Quantum Convolutional Codes Derived from Reed-Solomon and Reed-Muller Codes

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Abstract—Convolutional stabilizer codes promise to make quantum communication more reliable with attractive online encoding and decoding algorithms. This paper introduces a new approach to convolutional stabilizer codes based on direct limit constructions. Two families of quantum convolutional codes are derived from generalized Reed-Solomon codes and from Reed-Muller codes. A Singleton bound for pure convolutional stabilizer codes is given.

I. INTRODUCTION
A key obstacle to the communication of quantum information is decoherence, the spontaneous interaction of the environment with the information-carrying quantum system. The protection of quantum information with quantum error-correcting codes to reduce or perhaps nearly eliminate the impact of decoherence has led to a highly developed theory of quantum error-correcting block codes. Somewhat surprisingly, quantum convolutional codes have received less attention.

Ollivier and Tillich developed the stabilizer framework for quantum convolutional codes, and addressed encoding and decoding aspects of such codes [13], [14]. Almeida and Palazzo constructed a concatenated convolutional code of rate 1/4 with memory m = 3 [1]. Forney and Guha constructed quantum convolutional codes with rate 1/3 [5]. Also, in a joint work with Grassl, they derived rate (n − 2)/n convolutional stabilizer codes [4]. Grassl and Rötteler constructed quantum convolutional codes from product codes [8], and they gave a general algorithm to obtain non-catastrophic encoders [7].

In this paper, we give a new approach to quantum convolutional codes based on a direct limit construction, generalize some of the previously known results, and construct two families of quantum convolutional codes based on classical generalized Reed-Solomon and Reed-Muller codes.

II. BACKGROUND
In this section, we give some background concerning classical convolutional codes, following [9, Chapter 14] and [12].

Let \( F_q \) denote a finite field with \( q \) elements. An \( (n, k, \delta)_q \) convolutional code \( C \) is a submodule of \( F_q[D]^n \) generated by a right-invertible matrix \( G(D) \) defined as

\[
C = \{ u(D)G(D) \mid u(D) \in F_q[D]^{k \times n} \},
\]

such that \( \sum_{i=1}^k \nu_i = \max \{ \deg \gamma \mid \gamma \text{ is a } \delta\text{-min} \text{ of } G(D) \} \). Here, \( \nu_i = \max_{1 \leq j \leq n} \{ \deg g_{ij} \} \). We say \( \delta \) is the degree of \( C \). The memory \( \mu \) of \( G(D) \) is defined as \( \mu = \max_{1 \leq k} \nu_i \).

The weight \( \text{wt}(v(D)) \) of a polynomial \( v(D) \) in \( F_q[D] \) is defined as the number of nonzero coefficients of \( v(D) \), and the weight of an element \( u(D) \in F_q[D]^n \) is defined as \( \text{wt}(u(D)) = \sum_{i=1}^n \text{wt}(u_i(D)) \). The free distance \( d_f \) of \( C \) is defined as \( d_f = \text{wt}(C) = \min \{ \text{wt}(u) \mid u \in C, u \neq 0 \} \). We say that an \( (n, k, \delta, \mu, d_f)_q \) convolutional code with memory \( \mu \) and free distance \( d_f \) is an \( (n, k, \delta; \mu, d_f)_q \) convolutional code.

Let \( \mathbb{N} \) denote the set of nonnegative integers. Let \( \Gamma_q = \{ v: \mathbb{N} \to F_q \mid \text{all but finitely many coefficients of } v \text{ are } 0 \} \). We define a vector space isomorphism \( \sigma: \Gamma_q[D]^n \to \Gamma_q \) that maps an element \( u(D) \in \Gamma_q[D]^n \) to the coefficient sequence of the polynomial \( \sum_{i=0}^{\infty} D^i u_i(D^n) \), that is, an element in \( \Gamma_q[D]^n \) is mapped to its interleaved coefficient sequence. Frequently, we will refer to the image \( \sigma(C) \) of a convolutional code \( C \) again as \( C \), as it will be clear from the context whether we discuss the sequence or polynomial form of the code.

If \( G(D) \) is the generator matrix of a convolutional code \( C \), then one easily checks that \( \sigma(C) = \Gamma_qG \).

In the literature, convolutional codes are often defined in the form \( \{ p(D)G'(D) \mid p(D) \in F_q[D]^k \} \), where \( G'(D) \) is a matrix of full rank in \( F_q^{k \times n} \). In this case, one can obtain a generator matrix \( G(D) \) in our sense by multiplying \( G'(D) \) from the left with a suitable invertible matrix \( U(D) \) in \( F_q^{k \times k} \), see [9].

We define the Euclidean inner product of two sequences \( u \) and \( v \) in \( \Gamma_q \) by \( \langle u \mid v \rangle = \sum_{i \in \mathbb{N}} u_i v_i \), and the Euclidean dual of a convolutional code \( C \subseteq \Gamma_q \) by \( C^\perp = \{ u \in \Gamma_q \mid \langle u \mid v \rangle = 0 \text{ for all } v \in C \} \). A convolutional code \( C \) is called self-orthogonal if and only if \( C \subseteq C^\perp \). It is easy to see that a convolutional code \( C \) is self-orthogonal if and only if \( GG^T = 0 \).

Consider the finite field \( F_{q^2} \). The Hermitian inner product of two sequences \( u \) and \( v \) in \( \Gamma_{q^2} \) is defined as \( \langle u \mid v \rangle = \sum_{i \in \mathbb{N}} u_i v^*_i \). We have \( C^{q-k} = \{ u \in \Gamma_{q^2} \mid \langle u \mid v \rangle = 0 \text{ for all } v \in C \} \). As before, \( C \subseteq C^{q-k} \) if and only if \( GG^T = 0 \), where the Hermitian transpose \( \dagger \) is defined as \( \langle a_{ij} \rangle^\dagger = (a_{ji}^*) \).
III. QUANTUM CONVOLUTIONAL CODES

The state space of a $q$-ary quantum digit is given by the complex vector space $C^q$. Let $\{ |x\rangle \mid x \in F_q \}$ denote a fixed orthonormal basis of $C^q$, called the computational basis. For $a, b \in F_q$, we define the unitary operators

$$X(a)|x\rangle = |x + a\rangle \quad \text{and} \quad Z(b)|x\rangle = \exp(2\pi i \text{tr}(bx)/p)|x\rangle,$$

where the addition is in $F_q$, $p$ is the characteristic of $F_q$, and $\text{tr}(x) = x^0 + x^p + \cdots + x^{p^q-1}$ is the absolute trace from $F_q$ to $F_p$. The set $E = \{ X(a), Z(b) \mid a, b \in F_q \}$ is a basis of the algebra of $q \times q$ matrices, called the error basis.

A quantum convolutional code encodes a stream of quantum digits. One does not know in advance how many qudits $i.e.$, quantum digits will be sent, so the idea is to impose structure on the code that simplifies online encoding and decoding. Let $n, m$ be positive integers. We will process $n + m$ qudits at a time, $m$ qudits will overlap from one step to the next, and $n$ qudits will be output.

For each $t$ in $\mathbb{N}$, we define the Pauli group $P_t = \langle |M, M \in E^{(t+1)n+m} \rangle \rangle$ as the group generated by the $(t + 1)n + m$-fold tensor product of the error basis $E$. Let $I = X(0)$ be the $q \times q$ identity matrix. For $i, j \in \mathbb{N}$ and $i \leq j$, we define the inclusion homomorphism $\iota_{ij}$: $P_i \rightarrow P_j$ by $\iota_{ij}(M) = M \otimes I^{(j-i)}$. We have $\iota_{ij}(M) = M$ and $\iota_{ik} \circ \iota_{jk}$ holds for $i \leq j \leq k$. Therefore, there exists a group

$$P_\infty = \lim_{t \rightarrow \infty} (P_t, \iota_{ij}),$$

called the direct limit of the groups $P_t$ over the totally ordered set $(\mathbb{N}, \leq)$. For each nonnegative integer $i$, there exists a homomorphism $\iota_i: P_i \rightarrow P_\infty$ given by $\iota_i(M_i) = M_i \otimes I^{(\infty)}$ for $M_i \in P_i$, and $\iota_i = \iota_{ij} \circ \iota_{ij}$ holds for all $i \leq j \leq k$. We have $P_\infty = \bigcup_{j=0}^{\infty} \iota_j(P_j)$; put differently, $P_\infty$ consists of all infinite tensor products of matrices in $\langle |M \mid M \in E \rangle$ such that all but finitely many tensor components are equal to $I$. The direct limit structure that we introduce here provides the proper conceptual framework for the definition of convolutional stabilizer codes; see [16] for background on direct limits.

We will define the stabilizer of the quantum convolutional code also through a direct limit. Let $S_0$ be an abelian subgroup of $P_0$. For positive integers $t$, we recursively define a subgroup $S_t$ of $P_t$ by $S_t = \langle N \otimes I^{\otimes n}, I^{\otimes n} \otimes M \mid N \in S_{t-1}, M \in S_0 \rangle$. Let $Z_t$ denote the center of the group $P_t$. We will assume that (S1) $I^{\otimes n} \otimes M$ and $N \otimes I^{\otimes n}$ commute for all $N, M \in S_0$ and all positive integers $t$.

(S2) $S_t/Z_t$ is an $(t + 1)(n - k)$-dimensional vector space over $F_q$.

(S3) $S_t \cap Z_t$ contains only the identity matrix.

Assumption (S1) ensures that $S_t$ is an abelian subgroup of $P_t$. Assumption (S2) implies that $S_t$ is generated by $t + 1$ shifted versions of $n - k$ generators of $S_0$ and all these $(t + 1)(n - k)$ generators are independent, and Assumption (S3) ensures that the stabilizer (or +1 eigenspace) of $S_t$ is nontrivial as long as $k < n$.

The abelian subgroups $S_t$ of $P_t$ define an abelian group

$$S = \lim_{t \rightarrow \infty} (S_{t}, \iota_{ij}) = \langle \iota_{ik}(I^{\otimes n} \otimes M) \mid t \geq 0, M \in S_0 \rangle$$

generated by shifted versions of elements in $S_0$.

Definition 1: Suppose that an abelian subgroup $S_0$ of $P_0$ is chosen such that $S_1, S_2$, and $S_3$ are satisfied. Then the +1-eigenspace of $S = \lim_{t \rightarrow \infty} (S_{t}, \iota_{ij})$ in $\bigotimes_{i=0}^{\infty} C^q$ defines a convolutional stabilizer code with parameters $|(n, k, m)|_q$.

In practice, one works with a stabilizer $S_t$ for some large (but previously unknown) $t$, rather than with $S$ itself. We notice that the rate $k/n$ of the quantum convolutional stabilizer code defined by $S$ is approached by the rate of the stabilizer block code $S_t$ for large $t$. Indeed, $S_t$ defines a stabilizer code with parameters $|(t + 1)n + m, (t + 1)k + m)|_q$; therefore, the rates of these stabilizer block codes approach

$$\lim_{t \rightarrow \infty} \frac{(t + 1)k + m}{t} = \lim_{t \rightarrow \infty} \frac{k + m/(t + 1)}{n + m/(t + 1)} = \frac{k}{n}.$$

We say that an error $E$ in $P_\infty$ is detectable by a convolutional stabilizer code with stabilizer $S$ if and only if a scalar multiple of $E$ is contained in $S$ or if $E$ does not commute with some element in $S$. The weight $w$ of an element in $P_\infty$ is defined as its number of non-identity tensor components. A quantum convolutional stabilizer code is said to have free distance $d_f$ if and only if it can detect all errors of weight less than $d_f$, but cannot detect some error of weight $d_f$. Denote by $Z(P_\infty)$ the center of $P_\infty$ and by $C_{P_{\infty}}(S)$ the centralizer of $S$ in $P_\infty$. Then the free distance is given by $d_f = \min \{|w(e) \mid e \in C_{P_{\infty}}(S) \backslash Z(P_\infty)\}$.

Let $(\beta, \beta')$ denote a normal basis of $F_{q^2}/F_q$. Define a map $\tau: F_\infty \rightarrow \Gamma_{q^2}$ by $\tau(\omega^a X(a_0)Z(b_0) \cdots X(a_j)Z(b_j) \cdots) = (\beta a_0 + \beta' b_0, \beta a_1 + \beta' b_1, \ldots)$. For sequences $v$ and $w$ in $\Gamma_{q^2}$, we define a trace-alternating form

$$\langle v \mid w \rangle = \text{tr}_{q/p}(v \cdot w^q - v^q \cdot w)/\beta^2 - \beta'^2.$$

Lemma 2: Let $A$ and $B$ be elements of $P_\infty$. Then $A$ and $B$ commute if and only if $\tau(A) = \tau(B) \equiv 0$.

Proof: This follows from [11] and the direct limit structure.

Lemma 3: Let $Q$ be an $F_{q^2}$-linear $[(n, n, k, m)]_q$ quantum convolutional code with stabilizer $S$, where $S = \lim_{t \rightarrow \infty} (S_{t}, \iota_{ij})$ and $S_0$ an abelian subgroup of $P_0$ such that $S_1, S_2$, and $S_3$ hold. Then $C = \sigma^{-1} \tau(S)$ is an $F_{q^2}$-linear $(n, (n - k)/2; \mu \leq [n/m/n])_q$ convolutional code generated by $\sigma^{-1} \tau(S_0)$. Further, $C \subseteq C_{P_\infty}$. Proof: Recall that $\sigma: F_{q^2} \rightarrow \Gamma_{q^2}$, maps $u(D)$ in $F_{q^2}[D]^n$ to $\sum_{u=0}^{n-1} D^u u(D^n)$. It is invertible, thus $\sigma^{-1} \circ \tau(e)$ is well defined for any $e$ in $P_\infty$. Since $S$ is generated by shifted versions of $S_0$, it follows that $C = \sigma^{-1} \tau(S)$ is generated as the $F_\infty$ span of $\sigma^{-1} \tau(S_0)$ and its shifts, $i.e.$, $D^i \sigma^{-1} \tau(S_0)$, where $i \in N$. Since $Q$ is an $F_{q^2}$-linear $[(n, n, k, m)]_q$ quantum convolutional code, $S_0$ defines an $[(n + m, k + m)]_q$ stabilizer code with $(n - k)/2 F_{q^2}$-linear generators. Since the maps $\sigma$ and $\tau$ are linear $\sigma^{-1} \tau(S_0)$ is also $F_{q^2}$-linear. As $\sigma^{-1} \tau(e)$ is in $F_{q^2}[D]^n$ we can define an $(n - k)/2 \times n$ polynomial generator matrix that generates $C$. This generator matrix need not be right invertible, but we know
that there exists a right invertible polynomial generator matrix that generates this code. Thus $C$ is an $(n, (n-k)/2; \mu_q)$ code. Since $S$ is abelian, Lemma 2 and the $F_q^2$-linearity of $S$ imply that $C \subseteq C_{i,\mu}$.

Finally, observe that maximum degree of an element in $\sigma^{-1}(S\mu)$ is $\lceil m/n \rceil$ owing to $\sigma$. Together with [9, Lemma 14.3.8] this implies that the memory of $\sigma^{-1}(S\mu)$ must be $\mu \leq \lceil m/n \rceil$.

We define the degree of an $F_q^2$-linear $[(n, k, m), \mu; \delta, \eta]$ quantum convolutional code $Q$ with stabilizer $S$ as the degree of the classical convolutional code $\sigma^{-1}(S\mu)$. We denote an $[(n, k, m), \mu; \delta, \eta]$ quantum convolutional code with free distance $d_f$ and total constraint length $\delta$ as $[(n, k, m; \delta, d_f)]_q$. It must be pointed out this notation is at variance with the classical codes in not just the order but the meaning of the parameters.

**Corollary 4**: An $F_q^2$-linear $[(n, k, m; \delta, d_f)]_q$ quantum stabilizer code implies the existence of an $(n, (n-k)/2; \delta, \mu_q)$ convolutional code $C$ such that $d_f = \text{wt}(C_{i,\mu} \setminus C)$.

**Proof**: As before let $C = \sigma^{-1}(S\mu)$, by Lemma 2 we can conclude that $\sigma^{-1}(S\mu) \subseteq C_{i,\mu}$. Thus an undetectable error is mapped to an element in $C_{i,\mu} \setminus C$. While $\tau$ is injective on $S$ it is not the case with $S\mu$. However we can see that if $c$ is in $C_{i,\mu} \setminus C$, then surjectivity of $\tau$ on $(S\mu)$ implies that there exists an error $e$ in $C_{i,\mu} \setminus Z(S\mu)$ such that $\tau(e) = \sigma(e)$. As $\tau$ are isometric c is a undetectable error with $\text{wt}(c)$. Hence, we can conclude that $d_f = \text{wt}(C_{i,\mu} \setminus C)$. Combining with Lemma 3 we have the claim.

An $[(n, k, m; \delta, d_f)]_q$ code is said to be a pure code if there are no errors of weight less than $d_f$ in the stabilizer of the code. Corollary 4 implies that $d_f = \text{wt}(C_{i,\mu} \setminus C) = \text{wt}(C_{i,\mu})$.

**Theorem 5**: Let $C$ be an $(n, (n-k)/2; \delta, \mu_q)$ convolutional code such that $C \subseteq C_{i,\mu}$. Then there exists an $[(n, k, m; \delta, d_f)]_q$ quantum convolutional stabilizer code, where $d_f = \text{wt}(C_{i,\mu} \setminus C)$. The code is pure if $d_f = \text{wt}(C_{i,\mu})$.

**Proof**: [Sketch] Let $G(D)$ be the polynomial generator matrix of $C$, with the semi-infinite generator matrix $G$ defined as in equation 3. Let $C_t = \langle \sigma(G(D)), \ldots, \sigma(D^tG(D)) \rangle = \langle C_t-1, 1, \sigma(D^tG(D)) \rangle$, where $\sigma$ is applied to every row in $G(D)$. The self-orthogonality of $C_t$ implies that $C_t$ is also self-orthogonal. In particular $C_0$ defines an $[n + n\mu, (n-k)/2]_2$ self-orthogonal code. From the theory of stabilizer codes we know that there exists an abelian subgroup $S_0 \leq P_0$ such that $\tau(S_0) = C_0$, where $P_0$ is the Pauli group over $(t+1)n + m$ qudits; in this case $m = n\mu$. This implies that $\tau(I^{\otimes n} \otimes S_0) = \sigma(D^{t}G(D))$. Define $S_t = \langle S_{t-1}, I^{\otimes n} \otimes S_0 \rangle$, then $\tau(S_t) = \tau(S_{t-1}) \sigma(D^tG(D))$. Proceeding recursively, we see that $\tau(S_t) = \langle \sigma(G(D)), \ldots, \sigma(D^{t}G(D)) \rangle = C_t$. By Lemma 2 the self-orthogonality of $C_t$ implies that $S_t$ is abelian, thus $S_1$ holds. Note that $\tau(S_tZ_t/Z_t) = C_t$, where $Z_t$ is the center of $P_t$. Combining this with $F_q^2$-linearity of $C_t$ implies that $S_tZ_t/Z_t$ is a $(t+1)(n-k)$ dimensional vector space over $F_q^2$; hence $S_2$ holds. For $S_3$, assume that $z \neq \{1\}$ is in $S_t \cap Z_t$. Then $z$ can be expressed as a linear combination of the generators of $S_t$. But $\tau(z) = 0$ implying that the generators of $S_t$ are dependent. Thus $S = \text{lim}_{t \to \infty} (S_t \cap Z_t)$ and $S_3$ also holds. Thus $S = \text{lim}_{t \to \infty} (S_t \cap Z_t)$ defines an $[(n, k, n\mu; \delta)]_q$ stabilizer code. By definition the degree of the quantum code is the degree of the underlying classical code. As $\sigma^{-1}(C_{i,\mu}) = C_{i,\mu}$ and $d_f = \text{wt}(C_{i,\mu})$.

**Corollary 6**: Let $C$ be an $(n, (n-k)/2; \delta, \mu_q)$ code such that $C \subseteq C_{i,\mu}$. Then there exists an $[(n, k, n\mu; \delta, d_f)]_q$ code with $d_f = \text{wt}(C_{i,\mu} \setminus C)$. If it is pure if $\text{wt}(C_{i,\mu} \setminus C) = \text{wt}(C_{i,\mu})$.

**Proof**: Since $C \subseteq C_{i,\mu}$, its generator matrix $G$ as in equation 3 satisfies $GG^T = 0$. Thus we can obtain an $F_q^2$-linear $(n, (n-k)/2; \delta, \mu_q)$ code, $C'$ from $G$ as $C' = GRG^T$. Since $G_t \in F_q^{(n-k)/2 \times n}$ we have $GG^T = 0$. Thus $C' \subseteq C_{i,\mu}$. Further, it can checked that $\text{wt}(C_{i,\mu} \setminus C') = \text{wt}(C_{i,\mu}) \setminus C$. The claim follows from Theorem 5.

**Theorem 7 (Singleton bound)**: The free distance of an $[(n, k, m; \delta, d_f)]_q$ $F_q^2$-linear pure convolutional stabilizer code is bounded by

$$d_f \leq \frac{n-k}{2} \left( \left\lfloor \frac{2\delta}{n+k} + 1 \right\rfloor + 1 \right) + \delta + 1$$

**Proof**: By Corollary there exists an $(n, (n-k)/2, \delta, \mu_q)$ convolutional code such that $\text{wt}(C_{i,\mu} \setminus C) = d_f$ and the purity of the code implies that $\text{wt}(C_{i,\mu} \setminus C') = d_f$. The dual code $C^\perp$ or $C_{i,\mu}$ has the same degree as code [10, Theorem 2.66]. Thus, $C_{i,\mu}$ is an $(n, (n-k)/2, \delta, \mu_q)$ convolutional code with free distance $d_f$. By the generalized Singleton bound [17, Theorem 2.4] for classical convolutional codes, we have

$$d_f \leq \frac{n-(n+k)/2}{2} \left( \left\lfloor \frac{\delta}{(n+k)/2} + 1 \right\rfloor + 1 \right) + 1,$$

which implies the claim.

**IV. Convolutional RS Stabilizer Codes**

In this section we will use Piret’s construction of Reed-Solomon convolutional codes [15] to derive quantum convolutional codes. Let $\alpha$ be a primitive $n$th root of unity, where $n|q^2 - 1$. Let $w = (w_0, \ldots, w_{n-1})$, $\gamma = (\gamma_0, \ldots, \gamma_{n-1})$ be in $F_q^2$ where $w_i \neq 0$ and all $\gamma_i \neq 0$ are distinct. Then the generalized Reed-Solomon (GRS) code over $F_q^2$ is the code with the parity check matrix, (cf. [9, pages 175–178])

$$H_{\gamma, w} = \begin{bmatrix}
  w_0 & w_1 & \cdots & w_{n-1} \\
  w_0 \gamma_0 & w_1 \gamma_1 & \cdots & w_{n-1} \gamma_{n-1} \\
  w_0 \gamma_0^2 & w_1 \gamma_1^2 & \cdots & w_{n-1} \gamma_{n-1}^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  w_0 \gamma_0^{t-1} & w_1 \gamma_1^{t-1} & \cdots & w_{n-1} \gamma_{n-1}^{t-1}
\end{bmatrix}$$

The code is denoted by $\text{GRS}_{n-t}(\gamma, v)$, as its generator matrix is of the form $H_{\gamma, v}$ for some $v \in F_q^2$. It is an $[n, n-t, t+1]_q$ MDS code [9, Theorem 5.3.1]. If we choose $w_i = \alpha^i$, then $w_{i-t} \neq 0$. If gcd$(n, 2) = 1$, then $\alpha^2$ is also a primitive $n$th root of unity; thus $\gamma_i = \alpha^{2i}$ are all distinct and we have an $[n, n-t, t+1]_q$ GRS code with parity check matrix $H_0$, where

$$H_0 = \begin{bmatrix}
  1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\
  1 & \alpha^3 & \alpha^6 & \cdots & \alpha^{3(n-1)} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  1 & \alpha^{2t-1} & \alpha^{2(2t-1)} & \cdots & \alpha^{2(t-1)(n-1)}
\end{bmatrix}$$
Similarly if \( w_i = \alpha^{-i} \) and \( \gamma_i = \alpha^{-2i} \), then we have another \([n, n - t, t + 1]_{q^2}\) GRS code with parity check matrix

\[
H_1 = \begin{bmatrix}
1 & \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^{-(n-1)} \\
1 & \alpha^{-3} & \alpha^{-6} & \cdots & \alpha^{-3(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{-(2t-1)} & \alpha^{-2(2t-1)} & \cdots & \alpha^{-(2t-1)(n-1)} \\
\end{bmatrix}.
\]

The \([n, n - 2t, 2t+1]_{q^2}\) GRS code with \( w_i = \alpha^{-i(2t-1)} \) and \( \gamma_i = \alpha^{2i+1} \) has a parity check matrix \( H^* \) that is equivalent to \([H_1] \) up to a permutation of rows.

Our goal is to show that under certain restrictions on \( n \) the following semi-infinite coefficient matrix \( H \) determines an \( F_{q^2} \)-linear Hermitian self-orthogonal convolutional code

\[
H = \begin{bmatrix}
H_0 & H_1 & 0 & \cdots & 0 \\
0 & H_0 & H_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & H_0 \\
\end{bmatrix}.
\]

To show that \( H \) is Hermitian self-orthogonal, it is sufficient to show that \( H_0, H_1 \) are both self-orthogonal and \( H_0 \) and \( H_1 \) are orthogonal to each other. A portion of this result is contained in [6, Lemma 8], viz., \( n = q^2 - 1 \). We will prove a slightly stronger result.

**Lemma 8:** Let \( n|q^2 - 1 \) such that \( q + 1 < n \leq q^2 - 1 \) and \( 2 \leq \mu = 2t \leq [n/(q+1)] \), then

\[
\overline{H}_0 = (\alpha^{-ij})_{1 \leq i < \mu, 0 \leq j < n}
\]

and \( \overline{H}_1 = (\alpha^{-ij})_{1 \leq i < \mu, 0 \leq j < n} \) are self-orthogonal with respect to the Hermitian inner product. Further, \( \overline{H}_0 \) is orthogonal to \( \overline{H}_1 \).

**Proof:** Denote by \( \overline{H}_{0,j} = (1, \alpha^j, \alpha^{2j}, \ldots, \alpha^{(n-1)j}) \) and \( \overline{H}_{1,j} = (1, \alpha^{-j}, \alpha^{-2j}, \ldots, \alpha^{-(n-1)j}) \), where \( 1 \leq j \leq \mu - 1 \). The Hermitian inner product of \( \overline{H}_{0,j} \) and \( \overline{H}_{0,j} \) is given by

\[
\langle \overline{H}_{0,i}, \overline{H}_{0,j} \rangle_h = \sum_{l=0}^{n-1} \alpha^{il} \overline{\alpha}^{jl} = \frac{\alpha^{(i+j)n} - 1}{\alpha^{i+j} - 1},
\]

which vanishes if \( i + j \not\equiv 0 \mod n \). If \( 1 \leq i, j \leq \lceil n/(q+1) \rceil - 1 \leq q - 2 \), we have \( 1 \leq l \leq \lceil n/(q+1) \rceil - 1 \leq q - 2 \) while \( q \leq j \not\equiv q \mod n \). Thus \( i + j \not\equiv 0 \mod n \) and this inner product also vanishes, which proves the claim.

Since \( H_0 \) is contained in \( \overline{H}_0 \), we obtain the following:

**Corollary 9:** Let \( 2 \leq \mu = 2t \leq [n/(q+1)] \), where \( n|q^2 - 1 \) and \( q + 1 < n \leq q^2 - 1 \). Then \( H_0 \) and \( H_1 \) are Hermitian self-orthogonal. Further, \( H_0 \) is orthogonal to \( H_1 \) with respect to the Hermitian inner product.

Before we can construct quantum convolutional codes, we need to compute the free distances of \( C \) and \( C^{1,h} \), where \( C \) is the code generated by \( H \).

**Lemma 10:** Let \( 2 \leq 2t \leq [n/(q+1)] \), where \( \gcd(n, 2) = 1, n|q^2 - 1 \) and \( q + 1 < n \leq q^2 - 1 \). Then the convolutional code \( C = \Gamma_g H \) has free distance \( d_f \geq n - 2t + 1 \) while \( d_f = wt(C^{1,h}) \) is the free distance of \( C^{1,h} \).

**Proof:** Since \( d_f^2 = wt(C^{1,h}) = wt(C^1) \), we compute \( wt(C^1) \). Let \( c = (c_0, c_1, \ldots, c_n, c_{n+1}, \ldots) \) be a codeword in \( C^1 \) with \( c_i \in F_{q^2}^n \). By definition, we must have \( \langle c, c \rangle_h = \sum_{i,j} c_i \overline{c}_j = 0 \). If \( l \not\equiv n/\mu \), then \( c_i \) is in the dual of \( H^* \), which is an \([n, n - 2t, 2t+1]_{q^2} \) code. Thus \( wt(c) = wt(c_i) \geq 2t + 1 \) and \( d_f \geq 2t + 1 \). If \( \mu \not\equiv n/\mu \), then \( c_i \) is in the dual of \( H^* \). Thus \( \langle c, c \rangle_h = \sum_{i,j} c_i \overline{c}_j = 2t + 1 \). Let \( \langle c, c \rangle_h = (c_0, c_1, \ldots, c_n, \ldots) \) be a nonzero codeword in \( C^{1,h} \). Observing the structure of \( C^1 \), we see that any nonzero \( c_i \) must be in the span of \( H^* \). But \( H^* \) generates an \([n, 2t, n-2t+1]_{q^2} \) code. Hence \( d_f \geq n - 2t + 1 \). If \( 2t \leq [n/(q+1)] \), then \( d_f \geq n - 2t + 1 > 2t+1 = \frac{d_f}{2} \). If \( n \not\equiv n/\mu \), then \( wt(c) \leq \frac{d_f}{2} \). Thus \( d_f \geq n - 2t+1 \) holds.

The preceding proof generalizes [15, Corollary 4] where the free distance of \( C^{1,h} \) was computed for \( q = 2^m \).

**Theorem 11:** Let \( q \) be a power of a prime, \( n \) an odd divisor of \( q^2 - 1 \), such that \( q + 1 < n \leq q^2 - 1 \) and \( 2 \leq \mu = 2t \leq [n/(q+1)] \). Then there exists a pure quantum convolutional code with parameters \([n, n - \mu, n; \mu/2, \mu + 1]_q \). This code is optimal, since it attains the Singleton bound with equality.

**Proof:** The convolutional code generated by the coefficient matrix \( H \) in equation (4) has parameters \([n, \mu/2, \delta \leq \mu/2; 1, \mu/2]_q \). Inspecting the corresponding polynomial generator matrix shows that \( \delta \leq \mu/2 \), since \( n \not\equiv 1 \) and \( \mu/2 \). By Corollary 9 this code is Hermitian self-orthogonal; moreover, Lemma 10 shows that the distance of its dual code is given by \( d_f^2 = \mu - 1 < d_f \). By Theorem 9 we can conclude that there exists a pure convolutional stabilizer code with parameters \([n, n - \mu, n; \delta \leq \mu/2, \mu + 1]_q \). It follows from Theorem 7 that

\[
\mu + 1 \leq \lceil \mu/2 \rceil |(2\delta/(2n - \mu)) + 1 | + \delta + 1 \\
\leq \lceil \mu/2 \rceil |(2\delta/(2n - \mu)) + 1 | + \delta + 1.
\]

This gives \( \mu/(2n - \mu) = 0 \), the right hand side equals \( \mu/2 + \delta + 1 \), which implies \( \delta = \mu/2 \) and the optimality of the quantum code.

**V. Convolutional RM Stabilizer Codes**

In this section, we derive convolutional stabilizer codes from quasi-cyclic subcodes of binary Reed-Muller block codes [2], taking advantage of the framework developed by Esmaeili and Gulliver for classical convolutional codes [3].

Let \( u = (u_1, u_2, \ldots, u_n) \) and \( v = (v_1, v_2, \ldots, v_n) \) be vectors in \( F_2^n \); we define their boolean product as \( uv = (u_1v_1, u_2v_2, \ldots, u_nv_n) \). The product of \( i \) such \( n \)-tuples is said to have degree \( i \).

Let \( b_0 = (1, 1, \ldots, 1) \in F_2^m \). For \( m > 0 \) and \( 1 \leq i \leq m \), define \( b_i \in F_2^{2m-1} \) as concatenation of \( 2m-i \) blocks of the form \( 01 \in F_2^2 \), where \( 0 \) and \( 1 \) are the constant zero and one vectors in \( F_2^2 \), respectively. Let \( 0 \leq r < m \) and \( B = \{b_1, b_2, \ldots, b_m\} \subseteq F_2^{2m} \). Then the \( r \)th order Reed-Muller code
\( \mathcal{R}(r, m) \) is the linear span of \( b_0 \) and all products of elements in \( B \) of degree \( r \) or less. The code \( \mathcal{R}(r, m) \) has dimension \( k(r) = \sum_{i=0}^{r} (m_i) \) and minimum distance \( 2^{m-r} \); the dual of \( \mathcal{R}(r, m) \) is given by \( \mathcal{R}(r, m)^{\perp} = \mathcal{R}(m-1-r, m) \), and the dual distance of \( \mathcal{R}(r, m) \) is \( 2^{r+1} \), see [9] for details.

Let \( B_r \) denote the set of all products of elements in \( B \) of degree \( i \). For \( 0 \leq i \leq r < m \), a generator matrix \( G_r \) of \( \mathcal{R}(r, m) \) is given by (see [3] for details)

\[
G_r = \begin{bmatrix} B_r^c \\ B_r \end{bmatrix}.
\]

Let \( w_{ij} = (110 \ldots 0) \in F_{2^m}^a \). Let \( w_{ij} \) denote the vector obtained by concatenating \( l \) copies of \( w_{ij} \). For \( 0 \leq i \leq l-1 \), let \( M_{l,i} = (2^{l-i-1}w_{i+1}) \otimes B_r, \) which is a matrix of size \( (m-i) \times 2^m \), and let \( M_{l,0} = \left[ \begin{array}{cccc} G_{m-l} & 0 & \cdots & 0 \end{array} \right] \). One can derive a convolutional code as the rowspan of the semi-infinite matrix \( G \) given in (2), where \( \mu = 2^l-1 \) and the matrices \( G_i \), \( 0 \leq i < 2^l \), are defined by

\[
\begin{bmatrix} G_0 & G_1 & \cdots & G_{2^l-1} \end{bmatrix} = \begin{bmatrix} M_{l,0} & M_{l,1} & \cdots & M_{l,2^l-1} \end{bmatrix}.
\]

We note that \( G_0 = G_{m-l} \) and that the rows of \( G_i \), \( 1 \leq i \leq 2^l - 1 \), are a subset of the rows in \( G_0 \). The convolutional code generated by \( G \) is a \( (2m-l, \sum_{i=0}^{l} (m_i))_2 \) code with free distance \( 2^{m-r} \), see [3]. Even though \( G \) corresponds to a catastrophic encoder, we can conclude the following:

**Lemma 12:** Let \( C = \Gamma G \). Then \( d_f \), the free distance of the convolutional code \( C \) is \( 2^{r+1} \).

**Proof:** Let \( c_0 \) be a codeword in the dual of \( \mathcal{R}(r, m-1) \), i.e., \( c_0G_0^T = 0 \). As \( G_i \) are submatrices of \( G_0 \) we have \( c_0G_i^T = 0 \). It follows that \( c = (0, 0, c_0, 0, \ldots) \) satisfies \( CG = 0 \) and is in \( C \). Thus \( d_f \leq \min \{ \text{wt}(c_0) = \text{wt}(\mathcal{R}(r, m-l+1)) \} = 2^{r+1} \).

Let \( c = (\ldots, 0, 0, \ldots, 0) \) be a codeword of minimum weight in \( C \). Since \( CG = 0 \), we can infer that \( c_0G_0^T = 0 = c_0G_0^T \). Since \( c_0 \) is in the dual space of \( G_0 \), it has a minimum weight of \( 2^{r+1} \). Therefore, \( \min \{ \text{wt}(c_0) + 2^{r+1} \} \) \( \leq d_f \leq 2^{r+1} \); hence \( d_f = 2^{r+1} \).

**Lemma 13:** Let \( 1 \leq l \leq m \) and \( 0 \leq r \leq (m-l-1)/2 \), then the convolutional code generated by \( G \) is self-orthogonal.

**Proof:** It is sufficient to show that \( G_0G_j^T = 0 \) for \( 0 \leq i, j \leq 2^l - 1 \). Since the rows of \( G_i \) are a subset of the rows of \( G_0 \) it suffices to show that \( G_i \) is self-orthogonal. For \( G_i \) to be self-orthogonal we require that \( r \leq (m-l) - r \) which holds. Hence, \( G \) generates a self-orthogonal convolutional code.

**Theorem 14:** Let \( 1 \leq l \leq m \) and \( 0 \leq r \leq (m-l-1)/2 \), then there exist pure linear quantum convolutional codes with the parameters \( (2^{m-l}, 2^{m-l} - 2k(r), \leq 2^{m-l}(2^{l-1})/2) \) and free distance \( 2^{r+1} \), where \( k(r) = \sum_{i=0}^{l} (m_i) \).

**Proof:** By Lemma 13 \( G \) defines a linear self-orthogonal convolutional code with parameters \( (2^{m-l}, k(r), \leq 2^{r+1}) \).

and free distance \( 2^{r+1} \). By Corollary 6 there exists a linear \( (2^{m-l}, 2^{m-l} - 2k(r), \leq 2^{m-l}(2^{l-1})/2) \) convolutional stabilizer code. For \( 0 \leq r \leq (m-l-1)/2 \), the dual distance \( 2^{r+1} < 2^{r+1} \), hence the code is pure.

After circulating the first version of this manuscript, Grassl and Rötteler kindly pointed out that the convolutional codes in [3] that are used here have degree 0, hence, are a sequence of juxtaposed block codes disguised as convolutional codes. Consequently, the codes constructed in the previous theorem have parameters \( (2^{m-l}, 2^{m-l} - 2k(r), 0, 0, 2^{r+1})/2 \).

VI. CONCLUSION

We developed an approach to convolutional stabilizer codes that is based on a direct limit construction, formalizing the arguments given in [14]. We proved a Singleton bound for pure convolutional stabilizer codes, and derived an optimal family of quantum convolutional codes attaining this bound from generalized Reed-Solomon codes. We illustrated how to use quasi-cyclic subcodes of Reed-Muller codes to construct a family of convolutional stabilizer codes; this method can be applied to other quasi-cyclic codes as well.

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