PERCOLATION TIMES IN TWO–DIMENSIONAL MODELS FOR EXCITABLE MEDIA

JANKO GRAVNER
Mathematics Department, University of California, Davis, CA 95616
e-mail: gravner@feller.ucdavis.edu

Abstract. The three-color Greenberg–Hastings model (GHM) is a simple cellular automaton model for an excitable medium. Each site on the lattice $\mathbb{Z}^2$ is initially assigned one of the states 0, 1 or 2. At each tick of a discrete–time clock, the configuration changes according to the following synchronous rule: changes $1 \to 2$ and $2 \to 0$ are automatic, while an $x$ in state 0 may either stay in the same state or change to 1, the latter possibility occurring if there is at least one representative of state 1 in the local neighborhood of $x$. Starting from a product measure with just 1’s and 0’s such dynamics quickly die out (turn into 0’s), but not before 1’s manage to form infinite connected sets. A very precise description of this “transient percolation” phenomenon can be obtained when the neighborhood of $x$ consists of 8 nearest points, the case first investigated by S. Fraser and R. Kapral. In addition, first percolation times for related monotone models are addressed.

1991 Mathematics Subject Classification. Primary 60K35.

Keywords: additive growth dynamics, excitable media, Greenberg–Hastings model, percolation.

This work was partially supported by the research grant J1-6157-0101-94 from Slovenia’s Ministry of Science and Technology.

Submitted to EJP on February 20, 1996. Final version accepted on October 10, 1996.
PERCOLATION TIMES IN TWO–DIMENSIONAL MODELS FOR EXCITABLE MEDIA

Janko Gravner

1. Introduction.

The Greenberg-Hastings model (GHM) ([WR], [GH]) on $\mathbb{Z}^2$ tries to capture two essential properties of a two–dimensional excitable medium: a refractory phase and excitation by contact. In one of its simplest incarnations, GHM dynamics have two parameters: an integer number $\kappa \geq 3$, which we refer to as the number of colors (or states), and a finite set $N \subset \mathbb{Z}^2$, containing the origin, which is called the neighborhood. The neighborhood of an arbitrary site $x \in \mathbb{Z}^2$ is then obtained by translation: $N_x = x + N$. The state space of the GHM is $\{0, 1, \ldots, \kappa - 1\}^{\mathbb{Z}^2}$ and the update rule is given as follows:

\[
\gamma_{t+1}(x) = \begin{cases}
(\gamma_t(x) + 1) \mod \kappa, & \text{if } \gamma_t(x) > 0 \text{ or there is a } y \in N_x \text{ with } \gamma_t(y) = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

In words, all colors $1, \ldots, \kappa - 1$ update automatically to $2, \ldots, 0$, respectively, while 0 updates by contact with a least one color 1 in its neighborhood. Readers interested in mathematical analysis of this GHM rule and various generalizations may consult [DS], [FGG1], [FGG2], [DN], [Dur], [DG], and [Ste]. Researchers closer to applications have considered many classes of multi–parameter rules related to the GHM (see, for example, [WTW] and the references contained therein).

Assume for a moment that $N$ is the five point set $\{(0,0), (\pm 1, 0), (0, \pm 1)\}$, the nearest neighbor case. Assume moreover that $\gamma_0$ is a product measure with $P(\gamma_0(x) = 0) > 0$, $P(\gamma_0(x) = 1) > 0$ and $P(\gamma_0(x) = i) > 0$ for some $i \in \{2, \ldots, \kappa - 1\}$. Then it turns out that the dynamics is eventually run by “clocks,” i.e. objects like

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & \ldots & \kappa - 1 \\
\kappa - 1 & \kappa - 2 & \kappa - 3 & \kappa - 4 & \ldots & 0
\end{array}
\]

which repeat their pattern every $\kappa$ units of time and eventually cause other sites to do the same, giving rise to local periodicity (see [FGG2]). The main aim of this paper is to study the case when $\gamma_0$ contains only 0’s and 1’s, which results in a markedly different dynamics. To see why, start a nearest neighbor GHM with just a single 1 in a sea of 0’s. This initial state generates a diamond–shaped ring which keeps expanding forever. If the initial state contains just two 1’s, the two rings proceed to annihilate on the boundary and combine into one big ring, which again
expands without bound. We will show that such annihilating rings are the dominant feature of the dynamics started from a product measure consisting only of 0’s and 1’s. Indeed, one inevitably reaches such a conclusion by doing just a few computer simulations.

As in the simple example above, the number of colors turns out to be completely immaterial for the ring dynamics. Thus we assume that $\kappa = 3$ from now on. We also assume that the initial state $\gamma_0$ is the product measure given by $P(\gamma_0(x) = 1) = p = 1 - P(\gamma_0(x) = 0)$ for every $x$. Since our main results involve various percolation properties of subsets of $\mathbb{Z}^2$, we must start with a few definitions.

Fix a finite set $D \subset \mathbb{Z}^2$. We say that a set $G \subset \mathbb{Z}^2$ $D$–percolates if there exists an infinite sequence $x_1, x_2, \ldots$ of distinct sites in $G$ such that $x_{k+1} - x_k \in D$ for $k = 1, 2, \ldots$. The usual site percolation, which we call $\ell^1$–percolation, is obtained by taking $D = B_1(0, 1)$, while its dual, which we refer to as $\ell^\infty$–percolation, is given by $D = B_\infty(0, 1)$. (We adopt the usual notation for balls: $B_r(x, R) = \{ y : ||x - y||_r \leq R \}$.)

The study of ring growth and interaction in this simple model of an excitable medium fits into the general framework of investigating domain growth in physical systems: namely, how spatial structures arise far from equilibrium. Real–life phenomena that may be related to the ones discussed here include aspects of the Belousov–Zhabotinsky reaction and processes in cardiac tissue ([FK], [Kap], [KW], [SLC]). Indeed, ring dynamics may be the dominant feature of conduction of electrical impulses in a healthy heart, while spatial defects (e.g. scar tissue) can break the rings and cause, through the formation of clock–like excitation centers, atrial fibrillation. These natural phenomena were the main motivation of S. Fraser and R. Kapral, who studied ring dynamics in excitable media in [FK]. They chose the Moore neighborhood (i.e. $\mathcal{N} = B_\infty(0, 1)$) and combined computer simulations with various intuitively compelling but non-rigorous arguments, to arrive at the following three conclusions:

(1) Each point is in state 1 exactly once.

(2) The density of 1’s goes to 0 very quickly as time $t$ increases, and is of order $\sqrt{p}$ at its maximum.

(3) There exists a narrow time interval, lasting at most two time units, during which 1’s percolate. ([FK] is vague as to what type of percolation this is.)

Assuming (1), (2) follows readily, as shown in [Bra]. But (1) is not very hard to show either (see Corollary 2.2). The most interesting statement is of course (3); our main aim in the present paper is to give a rigorous and more detailed treatment of this issue.

Before we proceed, let us introduce some standard regularity assumptions on the neighborhood $\mathcal{N}$ which arose in our previous work ([GG]). We call $\mathcal{N}$ symmetric if $-\mathcal{N} = \mathcal{N}$ and
supercritical if the origin is an interior point of the convex hull \( \text{co}(\mathcal{N}) \). Moreover, denote the sum \( \mathcal{N} + \mathcal{N} + \cdots + \mathcal{N} \) of \( n \) copies of \( \mathcal{N} \) by \( \mathcal{N}^n \) (with the conventions \( \mathcal{N}^0 = \{0\}, \mathcal{N}^n = \emptyset \) when \( n < 0 \), and call \( \mathcal{N} \) irreducible if \( \cup_{n \geq 1} \mathcal{N}^n = \mathbb{Z}^2 \). Note that irreducibility implies supercriticality, but not vice versa. We assume throughout this paper that \( \mathcal{N} \) is both symmetric and irreducible.

**Theorem 1.** There exist sufficiently large integers \( a \) and \( b \) (depending only on \( \mathcal{N} \)) so that:

1. For every \( p > 0 \) there exists a time \( T(p) \) at which \( \{ \gamma_{T(p)} = 1 \} \) \( B_\infty(0,a) \)-percolates.
2. The set \( \{ \gamma_t = 1 \} \) may \( B_\infty(0,a) \)-percolate only for \( t \in [T(p), T(p) + b] \).
3. \( T(p) \sqrt{p} \) converges to a finite non-zero constant as \( p \to 0 \).

To prove a more precise theorem, we restrict to the special case of Moore neighborhood considered in [FK].

**Theorem 2.** Assume that \( \mathcal{N} = B_\infty(0,1) \).

1. For every \( p \in (0,1] \) there is a time \( T(p) \) such that the set \( \{ \gamma_t = 1 \} \) \( l^\infty \)-percolates for \( t = T(p) \) but does not \( l^\infty \)-percolate for \( t \notin [T(p), T(p) + 1] \).
2. There is a countable set \( S \subset (0,1) \) with \( 0 \) as its only limit point, such that the following is true. Let \( I_1, I_2, I_3, \ldots \) be the successive intervals in \( (0,1] \setminus S \), where \( I_1 \) is the one that includes \( 1 \). Let \( G_1 \) be the union \( I_1 \cup I_3 \cup I_5 \cup \ldots \) and \( G_2 = (0,1] \setminus S \setminus G_1 \). If \( p \in G_1 \), then 1's in \( \gamma_t \) \( \ell^1 \)-percolate at \( t = T(p) \) but do not \( \ell^\infty \)-percolate for \( t \neq T(p) \). If \( p \in G_2 \), then 1's in \( \gamma_t \) never \( \ell^1 \)-percolate, but they \( \ell^\infty \)-percolate for both \( t = T(p) \) and \( t = T(p) + 1 \). Finally, if \( p \in S \), then 1's in \( \gamma_t \) \( \ell^\infty \)-percolate only for \( t = T(p) \) and at this time they do not \( \ell^1 \)-percolate.

Figure 1 illustrates Theorem 2 for the density \( p = 0.01 \). The frames represent the state of the system on a \( 200 \times 200 \) square at times 3 and 4 (top), and 5 and 6 (bottom). At times 4 and 6 (resp. at time 5) the sites with color 1 that are connected via an \( l^\infty \)-path (resp. an \( l^1 \)-path) to the left edge of the rectangle are painted black. The remaining 1's are painted dark gray, while 2's are painted light gray. It is clear that at time 3 the rings of 1's are still too small to be able to \( \ell^\infty \)-percolate. This remains true at time 4, despite the existence of relatively long \( \ell^\infty \)-paths. At time 4, 1's form a left–right \( \ell^1 \)-connection of the square, suggesting \( \ell^1 \)-percolation. Finally, by time 5 there is enough annihilation between the rings so that 1's cannot even \( \ell^\infty \)-percolate. These figures leave little doubt that \( T(0.01) = 5 \) and \( 0.01 \in G_1 \).

It is natural to ask what happens to Theorems 1 and 2 in more general circumstances. One possible generalization is to consider an arbitrary excitation threshold \( \theta \); this means that \( x \) will change color from 0 to 1 iff the number of sites with color 1 in \( \mathcal{N}_x \) exceeds \( \theta \) ([FGG1], [FGG2],...
In this case “failed centers” can break up the rings and cause local periodicity; computer simulations show that this is indeed a likely scenario in many regimes, though a rigorous general statement is hard to prove. Another interesting class of questions concerns the expanding “shells” in dimensions 3 or higher. These higher dimensional models are considerably harder to deal with not only because of technical difficulties, but also because of qualitatively different “intermediate” regimes. On a different note, present methods can be extended to handle various related two–dimensional models with recurrent ring dynamics ([Gra3]).

Figure 1. Ring dynamics in the GHM.

The first step in the proof of the above theorems is a comparison between the GHM and a monotone 2–color model which is in fact the threshold 1 (often called additive) version of threshold growth dynamics ([GG]). This auxiliary monotone automaton with state space \( \{0, 1\}^\mathbb{Z}_2 \) is denoted by \( \xi_t \) and given by the update rule:

\[
\xi_{t+1}(x) = \max\{\xi_t(y) : y \in \mathcal{N}_x\}.
\]

Thus, \( \{\xi_t = 1\} = \{\xi_0 = 1\} + \mathcal{N}^t \). Assume that \( \xi_0 \) is a product measure of 0’s and 1’s, with a density of 1’s given by \( p > 0 \). Define \( T_a(p) \) to be the first time for which the set \( \{\xi_t = 1\} \) \( B_\infty(0,a) \)-percolates. By the ergodic theorem, \( T_a \) is a deterministic function of \( p \). It is also not
hard to prove the following result.

**Theorem 3.** As $p \to 0$, $T_{\omega}(p) \sqrt{p}$ converges to a finite non-zero constant $\lambda_c$, which depends on $\mathcal{N}$ but is independent of $\omega$.

Many models for crystal formation and growth, such as the famous Johnson–Mehl model ([JM]), include continuous seeding of nuclei. This inspired R. Kapral and M. Weinberg ([KW]) to propose a variant of $\xi_t$. This model, denoted by $\tilde{\xi}_t$, is a Markov chain with state space $\{0, 1\}^\mathbb{Z}^d$ and the following transition rule at every site $x$:

$$
\begin{align*}
0 &\to 1 \quad \text{if there is a 1 in } \mathcal{N}_x, \\
0 &\to 1 \quad \text{with probability } p, \text{ if there is no 1 in } \mathcal{N}_x, \\
0 &\to 0 \quad \text{with probability } 1 - p, \text{ if there is no 1 in } \mathcal{N}_x, \\
1 &\to 1 \quad \text{automatically.}
\end{align*}
$$

Hence this rule is a two–stage procedure: at the first stage the dynamics of $\xi_t$ are applied to a configuration of 0’s and 1’s, and at the second stage a proportion $p$ of the remaining 0’s are turned to 1’s.

![Figure 2. Threshold 1 growth rule with continuous nucleation.](image)

Start $\tilde{\xi}_t$ from the quiescent state: $\tilde{\xi}_0(x) = 0$ for all $x$. Now define $\tilde{T}_{\omega}(p)$ to be the first time at which $\{\tilde{\xi}_{t} = 1\} B_{\infty}(0, a)$-percolates. Our next theorem identifies the behavior of $\tilde{T}_{\omega}$ near $p = 0$. (We will see later that $\tilde{T}_{\omega}$ is a deterministic function of $p$ as well.)

**Theorem 4.** As $p \to 0$, $\tilde{T}_{\omega}(p)p^{1/3}$ converges to a finite non-zero constant.
This theorem is illustrated by Figure 2, which shows the state of the $500 \times 375$ system with $N = B_\infty(0, 1)$ and $p = 0.0005$ at time $t = 12$, the first time when there is an $\ell^\infty$-path connecting left and right edges of the rectangle. The picture suggests that $\hat{T}_1(0.0005) = 12$.

The rest of the paper is organized as follows. Section 2 contains a few results on additive growth, and a proof of Theorem 3. Theorem 4 is proved in Section 3. Section 4 states a few well known results about long-range percolation models, which we then use in Section 5 to prove Theorem 2. Finally, Section 6 completes the proof of Theorem 1.

2. Growth dynamics with initial nucleation.

We start with a simple lemma which establishes a connection between the growth automaton $\xi_t$ and the GHM $\gamma_t$.

**Lemma 2.1.** Assume that $\gamma_0 = \xi_0$. Then

$$\{ x : \gamma_{t+1}(x) = 1 \} = \{ x : \xi_t(x) = 0, \xi_{t+1}(x) = 1 \}.$$

**Proof.** Denote the set on the right of (2.1) by $A_t$. To prove (2.1) by induction, observe first that it is true at $t = 0$, and then assume that $\{ \gamma_s+1(x) = 1 \} = A_s$ for all $s < t$.

Firstly, we prove that $A_t \subseteq \{ \gamma_{t+1} = 1 \}$. Assume that $x \in A_t$. Then $x \notin A_s$ for any $s < t$. In particular, $x \notin A_{t-1}$, hence by hypothesis $\gamma_t(x) \neq 1$. Since $\gamma_t(x)$ cannot be 2 ($\gamma_t(x) = 2$ would imply $\gamma_{t-1}(x) = 1$ and thus $x \in A_{t-2}$, a contradiction), $\gamma_t(x) = 0$. Moreover, there is a $y \in N_x$ such that $\xi_t(y) = 1$. If $\xi_{t-2}(y) = 1$, then $\xi_{t-1}(x) = 1$, a contradiction. It follows that $y \in A_{t-1}$, $\gamma_t(y) = 1$ and, consequently, $\gamma_{t+1}(x) = 1$.

Secondly, we prove that $\{ \gamma_{t+1} = 1 \} \subseteq A_t$. Assume now that $\gamma_{t+1}(x) = 1$, so $\gamma_t(x) = 0$ and $\gamma_t(y) = 1$ for some $y \in N_x$. Hence $y \in A_{t-1}$, hence $\gamma_{t+1}(x) = 1$. Since $x \notin A_{t-1}$, we either have $\xi_t(x) = 0$ or $\xi_{t-1}(x) = 1$. But, since $x \notin A_{t-2}$, $\xi_{t-1}(x) = 1$ would imply that $\xi_{t-2}(x) = 1$ and hence $\xi_t(y) = 1$, a contradiction with $y \in A_{t-1}$. This leaves $\xi_t(x) = 0$ as the only possibility, proving that $x \in A_t$. □

**Corollary 2.2.** For each site $x$ there is a unique time $t$ at which $\gamma_t(x) = 1$.

Lemma 2.1 also immediately implies that $P(\gamma_t(x) = 1) = \left( 1 - (1 - p)^{|N_t \setminus N_{t-1}|} \right) (1 - p)^{|N_{t-1}|}$. To estimate the maximal value this expression can have, we first use the shape theorem; this result was first proved by Willson ([Wil]), but we give a short proof in Lemma 2.3 below. This theorem implies that $|N^n|/n^2 \to \text{Area}(co(N))$ as $n \to \infty$. (Here, and in the sequel, we denote by $|\cdot|$ the cardinality and by $\text{Area}$ the Lebesgue measure.)
Assume, for a moment, that the growth at the boundary is also regular enough so that

\[(2.2) \quad \frac{1}{n} \cdot |N^n \setminus N^{n-1}| \to c \quad \text{as } n \to \infty,\]

for some constant \(c > 0\). Then \(c = 2 \text{Area}(\text{co}(N))\), and a short computation gives

\[
\max_t P(\gamma_t(x) = 1) \sim \sqrt{2e^{-1} \text{Area}(\text{co}(N))} \cdot \sqrt{p} \quad \text{as } p \to 0.
\]

In fact, we show below (Lemma 2.4) that (2.2) is always true. Essentially, we show that the boundary of \(N^n\) is quickly trapped into a stable pattern. Figure 3 provides an illustration in which points in \(N^n \setminus N^{n-1}\) are periodically shaded.

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\ldots \quad \ldots \quad \ldots \\
\bullet \quad \bullet \quad \bullet \\
\ldots \quad \ldots \quad \ldots \\
\end{array}
\]

\(N = \begin{array}{c}
\bullet \quad 0 \\
\ldots \quad \ldots \quad \ldots \\
\bullet \\
\ldots \quad \ldots \\
\end{array}\)

Figure 3. \(N^{30}\) for the specified \(N\).

**Lemma 2.3.** There exists a constant \(M\) (depending only on \(N\)) such that the set \(N^n\) is included in \(n \cdot \text{co}(N)\) and includes all integer points in \((n - M) \cdot \text{co}(N)\).

**Proof.** The inclusion \(N^n \subset n \cdot \text{co}(N)\) is elementary. Next, it follows from the results in [Gra2] that there exists an \(\epsilon > 0\) so that if \(A\) is a subset of \(\mathbb{R}^2\) with \(C^2\) boundary whose curvature does not exceed \(\epsilon\), then \(A + N = A + \text{co}(N)\). Moreover, the curvature of the boundary of \(A + \text{co}(N)\) remains at most \(\epsilon\). Let \(A_0\) now be any set with boundary curvature at most \(\epsilon\), say \(A_0 = B_2(0, 1/\epsilon)\).

By irreducibility, there exists an \(n_0\) so that \(A_0 \cap \mathbb{Z}^2 \subset N^{n_0}\). A simple conjugacy argument ([GG]) yields \((A_0 + (n - n_0) \cdot \text{co}(N)) \cap \mathbb{Z}^2 \subset N^n\). \(\square\)
Lemma 2.4. (2.2) holds.

Proof. Take $m \geq 2$ and integers $x_1 < x_2 < \cdots < x_m$ and $M = \{x_1, \ldots, x_m\}$. If $a = \gcd(M - x_1)$, then elementary number theory implies that there exists a constant $M_0$ so that

\[(2.3) \quad [nx_1 + M_0, nx_m - M_0] \cap a\mathbb{Z} = [nx_1 + M_0, nx_m - M_0] \cap \mathbb{Z}^n.
\]

Let $e_1, e_2$ be the usual basis vectors and assume that $\beta_M$ is such that $N \subset \mathbb{Z}e_1 + (-\infty, \beta_M)[e_2]$. Moreover, assume that $N_l = \min\{\alpha : \alpha e_1 + \beta_M e_2 \in N\}$ and $N_r = \max\{\alpha : \alpha e_1 + \beta_M e_2 \in N\}$ are different, $N_l < N_r$. Then, we claim that, for any fixed $k \in \{0, 1, 2, \ldots\}$, either $(\mathbb{Z}e_1 + n(\beta_M - k)e_2) \cap N^n = \emptyset$ for all $n$, or else there exists positive integer constants $m_k, M_k$ and $a_1^{(k)}, \ldots, a_m^{(k)}$ so that

\[(2.4) \quad \left([-nN_l + M_k, nN_r - M_k] \cap \left( \bigcup_{i=1}^{m_k} a_i^{(k)} \mathbb{Z}\right) e_1 + n(\beta_M - k)e_2 \right. \]

\[= \left([-nN_l + M_k, nN_r - M_k] e_1 + n(\beta_M - k)e_2 \right) \cap N^n.
\]

Indeed, for $k = 0$, (2.4) follows immediately from (2.3). In words, we know that the last horizontal line of $N^n$ in the vertical direction is periodic, away from the edges. But then this argument shows that the same must be true for the next furthest line, and so on. Hence an induction argument with a repeated application of (2.3) establishes (2.4). By Lemma 2.3, for large enough $k$, $m_k = 1, a_1^{(k)} = 1$, and we can take $M_k = M$. This means that at least the boundary in the $e_2$ direction stabilizes away from the edges.

However, a similar argument (with a few more notational complications) will work in any direction $u$ such that a line with normal $u$ contains at least two extreme points of $\text{co}(N)$. Only finitely many such directions are determined by the boundary of $\text{co}(N)$ and it is easy to combine them to end the proof. $\square$

Next up is a well–known result about a continuous percolation model (for a proof, see [ZS] and [Roy]), which identifies the constant $\lambda_c$ in the statement of Theorem 3.

Lemma 2.5. Assume that $\mathcal{P} \subset \mathbb{R}^2$ is a Poisson random field with intensity 1. Attach to each $x \in \mathcal{P}$ a random variable $U_x > 0$; assume that $U_x$ are i.i.d., and bounded. Define, for any fixed $\lambda > 0$, the random set $W(\lambda) = \mathcal{P} + \lambda \cdot U_x \cdot \text{co}(N) \subset \mathbb{R}^2$. Let $W(\lambda, 0)$ be the connected component of $W(\lambda)$ which includes 0. Then there exists a $\lambda_c \in (0, \infty)$ such that

\[E(\text{Area}(W(\lambda, 0))) < \infty \quad \text{for } \lambda < \lambda_c\]

\[P(\text{Area}(W(\lambda, 0)) = \infty) > 0 \quad \text{for } \lambda > \lambda_c.
\]

The above statement remains unchanged (and so does $\lambda_c$) if $\text{Area}(W(\lambda, 0))$ is replaced by $|\mathcal{P} \cap W(\lambda, 0)|$, or the diameter of $W(\lambda, 0)$. Moreover, $W(\lambda)$ percolates, i.e. contains an unbounded
connected set, if \( \lambda > \lambda_c \), while all connected components are bounded if \( \lambda < \lambda_c \). Finally, \( W(\lambda)^c \) percolates if \( \lambda < \lambda_c \) and it does not if \( \lambda > \lambda_c \).

In this section, we only need the case when all the \( U_x \) are point masses at 1. To prove Theorem 3, we use a basic coupling between \( W(\lambda) \) and \( \xi_t \) which is similar to the one used by Zuev and Sidorenko in their pioneering papers ([ZS]).

**Proof of Theorem 3.** Fix the density \( p \) and define \( Q(x,p) = \{ y \in \mathbb{R}^2 : ||\sqrt{p} x - y||_\infty \leq \sqrt{p}/2 \} \). Let \( P \) be a Poisson random field with density 1. Declare a site \( x \in \mathbb{Z}^2 \) to be occupied if \( Q(x,p) \cap P \neq \emptyset \). Since Area\((Q(x,p)) = p \), and Area\((Q(x,p) \cap Q(y,p)) = 0 \) for \( x \neq y \), the sites in \( \mathbb{Z}^2 \) are independently occupied, each with probability \( 1 - e^{-p} \). Define \( \xi_0 \) by making \( \{ \xi_0 = 1 \} \) equal to the set of occupied sites.

We claim that there exists a constant \( N > 0 \) (again dependent only on \( N \)) so that,

\[
W((t - N)\sqrt{p}) \subset \sqrt{p}(\{\xi_t = 1\} + B_\infty(0,1/2)) \subset W((t + N)\sqrt{p}),
\]

for every \( t \) and \( p \).

To prove (2.5), assume first that a point \( x \in \mathbb{R}^2 \) is in the leftmost set of (2.5). Then there is a \( y \in P \) such that \( x \in y + (t - N)\sqrt{p} \cdot \text{co}(\mathcal{N}) \). Find a \( z \in \mathbb{Z}^2 \) so that \( ||y - \sqrt{p}z||_\infty \leq \sqrt{p}/2 \). This means that \( \xi_0(z) = 1 \) and hence the middle set in (2.5) covers \( \sqrt{p}z + \sqrt{p}(t - M) \cdot \text{co}(\mathcal{N}) \) by Lemma 3.3. Since \( \mathcal{N} \) is supercritical, \( B_\infty(0,1/2) \subset \text{co}(\mathcal{N}) \), and so \( x \) is in the middle set of (2.3) if we choose \( N = M + 1 \).

Assume now that \( x \) is in the middle set of (2.5). Hence there is a \( y \in \mathbb{Z}^2 \) so that \( \xi_t(y) = 1 \) and \( ||x - \sqrt{p}y||_\infty \leq \sqrt{p}/2 \). Then there exists a \( z \in \mathbb{Z}^2 \) so that \( \xi_0(z) = 1 \) and \( y \in z + t \cdot \text{co}(\mathcal{N}) \), and, in turn, some \( w \in \mathcal{P} \cap Q(z,p) \). Therefore \( x \in w + (t + 2)\sqrt{p} \cdot \text{co}(\mathcal{N}) \).

It is easy to translate (2.5) into a statement about \( T_a \). Namely, \( T_a(1 - e^{-p}) \in [\lambda_c/\sqrt{p} - 2a - N - 1, \lambda_c/\sqrt{p} + N + 1] \), finishing the proof. \( \square \)

3. Growth dynamics with continuous nucleation.

We start the analysis of \( \tilde{\xi}_t \) by associating to every site \( x \), independently of other sites, a geometric random variable \( S_x \) with \( P(S_x = k) = p(1 - p)^{k-1} \), \( k = 1, 2, \ldots \). Then,

\[
\{\xi_t = 1\} = \bigcup_{k \geq 1} \{x + N^t - S_x : S_x \leq t\}.
\]

Note that \( \tilde{T}_a(p) \) is a translation invariant function of the independent family \( \{S_x : x \in \mathbb{Z}^2\} \), hence a deterministic quantity. Moreover, (3.1) and (2.2) yield, after a straightforward computation,

\[
P(\tilde{\xi}_t(0) = 0) = \prod_{k=0}^{t} P(S_0 > t - k)^{|\mathcal{N}^t \setminus \mathcal{N}^{t-1}|} = (1 - p)^{(\text{Area}(\text{co}(\mathcal{N}))/3)t^2 + o(t^2)}.
\]
If we denote the first passage time to the origin by \( \tau \), i.e. \( \tau = \min\{ t : \xi_i(0) = 1 \} \), then (3.2) immediately implies that \( p^{1/3} \tau \) converges weakly as \( p \to 0 \) to a non-degenerate random variable. Therefore, most sites make the 0 to 1 transition at times of the order \( p^{1/3} \), which certainly makes Theorem 4 plausible.

In this section, we use Lemma 2.5 with \( U_x \) uniformly distributed on \([0, 1]\). Before proceeding with the proof of Theorem 3 we need a few preliminary results. Denote \( W^\varepsilon(\lambda) = W(\lambda) + \varepsilon \cdot \text{co}(\mathcal{N}) \), and \( W^{-\varepsilon}(\lambda) = \mathcal{P} + \max\{\lambda U_x - \varepsilon, 0\} \cdot \text{co}(\mathcal{N}) \).

**Lemma 3.1.** If \( \lambda < \lambda_c \) then there exists an \( \varepsilon > 0 \) so that the \( \varepsilon \)-fattening \( W^\varepsilon(\lambda) \) does not percolate.

**Proof.** For \( 0 \leq a < b \), let \( W'([a, b], \mu) = \bigcup\{x + U_x' \cdot \text{co}(\mathcal{N}) : x \in \mathcal{P}(\mu)\} \), where \( U_x' \) are now independent and uniform on \([a, b] \) and \( \mathcal{P}(\mu) \) is a Poisson point location with intensity \( \mu \). Note that \( W^\varepsilon(\lambda) = W'([\varepsilon, \lambda + \varepsilon], 1) \).

There exists a \( \delta > 0 \) so that \( W((\lambda + \delta)(1 + \delta)) \) does not percolate, hence (as is easily seen by shrinking the space by a factor of \( 1/(1 + \delta) \)) neither does \( W'([0, \lambda + \delta], 1 + \delta) \). However, by discarding points \( x \in \mathcal{P}(1 + \delta) \) for which \( U_x' < \varepsilon \), \( W'([0, \lambda + \delta], 1 + \delta) \) can be seen to dominate \( W'([\varepsilon, \lambda + \delta], (1 + \delta)(1 - \varepsilon/(\lambda + \delta))) \). If \( \varepsilon \) is chosen small enough so that \( \varepsilon < \delta \) and \((1 + \delta)(1 - \varepsilon/(\lambda + \delta)) \geq 1 \), then \( W'([\varepsilon, \lambda + \varepsilon], 1) \) cannot percolate. \( \square \)

**Lemma 3.2.** If \( \lambda > \lambda_c \) then there exists an \( \varepsilon > 0 \) so that \( W^{-\varepsilon}(\lambda) \) percolates.

**Proof.** The random variable \( \max\{\lambda U_x - \varepsilon, 0\} \) dominates a uniform random variable on \([0, \lambda - \delta] \) with probability \( 1 - \varepsilon/\delta \). If one chooses \( \varepsilon = \delta^2 \), then \( W^{-\varepsilon}(\lambda) \) dominates \( W'([0, \lambda - \delta], 1 - \delta) \), which does percolate for small enough \( \delta \). \( \square \)

**Lemma 3.3.** Given \( p \in [0, 1] \) and any integer \( t > 0 \) let

\[
ur = \frac{p(1 - p)^{t-r}}{1 - (1 - p)^t}, \quad vr = \sum_{i=1}^{r} ur = \frac{(1 - p)^{-r} - 1}{(1 - p)^{-t} - 1},
\]

for \( r = 1, \ldots, t \). (Note that \( v_t = 1 \).) Define a real-valued function \( f \) on \([0, 1] \) by \( f(0) = 0 \) and \( f(v) = rp^{1/3} \) for \( v \in (v_{r-1}, v_r), r = 1, \ldots, t \). If there exists a constant \( \lambda > 0 \) so that \( tp^{1/3} \to \lambda \) as \( p \to 0 \), then \( \sup_{v \in [0,1]} |f(v) - \lambda v| \to 0 \) as \( p \to 0 \).

**Proof.** Since \( tp \to 0 \), \( (1 - p)^{-r} = 1 + rp + a_r \), where \( |a_r| \leq C(rp)^2 \), for some constant \( C \). Therefore,

\[
|f(v_r) - \lambda v_r| = \left| rp^{1/3} - \lambda \frac{r}{t} \frac{1 + a_r/(rp)}{1 + a_t/(tp)} \right| \leq |tp^{1/3} - \lambda| + 2C\lambda tp.
\]
Moreover, a simple calculation shows that \( \sup_r (v_r - v_{r-1}) = \sup_r u_r \sim p^{1/3}/\lambda \) as \( p \to 0 \). Finally, if \( v \in (v_{r-1}, v_r] \), then \( |f(v) - \lambda v| \leq |f(v_r) - \lambda v_r| + \lambda(v_{r+1} - v_r) \leq |tp^{1/3} - \lambda| + Cp^{1/3} \), for a (different) constant \( C \). \( \square \)

**Proof of Theorem 3.** We prove that \( T_2(p) p^{1/3} \to \lambda_c^{2/3} \) as \( p \to 0 \).

Let \( R_t(x) \) equal 0 if \( S_x \geq t \) and \( t - S_x \) otherwise. This is the radius of the growing set which was generated by a seed at \( x \) during the time interval \([0, t]\). Observe that \( P(R_{t+1}(x) = r \mid R_{t+1}(x) > 0) = u_r, r = 1, \ldots, t \), where \( u_r \) are defined in the statement of Lemma 3.3. Now, assume that \( \mathcal{P} \) is an intensity 1 Poisson random field and declare a site \( x \in \mathbb{Z}^2 \) open at time \( t + 1 \) if

\[
B_\infty(x\sqrt{1-(1-p)^t}, \frac{1}{2}\sqrt{1-(1-p)^t}) \cap \mathcal{P} \neq \emptyset.
\]

This makes sites open independently with probability \( P(R_{t+1}(x) > 0) \). Adjoin to each site \( x \) that is open at time \( t \) a random variable \( V_{t+1}(x) \) with \( P(V_{t+1}(x) = r) = u_r, r = 1, \ldots, t \), independently of other points. We get, by (3.1),

\[
\{\tilde{\xi}_{t+1} = 1\} = \bigcup_{x \text{ open at time } t+1} \left( x + N_{V_{t+1}(x)} \right)
\]

Assume that the claim of the theorem is false. Then, we can find some sequence \( p_n \) converging to 0, and \( \lambda \in [0, \infty] \setminus \{\lambda_c\} \), such that \( \tilde{T}_a(p_n)p_n^{1/3} \to \lambda_c^{2/3} \) as \( n \to \infty \). We will assume that \( \lambda \in (0, \infty) \); the other two cases are then treated easily.

The first possibility is that \( \lambda < \lambda_c \). Take \( t = \tilde{T}_a(p_n) \). We claim that, if \( p_n \) is small enough, then the subset of \( \mathbb{R}^2: \sqrt{1-(1-p_n)^t} \cdot \left( \{\tilde{\xi}_{t+1} = 1\} + B_\infty(0, (a+1)/2) \right) \) does not percolate (i.e. contains no unbounded connected set). This will certainly show that \( \{\tilde{\xi}_{t+1} = 1\} \) does not \( B_\infty(0, a) \)-percolate, implying \( t < \tilde{T}_a(p_n) \), a contradiction. To prove the claim, note that \( \sqrt{1-(1-p_n)^t}/p_n^{1/3} \to \lambda^{1/3} \), so (3.3) and Lemma 3.3 imply that for every \( \epsilon > 0 \) there exists a large enough \( n \) so that the set \( \sqrt{1-(1-p_n)^t} \cdot \left( \{\tilde{\xi}_{t+1} = 1\} + B_\infty(0, (a+1)/2) \right) \) is included in the fattening \( W(\lambda)^\epsilon \). Now, use the \( \epsilon \) from Lemma 3.1.

On the other hand, if \( \lambda > \lambda_c \), then take \( t = \tilde{T}_a(p_n) - M - 2 \), with \( M \) as in Lemma 2.3. If \( p_n \) is small enough, then it follows from Lemma 2.3, (3.3), Lemma 3.3, and Lemma 3.2 that \( \sqrt{1-(1-p_n)^t} \cdot \left( \{\tilde{\xi}_{t+M+1} = 1\} + B_\infty(0, (a+1)/2) \right) \subset \mathbb{R}^2 \) percolates. Therefore, \( \{\tilde{\xi}_{t+M+1} = 1\} \) \( B_\infty(0, a) \)-percolates, and so \( t \geq T_2(p_n) - M - 1 \), a contradiction. \( \square \)

**4. Some percolation preliminaries.**

In this section, we state the results needed to prove Theorem 2. These are either applications of general percolation theory, or else follow by minor modifications of well–known arguments, so detailed proofs will be omitted.
Fix a positive integer $k$. Recall that $\gamma_0$ is an i.i.d. collection of Bernoulli random variables with density $p$. Call the sites of the set $\{\gamma_0 = 1\} + B_{\infty}(0, k)$ $k$–occupied, and other sites in $\mathbb{Z}^2$ $k$–vacant. Let $C_k^1(x)$ (resp. $C_k^\infty(x)$) be the $\ell^1$–cluster (resp. $\ell^\infty$–cluster) of occupied sites which includes $x$, i.e. the set of all occupied points which can be connected to $x$ by an $\ell^1$–path (resp. $\ell^\infty$–path) of occupied sites. A self–avoiding $\ell^1$–path (resp. $\ell^\infty$–path) which circles once around the origin is called an $\ell^1$–circuit (resp. an $\ell^\infty$–circuit). Define:

\[ p_k^c = \inf\{p : P(|C_k^1(0)| = \infty) > 0\}, \]
\[ \tilde{p}_k^c = \inf\{p : P(|C_k^\infty(0)| = \infty) > 0\}. \]

Note that $p_0^c = p_c \approx 0.593$ is the critical density for the usual site percolation on $\mathbb{Z}^2$, while $\tilde{p}_0^c = 1 - p_c$ is the critical density for its dual.

**Lemma 4.1.**

1. If $p \leq p_k^c$, then there exists an infinite sequence of disjoint $k$–vacant $\ell^\infty$–circuits.
2. If $p \geq p_k^c$, then there exists an infinite sequence of disjoint $k$–occupied $\ell^1$–circuits.
3. The $k$–occupied sites $\ell^1$–percolate iff $p > p_k^c$, while $k$–vacant sites $\ell^\infty$–percolate iff $p < p_k^c$.

**Proof.** The statements for $p \neq p_c$ follow by straightforward generalizations of the arguments in [Roy], together with standard geometric facts. (For example, an infinite $k$–occupied $\ell^\infty$–path does not exist iff there is an infinite sequence of disjoint $k$–vacant $\ell^1$–circuits.) In fact, away from the critical values the necessary version of the Russo–Seymour–Welsh (RSW) lemma ([Gri]) can be proved in a more direct fashion ([Gra1]).

If $p = p_c$, then the $k$–vacant sites again cannot $\ell^\infty$–percolate by the RSW lemma for $k$–vacant sites ([Roy]) and the standard form of Russo’s argument ([Gri]). To see that the $k$–occupied sites cannot $\ell^1$–percolate, assume they do and fix a small $\epsilon > 0$. Then, for a large enough $n$, the probability that there is an $\ell^1$–path consisting of $k$–occupied sites which stays inside the rectangle $[0, 1.2n] \times [0, n]$ and connects the left edge with the right edge is at least $1 - \epsilon$. (Otherwise infinitely many $k$–vacant $\ell^\infty$–circuits could be constructed using the argument of [Roy] and Chapter 6 of [Kes].) This enables us to decrease $p$ a little while keeping the probability of a left–right crossing of $[0, 1.2n] \times [0, n]$ larger than $1 - 2\epsilon$. A standard oriented percolation scheme (together with a Peierls argument) can now be utilized to show that, for a small enough $\epsilon$, an infinite $\ell^1$–path of $k$–occupied sites emanating from the origin exists with positive probability. This contradicts the fact that $p = p_c$. □
Of course, almost exactly the same arguments handle the $\ell^\infty$–percolation case:

**Lemma 4.2.**

1. If $p \leq \hat{p}_c^k$, then there exists an infinite sequence of disjoint $k$–vacant $\ell^1$–circuits.
2. If $p \geq \hat{p}_c^k$, then there exists an infinite sequence of disjoint $k$–occupied $\ell^\infty$–circuits.
3. The $k$–occupied sites $\ell^\infty$–percolate iff $p > \tilde{p}_c^k$, while $k$–vacant sites $\ell^1$–percolate iff $p < \tilde{p}_c^k$.

It is obvious that $p_{k+1} \leq \hat{p}_k \leq p_k$ for all $k \geq 0$. For many of our arguments it is however crucial that strict inequalities hold.

**Lemma 4.3.** $\cdots < p_c^3 < \hat{p}_c^2 < p_c^2 < \hat{p}_c^1 < p_c^1 < \hat{p}_c^0 = 1 - p_c < p_c^0 = p_c$.

**Proof.** To apply the arguments from Section 3 of [AG], note that $\ell^1$–percolation (resp. $\ell^\infty$–percolation) of $k$–occupied sites is equivalent to the site percolation on the graph with vertices $\mathbb{Z}^2$ and edges between any two sites $x$ and $y$ for which $y - x \in B^\infty(0, 2k) + B^1(0, 1)$ (resp. $y - x \in B^\infty(0, 2k + 1)$).

5. Ring dynamics in the GHM: the Moore neighborhood case.

We begin with some notation. By dist, we mean distance taken in the norm $||\cdot||_r$. Choose any set $A \subset \mathbb{Z}^d$. Define its $\ell^\infty$–boundary $\partial^\infty A$ to consist of all sites $x \in A$ for which $\text{dist}_\infty(x, A^c) = 1$, and the outside $\ell^\infty$–boundary by $\partial_0^\infty A = \partial^\infty(A^c)$. Note that Lemma 2.1 then says that $\{\gamma_{t+1} = 1\} = \partial_0^\infty(\{\xi_t = 1\})$.

The proof of Theorem 2 starts with a geometrical lemma which is probably well–known.

**Lemma 5.1.** Let $A \subset \mathbb{Z}^2$. Let $x, y \in \partial_0^\infty A$. Let $x$ and $y$ be connected by an $\ell^1$–path (resp. $\ell^\infty$–path) contained in $A^c$ and connected by an $\ell^1$–path (resp. $\ell^\infty$–path) contained in $A \cup \partial_0^\infty A$. Then $x$ and $y$ can be connected by an $\ell^1$–path (resp. $\ell^\infty$–path) contained in $\partial_0^\infty A$.

**Proof.** A path (an $\ell^1$–path or an $\ell^\infty$–path) will be called self–avoiding if the induced path in $\mathbb{R}^2$ (obtained by adding line segments between successive points) does not cross itself. Moreover, for two self–avoiding paths with the same endpoints $\pi_1$ and $\pi_2$, we define $K(\pi_1, \pi_2)$ to be the subset of $\mathbb{R}^2$ consisting of the union of the two induced paths together with all bounded components of the complement of this union.

We start by proving the $\ell^1$ version. Let $\pi_1$ be a self–avoiding $\ell^1$–path connecting $x$ to $y$ through $A \cup \partial_0^\infty A$ and let $\pi_2$ be a self–avoiding $\ell^1$–path connecting $x$ to $y$ through $A^c$. 

Assume first that the only common sites of \( \pi_1 \) and \( \pi_2 \) are \( x \) and \( y \). Then we can, without loss of generality, assume that \( K(\pi_1, \pi_2) \) is seen on the left while moving from \( x \) to \( y \) on \( \pi_2 \). Let \( \Pi \) be the set of all self-avoiding \( \ell^1 \)-paths which connect \( x \) to \( y \) through \( A^c \) and have \( K(\pi_1, \pi) \) on their left (but allow \( \pi \) to possibly intersect \( \pi_1 \)). We let \( \Pi \) be the set of all self-avoiding \( \ell^1 \)-paths which connect \( x \) to \( y \) through \( A^c \) and have \( K(\pi_1, \pi) \) on their left (but allow \( \pi \) to possibly intersect \( \pi_1 \)). Let \( \Pi \) be the set of all self-avoiding \( \ell^1 \)-paths which connect \( x \) to \( y \) through \( A^c \) and have \( K(\pi_1, \pi) \) on their left (but allow \( \pi \) to possibly intersect \( \pi_1 \)).

If not, then there is a \( z \) on \( \rho \) which is not on \( \partial^\infty_o A \). Figure 4 identifies (up to rotations) all possible local configurations around \( z \); we assume that the previous site on \( \rho \) was below \( z \). The symbol □ then indicates sites in \( \rho \), and \( \bullet \) indicates sites which are not in \( A \) (since \( z \notin \partial^\infty_o A \)). Eliminate \( z \) from \( \rho \), transforming the path in the indicated way. If this transformation creates a loop in the path, simply discard it. We thereby obtain a path \( \rho' \in \Pi \) such that \( \rho' \leq \rho \), but \( \rho' \neq \rho \), a contradiction.

Figure 4. Transformation of \( \rho \) in the \( \ell^1 \) case (left) and \( \ell^\infty \) case (right).
If $\pi_1$ and $\pi_2$ do meet at sites other than $x$ and $y$, let $x = v_0, v_1, \ldots, v_n = y$ be all points of intersection. Then do the above for $x$ replaced by $v_k$ and $y$ replaced by $v_{k+1}; k = 0, \ldots, n - 1.$

Finally, we prove the $\ell^\infty$ version. Let $\pi_1$ be a self-avoiding $l^\infty$-path connecting $x$ and $y$ through $A \cup \partial_x^\infty A$, and let $\pi_2$ be a self-avoiding $l^\infty$-path which connects $x$ and $y$ through $A^c$. Then $\rho$ is defined in a way analogous to the $\ell^1$ case. Again, to prove that $\rho$ consists of sites on $\partial_x^\infty A$, one needs to define a transformation that will eliminate $z \notin \partial_x^\infty A$ from $\rho$. The transformation is given by Figure 4 (all other cases are rotations or reflections of those drawn). Note that every new point on $\rho$ is within $\ell^1$-distance 1 of $z$, and, because $z$ is not on $\pi_1$, this implies that new points cannot be on the right of $\pi_1$. Apart from this issue, the argument is the same as before. $\square$

Proof of Theorem 2. Start by defining $S = \cup_{k \geq 0} \{p^k_c, \bar{p}^k_c\}$.

Assume first that $\bar{p}^k_c < p < p^k_c$ for some $k \geq 0$. Then $\{\xi_k = 1\}$ $\ell^\infty$–percolates at $t = k$, but neither $\ell^1$–percolates for $t \leq k$ nor $\ell^\infty$–percolates for $t < k$. We first prove that $\{\gamma_t = 1\}$ $\ell^\infty$–percolates for $t = k$. This is obvious for $k = 0$, so we can assume that $k \geq 1$.

Assume that $v_0, v_1, v_2, v_3, \ldots$ is a self-avoiding infinite $\ell^1$–path in $\{\xi_{k-1} = 0\}$ (which exists by Lemma 4.2(3)). Let $i_n, n = 1, 2, \ldots$ be such that all $v_{i_n}$ belong to the infinite $\ell^\infty$–cluster of 1’s in $\xi_k$. (Such indices must exist by Lemma 4.2(2), because 1’s in $\xi_k$ form an infinite sequence of disjoint $\ell^\infty$–circuits.) Then $v_{i_n}$ and $v_{i_{n+1}}$ can be connected by an $\ell^\infty$–path in $\{\xi_k = 1\} = \{\xi_{k-1} = 1\} \cup \partial_x^\infty \{\xi_{k-1} = 1\}$. By Lemma 5.1, we can connect $v_{i_n}$ and $v_{i_{n+1}}$ via an $\ell^\infty$–path $\pi_n$ in $\partial_x^\infty \{\xi_{k-1} = 1\}$. The union of paths $\pi_n$ is an infinite $\ell^\infty$–connected set, and consists of sites $v \in \partial_x^\infty \{\xi_{k-1} = 1\}$, hence such that $\xi_{k-1}(v) = 0$ and $\xi_k(v) = 1$, hence $\gamma_k(v) = 1$. Therefore $\{\gamma_k = 1\} \ell^\infty$–percolates.

Moreover, by Lemma 4.1(3), there is an $\ell^\infty$–path of 0’s in $\xi_k$. Since 1’s in $\xi_{k+1}$ $\ell^1$–percolate and form $\ell^1$–circuits (by Lemma 4.1(2)), this path must have infinitely many sites in the infinite $\ell^1$–cluster of $\{\xi_{k+1} = 1\}$. As in the preceding paragraph, Lemma 5.1 assures that 1’s in $\gamma_{k+1}$ $\ell^\infty$–percolate.

Finally, 0’s in $\xi_{k+1}$ do not $\ell^\infty$–percolate (Lemma 4.1(3)). Hence 1’s in $\gamma_{k+2}$ cannot $\ell^\infty$–percolate and neither can they at any succeeding time. Of course $\gamma_t \subset \xi_t$, so $\{\gamma_t = 1\}$ cannot $\ell^\infty$–percolate for $t < k$. As neither $\{\xi_k = 1\}$ nor $\{\xi_{k+1} = 0\}$ $\ell^1$–percolate, $\{\gamma_t = 1\}$ cannot $\ell^1$–percolate for $t \in \{k, k + 1\}$. Therefore we have proved the claim of the theorem with $T(p) = k$ in this case.

Assume now that $\bar{p}^{k+1}_c < p < \bar{p}^k_c$. Then $\{\xi_t = 1\} \ell^1$–percolates at time $t = k + 1$, but it does not $\ell^\infty$–percolate for $t \leq k$. There is an infinite $\ell^1$–path of 0’s in $\xi_k$ (Lemma 4.2(3)),
which intersects the infinite \( \ell^1 \)–cluster of 1’s in \( \xi_{k+1} \) infinitely many times (Lemma 4.1(2)(3)). A procedure similar to the one above, \( \ell^1 \) version of Lemma 5.1 shows that 1’s in \( \gamma_{k+1} \) \( \ell^1 \)–percolate. They cannot \( \ell^\infty \)–percolate in \( \gamma_t \) for \( t \geq k + 2 \) since 0’s in \( \xi_{k+1} \) do not \( \ell^\infty \)–percolate (Lemma 4.1(3)). This proves the claim of the theorem, with \( T(p) = k + 1 \) in this case.

Finally, we consider the case where \( p \) equals some critical value, assuming first that \( p = p^k_c \) for some \( k \). We then know that 1’s in \( \xi_k \) \( \ell^\infty \)–percolate and form infinitely many disjoint \( \ell^\infty \)–circuits (Lemma 4.2(2)(3)), while 0’s in \( \xi_{k-1} \) \( \ell^1 \)–percolate. As we have seen, this implies that 1’s in \( \gamma_k \) \( \ell^\infty \)–percolate. An infinite collection of disjoint \( \ell^1 \)–circuits of 1’s in \( \xi_k \) (Lemma 4.1(2)) prevents \( \ell^\infty \)–percolation of \( \{\gamma_{k+1} = 1\} \). As \( \{\xi_k = 1\} \) does not \( \ell^1 \)–percolate (Lemma 4.1(3)), neither can \( \{\gamma_k = 1\} \). Hence \( T(p) = k \) in this case.

If \( p = \tilde{p}^k_c \), then there is an \( \ell^\infty \)–path of 0’s in \( \xi_k \) (Lemma 4.1(3)), which is intersected infinitely many times by the infinite \( \ell^1 \)–cluster of 1’s in \( \xi_{k+1} \) (Lemma 4.1(2)). Again, this implies that \( \{\gamma_{k+1} = 1\} \) \( \ell^\infty \)–percolates for \( t = k + 1 \). Moreover, \( \{\gamma_k = 1\} \) does not \( \ell^\infty \)–percolate (as \( \{\xi_k = 1\} \) does not, by Lemma 4.2(3)), and neither can \( \{\gamma_{k+2} = 1\} \) (as \( \{\xi_{k+1} = 0\} \) does not). Since \( \{\xi_k = 0\} \) does not \( \ell^1 \)–percolate (Lemma 4.2(3)), neither does \( \{\gamma_{k+1} = 1\} \). Therefore, \( T(p) = k + 1 \) in this final case. □

6. Ring dynamics in the GHM: the general neighborhood case.

Proof of Theorem 1. Throughout this proof \( C \) is a generic positive integer constant, which may change value from one appearance to another, and depends only on \( N \). Recall the coupling of \( \xi_t \) and \( W \) from the proof of Theorem 3 which establishes (2.5). That proof and Lemma 2.5 then imply that there exists a \( C \) so that if \( t_2 > \lambda C^{-1/2} + C \), then \( \{\xi_{t_2} = 1\} \) \( \ell^1 \)–percolates and the infinite \( \ell^1 \)–cluster contains an infinite sequence of disjoint \( \ell^1 \)–circuits. On the other hand, if \( t_1 < \lambda C^{-1/2} - C \), then \( \{\xi_{t_1} = 0\} \) \( \ell^\infty \)–percolates.

For every \( x \in \mathbb{Z}^2 \), let \( e(x) \) be the “excitation time” of \( x \), when \( \gamma_{e(x)}(x) = 1 \). Assume that \( t_1, t_2 \) are as above, with \( t_2 - t_1 \leq C \). Let \( v_0, v_1, \ldots \) be a self–avoiding \( \ell^\infty \)–path inside \( \{\xi_{t_1} = 0\} \). Let \( i_k, k = 1, 2, \ldots \) be such that \( v_{i_k} \) are in the infinite \( \ell^1 \)–cluster of \( \{\xi_{t_2} = 1\} \). By Lemma 5.1 we can connect \( v_{i_k} \) and \( v_{i_{k+1}} \) by an \( \ell^\infty \)–path in \( \partial^\infty \{\xi_{t_2} = 1\} \). However, Lemma 2.3 implies that there exists a large enough \( C \) so that \( \xi_{t_2+C}(x) = 1 \) for every \( x \in \partial^\infty \{\xi_{t_2} = 1\} \). Thus we produce a self–avoiding infinite \( \ell^\infty \)–path \( v'_0, v'_1, \ldots \) inside \( \{\xi_{t_1} = 0\} \cap \{\xi_{t_1+C} = 1\} \). By Lemma 2.1, \( e(v'_k) \in [t_1, t_1 + C] \) for every \( k \), and thus there exists an \( x_k \in B^\infty(v'_k, C) \) so that \( \gamma_{t_1}(x_k) = 1 \). However, if \( a \) is large enough, then \( \{x_k\} \) forms an infinite \( B^\infty(0, a) \)–percolating set, proving part (1).

Parts (2) and (3) follow easily from Theorem 3 and its proof. □
Acknowledgements. The author expresses his utmost gratitude and appreciation to David Griffeath for guidance and encouragement during the writing of his thesis and this paper. The development of theory was greatly assisted by WinCA, a wonderful Windows–based CA simulation package, written by Robert Fisch and David Griffeath. (WinCA can be downloaded from http://math.wisc.edu/~griffeat/sink.html.) The author is also thankful to Ray Kapral for sharing his knowledge about physical systems modeled by the GHM, and to Yu Zhang for many very illuminating e–mail messages on percolation issues.

REFERENCES

[AG] M. Aizenman, G. Grimmett, Strict monotonicity for critical points in percolation and ferromagnetic models, J. Stat. Phys. 63 (1991), 817–835.

[Bra] R. M. Bradley, A Comment on “Ring Dynamics and percolation in an excitable medium,” J. Chem. Phys. 86 (1987), 7245–7246.

[DG] R. Durrett, D. Griffeath, Asymptotic behavior of excitable cellular automata, Experimental Math. 2 (1993), 184–208.

[DN] R. Durrett, C. Neuhauser, Epidemic with recovery in $d = 2$, Ann. Appl. Prob. 1 (1991), 189–206.

[DS] R. Durrett, J. E. Steif, Some rigorous results for the Greenberg-Hastings model, J. Theor. Prob. 4 (1991), 669–690.

[Dur] R. Durrett, Multicolor particle systems with large threshold and range, J. Theor. Prob. 5 (1992), 127–152.

[FGG1] R. Fisch, J. Gravner, D. Griffeath, Threshold-range scaling of excitable cellular automata, Statistic and Computing 1 (1991), 23–39.

[FGG2] R. Fisch, J. Gravner, D. Griffeath, Metastability in the Greenberg–Hastings model, Ann. Appl. Prob. 3 (1993), 935–967.

[FK] S. Fraser, R. Kapral, Ring dynamics and percolation in an excitable medium, J. Chem. Phys. 85 (1986), 5682–5688.

[GH] J. M. Greenberg, S. P. Hastings, Spatial patterns for discrete models of diffusion in excitable media, SIAM J. Appl. Math. 34 (1978), 515–523.

[Gra1] J. Gravner, Mathematical aspects of excitable media, Ph. D. Thesis, University of Wisconsin, 1991.

[Gra2] J. Gravner, The boundary of iterates in Euclidean growth models, Trans. Amer. Math. Soc., to appear.

[Gra3] J. Gravner, Recurrent ring dynamics in two-dimensional excitable cellular automata, submitted.

[GG] J. Gravner, D. Griffeath, First passage times for threshold growth dynamics on $\mathbb{Z}^2$, Ann. Prob., to appear.

[Gri] G. Grimmett, “Percolation,” Springer-Verlag, 1989.

[JM] W.A. Johnson and R.F. Mehl, Reaction kinetics in processes of nucleation and growth, Trans.
A.I.M.M.E. 135 (1939), 416–458.

[KW] R. Kapral, M. Weinberg, *Phase transformation kinetics in finite inhomogeneously nucleated systems*, J. Chem. Phys. 11 (1989), 7146–7152.

[Kap] R. Kapral, *Discrete models for chemically reacting systems*, J. Math. Chem. 6 (1991), 113–163.

[Kes] H. Kesten, “Percolation Theory for Mathematicians,” Birkhäuser, 1982.

[Pen] M. D. Penrose, *Single linkage clustering and continuum percolation*, J. of Multivariate Analysis 53 (1995), 90–104.

[Roy] R. Roy, *The Russo–Seymour–Welsh theorem and the equality of critical densities and the “dual” critical densities for continuum percolation on \( \mathbb{R}^2 \)*, Ann. Prob. 18 (1990), 1563–1575.

[SRC] J. M. Smith, A. L. Ritzenberg, R. J. Cohen, *Percolation theory and cardiac conduction*, Computers in Cardiology (1984), 175–178.

[Ste] J. Steif, *Two applications of percolation to cellular automata*, J. Stat. Phys. 78 (1995), 1325–1335.

[Wil] S. J. Willson, *On convergence of configurations*, Discrete Mathematics 23 (1978), 279–300.

[WR] N. Wiener, A. Rosenbluth, *The mathematical formulation of the problem of conduction of impulses in a network of connected excitable elements, specifically in cardiac muscle*, Arch. Inst. Cardiol. Mexico 16 (1946), 205–265.

[WTW] J. R. Weimar, J. J. Tyson, L. T. Watson, *Third generation cellular automaton for modeling excitable media*, Physica D 55 (1992), 328–339.

[ZS] S. A. Zuev, A. T. Sidorenko, *Continuous models of percolation theory I, II*, Theoretical and Mathematical Physics 62 (1985), 51–58, 171–177.