Some Properties of Posynomial Rings

Žarko Mijajlović, Miloš Milošević, Aleksandar Perović

Abstract

In this article we shall study some basic properties of posynomial rings with particular emphasis on rings $\text{Pos}(K, Q)[\bar{x}]$, and $\text{Pos}(K, Z)[\bar{x}]$. The latter ring is the well known ring of Laurent polynomials.

1 Introduction

The notion of a posynomial (positive polynomial) appeared in geometric programming as a generalization of a polynomial. Zener introduced posynomial functions about forty years ago in order to compute minimal costs (see [7]). Aside from economy and management, in the last decade posynomials have been used in optimal integral circuit design (see [5], [6] and [8]).

The applicability of posynomials essentially relies on definability of root functions in the theory of real closed fields (RCF) and on realtime procedures for quantifier elimination in RCF based on the partial cylindrical algebraic decomposition.

We shall study here some algebraic and computational properties of rings of posynomials over a commutative domain. In particular, it is proved that a posynomial ring $\text{Pos}(R, G)[\bar{x}]$ is not noetherian and it is not UFD (unique factorization domain) if $R$ is a domain and $G$ is an abelian group such that $\bigcup_{n \geq 1} G_n \neq \{0\}$, where $G_n = \bigcap_{k \in \mathbb{N}} n^k G$. Further, we introduce the posynomial Zariski topology and prove the analogues to the Hilbert’s Nullstellensatz and the real Nullstellensatz. Finally, we shall study the ideal membership problem in the posynomial rings $\text{Pos}(K, Z)[\bar{x}]$ and $\text{Pos}(K, Q)[\bar{x}]$ under assumption that $K$ is a computable domain.

2 Preliminaries and notation

Symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote respectively the sets of natural, integer, rational, real and complex numbers. Throughout this paper, we assume that $R = (R, +, \cdot, 0, 1)$ is a commutative domain with the multiplicative unit 1, $S = (S, +, 0)$ is a commutative semigroup, $G = (G, +, 0)$ is an abelian group and $K = (K, +, \cdot, 0, 1)$ is a field.
For the given function $f : S \to R$ we define its support by
$$\text{supp}(f) = \{ s \in S \mid f(s) \neq 0 \}.$$ 

The set of all functions $f : S \to R$ with finite supports we denote by $R[S]$. If $f, g \in R[S]$ and $s \in S$, an addition and a multiplication on $R[S]$ are defined by
$$(f + g)(s) = f(s) + g(s), \quad (fg)(s) = \sum_{u, v \in S, u + v = s} f(u)g(v).$$

If $0$ and $1$ are functions defined by
$$0(s) = 0, \quad 1(s) = \begin{cases} 1, & s = 0 \\ 0, & s \neq 0 \end{cases},$$
the structure $R[S] = (R[S], +, \cdot, 0, 1)$ is a commutative ring and it is called a semigroup ring (see [2] and [13]).

The ideal $I$ of the ring $R$ generated by $S \subseteq R$ will be denoted by $(S)_R$; we omit $R$ if the context is clear.

An ideal $I \subseteq R$ is real if for each sequence $r_1, \ldots, r_n$ of elements of $R$ we have that if $r_1^2 + \cdots + r_n^2 \in I$ than each $r_i$ is in $I$. For the rest of notation and definitions on real algebra we shall follow [3].

The dimension of $R$ is the maximal length of strictly increasing chains of prime ideals in $R$. More on dimension and integral elements can be found in [9] and [11].

3 Definition and basic properties

We introduce the notion of posynomial over $R$ and $S$ as a term of the form
$$\sum_{i=1}^{n} r_i x^{s_i}, \quad n \in \mathbb{N}, r_i \in R, s_i \in S,$$
where $x^0 = 1$, $x^{s_1} \cdot x^{s_2} = x^{s_1 + s_2}$. The posynomial ring over $R$ and $S$ is denoted by $\text{Pos}(R, S)[x]$, and we see that this ring is isomorphic to the semigroup ring $R[S]$. Posynomials in multiple variables are defined by induction:
$$\text{Pos}(R, S)[x_1, \ldots, x_{n+1}] = \text{Pos} (\text{Pos}(R, S)[x_1, \ldots, x_n], S)[x_{n+1}].$$

The following lemma is an easy fact on semigroup rings.

**Lemma 3.1** Let $R$ be a commutative ring, let $S$ be a commutative semigroup and suppose that $S$ has a finite cyclic subgroup. Then the ring $\text{Pos}(R, S)[\bar{x}]$ is not a domain.
Therefore, if $S$ is a finite group or if $S$ has an element of finite order, then $\text{Pos}(\mathbb{R}, S)[\bar{x}]$ is not a domain.

Let $S = (S, +, <, 0)$ be an ordered semigroup. We say that a posynomial $f(x) = \sum_{i=1}^{n} r_i x^{s_i}$, $r_i \neq 0$ is in ordered form if $s_1 < \cdots < s_n$. In particular, let $\deg(f) = s_n$ be a degree of the posynomial $f$.

**Lemma 3.2** Let $R$ be a domain and let $S$ be an ordered semigroup. Then the ring $\text{Pos}(\mathbb{R}, S)[\bar{x}]$ is a domain.

**Proof.** Observe that the product of two monomials with nonzero coefficients is not 0. Let $f = r_1 x^{s_1} + \cdots + r_n x^{s_n}$, $g = r'_1 x^{s'_1} + \cdots + r'_m x^{s'_m}$, $n > 1$ or $m > 1$, $r_i, r'_j \neq 0$, $s_1 < \cdots < s_n$ and $s'_1 < \cdots < s'_m$. Then

$$fg = r_1 r'_1 x^{s_1+s'_1} + a_n r'_m x^{s_n+s'_m} \neq 0$$

since $s_1 + s'_1 < s_n + s'_m$. We use induction to complete the claim. \[\square\]

**Corollary 3.1** Let the ring $R$ be a domain and let $G$ be a torsion free abelian group. Then $\text{Pos}(\mathbb{R}, G)[\bar{x}]$ is a domain.

**Proof.** Using the Malcev’s compactness theorem one can prove that each torsion free abelian group can be ordered, so by the previous lemma the claim follows. \[\square\]

Therefore, $\text{Pos}(\mathbb{R}, G)[\bar{x}]$ is a domain if and only if the abelian group $G$ is torsion free.

We use the same argument as in lemma 3.2 to prove:

**Theorem 3.1** Let $R$ be a domain and let $G$ be an ordered abelian group. Then units in $\text{Pos}(\mathbb{R}, G)[\bar{x}]$ are exactly monomials $r x^{s_1} \cdots x^{s_n}$, where $r$ is an invertible element of $R$.

For the given abelian group $G$ and an integer $n > 1$ let $G_n = (G_n, +, 0)$ be a subgroup of $G$ defined by

$$G_n = \bigcap_{k \in \mathbb{N}} n^k G.$$

**Theorem 3.2** Let $R$ be a domain and let $G$ be an ordered abelian group. If

$$\bigcup_{n>1} G_n \neq \{0\},$$

then $\text{Pos}(\mathbb{R}, G)[\bar{x}]$ is not noetherian.
Proof. Let \( s \in \bigcup_{n>1} G_n \setminus \{0\} \). Then there are an integer \( n > 1 \) and a sequence \( s_0, s_1, s_2, \ldots \) in \( G \) such that
\[
s = s_0 = ns_1 = n^2s_2 = n^3s_3 = \cdots .
\]
We claim that the chain
\[
\langle x_i^{s_0} - 1 \rangle \subseteq \langle x_i^{s_1} - 1 \rangle \subseteq \langle x_i^{s_2} - 1 \rangle \subseteq \cdots
\]
is strictly increasing. Note that
\[
x_i^{s_i} - 1 = x_i^{ns_i + 1} - 1 = (x_i^{s_i} - 1)(x_i^{(n-1)s_i+1} + \cdots + 1).
\]
Otherwise, let \( x_i^{s_i+1} - 1 = (x_i^{s_i} - 1) \cdot f \), where \( f \in \text{Pos}(R, G)[\bar{x}] \). Then,
\[
x_i^{s_i+1} - 1 = (x_i^{s_i} - 1) \cdot (x_i^{(n-1)s_i+1} + \cdots + 1) \cdot f,
\]
which yields that
\[
(x_i^{(n-1)s_i+1} + \cdots + 1) \cdot f = 1.
\]
This is a contradiction, since \( x_i^{(n-1)s_i+1} + \cdots + 1 \) is not a unit in the ring \( \text{Pos}(R, G)[\bar{x}] \). \( \square \)

Note that converse implication doesn’t hold. For instance, let \( G \) be a countable direct sum of copies of \( \mathbb{Z} \). Then
\[
\bigcup_{n>1} G_n = \{0\},
\]
since for each \( s \in \mathbb{Z} \) we have that \( |s| < n^{|s|} \). \( \text{Pos}(R, G)[\bar{x}] \) is isomorphic to the ring of Laurent polynomials with \( \aleph_0 \) variables, so it is not noetherian.

By the proof of the previous theorem we can conclude that \( \text{Pos}(R, Q)[\bar{x}] \) does not satisfy the ACC for principal ideals, so it cannot be UFD nor noetherian.

Let \( f = \sum_{i=1}^k c_i x_1^{s_1} \cdots x_n^{s_n} \in \text{Pos}(R, \mathbb{Z})[\bar{x}] \). We define the polynomial \( F(f) \in R[\bar{x}] \) by
\[
F(f) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \cdot f,
\]
where \( \alpha_i = \max\{-s_{1i}, \ldots, -s_{ki}\} \).

Note that \( F \) is compatible with \( \cdot \) (i.e. \( F(fg) = F(f)F(g) \)), but it is not compatible with + (for instance \( F(x+1) = x + 1 \) and \( F(1) = F(x) = 1 \)). It is easy to see that \( F(f) \) is irreducible in \( R[\bar{x}] \) if and only if \( f \) is irreducible in \( \text{Pos}(R, \mathbb{Z})[\bar{x}] \).

For an arbitrary positive integer \( m \) let us define a ring monomorphism \( \Phi_m : \text{Pos}(R, Q)[\bar{x}] \rightarrow \text{Pos}(R, Q)[\bar{x}] \) by
\[
\Phi_m(\sum_{i=1}^k c_i x_1^{s_1} \cdots x_n^{s_n}) = \sum_{i=1}^k c_i x_1^{ms_i} \cdots x_n^{ms_n}.
\]

4
Further, if \( f_1, \ldots, f_k \) are arbitrary posynomials from \( \text{Pos}(\mathbb{R}, \mathbb{Q})[x] \), then let \( \pi(f_1, \ldots, f_k) \) be the least positive integer \( m \) such that each \( \Phi_m(f_i) \) is a Laurent polynomial. It is easy to see that \( \Phi_m(f) \in \text{Pos}(\mathbb{R}, \mathbb{Z})[x] \) iff \( \pi(f)|m \), and thus

\[
\pi(f_1, \ldots, f_k) = \text{LCM}(\pi(f_1), \ldots, \pi(f_k)).
\]

Let \( f \in \text{Pos}(\mathbb{R}, \mathbb{Q})[x] \) and let \( m = \pi(f) \). Then \( f \) is atomic iff for each positive integer \( n \) the polynomial \( F(\Phi_m(f)) \) is irreducible in \( \mathbb{R}[x] \).

For example, there are no atomic elements in \( \text{Pos}(\mathbb{R}, \mathbb{Q})[x] \) and \( \text{Pos}(\mathbb{C}, \mathbb{Q})[x] \), since each polynomial of degree greater than 2 is reducible in \( \mathbb{R}[x] \), and each polynomial of degree greater than 1 is not atomic in \( \mathbb{C}[x] \). On the other hand, the posynomial \( x+2 \) is atomic in \( \text{Pos}(\mathbb{Q}, \mathbb{Q})[x] \), since each polynomial \( F(\Phi_n(x+2)) = x^n + 2 \) is by Eisenstein criterion irreducible in \( \mathbb{Q}[x] \).

Since \( \langle f_1, f_2 \rangle_{\text{Pos}(\mathbb{R}, \mathbb{Q})[x]} = \langle g \rangle_{\text{Pos}(\mathbb{R}, \mathbb{Q})[x]} \), where

\[
F(\Phi_{\pi(f_1, f_2)}(g)) = \text{GCD}(F(\Phi_{\pi(f_1, f_2)}(f_1)), F(\Phi_{\pi(f_1, f_2)}(f_2))),
\]

we see that every finitely generated ideal in \( \text{Pos}(\mathbb{K}, \mathbb{Q})[x] \) can be generated by one element.

**Example.** The ideal \( I = \langle x^{\frac{1}{n}} - 1 \mid n \in \mathbb{N} \rangle_{\text{Pos}(\mathbb{R}, \mathbb{Q})[x]} \) is prime:

suppose that \( fg \in I \); then there is a positive integer \( n \) such that \( fg \in \langle x^{\frac{1}{n}} - 1 \rangle_{\text{Pos}(\mathbb{R}, \mathbb{Q})[x]} \). Further, there is \( h \in \text{Pos}(\mathbb{R}, \mathbb{Q})[x] \) such that \( fg = h(x^{\frac{1}{n}} - 1) \). Let \( m = \pi(f, g, h, x^{\frac{1}{n}} - 1) \). Then

\[
F(\Phi_m(f))F(\Phi_m(g)) = F(\Phi_m(h))F(\Phi_m(x^{\frac{1}{n}} - 1)),
\]

so \( x - 1 \) divides at least one of polynomials \( F(\Phi_m(f)) \) and \( F(\Phi_m(g)) \); say \( F(\Phi_m(f)) \). We conclude that \( f \in \langle x^{\frac{1}{n}} - 1 \rangle_{\text{Pos}(\mathbb{R}, \mathbb{Q})[x]} \).

**Theorem 3.3** If \( \mathbb{K} \) is a field, then \( \dim(\text{Pos}(\mathbb{K}, \mathbb{Q})[x_1, \ldots, x_n]) = n \).

**Proof.** Note that for a nonzero integer \( n \) each posynomial \( x_i^{\frac{1}{n}} \) is a zero of a monic polynomial \( x_i^{\text{sgn}(n)} - x_i^{\text{sgn}(n)} \) over \( \text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}] \), so \( \text{Pos}(\mathbb{K}, \mathbb{Q})[\bar{x}] \) is an integral extension of \( \text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}] \). Hence, the dimension of the posynomial ring \( \text{Pos}(\mathbb{K}, \mathbb{Q})[\bar{x}] \) is equal to the dimension of \( \text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}] \) and since dimension is a local property we have that

\[
\dim(\text{Pos}(\mathbb{K}, \mathbb{Q})[x_1, \ldots, x_n]) = \dim(\mathbb{K}[x_1, \ldots, x_n]) = n.
\]

We observe that posynomials from \( \text{Pos}(\mathbb{K}, \mathbb{R})[\bar{x}] \) which annul some polynomial with coefficients from \( \mathbb{K}[\bar{x}] \) are exactly the elements of the ring \( \text{Pos}(\mathbb{K}, \mathbb{Q})[\bar{x}] \).

At the end of this section we discuss the possibility of functional representation of posynomials with positive rational exponents. Let \( \mathbb{K} \) be a finite field of prime characteristic \( p \). The inverse of the Frobenius automorphism \( x \mapsto x^{p^n} \).
is a unique function on $K$ which satisfies natural equalities for the $p^n$-th root function $\phi$:

$$(\phi(x))^{p^n} = x \quad \text{and} \quad \phi(xy) = \phi(x)\phi(y).$$

Let $K$ be an algebraic extension of the prime field $\mathbb{Z}_p$. Since each $a \in K$ is contained in some finite field $L$, again we conclude that there is a unique $b \in K$ such that $b^{p^n} = a$ and the corresponding $p^n$-th root function is compatible with multiplication. Note that the same is true for an algebraically closed field of characteristic $p$, since the polynomial $x^{p^n} - a$ has exactly one zero in that field.

Thus, if $K$ is an algebraically closed field of characteristic $p > 0$ or an algebraic extension of prime field $\mathbb{Z}_p$, then each posynomial $f$ in one variable over $K$ of the form

$$a_1 x^{l_1/n_1} + \cdots + a_k x^{l_k/n_k}, \ l_i, n_i \in \mathbb{N}$$

has a natural functional representation $f : K \to K$. Further, if

$$S = \{ \frac{l}{p^n} \mid l, n \in \mathbb{N} \},$$

then by $f \mapsto f$ is defined a ring homomorphism from $\text{Pos}(K, S)[\bar{x}]$ into $K^K$.

Observe that the functional representation of $x^{1/n} \in \text{Pos}(\mathbb{C}, \mathbb{Q})[x]$ determined by some branch of the $n$-th root is not compatible with multiplication in $\mathbb{C}$.

### 4 Laurent polynomials

Let $K$ be an arbitrary field of characteristic 0. The ring of Laurent polynomials over $K$ (in variables $x_1, \ldots, x_n$) is the ring $\text{Pos}(K, \mathbb{Z})[\bar{x}]$. Note that $\text{Pos}(K, \mathbb{Z})[\bar{x}]$ is just the localization of $K[\bar{x}]$ at $x_1 \cdots x_n$, so it is noetherian, UFD and a graded ring.

We define the Zariski topology for Laurent polynomials in a similar way as in the case of polynomial Zariski topology. Let

$$K^n_{\neq 0} = \{ (a_1, \ldots, a_n) \in K^n \mid a_1 \cdots a_n \neq 0 \}.$$ 

Each Laurent polynomial $f = \sum_{i=1}^k c_1 x_1^{s_1} \cdots x_n^{s_n}$ defines an unique function $f : K^n_{\neq 0} \to K$ in a quite natural way:

$$f(a_1, \ldots, a_n) = \sum_{i=1}^k c_1 a_1^{s_1} \cdots a_n^{s_n}.$$ 

Note also that the mapping $f \mapsto f$ is an embedding of the ring $\text{Pos}(K, \mathbb{Z})[\bar{x}]$ into the ring $K^{K^n}_{\neq 0}$.

Let $S \subseteq \text{Pos}(K, \mathbb{Z})[\bar{x}]$ be an arbitrary set of Laurent polynomials. A posynomial set in $K^n_{\neq 0}$ generated by $S$ is the set

$$V_{\text{Pos}}(S) = \{ (a_1, \ldots, a_n) \in K^n_{\neq 0} \mid (\forall f \in S) f[\bar{a}] = 0 \}.$$
First, let us observe that $K^n_{\neq 0}$ is the Zariski open set in affine space $K^n$ given as the complement of the Zariski closed set $V(x_1 \cdots x_n)$. Further,

$$V_{\text{Pos}}(S) = \bigcap_{f \in S} V(F(f)) \cap K^n_{\neq 0},$$

so the posynomial sets (which are the base closed sets in the posynomial Zariski topology) are closed in the induced topology on the open subset $K^n_{\neq 0}$ of the Zariski topology on $K^n$. Thus we can immediately conclude that $K^n_{\neq 0}$ is a Fréchet space (in the posynomial Zariski topology) and each posynomial function is continuous. Further, since each two nonempty Zariski open sets meet each other, the same will obviously hold for each two nonempty posynomial Zariski open sets, thus $K^n_{\neq 0}$ is not a Hausdorff space. The compactness of $K^n_{\neq 0}$ can be shown exactly in the same way as for $K^n$ with polynomial Zariski topology.

As dual notion to posynomial sets, for an arbitrary set $X \subseteq K^n_{\neq 0}$ let

$$I_{\text{Pos}}(X) = \{ f \in \text{Pos}(K, \mathbb{Z})[\mathbf{x}] \mid (\forall (a_1, \ldots, a_n) \in X)f[\mathbf{a}] = 0 \}.$$

The ring $\text{Pos}(K, \mathbb{Z})[\mathbf{x}]/I_{\text{Pos}}(X)$ is reduced. In particular, $I_{\text{Pos}}(X)$ is a radical ideal. The next two results are analogues of the corresponding polynomial theorems. The argument is similar, so we give only the proof of real Nullstellensatz.

**Theorem 4.1 (Nullstellensatz for Laurent Polynomials)** Let $K$ be an algebraically closed field and let $I$ be an arbitrary ideal in $\text{Pos}(K, \mathbb{Z})[\mathbf{x}]$. Then $V_{\text{Pos}}(I) \neq \emptyset$ if and only if $I$ is a proper ideal.

**Remark.** The Hilbert’s Nullstellensatz does not hold in $\text{Pos}(\mathbb{C}, \mathbb{Q})[\mathbf{x}]$.

First let us observe that the function which maps $x^\frac{1}{n}$ to the principal branch of the $n$-th root function is a ring embedding of $\text{Pos}(\mathbb{C}, \mathbb{Q})[\mathbf{x}]$ into $\mathbb{C}[x^0]$.

Then $I(V((x^\frac{1}{n} + 1))) = I(\emptyset) = \text{Pos}(\mathbb{C}, \mathbb{Q})[\mathbf{x}]$, but $1 \notin \text{rad}(x^\frac{1}{n} + 1)$.

**Theorem 4.2 (Real Nullstellensatz for Laurent polynomials)** Let $K$ be a real closed field and let $I$ be an ideal in $\text{Pos}(K, \mathbb{Z})[\mathbf{x}]$. Then

$$I = I_{\text{Pos}}(V_{\text{Pos}}(I))$$

if and only if $I$ is a real ideal.

**Proof.** We will consider only nontrivial direction. Suppose that $I$ is a real ideal; then it is a radical ideal and can be represented as a finite intersection of prime ideals $I_1, \ldots, I_k$ in $\text{Pos}(K, \mathbb{Z})[\mathbf{x}]$. Clearly,

$$I \subseteq I_{\text{Pos}}(V_{\text{Pos}}(I)).$$

Let $f \in I_{\text{Pos}}(V_{\text{Pos}}(I)) \setminus I$; for instance, let $f \notin I_1$. The ring $\text{Pos}(K, \mathbb{Z})[\mathbf{x}]$ is noetherian, so there are $f_1, \ldots, f_k \in I_1$ such that $I_1 = \langle f_1, \ldots, f_k \rangle$. Since each prime ideal is real, the field

$$K_1 = \mathbb{Q}(\text{Pos}(K, \mathbb{Z})[\mathbf{x}]/I_1)$$


is real. Let $K_2$ be a real closure of $K_1$. Each $x_i$ is invertible in $\text{Pos}(K, \mathbb{Z})[\bar{x}]$, so $x_i + I_1 \neq I_1$ and $(x_1 + I_1, \ldots, x_n + I_1)$ is a witness for

$$K_2 \models \exists \bar{v}(F(f)(\bar{v}) \neq 0 \land \bigwedge_{i=1}^{n} F(f_i)(\bar{v}) = 0 \land \bigwedge_{i=1}^{n} v_i \neq 0).$$

The submodel completeness of the theory of real closed fields yields

$$K \models \exists \bar{v}(F(f)(\bar{v}) \neq 0 \land \bigwedge_{i=1}^{n} F(f_i)(\bar{v}) = 0 \land \bigwedge_{i=1}^{n} v_i \neq 0),$$

which contradicts the fact that $V_{\text{Pos}}(f) \supseteq V_{\text{Pos}}(I) \supseteq V_{\text{Pos}}(I_1)$.

\section{5 Posynomials over computable fields}

From now on we will assume that $K$ is a computable field of characteristic 0.

\textbf{Lemma 5.1} Let $p_0, p_1, \ldots, p_n$ be arbitrary distinct prime numbers and let $f_i = x^{p_i} - 1$. Then

$$f_0 \notin \langle f_1, \ldots, f_n \rangle_{\text{Pos}(K, \mathbb{Q})[x]}.$$

\textbf{Proof.} Otherwise, there are posynomials $g_1, \ldots, g_k \in \text{Pos}(K, \mathbb{Q})[x]$ such that

$$f_0 = g_1 f_1 + \cdots + g_n f_n.$$

Let $m = \pi(f_0, f_1, \ldots, f_n, g_1, \ldots, g_n)$. Then there are unique positive integers $s$ and $d$ such that $m = p_0^s d$ and $\text{GCD}(p_0, d) = 1$. For an arbitrary $i > 0$ we have that

$$\Phi_m(f_i) = x^{s d^i/p_i} - 1 = (x^{p_0} - 1)(x^{p_0}(s - 1) + x^{p_0}(s - 2) + \cdots + 1)$$

and each $s^{i - j}$ is an integer, so $\Phi_m(f_i)$ is divisible by $x^{p_0} - 1$ in $\text{Pos}(K, \mathbb{Z})[x]$. But $\Phi_m(f_0)$ is not divisible by $x^{p_0} - 1$, and we obtain a contradiction. \qed

We see that $x^{s d^i/p_i} - 1$ is not a member of the posynomial ideal generated by the set

$$B = \{x^{s d^i/p_i} - 1 \mid p \in A\},$$

where $p_0 \notin A$ and each member of $A$ is a prime number. This is a consequence of the fact that for each ideal $I$, $a \in I$ if and only if $a$ can be represented as a finite sum of the form $\sum_{i=1}^{k} b_i a_i$, where $a_i$ belong to the set of generators for $I$.

\textbf{Theorem 5.1} The problem of ideal membership in $\text{Pos}(K, \mathbb{Q})[x]$ (for the given computable field $K$) is not decidable, i.e. there is a nonrecursive ideal in the ring $\text{Pos}(K, \mathbb{Q})[x]$. 

8
**Proof.** Let $A$ be a nonrecursive subset of $\mathbb{N}$ and let $I$ be a posynomial ideal generated by the set

$$B = \{x^i - 1 \mid i \in A\},$$

where $p_0, p_1, p_2, \ldots$ is an increasing enumeration of prime numbers. Then, by the previous lemma

$$x^i - 1 \in I$$

if and only if $i \in A$.

So, any algorithm which decides the predicate “$x^i - 1 \in I$” will also decide the predicate “$i \in A$” contradicting the fact that $A$ is a nonrecursive set. □

In the rest of this section we will describe one test for the membership to finitely generated ideals in $\text{Pos}(\mathbb{K}, \mathbb{Q})[\bar{x}]$.

**Theorem 5.2** Let $\mathbb{K}$ be a computable field. The question of ideal membership in the ring of Laurent polynomials $\text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}]$ is decidable. Moreover, there is an algorithm for testing the membership to finitely generated ideals in $\text{Pos}(\mathbb{K}, \mathbb{Q})[\bar{x}]$.

**Proof.** Let $I = \langle f_1, \ldots, f_n \rangle_{\text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}]}$ be an ideal in $\text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}]$. We notice that:

$$g \in \langle f_1, \ldots, f_k \rangle_{\text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}]}$$

if and only if

$$(x_1 \cdots x_n)^\lambda F(g) \in \langle F(f_1), \ldots, F(f_k) \rangle_{\mathbb{K}[\bar{x}]},$$

for some $\lambda \in \mathbb{N}$.

We can write (*) using the saturation ideal of $\langle F(f_1), \ldots, F(f_n) \rangle_{\mathbb{K}[\bar{x}]}$ by $x_1 \cdots x_n$:

$$g \in \langle f_1, \ldots, f_k \rangle_{\text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}]} \iff F(g) \in \langle F(f_1), \ldots, F(f_k) \rangle_{\mathbb{K}[\bar{x}]} \cap (x_1 \cdots x_n)$$

If $J$ is an ideal in $\mathbb{K}[x_1, \ldots, x_n]$, $h \in \mathbb{K}[\bar{x}]$ and $y$ a new variable, then

$$J : h^\infty = \langle J, 1 - yh \rangle \cap \mathbb{K}[\bar{x}],$$

where $J : h^\infty$ is an ideal in $\mathbb{K}[\bar{x}]$ and $\langle J, 1 - yh \rangle$ is an ideal in $\mathbb{K}[\bar{x}, y]$. The Gröbner basis of $J : h^\infty$ with respect to the lexicographical order $x_1 < \cdots < x_n$ is equal to the intersection of $\mathbb{K}[\bar{x}]$ and the Gröbner basis $B$ of $\langle J, 1 - yh \rangle$ with respect to the lexicographical order $x_1 < \cdots < x_n < y$ (see [2]).

Now we have an algorithm for testing whether $g \in \langle f_1, \ldots, f_k \rangle_{\text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}]}$ or not:

First we will find the Gröbner basis $B$ (with the respect to the lexicographical order) of $\langle F(f_1), \ldots, F(f_n), 1 - yx_1 \cdots x_n \rangle \subseteq K[\bar{x}, y]; B \cap K[\bar{x}] = B_1$ will be the Gröbner basis of

$$\langle F(f_1), \ldots, F(f_n) \rangle : (x_1 \cdots x_n)^\infty.$$

We divide $F(g)$ by $B_1$ in the lexicographical order; if the remainder is 0 then $g \in \langle f_1, \ldots, f_k \rangle_{\text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}]}$, otherwise $g \notin \langle f_1, \ldots, f_k \rangle_{\text{Pos}(\mathbb{K}, \mathbb{Z})[\bar{x}]}$.

We prove the existence of a procedure for testing ideal membership to finitely generated ideal in $\text{Pos}(\mathbb{K}, \mathbb{Q})[\bar{x}]$, where $\mathbb{K}$ is a computable field. Let the ideal
$J \subseteq \text{Pos}(K, \mathbb{Q})[x_1, \ldots, x_n]$ be generated by $f_1, \ldots, f_m$ and let $g$ be an arbitrary posynomial in $\text{Pos}(K, \mathbb{Q})[\bar{x}]$.

Suppose that $g \in J$. There exist $h_1, \ldots, h_m \in \text{Pos}(K, \mathbb{Q})[\bar{x}]$ such that

$$g = h_1 f_1 + \ldots + h_m f_m.$$  \hfill (1)

We will write down all exponents which appear in $g, f_1, \ldots, f_m$ in the form $\frac{p_i}{q_i}$, GCD($p_i, q_i$) = 1, $p_i \in \mathbb{Z}$, $q_i \in \mathbb{N}$, the exponents which appear in $h_1, \ldots, h_m$ in the form $\frac{k_i}{l_i}$, where $k_i$ and $l_i$ are relatively prime, and we will denote the least common multiple of denominators $q_i$ by $s$ (note that $s = \pi(g, f_1, \ldots, f_m)$).

Now, we rewrite exponents $\frac{p_i}{q_i}$ in the form $\frac{t_i}{s}$. Assume that the posynomial $h_1$ contains a monomial $M$ with variable $x_i$ to the power $\frac{k_{i_0}}{l_{i_0}}$, $l_{i_0} \nmid s$. Then, the product $h_1 f_1$ contains monomial $M_1$ with variable $x_i$ to the power $a = \frac{k_{i_0}}{l_{i_0}} + \frac{t_{j_0}}{s} = \frac{k_{i_0} s + l_{i_0} t_{j_0}}{l_{i_0} s}$.

Since $l_{i_0} \nmid s$ and GCD($k_{i_0}, l_{i_0}$) = 1 we conclude that $a$ is not of the form $\frac{t}{s}$ and that the monomial $M_1$ cannot appear on the left side of the equation (1). We thus obtain that all monomials with the same property as $M_1$ must cancel and that $g$ can be expressed as

$$g = \tilde{h}_1 f_1 + \ldots + \tilde{h}_m f_m,$$

where all denominators $l_i$ of exponents $\frac{k_i}{l_i}$ which occur in $\tilde{h}_1, \ldots, \tilde{h}_m$ divide $s$. Thus

$$g \in \langle f_1, \ldots, f_m \rangle_{\text{Pos}(K, \mathbb{Q})[\bar{x}]} \text{ iff } \Phi_s(g) \in \langle \Phi_s(f_1), \ldots, \Phi_s(f_m) \rangle_{\text{Pos}(K, \mathbb{Q})[\bar{x}]}.$$  \hfill \Box

References

[1] M. Aschenbrenner, *Ideal Membership in Polynomial Rings over the Integers*, J. Amer. Math. Soc 17(2004), 407-441

[2] T. Becker, V. Weispfening, *Gröbner Bases - a Computational Approach to Commutative Algebra*, Springer-Verlag, second printing 1998.

[3] J. Bochnak, M. Coste, M-F. Roy, *Real Algebraic Geometry*, Springer-Verlag 1998

[4] C. C. Chang, H. J. Keisler, *Model Theory*, Third Edition, North–Holland 1990

[5] J. Cong, *An interconnect-centric design flow for nanometer technologies*, Proc. of the IEEE, vol 89, no.4, 2001
[6] J. Dawson, S. Boys, T. Lee, M. Hershenson, *Optimal allocation of local feedback in multistage amplifiers via geometric programming*, IEEE transactions on circuits and systems I, 2000

[7] R. J. Duffin, C. Zener, E. L. Peterson, *Geometric Programming: Theory and Application*, John Wiley & Sons, 1967.

[8] T. Eeckelaert, W. Daems, G. Gielen, W. Sansen, *Generalized Posynomial Performance Modeling*, DATE 2003, IEEE Computer Society, 2003

[9] D. Eisenbud, *Commutative Algebra with a view Toward Algebraic Geometry*, Springer–Verlag, 1995

[10] D. Marker, *Model Theory: An Introduction*, Springer-Verlag, 2002

[11] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, 1986

[12] Ž. Mijajlović, Z. Marković, K. Došen, *Hilbertovi problemi i logika*, Zavod za udžbenike i nastavna sredstva–Beograd 1986

[13] D. Passman, *The Algebraic Structure of Group Rings*, John Wiley and Sons, 1977

[14] V. V. Prasolov, *Polynomials*, MCNMO 2003 (in Russian)

ŽARKO MIJAJLOVIĆ  
FACULTY OF MATHEMATICS  
UNIVERSITY OF BELGRADE  
STUDENTSKI TRG 16, 11000 BEOGRAD  
SERBIA AND MONTENEGRO  
E-mail: zarkom@eunet.yu

MILOŠ MILOŠEVIĆ  
MATHEMATICAL INSTITUTE  
SERBIAN ACADEMY OF SCIENCES AND ARTS  
KNEZA MIHAILA 35, 11001 BEOGRAD  
SERBIA AND MONTENEGRO  
E-mail: mionamil@eunet.yu

ALEKSANDAR PEROVIĆ  
MATHEMATICAL INSTITUTE  
SERBIAN ACADEMY OF SCIENCES AND ARTS  
KNEZA MIHAILA 35, 11001 BEOGRAD  
SERBIA AND MONTENEGRO  
E-mail: peramail314@yahoo.com