Dynamo experiments in torsioned toroidal devices

by

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Abstract

Recently Shukurov, Stepanov and Sokoloff [Phys. Rev. E 69 (2008)] have suggested that Moebius flows can support dynamo action. In this report, it is shown that a steady perturbation of a magnetic field in a general twisted Riemannian flux tube may support dynamo action. Instead of the twist number used in the above reference, the focus here is on the Frenet torsion of the magnetic flux tube. A relation between the constant torsion of the screw dynamo torus, its internal radius, and the ratio between toroidal and poloidal flow. Solution of self-induction equation for the screw dynamo torsioned flow, can be solved to yield a Frenet torsion as high as $\tau_0 \approx 7.5 \times \Omega^\theta H z^{-1} m^{-1}$, for an applied random magnetic field of $< B^0 > \approx 45 G$ for an induced steady perturbation of $B_1 \approx 0.03 G$, as in the Perm Riemannian torus experiment. The Moebius strip plays the role of the propeller divertor which imprints the angular velocity $\Omega^\theta$, around the torus axis, to the dynamo flow, here this role is played by torsion. Actually this situation is already familiar to plasma physicists, where in stellarators torsion substitutes very well the role played by external magnetic fields in tokamaks making magnetic fields twist along plasma toroidal devices. Weak torsion of the torus channel is assumed. Solution of the equation for the random flow, yields a magnetic field maximum growth rate of the order of $\gamma_{max} \approx 6 \times 10^{-2} \Omega^\theta$. Just to give an idea of how weak this is a galactic dynamo, may give rise to a growth rate of $\gamma_G \approx 10^{-5} s^{-1}$. In the previous expression, one notes that the torsion contributes to dynamo action. Actually a small torsion may be used to achieve galactic dynamos growth rate. For magnetic Reynolds number of $R_m \approx 16$ the torsion is shown to act as a breaking mechanism of the rotating Perm torus dynamo.

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I Introduction

The investigation of the kinematic screw (helical) dynamo action [1] in the flow of a cylindrical periodic tube with conducting wall, has been addressed by Dobler, Frick and Stepanov [2], by making use of an eigenvalue analysis of the steady regime, and the three-dimensional solution of the time-dependent self-induction equation. In the Perm dynamo experiment [3, 4, 5] in Russia, a torus device is used. The first type of dynamo flow on a Riemannian toroidal space, where a uniform stretched flow presented dynamo action has been proposed for the first time by Arnold, Zeldovich, Ruzmaikin and Sokoloff [6]. More recently two examples of dynamo action in Riemannian space [7, 8] has been presented. The first example was a stretch-twist and fold fast dynamo action in conformal Riemannian manifolds. The second example is application of the anti-fast-dynamo theorem by Vishik [9] to the plasma devices. Yet more recently, Shukurov, Stepanov and Sokoloff [10], has been proposed a Moebius strip flow, may support dynamo action, where the natural twist of the Moebius strip is used to substitute the divertor fan of the Perm torus dynamo experiment, in the creation of the screw of the dynamo flow. This technique is actually similar to the one used by plasma physicists in the stellarators devices, where the presence of the Frenet torsion plays the role of external magnetic fields in tokamaks, to induce the existence of helical highly conducting magnetic fields. Since Frenet torsion of the magnetic flux tube is part of the twist [11] of the tubes, a natural extension of the Moebius strip dynamo flow, would be to consider the torsioned dynamo flow action. In this paper such an enterprise is undertaken. As in the Moebius dynamo flow experiment proposal, here one assumes that the magnetic Reynolds $R_m$ is small as $R_m \approx 16$. This paper is organized as follows: In section 2 the perturbed equation is solved and the equation between torsion of the dynamo flow and the ratio between toroidal and poloidal components of the flow is deduced. In section 3, the random time dependent self-induced equation is solved and the dynamo growth rate is determined. Section 4 presents discussions and conclusions.
II Screw dynamo flows in Riemannian space

Let us start this section, by defining the perturbed random magnetic flow field \(< B^0 >\) as

\[
B = < B^0 > + B_1
\]  

(II.1)

where \(B_1\) is the magnetic field stationary perturbation, while \(< B^0 >\) is the random applied field. By substitution of this expression into the self-induction equation

\[
\partial_t B = \nabla \times (v \times < B >) + \lambda \Delta B
\]

(II.2)

yields

\[
\partial_t < B^0 > = \gamma < B^0 > = \nabla \times (v \times < B_1 >) + \lambda \Delta < B^0 >
\]

(II.3)

where \(\lambda\) is the diffusive coefficient, \(< B^0 > = e^{\gamma t} B^0 t\) and \(\Delta = \nabla^2\) is the Laplacian operator, which gradient operator \(\nabla\) is given in Riemannian curvilinear coordinates by

\[
\nabla = e_r \partial_r + e_\theta \frac{1}{r} \partial_\theta + t \partial_s
\]

(II.4)

where thin tube metric can be obtained from the Riemannian metric of the twisted magnetic flux tube [8]

\[
dl^2 = dr^2 + r^2 d\theta_R^2 + K^2(r, s) ds^2
\]

(II.5)

by taking \(K(r, s) = (1 - \kappa r \cos \theta) := 1\), where \(\kappa\) is the Frenet curvature and the twist transformation angle is given by

\[
\theta(s) := \theta_R - \int \tau(s) ds
\]

(II.6)

one obtains the Riemannian line element of the thin flux tube

\[
dl^2 = dr^2 + r^2 d\theta_R^2 + ds^2
\]

(II.7)

which gives rise the above gradient del operator \(\nabla\). Along with the Frenet frame \((t, n, b)\) one is able to solve the above equations. The perturbation first order equation becomes [12]

\[
\partial_t < B_1 > = \gamma < B^0 > = \nabla \times (v \times B_1 - < v \times B_1 >) + \lambda \Delta B_1
\]

(II.8)
The other random equation can be expressed in operator form as

\[(\gamma - \lambda \Delta) < B^0 > = \nabla \times (< v \times B_1 >) \quad \text{(II.9)}\]

As one shall see in the next section \(B_1\) can be expressed in terms of \(< B^0 >\), which shows that the last equation is an eigenvalue equation. Actually the dynamo operator

\[L = \gamma - \lambda \Delta \quad \text{(II.10)}\]

has been studied in compact Riemannian manifolds by Chiconne and Latushkin [13]. In the next section one shall solve this equation to obtain the values of the growth rate. The relation between \(\lambda\) and \(R_m\) is given by

\[\lambda = \frac{vl}{R_m} \quad \text{(II.11)}\]

where \(v\) and \(l\) are respectively the typical velocity and length scales involved in the dynamo twisted torus experiment. Note here that, as in the mean-field-magnetohydrodynamics [14], the random vector magnetic field here \(< B^0 > = < B^0 > \cdot t\). In this section one can write the equation for the perturbed field by assuming that it is a steady perturbation or \(\partial_t B_1 = 0\) which reduces the self-induction equation to

\[\Delta B_1 = -\frac{1}{\lambda}(< B^0 > \cdot \nabla)v \quad \text{(II.12)}\]

By computing the Laplacian operator \(\Delta\) in Riemannian space yields

\[\Delta = [\partial_r^2 + (1 - \frac{\tau_0^2}{r^2})\partial_s^2 - \tau_0 \cos \theta \partial_r + \frac{1}{r}(\sin \theta - \cos \theta)\partial_s] \quad \text{(II.13)}\]

where \(\tau_0\) is the constant Frenet torsion of screw dynamo. Splitting the magnetic fields into its toroidal and poloidal components yields

\[B_1 = B_s t + B_\theta e_\theta \quad \text{(II.14)}\]

Assuming that \(\partial_s B_s = B_r = \partial_t B_1 = 0\), and applying this Riemannian operator \(\Delta\) into the equation (II.12), after a long computation yields

\[\Delta B_1 = [(1 - \frac{\tau_0^2}{r^2})\partial_s^2 B_1 + \frac{1}{r}(\sin \theta - \cos \theta)\partial_s B_1] = -\frac{1}{\lambda}(< B^0 > \cdot \nabla)v \quad \text{(II.15)}\]
Therefore, splitting of this equation along the Frenet frame one obtains

\[
\frac{B^s_1}{<B^0>} = -\frac{1}{\lambda}(1 + \frac{\tau_0^2}{r^2})sin\theta v_\theta 
\]

(II.16)

\[
\frac{B^\theta_1}{<B^0>} = \frac{1}{\lambda}\frac{r\tan\theta}{\tau_0^3}v_\theta 
\]

(II.17)

and

\[
\frac{1}{r}(\sin\theta - \cos\theta)[\frac{B^s_1}{<B^0>}\tau_0 + \tau_0^2 r\sin^2\theta \frac{B^\theta_1}{<B^0>}] = \frac{1}{\lambda}(-v_\theta \tau_0^2 r^2 \sin\theta + v_s \tau_0) 
\]

(II.18)

By assuming another approximation of weak torsion and performing algebraic manipulations with those three last equations yields

\[\tau_0 = \frac{4}{\sqrt{2}} \frac{v_s}{v_\theta}\]

(II.19)

Since the toroidal and poloidal components of the flow are given by \(v_s = \Omega_s R\) and \(v_\theta = \Omega_\theta r\), where \(r\) and \(R\) are respectively the internal cross-section radius and external torus radius, which are given in the Perm torus dynamo experiment [1] by \(r = 0.02 m\) and \(R = 0.08 m\), where the torus Frenet curvature is given by \(\kappa_0 = \tau_0 = \frac{1}{R}\), yields a torsion value by

\[\tau_0 = \approx 7.5 \times \Omega_\theta^2 Hz^{-1} m^{-1}\]

(II.20)

The magnetic fields are given by

\[B^s_1(Na) = -R_m \frac{v_\theta}{\nu} <B^0>\]

(II.21)

which by making use of the Liquid sodium (Na) torus dynamo data for the applied field of \(<B^0> \approx 45 G\) and the induced field of \(B_1 \approx 0.3 G\) and a \(R_m \approx 16\), one obtains

\[\frac{v_\theta}{\nu} \approx 1.0 \times 10^2\]

(II.22)

Note that poloidal velocities induced by torsion of the twisted small-scale dynamo flow is much higher than the typical velocities scales. This is interesting cause the torsion seems to be able to damp toroidal velocities and work as a brake in the torus dynamo experiment to substitute the mechanical brake of the Perm torus dynamo experiment.
In the large-scale astrophysical dynamos where typical magnetic Reynolds numbers are $R_m \approx 10^3$ this same computation yields

$$\frac{v_\theta}{v} \approx 10^{-3} \quad (\text{II.23})$$

showing clearly that the poloidal flows are much slower than the typical velocities in the large-scale dynamo flow. This result is very well known in solar physics [15].

## III Random dynamo twisted flows in torus devices

In this section a straightforward but long analytical computation shall be performed to solve the remaining random applied field induced field equation

$$[\gamma - \lambda \Delta] (<B^0> \cdot t) = \nabla \times (\mathbf{v} \times <B_1>) \quad (\text{III.24})$$

This equation yields the following expressions

$$<B^0> [\gamma + \lambda (1 - \frac{\tau_0^2}{r^2}) + T_0 \tau_0] t + (\tau_0 v_s - \frac{v_\theta}{r} - \tau_0^2) n + b (1 - \frac{\tau_0^2}{r^2}) \tau_0^2] = \frac{1}{r} [e_r \cos \theta \beta^\theta_s + e_\theta \partial_s \beta^\theta_s] \quad (\text{III.25})$$

where $\beta^\theta_s := (v^s B^\theta_1 - v^\theta B^s_1)$ and $T_0 := \frac{1}{r} (\sin \theta - \cos \theta) \tau_0$. In the above computations use has been made of the solenoidal properties of vectors $\mathbf{B}$ and $\mathbf{v}$ as

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0 \quad (\text{III.26})$$

which in Riemannian curvilinear coordinates yields

$$\partial_s v_\theta = v_\theta \tau_0^2 r \sin \theta \quad (\text{III.27})$$

the same is valid for the magnetic field toroidal component. The equations above leads to the following expression for the growth rate

$$\gamma = (\tau_0 r \sin \theta + \cos^2 \theta) [v^s \frac{B^\theta_1}{<B^0> - v^\theta \frac{B^s_1}{<B^0>}]} \quad (\text{III.28})$$

This equation yields the maximum growth rate of the magnetic random flow as

$$\gamma_{max} = (R_m^{-1} + R_m r) \Omega^\theta \quad (\text{III.29})$$
where \( \sin \theta = \cos \theta = 1 \) has been used as the maximum of trigonometric functions. When the Reynolds number is small as in small-scale dynamos, \( R_m \approx 16 \), and by considering dynamo flows close to the dynamo torus torsioned axis (r=0), this expression reduces to

\[
\gamma_{\text{max}} = R_m^{-1} \Omega^\theta
\]  

(III.30)

which yields \( \gamma_{\text{max}} = 6 \times 10^{-2} \Omega^\theta \). Just to give an idea of how big this value is, the galactic dynamo \[16\], yields a growth rate of \( \gamma_G \approx 10^{-5} \text{s}^{-1} \).

IV Conclusions

A simple proposal to substitute divertors and breakings of Perm torus dynamo experiment, by a torsioned (twisted) flux tube in the Riemannian context. Though this solution is similar to recent proposal done by Shukurov et al [10], the basic advantage of the present proposal is that it allows analytical solutions, of the self-induction equation constraint to the weak torsion approximation of the torus dynamo device, instead of the numerical simulations undertaken by Shukurov et al. A Riemannian twisted geometry allows us to further investigate the Lyapunov exponents in a further generalization of the present model for a thick cross-section torus instead of the thin tube used here. The future analysis of the dynamo operator in the case of the thick dynamo torus device allows us also to preview physical processes that may happen in Perm torus dynamo experiment. Another advantage of the use of the twisted dynamo torus instead of the twisted Moebius strip is technological, since the technology of building a twisted torus device is already known from the stellarator plasma devices. Another motivation for the proposed dynamo magnetic flux tube torus device is that the idea of the magnetic flux tube as a dynamo has already been applied by discussed and developed by Schuessler [15] in the context of solar dynamos. Another motivation stems from the work of Wang et al [16] in the laminar plasma dynamos in cylinders.
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