Obstructions to extension of vector bundles

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September 30, 2022

Abstract

In the holomorphic or algebraic setting we consider a vector bundle $E$ on a smooth subvariety $X$ in a smooth variety $Y$ over a field of characteristic zero. Assuming $E$ extends to the $l$-th formal neighborhood of $X$ in $Y$, we study cohomological obstructions to extending it further to the $k$-th neighborhood, for $k > l$.

Introduction

We consider the setup of a smooth closed algebraic subvariety $X$ of dimension $p$, in a smooth algebraic variety $Y$ of dimension $p + q$ over a field of characteristic zero $\kappa$, and a vector bundle $E$ of rank $e$ on $X$. The $k$-th neighborhood $Y^{(k)}$ of $X$ in $Y$ is the scheme with the structure sheaf $\mathcal{O}^{(k)} = \mathcal{O}_Y/I_X^{k+1}$ where $\mathcal{O}_Y$ is the structure sheaf of $Y$ and $I_X$ is the ideal sheaf of $X$ (and the underlying topological space of $Y^{(k)}$ is homeomorphic to that of $X$).

We are interested in the question of existence of a locally free sheaf $E^{(k)}$ of $\mathcal{O}^{(k)}$-modules, such that $E^{(k)}/I_X E^{(k)}$ is isomorphic to $E$ as a sheaf of $\mathcal{O}_X$-modules. We will call such an $E^{(k)}$ an order $k$ extension of $E$. In a similar $C^\infty$ situation the problem would be trivial due to existence of a tubular neighborhood $U \subset Y$ of $X$ which admits a projection $U \to X$ so we can extend $E$ even to $U$ by taking the pullback with respect to the projection.

In the algebraic situation though we only have a formal analogue of the tubular neighborhood theorem valid when $Y$ and $X$ are affine: in this case the limit $\mathcal{O}^{(\infty)} = \lim_k \mathcal{O}^{(k)}$ is non-canonically isomorphic to the completed symmetric algebra of the conormal bundle $N^*$ of $X$ in $Y$. As the isomorphism is non-canonical, it will not hold for general $(X,Y)$ and instead we have a twisted version of the statement, in which $\mathcal{O}^{(\infty)}$ is described via a structure of an $L_\infty$-algebroid on the shift $N[-1]$ of the normal bundle.

This makes our question about existence of $E^{(k)}$ non-trivial. We apply the standard machinery of formal geometry and obstruction theory to find cohomology classes that must vanish for a choice of $E^{(l)}$ to admit an extension to some $E^{(k)}$. For $k \leq 2l + 1$ this is also a sufficient condition while for $k \geq 2l + 2$ one is dealing with abelian "shadows" of non-abelian cohomology classes.

In more detail, for $k \leq 2l$ we find a cohomology class in $H^2(Y^{(l)},(\mathcal{J}^{l+1}/\mathcal{J}^{k+1}) \otimes_{\mathcal{O}(l)} E^{(l)})$ (where $\mathcal{J} \subset \mathcal{O}^{(\infty)}$ is the ideal sheaf of $X$, i.e. completion of $I_X$) that must vanish if $E^{(k)}$ exists. If the vanishing does hold then the set of isomorphism classes of such bundle extensions is a nonempty torsor over $H^1(Y^{(l)},(\mathcal{J}^{l+1}/\mathcal{J}^{k+1}) \otimes_{\mathcal{O}(l)} E^{(l)})$.
We describe three different approaches to this problem. The first is based on formal geometry, works over a field of characteristic zero. It involved infinite dimensional torsors which makes it perhaps less explicit than the other two approaches. The second approach works over $\kappa = \mathbb{C}$ and involves the usual machinery of connections and similar operators. The third approach involves affine open covers and Cech complexes, although we only make it explicit for $k = 1, 2$ (this is the situation when a relevant Cech complex has a structure of a dg Lie algebra, rather than an $L_\infty$ algebra).

Remarks.

1. Although we will not use this, but constructing an infinite order extension $E^{(\infty)}$ is equivalent to giving $E$ the structure of a module over the $L_\infty$-algebroid, i.e. an $L_\infty$-morphism $N[-1] \to \text{At}(E)$ into the Atiyah algebra sheaf of $E$, such that the composition with the canonical Lie morphism $\text{At}(E) \to T_X$ equals the anchor $L_\infty$-morphism of $N[-1]$. In fact, the operators $\mathfrak{M}_i$ in Section 4 below are adjoint to the components of such an $L_\infty$ morphism.

2. A special case of this situation, when $Y = X \times X$ and $X \to Y$ is the diagonal morphism, was studied in [Ka]. In this case $E$ admits an extension to $Y$ by pullback with respect to either of the two projections, hence bundle $E$ admits a structure of an $L_\infty$-module over $N[-1] \simeq T[-1]$, as discussed in Section 2.7 of [Ka]. Note that the anchor morphism is trivial in this case.

This paper is organized as follows. In Section 1 we consider the affine case and show that the pair $(\mathcal{O}^{(\infty)}, E^{(k)})$ is can be trivialized for any choice of $E^{(k)}$. Section 2 outlines our strategies for dealing with the general case, as long as a simple computation involving lift of Maurer-Cartan elements, on which most identities of the paper are based. In Section 3 we outline the basic constructions of formal geometry, applications to lifts of torsors and do Lie cocycle computations to make our obstruction theory statements more explicit. In Section 4 we discuss the Dolbeault approach and, in particular, Yu’s formulation for the data describing $\mathcal{O}^{(\infty)}$, and give our description for the data describing $E^{(k)}$. Finally, in Section 5, we give a Cech version of $k = 1, 2$ and briefly indicate how the approach can be extended to arbitrary $k > 0$.

Notation. We will suppress $\mathcal{O}_X$ from notation in expressions like $\otimes_{\mathcal{O}_X}$, $\text{Hom}_{\mathcal{O}_X}$, $\text{End}_{\mathcal{O}_X}$ or $\text{Sym}^s_{\mathcal{O}_X}$, while keeping the reference to the base ring in all other cases, such as $\otimes_\kappa$.

## 1 Formal completion: the affine case.

Here we discuss the case of a closed embedding of smooth affine varieties

**Lemma 1** Given two smooth affine varieties $X = \text{Spec} R/I \hookrightarrow Y = \text{Spec} R$ there exists a filtered $\kappa$-algebra isomorphism between

$$\lim_{\leftarrow} R/I^{u+1} \simeq \text{Sym}^\bullet_R N^\vee = \prod_{u \geq 0} \text{Sym}^u_R N^\vee$$

where $N^\vee := I/I^2$ is the conormal module. This isomorphism may be assumed to induce the canonical isomorphism on associated graded factors: $I^u/I^{u+1} \simeq \text{Sym}^u_R N^\vee$. 


Moreover, given a projective $R/I$-module $E$ and a projective $R/I^{k+1}$-module $E^{(k)}$ with an isomorphism of $R/I$-modules $E \cong E^{(k)}/IE^{(k)}$, there exists an isomorphism

$$E^{(k)} \cong \left( \prod_{u=0}^{k} \text{Sym}^{u}_{R}N^{v} \right) \otimes_{R} E \quad (2)$$

of filtered $\kappa$-vector spaces, which is compatible with the module actions of the filtered $K$-algebras in $\mathbb{B}$

**Proof:** The isomorphism of filtered algebras is proved, e.g., in Lemma 5.2 of [CCT]. Once the filtered algebras are identified we can view $\text{Sym}^{u}_{R}N^{v}$ as a subset of $R/I^{k+1}$. For the case of projective modules we proceed by induction on $k \geq 0$. For $k = 0$ there is nothing to prove. Assuming the isomorphism exists for a particular $k > 0$, observe that

$$I^{k+1}E^{(k+1)}/I^{k+2}E^{(k+1)} \cong I^{k+1}/I^{k+2} \otimes_{R/I^{k+2}} E^{(k+1)} \cong \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E$$

Hence we have an exact sequence of $R/I^{k+1}$ modules

$$0 \to \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E \to E^{(k+1)} \to E^{(k)} \to 0$$

and by inductive assumption the quotient is isomorphic to $\left( \prod_{u=0}^{k} \text{Sym}^{u}_{R}N^{v} \right) \otimes_{R} E$. Thus we can choose a vector space splitting

$$\Phi : \left( \prod_{u=0}^{k+1} \text{Sym}^{u}_{R}N^{v} \right) \otimes_{R} E \cong E^{(k+1)}$$

and set for $x \in R^{\leq k+1} = \prod_{u=0}^{k+1} \text{Sym}^{u}_{R}N^{v}$ and $m \in R^{\leq k} \otimes_{R} E$

$$\psi(x, m) = x\Phi(m) - \Phi(xm) \in \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E.$$

It is immediate that the Hochschild cochain condition $\psi(xy, m) - x\psi(y, m) - \psi(x, ym) = 0$ is satisfied. By Lemma 9.1.9 in [We] the Hochschild complex of $R^{\leq k+1}$ with values in the bimodule $\text{Hom}_{\kappa}(R^{\leq k} \otimes_{R} E, \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E)$ computes $\text{Ext}^{r}_{R^{\leq k+1}}(R^{\leq k} \otimes_{R} E, \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E)$. Using the short exact sequence

$$0 \to \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E \to R^{\leq k+1} \otimes_{R} E \to R^{\leq k} \otimes_{R} E \to 0$$

and the fact that the natural pullback maps give isomorphisms

$$\text{Ext}^{0}_{R^{\leq k+1}}(R^{\leq k} \otimes_{R} E, \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E) \cong \text{Ext}^{0}_{R^{\leq k+1}}(E, \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E) \cong \text{Ext}^{0}_{R^{\leq k+1}}(E, \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E) \cong \text{Ext}^{0}_{R^{\leq k+1}}(R^{\leq k+1} \otimes_{R} E, \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E)$$

we conclude that

$$\text{Ext}^{1}_{R^{\leq k+1}}(R^{\leq k} \otimes_{R} E, \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E) \cong \text{Hom}_{R^{\leq k+1}}(\text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E, \text{Sym}^{k+1}_{R}N^{v} \otimes_{R} E) \quad (3)$$

Explicitly, the isomorphism is induced by restricting a Hochschild cocycle to $\text{Sym}^{k+1}_{R}N^{v} \subset R^{\leq k+1}$ and then observing that this restriction factors through the surjection to $E$. In particular, $E^{(k+1)}$ is isomorphic to $\left( \prod_{u=0}^{k+1} \text{Sym}^{u}_{R}N^{v} \right) \otimes_{R} E$ via a filtered isomorphism if both define the same class in the left hand side of (3). But both $\psi$ and a similar element for $\left( \prod_{u=0}^{k+1} \text{Sym}^{u}_{R}N^{v} \right) \otimes_{R} E$ give the identity element on the right hand side of (3), finishing the proof. $\square$
2 General strategy

2.1 Three approaches

In the general case the isomorphism proved in Lemma 1 does not exist. This can be remediated in three different ways

1. **Formal geometry.** We can replace $X$ by a torsor $P(k) \to X$ over a prounipotent group, understood as a generalized cover of $X$. The fiber of $P(k)$ over a point $x \in X$ parameterizes isomorphisms of formal completions at the point $x$

$$\hat{\mathcal{O}}_{X,x} \simeq \mathcal{A}, \quad \hat{\mathcal{O}}_x^\infty \simeq \prod_{u=0}^{\infty} \text{Sym}^u \mathcal{A} \mathcal{S}, \quad \hat{E}_x^{(k)} \simeq \bigoplus_{u=0}^{k} \text{Sym}^u \mathcal{A} \mathcal{S} \otimes \mathcal{E}$$

with the “standard formal models” built out of a power series algebra $\mathcal{A}$ and two projective (hence free) modules $\mathcal{S}$, $\mathcal{A}$ over it. $P(k)$ has a structure of a torsor over infinite dimensional prounipotent algebraic group $G(k)$. In the torsor is non-trivial, so a global algebraic section does not exist. Instead of working on $X$ we work with $G(k)$-equivariant objects on $P(k)$, or a certain quotient space $Q(k)$ of it which is torsor over a sheaf of unipotent groups on $X$.

2. **Dolbeault model.** When $\kappa = \mathbb{C}$ we can choose a $\mathbb{C}^\infty$-section $s : X \to Q(k)$ and consider sections various bundles and sections that reflect deviation of $s$ from being holomorphic.

3. **Cech model.** We can choose an open cover $X = \bigcup U_i$ and then local sections $s_i : U_i \to Q(k)|_{U_i}$, paying attention to what happens on double and triple intersections.

2.2 Reminder on lift of Maurer-Cartan elements.

Eventually, all three approaches to extending the bundle $E$ to from an order $k$ to an order $k + 1$ will rely on a fairly easy formalism of lifting Maurer-Cartan solutions, which we explain below to fix notation. Consider an abelian extension of differential graded Lie algebras

$$0 \to \mathcal{E} \to \hat{\mathcal{L}} \to \mathcal{L} \to 0.$$ 

where $\mathcal{E}$ is a dg module over $\mathcal{L}$. We are interested in the question of lifting a (degree one) Maurer-Cartan element in $\varphi \in \mathcal{L}^1$ to $\hat{\mathcal{L}}^1$. Suppose we have a linear section $s : \mathcal{L} \to \hat{\mathcal{L}}$ which we do not assume to commute with differential or bracket. Thus we have possibly nontrivial maps of degree 1 and 0, respectively

$$\Delta_1 : \mathcal{L} \to \mathcal{E}; \quad \Delta_1(x) = (d_{\hat{\mathcal{L}}}s - sd_{\mathcal{L}})(x),$$

$$\Delta_2 : \Lambda^2 \mathcal{L} \to \mathcal{E}; \quad \Delta_2(x_1 \wedge x_2) = [s(x_1), s(x_2)]_{\hat{\mathcal{L}}} - s[x_1, x_2]_{\mathcal{L}}$$

Now suppose that $\varphi$ is lifted to a Maurer-Cartan solution $\hat{\varphi} = s(\varphi) + \alpha \in \mathcal{L}^1$ with $\alpha \in \mathcal{E}^1$. This leads to the equations

$$0 = d(s\varphi + \alpha) + \frac{1}{2}[s\varphi + \alpha, s\varphi + \alpha] =$$

$$= s(d(\varphi) + \frac{1}{2}[\varphi, \varphi]) + \Delta_1(\varphi) + d\alpha + [s\varphi, \alpha] + \frac{1}{2}\Delta_2(\varphi, \varphi)$$
In other words, using the Maurer-Cartan equation for \( \varphi \) and the fact that the commutator with \( s \varphi \) agrees with the action of \( \mathcal{L} \) on \( \mathcal{E} \) we can write

\[-(d + \varphi)\alpha = \Delta_1(\varphi) + \frac{1}{2}\Delta_2(\varphi, \varphi)\] (4)

This can be reformulated in the usual way: a choice of \( \alpha \) is possible if the right hand side of the second equation in \([4]\) gives a zero class in the cohomology of \((\mathcal{E}, d + \varphi)\).

3 The formal geometry model

3.1 Torsors and Lie cohomology classes.

We give here a short introduction to Formal Geometry, a specific version needed for our purposes. Suppose that \( G \) is a proalgebraic group which splits into a semidirect product \( G \cong U \rtimes F \) of a pronunipotent normal subgroup \( U \) and a group \( F \). Let \( P \to X \) be a torsor over \( G \) (in our case, in the Zariski topology) and \( \pi : Q = P/G \to X \). For any coherent \( \mathcal{O}_X \)-module \( M \) its pullback \( \pi^*M \) has a canonical flat connection along the fibers and the relative de Rham complex \( \pi^*M \otimes_{\mathcal{O}_X} \Omega_X^* \) is well defined. Since \( Q \) is a limit of fibrations with finite dimensional affine fibers, the canonical map

\[M \to \pi_*(\pi^*M \otimes_{\mathcal{O}_Q} \Omega_P^*)\]

is a quasi-isomorphism. A slightly more general setting is to enlarge \( g \) to a Lie algebra \( \mathfrak{h} \) in such a way that \((G, \mathfrak{h}) \) forms a Harish-Chandra pair, see Section 2 of \([BK]\). If \( P \) has an additional transitive Harish-Chandra structure, cf. \([BK]\), and the pullback of \( M \) to \( P \) is isomorphic to a trivial vector bundle \( \mathcal{O}_P \otimes \kappa M \) associated with a module \( M \) over \((G, \mathfrak{h})\) then \( M \) has a flat connection and its de Rham complex is resolved by the pushforward of the de Rham complex of \( \mathcal{O}_P \otimes \kappa M \) to \( X \). The main source of examples is when \( M \) is a jet bundle of a certain vector bundle \( L \) and its local de Rham cohomology is only nontrivial in degree 0 where it is equal to \( L \). In this case the global de Rham cohomology of \( M \) is isomorphic to coherent cohomology of \( L \).

We will consider the complex of relative Lie cochains on the algebra \( g = Lie(G) \), with respect to the Lie subalgebra \( \mathfrak{f} = Lie(F) \), with coefficients in a \( g \)-module \( M \). In degree \( j \) its component \( C^j(\mathfrak{g}, \mathfrak{f}; M) \) is formed by \( \mathfrak{f} \)-invariant cochains \( \alpha : \Lambda^j(\mathfrak{g}/\mathfrak{f}) \to M \) with the Lie cochain differential \( \delta_{Lie} \) as in Section 1.3 of \([Fu]\).

Since \( g \) can be identified with the fiber of relative tangent bundle for \( P \to X \), a Lie cochain gives a differential form on \( P \) with values in \( \mathcal{O}_P \otimes \kappa M \), and relative cochains may be identified with forms on \( Q \) with values in \( \mathcal{O}_Q \otimes \kappa M \). The following construction is described in slightly different terms in \([Fu]\).

**Lemma 2** Let \( M \) be a locally free sheaf on \( X \) and \( M \) a module over \( G \) such that the pullback of \( M \) to \( P \) is isomorphic to \( \mathcal{O}_P \otimes \kappa M \) as a \( G \)-equivariant \( \mathcal{O}_P \)-module.

Let \( \alpha \in C^j(\mathfrak{g}, \mathfrak{f}; M) \) be a relative Lie cochain. Then \( \alpha \) defines a global section of \( \pi^*M \otimes_{\mathcal{O}_P} \Omega_P^j \) and hence of \( \pi_*(\pi^*M \otimes_{\mathcal{O}_P} \Omega_P^j) \). If \( \alpha \) is closed it defines the Gelfand-Fuks cohomology class \( GF(P, \alpha) \in H^j(X, M) \) (depending on the choice of \( P \)).

In the Harish-Chandra setting, for \( \alpha \in C^j(\mathfrak{h}, \mathfrak{f}; M) \) the class \( GF(P, \alpha) \) is defined in the de Rham cohomology \( H^j_{DR}(X, M) \).
Now suppose we have a morphism of pro-algebraic groups \( \tilde{G} \to G \) with a unipotent splitting as above, and assume that it induces an isomorphism \( \tilde{F} \to F \) and a morphism of unipotent parts \( \tilde{U} \to U \). If a \( G \)-torsor \( P \to X \) lifts to a \( G \)-torsor \( \tilde{P} \to P \to X \), then the induced projection \( \tilde{\pi} : \tilde{Q} = \tilde{P}/F \to X \) factors through \( \pi \) which allows to compare the two resolutions of \( M \) on \( X \):

\[
M \to \pi_* (\pi^* M \otimes_{O_X} \Omega^*_x) \to \tilde{\pi}_* (\tilde{\pi}^* M \otimes_{i^* O_{\tilde{Q}}} \Omega^*_{\tilde{Q}})
\]

**Lemma 3** Let \( \alpha \in C^j(\mathfrak{g}, f; M) \) be a relative Lie cocycle that has an exact pullback \( \tilde{\alpha} \in C^j(\tilde{\mathfrak{g}}, \tilde{f}; \tilde{M}) \). Then the cohomology class of \( GF(P, \alpha) \) is zero. If \( \tilde{G} \) is an abelian extension that splits over \( F \subset G \) with \( M \) viewed as abelian group; and \( \alpha \in C^2(\mathfrak{g}, f; M) \) is the extension cocycle for the induced extension of Lie algebras, then the vanishing of \( GF(P, \alpha) \) is equivalent to existence of \( \tilde{P} \). Similar statements hold in Harish-Chandra setting.

**Proof.** The first statement follows from the comparison of resolutions before the lemma. The second statement is proved in Proposition 2.7 of [BK]. \( \square \)

### 3.2 Formal derivations and cocycles.

We now proceed to discuss the Lie algebras to which the above theory will be applied, delaying the description of the proalgebraic groups and their torsors until later. Let \( \mathcal{A} = \mathbb{C}[[x_1, \ldots, x_p]] \) and let \( \mathcal{S}, \mathcal{E} \) be fixed free \( \mathcal{A} \)-modules of ranks \( q, e \), respectively. Define

\[
\mathcal{S}^u = \text{Sym}_{\mathcal{A}} \mathcal{S}, \quad \mathcal{A}_S = \prod_{u \geq 0} \mathcal{S}^u, \quad \mathcal{E}^{\leq k} = \bigoplus_{u=0}^k \mathcal{S}^u \otimes_{\mathcal{A}} \mathcal{E},
\]

viewing \( \mathcal{E}^{\leq k} \) as a module over \( \mathcal{A}_S \). Consider the Lie algebra of filtration-preserving pair derivations of

\[
\text{Der}_k = \text{Der}(\mathcal{A}_S, \mathcal{E}^{\leq k}),
\]

i.e. pairs of maps \( \varphi : \mathcal{A}_S \to \mathcal{A}_S, \psi : \mathcal{E}^{\leq k} \to \mathcal{E}^{\leq k} \) such that \( \varphi \) is a derivation and

\[
\psi(a \cdot e) = \varphi(a) \cdot e + a \cdot \psi(e).
\]

We require that \( \varphi \) is continuous in the topology defined by the powers of maximal ideal \( \mathfrak{m} \subset \mathcal{A} \) generated by the variables \( x_1, \ldots, x_p \), and also the filtration by powers of the ideal in \( \mathcal{A}_S \) generated by \( \mathcal{S} \). Similar condition is imposed on \( \psi \). Then \( \text{Der}_{-1} =: \text{Der}(\mathcal{A})_S \) is the Lie algebra of derivations \( \varphi \) satisfying the above properties.

Due to the Leibniz rule, such a pair can be described by operators \( A_v : \mathcal{A} \to \mathcal{S}^v, \) \( L_v : \mathcal{S} = \mathcal{S}^1 \to \mathcal{S}^{v+1}, \) \( M_v : \mathcal{E} \to \mathcal{S}^v \otimes_{\mathcal{A}} \mathcal{E} \) for \( v \geq 0 \). Here each \( A_v \) is a derivation, which can also be described by an \( \mathcal{A} \)-linear map \( a_v : \Omega^1_{\mathcal{A}} \to \mathcal{S}^v \), and \( L_v, M_v \) are first order operators with scalar symbols given by \( a_v \) (composed with the symmetric algebra product). We can also write

\[
\text{Der}_k = \prod_{u \geq 0} \text{Der}_k^u
\]
where $\text{Der}_k^u$ is spanned by triples $(A_u, L_u, M_u)$ with fixed $u$. Defining $\text{Der}_k^+$ as the infinite product $\prod_{u \geq 1} \text{Der}_k^u$, we have a semidirect product splitting:

$$\text{Der}_k \simeq \text{Der}_k^+ \times \text{Der}_k^0.$$ 

We would like to focus on the extension of Lie algebras

$$0 \rightarrow \text{End}^{l,k} = \bigoplus_{u = l+1}^{k} \text{End}^u \rightarrow \text{Der}_k \rightarrow \text{Der}_l \rightarrow 0, \quad \text{End}^u := S^u \otimes_A \text{End}_A(E) \quad (5)$$

for $k > l$. This extension does split into a semi-direct product, but we would like to make the splitting more explicit. Choose and fix a connection $\nabla^E$ on $E$ which exists since $E$ is free over $A$. A derivation $\varphi_v : \bigoplus_{u \geq 0} S^u \rightarrow \bigoplus_{u \geq 0} S^{u+v}$ defines operators $a_v \nabla^E : E \rightarrow S^v \otimes_A E$ which we extend to

$$s\varphi_v : \bigoplus_{u = 0}^{k-v} S^u \otimes_A E \rightarrow \bigoplus_{u = 0}^{k-v} S^{u+v} \otimes_A E, \quad u = 0, \ldots, k$$

Leibniz rule and restriction of $\varphi_v : S^u \rightarrow S^{u+v}$. Summing over $v = 0, \ldots, k$ we obtain a derivation of $E^{\leq k}$ that we denote by $s\varphi$. A general element in $\text{Der}_k$ will have components $\psi_v = s\varphi_v + e_v$ where $e_0, \ldots, e_k$ are $A$-linear. When $k > l$ we use the same notation $s$ for a linear splitting $s : \text{Der}_l \rightarrow \text{Der}_k$ which takes

$$((\varphi_0, \varphi_1, \ldots), (s\varphi_0 + e_0, \ldots, s\varphi_l + e_l)) \mapsto ((\varphi_0, \varphi_1, \ldots), (s\varphi_0 + e_0, \ldots, s\varphi_l + e_l, s\varphi_{l+1}, \ldots, s\varphi_k))$$

**Lemma 4** The splitting $s : \text{Der}_l \rightarrow \text{Der}_k$ induced by $\nabla^E$ is compatible with the Lie bracket if and only if $\nabla^E$ is flat.

**Proof.** First assume that the splitting is compatible with the bracket. Restricting to the degree 0 part, we get a Lie splitting of the Atiyah Lie algebra $\text{Der}(A, E)$ of filtration preserving derivations of the pair $(A, E)$

$$0 \rightarrow \text{End}^0 \rightarrow \text{Der}(A, E) \rightarrow T_A \rightarrow 0$$

given by the connection $\nabla^E$ and its compatibility with the bracket is equivalent to the vanishing of the curvature by definition of curvature.

For the other direction, it suffices to assume $l = -1$. Let $E_0$ be the quotient of $E$ modulo the maximal ideal of $A$. As $\nabla^E$ is flat, there is a unique section of $E \rightarrow E_0$ with image in the subspace of flat sections. Moreover, this gives an isomorphism of $A$-modules

$$E \simeq E_0 \otimes_\kappa A \quad (6)$$

such that $\nabla^E$ corresponds to the usual vector field action on the second tensor factor. Then this isomorphism sends $\nabla^E(\varphi)$ to $Id_{E_0} \otimes \varphi$ and the lemma follows. $\Box$

The Lie algebra extension (5) will have abelian kernel if $k \leq 2l + 1$. In full generality, we can obtain a quotient extension that has abelian kernel. To give more detail, write $\text{End}_0^l \subset \text{End}_0^l$ for the subalgebra
of endomorphisms of trace zero (twisted by appropriate symmetric power of \(S\)). Then, for \(k > 2l + 1\), the subalgebra spanned by the commutators in \(\text{End}^{l,k}\) is

\[
[\text{End}^{l,k}, \text{End}^{l,k}] = \bigoplus_{i=2l+2}^{k} \text{End}_{0}^{i}
\]

It is easy to see that this is an ideal in \(\text{Der}_{k}\) and, denoting

\[
\text{End}_{ab}^{l,k} := \text{End}^{l,k} / [\text{End}^{l,k}, \text{End}^{l,k}] \cong \bigoplus_{i=l+1}^{2l+1} \text{End}^{i} \oplus S^{2l+2} \oplus \ldots \oplus S^{k}
\]

we obtain an abelian extension (that splits into a semidirect product, once a flat connection on \(E\) is chosen):

\[
0 \to \text{End}_{ab}^{l,k} \to \text{Der}_{k}^{l} \to \text{Der}_{l} \to 0
\]

Fix a flat connection \(\nabla^{E}\) on \(E\). The Lie cocycle \(c(l,k)\) corresponding to the abelian extension \((7)\) has homogeneous components in degree \(i \in \{l+1, \ldots, k\}\). It measures incompatibility of the section with the Lie bracket in \(\text{Der}_{k}^{l}\). For \(l+1 \leq v \leq 2l+1\) the component is

\[
c(l,k)_{v}((\varphi, \psi) \wedge (\tilde{\varphi}, \tilde{\psi})) = \sum_{p=0}^{l} \left( [s_{\varphi_{v-p}} \tilde{e}_{p}] + [s_{\tilde{\varphi}_{v-p}} e_{p}] \right) + \sum_{p=v-l}^{l} [e_{v-p}, \tilde{e}_{p}]
\]

We observe that the second sum has \(l\) terms for \(v = l + 1\) and 0 terms for \(v = 2l + 1\). For \(v \in \{2l+2, \ldots, k\}\) (if that range is non-empty) the second sum is trivial and the first is replaced by its trace. The simplest statement can be made under the flatness assumption.

**Proposition 5** If the splitting of \((7)\) is induced by a flat connection \(\nabla^{E}\), the extension cocycle \(c(k,l)\) of \((7)\) is relative with respect to the subalgebra \(\text{Der}_{0}^{l} \subset \text{Der}_{0}\) formed by all derivations \((\varphi_{0} + \varphi_{1} + \ldots, s\varphi_{0} + e_{0})\) with constant \(e_{0}\) (which is well defined by flatness, cf. \((6)\)). Its image in \(C^{2}(\text{Der}_{k}^{l}, \text{Der}_{0}^{l}; \text{End}_{ab}^{l,k})\) is equal to \(-\delta \beta\) where \(\beta : \text{Der}_{k}^{l} \to \text{End}_{ab}^{l,k}\) is the projection (viewed as 1-cochain) induced by the splitting of \((7)\). In particular, the cohomology class \([\alpha]\) is in the kernel of the map

\[
H^{2}(\text{Der}_{l}, \text{Der}_{0}^{l}; \text{End}_{ab}^{l,k}) \to H^{2}(\text{Der}_{k}^{l}, \text{Der}_{0}^{l}; \text{End}_{ab}^{l,k})
\]

For \(l+1 \leq v \leq 2l+1\) its degree \(j\) component is given by \((8)\) while for \(2l+2 \leq v \leq k\) it is given by

\[
c(k,l)_{v}((\varphi, e) \wedge (\tilde{\varphi}, \tilde{e})) = \sum_{p=0}^{l} \left( \varphi_{j-p}(\text{Tr}(\tilde{e}_{p})) + \tilde{\varphi}_{j-p}(\text{Tr}(e_{p})) \right)
\]

**Proof.** To show that the cocycle is relative observe that the second sum in \((8)\) does not involve \(e_{0}\) as \(v - l \geq 0\) the first sum does involve terms like \([s\tilde{\varphi}, e_{0}]\) but those only depend on the covariant derivative of \(e_{0}\) which vanishes by assumption. \(\Box\)
3.3 Torsors, associated to order $k$ extensions.

We continue with the notation in the introduction. For $k \geq 1$, we would like to find an order $k$ extension of $E$. Given such extension $E^{(k)}$, by (1) on any affine open set $U \subset X$ there exist isomorphisms of filtered sheaves of algebras and modules

$$O^{(\infty)} \simeq \widehat{\text{Sym}}_{O_X} N^*, \quad E^{(k)} \simeq \text{Sym}^{\leq k}_{O_X} N^* \otimes O_X E$$

where $\widehat{\text{Sym}}$ stands for the completed symmetric power, i.e. the infinite product of $\text{Sym}^i$ for all $i \geq 0$. In particular, the respective isomorphisms hold for stalks at any closed point $x \in X$. Completing at the maximal ideal $m_x$ of $x$ and denoting $E_{x,0} = E_x/m_x E_x$ we conclude that, possibly shrinking $U$ to find a flat connection on $E|_U$ we can find isomorphisms

$$\widehat{O}_x^{(\infty)} \simeq \widehat{\text{Sym}}_{O_{X,x}} \hat{N}_x^*, \quad \widehat{E}_x^{(k)} \simeq (\text{Sym}^{\leq k}_{O_{X,x}} \hat{N}_x^*) \otimes_{\kappa} E_{x,0}$$

In addition, we can also find isomorphisms between completed objects

$$\widehat{O}_{X,x} \simeq A, \quad \hat{N}_x^* \simeq S, \quad E_{x,0} \simeq E_0$$

(10)

Combining (9) and (10), we get filtered isomorphisms

$$\widehat{O}_x^{(\infty)} \simeq A_S, \quad \widehat{E}_x^{(k)} \simeq E^{\leq k}$$

(11)

Working at first informally, we define three infinite dimensional torsors

$$Q(k) \rightarrow X, \quad R \rightarrow X, \quad P(k) \rightarrow X$$

such that the fiber over $x \in X$ consists of all filtered isomorphisms as in (9), (10), (11), respectively. We assume that isomorphisms of algebras are multiplicative and unital, and that isomorphisms of modules are compatible with the action of algebras. We also assume that in all three cases the isomorphisms are compatible with filtrations at the maximal ideals, and that for (9), (11) the filtration by the ideal sheaf of $X$ maps isomorphically to the degree filtration on the symmetric algebra. It follows from the definitions that

$$P(k) = Q(k) \times_X R,$$

By definition, the three (infinite dimensional) fiber bundles are torsors over the automorphism groups of the objects on the right hand side of (9), (10), (11), respectively. In the case of $Q(k)$ we are dealing with the sheaf of algebras, which is a twisted form of functions with values in a pro-unipotent group over $k$. We denote the three groups by $U(k)$, $F$ and $G(k)$, respectively.

First we describe the pro-agebraic group $G(k)$ as the group of filtered automorphisms of the pair $(A_S, E^{\leq k})$. We assume that the automorphisms preserve the filtration by powers of the ideal $S \cdot A_S$, and that the induced automorphisms of associated graded objects are compatible with the filtration by powers of the maximal ideal $m \subset A$. The group $F$ corresponding to (10), is the product of the automorphisms of $(A, S)$ which are compatible with the filtration induced by $m$, by the general linear group $GL(E_0)$. Finally, the normal pro-unipotent subgroup $U(k) \subset G(k)$ is formed by the filtered automorphisms which induce identity on the associated graded quotient of the filtration by powers of $S \cdot A_S$. We have a semidirect decomposition $G(k) = U(k) \rtimes F$. 

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We emphasize that $Q_k$ is not a $U_k$-torsor since $F$ is not a normal subgroup. In fact, the automorphisms of the two objects on the right hand side of (9) form a Zariski sheaf of pro-unipotent groups. We will not use this approach, just viewing $Q(k)$ as $P(k)/F$. If $P(k)$ is trivialized on a Zariski open subset then $\pi_k : Q(k) \to X$ is a projective limit of morphisms with affine fibers. Therefore we can apply Lemma 3.

A more rigorous way to define it is to consider the sheaves $\text{Jets}(\mathcal{O}(\infty))$, $\text{Jets}(E^{(k)})$ of jets of sections. By definition, $\text{Jets}(\mathcal{O}(\infty))$ is the completion of $\mathcal{O}_X \otimes \kappa \mathcal{O}(\infty)$ at the ideal sheaf given by the kernel of

$$\mathcal{O}_X \otimes \kappa \mathcal{O}(\infty) \to \mathcal{O}_X \otimes \kappa \mathcal{O}_X \to \mathcal{O}_X$$

where the first arrow is the reduction of the second tensor factor modulo the ideal sheaf of $X$ in $Y$, and the second arrow is the product map. The jet bundle $\text{Jets}(E^{(k)})$ is obtained similarly by completing the product $\mathcal{O}_X \otimes \kappa E^{(k)}$ along the action of the same ideal sheaf.

Then we would like to construct a pro-algebraic scheme $P(k) \to X$ such that giving a morphism $S \to P(k)$ is equivalent to giving a morphism $\rho : S \to X$ and filtered isomorphisms

$$\rho^* \text{Jets}(\mathcal{O}(\infty)) \simeq \mathcal{O}_S \otimes \kappa \mathcal{A}; \quad \rho^* \text{Jets}(E^{(k)}) \simeq \mathcal{O}_S \otimes \kappa E^{\leq k}$$

which are compatible with multiplicative structures. For trivial $E$ and the diagonal embedding $X \to Y = X \times X$ such a torsor in constructed in Section 6 of [VdB] and that construction may be repeated almost verbatim (the only extra ingredient is Lemma 1) in our case as well. Same applies to the case of $Q(k)$ and $R$.

Remark. In fact, $P(k)$ and $R$ have a richer structure of transitive Harish-Chandra torsors in the sense explained in [BK]. This means that the actions of Lie algebras of $G(k)$ and $F$, respectively, by the vector fields on the total space of the corresponding torsor, can be extended to a larger algebras of all derivations, not necessarily preserving the subspace defined by the maximal ideal at each point. Once standard consequence of having such a structure is that every associated vector bundle of such a torsor automatically has a flat connection. We will only use this when proving that $P(k)$ allows to recover $E^{(k)}$ by taking flat sections of the associated vector bundles that happens to be isomorphic to $\text{Jets}(E^{(k)})$.

3.4 Obstructions in the Lie cohomology model.

If an order $l$ extension $E^{(l)}$ admits a further order $k$ extension $E^{(k)}$, we have lifts of torsors

$$P(k) \to P(l) \to X; \quad Q(k) \to Q(l) \to X.$$  

We want to stress that each individual component $c(l,k)_v$ is not a cocycle as action of $\text{Der}_0$ on $\text{End}_{ab}^{l,k}$ does not preserve the grading by symmetric powers of $S$. Consequently, the resulting obstruction classes on $X$ will not take values in a direct sum of $\text{Sym}^v N^v \otimes \text{End}(E)$ but rather in a twisted version that we define now. For $l + 1 \leq k \leq 2l + 2$ define

$$\mathcal{E}nd^{h,l} := (J^{l+1}/J^{k+1}) \otimes \mathcal{O}(l) \text{ End}_{\mathcal{O}(l)}(E^{(l)})$$  (12)
where \( J \subset \mathcal{O}(\infty) \) is the completion of the ideal sheaf \( I_X \subset \mathcal{O}_X \). Note that \( k - l \leq l + 1 \) implies that the tensor factor on the left is indeed a module over \( \mathcal{O}(l) \). For \( k \geq 2l + 2 \) such a definition would not work and in this case we define

\[
\mathcal{E}nd_{ab}^{k,l} := \text{Ker}(\mathcal{E}nd^{2l+1,j} \oplus \mathcal{O}(k) \rightarrow \mathcal{O}^{2l+1})
\]

where the arrow is the difference of the trace map and the canonical surjection.

**Proposition 6** For \( k > l \), the pullback of \( \mathcal{E}nd_{ab}^{k,l} \) to the total space of \( P(l) \) is isomorphic to the trivial bundle obtained from the \( \text{Der}_k \)-module \( \mathcal{E}nd_{ab}^{k,l} \). If a choice of \( E(l) \) is fixed then existence of its degree \( k \) extension \( E(k) \) implies the vanishing of the class

\[
\text{GF}(P_1, c(l, k)) \in H^2(X, \mathcal{E}nd_{ab}^{k,l});
\]

with \( c(l, k) \) as in Proposition 3. If \( l < k < 2l + 2 \) and the vanishing occurs then the set of isomorphism classes of possible extensions \( E(k) \) is nonempty and forms a torsor over \( H^1(X, \mathcal{E}nd_{ab}^{k,l}) \).

**Proof.** Indeed, after pullback to the total space of \( P(l) \) we have trivializations \( \{\text{Id}\} \) which induce an isomorphism of the pullback of \( \text{Jets}(\mathcal{E}nd_{ab}^{k,l}) \) and the \( G(l) \)-equivariant bundle \( O_P \otimes \mathcal{E}nd_{ab}^{k,l} \). Since the de Rham cohomology of \( \text{Jets}(\mathcal{E}nd_{ab}^{k,l}) \) agrees with coherent cohomology of \( \mathcal{E}nd_{ab}^{k,l} \), the statement about the class follows.

If \( E(k) \) exists then the Gelfand-Fuchs class in question vanishes by Lemma 3.

By the same Lemma 3 the vanishing of the obstruction class for \( l + 1 < k \leq 2l + 1 \) implies that the torsor \( P(l) \) may be lifted to a torsor \( P(k) \). Now we finish as in Section 2 of [BK]: the \( (G(k), \text{Der}_k) \)-module \( \mathcal{E} \leq k \) gives an associated bundle on \( X \) with a flat connection and \( E(k) \) can be recovered as a sheaf of its flat sections. \( \square \)

**Examples.**

1. For \( k = l + 1 \) the obstruction bundle \( \mathcal{E}nd_{ab}^{l,k} \simeq \text{Sym}^{l+1}N^\vee \otimes \mathcal{E}nd(E) \) does not depend on the choice of \( E(l) \) although the cohomology class in general does.

2. For \( l = 0, k = 1 \) there are no commutators on the right hand side of \( \mathcal{E} \) and it can be re-written in terms of familiar classes. The Lie version of the Atiyah class is represented by the cocycle in \( C^1(\text{Der}_1, \text{Der}_0; \Omega_{A/\kappa} \otimes_A \mathcal{E}nd_A) \) which sends \( (\varphi, \psi) \) to \( \nabla^A(e_0) \). On the other hand, a derivation \( \varphi_v : A \rightarrow S \) may be decomposed into a de Rham differential \( A \rightarrow \Omega^1_{A/\kappa} \) and an \( A \)-linear morphism \( a_v : \Omega^1_{A/\kappa} \rightarrow S \). This gives a Lie algebraic version of the Kodaira-Spencer cocycle \( KS \in C^1(\text{Der}_k, \text{Der}_0; \text{Hom}_A(\Omega^1_{A/\kappa}, S)) \). The Gelfand-Fuchs classes of these cocycles are the usual Atiyah class \( At(E) \in H^1(X, \Omega^1 \otimes \mathcal{E}nd(E)) \) and the Kodaira-Spencer class \( KS(X,Y) \in H^1(X, \text{Hom}(\Omega^1, N^\vee)) \). The obstruction class is simply their cup product, taking values in \( H^2(X, N^\vee \otimes \mathcal{E}nd(E)) \). This first order obstruction formula is well-known, cf. [HT].

3. When \( E \) has rank 1, \( \mathcal{E}nd_{ab}^{k,l} \simeq J^{l+1}/J^{l+1} \simeq I^{l+1}_X/I^{l+1}_X \).

4. If \( E \) has rank 1 and \( H^j(X, \text{Sym}^k N^\vee) = 0 \) for \( i = 1, 2 \) and \( k \geq 1 \), we can conclude that there exists a unique extension of every finite order \( k \) (another known result, due to Grothendieck, cf. Proposition 3.12 in [SGA2]). In some cases we can only guarantee the vanishing for \( k \geq l + 1 \) with fixed \( l \) and in that case each choice of \( E(l) \) admits a unique extension of any order \( k > l \).

5. When \( X \) is a proper open subvariety in a smooth projective surface, the second cohomology vanishes since \( X \) has cohomological dimension 1. Thus, extensions \( E(k) \) always exist but they may not be unique.
4 The Dolbeault model

In this section we assume that the base field \( \kappa \) is the field \( \mathbb{C} \) of complex numbers and that \( X \to Y \) is an embedding of Kahler manifolds (of course, it suffices to have a Kahler metric on some open neighborhood of \( X \) in \( Y \)). For any holomorphic bundle \( V \) on \( X \) denote by \( A_X(V) \) the Dolbeault complex \( (\Lambda^\bullet(\Omega^{0,1}_X) \otimes V, \overline{\partial}) \) of \( V \). We can also define the Dolbeault algebra \( A_X(O_X(\infty)) \) and a module \( A_X(E(k)) \) over it using the completion procedure explained in Section 4 of [Yu].

This section is based on a general idea of [Ka] and [CCT]: that the sheaf of algebras \( O_X(\infty) \) may be described by replacing \( \prod_{u \geq 0} \text{Sym}^u N^\lor \) by a quasi-isomorphic sheaf of dg algebras, then deforming the differential, then passing to cohomology. One can develop a Cech version of this procedure using Thom-Whitney normalization instead of Cech complex, as done in [CCT], but the globalized Dolbeault version is a more explicit. In the case of the diagonal embedding \( X \to X \times X \) it is due to Kapranov.

4.1 Yu’s Dolbeault model of the infinitesimal neighborhood.

Proposition 7 The choice of the Kaehler metric induces an isomorphism of filtered algebras

\[ A_X(O^{(\infty)}) \simeq A_X(\widehat{\text{Sym}}^N) \]

which does not agree with the differentials. The Dolbeault differential \( \overline{\partial} \) on the left hand side, when transferred to the right hand side via the above isomorphism, takes the form \( \overline{\partial} + \mathcal{D} \) were \( \mathcal{D} = D_1 + D_2 + \ldots \) is an algebra derivation, with derivations

\[ \mathcal{D}_v : A_X(\widehat{\text{Sym}}^N) \to A_X(\widehat{\text{Sym}}^{N+1}) \]

obtained by the Leibniz rule extension from first order operators

\[ A_v : A_X(O_X) \to A_X^{+1}(\text{Sym}^N), \quad L_v : A_X(N^\lor) \to A_X^{+1}(\text{Sym}^{N+1}) \]

The explicit formulas for \( A_v, L_v \) in terms of the metric connection of a Kahler metric in the neighborhood of \( X \subset Y \), may be found in Theorem 5 of [Yu]. In fact, a choice of Kahler metric may be replaced by a choice of weaker structure. We only note here that \( A_i \) is a composition of a holomorphic de Rham differential \( \partial \) and a product with a smooth section \( a_v \in A_X(\Omega_X^{1,v}) \), and that the first order operator \( \mathcal{L}_v \) has symbol given by (the symmetrization of) \( a_v \), as follows immediately from the Leibniz rule.

Remarks.

1. The operators \( A_v, L_v \) are adjoint to the anchor maps and higher brackets, respectively, of the \( L_\infty \) algebroid structure on the homological shift \( N[-1] \) (realized by shifting its Dolbeault complex), as studied in [Yu]. We will not use this language although everything in our paper may be restated in such terms.

2. In an earlier work, cf. [Ka], Kapranov considered an important special case of the diagonal embedding \( X \to X \times X \). In this case the situation simplifies greatly: the operators \( L_i \) are obtained by symmetrizing the higher covariant derivatives of the curvature tensor. The ”adjoint anchor maps” \( A_i \) are all zero (this is obvious for \( A_1 \), for example).

Note that in this case the question of extending a vector bundle \( E \) on \( X \) to the formal neighborhood of the diagonal is trivial: the pullback with respect to either projection \( X \times X \to X \) is actually the extension to the whole of \( X \times X \). In particular, all obstruction classes vanish.
4.2 Dolbeault model for extended bundles.

Now assume we have fixed an order $k$ extension $E^{(k)}$ and an isomorphism $A^\bullet_X(\mathcal{O}^{(\infty)}) \simeq A^\bullet_X(\text{Sym}^k N^\vee)$ as in Proposition 7.

**Proposition 8** There exists an isomorphism compatible with the filtered module structures

$$(A^\bullet_X(E^{(k)}), \nabla) \simeq (A^\bullet_X(\text{Sym}^{\leq k} N^\vee \otimes E), \nabla + \mathcal{D})$$

where $\nabla + \mathcal{D}$ is a module derivation compatible with the algebra derivation $\nabla + \mathcal{D}$ from Proposition 7 and $\mathcal{D}E = \mathcal{D}_1E + \mathcal{D}_2E + \ldots + \mathcal{D}_kE$ where $\mathcal{D}_v : A^\bullet_X(\bigoplus_{u=0}^{k-v} \text{Sym}^u N^\vee \otimes E) \to A^{\bullet+1}_X(\bigoplus_{u=0}^{k-v} \text{Sym}^{u+v} N^\vee \otimes E)$ is obtained by Leibniz rule extension from a first order operator

$$\mathfrak{M}_v : A^\bullet_X(E) \to A^{\bullet+1}_X(\text{Sym}^v N^\vee \otimes E)$$

with the scalar symbol $a_i$. If an order $k$ extension $E^{(k)}$ restricts to an order $l$ extension $E^{(l)}$ then any choice of such isomorphism for $E^{(l)}$ may be extended to $E^{(k)}$ without changing the operators $\mathfrak{M}_1, \ldots, \mathfrak{M}_l$.

**Proof.** As $\mathcal{O}^{(k)}$ is identified (by a $C^\infty$ isomorphism) with $\text{Sym}^{\leq k} N^\vee$, this allows to view $E^{(k)}$ as a locally free sheaf of finite rank over $\mathcal{O}_X$. We can choose a $C^\infty$-splitting

$$E^{(k)} \simeq E \oplus I_X E^{(k)}$$

(for example by taking an orthogonal complement to the second factor, with respect to any Hermitian metric). Thus $\text{Sym}^v N^\vee \cdot E$ is well-defined as a sub-bundle of $E^{(k)}$ for $v = 1, \ldots, k$, and we obtain a required direct sum splitting. In particular, we can transfer the differential $\nabla$ from $A^\bullet_X(E^{(k)})$ to $A^\bullet_X(\text{Sym}^{\leq k} N^\vee \otimes E)$ and it will be automatically a module derivation compatible from the differential transferred from $A^\bullet_X(\mathcal{O}^{(k)})$.

By the derivation property, it is uniquely determined by its restriction to $E$ where it splits into components $\mathfrak{M}_v : A^\bullet_X(E) \to A^{\bullet+1}_X(\text{Sym}^v N^\vee \otimes E)$ for $v = 0, \ldots, k$. Since the isomorphism of bundles restricts to identity on the associated graded of the filtration, we see that $E_0 = \overline{\nabla}$. The symbol property is an immediate consequence of the derivation property, and the extension assertion is a direct consequence of the recursive nature of the above proof. $\square$

4.3 Construction of extensions and obstructions.

Conversely, assume that we have the operators $\mathfrak{M}_v, v = 1, \ldots, k$ such that the corresponding operator $\overline{\nabla} + \mathcal{D}$ squares to zero and is multiplicatively compatible with $\overline{\nabla} + \mathcal{D}$. Then the kernel of this operator is a module over $\text{Ker}(\overline{\nabla} + \mathcal{D}) = \mathcal{O}^{(\infty)}$ due to the multiplicative agreement of the two differentials. This kernel also have a filtration by submodules with associated graded $\text{Sym}^{\leq k} N^\vee \otimes E$. It follows that $E^{(k)} := \text{Ker}(\overline{\nabla} + \mathcal{D})$ is projective over $\mathcal{O}^{(k)}$ and thus gives an order $k$ extension.

To construct the operators $E_i$ we fix an Hermitian metric $h$ on $E$ and consider the unique Chern connection $\nabla^E = \nabla + \overline{\nabla}$ compatible with $h$, where $\nabla$ is the $(1,0)$ part of the connection. Its curvature $R_\nabla = \overline{\nabla} \nabla + \nabla \overline{\nabla}$ is a $(1,1)$-form twisted by endomorphisms of $E$. We now set

$$\mathfrak{M}_v = a_v \nabla + m_v$$

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Lemma 9 The following lemma is established by direct computation beault differential is deformed by the differential
As in the case of Lie cohomology model, it admits a section where the missing operators get filled in by the operators (sϕ)operator. This section is not compatible with the differential or the bracket. This means that
get filled in by the operators (sϕ)
Now we assume that k ≤ 2l + 1. Then there is an abelian extension
As in the case of Lie cohomology model, it admits a section where the missing operators get filled in by the operators (sϕ)operator. This section is not compatible with the differential or the bracket. This means that
This obstruction class is a degree two element of

Proposition 10 The element ψk satisfies the Maurer-Cartan solution if and only if

\[ -\overline{\partial}m_v - \sum_{p=l+1}^{v-1} [s\Delta_{v-p}, m_p] = a_v R\nabla + \frac{1}{2} \sum_{p=v-l}^{l} [m_{v-p}, m_p] + \sum_{p=v-l}^{l} [s\Delta_{v-p}, m_p] \]
for v = l + 1, ..., k, where \( R\nabla = \overline{\partial}\nabla + \nabla\overline{\partial} \in A_X^1(\Omega^1_X \otimes \text{End}(E)) \) is the curvature of the connection \( \nabla + \overline{\partial} \). In other words, the right hand side should define a trivial cohomology class in \( H^2(Y^{(l)}, \text{End}^{d,k}) \).
Proof. We apply last equation in Section 2.2. The claimed formula follows once we prove that
\[ \Delta_2(sD_p, sD_q) = 0, \quad \Delta_1(sD_v) = -a_v R_{\nabla} \]
The first assertion is derived from the identity \( \nabla^2 = 0 \) that holds for our metric connection. This is established by a tedious local computation in which we first reduce to the case when \( N^\nabla \) is trivial, then to the case when \( \mathcal{L}_1 = a_1 \Delta \) and finally to the case \( N^\nabla = \mathcal{O}_X \). The expression \( \Delta_1(sD_v) \) is just \( [\nabla, sD_v] \) and the Leibniz rule allows to reduce to the computation of the restriction to \( A^*_X(E) \) where the assertion follows from \( [\nabla, a_v] = 0 \) (which is a consequence of \( (\nabla + \Delta)^2 = 0 \)). \( \square \)

**Proposition 11** For \( l + 1 \leq k \leq 2l + 1 \) an order \( l \) extension \( E^{(l)} \) with a fixed Dolbeault model as in Proposition \( \Box \) can be further extended to \( E^{(k)} \) if and only if
\[ a_i R_{\nabla} + \frac{1}{2} \sum_{p=i-l}^{l} [m_{i-p}, m_p] + \sum_{p=i-l}^{l} [sD_{i-p}, m_p] \]
defines a zero class in \( H^2(Y^{(l)}, \mathcal{E}nd_{d^k,l}^k) \). If that is the case, the set of isomorphism classes of all extensions \( E^{(k)} \) is a torsor over \( H^1(Y^{(l)}, \mathcal{E}nd_{d^k,l}^k) \)

**Proof.** By Proposition \( \Box \) the Dolbeault model for \( E^{(l)} \) can be extended to a Dolbeault model for \( E^{(k)} \). By the previous proposition, existence of \( m_{l+1}, \ldots, m_k \) is equivalent to the vanishing of the cohomology class, as claimed. A different Dolbeault model for the same \( E^{(k)} \) will adjust these operators by an operator in the image of the differential, while in general a choice of \( m_v \) may be adjusted by an element in the kernel of the differential. Hence isomorphism classes of order \( k \) extensions are in bijective correspondence with the first cohomology group, as claimed. \( \square \)

For \( k \geq 2k + 2 \) we proceed as before, establishing a necessary condition. In this case, the obstruction is degree 2 element in
\[ A^2_X(\bigoplus_{u=l+1}^{2l+1} \text{Sym}^u N^\nabla \otimes \text{End}(E)) \oplus A^2_X(\bigoplus_{v=2l+2}^{k} \text{Sym}^u N^\nabla) \]
and the complex involved is isomorphic to the Dolbeault complex of \( \text{End}_{dab}^{d^k,l} \). The same argument as for \( k \leq 2k + 1 \) gives

**Proposition 12** If for \( k > 2l + 1 \) an order \( l \) extension \( E^{(l)} \) can be further extended to \( E^{(k)} \) then
\[ \bigoplus_{v=l+1}^{k} (a_v R_{\nabla} + \frac{1}{2} \sum_{p=v-l}^{l} [m_{i-p}, m_p] + \sum_{p=i-l}^{l} [sD_{v-p}, m_p]) \oplus \bigoplus_{v=2l+2}^{k} (a_v Tr(R_{\nabla}) + \sum_{p=1}^{l} D_{v-p} Tr(m_p)) \]
defines a zero class in \( H^2(Y^{(l)}, \mathcal{E}nd_{d^k,l}^{d^k,l}) \).

Remarks.

1. Note that \( Tr(R_{\nabla}) \in A^1_X(\Omega^1_X) \) represents the first Chern class of \( E \)
2. We could ask whether the vanishing of the class in the previous proposition ensures existence of some geometric object. It certainly gives an order \((2l+1)\) extension of \( E \), since we can simply truncate the statement. But if we look closer, we see that it also gives an order \( k \) extension of the determinant line bundle \( \Lambda^c E \). This is perhaps the best we can expect from looking at the usual - abelian - cohomology groups.
5 The Cech model

In this section we consider the Cech model of the situation. We choose an open covering \( \{U_i\}_{i \in I} \) such that \( \mathcal{O}^{(\infty)} \) can be identified with the completed symmetric algebra of the conormal bundle on each \( U_i \) (by Lemma 1 this works for any affine covering) and on which \( E \) admits a flat connection (e.g. because \( E \) is trivial on each \( U_i \), which holds for \( U_i \) small enough).

Unlike in the previous two models, we only consider extensions of first and second order here. There reason is that, for a sheaf of Lie algebras \( \mathcal{L} \) on \( X \), the Cech complex of the on the covering with coefficients in \( \mathcal{L} \) does not in general have the structure of a dg Lie algebra, only a homotopy Lie algebra, also called \( L_\infty \)-structure. An alternative would be to replace the Cech complex by Thom-Sullivan normalization but that makes the whole statement less transparent and computable.

For extensions of second order we can reduce to the case of a homotopy Lie algebra where all higher \( n \)-ary operations vanish for \( n \geq 3 \). In fact, for first order extensions even the Lie bracket is equal to zero.

Although the general case is not that difficult, we leave it to a motivated reader, observing here that a general group cocycle condition below is equivalent to a Maurer-Cartan condition if we use the Baker-Campbell-Hausdorff formula and then observe that its degree \( n \) term will match precisely the degree \( n \) term of the Maurer-Cartan equation. This nontrivial result is implicit in [Ge], and is proved explicitly in [FMM]. Note that both the \( L_\infty \)-structure and the BCH formula are defined up to certain ambiguity (in the case of BCH formula this is a consequence of the Jacobi identity), but the formalism developed in [Ge] implies that a choice of Dupont homotopy allows to fix both the \( L_\infty \)-structure and the CBH formula in a coherent way, which ensures agreement of the Maurer-Cartan equation and the cocycle condition.

5.1 Derivations on double intersections.

Fix an affine open covering \( X = \bigcup U_i \) and filtered isomorphisms

\[
\Phi_i : \hat{\mathcal{O}}^{(\infty)}|_{U_i} \to \hat{\text{Sym}}^\bullet N^\vee|_{U_i}
\]

(14)

which restrict to identity on associated graded quotients. These exist by Lemma 1. On a double intersection \( U_{ij} = U_i \cap U_j \) the disagreement of two isomorphisms \( \Phi_{ij} = \Phi_j \Phi_i^{-1} \) is a filtered algebra automorphism of \( \hat{\text{Sym}}_{\mathcal{O}_X} N^\vee|_{U_i} \) which restricts to identity on associated graded quotients. Therefore its logarithm is well defined on our completed objects and we can write

\[
\Phi_{ij} = \exp(\varphi_{ij})
\]

Lemma 13 The map

\[
\varphi_{ij} : \hat{\text{Sym}}_{\mathcal{O}_X} N^\vee|_{U_{ij}} \to \hat{\text{Sym}}_{\mathcal{O}_X} N^\vee|_{U_{ij}}
\]

is a filtered \( \kappa \)-algebra derivation which vanishes on associated graded quotients. In particular, it is uniquely determined by its restrictions

\[
A_{ij} : \mathcal{O}_X|_{U_{ij}} \to \hat{\text{Sym}}_{\mathcal{O}_X}^{\geq 1} N^\vee|_{U_{ij}}, \quad L_{ij} : N^\vee|_{U_{ij}} \to \hat{\text{Sym}}_{\mathcal{O}_X}^{\geq 2} N^\vee|_{U_{ij}}, \square
\]
We split the above two operators into homogeneous components:

\[ A^s_{ij} : \mathcal{O}_X|_{U_{ij}} \to \text{Sym}^s_{\mathcal{O}_X}N^\vee|_{U_{ij}}, \quad L^{s-1}_{ij} : N^\vee|_{U_{ij}} \to \text{Sym}^s_{\mathcal{O}_X}N^\vee|_{U_{ij}}. \]

The following lemma is proved by a straightforward computation

**Lemma 14** Each \( A^s_{ij} \) is a derivation and can be represented as a composition of \( d : \mathcal{O}_X|_{U_{ij}} \to \Omega^1_X|_{U_{ij}} \) and an \( \mathcal{O}_X \)-linear operator \( a^s_{ij} : \Omega^1_X|_{U_{ij}} \to \text{Sym}^s_{\mathcal{O}_X}N^\vee|_{U_{ij}} \). Each \( L^{s-1}_{ij} \) is a first order differential operator with the symbol \( 1_N \cdot a^{s-1}_{ij} \).

Similar arguments apply to extensions \( E^{(k)} \). As before, we can choose isomorphisms

\[ \tilde{\Psi}_i : E^{(k)}|_{U_i} \to \text{Sym}^{\leq k}N^\vee \otimes E|_{U_i} \quad (15) \]

which restrict to identity on associated graded quotients, then form automorphisms \( \Psi_{ij} = \tilde{\Psi}_j \tilde{\Psi}_{i}^{-1} \) of the term on the right, restricted to \( U_{ij} \). As in the case of functions, we have

\[ \Psi_{ij} = \exp(\psi_{ij}) \]

Again, the following lemma is a direct consequence of definitions.

**Lemma 15** The map

\[ \psi_{ij} : \text{Sym}^{\leq k}N^\vee \otimes E|_{U_{ij}} \to \text{Sym}^{\leq k}N^\vee \otimes E|_{U_{ij}} \]

agrees with the natural filtration on the truncated symmetric algebra and induces the zero map on the associated graded quotient. For sections \( f \), resp. \( e \), of \( \text{Sym}^{\leq k}N^\vee \), resp. \( \text{Sym}^{\leq k}N^\vee \otimes E \) on \( U_{ij} \), one has the following module derivation property:

\[ \psi_{ij}(f \cdot e) = \varphi_{ij}(f) \cdot e + f \cdot \psi_{ij}(e) \]

In particular, \( \psi_{ij} \) is uniquely determined by the restriction

\[ M_{ij} : E \to \bigoplus_{u=1}^{k} \text{Sym}^uN^\vee \otimes E \]

Each component \( M^u_{ij} : E \to \text{Sym}^uN^\vee \otimes E \) is a first order operator with the symbol \( 1_E \otimes a^u_{ij} \). For a connection \( \nabla_i \) on \( E|_{U_i} \) one can write

\[ M^v_{ij} = a^v_{ij} \nabla_j + m^v_{ij} \]

where \( m^v_{ij} : E \to \text{Sym}^vN^\vee \otimes E \) is an \( \mathcal{O}_X \)-linear operator. \( \Box \)

As in the Dolbeault case, the operator \( a^v_{ij} \nabla_j \) admits a Leibniz rule extension to a derivation of \( \text{Sym}^{\leq k}N^\vee \otimes E \), increasing the symmetric power degree by \( v \). We denote this operator by \( s\varphi^v_{ij} \).
5.2 The group cocycle condition and the Maurer-Cartan equation.

In addition to the multiplicative properties of our operators on each $U_{ij}$ we have the cocycle condition on triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$. We fix a total ordering on the index set of the covering and assume that $i < j < k$. The group valued cocycle conditions are, as usual,

$$\exp(\varphi_{ih}) = \exp(\varphi_{ij})\exp(\varphi_{jh}), \quad \exp(\psi_{ih}) = \exp(\psi_{ij})\exp(\psi_{jh})$$

where $(\varphi_{ij}, \psi_{ij})$ is considered a section over $U_{ij}$ of the sheaf of Lie algebras $\text{Der}^+_k$ formed by pair derivations $(\varphi, \psi)$ that satisfy the multiplicative conditions on Lemmas 13 and 15, agree with filtrations and induce a zero map on associated graded quotients.

If $\mathfrak{g}$ is the pro-nilpotent Lie algebra of a pro-unipotent algebraic group $G$, then for $x, y, z \in \mathfrak{g}$ then the group valued equation $\exp(z) = \exp(x)\exp(y)$ can be rewritten as

$$z = \log(\exp(x)\exp(y)) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] - [y, [x, y]]) - \frac{1}{24}[y, [x, [x, y]]] + \ldots$$

where the right hand side is given by the Baker-Campbell-Hausdorff formula, cf. [Se].

Note that for $\text{Der}^+_k$ we also have the grading $\text{Der}^+_k = \prod_{t \geq 1} \text{Der}^t_k$ where $\text{Der}^t_k$ is formed by the elements $(\varphi^t, \psi^t)$ the increase the symmetric power degree in $\text{Sym}^N \otimes F$ by $t$. As mentioned above, we are interested in the case when $t = 1, 2$. Since the Lie brackets are compatible with this grading, in degrees 1 and 2 the above cocycle conditions gives

$$\varphi^1_{ih} = \varphi^1_{ij} + \varphi^1_{jh}, \quad \varphi^1_{jh} = \psi^1_{ij} + \psi^1_{jh}$$

$$\varphi^2_{ih} = \varphi^2_{ij} + \varphi^2_{jh} + \frac{1}{2}[\varphi^1_{ij}, \varphi^1_{jh}], \quad \psi^2_{ih} = \psi^2_{ij} + \psi^2_{jh} + \frac{1}{2}[\psi^1_{ij}, \psi^1_{jh}]$$

We would like to rephrase these equations as cohomology conditions on $\mathcal{O}_X$-linear operators $a^1_{ij}, a^2_{ij}, m^1_{ij}, m^2_{ij}$.

**Proposition 16** A first order extension $E^{(1)}$ exists if and only if there exists a 1-cochain $m^1$ with values in $N^\vee \otimes \text{End}(E)$ such that

$$-\delta m^1 = a^1_{ij} At(E)_{jh}$$

where $At(E)_{jh} = (\nabla_j - \nabla_k)$ is the Atiyah cocycle of $E$. Two choices of a resolving cochain $m^1$ give isomorphic first order extensions if and only if their difference is exact.

If a choice of $m^1$ is fixed and the local connections $\nabla_1$ are flat, an extension of $E^{(1)}$ to $E^{(2)}$ exists if and only if there is a 1-cochain $m^2$ with values in $\text{Sym}^2 N^\vee \otimes E$ such that

$$-\delta m^2 = a^2_{ij} At(E)_{jh} + \frac{1}{2}[m^1_{ij}, m^1_{jk}] + \frac{1}{2}[s\varphi^1_{ij}, m^1_{ij}] + \frac{1}{2}[s\varphi^1_{ij}, m^1_{jk}]$$

In other words the right hand side defines the zero class in $H^2(X, \text{Sym}^2 N^\vee \otimes E)$. Two choices of a resolving cochain $m^2$ give isomorphic second order extensions if and only if their difference is exact.

Finally, if any choice of $E^{(2)}$ exists (i.e. without fixing $E^{(1)}$) then $(a^1_{ij} At(E)_{jh}, a^2_{ij} \text{Tr}(At(E)_{jh}))$ defines a zero cohomology class in $H^2(Y^2, \text{End}^{0,2})$.
We recall that the obstruction bundle $\mathcal{E}nd_{ab}^{0,2}$ is a non-trivial extension

$$0 \to \text{Sym}^2 N^\vee \to \mathcal{E}nd_{ab}^{0,2} \to N^\vee \otimes \text{End}(E) \to 0$$

of sheaves of $\mathcal{O}^{(2)}$-modules (in fact, even $\mathcal{O}^{(1)}$-modules). We also note that $Tr(At(E))_{ij}$ represents the first Chern class of $E$ and, while $a^2$ is not a cocycle, the product $a^2_{ij} Tr(At(E))_{jh}$ can be seen to be a cocycle once we know that the class of $a^1_{ij} At(E)_{jh}$ is trivial.

**Proof.** By Leibniz rule the cocycle condition on $\psi^1$ reduces to the case of $M^1_{ij}$ (once we know it for $\phi^1$). Hence

$$-\delta m^2 - \frac{1}{2}[s\varphi^1_{jh}, m^1_{ij}] - \frac{1}{2}[s\varphi^1_{ij}, m^1_{jh}] = a^2_{ij} At(E)_{jh}$$

since $\delta a^1 = 0$. Observe that, as in the Dolbeault setting, $a^1_{ij}$ is a Cech cocycle of the Kodaira-Spencer class (the extension class of the conormal short exact sequence) and $At(E)_{jh}$ represents the Atiyah class of the bundle $E$. This defines $m^1$ up to a cohomology class in $H^1(X, N^\vee \otimes \text{End}(E))$.

For the second order extension, we use Section 2.2. Then $\Delta_1 (s\varphi^2)$ gives $a^2_{ij} At(E)_{jh}$. For a flat connection the terms $\Delta_2 (s\varphi^1_{ij})$ vanish and the remaining terms give the second formula claimed in the proposition. Finally, for $l = 0$ and $k = 2$ we rearrange the terms of the second order equation.

$$-\delta m^2 - \frac{1}{2}[s\varphi^1_{jh}, m^1_{ij}] - \frac{1}{2}[s\varphi^1_{ij}, m^1_{jh}] = a^2_{ij} At(E)_{jh} + \frac{1}{2}[m^1_{ij}, m^1_{jk}]$$

then apply traces, keeping in mind that $Tr([s\varphi^1, m^1]) = \varphi^1(Tr(m^1))$ and that the trace of the commutator is zero. Finally, we observe that this procedure will turn the left hand side into a complete differential of the second order element in the Cech complex for $\mathcal{E}nd_{ab}^{0,2}$. □

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