Transition to Instability in a Kicked Bose-Einstein Condensate

Chuanwei Zhang\textsuperscript{1,2}, Jie Liu\textsuperscript{1,3}, Mark G. Raizen\textsuperscript{1,2}, and Qian Niu\textsuperscript{1}

\textsuperscript{1}Department of Physics, The University of Texas, Austin, Texas 78712-1081
\textsuperscript{2}Center for Nonlinear Dynamics, The University of Texas, Austin, Texas 78712-1081
\textsuperscript{3}Institute of Applied Physics and Computational Mathematics, P.O.Box 100088, Beijing, P. R. China

A periodically kicked ring of a Bose-Einstein condensate is considered as a nonlinear generalization of the quantum kicked rotor. For weak interactions between atoms, periodic motion (anti-resonance) becomes quasiperiodic (quantum beating) but remains stable. There exists a critical strength of interactions beyond which quasiperiodic motion becomes chaotic, resulting in an instability of the condensate manifested by exponential growth in the number of noncondensed atoms. Similar behavior is observed for dynamically localized states (essentially quasiperiodic motions), where stability remains for weak interactions but is destroyed by strong interactions.

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The classical kicked rotor is a textbook paradigm for dynamical chaos\textsuperscript{[1]}. The quantum kicked rotor has play an equally important role for the study of quantum chaos, for which a wide range of effects have been predicted\textsuperscript{[2]} and observed in experiments\textsuperscript{[3]}. In recent years, the realization of Bose-Einstein condensation (BEC) of dilute gases\textsuperscript{[4]} has opened new opportunities for studying dynamical systems in the presence of many-body interactions. A natural question to ask is how the physics of the quantum kicked rotor is modified by interactions\textsuperscript{[5]}.\textsuperscript{[6]}

In the classical regime, chaotic motion leads to diffusive growth in the kinetic energy. In quantum dynamics, where chaos is no longer possible because of the linearity of the Schrödinger equation, the motion becomes periodic (anti-resonance), quasiperiodic (dynamical localization), or resonant (quantum resonance)\textsuperscript{[7]}\textsuperscript{[8]}. In the mean field approximation, many-body interactions in BEC are represented by adding a nonlinear term in the Schrödinger equation\textsuperscript{[9]}. This nonlinearity makes it possible to bring chaos back into the system, leading to instability (in the sense of exponential sensitivity to initial conditions) of the condensate wave function\textsuperscript{[10]}. The onset of instability of the condensate can cause rapid proliferation of thermal particles\textsuperscript{[11]} that can be observed in experiments\textsuperscript{[12]}. It is therefore important to understand the route to chaos with increasing interactions. This problem has recently been studied for the kicked BEC in a harmonic oscillator\textsuperscript{[13]}.

In this Letter, we investigate the quantum dynamics of a BEC with repulsive interaction that is confined on a ring and kicked periodically. This system is a nonlinear generalization of the quantum kicked rotor. From the point of view of dynamical theory, the kicked rotor is more generic than the kicked harmonic oscillator, because it is a typical low dimensional system that obeys the KAM theorem\textsuperscript{[14]}. It is very interesting to understand how both quantum mechanics and mean field interaction affect the dynamics of such a generic system. We will focus most of our attention on the case of anti-resonance because it is the simplest periodic motion. Here we find, with both analytic and numerical calculations, that weak interactions will make the motion quasiperiodic in the form of quantum beating. For strong interactions, quasiperiodic motions are destroyed and we observe a transition to instability of the BEC that is also characterized by an exponential growth in the number of noncondensed atoms. Universal critical behavior for the transition is found. We have also studied nonlinear effect on dynamically localized states that may be regarded as quasiperiodic. Similar results are obtained in that localization remains for sufficiently weak interactions but become unstable beyond a critical strength of interactions.

Consider condensed atoms confined in a toroidal trap of radius $R$ and thickness $r$, where $r \ll R$ so that lateral motion is negligible and the system is essentially one-dimensional\textsuperscript{[15]}. The dynamics of the BEC is described by the dimensionless nonlinear Gross-Pitaveskii (GP) equation,

\begin{equation}
\frac{\partial}{\partial t} \psi = \left( -\frac{1}{2} \frac{\partial^2}{\partial \theta^2} + g |\psi|^2 + K \cos \theta \delta_{\epsilon}(T) \right) \psi, \quad (1)
\end{equation}

where $g = 8 \pi a R / r^2$ is the scaled strength of nonlinear interaction, $N$ is the number of atoms, $a$ is the s-wave scattering length, $K$ is the kick strength, $\delta_{\epsilon}(T)$ represents $\sum \delta(t - nT)$, $T$ is the kick period, and $\theta$ denotes the azimuthal angle. The length and the energy are measured in units $R$ and $\frac{\hbar^2}{2m}$, respectively. The wavefunction $\psi(\theta, t)$ has the normalization $\int_0^{2\pi} |\psi|^2 d\theta = 1$ and satisfies periodic boundary condition $\psi(\theta, t) = \psi(\theta + 2\pi, t)$. Experimentally, the ring-shape potential may be realized using two 2D circular “optical billiards” with the lateral dimension being confined by two plane optical billiards\textsuperscript{[16]}.

The $\delta$-kick may be realized by adding potential points along the ring with an off-resonant laser\textsuperscript{[17]}. The interaction strength $g$ may be adjusted using a magnetic field-dependent Feshbach resonance\textsuperscript{[18]}.

The mean energy of each particle is $E(t) = \int_0^{2\pi} d\theta \psi^* \left( -\frac{1}{2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{2} g |\psi|^2 \right) \psi$. To determine the evolu-
Figure 1: Plots of average energy $E(t)$ versus the number of kicks $t$ for three values of $g$. The kick strength $K = 0.8$.

domination of the energy, we numerically integrate Eq.(1) over a time span of 100 kicks, using a split-operator method [17], with the initial wavefunction $\psi$ being the ground state $\psi(\theta,0) = 1/\sqrt{2\pi}$. The kick period is chosen as $T = 2\pi$ to match the condition for anti-resonance. After each kick, the energy $E(t)$ is calculated and plotted as shown in Fig.1. We see a remarkable difference among noninteraction (Fig.1(a)), weak interaction (Fig.1(b)), and strong interaction (Fig.1(c)) cases. For noninteraction, the energy oscillates between two values (anti-resonance) and the period is $2T$; While in the weak interaction case, the amplitude of the oscillation decreases gradually to zero and then revives, similar to the phenomena of beating in classical waves. The values of the oscillation and beat frequencies are obtained by Fourier Transform and the results are presented in Fig.2. It is shown that, besides the appearance of the beat frequency, the interaction also modifies the oscillation frequency; For stronger interaction (Fig.1(c)), i.e. $g \geq 1.96$, we find that the energy’s evolution demonstrates an irregular pattern, clearly indicating the collapse of the quasiperiodic motion and the occurrence of instability.

For the case of weak interaction, the system is pre-
This expression as well as the relation between the oscillation frequency and the beat frequency, \( f_{\text{osc}} = \frac{1}{2} f_{\text{beat}}, \)

is in very good agreement with the numerical results as shown in Fig.2. Therefore the beating provides a method to measure interaction strength in an experiment.

Tuning the interaction strength still largely enhances further the nonlinearity of the system. From our general understanding of nonlinear systems, we expect that the solution will be driven towards chaos, in the sense of exponential sensitivity to initial condition and random evolution in the temporal domain. The latter character has been clearly displayed by the irregular pattern of the energy evolution in Fig.1(c). On the other hand, the onset of instability (or chaotic motion) of the condensate is accompanied with the rapid proliferation of thermal particles. Within the formalism of Castin and Dum [10], the growth of the number of the noncondensed atom will be exponential, similar to the exponential divergence of nearby trajectories in phase space of classical system. The growth rate of the noncondensed atoms is similar to the Lyapounov exponent, turning from zero to nonzero as instability occurs.

In Castin and Dum’s formalism, the mean number of noncondensed atoms at zero temperature is described by

\[
\langle \delta N(t) \rangle = \sum_{k=1}^{\infty} \langle v_k(t)|v_k(t) \rangle,
\]

where \( |v_k(t) \rangle \) are governed by

\[
\frac{d}{dt} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} H + gQ|\psi|^2Q & gQ\psi^2Q^* \\ -gQ^*\psi^2Q^* & -H - gQ^*|\psi|^2Q^* \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix},
\]

where \( H = \frac{\hat{\mathcal{E}}}{\hbar^2} + g|\psi|^2 - \mu, \mu \) is the chemical potential, \( \psi \) is the ground state of GP equation and the projection operators \( Q \) are given by \( Q = 1 - |\psi\rangle\langle\psi| \).

We numerically integrate Eq.(4) for the \( u_k, v_k \) pairs over a time span of 100 kicks, using a split operator method, parallel to numerical integration of GP equation. The initial conditions \( |u_k(0)\rangle, |v_k(0)\rangle \), for initial ground state wavefunction \( \psi(\theta) = 1/\sqrt{2\pi} \), are obtained by diagonalizing the linear operator in Eq.(4) [21]. After each kick the mean number of noncondensed atoms is calculated and plotted versus time in Fig.4(a).

We find that there exists a critical value for the interaction strength, i.e., \( g_c = 1.96 \), above which, the mean number of noncondensed atoms increases exponentially, indicating the instability of BEC. Below the critical point, the mean number of noncondensed atoms increases polynomially. As the nonlinear parameter crosses over the critical point, the growth rate turns from zero to nonzero, following a square-root law (inset in Fig.4(a)). This scaling law may be universal for Bogoliubov excitation as confirmed by recent experiments [11].

The critical value of the interaction strength depends on the kick strength. For very small kick strength, the critical interaction is expected to be large, because the ground state of the ring-shape BEC with repulsive interaction is dynamically stable [22]. For large kick strength, to induce chaos, the interaction strength must be large enough to compete with the external kick potential. So, in the parameter plane of \((g,k)\), the boundary of instability shows a “U” type curve (Fig.4(b)).

Across the critical point, the density profiles of both condensed and noncondensed atoms change dramatically. In Fig.5, we plot the temporal evolution of the density distributions of condensed atoms as well as noncondensed atoms. In the stable regime, the condensate density oscillates regularly with time and shows clear beating pattern (Fig.5(a)), whereas the density of the noncondensed atoms grows slowly and shows main peaks around \( \theta = \pm \pi \) and \( \theta \), besides some small oscillations (Fig.5(b)). In the unstable regime, the temporal oscillation of the condensate density is irregular (Fig.5(c)), whereas the density of noncondensed atoms grows explosively with the main concentration peaks at \( \theta = \pm \pi/2 \) where the gradient density of the condensed part is maximum (Fig.5(d)). Moreover, our numerical explorations show that the \( \cos^2 \theta \) mode (Fig5.(b)) dominates the density distribution of the noncondensed atoms as the interaction strength is less than 1.8. Thereafter, the \( \sin^2 \theta \) mode grows while \( \cos^2 \theta \)
mode decays, and finally $\sin^2 \theta$ mode become dominating in the density distribution of noncondensed atoms above the transition point (Fig 5(d)). Since the density distribution can be measured in experiment, this effect can be used to identify the transition to instability.

Finally, although the above discussions have been focused on a periodic state of anti-resonance, the transition to instability due to strong interactions also follows a similar path for a dynamically localized state \[23\]. The only difference is that we start out with a quasiperiodic rather than periodic motion in the absence of interaction. This means that it will generally be easier to induce instability but still requires a finite strength of interaction. In Fig 6, we show the nonlinear effect on a dynamically localized state \[24\]. The only difference is that we start out with a quasiperiodic rather than periodic motion in the absence of interaction. This means that chaos brought back by interaction in energy is much slower than the classical diffusion rate, accompanied with exponential growth of noncondensed atoms that clearly indicates the instability of the BEC. Notice that the rate of growth in energy is much slower than the classical diffusion rate, which means that chaos brought back by interaction in this quantum system is still much weaker than pure classical chaos.

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