A note on modified generalized Riccati equation method combined with new algebra expansion

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Abstract: In this present work, we have proposed modified generalized Riccati equation method for finding the exact traveling wave solutions of nonlinear evolution equations (NLEEs) via the mKdV equation. It has been shown that the proposed method is effective and can be used for many other NLEEs in mathematical physics. The obtained results show that the proposed method is very powerful and convenient mathematical tool for NLEEs in science and engineering.

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1. Introduction

In the nonlinear sciences, it is well known that many nonlinear evolution equations (NLEEs) are widely used to describe the complex phenomena such as fluid mechanics, meteorology, plasma physics, optical fibers, biology, solid-state physics, chemical kinematics, chemical physics, and so on. The powerful and efficient methods to find analytic solutions and numerical solutions of nonlinear equations have drawn a lot of interest by diverse group of researchers. Many efficient analytic and numerical methods have been presented so far. During the research, searching for traveling wave solutions of NLEEs has been the main goal of many researchers, and many powerful methods for constructing exact solutions of NLEEs have been established and developed. In order to better understand the nonlinear phenomena as well as further practical applications, it is important to seek their more exact traveling wave solutions. Even those special exact solutions that do not have a clear physical meaning can be used as test problems to verify the consistency and estimate errors of

ABOUT THE AUTHORS

The author and the research group are currently focused on investigation of new exact solitary wave solution of nonlinear evolution equations (NLEEs) through analytical methods using symbolic computation software, like Mathematica or Maple. To this end, we have used the latest methods alongside we present some extensions of the existing methods. In continuation of this endeavor, in this article a new general ansatz called the modified generalized Riccati equation method has been presented. It is our anticipation that the proposed extension will be effective, straightforward, and suitable for handling NLEEs and thus it deserves more utilization as well as spread.

PUBLIC INTEREST STATEMENT

The choice of an appropriate ansatz is of great importance when using the direct methods. In this article, a new general ansatz called the modified generalized Riccati equation method has been presented with computer algebra system which can be used to obtain explicit solutions to NLEEs. By means of this method, the mKdV equation, a very important equation in mathematical physics has been examined and trigonometric (periodic), hyperbolic (soliton), and rational function solutions are found. It has been established that the presented modified generalized Riccati equation method is effective, straightforward, and suitable for the treatment of other NLEEs and thus it deserves further application and studying as well.
various numerical, asymptotic, and approximate analytical methods. Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving NLEEs. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions. Therefore, investigation of exact traveling wave solutions is becoming successively attractive in nonlinear sciences day by day. However, not all equations posed of these models are solvable. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as, the homotopy perturbation method (Mohyud-Din, 2007; Mohyud-Din & Noor, 2009; Tauseef Mohyud-Din, Yıldırım, & Anıl Sezer, 2011; Tauseef Mohyud-Din, Yıldırım, & Demirli, 2011; Tauseef Mohyud-Din, Yıldırım, & Saraydın, 2011), the \((G'/G)\)-expansion method (Guo & Zhou, 2010; Wang, Li, & Zhang, 2008; Zayed & Gepreel, 2009), the exp-function method (Akbar & Ali, 2011; Bekir & Boz, 2008; Naher, Abdullah, & Akbar, 2011), the modified simple equation method (Khan & Akbar, 2013a, 2014; Mohamad Jawad, Petković, & Biswas, 2010; Zayed, 2011), the Hirota’s bilinear transformation method (Hirota, 1973; Hirota & Satsuma, 1981), the \(\exp(-\phi(z))\)-expansion method (Khan & Akbar, 2013b), the enhanced \((G'/G)\)-expansion method (Khan, Akbar, Rashidi, & Zamanpour, 2015), the tanh-function method (Parkes & Duffy, 1996; Wazwaz, 2005), the functional variable method (Khan & Akbar, 2015; Zerarka, Ouamane, & Attaf, 2010), generalized Riccati equation expansion method (Biao, Yong, Hengnong, & Hongqing, 2004), and so on.

Various ansatz have been proposed for seeking traveling wave solutions of nonlinear differential equations. The choice of an appropriate ansatz is of great importance when using the direct methods. In this paper, based on a new general ansatz, we propose the modified generalized Riccati equation method with new algebra expansion, which can be used to obtain explicit solutions of NLEEs.

The objective of this article is to present modified generalized Riccati equation method with new algebra expansion to construct the exact traveling wave solutions for NLEEs in mathematical physics via the mKdV equation.

The article is arranged as follows: In Section 2, the methodology is discussed. In Section 3, we apply this method to the NLEE pointed out above and in Section 4, conclusions are given.

2. Methodology

Suppose the general nonlinear partial differential equation,

\[
G(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0,
\]

where \(u = u(x, t)\) is an unknown function, \(P\) is a polynomial in \(u(x, t)\) and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved. The main steps of the modified generalized Riccati equation method combined with the algebra expansion are as follows:

**Step 1:** The traveling wave variable ansatz

\[
\xi = x \pm \omega t \quad u(x, t) = u(\xi)
\]

where \(\omega \in \mathbb{R} \setminus \{0\}\) is the speed of the traveling wave, permits us to transform the Equation (1) into the following ODE:

\[
H(u, u', u'', \ldots) = 0,
\]

where the superscripts stands for the ordinary derivatives with respect to \(\xi\).
Step 2: Suppose the traveling wave solution of Equation (3) can be expressed by a polynomial in \(F(\xi)\) as follows:

\[
u(\xi) = \sum_{i=0}^{n} a_i(m + F(\xi))^i + \sum_{j=1}^{n} b_i(m + F(\xi))^{-j},
\]

where \(a_n\) and \(b_n\) are not zero simultaneously. Also, \(F = F(\xi)\) satisfies the generalized Riccati equation,

\[
F' = r + pF + qF^2,
\]

where \(a_n, b_n, r, p,\) and \(q\) are arbitrary constants to be determined later.

The generalized Riccati Equation (5) has 27 solutions as follows (Zhu, 2008):  

**Family 1:** When \(\Omega = p^2 - 4qr < 0\) and \(pq \neq 0\) (or \(rq \neq 0\)), the solutions of Equation (5) are,

\[
F_1 = \frac{1}{2q} \left(-p + \sqrt{-\Omega} \tan \left(\frac{1}{2} \sqrt{-\Omega} \xi\right)\right),
\]

\[
F_2 = -\frac{1}{2q} \left(p + \sqrt{-\Omega} \cot \left(\frac{1}{2} \sqrt{-\Omega} \xi\right)\right),
\]

\[
F_3 = \frac{1}{2q} \left(-p + \sqrt{-\Omega} \tan \left(\sqrt{-\Omega} \xi\right) \pm \sec \left(\sqrt{-\Omega} \xi\right)\right),
\]

\[
F_4 = -\frac{1}{2q} \left(p + \sqrt{-\Omega} \cot \left(\sqrt{-\Omega} \xi\right) \pm \csc \left(\sqrt{-\Omega} \xi\right)\right),
\]

\[
F_5 = \frac{1}{4q} \left(-2p + \sqrt{-\Omega} \left(\tan \left(\frac{1}{4} \sqrt{-\Omega} \xi\right) - \cot \left(\frac{1}{4} \sqrt{-\Omega} \xi\right)\right)\right),
\]

\[
F_6 = \frac{1}{2q} \left(-p + \frac{\sqrt{(A^2 - B^2)} \sqrt{-\Omega} - A \sqrt{-\Omega} \cos \left(\sqrt{-\Omega} \xi\right)}{A \sin \left(\sqrt{-\Omega} \xi\right) + B}\right),
\]

\[
F_7 = \frac{1}{2q} \left(-p + \frac{\sqrt{(A^2 - B^2)} \sqrt{-\Omega} + A \sqrt{-\Omega} \cos \left(\sqrt{-\Omega} \xi\right)}{A \sin \left(\sqrt{-\Omega} \xi\right) + B}\right),
\]

where \(A\) and \(B\) are two non-zero real constants and satisfies the condition \(A^2 - B^2 > 0\).

\[
F_8 = -2r \left(\sqrt{-\Omega} \tan \left(\frac{1}{2} \sqrt{-\Omega} \xi\right) + p\right)^{-1},
\]

\[
F_9 = 2r \left(\sqrt{-\Omega} \cot \left(\frac{1}{2} \sqrt{-\Omega} \xi\right) - p\right)^{-1},
\]
Family 2: When \( \Omega = p^2 - 4qr > 0 \) and \( pq \neq 0 \) (or \( rq \neq 0 \)), the solutions of Equation (5) are, where \( A \) and \( B \) are two non-zero real constants and satisfies the condition \( B^2 - A^2 > 0 \).

\[
F_{10} = -2r \left( \sqrt{-\Omega} \tan \left( \sqrt{-\Omega} \xi \right) \pm \sqrt{-\Omega} \sec \left( \sqrt{-\Omega} \xi \right) + p \right)^{-1},
\]

\[
F_{11} = 2r \left( \sqrt{-\Omega} \cot \left( \sqrt{-\Omega} \xi \right) \pm \sqrt{-\Omega} \csc \left( \sqrt{-\Omega} \xi \right) - p \right)^{-1},
\]

\[
F_{12} = 4r \left( 2 \sqrt{-\Omega} \cot \left( \frac{1}{4} \sqrt{-\Omega} \xi \right) - \sqrt{-\Omega} \csc \left( \frac{1}{4} \sqrt{-\Omega} \xi \right) \sec \left( \frac{1}{4} \sqrt{-\Omega} \xi \right) - 2p \right)^{-1},
\]

\[
F_{13} = -\frac{1}{2q} \left( p + \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right),
\]

\[
F_{14} = -\frac{1}{2q} \left( p + \sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right),
\]

\[
F_{15} = -\frac{1}{2q} \left( p + \sqrt{\Omega} \left( \tanh \left( \sqrt{\Omega} \xi \right) \pm i \sec h \left( \sqrt{\Omega} \xi \right) \right) \right),
\]

\[
F_{16} = -\frac{1}{2q} \left( p + \sqrt{\Omega} \left( \coth \left( \sqrt{\Omega} \xi \right) \pm csc h \left( \sqrt{\Omega} \xi \right) \right) \right),
\]

\[
F_{17} = -\frac{1}{4q} \left( 2p + \sqrt{\Omega} \left( \tanh \left( \frac{1}{4} \sqrt{\Omega} \xi \right) + \coth \left( \frac{1}{4} \sqrt{\Omega} \xi \right) \right) \right),
\]

\[
F_{18} = \frac{1}{2q} \left\{ -p + \frac{\sqrt{\left( A^2 + B^2 \right) \sqrt{\Omega} - A \sqrt{\Omega} \cosh \left( \sqrt{\Omega} \xi \right)}}{A \sinh \left( \sqrt{\Omega} \xi \right) + B} \right\},
\]

\[
F_{19} = \frac{1}{2q} \left\{ -p - \frac{\sqrt{\left( B^2 - A^2 \right) \sqrt{\Omega} + A \sqrt{\Omega} \cosh \left( \sqrt{\Omega} \xi \right)}}{A \sinh \left( \sqrt{\Omega} \xi \right) + B} \right\},
\]

where \( A \) and \( B \) are two non-zero real constants and satisfies the condition \( B^2 - A^2 > 0 \).

\[
F_{20} = 2r \left( \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) - p \right)^{-1},
\]

\[
F_{21} = 2r \left( \sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} \xi \right) - p \right)^{-1},
\]

\[
F_{22} = 2r \left( \sqrt{\Omega} \tanh \left( \sqrt{\Omega} \xi \right) \pm i \sqrt{\Omega} \sec h \left( \sqrt{\Omega} \xi \right) - p \right)^{-1},
\]
When \( r = 0 \) and \( pq \neq 0 \), the solutions of Equation (5) are,

\[
F_{23} = 2r \left( \sqrt{\Omega} \coth \left( \sqrt{\Omega} \xi \right) \pm \sqrt{\Omega} \csc h \left( \sqrt{\Omega} \xi \right) - p \right)^{-1},
\]

\[
F_{24} = 4r \left( 2 \sqrt{\Omega} \coth \left( \frac{1}{4} \sqrt{\Omega} \xi \right) - \sqrt{\Omega} \csc h \left( \frac{1}{4} \sqrt{\Omega} \xi \right) \sec h \left( \frac{1}{4} \sqrt{\Omega} \xi \right) - 2p \right)^{-1},
\]

Family 3: When \( r = 0 \) and \( pq \neq 0 \), the solutions of Equation (5) are,

\[
F_{25} = -pd \frac{q}{q(\rho(x) - \sinh(p(x)))},
\]

\[
F_{26} = -\frac{p(\rho(x) + \sinh(p(x)))}{q(\rho(x) + \sinh(p(x)))},
\]

where \( d \) is an arbitrary constant.

Family 4: When \( q \neq 0 \) and \( r = p = 0 \), the solution of Equation (5) is,

\[
F_{27} = -\frac{1}{q(x) + c_1},
\]

where \( c_1 \) is an arbitrary constant.

Step 3: The positive integer \( n \) can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Equations (1) or (3). Moreover, precisely, we define the degree of \( u(\xi) \) as \( D(u(\xi)) = n \) which gives rise to the degree of other expression as follows:

\[
D \left( \frac{d^n u}{d \xi^n} \right) = n + q, \quad D \left( u^p \left( \frac{d^n u}{d \xi^n} \right)^s \right) = np + s(n + q).
\]

Therefore, we can find the value of \( n \) in Equation (4), using Equation (6).

Step 4: Substituting Equation (4) along with Equation (5) into Equation (3) together with the value of \( n \) obtained in step 3, we obtain polynomials in \( F_i \) and \( F_{-i} \) \((i = 0, 1, 2, 3, \ldots)\), then setting each coefficient of the resulted polynomial to zero, yields a system of algebraic equations for \( a_n, b_n, p, q, r, \) and \( \omega \).

Step 5: Suppose the value of the constants \( a_n, b_n, p, q, r, \) and \( \omega \) can be determined by solving the system of algebraic equations obtained in step 4. Since the general solutions of Equation (5) are known, substituting, \( a_n, b_n, p, q, r, \) and \( \omega \) into Equation (4), we obtain some new exact traveling wave solutions of the nonlinear evolution Equation (1).

3. Application

In the present work, we consider the following mKdV equation with parameters of the form,

\[
u_t - u^2 u_x + \delta u_{xxx} = 0.
\]

where \( \delta \) is a non-zero constant. The mKdV equation appears in electric circuits and multi-component plasmas. This equation describes nonlinear wave propagation in systems with polarity symmetry. The mKdV equation is used in electrodynamics, wave propagation in size quantized films, and in elastic media (Wazwaz, 2009).
The traveling wave transformation equation \( u(\xi) = u(x, t), \xi = x - \omega t \) transform Equation (7) to the following ordinary differential equation:

\[-\omega u' - u^2 u' + \delta u''' = 0. \tag{8}\]

Now integrating Equation (8) with respect to \( \xi \) once, we have

\[\delta u'' - \omega u - \frac{u^3}{3} + C = 0, \tag{9}\]

where \( C \) is a constant of integration. Balancing the highest order derivative term \( u'' \) and the nonlinear term \( u^3 \) from Equation (9), yields \( 3n = n + 2 \), which gives \( n = 1 \).

Hence, for \( n = 1 \) Equation (4) reduces to

\[u(\xi) = a_0 + a_1 (m + F) + b_1 (m + F)^{-1}. \tag{10}\]

Now substituting Equation (10) into Equation (5) into Equation (9), we get a polynomial in \( F(\xi) \). Equating the coefficient of same power of \( F(\xi) \), we attain the following system of algebraic equations:

\[-a_1^2 + 6\delta a_1 q^2 = 0.\]

\[18\delta a_1 q^2 m - 6a_1^2 m - 3a_0 a_1^2 + 9\delta a_1 pq = 0.\]

\[-15a_1^2 m^2 - 3a_0 a_1^2 - 15a_0 a_1^2 m - 3\omega a_1 + 18\delta a_1 q^2 m^2 + 6\delta a_1 q r + 3\delta a_1 p^2 + 27\delta a_1 pqm - 3a_1^2 b_1 = 0.\]

\[-3\omega a_0 - 6\delta b_1 q^2 m - 6a_0 a_1 b_1 - 12\omega a_2 m - 30a_0 a_1^2 m^2 + 3C - 20a_1^2 m^2 - 12 a_1^2 b_1 m + 18\delta a_1 qrm + 3\delta b_1 q r + 9\delta a_1 p q r - 12a_0 a_1 m + 3\delta a_1 p r - a_0^2 + 6\delta a_1 q^2 m^3 + 27\delta a_1 pqm^2 = 0.\]

\[3\delta b_1 p^2 - 30a_0 a_1^2 m^3 - 15a_1 m^4 - 9\omega a_0 m - 18a_0 a_1 m b_1 + 6\delta b_1 r q - 3a_1^2 b_1 + 9\delta a_1 pqm^3 + 9Cm + 9\delta a_1 qrm - 3a_1 b_1^2 - 3a_1^2 m\]

\[+9\delta a_1 q^2 m^2 + 18\delta a_1 qrm^2 - 9\delta b_1 pqm - 18\omega a_1 m^2 - 18a_0 a_1 m^2 - 3\omega b_1 - 18a_2^2 m^2 b_1 = 0.\]

\[-9\omega a_0 m^2 - 18\omega a_1 a_1 m^3 - 6a_0 m^2 - 6a_1 m b_1^2 + 9Cm^2 + 9\delta b_1 p r + 3\delta a_1 p q r - 12a_0 a_1 m^2 - 6a_0^2 m^2 - 6a_0^2 m^5 - 6\omega b_1 m - 12a_0^2 a_1 m^3 + 9\delta a_1 q r m^2 - 3a_0 a_1^2 m^4 - 12a_1 m^3 - 12a_1^2 m b_1\]

\[-a_0^2 a_1^2 + 6\delta a_1 q r m^2 - 3a_1^2 m^3 - 3\omega b_1 p^2 m - 3\omega a_1 q r m = 0.\]

\[-a_0^2 a_1^2 m^4 - 3a_0 a_1 m^5 - b_1 - a_0^2 m^3 - 3a_1 m^2 b_1^2 + 3Cm^3 - 3a_0 a_1 m^4 + 6\delta b_1 r^2 - a_1^2 m^6 - 6a_0 a_1 m^3 b_1 - 3a_0 b_1^2 m - 3a_1^2 m b_1 - 3\delta b_1 p r m - 3\delta b_1 b_1^2 m^2 - 3\omega a_1 m^3 + 3\delta a_1 p r m^2 = 0.\]

Solving the above system of equations for \( a_0, a_1, b_1, \omega, m, \) and \( C \), we get the following values:

**Case 1:** \( C = 0, \omega = -\frac{1}{2} (p^2 - 4qr)^\delta, m = m, a_0 = \pm \frac{3(2m - p)}{\sqrt{6}} r, a_1 = \pm \sqrt{6}q b_1 = 0.\)

**Case 2:** \( C = 0, \omega = -\frac{1}{2} (p^2 - 4qr)^\delta, m = m, a_0 = \pm \frac{3(2m - p)}{\sqrt{6}} r, a_1 = 0, b_1 = 1 = \pm \sqrt{6}(r + q m^2 - pm),\)
Case 3: \( C = 0, \omega = -2\delta p^2 + 8\delta qr, m = \frac{p}{2q}, a_0 = 0, a_1 = \pm \sqrt{\delta} q, b_1 = \pm \sqrt{\delta}(\frac{p^2 - 4qr}{4q}). \)

Case 4:
\[ C = \pm 2\sqrt{\delta} q\left(-3m^2pq + 2qmp + p^2m + 2m^3q^2 - pr\right). \]
\[ a_0 = \mp \frac{3\delta(2qm-p)}{\sqrt{\delta}}, a_1 = \pm \sqrt{\delta} q, \omega = -\frac{1}{2}\delta\left(12q^2m^2 - 12qmp + p^2 + 8qr\right), \]
\[ b_1 = \pm \sqrt{6\delta}(r + qm^2 - pm). \]

For Case 1, we get the following exact solutions in terms of hyperbolic, trigonometric, and rational functions:

When \( \Omega = p^2 - 4qr < 0 \) and \( pq \neq 0 \) (or \( rq \neq 0 \)), the solutions are,

\[ u_1(\xi) = \pm \frac{3\delta(2qm-p)}{\sqrt{6\delta}} \pm \sqrt{6\delta} q\left(m + \frac{1}{2}\left(-p + \sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right), \]

\[ u_2(\xi) = \pm \frac{3\delta(2qm-p)}{\sqrt{6\delta}} \pm \sqrt{6\delta} q\left(m - \frac{1}{2}\left(p + \sqrt{\Omega}\cot\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right)\right), \]

\[ u_3(\xi) = \pm \frac{3\delta(2qm-p)}{\sqrt{6\delta}} \pm \sqrt{6\delta} q\left(m + \frac{1}{2}\left(-p + \sqrt{\Omega}\tan\left(\sqrt{\Omega}\xi\right)\right) \pm \sec\left(\sqrt{\Omega}\xi\right)\right), \]

\[ u_4(\xi) = \pm \frac{3\delta(2qm-p)}{\sqrt{6\delta}} \pm \sqrt{6\delta} q\left(m - \frac{1}{2}\left(p + \sqrt{\Omega}\cot\left(\sqrt{\Omega}\xi\right)\right) \pm \csc\left(\sqrt{\Omega}\xi\right)\right), \]

\[ u_5(\xi) = \pm \frac{3\delta(2qm-p)}{\sqrt{6\delta}} \pm \sqrt{6\delta} q\left(m + \frac{1}{4}\left(-2p + \sqrt{\Omega}\tan\left(\frac{1}{4}\sqrt{\Omega}\xi\right) - \cot\left(\frac{1}{4}\sqrt{\Omega}\xi\right)\right)\right), \]

\[ u_6(\xi) = \pm \frac{3\delta(2qm-p)}{\sqrt{6\delta}} \pm \sqrt{6\delta} q\left(m + \frac{1}{2}\left(-p + \sqrt{\Omega}\cot\left(\sqrt{\Omega}\xi\right) - \csc\left(\sqrt{\Omega}\xi\right)\right)\right), \]

\[ u_7(\xi) = \pm \frac{3\delta(2qm-p)}{\sqrt{6\delta}} \pm \sqrt{6\delta} q\left(m + \frac{1}{4}\left(2p + \sqrt{\Omega}\tan\left(\frac{1}{4}\sqrt{\Omega}\xi\right) + \cot\left(\frac{1}{4}\sqrt{\Omega}\xi\right)\right)\right), \]

\[ u_8(\xi) = \pm \frac{3\delta(2qm-p)}{\sqrt{6\delta}} \pm \sqrt{6\delta} q\left(m + \frac{1}{2}\left(-p + \sqrt{\Omega}\cot\left(\sqrt{\Omega}\xi\right) + \csc\left(\sqrt{\Omega}\xi\right)\right)\right), \]

where \( \xi = x + \frac{1}{2}(p^2 - 4qr)\delta t \) and \( A, B \) are two non-zero real constants which satisfies the condition \( A^3 - B^3 > 0 \).
where

\[ u_{1_0}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m - 2r \left( \sqrt{-\Omega} \tan \left( \sqrt{-\Omega} \xi \right) \pm \sqrt{-\Omega} \sec \left( \sqrt{-\Omega} \xi \right) + p \right)^{-1} \right), \]

\[ u_{1_{10}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m + 2r \left( \sqrt{-\Omega} \cot \left( \sqrt{-\Omega} \xi \right) \pm \sqrt{-\Omega} \csc \left( \sqrt{-\Omega} \xi \right) - p \right)^{-1} \right), \]

\[ u_{1_{11}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m + 4r \left( \frac{2}{\sqrt{-\Omega}} \cot \left( \frac{1}{\sqrt{-\Omega}} \xi \right) - \sqrt{-\Omega} \csc \left( \frac{1}{\sqrt{-\Omega}} \xi \right) \sec \left( \frac{1}{\sqrt{-\Omega}} \xi \right) - 2p \right)^{-1} \right), \]

where \( \xi = x + \frac{1}{\delta}(p^2 - 4qr) \delta t. \)

When \( \Omega = p^2 - 4qr > 0 \) and \( pq \neq 0 \) (or \( qr \neq 0 \)), the solutions are,

\[ u_{1_{12}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m - \frac{1}{2q} \left( p + \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right), \]

\[ u_{1_{13}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m - \frac{1}{2q} \left( p + \sqrt{\Omega} \coth \left( \frac{1}{2} \sqrt{\Omega} \xi \right) \right) \right), \]

\[ u_{1_{14}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m - \frac{1}{2q} \left( p + \sqrt{\Omega} \left( \tanh \left( \sqrt{\Omega} \xi \right) \pm i \sec h \left( \sqrt{\Omega} \xi \right) \right) \right) \right), \]

\[ u_{1_{15}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m + \frac{1}{2q} \left( p + \sqrt{\Omega} \left( \coth \left( \sqrt{\Omega} \xi \right) \pm \csc h \left( \sqrt{\Omega} \xi \right) \right) \right) \right), \]

\[ u_{1_{16}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m + \frac{1}{4q} \left( 2p + \sqrt{\Omega} \left( \tanh \left( \frac{1}{4} \sqrt{\Omega} \xi \right) \right) \right) \right), \]

\[ u_{1_{17}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m + \frac{1}{4q} \left( 2p + \sqrt{\Omega} \left( \coth \left( \frac{1}{4} \sqrt{\Omega} \xi \right) \right) \right) \right), \]

\[ u_{1_{18}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m + \frac{1}{2q} \left( -p + \frac{\sqrt{A^2 + B^2}}{A \sinh \left( \sqrt{\Omega} \xi \right) + B} \right) \right), \]

\[ u_{1_{19}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m + \frac{1}{2q} \left( -p - \frac{\sqrt{B^2 - A^2}}{A \sinh \left( \sqrt{\Omega} \xi \right) + B} \right) \right), \]

where \( \xi = x + \frac{1}{\delta}(p^2 - 4qr) \delta t \) and \( A, B \) are two non-zero real constants which satisfies the condition \( B^2 - A^2 > 0. \)

\[ u_{1_{20}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m + 2r \left( \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) - p \right)^{-1} \right), \]

\[ u_{1_{21}}(\xi) = \mp \frac{3 \delta (2qm - p)}{\sqrt{6} \delta} \pm \sqrt{6} \delta q \left( m + 2r \left( \sqrt{\Omega} \cot \left( \frac{1}{2} \sqrt{\Omega} \xi \right) - p \right)^{-1} \right). \]
\[ u_{e_2}(\xi) = \pm \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta q \left( m + 2r \left( \sqrt{\Omega} \tanh \left( \sqrt{\Omega} \xi \right) \pm i \sqrt{\Omega} \sec h \left( \sqrt{\Omega} \xi \right) - p \right)^{-1} \right), \]

\[ u_{e_3}(\xi) = \pm \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta q \left( m + 2r \left( \sqrt{\Omega} \coth \left( \sqrt{\Omega} \xi \right) \pm \sqrt{\Omega} \csc h \left( \sqrt{\Omega} \xi \right) - p \right)^{-1} \right), \]

\[ u_{e_4}(\xi) = \pm \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta q \left( m + 4r \left( 2 \sqrt{\Omega} \coth \left( \frac{1}{\delta} \sqrt{\Omega} \xi \right) - \sqrt{\Omega} \csc h \left( \frac{1}{\delta} \sqrt{\Omega} \xi \right) \sec h \left( \frac{1}{\delta} \sqrt{\Omega} \xi \right) - 2p \right)^{-1} \right), \]

where \( \xi = x + \frac{1}{\delta} (p^2 - 4qr) \delta t. \)

When \( r = 0 \) and \( pq \neq 0 \), the solutions are,

\[ u_{e_5}(\xi) = \pm \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta q \left( m - \frac{pd}{q(\cosh(p\xi) - \sinh(p\xi))} \right), \]

\[ u_{e_6}(\xi) = \pm \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta q \left( m - \frac{p(\cosh(p\xi) + \sinh(p\xi))}{q(\cosh(p\xi) + \sinh(p\xi))} \right), \]

where \( \xi = x + \frac{1}{\delta} p^2 \delta t \) and \( d \) is an arbitrary constant, but not equal to zero.

When \( q \neq 0 \) and \( r = p = 0 \), we get

\[ u_{e_7}(\xi) = \pm \frac{\sqrt{6}\delta q}{qx + c_1}, \]

which is independent of variable \( t \), and \( c_1 \) is an arbitrary constant.

For Case 2, we get the following exact solutions in terms of hyperbolic, trigonometric, and rational functions:

When \( \Omega = p^2 - 4qr < 0 \) and \( pq \neq 0 \) (or \( rq \neq 0 \)), we obtain

\[ u_{2_1}(\xi) = \pm \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta(r + qm^2 - pm) \left( m + \frac{1}{2q} \left( -p + \sqrt{-\Omega} \tan \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right)^{-1}, \]

\[ u_{2_2}(\xi) = \pm \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta(r + qm^2 - pm) \left( m - \frac{1}{2q} \left( p + \sqrt{-\Omega} \cot \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right)^{-1}, \]

\[ u_{2_3}(\xi) = \pm \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta(r + qm^2 - pm) \left( m + \frac{1}{2q} \left( -p + \sqrt{-\Omega} \left( \tan \left( \sqrt{-\Omega} \xi \right) \pm \sec \left( \sqrt{-\Omega} \xi \right) \right) \right) \right)^{-1}, \]
where \( \xi = x + \frac{1}{2}(p^2 - 4qr)\delta t \).

When \( \Omega = p^2 - 4qr > 0 \) and \( pq \neq 0 \) (or \( rq \neq 0 \)), we get

\[
\begin{align*}
u_{2\alpha}(\xi) &= \pm \frac{3\delta(2qm-p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta(r + qm^2 - pm) \\
\times \left( m - \frac{1}{2q} \left( p + \sqrt{-\Omega} \left( \tan \left( \frac{1}{4} \sqrt{-\Omega} \xi \right) - \cot \left( \frac{1}{4} \sqrt{-\Omega} \xi \right) \right) \right) \right)^{-1},
\end{align*}
\]

where \( \xi = x + \frac{1}{2}(p^2 - 4qr)\delta t \).

When \( r = 0 \) and \( pq \neq 0 \), the solutions of Equation (5) are,

\[
\begin{align*}
u_{2\alpha}(\xi) &= \pm \frac{3\delta(2qm-p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta(qm^2 - pm) \left( m + 2r \left( \sqrt{-\Omega} \tanh \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) - p \right) \right)^{-1},
\end{align*}
\]

where \( \xi = x + \frac{1}{2}p^2\delta t \) and \( d \) is an arbitrary constant.

When \( q \neq 0 \) and \( r = p = 0 \), the solution of Equation (5) is,

\[
\begin{align*}
u_{2\gamma}(\xi) &= \pm \sqrt{6}\delta q \pm \sqrt{6}\delta qm^2 \left( m - \frac{1}{q x + c_1} \right)^{-1},
\end{align*}
\]

which is independent of variable \( t \), and \( c_1 \) is an arbitrary constant.

Likewise, we can write down the other families of exact solutions which have omitted for convenience.

For Case 3, we get the following exact solutions in terms of hyperbolic, trigonometric, and rational functions:

When \( \Omega = p^2 - 4qr < 0 \) and \( pq \neq 0 \) (or \( rq \neq 0 \)), we have

\[
\begin{align*}
u_{3\alpha}(\xi) &= \pm \sqrt{6}\delta q \left( \frac{p}{2q} + \frac{1}{2q} \left( -p + \sqrt{-\Omega} \tan \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right) \pm \frac{\sqrt{6}\delta (p^2 - 4qr)}{4q} \\
\times \left( \frac{p}{2q} + \frac{1}{2q} \left( -p + \sqrt{-\Omega} \tan \left( \frac{1}{2} \sqrt{-\Omega} \xi \right) \right) \right)^{-1},
\end{align*}
\]
where \( \xi = x + \left(2 \delta p^2 - 8 \delta qr\right) t. \)

When \( \Omega = p^2 - 4qr > 0 \) and \( pq \neq 0 \) (or \( rq \neq 0 \)), we get

\[
\begin{align*}
\nu_{1s}(\xi) &= \pm \sqrt{6 \delta} q \left( \frac{p}{2q} + \frac{1}{2q} \left( \frac{1}{2} \sqrt{\Omega} \right) \right) \left( \frac{p}{2q} + \frac{1}{2q} \left( \frac{1}{2} \sqrt{\Omega} \right) \right)^{-1}, \\
\end{align*}
\]

where \( \xi = x + \left(2 \delta p^2 - 8 \delta qr\right) t. \)

When \( r = 0 \) and \( pq \neq 0 \), we get

\[
\begin{align*}
\nu_{2s}(\xi) &= \pm \sqrt{6 \delta} q \left( \frac{p}{2q} - \frac{pd}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right) \left( \frac{p}{2q} - \frac{pd}{q(d + \cosh(p\xi) - \sinh(p\xi))} \right)^{-1}, \\
\end{align*}
\]

where \( \xi = x + 2 \delta p^2 t. \)

When \( q \neq 0 \) and \( r = p = 0 \), we obtain

\[
\begin{align*}
\nu_{3s}(\xi) &= \pm \sqrt{6 \delta} q \left( -\frac{1}{qx + c_1} \right) \pm \sqrt{6 \delta} q \left( -\frac{1}{qx + c_1} \right)^{-1}, \\
\end{align*}
\]

which is independent of variable \( t \), and \( c_1 \) is an arbitrary constant.

In the same way, we can write down the other families of exact solutions which have omitted for amenities.

For Case 4, we get the following traveling exact solutions in terms of hyperbolic, trigonometric, and rational functions:

When \( \Omega = p^2 - 4qr < 0 \) and \( pq \neq 0 \) (or \( rq \neq 0 \)), we get

\[
\begin{align*}
\nu_{4s}(\xi) &= \pm \frac{3\delta (2qm - p)}{\sqrt{6 \delta}} \pm \sqrt{6 \delta} q \left( m + \frac{1}{2q} \left[ -p + \sqrt{A^2 - B^2} \right. \right. \\
&\left. \left. \frac{A \sqrt{-\Omega} + \sqrt{-\Omega} \cos \left( \sqrt{-\Omega} \xi \right)}{A \sin \left( \sqrt{-\Omega} \xi \right) + B} \right] \right) \\
\end{align*}
\]

where \( \xi = x + \frac{1}{2} \delta \left( 12q^2 m^2 - 12qmp + p^2 + 8qr \right) t \) and \( A, B \) are two non-zero real constants and satisfies the condition \( A^2 - B^2 > 0 \).

When \( \Omega = p^2 - 4qr > 0 \) and \( pq \neq 0 \) (or \( rq \neq 0 \)), we get
represented in the following figures with the aid of Maple: (Figures 1–4).

\[ u_{\pm_1}(\xi) = \mp \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta q \left( m + 2r \left( \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) - p \right)^{-1} \right) \]

\[ \pm \sqrt{6}\delta (r + qm^2 - pm) \left( m + 2r \left( \sqrt{\Omega} \tanh \left( \frac{1}{2} \sqrt{\Omega} \xi \right) - p \right)^{-1} \right)^{-1}, \]

where \( \xi = x + \frac{1}{\lambda} \delta \left( 12q^2m^2 - 12qm + p^2 + 8qr \right) t. \)

When \( r = 0 \) and \( pq \neq 0 \), we get

\[ u_{\pm_2}(\xi) = \pm \frac{3\delta(2qm - p)}{\sqrt{6}\delta} \pm \sqrt{6}\delta q \left( m - \frac{p(\cosh(p\xi) + \sinh(p\xi))}{q(d + \cosh(p\xi) + \sinh(p\xi))} \right) \]

\[ \pm \sqrt{6}\delta (qm^2 - pm) \left( m - \frac{p(\cosh(p\xi) + \sinh(p\xi))}{q(d + \cosh(p\xi) + \sinh(p\xi))} \right)^{-1}, \]

where \( \xi = x + \frac{1}{\lambda} \delta \left( 12q^2m^2 - 12qm + p^2 \right) t. \)

When \( q \neq 0 \) and \( r = p = 0 \), we obtain

\[ u_{\pm_3}(\xi) = \pm \sqrt{6}\delta qm \pm \sqrt{6}\delta q \left( m - \frac{1}{q\xi + c_1} \right) \pm \sqrt{6}\delta qm^2 \left( m - \frac{1}{q\xi + c_1} \right)^{-1}, \]

which is independent of variable \( t \), and \( c_1 \) is an arbitrary constant.

Similarly, we can write down the other families of exact solutions which are omitted for convenience.

**Remark**  All the obtained solutions have been checked with Maple by putting them back into the original equations and found correct.

4. Results and discussion

In this section, we will give explanation of obtained solutions of mKdV equation. The introduction of dispersion without introducing nonlinearity destroys the solitary wave as different Fourier harmonics start propagating at different group velocities. On the other hand, introducing nonlinearity without dispersion also prevents the formation of solitary waves, because the pulse energy is frequently pumped into higher frequency modes. However, if both dispersion and nonlinearity are present, solitary waves can be sustained. Similar to dispersion, dissipation can also give rise to solitary waves when combined with nonlinearity. Hence, it is interesting to point out that the delicate balance between the nonlinearity effect of \( u' u \), and the dispersion effect of \( u_{\pm_1} \) gives rise to solitons, that after a fully interaction with others, the solitons come back retaining their identities with the same speed and shape. The mKdV equation has solitary wave solutions that have exponentially decaying wings. If two solitons of the mKdV equation collide, the solitons just pass through each other and emerge unchanged.

When \( \Omega = p^2 - 4qr < 0 \) and \( pq \neq 0 \) (or \( r \neq 0 \)), the obtained solutions are trigonometric function solutions which are periodic in nature. When \( \Omega = p^2 - 4qr > 0 \), \( pq \neq 0 \) (or \( r \neq 0 \)) and when \( r = 0, pq \neq 0 \) we got hyperbolic function solutions. When \( q = 0 \) and \( r = p = 0 \), the solutions are rational.

We have obtained traveling wave solutions with various profiles in terms of some unknown parameters. Solitary waves can be obtained from each traveling wave solution by setting particular values to its unknown parameters. Some of our obtained traveling and solitary wave solutions are represented in the following figures with the aid of Maple: (Figures 1–4).
Figure 1. Periodic solution $u_1(\xi)$ for $p = 1, q = r = 2, \delta = 1, m = 0$ within $-3 \leq x, t \leq 3$.

Figure 2. Periodic solution $u_1(\xi)$ for $p = q = 0.5, r = 7, \delta = 2, m = 0$ within $-3 \leq x, t \leq 3$.

Figure 3. Soliton solution $u_1(\xi)$ for $p = 5, q = r = \delta = 1, m = 0$ within $-3 \leq x, t \leq 3$. 
5. Comparisons

With the \( \left( G'/G \right) \)-expansion method: Wang et al. (2008) investigated exact solutions of the mKdV equation using the \( \left( G'/G \right) \)-expansion method and obtained only three solutions (see Appendix A). On the contrary using the modified generalized Riccati equation method with new algebra expansion, in this article, we have obtained many solutions from which some are presented in this article.

If we put \( C_2 = 0, \ C_1 \neq 0 \) in solution \( u_{1,2}(\xi) \) obtained by Wang et al. (2008) and \( p = \lambda = 2qm, \ q = 1, \ r = \mu \) in our solution \( u_{1,11}(\xi) \), then the solution \( u_{1,2}(\xi) \) obtained by Wang et al. (2008) is identical to our solution \( u_{1,11}(\xi) \).

Again, if we set \( C_1 = 0, \ C_2 \neq 0 \) in solution \( u_{1,2}(\xi) \) obtained by Wang et al. (2008) and \( p = \lambda = 2qm, \ q = 1, \ r = \mu \) in our solution \( u_{1,14}(\xi) \), then the solution \( u_{1,2}(\xi) \) obtained by Wang et al. (2008) is identical to our solution \( u_{1,14}(\xi) \).

Similarly, our solutions \( u_{1,1}(\xi) \) and \( u_{1,2}(\xi) \) are identical to the solution \( u_{1,6}(\xi) \) obtained by Wang et al. (2008).

If we put \( q = C_1 \) in our solution \( u_{1,2}(\xi) \), then our solution coincides with Wang et al.'s (2008) solution \( u_{5,6}(\xi) \).

6. Conclusions

In summary, the modified generalized Riccati equation method with new algebra expansion has been proposed and used to find out exact solutions of nonlinear equations with the aid of computer software Maple. This method allows to carry out the solution process of nonlinear wave equations more thoroughly and conveniently by computer algebra systems such as Maple and Mathematica. We have successfully obtained some exact traveling wave solutions of the mKdV equation with parameters. When the parameters are taken as special values, the solitary wave solutions and periodic wave solutions are originated from the exact solutions. We believe that these solutions will be of great importance to investigate the nonlinear phenomena arising in Mathematical Physics and Engineering fields. This work shows that the proposed modified generalized Riccati equation method is sufficient, effective, and suitable for solving other NLEEs, it deserves further applying and studying as well.
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Appendix A

Using the \((G'/G)-expansion method, Wang et al. (2008) obtained the following three types of traveling wave solutions:

When \(\lambda^2 - 4\mu > 0\),

\[
    \begin{align*}
    u(\xi) &= \pm \frac{1}{2} \sqrt{6\delta(\lambda^2 - 4\mu)} \left( C_1 \sinh \left( \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi \right) + C_2 \cosh \left( \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi \right) \right) \\
    &\quad \left( C_1 \cosh \left( \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi \right) + C_2 \sinh \left( \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi \right) \right),
    \end{align*}
\]

(A)

where \(\xi = x - \left( 2\delta - \frac{1}{2} \delta \lambda^2 \right) t\), \(C_1\) and \(C_2\) are arbitrary constants.

When \(\lambda^2 - 4\mu < 0\),

\[
    \begin{align*}
    u, \psi_1(\xi) &= \pm \frac{1}{2} \sqrt{6\delta(\lambda^2 - 4\mu)} \left( -C_1 \sin \left( \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi \right) + C_2 \cos \left( \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi \right) \right) \\
    &\quad \left( C_1 \cos \left( \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi \right) + C_2 \sin \left( \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi \right) \right),
    \end{align*}
\]

(B)

where \(\xi = x - \left( 2\delta - \frac{1}{2} \delta \lambda^2 \right) t\), \(C_1\) and \(C_2\) are arbitrary constants.

When \(\lambda^2 - 4\mu = 0\),

\[
    \begin{align*}
    u, \psi_6(\xi) &= \pm \frac{\sqrt{6\delta}}{C_1 + C_2} \xi,
    \end{align*}
\]

(C)

which is independent of variable \(t\), \(C_1\), and \(C_2\) are arbitrary constants.