GEOMETRIC INTERPRETATION OF THE INVARIANTS OF A SURFACE IN $\mathbb{R}^4$ VIA THE TANGENT INDICATRIX AND THE NORMAL CURVATURE ELLIPSE

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Abstract. At any point of a surface in the four-dimensional Euclidean space we consider
the geometric configuration consisting of two figures: the tangent indicatrix, which is a conic
in the tangent plane, and the normal curvature ellipse. We show that the basic geometric
classes of surfaces in the four-dimensional Euclidean space, determined by conditions on
their invariants, can be interpreted in terms of the properties of the two geometric figures.
We give some non-trivial examples of surfaces from the classes in consideration.

1. Introduction

In this paper we deal with the theory of surfaces in the four-dimensional Euclidean space
$\mathbb{R}^4$.

Let $M^2$ be a surface in $\mathbb{R}^4$ with tangent space $T_p M^2$ at any point $p \in M^2$. In [4] we
introduced the linear map $\gamma$ of Weingarten type at any $T_p M^2$ and sketched out the invariant
theory of surfaces on the base of $\gamma$.

We show that the role of the map $\gamma$ in the theory of surfaces in $\mathbb{R}^4$ is similar to that of
the Weingarten map in the theory of surfaces in $\mathbb{R}^3$.

First, the map $\gamma$ generates two invariant functions $k$ and $\varkappa$, analogous to the Gauss
curvature and the mean curvature in $\mathbb{R}^3$. Here again the sign of the function $k$ is a geometric
invariant and the sign of $\varkappa$ is invariant under the motions in $\mathbb{R}^4$. However, the sign of $\varkappa$
changes under symmetries with respect to a hyperplane in $\mathbb{R}^4$. The invariants $k$ and $\varkappa$ divide
the points of $M^2$ into four types: flat, elliptic, hyperbolic and parabolic points. In [4] we
gave a constructive classification of the surfaces consisting of flat points, i.e. satisfying the
condition $k = \varkappa = 0$. Everywhere, in the present considerations we exclude the points at
which $k = \varkappa = 0$.

Further, the map $\gamma$ generates the second fundamental form $II$ at any point $p \in M^2$.
The notions of a normal curvature of a tangent, conjugate and asymptotic tangents are
introduced in the standard way by means of $II$. The asymptotic tangents are characterized
by zero normal curvature.

The first fundamental form $I$ and the second fundamental form $II$ generate principal
tangents and principal lines, as in $\mathbb{R}^3$. Here, the points at which any tangent is principal
("umbilical" points) are characterized by zero mean curvature vector, i.e. the surfaces
consisting of "umbilical" points are exactly the minimal surfaces in $\mathbb{R}^4$. The principal normal
curvatures $\nu'$ and $\nu''$ arise in the standard way and the invariants $k$ and $\varkappa$ satisfy the
equalities

$$k = \nu' \nu''; \quad \varkappa = \frac{\nu' + \nu''}{2}.$$ 

The indicatrix of Dupin at an arbitrary (non-flat) point of a surface in $\mathbb{R}^3$ is introduced
by means of the second fundamental form. Here, using the second fundamental form $II$, we

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introduce the indicatrix $\chi$ at any point $p \in M^2$ in the same way:
\[ \chi : \nu'X^2 + \nu''Y^2 = \varepsilon, \quad \varepsilon = \pm 1. \]

Then the elliptic, hyperbolic and parabolic points of a surface $M^2$ are characterized in terms of the indicatrix $\chi$ as in $\mathbb{R}^3$. The conjugacy in terms of the second fundamental form coincides with the conjugacy with respect to the indicatrix $\chi$.

In [1, 5] we proved that the surface $M^2$ under consideration is with flat normal connection if and only if $\varepsilon = 0$. In Section 3 we prove that:

The surface $M^2$ is minimal if and only if the indicatrix $\chi$ is a circle.

The surface $M^2$ is with flat normal connection if and only if the indicatrix $\chi$ is a rectangular hyperbola (a Lorentz circle).

We also characterize the surfaces with flat normal connection in terms of the properties of the normal curvature ellipse.

In Section 4 we give examples of surfaces with $\varepsilon = 0$.

In Section 5 we give examples of surfaces with $k = 0$.

2. An Interpretation of the Second Fundamental Form

Let $M^2 : z = z(u, v)$, $(u, v) \in D (D \subset \mathbb{R}^2)$ be a 2-dimensional surface in $\mathbb{R}^4$. The tangent space $T_p M^2$ to $M^2$ at an arbitrary point $p = z(u, v)$ of $M^2$ is span$\{z_u, z_v\}$. We choose an orthonormal normal frame field $\{e_1, e_2\}$ of $M^2$ so that the quadruple $\{z_u, z_v, e_1, e_2\}$ is positive oriented in $\mathbb{R}^4$. Then the following derivative formulas hold:
\[ \nabla'_{z'u} z_u = z_{uu} = \Gamma^1_{11} z_u + \Gamma^2_{11} z_v + c^1_{11} e_1 + c^2_{11} e_2, \]
\[ \nabla'_{z'u} z_v = z_{uv} = \Gamma^1_{12} z_u + \Gamma^2_{12} z_v + c^1_{12} e_1 + c^2_{12} e_2, \]
\[ \nabla'_{z'v} z_v = z_{vv} = \Gamma^1_{22} z_u + \Gamma^2_{22} z_v + c^1_{22} e_1 + c^2_{22} e_2, \]

where $\Gamma^k_{ij}$ are the Christoffel’s symbols and $c^k_{ij}, i, j, k = 1, 2$ are functions on $M^2$.

We use the standard denotations $E(u, v) = g(z_u, z_u)$, $F(u, v) = g(z_u, z_v)$, $G(u, v) = g(z_v, z_v)$ for the coefficients of the first fundamental form and set $W = \sqrt{EG - F^2}$. Denoting by $\sigma$ the second fundamental tensor of $M^2$, we have
\[ \sigma(z_u, z_u) = c^1_{11} e_1 + c^2_{11} e_2, \]
\[ \sigma(z_u, z_v) = c^1_{12} e_1 + c^2_{12} e_2, \]
\[ \sigma(z_v, z_v) = c^1_{22} e_1 + c^2_{22} e_2. \]

In [4] we introduced a geometrically determined linear map $\gamma$ in the tangent space at any point of a surface $M^2$ and found invariants generated by this map.

We consider the functions
\[ L = \frac{2}{W} \begin{vmatrix} c^1_{11} & c^1_{12} \\ c^2_{11} & c^2_{12} \end{vmatrix}, \quad M = \frac{1}{W} \begin{vmatrix} c^1_{11} & c^2_{12} \\ c^2_{11} & c^2_{22} \end{vmatrix}, \quad N = \frac{2}{W} \begin{vmatrix} c^1_{12} & c^1_{11} \\ c^2_{12} & c^2_{22} \end{vmatrix}. \]

Denoting
\[ \gamma^1 = \frac{FM - GL}{EG - F^2}, \quad \gamma^2 = \frac{FL - EM}{EG - F^2}, \quad \gamma^1 = \frac{FN - GM}{EG - F^2}, \quad \gamma^2 = \frac{FM - EN}{EG - F^2}, \]
we obtain the linear map
\[ \gamma : T_p M^2 \rightarrow T_p M^2, \]
determined by the equalities
\[
\gamma(z_u) = \gamma_1^1 z_u + \gamma_2^1 z_v, \\
\gamma(z_v) = \gamma_1^2 z_u + \gamma_2^2 z_v.
\]

The linear map \(\gamma\) of Weingarten type at the point \(p \in M^2\) is invariant with respect to changes of parameters on \(M^2\) as well as to motions in \(\mathbb{R}^4\). This implies that the functions
\[
k = \frac{LN - M^2}{EG - F^2}, \quad \kappa = \frac{EN + GL - 2FM}{2(EG - F^2)}
\]
are invariants of the surface \(M^2\).

The invariant \(\kappa\) is the curvature of the normal connection of the surface \(M^2\) in \(\mathbb{E}^4\).

The invariants \(k\) and \(\kappa\) divide the points of \(M^2\) into four types [4]: flat, elliptic, parabolic and hyperbolic. The surfaces consisting of flat points satisfy the conditions
\[
k(u, v) = 0, \quad \kappa(u, v) = 0, \quad (u, v) \in \mathcal{D},
\]
or equivalently \(L(u, v) = 0, M(u, v) = 0, N(u, v) = 0, (u, v) \in \mathcal{D}\). These surfaces are either planar surfaces (there exists a hyperplane \(\mathbb{R}^3 \subset \mathbb{R}^4\) containing \(M^2\)) or developable ruled surfaces.

Further we consider surfaces free of flat points, i.e. \((L, M, N) \neq (0, 0, 0)\).

Let \(X = \alpha z_u + \beta z_v\), \((\alpha, \beta) \neq (0, 0)\) be a tangent vector at a point \(p \in M^2\). The Weingarten map \(\gamma\) determines a second fundamental form of the surface \(M^2\) at \(p \in M^2\) as follows:
\[
\Pi(\alpha, \beta) = -g(\gamma(X), X) = 2M\alpha\beta + N\beta^2, \quad \alpha, \beta \in \mathbb{R}.
\]

As in the classical differential geometry of surfaces in \(\mathbb{R}^3\) the second fundamental form \(\Pi\) determines conjugate tangents at a point \(p\) of \(M^2\).

Two tangents \(g_1 : X = \alpha_1 z_u + \beta_1 z_v\) and \(g_2 : X = \alpha_2 z_u + \beta_2 z_v\) are said to be conjugate tangents if \(\Pi(\alpha_1, \beta_1; \alpha_2, \beta_2) = 0\), i.e.
\[
L\alpha_1\alpha_2 + M(\alpha_1\beta_2 + \alpha_2\beta_1) + N\beta_1\beta_2 = 0.
\]

A tangent \(g : X = \alpha z_u + \beta z_v\) is said to be asymptotic if it is self-conjugate, i.e. \(L\alpha^2 + 2M\alpha\beta + N\beta^2 = 0\).

A tangent \(g : X = \alpha z_u + \beta z_v\) is said to be principal if it is perpendicular to its conjugate. The equation for the principal tangents at a point \(p \in M^2\) is
\[
\begin{vmatrix}
E & F \\
L & M
\end{vmatrix} \alpha^2 + \begin{vmatrix}
E & G \\
L & N
\end{vmatrix} \alpha\beta + \begin{vmatrix}
F & G \\
M & N
\end{vmatrix} \beta^2 = 0.
\]

A line \(c : u = u(q), \ v = v(q); \ q \in J\) on \(M^2\) is said to be a principal line (a line of curvature) if its tangent at any point is principal. The surface \(M^2\) is parameterized with respect to the principal lines if and only if
\[
F = 0, \quad M = 0.
\]

Let \(M^2\) be parameterized with respect to the principal lines and denote the unit vector fields \(x = \frac{z_u}{\sqrt{E}}, \ y = \frac{z_v}{\sqrt{G}}\). The equality \(M = 0\) implies that the normal vector fields \(\sigma(x, x)\) and \(\sigma(y, y)\) are collinear. We denote by \(b\) a unit normal vector field collinear with \(\sigma(x, x)\) and \(\sigma(y, y)\), and by \(l\) the unit normal vector field such that \(\{x, y, b, l\}\) is a positive oriented orthonormal frame field of \(M^2\) (the two vectors \(\{b, l\}\) are determined up to a sign). Thus we
obtain a geometrically determined orthonormal frame field \( \{ x, y, b, l \} \) at each point \( p \in M^2 \).

With respect to the frame field \( \{ x, y, b, l \} \) we have the following formulas:

\[
\begin{align*}
\sigma(x, x) &= \nu_1 b; \\
\sigma(x, y) &= \lambda b + \mu l; \\
\sigma(y, y) &= \nu_2 b,
\end{align*}
\]

where \( \nu_1, \nu_2, \lambda, \mu \) are invariant functions, whose signs depend on the pair \( \{ b, l \} \).

Hence the invariants \( k, \kappa \), and the Gauss curvature \( K \) of \( M^2 \) are expressed as follows:

\[
\begin{align*}
k &= -4\nu_1 \nu_2 \mu^2, \\
\kappa &= (\nu_1 - \nu_2)\mu, \\
K &= \nu_1 \nu_2 - (\lambda^2 + \mu^2).
\end{align*}
\]

The normal mean curvature vector field \( H \) of \( M^2 \) is

\[
H = \sigma(x, x) + \sigma(y, y) = \nu_1 + \nu_2 b.
\]

Let \( M^2 \) be a surface parameterized by principal tangents and \( g : X = \alpha z_u + \beta z_v \) be an arbitrary tangent of \( M^2 \). We call the function \( \nu_g = \frac{II(\alpha, \beta)}{I(\alpha, \beta)} \) the normal curvature of \( g \).

Obviously, a tangent \( g \) is asymptotic if and only if its normal curvature is zero.

The normal curvatures \( \nu' = \frac{L}{E} \) and \( \nu'' = \frac{N}{G} \) of the principal tangents are said to be principal normal curvatures of \( M^2 \). If \( g \) is an arbitrary tangent with normal curvature \( \nu_g \), and \( \varphi = \angle(g, z_u) \), then the following Euler formula holds

\[
\nu_g = \cos^2 \varphi \nu' + \sin^2 \varphi \nu''.
\]

The invariants \( k \) and \( \kappa \) of \( M^2 \) are expressed by the principal normal curvatures \( \nu' \) and \( \nu'' \) as follows:

\[
\begin{align*}
k &= \nu' \nu''; \\
\kappa &= \frac{\nu' + \nu''}{2}.
\end{align*}
\]

Hence, the invariants \( k \) and \( \kappa \) of \( M^2 \) play the same role in the differential geometry of surfaces in \( \mathbb{R}^4 \) as the Gaussian curvature and the mean curvature in the classical differential geometry of surfaces in \( \mathbb{R}^3 \).

As in the theory of surfaces in \( \mathbb{R}^3 \), we consider the indicatrix \( \chi \) in the tangent space \( T_pM^2 \) at an arbitrary point \( p \) of \( M^2 \), defined by

\[
\chi : \nu' X^2 + \nu'' Y^2 = \varepsilon, \quad \varepsilon = \pm 1.
\]

If \( p \) is an elliptic point (\( k > 0 \)), then the indicatrix \( \chi \) is an ellipse. The axes of \( \chi \) are collinear with the principal directions at the point \( p \), and the lengths of the axes are \( \frac{2}{\sqrt{|\nu'|}} \) and \( \frac{2}{\sqrt{|\nu''|}} \).

If \( p \) is a hyperbolic point (\( k < 0 \)), then the indicatrix \( \chi \) consists of two hyperbolas. For the sake of simplicity we say that \( \chi \) is a hyperbola. The axes of \( \chi \) are collinear with the principal directions, and the lengths of the axes are \( \frac{2}{\sqrt{|\nu'|}} \) and \( \frac{2}{\sqrt{|\nu''|}} \).

If \( p \) is a parabolic point (\( k = 0 \)), then the indicatrix \( \chi \) consists of two straight lines parallel to the principal direction with non-zero normal curvature.

The following statement holds good:

**Proposition 2.1.** Two tangents \( g_1 \) and \( g_2 \) are conjugate tangents of \( M^2 \) if and only if \( g_1 \) and \( g_2 \) are conjugate with respect to the indicatrix \( \chi \).
3. Classes of surfaces characterized in terms of the tangent indicatrix and the normal curvature ellipse

Each surface $M^2$ in $\mathbb{R}^4$ satisfies the following inequality:

$$\kappa^2 - k \geq 0.$$ 

The minimal surfaces in $\mathbb{R}^4$ are characterized by

**Proposition 3.1.** Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is minimal if and only if

$$\kappa^2 - k = 0.$$ 

The surfaces with flat normal connection are characterized by

**Proposition 3.2.** Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is a surface with flat normal connection if and only if

$$\kappa = 0.$$ 

We note that the condition $\kappa = 0$ implies that $k < 0$ and the surface $M^2$ has two families of orthogonal asymptotic lines.

Now we shall characterize the minimal surfaces and the surfaces with flat normal connection in terms of the tangent indicatrix of the surface.

**Proposition 3.3.** Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is minimal if and only if at each point of $M^2$ the tangent indicatrix $\chi$ is a circle.

**Proof:** Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. From equalities (2.3) it follows that

$$\kappa^2 - k = \left(\frac{\nu' - \nu''}{2}\right)^2.$$ 

Obviously $\kappa^2 - k = 0$ if and only if $\nu' = \nu''$. Applying Proposition 3.1, we get that $M^2$ is minimal if and only if $\chi$ is a circle. □

**Proposition 3.4.** Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is a surface of flat normal connection if and only if at each point of $M^2$ the tangent indicatrix $\chi$ is a rectangular hyperbola (a Lorentz circle).

**Proof:** Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. From (2.3) it follows that $\kappa = 0$ if and only if $\nu'' = -\nu'$.

If $M^2$ is a surface with flat normal connection, then $k < 0$, and hence $\chi$ is a hyperbola. From $\nu'' = -\nu'$ it follows that the semi-axes of $\chi$ are equal to $\frac{1}{|\nu'|}$, i.e. $\chi$ is a rectangular hyperbola.

Conversely, if $\chi$ is a rectangular hyperbola, then $\nu'' = -\nu'$, which implies that $M^2$ is a surface with flat normal connection. □

The minimal surfaces and the surfaces with flat normal connection can also be characterized in terms of the ellipse of normal curvature.

Let us recall that the *ellipse of normal curvature* at a point $p$ of a surface $M^2$ in $\mathbb{R}^4$ is the ellipse in the normal space at the point $p$ given by $\{\sigma(x, x) : x \in T_pM^2, g(x, x) = 1\}$ [7, 8]. Let $\{x, y\}$ be an orthonormal base of the tangent space $T_pM^2$ at $p$. Then, for any $v = \cos \psi x + \sin \psi y$, we have

$$\sigma(v, v) = H + \cos 2\psi \frac{\sigma(x, x) - \sigma(y, y)}{2} + \sin 2\psi \sigma(x, y),$$

where

$$\sigma(x, y) = \frac{\nu'' - \nu'}{2},$$

and

$$H = \frac{\nu'' + \nu'}{2}.$$
where $H = \frac{\sigma(x, x) + \sigma(y, y)}{2}$ is the mean curvature vector of $M^2$ at $p$. So, when $v$ goes once around the unit tangent circle, the vector $\sigma(v, v)$ goes twice around the ellipse centered at $H$. The vectors $\frac{\sigma(x, x) - \sigma(y, y)}{2}$ and $\sigma(x, y)$ determine conjugate directions of the ellipse.

A surface $M^2$ in $\mathbb{R}^4$ is called super-conformal [3] if at any point of $M^2$ the ellipse of curvature is a circle. In [3] it is given an explicit construction of any simply connected super-conformal surface in $\mathbb{R}^4$ that is free of minimal and flat points.

Obviously, $M^2$ is minimal if and only if for each point $p \in M^2$ the ellipse of curvature is centered at $p$.

The minimal surfaces in $\mathbb{R}^4$ are divided into two subclasses:

- the subclass of minimal super-conformal surfaces, characterized by the condition that the ellipse of curvature is a circle;
- subclass of minimal surfaces of general type, characterized by the condition that the ellipse of curvature is not a circle.

In [3] it is proved that on any minimal surface $M^2$ the Gauss curvature $K$ and the normal curvature $\kappa$ satisfy the following inequality

$$K^2 - \kappa^2 \geq 0.$$ 

The two subclasses of minimal surfaces are characterized in terms of the invariants $K$ and $\kappa$ as follows:

- the class of minimal super-conformal surfaces is characterized by $K^2 - \kappa^2 = 0$;
- the class of minimal surfaces of general type is characterized by $K^2 - \kappa^2 > 0$.

The class of minimal super-conformal surfaces in $\mathbb{R}^4$ is locally equivalent to the class of holomorphic curves in $\mathbb{C}^2 \equiv \mathbb{R}^4$.

The surfaces with flat normal connection are characterized in terms of the ellipse of normal curvature as follows

**Proposition 3.5.** Let $M^2$ be a surface in $\mathbb{R}^4$ free of flat points. Then $M^2$ is a surface with flat normal connection if and only if for each point $p \in M^2$ the ellipse of normal curvature is a line segment, which is not collinear with the mean curvature vector field.

**Proof:** In [1] it is proved that the curvature of the normal connection $\kappa$ of a surface $M^2$ in $\mathbb{R}^4$ is the Gauss torsion $\kappa_G$ of $M^2$. The notion of the Gauss torsion is introduced by É. Cartan [2] for a $p$-dimensional submanifold of an $n$-dimensional Riemannian manifold and is given by the Euler curvatures. In case of a 2-dimensional surface $M^2$ in $\mathbb{R}^4$ the Gauss torsion at a point $p \in M^2$ is equal to $2ab$, where $a$ and $b$ are the semi-axis of the ellipse of normal curvature at $p$. Hence, $\kappa = 0$ if and only if the ellipse of curvature is a line segment.

Let $M^2$ be a surface with flat normal connection, i.e. $\kappa = 0, k \neq 0$. From (2.2) it follows, that $\nu_1 = \nu_2$. Further, equalities (2.1) imply that for each $v = \cos \psi x + \sin \psi y$, we have $\sigma(v, v) = H + \sin 2\psi (\lambda b + \mu l)$. So, when $v$ goes once around the unit tangent circle, the vector $\sigma(v, v)$ goes twice along the line segment collinear with $\lambda b + \mu l$ and centered at $H$. The mean curvature vector field is $H = \nu_1 b$. Since $k \neq 0$ then $\mu \neq 0$, and the line segment is not collinear with $H$.

In case of $\lambda = 0$ the mean curvature vector field $H$ is orthogonal to the line segment, while in case of $\lambda \neq 0$ the mean curvature vector field $H$ is not orthogonal to the line segment. The length $d$ of the line segment is

$$d = \sqrt{\lambda^2 + \mu^2} = \sqrt{H^2 - K}.$$
So, there arises a subclass of surfaces with flat normal connection, characterized by the conditions:

\[ K = 0 \quad \text{or} \quad d = \|H\|. \]

Proposition 3.4 and Proposition 3.5 give us the following

**Corollary 3.6.** Let \( M^2 \) be a surface in \( \mathbb{R}^4 \) free of flat points. Then the tangent indicatrix \( \chi \) is a rectangular hyperbola (a Lorentz circle) if and only if the ellipse of normal curvature is a line segment, which is not collinear with the mean curvature vector field.

4. **Examples of Surfaces with Flat Normal Connection**

In this section we construct a family of surfaces with flat normal connection lying on a standard rotational hypersurface in \( \mathbb{R}^4 \).

Let \( \{e_1, e_2, e_3, e_4\} \) be the standard orthonormal frame in \( \mathbb{R}^4 \), and \( S^2(1) \) be a 2-dimensional sphere in \( \mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\} \), centered at the origin \( O \). We consider a smooth curve \( c : l = l(v), v \in J, J \subset \mathbb{R} \) on \( S^2(1) \), parameterized by the arc-length \( l^2(v) = 1 \). We denote \( t = l' \) and consider the moving frame field span\( \{t(v), n(v), l(v)\} \) of the curve \( c \) on \( S^2(1) \).

With respect to this orthonormal frame field the following Frenet formulas hold good:

\[
\begin{align*}
l' &= t; \\
t' &= \kappa n - l; \\
n' &= -\kappa t,
\end{align*}
\]

where \( \kappa \) is the spherical curvature of \( c \).

Let \( f = f(u), g = g(u) \) be smooth functions, defined in an interval \( I \subset \mathbb{R} \), such that \( f^2(u) + g^2(u) = 1, u \in I \). Now we construct a surface \( M^2 \) in \( \mathbb{R}^4 \) in the following way:

\[ M^2 : z(u, v) = f(u) l(v) + g(u) e_4, \quad u \in I, v \in J. \]

The surface \( M^2 \) lies on the rotational hypersurface \( M^3 \) in \( \mathbb{R}^4 \) obtained by the rotation of the meridian curve \( m : u \rightarrow (f(u), g(u)) \) around the \( Oe_4 \)-axis in \( \mathbb{R}^4 \). Since \( M^2 \) consists of meridians of \( M^3 \), we call \( M^3 \) a *meridian surface*.

The tangent space of \( M^2 \) is spanned by the vector fields:

\[ z_u = \ddot{f} l + \ddot{g} e_4; \]
\[ z_v = ft, \]

and hence the coefficients of the first fundamental form of \( M^2 \) are \( E = 1; F = 0; G = f^2(u) \).

Taking into account (4.1), we calculate the second partial derivatives of \( z(u, v) \):

\[ z_{uu} = \dddot{f} l + \dddot{g} e_4; \]
\[ z_{uv} = \dot{f} t; \]
\[ z_{vv} = f\kappa n - f l. \]

Let us denote \( x = z_u, y = \frac{z_v}{f} = t \) and consider the following orthonormal normal frame field of \( M^2 \):

\[ n_1 = n(v); \quad n_2 = -\dot{g}(u) l(v) + \dot{f}(u) e_4. \]

Thus we obtain a positive orthonormal frame field \( \{x, y, n_1, n_2\} \) of \( M^2 \). If we denote by \( \kappa_m \) the curvature of the meridian curve \( m \), i.e. \( \kappa_m(u) = \dot{f}(u)\ddot{g}(u) - \ddot{g}(u)\dot{f}(u) = \frac{-\dddot{f}(u)}{\sqrt{1 - f^2(u)}}, \)
then we get the following derivative formulas of $M^2$:

\begin{align}
\nabla_x'x &= \kappa m n_2; & \nabla_x' n_1 &= 0; \\
\nabla_x' y &= 0; & \nabla_y' n_1 &= -\frac{\kappa}{f} y; \\
\nabla_y' x &= \frac{\dot{f}}{f} y; & \nabla_y' n_2 &= -\kappa m x; \\
\nabla_y' y &= -\frac{\dot{f}}{f} x + \frac{\kappa}{f} n_1 + \frac{\dot{g}}{f} n_2; & \nabla_y' n_2 &= -\frac{\dot{g}}{f} y.
\end{align}

(4.3)

The coefficients of the second fundamental form of $M^2$ are $L = N = 0, M = -\kappa_m(u) \kappa(v)$. Taking into account (4.3), we find the invariants $k, \kappa, K$:

\begin{align}
\kappa &= -\frac{\kappa^2_m(u) \kappa^2(v)}{f^2(u)}; & \kappa &= 0; & K &= -\frac{\kappa_m(u) \dot{g}(u)}{f(u)}.
\end{align}

(4.4)

The equality $\kappa = 0$ implies that $M^2$ is a surface with flat normal connection. The mean curvature vector field $H$ is given by

\begin{align}
H &= -\frac{\kappa}{2f} n_1 + \frac{\dot{g} + f \kappa_m}{2f} n_2.
\end{align}

(4.5)

There are three main classes of meridian surfaces:

I. $\kappa = 0$, i.e. the curve $c$ is a great circle on $S^2(1)$. In this case $n_1 = \text{const}$, and $M^2$ is a planar surface lying in the constant 3-dimensional space spanned by $\{x, y, n_2\}$. Particularly, if in addition $\kappa_m = 0$, i.e. the meridian curve lies on a straight line, then $M^2$ is a developable surface in the 3-dimensional space spanned by $\{x, y, n_2\}$.

II. $\kappa_m = 0$, i.e. the meridian curve is part of a straight line. In such case $k = \kappa = K = 0$, and $M^2$ is a developable ruled surface. If in addition $\kappa = \text{const}$, i.e. $c$ is a circle on $S^2(1)$, then $M^2$ is a developable ruled surface in a 3-dimensional space. If $\kappa \neq \text{const}$, i.e. $c$ is not a circle on $S^2(1)$, then $M^2$ is a developable ruled surface in $\mathbb{R}^4$.

III. $\kappa_m \kappa \neq 0$, i.e. $c$ is not a great circle on $S^2(1)$, and $m$ is not a straight line. In this general case the invariant function $k < 0$, which implies that there exist two systems of asymptotic lines on $M^2$. The parametric lines of $M^2$ given by (4.2) are orthogonal and asymptotic.

Let $M^2$ be a meridian surface of the general class. Now we are going to find the meridian surfaces with:

- constant Gauss curvature $K$;
- constant mean curvature;
- constant invariant function $k$.

**Proposition 4.1.** Let $M^2$ be a meridian surface in $\mathbb{R}^4$. Then $M^2$ has constant non-zero Gauss curvature $K$ if and only if the meridian $c$ is given by

\begin{align}
f(u) &= \alpha \cos \sqrt{K} u + \beta \sin \sqrt{K} u, & K > 0; \\
f(u) &= \alpha \cosh \sqrt{-K} u + \beta \sinh \sqrt{-K} u, & K < 0,
\end{align}

where $\alpha$ and $\beta$ are constants.

**Proof:** Using (4.4) and $\dot{f}^2 + \dot{g}^2 = 1$, we obtain that $M^2$ has constant Gauss curvature $K \neq 0$ if and only if the meridian $m$ satisfies the following differential equation

$$\ddot{f}(u) + K f(u) = 0.$$
The general solution of the above equation is given by
\[ f(u) = \alpha \cos \sqrt{K} u + \beta \sin \sqrt{K} u, \quad \text{in case } K > 0; \]
\[ f(u) = \alpha \cosh \sqrt{-K} u + \beta \sinh \sqrt{-K} u, \quad \text{in case } K < 0, \]
where \( \alpha \) and \( \beta \) are constants. The function \( g(u) \) is determined by \( \dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}. \)

The equality (4.5) implies that the mean curvature of \( M^2 \) is given by
\[ ||H|| = \sqrt{\frac{\kappa^2(v) + (\dot{g}(u) + f(u)\kappa_m(u))^2}{4f^2(u)}}. \]

The meridian surfaces with constant mean curvature (CMC meridian surfaces) are described in

**Proposition 4.2.** Let \( M^2 \) be a meridian surface in \( \mathbb{R}^4 \). Then \( M^2 \) has constant mean curvature \( ||H|| = a = \text{const} \), \( a \neq 0 \) if and only if the curve \( c \) on \( S^2(1) \) is a circle with constant spherical curvature \( \kappa = \text{const} = b, \ b \neq 0 \), and the meridian \( m \) is determined by the following differential equation:
\[ (1 - \dot{f}^2 - f\ddot{f})^2 = (1 - \dot{f}^2)(4a^2f^2 - b^2). \]

**Proof:** From (4.6) it follows that \( ||H|| = a \) if and only if
\[ \kappa^2(v) = 4a^2f^2(u) - (\dot{g}(u) + f(u)\kappa_m(u))^2, \]
which implies
\[ \kappa = \text{const} = b, \ b \neq 0; \]
\[ 4a^2f^2(u) - (\dot{g}(u) + f(u)\kappa_m(u))^2 = b^2. \]

The first equality of (4.7) implies that the spherical curve \( c \) has constant spherical curvature \( \kappa = b \), i.e. \( c \) is a circle. Using that \( \dot{f}^2 + \ddot{f}^2 = 1 \), and \( \kappa_m = f\ddot{g} - \dot{g}\dot{f} \) we calculate that
\[ \dot{g} + f\kappa_m = \frac{1 - \dot{f}^2 - f\ddot{f}}{\sqrt{1 - \dot{f}^2}}. \]
Hence, the second equality of (4.7) gives the following differential equation for the meridian \( m \):
\[ (1 - \dot{f}^2 - f\ddot{f})^2 = (1 - \dot{f}^2)(4a^2f^2 - b^2). \]

Further, if we set \( \dot{f} = y(f) \) in equation (4.8), we obtain that the function \( y = y(t) \) is a solution of the following differential equation
\[ 1 - y^2 - \frac{t}{2}(y^2) = \sqrt{1 - y^2\sqrt{4a^2t^2 - b^2}}. \]
The general solution of the above equation is given by
\[ y(t) = \sqrt{1 - \frac{1}{t^2}} \left( C + \frac{t}{2}\sqrt{4a^2t^2 - b^2} - \frac{b^2}{4a} \ln \left| 2at + \sqrt{4a^2t^2 - b^2} \right| \right)^2; \quad C = \text{const}. \]
The function \( f(u) \) is determined by \( \dot{f} = y(f) \) and (4.9). The function \( g(u) \) is defined by \( \dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}. \)

At the end of this section we shall find the meridian surfaces with constant invariant \( k \).
Proposition 4.3. Let $M^2$ be a meridian surface in $\mathbb{R}^4$. Then $M^2$ has a constant invariant $k = \text{const} = -a^2$, $a \neq 0$ if and only if the curve $c$ on $S^2(1)$ is a circle with spherical curvature $\kappa = \text{const} = b$, $b \neq 0$, and the meridian $m$ is determined by the following differential equation:

$$\ddot{f}(u) = \mp \frac{a}{b} f(u) \sqrt{1 - \dot{f}^2(u)}.$$

Proof: Using (4.4) we obtain that $k = \text{const} = -a^2$, $a \neq 0$ if and only if $\kappa^2(v)\kappa_m^2(u) = a^2 f'^2(u)$. Hence,

$$\kappa(v) = \pm a \frac{f(u)}{\kappa_m(u)}.$$

The last equality implies

$$\kappa = \text{const} = b, b \neq 0;$$

(4.10)

$$\pm a \frac{f(u)}{\kappa_m(u)} = b.$$  

The first equality of (4.10) implies that the spherical curve $c$ has constant spherical curvature $\kappa = b$, i.e. $c$ is a circle. The second equality of (4.10) gives the following differential equation for the function $f(u)$:

(4.11)

$$\frac{\ddot{f}(u)}{\sqrt{1 - \dot{f}^2(u)}} = \mp \frac{a}{b} f(u).$$

Again setting $\dot{f} = y(f)$ in equation (4.11), we obtain that the function $y = y(t)$ is a solution of the following differential equation

$$\frac{yy'}{\sqrt{1 - y^2}} = \mp \frac{a}{b} t.$$  

The general solution of the above equation is given by

(4.12)

$$y(t) = \sqrt{1 - \left(C \pm \frac{a t^2}{b} \right)^2}; \quad C = \text{const.}$$

The function $f(u)$ is determined by $\dot{f} = y(f)$ and (4.12). The function $g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$. \hfill \Box

5. Examples of surfaces consisting of parabolic points

In this section we shall find the generalized (in the sense of C. Moore) rotational surfaces in $\mathbb{R}^4$, consisting of parabolic points.

We consider a surface $M^2$ in $\mathbb{R}^4$ given by

(5.1) $z(u, v) = (f(u) \cos \alpha v, f(u) \sin \alpha v, g(u) \cos \beta v, g(u) \sin \beta v); \quad u \in J \subset \mathbb{R}, \quad v \in [0; 2\pi)$,

where $f(u)$ and $g(u)$ are smooth functions, satisfying $\alpha^2 f'^2(u) + \beta^2 g'^2(u) > 0$, $f'^2(u) + g'^2(u) > 0$, $u \in J$, and $\alpha, \beta$ are positive constants.

Each parametric curve $u = u_0 = \text{const}$ of $M^2$ is given by

$$c_v : z(v) = (a \cos \alpha v, a \sin \alpha v, b \cos \beta v, b \sin \beta v); \quad a = f(u_0), \quad b = g(u_0)$$
and its Frenet curvatures are

$$\kappa_{c_u} = \sqrt{\frac{a^2\alpha^4 + b^2\beta^4}{a^2\alpha^2 + b^2\beta^2}}; \quad \tau_{c_u} = \frac{ab\alpha\beta(\alpha^2 - \beta^2)}{\sqrt{a^2\alpha^4 + b^2\beta^4}}; \quad \sigma_{c_u} = \frac{\alpha\beta\sqrt{a^2\alpha^4 + b^2\beta^4}}{\sqrt{a^2\alpha^4 + b^2\beta^4}}.$$ 

Hence, in case of $\alpha \neq \beta$ each parametric curve $u = \text{const}$ is a curve in $\mathbb{R}^4$ with constant curvatures, and in case of $\alpha = \beta$ each parametric curve $u = \text{const}$ is a circle.

Each parametric curve $v = v_0 = \text{const}$ of $M^2$ is given by

$$c_u : z(u) = (A_1 f(u), A_2 f(u), B_1 g(u), B_2 g(u)),$$

where $A_1 = \cos \alpha v_0$, $A_2 = \sin \alpha v_0$, $B_1 = \cos \beta v_0$, $B_2 = \sin \beta v_0$. The Frenet curvatures of $c_u$ are expressed as follows:

$$\kappa_{c_u} = \frac{|g'f'' - f'g''|}{(f'^2 + g'^2)^{3/2}}; \quad \tau_{c_u} = 0.$$ 

Hence, $c_u$ is a plane curve with curvature $\kappa_{c_u} = \frac{|g'f'' - f'g''|}{(f'^2 + g'^2)^{3/2}}$. So, for each $v = \text{const}$ the parametric curves $c_u$ are congruent in $\mathbb{R}^4$. We call these curves meridians of $M^2$.

Considering general rotations in $\mathbb{R}^4$, C. Moore introduced general rotational surfaces [6] (see also [7, 8]). The surface $M^2$, given by (5.1) is a general rotational surface whose meridians lie in two-dimensional planes.

The tangent space of $M^2$ is spanned by the vector fields

$$z_u = (f' \cos \alpha v, f' \sin \alpha v, g' \cos \beta v, g' \sin \beta v);$$
$$z_v = (-\alpha f \sin \alpha v, \alpha f \cos \alpha v, -\beta g \sin \beta v, \beta g \cos \beta v).$$

Hence, the coefficients of the first fundamental form are $E = f'^2(u) + g'^2(u)$; $F = 0$; $G = \alpha^2 f^2(u) + \beta^2 g^2(u)$ and $W = \sqrt{(f'^2 + g'^2)(\alpha^2 f^2 + \beta^2 g^2)}$. We consider the following orthonormal tangent frame field

$$x = \frac{1}{\sqrt{f'^2 + g'^2}} (f' \cos \alpha v, f' \sin \alpha v, g' \cos \beta v, g' \sin \beta v);$$
$$y = \frac{1}{\sqrt{\alpha^2 f^2 + \beta^2 g^2}} (-\alpha f \sin \alpha v, \alpha f \cos \alpha v, -\beta g \sin \beta v, \beta g \cos \beta v).$$

The second partial derivatives of $z(u, v)$ are expressed as follows

$$z_{uu} = (f'' \cos \alpha v, f'' \sin \alpha v, g'' \cos \beta v, g'' \sin \beta v);$$
$$z_{uv} = (-\alpha f' \sin \alpha v, \alpha f' \cos \alpha v, -\beta g' \sin \beta v, \beta g' \cos \beta v);$$
$$z_{vv} = (-\alpha^2 f \cos \alpha v, -\alpha^2 f \sin \alpha v, -\beta^2 g \cos \beta v, -\beta^2 g \sin \beta v).$$

Now let us consider the following orthonormal normal frame field

$$n_1 = \frac{1}{\sqrt{f'^2 + g'^2}} (g' \cos \alpha v, g' \sin \alpha v, -f' \cos \beta v, -f' \sin \beta v);$$
$$n_2 = \frac{1}{\sqrt{\alpha^2 f^2 + \beta^2 g^2}} (-\beta g \sin \alpha v, \beta g \cos \alpha v, \alpha f \sin \beta v, -\alpha f \cos \beta v).$$

It is easy to verify that $\{x, y, n_1, n_2\}$ is a positive oriented orthonormal frame field in $\mathbb{R}^4$. 

GEOMETRIC INTERPRETATION OF THE INVARIANTS OF A SURFACE IN $\mathbb{R}^4$
We calculate the functions $c^k_{ij}, \ i, j, k = 1, 2$:

$$
c_{11} = g(z_{uu}, n_1) = \frac{g''_{f} - f''_g}{\sqrt{f'^2 + g'^2}}; \quad c_{11} = g(z_{uu}, n_2) = 0;
$$

$$
c_{12} = g(z_{uv}, n_1) = 0; \quad c_{12} = g(z_{uv}, n_2) = ds\frac{\alpha\beta(g' - f'g')}{\sqrt{\alpha^2f'^2 + \beta^2g'^2}}.
$$

$$
c_{22} = g(z_{vv}, n_1) = \frac{\beta^2gf' - \alpha^2fg'}{\sqrt{f'^2 + g'^2}}; \quad c_{22} = g(z_{vv}, n_2) = 0.
$$

Therefore the coefficients $L, M$ and $N$ of the second fundamental form of $M^2$ are expressed as follows:

$$
L = \frac{2\alpha\beta(gf' - fg')(gf'' - f'g'')}{(\alpha^2f'^2 + \beta^2g'^2)(f'^2 + g'^2)}; \quad M = 0; \quad N = -\frac{2\alpha\beta(gf' - fg')(\beta^2gf' - \alpha^2fg')}{(\alpha^2f'^2 + \beta^2g'^2)(f'^2 + g'^2)}.
$$

Consequently, the invariants $k, \kappa$ and $K$ of $M^2$ are:

$$
k = \frac{-4\alpha^2\beta^2(gf' - fg')(gf'' - f'g'')(\beta^2gf' - \alpha^2fg')}{(\alpha^2f'^2 + \beta^2g'^2)(f'^2 + g'^2)^3};
$$

$$
\kappa = \frac{\alpha\beta(gf' - fg')}{(\alpha^2f'^2 + \beta^2g'^2)^2(f'^2 + g'^2)^2} \left((\alpha^2f'^2 + \beta^2g'^2)(gf'' - f'g'') - (f'^2 + g'^2)(\beta^2gf' - \alpha^2fg')\right);
$$

$$
K = \frac{(\alpha^2f'^2 + \beta^2g'^2)(\beta^2gf' - \alpha^2fg')(gf'' - f'g'') - \alpha^2\beta^2(f'^2 + g'^2)(gf' - fg')^2}{(\alpha^2f'^2 + \beta^2g'^2)^2(f'^2 + g'^2)^2}.
$$

Now we shall find the generalized rotational surfaces with $k = 0$. Without loss of generality we assume that the meridian $m$ is defined by $f = u; \ g = g(u)$. Then

$$
k = \frac{4\alpha^2\beta^2(g - ug')^2g''(\beta^2g - \alpha^2ug')}{(\alpha^2u'^2 + \beta^2g'^2)^3(1 + g'^2)^3};
$$

The invariant $k$ is zero in the following three cases:

1. $g(u) = au, \ a = \text{const} \neq 0$. In that case $k = \kappa = K = 0$, and $M^2$ is a developable surface in $\mathbb{R}^4$.

2. $g(u) = au + b, \ a = \text{const} \neq 0, b = \text{const} \neq 0$. In this case $k = 0$, but $\kappa \neq 0, K \neq 0$. Consequently, $M^2$ is a non-developable ruled surface in $\mathbb{R}^4$.

3. $g(u) = cu^{\beta/2}, \ c = \text{const} \neq 0$. In case of $\alpha \neq \beta$ we get $k = 0$, and the invariants $\kappa$ and $K$ are given by

$$
\kappa = \frac{c^2\beta^3(\beta^2 - \alpha^2)^2u^{2\beta^2 - \alpha^2}}{\alpha^5 \left(\alpha^2u'^2 + \beta^2c^2u^{2\beta^2} - \alpha^2\right)^2} \left(1 + c^2\beta^4u^{2\beta^2 - \alpha^2}\right)^2;
$$

$$
K = -\frac{c^2\beta^2(\beta^2 - \alpha^2)u^{2\beta^2}}{\alpha^2 \left(\alpha^2u'^2 + \beta^2c^2u^{2\beta^2} - \alpha^2\right)^2} \left(1 + c^2\beta^4u^{2\beta^2 - \alpha^2}\right).
$$

Hence, $\kappa \neq 0, K \neq 0$. In this case the parametric lines $u = \text{const}$ and $v = \text{const}$ are not straight lines. This is a non-trivial example of generalized rotational surfaces with $k = 0$.

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