A RECURSION ON MAXIMAL CHAINS IN THE TAMARI LATTICES

LUKE NELSON

Abstract. The Tamari lattices have been intensely studied since their introduction by Dov Tamari around 1960. However oddly enough, a formula for the number of maximal chains is still unknown. This is due largely to the fact that maximal chains in the \( n \)-th Tamari lattice \( T_n \) range in length from \( n - 1 \) to \( \binom{n}{2} \). In this note, we treat vertices in the lattice as Young diagrams and identify maximal chains as certain tableaux. For each \( i \geq -1 \), we define \( S_i(n) \) as the set of maximal chains in \( T_n \) of length \( n + i \). We give a recursion for \( \#S_i(n) \) and an explicit formula based on predetermined initial values. The formula is a polynomial in \( n \) of degree \( 3i + 3 \). For example, the number of maximal chains of length \( n \) in \( T_n \) is \( \#S_0(n) = \binom{n}{3} \). The formula has a combinatorial interpretation in terms of a special property of maximal chains.

1. Introduction

The Tamari lattices \( \{T_n\} \) have been intensely studied since their introduction by Dov Tamari \cite{28}. He defined \( T_n \) on bracketings of a set of \( n + 1 \) objects, with a cover relation based on the associativity rule in one direction. Friedman and Tamari later proved the lattice property \cite{16}. \( T_n \) is both a quotient and a sublattice of the weak order on the symmetric group \( S_n \), and its Hasse diagram is the 1-skeleton of the associahedron (or Stasheff polytope) \cite{3, 23}.

As usual, \([n]\) denotes the set \( \{1, \ldots, n\} \) of the first \( n \) positive integers, and we adopt the convention that \([0]\) denotes the empty set. The number of vertices in \( T_n \) is the \( n \)-th Catalan number, \( C_n = \frac{1}{n+1} \binom{2n}{n} \). Triangulations of a convex \((n+2)\)-gon, noncrossing partitions of \([n]\), and Dyck paths of length \( 2n \) are examples of over 200 combinatorial structures counted by the Catalan sequence \cite{26}. The Kreweras and Stanley lattices are two other noted lattices defined on Catalan sets. The Stanley lattice is a refinement of the Tamari lattice, which is a refinement of the Kreweras lattice \cite{20, Exercises 7.2.1.6-27, 28} \cite{2}.

The search for enumeration formulas, whether for maximal chains or intervals, etc., are classic problems for any family of posets. The pursuit of solutions often leads to relationships with other combinatorial structures and a better understanding of the poset at hand.

Definitions concerning posets may be found in \cite[Chapter 3]{27}. If \( x \leq y \) in a poset \( P \), the subposet \([x,y] = \{ z \in P \mid x \leq z \leq y \} \) of \( P \) is called a (closed) interval. Intervals in the Kreweras lattice are in bijection to ternary trees \cite{21, 12}, while those in the Stanley lattice are pairs of noncrossing Dyck paths \cite{9}. Chapoton enumerated the intervals in \( T_n \), finding this to be the number of planar triangulations (i.e., maximal planar graphs) \cite{6}. Bergeron and Préville-Ratelle generalized the Tamari posets to the \( m \)-Tamari posets \( T_n^m \) \cite{1} (the case \( m = 1 \) is \( \{T_n\} \)). Shortly after, Bousquet-Méhou, Fusy and Préville-Ratelle proved the \( m \)-Tamari posets are lattices and generalized Chapoton’s formula to \( T_n^m \) \cite{4}. Its formula has a simple factorized form (as in Chapoton’s formula) and a combinatorial interpretation; see \cite{2, 7, 8}.

A subset of a poset in which any two elements are comparable is called a chain. The length of a finite chain \( C \) is the number of its elements minus one, denoted \( l(C) \). If \( x < y \) and there does not exist \( z \) such that \( x < z < y \), then we say that \( y \) covers \( x \), or \( x \) is covered by \( y \), and denote \( y \triangleright x \), or \( x \triangleleft y \). A saturated chain in a finite poset is a sequence of elements \( x_0 \triangleleft x_1 \triangleleft \cdots \triangleleft x_{l-1} \triangleleft x_l \). If a poset has an element \( x \) with the property that \( x \triangleleft y \) for all \( y \) in the poset, we denote that element by \( \hat{0} \). Similarly, \( \hat{1} \) is the element above all others if it exists. In a finite poset with a \( \hat{0} \) and \( \hat{1} \), a maximal chain is a sequence of elements \( \hat{0} = x_0 \triangleleft x_1 \triangleleft \cdots \triangleleft x_{l-1} \triangleleft x_l = \hat{1} \).

Maximal chains in the Kreweras lattice, of noncrossing partitions of \([n]\), are in bijection to factorizations of an \( n \)-cycle as the product of \( n - 1 \) transpositions \cite{21, 20, Exercise 7.2.1.6-33}, the number of which is \( n^{n-2} \) \cite{10}. This is also the number of parking functions of length \( n - 1 \) \cite{14, 25}, and the number of

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trees on $n$ labeled vertices [5]. There is a simple bijection between maximal chains in the Stanley lattice of order $n$ and standard Young tableaux of shape $(n - 1, n - 2, \ldots, 1)$ (see [20, Exercise 7.2.1.6-34] for one), whose number is given by the hook length formula [15]:

$$\frac{(n)}{1^{n-1}3^{n-2}\cdots(2n-5)^2(2n-3)!}.$$  

This is again the number of maximal chains in the weak order on $S_n$ [11, 24].

A natural question arises: What is the number of maximal chains in the Tamari lattice? Quoting Knuth, “The enumeration of such paths in Tamari lattices remains mysterious.” Despite their interesting aspects and the attention they have received, a formula for the number of maximal chains in the Tamari lattices is still unknown. The complexity of the problem is largely due to the fact that maximal chains in the Tamari lattices vary over a great range. The longest chains in $T_n$ have length $\binom{n}{2}$ and the unique shortest has length $n - 1$ [22] [20, Exercise 7.2.1.6-27(h)]. We determined a bijection between longest chains in $T_n^{(m)}$ and standard $m$-shifted tableaux of shape $(m(n - 1), m(n - 2), \ldots, m)$ in [13] and, as a corollary, a formula for the number of longest chains in $T_n$:

$$\binom{n}{2} \cdot \frac{(n - 2)!((n - 3)! \cdots (2)!1)!}{(2n - 3)!((2n - 5)! \cdots (3)!1)!}.$$  

Keller introduced green mutations and maximal sequences of such mutations, called maximal green sequences [19]. In certain cases, maximal green sequences are in bijection with maximal chains in the Tamari lattice or the Cambrian lattice (a generalization of the Tamari lattice [23]) [18, 17]. Garver and Musiker list several applications of maximal green sequences to representation theory and physics [17]. The problems of enumeration and classification of such sequences are noted interests.

Aside from these cases, we are unaware of any results pertaining to the enumeration of maximal chains in the Tamari lattices. In this note, our focus pertains to the following definition.

**Definition 1.1.** Let $i \geq -1$ and $n \geq 1$. $C_i(n)$ is the set of all maximal chains of length $n + i$ in $T_n$.

The main result of this note is Theorem 5.9: We give a recursion for $\#C_i(n)$ and an explicit formula based on predetermined initial values. The formula is a polynomial in $n$ of degree $3i + 3$. For example, the number of maximal chains of length $n - 1$ in $T_n$ is $\#C_{-1}(n) = 1$, while the number of length $n$ is $\#C_0(n) = \binom{n}{2}$. Table 1.1 is a computer based compilation of the numbers of maximal chains by length in $T_n$ up through $T_9$. The numbers of lengths 3, 4, 5 and 6 in $T_9$ are $\#C_{-1}(4) = 1$, $\#C_0(4) = 4$, $\#C_1(4) = 2$ and $\#C_2(4) = 2$, respectively. The bottom entry of a column is the number of longest chains in $T_n$ (given by equation (1.2)). For example, the number of longest chains in $T_6$ is 286.

Bernardi and Bonichon rewrote the covering relation in $T_n$ in terms of Dyck paths [2]. We find it useful to work mainly from the perspective of Young diagrams, but rely on properties of both sets. We present basic terminology and the covering relation in Section 2.

We rely on two main maps: $\psi$ and $\phi_{i,n}^\ast$. We use $\psi$ to identify maximal chains in $T_n$ with certain tableaux. We obtain an expression for the number of maximal chains using $\phi_{i,n}^\ast$, which takes a maximal chain in $C_i(n)$ to one in $C_i(n + 1)$. Because of $\psi$, we may express $\phi_{i,n}^\ast$ as a map on tableaux. In Section 3, we define $\psi$, establish basic properties, and conclude with Theorem 3.11, by defining the first part of $\phi_{i,n}^\ast$. A maximal chain in the image of $\psi$ may or may not possess a “plus-full-set”; see Definition 4.1. In Section 4, we establish the other parts of $\phi_{i,n}^\ast$ in Theorems 4.3 and 4.6. We specialize Theorem 4.6 to Theorem 4.8, thereby defining $\phi_{i,n}^\ast$, where $r$ determines the domain and codomain in terms of plus-full-sets. The main focus of Section 4 and a key ingredient leading up to our main objective is the fact that this map is bijective.

In Section 5, we gather more on properties and consequences of $\phi_{i,n}^\ast$ and tie our results together to write a recursive formula for $\#C_i(n)$. $\phi_{i,n}^\ast$ takes a maximal chain to one with one more plus-full-set; see Proposition 5.1. This enables us to write every maximal chain, which has a plus-full-set, uniquely in terms of one with one less plus-full-set. We extend this to a unique representation in terms of a maximal chain with no plus-full-sets in Corollary 5.3. By relating this representation to specific plus-full-sets that a maximal chain contains in Proposition 5.5, we obtain an expression for $\#C_i(n)$; see equation (5.5). For each $i \geq -1$, there exists a maximal chain in $C_i(2i + 3)$ containing no plus-full-sets (Lemma 5.7), but
A recursion on maximal chains in the Tamari lattices

| Length | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| $n-1$  | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| $n$    | 1     | 4     | 10    | 20    | 35    | 56    | 84    |
| $n+1$  | 2     | 22    | 112   | 392   | 1,092 | 2,604 |
| $n+2$  | 2     | 22    | 232   | 1,744 | 9,220 | 37,444|
| $n+3$  | 18    | 362   | 4,474 | 40,414| 280,214|
| $n+4$  | 13    | 348   | 8,435 | 123,704| 1,321,879|
| $n+5$  | 12    | 456   | 12,732| 550,932| 12,512,827|
| $n+6$  | 390   | 17,337| 232   | 1,744 | 9,220 | 37,444|
| $n+7$  | 420   | 21,158| 232   | 1,744 | 9,220 | 37,444|
| $n+8$  | 286   | 33,940| 4,474 | 40,414| 280,214|
| $n+9$  | 40    | 41,230| 3,316,121,272| 994,441,978,397|
| $n+10$ | 41,230| 3,316,121,272| 994,441,978,397|
| $n+11$ | 45,048| 4,810,150| 15,969,449,634|
| $n+12$ | 50,752| 7,264,302| 221,484,557|
| $n+13$ | 50,752| 7,264,302| 221,484,557|
| $n+14$ | 33,592| 8,435   | 123,704| 1,321,879|
| $n+15$ | 27,502,220| 5,604,687,775|
| $n+16$ | 41,826| 10,435,954| 1,134,705,692|
| $n+17$ | 382   | 17,337 | 232   | 1,744 | 9,220 | 37,444|
| $n+18$ | 420   | 21,158 | 232   | 1,744 | 9,220 | 37,444|
| $n+19$ | 286   | 33,940 | 4,474 | 40,414| 280,214|
| $n+20$ | 40    | 41,230 | 3,316,121,272| 994,441,978,397|
| $n+21$ | 45,048| 4,810,150| 15,969,449,634|
| $n+22$ | 50,752| 7,264,302| 221,484,557|
| $n+23$ | 33,592| 8,435   | 123,704| 1,321,879|
| $n+24$ | 27,502,220| 5,604,687,775|
| $n+25$ | 41,826| 10,435,954| 1,134,705,692|
| $n+26$ | 382   | 17,337 | 232   | 1,744 | 9,220 | 37,444|
| $n+27$ | 420   | 21,158 | 232   | 1,744 | 9,220 | 37,444|
| Totals | 1     | 1     | 2     | 9     | 98    | 2,981 | 340,549| 216,569,887| 994,441,978,397|

Table 1.1. $\#C_i(n)$: Number of maximal chains in $T_n$ of length $n + i$

surprisingly, for all $n \geq 2i + 4$, each maximal chain in $C_i(n)$ has a plus-full-set (Theorem 5.8). We utilize these latter two facts to refine our expression for $\#C_i(n)$ and achieve our main objective in Theorem 5.9.

2. Preliminaries

In this section, we present basic terminology of Young diagrams and Dyck paths and the covering relation in the Tamari lattice.

A partition of a positive integer $l$ is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers summing to $l$. A Young diagram $Y$ of shape $\lambda$ is a left-justified collection of boxes having $\lambda_j$ boxes in row $j$, for $1 \leq j \leq k$. The shape of $Y$ is denoted $\text{sh}(Y)$. Rows and columns of the diagram begin with an index of one. The length of a row (or of a column) is its number of boxes. We denote the box in row $x$ and column $y$ by $(x, y)$. We adopt the English notation, in which rows are indexed downward. The empty partition $\lambda = (0)$ is associated with the null diagram $\emptyset$ having no boxes. The staircase shape $(n, n-1, n-2, \ldots, 1)$ is denoted $\delta_n$, where we set $\delta_0 = (0)$ if $n \leq 0$. Often times, which will be clear by the context, we abuse notation by identifying a partition $\lambda$ with its associated Young diagram also denoted $\lambda$ or vice versa. For example, $\delta_3$ is the shape $(3, 2, 1)$ or it is the Young diagram of that shape.

A Dyck path of length $2n$ is a path on the square grid of north and east steps from $(0, 0)$ to $(n, n)$ which never goes below the line $y = x$. Necessarily, every Dyck path begins with a north step and ends with an east step, and has an equal number of both types of steps. The height of a Dyck path is its
number of north steps. In [2], vertices in $T_n$ are interpreted as the set of Dyck paths of length $2n$, the number of which is the $n$-th Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.

There is a natural bijective correspondence between the set of Dyck paths of length $2n$ and a set of Young diagrams, to which we identify the set of vertices in $T_n$: Roughly speaking, a Dyck path gives the silhouette of the Young diagram. This is the set of Young diagrams contained in $\delta_{n-1}$. $T_1$ is comprised of a single vertex, the null diagram. Figure 2.1 is the set of $C_4 = 14$ Dyck paths of length 8 and corresponding Young diagrams.

![Figure 2.1. The vertices of $T_4$ in terms of Dyck paths and Young diagrams](image)

**Remark 2.1.** Let $Y \in T_n$ be a Young diagram. For each $m \geq n$, $Y \in T_m$ and $Y$ corresponds to exactly one Dyck path of length $2m$. In the other direction, any Dyck path (regardless of length) corresponds to exactly one Young diagram.

**Definition 2.2.** If $P$ is a Dyck path and $L$ is the line segment (of slope one) that joins the endpoints of $P$, then $P$ is said to be prime if $P$ intersects $L$ only at the endpoints of $P$. (We word the definition of prime Dyck paths differently than in [2], but it has the same meaning.)

Of the Dyck paths in Figure 2.1, only the last five in the second row are prime, corresponding to the Young diagrams contained in (2,1). A Dyck path of length $2n$ has exactly $n$ prime Dyck subpaths, each uniquely determined by its beginning north step. In Figure 2.2, for the given Dyck path of length 8, its 4 prime Dyck subpaths are bolded. The line segment joining the endpoints of each prime Dyck subpath is drawn. As required in Proposition 3.7, we characterize pairs of prime Dyck subpaths.

**Lemma 2.3.** If $Q$ and $R$ are two prime Dyck subpaths of a Dyck path, then exactly one of the following characterizes $Q$ and $R$:

1. $Q \cap R = \emptyset$, i.e., $Q$ and $R$ have no common points.
2. $Q$ and $R$ intersect in a single point.
3. $Q \subset R$ or $R \subset Q$, i.e., $Q$ is a proper subpath of $R$, or $R$ is a proper subpath of $Q$.
4. $Q = R$.

**Proof.** Suppose neither (1) nor (2). Then $Q$ and $R$ must have a step in common. If the endpoints of $Q$ and of $R$ all lie on the same line, then $Q = R$. If, without loss of generality, the line containing the endpoints of $Q$ lies above the line containing the endpoints of $R$, then $Q$ is a proper subpath of $R$. □

![Figure 2.2. Prime Dyck subpaths](image)  ![Figure 2.3. Prime paths](image)

For an example of Lemma 2.3(1), take $Q$ and $R$ to be the prime Dyck subpaths in the first and third examples of Figure 2.2. For 2.3(2), let $Q$ and $R$ be the subpaths in the first and second examples. For $Q \subset R$, let $Q$ and $R$ be the subpaths in the third and second examples, respectively.

The notion of prime is intimately tied to the covering relation in the Tamari lattices. We need to extend this notion.
Definition 2.4. Let $Y$ be a Young diagram and $d \geq 1$. Let $e$ be the vertical edge at the end of row $d$ in $Y$ for which $e$ is on a corresponding Dyck path $P$ to $Y$ (row $d$ may be empty). The prime path of row $d$ is the prime Dyck subpath of $P$ beginning with $e$.

Suppose $B$ is the last box of its row in $Y$. The prime path of $B$ is the prime path of its row. The $B$-strip is the set of all boxes in $Y$ with the right vertical edge on the prime path of $B$. If $B$ is the last box of its row and the lowest box of its column, then $B$ is a corner box.

In Figure 2.3, prime paths are bolded, and $B$-strips are grayed. The first example is the prime path of the third row in the Young diagram of shape $(1)$. Each of the second and third examples is the prime path of a box $B$ in the Young diagram of shape $(2,1,1)$. $B$ is a corner box in the third example.

We give two versions of the covering relation. The second, in terms of Young diagrams, follows from the correspondence of Dyck paths.

Proposition 2.5. [2, Proposition 2.1] Covering relation in the Tamari lattices: Dyck paths. Let $P$ and $P'$ be Dyck paths. Then $P'$ covers $P$ ($P' \triangleright P$) if and only if there exists an east step $e$ in $P$ followed by a north step, such that $P'$ is obtained from $P$ by swapping $e$ and the prime Dyck subpath following it.

Proposition 2.6. Covering relation in the Tamari lattices: Young diagrams. Let $Y$ and $Y'$ be Young diagrams. Then $Y'$ covers $Y$ ($Y' \triangleright Y$) if and only if there exists a corner box $B$ in $Y$, such that $Y'$ is obtained from $Y$ by removing the $B$-strip.

Examples of the covering relation for Dyck paths in Figure 2.4 correspond to the examples for Young diagrams in Figure 2.5. In the Dyck path examples, the east step of $P$ referenced in the proposition is grayed, and the prime Dyck subpath following it is bolded. In the Young diagram examples, the prime path of the corner box $B$ in $Y$ is bolded, and the $B$-strip is grayed.

The Hasse diagrams for $T_3$ and $T_4$ are shown in Figure 2.6. The maximum element $\hat{1}$ in $T_n$ is the null diagram, and the minimum element $\hat{0}$ is $\delta_{n-1}$.

3. Representation of maximal chains

In this section, we relate an efficient approach to work with certain saturated chains in the Tamari lattices through the map $\psi$. In particular, $\psi$ assigns each maximal chain to a unique tableau. We establish basic properties and explicitly characterize chains in terms of tableaux. We then enter into more technical material and conclude with Theorem 3.11, where we define the first piece of $\phi_{r,n}$.

Definition 3.1. For our purposes, a tableau $T$ of shape $\lambda$ is obtained by filling each box of the Young diagram of shape $\lambda$ with a positive integer, where each row strictly increases when read left to right, and each column weakly increases when read top to bottom. The length of $T$, denoted $l(T)$, is the number of its distinct labels. We also require that its set of labels is precisely $[l(T)]$. (Often in the literature, this is the definition of a row-strict tableau.)
For $r \in [l(T)]$, the $r$-set is the set of all boxes in $T$ labeled with $r$. The label in the box $(x, y)$ is denoted $T(x, y)$. For $r \in \{0, 1, \ldots, l(T)\}$, the tableau obtained from $T$ made of all the elements less than or equal to $r$ is denoted $T(r)$.

**Definition 3.2.** Let $C = (\emptyset = Y_0 \succ Y_1 \succ \cdots \succ Y_l)$ be a saturated chain under the Tamari order in terms of Young diagrams. As $C$ is traversed upwards in the Hasse diagram, boxes are removed from $Y_l$. For each $r \in [l]$, label the boxes removed in the transition $Y_{r-1} \succ Y_r$ with $r$. The resulting tableau, of the shape of $Y_l$, is $\psi(C)$. A $\psi$-tableau is an element in the image of $\psi$.

Examples of $\psi$-tableaux are shown in Figure 3.1. The nine maximal chains of $\mathcal{T}_4$ are shown in Figure 3.2. We defined $\psi$ on maximal chains in [13, Definition 3.1].

**Proposition 3.3.** $\psi$ is injective.

**Proof.** Let $T$ be a $\psi$-tableau. Then $T = \psi(C)$ for some $C = (\emptyset = Y_0 \succ Y_1 \succ \cdots \succ Y_{l(T)})$. The length of $T$ is the length of $C$. For each $r \in \{0, 1, \ldots, l(T)\}$, $Y_r = \{(x, y) \in T(r)\}$. □

Because $\psi$ is injective, for each $n \geq 1$ and for each Young diagram $Y \in \mathcal{T}_n$, $\psi$ extends to a bijection of sets between saturated chains in $\mathcal{T}_n$ of length $l$ whose minimal element is $Y$ and maximal element is $\emptyset$, and $\psi$-tableaux of length $l$ and the shape of $Y$. Since we identify vertices in $\mathcal{T}_n$ as Young diagrams contained in $\delta_{n-1}$, $\psi$ induces the following examples of bijective correspondences between:

- elements of $\mathcal{C}_i(n)$ and $\psi$-tableaux of length $n + i$ and shape $\delta_{n-1}$.
• maximal chains in \( T_n \) and \( \psi \)-tableaux of shape \( \delta_{n-1} \), and
• saturated chains in \( T_n \) whose maximal element is \( \emptyset \) and \( \psi \)-tableaux contained in \( \delta_{n-1} \).

**Definition 3.4.** A nonempty subset of boxes of a Young diagram begins and ends in its rows of minimum and maximum index, respectively. Similarly, we may refer to the begin-box or end-box of that subset.

If \( Y \neq \emptyset \), then the outer diagonal of \( Y \) is the set of boxes \( \{(x, y) \in Y : x + y = m\} \) where \( m \) is the maximum of \( \{x + y \mid (x, y) \in Y\} \); otherwise, the outer diagonal of \( Y = \emptyset \) is the empty set.

The outer diagonal of a \( \psi \)-tableau of shape \( \delta_{n-1} \), for \( n \geq 1 \), is the set of boxes \( \{(k, n-k) \mid k \in [n-1]\} \).

Next we characterize \( \psi \)-tableau. Statements 3.5(2)-(3), listed for convenience, follow from 3.5(1).

**Proposition 3.5.** Characterization of \( \psi \)-tableaux. Let \( T \) be a tableau and \( l = l(T) \). For each \( k \in [l] \), let \( B_k \) be the end-box of the \( k \)-set. Then:

1. \( T \) is a \( \psi \)-tableau if and only if for each \( k \in [l] \), the \( k \)-set is the \( B_k \)-strip in \( T(k) \).
2. Fix \( r \in \{0,1,\ldots,l\} \). \( T \) is a \( \psi \)-tableau if and only if \( T(r) \) is a \( \psi \)-tableau, and for each \( k \in \{r+1,r+2,\ldots,l\} \), the \( k \)-set is the \( B_k \)-strip in \( T(k) \).
3. If \( l > 0 \), then \( T \) is a \( \psi \)-tableau if and only if \( T(l-1) \) is a \( \psi \)-tableau, and the \( l \)-set is the \( B_l \)-strip in \( T \).

**Proof.** (1). For each \( 0 \leq k \leq l \), let \( Y_k = \{(x, y) \in T(k)\} \). Then \( T \) is a \( \psi \)-tableau if and only if \( \emptyset = Y_0 \succ T_1 \succ \cdots \succ T_l \). Each \( Y_k \) has the shape of \( T(k) \), and \( Y_0 = \emptyset \). Furthermore, for each \( k \in [l] \), \( B_k \) is a corner box in \( Y_k \) and is the unique box of maximum row index of all the boxes removed from \( Y_k \) to obtain \( Y_{k-1} \). Thus, by Proposition 2.6, \( Y_{k-1} \succ Y_k \) if and only if the set of boxes removed from \( Y_k \) to obtain \( Y_{k-1} \) is the \( B_k \)-strip in \( Y_k \) if and only if the \( k \)-set is the \( B_k \)-strip in \( T(k) \).

In each tableau of Figure 3.3, the \( (3,2) \)-strip, \( \{(1,3),(2,2),(3,2)\} \), is grayed and the prime path of \( (3,2) \) is bolded. In the first tableau (of length 4), the 4-set, \( \{(2,2),(3,2)\} \), does not agree with the \( (3,2) \)-strip, so is not a \( \psi \)-tableau. The second tableau is a \( \psi \)-tableau.

![Figure 3.3](image)

**Figure 3.3.** The first tableau is not a \( \psi \)-tableau; the second one is.

The warm up properties in the following proposition come directly from basic definitions, the covering relation (Proposition 2.6) and the definition of \( \psi \) (Definition 3.2).

**Proposition 3.6.** Basic Properties.

Let \( Y \) be a Young diagram, and suppose \( B \in Y \) is the last box of the \( x_B \)-th row. Then:

1. Each box in the outer diagonal is a corner box.
2. Let \( h \geq 1 \). The height of the prime path of \( B \) is \( h \) if and only if the \( B \)-strip is the set of last boxes of all rows \( j \) for which \( x_B - h + 1 \leq j \leq x_B \).
3. Suppose \( B' \in Y \) is the last box of the \( x_{B'} \)-th row. Then:
   (a) \( B' \) is not in the \( B \)-strip if and only if the \( B \)-strip is entirely above or entirely below row \( x_{B'} \).
   (b) \( B' \) is in the \( B \)-strip if and only if the prime path of \( B' \) is a subpath of the prime path of \( B \).
   (c) Suppose that \( B \) is in the outer diagonal and that \( x_{B'} > x_B \). Then the \( B' \)-strip begins in a row of index greater than \( x_B \).

Let \( T \) be a \( \psi \)-tableau and \( l = l(T) \). Then:

4. Each box in the outer diagonal is the end-box of a \( j \)-set, for some \( j \in [l] \). Thus there are no repeat labels in the outer diagonal.
5. For each \( r \in \{0,1,\ldots,l\} \), \( T(r) \) is a \( \psi \)-tableau of length \( r \). Also \( \emptyset = T(0) \succ sh(T(1)) \succ \cdots \succ sh(T(l-1)) \succ sh(T(l)) = sh(T) \).

The remaining material of this section is more technical and is necessary to verify properties of the map \( \phi_{l,m}^i \) defined in the next section.
Proposition 3.7. Let \( C = (P_0 \succ P_1 \succ \cdots \succ P_l) \) be a saturated chain in \( T_n \) in terms of Dyck paths (of length \( 2n \)). Let \( k \in [n] \). For each \( 0 \leq j \leq l \), let \( h_j \) be the height of the prime Dyck subpath beginning with the \( k \)-th north step of \( P_j \). Then \( h_0 \geq h_1 \geq \cdots \geq h_l \).

Proof. By induction on \( l \), it suffices to show \( h_{l-1} \geq h_l \) for \( l > 0 \). Let \( Q \) be the prime Dyck subpath of \( P_l \) beginning with its \( k \)-th north step. Let \( R \) be the prime Dyck subpath of \( P_l \) that shifts to the left one unit in the transition \( P_{l-1} \succ P_l \) (see Proposition 2.5). We have a few cases to check as outlined in Lemma 2.3. If \( Q \cap R = \emptyset \) or \( Q \subseteq R \) or \( R \subseteq Q \) or \( Q = R \), then \( h_{l-1} = h_l \). One case remains: \( Q \cap R \) is a single point. If \( R \) ends where \( Q \) begins, then \( h_{l-1} = h_l \). If \( Q \) ends where \( R \) begins, then the trailing east step of \( Q \) swaps with \( R \) in the transition \( P_{l-1} \succ P_l \). In this case, \( h_{l-1} > h_l \). \( \square \)

Figure 3.4 is an example of a maximal chain in terms of Dyck paths \( P_j \) of length 8. In the context of Proposition 3.7, the sequences \( (h_0, h_1, h_2, h_3, h_4) \), for \( k \in \{1, 2, 3, 4\} \), are \( (4, 4, 2, 1, 1) \), \( (3, 1, 1, 1, 1) \), \( (2, 2, 2, 1) \) and \( (1, 1, 1, 1, 1) \), respectively.

![Figure 3.4. Example of Proposition 3.7](image)

Remark 3.8. Knuth shows there are the Catalan number \( C_n \) of forests on \( n \) nodes, by creating a bijection between the sets of forests and scope sequences of length \( n \) [20]. Scope sequences also appear in [3, Definition 9.1]. Proposition 3.7 follows from both references directly from the covering relation in terms of scope sequences: Let \( P \) be a Dyck path of length \( 2n \). For each \( 1 \leq k \leq n \), let \( h_k \) be the height of the prime Dyck subpath of \( P \) beginning with its \( k \)-th north step. Then the scope sequence of the forest associated to \( P \) is \((H_1 - 1, H_2 - 1, \ldots, H_n - 1)\).

Often times we will need to determine if a given tableau is a \( \psi \)-tableau. The following definition and lemma will be utilized in this regard.

Definition 3.9. We say that a set of boxes \( S \) translates to a set of boxes \( S' \) if each box in \( S \) differs from one in \( S' \) by the same horizontal and vertical amounts, and both sets are equal in size, i.e., if for some constants \( p \) and \( q \). \( S' = \{(x + p, y + q) | (x, y) \in S\} \). Similarly, a subset of elements of a tableau may translate, or a path of north and east steps may translate.

Suppose \( B = (x_B, y_B) \) is the last box of its row in a Young diagram \( Y \). Let \( h \) be the height of the prime path of \( B \). The enclosure of the \( B \)-strip in \( Y \) is the set of boxes

\[ \{(x, y) \in Y | x_B - h \leq x \leq x_B \text{ and } y_B \leq y \leq y_B + h\}, \]

where we consider row 0 to be an infinite row of boxes.

In Figure 3.5, the borders of \( B \)-strips and \( B' \)-strips are bolded and their enclosures are grayed. \( Y_2 \) has shape \((3, 2, 1, 1)\), but the enclosure of its \( B \)-strip has shape \((5, 3, 2, 1, 1)\); it contains 5 boxes from row 0.

![Figure 3.5. Examples of the Translation Lemma](image)

In the context of Definition 3.9, notice that the \( B \)-strip is a subset of its enclosure, and that the \( B \)-strip begins in a row of index one more than where its enclosure begins. The shape of the enclosure is the shape of a Young diagram. Furthermore, the \( B \)-strip determines its enclosure and vice versa.
Lemma 3.10. Translation Lemma. Suppose \( B \) and \( B' \) are the last boxes of their rows in Young diagrams \( Y \) and \( Y' \), respectively. Then the following conditions are equivalent.
- The enclosures of the \( B \)-strip in \( Y \) and of the \( B' \)-strip in \( Y' \) have the same shape.
- The enclosure of the \( B \)-strip in \( Y \) translates to the enclosure of the \( B' \)-strip in \( Y' \).
- The \( B \)-strip in \( Y \) translates to the \( B' \)-strip in \( Y' \).
- The prime path of \( B \) in \( Y \) translates to the prime path of \( B' \) in \( Y' \). (The two prime paths are the same sequences of north and east steps.)

If any of the conditions are satisfied, then the translations of the various entities in the last three conditions are by the same horizontal and vertical amounts as \( B \) translates to \( B' \).

In Figure 3.5, the \( B \)-strip in \( Y_1 \) translates 3 units to the left and 1 unit down to the \( B' \)-strip in \( Y_1' \). The \( B \)-strip in \( Y_2 \) translates 1 unit to the right and 2 units down to the \( B' \)-strip in \( Y_2' \). In both cases, the four conditions in the lemma may be verified.

In what follows, we define \( \alpha_d \) as a map on tableaux. Theorem 3.11(4) is a result specific to \( \psi \)-tableaux. \( \alpha_d \) is part of the map \( \phi^{p,N}_{d,u} \) defined in Section 4.

Theorem 3.11. Let \( \mathcal{X} \) be the set of all tableaux and \( d \geq 1 \). Let \( \mathcal{Y}_d \) be the set of all tableaux with row \( d \) identical to row \( d + 1 \). Define \( \alpha_d(T) \) to be the tableau obtained from \( T \in \mathcal{X} \) by shifting any rows of index greater than \( d \) down one row and repeating row \( d \) in row \( d + 1 \). Then the map \( T \mapsto \alpha_d(T) \) is a bijection from \( \mathcal{X} \) to \( \mathcal{Y}_d \) (is clear). (Obtain \( \alpha_d^{-1}(T) \) from \( \tilde{T} \in \mathcal{Y}_d \) by deleting row \( d + 1 \) and shifting any rows of index greater than \( d + 1 \) up one row.) Let \( T \in \mathcal{X} \) and \( \tilde{T} = \alpha_d(T) \) (equivalently, let \( \tilde{T} \in \mathcal{Y}_d \) and \( T = \alpha_d^{-1}(T) \)). Let \( l = l(T) = l(\tilde{T}) \), and \( b \) the length of row \( d \) in both \( T \) and \( \tilde{T} \). Then:

1. The first \( d \) rows in \( T \) are identical to those rows in \( \tilde{T} \). In particular, \( b = 0 \) if and only if \( T = \tilde{T} \).
2. The rows of index at least \( d \) in \( T \) are identical to the rows of index at least \( d + 1 \) in \( \tilde{T} \). Each of the first \( b \) columns of \( \tilde{T} \) has column length one more than its respective column in \( T \).
3. Suppose \( b > 0 \). Let \( B \) and \( \tilde{B} \) be the end-boxes of the \( l \)-sets in \( T \) and in \( \tilde{T} \), respectively, and \( x_B \) the row index of \( B \). Then:
   a. \( x_B < d \) if and only if \( B = \tilde{B} \) if and only if the \( l \)-sets in \( T \) and in \( \tilde{T} \) are equal.
   b. \( x_B \geq d \) if and only if \( B \) translates down one unit to \( \tilde{B} \).
   c. The \( l \)-set in \( T \) begins in a row of index greater than \( d \) if and only if the \( l \)-set in \( \tilde{T} \) begins in a row of index greater than \( d + 1 \) if and only if the \( l \)-set in \( T \) translates down one unit to the \( l \)-set in \( \tilde{T} \).
   d. Suppose \( x_B \geq d \). Then the \( l \)-set in \( T \) is the set of last boxes of all rows \( j \) for \( 1 \leq j \leq x_B \) if and only if the \( l \)-set in \( \tilde{T} \) is the set of last boxes of all rows \( j \) for \( 1 \leq j \leq x_B + 1 \).
4. Suppose the height of the prime path of row \( d \) in \( T \) is \( d \) (equivalently, the height of the prime path of row \( d \) in \( \tilde{T} \) is \( d \)). Then \( T \) is a \( \psi \)-tableau if and only if \( \tilde{T} \) is a \( \psi \)-tableau.

\( T_1 \) and \( T_2 \) are both \( \psi \)-tableaux in Figure 3.6. For \( d = 5 \), \( T_1 \) satisfies the assumptions of Theorem 3.11(4); so \( \alpha_5(T_1) \) is a \( \psi \)-tableau (the prime paths of row 5 are bolded). For \( d = 3 \), \( T_2 \) does not satisfy the assumptions of Theorem 3.11(4); the prime path of row 3 (in bold) has height 1. In this case, \( \alpha_3(T_2) \) is not a \( \psi \)-tableau; the (4,2)-strip (grayed) does not equal the 4-set.

\[ \begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|}
\hline
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array} \quad \begin{array}{|c|c|c|c|}
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array} \end{array} \]

Figure 3.6. Examples of \( \alpha_d \)

Proof of Theorem 3.11. (1)-(3) follow easily from the definition of \( \alpha_d \).
The proof of (4) is by induction on \( l \). The base case \( l = 0 \) is handled in (1), so we may assume \( b > 0 \) (and \( l > 0 \)). Assume the notation in (3). We claim that

\begin{equation}
(3.1)
\end{equation}

the \( l \)-set is the \( B \)-strip in \( T \) if and only if the \( l \)-set is the \( \bar{B} \)-strip in \( \bar{T} \).

This fact implies (4) as follows. Suppose \( T \) is a \( \psi \)-tableau for which the height of the prime path of row \( d \) is \( d \). Then \( T^{(l-1)} \) is a \( \psi \)-tableau for which the height of the prime path of row \( d \) is \( d \) (by Proposition 3.7). By induction, it follows that \( \bar{T}^{(l-1)} \) is a \( \psi \)-tableau, and (3.1) implies that \( \bar{T} \) is a \( \psi \)-tableau (by Proposition 3.5(3)). The other direction is similar.

To prove (3.1), first suppose \( x_B < d \). Then the conditions in (3a) hold. It follows from (1), that the \( B \)-strip in \( T \) equals the \( \bar{B} \)-strip in \( \bar{T} \). (3.1) follows from the translation lemma and (3c).

Now suppose \( x_B \geq d \). Let \( P \) be the path along the contour of \( T \) between the bottom right corners of \( B \) and \( (d,b) \) (if \( B = (d,b) \), then \( P \) is a point; otherwise, assume \( P \) is made of north and east steps from \( B \) to \( (d,b) \)). Let \( \bar{P} \) be the like path associated to \( \bar{T} \) between \( \bar{B} \) and \( (d+1,b) \). It follows from (2) and (3b) that \( \bar{B} \) has row index \( x_B + 1 \) (and the same column index as \( B \)) and that \( P \) translates down one unit to \( \bar{P} \). There are two subcases.

In the first subcase, the prime path of \( B \) in \( T \) is a subpath of \( P \) (if and only if the prime path of \( \bar{B} \) in \( \bar{T} \) is a subpath of \( \bar{P} \)). Then the prime path of \( B \) in \( T \) translates down one unit to the prime path of \( \bar{B} \) in \( \bar{T} \). (3.1) follows from the translation lemma and (3c).

In the second subcase, the prime path of \( B \) in \( T \) is a subpath of the prime path of \( B \) (if and only if the prime path of \( \bar{B} \) in \( \bar{T} \) is a subpath of the prime path of \( \bar{B} \)). By assumption, the prime path of \( (d,b) \) in \( T \) has height \( d \). It follows that the prime path of \( B \) in \( T \) has height \( x_B \). Thus the \( B \)-strip in \( T \) is the set of last boxes of all rows \( j \) for \( 1 \leq j \leq x_B \). The prime path of \( (d,b) \) in \( \bar{T} \) has height \( d \), and is a subpath of the prime path of \( (d+1,b) \), which in turn (we said) is a subpath of the prime path of \( \bar{B} \). It follows that the prime path of \( \bar{B} \) in \( \bar{T} \) has height \( x_B + 1 \) (\( \bar{B} \) has row index \( x_B + 1 \)). Thus the \( \bar{B} \)-strip in \( \bar{T} \) is the set of last boxes of all rows \( j \) for \( 1 \leq j \leq x_B + 1 \). (3.1) then follows by (3d).

Going forward, we identify chains by corresponding \( \psi \)-tableaux.

4. Plus-full-sets and a map on maximal chains

The main focus of this section is to relate the map \( \phi_{l,n}^r \) and show that it is bijective; see Theorem 4.8. A maximal chain in the image of \( \psi \) may or may not possess a plus-full-set. \( \phi_{l,n}^r \) takes a maximal chain in \( C_l(n) \) to one in \( C_l(n+1) \), where \( r \) determines the domain and codomain in terms of plus-full-sets. We first make basic definitions and then build on \( \phi_{l,n}^r \) in Theorems 4.3 and 4.6. We then specialize the map defined in Theorem 4.6 by modifying its domain and codomain to define \( \phi_{l,n}^r \) in Theorem 4.8.

Recall that a maximal chain \( C \in C_l(n) \) satisfies \( s_h(C) = \delta_{n-1} \) and \( l(C) = n+i \). The outer diagonal of \( C \) is the set of boxes \( \{(k,n-k) \mid k \in [n-1]\} \). The label in the box \( (x,y) \) is denoted \( C(x,y) \).

**Definition 4.1.** Let \( C \) be a \( \psi \)-tableau of shape \( \delta_{n-1} \), for some \( n \geq 1 \). For \( r \in [l(C)] \), if the \( r \)-set begins in its first row and ends in its outer diagonal, then we call the \( r \)-set an \( r \)-full-set, or more generally a full-set. In this case, there is a unique box in the outer diagonal labeled with \( r \) (by property 3.6(4)), i.e., there is a unique \( k \in [n-1] \) satisfying \( C(k,n-k) = r \). The \( r \)-set is an \( r^+ \)-full-set, or more generally a plus-full-set, if:

- The \( r \)-set is a full-set, and
- its end-box \( (k,n-k) \) satisfies \( k = n-1 \), or \( k \in [n-2] \) and \( C(k+1,n-k-1) < C(k,n-k) \) (the southwest neighbor of \( (k,n-k) \) has a label less than \( r \)).

For each \( r \in [n+i] \), \( S_r^l(n) \) is the set of all \( C \in C_l(n) \) satisfying:

- The \( r \)-set is a plus-full-set, and
- for each \( j \in [r-1] \), the \( j \)-set is not a plus-full-set.

**Remark 4.2.** The \( S_r^l(n) \) are disjoint subsets of \( C_l(n) \). We denote a disjoint union of sets with \( \bigsqcup \).

\( C_0(3) \) consists of a single maximal chain, call it \( C \), shown in Figure 4.1. Neither its 1-set nor 3-set is a full-set, so neither is a plus-full-set. Its 2-set is a full-set, but is not a plus-full-set (\( 3 = C(2,1) > C(1,2) = 2 \)). Thus each of the subsets \( S_0^1(3), S_0^2(3) \) and \( S_0^3(3) \) of \( C_0(3) \) is the empty set.
Figure 4.1. \( C_0(3) \)  

Figure 4.2. \( C_0(4) = \biguplus_{r \in \mathbb{R}^2} S_r(4) \)

Each of the subsets \( S_r^1(4), S_r^2(4), S_r^3(4) \) and \( S_r^4(4) \) of \( C_0(4) \) consists of exactly one maximal chain, listed in Figure 4.2. The plus-full-set that qualifies each maximal chain is grayed. \( C_0(4) \) is the disjoint union of these subsets. It is shown in Theorem 5.8 that for all \( n \geq 2i + 4 \), \( C_i(n) \) is the disjoint union of nonempty subsets \( S_r^i(n) \) for which \( r \in [3i + 4] \).

**Theorem 4.3.** Let \( n \geq d \geq 1 \) and \( r \geq 0 \). Define \( \beta_d(Y) \) to be the tableau obtained from a tableau \( Y \) of length \( r \) by appending a box labeled with \( r + 1 \) to the end of all rows \( j \) for which \( 1 \leq j \leq d \).

Define \( X_d^r, n - 1 \) to be the set of all \( \psi \)-tableaux \( X \) of length \( r \) satisfying (1), \( Z \) is contained in \( \delta_r - 1 \), (2) the length of row \( d \) is \( n - d \), and (3) the prime path of row \( d \) has height \( d \).

Define \( Z_d^r, n \) to be the set of all \( \psi \)-tableaux \( Z \) of length \( r \) satisfying (1) \( Z \) is contained in \( \delta_r \), (2) \( r \) and \( d + 1 \) of \( Z(r) \) are identical, (3) the end-box of \( (r + 1) \)-set is \( (d, n - d + 1) \), and (4) the prime path of \( (d, n - d + 1) \) has height \( d \). (1) Suppose \( X \) is a tableau of length \( r \) which satisfies (1), and let \( Z = \beta_d(\alpha_d(X)) \). Then each of the first \( d \) rows in \( Z \) has row length one more than its respective row in \( X \), and each of the first \( n - d \) columns in \( Z \) has column length one more than its respective column in \( X \).

(2) The map \( X \mapsto \beta_d(\alpha_d(X)) \) is a bijection from \( X_d^r, n - 1 \) to \( Z_d^{r + 1}, n \) (see Figure 4.3).

**Proof.** (1) follows immediately by Theorem 3.11(1). (2) and the definition of \( \beta_d \).

To prove that this map is well-defined, suppose \( X \in X_d^r, n - 1 \). Let \( Y = \alpha_d(X) \) and \( Z = \beta_d(Y) \). \( Y \) and \( Z \) have lengths \( r \) and \( r + 1 \), respectively. Since \( X \) satisfies (1), it follows from (1) that \( Z \) satisfies (1). \( Y \) satisfies (2) and \( Z \) satisfies (2) and (3). \( Y \) is a \( \psi \)-tableau by Theorem 3.11(4). The prime path of row \( d \) in \( Y \) has height \( d \) and translates one unit to the right to the prime path of \( (d, n - d + 1) \) in \( Z \) (as implied by the labeling of the \( (r + 1) \)-set); thus \( Z \) satisfies \( (4) \), and the \( (r + 1) \)-set is the \( (d, n - d + 1) \)-strip in \( Z \). \( Z \) is a \( \psi \)-tableau by Proposition 3.5(3), and thus \( Z \in Z_d^{r + 1}, n \). From the fact that \( X = \alpha_d^{-1}(Z(r)) \), this map is injective.

To prove that this map is surjective, suppose \( Z \in Z_d^{r + 1}, n \). Because \( Z \) satisfies (2), we can let \( X = \alpha_d^{-1}(Z(r)) \). Evidently \( \beta_d(\alpha_d(X)) = \beta_d(Z(r)) = Z \). We only must show that \( X \in X_d^r, n - 1 \). Since \( Z \) satisfies (3), we have that \( Z(r) \) satisfies (1), and thus also \( X \) satisfies (1). Since \( Z \) satisfies (1), it follows from (1) that \( X \) satisfies (1). Since \( Z \) satisfies (4), it follows from Proposition 3.7 that \( Z(r) \) satisfies (4). \( X \) satisfies (3) and is a \( \psi \)-tableau by Theorem 3.11(4). Thus \( X \in X_d^r, n - 1 \). \( \square \)

**Corollary 4.4.** Let \( n \geq d \geq 1 \) and \( r \geq 0 \). Suppose \( X \in X_d^r, n - 1 \), and let \( Z = \beta_d(\alpha_d(X)) \) (equivalently, suppose \( Z \in Z_d^{r + 1}, n \), and let \( X = \alpha_d^{-1}(\beta_d^{-1}(Z(r))) \)). Then:

(1) If \( d = n \), then row \( d \) in \( X \) is empty, and \( (d, n - d + 1) = (n, 1) \) is in the outer diagonal of \( \delta_n \). If \( d \in [n - 1] \), then \( (d, n - d) \in X \) is in the outer diagonal of \( \delta_{n - 1} \), and \( (d + 1, n - d) \), \( (d, n - d + 1) \in Z \) are in the outer diagonal of \( \delta_n \).
Overlay $X$ on top of the Young diagram of shape $\delta_{n-1}$ so that $X$ is positioned to the top left, and call this construction $\bar{X}$. Overlay $Z$ on top of $\delta_n$ in the like manner to obtain $\bar{Z}$ (see Figure 4.3).

(2) Any unlabeled boxes in $\bar{X}$ of row index less than $d$ (equivalently, to the right of column $n - d$) translate to the right one unit to (and have the same skew shape as) any unlabeled boxes in $\bar{Z}$ of row index less than $d$ (equivalently, to the right of column $n - d + 1$). Any unlabeled boxes in $\bar{X}$ to the left of column $n - d$ (equivalently, of row index greater than $d$) translate down one unit to (and have the same skew shape as) any unlabeled boxes in $\bar{Z}$ to the left of column $n - d$ (equivalently, of row index greater than $d + 1$).

(3) Any unlabeled boxes in $\bar{X}$ and $\bar{Z}$ are accounted for in (2).

Proof. (1) is clear. (2) and (3) follow from (1) and Theorem 4.3(1). \qed

**Lemma 4.5.** Let $d \geq 1$, and suppose $T$ is a $\psi$-tableau such that rows $d$ and $d + 1$ have equal lengths. Then those rows are identical.

Proof. Let $b$ be the length of row $d$. If $b = 0$, then this is clear, so assume $b > 0$. By induction it suffices to show that $T(d, b) = T(d + 1, b)$. Let $r = T(d + 1, b)$ and $B$ be the end-box of the $r$-set. $T(r)$ contains $(d + 1, b)$, so must contain $(d, b)$. The prime path of $(d, b)$ in $T(r)$ is a subpath of the prime path of $(d + 1, b)$ which in turn is a subpath of the prime path of $B$. By property 3.6(3b), $T(d + 1, b) = T(d, b)$. \qed

**Theorem 4.6.** Let $i \geq -1$, $n \geq 1$ and $0 \leq r \leq n + i$. For $C \in C_i(n)$, define $\hat{C}(r)$ as follows. If there exists $k \in \{n - 1\}$ such that $C(k, n - k) \leq r$, then let $d$ be minimal for $k$; otherwise, set $d = n$. $\hat{C}(r)$ is the tableau obtained after performing the following iterative steps:

1. Start with $\alpha_d(C(r))$.
2. Obtain $\beta_d(\alpha_d(C(r)))$.
3. For elements greater than $r$ in $C$:
   a. For all those of row index less than $d$, translate them to the right one unit to our construction, and increment their labels by one.
   b. For all those of row index greater than $d$, translate them down one unit to our construction, and increment their labels by one.

Then the map $C \mapsto \hat{C}(r)$ is a bijection from $C_i(n)$ to $\{\hat{C} \in C_i(n + 1) \mid \hat{C}$ has an $(r + 1)^+$-full-set$\}$.

| $r$ | $d = 0$ | $r = 1$ | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ | $r = 6$ |
|-----|---------|---------|---------|---------|---------|---------|---------|
| $d = 1$ | 1 2 3 5 | 1 2 3 5 | 1 2 3 5 | 1 2 3 5 | 1 2 3 5 | 1 2 3 5 | 1 2 3 5 |
| $d = 2$ | 1 3 4 6 | 1 3 4 6 | 1 3 4 6 | 1 3 4 6 | 1 3 4 6 | 1 3 4 6 | 1 3 4 6 |
| $d = 3$ | 1 4 5 7 | 1 4 5 7 | 1 4 5 7 | 1 4 5 7 | 1 4 5 7 | 1 4 5 7 | 1 4 5 7 |

**Figure 4.4.** $\hat{C}(r)$ is computed for each $0 \leq r \leq 6$ for a $C \in C_1(5)$.
Examples for a maximal chain $C \in \mathcal{C}_1(5)$ are shown in Figure 4.4. $\tilde{C}(r)$ is computed for each $0 \leq r \leq 6$. The outline of $C^{(r)}$ is bolded in the first row of the figure, and its image under $\alpha_d$ is shown in Step 1. Boxes labeled with $r+1$ resulting from $\tilde{B}_2$ are grayed in Step 2. Elements greater than $r$ in $C$ are grayed in the first row of the figure. Their translated counterparts are grayed in Step 3, and the borders of $(d,n-d+1)$ is bolded.

Proof of Theorem 4.6. Suppose $C \in \mathcal{C}_i(n)$. Our choice of $d$ is unique for $C$ and $r$. If $d = n$, then row $d$ in $C$ is empty, and $C^{(r)}$ is contained in $\delta_{n-2}$. If $d \in [n-1]$, then $C(d,n-d) \leq r$, and $(d,n-d)$ is the box of least row index in the outer diagonal of $C^{(r)}$. In either case, the height of the prime path of row $d$ in $C^{(r)}$ is $d$. Thus $C^{(r)} \in \mathcal{C}_d^{(r)}$, and evidently $\tilde{\beta}_d(\alpha_d(C^{(r)})) \in \mathcal{Z}_d^{(r+1)}$. It also follows from Corollary 4.4(2)-(3) and by way of steps (3a)-(3b) that $\tilde{C}(r)$ is a tableau of shape $\tilde{\delta}_n$ and of length $l(C) + 1 = n + i + 1$. Moreover, the $(r+1)$-set in $\tilde{C}(r)$ begins in the first row, and its end-box $(d,n-d+1)$ is in the outer diagonal (see Corollary 4.4(1)); so it is a full-set. If $d \in [n-1]$, then $(d+1,n-d) \in \alpha_d(C^{(r)}) = (\tilde{C}(r))^{(r)}$ which implies that $(\tilde{C}(r))(d+1,n-d) \leq r < r+1 = (\tilde{C}(r))(d,n-d+1)$. Thus the $(r+1)$-set is a plus-full-set.

On the other hand suppose $\bar{C} \in \mathcal{C}_i(n+1)$ has an $(r+1)^+$-full-set. The $(r+1)^+$-full-set in $\bar{C}$ begins in the first row and there is a unique $d \in [n]$ satisfying $\bar{C}(d,n-d+1) = r+1$. The height of the prime path of $(d,n-d+1)$ in $C^{(r+1)}$ is $d$. We claim that rows $d$ and $d+1$ in $\tilde{C}^{(r)}$ are identical. By Lemma 4.5, it suffices to show that those rows have equal length. If $d = n$, then $\bar{C}(d,n-d+1) = \tilde{C}(n,1) = r+1$, which implies that rows $d$ and $d+1$ are empty in $\tilde{C}(r)$. Suppose $d \in [n-1]$. We have that $\bar{C}(d+1,n-d) < \bar{C}(d,n-d+1) = r+1$ (by Definition 4.1) and that $\bar{C}(d,n-d) < \bar{C}(d,n-d+1) = r+1$ (rows strictly increase). Thus $(d+1,n-d)$ and $(d,n-d)$ are the last boxes of their rows in $\tilde{C}(r)$. Our claim is proved. Thus $\tilde{C}(r+1) \in \mathcal{Z}_d^{(r+1)}$, and $\alpha_d^{-1}(\tilde{C}(r+1)) \in \mathcal{X}_d^{(r+1)}$. It also follows from Corollary 4.4(2)-(3) that the construction obtained from $\bar{C}$ by reversing steps (1)-(3) is a tableau of shape $\tilde{\delta}_{n-1}$ and of length $l(C) - 1 = n + i$.

If $r = n + i$, we are done; otherwise, assume $0 \leq r < n + i$. Suppose that $\bar{C}$ is obtained from $C \in \mathcal{C}_i(n)$ by way of steps (1)-(3), or that $C$ is obtained from $\bar{C} \in \mathcal{C}_i(n+1)$ having an $(r+1)^+$-full-set by reversing those steps. We will show that $C$ is a $\psi$-tableau if and only if $\bar{C}$ is a $\psi$-tableau. Based on this fact, the map $C \mapsto \bar{C}(r)$ is well-defined and surjective, and it is clearly injective. Let $r < k \leq n + i$, $B_k$ the end-box of the $k$-set in $C$, and $\bar{B}_{k+1}$ the end-box of the $(k+1)$-set in $\bar{C}$. We claim that

\begin{equation}
(4.1) \quad \text{the $k$-set is the $B_k$-strip in $C^{(k)}$ if and only if the $(k+1)$-set is the $\bar{B}_{k+1}$-strip in $\bar{C}^{(k+1)}$}.
\end{equation}

Since $C^{(r)}$ and $\bar{C}^{(r+1)}$ are $\psi$-tableaux, (4.1) implies that $C$ is a $\psi$-tableau if and only if $\bar{C}$ is a $\psi$-tableau by way of Proposition 3.5(2). The proof of (4.1) relies on Corollary 4.4. There are two cases.

In the first case, the following three equivalent conditions hold due to step (3a) (or its reverse): $B_k \in C$ has row index less than $d$, $\bar{B}_{k+1} \in \bar{C}$ has row index less than $d$, and $B_k$ translates to the right one unit to $\bar{B}_{k+1}$. Its clear that the $B_k$-strip in $C^{(k)}$ translates to the right one unit to the $\bar{B}_{k+1}$-strip in $\bar{C}^{(k+1)}$. (4.1) then follows by way of step (3a).

In the second case, the following three equivalent conditions hold due to step (3b) (or its reverse): $B_k \in C$ has row index greater than $d$, $\bar{B}_{k+1} \in \bar{C}$ has row index greater than $d + 1$, and $B_k$ translates down one unit to $\bar{B}_{k+1}$. Since $(d,n-d)$ is in the outer diagonal of $C^{(k)}$, it follows by property 3.6(3c) that the $B_k$-strip in $C^{(k)}$ begins in a row of index greater than $d$. Likewise, since $(d+1,n-d)$ is in the outer diagonal of $\bar{C}^{(k+1)}$, the $\bar{B}_{k+1}$-strip in $\bar{C}^{(k+1)}$ begins in a row of index greater than $d + 1$. The shape of $\{(x,y) \in C^{(k)} | x \geq d\}$ is the shape of $\{(x,y) \in \bar{C}^{(k+1)} | x \geq d + 1\}$ and those sets contain the enclosures of the $B_k$-strip in $C^{(k)}$ and of the $\bar{B}_{k+1}$-strip in $\bar{C}^{(k+1)}$, respectively. By the translation lemma, the $B_k$-strip in $C^{(k)}$ translates down one unit to the $\bar{B}_{k+1}$-strip in $\bar{C}^{(k+1)}$. (4.1) then follows by way of step (3b).

The map in Theorem 4.6 preserves full-sets and plus-full-sets in the following sense.

Proposition 4.7. Let $i \geq -1$, $n \geq 1$ and $0 \leq r \leq n + i$. Suppose $C \in \mathcal{C}_i(n)$. Then:
(1) For $j \in [r]$, $C$ has a $j$-full-set (respectively, $j^+$-full-set) if and only if $\hat{C}(r)$ has a $j$-full-set (respectively, $j^+$-full-set).

(2) For $r < j \leq n + i$, $C$ has a $j$-full-set (respectively, $j^+$-full-set) if and only if $\hat{C}(r)$ has a $(j+1)$-full-set (respectively, $(j+1)^+$-full-set).

Proof. By our choice of $d$ in Theorem 4.6, $d \in [n]$ and the following two items hold.

- Each box of row index less than $d$ in the outer diagonal of $C$ has a label greater than $r$. If $d \in [n-1]$, then $C(d,n-d) \leq r$.
- $(\hat{C}(r))(d,n-d+1) = r + 1$. Each box of row index less than $d$ in the outer diagonal of $\hat{C}(r)$ has a label greater than $r + 1$. If $d \in [n-1]$, then $(\hat{C}(r))(d+1,n-d) \leq r$.

1. Suppose $j \in [r]$. If the $j$-set is a full-set in $C$, it ends in a row of index at least $d$. If the $j$-set is a full-set in $\hat{C}(r)$, it ends in a row of index at least $d+1$. It then follows from step 4.6(1) that $C$ has a $j$-full-set if and only if $\hat{C}(r)$ has a $j$-full-set.

   In that case, we have $C(k,n-k) = j = (\hat{C}(r))(k+1,n-k)$ for a unique $k$ satisfying $d \leq k \leq n-1$. Assume that case. If $k = n-1$, then both $C$ and $\hat{C}(r)$ have $j$-full-sets, so suppose $k \in [n-2]$. Then it also follows from step 4.6(1) that $C(k+1,n-k-1) = C(k,n-k)$ if and only if $(\hat{C}(r))(k+1,n-k-1) = (\hat{C}(r))(k+1,n-k)$ (in that case, $C(k+1,n-k-1) = (\hat{C}(r))(k+1,n-k-1)$), finishing the proof.

2. Suppose $r < j \leq n+i$. If the $j$-set is a full-set in $C$, it ends in a row of index less than $d$ (otherwise, row $d$ in $C$ would have the label $j > r$). If the $(j+1)$-set is a full-set in $\hat{C}(r)$, it ends in a row of index less than $d$ (otherwise, row $d$ in $\hat{C}(r)$ would have the label $j + 1 > r + 1$). It then follows from step 4.6(3a) that $C$ has a $j$-full-set if and only if $\hat{C}(r)$ has a $(j+1)$-full-set. In that case, we have $C(k,n-k) = j$ and $\hat{C}(r))(k,n-k+1) = j + 1$ for a unique $k$ satisfying $1 \leq k < d$. Supposing that case, there are three subcases.

   In the first subcase, suppose that $k = n-1$, so $d = n$. Thus $C(n-1,1) = j$, and $(\hat{C}(r))(n,1) = r + 1 < j + 1 = (\hat{C}(r))(n-1,2)$, so that $C$ has a $j^+$-full-set and $\hat{C}(r)$ has a $(j+1)^+$-full-set.

   In the second subcase, suppose that $k \in [n-2]$ and that $k + 1 = d$. Then $C(k+1,n-k-1) = C(d,n-d) \leq r < j = C(k,n-k)$ and $(\hat{C}(r))(k+1,n-k) = (\hat{C}(r))(d,n-d+1) = r + 1 < j + 1 = (\hat{C}(r))(k,n-k+1)$, so that $C$ has a $j^+$-full-set and $\hat{C}(r)$ has a $(j+1)^+$-full-set.

   In the third subcase, suppose that $k \in [n-2]$ and that $k + 1 < d$. Then $C(k+1,n-k-1) > r$ and $(\hat{C}(r))(k+1,n-k) > r + 1$. It follows from step 4.6(3a) that $C(k+1,n-k-1) + 1 = (\hat{C}(r))(k+1,n-k)$, and thus that $C(k+1,n-k-1) < C(k,n-k) = j$ if and only if $(\hat{C}(r))(k+1,n-k) < (\hat{C}(r))(k,n-k+1) = j + 1$. Therefore $C$ has a $j^+$-full-set if and only if $\hat{C}(r)$ has a $(j+1)^+$-full-set. □

Theorem 4.8. Let $i \geq -1$, $n \geq 1$ and $0 \leq r \leq n + i$. The map

$$\phi_{r,n} : \{C \in C_i(n) \mid \forall j \in [r], C \notin S_j^r(n)\} \rightarrow S_{r+1}^r(n+1)$$

$$C \mapsto \hat{C}(r)$$

is a bijection.

Remark 4.9. The condition on the domain of $\phi_{r,n}$, $\forall j \in [r], C \notin S_j^r(n)$, is equivalent to $\forall j \in [r]$, the $j$-set in $C$ is not a plus-full-set.

In the examples in Figure 4.4, $C$ has no plus-full-sets. Thus for each $0 \leq r \leq 6$, $C$ is in the domain of $\phi_{r,5}$.

Proof of Theorem 4.8. In lieu of Theorem 4.6, this follows from Proposition 4.7(1). □

Corollary 4.10. Let $i \geq -1$, $n \geq 1$ and $0 \leq r \leq n + i$. Then

$$\#S_{r+1}^r(n+1) = \#C_i(n) - \sum_{j=1}^{r} \#S_j^r(n).$$

Proof. This is a direct implication of Theorem 4.8, recalling Remark 4.2. □
A recursion on maximal chains in the Tamari lattices

5. A formula for the number of maximal chains of length \( n + i \) in \( \mathcal{T}_n \)

The technical work in verifying the bijectivity of \( \phi_{i,n}^r \) is complete! In this section, we gather more on properties and consequences of this map and tie our results together to write a recursive formula for \( \#C_i(n) \). \( \phi_{i,n}^r \) maps a maximal chain to one with one more plus-full-set (Proposition 5.1). We may then write each maximal chain having a plus-full-set uniquely in terms of one with no plus-full-sets (Corollary 5.3). By relating this unique representation for a maximal chain to specific plus-full-sets that it contains (Proposition 5.5), we obtain an expression for \( \#C_i(n) \); see equation (5.5). For each \( i \geq -1 \), there exists a maximal chain in \( C_i(2i + 3) \) containing no plus-full-sets (Lemma 5.7), but surprisingly, for all \( n \geq 2i + 4 \), each maximal chain in \( C_i(n) \) has a plus-full-set (Theorem 5.8). We utilize these latter two facts to refine our expression for \( \#C_i(n) \) and show that it is a polynomial of degree \( 3i + 3 \) in Theorem 5.9.

**Proposition 5.1.** Each \( C \) in the domain of \( \phi_{i,n}^r \) has one less plus-full-set than its image.

**Proof.** Suppose \( C \) is in the domain of \( \phi_{i,n}^r \). By definition, for each \( j \in [r] \), neither \( C \) nor \( \phi_{i,n}^r(C) \) has a \( j^+ \)-full-set. Of course, \( \phi_{i,n}^r(C) \) has the \( (r + 1)^+ \)-full-set. The proof follows from Proposition 4.7(2). \( \square \)

**Definition 5.2.** \( \mathcal{N}_i(n) \) is the set of all maximal chains in \( C_i(n) \) having no plus-full-sets.

\[
C_i(n) \text{ is a disjoint union of the } n+i+1 \text{ subsets, } \mathcal{N}_i(n) \text{ and } S_i^j(n), j \in [n+i], \text{ i.e.,}
\]

\[
C_i(n) = \mathcal{N}_i(n) \bigcup \left( \bigcup_{j \in [n+i]} S_i^j(n) \right).
\]

**Corollary 5.3.** Suppose the number of plus-full-sets of some \( C \in C_i(n) \) is \( t > 0 \). Then there exists a unique \( \tilde{C}_1 \in C_i(n-1) \) and a unique \( r_1 \), such that \( C = \phi_{i,n-1}^{r_1}(\tilde{C}_1) \). This representation may be extended to obtain unique representations

\[
C = \phi_{i,n-1}^{r_1}(\tilde{C}_1) = \phi_{i,n-1}^{r_2}(\phi_{i,n-2}^{r_1}(\tilde{C}_2)) = \cdots = (\phi_{i,n-1}^{r_1} \circ \phi_{i,n-2}^{r_2} \circ \cdots \circ \phi_{i,n-t}^{r_1})(\tilde{C}_t),
\]

until we arrive at \( \tilde{C}_t \in \mathcal{N}_i(n-t) \).

**Proof.** The codomain of \( \phi_{i,n-1}^{r_1} \) is \( S_i^{r+1}(n) \). As \( r \) varies, \( 0 \leq r \leq n-1+i \), the \( S_i^{r+1}(n) \) are disjoint subsets of \( C_i(n) \). \( C \) is an element of exactly one of the \( S_i^{r+1}(n) \), so there exists a unique \( r_1 \), such that \( \phi_{i,n-1}^{r_1} \) has \( C \) in its codomain. Since \( \phi_{i,n-1}^{r_1} \) is bijective, there exists a unique \( \tilde{C}_1 \in C_i(n-1) \), such that \( C = \phi_{i,n-1}^{r_1}(\tilde{C}_1) \). By Proposition 5.1, the number of plus-full-sets in \( \tilde{C}_1 \) is \( t-1 \). If \( t = 1 \), then \( \tilde{C}_1 \in \mathcal{N}_i(n-1) \); otherwise, \( t-1 > 0 \), and we may repeat until we arrive at \( \tilde{C}_t \in \mathcal{N}_i(n-t) \). \( \square \)

**Remark 5.4.** A maximal chain in \( \mathcal{T}_n \) has at most \( n-1 \) plus-full-sets, as bounded by the \( n-1 \) strictly increasing labels in its first row.

By Corollary 5.3, the number of maximal chains in \( C_i(n) \) with exactly \( t \) plus-full-sets, \( 1 \leq t \leq n-1 \), is the number of representations

\[(\phi_{i,n-1}^{r_1} \circ \phi_{i,n-2}^{r_2} \circ \cdots \circ \phi_{i,n-t}^{r_1})(\tilde{C})\]

over \( t \)-tuples \((r_1, r_2, \ldots, r_t)\) and over \( \tilde{C} \in \mathcal{N}_i(n-t) \). Each \( t \)-tuple \((r_1, r_2, \ldots, r_t)\) must satisfy restrictions imposed on the \( r_j \) in Theorem 4.8:

**Proposition 5.5.** Suppose that \( \tilde{C} \in \mathcal{N}_i(n-t) \) for some \( n \) and \( t \), satisfying \( 1 \leq t \leq n-1 \). A \( t \)-tuple \((r_1, r_2, \ldots, r_t)\) for the representation (5.2), must only satisfy \( 0 \leq r_1 \leq r_2 \leq \cdots \leq r_t \leq n-t+i \). The number of these, hence the number of representations (5.2), is \( \binom{n+i}{t} \).
For a $t$-tuple $(r_1, r_2, \ldots, r_t)$ which satisfies the criteria, let $C = (\phi_{i,n-1}^1 \circ \phi_{i,n-2}^2 \circ \cdots \circ \phi_{i,n-t}^1)(\tilde{C})$. The set of specific plus-full-sets in $C$ is

\[
\{ j \mid C \text{ has a } j^+\text{-full-set} \} = \{ r_1 + 1, r_2 + 2, \ldots, r_t + t \},
\]

which is a $t$-element subset of $[n + i]$ unique to $(r_1, r_2, \ldots, r_t)$.

**Proof.** Consider a $t$-tuple $(r_1, r_2, \ldots, r_t)$ for the representation (5.2). By Theorem 4.8, since $\tilde{C} \in \mathcal{N}_t(n - t)$, $r_t$ for $\phi_{i,n-1}^1$ must only satisfy $0 \leq r_t \leq n - t + i$. We obtain $\phi_{i,n-t}^1(\tilde{C}) \in S_{i,n+1}^{t+1}(n - t + 1)$. The $(r_t + 1)$-set in $\phi_{i,n-t}^1(\tilde{C})$ is its only plus-full-set. By definition, $r_{t-1}$ for $\phi_{i,n-(t-1)}^1$ must only satisfy $0 \leq r_{t-1} \leq r_t$. Continuing in this manner, we find that $(r_1, r_2, \ldots, r_t)$ must only satisfy $0 \leq r_1 \leq r_2 \leq \cdots \leq r_t \leq n + i$. The standard trick is to make the substitution $r_k = u_k - k$, obtaining $0 \leq u_1 < u_2 < \cdots < u_t \leq n + i$. The $t$-tuples $(u_1, u_2, \ldots, u_t)$ which satisfy this are the $t$-element subsets of $[n + i]$. Moreover, by 4.7(2), $\{ u_1, u_2, \ldots, u_t \} = \{ r_1 + 1, r_2 + 2, \ldots, r_t + t \}$ is the set $\{ j \mid C \text{ has a } j^+\text{-full-set} \}$ for $C = (\phi_{i,n-1}^1 \circ \phi_{i,n-2}^2 \circ \cdots \circ \phi_{i,n-t}^1)(\tilde{C})$. 

**Corollary 5.6.** There is equal representation in $C_t(n)$ over equal size subsets of $[n + i]$ in terms of specific plus-full-sets found in maximal chains: For each $t$-element subset $U \subseteq [n + i]$, such that $0 \leq t \leq n - 1$,

\[
\#\{ C \in \mathcal{C}_t(n) \mid U = \{ j \mid C \text{ has a } j^+\text{-full-set} \} \} = \#\mathcal{N}_t(n - t), \tag{5.3}
\]

\[
\#\{ C \in \mathcal{C}_t(n) \mid U \subseteq \{ j \mid C \text{ has a } j^+\text{-full-set} \} \} = \#\mathcal{C}_t(n - t). \tag{5.4}
\]

**Proof.** The case $t = 0$ is trivial so assume $1 \leq t \leq n - 1$.

For equation (5.3), let $S_1 = \{ C \in \mathcal{C}_t(n) \mid U = \{ j \mid C \text{ has a } j^+\text{-full-set} \} \}$. Suppose $U = \{ u_1, u_2, \ldots, u_t \}$, where $0 < u_1 < u_2 < \cdots < u_t \leq n + i$. By Corollary 5.3 and Proposition 5.5, each $C \in S_1$ has the representation

\[
(\phi_{i,n-1}^{u_1-1} \circ \phi_{i,n-2}^{u_2-2} \circ \cdots \circ \phi_{i,n-t}^{u_t-t})(\tilde{C}),
\]

for a unique $\tilde{C} \in \mathcal{N}_t(n - t)$, and this is an element of $S_1$ for every $\tilde{C} \in \mathcal{N}_t(n - t)$.

For equation (5.4), let $S_2 = \{ C \in \mathcal{C}_t(n) \mid U \subseteq \{ j \mid C \text{ has a } j^+\text{-full-set} \} \}$. Because of equation (5.3), $\#S_2$ depends only on $\#U = t$. It suffices to consider $U = [t]$. Suppose $C \in S_2$ has exactly $s \geq t$ plus-full-sets. Again by Corollary 5.3 and Proposition 5.5, there exists a unique $\tilde{C} \in \mathcal{N}_t(n - s)$ and a unique $s$-tuple $(r_1, r_2, \ldots, r_s)$, such that

\[
C = (\phi_{i,n-1}^{r_1} \circ \phi_{i,n-2}^{r_2} \circ \cdots \circ \phi_{i,n-s}^{r_s})(\tilde{C}),
\]

where $0 \leq r_1 \leq r_2 \leq \cdots \leq r_s \leq n - s + i$ and $[t] \subseteq \{ r_1 + 1, r_2 + 2, \ldots, r_s + s \}$. For each $k \in [t]$, we must have $r_k = 0$. Thus, each $C \in S_2$ has the representation

\[
(\phi_{i,n-1}^0 \circ \phi_{i,n-2}^0 \circ \cdots \circ \phi_{i,n-t}^0)(\tilde{C}),
\]

for a unique $\tilde{C} \in \mathcal{C}_t(n - t)$, and this is an element of $S_2$ for every $\tilde{C} \in \mathcal{C}_t(n - t)$. 

An expression for $\#\mathcal{C}_t(n)$ is acquired from equation (5.3). (Note, the expression may be obtained directly from Proposition 5.5.)

\[
\#\mathcal{C}_t(n) = \sum_{i=0}^{n-1} \binom{n + i}{t} \#\mathcal{N}_t(n - t) = \sum_{i=1}^{n} \binom{n + i}{t + i} \#\mathcal{N}_t(t). \tag{5.5}
\]

The second expression follows from the first by reindexing and is refined in Theorem 5.9.

An expression for $\#\mathcal{N}_t(n)$ is obtained from equations (5.3), (5.4) and the principle of inclusion and exclusion.

\[
\#\mathcal{N}_t(n) = \sum_{i=0}^{n-1} (-1)^i \binom{n + i}{t} \#\mathcal{C}_t(n - t) = \sum_{i=1}^{n} (-1)^{n-i} \binom{n + i}{t + i} \#\mathcal{C}_t(t). \tag{5.6}
\]

The second expression follows from the first by reindexing.
In Theorem 5.8, the surprising fact is that for all \( n \geq 2i + 4 \), every maximal chain in \( C_i(n) \) has a plus-full-set. For this condition on \( C_i(n) \), we can do no better (the following lemma). These facts enable us to reach our main objective in Theorem 5.9.

**Lemma 5.7.** For each \( i \geq -1 \), \( \#N_i(2i + 3) > 0 \).

**Proof.** For \( i = -1 \), \( C_{i-1}(1) \) consists of only the null diagram, so assume \( i \geq 0 \). A qualifying maximal chain in \( T_n, n = 2i + 3 \), will have the shape \( \delta_{n-1} = (2i + 2, \ldots, 1) \) and length \( n + i = 3i + 3 \). Let \( C \) be a Young diagram of the desired shape.

For all \( 0 \leq k \leq i \), let \( C(n - (2k + 1), 2k + 1) = n + i - k \). Each \( r \)-set, for \( r \in [n,n+i] \), is labeled here and consists of a single box in the outer diagonal. Specifically, \( C(n-1,1) = n + i \) and \( C(2,n-2) = n \), and every other box between \((n-1,1)\) and \((2,n-2)\) gets a label.

For all \( k \in [n-1] \), label the remaining unlabeled boxes in column \( k \) with \( k \).

The resulting diagram is a \( \phi \)-tableau of the desired shape and length (see Figure 5.1). Moreover, the full-sets of \( C \) end in the boxes \((n - (2k + 2), 2k + 2), 0 \leq k \leq i \), such that \( C(n - (2k + 1), 2k + 1) > C(n - (2k + 2), 2k + 2) \), so that each full-set is not a plus-full-set. 

\[ \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array} \]

**Figure 5.1.** Examples of maximal chains in \( N_i(2i + 3), i \in \{0,1,2,3\} \)

**Theorem 5.8.** For each \( i \geq -1 \) and for all \( n \geq 2i + 4 \),

1. \( \#C_i(n) > 0 \).
2. every element of \( C_i(n) \) has a plus-full-set, i.e., \( \#N_i(n) = 0 \).
3. \( C_i(n) = \bigcup_{j \in [3i+4]} S_i^j(n) \), where each \( S_i^j(n) \) is nonempty.

**Proof.** (1). Since maximal chains in \( T_n \) range in length from \( n-1 \) to \( \binom{n}{2} \), it suffices to show that \( n-1 \leq n+i \leq \binom{n}{2} \). Since \( i \geq -1 \), \( n-1 \leq n+i \). Since \( n \geq 2i+4 \geq 2 \),

\[ n+i \leq \binom{n}{2} \iff 0 \leq n^2 - 3n - 2i, \]

and

\[ n^2 - 3n - 2i \leq n^2 - 3n + 4 - n = (n-2)^2 \geq 0. \]

(2). Let \( C \in C_i(n) \). Since \( n \geq 2 \), \( C \) is not the null diagram. Suppose \( n = 2i + 4 + l \) for some \( l \geq 0 \). The length of \( C \) is \( n+i = 3i+4+l \). There are no repeat labels in the first row of \( n-1 = 2i+3+l \) boxes. Likewise, there are no repeat labels in the \( n-1 \) boxes in the outer diagonal. Let \( x \) be the number of full-sets in \( C \) and note that \( (1,n-1) \) constitutes a full set. The combined number of distinct labels in the first row and outer diagonal is \( x+2(n-1-x) \leq n+i \), thus

\[ x \geq n-2-i = i+2+l. \]

Suppose two full-sets end in boxes in adjacent columns, say in \((k,n-k)\) and \((k+1,n-k-1)\). Then \( C(k+1,n-k-1) < C(k,n-k) \), and thus \( C \) has an \( r^+ \)-full-set for \( r = C(k,n-k) \). On the other hand suppose no two full-sets end in adjacent columns. Then we require at least \((i+2+l) +(i+1+1) = 2i + 3 + 2l \) boxes in the outer diagonal. Thus, \( l = 0 \), \( n = 2i+4 \), and of the \( 2i+3 \) boxes in the outer diagonal, full-sets end in \( i+2 \) of them. But then one must end in \((n-1,1)\), resulting in an \( r^+ \)-full-set for \( r = C(n-1,1) \).

(3). Let \( n = 2i+4+l \) for some \( l \geq 0 \). Let \( C \in C_i(n) \) and suppose its number of plus-full-sets is \( t \). By Corollary 5.3, there exists a unique \( \tilde{C} \in N_i(n-t) \) and a unique \( t \)-tuple \((r_1,r_2,\ldots,r_t)\), such that

\[ C = (\phi_{i,n-1}^t \circ \phi_{i,n-2}^t \circ \cdots \circ \phi_{i,n-t}^t)(\tilde{C}) \].

By (2), \( n-t \leq 2i+3 \), thus \( t \geq l+1 \). Since the length of \( C \)
is $n + i = 3i + 4 + l$, there exists a $j^+$-full-set in $C$ satisfying $j \leq 3i + 4$. Thus, $C \in S_{i}^{j}(n)$ for some $r \leq j \leq 3i + 4$.

We show by induction on $n$, that for each $j \in [3i + 4]$, $S_{i}^{j}(n)$ is nonempty. For the base case $n = 2i + 4$, let $C \in \mathcal{N}_{i}(2i + 3)$. Then for each $r$ (as in Theorem 4.8), satisfying $0 \leq r \leq 3i + 3$, $\phi^{2i+3}_{i}(C) \in S_{i}^{r+1}(2i + 4)$. Now suppose the statement is true for $n$. By the inductive hypothesis, there exists $C \in S_{i}^{r+1}(n)$. We may choose any value of $r$ (as in Theorem 4.8), satisfying $0 \leq r \leq 3i + 3$, to obtain $\phi_{i}(C) \in S_{i}^{r+1}(n + 1)$. □

**Theorem 5.9.** For each $i \geq -1$ and for all $n \geq 1$, the number of maximal chains in $T_{n}$ of length $n + i$ is

$$
(5.7) \quad \#C_{i}(n) = \sum_{t=1}^{2i+3} \binom{n+i}{t+i} \#N_{i}(t),
$$

a polynomial in $n$ of degree $3i + 3$. The initial values of $\#N_{i}(n)$, $n \in [2i + 3]$, are

$$
(5.8) \quad \#N_{i}(n) = \sum_{t=1}^{n} (-1)^{n-t} \binom{n+i}{t+i} \#C_{i}(t).
$$

**Proof.** If $n \geq 2i + 4$, then for all $t$ satisfying $2i + 4 \leq t \leq n$, $\#N_{i}(t) = 0$, thus equation (5.5) reduces to equation (5.7). On the other hand, suppose $1 \leq n \leq 2i + 3$. Then for all $t$ satisfying $n < t \leq 2i + 3$, $\binom{n+i}{t+i} = 0$, thus equation (5.7) reduces to equation (5.5). Equation (5.8) is equation (5.6).

The summand for $t = 2i + 3$ in equation (5.7) is the one containing the term with the largest power of $n$, the term being

$$
\frac{n^{3i+3}}{(3i + 3)!} \#N_{i}(2i + 3).
$$

By Lemma 5.7, this term is nonzero, thus $\#C_{i}(n)$ is a polynomial of degree $3i + 3$. □

| Length | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ | $T_9$ | $T_{10}$ | $T_{11}$ | $T_{12}$ | $T_{13}$ |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|----------|----------|
| $n - 1$ | 1     |       |       |       |       |       |       |       |       |          |          |          |          |
| $n$   | 1     |       |       |       |       |       |       |       |       |          |          |          |          |
| $n + 1$ | 2     | 10    |       |       |       |       |       |       |       |          |          |          |          |
| $n + 2$ | 2     | 8     | 112   | 280   |       |       |       |       |       |          |          |          |          |
| $n + 3$ | 18    | 220   | 1,464 | 9,240 | 15,400 |       |       |       |       |          |          |          |          |
| $n + 4$ | 13    | 218   | 5,322 | 42,592 | 281,424 | 1,121,120 | 1,401,400 |       |       |          |          |          |          |
| $n + 5$ | 12    | 324   | 8,052 | 142,944 | 1,714,700 | 12,180,168 | 65,985,920 | 190,590,400 | 190,590,400 |          |          |          |          |

**Table 5.1.** $\#N_{i}(n)$: Number of maximal chains in $T_{n}$ of length $n + i$ with no plus-full-sets

Table 5.1 is a computer based compilation of the numbers of maximal chains in $\mathcal{N}_{i}(n)$ for $-1 \leq i \leq 5$. The problem of enumerating $C_{i}(n)$ is reduced to computing $\#N_{i}(n)$, for $n \in [2i + 3]$. For example, the number of maximal chains of length 14 in $T_{11}$ is

$$
\#C_{3}(11) = 18 \binom{14}{8} + 220 \binom{14}{9} + 1464 \binom{14}{10} + 9240 \binom{14}{11} + 15400 \binom{14}{12},
$$

(5.9)

$$
= 18 \binom{14}{6} + 220 \binom{14}{5} + 1464 \binom{14}{4} + 9240 \binom{14}{3} + 15400 \binom{14}{2}.
$$

According to Theorem 5.8, for all $n \geq 2i + 4 = 10$, each maximal chain in $C_{3}(n)$ has a plus-full-set, so this follows for $n = 11$. The interpretation for equation (5.9) is that in $C_{3}(11)$, the numbers of maximal chains having exactly 2, 3, 4, 5 and 6 plus-full-sets are 15400(14), 9240(13), 1464(14), 220(14) and 18(16), respectively. Moreover, the subset of $C_{3}(11)$ of maximal chains containing exactly $j$ plus-full-sets, $2 \leq j \leq 6$, has an equal number of maximal chains over all $j$-element subsets of [14] of particular sets of plus-full-sets.
Interpreting maximal chains in the Tamari lattice as \(\psi\)-tableaux has proven an efficient method of study. The pursuit of the formula for \(\#C_i(n)\) led to the plus-full-set property and some interesting combinatorics. Based on numerical evidence, we conclude this note with a conjecture.

**Conjecture 5.10.** For all \(i \geq -1\),

\[
\#N_i(2i + 3) = \prod_{j=1}^{i+1} \binom{3j - 1}{2},
\]

and for all \(i \geq 0\),

\[
\#N_i(2i + 2) = \frac{i+1}{3} \prod_{j=1}^{i+1} \binom{3j - 1}{2},
\]

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