STUDIES ON THE GREEN-GRIFFITHS-LANG CONJECTURE

MOHAMMAD REZA RAHMATI

Abstract. In this short note we explain some generalizations of the works by J. P. Demailly, J. Merker, Y. T. Siu on Green-Griffiths-Lang conjecture for the entire curve locus. We try to apply the Holomorphic Morse inequalities for the invariant jet differentials. Also we apply the similar argument to examine a generalization of a result of J. Merker and Y. T. Siu.

1. Introduction

Demaily-Semple tower associated to a pair \((X, V)\) where \(X\) is a complex projective manifold (probably singular) and \(V\) is a holomorphic subbundle of \(T_X\) the tangent bundle of \(X\) of rank \(r\), is a prolongation sequence of projective bundles

\[
P^{r-1} \to X_k = P(V_{k-1}) \to X_{k-1}, \quad k \geq 1
\]

obtained inductively making \(X_k\) a weighted projective bundle over \(X\). The sequence provides a tool to study the locus of nonconstant holomorphic maps \(f : \mathbb{C} \to X\) such that \(f'(t) \in V\). It is a conjecture due to Green-Griffiths-Lang that the total image of all these curves is included in a proper subvariety of \(X\); if \(X\) is of general type. Another way to look at the above prolongation sequence is to consider the vector bundles \(E_{k,m}^{\text{GG}}\) of germs of weighted homogeneous polynomials along the fibers in \(X_k \to X\). These polynomials can be definitely considered in the germs of indeterminates \(z = f, \xi(1) = f', ..., \xi(k) = f^{(k)}\). There is a well defined differential operator induced by formal differentiation over polynomials \(P(f, f', ..., f^{(k)}) \in \Gamma(X, E_{k,m}^{\text{GG}})\);

\[
(\cdot)' = \delta : E_{k,m}^{\text{GG}} \to E_{k+1,m+1}^{\text{GG}}; \quad \delta = \xi_1 \frac{\partial}{\partial z} + \sum_i \xi_{i+1} \nabla \xi_i
\]

One also has \(E_{k,m}^{\text{GG}} V^* = J_k V / \mathbb{C}^*\) where \(J_k V\) is the bundle of \(k\)-jets of \(V\). The symbol \(V^*\) used in \(E_{k,m}^{\text{GG}} V^*\) is only a notation reminding the duality appearing in the fact that \(S^*V = \text{Sym}^*V\) can be understood as polynomial algebra on a basis of \(V^*\) (here \(V\) is a vector space). The analytic structure on \(E_{k,m}^{\text{GG}} V^*\) comes from the fact that, one can arrange the germs to depend holomorphically on their initial values at

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0. The vector bundles $E_{k,m}^{GG}$ are naturally $D_{X_k}$-modules. One can define a decreasing filtration on $E_{k,m}^{GG}$ by

$$F^p E_{k,m}^{GG} = \{ Q(z, \xi) \mid Q \text{ homogeneous of weighted degree } m \geq p \}$$

Then we may set

$$E_{k,\bullet}^{GG} V^* = \bigoplus_m E_{k,m}^{GG} V^*, \quad X_k^{GG} := \text{Proj}(E_{k,\bullet}^{GG} V^*) = \text{Proj}(gr_F E_{k,\bullet}^{GG} V^*)$$

Moreover if we define the increasing filtration

$$W_k E_{\bullet,\bullet}^{GG} = \bigcup_{i=1}^k E_{i,\bullet}^{GG}, \quad E_{\bullet,\bullet}^{GG} = \bigcup E_{k,\bullet}^{GG}$$

Then the filtration induced from $F^p$ in (19) on $Gr_k^W E_{\bullet,\bullet}$ defines a Hodge like filtration except that is infinite. The operator $\delta$ as mentioned satisfies a Griffiths transversality

$$\delta F^p Gr_k^W E_{\bullet,\bullet} \subset F^{p+1} Gr_k^W E_{\bullet,\bullet}$$

The 4-tuple $(E_{\bullet,\bullet}, W, F, \delta)$ is an inductive $D$-module.

2. The Green-Griffiths-Lang (GGL)-Conjecture

In [D] Demailly presents a strategy which proves the Green-Griffiths-Lang conjecture. The precise statement of the conjecture is

**Green-Griffiths-Lang Conjecture**: Let $(X, V)$ be a pair where $X$ is a projective variety, and $V$ is a holomorphic subbundle of the tangent bundle $T_X$. Then there should exists an algebraic subvariety $Y \subseteq X$ such that every nonconstant entire curve $f : \mathbb{C} \to X$ tangent to $V$ is contained in $Y$.

This conjecture connects many other conjectures in Kahler geometry (see the reference). Lets $E_{k,m}^{GG}$ is the Green-Griffiths bundle of germs of weighted homogeneous polynomials along the fibers in the Semple tower. A basic observation due to Demailly about the Green-Griffiths bundles is the following vanishing criteria;

$$H^q(X, E_{k,m}^{GG} \otimes A^{-1}) = 0, \quad q > 0, \ m >> k >> 0$$

where $A$ should be ample. In fact the asymptotic result will hold even with $A$ arbitrary hermitian bundle.
Theorem 2.1. (J. P. Demaily) \[D\] Let \((X, V)\) be a directed projective variety such that \(K_V\) is big, and let \(A\) be an ample divisor. Then for \(k \gg 1\) and \(\delta \in \mathbb{Q}_+\) small enough, and \(\delta \leq c(\log k)/k\), the number of sections \(h^0(X, E_{k,m}^{GG} \otimes \mathcal{O}(-m\delta A))\) has maximal growth, i.e is larger than \(c_k m^{n+kr-1}\) for some \(m \geq m_k\), where \(c, c_k > 0\), \(n = \dim(X)\), \(r = \text{rank}(V)\). In particular entire curves \(f : \mathbb{C} \to X\) satisfy many algebraic differential equations.

In fact any section of \(P \in H^0(X, E_{k,m}^{GG} \otimes A^{-1}) = H^0(X_k; \mathcal{O}_{X_k}(m) \otimes \pi_0^* A^{-1})\) with \(A\) being an ample line bundle on \(X\), satisfies an algebraic differential equation \(P(f, f', ..., f^{(k)}) = 0\). He computes the a precise asymptotic for the curvature volume of the \(k\)-jet metric

\[
|z; \xi| = \left( \sum_s \left\| \xi_s \right\|^2 h_s^{2p/s} \right)^{1/p}, \quad \left\| \xi_s \right\|^2 = \sum_{\alpha} \left| \xi_{s\alpha} \right|^2 + \sum_{ij\alpha\beta} c_{ij\alpha\beta} \bar{z}_i z_j \xi_{s\alpha} \bar{\xi}_{s\beta}
\]

That is the Fubini-Study metric with its curvature in level \(s\) of the jet bundle. Denote the second term in (9) by \(\Theta_{FS}^s\). Therefore

\[
\log |z; \xi| = \frac{1}{p} \log \left( \sum_s \left\| \xi_s \right\|^2 + \left( \sum_s \left\| \xi_s \right\|^2 \right)^{p/s} \frac{P_s}{\left\| \xi_s \right\|^2} \Theta_{FS}^s \right)
\]

Then we need to look at the integral

\[
\int_{X_k,q} \Theta^{n+kr-1} = \frac{(n + kr - 1)!}{n!(kr - 1)!} \int_X \int_{P_{(1, ..., kr)}} w_{a,r,p}^{kr-1}(\eta) \gamma_k(z, \eta) \gamma_k(z, \eta)^n
\]

where

\[
\gamma_k(z, \eta) = \frac{i}{2\pi} \sum_{s \geq 1} \frac{1}{s} \sum_t \left| \eta_s \right|^{2p/s} \left| \eta_t \right|^{2p/t} \sum_{ijkl\mu} c_{ij\alpha\beta} \eta_{s\alpha} \eta_{s\beta} \left| \eta_s \right|^2 d\bar{z}_i \land d\bar{z}_j
\]

By a duality argument one tries to find sufficient differential operators acting on these asymptotic global sections. A basic question that may helps toward GGL-conjecture is

Generalization-Question: Assume \(X\) is a projective variety and \(B\) an ample line bundle on \(X\). Then in the Demailly-Semple prolongation

\[
H^0(X_k, D_{X_k/X}(m') \otimes \mathcal{O}_{X_k}(m) \otimes \pi_0^* B) \neq 0, \quad m' \gg m \gg k
\]

The fact is if this holds then the global sections of the twisted \(D\)-module will act on the sections in Demailly’s theorem (2.1). Arranging \(m', m, k\) suitably one can produce equations of the form
which proves the Green-Griffiths-Lang conjecture. Thus one is encouraged to apply
Demailly method of using Holomorphic Morse inequalities to (13). By the analogy
between microlocal differential operators and formal polynomials on the symmetric
tensor algebra it suffices to show

\begin{equation}
H^0(X_k, \text{Sym} \leq m' \tilde{V}_k \otimes \mathcal{O}_{X_k}(m) \otimes \pi_{0k}^* B) \neq 0, \quad m' \gg m \gg k
\end{equation}

where \( \tilde{V}_k \) is the in-homogenized \( V_k \) as acting as differential operators in first order.
We also wish to work over the Demailly-Semple bundle of invariant jet truncations
(15). In order to produce an invariant metric on \( X_k \) with respect to the action of
the both Lie groups of local automorphisms of \( \mathbb{C} \) and the manifold \( X \) a candidate is

\begin{equation}
| \xi | = \sum_{s=1}^{k} \epsilon_s \left( \sum_{\alpha} | P_{s}^{2p/w(\alpha)} \right)^{1/p}
\end{equation}

where \( P_{\alpha} \) runs over a set of generators of the algebra \( R_k = \bigoplus_m E_{k,m} V^* \). Along this
one also needs to check out if this algebra is finitely generated (see [?] for details of
a discussion on this).

In [M] J. Merker proves the Green-Griffiths-Lang conjecture for a generic hyper-
surface in \( \mathbb{P}^{n+1} \). He proves for \( X \subset \mathbb{P}^{n+1}(\mathbb{C}) \) of degree \( d \) as a generic member in the
universal family

\begin{equation}
\mathfrak{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{(n+1+d)!/(n+1)!d!-1}
\end{equation}

parametrizing all such hypersurfaces, the GGL-conjecture holds. His method uses a
theorem of Y. T. Siu, as the following.

**Theorem 2.2.** (Y. T. Siu (2004) [S], [D]) Let \( X \) be a general hypersurface in \( \mathbb{P}^{n+1} \)
as explained above. Then, there are two constants \( c_n \geq 1 \) and \( c'_n \geq 1 \) such that the
twisted tangent bundle

\begin{equation}
T_{J_{\text{vert}}(X)} \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{n+1}}(c_n) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{(n+1+d)!/(n+1)!d!-1}}(c'_n)
\end{equation}

is generated at every point by its global sections.

The proof by Merker, the above Theorem is established outside a certain exception-
algebraic subset \( \Sigma \subset J_{\text{vert}}(\mathfrak{X}) \) defined by vanishing of certain Wronskians.
In order to give a similar proof of GGL conjecture for general \( X \) one may use the
following generalization.
Generalization-Question: If $X \subset \mathbb{P}^{n+1}$ be a generic member of a family $X$ of projective varieties, then there are constants $c_n$ and $c'_n$ such that

$$T_{\text{vert}}^n(X) \otimes O_X(c_n) \otimes \pi_0^* L^{c'_n}$$

is generated at every point by its global sections, where $L$ is an ample line bundle on $X$. To this end by a similar procedure as the former case one may check the holomorphic Morse estimates applied to the following metric on the symmetric powers.

$$|(z, \xi)| = \left( \sum_{s=1}^{k} \epsilon_s \left( \sum_{u_i \in S^s V^*} |W_{u_1, \ldots, u_s}^s|^2 + \sum_{ij\alpha u\beta} C_{ij\alpha u\beta} z_i \bar{z}_j u_\alpha \bar{u}_\beta \right)^{p/(s+1)} \right)^{1/p}$$

where $W_{u_1, \ldots, u_s}^s$ is the Wronskian.

$$W_{u_1, \ldots, u_s}^s = W(u_1 \circ f, \ldots, u_s \circ f)$$

and we regard the summand front the $\epsilon_s$ as a metric on $S^s V^*$. The coefficient $C_{ij\alpha u\beta}$ we are going to compute. Moreover the frame $\langle u_i \rangle$ is chosen of monomials to be holomorphic and orthonormal at 0 dual to the frame $\langle e_{\alpha} = \sqrt{l!/\alpha!} e_1^{\alpha_1} \cdots e_r^{\alpha_r} \rangle$.

The scaling of the basis in $S^l V^*$ is to make the frame to be orthonormal and are calculated as follows;

$$\langle e_\alpha, e_\beta \rangle = \langle \sqrt{l!/\alpha!} e_1^{\alpha_1} \cdots e_r^{\alpha_r}, \sqrt{l!/\alpha!} e_1^{\beta_1} \cdots e_r^{\beta_r} \rangle = \sqrt{1/\alpha!} 1/\beta! \prod_{i=1}^{l} e_{\eta(i)}, \prod_{\sigma \in S_l} \prod_{i=1}^{l} e_{\eta \sigma(i)}$$

via the embedding $S^l V^* \hookrightarrow V^l$ and the map $\eta : \{1, \ldots, l\} \rightarrow \{1, \ldots, r\}$ taking the value $i$ the $\alpha_i$ times.

To compute the coefficients of the curvature tensor we proceed as follows. Because the frame $\langle e_\lambda \rangle$ of $V$ were chosen to be orthonormal at the given point $x \in X$, substituting

$$\langle e_\lambda, e_\mu \rangle = \delta_{\lambda \mu} + \sum_{ij\lambda \mu} C_{ij\lambda \mu} z_i \bar{z}_j + \ldots$$

It follows that

$$\langle e_\alpha, e_\beta \rangle = \sqrt{1/\alpha!} 1/\beta! (\delta_{\alpha \beta} + \sum_{\eta \sigma(i) = \eta(i)} c_{ij\alpha \eta(i) \beta \eta(i)} z_i \bar{z}_j + \ldots)$$
which explains the scalars $C_{iju,au_b}$ in terms of curvature of the metric on $V$. In the estimation of the volume

\begin{equation}
\int_X \Theta^{n+k(r-1)} = \int_X (\Theta_{\text{vert}} + \Theta_{\text{hor}})^{n+k(r-1)} = \frac{(n + k(r - 1))!}{n!(k(r - 1))!} \int_X \int_{F[1^{r}],...,k^{[r]}} \Theta^{k(r-1)}\Theta^{n}_{\text{vert}}\Theta^{n}_{\text{hor}}
\end{equation}

However the calculations with $\Theta$ involves more complicated estimates. The above metric reflects some ideas of D. Brotbek [?], on Wronskian ideal sheaves. It would be interesting to check out if the Monte Carlo process would converge with this metric.

**Remark 2.3.** Forgetting about the complications in the metrics in order to define positive $k$-jet metrics on Green-Griffths bundle one can proceed as follows. It is possible to choose the metric $h$ on $X$ such that the curvature of the canonical bundle $K_V = \bigwedge^r V^*$ is given by a Kahler form, i.e.

\begin{equation}
\Theta_{K_V,\det(h^*)} = w > 0
\end{equation}

Then $\Theta_{\det(V),\det(h)} = -w$. In the fibration sequence

\begin{equation}
\begin{array}{c}
\mathcal{O}_{X_1}(-1) \hookrightarrow \pi_1^* V \\
\downarrow \\
\downarrow \\
X_1 \\
\downarrow \\
X
\end{array}
\end{equation}

One can consider a metric with curvature $\epsilon_1 \Theta_{\mathcal{O}_{X_1}(-1)} + \pi_1^* w > 0$ for $\epsilon_1 << 1$ on $\mathcal{O}_{X_1}(-1)$. Repeating this argument inductively, one obtains a metric on $\mathcal{O}_{X_k}(-1)$ with curvature

\begin{equation}
\Theta_{\mathcal{O}_{X_k}(-1),\hat{\vartheta}} = \sum_j \epsilon_1 \cdots \epsilon_j \pi_{jk}^* \Theta_{\mathcal{O}_{X_j}(-1)} + \pi_{0k}^* w > 0, \quad \ldots \ll \epsilon_2 \ll \epsilon_1 \ll 1
\end{equation}

However the metric $\hat{\vartheta}$ is hard to analyze, because of adding the curvature to the metric at each step.

3. **The case of Invariant jet differentials**

We come back to the question that if a similar theorem to (2.1) can be proved for invariant jet bundles. The fact is that the bundles $E^{GG}_{k,m}$ can be acted by the non-reductive group of holomorphic automorphisms of $(\mathbb{C},0)$. The result are the bundles $E_{k,m}$ studied by J. P. Demailly. In this case one needs to consider the metrics that are
invariant under the action of this group. Considering the discussion at the beginning of this article one may consider metrics of the form

\[ \sum_{s=1}^{k} \epsilon_s ( \sum_{\alpha} | P_{\alpha}(\xi) |^2)^{p/(2s-1)} )^{1/p} \]

where the \( P_\alpha \) are a set of invariant polynomials in jet coordinates. The effect of this is then, the Demailly-Semple locus of the lifts of entire curves should be contained in

\[ P_\alpha = 0, \quad \forall \alpha \]

For instance a choice of \( P_\alpha \)'s could be by the Wronskians. However we slightly try to do some better choice. First lets make some correspondence between invariant jets with non-invariant ones. Lets consider a change of coordinates on the \( \mathbb{C} \) by

\[ \xi = (f_1, ..., f_r) \mapsto (f_1 \circ f_1^{-1}, ..., f_r \circ f_1^{-1}) = (t, g_2, ..., g_r) = \eta \]

This makes the first coordinate to be the identity and the other components to be invariant by any change of coordinates on \( \mathbb{C} \). If we differentiate in the new coordinates successively, then all the resulting fractions are invariant of degree 0

\[ g'_2 = \frac{f'_2}{f^2_2}, \quad g''_2 = \frac{P_2 = f'_1f''_2 - f'_2f''_1}{f^3_2} = W(f_1, f_2), \ldots \]

We take the \( P_\alpha \)'s to be all the polynomials that appear in the numerators of the components when we successively differentiate \( (36) \) with respect to \( t \). An invariant metric in the first coordinates corresponds to a usual metric in the second one subject to the condition that we need to make the average under the unitary change of coordinates in \( V \). That is the effect of the change of variables in \( X \) has only effect as the first derivative by composition with a linear map, up to the epsilon factors. Therefore the above metric becomes similar to the metric used in \( [?] \) in the new coordinates produced by \( g \)'s,

\[ | (z; \xi) | \sim ( \sum_s \epsilon_s \| \eta_s \| \eta_{11} )^{2s-1} \| \eta_{11} \|_{h}^{p/(2s-1)} )^{1/p} = ( \sum_s \epsilon_s \| \eta_s \|_{h}^{p/(2s-1)} )^{1/p} | \eta_{11} | \]

where the weight of \( \eta_s \) can be seen by differentiating \( (36) \) to be equal \( (2s - 1) \) inductively. We need to modify the metric in \( (39) \) slightly to be invariant under hermitian transformations of the vector bundle \( V \). In fact the role of \( \eta_{11} \) can be done by any other \( \eta_{1i} \) or even any other non-sero vector. To fix this we consider
where the integration only affects the last factor making average over all vectors in $v \in V$. This will remove the the former difficulty. The curvature is the same as for the metric in (17) but only an extra contribution from the last factor,

\begin{equation}
\gamma_k(z, \eta) = \frac{i}{2\pi} (w_{r,p}(\eta) + \sum_{l=1}^{m} b_{lma} (\int_{v||v||=1} v_{\alpha}) d\bar{z}_l \wedge d\bar{z}_m + \sum_{s} \frac{1}{s} \sum_{t} |\eta_s|^{2p/s} \sum_{c_{ij\lambda\mu}} |\eta_{s\alpha}|^{2p/s} \sum_{c_{ij\lambda\mu}} |\eta_{s\beta}|^{2p/s} dz_i \wedge d\bar{z}_j )
\end{equation}

The contribution of the factor $|\eta_{11}|$ can be understood as the curvature of the sub-bundle of $V$ which is orthogonal complement to the remainder. Thus

\begin{equation}
b_{lma} = c_{lmaa}
\end{equation}

where $c_{lm1}$ is read from the coefficients of the curvature tensor of $(V, w^{FS})$ the Fubini-Study metric on $V$ (the second factor in (8). In the course of evaluating with the Morse inequalities the curvature form is replaced by the trace of the above tensor in raising to the power $n = \dim X$, then if we use polar coordinates, the value of the curvature when integrating over the sphere yields the following

\begin{equation}
\gamma_k = \frac{i}{2\pi} (\sum_{l=1}^{m} b_{lma} d\bar{z}_l \wedge d\bar{z}_m + \sum_{s} \frac{1}{s} \sum_{c_{ij\lambda\mu}} u_{s\lambda} \bar{u}_{s\lambda} dz_i \wedge d\bar{z}_j )
\end{equation}

Because the first term is a finite sum with respect to $s$, the estimates for this new form would be essentially the same as those in [D]. Therefore one expects

\begin{equation}
\int_{X_{k,q}} \Theta^{n+k(r-1)} = \frac{(\log k)^n}{n!(k!)^r} (\int_{X} 1_{\gamma_k} \gamma^n + O((\log k)^{-1})
\end{equation}

similar estimates for non-invariant case. We have proved the following.

**Corollary 3.1.** The analogue of Theorem (2.1) holds for the bundle $E^{GG}_{k,m}$ replaced by $E_{k,m}$ (subject to the finite generation of fiber rings in $E_{k,m}$).

**Remark 3.2.** If $P = P(f, f', ..., f^{(k)})$ and $Q = Q(f, f', ..., f^{(k)})$ are two local sections of the Green-Griffiths bundle, then the first invariant operator is probably

\begin{equation}
\nabla_j : f \mapsto f^{(j)}
\end{equation}

Define a bracket operation as follows
\[ [P, Q] = (d \log \frac{P_{1/\deg(P)}}{Q_{1/\deg(Q)}}) \times PQ = \frac{1}{\deg(P)}PdQ - \frac{1}{\deg(Q)}QdP \]

Later one successively defines the brackets
\[ [\nabla_j, \nabla_k] = f_j^l f_k^l - f_j^f f_k^f \]
\[ [\nabla_j, [\nabla_k, \nabla_l]] = f_j^l (f_k^l f_l^j - f_l^l f_k^j) - 3f_j^l (f_k^l f_l^f - ...) \]

IF \((V, h)\) is a Hermitian vector bundle, the equations in (43) define inductively \(G_k\)-equivariant maps
\[ Q_k : J_kV \rightarrow S^{k-2}V \otimes \bigwedge^2 V, \quad Q_k(f) = [f', Q_{k-1}(f)] \]

The sections produced by \(Q_k(f)\) generate the fiber rings of Demailly-Semple tower \([?]\).

**References**

[D] J. P. Demailly, Hyperbolic algebraic varieties and holomorphic differential equations, Acta Math. Vietnam, 37(4) 441-512, 2012

[GG] M. Green, P. Griffiths, Two applications of algebraic geometry to entire holomorphic mappings, The Chern Symposium 1979. (Proc. Intern. Sympos., Berkeley, California, 1979) 41-74, Springer, New York, 1980.

[M] J. Merker, low pole order frameson vertical jets of the universal hypersurface, Ann. Institut Fourier (Grenoble), 59, 2009, 861-933

[S] Y. T. Siu, Hyperbolicity in complex geometry, The legacy of Niels Henrick Abel, Springer, Berlin, 2004, 543-566

Institut Fourier, Universite Grenoble Alpes, France

E-mail address: mrahmati@cimat.mx