas and supernovae J. Exp. Theor. Phys. 81 135
[43] Kurokawa I 2018 Compatibility complexes of overdetermined PDEs of finite type, with applications to the Killing equation (arXiv:1805.03751)
[44] Stix T H 1992 Waves in Plasmas (New York: AIP)
[45] Dodin I Y, Zhmoginov A I and Ruiz D E 2017 Variational principles for dissipative (sub)systems, with applications to the theory of linear dispersion and geometrical optics Phys. Lett. A 381 1411
[46] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman)
[47] Oancea M A, Joudioux J, Dodin I Y, Ruiz D E, Paganini C F and Andersson L 2020 Gravitational spin Hall effect of light Phys. Rev. D 102 024075
[48] Fröb M B and Tehrani M T 2018 Green’s functions and Hadamard parametrices for vector and tensor fields in general linear covariant gauges Phys. Rev. D 97 025022
[49] Dodin I Y 2022 Quasilinear theory for inhomogeneous plasma J. Plasma Phys. 88 905880407
[50] Fröb M B 2018 Gauge-invariant quantum gravitational corrections to correlation functions Class. Quantum Grav. 35 055006
[51] Fröb M B and Lima W C C 2018 Propagators for gauge-invariant observables in cosmology Class. Quantum Grav. 35 095010
[52] Garg D and Dodin I Y 2021 Gauge-invariant gravitational waves in matter beyond linearized gravity (arXiv:2106.05062)
[53] Baer C, Ginoux N and Pfaeffle F 2008 Wave equations on Lorentzian manifolds and quantization (arXiv:0806.1036)
[54] Our calculations were facilitated by Mathematica © 1988–2019 Wolfram Research, Inc. version number 12.0.0.0
[55] Schutz B 2009 A First Course in General Relativity (New York: Cambridge University Press) (equation (8.36))
[56] Isaacson R A 1968 Gravitational radiation in the limit of high frequency. I. The linear approximation and geometrical optics Phys. Rev. 166 1263
[57] MacCallum M A H and Taub A H 1973 The averaged Lagrangian and high-frequency gravitational waves Commun. Math. Phys. 30 153
[58] Dodin I Y, Ruiz D E, Yanagihara K, Zhou Y and Kubo S 2019 Quasioptical modeling of wave beams with and without mode conversion. I. Basic theory Phys. Plasmas 26 072110
[59] McDonald S W 1988 Phase-space representations of wave equations with applications to the eikonal approximation for short-wavelength waves Phys. Rep. 158 337
[60] Ruiz D E and Dodin I Y 2017 Ponderomotive dynamics of waves in quasiperiodically modulated media Phys. Rev. A 95 032114
[61] Zhu H and Dodin I Y 2021 Wave-kinetic approach to zonal-flow dynamics: recent advances Phys. Plasmas 28 032303
[62] Brewin L 1998 Riemann normal coordinates, smooth lattices and numerical relativity Class. Quantum Grav. 15 3085