Context-Aware Local Differential Privacy

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Abstract

Local differential privacy (LDP) is a strong notion of privacy for individual users that often comes at the expense of a significant drop in utility. The classical definition of LDP assumes that all elements in the data domain are equally sensitive. However, in many applications, some symbols are more sensitive than others. This work proposes a context-aware framework of local differential privacy that allows a privacy designer to incorporate the application’s context into the privacy definition. For binary data domains, we provide a universally optimal privatization scheme and highlight its connections to Warner’s randomized response (RR) and Mangat’s improved response. Motivated by geolocation and web search applications, for \( k \)-ary data domains, we consider two special cases of context-aware LDP: block-structured LDP and high-low LDP. We study discrete distribution estimation and provide communication-efficient, sample-optimal schemes and information theoretic lower bounds for both models. We show that using contextual information can require fewer samples than classical LDP to achieve the same accuracy.

1 Introduction

Differential privacy (DP) is a rigorous notion of privacy that enforces a worst-case bound on the privacy loss due to release of query results \cite{Dwork2006}. Its local version, local differential privacy (LDP) (Definition\textsuperscript{1}), provides context-free privacy guarantees even in the absence of a trusted data collector \cite{Dwork2006a,Dwork2006b,Dwork2014}. Under LDP, all pairs of elements in a data domain are assumed to be equally sensitive, leading to harsh privacy-utility trade-offs in many learning applications. In many settings however, some elements are more sensitive than others. For example, in URL applications, users may want to hide sensitive URLs from non-sensitive ones, and in geolocation applications, users may want to hide their precise location within a city, but not the city itself.

This work introduces a unifying context-aware notion of LDP where different pairs of domain elements can have “arbitrary” sensitivity levels. For binary domains, we provide a universal optimality result and highlight interesting connections to Warner’s response and Mangat’s improved response. For \( k \)-ary domains, we look at two canonical examples of context-aware LDP: block-structured LDP and high-low LDP. For block-structured LDP, the domain is partitioned into \( m \) blocks and the goal is to hide the identity of elements within the same block but not the block identity. This is motivated by geolocation applications where users can be in different cities, and it is not essential to hide the city of a person but rather where exactly within that city a person is (i.e., which bars, restaurants, or businesses they visit). In other words, in this context, users would like to hide which element in a given partition of the domain their data belongs to – and not necessarily which partition their data is in, which may be known from side information or application context. For high-low LDP, we assume there is a set of sensitive elements and we only require the information about possessing sensitive elements to be protected. This can be applied in web browsing services, where there are a large number of URLs and not all of them contain sensitive information.

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1.1 Related work

Recent works consider relaxations of DP [5, 6, 7] and LDP [8, 9, 10] that incorporate the context and structure of the underlying data domain. Indeed, [6, 10] investigate settings where a small fraction of the domain elements are secrets and require that the adversary’s knowledge about whether or not a user holds a secret cannot increase much upon the release of data. [5, 7, 8, 9] model the data domain as a metric space and scale the privacy parameter between pairs of elements by their distance. [11] considers categorical high dimensional count models and define privacy measures that help privatize counts that are close to each other in $\ell_1$ distance.

To investigate the utility gains under this model of privacy, we consider the canonical task of distribution estimation [12, 13, 14, 15]: Estimate an unknown discrete distribution given (privatized) samples from it. The trade-off between utility and privacy in distribution estimation has received recent attention, and optimal rates have been established [10, 17, 18, 19, 20, 21]. These recent works show that the sample complexity for $k$-ary distribution estimation up to accuracy $\alpha$ in total variation distance increases from $\Theta(\frac{1}{\alpha^2})$ (without privacy constraints) to $\Theta\left(\frac{1^2}{\alpha^2\varepsilon^2}\right)$ (for $\varepsilon = O(1)$) when we impose $\varepsilon$-LDP constraints. For both the block-structured, and high-low models of LDP, we will characterize the precise sample complexity by providing privatization and estimation schemes that are both computation- and communication-efficient.

Another line of work considers a slightly different problem of heavy hitter estimation where there is no distributional assumption under the data samples that the users have and the main focus is on reducing the computational complexity and communication requirements [22, 23, 24, 25, 26, 27].

1.2 Our contributions

Motivated by the limitations of LDP, and practical applications where not all domain elements are equally sensitive, we propose a general notion of context-aware local differential privacy (Definition 2).

When the underlying data domain is binary, we provide a complete characterization of a universally optimal scheme, which interpolates between Warner’s randomized response and Mangat’s improved response. This result is given in Theorem 2.

We then consider general data domains and investigate two practically relevant settings. The first is the block structured model and is motivated by applications in geo-location and census data collection. In it, we assume that the underlying data domain is divided into a number of groups, and the symbols within a group are sensitive. For example, when the underlying domain is a set of geographical locations, each block can correspond to the locations within a city. In this case, we would like to privatize the precise value within a particular block, but the privacy of the identity of the block in which the sample lies is not of great concern (Definition 4). In Theorem 4 we characterize the sample complexity of estimating $k$-ary discrete distributions in this model of privacy. We propose a privatization scheme based on the recently proposed Hadamard Response (HR), which is both computation- and communication-efficient. We then prove the optimality of these bounds by proving a matching information-theoretic lower bound. This is achieved by casting the problem as an information constrained setting and invoking the lower bounds from [29, 30]. We show that when all the blocks have roughly the same number of symbols, the sample complexity can be saved by a factor of the number of blocks compared to that under the classic LDP. See Theorem 4 for a formal statement of this result.

The second model we consider is the high-low model, where there are a few domain elements that are sensitive while the others are not. A form of this high-low notion of privacy was proposed in [10], which also considered distribution estimation and proposed an algorithm based on RAPPOR [31]. We propose a new privatization scheme based on HR that has the same sample complexity as [10] with a much smaller communication budget. We also prove a matching information theoretic lower bound that matches the matching lower bound showing the optimality of these schemes [10]. In this case, the sample complexity only depends quadratically in the number of sensitive elements and linear in the domain size compared to the quadratic dependence on the domain size under the classic LDP. See Theorem 5 for a formal statement of this result.
As a consequence of these results, we observe that the sample complexity of distribution estimation can be significantly less than that in the classical LDP. Thus contextual privacy should be viewed as one possible method to improve the trade-off between the utility and privacy in LDP when different domain elements have different levels of privacy.

1.3 Organization

In Section 2 we define LDP and the problem of distribution estimation. In Section 3 we define context-aware LDP and provide an operational definition along with some of its specializations. In Section 4 we provide the optimal privatization scheme for binary domains. In Section 5 and Section 6 we derive the sample complexity of distribution learning in the high-low model, and block-structured model respectively. We conclude in Section 7.

2 Preliminaries

Let \( X \) be the \( k \)-ary underlying data domain, wlog let \( X = [k] := \{1, \ldots, k\} \). There are \( n \) users, and user \( i \) has a (potentially) sensitive data point \( X_i \in X \). A privatization scheme is a mechanism to add noise to mask the true \( X_i \)'s, and can be represented by a conditional distribution \( Q : X \to Y \), where \( Y \) denotes the output domain of the privatization scheme. For \( y \in Y \) and \( x \in X \), \( Q(y|x) \) is the probability that the privatization scheme outputs \( y \), upon observing \( x \). If we let \( X \) and \( Y \) denote the input and output of a privatization scheme, then \( Q(y|x) = \Pr(Y = y|X = x) \).

**Definition 1** (Local Differential Privacy. [2, 4]). Let \( \varepsilon > 0 \). A conditional distribution \( Q \) is \( \varepsilon \)-locally differentially private (\( \varepsilon \)-LDP) if for all \( x, x' \in X \) and all \( S \subset Y \),

\[
Q(S|x) \leq e^{\varepsilon} Q(S|x'),
\]

where \( Q(S|x) := \Pr(Y \in S|X = x) \). Let \( Q_\varepsilon \) be the set of all \( \varepsilon \)-LDP conditional distributions with input \([k]\).

**Distribution estimation.** Let

\[
\Delta_k := \{(p_1, \ldots, p_k) : p_x \geq 0 \ \forall x \in [k], \text{ and } p_1 + \ldots + p_k = 1\}
\]

be all discrete distributions over \([k]\). We assume that the user's data \( X^n := X_1, \ldots, X_n \) are independent draws from an unknown \( p \in \Delta_k \). User \( i \) passes \( X_i \) through the privatization channel \( Q \) and sends the output \( Y_i \) to a central server. Upon observing the messages \( Y^n := Y_1, \ldots, Y_n \), the server then outputs \( \hat{p} \) as an estimate of \( p \). Our goal is to select \( Q \) from a set of allowed channels \( Q \) and to design an estimation scheme \( \hat{p} : Y^n \to \Delta_k \) that achieves the following min-max risk

\[
r(k, n, d, Q) = \min_{Q \in Q} \min_{\hat{p}} \max_{p \in \Delta_k} \mathbb{E}[d(p, \hat{p})],
\]

where \( d(\cdot, \cdot) \) is a measure of distance between distributions. In this paper we consider the total variation distance, \( d_{TV}(p, q) := \frac{1}{2} \sum_{i=1}^{k} |p_i - q_i| \). For a parameter \( \alpha > 0 \), the sample complexity of distribution estimation to accuracy \( \alpha \) is the smallest number of users for the min-max risk to be smaller than \( \alpha \),

\[
n(k, \alpha, Q) := \arg \min_{n} \{r(k, n, d_{TV}, Q) < \alpha\}.
\]

When \( Q \) is \( Q_\varepsilon \), the channels satisfying \( \varepsilon \)-LDP constraints with input domain \([k]\), it is now well established that for \( \varepsilon = O(1) \) \([16, 18, 20, 19, 21]\),

\[
n(k, \alpha, Q_\varepsilon) = \Theta\left(\frac{k^2}{\alpha^2 \varepsilon^2}\right).
\]
Hadamard matrix (Sylvester’s construction). Let $H_1 = [1]$, then Sylvester’s construction of Hadamard matrices is a sequence of square matrices of size $2^i \times 2^i$ recursively defined as

$$H_{2m} = \begin{bmatrix} H_m & H_m \\ H_m & -H_m \end{bmatrix}$$

Let $S_i = \{y \mid y \in \{S \mid H(i+1, y) = +1\}$ be the column indices of +1’s in the $(i+1)$th row of $H_m$, we have

- $\forall i \in [m - 1], |S_i| = \frac{m}{2}$
- $\forall i \neq j \in [m - 1], |S_i \cap S_j| = \frac{m}{4}$

3 Context-aware LDP

In local differential privacy, all elements in the data domain are assumed to be equally sensitive, and the same privacy constraint is enforced on all pairs of them (see (1)). However, in many settings some domain elements might be more sensitive than others. For example, in geo-location applications, the precise street number of a person’s location might be more sensitive information than the name of the city they are in. To capture this, we present a context-aware notion of privacy. Let $E \in \mathbb{R}_{\geq 0}^{k \times k}$ be a matrix of non-negative entries, where for $x, x' \in [k]$, $\varepsilon_{x,x'}$ is the $(x, x')$th entry of $E$.

**Definition 2** (Context-Aware LDP). A conditional distribution $Q$ is $E$-LDP if for all $x, x' \in \mathcal{X}$ and $S \subseteq \mathcal{Y}$,

$$Q(S|x) \leq e^{\varepsilon_{x,x'}} Q(S|x').$$

(4)

This definition allows us to have a different sensitivity level between each pair of elements to incorporate context information. For a privacy matrix $E$, the set of all $E$-LDP mechanisms is denoted by $Q_E$. When all the entries of $E$ are $\varepsilon$, we obtain the classical LDP. We provide an operational definition of context-aware LDP in Section 3.1, and give a single optimal privatization scheme for all $E$ matrices for binary domains, namely when $k = 2$, in Section 4.

For general $k$, it is unclear whether there is a simple characterization of the optimal schemes for all $E$ matrices. First note that if $\varepsilon^* = \min_{i,j} \varepsilon_{i,j}$ is the smallest entry of $E$, then any $\varepsilon^*$-LDP algorithm is also $E$-LDP. However, this is not helpful since it does not help us capture the context of the application and consequently get rid of the stringent requirements of standard LDP. Motivated by applications in geolocation and web search, we consider structured $E$ matrices that are both practically motivating, and are parameterized by a few parameters (or a single parameter) so that they are easier to analyze and implement.

**Metric based LDP** [9]. Metric-based LDP introduced by [9] considers $\mathcal{X}$ to be a metric space endowed with a distance $d_X$. They consider schemes where close-by symbols have similar output distributions. To do this, they consider the $E$ matrix given by

$$\varepsilon_{x,x'} = d_X(x, x').$$

Notice that this definition is symmetric with respect to any two elements in the domain, which is not required in our proposed definition.

**High-Low LDP (HLLDP).** The high-low model captures applications where there are certain domain elements that are private, and the remaining elements are non-private. We only want to protect the privacy of private elements. This is formalized below.

**Definition 3.** Let $A = \{x_1, \cdots, x_s\} \subset \mathcal{X}$ denote the set of sensitive domain elements, and all symbols in $B := \mathcal{X} \setminus A$ are non-private. A privatization scheme is said to be $(A, \varepsilon)$-HLLDP if $\forall S \subset \mathcal{Y}$, and $x \in A, x' \in \mathcal{X}$,

$$\frac{\Pr(Y \in S | X = x)}{\Pr(Y \in S | X = x')} \leq e^\varepsilon.$$

(5)
This implies that when the input symbol is in $A$, the output distribution cannot be multiplicatively larger than the output distribution for any other symbol, but there is no such restriction for symbols in $B$. HLLDP was also defined in [10]. We solve the problem of minimax distribution estimation under this privacy model in Section 5.

**Block Structured LDP (BSLDP).** In applications such as geo-location, it is important to preserve the privacy of symbols that are close to each other. We consider this model where the data domain is divided into various partitions (e.g., the cities), and we would like the symbols within a partition (e.g., the various locations within a city) to be hard to distinguish. This is formalized below.

**Definition 4.** Suppose there is a partition of $\mathcal{X}$, which is $P = \{X_1, X_2, ..., X_m\}$. Then $\forall \varepsilon > 0$, a privatization scheme is said to be $(P, \varepsilon)$ - BSLDP if it satisfies $\forall i \in [m], x, x' \in X_i$, and any $S \subset Y$, we have

$$\frac{\Pr(Y \in S | X = x)}{\Pr(Y \in S | X = x')} \leq e^{\varepsilon}.$$  

(6)

This definition relaxes the local version of differential privacy in the following way: Given a partition of the input set $P = \{X_1, X_2, ..., X_m\}$, it requires different levels of indistinguishability for element pairs in the same block and those in different blocks. We solve the problem of minimax distribution estimation under this privacy model in Section 6.

### 3.1 Operational definition of context aware LDP

Recall that the entries of $E$ denote the amount of privacy across the corresponding row and column symbols. We provide an operational interpretation of $E$-LDP by considering a natural hypothesis testing problem. Suppose we are promised that the input is in $\{x, x'\}$ for some symbols $x, x' \in [k]$, and an $E$-LDP scheme outputs a symbol $Y \in Y$. Given $Y$, the goal is to test whether the input is $x$ or $x'$.

**Theorem 1** (Operational Interpretation of Context-Aware LDP). A conditional distribution $Q$ is $E$-locally differentially private if and only if for all $x, x' \in \mathcal{X}$ and all decision rules $\hat{X} : Y \rightarrow \{x, x'\}$,

$$P_{FA}(x, x') + e^{\varepsilon_{x,x'}} P_{MD}(x, x') \geq 1,$$

(7)

and

$$e^{\varepsilon_{x,x'}} P_{FA}(x, x') + P_{MD}(x, x') \geq 1,$$

(8)

where

$$P_{FA}(x, x') = \Pr(\hat{X} = x | X = x'), \quad P_{MD}(x, x') = \Pr(\hat{X} = x' | X = x).$$

**Proof.** Let $Q$ be an $E$-LDP scheme from $\mathcal{X}$ to $Y$. Note that $X - Y - \hat{X}$ form a Markov chain.

$$P_{MD}(x, x') = \Pr(\hat{X} = x' | X = x)$$

$$= \sum_{y \in Y} \Pr(\hat{X} = x' | Y = y) \Pr(Y = y | X = x)$$

$$= \sum_{y \in Y} \Pr(\hat{X} = x' | Y = y) Q(y | x)$$

$$\geq \sum_{y \in Y} \Pr(\hat{X} = x' | Y = y) Q(y | x') e^{-\varepsilon_{x,x'}}$$

$$= e^{-\varepsilon_{x,x'}} \left( \sum_{y \in Y} \Pr(\hat{X} = x' | Y = y) \Pr(Y = y | X = x') \right)$$

$$= e^{-\varepsilon_{x,x'}} \Pr(\hat{X} = x' | X = x')$$

$$= e^{-\varepsilon_{x,x'}} (1 - P_{FA}(x, x')).$$

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Figure 1: Error region for hypothesis testing between \( i \) and \( j \) under DP constraints.

(a) Error Region for Standard LDP.
(b) Error Region for Context-Aware LDP.

Rearranging the terms gives (7). (8) can be obtained similarly starting with \( P_{FA}(x, x') \).

Now for the other direction, consider a decision rule that satisfies (7) and (8). For any \( x, x' \in \mathcal{X} \) and \( S \subset \mathcal{Y} \), consider a decision rule that outputs \( \hat{X} = x \) if \( Y \in S \) and \( \hat{X} = x' \) otherwise. Then we get,

\[
\frac{Q(S|x)}{Q(S|x')} = \frac{\Pr(\hat{X} = x | X = x)}{\Pr(\hat{X} = x | X = x')} = \frac{1 - P_{MD}(x, x')}{P_{FA}(x, x')} \leq e^{\varepsilon_{x,x'}},
\]

showing that the scheme is \( E \)-LDP.

Consider a test for distinguishing \( x \) and \( x' \) given above in Theorem 1. Figure 1 shows the effective error regions for any estimator \( \hat{X} \) under the privacy constraints \( \varepsilon_{x,x'} \) and \( \varepsilon_{x',x} \). We can see that unlike the symmetric region under LDP, we are pushing the miss detection rate to be higher when \( \varepsilon_{x,x'} < \varepsilon_{x',x} \). This shows that symbol \( x \) is more private than symbol \( x' \), namely we want to protect the identity of symbol being \( x \) more than we want to protect the identity being \( x' \).

4 Binary domains

Consider a binary domain, namely \( k = 2 \) and domain elements \( \{1, 2\} \). In this case, we have

\[
E = \begin{bmatrix}
0 & \varepsilon_{1,2} \\
\varepsilon_{2,1} & 0
\end{bmatrix}.
\]

Perhaps the oldest and simplest privatization mechanism is Warner’s randomized response for confidential survey interviews [2]. In this section, we give the optimal scheme for all utility functions that obey the data processing inequality under all possible binary constraints. We prove that when \( \varepsilon_{1,2} = \varepsilon_{2,1} \), the optimal scheme is Warner’s randomized response. Consider the composition of two privatization mechanisms \( Q \circ W \) where the output of the first mechanism \( Q \) is applied to another mechanism \( W \). We say that a utility function \( U(\cdot) \) obeys the data processing inequality if for all \( Q \) and \( W \)

\[
U(QW) \leq U(Q).
\]

In other words, further processing of the data can only reduce the utility. Such utility functions are ubiquitous. For example, in the minimax distribution learning context of this paper, \( U(Q) \) may be chosen as \( -\min_p \max_{\tilde{p}} \mathbb{E}[d(p, \tilde{p})] \) (i.e., the negative of the minimax risk under a fixed mechanism \( Q \)) with \( d \) being any \( \ell_p \) distance or \( f \)-divergence.
Theorem 2 (Optimal Mechanism for Binary Domains). Let $X$ be a binary input alphabet and $U(Q)$ be any utility function that obeys the data processing inequality. Then for any $\varepsilon_{1,2}, \varepsilon_{2,1} \geq 0$, the following privatization mechanism

$$Q^* = \frac{1}{e^{\varepsilon_{2,1}} - e^{\varepsilon_{2,1}}} \left[ e^{\varepsilon_{2,1}} (1 - e^{-\varepsilon_{2,1}}) \quad e^{-\varepsilon_{1,2}} (e^{\varepsilon_{2,1}} - 1) \right],$$

solves

$$\max_Q U(Q) \text{ subject to } Q \in \mathcal{Q}_E,$$

As a special case of the above theorem, if we consider the original local differential privacy setup where $\varepsilon_{1,2} = \varepsilon_{2,1} = \varepsilon$, then the optimal mechanism for binary alphabets is

$$Q^* = \frac{1}{e^\varepsilon + 1} \left[ e^\varepsilon \quad 1 \quad 1 \quad e^\varepsilon \right].$$

This is Warner’s randomized response model in confidential structured survey interview with $p = e^\varepsilon / (e^\varepsilon + 1)$. Warner’s randomized response was shown to be optimal for binary alphabets in [2]. Another interesting special case of the above theorem is Mangat’s improved randomized response strategy [33]. To see this, let $\varepsilon_{2,1} = \infty$ and $p = e^{-\varepsilon_{1,2}}$. Then

$$Q^* = \frac{1}{1 - p} \left[ 1 - p \quad p \quad 1 \right].$$

This is exactly Mangat’s improved randomized response strategy. Thus Mangat’s randomized response with $p = e^{-\varepsilon_{1,2}}$ is optimal for all utility functions obeying the data processing inequality under this generalized differential privacy framework with $\varepsilon_{2,1} = \infty$.

5 Distribution estimation in the high-low model

We characterize the optimal sample complexity of distribution estimation under the high-low model (see Definition 3).

Theorem 3. Let $A \subset X$, with $|A| = s < k/2$, and $\varepsilon = O(1)$. Let $\mathcal{Q}_{A,\varepsilon}$ be the set of all possible channels satisfying $(A, \varepsilon)$-HLLDP, then:

$$n(k, \alpha, \mathcal{Q}_{A,\varepsilon}) = \Theta \left( \frac{s^2}{\alpha^2 \varepsilon^2} + \frac{k}{\alpha^2 \varepsilon} \right),$$

where $s = |A|$.

This question was considered in [10], which gave an algorithm based on RAPPOR [31] that has the optimal sample complexity, but requires $\Omega(k)$ bits of communication from each user, which is prohibitive in settings where the uplink capacity is limited for users. We design a scheme based on Hadamard Response(HR), which is also sample-optimal but requires only $\log k$ bits of communication from each user.

[10] noted that a lower bound of $\frac{s^2}{\alpha^2 \varepsilon^2}$ (the first term in (12)) is immediately implied by previously known results on distribution estimation under standard LDP (3 for $k = s$). However, obtaining a lower bound equalling the second term was still open. Using the recently proposed technique in [29], we prove this lower bound in Section 5.2.

5.1 Upper bound using a variant of Hadamard response

We propose an algorithm based on Hadamard response [21], which gives us a tight upper bound for $k$-ary distribution estimation under $(A, \varepsilon)$-high-low LDP.

Let $s = |A|$ and $S$ be the smallest power of 2 larger than $s$, i.e., $S := 2^{\lceil \log(s + 1) \rceil}$. Let $t := k - s$ be the number of all non-sensitive elements. Then we have $S + t \leq 2(s + t) = 2k$. Let $H_S$ to be the $S \times S$ Hadamard
matrix using Sylvester’s construction \cite{34}. Define the output alphabet to be \( [S + t] = \{1, ..., S + t\} \). Then the channel is defined as the following: When \( x \in A = [s] \), we have
\[
\Pr(Y = y|X = x) = \begin{cases} \frac{2e^\varepsilon}{S(e^\varepsilon + 1)} & \text{if } y \in [S] \text{ s.t. } H_S(x, y) = 1, \\ \frac{2}{S(e^\varepsilon + 1)} & \text{if } y \in [S] \text{ s.t. } H_S(x, y) = -1, \\ 0, & \text{if } y \notin [S]. \end{cases}
\] (13)

Else if \( x \notin A \), we have
\[
\Pr(Y = y|X = x) = \begin{cases} \frac{2}{S(e^\varepsilon + 1)}, & \text{if } y \in [S], \\ \frac{e^\varepsilon - 1}{e^\varepsilon + 1}, & \text{if } y = x + S - s, \\ 0, & \text{otherwise}. \end{cases}
\] (14)

It is easy to verify that this scheme satisfies \((A, \varepsilon)\)-high-low LDP. Next we show how the output distribution is related to the input distribution and construct an estimator based on them. For all \( i \in [s] \), define set \( S_i = \{y|y \in [S], H(i + 1, y) = +1\} \). Then using properties of Hadamard matrices in Section 2
\[
p(S_i)_i := \Pr(y \in S_i) = \sum_{x \in [k]} \Pr(Y \in S_i|X = x)p_x \\
= p_i|S_i| \frac{2e^\varepsilon}{S(e^\varepsilon + 1)} + \sum_{x \in A, x \neq i} p_x \left( |S_i \cap S_x| \frac{2e^\varepsilon}{S(e^\varepsilon + 1)} + |S_i \cap S| \frac{2}{S(e^\varepsilon + 1)} \right) + \sum_{x \notin A} p_x|S_i| \frac{2}{S(e^\varepsilon + 1)} \\
= \frac{1}{e^\varepsilon + 1} + \frac{e^\varepsilon - 1}{2(e^\varepsilon + 1)} \Pr(x \in A) + \frac{e^\varepsilon - 1}{2(e^\varepsilon + 1)}p_i, \\
p([S]) := \Pr(y \in [S]) = \Pr(x \in A) + S \frac{2e^\varepsilon}{S(e^\varepsilon + 1)}(1 - \Pr(x \in A)) = \frac{2}{e^\varepsilon + 1} + \frac{e^\varepsilon - 1}{e^\varepsilon + 1} \Pr(x \in A). \] (15)

Once we observe \( Y_1, Y_2, \ldots, Y_n \), we can get the following unbiased empirical estimates: for all \( i \in [s] \),
\[
\hat{p}(S_i) = \frac{1}{n} \left( \sum_{m=1}^{n} \mathbb{1}\{Y_m \in S_i\} \right), \\
\hat{p}([S]) = \frac{1}{n} \left( \sum_{m=1}^{n} \mathbb{1}\{Y_m \in S\} \right).
\]

Our estimates for these \( p_i \)'s will be
\[
\hat{p}_i = \frac{2(e^\varepsilon + 1)}{e^\varepsilon - 1} \left( \hat{p}(S_i) - \frac{1}{e^\varepsilon + 1} \hat{p}(A) \right), \quad \text{where } \hat{p}(A) = \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \left( \hat{p}([S]) - \frac{2}{e^\varepsilon + 1} \right). \] (17)

For all \( i \notin [s] \), we simply use the empirical estimates
\[
\hat{p}_i = \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \frac{1}{n} \left( \sum_{m=1}^{n} \mathbb{1}\{Y_m = i + S - s\} \right). \] (18)

Then by \cite{14}, \cite{15} and \cite{16}, \( \hat{p}_i \)'s are unbiased estimates for all \( i \in [k] \). By bounding the variance, it can be showed that the estimator proposed in (17) and (18) achieves the following:
\[
\mathbb{E}[\ell_1(\hat{p}, p)] \leq \sqrt{\frac{3s^2}{n} \left( \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \right)^2} + \sqrt{\frac{e^\varepsilon + 1}{e^\varepsilon - 1}}. \] (19)

Since \( \frac{e^\varepsilon + 1}{e^\varepsilon - 1} = O(\frac{1}{\varepsilon}) \) when \( \varepsilon = O(1) \), setting the right hand side to be smaller than \( \alpha \) gives us the upper bound part of Theorem \cite{3}. For the proof of (19), please see Section \cite{B.1}. In this scheme, each user only needs to communication at most \( \log k + 1 \) bits since \( K \leq 2k \) while the scheme proposed in \cite{10} needs \( \Omega(k) \) bits from each user.
5.2 Lower bound

We now prove a lower bound of

$$\Omega \left( \frac{s^2}{\alpha^2 \varepsilon^2} + \frac{k}{\alpha^2 \varepsilon} \right)$$

on the sample complexity when $\varepsilon = O(1)$, and $s < k/2$. A lower bound of $\Omega(s^2/\varepsilon^2 \alpha^2)$ follows from [3], which follows directly from the lower bounds on sample complexity of distribution under standard $\varepsilon$-LDP (e.g., Theorem IV.1 in [20]).

To prove a lower bound equaling the second term, we use the framework developed in [29] to prove lower bounds for distributed inference under various information constraints (e.g., privacy and communication constraints). Their key idea is to design a packing of distributions around the uniform distribution and show that the amount of information that can be gleaned from the output of these schemes is insufficient for distribution estimation. In particular, we will use their following result.

**Lemma 1.** [29, Lemma 13] Let $u$ be the uniform distribution over $[k]$ and $P$ be a family of distributions satisfying the following two conditions.

1. $\forall p \in P$, we have
   $$d_{TV}(p, u) \geq \alpha.$$

2. $\forall p_1 \in P$, we have
   $$|\{p_2 \in P | d_{TV}(p_1, p_2) \leq \frac{\alpha}{3}\}| \leq C_\alpha.$$

Suppose $Q$ is the set of all channels we can use to get information about $X$, then we have the sample complexity of $k$-ary distribution learning up to TV distance $\pm \alpha/3$ under channel constraints $Q$ is at least

$$\Omega \left( \frac{|P| - \log C_\alpha}{\max_{Q \in Q} \chi^2(Q|P)} \right),$$

where $p^Q(u^Q)$ is the distribution of $Y$ when $X \sim p(u)$, and

$$\chi^2(\mathcal{P}) := \frac{1}{|P|} \sum_{p \in \mathcal{P}} d_{\chi^2}(p^Q, u^Q),$$

and

$$d_{\chi^2}(p, q) := \sum_{y \in \mathcal{Y}} \frac{(p(y) - q(y))^2}{q(y)}.$$

Let $k' = k - s \geq k/2$ be the number of non-sensitive elements. Let $\mathcal{Z} = \{+1, -1\}^{k'}$ be the set of $k'$ bits. For all $z \in \mathcal{Z}$, define $p_z$ as the following

$$p_z(i) = \begin{cases} \frac{1}{k'} + \frac{\alpha}{k} \sum_{j=1}^{s} z_j & i = 1, \\ \frac{1}{k'} + \frac{\alpha}{k} z_i & i = 2, 3, ..., s, \\ \frac{1}{k} + \frac{\alpha}{k} & i = s + 1, s + 2, ..., k. \end{cases}$$

Let $\mathcal{P}_z = \{p_z | z \in \mathcal{Z}\}$ be the set of all distributions defined by $z \in \mathcal{Z}$. Let $U_{\mathcal{Z}}$ be a uniform distribution over set $\mathcal{Z}$. Then we have

$$\chi^2(Q|\mathcal{P}_z) = \mathbb{E}_{z \sim U_{\mathcal{Z}}} \left[ \sum_{y \in \mathcal{Y}} \frac{(p^Q_z(y) - u^Q(y))^2}{u^Q(y)} \right] = \frac{k\alpha^2}{k^2} \mathbb{E}_{z \sim U_{\mathcal{Z}}} \left[ \sum_{y \in \mathcal{Y}} \frac{(\sum_{i=1}^{k'} (Q(y|1) - Q(y|s+i)) z_i)^2}{\sum_{j \in [k]} Q(y|j)} \right].$$

To bound this quantity, we have the following claim, which we prove in Section C.1.
Claim 1. If $\forall Q \in \mathbb{Q}$, we have: $\forall i \in \{s + 1, s + 2, ..., s + k\}, y \in \mathcal{Y}$,
\[
Q(y|1) \leq e^\varepsilon Q(y|i),
\]
then
\[
\sum_{i=1}^{K'} \sum_{y \in \mathcal{Y}} \frac{(Q(y+s+i) - Q(y|1))^2}{\sum_{j \in [k]} Q(y|j)} = O(\varepsilon k). \tag{20}
\]

Moreover, we have $|\mathbb{P}| = 2^{K'}$ and
\[
C_\alpha \leq 2^{(1-h(1/3)K')},
\]
where $h(x) = x \log(1/x) + (1-x) \log(1/(1-x))$. Combining these results and using Lemma 1, we get the sample complexity is at least
\[
\Omega\left(\frac{\log|\mathbb{P}| - \log C_\alpha}{\max_{Q \in \mathbb{Q}} \chi^2(Q|\mathbb{P})}\right) = \Omega\left(\frac{k}{\alpha^2 \varepsilon}\right).
\]

6 Distribution estimation under block-structured model

For distribution estimation under block-structured LDP constraints we prove the following theorem.

**Theorem 4.** Let $\varepsilon = O(1)$, and $\mathbb{P} = \{X_1, X_2, ..., X_m\}$ be a partition of $\mathcal{X}$ and $\mathbb{Q}_{\mathbb{P}, \varepsilon}$ be the set of all possible channels that satisfy $(\mathbb{P}, \varepsilon)$-BSLDP, then
\[
n(k, \alpha, \mathbb{Q}_{\mathbb{P}, \varepsilon}) = \Theta\left(\frac{\sum_{i=1}^{m} k_i^2}{\alpha^2 \varepsilon^2}\right),
\]
where $\forall i \in [m], |X_i| = k_i$ and $|\mathcal{X}| = k = \sum_{i=1}^{m} k_i$.

In Section 6.1, we describe an algorithm based on HR that is $(\mathbb{P}, \varepsilon)$ - BSLDP and can estimate the underlying distribution up to TV distance
\[
O\left(\sqrt{\frac{\sum_{i=1}^{m} k_i^2}{nc^2}}\right) \tag{21}
\]
Setting the right hand side to be smaller than $\alpha$, we get the upper bound part of Theorem 4. Moreover, it only uses $O(\log k)$ bits of communication from each user. We also prove a matching lower bound which shows that our algorithm is information theoretically optimal, presented in Section 6.2.

6.1 Block Hadamard Response

The idea of the algorithm is to perform Hadamard Response proposed in [21] within each block. Without loss of generality we assume each block $\mathcal{X}_j = \{(j, i)|i \in [k_j]\}$. For each block $\mathcal{X}_j, j \in [m]$, we associate a Hadamard matrix $H_{K_j}$ with $K_j = 2^{[\log(k_j + 1)]}$. Let $\mathcal{Y}_j = \{(j, i)|i \in [K_j]\}$. For each $x = (j, i) \in \mathcal{X}_j$, we assign the $(i+1)$th row of $H_{K_j}$ to $x$. Define the set of locations of ‘+1’s at the $(i+1)$th row of $H_{K_j}$ to be $S_x$. Then the output domain is $\mathcal{Y} := \cup_{j=1}^{m} \mathcal{Y}_j$. The privatization scheme is given as
\[
Q(Y = (j, i)|X) = \begin{cases} 2e^\varepsilon \frac{k_j}{k_j+1}, & X \in \mathcal{X}_j, i \in S_X, \\ \frac{k_j^2}{k_j+1}, & X \in \mathcal{X}_j, i \notin S_X, \\ 0, & \text{elsewhere}. \end{cases}
\]

It is easy to verify that this scheme satisfies the privacy constraints. Let $Y(1), Y(2)$ be the two coordinates of each output $Y$. Then for each block $X_j$ and $x \in X_j$, define
\[
p(X_j) := Pr(X \in X_j), \quad p(S_x) := Pr(Y(1) = j, Y(2) \in S_x).
\]
Using properties of Hadamard matrices in Section 2, \( \forall j \in [m] \) and \( x \in X_j \),

\[
p(S_x) = \Pr(X \in X_j, X \neq x) \left( |S_x \cap X_j| \frac{2e^n}{1 + e^n} + |S_x \cap \overline{X_j}| \frac{2}{K_j(1 + e^n)} \right) + p_x \frac{2e^n}{K_j(1 + e^n)}
\]

By observing \( Y_1, Y_2, \ldots, Y_n \), we obtain empirical estimates for \( p(X_j) \) and \( p(S_x) \) as following. For each \( j \in [m] \) and \( x \in X_j \),

\[
\hat{p}(X_j) = \frac{1}{n} \sum_{t=1}^{n} 1\{Y_t(1) = j\}, \quad \hat{p}(S_x) = \frac{1}{n} \sum_{t=1}^{n} 1\{Y_t(1) = j, Y_t(2) \in S_x\}.
\]

Then from (22),

\[
\hat{p}_x = \frac{2(e^n + 1)}{e^n - 1} \left( \hat{p}(S_x) - \frac{p(X_j)}{2} \right)
\]

is an unbiased estimate for \( p_x \)'s. By bounded the variance of the estimator, we can show that:

\[
\mathbb{E}[\ell_2^2(\hat{p}, p)] \leq \frac{12 \max_i k_i}{n} \left( \frac{e^n + 1}{e^n - 1} \right)^2, \quad \mathbb{E}[\ell_1(p, \hat{p})] \leq \frac{2(e^n + 1)}{e^n - 1} \sqrt{\frac{3 \sum_{j=1}^{m} k_j^2}{n}}.
\]

Since \( \frac{e^n + 1}{e^n - 1} = O(1) \) when \( \varepsilon = O(1) \), we get desired bounds in (24). For the proof of (24), see Section 3.2.

We note here that our algorithm also gives the optimal bound in terms of \( \ell_2 \) distance. A matching lower bound can shown using well established results \[16\] on LDP by considering the maximum of expected loss if we put all the mass on each single block.

### 6.2 Lower bound

We now prove the lower bound part of Theorem 4. The general idea is similar to the proof in Section 5.2 which is based on Lemma 1. Without loss of generality, we assume all the \( k_i \)'s are even numbers. We construct a family of distributions as following: Let \( Z = Z_1 \times Z_2 \times \cdots \times Z_m \) and \( \forall j \in [m], Z_j = \{+1, -1\}^{k_j/2} \).

\( \forall z \in Z \), we denote \( j \)th entry of \( z \) as \( z_j \) where \( z_j \in Z_j \). Define \( z_{j,i} \) to be the \( i \)th bit of \( z_j \). \( \forall j \in [m] \) and \( i \in [k_j/2] \), we have

\[
p_z(X = (j, 2i - 1)) = \frac{1}{k} + \frac{2z_{j,i}k_j\alpha}{\sum_{i=1}^{k_j} k_i^2}, \quad p_z(X = (j, 2i)) = \frac{1}{k} - \frac{2z_{j,i}k_j\alpha}{\sum_{i=1}^{k_j} k_i^2}
\]

Note that \( \forall z \in Z \), \( d_{TV}(p_z, u) = \alpha \). Moreover, \( |\mathcal{P}| = 2^{\frac{h}{2}} \) and \( |C_{\alpha}| \leq 2^{\frac{h}{2} h(1/3)} \) where \( h \) is the binary entropy function. By Lemma 1, let \( \mathcal{Q} \) be the set of channels that satisfy \( (\mathcal{P}, \varepsilon) \)-LDP and \( \mathcal{P}_Z = \{p_z | z \in Z\} \), it would suffice to show that

\[
\max_{Q \in \mathcal{W}} \chi^2(Q|\mathcal{P}_Z) = O\left(\frac{k\alpha^2 \varepsilon^2}{\sum_{i=1}^{k_j} k_i^2}\right).
\]

The proof of (26) is technical and is presented in Section 6.2.

### 7 Conclusion

In this work, we presented a general context-aware LDP framework and investigated communication, privacy, and utility trade-offs under both binary and \( k \)-ary data domains for minimax distribution estimation. We
derived a universally optimal mechanism for binary data domains and identified a family of communication-efficient and utility-optimal schemes under two canonical models of context-aware LDP: block-structured LDP and high-low LDP. For both models of privacy, we showed that using contextual information requires fewer samples than classical LDP to achieve the same accuracy. Our analysis is a worst-case one (i.e., maximal utility under the worst-case distribution). More work needs to be done to investigate the sample complexity gains under real life distributions and datasets.

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A Proof of Theorem 2

In Section 3.1, we give the operational meaning of context-aware LDP in Theorem 1. For binary alphabet, if \( Y \) is generated using an \( E \)-LDP mechanism, the error region for all binary hypothesis testing rule \( \hat{X} : Y \rightarrow \{1, 2\} \), denoted by \( \mathcal{R}_{\varepsilon_1, \varepsilon_2, 1} \), can be expressed by the convex hull defined by the following three points:

\[
(P_{FA}^1, P_{MD}^1) = (1, 0), \quad (P_{FA}^2, P_{MD}^2) = (0, 1), \quad (P_{FA}^3, P_{MD}^3) = \left( \frac{e^{\varepsilon_2, 1} - 1}{e^{\varepsilon_2, 1} - e^{-\varepsilon_1, 2}}, \frac{e^{\varepsilon_2, 1} - 1}{e^{\varepsilon_2, 1} - e^{-\varepsilon_1, 2}} \right),
\]

where \( P_{FA} = \Pr(\hat{X} = 1|X = 2) \) and \( P_{MD} = \Pr(\hat{X} = 2|X = 1) \).

Next we show if we use the scheme expressed in (9), the error region \( \mathcal{R}_{Q^*} = \mathcal{R}_{\varepsilon_1, \varepsilon_2, 1} \). Hence we only need to show the reverse direction. More specifically, we only need to show the three vertices expressed in (27) and (28) can be achieved. The two vertices in (27) can be achieved by trivial rules \( \hat{X} = 1 \) and \( \hat{X} = 2 \). Next we show the decision rule \( \hat{X}(Y) = Y \) achieves the error in (28). Using this decision rule, we have:

\[
P_{FA} = \Pr(\hat{X} = 1|X = 2) = \frac{e^{-\varepsilon_1, 2}(e^{\varepsilon_2, 1} - 1)}{e^{\varepsilon_2, 1} - e^{-\varepsilon_1, 2}} = \frac{e^{\varepsilon_2, 1} - 1}{e^{\varepsilon_2, 1} - e^{-\varepsilon_1, 2}},
\]

\[
P_{MD} = \Pr(\hat{X} = 2|X = 1) = \frac{1 - e^{-\varepsilon_1, 2}}{e^{\varepsilon_2, 1} - e^{-\varepsilon_1, 2}} = \frac{e^{\varepsilon_1, 2} - 1}{e^{\varepsilon_2, 1} - e^{-\varepsilon_1, 2}},
\]

which completes the proof.

For any other scheme \( Q \in Q_{\varepsilon_1} \), we have \( \mathcal{R}_Q \subset \mathcal{R}_{\varepsilon_1, \varepsilon_2, 1} = \mathcal{R}_{Q^*} \). Hence by Theorem 20 in [32], we have if \( Y_Q \) and \( Y_{Q^*} \) are the outputs of schemes \( Q \) and \( Q^* \) respectively, there exists a coupling between \( Y_Q \) and \( Y_{Q^*} \) such that \( X - Y_{Q^*} - Y_Q \) forms a Markov chain, which, by data processing inequality, implies

\[
U(Q) \leq U(Q^*).
\]

B Error bound proofs

B.1 Error bound proof for the high-low model (see [19])

In this section, we bound both the expected \( \ell_1 \) risk and \( \ell_2 \) risk of the estimator proposed in (17) and (18).

\[
\mathbb{E} \left[ \ell_2^2(\hat{p}, p) \right] = \sum_{i=1}^{k} \text{Var}(\hat{p}_i) = \sum_{i=1}^{s} \text{Var}(\hat{p}_i) + \sum_{i=s+1}^{k} \text{Var}(\hat{p}_i)
\]

\[
\leq \sum_{i=1}^{s} \left[ 2 \left( \frac{2e + 1}{e^2 - 1} \right)^2 \text{Var}(p(S_i)) + 2 \text{Var}(p(A)) \right] + \sum_{i=s+1}^{k} \text{Var}(\hat{p}_i)
\]

\[
= \sum_{i=1}^{s} \left[ 2 \left( \frac{2e + 1}{e^2 - 1} \right)^2 \frac{n}{n} + 2 p(A)(1 - p(A)) \right] + \sum_{i=s+1}^{k} \text{Var}(\hat{p}_i)
\]

\[
\leq \frac{1}{n} \sum_{i=1}^{s} 3 \left( \frac{e + 1}{e^2 - 1} \right)^2 + \sum_{i=s+1}^{k} \left( \frac{e + 1}{e^2 - 1} \right)^2 \frac{e^2 - 1}{e^2 + 1} \rho_i (1 - \frac{e^2 - 1}{e^2 + 1} \rho_i)
\]

\[
\leq \frac{1}{n} \left( 3s \left( \frac{e + 1}{e^2 - 1} \right)^2 + \frac{e + 1}{e^2 - 1} \right). \quad (29)
\]
Similarly, we get

$$E[\ell_1(\hat{p}, p)] \leq \sqrt{s \sum_{i=1}^{s} \text{Var}(\hat{p}_i)} + \sqrt{k \sum_{i=s+1}^{k} \text{Var}(\hat{p}_i)}$$

$$\leq \sqrt{3s^2 \left( \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \right)^2} + \sqrt{k \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \frac{k}{n}}. \quad (30)$$

### B.2 Error bound proof for the block-structured model (see (24))

In this section, we bound both the expected $\ell_1$ risk and $\ell_2$ risk of the estimator proposed in (23).

$$E[\ell_2^2(\hat{p}, p)] = \sum_{x \in X} \text{Var}(\hat{p}_x) = \sum_{j=1}^{m} \sum_{x \in X_j} \text{Var}(\hat{p}_x)$$

$$\leq \sum_{j=1}^{m} \sum_{x \in X_j} \left( \frac{2(e^\varepsilon + 1)}{e^\varepsilon - 1} \right)^2 \left( 2\text{Var}(\hat{p}_x) + \frac{1}{2} \text{Var}(p(X_j)) \right)$$

$$= \left( \frac{2(e^\varepsilon + 1)}{e^\varepsilon - 1} \right)^2 \sum_{j=1}^{m} \sum_{x \in X_j} \frac{1}{n} \left( 2p(S_x)(1 - p(S_x)) + \frac{1}{2}p(X_j)(1 - p(X_j)) \right)$$

$$\leq \left( \frac{2(e^\varepsilon + 1)}{e^\varepsilon - 1} \right)^2 \sum_{j=1}^{m} \sum_{x \in X_j} \frac{3}{n}p(X_j)$$

$$\leq \frac{12}{n} \left( \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \right)^2 \sum_{j=1}^{m} k_j p(X_j)$$

$$\leq \frac{12 \max_j k_j}{n} \left( \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \right)^2. \quad (31)$$

For $\ell_1$ error, using similar steps, we get:

$$E[\ell_1(\hat{p}, p)] = \sum_{j=1}^{m} \sum_{x \in X_j} |p_x - \hat{p}_x| \leq \sum_{j=1}^{m} \sqrt{k_j \sum_{x \in X_j} \text{Var}(\hat{p}_x)} \leq \sum_{j=1}^{m} \frac{2(e^\varepsilon + 1)}{e^\varepsilon - 1} \sqrt{\frac{3k_j^2}{n}p(X_j)}$$

$$\leq \frac{2(e^\varepsilon + 1)}{e^\varepsilon - 1} \sqrt{\frac{3}{n} \sum_{j=1}^{m} k_j^2 \sum_{x \in X_j} p(X_j)}$$

$$= \frac{2(e^\varepsilon + 1)}{e^\varepsilon - 1} \sqrt{\frac{3}{n} \sum_{j=1}^{m} k_j^2}.$$ 

The second last inequality comes from Cauchy-Schwarz inequality.
C Lower bound proofs

C.1 Proof of Claim 1

By definition, we have

\[
\sum_{y \in Y} \sum_{i=1}^{k'} \frac{(Q(y|s + i) - Q(y|1))^2}{\sum_{1 \leq i' \leq k} Q(y|i')} \leq \sum_{y \in Y} \sum_{i=1}^{k'} \frac{(Q(y|s + i) - Q(y|1))^2}{\sum_{1 \leq i' \leq k} Q(y|s + i') + Q(y|1)}
\]

\[
= \sum_{y \in Y} \left( \sum_{i=1}^{k'} Q(y|s + i)^2 + Q(y|1)^2 \sum_{1 \leq i' \leq k} Q(y|s + i') + Q(y|1) \right) + \sum_{i=1}^{k'} Q(y|1)(Q(y|1) - Q(y|s + i)) \sum_{1 \leq i' \leq k} Q(y|s + i') + Q(y|1) - Q(y|1)
\]

\[
\leq \sum_{y \in Y} \left( \max_i Q(y|i) + \frac{\sum_{i=2}^{k'} Q(y|1)(e^\varepsilon - 1)Q(y|s + i)}{\sum_{1 \leq i' \leq k} Q(y|s + i') + Q(y|1)} - Q(y|1) \right)
\]

\[
\leq \sum_{y \in Y} \left( \max_i Q(y|i) + Q(y|1)(e^\varepsilon - 1) - Q(y|1) \right).
\] (32)

To proceed, we need the following lemma.

**Lemma 2.** For all set \( M \subset Y \), \( \forall i \in [k] \), we have:

\[
\sum_{y \in M} Q(y|i) \leq 1 - e^{-\varepsilon} \left( 1 - \sum_{y \in M} Q(y|1) \right).
\]

**Proof.**

\[
\sum_{y \in M} Q(y|i) = 1 - \sum_{y \in M^c} Q(y|i) \leq 1 - e^{-\varepsilon} \sum_{y \in M^c} Q(y|1) = 1 - e^{-\varepsilon} \left( 1 - \sum_{y \in M} Q(y|1) \right).
\]

Next, we partition output set \( Y \) into \( k \) subsets, where \( \forall t \in [k], \)

\[
M_t = \{ y \in Y | \arg \max_{i \in [k]} Q(y|i) = t \}.
\]

Then combining (32) and Lemma 2 we have:

\[
\sum_{y \in Y} \sum_{i=1}^{k} \frac{(Q(y|i) - Q(y|1))^2}{\sum_{1 \leq i' \leq k} Q(y|i')} \leq (e^\varepsilon - 1) + \sum_{t \in [k]} \sum_{y \in M_t} (Q(y|t) - Q(y|1))
\]

\[
\leq (e^\varepsilon - 1) + \sum_{t \in [k]} \left( 1 - e^{-\varepsilon} \left( 1 - \sum_{y \in M_t} Q(y|1) \right) - \sum_{y \in M_t} Q(y|1) \right)
\]

\[
= (e^\varepsilon - 1) + \sum_{t \in [k]} (1 - e^{-\varepsilon})(1 - \sum_{y \in M_t} Q(y|1)) = O(k\varepsilon).
\]

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C.2 Proof of (26)

Define \( Q(y|j, i) = \Pr (Y = y|X = (j, i)) \), we have:

\[
\chi^2(Q|P) = \sum_{y \in Y} \frac{\sum_{j=1}^{m} \sum_{i \in [k_j/2]} (Q(y|j, 2i) - Q(y|j, 2i - 1))^2}{\frac{1}{16} \sum_{j=1}^{m} \sum_{i \in [k_j/2]} Q(y|j, i)}
\]

\[
\leq \frac{k \alpha^2}{\sum_{t=1}^{m} k_t^2} \sum_{y \in Y} \frac{\sum_{j=1}^{m} \sum_{i \in [k_j/2]} (Q(y|j, 2i) - Q(y|j, 2i - 1))^2}{\sum_{j=1}^{m} \sum_{i \in [k_j]} Q(y|j, i)}.
\]

(33)

Within each block \( \mathcal{X}_j \), the elements satisfy classic \( \varepsilon \)-LDP. It is proved in [29] that \( \forall j \in [m], i \in [k_j], \)

\[
(Q(y|j, 2i) - Q(y|j, 2i - 1))^2 \leq \left(\frac{e^\varepsilon - 1}{k_j}\right)^2 \sum_{i \in [k_j]} Q(y|j, i).
\]

Hence we have

\[
\chi^2(Q|P) \leq \frac{k \alpha^2 (e^\varepsilon - 1)^2}{\sum_{t=1}^{m} k_t^2} \sum_{y \in Y} \frac{\sum_{j=1}^{m} \sum_{i \in [k_j]} Q(y|j, i)^2}{\sum_{j=1}^{m} \sum_{i \in [k_j]} Q(y|j, i)}
\]

\[
\leq \frac{k \alpha^2 (e^\varepsilon - 1)^2}{\sum_{t=1}^{m} k_t^2} \sum_{y \in Y} \frac{\sum_{j=1}^{m} \sum_{i \in [k_j]} Q(y|j, i)^2}{\sum_{i \in [k_j]} Q(y|j, i)}
\]

\[
= \frac{k \alpha^2 (e^\varepsilon - 1)^2}{\sum_{t=1}^{m} k_t^2} \sum_{j=1}^{m} \sum_{i \in [k_j]} Q(y|j, i)^2 \times k_j
\]

\[
\leq \frac{8 k \alpha^2 (e^\varepsilon - 1)^2}{\sum_{t=1}^{m} k_t^2}
\]

\[
= O\left(\frac{k \alpha^2 \varepsilon^2}{\sum_{t=1}^{m} k_t^2}\right).
\]

(34)