Supersymmetric Quantum Spherical Model: A Model for Hodge Theory

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We discuss various symmetry properties of the $\mathcal{N} = 2$ supersymmetric quantum spherical model in one (0 + 1)-dimension of spacetime and provide their relevance in the realm of the mathematics of differential geometry. We show one-to-one mapping between the continuous symmetry transformations (and corresponding generators) and de Rham cohomological operators of differential geometry. One of the novel observations is the existence of discrete symmetry transformations which play a crucial role in providing the physical realization of the Hodge duality ($\star$) operation. Thus, the present model provides a toy model for the Hodge theory.

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I. INTRODUCTION

Over the years, it has always been a good idea in theoretical physics to have a toy model as simple as possible that exhibits a certain subtle phenomenon and it is often the case when we study very complicated physical systems. One of the prime examples is the study of phase transition in ferromagnetic materials through Ising model [1]. Ising model is an integrable model and has led to develop the integrable hierarchy. The Ising model has been generalized to the spherical model where the spin of a set of particles on lattice interacts with the nearest neighbors with a spherical constraint [2]. The classical spherical model is exactly solvable for the nearest neighbor interactions, long range power law interaction and random interactions [2–4]. The beauty of this model is that it can be useful to explain many properties of different kind of physical systems such as anti-ferromagnet with competing interaction and infinite ranged spin glasses [3–7].

The spherical model also plays a pivotal role in studying the finite temperature critical behavior and phase transition [8–9]. Motivated by the simplicity and its myriad applications, the spherical model has been extended to its quantum version by performing the canonical quantization [10]. The quantum spherical model (QSM) has led to investigate the phase transitions at finite as well as zero temperatures in ferromagnetic materials and it is also useful to study the finite temperature critical behavior of quantum spherical spin-glass [11]. It is to be noted that the QSM can also be obtained by using the path integral quantization of its classical version [12].

In addition, the QSM with complex spin variables having continuous local gauge symmetry is shown to be exactly solvable in thermodynamic limit and it exhibits very interesting properties. For instance, the local symmetry is spontaneously broken at finite as well as zero temperatures which implies the existence of classical and quantum phase transitions with a non-trivial critical behavior. Furthermore, in the continuum limit of the partition function of QSM with local symmetry is equivalent to the well-known $CP^{n-1}$ model in the limit $n \to \infty$ [13].

Recently, the supersymmetric version of QSM (within the framework of superspace formalism) has been discussed where two nilpotent symmetry transformations are found to exist [14]. Moreover, by using the path integral formalism, the critical behavior of this model has been discussed at finite and zero temperatures where the supersymmetric QSM model shows critical behavior when the supersymmetry is broken [15–17].

Quantum spherical model (QSM) is one of the simplest models which is physically as well as mathematically very rich. One should note that the QSM has played a key role in explaining the various properties/aspects of the physical systems but this model is mathematically less explored especially from the symmetry point of view. It is to be pointed out that the Ising model of statistical mechanics, by which the quantum spherical model is originated, has an interesting mathematical connection with $\mathcal{N} = 2$ supersymmetry quantum mechanical models and Hodge theory [18]. It is well-known that supersymmetry is a key ingredient of high energy physics and it predicts that there is a fermionic superpartner corresponding to each bosonic partner and vice-versa.

In the recent spate of papers [19–22], in addition to the usual (anti-)BRST transformations, the nilpotent (anti-)co-BRST transformations also exist for the Abelian $p$-form ($p = 1, 2, 3$) gauge theories in $D = 2p$-dimensions of spacetime within the framework of BRST formalism. As far as the models of Hodge theory are concerned, within the framework of BRST formalism, Abelian $p$-form (i.e. $p = 1, 2, 3$) gauge theories have been shown to be the tractable field-theoretic model for the Hodge theory in $D = 2p$-dimensions of spacetime where all the de Rham cohomological operators of differential geometry find their physical realizations in terms of the continuous symmetry transformations (and their corresponding generators) and the discrete symmetry of the theory provides the analogue of Hodge duality operation. In addition, ($0 + 1$)-dimensional toy models (e.g. rigid rotor
and Christ-Lee model) are also shown to be the models for Hodge theory [23, 24]. By exploiting the supervariable technique, the supersymmetric quantum mechanical models with harmonic and general potentials have also been proven to be the model for Hodge theory [23–28]. It is important to mention that the (anti-)BRST and supersymmetric transformations are nilpotent of order two but BRST and anti-BRST transformations anticommute whereas the anticommutator of two supersymmetric transformations is non-zero and equivalent to the time-translation.

Our present investigation is motivated by the following factors. First, we explore the various continuous symmetries posses by our present system. In fact, we show that in addition to the supersymmetric transformations \( (s_1, s_2) \), there also exists one more continuous transformation \( s_w \) under which Lagrangian for QSM remains invariant. The latter symmetry transformation \( s_w \) is bosonic in nature. Second, we prove that nilpotent supersymmetric transformations \( (s_1, s_2) \) and bosonic transformation \( s_w \) follow a similar algebra as satisfied by the de Rham cohomological operators of differential geometry. Third, we also show the existence of novel discrete symmetry transformations under which the Lagrangian for the supersymmetric QSM remains invariant. These bosonic and discrete transformations were not reported in [13]. It is important to mention here that these discrete symmetry transformations play very important role because these transformations provides physical realization of the Hodge duality (*). Finally, we show that the \( N = 2 \) supersymmetric QSM provides a simple toy model for the Hodge theory in its own right.

The outline of our present investigation is as follows. In section II, we discuss very brief about the QSM. Section III deals with the \( N = 2 \) supersymmetrization of the QSM. The various continuous symmetries (and corresponding charges) and discrete symmetries are discussed in section IV and section V, respectively. We capture the physical realization of the abstract de Rham cohomological operators of differential geometry in terms of the symmetry properties of the present model in section VI. The cohomological aspects are discussed in this section, too. Finally, in section VII, we provide the concluding remarks.

In Appendix A, we discuss about the on-shell nilpotent symmetries for the supersymmetric QSM model.

II. QUANTUM SPHERICAL MODEL: A BRIEF INTRODUCTION

The quantum version of the classical spherical model is described by the following Hamiltonian:

\[
H_0 = \frac{g}{2} \sum_r P_r^2 + \frac{1}{2} \sum_{r,r'} J_{r,r'} S_r S_{r'} + h \sum_r S_r,
\]

where \( \{ S_r \} \) is a set of continuous spin variables associated to each lattice site \( r \) on a \( d \)-dimensional hypercubic lattice and ranging from \(-\infty\) to \(+\infty\), the interaction energy \( J_{r,r'} = J(|r - r'|) \) depends upon the distance between the lattice sites \( r \) and \( r' \) where the interaction energy is assumed to be translationally invariant. Here \( P_r \) is the canonical conjugate momentum corresponding to the dynamical spin variable \( S_r \), \( h \) is an external field and \( g \) defines the quantum coupling constant. The spin variables \( S_r \) are subject to a spherical constraint

\[
\sum_r S_r^2 - N = 0,
\]

where \( N \) is the total number of lattice sites. In the limit \( g \to 0 \), the QSM reduces to the classical model.

Using the Legendre transformations and switching-off external field (i.e. \( h = 0 \)), we obtain the following Lagrangian

\[
L_0 = \frac{1}{2g} \sum_r \dot{S}_r^2 - \frac{1}{2} \sum_{r,r'} J_{r,r'} S_r S_{r'},
\]

where \( \dot{S}_r = \frac{d}{dt} S_r \) is the generalized velocity in the configuration space. The canonical conjugate variables \( S_r \) and \( P_r \) obey the following commutations relations (with \( h = 1 \)):

\[
[S_r, P_{r'}] = i \delta_{r,r'}, \quad [S_r, S_{r'}] = 0, \quad [P_r, P_{r'}] = 0,
\]

where \( P_r = \frac{1}{g} \dot{S}_r \) is the canonical conjugate momentum corresponding to the dynamical variable \( S_r \).

In our next sections, we supersymmetrize the above Lagrangian by means of \( N = 2 \) superspace formalism and discuss the various continuous and discrete symmetries.

III. \( N = 2 \) SUPERSYMMETRIZATION OF QUANTUM SPHERICAL MODEL

We consider the \( N = 2 \) real supervariable \( \Phi_r(t, \theta, \bar{\theta}) \) associated to each lattice site \( r \) defined on \( (1, 2) \)-dimensional supermanifold. The \( (1, 2) \)-dimensional superspace is parameterized by the superspace coordinates \( (t, \theta, \bar{\theta}) \). Here \( t \) is the bosonic time-evolution parameter and \( \theta, \bar{\theta} \) are the Grassmannian variables (with \( \theta^2 = \bar{\theta}^2 = 0, \quad \theta \bar{\theta} + \bar{\theta} \theta = 0 \)). The supervariable \( \Phi_r(t, \theta, \bar{\theta}) \) can be expanded along the Grassmanniann directions as:

\[
\Phi_r(t, \theta, \bar{\theta}) = S_r(t) + i \theta \bar{\psi}_r(t) + i \bar{\theta} \psi_r(t) + \theta \bar{\theta} F_r(t),
\]

where \( S_r, \psi_r, \bar{\psi}_r \) are the basic dynamical variables and \( F_r \) is an auxiliary variable for our \( N = 2 \) SUSY quantum spherical system. The fermionic variables \( \psi_r, \bar{\psi}_r \) (with \( \psi_r^2 = \bar{\psi}_r^2 = 0, \quad \psi_r \bar{\psi}_r + \bar{\psi}_r \psi_r = 0 \)) are the \( N = 2 \) supersymmetric counterparts of the bosonic variable \( S_r \). All the basic and auxiliary variables are the function of time-evolution parameter \( t \) only.

The spherical constraint (2) can be imposed on the supervariable (5) as supersymmetric spherical constraint on the system in the following fashion:

\[
\sum_r \Phi_r(t, \theta, \bar{\theta}) \Phi_r(t, \theta, \bar{\theta}) - N = 0.
\]
The above supersymmetric spherical constraint leads to the following constraints, namely:

\[ \sum_{r} S_{r}^2 - N = 0, \quad \sum_{r} \left( \bar{\psi}_{r} \psi_{r} + S_{r} F_{r} \right) = 0, \]
\[ \sum_{r} \bar{\psi}_{r} S_{r} = 0, \quad \sum_{r} \bar{\psi}_{r} S_{r} = 0. \]  

(7)

It is evident that the first two constraints are bosonic whereas last two are fermionic in nature. These (bosonic)fermionic constraints can be implemented in the supersymmetric theory with the help of (bosonic)fermionic Lagrange multipliers.

Now we introduce two supercharges \( Q \) and \( \bar{Q} \) (with \( Q^2 = 0, \bar{Q}^2 = 0 \)) for the above \( N = 2 \) supersymmetric quantum spherical model as

\[ Q = \frac{\partial}{\partial \bar{\theta}} + i \theta \frac{\partial}{\partial t}, \quad \bar{Q} = \frac{\partial}{\partial \theta} + i \bar{\theta} \frac{\partial}{\partial t}, \]  

(8)

where the partial derivatives \( \partial/\partial t, \partial/\partial \theta, \partial/\partial \bar{\theta} \) are defined on the (1, 2)-dimensional supermanifold. The above fermionic charges turn out to be the generators of the following translations in the superspace:

\[ t \rightarrow t' = t + i (\epsilon \bar{\theta} + \bar{\epsilon} \theta), \quad \theta \rightarrow \theta' = \theta + \epsilon, \quad \bar{\theta} \rightarrow \bar{\theta}' = \bar{\theta} + \bar{\epsilon}, \]  

(9)

where \( \epsilon \) and \( \bar{\epsilon} \) (with \( \epsilon^2 = 0, \bar{\epsilon}^2 = 0, \epsilon \bar{\epsilon} + \bar{\epsilon} \epsilon = 0 \)) are the global and infinitesimal shift transformation parameters along the Grassmannian directions on the (1, 2)-dimensional supermanifold.

The supersymmetric transformation (\( \delta \)) on the supervariable can be expressed in terms of the supercharges \( Q \) and \( \bar{Q} \) as illustrated below

\[ \delta \Phi_{r}(t, \theta, \bar{\theta}) = \delta S_{r}(t) + i \theta \delta \bar{\psi}_{r}(t) + i \bar{\theta} \delta \psi_{r}(t) + \theta \bar{\theta} \delta F_{r}(t) \]
\[ = (\epsilon Q + \bar{\epsilon} \bar{Q}) \Phi_{r}(t, \theta, \bar{\theta}) \equiv \Phi_{r}(t', \theta', \bar{\theta}') - \Phi_{r}(t, \theta, \bar{\theta}) \]  

(10)

The transformation (\( \delta \)) can be divided into two infinitesimal transformations \( \delta_1 \) and \( \delta_2 \) because of the presence of \( N = 2 \) supersymmetry. These are listed as follows:

\[ \delta_1 S_{r} = i \bar{\epsilon} \psi_{r}, \quad \delta_1 \bar{\psi}_{r} = -\bar{\epsilon} (\dot{S}_{r} + i F_{r}), \]
\[ \delta_1 F_{r} = -\bar{\epsilon} \bar{\psi}_{r}, \quad \delta_1 \psi_{r} = 0, \]
\[ \delta_2 S_{r} = i \epsilon \bar{\psi}_{r}, \quad \delta_2 \bar{\psi}_{r} = -\epsilon (\dot{S}_{r} - i F_{r}), \]
\[ \delta_2 F_{r} = \epsilon \dot{\psi}_{r}, \quad \delta_2 \psi_{r} = 0. \]  

(11)

where we have defined \( \dot{S}_{r} = dS_{r}/dt, \dot{\psi}_{r} = d\psi_{r}/dt, \dot{\bar{\psi}}_{r} = d\bar{\psi}_{r}/dt \). It can be readily checked that \( \delta_1 \) and \( \delta_2 \) are off-shell nilpotent of order two (i.e. \( \delta_1^2 = 0, \delta_2^2 = 0 \)).

With the help of super-covariant derivatives \( \mathcal{D} \) and \( \bar{\mathcal{D}} \) defined as

\[ \mathcal{D} = \frac{\partial}{\partial \theta} - i \theta \frac{\partial}{\partial t}, \quad \bar{\mathcal{D}} = \frac{\partial}{\partial \bar{\theta}} - i \bar{\theta} \frac{\partial}{\partial t}, \]  

(12)

we write the general Lagrangian for the \( N = 2 \) supersymmetric QSM as

\[ L = \int d\theta d\bar{\theta} \left[ \frac{1}{2g} \sum_{r} \mathcal{D} \Phi_{r} (t, \theta, \bar{\theta}) \bar{\mathcal{D}} \Phi_{r} (t, \theta, \bar{\theta}) \right. \]
\[ + \frac{1}{2} \sqrt{g} \sum_{r, r'} U_{r, r'} \Phi_{r} (t, \theta, \bar{\theta}) \Phi_{r'} (t, \theta, \bar{\theta}) \right]. \]  

(13)

After performing the Grassmanian integrations, we obtain the desired Lagrangian for the \( N = 2 \) supersymmetric QSM:

\[ L = \frac{1}{2g} \sum_{r} \dot{S}_{r}^2 + \frac{i}{g} \sum_{r} \bar{\psi}_{r} \dot{\psi}_{r} + \frac{1}{2g} \sum_{r} F_{r}^2 \]
\[ - \frac{1}{\sqrt{g}} \sum_{r, r'} U_{r, r'} S_{r} F_{r'} - \frac{i}{\sqrt{g}} \sum_{r, r'} U_{r, r'} \bar{\psi}_{r} \psi_{r'}. \]  

(14)

In the above, we can re-scale the dynamic fermionic variables \( \psi_{r} \rightarrow g^{1/4} \Psi_{r}, \bar{\psi}_{r} \rightarrow g^{1/4} \bar{\Psi}_{r}, \epsilon \rightarrow g^{1/4} \bar{\epsilon} \) as well as global Grassmannian parameters \( \bar{\epsilon} \rightarrow g^{1/4} \bar{\epsilon} \) without any loss of generality. As a consequence, the Lagrangian (14) for the supersymmetric QSM takes the following form:

\[ L = \frac{1}{2g} \sum_{r} \dot{S}_{r}^2 + \frac{i}{g} \sum_{r} \bar{\Psi}_{r} \dot{\Psi}_{r} + \frac{1}{2g} \sum_{r} F_{r}^2 \]
\[ - \frac{1}{g} \sum_{r, r'} U_{r, r'} S_{r} F_{r'} - \frac{i}{g} \sum_{r, r'} U_{r, r'} \bar{\Psi}_{r} \Psi_{r'}. \]  

(15)

and the supersymmetric transformations (11) read:

\[ \delta_1 S_{r} = i \bar{\epsilon} \bar{\Psi}_{r}, \quad \delta_1 \bar{\Psi}_{r} = -\bar{\epsilon} (\dot{S}_{r} + i F_{r}), \]
\[ \delta_1 F_{r} = -\bar{\epsilon} \bar{\Psi}_{r}, \quad \delta_1 \Psi_{r} = 0, \]
\[ \delta_2 S_{r} = i \epsilon \dot{\Psi}_{r}, \quad \delta_2 \Psi_{r} = -\epsilon (\dot{S}_{r} - i F_{r}), \]
\[ \delta_2 F_{r} = \epsilon \dot{\Psi}_{r}, \quad \delta_2 \bar{\Psi}_{r} = 0. \]  

(16)

For our further discussions, we shall work with the above Lagrangian (15). Moreover, the supersymmetric spherical constraints in (7) can be written in terms of new re-scaled variables as

\[ \sum_{r} S_{r}^2 - N = 0, \quad \sum_{r} (\sqrt{g} \bar{\Psi}_{r} \Psi_{r} + S_{r} F_{r}) = 0, \]
\[ \sum_{r} \Psi_{r} S_{r} = 0, \quad \sum_{r} \bar{\Psi}_{r} S_{r} = 0. \]  

(17)

By exploiting the Legendre transformations, the Hamiltonian for the above system is given by

\[ H = \frac{g}{2} \sum_{r} P_{r}^2 + \frac{1}{2g} \sum_{r, r'} U_{r, r'} S_{r} F_{r'} - \frac{1}{2} \sum_{r} F_{r}^2 \]
\[ + \sum_{r, r'} U_{r, r'} \bar{\Psi}_{r} \Psi_{r'}. \]  

(18)

The canonical conjugate momenta \( P_{r} = \frac{1}{g} \dot{S}_{r} \) and \( \Pi_{r} = -\frac{i}{\sqrt{g}} \bar{\Psi}_{r} \) (derived from Lagrangian (15)) corresponding
to the dynamical variables $S_r$ and $Ψ_r$, respectively, obey the following non-vanishing canonical (anti)commutation relations:

$$[S_r, P_{r'}] = i δ_{r,r'},$$

$$\{Ψ_r, Π_{r'}\} = -\frac{i}{\sqrt{g}} δ_{r,r'} \Rightarrow \{Ψ_r, \bar{Ψ}_{r'}\} = δ_{r,r'},$$

where rest of the (anti)commutators are turn out to be zero.

### IV. CONTINUOUS SYMMETRIES AND CONSERVED CHARGES

In this section, we discuss various continuous symmetries of the present model. The Lagrangian respects the following off-shell nilpotent (i.e. $s_1^2 = 0$ and $s_2^2 = 0$) $N = 2$ supersymmetric transformations:

$$s_1 S_r = i \sqrt{g} Ψ_r, \quad s_1 \bar{Ψ}_r = -(\bar{S}_r + i F_r),$$

$$s_2 F_r = -\sqrt{g} Ψ_r, \quad s_1 Ψ_r = 0,$$

$$s_2 S_r = i \sqrt{g} \bar{Ψ}_r, \quad s_2 \bar{Ψ}_r = -(\bar{S}_r - i F_r),$$

$$s_2 F_r = \sqrt{g} \bar{Ψ}_r, \quad s_2 Ψ_r = 0,$$

where we have captured the global Grassmannian parameters $ε$ and $\bar{ε}$ in the supersymmetric transformations $s_1$ and $s_2$. For the sake brevity, we have chosen: $δ_1 = \bar{ε} s_1$ and $δ_2 = ε s_2$ (cf. [16]). Under these off-shell nilpotent and continuous supersymmetric transformations, the Lagrangian remains quasi-invariant (i.e. transforms to a total time derivative) as:

$$s_1 L = \frac{d}{dt} \left[ \sum_{r,r'} U_{r,r'} S_r Ψ_{r'} \right],$$

$$s_2 L = \frac{d}{dt} \left[ \frac{1}{\sqrt{g}} \sum_r Ψ_r \left( i \dot{S}_r + F_r \right) - \sum_{r,r'} U_{r,r'} \bar{Ψ}_{r'} S_r \right].$$

As a consequence, the action integral ($\int dt L$) remains invariant (i.e. $s_1 \int dt L = 0$ and $s_2 \int dt L = 0$).

It is clear that even though continuous transformations $s_1$ and $s_2$ are off-shell nilpotent of order two (i.e. $s_1^2 = 0$ and $s_2^2 = 0$) but they do not anticommute i.e. $\{s_1, s_2\} \neq 0$ for any generic variable. In fact, the anti-commutator leads to the time translation (modulo a constant factor). This is one of the characteristic features of the $N = 2$ supersymmetric models. As a consequence, we define a continuous bosonic symmetry $s_w = \{s_1, s_2\}$. For any generic variable $χ$, we obtain the bosonic symmetry transformation:

$$s_w χ = -2i \sqrt{g} χ, \quad χ(t) = S_r, \quad Ψ_r, \quad F_r,$$

under which the Lagrangian transforms to a total time derivative as

$$s_w L = \frac{d}{dt} (-2i \sqrt{g} L).$$

Thus, the action integral remains invariant under bosonic symmetry transformation ($s_w$).

According to Noether’s theorem, the invariance of the action under the continuous symmetry transformations $s_1$, $s_2$ and $s_w$ lead to the conserved charges. These are listed as follows:

$$Q = \frac{1}{\sqrt{g}} \sum_r (i \dot{S}_r - F_r) Ψ_r,$$

$$\bar{Q} = \frac{1}{\sqrt{g}} \sum_r Ψ_r (i \dot{S}_r + F_r),$$

$$Q_w = -2i \sqrt{g} \left( \frac{1}{2} \sum_r \dot{S}_r^2 + \frac{1}{\sqrt{g}} \sum_{r,r'} U_{r,r'} S_r F_{r'} \right.$$

$$- \frac{1}{2g} \sum_r F_r^2 + \sum_{r,r'} U_{r,r'} \bar{Ψ}_r Ψ_{r'} \right) = -2i \sqrt{g} \bar{H}. \quad (24)$$

It is to be noted that the bosonic charge $Q_w$ is nothing but the Hamiltonian of system (multiplied by a constant factor $-2i \sqrt{g}$).

The conservation of the above charges can be proven by exploiting the following Euler-Lagrange equations of motion:

$$\dot{S}_r + \sqrt{g} \sum_{r'} U_{r,r'} F_{r'} = 0,$$

$$\dot{Ψ}_r + i \sqrt{g} \sum_{r'} U_{r,r'} Ψ_{r'} = 0,$$

$$\dot{Ψ}_r - i \sqrt{g} \sum_{r'} U_{r,r'} \bar{Ψ}_{r'} = 0,$$

$$F_r = \sqrt{g} \sum_{r'} U_{r,r'} S_{r'},$$

which have been derived from the Lagrangian [15].

### V. DISCRETE SYMMETRIES

In addition to the above continuous symmetries, the Lagrangian [15] also respects the following discrete symmetry transformations

$$t \rightarrow -t, \quad S_r \rightarrow \mp S_r, \quad Ψ_r \rightarrow \pm Ψ_r,$$

$$\bar{Ψ}_r \rightarrow \mp \bar{Ψ}_r, \quad F_r \rightarrow \mp F_r.$$

It is to be noted that we have time-reversal symmetry (i.e. $t \rightarrow -t$) in the theory. In addition to the above discrete symmetry, we also have another set of discrete symmetry transformations

$$t \rightarrow -t, \quad S_r \rightarrow \pm S_r, \quad Ψ_r \rightarrow \pm i Ψ_r,$$

$$\bar{Ψ}_r \rightarrow \pm i \bar{Ψ}_r, \quad F_r \rightarrow \pm F_r,$$

(27)
which leave the Lagrangian invariant. Under both set of
discrete symmetry transformations the conserved charges
transform in the following fashion:
\*Q = +Q, \*Q = −Q,
\*Q_w = +Q_w, \*H = +H,   \tag{28}

where * corresponds to the discrete transformations \[26\]
and \[27\]. It is clear from the above that under the dis-
screte symmetry transformations \(Q \rightarrow +Q\) and \(Q \rightarrow −Q\).
This is like the case of electromagnetic duality for the
source free Maxwell’s theory where \(E \rightarrow +B\) and \(B \rightarrow −E\). We note that the two successive operations of dis-
crete symmetry transformations on any generic variable yields
\(\*(\*S_r) = +S_r, \*(\*F_r) = +F_r, \tag{29}\)
\(\*(\*Ψ_r) = −Ψ_r, \*(\*Ψ_r) = −Ψ_r.\)

Similarly, under the two successive transformations \[26\]
and \[27\], the conserved charges transform as: \(\*(\*Q) = −Q, \*(\*Q) = −Q, \*(\*Q_w) = +Q_w\) and \(*H = +H\).
Furthermore, the interplay of fermionic and discrete sym-
metry transformations provides an interesting relation-
ship:
\[s_2χ = ± * s_1 * χ, \tag{30}\]
where \((±)\) signs in the r.h.s. of the above expres-
sion correspond to the generic variable being (bosonic)
fermionic in nature. In fact, the \((±)\) signs are dictated by
two successive operations of discrete transformations
on the generic variable (cf. \[29\]). Furthermore, for any
generic variable there also exists a reverse relationship
\((i.e. \; s_1χ = ± * s_2 * χ)\) corresponding to the above rela-
tionship \[30\].

VI. SYMMETRY ALGEBRA AND
COHOMOLOGICAL ASPECTS

In our previous sections, we have discussed two
fermionic symmetries and a bosonic symmetry. The oper-
ator form of the continuous and discrete symmetry trans-
formations obey the following interesting algebra \[29\] [31]:
\[s_1^2 = 0, \quad s_2^2 = 0, \quad \{s_1, s_2\} = s_w,\]
\[\{s_w, s_1\} = 0, \quad \{s_w, s_2\} = 0, \quad s_2 = ± * s_1 * . \tag{31}\]
The above algebra among the continuous symmetry trans-
formations is reminiscent of the algebra obeyed by
the de Rham cohomological operators of differential
geometry. The latter algebra is given as follows \[29\] [32]:
\[d^2 = 0, \quad δ^2 = 0, \quad \{d, \; δ\} = Δ,\]
\[\{δ, \; Δ\} = 0, \quad \{Δ, \; δ\} = 0, \quad δ = ± * d* . \tag{32}\]
where \(d, \; δ, \; Δ\) and \(*)\) are the exterior derivative, co-
exterior derivative, Laplacian operator and Hodge duality
operation defined on a D-dimensional compact man-
ifold without boundary. The \((±)\) signs in the last rela-
tion depend on the dimensionality of the space as well
as degree of a given form. It is clear that there is one-to-one mapping between the continuous trans-
formations \((s_1, s_2, s_w)\) and de Rham cohomological operators
\((d, δ, Δ)\). In fact, one can identify exterior derivative \(d\) with
fermionic transformation \(s_1\), co-exterior derivative
\(δ\) with \(s_2\) and Laplacian operator \(Δ\) with bosonic
transformation \(s_w\). The discrete symmetry transformations \(*\)
in \[26\] and \[27\] provide the analogue of Hodge duality \(*\)
operation.

As far as the properties of the differential operators are
concerned, we note that the exterior derivative, when acts
on a given form \(f_n\), raises the degree of the form by one
\((i.e. \; df_n \sim f_{n+1})\) where \(n\) is the degree of form whereas
co-exterior derivative decreases the degree by one \((i.e. \; δf_n \sim f_{n-1})\). The Laplacian operator does not effect
the degrees of the form \((i.e. \; Δf_n \sim f_n)\). We shall captures
these properties in terms of the conserved charges.

Before going into details, we first point out that the
conserved charges corresponding to the continuous sym-
metry transformations obey the standard \(N = 2\) super-
symmetric algebra:
\[Q^2 = 0, \quad Q^2 = 0, \quad \{Q, \; Q\} = Q_w = −2i\sqrt{g}H, \tag{33}\]
\[[H, \; Q] = 0, \quad [H, \; Q] = 0,\]
The above algebra is exactly similar to the algebra fol-
lowed by the de Rham cohomological operators (cf. \[32\]).
It is clear that \(Q_w\) is the Casimir operator of the above
algebra. In other words, one can also say that the Hamil-
tonian of the theory is the Casimir operator. Due to the
validity of last two relationships in \[33\] and for the non-
singular Hamiltonian, we equivalently have \([H^{-1}, \; Q] = 0\)
and \([H^{-1}, \; Q] = 0\). As a result, \(H^{-1}\) would also be the
Casimir operator for the above algebra.

Form the above arguments, the following algebraic rela-
tions among the conserved charges are true:
\[\left[\frac{QQ}{H}, \; Q\right] = +Q, \quad \left[\frac{Q}{H}, \; Q\right] = +Q, \tag{34}\]
\[\left[\frac{QQ}{H}, \; Q\right] = −Q, \quad \left[\frac{Q}{H}, \; Q\right] = −Q,\]
\[\left[\frac{QQ}{H}, \; H\right] = 0, \quad \left[\frac{Q}{H}, \; H\right] = 0.\]

In view of the above, let us define an eigenstate \(|α\rangle_p\) with
respect to an operator \(\frac{QQ}{H}\) having eigenvalue \(p\) such that
\(\frac{QQ}{H}|α\rangle_p = p|α\rangle_p\). Using the above algebraic relationship,
we obtain
\[\left(\frac{QQ}{H}\right) Q|α\rangle_p = (p + 1)|α\rangle_p, \tag{35}\]
\[\left(\frac{QQ}{H}\right) \tilde{Q}|α\rangle_p = (p - 1)|α\rangle_p,\]
\[\left(\frac{QQ}{H}\right) H|α\rangle_p = p|α\rangle_p.\]

These equations reflect the fact that the eigenstates
\(Q|α\rangle_p, \tilde{Q}|α\rangle_p\) and \(H|α\rangle_p\) have the eigenvalues \((p + 1),\)
given form by the eigenvalue \( p \), and correspondingly with respect to an operator \( (d, \delta, \Delta) \) find their physical realization in terms of the discrete symmetries. As a consequence, the two supersymmetric QSM turns out to be a model for Hodge theory.

In the known literature, within the framework of BRST formalism, we have field-theoretic models which happen to be the models for Hodge theory in specific \( D = 2p \)-dimensions of spacetime where \( p \) is the degree of a given differential form \( 19,22 \). In these models, we have two-to-one mapping between the continuous symmetry transformations and de Rham cohomological operators where the continuous and nilpotent (anti-)BRST \((s^{(a)}b)\), (anti-)co-BRST \((s^{(a)}b)\) transformations and a unique bosonic transformation (and their corresponding generators) provide the physical realizations of the de Rham cohomological operators.

We note that, even though, the (anti-)BRST, (anti)-co-BRST and \( \mathcal{N} = 2 \) supersymmetric transformations are global and nilpotent of order two (i.e. \( s^{2}_{a} = s^{2}_{a} = 0 \) and \( s^{1}_{1} = s^{2}_{2} = 0 \)), there are some glaring differences between them. The (anti-)BRST and (anti)-co-BRST are anticommuting (i.e. \( \{s_{b}, s_{ab}\} = \{s_{d}, s_{ad}\} = 0 \) whereas \( \mathcal{N} = 2 \) supersymmetric transformations do not anticommute (i.e. \( \{s_{1}, s_{2}\} \neq 0 \)). On the other hand, the anticommutators \( \{s_{b}, s_{d}\} = \{s_{ab}, s_{ad}\} = s_{w} \) lead to a unique bosonic symmetry \( s_{w} \). Similarly, the anticommutator \( \{s_{1}, s_{2}\} = s_{w} \) of supersymmetric transformations \( s_{1} \) and \( s_{2} \) also defines a bosonic symmetry transformation. But, \( s_{w} \) is different from \( s_{w} \) in the sense that the former transformation produces the time translation (i.e. \( s_{w} \chi \approx \dot{\chi} \)) and, thus, Hamiltonian is the generator of \( s_{w} \) whereas the latter symmetry \( s_{w} \) does not emerge form Hamiltonian.

In view of our present work, it is an interesting piece of work to supersymmetrize the quantum spherical spin model with local symmetry \( \mathbb{R}^{2} \) and apply our present idea. Further, our idea can also be applied to the \( \mathcal{N} = 2,4 \) supersymmetric field theories having local gauge symmetries. These works are under investigation and we shall report elsewhere.

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Appendix A: On-shell supersymmetric transformations

The Lagrangian (15) can be simplified by eliminating an auxiliary variable $F_r$. Using the equation of motion for $F_r$ (cf. (25)) in (15), we obtain

$$L_0 = \frac{1}{2g} \sum_r S_r^2 + \frac{i}{\sqrt{g}} \sum_r \bar{\Psi}_r \psi_r$$

$$- \sum_{r,r'} U_{r,r'} \bar{\Psi}_r \Psi_{r'} - \frac{1}{2} \sum_{r,r'} J_{r,r'} S_r S_{r'}, \quad (A1)$$

where we have defined

$$\sum_s U_{r,s} U_{s,r'} \equiv J_{r,r'}. \quad (A2)$$

It is to be noted that the above Lagrangian is coincide with the Lagrangian given in [14]. Similarly, one can eliminate $F_r$ from the supersymmetric transformations [20] and obtain the following on-shell nilpotent transformations, namely;

$$s_1 S_r = i \sqrt{g} \Psi_r, \quad s_1 \Psi_r = 0,$$

$$s_2 S_r = i \sqrt{g} \bar{\Psi}_r, \quad s_2 \bar{\Psi}_r = 0,$$

$$s_2 S_r = - (\dot{S}_r + i \sqrt{g} \sum_{r'} U_{r,r'} S_{r'}), \quad (A3)$$

where the nilpotency of $s_1$ and $s_2$ can be shown in a straightforward by using the Euler-Lagrange equations of motion derived from (A1).

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