Comments on Duality in MQCD

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Abstract

We clarify some ambiguous points in a derivation of duality via brane exchange using M-theory language, and propose a “proof” of duality in MQCD. Actually, duality in MQCD is rather trivial and does not need a complicated proof. The problem is how to interpret it in field theory language. We examine BPS states in $\mathcal{N} = 2$ theory and find the particle correspondence under duality. In the process, we also find some exotic particles in $\mathcal{N} = 2$ MQCD, and we observe an interesting phenomenon in type IIA string theory, namely, that fundamental strings are converted into D2-branes via the exchange of two NS5-branes. We also discuss how we should understand Seiberg’s $\mathcal{N} = 1$ duality from exact duality in MQCD.

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§1. Introduction

Since Seiberg’s duality in $\mathcal{N} = 1$ SQCD appeared, various examples of duality have been found, and several attempts to expose the mysterious nature of duality have been made. However, there is still no proof of the duality, and we do not really know why and when there is a dual description.

One of the most fancy derivations of Seiberg’s duality is obtained via brane exchange in type IIA string theory, extending the idea given in Ref. 5). The authors of that paper argued that the electric theory in Seiberg’s duality (i.e. $\mathcal{N} = 1$ $SU(N_c)$ SQCD with $N_f$ flavors) can be realized as an effective world volume theory setting two NS5-branes, $N_c$ D4-branes and $N_f$ D6-branes, in a certain configuration. Then, if one gradually deform the configuration and exchanges the two NS5-branes, the effective theory turns out to be the magnetic theory (i.e. $\mathcal{N} = 1$ $SU(N_f - N_c)$ SQCD with $N_f$ flavors and a gauge singlet meson)! This argument explains the fact that the vacuum moduli spaces in the electric and magnetic theories are the same. However, since the brane configurations are not the same, we think that it is still unclear whether or not the electric and magnetic theories are really equivalent.

Recently it has become clear that four-dimensional field theory can be analyzed via M-theory technology. The brane configurations for the electric and magnetic theories in type IIA string theory can be lifted to M5-brane configurations in M-theory. In the M-theory language, SQCD is realized as an effective field theory on the M5-brane world volume wrapped on a Riemann surface, which is often called MQCD.

It is not difficult to lift the above arguments to M-theory. Many results are beautifully reproduced using M-theory language. The key ingredient we think is that the electric and magnetic theories should be understood as the effective world volume theory of the identical M5-brane. In this context, the electric and magnetic theories are, by definition, exactly equivalent. Our motivation for this work is to understand duality in this way and make a dictionary translating between the electric and magnetic descriptions. We wish to emphasize that MQCD is a nice formulation of gauge theories, which is quite compatible with duality.

In §2, we review some field theory results given in Ref. 10),11). In §3 we propose a “proof” of duality in MQCD, and show that the results in §2 can be reproduced consistently. We also discuss how we should understand Seiberg’s duality from the exact duality in MQCD. Section §4 is devoted to exploring the correspondence of BPS states under duality in $\mathcal{N} = 2$ theory. It will become clear that the magnetic theory can be understood as the soliton sector of the electric theory. Interpreting this result in the IIA picture, we observe the interesting
phenomenon that fundamental strings are converted into D2-branes via the exchange of two NS5-branes. Here we give some explicit examples of BPS states, finding holomorphic surfaces embedded in a multi-Taub-NUT space. We find W-bosons and quarks, which are the elementary particles in the field theory, together with many exotic particles, which can never be obtained in the perturbative field theory. We also make some comments on the realization of BPS states via geodesics on the Riemann surface. In §5, we make our conclusion and discuss future directions to be pursued.

§2. The field theory approach

2.1. Duality in $\mathcal{N} = 2$ SQCD

In this subsection, we review some semi-classical results in $\mathcal{N} = 2$ SQCD given in Ref. 10, 11. Here we consider $\mathcal{N} = 2$ $SU(N_c)$ SQCD with $2N_c$ flavors. The theory can be described in terms of $\mathcal{N} = 1$ superfields: $W_\alpha$ (a field strength chiral superfield), $\Phi$ (a chiral superfield in the adjoint representation of the gauge group), $Q^i$ and $\tilde{Q}^i$ (chiral superfields in the $N_c$ and $\overline{N_c}$ representation of the gauge group, respectively), where $i = 1, \cdots, 2N_c$ are flavor indices. The superpotential is

$$W_{\text{ele}} = Q^i \Phi \tilde{Q}^i + m^i_j Q^i \tilde{Q}^j,$$

(2.1)

where $m = (m^i_j) = \text{diag}(m_1, \cdots, m_{2N_c})$ is a quark mass matrix. The bare gauge coupling constant is denoted as $\tau = \frac{8\pi g^2}{\alpha} + \frac{\theta}{\pi}$. We call this theory the ‘electric theory’.

The basic holomorphic gauge invariant operators which parameterize the vacuum moduli space are meson and baryons defined as follows:

$$M^i_j \equiv Q^i \tilde{Q}^j,$$

(2.2)

$$B^{i_1 \cdots i_{N_c}} \equiv Q^{i_{N_c}}_{a_{N_c}} \epsilon^{a_1 \cdots a_{N_c}},$$

(2.3)

$$\tilde{B}^{i_1 \cdots i_{N_c}} \equiv \tilde{Q}^{i_{N_c}}_{a_{N_c}} \epsilon^{a_1 \cdots a_{N_c}}.$$

(2.4)

The dual theory has the same matter content and superpotential as in the electric theory,

$$W_{\text{mag}} = q_i \varphi \tilde{q}^i + m^i_j q_i \tilde{q}^j,$$

(2.5)

but coupling constants are different. The quark mass matrix and gauge coupling constant for the dual theory are given as $m' = m - \frac{1}{N_c} \text{Tr} m$ and $\tau' = -1/\tau$, respectively. We call this theory the ‘magnetic theory’. Meson and baryons for the magnetic theory are defined as

$$N^i_j \equiv q_{ij} \varphi \tilde{q}^i,$$

(2.6)

$$b_{i_1 \cdots i_{N_c}} \equiv q_{a_{N_c}i_{N_c}} \epsilon^{a_1 \cdots a_{N_c}},$$

(2.7)

$$\tilde{b}^{i_1 \cdots i_{N_c}} \equiv \tilde{q}_{a_{N_c}i_{N_c}} \epsilon^{a_1 \cdots a_{N_c}}.$$

(2.8)
The electric and magnetic theories are conjectured to be equivalent under the correspondence\textsuperscript{[2,3,11]}

\begin{equation}
\text{electric} \leftrightarrow \text{magnetic}
\end{equation}

\begin{align}
M & \leftrightarrow N', \\
(M' & \leftrightarrow N), \quad (2.9) \\
B & \leftrightarrow (-1)^{N_c} \ast b, \\
\tilde{B} & \leftrightarrow \ast \tilde{b}, \quad (2.10, 2.11, 2.12)
\end{align}

where $N' \equiv N - \frac{1}{N_c} \text{Tr} \, N$, $M' \equiv M - \frac{1}{N_c} \text{Tr} \, M$, and “$\ast$” indicates the contraction of all flavor indices with the totally antisymmetric tensors $\epsilon_{i_1 \cdots i_{2N_c}}$ or $\epsilon^{i_1 \cdots i_{2N_c}}$.

Now we set the quark mass matrix as

\begin{equation}
m = \text{diag}(0, \cdots, 0, m_{N_f+1}, \cdots, m_{2N_c}), \quad (2.13)
\end{equation}

\begin{align}
m' & \equiv m - \frac{1}{N_c} \text{Tr} \, m, \\
& = \text{diag}(-2m_S, \cdots, -2m_S, m_{N_f+1} - 2m_S, \cdots, m_{2N_c} - 2m_S), \quad (2.14)
\end{align}

where we have defined $m_S \equiv \frac{1}{2N_c} \text{Tr} \, m$ and the $m_i$ are chosen to be generic.

Let us consider the baryonic branch on which $B$ or $\tilde{B}$ is non-zero. It exists only for the case $N_c \leq N_f$. It is easy to derive semi-classical vacuum expectation values of the adjoint fields on the baryonic branch, using classical F-term equations\textsuperscript{[8,11]}. The results are

\begin{align}
\Phi & = 0, \quad (2.15) \\
\varphi & = \text{diag}(2m_s, \cdots, 2m_s, 2m_s - m_{N_f+1}, \cdots, 2m_s - m_{2N_c}). \quad (2.16)
\end{align}

From these forms, we see that the unbroken gauge groups of the electric and magnetic theories are $SU(N_c)$ and $SU(N_f - N_c) \times U(1)^{2N_c - N_f}$, respectively. Note that since there are no symmetries which restrict the $\tau$ dependence in the F-term equations, we cannot exclude the possibility that these results are corrected due to some non-perturbative effects. Actually, we will see later, using M-theory language, that there are such non-perturbative corrections.

2.2. Duality in $\mathcal{N} = 1$ SQCD

Here we summarize some semi-classical results in $\mathcal{N} = 1$ theory given in Ref.\textsuperscript{[11]}. Let us consider $\mathcal{N} = 1$ deformed theory, adding a mass term for the adjoint chiral multiplet to the electric theory (2.1):

\begin{equation}
W_{\text{ele}} = Q^i \Phi \tilde{Q}_i + m^i_j Q^i \tilde{Q}_j + \frac{\mu}{2} \text{tr} \, \Phi^2. \quad (2.17)
\end{equation}
It was argued in Ref. [11] that the magnetic theory (2.5) should be deformed as

\[ W_{\text{mag}} = q_i \bar{\varphi} \tilde{q}_i + m'_{ij} q_i \bar{q}_j - \frac{\mu}{2} \text{tr} \varphi^2. \] (2.18)

The correspondence of the meson operators is also deformed as

\[
\text{electric} \leftrightarrow \text{magnetic} \\
M \leftrightarrow \tilde{N},
\]
(2.19)

\[
(\tilde{M} \leftrightarrow N),
\]
(2.20)

where we have defined \( \tilde{N} \equiv N' + \mu m' \) and \( \tilde{M} \equiv M' - \mu m \).

We set the quark mass as in (2.13) again and consider the baryonic branch. The vacuum expectation values of the adjoint fields are the same as (2.15) and (2.16). The meson VEV is, up to complexified flavor symmetry transformations,

\[
M = \begin{pmatrix}
\rho & \cdots & \\
\cdots & \rho & \\
\end{pmatrix},
\]
(2.21)

\[
N = \tilde{M} = \begin{pmatrix}
\rho & \cdots & \\
\cdots & \rho & \\
\cdots & -\sigma_{N_f+1} & \\
\cdots & \cdots & -\sigma_{2N_c}
\end{pmatrix},
\]
(2.22)

where we have defined \( \sigma_i \equiv \rho + \mu m_i \).

We can derive Seiberg’s duality from the relation \( N = \tilde{M} \). Let us consider the baryonic root, where we have \( M = Q \bar{Q} = 0 \) and \( N = q \bar{q} = -\mu m \). Taking the D-term equations into
account, we obtain

\[ Q = \tilde{Q} = 0, \tag{2.23} \]

\[ q = \tilde{q} = \begin{pmatrix} \kappa_{N_f+1} \\ \vdots \\ \kappa_{2N_c} \end{pmatrix}, \tag{2.24} \]

where \( \kappa_i \equiv \sqrt{-\mu m_i} \). From these forms we expect that the \( SU(N_c) \) gauge symmetry is unbroken in the electric theory, while \( SU(N_c) \) is broken to \( SU(N_f - N_c) \) in the magnetic theory.

Of course, there are many ambiguities in the semi-classical arguments given above. In the following sections, we will re-derive these results in the M-theory language, which is reliable even in the strong coupling regime, and try to clarify how to understand duality in this context.

§3. A “Proof” of duality in MQCD

3.1. Duality in \( \mathcal{N} = 2 \) MQCD

As in Ref. \[13\], we define \( \mathcal{N} = 2 \) MQCD as an effective world volume theory on an M5-brane wrapped on a Riemann surface \( \Sigma \) embedded holomorphically in \( \mathbb{C} \times \mathbb{C}^* = \{(v, t) \mid v = x^4 + x^5i, t = e^{-s}, s = x^6 + x^{10}i\} \). In order to obtain the electric theory in §2.1 (i.e. \( \mathcal{N} = 2 \) \( SU(N_c) \) MQCD with \( 2N_c \) flavors), we choose \( \Sigma \) to be the Seiberg-Witten curve given in Ref. \[13\]

\[ t^2 - 2 \prod_{a=1}^{N_c} (v + \phi_a) t + (1 - h(\tau)^2) \prod_{j=1}^{2N_c} (v - m_j + (1 - h(\tau))m_S) = 0, \tag{3.1} \]

where we have defined \( h(\tau) \equiv \frac{\varphi_3(\tau)^4}{\varphi_4(\tau)^3 - \varphi_2(\tau)^5} \).

Similarly, we can define the magnetic theory by choosing \( \Sigma \) to be

\[ t^2 - 2 \prod_{a=1}^{N_c} (v + \phi_a) t + (1 - h'(\tau')^2) \prod_{j=1}^{2N_c} (v - m'_j + (1 - h'(\tau'))m'_S) = 0, \tag{3.2} \]

where \( \tau' = -1/\tau \), \( m' = m - \frac{1}{N_c} \text{Tr} m \) and \( m'_S = \frac{1}{2N_c} \text{Tr} m' \), which are the gauge coupling constant and the quark mass parameters for the magnetic theory suggested in §2.1.
Using the relation $h(-1/\tau) = -h(\tau)$, it is easy to see that (3.1) and (3.2) are the same. Hence the electric and magnetic theories in MQCD are exactly equivalent, by definition. The duality transformation is merely a change of description. It may be suitable to refer to the electric and magnetic theories as the electric and magnetic description of MQCD, respectively.

Let us make some comments on the brane exchange in IIA picture. In the electric description, the weak coupling region is $\tau \sim i\infty$. The asymptotic positions of the two NS5-branes in the $s = x^6 + x^{10}i$ plane can be read from the value of $t = e^{-s}$ at $v \to \infty$, and we obtain

$$\delta s \equiv s|_{NS5(1)} - s|_{NS5(2)}, \quad (3.3)$$

$$= \log(1 + h) - \log(1 - h), \quad (3.4)$$

$$\sim -i\pi\tau. \quad (\tau \sim i\infty) \quad (3.5)$$

If we fix the quark mass parameters while moving $\tau$ to the strong coupling region, and set $\tau \sim 0$, we can see from (3.4) that the two NS5-branes are exchanged as argued in Ref. [19]. In this region, the magnetic description is weak coupling. The brane exchange changes the weak coupling electric theory into the strong coupling electric theory, which is equivalent to the weak coupling magnetic theory. Notice that the brane exchange does not change one theory into another equivalent theory, since the weak coupling electric theory and the weak coupling magnetic theory are, in general, not equivalent. For example, when we use $m_S \neq 0$, the Riemann surface (3.1) is not invariant under brane exchange, and the effective action computed as in Ref. [14] is explicitly deformed via brane exchange. If we tune $m_S$ to zero, the brane exchange takes one theory to the same theory. However, it is not worth referring to this as ‘duality’, since the brane exchanged theory has the same matter content and couplings as the original one. We emphasize that we refer to the exact equivalence of the weak (strong) coupling electric theories and the strong (weak) coupling magnetic theories (respectively) as ‘duality’. This is highly non-trivial in usual field theory, but trivial in MQCD.

In order to compare the results in MQCD with the semi-classical results given in the previous section, we should analyze the weak coupling region. Let us check that the results given in the last section can be reproduced in the weak coupling limits in the electric and magnetic descriptions of MQCD.

Now we set the quark mass matrix as in (2.13) and consider the baryonic branch root as considered in Ref. [10], [15], [1]. On the baryonic branch, the curve (3.1) is factorized as

$$(t - \alpha(v))(t - \beta(v)) = 0, \quad (3.6)$$
where $\alpha(v)$ and $\beta(v)$ are some polynomials satisfying

$$\alpha(v) + \beta(v) = 2 \prod_{a=1}^{N_c} (v + \phi_a),$$  \hspace{1cm} (3.7)$$

$$\alpha(v)\beta(v) = (1 - h^2) \prod_{j=1}^{2N_c} (v - m_j + (1 - h)m_S).$$  \hspace{1cm} (3.8)

The solution exists if and only if the $\phi_a$ satisfy

$$2 \prod_{a=1}^{N_c} (v + \phi_a) = (1 + h) \hat{v}^{N_c} + (1 - h) \hat{v}^{N_f-N_c} \prod_{j=N_f+1}^{2N_c} (\hat{v} - m_j),$$  \hspace{1cm} (3.9)

where we have defined $\hat{v} \equiv v + (1 - h)m_S$. Then the solution is

$$\alpha(v) = (1 + h) \hat{v}^{N_c},$$  \hspace{1cm} (3.10)$$

$$\beta(v) = (1 - h) \hat{v}^{N_f-N_c} \prod_{j=N_f+1}^{2N_c} (\hat{v} - m_j).$$  \hspace{1cm} (3.11)

From (3.9), we can read off non-perturbative corrections to (2.15) and (2.16). At the weak coupling limit $\tau \to i\infty$ ($h(\tau) \to 1$), (3.9) becomes

$$\prod_{a=1}^{N_c} (v + \phi_a) = v^{N_c},$$  \hspace{1cm} (3.12)

which agrees with the semi-classical result (2.15). On the other hand, in the strong coupling limit $\tau \to 0$ ($h(\tau) \to -1$), (3.9) becomes

$$\prod_{a=1}^{N_c} (v + \phi_a) = (v + 2m_S)^{N_f-N_c} \prod_{j=N_f+1}^{2N_c} (v - m_j + 2m_S),$$  \hspace{1cm} (3.13)

which is consistent with the semi-classical result in the magnetic theory (2.16).

3.2. Duality in $\mathcal{N} = 1$ MQCD

In order to obtain $\mathcal{N} = 1$ MQCD, we rotate one of the NS5-branes in the $\mathcal{N} = 2$ brane configuration to $w = x^8 + x^9 i$ direction. Here we only consider the baryonic branch on which the curve is factorized into the two components of (3.6):

$$C_R : t = \alpha(v),$$  \hspace{1cm} (3.14)$$

$$C_L : t = \beta(v).$$  \hspace{1cm} (3.15)$$

The functions $\alpha(v)$ and $\beta(v)$ are given in (3.10) and (3.11). The rotated curve is given as in Ref. [13]:

$$C_R : t = \alpha(v), \quad w = \mu \hat{v} + \rho.,$$  \hspace{1cm} (3.16)$$

$$C_L : t = \beta(v), \quad w = 0.$$  \hspace{1cm} (3.17)
As argued in Ref. [15], the meson VEV can be obtained from the value of \( w \), setting \( t = 0 \) in (3.16). Eliminating \( \hat{v} \) in (3.16), we obtain
\[
t = (1 + h) \left( \frac{1}{\mu} (w - \rho) \right)^{N_c},
\]
which means that the meson VEV is
\[
M = \text{diag}(\rho, \ldots, \rho, 0, \ldots, 0). \tag{3.19}
\]
This result agrees with the field theory result (2.21). Note that this is not a good variable in the strong coupling limit \( h \to -1 \), since the right hand side of (3.18) vanishes.

Then, how can we obtain the meson VEV in the magnetic theory? As explained in the last subsection, we should read it from the same Riemann surface as above. Actually there is another candidate for the meson VEV.

Let \( \hat{w} \equiv w - \mu \hat{v} - \rho \), and rewrite the curve as
\[
C_R : t = \alpha(v), \quad \hat{w} = 0, \tag{3.20}
\]
\[
C_L : t = \beta(v), \quad \hat{w} = -\mu \hat{v} - \rho. \tag{3.21}
\]
We propose that the meson VEV in the magnetic theory can be read from the value of \( \hat{w} \) setting \( t = 0 \) in (3.21). Eliminating \( \hat{v} \) in (3.21), we obtain
\[
t = (1 - h) \left( \frac{-1}{\mu} (\hat{w} + \rho) \right)^{N_j - N_c} \prod_{j=N_f+1}^{2N_c} \left( \frac{-1}{\mu} (\hat{w} + \rho) - m_j \right), \tag{3.22}
\]
from which we interpret the meson VEV in the magnetic theory as
\[
N = \text{diag}(0, \ldots, 0, -\rho, \ldots, -\rho, -\sigma_{N_f+1}, \ldots, -\sigma_{2N_c}), \tag{3.23}
\]
where \( \sigma_i \equiv \rho + \mu m_i \). This form is exactly what we have expected in (2.22).

3.3. Toward Duality in Field Theory

What we have shown in the last subsection is that the meson VEV for both the electric and magnetic theories can be read from the identical Riemann surface. Here we want to try to explain how to interpret this result in the field theory.

We proposed in §2.2 that the electric and magnetic theories have the same matter content with couplings chosen as follows.

|            | electric | magnetic |
|------------|----------|----------|
| gauge coupling | \( \tau \) | \(-1/\tau\) |
| quark mass | \( m_i \) | \( m'_i \) |
| adjoint mass | \( \mu \) | \(-\mu\) |
Notice that we should always consider the weak coupling region whenever we compare the M-theory description with the field theory description.

First we set $\tau \sim i\infty$ and consider the brane configuration for the weak coupling electric theory. As in (3.16), we rotate $C_R$ to satisfy the asymptotic condition $w \sim \mu v$. (Fig. 1)

On the other hand, if we wish to obtain the brane configuration for the proposed magnetic theory in the same way as above, we should move $\tau$ to $-1/\tau = \tau' \sim i\infty$, i.e. exchange the two NS5-branes, and then rotate $C_L$ with the asymptotic condition $w \sim -\mu v$. (Fig. 2)

In order to compare this theory with the weak coupling electric theory, we restore the value of $\tau$ to $\tau \sim i\infty$. As a result, we obtain the brane configuration for the magnetic theory proposed in the field theory approach, in which $C_L$ is rotated in the opposite direction as $C_R$ in Fig. 1. (Fig. 3)
This brane configuration is nothing but that given in (3.20) and (3.21), replacing \( \hat{w} \) with \( w \). Now it has become clear the reason why we have interpreted the meson VEV in the magnetic theory as the value of \( \hat{w} \) instead of \( w \) in the last subsection, in order to reproduce the field theory results. However, this brane configuration is not the same as that in the electric theory, and this theory is \textit{not} equivalent to the electric theory. If we set \( \hat{w} = x^8 + x^9i \) instead of setting \( w = x^8 + x^9i \), the Kähler potential and the higher derivative terms in MQCD would be changed, since they pick up the background space-time metric, which is sensitive to the coordinate change \( \hat{w} \rightarrow w \). The two configurations in Fig. 1 and Fig. 3 are actually the same as the complex manifolds. So only the holomorphic structures, such as VEVs for the holomorphic gauge invariant operators, are expected to be correspondent.

If we wish to obtain an exact duality, we should again rotate the two NS5-branes in Fig. 3 to obtain the same brane configuration as in the electric theory Fig. 1. Therefore the magnetic theory which is exactly equivalent to the electric theory will have complicated non-holomorphic terms in its Lagrangian.

Now consider taking the limit \( \mu \rightarrow \infty \). In this limit, \( \hat{w} \) is no longer a good coordinate to parameterize the \( w = x^8 + x^9i \) direction, and therefore the meson VEV \( N \) in the magnetic theory loses its physical meaning. Hence we should introduce a new variable instead of \( N \) to parameterize the vacuum moduli space in the magnetic description. This suggests that we must include the gauge singlet meson field \( M \), which is needed in Seiberg’s duality, in the massless matter content of the magnetic theory.

Let us make some comments on the coupling flow (see also Ref. 11). Seiberg’s duality is believed to be true only in the low energy limit. The reason is that both the electric and magnetic theories seem to be asymptotically free (for \( 3/2N_c < N_f < 3N_c \)), so we will be able to observe the differences in the high energy region. On the other hand, we have seen that the electric and magnetic theories are exactly equivalent in MQCD. How can these situations be consistent? The resolution is as follows. It is believed that there is an IR fixed point in the gauge coupling flow. So if we tune the bare coupling to be stronger than the IR fixed point, the theory becomes asymptotically non-free. In our situation, both the electric and magnetic theories are regularized by the finite \( \mathcal{N} = 2 \) theory at an energy scale higher than \( \mu \) and \( m_i \). The gauge coupling is fixed to be the bare value \( \tau \) at this high energy region. As we have discussed in (3.2), the bare couplings for the electric and magnetic theories are related as \( \tau = -1/\tau' \). This fact suggests that if the electric gauge coupling is weaker than the IR fixed point, then the magnetic gauge coupling is stronger than the IR fixed point. Therefore if the electric theory is an asymptotically free theory, the magnetic theory should be asymptotically non-free. When we move to the high energy region, where the electric theory is weak coupling, the magnetic coupling becomes stronger and stronger, and so there
is no apparent discrepancy in Seiberg’s duality even in the high energy region.

§ 4. BPS states and duality

In the last section, we defined the electric and magnetic theories as an effective world volume theory of an M5-brane wrapped on the same Riemann surface. In this formulation, duality is indeed manifest, but the interpretation in the usual field theory language is not so clear. Hence, in order to understand more detailed structures of duality, it is important to examine the particle content of the theory and determine the correspondence of the particles under duality.

In this section, we return to the analysis in $\mathcal{N} = 2$ $SU(N_c)$ MQCD with $2N_c$ flavors, and examine BPS states in the theory. We will see that there are W-bosons (massive vector multiplets) and quarks (hypermultiplets) in the weak coupling electric theory, as expected, and find that the elementary particles in the magnetic theory appear as magnetic monopoles in the electric theory. Moreover, we find many exotic states which cannot be obtained in the perturbative field theory. We will also discuss the realizations of BPS states via geodesics in the Riemann surface $\Sigma$, and apply the technique to the $N_c = 2$ case.

In these analyses, M-theoretical viewpoints are essential, and it is quite non-trivial to interpret the results in the IIA picture. As an example, we will consider an interesting phenomenon in type IIA string theory in §4.5.

4.1. Multi Taub-NUT space

In this section we embed the Riemann surface $\Sigma$, on which the M5-brane is wrapped, into multi-Taub-NUT space $Q$. $Q$ is a hyper-Kähler manifold, which possesses complex structures $I$, $J$ and $K$ satisfying

$$I^2 = J^2 = K^2 = -1,$$
$$IJ = -JI = K, \ JK = -KJ = I, \ KI = -IK = J.\ (4.1)$$

As a complex manifold with respect to the complex structure $I$, $Q$ can be written as

$$YZ = \prod_{j=1}^{2N_c} (\hat{v} - m_j) \equiv \prod_{j=1}^{2N_c} (v - m_j + (1 - h(\tau))m_S),\ (4.3)$$

where $Y$, $Z$ and $v$ are complex variables which can be related to the real coordinate $(x^4, x^5, x^6, x^{10})$ as

$$Y = e^{-(x^6 + x^{10})} \prod_{j=1}^{2N_c} \sqrt{|\vec{x} - \vec{x}_j^*| - (x^6 - x_{j^*}^6)},\ (4.4)$$
\[ Z = e^{x_6 + x^{10}i} \prod_{j=1}^{2N_c} \left( \sqrt{\left| \vec{x} - \vec{x}_j \right|} + (x^6 - x^6_j) \left( \frac{\hat{v} - m_j}{|\hat{v} - m_j|} \right) \right), \] (4.5)

\[ \hat{v} = x^4 + x^5i, \] (4.6)

where \( \vec{x} = (x^4, x^5, x^6) \), \( m_j = x^4_j + x^5_ji \). \( \vec{x}_j = (x^4_j, x^5_j, x^6_j) \) is the position of NUT singularities, which can be interpreted as the position of D6-branes in the type IIA picture. In these coordinates, the metric on \( Q \) is given as

\[ ds^2 = V d\vec{x}^2 + \frac{1}{V} (dx^{10} + \vec{\omega} \cdot d\vec{x})^2, \] (4.7)

where

\[ V = 1 + \sum_{j=1}^{2N_c} \frac{1}{2|x - \vec{x}_j|} \] (4.8)

\[ \vec{\nabla} \times \vec{\omega} = \vec{\nabla} V. \] (4.9)

The Kähler form \( K \) and the holomorphic 2-form \( \Omega \) are

\[ K = i V d\hat{v} \wedge d\bar{\hat{v}} + \frac{i}{V} \left( \frac{dY}{Y} - \delta d\hat{v} \right) \wedge \left( \frac{dY}{Y} - \delta d\hat{v} \right), \] (4.10)

\[ \delta \equiv \frac{1}{2} \sum_{j=1}^{2N_c} \frac{x^6 - x^6_j + |\vec{x} - \vec{x}_j|}{|\vec{x} - \vec{x}_j| (\hat{v} - m_j)}, \] (4.11)

\[ \Omega = 2 d\hat{v} \wedge \frac{dY}{Y}. \] (4.12)

We take \( \Sigma \) to be

\[ \sqrt{1 - h(\tau)^2} (Y + Z) = 2 \prod_{a=1}^{N_c} (v + \phi_a), \] (4.13)

which is equivalent, as a Riemann surface, to the curve in (3.1).

For later use, let us express \( Q \) as a complex manifold with respect to another complex structure which is orthogonal to \( I \). The holomorphic variables with respect to the complex structure \( e^{i\theta}J \) are

\[ \tilde{Y} = e^{-\left(\bar{x}^4 + (x^{10} - \gamma)i\right)} \prod_{j=1}^{2N_c} \sqrt{|\vec{x} - \vec{x}_j|} - (\bar{x}^4 - \bar{x}_j^4), \] (4.14)

\[ \tilde{Z} = e^{\bar{x}^4 + (x^{10} - \gamma)i} \prod_{j=1}^{2N_c} \left( \sqrt{|\vec{x} - \vec{x}_j|} + (\bar{x}^4 - \bar{x}_j^4) \left( \frac{\bar{v} - \bar{m}_j}{|\bar{v} - m_j|} \right) \right), \] (4.15)

\[ \bar{v} = -x^6 + \bar{x}^5i, \] (4.16)
where we have defined
\[
\tilde{x}^4 + \tilde{x}^5 i = e^{-i\theta}(x^4 + x^5 i),
\]
\[
\gamma = \sum_{j=1}^{2N_c} \arg \left( \tilde{v} - \tilde{m}_j + |\tilde{x} - \tilde{x}_j| - (\tilde{x}^4 - \tilde{x}_j^4) \right),
\]
\[
\tilde{m}_j = -x_j^6 + \tilde{x}_j^5 i.
\]

Using these variables, \( Q \) can be written as
\[
\tilde{Y} \tilde{Z} = \prod_{j=1}^{2N_c} (\tilde{v} - \tilde{m}_j).
\]

The Kähler form \( \tilde{K} \) and the holomorphic 2-form \( \tilde{\Omega} \) are
\[
\tilde{K} = i V d\tilde{v} \wedge d\bar{\tilde{v}} + \frac{i}{V} \left( \frac{dY}{Y} - \tilde{\delta} d\bar{v} \right) \wedge \left( \frac{dY}{Y} - \tilde{\delta} d\bar{v} \right),
\]
\[
\tilde{\delta} \equiv \frac{1}{2} \sum_{j=1}^{2N_c} \frac{|\tilde{x}^4 - \tilde{x}_j^4| + |\tilde{x} - \tilde{x}_j|}{|\tilde{x} - \tilde{x}_j|} (\tilde{v} - \tilde{m}_j),
\]
\[
\tilde{\Omega} = 2 d\tilde{v} \wedge \frac{dY}{Y}.
\]

It is straightforward to check the following relations:
\[
\tilde{K} = \text{Im} (e^{-i\theta} \Omega),
\]
\[
\text{Re} \tilde{\Omega} = \text{Re} (e^{-i\theta} \Omega),
\]
\[
\text{Im} \tilde{\Omega} = -K.
\]

4.2. \textit{BPS states in MQCD}

BPS states in MQCD are realized as M2-branes ending on the M5-brane. The M2-brane is decomposed as \( \mathbb{R} \times \Sigma' \), where \( \mathbb{R} \) is the world line of the particle and \( \Sigma' \) is a Riemann surface holomorphically embedded in \( Q \) with respect to the complex structure \( e^{i\theta} J \), and the boundary of \( \Sigma' \) lies in \( \Sigma \).

The ele-mag charges for the BPS states can be read from the homology class of the boundary \( \partial \Sigma' \) in \( \Sigma \). Let us explain this fact explicitly.

There is a self-dual 2-form field \( B_2^+ \), whose field strength \( H_3^+ = dB_2^+ \) is self-dual, on the M5-brane world volume. When the M5-brane is wrapping a Riemann surface \( \Sigma \), \( B_2^+ \) should be expanded via harmonic forms on \( \Sigma \) in order to pick up the massless modes in the four dimensional effective theory. If the genus of \( \Sigma \) is \( l \), we have \( l \) holomorphic 1-forms.
\( \omega^J (J = 1, \cdots, l) \) on \( \Sigma \). The harmonic 1-forms are given by the real and imaginary parts of the holomorphic 1-forms. Setting \( \omega_{\text{ele}}^J \equiv \text{Re} \, \omega^J \) and \( \omega_{\text{mag}}^J \equiv \text{Im} \, \omega^J \), we have

\[
B^+_2 = \omega_{\text{ele}}^J \wedge A_{\text{ele}}^J + \omega_{\text{mag}}^J \wedge A_{\text{mag}}^J ,
\]

where \( A_{\text{ele}}^J \) and \( A_{\text{mag}}^J \) are 1-form fields in \( \mathbb{R}^4 \). Since the field strength of \( B^+_2 \) is self dual, using the relation \( \star \omega_{\text{ele}} = \omega_{\text{mag}} \), it follows that the field strengths of \( A_{\text{ele}} \) and \( A_{\text{mag}} \) are dual to each other: \( \star F_{\text{ele}} = F_{\text{mag}} \).

We choose a symplectic basis \( \{ \alpha_I, \beta_I \} \) of \( H_1(\Sigma, \mathbb{Z}) \), and we take \( \omega^J \) to satisfy

\[
\int_{\alpha_I} \omega^J = \delta^{IJ} .
\]

Then, the matrix with entries

\[
\int_{\beta_I} \omega^J \equiv \tau^{IJ} \equiv \left( \frac{8\pi i}{g^2} \right)^{IJ} + \frac{\theta^{IJ}}{\pi} ,
\]

becomes the effective gauge coupling constant for the \( U(1)^l \) theory, as in the Seiberg-Witten theory.\(^{14}\),\(^{26}\),\(^{6}\)

The boundary of M2-branes (\( \simeq \mathbb{R} \times \partial \Sigma' \)) are strings on the M5-brane and couple to the 2-form field\(^{27}\) as

\[
S_{\text{int}} \sim \int_{\mathbb{R} \times \partial \Sigma'} B^+_2 .
\]

If the homology class of \( \partial \Sigma' \) is \( n_{e}^I \alpha_I + n_{m}^I \beta_I \in H_1(\Sigma, \mathbb{Z}) \), putting all these data together, we obtain

\[
S_{\text{int}} \sim \left( n_{e}^I + n_{m}^I \frac{\theta^{IJ}}{\pi} \right) \int_{\mathbb{R}} A_{\text{ele}}^I + n_{m}^I \left( \frac{8\pi i}{g^2} \right)^{IJ} \int_{\mathbb{R}} A_{\text{mag}}^J .
\]

From this, we can interpret \( n_{e} + n_{m} \frac{\theta}{\pi} \) as the electric charges and \( n_{m} \) as the magnetic charges of the BPS state. Thus we have obtained the standard charge assignment in Seiberg-Witten theory.\(^{14}\) Note that when \( \theta \neq 0 \), the electric charges are shifted from \( n_{e} \) by \( n_{m} \frac{\theta}{\pi} \). This phenomenon is well known in the field theory as the ‘Witten effect’.\(^{28}\)

4.3. Construction of BPS states

In this subsection, we first consider the weak coupling limit, and find \( \Sigma' \) for the W-bosons and quarks in the multi-Taub-NUT space \( Q \). We also show that there are no other particles in this limit. These result shows that the theory is really \( SU(N_c) \) gauge theory with \( 2N_c \)

\(^{14}\) Note that there are no normalizable harmonic 0,2-forms, since the Riemann surface \( \Sigma \) is not compact.
flavors in the weak coupling region. Then we turn on the gauge coupling and find more examples of BPS states.

Let us first consider the weak coupling limit \( \tau \to i \infty, h(\tau) \to 1 \). In this limit, the M5-brane (4.13) reduces to

\[
0 = N_c \prod_{a=1}^{N_c} (v + \phi_a). \tag{4.32}
\]

Thus, \( x^4 \) and \( x^5 \) are fixed to be constant on each connected component of \( \partial \Sigma' \). As mentioned above, \( \Sigma' \) is a Riemann surface holomorphically embedded in \( Q \) with respect to the complex coordinates \( \tilde{Y}, \tilde{Z}, \tilde{v} \) in (4.14) \( \sim \) (4.16). Let \( p \) be a projection map from \( \Sigma' \) to the \( \tilde{v} \)-plane induced by the canonical projection \( (\tilde{Y}, \tilde{Z}, \tilde{v}) \to \tilde{v} \). Then \( p(\partial \Sigma'_0) \) lies on a fixed line which is parallel to the real axis, since \( \bar{x}^5 \) takes a fixed value. Here \( \partial \Sigma'_0 \) denotes a connected component of \( \partial \Sigma' \). Now we restrict our discussion on BPS states with finite mass. Since the mass is proportional to the area of \( \Sigma' \), the closure of \( \Sigma' \) should be compact. Let us suppose that \( p \) is not a constant map. Then, since \( p \) is holomorphic, \( p \) is an open map. Using a standard technique in topology, we can show that the boundary of \( p(\Sigma') \) lies in \( p(\partial \Sigma') \). But then, since \( p(\partial \Sigma') \) is a union of parallel lines, the image of \( p \) will inevitably extend infinitely in the \( x^6 \) direction, contradicting the compactness of the closure of \( p(\Sigma') \). Thus we conclude that \( p \) must be a constant map.

We classify \( \Sigma' \) in two cases:

(i) \( \tilde{v} = \tilde{m}_j \) for some \( j \)

In this case, (4.20) implies \( \bar{Y} = 0 \) or \( \bar{Z} = 0 \). From (4.14) and (4.15), \( \bar{Y} = 0 \) implies \( \bar{x}^4 > \bar{x}_4^i \) and \( \bar{Z} = 0 \) implies \( \bar{x}^4 < \bar{x}_4^i \). We can tune \( \theta \) in (4.17), in order that \( \Sigma' \) intersects \( \Sigma \). In this case, \( \Sigma' \) is a disk and the BPS state in four-dimensional effective theory makeup a hypermultiplet. As a result, we have obtained \( 2N_c \) flavors of quarks with mass \( \propto |\phi_a + m_j| \). (Fig. 4)

(ii) \( \tilde{v} = \text{constant} \neq \tilde{m}_j \) for all \( j \)

In this case, (4.20) implies \( \bar{Y} \neq 0 \) and \( \bar{Z} \neq 0 \). We must tune \( \theta \) in (4.17) in order that \( \Sigma' \) can intersect with two cycles in \( \Sigma \). In this case, \( \Sigma' \) is a cylinder and the corresponding BPS state constitutes a vector multiplet. As a result, we have obtained W-bosons with mass \( \propto |\phi_a - \phi_b| \). When we take \( \forall \phi_a = 0 \), they will become massless and form an \( SU(N_c) \) gauge multiplet together with the \( U(1)^{N_c-1} \) fields given in \( \S 4.2 \). (Fig.)

\[\text{\footnote{\(\text{We ignore Kaluza-Klein modes in our analysis.}\)}}\]

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Fig. 4. Here $\otimes$ represents the position of a NUT singularity, and the cylinder stretching in the $x^6$ direction represents a part of the M5-brane $\Sigma$. The disk which is caught by a NUT and winding around a cycle in $\Sigma$ is the M2-brane $\Sigma'$ for a quark.

Next we switch on the gauge coupling. It is not easy to find $\Sigma'$ explicitly in a generic brane configuration. Thus we only give several examples with a specific choice of $\phi_a$ and $\hat{x}_j$.

We use the following notation:

$$C(\hat{v}) \equiv \prod_{a=1}^{N_c} (v + \phi_a) \equiv \hat{v}^{N_c} + s_1 \hat{v}^{N_c-1} + \cdots + s_{N_c},$$  \hspace{0.5cm} (4.33)

$$B(\hat{v}) \equiv \prod_{j=1}^{2N_c} (\hat{v} - m_j) \equiv \hat{v}^{2N_c} + w_1 \hat{v}^{2N_c-1} + \cdots + w_{2N_c},$$  \hspace{0.5cm} (4.34)

$$\tilde{B}(\hat{v}) \equiv \sqrt{(1 - h^2) B(\hat{v})}.$$  \hspace{0.5cm} (4.35)

We consider the case in which $\forall x^6_j = 0$, $-1 < h(\tau) < 1$ and $\forall s_a, \forall w_j \in \mathbb{R}$. We take the complex structure with $\theta = 0$ for the M2-brane and consider $\Sigma'$ satisfying $\tilde{v} = 0$, that is, $x^5 = x^6 = 0$. Then the intersection of $\Sigma$ and $\Sigma'$ is given as

$$x^5 = x^6 = 0,$$  \hspace{0.5cm} (4.36)

$$C(x^4) = \tilde{B}(x^4) \cos x^{10}.$$  \hspace{0.5cm} (4.37)

Fig. 5. W-boson
We can find various states by constructing graphs of $C(x^4)$ and $\pm \tilde{B}(x^4)$. For example, typical graphs for dyons, W-bosons and quarks are given as in Fig. 6.

![Graphs of $C(x^4)$ and $\pm \tilde{B}(x^4)$ for different particles](image)

Fig. 6.

Note that solutions of $C(x^4) = \pm \tilde{B}(x^4)$ are branch points on the real axis of the $\hat{v}$-plane, and zero points of $\tilde{B}$ are the NUT singularities. We can see from these figures that when we tune the moduli parameters $\phi_a$ or the mass parameters $m_j$ to obtain massless states, two branch points must meet, and the Riemann surface $\Sigma$ becomes singular. These figures also allow for the visualization of decay patterns of the particles by moving $\phi_a$ and $m_j$. For example, the W-boson in Fig. 6 will decay into two quarks, as in Fig. 6, tuning $m_j$ such that the two NUTs meet on the $\hat{v} = 0$ plane. Similarly, tuning $\phi_a$, the W-boson in Fig. 6 will decay into the dyon in Fig. 6, and some other particles which are required in order to be consistent with charge conservation. We can never find all the particles to decay in this picture, since they are, as argued in Ref. 22, mutually non-local.

There are much more complicated and physically interesting examples, as shown in Fig. 7.

![Graphs showing exotic W-bosons and particles](image)

Fig. 7.

MQCD predicts the existence of these exotic particles in $\mathcal{N} = 2$ SQCD. There are examples of $\Sigma'$ with more than two holes, which may correspond to multiplets with spin $3/2$ or higher. They carry non-parallel electric and magnetic charges and thus there seem to be no classical solutions corresponding to these particles in the $\mathcal{N} = 2$ field theory. 29, 30

Although it would be quite interesting to explore further, we wish to return to the original subject in order to be faithful to the title. (See Ref. 31 for a recent progress in $\mathcal{N} = 4$ SYM.)
4.4. Magnetic theory in electric theory

In the usual field theory description, one defines a theory giving a Lagrangian. We refer to fields in the Lagrangian as elementary fields, and refer to the corresponding particles as elementary particles. The elementary particles in the electric theory constitute an SU($N_c$) vector multiplet and $2N_c$ flavors of hypermultiplets. In the last subsection, we have found those particles at the weak coupling limit in the electric description of MQCD. If we consider the strong coupling limit, we find the elementary particles in the magnetic theory, since the theory is weak coupling in the magnetic description. It is obvious that the elementary particles in the electric theory and the magnetic theory are not identical. So, if one chooses the elementary particles in the electric theory to write down the Lagrangian, we should interpret the elementary particles in the magnetic theory as solitons in the field theory.

What is remarkable in MQCD is that the elementary particles and solitons are treated on an equal footing. Both the electric and magnetic particles appear as M2-branes ending on the M5-brane. Thanks to this feature, we can treat the electric and magnetic theories simultaneously, and it becomes possible to analyze the particle correspondence under duality.

Now let us analyze the correspondence of the ele-mag charges and make sure that the elementary particles in the magnetic theory are indeed magnetic monopoles in the electric description. For simplicity, we set all the quark mass parameters to zero and take $\phi_a = \phi e^{2\pi i a/N_c}$. Then the Seiberg-Witten curve (3.1) becomes

$$t^2 - 2(v^{N_c} + \phi^{N_c}) t + (1 - h^2) v^{2N_c} = 0. \tag{4.38}$$

The branch points in the $v$-plane are given by

$$v_{a\pm} = \phi_a \left( -1 \pm \sqrt{1 - h^2} \right)^{N_c^+} \frac{1}{h^2} \quad (a \in \mathbb{Z} \text{ (mod } N_c)) \tag{4.39}$$

We define the $A_a$-cycle, $B_a$-cycle and $C_a$-cycle to be cycles which encircle $(v_{a+}, v_{a-})$, $(v_{a-}, v_{a+})$, and $(v_{a+}, v_{a-})$ in the $v$-plane, respectively, as shown in Fig. 8.

The intersection numbers for these cycles are given as

$$A_a \cdot B_a = 1, \quad A_{a-1} \cdot B_a = -1, \quad A_a \cdot C_a = 1, \quad A_{a-1} \cdot C_a = -1, \quad B_{a-1} \cdot B_a = -1, \quad C_{a-1} \cdot C_a = 1, \tag{4.40}$$

and the others are all zero. Of course, they are not all independent, satisfying the following relations:

$$\sum_{a=1}^{N_c} A_a = \sum_{a=1}^{N_c} B_a = \sum_{a=1}^{N_c} C_a = 0, \quad (4.41)$$

$$A_{a-1} + A_a = B_a - C_a. \quad (4.42)$$
Fig. 8. The $A_a, B_a, C_a$-cycles for the case $N_c = 3$ and $0 < h < 1$ in $v$-plane. The wavy lines are branch cuts stretched between the branch points $v_{a+}$ and $v_{a-}$.

We choose the symplectic basis \( \{\alpha_I, \beta_J | I = 1, \cdots, N_c - 1\} \) of \( H_1(\Sigma, \mathbb{Z}) \) as

\[ \alpha_I \equiv A_I, \quad \beta_I \equiv \sum_{a=1}^{I} (B_a - A_{a-1}), \]  

where \( A_0 \equiv A_{N_c} \). It is easy to check that they satisfy

\[ \alpha_I \cdot \alpha_J = \beta_I \cdot \beta_J = 0, \quad \alpha_I \cdot \beta_J = \delta_{IJ}. \]

At the weak coupling limit in the electric description \((h \to 1)\), the branch points $v_{a+}$ and $v_{a-}$ approach each other, and $\alpha$-cycles become the vanishing cycle. The quarks and $W$-bosons considered in the last subsection are constructed with M2-branes, whose boundary is homotopic to the linear combinations of the $\alpha$-cycles in $\Sigma$ and thus is electrically charged in the convention given in §4.2.

The elementary particles in the magnetic theory are constructed similarly in the strong coupling limit in the electric description \((h \to -1)\). So let us move on to the strong coupling region; that is, we exchange the two NS5-branes in the type IIA picture. We consider moving $h$ from the weak coupling region \((h \sim 1)\) to the strong coupling region \((h \sim -1)\) along the real axis. Since there is a singularity at $h = 0$, which is due to the collision of two NS5-branes, we should deform the path near $h \sim 0$ to avoid the singularity.

It would be easier to consider this in the $h^2$-plane. The above prescription is equivalent to moving $h^2$ once around 0. When $h^2$ is nearly 0, the branch points are at $v_{a+} \sim \phi_a(-1/2)^{\frac{1}{N_c}}$.
and \( v_{a-} \sim \phi(a(-2/h^2)\frac{1}{\pi c} \). While moving \( h^2 \) around 0, \( v_{a-} \) moves to \( v_{(a-1)-} \), and the \( v_{a+} \) are fixed.

\[ v_{a-} \sim \phi(a(-2/h^2)\frac{1}{\pi c} \]

\[ v_{a-} \rightarrow v_{(a-1)-} \]

Fig. 9. The motion of \( \alpha \)-cycles and \( \beta \)-cycles via brane exchange. It can be easily seen that the role of \( \alpha \)-cycles and \( \beta \)-cycles are exchanged.

We define the \( A'_a \)-cycle, \( B'_a \)-cycle and \( C'_a \)-cycle as in Fig. 10. These play the same role as \( A_a, B_a, C_a \)-cycle after brane exchange.

\[ A'_a = A'_a - B'_a \]
\[ B'_a = -B'_{a-1} \]
\[ C'_a = C'_{a-1} \]

The \( A_a, B_a, C_a \)-cycles are expressed as

The symplectic basis which is canonical for the magnetic description is

\[ \alpha'_I = A'_I \]
\[ \beta'_I \equiv \sum_{a=1}^{I}(B'_a - A'_{a-1}), \]  
(4.51)

where \( I = 1, \cdots, N_c - 1 \) and \( A'_0 \equiv A'_{N_c} \). The relation between the two symplectic bases is

\[
\alpha_I = \alpha'_I - \alpha'_{I-1} - \beta'_I + \beta'_{I-1},
\]
(4.52)

\[
\beta_I = \sum_{J=I}^{N_c-1} \alpha'_J,
\]
(4.53)

where \( \beta'_0 \equiv 0, \alpha'_0 \equiv -\sum_{J=1}^{N_c-1} \alpha'_J \). The charge assignments in these bases are related as

\[
n_e^I = -\sum_{J=I}^{N_c-1} n'_J,
\]
(4.54)

\[
n_m^I = n_e^I - n'_I + n'_1 - n'_m - n'_m - n'_m - n'_m - n'_m,
\]
(4.55)

where we have defined \( n'_0 \equiv 0, n'_m \equiv -\sum_{J=1}^{N_c-1} n'_J \).

The elementary particles in the magnetic description are dual quark s and dual W-bosons whose boundaries in the M2-branes are \( \partial \Sigma' = \pm A'_a \) and \( \partial \Sigma' = A'_a - A'_b \), respectively. So they have \( U(1)^{N_c-1} \) charges with \( n'_e \neq 0 \) and \( n'_m = 0 \). Equations (4.54) and (4.55) imply that these particles are magnetic monopoles in the electric description.

As an example, let us consider the \( N_c = 3 \) case. Quarks and W-bosons in the electric theory have charges as follows:

|       | \( n_1^1 \) | \( n_2^1 \) | \( n_1^2 \) | \( n_2^2 \) |
|-------|-------------|-------------|-------------|-------------|
| quarks | \( \pm 1 \)  | \( 0 \)      | \( 0 \)      | \( 0 \)      |
|        | \( \pm 1 \)  | \( \pm 1 \)  | \( 0 \)      | \( 0 \)      |
| W-bosons | \( \pm 1 \)  | \( \pm 2 \)  | \( 0 \)      | \( 0 \)      |
|        | \( \pm 2 \)  | \( \pm 1 \)  | \( 0 \)      | \( 0 \)      |
(4.56)

The dual quarks and dual W-bosons in the magnetic theory can be interpreted in the electric theory as magnetic monopoles with charges given by (4.55):

|       | \( n_1^1 \) | \( n_2^1 \) | \( n_1^2 \) | \( n_2^2 \) |
|-------|-------------|-------------|-------------|-------------|
| dual  | 0           | 0           | \( \pm 1 \)  | 0           |
| quarks | 0           | 0           | 0           | \( \mp 1 \)  |
|        | 0           | 0           | \( \pm 1 \)  | \( \mp 1 \)  |
| W-bosons | 0           | 0           | \( \pm 1 \)  | \( \mp 2 \)  |
|        | 0           | 0           | \( \pm 2 \)  | \( \mp 1 \)  |
(4.57)
We can make a prediction for the field theory that there exist these solitonic states in the electric theory at least in the strong coupling region. These solitons will dominate in the strong coupling region and form the dual $SU(N_c)$ multiplets.

Note, however, that we do not know whether these magnetic particles also exist in the weak coupling region, since we may cross the curves of marginal stability during the brane exchange, and they may decay into some other particles. For the case $N_c = 2$, there are no curves of marginal stability, and the particle spectrum is predicted to be invariant under the duality group $SL(2,\mathbb{Z})$. But, as far as we know, it is still unclear whether or not the particle spectrum is invariant under the duality group in the higher rank gauge theories, (see Ref. [29] and [33]). In §4.6, we make a brief argument about these points from the M-theoretical point of view.

4.5. *Brane conversion via brane exchange*

As a by-product of the analysis in the last subsection, we observe an interesting phenomenon in type IIA string theory.

Recall that $\alpha$-cycles transform to $\beta$-cycles via brane exchange.

Let us consider a quark in the weak coupling electric theory. As constructed in §4.3, the Riemann surface $\Sigma'$ for the quark is a disk caught by a NUT, whose boundary winds around the $\alpha$-cycle of $\Sigma$ (Fig. 4). It can be interpreted as a fundamental string stretched between a D4-brane and a D6-brane in the IIA picture. Then, we move $\tau$ to the strong coupling region and exchange the two NS5-branes. The quark is then interpreted as a solitonic state in the
weak coupling magnetic theory; that is, the boundary of $\Sigma'$ winds around the $\beta$-cycle of the brane exchanged Riemann surface $\Sigma$.

![Diagram](image1)

Fig. 12.

Therefore the M2-brane for the quark in the strong coupling electric theory is interpreted as a D2-brane in the type IIA picture. As a result, the fundamental string is converted into a D2-brane via the exchange of two NS5-branes, as shown in Fig. 13.

![Diagram](image2)

Fig. 13

4.6. Geodesics

In Ref. 24, it is argued that the BPS states are represented by the geodesics on $\Sigma$ with the metric $ds^2 = |\lambda|^2$, where $\lambda$ is the Seiberg-Witten 1-form. However, it was found in Ref. 23 that several examples of the BPS states in MQCD do not correspond to the geodesics. In this subsection, we first resolve this discrepancy and then apply the technique in Ref. 24, 34 and 35 to our situation. Related arguments can also be found in a recent interesting paper Ref. 25).

The mass of a particle in MQCD is proportional to the area of the membrane:

$$\text{mass} \propto \int_{\Sigma'} (\text{vol}) = \int_{\Sigma'} \sqrt{K^2 + |\Omega|^2}. \quad (4.58)$$
Here (vol) is the volume form on $\Sigma'$, and $K$ and $\Omega$ are the Kähler form (4.10) and holomorphic 2-form (4.12) on $Q$ pulled back to $\Sigma'$. As shown in Ref. [21], [22], [23], the BPS condition implies that $\Omega$ has a constant phase and $K = 0$ on $\Sigma'$, which is equivalent to the condition that $\Sigma'$ is holomorphically embedded in $Q$ with respect to a complex structure which is orthogonal to the original one. Therefore,

$$\text{mass} \propto \int_{\Sigma'} |\Omega| = \left| \int_{\Sigma'} \Omega \right| = \left| \int_{\partial \Sigma'} \lambda \right|,$$

(4.59)

where $\lambda \equiv 2 \dot{v} \frac{dY}{Y}$ is a 1-form satisfying $\Omega = d\lambda$. The pull back of $\lambda$ to $\Sigma$ is proportional to the Seiberg-Witten 1-form. Note that if $\Sigma$ passes through the points with $Y = 0$, we should also include integration around these points. Hence $\partial \Sigma'$ in (4.59) may have components which are not included in $\Sigma$. Note also that at the point with $Y = 0$, (4.3) implies $\dot{v} = m_j$ for some $j$, and thus the integration of $\lambda$ around these points is proportional to $m_j$. [23]

The statement that $\partial \Sigma'$ is a geodesic on $\Sigma$ with the metric $ds^2 = |\lambda|^2$ is equivalent to the statement that $\lambda$ has a constant phase on $\partial \Sigma'$. However, (4.59) does not necessarily imply $\int f |\lambda| = \int |\lambda|$. The problem is that there is an ambiguity in the definition of $\lambda$. We can use $\lambda + df$ instead of $\lambda$ without changing the entire story. Here $f$ is a function on $Q$ which is well-defined at least locally. Let us show that $\partial \Sigma'$ is indeed a geodesic on $\Sigma$ with the metric $ds^2 = |\lambda + df|^2$ for some function $f$. Since $\Omega$ has a constant phase, we can assume $\mathrm{Re} \, \Omega = 0$ on $\Sigma'$. Then the relations $0 = \mathrm{Re} \, \Omega = d \mathrm{Re} \, \lambda$ imply that there is a real function $f_R$ which satisfies $\mathrm{Re} \, \lambda = -df_R$ on $\Sigma'$. Therefore if we choose $f$ such that $\mathrm{Re} \, f = f_R$ on $\Sigma'$, $\lambda + df$ has a constant phase on $\Sigma'$.

As an example, let us consider $\mathcal{N} = 2$ $SU(2)$ SYM with 4 massless flavors and consider the case in which $\Sigma'$ is homeomorphic to a disk. We define a 1-form $\lambda_f$ on $\Sigma$ as

$$\lambda_f = \lambda_f \, dx^i \equiv \lambda + df,$$

(4.60)

where $f$ is a function on $\Sigma$. Taking a suitable affine parameter $t$, a geodesic on $\Sigma$ with the metric $ds^2 = |\lambda_f|^2$ satisfies the geodesic equation

$$\lambda_f \frac{dx^i}{dt} = \alpha,$$

(4.61)

where $\alpha$ is a constant. Integrating this equation, we obtain

$$\int_{x(0)}^{x(t)} \lambda + f(x(t)) - f(x(0)) = \alpha \, t.$$

(4.62)

Now, explicit calculation shows that $\lambda$ is proportional to a holomorphic 1-form on $\Sigma$ up to an exact 1-form, and so, redefining $f$, we can set $\lambda \propto \frac{dx}{y}$. Here we define the map

$$\varphi_f : \Sigma \to J(\Sigma)$$

(4.63)
by choosing a point $P_0 \in \Sigma$ and setting

$$\varphi_f(P) = \int_{P_0}^{P} \lambda + f(P) - f(P_0).$$

(4.64)

Here, $J(\Sigma)$ is the Jacobian of $\Sigma$ which is defined as $J(\Sigma) \equiv \mathbb{C}/\Gamma$, where $\Gamma \equiv \mathbb{Z} \int_\alpha \lambda \oplus \mathbb{Z} \int_\beta \lambda$ is a lattice in $\mathbb{C}$. Equation (4.62) implies that the image of the geodesic under the map $\varphi_f$ is a straight line on $J(\Sigma)$.

Let us first consider the $f = 0$ case. A closed straight line on $J(\Sigma)$ is a line winding $p$ times around the $\alpha$ cycle and $q$ times around the $\beta$ cycle, where $p$ and $q$ are relatively prime integers. As is well known (see for example Ref. 36), the map $\varphi_0$ is biholomorphic, and thus all the $(p,q)$ lines correspond to geodesics on $\Sigma$. This fact strongly suggests that there are hypermultiplets with $(n_e,n_m) = (p,q)$ charge, and moreover, that there are no hypermultiplets carrying $(np,nq)$ charge with $n \geq 2$. It is clear that we cannot find further states, even if we take into account the function $f$ in (4.64). This result is consistent with the prediction given in Ref. 14, that the BPS spectrum is invariant under the duality group $SL(2,\mathbb{Z})$. Note that this result has already been derived using different methods in Ref. 37, 38.

If we naively generalize this argument to the case with $N_c > 2$, the Jacobian $J(\Sigma)$ is replaced by $\mathbb{C}/\Gamma$, where $\Gamma = (\oplus_i \int_{\alpha_i} \lambda) \oplus (\oplus_i \int_{\beta_i} \lambda)$. However, $\Gamma$ is in general a dense set in $\mathbb{C}$, and $\mathbb{C}/\Gamma$ is no longer a manifold. We do not know how to resolve this problem, and leave it for future study.

§5. Discussions

As we have emphasized, MQCD is a democratic theory in which there is no discrimination between elementary particles and solitons. This is why we can treat the electric and magnetic theories simultaneously, and duality becomes manifest. This is quite analogous to the S-duality in type IIB string theory, which can be understood as a redefinition of a symplectic basis of $T^2$ in M-theory 39, 40.

On the other hand, in order to describe the theory in field theory language, namely, in order to write down the Lagrangian, we must pick up the elementary particles, which dominate in the weak coupling limit, from the particle spectrum. There may be another choice of elementary particles which dominate in another limit of the couplings. If we choose another set of particles as the elementary particles, the Lagrangian will become totally different from the original one. This is the reason that duality in field theory is so mysterious and difficult to prove exactly. Note that the brane exchange in Ref. 3, 2, 3, 19 is not a
duality transformation, but a prescription to find another set of the elementary particles, moving the gauge coupling from weak to strong.

We have argued in the $\mathcal{N} = 2$ duality that the elementary particles of the magnetic theory appear as solitons in the electric theory. Since Seiberg’s $\mathcal{N} = 1$ duality can be obtain as a deformation of $\mathcal{N} = 2$ duality, we can extrapolate to conjecture that the magnetic theory in Seiberg’s duality is also a theory of solitons, as suggested in Seiberg’s paper\cite{1}. We did not examine the spectrum of the $\mathcal{N} = 1$ theory, but at least in principle, we can analyze it finding stable minimal surfaces. It would be interesting to extend our arguments to non-BPS states (see Ref. [4]).

In this paper, MQCD is defined as an effective field theory on an M5-brane world volume. Hence, we must abandon the democracy once we write down the effective Lagrangian. Moreover, we do not really know whether or not it is a consistent theory. Therefore it is necessary to construct a microscopic definition of MQCD, which may be easier to construct than that of M-theory. We think that the construction of MQCD might be a first step to go beyond the Lagrangian description of particle physics.

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