Aspects of the inverse problem for the Toda chain.

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Abstract

We generalize Babelon’s approach to equations in dual variables so as to be able to treat new types of operators which we build out of the sub-constituents of the model’s monodromy matrix. Further, we also apply Sklyanin’s recent monodromy matrix identities so as to obtain equations in dual variables for yet other operators. The schemes discussed in this paper appear to be universal and thus, in principle, applicable to many models solvable through the quantum separation of variables.

Introduction

The Toda chain refers to a quantum mechanical \(N + 1\)-body Hamiltonian in one spatial dimension

\[
H_\kappa = -\sum_{a=1}^{N+1} p_n^2 + \kappa e^{x_{a+1} - x_a} + \sum_{a=1}^{N} e^{x_a - x_{a+1}} \quad \text{with} \quad p_n = \frac{\hbar}{i} \frac{\partial}{\partial x_n}.
\]

There, \(p_n\) and \(x_n\) are pairs of conjugated variables satisfying the canonical commutation \([x_k, p_\ell] = i\hbar\). Also, the index \(n\) refers to the quantum space where these operators act non-trivially. When \(\kappa = 1\), one deals with the so-called closed Toda chain whereas the model at \(\kappa = 0\) is referred to as the open Toda chain.

The classical counterpart of the model has been introduced by Toda \([36]\). Its classical integrability has been established in \([7, 16]\). Explicit formulae for the inverse action-angle map have been obtained first by Ruijsenaars \([30]\) and recently rederived by Fehér \([6]\) by means of a much simpler setting. Further, Olshanetsky and Perelomov \([28]\) constructed the quantum integrals of motion inductively whereas Kostant \([20]\) identified eigenfunctions of the open chain with Whittaker functions for \(GL(N, \mathbb{R})\). The explicit characterization of the spectrum of \(H_{\kappa=1}\) has been first investigated by Gutzwiller \([13, 14]\) for small values of \(N\) \((N=1, 2, 3)\) through a direct analysis of the partial differential equation. His main achievement was to express the eigenfunctions of the \(N+1\)-body periodic chain in terms of an integral transform whose kernel corresponds to the generalized eigenfunctions of the \(N\)-body open Toda chain. This integral transform also involved a function solving a second order difference equation in one variable, the \(T - Q\) equation \([4]\). Gaudin and Pasquier were the first to obtain the operator valued \(T - Q\) equations associated with this model, this for any value of \(N\). Then, Sklyanin \([34]\) introduced the so-called

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quantum separation of variables what allowed him to derive the scalar form of the \( T - Q \) equations for the Toda chain this, as well, for any value of \( N \). In fact, the quantum separation of variables is realised by means of an integral transform. Namely, define the transform

\[
\Phi(x_{N+1}) = \int_{\mathbb{R}^{N+1}} \Psi_{y_N;x}(x_{N+1}) \tilde{\Phi}(y_N;\epsilon) \cdot \frac{d\mu(y_N)}{\sqrt{N!}} \mathrm{d}\epsilon ,
\]  

(0.2)

where the subscript indicates the dimensionality of the vectors, \( x_{N+1} = (x_1, \ldots, x_{N+1}) \) and \( y_N = (y_1, \ldots, y_N) \). Finally, \( d\mu(y_N) \) is the Sklyanin measure. The purpose of this transform is to map the multidimensional spectral problem associated with \( H_{k=1} \) onto a one dimensional spectral problem. It is in this sense that one speaks of separation of variables.

As observed by Gutzwiller, the correct object for defining the kernel of the integral transform are eigenfunctions \( \varphi_{y_N}(x_N) \) of the open Toda chain -\( H_{k=0} \)- with \( N \)-particles. Namely one should take

\[
\Psi_{y_N;x}(x_{N+1}) = e^{\epsilon \left( \tilde{\Phi}_0 \right)_N} \cdot \varphi_{y_N}(x_N)
\]  

(0.3)

The map \( \tilde{\Phi} \rightarrow \Phi \), defined\(^1\) on \( L^2_{\text{sym}}(\mathbb{R}^N \times \mathbb{R}, d\mu(y_N) \otimes \mathrm{d}\epsilon) \), extends to an unitary map from \( L^2_{\text{sym}}(\mathbb{R}^N \times \mathbb{R}, d\mu(y_N) \otimes \mathrm{d}\epsilon) \) onto \( L^2(\mathbb{R}^{N+1}, d^{N+1}x) \). This unitarity has been first established, within group theoretical based arguments. The work \(^{[31]}\) proved the orthogonality that is to say the isometric character of the inverse transform whereas completeness follows from the arguments that can be found in \(^{[37]}\). Then, in \(^{[32]}\) a formal proof, based on techniques developed in \(^{[5]}\), of the orthogonality of the transform has been proposed. Finally, the author \(^{[22]}\) gave recently a new proof of the transform’s unitarity. The proof given by the author was based, on the one hand, on bringing rigour to the arguments of \(^{[32]}\) and, on the other hand, in developing a new technique allowing one to prove completeness this solely by using the quantum inverse scattering framework. Unitarity being established, the characterization of the spectrum boils down to solving the model’s \( T - Q \) scalar equations as shown by An \(^{[1]}\). The latter’s solution can be described either on the algebraic \(^{[8]}\) or non-linear integral equation \(^{[23]}\) levels quite explicitly. Hence, it is quite fair to state that, as of today, the understanding of the structure of the space of states and of the model’s spectrum are quite good.

However, from the perspective of applications to physics, it is the access to a model’s correlation functions that is the most interesting. Taking into account the natural simple description of the closed Toda chain’s eigenfunction on the space \( L^2_{\text{sym}}(\mathbb{R}^N \times \mathbb{R}, d\mu(y_N) \otimes \mathrm{d}\epsilon) \), it appears most reasonable to compute correlation functions by solving the so-called inverse problem, \( \epsilon \mathrm{c} \mathrm{m} \) compute the expectation values \( \langle \Phi_1 | O | \Phi_2 \rangle \) by passing to the representation of the model’s Hilbert space on \( L^2_{\text{sym}}(\mathbb{R}^N \times \mathbb{R}, d\mu(y_N) \otimes \mathrm{d}\epsilon) \). This operation means that one should manage to express how operators \( O \) having a simple (\( \epsilon \mathrm{c} \mathrm{m} \) local) action on the model’s original space act directly on the space of functions where the separation of variables occurs. Due to the structure of the transform \(^{[0.2]}\) it is, in fact, enough to determine how the action of such operators translates itself on the dual \( (y_N;\epsilon) \) variables of the kernel functions \( \Psi_{y_N;x}(x_{N+1}) \), \( \epsilon \mathrm{c} \mathrm{m} \) obtain an equation

\[
O \cdot \Psi_{y_N;x}(x_{N+1}) = \tilde{O} \cdot \Psi_{y_N;x}(x_{N+1})
\]  

(0.4)

in which the operator \( O \) acts on the space variables \( x_{N+1} \) whereas its dual operator \( \tilde{O} \) acts on the dual ones \( y_N;\epsilon \).

The inverse problem for integrable models solvable by the algebraic Bethe Ansatz method has been first solved in \(^{[19]}\) and further developed in \(^{[25]}\)\(^{[29]}\). One can in fact say that, within today’s state of the art, the resolution of the inverse problem for models solvable by the algebraic Bethe Ansatz is quite well understood. The situation is however not so well established in what concerns models solvable by the quantum separation of variables method.

\(^1\)the subscript \( \text{sym} \) indicates that the functions are symmetric in the first set of \( N \) variables.
In [3], Babelon conjectured, on the basis of semi-classical arguments, the form certain local operators associated with the Toda chain take on $L^2_{\text{sym}}(\mathbb{R}^N \times \mathbb{R}, d\mu(y_N) \otimes de)$. He then justified [2] one set of his formulae by computing the action of these operators on Whittaker functions. Recently, Sklyanin [33] managed to reproduce Babelon's formulae through simple algebraic arguments based on the quantum inverse scattering approach to the quantum Toda chain.

A different route to solving inverse problems for certain quantum separation of variables models - those associated with finite dimensional representations attached to each lattice node - has been proposed in [12]. This method builds on Oota’s [29] ideas for solving the inverse problem for algebraic Bethe Ansatz solvable models as well as on certain properties associated with the finite dimensionality of the representations. It was applied to other models in a subsequent series of works, see eg [27]. However, the method works only, per se, for inhomogeneous deformations of an integrable model of interest. Although, within such an approach, the final expression for the correlation functions have a well-defined homogeneous limit, its characterisation in a convenient form still remain an open problem.

In the present paper, we push forward the techniques developed by Babelon [2, 3] and demonstrate that one can derive equations in dual variables for more general operators. Such operators are built out of certain sub-components of the model’s monodromy matrix. Due to the natural quantum inverse scattering method interpretation of these operators, we believe that our construction can be generalized to other, more complex models. Furthermore, in fact, we show that Sklyanin’s recent observations [33] allow one to obtain equations in dual variables for an even larger class of operators.

The paper is organised as follows. In section 1 we revisit certain aspects of the quantum integrability of the Toda chain. After recalling the main ingredients of the quantum inverse scattering method approach to this model, we build new types of Mellin-Barnes multiple integral representations for the function $\Psi_{y_N}(x_{N+1})$. In section 2 we gather the main results of this paper, namely, a set of equations in dual variables for certain classes of operators built out of sub-components of the monodromy matrix. Most of the proofs and technical details are gathered in three appendices.

1 Integrability of the quantum Toda chain

1.1 The Lax matrix formulation

The quantum integrability of the Toda chain can be described within the framework of the quantum inverse scattering method. The central object in this approach is a $2 \times 2$ Lax matrix

$$L_{0n}(\lambda) = \begin{pmatrix} \lambda - p_n & e^{-x_n} \\ -e^{x_n} & 0 \end{pmatrix} \quad \text{with} \quad [x_k, p_\ell] = i\hbar \delta_{k\ell}. \quad (1.1)$$

It is straightforward to check that the latter satisfies the below quadratic algebra

$$R_{00'}(\lambda - \mu)L_{0n}(\lambda)L_{0'n}(\mu) = L_{0'n}(\mu)L_{0n}(\lambda)R_{00'}(\lambda - \mu) \quad (1.2)$$

where the $4 \times 4$ $R$-matrix reads:

$$R_{00'}(\lambda) = \begin{pmatrix} \lambda + i\hbar & 0 & 0 & 0 \\ 0 & \lambda & i\hbar & 0 \\ 0 & i\hbar & \lambda & 0 \\ 0 & 0 & 0 & \lambda + i\hbar \end{pmatrix}. \quad (1.3)$$
Out of such matrices, one builds the so-called monodromy matrix of the model as an ordered product of local Lax matrices:

$$T_{0;1,N+1}(\lambda) = L_0(\lambda) \ldots L_{0N+1}(\lambda) = \begin{pmatrix} A_{1,N+1}(\lambda) & B_{1,N+1}(\lambda) \\ C_{1,N+1}(\lambda) & D_{1,N+1}(\lambda) \end{pmatrix}_{[0]} .$$ \hspace{1cm} (1.4)

The ultra-local algebra \((1.2)\) satisfied by the Lax matrices raises to a quadratic algebra, the so-called Yang–Baxter algebra, relating the entries of the monodromy matrix:

$$R_{0\nu}(\lambda - \mu)T_{0;1,N+1}(\lambda)T_{0';1,N+1}(\mu) = T_{0';1,N+1}(\mu)T_{0;1,N+1}(\lambda)R_{0\nu}(\lambda - \mu) .$$ \hspace{1cm} (1.5)

The relations encoded in the above algebra are sufficiently rich so as to provide one with the full spectrum and complete set of eigenfunctions of the \(N\)-body closed Toda chain. The simplest, yet by no means less important, consequence of the above Yang–Baxter algebra for the monodromy matrix is the possibility to provide a set of \(N + 1\) Hamiltonians in involution, which, in particular, contains \(H_{|\kappa=1|}\). In order to do so, one defines the so-called transfer matrix \(\tau(\lambda) = \text{tr}_0[T_{0;1,N+1}(\lambda)]\) of the model. The Yang–Baxter equation ensures that \(\tau(\lambda)\) gives rise to a one parameter \(\lambda\) commutative subalgebra of operators on \(L^2(\mathbb{R}^{N+1}, d^{N+1}\lambda)\). Since \(\tau(\lambda)\) is a monoic operator valued polynomial in \(\lambda\) of degree \(N + 1\), the transfer matrix gives rise to a set of \(N + 1\) Hamiltonians in involution. These can be, for instance, defined as the coefficients arizing in the \(\lambda\)-expansion of \(\tau(\lambda)\):

$$\tau(\lambda) = \lambda^{N+1} + \sum_{k=1}^{N+1} (-1)^k \lambda^{N-k+1} \tau_k \quad \text{with} \quad \tau_1 = \sum_{a=1}^{N+1} p_a \quad \text{and} \quad \tau_2 = \tau_1 - H_{|\kappa=1} .$$ \hspace{1cm} (1.6)

The Lax matrix given in \((1.1)\) can be explicitly inverted with the help of the quantum determinants relation

$$[L_{0\nu}(\lambda)]^{-1} = \begin{pmatrix} 0 & e^{-\lambda n} \\ e^{\lambda n} & \lambda - i\hbar - p_n \end{pmatrix}_{[0]} = \sigma_0^y \cdot L_{0\nu}^h(\lambda - i\hbar) \cdot \sigma_0^y .$$ \hspace{1cm} (1.7)

Just as in the case of the local Yang–Baxter algebra, this local inversion formula can be raised to the level of the monodromy matrix leading to

$$[T_{0;1,N+1}(\lambda)]^{-1} = \sigma_0^y T_{0;1,N+1}(\lambda - i\hbar) \sigma_0^y .$$ \hspace{1cm} (1.8)

This explicit realisation for the inverse of the monodromy matrix leads to the so-called quantum determinant relations. By reading out the various entries of the matrix product in the identity \([T_{0;1,N+1}(\lambda)] \cdot [T_{0;1,N+1}(\lambda)]^{-1} = I_2 \otimes \text{id}_{L^2(\mathbb{R}^N)}\) one obtains

$$\begin{cases} A_{1,N+1}(\lambda)D_{1,N+1}(\lambda - i\hbar) - B_{1,N+1}(\lambda)C_{1,N+1}(\lambda - i\hbar) = \text{id}_{L^2(\mathbb{R}^{N+1})} \\
D_{1,N+1}(\lambda)A_{1,N+1}(\lambda - i\hbar) - C_{1,N+1}(\lambda)B_{1,N+1}(\lambda - i\hbar) = \text{id}_{L^2(\mathbb{R}^{N+1})} \end{cases}$$ \hspace{1cm} (1.9)

and also

$$\begin{cases} C_{1,N+1}(\lambda)D_{1,N+1}(\lambda - i\hbar) = D_{1,N+1}(\lambda)C_{1,N+1}(\lambda - i\hbar) \\
B_{1,N+1}(\lambda)A_{1,N+1}(\lambda - i\hbar) = A_{1,N+1}(\lambda)B_{1,N+1}(\lambda - i\hbar) \end{cases}.$$ \hspace{1cm} (1.10)

It is readily seen that \(B_{1,N+1}(\lambda) = e^{-\lambda x_{N+1}}A_{1,N}(\lambda)\) so that \(B_{1,N+1}(\lambda)\) is realized as a multiplication operator in respect to \(x_{N+1}\). Furthermore, the operator \(A_{1,N}(\lambda)\) is the generating function of the integrals of motion for the open \(N\)-particle Toda chain.
In fact, one way of defining the integral kernel \( \Psi_{y,N,x}(x_{N+1}) \) of the SoV transform (1.2) is as the functions satisfying to the equation

\[
B_{1,N+1}(\lambda)\Psi_{y,N,x}(x_{N+1}) = \prod_{a=1}^{N} (\lambda - y_a) \cdot \Psi_{y_N,x+j\hbar}(x_{N+1}) .
\] (1.11)

The latter equation, along with a multiplicity one theorem \([11, 22, 31, 37]\), i.e. the completeness and orthogonality of the system \( \Psi_{y_N,x}(x_{N+1}) \), completely characterises these functions.

In order to establish the equations in dual variables for \( \Psi_{y_N,x}(x_{N+1}) \) that are at the core of the present paper, we shall construct a generalization of the Mellin-Barnes multiple integral representation for \( \Psi_{y_N,x}(x_{N+1}) \) that has been initially obtained by Kharchev and Lebedev \([17, 18]\) following the procedure suggested by Sklyanin in \([35]\). More precisely, the Mellin-Barnes integral representation is obtained recursively by demanding that equation (1.11) holds and acting with the \( B_{1,N+1} \)-operator by means of its decomposition associated with the splitting of the original chain of length \( N + 1 \) into two-sub chains of respective lengths \( r \) and \( N + 1 - r \). The case originally dealt with by Kharchev and Lebedev corresponds to \( r = 1 \). However many properties of the Whittaker functions and in particular the equations in the dual variables are more apparent to see from representations subordinate to a splitting with a general \( r \). Indeed, as observed by Babelon \([2]\), the Mellin-Barnes representation is perfectly adapted for solving the inverse problem. The main technical reason for this is that this multiple integral representation, as opposed to the Gauss–Givental one \([10]\), does not treat all the entries of the vector \( x_{N+1} \) on an equal footing but rather singles out a portion thereof.

### 1.2 Inductive construction of the Mellin-Barnes representation and its basic properties

It will be convenient, in the following, to introduce the notation \( \bar{z}_k \), which means that for any \( k \)-dimensional vector \( z_k = (z_1, \ldots, z_k), k \in \mathbb{N}, \)

\[
\bar{z}_k = \sum_{s=1}^{k} z_s .
\] (1.12)

Sometimes, when there will be no ambiguity on the dimensionality of the vector, we will simply drop the associated subscript, e.g.

\[
\bar{y} = \sum_{p=1}^{N} y_p .
\] (1.13)

**Definition 1.1** Let \( y_N \in \mathbb{C}^N \) and \( \varepsilon \in \mathbb{C} \) be given. Then, define the function \( \Psi_{y_N,x}(x_{N+1}) \) inductively as follows. First set

\[
\Psi_{0,x}(x) = e^{\frac{\text{i} \varepsilon}{\hbar} x}
\] (1.14)

and then define the collection of functions \( \Psi_{y_N,x}(x_{N+1}) \) by

\[
\Psi_{y_N,x}(x_{N+1}) = \int_{\mathbb{C}^{N-r}} \Psi_w \Psi_{x-N-y}(x_1) \Psi_{z-x-y}(x_2) \sigma(w, z \mid y_N) \cdot \prod_{a=1}^{r-1} \int_{\mathbb{C}^{N-r}} dw_a \cdot \prod_{a=r}^{N-1} dz_a .
\] (1.15)

There the integration runs through two sets of variables \( w_a \) and \( z_b \) which are collected in a vector notation

\[
w = (w_1, \ldots, w_{r-1}, 0, \ldots, 0) \quad \text{and} \quad z = (0, \ldots, 0, z_{r}, \ldots, z_{N-1}) .
\] (1.16)
The vectors
\[ x_1 = (x_1, \ldots, x_r) \quad \text{and} \quad x_2 = (x_{r+1}, \ldots, x_{N+1}) \]
(1.17)

correspond to a splitting of the coordinates of the position vector \( x_{N+1} = (x_1, \ldots, x_{N+1}) \). Further, the integration runs through the domain
\[ \mathcal{C}_{r-1,N-r} = (\mathbb{R} - i\alpha)^{-1} \times (\mathbb{R} + i\alpha)^{N-r} \]
where \( \alpha \) is such that \( \alpha > \max_{k \in \{1; N\}} |\mathcal{S}(y_k)| \).
(1.18)

Finally, the weight \( \sigma(w, z \mid y_N) \) arising under the integral sign is given by
\[
\sigma(w, z \mid y_N) = \frac{(2\pi\hbar)^{1-N}}{(r-1)!(N-r)!} \cdot \prod_{a=1}^{r-1} \prod_{b=1}^N \left\{ \Gamma \left( \frac{y_b - w_a}{\hbar} \right) \right\} \cdot \prod_{a=r}^N \left\{ \Gamma \left( \frac{z_a - y_b}{\hbar} \right) \right\}.
\]
(1.19)

There are several points that ought to be addressed in respect to this definition.

- One should check that it is clean-cut, \( \text{i.e.} \) that the integrals (1.15) are indeed convergent;
- one should establish the natural properties of the functions defined through (1.15) such as their asymptotic behaviour in the \( y_a \)'s and their regularity properties in respect to the variables \( (y_N, \varepsilon) \) and \( x_{N+1} \);
- one should establish that it is consistent, \( \text{i.e.} \) that the functions \( \Psi_{y_N \varepsilon}(x_{N+1}) \) do not depend on the value of the integer \( r \) used in the splitting of the integration variables (1.16).

We shall investigate the first two points in the proposition below.

**Proposition 1.1** Let \( \eta > 0 \) be fixed and such that \( |\mathcal{S}(y_k)| < \eta \) for all \( k = 1, \ldots, N \) as well as \( |\mathcal{S}(\varepsilon)| < \eta \). Then, given \( N \), there exist constants \( C_1, C_2 \) and \( M \) such that the functions defined through (1.15) satisfy to the bounds
\[
|\Psi_{y_N \varepsilon}(x_{N+1})| \leq C_1 e^{C_2|y_{N+1}|} \cdot (1 + |y_N|)^M \cdot \exp \left\{ -\frac{\pi}{2\hbar} \sum_{a,b}^{N} |\mathcal{S}(y_a - y_b)| \right\}.
\]
(1.20)

In particular, the multiple integral -independently of the value of \( r \) - in (1.15) is convergent and defines an entire functions of \( (y_N, \varepsilon, x_{N+1}) \). Note that \( \| \cdot \| \) refers to the \( L^1 \) norm on \( \mathbb{R}^k \), with \( k \)-being the dimensionality of the vector, namely, for a \( p \)-dimensional vector \( v_p \), one has
\[
\|v_p\| = \sum_{a=1}^p |v_a|.
\]
(1.21)

Finally, the constants \( C_1, C_2 \) in (1.20) do not depend on the chain of splittings \( r \) used in the application of the definition (1.17) so as to recursively build the function \( \Psi_{y_N \varepsilon}(x_{N+1}) \).

We postpone the proof of proposition (1.1) to appendix A and continue by listing the properties of the \( \Psi_{y_N \varepsilon}(x_{N+1}) \) functions in respect to the action of matrix entries of the monodromy matrix as well as by establishing the independence on the splitting \( r \) of the whole construction.
Proposition 1.2 The part of construction (1.15) for the function \( \Psi_{\gamma, \varepsilon}(x_{N+1}) \) does not depend on \( r \) and the latter function satisfies to the identities:

\[
[B_{1,N+1}(\lambda)\Psi_{\gamma, \varepsilon}(x_{N+1})] = \prod_{a=1}^{N} (\lambda - y_a) \cdot \Psi_{\gamma, \varepsilon + \imath \hbar}(x_{N+1}) .
\] (1.22)

Also, \( \Psi_{\gamma, \varepsilon} \) fulfils

\[
[D_{1,N+1}(\lambda)\Psi_{\gamma, \varepsilon}(x_{N+1})] = \sum_{p=1}^{N} (-i)^{N+1} \prod_{s=1}^{N} N_{p \neq s} (\lambda - y_s) \cdot \Psi_{\gamma, \varepsilon - i\hbar \epsilon_{p}, \varepsilon}(x_{N+1}) ,
\] (1.23)

as well as

\[
[A_{1,N+1}(\lambda)\Psi_{\gamma, \varepsilon}(x_{N+1})] = (\lambda - \varepsilon + \sum_{p} \hbar \epsilon_{p}) \prod_{\ell=1}^{N} (\lambda - y_{\ell}) \Psi_{\gamma, \varepsilon}(x_{N+1}) + \sum_{p=1}^{N} (-i)^{N+1} \prod_{s=1}^{N} N_{p \neq s} (\lambda - y_s) \cdot \Psi_{\gamma, \varepsilon - i\hbar \epsilon_{p}, \varepsilon}(x_{N+1}) .
\] (1.24)

There \( \epsilon_{p} \) stands for the unit vector in \( \mathbb{R}^N \) with a 1 solely in its \( p \)th entry:

\[
\epsilon_{p} = (0, \ldots, 0, 1, 0, \ldots, 0) .
\] (1.25)

Finally, one also has the identities

\[
\prod_{a=1}^{N} e^{i \lambda} \cdot \Psi_{\gamma, \varepsilon}(x_{N+1}) = \Psi_{\gamma, \varepsilon - i\hbar \epsilon_{p}, \varepsilon}(x_{N+1}) \quad \text{and} \quad e^{-\ell \lambda} \cdot \Psi_{\gamma, \varepsilon}(x_{N+1}) = \Psi_{\gamma, \varepsilon + i\hbar \epsilon_{p}, \varepsilon}(x_{N+1}) .
\] (1.26)

The vector \( \epsilon \) introduced above takes the form \( \epsilon = \sum_{p=1}^{N} \epsilon_{p} . \)

The proof of the proposition is postponed to appendix [B].

We remind that the action of the \( C_{1, \ldots, N+1}(\lambda) \) operator on \( \Psi_{\gamma, \varepsilon}(x_{N+1}) \) can be obtained from the quantum determinant relation

\[
C_{1, N+1}(\lambda) = (D_{1, N+1}(\lambda) \cdot A_{1, N+1}(\lambda - i\hbar) - 1) \cdot B_{1, N+1}^{-1}(\lambda - i\hbar) .
\] (1.27)

Corollary 1.1 The action of the operator \( C_{1, N+1}(\lambda) \) on the function \( \Psi_{\gamma, \varepsilon}(x_{N+1}) \) takes the form

\[
C_{1, N+1}(\lambda) \cdot \Psi_{\gamma, \varepsilon}(x_{N+1}) = (\lambda - \varepsilon + \sum_{p} \hbar \epsilon_{p}) \prod_{a=1}^{N} (-i)^{N+1} \prod_{s=1}^{N} N_{p \neq s} (\lambda - y_s) \cdot \Psi_{\gamma, \varepsilon - i\hbar \epsilon_{p}, \varepsilon}(x_{N+1})
\]
\[
+ \sum_{p, r=1}^{N} \sum_{s=1}^{N} \prod_{a=p}^{N} (y_{a} - y_{s}) \prod_{b=p, r \neq s}^{N} \frac{1}{y_{a} - y_{b}} \cdot \Psi_{\gamma, \varepsilon - i\hbar \epsilon_{p}, \varepsilon}(x_{N+1}) + W((y_{a})_{N}^{*}, \lambda) \cdot \Psi_{\gamma, \varepsilon - i\hbar \epsilon_{p}, \varepsilon}(x_{N+1}) ,
\] (1.28)

where \( W((y_{a})_{N}^{*}, \lambda) \) is given by

\[
W((y_{a})_{N}^{*}, \lambda) = - \sum_{p=1}^{N} \prod_{s=1}^{N} \frac{\lambda - y_{s}}{y_{p} - y_{s}} \cdot \sum_{\epsilon = \pm} \prod_{r=1}^{N} \frac{1}{y_{p} - y_{r} + i\epsilon \hbar} .
\] (1.29)
Proof —
A direct calculation allows one to recast the action of the operator \( C_{1\ldots N+1}(\lambda) \) in the form (1.28) with

\[
W(y_{a}^{N+1}; \lambda) = \sum_{p=1}^{N} \frac{1}{\lambda - y_{p} - i\hbar} \cdot \prod_{s \neq p}^{N} \frac{\lambda - y_{s}}{(y_{p} - y_{s}) (y_{p} - y_{s})} - \prod_{\ell=1}^{N} \left[ \frac{1}{\lambda - y_{\ell} - i\hbar} \right].
\] (1.30)

In order to relate (1.30) to (1.29), we observe that \( W(y_{a}^{N+1}; \lambda) \) is a rational function of \( \lambda \) such that \( W(y_{a}^{N+1}; \lambda) = O(\lambda^{-2}) \) when \( \lambda \to \infty \) and with potential poles at \( \lambda = y_{\ell} + i\hbar, \ell = 1, \ldots, N \). However, one has that

\[
\text{Res}(W(y_{a}^{N+1}; \lambda) \cdot d\lambda, \lambda = y_{p} + i\hbar) = \prod_{\ell=1}^{N} \left[ \frac{1}{y_{p} - y_{\ell}} \right] - \prod_{\ell=1}^{N} \left[ \frac{1}{y_{p} - y_{\ell}} \right] = 0.
\] (1.31)

It thus follows that \( W(y_{a}^{N+1}; \lambda) \) is a polynomial in \( \lambda \) of degree \( N - 2 \). As such it can be reconstructed by an interpolation at the points \( \lambda = y_{a}, a = 1, \ldots, N \). It is readily seen that one has

\[
W(y_{a}^{N+1}; y_{r}) = - \prod_{\ell=1}^{N} \left[ \frac{1}{y_{r} - y_{\ell} + i\hbar} \right] - \prod_{\ell=1}^{N} \left[ \frac{1}{y_{r} - y_{\ell} - i\hbar} \right].
\] (1.32)

Hence the expression (1.29) for \( W(y_{a}^{N+1}; \lambda) \) follows.

\[\blackslug\]

2 Resolution of the inverse problem

In this section, we show how to express the action of certain specific local operators on the kernel \( \Psi_{y_{a},x}(x_{N+1}) \) of the SoV transform in terms of an action on its dual variables. In their turn, these dual equations show that the \( L_{2}^{2}(\mathbb{R}^{N} \times \mathbb{R}, d\mu(y_{N}) \otimes dx) \) where the quantum separation of variables occurs.

2.1 The position operators

Having established a specific representation for the system of eigenfunctions of the \( B_{1\ldots N+1}(\lambda) \) operator, we are in position to compute the action of a certain class of operators on the kernel of the SoV transform \( \Psi_{y_{a},x}(x_{N+1}) \). This result allows us to deduce the action on \( \Psi_{y_{a},x}(x) \) of a complete system of local operators. This provides a solution to the inverse problem.

Proposition 2.1 The operator

\[
O_{r}(\lambda) = \prod_{a=1}^{r} \left\{ e^{x_{a} - y_{N+1}} \right\} \cdot D_{r+1,N+1}(\lambda)
\] (2.1)

has the below action on the kernel \( \Psi_{y_{a},x}(x_{N+1}) \) of the SoV transform

\[
O_{r}(\lambda) \cdot \Psi_{y_{a},x}(x_{N+1}) = - \sum_{I_{N}=\sigma \cup \tau + 1} \prod_{b \geq a} \left( \frac{-i}{y_{a} - y_{b}} \right) \prod_{b \geq \sigma} (\lambda - y_{b}) \cdot \Psi_{y_{a} - i\hbar, x_{a}, \ldots, x_{\sigma}}(x_{N+1}),
\] (2.2)

where \( I_{N} = \{ 1 ; N \} \) and the sum runs through all partitions \( \sigma \cup \tau \) of \( I_{N} \) under the constraint \( \# \sigma = r + 1 \).
We postpone the proof of the above proposition to appendix \[C\].

Note that the formula for the action of the operator \(O_r(\lambda)\) given in \[(2.2)\] allows one to access to the one for products of position operators. Indeed, one has the

**Corollary 2.1** The operator \(\prod_{a=1}^{r} \left( e^{y_a x_{N+1}} \right)^{-1} \) has the below action on \(\Psi_{y_N, x}(x_{N+1})\)

\[
\prod_{a=1}^{r} \left( e^{y_a x_{N+1}} \right)^{-1} \cdot \Psi_{y_N, x}(x_{N+1}) = \sum_{y_N = \sigma N}^{\sigma N} \prod_{a \neq c} \left( \frac{-i}{y_a - y_c} \right) \cdot \Psi_{y_N, x}(x_{N+1}).
\]

(2.3)

Note that \[(2.3)\] is precisely the equation in dual variables obtained by Babelon in \[2\].

**Proof** —
This is a straightforward consequence of formula \[(2.2)\] as soon as one observes that

\[
D_{r, N+1}(\lambda) = -\lambda^{N-r} e^{y_{r+1} x_{N+1}} + O(\lambda^{N-r-1}).
\]

(2.4)

\[\square\]

### 2.2 Reconstruction of the momentum operators

**Proposition 2.2** The following action in dual variables holds

\[
\prod_{\ell=1}^{r} \left( e^{y_{\ell} x_{N+1}} \right)^{-1} \cdot \left( \sum_{\ell=1}^{r} p_{\ell} \right) \cdot \Psi_{y_N, x}(x_{N+1}) = \sum_{y_N = \sigma N}^{\sigma N} \prod_{a \neq c} \left( \frac{-i}{y_a - y_c} \right) \cdot \Psi_{y_N, x}(x_{N+1}).
\]

(2.5)

The form of the above action, expressed directly in terms of the action of the operator on \(L_{\text{sym} - \mathbb{R}^N \times \mathbb{R}}^2 \otimes \text{d} \mu(y_N) \otimes \text{d} \epsilon\), has been conjectured by Babelon in \[3\]. Recently, the formula \[(2.5)\] has been established by Sklyanin through a slightly different procedure \[33\].

**Proof** —
A Lax matrix acting on any quantum site \(r\) of the full chain can be inverted with the help of the quantum determinant relation \[(1.7)\]. This relation allows one to obtain a recursive reconstruction for the entries of the monodromy matrix \(T_{0, r+1, \ldots, N+1}(\lambda)\) in terms of the entries of the Lax matrix on site \(r\) and the monodromy matrix \(T_{0, r, \ldots, N}(\lambda)\). Indeed, one has

\[
T_{0, r+1, N+1}(\lambda) = [L_0(\lambda)]^{-1} \cdot T_{0, r, N+1}(\lambda).
\]

(2.6)

The idea of such a recursive reconstruction has been first proposed by Kuznetzov \[24\] when he considered the inverse problem for classical \(\text{sl}(2)\)-type integrable lattices. Relations such as \[(2.6)\] allowed him to provide a recursive construction of the canonical transformation between the classical separated variables for a chain of length \(N\) and \(N + 1\).

In our case of interest, the above identity allows us to access to the action of the combination of position and momenta of interest on the function \(\Psi_{y_N, x}(x_{N+1})\). Namely, the equality between the second columns in \[(2.6)\] leads to a three-term recurrence relation satisfied by \(D\)-operators associated with chains of different lengths:

\[
D_{r, N+1}(\lambda) = -e^{y_{r+1}} (\lambda - i\hbar - p_r) D_{r-1, N+1}(\lambda) \quad \text{for} \quad r = 2, \ldots, N,
\]

(2.7)
and the initiation condition
\[ D_{2,N+1}(\lambda) = e^{x_1} B_{1,N+1}(\lambda) + (\lambda - i\hbar - p_1) D_{1,N+1}(\lambda) \, . \tag{2.8} \]

The operator \( D_{r,N+1}(\lambda) \) is an operator-valued polynomial in \( \lambda \) of degree \( N - r \):
\[ D_{r,N+1}(\lambda) = \sum_{p=0}^{N-r} \lambda^{N-r-p} \cdot d_p^{(r)} \, . \tag{2.9} \]

It is easy to check that
\[ \lim_{\lambda \to \infty} \left[ \lambda^{-N} D_{r,N+1}(\lambda) \right] = -e^{x_r - x_{N+1}} \quad \text{so that} \quad d_0^{(r)} = -e^{x_r - x_{N+1}} \, . \tag{2.10} \]

The \( \text{lhs} \) of equation (2.2) is an operator-valued polynomial in \( \lambda \) of degree \( N - r - 1 \). Since, \( \text{a priori} \), the \( \text{rhs} \) of this equation is an operator valued polynomial in \( \lambda \) of degree \( N - r + 1 \), its coefficients associated with \( \lambda^{N-r+1} \) and \( \lambda^{N-r} \) have to vanish. More precisely, (2.2) implies that
\[ O(\lambda^{N-r-1}) = e^{x_r - x_{N+1}} \lambda^{N-r+1} - e^{x_r - x_{N+1}} d_1^{(r-1)} \lambda^{N-r} - \lambda^{N-r+1} e^{x_r - x_{N+1}} \]
\[ + \lambda^{N-r} d_1^{(r)} - (i\hbar + p_r) \times -e^{x_r - x_{N+1}} \lambda^{N-r} + O(\lambda^{N-r-1}) \, . \tag{2.11} \]

The leading terms cancel out explicitly whereas the cancellation of the first sub-leading ones leads to the relation
\[ e^{x_r - x_{N+1}} p_r + d_1^{(r)} - e^{x_r - x_{N+1}} d_1^{(r-1)} = 0 \quad \text{for} \quad r = 2, \ldots, N \, . \tag{2.12} \]

We also need the relation at \( r = 1 \) so as to have a closed system. The operator \( B_{1,N+1}(\lambda) \) is a polynomial in \( \lambda \) of degree \( N \) with leading asymptotics at infinity \( e^{-x_{N+1}} \lambda^N \). It has the representation
\[ B_{1,N+1}(\lambda) = e^{-x_{N+1}} \sum_{p=0}^{N} \lambda^{N-p} b_p \, . \tag{2.13} \]

Inserting this decomposition into (2.8) and using that the coefficients in front of \( \lambda^N \) and \( \lambda^{N-1} \) in (2.8) both vanish leads to
\[ e^{x_1 - x_{N+1}} p_1 + d_1^{(1)} + e^{x_1 - x_{N+1}} b_1 = 0 \, . \tag{2.14} \]

A decreasing induction shows that
\[ \prod_{\ell=t-1}^{r} \{ e^{x_{\ell} - x_{N+1}} \} \cdot \left( \sum_{\ell=t}^{r} p_\ell \right) + \prod_{\ell=t-1}^{r} \{ e^{x_{\ell} - x_{N+1}} \} \cdot d_1^{(r)} - \prod_{\ell=t}^{r} \{ e^{x_{\ell} - x_{N+1}} \} \cdot d_1^{(r-1)} = 0 \quad \text{for} \quad r \geq t \geq 2 \, . \tag{2.15} \]

Hence, setting \( t = 2 \) and replacing \( d_1^{(1)} \) with the help of (2.14) gives
\[ \prod_{\ell=1}^{r} \{ e^{x_{\ell} - x_{N+1}} \} \cdot \left( \sum_{\ell=1}^{r} p_\ell \right) = - \prod_{\ell=1}^{r} \{ e^{x_{\ell} - x_{N+1}} \} \cdot b_1 - \prod_{\ell=1}^{r-1} \{ e^{x_{\ell} - x_{N+1}} \} \cdot d_1^{(r)} \, . \tag{2.16} \]

Equation (2.16) above recasts the combination of position and momentum operators of interest solely in terms of operators whose action on the function \( \Psi_{y, \alpha, t}(x_{N+1}) \) is known. Indeed, the action of the product of exponents in
position operators in given by \((2.3)\), whereas the action of the coefficients \(b_1\) and \(d_1^{(r)}\) can be deduced, respectively, from \((1.22)\) and \((2.2)\) which yields

\[
b_1 \cdot \Psi_{\gamma, \epsilon}^{\gamma, \epsilon}(x_{N+1}) = -\overline{y}_{N} \cdot \Psi_{\gamma, \epsilon}^{\gamma, \epsilon}(x_{N+1})
\]  
(2.17)

and

\[
\prod_{\ell=1}^{r-1} \left\{ e^{x_{\ell-N_{N+1}}} \right\} \cdot d_{1}^{(r)} \cdot \Psi_{\gamma, \epsilon}^{\gamma, \epsilon}(x_{N+1}) = \sum_{I_{N-1}} \prod_{\ell \in \gamma} \left\{ \frac{-i}{y_{a} - y_{b}} \right\} \cdot \left( \sum_{b \notin \gamma} y_{b} \right) \cdot \Psi_{\gamma, \epsilon}^{\gamma, \epsilon}(x_{N+1}) .
\]  
(2.18)

It only remains to carry out a straightforward replacement.

\[\blacksquare\]

### 2.3 Reconstruction of other operators

The below lemma has been established recently by Sklyanin [33].

**Lemma 2.1** Given any \(N\) and \(r \in \mathbb{N} ; N - 1 \) has the below operator identities

\[
A_{1,r}(\lambda) = D_{r+1,N+1}(\lambda + i\hbar) \cdot A_{1,N+1}(\lambda) - C_{r+1,N+1}(\lambda + i\hbar) \cdot B_{1,N+1}(\lambda)
\]  
(2.19)

\[
B_{1,r}(\lambda) = A_{r+1,N+1}(\lambda) - B_{r+1,N+1}(\lambda) \cdot A_{1,N+1}(\lambda)
\]  
(2.20)

\[
C_{1,r}(\lambda) = D_{r+1,N+1}(\lambda + i\hbar) \cdot C_{1,N+1}(\lambda) - C_{r+1,N+1}(\lambda + i\hbar) \cdot D_{1,N+1}(\lambda)
\]  
(2.21)

\[
D_{1,r}(\lambda) = A_{r+1,N+1}(\lambda + i\hbar) \cdot D_{1,N+1}(\lambda) - B_{r+1,N+1}(\lambda + i\hbar) \cdot C_{1,N+1}(\lambda)
\]  
(2.22)

**Proof** —

We only establish the first identity. The other ones are proved in much the same way. One has, due to the quantum determinant relation,

\[
A_{1,r}(\lambda) = A_{1,r}(\lambda) \cdot \left( D_{r+1,N+1}(\lambda + i\hbar) - C_{r+1,N+1}(\lambda + i\hbar) \right) \cdot B_{r+1,N+1}(\lambda)
\]

\[
= D_{r+1,N+1}(\lambda + i\hbar) \cdot \left( A_{1,N+1}(\lambda) - B_{1,r}(\lambda) \right) \cdot C_{r+1,N+1}(\lambda) + C_{r+1,N+1}(\lambda + i\hbar) \cdot A_{1,r}(\lambda) \cdot B_{r+1,N+1}(\lambda)
\]

\[
= D_{r+1,N+1}(\lambda + i\hbar) \cdot A_{1,N+1}(\lambda) - C_{r+1,N+1}(\lambda + i\hbar) \cdot B_{1,N+1}(\lambda)
\]  
(2.23)

where, to get the last line, we have used the off-diagonal quantum-determinant issued equations.

**Proposition 2.3** The operator

\[
\hat{\mathcal{A}}_{r}(\lambda) = \prod_{\ell=1}^{r} \left\{ e^{x_{\ell-N_{N+1}}} \right\} \cdot A_{1,r}(\lambda)
\]  
(2.24)

has the below action on the kernel \(\Psi_{\gamma, \epsilon}^{\gamma, \epsilon}(x_{N+1})\) of the SoV transform

\[
\hat{\mathcal{A}}_{r}(\lambda) \cdot \Psi_{\gamma, \epsilon}^{\gamma, \epsilon}(x_{N+1}) = \sum_{I_{N}} \prod_{\ell \in \gamma} \left\{ \frac{-i}{y_{a} - y_{b}} \right\} \cdot \prod_{b \notin \gamma} (\lambda - y_{b}) \cdot \Psi_{\gamma, \epsilon}^{\gamma, \epsilon}(x_{N+1})
\]  
(2.25)

where \(I_{N} = \mathbb{N} ; N \).
Note that equation (2.5) can be readily deduced from the above action of \( \mathcal{A}_r(\lambda) \) by identifying the \( O(\lambda^{-1}) \) part of both side’s \( \lambda \to \infty \) asymptotics.

**Proof** —

In order to recast \( \mathcal{A}_r(\lambda) \cdot \Psi_{y,\varepsilon}(x_{N+1}) \) in terms of an action on dual variables, we use that it is a polynomial in \( \lambda \) of degree \( r \). It can thus be reconstructed by interpolation at \( r + 1 \) points. For \( r = N \) one has

\[
\mathcal{A}_N(\lambda) = \prod_{n=1}^{N} \left( e^{x_n - x_{N+1}} \right) \cdot e^{x_{N+1}} \cdot B_{1,N+1}(\lambda),
\]

(2.26)

and the action of all the operators is known. Hence, it remains to consider the case \( r < N \). In such a case, one obtains \( N \) interpolation points \( y_1, \ldots, y_N \) by acting on \( \Psi_{y,\varepsilon}(x_{N+1}) \) with the operator

\[
\mathcal{A}_r(y_a) = O_r(y_a + \imath \varepsilon)A_{1,N+1}(y_a) - \prod_{n=1}^{N} \left( e^{x_n - x_{N+1}} \right) \cdot C_{r+1,N+1}(y_a + \imath \varepsilon) \cdot B_{1,N+1}(y_a)
\]

(2.27)

and using that the second operator produces vanishing contributions. This leads to

\[
\mathcal{A}_r(y_a) \cdot \Psi_{y,\varepsilon}(x_{N+1}) = (-1)^r \sum_{I_r=\mathcal{O}_{\sigma} \setminus \mathcal{B}_{r+1}} \prod_{a \in \mathcal{O}} \left( \frac{-i}{y_a - y_i} \right) \prod_{b \in \mathcal{B}} (y_a - y_b) \cdot \Psi_{y_{\varepsilon},\varepsilon}(x_{N+1}).
\]

(2.28)

Hence, reconstructing the polynomial of interest through interpolation, we get

\[
\mathcal{A}_r(\lambda) \cdot \Psi_{y,\varepsilon}(x_{N+1}) = \sum_{I_r=\mathcal{O}_{\sigma} \setminus \mathcal{B}_{r+1}} \prod_{a \in \mathcal{O}} \left( \frac{\lambda - y_a}{y_a - y_p} \right) \cdot \prod_{b \in \mathcal{B}} (\lambda - y_b) \cdot \Psi_{y_{\varepsilon},\varepsilon}(x_{N+1}),
\]

(2.29)

where

\[
P_{\mathcal{O}_r}(\lambda) = \sum_{a \in \mathcal{O}} \prod_{p \neq a} \left( \frac{\lambda - y_p}{y_a - y_p} \right).
\]

(2.30)

Again, by interpolation, it is readily seen that \( P_{\mathcal{O}_r}(\lambda) = 1 \).

**Corollary 2.2**

\[
\prod_{a=1}^{r} \left( e^{x_a - x_{N+1}} \right) C_{r+1,N+1}(\lambda) \cdot \Psi_{y,\varepsilon}(x_{N+1}) = -(\lambda - \varepsilon + \imath \varepsilon) \sum_{I_r=\mathcal{O}_{\sigma} \setminus \mathcal{B}_{r+1}} \prod_{a \in \mathcal{O}} \left( \frac{-i}{y_a - y_b} \right) \prod_{b \in \mathcal{B}} (\lambda - y_b) \cdot \Psi_{y_{\varepsilon},\varepsilon}(x_{N+1})
\]

\[
+ \sum_{I_r=\mathcal{O}_{\sigma} \setminus \mathcal{B}_{r+1}} \prod_{a \in \mathcal{O}} \left( \frac{-i}{y_a - y_b} \right) \cdot \mathcal{W}(y_a \in \mathcal{O} ; \lambda) \cdot \Psi_{y_{\varepsilon},\varepsilon}(x_{N+1})
\]

(2.31)

in which \( \mathcal{W} \) is as given by (1.29). Also, we do stress that the last sum over partitions runs with the integer \( p \) not being fixed. Finally, we have introduced the notation

\[
e_{\sigma} \equiv \sum_{a \in \sigma} e_a.
\]

(2.32)
This formula is a mere conjunction of (2.25) and (2.2) and the relation (2.19) followed by an application of
the two representation (1.30) and (1.29) for $W((y_{\alpha})_N^1, \lambda)$. Note that a similar formula can be derived for the action of
the operator

$$
\prod_{a=1}^{r} \left\{ e^{x_a - y_{N+1}} \right\} C_{1,r}(\lambda)
$$

(2.33)

by using equation (2.21) along with the aforeobtained results.

**Conclusion**

In this paper we have recast the action of various local and non-local operators of the closed Toda on the SoV transform’s kernel $\Psi_{y,x,e}(x_{N+1})$ in terms of an action on the dual variables $y_N$ and $e$. Our approach builds on Babelon’s idea relative to acting with products of positions operators on the integral kernel of a fully iterated Mellin-Barnes multiple integral representation for $\Psi_{y,x,e}(x_{N+1})$ [2]. In the present paper, we have managed to make
the method more efficient by using $r$-split Mallin-Barnes representations that are adapted to the type of operators
for which one wants to set an equation in dual variables. The Mellin-Barnes based approach appears to be quite systematic and the obtained formulae take a quite universal form. It is thus plausible that the present results should extend quite easily to the case of other quantum integrable models solvable by the quantum separation of variables method. Furthermore, when combined with Sklyanin’s quantum determinant based identities, it allows one to
obtain equations in dual variables for an even larger family of operators associated with the model.

**Acknowledgements**

K. K. K. would like to thank E. K. Sklyanin for stimulating discussions relative to various topics treated in this paper. K. K. K. is supported by CNRS, the Burgundy region PARI 2013 FABER grant "Structures et asymptotiques d’intégrales multiples” and the PEPS-PTI 2012”Asymptotic d’intégrales multiples” grant. This research has
been initiated when the author was supported by the EU Marie-Curie Excellence Grant MEXT-CT-2006-042695
and DESY. K. K. K. is indebted to the Laboratoire de Physique Théorique d’Annecy-Le-Vieux for its warm
hospitality when part of the research has been carried out.

**A Proof of proposition 1.1**

**Proof —**

We prove the claim by induction. Since the independence of the construction on $r$ will follow from the next
proposition, we take this fact for granted here. It is clear that all the claims of the proposition are satisfied for
$N = 0$. Assume that they hold true up to some $N$, this for any $r \in \{ 1 ; N - 1 \}$. It is well know that there exists $C_{\epsilon,\eta} > 0$ such that one has the bounds

$$
|\Gamma(x + iy)| \leq C_{\epsilon,\eta} \cdot |y|^x e^{-\frac{\pi}{2}|y|} \quad \text{ uniformly in } y \in \mathbb{R} , \quad \eta > |x| > \epsilon > 0 .
$$

(A.1)

Likewise, there also exists $C_{\eta} > 0$ such that

$$
|\Gamma(x + iy)|^{-1} \leq C_{\eta} \cdot |y|^{\frac{1}{2} - x} e^{\frac{\pi}{2}|y|} \quad \text{ uniformly in } y \in \mathbb{R} \text{ and } |x| < 2\eta .
$$

(A.2)
Thus we pick an \( \eta > 0 \) such that \( \eta > \max_{k \in [1:N]} (|\mathfrak{I}(y_k)|) \), and we fix the shift \( \alpha \) occurring in (1.18). It is then readily seen that there exists \( \eta \)-dependent constants \( C > 0 \) and \( M, M' \) such that

\[
|\varpi(w, z | y_N)| \leq C(1 + ||w|| + ||z||)^M \cdot (1 + ||y_N||)^M \prod_{b=1}^N \left\{ e^{-\frac{r-1}{\lambda_b} |\sum_{a=1}^{N-1} |\mathfrak{R}(w_a-y_b)| + \sum_{a=1}^{N-1} |\mathfrak{I}(z_a-y_b)|} \right\} \\
\times \prod_{a=1}^{r-1} \prod_{b=1}^{N-1} \left\{ e^{-\frac{r-1}{\lambda_b} |\sum_{a=1}^{N-1} |\mathfrak{R}(w_a-y_b)| + \sum_{a=1}^{N-1} |\mathfrak{I}(z_a-y_b)|} \right\}. \tag{A.3}
\]

Note that above and in the following, the dimensionality of the \( L^1 \)-norms, \( \textit{cf} \ (1.21) \), is undercurrent by the context. Since the induction hypothesis does not depend on \( r \), one does not have to discuss which pattern of decomposition has been used so as to construct the functions appearing in the integral representation (1.15). Thus choosing some \( r \) and applying the induction hypothesis to these two functions, one gets that there exist constants \( \kappa_1, \kappa_2, \kappa_3 \) and \( \kappa_4 \) such that

\[
\left| \Psi_{w, \vec{y}_N, \vec{z}(x_1)} \Psi_{z, \vec{y}_N, \vec{z}(x_2)} \varpi(w, z | y_N) \right| \leq \kappa_1 e^{\kappa_2 ||y_N||^\kappa_3 (1 + ||y||)^\kappa_4} \cdot e^{-\frac{r-1}{\lambda_b} \sum_{a=1}^{N-1} |\mathfrak{R}(y_a-y_b)|} \tag{A.4}
\]

Above, we have reparameterized the variables \( y_N, w \) and \( z \)

\[
\lambda_b = \mathfrak{R}(y_b) \quad \text{and} \quad \gamma = (\mathfrak{R}(w_1), \ldots, \mathfrak{R}(w_{r-1}), \mathfrak{R}(z_r), \ldots, \mathfrak{R}(z_{N-1})). \tag{A.5}
\]

Also, the eventual dependence on \( r \) of the constants \( \kappa_3, a = 1, \ldots, 4 \) was removed by taking the supremum when \( r \) runs through \( \|1 : N - 1\| \).

Since the upper bound above is already symmetric in respect to the \( \lambda_a \)'s and \( \gamma_a \)'s, it is enough to show that it belongs to \( L^1(\Omega_{N-1}) \) where

\[
\Omega_{N-1} = \{ y_{N-1} \in \mathbb{R}^{N-1} : \gamma_1 < \cdots < \gamma_{N-1} \}. \tag{A.6}
\]

Further, again in virtue of the symmetry, in doing so, one may also assume the ordering \( \lambda_1 < \cdots < \lambda_N \). For such an ordering of both sets of variables, one has the identity\(^2\) \( (1.15) \)

\[
\sum_{k=1}^{N} \sum_{j=1}^{N-1} |y_j - \lambda_k| - \sum_{a<b}^{N} |\lambda_a - \lambda_b| - \sum_{a<b}^{N-1} |y_a - y_b| = \sum_{j=1}^{N} \phi_j(y_j | \lambda_N) \tag{A.7}
\]

where

\[
\phi_j(y_j | \lambda_N) = \sum_{k=1}^{j} \left[ |y_j - \lambda_k| - (\lambda_k - y_j) \right] + \sum_{k=j+1}^{N} \left[ |y_j - \lambda_k| - (y_j - \lambda_k) \right]. \tag{A.8}
\]

Its use leads to

\[
\left| \Psi_{w, \vec{y}_N, \vec{z}(x_1)} \Psi_{z, \vec{y}_N, \vec{z}(x_2)} \varpi(w, z | y_N) \right| \leq \kappa_1 e^{\kappa_2 ||y_N||^\kappa_3 (1 + ||y||)^\kappa_4} \cdot e^{-\frac{r-1}{\lambda_b} \sum_{a=1}^{N-1} |\mathfrak{R}(y_a-y_b)|} \prod_{a=1}^{N-1} \left\{ (1 + ||y||)^\kappa_3 e^{-\frac{r-1}{\lambda_b} \phi(y_j | \lambda_N)} \right\} \tag{A.9}
\]

\(^2\) Note that the identity when the \textit{rhs} has been replaced with \( \geq 0 \) has been first used in \( [2] \) so as to prove the convergence properties of the integral in question.
for \( \gamma_{N-1} \in \Omega_{N-1} \). Hence, we consider the integral

\[
\mathcal{J}_N = \int_{\mathbb{R}^{N-1}} (1 + ||\gamma||)^{\kappa_3} \prod_{j=1}^{N} \exp \left\{ - \frac{\pi}{2h} \phi_j(\gamma_j | A_N) \right\} \mathbf{1}_{\Omega_{N-1}}(\gamma) \cdot d^N \gamma .
\]

(A.10)

and compute the integrals successively starting from the one over \( \gamma_{N-1} \). Since the integrand is positive and piece-wise continuous on \( \mathbb{R}^{N-1} \) and Lebesgue’s measure is \( \sigma \)-finite, by Fubini-Tonelli-Lebesgue theorem this is enough so as to guarantee the \( L^1(\mathbb{R}^{N+1}) \) character of the integral. For this purpose, given any \( j \) and \( m_j \in \mathbb{R} \), observe that one has the chain of majorations:

\[
\int_{\gamma_{j-1}}^{\infty} (1 + |\gamma_j|)^{m_j} \cdot e^{- \frac{\pi}{2h} \phi_j(\gamma_j | A_j)} \cdot d\gamma_j \leq \int_{\min(\gamma_{j-1}, -1)}^{\max(\gamma_{j-1}, -1)} (1 + |\gamma_j|)^{m_j} \cdot d\gamma_j + \int_{\max(\gamma_{j-1}, -1)}^{\infty} (1 + |\gamma_j|)^{m_j} \cdot e^{- \frac{\pi}{2h} \phi_j(\gamma_j | A_j)} \cdot d\gamma_j \\
\leq \frac{1}{m_j + 1} \left[ (1 + \max(\gamma_{j-1}, -1))^{|\gamma_j|} - (1 + \min(\gamma_{j-1}, -1))^{|\gamma_j|} \right] + C_{m_j,N}^j \prod_{k=1}^{N} e^{\pi} \leq C_{m_j,N}^j (1 + |\gamma_{j-1}|)^{|\gamma_j|},
\]

(A.11)

for some constants \( C_{m_j,N}^j, C_{m_j,N}^j > 0 \). Hence, when carrying out the succesive chain of integrations in (A.10) and dealing with the integral in respect to \( \gamma_j \) it is readily seen that the sole e\( \text{ffect of integrating in respect to} \gamma_N, \ldots, \gamma_{j+1} \) was to increase the original exponent \( \kappa_3 \) by some sufficiently large integer, thus, effectively, reducing the integration versus \( \gamma_j \) to the model integral that was written in the rhs of (A.11).

One can continue in such a way up to integrating over \( \gamma_1 \). Then one deals with an integration over \( \mathbb{R} \). It is then easy to see that there exists an \( A_N \)-dependent constant \( C \) such that

\[
\phi_1(\gamma_1 | A_N) \geq (2|\gamma_1| - \tilde{C}) .
\]

(A.12)

This last estimate ensures that the integral over \( \gamma_1 \) is convergent as well.

It solely remains to establish that \( \Psi_{x_{N+1},y_{N+1}}(x_{N+1}) \) is an entire function of \( (x_{N+1}, y_{N}, y_{N+1}) \). This is clear for \( N = 0 \). Further, assume that this has been established up to some \( N - 1 \). The estimates for the convergence of the integral readily lead to the fact that it defines a continuous function of \( (x_{N+1}, y_{N}, y_{N+1}) \).

Let \( \mathcal{C} \) be a closed loop in \( \mathbb{C} \) lying in the strip \( |\Im(z)| < \eta \). Then, since \( \mathcal{C} \) is compact and the recursive integrand converges uniformly in \( (y_{N}, y_{N+1}) \) and \( x_{N+1} \) bounded, one gets that, for any \( a = 1, \ldots, N + 1 \)

\[
\oint_{\mathcal{C}} \Psi_{y_{N+1}}(x_{N+1}) \cdot dx_a = \int_{\mathcal{C}} \left\{ \oint_{\mathcal{C}} \Psi_{w; y_{N+1}} \cdot \tau(x_1, y_{N+1}) \cdot \mathcal{E}(w, z | y_{N}) \cdot dy_a \right\} \cdot \prod_{a=1}^{N-1} \int_{z_a}^{z_a} dw_a \cdot \prod_{a=r}^{N} dz_a = 0 ,
\]

and likewise

\[
\oint_{\mathcal{C}} \Psi_{y_{N+1}}(x_{N+1}) \cdot dx_a = \int_{\mathcal{C}} \left\{ \oint_{\mathcal{C}} \Psi_{w; y_{N+1}} \cdot \tau(x_1, y_{N+1}) \cdot \mathcal{E}(w, z | y_{N}) \cdot dy_a \right\} \cdot \prod_{a=1}^{N-1} \int_{z_a}^{z_a} dw_a \cdot \prod_{a=r}^{N} dz_a = 0 .
\]

Hence, by Morera’s theorem the function is holomorphic in each of the variables taken singly. Thus, by Hartog’s theorem, it is a holomorphic function of \( (y_{N}, y_{N+1}) \) belonging to the poly-strip \( |\Im(y_a)| < \eta , a = 1, \ldots, N + 1 \) and \( |\Im(x_a)| < \eta , a = 1, \ldots, N + 1 \). Since \( \eta \) was arbitrary, the claim follows. 

\[\blacksquare\]
B Proof of proposition 1.2

It is clear from the explicit definition of \( \Psi_{0,\varepsilon}(\chi) \) and the expressions for the one-site operators that the equations (1.22)-(1.24) and the two relations given in (1.26) are indeed satisfied. We now prove the statement by induction. Thus we assume having built all of the lower number of variables functions \( \Psi_{y,\varepsilon}(x_{k+1}), k = 0, \ldots, N - 1 \) which

- are independent of the splitting \( r \) used for their construction
- satisfy the appropriate analogues of the relations (1.22)-(1.24) and (1.26).

We shall now proceed in two steps. First, we shall establish the form of the action of operators on \( \Psi_{y,\varepsilon}(x_{N+1}) \) and then we shall prove the independence of the construction on \( r \).

Action of the operators

We now check that the action of the \( N + 1 \)-site operators \( B, D \) and \( A \) takes the desired form (1.22)-(1.24) and that the two relations given in (1.26) hold. For this, we split the matrix product defining the \( N+1 \) site monodromy matrix into a product of the monodromy matrices associated with subchains of sites \( 1, \ldots, r \) and \( r + 1, \ldots, N + 1 \) respectively:

\[
T_{0,1,N+1}(\lambda) = T_{0,1,r}(\lambda) \cdot T_{0,r+1,N+1}(\lambda)
\]  

We first start by computing the action of the \( B_{1,\ldots,N+1}(\lambda) \) operator. It follows from the explicit representation for the local Lax matrices (1.1) that \( B_{1,N+1}(\lambda) \) is given by a finite linear combination of at most first order differential operators in each of the \( \chi_k \)'s. Since, according to proposition (1.1) \( \Psi_{y,\varepsilon}(x_{N+1}) \) is holomorphic in respect to \( \chi_k \), one can represent \( \partial_{\chi_k} \) in terms of a compactly supported contour integral operator. Then, the absolute convergence of the integral in (1.15) allows one to apply Fubini’s theorem and exchange the orders of integration. In other words it is allowed to exchange the integration in (1.15) with differentiations in respect to any \( \chi_k \). As a consequence, one can move the operator \( B_{1,N+1}(\lambda) \) under the integral sign when computing its action. This last step allows one to use the split-like representation for this operator given in (B.1) along with the formulae for the action of the appropriate operators on lower rank functions so as to compute the effect of the action. This produces sums involving various combinations of functions \( \Psi_{z-i\hbar\gamma_{y},x_{N}^{+},z+\hbar\gamma_{y}}(x_{2}) \) or \( \Psi_{w+i\hbar\gamma_{y},z_{1}}(x_{1}) \) with shifts in their dual variables \( w \) or \( z \). One can then split the resulting integral into several sums since all of the individual terms converge absolutely in virtue of the bounds established in proposition 1.1. Then, one can shift the various integration contours by \( \pm \hbar \) so as to recover, in all integrals, the product of functions \( \Psi_{w_{\gamma_{y}-z_{1}}}(x_{1}) \cdot \Psi_{z_{1}-z_{2}}(x_{2}) \) in each term under the integration sign. Note that the apparent poles at \( z_{p} = z_{a} \) for \( a \in \{r, \ldots, N - 1\} \setminus \{p\} \) and \( w_{p} = w_{a} \) for \( a \in \{1, \ldots, r - 1\} \setminus \{p\} \) which arise in the intermediate calculations are, in fact, cancelled out by the zeroes of the weight function \( \sigma(w, z | y_{N}) \). All in all, these shifts of contours lead to the integral representation

\[
B_{1,N+1}(\lambda) \cdot \Psi_{y,\varepsilon}(x_{N+1}) = \int_{C_{r+1,N-r}} g_{\lambda}(w, z | y_{N})\Psi_{w_{\gamma_{y}-z_{1}}(x_{1})} \Psi_{z_{1}-z_{2}}(x_{2}) \cdot \prod_{a=1}^{r-1} dw_{a} \cdot \prod_{a=r}^{N-1} dz_{a}
\]  

(B.2)
where the function $g_A(w, z | y_N)$ is given by
\[
g_A(w, z | y_N) = (\lambda - \overline{y}_N + \overline{z} + \overline{w}) \prod_{a=1}^{r-1} (\lambda - w_a) \cdot \prod_{a=r}^{N-1} (\lambda - z_a) \cdot \varpi(w, z | y_N) \\
+ \sum_{p=1}^{r-1} (i)^p \prod_{a=1, \neq p}^{r-1} \left( \frac{\lambda - w_a}{w_p - w_a - i\hbar} \right) \prod_{a=r}^{N-1} (\lambda - z_a) \cdot \varpi(w - i\hbar p, z | y_N) \\
+ \prod_{a=1}^{r-1} (\lambda - w_a) \sum_{p=r}^{N-1} (-i)^{N-r+1} \prod_{a=r, \neq p}^{N-1} \left( \frac{\lambda - z_a}{z_p - z_a + i\hbar} \right) \cdot \varpi(w, z + i\hbar p | y_N). \quad (B.3)
\]

The action of the operator $B_{1,N+1}(\lambda)$ will take the form given in (1.22) as soon as we establish that
\[
r^{(1)}_A(w, z | y_N) = g_A(w, z | y_N) - \prod_{a=1}^{N} (\lambda - y_a) \cdot \varpi(w, z | y_N) \quad (B.4)
\]
vanishes. Since $r^{(1)}_A(w, z | y_N)$ is a polynomial in $\lambda$ of degree at most $N$, it is enough to show that it vanishes at $N + 1$ points. Hence, we interpolate at $\lambda = w_\ell, \ell = 1, \ldots, r - 1$ and $\lambda = z_a, a = r, \ldots, N - 1$ and then showing that, it has large $\lambda$ asymptotics $r^{(1)}_A(w, z | y_N) = O(\lambda^{N-2})$. It follows from the system of equations satisfied by the weight factor $\varpi(w, z | y_N)$:
\[
\prod_{a=1}^{N} (w_\ell - y_a) \cdot \varpi(w, z | y_N) = (i)^{r-1} \prod_{a=1, \neq \ell}^{r-1} \left( \frac{w_\ell - w_a}{w_p - w_a - i\hbar} \right) \cdot \prod_{a=r}^{N-1} (w_\ell - z_a) \cdot \varpi(w - i\hbar \ell, z | y_N) \quad (B.5)
\]
for $\ell = 1, \ldots, r - 1$ and
\[
\prod_{a=1}^{N} (z_\ell - y_a) \cdot \varpi(w, z | y_N) = (-i)^{N-r+1} \prod_{a=1}^{r-1} (z_\ell - w_a) \prod_{a=r, \neq \ell}^{N-1} \left( \frac{z_\ell - z_a}{z_p - z_a + i\hbar} \right) \cdot \varpi(w, z + i\hbar \ell | y_N) \quad (B.6)
\]
for $\ell = r, \ldots, N - 1$, that $r^{(1)}_A(w, z | y_N)$ vanishes at the aforementioned interpolation points.

Further, one has that
\[
\prod_{a=1}^{N} (\lambda - y_a) \cdot \varpi(w, z | y_N) = (\lambda^N - \overline{y}_N \cdot \lambda^{N-1} + O(\lambda^{N-2})) \cdot \varpi(w, z | y_N). \quad (B.7)
\]
It is also readily seen that the sums occuring in the second and third lines of equation (B.3) are a $O(\lambda^{N-2})$. Hence, the leading and first sub-leading terms at $\lambda \to \infty$ issue from the first line of (B.3) and thus read
\[
g_A(w, z | y_N) = (\lambda^N - \overline{y}_N \cdot \lambda^{N-1} + O(\lambda^{N-2})) \cdot \varpi(w, z | y_N). \quad (B.8)
\]
Accordingly, we get that, indeed $r^{(1)}_A(w, z | y_N) = O(\lambda^{N-2})$, so that, all in all $r^{(1)}_A(w, z | y_N) = 0$.

We now check that the action of the operator $D_{1,N+1}(\lambda)$ takes the form (1.22). For this purpose we use the representation for this operator in terms of operators associated with various sub-chains of the model that follows from (B.1) and proceeding exactly as in the case of the action of the operator $B_{1,N+1}(\lambda)$ so as to compute the effect of its action by acting with the appropriate lower number of sites operators on the functions $\Psi_{w,\overline{y}_N - z - i\hbar}(x_1)$
or $\Psi_{z,\epsilon - \Psi_0 \cdot \Psi_0 + \Psi}(x_2)$ arising under the integral sign in (1.15). Then we reorganize the expression by splitting the integrand and shifting the contours in appropriate expressions so as to solely integrate the product of functions $\Psi_{w,\epsilon - \Psi_0 \cdot \Psi_0 + \Psi}(x_1) \Psi_{z,\epsilon - \Psi_0 \cdot \Psi_0 + \Psi}(x_2)$ at the very end of this procedure. It then remains to use the recurrence equations under shifts of its variables satisfied by the weight factor $\sigma(w, z | y_N)$ so as to get that

$$D_{1,N+1}(\lambda) \cdot \Psi_{y_N,\epsilon}(x_{N+1}) = \int u_\lambda(w, z | y_N) \Psi_{w,\epsilon - \Psi_0 \cdot \Psi_0 + \Psi}(x_1) \Psi_{z,\epsilon - \Psi_0 \cdot \Psi_0 + \Psi}(x_2) \, \sigma(w, z | y_N) \, \prod_{a=1}^{r-1} dw_{a} \prod_{a=r}^{N-1} dz_{a} \cdot (B.9)$$

where we have set

$$u_\lambda(w, z | y_N) = \sum_{p=1}^{r-1} \sum_{q=r}^{N-1} (-1)^q \prod_{a=r}^{r-1} \frac{(\lambda - w_a)}{(z_a - w_a)} \prod_{b=r}^{N-1} \frac{(z_b - w_p - i\hbar)}{(z_b - w_a)} \prod_{a=r}^{N-1} \frac{(z_q - y_b)}{(y_b - w_p - i\hbar)}$$

Then, taking into account that

$$\sum_{p=1}^{N} (-1)^{N+1} \prod_{a=1}^{N} \frac{(\lambda - y_a)}{(y_p - y_a)} \cdot \sigma(w, z | y_N - i\hbar e_p) = \sigma(w, z | y_N) \cdot f_\lambda(w, z | y_N) \cdot (B.11)$$

with

$$f_\lambda(w, z | y_N) = - \sum_{p=1}^{N} \prod_{a=r}^{r-1} \frac{(y_p - z_a)}{(y_p - w_a - i\hbar)} \prod_{a=r}^{N} \frac{(\lambda - y_a)}{(y_p - y_a)} \cdot (B.12)$$

we infer that the action of the operator $D_{1,N+1}(\lambda)$ will take the form (1.23) as soon as we prove that

$$i_\lambda^{(2)}(w, z | y_N) = f_\lambda(w, z | y_N) - u_\lambda(w, z | y_N) \quad \text{vanishes} \cdot (B.13)$$

Since it is a polynomial of degree $N - 1$, it is enough to prove that it vanishes at $N$ points. Below, we show that, indeed, it vanishes at

- $\lambda = w_p$ with $p = 1, \ldots, r - 1$;
- $\lambda = z_p$ with $p = r, \ldots, N - 1$;

and that it behaves as $i_\lambda^{(2)}(w, z | y_N) = O(\lambda^{N-1})$ in the $\lambda \to \infty$ regime.
• **Behaviour at \( \infty \)**

It follows from an immediate inspection that

\[
u_{\lambda}(w, z \mid y_N) = \lambda^{N-1} \kappa_{N-1} + O(\lambda^{N-2}) \quad \text{with} \quad \kappa_{N-1} = (-1)^r \sum_{p=1}^{r-1} \left\{ \prod_{b=r}^{N-1} \frac{z_b - w_p - i\hbar}{\prod_{b=1}^{N-1} (y_b - w_p - i\hbar)} \right\} \cdot \prod_{a=1}^{N-1} \left\{ \frac{1}{w_p - w_a} \right\} \cdot \prod_{b=1}^{r-1} \left\{ \frac{1}{w_p - w_a} \right\} \cdot \prod_{l=1}^{r-1} \left\{ \frac{1}{w_p - w_a} \right\} \cdot \prod_{b=1}^{N-1} \left\{ \frac{1}{y_b - y_l} \right\} \cdot \prod_{b=1}^{r-1} \left\{ \frac{1}{w_p - w_a} \right\} . \tag{B.14}\]

The leading coefficient can be recast as a contour integral which, then, can be computed by taking the residues outside of the integration contour. This yields

\[
\kappa_{N-1} = (-1)^r \int_{\Gamma'(w_1^{-1})} \prod_{b=r}^{N-1} \left\{ \frac{1}{(y_b - \omega - i\hbar)} \right\} \cdot \prod_{a=1}^{r-1} \left\{ \frac{1}{(\omega - w_a)} \right\} \cdot \frac{d\omega}{2\pi i} = (-1)^r \sum_{l=1}^{r-1} \left\{ \prod_{b=r}^{N-1} \frac{z_b - y_l}{\prod_{b=1}^{N-1} (y_b - w_a - i\hbar)} \right\} \cdot \prod_{b=1}^{r-1} \left\{ \frac{1}{w_p - w_a} \right\} \cdot \prod_{l=1}^{r-1} \left\{ \frac{1}{y_b - y_l} \right\} . \tag{B.15}\]

There \( \Gamma'(w_1^{-1}) \) refers to a counterclockwise loop of index 1 around the points \( w_a \), \( a = 1, \ldots, r - 1 \) but not encircling any other singularity of the integrand.

It is immediate to see from here that, indeed,

\[
\kappa_{N-1} = \lim_{\lambda \to \infty} \left\{ \lambda^{1-N} f_{\lambda}(w, z \mid y_N) \right\} . \tag{B.16}\]

• **Interpolation at \( \lambda = z_q \) with \( q = r, \ldots, N - 1 \)**

Setting \( \lambda = z_q \) with \( q = r, \ldots, N - 1 \) in (B.10) leads to

\[
u_{z_q}(w, z \mid y_N) = \sum_{p=1}^{r-1} (-1)^r \prod_{a=1}^{r-1} \left\{ \frac{1}{w_p - w_a} \right\} \cdot \prod_{b=r}^{N-1} \left\{ \frac{z_q - y_l}{\prod_{b=1}^{N-1} (y_b - w_a - i\hbar)} \right\} \cdot \prod_{l=1}^{r-1} \left\{ \frac{1}{y_b - y_l} \right\} \cdot \prod_{b=1}^{N-1} \left\{ \frac{1}{y_b - y_l} \right\} \cdot \prod_{l=1}^{r-1} \left\{ \frac{1}{y_b - y_l} \right\} \cdot \prod_{b=1}^{r-1} \left\{ \frac{1}{y_b - y_l} \right\} \cdot \prod_{l=1}^{r-1} \left\{ \frac{1}{y_b - y_l} \right\} . \tag{B.17}\]

• **Interpolation at \( \lambda = w_p \) with \( p = 1, \ldots, r - 1 \)**

A straightforward calculation shows that

\[
u_{w_q}(w, z \mid y_N) = L_1 + \cdots + L_5 , \tag{B.18}\]
where we agree upon

\[
\mathcal{L}_1 = (-1)^r \sum_{q=r}^{N-1} \prod_{a=q+1}^{N-1} \left\{ \frac{w_p - z_a}{z_q - z_a} \right\} \prod_{b=r}^{N-1} \left( z_b - w_p - i\hbar \right) \prod_{a=1}^{r-1} \left\{ \frac{1}{z_q - w_a} \right\} \prod_{b=1}^{N} \left( z_q - y_b \right) ,
\]

(B.19)

\[
\mathcal{L}_2 = (-1)^r \prod_{b=1}^{N} \left( y_b - w_p - i\hbar \right) \prod_{a=r}^{N-1} \left\{ (w_p - z_a)(z_a - w_p - i\hbar) \right\},
\]

(B.20)

\[
\mathcal{L}_3 = \prod_{a=r}^{N-1} (w_p - z_a) \sum_{\ell=1}^{r-1} \frac{1}{(w_p - w_\ell)(w_\ell - w_p + i\hbar)} \prod_{b=r}^{N-1} \left( \frac{1}{z_b - w_p} \right) \prod_{b=1}^{N} \left( \frac{y_b - w_p}{y_b - w_\ell - i\hbar} \right) ,
\]

(B.21)

\[
\mathcal{L}_4 = \prod_{a=r}^{N-1} (w_p - z_a) \sum_{\ell=1}^{r-1} \frac{1}{(w_p - w_\ell)(w_\ell - w_p + i\hbar)} \prod_{b=r}^{N-1} \left( \frac{1}{z_b - w_\ell} \right) \prod_{b=1}^{N} \left( \frac{y_b - w_\ell}{y_b - w_p - i\hbar} \right) ,
\]

(B.22)

and

\[
\mathcal{L}_5 = - \prod_{a=r}^{N-1} (w_p - z_a) \cdot \sum_{\ell=1}^{r-1} \prod_{a=1}^{r-1} \left\{ \frac{1}{w_p - w_a + i\hbar} \right\}.
\]

(B.23)

Most of the sums can be re-expressed in terms of contour integrals. Namely, one has

\[
\mathcal{L}_1 = (-1)^r \prod_{b=1}^{N} \left( y_b - w_p - i\hbar \right) \cdot \mathcal{S}_1
\]

(B.24)

where

\[
\mathcal{S}_1 = \sum_{q=r}^{N-1} \frac{-1}{z_q - w_p - i\hbar} \cdot \prod_{a=r}^{N-1} \left\{ \frac{1}{z_q - z_a} \right\} \prod_{a=1}^{r-1} \left\{ \frac{1}{z_q - w_a} \right\} \prod_{b=1}^{N} \left( z_q - y_b \right) .
\]

(B.25)

Thus it follows that

\[
\mathcal{S}_1 = \int_{\mathcal{C}} \frac{d\omega}{2i\pi} \left( \frac{1}{\omega - w_p + i\hbar} \right) \prod_{a=r}^{N-1} \left\{ \frac{1}{\omega - z_a} \right\} \prod_{a=1}^{r-1} \left\{ \frac{1}{\omega - w_a} \right\} \prod_{b=1}^{N} \left( \omega - y_b \right)
\]

\[
= \frac{1}{N-1} \prod_{a=r}^{N} (w_p - y_b + i\hbar) \prod_{a=1}^{r-1} \frac{1}{w_p - w_a + i\hbar} + \frac{1}{N} \prod_{b=1}^{N} (w_p - y_b) \prod_{a=1}^{r-1} \frac{1}{w_p - w_a} + \prod_{\ell=1}^{r-1} \frac{1}{w_p - w_\ell} \prod_{a=1}^{N-1} \frac{1}{w_p - w_a} + \prod_{\ell=1}^{r-1} \frac{1}{w_\ell - z_a} \prod_{a=1}^{N-1} \frac{1}{w_\ell - w_a} + \prod_{\ell=1}^{r-1} \frac{1}{w_\ell - y_b} \prod_{a=1}^{N-1} \frac{1}{w_\ell - w_a}
\]

\[
- (w_p + i\hbar + \bar{w} + \bar{z} - \bar{y}_N) ,
\]

(B.26)
where we have taken the integral by computing the residues at the poles lying outside of the contour. Note that there was a non-vanishing residue at $\infty$.

As a consequence, we get

\[
L_1 + L_2 + L_5 = \sum_{a=1}^{N-1} \left( (w_p - z_a)(z_a - w_p - i\hbar) \right) \prod_{b=1}^{N-1} (y_b - w_p - i\hbar) + \sum_{\ell=1}^{r-1} \prod_{a=1}^{N-1} \left( \frac{1}{z_a - w_\ell - i\hbar} \right) \prod_{a=1}^{r} \left( \frac{1}{w_\ell - w_a - i\hbar} \right)
\]

\[
- \sum_{a=1}^{N-1} \frac{(w_p - z_a)}{\prod_{\ell=1}^{r-1} (w_p - w_\ell - i\hbar)} . \quad (B.27)
\]

We now rewrite $L_3$:

\[
L_3 = -\frac{i}{\hbar} \prod_{a=1}^{N-1} \left( \frac{1}{w_p - w_a} \right) \cdot \prod_{b=1}^{r-1} \left( w_p - z_b + i\hbar \right) \cdot \prod_{b=1}^{N-1} \left( \frac{y_b - w_p}{y_b - w_\ell - i\hbar} \right) + \frac{N-1}{\hbar} \prod_{b=1}^{N-1} \left( w_p - z_b \right) \cdot \prod_{b=1}^{r-1} \left( \frac{1}{y_b - w_\ell - i\hbar} \right) \cdot \prod_{b=1}^{N} \left( \frac{y_b - w_\ell}{y_b - w_p - i\hbar} \right) \cdot \prod_{b=1}^{r-1} \left( y_\ell - w_\ell - i\hbar \right) \cdot \prod_{b=1}^{N} \left( \frac{y_b - w_\ell}{y_b - w_p - i\hbar} \right)
\]

Finally, it is readily seen that

\[
L_4 = \frac{i}{\hbar} \prod_{b=1}^{N-1} \left( w_p - z_b + i\hbar \right) \cdot \prod_{a=1}^{r-1} \left( \frac{1}{w_p - w_a} \right) \cdot \prod_{b=1}^{N-1} \left( \frac{y_b - w_p}{y_b - w_\ell - i\hbar} \right) + \prod_{a=1}^{r-1} \left( w_p - z_a \right) \cdot \prod_{b=1}^{N} \left( \frac{z_b - w_\ell - i\hbar}{z_b - w_p - i\hbar} \right) \cdot \prod_{b=1}^{r-1} \left( y_\ell - w_\ell - i\hbar \right) \cdot \prod_{b=1}^{N} \left( \frac{y_b - w_\ell}{y_b - w_p - i\hbar} \right) . \quad (B.29)
\]

Hence, adding up all the partial representations for the $L_k$’s, we get that

\[
\sum_{p=1}^{5} L_p = -\sum_{\ell=1}^{N-1} \prod_{b=1}^{r-1} \left( \frac{1}{y_\ell - w_\ell - i\hbar} \right) \cdot \prod_{b=1}^{N} \left( \frac{y_b - w_\ell}{y_b - y_\ell} \right) \cdot \prod_{b=1}^{r-1} \left( y_\ell - w_\ell - i\hbar \right) \cdot \prod_{b=1}^{N} \left( \frac{y_b - w_\ell}{y_b - w_p - i\hbar} \right) . \quad (B.30)
\]
The form of the action of the operator $A_{1,N+1}(\lambda)$ can be readily inferred by evaluating at $\lambda = y_k$ the action of the below form of the quantum determinant relation

$$A_{1,N+1}(\lambda - i\hbar)D_{1,N+1}(\lambda) \equiv C_{1,N+1}(\lambda - i\hbar)B_{1,N+1}(\lambda) = 1.$$  \hspace{1cm} (B.31)

on $\Psi_{y_k,\ell}(x_{N+1})$. This removes the contribution coming from $B_{1,N+1}(y_k)$ whereas the action of $D_{1,N+1}(y_k)$ is known. This allows one to interpolate the action of $A_{1,N+1}(\lambda)$ at $N$ points. Furthermore, one has the following asymptotic behavior

$$A_{1,N+1}(\lambda) = \lambda^{N+1} - \lambda^N \left( \sum_{a=1}^{N+1} p_a \right) + O(\lambda^{N-2})$$  \hspace{1cm} (B.32)

Since, it is readily seen from the induction hypothesis that

$$\left( \sum_{a=1}^{N+1} p_a \right) \cdot \Psi_{y_k,\ell}(x_{N+1}) = e \cdot \Psi_{y_k,\ell}(x_{N+1}),$$  \hspace{1cm} (B.33)

the full form for the action of the operator $A_{1,N+1}(\lambda)$ follows.

- **Independence on the splitting parameter $r$**

It follows from the joint results of [9, 11, 21] or, more directly, from [22] that, up to a multiplicative constant, there exist a unique function $f_{y_k,\ell}(x_{N+1})$ such that

$$B_{1,N+1}(\lambda) \cdot f_{y_k,\ell}(x_{N+1}) = \prod_{a=1}^{N} (\lambda - y_a) \cdot f_{y_k,\ell}(x_{N+1}) \quad \text{for any} \quad \lambda \in \mathbb{R}.$$  \hspace{1cm} (B.34)

Let $\Psi^{(r)}_{y_k,\ell}(x_{N+1})$ be as constructed through (1.15) with a splitting $r$. Since, independently of the value of $r$ these functions satisfy (B.34), there exists a constant $c_{r,\ell}$ such that

$$\Psi^{(r)}_{y_k,\ell}(x_{N+1}) = c_{r,\ell} \cdot \Psi^{(r)}_{y_k,\ell}(x_{N+1}).$$  \hspace{1cm} (B.35)

It remains to fix this constant by taking the $x_{N+1} \to \infty$ asymptotics of both expressions. In fact, it is enough to compute the leading asymptotic expansion in one direction of $\mathbb{R}^{N+1}$. For technical reasons, we shall focus on the following limit

$$x_a \to +\infty \quad a = 1, \ldots, N + 1 \quad \text{such that} \quad x_{a+1} - x_a \to +\infty.$$  \hspace{1cm} (B.36)

Taken into account the independence on the splitting part in the construction of $\Psi$ functions in less than $N$ variables, it follows from the results obtained in [22], equations (A.7)-(A.8), that $\Psi^{(1)}_{y_k,\ell}(x_{N+1})$ admits the integral representation

$$\Psi^{(1)}_{y_k,\ell}(x_{N+1}) = e^{\frac{i}{\hbar}(\mathcal{C}_{N+1})_{y_k,\ell}} \sum_{r \in \mathbb{Z}_N} J_N(y_{N,\tau}, x_N) \quad \text{with} \quad y_{N,\tau} = (y_{\tau(1)}, \ldots, y_{\tau(N)})$$  \hspace{1cm} (B.37)

where

$$J_N(w_{N}^{(0)}, x_N) = e^{\frac{i}{\hbar}W_N^{(0)}(x_N)} \prod_{s=1}^{N-1} \int_{(R-i\alpha)_N^{s,s}} d^{N-s} w^{(s)}(s) \prod_{a=1}^{N-1} \left\{ e^{\frac{i}{\hbar}(y_{N_{a+1}} - y_{N_a})} W_N^{(s)}(x_{N_{a+1}}) \right\} \cdot \frac{W_N(w_{N-s}^{(s)})_{N-s}^{N-1}}{\prod_{s=1}^{N-s} \prod_{a=1}^{N-1} (w_{a+1}^{(s)} - w_a^{(s-1)}),}$$  \hspace{1cm} (B.38)
It is readily seen that this leads, all-in-all, to the following contiguous poles in the direction \( (B.36) \). Thus, the asymptotics of \( \epsilon > 0 \)

\[
W_N(\psi_{N-s,0}^{(s)})^{N-1} = \frac{\tau^{\frac{N}{2}(1-n)} \psi_{N}^{(0)}}{\prod_{b=1}^{N} (w_{a}^{(0)} - w_{b}^{(0)})} \cdot \prod_{a=1}^{N} \left\{ \frac{(-ih)^{(N-a)} \psi_{N}^{(a)}}{(2\pi)^{N-a}} \right\} \cdot \prod_{a=1}^{N-s} \cdot \prod_{b=1}^{N-s} \frac{\Gamma \left( \frac{w_{a}^{(s)}}{ih} - \frac{w_{b}^{(s)}}{ih} + 1 \right)}{\Gamma \left( \frac{w_{a}^{(s)}}{ih} - \frac{w_{b}^{(s)}}{ih} + 1 \right)}. \tag{B.39}
\]

The leading \( x_N \to \infty \) asymptotic behaviour of \( J_N(\psi_{N}^{(0)}, x_N) \) can be extracted from \( \psi_{N}^{(0)} \) by appropriate shifts (to the upper or lower half-planes) of the integration contours as described in \( \psi_{N}^{(0)} \). For the limit of interest \( (B.36) \), it is enough to move all contours to the upper half-plane. The sole contribution that will not lead to exponentially small corrections in respect to \( x_{a+1} - x_{a} \), for some \( a = 1, \ldots, N - 1 \), corresponds to computing the residues at all contiguous poles \( i \epsilon \) at

\[
\psi_{N}^{(0)} = \psi_{N}^{(0)} \quad \text{and further successively} \quad \psi_{N-s}^{(s)} = \psi_{N-s}^{(0)}.
\tag{B.40}
\]

It is readily seen that this leads, all-in-all, to the following \( x_{N+1} \to \infty \) asymptotics

\[
J_N(\psi_{N}^{(0)}, x_N) = \sum_{r=1}^{N} \int_{a=1}^{N} \frac{\psi_{N}^{(0)} \cdot \Gamma \left( \frac{w_{a}^{(0)}}{ih} - \frac{w_{b}^{(0)}}{ih} + 1 \right)}{\sum_{a=1}^{N} \psi_{N}^{(0)}} + O \left( \sum_{k=1}^{N-1} [x_{a+1} - x_{a}]^{-\infty} \right). \tag{B.41}
\]

in the direction \( (B.36) \). Thus, the asymptotics of \( \psi_{y_N}^{(l)}(x_{N+1}) \) in the very same direction read

\[
\psi_{y_N}^{(l)}(x_{N+1}) = \sum_{r=1}^{N} \int_{a=1}^{N} \frac{\psi_{N}^{(0)} \cdot \Gamma \left( \frac{w_{a}^{(0)}}{ih} - \frac{w_{b}^{(0)}}{ih} + 1 \right)}{\sum_{a=1}^{N} \psi_{N}^{(0)}} + O \left( \sum_{k=1}^{N-1} [x_{a+1} - x_{a}]^{-\infty} \right). \tag{B.42}
\]

By the induction hypothesis, for any value of \( r \), the integrand in \( (1.15) \) can be recast by means of the lower number of variables version of formula \( (B.37) \), namely

\[
\psi_{y_N}^{(l)}(x_{N+1}) = \frac{\psi_{y_N}^{(0)} \cdot \Gamma \left( \frac{w_{a}^{(0)}}{ih} - \frac{w_{b}^{(0)}}{ih} + 1 \right)}{\sum_{a=1}^{N} \psi_{N}^{(0)}} + O \left( \sum_{k=1}^{N-1} [x_{a+1} - x_{a}]^{-\infty} \right). \tag{B.43}
\]

where we agree upon

\[
x_{r-1} = (x_{1}, \ldots, x_{r-1}) \quad \text{and} \quad \tilde{x}_{N-r} = (x_{r}, \ldots, x_{N}). \tag{B.44}
\]

In order to extract the leading \( O(1) \), \( x_{N+1} \to \infty \) as in \( (B.36) \), contributions of \( (B.43) \) one should move the \( w \)-integration to the upper half-plane and the \( z \)-integration slightly to the lower half-plane. The sole contribution not leading to exponentially small term corresponds to taking the residues at

\[
w_{a} = y_{r(a)} \quad \text{for} \quad a = 1, \ldots, r - 1 \quad \text{and} \quad z_{a} = y_{r(a+1)} \quad \text{for} \quad a = r, \ldots, N - 1 \tag{B.45}
\]

when \( \sigma \) runs through \( \Xi_N \). A straightforward computation then shows that for \( x_{N+1} \to \infty \) as in \( (B.36) \),

\[
\psi_{y_N}^{(l)}(x_{N+1}) = \frac{\psi_{y_N}^{(0)} \cdot \Gamma \left( \frac{w_{a}^{(0)}}{ih} - \frac{w_{b}^{(0)}}{ih} + 1 \right)}{\sum_{a=1}^{N} \psi_{N}^{(0)}} + O \left( \sum_{k=1}^{N-1} [x_{a+1} - x_{a}]^{-\infty} \right). \tag{B.46}
\]

Hence \( c_{r,\ell} = 1 \) and the independence of \( \psi_{y_N}^{(l)}(x_{N+1}) \) on the splitting \( r \) used in the integral representation \( (1.15) \) follows.
C  Proof of proposition 2.1

The starting point for computing the action of the operator \( O_r(\lambda) \) on \( \Psi_{y_N, \mathbf{r}}(\mathbf{x}_{N+1}) \) is to use the integral representation (1.15) for \( \Psi_{y_N, \mathbf{r}}(\mathbf{x}_{N+1}) \). This has the advantage of allowing one to act directly, under the integral sign,

- with the product of position operators on \( \Psi_{w, \mathbf{r}}(\mathbf{x}_1) \) by means of (1.26)
- with the operator \( D_{r+1, N+1}(\lambda) \) on \( \Psi_{z, \mathbf{r}}(\mathbf{x}_2) \) by means of (1.23).

Shifting the integration contours along the lines described in appendix B recasts the action in the form

\[
O_r(\lambda) \cdot \Psi_{y_N, \mathbf{r}}(\mathbf{x}_{N+1}) = \int \Psi_{w, \mathbf{r}}(\mathbf{x}_1) \Psi_{z, \mathbf{r}}(\mathbf{x}_2) \left[ \prod_{a=1}^{N-1} \frac{1}{y_b - w_a - i\hbar} \right] \times h_\lambda(\mathbf{w}, \mathbf{z} | y_N) \cdot \sigma(\mathbf{w}, \mathbf{z} | y_N) \prod_{a=r}^{N-1} \frac{1}{w_a - y_a},
\]

where we have set

\[
h_\lambda(\mathbf{w}, \mathbf{z} | y_N) = \left( i \right)^{r(r-1)} \prod_{a=1}^{N-1} \frac{1}{y_b - w_a - i\hbar} \cdot \prod_{b \neq r} (\lambda - y_b) \cdot \Psi_{y_N, \mathbf{r}}(\mathbf{x}_{N+1})
\]

Similarly, one can re-express the form of the action given in (2.2) by moving the multiple sum under the integral sign leading to

\[
h_\lambda(\mathbf{w}, \mathbf{z} | y_N) = \left( -i \right)^{r+1(r-2)} \prod_{a=r+1}^{N} \frac{1}{y_a - y_b} \cdot \prod_{b \neq r} (\lambda - y_a) \cdot \Psi_{y_N, \mathbf{r}}(\mathbf{x}_{N+1})
\]

where

\[
t_\lambda(\mathbf{w}, \mathbf{z} | y_N) = \left( -i \right)^{r+1(r-2)} \prod_{a=r+1}^{N} \frac{1}{y_a - y_b} \cdot \prod_{b \neq r} (\lambda - y_a) \cdot \Psi_{y_N, \mathbf{r}}(\mathbf{x}_{N+1})
\]

Note that, the multiple sum in the definition of \( t_\lambda(\mathbf{w}, \mathbf{z} | y_N) \) can be re-cast in terms of a single \((r+1)\)-fold contour integral

\[
f_\lambda(\mathbf{w}, \mathbf{z} | y_N) = \frac{\left( -i \right)^{r+1(r+1)}}{(r+1)!} \oint_{\gamma(\gamma_{r+1})} \prod_{b=1}^{N} \frac{1}{\lambda - y_a} \cdot \prod_{a \neq b}^{r+1} (s_a - s_b) \prod_{a=1}^{N} \frac{1}{y_b - s_a} \cdot \prod_{b=1}^{r+1} \frac{1}{s_b - w_a - i\hbar} \cdot \prod_{a=r}^{N-1} \frac{1}{s_a - s_b} \prod_{a=1}^{N} \frac{1}{s_b - w_a - i\hbar} \cdot \frac{d^{r+1}s}{(2\pi i)^{r+1}}.
\]
The contour $\mathcal{C}(\{y_a\}_1^N)$ appearing above is a counterclockwise loop of index 1 around each of the $y$’s that does not surround any of the other poles in the integration variables $s_a$, with $a = 1, \ldots, r + 1$.

As a consequence, one gets that equations (C.2) will follow as soon as we have shown that

$$t^{(3)}(w, z | y_N) = h_A(w, z | y_N) - t_A(w, z | y_N) \tag{C.6}$$

vanishes.

Notice that the functions defined in (C.2) and (C.4) are both polynomials in $\lambda$ of degree $N - 1 - r$. Hence, it is enough to show that $t^{(3)}(w, z | y_N)$

- vanishes at the points $\lambda = z_p$ with $p = r, \ldots, N - 1$ ;
- has $O(\lambda^{N-2-r})$ leading asymptotics around the point $\lambda = \infty$, i.e. $r^{(3)}(w, z | y_N) = O(\lambda^{N-2-r})$.

For this purpose, we evaluate the integral representation (C.5) by the residues lying outside of the original contour $\mathcal{C}(\{y_a\}_1^N)$. This demands a little care as, individually in each variable $s_a$, the integrand behaves as a constant when $s_a \to \infty$. Hence one has to take into account the contribution of the residue at $s_a = \infty$. Still, the explicit computation of this residue’s contribution can be avoided first by regularizing the integral (so that it has a faster decay at infinity) and then, once all the calculations are finished, removing the regularization parameter.

More precisely, in order to show the equality at the point $\lambda = z_p$ with $p = r, \ldots, N - 1$ we regularize the integral representation for $t^{(3)}_\lambda(w, z | y_N)$ in the form

$$\tilde{t}_p(w, z | y_N) = -\frac{(i)^{r+1}r}{(r+1)!} \prod_{b=1}^N (z_p - y_b)$$

$$\times \int_{\mathcal{C}(\{y_a\}_1^N)} \prod_{a \neq b}^N (s_a - s_b) \prod_{b=1}^{r+1} [\prod_{a=1}^N \frac{1}{y_b - s_a}] \prod_{a \neq r}^{N-1} (z_a - s_b) \prod_{a=1}^{r+1} \left[ \prod_{b=1}^N \left( \frac{1}{s_b - w_a + i\hbar} \right) \right] \cdot \frac{dr+1}{(2\pi r+1)} \tag{C.7}$$

The above function is built in such a way that

$$\tilde{t}_p(w, z | y_N) \xrightarrow{\omega_r, w_{r+1} \to \infty} t^{(3)}_\lambda(w, z | y_N) \tag{C.8}$$

The integral representation (C.7) can be easily evaluated by taking the residues outside of the contour $\mathcal{C}(\{y_a\}_1^N)$. The integrand has poles in respect to the variables $s_a$ at points

$$s_a = w_a + i\hbar \quad \text{with} \quad b = 1, \ldots, r + 1 \tag{C.9}$$

Yet, because of the presence of the squared Van-der-Monde determinant, solely the residues computed at distinct points for distinct variables lead to non-zero contributions. Further, the symmetry of the integrand implies that the integral will be given by $(r+1)!$ times the residues of the integrand at the points

$$s_a = w_a + i\hbar \quad \text{with} \quad a = 1, \ldots, r + 1 \tag{C.10}$$

As a consequence,

$$\tilde{t}_p(w, z | y_N) = -(-1)^{r+1} (i)^{r+1} \prod_{b=1}^N (z_p - y_b) \prod_{a=1}^{r+1} \prod_{b=1}^N \frac{1}{s_a - w_a + i\hbar} \cdot \prod_{b=1}^{r+1} \prod_{a=1}^N \left( \frac{1}{w_a - y_b + i\hbar} \right) \tag{C.11}$$
Thus, by taking in the $w_r, w_{r+1} \to \infty$ limit, in virtue of (C.8),

$$f_{\infty}(w, z \mid y_N) = (i)^{(r-1)} \sum_{a=1}^{N} \prod_{b \neq a}^{r-1} \prod_{b=1}^{r-1} (z_a - w_b - i\hbar) \prod_{a=1}^{r-1} \left( \frac{1}{z_a - w_a - i\hbar} \right) = u_{r}(w, z \mid y_N). \quad (C.12)$$

As a consequence, it solely remains to show the equality of the leading asymptotics at $\lambda = \infty$. Sending $\lambda \to \infty$ in (C.5) and restricting to the leading asymptotics makes the behaviour of the integrand at $s_a = \infty$ even worse than in the previous case (the latter grows in this situation linearly in $s_a$ at $s_a = \infty$). In thus appears convenient, for the purpose of intermediate calculations, to regularize the integrand by adding three auxiliary parameters $w_r, w_{r+1}, w_{r+2}$. The result of interest will then be recovered by sending the three variables to $\infty$. More precisely, we set

$$\tilde{I}_{\infty}(w, z \mid y_N) = -\frac{(i)^{(r+1)}}{r+1} \left( \prod_{a=1}^{r+2} (-w_a) \right)^{r+1} \int \prod_{a,b}^{r+1} (s_a - s_b) \prod_{a=1}^{r+1} \prod_{b=1}^{N} \left( \frac{1}{y_b - s_a} \right) \prod_{b=1}^{r+1} \left( \frac{1}{s_b - w_a - i\hbar} \right) \prod_{a=1}^{r+2} \prod_{b \neq a}^{r+2} (s_a - s_b) \frac{1}{y_b - w_a - i\hbar}, \quad (C.13)$$

so that

$$\tilde{I}_{\infty}(w, z \mid y_N) \left. \right|_{w_r, w_{r+1}, w_{r+2} \to \infty} = \lim_{\lambda \to \infty} \left\{ \lambda^{r+1} \tilde{I}_{\lambda}(w, z \mid y_N) \right\}. \quad (C.14)$$

The integral in (C.13) can be estimated by the residues outside of $\mathcal{C}(y_{a_1}^N)$. These are located at

$$s_a = w_b + i\hbar \quad \text{with} \quad b = 1, \ldots, r + 2. \quad (C.15)$$

Again, due to the presence of the Van-der-Monde determinants, the only choice of residues giving non-zero contribution corresponds to computing the residues at

$$\{s_a\}_{a=1}^{r+1} = \{w_a + i\hbar\}_{a=p}^{r+2} \quad \text{for} \quad p = 1, \ldots, r + 2. \quad (C.16)$$

Each of these contributions ought to be weighted by the factor $(r+1)!$ originating from the symmetry of the integrand. Hence,

$$\tilde{I}_{\infty}(w, z \mid y_N) = (i)^{(r-1)} \left( \prod_{a=1}^{r+2} (-w_a) \right)^{r+1} \sum_{p=1}^{r+2} \prod_{b=1}^{N} \prod_{a=1}^{r+2} \left( \frac{1}{y_b - w_a - i\hbar} \right) \prod_{a=1}^{r+2} \prod_{b \neq a}^{r+2} \left( \frac{1}{z_b - w_a - i\hbar} \right) \prod_{a=1}^{r+2} \prod_{b=1}^{N} \left( \frac{1}{(y_b - w_a - i\hbar)} \right) \prod_{b=1}^{N} \left( \frac{1}{y_b - w_a - i\hbar} \right) \cdot \mathcal{S}. \quad (C.17)$$
There, we have set

\[
S = \sum_{p=1}^{r+2} \prod_{a=1}^{r+2} \left\{ \frac{1}{w_a - w_p} \right\} \cdot \prod_{b=1}^{N} (y_b - w_p - i\hbar) \prod_{r=1}^{N-1} \left\{ \frac{1}{z_b - w_p - i\hbar} \right\}.
\]  

(C.18)

This sum can be recast as a contour integral over the counter-clockwise loop $\mathcal{C}(\{w_a\}_{1}^{r+2})$ surrounding the points $\{w_a\}_{1}^{r+2}$ but not any other singularity of the integrand.

\[
S = -\oint_{\mathcal{C}(\{w_a\}_{1}^{r+2})} \prod_{a=1}^{r+2} \left\{ \frac{1}{w_a - \tau} \right\} \cdot \prod_{b=1}^{N} (y_b - \tau - i\hbar) \prod_{r=1}^{N-1} \left\{ \frac{1}{z_b - \tau - i\hbar} \right\} \cdot \frac{d\tau}{2i\pi}.
\]  

(C.19)

The integrand decays as $\tau^{-2}$ at infinity, so that there is no residue at $\infty$ and the only poles of the integrand lying outside of $\mathcal{C}(\{w_a\}_{1}^{r+2})$ are at

\[
\tau = z_b - i\hbar \quad \text{for} \quad b = r, \ldots, N - 1.
\]  

(C.20)

Thence, taking the integral by the residues lying outside of the contour of integration yields

\[
S = -\sum_{p=1}^{N-1} \prod_{b=1}^{N} \left\{ \frac{1}{z_b - z_p} \right\} \cdot \prod_{r=1}^{r+2} \prod_{a=1}^{r+2} (w_a - z_p + i\hbar).
\]  

(C.21)

Taking the $w_a \to \infty$, $a = r, \ldots, r + 2$, limit on the level of the last formula is straightforward. One ultimately gets that

\[
\lim_{\lambda \to \infty} \left\{ A^{r+1-N} t_\lambda(w, z | y) \right\} = (i)^{(r-1)} \prod_{a=1}^{r-1} \left\{ \frac{N}{N-1} \prod_{b=1}^{N} (y_b - w_a - i\hbar) \right\} \times \sum_{p=1}^{N} \prod_{b=1}^{N-1} \left\{ \frac{1}{z_b - z_p} \right\} \prod_{r=1}^{N-1} \prod_{a=1}^{r+2} (w_a - z_p + i\hbar).
\]  

(C.22)

As a consequence, the leading asymptotics at $\lambda \to \infty$ of the polynomials $t_\lambda(w, z | y)$ and $h_\lambda(w, z | y)$ coincide. Thus, we have provided the equality at enough interpolation points so as to ensure that the two polynomials are equal.

\[\blacksquare\]

References

[1] D. An, "Complete Set of Eigenfunctions of the Quantum Toda Chain.", Lett. Math. Phys. 87 (2009), 209–223.

[2] O. Babelon, "Equations in Dual Variables for Whittaker Functions.", Lett. Math. Phys. 65 (2003), 229–240.

[3] , "On the Quantum Inverse Problem for the Closed Toda Chain.", J. Phys. A 37 (2004), 303–316.

[4] R.J. Baxter, "Partition function of the eight vertex lattice model.", Ann. Phys. 70 (1972), 193–228.

[5] S. E. Derkachov, G. P. Korchemsky, and A. N. Manashov, "Noncompact Heisenberg spin magnets from high-energy QCD: I. Baxter Q-operator and Separation of Variables.", Nucl. Phys. B617 (2001), 375–440.
[6] L. Fehér, "Action-angle map and duality for the open Toda lattice in the perspective of Hamiltonian reduction.",

[7] H. Flaschka, "The Toda lattice II: Existence of integrals.", Phys. Rev. B 9 (1974), 1924–1925.

[8] M. Gaudin and V. Pasquier, "The periodic Toda chain and a matrix generalization of the Bessel function recursion relations.", J. Phys. A: Math. Gen. 25 (1992), 5243–5252.

[9] A. Gerasimov, S. Kharchev, and D. Lebedev, "Representation Theory and Quantum Inverse Scattering Method: The Open Toda Chain and the Hyperbolic Sutherland Model.", Int. Math. Res. Notices 17 (2004), 823–854.

[10] A. Givental, "Stationary Phase Integrals, Quantum Toda Lattices, Flag Manifolds and the Mirror Conjecture.", AMS Trans. (2) 180 (1997), 103–115.

[11] R. Goodman and N.R. Wallach, "Classical and quantum-mechanical systems of Toda lattice type. III", Comm. Math. Phys. 105 (1986), 473–509.

[12] N. Grosjean, J. M. Maillet, and G. Niccoli, "On the form factors of local operators in the lattice sine-Gordon model.", J. Stat. Mech.: Th. and Exp. (2012), P10006.

[13] M. C. Gutzwiller, "The quantum mechanical Toda lattice.", Ann. Phys. 124 (1980), 347–387.

[14] _____, "The quantum mechanical Toda lattice II.", Ann. Phys. 133 (1981), 304–331.

[15] N. Z. Iorgov and V. N. Shadura, "Wave functions of the Toda chain with boundary interaction.", Theor. Math. Phys. 142 (2005), 289–305.

[16] M. Kac and P. Van Moerbeke, "A complete solution of the periodic Toda chain.", Proc. Nat. Acad. Sci. 72 (1975), 2879–2880.

[17] S. Kharchev and D. Lebedev, "Eigenfunctions of $GL(N,\mathbb{R})$ Toda chain: The Mellin-Barnes representation.", JETP Lett. 71 (2000), 235–238.

[18] _____, "Integral representations for the eigenfunctions of quantum open and periodic Toda chains from QISM formalism.", J.Phys.A 34 (2001), 2247–2258.

[19] N. Kitanine, J.-M. Maillet, and V. Terras, "Form factors of the XXZ Heisenberg spin-1/2 finite chain.", J. Phys. A: Math. Gen. 35 (2002), L753–10502.

[20] B. Kostant, "Quantization and representation theory.", In "Representation theory of Lie groups", Proc. SCR/LMS res. symp. on rep. of Lie groups, London Math. Soc. Lect. Note 34 (1979), 287–316.

[21] _____, "The solution to a generalized Toda lattice and representation theory.", Adv. in Math. 34 (1979), 195–338.

[22] K. K. Kozlowski, "Unitarity of the SoV transform for the Toda chain.", math-ph:1306.4967.

[23] K. K. Kozlowski and J. Teschner, "TBA for the Toda chain.", Festschrift volume for Tetsuji Miwa, "Infinite Analysis 09: New Trends in Quantum Integrable Systems". (math-ph/10062905).

[24] V. B. Kuznetsov, "Inverse problem for $sl(2)$ lattices.", Proc. Int. Conf., Symmetry and Perturbation Theory, World Scientific (2002), 136–152.
[25] J.-M. Maillet and V. Terras, "On the quantum inverse scattering problem.", Nucl. Phys. B 575 (2000), 627–644.

[26] N. A. Nekrasov and S. L. Shatashvili, "Quantization of Integrable Systems and Four Dimensional Gauge Theories.", Proc. 16th Int. Congr. Math. Phys., Prague, Editor : P. Exner, World Scientific 2010 (2009), 265–289.

[27] G. Niccoli, "Non-diagonal open spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and matrix elements of some quasi-local operators.", J.Stat.Mech. (2012), P10025.

[28] M.A. Olshanetsky and A.M. Perelomov, "Quantum completely integrable systems connected with semi-simple Lie algebras.", Lett. Math. Phys. 2 (1977), 7–13.

[29] T. Oota, "Quantum projectors and local operators in lattice integrable models.", J. Phys. A: Math. Gen. 37 (2004), 441–452.

[30] S. Ruijsenaars, "Action-angle maps and scattering theory for some finite-dimensional integrable systems III. Sutherland type systems and their duals.", Publ. Resch. Inst. Math. Sci. 31 (1995), 247–353.

[31] M. A. Semenov-Tian-Shansky, "Quantization of Open Toda Lattices.", Encycl. Math. Sci., Vol 16 Dynamical Systems VII, Edts V. I. Arnol’d and S. P. Novikov, Springer-Verlag, Berlin (1994), 226–259.

[32] A. V. Silantyev, "Transition function for the Toda chain.", Theor. Math. Phys. 150 (2007), 315–331.

[33] E. K. Sklyanin, "Bispectrality for the quantum open Toda chain.", nlin.SI://1306.0454.

[34] ______, "The quantum Toda chain.", Lect. Notes in Phys. 226 (1985), 196–233.

[35] ______, "Quantum Inverse Scattering Method. Selected Topics.", in "Quantum Group and Quantum Integrable Systems" (Nankai Lectures in Mathematical Physics), ed. by Mo-Lin Ge, Singapore: World Scientific (1992), 63–97.

[36] W. Toda, "Wave propagation in anharmonic lattices.", J. Phys. Soc. Jap. 23 (1967), 501–506.

[37] N. R. Wallach, "Real reductive groups II.", Pure and applied mathematics, vol. 132-II, Academic Press, inc., 1992.