Kelvin-Helmholtz instability of anisotropic magnetized plasma using generalized polytrope laws

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Abstract. The problem of Kelvin-Helmholtz (K-H) instability of two superposed compressible magnetized anisotropic pressure plasmas is investigated using generalized polytrope laws. The relevant magnetohydrodynamic (MHD) equations of the problem have been modified using generalized polytrope laws in terms of polytropic indices. The general dispersion relation is obtained using normal mode analysis by applying the appropriate boundary conditions. The conditions for K-H stability, instability and overstability are obtained for MHD and Chew-Goldberger and Low (CGL) set of equations. It is found that the conditions of K-H stability, instability and overstability depend on polytropic indices and magnetic field. We find that in general overstability is not possible unless the new conditions in terms of polytropic indices are not satisfied. The weak magnetic field changes the criteria of K-H instability. The effect of pressure anisotropy is studied on the growth rate of K-H instability. We conclude that increase in pressure anisotropy causes increase in the region of K-H instability.

1. Introduction

The Kelvin-Helmholtz (K-H) instability of an interface caused by tangential velocity discontinuities in homogeneous plasma plays an important role in various problems of space, and astrophysical research with sheared plasma flow. The K-H instability is widely used to discuss the stability of the Earth’s magnetopause and interaction between flow of different velocities in the solar wind [1,2]. Rao et al. [3] have discussed the slipping stream instability in the context of K-H instability in anisotropic plasma with magnetic shear considering magnetic field both transverse and longitudinal to the direction of the magnetic field. Nagano [4] has investigated the problem of K-H instability of isotropic MHD fluids for both compressible and incompressible plasmas taking the effects of finite Larmor radius (FLR) corrections. Fujimoto and Terasawa [5] have carried out the study of ion inertia effect on the K-H instability of two fluid plasmas arising from the Hall term. Gonjalez and Gratton [6] have discussed the K-H instability in compressible plasma taking the orientation of the magnetic field with respect to the flow. In order to examine the stability in collisionless regions we use the Chew-Goldberger and Low (CGL) approximation for collisionless anisotropic pressure plasmas. In this direction, Duhau and Gratton [7] have investigated the K-H instability of anisotropic pressure plasma in presence of magnetic field using CGL approximation. Singh and Talwar [8] have investigated the K-H instability of anisotropic pressure plasma using CGL approximation in a slab model of a jet moving in external plasma. Talwar [9] has presented a study of K-H instability of homogeneous
anisotropic pressure plasma. Recently, Ruokola and Kopu [10] have investigated the condition of interfacial K-H instability for a system of two anisotropic super fluid flows.

In recent years, many authors have tried to explain the K-H instability as well as other hydromagnetic instability viz. self-gravitational instability, “mirror” instability and “firehose” instability etc. using generalized polytrope laws. In this connection, Abraham-Shrauner [11] has proposed a generalized polytrope model for the MHD set of equations for considering the collisional, collisionless and transitional regions of the flow problem. Sharma and Chhajlani [12] have discussed the K-H instability using polytrope pressure laws and assuming magnetic field perpendicular to the direction of the flow. Recently, Prajapati et al. [13] have investigated the self-gravitational instability of anisotropic pressure plasma with rotation, Hall current and finite electrical resistivity using generalized polytrope laws.

Thus in the present paper, we have investigated the K-H instability of anisotropic pressure plasma using generalized polytrope laws. We have also discussed the effect of direction of magnetic filed on the dispersion relation and conditions of K-H stability and instability. The magnetic filed is assumed parallel to the direction of the flow.

2. Linearized perturbation equations of the problem

Let us consider a plane surface of discontinuity of tangential velocity existing at the interface ($z = 0$) between the two semi-infinite regions of homogeneous anisotropic pressure plasma, which is permeated with a uniform magnetic field $\mathbf{B}(0, 0, B_z)$ along the $z$-direction. $U_1$ and $U_2$ be the uniform velocities along $x$-direction in two regions $z < 0$ and $z > 0$ respectively (see figure 1).

Thus the relevant linearized MHD equations of the motions are

$$\rho \left( \frac{\partial}{\partial t} + U \cdot \nabla \right) \delta \mathbf{v} = -\nabla \cdot \delta \mathbf{P} + \frac{1}{4\pi} (\nabla \times \delta \mathbf{B}) \times \mathbf{B},$$

$$\frac{\partial \delta \rho}{\partial t} = -\rho \nabla \cdot \delta \mathbf{v} - \delta \rho \nabla \cdot U,$$

$$\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (U \times \delta \mathbf{B}) + \nabla \times (\delta \mathbf{v} \times \mathbf{B}),$$
The perturbed pressure tensor is given as
\[ \delta P = (\delta \rho, \delta p_\perp, \delta p_\parallel) \]
and
\[ \delta P = (\delta \rho, \delta p_\perp, \delta p_\parallel) (\delta n + n \delta n) \]
where \( \delta \rho, \delta p_\perp, \delta p_\parallel \) are the perturbations in density, pressure perpendicular and parallel to the direction of magnetic field, magnetic field and fluid velocity respectively. \( U_0(u, 0, w) \) is the initial unperturbed flow velocity along \( x \)-axis. \( n \) be the unit vector along the magnetic field and \( I \) is the unit dyadic.

3. Boundary conditions and dispersion relation

We assume that all the perturbed quantities are varying exponentially and given with multiplication of some function of \( z \). i.e.
\[ f(z) \exp(i k x + n t) \]
On applying (8) into (1)-(7) we obtain two equations for \( \delta v \),
\[ \frac{\partial \delta V}{\partial t} + \nabla \cdot \delta \mathbf{V} = \left[ (n^2 + (\alpha + 1) s_\parallel^2 k^2 - s_\perp^2 k^2) \right] \mathbf{D} w, \]
and
\[ \frac{\partial \delta \mathbf{V}}{\partial t} + \nabla \cdot \delta \mathbf{V} = \left[ (n^2 + (\alpha + 1) s_\parallel^2 k^2 - s_\perp^2 k^2) \right] \mathbf{D} w, \]
where \( s_\parallel = p_\parallel / \rho_\parallel \), \( s_\perp = p_\perp / \rho_\perp \), \( V_\parallel^0 = B_\parallel^0 / 4 \pi \rho_\parallel \), \( A^2 = s_\perp^2 - s_\parallel^2 - V_\parallel^0 \), \( n^0 = (n + ikU_\parallel) \).

Equation (11) represents the relation for velocity component longitudinal to the direction of the magnetic field (i.e. \( w \)) in terms of polytropic indices. The general dispersion relation for K-H instability may be obtained by applying the appropriate boundary conditions to (11).
(iii) The normal stress should be continuous across the face i.e.
\[ \delta p_{1}^{(1)} + \frac{B_0}{4\pi} \delta B_{1} = \delta p_{2}^{(2)} + \frac{B_0}{4\pi} \delta B_{2}. \] (16)

Using (16) we obtain the ratio of \( m_1 \) and \( m_2 \) as
\[ \frac{m_1}{m_2} = \frac{-(n_1^2 + \gamma s_1^2 k^2)}{(n_2 + \gamma s_2^2 k^2)(V_0^2 + \beta s_0^2)(n_2 + \gamma s_2^2 k^2) + \varepsilon (s_1^2 + (\alpha + 1)s_1^2 k^2 - s_2^2 k^2)}. \] (17)

Substituting for \( m_1 \) and \( m_2 \) from (13) in (17) and simplifying we finally obtain the dispersion relation
\[ U_p^4 - 2(U_1 + U_2)U_p^2 + U_p^2[(U_1 + U_2)^2 + 2U_1 U_2 - 2\gamma s_0^2] - U_p^2[2(U_1 + U_2)U_1 U_2 - \gamma s_1^2] \]
\[ + U_1^2 U_2^2 - \gamma s_0^2 (s_1^2 - s_2^2 - V_0^2 + U_1^2 + U_2^2) + \frac{(\gamma s_1^2 + A^2)[(\beta s_0^2 + V_0^2) + \varepsilon (\alpha + 1)s_1^2 - \varepsilon s_1^2]}{(\epsilon + \beta)^2 + V_0^2} \]
\[ = 0. \] (18)

where \( n_i = i(U_1 - U_p), n_2 = i(k(U_2 - U_p)) \) and \( U_p = \text{in} / k \) is the phase velocity.

Equation (18) represents the general dispersion relation for K-H instability of two superposed magnetized anisotropic plasma with polytropic indices. The streams are slipping past over each other with velocity \( U_1 \) and \( U_2 \). The general dispersion relation (18) is obtained in simple form by assuming that both plasma fluids are moving relative to each other with a constant flow velocity i.e. we put \( U_1 = -U_2 = U \). Then (18) reduces to
\[ U_p^4 - 2U_p^2(U^2 + \gamma s_1^2) + U^4 - 2\gamma s_0^2 U^2 + \gamma^2 s_1^4 \]
\[ - \varepsilon s_1^2 [(\gamma s_1^2 + s_0^2 + s_0^2 - \gamma - \alpha - 1) s_1^2] \]
\[ = 0. \] (19)

This dispersion relation can be also reduced for collisional isotropic MHD approximation and collisionless anisotropic CGL approximation. If we substitute \( \varepsilon = 5/3 \) and \( \alpha = \beta = 0 \) then we obtain the dispersion relation for isotropic MHD approximation. If we put \( \gamma = 3, \alpha = 2 \) and \( \varepsilon = \beta = 1 \) then we recover the dispersion relation for anisotropic CGL approximation, which is identical to Talwar [9].

4. Discussions of the dispersion relation

The conditions of K-H stability and K-H instability and the character of the roots of the dispersion relation can be explained with the help of (19). The dispersion relation (19) gives four roots may be real or complex. The K-H stability and instability of the configuration depends upon the nature of the roots of (19). If all the roots of \( U_p^4 \) obtained from (19) are real, then the considered configuration will obviously stable. If \( U_p^4 \) is negative real, it leads to monotonic instability. In case if \( U_p^4 \) has complex value, it corresponds to an overstable situation. If the discriminant of (19) is negative, \( U_p^4 \) will have complex value leading to an over stability. Thus the condition for over stability is
\[ U_p^4 < \frac{\varepsilon s_1^2 [(\gamma - \alpha - 1) s_1^2] (s_1^2 + V_0^2)}{4\gamma s_1^4 [(\epsilon + \beta)^2 s_0^2 + V_0^2]} \]. (20)

We note that the condition of over stability depends upon polytropic indices and magnetic field for all values of relative speed \( U \). We find that the Alfven velocity couples with the sound velocities parallel and perpendicular to the direction of magnetic field.

Now we derive the condition of overstability for special cases of collisional MHD and collisionless CGL set of equations by substituting the relevant values of polytropic indices in (20).

For collisionless CGL anisotropic plasma we substitute \( (\gamma = 3, \alpha = 2, \varepsilon = \beta = 1) \)
$U^2 < \frac{s_\perp^2(s_\perp^2 + V_\perp^2 - 4s_\parallel^2)}{12s_\parallel^2(2s_\parallel^2 + V_\parallel^2)}$.  

(21)

For collision dominated isotropic plasma we put $(\epsilon = \gamma = 5/3, \alpha = \beta = 0, s_\perp^2 = s_\parallel^2 = s^2)$,

$U^2 < \frac{5/12(V_\perp^2 - 5/3 s^2)s^2}{(5/3 s^2 + V_\parallel^2)}$.  

(22)

The condition for overstability of anisotropic plasma has been discussed by Talwar [9].

It is obvious that, the configuration of two superposed slipping streams of anisotropic pressure plasma is overstable for relative speeds (2 U ) less than a critical speed given by expression (20). It is clear that overstability is not possible for any relative speed provided (23). We obtain relative speed higher than the value for the K-H stability. The comparison of condition (23) with (24) and (25) shows that numerators on the right side are same but the denominators are always larger than the denominators of instability and K-H stability. The terms due to magnetic field comes into the form of Alfven velocity in the conditions of K-H instability and K-H stability. The comparison of condition (23) with (24) and (25) shows that K-H stability or K-H instability the system is obtained from the constant term of overstability provided the relative speed 2\(\frac{s_\perp^2}{s_\parallel^2}\) i.e.

$$V_\perp^2 + (\epsilon + \beta) s_\parallel^2 > \frac{\epsilon s_\perp^2 s_\parallel^2 + (\gamma - \alpha - 1) s_\perp^2 [(\gamma + 1) s_\perp^2 + (\epsilon + \beta) s_\parallel^2] - 4 s_\parallel^2 U^2 s_\parallel^2}{s_\parallel^2 + (\gamma - \alpha - 1) s_\perp^2}.$$  

(23)

From this condition, we find that the slipping streams of anisotropic plasma can show instability through overstability provided the relative speed 2 U is less than the critical value 

$$(s_\parallel^2 / s_\perp^2)\sqrt{\frac{\epsilon}{s_\perp^2 + (\gamma - \alpha - 1) s_\perp^2}}/\gamma$$

and the magnetic field is sufficiently strong to satisfy condition (23). If the magnetic field is not to strong to satisfy the above inequality, or if the relative speed be more than the above mentioned critical value, there is no overstability and the configuration is either monotonically unstable or stable depending upon the strength of the prevailing magnetic field. The conditions for the K-H stability or K-H instability the system is obtained from the constant term of (19) i.e.

$$V_\perp^2 + (\epsilon + \beta) s_\parallel^2 < \frac{\epsilon s_\perp^2 s_\parallel^2 + (\gamma - \alpha - 1) s_\perp^2 [(\gamma + 1) s_\perp^2 + (\epsilon + \beta) s_\parallel^2]}{(U^2 - \gamma s_\perp^2)^2 + \epsilon [s_\perp^2 + (\gamma - \alpha - 1) s_\perp^2]s_\parallel^2}.  

(Unstable)  

(24)

$$V_\perp^2 + (\epsilon + \beta) s_\parallel^2 > \frac{\epsilon s_\perp^2 s_\parallel^2 + (\gamma - \alpha - 1) s_\perp^2 [(\gamma + 1) s_\perp^2 + (\epsilon + \beta) s_\parallel^2]}{(U^2 - \gamma s_\perp^2)^2 + \epsilon [s_\perp^2 + (\gamma - \alpha - 1) s_\perp^2]s_\parallel^2}.  

(Stable)  

(25)

Equations (24) and (25) gives the conditions of K-H instability and K-H stability. It is clear that these conditions depend upon magnetic field and polytropic indices. These conditions can also be reduces for MHD and CGL set of equations by substituting the relevant values of polytropic indices. The terms due to magnetic field comes into the form of Alfven velocity in the conditions of K-H instability and K-H stability. The comparison of condition (23) with (24) and (25) shows that numerators on the right side are same but the denominators are always larger than the denominators of (23), so that the critical value of $V_\perp^2 + (\epsilon + \beta) s_\parallel^2$ above which the configuration is overstable, is higher than the value for the K-H stability.

The critical value of the relative speed for both MHD and CGL approximations can be derived from the condition (23). We obtain relative speed $\sqrt{5/3} s$ for isotropic MHD approximation and $s_\perp^2 / \sqrt{3} s_\parallel^2$ for anisotropic CGL approximation. It is clear that isotropic plasma with small relative speed changes from unstable state to stable state by increasing in the magnetic field and further increase in magnetic field makes the system overstable. In the case of anisotropic plasma this is not possible because the minimum relative speed required for monotonic instability is always larger than the maximum relative speed for overstability.

\[5\]
In figures 2 and 3, we have plotted the critical hydromagnetic speed $V'_0$ against the tangential speed $V^*$ (both normalized in terms of $s_\perp$) for MHD and CGL approximations. The curves depict the overstable-stable-unstable regimes for two different values of $\delta = s_\perp / s_s = 0.8$ and 1.2 respectively. For CGL approximations we find that an increase in pressure anisotropy increases the region of instability and shifts it towards the higher value of tangential speed. In the case of MHD approximation there is no effect of pressure anisotropy on the growth rate of the K-H instability. We also find that on increasing the pressure anisotropy the value of hydromagnetic speed also increases. In case of MHD approximation there is no effect of pressure anisotropy on the peak value of hydromagnetic speed.
5. Conclusion
We have investigated the K-H instability problem of two semi-infinite regions of homogeneous, magnetized anisotropic pressure plasma considering generalized polytrope pressure laws. The general dispersion relation is obtained in terms of magnetic field and polytropic indices using the appropriate boundary conditions. This dispersion relation is simplified by taking the assumption that both fluids are moving relative to each other with a constant flow velocity. The dispersion relation is also reduces for MHD and CGL approximations. The conditions of K-H stability, instability and overstability are obtained. It is found that the conditions for instability, stability and overstability depend upon polytropic indices and direction of magnetic field. We also find that, in general overstability is not possible unless the conditions \( V_0^2 > (\gamma + 1) s_t^2 - s_l^2 \) and \( s_t^2 > (\alpha + 1 - \gamma) s_l^2 \) or alternatively \( V_0^2 < (\gamma + 1) s_t^2 - s_l^2 \) and \( s_t^2 < (\alpha + 1 - \gamma) s_l^2 \) are not satisfied. However, for the weak magnetic field the conditions change to \( s_t^2 > (\gamma + 1) s_l^2 \) and \( s_t^2 > (\alpha + 1 - \gamma) s_l^2 \). In isotropic plasma the condition of instability easily passes to stability and then to overstability by increasing the magnetic field but this is not possible in the case of anisotropic pressure plasmas. From the graphically representations it is concluded that increase in the pressure anisotropy causes also increase in the growth rate of K-H instability for CGL approximation. In the case of MHD approximation the growth rate is uninfluenced by the presence of pressure anisotropy. Hence the presence of pressure anisotropy has destabilizing influence on the growth rate of the K-H instability of CGL system.

Thus we have investigated the K-H instability of two superposed anisotropic magnetized pressure plasma considering the parallel magnetic field to the direction of the flow using generalized polytrope laws.

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