The incompressible limit for the Rayleigh–Bénard convection problem

Peter Bella\textsuperscript{1,*} Eduard Feireisl\textsuperscript{2,†} Florian Oschmann\textsuperscript{1,*}

\textsuperscript{1} TU Dortmund, Fakultät für Mathematik
Vogelpothsweg 87, 44227 Dortmund, Germany

\textsuperscript{2} Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Praha 1, Czech Republic

Abstract

We consider a general compressible viscous and heat conducting fluid confined between two parallel plates and heated from the bottom. The time evolution of the fluid is described by the Navier–Stokes–Fourier system considered in the regime of low Mach and Froude numbers suitably interrelated. The asymptotic limit is identified as the Oberbeck–Boussinesq system supplemented with non–local boundary conditions for the temperature deviation.

Keywords: Navier–Stokes–Fourier system, Oberbeck–Boussinesq system, stratified fluids, incompressible limit, non–local boundary conditions

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1 Introduction

The Rayleigh–Bénard problem concerns the motion of a compressible fluid confined between two parallel plates heated from the bottom. For the sake of simplicity, we suppose the motion is space–periodic with respect to the horizontal variable. Accordingly, the underlying spatial domain may be identified with the flat torus in the horizontal plane:

$$\Omega = \mathbb{T}^d \times (0,1), \quad \mathbb{T}^d = \left[0,1\right]_0^d, \quad d = 1, 2.$$ 

The state of the fluid at a given time $t$ and a spatial position $x \in \Omega$ is characterized by the mass density $\varrho = \varrho(t, x)$, the absolute temperature $\vartheta = \vartheta(t, x)$, and the velocity $\mathbf{u} = \mathbf{u}(t, x)$. We consider the motion in the asymptotically incompressible and stratified regime. Accordingly, the time evolution of the flow is governed by the scaled Navier–Stokes–Fourier (NSF) system:

\begin{align*}
\partial_t \varrho + \text{div}_x (\varrho \mathbf{u}) &= 0, & (1.1) \\
\partial_t (\varrho \mathbf{u}) + \text{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) &= \text{div}_x S(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon} \varrho \nabla_x G, & (1.2) \\
\partial_t (\varrho s(\varrho, \vartheta)) + \text{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \text{div}_x \left( \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\varrho} \right) &= \frac{1}{\varrho} \left( \varepsilon^2 S : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\varrho} \right), & (1.3) \\
\text{supplemented with the boundary conditions} \\
\mathbf{u}|_{\partial\Omega} &= 0. & (1.4)
\end{align*}
The viscous stress is given by Newton’s rheological law

\[ S(\vartheta, \nabla_x u) = \mu(\vartheta) \left( \nabla_x u + \nabla_x^T u - \frac{2}{d} \text{div}_x u I \right) + \eta(\vartheta) \text{div}_x u I, \tag{1.6} \]

while the internal energy flux satisfies Fourier’s law

\[ q(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta. \tag{1.7} \]

Here, the more conventional internal energy (heat) equation

\[ \partial_t (\rho e(\rho, \vartheta)) + \text{div}_x (\rho e(\rho, \vartheta)) + \text{div}_x q(\vartheta, \nabla_x \vartheta) = \varepsilon^2 S(\vartheta, \nabla_x u) - p(\rho, \vartheta) \text{div}_x u \tag{1.8} \]

is replaced by the entropy balance equation (1.3). The equations (1.3), (1.8) are equivalent in the framework of classical solutions as long as the thermodynamic functions \( p(\rho, \vartheta), e(\rho, \vartheta), s(\rho, \vartheta) \) are interrelated through Gibbs’ equation

\[ \vartheta Ds = De + pD \left( \frac{1}{\vartheta} \right). \tag{1.9} \]

Besides Gibbs’ equation, we impose the hypothesis of thermodynamic stability written in the form

\[ \frac{\partial p(\rho, \vartheta)}{\partial \rho} > 0, \quad \frac{\partial e(\rho, \vartheta)}{\partial \vartheta} > 0 \text{ for all } \rho, \vartheta > 0. \tag{1.10} \]

The scaling in (1.1)–(1.3) represented by a small parameter \( \varepsilon > 0 \) corresponds to the Mach number \( \text{Ma} = \varepsilon \) and the Froude number \( \text{Fr} = \varepsilon^{\frac{1}{2}} \), see e.g. the survey paper by Klein et al. [15]. Our goal is to identify the asymptotic limit of solutions for \( \varepsilon \to 0 \).

### 1.1 Asymptotic limit

The zero–th order terms in the asymptotic limit are determined by the stationary problem

\[ \nabla_x p(\rho, \vartheta) = \varepsilon \vartheta \nabla_x G. \tag{1.11} \]

Accordingly, a solution of the Navier–Stokes–Fourier system can be written in the form

\[ \rho_\varepsilon = \overline{\rho} + \varepsilon \mathcal{R}_\varepsilon, \quad \vartheta_\varepsilon = \overline{\vartheta} + \varepsilon \mathcal{I}_\varepsilon, \]

where \( \overline{\rho}, \overline{\vartheta} \) are positive constants and \( \mathcal{R}_\varepsilon \to \mathcal{R}, \mathcal{I}_\varepsilon \to \mathcal{I} \) satisfy in the asymptotic limit \( \varepsilon \to 0 \) the so called Boussinesq relation

\[ \frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\rho}} \nabla_x \mathcal{R} + \frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\vartheta}} \nabla_x \mathcal{I} = \overline{\rho} \nabla_x G. \tag{1.12} \]
Without loss of generality, we suppose
\[ \int_{\Omega} G \, dx = 0, \quad \overline{v} = \int_{\Omega} \rho_\varepsilon \, dx \equiv \frac{1}{|\Omega|} \int_{\Omega} \rho_\varepsilon \, dx, \]
meaning, in particular, the total mass of the fluid is constant independent of \( \varepsilon \).

Anticipating convergence of the temperature deviations
\[ T_\varepsilon = \vartheta_\varepsilon - \vartheta \to T \]
we impose the boundary condition
\[ \vartheta_\varepsilon |_{\partial \Omega} = \vartheta_B = \vartheta + \varepsilon \Theta_B, \quad \Theta_B = \Theta_B(x), \] (1.13)
where the perturbation \( \Theta_B \) is not necessarily positive.

### 1.2 Limit system

Formally, the limit problem is expected to be the Oberbeck–Boussinesq (OB) system:

\[
\begin{align*}
\text{div}_x U &= 0, \\
\mathcal{M} \left( \partial_t U + U \cdot \nabla_x U \right) + \nabla_x \Pi &= \nabla_x S(\vartheta, \nabla_x U) + \mathcal{M} \nabla_x G, \\
\mathcal{N} c_p(\vartheta, \vartheta) \left( \partial_t \overline{\vartheta} + U \cdot \nabla_x \overline{\vartheta} \right) - \vartheta \nabla \alpha(\vartheta, \vartheta) U \cdot \nabla_x G &= \kappa(\vartheta) \Delta_x \overline{\vartheta},
\end{align*}
\] (1.14)

where
\[
\alpha(\vartheta, \vartheta) \equiv \frac{1}{\vartheta} \frac{\partial p(\vartheta, \vartheta)}{\partial \vartheta}, \quad c_p(\vartheta, \vartheta) \equiv \frac{\partial e}{\partial \vartheta} + \vartheta^{-1} \vartheta \alpha(\vartheta, \vartheta) \frac{\partial p(\vartheta, \vartheta)}{\partial \vartheta}
\] (1.15)
stand for the coefficient of thermal expansion, and the specific heat at constant pressure, respectively. The quantities \( \mathcal{M}, \mathcal{N} \) satisfy the Boussinesq relation (1.12).

We refer to the survey by Zeytounian [19] and the list of references therein for a formal derivation of the Oberbeck–Boussinesq system. A rigorous proof of convergence of a family of (weak) solutions to the Navier–Stokes–Fourier system, meaning
\[ u_\varepsilon \to U, \quad \frac{\partial_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \overline{\vartheta} \text{ in some sense,} \]
was obtained in [9] (see also [10, Chapter 5]) for the conservative boundary conditions
\[ u_\varepsilon \cdot n|_{\partial \Omega} = 0, \quad (S(\partial_\varepsilon, \nabla_x u_\varepsilon) \cdot n) \times n|_{\partial \Omega} = 0, \]
Rather surprisingly, we show that the limit system is in fact different if the Dirichlet boundary conditions (1.4), (1.5) are imposed. In accordance with (1.4), (1.13), we consider the scaled Navier–Stokes–Fourier system with the boundary conditions

\[ u_\varepsilon|_{\partial \Omega} = 0, \quad \vartheta_\varepsilon|_{\partial \Omega} = \vartheta_B = \overline{\vartheta} + \varepsilon \Theta_B. \]  

(1.17)

Imposing (1.17) in place of (1.16) drastically changes the behavior of the fluid as the resulting system is no longer energetically closed and the dynamics are driven by the temperature gradient. Indeed, the relevance of the Oberbeck–Boussinesq approximation for the Rayleigh–Bénard convection has been questioned by several authors, see Borman [5], Fröhlich, Laure, and Peyret [13], Nadolin [17], among others. The main issue is the violation of mass conservation. More precisely, in [16], the author shows by a simple argument that the usual Boussinesq approximation is incompatible with the principle of conservation of mass. Later and also very recently, in [2] the authors gave three explicit examples for this phenomenon. A quote out of this paper regarding applicability of the usual OB system is as follows: “It is also possible that the existing approaches that define the Oberbeck–Boussinesq approximation as a limiting case of the fully compressible set of local balance equations are intrinsically biased. It is the authors’ opinion that, until a rigorous and nonsingular theoretical scheme will be set up to justify the approximation as an asymptotic regime, its validity relies entirely on the widely–documented experimental validations available in the literature for a very broad range of flow regimes.” In contrast to the conventional OB system, the limiting system we obtain here is, indeed, consistent with mass conservation. Thus, it seems to resolve the question of applicability of the Boussinesq approximation in a physically meaningful way.

We show that the asymptotic limit of a family of (weak) solutions to the scaled NSF system in the Rayleigh–Bénard regime satisfies a modified Oberbeck–Boussinesq approximation at least if the initial data are well prepared. Specifically, we show the existence of the limits

\[ u_\varepsilon \to U, \quad \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \vartheta, \quad \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to r, \]

solving the modified Oberbeck–Boussinesq system

\[
\begin{align*}
\text{div}_x U &= 0, \\
\overline{\vartheta} \left( \partial_t U + U \cdot \nabla_x U \right) + \nabla_x \Pi &= \text{div}_x \mathfrak{S}(\overline{\vartheta}, \nabla_x U) + r \nabla_x G, \\
\mathfrak{S}_p(\overline{\vartheta}, \overline{\vartheta}) \left( \partial_t \mathfrak{S} + U \cdot \nabla_x \mathfrak{S} \right) - \overline{\vartheta} \partial \alpha(\overline{\vartheta}, \overline{\vartheta}) U \cdot \nabla_x G &= \kappa(\overline{\vartheta}) \Delta_x \mathfrak{S} + \overline{\vartheta} \partial \alpha(\overline{\vartheta}, \overline{\vartheta}) \frac{\partial p(\overline{\vartheta}, \overline{\vartheta})}{\partial \overline{\vartheta}} \partial_t \int_\Omega \mathfrak{S} \, dx,
\end{align*}
\]

(1.18)
together with the Boussinesq relation

$$\frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\vartheta}} \nabla_x r + \frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\vartheta}} \nabla_x \Theta = \overline{\rho}\nabla_x G, \quad \int_{\Omega} r \, dx = 0,$$

(1.19)

and the boundary conditions

$$U|_{\partial \Omega} = 0, \quad \Theta|_{\partial \Omega} = \Theta_B.$$

(1.20)

Let us emphasize that after the change of variables

$$\Theta = \overline{\Theta} - \lambda(\overline{\rho}, \overline{\vartheta}) \int_{\Omega} \overline{\Theta} \, dx, \quad \lambda(\overline{\rho}, \overline{\vartheta}) = \frac{\overline{\rho}\alpha(\overline{\rho}, \overline{\vartheta})}{\overline{\rho}c_p(\overline{\rho}, \overline{\vartheta})} \in (0, 1),$$

system (1.18)-(1.20) can be equivalently stated as the conventional Oberbeck–Boussinesq system

$$\text{div}_x U = 0,$$

$$\overline{\varrho}(\partial_t U + U \cdot \nabla_x U) + \nabla_x \Pi = \text{div}_x \mathcal{S}(\overline{\varrho}, \nabla_x U) + r \nabla_x G,$$

$$\overline{\varrho}c_p(\overline{\rho}, \overline{\vartheta}) \left( \partial_t \Theta + U \cdot \nabla_x \Theta \right) - \overline{\varrho}\alpha(\overline{\rho}, \overline{\vartheta}) U \cdot \nabla_x G = \kappa(\overline{\varrho}) \Delta_x \Theta,$$

$$\frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\vartheta}} \nabla_x r + \frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\vartheta}} \nabla_x \Theta = \overline{\rho}\nabla_x G$$

(1.21)

with the no-slip boundary condition for the velocity

$$U|_{\partial \Omega} = 0,$$

(1.22)

and a nonlocal boundary condition for the temperature deviation

$$\Theta|_{\partial \Omega} = \Theta_B - \frac{\lambda(\overline{\rho}, \overline{\vartheta})}{1 - \lambda(\overline{\rho}, \overline{\vartheta})} \int_{\Omega} \Theta \, dx.$$

(1.23)

Note that $\lambda \in (0, 1)$ follows directly from (1.15) and (1.10). Parabolic equations with the non-local boundary terms similar to (1.23) have been recognized by Day [8] in the context of some models in thermoelasticity, and subsequently studied by a number of authors, see Chen and Liu [7], Friedman [12], Gladkov and Nikitin [14], Pao [18], among others. It is worth noting that the last equation in (1.21) is nothing other than the Boussinesq relation (1.12). In particular, there is a spatially homogeneous function $\chi = \chi(t)$ such that

$$\frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\vartheta}} r + \frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\vartheta}} \Theta = \overline{\rho}G + \chi.$$

(1.24)
As the spatial average of \( r \) does not change the momentum equation, specifically it can be included in the pressure \( \Pi \), we may assume
\[
\int_{\Omega} r \, dx = 0 \Rightarrow \chi = \frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \vartheta} \int_{\Omega} \Theta \, dx.
\]

Here and hereafter, we always suppose that the coefficient of thermal expansion \( \alpha(\overline{\rho}, \overline{\vartheta}) \) at the reference state \((\overline{\rho}, \overline{\vartheta})\) is strictly positive, for more specific assumptions concerning the form of the equation of state see Section \( \ref{sec:equation_of_state} \) below.

### 1.3 The strategy of the convergence proof

Our approach is based on the concept of \textit{weak solutions} to the NSF system with energetically open boundary conditions developed recently in \cite{6}, \cite{11}. In particular, the relative energy inequality based on the ballistic energy balance is used to measure the distance between the solutions of the primitive and target systems. In contrast with \cite{9}, \cite{10, Chapter 5}, strong convergence is obtained with certain explicit estimates of the error depending on how close are the initial data to their limit values.

The paper is organized as follows. We start with the concept of weak solutions for the NSF system with Dirichlet boundary conditions introduced in \cite{6}. In particular, we recall the ballistic energy and the associated relative energy inequality in Section \( \ref{sec:weak_solutions} \). Then, in Section \( \ref{sec:strong_convergence} \) we record the available results on solvability of the OB system in the framework of strong solutions. The main results on convergence to the target OB system are stated in Section \( \ref{sec:convergence} \). The rest of the paper is then devoted to the proof of convergence. In Section \( \ref{sec:proof_of_convergence} \) we derive the basic energy estimates that control the amplitude of the fluid velocity as well as the distance of the density and the temperature profiles from their limit values independent of the scaling parameter \( \varepsilon \). The proof of convergence to the OB system is completed in Section \( \ref{sec:final_conclusion} \) by means of the relative energy inequality.

### 2 Weak solutions to the primitive system

Following \cite{6}, \cite{11}, we introduce the concept of weak solutions to the NSF system.

**Definition 2.1 (Weak solution to the NSF system).** We say that a trio \((\rho, \vartheta, u)\) is a weak solution of the NSF system (1.1)–(1.7), with the initial data
\[
\rho(0, \cdot) = \rho_0, \quad (\rho u)(0, \cdot) = \rho_0 u_0, \quad (\rho s)(0, \cdot) = \rho_0 s(\rho_0, \vartheta_0),
\]
if the following holds:

- The solution belongs to the \textit{regularity class}:
  \[
  \rho \in L^\infty(0, T; L^\gamma(\Omega)) \text{ for some } \gamma > 1, \quad \rho \geq 0 \text{ a.a. in } (0, T) \times \Omega,
  \]
\[ u \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^d)), \]
\[ \varrho^{3/2}, \log(\varrho) \in L^2(0, T; W_1^2(\Omega)) \text{ for some } \beta \geq 2, \varrho > 0 \text{ a.a. in } (0, T) \times \Omega, \]
\[ (\varrho - \varrho_B) \in L^2(0, T; W_0^{1,2}(\Omega)). \] (2.1)

- The **equation of continuity** (1.1) is satisfied in the sense of distributions, more specifically,
\[
\int_0^T \int_\Omega \left[ \varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \, dx \, dt = - \int_\Omega \varrho(0) \varphi(0, \cdot) \, dx
\] for any \( \varphi \in C^1_c([0, T) \times \overline{\Omega}). \)

- The **momentum equation** (1.2) is satisfied in the sense of distributions,
\[
\int_0^T \int_\Omega \left[ \varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \frac{1}{\varrho} \mathbf{q}(\varrho, \vartheta) \text{div}_x \varphi \right] \, dx \, dt
\] \[
= \int_0^T \int_\Omega \left[ \mathbf{S}(\varrho, \nabla_x \mathbf{u}) : \nabla_x \varphi - \frac{1}{\varrho} \mathbf{q}(\varrho, \vartheta) \cdot \nabla_x \varphi \right] \, dx \, dt - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \, dx
\] (2.3) for any \( \varphi \in C^1_c([0, T) \times \Omega; \mathbb{R}^d). \)

- The **entropy balance** (1.3) is replaced by the inequality
\[
- \int_0^T \int_\Omega \left[ \varrho s(\varrho, \vartheta) \partial_t \varphi + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}(\varrho, \vartheta) \cdot \nabla_x \varphi}{\varrho} \right] \, dx \, dt
\] \[
\geq \int_0^T \int_\Omega \frac{\varphi}{\varrho} \left[ \varepsilon^2 \mathbf{S}(\vartheta, \nabla_x \mathbf{u}) : D_x \mathbf{u} - \frac{\mathbf{q}(\varrho, \vartheta) \cdot \nabla_x \varphi}{\varrho} \right] \, dx \, dt + \int_\Omega \varrho_0 s(\varrho_0, \vartheta_0) \varphi(0, \cdot) \, dx
\] (2.4) for any \( \varphi \in C^1_c([0, T) \times \Omega), \varphi \geq 0, \) where \( D_x \mathbf{u} = \frac{1}{2}(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}). \)

- The **ballistic energy balance**
\[
- \int_0^T \partial_t \psi \int_\Omega \int_\Omega \varepsilon^2 \varrho |\mathbf{u}|^2 + \varrho \mathbf{c}(\varrho, \vartheta) - \vartheta \varrho s(\varrho, \vartheta) \right] \, dx \, dt
\] \[
+ \int_0^T \psi \int_\Omega \frac{1}{\varrho} \left[ \varepsilon^2 \mathbf{S}(\vartheta, \nabla_x \mathbf{u}) : D_x \mathbf{u} - \frac{\mathbf{q}(\varrho, \vartheta) \cdot \nabla_x \varphi}{\varrho} \right] \, dx \, dt
\] \[
\leq \int_0^T \psi \int_\Omega \left[ \varrho \mathbf{u} \cdot \nabla_x \mathbf{c} - \varrho s(\varrho, \vartheta) \partial_t \vartheta - \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \vartheta + \frac{\mathbf{q}(\varrho, \vartheta) \cdot \nabla_x \varphi}{\varrho} \right] \, dx
\] \[
+ \psi(0) \int_\Omega \left[ \frac{1}{2} \varepsilon^2 \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 \mathbf{c}(\varrho_0, \vartheta_0) - \vartheta(0, \cdot) \varrho_0 s(\varrho_0, \vartheta_0) \right] \, dx
\] (2.5) holds for any \( \psi \in C^1_c([0; T]), \psi \geq 0, \) and any \( \vartheta \in C^1([0; T] \times \overline{\Omega}) \) satisfying
\[ \vartheta > 0, \vartheta|_{\partial \Omega} = \vartheta_B. \]
Although quite general, the weak solutions in the sense of Definition 2.1 comply with the weak–strong uniqueness principle, meaning they coincide with the strong solution as long as the latter exists, see [11] Chapter 4.

2.1 Relative energy inequality

Following [6], [11], we introduce the scaled relative energy

\[
E_\varepsilon\left(\varrho, \vartheta, \mathbf{u} \big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right) = \frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + \frac{1}{\varepsilon^2} \left[ \varrho c - \tilde{\vartheta} \left( \varrho s - \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) \right) - \left( \epsilon(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\vartheta} s(\tilde{\varrho}, \tilde{\vartheta}) + \frac{p(\tilde{\varrho}, \tilde{\vartheta})}{\tilde{\varrho}} \right) (\varrho - \tilde{\varrho}) - \tilde{\varrho} c(\tilde{\varrho}, \tilde{\vartheta}) \right].
\]

The hypothesis of thermodynamic stability (1.10) can be equivalently rephrased as (strict) convexity of the total energy expressed with respect to the conservative entropy variables

\[
E_\varepsilon\left(\varrho, S = \varrho s(\varrho, \vartheta), \mathbf{m} = \varrho \mathbf{u}\right) = \frac{1}{2} \frac{1}{\varrho} |\mathbf{m}|^2 + \frac{1}{\varepsilon^2} \varrho c(\varrho, S),
\]

whereas the relative energy can be written as Bregmann distance

\[
E_\varepsilon\left(\varrho, S, \mathbf{m} \big| \tilde{\varrho}, \tilde{S}, \tilde{\mathbf{m}}\right) = E_\varepsilon(\varrho, S, \mathbf{m}) - \left\langle \partial_{\varrho, S, \mathbf{m}} E_\varepsilon(\tilde{\varrho}, \tilde{S}, \tilde{\mathbf{m}}); (\varrho - \tilde{\varrho}, S - \tilde{S}, \mathbf{m} - \tilde{\mathbf{m}}) \right\rangle - E_\varepsilon(\tilde{\varrho}, \tilde{S}, \tilde{\mathbf{m}})
\]

associated to the convex functional \((\varrho, S, \mathbf{m}) \mapsto E_\varepsilon(\varrho, S, \mathbf{m})\), see [11] Chapter 3, Section 3.1. Finally, as stated in [6], any weak solution in the sense of Definition 2.1 satisfies the relative energy inequality

\[
\left[ \int_\Omega E_\varepsilon\left(\varrho, \vartheta, \mathbf{u} \big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}\right) \, d\mathbf{x} \right]_{t=0}^{t=\tau} + \int_0^{\tau} \int_\Omega \frac{\varrho}{\vartheta} \left( S(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{1}{\varepsilon^2} \kappa(\vartheta) |\nabla_x \vartheta|^2 \right) \, d\mathbf{x} \, dt \leq -\frac{1}{\varepsilon^2} \int_0^{\tau} \int_\Omega \left( \kappa(\vartheta) \frac{\nabla_x \vartheta}{\vartheta} \right) \cdot \nabla_x \tilde{\vartheta} \, d\mathbf{x} \, dt
\]

\[
- \int_0^{\tau} \int_\Omega \left[ \varrho (\mathbf{u} - \tilde{\mathbf{u}}) \otimes (\mathbf{u} - \tilde{\mathbf{u}}) + \frac{1}{\varepsilon^2} p(\varrho, \vartheta) \right] : \nabla_x \tilde{\varrho} \, d\mathbf{x} \, dt + \int_0^{\tau} \int_\Omega \left[ \frac{1}{\varepsilon^2} \nabla_x G - \partial_t \tilde{\mathbf{u}} - (\tilde{\mathbf{u}} \cdot \nabla_x) \tilde{\mathbf{u}} \right] \cdot (\mathbf{u} - \tilde{\mathbf{u}}) \, d\mathbf{x} \, dt
\]

\[
+ \frac{1}{\varepsilon^2} \int_0^{\tau} \int_\Omega \left[ \left( 1 - \frac{\vartheta}{\tilde{\vartheta}} \right) \partial_t p(\tilde{\varrho}, \tilde{\vartheta}) - \frac{\vartheta}{\tilde{\vartheta}} \mathbf{u} \cdot \nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) \right] \, d\mathbf{x} \, dt (2.6)
\]

for a.a. \(\tau > 0\) and any trio of continuously differentiable functions \((\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})\) satisfying

\[
\tilde{\varrho} > 0, \quad \tilde{\vartheta} > 0, \quad \tilde{\vartheta}|_{\partial \Omega} = \vartheta_B, \quad \tilde{\mathbf{u}}|_{\partial \Omega} = 0. \quad (2.7)
\]
2.2 Constitutive relations

The existence theory developed in [6], [11] is conditioned by certain restrictions imposed on the constitutive relations (state equations) similar to those introduced in the monograph [10, Chapters 1,2]. Specifically, the equation of state takes the form

\[ p(\varrho, \vartheta) = p_m(\varrho, \vartheta) + p_{rad}(\vartheta), \] (2.8)

where \( p_m \) is the pressure of a general monoatomic gas,

\[ p_m(\varrho, \vartheta) = \frac{2}{3} \varrho e_m(\varrho, \vartheta), \] (2.9)

enhanced by the radiation pressure

\[ p_{rad}(\vartheta) = \frac{a}{3} \varrho^4, \quad a > 0. \]

Accordingly, the internal energy reads

\[ e(\varrho, \vartheta) = e_m(\varrho, \vartheta) + e_{rad}(\varrho, \vartheta), \quad e_{rad}(\varrho, \vartheta) = \frac{a}{\varrho} \varrho^4. \]

Moreover, using several physical principles it was shown in [10, Chapter 1]:

- **Gibbs’ relation** together with (2.9) yield

\[ p_m(\varrho, \vartheta) = \varrho^{\frac{5}{3}} P\left( \frac{\varrho}{\varrho^{\frac{5}{3}}} \right) \]

for a certain \( P \in C^1[0,\infty) \). Consequently,

\[ p(\varrho, \vartheta) = \varrho^{\frac{5}{3}} P\left( \frac{\varrho}{\varrho^{\frac{5}{3}}} \right) + \frac{a}{3} \varrho^4, \quad e(\varrho, \vartheta) = \frac{3}{2} \varrho^{\frac{5}{3}} P\left( \frac{\varrho}{\varrho^{\frac{5}{3}}} \right) + \frac{a}{\varrho} \varrho^4, \quad a > 0. \] (2.10)

- **Hypothesis of thermodynamic stability** [1,10] expressed in terms of \( P \) gives rise to

\[ P(0) = 0, \quad P'(Z) > 0 \text{ for } Z \geq 0, \quad 0 < \frac{\frac{5}{3} P(Z) - P'(Z)Z}{Z} \leq C \text{ for } Z > 0. \] (2.11)

In particular, the function \( Z \mapsto P(Z)/Z^{\frac{5}{3}} \) is decreasing, and we suppose

\[ \lim_{Z \to \infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \] (2.12)
\[ s(\rho, \theta) = s_m(\rho, \theta) + s_{\text{rad}}(\rho, \theta), \quad s_m(\rho, \theta) = S\left(\frac{\rho}{\rho^\gamma}\right), \quad s_{\text{rad}}(\rho, \theta) = \frac{4a}{3} \theta^3, \quad (2.13) \]

where \[ S'(Z) = -\frac{3}{2} \frac{\delta^5 P(Z) - P'(Z) Z}{Z^2} < 0. \quad (2.14) \]

- In addition, we impose the Third law of thermodynamics, cf. Belgiorno [3], [4], requiring the entropy to vanish when the absolute temperature approaches zero,
\[ \lim_{Z \to \infty} S(Z) = 0. \quad (2.15) \]

Finally, we suppose the transport coefficients are continuously differentiable functions satisfying
\[ 0 < \mu(1 + \theta) \leq \mu(\theta), \quad |\mu'(\theta)| \leq \overline{\mu}, \]
\[ 0 \leq \eta(\theta) \leq \overline{\eta}(1 + \theta), \]
\[ 0 < \kappa(1 + \theta^\beta) \leq \kappa(\theta) \leq \overline{\kappa}(1 + \theta^\beta), \quad \text{where} \ \beta > 6. \quad (2.16) \]

As a consequence of the above hypotheses, we get the following estimates:
\[ \rho^\gamma + \theta^4 \lesssim \rho e(\rho, \theta) \lesssim 1 + \rho^\gamma + \theta^4, \quad (2.17) \]
\[ s_m(\rho, \theta) \lesssim \left(1 + \log(\rho) + [\log(\theta)]^+\right) \rho. \quad (2.18) \]

see [10] Chapter 3, Section 3.2. Here and hereafter, the symbol \( a \lesssim b \) means there is a positive constant \( C > 0 \) such that \( a \leq Cb \).

We report the existence result proved in [6, Theorem 4.2].

**Proposition 2.2 (Primitive system, global existence).** Let the thermodynamic functions \( p, e, s \) and the transport coefficients \( \mu, \eta, \kappa \) satisfy the hypotheses (2.8)–(2.16). Let \( \vartheta_B \in C^2(\partial\Omega) \).
Suppose the initial data \( \rho_0, \vartheta_0, u_0 \) satisfy
\[ \rho_0 > 0, \ \vartheta_0 > 0, \ \int_\Omega E_\varepsilon(\rho_0, \vartheta_0, u_0) \ dx < \infty. \]

Then for any \( T > 0 \), the Navier–Stokes–Fourier system (1.1)–(1.7) admits a weak solution \( (\rho, \vartheta, u) \) in \( (0, T) \times \Omega \) in the sense specified in Definition 2.7.

### 3 Strong solutions to the target system

Our analysis requires the existence of regular solutions to the Oberbeck–Boussinesq system (1.21)–(1.23). The relevant result was proved in [11] Theorem 2.3, Theorem 3.1].
Proposition 3.1 (Strong solutions to target system). Suppose that
\[ G \in W^{1,\infty}(\Omega), \ \Theta_B \in C^2(\overline{\Omega}), \]  
and
\[ \Theta_0 \in W^{2,p}(\Omega), \ U_0 \in W^{2,p}(\Omega; \mathbb{R}^d), \ \text{div}_x U_0 = 0, \] for any \( 1 \leq p < \infty, \)
together with the compatibility conditions
\[ U_0 = 0, \ \Theta_0 + \frac{\lambda}{1 - \lambda} \int_{\Omega} \Theta_0 \, dx = \Theta_B \text{ on } \partial \Omega. \]  

Then there exists \( T_{\text{max}} > 0, \ T_{\text{max}} = \infty \text{ if } d = 2, \) such that the OB system (1.21)–(1.23) with the initial data
\[ U(0, \cdot) = U_0, \ \Theta(0, \cdot) = \Theta_0, \] admits a strong solution \( U, \Theta \) in the regularity class
\[ U \in L^p(0,T; W^{2,p}(\Omega; \mathbb{R}^d)), \ \partial_t U \in L^p(0,T; L^p(\Omega; \mathbb{R}^d)), \ \Pi \in L^p(0,T; W^{1,p}(\Omega)), \]
\[ \Theta \in L^p(0,T; W^{2,p}(\Omega)), \ \partial_t \Theta \in L^p(0,T; L^p(\Omega; \mathbb{R}^d)) \]  
for any \( 1 \leq p < \infty \text{ and any } 0 < T < T_{\text{max}}. \)

Given the parabolic character of the OB system, the strong solutions are in fact classical if higher regularity of the data is required, cf. [1, Theorem 4.1].

4 Asymptotic limit, main result

We are ready to state our main result concerning the singular limit \( \varepsilon \to 0 \) in the primitive NSF system.

**Theorem 4.1 (Singular limit).** Let the constitutive hypotheses (2.8)–(2.18) be satisfied. Let the data belong to the regularity class
\[ G \in W^{1,\infty}(\Omega), \ \Theta_B \in C^2(\overline{\Omega}). \]
Let \( (\varrho_\varepsilon, \vartheta_\varepsilon, u_\varepsilon)_{\varepsilon > 0} \) be a family of weak solutions to the NSF system (1.1)–(1.7), with the boundary data
\[ u_\varepsilon|_{\partial \Omega} = 0, \ \vartheta_\varepsilon|_{\partial \Omega} = \overline{\nu} + \varepsilon \Theta_B, \ \overline{\nu} > 0 \text{ constant}, \]  
and the initial data
\[ \varrho_\varepsilon(0, \cdot) = \overline{\varrho} + \varepsilon \varrho_0, \ \overline{\varrho} > 0 \text{ constant}, \int_{\Omega} \varrho_0 \, dx = 0, \ \vartheta_\varepsilon(0, \cdot) = \overline{\nu} + \varepsilon \vartheta_0, \ u_\varepsilon(0, \cdot) = u_0, \]
where, in addition,

\[
\|\varrho_0,\varepsilon\|_{L^\infty(\Omega)} \lesssim 1, \quad \varrho_0,\varepsilon \to r_0 \text{ in } L^1(\Omega),
\]

\[
\|\vartheta_0,\varepsilon\|_{L^\infty(\Omega)} \lesssim 1, \quad \vartheta_0,\varepsilon \to \xi_0 \text{ in } L^1(\Omega),
\]

\[
\|u_0,\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^d)} \lesssim 1, \quad u_0,\varepsilon \to U_0 \text{ in } L^1(\Omega;\mathbb{R}^d),
\]

and

\[
\xi_0 \in W^{2,p}(\Omega), \quad U_0 \in W^{2,p}(\Omega;\mathbb{R}^d), \text{ for any } 1 \leq p < \infty, \quad \text{div}_x U_0 = 0,
\]

\[
U_0 = 0, \quad \xi_0 = \Theta_B \text{ on } \partial \Omega,
\]

\[
\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} \nabla_x r_0 + \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \nabla_x \xi_0 = \widehat{\varrho} \nabla_x G.
\]

Then

\[
\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to r \text{ in } L^\infty(0,T;L^1(\Omega)),
\]

\[
\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \xi \text{ in } L^\infty(0,T;L^1(\Omega)),
\]

\[
\sqrt{\varrho_\varepsilon} u_\varepsilon \to \sqrt{\varrho} U \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d)),
\]

as \(\varepsilon \to 0\) for any \(T < T_{\text{max}}\), where

\[
r, \quad \Theta = \xi - \lambda(\overline{\varrho}, \overline{\vartheta}) \int_\Omega \xi \, dx, \quad U
\]

is the strong solution of the OB system \((1.21) - (1.23)\) with the initial data \(U(0,\cdot) = U_0, \quad \Theta_0 = \xi_0 - \lambda \int_\Omega \xi_0 \, dx\) defined on \([0,T_{\text{max}}]\).

Note that hypotheses \((4.3) - (4.5)\) correspond to well prepared data. The rest of the paper is devoted to the proof of Theorem 4.1.

5 Basic energy estimates

In order to perform the limit claimed in Theorem 4.1 we need bounds on the sequence \((\varrho_\varepsilon, \vartheta_\varepsilon, u_\varepsilon)_{\varepsilon > 0}\) independent of \(\varepsilon\). We start by introducing the notation borrowed from [10] to distinguishing the “essential” and “residual” range of the thermostatic variables \((\varrho, \vartheta)\). Specifically, given a compact set

\[
K \subset \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \mid \varrho > 0, \vartheta > 0 \right\}
\]
we introduce
\[ g_{\text{ess}} = g 1_{(\varrho, \vartheta) \in K}, \quad g_{\text{res}} = g - g_{\text{ess}} = g 1_{(\varrho, \vartheta) \in \mathbb{R}^2 \setminus K} \]
for any measurable \( g \). As a matter of fact, the characteristic function \( 1_{(\varrho, \vartheta) \in K} \) can be replaced by its smooth regularization by a suitable convolution kernel. As shown in [10, Chapter 5, Lemma 5.1], the relative energy enjoys the following coercivity properties:

\[
E_{\varepsilon} \left( \varrho, \vartheta, u \big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{u} \right) \geq E_{\varepsilon} \left( \varrho, \vartheta, u \big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{u} \right)_{\text{ess}} \geq C \left( \frac{|\varrho - \tilde{\varrho}|^2}{\varepsilon^2} + \frac{|\vartheta - \tilde{\vartheta}|^2}{\varepsilon^2} + |u - \tilde{u}|^2 \right)_{\text{ess}}, \quad (5.1)
\]
\[
E_{\varepsilon} \left( \varrho, \vartheta, u \big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{u} \right) \geq E_{\varepsilon} \left( \varrho, \vartheta, u \big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{u} \right)_{\text{res}} \geq C \left( \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^2} \varrho e(\varrho, \vartheta) + \frac{1}{\varepsilon^2} \varrho s(\varrho, \vartheta) + \varrho |u|^2 \right)_{\text{res}}, \quad (5.2)
\]
whenever \( (\tilde{\varrho}, \tilde{\vartheta}) \in \text{int}[K] \), where the constant \( C \) depends on \( K \) and the distance

\[
\sup_{t,x} \text{dist} \left[ (\tilde{\varrho}(t, x), \tilde{\vartheta}(t, x)); \partial K \right].
\]

In the subsequent analysis, we consider

\[
K = \mathcal{U}(\overline{\varrho}, \overline{\vartheta}) \subset (0, \infty)^2, \quad \mathcal{U}(\overline{\varrho}, \overline{\vartheta}) - \text{an open neighborhood of } (\overline{\varrho}, \overline{\vartheta}).
\]

### 5.1 Energy estimates

In agreement with hypothesis \([1.2]\), we have

\[
\int_{\Omega} E_{\varepsilon} \left( \varrho_{\varepsilon}(0, \cdot), \vartheta_{\varepsilon}(0, \cdot), u_{\varepsilon}(0, \cdot) \big| \overline{\varrho}, \overline{\vartheta} + \varepsilon \Theta_B, 0 \right) \, dx \lesssim 1 \text{ independently of } \varepsilon \to 0, \quad (5.3)
\]
where \( \Theta_B = \Theta_B(x) \) is a suitable extension of the temperature boundary condition inside \( \Omega \). Plugging this ansatz in the relative energy inequality \([2.6]\) we obtain

\[
\left[ \int_{\Omega} E_{\varepsilon} \left( \varrho_{\varepsilon}, \vartheta_{\varepsilon}, u_{\varepsilon} \big| \overline{\varrho}, \overline{\vartheta} + \varepsilon \Theta_B, 0 \right) \, dx \right]_{t=0}^{t=\tau}
\]
\[
+ \int_{t=0}^{t=\tau} \int_{\Omega} \left( \frac{\overline{\vartheta} + \varepsilon \Theta_B}{\vartheta_{\varepsilon}} \left( \mathcal{S}(\varrho_{\varepsilon}, \nabla x u_{\varepsilon}) : \nabla x u_{\varepsilon} + \frac{1}{\varepsilon^2} \varrho_{\varepsilon} (\nabla x \varrho_{\varepsilon})^2 \right) \right) \, dx \, dt
\]
\[
\leq -\frac{1}{\varepsilon} \int_{t=0}^{t=\tau} \int_{\Omega} \left( \varrho_{\varepsilon} (s(\varrho_{\varepsilon}, \vartheta_{\varepsilon}) - s(\overline{\varrho}, \overline{\vartheta} + \varepsilon \Theta_B)) u_{\varepsilon} \cdot \nabla x \Theta_B - \frac{\varrho_{\varepsilon} (\nabla x \varrho_{\varepsilon})}{\vartheta_{\varepsilon}} \cdot \nabla x \Theta_B \right) \, dx \, dt
\]
\[
+ \int_{t=0}^{t=\tau} \int_{\Omega} \frac{1}{\varepsilon} \nabla x G \cdot u_{\varepsilon} \, dx \, dt
\]
\[
- \frac{1}{\varepsilon} \int_{t=0}^{t=\tau} \int_{\Omega} \left( \frac{\partial p(\overline{\varrho}, \overline{\vartheta} + \varepsilon \Theta_B)}{\partial \varrho} - \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \right) \frac{\varrho_{\varepsilon}}{\overline{\varrho}} u_{\varepsilon} \cdot \nabla x \Theta_B \, dx \, dt
\]
\[-\frac{1}{\varepsilon} \int_0^T \int_\Omega \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla x \Theta_B \; dx \; dt. \tag{5.4}\]

Our goal is to control the integrals on the right-hand side to apply Grönwall’s argument. To this end, we fix the compact set \(K\) determining the essential and residual component to contain the point \((\bar{\varrho}, \bar{\vartheta})\) in its interior. In particular, the same is true for the range of the function \((\bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B)\) as soon as \(\varepsilon > 0\) is small enough. Accordingly, we will systematically use the coercivity of the relative energy \(E_\varepsilon\) stated in (5.1), (5.2) in the estimates below.

5.1.1 Estimates

Step 1: First,

\[
\frac{1}{\varepsilon} \int_\Omega \left| \varrho_\varepsilon(s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B)) \mathbf{u}_\varepsilon \cdot \nabla x \Theta_B \right| \; dx
\leq \frac{1}{\varepsilon} \int_\Omega \left| \left[ \varrho_\varepsilon(s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B)) \mathbf{u}_\varepsilon \right]_{\text{ess}} \right| \; dx
+ \frac{1}{\varepsilon} \int_\Omega \left| \left[ \varrho_\varepsilon(s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B)) \mathbf{u}_\varepsilon \right]_{\text{res}} \right| \; dx,
\]

where

\[
\frac{1}{\varepsilon} \int_\Omega \left| \left[ \varrho_\varepsilon(s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B)) \mathbf{u}_\varepsilon \right]_{\text{ess}} \right| \; dx \lesssim \frac{1}{\varepsilon^2} \int_\Omega \left| \left[ (s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B)) \right]_{\text{ess}} \right|^2 \; dx
+ \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \; dx \lesssim \int_\Omega E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \bigg| \bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B, 0 \right) \; dx, \tag{5.5}\]

and

\[
\frac{1}{\varepsilon} \int_\Omega \left| \left[ \varrho_\varepsilon(s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B)) \mathbf{u}_\varepsilon \right]_{\text{res}} \right| \; dx
\leq \frac{1}{\varepsilon} \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon| \; dx + \frac{1}{\varepsilon} \int_\Omega \varrho_\varepsilon s_m(\varrho_\varepsilon, \vartheta_\varepsilon) |\mathbf{u}_\varepsilon| \; dx + \frac{1}{\varepsilon} \int_\Omega \varrho_\varepsilon^2 |\mathbf{u}_\varepsilon| \; dx.
\]

Furthermore,

\[
\frac{1}{\varepsilon} \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon| \; dx \lesssim \frac{1}{\varepsilon^2} \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon| \; dx + \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \; dx \lesssim \int_\Omega E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \bigg| \bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B, 0 \right) \; dx, \tag{5.6}\]

since the total mass is constant independent of \(\varepsilon\). In view of the bounds (2.17), (2.18),

\[
\frac{1}{\varepsilon} \int_\Omega \varrho_\varepsilon s_m(\varrho_\varepsilon, \vartheta_\varepsilon) |\mathbf{u}_\varepsilon| \; dx \lesssim \frac{1}{\varepsilon^2} \int_\Omega \varrho_\varepsilon s_m(\varrho_\varepsilon, \vartheta_\varepsilon) |\mathbf{u}_\varepsilon| \; dx + \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \; dx
\lesssim \int_\Omega E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \bigg| \bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B, 0 \right) \; dx. \tag{5.7}\]
Finally,
\[
\frac{1}{\varepsilon} \int_{\Omega} \frac{\kappa(\vartheta_{\varepsilon}) |\nabla_{\varepsilon} \vartheta_{\varepsilon}|^2}{\vartheta_{\varepsilon}^2} \, dx \lesssim \int_{\Omega} |\nabla_{\varepsilon} \log(\vartheta_{\varepsilon})|^2 + |\nabla_{\varepsilon} \vartheta_{\varepsilon}^\beta|^2 \, dx, \quad \beta > 6.
\] (5.8)

Consequently, as the measure of the residual set is small (cf. (5.2)), we get
\[
\frac{1}{\varepsilon^2} \int_{\Omega} \delta \left[ |\vartheta_{\varepsilon}|_{W^{1,2}(\Omega; \mathbb{R}^d)} \right]_{\text{res}} \, dx \lesssim \int_{\Omega} |\nabla_{\varepsilon} \log(\vartheta_{\varepsilon})|^2 + |\nabla_{\varepsilon} \vartheta_{\varepsilon}^\beta|^2 \, dx \lesssim \frac{1}{\varepsilon^2} \int_{\Omega} |\nabla_{\varepsilon} \vartheta_{\varepsilon}^\beta|^2 \, dx + C(\delta) \int_{\Omega} E_{\varepsilon} \left( \vartheta_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon}; \vartheta, \vartheta + \varepsilon \Theta_B, 0 \right) \, dx \tag{5.9}
\]
for any \( \delta > 0 \).

**Step 2:** In accordance with hypothesis (2.16),
\[
\frac{1}{\varepsilon} \left| \int_{\Omega} \frac{\kappa(\vartheta_{\varepsilon})}{\vartheta_{\varepsilon}} \nabla_{\varepsilon} \vartheta_{\varepsilon} \, dx \right| \lesssim \frac{1}{\varepsilon} \int_{\Omega} |\nabla_{\varepsilon} \log(\vartheta_{\varepsilon})| + \vartheta_{\varepsilon}^{\beta - 1} |\nabla_{\varepsilon} \vartheta_{\varepsilon}| \, dx,
\]
where
\[
\frac{1}{\varepsilon} \int_{\Omega} |\nabla_{\varepsilon} \log(\vartheta_{\varepsilon})| \, dx \lesssim \frac{\delta}{\varepsilon^2} \int_{\Omega} |\nabla_{\varepsilon} \log(\vartheta_{\varepsilon})|^2 \, dx + C(\delta) \tag{5.10}
\]
for any \( \delta > 0 \); hence the integral is controlled by dissipation.

Next,
\[
\frac{1}{\varepsilon} \int_{\Omega} \vartheta_{\varepsilon}^{\beta - 1} |\nabla_{\varepsilon} \vartheta_{\varepsilon}| \, dx = \frac{1}{\varepsilon} \int_{\Omega} \vartheta_{\varepsilon}^\beta \nabla_{\varepsilon} \vartheta_{\varepsilon}^\beta \, dx \leq \frac{\delta}{\varepsilon^2} \int_{\Omega} |\nabla_{\varepsilon} \vartheta_{\varepsilon}^\beta|^2 \, dx + C(\delta) \int_{\Omega} |\vartheta_{\varepsilon}^\beta|^2 \, dx, \tag{5.11}
\]
where the first term is controlled by dissipation and the second one by Poincaré’s inequality
\[
\int_{\Omega} |\vartheta_{\varepsilon}^\beta|^2 \, dx \lesssim \int_{\Omega} |\nabla_{\varepsilon} \vartheta_{\varepsilon}^\beta|^2 \, dx + \int_{\partial\Omega} (\vartheta + \varepsilon \Theta_B)^\beta \, d\sigma_x. \tag{5.12}
\]

**Step 3:** We have
\[
\frac{1}{\varepsilon} \int_{0}^{\tau} \int_{\Omega} \vartheta_{\varepsilon} \nabla_{\varepsilon} G \cdot \mathbf{u}_{\varepsilon} \, dx \, dt = -\frac{1}{\varepsilon} \int_{0}^{\tau} \int_{\Omega} \nabla_{\varepsilon} \vartheta_{\varepsilon} \cdot \mathbf{u}_{\varepsilon} \, dx \, dt = \int_{0}^{\tau} \int_{\Omega} \frac{1}{\varepsilon} (\vartheta_{\varepsilon} - \vartheta) G \, dx \, dt
\]
\[
\left[\int_{\Omega} \frac{\varrho_\varepsilon - \vartheta}{\varepsilon} G \, dx\right]_{t=0}^{t=\tau}.
\]

Seeing that
\[
E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \bigg| \vartheta, \vartheta' + \varepsilon \Theta_B, 0 \right) - c_1 \lesssim E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \bigg| \vartheta, \vartheta' + \varepsilon \Theta_B, 0 \right) - \int_{\Omega} \frac{\varrho_\varepsilon - \vartheta}{\varepsilon} G \, dx
\]
\[
\lesssim E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \bigg| \vartheta, \vartheta' + \varepsilon \Theta_B, 0 \right) + 1
\]

we can add this term to the relative energy on the left–hand side of (2.6).

**Step 4:**
\[
\frac{1}{\varepsilon} \int_{\Omega} \left( \frac{\partial p(\varrho, \vartheta' + \varepsilon \Theta_B)}{\partial \vartheta} - \frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \right) \frac{\varrho_\varepsilon}{\vartheta} \mathbf{u}_\varepsilon \cdot \nabla x \Theta_B \, dx
\]
\[
\lesssim \int_{\Omega} \vartheta_\varepsilon \left| \mathbf{u}_\varepsilon \right| \, dx \lesssim \int_{\Omega} \varrho_\varepsilon \, dx + \int_{\Omega} \varrho_\varepsilon \left| \mathbf{u}_\varepsilon \right|^2 \, dx \lesssim \int_{\Omega} E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \bigg| \vartheta, \vartheta' + \varepsilon \Theta_B, 0 \right) \, dx + 1.
\]

**Step 5:** The last integral on the right–hand side of (5.4) can be handled exactly as in Step 3.

### 5.1.2 Conclusion, uniform bounds

In view of the estimates obtained in the previous section, we may apply Grönwall’s lemma to the relative energy inequality (5.4). As the initial data satisfy (5.3), we deduce the following bounds independent of the scaling parameter \( \varepsilon \to 0 \):

\[
\text{ess sup}_{t \in (0, T)} \int_{\Omega} E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \bigg| \vartheta, \vartheta' + \varepsilon \Theta_B, 0 \right) \, dx \lesssim 1,
\]

\[
\int_{0}^{T} \left\| \mathbf{u}_\varepsilon \right\|^2_{W^{1,2}_{0}(\Omega; \mathbb{R}^d)} \, dt \lesssim 1,
\]

\[
\frac{1}{\varepsilon^2} \int_{0}^{T} \left( \| \nabla x \log(\vartheta_\varepsilon) \|^2_{L^2(\Omega; \mathbb{R}^d)} + \| \nabla x \vartheta_\varepsilon^\beta \|^2_{L^2(\Omega; \mathbb{R}^d)} \right) \lesssim 1.
\]

Next, it follows from (5.15) that the measure of the residual set shrinks to zero, specifically

\[
\frac{1}{\varepsilon^2} \text{ess sup}_{t \in (0, T)} \int_{\Omega} \left[ 1 \right]_{\text{res}} \, dx \lesssim 1.
\]

In addition, we get from (5.15):

\[
\text{ess sup}_{t \in (0, T)} \int_{\Omega} \varrho_\varepsilon \left| \mathbf{u}_\varepsilon \right|^2 \, dx \lesssim 1,
\]

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\[
\begin{align*}
\text{ess sup}_{t \in (0,T)} \left\| \frac{\varphi_{\epsilon} - \bar{\varphi}}{\epsilon} \right\|_{L^2(\Omega)} & \lesssim 1, \\
\text{ess sup}_{t \in (0,T)} \left\| \frac{\vartheta_{\epsilon} - \bar{\vartheta}}{\epsilon} \right\|_{L^2(\Omega)} & \lesssim 1, \\
\frac{1}{\epsilon^2} \text{ess sup}_{t \in (0,T)} \left\| [\varphi_{\epsilon}]_{\text{res}} \right\|_{L^2(\Omega)} + \frac{1}{\epsilon^2} \text{ess sup}_{t \in (0,T)} \left\| [\vartheta_{\epsilon}]_{\text{res}} \right\|_{L^4(\Omega)} & \lesssim 1. \tag{5.19}
\end{align*}
\]

Combining (5.17), (5.18), and (5.19), we conclude
\[
\int_0^T \left\| \frac{\log(\vartheta_{\epsilon}) - \log(\bar{\vartheta})}{\epsilon} \right\|_{W^{1,2}(\Omega)}^2 \, dt + \int_0^T \left\| \frac{\vartheta_{\epsilon} - \bar{\vartheta}}{\epsilon} \right\|_{W^{1,2}(\Omega)}^2 \, dt \lesssim 1. \tag{5.20}
\]

Finally, we claim the bound on the entropy flux
\[
\int_0^T \left\| \left[ \frac{\kappa(\vartheta_{\epsilon})}{\vartheta_{\epsilon}} \frac{\nabla_x \vartheta_{\epsilon}}{\epsilon} \right]_{\text{res}}^q \right\|_{L^q(\Omega;\mathbb{R}^d)} \, dt \lesssim 1 \text{ for some } q > 1. \tag{5.21}
\]

Indeed we have
\[
\left\| \left[ \frac{\kappa(\vartheta_{\epsilon})}{\vartheta_{\epsilon}} \frac{\nabla_x \vartheta_{\epsilon}}{\epsilon} \right]_{\text{res}} \right\|_{L^q(\Omega;\mathbb{R}^d)} \lesssim \frac{1}{\epsilon} \left| \nabla_x \log(\vartheta_{\epsilon}) \right| + \frac{1}{\epsilon} \left| \vartheta_{\epsilon}^\beta \nabla_x \vartheta_{\epsilon}^\beta \right|,
\]
where the former term on the right–hand side is controlled via (5.20). As for the latter, we deduce from (5.17) that
\[
\left\| \frac{1}{\epsilon} \nabla_x \vartheta_{\epsilon}^\beta \right\|_{L^2((0,T) \times \Omega;\mathbb{R}^d)} \lesssim 1;
\]
hence it is enough to check
\[
\left\| \vartheta_{\epsilon}^\beta \right\|_{L^r((0,T) \times \Omega)} \lesssim 1 \text{ for some } r > 2. \tag{5.22}
\]

To see (5.22), first observe that
\[
\text{ess sup}_{t \in (0,T)} \left\| \vartheta_{\epsilon} \right\|_{L^4(\Omega)} \lesssim 1,
\]
and, in view of (5.17) and Poincaré inequality,
\[
\left\| \vartheta_{\epsilon}^\beta \right\|_{L^2((0,T);L^6(\Omega))} \lesssim 1 \text{ (for } d = 3).$

Consequently, (5.22) follows by interpolation.

6 Convergence to the target system
Our ultimate goal is to perform the limit $\epsilon \to 0$. We proceed in two steps.

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6.1 Weak convergence

In view of the uniform bounds established in Section 5.1.2, we may infer

\[ \rho_\varepsilon \to \overline{\rho} \text{ in } L^\frac{3}{2}(\Omega) \text{ uniformly for } t \in (0, T), \] (6.1)

\[ \vartheta_\varepsilon \to \overline{\vartheta} \text{ in } L^2(0, T; W^{1,2}(\Omega)), \] (6.2)

\[ u_\varepsilon \to u \text{ weakly in } L^2(0, T; W^{1,2}_0(\Omega; \mathbb{R}^d)), \] (6.3)

where (6.3) may require extraction of a suitable subsequence. As we shall eventually see, the limit velocity \( u = U \) is unique so that the convergence is, in fact, unconditional. In addition, we may let \( \varepsilon \to 0 \) in the weak formulation of the equation of continuity (2.2) to deduce

\[ \text{div}_x u = 0. \] (6.4)

Next, we use (5.19), (5.20) to obtain (a priori for suitable subsequences),

\[
\begin{align*}
\frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon} &= \left[ \frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon} \right]_{\text{ess}} + \left[ \frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon} \right]_{\text{res}}, \\
\left[ \frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon} \right]_{\text{ess}} &\to \mathfrak{R} \text{ weakly-(*) in } L^\infty(0, T; L^2(\Omega)), \\
\left[ \frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon} \right]_{\text{res}} &\to 0 \text{ in } L^\infty(0, T; L^\frac{3}{2}(\Omega)),
\end{align*}
\] (6.5)

\[ \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \mathfrak{F} \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)) \text{ and weakly-(*) in } L^\infty(0, T; L^2(\Omega)). \] (6.6)

Moreover, in view of (1.1),

\[ \mathfrak{F}|_{\partial \Omega} = \Theta_B. \] (6.7)

Finally, we perform the limit in the rescaled momentum equation (1.2) to deduce

\[
\frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\rho}} \nabla_x \mathfrak{R} + \frac{\partial p(\overline{\rho}, \overline{\vartheta})}{\partial \overline{\vartheta}} \nabla_x \mathfrak{F} = \overline{\rho} \nabla_x G
\] (6.8)

in the sense of distributions. In particular, it follows from (6.8) that

\[ \mathfrak{R} \in L^2(0, T; W^{1,2}(\Omega)). \] (6.9)

6.2 Strong convergence

First, it is more convenient to rewrite the target OB system in terms of the variable

\[ T, \text{ where } T = \lambda(\overline{\rho}, \overline{\vartheta}) \int_\Omega T \, dx = \Theta. \]
Accordingly, we get
\[
\text{div}_x U = 0,
\]
\[
\bar{\tau} \left( \partial_t U + U \cdot \nabla_x U \right) + \nabla_x \Pi = \text{div}_x S(\bar{\tau}, \nabla_x U) + r \nabla_x G,
\]
\[
\bar{\varphi}_p(\bar{\tau}, \bar{\vartheta}) \left( \partial_t \mathcal{T} + U \cdot \nabla_x \mathcal{T} \right) - \bar{\varphi} \partial_\vartheta(\bar{\tau}, \bar{\vartheta}) U \cdot \nabla_x G = \kappa(\bar{\vartheta}) \Delta_x \mathcal{T} + \bar{\varphi}_p(\bar{\tau}, \bar{\vartheta}) \frac{\partial \bar{p}(\bar{\tau}, \bar{\vartheta})}{\partial \varphi} \partial_t \int_\Omega \mathcal{T} \, dx,
\]
\[
(6.10)
\]
and the Boussinesq relation
\[
\frac{\partial \bar{p}(\bar{\tau}, \bar{\vartheta})}{\partial \varphi} \nabla_x r + \frac{\partial \bar{p}(\bar{\tau}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \mathcal{T} = \bar{\varphi} \nabla_x G, \quad \int_\Omega r \, dx = 0,
\]
\[
(6.11)
\]
the boundary conditions
\[
U|_{\partial \Omega} = 0, \quad \mathcal{T}|_{\partial \Omega} = \Theta_B,
\]
\[
(6.12)
\]
and the initial conditions
\[
U(0, \cdot) = U_0, \quad \mathcal{T}(0, \cdot) = \mathcal{\Omega}_0.
\]
\[
(6.13)
\]
In accordance with Proposition 3.1 and hypotheses (4.4), (4.5), the problem (6.10)–(6.13) admits a unique regular solution on a time interval \([0, T_{\text{max}}])\), where \(T_{\text{max}} > 0\) and \(T_{\text{max}} = \infty\) if \(d = 2\).

### 6.3 Relative energy

To complete the proof of Theorem 4.1 we use the relative energy inequality (2.6), with the ansatz
\[
E(\varrho, \vartheta, \mathbf{u} | \mathcal{\varphi} + \varepsilon r, \mathcal{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U}).
\]
In accordance with our choice of the initial data,
\[
\int_\Omega E(\varrho, \vartheta, \mathbf{u} | \mathcal{\varphi} + \varepsilon r, \mathcal{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U}) (0, \cdot) \, dx \to 0 \text{ as } \varepsilon \to 0.
\]
\[
(6.14)
\]
Our goal is to show that \(E(\varrho, \vartheta, \mathbf{u} | \mathcal{\varphi} + \varepsilon r, \mathcal{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U}) \to 0\), which finally yields
\[
\mathcal{T} = \lim_{\varepsilon \to 0} \frac{\vartheta - \mathcal{\vartheta}}{\varepsilon} = \mathcal{\Omega}, \quad r = \lim_{\varepsilon \to 0} \frac{\varrho - \mathcal{\varphi}}{\varepsilon} = \mathfrak{R}, \quad \lim \mathbf{u} = \mathbf{U},
\]
and the trio \((\mathfrak{R}, \mathcal{\Omega}, \mathbf{U}) = (r, \mathcal{T}, \mathbf{U})\) is the strong solution to the OB system (1.21)–(1.23).

**Step 1:** Plugging our ansatz in the relative energy inequality (2.6) and using \(\text{div}_x U = 0\) we get
\[
\left[ \int_\Omega E(\varrho, \vartheta, \mathbf{u} | \mathcal{\varphi} + \varepsilon r, \mathcal{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U}) \, dx \right]_{t=0}^{t=\tau}
\]
\[ + \int_0^T \int_\Omega \frac{\vartheta + \varepsilon T}{\vartheta_\varepsilon} \left( S(\vartheta_\varepsilon, D_x u_\varepsilon) : D_x u_\varepsilon + \frac{1}{\varepsilon^2} \kappa(\vartheta_\varepsilon) \nabla_x \vartheta_\varepsilon \cdot \nabla_x \vartheta_\varepsilon \right) \, dx \, dt \]

\[ \leq -\frac{1}{\varepsilon} \int_0^T \int_\Omega \varrho_\varepsilon \left[ s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\vartheta + \varepsilon T, \vartheta + \varepsilon \vartheta) \right] \partial_t \vartheta \, dx \, dt \]

\[ - \frac{1}{\varepsilon} \int_0^T \int_\Omega \varrho_\varepsilon \left[ s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\vartheta + \varepsilon T, \vartheta + \varepsilon \vartheta) \right] u_\varepsilon \cdot \nabla_x \vartheta \, dx \, dt \]

\[ + \frac{1}{\varepsilon} \int_0^T \int_\Omega \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon \cdot \nabla_x \vartheta \, dx \, dt \]

\[ - \int_0^T \int_\Omega \left[ \varrho_\varepsilon (u_\varepsilon - U) \otimes (u_\varepsilon - U) - S(\varrho_\varepsilon, D_x u_\varepsilon) \right] : D_x U \, dx \, dt \]

\[ + \int_0^T \int_\Omega \varrho_\varepsilon \left[ \frac{1}{\varepsilon} \nabla_x G - \partial_t U - (U \cdot \nabla_x) U \right] \cdot (u_\varepsilon - U) \, dx \, dt \]

\[ + \frac{1}{\varepsilon^2} \int_0^T \int_\Omega \left[ \left( 1 - \frac{\varrho_\varepsilon}{\vartheta + \varepsilon T} \right) \partial_t p(\vartheta + \varepsilon T, \vartheta + \varepsilon \vartheta) - \frac{\varrho_\varepsilon}{\vartheta + \varepsilon T} u_\varepsilon \cdot \nabla_x p(\vartheta + \varepsilon T, \vartheta + \varepsilon \vartheta) \right] \, dx \, dt. \]  

\[ (6.15) \]

**Step 2:** As \( r, U \) satisfy the momentum equation

\[-\varrho (\partial_t U + U \cdot \nabla_x U) = \nabla_x \Pi - \text{div}_x S(\vartheta, \nabla_x U) - r \nabla_x G,\]

we get

\[ \int_\Omega \varrho_\varepsilon \left[ \frac{1}{\varepsilon} \nabla_x G - \partial_t U - (U \cdot \nabla_x) U \right] \cdot (u_\varepsilon - U) \, dx \]

\[ = \int_\Omega \frac{\varrho_\varepsilon}{\vartheta} \left[ \frac{1}{\varepsilon} \nabla_x G + \nabla_x \Pi - \text{div}_x S(\vartheta, \nabla_x U) - r \nabla_x G \right] \cdot (u_\varepsilon - U) \, dx. \]

Thus we can use the convergence established in \((6.1) - (6.3)\) to rewrite \((6.15)\) in the form

\[ \int_\Omega E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, u_\varepsilon \, | \, \vartheta + \varepsilon T, U \right) (\tau, \cdot) \, dx \]

\[ + \int_0^T \int_\Omega \left( S(\vartheta, D_x u_\varepsilon) - S(\vartheta, D_x U) \right) : \left( D_x u_\varepsilon - D_x U \right) \, dx \, dt \]

\[ + \int_0^T \int_\Omega \left( \frac{\vartheta + \varepsilon T}{\vartheta_\varepsilon} \right) \frac{\kappa(\vartheta_\varepsilon) \nabla_x \vartheta_\varepsilon \cdot \nabla_x \vartheta_\varepsilon}{\varepsilon^2} \, dx \, dt \]

\[ - \int_0^T \int_\Omega \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon \cdot \nabla_x \vartheta \, dx \, dt \]

\[ \leq -\frac{1}{\varepsilon} \int_0^T \int_\Omega \varrho_\varepsilon \left[ s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\vartheta + \varepsilon T, \vartheta + \varepsilon \vartheta) \right] \partial_t \vartheta \, dx \, dt \]

\[ - \frac{1}{\varepsilon} \int_0^T \int_\Omega \varrho_\varepsilon \left[ s(\varrho_\varepsilon, \vartheta_\varepsilon) - s(\vartheta + \varepsilon T, \vartheta + \varepsilon \vartheta) \right] u_\varepsilon \cdot \nabla_x \vartheta \, dx \, dt \]
+ \int_0^\tau \int_\Omega \frac{\partial \epsilon}{\partial \Theta} \nabla_x \Pi \cdot (\mathbf{u}_\epsilon - \mathbf{U}) \, dx \, dt \\
+ \int_0^\tau \int_\Omega \frac{\partial \epsilon}{\partial \Theta} \left[ \frac{1}{\epsilon} \Theta \nabla_x G - r \nabla_x G \right] \cdot (\mathbf{u}_\epsilon - \mathbf{U}) \, dx \, dt \\
+ \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega \left[ \left( 1 - \frac{\partial \epsilon}{\partial \Theta + \epsilon r} \right) \partial_r (\overline{\Theta} + \epsilon r, \overline{\Theta} + \epsilon T) - \frac{\partial \epsilon}{\partial \Theta + \epsilon r} \mathbf{u}_\epsilon \cdot \nabla_x p(\overline{\Theta} + \epsilon r, \overline{\Theta} + \epsilon T) \right] \, dx \, dt \\
+ C \int_0^\tau \int_\Omega E_\epsilon \left( \frac{\partial \epsilon}{\partial \Theta}, \vartheta, \mathbf{u}_\epsilon \right) (\overline{\Theta} + \epsilon r, \overline{\Theta} + \epsilon T, \mathbf{U}) \, dx \, dt + O(\epsilon),

where the symbol $O(\epsilon)$ denotes a generic error, $O(\epsilon) \to 0$ as $\epsilon \to 0$.

In addition, in view of the convergences (6.11) – (6.3), we conclude

$$
\int_0^\tau \int_\Omega \frac{\partial \epsilon}{\partial \Theta} \nabla_x \Pi \cdot (\mathbf{u}_\epsilon - \mathbf{U}) \, dx \, dt = O(\epsilon);
$$

hence

$$
\int_\Omega E_\epsilon \left( \frac{\partial \epsilon}{\partial \Theta}, \vartheta, \mathbf{u}_\epsilon \right) (\overline{\Theta} + \epsilon r, \overline{\Theta} + \epsilon T, \mathbf{U}) (\tau, \cdot) \, dx
$$

$$
+ \int_0^\tau \int_\Omega \left( \Theta \left( \frac{\partial \epsilon}{\partial \Theta}, \vartheta, \mathbf{u}_\epsilon \right) - \Theta (\overline{\Theta}, \overline{\Theta}, \mathbf{U}) \right) : \left( \Theta \mathbf{u}_\epsilon - \Theta \mathbf{U} \right) \, dx \, dt
$$

$$
+ \int_0^\tau \int_\Omega \left( \frac{\partial \epsilon}{\partial \Theta} + \epsilon T \right) \kappa (\vartheta, \vartheta) \nabla_x \vartheta_\epsilon \cdot \nabla_x \vartheta_\epsilon \, dx \, dt - \int_0^\tau \int_\Omega \kappa (\vartheta_\epsilon) \nabla_x \vartheta_\epsilon \cdot \nabla_x T \, dx \, dt
$$

$$
\leq -\frac{1}{\epsilon} \int_0^\tau \int_\Omega \vartheta \left[ \left( s (\vartheta, \vartheta) - s (\overline{\Theta} + \epsilon r, \overline{\Theta} + \epsilon T) \right) \partial_r T \right] \, dx \, dt
$$

$$
- \frac{1}{\epsilon} \int_0^\tau \int_\Omega \vartheta \left[ \left( s (\vartheta, \vartheta) - s (\overline{\Theta} + \epsilon r, \overline{\Theta} + \epsilon T) \right) \mathbf{u}_\epsilon \cdot \nabla_x T \right] \, dx \, dt
$$

$$
+ \int_0^\tau \int_\Omega \frac{\partial \epsilon}{\partial \Theta} \left[ \frac{1}{\epsilon} \Theta \nabla_x G - r \nabla_x G \right] \cdot (\mathbf{u}_\epsilon - \mathbf{U}) \, dx \, dt
$$

$$
+ \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega \left[ \left( 1 - \frac{\partial \epsilon}{\partial \Theta + \epsilon r} \right) \partial_r (\overline{\Theta} + \epsilon r, \overline{\Theta} + \epsilon T) - \frac{\partial \epsilon}{\partial \Theta + \epsilon r} \mathbf{u}_\epsilon \cdot \nabla_x p(\overline{\Theta} + \epsilon r, \overline{\Theta} + \epsilon T) \right] \, dx \, dt
$$

$$
+ C \int_0^\tau \int_\Omega E_\epsilon \left( \frac{\partial \epsilon}{\partial \Theta}, \vartheta, \mathbf{u}_\epsilon \right) (\overline{\Theta} + \epsilon r, \overline{\Theta} + \epsilon T, \mathbf{U}) \, dx \, dt + O(\epsilon).
$$

(6.16)

**Step 3:** At this stage, we use the Boussinesq relation (6.11) to obtain

$$
\frac{\partial p(\overline{\Theta}, \overline{\Theta})}{\partial \vartheta} r + \frac{\partial p(\overline{\Theta}, \overline{\Theta})}{\partial \vartheta} T = \overline{\Theta} G + \chi(t),
$$

where

$$
\chi = \frac{\partial p(\overline{\Theta}, \overline{\Theta})}{\partial \vartheta} \int_\Omega T \, dx - \overline{\Theta} \int_\Omega G \, dx = \frac{\partial p(\overline{\Theta}, \overline{\Theta})}{\partial \vartheta} \int_\Omega T \, dx
$$

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since \( \int_{\Omega} G \ dx = 0 \). Consequently,

\[
\frac{1}{\varepsilon^2} \int_{\Omega} \left( 1 - \frac{\varrho_e}{\varrho + \varepsilon r} \right) \partial_t p(\varrho + \varepsilon r, \varrho + \varepsilon T) \ dx \\
= \frac{1}{\varepsilon} \int_{\Omega} \left( 1 - \frac{\varrho_e}{\varrho + \varepsilon r} \right) \left( \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \varrho} \partial_r r + \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \vartheta} \partial_t T \right) \ dx \\
= \int_{\Omega} \frac{1}{\varepsilon} \left( 1 - \frac{\varrho_e}{\varrho + \varepsilon r} \right) \left( \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \varrho} - \frac{\partial p(\varrho, \varrho)}{\partial \varrho} \right) \partial_r r \ dx \\
+ \int_{\Omega} \frac{1}{\varepsilon} \left( 1 - \frac{\varrho_e}{\varrho + \varepsilon r} \right) \left( \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \vartheta} - \frac{\partial p(\varrho, \varrho)}{\partial \vartheta} \right) \partial_t T \ dx \\
+ \frac{1}{\varepsilon} \int_{\Omega} \left( \frac{\varrho + \varepsilon r - \varrho_e}{\varrho + \varepsilon r} \right) \partial_t \chi \ dx,
\]

(6.17)

where

\[
\int_{\Omega} \frac{1}{\varepsilon} \left( 1 - \frac{\varrho_e}{\varrho + \varepsilon r} \right) \left( \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \varrho} - \frac{\partial p(\varrho, \varrho)}{\partial \varrho} \right) \partial_r r \ dx \\
+ \int_{\Omega} \frac{1}{\varepsilon} \left( 1 - \frac{\varrho_e}{\varrho + \varepsilon r} \right) \left( \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \vartheta} - \frac{\partial p(\varrho, \varrho)}{\partial \vartheta} \right) \partial_t T \ dx = O(\varepsilon).
\]

Moreover,

\[
\frac{1}{\varepsilon} \frac{\varrho + \varepsilon r - \varrho_e}{\varrho + \varepsilon r} = - \frac{\varrho_e - \varrho}{\varepsilon(\varrho + \varepsilon r)} + \frac{r}{\varrho + \varepsilon r} \to \frac{1}{\varepsilon}(r - \mathcal{R}).
\]

Seeing that

\[
\int_{\Omega} r \ dx = \int_{\Omega} \mathcal{R} \ dx = 0,
\]

we may infer

\[
\frac{1}{\varepsilon} \int_{\Omega} \left( \frac{\varrho + \varepsilon r - \varrho_e}{\varrho + \varepsilon r} \right) \partial_t \chi \ dx = O(\varepsilon).
\]

Similarly,

\[
- \frac{1}{\varepsilon^2} \int_{0}^{T} \int_{\Omega} \frac{\varrho_e}{\varrho + \varepsilon r} \mathbf{u}_e \cdot \nabla_x p(\varrho + \varepsilon r, \varrho + \varepsilon T) \ dx \ dt \\
= - \frac{1}{\varepsilon} \int_{0}^{T} \int_{\Omega} \frac{\varrho_e}{\varrho + \varepsilon r} \mathbf{u}_e \cdot \left( \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \varrho} \nabla_x r + \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \vartheta} \nabla_x T \right) \ dx \ dt \\
= - \int_{0}^{T} \int_{\Omega} \frac{1}{\varepsilon} \frac{\varrho_e}{\varrho + \varepsilon r} \mathbf{u}_e \cdot \nabla_x r \left( \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \varrho} - \frac{\partial p(\varrho, \varrho)}{\partial \varrho} \right) \ dx \ dt \\
- \int_{0}^{T} \int_{\Omega} \frac{1}{\varepsilon} \frac{\varrho_e}{\varrho + \varepsilon r} \mathbf{u}_e \cdot \nabla_x T \left( \frac{\partial p(\varrho + \varepsilon r, \varrho + \varepsilon T)}{\partial \vartheta} - \frac{\partial p(\varrho, \varrho)}{\partial \vartheta} \right) \ dx \ dt
\]

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\[-\frac{1}{\varepsilon} \int_0^r \int_{\Omega} \frac{\partial \varepsilon}{\partial \bar{\theta} + \varepsilon r} \mathbf{u}_x \cdot \nabla_x G \; dx \; dt. \tag{6.18}\]

Using (6.1), (6.3), we perform the limit in the first integral obtaining

\[
\int_0^r \int_{\Omega} \frac{1}{\varepsilon + \varepsilon r} \mathbf{u}_x \cdot \nabla_x r \left( \frac{\partial p(\bar{\theta} + \varepsilon r, \bar{\vartheta} + \varepsilon T)}{\partial \bar{\theta}} - \frac{\partial p(\bar{\vartheta}, \bar{\vartheta})}{\partial \bar{\vartheta}} \right) \; dx \; dt
\]

\[
+ \int_0^r \int_{\Omega} \frac{1}{\varepsilon + \varepsilon r} \mathbf{u}_x \cdot \nabla_x T \left( \frac{\partial p(\bar{\theta} + \varepsilon r, \bar{\vartheta} + \varepsilon T)}{\partial \bar{\vartheta}} - \frac{\partial p(\bar{\theta}, \bar{\theta})}{\partial \bar{\theta}} \right) \; dx \; dt
\]

\[
= \int_0^r \int_{\Omega} \mathbf{u} \cdot \left( \frac{\partial^2 p(\bar{\theta}, \bar{\vartheta})}{\partial \bar{\vartheta}^2} \nabla_x r + \frac{\partial^2 p(\bar{\theta}, \bar{\vartheta})}{\partial \bar{\vartheta} \partial \vartheta} \nabla_x (r T) + \frac{\partial^2 p(\bar{\theta}, \bar{\vartheta})}{\partial \vartheta^2} T \nabla_x T \right) \; dx + O(\varepsilon)
\]

as \( \text{div}_r \mathbf{u} = 0 \).

Finally,

\[
-\frac{1}{\varepsilon} \int_0^r \int_{\Omega} \frac{\partial \varepsilon}{\partial \bar{\theta} + \varepsilon r} \mathbf{u}_x \cdot \nabla_x G \; dx \; dt
\]

\[
= -\frac{1}{\varepsilon} \int_0^r \int_{\Omega} \partial \varepsilon \mathbf{u}_x \cdot \nabla_x G \; dx \; dt + \frac{1}{\varepsilon} \int_0^r \int_{\Omega} \left( 1 - \frac{\bar{\theta}}{\bar{\theta} + \varepsilon r} \right) \partial \varepsilon \mathbf{u}_x \cdot \nabla_x G \; dx \; dt
\]

\[
= -\frac{1}{\varepsilon} \int_0^r \int_{\Omega} \partial \varepsilon \mathbf{u}_x \cdot \nabla_x G \; dx \; dt + \int_0^r \int_{\Omega} -\frac{r}{\bar{\theta} + \varepsilon r} \partial \varepsilon \mathbf{u}_x \cdot \nabla_x G \; dx \; dt
\]

\[
= -\frac{1}{\varepsilon} \int_0^r \int_{\Omega} \partial \varepsilon \mathbf{u}_x \cdot \nabla_x G \; dx \; dt + \int_0^r \int_{\Omega} \mathbf{r} \cdot \nabla_x G \; dx \; dt + O(\varepsilon).
\]

Consequently, abbreviating \( s_\varepsilon = s(\partial \varepsilon, \vartheta_\varepsilon) \), relation (6.16) can be rewritten as

\[
\int_{\Omega} E_\varepsilon \left( \partial \varepsilon, \vartheta_\varepsilon, \mathbf{u}_x \! \left| \! \begin{array}{c} \bar{\theta} + \varepsilon r, \bar{\vartheta} + \varepsilon T \end{array} \right. \! \right) (\tau, \cdot) \; dx
\]

\[
+ \int_0^r \int_{\Omega} \left( S(\bar{\vartheta}, D_x \mathbf{u}_x) - S(\bar{\vartheta}, D_x \mathbf{U}) \right) : \left( D_x \mathbf{u}_x - D_x \mathbf{U} \right) \; dx \; dt
\]

\[
+ \int_0^r \int_{\Omega} \left( \frac{\bar{\vartheta} + \varepsilon T}{\bar{\vartheta}^2} \kappa(\vartheta_\varepsilon) \nabla \vartheta_\varepsilon \cdot \nabla \vartheta_\varepsilon \right) \; dx \; dt - \int_0^r \int_{\Omega} \kappa(\vartheta_\varepsilon) \nabla \vartheta_\varepsilon \cdot \nabla x T \; dx \; dt
\]

\[
\leq -\frac{1}{\varepsilon} \int_0^r \int_{\Omega} \left( \partial \varepsilon(s_\varepsilon - s(\bar{\vartheta} + \varepsilon r, \bar{\vartheta} + \varepsilon T)) \partial_t T + \partial \varepsilon(s_\varepsilon - s(\bar{\vartheta} + \varepsilon r, \bar{\vartheta} + \varepsilon T)) \mathbf{u}_x \cdot \nabla x T \right) \; dx \; dt
\]

\[
- \int_0^r \int_{\Omega} \frac{\partial \varepsilon}{\partial \vartheta} \nabla x G \cdot \mathbf{U} \; dx \; dt + \int_0^r \int_{\Omega} r \nabla x G \cdot \mathbf{U} \; dx \; dt
\]

\[
+ C \int_0^r \int_{\Omega} E_\varepsilon \left( \partial \varepsilon, \vartheta_\varepsilon, \mathbf{u}_x \! \left| \! \begin{array}{c} \bar{\theta} + \varepsilon r, \bar{\vartheta} + \varepsilon T \end{array} \right. \! \right) \; dx \; dt + O(\varepsilon). \tag{6.19}\]
Step 4: In view of solenoidality \( \text{div}_x U = 0 \), we have

\[
- \int_0^T \int_\Omega \frac{\rho_e}{\varepsilon} \nabla_x G \cdot U \, dx \, dt + \int_0^T \int_\Omega r \nabla_x G \cdot U \, dx \, dt = - \int_0^T \int_\Omega \frac{(\rho_e - (\bar{\rho} + \varepsilon r))}{\varepsilon} U \cdot \nabla_x G \, dx \, dt. \tag{6.20}
\]

Consequently, we may use the bounds (5.20), (5.21) along with the convergences established in (6.1)–(6.6) to rewrite (6.19) in the form

\[
\int_\Omega E_\varepsilon \left( \rho_\varepsilon, \theta_\varepsilon, u_\varepsilon \big| \bar{\rho} + \varepsilon r, \bar{\theta} + \varepsilon \mathcal{T}, U \right) (\tau, \cdot) \, dx \\
+ \int_0^T \int_\Omega \left( S(\bar{\rho}, D_x u_\varepsilon) - S(\bar{\rho}, D_x U) \right) : \left( D_x u_\varepsilon - D_x U \right) \, dx \, dt \\
+ \int_0^T \int_\Omega \left( \frac{\bar{\theta} + \varepsilon \mathcal{T}}{\bar{\theta}^2} \right) \kappa(\theta_\varepsilon) \nabla_x \theta_\varepsilon \cdot \nabla_x \bar{\theta} \, dx \, dt - \int_0^T \int_\Omega \frac{\kappa(\bar{\theta})}{\bar{\theta}} \nabla_x \bar{\theta} \cdot \nabla_x \mathcal{T} \, dx \, dt \\
\leq - \frac{1}{\varepsilon} \int_0^T \int_\Omega \rho_\varepsilon \left[ (s(x_\varepsilon, \theta_\varepsilon) - s(\bar{\rho} + \varepsilon r, \bar{\theta} + \varepsilon \mathcal{T})) \right] \partial_\mathcal{T} \, dx \, dt \\
- \frac{1}{\varepsilon} \int_0^T \int_\Omega \rho_\varepsilon \left[ s(x_\varepsilon, \theta_\varepsilon) - s(\bar{\rho} + \varepsilon r, \bar{\theta} + \varepsilon \mathcal{T}) \right] u_\varepsilon \cdot \nabla_x \mathcal{T} \, dx \, dt \\
+ \frac{1}{\varepsilon} \int_0^T \int_\Omega \rho_\varepsilon (s(x_\varepsilon, \theta_\varepsilon) - s(\bar{\rho} + \varepsilon r, \bar{\theta} + \varepsilon \mathcal{T})) (U - u_\varepsilon) \cdot \nabla_x \mathcal{T} \, dx \, dt \\
+ \int_0^T \int_\Omega (r - \mathcal{A}) \nabla_x G \cdot U \, dx \, dt \\
+ C \int_0^T \int_\Omega E_\varepsilon \left( \rho_\varepsilon, \theta_\varepsilon, u_\varepsilon \big| \bar{\rho} + \varepsilon r, \bar{\theta} + \varepsilon \mathcal{T}, U \right) \, dx \, dt + O(\varepsilon). \tag{6.21}
\]

Step 5: Now we use the fact that \( \mathcal{T} \) solves the modified heat equation (6.10), specifically,

\[
\partial_\mathcal{T} + U \cdot \nabla_x \mathcal{T} = \frac{\bar{\theta} \alpha(\bar{\rho}, \bar{\theta})}{c_p(\bar{\rho}, \bar{\theta})} \nabla_x G \cdot U + \frac{\kappa(\bar{\theta})}{\bar{c}_p(\bar{\rho}, \bar{\theta})} \Delta_\mathcal{T} + \frac{1}{\bar{c}_p(\bar{\rho}, \bar{\theta})} \Lambda(t),
\]

\[
\Lambda = \frac{\bar{\theta} \alpha(\bar{\rho}, \bar{\theta})}{c_p(\bar{\rho}, \bar{\theta})} \frac{\partial \theta(\bar{\rho}, \bar{\theta})}{\partial \theta} \partial_\mathcal{T} \int_\Omega \mathcal{T} \, dx. \tag{6.22}
\]

Thus we may perform the limit in several integrals in (6.21) obtaining

\[
\int_\Omega E_\varepsilon \left( \rho_\varepsilon, \theta_\varepsilon, u_\varepsilon \big| \bar{\rho} + \varepsilon r, \bar{\theta} + \varepsilon \mathcal{T}, U \right) (\tau, \cdot) \, dx \\
+ \int_0^T \int_\Omega \left( S(\bar{\rho}, D_x u_\varepsilon) - S(\bar{\rho}, D_x U) \right) : \left( D_x u_\varepsilon - D_x U \right) \, dx \, dt \\
+ \int_0^T \int_\Omega \left( \frac{\bar{\theta} + \varepsilon \mathcal{T}}{\bar{\theta}^2} \right) \kappa(\theta_\varepsilon) \nabla_x \theta_\varepsilon \cdot \nabla_x \bar{\theta} \, dx \, dt - \int_0^T \int_\Omega \frac{\kappa(\bar{\theta})}{\bar{\theta}} \nabla_x \bar{\theta} \cdot \nabla_x \mathcal{T} \, dx \, dt
\]
In view of the Boussinesq relations (6.8), (6.11), the expression
\[ \frac{\partial s(\overline{\varrho}, \overline{v})}{\partial q} (\mathcal{R} - r) + \frac{\partial s(\overline{\varrho}, \overline{v})}{\partial \vartheta} (\mathcal{S} - \mathcal{T}) \] 
where we have used
\[ \frac{\partial \varrho}{\partial q} (\mathcal{R} - r) + \frac{\partial \varrho}{\partial \vartheta} (\mathcal{S} - \mathcal{T}) \]
is spatially homogeneous, meaning it depends on \( t \),
Similarly, we can rewrite the second integral on the right-hand side of (6.23) as
\[ \frac{\kappa(\overline{v})}{\bar{\varrho} c_p(\overline{\varrho}, \overline{v})} \Delta_s \mathcal{T} \]
dx dt
\[ + \int_0^\tau \int_\Omega \partial p(\overline{\varrho}, \overline{v}) \left[ \frac{\partial p(\overline{\varrho}, \overline{v})}{\partial \vartheta} \right]^{-1} (\mathcal{R} - r) + (\mathcal{S} - \mathcal{T}) \]
\[ \frac{\kappa(\overline{v})}{\bar{\varrho} c_p(\overline{\varrho}, \overline{v})} \Delta_s \mathcal{T} \]
dx dt
\[ + C \int_0^\tau \int_\Omega E_\varepsilon \left( q_\varepsilon, \vartheta_\varepsilon, u_\varepsilon \right) \overline{\varrho} + \varepsilon r, \overline{v} + \varepsilon \mathcal{T}, \mathcal{U} \] dx dt + \mathcal{O}(\varepsilon),
(6.23)
where we have used
\[ \frac{1}{\varepsilon} \int_0^\tau \int_\Omega q_\varepsilon (s(q_\varepsilon, \vartheta_\varepsilon) - s(\overline{\varrho} + \varepsilon r, \overline{v} + \varepsilon \mathcal{T})) (\mathcal{U} - u_\varepsilon) \cdot \nabla_s \mathcal{T} \] dx dt
\[ \lesssim \int_0^\tau \int_\Omega E_\varepsilon (q_\varepsilon, \vartheta_\varepsilon, u_\varepsilon \right) \overline{\varrho} + \varepsilon r, \overline{v} + \varepsilon \mathcal{T}, \mathcal{U} \] dx dt.

Now, we use
\[ \int_\Omega (r - \mathcal{R}) \] dx = 0
to rewrite the third integral on the right-hand side of (6.23) as
\[ - \int_0^\tau \int_\Omega \frac{\partial s(\overline{\varrho}, \overline{v})}{\partial q} (\mathcal{R} - r) + \frac{\partial s(\overline{\varrho}, \overline{v})}{\partial \vartheta} (\mathcal{S} - \mathcal{T}) \]
\[ \frac{1}{\bar{\varrho} c_p(\overline{\varrho}, \overline{v})} \Delta_s \mathcal{T} \]
dx dt
\[ = - \int_0^\tau \int_\Omega \frac{\partial p(\overline{\varrho}, \overline{v})}{\partial \vartheta} \left[ \frac{\partial p(\overline{\varrho}, \overline{v})}{\partial \vartheta} \right]^{-1} (\mathcal{R} - r) + (\mathcal{S} - \mathcal{T}) \]
\[ \frac{1}{c_p(\overline{\varrho}, \overline{v})} \Delta_s \mathcal{T} \] dx dt.
(6.24)
In view of the Boussinesq relations (6.8), (6.11), the expression
\[ \left[ \frac{\partial p(\overline{\varrho}, \overline{v})}{\partial \vartheta} \right]^{-1} (\mathcal{R} - r) + (\mathcal{S} - \mathcal{T}) \]
is spatially homogeneous, meaning it depends on \( t \) only.

Similarly, we can rewrite the second integral on the right-hand side of (6.23) as
\[ - \int_0^\tau \int_\Omega \frac{\partial s(\overline{\varrho}, \overline{v})}{\partial q} (\mathcal{R} - r) + \frac{\partial s(\overline{\varrho}, \overline{v})}{\partial \vartheta} (\mathcal{S} - \mathcal{T}) \]
\[ \frac{\kappa(\overline{v})}{\bar{\varrho} c_p(\overline{\varrho}, \overline{v})} \Delta_s \mathcal{T} \]
dx dt
\[ = - \int_0^\tau \int_\Omega \frac{\partial s(\overline{\varrho}, \overline{v})}{\partial q} \left[ (\mathcal{R} - r) + \frac{\partial p(\overline{\varrho}, \overline{v})}{\partial \vartheta} \right]^{-1} (\mathcal{S} - \mathcal{T}) \]
\[ \frac{\kappa(\overline{v})}{c_p(\overline{\varrho}, \overline{v})} \Delta_s \mathcal{T} \] dx dt
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\[ + \int_{0}^{T} \int_{\Omega} \left( \frac{\partial s(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1} (\overline{x} - T) - \frac{\partial s(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} (\overline{x} - T) \right) \frac{\kappa(\overline{\vartheta})}{c_{p}(\overline{\varphi}, \overline{\vartheta})} \Delta x \right) dx \, dt. \]

(6.25)

Similarly to the above, the quantity

\[
\left[ (R - r) + \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1} (\overline{x} - T) \right]
\]

is independent of \( x \).

Now, integrating equation (6.22) in \( x \) we obtain the identity

\[
\left[ \left( \frac{\partial \alpha(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1} - \frac{1}{\kappa(\overline{\vartheta})} \right] [\Omega] \Lambda(t) = \int_{\partial \Omega} \frac{\kappa(\overline{\vartheta})}{\nu c_{p}(\overline{\varphi}, \overline{\vartheta})} \nabla x T \cdot n \, d\sigma. \]

(6.26)

Finally, plugging (6.20) in (6.25) we can compute the sum of (6.24) with the first integral in (6.25) obtaining

\[
- \int_{\Omega} \frac{\partial s(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \left[ \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1} (R - r) + (\overline{x} - T) \right] \frac{1}{c_{p}(\overline{\varphi}, \overline{\vartheta})} \Lambda(t) \, dx
- \int_{\Omega} \frac{\partial s(\overline{\varphi}, \overline{\vartheta})}{\partial q} \left[ (R - r) + \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1} (\overline{x} - T) \right] \times
\times \left[ \frac{\partial}{\partial \vartheta} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1} - \frac{1}{c_{p}(\overline{\varphi}, \overline{\vartheta})} \right] \Lambda(t) \, dx. \]

(6.27)

Now, in accordance with Gibbs' relation and the definitions of \( \alpha \) and \( c_{p} \) in (1.15),

\[
- \frac{\partial s(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1} \frac{c_{p}(\overline{\varphi}, \overline{\vartheta})}{\overline{\varphi}} \overline{\varphi} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \right)^{-1} - 1
= - \frac{1}{\overline{\varphi}} \frac{\partial e(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1}
- \frac{\partial s(\overline{\varphi}, \overline{\vartheta})}{\partial q} \left[ \frac{\overline{\varphi}}{\partial \vartheta} \left( \frac{\partial e(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \right) + \frac{1}{\overline{\varphi}} \frac{\partial e(\overline{\varphi}, \overline{\vartheta})}{\partial q} \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1} - 1 \right]
= - \frac{1}{\overline{\varphi}} \frac{\partial e(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial q} \right)^{-1}
- \frac{\partial s(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \left[ \frac{\partial e(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \left( \frac{\partial p(\overline{\varphi}, \overline{\vartheta})}{\partial \vartheta} \right)^{-1} \right] = 0. \]

(6.28)
Thus, the coefficient multiplying $\mathcal{R} - r$ vanishes. By the same token, we deduce that the coefficient multiplying $\Xi - T$ vanishes.

Next, we handle the second integral in (6.25). Using Gibbs’ relation and the constitutive relations obtained in Section 2.2, specifically,

$$\frac{\partial s(\bar{\varphi}, \bar{\vartheta})}{\partial \vartheta} = -\frac{1}{\bar{\vartheta}} \frac{\partial p(\bar{\varphi}, \bar{\vartheta})}{\partial \vartheta}. $$

Thus, we get

$$\left[ \frac{\partial s(\bar{\varphi}, \bar{\vartheta})}{\partial \vartheta} \frac{\partial p(\bar{\varphi}, \bar{\vartheta})}{\partial \vartheta} \left( \frac{\partial p(\bar{\varphi}, \bar{\vartheta})}{\partial \vartheta} \right)^{-1} \right] \frac{\kappa(\bar{\vartheta})}{c_p(\bar{\varphi}, \bar{\vartheta})} = -\frac{\kappa(\bar{\vartheta})}{\bar{\vartheta}}. \quad (6.29)$$

Finally, we regroup terms containing $\nabla_x G$:

$$- \int_{\Omega} \bar{\vartheta} \left( \frac{\partial s(\bar{\varphi}, \bar{\vartheta})}{\partial \vartheta} (\mathcal{R} - r) + \frac{\partial s(\bar{\varphi}, \bar{\vartheta})}{\partial \vartheta} (\Xi - T) \right) \frac{\bar{\vartheta}(\bar{\varphi}, \bar{\vartheta})}{c_p(\bar{\varphi}, \bar{\vartheta})} \nabla_x G \cdot \mathbf{U} \, dx + \int_{\Omega} (r - \mathcal{R}) \nabla_x G \cdot \mathbf{U} \, dx = \int_{\Omega} \nabla_x (r - \mathcal{R}) G \cdot \mathbf{U} \, dx. \quad (6.30)$$

Using Boussinesq relation, we deduce

$$\bar{\vartheta} \left( \frac{\partial s(\bar{\varphi}, \bar{\vartheta})}{\partial \vartheta} \nabla_x (r - r) + \frac{\partial s(\bar{\varphi}, \bar{\vartheta})}{\partial \vartheta} \nabla_x (\Xi - T) \right) \frac{\bar{\vartheta}(\bar{\varphi}, \bar{\vartheta})}{c_p(\bar{\varphi}, \bar{\vartheta})} \nabla_x (r - r) \frac{\bar{\vartheta}(\bar{\varphi}, \bar{\vartheta})}{c_p(\bar{\varphi}, \bar{\vartheta})} = -\nabla_x (r - r). \quad (6.31)$$

Thus, rearranging terms and using $\Xi|\partial \Omega = T|\partial \Omega$, (6.23) reduces to the desired inequality

$$\int_{\Omega} E_{\varepsilon} \left( \varrho_{\varepsilon}, \vartheta_{\varepsilon}, \mathbf{u}_{\varepsilon} \right) (\bar{\vartheta} + \varepsilon r, \bar{\vartheta} + \varepsilon T, \mathbf{U}) \, d\tau, \quad dx$$

$$+ \int_0^T \int_{\Omega} \left( \mathcal{S}(\bar{\vartheta}, \mathbb{D}_x \mathbf{u}_{\varepsilon}) - \mathcal{S}(\bar{\vartheta}, \mathbb{D}_x \mathbf{U}) \right) : \left( \mathbb{D}_x \mathbf{u}_{\varepsilon} - \mathbb{D}_x \mathbf{U} \right) \, dx \, dt$$
\begin{align*}
  &+ \int_0^\tau \int_\Omega \frac{\kappa(\overline{v})}{\overline{v}} \left|\nabla_x \left( \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right) - \nabla_x \vartheta \right|^2 \, dx \, dt \\
  \lesssim & \int_0^\tau \int_\Omega E_\varepsilon \left( \vartheta_\varepsilon, \vartheta_\varepsilon, u_\varepsilon, \overline{\vartheta} + \varepsilon r, \overline{\vartheta} + \varepsilon \vartheta, U \right) \, dx \, dt + O(\varepsilon).
\end{align*}

Using Grönwall lemma and letting \( \varepsilon \to 0 \) we obtain the conclusion claimed in Theorem 4.1.

References

[1] A. Abbatiello and E. Feireisl. The Oberbeck–Boussinesq system with non–local boundary conditions. 2022. Preprint.

[2] Barletta, A and Celli, M and Rees, DAS. On the use and misuse of the Oberbeck-Boussinesq approximation. 2022. arXiv preprint arXiv:2202.10981.

[3] F. Belgiorno. Notes on the third law of thermodynamics, I. J. Phys. A, 36:8165–8193, 2003.

[4] F. Belgiorno. Notes on the third law of thermodynamics, ii. J. Phys. A, 36:8195–8221, 2003.

[5] A.S. Bormann. The onset of convection in the Rayleigh-Bénard problem for compressible fluids. Continuum Mech. Thermodyn, 13:9–23, 2001.

[6] N. Chaudhuri and E. Feireisl. Navier–Stokes–Fourier system with Dirichlet boundary conditions. arXiv preprint No. 2106.05315, 2021.

[7] Y. Chen and L. Liu. Global blow-up for a localized nonlinear parabolic equation with a nonlocal boundary condition. J. Math. Anal. Appl., 384(2):421–430, 2011.

[8] W. A. Day. Extensions of a property of the heat equation to linear thermoelasticity and other theories. Quart. Appl. Math., 40(3):319–330, 1982/83.

[9] E. Feireisl and A. Novotný. The Oberbeck-Boussinesq approximation as a singular limit of the full Navier-Stokes-Fourier system. J. Math. Fluid Mech., 11(2):274–302, 2009.

[10] E. Feireisl and A. Novotný. Singular limits in thermodynamics of viscous fluids. Advances in Mathematical Fluid Mechanics. Birkhäuser/Springer, Cham, 2017. Second edition.

[11] E. Feireisl and A. Novotný. Mathematics of open fluid systems. Birkhäuser–Verlag, Basel, 2022.

[12] A. Friedman. Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions. Quart. Appl. Math., 44(3):401–407, 1986.

[13] J. Fröhlich, P. Laure, and R. Peyret. Large departures from Boussinesq approximation in the Rayleigh-Bénard problem. Physics of Fluids A: Fluid Dynamics, 4:1355, 1992.
[14] A. L. Gladkov and A. I. Nikitin. On the existence of global solutions of a system of semilinear parabolic equations with nonlinear nonlocal boundary conditions. *Differ. Equ.*, **52**(4):467–482, 2016. Translation of *Differ. Uravn.* **52** (2016), no. 4, 490–505.

[15] R. Klein, N. Botta, T. Schneider, C.D. Munz, S. Roller, A. Meister, L. Hoffmann, and T. Sonar. Asymptotic adaptive methods for multi-scale problems in fluid mechanics. *J. Engrg. Math.*, **39**:261–343, 2001.

[16] K. Maruyama. Rational derivation of the Boussinesq approximation. Zenodo. https://doi.org/10.5281/zenodo.3757250

[17] K. A. Nadolin. Boussinesq approximation in the Rayleigh-Benard problem. *Izv. Ross. Akad. Nauk Mekh. Zhidk. Gaza*, **5**:3–10, 1995.

[18] C. V. Pao. Asymptotic behavior of solutions of reaction-diffusion equations with nonlocal boundary conditions. volume **88**, pages 225–238. 1998. Positive solutions of nonlinear problems.

[19] R. Kh. Zeytounian. Joseph Boussinesq and his approximation: a contemporary view. *C.R. Mecanique*, **331**:575–586, 2003.