Rare Radiative $B_c \to D_{s1}(2460)\gamma$ Transition in QCD

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Abstract

We investigate the radiative $B_c \to D_{s1}\gamma$ transition in the framework of QCD sum rules. In particular, we calculate the transition form factors responsible for this decay in both weak annihilation and electromagnetic penguin channels using the quark condensate, mixed and two-gluon condensate diagrams as well as propagation of the soft quark in the electromagnetic field as non-perturbative corrections. These form factors are then used to estimate the branching ratios of the channels under consideration. The total branching ratio of the $B_c \to D_{s1}\gamma$ transition is obtained to be in order of $10^{-5}$, and the dominant contribution comes from the weak annihilation channel.

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I. INTRODUCTION

The $B_c$ is the only heavy meson consisting of two heavy quarks with different flavors, hence the decay properties of this meson are of special interest. The difference in heavy quark flavors forbids annihilation of this meson into gluons, so the excited $B_c$ states undergo pionic or radiative transition to the pseudoscalar (PS) ground state when these states lie below the threshold of the decay into the pair of heavy $B$ and $D$ mesons. The resulting PS ground state is more stable compared to the corresponding quarkonia and decays mostly weakly. Because of this phenomenon, it is expected that the experimental study of the $B_c$ meson and its decay properties will constitute an important part of the physics program at LHCb. The study of the heavy mesons will not only provide a window in extracting the most accurate values of the Cabbibo- Kobayashi- Maskawa (CKM) matrix elements as the sources of the $CP$-violation in the Standard Model (SM) but also will help us better understand the perturbative and non-perturbative aspects of QCD.

In the present study, we work out the rare radiative $B_c \to D_{s1}(2460)\gamma$ transition in the framework of the QCD sum rules [1, 2]. Here, the $D_{s1}(2460)$ is the axial vector charmed-strange meson with quantum numbers $J^P = 1^+$ and the interpolating current $\eta_\nu = \bar{s}\gamma_\nu\gamma_5 c$. This transition proceeds via both weak annihilation (WA) and electromagnetic penguin (EP) of flavor changing neutral current (FCNC) transition, based on the $b \to s\gamma$ at quark level. We calculate the transition form factors responsible for this decay in both WA and EP modes using the quark condensate, mixed and two-gluon condensate diagrams, as well as propagation of the soft quark in the electromagnetic field as non-perturbative corrections. We then use these form factors to estimate the branching ratios in both modes as well as the total branching fraction of the $B_c \to D_{s1}(2460)\gamma$ transition. As expected, the dominant contribution comes from the weak annihilation channel. Note that similar decays like the $B_c \to D_s^*\gamma$ transition have been studied in the same framework [3]. Some other radiative channels of the $B_c$ meson like, $B_c \to l\bar{\nu}\gamma$ and $B_c \to B_u^*\gamma$ have also been previously studied using the QCD sum rules technique [4, 5]. For analysis of other decay channels of the $B_c$ meson see, for instance, [6–9].

The outline of the paper is as follows. In Section II, we consider the radiation of the photon from both $B_c$ and $D_{s1}$ mesons, to construct the transition amplitude for the WA channel in terms of four relevant form factors. Two of the form factors $F_V^{(B_c)}$ and $F_A^{(B_c)}$,
responsible for the emission of the photon from the initial state, are calculated in [4], and the remaining two form factors $F_{V}^{(D_{s1})}$ and $F_{A}^{(D_{s1})}$, representing the emission of the photon from $D_{s1}$ meson, are calculated in Section III. In Section IV, we consider the two gluon condensate contributions, to calculate the transition form factors responsible for the EP mode. Finally, Section V is devoted to the numerical analysis of the form factors and calculation of the decay rates and branching ratios for the modes under consideration. We also present results for the total decay rate and branching ratio of the $B_{c} \to D_{s1}(2460)\gamma$ transition. This Section also contains our concluding remarks.

II. WEAK ANNIHILATION AMPLITUDE

In this section, we construct the WA amplitude for the radiative $B_{c} \to D_{s1}\gamma$ transition. Considering the quark contents of the initial and final mesonic states, the possible diagrams are shown in figure 1. Taking into account these diagrams, the transition amplitude for the radiative decay under consideration is written as

$$M^{WA}(B_{c} \to D_{s1}\gamma) = \frac{G_{F}}{\sqrt{2}}V_{cb}V_{cs}^{*}\langle D_{s1}(p)|\gamma(q)|B_{c}(p+q)\rangle, \quad (1)$$

where $G_{F}$ is the Fermi weak coupling constant, $V_{ij}$ are elements of the CKM matrix, $\Gamma_{\nu} = \gamma_{\nu}(1 - \gamma_{5})$; and $p$, $q$ and $p + q$ are the momenta of the $D_{s1}$ meson, photon and $B_{c}$ meson, respectively. To proceed further, we use the factorization hypothesis and write the transition
matrix element in Eq. (1) as

\[
\langle D_{s1}(p)\gamma(q)|(\bar{s}\Gamma_{c})(\bar{c}\Gamma_{b})|B_{c}(p+q)\rangle = -e\varepsilon^{\mu}(D_{s1})\nu f_{D_{s1}}m_{D_{s1}}T_{\mu\nu}^{(B_{c})} - i\varepsilon^{\mu}(p+q)\nu f_{B_{c}}T_{\mu\nu}^{(D_{s1})},
\]

(2)

where we have divided the matrix element into two separate parts: the emission of the photon from the \(B_{c}\) meson (diagrams (i) and (ii) in figure 1), represented by the covariant tensor \(T_{\mu\nu}^{(B_{c})}\) and the emission of the photon from the \(D_{s1}\) meson denoted by the tensor \(T_{\mu\nu}^{(D_{s1})}\) (see diagrams (iii) and (iv) in figure 1). In Eq. (2), \(f_{B_{c}}\) (\(f_{D_{s1}}\)) is the decay constant of the \(B_{c}\) (\(D_{s1}\)) meson and \(\varepsilon^{\mu}(\varepsilon^{(D_{s1})}\nu)\) is the polarization vector of the photon (\(D_{s1}\) meson). The covariant tensors \(T_{\mu\nu}^{(B_{c})}\) and \(T_{\mu\nu}^{(D_{s1})}\) are defined as

\[
T_{\mu\nu}^{(B_{c})}(p,q) \equiv i \int d^{4}x e^{iqx} \langle 0| T \left\{ j_{\mu}^{em}(x)\bar{s}(0)\Gamma_{\nu}b(0) \right\} |B_{c}(p+q)\rangle,
\]

(3)

\[
T_{\mu\nu}^{(D_{s1})}(p,q) \equiv i \int d^{4}x e^{iqx} \langle D_{s1}(p)| T \left\{ j_{\mu}^{em}(x)\bar{s}(0)\Gamma_{\nu}c(0) \right\} |0\rangle,
\]

(4)

where \(j_{\mu}^{em}\) is the electromagnetic current and \(T\) is the time ordering operator. Applying the Ward identity for the electromagnetic current, using \(q^{2} = 0\) for the real photon, \(\varepsilon.q = 0\) and \(\varepsilon^{(D_{s1})}.p = 0\), similar to what is done in [3, 10, 11], we get the following results corresponding to the emission of the photon from the initial and final mesonic states in terms of form factors:

\[
e\varepsilon^{\mu}(D_{s1})\nu f_{D_{s1}}m_{D_{s1}}T_{\mu\nu}^{(B_{c})} = e f_{D_{s1}}m_{D_{s1}} \left\{ \left[ (\varepsilon,\varepsilon^{(D_{s1})})(p,q) - (\varepsilon.p)(\varepsilon^{(D_{s1})}.q) \right] iF_{A}^{(B_{c})} \right. \\
+ \left. i f_{B_{c}}(\varepsilon,\varepsilon^{(D_{s1})}) + \varepsilon_{\nu\mu\lambda\sigma}(\varepsilon^{(D_{s1})}\nu)\varepsilon^{\mu}p^{\lambda}q^{\sigma} F_{V}^{(B_{c})} \right\},
\]

(5)

\[
i\varepsilon^{\mu}(p+q)\nu f_{B_{c}}T_{\mu\nu}^{(D_{s1})} = i f_{B_{c}} \left\{ \left[ (\varepsilon,\varepsilon^{(D_{s1})})(p,q) - (\varepsilon.p)(\varepsilon^{(D_{s1})}.q) \right] iF_{A}^{(D_{s1})} \right. \\
+ \left. f_{D_{s1}}m_{D_{s1}}(\varepsilon,\varepsilon^{(D_{s1})}) + \varepsilon_{\nu\mu\lambda\sigma}(\varepsilon^{(D_{s1})}\nu)\varepsilon^{\mu}p^{\lambda}q^{\sigma} F_{V}^{(D_{s1})} \right\},
\]

(6)
where $F^{(B_c)}_{V(A)}$ and $F^{(D_{s1})}_{V(A)}$ are the transition form factors. Using Eqs (5), (6) and (2) we find the WA transition amplitude to be

$$M^{WA}(B_c \rightarrow D_{s1}\gamma) =$$

$$eG_F \sqrt{2} V_{cb} V_{cs}^* \left\{ -f_{D_{s1}} m_{D_{s1}} \left[ (\varepsilon^{(D_{s1})}) (p,q) - (\varepsilon^{(D_{s1})}) (\varepsilon^{(D_{s1})}) \cdot q \right] iF^{(B_c)}_A + i f_{B_c} (\varepsilon^{(D_{s1})}) \right\} +$$

$$+ f_{D_{s1}} m_{D_{s1}} (\varepsilon^{(D_{s1})}) + \varepsilon_{\mu\lambda\sigma} (\varepsilon^{(D_{s1})}) \varepsilon^\mu \lambda^\sigma F^{(D_{s1})}.$$

(7)

As mentioned in Section I, the form factors $F^{(B_c)}_{V}$ and $F^{(B_c)}_{A}$ are calculated in [4], so what remains to be calculated are the form factors $F^{(D_{s1})}_{V}$ and $F^{(D_{s1})}_{A}$, which we discuss in the next Section.

### III. QCD SUM RULES FOR THE FORM FACTORS $F^{(D_{s1})}_{V}$ AND $F^{(D_{s1})}_{A}$

To calculate the transition form factors $F^{(D_{s1})}_{V}$ and $F^{(D_{s1})}_{A}$ via QCD sum rules formalism, we start considering the following correlation function:

$$\Pi_{\mu\nu}(p,q) = i \int d^4x e^{iqx} \langle \gamma(q) | T \left\{ \bar{c}(x) \gamma^\mu (1 - \gamma_5) s(x) \bar{s}(0) \gamma_5 c(0) \right\} | 0 \rangle,$$

(8)

where $Q = p + q$. The basic idea in this method is to calculate this correlation function first in hadronic language, called phenomenological or physical side and second in terms of the QCD degrees of freedom using the operator product expansion in deep Euclidean space, called the theoretical or QCD side. The two representations are then matched in order to get the QCD sum rules for the form factors. To suppress the contributions coming from the higher energy states and continuum we apply a Borel transformation as well as continuum subtraction which bring two auxiliary parameters: namely the Borel mass parameter and the continuum threshold. We shall find their working regions requiring that the physical observables be independent of these parameters.

First, we focus on calculation of the phenomenological side. For this aim, we insert a full set of hadronic $D_{s1}$ state into Eq.(8) and perform the four-integral over $x$ to get

$$\Pi_{\mu\nu}(p,q) = \frac{\langle \gamma(q) | T \left\{ \bar{c}(x) \gamma^\mu (1 - \gamma_5) s(x) \bar{s}(0) \gamma_5 c(0) \right\} | 0 \rangle}{m^2_{D_{s1}} - p^2}.$$
The matrix element, \( \langle D_{s1}(p)|\bar{s}\gamma_\nu\gamma_5 c|0 \rangle \) is defined in terms of the decay constant and the polarization vector of the \( D_{s1} \) meson as
\[
\langle D_{s1}(p)|\bar{s}\gamma_\nu\gamma_5 c|0 \rangle = f_{D_{s1}} m_{D_{s1}} \varepsilon^{(D_{s1})}_\nu, \tag{10}
\]
while the transition matrix element is parametrized in terms of form factors,
\[
\langle \gamma(q)|\bar{c}\gamma_\mu (1 - \gamma_5) s|D_{s1}(p) \rangle = i \varepsilon_{\mu\nu\alpha\sigma} q^\alpha \frac{\varepsilon^{(D_{s1})}_\beta F^{D_{s1}}(Q^2)}{m_{D_{s1}}^2} \left[ \varepsilon_\mu (\varepsilon^{(D_{s1})})_\nu - (\varepsilon_\nu (\varepsilon^{(D_{s1})})_\mu q_\nu \frac{F^{D_{s1}}(Q^2)}{m_{D_{s1}}^2} - (\varepsilon_\mu (\varepsilon^{(D_{s1})})_\nu q_\mu \frac{F^{D_{s1}}(Q^2)}{m_{D_{s1}}^2} \right]. \tag{11}
\]
Substituting Eqs. (10) and (11) into Eq. (9) and summing over the polarization vector of the \( D_{s1} \) meson, we find the following result for the phenomenological part of the correlation function:
\[
\Pi_{\mu\nu}(p,q) = i f_{D_{s1}} m_{D_{s1}} \left[ i \varepsilon_{\mu\nu\alpha\sigma} q^\alpha \frac{F^{D_{s1}}(Q^2)}{m_{D_{s1}}^2} + \left[ \varepsilon_\mu (\varepsilon^{(D_{s1})})_\nu - (\varepsilon_\nu (\varepsilon^{(D_{s1})})_\mu q_\nu \frac{F^{D_{s1}}(Q^2)}{m_{D_{s1}}^2} + (\varepsilon_\mu (\varepsilon^{(D_{s1})})_\nu q_\mu \frac{F^{D_{s1}}(Q^2)}{m_{D_{s1}}^2} \right]. \tag{12}
\]
We now compute the QCD side of the correlation function within the deep Euclidean region in terms of the QCD parameters. We start by writing the correlation function in terms of the two selected structures as
\[
\Pi_{\mu\nu}(p,q) = i \varepsilon_{\mu\nu\alpha\sigma} q^\alpha \Pi_1 + \left[ q_\mu \varepsilon_\nu - \varepsilon_\mu q_\nu \right] \Pi_2, \tag{13}
\]
where each function \( \Pi_i \) \((i = 1 \text{ or } 2)\) has perturbative and non-perturbative parts, i.e.,
\[
\Pi_i = \Pi_i^{\text{pert}} + \Pi_i^{\text{non-pert}}. \tag{14}
\]
To calculate the perturbative parts, we consider diagrams (a) and (b) in figure 2 where the photon can be radiated from both the charm and strange quarks. For the non-perturbative parts, we take into account the quark condensate and mixed diagrams \( \text{[diagrams 2 (c), 2 (d) and 2 (e)] as well as diagram 2 (f) for the interaction of the photon with the soft quark.} \]

The perturbative part in each case can be written via the dispersion relation as
\[
\Pi_i^{\text{pert}} = \int ds \frac{\rho_i(s,Q^2)}{s - p^2} + \text{subtraction terms.} \tag{15}
\]
where \( \rho_i \) are the spectral densities. Our main task is now to calculate these spectral densities using the diagrams (a) and (b) in figure 2. Here, we use a method based on both Feynman
and Schwinger parameterizations with several Borel transformations (see also [12]). The Feynman amplitude for the diagram (a) can be written as

$$\Pi_{\mu\nu,(a)} = e N_c Q_s \int \frac{d^4k}{(2\pi)^4} \left\{ Tr \left[ i(k + m_c) \gamma_\nu \gamma_5 \left( p + k + m_s \right) \frac{i(Q + k + m_s)}{(p + k)^2 - m_s^2} \frac{i(Q)\gamma_\mu(1 - \gamma_5)}{(Q + k)^2 - m_s^2} \right] \right\},$$

where $N_c = 3$ is the number of colors and $Q_s$ is the charge of the strange quark. Using the Feynman parameterization, we perform the four-integral over $k$ and then we use the Schwinger parameterization

$$\frac{1}{\Delta_n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha \alpha^{n-1} e^{-\alpha\Delta},$$

FIG. 2. Diagrams for bare-loop [(a), (b)], quark and mixed condensates [(c), (d), (e)] and propagation of the soft quark in the electromagnetic field (f).
to write the denominators in exponential forms. As a result, we get

\[ \Pi_{1,a}^{pert} = \frac{eN_c Q_s}{4\pi^2} \left\{ \int_0^1 dx \int_0^1 dy \left[ m_s (m_c + m_s xy) + p^2 x \mathbf{r}(1 - \mathbf{r} y) + 2p q x^2 y^2 \right] \int_0^\infty d\alpha e^{-\alpha \Delta} \right\} , \]

\[ \Pi_{2,a}^{pert} = \frac{eN_c Q_s}{4\pi^2} \left\{ \int_0^1 dx \int_0^1 dy \left[ m_s (m_c - m_s xy) - p^2 x \mathbf{r}(1 - \mathbf{r} y) - 2p q x^2 y^2 \right] \int_0^\infty d\alpha e^{-\alpha \Delta} \right\} , \]

where \( \mathbf{r}(y) = 1 - x(y) \), and \( \Delta = m_c^2 \mathbf{r} + m_s^2 x - p^2 x \mathbf{r} y - Q^2 x \mathbf{r} y \).

Applying a double Borel transformation \( \hat{B}(M_1^2) \hat{B}(M_2^2) \) on \( \Pi_i^{pert} \), that transforms \( Q^2 \to M_1^2 \) and \( p^2 \to M_2^2 \), we obtain

\[ \hat{\Pi}_{1,a}^{pert} = \frac{eN_c Q_s}{4\pi^2} \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} \int_0^1 dx \frac{1}{x} \frac{e^{(m_c^2 + m_s^2)(\sigma_1 + \sigma_2)}}{x} \]

\[ \times \left\{ m_c m_s + m_s^2 x \frac{\sigma_1}{\sigma_1 + \sigma_2} + 2x(1 - x^2) \frac{\sigma_1}{(\sigma_1 + \sigma_2)^2} \right\} , \]

\[ \hat{\Pi}_{2,a}^{pert} = \frac{eN_c Q_s}{4\pi^2} \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} \int_0^1 dx \frac{1}{x} \frac{e^{(m_c^2 + m_s^2)(\sigma_1 + \sigma_2)}}{x} \]

\[ \times \left\{ m_c m_s - m_s^2 x \frac{\sigma_1}{\sigma_1 + \sigma_2} - 2x(1 - x^2) \frac{\sigma_1}{(\sigma_1 + \sigma_2)^2} \right\} , \]

where \( \sigma_{1,2} = 1/M_{1,2}^2 \) and we have used

\[ \hat{B}_{p^2}(M^2) e^{-\alpha p^2} = \delta(1 - \alpha M^2) , \]

\[ \hat{B}_{p^2}(M^2) p^2 e^{-\alpha p^2} = -\frac{d}{d\alpha} \hat{B}_{p^2}(M^2) e^{-\alpha p^2} = -\frac{d}{d\alpha} \delta(1 - \alpha M^2) . \]

Now, we perform a second double Borel transformation on \( \hat{\Pi}_i^{pert} \) in order to transform \( \sigma_1 \) and \( \sigma_2 \) to the new variables \( w \) and \( s \) using

\[ g_1(w, s) = \frac{1}{ws} \hat{B} \left( \frac{1}{w} \sigma_1 \right) \hat{B} \left( \frac{1}{s} \sigma_2 \right) \frac{\hat{\Pi}_i^{pert}}{\sigma_1 \sigma_2} . \]

In the calculations, we also use the relations

\[ \hat{B} \left( \frac{1}{w} \sigma_1 \right) \hat{B} \left( \frac{1}{s} \sigma_2 \right) e^{-\alpha(\sigma_1 + \sigma_2)} = \delta(1 - \frac{\alpha}{w}) \delta(1 - \frac{\alpha}{s}) , \]

and

\[ \sigma^n e^{-\alpha \sigma} = \left( -\frac{d}{d\alpha} \right)^n e^{-\alpha \sigma} . \]
The final expressions for the spectral densities are then calculated via the following formula:

\[ \rho_i(s, Q^2) = \int dw \frac{q_i(w, s)}{w - Q^2}. \]  

(26)

After lengthy calculations, we get the following spectral densities corresponding to the diagram (a):

\[ \rho_{1a}(s, Q^2) = \frac{eN_c Q_s}{16\pi^2} \frac{1}{(s - Q^2)^2} \int_{x_0}^{x_1} dx \frac{1}{x \bar{x}^2} \left\{ m_c^4 x^2 (x - 5) + m_s^4 x^2 (x - 6) - m_c^2 m_s^2 x \bar{x} (2x - 11) 
+ 4m_c m_s x \bar{x}(s - Q^2) + m_s^2 x \bar{x} \left[ (\bar{x} - 8)Q^2 + \bar{x}(s - Q^2) \right] - m_s^2 x \bar{x} \left[ (x - 10)Q^2 
+ (x - 2)(s - Q^2) \right] - 4x^2 \bar{x}^2 Q^2 (s - Q^2) - 4x^2 \bar{x}^2 Q^4 \right\}, \]

(27)

\[ \rho_{2a}(s, Q^2) = \frac{eN_c Q_s}{16\pi^2} \frac{1}{(s - Q^2)^2} \int_{x_0}^{x_1} dx \frac{1}{x \bar{x}^2} \left\{ - m_c^4 x^2 (x - 5) - m_s^4 x^2 (-6 + x) + m_c^2 m_s^2 x \bar{x} (2x - 11) 
+ 4m_c m_s x \bar{x}(s - Q^2) - m_s^2 x \bar{x} \left[ (\bar{x} - 8)Q^2 + \bar{x}(s - Q^2) \right] + m_s^2 x \bar{x} \left[ (x - 10)Q^2 
+ (x - 2)(s - Q^2) \right] + 4x^2 \bar{x}^2 Q^2 (s - Q^2) + 4x^2 \bar{x}^2 Q^4 \right\}, \]

(28)

where the integral boundaries \( x_0 \) and \( x_1 \) satisfy the following inequality:

\[ sx \bar{x} - (m_c^2 \bar{x} + m_s^2 x) \geq 0, \]

(29)

which comes from the definition of the Heaviside-Theta function arising in these calculations.

Similarly, we calculate the contribution of the diagram 2(b). The final expressions for the spectral densities corresponding to the two selected structures are

\[ \rho_1(s, Q^2) = \frac{eN_c}{32\pi^2} \frac{1}{(s - Q^2)^2} \left\{ Q_s \left[ \lambda \left( 4(5\alpha - 5\beta - 1)sQ^2 + s^2 \left[ 3\alpha(\alpha - 3) - \beta(6\alpha + 1) + 3\beta^2 \right] \right) 
+ 2 \left( 4m_c m_s (s - Q^2) + 8\alpha sQ^2 + \alpha(1 - 4\alpha + 9\beta)s^2 \right) \ln \left( \frac{1 + \alpha - \beta - \lambda}{1 + \alpha - \beta + \lambda} \right) 
+ Q_c \left[ \lambda \left( 4(5\beta - 5\alpha - 1)sQ^2 + s^2 \left[ 3\beta(\beta - 3) - \alpha(6\beta + 1) + 3\alpha^2 \right] \right) 
+ 2 \left( 4m_c m_s (s - Q^2) + 8\beta sQ^2 + \beta(1 - 4\beta + 9\alpha)s^2 \right) \ln \left( \frac{1 + \beta - \alpha - \lambda}{1 + \beta - \alpha + \lambda} \right) \right] \right\}, \]

(30)
where $\alpha = \frac{m_q^2}{s}$, $\beta = \frac{m_s^2}{s}$ and $\lambda = \sqrt{1 + \alpha^2 + \beta^2 - 2\alpha - 2\beta - 2\alpha\beta}$.

For the non-perturbative parts, we begin by calculating contributions of the quark condensate and mixed diagrams (diagrams (c), (d) and (e) in figure 2) and obtain

\[
\Pi_{1(c,d,e)}^{\text{non-pert}} = \frac{m_c}{r^2 R^2} \langle \bar{s}s \rangle + \frac{m_s}{2} \langle \bar{s}s \rangle \left[ \frac{2}{r^2 R^2} + \frac{m_c^2}{r^4 R^4} - \frac{7m_c^2}{r^4 R^4} - \frac{4m_c^4}{r^4 R^4} \right]
\]

\[
+ \frac{m_s^2}{2} \langle \bar{s}s \rangle \left[ \frac{2m_c^3}{r^2 R^6} - \frac{8m_c^5}{r^6 R^4} + \frac{2m_c^3}{r^4 R^4} - \frac{3m_c^6}{r^4 R^4} + \frac{2m_c^5}{r^6 R^4} - \frac{m_c^8}{r^4 R^4} \right]
\]

\[
+ \frac{m_0}{12} \langle \bar{s}s \rangle \left[ -\frac{6m_c^3}{r^2 R^6} + \frac{24m_c^5}{r^6 R^4} - \frac{6m_c^3}{r^4 R^4} + \frac{8m_c^6}{r^4 R^4} - \frac{6m_c^6}{r^6 R^4} + \frac{3m_c^8}{r^4 R^4} \right] - \frac{4m_c^3}{r^2 R^4} \langle \bar{s}s \rangle,
\]

(32)

\[
\Pi_{2(c,d,e)}^{\text{non-pert}} = -\frac{m_c}{r^2 R^2} \langle \bar{s}s \rangle + \frac{m_s}{2} \langle \bar{s}s \rangle \left[ \frac{1}{r^2 R^2} + \frac{1}{R^4} - \frac{4m_c^4}{r^4 R^4} - \frac{3m_c^6}{r^4 R^4} + \frac{2m_c^8}{2r^4 R^2} \right]
\]

\[
+ \frac{m_s^2}{2} \langle \bar{s}s \rangle \left[ -\frac{2m_c^3}{r^2 R^6} + \frac{8m_c^5}{r^6 R^4} - \frac{2m_c^3}{r^4 R^4} + \frac{3m_c^6}{r^4 R^4} - \frac{2m_c^5}{r^6 R^4} + \frac{m_c^8}{r^4 R^4} \right]
\]

\[
+ \frac{m_0^2}{4} \langle \bar{s}s \rangle \left[ \frac{2m_c^3}{r^2 R^6} - \frac{8m_c^5}{r^6 R^4} + \frac{2m_c^3}{r^4 R^4} - \frac{4m_c^6}{r^4 R^4} + \frac{2m_c^5}{r^6 R^4} - \frac{m_c^8}{r^4 R^4} \right] + \frac{4m_c^3}{r^2 R^4} \langle \bar{s}s \rangle,
\]

(33)

where $r^2 = p^2 - m_c^2$ and $R^2 = Q^2 - m_c^2$.

The final contribution to the WA mode is that of diagram (f). This diagram corresponds to the propagation of the soft quark in the external electromagnetic field. Here we need to make use of the light-cone version of the QCD sum rules and photon distribution amplitudes (DAs). The relevant correlation function is of the form:

\[
\Pi_{\mu\nu,(f)}(p, q) = i \int d^4x e^{-iQx} \langle \gamma(q) | T \left\{ \bar{\sigma}(0) \gamma_\mu \gamma_5 c(0) \bar{c}(x) \gamma_\nu (1 - \gamma_5) s(x) \right\} | 0 \rangle.
\]

(34)

Contracting the $c$-quark lines in Eq. (31) and using the propagator of the heavy quark in momentum space, we obtain

\[
\Pi_{\mu\nu,(f)}(p, q) = i^2 \int d^4x \frac{d^4k}{(2\pi)^4} \frac{e^{-i(Q-k)x}}{m_c^2 - k^2} \langle \gamma(q) | \bar{\sigma} \gamma_\mu \gamma_5 (k + m_c) \gamma_\nu (1 - \gamma_5) s | 0 \rangle.
\]

(35)
To relate the matrix element in the above equation to the photon DAs, we use the identities
\[ \gamma_{\mu} \gamma_{\nu} = g_{\mu \nu} + i \sigma_{\mu \nu}, \]
\[ \gamma_{\mu} \gamma_{5} = g_{\mu 5} - 2 \varepsilon_{\mu \omega \alpha \beta} \sigma_{\alpha \beta}, \]
\[ \gamma_{\mu} \gamma_{\nu} = g_{\mu \nu} + \gamma_{\mu} \gamma_{\nu} - g_{\mu \nu} \gamma_{\alpha} + i \varepsilon_{\mu \omega \alpha \lambda} \gamma_{\lambda} \gamma_{5}. \]

The relevant photon DAs of twist 2, 3, and 4 \cite{13, 14} are
\[ \langle \gamma (q) | s \gamma_{\nu} | 0 \rangle = -\frac{Q_s}{2} f_{3\gamma} \int_{0}^{1} du \bar{\psi}^{(V)} (u) x^{\theta} F_{\theta \nu} (ux), \]
\[ \langle \gamma (q) | s \gamma_{\alpha} \gamma_{5} | 0 \rangle = -\frac{iQ_s}{4} f_{3\gamma} \int_{0}^{1} du \bar{\psi}^{(A)} (u) x^{\theta} \tilde{F}_{\theta \alpha} (ux), \]
\[ \langle \gamma (q) | s \sigma_{\alpha \beta} | 0 \rangle = Q_s \langle ss \rangle \int_{0}^{1} du \phi (u) F_{\alpha \beta} (ux) \]
\[ + \frac{Q_s \langle ss \rangle}{16} \int_{0}^{1} dx x^{\alpha} A (u) F_{\alpha \beta} (ux) \]
\[ + \frac{Q_s \langle ss \rangle}{8} \int_{0}^{1} du B (u) x^{\beta} (x_{\beta} F_{\alpha \rho} (ux) - x_{\alpha} F_{\beta \rho}), \]

where \( F_{\mu \nu} \) is the field strength tensor of the electromagnetic field and is defined by
\[ F_{\mu \nu} (x) = -i (\varepsilon_{\mu q_{\nu}} - \varepsilon_{\nu q_{\mu}}) e^{iqx}, \]

and
\[ \tilde{F}_{\mu \nu} (x) = \frac{1}{2} \varepsilon_{\mu \nu \alpha \beta} F_{\alpha \beta} (x). \]

The wave function \( \phi (u) \) is defined in terms of the magnetic susceptibility \( \chi (\mu) \) at a renormalization scale (\( \mu = 1 \text{GeV}^2 \)) in the following manner:
\[ \phi (u) = \chi (\mu) u (1 - u). \]

The remaining functions \( \bar{\psi}^{(V)} (u), \bar{\psi}^{(A)} (u), A (u), \) and \( B (u) \) are also defined as \cite{13, 14}
\[ \bar{\psi}^{(V)} (u) = -20 u (1 - u) (2 u - 1) + \frac{15}{16} (\omega_{\gamma} A - 3 \omega_{\gamma} V) u (1 - u) (2 u - 1) (2 u - 1)^2 - 3, \]
\[ \bar{\psi}^{(A)} (u) = (1 - (2 u - 1)^2) \left( 5 (2 u - 1)^2 - 1 \right) \frac{5}{2} (1 + \frac{19}{16} \omega_{\gamma} V - \frac{3}{16} \omega_{\gamma} A), \]
\[ A (u) = 40 u (1 - u) (3 k^+ + 1) + 8 (\xi_2^+ - 3 \xi_2) \]
\[ \times [u (1 - u) (2 + 13 u (1 - u)) + 2 u^3 (10 - 15 u + 16 u^2) \ln u \]
\[ + 2 (1 - u)^3 (10 - 15 (1 - u) + 6 (1 - u^2)) \ln (1 - u)], \]
\[ B (u) = 40 \int_{0}^{u} d\alpha (4 - \alpha) (1 + 3 k^+) \left[ -\frac{1}{2} + \frac{3}{2} (2 \alpha - 1)^2 \right], \]
where $k$, $k^+$, $\xi_2$, $\xi_2^+$ and $f_{3\gamma}$ are constants (see [13, 14]). Putting the above equations all together and after performing the four-integrals over $x$ and $k$, the coefficients of the corresponding structures, $i\varepsilon_{\mu\alpha\beta} \varepsilon^\alpha q^\beta$ and $[q_\mu \varepsilon^\nu - \varepsilon_\mu q^\nu]$ are obtained as follows:

$$
\Pi_{1f}^{non-pert}(p, q) = \frac{Q_s}{2(m_c^2 - p^2)^2} \int_0^1 du \left\{ m_c^3 \langle \bar{s}s \rangle \mathbf{A}(u) \\
+ (m_c^2 - p^2) \left[ m_c \langle \bar{s}s \rangle \mathbf{B}(u) - 2(5m_c^2 - p^2)(m_c\langle \bar{s}s \rangle \phi(u) - f_{3\gamma} \psi^{(V)}(u)) \right] \right\},
$$

(42)

$$
\Pi_{2f}^{non-pert}(p, q) = \frac{m_cm_s Q_s}{2(m_c^2 - p^2)^2} \int_0^1 du \left\{ \mathbf{A}(u)m_c^2 \langle \bar{s}s \rangle + 2(-5m_c^2 + p^2)\langle \bar{s}s \rangle \phi(u) \\
+ (m_c^2 - p^2) \left[ \mathbf{B}(u)\langle \bar{s}s \rangle + f_{3\gamma}m_c \psi^{(A)}(u) \right] \right\}.
$$

(43)

Now, to find the QCD sum rules for the form factors we match the coefficients of the selected structures from both phenomenological and QCD sides and perform the Borel transformation with respect to the momentum of $D_{s1}$ meson ($p^2 \rightarrow M_B^2$). To further suppress the contributions of the higher energy states and continuum we also perform the continuum subtraction and use the quark-hadron duality assumption and find

$$
F_{V,A}^{(D_{s1})}(Q^2) = \frac{m_{D_{s1}}}{\int_{D_{s1}}} e^{m_{D_{s1}}/M_B^2} \hat{B} \left\{ \int_{(m_s + m_c)^2}^{s_0} ds \frac{P_{1,2}(s, Q^2)}{s - p^2} + \Pi_{1,2(c+d+e+f)}^{non-pert} \right\},
$$

(44)

where $s_0$ is the continuum threshold and the $V$ ($A$) on the left-hand side corresponds to the 1 (2) on the right-hand side. To obtain the expressions for the above sum rules in the Borel scheme, we perform the Borel transformation using the standard rule

$$
\frac{\hat{B}}{(p^2 - s)^n} = (-1)^n \frac{e^{-s/M_B^2}}{\Gamma(n)(M_B^2)^{n-1}}.
$$

(45)

IV. QCD SUM RULES FOR THE FORM FACTORS RESPONSIBLE FOR THE ELECTROMAGNETIC PENGUIN MODE

At the quark level, the FCNC based EP transition of the $B_c \rightarrow D_{s1}\gamma$ proceeds via $b \rightarrow s\gamma$ whose effective Hamiltonian is written as

$$
H^{eff} = -\frac{G_F e}{4\pi\sqrt{2}} V_{tb}^* V_{ts} C_7(\mu) \bar{s}s [m_b \frac{1 + \gamma_5}{2} + m_s \frac{1 - \gamma_5}{2}] bF^{\mu\nu}.
$$

(46)

The amplitude of this mode is obtained from

$$
M^{EP} = \langle D_{s1}(p)|H^{eff}|B_c(Q) \rangle,
$$

(47)
hence to proceed further, we need to calculate the following matrix elements:

\[ \langle D_{s1} | \overline{s} \sigma_{\mu \nu} (1 \pm \gamma_5) q^\nu b | B_c \rangle, \]  \hspace{1cm} (48)

which can be parametrized in terms of two gauge invariant form factors \( T_1(q^2) \) and \( T_2(q^2) \) in the case of real photon, i.e.

\[ \langle D_{s1}(p, \varepsilon^{(D_{s1})}) | \overline{s} \sigma_{\mu \nu} q^\nu b | B_c(Q) \rangle = i \varepsilon_{\mu \alpha \beta \lambda} \varepsilon^{(D_{s1})\alpha} p^\beta Q^\lambda T_1(0), \]
\[ \langle D_{s1}(p, \varepsilon^{(D_{s1})}) | \overline{s} \sigma_{\mu \nu} q^\nu b | B_c(Q) \rangle = \left[ (m_{B_c}^2 - m_{D_{s1}}^2) \varepsilon^{(D_{s1})\mu} - (\varepsilon^{(D_{s1}).q}(p + Q)_\mu \right] T_2(0), \]  \hspace{1cm} (49)

where these two from factors are not independent from each other. Using the relation, \( \sigma_{\mu \nu} \gamma_5 = -\frac{i}{2} \varepsilon_{\mu \nu \alpha \beta} \sigma^{\alpha \beta} \), we see \( T_1(0) = \frac{1}{2} T_2(0) \). Therefore, we need to calculate just one of them, and we choose to calculate the form factor \( T_2(0) \). The corresponding correlation function is chosen as

\[ \Pi_{\mu \nu}(p^2, Q^2) = i^2 \int \frac{d^4x d^4y}{e^{-i(Q \cdot p_y)}} \langle 0 | T \left\{ \overline{c}(y) \gamma_\nu \gamma_5 s(y) \overline{s}(0) \sigma_{\mu \alpha} q^\alpha b(0) \gamma_5 c(x) \right\} | 0 \rangle, \]  \hspace{1cm} (50)

where \( \overline{c} \gamma_5 c \) and \( \overline{s} \gamma_\nu \gamma_5 s \) are the interpolating currents of the initial and final mesonic states, respectively, and \( \overline{s} \sigma_{\mu \alpha} q^\alpha b \) is the transition current. Using the general philosophy of the QCD sum rules we calculate this correlation function in two different languages: namely the hadronic language and the quark-gluon language. For the hadronic, or phenomenological side, we get

\[ \Pi_{\mu \nu}(p^2, Q^2) = \frac{i f_{D_{s1}} f_{B_c} m_{D_{s1}} m_{B_c}^2}{(m_{B_c}^2 - Q^2)(m_{D_{s1}}^2 - p^2)(m_b + m_c)} \left\{ (m_{B_c}^2 - m_{D_{s1}}^2) g_{\mu \nu} T_2(0) \right. \]
\[ - \left. \left( \frac{m_{B_c}^2 - m_{D_{s1}}^2}{m_{D_{s1}}^2} \right) p_\mu p_\nu T_2(0) + (p + Q)_\mu \left[ \frac{p \cdot q}{m_{D_{s1}}^2} p_\nu - q_\nu \right] T_2(0) \right\} + ...., \]  \hspace{1cm} (51)

where ... denotes contributions of the higher energy states and continuum which will be suppressed by applying the Borel transformation as well as the continuum subtraction, and we have used the following definition of the decay constant of the \( B_c \) meson:

\[ \langle B_c | \overline{c} \gamma_5 c | 0 \rangle = i \frac{f_{B_c} m_{B_c}^2}{m_b + m_c}. \]  \hspace{1cm} (52)

To calculate the form factor \( T_2(0) \), we choose the structure \( g_{\mu \nu} \).

In the QCD side, the correlation function is written in terms of the selected structure as

\[ \Pi_{\mu \nu} = g_{\mu \nu} \Pi(p^2, Q^2), \]  \hspace{1cm} (53)
where
\[ \Pi(p^2, Q^2) = \Pi^{pert}(p^2, Q^2) + \Pi^{non-pert}(p^2, Q^2). \] (54)

Here, the perturbative part is related to the spectral density, \( \rho^{pert}(s', t) \) by a double dispersion integral,
\[ \Pi^{pert}(p^2, Q^2) = -\frac{1}{(2\pi)^2} \int \int ds' dt \frac{\rho^{pert}(s', t)}{(s' - Q^2)(t - p^2)} + \text{subtraction terms}, \] (55)
and for the non-perturbative contributions we will calculate the two-gluon condensate diagrams.

Now, we focus our attention on calculating the spectral density. Using the Cutkosky method \[15\], we get
\[ \rho^{pert}(s', t) = 2N_c \left\{ I_0 \left[ \Delta[(m_b - m_c)(m_c + m_s) + t] - \Delta'[t] \right] \right. \]
\[ + \left. 2m_c[(m_c + m_s)s' + m_b t - m_c t] - m_c(m_b + m_s)u \right\} + 2A1(-2s' + u), \] (56)
where
\[ I_0 = \frac{1}{4\sqrt{\chi(s', t, q^2)}}, \]
\[ \chi(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc, \]
\[ A_1 = \frac{-I_0}{(-4s't + u^2)^2} \left[ \Delta^2 t + \Delta'^2 s' - \Delta\Delta' u + \frac{m_c^2(-4s't + u^2)}{2} \right], \]
\[ \Delta = s' + m_c^2 - m_b^2, \]
\[ \Delta' = t + m_c^2 - m_s^2, \]
\[ u = t + s' - q^2. \] (57)

Note that, to obtain the above spectral density, we have performed the integrals over the delta-functions which restricts the boundaries of the integrals on the \( s' \) and \( t \) as:
\[ m_c^2 \leq t \leq t_0, \]
\[ t - \frac{tm_b^2}{m_c^2 - t} \leq s' \leq s'_0, \] (58)
where \( s'_0 \) and \( t_0 \) are the continuum thresholds in the initial and final channels in the case of EP mode. There are several sources for non-perturbative contributions, such as quark, quark-gluon, and gluon condensates, however, the quark-quark and quark-gluon condensates give zero contributions after applying the double Borel transformation with respect to \( Q^2 \) (\( Q^2 \to M_1^2 \)) and \( p^2 \) (\( p^2 \to M_2^2 \)). Therefore, the remaining source of the non-perturbative
contributions would be the gluon condensates [see Fig. (3)]. The calculation of such contributions is lengthy but standard. For the non-perturbative part in the Borel scheme, we get

\[ \Pi^{non-pert} = M_1^2 M_2^2 \left( \frac{\alpha_s}{\pi} G^2 \right) C_{G^2}, \]  

(59)

where \( C_{G^2} \) is the Wilson coefficient of the gluon condensates that is defined as

\[ C_{G^2} = C_G^{a_2} + C_G^{b_2} + C_G^{c_2} + C_G^{d_2} + C_G^{e_2} + C_G^{f_2}. \]  

(60)

The explicit expressions of the \( C_G^i \) are given in the Appendix.

Using a similar procedure to that presented in the previous section, we get the sum rule

FIG. 3. Feynman diagrams for gluon condensates corrections.
The physical observables are practically independent of them. The continuum thresholds are small, i.e., the series of contributions of the higher states and continuum are effectively suppressed, but also the contributions of the higher order operators and higher twist DAs are small, for the decay $B^+ \to D_{s1}(2460)^+ \gamma$ transition. For this aim, we use the quark and mesons’ masses as $m_c = (1.275 \pm 0.015) \text{GeV}$, $m_\pi \simeq 142 \text{MeV}$, $m_b = (4.7 \pm 0.1) \text{GeV}$, $m_{D_{s1}} = (2459.6 \pm 0.6) \text{MeV}$, $m_{B_c} = (6.277 \pm 0.006) \text{GeV}$, and $f_{D_{s1}} = (225 \pm 25) \text{MeV}$ and $f_{B_c} = (350 \pm 25) \text{MeV}$. The values of the decay constants, we use $f_{D_{s1}} = (225 \pm 25) \text{MeV}$ and $f_{B_c} = (350 \pm 25) \text{MeV}$. The parameters entered the photon DAs are also taken as $\chi = (3.15 \pm 0.30) \text{GeV}^{-2}$, $k = 0.2$, $k^+ = 0$, $\zeta_1 = 0.4$, $\zeta_2 = 0.3$, $\zeta_2^+ = 0$, $f_{3\gamma} = -(4 \pm 2) \times 10^{-3} \text{GeV}^2$, $\omega^A = -2.1 \pm 1.0$ and $\omega^V = 3.8 \pm 1.8$. The remaining parameters are chosen as $|V_{cs}| = 0.957 \pm 0.017$, $|V_{cb}| = 0.0416 \pm 0.0006$, $|V_{tb}| = 0.77^{+0.18}_{-0.24}$, $|V_{ts}| = (40.6 \pm 2.7) \times 10^{-3}$, $C_\gamma(\mu = m_\tau) = -0.0068 - 0.02i$, and $\tau_{B_c} = 0.52 \times 10^{-12} s$.

The sum rules for the form factors contain also the continuum thresholds and the Borel mass parameters as auxiliary objects. We find working regions for these parameters such that the physical observables are practically independent of them. The continuum thresholds are not completely arbitrary but are correlated with the energy of the first excited states in the initial and final mesonic channels. Our numerical results show that the results depend weakly on the thresholds in the intervals $s_0 = t_0 = (6 - 8) \text{GeV}^2$ and $s'_0 = (45 - 50) \text{GeV}^2$. The working regions for the Borel parameters are obtained by demanding that not only the contributions of the higher states and continuum are effectively suppressed, but also the contributions of the higher order operators and higher twist DAs are small, i.e., the series of the sum rules converge. These conditions lead to the intervals, $6 \text{GeV}^2 \leq M_1^2 \leq 12 \text{GeV}^2$, $10 \text{GeV}^2 \leq M_2^2 \leq 30 \text{GeV}^2$ and $5 \text{GeV}^2 \leq M_3^2 \leq 12 \text{GeV}^2$ for the Borel mass parameters.

Now, we proceed to find the fit functions of the form factors using the aforesaid working regions for the auxiliary parameters. Here we would like to mention that, for the decay
rates, we need only the values of the form factors $F_{V}^{(D_{s1})}$ and $F_{A}^{(D_{s1})}$ at $Q^2 = m_{B_c}^2$, $F_{V}^{(B_c)}$ and $F_{A}^{(B_c)}$ at $p^2 = m_{D_{s1}}^2$ and $T_2$ at $q^2 = 0$. However, we determine their fit functions in general and give their values at these fixed points. The fit functions for the form factors $F_{V}^{(D_{s1})}$ and $F_{A}^{(D_{s1})}$ are

$$f(Q^2) = \frac{f(0)}{1 + a \frac{Q^2}{m_{D_{s1}}^2} + b \left( \frac{Q^2}{m_{D_{s1}}^2} \right)^2},$$

(62)

where $f(0)$, $a$ and $b$ are the fit parameters whose values are

| form factors | $f(0)$   | $a$     | $b$     |
|--------------|----------|---------|---------|
| $F_{V}^{(D_{s1})}(Q^2)$ | 0.098    | 0.171   | -0.008  |
| $F_{A}^{(D_{s1})}(Q^2)$ | -2.478   | 3.644   | -0.005  |

The values of these form factors at $Q^2 = m_{B_c}^2$ are

$$F_{V}^{(D_{s1})}(Q^2 = m_{B_c}^2) = 0.055 \pm 0.016,$$

$$F_{A}^{(D_{s1})}(Q^2 = m_{B_c}^2) = -0.102 \pm 0.030,$$

(63)

where the errors on the values are due to the uncertainties in determination of the working regions for the auxiliary parameters as well as those coming from the DAs and other input parameters.

Also the fit functions for the form factors $F_{A,V}^{(B_c)}$ are [4]

$$F_{V}^{(B_c)}(p^2) = \frac{F_{V}(0)}{1 - p^2/m_1^2},$$

$$F_{A}^{(B_c)}(p^2) = \frac{F_{A}(0)}{1 - p^2/m_2^2},$$

(64)

where the fit parameters are

| $F_{V}(0)$ | 0.44GeV | $m_1^2 = 43.10GeV^2$ |
|------------|---------|----------------------|
| $F_{A}(0)$ | 0.21GeV | $m_2^2 = 48.00GeV^2$ |

The values of the form factors $F_{V,A}^{(B_c)}$ calculated at $p^2 = m_{D_{s1}}^2$ are

$$F_{V}^{(B_c)}(p^2 = m_{D_{s1}}^2) = (0.51 \pm 0.14)GeV,$$

$$F_{A}^{(B_c)}(p^2 = m_{D_{s1}}^2) = (0.24 \pm 0.07)GeV.$$

(65)

For the form factor induced by the EP at $q^2 = 0$, we obtain

$$T_2(0) = -0.298 \pm 0.085.$$

(66)
At the end of this section we would like to calculate the decay widths and branching ratios. Using the amplitudes of each decay mode, we find the following expressions for the decay rates at fixed points in WA and EP channels as well as for the total decay rate of the transition under consideration:

\[
\Gamma^{(WA)}(B_c \rightarrow D_{s1}\gamma) = \frac{G_F^2\alpha|V_{cb}V_{cs}^*|^2}{16} \left(\frac{m_{B_c}^2 - m_{D_{s1}}^2}{m_{B_c}}\right)^3 \times \left\{ f_{B_c}^2 \left[ (F_A^{(D_{s1})})^2 + (F_V^{(D_{s1})})^2 \right] + 2f_{B_c} f_{D_{s1}} F^{(B_c)}_V F^{(D_{s1})}_V \frac{m_{D_{s1}}}{m_{B_c}} \right\} \\
+ f_{D_{s1}}^2 m_{D_{s1}}^2 \left[ \frac{(F_A^{(B_c)})^2}{m_{B_c}^4} + \frac{(F_V^{(B_c)})^2}{m_{B_c}^4} \right],
\]

(67)

\[
\Gamma^{(EP)}(B_c \rightarrow D_{s1}\gamma) = \frac{G_F^2\alpha|C_7|^2|V_{tb}V_{ts}^*|^2}{1024\pi^4} \left(\frac{m_{B_c}^2 - m_{D_{s1}}^2}{m_{B_c}}\right)^3 \times \left(16(m_b + m_s)^2 + (m_b - m_s)^2 \right)[T_2(0)]^2,
\]

(68)

\[
\Gamma^{(total)}(B_c \rightarrow D_{s1}\gamma) = \frac{G_F^2\alpha}{1024\pi^4} \left(\frac{m_{B_c}^2 - m_{D_{s1}}^2}{m_{B_c}}\right)^3 \times \left\{ 64\pi^4|V_{cb}V_{cs}^*|^2 \left[ F_{B_c}^2 \left( (F_A^{(D_{s1})})^2 + (F_V^{(D_{s1})})^2 \right) \right] \\
+ 2f_{B_c} f_{D_{s1}} F^{(B_c)}_V F^{(D_{s1})}_V \frac{m_{D_{s1}}}{m_{B_c}} + f_{D_{s1}}^2 m_{D_{s1}}^2 \left[ \frac{(F_A^{(B_c)})^2}{m_{B_c}^4} + \frac{(F_V^{(B_c)})^2}{m_{B_c}^4} \right] \right\} \\
+ |C_7|^2|V_{tb}V_{ts}^*|^2 \left(16(m_b - m_s)^2 + (m_b + m_s)^2 \right)[T_2(0)]^2 \\
+ 16\pi^2 T_2(0)|V_{cb}V_{cs}^*||V_{tb}V_{ts}^*| f_{D_{s1}} m_{D_{s1}} X \left\{ \frac{4 F^{(B_c)}_V}{m_{B_c}^2} (m_b - m_s) \\
+ \frac{F^{(B_c)}_V}{m_{B_c}^2} (m_b + m_s) \right\} + f_{B_c} \left\{ F^{(D_{s1})}_V (m_b + m_s) X \\
- 4 F^{(D_{s1})}_A (m_b - m_s) Y \right\} \right\},
\]

(69)

where \(X\) and \(Y\) are the real and imaginary parts of the Wilson coefficient \(C_7\), respectively. In these formulas, the fixed point values of the form factors are used.

Finally the numerical values of the corresponding branching ratios for the radiative decay under consideration are obtained as follows:

\[
B^{(EP)}(B_c \rightarrow D_{s1}\gamma) = (1.769 \pm 0.582) \times 10^{-8},
\]

\[
B^{(WA)}(B_c \rightarrow D_{s1}\gamma) = (2.243 \pm 0.736) \times 10^{-5},
\]

\[
B^{(total)}(B_c \rightarrow D_{s1}\gamma) = (2.351 \pm 0.795) \times 10^{-5},
\]

(70)
where the dominant contribution to each channel comes from the perturbative part. From these values, we also see that the $B_c \to D_{s1}(2460)\gamma$ transition proceeds mostly via the WA mode. The order of the total branching ratio indicates that this decay channel can be detected at LHCb in near future. Any measurement on this decay and the comparison of the obtained data with our predictions in the present work can give valuable information about the nature and internal structure of the participating particles, especially the $D_{s1}$ meson.

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VI. APPENDIX

The explicit expressions for $C_{G^2}^i$ are given as follows:

$$C_{G^2}^a = m_b \left\{ -2m_b m_c^2 [I_0[1, 1, 1] - 3m_b^2 I_0[1, 4, 1] + 2m_b^2 (m_b - m_s) (m_b + m_s) I_0[1, 5, 1]] 
+ 2m_c (I_0[1, 2, 1] - 4m_b^2 I_0[1, 3, 1] - m_b m_s I_0[1, 3, 1] + 5m_b I_0[1, 4, 1] 
+ 3m_b^2 m_s I_0[1, 4, 1] - 3m_b^2 m_s^2 I_0[1, 4, 1] + 2m_b^2 (-m_b + m_s) (m_b + m_s)^2 I_0[1, 5, 1]) 
+ 2m_s [I_0[1, 2, 1] - 2m_b^6 I_0[1, 5, 1] + m_b^4 (5I_0[1, 4, 1] + 2I_0^{[0,1]}[1, 5, 1]) 
+ m_b^2 (-4I_0[1, 3, 1] - 3I_0^{[0,1]}[1, 4, 1] + 3I_0^{[1,0]}[1, 3, 1]) 
+ m_b (3I_0^{[0,1]}[1, 3, 1] + I_0^{[0,2]}[1, 4, 1] + I_0^{[1,0]}[1, 3, 1] + 2m_b^4 (3I_0^{[0,1]}[1, 5, 1] + I_0^{[1,0]}[1, 5, 1]) 
- m_b^2 (9I_0^{[0,1]}[1, 4, 1] + 2I_0^{[0,2]}[1, 5, 1] + 3I_0^{[1,0]}[1, 4, 1] - 2I_0^{[2,0]}[1, 4, 1]) 
- 4I_3^{[0,1]}[1, 5, 1] + 4I_3^{[1,0]}[1, 3, 1]) \right\},$$

(71)
\[\begin{align*}
C_{G2}^b &= 7m_b^2m_c^2m_s^2I_0[1, 1, 3] + m_b^2I_0[1, 1, 4] + m_cI_0[1, 1, 4] + m_bm_cI_0[1, 1, 4] \\
&\quad - m_b^2m_cI_0[1, 1, 4] - m_b^3m_sI_0[1, 1, 4] + m_cI_0[1, 1, 4] + 2m_bm_cI_0[1, 1, 4] \\
&\quad + m_b^2m_cI_0[1, 1, 4] - m_cI_0[1, 1, 5] - m_bm_cI_0[1, 1, 5] + m_b^3m_cI_0[1, 1, 5] - m_b^2m_cI_0[1, 1, 5] \\
&\quad + m_cI_0[1, 1, 5] - 2m_bm_cI_0[1, 1, 5] + 2m_b^3m_cI_0[1, 1, 5] - 2m_b^2m_cI_0[1, 1, 5] \\
&\quad - m_bI_0[0, 1][1, 1, 4] + m_sI_0[0, 1][1, 1, 4] - 3/2I_0[0, 1][1, 1, 5] + 3/2m_bI_0[0, 1][1, 1, 5] \\
&\quad + m_bI_0[0, 1][1, 1, 5] - m_sI_0[0, 1][1, 1, 2] + 3/2I_0[1, 0][1, 1, 3] + m_bI_0[1, 0][1, 1, 3] \\
&\quad + 3m_sI_0[1, 0][1, 1, 3] + 1/2m_b^2I_0[1, 0][1, 1, 4] - m_bm_sI_0[1, 1, 4] + 1/2I_0[2, 0][1, 1, 4] \\
&\quad + m_sI_0[2, 0][1, 1, 4],
\end{align*}\]

\[\begin{align*}
C_{G2}^c &= 1/6 \left\{ 2m_b^5[m_s(-I_0[3, 1, 1] + m_cI_0[3, 1, 2] + I_0[0, 1][3, 1, 1]) \\
&\quad + m_c(m_cI_0[3, 2, 2] + I_0[0, 1][3, 1, 2]) - 3I_0[0, 1][3, 2, 1] + I_0[0, 1][3, 2, 2] + 3I_0[0, 2][3, 1, 2] \\
&\quad - 2I_0[0, 2][3, 2, 1] + I_0[0, 2][3, 2, 2] + 3I_0[1, 0][3, 1, 1] - I_0[1, 0][3, 1, 2] \\
&\quad + 2m_b^3I_0[3, 1, 2] - I_0[3, 2, 2] - 4I_0[0, 1][3, 1, 2] + 3I_0[1, 0][3, 1, 2] + 3I_0[1, 0][3, 1, 1] \\
&\quad - 2I_0[0, 1][3, 2, 1] - I_0[0, 1][3, 1, 2] + 2m_b(-2I_0[0, 1][3, 2, 2] + I_0[1, 0][3, 1, 1]) \\
&\quad + m_c((1 + 2m_c)I_0[3, 2, 2] + 2I_0[0, 1][3, 1, 1] - 2I_0[0, 1][3, 2, 1] + 4m_cI_0[0, 1][3, 2, 2] \\
&\quad + I_0[0, 2][3, 1, 2] + I_0[0, 2][3, 1, 1]) - m_b^2(2m_c^2I_0[3, 2, 2] + I_0[0, 1][3, 1, 2] - 6I_0[0, 1][3, 1, 2] \\
&\quad - 2I_0[0, 2][3, 1, 2] + 6I_0[0, 2][3, 1, 2] + I_0[0, 2][3, 1, 2] + 3I_0[1, 0][3, 1, 2] + 2m_cI_0[3, 1, 2] - 5I_0[1, 0][3, 1, 2] \\
&\quad - I_0[1, 0][3, 2, 1] + 3I_0[1, 0][3, 2, 1] - 2I_0[1, 0][3, 2, 2] - 7m_cI_0[0, 1][3, 2, 2] \\
&\quad + 2m_cI_0[3, 1, 2] - I_0[3, 2, 1] + 2I_0[0, 2][3, 1, 2] - I_0[1, 1][3, 2, 1]) + 2I_0[1, 1][3, 2, 1] \\
&\quad - I_0[0, 2][3, 2, 2] + 2I_0[0, 1][3, 2, 1] - 2I_0[2, 1][3, 2, 2] + I_0[2, 1][3, 2, 2] \\
&\quad + m_c^2(-2I_0[0, 1][3, 2, 2] + 10I_0[0, 2][3, 1, 2] - 2I_0[0, 2][3, 2, 1] + I_0[1, 0][3, 2, 1] + I_0[1, 0][3, 2, 2] \\
&\quad - 14I_0[1, 1][3, 1, 2] + 14I_0[0, 0][3, 2, 2] + 3I_0[1, 2][3, 1, 2] + 2I_0[2, 0][3, 1, 2] - 10I_0[2, 0][3, 2, 1] \\
&\quad - 3I_0[2, 0][3, 2, 2] - 3I_0[0, 2][3, 2, 2] = I_0[3, 0][3, 2, 2]) \right\},
\end{align*}\]
\[ C^d_{G^2} = \frac{1}{12} \left\{ 2m^4_c I_0[3, 2, 1] + 2m^3_c m_s I_0[3, 1, 1] + 24m^7_b (m_c + m_s) I_0[3, 2, 2] + I_0^{[1,0]}[3, 2, 2] + 2m^6_b (8m_c m_s I_0[3, 1, 1] + 8m^2_b I_0[3, 2, 2] + 3I_0^{[0,1]}[3, 1, 2] + I_0^{[1,0]}[3, 1, 1]) \\
+ 6m^5_b (2m^3_c I_0[3, 1, 1] + 2m^2_c m_s I_0[3, 2, 2] - 2m_s (-2I_0[3, 1, 2] + 2m^2_s I_0[3, 2, 1]) \\
+ 2I_0^{[0,1]}[3, 2, 2]) + m_c - 9I_0^{[0,1]}[3, 2, 1] + I_0^{[1,0]}[3, 1, 2]) + 3I_0^{[1,0]}[3, 2, 1] - I_0^{[1,0]}[3, 2, 2] \\
+ 2m_c m_s (2I_0^{[0,1]}[3, 1, 2] - I_0^{[1,0]}[3, 1, 1] - I_0^{[1,0]}[3, 1, 2]) + 2I_0^{[1,1]}[3, 2, 2] + I_0^{[1,2]}[3, 1, 2] \\
- m^2_c (-2I_0[3, 1, 2] + 2I_0[3, 2, 2] + 2I_0^{[0,1]}[3, 2, 2] + I_0^{[0,2]}[3, 2, 2] - 5I_0^{[1,0]}[3, 2, 1] \\
+ 3I_0^{[1,0]}[3, 2, 2] - I_0^{[2,0]}[3, 2, 2]) - I_0^{[2,0]}[3, 2, 2] + m^4_b (4m^4_b I_0[3, 2, 1] + 4m^2_c m_s I_0[3, 2, 2] \\
+ 2I_0^{[0,1]}[3, 2, 1] - 15I_0^{[0,1]}[3, 2, 2] - 2I_0^{[0,2]}[3, 1, 1] + 5I_0^{[0,1]}[3, 2, 2] - 4I_0^{[1,0]}[3, 1, 2]) \\
+ 6I_0^{[1,0]}[3, 2, 2] + 3(I_0^{[0,1]}[3, 2, 2] + I_0^{[1,0]}[3, 2, 2]) + I_0^{[1,1]}[3, 1, 2] + 4I_0^{[2,0]}[3, 2, 2] \\
+ 3m_b (2m^2_c m_s (-I_0[3, 2, 1] - 2I_0^{[0,1]}[3, 2, 2] + 2I_0^{[1,0]}[3, 2, 2]) - 4I_0[3, 2, 2] \\
+ 3I_0^{[0,1]}[3, 2, 1] - 9I_0^{[0,1]}[3, 2, 2] - 3I_0^{[0,2]}[3, 1, 2] + 5I_0^{[1,0]}[3, 1, 2] + I_0^{[1,0]}[3, 2, 1] \\
+ 3I_0^{[2,0]}[3, 1, 2]) - 4m_s(I_0[3, 1, 2] + I_0^{[0,1]}[3, 2, 2] + I_0^{[1,0]}[3, 1, 2] - 2I_0^{[1,0]}[3, 2, 1] \\
- I_0^{[1,1]}[3, 2, 1] + I_0^{[2,0]}[3, 2, 2]) - 3m^3_b (-4m^2_b I_0[3, 2, 2] + 2m^2_c m_s - 4I_0^{[1,0]}[3, 2, 2] \\
+ 4I_0^{[1,0]}[3, 1, 2]) - 4m_s(-3I_0[3, 2, 1] + 4I_0[3, 2, 2] + 3I_0^{[0,1]}[3, 2, 2] - 4I_0^{[1,0]}[3, 2, 2] \\
- 2I_0^{[1,1]}[3, 2, 2] + 2I_0^{[2,0]}[3, 2, 2]) + m_c(-16I_0[3, 1, 2] + 12I_0[3, 2, 2] - 27I_0^{[0,1]}[3, 1, 2] \\
+ 6I_0^{[1,0]}[3, 2, 1] - 6I_0^{[2,0]}[3, 2, 1] + 3I_0^{[1,0]}[3, 1, 1] + 10I_0^{[1,0]}[3, 2, 2] + 6I_0^{[2,0]}[3, 2, 2] \\
+ m^2_b (m^4_c(-6I_0[3, 1, 2] + 4I_0[3, 2, 2])) + m^3_b (4m_s I_0[3, 1, 2] - 6m_s I_0[3, 2, 1]) \\
- 3I_0^{[0,1]}[3, 1, 1] + 12I_0^{[1,0]}[3, 1, 2] - 9I_0^{[1,0]}[3, 1, 2] + 4I_0^{[1,0]}[3, 2, 1] \\
+ 2m_c m_s (7I_0[3, 1, 1] + 9I_0^{[0,1]}[3, 2, 2] - 2I_0^{[1,0]}[3, 2, 1] - 6I_0^{[1,0]}[3, 2, 2]) - 9I_0^{[1,1]}[3, 1, 1] \\
- 2I_0^{[1,2]}[3, 2, 1] + m^2_c (14I_0[3, 1, 2] - 12I_0[3, 2, 2] + 9I_0^{[0,1]}[3, 1, 2] \\
- 2I_0^{[0,1]}[3, 2, 2] + 2I_0^{[2,0]}[3, 1, 2] - 10I_0^{[1,0]}[3, 1, 2] + 9I_0^{[1,0]}[3, 2, 1] - 2I_0^{[2,0]}[3, 1, 2] \\
- 6I_0^{[2,0]}[3, 1, 2] + 6I_0^{[2,0]}[3, 2, 2] + 2I_0^{[3,0]}[3, 1, 0] - I_0^{[3,0]}[3, 2, 2] \right\}, \quad (74)
\[ C_{G^2}^c = 1/6 \left\{ -2m_c m_s (-I_0[1, 3, 2] + I_0[1, 3, 3]) + 4m_b^5 (-m_c I_0[1, 2, 3] - 2m_s I_0[1, 3, 3]) \right. \\
-2m_c^2[I_0[1, 2, 2] - I_0[1, 3, 3]] - 3I_0^{[0,1]}[1, 1, 3] + I_0^{[0,1]}[1, 2, 2] - I_0^{[0,2]}[1, 2, 3] \right. \\
-I_0^{[1,0]}[1, 2, 2] + 3I_0^{[1,0]}[1, 3, 1] - 2m_b^4 (-m_c m_s I_0[1, 2, 1] - 2m_c^2 I_0[1, 3, 3]) \right. \\
+3I_0^{[0,1]}[1, 3, 3] + I_0^{[1,0]}[1, 2, 2]) - 2m_b^3 [m_c (-3I_0[1, 2, 3] + 2I_0[1, 3, 3])] \\
\left. + 2m_s (-I_0[1, 3, 3] - 2I_0^{[0,1]}[1, 2, 2] + 2I_0^{[1,0]}[1, 3, 3]) \right] + I_0^{[2,0]}[1, 3, 2] \\
+ I_0^{[0,1]}[1, 1, 3] + I_0^{[1,0]}[1, 2, 3] - 3I_0[1, 3, 1] + 2I_0^{[0,1]}[1, 2, 3] - 2I_0^{[1,0]}[1, 3, 3]) \right\} + I_0^{[2,0]}[1, 3, 2] \\
+ m_b^2 (2m_c^2 (2I_0[1, 2, 3] - 3I_0[1, 3, 2]) + 2m_c m_s (2I_0[1, 1, 2] - 3I_0[1, 3, 3]) + 9I_0^{[0,1]}[1, 2, 3] \\
-2I_0^{[0,1]}[1, 3, 2] + 2I_0^{[0,2]}[1, 3, 3] - 6I_0^{[1,0]}[1, 1, 3] + 3I_0^{[1,0]}[1, 3, 2] - 2I_0^{[2,0]}[1, 3, 3]) \right\} , \quad (75) \\
\right. \\
\left. \\
C_{G^2}^f = 2/3m_s \left\{ m_b^3 (-m_c^2 I_0[2, 1, 4]) + m_b^2 m_c (m_c I_0[2, 1, 4] - 2I_0^{[0,1]}[2, 1, 4]) \right. \\
+m_b (I_0^{[0,1]}[2, 1, 3] - 2I_0^{[1,0]}[2, 1, 4] + I_0^{[0,2]}[2, 1, 2] + I_0^{[1,0]}[2, 1, 3] + m_c I_0^{[1,0]}[2, 1, 3] \right. \\
\left. - I_0^{[1,1]}[2, 1, 4] + m_c (2I_0^{[0,1]}[2, 1, 3] + I_0^{[0,2]}[2, 1, 3] - 3I_0^{[1,0]}[2, 1, 4]) \right\} , \quad (76) \\
\right. \\
\end{align*}

where we have ignored terms with higher powers of the strange quark mass. The functions, \(I_n[a, b, c]\) and \(I_n^{[i,j]}[a, b, c]\) are defined as:

\[ 
I_0[a, b, c] = \frac{(-1)^{a+b+c}}{16\pi^2 \Gamma(a) \Gamma(b) \Gamma(c)} (M_1^2)^{2-a-b} (M_2^2)^{2-a-c} U_0(a + b + c - 4, 1 - c - b), \\
I_1[a, b, c] = \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a) \Gamma(b) \Gamma(c)} (M_1^2)^{2-a-b} (M_2^2)^{3-a-c} U_0(a + b + c - 5, 1 - c - b), \\
I_2[a, b, c] = \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a) \Gamma(b) \Gamma(c)} (M_1^2)^{3-a-b} (M_2^2)^{2-a-c} U_0(a + b + c - 5, 1 - c - b), \\
I_n^{[i,j]}[a, b, c] = [M_1^2]^i [M_2^2]^j \frac{d^i}{d(M_1^2)^i} \frac{d^j}{d(M_2^2)^j} [M_1^2]^i [M_2^2]^j I_n[a, b, c]. \quad (77) 
\]

where \(U_0(a, b)\) is given by

\[ 
U_0(a, b) = \int_0^1 dy (y + M_1^2 + M_2^2)^a y^b \exp[-\frac{B_1}{y} - B_0 - B_1 y], \quad (78) 
\]
and

\[
B_{-1} = \frac{m_b^2}{M_1^2} [M_1^2 + M_2^2],
\]

\[
B_0 = \frac{1}{M_1^2 M_2^2} [M_1^2 m_c^2 + M_2^2 (m_c^2 + m_b^2)],
\]

\[
B_1 = \frac{m_c^2}{M_1^2 M_2^2}. \tag{79}
\]

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