EQUIVARIANT CELLULAR HOMOLOGY AND ITS APPLICATIONS

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Abstract. In this work we develop a cellular equivariant homology functor and apply it to prove an equivariant Euler-Poincaré formula and an equivariant Lefschetz theorem.

1. Introduction

Let $D$ be an arbitrary small topologically enriched category. In this paper we develop a $D$-CW-homology functor which allows for easy computation of the ordinary $D$-equivariant homology defined by E. Dror Farjoun in [4]. Our approach is a generalization of the $G$-CW-(co)homology functor constructed by S.J. Willson in [13] for the case of $G$ being compact Lie group.

Then we apply the $D$-CW-homology functor to obtain:

(i) Equivariant Euler-Poincaré formula:

$$\chi^D(X) = \sum_{n=0}^{\infty} (-1)^n \tilde{rk}_{HS}(H^D_n(X;\mathcal{I}))$$

This formula establishes a connection between the equivariant homology and an equivariant Euler characteristic; $\tilde{rk}_{HS}(-)$ is a slight modification of Hattori-Stallings rank (originally defined in [8],[11]).

(ii) Equivariant Lefschetz theorem: Let $X$ be a triangulated $D$-space, $f : X \to X$ an equivariant map. If the equivariant Lefschetz number

$$\Lambda_D(f) = \sum_{n=0}^{\infty} (-1)^n \tilde{tr}_{HS}(H^D_n(f;\mathcal{I}))$$

is not equal to zero, then there are $f$-invariant orbits in $X$, moreover the orbit types of the invariant orbits may be recovered from $\Lambda_D(f)$.

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2. Preliminaries

2.1. $D$-spaces. Let $\mathcal{T}$ denote the category of the compactly generated Hausdorff topological spaces. Fix an arbitrary small category $D$ enriched over $\mathcal{T}$. We work in the category $\mathcal{T}^D$ of functors from $D$ to $\mathcal{T}$. The objects of this category

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are called topological diagrams or just $D$-spaces. The arrows in $\mathcal{T}op^D$ are natural transformations of functors or equivariant maps.

2.2. $D$-homotopy. An equivariant homotopy between two $D$-maps $f, g : X \to Y$, where $X, Y$ are $D$-diagrams, is a $D$-map $H : X \times I \to Y$, where $I$ denotes the constant $D$-space $I(d) = [0, 1]$. A homotopy equivalence $f : X \to Y$ is a map with a (two sided) $D$-homotopy inverse.

2.3. $D$-orbits. We recall now the central concept of the $D$-homotopy theory (introduced in [1],[3]) – that of $D$-orbit. A $D$-orbit is a $D$-space $T : D \to \mathcal{T}op$, such that $\text{colim}_D T = \{\ast\}$. A free $D$-orbite generated in $d \in \text{ob}(D)$ is $\mathcal{T}op^D \ni F^d = \text{hom}_D(d, \ast)$, i.e. $F^d(d') = \text{hom}_D(d, d')$ and $F^d(d' \to d'')$ is given by the composition. Clearly $F^d$ is a $D$-orbit. A $D$-space $X$ is called free iff for any $s \in \text{colim}_D X$ the full orbit $T_s$ lying over $s$ is free.

2.4. $D$-CW-complexes. A $D$-cell is a $D$-space of the form $T \times e^n$, where $T$ is a $D$-orbit and $e^n$ is the standard $n$-cell. An attaching map of this $D$-cell to some $D$-space $X$ is a map $\phi : T \times \partial e^n \to X$.

A (relative) $D$-CW-complex $(X, X_{-1})$ is a $D$-space $X$ together with a filtration $X_{-1} \subset X_0 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots \subset X = \text{colim}_n X_n$, such that $X_{n+1}$ is obtained from $X_n$ by attaching a set of $n$-dimensional $D$-cells. Namely one has a push-out diagram of $D$-spaces:

$$
\begin{align*}
\coprod_i (T_i \times \partial e^n) & \xrightarrow{\phi} X_{n-1} \\
\downarrow & \downarrow \\
\coprod_i (T_i \times e^n) & \xrightarrow{\phi} X_n
\end{align*}
$$

If $X_{-1} = \emptyset$ we call the $D$-CW-complex absolute.

Let $X$ be a $D$-CW-complex. A $D$-subspace $Y \subset X$ is called the cellular subspace if $Y$ has a $D$-CW-structure such that each cell of $Y$ is also a cell of $X$.

2.5. The category of orbits. The category of orbits $\mathcal{O}$ is a full topological subcategory of $\mathcal{T}op^D$ generated by all $D$-orbits.

Usually $\mathcal{O}$ is not a small category. For example for $D = J = (\bullet \rightarrow \bullet)$, then $\mathcal{O} \cong \mathcal{T}op$. A model category has been constructed for the $D$-spaces of arbitrary orbit type in [3]. We will be interested in the diagrams which are homotopy equivalent to the finite $D$-CW-complexes, i.e. only finite number of orbit types appear in such diagram. We collect those orbits into the a full subcategory $\mathcal{O}'$ of $\mathcal{O}$ with a finite amount of objects.

2.6. Orbit point $(\cdot)^\mathcal{O}$ and realization $| \cdot |_D$ functors. Suppose $\mathcal{O}$ is a small category of $D$-orbits. An orbit point functor $(\cdot)^\mathcal{O} : \mathcal{T}op^D \to \mathcal{T}op^{\mathcal{O}^{op}}$ is a generalization to the diagram case of Bredon’s fixed point functor. For any $D$-space $X$ (usually of type $\mathcal{O}$) $(X)^\mathcal{O}$ is a $\mathcal{O}^{op}$ diagram such that $(X)^\mathcal{O}(T) = \text{hom}_D(T, X)$ for all $T \in \text{ob}(\mathcal{O})$ and the arrows of the diagram are induced by composition with the maps between orbits.

If $f : X \to Y$ is an equivariant map between two $D$-spaces, then there exist an $\mathcal{O}^{op}$-equivariant map $f^\mathcal{O} : X^\mathcal{O} \to Y^\mathcal{O}$, which is obtained from $f$ by composition:

$$X^\mathcal{O}(T) = \text{hom}_D(T, X) \ni g \mapsto f \circ g \in \text{hom}_D(T, Y) = Y^\mathcal{O}(T)$$
The fundamental property of $(\cdot)^O$ functor is that for any $D$-space $X$ the $O^{op}$-space $(X)^O$ is $O^{op}$-free [3, 3.7].

**Example 2.1.** Consider a free $D$-space $X$. And let the orbit category $O$ consists of all the free orbits. Then $O$ is isomorphic to $D^{op}$ as a category and $(X)^O \cong X$ (Yoneda’s lemma).

Another easy case occurs then $X$ is a $D$-CW-space. We shall discuss it in the next section.

There exist a left adjoint to $(\cdot)^O$. It is called realization functor $|\cdot|_D$, since it takes an $O^{op}$-space and produce a $D$-space with the prescribed orbit point data (up to local weak equivalence). Realization functor in the group case has been constructed by A.D.Elmendorf in [10, I.5] \((U(D), \chi^D)\). Or, equivalently, we say that $\chi^D(X) \in U(D)$ is equal to the alternating sum of the orbit types over the dimensions of the cells of $X$ in the free abelian group $U(D) = \bigoplus_{T \in Isol(hO)} \mathbb{Z}$ generated by the homotopy types of orbits.

**3. Equivariant cellular homology**

**3.1. $O^{op}$-CW structure on the orbit point space of a $D$-CW-complex.** Let $C$ denote a subcategory of $\mathcal{T}op^{O^{op}}$ which is obtained as the image of $\mathcal{T}op^D$ under the functor $(\cdot)^O$. Recall that there is the inclusion of the categories $u : D \hookrightarrow O^{op}$, where $u(d) = F^d$, for each $d \in ob(D)$. Hence there is a functor $\text{Res} : \mathcal{T}op^{O^{op}} \to \mathcal{T}op^D$. By abuse of notation we denote by $\text{Res}$ also $\text{Res}|_C$.

**Lemma 3.1.** The functor $(\cdot)^O$ is fully faithful.

**Proof.** The faithfulness is clear. We have to show only that for any map $f_0 : X^O \to Y^O$ there exists a map $f : X \to Y$ s.t. $f_0 = f^O$. Take $f = \text{Res}(f_0)$, then if $X$ and $Y$ were orbits the result follows from the bijective correspondence induced by the Yoneda’s lemma: $\text{hom}_D(X, Y) = hom_{O^{op}}(X^O, Y^O)$. The general claim will follow from the comparison of $f_0$ and $f^O$ orbitwise, i.e. by their action on each full orbit. Fortunately the functor $(\cdot)^O$, being right adjoint, commutes with taking full orbit (pullback).

**Lemma 3.2.** The pair of functors $\text{Res}(\cdot) : C \leftrightarrow \mathcal{T}op^D : (\cdot)^O$ induce the equivalence of the categories $C$ and $\mathcal{T}op^D$.

**Proof.** We need to construct the natural isomorphisms of the functors $id_{\mathcal{T}op^O} \cong \text{Res}((\cdot)^O)$ and $id_C \cong (\text{Res}(\cdot))^O$.

Let $X \in \mathcal{T}op^D$, then $\text{Res}(X^O) \cong X$ because the generalized lemma of Yoneda [9] induces the objectwise homeomorphisms and the equivariance is preserved by the naturality of the Yoneda’s isomorphism. But an equivariant map which is the objectwise homeomorphism is an isomorphism of $D$-spaces, hence the first isomorphism of functors.

Let $X^O \in C$, then $\text{Res}(X^O) \cong X$ by the first homeomorphism, then $(\text{Res}(X^O))^O \cong X^O$. Hence the second isomorphism.
Proposition 3.3. Let \( X \) be a (pointed) \( D \)-CW–space of orbit type \( O \), where \( O \) is a small category of orbits. Consider the \( O^{op} \)–space \( X^{O} \) to be the orbit point space of \( X \). Then \( X^{O} \) has \( O^{op} \)-CW structure which corresponds to the \( D \)-CW structure of \( X \) in the following sense: let \( pt_{D} \subseteq X_{0} \subseteq X_{1} \subseteq \cdots \subseteq X_{n} \subseteq \cdots \subseteq X = \text{colim}_{n} X_{n} \) be a \( D \)-CW filtration of \( X \), such that each \( X_{n} \) is a push-out:

\[
\bigsqcup_{i} T_{i} \times S^{n-1} \xrightarrow{\phi} X_{n-1} \\
\downarrow \\
\bigsqcup_{i} T_{i} \times D^{n} \xrightarrow{\phi} X_{n}
\]

then there exist a \( O^{op} \)-CW–filtration: \( pt_{O^{op}} \subseteq X_{0}^{O} \subseteq X_{1}^{O} \subseteq \cdots \subseteq X_{n}^{O} \subseteq \cdots \subseteq X^{O} = \text{colim}_{n} X_{n}^{O} \), such that \( X_{n}^{O} = (X_{n})^{O} \), and

\[
\bigsqcup_{i} F_{T_{i}} \times S^{n-1} \xrightarrow{\phi^{O}} X_{n-1}^{O} \\
\downarrow \\
\bigsqcup_{i} F_{T_{i}} \times D^{n} \xrightarrow{\phi^{O}} X_{n}^{O}
\]

is a push-out square.

Proof. We proceed by the induction on the skeleton of \( X \).

\[
X_{0}^{O}(T) = (\prod T_{i})^{O}(T) = \prod ((T_{i})^{O}(T)) = \prod \text{hom}_{D}(T, T_{i}) = \\
\prod \text{hom}_{O^{op}}(F_{T_{i}}, F_{T}) = \prod F_{T}(T).
\]

Hence the base of the induction.

Suppose we know the claim for \( X_{n} \). Then it follows for \( X_{n+1} \) since \((\cdot)^{O}\) is both left and right adjoint, so commutes both with push-outs and products.

3.2. \( D \)-CW-homology functor. The construction of the (co)homology functor in \([1, 4.16]\) depends on the specific \( D \)-CW-decomposition of \( X^{O} \). We apply this construction to the cellular structure of \( X^{O} \), which was constructed in 3.3 and obtain the required \( D \)-CW-homology functor.

3.3. Isotropy ring \( \mathcal{I} \). In \([13]\) a universal coefficient system for the \( G \)-equivariant homology have been developed where \( G \) is a compact Lie group. Let us generalize this approach to the coefficient systems for the classical \( D \)-homology theory. Suppose \( O' \) is a small, full subcategory of the orbit category \( O \). Let \( X \) be a \( D \)-space of orbit type \( O' \). Then a coefficient system for the ordinary (co)homology is a homotopy (co)functor \( M : O' \to (R - \text{mod}) \).

Definition 3.4. Let \( R \) be a commutative ring. An isotropy ring \( \mathcal{I} = I_{D}^{R, O'} \) is generated by \( \text{mor}(hO') \) as a free \( R - \text{mod} \). Define the multiplication on the generators by

\[
fg = \begin{cases} 
    f \circ g, & \text{if } \text{codom}(g) = \text{dom}(f) \\
    0, & \text{otherwise}
\end{cases}
\]

and extend the definition to the rest of the elements of \( \mathcal{I} \) by linearity.
Proposition 3.5. The category $\mathcal{M}$ of the left $\mathcal{I}$-modules which satisfy:

$\forall M \in \text{ob}(\mathcal{M}), M = \bigoplus_{T \in \text{ob}(\mathcal{hO}')} 1_T M$

(where $\{1_T M\}_{T \in \text{ob}(\mathcal{hO}')}^\perp$ are left $R$-modules) and the category of $R(h\mathcal{O}')$-mod of functors from $h\mathcal{O}'$ to the category of left $R$-modules are equivalent.

Proof. Let us define a pair of functors which induce the required equivalence:

$\zeta: \mathcal{M} \rightsquigarrow \mathcal{R}(h\mathcal{O}')$ mod $\xi$.

Let $M \in \text{ob}(\mathcal{M})$, $T \in \text{ob}(\mathcal{hO}')$, then define

$\zeta M(T) = 1_T M$.

If $\text{mor}(h\mathcal{O}') \ni f: T_1 \rightarrow T_2$, then define

$\zeta M(f)(1_{T_1} m) = f 1_{T_1} m = (1_{T_2} f) 1_{T_1} m \in 1_{T_2} M$.

Obviously the morphisms of the left $\mathcal{I}$-modules correspond to the natural transformations of the functors.

Given a $\mathcal{R}(h\mathcal{O}')$-module $N$, then

$\xi N = \bigoplus_{T \in \text{ob}(h\mathcal{O}')} N(T)$, as a left $R$-module.

Define the left $\mathcal{I}$-module structure on $\xi N$ by $f(\ldots, n, \ldots) = (\ldots, fn, \ldots)$, where

$N(\text{codom}(f)) \ni fn = \begin{cases} f(n), & \text{if } n \in N(\text{dom}(f)) \\ 0, & \text{otherwise} \end{cases}$

Now it is clear that the defined functors provide the equivalence of the categories.

Remark 3.6. The ring $\mathcal{I}$ considered as a left $\mathcal{I}$-module is an object of $\mathcal{M}$, because $\mathcal{I} \cong \bigoplus_{T \in \text{ob}(h\mathcal{O}')} 1_T \mathcal{I}$ (as left $R$-modules) by the construction. But it also carries an obvious structure of the right $\mathcal{I}$ modules correspond to the natural transformations of the functors.

Given a $\mathcal{R}(h\mathcal{O}')$-module $N$, then

$\xi N = \bigoplus_{T \in \text{ob}(h\mathcal{O}')} N(T)$, as a left $R$-module.

Remark 3.7. If $\text{ob}(h\mathcal{O}')$ is finite then the ring $\mathcal{I}$ has a two-sided identity element $1 = \sum_{T \in \text{ob}(h\mathcal{O}')} 1_T \mathcal{I}$ together with its decomposition into the sum of the orthogonal idempotents and the condition (3) is redundant.

Definition 3.8. The augmentation $\varphi: \mathcal{I} \rightarrow \bigoplus_{T \in \text{Isob}(\text{ob}(h\mathcal{O}'))} R$ is defined for any

$I \ni g = \sum_{T \in \text{ob}(h\mathcal{O}')} \sum_{f \in \text{mor}(T, T)} r_f + \sum_{h \in \text{mor}(T_1, T_2, T_1 \neq T_2)} s_h h$

(only a finite number of $r_f, s_h \in R$ is non equal to zero) to be

$\varphi(g) = (\ldots, \sum_{f \in \text{mor}(T, T)} r_f, \ldots) \in \bigoplus_{T \in \text{Isob}(\text{ob}(h\mathcal{O}'))} R$

Remark 3.9. The idempotents in $\mathcal{I}$ which correspond to the $D$-homotopy equivalent orbits are identified under $\varphi$. Apparently, $\varphi$ is an epimorphism of rings. Consider the abelinization functor $\text{Ab}: (\text{Rings}) \rightarrow \text{Ab}$ which corresponds to a ring its additive group divided by the commutator subgroup. Then $\text{Ab}(\varphi): \text{Ab}(\mathcal{I}) \rightarrow \bigoplus_{T \in \text{Isob}(\text{ob}(h\mathcal{O}'))} R$. The last map will be used to obtain a generalization of the Euler-Poincaré formula.
4. Applications

Let $X$ be a finite $D$-CW-complex of type $\mathcal{O}'$ for some orbit category $\mathcal{O}'$ with $\text{obj}(\mathcal{O}')$ a finite set.

4.1. Equivariant Euler-Poincaré formula. We remind that the equivariant Euler characteristic lies in the abelian group $U(D) \cong \bigoplus_{\text{iso}(\text{obj}(h\mathcal{O}'))} \mathbb{Z}$, so in order to apply Hattori–Stallings machinery we need to choose a coefficient system for the equivariant homology such that the resulting chain complex and homology groups will be endowed with the module structure over some ring $S$ which allows an epimorphism $\varepsilon : \text{Ab}(S) \twoheadrightarrow U(D)$.

Our choice of the coefficient system for the equivariant homology will be the isotropy ring $\mathcal{I} = I_D^{\mathcal{O}'}$ taken over itself as a left module.

Lemma 4.1. Let $X$ be a finite $D$-CW-complex. Suppose $X$ has $n_q$ $q$-dimensional cells and $t_1 + \cdots + t_s = n_q$, $t_i$ is the number of $q$-dimensional cells of the same homotopy type $T_i \in \text{Iso}(\text{obj}(h\mathcal{O}'))$. Then $C_q(X) \otimes_{\mathcal{O}' \mathcal{I}} \mathcal{I} \cong C_q(T_1) t_1 + \cdots + C_q(T_s) t_s$ as a left $\mathbb{Z}$-module.

Proof. Let $t_i = r_{i1} + \cdots + r_{ik}$, where $r_{ij}$ is the number of $q$-dimensional cells of type $T_{ij} \in \text{obj}(\mathcal{O})$ of homotopy type $T_i$. By the construction of the equivariant homology $C_q(X) = \bigoplus_{s=1}^k (\oplus_{j=1}^k \mathbb{Z}(\text{hom}_{\mathcal{O}}(?, T_{ij})^{r_{ij}}))$. The dual Yoneda isomorphism [3, p.74] implies:

$$C_q(X) \otimes_{\mathcal{O}' \mathcal{I}} \mathcal{I} \cong \bigoplus_{i=1}^k (\oplus_{j=1}^k \mathcal{I}(T_{ij})^{r_{ij}}) \cong \bigoplus_{i=1}^k \oplus_{j=1}^k (1 T_i \mathcal{I})^{r_{ij}},$$

If $T_{ij}$ is isomorphic to $T_{ij'}$ in $h\mathcal{O}'$ then there is an obvious isomorphism of the left $\mathbb{Z}$-modules and right $\mathcal{I}$-modules $1 T_{ij} \mathcal{I} \cong 1 T_{ij'} \mathcal{I}$. Let us choose a representative $T_i$ of each isomorphism class of objects in $h\mathcal{O}'$, then

$$C_q(X) \otimes_{\mathcal{O}' \mathcal{I}} \mathcal{I} \cong \bigoplus_{i=1}^k (1 T_i \mathcal{I})^{(\sum_{j=1}^k r_{ij})} \cong \bigoplus_{i=1}^k (1 T_i \mathcal{I})^{t_i} \cong \bigoplus_{i=1}^k (\mathcal{I}(T_i))^{t_i}$$

Because of [3, 4] the equivariant chain complex $\{C_q(X) \otimes_{\mathcal{O}' \mathcal{I}} \mathcal{I}\}_{q=0}^{\dim X}$ is a complex of projective right $\mathcal{I}$-modules and the equivariant homology is endowed with the right $\mathcal{I}$-module structure.

Notation: $\chi_{\text{HS}}(\cdot)$ means Euler characteristic of a $\mathcal{I}$ differential complex with respect to $r_{k\text{HS}}(\cdot)$.

Proposition 4.2. Let $K_* = C_*(X) \otimes_{\mathcal{O}' \mathcal{I}} \mathcal{I}$ be a right $\mathcal{I}$-complex, then $\chi^D(X) = \text{Ab}(\phi)(\chi_{\text{HS}}(K_*))$ whenever left side is defined.

Proof. It is easy to see that $r_{k\text{HS}}(1 T \mathcal{I}) = 1 T \in \text{Ab}(\mathcal{I})$. Lemma 4.1 together with [4] completes the proof.

Now we combine 4.2 with the additivity properties of the Hattori-Stallings rank and obtain the following

Theorem 4.3. $\chi^D(X) = \text{Ab}(\phi)((\sum_{n=0}^{\infty} (-1)^n r_{k\text{HS}} H_n^D(X; \mathcal{I})), \text{ whenever the left side is defined.}$

Example 4.4. Consider the $J$-diagram:
Z has two 0-cells of type $T_2 = [\downarrow\uparrow]$ and one 1-cell of type $T_3 = [\downarrow\uparrow\uparrow]$, hence
\[ \chi'(Z) = 2[\downarrow\uparrow\uparrow] - [\downarrow\uparrow\uparrow\uparrow]. \]

The category $\mathcal{O}'$ of orbits contains two objects: $T_2, T_3$. The cellular chain complex tensored with the coefficients $I = I_Z, \{T_2, T_3\}$ becomes:
\[ \cdots \to 0 \to 1_{T_3} \mathcal{I} \to (1_{T_2} \mathcal{I})^2 \to \cdots \]
and $\partial_1 = 0$ from the orbit type considerations.

Let us, for comparison, calculate the $J$-equivariant homology of $Z$ with $Z_{\mathcal{O}'}$ coefficients:
\[ H_i^J(Z, Z_{\mathcal{O}'}) = H_i(\text{colim}_J Z, Z, \mathcal{I}) \]
(see [1, 5.2]). Then colim$_J Z = I = [0, 1]$ and
\[ H_i^J(Z, Z_{\mathcal{O}'}) = \begin{cases} Z, & i = 0 \\ 0, & \text{otherwise} \end{cases} \]

We can see that $Z_{\mathcal{O}'}$ coefficients are inappropriate to the Euler-Poincaré formula.

4.2. Equivariant Lefschetz theorem. Using cellular equivariant homology functor we are able now to prove a version of the equivariant Lefschetz theorem.

Some result of the Lefschetz type in the equivariant setting may be obtained already by applying the ordinary Lefschetz theorem: consider an equivariant map $f : X \to X$, where $X$ is a diagram over small category $D$, then if the Lefschetz number $\Lambda(\text{colim}_D X) \neq 0$ there are $f$-invariant $D$-orbits in $X$. However the advantage of using the equivariant homology and equivariant Lefschetz number $\Lambda_D(X) \in U(D)$ is that we obtain the specific information about orbit type of the invariant orbit.

First we give a technical

**Definition 4.5.** A $D$-CW-complex $X$ will be called the triangulated $D$-space if the natural CW-structure of colim $X$ also triangulates colim $X$.

The following lemma will be used in the proof of the equivariant Lefschetz theorem.

**Lemma 4.6.** Let $X$ be a triangulated diagram, then for any refinement $Y$ of the triangulation of colim $X$, there exists a $D$-CW-complex $X'$, such that $X'$ is $D$-homeomorphic to $X$ and colim $X' = Y$ (as the triangulated spaces). $X'$ will be called the refinement of $X$. 
Proof. Consider a new simplex $\Delta$ in the triangulation of $Y$. It lies in some old simplex of $\text{colim} \; X : \Delta \in \Delta'$. Then consider the pull-back:

$$\lim \left( \begin{array}{c} X \\ \Delta \hookrightarrow \text{colim}_D X \end{array} \right) = T \times \Delta,$$

where $T$ is the orbit which lies over $\Delta'$.

We’ve obtained the cell of the new $D$-CW–complex $X'$. Continuing in the same way for the rest of the simplices of $Y$ completes the construction of $X'$. Hence $D$-CW–complex $X'$ has the same underlying topological diagram as $X$, therefore they are $D$-homeomorphic.

\[ \square \]

**Definition 4.7.** Let $f : X \to X$ be a map of the finite triangulated $D$-space $X$ of orbit type $O'$, where $O'$ is an orbit category with the finite number $n$ of objects. Let $I = I_D^{O'}$. Then the equivariant Lefschetz number of $f$:

$$U(D) \ni (\lambda_1, \ldots, \lambda_n) = \Lambda_D(f) = \text{Ab}(\varphi)(\sum_{k=0}^{\infty} (-1)^k \text{tr}_{\text{HS}}(H_k(f; I)))$$

**Theorem 4.8.** Let $X$ be a finite triangulated diagram over $D$. $f : X \to X$ be a $D$-map. $\Lambda_D(f) = (\lambda_1, \ldots, \lambda_n) \in U(D)$ - Lefschetz number of $f$. Then if there is no $f$-invariant orbit of type $T_m$, $\lambda_m = 0$.

Proof. A simplex in $\text{colim} \; X$ will be called of type $T$ if the overlaying orbit is of type $T$ in $X$. Then the condition that there are no invariant orbits of type $T_m$ is equivalent to the condition that there are no fixed points in the simplices of type $T_m$.

Since $X$ is a finite triangulated diagram, $\text{colim} \; X$ is a finite triangulated space, hence it is a compact metric space. If there are no fixed points of type $T_m$, then there exists a refinement $Y'$ of the triangulation such that if $\Delta$ is a simplex of type $T_m$ in $Y$, $\Delta \cap (\text{colim} \; f)(\Delta) = \varnothing$.

Consider the refinement $X'$ of $X$, which exists by lemma 4.4. Since $X' \cong X$, $H_k^D(X'; I) = H_k^D(X; I)$, $\Lambda(f') = \Lambda(f)$, where $f' : Y' \to X'$ is equal to $f$, $\lambda'_m = \lambda_m$. Therefore, it is enough to show that $\lambda'_m = 0$.

Now,

$$\Lambda_D(f') = \text{Ab}(\varphi)(\sum_{k=0}^{\infty} (-1)^k \text{tr}_{\text{HS}}(H_k(f'; I))) = \text{Ab}(\varphi)(\sum_{k=0}^{\infty} (-1)^k \text{tr}_{\text{HS}}(C_k(f'; I))),$$

where $C_k(f'; I)$ is the map induced by $f$ on the chains $C_k(X; I) = C_k(X) \otimes_{O'} I = (1_T I)^{t_1} \oplus \ldots \oplus (1_T I)^{t_n}$ as $I$-module. Because of the property: $\Delta \cap (\text{colim} \; f)(\Delta) = \varnothing$ for any simplex $\Delta$ of type $T_m$, the induced map on $C_k(X; I)$ will take the generator $1_T$ corresponding to $\Delta$ outside the submodule $1_T I$, that it generates. Then the $m$-th entry of $\text{Ab}(\varphi)(\sum_{k=0}^{\infty} (-1)^k \text{tr}_{\text{HS}}(C_k(f'; I)))$ will be zero. This is true for all $k$, hence $\lambda'_m = 0$.

\[ \square \]

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