Abstract. We prove in Theorem 2.2 that the multiplicatively closed subset generated by at most two elements in the set of natural numbers \( \mathbb{N} \) has arbitrarily large gaps by explicitly constructing large integer intervals with known prime factorization for the end points, which do not contain any element from the multiplicatively closed set apart from the end points, which belong to the multiplicatively closed set. An Example 4.6 is also illustrated.

We also give a criterion in Theorems 7.8, 7.12 by using a geometric correspondence between maximal singly generated multiplicatively closed sets and points of the space \( \mathbb{PF}^{\infty}_{\mathbb{Q}^{\geq 0}} \) (refer to Theorem 7.5) as to when a finitely generated multiplicatively closed set gives rise to a doubly multiplicatively closed line (refer to Definition 7.4). We answer a similar Question 5.1 partially about gaps in a multiply-generated multiplicative closed set, when it is contained in a doubly multiplicative closed set using Theorem 7.8 and Theorem 7.17.

In the appendix Section 8 we discuss another constructive proof (refer to Theorem 8.6) for arbitrarily large gap intervals, where the prime factorization is not known for the right end-point unlike the constructive proof of the main result of the article in the case of multiplicatively closed set \( \{p_i p_j^2 \mid i, j \in \mathbb{N} \cup \{0\}\} \) with \( p_1 < p_2 \), \( \log p_1(p_2) \) irrational for which the prime factorization is known for both the end-points of the gap interval via the stabilization sequence of the irrational \( \frac{1}{\log p_1 p_2} \).

1. Introduction

Historically we have seen more of existence proofs of arbitrary large gaps in certain subsets of integers that are present in the literature. A short survey below mentions such results. However constructive proofs in particular those which give the formulae for the end points of the arbitrary large gap intervals have not been there. Here in this article we will be interested in one such constructive proof.

1.1. Short Survey. The distribution of integers with exactly \( k \) distinct prime factors has been studied by many authors. It was first shown by Landau [3] that for a fixed \( k \geq 1 \), the function defined by

\[
\pi(x, k) = \sum_{n \leq x} f_k(n),
\]

where \( f_k(n) = 1 \) if \( n \) has exactly \( k \)-prime factors and 0 otherwise satisfies

\[
\pi(x, k) = \left( \frac{x}{\log x} \right) \frac{(\log \log x)^{k-1}}{(k-1)!} (1 + o(1)).
\]

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Among the other authors who have obtained similar or better asymptotic expressions are Sathe [4, 5], Selberg [6], Hensely [1], Hildebrand and Tenenbaum [2].

Let \( \{p_1, p_2, \ldots, p_k\} \) be any set of \( k \)-distinct primes. Let \( S_{\{p_1, p_2, \ldots, p_k\}} \) be the multiplicatively closed set generated by 1 and numbers, which have exactly and all the factors from \( \{p_1, p_2, \ldots, p_k\} \). Let \( C \) be the collection of all \( k \)-subsets of prime numbers. Consider the set

\[ S_k = \bigcup_{c \in C} S_c. \]

Using any of the results say the result by Landau [3] about asymptotics of \( \pi(x, k) \) we conclude that there are arbitrarily large gaps in \( S \). We observe here that using Equation 1.1 we have

\[ \lim_{x \to \infty} \frac{\pi(x, k)}{x} = 0. \]

If the gaps were bounded then we have that

\[ \liminf_{x \to \infty} \frac{\pi(x, k)}{x} > 0 \]

would be a non-zero constant. Hence the gaps must be arbitrarily large in the set \( S_k \).

With an additional bit of effort on the result of Landau [3] we can extend and conclude arbitrary large gaps for the set

\[ \bigcup_{i=1}^{k} S_i. \]

Now choose a base say \( b = 2 \). If we use asymptotics for a multiplicatively closed set \( T \) generated by primes \( \{p_1, p_2, \ldots, p_k\} \) then we get for large \( x \) the following inequality

\[ \left\lfloor \frac{\log_b x}{k} \sum_{i=1}^{k} \log_b p_i \right\rfloor \leq \#(T \cap [1, x]) \leq \prod_{i=1}^{k} \left\lfloor \frac{\log_b x}{\log_b p_i} \right\rfloor. \]

Hence again we have

\[ \lim_{x \to \infty} \frac{\#(T \cap [1, x])}{x} = 0 \]

from which we will be able to conclude that there are arbitrarily large gaps in \( T \).

However here in this article we give a constructive proof for multiplicatively closed sets, which are contained in doubly generated multiplicatively closed sets with known generators. First we consider multiplicatively closed sets generated by two primes or more generally two positive integers (> 1), which are not Log-Rational to each other. We note here that the multiplicatively closed set can contain numbers with single prime factor unlike the set, which is considered in the result by Landau [3]. Using the technique of rational approximation and stabilization of the sequences of approximate inverses and increasing gaps between two such successive ones we explicitly construct by locating large intervals of natural numbers, which do not contain any element in the given multiplicatively closed set there by proving the main result Theorem 2.2 given below.

2. The main result and method of Proof

Let \( \mathbb{N} \) denote the set of natural numbers. Let \( \mathbb{P} = \{2, 3, 5, \ldots, \} \) denote the set of primes. Here we give using techniques from number theory, geometry and finite fields, a constructive proof of the main result 2.2, where the prime factorization of the end-points of the gap intervals are known and also the end-points belong to the multiplicatively closed set itself. Before we state the main result we need a definition.
**Definition 2.1** (Stabilization sequence of an irrational using sequences of approximate inverses). Let \( 0 < \alpha < 1 \) be an irrational. Let \( \frac{p_n}{q_n}, \gcd(p_n, q_n) = 1, p_n < q_n \) be any sequence of positive rationals converging to \( \alpha \). Now consider the arithmetic progressions \( p_n \mathbb{Z}^+ \) and \( q_n \mathbb{Z}^+ \). Consider the sequence

\[
(p_n \mathbb{Z}^+ \cup q_n \mathbb{Z}^+ \cup \{0\}) \cap \{0, 1, 2, \ldots, q_n p_n\} \subset \{0, 1, 2, \ldots, q_n p_n\}
\]
given as follows.

\[
\begin{align*}
l_0(n) &= 0, p_n, 2p_n, 3p_n, \ldots, l_1(n) p_n, q_n, \\
(l_1(n) + 1)p_n, (l_1(n) + 2)p_n, \ldots, l_2(n) p_n, 2q_n, \\
(l_2(n) + 1)p_n, \ldots, l_i(n) p_n, iq_n, \\
(l_i(n) + 1)p_n, \ldots, (q_n - 1)p_n, p_n q_n.
\end{align*}
\]

For every \( n \in \mathbb{N} \), define the sequence of numbers

\[
\{l_{j_1(n)}(n), l_{j_2(n)}(n), \ldots, l_{j_{r_n}(n)}(n)\}
\]
given as follows. We define \( j_1(n) = 0, l_0(n) = 0 \). Now let \( j(n) \in \{j_1(n), j_2(n), \ldots, j_{r_n}(n)\} \). The defining/characterizing property for \( l_{j(n)}(n) \) is given by

\[
q_n \geq j(n) \geq 1, (l_{j(n)}(n) + 1)p_n - j(n) q_n < \min\{l_i(n) p_n - iq_n \mid 0 \leq i < j(n)\}.
\]

We have \( \{l_{j_1(n)}(n), l_{j_2(n)}(n), \ldots, l_{j_{r_n}(n)}(n)\} = \)

\[
\begin{cases} 
0 = l_0(n) = l_{j_1(n)}(n) = p_n^{-1} - 1 \mod q_n \text{ if } p_n = 1 \\
0 = l_{j_1(n)}(n) = l_0(n) = l_{j_2(n)}(n) = l_1(n) < l_{j_3(n)}(n) < \ldots \\
< l_{j_{r_n}(n)}(n) = p_n^{-1} - 1 \mod q_n \text{ if } p_n \neq 1.
\end{cases}
\]

The sequence

\[
\{1 = l_0 + 1 = l_{j_1(n)}(n) + 1, l_{j_2(n)}(n) + 1, \ldots, l_{j_{r_n}(n)}(n) + 1\}
\]
is the sequence of approximate inverses of \( p_n \mod q_n \). By using Theorems 3.1, 3.2, 3.3 we conclude that the gaps

- \( l_{j_{i+1}(n)}(n) - l_{j_i(n)}(n) \) in the above sequence is increasing.
- The values \( j_i(n) \) stabilize and also \( l_{j_i(n)}(n) \) is eventually a constant as \( n \to \infty \) for a stabilized \( j_i \). (Let the stabilized constant be denoted by \( l_{j_i} \)).
- We have \( \lim_{i \to \infty} (l_{j_{i+1}} - l_{j_i}) \geq \infty \).

This stabilized approximate inverse sequence \( \{l_{j_i} + 1 : i \in \mathbb{N}\} \) is called the stabilization sequence of the irrational \( \alpha \).

Now we state the main result.

**Theorem 2.2.** Let \( \mathbb{P} = \{p_1 < p_2\} \) be a set of two integers, which are not log-rational to each other. Let \( \alpha = \frac{1}{\log p_2(p_2)} \) be the associated irrational less than one. Let \( \{s_i : i \in \mathbb{N}\} \) be the stabilization sequence of \( \alpha \). Let \( t_i = \lfloor s_i \alpha \rfloor \). Then

1. The integer interval

\[
(p_2^{t_i+1-1}, \ldots, p_2^{t_i})
\]

contains no element in the multiplicatively closed set generated by \( \mathbb{P} \) apart from the end-points.

2. The limit of the above gaps (length of the above interval) tends to infinity better than a geometric progression with common ratio \( p_2 \).

This theorem is illustrated with the Example 4.6. As an application of Theorem 2.2 we have the following corollary.
Corollary 2.3. Let \( A \) be a finite set of positive natural numbers. Let \( \mathbb{P} \neq \{1\} \) be a nonempty set of at most two natural numbers. Let \( S = \{1 < a_1 < a_2 < \ldots < a_n < \ldots \} \subset \mathbb{N} \) be the infinite multiplicatively closed set generated by \( A \). Suppose the multiplicatively closed set \( S \subset \langle \mathbb{P} \rangle \) multiplicatively closed set generated by the set \( \mathbb{P} \). Then we have

- \( \limsup_{n \to \infty} (a_{n+1} - a_n) = \infty \).
- We have explicit expressions for the end points of certain arbitrarily large gap intervals in the set \( S \) using the generators of \( \mathbb{P} \).

2.1. Summary of the Proof.

We summarize the method of proof and the structure of the paper in this section.

In Section 3 we first show that for any two relatively prime numbers \( 1 < p < q \) the gaps between successive approximate inverses of \( p \mod q \) is increasing in Theorem 3.1. In Theorems 3.2, 3.3 we prove for a sequence of positive rationals converging to an irrational in \([0,1] \), the sequence of approximate inverses eventually stabilize and the gaps between successive approximate inverses increase.

In Section 4 we prove our main Theorem 2.2. We consider a multiplicatively closed set \( S \) generated by two positive numbers \( p_1, p_2 \), which are log irrational to each other i.e. \( \log p_1(p_2) \) is irrational. We apply Theorems 3.2, 3.3 to \( \frac{1}{\log p_1(p_2)} \) for a suitable sequence of positive rationals obtained in Lemma 4.3 and conclude increasing gaps for the stabilized sequence. Then we locate integer intervals in Lemma 4.4 of arbitrarily large size, which has no elements from the multiplicatively closed set \( S \). This finally proves our main Theorem 2.2 and Corollary 2.3.

This Theorem 2.2 leads to an open Question 5.1. We state this open question in Section 5.

In an attempt to answer this open Question 5.1 in Section 6 we mention that a generalization of the proof of Theorem 2.2 is not directly feasible by proving Lemma 6.2 via an example.

In Section 7 we associate to every multiplicatively closed set a point in the projective space \( \mathbb{P}F_{\mathbb{Q}^\infty} \) and conversely to every point, a maximal singly generated multiplicatively closed set in Theorem 7.5. Then we characterize when two points \( P_1, P_2 \in \mathbb{P}F_{\mathbb{Q}^\infty} \) give rise to the same point in terms of Log-Rationality in Theorem 7.6. In Theorem 7.8 we give a criterion for when a finitely generated multiplicatively closed set is contained in doubly generated multiplicatively closed set and in Theorem 7.12 we classify doubly multiplicatively closed lines (refer to Definition 7.4).

In view of Question 5.1 if a multiplicatively closed set \( S \) is generated by \( r \) elements and these generators give rise to \( s \)—distinct points in the projective space \( \mathbb{P}F_{\mathbb{Q}^\infty} \) (refer to Definition 7.1) with \( s \leq r \) then \( S \) is contained in a multiplicatively closed set \( T \), which is generated by \( s \)—elements. So Theorem 2.2 can be used to answer Question 5.1 whenever \( s \leq 2 \) with a known single generator or pair of generators in the affirmative using the same construction (refer to Section 4). Even otherwise also, if these \( s \)—points generate a doubly multiplicative closed line (refer to Definition 7.4 and Theorems 7.8, 7.12) then Theorem 2.2 can be used to answer Question 5.1 in the affirmative using the same construction (refer to Theorem 7.17).

In Section 7, Theorem 7.8 and Example 7.9 leads to the following interesting question, which is answered completely in Theorem 7.12.

Question 2.4. Classify all lines \( L \) obtained by joining two points \( P_1, P_2 \in \mathbb{P}F_{\mathbb{Q}^\infty} \subset \mathbb{P}F_{\mathbb{Q}^\infty} \), which are doubly multiplicatively closed lines.
3. Irrationals and behaviour of rational approximations, arithmetic progressions, stabilization

We start this section by proving a theorem below on increasing gaps for the successive approximate inverses.

**Theorem 3.1** (Increasing gaps between successive approximate inverses). Let \( p, q \) be two positive integers with \( \gcd(p, q) = 1, p < q \). Consider the arithmetic progressions \( p\mathbb{Z}^+ \) and \( q\mathbb{Z}^+ \). Consider the sequence \( (p\mathbb{Z}^+ \cup q\mathbb{Z}^+ \cup \{0\}) \cap \{0, 1, 2, \ldots, qp\} \) in the set \( \{0, 1, 2, \ldots, qp\} \).

\[
\begin{align*}
    l_0 &= 0, p, 2p, 3p, \ldots, l_1p, q, \\
    (l_1 + 1)p, (l_1 + 2)p, \ldots, l_2p, 2q, \\
    (l_2 + 1)p, \ldots, l_ip, iq, \\
    (l_i + 1)p, \ldots, (q - 1)p, qp.
\end{align*}
\]

Now consider the sequence of numbers

\[
\{l_0 = 0\} \cup \{l_j \mid q \geq j \geq 1, (l_j + 1)p - jq < \min_{0 \leq i < j} \{(li + 1)p - iq\}\}
\]

\[
= \{l_{j_1}, l_{j_2}, \ldots, l_{j_r}\}
\]

\[
\begin{cases}
    = \{0 = l_0 = l_{j_1} = p^{-1} - 1 \mod q\} & \text{if } p = 1 \\
    = \{0 = l_{j_1} = l_0 < l_{j_2} = l_1 < l_{j_3} < \ldots < (l_{j_r} = p^{-1} - 1 \mod q)\} & \text{if } p \neq 1.
\end{cases}
\]

Then the gaps \( l_{j_{i+1}} - l_{j_i} \) in the above sequence is increasing.

**Proof.** If \( p = 1 \) then there is nothing to prove. So assume \( p > 1 \). First we observe that \( p \) is a unit in \( \mathbb{Z}/q\mathbb{Z} = \{0, 1, 2, \ldots, q - 1\} \). The values \( (l_i + 1) \) tend to the inverse of \( p \) because the least possible value for \( (l_i + 1)p - jq \) is one. If we consider the sequence of multiples \( \{(l_j + 1)p \mod q, (l_{j_2} + 1)p \mod q, \ldots, (l_{j_r} + 1)p \mod q\} \) then the values are distinct and decrease to 1 as multiplies of \( p \) given by \( 0, p, 2p, \ldots, (q - 1)p \) gives rise to all residue classes modulo \( q \). Now suppose we consider three consecutive elements in the sequence \( l_{j_1}, l_{j_1+1}, l_{j_1+2} \) then we have

\[
\begin{align*}
    (l_{j_1} + 1)p &= k_{j_1}q + x_{j_1} \\
    (l_{j_1+1} + 1)p &= k_{j_1+1}q + x_{j_1+1} \\
    (l_{j_1+2} + 1)p &= k_{j_1+2}q + x_{j_1+2}
\end{align*}
\]

and the residue classes satisfy \( x_{j_1} > x_{j_1+1} > x_{j_1+2} \) and moreover for any \( t < l_{j_1+1} - l_{j_1} \) we have

\[
\text{if } (l_{j_1} + 1 + t)p = kq + x \text{ then } x > x_{j_1}
\]

because of the minimality condition on \( (l_{j_1} + 1)p - jq \) as the lesser than \( (l_{j_1} + 1) \) multiples of \( p \) are not as close to multiples of \( q \), where we compare multiples of \( p \) to numbers, which are smaller and multiples of \( q \). So we have

\[
(l_{j_1+1} + 1 + t)p = (l_{j_1+1} - l_{j_1})p + (l_{j_1} + 1 + t)p = (k_{j_1+1} - k_{j_1} + k)q + x_{j_1+1} - x_{j_1} + x.
\]

Now note in the right hand side we have the following inequalities for the residue classes \( \mod q \).

\[
\begin{align*}
    0 < x_{j_1} < q \\
    0 < x_{j_1+1} < q \\
    0 < x_{j_1} - x_{j_1+1} < q \\
    0 < x_{j_1+1} < x_{j_1+1} - x_{j_1} + x < x < q
\end{align*}
\]
Theorem 3.2

Let \( p_n, q_n \) be a sequence of positive integers with \( \gcd(p_n, q_n) = 1 \) and suppose \( \frac{p_n}{q_n} \) is a Cauchy sequence converging to an irrational number \( 0 < \alpha < 1 \). Define as in the previous lemma the sequence \( l_i(n) \) and consider the set

\[
\{0 = l_{j_1(n)}(n) < l_{j_2(n)}(n) = l_1(n) < l_{j_3(n)}(n) < \ldots < l_{j_r(n)}(n) = p_n^{-1} - 1 \mod q_n\}.
\]

The values \( j_i(n) \) stabilize and also \( l_{j_i(n)}(n) \) is eventually a constant as \( n \to \infty \) for a stabilized \( j_i \).

Proof. We can assume that \( p_n < q_n \) and \( p_n \neq 1 \). If \( p_n = 1 \) for infinitely many positive integers \( n > 0 \) then \( \frac{p_n}{q_n} \to 0 \), which is a contradiction. We observe that \( l_i(n) = \lfloor \frac{i}{\alpha} \rfloor \) and for fixed \( i \), \( l_i(n) \) is eventually \( \lfloor \frac{i}{\alpha} \rfloor \) as \( n \to \infty \). Also we have the sequence \( j_i(n) \) stabilizes as \( n \to \infty \) because in the inductive definition, we have \( j_i(n) \) satisfies the property that

\[
(l_{j_i(n)}(n) + 1)p_n - j_i(n)q_n < \min_{0 \leq i < j_i(n)} \{(l_i(n) + 1)p_n - iq_n\}
\]

or equivalently that

\[
(l_{j_i(n)}(n) + 1)p_n - j_i(n) < \min_{0 \leq i < j_i(n)} \{(l_i(n) + 1)p_n - iq_n\}.
\]

Now if \( n \to \infty \) then we get that \( (l_i(n) + 1)p_n - i \to (\lfloor \frac{i}{\alpha} \rfloor + 1)\alpha - i \), which is independent of \( n \). Now the independence of \( n \) here implies the stabilization of \( j_i(n) \) follows as \( n \to \infty \). This completes the proof of Theorem 3.2.

Theorem 3.3

Let \( p_n, q_n \) be a sequence of positive integers with \( \gcd(p_n, q_n) = 1 \) and suppose \( \frac{p_n}{q_n} \) is a Cauchy sequence converging to an irrational number \( 0 < \alpha < 1 \). Define as in the previous lemma the sequence \( l_i(n) \) and consider the set

\[
\{0 = l_{j_1(n)}(n) < l_{j_2(n)}(n) = l_1(n) < l_{j_3(n)}(n) < \ldots < l_{j_r(n)}(n) = p_n^{-1} - 1 \mod q_n\}.
\]
Using the previous lemma let $j_i = \lim_{n \to \infty} j_i(n), i_i = \lim_{n \to \infty} i_i(n)$. Then we have
\[
\lim_{i \to \infty} l_{j_{i+1}} - l_{j_i} = \infty.
\]

**Proof.** We can assume that $p_n \neq 1$ eventually. We observe that using the previous Theorem\ref{3.2} we have for every $i \in \mathbb{N}, l_{j_{i+2}} - l_{j_{i+1}} = l_{j_{i+1}} - l_{j_i}$.

If the above limit is not infinity (say equal to $\infty$) then eventually $l_{j_i}$ form an arithmetic progression with common difference $d$. Then $(l_{j_i} + 1) = \lceil \frac{j_i}{\alpha} \rceil$ is in arithmetic progression with common difference $d$. On the one hand the sequence
\[
\lceil \frac{j_i}{\alpha} \rceil - j_i \not\to 0
\]
On the other hand the sequence has a distribution if $l_{j_i}$ are in arithmetic progression. Because if $l_{j_i} = l_{j_{i_0}} + kd$ with $k \in \mathbb{N}$ and fractional parts $z_{j_i}$ are such that $\frac{j_i}{\alpha} + z_{j_i} = \lceil \frac{j_i}{\alpha} \rceil = l_{j_i} + 1$. Then we get $(l_{j_{i_0}} + kd + 1)\alpha - j_i = z_j\alpha \not\to 0$. However the fractional parts $\{l_{j_{i_0}} + kd + 1\alpha - j_i\} = \{l_{j_{i_0}} + kd + 1\alpha\}$ are distributed in the unit interval uniformly as $k \in \mathbb{N}$ by Weyl’s Criterion, Theorem \ref{3.3}. So this is a contradiction and Theorem \ref{3.3} follows.

We mention Weyl’s Equidistributive Criterion here (See also \cite{2}.)

**Theorem 3.4.** Let $\alpha$ be a positive irrational. Let $0 \leq a \leq b \leq 1$. For $x \in \mathbb{R}^+$, let $\{x\}$ denote the fractional part of $x$. Then we have
\[
\frac{\#\{n \mid a \leq \{n\alpha\} \leq b, 1 \leq n \leq N\}}{N} \to (b-a) \text{ as } N \to \infty.
\]

4. The main theorem and construction of arbitrarily large gaps

Before we prove the main Theorem \ref{2.2} of this section we prove the following three Lemmas \ref{4.1} \ref{4.3} \ref{4.4}.

**Lemma 4.1.** Let $p_1 < p_2$ be two natural numbers such that $gcd(p_1, p_2) = 1$. Then
- Either $p_1 = 1$.
- Or $Log(p_1), Log(p_2)$ are both simultaneously irrationals.

**Proof.** If $p_1 = 1$ then there is nothing to prove. Suppose $Log(p_1) = \frac{m}{n}$ for some positive integers $m, n > 0$. Then we have $p_2^n = p_1^m$ a contradiction to unique factorization into primes. So $Log(p_1)$ is irrational.

**Definition 4.2.** We say a pair $(p_1, p_2) \in \mathbb{N}^2$ is an irrational pair if $p_1 \neq 1$ and $p_2 \neq 1$ and both $Log(p_1), Log(p_2)$ are irrationals. For example a GCD—one pair $(p_1, p_2) \in \mathbb{N}^2$, where $p_1 \neq 1 \neq p_2$ is an irrational pair.

**Lemma 4.3.** Let $(p_1, p_2) \in \mathbb{N}^2$ be such that $p_1 < p_2$ and is an irrational pair. Let $\alpha = Log(p_2)(p_1) < 1$. Let $x_2(i) = \lceil \frac{i}{\alpha} \rceil$. For every positive integer $i$ let $z_i = -i + x_2(i)\alpha$.

Define a subsequence with the property that
\[
z_{k_j} < z_{k_{j-1}} = \min\{z_1, z_2, \ldots, z_{k_j-1}\}.
\]

Then
1. $z_{k_j} \not\to 0$.
2. $k_j - k_{j-1}$ is increasing.
3. $\lim_{j \to \infty} (k_j - k_{j-1}) = \infty.$
Proof. First we define a sequence of number parts \(0 < y_i < 1\) defined by the equation
\[y_i + \frac{i}{\alpha} = \left\lfloor \frac{i}{\alpha} \right\rfloor = x_2(i) .\]
Define a subsequence with the property that
\[y_{k_j} < y_{k_{j-1}} = \min\{y_1, y_2, \ldots, y_{k_j} \} .\]
Since the number parts of \(\left\lfloor \frac{i}{\alpha} \right\rfloor\) is also dense in \([0, 1]\) we have that \(y_{k_j} \searrow 0\).
We also have for every \(i\), \(z_i = y_i \alpha\). So \(z_{k_j}\) also satisfies the property that
\[z_{k_j} < z_{k_{j-1}} = \min\{z_1, z_2, \ldots, z_{k_j} \} .\]
Now we have \(x_2(k_j) = \frac{k_j}{\alpha} + y_{k_j}\) and \(y_{k_j} \searrow 0\). Since \(y_{k_j} \alpha < 1\) then
\[\lfloor x_2(k_j) \alpha \rfloor = k_j .\]
Now we apply the previous Theorems 3.2, 3.3 as follows. The sequence
\[\frac{k_j}{x_2(k_j)} = \alpha - \frac{z_{k_j}}{x_2(k_j)} \longrightarrow \alpha \text{ as } j \longrightarrow \infty .\]
In Theorems 3.2, 3.3 we choose \(\alpha\), which is an irrational satisfying the property that \(0 < \alpha < 1\) and the sequence of rationals
\[\frac{k_j}{x_2(k_j)} = \frac{p_j}{q_j} \longrightarrow \alpha \text{ as } j \longrightarrow \infty ,\text{ where } \gcd(p_j, q_j) = 1 .\]
Now by the very definition of \(z_{k_j}\) and using the properties of stabilization and eventual invariance we have
- \(x_2(k_j) - x_2(k_{j-1})\) is increasing.
- \(\lim_{j \to \infty} (x_2(k_j) - x_2(k_{j-1})) = \infty .\)
This implies we also have
- \(k_j - k_{j-1}\) is increasing.
- \(\lim_{j \to \infty} (k_j - k_{j-1}) = \infty .\)
This proves Lemma 4.3 \(\Box\)

Now we prove the following Lemma 4.4.

Lemma 4.4. Let \(p_1 < p_2\) be two integers such that \((p_1, p_2)\) is an irrational pair. Using the notations of the previous Lemma 4.3, we have for any integer \(0 \leq t < k_{j+1} - k_j\) there are no numbers of the form \(p_1^{k_j} p_2^t\) in the integer interval excluding the end-points.
\[(p_2^{k_j+t}, \ldots, p_1^{x_2(k_j)+t}) .\]

Proof. Let \(\alpha = \log_{p_2}(p_1) < 1\). Here we use the following fact. We have \(\lfloor x_2(k_j) \alpha \rfloor = k_j .\)
Suppose if there exists such a number \(p_2^{k_j+t} < p_1 p_2^a < p_1^{x_2(k_j)+t}\) then we have
\[k_j + t < a + \alpha < t + x_2(k_j) \alpha < t + k_j + 1 .\]
\[\rightarrow \alpha \leq t + a + \alpha < x_2(k_j) \alpha < k_j + 1 .\]
\[\rightarrow k_j + t - a < \alpha < k_j + t - a + 1 .\]
So we have that \(b \neq 0\). Similarly \(b \neq x_2(k_j)\). If \(b = x_2(k_j)\) then we get that \(k_j = k_j + t - a\), which implies \(t = a\). Hence \(p_1^{k_j} p_2^t\) is an end-point, which is not considered.
Let \(b \alpha = k_j + t - a + z\). Consider the case \(k_j + t - a < k_{j+1}\). Then by definition of \(z_{k_j}\) and \(z_{k_{j+1}}\) and since \(b \neq x_2(k_j)\) we have \(z \geq z_{k_j} > z_{k_{j+1}}\). Hence
\[k_j < k_j + z = -t + a + \alpha < x_2(k_j) \alpha < k_j + 1 .\]
Hence we get \(z < z_{k_j}\), which is a contradiction. Hence we must have \(k_j + t - a \geq k_{j+1}\), which implies \(t \geq k_{j+1} - k_j + a \geq k_{j+1} - k_j\), which is again a contradiction to the hypothesis \(0 \leq t < k_{j+1} - k_j\). This proves Lemma 4.4 \(\Box\)
Using these three Lemmas 4.1, 4.3, 4.4 we prove our main Theorem 2.2 of this article and its Corollary 2.3.

Proof. Suppose $S = \{1, f, f^2, \ldots \}$ a singly generated multiplicatively closed set then we immediately have $\lim_{j \to \infty} (f^{j+1} - f^j) = \infty$.

Now suppose $S = \{g_1^i g_2^j \mid i, j \geq 0 \}$ and $\text{Log}_{p_1}(g_2)$ is rational then $S$ is contained in a singly generated multiplicatively closed set $\mathbb{T}$ using Theorem 7.6. So there exists arbitrarily large gaps in $S$ as well.

Now suppose $S = \{p_1^i p_2^j \mid i, j \geq 0 \}$ and $\text{Log}_{p_1}(p_2)$ is irrational. Then in Lemma 4.3 we substitute $t = k_{j+1} - k_j - 1$ and we obtain a gap of size

$$0 < p_1^{x_2(k_j)} p_2^{k_j+t} - p_2^{x_2(k_j)} = p_2^{x_2(k_j) - k_j} \geq p_2^{k_j+1-k_j-1}.$$ 

Hence the limit superior of the gaps tend to infinity in the multiplicatively closed set $S$ using Lemma 4.3. Now Theorem 2.2 follows.

Note 4.5. Via the sequence $k_j$ we know the prime factorization of the end points of the intervals $(p_1^{x_2(k_j)}, p_2^{k_j+t}, p_1^{x_2(k_j)}, p_2^{k_j})$ for $0 < t < k_{j+1} - k_j$, which are all gap intervals.

To prove Corollary 2.3 we can use Theorem 2.2 by observing that using Lemma 4.1 the pair $(p_1, p_2)$ is an irrational pair if both $p_1, p_2$ are primes, which also implies that both $\text{Log}_{p_1}(p_2), \text{Log}_{p_2}(p_1)$ are irrational.

Here we give an example illustrating the ideas used to prove Theorem 2.2.

Example 4.6 (Main example). Consider the irrational $\frac{1}{\text{Log}_{p_2}(p_1)}$. The first few terms of the sequence $k_j$, which is defined by the fractional parts

$$z_{k_j} < z_{k_j-1} = \text{min}\{z_1, z_2, \ldots, z_{k_j-1}\}$$

is given by

$$\{1, 3, 5, 17, 29, 41, 94, 147, 200, 253, 306, 971, 1636, 2301, 2966, 3631, 4296, 4961, 5626, 6291, 6956, 7621, 8286, 8951, 9616, 10281, 10946, 11611, 12276, 12941, 13606, 14271, 14936, 15601, 16374, 17039, 18094, 19457, 20511, 21565, 22619, 23673, 24727, 25735, 301994\}.$$

The corresponding first few terms of the sequence $x_2(k_j)$ is given by

$$\{2, 5, 8, 27, 46, 65, 149, 233, 317, 401, 485, 1539, 2593, 3647, 4701, 5755, 6809, 7863, 8917, 9971, 11025, 12079, 13133, 14187, 15241, 16295, 17349, 18403, 19457, 20511, 21565, 22619, 23673, 24727, 25735, 301994\}.$$

The first few terms of the rational approximation sequence to $\alpha$ is given by

$$\frac{1}{2}, \frac{3}{5}, \frac{17}{29}, \frac{29}{41}, \frac{94}{147}, \frac{200}{253}, \frac{253}{306}, \frac{971}{1636}, \frac{1636}{2301}, \frac{2301}{2966}, \frac{2966}{3631}, \frac{3631}{4296}, \frac{4296}{4961}, \frac{4961}{5626}, \frac{5626}{6291}, \frac{6291}{6956}, \frac{6956}{7621}, \frac{7621}{8286}, \frac{8286}{8951}, \frac{8951}{9616}, \frac{9616}{10281}, \frac{10281}{10946}, \frac{10946}{11611}, \frac{11611}{12276}, \frac{12276}{12941}, \frac{12941}{13606}, \frac{13606}{14271}, \frac{14271}{14936}, \frac{14936}{15601}, \frac{15601}{16295}, \frac{16295}{17039}, \frac{17039}{18094}, \frac{18094}{19457}, \frac{19457}{20511}, \frac{20511}{21565}, \frac{21565}{22619}, \frac{22619}{23673}, \frac{23673}{24727}, \frac{24727}{25735}, \frac{25735}{301994}.$$

This sequence for approximate inverses for the fraction $190537$ is given by

$$\{1, 2, 5, 8, 27, 46, 65, 149, 233, 317, 401, 485, 1539, 2593, 3647, 4701, 5755, 6809, 7863, 8917, 9971, 11025, 12079, 13133, 14187, 15241, 16295, 17349, 18403, 19457, 20511, 21565, 22619, 23673, 24727, 25735, 301994\}.$$

\[\]
We note that it matches with \( x_2(k_j) \). Actually this can be obtained for any suitable rational approximation sequence for \( \alpha \). The first few gaps of intervals with the prime factorization of end-points of the gap intervals of the form
\[
(p_2^{k_{j+1}-1} \ldots p_1^{k_{j+1}} p_2^{k_{j+1}})
\]
using this method is given by
\[
(3^2 \ldots 2^31), (3^4 \ldots 2^531), (3^16 \ldots 2^8311), (3^{28} \ldots 2^{27}3^{11}), (3^{40} \ldots 2^{46}3^{11}),
\]
\[
(3^{93} \ldots 2^653^{52}), (3^{146} \ldots 2^1493^{52}), (3^{199} \ldots 2^{233}3^{52}), (3^{252} \ldots 2^{317}3^{52}),
\]
\[
(3^{305} \ldots 2^4013^{52}), (3^{970} \ldots 2^4853^{64}), (3^{1635} \ldots 2^15393^{64}),
\]
\[
(3^{2300} \ldots 2^25933^{64}), (3^{2965} \ldots 2^36473^{64}), (3^{3630} \ldots 2^47013^{64}),
\]
\[
(3^{4295} \ldots 2^57553^{64}), (3^{4960} \ldots 2^68093^{64}), (3^{5625} \ldots 2^78633^{64}),
\]
\[
(3^{6290} \ldots 2^89173^{64}), (3^{6955} \ldots 2^99713^{64}), (3^{7620} \ldots 2^{11025}3^{64}),
\]
\[
(3^{8285} \ldots 2^120793^{64}), (3^{8950} \ldots 2^131333^{64}), (3^{9615} \ldots 2^141873^{64}),
\]
\[
(3^{10280} \ldots 2^152413^{64}), (3^{10945} \ldots 2^162953^{64}), (3^{11610} \ldots 2^173493^{64}),
\]
\[
(3^{12275} \ldots 2^184033^{64}), (3^{12940} \ldots 2^194573^{64}), (3^{13605} \ldots 2^{20511}3^{64}),
\]
\[
(3^{14270} \ldots 2^215653^{64}), (3^{14935} \ldots 2^226193^{64}), (3^{15600} \ldots 2^236733^{64}),
\]
\[
(3^{17467} \ldots 2^247273^{11866}), (3^{19034} \ldots 2^752353^{11866}), (3^{19053} \ldots 2^1257433^{111201}).
\]

5. An open question

In this section we state an open question arising from Theorem 2.2

**Question 5.1.** Let \( S = \{1 < a_1 < a_2 < \ldots \} \subset \mathbb{N} \) be a finitely generated multiplicatively closed infinite set generated by positive integers \( d_1, d_2, \ldots, d_n \). How do we construct explicitly arbitrarily large integer intervals with known prime factorization of the end points, which do not contain any elements from the set \( S \) using the positive integer \( d_1, d_2, \ldots, d_n \)?

In the later sections we consider some implications of our results regarding this open question and partially answer this question in the affirmative.

6. On a generalization of this method to more than two generators

In the proof of the main Theorem 2.2 we know the prime factorizations of both the end points of the gap interval via the stabilization sequence. Sometimes knowing factorizations is helpful because of the following note.

**Note 6.1.** If a large number \( N \) has exactly has two large prime factor pair say \( \{q_1, q_2\} \) and if \( N \) lies in a gap interval of multiplicatively closed set generated by \( p_1, p_2 \) then we can postively conclude that the factor pair of \( N, \{q_1, q_2\} \neq \{p_1, p_2\} \). The gap intervals in Theorem 2.2 are easy to generate for any pair \( \{p_1, p_2\} \) such that \( \text{Log} p_1 (p_2) \) is irrational.

In this section we point out that a certain generalization of this method of proof of Theorem 2.2 to more than two generators is not directly feasible. In particular in an attempt to answer Question 5.1 we prove a lemma, which says that the same technique may or may not be extendable for more than two generators.

**Lemma 6.2.** Let \( G = \{p_1 < p_2 < \ldots < p_l\} \) be a finite set of primes. Let \( k \) be any positive integer. Consider the monoid \( T = \left\{ \sum_{i=1}^{l-1} x_i \text{Log} p_i | x_i \in \mathbb{N} \cup \{0\} \right\} \). Consider the set
We can show the inequalities $z_{\limsup}$ that are not increasing. This proves the lemma. However we mention that it is possible showing inequalities we obtain

\[ z_{k_j} < z_{k_{j-1}} = \min\{z_1, z_2, \ldots, z_{k_j-1}\} \]

then the sequence of integers \(\{k_{j+1} - k_j : j \in \mathbb{N}\} \) need not be increasing.

\textbf{Proof.} Consider the following example. Let \(\{p_1 = 2 < p_2 = 3 < p_3 = 5\}\). By calculating the logarithm of numbers to the base 5 in the sequence \(\{2^i3^j \mid 0 \leq i, j \leq 50\}\) or by actually showing inequalities we obtain

- \(k_0 = 0, z_{k_0} = z_0 = \log_5(2) - 0\).
- \(k_1 = 1, z_{k_1} = z_1 = \log_5(23) - 1\).
- \(k_2 = 2, z_{k_2} = z_2 = \log_5(3^3) - 2\).
- \(k_3 = 3, z_{k_3} = z_3 = \log_5(2^7) - 3\).
- \(k_4 = 7, z_{k_4} = z_7 = \log_5(2^23^9) - 7\).
- \(k_5 = 8, z_{k_5} = z_8 = \log_5(2^{17}3) - 8\).
- \(k_6 = 13, z_{k_6} = z_{13} = \log_5(2^83^14) - 13\).
- \(k_7 = 14, z_{k_7} = z_{14} = \log_5(2^{23}3^6) - 14\).

We can show the inequalities $z_{k_0} > z_{k_1} > z_{k_2} > z_{k_3} > z_{k_4} > z_{k_5} > z_{k_6} > z_{k_7}$ and

\[ k_1 - k_0 = k_2 - k_1 = k_3 - k_2 = 1 < k_4 - k_3 = 4 > k_5 - k_4 = 1 < k_6 - k_5 = 5 > k_7 - k_6 = 1, \]

which is not increasing. This proves the lemma. However we mention that it is possible that $\limsup_{j \to \infty}(k_{j+1} - k_j) = \infty$, which additionally requires a proof. \(\square\)

### 7. Geometry of singly and doubly generated multiplicatively closed sets

In this section we partially answer Question 5.1 using Theorems 7.8,7.12 in Theorem 7.17. First we begin with a few definitions.

\textbf{Definition 7.1.} Let \(\mathbb{Q}\) denote the field of rational numbers. Let \(\mathbb{Q}_{\geq 0}\) denote the set of non-negative rationals. Define an equivalence relation \(\sim_R\) on

\[ \mathbb{Q}_{\geq 0}^\infty \setminus \{0\} = \bigoplus_{i=1}^\infty \mathbb{Q}_{\geq 0} \setminus \{0\}. \]

We say \((a_1, a_2, \ldots, a_i) \sim_R (b_1, b_2, \ldots, b_i) \in \mathbb{Q}_{\geq 0}^\infty \setminus \{0\}\) if there exists \(\lambda \in \mathbb{Q}^+\) such that \(a_i = \lambda b_i\) for all \(i \geq 1\). Let \(\mathbb{PF}_{\mathbb{Q}_{\geq 0}}^\infty\) denote the projective space

\[ \mathbb{PF}_{\mathbb{Q}_{\geq 0}}^\infty = \frac{\mathbb{Q}_{\geq 0}^\infty \setminus \{0\}}{\sim_R}. \]

\textbf{Definition 7.2.} Let \(\mathbb{Q}\) denote the field of rational numbers. Define an equivalence relation on

\[ \bigoplus_{i \geq 1} \mathbb{Q} \setminus \{0\}. \]

We say \((a_1, a_2, \ldots, a_n) \sim_R (b_1, b_2, \ldots, b_n)\) if \(a_i = \lambda b_i\) for some \(\lambda \in \mathbb{Q}^+\). Let \(\mathbb{PF}_{\mathbb{Q}}^\infty\) denote the space

\[ \mathbb{PF}_{\mathbb{Q}}^\infty = \frac{\bigoplus_{i \geq 1} \mathbb{Q} \setminus \{0\}}{\sim_R}. \]

The space \(\mathbb{PF}_{\mathbb{Q}_{\geq 0}}^\infty \subset \mathbb{PF}_{\mathbb{Q}}^\infty\) as the subset of points, which have all non-negative and at least one positive integer representatives. We note that if two finite tuples, which have positive
coordinates are rational multiple of each other then they are positive rational multiple of each other.

**Definition 7.3.** Let \( \mathbb{P} = \{ p_1 = 2, p_2 = 3, p_3 = 5, \ldots \} \subset \mathbb{N} \) be the set of primes, where \( p_i \) denote the \( i \)-th prime. We say a set \( S \subset \mathbb{N} \) is singly generated multiplicatively closed if \( S = \{ 1, f, f^2, \ldots \} \) for some \( f \in \mathbb{N}, f \neq 1 \). We say \( S \) is a singly generated maximal multiplicatively closed set if \( T \) is any singly generated multiplicatively closed set and \( T \supset S \) then \( T = S \).

**Definition 7.4.** Let \( L \) be a line obtained by joining two points \( P_1, P_2 \in \mathbb{P} \mathbb{F}_{\mathbb{Q}_{\geq 0}} \subset \mathbb{P} \mathbb{F}_{\mathbb{Q}} \). We say \( L \) is a doubly multiplicatively closed line, if we consider only integers and (not elements of \( \mathbb{Q}_{\geq 0} \backslash \mathbb{Z}_{\geq 0} \)) associated to all tuples whose equivalence classes are points that lie on \( L \) (refer to the proof of Theorem 7.5) then it gives rise to a doubly generated multiplicatively closed set.

In view of Example 7.9 not all lines \( L \) are doubly multiplicatively closed lines. However we note that each point \( P \in \mathbb{P} \mathbb{F}_{\mathbb{Q}_{\geq 0}} \) gives rise to a unique maximal singly generated multiplicatively closed set (See Theorem 7.5).

Now we state a correspondence theorem.

**Theorem 7.5.** Let 
\[
\mathcal{S} = \{ S \subset \mathbb{N} \mid \text{such that } S \text{ is a maximal singly generated multiplicatively closed set.} \}
\]

Then there is a bijective correspondence between

\[
\mathcal{S} \leftrightarrow \mathbb{P} \mathbb{F}_{\mathbb{Q}_{\geq 0}}
\]
i.e. between maximal singly generated multiplicatively closed sets and the points of the space \( \mathbb{P} \mathbb{F}_{\mathbb{Q}_{\geq 0}} \) given by

\[
S = \{ f^n \mid 0 \leq n \in \mathbb{N} \cup \{0\}, \]

\[
f = \prod_{j=1}^{k} p_{i_j}^{r_{i_j}} \text{ with } p_{i_1} < p_{i_2} < \ldots < p_{i_k} \in \mathbb{P}, r_{i_1}, \ldots, r_{i_k} \in \mathbb{N}
\]

\[
\rightarrow P = [\ldots : r_{i_1} : \ldots : r_{i_2} : \ldots : \ldots : r_{i_k} : \ldots ] \in \mathbb{P} \mathbb{F}_{\mathbb{Q}_{\geq 0}}
\]

where the coordinates of any point in \( \mathbb{P} \mathbb{F}_{\mathbb{Q}_{\geq 0}} \) are ordered according to increasing sequence of primes in the set \( \mathbb{P} \).

**Proof.** The bijection is given as follows. Let 
\[
S = \{ 1, f, f^2, \ldots \}
\]

be any singly generated multiplicatively closed set. Let

\[
f = \prod_{j=1}^{k} p_{i_j}^{r_{i_j}} \text{ with } p_{i_1} < p_{i_2} < \ldots < p_{i_k} \in \mathbb{P}, r_{i_1}, \ldots, r_{i_k} \in \mathbb{N}.
\]

To this multiplicatively closed set we associate the point

\[
P = [\ldots : r_{i_1} : \ldots : r_{i_2} : \ldots : \ldots : r_{i_k} : \ldots ] \in \mathbb{P} \mathbb{F}_{\mathbb{Q}_{\geq 0}}.
\]

The condition that \( S \) is maximal is equivalent to the condition

\[
gcd(r_{i_1}, r_{i_2}, \ldots, r_{i_k}) = 1.
\]

Also given any point \( P \) in \( \mathbb{P} \mathbb{F}_{\mathbb{Q}_{\geq 0}} \) there is a unique non-negative integer coordinate representative of \( P \) with \( gcd \) of the coordinates equal to one, which gives rise to the integer \( f \in \mathbb{N} \) with \( f \neq 1 \).
This establishes the bijection and hence Theorem 7.5 follows. \(\square\)

**Theorem 7.6 (Log-Rationality).** Let \(P_1, P_2\) be two points (possibly the same point) in \(\mathbb{P}^\infty_{\mathbb{Q}_{\geq 0}}\). Let

\[
g_1 = \prod_{j=1}^{t} p_{ij}^{r_{ij}}, \quad g_2 = \prod_{j=1}^{t} d_{ij}^{s_{ij}}
\]

two positive integers (> 1) with their unique prime factorizations such that

\[
P_1 = [\ldots : r_{i1} : \ldots : r_{i2} : \ldots : \ldots : \ldots : r_{it} : \ldots ]
\]

\[
P_2 = [\ldots : s_{i1} : \ldots : s_{i2} : \ldots : \ldots : s_{iu} : \ldots ].
\]

Let \(f_1, f_2\) be the corresponding positive integers (> 1) under the bijection given in Theorem 7.5 then the following are equivalent.

1. (Log-Rationality:) \(\log g_1, g_2\) is rational.
2. \(P_1 = P_2\)
3. \(f_1 = f_2\).
4. The multiplicatively closed set \(\mathbb{T} = \{g_1^i g_2^j \mid i, j \geq 0\}\) is contained in a singly generated maximal multiplicatively closed set.

**Proof.** Suppose \(\log g_1, g_2 = \frac{m}{n}\) is rational. Then we have \(g_2^n = g_1^m\). So the distinct prime factors of \(g_1, g_2\) agree and we also have that their exponents are projectively equivalent. Hence we get \(P_1 = P_2\). So this implies \(f_1 = f_2 = f\) say. Then we get that \(\mathbb{T} \subseteq \{1, f, f^2, \ldots \}\).

For the converse if \(\mathbb{T} \subseteq \{1, f, f^2, \ldots \}\) for some \(1 \neq f \in \mathbb{N}\) then \(g_1 = f^n, g_2 = f^m\) and we have \(g_2^n = g_1^m\). Hence \(\log g_1, g_2 = \frac{m}{n}\) is rational. This completes the equivalence of the statements (1), (2), (3), (4) and also proves Theorem 7.5. \(\square\)

Now we have the following corollary.

**Corollary 7.7.** A multiplicatively closed set \(\mathbb{T} = \{g_1^i g_2^j \mid i, j \geq 0, g_1, g_2 \in \mathbb{Q}_{\geq 0}\}\) is not contained in a singly generated multiplicatively closed set if and only if \(\log g_1, g_2\) are both irrational if and only if \(g_1, g_2\) represent two distinct points in the projective space \(\mathbb{P}^\infty_{\mathbb{Q}_{\geq 0}}\).

In the theorem that follows we give a criterion as to when a multiplicatively closed set is contained in a doubly generated multiplicatively closed set.

**Theorem 7.8.** Let \(\mathbb{S} = \{g_1^{i_1} g_2^{i_2} \ldots g_r^{i_r} \mid i_1, i_2, \ldots, i_r \in \mathbb{N} \cup \{0\}\}\) be a multiplicatively closed set generated by \(r\) elements. Suppose corresponding to these positive integers \(g_i : 1 \leq i \leq r\) the points \([g_i] : 1 \leq i \leq r \in \mathbb{P}^\infty_{\mathbb{Q}_{\geq 0}} \subset \mathbb{P}^\infty_{\mathbb{Q}}\) lie on a projective line \(L\) (i.e. a rank-1, 2 condition on the matrix of powers of primes in the prime factorizations of \(g_i\) obtained by joining two points of \(\mathbb{P}^\infty_{\mathbb{Q}_{\geq 0}}\) whose corresponding integers are relatively prime. Then \(\mathbb{S}\) is contained in a doubly generated multiplicatively closed set.

**Proof.** If \(\mathbb{S}\) gives rise to a single point then there is nothing to prove. So let \(P_1, P_2 \in \mathbb{P}^\infty_{\mathbb{Q}_{\geq 0}}\) be any two distinct points, which gives rise to the projective line \(L\). Let \(p_1, p_2\) be the positive integers, which represent these points \(P_1, P_2\) with \(gcd(p_1, p_2) = 1\). Then the hypothesis that the points \([g_i]\) lie on the projective line \(P_1P_2\) implies that there exists integers \(a_i, b_i, c_i \geq 0\) such that \(p_1^{a_i} p_2^{b_i} = g_i^{c_i}\) for \(1 \leq i \leq r\). Consider the unique prime factorization of

\[
p_1 = q_1^{s_1} q_2^{s_2} \cdots q_l^{s_l}, \quad p_2 = q_1^{t_1} q_2^{t_2} \cdots q_l^{t_l},
\]

where we assume without loss of generality that \(gcd(s_1, s_2, \ldots, s_l) = 1, gcd(t_1, t_2, \ldots, t_l) = 1\).

If in addition we have \(gcd(p_1, p_2) = 1\) then we have \(s_j t_j = 0 : 1 \leq j \leq l\) but one of \(s_j\) and \(t_j\) is non-zero for each \(j\). In all cases we conclude that \(c_i \mid s_j a_i, c_i \mid t_j b_i\) for \(1 \leq j \leq l\). So
In Example 7.9, for all integer representatives $r$ be a multiplicatively closed set generated by $p$ exists two positive integers $\{i,j \geq 0\} \in S$ and this proves Theorem 7.8

**Example 7.9.** Let $g_1 = 45, g_2 = 20, g_3 = 30$. Then we have $g_1g_2 = g_3^2$. So the doubly generated multiplicatively closed set generated by $g_1, g_2$ contains $g_3^2$ but not $g_3$. However there is no doubly generated multiplicatively closed set containing all $g_1, g_2, g_3$ because there are no two distinct non-trivial common factors of $g_1, g_2, g_3$ as $gcd(g_1, g_2, g_3) = 5$, which is prime. Now the corresponding exponents satisfy

$$(0, 2, 1, 0, \ldots) + (2, 0, 1, 0, \ldots) = 2(1, 1, 1, 0, \ldots).$$

since $g_1g_2 = g_3^2$ and the exponent vectors lie on a projective line $L \subset \mathbb{F}_Q^\infty$.

**Note 7.10.** Theorem 7.8 can be generalized as follows. Let $S = \{g_1^{i_1}g_2^{i_2}g_3^{i_3}| i_1, i_2, i_3 \in \mathbb{N} \cup \{0\}\}$ be a multiplicatively closed set generated by $r$ elements. Fix a prime $p = 2$. Suppose there exists two positive integers $p_1, p_2$ such that the monoid $\{aLog_p(p_1) + bLog_p(p_2) | a, b \in \mathbb{Z}_{\geq 0}\}$ contains the set $\{Log_p(g_i) | 1 \leq i \leq r\}$ then the set $S \subset T = \{p_1^{i_1}p_2^{i_2} | i, j \geq 0\}$.

**Note 7.11.** In Example 7.9 for all integer representatives $f \in \mathbb{N}$ such that $[f] \in L$ we have $5 \mid f$. This is the only prime with this property for the line $L$, which is not a doubly multiplicatively closed line. So does there exist a doubly multiplicatively closed line with such a prime? Definitely not when there are only two primes involved with the line $L$.

In Example 7.9 we have the following properties holding true.

- For all integer representatives $f \in \mathbb{N}$ such that $[f] \in L$ we have $5 \mid f$ and this is the only such prime. Neither of the primes $2, 3$ satisfy this property.
- There exists numbers $g_1 = 45, g_2 = 20$ whose points lie on $L$ and two primes $2, 3$ such that

$$3 \mid 45, 3 \mid 20, 2 \mid 20, 2 \mid 45.$$  

- The lattice $M$ corresponding to $L$ is a two dimensional lattice, which does not possess a basis $\{x, y\}$ such that $M \cap \mathbb{Z}_{\geq 0}^2$ satisfies the monoid addition property. i.e.

$$ax + by \in M \cap \mathbb{Z}_{\geq 0}^2 \iff a \geq 0, b \geq 0.$$

In the following theorem we classify doubly multiplicatively closed lines.

**Theorem 7.12.** A line $L$ joining two points $P_1, P_2 \in \mathbb{PF}_Q^{\infty} \subset \mathbb{PF}_Q^\infty$ is a doubly multiplicatively closed line if and only if there exists two points $Q_1 = [q_1], Q_2 = [q_2] \in \mathbb{PF}_Q^{\infty} \subset \mathbb{PF}_Q^\infty$ with positive integers

$$q_1 = p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}, q_2 = p_1^{b_1}p_2^{b_2} \ldots p_r^{b_r}.$$  

with the following properties.

1. **Trivial Index Property** (Alternative, refer [A]): The gcd of two by two minors of

$$\begin{pmatrix}
    a_1 & a_2 & \ldots & a_r \\
    b_1 & b_2 & \ldots & b_r
\end{pmatrix}$$

is one.

2. **Monoid Addition Property** (Alternative, refer [B]): There exists two subscripts $i, j$ such that $a_i b_i = 0 = a_j b_j$ and either $a_i b_j \neq 0$ or $a_j b_i \neq 0$.

In particular if there exists two points $Q_1, Q_2 \subset L \cap \mathbb{PF}_Q^{\infty}$ such that their corresponding integer representatives are relatively prime then $L$ is a doubly multiplicatively closed line.
Proof. First we prove the last assertion. Suppose there exists such points \( Q_1, Q_2 \) on \( L \) and let \( q_1, q_2 \) be the corresponding integers. Let

\[
q_1 = p_1^{s_1} p_2^{s_2} \ldots p_l^{s_l}, q_2 = p_{l+1}^{s_{l+1}} p_{l+2}^{s_{l+2}} \ldots p_n^{s_n}.
\]

be their unique prime factorizations with \( s_i \in \mathbb{N} : 1 \leq i \leq n, \gcd(q_1, q_2) = 1 \). Now we choose \( q_1, q_2 \) such that

\[
\gcd(s_1, s_2, \ldots, s_l) = 1 = \gcd(s_{l+1}, s_{l+2}, \ldots, s_n).
\]

If \( g \in \mathbb{N} \) such that \([g] \in L\) then there exist \( a, b, c \in \mathbb{N} \) such that \( q_1^a q_2^b = g^c \). This implies

\[
c | a s_i : 1 \leq i \leq l, c | b s_i : l + 1 \leq i \leq n \Rightarrow c | a, c | b.
\]

So \( g \in \mathbb{T} = \{ q_1^a q_2^b \mid i, j \in \mathbb{N} \cup \{0\} \} \). In this particular case we also have in the matrix

\[
\begin{pmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
0 & \ldots & 0
\end{pmatrix}
\]

has the property that its \( \gcd \) of two minors equal \( \gcd(s_i s_j : 1 \leq i \leq l, l + 1 \leq j \leq n) = 1 \). This proves the last assertion.

Now we prove the first assertion. In particular the implication (\( \Leftarrow \)). Assume \( L \) has such points \( Q_1 = [q_1], Q_2 = [q_2] \).

Let \( V = \mathbb{Q} - \operatorname{span} \) of the vectors \( \{ a = (a_1, \ldots, a_r), (b_1, \ldots, b_r) \} \). Let \( M = \mathbb{Z} - \operatorname{span} \) of the vectors \( \{ a, b \} \). Then we have the following properties.

(A) Trivial Index Property:

\[
V \cap \mathbb{Z}^r = M.
\]

This follows because of the \( \gcd \) of the \( 2 \times 2 \) minors is one. i.e. \( M \) has a trivial index in \( V \cap \mathbb{Z}^r \). Using the theorem for sublattices we get that for the tower of sublattices

\[
M \subset V \cap \mathbb{Z}^r \subset \mathbb{Z}^r
\]

that there exists a basis of \( \mathbb{Z}^r \) given by \( \{ u_1, u_2, \ldots, u_r \} \) and positive integers \( d_1, d_2 \) such that \( \{ u_1, u_2 \} \) is a basis of \( V \cap \mathbb{Z}^r \) and \( \{ u_1, u_2 u_2 \} \) is a basis of \( M \), which has therefore index \( d_1 d_2 \), which is also \( \gcd \) of the \( 2 \times 2 \) minors of \( \{ u_1, u_1 u_2 \} \). Since \( \{ d_1 u_1, d_2 u_2 \} \) differ from the basis \( \{ a, b \} \) of \( M \) by an \( SL_2(\mathbb{Z}) \) matrix we have \( d_1 d_2 = 1 \).

(B) Monoid Addition Property:

\[
\alpha a + \beta b \in M \cap \mathbb{Z}_0^r \iff \alpha \geq 0, \beta \geq 0.
\]

This follows because there exists two subscripts \( i, j \) such that \( a_i b_i = 0 = a_j b_j \) and either \( a_i b_j \neq 0 \) or \( a_j b_i \neq 0 \) and the coordinate entries of both \( a, b \) are non-negative.

So we get that if \( g \in \mathbb{N} \) such that \([g] \in L\) then \( g = q_1^i q_2^j \) for some \( i, j \in \mathbb{N} \cup \{0\} \). So the required multiplicatively closed set representing the line is \( \mathbb{T} = \{ q_1^i q_2^j \mid i, j \in \mathbb{N} \cup \{0\} \} \).

Now we prove the implication (\( \Rightarrow \)).

Suppose \( L \) is a multiplicatively closed line with the multiplicatively closed set being \( \mathbb{T} = \{ q_1^i q_2^j \mid i, j \in \mathbb{N} \cup \{0\} \} \). So if \( g \in \mathbb{N} \) such that \([g] \in L\) then \( g \in \mathbb{T} \). Let the prime exponent vectors of \( q_1, q_2 \) be \( s, t \) with \( s = (s_1, s_2, \ldots, s_r), t = (t_1, t_2, \ldots, t_r), \gcd(s_1, s_2, \ldots, s_r) = 1 = \gcd(t_1, t_2, \ldots, t_r) \). This proof is a bit long. We prove both the Trivial Index Property and the Monoid Addition Property for \( \{ s, t \} \).

Claim 7.13. Let \( V \) be a two dimensional \( \mathbb{Q} \)-vector space spanned by \( s = (s_1, s_2, \ldots, s_r), t = (t_1, t_2, \ldots, t_r) \). Let \( M = V \cap \mathbb{Z}^r \). Then there exists \( w \in M \) with all its coordinate entries non-negative such that \( \{ s, w \} \) is a basis for \( M \).
Hence there exists $Z$ say
\[ u = (u_1, u_2, \ldots, u_r), v = (v_1, v_2, \ldots, v_r), w^1, w^2, \ldots, w^{r-2} \]
and positive integers $d_1, d_2$ such that \{d_1u, d_2v\} is a basis of $M$ with $d_1 \mid d_2$. Since $M$ contains a gcd one vector on every line we have $d = d_2$. If $\alpha u + \beta v = s$ then $gcd(\alpha, \beta) = 1$ because $(\alpha) + (\beta) = (\alpha u_1, \alpha u_2, \ldots, \alpha u_r) + (\beta v_1, \beta v_2, \ldots, \beta v_r) \supset (\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2, \ldots, \alpha u_r + \beta v_r) = (s_1, s_2, \ldots, s_r) = Z$.
Hence there exists $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(Z)$ such that
\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0_{2 \times (r-2)} \\ I_{(r-2) \times (r-2)} \end{pmatrix} = \begin{pmatrix} u \\ v \\ w^1 \\ \vdots \\ w^{r-2} \end{pmatrix},
\]
where $w = \gamma u + \delta v$. Apriori $w$ need not have non-negative entries. Using a unipotent lower triangular matrix over $Z$ we need to consider only those entries of $w$ whose corresponding entries in $s$ are zero. Now the vector $t$, which has non-negative entries lies in the span $M$ of $s, w$. i.e $t = es + \mu w$ with $e \in Z, \mu \in Z^* = Z \setminus \{0\}$. Now if $s_i = 0, w_i \neq 0$ then $sign(w_i) = sign(\mu)$. If $sign(\mu)$ is negative then we consider $-w$ instead of $w$. Then we get that the $w_i$ has non-negative sign whenever $s_i$ is zero. Now again using unipotent lower triangular matrix over $Z$ we make the sign of the remaining entries of $w$ non-negative. Hence we arrive at a basis \{s, w\} of $M$ such that both have non-negative integer entries. We have obtained \{s, w\} from \{u, v\} by an $SL_2(Z)$ transformation with determinant $\pm 1$.

This proves Claim \[7.13\] \[\square\]

**Claim 7.14 (Trivial index property).** Let \{s, w\} be the basis of $M$ obtained from Claim 7.13

Let 
\[ q_1 = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}, e = p_1^{w_1} p_2^{w_2} \cdots p_r^{w_r}. \]

Since $M \cap Z_{\geq 0}^r$ corresponds to a doubly multiplicatively closed set $T = \{q_1 q_2^j \mid i, j \in N \cup \{0\}\}$ we have
\begin{itemize}
  \item \{q_1 q_2^j \mid i, j \in Z\} = \{q_1 q_2^j \mid i, j \in Z\}.
  \item The $Z$–span of \{s, t\} is the same as $Z$–span of \{s, w\}.
  \item If for some $\alpha, \beta \in Q, \alpha s + \beta t \in Z^r$ then $\alpha, \beta \in Z$.
\end{itemize}

**Proof of Claim.** Since $q_1, e$ corresponds to points in $M \cap Z_{\geq 0}^r$ we have $e = q_1^{m_1} q_2^{m_2}, m_1, m_2 \in N \cup \{0\}$.

So we have \{q_1 q_2^j \mid i, j \in Z\} $\subset$ \{q_1 q_2^j \mid i, j \in Z\}. The other way containment is immediate since $Z$–span of \{s, w\} contains $Z$–span of \{s, t\}. Now the rest of the claim for exponents follows as \{s, w\} is a $Z$–basis for $M = V \cap Z^r$ and $Q$–basis for $V$. This proves the trivial index property for \{s, t\}. \[\square\]

**Claim 7.15 (Monoid addition property).** The basis \{s, t\} has monoid addition property.

**Proof of Claim.** Now suppose if all the coordinate entries of $s$ is positive. Then for some large $m \in N$ we have $ms - t$ has non-negative entries, which is a contradiction. Hence there exist a subscript $i$ such that $s_i = 0, t_i \neq 0$. Similarly there exist a subscript $j$ such that $t_j = 0, s_j \neq 0$. This proves the monoid addition property that 
\[ \alpha s + \beta t \in M \cap Z_{\geq 0}^r \Leftrightarrow \alpha \geq 0, \beta \geq 0. \]
This completes the proof of Theorem 7.12.

Example 7.16. Let \( g_1 = 10, g_2 = 15 \). Then the line joining the points \([g_1], [g_2]\) is a multiplicatively closed line using Theorem 7.12 where as the line in Example 7.9 joining \([g_1 = 20], [g_2 = 45]\) is not multiplicatively closed. Now we could also use Theorem 7.12 to prove this fact in another way.

Theorem 7.17. Let \( S \subset \mathbb{N} \) be a finitely generated multiplicatively closed set whose corresponding points lie on a double multiplicatively closed line \( L \) containing points \( Q_1 = [q_1], Q_2 = [q_2] \) satisfying trivial index property and monoid addition property then we have explicit expressions for the end points of certain arbitrarily large gap intervals in the set \( S \) using the generators \( q_1, q_2 \).

Proof. This theorem follows because the set \( S \subset \{q_i q_j \mid i, j \in \mathbb{N} \cup \{0\}\} \) and then we use main result 2.2.

8. Appendix

In this appendix section we prove some interesting lemmas about gaps, also present some motivating examples and give another constructive proof and discuss advantages and disadvantages with respect to the above given constructive proof.

We begin with a lemma.

Lemma 8.1. (1) Let \( S \subset \mathbb{N} \) be an infinite set. If

\[
\liminf_{n \to \infty} \frac{\#(S \cap [1, \ldots, n])}{n} = 0
\]

then there are arbitrarily large gaps in \( S \).

(2) Let \( S_i \subset \mathbb{N} : 1 \leq i \leq k \) be \( k \)-infinite subsets. If for each \( 1 \leq i \leq k \)

\[
\lim_{n \to \infty} \frac{\#(S_i \cap [1, \ldots, n])}{n} = 0
\]

then there are arbitrarily large gaps in \( S = \bigcup_{i=1}^{k} S_i \).

Proof. To prove (1) we observe that if the gaps were bounded then

\[
\liminf_{n \to \infty} \frac{\#(S \cap [1, \ldots, n])}{n} > 0.
\]

To prove (2) we have

\[
0 \leq \lim_{n \to \infty} \frac{\#(S \cap [1, \ldots, n])}{n} \leq \lim_{n \to \infty} \sum_{i=1}^{k} \frac{\#(S_i \cap [1, \ldots, n])}{n} = 0.
\]

Hence using (1) the gaps in \( S \) is unbounded.

Example 8.2. The following sets have arbitrarily large gaps.

- A multiplicatively closed set generated by finitely many positive integers \( > 1 \).
- The set of all integers, which have exactly \( k \)-prime factors.
- The set of all integers, which have at most \( k \)-prime factors.

Theorem 8.3. Let \( S_1, S_2 \) be two infinite subsets of \( \mathbb{N} \). Let \( S_3 = S_1 \cup S_2, S_4 = S_1 S_2 = \{s_1 s_2 \mid s_i \in S_i, i = 1, 2\} \). Let \( S_i = \{1 < a_{i1} < a_{i2} < \ldots\} \) for \( i = 1, 2, 3, 4 \). Then

(1) \( \limsup_{j \to \infty} (a_{i(j+1)} - a_{ij}) = \infty \) for \( i = 1, 2 \neq \limsup_{j \to \infty} (a_{i(j+1)} - a_{ij}) = \infty \) for \( i = 3, 4 \).
\[(2) \lim_{j \to \infty} (a_{1(j+1)} - a_{1j}) = \infty, \limsup_{j \to \infty} (a_{2(j+1)} - a_{2j}) = \infty \text{ then} \limsup_{j \to \infty} (a_{3(j+1)} - a_{3j}) = \infty \text{ and does not imply} \limsup_{j \to \infty} (a_{4(j+1)} - a_{4j}) = \infty.\]

\[(3) \lim_{j \to \infty} (a_{i(j+1)} - a_{ij}) = \infty \text{ for} i = 1, 2 \Rightarrow \limsup_{j \to \infty} (a_{4(j+1)} - a_{4j}) = \infty.\]

**Proof.** Let us prove (1) by giving a counter example.

- Consider the set of natural numbers \(\mathbb{N}\). Decompose \(\mathbb{N}\) into two sets \(S_1, S_2\) as follows. Keep the first element of \(\mathbb{N}\) in \(S_1\). The next two elements in \(S_2\). The next three elements in \(S_1\) and so on i.e.

\[
S_1 = \bigcup_{i \geq 0} \{(2i + 1)(i + 1) - 2i, \ldots, (2i + 1)(i + 1)\}
\]

\[
S_2 = \bigcup_{i \geq 1} \{i(2i + 1) - 2i + 1, \ldots, i(2i + 1)\}
\]

Then \(S_1 \cup S_2 = \mathbb{N}\).

- Partition the set of primes \(\mathbb{P}\) into two infinite subsets of primes \(\mathbb{P}P_1, \mathbb{P}P_2\). Let \(S_i\) be the multiplicatively closed set generated by \(\mathbb{P}P_i\) for \(i = 1, 2\). Then \(S_1S_2 = \mathbb{N}\) and \(\limsup_{j \to \infty} (a_{i(j+1)} - a_{ij}) = \infty\) for \(i = 1, 2\) by an application of chinese remainder theorem.

Let us prove (2). Given any \(N > 0\) there exists \(M\) such that \(a_{1(k+1)} - a_{1k} > N\) for all \(k > M\) and there exists infinitely many \(l > M\) such that \(a_{2(l+1)} - a_{2l} > N\). Also choose large enough \(l = l_0 > M\) such that if \(a_{1k_0} > a_{2l_0}\) then \(k_0 > M\). If \(a_{2l_0} < a_{2(l_0+1)}\) are consecutive in \(S_1 \cup S_2\) then we have produced a gap more than \(N\). If \(a_{2l_0} < a_{1k_0}\) are consecutive then

- We have either \(a_{2l_0} < a_{1k_0} < a_{1(k_0+1)}\) as consecutive integers in \(S_1 \cup S_2\).

- Or \(a_{2l_0} < a_{1k_0} < a_{2(l_0+1)}\) as consecutive integers in \(S_1 \cup S_2\).

In the first case we are done again. In the second case we have either \(a_{1k_0} - a_{2l_0} > \frac{N}{2}, a_{2(l_0+1)} - a_{1k_0} > \frac{N}{2}\). Hence we have produced a gap more than \(\frac{N}{2}\). Moreover these gaps can be produced arbitrary number of times by choosing \(M\) larger and larger for any positive integer \(N\). So we have \(\limsup_{j \to \infty} (a_{3(j+1)} - a_{3j}) = \infty\).

Now for second part of (2) we give a counter example. Let \(S_1 = \{n^2 \mid n \in \mathbb{N}\}\). Let \(S_2 = \{n \in \mathbb{N} \mid n \text{ is square free}\}\). Then \(S_1 \cup S_2 = \mathbb{N}\). We have \(\lim_{j \to \infty} (a_{1(j+1)} - a_{1j}) = \infty\). Also by an application of chinese remainder theorem we have \(\limsup_{j \to \infty} (a_{2(j+1)} - a_{2j}) = \infty\).

Let us prove (3). Fix a large integer \(K\). Let \(T_1 = \{1 < a_{11} < a_{12} < \ldots < a_{1N}\}, T_2 = \{1 < a_{21} < a_{22} < \ldots < a_{2M}\}\). Suppose \(a_{1(t+1)} - a_{1t} \geq K\) for all \(t \geq N - 1\) and \(a_{2(t+1)} - a_{2t} \geq K\) for all \(t \geq M - 1\). Let \(a_{1N}a_{2M}, a_{1N}a_{2M}\) be two successive numbers in the set \(S_1 \cup S_2\). Then we have either \(\bar{N} > N\) or \(\bar{M} > M\). We note that for \(\bar{N} > N\) we have

\[
a_{1\bar{N}}a_{2\bar{M}} - a_{1N}a_{2M} \geq (a_{1\bar{N}} - a_{1N})a_{2\bar{M}} \geq K \text{ if} \bar{M} \geq M.
\]

For

\[
a_{1\bar{N}}a_{2\bar{M}} - a_{1N}b_{2M} \geq (a_{1\bar{N}} - a_{1N})a_{2M} \geq K \text{ if} M > \bar{M}.
\]

The argument is similar if \(\bar{M} > M\). This holds for any large \(K\). So \(\limsup_{j \to \infty} (a_{4(j+1)} - a_{4j}) = \infty\).

Hence we have completed the proof of this theorem. \(\square\)

**Theorem 8.4.** Let \(S_i : 1 \leq i \leq n\) be finitely many infinite subsets of \(\mathbb{N}\). Let \(S_{n+1} = \bigcup_{i=1}^n S_i, S_{n+2} = \prod_{i=1}^n S_i = \{s_1s_2\ldots s_n \mid s_i \in S_i, 1 \leq i \leq n\}\). Let \(S_i = \{1 < a_{i1} < a_{i2} < \ldots\} : 1 \leq
Let \( a \) be an arbitrary positive integer. Let \( \mathbb{N} \) be the set of natural numbers.

\[
\lim_{j \to \infty} (a_{i(j+1)} - a_{ij}) = \infty \text{ for } i, \ldots, n \text{ then } \limsup_{j \to \infty} (a_{i(j+1)} - a_{ij}) = \infty \text{ for } i = n + 1, n + 2.
\]

**Proof.** The proof of this theorem is left to the interested reader. \( \square \)

**Corollary 8.5.**
(1) The set of natural numbers \( \mathbb{N} \) cannot be written as a finite product of sets \( S_1 S_2 \ldots S_n \), where the gaps in \( S_i \) diverges to \( \infty \) for \( 1 \leq i \leq n \).

(2) The set of natural numbers \( \mathbb{N} \) cannot be written as a finite union of sets \( S_1 \cup S_2 \cup \ldots \cup S_n \), where the gaps in \( S_i \) diverges to \( \infty \) for \( 1 \leq i \leq n \).

(3) The multiplicatively closed subset \( S \) of \( \mathbb{N} \) generated by finitely many positive integers \( > 1 \) has arbitrarily large gaps.

**Theorem 8.6 (Another constructive proof).** The multiplicatively closed subset of \( \mathbb{N} \) generated by finitely many positive integers \( S \) has arbitrarily large gaps.

**Proof.** We give here another constructive proof in this Theorem. Let \( K \) be an arbitrary positive integer. Let \( n_1, n_2, \ldots, n_k \) be the generators of the multiplicatively closed set.

Define \( \lfloor \log_{n_1}(K) \rfloor = a_i \). Then we have for all

\[
t_i \geq a_i, t_i \in \mathbb{N}, n_i^{t_i+1} - n_i^{t_i} = n_i^t (n_i - 1) \geq n_i^1 \geq K.
\]

The gap between \( n_1^{t_i} n_2^{t_i} \ldots n_k^{t_i} \) and the next number \( l \) in the set \( S_1 S_2 \ldots S_k \) is at least \( K \).

Let \( l = n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k} \) be the next number. Then there is at least one \( i = i_0 \) such that \( s_i > t_i \).

Let \( a = \frac{n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k}}{n_1^{t_i} n_2^{t_i} \ldots n_k^{t_i}}, b = \frac{n_1^{t_i} n_2^{t_i} \ldots n_k^{t_i}}{n_1^{t_1} n_2^{t_1} \ldots n_k^{t_1}}. \) So we get that \( n_1^{s_1} n_2^{s_2} \ldots n_k^{s_k} - n_1^{t_1} n_2^{t_2} \ldots n_k^{t_k} = n_i^{s_i} a - n_i^{t_i} b = n_i^{t_i} (n_i^{s_i-t_i} a - b) \geq n_i^{t_i} \geq K. \)

**Note 8.7.** The difference between this constructive proof and the other constructive proof is that we do not exactly know the right end point \( l \) of this Gap-Interval as we do not know its prime factorization exactly. However we were able to locate a point \( n_1^{a_1} n_2^{a_2} \ldots n_k^{a_k} \) and a gap interval of size at least \( K \) with this integer as the left end point for every positive integer \( K > 0 \).

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