HINDMAN’S THEOREM, ELLIS’S LEMMA, AND THOMPSON’S GROUP $F$

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Abstract. The purpose of this article is to formulate generalizations of Hindman’s Theorem and Ellis’s Lemma for non associative groupoids. A relation between these conjectures is proved and it is shown that they imply the amenability of Thompson’s group $F$. In fact the amenability of $F$ is equivalent to a finite form of the conjectured extension of Hindman’s Theorem.

1. Introduction

In [13] a connection was established between the amenability of discrete groups and structural Ramsey theory. The results presented there grew out of the analysis of the amenability problem$^1$ for Richard Thompson’s group $F$ and in particular out of the realization that it was closely related to the generalization of Hindman’s Theorem and Ellis’s Lemma to non associative groupoids.$^2$ This analysis is the content of the present article.

Richard Thompson’s group $F$ can be defined abstractly as the group on the set of generators $x_n \ (n \in \omega)$ subject to the relations $x_ix_nx_i^{-1} = x_{n+1}$ for each $i < n$; a more informative model of this group will be developed below. The question of its amenability was considered by

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1 In spite of the assertions made by Shavgulidze in [17], [18], this problem remains open (see [14]).

2 A groupoid is a set equipped with a binary operation.
R. Thompson himself \cite{19} but was rediscovered and popularized by Geoghegan in 1979. It first appeared in print in \cite{4} p. 549.

Recall that a group $G$ is amenable\cite{5} if there is a finitely additive translation invariant probability measure $\mu$ which measures all subsets of $G$. I will not further motivate this problem here; a semi-expository paper will be published separately \cite{11}. A general introduction to $F$ and Thompson’s other groups can be found in \cite{2}.

We will now turn to Hindman’s Theorem. Building on work of Rado and Shur and confirming a conjecture of Graham and Rothschild \cite{6}, Hindman proved the following result.

**Hindman’s Theorem.** \cite{9} If $c : \mathbb{N} \rightarrow k$ is a coloring of $\mathbb{N}$ with finitely many colors, then there is an infinite $X \subseteq \mathbb{N}$ such that $c$ is monochromatic on the sums of finite subsets of $X$.

Hindman’s original proof of this theorem was elementary and combinatorial but quite complex. Galvin and Glazer later gave a simple proof of this theorem using topological dynamics, which I will now describe (see \cite{10} p. 102-103). The operation of addition on $\mathbb{N}$ can be extended to its Čech-Stone compactification $\beta \mathbb{N}$ to yield a compact left topological semigroup. Galvin realized that the existence of an idempotent $U$ in $\beta \mathbb{N}$ allowed for a simple recursive construction of infinite monochromatic sets as in the conclusion of Hindman’s Theorem. Glazer then observed that the existence of such idempotents follows immediately from the following lemma of Ellis.

**Ellis’s Lemma.** \cite{3} If $(S, \star)$ is a compact left topological semigroup, then $S$ contains an idempotent.

We will now examine to what extent both Hindman’s Theorem and Ellis’s Lemma can be generalized to a non associative setting. Let $(T, \hat{+}, 1)$ denote the free groupoid on one generator. The algebra $(T, \hat{+}, 1)$ can be represented in the following manner which will be useful later when defining our model for Thompson’s group $F$. If $A$ and $B$ are subsets of $[0, 1]$, define

$$A \hat{+} B = \frac{1}{2} A \cup \frac{1}{2} (B + 1).$$

Observe that, as a function, $\hat{+}$ is injective. Consequently, the groupoid generated by $1 = \{1\}$ is free; we will take this as our model of $T$. Just as in the case of addition on $\mathbb{N}$, a binary operation $\star$ on a set $S$ can be

\footnote{Throughout this paper, the adjective “left” is implicit in the usage of action, amenable, Følner, invariant, and translation unless otherwise stated.}
extended to $\beta S$ as follows: $W$ is in $U \star V$ if and only if
$$\{u \in S : \{v \in S : u \star v \in W\} \in V\} \in U$$

Let us first observe that $(\beta \mathbb{T}, \mathbb{T}^\times)$ does not contain an idempotent and
that Hindman’s Theorem are false if we replace $\mathbb{N}$ by $\mathbb{T}$. To see this,
define $c : \mathbb{T} \to \{0, 1\}$ recursively by:
$$c(1) = 0 \quad \text{and} \quad c(A \mathbb{T} B) = 1 - c(B).$$

If $U$ is an ultrafilter on $\mathbb{T}$, then
$$\{T \in \mathbb{T} : c(T) = 0\} \in U \iff \{T \in \mathbb{T} : c(T) = 1\} \in U \mathbb{T} U$$
and in particular, $U \mathbb{T} U \neq U$ for any $U \in \beta \mathbb{T}$. Similarly, if $A, B, C \in \mathbb{T}$,
then
$$c(A \mathbb{T} (B \mathbb{T} C)) = 1 - c((A \mathbb{T} B) \mathbb{T} C)$$
and hence the simplest non associative instance of Hindman’s Theorem
fails for $\mathbb{T}$.

It is informative to compare this situation to a reformulation of Hindman’s Theorem due to Baumgartner.

**Theorem 1.1.** If $c : \text{FIN} \to \{0, \ldots, k - 1\}$ is a coloring of FIN with finitely many colors,
then there is an infinite sequence $x_0 < x_1 < \ldots$ of elements of FIN such that $c$ is monochromatic
on all finite unions of members of the sequence.

Here FIN denotes the non empty finite subsets of $\mathbb{N}$ and $x < y$ abbreviates $\max(x) < \min(y)$.
This can be regarded as the corrected form the following false statement: If $c : \text{FIN} \to \{0, \ldots, k - 1\}$, then
there is an infinite set $X$ such that $c$ is monochromatic on all non empty finite subsets of $X$.
The reason this statement is false is that every infinite subset of $\mathbb{N}$ contains finite non empty subsets of both
even and odd cardinalities. Observe that this statement is equivalent
to the modification of Baumgartner’s Theorem where we require $x_i$ to be a singleton for all $i$.
Thus we can avoid this trivial counterexample by allowing singletons to be “glued” together into blocks.

For the non associative analog of Hindman’s Theorem, I propose
a different form of “gluing.” Define $T_n$ to be all elements of $\mathbb{T}$
of cardinality $n$. These are the ways to associate the addition in a sum
of $n$ ones. In particular, each $T_n$ is finite and in fact the cardinalities
of these sets are given by the Catalan numbers. Let $A_n$ denote the
collection of all probability measures on $T_n$. Notice that $A_n$ can be
viewed as a convex subset of the vector space generated by $T_n$ and
$T_n$ can be regarded as the set of extreme points of $A_n$. In particular,
if $c : T_n \to \mathbb{R}$ is any function, then $c$ extends linearly to a function
which maps $\mathbb{A}_n$ into $\mathbb{R}$; such extensions will be taken without further mention. Define $\mathbb{A}$ to be the (disjoint) union of the sets $\mathbb{A}_n$ and define $\# : \mathbb{A} \to \mathbb{N}$ by $\#(\nu) = n$ if $\nu \in \mathbb{A}_n$. The operation of $\hat{+}$ on $\mathbb{T}$ extends bilinearly to a function defined on $\mathbb{A}$:

$$\mu \hat{+} \nu(Z) = \sum_{A \hat{+} B \in \mathbb{Z}} \mu(\{A\}) \nu(\{B\})$$

Observe that $\#$ is a homomorphism from $(\mathbb{A}, \hat{+})$ to $(\mathbb{N}, +)$. Also, if $T$ is in $\mathbb{T}_m$, then $T$ defines a function from $\mathbb{T}_m \to \mathbb{T}$ by substitution: $T(U_0, \ldots, U_{m-1})$ is obtained by substituting $U_i$ for the $i$th occurrence of $\mathbf{1}$ in the term corresponding to $T$. This is the non associative analog of $\sum_{i<m} U_i$. This operation extends to an $m$-multilinear function which maps $\mathbb{A}^m$ into $\mathbb{A}$. We are now ready to state the conjectured generalization of Hindman’s Theorem.

**Conjecture 1.2.** If $c : \mathbb{T} \to [0, 1]$ and $\epsilon > 0$, then there is an $r \in [0, 1]$ and an infinite sequence $\mu_i (i < \infty)$ of elements of $\mathbb{A}$ such that whenever $T$ is in $\mathbb{T}_m$ and $i_0 < \ldots < i_{m-1}$ is admissible for $T$

$$|c(T(\mu_{i_0}, \ldots, \mu_{i_{m-1}})) - r| < \epsilon$$

Admissibility is a technical conditions which will be defined later. For now it is sufficient to say that if $m - 1 \leq i_0 < \ldots < i_{m-1}$, then $i_0 < \ldots < i_{m-1}$ is admissible for any element of $\mathbb{T}_m$. Additionally any increasing sequence of $m$ integers is admissible for some element of $\mathbb{T}_m$. In particular, if $c(T)$ is required to depend only on $\#(T)$, then the above conjecture reduces to Hindman’s Theorem. Notice that if $c : \mathbb{T} \to \{0, 1\}$ is the function defined above, then the extension of $c$ will have the property that there will be elements $\alpha \hat{+} (\beta \hat{+} \gamma)$ of $\mathbb{A}$ such that

$$c(\alpha \hat{+} (\beta \hat{+} \gamma)) = 1/2 = 1 - 1/2 = c((\alpha \hat{+} \beta) \hat{+} \gamma).$$

The extension of a binary operation $\star$ on a set $S$ mentioned above can be generalized so as to extend $\star$ to the space of all finitely additive probability measures on $S$:

$$\mu \star \nu(Z) = \int \int \chi_Z(x \star y) d\nu(y) d\mu(x)$$

I also conjecture the following non associative form of Ellis’s Lemma.

**Conjecture 1.3.** If $(S, \star)$ is a groupoid and $C \subseteq \mathcal{P}(S)$ is a compact convex subgroupoid, then there is a $\mu$ in $C$ such that $\mu \star \mu = \mu$.

In this article I will prove that Conjecture 1.3 implies Conjecture 1.2. If the Continuum Hypothesis is assumed, a partial converse can also be established: Conjecture 1.2 implies the existence of an idempotent
in $P(T)$. Idempotent measures will be shown to be invariant with respect to the partial action of $F$ on $T$. Moreover, I will prove that the amenability of $F$ is equivalent to a weakening of the finite form of Conjecture \[12\]

2. Notation

Before beginning, let us fix some notational conventions. In this paper, $\mathbb{N}$ will be taken to be the positive natural numbers and $\omega$ will denote $\mathbb{N} \cup \{0\}$. Elements of $\omega$ are identified with the set of their predecessors: $0 = \emptyset$ and $n = \{0, \ldots, n - 1\}$. If $S$ is a set, then the powerset of $S$ will be denoted by $\mathcal{P}(S)$. The collection of all finitely additive probability measures on $S$ will be denoted $P(S)$. This is viewed as a compact topological space by regarding it as a subspace of the product space $[0, 1]^\mathcal{P}(S)$ or, equivalently, as a subspace of $\ell^\infty(S)^*$ with the weak* topology (see \[15\]).

Thompson’s group $F$ can be described as follows. If $A, B \in T$ have equal cardinality, then the increasing function from $A$ to $B$ extends linearly to an automorphism of $([0, 1], \leq)$. We will write $(A \to B)$ to denote this map. The collection of all such functions with the operation of composition is $F$. The group $F$ acts partially on $T$ by setwise application with the stipulation that $f \cdot T$ is only defined when $f$ is linear on each interval contained in the complement of $T$. The standard generators for $F$ are given by:

$$x_0 = ((1 \hat{\to} 1) \hat{\to} 1 \to 1 \hat{\to} (1 \hat{\to} 1))$$

$$x_1 = (1 \hat{\to} ((1 \hat{\to} 1) \hat{\to} 1) \to 1 \hat{\to} (1 \hat{\to} (1 \hat{\to} 1)))$$

If we view elements of $T$ as terms, then the action of $F$ on $T$ is by re-association:

$$x_0 \cdot ((A \hat{\to} B) \hat{\to} C) = A \hat{\to} (B \hat{\to} C)$$

$$x_1 \cdot (S \hat{\to} ((A \hat{\to} B) \hat{\to} C)) = S \hat{\to} (A \hat{\to} (B \hat{\to} C))$$

The partial action of $F$ on $T$ canonically corresponds to the action of $F$ on its positive elements with respect to the generating set $x_k$ ($k \in \omega$). It is well known that $F$ is amenable if and only if this partial action is amenable (details can be found in, e.g., \[12\]). By amenability of the partial action, we mean that there is a $\mu$ in $P(T)$ such that

$$\mu(\{T \in T : x_0 \cdot T \text{ and } x_1 \cdot T \text{ are defined}\}) = 1,$$

$$\mu(x_0 \cdot Z) = \mu(x_1 \cdot Z) = \mu(Z)$$

whenever $Z \subseteq T$. 
3. Products of finitely additive measures

In this section we will recall the definition of a product operation on finitely additive probability measures which is an extension of the Fubini product of filters. Proofs are sketched for completeness.

First I will recall some standard definitions from functional analysis; further reading and background can be found in [15] [16]. If \( X \) is a Banach space, let \( X^* \) denote the collection of continuous linear functionals on \( X \). If \( S \) is a set, \( \ell^\infty(S) \) denotes the space of bounded functions from \( S \) into \( \mathbb{R} \) with the supremum norm. The space \( \ell^\infty(S)^* \) will primarily be given the weak* topology: the weakest topology which makes the evaluation maps \( f \mapsto f(g) \) continuous for each \( g \) in \( \ell^\infty(S) \).

We will identify \( P(S) \) with the subspace of \( \ell^\infty(S)^* \) consisting of those \( f \) such that \( f(g) \geq 0 \) for all \( g \geq 0 \) and such that \( f(\bar{1}) = 1 \), where \( \bar{1} \) is the function which is constantly 1. The elements of \( P(S) \) with finite support are dense in \( P(S) \) in the weak* topology and this will be used frequently without further mention.

Suppose that \( S_0 \) and \( S_1 \) are non-empty sets. Define \( \otimes : P(S_0) \times P(S_1) \to P(S_0 \times S_1) \) by

\[
\mu \otimes \nu(Z) = \int \int \chi_Z(x, y) d\nu(y) d\mu(x).
\]

It should be noted that the order of integration is significant when measures are only required to be finitely additive. We will discuss this in the concluding remarks since it has some significance to the study of the amenability problem for \( F \).

**Proposition 3.1.** If \( S_0 \) and \( S_2 \) are nonempty sets, then for every \( \nu \in P(S_0) \), \( \mu \mapsto \mu \otimes \nu \) is continuous. Moreover if \( \nu \) is finitely supported, then the map \( \mu \mapsto \nu \otimes \mu \) is continuous.

**Proof.** Let \( \tau_{\text{norm}} \) and \( \tau_{\text{ws}} \) denote the norm and weak* topologies on \( P(S) \), regarded as a subspace of \( \ell^\infty(S)^* \). It follows from the basic properties of integration that \( \otimes : P(S_0) \times P(S_1) \to P(S_0 \times S_1) \) is continuous when \( P(S_0 \times S_1) \) is given the weak* topology and \( P(S_0) \times P(S_1) \) is given the \( \tau_{\text{ws}} \times \tau_{\text{norm}} \)-topology. Moreover, if \( \mu \) is finitely supported, then every \( \tau_{\text{norm}} \)-neighborhood contains a \( \tau_{\text{ws}} \)-neighborhood. The proposition follows immediately. \( \square \)

**Proposition 3.2.** If \( S_0, S_1, \) and \( S_2 \) are nonempty sets and \( \mu_i \in S_i \) for \( i < 3 \), then \( (\mu_0 \otimes \mu_1) \otimes \mu_2 = \mu_0 \otimes (\mu_1 \otimes \mu_2) \) up to the identification of \( (S_0 \times S_1) \times S_2 \) with \( S_0 \times (S_1 \times S_2) \).
Proof. Let \( \mu_i \ (i < 3) \) be given. It is sufficient to show that for every \( Z \subseteq S_0 \times S_1 \times S_2 \) and every \( \epsilon > 0 \) that
\[
|\mu_0 \otimes (\mu_1 \otimes \mu_2)(Z) - \mu_0 \otimes (\mu_1 \otimes \mu_2)(Z)| < \epsilon.
\]
Applying Proposition 3.1 repeatedly, find finitely supported measures \( \mu'_i \ (i < 3) \) such that
\[
|\mu_0 \otimes (\mu_1 \otimes \mu_2)(Z) - (\mu'_0 \otimes (\mu'_1 \otimes \mu'_2))(Z)| < \epsilon/2,
\]
\[
|\mu_0 \otimes (\mu_1 \otimes \mu_2)(Z) - (\mu'_0 \otimes (\mu'_1 \otimes \mu'_2))(Z)| < \epsilon/2.
\]
Since \( \mu'_i \ (i < 3) \) are finitely supported,
\[
(\mu'_0 \otimes (\mu'_1 \otimes \mu'_2)) = (\mu'_0 \otimes (\mu'_1 \otimes \mu'_2))
\]
and therefore
\[
|\mu_0 \otimes (\mu_1 \otimes \mu_2)(Z) - \mu_0 \otimes (\mu_1 \otimes \mu_2)(Z)| < \epsilon.
\]
\[\Box\]

Now suppose that \((S, \ast)\) is a groupoid. Extend \( \ast \) to \( P(S) \) as follows:
\[
\mu \ast \nu(Z) = \mu \otimes \nu(\{(x, y) \in S^2 : x \ast y \in Z\})
\]
It follows from Proposition 3.1 that if \( \nu \in P(S) \), then \( \mu \mapsto \mu \ast \nu \) is continuous. If \( \nu \) is finitely supported, then moreover \( \mu \mapsto \nu \ast \mu \) is continuous.

4. THE RELATIONSHIP BETWEEN CONJECTURES 1.2 AND 1.3 AND THE AMENABILITY OF \( F \)

In this section I will prove that Conjecture 1.3 implies Conjecture 1.2 and that if the Continuum Hypothesis is assumed, then Conjecture 1.2 implies the existence of an idempotent in \( P(\mathbb{T}) \). Conjecture 1.2 will also be shown to imply that \( F \) is amenable. In fact any idempotent measure in \( P(\mathbb{T}) \) is invariant with respect to the partial action of \( F \) on \( \mathbb{T} \).

Before proceeding, it is necessary to define the notion of admissibility from the statement of Conjecture 1.2. This definition is technical, but it seems unavoidable in the arguments which follow. For each \( T \) in \( \mathbb{T} \) and \( k < \#(T) \), define \( d_k : \mathbb{T} \to \{-1\} \cup \omega \) by
\[
d_k(T) = \begin{cases} 
-1 & \text{if } T = 1 \text{ and } k = 0 \\
d_k(A) + 1 & \text{if } T = A \uparrow B \text{ and } k < \#(A) \\
\#(A) + d_k(A)(B) & \text{if } T = A \uparrow B \text{ and } \#(A) \leq k < \#(T)
\end{cases}
\]
If \( T \) is in \( \mathbb{T}_m \), then \( F = \{i_0 < \ldots < i_{m-1}\} \) is admissible for \( T \) if for all \( k < m, d_k(T) \leq i_k \). If \( \tau \) is in \( \mathbb{A} \), then we say that a sequence \( i_k \)
Lemma 4.1. If $T$ is in $\mathbb{T}_m$, then whenever $\nu_k (k < l)$ are in $A$ and $\mu$ is an idempotent in $P(\mathbb{T})$, $T(\nu_0, \ldots , \nu_{k-1}, \mu)$ is equivalent to $\hat{T}(\nu_0, \ldots , \nu_{k-1}; \mu)$ for some $\hat{T}$ with $(\hat{T}) = d_k(T) + 2$.

Proof. Before proceeding, it will be helpful to define some notation. If $T$ is in $\mathbb{T}_m$, let $T_k (k < m)$ be the sequence of elements of $\mathbb{T}$ defined recursively as follows:

$$T_k = \begin{cases} 1 & \text{if } T = 1 \\ (A_k)^{\hat{1}} & \text{if } T = A^{\hat{1}}B \text{ and } k < #(A) \\ A^{\hat{1}}(B_{k-#(A)}) & \text{if } T = A^{\hat{1}}B \text{ and } #(A) \leq k \end{cases}$$

It is sufficient to show that for all $k < m$, $(T_k) = d_k(T) + 2$ and for any $k \leq l < m$ and $\nu_i (i < k)$,

$$T_l(\nu_0, \ldots , \nu_{k-1}, \mu) = T(\nu_0, \ldots , \nu_{k-1}; \mu).$$

This is proved by induction on $m$. If $m = 1$, then conclusion follows from the observations that $(T_0) = 1 = -1 + 2$ and $T(\mu) = T_0(\mu)$. If $m > 1$, then $T = A^{\hat{1}}B$ for some $A$ and $B$ in $\mathbb{T}$. Setting $m_0 = #(A)$, it follows from the definitions that if $k < m_0$ then

$$(T_k) = #(A_k) + 1 = d_k(A) + 3 = d_k(T) + 2$$

and if $m_0 \leq k < m$ then

$$(T_k) = m_0 + #(B_{k-m_0}) = m_0 + d_{k-m_0}(B) + 2 = d_k(T) + 2.$$
Lemma 4.2. For every $k < m$. If $m_0 \leq k < l$, then
\[ T_l(n_0, \ldots, n_{m-1}; \mu) = A(n_0, \ldots, n_{m-1}) + B(n_m, \ldots, n_{k-1}; \mu) \]
\[ = A(n_0, \ldots, n_{m-1}) + B(n_m, \ldots, n_{k-1}; \mu) = T(n_0, \ldots, n_{k-1}; \mu). \]
\[ \square \]

Proof. Let $A$, $B$, and $i_k (k < m + n)$ be given as in the statement of the lemma and let $k < m + n$. If $k < m$, then
\[ d_k(A + B) = d_k(A) + 1 \leq i_k - 1 + 1 = i_k. \]
If $m \leq k < m + n$, then
\[ d_k(A + B) = m + d_k(A) = m + i_k - m = i_k. \]
\[ \square \]

Theorem 4.3. Conjecture 1.3 implies Conjecture 1.2.

Proof. Fix an arbitrary function $c : T \to 2$. Let $U$ be an ultrafilter on $\mathbb{N}$ such that $U + U = U$ and define $C$ be the collection of all $\mu$ in $P(T)$ such that if $W \subseteq P(T)$ is an open neighborhood of $\mu$, then
\[ \{ n \in \mathbb{N} : W \cap A_n \neq \emptyset \} \in U. \]

Claim 4.4. $C$ is a compact convex subgroupoid of $(P(T), +)$.

Proof. It follows immediately from the definition that $C$ is closed and hence compact. To see that $C$ is convex, let $\mu, \nu \in C$ be given and let $0 < \lambda < 1$. Let $W$ be an open set about $\lambda \mu + (1 - \lambda) \nu$ and fix open $U$ and $V$ about $\mu$ and $\nu$ respectively such that if $\mu' \in U$ and $\nu' \in V$, then $\lambda \mu' + (1 - \lambda) \nu'$ is in $W$. Since $U$ is closed under finite intersections,
\[ X = \{ n \in \mathbb{N} : (U \cap A_n) \neq \emptyset \wedge (V \cap A_n) \neq \emptyset \} \]
is in $U$. If $n$ is any element of $X$, $\mu' \in U \cap A_n$, and $\nu' \in V \cap A_n$, then $\lambda \mu' + (1 - \lambda) \nu'$ is in $A_n \cap W$ and therefore $\{ n \in \mathbb{N} : W \cap A_n \neq \emptyset \}$ is in $U$ as desired.

To see that $C$ is closed under $+$, let $W$ be open about $\mu + \nu$ for $\mu$ and $\nu$ in $C$. Let $Z = \{ p \in \mathbb{N} : W \cap A_p \neq \emptyset \}$. Since $U$ is an idempotent, we need to prove that there is a set $X$ in $U$ such that for every $m$ in $X$,
\[ \{ n \in \mathbb{N} : m + n \in Z \} \in U \]
Applying Proposition 3.1, there is an open $U$ about $\mu$ such that if $\mu'$ is in $U$, then $\mu' + \nu$ is in $W$. By assumption, there is an $X$ in $\mathcal{U}$ such that if $m$ is in $X$, then $U \cap \mathbb{A}_m \neq \emptyset$. For each $m$ in $X$, let $\mu_m$ be an element of $U \cap \mathbb{A}_m$. Again by Proposition 3.1, there is, for each $m$ in $X$, an open $V_m$ about $\nu$ such that if $\nu'$ is in $V_m$, then $\mu_m + \nu'$ is in $W$. By our assumption, for each $m$ in $X$ we have that

$$Y_m = \{ n \in \mathbb{N} : V_m \cap \mathbb{A}_n \neq \emptyset \} \in \mathcal{U}$$

Now, if $n$ is in $Y_m$, let $\nu'$ be in $\mathbb{A}_n \cap V_m$. It follows that $\mu_m + \nu'$ is in $W \cap \mathbb{A}_{m+n}$. Thus $Y_m \subseteq \{ n \in \mathbb{N} : m+n \in Z \}$ and hence $\{ n \in \mathbb{N} : m+n \in Z \}$ is also in $\mathcal{U}$.

By applying the claim and our assumption, it is possible to choose an idempotent $\mu$ in $C$. Define $r = c(\mu)$. Construct $\mu_i$ ($i \in \omega$) in $\mathbb{A}$ by induction such that, if $\mu_i$ ($i < n$) have been constructed, then for all $k \leq m-2 < i_0 < \ldots < i_{k-1} < n$ and $T$ in $\mathbb{T}_m$,

$$|c(T(\mu_{i_0}, \ldots, \mu_{i_{k-1}}, \mu)) - c(T(\mu_{i_0}, \ldots, \mu_{i_{k-2}}, \mu))| < \epsilon/m.$$  

This is possible by applying the definition of $C$ and the following claim.

**Claim 4.5.** If $T$ is in $\mathbb{T}_m$ and $\nu_i$ ($i < m$) are such that $\nu_i$ is in $P(\mathbb{T})$ and has finite support if $i < k$, then the function $F(\zeta)$ defined by

$$\zeta \mapsto T(\nu_0, \ldots, \nu_{k-2}, \zeta, \nu_k, \ldots, \nu_{m-1})$$

is continuous.

**Proof.** The proof is by induction on $m$. If $m = 1$, then there is nothing to show since then $F$ is just the identity. If $m > 1$ and $T$ is in $\mathbb{T}_m$, then there are $A$ and $B$ such that $T = A + B$. If $\#(A) = l \leq k$, then

$$T(\nu_0, \ldots, \nu_{k-2}, \zeta, \nu_k, \ldots) = A(\nu_0, \ldots, \nu_{l-1}) + B(\nu_l, \ldots, \nu_{k-1}, \zeta, \nu_k, \ldots)$$

Letting $\nu = A(\nu_0, \ldots, \nu_{l-1})$, we have that

$$F(\zeta) = \nu + B(\nu, \ldots, \nu_{k-1}, \zeta, \nu_k, \ldots, \nu_{m-1}).$$

Continuity of $F$ now follows from Proposition 3.1 and the induction hypothesis applied to $B$. A similar argument handles the case $\#(A) > k$.

Now we will verify that $\mu_k$ ($k \in \omega$) satisfies the conclusion of Conjecture 1.2. To this end, let $T$ be an element of $\mathbb{T}_m$ and let $i_0 < \ldots < i_{m-1}$ be admissible for $T$. Applying the claim yields

$$|c(T(\mu_{i_0}, \ldots, \mu_{i_{m-1}})) - c(T(\mu, \ldots, \mu))| \leq \sum_{k<m} |c(T(\mu_{i_0}, \ldots, \mu_{k-1}; \mu)) - c(T(\mu_{i_0}, \ldots, \mu_{k-2}; \mu))|$$
\[
= \sum_{k<m} |c(T_k(\mu_{i_0}, \ldots, \mu_{i_{k-1}}; \mu)) - c(T_k(\mu_{i_0}, \ldots, \mu_{i_{k-2}}; \mu))|.
\]

Also by the claim and by admissibility, \(\#(T_k) \leq d_k(T) + k + 1 \leq i_k + 1\). Thus \(\mu_{i_k}\) was chosen such that

\[
|c(T_k(\mu_{i_0}, \ldots, \mu_{i_k}; \mu)) - c(T_k(\mu_{i_0}, \ldots, \mu_{i_{k-1}}; \mu))| < \epsilon/m.
\]

Recalling that \(r = c(T(\mu, \ldots, \mu))\) and putting this all together we have that
\[
|c(T(\mu_{i_0}, \ldots, \mu_{i_{m-1}})) - r| < m\epsilon/m = \epsilon.
\]

□

In the presence of the Continuum Hypothesis, it is possible to prove a partial converse to the previous theorem. This is an adaptation of an observation of Hindman that the existence of an idempotent ultrafilter in the semigroup \((\beta\mathbb{N}, +)\) follows from the Continuum Hypothesis (this predates the Galvin-Glazer proof of Hindman’s Theorem [10, p. 102-103]). I would like to thank Stevo Todorcevic for bringing this observation to my attention and suggesting that it should be adaptable to the present context.

**Theorem 4.6.** (CH) Conjecture 1.2 implies there is an idempotent measure in \(P(T)\).

**Proof.** Let \(\mathbb{A}^{[\infty]}\) denote the collection of all infinite sequences \(\bar{\mu}\) of elements of \(\mathbb{A}\) such that for all \(k \in \omega\), \(\sum_{i<k} \#(\mu_i) < \#(\mu_k)\). \(\mathbb{A}^{[\infty]}\) is equipped with the following order: \(\bar{\nu} \leq \bar{\mu}\) if there are finite sets \(F_i (i \in \omega)\) and \(\tau_i (i \in \omega)\) in \(\mathbb{A}\) such that

1. if \(i < j\), then \(F_i < F_j\);
2. \(\#(\tau_i) = |F_i|\);
3. \(\nu_i = \tau_i(\bar{\mu} \upharpoonright F_i)\).

(Here we are abusing notation and writing \(\bar{\mu} \upharpoonright F_i\) when we mean the sequence \(\mu_{j_k} (k < m)\) where \(j_k (k < m)\) is the increasing enumeration of \(F_i\). We leave it to the reader to verify that \(\leq\) is a transitive relation, as well as the following claim.

**Claim 4.7.** If \(\bar{\mu}\) is in \(\mathbb{A}^{[\infty]}\) and \(F, F' \subseteq \omega\) are finite, \(\tau \in \mathbb{A}_{|F|}\), \(\tau' \in \mathbb{A}_{|F'|}\), and \(\tau(\bar{\mu} \upharpoonright F) = \tau'(\bar{\mu} \upharpoonright F')\), then \(\tau = \tau'\) and \(F = F'\).

(This follows easily from the observation that if \(n_i (i \in \omega)\) is a sequence such that for all \(k\), \(\sum_{i<k} n_i < n_k\), then all finite sums of terms from this sequence are distinct.) The set \(F\) is said to be the support of \(\tau(\bar{\mu} \upharpoonright F)\) with respect to \(\bar{\mu}\). Define \(\bar{\nu} \leq_n \bar{\mu}\) if \(\langle \nu_k : n \leq k < \infty\rangle \leq \bar{\mu}\). The order \(\bar{\nu} \leq_n \bar{\mu}\) is defined to mean that there is an \(n\) such that \(\bar{\nu} \leq_n \bar{\mu}\).
Fix an arbitrary \( \bar{\lambda} \) in \( A^{[\infty]} \) for the remainder of the proof. An \( \alpha \) in \( A \) is \textit{\( p \)-admissible} (with respect to our choice of \( \bar{\lambda} \)) if there is a \( \tau \) in \( A_m \) and an \( m \)-element set \( F \subseteq \omega \) such that \( F - p \) is admissible for \( \tau \) and \( \alpha = \tau(\bar{\lambda} \upharpoonright F) \). If \( \bar{\mu} \leq \bar{\lambda} \), define \( A^{\bar{\mu} - p} \) to be the collection of all \( \tau(\bar{\mu} \upharpoonright F) \) which are \( p \)-admissible. If \( \bar{\mu} \leq \ast \bar{\lambda} \), let \( C_{\bar{\mu}} \) be the set of all \( \nu \in P(\mathbb{T}) \) which are in the weak* closure of \( A^{\bar{\mu} - p} \) for each \( p \). Observe that if \( \bar{\nu} \leq \ast \bar{\mu} \leq \ast \bar{\lambda} \), then there is an \( n \) such that for every \( p \geq n \), \( A^{\bar{\mu} - p} \subseteq A^{\bar{\mu} - p} \) and therefore \( C_{\bar{\nu}} \subseteq C_{\bar{\mu}} \).

We will now prove a number of claims.

**Claim 4.8.** Any countable \( \leq \ast \)-descending sequence \( \bar{\nu} \) in \( A^{[\infty]} \) such that for all \( k \), \( \bar{\nu}_k \leq \bar{\lambda} \), there is a \( \bar{\nu}_\omega \leq \bar{\lambda} \) such that for all \( k \), \( \bar{\nu}_\omega \leq \ast \bar{\nu}_k \).

**Proof.** Let \( \bar{\nu}_k \ (k \in \omega) \) be given as in the statement of the claim. Let \( n_k \ (k \in \omega) \) be a strictly increasing sequence such that if \( i < k \), then \( \nu_i \leq n_k \nu_i \) and moreover if \( i \leq j < k \), then the maximum of the support of \( \nu_i(n_j) \) in \( \bar{\lambda} \) is less than the minimum of the support of \( \nu_k(n_k) \) in \( \bar{\lambda} \). Define \( \nu_\omega(k) = \nu_{n_k+1}(k) \). It follows that \( \bar{\nu}_\omega(n_k \bar{\nu}_k \) for all \( k \in \omega \) and that \( \bar{\nu}_\omega \leq \bar{\lambda} \). \( \square \)

**Claim 4.9.** For every \( \bar{\mu} \leq \bar{\lambda} \), \( C_{\bar{\mu}} \) is a subgroupoid of \( P(\mathbb{T}) \).

**Proof.** Let \( \alpha, \beta \) be elements of \( C_{\bar{\mu}} \), let \( W \) be an arbitrary open sets about \( \alpha + \beta \), and let \( p \) be arbitrary. By Proposition 3.4, there is an open \( U \) about \( \alpha \) such that if \( \alpha' \) is in \( U \), then \( \alpha' + \beta \) is in \( W \). Let \( \alpha' \) be any element of \( A^{\bar{\mu} - p + 1} \). Let \( V \) be open about \( \beta \) such that if \( \beta' \) is in \( V \), then \( \alpha' + \beta' \) is in \( W \). If \( q = \#(\alpha') \), then by Lemma 4.2, if \( \beta' \) is in \( A^{\bar{\mu} - q} \), \( \alpha' + \beta' \) is in \( A^{\bar{\mu} - p} \). If \( \beta' \) is any element of \( V \cap A^{\bar{\mu} - q} \), then we have that \( \alpha' + \beta' \) is in \( W \cap A^{\bar{\mu} - p} \) as desired. \( \square \)

**Claim 4.10.** Conjecture [1,2] implies that whenever \( \bar{\mu} \leq \bar{\lambda} \) is in \( A^{[\infty]} \) and \( E \subseteq \mathbb{T} \), there is a \( \bar{\nu} \leq \ast \bar{\mu} \) such that \( \zeta(E) = \zeta'(E) \) for every \( \zeta, \zeta' \in C_{\bar{\nu}} \).

**Proof.** Since countable descending sequences in \( (A^{[\infty]}, \leq \ast) \) have lower bounds, it is sufficient to prove that for every \( \epsilon > 0 \), there is a \( \bar{\nu} \leq \bar{\mu} \) such that \( |\zeta(E) - \zeta'(E)| < \epsilon \) for every \( \zeta, \zeta' \in C_{\bar{\nu}} \). Let \( \mu \) be a weak* accumulation point of \( \{\mu_i : i \in \omega\} \). Define \( c : \mathbb{T} \to [0, 1] \) by \( c(T) = T(\mu, \ldots, \mu)(E) \). By Conjecture [1,2], there is a sequence \( \bar{\eta} \) in \( A^{[\infty]} \) and an \( r \) such that if \( \tau \) in \( A_m \) and \( i_k \ (k < m) \) is admissible for \( \tau \), then

\[
|c(\tau(\eta_{i_0}(\mu, \ldots, \mu), \ldots, \eta_{i_{m-1}}(\mu, \ldots, \mu))) - r| \leq \epsilon/2.
\]

Using Claim 4.5 and an argument similar to that used at the end of the proof of Theorem 4.3, inductively construct a finite sets \( F_i \ (i \in \omega) \)
such that \( |F_i| = \#(\eta_i)\), \( F_i < F_{i+1} \) and
\[
|c(\tau(\eta_0(\bar{\mu} \upharpoonright F_{i0}), \ldots, \eta_{m-1}(\bar{\mu} \upharpoonright F_{im-1}))) - r| \leq \epsilon/2
\]
whenever \( \tau \) is in \( A_m \) and \( i_k (k < m) \) is admissible for \( \tau \).

It suffices to show that if we define \( \nu_i = \eta_i(\bar{\mu} \upharpoonright F_i) \), then for all \( \zeta \) in \( C_{\bar{\mu}} \), \( |\zeta(E) - r| \leq \epsilon/2 \), by Claim 4.9. Suppose that this is not the case and let \( \zeta \) witness this. Then there is a \( \tau \) in \( A_m \) for some \( m \) and an \( i_k (k < m) \) which is admissible for \( \tau \) such that
\[
|\tau(\nu_0, \ldots, \nu_{m-1})(E) - r| > \epsilon,
\]
which contradicts our construction of \( \bar{\nu} \).

In order to complete the proof, let \( E_\xi (\xi < \omega_1) \) be an enumeration of the subsets of \( T \) as provided by the Continuum Hypothesis. Construct \( \bar{\mu}_\xi (\xi < \omega_1) \) by induction which is \( \leq^* \)-decreasing, satisfies that \( \bar{\mu}_\xi \leq \lambda \) for all \( \xi \) and is such that \( \zeta(E_\xi) = \zeta'(E_\xi) \) for every \( \zeta, \zeta' \in C_{\bar{\mu}_{\xi+1}} \). This is possible by Claims 4.8 and 4.10. We are finished with the observation that we have arranged that
\[
\bigcap_{\xi < \omega_1} C_{\bar{\mu}_\xi}
\]
consists of a single element \( \nu \). (The intersection is non empty since \( C_{\bar{\mu}_\xi} (\xi < \omega_1) \) is a \( \subseteq \)-descending sequence of compact sets). By Claim 4.9 each \( C_{\bar{\mu}_\xi} \) is closed under \( \hat{+} \) and therefore \( \nu \hat{+} \nu = \nu \).

Next we will prove the following theorem.

**Theorem 4.11.** If \( \mu \in P(T) \) is an idempotent measure, then \( \mu \) is \( F \)-invariant.

**Proof.** Suppose that \( \mu \in P(T) \) satisfies \( \mu \hat{+} \mu = \mu \); we need to show that \( \mu \) is \( F \)-invariant. First observe that
\[
\mu(\{1\}) = \hat{\mu} \mu(\{1\}) = \mu \otimes \mu(\emptyset) = 0.
\]

Also
\[
\mu \hat{+} \mu(T \in T : \exists A(T = A \hat{+} 1)) = \mu(T) \cdot \mu(\{1\}) = 1 \cdot 0 = 0.
\]

Since \( \mu \) is an idempotent, following identities hold:
\[
\mu = \mu \hat{+} (\mu \hat{+} \mu) = (\mu \hat{+} \mu) \hat{+} \mu
\]
\[
\mu = \mu \hat{+} (\mu \hat{+} (\mu \hat{+} \mu)) = \mu \hat{+} ((\mu \hat{+} \mu) \hat{+} \mu)
\]

Now suppose that \( Z \subseteq T \).
\[
\mu(Z) = \mu(\{T \in Z : \exists A \exists B \exists C (T = (A \hat{+} B) \hat{+} C)\})
\]
\[
= (\mu \hat{+} \mu) \hat{+} \mu(\{T \in Z : \exists A \exists B \exists C (T = (A \hat{+} B) \hat{+} C)\})
\]
closed under the operation of $F$ generating set $x$.

**Proof.** This is essentially Theorem 2.1 of [13], modulo unraveling definitions. Define $I_n = \{2^{-i} : i < n-1\} \cup \{1\}$. Let $P_n$ denote the elements of $F$ of the form $g_T = (T \to I_n)$, where $T$ is in $T_n$ and set $P = \bigcup_n P_n$. The set $P$ consists of the positive elements of $F$ with respect to the generating set $x_n$ ($n \in \omega$) where $x_{n+1} = x_0^nx_1x_0^{-n}$. In particular, $P$ is closed under the operation of $F$.

If $S$ and $T$ are in $T$ and $g_S \circ g_T$ is in $P$, define $S \cdot T$ by the equation $g_{ST} = g_S \circ g_T$. Unraveling the definitions reveals that a copy of $T_m$ in $A_n$ is a set of the form $\{T \cdot \nu : T \in T_m\}$ where $\nu$ is some element of $A_n$ such that for all $T$ in $T_m$, $T \cdot \nu$ is defined. Thus (2) is equivalent to saying that for every $m$ there is an $n$ such that $P_n$ is $1/2$-Ramsey with respect to $P_m$. It now follows from Theorem 2.1 of [13] applied to $G = F$ and $H = P$ that (2) implies (1).

We finish this section by remarking that a weak form of the finitary version of Conjecture [12] is in fact equivalent to the amenability of $F$. If $m \in \mathbb{N}$, an embedding of $T_m$ in $T$ is function of the form

$$T \mapsto T(U_0, \ldots, U_{m-1})$$

for some sequence $U_i$ ($i < m$) of elements of $T$. Observe that such an embedding maps into $T_n$ for some $n$. An embedding of $T_m$ into $A_n$ is a convex combination of embeddings of $T_m$ into $T_n$. The range of such an embedding is a copy of $T_m$ in $A_n$.

**Theorem 4.12.** The following are equivalent:

1. Thompson’s group $F$ is amenable.
2. For every $m$ there is an $n$ such that if $E \subseteq T_n$, then there is a copy $\mathcal{X}$ of $T_m$ in $A_n$ such that $|\nu(E) - \nu'(E)| \leq 1/2$ whenever $\nu, \nu' \in \mathcal{X}$.

**Remark 4.13.** A more restrictive notion of copy is the following: a strong copy of $T_m$ in $A$ is the range of a function of the form

$$T \mapsto T(\mu_0, \ldots, \mu_{m-1})$$

where $\mu_i$ ($i < m$) are elements of $A$. The finite form of Conjecture [12] would assert that for every $m$ there is an $n$ such that if $f : T_n \to [0, 1]$, then there is a strong copy of $T_m$ in $A_n$ on which $f$ is within $1/2$ of being constant. It is unclear if this assertion is equivalent to the amenability of $F$. 

**Proof.** This is essentially Theorem 2.1 of [13], modulo unraveling definitions. Define $I_n = \{2^{-i} : i < n-1\} \cup \{1\}$. Let $P_n$ denote the elements of $F$ of the form $g_T = (T \to I_n)$, where $T$ is in $T_n$ and set $P = \bigcup_n P_n$. The set $P$ consists of the positive elements of $F$ with respect to the generating set $x_n$ ($n \in \omega$) where $x_{n+1} = x_0^n x_1 x_0^{-n}$. In particular, $P$ is closed under the operation of $F$.

If $S$ and $T$ are in $T$ and $g_S \circ g_T$ is in $P$, define $S \cdot T$ by the equation $g_{ST} = g_S \circ g_T$. Unraveling the definitions reveals that a copy of $T_m$ in $A_n$ is a set of the form $\{T \cdot \nu : T \in T_m\}$ where $\nu$ is some element of $A_n$ such that for all $T$ in $T_m$, $T \cdot \nu$ is defined. Thus (2) is equivalent to saying that for every $m$ there is an $n$ such that $P_n$ is $1/2$-Ramsey with respect to $P_m$. It now follows from Theorem 2.1 of [13] applied to $G = F$ and $H = P$ that (2) implies (1).
On the other hand, if $F$ is amenable, then it is well known (see, e.g., [12]) that for every $\epsilon > 0$, there is a finite subset of $P$ which is $\epsilon$-Følner. This readily implies that for every $m$ there is an $n$ such that $P_n$ is $1/2$-Ramsey with respect to $P_m$. □

5. Concluding remarks

We still do not know if $F$ is amenable, but I feel Conjectures 1.2 and 1.3 are based on sound heuristics from Ramsey theory. It is rare in the Ramsey theory of countably infinite sets that there are difficult counterexamples to Ramsey-type theorems. On the other hand, there are many deep and often difficult examples of Ramsey-type theorems: Hindman’s Theorem [9], Gowers’s generalization of Hindman’s Theorem to $\text{FIN}_k$ [5], the Hales-Jewett Theorem [7], and the Halpern-Läuchli Theorem [8]. See [20] for further reading on these theorems as well as many others.

Also, while we do not know whether $(P(S), \star)$ contains an idempotent if $(S, \star)$ is an arbitrary groupoid, we do know that there are quite different examples of groupoids which admit idempotent measures: semigroups (Ellis’s Lemma), finite groupoids (Brouwer’s Fixed Point Theorem), and groupoids depending only on one variable (the Schauder-Tychonoff Fixed Point Theorem). In fact these classes of groupoids can be used to establish that many easily definable subsets of $\mathbb{T}$ are automatically invariantly measurable. A further discussion of this can be found in the forthcoming [11].

It was noted above that the order of integration used in the definition of $\otimes$ was significant. Suppose that $\mu \in P(\mathbb{T})$ is such that $\mu(F) = 0$ whenever $F$ is a finite subset of $\mathbb{T}$. Under our choice of the order of integration, $\mu \hat{+} \mu(\{A \hat{+} B : |A| < |B|\}) = 1$. This is because for any fixed $A$, there are only finitely many $B$ such that $|B| \leq |A|$ and therefore $\mu(\{B \in \mathbb{T} : |A| < |B|\}) = 1$. If the opposite order of integration is used in defining $\hat{+}$, we would have that $\mu \hat{+} \mu(\{A \hat{+} B : |A| > |B|\}) = 1$. (Under either choice, $\mu \hat{+} \mu(\{A \hat{+} B : |A| = |B|\}) = 0$.) If $\sigma$ is a finite binary sequence and $T$ is an element of $\mathbb{T}$ regarded as a term, let $T/\sigma$ denote the subterm of $T$ with address $\sigma$ (recursively this is defined by $T/() = T$, $(A \hat{+} B)/0 \hat{\sigma} = A/\sigma$, and $(A \hat{+} B)/1 \hat{\sigma} = B/\sigma$). Notice that if $\mu$ is an idempotent, then $\mu(\{T \in \mathbb{T} : |T/001| < |T/01| < |T/10|\}) = 1$. 

\[\mu(\{T \in \mathbb{T} : |T/001| < |T/01| < |T/10|\}) = 1.\]
In [12], it was demonstrated that if \( \mu \) is an \( F \)-invariant element of \( P(\mathbb{T}) \), then \( \mu \)-a.e. \( T \) satisfy one of the following inequalities:

\[
|T/001| < |T/01| < |T/10| \\
|T/001| > |T/01| > |T/10|.
\]

It is tempting to speculate that not only do idempotent measures exist in \( P(\mathbb{T}) \) — and hence that \( F \) is amenable — but that if we let \( I \) denote the collection of all \( \mu \) which are idempotents in an extension of \( \hat{F} \) defined using either choice of the order of integration, then the closed convex hull of \( I \) consists of all invariant measures.

References

[1] J. E. Baumgartner. A short proof of Hindman’s theorem. *J. Combinatorial Theory Ser. A.*, 17:384–386, 1974.
[2] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson’s groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.
[3] R. Ellis. Distal transformation groups. *Pacific J. Math.*, 8:401–405, 1958.
[4] S. M. Gersten and J. R. Stallings, editors. *Combinatorial group theory and topology*, volume 111 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1987. Papers from the conference held in Alta, Utah, July 15–18, 1984.
[5] W. T. Gowers. Lipschitz functions on classical spaces. *European J. Combin.*, 13(3):141–151, 1992.
[6] R. L. Graham and B. L. Rothschild. Ramsey’s theorem for \( n \)-parameter sets. *Trans. Amer. Math. Soc.*, 159:257–292, 1971.
[7] A. W. Hales and R. I. Jewett. Regularity and positional games. *Trans. Amer. Math. Soc.*, 106:222–229, 1963.
[8] J. D. Halpern and H. Läuchli. A partition theorem. *Trans. Amer. Math. Soc.*, 124:360–367, 1966.
[9] N. Hindman. Finite sums from sequences within cells of a partition of \( N \). *J. Combinatorial Theory Ser. A.*, 17:1–11, 1974.
[10] N. Hindman and D. Strauss. *Algebra in the Stone-Čech compactification*, volume 27 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1998. Theory and applications.
[11] J. Tatch Moore. Analysis of the amenability problem for Thompson’s group \( F \). In preparation.
[12] J. Tatch Moore. Fast growth in Følner function for Thompson’s group \( F \). ArXiv preprint 0905.1118, August 2009.
[13] J. Tatch Moore. Amenability and Ramsey theory. ArXiv Preprint 1106.3127, June 2011.
[14] J. Tatch Moore. A note on Shavgulidze’s papers concerning the amenability problem for Thompson’s group \( F \). ArXiv preprint 1102.0747, February 2011.
[15] A. L. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.
[16] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.
[17] E. T. Shavgulidze. Amenability of discrete subgroups of the group of diffeomorphisms of the circle. *Russ. J. Math. Phys.*, 16(1):130–132, 2009.

[18] E. T. Shavgulidze. The Thompson group $F$ is amenable. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 12(2):173–191, 2009.

[19] Letter from Richard Thompson to George Francis, dated September 26, 1973.

[20] S. Todorcevic. *Introduction to Ramsey spaces*, volume 174 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2010.

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