SMALLEST AND LARGEST GENERALIZED EIGENVALUES OF LARGE MOMENT MATRICES AND SOME APPLICATIONS.

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Abstract. The main aim of this work is to compare two Borel measures through their moment matrices using a new notion of smallest and largest generalized eigenvalues. With this approach we provide information in problems as the localization of the support of a measure. In particular, we prove that if a measure is comparable in an algebraic way with a measure in a Jordan curve then the curve is contained in its support. We obtain a description of the convex envelope of the support of a measure via certain Rayleigh quotients of certain infinite matrices. Finally some applications concerning polynomial approximation in means squares are given, generalizing the results in [9].

Keywords. Hermitian moment problem, orthogonal polynomials, smallest eigenvalue, measures, approximation by polynomials

1. Introduction

Let \( M = (c_{i,j})_{i,j=0}^{\infty} \) be an infinite Hermitian matrix, i.e., \( c_{i,j} = \overline{c_{j,i}} \) for all \( i, j \) non-negative integers. Following [9] we say that an infinite hermitian matrix \( M \) is positive definite, in short an HPD matrix (resp. positive semidefinite, in short HSPD matrix) if \( |M_n| > 0 \) (resp. \( |M_n| \geq 0 \) for all \( n \geq 0 \)), where \( M_n \) is the truncated matrix of size \( (n+1) \times (n+1) \) of \( M \). Following the notation in [9] and [11] an infinite HPD matrix defines an inner product, denoted by \( \langle \cdot, \cdot \rangle_M \), in the space \( \mathbb{C}_0 \) of all polynomials with complex coefficients in the following way: if \( p(z) = \sum_{k=0}^{n} v_k z^k \) and \( q(z) = \sum_{k=0}^{m} w_k z^k \) then

\[
\langle p(z), q(z) \rangle_M = v^T M w^*
\]

being \( v = (v_0, \ldots, v_n, 0, 0, \ldots) \), \( w = (w_0, \ldots, w_m, 0, 0, \ldots) \) \( \in \mathbb{C}_0 \) where \( \mathbb{C}_0 \) is the space of all complex sequences with only finitely many non-zero entries and the symbol \( * \) denotes the conjugate transpose for matrices of any size. The associated norm is denoted by \( \|p(z)\|_M^2 = \langle p(z), p(z) \rangle_M \) for every \( p(z) \in \mathbb{P}[z] \), and the vector space \( \mathbb{P}[z] \) is a pre-hilbertian space endowed with such norm and its completion will be denoted by \( P^2_M \). In this way, \( P^2_M \) is a Hilbert space associated to the inner product \( \langle \cdot, \cdot \rangle_M \).

Of course, the most interesting class of infinite HPD (and also HSPD) matrices are those which are moment matrices with respect to a certain positive measure (see e.g. [25]), i.e., matrices \( M = (c_{i,j})_{i,j=0}^{\infty} \) such that there exists a positive Borel measure \( \mu \)

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with support in the complex plane, which is called a representing measure of $\mathbf{M}$, with finite moments for all $i, j \geq 0$

$$c_{i,j} = \int z^i \overline{z}^j d\mu. \tag{2}$$

Thorough the paper we always consider positive Borel measures $\mu$ supported in the complex plane and with finite moments. The associated moment matrix to a measure $\mu$, denoted by $\mathbf{M}(\mu)$, is always an infinite HSPD, being an HPD matrix if and only if the support of $\mu$ is an infinite set. For more information concerning the characterization of infinite HPD matrices which are moment matrices with respect to a certain measure $\mu$ with support in $\mathbb{C}$ see among others [1], [6] and [19].

Our aim here is to study certain geometrical and localization properties of the support of measures thorough their associated moment matrices. This will be done in the context of matrix algebra using as essential tool the infinite matrices and the generalizations of some well known notions as generalized eigenvalues. The approach is the same as in [21], [3], [9] and [11] and the essential key is the following identity: if $p(z) = v_0 + v_1 z + \cdots + v_n z^n \in \mathbb{P}[z]$ and $(v_0, v_1, \ldots, v_n, 0, 0, 0, \ldots) \in c_{00}$,

$$v\mathbf{M}(\mu)v^* = \int |p(z)|^2 d\mu. \tag{3}$$

In the case of Toeplitz definite positive matrices, which are moment matrices associated to measures with support in the unit circle, the behaviour of various spectral characteristics, in particular eigenvalues, eigenvectors, and extreme eigenvalues, has been an object of active investigation (see e.g. the classical Szego’s paper [21] or [16]). In the case of Hankel positive moment matrices in [5] a characterization of the uniqueness of the representing measure is given in terms of the asymptotic behaviour of the smallest eigenvalues of the truncated matrices of the moment matrix. The analysis of the largest and smallest eigenvalues plays an important role since they provide useful information about the nature of Hankel matrices generated for weight functions and is a topic of great interest and activity (see e.g. [3], [4], and [24] among others). We here are concerned with general moment matrices associated with measures supported in the complex plane. In this direction in [9] a sufficient condition is given to assure polynomial approximation in the corresponding space of measure $L^2(\mu)$ (for compactly supported measures) also in terms of the asymptotic behaviour of the smallest eigenvalues. With the same approach in [11], for measures supported in Jordan curves, a characterization of polynomial approximation is given.

The main novelty in this work is the analysis of generalized eigenvalues of large hermitian matrices as a new tool in order to compare two different measures or even sets (suggested in [21]). Our motivation was to realize that the analysis of eigenvalues can be seen as a way to compare any infinite hermitian matrix with the identity matrix $\mathbf{I}$, which is indeed the moment matrix associated with the uniform Lebesgue measure in the unit circle. Such comparison does not provide, in general, any information concerning density polynomial when we consider measures supported in other disks. Nevertheless, we can make such comparison between any two measures
via *generalized eigenvalues* as we will do; with this approach we may describe the support of one measure in relation with the support of another.

In the first section, given two infinite HPD matrices we introduce two indexes that describe the asymptotic behaviour of the smallest and largest generalized eigenvalues of large hermitian matrices. In the particular case of moment matrices these indexes provide an algebraic tool to compare measures. Some properties of these indexes will be given.

In the second section we see the impact of the largest generalized eigenvalue of a measure with respect to other in the support of the first in relation with the support of the second. The main result of this section asserts that if such largest generalized eigenvalue is finite then the support of the first measure is contained in the polynomially convex hull of the support of the second. In particular, for measures with polynomially convex support, that means without *holes* in their supports, that are *well comparable* between them (with positive smallest generalized eigenvalue and finite largest generalized eigenvalue) the supports coincide. In the particular case of a good comparison of a certain measure with a measure supported in a Jordan curve we obtain that the Jordan curve must be contained in the support of such measure; this lets to find Jordan curves in the supports of a measure using this algebraic approach.

The third section is devoted to the description of the convex envelope of the support of a measure. Our main result is that the convex hull of the support of a measure can be obtained using Rayleigh quotients of a certain infinite matrix (mainly, the moment matrix without the first row) with the moment matrix. In order to prove this result we use theory of normal operators in Hilbert spaces and results concerning Hessemberg matrices in the theory of orthogonal polynomials.

In the last section we generalize the results of polynomial approximation in mean square in [9]. This generalization is obtained when we replace the uniform Lebesgue measures for other uniform Lebesgue measures in circles, which are, in particular, non complete. We do not know if we can replace these measures by any measure for which there is no polynomial approximation in mean square. In this direction, we give a partial answer when the first measure is supported in a Jordan curve.

We begin with some notation and definitions:

Thorough all the paper we denote by \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) and \( \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \} \). For general disks we denote \( \mathbb{D}_r(z_0) = \{ z \in \mathbb{C} : |z - z_0| < r \} \) and \( \mathbb{S}_r(z_0) = \{ z \in \mathbb{C} : |z - z_0| = r \} \). We denote by \( \mathbf{m} \) the uniform Lebesgue measure in the unit sphere; for general spheres we denote by \( \mathbf{m}_{z_0,r} \) the uniform Lebesgue measure in the sphere \( \mathbb{S}_r(z_0) \) and by simplicity in the particular case of spheres with center 0 we denote \( \mathbf{m}_r = \mathbf{m}_{0,r} \).
2. Generalized eigenvalues associated to infinite SHPD matrices.

Our aim in this section is to extend the notion of generalized eigenvalues in the context of infinite dimensional matrices. Recall that (see e.g. [2])

**Definition 1.** Given two finite matrices of size \( n \times n \), \( A \) and \( B \), we say that \( \lambda \) is an eigenvalue of \( A \) with respect to \( B \) if there exists a nonzero vector \( v \in \mathbb{C}^n \) such that \( Av = \lambda Bv \). These are called the *generalized eigenvalues* of \( A \) with respect to \( B \).

It is well known that in the case of hermitian matrices the generalized eigenvalues are real numbers. This lets to define the following indexes:

**Definition 2.** Let \( A, B \) be Hermitian matrices of size \( n \), we define \( \lambda_n(A, B) \) as the smallest eigenvalue of \( A \) with respect to \( B \) and \( \beta_n(A, B) \) as the largest eigenvalue of \( A \) with respect to \( B \).

It is obvious that \( \lambda_n(A, B) \leq \beta_n(A, B) \). Moreover, in the case of Hermitian semi-definite positive matrices \( \lambda_n(A, B) \geq 0 \).

Of course, the notion of generalized eigenvalue is a generalization of the notion of eigenvalues. Indeed, in the particular case of \( B = I \) (the identity matrix of size \( n \)) it is obvious that \( \lambda_n(A, I) = \lambda_n(A) \) and \( \beta_n(A, I) = \beta_n(A) \) with the notation introduced in [9]. It is well known by the classical Rayleigh quotient that for finite hermitian matrices it follows

\[
\lambda_n(A) = \inf \left\{ \frac{vAv^*}{vv^*}, v \in \mathbb{C}^n \setminus \{0\} \right\} = \inf \left\{ vAv^* : v \in \mathbb{C}^n, vv^* = 1 \right\},
\]

\[
\beta_n(A) = \sup \left\{ \frac{vAv^*}{vv^*}, v \in \mathbb{C}^n \setminus \{0\} \right\} = \sup \left\{ vAv^* : v \in \mathbb{C}^n, vv^* = 1 \right\}.
\]

This criteria also holds for the generalized eigenvalues replacing the identity matrix by an HPD matrix \( B \) (see e.g. [15]).

(*Generalized Rayleigh criteria*) Let \( A, B \) be HSPD matrices of size \( n \) being \( B \) positive definite, then:

\[
\lambda_n(A, B) = \inf \left\{ vAv^* : v \in \mathbb{C}^n, vBv^* = 1 \right\} = \inf \left\{ \frac{vAv^*}{vBv^*} : v \in \mathbb{C}^n \setminus \{0\} \right\},
\]

\[
\beta_n(A, B) = \sup \left\{ vAv^* : v \in \mathbb{C}^n, vBv^* = 1 \right\} = \sup \left\{ \frac{vAv^*}{vBv^*} : v \in \mathbb{C}^n \setminus \{0\} \right\}.
\]

The definition of the Rayleigh quotients can be given in the more general context of infinite matrices (not necessarily hermitian) in the following way:

**Definition 3.** Given an infinite matrix \( A \) and an infinite HPD matrix \( B \) a Rayleigh quotient of \( A \) related to \( B \) is \( \frac{vAv^*}{vBv^*} \) with \( v \in c_{00} \setminus \{0\} \) and the set of Rayleigh quotients is the following set in the complex plane:

\[
\left\{ \frac{vAv^*}{vBv^*} : v \in c_{00} \setminus \{0\} \right\} = \left\{ vAv^* : v \in c_{00}, vBv^* = 1 \right\}.
\]
We generalize the indexes $\lambda$ and $\beta$ introduced for finite matrices in the context of infinite dimensional matrices in the same lines as in [9], [11]. In order to do it, consider two HSPD matrices $A, B$ with $B$ being positive definite, and let $A_n, B_n$ the their truncated matrices of size $(n+1) \times (n+1)$. It is easy to check that the sequence of real numbers $\{\lambda_n(A_n, B_n)\}_{n=0}^{\infty}$ is a non increasing sequence and the sequence $\{\beta_n(A_n, B_n)\}_{n=0}^{\infty}$ is a non decreasing sequence. Therefore, $\lim_{n \to \infty} \lambda_n(A_n, B_n)$ exists, and either is zero or a positive number. On the other hand, $\lim_{n \to \infty} \beta_n(A_n, B_n)$ exists (is a real number) or it is infinite. Thus, we introduce the following indexes

**Definition 4.** Let $A, B$ two infinite HSPD matrices with $B > 0$. We define:

$$0 \leq \lambda(A, B) := \lim_{n \to \infty} \lambda_n(A_n, B_n)$$

$$\beta(A, B) := \lim_{n \to \infty} \beta_n(A_n, B_n) \leq \infty$$

where $A_n, B_n$ are the corresponding truncated matrices of size $(n+1) \times (n+1)$ of $A, B$ respectively.

Moreover, using the generalized Rayleigh criteria it follows:

$$\lambda(A, B) = \inf \left\{ \frac{v^*Av}{v^*Bv} : v \in c_{00} \setminus \{0\} \right\} = \inf \{v^*Av : v \in c_{00}, \ v^*Bv = 1\},$$

$$\beta(A, B) = \sup \left\{ \frac{v^*Av}{v^*Bv} : v \in c_{00} \setminus \{0\} \right\} = \sup \{v^*Av : v \in c_{00}, \ v^*Bv = 1\}.$$

**Remark 1.** In the case that $A > 0$ and $B > 0$ the following relationship between these indexes is given:

$$\lambda(A, B) = \frac{1}{\beta(B, A)} \tag{4}$$

The particularization of the indexes $\lambda$ and $\beta$ for infinite moment matrices lets to introduce these indexes between measures in the following way:

**Definition 5.** Given two supported in the complex plane measures $\mu_1, \mu_2$ with $\mu_2$ infinitely supported we define the following indexes

$$\lambda(\mu_1, \mu_2) := \lambda(M(\mu_1), M(\mu_2)) \quad \beta(\mu_1, \mu_2) := \beta(M(\mu_1), M(\mu_2)).$$

**Remark 2.** By remark 1 in the case that both measures $\mu_1, \mu_2$ have infinite support the following relationship between the indexes follows:

$$\lambda(\mu_1, \mu_2) = \frac{1}{\beta(\mu_2, \mu_1)}.$$

**Example 1.** For uniform Lebesgue measures $m_r$ with $r > 0$ it holds
i) If \( r < 1, \lambda(m_r, m) = 0, \quad \beta(m_r, m) = 1. \)

ii) If \( r > 1, \lambda(m_r, m) = 1, \quad \beta(m_r, m) = \infty. \)

Indeed, denote by \( M_r \) the associated moment matrix with \( m_r \) which is given by:

\[
M_r = \begin{pmatrix}
1 & 0 & 0 & \ldots \\
0 & r^2 & 0 & \ldots \\
0 & 0 & r^4 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Then the truncated matrices of size \((n + 1) \times (n + 1)\) are diagonals and the result follows easily.

Example 2. The Pascal matrix \( P \) is the moment matrix associated with the normalized Lebesgue measure \( m_{1,1} \) in \( S_1(1) \) (see e.g. \[9\])

\[
P = \begin{pmatrix}
1 & 1 & 1 & 1 & \ldots \\
1 & 2 & 3 & 4 & \ldots \\
1 & 3 & 6 & 10 & \ldots \\
1 & 4 & 10 & 20 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

As it was pointed out in \[9\] \( \lim_{n \to \infty} \lambda_n(P) = \lambda(m_{1,1}, m) = 0. \) On the other hand, \( \beta(m_{1,1}, m) = \infty; \) indeed, by Rayleigh criterion \( \beta(\mu, m) = \beta(P,I) \geq e_n P^* e_n = c_{n,n} \to \infty \) whenever \( n \to \infty \) (where \( e_n \) is the usual basis in the space \( c_00 \)).

The following result, which is a generalization of the results in \[9\], is the key to establish the comparison of measures via generalized eigenvalues of the associated moment matrices:

Proposition 1. Let \( \mu_1, \mu_2 \) be two measures supported in the complex plane with \( \mu_2 \) infinitely supported and let \( M_i = M(\mu_i), i = 1, 2, \) their associated moment matrices. Then,

(i) \( \beta(\mu_1, \mu_2) < \infty \) if and only if there exists a constant \( c > 0 \) such that for every polynomial \( p(z) \)

\[
\int |p(z)|^2d\mu_1 \leq c \int |p(z)|^2d\mu_2.
\]

(ii) \( \lambda(\mu_1, \mu_2) > 0 \) if and only if there is a constant \( c > 0 \) such that for every polynomial \( p(z) \)

\[
\int |p(z)|^2d\mu_1 \geq c \int |p(z)|^2d\mu_2.
\]

Proof. Assume \( \beta(\mu_1, \mu_2) < \infty \) is true, then for every polynomial \( p(z) = \sum_{k=0}^{n} v_k z^k \) and \( v = (v_0, v_1, \ldots, v_n, 0, 0, 0, \ldots) \in c_00, \) with \( n \in \mathbb{N}, \) via the identification \[3\] we have that

\[
v M_i v^* = \int |p(z)|^2d\mu_i, \quad i = 1, 2.
\]
and therefore
\[ \beta(\mu_1, \mu_2) = \sup \left\{ \frac{v^* M_1 v}{v^* M_2 v}, v \in \mathcal{c}_0 \setminus \{0\} \right\}. \]
By taking \( c = \beta(\mu_1, \mu_2) > 0 \), it follows that for every \( p(z) \in \mathbb{P}[z] \)
\[ \frac{\int |p(z)|^2 d\mu_1}{\int |p(z)|^2 d\mu_2} \leq c. \]
Then, for every \( p(z) \in \mathbb{P}[z] \),
\[ \int |p(z)|^2 d\mu_1 \leq c \int |p(z)|^2 d\mu_2. \]
On the other hand, assume now that there exists \( c > 0 \) verifying that for every \( p(z) \in \mathbb{P}[z] \),
\[ \int |p(z)|^2 d\mu_1 \leq c \int |p(z)|^2 d\mu_2. \]
By using again the identification (3) we have
\[ \beta(\mu_1, \mu_2) = \sup \left\{ v^* M_1 v : v^* M_2 v = 1, v \in \mathcal{c}_0 \right\} = \sup \left\{ \int |p(z)|^2 d\mu_1 : \int |p(z)|^2 d\mu_2 = 1, p(z) \in \mathbb{P}[z] \right\} \leq c; \]
consequently \( \beta(\mu_1, \mu_2) < \infty \) as we required.
The proof for the index \( \lambda \) is analogous. \( \square \)

3. IMPACT OF GENERALIZED EIGENVALUES IN THE SUPPORT OF THE MEASURES

First of all we begin with an easy result that shows the impact of the behaviour of the diagonal sequence \( \{c_{n,n}\}_{n=0}^\infty \) of a certain moment matrix \( M = (c_{i,j})_{i,j=0}^\infty \) in the boundedness of the support of the representing measure \( \mu \). Note that the sequence \( \{c_{n,n}\}_{n=0}^\infty \) verifies the property:
\[ c_{n,n}^2 \leq c_{n-1,n-1} c_{n+1,n+1} \quad n \geq 1 \]
Indeed, this is a consequence of the Cauchy-Schwartz inequality of the inner product in the Hilbert space \( L^2(\mu) \). In this case, \( \{c_{n,n}/c_{n+1,n+1}\}_{n=0}^\infty \) is a non decreasing sequence and consequently, by elementary results either \( \lim_{n \to \infty} |c_{n,n}|^{1/n} < \infty \) or \( \lim_{n \to \infty} |c_{n,n}|^{1/n} = \infty \). The following characterization of boundedness of the support of the measure is proved in [23]. We prove it for the sake of completeness.

**Proposition 2.** [23] Let \( M = (c_{i,j})_{i,j=0}^\infty \) be a positive definite moment matrix and let \( \mu \) be any representing measure. Then, the following are equivalent:

i) \( \text{supp}(\mu) \) is compact.

ii) \( \lim_{n \to \infty} |c_{n,n}|^{1/n} < \infty \)

Moreover, in this case if \( R = \lim_{n \to \infty} |c_{n,n}|^{1/n} \) then \( R = \sup \{|z| : z \in \text{supp}(\mu)\} \).
Proof. Assume first that \( \text{supp}(\mu) \) is a bounded set and \( R = \sup\{|z| : z \in \text{supp}(\mu)\} \) (note that \( R > 0 \) since the support is infinite). Then \( \text{supp}(\mu) \subset \mathbb{D}_R(0) \) and

\[
|c_{n,n}| = \int |z|^{2n} \, d\mu \leq R^{2n} \mu(\mathbb{D}_R(0)) = c_0 R^{2n}
\]

Therefore,
\[
\lim_{n \to \infty} |c_{n,n}|^{\frac{1}{2n}} \leq R < \infty.
\]

On the other hand, assume that \( \lim_{n \to \infty} |c_{n,n}|^{\frac{1}{2n}} = R \), we prove that \( \text{supp}(\mu) \subset \mathbb{D}_R(0) \).

Assume the contrary, then there exists \( z_0 \in \text{supp}(\mu) \) such that \( |z_0| > R \). Thus we may choose \( r > 0 \) and \( \epsilon > 0 \) such that \( |z| \geq R + \epsilon \) for every \( z \in \mathbb{D}_r(z_0) \). Therefore, for all \( n \in \mathbb{N} \)

\[
\int |z|^{2n} \, d\mu \geq \int_{\mathbb{D}_r(z_0)} |z|^{2n} \, d\mu \geq (R + \epsilon)^{2n} \mu(\mathbb{D}_r(z_0)) = c(R + \epsilon)^{2n}.
\]

Then,
\[
\lim_{n \to \infty} |c_{n,n}|^{\frac{1}{2n}} \geq R + \epsilon \quad \text{which is not possible.} \quad \square
\]

As a consequence of the above result we obtain the following result:

**Corollary 1.** Let \( \mu \) be a measure such that \( \beta(\mu, m_r) < \infty \) for some \( r > 0 \). Then, \( \lim_{n \to \infty} |c_{n,n}|^{\frac{1}{2n}} \leq r \) and consequently \( \text{supp}(\mu) \subset \mathbb{D}_r(0) \).

**Proof.** Assume \( \beta(\mu, m_r) < \infty \), by Proposition 1 there exists a constant \( c > 0 \) verifying that for every polynomial, and in particular for polynomials \( z^n \) with \( n \in \mathbb{N}_0 \)

\[
c_{n,n} = \int |z|^{2n} \, d\mu \leq c \int |z|^{2n} \, d\mu = c \, r^{2n}.
\]

Therefore \( \lim_{n \to \infty} |c_{n,n}|^{\frac{1}{2n}} \leq r \) and by Proposition 2 \( \text{supp}(\mu) \subset \mathbb{D}_r(0) \). \quad \square

**Remark 3.** Note that \( \lim_{n \to \infty} |c_{n,n}|^{\frac{1}{2n}} = r \) is a weaker condition than \( \beta(\mu, m_r) < \infty \).

Indeed, consider a measure with support in \( \mathbb{T} \) given by \( d\mu(\theta) = w(\theta) \, d\theta \) with \( w(\theta) \in L^1(-\pi, \pi) \) with \( \text{esssupp} \, w(\theta) = \infty \) being \( \text{esssupp} \, w(\theta) \) the essential upper bound, that is, the smallest number \( C > 0 \) such that \( w(\theta) \leq C \) a.e. By proposition 2 it follows that \( \lim_{n \to \infty} |c_{n,n}|^{\frac{1}{2n}} = 1 \). Nevertheless, \( \beta(\mu, m) = \infty \); indeed, by the results in [16] \( \lim_{n \to \infty} \beta_n = \text{esssupp} \, w(\theta) = \infty \) being \( \beta_n \) the largest eigenvalue of the truncated matrix of \( M(\mu) \) of size \( (n+1) \times (n+1) \).

As it is established in Corollary 1 if \( \beta(\mu, m_r) < \infty \) then \( \text{supp}(\mu) \subset \mathbb{D}_r(0) \) which is not the support of \( m_r \) although it is, in particular, its convex envelope. This result motivates the following problem: Consider two compactly supported \( \mu_1, \mu_2 \) measures verifying that \( \beta(\mu_1, \mu_2) < \infty \), we wonder if there is a relationship between the corresponding supports of \( \mu_1, \mu_2 \). Of course, it is not true in general that \( \text{supp}(\mu_1) \subset \text{supp}(\mu_2) \). Indeed, consider the Lebesgue measures \( m \) and \( m_{\frac{1}{2}} \) in Example 2 verifying \( \beta(m_{\frac{1}{2}}, m) = 1 \), nevertheless \( \text{supp}(m_{\frac{1}{2}}) = \frac{1}{2} \mathbb{T} \).
As an answer of the above problem our main result in this section establishes the following relationship: if \( \beta(\mu_1, \mu_2) < \infty \) then the support of \( \mu_1 \) must be contained in the polynomially convex hull of the support of \( \mu_2 \).

We recall that (see e.g. [8]) for a compact set \( K \) in the complex plane the polynomially convex hull of \( K \), denoted \( \text{Pc}(K) \), is defined as:

\[
\text{Pc}(K) := \{ z \in \mathbb{C} : |p(z)| \leq \max_{\xi \in K} |p(\xi)|, \text{ for all } p(z) \in \mathbb{P}[z] \}.
\]

A compact set is said to be polynomially convex if \( K = \text{Pc}(K) \). Obviously, \( K \subseteq \text{Pc}(K) \). In the complex plane polynomial convexity turns to be a purely topological notion. Indeed, using the maximum modulus principle and the Runge approximation theorem, one proves that a compact set is polynomially convex if and only if \( \mathbb{C} \setminus K \) is connected.

**Theorem 1.** Let \( \mu_1, \mu_2 \) be compactly supported measures verifying \( \beta(\mu_1, \mu_2) < \infty \). Then, the support of \( \mu_1 \) is contained in the polynomially convex hull of the support of \( \mu_2 \), i.e.,

\[
\text{supp}(\mu_1) \subseteq \text{Pc}(\text{supp}(\mu_2)).
\]

**Proof.** Denote by \( K_i = \text{supp}(\mu_i) \), \( i = 1, 2 \). By [1] it follows that for every polynomial \( p(z) \) it follows that

\[
\int |p(z)|^2 d\mu_1 \leq \int |p(z)|^2 d\mu_2. \tag{6}
\]

Assume that \( K_1 \) is not a subset of \( \text{Pc}(K_2) \). Then, there exists \( z_1 \in K_1 \) such that \( z_1 \notin \text{Pc}(K_2) \). Since \( \text{Pc}(K_2) \) is polynomially convex from the results in [13] it follows that there is a polynomial \( p_0(z) \in \mathbb{P}[z] \) such that \( p_0(z_1) = 1 \) and \( |p_0(z)| < 1 \) for every \( z \in \text{Pc}(K_2) \). By compactness of \( \text{Pc}(K_2) \) there exists \( 0 \leq \alpha < 1 \) such that \( |p_0(z)| \leq \alpha \) for all \( z \in \text{Pc}(K_2) \). Next, since \( p(z_1) = 1 \) there exists \( r > 0 \) such that \( |p_0(z)| > \frac{1+\alpha}{2} \) for all \( z \in B_r(z_1) \). On the other hand, since \( z_1 \in K_1 \) it follows that \( \mu_1(\text{supp}(\mu_2)) > 0 \). Therefore, by applying inequality (6) to the polynomials \( p^n_0(z) \) with \( n \in \mathbb{N} \) we obtain that

\[
\left( \frac{1+\alpha}{2} \right)^n \mu_1(B_r(z_1)) \leq \int_{K_1} |p_0(z)|^n d\mu_1 \leq \int_{K_2} |p_0(z)|^n d\mu_2 \leq \alpha^n \mu_2(\text{supp}(\mu_2)).
\]

That means that there exists \( C > 0 \) such that \( \left( \frac{(1+\alpha)/2}{\alpha} \right)^n \leq C \) for every \( n \in \mathbb{N} \), which is not possible since \( \frac{1+\alpha}{2} > \alpha \).

**Remark 4.** The converse of this result is not true. Indeed, consider \( \mu_1 = m \) and the measure in the unit circle \( \mu_2(\theta) = (1 + \cos \theta)d\theta \) which appears in [9]. It is clear that \( \text{supp}(\mu_1) \subseteq \text{Pc}(\text{supp}(\mu_2)) \) and nevertheless \( \beta(\mu_1, \mu_2) = \infty \). Indeed, by [9] if \( T \) is the Toeplitz matrix associated with \( \mu_2 \) then \( \lambda(\mu_2, \mu_1) = \lim_{n \to \infty} \lambda_n(T_n) = 0 \) and therefore \( \beta(\mu_1, \mu_2) = \infty \) by Remark 2.

As a consequence of theorem [1] and remark 2 we have the following result
Corollary 2. Let $\mu_1, \mu_2$ two compactly supported measures with infinite support with polynomially convex supports. If $0 < \lambda(\mu_1, \mu_2) \leq \beta(\mu_1, \mu_2) < \infty$ then the supports of both measures coincide.

It is interesting to point out that this result provides a tool to recognize the support of a measure comparing this measure, with our indexes, with another well known measure. In particular, we obtain an application to find Jordan curves in the support of a measure:

Corollary 3. Let $\Gamma$ be a Jordan curve and $\nu_\Gamma$ any measure such that $\supp(\nu_\Gamma) = \Gamma$. Let $\mu$ be other measure such that verifies

$$0 < \lambda(\mu, \nu_\Gamma) \leq \beta(\mu, \nu_\Gamma) < \infty.$$

Then, $\Gamma \subset \supp(\mu) \subset Pc(\Gamma)$.

Proof. Since $\beta(\mu, \nu_\Gamma) < \infty$ by Theorem 1 it follows that $\supp(\mu) \subset Pc(\nu_\Gamma) = \hat{\Gamma} \cup \Gamma$. On the other hand by Remark 2 it follows that $\beta(\nu_\Gamma, \mu) = \frac{1}{\lambda(\mu, \nu_\Gamma)} < \infty$, and again by using Theorem 1

$$\Gamma = \supp(\nu_\Gamma) \subset Pc(\supp(\mu)).$$

Assume that $\Gamma$ is not contained in $\supp(\mu)$. Then there existe $z_1 \in \Gamma$ such that $z_1 \notin \supp(\mu)$. Then, $\supp(\mu) \subset \Gamma \setminus D_s(z_1)$ for some $s > 0$. Then, by the definition of the polinomially convex hull

$$\Gamma \subset Pc(\supp(\mu)) \subset Pc \left( (\hat{\Gamma} \cup \Gamma) \setminus D_s(z_1) \right) = (\hat{\Gamma} \cup \Gamma) \setminus D_s(z_1)$$

which is not possible. \qed

4. An algebraical approach of the convex envelope of a measure

In the sequel we are concerned with the convex hull of the support of a measure.

We obtain an algebraic description of such set in terms of Rayleigh quotients of a certain matrix (not necessarily Hermitian) related with the associated moment matrix. In order to obtain the main result in this section we need some definitions and lemmas.

Definition 6. Let $T$ be a bounded operator in a Hilbert space $\mathcal{H}$. The numerical range of $T$ is defined as

$$W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}$$

Lemma 1. Let $T$ be a bounded operator in a Hilbert space $\mathcal{H}$. Assume that $Y$ is dense set in $\mathcal{H}$, then

$$\overline{W}(T) = \{ \langle Ty, y \rangle : y \in Y, \|y\| = 1 \}.$$

Proof. Obviously

$$\{ \langle Ty, y \rangle : y \in Y, \|y\| = 1 \} \subset \overline{W}(T).$$

On the other hand, assume $z_0 \in \overline{W}(T)$, then there is a sequence $\{x_n\}_{n=1}^\infty$ with $\|x_n\| = 1$, such that $z_0 = \lim_{n \to \infty} \langle Tx_n, x_n \rangle$. Since $Y$ is a dense set in $\mathcal{H}$, there is a
sequence \( \{y_n\}_{n=1}^\infty \) such that \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \); in particular, \( \{y_n\}_{n=1}^\infty \) is a bounded sequence. Therefore,

\[
\|Tx_n - y_n\| \leq \|T\| \|x_n - y_n\| \|y_n\| \to 0 \text{ if } n \to \infty.
\]

Consequently, \( z_0 = \lim_{n \to \infty} \langle Tx_n, y_n \rangle = \mathcal{C}(\sup(\mu)). \)

**Theorem 2.** Let \( M = (c_{i,j})_{i,j=0}^\infty \) be a moment matrix with \( \lim_{n \to \infty} |c_{n,n}|^{\frac{1}{n}} < \infty \) and let \( \mu \) be the representing measure. Let \( M' \) be the infinite matrix obtained by removing the first row in \( M \), then

\[
\left\{ \frac{vM'v^*}{vMv^*} : v \in c_{00} \setminus \{0\} \right\} = Co(\sup(\mu)).
\]

**Proof.** Let’s see first in (7) the inclusion from left to right. We will call \( \Omega \subset \mathbb{C} \) the set of complex numbers of the form

\[
\Omega = \left\{ \frac{vM'v^*}{vMv^*} : v \in c_{00} \setminus \{0\} \right\}.
\]

By using (3), if \( p(z) = v_0 + v_1 z + \cdots + v_n z^n \), and \( v = (v_0, v_1, \ldots, 0, \ldots) \in c_{00} \),

\[
\int z |p(z)|^2 d\mu = \int z p(z) \overline{p(z)} \ d\mu = v S_L M v^* = vM'v^*
\]

where \( S_L \) is the shift left matrix and \( M' = S_L M \) is the infinite matrix obtained by removing the first row in \( M \). Therefore, \( \Omega \) can be rewritten as

\[
\Omega = \left\{ \frac{\int z |q(z)|^2 d\mu}{\int |q(z)|^2 d\mu} : q(z) \in \mathbb{P}[z] \setminus \{0\} \right\}.
\]

Let be \( z_0 \in \Omega \), by definition there exists a polynomial \( q(z) \in \mathbb{P}[z] \) such that

\[
z_0 = \frac{\int z |q(z)|^2 d\mu}{\int |q(z)|^2 d\mu}.
\]

This means that \( z_0 \) can be interpreted as the center of gravity of the weight function \( w_q(z) = |q(z)|^2 d\mu \). In particular, if we consider the associated moment matrix with respect to such measure denoted by \( M(w_q) = (\widetilde{c}_{i,j})_{i,j=0}^\infty \) it follows that \( z_0 \) is the zero of the orthogonal polynomial of degree one associated to the measure \( w_q \) which is

\[
\det \begin{pmatrix} \widetilde{c}_{00} & 1 \\ \widetilde{c}_{10} & z \end{pmatrix} = z \widetilde{c}_{00} - \widetilde{c}_{10} = 0,
\]

being

\[
z_0 = \frac{\widetilde{c}_{10}}{\widetilde{c}_{00}} = \frac{\int z |q(z)|^2 d\mu}{\int |q(z)|^2 d\mu}.
\]
As it is well known by Fejer’s Theorem (see e.g. [14]) the zeros of the orthogonal polynomials are included in the convex envelope of the support of the measure, and this proves the inclusion \( \Omega \subset \text{Co}(\text{supp}(\mu)) \).

It remains to prove the reverse inclusion. Consider the multiplication by \( z \)-operator \( \Delta_z : L^2(\mu) \to L^2(\mu) \), given by \( \Delta_z(p(z)) = zp(z) \). This is a normal operator and it is well known (see e.g. [7]) that the spectrum of this operator is the support of the measure \( \mu \); i.e. \( \sigma(\Delta_z) = \text{supp}(\mu) \).

Consider now the restriction operator of \( \Delta_z \) to \( P^2(\mu) \), \( \mathcal{D} : P^2(\mu) \to P^2(\mu) \); this is a subnormal operator which minimal normal extension is \( \Delta_z \). By one of the basic properties relating the spectrum of a subnormal operator (see e.g. [7]), the spectrum of the normal extension of an operator is contained in the spectrum of the operator, consequently

\[
\text{supp}(\mu) \subset \sigma(\Delta_z) \subset \sigma(\mathcal{D}).
\]

Now we use that spectrum of an operator lies in the closure of its numerical range, that is \( \sigma(\mathcal{D}) \subset \overline{W(\mathcal{D})} \). On the other hand, the famous Toeplitz-Hausdorff theorem (see e.g. [17]) asserts that the numerical range of a bounded operator is always a convex set (no necessarily closed). Thus, the convex hull of the spectrum of an operator lies in the closure of the numerical range, therefore

\[
\text{Co}(\text{supp}(\mu)) \subset \text{Co}(\sigma(\mathcal{D})) \subset \overline{W(\mathcal{D})}.
\]

Then, by lemma above, since \( \mathbb{P}[z] \) is dense in the Hilbert space \( P^2(\mu) \) it follows that

\[
\overline{W(\mathcal{D})} = \overline{\{ \langle \mathcal{D}(p), p \rangle : \langle p, p \rangle = 1, p \in P^2(\mu) \}} = \overline{\{ \langle zp, p \rangle : p \in \mathbb{P}[z], \langle p, p \rangle = 1 \}},
\]

and again via the identification (3) it follows

\[
\overline{W(\mathcal{D})} = \overline{\left\{ \frac{v M^\prime v^\ast}{v M v^\ast} : v \in c_{00} \setminus \{0\} \right\}}.
\]

Consequently

\[
\text{Co}(\text{supp}(\mu)) \subset \overline{W(\mathcal{D})} = \overline{\left\{ \frac{v M^\prime v^\ast}{v M v^\ast} : v \in c_{00} \setminus \{0\} \right\}},
\]

as we required. \( \square \)

Given a certain compactly supported measure \( \mu \) the matrix representation of the operator \( \mathcal{D} \), introduced in the proof of Theorem 2, in the space \( P^2(\mu) \) with respect to the associated basis of orthonormal polynomials is an upper Hesseberg matrix (see e.g. [14]). The following result provides a way to obtain the convex envelope of the support of a measure via such upper Hesseberg matrix

**Corollary 4.** Let \( \mu \) be a compactly supported measure and let \( \mathcal{D} \) be the Hesseberg matrix associated to \( \mu \). Then:

\[
\text{Co}(\text{supp}(\mu)) = \left\{ v \mathbb{D} v^\ast : v \in c_{00}, vv^\ast = 1 \right\} = \bigcup_{n=0}^{\infty} W(\mathbb{D}_n).
\]
where $D_n$ is the $n+1$-section of the matrix $D$.

**Example 3.** As it is known the identity matrix is the moment matrix associated with $m$. On the other hand the matrix representation of the multiplication operator by $z$, in the canonical basis of $\ell^2$, is

$$D = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$  

We will get the value of $W(D_n)$. The projection on $\mathbb{R}$ of field of values $W(D_n)$ is

$$\Re(D_n) = \frac{D_n + D_n^*}{2} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \ldots & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \ldots & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \ldots & \frac{1}{2} & 0 \end{pmatrix}.$$  

It is well known that $\Re(D_n)$ is the $n \times n$ tridiagonal section of $U_n(x)$, i.e. the Jacobi matrix of orthogonal Tchebyschev polynomials of the second class on $[-1, 1]$. We have that the eigenvalues of $\Re(D_n)$ are the zeros of $U_n(x)$. It is well known that $U_n(x) = \sin((n+1) \arccos(x))/\sin(n \arccos(x))$. We set $n \in \mathbb{N}$, and obtain

$$\lambda_{\text{max}}(\Re(D_n)) = \max \{ U_n(x) = 0 \} = \max \{ \sin((n+1) \arccos(x)) = 0 \} = \max \{ \cos(\frac{\pi}{n+1}) \}.$$  

$\Re(D_n)$ is a symmetric matrix hence $W(\Re(D_n)) = [-\cos(\frac{\pi}{n+1}), \cos(\frac{\pi}{n+1})]$, and by the symmetry with respect to the origin we finally have

$$W(D_n) = \{ z \in \mathbb{C} : |z| \leq \cos(\frac{\pi}{n+1}) \}.$$

Therefore

$$\bigcup_{n=1}^{\infty} W(D_n) = \{ z \in \mathbb{C} : |z| \leq 1 \},$$

and the corollary is fulfilled since the convex envelope of support of the Lebesgue measure in the unit circumference, $\text{supp}(m) = \mathbb{T}$ and $\text{Co}(\text{supp}(m)) = \overline{D}_1(0)$.

As a consequence of the above results we obtain the following identity that verifies infinite moment matrices and which provides a necessary condition to be a moment matrix:

**Corollary 5.** Let $M = (c_{i,j})_{i,j=0}^{\infty}$ be a positive definite moment matrix and let $\widehat{M}$ be the matrix obtained by removing the first row and the first column. Then,

$$\beta(\widehat{M}, M) = \lim_{n \to \infty} |c_{n,n}|^{1/n}$$
Proof. First of all we show that the matrix $\tilde{M}$ is also a moment matrix. Indeed, if $\mu$ is a representing measure for $M$ and we compute the moments of the measure $|z|^2\mu$ we have

$$c_{i,j}(|z|^2\mu) = \int |z|^2 z^i z^j d\mu = \int z^{i+1} z^{j+1} d\mu.$$ 

This means that the matrix obtained by removing the first row and column in $M$, that we denote $\hat{M}$, is the moment matrix associated with the measure $|z|^2\mu$.

Therefore, if $\Delta_z : P^2(\mu) \to P^2(\mu)$ is the multiplication by-$z$-operator,

$$\|\Delta_z\|^2 = \sup \{ \int z p(z)z \overline{p(z)} d\mu : \int |p(z)|^2 d\mu = 1, p \in P[z] \} = \sup \{ \int |p(z)|^2 |z|^2 d\mu : \int |p(z)|^2 d\mu = 1, p \in P[z] \}.$$ 

By using (3) we have

$$\|\Delta_z\|^2 = \sup \{ v \overline{\tilde{M} v}^* : v \in c_{00}, v \overline{M v}^* = 1 \} = \beta(|z|^2 d\mu, d\mu) = \sup \{|z|^2 : z \in \text{supp}(\mu) \} = \beta(\tilde{M}, M).$$

Now the result follows from Proposition 1.

Remark 5. Note that for an HPD Toeplitz matrix $T$ it is obvious that $\beta(\tilde{T}, T) = 1$. Nevertheless, this identity is not true in general for infinite HPD matrices; indeed, consider the infinite diagonal matrix $M = \text{diag}(1, 2, 1, 2^2, 1, 2^3, 1, 2^4, \ldots)$ which is an HPD matrix such that $\lim_{n \to \infty} |c_{n,n}|^{1/n}$ does not exist. Note that $M$ is not a moment matrix since the elements of the diagonal does not fulfill Cauchy-Schwarz inequality since $(2^n)^2 \not\leq 1 \cdot 1$.

5. A generalization of a sufficient condition for completeness of polynomials.

In [9] a sufficient condition for completeness of polynomials is given in the following terms: if polynomials are dense in a certain measure space $L^2(\mu)$ for a compactly supported measure then the smallest eigenvalue of the truncated matrices of the moment matrices $\lambda_n \to 0$, as $n \to \infty$; using our notation $\lambda(\mu, m) = 0$. There is proved that this is not necessary condition for completeness of polynomials. Indeed, consider the measures $m_r$ with $r < 1$; by (1) $\lambda(m_r, m) = 0$ and nevertheless polynomials are not dense in $L^2(m_r)$. We here generalize this result comparing with the Lebesgue measures in circles with different center and radius. The proof is based in the following matrix version of the change of variable theorem:

Recall that if $\varphi(z) = \alpha z + z_0$ is a similarity map onto $\mathbb{C}$ with $\alpha, z_0 \in \mathbb{C}$ it is proved in [11] and [12] that the moment matrix of the image measure $\mu \circ \varphi^{-1}$, denoted by $M_\varphi^\varphi(\mu)$, has finite section of size $(n+1) \times (n+1)$

\begin{equation}
M_\varphi^\varphi(\mu) = A_n(\alpha, \beta)M_n(\mu)A_n^*(\alpha, \beta)
\end{equation}
where \( A_n(\alpha, \beta) \) is defined as in \([12]\) (with certain modifications due to the different expression for the inner product)

\[
A_n(\alpha, \beta) = \begin{pmatrix}
    a_0^0 b_0^0 & 0 & 0 & \cdots & 0 \\
    a_0^0 b_1^1 & a_1^1 b_0^0 & 0 & \cdots & 0 \\
    a_0^0 b_2^2 & a_1^1 b_1^1 & a_2^2 b_0^0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_0^n b_n^n & a_1^1 b_{n-1}^n & a_2^2 b_{n-2}^n & \cdots & a_n^n b_0^0
\end{pmatrix}
\]

In the particular case of \( \alpha = r \in \mathbb{R} \) we obtain the following result:

**Lemma 2.** Let \( \varphi(z) = rz + z_0 \) be a certain similarity map with \( r > 0 \) and \( \mu \) a measure. Consider the image measure \( \mu \circ \varphi^{-1} \) obtained after similarity map. Then,

\[
\lambda(\mu, m) = \lambda(\mu \circ \varphi^{-1}, m_{z_0;r}), \quad \beta(\mu, m) = \beta(\mu \circ \varphi^{-1}, m_{z_0;r})
\]

**Proof.** We prove the result for the index \( \lambda \) (for the other index the proof is analogous). Denote by \( M_1 = M(\mu) \) and \( M_2 = M^r(\mu) \). We have seen in \([8]\) that for all \( n \in \mathbb{N}_0 \)

\[
M_n^r = A_n(r; z_0)M_nA_n^*(r; z_0).
\]

Consequently for every \( v \in \mathbb{C}^{n+1} \) it follows

\[
vM_n^r v^* = vA_n(r; z_0)M_nA_n^*(r; z_0)v^*
\]

and we have

\[
\lambda_n(\mu \circ \varphi^{-1}, m_{z_0;r}) = \inf \{vM_n^r v^* : vM_n(m_{z_0;r})v^* = 1\}
\]

\[
= \inf \{vA_n(r; z_0)M_1 nA_n^*(r; z_0)v^* : vA_n(r; z_0)I_nA_n^*(r; z_0)v^* = 1\}.
\]

By making the change \( w = vA_n(r; z_0) \) in the last equality, it follows

\[
\lambda_n(\mu \circ \varphi^{-1}, m_{z_0;r}) = \inf \{wM_n w^* : wI_n w^* = 1\},
\]

and by taking limits we obtain

\[
\lambda(\mu \circ \varphi^{-1}, m_{z_0;r}) = \inf \{wM(\mu) w^* : ww^* = 1\} = \lambda(\mu, m)
\]

as we required. \( \square \)

**Remark 6.** The above lemma can be obviously extended for any infinitely supported measures in the complex plane \( \mu_1, \mu_2 \) in the following way: let \( \varphi \) be a similarity map then

\[
\lambda(\mu_1, \mu_2) = \lambda(\mu_1 \circ \varphi^{-1}, \mu_2 \circ \varphi^{-1}) \quad \beta(\mu_1, \mu_2) = \beta(\mu_1 \circ \varphi^{-1}, \mu_2 \circ \varphi^{-1}).
\]

**Theorem 3.** Let \( \mu \) be a compactly supported measure. If \( P^2(\mu) = L^2(\mu) \), then for every \( z_0 \in \mathbb{C} \) and \( r > 0 \), it holds

\[
\lambda(\mu, m_{z_0;r}) = 0.
\]
Proof. Assume the contrary. Then, there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that
\[
\lambda(\mu, m_{z_0; r}) > 0.
\]
Then, consider the similarity $\varphi(z) = \frac{1}{r}z - z_0$ and the image measure $\mu \circ \varphi^{-1}$ obtained by applying the similarity $\varphi$ to $\mu$. Then, by lemma (2) it follows that
\[
\lambda(\mu \circ \varphi^{-1}, m) = \lambda(\mu, m_{z_0; r}) > 0.
\]
Then by the results in [9] it follows that $P^2(\mu \circ \varphi^{-1}) \neq L^2(\mu \circ \varphi^{-1})$ and consequently $L^2(\mu) \neq P^2(\mu)$.

Above theorem can be rewritten as follows: if a measure $\mu$ verifies that $\lambda(\mu, m_{z_0; r}) > 0$ for a certain Lebesgue measure $m_{z_0; r}$, obviously non complete (i.e. polynomials are not dense) then also $\mu$ is not complete. One can wonder if we may replace measures $m_{z_0; r}$ by any non complete measure and this motivates the following problem:

**Problem:** Assume $\mu_1, \mu_2$ two compactly supported measures verifying:

1. $\mu_2$ is not complete, i.e., $P^2(\mu_2) \neq L^2(\mu_2)$.
2. $\lambda(\mu_1, \mu_2) > 0$.

Is it true that $L^2(\mu_1) \neq P^2(\mu_1)$?

We obtain a partial answer when the first measure is supported in a Jordan curve:

**Proposition 3.** Let $\mu_1, \mu_2$ be infinitely supported measures in the complex plane such that $P^2(\mu_2) \neq L^2(\mu_2)$ and $\lambda(\mu_1, \mu_2) > 0$. Then, the following holds:

If $\mu_1$ is supported on a Jordan curve $\Gamma$, then $P^2(\mu_1) \neq L^2(\mu_1)$.

**Proof.** By Thomson’s theorem [22] since $L^2(\mu_2) \neq P^2(\mu_2)$ it follows the existence of a bounded point evaluation for $\mu_2$, say $\xi$. Recall that $\xi \in \mathbb{C}$ is a bounded point evaluation for the measure $\mu_2$ if there exists a constant $c > 0$ such that for every polynomial $p(z) \in \mathbb{P}[z]$,
\[
|p(\xi)|^2 \leq c \int |p(z)|^2 d\mu_2.
\]
Since $\lambda(\mu_1, \mu_2) > 0$ it easily follows that $\xi$ is a bounded point evaluation of $\mu_1$. Indeed, by Proposition [1] there exists $C > 0$ such that for every polynomial $p(z)$
\[
\int |p(z)|^2 d\mu_1 \geq C \int |p(z)|^2 d\mu_2.
\]
Consequently, combining both inequalities we have that for every polynomial $p(z)$
\[
|p(\xi)|^2 \leq C \int |p(z)|^2 d\mu_2 \leq \frac{c}{C} \int |p(z)|^2 d\mu_1
\]
and $\xi$ is a bounded point evaluation for $\mu_1$. The conclusion now follows by [11] since for supported in Jordan curves measures the existence of a bounded point evaluation implies that the measure is not complete.

□
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