Abstract: Medical diagnosis has been recognized as one of the key processes in clinical medicine, which determines diseases from some given symptoms. Nonetheless, previous works about medical diagnosis have some drawbacks because medical data are usually fuzzy, uncertain, incomplete and imprecise. To achieve the optimal medical diagnosis decision by reducing cost and enhancing accuracy, this paper develops a new method named interval three-way concept lattice model. Firstly, we redefine the decision rules and metric function of three-way decision based on interval concept lattice. Secondly, we build a visualized hierarchical structure of relationship between concepts through interval concept construction algorithm which helps us to make decision preferably and clearly. Finally, we establish a dynamic strategy optimization model for medical diagnosis decision making. In addition, a medical case demonstrates the effectiveness and feasibility of this proposed model.

Keywords: medical diagnosis; formal concept analysis; three-way decision; interval fuzzy concept lattice

1. Introduction

Recently, the government’s new medical reform program has boosted the construction of digital hospital and regional medical network construction. With the standardization and widely used of Electronic Health Records (EHR) and Hospital Information System (HIS), the Health care industry has accumulated a lot of medical data, including the patient’s medical records, patient diagnosis records, medical image data and so on [1]. It will provide a great convenience for the hospital diagnosis and management of the process when we use medical data effectively. However, now most hospitals, in order to process the data, are mostly confined to adding, deleting, changing, and checking operations, which belongs to the low level of application of medical data without deep analysis.

Medical diagnosis has been recognized as one of the key processes in clinical medicine, which determines diseases from some given symptoms. Many researchers centralized focus on enhancing the accuracy of disease diagnosis by using computerized techniques. Nowadays, Computerized techniques and theories, for example, fuzzy set theories, statistical tools, machine learning algorithms, and recommender systems, have been being widely applied for medical diagnosis [2].

As early as 1959, Ledley’s [3] study aroused widespread concern by using of the Bayesian formula to estimate the probability of medical disease diagnosis. Subsequently, Warner [4] used the naïve Bayes theory to establish a probabilistic model to diagnose congenital heart disease and promoted the development of medical decision support information system. However, the hypothesis of Naive Bayes theory is too simple which assumes that symptoms and diseases are independent of each other. From this point of view, scholars explored other, better methods of a medical decision-making model.
Medical data are usually fuzzy, uncertain, incomplete and imprecise, such as symptoms data, the relation between patient and symptoms, and the relation between symptoms, diseases, etc. [5]. Zadeh [6], the pioneer of fuzzy sets theory, firstly applied the fuzzy sets to analyze the relationship between symptoms and diseases in 1969. In addition, fuzzy set theory with extensions was developed, such as intuitionistic fuzzy sets [7,8], 2-tuple fuzzy set [9], fuzzy soft sets [10], probabilistic linguistic term sets [11], interval-valued fuzzy sets [12–14], neutrosophic fuzzy sets [15–19] and Z-numbers [20]. Although the research results of fuzzy sets theories have been very rich [21], due to the complexity and uncertainty of disease diagnosis, it is still a hot topic of the medical diagnosis with fuzzy set theory is used to deal with the problem of medical diagnosis decision.

Three-way decision model (TWDM), which consists of positive, uncertain, and negative regions, is an useful decision-making approach and it has been widely applied to many domains [22]. The application of TWDM in medical diagnosis should be noted. The three-way decision model has firstly introduced a generic framework and applied to the medical diagnosis domain by Yao [23]. In the medical field, Hu [24] developed a novel model for making treatment decisions. Besides, Yao [25] propose a web-based medical decision support system in TWDM on the basis of game theoretic rough sets, which can significantly decrease the uncertainty of medical diagnosis.

Nowadays, lattice theory has been richly used in analyzing biomedical data [26,27]. In 1982, Wille earliest promoted formal concept analysis (FCA) and applied it to knowledge processing tasks [28]. FCA builds a mathematical framework for scholars to analyze data and process tasks [29]. In addition, one of the key benefits of FCA is its capability to predict alternative or potential cases that are similar to those examined (but not present) in each case study [30], therefore endowing it with also generalizing predictive character and a further outreach. Now FCA has been used in various domains successfully especially in medical decision field [31]. Recently, to represent precisely the vagueness and uncertainty of information, FCA has been extended to fuzzy concept lattice analysis based on formal context which combined FCA with fuzzy set theory [32]. In addition, many studies augmented fuzzy formal concept analysis with intuitionistic fuzzy set [33], Bipolar fuzzy sets [34], neutrosophic set [35], rough set [36] and so on. Furthermore, the extension of fuzzy concept lattice and three-way decision theory has attracted the attention of some scholars [37]. There are future trends to extend FCA in a defined fuzzy universe due to their suitable applicability and it is significant for us to use fuzzy concept lattice effectively [31].

However, there are still some drawbacks to previous works in medical diagnosis decision, Firstly, it is not properly appropriate for the missing data situation, which is almost universal. Original information of medical data is composed of different types of data obtained from different times and give predictive values of diseases. Secondly, the previous works will lead to the loss or distortion of information about the development tendency of the values by the time. Last but certainly not least, the diagnosis process is dependent on the defuzzification method [7]. All limitations above of previous methods reduce the accuracy of medical diagnosis. Therefore, a novel method which combined three-way decision and interval-valued fuzzy concept lattice is a great option to eliminate the defects of the previous researches [38]. The IVIFS can describe properly the uncertain and incomplete medical data which represents the membership and non-membership degree of attributes in the interval [0,1]. Meanwhile, the object of universe can be divided into acceptance, uncertain, and rejection regions according to the conditional attributes. In addition, the generation of interval-valued concept lattices is a clustering process. Thus, the parent-child relationship in lattice structure can deal with boundary domain samples and reduce decision losses.

The intention of this article is to develop the interval three-way concept lattice model in medical diagnosis decisions, which is based on the three-way decision and interval fuzzy concept lattice. The proposed model forms a clear and visualized hierarchical structure of the relationship between concepts through formal concept analysis. In addition, a medical diagnosis case validates that we can make decision quickly and preferably by the model.
The rest of this article is constituted as follows: Section 2 introduces medical diagnosis problem related works including interval-valued intuitionistic fuzzy sets, three-way decision and formal concept analysis. Section 3 presents the proposed approach which is called the interval three-way concept lattice model. Section 4 validates the model through an illustrative case based on medical diagnosis information. Finally, Section 5 includes the conclusions and the future research directions.

2. Preliminaries

In this section, we first introduce the medical diagnosis problem, and then briefly present some basic concepts, properties and notations including the three-way decision, formal concept analyze.

2.1. The Medical Diagnosis Problem

Medical diagnosis has been widely recognized as one of the key processes in clinical medicine, which determines diseases from some given symptoms [8]. Nonetheless, medical data are usually fuzzy, uncertain, incomplete and imprecise, such as symptoms data, the relation between patients and symptoms, and the relation between symptoms and diseases etc. Under the circumstances, we state its definition as follows.

Definition 1 [39]. The number of Patients, symptoms, and diseases are represented respectively as $P = \{P_1, P_2, \cdots , P_m\}$, $S = \{S_1, S_2, \cdots , S_k\}$, $D = \{D_1, D_2, \cdots , D_n\}$, where $m, k, n \in N^+$. Meanwhile, the set $R_{PS} = \{R(P_i, S_j) | \forall i, j \in N^+\}$ denotes the relationship between patients and symptoms which indicates the possibility of the patient $P_i$ has the symptom $S_j$. The relationship between symptoms and diseases is characterized by the set $R_{SD} = \{R(S_j, D_h) | \forall j, h \in N^+\}$ where $R(S_j, D_h)$ shows that the probability of the symptom $S_j$ would cause the disease $D_h$. Last but certainly not least, the relation set $R_{PD} = \{R(P_i, D_h) | \forall i, h \in N^+\}$ which reflects the patient $P_i$ acquires the disease $D_h$ or not. Finally, we can simplify the medical diagnosis problem as $\{R_{PS}, R_{SD}\} \rightarrow R_{PD}$.

Nowadays, computerized techniques and theories, for example, fuzzy set theories, statistical tools, machine learning algorithms, and recommender systems, have been being widely applied to medical diagnosis [40]. Sanchez [41] put forward the earliest fuzzy set method in medical diagnosis decision-making. From then on, IFS with its extension fuzzy sets has been applied into medical decision-making field. Meanwhile, interval-valued IFS (IVIFS) theory is one of an effective way to increase the accuracy of medical diagnosis among the vague incomplete and missing medical data. The IVIFS is denoted as follows.

Definition 2 [39]. An interval-valued intuitionistic fuzzy set (IVIFS) in a non-empty set $X$ is,

$$\tilde{A} = \{ (x, \mu_{\tilde{A}}, \nu_{\tilde{A}}, \tilde{\pi}_{\tilde{A}}) | x \in X \},$$

where the functions $\mu_{\tilde{A}} : X \rightarrow \text{int}([0, 1])$ and $\nu_{\tilde{A}} : X \rightarrow \text{int}([0, 1])$ are the interval membership and interval non-membership degrees of each element $x \in X$ to the IVIFS $\tilde{A}$, $\tilde{\pi}_{\tilde{A}} : X \rightarrow \text{int}([0, 1])$ indicates the interval hesitation degree of $x \in X$ respectively,

$$\sup(\mu_{\tilde{A}}(x)) + \sup(\nu_{\tilde{A}}(x)) + \inf(\tilde{\pi}_{\tilde{A}}(x)) = 1,$$

$$\inf(\mu_{\tilde{A}}(x)) + \inf(\nu_{\tilde{A}}(x)) + \sup(\tilde{\pi}_{\tilde{A}}(x)) = 1,$$

$$\tilde{\pi}_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x).$$

For convenience, the IVIFS $\tilde{A}$ can be represented as follows.

$$\tilde{A} = \left( [a^l(x), a^u(x)], [b^l(x), b^u(x)], [c^l(x), c^u(x)] \right).$$
Some notions and properties of IVIFS such as computing distance or similarities are extended to model uncertainty in the medical diagnosis. However, there are still some limitations to the previous method. Firstly, it is not properly appropriate for the missing data situation, which is almost universal. Secondly, historic diagnosis data of patients could not be used effectively [8]. Last but certainly not least, the diagnosis process is dependent on the defuzzification method [7]. All limitations above of previous methods decrease the accuracy of decision-making in medical diagnosis filed. Therefore, a proper approach should be introduced.

2.2. Three-Way Decision Model

TWDM has been generally applied to lots of domains, such as environmental precaution, information filtering, and investment decision, etc. [22]. By the way, the application of TWDM in medical diagnosis should be noted. The three-way decision model has firstly introduced a generic framework and applied to the medical diagnosis domain by Yao [23]. TWDM is constituted by two states and three actions. In addition, it classifies objects into disjoint positive, negative, and uncertain regions.

**Definition 3** [23]. Let \( U = \{ C, \neg C \} \) denotes weather element \( X \) is in \( C \) or not. The decision-making state values are expressed as \( R(X) \). Meanwhile, the set of actions is given by \( B = \{ B_a, B_d, B_f \} \), where \( B_a, B_d, \) and \( B_f \) respectively represent acceptance, delay, and refusal actions. As well time, \( \alpha \) and \( \beta \) are the suitable threshold to classify \( X \) into different regions. Finally, the decision rules are shown below.

- **(B_a)** If \( R(X) \geq \alpha \), decide \( X \in \) The III region and take acceptance actions.
- **(B_d)** If \( \beta \leq R(X) \leq \alpha \), decide \( X \in \) The II region and take delay actions.
- **(B_f)** If \( R(X) \leq \beta \), decide \( X \in \) The I region and take refusal actions.

2.3. Formal Concept Analysis

Formal concept analysis (FCA) consists of concept lattices and formal concepts, and is a powerful tool to analyse information with fuzzy attributes. A formal concept lies in a pair of elements. In addition, concept lattice is an ordered hierarchy derived by formal context with object space and attribute space [42]. Here are some notions and properties of FCA as follows.

**Definition 4** [43]. A formal context is a triple \( A = (U, V, R) \), consisted by the object set \( U \), attributes set \( V \), relationship \( R \) between \( U \) and \( V \). If \( x_i \in U \) and \( y_j \in V \) \((i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)\), \((x_i, y_j) \in R\) is denoted as \( xRy \) which means the object \( x_i \) owned attribute. Let \( V_j \) be the subset of attributes, then \( |V_j| = 2^m \) and a three-way quotient set is denoted as \( \Omega(V) = \{ V_j | j \in V \} \). For short, we denote the power set of \( \Omega(V), y_j \) by \( 2^{\Omega(V)} \).

And every \( B \in 2^{\Omega(V)} \) can be considered as a knowledge group jointly inducing three-way decisions.

Residual lattice with some simple properties is a principal component of truth degrees. We denote \( L = (L, \wedge, \lor, \otimes, \rightarrow, 0, 1) \) as the residual lattice where 0 and 1 respectively indicate minimum and maximum objects. The operators \( \otimes \) and \( \rightarrow \) are adjoint operators which satisfied \( x \otimes y = \min(x, y) \) and \( x \rightarrow y = 1 \) if \( x \leq y \), otherwise \( b \) [44].

Under the background of formal context \( A = (U, V, R) \), let the formal fuzzy concept denotes as an ordered pair of \((X, Y)\) from \( A \). We can define following a pair of operators.

- If \( P(U) \rightarrow P(V) \), then \( X = \{ y \in V | \forall x \in X, (x, y) \in R \} \).
- If \( P(V) \rightarrow P(U) \), then \( Y = \{ x \in U | \forall y \in Y, (x, y) \in R \} \).
Once \( X^\uparrow = Y \) and \( Y^\downarrow = X \), we call \( X \) and \( Y \) as the extent and the intent of concept \((X, Y)\) respectively. \( S(U, V, R) \), generated from the formal context \( A = (U, V, R) \), is denoted as universal set of all formal fuzzy concepts [45]. For each formal concept \((X_1, Y_1), (X_2, Y_2) \in S(U, V, R)\), we define it satisfied the partial ordering relation:

\[
(X_1, Y_1) \leq (X_2, Y_2) \iff X_1 \subset X_2 \iff Y_1 \supset Y_2.
\]

At the same time, the infimum and the supremum for compete lattice are respectively denoted as follows.

\[
(X_1, Y_1) \wedge (X_2, Y_2) = \left( X_1 \cap X_2, (Y_1 \cup Y_2)^\downarrow \right),
\]

\[
(X_1, Y_1) \vee (X_2, Y_2) = \left( (X_1 \cup X_2)^\downarrow, Y_1 \cap Y_2 \right).
\]

Expanding it to the general form is expressed as:

\[
\land_{i \in I} (X_i, Y_i) = \left( \cap_{i \in I} X_i, (\cup_{i \in I} Y_i)^\downarrow \right),
\]

\[
\lor_{i \in I} (X_i, Y_i) = \left( (\cup_{i \in I} X_i)^\downarrow, \cap_{i \in I} Y_i \right).
\]

### 3. Interval Three-Way Concept Lattice Model

In this section, the proposed interval three-way concept lattice model was presented. Let \( P \), \( S \) and \( D \) denotes the set of patients, symptoms, and diseases as Definition 1. The patient \( P_i (\forall i \in 1, 2, \cdots, m) \) has some characteristics of symptoms \( S_j (\forall j \in 1, 2, \cdots, k) \). In addition, the diseases \( D_h (\forall h \in 1, 2, \cdots, n) \) also contain intuitionistic linguistic labels. It is significant to translate medical information into the interval-valued intuitionistic fuzzy set.

#### 3.1. Algorithm for Three-Way Concepts Construction Using IVIFS

Let us denote a fuzzy formal context as \( A = (U, V, R) \) where \(|U| = m, |V| = n\) and, \( R \) shows three-way relation among them using IVIFS for expressing much clearly. Then interval three-way concept lattice is derived as follows.

**Definition 5.** Let suppose an IVIFS for attributes of symptoms i.e.,

\[
V(x_j) = \left\{ x_j, \left[ a_u^v(x_j), a_u^a(x_j) \right], \left[ b_u^e(x_j), b_u^b(x_j) \right], \left[ c_u^l(x_j), c_u^h(x_j) \right] \right\} \in [0, 1]^3 \forall x_j \in V
\]

Membership function, non-membership function and hesitation degree is respectively denoted as interval-valued \( [a_u^v(x_j), a_u^a(x_j)] \), \( [b_u^e(x_j), b_u^b(x_j)] \) and \( [c_u^l(x_j), c_u^h(x_j)] \). The same procedure may be easily adapted to obtain IVIFS for \( U \).

The obtained pair \((U, V)\) is taken for the formal fuzzy concept iff: \( U^\uparrow = V \) and \( V^\downarrow = U \). We can interpret it as IVIFS of an object which has maximal membership, minimum non-membership and hesitation value in the interval space \([0, 1]^3\) using residual lattice. The integrating of \((U, V)\) can be defined as.

\[
U \cup V = \left\{ x_j, \left[ \max\left( a_u^v(x_j), a_u^a(x_j) \right), \max\left( a_u^v(x_j), a_u^a(x_j) \right) \right], \right.
\]

\[
\left. \left[ \min\left( b_u^e(x_j), b_u^b(x_j) \right), \min\left( b_u^e(x_j), b_u^b(x_j) \right) \right], \right.
\]

\[
\left. \left[ \min\left( c_u^l(x_j), c_u^h(x_j) \right), \min\left( c_u^l(x_j), c_u^h(x_j) \right) \right] \right\}, \quad (1)
\]
where $x_j \in V$.

$$U \cap V = \left\{ x_j \left[ \min \left( a_a^u (x_j), a_v^u (x_j) \right), \min \left( a_a^l (x_j), a_v^l (x_j) \right) \right], \max \left( b_a^u (x_j), b_v^u (x_j) \right), \max \left( b_a^l (x_j), b_v^l (x_j) \right) \right\},$$

where $x_j \in V$.

Until the membership value getting no longer bigger, then the pair of IVIFS $(U, V)$ is referred to a formal concept, where $U$ is known as extent and $V$ is known as intent. Above these, we propose an algorithm for interval three-way concepts construction as Algorithm 1 shows.

**Algorithm 1.** Three-way concept lattice using IVIFS.

1. Input a fuzzy formal context $A = (U, V, R)$, where $|U| = m$, $|V| = n$.
2. Find all the subsets of attributes set $V$ and represent as $k_i$.
3. for each $i = 1$ to $2^n$ do
4. Set up the membership to maximum value for subset $(k_i) = \max \{ \{1, 1\}, \{0, 0\}, \{0, 0\} \}$
5. Apply down operator $\downarrow$ to the set $V$
6. Figure the membership, hesitation and non-membership value for the derived objects set:
7. Apply up operator $\uparrow$ to the set $U$
8. if $U(X) \uparrow = V(X) \downarrow$ then for intent represent in the set of attributes.
9. Then it is a concept.
10. else Other attribute $z \in V$ covers the obtained objects
11. Then $k_i = (z \cup k_i)$, / add the novel attribute as an intent
12. The new concept $(X_k, Y_k)$ is generated.
13. end if
14. end for
15. Set the membership of last concept set $= \{ \{0, 0\}, \{0, 0\}, \{1, 1\} \}$ for $V(Y)$. 
16. Construct the concept lattice via Galois connection.
17. Return the set of interval-three-way concepts
18. $U(x_i) = \{ x_i \left[ \min \left( a_a^u (x_i), a_v^u (x_i) \right), \min \left( a_a^l (x_i), a_v^l (x_i) \right) \right], \max \left( b_a^u (x_i), b_v^u (x_i) \right), \max \left( b_a^l (x_i), b_v^l (x_i) \right) \right\}$
19. $V(y_j) = \{ y_j \left[ \min \left( a_a^u (y_j), a_v^u (y_j) \right), \min \left( a_a^l (y_j), a_v^l (y_j) \right) \right], \max \left( b_a^u (y_j), b_v^u (y_j) \right), \max \left( b_a^l (y_j), b_v^l (y_j) \right) \right\}$.

The Algorithm 1 generates interval-values intuitionistic fuzzy concepts via subset of attributes. In addition, through Galois connection, these interval concepts can be constructed as a visual hierarchical structure which makes the process of decision-making more intuitive.
3.2. A Novel Interval Three-Way Concept Lattice Model

After computing by the Algorithm 1, we can get an interval three-way concept lattice as $A^\beta = (U, V, R)$. $G = (M^a, M^\beta, Y)$ is one of interval concept of $A^\beta$. Set $U$ was classified into three parts by the upper and lower extent of interval-valued intuitionistic fuzzy concept [38].

$$POS^\beta_a (x) = M^\beta = \left\{ x \in U \left\vert \frac{|f(x \cap Y)|}{|Y|} \geq \beta \right\} \right.$$  

$$BND^\beta_a (x) = M^\beta - M^a = \left\{ x \in U \left\vert \alpha \leq \frac{|f(x \cap Y)|}{|Y|} < \beta \right\} \right.$$  

$$NEG^\beta_a (x) = U - M^a = \left\{ x \in U \left\vert \frac{|f(x \cap Y)|}{|Y|} < \alpha \right\} \right.$$  

If $x \in POS^\beta_a (x)$, then we take acceptance actions. If $x \in NEG^\beta_a (x)$, then we take refusal actions. Otherwise, we delayed the decision actions. $B_i, B_d,$ and $B_f$ represent respectively the acceptance action, delay action, and refusal action. The score function is denoted as follows.

**Definition 6.** Let $C = \{C, \neg C\}$ denotes weather element $X$ is in $C$ or not. Then the proportion of each action are,

- **Correct acceptance rate:** 
  $$\frac{|C \cap POS^\beta_a (x)|}{|POS^\beta_a (x)|} = \frac{|C \cap M^\beta|}{|M^\beta|} \tag{3}$$

- **False acceptance rate:** 
  $$\frac{|C^c \cap POS^\beta_a (x)|}{|POS^\beta_a (x)|} = \frac{|C^c \cap M^\beta|}{|M^\beta|} \tag{4}$$

- **Correct delay rate:** 
  $$\frac{|C \cap BND^\beta_a (x)|}{|BND^\beta_a (x)|} = \frac{|C \cap (M^a - M^\beta)|}{|M^a - M^\beta|} \tag{5}$$

- **False delay rate:** 
  $$\frac{|C^c \cap BND^\beta_a (x)|}{|BND^\beta_a (x)|} = \frac{|C^c \cap (M^a - M^\beta)|}{|M^a - M^\beta|} \tag{6}$$

- **Correct refusal rate:** 
  $$\frac{|C \cap NEG^\beta_a (x)|}{|NEG^\beta_a (x)|} = \frac{|C \cap (U - M^a)|}{|U - M^a|} \tag{7}$$

- **False refusal rate:** 
  $$\frac{|C^c \cap NEG^\beta_a (x)|}{|NEG^\beta_a (x)|} = \frac{|C^c \cap (U - M^a)|}{|U - M^a|} \tag{8}$$

**Definition 7.** Let $C = \{C, \neg C\}$ denotes weather element $X$ is in $C$ or not. We can divide the receiving rate, delaying rate, and refusing rate into $C$ or $\neg C$ respectively. From the point of view, the score functions can be considered as,
The interval three-way decision domain can be used as a new object of making the decision. In addition, the decision space relationship can decide element \( x \) of action \( B \) lattice as \( A \). Let the fuzzy formal context denote as \( A \). Definition 8 [46].

As depicted in Table 1, the loss expectation function is respectively denoted as.

| Table 1. The loss expectation matrix. |
|--------------------------------------|
| \( C(P) \) | \( C(N) \) |
| \( B_t \) | \( l_{tp} \) | \( l_{tn} \) |
| \( B_i \) | \( l_{ip} \) | \( l_{in} \) |
| \( B_f \) | \( l_{fp} \) | \( l_{fn} \) |

Generally, TWDM focus on cost losses including misclassification costs and delay costs in different states. In Table 1, \( l_{tp}, l_{ip}, \) and \( l_{fp} \) indicates the costs for each action of \( B_t, B_i, \) and \( B_f \) respectively represent acceptance, delay, and refusal action when an element pertains to \( C \), and vice versa [22].

As depicted in Table 1, the loss expectation function is respectively denoted as.

\[
R(B_t) = l_{tp} \left( \frac{|X \cap M^\beta|}{|M^\beta|} \right) + l_{in} \left( \frac{|X \cap M^\beta|}{|M^\beta|} \right)
\]

\[
R(B_i) = l_{ip} \left( \frac{|X \cap (M^\alpha - M^\beta)|}{|M^\alpha - M^\beta|} \right) + l_{in} \left( \frac{|X \cap (M^\alpha - M^\beta)|}{|M^\alpha - M^\beta|} \right)
\]

\[
R(B_f) = l_{fp} \left( \frac{|X \cap (U - M^\alpha)|}{|U - M^\alpha|} \right) + l_{fn} \left( \frac{|X \cap (U - M^\alpha)|}{|U - M^\alpha|} \right)
\]

| Definition 8 [46]. | Let the fuzzy formal context denotes as \( A = (U, V, R) \) with decision interval \([\alpha, \beta]\). After computing by the algorithm of Table 1, we can get an interval-valued intuitionistic concept lattice as \( A^\beta_a = (U, V, R) \). \( G = (M^\alpha, M^\beta, Y) \) is one of interval concept of \( A^\beta_a \). Then \( \tilde{A} = (M^\alpha, M^\beta, Y; R(B_i), R(B_t), R(B_f)) \) represents interval three-way fuzzy concept lattice. For each decision element \( x \) of action \( B_a (a \in t, i, f) \) with loss \( R \), it can be repressed as \( I = (B_a, R(B_a)) \). |

Three-way decision space is constituted by interval concept lattice based on prior formal context. The interval three-way decision domain \( A^\beta_a(U, V, R) \) which is divided by interval concept lattice can be used as a new object of making the decision. In addition, the decision space relationship can decide
the next action decision to reduce losses. For a new object $x$ with attribute set $V' \subseteq V$, we search for the interval three-way fuzzy concept owned intent of $V'$. Here are the procedures of the proposed interval three-way concept lattice model.

**Step 1.** Suppose an action strategy $J_k \in JS = (J_1, J_2, \ldots, J_n)$ is generated by interval three-way fuzzy concept $\tilde{A} = (M^a, M^b, Y; R(B_i), R(B_j), R(B_f))$.

**Step 2.** Computing the loss $R_{ik}$ of each $J_k$ by loss function above.

**Step 3.** If there is the only action $J_k$ which is satisfied $R_{ik} = \min(P_{t1}, P_{t2}, \ldots, P_{tn})$ then accept strategy $J_k$ and refuse other strategies.

**Step 4.** If there are multiple actions $J$ what are satisfied the formula above, then take delay decision and add newly attributes until $R_{ik} = \min(P_{t1}, P_{t2}, \ldots, P_{tn})$ holds.

**Step 5.** The action strategy $J_n$ and $J_m$ is decided respectively by interval concept lattice $\tilde{A}_n = (M^n, M^b_n, Y; R(B_i), R(B_j), R(B_f))$ and $\tilde{A}_m = (M^m, M^b_m, Y; R(B_i), R(B_j), R(B_f))$.

At the same time, $R_{in} = R_{im} = \min(R_{t1}, R_{t2}, \ldots, R_{tn})$ holds. Then we should search the subset of $\tilde{A}_n$ and $\tilde{A}_m$. After adding attribute according to those intents of subsets, we search the minimum loss for taking acceptance action [37].

Once there is none concept, we should search for the concept which possesses the intent $Y'$. It exists that $Y' \subseteq Y$ and $|Y' - Y| = 1$. Analogously, we can get the decided action by repeating the above steps and same criteria.

4. **Application of the Proposed Model in Medical Diagnosis**

In this part, we conduct an illustrative experiment in medical diagnosis to validate the performance of interval three-way concept lattice model. As Table 2 shows, there are six patients with different symptoms. The decision is based on the following:

1. $S_1$ as fever,
2. $S_2$ as cough,
3. $S_3$ as headache,
4. $S_4$ as fatigue,
5. $S_5$ as nausea,
6. $S_6$ as inappetence, and
7. $S_7$ as chest tightness

Diseases of these patients are represed as,

1. $D_1$ as flu,
2. $D_2$ as chicken pox,
3. $D_3$ as measles,
4. $D_4$ as heatstroke

Suppose loss function value respectively as, $l_{ip} = 0, l_{ip} = 9, l_{ip} = 15, l_{tn} = 17, l_{tn} = 2, l_{tn} = 0$
Table 2. The medical diagnosis formal context.

| P  | S₁ | S₂ | S₃ | S₄ | S₅ | S₆ | S₇ | D  |
|----|----|----|----|----|----|----|----|----|
| 1  | 1  | 0  | 1  | 1  | 0  | 0  | 0  | D₁ |
| 2  | 1  | 1  | 0  | 0  | 1  | 0  | 0  | D₂ |
| 3  | 1  | 1  | 1  | 0  | 0  | 0  | 0  | D₃ |
| 4  | 0  | 0  | 1  | 0  | 1  | 1  | 1  | D₄ |
| 5  | 1  | 0  | 1  | 0  | 0  | 0  | 0  | D₅ |
| 6  | 0  | 0  | 1  | 1  | 0  | 0  | 0  | D₆ |

Decision interval [α, β] which reflects the risk preference of decision-makers, is decided by decision-makers. In this case, we suppose decision interval is [0.6, 0.8]. After using the proposed algorithm in Algorithm 1, the interval three-way concept is obtained as shown in Table 3.

Table 3. Interval three-way concept.

| Concept | Upper Extent | Lower Extent | Intent | Rᵣ | Rᵢ | Rᶠ |
|---------|--------------|--------------|--------|----|----|----|
| A₁      | Φ            | Φ            | S₁S₂S₃S₄S₅S₆S₇D₁D₂D₃D₄ | —  | —  | —  |
| A₂      | 1235         | 1235         | S₁D₁   | 8.50 | 0.00 | 7.50 |
| A₃      | 1346         | 1346         | S₃D₁   | 8.50 | 0.00 | 7.50 |
| A₄      | 156          | 156          | S₄D₁   | 0.00 | 0.00 | 0.00 |
| A₅      | 13           | 13           | S₁S₂D₁ | 8.50 | 0.00 | 7.50 |
| A₆      | 15           | 15           | S₁S₃D₁ | 0.00 | 0.00 | 3.75 |
| A₇      | 16           | 16           | S₂S₃D₁ | 0.00 | 0.00 | 3.75 |
| A₈      | 1356         | 1            | S₁S₂S₄D₃ | 0.00 | 6.67 | 0.00 |
| A₉      | 1235         | 1235         | S₁D₂   | 12.75 | 0.00 | 0.00 |
| A₁₀     | 23           | 23           | S₁S₂D₂ | 8.50 | 0.00 | 0.00 |
| A₁₁     | 2            | 2            | S₁S₃D₂ | 0.00 | 0.00 | 0.00 |
| A₁₂     | 2            | 2            | S₂S₃D₂ | 0.00 | 0.00 | 0.00 |
| A₁₃     | 23           | 2            | S₁S₂S₃D₂ | 0.00 | 2.00 | 0.00 |
| A₁₄     | 1235         | 1235         | S₁D₃   | 12.75 | 0.00 | 0.00 |
| A₁₅     | 1346         | 1346         | S₃D₃   | 12.75 | 0.00 | 0.00 |
| A₁₆     | 23           | 23           | S₁S₂D₃ | 8.50 | 0.00 | 0.00 |
| A₁₇     | 13           | 13           | S₁S₃D₃ | 8.50 | 0.00 | 0.00 |
| A₁₈     | 3            | 3            | S₂S₃D₃ | 0.00 | 0.00 | 0.00 |
| A₁₉     | 123          | 3            | S₁S₂S₃D₃ | 0.00 | 2.00 | 0.00 |
| A₂₀     | 1346         | 1346         | S₃D₄   | 12.75 | 0.00 | 0.00 |
| A₂₁     | 4            | 4            | S₃S₆S₇D₄ | 0.00 | 0.00 | 0.00 |
| A₀      | 123456       | 123456       | Φ      | 0.00 | 0.00 | 0.00 |

According to Table 3 and Algorithm 1, Hasse diagram of interval fuzzy concept lattice is derived through Galois connection as shown in Figure 1. In addition, it also represents three-way decision space \( \mathcal{A}_p^\beta(U, V, R) \). In Figure 1, decision-makers can intuitively realize the meaning of each concept and relationships between concepts. For example, concept \( A₀(123456, 123456, \phi, 0.00, 0.00, 0.00) \) expresses the extent is 123456 and intent is \( \phi \), it indicates all the patients do not have the same symptoms. Sub-concept \( A₂(1235, 1235, S₁D₁, 8.50, 0.00, 7.50) \) of \( A₀ \) depicts patients 1, 2, 3, and 5 can be classified into the same class which has a fever and get the flu, and the cost of taking acceptation, delay, and rejection action is 8.50, 0.00, and 7.50 respectively. For sub-concept \( A₁(\phi, \phi, S₁S₂S₃S₄S₅S₆S₇D₁D₂D₃D₄, -, -, -) \), it signifies there is no disease with all the symptoms of a fever, cough, headache, fatigue, nausea, lack of appetite, and chest tightness.
Supposed there comes a new patient $x$ with symptoms of fever $S_1$ and headache $S_3$. First of all, we should search the interval three-way concept for attributes as $S_1, S_3$. Then it can be obtained as:

$$A_5 = (13, 13, S_1S_3D_1; 8.50, 0.00, 0.75),$$

$$A_{17} = (13, 13, S_1S_3D_3; 8.50, 0.00, 0.00).$$

$A_5$ and $A_{17}$ respectively indicated $D_1$ taking action $J_1 = (a_{i1}, R(a_{p1}) = 8.50)$ and $D_3$ taking action $J_2 = (a_{i2}, R(a_{p2}) = 8.50)$ which constitutes the binary decision stratagem space $JS = (J_1, J_2)$. It equals to $R(a_{i1}) = R(a_{i2}) = 8.50$ that is to say that the loss of acceptance of decision making by $D_1$ and $D_3$ for $x$ is the same.

For further confirmed what disease $x$ acquires, sub-concepts of $A_5$ and $A_{17}$ are searched as:

$$A_8 = (1356, 1, S_1S_2S_4D_1; 0.00, 6.67, 0.00),$$

$$A_{19} = (123, 3, S_1S_2S_3D_3; 0.00, 2.00, 0.00).$$

It needs to add decision attributes respectively, which means we should enquire about the patient’s symptoms for an accurate decision. If he or she has the symptom of fatigue $S_4$, then the diagnosis result is $D_1$ which means the patient acquires flu. Else, the diagnosis result is measles $D_3$, once he has the symptom of cough $S_2$.

In this section, we perform the proposed interval three-way concept lattice method on an illustrative medical diagnosis case. When applied the approach described into the example, the medical information was collected in IVIFS which effectively reflected the semantics in actual. To reflect the risk preference of decision-makers and make the computing briefly, we take cut sets on decision interval $[a, \beta]$. Using the interval three-way concepts construction algorithm, a clear and visualized Hasse diagram of interval fuzzy concept lattice can be generated which helps us to make decision preferably. Through the proposed score function, we computed the losses of acceptation, delay, and rejection strategy respectively. In addition, the decision can be made by comparing the loss
of each strategy. While taking the delay strategy, we can wait for further observation until adding new attribute. Then we should search for the sub-concept and take the strategy of action minimum loss as the final decision. Thus, the proposed model can handle the medical diagnosis problem properly, and it can be applied to various fields.

5. Conclusions

In this paper, we put forward an interval three-way concept lattice model, which can provide decision making for the objects in the boundary domain and reduce the loss of decision making. According to the concept extension, we divide the decision domains into three decision regions as the new positive, boundary and negative region grounded on given decision rules. We redefine interval three-way decision measurements and decision loss functions based on IVIFS, TWDM and FCA. In addition, we put forward an algorithm for the interval three-way concepts construction, which can build a clear and visualized hierarchical relationship between concepts. Through the concept lattice hierarchy, we establish a dynamic strategy optimization model for medical diagnosis field. In addition, an example of medical diagnosis validates the effectiveness of the model and the operator. The interval three-way concept lattice model can apply to other fields beside medical diagnosis.

The proposed model provides solutions for medical diagnosis in the fuzzy environment. Firstly, the IVIFS can describe the uncertain and incomplete medical data properly and then improve the accuracy of diagnosis. Secondly, an algorithm is presented for the interval three-way concepts construction. In addition, a visualized hierarchical structure can be derived which helps us to make decisions, preferably such as in Figure 1. Thirdly, loss functions are redefined and a dynamic strategy optimization model is established which can be effectively used in many domains. However, there are still some issues which are not considered in this paper. First is the complexity of construction interval fuzzy concept lattice. Second is the data of illustration example is small.

Thus, further studies can be focused on the following aspects. Firstly, the reduction of interval three-way concept lattice can be presented which reduce the complexity of construction concept lattice. Secondly, it might optimize the proposed model and apply it to handle large-sized data of symptoms and disease in medical diagnosis field. Thirdly, it can be further investigated to combine the three-way concept lattice theory and other fuzzy sets, such as neutrosophic sets, picture fuzzy sets, and Z-numbers.

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