ON THE NORMS OF CIRCULANT, \( r \)-CIRCULANT, SEMI-CIRCULANT AND HANKEL MATRICES WITH TRIBONACCI SEQUENCE

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Abstract. In this paper, some special matrices in terms of tribonacci sequence \( \mathcal{M}_n \) are considered. More precisely, the norms of circulant, \( r \)-circulant, semi-circulant and Hankel matrices are calculated. In particular, the Euclidean norms and their bounds for the spectral norm of above mention matrices are obtained.

1. Introduction and Preliminaries

In the recent years, remarkable work has been done on the Fibonacci and Lucas numbers \[3\]. Many authors tried to investigate their properties and opened some new directions towards these numbers. In 1963, Feinberg discussed some properties of tribonacci sequence \[2\]. Catalani investigated tribonacci-Lucas numbers \[1\]. Solak and Bozkurt \[8\] established bounds for the special norms of circulant matrices and they also found the upper and lowers bounds for Cauchy-Toeplitz and Cauchy Hankel matrices \[9\].

The tribonacci sequence is defined as:

(1.1) \[ \mathcal{M}_n = \mathcal{M}_{n-1} + \mathcal{M}_{n-2} + \mathcal{M}_{n-3} \]

with initial conditions \( \mathcal{M}_0 = 0, \mathcal{M}_1 = \mathcal{M}_2 = 1 \).

A matrix \( \mathcal{U} = (\mu_{ij}) \in M_{n,n}(\mathbb{C}) \) is called \( r \)-circulant on tribonacci sequence if it is of the form

(1.2) \[ \mu_{ij} = \begin{cases} \mathcal{M}_{j-i} & j \geq i \\ r \mathcal{M}_{n+j-i} & j < i \end{cases} \]

where \( r \in \mathbb{C} \). If \( r=1 \), then matrix \( \mathcal{U} \) is called circulant.

A matrix \( \mathcal{U} = (\mu_{ij}) \in M_{n,n}(\mathbb{C}) \) is called semi-circulant on tribonacci sequence if it is of the form

\[ \mu_{ij} = \begin{cases} \mathcal{M}_{j-i} & i \leq j \\ 0 & \text{otherwise} \end{cases} \]

Similarly, a Hankel matrix on tribonacci sequence is defined as \( \mathcal{H} = (h_{ij}) \in M_{n,n}(\mathbb{C}) \), where \( h_{ij} = \mathcal{M}_{i+j-1} \).

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The $\ell_q$ norm of a matrix $\mathbf{U} = (\mu_{ij}) \in M_{n,n}(\mathbb{C})$ is defined by

$$\|\mathbf{U}\|_q = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |\mu_{ij}|^q\right)^{1/q} \quad (1 \leq q \leq \infty)$$

If $q = \infty$, then $\|\mathbf{U}\|_\infty = \lim_{q \to \infty} \|\mathbf{U}\|_q = \max_{i,j} |\mu_{ij}|$.

The Euclidean (Frobenius) norm of the matrix $\mathbf{U}$ is defined as

$$\|\mathbf{U}\|_E = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |\mu_{ij}|^2\right)^{1/2}$$

The spectral norm of the matrix $\mathbf{U}$ is given as

$$\|\mathbf{U}\|_2 = \sqrt{\max_{1 \leq i \leq n} |\beta_i|},$$

where $\beta_i$ are the eigenvalues of the matrix $\mathbf{U}^t \mathbf{U}$.

The following inequality between Euclidean and spectral norm holds [10]

$$\frac{1}{\sqrt{n}} \|\mathbf{U}\|_E \leq \|\mathbf{U}\|_2 \leq \|\mathbf{U}\|_E$$

**Definition 1.1.** [7] Let $M = (m_{ij})$ and $N = (n_{ij})$ be $m \times n$ matrices. Then, the Hadamard product of $M$ and $N$ is given by

$$M \odot N = (m_{ij} n_{ij})$$

**Definition 1.2.** [9] The maximum column length norm $c_1(\cdot)$ and maximum row length norm $r_1(\cdot)$ for $m \times n$ matrix $\mathbf{U} = (\mu_{ij})$ is defined $c_1(\mathbf{U}) = \sqrt{\max_{j} \sum_{i} |\mu_{ij}|^2}$ and $r_1(\mathbf{U}) = \sqrt{\max_{i} \sum_{j} |\mu_{ij}|^2}$ respectively.

**Theorem 1.3.** [6] Let $A = (a_{ij})$, $B = (b_{ij})$ and $\mathbf{U} = (\mu_{ij})$ be $p \times q$ matrices. If $\mathbf{U} = A \circ B$, then $\|\mathbf{U}\|_2 \leq r_1(A)c_1(B)$.

The following lemmas describe the properties of tribonacci sequence.

**Lemma 1.4.** [5] The sum of square of first $n$ term of tribonacci sequence is given by

$$\sum_{k=1}^{n} \mathcal{M}_k^2 = S_n = \frac{1 + 4\mathcal{M}_n \mathcal{M}_{n+1} - (\mathcal{M}_{n+1} - \mathcal{M}_{n-1})^2}{4}$$

**Lemma 1.5.** [5] For all $n \geq 1$

$$\sum_{k=1}^{n} \mathcal{M}_{k+1} \mathcal{M}_{k-1} = A_n + \mathcal{M}_{n+1} \mathcal{M}_{n-1},$$

where $A_n = \frac{(\mathcal{M}_{n+1} - \mathcal{M}_{n-1})^2 - 1}{4}$. 

Lemma 1.6. The sum of first \( n \) terms of tribonacci numbers is given by
\[
\sum_{k=1}^{n} m_k = \frac{m_{n+2} + m_n - 1}{2}.
\]

Lemma 1.7. The sum of product of first \( n \) consecutive tribonacci numbers is given by
\[
\sum_{k=1}^{n} m_k m_{k+1} = M_n = \frac{A_n - 9m_n m_{n-2} - S_n + 2m_n^2 + m_{n-1}^2 + 2m_{n-2} + 2m_{n-3} + 2m_n m_{n-1} + 2m_{n-2} + 2m_n m_{n-1} + 2m_{n+1} m_n}{2}.
\]

Proof. Let us take
\[
\begin{align*}
M_{k-1} m_{k-3} &= (m_{k-2} + m_{k-3} + m_{k-4}) M_{k-3} \\
M_{k-1} m_{k-3} &= M_{k-2} M_{k-3} + M_{k-3}^2 + M_{k-4} M_{k-3} \\
\sum_{k=3}^{n} m_{k-1} m_{k-3} &= \sum_{k=3}^{n} m_{k-2} m_{k-3} + \sum_{k=3}^{n} m_{k-4} m_{k-3} \\
(A_n - m_n m_{n-2} - S_n - m_n^2 + m_{n-1}^2 - m_{n-2}^2) &= 2 \sum_{k=3}^{n} m_{k-2} m_{k-3} - m_{n-2} m_{n-3}
\end{align*}
\]
(1.4)
\[
\sum_{k=3}^{n} m_{k-2} m_{k-3} = \frac{A_n - m_n m_{n-2} - S_n + m_n^2 + m_{n-1}^2 + m_{n-2}^2 + m_{n-2} m_{n-3}}{2}
\]

On the other hand, we know that,
(1.5)
\[
\sum_{k=3}^{n} m_{k-2} m_{k-3} = \sum_{k=1}^{n} m_k m_{k+1} - m_{n-1} m_{n-2} - m_n m_{n-1} - m_{n+1} m_n
\]

using equations (1.4) and (1.5), we have
\[
\sum_{k=1}^{n} m_k m_{k+1} = \frac{A_n - 9m_n m_{n-2} - S_n + 2m_n^2 + m_{n-1}^2 + 2m_{n-2} + 2m_{n-3} + 2m_n m_{n-1} + 2m_{n-2} + 2m_n m_{n-1} + 2m_{n+1} m_n}{2}
\]

Lemma 1.8. The following identity hold for the tribonacci sequence:
\[
\sum_{k=1}^{n} m_k^2 = R_n = \frac{1}{4} \left( n + 1 + 4M_n - 2S_n - m_n^2 + m_{n+1}^2 + 2A_n + 2m_{n+1} m_{n-1} \right)
\]
where \( S_n, A_n \text{ and } M_n \) defined in lemma (1.4), (1.5)and (1.7) respectively.

2. Main Theorems

In this section, we will give main results about the norms of \( r - \)circulant, circulant, semi-circulant and Hankel matrix with tribonacci sequence.

Theorem 2.1. Let \( \mathcal{U} = \mathcal{U}(m_0, m_1 \ldots m_{n-1}) \) be \( r - \)circulant matrix.

If \( |r| \geq 1 \), then,
\[
|\mathcal{U}|_2 \leq \sqrt{\frac{1 + 4S_n m_{n-1} - (m_{n} - m_{n-2})^2}{4}} \leq |\mathcal{U}|_2 \leq |r| \left[ \frac{1 + 4S_n m_{n-1} - (m_{n} - m_{n-2})^2}{4} \right]
\]

If \( |r| < 1 \), then,
\[
|\mathcal{U}|_2 \leq \sqrt{\frac{1 + 4S_n m_{n-1} - (m_{n} - m_{n-2})^2}{4}} \leq |\mathcal{U}|_2 \leq (n - 1) \left[ \frac{1 + 4S_n m_{n-1} - (m_{n} - m_{n-2})^2}{4} \right]
\]
Proof. The $r-$circulant matrix $\mathbf{U}$ with tribonacci sequence is given as:

$$
\mathbf{U} = \begin{bmatrix}
M_0 & M_1 & M_2 & \cdots & M_{n-1} \\
rM_{n-1} & M_0 & M_1 & \cdots & M_{n-2} \\
rM_{n-2} & rM_{n-1} & M_0 & \cdots & M_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
rM_1 & rM_2 & rM_3 & \cdots & M_0
\end{bmatrix}
$$

and by the definition of Euclidean norm, we have

$$
(2.1) \quad \|\mathbf{U}\|_E^2 = \sum_{k=0}^{n-1} (n-k) M_k^2 + \sum_{k=1}^{n-1} k|r|^2 M_k^2.
$$

Here, we have two cases depending on $r$.

Case 1. If $|r| \geq 1$, then from equation (2.1), we have

$$
\|\mathbf{U}\|_E^2 \geq \sum_{k=0}^{n-1} (n-k) M_k^2 + \sum_{k=1}^{n-1} kM_k^2 = n \sum_{k=0}^{n-1} M_k^2
$$

and by lemma (1.2), we get,

$$
\|\mathbf{U}\|_E^2 \geq n \sum_{k=0}^{n-1} M_k^2 = n \left[ \frac{1 + 4M_n M_{n-1} - (M_n - M_{n-2})^2}{4} \right].
$$

Now by inequality (1.3), we obtain

$$
(2.2) \quad \|\mathbf{U}\|_2 \geq \sqrt{\frac{1 + 4M_n M_{n-1} - (M_n - M_{n-2})^2}{4}}.
$$

On the other hand, let us define matrices $A$ and $B$ as:

$$
A = \begin{bmatrix}
rM_0 & 1 & 1 & \cdots & 1 \\
rM_{n-1} & rM_0 & 1 & \cdots & 1 \\
rM_{n-2} & rM_{n-1} & rM_0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
rM_1 & rM_2 & rM_3 & \cdots & rM_0
\end{bmatrix} \quad B = \begin{bmatrix}
M_0 & M_1 & M_2 & \cdots & M_{n-1} \\
1 & M_0 & M_1 & \cdots & M_{n-2} \\
1 & 1 & M_0 & \cdots & M_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & M_0
\end{bmatrix}
$$

It is easy to see that $\mathbf{U} = A \circ B$, then from definition (1.2), we have

$$
r_1(A) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{nj}|^2} = \sqrt{|r|^2 \sum_{k=0}^{n-1} M_k^2} = |r| \sqrt{\frac{1 + 4M_n M_{n-1} - (M_n - M_{n-2})^2}{4}}.
$$

$$
c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{nj}|^2} = \sqrt{\sum_{k=0}^{n-1} M_k^2} = \sqrt{\frac{1 + 4M_n M_{n-1} - (M_n - M_{n-2})^2}{4}}.
$$
Now, using theorem (1.3), we obtain,

$$\|\mathcal{U}\|_2 \leq r_1(A)c_1(B) = |r| \left[ \frac{1 + 4M_nM_{n-1} - (M_n - M_{n-2})^2}{4} \right]$$

Combine equations (2.2) and (2.3), we get the required results as

$$\sqrt{\left[ \frac{1 + 4M_nM_{n-1} - (M_n - M_{n-2})^2}{4} \right]} \leq \|\mathcal{U}\|_2 \leq |r| \left[ \frac{1 + 4M_nM_{n-1} - (M_n - M_{n-2})^2}{4} \right].$$

Case 2. If $|r| \leq 1$, then we have

$$\|\mathcal{U}\|_E \geq \sum_{k=0}^{n-1} (n - k) |r|^2M_k^2 + \sum_{k=0}^{n-1} k|r|^2M_k^2 = n \sum_{k=0}^{n-1} |r|^2M_k^2$$

$$ \frac{1}{\sqrt{n}} |\mathcal{U}|_E \geq |r| \sqrt{\left[ \frac{1 + 4M_nM_{n-1} - (M_n - M_{n-2})^2}{4} \right]}.$$

We obtain the following inequality by using (1.3)

$$\|\mathcal{U}\|_2 \geq |r| \sqrt{\left[ \frac{1 + 4M_nM_{n-1} - (M_n - M_{n-2})^2}{4} \right]}$$

Now, consider the matrices $A'$ and $B'$

$$A' = \begin{bmatrix} M_0 & 1 & 1 & \cdots & 1 \\ r & M_0 & 1 & \cdots & 1 \\ r & r & M_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & M_0 \end{bmatrix}, \quad B' = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots & M_{n-1} \\ M_{n-1} & M_0 & M_1 & \cdots & M_{n-2} \\ M_{n-2} & M_{n-1} & M_0 & \cdots & M_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_1 & M_2 & M_3 & \cdots & M_0 \end{bmatrix}$$

such that $\mathcal{U} = A' \circ B'$, so

$$r_1(A') = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |a'_{ij}|^2} = \sqrt{M_0^2 + (n - 1)} = \sqrt{n - 1}$$

$$c_1(B') = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |b'_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} M_k^2} = \sqrt{\left[ \frac{1 + 4M_nM_{n-1} - (M_n - M_{n-2})^2}{4} \right]}$$

Again by theorem (1.3), we have

$$\|\mathcal{U}\|_2 \leq r_1(A')c_1(B') = \sqrt{n - 1} \sqrt{\left[ \frac{1 + 4M_nM_{n-1} - (M_n - M_{n-2})^2}{4} \right]}$$

(2.5) $$\|\mathcal{U}\|_2 \leq \sqrt{n - 1} \sqrt{\left[ \frac{1 + 4M_nM_{n-1} - (M_n - M_{n-2})^2}{4} \right]}.$$
from (2.4) and (2.5), we have

\[ |r| \sqrt{\frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4}} \leq \|U\|_2 \leq \sqrt{n - 1} \left[ \frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4} \right]. \]

\[ \Box \]

In the following theorem, we obtained the bounds of spectral norm of circulant matrix.

**Theorem 2.2.** Let \( U \) be the circulant matrix on tribonacci sequence, then

\[ \|U\|_E = \sqrt{n \left[ \frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4} \right]} \]

and

\[ \sqrt{\frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4}} \leq \|U\|_2 \leq \sqrt{\frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4}} \left| 1 + \frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4} \right]. \]

**Proof.** Since by definition of circulant matrix, the matrix \( U \) has of the form

\[ U = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ m_{n-1} & m_0 & m_1 & \cdots & m_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & m_0 \end{bmatrix} \]

Let matrices \( C \) and \( D \) be defined as

\[ C = \begin{cases} c_{ij} = m_{(j-i, \mod n)} & i \geq j \\ c_{ij} = 1 & i < j \end{cases}, \quad D = \begin{cases} d_{ij} = m_{(j-i, \mod n)} & i < j \\ d_{ij} = 1 & i \geq j \end{cases} \]

Then it is easy to see that the row norm and column norm of \( C \) and \( D \) are given as:

\[ r_1(C) = \max_i \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} m_i^2} = \sqrt{\frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4}} \]

\[ c_1(D) = \max_j \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{1 + \sum_{i=1}^{n-1} m_i^2} = \sqrt{1 + \frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4}} \]

Using the theorem (1.3), we have

(2.6)

\[ \|U\|_2 \leq \sqrt{\frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4}} \left| 1 + \frac{1 + 4m_n m_{n-1} - (m_n - m_{n-2})^2}{4} \right]. \]

By definition of \( \|\cdot\|_E^2 \), we have

\[ \|U\|_E^2 = n \sum_{k=0}^{n-1} m_k^2 \]
Using lemma (1.8), we have

(2.7) \[ \|\mathbf{U}\|_E = \sqrt{n \left[ \frac{1 + 4M_n M_{n-1} - (M_n - M_{n-2})^2}{4} \right]} \]

so using equation (1.3), we have

(2.8) \[ \sqrt{\left[ \frac{1 + 4M_n M_{n-1} - (M_n - M_{n-2})^2}{4} \right]} \leq \|\mathbf{U}\|_2 \]

and combining the results (2.6) and (2.8), we have

\[ \sqrt{\left[ \frac{1 + 4M_n M_{n-1} - (M_n - M_{n-2})^2}{4} \right]} \leq \|\mathbf{U}\|_2 \leq \sqrt{\left[ \frac{1 + 4T_n T_{n-1} - (T_n - T_{n-2})^2}{4} \right]} + 1 + \left[ \frac{1 + 4M_n M_{n-1} - (M_n - M_{n-2})^2}{4} \right] \]

\[ \square \]

**Theorem 2.3.** If \( \mathbf{U} \) is an \( n \times n \) semi-circulant matrix \( \mathbf{U} = (\mu_{ij}) \) with the tribonacci numbers then,

\[ \|\mathbf{U}\|_E = \sqrt{\frac{1}{4} \left( n + 1 + 4M_n - 2S_n - M^2_{n+1} + M^2_n + 2A_n + 2M_{n+1}M_{n-1} \right)} \]

where \( S_n, A_n \) and \( M_n \) defined in lemma (1.4), (1.5) and (1.7) respectively.

**Proof.** For the semi-circulant matrix \( \mathbf{U} = (\mu_{ij}) \) with the tribonacci numbers we have

\[ \mu_{ij} = \begin{cases} M_{j-i+1} & i \leq j \\ 0 & \text{otherwise} \end{cases} \]

Now from the definition of Euclidean norm, we have

\[ \|\mathbf{U}\|_E^2 = \sum_{j=1}^{n} \sum_{i=1}^{j} (\mu_{j-i+1})^2 = \sum_{j=1}^{n} \left( \sum_{k=1}^{j} M_{k}^2 \right) \]

Using lemma (1.8), we have

\[ \|\mathbf{U}\|_E = \sqrt{\left( \frac{1}{4} \right) \left( n + 1 + 4M_n - 2S_n - M^2_{n+1} + M^2_n + 2A_n + 2M_{n+1}M_{n-1} \right)} \] \[ \square \]

**Theorem 2.4.** If \( \mathbf{U} \) is an \( n \times n \) Hankel matrix \( \mathbf{U} = (\mu_{ij}) \) with \( \mu_{ij} = M_{i+j-1} \), then

\[ \|\mathbf{U}\|_E^2 = \left( R_{2n-1} - 2R_{n-1} \right)^2 \]

where \( R_n = \sum_{k=1}^{n} \sum_{i=1}^{k} M_i^2 \).

**Proof.** By definition of Hankel matrix, the matrix \( \mathbf{U} \) is of the form

\[ \mathbf{U} = \begin{bmatrix} M_1 & M_2 & M_3 & \cdots & M_{n-1} & M_n \\ M_2 & M_3 & M_4 & \cdots & M_n & M_{n+1} \\ M_3 & M_4 & M_5 & \cdots & M_{n+1} & M_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n-1} & M_n & M_{n+1} & \cdots & M_{2n-3} & M_{2n-2} \\ M_n & M_{n+1} & M_{n+2} & \cdots & M_{2n-2} & M_{2n-1} \end{bmatrix} \]

so,

\[ \|\mathbf{U}\|_E = \left( \sum_{k=1}^{n} M_k^2 + \sum_{k=2}^{n+1} M_k^2 + \sum_{k=3}^{n+2} M_k^2 + \cdots + \sum_{k=n}^{2n-1} M_k^2 \right)^{\frac{1}{2}} \].
\[
\|U\|_E = \left( \sum_{k=1}^{n} \mathcal{M}_k^2 + \sum_{k=1}^{n+1} \mathcal{M}_k^2 + \sum_{k=1}^{n+2} \mathcal{M}_k^2 + \cdots + \sum_{k=1}^{2n-1} \mathcal{M}_k^2 \right) - \sum_{k=1}^{n-1} k \sum_{i=1}^{\mathcal{M}_i^2} \right)^{\frac{1}{2}}
\]

\[
\|U\|_E = (S_n + S_{n+1} + \cdots + S_{2n-1} - R_{n-1})^{\frac{1}{2}}
\]

\[
\|U\|_E = \left( \sum_{k=1}^{2n-1} S_k - \sum_{k=1}^{n-1} S_k - R_{n-1} \right)^{\frac{1}{2}}
\]

\[
\|U\|_E = (R_{2n-1} - 2R_{n-1})^{\frac{1}{2}}.
\]

In next theorem, we give bounds for the spectral norm of Hankel matrix.

**Theorem 2.5.** If \(U\) is an \(n \times n\) Hankel matrix \(U = (\mu_{ij})\) with \(\mu_{ij} = \mathcal{M}_{i+j-1}\) then,

\[
\frac{1}{\sqrt{n}} \|U\|_E \leq \|U\|_2 \leq \frac{1 + 4\mathcal{M}_n \mathcal{M}_{n+1} - (\mathcal{M}_{n+1} - \mathcal{M}_{n-1})^2}{4}.
\]

**Proof.** From theorem (2.4), and inequality (1.3), we have

\[
\frac{1}{\sqrt{n}} \|U\|_E \leq \|U\|_2
\]

On the other hand, let us define two new matrices

\[
G = \begin{cases} 
\mu_{ij} = \mathcal{M}_{i+j-1} & i \leq j \\
\mu_{ij} = 1 & i > j
\end{cases}
\]

and

\[
K = \begin{cases} 
k_{ij} = \mathcal{M}_{i+j-1} & i > j \\
k_{ij} = 1 & i \leq j
\end{cases}
\]

It can easily be seen that \(U = G \circ K\), and thus we obtain

\[
r_1 (G) = \max_i \sqrt{\sum_j |g_{ij}|^2} = \sqrt{\sum_{i=1}^{n} \mathcal{M}_i^2} = \sqrt{\frac{1 + 4\mathcal{M}_n \mathcal{M}_{n+1} - (\mathcal{M}_{n+1} - \mathcal{M}_{n-1})^2}{4}}
\]

and

\[
c_1 (K) = \max_j \sqrt{\sum_i |k_{ij}|^2} = \sqrt{1 + \sum_{i=2}^{n} \mathcal{M}_i^2} = \sqrt{\frac{1 + 4\mathcal{T}_n \mathcal{M}_{n+1} - (\mathcal{M}_{n+1} - \mathcal{M}_{n-1})^2}{4}}
\]

Using the theorem (1.3), we have

\[
\|U\|_2 \leq \frac{1 + 4\mathcal{M}_n \mathcal{M}_{n+1} - (\mathcal{M}_{n+1} - \mathcal{M}_{n-1})^2}{4}
\]

**Theorem 2.6.** If \(U\) be an \(n \times n\) Hankel matrix with \(\mu_{ij} = \mathcal{M}_{i+j-1}\), then

\[
\|U\|_1 = \|U\|_\infty = \frac{\mathcal{M}_{2n+1} - \mathcal{M}_{n+1} + \mathcal{M}_{n-1} - \mathcal{M}_{2n-1}}{2}
\]
Proof. From the definition of the matrix $U$, we can write
\[
\|U\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| = \max_{1 \leq j \leq n} \{|a_{1j}| + |a_{2j}| + |a_{3j}| \ldots |a_{nj}|\}
\]
\[
\|U\|_1 = M_n + M_{n+1} + M_{n+2} + \cdots + M_{2n-1}
\]
by lemma(1.6), we have
\[
\|U\|_1 = \frac{M_{2n+1} - M_{n+1} + M_{n-1} - M_{2n-1}}{2}.
\]
Similarly, the row norm of the matrix $M$ can be computed as:
\[
\|U\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| = \frac{M_{2n+1} - M_{n+1} + M_{n-1} - M_{2n-1}}{2}.
\]

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