On the Calculation of QED Amplitudes in a Constant Field

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Abstract

It is explained how first-quantized worldline path integrals can be used as an efficient alternative to Feynman diagrams in the calculation of QED amplitudes and effective actions. The examples include the one-loop photon splitting amplitude, the two-loop Euler-Heisenberg Lagrangian, and the one-loop axial vacuum polarization tensor in a general constant electromagnetic field.
1 Introduction

During the last few years it has been established that techniques originally developed in string perturbation theory can be used to improve on the efficiency of calculations in ordinary perturbative quantum field theory. This fact was first demonstrated in the calculation of tree-level and one-loop QCD scattering amplitudes [1] but also for one-loop gravity [2] and higher-loop supergravity amplitudes [3]. Those advantages are due to the superior organisation of string theory as compared to field theory amplitudes. To some extent they can be recaptured in a more elementary approach [4] based on the representation of one-loop effective actions in terms of first-quantized path integrals [5, 6], which one evaluates by a technique analogous to the one which is used for the calculation of the string path integral. This approach was found to be particularly well-suited to the calculation of amplitudes and effective actions in QED [4, 7, 8]. Here it has been successfully used for calculations involving constant external fields [9, 10, 11, 12, 13, 14], and also as an easy alternative route to the construction of multiloop Bern-Kosower type formulas [8].

In the present report we focus on the constant external field case, and present the following calculations: i) One-loop photon splitting in a constant magnetic field [12]; ii) The two-loop Euler-Heisenberg Lagrangian [13, 14]; iii) The axial vacuum polarization tensor in a general constant external field.

2 The Worldline Path Integral Technique in QED

Our starting point for QED calculations is the following path integral representing the one-loop effective action for the Maxwell field due to an electron loop:

\[
\Gamma[A] = -2 \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int DxD\psi \exp \left[ - \int_0^T d\tau \left( \frac{1}{4} \dot{x}^2 + \frac{1}{2} \dot{\psi}^2 + i e A_\mu \dot{x}^\mu - i e \dot{\psi}^\mu F_{\mu\nu} \psi^\nu \right) \right]
\]

Here \( T \) is the usual Schwinger proper–time parameter, the \( x^\mu(\tau) \)'s are the periodic functions from the circle with circumference \( T \) into \( D \)-dimensional spacetime, and the \( \psi^\mu(\tau) \)'s their antiperiodic Grassmannian supersymmetric partners. The analogous formula for scalar QED is obtained simply by discarding the global factor of \(-2\) (which is for statistics and degrees of freedom) and the Grassman path integral (which represents the electron spin). In the “string–inspired” approach, the path integrals over \( y \) and \( \psi \) are evaluated by one-dimensional perturbation theory, using the Green functions

\[
\langle y^\mu(\tau_1)y^\nu(\tau_2) \rangle = -g^{\mu\nu} G_B(\tau_1, \tau_2) = -g^{\mu\nu} \left[ |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} \right]
\]

\[
\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2) \rangle = \frac{1}{2} g^{\mu\nu} G_F(\tau_1, \tau_2) = \frac{1}{2} g^{\mu\nu} \operatorname{sign}(\tau_1 - \tau_2)
\]

We abbreviate \( G_B(\tau_1, \tau_2) =: G_{B12} \) etc. The bosonic Wick contraction is actually carried out in the relative coordinate \( y(\tau) = x(\tau) - x_0 \) of the closed loop, while the (ordinary) integration over
the average position \( x_0 = \frac{1}{T} \int_0^T d\tau \ x(\tau) \) yields energy–momentum conservation. The result of this evaluation is the one-loop effective Lagrangian \( \mathcal{L}(x_0) \).

One–loop scattering amplitudes are obtained by specializing the background to a finite sum of plane waves of definite polarization. In the case of scalar QED this leads to the following extremely compact “Bern-Kosower master formula” for the one-loop (off-shell) N-photon amplitude,

\[
\Gamma[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N] = (-ie)^N (2\pi)^D\delta(\sum k_i) \int_0^\infty\frac{dT}{T} [4\pi T]^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \times \exp\left\{ \sum_{i,j=1}^N \left[ \frac{1}{2} G_{Bij} k_i \cdot k_j - i\dot{G}_{Bij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j \right] \right\} \bigg|_{\text{multi-linear}} (2.3)
\]

Here it is understood that only the terms linear in all the \( \varepsilon_1, \ldots, \varepsilon_N \) have to be taken. Besides the Green’s function \( G_B \) also its first and second derivatives appear, \( \dot{G}_B(\tau_1, \tau_2) = \text{sign}(\tau_1 - \tau_2) - 2(\tau_1 - \tau_2) - \frac{2}{T} \). Dots generally denote a derivative acting on the first variable, \( \dot{G}_B(\tau_1, \tau_2) \equiv \frac{\partial}{\partial \tau_1} G_B(\tau_1, \tau_2) \), and we abbreviate \( G_{Bij} \equiv G_B(\tau_i, \tau_j) \) etc. The factor \([4\pi T]^{-\frac{D}{2}}\) represents the free Gaussian path integral determinant factor.

For the fermion QED case an analogous formula can be written using a superfield formalism \[8, 13\]. Alternatively the additional terms from the Grassmann path integration can also be generated by performing a certain partial integration algorithm on the above expression, and then applying a simple “substitution rule” on the result \[1\]. As an additional benefit of this procedure one obtains a permutation symmetric gauge invariant decomposition of the \( N \)-photon (scalar and fermion QED) amplitudes \[15\].

3 QED in a Constant Field (1 - Loop)

The presence of an additional constant external field, taken in Fock-Schwinger gauge centered at \( x_0 \) \[7\], changes the path integral Lagrangian in eq.(2.1) only by a term quadratic in the fields, \( \Delta \mathcal{L} = \frac{1}{2} ie \gamma^\mu F_{\mu\nu} \dot{\gamma}^\nu - ie \gamma^\mu F_{\mu\nu} \gamma^\nu \). The field can therefore be absorbed by a change of the free worldline propagators, replacing \( G_B, G_F \) by \[13\]

\[
G_B(\tau_1, \tau_2) = \frac{1}{2(eF)^2} \left( \frac{eF}{\sin(eTF)} e^{-ieTF\dot{G}_{B12}} + ie F\dot{G}_{B12} - \frac{1}{T} \right) \quad \text{(3.1)}
\]

\[
G_F(\tau_1, \tau_2) = G_{F12} \frac{e^{-ieTF\dot{G}_{B12}}}{\cos(eTF)} \quad \text{(3.2)}
\]

Thus eq.(2.3) generalizes to the case of the scalar QED \( N \)-photon scattering amplitude in a constant field as follows,
\[ \Gamma[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N] = (-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} |4\pi T|^{-\frac{D}{2}} e^{-m^2 T} \det^{-\frac{1}{2}} \left[ \frac{\sin(eFT)}{eFT} \right] \\
\times \prod_{i=1}^N \int_0^T d\tau_i \exp \left\{ \sum_{i,j=1}^N \left[ \frac{1}{2} k_i \cdot G_{Bij} \cdot k_j - i\varepsilon_i \cdot \dot{G}_{Bij} \cdot k_j + \frac{1}{2} \varepsilon_i \cdot \ddot{G}_{Bij} \cdot \varepsilon_j \right] \right\}_{\text{multi-linear}} \] 

(3.3)

The determinant factor appearing here takes the change of the path integral determinant due to the constant field into account. Since by itself it just describes the vacuum amplitude in a constant field one finds it to be, of course, just the proper-time integrand of the one-loop Euler-Heisenberg Lagrangian.

In [12] this formalism has been applied to a recalculation of the QED photon splitting amplitude in a constant magnetic field \( B \) [16]. Using the modified master formula above with \( N = 3 \) as a starting point one finds, in the spinor QED case, the following parameter integral representation for this amplitude \( (z = eBT) \),

\[ C[\omega, \omega_1, \omega_2, B] = \frac{m^8}{4\omega \omega_1 \omega_2} \int_0^\infty \frac{dT}{z^2 \sinh(z)} e^{-m^2 T} \\
\times \int_0^T d\tau_1 d\tau_2 \exp \left\{ \frac{1}{2} \sum_{i,j=0}^2 \bar{\omega}_i \bar{\omega}_j \left[ G_{Bij} + \frac{T}{2z} \frac{\cosh(z \dot{G}_{Bij})}{\sinh(z)} \right] \right\} \\
\times \left\{ \left[ -\cosh(z) \dot{G}_{B12} + \omega_1 \omega_2 \left( \cosh(z) - \cosh(z \dot{G}_{B12}) \right) \right] \right\} \\
\times \left[ \frac{\omega}{\sinh(z) \cosh(z)} - \omega_1 \frac{\cosh(z \dot{G}_{B01})}{\sinh(z)} - \omega_2 \frac{\cosh(z \dot{G}_{B02})}{\sinh(z)} \right] \\
+ \frac{\omega_1 \omega_2 G_{F12}}{\cosh(z)} \left[ \sinh(z \dot{G}_{B01}) \left( \cosh(z) - \cosh(z \dot{G}_{B02}) - (1 \leftrightarrow 2) \right) \right] \right\} 
\right. 

(3.4)

Here \( \omega (\omega_1, \omega_2) \) denotes the energy of the incoming (outgoing) photon(s), and we have defined \( \bar{\omega}_0 = \omega, \bar{\omega}_1, \bar{\omega}_2 = -\omega_1, \omega_2 \). This integral formula is exact for arbitrary magnetic field strengths and for photon energies up to the pair creation threshold. Of the various known integral representations for this amplitude it is the most compact one.

### 4 QED in a Constant Field (2 - Loop)

Since the one-loop master formulas eqs. (2.3), (3.3) and their fermion loop generalizations are valid off-shell, they can be used also for the construction of higher-loop amplitudes [8, 13]. In [13] we
obtained along these lines the following result for the on-shell renormalized two-loop correction to the spinor QED Euler-Heisenberg Lagrangian in a constant magnetic field,

\[ L(2) = \frac{\alpha}{(4\pi)^3} \int_0^\infty \frac{d\tau}{\tau^3} e^{-m^2\tau} \int_0^1 du_a \left[ L(z, u_a, 4) - L_{02}(z, u_a, 4) - \frac{g(z, 4)}{G_{Bab}} \right] - \frac{\alpha}{(4\pi)^3} m^2 \int_0^\infty \frac{d\tau}{\tau^2} e^{-m^2\tau} \left[ \frac{z}{\tanh(z)} - \frac{z^2}{3} - 1 \right] \left[ 18 - 12\gamma - 12 \ln(m^2\tau) + \frac{12}{m^2\tau} \right] \]

(3.1)

where \( u_a \equiv \frac{|\tau_a - \tau_b|}{T} \)

\( G_z^{z_{Bab}} \equiv \frac{T}{2} \left[ \cosh(z) - \cosh(zG_{Bab}) \right] \)

\[ L(z, u_a, 4) = \frac{z}{\tanh(z)} \left\{ B_1 \ln\left(\frac{G_{Bab}/G_{Bab}^{z}}{G_{Bab} - G_{Bab}^{z}}\right) + \frac{B_2}{G_{Bab}(G_{Bab} - G_{Bab}^{z})} + \frac{B_3}{G_{Bab}(G_{Bab} - G_{Bab}^{z})} \right\} \]

\( B_1 = 4z \left( \coth(z) - \tanh(z) \right) G_{Bab}^z - 4G_{Bab} \)

\( B_2 = 2G_{Bab}\dot{G}_{Bab} + z(8\tanh(z) - 4\coth(z))G_{Bab}^z - 2 \)

\( B_3 = 4G_{Bab} - 2\dot{G}_{Bab}G_{Bab}^z - 4z \tanh(z)G_{Bab}^z + 2 \)

\( L_{02}(z, u_a, 4) = -\frac{12}{G_{Bab}} + 2z^2 \), \( g(z, 4) = -6 \left[ \frac{z^2}{\sinh(z)^2} + z \coth(z) - 2 \right] \).

(3.2)

Comparing with the two existing previous calculations of this amplitude we found agreement with [17] to order of \( O(B^{20}) \) in the weak-field expansion in \( B \), and term by term agreement with [18] after performance of an appropriate finite electron mass renormalization. This calculation was further generalized to the case of a general constant field in [14].

5 Inclusion of Axial Vectors

Finally, we mention the following recent generalization of eq.(2.1) to the case of a spinor loop coupled both to a vector field \( A \) and an axial vector field \( A_5 \) [19]:

\[ \Gamma[A, A_5] = -2 \int_0^\infty \frac{dT}{T} e^{-m^2T} \int dx \int D\psi e^{-i\int_0^T d\tau L(\tau)} \]

\[ L(\tau) = \frac{1}{4} \dot{x}^2 + \frac{1}{2} \psi \cdot \dot{\psi} + ie\dot{x}^\mu A_\mu - ie\psi^\mu F_{\mu\nu} \psi^\nu - 2ie_5 \gamma_5 \dot{x}^\mu \psi_\mu \psi_\nu A_5^\nu + ie_5 \gamma_5 \partial_\mu A_5^\mu + 2e_5^2 A_5^2 \]

(4.1)
The periodicity properties of the Grassmann path integral \( \int \mathcal{D}\psi \) are now determined by the operator \( \hat{\gamma}_5 \). After expansion of the interaction exponential a given term in the integrand will have to be evaluated using antiperiodic (periodic) boundary conditions on \( \psi \) if it contains \( \hat{\gamma}_5 \) at an even (odd) power. (After the boundary conditions are determined \( \hat{\gamma}_5 \) can be replaced by unity.) Thus for amplitudes involving an odd number of axial vectors \( \int \mathcal{D}\psi \) will have zero-modes, which upon integration produce the expected \( \varepsilon \)-tensor. We have applied this path integral to a calculation of the axial vacuum polarization tensor \( \langle AA_5 \rangle \) in a general constant field, with the result \[20\]

\[
\Pi_{\mu\nu}^5(k) = \frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} (4\pi T)^{-2} \int_0^T d\tau_1 \int_0^T d\tau_2 J_{\mu\nu} e^{-T k \cdot V \cdot k} \tag{4.2}
\]

where

\[
J_{\mu\nu} = 4iT \left\{ \tilde{F}^{\mu\nu} k U k + \left[ (\tilde{F} k)^{\mu}(U k)_{\nu} + (\mu \leftrightarrow \nu) \right] + T^2 F \cdot \tilde{F} \left\{ -S^{\mu\nu} k U k - \left[ (S k)^{\mu}(U k)_{\nu} + (\mu \leftrightarrow \nu) \right] + (A_{12} S_{22})^{\mu\nu} k A_{12} k + \left[ (A_{12} k)^{\mu}(S_{22} A_{12} k)_{\nu} + (\mu \leftrightarrow \nu) \right] \right\} \right\}
\]

\[
S_{12} = i \left( \frac{\cos(z \hat{G}_{B12})}{\sin(z)} \right) - \frac{1}{z}, \quad S_{22} = i \left( \cot(z) - \frac{1}{z} \right), \quad A_{12} = \frac{\sin(z \hat{G}_{B12})}{\sin(z)}
\]

\[
U = \frac{1 - \cos(z \hat{G}_{B12}) \cos(z)}{\sin^2(z)}, \quad V = \frac{\cos(z \hat{G}_{B12}) - \cos(z)}{2z \sin(z)} \tag{4.3}
\]

(\( \tilde{F} \) denotes the dual field strength tensor). For the magnetic special case this parameter integral agrees with the known field theory result \[21\]. However in contrast to those field theory calculations the path integral calculation is manifestly gauge invariant.

### 6 Summary

From the examples given here it should be evident that the string-inspired technique is an elegant and efficient tool for the calculation of the QED photon S-matrix. Some other quantities of interest which we hope to obtain along these lines are the off-shell photon-photon scattering amplitude, and gradient corrections to the photon splitting amplitude. Also we are currently working on a complete automatization of the calculation of the QED effective action in the higher derivative expansion \[22\].
References

[1] Z. Bern and D. A. Kosower, Nucl. Phys. B379 (1992) 451.
[2] Z. Bern, D. C. Dunbar, and T. Shimada, Phys. Lett. B312 (1993) 277 (hep-th/9307001).
[3] Z. Bern, L. Dixon, D.C. Dunbar, M. Perelstein, and J.S. Rozowsky, SLAC-PUB-7751 (hep-th/9802162).
[4] M. J. Strassler, Nucl. Phys. B385 (1992) 145.
[5] R. P. Feynman, Phys. Rev. 80 (1950) 440.
[6] E. S. Fradkin, Nucl. Phys. 76 (1966) 588.
[7] M. G. Schmidt and C. Schubert, Phys. Lett. B318 (1993) 438 (hep-th/9309055).
[8] M.G. Schmidt and C. Schubert, Phys. Rev. D53 (1996) 2150 (hep-th/9410100).
[9] D. Cangemi, E. D’Hoker, and G. Dunne, Phys. Rev. D51 (1995) 2513 (hep-th/9409113).
[10] V.P. Gusynin and I.A. Shovkovy, Can. J. Phys. 74 (1996) 282 (hep-ph/9509383); UCTP-106-98 (hep-th/9804143).
[11] R. Shaisultanov, Phys. Lett. B 378 (1996) 354 (hep-th/9512142).
[12] S. L. Adler and C. Schubert, Phys. Rev. Lett. 77 (1996) 1695 (hep-th/9603033).
[13] M. Reuter, M.G. Schmidt, and C. Schubert, Ann. Phys. (N.Y.) 259 (1997) 313 (hep-th/9610191); D. Fliegner, M. Reuter, M.G. Schmidt, and C. Schubert, Theor. Math. Phys. 113 (1997) 1442 (hep-th/9704194).
[14] B. Körös and M.G. Schmidt (hep-th/9803144).
[15] C. Schubert, ANL-HEP-PR-97-83 (hep-th/9710067) Eur. Phys. J. C, in print.
[16] S.L. Adler, Ann. Phys. 67 (1971) 599.
[17] V. I. Ritus, Sov. Phys. JETP 42 (1975) 774.
[18] W. Dittrich and M. Reuter, Effective Lagrangians in Quantum Electrodynamics, Springer 1985.
[19] D.G.C. McKeon and C. Schubert, LAPTH-686/98.
[20] A.N. Ioannisian and C. Schubert, in preparation.
[21] L.L. DeRaad Jr., K.A. Milton, and N.D. Hari Dass, Phys. Rev. D 14 (1976) 3326; A.N. Ioannisian and G. Raffelt, Phys. Rev. D 55 (1997) 7038 (hep-ph/9612285).
[22] D. Fliegner and C. Schubert, work in progress.