Two-fluid theory for superfluid system with anisotropic effective masses

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In this work, we generalize the two-fluid theory to a superfluid system with anisotropic effective masses along different principal axis directions. As a specific example, such a theory can be applied to spin-orbit coupled Bose-Einstein condensate (BEC) at low temperature. The normal density from phonon excitations and the second sound velocity are obtained analytically. Near the phase transition from the plane wave to zero-momentum phases, due to the effective mass divergence, the normal density from phonon excitation increases greatly, while the second sound velocity is suppressed significantly. With quantum hydrodynamic formalism, we give unified derivations for suppressed superfluid density and Josephson relation. At last, the momentum distribution function and fluctuation of phase for the long wave length are also discussed.

I. INTRODUCTION

At low temperature, Bose-Einstein condensation and superfluidity would occur in bosonic system. Tissa [1] and Landau [2, 3] propose two-fluid theory to explain the superfluid phenomena in Helium-4. Comparing with usual classic fluid, due to an extra degrees of freedom (existence of condensate), the existence of second sound is an evident signal of superfluidity. With realizations of Bose-Einstein condensate (BEC) in dilute atomic gas, the second sound and related superfluid phenomena in atomic gas have attracted great interests. For example, sound velocities at zero temperature as a function of density in BEC [4] and Fermi superfluid gas [5] have been measured experimentally. The application of two-fluid theory for sound propagations in cold atomic gas has been proposed [6, 7]. For uniformly strongly interacting fermion gas, the sound velocities for first sound, second sound[8, 9] and quenched momentum of inertia[10] have been calculated. For trapped system, in order to solve Landau-Khalataikov’s two-fluid equations, a variational method has been widely used [11–15]. The predictions on the sound velocities and the temperature dependence of collective modes for trapped gas from Landau’s two-fluid theory has been confirmed in cold atom experiments [16–18]. In additions, the quenched moment of inertia resulting from superfluidity in cold atoms has been observed experimentally[19].

Recently spin-orbit coupled BEC has been realized experimentally [20–25]. It is shown that, even at zero temperature, there exists finite normal density, and even all the total density becomes normal at some conditions although the condensate fraction is finite [26]. At the zero temperature, due to finite normal density, there is finite momentum of inertia in the spin-orbit coupled BEC [27]. It is shown that the suppressing of superfluid density is closely related the enhancements of effective masses near the ground state. Due to the enhancements of effective masses, the expansion behaviors of spin-orbit coupled gas shows anisotropy [28, 29] and self-trapping [30].

It is expected that due to enhancements of effective masses in spin-orbit coupled BEC, the corresponding two-fluid theory at finite temperature need to be revised greatly. In this paper, we show that a lot of superfluid properties of spin-orbit coupled BEC, e.g., the decreasing of superfluid density, the suppressed anisotropic sound velocities, etc., can be described by an anisotropic boson system with different effective masses along the three principal axis directions. Therefore, at low temperature, the two-fluid theory for anisotropic bosonic system finds an immediate application in spin-orbit coupled superfluid system.

In this work, we generalize the two-fluid theory to an anisotropic system with different effective masses and use it to explain the superfluid properties of spin-orbit coupled bosons. The paper is organized as follows. In Sec. II, we review the thermodynamic relations for superfluid system. In Sec. III, based on the entropy equation, we give a derivation for dissipationless two-fluid equations in anisotropic system. In Sec. IV, as an application of the two-fluid theory for anisotropic system, we give a specific example, namely, spin-orbit coupled BEC, to illustrate the above results. A summary is given in Sec. IV.

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II. THERMODYNAMIC RELATIONS FOR SUPERFLUID SYSTEM

First of all, we consider an original system $K_0$ with particle mass $m$, in which the many-particle Hamiltonian

$$H_0 = \sum_{i} \frac{p_{0i}^2}{2m} + \frac{1}{2} \sum_{i \neq j} V(r_i - r_j),$$

where $p_0$ is the momentum for $K_0$ and $V(r_i - r_j)$ is the interaction energy between particles. In the following, we mainly investigate the effects arising from enhancements of effective masses, i.e., $m \to zm$ with $z > 1$. For this purpose, we consider another system $K$ with effective mass $m' = zm$. The corresponding Hamiltonian and Lagrangian are written as

$$H = \sum_{i} \frac{p_i^2}{2zm} + \frac{1}{2} \sum_{i \neq j} V(r_i - r_j),$$

$$L = \sum_{i} \frac{zmv_i^2}{2} - \frac{1}{2} \sum_{i \neq j} V(r_i - r_j),$$

where $p$ and $v$ are the particle momentum and velocity for $K$. When we quantize the above two Hamiltonians $H_0$ and $H$, the momentum should be replaced by an operator, i.e., $p_0 = p = -i\hbar \nabla$. From the relations $p_0 = mv_0$ and $p = zmv$, we get the velocity for $K$ in terms of that of $K_0$, i.e.,

$$v = v_0/z,$$

where $v_0$ is the particle velocity for $K_0$ with mass $m$. Equation (2) shows that enhancements of masses would result in the decreasing of velocity. In the following, the velocity appearing in expressions is always referred to the velocity of the original system $K_0$, which has mass $m$, rather $zm$. The Lagrangian for $K$ can also be expressed in terms of $v_0$, i.e.,

$$L = \sum_{i} \frac{zmv_0^2}{2} - \frac{1}{2} \sum_{i \neq j} V(r_i - r_j).$$

Now we consider a moving reference frame with the relative velocity $u$ with respect to the laboratory reference frame. The velocities have a relation

$$v_0 = v' + u,$$

where $v'$ is the velocity in the moving frame and the Lagrangian is rewritten as

$$L = \sum_{i} \frac{m(v'_i + u)^2}{2z} - \frac{1}{2} \sum_{i \neq j} V(r_i - r_j).$$

The canonical momentum and Hamiltonian in the moving frame are thus given by

$$p' = \frac{\partial L}{\partial v'} = m(v' + u)/z = p_0/z = p/z.$$

$$H' = \sum_{i} p'_i \cdot v'_i - L = H - \frac{u \cdot P}{z},$$

where the total momentum $P = \sum_j p_j$. The free energy $F$ in the moving frame is related to the above Hamiltonian $H'$ through

$$Z = \text{tr} e^{-\beta H'} = e^{-\beta F} = e^{-\beta(E-TS-\mathbf{u} \cdot \mathbf{P}/z)},$$

where $Z$ is the partition function, $E$ is the energy, $S$ is the entropy, $\beta = 1/T$ is the inverse temperature, and the free energy

$$F = E - TS - \mathbf{u} \cdot \mathbf{P}/z.$$

The grand potential is

$$\Omega = -pV = F - \mu N = E - TS - \mathbf{u} \cdot \mathbf{P}/z - \mu N,$$

where $p$ is the pressure, $V$ is the system volume, $\mu$ is the chemical potential, and $N$ is the total particle number. By defining the energy density $\epsilon = E/V$, the entropy density $s = S/V$, the momentum density $g = \mathbf{P}/V$, and the particle number density $n = N/V$, the pressure $p$ is

$$p = -\epsilon + Ts + \mathbf{u} \cdot \mathbf{g}/z + \mu n.$$

Since the free energy is a function of $\{T, V, u, N\}$, e.g., $F = F(T, V, u, N)$, using Eq. (5), we obtain

$$dF = -SdT - pdV + \mu dN - \mathbf{P} \cdot d\mathbf{u}/z$$

$$= dE - TdS - SdT - \mathbf{u} \cdot d\mathbf{P}/z - \mathbf{P} \cdot d\mathbf{u}/z,$$

which leads to a fundamental thermodynamic relation

$$Tds = dE + pdV - \mu dN - \mathbf{u} \cdot d\mathbf{P}/z.$$  

For a fixed volume, $dV = 0$. For the unit volume, Eq. (8) turns into

$$Tds = de - \mu dn - \mathbf{u} \cdot d\mathbf{j},$$

where $j = g/(zm)$ is the particle number current.

For a classic system, the trace in Eq. (4) can be replaced by an integral in phase space, i.e., $\text{tr}(...) \to \int \prod_i dp_i dr_i (...)$. Using $p_i^2/(2zm) - u \cdot p_i/z = (p_i - mu)^2/(2zm) - mu^2/(2z)$, we have

$$Z = e^{\beta Nmu^2/(2z)} Z_0 = e^{\beta Nmu^2/(2z)} \int \prod_i dp_i dr_i e^{-\beta H},$$

where $Z_0 = e^{-\beta F_0}$ is the partition function and $F_0$ is free energy in rest frame of fluid. So the free energy $F$ is given by

$$F = F_0 - Nmu^2/(2z).$$

When temperature $T$ is below the superfluid transition temperature, Eq. (10) can be extended to case in which the superfluid and normal parts move respectively with the velocities $\mathbf{v}_s = \hbar \nabla \theta/m$ and $\mathbf{v}_n = u$ [32]. In this case,
the free energy for the unit volume, \( f = F/V \), is given by
\[
f = f_0 - \frac{nmv^2}{2z} + n_s m (v_s - v_n)^2/(2z),
\]
where \( \theta \) is the phase of the condensate order parameter, the term \( n_s m (v_s - v_n)^2/(2z) \) describes extra energy due to the moving of the superfluid part relative to the normal part, and \( n_s \) is the particle number density of the superfluid part. We should remind that the velocities for \( K \) with mass \( zm \) are \( v_s(n)/z \).

In principle, the free energy (11) in the superfluid system can be obtained from the following processes. First we get the modified free energy density \( f \) by the modified Hamiltonian \( H \), e.g.,
\[
\tilde{H} = H' - H \cdot \tilde{v}_s, \\
\tilde{Z} = e^{-\beta f} = \text{tr} e^{-\beta (H' - H \cdot \tilde{v}_s)}, \\
\tilde{f} = \epsilon - Ts - \mu n, j - h \cdot v_s,
\]
\[\text{Eq. (12)}\]
where \( v_n = u \), the superfluid velocity operator \( \tilde{v}_s = h \nabla \theta/m \) is related the phase of the field operator \( \psi(r) = \sqrt{n_0} e^{i \theta} \) with the condensate density \( n_0 \), \( H \) is the conjugate variable of \( \tilde{v}_s \) and its density \( h \equiv H/V \), and \( v_s = \langle \tilde{v}_s \rangle \). The free energy density \( \tilde{f} \) would be a function of variables \( \{T, n, v_n, v_s\} \) and the variation of \( \tilde{f} \) with respect to \( dh \) is \( df = -v_s \cdot dh \).

The free energy density
\[
f = \tilde{f} + h \cdot v_s = \epsilon - Ts - m v_n \cdot j.
\]
which is a function of variables \( \{T, n, v_n, v_s\} \) and the variation of \( f \) with respect to \( d\nu_s \) is
\[
df = h \cdot d\nu_s.
\]
Similar to Eq. (7), the variation of the free energy density becomes
\[\text{Eq. (13)}\]
\[
\frac{df}{ds}T + \mu dn - m j \cdot d\nu_n + h \cdot d\nu_s.
\]

Form Eqs. (11) and (13), the particle number current \( j \) and the field conjugate to the superfluid velocity \( v_s \) are given respectively by
\[
j = -\frac{\partial f}{\partial \nu_n} = \frac{n_n v_n + n_s v_s}{z},
\]
\[
h = \frac{\partial f}{\partial \nu_s} = \frac{n_s m (v_s - v_n)}{z},
\]
\[\text{Eq. (14)}\]
where \( n_n = n - n_s \) is the particle number density of the normal part. From Eqs. (7) and (14), the thermodynamic relations are generalized as
\[
p = -\epsilon + Ts + m v_n \cdot j + \mu n, \\
Ts = dc - \mu dn - m v_n \cdot j - h \cdot d\nu_s.
\]
\[\text{Eq. (15)}\]
Equation (15) also holds for the anisotropic superfluid system.

III. TWO-FLUID EQUATIONS FOR BOSONS WITH ANISOTROPIC EFFECTIVE MASSES

For an anisotropic system \( K \) with different effective masses along three different principal axis directions, the Hamiltonian
\[
H = H_0 + H_{\text{int}},
\]
\[
H_0 = \int d^3r \psi^\dagger \left[ \frac{p^2}{2m_1} + \frac{p^2}{2m_2} + \frac{p^2}{2m_3} \right] \psi,
\]
\[
H_{\text{int}} = \frac{1}{2} \int d^3r_1 d^3r_2 \psi^\dagger (r_1) \psi^\dagger (r_2) V(r_1 - r_2) \psi (r_2) \psi (r_1),
\]
\[\text{Eq. (16)}\]
where \( m_i \) is the effective mass along the ith principal axis and \( \psi \) is the bosonic field operator. We should note that, although existence of anisotropic effective masses, the Hamiltonian (16) still has Galilean transformation invariance [31], and can describe the spin-orbit coupled BEC near ground states realized in recent experiments [20]. In specific, we write the effective mass as
\[
m_i = m z_i,
\]
where \( z_i = 1, 2, 3 \geq 1 \) characterize the enhancements of masses and \( m \) is the particle mass for the original system \( K \) with \( z_1 = z_2 = z_3 = 1 \).

A. Two-fluid equations for the anisotropic system

To obtain the two-fluid equations for the anisotropic system, we generalize the free energy density in Eq. (11) as
\[
f = f_0(T, n) - \sum_{i=1,2,3} \frac{n m v_{ni}^2}{2 z_i} + \sum_{i=1,2,3} \frac{n_s m (v_{si} - v_{ni})^2}{2 z_i},
\]
\[\text{Eq. (17)}\]
where \( f_0 \) is the free energy density in the rest frame of fluid.

Based on Eq. (17), similarly to Eq. (14), the particle number current of the ith axis direction is given by
\[
j_i = \frac{\partial f}{\partial v_{ni}} = \frac{n_n v_{ni} + n_s v_{si}}{z_i}.
\]
\[\text{Eq. (18)}\]
The conjugate variable of the superfluid velocity \( v_{si} \) is given by
\[
h_i = \frac{\partial f}{\partial v_{si}} = \frac{n_s m (v_{si} - v_{ni})}{z_i}.
\]
Although existence of anisotropy, the particle number, momentum and energy are still conserved. The corre-
sponding continuity equations are given by

\[
\frac{\partial n}{\partial t} + \sum_i \partial_i j_i = 0,
\]

\[
\frac{\partial g_i}{\partial t} + \sum_j \partial_j \pi_{ij} = 0,
\]

\[
\frac{\partial \varepsilon}{\partial t} + \sum_i \partial_i j_i^\varepsilon = 0,
\]

where \(g_i = m_i j_i = z_i m_j\), \(\pi_{ij}\) is the momentum current tensor, \(\varepsilon\) is the energy density, and \(j_i^\varepsilon\) is the energy current. The superfluid velocity can be written as a gradient of a condensate phase, i.e., \(\mathbf{v}_s = \hbar \nabla \theta / m\). Therefore, the superfluid velocity \(\mathbf{v}_s\) is irrotational and satisfies the equation [33]

\[
\frac{m \partial v_{si}}{\partial t} + \partial_i (\mu + X) = 0,
\]

where \(\mu\) is the chemical potential and \(X\) is the scalar function which need to be determined by an entropy equation (see the following). The irrotationality condition is

\[
\partial_i v_{sj} = \partial_j v_{si}.
\]

We should note that the superfluid velocity for the anisotropic system \(K\) with mass \(z_i m\), i.e., \(v_{si} / z_i\) would have no irrotationality [27] due to \(z_i \neq z_j\) in general.

Using the thermodynamic relations in Eq. (15), continuity Eqs. (19)-(21), Eqs. (22) and (23), we get

\[
T \left[ \frac{\partial s}{\partial t} + \sum_i \partial_i \left( \frac{sv_{ni}}{z_i} + \frac{Q_i}{T} \right) \right] = -\sum_i Q_i \frac{\partial T}{T} - \sum_i \left( \frac{g_i}{z_i m} - \frac{nv_{ni}}{z_i} - \frac{h_i}{m} \right) \partial_i \mu - \sum_{ij} \left( \frac{\pi_{ij}}{z_j} - \frac{p}{z_i} \delta_{ij} - \frac{m_j v_{ni}}{z_i} - \frac{v_{sj} h_i}{z_j} \right) \partial_i v_{nj} - \sum_i \left( \frac{X}{m} - \frac{v_{sj} v_{nj}}{z_j} \right) \partial_i h_i.
\]

In the deriving Eq. (24), we have introduced the heat current vector \(\mathbf{Q}\) with

\[
Q_i = j_i^\varepsilon - \frac{\mu (g_i / m - nv_{ni})}{z_i} - \sum_j \frac{v_{nj} \pi_{ij}}{z_j} - \frac{\varepsilon v_{ni}}{z_i} + \sum_j \frac{mv_{ni} v_{nj} j_j}{z_i} + \left( \frac{v_{nj} v_{nj}}{z_j} - \frac{X}{m} \right) h_i,
\]

and used the thermodynamic relation \(p = -\varepsilon + Ts + m v_{nj} J + \mu n\). The right-hand side of Eq. (24) is a form of “current” product “force” for entropy production. For dissipationless process, entropy production should be zero, so the right-hand side should vanish, i.e.,

\[
Q_i = 0, \quad \frac{g_i}{z_i m} - \frac{nv_{ni}}{z_i} - \frac{h_i}{m} = 0, \quad \pi_{ij} - \frac{p \delta_{ij}}{z_j} + \frac{m_j v_{ni}}{z_i} - \frac{v_{sj} h_i}{z_j} = 0, \quad \frac{X}{m} - \frac{v_{sj} v_{nj}}{z_j} = 0.
\]

From Eq. (25), we get constitutive relations

\[
X = \sum_j \frac{mv_{kj} v_{nj}}{z_j}, \quad g_i = mnv_{ni} + z_i h_i = n_i m v_{ni} + n_i m v_{si} = z_i m j_i, \quad \pi_{ij} = p \delta_{ij} + \frac{z_j m_j v_{ni}}{z_i} + v_{sj} h_i, \quad j_i^\varepsilon = \frac{\mu (g_i / m - nv_{ni})}{z_i} + \frac{v_{nj} \pi_{ij}}{z_j} + \frac{\varepsilon v_{ni}}{z_i} - \frac{mv_{ni} v_{nj} j_j}{z_i}.
\]

The entropy equation (24) becomes its conservation equation

\[
\frac{\partial s}{\partial t} + \sum_i \partial_i \left( \frac{sv_{ni}}{z_i} \right) = 0.
\]

The energy conservation equation can be replaced by the entropy conservation equation. So we have four complete equations for the two-fluid theory

\[
\frac{\partial n}{\partial t} + \sum_i \partial_i j_i = 0, \quad \frac{\partial g_i}{\partial t} + \sum_j \partial_j \pi_{ij} = 0, \quad \frac{\partial s}{\partial t} + \sum_i \partial_i \left( \frac{sv_{ni}}{z_i} \right) = 0, \quad \frac{m \partial v_{si}}{\partial t} + \partial_i (\mu + \sum_j \frac{mv_{kj} v_{nj}}{z_j}) = 0,
\]

with constitutive relations

\[
j_i = \frac{n_i v_{ni} + n_s v_{si}}{z_i}, \quad g_i = z_i m j_i = mn_i v_{ni} + mn_s v_{si}, \quad \pi_{ij} = p \delta_{ij} + \frac{mn_i v_{nj} v_{ni} + mn_s v_{sj} v_{si}}{z_i}.
\]

Eqs. (28)-(31) are the main results of this paper. These equations have several important properties. Firstly, we see the first prominent character of the above two-fluid equations (28)-(31) is that, due to \(z_i \neq z_j\) in general, the momentum current tensor \(\pi_{ij}\) would not be a symmetrical tensor in the anisotropic system, i.e., \(\pi_{ij} \neq \pi_{ji}\).
Secondly, when \( z_1 = z_2 = z_3 = 1 \), using the relation between the energy (\( \epsilon \)) in the laboratory frame which is at rest and that (\( \epsilon_0 \)) in another reference frame where the superfluid part is at rest [33], i.e.,

\[
\epsilon = \frac{n m v^2}{2} + g_0 \cdot v_s + \epsilon_0
\]

with \( g_0 = n n m (v_n - v_s) \), and further comparing the thermodynamic relation Eq. (15) with that in [33], i.e.,

\[
d\epsilon_0 = T ds + \mu_0 dn + (v_n - v_s) \cdot dg_0,
\]

we immediately get the relation for two chemical potentials \( \mu \) and \( \mu_0 \) through

\[
\mu_0 + \frac{m v^2}{2} = \mu + m v_s \cdot v_n.
\]

Here \( \mu_0 = \partial \epsilon_0 / \partial n \) denotes the chemical potential for the reference frame in which the superfluid part is at rest, while \( \mu = \partial \epsilon_0 / \partial n \) is the chemical potential for the laboratory frame. So Eqs. (28-31) recover the famous Landau-Khalatikov’s two-fluid equations [3, 33], by replacing of \( \mu + m v_s \cdot v_n \to \mu_0 + m v^2/2 \) in Eq. (31) and with constitutive relations \( j_i = n n v_{n i} + n s v_{s i}, g_i = m j_i, \) and \( \pi_j = p \delta_{i j} + m n n v_{n j} v_{n i} + m n s v_{s j} v_{s i} \). For the anisotropic case, the relation between two chemical potential is given by

\[
\mu_0 + \sum_j \frac{m v^2}{2 z_j} = \mu + \sum_j \frac{m v s j v_{n j}}{z_j}.
\]

Thirdly, at \( T = 0 \) \( (n_s = n, n_n = 0, s = 0, v_n = 0) \), the entropy in Eq. (30) can be neglected and the constitutive relations become \( j_i = n n v_{n i} / z_i; g_i = n m v_{s i}, \) and \( \pi_j = p \delta_{i j} + m n n v_{n j} v_{s i} / z_i \). Using the thermodynamic relation in Eq. (15) (Gibbs-Duhem relation for superfluid system at \( T = 0 \)), i.e., \( dp = nd \mu - h \cdot dv \) and irrotational condition \( \partial_t v_{s j} = \partial_j v_{s i} \), one can show that Eqs. (29) and (31) are equivalent. Taking \( \mu_0 = \sum_j m v^2 / (2 z_j) = \mu + \sum_j \frac{m v s j v_{n j}}{z_j} \) \( (v_s = 0) \) into account, the two-fluid equations (28-31) at \( T = 0 \) reduce to

\[
\frac{dn}{dt} + \sum_i \partial_t j_i = 0,
\]

\[
\frac{m \partial v_{n i}}{dt} + \partial_i \left( \mu_0 + \sum_j \frac{m v^2}{2 z_j} \right) = 0,
\]

which is consistent with Eqs. (8-10) for hydrodynamics of spin-orbit coupled BEC in Ref. [29], after replacements of \( \mu_0 \to g n + V_{ext} \) and \( v_{s i} \to z_i v_{s i} \) (replaced by the velocities for \( K \)). Therefore, in this sense, we can use the model of anisotropic effective mass [Eq. (16)] to describe the dynamics of the spin-orbit coupled BEC.

At last, we can also use the velocities for \( K \) [see Eq. (2)] to express the two-fluid equations, i.e., with replacements of \( v_{s i} \to z_i v_{s i} \) and \( v_{n i} \to z_i v_{n i} \) in the two-fluid equations (28-31).

### B. First sound and second sound

It is known that the existence of second sound is an important character for superfluidity. With the two-fluid equations, we can investigate the sound propagations for the anisotropic system. If the amplitudes of sound oscillations are small and the velocity fields \( v_{s i} \) is also small, then we can neglect the second order terms of velocities in the two-fluid equation, i.e.,

\[
\frac{\partial n}{\partial t} + \sum_i \partial_t j_i = 0,
\]

\[
\frac{\partial g_i}{\partial t} + \partial_t p = 0,
\]

\[
\frac{\partial s}{\partial t} + \sum_i \partial_i \left( s v_{n i} \right) = 0,
\]

\[
n \frac{\partial v_{s i}}{\partial t} + \partial_t \mu = 0,
\]

with \( g_i = z_i m j_i = m n n v_{n i} + m n s v_{s i} \).

From the first two equations, we get

\[
\frac{\partial^2 n}{\partial t^2} = \sum_i \frac{\partial^2 p}{z_i m n}.
\]

From equation \( g_i = m n n v_{n i} + m n s v_{s i} \), we get \( v_{n i} = (g_i - m n s v_{s i}) / (m n n) \) and

\[
\frac{\partial s}{\partial t} \simeq \sum_i \frac{s}{z_i m n} (\partial_t g_i - m n s \partial_t v_{s i}).
\]

By introducing the entropy for the unit mass, \( \tilde{s} = s / (n m) \), and \( ds = m s d n + m n d \tilde{s} \), we get

\[
m s \frac{\partial^2 n}{\partial t^2} + n m \frac{\partial^2 \tilde{s}}{\partial t^2} = \sum_i \frac{\tilde{s}}{z_i} \frac{\partial^2 p}{n m} + n m \frac{\partial^2 \tilde{s}}{\partial t^2}
\]

\[
= \sum_i \frac{n s}{z_i n} \left( \frac{\partial^2 p}{n m} - n s \frac{\partial^2 \tilde{s}}{\partial t^2} \right).
\]

Using the thermodynamic relation (Gibbs-Duhem relation) \( dp = nd \mu + s d T \) and \( n = n_n + n_s \), we get

\[
n m \frac{\partial^2 \tilde{s}}{\partial t^2} = \sum_i \left[ \frac{n s}{z_i n} (n \partial^2 \tilde{s} + s \partial^2 T) - \frac{n s}{z_i n} \frac{\partial^2 \tilde{s}}{\partial t^2} \right]
\]

\[
= \sum_i \frac{n s}{z_i n} \partial^2 T = \sum_i \frac{n n m s^2}{z_i n} \partial^2 T.
\]

Therefore, we obtain

\[
\frac{\partial^2 \tilde{s}}{\partial t^2} = \sum_i \frac{n n m s^2}{z_i m n} \partial^2 T.
\]

\[
\frac{\partial^2 n}{\partial t^2} = \sum_i \frac{\partial^2 p}{z_i m n} \partial^2 T.
\]
Choosing \((n, \bar{s})\) as independent variables, we have
\[
dp = \frac{\partial p}{\partial n} \bar{s} \, dn + \frac{\partial p}{\partial \bar{s}} \, n \, d\bar{s},
\]
\[
dT = \frac{\partial T}{\partial n} \bar{s} \, dn + \frac{\partial T}{\partial \bar{s}} \, n \, d\bar{s}.
\]
If the sound oscillations have the plane wave forms, i.e.,
\[
\left( \frac{\delta \bar{s}}{\delta n} \right) = \left( \begin{array}{c} A \\ B \end{array} \right) e^{i(q \cdot r - \omega t)},
\]
substituting it in Eq. (34), we get
\[
\frac{\omega^2}{\left( \begin{array}{c} A \\ B \end{array} \right)} = \left( \begin{array}{c} W(\alpha, \phi) \left( \frac{\partial T}{\partial \bar{s}} \right)_n \, W(\alpha, \phi) \left( \frac{\partial T}{\partial \bar{s}} \right)_s \end{array} \right) \left( \begin{array}{c} A \\ B \end{array} \right) q^2,
\]
where \(q = q [\cos(\alpha), \sin(\alpha) \cos(\phi), \sin(\alpha) \sin(\alpha)]\) and
\[
1 = \frac{\cos^2(\alpha)}{z_1} + \frac{\sin^2(\alpha) \cos^2(\phi)}{z_2} + \frac{\sin^2(\alpha) \sin^2(\phi)}{z_3}.
\]
We define the sound velocity \(c^2 = \omega^2/q^2\). In order to achieve non-trivial solutions in Eq. (35), we get
\[
\text{Det} \left[ \left( W(\alpha, \phi) \left( \frac{\partial T}{\partial \bar{s}} \right)_n + c^2 \frac{\partial W(\alpha, \phi) \left( \frac{\partial T}{\partial \bar{s}} \right)_s}{mZ(\alpha, \phi) \left( \frac{\partial n}{\partial \bar{s}} \right)_s} \right) \right] = 0.
\]
Further using the fact that \(\left( \frac{\partial T}{\partial m} \right)_s \left( \frac{\partial n}{\partial m} \right)_s - \left( \frac{\partial T}{\partial n} \right)_s \left( \frac{\partial n}{\partial m} \right)_s = \frac{1}{c^2} \frac{\partial n}{\partial \bar{s}} \frac{\partial n}{\partial \bar{s}} \right) \), where we have introduced the heat capacity at constant volume \(C_V = T \left( \frac{\partial s}{\partial T} \right)_V\), and used the relation \(\frac{\partial n}{\partial \bar{s}} = -\frac{\partial n}{\partial \bar{s}} \frac{\partial T}{\partial \bar{s}} \), the sound velocity equation becomes
\[
c^4 = \left[ \frac{TW}{C_V} + \frac{1}{Z} \left( \frac{\partial p}{\partial n} \right)_s \right] c^2 + \frac{TW}{C_V Z} \left( \frac{\partial p}{\partial n} \right)_T = 0,
\]
whose compressibility \(\partial p/\partial n = \partial p/ (m \, \partial n)\). From Eq. (37), we can get the first sound velocity \(c_1\) and the second sound velocity \(c_2\) [34]. We see the effects of enhancements of effective masses for sound velocities are to produce a factor \(\sqrt{1/Z(\alpha, \phi)} \leq 1\).

At zero temperature \(T = 0\) (\(s = 0, n_n = 0, s_n = n, v_n = 0\)), the linear Equation (33) reduces to
\[
\frac{\partial n}{\partial t} + \frac{n \, \partial v_{xx}}{z_1} + \frac{n \, \partial v_{yy}}{z_2} + \frac{n \, \partial v_{zz}}{z_3} = 0,
\]
\[
m \frac{\partial v_{xx}}{\partial t} + \nabla \cdot \mu = 0,
\]
where \(n\) is the average particle number density in the ground state.

The first and second sounds may be probed by measuring the density response functions. In order to get the density response function, we need to add an external perturbation potential \(\delta U e^{i(q \cdot r - \omega t)}\) in Eq. (34) of the sound propagations, e.g.,
\[
\frac{\partial^2 \bar{s}}{\partial t^2} = \sum_i \frac{n_i \, s_i^2}{z_i} \, \partial_i^2 T,
\]
\[
\frac{\partial^2 n}{\partial t^2} = \sum_i \frac{1}{z_i \, m} \, \partial_i^2 \left[ \frac{p_i + n \delta U e^{i(q \cdot r - \omega t)}}{q^2} \right].
\]
The density response function is defined as
\[
\chi(q, \omega) = \frac{\delta n}{\delta U e^{i(q \cdot r - \omega t)}}.
\]
From
\[
\frac{\omega^2}{\left( \frac{\delta \bar{s}}{\delta n} \right)} = \left( \begin{array}{c} W(\alpha, \phi) \left( \frac{\partial T}{\partial \bar{s}} \right)_n \, W(\alpha, \phi) \left( \frac{\partial T}{\partial \bar{s}} \right) \end{array} \right) \left( \begin{array}{c} \delta \bar{s} \\ \delta n \end{array} \right) q^2
\]
\[
+ \left( \begin{array}{c} 0 \\ \frac{\partial s}{\partial m} \end{array} \right) q^2 e^{i(q \cdot r - \omega t)},
\]
we get
\[
\delta \bar{s} = \frac{n \left[ \omega^2 q^2 - q^4 W(\alpha, \phi) \frac{\partial T}{\partial m} \right]}{mZ(\alpha, \phi) \left[ \omega^4 - (c_1^2 + c_2^2) \omega^2 q^2 + c_1^2 c_2^2 q^4 \right]} \delta U e^{i(q \cdot r - \omega t)}.
\]
The density response function is given by
\[
\chi(q, \omega) = \frac{n \left[ \omega^2 q^2 - q^4 W(\alpha, \phi) \frac{\partial T}{\partial m} \right]}{mZ(\alpha, \phi) \left[ \omega^4 - (c_1^2 + c_2^2) \omega^2 q^2 + c_1^2 c_2^2 q^4 \right]} \delta U e^{i(q \cdot r - \omega t)}.
\]
So the density response function is given by
\[
\chi(q, \omega) = \frac{n \left[ \omega^2 q^2 - q^4 W(\alpha, \phi) \frac{\partial T}{\partial m} \right]}{mZ(\alpha, \phi) \left[ \omega^4 - (c_1^2 + c_2^2) \omega^2 q^2 + c_1^2 c_2^2 q^4 \right]} \delta U e^{i(q \cdot r - \omega t)}.
\]
In Eq. (41), we have introduced the weights \(w_1(2)\) for the first (second) sound in the density response functions, which satisfies
\[
\frac{w_1 + w_2}{c_1^2 + c_2^2} = \frac{1}{c_1^2 + c_2^2},
\]
\[
\frac{w_1}{c_1^2} + \frac{w_2}{c_2^2} = \frac{1}{\left( \frac{\partial p}{\partial n} \right)_T}.
\]
The imaginary part of the density response function is
\[
\chi''(q, \omega) = \text{Im}[\chi(q, \omega + i \omega)]
\]
\[
= -\frac{\pi n}{2m} \left\{ \frac{w_1 q}{c_1} \left[ \delta(\omega - c_1 q) - \delta(\omega + c_1 q) \right] \right. \]
\[
+ \frac{w_2 q}{c_2} \left[ \delta(\omega - c_2 q) - \delta(\omega + c_2 q) \right] \right\} .
\]
The f-sum rule and the compressibility sum rules (for unit volume) [35, 36] are obtained
\[
-\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \omega \chi''(q, \omega) = \frac{m q^2}{m Z(\alpha, \phi)},
\]
\[
\lim_{q \to 0} \left\{ -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{\omega \chi''(q, \omega)}{\omega} \right\} = \frac{n}{\left( \frac{\partial p}{\partial n} \right)_T}.
\]
or in terms of the dynamic structure factor \( S(q, \omega) = \frac{1}{\pi(1 - e^{-\omega/T})} \text{Im}[\chi(q, \omega + i0)] \) [set the Boltzmann constant \( k_B = 1 \)],

\[
\int_{-\infty}^{\infty} d\omega S(q, \omega) = \frac{nq^2}{2mZ(\alpha, \phi)}, \\
\lim_{q \to 0} \int_{-\infty}^{\infty} \frac{d\omega S(q, \omega)}{\omega} = \frac{n}{2(\partial\rho/\partial p)_T}.
\] (45)

We see that in the anisotropic effective mass system, the weight of sound oscillations in the density response decrease due to enhancement of effective mass.

### C. Normal density from thermal phonon excitations near \( T=0 \)

At low temperature, the gapless phonon excitations would dominate the thermodynamics and result in the normal density. The normal density can be calculated from phonon excitations by using the Landau’s theory [37]. Assuming a thin tube filled with liquid is moving with velocity \( u \) along the \( i \)th axis direction. The normal part is moving due to dragging by the tube and in equilibrium with tube wall, while the superfluid part is at rest. The current associated normal part is given by

\[
j_i = \sum_q q_i n(q),
\] (46)

where \( q_i=x,y,z \) is the \( i \)th component of vector \( q \), \( n(q) = 1/[e^{(\omega(q)+\nu)/\hbar T} - 1] \) is the Bose distribution for phonon, the phonon energy \( \omega(q) = c(q)q \), the sound velocity \( c(q) = c_0\sqrt{q^2/z_1 + q^2/z_2 + q^2/z_3} \), and \( c_0 = \sqrt{\partial\rho/\partial p} \) is the sound velocity determined by compressibility at zero temperature. The average drift velocity of the phonon gas is exactly given by

\[
\bar{v} = \frac{\sum_q v_i n(q)}{\sum_q n(q)} = u,
\] (47)

with the phonon group velocity \( v_i = \partial\omega(q)/\partial q_i \).

On the other hand, the current from the normal part is given by \( j_i = \rho_n \bar{v} \) with the normal density \( \rho_n \). From Eqs. (46) and (47) and taking the limit of \( u \to 0 \), we get

\[
\rho_n = z_i \sqrt{z_1 z_2 z_3} \frac{2\pi^2 T^4}{45\hbar^3 c_0^3}.
\] (48)

We see the normal density satisfies relation \( \rho_n : \rho_{ny} : \rho_{nz} = z_1 : z_2 : z_3 \). When \( z_i = 1 \), the normal density is reduced to the Landau’s result \( \rho_n,\text{Landau} = 2\pi^2 T^4 / (45\hbar^3 c_0^3) \) [2, 34]. The corrections of the normal density relative to the usual Landau’s result is given by

\[
\beta_i \equiv \rho_n / \rho_n,\text{Landau} = z_i \sqrt{z_1 z_2 z_3}.
\]

The normal particle number density is given by

\[
n_n = \rho_n/(z_i m) = \sqrt{z_1 z_2 z_3} \frac{2\pi^2 T^4}{45\hbar^3 c_0^3}.
\] (49)

We see that when the effective masses increase, the normal density from phonon excitations also increases. This is because that as \( z_i > 0 \), the phonon excitation energy \( \omega(q) \) decreases for a fixed momentum \( q \), then the phonon number also increases for a given temperature \( T \).

Near the zero temperature, the free energy is given by

\[
F = E_0 + F_{\text{phonon}},
\]

\[
F_{\text{phonon}} = -T \sum q \ln\left(\frac{1}{1 - e^{-\omega(q)/T}}\right)
\]

\[
= \frac{TV}{(2\pi\hbar)^3} \int dq \ln[1 - e^{-\omega(q)/T}] = -\sqrt{z_1 z_2 z_3} \frac{\sqrt{\pi^2} T^4}{90\hbar^3 c_0^3},
\]

where \( E_0 \) is the ground state energy. The entropy and heat capacity are given by

\[
\dot{s} = -\frac{\partial F}{n m \partial T} = \sqrt{z_1 z_2 z_3} \frac{2\pi^2 T^3}{45\hbar^3 c_0^3},
\]

\[
C_V = T \frac{\partial \dot{s}}{\partial T} = \sqrt{z_1 z_2 z_3} \frac{2\pi^2 T^3}{15\hbar^3 c_0^3}.
\]

Near \( T = 0 \), the adiabatic compressibility would equal with the isothermal compressibility, i.e., \( \frac{\partial p}{\partial \rho} \approx (\frac{\partial p}{\partial T})_T \), so we get first and second sound velocities from Eq. (37)

\[
c_1 = \sqrt{\frac{1}{Z(\alpha, \phi)} \left(\frac{\partial\rho}{\partial p}\right)_\phi} = \sqrt{\frac{1}{Z(\alpha, \phi)} c_0},
\]

\[
c_2 = \sqrt{\frac{TW(\alpha, \phi)}{C_V}} = \frac{1}{(z_1 z_2 z_3)^{1/4}} \frac{c_1}{\sqrt{3}}.
\] (50)

For isotropic system \( (z_1 = z_2 = z_3) \), the above formula [Eq. (50)] for second sound recovers the famous Landau’s result, i.e., \( c_2 = c_1/\sqrt{3} \) [2]. Comparing with usual case, the first sound velocity is suppressed by a factor \( \sqrt{1/Z(\alpha, \phi)} \), while the second sound is suppressed by a factor \( \sqrt{1/Z(\alpha, \phi)} \sqrt{z_1 z_2 z_3} \). The correction of the second sound along the \( i \)th axis direction is

\[
\gamma_i \equiv \frac{c_2}{(c_0/\sqrt{3})} = \sqrt{1/(z_i \sqrt{z_1 z_2 z_3})}.
\]

As \( T \to 0 \), the weight of second in the density response is proportional to difference between two compressibility, i.e., \( \Delta (\frac{\partial p}{\partial \rho}) \equiv (\frac{\partial p}{\partial T})_T - (\frac{\partial p}{\partial T})_T \propto T^4 \to 0 \). So, the weight of first sound \( w_1 \to 1/Z(\alpha, \phi) \) and weight for second sound \( w_2 \to 0 \) [see Eq. (42)].
IV. SPIN-ORBIT COUPLED BEC

A. Normal density from phonon excitations and sound velocities

The Hamiltonian of spin-orbit coupled BEC is given by [21–25, 38]

\[
H = H_0 + H_{\text{int}}, \quad H_0 = \int d^3r \psi^\dag \left[ \frac{(p_x - k_0 \sigma_x)^2 + p_y^2 + p_z^2}{2m} + \frac{\Omega}{2} \sigma_x \right] \psi,
\]

\[
H_{\text{int}} = \frac{1}{2} \int d^3r \left[ g' \psi_1^\dag(r) \psi_2^\dag(r) \psi_2(r) \psi_1(r) + \text{H.c.} \right] + g \psi_1^\dag(r) \psi_2^\dag(r) \psi_2(r),
\]

(51)

where \( k_0 \) and \( \Omega \) are the strengths of the spin-orbit and Raman couplings, \( \psi_{1(2)} \) is the bosonic field operator, \( \psi^\dag = [\psi_1^\dag, \psi_2^\dag] \) is spinor form, \( m \) is the atomic mass, \( g = 4\pi \hbar^2 a_s / m \) and \( g' = 4\pi \hbar^2 a'_s / m \) are the strengths of the intra- and inter-species interactions with \( a_s \) and \( a'_s \) being the s-wave scattering lengths. The above Hamiltonian breaks the Galilean transform invariance [39], however we will see that the effective low energy hydrodynamics for sound oscillations restore the Galilean invariance [31]. In the following, we focus on the case of \( U(2) \) invariant interaction, i.e., \( g' = g \). In this section, we set \( m = 1 \) and \( \hbar = 1 \) for simplification.

At zero temperature, the mean-field ground state wave function is written as [28, 40–44]

\[
|0\rangle = \sqrt{n_0} \left( \cos(\theta) s - \sin(\theta) c_{0x} \right) e^{ip_0 x}.
\]

When \( \Omega < 2 k_0^2 \), \( p_0 = k_0 \sqrt{1 - \Omega^2 / (4 k_0^2)} \) and \( \cos(2\theta) = p_0 / k_0 \); while for \( \Omega > 2 k_0^2 \), \( p_0 = 0 \) and \( \theta = \pi / 4 \). A quantum phase transition occurs at \( \Omega = 2k_0^2 \) where the sound speed along the x-axis direction becomes zero [25, 40, 45]. \( n_0 \) is atom number density in condensates. For weakly interacting Boson gas, \( n_0 \approx n \) (total particle number density).

Here we try to give a derivation of hydrodynamics for low energy phonon excitation. Our starting point is the microscopic equation of the order parameter, i.e., the time-dependent Gross-Pitaevskii (GP) equation. We assume the order parameter is

\[
|\psi\rangle = \left( \frac{n_1 e^{i\theta_1}}{n_2 e^{i\theta_2}} \right),
\]

which satisfies the time-dependent GP equation [46]

Around the ground state, we expand the GP equations in terms of small fluctuations \( \delta n_s \) and \( \delta \theta_s \) and get the linear equations

\[
\partial_t \delta n_1 = -\left( (p_0 - k_0) \partial_x \delta n_1 + n_1 \nabla^2 \delta \theta_1 \right) + \Omega \sqrt{n_1} n_2 (\delta \theta_1 - \delta \theta_2),
\]

\[
\partial_t \delta n_2 = -\left( (p_0 + k_0) \partial_x \delta n_2 + n_2 \nabla^2 \delta \theta_2 \right) - \Omega \sqrt{n_1} n_2 (\delta \theta_1 - \delta \theta_2),
\]

\[
\partial_t \delta \theta_1 = -\frac{\Omega \sqrt{n_1}}{4 n_1} + (p_0 - k_0) \partial_x \delta \theta_1 + (g \delta n_2 + g \delta n_1),
\]

\[
\partial_t \delta \theta_2 = -\frac{\Omega \sqrt{n_2}}{4 n_2} + (p_0 + k_0) \partial_x \delta \theta_2 + (g \delta n_2 + g \delta n_1),
\]

(52)

where \( \bar{n}_1(2) \) denotes its average value in the ground state.

Next we introduce the total density fluctuation \( \delta n = \delta n_1 + \delta n_2 \), the spin polarization \( \delta S_z = \delta n_1 - \delta n_2 \), the common phase \( \delta \theta = (\delta \theta_1 - \delta \theta_2) / 2 \), and the relative phase fluctuation \( \delta \theta_R = \delta \theta_1 - \delta \theta_2 \). At low energy \( \omega \to 0 \) and \( q \to 0 \), we can adiabatically eliminate the spin part of fluctuation. Then we get effective hydrodynamic equation for the total density fluctuation \( \delta n \) and common phase \( \delta \theta \) at \( T = 0 \) [29] i.e.,

\[
\partial_t \delta n = -n \frac{\partial^2 \delta \theta}{z_1} + (\partial_y^2 + \partial_z^2) \delta \theta],
\]

\[
- \partial_t \delta \theta = g \delta n,
\]

where \( n = \bar{n}_1 + \bar{n}_2 \) is the average particle density in the ground state, \( z_1 = 1 / \cos^2(2\theta) = 1 / \left[ 1 - \Omega^2 / (4k_0^2) \right] \) describes the enhancements of effective masses for the plane-wave phase and \( z_1 = 1 / (1 - 2k_0^2 / \Omega) \) for the zero-momentum phase. Near the phase transition (\( \Omega \to 2k_0^2 \)), \( z_1 \to \infty \). From Eq. (52), we get the sound velocity

\[
\omega_{\pm q} = c(\hat{q}) q,
\]

where \( c(\hat{q}) \equiv \sqrt{c_0^2 / z_1 + \sin^2(\alpha) c_0}, \quad c_0 = \sqrt{g n / m} = \sqrt{\partial p / \partial n} = \sqrt{n \partial \mu / \partial n} \) with \( \mu = g n - \Omega^2 / (8 k_0^2) \) for the plane-wave phase and \( \mu = g n + (k_0^2 - \Omega^2) / 2 \) for the zero-momentum phase [28], \( \hat{q} = q / q = (\cos(\alpha), \sin(\alpha) \cos(\phi), \sin(\alpha) \sin(\phi)) \), and \( \alpha \) is angle between \( \hat{q} \) and x-axis. Taking the spatial derivatives of the second equation and identifying \( g \delta n \to \mu \) (deviations relative to the ground state values), Eq. (52) becomes Eq. (38) of the linear two-fluid equation with \( z_{1L} = 1 / z_2 = 1 / z_3 = 1 \).

From Eqs. (48) and (50), we get the normal density from thermal phonon excitations and the first and second
sound velocities in spin-orbit coupled BEC, i.e.,
\[
\rho_n(\hat{x}) = \frac{3}{2} \frac{2\pi^2 T^4}{4\hbar^3 c_0^2}, \\
\rho_n(\hat{y}) = \rho_n(\hat{z}) = \frac{3}{2} \frac{2\pi^2 T^4}{4\hbar^3 c_0^2}, \\
c_1(\hat{q}) = c_0 \sqrt{\cos^2(\beta)/z_1 + \sin^2(\beta)}, \\
c_2(\hat{q}) = \frac{1}{(z_1)^{1/4}} \frac{c_0(\hat{q})}{\sqrt{3}} = \frac{1}{(z_1)^{1/4}} \frac{c_0(\hat{q})}{\sqrt{3}}.
\]
(53)

Along the \( x \)-direction, the corrections of the normal density and the second sound velocity are
\[
\beta_1 = \frac{3}{z_1}, \\
\gamma_1 = \frac{1}{z_1}.
\]
(54)

From Eqs. (53) and (54), we see that with increasing the effective mass, the normal density increases; while the second sound velocity decreases. Especially, when \( \Omega \to 2k_0^2 \), near the phase transition \( z_1 = 1/\sqrt{1 - \Omega^2/(4k_0^2)} \) or \( 1/[1 - 2k_0^2/\Omega] \to \infty \) of the plane-wave and zero-momentum phases in spin-orbit coupled BEC, the effective mass diverges along the \( x \)-axis direction. The normal density \( \rho_n(\hat{x}) \) from phonon excitations would be enhanced greatly (see Fig.1), while the second sound velocity along the \( x \)-axis direction \( c_2(\hat{x}) \to 0 \).

![Graph](image)

**FIG. 1**: The corrections of the normal density [panel (a)] and the second sound velocity [panel (b)] in spin-orbit coupled BEC (along the \( x \)-axis direction). Note that near the phase transition \( (\Omega/k_0^2 \to 2) \), the effective mass would diverge, i.e., \( z_1 \to \infty \).

### B. Superfluid density and Josephson relation

We should remark that one can take two different viewpoints on the effects of enhancement of effective mass in Eq. (52). The first one is that the particle density does not change, while the superfluid velocity decreases by a factor \( 1/z_1 \), which is adopted by previous sections in this paper. The other one is that the superfluid density decreases by a factor \( 1/z_1 \), while the superfluid velocity does not change, which would be adopted in following part in this subsection.

Here we introduce superfluid density along the \( \hat{q} \)-direction \( \rho_s(\hat{q}) \), i.e.,
\[
\rho_s(\hat{x}) = n/z_1, \quad \rho_s(\hat{y}) = \rho_s(\hat{z}) = \rho_s = n, \\
\rho_s(\hat{q}) = \rho_{sx} \cos^2(\alpha) + \rho_{sz} \sin^2(\alpha).
\]
(55)

In this case, Eq. (52) becomes
\[
\partial_t \delta n = -\rho_s(\hat{x}) \partial_{\hat{x}}^2 \delta \theta - \rho_s \left( \partial_{\hat{q}}^2 + \partial_{\hat{z}}^2 \right) \delta \theta, \\
- \partial_\theta \delta \theta = g \delta n.
\]
(56)

In the following, we will show that \( \rho_s(\hat{q}) \) indeed is the superfluid density. Due to influences of upper branch, in the cases of spin-orbit coupled BEC, the superfluid density \( \rho_s(\hat{q}) \) would be smaller than total density, i.e., \( \rho_s(\hat{q}) < n \) [26]. In this sense, we can interpret the suppression of superfluid density in spin-orbit coupled BEC is due to the enhancement of effective mass by a factor \( z_1 \geq 1 \).

Further more, we can write down an effective Hamiltonian for the hydrodynamic equation (56), i.e.,
\[
H_{\text{eff}} = \frac{1}{2} \int d^3r \left\{ \rho_s(\hat{x}) \partial_{\hat{x}}^2 \delta \theta^2 + \rho_s + \delta \left[ (\partial_{\hat{q}}^2 + \partial_{\hat{z}}^2) \delta \theta^2 + g(\delta n)^2 \right] \right\}.
\]
(57)

Assuming the commutator relation \( \{\delta \theta(r), \delta n(r')\} = -i\delta^3(r-r') \) (Poisson brackets), we can easily get the above hydrodynamic equation (56) from the Hamilton’s equations, i.e.,
\[
\partial_t \delta n(r) = \{\delta n(r), H_{\text{eff}}\}, \partial_\theta \delta \theta(r) = \{\delta \theta(r), H_{\text{eff}}\}.
\]

By means of the quantized commutator relation \( [\delta \theta(r), \delta n(r')] = -i\delta^3(r-r') \) [47], the phase \( \delta \theta \) and density \( \delta n \) can be expanded by the phonon’s annihilation and creation operators, e.g.,
\[
\delta \theta(r,t) = \sum_q [A_q C_q e^{-i(q \cdot r - \omega_q t)} + A_q^* C_q^\dagger e^{i(q \cdot r - \omega_q t)}], \\
\delta n(r,t) = \sum_q [B_q C_q e^{-i(q \cdot r - \omega_q t)} + B_q^* C_q^\dagger e^{i(q \cdot r - \omega_q t)}],
\]

where \( A_q(B_q) \) is coefficient to be determined and \( C_q \) is the annihilation operator for phonon. From the continuity equation
\[
\partial_t \delta n = -\rho_s(\hat{x}) \partial_{\hat{x}}^2 \delta \theta - \rho_s \left( \partial_{\hat{q}}^2 + \partial_{\hat{z}}^2 \right) \delta \theta,
\]
we get
\[
-i\hat{q} B_q = q \rho_s(\hat{q}) A_q.
\]
From the commutator relation \([\delta \theta (\mathbf{r}), \delta n (\mathbf{r}')] = -i \delta^3 (\mathbf{r} - \mathbf{r}')\), we get \(A_Q B_Q^* = -i/2\) and then \(A_Q = \sqrt{c(\mathbf{q})/2 \rho_s(\mathbf{q}) q}\), \(B_Q = i \sqrt{\rho_s(\mathbf{q}) q/2 c(\mathbf{q})}\). Finally, we have

\[
\delta \theta (\mathbf{r}, t) = \sum_q \sqrt{\frac{c(\mathbf{q})}{2 \rho_s(\mathbf{q}) q}} [C_q e^{i (\mathbf{q} \cdot \mathbf{r} - \omega_q t)} + C_q^* e^{-i (\mathbf{q} \cdot \mathbf{r} - \omega_q t)}],
\]

\[
\delta n (\mathbf{r}, t) = \sum_q \sqrt{\frac{\rho_s(\mathbf{q}) q}{2 c(\mathbf{q})}} [C_q e^{i (\mathbf{q} \cdot \mathbf{r} - \omega_q t)} - C_q^* e^{-i (\mathbf{q} \cdot \mathbf{r} - \omega_q t)}].
\]

From Eq. (58), we get density and phase fluctuations in terms of phonon’s operators, i.e.,

\[
n_q = i \sqrt{\frac{\rho_s(\mathbf{q}) q}{2 c(\mathbf{q})}} (C_q - C_q^*),
\]

\[
\theta_q = \sqrt{\frac{c(\mathbf{q})}{2 \rho_s(\mathbf{q}) q}} (C_q + C_q^*),
\]

(59)

From Eq. (59), we can verify that \(\rho_s(\mathbf{q})\) is indeed the superfluid density. For example, the superfluid density can be written as [26]

\[
\rho_s(\mathbf{q}) = c^2 (\mathbf{q}) \kappa (\mathbf{q}) = \lim_{q \to 0} \frac{|(\mathbf{q} | n_{-\mathbf{q}} | 0)|^2 \omega_q + |(-\mathbf{q} | n_{\mathbf{q}} | 0)|^2 \omega_q}{q^2},
\]

(60)

where

\[
\kappa (\mathbf{q}) = \lim_{q \to 0} \left[ \frac{|(\mathbf{q} | n_{+\mathbf{q}} | 0)|^2}{c(\mathbf{q}) q} + \frac{|(-\mathbf{q} | n_{-\mathbf{q}} | 0)|^2}{c(\mathbf{q}) q} \right]
\]

is the compressibility, \(0\) is the ground state, and \(|\mathbf{q}\rangle\rangle = C_q |0\rangle\rangle\) is the single-phonon state. In Eq. (60), we have used the fact that the single-phonon’s contributions is dominant in the compressibility [26] and \(\omega_q = eq\).

On the other hand, as \(q \to 0\) and at low energy, the boson field operator can be written as [47]

\[
\psi_\sigma (\mathbf{r}) = \langle \psi_\sigma \rangle e^{i \delta \theta (\mathbf{r})} \simeq \langle \psi_\sigma \rangle [1 + i \delta \theta (\mathbf{r})].
\]

So we get

\[
\psi_\sigma, \mathbf{q} = i \langle \psi_\sigma \rangle \theta_q = i \langle \psi_\sigma \rangle \sqrt{\frac{c(\mathbf{q})}{2 \rho_s(\mathbf{q}) q}} (C_q + C_q^*).\]

(61)

With Eq. (61), the matrix element \(\langle 0 | \psi_\sigma, \mathbf{q} | \mathbf{q} \rangle = i \langle \psi_\sigma \rangle \sqrt{\frac{c(\mathbf{q})}{2 \rho_s(\mathbf{q}) q}}\) and the Green’s function \(G_{\sigma \sigma} (\mathbf{q}, 0) = -\frac{|\langle \psi_{\sigma \sigma} | c^2 (\mathbf{q}) q | \mathbf{q} \rangle |^2}{\rho_s(\mathbf{q}) c(\mathbf{q}) q} + \frac{|\langle \psi_{\sigma \sigma} | c^2 (\mathbf{q}) q | \mathbf{q} \rangle |^2}{\rho_s(\mathbf{q}) c(\mathbf{q}) q} \) as \(q \to 0\) [48]. Using \(n_0 = \sum_{\sigma = 1, 2} |\langle \psi_{\sigma \sigma} \rangle|^2\), the Josephson relation is obtained [48]

\[
\rho_s(\mathbf{q}) = - \lim_{q \to 0} \frac{n_0}{q^2 eG(\mathbf{q}, 0)}.\]  

The superfluid density from the Josephson relation [Eq. (62)] is also consistent with the current-current correlation calculations [26].

From Eq. (61) of \(\psi_\sigma, \mathbf{q}\), we get the momentum distribution function as \(q \to 0\),

\[
N_q = \sum_{\sigma = 1, 2} \langle \psi_\sigma^\dagger q | \psi_\sigma q \rangle = \frac{n_0 c(\mathbf{q})}{2 \rho_s(\mathbf{q}) q} (2 n_q + 1),
\]

where \(n_q = 1/(e^{\omega_q/T} - 1)\) is the phonon Bose distribution function for the rest frame. Specially, at \(T = 0\) and as \(q \to 0\), \(N_q = n_0 c(\mathbf{q})/[2 \rho_s(\mathbf{q}) q] \propto 1/q^2\); when \(\omega_q \ll T\), \(N_q = n_0 T/[\rho_s(\mathbf{q}) q^2] \propto 1/q^2\), which is an anisotropic generalization of the isotropic result [47, 49].

Using the effective Hamiltonian, i.e., Eq. (57), we can calculate the phase or density fluctuations within the hydrodynamic formalism [34]. The energy in the momentum space is

\[
\delta E = H_{\text{eff}} = \frac{1}{2} \int d^3 \mathbf{r} \{ \rho_s(\mathbf{q}) (\partial_s \delta \theta)^2 + \rho_s \langle \partial_q \delta \theta^2 + (\partial_q \delta \theta)^2 \} + g (\delta n)^2,\]

(63)

Because the thermal probability distribution is

\[
P = e^{-\delta E/T} = e^{-\frac{1}{2} \sum_q \rho_s(\mathbf{q}) q^2 |\delta \theta_q|^2 + g |\delta n_q|^2}/T.
\]

so for the long wave length fluctuation (\(\omega_q \ll T\)), the thermal fluctuations of the phase and density are

\[
\langle |\delta \theta_q|^2 \rangle = \frac{T}{\rho_s(\mathbf{q}) q^2}, \langle |\delta n_q|^2 \rangle = \frac{T}{g}.
\]

Along the \(x\)-axis direction (\(\mathbf{q} = \mathbf{\hat{x}}\)), we see the fluctuation of phase near phase transition point \(|\rho_s(\mathbf{\hat{x}}) | \rightarrow 0\) is very dramatic and diverges, while the fluctuation of the density is finite.

V. CONCLUSION

In summary, we have generalized the two-fluid theory to anisotropic superfluid system with several different effective masses along different axes. In general, the weights of sound oscillations in the density response functions and the sound velocities are suppressed due to the enhancements of effective masses. For spin-orbit coupled BEC, we construct hydrodynamics for low energy phonons’ excitation by eliminating the up branch gapped spin excitations. With hydrodynamics, due to the enhancements of effective masses, the normal density arising from the phonon excitations is enhanced, while the second sound velocity is suppressed. Such effects can be manifested near the phase transition from the plane-wave to zero-momentum phases in spin-orbit coupled BEC.
where the effects of effect mass enhancements are most obvious. With the quantum hydrodynamics, the suppressed superfluid density, Josephson relation, momentum distributions and phase fluctuations for long wave length are also investigated. We have found that near the phase transition, the phase fluctuations are most obvious. The experimental measurements on the dynamical structure factors would provide the anisotropic dynamic information for spin-orbit coupled BEC.

The anisotropic two-fluid model can be used to describe the spin-coupled BEC near zero temperature, while how to construct corresponding hydrodynamic theory at higher temperature where the up gapped excitation would play an important role, still needs further investigations.

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