Presentations of subgroups of the braid group generated by powers of band generators

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Abstract

According to the Tits conjecture proved by Crisp and Paris, [CP], the subgroups of the braid group generated by proper powers of the Artin elements \( \sigma_i \) are presented by the commutators of generators which are powers of commuting elements. Hence they are naturally presented as right-angled Artin groups.

The case of subgroups generated by powers of the band generators \( a_{ij} \) is more involved. We show that the groups are right-angled Artin groups again, if all generators are proper powers with exponent at least 3. We also give a presentation in cases at the other extreme, when all generators occur with exponent 1 or 2, which is far from being that of a right-angled Artin group.

1 Introduction

Inspired by the Tits conjecture and its solution by Crisp and Paris, [CP], we investigate subgroups of the braid groups \( \text{Br}_n \) generated by powers of the band generators \( a_{ij} \). They are the generators in the ‘dual’ or BKL presentation of \( \text{Br}_n \):

\[
\left\langle a_{ij}, 1 \leq i < j \leq n \right| a_{ij}a_{kl} = a_{kl}a_{ij} \quad \text{if} \quad (k-i)(k-j)(l-i)(l-j) > 0 \\
\quad a_{ij}a_{ik} = a_{jk}a_{ij} = a_{ik}a_{jk} \quad \text{if} \quad 1 \leq i < j < k \leq n. \right.
\]

and can be identified with the following braid diagrams:

\[
\begin{array}{ccccccccccc}
n & n-1 & j+1 & j & j-1 & i+1 & i & i-1 & 2 & 1 \\
a_{ij} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

So for a given symmetric matrix \( M = (m_{ij}) \) of exponents, the Coxeter datum, we define

\[
E_{(m_{ij})} = \langle a_{ij}^{m_{ij}}, 1 \leq i < j \leq n \rangle \subset \text{Br}_n
\]
As usual a vanishing exponent $m_{ij} = 0$ yields the identity element, thus $E_M$ is in fact generated by the powers of band generators $a_{ij}$ with $m_{ij} \neq 0$.

In that generality the groups $E_M$ were introduced by Kluitmann, cf. [Kl], in conjunction with the more prominent Coxeter and Artin group associated with $M$:

$$C_M \cong \langle s_1, \ldots, s_n \mid s_i^2, (s_is_j)^{m_{ij}} \rangle$$

$$A_M \cong \langle t_1, \ldots, t_n \mid \underbrace{t_it_jt_i \ldots}^{m_{ij} \text{ factors}}, \underbrace{t_jt_it_j \ldots}^{m_{ij} \text{ factors}} \rangle$$

(Note that our entries $m_{ij} = 0$ in the Coxeter datum serve the same purpose as entries $m_{ij} = \infty$, which are more commonly used.)

For $M$ of large type, ie. $m_{ij} \neq 1, 2$, Kluitmann proved (and conjectured for non-redundant $M$, ie. $m_{ij} \neq 1$) that $E_M$ is the stabiliser of $(s_1, \ldots, s_n)$ for the natural action of $\text{Br}_n$ on $C_M \times \cdots \times C_M$. If $m_{ij} = 1$ is not excluded, $E_M$ may only be expected to be the stabiliser of $(t_1, \ldots, t_n)$ for the action on $A_M \times \cdots \times A_M$, cf. [Lo1].

For $M$ of ADE-type Dörner [Dö] proved Kluitmanns conjecture. The stabiliser group and hence $E_M$ has been shown by Looijenga, cf [Lo] and others to be the fundamental group of the complement of the bifurcation set in the truncated versal unfolding of the hypersurface singularity of the same type (ADE).

In any case $E_M$ as a subgroup of a braid group $\text{Br}_n$ acts freely and discontinuously on a contractible CW-complex of dimension $n-1$, and therefore deserves our special interest.

While we would like to find a presentation of $E_M$ in terms of its natural generators in general, we can provide two partial results in this paper.

Our first result is to give the presentation of $E_M$ in case of matrices $M$ of large type:

**Theorem 1.1** In case of matrices $M$ of large type, ie. $m_{ij} \neq 1, 2$ for all $1 \leq i < j \leq n$, the elements $b_{ij} = a_{ij}^{m_{ij}}$ with $m_{ij} \geq 3$ generate the subgroup $E_M$ with presentation

$$\left\langle b_{ij}, m_{ij} \neq 0 \mid b_{ij}b_{kl} = b_{kl}b_{ij}, \text{ if } a_{ij}a_{kl} = a_{kl}a_{ij} \right\rangle.$$

If $m_{ij} = 0$ for all $i, j$ such that $|i - j| \geq 2$ this covers the braid group case of the theorem of Crisp and Paris.

Second we determine a presentation in the cases at the other extreme, when only exponents 1 and 2 are allowed. In this case several distinct matrices may lead to the same subgroup $E_M$. Therefore we include a condition on $M$ which give a distinguished matrix in each case.

**Theorem 1.2** Given a symmetric matrix $M$ with entries $m_{ij} \in \{1, 2\}$ such that $m_{ik}$ is equal 1 if $m_{ij}$ and $m_{jk}$ are, a presentation of $E_M$ generated by $b_{ij} = a_{ij}^{m_{ij}}$ is obtained imposing the following relations:
i) \[ b_i b_j = b_j b_i \] for \( i < j < k < l \) or \( j < k < i < l \)

ii) \[ b_j a_{kl}^2 b_{ik} a_{kl}^{-2} = a_{kl}^2 b_{ik} a_{kl}^{-2} b_{jl} \] for \( i < j < k < l \) and \( a_{kl}^2 = b_{kl} \) resp. \( b_{kl}^2 \), if \( m_{kl} = 2 \) resp. 1.

iii) \[ b_j b_i = b_j b_i, b_{ij} b_{ik} b_{jk} = b_{ik} b_{jk} b_{ij} \] for \( i, j, k \) in cyclic order and \( m_{ij} = 1 \), \( m_{ik} = m_{jk} = 2 \).

iv) \[ b_j b_{ik} b_{jk} = b_j b_{ik} b_{jk} = b_{ik} b_{jk} b_{ij} \] for \( i < j < k \) and \( m_{ij} = m_{ik} = m_{jk} = 2 \),

v) \[ b_{ij} b_{ik} = b_{jk} b_{ij} = b_{ik} b_{jk}, \] for \( i < j < k \) and \( m_{ij} = m_{ik} = m_{jk} = 1 \),

Such groups \( E_M \) can be identified with subgroups of \( \text{Br}_n \) preserving a partition \( P \) of \( \{1, ..., n\} \) via the natural surjection \( \text{Br}_n \to S_n \). These groups were considered before by Manfredini [Man] who gave presentations which are more economical in terms of generators than ours. In contrast it is much easier to state and prove the relations of our presentations.

In particular the pure braid group \( \text{PBr}_n \) corresponds to the matrix \( M \) with all entries \( m_{ij} = 2 \). We thus get a presentation of \( \text{PBr}_n \), which differs from the classical presentation of Artin [Ar] but recovers in fact the earlier one by Burau [Bu].

Of the following three sections we devote the first two to the proofs of the two theorems, while in the last sections we want to suggest an approach in the case with \( M \) of ADE-type.

## 2 The case of matrices of large type

In their approach Crisp and Paris exploit a geometric action of Artin groups on a fundamental groupoid and the solution of the word problem for right-angled Artin groups.

We follow their strategy but in place of their action we pick a natural geometric representation of \( \text{Br}_n \), the identification with the mapping class group of the punctured disc. Then we get an induced action on the fundamental group of the punctured disc, which we identify with the free group \( F_n = \langle t_1, ..., t_n \mid \rangle \). The right Hurwitz actions of \( \text{Br}_n \) on \( F_n \) and the universal Coxeter quotient \( \tilde{C}_n = \langle s_1, ..., s_n \mid s_i^2 = 1 \rangle \) are given by

\[
(t_i)\sigma_j = \begin{cases} 
  t_{i-1} & \text{if } i = j + 1 \\
  t_i t_{i+1} t_i^{-1} & \text{if } i = j \\
  t_i & \text{else}
\end{cases} \quad \text{and} \quad (s_i)\sigma_j = \begin{cases} 
  s_{i-1} & \text{if } i = j + 1 \\
  s_i s_{i+1} s_i & \text{if } i = j \\
  s_i & \text{else}
\end{cases}
\]

We must now define the terms to describe some decisive features of the action of generators \( a_{jk}^a \) on elements in \( \tilde{C}_n \), which we represent by words without repetition in the alphabet \( s_1, ..., s_n \). Occasionally a transformation rule will result in more general words, but reduction, the process of removing a pair of adjacent identical letters, will unambiguously stop at the representing word without repetitions.
Definition A subword $w_0$ of a word $w$ in the alphabet $s_1, \ldots, s_n$ is called a $jk$-subword if it consists of letters $s_j$ and $s_k$ only, it is called long if it contains at least four letters.

Definition The $jk$-factorisation of a word $w$ is given by the unique factorisation

$$w = w_0s_{i_1}w_1 \cdots s_{i_\ell}w_{\ell},$$

such that each $w_\nu$ is a $jk$-subword of $w$, possibly of length 0, and such that $1 \leq i_\nu \leq n$, $i_\nu \neq j, k$ for all $\nu$.

Definition A $jk$-subword $w_\nu$ is called critical if either of the following condition holds

i) the length $|w_\nu|$ is odd and $w_\nu$ has two identical adjacent letters, distinct from $s_j, s_k$,

ii) the length $|w_\nu|$ is even, possibly 0, and $w_\nu$ has adjacent letters $s_i$ and $s_l$ with $il$ and $jk$ crossing, that is $j, k$ are either in the even or the odd positions for the natural order of the four numbers $i, j, k, l$.

Note that being critical is not an absolute notion but depends on the 'embedding' as a subword, more precisely on the adjacent letters.

Proposition 2.1 Suppose that a word $w$ has a $jk$-factorisation $w_0s_{i_1}w_1 \cdots s_{i_\ell}w_{\ell}$ with no $jk$-subword $w_\nu$ of length $2|m|$. Then the $jk$-factorisation of $(w)a_j^m$ can be written as

$$w'_0s_{i_1}w'_1 \cdots s_{i_\ell}w'_{\ell},$$

where $w'_\nu$ is critical with $|w'_\nu| + |w_\nu| \geq 2|m|$ if $w_\nu$ is critical.

Proof: From the Hurwitz action we deduce the following action of a band generator and its powers on any letter using the identity $(s_j s_k) = (s_k s_j)^{-1}$:

$$(s_i)a_j^m = \begin{cases} s_i, & i < j \text{ or } i > k \\ s_j(s_j s_k)^{-m} = (s_j s_k)^m s_j, & i = j \\ (s_j s_k)^m s_i (s_j s_k)^{-m}, & j < i < k \\ (s_j s_k)^m s_k = s_k (s_j s_k)^{-m}, & i = k \end{cases}$$

They imply the following transformation of various $jk$-subwords, $p \geq 0$:

$$(s_j s_k)^p a_j^m = (s_j s_k)^p$$

$$(s_k s_j)^p a_j^m = (s_j s_k)^{-p}$$

$$(s_j s_k)^p s_j a_j^m = (s_j s_k)^p (s_j s_k)^m s_j = (s_j s_k)^{p+m} s_j$$

$$(s_k s_j)^p s_k a_j^m = (s_j s_k)^{-p} (s_j s_k)^m s_k = (s_j s_k)^{-p+m} s_k$$

Hence we obtain the word $w' = (w)a_j^m$ by first applying these transformation to the factors $w_\nu, s_\nu$ and then removing adjacent pairs of identical letters.
By the first step we get in general a non-reduced word. Its $jk$-factorisation has the same number of terms as that of $w$, since the number of letters distinct from $s_j, s_k$ is preserved.

In the next step we want to remove all repetitious pairs in the $jk$-subwords. Each $jk$-subword between $s_{i_\nu}$ and $s_{i_{\nu+1}}$ consist of contributions from three sources:

i) the first from $s_{i_\nu}$: $(s_j s_k)^{-m}$ if $j < i_\nu < k$, nothing otherwise.

ii) the second from $w_{\nu}$: $w_{\nu}$ if $|w_{\nu}|$ is even, $(s_j s_k)^m w_{\nu}$ if $|w_{\nu}|$ is odd.

iii) the third from $s_{i_{\nu+1}}$: $(s_j s_k)^m$ if $j < i_{\nu+1} < k$, nothing otherwise.

So the $jk$-subwords $w'_{\nu}$ may be written as either $(s_j s_k)^m w_{\nu}$, $(s_j s_k)^{-m} w_{\nu}$ or $w_{\nu}$. Reducing further we get eventually the $jk$-word $w'_{\nu}$, which may be empty only if $w_{\nu}$ is since $|w_{\nu}| \neq 2|m|$ is assumed.

But then also the word $w'_0 s_{i_1} w'_1 \cdots s_{i_\ell} w'_\ell$ has no repetitions, thus our claim on the $jk$-factorisation of $w'$ is proved.

We are left to consider the two particular cases when $w_{\nu}$ is critical. If $i_\nu = i_{\nu+1}$ and $|w_{\nu}|$ is odd, then $w'_{\nu} = (s_j s_k)^{\pm m} w_{\nu}$, hence $w'_{\nu}$ is critical and $|w_{\nu}| + |w'_{\nu}| \geq 2|m|$. The same conclusion holds if $i_\nu i_{\nu+1}$ and $jk$ are crossing and $|w_{\nu}|$ is even.

**Proposition 2.2** Suppose that $w$ is without long $il$-subword while $(w) a^m_{il}$, with $|m| \geq 3$ has a long $jk$-subword. Then one of the following two conditions holds:

i) $il$ and $jk$ are identical,

ii) $il$ and $jk$ are non-crossing and $w$ has a long $jk$-subword.

**Proof:** By the previous proposition we may write the $il$-factorisations of $w$ and its image $w' = (w) a^m_{il}$ as

$$w_0 s_{i_1} w_1 \cdots s_{i_\ell} w_\ell \quad \text{resp.} \quad w'_0 s_{i_1} w'_1 \cdots s_{i_\ell} w'_\ell.$$  

Any long $jk$-subword in $w'$ is either

i) a long $il$-word, if $il$ and $jk$ are identical,

ii) an alternating sequence of at least two $il$-words of length 1 and two copies of a letter not in $\{s_i, s_l\}$, if $il$ and $jk$ are neither identical nor disjoint,

iii) an alternating sequence of letters $s_j, s_k$ interspersed by $il$-subwords of length 0, if $il$ and $jk$ are disjoint.

The first case is accounted for by the claim.

In the second case $w'$ contains a maximal $il$-subword $w'_{\nu}$ of length 1 with identical adjacent letters. Hence $w'_{\nu}$ is critical. But then by prop. 2.1 also $w_{\nu}$ must be critical and of length $|w_{\nu}| \geq 2|m| - |w'_{\nu}| \geq 5$ in violation of our assumptions.
In the last case each interspersed $il$-subword $w'_v$ may not be critical, otherwise it would originate in a critical $il$-subword $w_v$ of length $|w_v| \geq 2|m| - |w'_v| \geq 6$ which again is excluded by our assumption. Since the adjacent letters are $s_j$ and $s_k$, we conclude that $il$ and $jk$ are non-crossing, as claimed.

For the ensuing argument we introduce some notation to handle elements of the right-angled Artin groups $G_M$ defined on generators in bijection to elements of $T_M$:

\[ G_M = \{ b_\tau, \tau \in T_M \mid b_\sigma b_\tau = b_\sigma b_\tau \text{ if } \tau, \sigma \text{ are non-crossing} \} \]

\[ T_M = \{ ij \mid 1 \leq i < j \leq n, m_{ij} \neq 0 \} \]

An expression in the letters $b_\sigma$ is a sequence with non-vanishing exponents $p_i$

\[ W = (b_{\sigma_1}^{p_1}, b_{\sigma_2}^{p_2}, \ldots, b_{\sigma_\ell}^{p_\ell}) \]

The index $\ell = \ell(W)$ is called the length of the expression $W$. Given $\beta$ in $E_M$ we call $W$ an expression for $\beta$ if

\[ \beta = a_{\sigma_1}^{p_1m_{\sigma_1}} \cdots a_{\sigma_\ell}^{p_\ell m_{\sigma_\ell}} \]

As in [CP] the following terminology is based on Brown. Consider an expression $W$ as above. Suppose that there exists $i \in \{1, \ldots, \ell\}$ such that $b_{\sigma_i} = b_{\sigma_{i+1}}$, put

\[ W' = \begin{cases} (b_{\sigma_1}^{p_1}, b_{\sigma_2}^{p_2}, \ldots, b_{\sigma_{i-1}}^{p_{i-1}}, b_{\sigma_i}^{p_i+p_{i+1}}, b_{\sigma_{i+2}}^{p_{i+2}}, \ldots, b_{\sigma_\ell}^{p_\ell}) & \text{if } p_i + p_{i+1} \neq 0, \\ (b_{\sigma_1}^{p_1}, b_{\sigma_2}^{p_2}, \ldots, b_{\sigma_{i-1}}^{p_{i-1}}, b_{\sigma_{i+2}}^{p_{i+2}}, \ldots, b_{\sigma_\ell}^{p_\ell}) & \text{if } p_i + p_{i+1} = 0. \end{cases} \]

We say that $W'$ is obtained from $W$ via an elementary operation of type $I$. This operation shortens the length of an expression by 1 or 2.

Suppose that there exists $i \in \{1, \ldots, \ell-1\}$ such that $\sigma_i, \sigma_{i+1}$ are non-crossing. Put

\[ W'' = (b_{\sigma_1}^{p_1}, b_{\sigma_2}^{p_2}, \ldots, b_{\sigma_{i-1}}^{p_{i-1}}, b_{\sigma_{i+1}}^{p_{i+1}}, b_{\sigma_i}^{p_i}, b_{\sigma_{i+2}}^{p_{i+2}}, \ldots, b_{\sigma_\ell}^{p_\ell}). \]

We say that $W''$ is obtained from $W$ by an elementary operation of type $II$. This operation leaves the length of an expression unchanged.

We shall say that $W$ is $M$-reduced if the length of $W$ can not be reduced by applying a sequence of elementary operations. Clearly every element of the right-angled presented group has a reduced expression.

A reduced expression $W$ is said to end in $\tau \in T_M$ if it is related by a sequence of operations of type $II$ to an expression $(b_{\sigma_1}^{p_1}, b_{\sigma_2}^{p_2}, \ldots, b_{\sigma_\ell}^{p_\ell})$ in which $\sigma_\ell = \tau$.

**Proposition 2.3** Suppose $X = (x_1, \ldots, x_\ell), x_\nu = b_{\sigma_\nu}^{p_\nu}$, is a $M$-reduced expression for $\beta \in E_M \subset Br_n$. If $(s_i)\beta$ contains a long $jk$-subword, then $X$ ends in $\tau = jk$.

**Proof:** We argue by induction on the length $\ell$ of $X$; obviously for trivial $X$ there is nothing to be proved.

Hence given an expression $X$ of length $\ell > 0$ we may assume that the claim holds for $X' = (x_1, \ldots, x_{\ell-1})$, which is a $M$-reduced expression for $\beta'$, where $\beta = \beta'a_{\sigma_\ell}^{mp_{\ell}}, m = m_{\sigma_\ell}$.
Note that \((s_i)\beta'\) has no long \(\sigma_\ell\)-subword, for otherwise \(X'\) ends with \(\sigma_\ell\) by induction, contrary to our assumption that \(X\) is \(M\)-reduced.

Since \((s_i)\beta = ((s_i)\beta') a_{\sigma_\ell}^{mp}\) contains a long \(\tau\)-subword, we conclude with prop. 2.2 that either \(\tau = \sigma_\ell\) or \(\tau, \sigma_\ell\) are non-crossing and \((s_i)\beta'\) contains a long \(\tau\)-subword.

In the first case the claim is obviously true. In the second case, by induction, \(X'\) ends in \(\sigma_\ell\) by induction, contrary to our assumption that \(X\) is \(M\)-reduced.

Since \((s_i)\beta = ((s_i)\beta') a_{\sigma_\ell}^{mp}\) contains a long \(\tau\)-subword, we conclude with prop. 2.2 that either \(\tau = \sigma_\ell\) or \(\tau, \sigma_\ell\) are non-crossing and \((s_i)\beta'\) contains a long \(\tau\)-subword.

In the first case the claim is obviously true. In the second case, by induction, \(X'\) ends in \(\tau\), hence there is an expression \((x_1', ..., x'_{\ell-1})\) of \(\beta'\) with \(\sigma_\ell' = \tau\), which is obtained by operations of type \(II\) from \(X'\). Hence \(X\) transforms into \((x_1', ..., x'_{\ell-1}, x_\ell)\).

Since \(\tau, \tau_\ell\) are non-crossing a further operation of type \(II\) yields

\[(x_1', ..., x'_{\ell-2}, x_\ell, x'_{\ell-1})\]

which shows that \(X\) ends in \(\tau\) as claimed. \(\Box\)

**Proposition 2.4** Suppose \(X = (x_1, ..., x_\ell)\) is a non-trivial \(M\)-reduced expression for \(\beta\) with \(\sigma_\ell = jk\), \(x_\ell = b_{jk}^{p}\). Then \((s_j)\beta \neq s_j\), in particular \(G_M\) injects into \(B_n\).

**Proof:** Again \(X' = (x_1, ..., x_{\ell-1})\) is an expression for \(\beta' = \beta a_{jk}^{-p_{mj}}\). Suppose contrary to our claim that \((s_j)\beta = s_j\), so

\[(s_j)\beta' = (s_j)\beta a_{jk}^{-p_{mj}} = (s_j) a_{jk}^{-p_{mj}}.\]

By the transformation rules in proposition 2.1 the word on the right contains a long \(jk\)-subword, hence the same is true on the left hand side. With proposition 2.3 we conclude that \(X'\) ends in \(\sigma_\ell = jk\). Now we have reached a contradiction to the hypothesis that \(X\) is \(M\)-reduced, thus our claim holds true. \(\Box\)

The observation of the proposition concludes the proof of theorem 1.1 since \(G_M\) now maps isomorphically onto \(E_M\) identifying the respective sets of generators.

Our argument give also a new prove of the Tits conjecture in case of the braid group and large exponents, i.e. at least 3.

### 3 partition preserving braid subgroups

We want to study the groups \(E_M\) of theorem 1.2 in terms of corresponding partitions \(P\) of the set \(\{1, ..., n\}\). Given such a partition we define its stabiliser in the symmetric group \(S_n\) to be

\[S_{n,P} = \langle (ij)|i \sim_P j \rangle \subset S_n,\]

where \(\sim_P\) is the unique equivalence relation with set of equivalence classes equal to \(P\).

The significance of these notions for our problem lies in the following correspondence:
Proposition 3.1 There is a natural bijection between matrices $M$ as in theorem 1.2 and the partitions $P$ of $\{1, ..., n\}$ such that there is a natural diagram with exact rows:

$$
1 \to \mathbb{PBr}_n \to E_M \to S_{n,P} \to 1
$$

Proof: To a partition $P$ we associate the unique matrix $M$ with $m_{ij} = 1 \Leftrightarrow i \sim_P j$ and $m_{ij} = 2$ otherwise. The additional condition, $m_{ij} = m_{jk} = 1 \Rightarrow m_{ik} = 1$, holds for $M$ since $\sim_P$ is transitive. Reversely with $M$ we may define the relation $i \sim_M j :\iff m_{ij} = 1 \vee i = j$, which obviously is reflexive and symmetric, but also transitive due to the additional condition on $M$. The set of equivalence classes yields the partition $P$ associated with $M$.

For the second claim we first convince ourselves that $S_{n,P}$ is generated by all transpositions $(ij)$ with $i \sim_P j$. For the associated $M$ obviously $E_M$ contains $\mathbb{PBr}_n$. Moreover $(ij)$ has a preimage $b_{ij}$ in $E_M$ if and only if $m_{ij} = 1$, hence $E_M$ surjects onto $S_{n,P}$. □

In order to prove theorem 1.2 we want to set up an induction over $n$ which is well-based, since the cases $n = 1, n = 2$ are obviously true. Given a partition $P$ of $\{1, ..., n\}$ we define the induced partition $P'$ of $\{1, ..., n-1\}$ and we denote by $P'n$ the partition of $\{1, ..., n\}$ for which $\{n\}$ is a part on its own and all other part are those of $P'$.

Moreover we use $a_i$ as a shorthand for $a_{i,n}$

Lemma 3.2 Suppose $E_{P'}$ is presented as $\langle b_{jk}, 1 \leq j < k < n | \mathcal{R}' \rangle$, then $E_{P'n}$ has a presentation by generating elements $a_i^2, b_{jk}$ subject to relations in $\mathcal{R}'$ and

i) $a_i^2 b_{jk} = b_{jk} a_i^2$ for $i < j < k < n$ or $j < k < i < n$

ii) $a_i^2 a_j^2 b_{ik} a_k^{-2} = a_k^2 b_{ik} a_k^{-2} a_j^2$. for $i < j < k < n$

iii) $b_{ij} a_i^2 = a_j^2 b_{ij}, b_{ij} a_i^2 a_j^2 = a_i^2 a_j^2 b_{ij}$ for $i < j < n$ and $m_{ij} = 1$,

iv) $a_i^2 b_{ij} a_i^2 = b_{ij} a_i^2 a_j^2 = a_j^2 a_i^2 b_{ij}$ for $i < j < n$ and $m_{ij} = 2$,

In particular the theorem 1.2 holds true for $E_{P'n}$, if it holds true for $E_{P'}$.

Proof: All given relations are shown to hold by a straightforward calculation. On the other hand we can argue with the following natural diagram of split exact rows:

$$
1 \to \langle a_{i,n}^2 \rangle \to E_{P'n} \to E_{P'} \to 1
$$

Hence it suffices to show that all relations obtained from the action of the extension can be deduced from those given in the claim. These relations are easily obtained by 'combing' the braid obtained by conjugation. The following list – with $i < j < k < n$ and $m_{ij} = 1$ in case 3, 4, resp. $m_{ij} = 2$ in case 5, 6, $m_{ik} = 1$ in case 7 and $m_{ik} = 2$ in case 8 – is exhaustive.
We may now apply the claim of the main part of the lemma to the particular case in which $\mathcal{R}'$ is the set of relations associated to $E_{P'}$ and the generators $b_{jk}, 1 \leq j < k < n$ in the claim of the theorem and $\langle b_{jk}, j, k < n | \mathcal{R}' \rangle$ is a presentation of $E_{P'}$.

Then of course $E_{P'n}$ is presented by generators $b_{ij}, i, j < n$ and $b_{in} = a_{in}^2$ subject to the relations $\mathcal{R}'$ and those of the list, we just derived. Since they correspond bijectively to relations of the theorem $[1,2]$ in case $P = P'n$ we have concluded our proof. \hfill \Box

Now given a general $P$ we consider the group $G_M$ given by the presentation of theorem $[1,2]$ in terms of the matrix $M$ associated to $P$ by $\mathcal{B}$ but with elements $\tilde{b}_{ij}$ to mark the distinction with the subgroups of $B_n$. Let $I \subset \{1, \ldots, n\}$ be the equivalence class of $n$ under $\sim_P$. Then we denote by $H_M$ the subgroup of $G_M$ generated by elements $\{\tilde{b}_{ij}, i < j < n\}, \{\tilde{b}_i := \tilde{b}_{in}, i \notin I\}$ and $\{\tilde{b}_i^2 := \tilde{b}_{in}^2, i \in I\}$, where we take $b_n = 1$.

**Lemma 3.3** $G_M$ can be given as a finite union of right cosets of $H_M$ (with $\tilde{b}_n = 1$):

$$G_M = \bigcup_{i \in I} \tilde{b}_i H_M.$$
\textbf{Proof:} Since each generator is either in $H_M$ or in $\{\tilde{b}_t | t \in I\}$, all generators of $G_M$ belong to this union of cosets and multiplication by any generator on the left maps $H_M$ to one of the given cosets. Hence it remains to prove that multiplication on the left by any generator $\tilde{b}$ of $G_M$ maps an element of any of the non-trivial given cosets of $H_M$ into one of the given cosets. We refer by roman numbers to the relations of the theorem which are used and freely move factors from and into $H_M$:

i) $\tilde{b} = \tilde{b}_t$, $i \in I$

(a) $i < t$

$$\tilde{b}_i \tilde{b}_t \in \tilde{b}_t H_M$$

(b) $i = t$

$$\tilde{b}_t \in H_M$$

(c) $i > t$

$$\tilde{b}_i \tilde{b}_t \in \tilde{b}_i \tilde{b}_t^{-1} H_M = \tilde{b}_i \tilde{b}_t^{-1} H_M \equiv \tilde{b}_i \tilde{b}_t H_M = \tilde{b}_i \tilde{b}_t^{-2} H_M \in \tilde{b}_t H_M$$

ii) $\tilde{b} = \tilde{b}_t$, $i \notin I$

(a) $i < t$

$$\tilde{b}_i \tilde{b}_t \in \tilde{b}_t H_M$$

(b) $i = t$ not possible since $t \in I, i \notin I$

(c) $i > t$

$$\tilde{b}_i \tilde{b}_t \in \tilde{b}_i \tilde{b}_t H_M \equiv \tilde{b}_i \tilde{b}_t \tilde{b}_t H_M \equiv \tilde{b}_i \tilde{b}_t \tilde{b}_t H_M = \tilde{b}_i \tilde{b}_t H_M$$

iii) $\tilde{b} = \tilde{b}_{ij}$

(a) $t < i < j$ or $i < j < t$

$$\tilde{b}_{ij} \tilde{b}_t \equiv \tilde{b}_{ij} \tilde{b}_t \in \tilde{b}_t H_M$$

(b) $t = i < j, j \in I$, ie. $m_{ij} = 1$

$$\tilde{b}_{ij} \tilde{b}_t \equiv \tilde{b}_{ij} \tilde{b}_t \in \tilde{b}_j H_M$$

(c) $i < j = t, i \in I$, ie. $m_{ij} = 1$

$$\tilde{b}_{ij} \tilde{b}_t \equiv \tilde{b}_{ij} \tilde{b}_t \in \tilde{b}_i H_M$$

(d) $t = i < j, j \notin I$, ie. $m_{ij} = 2$

$$\tilde{b}_{ij} \tilde{b}_t \equiv \tilde{b}_{ij} \tilde{b}_t \in \tilde{b}_t H_M$$

(e) $i < j = t, i \notin I$, ie. $m_{ij} = 2$

$$\tilde{b}_{ij} \tilde{b}_t \equiv \tilde{b}_{ij} \tilde{b}_t \tilde{b}_t H_M \equiv \tilde{b}_{ij} \tilde{b}_t \tilde{b}_t H_M \equiv \tilde{b}_{ij} \tilde{b}_t \tilde{b}_t H_M = \tilde{b}_t H_M$$
iv) \( \tilde{b} = b_{ik}, \ i < t < k, \)

(a) \( i, t \in I, \ k \not\in I, \) i.e. \( m_{ik} = 2, \) then with \(*\): cases (a) of \( i) \) and (c) of \( ii) \) above

\[
\tilde{b}_{ik}\tilde{b}_t = \tilde{b}_{ik}\tilde{b}_t \tilde{b}_k\tilde{b}_i H_M = \tilde{b}_{ik}\tilde{b}_i \tilde{b}_k\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_{ik}\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_t H_M
\]

(b) \( t \in I, \ i, k \not\in I, \) \( m_{ik} = 2, \) then with \(*\): cases (a) and (c) of \( ii) \) above

\[
\tilde{b}_{ik}\tilde{b}_t = \tilde{b}_{ik}\tilde{b}_t \tilde{b}_k\tilde{b}_i H_M = \tilde{b}_{ik}\tilde{b}_i \tilde{b}_k\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_{ik}\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_t H_M
\]

(c) \( k, t \in I, \ i \not\in I, \) i.e. \( m_{ik} = 2, \) then with \(*\): cases (c) of \( i) \) and (a) of \( ii) \) above

\[
\tilde{b}_{ik}\tilde{b}_t = \tilde{b}_{ik}\tilde{b}_t \tilde{b}_k\tilde{b}_i H_M = \tilde{b}_{ik}\tilde{b}_i \tilde{b}_k\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_{ik}\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_t H_M
\]

(d) \( t \in I, \ i, k \not\in I, \) \( m_{ik} = 1, \) then with \(*\): cases (a) and (c) of \( ii) \) above

\[
\tilde{b}_{ik}\tilde{b}_t = \tilde{b}_{ik}\tilde{b}_t \tilde{b}_k\tilde{b}_i H_M = \tilde{b}_{ik}\tilde{b}_i \tilde{b}_k\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_{ik}\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_t H_M
\]

(e) \( t, k \in I, \) i.e. \( m_{ik} = 1, \) then with \(*\): cases (a) and (c) of \( i) \) above

\[
\tilde{b}_{ik}\tilde{b}_t = \tilde{b}_{ik}\tilde{b}_t \tilde{b}_k\tilde{b}_i H_M = \tilde{b}_{ik}\tilde{b}_i \tilde{b}_k\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_{ik}\tilde{b}_t H_M = \tilde{b}_k\tilde{b}_t H_M
\]

\( \square \)

We are now ready to give the proof of our second theorem.

\textit{Proof of thm.} \( \square \) The case \( n = 1 \) is void and the case \( n = 2 \) consists of two subcases with group \( E \) freely generated by \( \sigma_1 \) resp. \( \sigma_1^2 \) in accordance with the claim. Hence it suffices to prove the claim for matrices \( M \) of size \( n \) relying on the induction hypothesis. In fact we will show that there is an isomorphism \( G_M \to E_M \) induced by the natural bijection \( \tilde{b}_{ij} \to b_{ij} \) on generators.

First we observe that we get in fact a surjection, since the relations in \( G_M \) map to relations in \( B_n \), which can be checked by straightforward calculations. Moreover we note that the subgroup \( H_M \) maps onto \( E_{P'n} \) where \( P'n \) is obtained as before from the partition \( P \) associated to \( M \).

We will now exploit the induction hypothesis and lemma \( \square \) which provide a presentation of \( E_{P'n} \), to get at least a partial inverse \( E_{P'n} \to H_M \). Since the relations in \( \mathcal{R}' \) pose no problems we restrict to enumerate the relations in \( G_M \) from which the relations of \( E_{P'n} \) in the corresponding enumeration of lemma \( \square \) follow.

i) \( \tilde{b}_t \tilde{b}_{jk} = \tilde{b}_{jk} \tilde{b}_t \)
ii) according to $m_{kn} = 2$ or $1$ either $a_k^2 \mapsto \tilde{b}_k$ (a), or $a_k^2 \mapsto \tilde{b}_k^2$ (b).
   
   (a) $\tilde{b}_j \tilde{b}_k \tilde{b}_k \tilde{b}_k^{-1} = \tilde{b}_j \tilde{b}_k \tilde{b}_k^{-1} \tilde{b}_j$
   (b) $\tilde{b}_j \tilde{b}_k^2 \tilde{b}_k \tilde{b}_k^{-2} = \tilde{b}_k^2 \tilde{b}_k \tilde{b}_k^{-2} \tilde{b}_j$

iii) since $m_{ij} = 1$ either $a_i, a_j \notin E_M$ and $a_i^2 \mapsto \tilde{b}_i$, $a_j^2 \mapsto \tilde{b}_j$ (a),
   or $a_i, a_j \in E_M$ and $a_i^2 \mapsto \tilde{b}_i^2$, $a_j^2 \mapsto \tilde{b}_j^2$ (b).
   
   (a) $\tilde{b}_{ij} \tilde{b}_j = \tilde{b}_j \tilde{b}_{ij}$, $\tilde{b}_j \tilde{b}_j \tilde{b}_j = \tilde{b}_j \tilde{b}_j \tilde{b}_j$
   (b) $\tilde{b}_{ij} \tilde{b}_j = \tilde{b}_j \tilde{b}_j$ implies $\tilde{b}_{ij} \tilde{b}_j^2 = \tilde{b}_j^2 \tilde{b}_j$ and
       $\tilde{b}_{ij} \tilde{b}_j \tilde{b}_j = \tilde{b}_j \tilde{b}_j \tilde{b}_j = \tilde{b}_j \tilde{b}_j \tilde{b}_j$
       $\tilde{b}_{ij} \tilde{b}_j \tilde{b}_j = \tilde{b}_j \tilde{b}_j \tilde{b}_j$

iv) since $m_{ij} = 2$ either $a_i, a_j \notin E_M$ and $a_i^2 \mapsto \tilde{b}_i$, $a_j^2 \mapsto \tilde{b}_j$ (a),
   or $a_i \in E_M, a_j \notin E_M$ and $a_i^2 \mapsto \tilde{b}_i^2$, $a_j^2 \mapsto \tilde{b}_j^2$ (b),
   or $a_i \notin E_M, a_j \in E_M$ and $a_i^2 \mapsto \tilde{b}_i^2$, $a_j^2 \mapsto \tilde{b}_j^2$ (c).
   
   (a) $\tilde{b}_i \tilde{b}_i \tilde{b}_i = \tilde{b}_i \tilde{b}_i \tilde{b}_i$
   (b) $\tilde{b}_i \tilde{b}_i \tilde{b}_i = \tilde{b}_i \tilde{b}_i \tilde{b}_i$
   (c) $\tilde{b}_i \tilde{b}_i \tilde{b}_i = \tilde{b}_i \tilde{b}_i \tilde{b}_i$

Next we consider the following diagram with exact rows:

\[
\begin{array}{ccc}
1 & \rightarrow & \text{PBr}_n \rightarrow E_{P^*n} \rightarrow S_{n,P^*n} \rightarrow 1 \\
1 & \rightarrow & \text{PBr}_n \rightarrow E_P \rightarrow S_{n,P} \rightarrow 1 \\
\end{array}
\]

Hence $E_{P^*n}$ is of finite index $|I| = \#\{i \leq n \mid i \sim_P n\}$ in $E_P$. This injection factors as $E_{P^*n} \cong H_M \hookrightarrow G_M \twoheadrightarrow E_P$. Since by Lemma 3.3, the index of $H_M$ in $G_M$ is at most $|I|$, we can conclude that $E_P$ is isomorphic to $G_M$. \(\square\)

## 4 further prospects

In this last section we want to address two directions of further studies. First we would like to mention the simplest instance of a structural result describing groups associated to

a matrix $M$ composed of several submatrices in terms of the groups associated to these submatrices.

**Proposition 4.1** Suppose $M$ is a block matrix with diagonal blocks $M_1, M_2$ and zero off-diagonal blocks. Then the groups associated to $M_1, M_2$ are free or direct factors:

$$A_M = A_{M_1} \ast A_{M_2}, \quad C_M = C_{M_1} \ast C_{M_2}, \quad E_M = E_{M_1} \times E_{M_2}$$
It should also be worthwhile to understand how the group $E_M$ changes under the action of a permutation matrix on $M$.

More emphasis we would like to put on the problem to find presentations for matrices with entries $m_{ij} \geq 2$, in particular with matrices of finite irreducible Coxeter groups.

The following result on relation between the generators of such groups $E_M$ is obtained by straightforward calculations.

**Proposition 4.2** In case of matrices $M$ with entries $m_{ij} \geq 2$, the group $E_M$ generated by $b_{ij} = a_{ij}^{m_{ij}}$ has relations

i) for $i < j < k < l$

$$b_{ij}b_{kl} = b_{kl}b_{ij}, \quad b_{il}b_{jk} = b_{jk}b_{il},$$

ii) for $i < j < k < l$ and $m_{jk} = 2$

$$b_{ik}b_{jk}b_{jl}^{-1} b_{jk} = b_{jk}b_{jl}^{-1} b_{ik}.$$

iii) for $i < j < k$, $j < k < i$ or $k < i < j$ with

(a) $m_{ij} = m_{ik} = 2$, $m_{jk} = 2\nu$

$$(b_{ij}b_{ik})^{\nu-1} b_{jk} b_{ij} b_{ik} = b_{ik} (b_{ij}b_{ik})^{\nu-1} b_{jk} b_{ij} = (b_{ij}b_{ik})^{\nu} b_{jk},$$

(b) $m_{ij} = m_{ik} = 2$, $m_{jk} = 2\nu + 1$

$$b_{ik} (b_{ij}b_{ik})^{\nu-1} b_{jk} b_{ij} b_{ik} = (b_{ij}b_{ik})^{\nu-1} b_{jk} b_{ij} = b_{ik} (b_{ij}b_{ik})^{\nu} b_{jk},$$

(c) $m_{ij} = m_{ik} = m_{jk} = 3$

$$b_{ij}b_{jk} b_{ij} b_{ik} b_{jk} = b_{jk} b_{ij} b_{ik} b_{jk} b_{ij} = b_{ij} b_{ik} b_{jk} b_{ij} b_{ik},$$

(d) $m_{ij} = m_{ik} = 3$, $m_{jk} = 4$

$$b_{ij} b_{ik} b_{jk} b_{ij} b_{ik} b_{jk} = b_{ik} b_{jk} b_{ij} b_{ik} b_{jk} = b_{jk} b_{ij} b_{ik} b_{jk} b_{ij} b_{ik},$$

(e) $m_{ij} = m_{ik} = 3$, $m_{jk} = 5$

$$b_{ij} b_{ik} b_{jk} b_{ij} b_{ik} b_{jk} b_{ij} b_{ik} b_{jk} b_{ij} = b_{jk} b_{ij} b_{ik} b_{jk} b_{ij} b_{ik} b_{jk} b_{ij}.$$
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