On topological actions of finite groups on $S^3$

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Abstract. We consider orientation-preserving actions of a finite group $G$ on the 3-sphere $S^3$ (and also on Euclidean space $\mathbb{R}^3$). By the geometrization of finite group actions on 3-manifolds, if such an action is smooth then it is conjugate to an orthogonal action, and in particular $G$ is isomorphic to a subgroup of the orthogonal group $\text{SO}(4)$ (or of $\text{SO}(3)$ in the case of $\mathbb{R}^3$). On the other hand, there are topological actions with wildly embedded fixed point sets; such actions are not conjugate to smooth actions but one would still expect that the corresponding groups $G$ are isomorphic to subgroups of the orthogonal groups $\text{SO}(4)$ (or of $\text{SO}(3)$, resp.). In the present paper, we obtain some results in this direction; we prove that the only finite, nonabelian simple group with a topological action on $S^3$, or on any homology 3-sphere, is the alternating or dodecahedral group $A_5$ (the only finite, nonabelian simple subgroup of $\text{SO}(4)$), and that every finite group with a topological, orientation-preserving action on Euclidean space $\mathbb{R}^3$ is in fact isomorphic to a subgroup of $\text{SO}(3)$.

1. Introduction

We consider topological actions of finite groups on the 3-sphere (i.e., by homeomorphisms); all actions in the present paper will be faithful and orientation-preserving. By the geometrization of finite group actions on 3-manifolds after Thurston and Perelman, each finite group $G$ acting smoothly on $S^3$ is conjugate to an orthogonal action; in particular, $G$ is isomorphic to a subgroup of the orthogonal group $\text{SO}(4)$. The last statement is no longer true for smooth actions on arbitrary homology 3-spheres, and a classification of such groups appears to be difficult (see [Z1], [MeZ1]). Also, for each dimension $n \geq 4$ there exist finite groups which admit a topological, orientation-preserving action on $S^n$ but are not isomorphic to a subgroup of $\text{SO}(n+1)$ (see [Z3]; this remains open for smooth actions).

On the other hand, not much is known on the finite groups $G$ admitting a topological action on $S^3$; some examples are known of actions with wildly embedded fixed point sets, mainly for cyclic groups. Such actions are not conjugate to smooth or orthogonal actions, but one would still expect that the groups $G$ are isomorphic to subgroups of $\text{SO}(4)$; in the present paper, we prove the following results in this direction.
**Theorem 1.** Let \( G \) be a finite group of orientation-preserving homeomorphism of \( S^3 \) or a homology 3-sphere such that each element of \( G \) has nonempty fixed point set; then \( G \) is isomorphic to a subgroup of the orthogonal group \( \text{SO}(3) \). In particular, this is the case if \( G \) has a global fixed point.

By extending actions on \( \mathbb{R}^3 \) to its 1-point compactification \( S^3 \), this implies:

**Corollary 1.** A finite group \( G \) of orientation-preserving homeomorphisms of Euclidean space \( \mathbb{R}^3 \) is isomorphic to a subgroup of \( \text{SO}(3) \).

The proof of Theorem 1 uses a purely algebraic characterization of the finite subgroups of the orthogonal group \( \text{SO}(3) \) by A. Miller (Theorem 3 in section 3). In section 3 we shall present a short, direct proof of Corollary 1 on the basis of Miller’s result; an independent proof is given in [KSu]. For smooth actions, Corollary 1 follows from the geometrization of finite group actions on \( \mathbb{R}^3 \) (see [KSc]). In particular, every smooth action of a finite group on \( \mathbb{R}^3 \) has a global fixed point. This remains true for topological actions of finite solvable groups on \( \mathbb{R}^3 \) (see [GMZ, proof of Theorem 1]); since the only nonsolvable subgroup of \( \text{SO}(3) \) is the alternating group \( A_5 \), this raises the following:

**Question.** Is there a topological action of \( A_5 \) on \( \mathbb{R}^3 \) without a global fixed point? Equivalently, is there a topological action of \( A_5 \) on \( S^3 \) with exactly one global fixed point?

We note that the group \( A_5 \) has an action on the Poincaré homology 3-sphere with exactly one global fixed point; the complement of the fixed point is an acyclic 3-manifold with an action of \( A_5 \) without a global fixed point, so the question has a positive answer for homology 3-spheres and acyclic 3-manifolds, in general.

**Corollary 2.** Let \( M \) be a 3-manifold whose universal covering is \( \mathbb{R}^3 \) or \( S^3 \). Let \( G \) be a finite group of orientation-preserving homeomorphisms of \( M \) with a global fixed point; then \( G \) is isomorphic to a subgroup of \( \text{SO}(3) \).

This follows from Corollary 1 and Theorem 1 by lifting \( G \) to an isomorphic group of homeomorphisms of the universal covering with a global fixed point. Note that, by considering invariant regular neighbourhoods, this is true for smooth actions for the case of arbitrary 3-manifolds, and also for topological actions it should remain true for arbitrary 3-manifolds.

**Theorem 2.** A finite, nonabelian simple group of homeomorphisms of \( S^3 \) or a homology 3-sphere is isomorphic to the alternating group \( A_5 \).

For smooth actions, this is proved in [Z2] and [MeZ2]; using Proposition 1 in section 2, we reduce the proof of Theorem 2 to the proof in [MeZ2], based on the Gorenstein-Walter...
classification of the finite simple groups with dihedral Sylow 2-subgroups. Actions of finite simple groups on spheres and homology spheres in higher dimensions are considered in [GZ], and again these results remain true for topological actions.

In section 2 we prove a technical key result which allows to generalize various known results about finite group actions on homology 3-spheres from smooth actions to topological actions.

2. A preliminary result

Our technical key result is the following:

**Proposition 1.** Let $G$ be a finite group with an orientation-preserving, topological action on $S^3$ (or on a homology 3-sphere). Suppose that the fixed point set of an element $g \in G$ is a (possibly wildly embedded) circle $K \cong S^1$. Then the normalizer $N = N_G(g)$ of $g$ in $G$ is isomorphic to a subgroup of a semidirect product $(\mathbb{Z}_a \times \mathbb{Z}_b) \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts dihedrally on the abelian group $A = \mathbb{Z}_a \times \mathbb{Z}_b$.

For smooth actions this follows easily, for the case of arbitrary 3-manifolds, from the existence of invariant regular neighbourhoods (considering rotations of minimal angle around a smoothly embedded circle, see [MeZ2, Lemma 1]), and also for topological actions it should remain true for arbitrary 3-manifolds. As in the case of smooth actions, Proposition 1 has various applications to the structure of finite groups admitting a topological action on $S^3$ (it is a basic tool for the partial characterization of the finite groups which admit a smooth action on a homology 3-sphere in [MeZ1], [Z1]).

**Proof of Proposition 1.** Each element of the normalizer $N$ maps the circle $K$ and its complement $M = S^3 - K$ to itself. By Alexander-Lefschetz duality, $S^3 - K$ is also a homology (and cohomology) circle (that is, has the homology or cohomology of the circle). Let $B$ be the subgroup of $N$ which fixes $K$ pointwise; then, by Smith fixed point theory, $B$ acts freely on the cohomology circle $S^3 - K$, therefore $B$ has periodic cohomology of period two and is a cyclic group $B = \mathbb{Z}_n$ (see [Br]). We call the elements of $B = \mathbb{Z}_n$ rotations around $K$. The factor group $C = N/B$ acts faithfully on the circle $K$ and hence is a cyclic or a dihedral group; its nontrivial elements are either reflections of $K$ (i.e., fixing exactly two points of $K$), or rotations along $K$, i.e. without fixed points on $K$.

The normalizer $N$ acts on $S^3 - K$, with the cyclic normal subgroup $B = \mathbb{Z}_n$ acting freely, and we consider the cyclic regular covering $M = S^3 - K \to \bar{M} = (S^3 - K)/B$ and the cohomology spectral sequence associated to this covering (see [McL]):

$$E_2^{i,j} = H^i(\mathbb{Z}_n; H^j(M)) \Rightarrow H^{i+j}(\bar{M}),$$
converging to the graded group associated to a filtration of $H^*(\tilde{M})$ (integer coefficients). Since $M$ is a cohomology circle, the spectral sequence is concentrated in the rows $j = 0$ and $j = 1$, so the only possibly nontrivial differentials, of bidegree $(2, -1)$, are
\[
d^i_{2,1} : E^{i,1}_2 = H^i(\mathbb{Z}_n; H^1(M)) \to E^{i+2,0}_2 = H^{i+2}(\mathbb{Z}_n; H^0(M)),
\]
where $B = \mathbb{Z}_n$ acts by the identity on $H^0(M) \cong \mathbb{Z}$ and, by duality, also on $H^1(M) \cong \mathbb{Z}$ (since $G$ acts orientation-preservingly on $S^3$ and also on $K$, it acts by the identity on the first homology and cohomology $Z$ of $K$, and hence by duality also on the first homology and cohomology $Z$ of $M = S^3 - K$). Summarizing, we have
\[
d^i_{2,1} : H^i(\mathbb{Z}_n; \mathbb{Z}) \to H^{i+2}(\mathbb{Z}_n; \mathbb{Z}),
\]
where $H^i(\mathbb{Z}_n; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ if $i = 0$, to $\mathbb{Z}_n$ if $i > 0$ is even, and trivial if $i$ is odd. In particular, $\mathbb{Z}_n$ has periodic cohomology of period two (for $i > 0$), and the duality isomorphism is given by the cup-product with an element $u \in H^2(\mathbb{Z}_n; \mathbb{Z}) \cong \mathbb{Z}_n$ (see [Br]).

Since $\tilde{M}$ is also a 3-manifold, its cohomology is trivial in dimensions larger than three; hence, passing to the limit of the spectral sequence, the differentials $d^i_{2,1}$ have to be isomorphisms for larger $i$. But then also the differential $d^i_{2,1}$ has to be an isomorphism, by dimension shifting with the cup-product with $u \in H^2(\mathbb{Z}_n; \mathbb{Z})$ and the multiplicative structure of the spectral sequence (see [S, section 9.4]). Replacing $H^0(\mathbb{Z}_n; \mathbb{Z}) \cong \mathbb{Z}$ by its quotient, the Tate cohomology group $\hat{H}^0(\mathbb{Z}_n; \mathbb{Z}) \cong \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, also $d^0_{2,1}$ becomes an isomorphism, and hence $d^0_{2,1} : H^0(\mathbb{Z}_n; \mathbb{Z}) \cong \mathbb{Z} \to H^2(\mathbb{Z}_n; \mathbb{Z}) \cong \mathbb{Z}_n$ has to be surjective.

Passing to the limit of the spectral sequence, this implies easily that $\tilde{M}$ is a cohomology circle, and hence also a homology circle; in particular, $H_1(\tilde{M}; \mathbb{Z}) \cong \mathbb{Z}$.

We consider the group extension $1 \to \pi_1(M) \to \pi_1(\tilde{M}) \to \mathbb{Z}_n \to 1$ associated to the regular covering $M \to \tilde{M}$, with covering group $\mathbb{Z}_n$. Abelianizing $\pi_1(M)$, we get an extension
\[
0 \to \pi_1(M)_{ab} = \pi_1(M)/[\pi_1(M), \pi_1(M)] \to \pi_1(\tilde{M})/[\pi_1(M), \pi_1(M)] \to \mathbb{Z}_n \to 0,
\]
with $\pi_1(M)_{ab} \cong H_1(M) \cong \mathbb{Z}$. Since, as before, the covering group $\mathbb{Z}_n$ acts trivially on $H_1(M) \cong \mathbb{Z}$, also $\pi_1(\tilde{M})/[\pi_1(M), \pi_1(M)]$ is an abelian group and hence isomorphic to the abelianized group $\pi_1(\tilde{M})_{ab} \cong H_1(M) \cong \mathbb{Z}$, so we have an exact sequence
\[
0 \to H_1(M) \cong \mathbb{Z} \to H_1(\tilde{M}) \cong \mathbb{Z} \to \mathbb{Z}_n \to 0.
\]
Let $g \in N$ be a rotation along $K$; since $g$ acts orientation-preservingly on $K$, it acts by the identity on the homology and cohomology of $K$, and by duality by the identity also on $H_1(M) \cong \mathbb{Z}$. Then also the homeomorphism $\tilde{g}$ induced by $g$ on $\tilde{M}$ acts by the identity on $H_1(\tilde{M}) \cong \mathbb{Z}$, and hence $g$ acts by the identity on the covering group $\mathbb{Z}_n$. It
follows that the subgroup $A$ of $N$ of rotations around and along $K$ is abelian, of rank one or two.

If $g \in N$ is a reflection of $K$ instead, then $g$ fixes exactly two points in $K$ and, by Smith theory, a circle in $S^3$ which implies $g^2 = 1$ (since a nontrivial element cannot fix two different circles). Also, since $g$ acts dihedrally on the homology and cohomology of $K$, by duality it acts dihedrally also on $H_1(M) \cong \mathbb{Z}$, and hence dihedrally also on $\mathbb{Z}_n$.

Concluding, the normalizer $N$ has the structure given in Proposition 1.

3. Proofs of Theorems 1 and 2

The proof of Theorem 1 uses the following purely algebraic characterization of the finite subgroups of the orthogonal group $SO(3)$.

**Theorem 3.** ([Mi]) A finite group $G$ is isomorphic to a subgroup of $SO(3)$ if and only if the normalizer of each nontrivial element $g \in G$ is a cyclic or dihedral group. This is in turn equivalent to the following condition: Distinct maximal cyclic subgroups of $G$ intersect trivially, and each maximal cyclic subgroup of $G$ has index one or two in its normalizer in $G$.

**Proof of Theorem 1.** By Smith fixed point theory, the fixed point set of a nontrivial element $g \in G$ is a circle $K$. By Proposition 1, the normalizer $N = N_G(g)$ is isomorphic to a subgroup of a semidirect product $(\mathbb{Z}_a \times \mathbb{Z}_b) \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts dihedrally on the abelian group $A = \mathbb{Z}_a \times \mathbb{Z}_b$. As in the proof of Proposition 1, the group $A$ consists of rotations around and along $K$; it has a cyclic subgroup $B$ of rotations around $K$ (i.e., fixing $K$ pointwise), with cyclic factor group $A/B$. We will show that $A$ is a cyclic group and apply Theorem 3.

Suppose that $A$ has a subgroup $\mathbb{Z}_p \times \mathbb{Z}_p$, for a prime $p$. We apply the Borel formula (which holds in a purely topological setting) to the subgroup $\mathbb{Z}_p \times \mathbb{Z}_p$ ([Bo]; see also [MeZ2] for such an application). If $p > 2$ then by the Borel formula, $\mathbb{Z}_p \times \mathbb{Z}_p$ has exactly two cyclic subgroups $\mathbb{Z}_p$ with nonempty fixed point set (two different circles), so some subgroup $\mathbb{Z}_p$ acts freely on $S^3$ contrary to the hypothesis of Theorem 1 i). If $p = 2$ then either there are again two involutions in $\mathbb{Z}_2 \times \mathbb{Z}_2$ with nonempty fixed point set and one free involution, or all three involutions have nonempty fixed point set (three circles intersecting in two points). In particular, in the second case some involution in $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts as a reflection on $K$ which, however, is not the case in the group $A$ of rotations of $K$.

So $A$ has no subgroups $\mathbb{Z}_p \times \mathbb{Z}_p$ and hence is cyclic, hence the normalizer of $g$ in $G$ is cyclic or dihedral; by Theorem 3, $G$ is isomorphic to a subgroup of $SO(3)$, concluding the proof of Theorem 1.
Proof of Corollary 1. We give a short direct proof of Corollary 1 which uses only a small part of the preceding results.

The action of $G$ on $\mathbb{R}^3$ extends to an action on its 1-point compactification $S^3$ with a global fixed point. We shall verify the conditions of the second part of Theorem 3 for this action of $G$ on $S^3$.

Let $g$ be an element of $G$ generating a maximal cyclic subgroup $M$ of $G$. By Smith fixed point theory, the fixed point set of $g$ and $M$ is a single circle $K$. The normalizer $N$ of $g$ in $G$ maps $K$ to itself, hence every nontrivial element of $N$ fixes $K$ pointwise or acts as a reflection on $K$, and the subgroup $B$ of $N$ fixing $K$ pointwise has index one or two in $N$. By the first paragraph of the proof of Proposition 1, $B$ is cyclic; since $M$ is maximally cyclic, $B = M$, and hence $M$ has index one or two in its normalizer.

Moreover, if two maximal cyclic subgroups of $G$ have nontrivial intersection then they have the same circle $K$ as fixed point set, hence generate a cyclic subgroup of $G$ and are equal. Theorem 3 now implies that $G$ is isomorphic to a subgroup of $SO(3)$.

This concludes the proof of Corollary 1.

Finally, the Proof of Theorem 2 is analogous to the proof of the main Theorem in [MeZ2] (which uses the Gorenstein-Walter classification of the finite simple groups with dihedral Sylow 2-subgroups). The proof of [MeZ2, Theorem] depends on two lemmas; Lemma 1 in [MeZ2] is the version for smooth actions of Proposition 1 of the present paper, and Lemma 2 is again a consequence of the Borel formula, so both Lemma 1 and Lemma 2 of [MeZ2] hold for purely topological actions. The proof of Theorem 2 is now completely analogous to the proof of [MeZ2, Theorem]: by Gorenstein-Walter one reduces first to the linear fractional groups $\text{PSL}_2(q)$, for an odd prime power $q$, or the alternating group $A_7$ (these are the finite, nonabelian simple groups with dihedral Sylow 2-subgroups), and finally to $A_5 \cong \text{PSL}_2(5)$ by Proposition 1 and the purely topological argument in [Z2] (using also [MeZ1, section 6] to exclude the small groups $\text{PSL}_2(5^2)$ and $\text{PSL}_2(3^2) \cong A_6$).

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