Dressing Symmetries of Holomorphic BF Theories

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Abstract

We consider holomorphic BF theories, their solutions and symmetries. The equivalence of Čech and Dolbeault descriptions of holomorphic bundles is used to develop a method for calculating hidden (nonlocal) symmetries of holomorphic BF theories. A special cohomological symmetry group and its action on the solution space are described.

PACS: 11.15.-q; 11.30.-j; 02.20.Tw

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1 Introduction

Let $Z$ be a complex $n$-dimensional manifold, $G$ a (complex) semisimple matrix Lie group, $\mathfrak{g}$ its Lie algebra, $P$ a principal $G$-bundle over $Z$, $A$ a connection 1-form on $P$ and $F_A = dA + A \wedge A$ its curvature. Consider the following action:

$$S_{hBF} = \int_Z \text{Tr} (B \wedge F_A^{0,2}),$$

(1.1)

where $B$ is an $\text{ad}P$-valued $(n, n-2)$-form on $Z$, $\text{ad}P := P \times_G \mathfrak{g}$, and $F_A^{0,2}$ is the $(0,2)$-component of the curvature tensor $F_A$. The field equations for the action (1.1) are

$$\bar{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = 0,$$

(1.2a)

$$\bar{\partial} B + A^{0,1} \wedge B - B \wedge A^{0,1} = 0,$$

(1.2b)

where $\bar{\partial}$ is the $(0,1)$-part of the exterior derivative $d = \partial + \bar{\partial}$ and $A^{0,1}$ is the $(0,1)$-component of a connection 1-form $A = A^{1,0} + A^{0,1}$ on $P$.

Notice that Eqs.(1.2a) coincide with the compatibility conditions $F_A^{0,2} = \bar{\partial}^2 = 0$ of Eqs.(1.2b), $\bar{\partial} A = \bar{\partial} + A^{0,1}$. It follows from Eqs.(1.2) that models (1.1) describe holomorphic structures $\bar{\partial} A$ on bundles over complex $n$-manifolds $Z$ and $\bar{\partial} A$-closed $\text{ad}P$-valued $(n, n-2)$-forms $B$ on $Z$. So, theories with the action (1.1) generalize topological BF theories [1, 2] which give a field-theoretic description of flat connections on bundles over real $n$-manifolds. Models (1.1) can also be considered as a generalization of holomorphic Chern-Simons-Witten theories [3, 4] defined in three complex dimensions. Theories with the action (1.1) have been introduced in [5] and called holomorphic BF theories. They can be useful in describing invariants of complex manifolds (Ray-Singer holomorphic torsion and the others) and compactified configurations in superstring theory [6]. We believe holomorphic BF theories (hBF) deserve further developing.

The purpose of the present paper is to describe a procedure of constructing solutions to Eqs.(1.2) and mappings of solutions into one another (dressing transformations). We describe a parametrization of solutions to Eqs.(1.2) by transition functions of topologically trivial holomorphic bundles and elements of Dolbeault cohomology groups. We show that all (dressing) symmetries of Eqs.(1.2) can be calculated with the help of homological algebra methods.

2 Field equations of hBF theories and their solutions

2.1 Flat (0,1)-connections and functional matrix equations

We consider a complex $n$-manifold $Z$, a principal $G$-bundle $P$ over $Z$ and a connection 1-form $A$ on $P$. The curvature $F_A = dA + A \wedge A$ of a connection $A$ splits into components,

$$F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2},$$

and the $(0,2)$-component of the curvature tensor is

$$F_A^{0,2} = \bar{\partial}^2 = \bar{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1}.$$
The (0,1)-component \( A^{0,1} \) of a connection 1-form \( A = A^{1,0} + A^{0,1} \) will be called the (0,1)-connection.

For simplicity we shall consider a trivial \( G \)-bundle \( P_0 \cong \mathbb{Z} \times G \). Then for the adjoint bundle \( \text{ad} \, P_0 = P_0 \times_G \mathfrak{g} \) we have \( \text{ad} \, P_0 \cong \mathbb{Z} \times \mathfrak{g} \), where \( \mathfrak{g} \) is the Lie algebra of a group \( G \). Generalization to the case of nontrivial bundles is straightforward and not difficult. We denote by

\[
\Omega^{p,q}(\mathbb{Z}, \mathfrak{g})
\]

the space of \( \mathfrak{g} \)-valued smooth \((p,q)\)-forms on \( \mathbb{Z} \). Taking a form \( B \in \Omega^{n,n-2}(\mathbb{Z}, \mathfrak{g}) \), we introduce the action (1.1) of holomorphic BF theory and consider field equations (1.2).

Solutions of Eqs.(1.2a) are flat \((0,1)\)-connections \( A^{0,1} \). They can be described in different ways. To show this, let us fix a covering \( \mathcal{U} = \{ U_\alpha \} \) of a complex manifold \( Z \), \( \alpha \in I \). Then consider a manifold

\[
\mathcal{U}^{(0)} \equiv Z = \bigcup_{\alpha \in I} U_\alpha,
\]

and the subsets

\[
\mathcal{U}^{(1)} = \bigcup_{\alpha, \beta \in I} U_\alpha \cap U_\beta,
\]

\[
\mathcal{U}^{(2)} = \bigcup_{\alpha, \beta, \gamma \in I} U_\alpha \cap U_\beta \cap U_\gamma
\]

of a manifold \( Z \). The summation in (2.1b) and (2.1c) is carried out over \( \alpha, \beta, ... \) for which \( U_\alpha \cap U_\beta \neq \emptyset \) and \( U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset \).

Let us consider a collection \( \psi = \{ \psi_\alpha \} \) of smooth \( G \)-valued functions \( \psi_\alpha \) defined on \( U_\alpha \), a collection \( f = \{ f_{\alpha \beta} \} \) of holomorphic matrices on nonempty intersections \( U_\alpha \cap U_\beta \) and suppose that \( \psi_\alpha \)'s satisfy the differential equations

\[
(\bar{\partial} \psi_\alpha) \psi_\alpha^{-1} = (\bar{\partial} \psi_\beta) \psi_\beta^{-1}
\]

defined on \( \mathcal{U}^{(1)} \), and \( f_{\alpha \beta} \)'s satisfy the functional equations

\[
f_{\alpha \beta | \gamma} f_{\beta \gamma | \alpha} f_{\gamma \alpha | \beta} = 1
\]

on \( \mathcal{U}^{(2)} \). Here \( f_{\alpha \beta | \gamma} \) means the restriction of \( f_{\alpha \beta} \) to an open set \( U_\alpha \cap U_\beta \cap U_\gamma \). Now, let us define a map of solutions \( \{ \psi_\alpha \} \) of Eqs.(2.2) into solutions \( \{ f_{\alpha \beta} \} \) of Eqs.(2.3a) by the formula \( \eta : \{ \psi_\alpha \} \mapsto \{ \psi_\alpha^{-1} \psi_\beta \} \), and denote by \( \mathcal{F} \) the subset of those solutions \( \{ f_{\alpha \beta} \} \) of Eqs.(2.3a) for which there exists a collection \( \psi = \{ \psi_\alpha \} \) of smooth \( G \)-valued functions \( \psi_\alpha \) on \( U_\alpha \) such that

\[
f_{\alpha \beta} = \psi_\alpha^{-1} \psi_\beta.
\]

Using equations \( \bar{\partial} f_{\alpha \beta} = 0 \), one can easily show that these \( \{ \psi_\alpha \} \) will satisfy Eqs.(2.2) and therefore \( \mathcal{F} = \text{Im} \, \eta \).

Denote by \( \mathcal{X} \) the space of solutions to Eqs.(2.2) and consider a map

\[
\eta : \mathcal{X} \rightarrow \mathcal{F}
\]
given for any $\psi = \{\psi_\alpha\} \in \mathcal{X}$ by the formula

$$\eta(\psi) = \eta(\{\psi_\alpha\}) = \{\psi_\alpha^{-1} \psi_\beta\} = f. \quad (2.4b)$$

It is easy to see that if $g = \{g_\alpha\}$ is an element of the gauge group $G$, then $\psi = \{\psi_\alpha\} \in \mathcal{X}$ and $g^{-1} \psi = \{g_\alpha^{-1} \psi_\alpha\} \in \mathcal{X}$ are projected by the map (2.4) into the same solution $f = \eta(\psi)$ of Eqs.(2.3). Therefore the space $\mathcal{F}$ of solutions to functional equations (2.3) can be identified with the space of orbits of the group $G$ in the set $\mathcal{X}$,

$$\mathcal{F} \simeq \mathcal{X}/G. \quad (2.5)$$

Now consider a (0,1)-connection $A^{0,1}$ on $P_0$, restrict $A^{0,1}$ to $U_\alpha$’s and consider a collection $A^{0,1} = \{A^{(\alpha)}\}$ of (0,1)-connections $A^{(\alpha)} := A^{0,1}|_{U_\alpha}$. In terms of $A^{(\alpha)}$’s, Eqs.(1.2a) have the form

$$\bar{\partial}A^{(\alpha)} + A^{(\alpha)} \wedge A^{(\alpha)} = 0 \quad (2.6a)$$

$$A^{(\alpha)} = A^{(\beta)} \text{ on } U_\alpha \cap U_\beta. \quad (2.6b)$$

Denote by $\mathcal{N}$ the space of solutions to Eqs.(2.6) and define a map

$$\bar{\delta}^0 : \mathcal{X} \to \mathcal{N} \quad (2.7a)$$

given by the formula

$$\bar{\delta}^0(\psi) = \psi \bar{\partial} \psi^{-1} = \{\psi_\alpha \bar{\partial} \psi_\alpha^{-1}\} = \{A^{(\alpha)}\} = A^{0,1}, \quad (2.7b)$$

where $\psi = \{\psi_\alpha\} \in \mathcal{X}$. It is clear that $\{A^{(\alpha)}\} = \{\psi_\alpha \bar{\partial} \psi_\alpha^{-1}\}$ satisfy Eqs.(2.6) if $\{\psi_\alpha\}$ satisfy Eqs.(2.2).

It is not difficult to see that for any $\{\bar{\psi}_\alpha\} \in \mathcal{X}$ we have $\{\psi_\alpha h_\alpha^{-1}\} \in \mathcal{X}$ if $h_\alpha$’s are holomorphic $G$-valued functions on $U_\alpha$’s,

$$\bar{\partial}h_\alpha = 0. \quad (2.8)$$

Moreover, such $\{\bar{\psi}_\alpha\}, \{\psi_\alpha h_\alpha^{-1}\} \in \mathcal{X}$ are mapped by $\bar{\delta}^0$ into the same flat (0,1)-connection $A^{0,1} = \{A^{(\alpha)}\} = \{\psi_\alpha \bar{\partial} \psi_\alpha^{-1}\}$. We denote by $\mathcal{H}$ the set of all collections $h = \{h_\alpha\}$ of $G$-valued locally defined holomorphic functions $h_\alpha : U_\alpha \to G$. The set $\mathcal{H}$ is a group under the pointwise multiplication: $h \chi = \{h_\alpha \chi_\alpha\}$ for $h = \{h_\alpha\}, \chi = \{\chi_\alpha\} \in \mathcal{H}$. So, it follows from (2.7) that the space $\mathcal{N}$ of flat (0,1)-connections can be identified with the space of orbits of the group $\mathcal{H}$ in the space $\mathcal{X}$ of solutions to Eqs.(2.2),

$$\mathcal{N} \simeq \mathcal{H}/\mathcal{X}. \quad (2.9)$$

We see that if we find $\psi = \{\psi_\alpha\}$ from Eqs.(2.2) or Eqs.(2.3), then we can obtain a flat (0,1)-connection $A^{0,1} = \psi \bar{\partial} \psi^{-1}$ with the help of the map (2.7). Thus, flat (0,1)-connections can be found not only by solving Eqs.(1.2a) on $U^{(0)}$ but also by solving Eqs.(2.2) on $U^{(1)}$ or Eqs.(2.3a) on $U^{(2)}$.

### 2.2 Moduli space of holomorphic structures

Consider a (trivial) $G$-bundle $P_0$ over a complex $n$-manifold $Z$ and a (0,1)-connection $A^{0,1}$ on $P_0$. The gauge group $G$ acts on $A^{0,1} = \{A^{(\alpha)}\}$ by the formula

$$A^{0,1} \mapsto \text{Ad}_{g^{-1}} A^{0,1} = g^{-1} A^{0,1} g + g^{-1} \bar{\partial}g = \{g_\alpha^{-1} A^{(\alpha)} g_\alpha + g_\alpha^{-1} \bar{\partial}g_\alpha\}, \quad (2.10)$$
where \( g = \{ g_\alpha \} \in \mathcal{G} \). Equations (1.2a) and (2.6) are invariant under the transformations (2.10).

We denote by \( \mathcal{M} \) the set of orbits of the gauge group \( \mathcal{G} \) in the set \( \mathcal{N} \) of solutions to Eqs.(2.6),

\[
\mathcal{M} = \mathcal{N} / \mathcal{G}.
\]

By definition, \( \mathcal{M} \) is the moduli space of flat \((0,1)\)-connections \( A^{0,1} \) parametrizing holomorphic structures \( \bar{\partial}_A \) on the bundle \( P_0 \). By introducing a projection

\[
\pi : \mathcal{N} \to \mathcal{M},
\]

we obtain a composite map

\[
\pi \circ \delta^0 : \mathfrak{X} \xrightarrow{\delta^0} \mathcal{N} \xrightarrow{\pi} \mathcal{M}
\]

of \( \mathfrak{X} \) onto \( \mathcal{M} \).

Recall that on the space \( \mathfrak{X} \) we have an action not only of the group \( \mathcal{G} \) but also of the group \( \mathfrak{H} \),

\[
\mathfrak{H} \ni h = \{ h_\alpha \} : \psi \mapsto \psi h^{-1} = \{ \psi_\alpha h^{-1}_\alpha \},
\]

where \( \psi = \{ \psi_\alpha \} \in \mathfrak{X} \). This action induces the following action of \( \mathfrak{H} \) on matrices \( f_{\alpha\beta} = \psi^{-1}_\alpha \psi_\beta \):

\[
\mathfrak{H} \ni h = \{ h_\alpha \} : f_{\alpha\beta} \mapsto \tilde{f}_{\alpha\beta} = h_\alpha f_{\alpha\beta} h^{-1}_\beta.
\]

Therefore one can introduce the space \( \mathfrak{H} \backslash \mathcal{F} \) of orbits of the group \( \mathfrak{H} \) in the space \( \mathcal{F} \) of solutions to Eqs.(2.3). Then, using the bijection (2.5), we obtain

\[
\mathfrak{H} \backslash \mathcal{F} \simeq \mathfrak{H} \backslash \mathfrak{X} / \mathcal{G}.
\]

By definition, \( \mathfrak{H} \backslash \mathcal{F} \) is the moduli space of solutions to functional equations (2.3). Comparing (2.9), (2.11) and (2.16), we obtain bijections

\[
\mathcal{M} \simeq \mathcal{N} / \mathcal{G} \simeq \mathfrak{H} \backslash \mathfrak{X} / \mathcal{G} \simeq \mathfrak{H} \backslash \mathcal{F},
\]

i.e. there is a one-to-one correspondence between the moduli spaces of solutions to Eqs.(2.3) and Eqs.(2.6). We identify these moduli spaces with the moduli space \( \mathcal{M} \) of holomorphic structures on the bundle \( P_0 \to Z \).

Let us denote by \( p \) a projection

\[
p : \mathcal{F} \to \mathcal{M}.
\]

Combining (2.4) and (2.18), we obtain a composite map

\[
p \circ \eta : \mathfrak{X} \xrightarrow{\eta} \mathcal{F} \xrightarrow{p} \mathcal{M}
\]

of \( \mathfrak{X} \) onto \( \mathcal{M} \) (cf.(2.13)).
Here $\mathcal{X}$ is the solution space of differential equations (2.2), $\mathcal{N}$ is the space of flat $(0, 1)$-connections on $P_0$, $\mathcal{F}$ is the solution space of functional equations (2.3), and $\mathcal{M}$ is the moduli space of holomorphic structures on the bundle $P_0 \to Z$.

### 2.3 Non-Abelian cohomology and holomorphic bundles

Results of Sect.2.2 can be reformulated in terms of homological algebra using sheaves of non-Abelian groups. Namely, let $\mathcal{G}$ be the sheaf of germs of smooth $G$-valued functions on $Z$, $\mathcal{H}$ its subsheaf of holomorphic $G$-valued functions and $\mathcal{A}^{0,1}$ the sheaf of flat $(0,1)$-connections on $P_0$ (germs of solutions to Eqs.(1.2a)). We fix a covering $\mathcal{U} = \{U_\alpha\}$ of a manifold $Z$ and introduce the following sets: the set $C^0(\mathcal{U}, \mathcal{G})$ of 0-cochains of the covering $\mathcal{U}$ with values in $\mathcal{G}$, the set $Z^0(\mathcal{U}, \mathcal{G})$ of 0-cocycles with values in $\mathcal{G}$, the set $C^1(\mathcal{U}, \mathcal{G})$ of 1-cochains with values in $\mathcal{G}$, the set $Z^1(\mathcal{U}, \mathcal{G})$ of 1-cocycles of the covering $\mathcal{U}$ with values in the sheaf $\mathcal{G}$ and the 1-cohomology set $H^1(\mathcal{U}, \mathcal{G})$. These sets contain the subsets $C^0(\mathcal{U}, \mathcal{H})$, $Z^0(\mathcal{U}, \mathcal{H})$, $C^1(\mathcal{U}, \mathcal{H})$, $Z^1(\mathcal{U}, \mathcal{H})$ and $H^1(\mathcal{U}, \mathcal{H})$, respectively. All the definitions can be found e.g. in [6, 7, 8, 9].

Recall that by definition $H^0(Z, \mathcal{G}) = \Gamma(Z, \mathcal{G}) = Z^0(\mathcal{U}, \mathcal{G})$, $H^0(Z, \mathcal{A}^{0,1}) = \Gamma(Z, \mathcal{A}^{0,1}) = Z^0(\mathcal{U}, \mathcal{A}^{0,1})$. Moreover, one can always choose a covering $\mathcal{U} = \{U_\alpha\}$ such that it will be $H^1(\mathcal{U}, \mathcal{G}) = H^1(Z, \mathcal{G})$, $H^1(\mathcal{U}, \mathcal{H}) = H^1(Z, \mathcal{H})$. This is realized, for instance, when $U_\alpha$ ‘s are Stein manifolds and we suppose that the chosen covering satisfies the above conditions. In cohomological terms some of spaces and groups introduced earlier are defined as follows:

\begin{align}
\mathcal{H} &= C^0(\mathcal{U}, \mathcal{H}), \\
\mathcal{N} &= H^0(Z, \mathcal{A}^{0,1}), \\
\mathcal{G} &= H^0(Z, \mathcal{G}), \\
\mathcal{M} &\simeq \mathcal{N}/\mathcal{G} = H^0(Z, \mathcal{A}^{0,1})/H^0(Z, \mathcal{G}).
\end{align}

Notice also that the space $\mathcal{X}$ is a subset of the set $C^0(\mathcal{U}, \mathcal{G})$, and the space $\mathcal{F}$ is a subset of the set $Z^1(\mathcal{U}, \mathcal{H})$. Namely, $\mathcal{F}$ is the set of those 1-cocycles $f \in Z^1(\mathcal{U}, \mathcal{H})$ that are smoothly equivalent to the cocycle $f_0 = \{\text{id}_{U_\alpha \cap U_\beta}\}$.

We denote by $i : \mathcal{H} \to \mathcal{G}$ an embedding of $\mathcal{H}$ into $\mathcal{G}$ and define a map $\delta^0 : \mathcal{G} \to \mathcal{A}^{0,1}$ given for any open set $U$ of the space $Z$ by the formula

$$\delta^0(\psi_U) := \psi_U \partial \psi_U^{-1},$$

(2.22)
where $\psi_U \in \Gamma(U, \mathcal{G})$ is a smooth $G$-valued function on $U$. Let us also introduce an operator $\bar{\delta}^1$ acting on $(0,1)$-connections $A^{0,1}$ by the formula

$$\bar{\delta}^1(A^{0,1}) := \bar{\partial}A^{0,1} + A^{0,1} \wedge A^{0,1}. \tag{2.23a}$$

By definition, $\bar{\delta}^1$ maps any flat $(0,1)$-connection into zero and therefore

$$\bar{\delta}^1(A^{0,1}) = 0 \iff A^{0,1} = \text{Ker} \bar{\delta}^1. \tag{2.23b}$$

Remember that locally Eqs.(1.2a) are solved trivially, and on any sufficiently small open set $U \subset Z$ we have $A^{0,1}_U = \psi_U \bar{\partial} \psi^{-1}_U$, where $A^{0,1}_U \in \Gamma(U, A^{0,1})$ and $\psi_U \in \Gamma(U, \mathcal{G})$ is a smooth $G$-valued function on $U$ (cf.(2.22)). It is easy to see that

$$A^{0,1}_U = \psi_U \bar{\partial} \psi^{-1}_U = (\psi_U h^{-1}_U) \bar{\partial} (\psi_U h^{-1}_U)^{-1}, \tag{2.24}$$

where $h_U \in \Gamma(U, \mathcal{H})$ is an arbitrary holomorphic $G$-valued function on $U$ (a section of the sheaf $\mathcal{H}$ over $U$). Therefore, the sheaf of germs of solutions to Eqs.(1.2a) is isomorphic to the quotient sheaf $\mathcal{H}/\mathcal{S}$. Notice that the (left) action of the sheaf $\mathcal{H}$ on $\mathcal{S}$ is described for any open set $U$ by the formula $\psi_U \mapsto \psi_U h^{-1}_U$, where $\psi_U \in \Gamma(U, \mathcal{G})$, $h_U \in \Gamma(U, \mathcal{H})$. Thus, we have the exact sequence of sheaves

$$e \to \mathcal{H} \to \mathcal{G} \to A^{0,1} \to e, \tag{2.25}$$

where $e$ is a marked element of the considered sets (the identity in the sheaf $\mathcal{H} \subset \mathcal{G}$ and zero in the sheaf $A^{0,1}$). From (2.25) we obtain the exact sequence of cohomology sets $[6, 7, 8]$,

$$e \to H^0(Z, \mathcal{H}) \to H^0(Z, \mathcal{G}) \to H^0(Z, A^{0,1}) \to H^1(Z, \mathcal{H}) \to H^1(Z, \mathcal{G}), \tag{2.26}$$

where the map $\rho$ coincides with the canonical embedding induced by the embedding of sheaves $i : \mathcal{H} \to \mathcal{G}$.

By definition the 1-cohomology sets $H^1(Z, \mathcal{H})$ and $H^1(Z, \mathcal{G})$ parametrize the sets of equivalence classes of holomorphic and smooth $G$-bundles over $Z$, respectively. The kernel $\text{Ker} \rho = \rho^{-1}(e)$ of the map $\rho$ coincides with a subset of equivalence classes of topologically trivial holomorphic bundles $P$. Therefore we have

$$\mathcal{H}/\mathcal{F} = \text{Ker} \rho, \tag{2.27}$$

where the space $\mathcal{H}/\mathcal{F}$ is the moduli space of solutions to Eqs.(2.3). By virtue of the exactness of the sequence (2.26), the space $\text{Ker} \rho = \mathcal{H}/\mathcal{F}$ is bijective to the quotient space (2.21d). So, the bijections (2.17) follow from the exact sequence (2.26) and we have

$$\mathcal{M} \simeq H^0(Z, A^{0,1})/H^0(Z, \mathcal{G}) \simeq \text{Ker} \rho. \tag{2.28}$$

Recall that $\mathcal{M}$ is the moduli space of holomorphic structures on the bundle $P_0$. 

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2.4 Algebra-valued forms and Dolbeault cohomology

Now let us consider Eqs.(1.2b) on globally defined $g$-valued $(n, n-2)$-forms $B$ on $Z$. These equations can be rewritten in the form

$$\bar{\partial}_A B = 0, \quad (2.29)$$

and Eqs.(1.2a) coincide with the compatibility conditions $F_A^{0,2} = \bar{\partial}_A^2 = 0$ of Eqs.(2.29).

The gauge group $\mathcal{G}$ acts on a field $B \in \Omega^{n,n-2}(Z, g)$ by the formula

$$B \mapsto \text{Ad}_{g^{-1}} B = g^{-1} B g, \quad (2.30)$$

where $g \in \mathcal{G}$. Notice that the action (1.1) and Eqs.(1.2) are invariant under the gauge transformations (2.10), (2.30) and under the following “cohomological” symmetry transformations:

$$B \mapsto B + \bar{\partial}_A \Phi, \quad (2.31)$$

where $\Phi \in \Omega^{n,n-3}(Z, g)$. By virtue of this invariance, solutions $B$ and $B + \bar{\partial}_A \Phi$ of Eqs.(1.2b) are considered as equivalent.

Equations (1.2b) are linear in $B$. For any fixed flat $(0,1)$-connection $A^{0,1}$ the space of nontrivial solutions to Eqs.(1.2b) is the $(n, n-2)$-th Dolbeault cohomology group

$$H^{n,n-2}_{\bar{\partial}_A; f_0}(Z) := \frac{\{ B \in \Omega^{n,n-2}(Z, g) : \bar{\partial}_A B = 0 \}}{\{ B = \bar{\partial}_A \Phi, \Phi \in \Omega^{n,n-3}(Z, g) \}}. \quad (2.32)$$

So, the space of nontrivial solutions to Eqs.(1.2b) forms the vector space $H^{n,n-2}_{\bar{\partial}_A; f_0}(Z)$ depending on a solution $A^{0,1}$ of Eqs.(1.2a).

For a fixed flat $(0,1)$-connection $A^{0,1} = \psi \bar{\partial}\psi^{-1} = \{ \psi_\alpha \bar{\partial}\psi^{-1}_\alpha \}$ any solution of Eqs.(1.2b) has the form

$$B = \psi B_0 \psi^{-1} = \{ \psi_\alpha B_0^{(\alpha)} \psi^{-1}_\alpha \} = \{ B^{(\alpha)} \}, \quad (2.33)$$

where $B_0 = \{ B_0^{(\alpha)} \}$ is an arbitrary solution of the equations

$$\bar{\partial} B_0 = 0. \quad (2.34a)$$

Here $B^{(\alpha)} := B|_{U_\alpha}$ and $B_0^{(\alpha)} := B_0|_{U_\alpha}$ are restrictions of $B$ and $B_0$ to an open set $U_\alpha$ from the covering $\mathfrak{U} = \{ U_\alpha \}, \alpha \in I$. On nonempty intersections $U_\alpha \cap U_\beta$ we have $B^{(\alpha)} = B^{(\beta)}$. These compatibility conditions for $B = \{ B^{(\alpha)} \}$ lead to the following compatibility conditions for $B_0 = \{ B_0^{(\alpha)} \}$:

$$B_0^{(\alpha)} = f_{\alpha \beta} B_0^{(\beta)} f^{-1}_{\alpha \beta}, \quad (2.34b)$$

where $f_{\alpha \beta} := \psi^{-1}_\alpha \psi_\beta$, and $\{ \psi_\alpha \}$ satisfy Eqs.(2.2). It follows from Eqs.(2.2) that $f_{\alpha \beta}$’s are holomorphic matrices on $U_\alpha \cap U_\beta$. Therefore, $\{ f_{\alpha \beta} \}$ can be chosen as transition functions in a holomorphic bundle $P$.

Notice that the space of nontrivial solutions to Eqs.(2.34) is the standard Dolbeault cohomology group

$$H^{n,n-2}\bar{\partial}P(Z) = \frac{\{ \bar{\partial}\text{-closed ad}P\text{-valued } (n, n-2)\text{-forms on } Z \}}{\{ \bar{\partial}\text{-exact ad}P\text{-valued } (n, n-2)\text{-forms on } Z \}}, \quad (2.35)$$

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where $P$ and $\text{ad}P = P \times_G \mathfrak{g}$ are holomorphic bundles defined by transition functions $\{f_{\alpha\beta}\} = \{\psi_{\alpha}^{-1}\psi_{\beta}\}$. Formula (2.33) defines an isomorphism of the vector spaces (2.32) and (2.35).

To sum up, one can easily construct solutions of Eqs.(1.2b) if one knows solutions of Eqs.(1.2a) and Eqs.(2.34). Moreover, the space of solutions to Eqs.(1.2) forms a vector bundle $\mathcal{T} \to \mathcal{N}$, the base space of which is the space $\mathcal{N}$ of solutions to Eqs.(1.2a), and fibres of the bundle $\mathcal{T}$ at points $A^{0,1} \in \mathcal{N}$ are the vector spaces $H^{0,n-2}_{\partial A^{0,1}P_0}(Z)$ of nontrivial solutions to Eqs.(1.2b). Recall that the gauge group $\mathcal{G}$ acts on solutions $(A^{0,1},B)$ of Eqs.(1.2) by formulae (2.10), (2.30). Therefore, identifying points $(A^{0,1},B) \in \mathcal{T}$ and $(g^{-1}A^{0,1}g + g^{-1}\partial g, g^{-1}Bg) \in \mathcal{T}$ for any $g \in \mathcal{G}$, we obtain the moduli space

$$\mathfrak{M} = \mathcal{T}/\mathcal{G}$$

(2.36)

of solutions to Eqs.(1.2). The space $\mathfrak{M}$ is a vector bundle over the moduli space $\mathcal{M}$ of flat $(0,1)$-connections with fibres at points $[A^{0,1}] \in \mathcal{M}$ isomorphic to the Dolbeault cohomology groups (2.32).

### 3 Dressing transformations in hBF theories

#### 3.1 Cohomological symmetry groups

In Sect.2 we have discussed the correspondence between flat $(0,1)$-connections $A^{0,1} = \{A^{(\alpha)}\}$ on a $G$-bundle $P_0 \to Z$ and 1-cocycles $f = \{f_{\alpha\beta}\}$ defining topologically trivial holomorphic bundles $P$ over $Z$. It follows from this correspondence that if we define an action of some group on the space $\mathcal{F}$ of transition functions $f$ of topologically trivial holomorphic bundles, then using the correspondence $A^{0,1} \leftrightarrow f$ (see the diagram (2.20)), we obtain an induced action of this group on the space $\mathcal{N}$ of flat $(0,1)$-connections on the bundle $P_0$. In this section we introduce a special cohomological group and describe its action on the space $\mathcal{F}$.

Consider a collection $h = \{h_{\alpha\beta}\} \in C^1(\mathcal{U}, \mathcal{H})$ of holomorphic matrices such that

$$h_{\alpha\beta|\gamma} = h_{\alpha\gamma|\beta}, \quad (3.1)$$

where $h_{\alpha\beta|\gamma}$ means the restriction of $h_{\alpha\beta}$ to an open set $U_\alpha \cap U_\beta \cap U_\gamma$. The constraints (3.1) are not severe. They simply mean that sections $h_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{H})$ of the sheaf $\mathcal{H}$ over $U_\alpha \cap U_\beta$ can be extended to sections of the sheaf $\mathcal{H}$ over the open set

$$\mathcal{U}^{(1)} = \bigcup_{\alpha,\beta \in I} U_\alpha \cap U_\beta,$$

where the summation is carried out in all $\alpha, \beta \in I$ for which $U_\alpha \cap U_\beta \neq \emptyset$. In other words, it follows from (3.1) that there exists a holomorphic map

$$h_{\mathcal{U}^{(1)}} : \mathcal{U}^{(1)} \to G \quad (3.2a)$$

such that

$$h_{\alpha\beta} = h_{\mathcal{U}^{(1)}(U_\alpha \cap U_\beta)} \quad (3.2b)$$
One can identify \( h = \{ h_{\alpha \beta} \} = \{ h_{U^{(1)}}|_{U_{\alpha} \cap U_{\beta}} \} \) and \( h_{\mathcal{U}^{(1)}} \). Such \( h \in \Gamma(\mathcal{U}^{(1)}, \mathcal{H}) \) form a subgroup

\[
\tilde{C}^1(\mathcal{U}, \mathcal{H}) := \{ h \in C^1(\mathcal{U}, \mathcal{H}) : h_{\alpha \beta|\gamma} = h_{\alpha \gamma|\beta} \text{ on } U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \neq \emptyset \}
\]

(3.3)
of the group \( C^1(\mathcal{U}, \mathcal{H}) \).

We consider \( \tilde{C}^1(\mathcal{U}, \mathcal{H}) \) as a local group, i.e. we choose a neighbourhood \( \mathcal{C} \) of the identity \( e \) in \( \tilde{C}^1(\mathcal{U}, \mathcal{H}) \) and take elements \( h \) only from \( \mathcal{C} \subset \tilde{C}^1(\mathcal{U}, \mathcal{H}) \). The local group \( \mathcal{C} \) is a representative of the germ of the group \( \tilde{C}^1(\mathcal{U}, \mathcal{H}) \) at the point \( e \in \tilde{C}^1(\mathcal{U}, \mathcal{H}) \). We define the following action of the group \( \mathcal{C} \) on the space \( \mathcal{F} \):

\[
T(h, f) : f_{\alpha \beta} \mapsto f_{\alpha \beta}^h = T(h, f)_{\alpha \beta} = h_{\alpha \beta} f_{\alpha \beta} h_{\beta \alpha}^{-1},
\]

(3.4a)
where \( h \in \mathcal{C} \), and a 1-cocycle \( f = \{ f_{\alpha \beta} \} = \{ \psi_{\alpha}^{-1} \psi_{\beta} \} \in \mathcal{F} \) defines a topologically trivial holomorphic bundle \( P \). It is easy to see that \( \{ f_{\alpha \beta}^h \} \) satisfy Eqs.(2.3a) by virtue of the definition (3.3) of the group \( \tilde{C}^1(\mathcal{U}, \mathcal{H}) \) and therefore \( \{ f_{\alpha \beta}^h \} \) is a 1-cocycle. Moreover, for \( h \in \mathcal{C} \subset \tilde{C}^1(\mathcal{U}, \mathcal{H}) \) there exists a 0-cochain \( \psi^h = \{ \psi^h_{\alpha} \} \in \mathcal{X} \subset C^0(\mathcal{U}, \mathcal{S}) \) such that

\[
f_{\alpha \beta}^h = (\psi^h_{\alpha})^{-1} \psi^h_{\beta},
\]

(3.4b)
since small enough deformations do not change the topological trivializability of holomorphic bundles. This well-known statement follows from the equality \( H^1(Z, s_P) = 0 \), where \( s_P \) is the sheaf of smooth sections of the bundle \( \text{ad}P \).

So, we have a map

\[
T : \mathcal{C} \times \mathcal{F} \to \mathcal{F},
\]

(3.5)
and to each \( h \in \mathcal{C} \) there corresponds a bijective transformation

\[
T_h : f \mapsto T(h, f)
\]

(3.6)
of the set \( \mathcal{F} \). The map \( t : h \mapsto T_h \) is a homomorphism of the group \( \mathcal{C} \) into the group \( \text{Bij}(\mathcal{F}) \) of all bijective transformations of the set \( \mathcal{F} \). Notice that maps (3.6) are connected with maps between bundles \( (P, f) \) and \( (P^h, f^h) \), where a bundle \( P^h \) is defined by transition functions \( \{ f^h_{\alpha \beta} \} \). These bundles are diffeomorphic but not biholomorphic. A diffeomorphism of \( P \) onto \( P^h \) is defined by a 0-cochain \( \psi^{-1} \psi^h = \{ \psi^{-1}_{\alpha} \psi^h_{\alpha} \} \in C^0(\mathcal{U}, \mathcal{S}) \). Moreover, the bundles \( P \) and \( P^h \) become biholomorphic after the restriction to \( \mathcal{U}^{(1)} : P_{\mathcal{U}^{(1)}} \sim P^h_{\mathcal{U}^{(1)}} \). In other words, a map \( T_h : f \mapsto f^h, h \in \mathcal{C} \), defines a local biholomorphism \( P_{\mathcal{U}^{(1)}} \to P^h_{\mathcal{U}^{(1)}} \) which does not extend up to the biholomorphism of \( P \) and \( P^h \) as holomorphic bundles over \( Z \).

More general transformations of the space \( \mathcal{F} \) of topologically trivial holomorphic bundles can be found by discarding the conditions (3.1) on matrices \( \{ h_{\alpha \beta} \} \). Namely, let us consider the transformations (3.4a) with an arbitrary element \( h = \{ h_{\alpha \beta} \} \) of the group \( C^1(\mathcal{U}, \mathcal{H}) \). Then consider the equations

\[
h_{\alpha \beta} f_{\alpha \beta} h_{\beta \alpha}^{-1} h_{\beta \gamma} f_{\beta \gamma} h_{\gamma \beta}^{-1} h_{\gamma \alpha} f_{\gamma \alpha} h_{\alpha \gamma}^{-1} = 1
\]

(3.7)
on \( \{ h_{\alpha \beta} \} \). These equations mean that \( f^h = T(h, f) = \{ h_{\alpha \beta} f_{\alpha \beta} h_{\beta \alpha}^{-1} \} \) is a 1-cocycle. For each solution \( h = \{ h_{\alpha \beta} \} \) of Eqs.(3.7) we obtain a map \( T_h \) of the space of holomorphic bundles into
itself. Moreover, solutions \( h = \{ h_{\alpha \beta} \} \) of Eqs.\((3.7)\) that are close to the identity correspond to transformations preserving topological triviality of bundles, and we obtain the transformations

\[
\mathcal{F} \ni f \xrightarrow{T_h} f^h \in \mathcal{F}.
\]

In principle, by solving Eqs.\((3.7)\) one can obtain all elements of the group of local bijections of the space \( \mathcal{F} \).

### 3.2 Actions of groups on the solution space

We consider the trivial \( G \)-bundle \( P_0 \) with the transition functions \( f_0 = \{ \text{id} \psi \}_{\alpha} \) and a flat \((0,1)\)-connection \( A^{0,1} = \{ A^{(\alpha)} \} \) on \( P_0 \). As it was discussed in Sect.\( 2 \), for any flat \((0,1)\)-connection \( A^{0,1} \) there exists a 0-cochain \( \psi = \{ \psi_{\alpha} \} \in \mathfrak{c} \subset C^0(\mathfrak{g}, \mathfrak{s}) \) such that \( A^{0,1} = \psi \partial \psi^{-1} = \{ \psi_{\alpha} \partial \psi_{\alpha}^{-1} \} \). If we denote by \( \varphi \) a section of the fibration \( \delta^0 : \mathfrak{c} \to \mathcal{N} \), \( \delta^0 \circ \varphi = \text{id} \), then \( \psi = \varphi(A^{0,1}) \), where \( A^{0,1} \in \mathcal{N} \). Notice that \( \varphi \) is a local section, i.e. we consider an open neighbourhood of the point \( A^{0,1} \in \mathcal{N} \). The choice of \( \varphi \) is not unique, and an element \( \psi = \varphi(A^{0,1}) \) is defined up to an element from the group \( \mathfrak{h} = C^0(\mathfrak{g}, \mathfrak{s}) \). Using the maps \( \varphi \) and \( \eta \), we obtain a map

\[
\eta \circ \varphi : (f_0, A^{0,1}) \mapsto (f, 0),
\]

where \( f = \{ f_{\alpha \beta} \} = \{ \psi_{\alpha}^{-1} \psi_{\beta} \} \) are transition functions of a \( G \)-bundle \( P \).

Conversely, denote by \( \zeta \) a local section of the fibration \( \eta : \mathfrak{c} \to \mathfrak{f} \), i.e. \( \eta \circ \zeta = \text{id} \) on an open neighbourhood of the point \( f = \{ \psi_{\alpha}^{-1} \psi_{\beta} \} \in \mathfrak{f} \). Of course, the choice of a section \( \zeta \) is not unique, and an element \( \psi = \zeta(f) \in \mathfrak{c} \) is defined up to an element from the gauge group \( \mathfrak{g} = H^0(Z, \mathfrak{s}) \). Using the maps \( \zeta \) and \( \delta^0 \), we obtain a map

\[
\delta^0 \circ \zeta : (f_0, 0) \mapsto (f_0, \tilde{A}^{0,1}),
\]

where \( \tilde{f} \) is an element from an open neighbourhood of \( f \in \mathfrak{f} \), and \( \tilde{A}^{0,1} \) is an element from an open neighbourhood of \( A^{0,1} \in \mathcal{N} \).

In Sect.\( 3.1 \) we have described maps \( T_h : f \mapsto f^h \), where \( f^h \in \mathfrak{f} \) if \( h \in \mathfrak{c} \). Then, using the map \((3.9)\) for \( \tilde{f} = f^h \), we obtain

\[
\tilde{A}^{0,1} = \psi^h \partial (\psi^h)^{-1} = \{ \psi_{\alpha} h \partial (\psi_{\alpha})^{-1} \},
\]

where \( \psi^h = \tilde{\psi} = \zeta(f^h) \) can be found from formula \((3.4b)\). By construction, \( \tilde{A}^{0,1} \) satisfies Eqs.\((1.2a)\). Thus, if we take a “seed” flat \((0,1)\)-connection \( A^{0,1} \) and carry out the sequence of transformations

\[
A^{0,1} \xrightarrow{\varphi} \psi \xrightarrow{T_h} f^h \xrightarrow{\zeta} \psi^h \partial^0 \xrightarrow{\tilde{A}^{0,1}},
\]

we obtain a new flat \((0,1)\)-connection \( \tilde{A}^{0,1} \) depending nonlocally on \( A^{0,1} \) and \( h \in \mathfrak{c} \).

Let us introduce

\[
\phi(h) := \psi^h \psi^{-1} = \{ \psi_{\alpha} h \psi_{\alpha}^{-1} \} = \{ \phi_{\alpha}(h) \} \in C^0(\mathfrak{g}, \mathfrak{s}).
\]
Then we have
\[ \tilde{A}^{0,1} = \phi(h)A^{0,1}\phi(h)^{-1} + \phi(h)\bar{\partial}\phi(h)^{-1} = \{ \phi_\alpha(h)A^{(\alpha)}\phi_\alpha(h)^{-1} + \phi_\alpha(h)\bar{\partial}\phi_\alpha(h)^{-1} \}. \] (3.13)

Formally, (3.13) looks like a gauge transformation. But actually the transformation
\[ \text{Ad}_{\phi(h)} : A^{0,1} \mapsto \text{Ad}_{\phi(h)}A^{0,1} = \phi(h)A^{0,1}\phi(h)^{-1} + \phi(h)\bar{\partial}\phi(h)^{-1} \] (3.14a)
defined by (3.12) consists of the sequence (3.11) of transformations and is not a gauge transformation since \( \phi_\alpha(h) \neq \phi_\beta(h) \) on \( U_\alpha \cap U_\beta \neq \emptyset \). Recall that for gauge transformations \( \text{Ad}_g : A^{0,1} \mapsto \text{Ad}_gA^{0,1} = gA^{0,1}g^{-1} + g\bar{\partial}g^{-1} \) one has \( g_\alpha = g_\beta \) on \( U_\alpha \cap U_\beta \) for \( g = \{ g_\alpha \} \in \mathcal{G} \). In other words, \( g = \{ g_\alpha \} \) is a \textit{globally} defined \( G \)-valued function on \( Z \) and \( \phi(h) = \{ \phi_\alpha(h) \} \) is a collection of \textit{locally} defined \( G \)-valued functions \( \phi_\alpha(h) : U_\alpha \to G \) which are constructed by the algorithm described above. So, to each \( h \in \mathcal{C} \) there corresponds a bijective transformation \( \text{Ad}_{\phi(h)} \) of the set \( \mathcal{N} \), and the map \( \gamma : h \mapsto \text{Ad}_{\phi(h)} \) is a homomorphism of the group \( \mathcal{C} \) into the group \( \text{Bij}(\mathcal{N}) \) of all bijections of the set \( \mathcal{N} \).

From formulae (2.33) and (3.12) it follows that the transformations \( \text{Ad}_{\phi(h)} \) act on any solution of Eqs.(1.2b) by the formula
\[ \text{Ad}_{\phi(h)} : B \mapsto \text{Ad}_{\phi(h)}B = \phi(h)B\phi(h)^{-1} = \{ \phi_\alpha(h)B^{(\alpha)}\phi_\alpha(h)^{-1} \}, \] (3.14b)
where \( \phi(h) = \{ \phi_\alpha(h) \} \) is defined in (3.12). As is shown above, the transformation \( \text{Ad}_{\phi(h)} \) is not a gauge transformation and therefore \( \text{Ad}_{\phi(h)}B \) is a new solution of Eqs.(1.2b). So, we have described a homomorphism of the group \( \mathcal{C} \) into the group \( \text{Bij}(\mathcal{T}) \) of bijective transformations of the space \( \mathcal{T} \) of solutions to equations of motion of holomorphic BF theory. The transformations (3.14) will be called the \textit{dressing transformations}. In this terminology we follow the papers [11, 12, 13], where analogous transformations were used for constructing solutions of integrable equations.

4 \hspace{1em} \textbf{Dressing symmetries and special hBF theories}

Let now \( Z \) be a Calabi-Yau \( n \)-manifold. This means that besides a complex structure, on \( Z \) there exist a Kähler 2-form \( \omega \), a Ricci-flat Kähler metric \( g \) and a nowhere vanishing holomorphic \((n,0)\)-form \( \theta \). We consider a (trivial) principal \( G \)-bundle \( P_0 \) over \( Z \) and the \((0,1)\)-component \( A^{0,1} \) of a connection 1-form \( A \) on \( P_0 \). The existence on \( Z \) of a nowhere degenerate holomorphic \((n,0)\)-form \( \theta \) permits one to introduce one more class of models describing holomorphic structures on bundles over Calabi-Yau manifolds. These models are called \textit{holomorphic }\( \theta \text{BF theories} \) \((h\theta\text{BF})\) and their action functional \( [5] \) have the form
\[ S_{h\theta\text{BF}} = \int_Z \theta \wedge \text{Tr}(B^{0,n-2} \wedge F_A^{0,2}), \] (4.1)
where \( B^{0,n-2} \) is a \( g \)-valued \((0,n-2)\)-form on \( Z \), and \( F_A^{0,2} \) is the \((0,2)\)-component of the curvature tensor of a connection \( A \) on \( P_0 \). The action (4.1) leads to the following equations of motion:
\[ \bar{\partial}A^{0,1} + A^{0,1} \wedge A^{0,1} = 0, \] (4.2a)
\[ \partial B^{0,n-2} + A^{0,1} \wedge B^{0,n-2} - (-1)^n B^{0,n-2} \wedge A^{0,1} = 0. \]  

(4.2b)

So, the action (4.1) provides us with a field-theoretic description of holomorphic structures on bundles over Calabi-Yau manifolds.

A description of solutions and symmetries of Eqs.(4.2) literally reproduce a description of solutions and symmetries of field equations (1.2) of holomorphic BF theories. In particular, the description of flat (0,1)-connections is not changed since Eqs.(4.2a) coincide with Eqs.(1.2a), and for a description of solutions to Eqs.(4.2b) it is sufficient to replace \( B \in \Omega^{n,n-2}(Z, \mathfrak{g}) \) by \( B \in \Omega^{0,n-2}(Z, \mathfrak{g}) \) in all formulae of Sect.2.4 and Sect.3.2.

One can also consider special holomorphic BF theories on twistor spaces of self-dual 4-manifolds \([3]\). To describe these models, we consider a Riemannian real 4-manifold \( M \) with self-dual Weyl tensor (a self-dual manifold) and the bundle \( \tau : Z \to M \) of complex structures on \( M \) (the twistor space of \( M \)) with \( \mathbb{CP}^1 \) as a typical fibre \([4, 14]\). The twistor space \( Z \) of a self-dual 4-manifold \( M \) is a complex 3-manifold \([13]\) which is the total space of a fibre bundle over \( M \) associated with the bundle of orthonormal frames on \( M \).

Using the complex structures on \( \mathbb{CP}^1 \hookrightarrow Z \) and \( Z \), one can split the complexified tangent bundle of \( Z \) into a direct sum

\[ T^C(Z) = T^{1,0} \oplus T^{0,1} = (Ver^{1,0} \oplus Hor^{1,0}) \oplus (Ver^{0,1} \oplus Hor^{0,1}) \]  

(4.3)

of subbundles of type \( (1, 0) \) and \( (0, 1) \). Here \( Ver^{0,1} \) is the distribution of vertical \( (0, 1) \)-vector fields. Analogously the complexified cotangent bundle of \( Z \) is splitted into a direct sum of subbundles \( T_{1,0} \) and \( T_{0,1} \).

Let \( E^{3,3} \) be a \( \mathfrak{g} \)-valued \( (3,3) \)-form on \( Z \), and \( V^{0,1} \) be an arbitrary \( (0,1) \)-vector field from the distribution \( Ver^{0,1} \). Consider a trivial \( G \)-bundle \( P_0 \) over \( Z \), the \( (0,1) \)-component \( A^{0,1} \) of a connection \( A \) and the \( (0,2) \)-component \( F^0_A \) of the curvature of a connection \( A \) on \( P_0 \). Denote by \( V^{0,1} \wedge A^{0,1} \) the contraction of \( V^{0,1} \) with \( A^{0,1} \) and consider the action \([3]\)

\[ S_{BBF} = \int_Z \text{Tr} [B^{3,1} \wedge F^0_A - \gamma (V^{0,1} \wedge A^{0,1}) E^{3,3}], \]  

(4.4)

where \( B^{3,1} \in \Omega^{3,1}(Z, \mathfrak{g}) \) and \( \gamma = \text{const.} \). This action leads to the following field equations:

\[ \partial A^{0,1} + A^{0,1} \wedge A^{0,1} = 0, \quad \gamma V^{0,1} \wedge A^{0,1} = 0, \]  

(4.5a)

\[ \bar{\partial} B^{3,1} + A^{0,1} \wedge B^{3,1} - B^{3,1} \wedge A^{0,1} = \gamma V^{0,1} \wedge E^{3,3}. \]  

(4.5b)

Equations (4.5a) on the twistor space \( Z \) of a self-dual 4-manifold \( M \) are equivalent to the self-dual Yang-Mills (SDYM) equations on \( M \) \([13, 15]\). So, the action (4.4) can be considered as an action for SDYM models. Dressing symmetries of Eqs.(4.5a) have been described in \([17, 18]\). These symmetries can be reduced to symmetries of integrable models in less than four dimensions.
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