Criteria for sharksfin and deltoid singularities from the plane into the plane and their applications

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Abstract
We give criteria for sharksfin and deltoid singularities from the plane into the plane. We also give geometric meanings for the conditions in the criterion of a sharksfin. As applications, we investigate such singularities appearing on an orthogonal projection of a Whitney umbrella, and a sharksfin appearing on planar motions with 2-degrees of freedom.

Keywords Criteria · Sharksfin · Deltoid · Whitney umbrella · Planar motions

Mathematics Subject Classification 57R45 · 58K05

1 Introduction
The singularities of maps from the plane into the plane have been one of the fundamental subjects in the singularity theory of the smooth maps. Classification of singularities of maps from the plane into the plane has been investigated by many researchers, and useful recognition criteria for main corank one singularities are given (see for example Bruce et al. 1987; Goryunov 1984; Kabata 2016; Ohmoto and Aicardi 2006; Rieger 1987; Rieger and Ruas 1991; Saji 2010; Whitney 1955). However, as long as the authors know, useful recognition criteria for corank two maps have not been given...
in the literature. In this paper, we give useful recognition criteria for “sharksfin” and “deltoid” singularities, which are the most generic singularities of corank two maps from the plane into the plane:

**Theorem 1.1** Let \(f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)\) be a map-germ satisfying \(\text{rank } df_0 = 0\), and let \(\lambda\) be an identifier of singularities of \(f\). Let \(\lambda\) have a non-degenerate critical point at 0, and let \(\eta_1, \eta_2\) be vector fields which satisfy that \(\eta_1(0)\) and \(\eta_2(0)\) are linearly independent solutions of the Hesse quadric of \(\lambda\) at 0. Then \(f\) is a sharksfin (respectively, deltoid) at 0 if and only if \(\det \text{Hess}(\lambda)(0) < 0\) (respectively, \(\det \text{Hess}(\lambda)(0) > 0\)),

\[
\det(\eta_1^2 f, \eta_1^3 f)(0) \neq 0 \quad \text{and} \quad \det(\eta_2^2 f, \eta_2^3 f)(0) \neq 0.
\]

Here, \(\eta f\) stands for the directional derivative of \(f\) by the vector field \(\eta\), and \(\eta^i f = \eta(\eta^{i-1} f)\). See Sect. 2 for the definitions of identifier of singularities and solutions of the Hesse quadric of a function at a non-degenerate critical point. We also give a geometric meaning of the condition (1.1) as follows: If an identifier of singularities has an index 1 non-degenerate critical point at 0, then the singular set \(S(f)\) consists of two transversal regular curves, say \(\gamma_i\) \((i = 1, 2)\). The condition (1.1) is then equivalent to both \(f \circ \gamma_i\) \((i = 1, 2)\) having \(3/2\)-cusps at 0 (Proposition 3.1).

As applications, we investigate the geometry of an orthogonal projection of a Whitney umbrella and singularities of planar motions with 2-degrees of freedom. Since maps \(\mathbb{R}^2 \to \mathbb{R}^2\) appear in several geometric situations (Gibson et al. 1998; Kabata 2016; Sinha and Tari 2018; Saji 2010; West 1995), the authors believe these criteria will work well if one wishes to find concrete conditions for a given map to be a sharksfin or a deltoid.

### 2 Preliminaries and proof of criteria

A map-germ \(f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)\) is called a sharksfin (respectively, deltoid) if it is \(\mathcal{A}\)-equivalent to the map-germ \((u, v) \mapsto (uv, u^2 + v^2 + u^3)\) (respectively, \((uv, -u^2 + v^2 + u^3)\)) at the origin 0. Here two map-germs \(f_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)\) \((i = 1, 2)\) are said to be \(\mathcal{A}\)-equivalent if there exist a diffeomorphism-germ \(\varphi\) on the source space of \(f_1\) and a diffeomorphism-germ \(\Phi\) on the target space of \(f_2\) such that \(\Phi \circ f_1 = f_2 \circ \varphi\) holds. We note that the map-germ \((u, v) \mapsto (uv, u^2 + v^2 + u^3)\) is \(\mathcal{A}\)-equivalent to \((u, v) \mapsto (u^2 + v^3, v^2 + u^3)\), and some of the literature uses this form for a sharksfin.

A fundamental classification of map-germs from the plane into the plane is given in Whitney (1955); Rieger (1987). In Rieger and Ruas (1991), it is shown that a sharksfin and a deltoid are the only singularities of rank zero and codimension equal to or less than four. See Kabata (2016); Saji (2010); Saji et al. (2009); Whitney (1955) for recognition criteria for fundamental rank one singularities.

Let \(f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)\) be a map-germ and let \(\text{rank } df_0 = 0\). Let \(\lambda\) be a non-zero functional multiple of the determinant of the Jacobi matrix \(J_f\) of \(f\). Since \(S(f) = \lambda^{-1}(0)\), we call \(\lambda\) an identifier of singularities. By the assumption \(\text{rank } df_0\) is zero, the function \(\lambda\) has a critical point at 0.
Let \( h : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0) \) be a function which has a non-degenerate critical point at 0. The solution of the Hesse quadric of \( h \) at 0 is a non-zero vector \((x, y)\) which satisfies
\[
h_{uu}(0)x^2 + 2h_{uv}(0)xy + h_{vv}(0)y^2 = 0,
\]
and \( \text{Hess}(h)(0) \) is the Hesse matrix of \( h \) at 0. If \( \text{det} \text{Hess}(h)(0) > 0 \), then there exist two \( \mathbb{C} \)-linearly independent \( \mathbb{C} \)-valued vectors \( v_1, v_2 \). If \( \text{det} \text{Hess}(h)(0) < 0 \), then there exist two \( \mathbb{R} \)-linearly independent \( \mathbb{R} \)-valued vectors \( v_1, v_2 \), and these are the tangent vectors of two branch curves of the zero set of \( h \) at 0. For a diffeomorphism \( \varphi : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0) \), since the Hesse matrix of \( h \circ \varphi^{-1} \) is \( J_{\varphi}(0)^{-1} \text{Hess}(h)(0)J_{\varphi}(0)^{-1} \), the two directions defined by solutions of the Hesse quadric of \( h \) at 0 do not depend on the choice of the coordinate system on the source space. Here, \( t_\varphi \) stands for the matrix transposition. We note that if solutions of the Hesse quadric \( \eta_i \) \((i = 1, 2)\) are complex vectors, then the left-hand sides of (1.1) may be complex numbers.

Now we give a proof of Theorem 1.1. We first simplify the expression of a rank zero-germ \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0) \) by coordinate changes. Keeping in mind that we will investigate geometric meanings, we restrict to using only particular coordinate changes in the following Lemma 2.1. The symbol \( O(i) \) stands for the terms whose degrees are greater than or equal to \( i \). A coordinate system is said to be positive if it has the same orientation as the standard \( \mathbb{R}^2 \).

**Lemma 2.1** If rank \( df_0 = 0 \) and 0 is a non-degenerate critical point of \( \lambda \), then there exist a positive local coordinate system \((u, v)\) near 0, and an orientation preserving isometry \( M \in \text{SO}(2) \) on \((\mathbb{R}^2, 0)\) such that \( M \circ f(u, v) = (uv, O(2)) \).

**Proof** Firstly we show that we may assume 0 is a non-degenerate critical point with index 1 of \( f_1 \), where \( f = (f_1, f_2) \). If \( f_1 = O(3) \) or \( f_2 = O(3) \), then the Jacobian of \( f \) is \( O(3) \). In particular, 0 will never be a non-degenerate critical point of an identifier of singularities. Thus, we may assume \( (f_1)_{uu}(0) \neq 0 \) by a suitable rotation on the target and by choosing a positive coordinate system on the source. By a rotation on the target, \( f \) can be written as \( f = (f_1, b_{11}uv + b_{02}v^2) + O(3) \). If \( b_{11} \neq 0 \), then the claim is easily proved. We assume \( b_{11} = 0 \). Since \( f_2 \neq O(3) \), it holds that \( b_{02} \neq 0 \). Thus \( f \) can be written as \( f = (a_{20}u^2 + 2a_{11}uv + a_{02}v^2, b_{02}v^2) + O(3) \). If \( a_{20}a_{02} - a_{11}^2 < 0 \), then the claim is proved, and if \( a_{20} = 0 \), then \( \lambda \) degenerates. So, we may assume \( a_{20}a_{02} - a_{11}^2 > 0 \) and \( a_{12} \neq 0 \). By taking the rotation by degree \( \theta \) on the target, the first component of \( f \) can be written as \( a_{20} \cos \theta u^2 + 2a_{11} \cos \theta uv + (a_{02} \cos \theta - b_{02} \sin \theta) v^2 \). One can find an angle \( \theta \) such that \( \cos \theta (\cos \theta (a_{20}a_{02} - a_{11}^2) - \sin \theta a_{20}b_{02}) < 0 \). This shows the claim.

Thus we may assume that 0 is a non-degenerate critical point with index 1 of \( f_1 \). By the Morse lemma, there exists a positive coordinate system \((u, v)\) such that \( f \) can be written as \( f = (\pm uv, O(2)) \). Taking a \( \pi \)-rotation on the target if necessary, this shows the assertion.

By Lemma 2.1, we see \( f \) is \( \mathcal{A} \)-equivalent to
\[
f(u, v) = \left( \begin{array}{c} \frac{u^2}{2} + \frac{v^2}{2} + \sum_{i+j=3} \frac{a_{ij}u^i v^j}{i!j!} + O(4) \end{array} \right) \quad (\varepsilon = \pm 1). \tag{2.1}
\]
Lemma 2.2 Let $f$ be a map-germ of the form (2.1). When $\varepsilon = 1$, then $f$ is a sharksfin if and only if $(a_{30} - 3a_{21} + 3a_{12} - a_{03})(a_{30} + 3a_{21} + 3a_{12} + a_{03}) \neq 0$. When $\varepsilon = -1$, then $f$ is deltoid if and only if $a_{30} - 3a_{12} \neq 0$ or $3a_{21} - a_{03} \neq 0$.

Proof Following Rieger and Ruas (1991)[Propositions 2.1.1 and 2.2.1], we will give a proof of this lemma. By the coordinate change $u = u_1 + a_{12}u_1^2/2 - \varepsilon a_{21}u_1v_1/2$, $v = v_1 - a_{12}u_1v_1/2 + \varepsilon a_{21}v_1^2/2$, we see that

$$f(u_1, v_1) = \left( \frac{\varepsilon u_1^2}{2} + \frac{v_1^2}{2} + \frac{a_{30}u_1^3}{6} + \frac{a_{03}v_1^3}{6} \right) + (O(4), O(4)), \quad (2.2)$$

where $\tilde{a}_{30} = a_{30} + 3\varepsilon a_{12}$ and $\tilde{a}_{03} = a_{03} + 3\varepsilon a_{21}$. Let $f$ be written as in (2.2), and $\varepsilon = 1$. By the coordinate change

$$u_1 = u_2 - v_2 + (\tilde{a}_{03} - 2\tilde{a}_{30})u_2^2/12 + \tilde{a}_{30}u_2v_2/4 - (\tilde{a}_{03} + 2\tilde{a}_{30})v_2^2/12,$$

$$v_1 = u_2 + v_2 + (\tilde{a}_{30} - 2\tilde{a}_{03})u_2^2/12 - \tilde{a}_{03}u_2v_2/4 - (2\tilde{a}_{03} + \tilde{a}_{30})v_2^2/12$$

and $\Phi(X, Y) = (X + Y, -X + Y)/2$, we see

$$\Phi \circ f(u_2, v_2) = \left( \frac{u_2^2 + \left( \frac{\tilde{a}_{03} - \tilde{a}_{30}}{12} \right) u_2^3}{2} + \frac{\tilde{a}_{03} + \tilde{a}_{30}}{12} v_2^3 \right) + (O(4), O(4)). \quad (2.3)$$

Since a sharksfin is 3-determined (Rieger and Ruas 1991[Proposition 2.1.1]), $f$ in the form (2.1) is a sharksfin if and only if $(\tilde{a}_{03} - \tilde{a}_{30})(\tilde{a}_{03} + \tilde{a}_{30}) \neq 0$, namely, $(a_{30} - 3a_{21} + 3a_{12} - a_{03})(a_{30} + 3a_{21} + 3a_{12} + a_{03}) \neq 0$.

Next, we show the case of a deltoid. Let $f$ be written as in (2.2) and $\varepsilon = -1$. Let $p(x)$ be the polynomial $p(x) = \tilde{a}_{03}x^5 - 5\tilde{a}_{30}x^4 - 10\tilde{a}_{03}x^3 + 10\tilde{a}_{30}x^2 + 5\tilde{a}_{03}x - \tilde{a}_{30}$, and let $x_0$ be one of the solutions of $p(x) = 0$. We consider the coordinate change defined by

$$u_1 = u_2 - x_0v_2 + (u_2^2(-5\tilde{a}_{30}(-3 + x_0^2) + \tilde{a}_{03}x_0(5 + x_0^2)) + u_2v_2(\tilde{a}_{03}x_0^2(7 - x_0^2)) + 5\tilde{a}_{30}x_0(-3 + x_0^2)) + v_2^2(\tilde{a}_{03}(-9 + 35x_0^2) + x_0(\tilde{a}_{03}(13 - 7x_0^2))))/96,$$

$$v_1 = u_2x_0 + v_2 + (u_2^2x_0(5\tilde{a}_{30}(-3 + x_0^2) - \tilde{a}_{03}x_0(5 + x_0^2)) - 4u_2v_2(\tilde{a}_{03}x_0(3 - 5x_0^2)) + \tilde{a}_{03}x_0(5 + x_0^2)) + v_2^2x_0(\tilde{a}_{30}(57 - 35x_0^2) + \tilde{a}_{03}x_0(-61 + 7x_0^2))))/96,$$

and we consider

$$\Phi(X, Y) = \left( \frac{(1 - x_0^2)X + 2x_0Y}{(1 + x_0^2)^2}, - \frac{2(x_0^2 - 1)(2x_0X + (x_0^2 - 1)Y)}{(1 + x_0^2)^2(x_0^2 - 1)} \right).$$
Then the terms of $\Phi \circ f$ whose orders are less than or equal to three are
\[
\begin{align*}
(u_2 v_2, -u_2^2 + v_2^2 + \frac{(\tilde{a}_{03}^2 + \tilde{a}_{30}^2)(1 - 10x_0^2 + 5x_0^4)}{3\tilde{a}_{03}(1 + x_0^2)^2}v_2) \\
+ \frac{p(x_0)}{96(1 + x_0^2)^2}\left(-2x_0u_2^3 + 3(x_0^2 - 1)u_2^2v_2 + 6x_0u_2v_2^2 + (9 + 7x_0^2)v_3^2,
2((x_0^2 - 1)u_2^3 + 6x_0u_2v_2^2 - 3(x_0^2 - 1)u_2v_2^2 + 2(8\tilde{a}_{30}/\tilde{a}_{03} + 7x_0)v_3^2)\right).
\end{align*}
\] (2.4)

The solutions of $5x^4 - 10x^2 + 1 = 0$ are $x_1 = \pm(1 \pm 2/\sqrt{5})^{1/2}$, and we see $5p(x_1) = \pm 16\tilde{a}_{03}(10 \pm 22/\sqrt{5})^{1/2}$ is not zero if $\tilde{a}_{03} \neq 0$. Since a deltoid is 3-determined (Rieger and Ruas 1991 [Proposition 2.2.1]), $f$ in the form (2.1) is a deltoid if and only if $\tilde{a}_{30} \neq 0$ or $\tilde{a}_{03} \neq 0$, namely, $a_{30} - a_{12} \neq 0$ or $3a_{21} - a_{03} \neq 0$.

**Lemma 2.3** Let $0$ be a non-degenerate critical point of an identifier of singularities $\lambda$ of $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$, and let $(\eta_{11}, \eta_{12}), (\eta_{21}, \eta_{22})$ (possibly complex) be two linearly independent solutions of the Hesse quadric of $\lambda$ at $0$. Let $\eta_i$ be a vector field satisfying $\eta_i(0) = \eta_{i1}\partial_u + \eta_{i2}\partial_v$ ($i = 1, 2$). Then the condition $\det(\eta_1^2 f, \eta_1^3 f)(0) \neq 0$ and $\det(\eta_2^2 f, \eta_2^3 f)(0) \neq 0$ does not depend on the choice of $\eta_1, \eta_2$.

**Proof** The condition clearly does not depend on the choice of the coordinate system on the source space, and the condition does not change by a linear coordinate transformation on the target space. As we remarked just after Lemma 2.1, we may assume $f$ is written as in (2.1). Then the identifier of singularities is $-\varepsilon u^2 + v^2 + O(3)$. Since the condition does not change under a constant multiplication of the solutions of the Hesse quadric $\eta_1$ and $\eta_2$ of an identifier of singularities, we may assume $\eta_1 = (1, 1), \eta_2 = (1, -1)$ if $\varepsilon = 1$, and $\eta_1 = (1, i), \eta_2 = (1, -i)$ if $\varepsilon = -1$, where $i = \sqrt{-1}$. By a direct calculation, $\det(\xi^2 f, \xi^3 f)(0)/2$ is
\[
\begin{align*}
\begin{cases}
 a_{30} + 3a_{21} + 3a_{12} + a_{03} & (\varepsilon = 1, \xi(0) = \eta_1 = (1, 1)) \\
-a_{30} + 3a_{21} - 3a_{12} + a_{03} & (\varepsilon = 1, \xi(0) = \eta_2 = (1, -1)) \\
 a_{30}i - 3a_{21} - 3ia_{12} + a_{03} & (\varepsilon = -1, \xi(0) = \eta_1 = (1, i)) \\
-a_{30}i - 3a_{21} + 3ia_{12} + a_{03} & (\varepsilon = -1, \xi(0) = \eta_2 = (1, -i)),
\end{cases}
\end{align*}
\] (2.5)

and these depend only on the value of $\xi(0)$. This shows the assertion.

**Proof of Theorem 1.1** The sufficiency follows by the independence of the choice of coordinate systems and vector fields. We show the necessity. We assume the assumption of Theorem 1.1. Since the condition does not depend on the choice of coordinate systems and vector fields, we may assume $f$ is written as in (2.1). Then by (2.5) in the proof of Lemma 2.3, the condition (1.1) in Theorem 1.1 is equivalent to $(a_{30} + 3a_{21} + 3a_{12} + a_{03})(-a_{30} + 3a_{21} - 3a_{12} + a_{03}) \neq 0$ when $\varepsilon = 1$, and $a_{30} - a_{12} \neq 0$ or $3a_{21} - a_{03} \neq 0$ when $\varepsilon = -1$. By Lemma 2.2, we have the assertion.
3 Geometry of a sharksfin and a deltoid

3.1 Geometric meanings of a criterion for a sharksfin

We here give a geometric interpretation of condition (1.1). Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) be a map-germ with rank \( df_0 = 0 \), and let an identifier of singularities \( \lambda \) have an index one critical point at 0. Then the set \( \lambda^{-1}(0) \) consists of images of two transversal regular curves passing through 0. We set these curves as \( (R, 0) \to (\mathbb{R}^2, 0) \). A curve-germ \( c : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0) \) at 0 is a 3/2-cusp if it is \( A \)-equivalent to \( t \mapsto (t^2, t^3) \). It is well-known that \( c : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0) \) is a 3/2-cusp if and only if \( c'(0) = 0 \), and \( \det (c''(0), c'''(0)) \neq 0 \). Let \( c : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0) \) be a curve and 0 a 3/2-cusp. The cuspidal direction of \( c \) at 0 is the direction defined by \( c''(0) \). The cuspidal direction bisects the cusp. Then the following proposition holds.

**Proposition 3.1** Under the above setting, \( f \) at 0 is a sharksfin if and only if \( f \circ \gamma_i \) \((i = 1, 2) \) at 0 are both 3/2-cusps.

**Proof** Since the condition and the assertion do not depend on the choice of the coordinate systems, we may assume that \( f \) is written as (2.1) with \( \varepsilon = 1 \). Then we may assume that \( \gamma_1 = \gamma_+ = (t, a_+(t)), \gamma_2 = \gamma_- = (t, a_-(t)) \). Since \( \lambda(t, a_0(t)) = 0 \), we have \( a'_\pm(0) = \pm 1 \). We set \( \gamma_i(t) = f \circ \gamma_i(t) \). Then we see \( \gamma''_\pm(0) = 2(\pm 1, 1), \gamma'''_i(0) = (3a''_\pm(0), 3a'''_\pm(0)) = 3a_0 + 3a_{21} \pm 3a_{12} + a_{03} \). By (2.5), we have the assertion.

**3.2 An SO(2)-normal form**

We give a simplified form of a given rank zero germ by using diffeomorphisms on the source and isometries on the target space. Since coefficients of such forms are differential geometric invariants, this is convenient for studying the differential geometry of singularities. See Bruce and West (1998); Fukui and Hasegawa (2012); Fukui (2020); Hasegawa et al. (2014); Martins and Saji (2016); Sinha and Saji (Sinha and Saji); Sinha and Tari (2018); Tari (2007); West (1995) for such studies, for example. Two map-germs \( f_j : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) are said to be \( \mathcal{R}_+ \times \text{SO}(2) \)-equivalent if there exist an orientation preserving diffeomorphism-germ \( \varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) and an orientation preserving isometry \( M \) on \( (\mathbb{R}^2, 0) \), namely, \( M \in \text{SO}(2) \), such that \( M \circ f_1 = f_2 \circ \varphi \) holds.

**Proposition 3.2** Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) be a map-germ with rank \( df_0 = 0 \), and let an identifier of singularities \( \lambda \) have a non-degenerate critical point at 0. Then \( f \) is \( \mathcal{R}_+ \times \text{SO}(2) \)-equivalent to the germ

\[
\left( uv, \frac{\varepsilon_1 a_2 u^2}{2} + \frac{\varepsilon_2 a_2 v^2}{2} + \frac{a_3 u^3}{6} + \frac{a_0 v^3}{6} + (O(4), O(4)), \right.\\
\left. (a_2 > 0, (\varepsilon_1, \varepsilon_2) \in \{(1, 1), (1, -1), (-1, 1)\}) \right) \quad (3.1)
\]
Lemma 3.3 Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ be a map-germ satisfying the condition in Proposition 3.2. Then $f$ is $\mathcal{R}_+ \times \text{SO}(2)$-equivalent to the germ $(uv, a_20u^2/2 + a_{02}v^2/2 + O(3))$ $(a_20a_{02} \neq 0)$.

Proof By Lemma 2.1 and by a dilatation for $u, v$, we may assume $f$ is given by $(uv, \varepsilon_1a_20u^2/2 + a_{11}uv + \varepsilon_2a_{20}v^2/2 + O(3))$ $(a_20 \neq 0, \varepsilon_1, \varepsilon_2 \in \{1, -1\})$. If $a_{11} = 0$, then no proof is needed. We assume $a_{11} \neq 0$. We set $u = u_1 + v_1, v = cu_1 + dv_1 (c, d \in \mathbb{R})$, and $M$ is the rotation matrix by degree $\theta$. Then $M \circ f(u, v)$ is

\[
(A_1u^2/2 + *uv + A_2v^2/2, *u^2 + A_3uv + *v^2) + (O(3), O(3)),
\]

where $*$ stands for a real number which will not be needed in later calculations, and

\[
A_1 = 2c \cos \theta - (a_{20} + 2a_{11}c + a_{20}c^2) \sin \theta, \quad A_2 = 2d \cos \theta - (a_{20} + 2a_{11}d + a_{20}d^2) \sin \theta, \quad A_3 = (a_{20} + a_{20}cd + a_{11}(c + d)) \cos \theta + (c + d) \sin \theta.
\]

To show the lemma, we need to solve the equation $A_1 = A_2 = A_3 = 0$ with respect to $c, d, \theta$. If $\sin \theta = 0$, then $c = d = 0$, and the equation $A_3 = 0$ is $a_{20} = 0$. Thus $\sin \theta = 0$ is not a solution. We assume $\sin \theta \neq 0$. Noticing the function $\cos \theta / \sin \theta$ takes values in $\mathbb{R}$, we set $t = \cos \theta / \sin \theta$. Then the equations $A_1 = 0$ and $A_2 = 0$ are the same. Thus the solutions $c, d$ are the two solutions of the equation

\[
-a_{20}x^2 + 2(-a_{11} + t)x - a_{20} = 0
\]

with respect to $x$. We set $c, d$ are these two solutions satisfying $d - c > 0$, where we will see (3.3) has distinct real roots. Hence $c + d = 2(-a_{11} + t)/a_{20}$ and $cd = 1$. Substituting this into $A_3 = 0$, we obtain one of the solutions

\[
t = \frac{-(1 - a_{11}^2 + a_{20}^2) + \sqrt{(1 - a_{11}^2 + a_{20}^2)^2 + 4a_{11}^2}}{2a_{11}},
\]

where $(1 - a_{11}^2 + a_{20}^2)^2 + 4a_{11}^2 > 0$ is obvious. Substituting this into $A_1$ and $A_2$, we obtain solutions $c$ and $d$. By the inequality $-(a_{11} + t)^2 - a_{20}^2 > (1 - a_{11}^2 + a_{20}^2)^2 + 4a_{11}^2)$, we see that the equation (3.3) has two distinct real roots under (3.4). Since 0 is a non-degenerate critical point of $\lambda$, neither the coefficients of $uv$ in the first component nor those of $u^2, v^2$ in the second component vanish. By a positive dilatation of $u$ and $v$ with a $\pi$-rotation in the target, we may assume the coefficient of $uv$ in the first component is one. So we may assume $f$ is written as $f = (uv + O(3), a_20u^2/2 + a_{02}v^2/2 + O(3))$ $(a_20a_{02} \neq 0)$. By the Morse lemma, there exists a coordinate change $u = au_1 + bv_1 + O(2), v = cu_1 + dv_1 + O(2)$ such that $u_1v_1$ is equal to the first component of $f$. Then we have $ac = bd = 0$, and we may assume $b = c = 0$, since the case $a = d = 0$ is similar. If the coordinate change reverses the orientation, we compose with $(u, v) \mapsto (-u, v)$. This proves the assertion.
Proof of Proposition 3.2 By Lemma 3.3, we may assume $f$ is written as $(uv, a_{20}u^2/2 + a_{02}v^2/2 + \sum_{i+j=3} a_{ij}u^i v^j / (i! j!) + O(4)) (a_{20}a_{02} \neq 0)$. By the coordinate change

$$u = u_1 + \frac{a_{12}u_1^2}{2a_{02}} - \frac{a_{21}u_1v_1}{2a_{20}}, \quad v = v_1 - \frac{a_{12}u_1v_1}{2a_{02}} + \frac{a_{21}v_1^2}{2a_{20}},$$

$f$ is written as $(u_1 v_1, a_{20}u_1^2/2 + a_{02}v_1^2/2 + \tilde{a}_{30}u_1^3/6 + \tilde{a}_{03}v_1^3/6) + (O(4), O(4))$. By setting $u_1 = |a_{02}/a_{20}|^{1/4}u_2$, $v_1 = |a_{20}/a_{02}|^{1/4}v_2$, and taking $(u, v) \mapsto (v, -u)$ and $\pi$-rotation on the target if $a_{20} < 0$ and $a_{02} < 0$, we see the assertion. \qed

We remark that the number of coefficients of the terms whose degrees are less than or equal to 3 in the form (3.1) in Proposition 3.2 is 3. These are $a_{20}$, $a_{30}$ and $a_{03}$. However, coefficients in the first component remain. When considering higher degree terms, it should be remarked that the form in Lemma 3.3 is convenient, since its first component is just $uv$.

Here we study geometric meaning of the coefficients of the SO(2)-normal form. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a sharksfin. In Proposition 3.1, we showed that the images of two branch curves of $S(f)$ are both $(2, 3)$-cusps. Here we give an interpretation of the coefficients $a_{20}, a_{30}, a_{03}$ in the SO(2)-normal form (3.1) by geometries of these $(2, 3)$-cusps. As in the proof of Proposition 3.1, we denote two branch curves of $S(f)$ by $\gamma_i (i = 1, 2)$ and set $\hat{\gamma}_i = f \circ \gamma_i$. By the proof of Proposition 3.1, the cuspidal directions of $\hat{\gamma}_i (i = 1, 2)$ at 0 are linearly independent. Thus the angle between two cuspidal directions of $\hat{\gamma}_i (i = 1, 2)$ is a geometric invariant of $f$. On the other hand, let $c : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a $3/2$-cusp. The cuspidal curvature of $c$ at 0 is defined by $\kappa_{cusp}(c) = \det(c''(0), c'''(0))|c'''(0)|^{-5/2}$. The cuspidal curvature measures the wideness of the $3/2$-cusp. See Saji et al. (2010); Umehara (2005) for details. Since $\hat{\gamma}_i (i = 1, 2)$ have $3/2$-cusps at 0, one can compute the cuspidal curvatures. We may assume that $f$ is written as in (3.1). Then in the same notation and by the same arguments as in the proof of Proposition 3.1, we have $a_{i\pm} = \pm t + (a_{30} - a_{03})t^2/(4a_{20}) + O(3)$. By a direct calculation, $\kappa_{cusp}(\hat{\gamma}_\pm) = 2(\pm a_{30} + a_{03})(4 + 4a_{20}^{-1})^{-5/4}$, and the angle $\theta_\gamma$ between the two cuspidal directions of $\hat{\gamma}_i (i = 1, 2)$ is $\cos^{-1}\left(|a_{20}^2 - 1|/(a_{20}^2 + 1)\right)$. By these formulas and $a_{20} > 0$, together with Proposition 3.2, the cuspidal curvatures of $\hat{\gamma}_\pm$ and the angle between the two cuspidal directions determine the sharksfin up to three degrees, namely, $\kappa_{cusp}(\hat{\gamma}_\pm)$ and $\theta_\gamma$ determine the $\mathcal{R}_+ \times \text{SO}(2)$-class of sharksfin up to 3-jets. It is known that $f(S(f))$ determines the $\mathcal{R}$-class of a sharksfin, since it is a critical normalization du Plessis et al. (1997); Wirzhmüller (1980). On the other hand, a deltoid is not a critical normalization, and we cannot find such invariants in terms of $f(S(f)) = \{0\}$.

3.3 Projection of a Whitney umbrella

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a Whitney umbrella or equivalently, a cross cap, namely, it is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, uv)$. A line generated by $df_0(T \mathbb{R}^2)$ is called the tangent line of $f$, and a plane $P$ perpendicular to the tangent line is called the normal plane. Let $\pi : \mathbb{R}^3 \to P$ be the orthogonal projection. It is known that...
if $f$ is a Whitney umbrella, then $\pi \circ f$ is a sharksfin or a deltoid, generically. More precisely, there exist a coordinate system $(u, v)$ and a rotation $\Phi$ on $\mathbb{R}^3$ such that

$$f(u, v) = \left(u, uv + \frac{c_3 v^3}{6}, \sum_{i+j=2}^3 \frac{d_{ij} u^i v^j}{i! j!}\right) + (0, O(4), O(4)), \quad (3.5)$$

where $c_3, d_{02} > 0$, $d_{ij} \in \mathbb{R}$. See West (1995) or Fukui and Hasegawa (2012). A Whitney umbrella is said to be elliptic (respectively, hyperbolic) if $d_{20} > 0$ (respectively, $d_{20} < 0$). See Bruce and West (1998); Fukui and Hasegawa (2012); Hasegawa et al. (2014); Tari (2007); West (1995) for the geometric meanings of other coefficients. Considering the orthogonal projection with respect to the tangent line into the normal plane, a rank zero singular point appears. A sharksfin (respectively, deltoid) appears on the projection of an elliptic (respectively, hyperbolic) Whitney umbrella. We give a precise condition for this fact in terms of the coefficients $c_3, d_{ij} (i + j = 2, 3)$ in (3.5), namely, geometric information of the Whitney umbrella.

**Theorem 3.4** Let $f$ be a Whitney umbrella written in the form (3.5) with $d_{20} \neq 0$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the orthogonal projection $(X_1, X_2, X_3) \mapsto (X_2, X_3)$. Then $\pi \circ f$ at 0 is a sharksfin if and only if $d_{20} > 0$ and

$$d_{30}\tilde{a}^3_{02} + 3\delta_{21}\tilde{a}_{20}\tilde{a}^2_{02} + (3d_{12}\tilde{a}^2_{20} - c_3\tilde{a}_{20}^4)\tilde{a}_{02} + \delta(d_{03} - d_{11}c_3)\tilde{a}_{20}^3 \neq 0 \quad (3.6)$$

hold for both $\delta = 1$ and $\delta = -1$, where $d_{20} = \tilde{d}_{20}^2$, $d_{02} = \tilde{d}_{02}^2$. On the other hand, $\pi \circ f$ at 0 is a deltoid if and only if $d_{20} < 0$ and

$$d_{30}\tilde{d}_{20}^3 - 3d_{12}\tilde{d}_{20}^2 - c_3\tilde{d}_{20}^4 \neq 0 \quad \text{or} \quad d_{03}\tilde{d}_{20}^2 - 3d_{21}\tilde{d}_{02}^2 - d_{11}c_3\tilde{d}_{20}^2 \neq 0, \quad (3.7)$$

where $d_{20} = -\tilde{a}_{20}^2$, $d_{02} = \tilde{d}_{02}^2$.

We remark that the $\pm$-ambiguity of $\tilde{a}_{20}^2$, $\tilde{d}_{02}^2$ is included by the condition (3.6) holding for both $\delta = \pm 1$.

**Proof** Let $\lambda$ be an identifier of singularities of $\pi \circ f$. If $d_{20} > 0$ (note that $d_{02} > 0$ in (3.5)) then $\lambda = -\tilde{a}_{20}^2u^2 + \tilde{d}_{02}^2 v^2$. In this case, we set $\eta_1 = \eta_+ = (\tilde{d}_{02}, \tilde{a}_{20})$, $\eta_2 = \eta_- = (-\tilde{d}_{02}, \tilde{a}_{20})$. Then we see

$$\eta_+ \eta_-(\pi \circ f)(0) = (2\tilde{d}_{20}\tilde{d}_{02}, 2\tilde{a}_{20}\tilde{d}_{02}(d_{11} + \tilde{a}_{20}\tilde{d}_{02})),
\eta_\pm \eta_\pm (\pi \circ f)(0) = (c_3\tilde{d}_{20}^3, d_{30}\tilde{d}_{02}^3 + 3d_{21}\tilde{d}_{02}\tilde{d}_{20}^2 + 3d_{12}\tilde{d}_{20}\tilde{d}_{02} + d_{03}\tilde{d}_{20}^3).$$

By a direct calculation, we have the assertion. One can obtain the case of $d_{20} < 0$, under a similar calculation by setting $\eta_1 = \eta_+ = (i\tilde{d}_{02}, \tilde{a}_{20})$, $\eta_2 = \eta_- = (-i\tilde{d}_{02}, \tilde{a}_{20})$, where $i = \sqrt{-1}$. \qed
The SO(2)-normal form of \( \pi \circ f \) is (we allow \((\varepsilon_1, \varepsilon_2) = (-1, -1) \) in (3.1))

\[
\begin{pmatrix}
    \omega v, & w_3 \\
    2|w_1| & w_3
\end{pmatrix}
\begin{pmatrix}
w_2 \\
w_2
\end{pmatrix}
\begin{pmatrix}
1/2 u^2 + w_2 & w_3 \left( \frac{1}{2} \omega^2 + \frac{1}{6} w_4 u^3 + \frac{5}{6} \omega^3 \right)
\end{pmatrix}
\begin{pmatrix}
w_1 = -2((d_{11} - \cot \theta) \cos \theta + (-d_{11}^2 + d_{02}d_{20} + d_{11} \cot \theta) \sin \theta)/d_{02}, \\
w_2 = 2(d_{11} + \cot \theta - x_1)(\cot \theta \cos \theta + \sin \theta)/d_{02}, \\
w_3 = 2(-d_{11} + \cot \theta + x_1)(\cot \theta \cos \theta + \sin \theta)/d_{02},
\end{pmatrix}
\]
\[
\cot \theta = \frac{-1 + d_{11}^2 - d_{02}d_{20} + \sqrt{4d_{11}^2 + (1 - d_{11}^2 + d_{02}d_{20})^2}}{2d_{11}}.
\]

One can obtain \(w_4\) and \(w_5\) by following the procedure of the proof of Proposition 3.2. However, it is quite complicated, and we do not state it here.

### 3.4 Planar motions

As an application of the criteria, we give a concrete condition that a singular point appearing on a planar motion is a sharksfin.

Let \(S^1\) be the 1-dimensional torus \(S^1 = R/2\pi \mathbf{Z}\), and let \(SE(2) = R^2 \times SO(2)\) be the 3-dimensional Lie group. We take a map-germ \(\alpha : (R^2, 0) \to SE(2)\), which is called a planar motion-germ with 2-degrees of freedom in the context here, see Gibson et al. (1998) for detail. Take a point \(\omega \in R^2\), and set a map-germ \(ev_\omega : (SE(2), (a_0, A_0)) \to R^2\) by \(ev_\omega(a, A) = A\omega + a\). Then the composition \(ev_\omega \circ \alpha : (R^2, 0) \to R^2, ev_\omega \circ \alpha(s) = A(s)\omega + a(s)\) traces the point \(w\) by the action of \(\alpha(s)\), and is called a trajectory of \(\omega\) by \(\alpha\), where \(\alpha(s) = (a(s), A(s)) \in SE(2)\). In Gibson et al. (1998), a generic classification of singularities of \(ev_\omega \circ \alpha\) at 0 is given when \(\alpha\) at 0 is a Whitney umbrella, namely, it is \(A\)-equivalent to \((u, v) \mapsto (u, v^2, uv)\).

Here, we consider a special case of the above motion. Let \(\alpha : (R, 0) \to SE(2)\) and \(\beta : (R, 0) \to SE(2)\) be two curves. Then the composite motion of \(\alpha\) and \(\beta\) is defined by \(\beta(v)\alpha(u) : (R, 0) \times (R, 0) \to SE(2)\) where the product \(\beta(v)\alpha(u)\) is that of \(SE(2)\). Composite motions are planar motions with 2-degrees of freedom. In Gibson et al. (1998), a generic classification of singularities of \(ev_\omega \circ \nu\) at 0 is given, where a sharksfin is in the classification. It is also shown that a deltoid never appears on \(ev_\omega \circ \nu\). We give a concrete condition for \(ev_\omega \circ \nu\) to be a sharksfin in terms of the geometry of \(\omega\) and \(\alpha, \beta\), by using our criterion (Theorem 1.1) when \(\alpha : (R, 0) \to SE(2, (0, E))\) and \(\beta : (R, 0) \to SE(2, (0, E))\) have singular points at 0. We identify \(SO(2)\) with \(S^1\) and we set \(\alpha(u) = ((\tilde{a}_1(u), \tilde{a}_2(u)), \tilde{p}(u))\) and \(\beta(u) = ((\tilde{b}_1(u), \tilde{b}_2(u)), \tilde{q}(u))\), where \((\tilde{a}_1(u), \tilde{a}_2(u)), (\tilde{b}_1(u), \tilde{b}_2(u)) \in R^2, \tilde{p}(u), \tilde{q}(v) \in S^1\).

By the assumption, we can write \(\alpha(u) = ((u^2a_1(u), u^2a_2(u), u^2p(u))\) and \(\beta(v) = (v^2b_1(v), v^2b_2(v)), v^2q(v))\).

**Theorem 3.5** We set \(W = \{(1, w_2, -w_1), V_1(u) = \{(p(u), a_1(u), a_2(u))\) and \(V_2(v) = \{(q(v), b_1(v), b_2(v))\). Under the above notation, 0 is a corank 2 singular point of \(f = ev_\omega \circ \nu\). Furthermore, \(ev_\omega \circ \nu\) at 0 is a sharksfin if and only if
\[ \text{det}(W, V_1(0), V_2(0)) \neq 0, \ \text{det}(W, V_1(0), V_1'(0)) \neq 0, \ \text{det}(W, V_2(0), V_2'(0)) \neq 0. \]

We remark that the curve \( \alpha = (u^2a_1(u), u^2a_2(u)) \) at 0 is a 3/2-cusp if and only if \((a_1(0), a_1'(0))\) and \((a_2(0), a_2'(0))\) are linearly independent.

**Proof** The first assertion is obvious by

\[
f = \left( v^2b_1(v) + u^2a_1(u) \cos(v^2q(v)) + w_1 \cos(u^2p(u) + v^2q(v)) \right. \\
- u^2a_2(u) \sin(v^2q(v)) - w_2 \sin(u^2p(u) + v^2q(v)), \\
v^2b_2(v) + u^2a_2(u) \cos(v^2q(v)) + w_2 \cos(u^2p(u) + v^2q(v)) \\
+ u^2a_1(u) \sin(v^2q(v)) + w_1 \sin(u^2p(u) + v^2q(v)) \right). 
\]

Since \( \det J_f \) has the factor \( uv \), we see \( \text{Hess}(\det J_f)(0,0) \) is anti-diagonal, and the both anti-diagonal components are \( \det(W, V_1(0), V_2(0)) \). Thus \( \eta_1 = \partial_u \) and \( \eta_2 = \partial_v \) are the solutions of the Hesse quadric. By \( f_{uuu}(0,0) = 2(a_1(0) - w_2p(0), a_2(0) + w_1p(0)) \), \( f_{uuu}(0,0) = 6(a_1'(0) - w_2p'(0), a_2'(0) + w_1p'(0)) \) and by the same calculation for \( f_{uv}, f_{uuv}, f_{uvu} \), we have \( \det(f_{uu}(0,0), f_{uuu}(0,0)), \det(f_{uv}(0,0), f_{uvu}(0,0)) \). This yields the assertion. \( \square \)

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