A description of characters on the infinite wreath product.

A.V. Dudko, N.I. Nessonov

Abstract

Let $S\infty$ be the infinity permutation group and $\Gamma$ an arbitrary group. Then $S\infty$ admits a natural action on $\Gamma\infty$ by automorphisms, so one can form a semidirect product $\Gamma\infty \times S\infty$, known as the wreath product $\Gamma \wr S\infty$ of $\Gamma$ by $S\infty$. We obtain a full description of unitary $II_1$–factor-representations of $\Gamma \wr S\infty$ in terms of finite characters of $\Gamma$. Our approach is based on extending Okounkov’s classification method for admissible representations of $S\infty \times S\infty$. Also, we discuss certain examples of representations of type $III$, where the modular operator of Tomita-Takesaki expresses naturally by the asymptotic operators, which are important in the characters-theory of $S\infty$.

1 Introduction

1.1. A definition of the wreath product. Let $\mathbb{N}$ stand for the natural numbers. A bijection $s : \mathbb{N} \to \mathbb{N}$ is called finite if the set $\{i \in \mathbb{N} | s(i) \neq i\}$ is finite. Define $S\infty$ as the group of all finite bijections $\mathbb{N} \to \mathbb{N}$ and set $S_n = \{s \in S\infty | s(i) = i \forall i > n\}$. For every group $\Gamma$, an element of $\Gamma^n$ can always be written as a sequenced collection $[\gamma_k]_{k=1}^n = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, where $\gamma_k \in \Gamma$. Let $e$ be the unit of $\Gamma$. For any $n > 1$ we identify the element $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}) \in \Gamma^{n-1}$ with $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, e) \in \Gamma^n$ and set $\Gamma e^\infty = \lim_{n \to -\infty} \Gamma^n$. One can view $\Gamma e^\infty$ as a group of infinite sequenced collections $[\gamma_k]_{k=1}^\infty$ such that there are finitely many elements $\gamma_k$ not equal to $e$. The wreath product $\Gamma \wr S_n$ is the semidirect product $\Gamma^n \rtimes S_n$ for the natural permutation action of $S_n$ on $\Gamma^n$ (see [4]). In the same way, we define the group $\Gamma \wr S\infty = \Gamma e^\infty \rtimes S\infty$. $\Gamma \wr S\infty$ can be also viewed as the inductive limit $\lim_{n \to -\infty} \Gamma \wr S_n$. Using the embeddings $\gamma \in \Gamma^n \to (\gamma, id) \in \Gamma \wr S_n$ and $s \in S_n \to (e(n), s) \in \Gamma \wr S_n$, where $e(n) = (e, e, \ldots, e)$ and $id$ is the identical

*Supported by the CRDF-grant UM1-2546
bijection, we may identify \( \Gamma^n \) and \( \mathcal{S}_n \) with the corresponding subgroups in \( \Gamma \wr \mathcal{S}_n \). If \( \Gamma \) is a topological group, then we equip \( \Gamma^n \) with the natural product-topology. Furthermore, we will always consider \( \Gamma^\infty \) as a topological group with the inductive limit topology. As a set, \( \Gamma \wr \mathcal{S}_\infty \) is just \( \Gamma^\infty \times \mathcal{S}_\infty \). Therefore, we equip \( \Gamma \wr \mathcal{S}_\infty \) with the product-topology, considering \( \mathcal{S}_\infty \) as a discrete topological space.

### 1.2. The Results.

In this paper we give a full classification of indecomposable characters (see Definitions (3) – (4)) on \( \Gamma \wr \mathcal{S}_\infty \) (Theorem 7). Our approach is based on the semigroup method of Olshanski [7] and the ideas of Okounkov used in the study of admissible representations of the groups related to \( \mathcal{S}_\infty \) [2], [3]. We have noticed that two double cosets containing the transposition or \( \gamma \in \Gamma \) are commutated, as the elements of Olshanski semigroup. (see Fig. 8, p.23 and Lemma 21). This observation enables one to develop Okounkov’s method for the group \( \mathcal{S}_\infty \wr \Gamma \) (see Section 5). In Section 3 we discuss certain examples of representations of type III. The corresponding positive definite functions (p.d.f.) \( \varphi \) are not characters, but the following holds:

\[
\varphi(sg) = \varphi(gs) \text{ for all } g \in \mathcal{S}_\infty \wr \Gamma \text{ and } s \in \mathcal{S}_\infty. \quad (1.1)
\]

Hence the restriction \( \varphi|_{\mathcal{S}_\infty} \) is a character. At that, the Okounkov’s asymptotic operators (see (4.7)) are naturally connected to the Tomita-Takesaki modular operator (see subsection 3.3). In fact, this observation is common for p.d.f. with the property (1.1). For those, we are going to produce a complete classification in a subsequent paper.

### 1.3. The basic definition and the conjugate classes.

Let \( \mathcal{H} \) be a Hilbert space, \( \mathcal{B}(\mathcal{H}) \) an algebra of all bounded operators in \( \mathcal{H} \), and \( \mathcal{I}_\mathcal{H} \) the identity operator in \( \mathcal{H} \). We denote by \( \mathcal{U}(\mathcal{H}) \) the unitary subgroup in \( \mathcal{B}(\mathcal{H}) \). By a unitary representation of the topological group \( G \) we always mean a continuous homomorphism of \( G \) into \( \mathcal{U}(\mathcal{H}) \), where \( \mathcal{U}(\mathcal{H}) \) is equipped with the strong operator topology.

**Definition 1.** A unitary representation \( \pi : G \rightarrow \mathcal{U}(\mathcal{H}) \) of \( G \) is called a factor-representation if the \( W^* \)-algebra \( \pi(G)'' \) generated by the operators \( \pi(g) \ (g \in G) \), is a factor.

**Definition 2.** A unitary representation \( \pi \) is called a factor-representation of finite type if \( \pi(G)'' \) is a factor of type II_1.
Let $\mathcal{M}$ be a factor of type $II_1$ and $\mathcal{M}$ a subalgebra of $\mathcal{B}(\mathcal{H})$. If $\pi(G) \subset U(\mathcal{M}) = \mathcal{M} \cap U(\mathcal{H})$ and $tr_\mathcal{M}$ is the unique normal, normalized ($tr(I) = 1$) trace on $\mathcal{M}$, then it determines a character $\phi_\pi^\mathcal{M}$ of $G$ by $\phi_\pi^\mathcal{M}(g) = tr_\mathcal{M}(\pi(g))$.

**Definition 3.** A continuous function $\phi$ on $G$ is called a character if it satisfies the following properties:

(a) $\phi$ is central, that is, $\phi(g_1g_2) = \phi(g_2g_1) \forall g_1, g_2 \in G$;

(b) $\phi$ is positive definite, that is, for all $g_1, g_2, \ldots, g_n$ the matrix $[\phi(g_jg_k^{-1})]_{j,k=1}^n$ is non-negatively definite;

(c) $\phi$ is normalized, that is, $\phi(e_G) = 1$, where $e_G$ is the unit of $G$.

**Definition 4.** A character $\phi$ is called indecomposable if the group representation corresponding to $\phi$ (according to the GNS construction) is a factor-representation.

In this paper we obtain a complete description of indecomposable characters on $\Gamma \wr S_\infty$ in the case when $\Gamma$ is a separable topological group.

First, let us describe the conjugacy classes in $\Gamma \wr S_\infty$. Recall that the conjugacy classes in $S_\infty$ are parameterized by partitions $\lambda$, that is by unordered $\infty$-tuples $\lambda_1, \lambda_2, \ldots$ of natural numbers such that there are finitely many elements $\lambda_k$ not equal to 1. Namely, $\lambda_1, \lambda_2, \ldots$ are the orders of cycles of a permutation $s \in S_\infty$. Furthermore, an element $\Gamma \wr S_\infty$ can be written as a product of an element of $S_\infty$ and an element of $\Gamma_e \wr S_\infty$, and the commutation rule between these two kinds of elements is as follows:

$$s \cdot \gamma = s \cdot (\gamma_1, \gamma_2, \ldots) = (\gamma_{s^{-1}(1)}, \gamma_{s^{-1}(2)}, \ldots) \cdot s,$$

(1.2)

where $s \in S_\infty$, $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e$. Let $\mathbb{N}/s$ be the set of orbits of $s$ on $\mathbb{N}$. Note that for $p \in \mathbb{N}/s$ the permutation $s_p$ given by

$$s_p(k) = \begin{cases} s(k) & \text{if } k \in p \\ k & \text{otherwise} \end{cases},$$

is a cycle of order $|p|$, where $|p|$ stand for the cardinality of $p$. For $\gamma = (\gamma_1, \gamma_2, \ldots) \in \Gamma_e$ define the element $\gamma(p) = (\gamma_1(p), \gamma_2(p), \ldots) \in \Gamma_e$ as follows:

$$\gamma_k(p) = \begin{cases} \gamma_k & \text{if } k \in p \\ e & \text{otherwise}. \end{cases}$$

(1.3)

Thus, using (1.2), we have

$$s \cdot \gamma = \prod_{p \in \mathbb{N}/s} s_p \cdot \gamma(p).$$

(1.4)
Denote by $c_G(g)$ the conjugacy class of $g \in G$. Finally, for $g = s \cdot \gamma \in \Gamma \wr \mathcal{S}_\infty$ define the invariant $i(g)$ given by unordered $\infty$-tuples of pairs 
\[ \{ [p], c_p \left( \gamma_k \cdot \gamma_{s(k)} \cdot \cdots \gamma_{s(l)}(k) \cdots \gamma_{s(|p|-1)}(k) \} \}_{k \in \mathbb{N}, p \in \mathbb{N}/s}, \]
where $s(l)$ is $l$-th iteration of $s$. The following statement can be easily proved.

**Proposition 5.** Let $g_1$ and $g_2$ be elements of $\Gamma \wr \mathcal{S}_\infty$. Then $c(g_1) = c(g_2)$ if and only if $i(g_1) = i(g_2)$.

### 1.4. The multiplicativity.

The following claim gives a useful characterization of the class of indecomposable characters:

**Proposition 6.** The following assumptions on a character $\phi$ of $\Gamma \wr \mathcal{S}_\infty$ are equivalent:

(a) $\phi$ is indecomposable;

(b) $\phi(g) = \prod_{p \in \mathbb{N}/s} \phi(s_p \cdot \gamma(p))$ for any $g = s \cdot \gamma = \prod_{p \in \mathbb{N}/s} s_p \cdot \gamma(p)$ (see [14]).

**Proof.** To prove the proposition, we consider the elements $g = s \cdot \gamma$ and $g' = s' \cdot \gamma'$ of $\Gamma \wr \mathcal{S}_\infty$ satisfying the following condition:

\[ \left( \bigcup_{p \in \{ q \in \mathbb{N}/s \mid |q| > 1 \}} p \bigcup \{ k \mid \gamma_k = e \} \right) \subset \left( \bigcup_{p \in \{ q \in \mathbb{N}/s' \mid |q| = 1 \}} p \bigcap \{ k \mid \gamma'_k = e \} \right). \]

Then there exists a sequence $\{ s_n \}_{n \in \mathbb{N}} \subset \mathcal{S}_\infty$ such that

\[ s_n \cdot g = g \cdot s_n \text{ and } s_n \cdot g' s_n^{-1} \cdot h = h \cdot s_n \cdot g' s_n^{-1} \text{ for all } h \in \Gamma \wr \mathcal{S}_n. \quad (1.5) \]

Suppose now that (a) holds. Using the GNS-construction, we produce the representation $\pi_\phi$ of $\Gamma \wr \mathcal{S}_\infty$ which acts in a Hilbert space $\mathcal{H}_\phi$ with a cyclic vector $\xi_\phi$ such that

\[ \phi(g) = (\pi_\phi(g) \xi_\phi, \xi_\phi). \]

Let $A = w - \lim_{n \to \infty} \pi_\phi(s_n \cdot g' s_n^{-1})$ be a limit of the sequence $\pi_\phi(s_n \cdot g' s_n^{-1})$ in the weak operator topology. Using [15], we deduce by Definition 3 that $A = aI$, where $I$ is the identity operator in $\mathcal{H}_\phi$ and $a$ a complex number. Therefore,

\[ \phi(g \cdot g') = \lim_{n \to \infty} \phi(s_n \cdot g' \cdot s_n^{-1}) = \phi(g) \cdot \lim_{n \to \infty} \phi(s_n \cdot g' \cdot s_n^{-1}) = \phi(g) \cdot \phi(g'). \]

4
Thus (b) follows from (a).

Conversely, suppose that (b) holds. For any subset \( S \) of \( \mathcal{B}(\mathcal{H}) \), define its commutant as follows:

\[
S' = \{ T \in \mathcal{B}(\mathcal{H}) \mid ST = TS \text{ for all } S \in S \}.
\]

If \( \pi_\phi(\Gamma \downarrow \mathfrak{S}_\infty) \cap \pi_\phi(\Gamma \downarrow \mathfrak{S}_\infty)' = \mathbb{Z} \) is larger than the scalars, then it contains a pair of orthogonal projections \( E \) and \( F \) with the properties:

\[
\phi(E) \neq 0, \phi(F) \neq 0 \text{ and } E \cdot F = 0.
\] (1.6)

By the von Neumann Double Commutant Theorem, for any \( \varepsilon > 0 \) there exist \( g_k^E, g_k^F \in \Gamma \downarrow \mathfrak{S}_n \subset \Gamma \downarrow \mathfrak{S}_\infty \) (\( n < \infty \)) and complex numbers \( c_k^E, c_k^F \) (\( k = 1, 2, \ldots, N < \infty \)) such that

\[
\left\| \sum_{k=1}^{N} c_k^E \pi_\phi( g_k^E ) \xi_\phi - E \xi_\phi \right\| < \varepsilon \phi(E),
\]

\[
\left\| \sum_{k=1}^{N} c_k^F \pi_\phi( g_k^F ) \xi_\phi - F \xi_\phi \right\| < \varepsilon \phi(F).
\] (1.7)

Consider the bijection

\[
\tau(j) = \begin{cases} 
  j + n & \text{if } j \leq n, \\
  j - n & \text{if } n < j \leq 2n, \\
  j & \text{otherwise.}
\end{cases}
\]

By Definition (3), use (1.7) to obtain

\[
\left\| \sum_{k=1}^{N} c_k^E \pi_\phi( \tau g_k^E \tau ) \xi_\phi - E \xi_\phi \right\| < \varepsilon \phi(E),
\]

\[
\left\| \sum_{k=1}^{N} c_k^F \pi_\phi( \tau g_k^F \tau ) \xi_\phi - F \xi_\phi \right\| < \varepsilon \phi(F).
\] (1.8)

Now, using (b), (1.6), (1.7) and (1.8), we have

\[
\varepsilon \sqrt{\phi(E) \phi(F)} \left( \varepsilon \sqrt{\phi(E) \phi(F)} + \sqrt{\phi(E)} + \sqrt{\phi(F)} \right)
\]

\[>
\left| \left( \sum_{k=1}^{N} c_k^E \pi_\phi( \tau g_k^E \tau ) \cdot \sum_{k=1}^{N} c_k^F \pi_\phi( g_k^F ) \xi_\phi, \xi_\phi \right) \right|
\]

\[=
\left| \left( \sum_{k=1}^{N} c_k^E \pi_\phi( \tau g_k^E \tau ) \xi_\phi, \xi_\phi \right) \cdot \left( \sum_{k=1}^{N} c_k^F \pi_\phi( \tau g_k^F ) \xi_\phi, \xi_\phi \right) \right|
\]

\[>
\phi(E) \phi(F) (\varepsilon + 1)^2.
\]
Hence
\[
\varepsilon > \left[ 1 - \frac{\sqrt{\phi(F)}}{\sqrt{\phi(E)}} + \frac{1 - \sqrt{\phi(E)}}{\sqrt{\phi(F)}} \right]^{-1}.
\]
Then, comparing this to (1.6), we get a contradiction. 

1.5. The main result. In [5], E. Thoma obtained the following remarkable description of all indecomposable characters of \( \mathcal{S}_\infty \). The characters of \( \mathcal{S}_\infty \) are labeled by pairs of non-increasing positive sequences \( \{\alpha_k\}, \{\beta_k\} \) \( (k \in \mathbb{N}) \) such that
\[
\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k \leq 1. \tag{1.9}
\]
The value of the corresponding character on a permutation with a single cycle of length \( l \) is
\[
\sum_{k=1}^{\infty} \alpha_k^l + (-1)^{l-1} \sum_{k=1}^{\infty} \beta_k^l.
\]
Its value on a permutation with several disjoint cycles equals to the product of values on each cycle.

Here is our main result.

**Theorem 7.** Let \( \phi \) be an indecomposable character of \( \Gamma \wr \mathcal{S}_\infty \). Then there exist a representation \( \varphi^0 \) of \( \Gamma \) of finite type, two non-increasing sequences of positive numbers \( \{\alpha_k\}, \{\beta_k\} \) \( (k \in \mathbb{N}) \), and two sequences \( \{\varphi_{\alpha}^k\} \) and \( \{\varphi_{\beta}^k\} \) of finite-dimensional irreducible representations of \( \Gamma \) such that for \( g = s \cdot \gamma \in \Gamma \wr \mathcal{S}_\infty \) (see (1.2) – (1.3)) one has
\[
\phi(g) = \prod_{p \in \mathbb{N}/s} \left\{ \delta_p \cdot \left[ 1 - \sum_{k \in \mathbb{N}} \left( \alpha_k \cdot \dim \varphi_{\alpha}^k + \beta_k \cdot \dim \varphi_{\beta}^k \right) \right] \cdot \prod_{j \in p} tr_0(\gamma_j) 
+ \sum_{k=1}^{\infty} \left[ \alpha_k^{|p|} • Tr_{\alpha_k}(\tilde{\gamma}(p)) + (-1)^{|p|-1} \beta_k^{|p|} • Tr_{\beta_k}(\tilde{\gamma}(p)) \right] \right\}, \tag{1.10}
\]
where \( \tilde{\gamma}(p) = \gamma_k \cdot \gamma_{s(k)} \cdots \gamma_{s(1)(k)} \cdots \gamma_{s(|p|-1)(k)} \) \( (k \in p) \); \( Tr_{\alpha_k}, Tr_{\beta_k} \) are characters corresponding to the representations \( \varphi_{\alpha}^k, \varphi_{\beta}^k \), \( tr_0 \) is the normalized character of the representation \( \varphi^0 \);
\[
\delta_p = \begin{cases} 
1 & \text{if } |p| = 1, \\
0 & \text{if } |p| > 1.
\end{cases}
\]
and \( \sum_k (\alpha_k \cdot \dim \varrho^\alpha_k + \beta_k \cdot \dim \varrho^\beta_k) \leq 1. \)

Now we formulate the main result in the case when \( \Gamma \) is a \textit{locally compact} abelian group.

Let \( \hat{\Gamma} \) stand for the \textit{dual} group of \( \Gamma \).

**Theorem 8.** Let \( \varphi \) be an indecomposable character of \( \Gamma \rtimes \mathbb{S}_\infty \). There exist a probability measure \( \mu \) on \( \hat{\Gamma} \), two non-increasing sequences of positive numbers \( \{\alpha_k\}, \{\beta_k\} \) \( (k \in \mathbb{N}) \), two sequences \( \{\hat{\alpha}_k\} \) and \( \{\hat{\beta}_k\} \) of elements of \( \hat{\Gamma} \), such that for \( g = s \cdot \gamma \in \Gamma \rtimes \mathbb{S}_\infty \) (see (1.2) – (1.3))

\[
\varphi(g) = \prod_{p \in \mathbb{N}_0} \left\{ \delta_p \left[ 1 - \sum_{k \in \mathbb{N}} (\alpha_k + \beta_k) \right] \cdot \int_{\hat{\Gamma}} \left( \prod_{j \in \mathbb{N}} \gamma_j(p) \right) \mathrm{d}\mu \right\} \\
+ \sum_{k=1}^{\infty} \left[ \alpha_k |p| \cdot \hat{\alpha}_k \left( \prod_{j \in \mathbb{N}} \gamma_j(p) \right) + (-1)^{|p|-1} \beta_k |p| \cdot \hat{\beta}_k \left( \prod_{j \in \mathbb{N}} \gamma_j(p) \right) \right],
\]

where \( \{\alpha_k\} \) and \( \{\beta_k\} \) satisfy (1.9).

## 2 Realizations of \( II_1 \)-factor-representations.

A complete family of \( II_1 \)-factor-representations of \( G \rtimes \Gamma \) can be constructed using the Vershik-Kerov [8] or Olshanski [7] realizations, found for the \( II_1 \)-factor-representations of the infinite symmetric group \( \mathbb{S}_\infty \). We follow the approach developed by Olshanski as it leads to less spadework.

### 2.1. A construction of representations. Let \( \{\alpha_k\}_{k \in \mathbb{N}}, \{\beta_k\}_{k \in \mathbb{N}} \) be two finite or infinite sets of numbers from \( (0,1) \) and suppose that \( \varrho^\alpha_k \) and \( \varrho^\beta_k \) are unitary irreducible finite dimensional representations of \( \Gamma \) that act in the Hilbert spaces \( \mathcal{H}^\alpha_k \) and \( \mathcal{H}^\beta_k \) respectively. Assume that

\[
\sum_k \alpha_k \cdot \dim \varrho^\alpha_k + \sum_k \beta_k \cdot \dim \varrho^\beta_k \leq 1.
\]

Set

\[
\delta = 1 - \sum_k \alpha_k \cdot \dim \varrho^\alpha_k - \sum_k \beta_k \cdot \dim \varrho^\beta_k.
\]
Let $\mathcal{H}^0$ stand for the (Hilbert) space of a unitary representation $\varrho^0$ of $\Gamma$ of finite type. We may assume without loss of generality that there exists a cyclic and separating unit vector $\xi^{(0)}$ for the pair $(\varrho^0(\Gamma), \mathcal{H}^0)$. To rephrase, $[\varrho^0(\Gamma)\xi^{(0)}] = [\varrho^0(\Gamma')\xi^{(0)}] = \mathcal{H}^0$, where $[\varrho^0(\Gamma)\xi^{(0)}]$ is the subspace generated by $\varrho^0(\Gamma)\xi^{(0)}$. Furthermore, the formula $tr^0(\gamma) = (\varrho^0(\gamma)\xi^{(0)}, \xi^{(0)})_{\mathcal{H}^0}$ determines a character on $\Gamma$. Thus, we associate a unitary representation $(\varrho^0)^{(2)}$ of $\Gamma \times \Gamma$ to the representation $\varrho^0$. Namely, $(\varrho^0)^{(2)}$ is defined as follows:

$$(\varrho^0)^{(2)}((\gamma_1, \gamma_2)) \left( \varrho^0(\gamma)\xi^{(0)} \right) = \varrho^0(\gamma_1)\varrho^0(\gamma)\varrho^0(\gamma_2^{-1})\xi^{(0)}.$$ 

Denote by $(\varrho^{0k}, \mathcal{H}^{0k}, \xi^{(0k)})$ the k-th copy of the triplet $(\varrho^0, \mathcal{H}^0, \xi^{(0)})$.

Let $\left\{ e_{(\alpha_k)}^j \right\}_{1 \leq j \leq \text{dim} \mathcal{H}^{\alpha_k}}$ be an orthonormal basis in $\mathcal{H}^{\alpha_k}$. Define the matrix elements of $\varrho^{\alpha_k}$ as follows:

$$\varrho^{\alpha_k}_{jl}(\gamma) = (\varrho^{\alpha_k}(\gamma)e_{(\alpha_k)}^j, e_{(\alpha_k)}^l),$$

where bar denotes the complex conjugation.

Let

$$\mathbf{H} = \left( \bigoplus_k \mathcal{H}^{\alpha_k} \bigoplus_k \mathcal{H}^{\beta_k} \right) \otimes \left( \bigoplus_k \mathcal{H}^{\alpha_k} \bigoplus_k \mathcal{H}^{\beta_k} \right) \bigoplus \mathcal{H}^{0k}$$

and

$$\eta^{(m)} = \sum_k \alpha_k \left( \sum_j e_{(\alpha_k)}^j \otimes e_{(\alpha_k)}^j \right) + \sum_k \beta_k \left( \sum_j e_{(\beta_k)}^j \otimes e_{(\beta_k)}^j \right) + \sqrt{\delta} \xi^{(0m)}.$$ 

Define the unitary representations $\varrho$ and $\bar{\varrho}$ of $\Gamma$ in $\mathbf{H}$ as follows

$$\varrho = \left( \bigoplus_k \varrho^{\alpha_k} \bigoplus_k \varrho^{\beta_k} \right) \otimes \left( \bigoplus_k \mathbb{I} \bigoplus_k \mathbb{I} \right) \otimes \varrho^{0k}$$

$$\bar{\varrho} = \left( \bigoplus_k \mathbb{I} \bigoplus_k \mathbb{I} \right) \otimes \left( \bigoplus_k \varrho^{\alpha_k} \bigoplus_k \varrho^{\beta_k} \right) \otimes \varrho^{0k}, \text{ where } \varrho^{0k}(\gamma) = (\varrho^0)^{(2)}((e_\Gamma, \gamma)).$$

We identify $\mathcal{H}^{\alpha_k} \otimes \mathcal{H}^{\alpha_k}, \mathcal{H}^{\beta_k} \otimes \mathcal{H}^{\beta_k}$, and $\mathcal{H}^{0k}$ to their images with respect to their natural embeddings to $\mathbf{H}$. Denote by $\mathbf{H}^m$ the m—th copy of the Hilbert space $\mathbf{H}$ and consider the infinite tensor product

$$\tilde{\mathbf{H}} = \bigotimes_m \left( \mathbf{H}^m, \eta^{(m)} \right).$$
It is convenient to represent \( \tilde{H} \) as the closure of linear span of vectors of the form
\[
\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{m-1} \otimes \eta^{(m)} \otimes \eta^{(m+1)} \otimes \cdots,
\]
where \( \zeta_j \) is any vector from \( H^j \).

Now fix the orthonormal basis
\[
B = \left\{ e_j^{(r)} \otimes e_l^{(s)} \in \left( \bigoplus_k H^\alpha_k \oplus H^\beta_k \right) \otimes \left( \bigoplus_k H^\alpha_k \oplus H^\beta_k \right), e_j \in \bigoplus_k H^0_k \right\}
\]
in \( H \) and assume below \( \zeta_j \in B \). By the vector \( \zeta = \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{m-1} \otimes \cdots \) we build a sequence \( j(\zeta) = \{ j_1 < j_2 < \cdots \} \) such, that
\[
\zeta_{j_1} = e_m^{(\beta_k)} \otimes f \text{ for some } \beta_k \text{ and } m.
\]

Define for \( s \in S_\infty \) a vector \( s(\zeta) = \vartheta_1 \otimes \vartheta_2 \otimes \cdots \otimes \vartheta_{m-1} \otimes \cdots \) as follows:
\[
\vartheta_k = \begin{cases} 
eq (r) \otimes e, & \text{if } \zeta_{s^{-1}(k)} = e_{l}^{(r)} \otimes e \text{ and } \zeta_k = e_j^{(s)} \otimes f \\ e_{i_k}^{(r)} \otimes e, & \text{if } \zeta_{s^{-1}(k)} = e_{i_k}^{(r)} \otimes e \text{ and } \zeta_k \in \bigoplus_l H^0_l \\ \zeta_{s^{-1}(k)}, & \text{if } \zeta_{s^{-1}(k)} \in \bigoplus_l H^0_l. \end{cases}
\]

For any \( j_1 \in j(\zeta) \) such that \( \zeta_{j_1} = e_m^{(\beta_k)} \otimes f \) there exists \( j_2 \in j(s(\zeta)) \) with the property
\[
\vartheta_{j_2} = e_m^{(\beta_k)} \otimes g.
\]

Let \( t \) be a permutation of the set \( \{ j_1^s, j_2^s, \ldots \} \) for which \( t(j_1^s) < t(j_2^s) < \cdots \).

Finally, set \( \psi(s, \zeta) = sgn(t) \). The corresponding representation \( \pi \) of \( S_\infty \Gamma \) can be realized in \( \tilde{H} \) as follows:
\[
\pi(\gamma) \left( \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{m-1} \otimes \eta^{(m)} \otimes \cdots \right) = \varrho(\gamma_1) \zeta_1 \otimes \varrho(\gamma_2) \zeta_2 \otimes \cdots \otimes \varrho(\gamma_{m-1}) \zeta_{m-1} \otimes \varrho(\gamma_m) \eta^{(m)} \otimes \cdots \]
\[
\text{and for } s \in S_\infty \quad \pi(s) \left( \zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_{m-1} \otimes \cdots \right) = \psi(s, \zeta) s(s(\zeta)). \tag{2.2}
\]
2.2. The character’s formula. Set \( \tilde{\eta} = \bigotimes_{m} \eta^{(m)} \). Assume that \( s \) is the cycle \( (1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow k-1 \rightarrow k) \). Let \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k, e_\Gamma, e_\Gamma, \ldots) \). Routine calculations provide that

\[
\left( \pi(\gamma) \tilde{\eta}, \tilde{\eta} \right) = \sum_{j} \alpha_j^k \text{Tr} \left( \varrho^{\alpha_j} (\gamma_1 \gamma_2 \cdots \gamma_k) \right) + \sum_{j} \text{Tr} \left( \varrho^{\beta_j} (\gamma_1 \gamma_2 \cdots \gamma_k) \right)
\]

(2.3)

where \( \text{Tr} (\varrho^r) = \dim \varrho \sum_{j=1}^{\dim \varrho} \varrho_{jj}^r (\gamma) \) and \( k > 1 \).

It is obvious, that

\[
\left( \pi(\gamma) \tilde{\eta}, \tilde{\eta} \right) = \prod_{j=1}^{k} \left( \sum_{i} \alpha_i \text{Tr} \left( \varrho^{\alpha_i} (\gamma_j) \right) + \sum_{i} \beta_i \text{Tr} \left( \varrho^{\beta_i} (\gamma_j) \right) + \left( \varrho^0 (\gamma_j), \xi^{(0)} \right) \right).
\]

Since \( tr^0 \) is a character on \( \Gamma \), one can use (2.3) and the multiplicativity property (see Proposition 6) to obtain the following

**Corollary 9.** Let \( \chi(g) = \left( \pi(g) \tilde{\eta}, \tilde{\eta} \right) \). Then \( \chi \) is an indecomposable character on \( \mathcal{S}_\infty \wr \Gamma \).

3. Another examples.

In this section we construct examples of infinite type representations of \( \mathcal{S}_\infty \wr \mathbb{Z}_2 \). The corresponding positive definite functions are not characters. On the other hand they satisfy the following condition:

\( \varphi(sg) = \varphi(gs) \) for all \( g \in G = \mathcal{S}_\infty \wr \Gamma \) and \( s \in \mathcal{S}_\infty \).

In the generic case the representation \( \pi_\varphi \) built by GNS-construction from \( \varphi \) is of type \( III \). Furthermore, the state \( \varphi \) on the \( W^* \)-algebra \( \pi_\varphi (G)'' \) is exact. These properties allow one to construct the Tomita-Takesaki modular operator \( \Delta_\varphi \). Surprisingly, \( \Delta_\varphi \) is naturally related to the Okounkov operator \( \mathcal{O}_k \) (see [4]), which is an important in the representation theory of symmetrical group (see [2], [3]).

3.1. A construction. Let \( X_i = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{0, 1\} \times \{0, 1\} \). Define a probability measure \( \nu_i \) on \( X_i \) by \( \nu_i((k, l)) = p_{kl} \). Let \( (X, \mu) = \prod_i (X_i, \nu_i) \)
and \( x = (x_i) \in X \), where \( x_i = (x_i^{(0)}, x_i^{(1)}) \in X_i \), \( x_i^{(k)} \in \{0, 1\} \). Define an action \( a \) of \( g = (s_0, s_1) \in S_\infty \times S_\infty \) on \((X, \mu)\) as follows:

\[
(a_g(x))_i^{(k)} = x_{s_k(i)}^{(k)} \quad (k = 0, 1).
\]

**Remark 1.** The measure \( \mu \) is \( S_\infty \times S_\infty \)-quasiinvariant if and only if \( p_{ij} \neq 0 \) for all \( i, j = 0, 1 \).

We are about to construct a unitary representation \( \pi_\mu \) of \( G \times G \) in \( L^2(X, \mu) \). With \( \varsigma \in L^2(X, \mu) \) set up

\[
\left( \pi_\mu ((s_0, s_1)) \varsigma \right)(x) = \left( \frac{d\mu(a_g(x))}{d\mu(x)} \right)^{\frac{1}{2}} \varsigma(a_g(x)),
\]

\[
\left( \pi_\mu \left( \gamma^{(0)}, \gamma^{(1)} \right) \right)(x) = (-1)^{\sum_{i,k} \gamma_{i}^{(k)} x_{i}^{(k)}} \varsigma(x),
\]

where \( \gamma^{(0)} = \left( \gamma_{i}^{(0)} \right) \in \mathbb{Z}_\infty^2 \), \( \gamma^{(1)} = \left( \gamma_{i}^{(1)} \right) \in \mathbb{Z}_2^\infty \), and \( \left( \gamma^{(0)}, \gamma^{(1)} \right) \in \mathbb{Z}_2^\infty \times \mathbb{Z}_2^\infty \).

Let \( \pi_\mu^{(0)}(g) = \pi_\mu ((g, e_G)) \) and \( \pi_\mu^{(1)}(g) = \pi_\mu ((e_G, g)) \).

**Proposition 10.** \( \pi_\mu \) is irreducible. Hence, \( \pi_\mu^{(0)} \) and \( \pi_\mu^{(1)} \) are factor-representations of \( S_\infty \wr \Gamma \).

**Proof.** Obvious. \( \square \)

### 3.2. A cyclic separating vector.

Let \( I \) be an element of \( L^2(X, \mu) \) given by the function identically equal to the unit.

**Theorem 11.** If \( \det [p_{ij}] \neq 0 \), then \( I \) is a cyclic separating vector for \( \pi_\mu^{(0)}(G)' \) and \( \pi_\mu^{(1)}(G)' \). That is,

\[
\left[ \pi_\mu^{(0)}(G)' I \right] = \left[ \pi_\mu^{(1)}(G)' I \right] = L^2(X, \mu).
\]

**Proof.** Let \((k, l)\) be a transposition from \( S_\infty \). First notice that the operator

\[
\mathcal{O}_k^{(j)} = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \pi_\mu^{(j)} ((k, l)) \quad (\text{see (4.7)})
\]

11
belongs to $\pi^{(j)}(G')$ ($j = 0, 1$). Since
\[ (L^2(X, \mu), \mathbb{I}) = \bigotimes_{i=1}^{\infty} (L^2(X_i, \nu_i), \mathbb{I}) \]
one can apply the law of large numbers to deduce that
\[ O^{(j)}_i = I \otimes I \otimes \ldots \otimes O^{(j,i)}_i \otimes I \otimes \ldots \]
Furthermore, if $\chi_{kl}^{(i)}$ is the indicator of the point $(k, l) \in X_i = \mathbb{Z}_2 \times \mathbb{Z}_2$, the matrices of $O^{(0,i)}_i$ and $O^{(1,i)}_i$ in the orthonormal basis $\{ e_{kl}^{(i)} = \frac{\chi_{kl}^{(i)}}{\sqrt{p_{kl}}} \}_{k,l=0,1}$ are as follows:
\[ O^{(0,i)}_i \leftrightarrow \begin{bmatrix} p_{00} + p_{01} & 0 & 0 & \sqrt{p_{00}p_{10} + p_{01}p_{11}} & 0 \\ 0 & p_{00} + p_{01} & 0 & 0 & \sqrt{p_{00}p_{10} + p_{01}p_{11}} \\ 0 & 0 & \sqrt{p_{00}p_{10} + p_{01}p_{11}} & p_{10} + p_{11} & 0 \\ 0 & \sqrt{p_{00}p_{10} + p_{01}p_{11}} & p_{00} + p_{01} + p_{10} + p_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{p_{00}p_{10} + p_{01}p_{11}} \end{bmatrix} \]
(3.2)
\[ O^{(1,i)}_i \leftrightarrow \begin{bmatrix} p_{00} + p_{10} & 0 & 0 & \sqrt{p_{00}p_{01} + p_{10}p_{11}} & 0 \\ \sqrt{p_{00}p_{10} + p_{01}p_{11}} & p_{00} + p_{01} + p_{10} + p_{11} & 0 & 0 & \sqrt{p_{00}p_{10} + p_{01}p_{11}} \\ 0 & 0 & 0 & \sqrt{p_{00}p_{01} + p_{10}p_{11}} & p_{00} + p_{01} + p_{10} + p_{11} \\ 0 & 0 & 0 & 0 & \sqrt{p_{00}p_{01} + p_{10}p_{11}} \end{bmatrix} \]
By the construction,
\[ \pi^{(k)}_\mu (\gamma_i^{(k)}) = \bigotimes_{i=1}^{\infty} \pi^{(k,i)}_\mu (\gamma_i^{(k)}), \]
where $\pi^{(0,i)}_\mu (\gamma_i^{(0)})$ and $\pi^{(1,i)}_\mu (\gamma_i^{(1)})$ are determined by the matrices
\[ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (-1)^{i^{(0)}} & 0 \\ 0 & 0 & 0 & (-1)^{i^{(0)}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1)^{i^{(1)}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (-1)^{i^{(1)}} \end{bmatrix} \]
(3.3)
Use the map
\[ \mathcal{J}_i : \sum_{m,n=0,1} a_{mn} e_{mn}^{(i)} \rightarrow \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}, \]
(3.4)
to identify $L^2(X_i, \nu_i)$ to the full matrix algebra $M_2(\mathbb{C})$, so that
\[ \mathcal{J}_i(\mathbb{I}) = \begin{bmatrix} \sqrt{p_{00}} & \sqrt{p_{01}} \\ \sqrt{p_{10}} & \sqrt{p_{11}} \end{bmatrix} \]
12
Equip $M_2(\mathbb{C})$ with the Hermitian form

$$\langle a, b \rangle_i = \text{Tr} (b^* a),$$

then $\mathcal{J}_i$ is a unitary and $\mathcal{J}_i L^2(X_i, \nu_i) = M_2(\mathbb{C})$. Now as an elementary consequence of (3.2) and (3.3) one has:

$$\mathcal{J}_i \mathcal{O}_i^{(0,i)} \mathcal{J}_i^{-1} a = \begin{bmatrix}
p_{00} + p_{01} & \sqrt{p_{00}p_{10}} + \sqrt{p_{01}p_{11}} \\
\sqrt{p_{00}p_{10}} + \sqrt{p_{01}p_{11}} & p_{10} + p_{11}
\end{bmatrix} a, \tag{3.5}
$$

$$\mathcal{J}_i \mathcal{O}_i^{(1,i)} \mathcal{J}_i^{-1} a = a \begin{bmatrix}
p_{00} + p_{10} & \sqrt{p_{00}p_{11}} + \sqrt{p_{10}p_{11}} \\
\sqrt{p_{00}p_{11}} + \sqrt{p_{10}p_{11}} & p_{01} + p_{11}
\end{bmatrix},$$

$$\mathcal{J}_i \pi_{\mu}^{(0,i)} (\gamma_i^{(0)}) \mathcal{J}_i^{-1} a = \begin{bmatrix}1 & 0 \\
0 & (-1)^i \gamma_i^{(0)}\end{bmatrix} a,$$

$$\mathcal{J}_i \pi_{\mu}^{(1,i)} (\gamma_i^{(1)}) \mathcal{J}_i^{-1} a = a \begin{bmatrix}1 & 0 \\
0 & (-1)^i \gamma_i^{(1)}\end{bmatrix},$$

where $a \in M_2(\mathbb{C})$.

Thus, in view of Remark (see p. 11), the algebra $\mathfrak{m}_n^k$ generated by the operators $\mathcal{J}_i \mathcal{O}_i^{(k,i)} \mathcal{J}_i^{-1}$ and $\mathcal{J}_i \pi_{\mu}^{(k,i)} (\gamma_i^{(k)}) \mathcal{J}_i^{-1}$ is just $M_2(\mathbb{C})$. Since det $(\mathcal{J}_i (\mathbb{I})) \neq 0$, one has finally $\mathfrak{m}_n^0 \mathcal{J}_i (\mathbb{I}) = \mathfrak{m}_n^1 \mathcal{J}_i (\mathbb{I}) = M_2(\mathbb{C}). \tag{\Box}$

3.3. The modular operator. Consider the Hilbert space $\mathcal{H} = \bigotimes_{i=1}^\infty (M_2(\mathbb{C}), \langle \cdot, \cdot \rangle_i, \mathcal{J}_i(\mathbb{I}))$. It is convenient to represent $\mathcal{H}$ as a closure of the linear span of the vectors $a_1 \otimes a_2 \otimes \ldots \otimes a_i \otimes \mathcal{J}_{i+1}(\mathbb{I}) \otimes \mathcal{J}_{i+2}(\mathbb{I}) \ldots$, where $a_i \in M_2(\mathbb{C})$. If $\mathcal{H} = \bigotimes_{i=1}^\infty \mathcal{J}_i$, one has by Theorem (11)

$$\mathcal{H} L^2(X, \mu) = \mathcal{H}.$$  

Let $\mathcal{L}(\mathcal{H})$ and $\mathcal{R}(\mathcal{H})$ be the $W^*$-algebras generated in $\mathcal{H}$ by the operators of left and right multiplication by elements of the form

$$a_1 \otimes a_2 \otimes \ldots \otimes a_i \otimes I_2 \otimes I_2 \otimes \ldots,$$

where $a_i \in M_2(\mathbb{C})$, $I_2 = \begin{bmatrix}1 & 0 \\
0 & 1\end{bmatrix}$.

**Proposition 12.** $\pi_{\mu}^{(0)} (G)'' = \mathcal{J}^{-1} \mathcal{L}(\mathcal{H}) \mathcal{J}$ and $\pi_{\mu}^{(1)} (G)'' = \mathcal{J}^{-1} \mathcal{R}(\mathcal{H}) \mathcal{J}$.

**Proof.** Let $\mathfrak{A}_n^{(j)}$ stand for the $W^*$-algebra generated by the operators $\{\mathcal{O}_i^{(j)}\}_{i=1}^n$ and $\{\pi_{\mu}^{(j)} (\Gamma^n)\}$ ($j = 0, 1$). In view of (3.5), $\mathfrak{A}_n^{(j)}$ is isomorphic $\bigotimes_{i=1}^n M_2(\mathbb{C})$. Therefore, $\pi_{\mu}^{(j)} (\mathcal{G}_n) \subset \mathfrak{A}_n^{(j)}$. Finally, use (3.5) deduce $\mathfrak{A}_n^{(0)} \subset \mathcal{L}(\mathcal{H})$ and $\mathfrak{A}_n^{(1)} \subset \mathcal{R}(\mathcal{H})$. \tag{\Box}
Lemma 13. Let $i(p)$ be an element of $p \in \mathbb{N}/s$. Given any $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots) \in \Gamma_c^\infty$, there exists $\tilde{\gamma} \in \Gamma_c^\infty$ with the property $\gamma \cdot s \cdot \tilde{\gamma} = s \cdot \gamma'$, where

$$\gamma'_s(l(i(p))) = e_l \text{ for all } l = 1, 2, \ldots, |p| - 1 \text{ and } p \in \mathbb{N}/s,$$

$$\gamma'_s(|p| - 1)(i(p)) = \gamma_s(|p| - 1)(i(p)) \cdot \gamma_s(|p| - 2)(i(p)) \cdots \gamma_s(i(p)).$$
Proof. Let the $\tilde{\gamma}$ be defined as follows:
\[
\tilde{\gamma}_i(p) = e^\Gamma, \quad \tilde{\gamma}_{s(i)}(i(p)) = \gamma_{s(i)}^{-1}, \quad \tilde{\gamma}_{s(i)}(2)(i(p)) = \gamma_{s(i)}^{-1} \cdot \gamma_{s(i)}^{-1}, \ldots,
\]
\[
\tilde{\gamma}_{s(|p|-1)}(i(p)) = \gamma_{s(i)}^{-1} \cdot \gamma_{s(i)}^{-1} \cdot \gamma_{s(|p|-2)}(i(p)) \quad \text{for all } p \in \mathbb{N}/s.
\]
Now our statement can be readily verified.

Lemma 14. Let $s$ be a cycle from $\mathfrak{S}_\infty$. Suppose that for $\beta, \gamma \in \Gamma_e^\infty$ the following relations hold:
\[
\beta_k = \gamma_k = e^\Gamma \quad \text{for all } k \in \{j \in \mathbb{N} \mid s(j) = j\}.
\]
If $s\beta$ and $s\gamma$ are in the same conjugate class, then there exists $\tilde{\gamma} \in \Gamma_e^\infty$ such that $s\gamma = \tilde{\gamma} \cdot s\beta \cdot \tilde{\gamma}^{-1}$.

Proof. One may assume without loss of generality that $s(k) = k + 1$ for $k = 1, 2, \ldots, m - 1$, $s(m) = 1$ and $s(l) = l$ for all $l > m$.

By Lemma 13 there exist $\tilde{\gamma}, \tilde{\beta} \in \Gamma_e^\infty$ with the properties
\[
\tilde{\gamma} \cdot s \cdot \gamma \cdot \tilde{\gamma}^{-1} = s \cdot \gamma', \quad \tilde{\beta} \cdot s \cdot \beta \cdot \tilde{\beta}^{-1} = s \cdot \beta', \quad \text{where}
\]
\[
\gamma_k' = \beta_k' = e^\Gamma \quad \text{for } k = 1, 2, \ldots, m - 1, m + 1, \ldots.
\]
Let $s \in \mathfrak{S}_\infty$ and $\delta \in \Gamma_e^\infty$ be such that
\[
(t\delta)s\gamma'(t\delta)^{-1} = s\beta'.
\]
One has the following relations:
\[
\delta_2 \gamma_1' &= \beta_1'(t(1)) \delta_1 \\
\delta_3 \gamma_2' &= \beta_1'(t(2)) \delta_2 \\
& \vdots \quad \vdots \\
\delta_m \gamma_{m-1}' &= \beta_1'(t(m-1)) \delta_{m-1} \\
\delta_1 \gamma_m' &= \beta_1'(t(m)) \delta_m.
\]
By assumptions of the Lemma, $t(\{1, 2, \ldots, m\}) = \{1, 2, \ldots, m\}$, and we may assume that $t(k) = k$ for all $k > m$. Hence, there exists a map $f$ from $\mathbb{N}$ to $\mathbb{N}$ such that
\[
t(k) = s^{f(k)}(k) \quad \text{for } k \in \mathbb{N}.
\]
Now use the relation \( ts = st \) to obtain
\[
f(k) = l \text{ for } k = 1, 2, \ldots, m. \tag{4.3}
\]
Since \( s^m \) is the identity, it suffices to consider the case \( l \in \{1, 2, \ldots, m - 1\} \).
Use \( \text{(4.2)} \) to obtain
\[
\delta_1 = \ldots = \delta_{m-l}, \quad \delta_{m-l+1} = \ldots = \delta_m,
\]
\[
\beta'_m = \delta_m \delta_1^{-1}, \quad \gamma'_m = \delta_1^{-1} \delta_m.
\]
These relations together with \( \text{(4.8)} \) yield the following relation:
\[
\delta' s \gamma' (\delta')^{-1} = s \delta', \text{ where } \delta' = (\delta_1^{-1}, \delta_1^{-1} \delta_1, \ldots, \delta_1^{-1} \delta_1, \ldots).
\]

In what follows, \( (\pi_\phi, \mathcal{H}_\phi, \xi_\phi) \) is the unitary representation of \( G = \Gamma \wr \mathfrak{S}_\infty \) that corresponds by GNS-construction to the character \( \phi \). In particular, the operators \( \pi(G) \) act in \( \mathcal{H}_\phi \) with cyclic separating vector \( \xi_\phi \). That is,
\[
[\pi_\phi (G) \xi_\phi] = [\pi_\phi (G)' \xi_\phi] = \mathcal{H}_\phi, \tag{4.4}
\]
where \([S]\) stands for the closed subspace in \( \mathcal{H}_\phi \) generated by \( S \). Moreover \( \phi(g) = (\pi_\phi (g) \xi_\phi, \xi_\phi) \) for all \( g \in G \).

The property \( \text{(4.4)} \) allows one to produce a unitary spherical representation \( \pi^{(2)}_\phi \) of the Olshanski pair \( (G \times G, K) \), where \( K = \text{diag} G = \{(g, g)\}_{g \in G} \).

Namely,
\[
\pi^{(2)}_\phi (g_1, g_2) x \xi_\phi = \pi_\phi (g_1) x \pi_\phi (g_2)^* \xi_\phi \text{ for all } x \in \pi_\phi (G)''. \tag{4.5}
\]

Let
\[
G_n(\infty) = \{g = s \cdot \gamma \in G \mid s(l) = l \text{ and } \gamma_l = e \text{ for all } l = 1, 2, \ldots, n\},
\]
\[
K_n(\infty) = K \cap (G_n(\infty) \times G_n(\infty)), \quad G_n = \Gamma \wr \mathfrak{S}_n, \quad K_n = (G_n \times G_n) \cap K.
\]
It follows from the definition that \( G_0(\infty) = G_\infty = G, \ K_0(\infty) = K_\infty = K \).

Set
\[
\mathcal{H}^{K_n(\infty)}_\phi = \{\eta \in \mathcal{H}_\phi \mid \pi^{(2)}_\phi (g) \eta = \eta \text{ for all } g \in K_n(\infty)\},
\]
and let \( P_n \) be an orthogonal projection onto \( \mathcal{H}^{K_n(\infty)}_\phi \).
Lemma 15. $\bigcup_{n=0}^{\infty} \mathcal{H}_{\phi}^{K_{n}(\infty)}$ is a dense subspace in $\mathcal{H}_{\phi}$. In different terms, $\lim_{n \to \infty} P_{n} = \mathcal{I}_{\mathcal{H}_{\phi}}$ in the strong operator topology.

Proof. It follows from the definition of $\pi_{\phi}^{(2)}$ (see (4.3)) that

$$\left[\pi_{\phi}^{(2)}(G_{n}) \xi_{\phi}\right] \subset \mathcal{H}_{\phi}^{K_{n}(\infty)}. \quad (4.6)$$

On the other hand, $\xi_{\phi}$ is a cyclic vector. That is, $\left[\bigcup_{n=1}^{\infty} \pi_{\phi}^{(2)}(G_{n}) \xi_{\phi}\right] = \mathcal{H}_{\phi}$. Now our statement follows from (4.6). \qed

Remind a construction of asymptotic operators as it appears in [2], [3]. Consider the transposition $(i, n) \in \mathcal{S}_{\infty}$ and the operator

$$\mathcal{O}_{k} = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \pi_{\phi}((k, l)). \quad (4.7)$$

The limit exists in the strong operator topology.

We follow the idea of Olshanski (see [2], [7] and [?] for the case of $\mathcal{S}_{\infty}$) in extending his approach to our setting for $\Gamma \wr \mathcal{S}_{\infty}$. An algebraic structure of the associated Olshanski semigroup has been used above to predict important relations between the operators $\mathcal{O}_{k}$ and $\pi_{\phi}(\Gamma_{n}^{\infty})$.

Below we sketch the basic algebraic constructions and hope, with some details being left to a reader. Hopefully, this will allow later to receive a complete classification of admissible representations for wreath products $\Gamma \wr \mathcal{S}_{\infty}$ in some reasonable cases.

For any $n \in \mathbb{N}$ consider the $W^{*}$–algebra $\left(P_{n} \pi_{\phi}^{(2)}(G \times G) P_{n}\right)^{\prime\prime}$ generated by the operators $P_{n} \pi_{\phi}^{(2)}(G \times G) P_{n}$, which act in $P_{n} \mathcal{H}_{\phi}$. Obviously, the map

$$\pi_{\phi}^{(2,n)} : (g_{1}, g_{2}) \in G \times G \to P_{n} \pi_{\phi}^{(2)}(g_{1}, g_{2}) P_{n} \in \left(P_{n} \pi_{\phi}^{(2)}(G \times G) P_{n}\right)^{\prime\prime} \quad (4.8)$$

is constant on the double cosets $K_{n}(\infty) \backslash G \times G / K_{n}(\infty)$.

We are about to equip $K_{n}(\infty) \backslash G \times G / K_{n}(\infty)$ with a structure of semigroup in such a way that $\pi_{\phi}^{(2,n)}$ becomes a homomorphism. First, we extend the idea of Olshanski who applied a diagram technic to describe algebraic operations on $\mathcal{S}_{\infty}$, onto the semigroup $K_{n}(\infty) \backslash G \times G / K_{n}(\infty)$. For that, for any double coset we construct a so-called an admissible graph, which carries an important information about this coset.

17
Let $\mathfrak{G}$ be any graph. Denote by $V(\mathfrak{G})$ and $E(\mathfrak{G})$ respectively the set of all vertices and edges of $\mathfrak{G}$. Consider a disjoint union

$$ \overline{(\mathbb{Z} \setminus 0)} \cup (\mathbb{Z} \setminus 0) = \{ \cdots, -2, -1, 1, 2, \cdots \} \cup \{ \cdots, -2, -1, 1, 2, \cdots \} $$

of two copies of $(\mathbb{Z} \setminus 0)$. A map $\psi_\mathfrak{G}$ from $V(\mathfrak{G})$ to this disjoint union such that $\psi_\mathfrak{G}(v) \neq \psi_\mathfrak{G}(v')$ for all pairs of different vertices $v$ and $v'$, is called a vertex-coloring of the graph $\mathfrak{G}$.

**Definition 16.** An admissible graph is a vertex-colored, directed graph $\mathfrak{G}$ with the properties:

(a) $V(\mathfrak{G})$ is the disjunct union of four finite sets $V(\mathfrak{G})_u^-, V(\mathfrak{G})_u^+, V(\mathfrak{G})_o^-$ and $V(\mathfrak{G})_o^+$, with

$$ \psi(\mathfrak{G}) (V(\mathfrak{G})_u^-) = \{ \pm 1, \pm 2, \cdots, \pm n \}, \psi(\mathfrak{G}) (V(\mathfrak{G})_o^-) = \{ \mp 1, \mp 2, \cdots, \mp n \}; $$

(b) if $i(e)$ and $t(e)$ are initial and terminal vertices of $e \in E(\mathfrak{G})$, then

$$ \left| \left\{ e \in E(\mathfrak{G}) \mid i(e) = v, t(e) = v' \text{ or } t(e) = v, i(e) = v' \right\} \right| = \begin{cases} 1 & \text{if } v \neq v' \\ 0 & \text{otherwise}; \end{cases} $$

(c) if $e \in E(\mathfrak{G})$ then either of the four cases holds:

$$ i(e) \in V(\mathfrak{G})_o^- \text{ and } t(e) \in V(\mathfrak{G})_u^-, $$
$$ i(e) \in V(\mathfrak{G})_u^+ \text{ and } t(e) \in V(\mathfrak{G})_u^-, $$
$$ i(e) \in V(\mathfrak{G})_o^+ \text{ and } t(e) \in V(\mathfrak{G})_o^-, $$
$$ i(e) \in V(\mathfrak{G})_o^- \text{ and } t(e) \in V(\mathfrak{G})_o^+; $$

(d) one has a well defined marking function $m_\mathfrak{G}$ from $E(\mathfrak{G})$ to $(\mathbb{N}/2, \{0\}) \times \Gamma$, where $\mathbb{N}/2$ is the set of positive half-integer numbers.

**Definition 17.** The disjunct sum of the admissible graph $\mathfrak{G}$ and the countable set $\mathfrak{C}$ of the circles is called the admissible diagram if there is a well defined marking function $m_\mathfrak{C}$ from $\mathfrak{C}$ to $\mathbb{N} \times c_\gamma(\Gamma)$, where $c_\gamma(\Gamma)$ is the set of conjugacy classes of $\Gamma$ with property: there are finitely many elements $m_\mathfrak{C}(c) \ (c \in \mathfrak{C})$ not equal to $(1, e_\Gamma)$.

Here is a graphic interpretation for the elements of $G \times G$. Let $g = (s_1 \gamma', s_2 \gamma'')$. Consider two copies $\overline{(\mathbb{Z} \setminus 0)}$ and $(\mathbb{Z} \setminus 0)$ of $\mathbb{Z} \setminus 0$. It is convenient to position the elements of $\overline{(\mathbb{Z} \setminus 0)}$ and $(\mathbb{Z} \setminus 0)$ on two horizontal lines, $(\mathbb{Z} \setminus 0)$
above \((\mathbb{Z} \setminus 0)\). Draw the edges \(e \left( i, s_2(i) \right)\) from \(i \in (\mathbb{Z} \setminus 0)\) to \(s_2(i) \in (\mathbb{Z} \setminus 0)\) and \(e \left( -i, -s_1(i) \right)\) from \(-i \in (\mathbb{Z} \setminus 0)\) to \(-s_1(i) \in (\mathbb{Z} \setminus 0)\) for \(i > 0\) (see Fig. 1). Finally, define in a similarity with Definition 16 the marking function \(m_{\text{gr}(g)}\) as follows:

\[
m_{\text{gr}(g)} \left( e \left( i, j \right) \right) = \begin{cases} 
(0, \gamma''_i) & \text{if } i > 0, \\
(0, \gamma'_i) & \text{if } i < 0.
\end{cases}
\]

A graph has been constructed, to be denoted by \(\text{gr}(g)\). Obviously, \(i \left( E \left( \text{gr}(g) \right) \right) = (\mathbb{Z} \setminus 0)\) and \(t \left( E \left( \text{gr}(g) \right) \right) = (\mathbb{Z} \setminus 0)\).

To produce the graph \(\text{gr}(gh)\), it is convenient to position \(\text{gr}(g)\) above \(\text{gr}(h)\). After the natural gluing the vertices \(\mathbf{i} \in t \left( E \left( \text{gr}(h) \right) \right)\) and \(\mathbf{i} \in i \left( E \left( \text{gr}(g) \right) \right)\) we receive \(\text{gr}(gh)\). It is clear that

\[
m_{\text{gr}(gh)} \left( e \left( \mathbf{i}, \mathbf{j} \right) \right) = m_{\text{gr}(g)} \left( e_g \left( \mathbf{k}, \mathbf{j} \right) \right) \cdot m_{\text{gr}(h)} \left( e_h \left( \mathbf{i}, \mathbf{k} \right) \right),
\]

where the edge \(e \left( \mathbf{i}, \mathbf{j} \right)\) is the joining of the edges \(e_h \left( \mathbf{i}, \mathbf{k} \right) \in E \left( \text{gr}(h) \right)\) and \(e_g \left( \mathbf{k}, \mathbf{j} \right) \in E \left( \text{gr}(g) \right)\). If \(h \in G \times G\) is defined by the graph on Fig. 2 then the graph \(\text{gr}(gh)\) on Fig. 3 corresponds to the product \(gh\). Besides that, we equip \(\left\{ \mathbb{N}/2, 0 \right\} \times \Gamma\) with a natural semigroup structure.

Pass to a construction of the admissible diagram (see Definition 17) which
corresponds to the coset $\theta_n(g) \in K_n(\infty) \setminus G \times G / K_n(\infty)$ containing $g \in G \times G$. It splits into four steps.

- At Step 1, draw in $\text{gr}(g)$ for $i > n$ the edges $e(-i,i)$ and $e(i,-i)$ that connect the vertices $-i$ to $i$ and $i$ to $-i$. Denote the new graph by $\text{gr}(g)$ (see Fig. 4) and extend the marking function $m_{\text{gr}(g)}$ on $\text{gr}(g)$ assuming that

$$m_{\text{gr}(g)}(e(-i,i)) = m_{\text{gr}(g)}(e(i,-i)) = \left(\frac{1}{2}, \epsilon_{\Gamma}\right) \in \{N/2, 0\} \times \Gamma.$$

- At Step 2, extend the marking function $m_{\text{gr}(g)}$ to the pathes of $\overline{\text{gr}(g)}$. First, it is reasonable to assume, that

$$m_{\overline{\text{gr}(g)}}(e(J,L)) = (l, \gamma^{-1}), \text{ when } m_{\overline{\text{gr}(g)}}(e(L,J)) = (l, \gamma).$$

If the path $p \in \overline{\text{gr}(g)}$ is formed by a sequence $\{e_1, e_2, \ldots, e_j\}$ of edges,
then
\[ m_{\text{gr}(g)}(p) = m_{\text{gr}(g)}(e_j) \cdot m_{\text{gr}(g)}(e_{j-1}) \cdots m_{\text{gr}(g)}(e_1). \]

- At Step 3, notice, that \( \text{gr}(g) \) is a disjoint union of its connected components, each of those being a non closed path or cycle. By the construction, the ends of any non-closed path belong to
\[ \mathcal{V}_n = \{-n, \ldots, -2, -1, 1, 2, \ldots, n\} \] 
Furthermore, define a coherent positive orientation on any non-closed path, which contains vertices \( i \) or \( j \) with \( |i| > n \), if we assume that its initial vertex belongs to the set
\[ \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}. \]
The condition that any cycle contains an edge of the form \( e(i, j) \), where \( i, j > 0 \), defines the corresponding orientation on closed pathes.

- Step 4. The set of vertices of the admissible graph \( \text{gr}(\theta_n(g)) \) (see Definition 16) is \( \mathcal{V}_n \). Each oriented non-closed path \( p \) (see Step 3) determines an oriented edge \( e(p) \) of \( \text{gr}(\theta_n(g)) \) as follows:
\[ i(p) = i(e(p)), \ t(p) = t(e(p)). \]
Finally, \( m_{\text{gr}(\theta_n(g))}(e(p)) = m_{\text{gr}(g)}(p) \) (see Definition 16 (d)). Similarly, each oriented cycle \( c \) defines the circle \( c(c) \) (see Definition 17). The value of the marking function on \( c(c) \) coincides with \( m_{\text{gr}(g)}(c) = (r, \gamma) \). It is clear that the conjugacy class of \( \gamma \) does not depend on a choice of initial vertex in \( c \). The visualization of this algorithm for \( g \in G_5 \) is depicted on Fig. 5. Notice that there are no circles on Fig. 5 which are marked through \( (1, e_\Gamma) \).

Hence, by Definition 17 we obtain the admissible diagram \( \mathfrak{D}(\theta_n(g)) \) of the coset \( \theta_n(g) \).

Using the diagram technic, we describe the algorithm of multiplication for cosets. Let \( \theta_n(g) \) and \( \theta_n(h) \) be two elements from \( K_n(\infty) \setminus G \times G / K_n(\infty) \). Let \( \theta_n(g) \circ \theta_n(h) \) stand for the product \( \theta_n(g) \) and \( \theta_n(h) \). Again, it is convenient to position \( \mathfrak{D}(\theta_n(g)) \) above \( \mathfrak{D}(\theta_n(h)) \). Later on, we paste the vertices
\[ \{-n, \ldots, -2, -1, 1, 2, \ldots, n\} \subset \mathfrak{D}(\theta_n(h)) \]
with the corresponding ones from
\[ \{-n, \ldots, -2, -1, 1, 2, \ldots, n \} \subset \mathfrak{d}(\theta_n(g)). \]
The resulting graph inherits naturally an orientation from the diagrams \( \mathfrak{d}(\theta_n(g)) \) and \( \mathfrak{d}(\theta_n(h)) \). Just as at Step 4, we replace the newly formed connected components (non closed paths or cycles) by the oriented edges or circles and define on those the marking function. The received diagram for \( g, h \in G_5 \) is represented on Fig. 7.

**Remark 2.** Let \((i, n + 1)\) be the transposition from \( \mathfrak{S}_\infty \subset G = \mathfrak{S}_\infty \wr \Gamma \), and let \( \gamma \) be any element from \( \Gamma \subset G \). It is easy to check up, following Steps 1-4,
Figure 8: $d(\theta_n((i, n + 1)))$ and $d(\theta_n(\gamma))$

that they are determined by the admissible diagrams depicted in Figure 8.

By the graphic interpretation of the multiplication for cosets one has

$$\theta_n(\gamma) \circ \theta_n((i, n + 1)) = \theta_n((i, n + 1)) \circ \theta_n(\gamma).$$ (4.9)

If $\pi^{(2,n)}_\phi$ (see (4.8)) is the representation of the semigroup of cosets, then

$$\pi^{(2,n)}_\phi(\theta_n((i, n + 1))) = P_n O_i P_n(see (4.7))$$ and $\pi^{(2,n)}_\phi(\theta_n(\gamma)) = \pi^{(2,n)}_\phi(\gamma)$.

Therefore, $P_n O_i P_n \pi^{(2)}_\phi(\gamma) P_n = P_n \pi^{(2)}_\phi(\gamma) P_n O_i P_n$. Using Lemma 15, we obtain $O_i \pi^{(2)}_\phi(\gamma) = \pi^{(2)}_\phi(\gamma) O_i$. This fact will be proved rigorously in Lemma 21.

Now we give a precise definition of multiplication and involution on the double cosets $K_n(\infty) \backslash G \times G / K_n(\infty)$. Denote by $\omega^{(n)}_m$ the permutation from $\mathfrak{S}_n(\infty)$, which acts as follows:

$$\omega^{(n)}_m(i) = \begin{cases} i & \text{if } i \leq n \text{ or } i > n + 2m \\ i + m & \text{if } n < i \leq n + m \\ i - m & \text{if } n + m < i \leq n + 2m. \end{cases}$$

**Proposition 18.** Let $\tilde{g}, \tilde{g}'$ be double cosets and $g, g'$ any elements from $\tilde{g}, \tilde{g}'$, respectively. There exist $M (g, g') \in \mathbb{N}$ such that for $m > M (g, g')$, $\theta_n\left(g \omega^{(n)}_m g'\right)$ does not depend on the choice of $g \in \theta_n(g) = \tilde{g}$ and $g' \in \theta_n(g') = \tilde{g}'$. The multiplication $\tilde{g} \circ \tilde{g}' = \theta_n\left(g \omega^{(n)}_m g'\right)$ and the involution $\tilde{g}^* = \theta_n\left(g^{-1}\right)$ determine a structure of $\ast$-semigroup on $K_n(\infty) \backslash G \times G / K_n(\infty)$ so that $\pi^{(2,n)}_\phi$ (see (4.8)) is a $\ast$-homomorphism.

Before proving the theorem, we discuss several auxiliary assertions. The following statement is a generalization of Theorem 2.5 from [6].

23
Lemma 19. Let $G \subset H$ be discrete groups with the property: there exists $\omega \in G$ such that for any set $\{g_1, g_2, \ldots, g_p\} \subset G$ ($p \in \mathbb{N}$), one can choose an element $g \in G$ for which $H\{gg_1, gg_2, \ldots, gg_p\}H = H\omega H$. If $T$ is a unitary representation of $G$ in a Hilbert space $H_T$ and $P_G$ the orthogonal projection onto the subspace $H_G^T = \{\xi \in H_T | T(g)\xi = \xi \forall g \in G\}$, then $P_G = P_H T(\omega) P_H$.

Proof. Let $\xi \in H_T$ and $\eta = P_H \xi$. Denote by $C_\eta$ the closure of the set of vectors of the form

$$\sum_{j=1}^m \alpha_j T(g_j) \eta,$$

where $g_j \in G$, $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$ and $\sum_{j=1}^m \alpha_j = 1$.

By our construction, $T(g)C_\eta = C_\eta$. Since there exists a unique vector $\zeta \in C_\eta$ with the property $\|\zeta\| = \min \{|\vartheta| | \vartheta \in C_\eta\}$, one has $T(g)\zeta = \zeta$ for all $g \in G$. Therefore, for any $\epsilon > 0$ there exist $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$ with the properties

$$\sum_{j=1}^m \alpha_j = 1 \quad \text{and} \quad \left\| \sum_{j=1}^m \alpha_j T(g_j) \eta - \zeta \right\| < \frac{\epsilon}{2}.$$

Hence, $\|P_G \eta - \zeta\| < \frac{\epsilon}{2}$ and

$$\left\| \sum_{j=1}^m \alpha_j T(g_j) \eta - P_G \eta \right\| < \epsilon \quad \text{for all} \quad g \in G. \quad (4.10)$$

If $g$ satisfies the assumptions of the Lemma, then, using (4.10), we have

$$\|P_G T(\omega) P_H \eta - P_G \eta\| < \epsilon.$$

Since $\xi$, $\epsilon$ are chosen arbitrarily and $\eta = P_H \xi$, then $P_G T(\omega) P_H = P_G$. $\square$

Lemma 20. For any $m,n \in \mathbb{N}$, $G = K_n(\infty)$, $H = K_{n+m}(\infty)$, $\omega = \begin{pmatrix} \omega_m(n) \\ \omega_m(n) \end{pmatrix}$ satisfy the assumptions of Lemma 19. Furthermore, $\pi_\phi^{(2)} \begin{pmatrix} \omega_m(n) \\ \omega_m(n) \end{pmatrix}$ converges to $P_n$ weakly as $m \to \infty$.

Proof. Define a permutation $\omega_{l,m}^n$ as follows:

$$\omega_{l,m}^n(i) = \begin{cases} 
  i & \text{if } i \leq n \text{ or } i > n + l + m \\
  i + m & \text{if } n < i \leq n + l \\
  i - l & \text{if } n + l < i \leq n + l + m.
\end{cases}$$
Finally, it is clear that \( P_{15} \) and \( P_{19} \). Thus the first statement is proved. The last statement follows from Lemmas \( \text{15} \) and \( \text{19} \).

Proof of Proposition \( \text{18} \). There exists \( m \in \mathbb{N} \) such that \( g, g' \in G_{n+m} \times G_{n+m} \). Let \( h \) be an element of \( K_{n+M} \cap K_{n}(\infty) \), where \( M \geq n \). Notice, that \( \omega_{M}^{(n)} h \omega_{M}^{(n)} \in K_{n+M}(\infty) \). Hence, using (4.11), we have

\[
\theta_{n} \left( g \omega_{m}^{(n)} g' \right) = \theta_{n} \left( g \omega_{M}^{(n)} g \omega_{M}^{(n)} h \omega_{M}^{(n)} \right) = \theta_{n} \left( g h \omega_{M}^{(n)} g' \right) = \theta_{n} \left( g h \omega_{m}^{(n)} g' \right). \tag{4.12}
\]

In a similar way, \( \theta_{n} \left( g \omega_{m}^{(n)} g' \right) = \theta_{n} \left( g \omega_{m}^{(n)} h g' \right) \). Thus the first statement is proved.

Since by Lemmas \( \text{19} \) and \( \text{20} \), \( g, g' \in G_{n+m} \times G_{n+m} \) and \( P_{n} = P_{n+m} \pi^{(2)}_{\phi} \left( \omega_{m}^{(n)} \right) P_{n+m} \), one has

\[
P_{n} \pi^{(2)}_{\phi} \left( g \right) P_{n} \pi^{(2)}_{\phi} \left( g' \right) P_{n} = P_{n+m} \pi^{(2)}_{\phi} \left( g \right) P_{n+m} \pi^{(2)}_{\phi} \left( \omega_{m}^{(n)} \right) P_{n+m} \pi^{(2)}_{\phi} \left( g' \right) P_{n} = P_{n} \pi^{(2)}_{\phi} \left( g \right) \pi^{(2)}_{\phi} \left( \omega_{m}^{(n)} \right) \pi^{(2)}_{\phi} \left( g' \right) P_{n+m} P_{n} = P_{n} \pi^{(2)}_{\phi} \left( g \omega_{m}^{(n)} g' \right) P_{n}.
\]

Finally, it is clear that \( P_{n} \pi^{(2)}_{\phi} \left( g \right) P^{*}_{n} = P_{n} \pi^{(2)}_{\phi} \left( g^{-1} \right) P_{n} \). \( \square \)

5 A proof of the main result.

The proof of Theorem \( \text{7} \) splits into a few lemmas.

Define for \( k \in \mathbb{N} \) the element \( \gamma \left( \{ k \} \right) = \left( \gamma_{1}(\{ k \}), \gamma_{2}(\{ k \}), \ldots, \gamma_{l}(\{ k \}), \ldots \right) \in \Gamma^{\infty} \) as follows:

\[
\gamma_{l}(\{ k \}) = \begin{cases} 
\gamma_{k} & \text{if } l = k \\
\varepsilon & \text{otherwise.}
\end{cases} \tag{5.1}
\]
For each indecomposable character \( \phi \) let \((\pi_\phi, \mathcal{H}_\phi, \xi_\phi)\) denote the cyclic representation of the group \( \mathfrak{G}_\infty \wr \Gamma \) associated to \( \phi \) via the GNS-construction.

**Lemma 21.** If a \( W^* \)-algebra \( \mathfrak{A} \) is generated by the operators \( \pi_\phi (\Gamma_e^\infty) \), \( \{ \mathcal{O}_j \}_{j \in \mathbb{N}} \), and \( C(\mathfrak{A}) \) is a center of \( \mathfrak{A} \), then \( \{ \mathcal{O}_j \}_{j \in \mathbb{N}} \subset C(\mathfrak{A}) \).

**Proof.** The relation \( \mathcal{O}_k \cdot \mathcal{O}_l = \mathcal{O}_l \cdot \mathcal{O}_k \) allows an easy verification by definition (4.7) (see [2] or [3]).

Now prove the relation
\[
\mathcal{O}_l \cdot \pi_\phi (\gamma) = \pi_\phi (\gamma) \cdot \mathcal{O}_l \quad \text{for all } \gamma \in \Gamma_e^\infty \text{ and } l \in \mathbb{N}.
\]

(5.2)

Let \( K_n^G(\infty) = K_n(\infty) \cap (\mathfrak{G}_\infty \times \mathfrak{G}_\infty) \) and \( K_n^G(m) = K_n^G(\infty) \cap (G_m \times G_m) \),

where \( m > n \). If \( P_n^G \) stands for the orthogonal projection onto \( \mathcal{H}_\phi^{K_n^G(\infty)} \), then
\[
P_n^G = \lim_{m \to \infty} \frac{1}{(m-n)!} \sum_{g \in K_n^G(m)} \pi_\phi^{(2)} (g)
\]

in the strong operator topology and \( P_n^G \geq P_n \). Hence, using (4.7) and (5.3), we obtain for \( i \leq n < k \)
\[
P_n^G \mathcal{O}_i P_n^G = P_n^G \pi_\phi ((i, k)) P_n^G \text{ and } P_n \mathcal{O}_i P_n = P_n \pi_\phi ((i, k)) P_n.
\]

(5.4)

In the case when \( \gamma_l = e \) the equality (5.2) easily follows from (4.7). Therefore, it suffices to prove (5.2) for the elements \( \gamma = \gamma(\{l\}) \) (see (5.1)).

If \( i \leq n < k \), then, using (4.7), we have
\[
P_n \pi_\phi (\gamma(\{l\})) \mathcal{O}_i P_n \{P_n \geq P_n\} = P_n P_n \pi_\phi (\gamma(\{i\})) \mathcal{O}_i P_n P_n
\]

(5.7)

\[
\begin{align*}
P_n \pi_\phi (\gamma(\{i\})) \mathcal{O}_i P_n & = P_n \pi_\phi (\gamma(\{i\})) \pi_\phi ((i, k)) P_n P_n \\
& = P_n \pi_\phi (\gamma(\{i\})) \pi_\phi (\gamma(\{k\})) P_n \\
& = P_n \pi_\phi (\gamma(\{k\})) \pi_\phi ((i, k)) P_n
\end{align*}
\]

(5.5)

\[
\begin{align*}
P_n \pi_\phi (\gamma(\{k\})) \pi_\phi ((i, k)) P_n & = P_n \pi_\phi (\gamma(\{k\})) \pi_\phi ((i, k)) P_n \\
& = P_n \pi_\phi (\gamma(\{k\})) \pi_\phi ((i, k)) P_n = P_n \pi_\phi (\gamma(\{i\})) P_n
\end{align*}
\]

(5.6)

\[
P_n \mathcal{O}_i \pi_\phi (\gamma(\{i\})) P_n
\]

Since \( \lim_{n \to \infty} P_n = \mathcal{I}_{\mathcal{H}_\phi} \) (see Lemma 13), the relation
\[
\pi_\phi (\gamma(\{i\})) \mathcal{O}_i = \mathcal{O}_i \pi_\phi (\gamma(\{i\}))
\]

follows.
We use the notation \((i_0, i_1, \ldots, i_{q-1})\) for the cyclic permutation \(s\) which acts as follows
\[
s(i) = \begin{cases} i_{k+1 \mod q} & \text{if } i = i_k \in \{i_0, i_1, \ldots, i_{q-1}\} \\ i & \text{otherwise,} \end{cases}
\]

**Lemma 22.** If \(O_i\) is defined as in (4.7) and
\[
D(m, n, q) = \left\{ \mathbf{k} = (k_1, k_2, \cdots, k_q) \in \mathbb{N} | k_i \neq k_j \text{ and } m < k_i \leq n \forall i, j = 1, \ldots, q \right\},
\]
then for every positive integer \(m\)
\[
O_i^q = \lim_{n \to \infty} \frac{1}{n^q} \sum_{\mathbf{k} \in D(m, n, q)} \pi_\phi((k_q, k_{q-1}, \ldots, k_1, i)).
\]

**Proof.** If we notice that
\[
(i, k_1) \cdot (i, k_2) \cdots (i, k_q) = (k_q, k_{q-1}, \ldots, k_1, i)
\]
for pairwise different \(i, k_1, k_2, \ldots, k_q\) and \(\text{Card}(D(m, n, q)) = (n - m - j)^{q-1}\), the proof becomes obvious.

**Lemma 23.** Let \(g = \prod_{p \in \mathbb{N}/s} s_p \cdot \gamma(p)\) be a decomposition of \(g = s \cdot \gamma \in \Gamma \wr \mathcal{S}_\infty\) (see (1.4)) and \(i(p)\) any element from \(p \in \mathbb{N}/s\). Define \(\gamma^{(i(p))} \in \Gamma_{e}^{\infty}\) as follows
\[
\gamma_k^{(i(p))} = \begin{cases} \gamma_{i(p)} \cdot \gamma_{s^{-1}(i(p))} \cdots \gamma_{s(-|p|+2)(i(p))} \cdot \gamma_{s(-|p|+1)(i(p))} & \text{if } k = i(p), \\ e & \text{otherwise} \end{cases}
\]

If \(\phi\) is an indecomposable character on \(\Gamma \wr \mathcal{S}_\infty\), then
\[
\left( \pi_\phi(s \cdot \gamma) \prod_j \mathcal{O}_{j}^{r_j} \xi_\phi, \xi_\phi \right) = \prod_{p \in \mathbb{N}/s} \left( \pi_\phi \left( \gamma^{(i(p))} \right) \mathcal{O}_{i(p)}^{[|p|-1+\sum_{j \in p} r_j]} \xi_\phi, \xi_\phi \right). \tag{5.6}
\]

**Proof.** By Proposition 6 we have
\[
\left( \pi_\phi(s \cdot \gamma) \prod_j \mathcal{O}_{j}^{r_j} \xi_\phi, \xi_\phi \right) = \prod_{p \in \mathbb{N}/s} \left( \pi_\phi(s_p \cdot \gamma(p)) \prod_{j \in p} \mathcal{O}_{j}^{r_j} \xi_\phi, \xi_\phi \right). \tag{5.7}
\]
Hence, by the relation \( \tau \cdot \gamma \cdot \gamma^{-1} = \gamma \), we find \( \tilde{\gamma} \in \Gamma_\infty^\gamma \) such that
\[
\tilde{\gamma} \cdot s \cdot \tilde{\gamma}^{-1} = s \cdot \gamma^{(i_1)}.
\] (5.8)

Thus, by Lemma 21,
\[
\left( \pi_\phi \left( s \cdot \gamma \right) \prod_{j \in p} O_j^{(i_j)} \xi_\phi, \xi_\phi \right) = \left( \pi_\phi \left( \gamma^{(i_1)} \right) \pi_\phi \left( s \right) \prod_{j \in p} O_j^{(i_j)} \xi_\phi, \xi_\phi \right). \tag{5.9}
\]

Let
\[
\mathcal{G}_\infty^j = \{ \tau \in \mathcal{G}_\infty | \tau(j) = j \}.
\]

Now use Lemma 22 to obtain

\[
\left( \pi_\phi \left( \gamma^{(i_1)} \right) \pi_\phi \left( s \right) \prod_{j \in p} O_j^{(i_j)} \xi_\phi, \xi_\phi \right) = \lim_{n \to \infty} \frac{1}{n^q} \sum_{\vec{k} \in D(m,n,q)} \left( \pi_\phi \left( \gamma^{(i_1)} \right) \pi_\phi \left( \left( k_{r_{i_1}}^{(i_1)}, k_{r_{i_1} - 1}^{(i_1)}, \ldots, k_1^{(i_1)} \right) \right) \right) \xi_\phi, \xi_\phi, \xi_\phi
\]

where
\[
\vec{k} = \left( k_{r_{i_1}}^{(i_1)}, k_{r_{i_1} - 1}^{(i_1)}, \ldots, k_1^{(i_1)} \right) = \sum_{j \in p} r_j.
\]

Hence, by the relation \( \tau \cdot \gamma^{(i_1)} \tau^{-1} = \gamma^{(i_1)} \), we have
\[
\left( \pi_\phi \left( \gamma^{(i_1)} \right) \pi_\phi \left( s \right) \prod_{j \in p} O_j^{(i_j)} \xi_\phi, \xi_\phi \right) = \lim_{n \to \infty} \frac{1}{n^q} \sum_{\vec{k} \in D(m,n,q')} \left( \pi_\phi \left( \gamma^{(i_1)} \right) \pi_\phi \left( \left( k_{r_{i_1}}^{(i_1)}, k_{r_{i_1} - 1}^{(i_1)}, \ldots, k_1^{(i_1)} \right) \right) \right) \xi_\phi, \xi_\phi
\]

28
Lemma 25. Let \( \Delta = [a, b] \) be an interval in \([-1, 0]\) or in \([0, 1]\) with the property \( \min \{|a|, |b|\} > \varepsilon > 0 \). If \( E^{(i)} \Delta \) is a spectral projection of \( O_i \) corresponding to \( \Delta \), then for any orthogonal projection \( E \) from \( A_i \) one has

\[
\left( E E^{(i)} \Delta \xi, \xi \right) \geq \varepsilon \left( E E^{(i)} \Delta \xi, \xi \right).
\]

Proof. Using Lemmas 24 and 25 we have

\[
\left| \left( \pi_{\phi} ((i, i + 1)) EE^{(i)} \Delta \xi, EE^{(i)} \Delta \xi \right) \right| = \left| \left( O_i EE^{(i)} \Delta \xi, EE^{(i)} \Delta \xi \right) \right| > \varepsilon \left| \left( EE^{(i)} \Delta \xi, \xi \right) \right|.
\]

On the other hand, under the assumption \( E^{(i+1)} = \pi_{\phi} ((i, i + 1)) E \pi_{\phi} ((i, i + 1)) \), one has

\[
EE^{(i)} \Delta \cdot E^{(i+1)} E^{(i+1)} \Delta \cdot \pi_{\phi} ((i, i + 1)) = \pi_{\phi} ((i, i + 1)) \cdot EE^{(i)} \Delta \cdot E^{(i+1)} E^{(i+1)} \Delta.
\]

This relation, in view of Lemma 22 implies the statement of Lemma 25. \( \square \)

We use the notation \( A_j \) for the \( W^* \)-algebra generated by \( \pi_{\phi} (\gamma) \), \( \gamma = (e, \ldots, e, \gamma_j, e, \ldots) \), and \( O_j \). Given an operator \( A \) from \( A_j \), denote by \( A^{(k)} \) its copy in \( A_k \):

\[
A^{(k)} = \pi_{\phi} ((j, k)) A \pi_{\phi} ((j, k)) \quad (A^{(j)} = A).
\]

The next assertion follows from Lemma 24.

Lemma 24. Let \( s, i(p) \) be the same as in Lemma 24. If \( A_j, B_j \in A_j \), then

\[
\left( \pi_{\phi} (s) \prod_j A_j \xi_\phi, \prod_j B_j \xi_\phi \right) = \prod_{p \in \mathbb{N}/s} \left( A_{i(p)}^{(i(p))} B_{i(p)}^{(i(p))} A_{s^{-1}(i(p))}^{(i(p))} B_{s^{-1}(i(p))}^{(i(p))} \right) \cdots (5.10)
\]

\[
\cdots A_{s^{-1}-i(p)}^{(i(p))} B_{s^{-1}-i(p)}^{(i(p))} \left( O_{i(p)} \right)^{|p|-1} \xi_\phi, \xi_\phi
\]

The following lemma is an analogue of Theorem 1 from [3].

Lemma 25. Let \( \Delta = [a, b] \) be an interval in \([-1, 0]\) or in \([0, 1]\) with the property \( \min \{|a|, |b|\} > \varepsilon > 0 \). If \( E^{(i)} \Delta \) is a spectral projection of \( O_i \) corresponding to \( \Delta \), then for any orthogonal projection \( E \) from \( A_i \) one has

\[
\left( EE^{(i)} \Delta \xi, \xi \right) \geq \varepsilon \left( E E^{(i)} \Delta \xi, \xi \right).
\]
Therefore,

\[
|\left( \pi_\phi \left( (i, i + 1) \right) EE_\Delta^{(i)} \xi_\phi, EE_\Delta^{(i)} \xi_\phi \right)| \\
= |\left( \pi_\phi \left( (i, i + 1) \right) E^{(i+1)} EE_\Delta^{(i)} \xi_\phi, \xi_\phi \right)| \\
\leq |\left( E^{(i+1)} EE_\Delta^{(i)} \xi_\phi, \xi_\phi \right)|^{Prop.15} \left( EE_\Delta^{(i)} \xi_\phi, \xi_\phi \right)^2.
\]

Hence, using (5.11), we obtain our statement.

The following statement is well known (see [3]) and also follows from Lemma 25.

**Corollary 26.** There exists at most countable set of numbers \( \alpha_i, \beta_i \) from (0, 1) and a set of pairwise orthogonal projections \( \{E^{(k)}(\alpha_i), E^{(k)}(\beta_i)\} \subset \mathfrak{A}_k \) such that

\[
O_k = \sum \alpha_i E^{(k)}(\alpha_i) - \sum \beta_i E^{(k)}(\beta_i).
\]

The following assertion is an analogue of Theorem 2 from [3].

**Lemma 27.** Let \( r \) be a number from \( \{\alpha, \beta\} \) and let \( E \) be any projection from \( \mathfrak{A}_k \). If \( (E \cdot E^{(k)}(r) \xi_\phi, \xi_\phi) = r\nu(r) \neq 0 \), then \( \nu(r) \in \mathbb{Z} \).

**Proof.** For completeness of the proof, we use the arguments of Kerov, Olshanski, Vershik and Okounkov from [1] and [3].

For any \( m \in \mathbb{N} \), define the projection \( e_m(r) \) as follows:

\[
e_m(r) = \prod_{j=1}^m E^{(j)}(r), \text{ where}
\]

\[
E^{(j)} = \pi_\phi((j, k)) E \pi_\phi((j, k)), \quad E^{(j)}(r) = \pi_\phi((j, k)) E^{(k)}(r) \pi_\phi((j, k)).
\]

Let \( \mathbb{P}_m(s) \) be the set of orbits \( s \) on \( \{1, 2, \ldots, m\} \). If \( s \in \mathbb{S}_m \), then by Lemma 24 we obtain

\[
(\pi_\phi(s)e_m(r)\xi_\phi, e_m(r)\xi_\phi) = \nu(r)^{\mathbb{P}_m(s)} \prod_{p \in \mathbb{P}_m(s)} r^{|p|}.
\]

Set \( \phi_r(s) = \frac{(\pi_\phi(s)e_m(r)\xi_\phi, e_m(r)\xi_\phi)}{(e_m(r)\xi_\phi, e_m(r)\xi_\phi)} \). Using (5.13), we have

\[
\phi_r(s) = \frac{\nu(r)^{\mathbb{P}_m(s)}}{\nu(r)^m}.
\]
Therefore, $\phi_r$ is an indecomposable character on $\mathfrak{S}_\infty$ in view of Proposition 6.

We following G. Olshanski (see [6]) in expounding the proof of the following formula:

$$\sum_{s \in \mathfrak{S}_m} \text{sgn}(s) t^{\|P_m(s)\|} = t(t-1) \cdots (t-m+1). \quad (5.15)$$

For that, we consider the canonical projection $p_{m,m-1}$ from $\mathfrak{S}_m$ onto $\mathfrak{S}_{m-1}$:

$$(p_{m,m-1}(s))(i) = \begin{cases} s(i) & \text{if } s(i) < m \\ s(m) & \text{if } s(i) = m. \end{cases}$$

Since $|P_{m-1}(p_{m,m-1}(s))| = |P_m(s)|$ when $s \not\in \mathfrak{S}_{m-1}$, and $|P_{m-1}(p_{m,m-1}(s))| = |P_m(s)| - 1$ when $s \in \mathfrak{S}_{m-1}$, then

$$\sum_{s \in \mathfrak{S}_m} \text{sgn}(s) t^{\|P_m(s)\|} = \sum_{s \in \mathfrak{S}_{m-1}} \sum_{\tilde{s} \in \mathfrak{S}_m: p_{m,m-1}(\tilde{s}) = s} \text{sgn}(s) t^{\|P_m(s)\|} =
\sum_{s \in \mathfrak{S}_{m-1}} t^{\|P_m(s)\|} - (m-1) \cdot \sum_{s \in \mathfrak{S}_{m-1}} t^{\|P_m(s)\|} = (t-m+1) \sum_{s \in \mathfrak{S}_{m-1}} t^{\|P_m(s)\|}.
$$

Hence (5.15) is now accessible by an elementary induction argument.

We follow the idea of A. Okounkov in considering the orthogonal projection

$$\text{Alt}_r(m) = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} \text{sgn}(s) \pi_{\phi_r}(s).$$

Since $\sum_{s \in \mathfrak{S}_m} \text{sgn}(s) \phi_r(s) \geq 0$, then, using (5.14) and (5.15), we obtain for $r > 0$

$$\nu(r) \cdot (\nu(r) - 1) \cdots (\nu(r) - m + 1) \geq 0 \text{ for all } m \in \mathbb{N}.$$ 

Thus, we get a contradiction in the case $\nu(r) > 0$. The opposite case $\nu(r) < 0$ can be considered in a similar way. For that, one should use the formula

$$\sum_{s \in \mathfrak{S}_m} t^{\|P_m(s)\|} = t(t+1) \cdots (t+m-1) \text{ (see [6])}$$

and consider the projection

$$\text{Sym}_r(m) = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} \pi_{\phi_r}(s).$$
Proof of Theorem 7. Let \( E_k(r) \) be the spectral projection of \( \mathcal{O}_k \) (see (1.7), (5.12)). By Lemma 27, for \( r \neq 0 \) the \( W^* \)-algebra \( E_k(r) \mathfrak{A}_k \) (see p. 29) is finite-dimensional. On the other hand, use Lemma 21 to obtain the unitary representation \( \left( E_k(r) \pi_{\phi} \big|_{\Gamma}, E_k(r) \mathcal{H}_\phi \right) \) of the group \( \Gamma \) in the space \( E_k(r) \mathcal{H}_\phi \). Thus, the representations \( \varphi^r \) for \( r \neq 0 \) as in Theorem 7 are the irreducible components of \( \left( E_k(r) \pi_{\phi} \big|_{\Gamma}, E_k(r) \mathcal{H}_\phi \right) \). The formula for characters follows from Lemmas 21 and 24. Finally, for each character as in Theorem 7 we construct the realization as in Section 2.

References

[1] S. Kerov, G. Olshanski, A. Vershik, *Harmonic analysis on the infinite symmetric group*, RT-0312270.

[2] A. Okounkov, *The Thoma theorem and representation of the infinite bisymmetric group*, Funct. Anal. Appl. 28 (1994), no. 2, 100 – 107.

[3] A. Okounkov, *On the representation of the infinite symmetric group*, RT-9803037.

[4] P. Etingof and S. Montarani *Finite dimensional representations of symplectic reflection algebras associated to wreath products*, math. RT-0403250.

[5] E. Thoma, *Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen symmetrischen Gruppe*, Math. Zeitschr. 85 (1964), no. 1, 40 – 61.

[6] G. Olshanski, *An introduction to harmonic analysis on the infinite symmetric group*, RT-0311369.

[7] G. Olshanski, *Unitary representations of \((G,K)\)-pairs connected with the infinite symmetric group \(S(\infty)\)*, Algebra i Analiz 1 (1989), no. 4, 178 – 209 (Russian); English translation in Leningrad Math. J. 1 (1990), no. 4, 983 – 1014.

[8] A. Vershik and S. Kerov, *Characters and factor representations of the infinite symmetric group*, Soviet Math. Dokl., 23 (1981), no. 2, 389 – 392.
[9] O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1. $C^*$-- and $W^*$--Algebras Symmetry Groups Decomposition of States*, 2nd Eduction Springer-Verlag Berlin Heidelberg New York, 1987, 520p.

Authors:

Nessonov Nikolay, Institute For Low Temperature Physics and Engineering, Department of Mathematics, 47 Lenin Avenue, Kharkiv, Ukraine, (0572)-30-85-85
nessonov@ilt.kharkov.ua

Dudko Artem, Kharkov National University,
artemdudko@rambler.ru