Hawkes process and Edgeworth expansion with application to maximum likelihood estimator

Masatoshi Goda\textsuperscript{1,2}

Received: 15 June 2020 / Accepted: 6 January 2021 / Published online: 18 January 2021
© The Author(s), under exclusive licence to Springer Nature B.V. part of Springer Nature 2021

Abstract
We provide a rigorous mathematical foundation of the theory for the higher-order asymptotic behavior of the one-dimensional Hawkes process with an exponential kernel. As an important application, we give the second-order asymptotic distribution for the maximum likelihood estimator of the exponential Hawkes process.

Keywords Hawkes process · Inferential statistics · Edgeworth expansion · Asymptotic expansion · Maximum likelihood estimator

1 Introduction
The Hawkes process was introduced by Hawkes (1971) in 1971. It has a self-exciting property and has been used to model earthquakes and their aftershocks (Ogata 1988), events in social media (Rizoiu et al. 2017), activity of financial markets (Vladimir Filimonov 2012) and so on. Furthermore, the multivariate Hawkes process has also been used for modeling the limit order book; for example, see Abergel et al. (2016). In many of these studies, the Hawkes process with an exponential kernel has been used.

Regarding statistical inference for the Hawkes process, the quasi maximum likelihood estimator (QMLE) and the quasi Bayesian estimator (QBE) are practical. The consistency, the asymptotic normality and the convergence of moments of these estimators were established for the multivariate Hawkes process with exponential kernels, see Clinet and Yoshida (2017). Roughly, the asymptotic normality of an estimator \( \hat{\theta}_T \) is characterized as follows.

\[
\left| E \left[ f \left( \sqrt{T} (\hat{\theta}_T - \theta_0) \right) \right] - \int f(x) \phi(x; g^{-1}) dx \right| = o(1)
\]

for some appropriate functions \( f \), where \( g \) is the Fisher information matrix, \( \phi(x; g^{-1}) \) is the probability density function of the normal distribution \( N(0, g^{-1}) \) and \( \theta_0 \) is the true parameter.
In practice, there are often situations where we cannot obtain data with sufficient observation time. In such cases, it is not appropriate to approximate the error distribution of an estimator by the normal distribution. Then, the derivation of confidence intervals and hypothesis testing cannot be performed accurately. If we establish a theory of Edgeworth expansion for the distribution of an estimator \( \hat{\theta}_T \), we get an improved error evaluation. Edgeworth expansion is obtained by formally expanding the characteristic function and applying the inverse Fourier transform. Roughly, we may have the following evaluation in the case of the second-order expansion.

\[
\left| E \left[ f \left( \sqrt{T} (\hat{\theta}_T - \theta_0) \right) \right] - \int f(x) q_{T,3}(x) \, dx \right| = o(T^{-1/2}),
\]  

(1.1)

for some appropriate functions \( f \), where \( q_{T,3}(x) \, dx \) is some signed measure. In addition, we also derive a Studentized asymptotic distribution for practical applications.

In this paper, we deal with the one-dimensional Hawkes process with an exponential kernel and establish the Edgeworth expansion for the distribution of its maximum likelihood estimator (MLE). The outline of the concrete proofs is as follows.

In the case of independent identical distribution, the validity of the Edgeworth expansion for an MLE is reduced to the expansion of a log-likelihood process. Same as the i.i.d. case, the asymptotic expansion are given for an M-estimator of a functional of an \( \varepsilon \)-Markov process with a mixing property in Sakamoto and Yoshida (2004). In the case of MLE, the essence of the theory of the asymptotic expansion lies in the analysis of a log-likelihood process. With this background, we deal with the asymptotic expansion for the class of functionals of the Hawkes process containing the derivatives of the log-likelihood process.

To prove the validity of (1.1), we prepare the theory of Edgeworth expansion for the distribution of a functional of a geometric mixing process. For discrete-time processes, the scheme of this theory was established by Götzé and Hipp (1983). It was extended to a continuous-time case in Kusuoka and Yoshida (2000). Moreover, a more general framework is established in Yoshida (2004). We will reduce their theory to a simple framework without the Cramér-condition. Furthermore, we have developed a framework that is not confined to the non-degeneracy of variance by appropriately modifying the variance of the random variables.

Second, we will apply the theory of the Edgeworth expansion to the derivatives of the log-likelihood process of the exponential Hawkes process. In this application, we introduce the Hawkes core process. In the proof of this theory, it is essential to confirm the conditions regarding the mixing property and the finiteness of moments of the Hawkes core process. The mixing property of the Hawkes core process follows from its Markovian property and geometric ergodicity. It is known that the exponential Hawkes intensity process has the Markovian property, see Oakes (1975). However, we introduce a new proof including a method applicable to the Hawkes core process. We investigate these properties by using the idea of the extended generator.

Finally, we give the second-order asymptotic distribution for the MLE of the exponential Hawkes process. In this regard, we confirm some conditions for the log-likelihood process.

Numerical calculations of the asymptotic distribution using the Monte Carlo method are also presented. Furthermore, the results of the simulations with R confirm that the asymptotic distribution we introduced is a better approximation than the approximation by the normal distribution. Comparisons with the bootstrap method are also done.

Section 2 presents a theory of the Edgeworth expansion for the distribution of a functional of a geometric mixing process. It also describes how to apply the Edgeworth expansion to an MLE. In Sect. 3, we see the properties of the one-dimensional Hawkes process with an
exponential kernel. It is difficult to directly express the derivative of the log-likelihood process as a functional of the Hawkes intensity. Therefore, we introduce the Hawkes core process and also investigate its properties. In Sect. 4, we will apply the theory of the Edgeworth expansion to a functional of the Hawkes core process. In particular, we give the concrete form of the second-order asymptotic distribution for the MLE of the exponential Hawkes process. Finally, Sect. 5 shows the simulation results about the second-order asymptotic distribution for the MLE of the exponential Hawkes process. The details of the proofs in each section are summarized in “Appendix”.

2 Asymptotic expansion

2.1 Asymptotic expansion under geometric mixing condition

In this section, we introduce the theory of Edgeworth expansion for the distribution of a functional of a geometric mixing process. The following framework is given by Theorem 2.10 in Götze and Hipp (1983) for discrete-time processes. It is extended to a continuous-time case under the conditional Cramér-condition in Yoshida (2004). In this paper, we rewrite this theory without using the Cramér-condition, referring to Götze and Hipp (1978).

Let \((\Omega, \mathcal{F}, P)\) be a probability space. Assume that we are given \(\sigma\)-fields \(\{\mathcal{B}_I\}\) indexed by intervals \(I \subset \mathbb{R}_+\). We consider a process \(Z = (Z_t)_{t \in \mathbb{R}_+} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d\) whose increment is adapted to \(\mathcal{B}_I\), namely \(Z_I = Z_t - Z_s \in \mathcal{B}_I\) for every closed interval \(I = [s, t] \subset \mathbb{R}_+\) with \(s < t\) and \(Z_0 \in \mathcal{B}_0\). We will derive the asymptotic expansion for the distribution of the normalized process \(S_T = Z_T / \sqrt{T}\). For this purpose, we assume that the following two conditions hold.

\[ [A1 \] (Geometric mixing property) \]
There exists a positive constant \(a\) such that for any \(s, t \in \mathbb{R}_+\) with \(s \leq t\), and for any \(f \in \mathcal{F}_{[0, s]}\) and \(g \in \mathcal{F}_{(t, \infty)}\) with \(\|f\|_\infty \leq 1\) and \(\|g\|_\infty \leq 1\),

\[ |E[f g] - E[f]E[g]| \leq a^{-1} e^{-a(t-s)}. \]

\[ [A2 \] (Moment property) \]
\[ \sup_{t \in \mathbb{R}_+, 0 \leq h \leq \Delta} \|Z_{[t, t+h]}\|_{L^p(P)} < \infty \text{ and } E[Z_{[t, t+\Delta]}] = 0 \text{ for any } \Delta > 0 \text{ and } p > 0. \]

Moreover, \(Z_0 \in \bigcap_{p>1} L^p(P)\) and \(E[Z_0] = 0\).

These conditions [A1] and [A2] are needed for the validity of the formal Edgeworth expansion. Before starting the statement of the asymptotic expansion, we prepare some notation under the condition [A2]. The \(r\)-th cumulant functions \(\chi_{T,r}(u)\) of \(S_T\) are defined by

\[ \chi_{T,r}(u) = \left( \frac{d}{d \varepsilon} \right)^r \bigg|_{\varepsilon=0} \log E \left[ e^{i \varepsilon u' S_T} \right], \]

where \(u'\) represents the transpose of \(u\). Then, since \(\chi_{T,1}(u) = E[iu' S_T] = 0\), the characteristic function of \(S_T\) is formally expressed as

\[ E \left[ e^{i u' S_T} \right] = \exp \left( \log E \left[ e^{i u' S_T} \right] \right) \bigg|_{\varepsilon=1} = \exp \left( \sum_{r=2}^{\infty} r!^{-1} \varepsilon^{r-2} \chi_{T,r}(u) \right) \bigg|_{\varepsilon=1}. \]

1 \(\mathcal{F}\) denote the set of \(\mathcal{B}\)-measurable functions.
Next, we define functions $\hat{P}_{T,r}(u)$ by the formal Taylor expansion at $\varepsilon = 0$:

$$\exp\left(\sum_{r=2}^{\infty} r!^{-1} \varepsilon^{r-2} \chi_{T,r}(u)\right) = \exp\left(\frac{1}{2} \chi_{T,2}(u)\right) + \sum_{r=1}^{\infty} \varepsilon^r T^{-\frac{r}{2}} \hat{P}_{T,r}(u),$$  \hspace{1cm} (2.1)

where

$$\hat{P}_{T,r}(u) = \exp\left(\frac{1}{2} \chi_{T,2}(u)\right) \sum_{l=1}^{r} \sum_{r_1, \ldots, r_l \in \mathbb{N}; \ r_1 + \cdots + r_l = r} \frac{\chi_{T,r_1+2}(u) \cdots \chi_{T,r_l+2}(u)}{l!(r_1 + 2)! \cdots (r_l + 2)!}.$$  \hspace{1cm} (2.2)

Let $\hat{\Psi}_{T,p}(u)$ be the partial sum of the right-hand side of (2.1) with $\varepsilon = 1$:

$$\hat{\Psi}_{T,p}(u) = \exp\left(\frac{1}{2} \chi_{T,2}(u)\right) + \sum_{r=1}^{p-2} T^{-\frac{r}{2}} \hat{P}_{T,r}(u).$$  \hspace{1cm} (2.3)

We want to define a signed measure $\Psi_{T,p}$ as the Fourier inversion of $\hat{\Psi}_{T,p}(u)$. However, when $\chi_{T,2}(u)$ is not negative definite, $\Psi_{T,p}$ does not have the density function. To overcome this problem, we set $\Sigma_{T,D} = \text{Var}[S_T] + T^{-D} I$ for a positive constant $D$, where $I$ is an identity matrix. Then, we define

$$\hat{\Psi}_{T,p,D}(u) = \exp\left(-\frac{1}{2} u' \Sigma_{T,D} u\right) + \sum_{r=1}^{p-2} T^{-\frac{r}{2}} \exp\left(-\frac{1}{2} u' \Sigma_{T,D} u\right) \sum_{l=1}^{r} \sum_{r_1, \ldots, r_l \in \mathbb{N}; \ r_1 + \cdots + r_l = r} \frac{\chi_{T,r_1+2}(u) \cdots \chi_{T,r_l+2}(u)}{l!(r_1 + 2)! \cdots (r_l + 2)!}.$$  \hspace{1cm} (2.4)

The cumulant functions also have the following representation,

$$\chi_{T,k}(u) = i^k \sum_{a_1 \cdots a_k = 1}^{d} u_{a_1} \cdots u_{a_k} \lambda_{T - (a_1 \cdots a_k)},$$  \hspace{1cm} (2.5)

where $u = (u_1, \ldots, u_d)$ and $\lambda_{T - (a_1 \cdots a_k)}$ is the $(a_1, \ldots, a_k)$-cumulant of $S_T$, i.e.,

$$\lambda_{T - (a_1 \cdots a_k)} = (-i)^k \frac{\partial^k}{\partial u_{a_1} \cdots \partial u_{a_k}} \log E\left[e^{iu'S_T}\right].$$

Consider the order of convergence, we put

$$\lambda_{T - (a_1 \cdots a_k)} = T^{(m-2)/2} \lambda_{T - (a_1 \cdots a_k)}.$$  \hspace{1cm} (2.6)

Let $h_{a_1 \cdots a_k}(z; \Sigma)$ be the Hermite polynomials, i.e.

$$h_{a_1 \cdots a_k}(z; \Sigma) = \frac{(-1)^k}{\phi(z; \Sigma)} \frac{\partial^k}{\partial z_{a_1} \cdots \partial z_{a_k}} \phi(z; \Sigma),$$  \hspace{1cm} (2.7)

where $\phi(x; \Sigma)$ is the probability density function of the normal distribution $N(0, \Sigma)$. In particular, we write $\phi(x) = \phi(x; I_d)$ and $h_{a_1 \cdots a_k}(z) = h_{a_1 \cdots a_k}(z; I_d)$ for identity matrix $I_d$.

We define a signed measure $\Psi_{T,p,D}$ as the Fourier inversion of $\hat{\Psi}_{T,p,D}(u)$. Then, from (2.4), (2.5), (2.6) and (2.7), the density function $p_{T,p,D}(z)$ of $\Psi_{T,p,D}$ is written as

$$p_{T,p,D}(z) = \phi(z; \Sigma_{T,D})$$
If there is no confusion, we write \( l T \), where \( A_k \) represents the index sequence \( a_1 \cdots a_k \) and if there are same index sequences, summing up them with respect to \( a_1 \cdots a_k \) in accordance with the Einstein summation convention. Furthermore if there are different multiple index sequences, we distinguish them, for example,

\[
K^{A_i};L^{A_j};M_{A_iA_j} = \sum_{a_1,...,a_i,a'_1,...,a'_j=1} K^{a_1...a_i};L^{a'_1...a'_j};M_{a_1...a_i,a'_1...a'_j}.
\]

For a vector of nonnegative integers \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( t \in \mathbb{R}^d \) and \( f \in C^{[\alpha]}(\mathbb{R}^d) \), let

\[
|\alpha| = \sum_{i=1}^d \alpha_i, \quad t^\alpha = \prod_{i=1}^d t_i^{\alpha_i} \quad \text{and} \quad \partial^\alpha f = \frac{\partial^{[\alpha]} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.
\]

For positive constants \( \Gamma, L_1, L_2 \), we denote by \( \mathcal{E}(\Gamma, L_1, L_2) \) a set of functions \( f \in C^\Gamma(\mathbb{R}^d) \) with \( \sup_{|\alpha| \leq \Gamma} |\partial^\alpha f(x)| \leq L_2(1 + |x|)^{L_1} \) for every \( x \in \mathbb{R}^d \). The following theorem is the main statement in this subsection. A proof can be found in “Appendix”.

**Theorem 2.1** Let \( p \in \mathbb{N} \) with \( p \geq 2 \) and \( L_1, L_2 > 0 \). Suppose that the conditions \([A1] \) and \([A2] \) are satisfied. Then, there exist \( D > 0 \) and \( \Gamma \in \mathbb{N} \) such that for any \( f \in \mathcal{E}(\Gamma, L_1, L_2) \),

\[
E[f(S_T)] - \int_{\mathbb{R}^d} f(z)p_{T, D}(z)dz = o\left(T^{-(p-2)/2}\right).
\]

### 2.2 Asymptotic expansion for maximum likelihood estimator

Applying Theorem 2.1, we consider getting the asymptotic expansion for a maximum likelihood estimator (MLE) up to the second order. We refer to Sakamoto and Yoshida (2004) to construct the following framework.

Let \((\Omega, \mathcal{F}, P)\) be a probability space, \( \Theta \subset \mathbb{R}^p \) be an open bounded convex set and \( N_t(\theta) \) be a \( \mathbb{R}^d \)-valued stochastic process parameterized by \( \theta \in \Theta \). We assume that the log-likelihood process of \( N_t(\theta_0) \) is given by \( l_T : \Theta \times \Omega \rightarrow \mathbb{R} \), where \( \theta_0 \in \Theta \) is the true parameter. Then, the MLE \( \hat{\theta}_T \) is defined by

\[
\hat{\theta}_T (\omega) = \operatorname{argmax}_{\theta \in \Theta} l_T (\theta, \omega) \quad \text{for} \quad \omega \in \Omega.
\]

If there is no confusion, we write \( l_T (\theta) = l_T (\theta, \omega) \). For \( r \in \mathbb{N} \) and a sequence of indexes \( \alpha = (a_1, \ldots, a_r) \in \{1, \ldots, p\}^r \), we write

\[
D^\alpha = \frac{\partial^r}{\partial \theta^{a_1} \cdots \partial \theta^{a_r}}, \quad \text{(2.9)}
\]

where \( \theta^{a_i} \) is the \( a_i \)-th component of \( \theta \). Moreover, \( \partial_\theta \) denotes the vector differential operator (the gradient operator). Let \( l_{a_1 \cdots a_k} (\theta) = D^{(a_1 \cdots a_k)} l_T (\theta) \) and \( v_{a_1 \cdots a_k} (\theta) = E \left[ \frac{1}{2} l_{a_1 \cdots a_k} (\theta) \right] \).

We write \( g_T = (g_{ab})_{a,b=1,\ldots,p} = (-v_{ab} (\theta_0))_{a,b=1,\ldots,p} \). We assume that

\[
[A3] |g_T - g| \rightarrow 0 \text{ as } T \rightarrow \infty, \text{ where the norm } | \cdot | \text{ is the Frobenius norm and } g \text{ is a non-singular matrix.}
\]
[B0] \( l_T(\theta) \) satisfies the following conditions.

(i) \( l_T \in C^4(\theta) \) a.s.

(ii) the score function \( \partial_\theta l_T(\theta) \) satisfies \( E[\partial_\theta l_T(\theta_0)] = 0 \).

(iii) \( \text{Var}\left[ \frac{1}{T} \partial_\theta l_T(\theta_0) \right] = g_T \).

(iv) \( \mathbb{D}^c v_{ab}(\theta) = v_{abc}(\theta) \).

Under the condition [A1], \( g_T \) is non-singular for sufficiently large \( T \). We write \( g_T^{-1} = (g^{abc})_{a,b=1,\ldots,p} \). The following conditions are assumed for some positive constants \( q_1, q_2, q_3 \) and \( \gamma \).

[B1] \( q_1 \sup_{T>0} \left\| T^{-\frac{1}{2}} l_a(\theta_0) \right\|_{L^q_1(P)} < \infty \) for \( a \in \{1, \ldots, p\} \).

[B2] \( q_2, \gamma \sup_{T>0, \theta \in \Theta} \left\| T^{\frac{\gamma}{2}} (T^{-1} l_{a_1 \cdots a_k}(\theta) - \nu_{a_1 \cdots a_k}(\theta)) \right\|_{L^{q_3}(P)} < \infty \) for \( k = 2, 3 \), \( a_1, \ldots, a_k \in \{1, \ldots, p\} \).

[B3] There exist an open set \( \hat{\Theta} \) including \( \theta_0 \) and a positive constant \( T_0 \) such that

\[
\inf_{T>T_0, \theta_0, \theta_1, \theta_2 \in \hat{\Theta}, \|x\|=1} \left| \int_0^1 v_{ab}(\theta_1 + s(\theta_2 - \theta_1)) \, ds \right| > 0.
\]

[B4] \( q_3 \sup_{T>0} \left\| \sup_{\theta \in \Theta} \left\| T^{-1} l_{a_1 \cdots a_4}(\theta) \right\|_{L^q_3(P)} < \infty \right\| \right\|_{L^q_3(P)} < \infty \) for \( a_1, \ldots, a_4 \in \{1, \ldots, p\} \).

To get the asymptotic expansion of an MLE, we approximate the MLE with the sum of log-likelihood processes. Let

\[
Z_a = \frac{1}{\sqrt{T}} l_a(\theta_0) \quad \text{and} \quad Z_{ab} = \sqrt{T} \left( \frac{1}{T} l_{ab}(\theta_0) - \nu_{ab}(\theta_0) \right).
\]

With the Einstein summation convention, for an index sequence \( A \), we write

\[
\nu_A^{a_1} = g^{abc} v_{bA}(\theta_0) \quad \text{and} \quad Z_A^{a_1} = g^{abc} Z_{bA}.
\]

Under the condition [B3], let \( \hat{\Theta} \) be the one in [B3]. Set \( \Omega_T = \{ \omega \in \Theta \mid \exists! \hat{\theta}_T(\omega) \in \hat{\Theta} \text{ s.t. } \partial_\theta l_T(\hat{\theta}_T(\omega), \omega) = 0 \} \). Write \( \hat{\theta}_a^{a_1 \cdots a_k} = T^{\frac{k}{2}} (\hat{\theta}_T - \theta_0)^{a_1 \cdots a_k} \cdot (\hat{\theta}_T - \theta_0)^{a_1 \cdots a_k} \). On the set \( \Omega_T \), from the Taylor expansion of \( \frac{1}{T} l_a(\theta) \) at \( \theta = \theta_0 \), we immediately get the following two stochastic expansions

\[
\sqrt{T} (\hat{\theta}_T - \theta_0)^{a_1} = Z_a^{a_1} + T^{-\frac{1}{2}} \left( Z_a^{a_1} \hat{\theta}_a^{a_1} + \frac{1}{2} v_a^{a_1} \hat{\theta}_a^{a_1} \right) + T^{-1} \tilde{R}_1^{a_1} = Z_a^{a_1} + T^{-\frac{1}{2}} \tilde{R}_1^{a_1};
\]

for any \( a = 1, \ldots, p \), where

\[
\tilde{R}_2^{a_1} = \frac{1}{2} Z_{a_1 a_2}^{a_1} \hat{\theta}_a^{a_1 a_2} + \frac{1}{2} \left\{ \int_0^1 (1 - u)^2 g^{ab} \left( \frac{1}{T} l_{b a_1 a_2 a_3} \left( \theta_0 + u(\hat{\theta}_T - \theta_0) \right) \right) \, du \right\} \hat{\theta}_a^{a_1 a_2 a_3},
\]

and

\[
\tilde{R}_4^{a_1} = Z_{a_1}^{a_1} \hat{\theta}_a^{a_1} + \frac{1}{2} v_{a_1 a_2}^{a_1} \hat{\theta}_a^{a_1 a_2} + T^{-\frac{1}{2}} \tilde{R}_2^{a_1}.
\]

From these two expressions, we get

\[
\sqrt{T} (\hat{\theta}_T - \theta_0)^{a_1} = Z_a^{a_1} + T^{-\frac{1}{2}} \left( Z_a^{a_1} Z_a^{a_1} + \frac{1}{2} v_{a_1 a_2}^{a_1} Z_a^{a_1} Z_a^{a_2} \right) + T^{-1} \tilde{R}_2^{a_1}, \quad (2.10)
\]
where
\[ \bar{R}_2 = Z_{a1} \bar{R}^{a1} + \bar{R}_2^{a2} + T^{-\frac{1}{2}} \left( \frac{1}{2} \nu_{a1a2}^{a1} \bar{R}_1^{a1} \bar{R}_1^{a2} \right). \]

We consider applying the transformation formula for the asymptotic expansion. Let
\[ Z_T^{(1)} = T^{\frac{1}{2}} (Z_1, \ldots, Z_p) \quad \text{and} \quad Z_T^{(2)} = T^{\frac{1}{2}} (Z_1, \ldots, Z_1p, Z_21, \ldots, Z_2p, \ldots, Z_{p1}, \ldots, Z_{pp}). \]
Moreover, we put
\[ Z_T = (Z_T^{(1)}, Z_T^{(2)}) \quad \text{and} \quad \Sigma_T = \text{Var} \left[ \frac{Z_T}{\sqrt{T}} \right] + T^{-D} I \]
for a positive constant $D$. Let a $(p + p^2) \times (p + p^2)$-matrix $C_T$ be
\[ C_T = \begin{pmatrix} g_T^{-1} & 0 \\ O & G_T \end{pmatrix}, \]
where
\[ G_T = \begin{pmatrix} G_{11} & \cdots & G_{1p} \\ \vdots & \ddots & \vdots \\ G_{p1} & \cdots & G_{pp} \end{pmatrix} \quad \text{and} \quad G_{ij} = \begin{pmatrix} g_{ij}^1 & 0 & \cdots & 0 \\ 0 & g_{ij}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & g_{ij}^p \end{pmatrix} : p \times p \text{-matrix for } i, j = 1, \ldots, p. \]

Then, we define
\[ \tilde{Z}_T = \frac{1}{\sqrt{T}} C_T Z_T = (Z^1, \ldots, Z^p; Z^{11}, \ldots, Z^{1p}; Z^{11}_1, \ldots, Z^{1p}_1; Z^{11}_2, \ldots, Z^{1p}_2; \ldots; Z^{11}_p, \ldots, Z^{1p}_p), \]
a and we write $\tilde{Z}_T^{(1)} = (Z^1, \ldots, Z^p)$ and $\tilde{Z}_T^{(2)} = (Z^{11}_1, \ldots, Z^{1p}_p)$. We define the polynomial $Q_1(z)$ for a $(p + p^2)$-dimensional vector $z = (z^1, \ldots, z^p, z^{11}_1, \ldots, z^{1p}_p)$ such that $a$-th element of $Q_1(z)$ is
\[ Q_1^a(z) = z^{a1}_1 z^{a1}_2 + \frac{1}{2} \nu_{a1a2}^{a1} z^{a1}_2 z^{a2}_2. \]

From (2.10), (2.11) and (2.12), we get
\[ \sqrt{T} (\hat{\theta}_T - \theta_0) = \tilde{Z}_T^{(1)} + T^{-\frac{1}{2}} Q_1 \left( \tilde{Z}_T^{(1)}, \tilde{Z}_T^{(2)} \right) + T^{-1} \bar{R}_2. \]

From (2.13), we can give the asymptotic expansion for the MLE by using the transformation formula. However, it is complicated to calculate the concrete form of the density function $g_{T,3,D}$ described later. In order to simplify this calculation, the orthogonalization of $\tilde{Z}_T$ is convenient. We put $\tilde{\Sigma}_T^{(i,j)} = \text{Cov} \left[ \tilde{Z}_T^{(i)}, \tilde{Z}_T^{(j)} \right]$ for $i, j = 1, 2$. Remark that $g_T^{-1} + T^{-D} (g_T^{-1})^2$ is non-singular for sufficiently large $T > 0$. Thus, we can define $\tilde{g}_T^{-1} = (\tilde{g}^{ab})_{a,b = 1, \ldots, p} = g_T^{-1} + T^{-D} (g_T^{-1})^2$ and $\tilde{g}_T = (\tilde{g}_{ab})_{a,b = 1, \ldots, p} = (\tilde{g}_T^{-1})^{-1}$. Let
\[ M_{T,3,D} = \begin{pmatrix} I & \tilde{\Sigma}_T^{(2,1)} \tilde{g}_T^{-1} & 0 \\ \tilde{\Sigma}_T^{(1,1)} & \tilde{g}_T^{-1} & \tilde{g}_T^{-1} \end{pmatrix}. \]
We set $\tilde{Z}_T = M_{T,D} \tilde{Z}_T = (\tilde{Z}_T^{(1)}, \tilde{Z}_T^{(2)} - \tilde{\Sigma}_T^{(2,1)} \tilde{g}_T \tilde{Z}_T^{(1)})$, and we write $\tilde{Z}_T^{(1)} = \tilde{Z}_T^{(1)}$ and $\tilde{Z}_T^{(2)} = \tilde{Z}_T^{(2)} - \tilde{\Sigma}_T^{(2,1)} \tilde{g}_T \tilde{Z}_T^{(1)}$. Then an elementary calculation yields

$$\tilde{\Sigma}_{T,D} = M_{T,D} C_T \Sigma_{T,D} C_T' M_{T,D}' = \begin{pmatrix} \tilde{\Sigma}_{T,D}^{(1,1)} & \tilde{O} \\ \tilde{O} & \tilde{\Sigma}_{T,D}^{(2,2)} \end{pmatrix},$$

where $\tilde{\Sigma}_{T,D}^{(1,1)} = \tilde{g}_T^{-1}$ and $\tilde{\Sigma}_{T,D}^{(2,2)} = \tilde{\Sigma}_T^{(2,2)} + T^{-D} \tilde{G}_T^2 - \tilde{\Sigma}_T^{(2,1)} \tilde{g}_T \tilde{\Sigma}_T^{(1,2)}$. In terms of $\tilde{Z}_T$, $\sqrt{T}(\hat{\theta}_T - \theta_0)$ is rewritten as

$$\sqrt{T}(\hat{\theta}_T - \theta_0) = \tilde{Z}_T^{(1)} + T^{-\frac{1}{2}} \tilde{Q}_1 \left( \tilde{Z}_T^{(1)}, \tilde{Z}_T^{(2)} \right) + T^{-\frac{1}{2}} \tilde{R}_2,$$  \tag{2.14}

where $\tilde{Q}_1$ is a polynomial for $\tilde{z}^{(1)} = (\tilde{z}_1^{(1)}, \ldots, \tilde{z}_p^{(1)})$ and $\tilde{z}^{(2)} = (\tilde{z}_1^{(2)}, \ldots, \tilde{z}_p^{(2)})$ satisfying

$$\tilde{Q}_1^{a_1} \left( \tilde{z}^{(1)}, \tilde{z}^{(2)} \right) = \tilde{Q}_1^{a_1} \left( \tilde{z}^{(1)}, \tilde{z}^{(2)} + \tilde{\Sigma}_T^{(2,1)} \tilde{g}_T \tilde{z}^{(1)} \right) = \tilde{z}_{a_1}^{(1)} + \tilde{z}_{a_1}^{(2)}; \tilde{z}_{a_1}^{(1)}; \tilde{z}_{a_1}^{(2)};$$

for $\tilde{V}_{bc} = \text{Cov}[Z^{a_1}_b, Z^{a_1}_c]_{\tilde{g}_{a_1}}$ and $\tilde{\mu}_{a_1,a_2} = (\tilde{V}_{a_1,a_2} + \tilde{V}_{a_2,a_1} + \tilde{v}_{a_1,a_2})/2$.

Focus on the main part of (2.14). We set $\tilde{S}_T = \tilde{Z}_T^{(1)} + T^{-\frac{1}{2}} \tilde{Q}_1 \left( \tilde{Z}_T^{(1)}, \tilde{Z}_T^{(2)} \right)$. The following proposition holds, see “Appendix” for a proof.

**Proposition 2.2** Let $L_1, L_2 > 0$. Suppose that the conditions [A1]–[A3] and [B0] hold. Then, there exist $D > 0$ and $\Gamma \in \mathbb{N}$ such that for any $f \in \mathcal{E}(\Gamma, L_1, L_2)$,

$$|E \left[ f(\tilde{S}_T) \right] - \int_{\mathbb{R}^d} f(z^{(1)}) \tilde{q}_{T,3,D}(z^{(1)}) dz^{(1)}| = o \left( T^{-1/2} \right),$$

where

$$\tilde{q}_{T,3,D}(z^{(1)}) = \phi(z^{(1)}, g^{-1}_T) + \frac{1}{\sqrt{T}} \left\{ \left( \frac{1}{6} \tilde{\kappa}_T \tilde{a}_1 \tilde{a}_2 ; \tilde{g}_T \tilde{a}_1 ; \tilde{g}_T \tilde{a}_2 \right) \tilde{h}_{a_1,a_2} \phi(z^{(1)}, g^{-1}_T) \right\},$$

We need to evaluate the remainder term $\tilde{R}_2$ in (2.14). The conditions [B0]–[B4] lead to the following proposition. We will give a proof in “Appendix”.

**Proposition 2.3** Let $L > 1$, $\gamma \in (0, 1)$ and $q_1, q_2, q_3 > 0$ with

$$q_1 > 3L, \quad q_2 > \max \left( p, \frac{3q_1L}{q_1 - 3L} \right),$$

$$q_3 > \frac{q_1L}{q_1 - 3L}, \quad \frac{2}{3} + \max \left( \frac{L}{q_2}, \frac{L}{3q_3} \right) < \gamma < 1 - \frac{L}{q_1}. \tag{2.15}$$

For these constants, we assume that the conditions [B0]–[B4] hold. Then, there exist $C > 0$ and $\varepsilon' \in (0, 1)$ such that

$$P \left[ \Omega_T \cap \left\{ \left| \tilde{R}_{2,\varepsilon'} \right| \leq CT^{-\frac{1}{4}+\varepsilon'} \right\}, a = 1, \ldots, p \right] = 1 - o \left( T^{-\frac{1}{2}} \right).$$

Finally, we assume the following condition.

[C1] \[ \sup_{T>0} \| \sqrt{T}(\hat{\theta}_T - \theta_0) \|_{L^k(P)} < \infty \] for any $k > 0.
From Propositions 2.2 and 2.3, we get the asymptotic expansion for the distribution of the MLE. Remark that, since $g_T^{-1}$ is non-singular for large $T > 0$ under the condition [A3], we can get rid of a constant $D$. Let $V_{bc}^{ai} = \text{Cov}[Z_{bc}^b, Z_{ai}^a]$ and $\mu_{a1a2}^{ai} = (V_{a1a2}^{ai} + V_{a2a1}^{ai} + \nu_{a1a2}^{ai})/2$. The following theorem is the conclusion of this section.

**Theorem 2.4** Let $L_1, L_2 > 0$. Suppose that the conditions [A1]–[A3] and [C1] are satisfied. Moreover, assume that there exist $L > 1$, $\gamma \in (0, 1)$ and $q_1, q_2, q_3 > 0$ such that [B0]–[B4] with (2.15) hold. Then, there exists $\Gamma \in \mathbb{N}$ such that for any $f \in \mathcal{S}(\Gamma, L_1, L_2)$,

$$
E \left[ f(\sqrt{T}(\hat{\theta}_T - \theta_0)) \right] - \int_{\mathbb{R}^p} f(z^{(1)}) q_{T,3}(z^{(1)}) dz^{(1)} = o(T^{-\frac{1}{2}}),
$$

where

$$
q_{T,3}(z^{(1)}) = \phi(z^{(1)}; g_T^{-1}) + \frac{1}{\sqrt{T}} \left\{ \left( \frac{1}{6} \kappa_T^{a1a2a3}; + \mu_{b1b2}^{a3}; g_{b2a2}^{a3}; h_{a1a2a3}(z^{(1)}); g_T^{-1} \right)
+ \mu_{b1b2}^{a1}; g_{b1b2}; h_{a1}(z^{(1)}); g_T^{-1} \right\} \phi(z^{(1)}; g_T^{-1}).
$$

In practice, we often want to compute coefficients from the data. However, if $g_T^{-1}$ is replaced with some estimator $\hat{g}_T^{-1}$ in Theorem 2.4, the order of the approximation is reduced since the error between $g_T^{-1}$ and $\hat{g}_T^{-1}$ is usually $O_p(T^{-1/2})$. One solution is to make the error of the estimator Studentized. For example, we consider an observed information $-\frac{1}{T} I_{ab}(\hat{\theta}_T)_{a,b=1,\ldots,p}$ as an estimator $\hat{g}_T$. Write $I_d = (I_{ab})_{a,b=1,\ldots,p}$, $g_T^{-1} = (\sqrt{g}_{ab})_{a,b=1,\ldots,p}$, $\hat{g}_T^{-1} = (\sqrt{\hat{g}}_{ab})_{a,b=1,\ldots,p}$ and $g_T^{-\frac{1}{2}} = (\sqrt{g}_{ab})_{a,b=1,\ldots,p}$ and $g_T^{-\frac{1}{2}} = (\sqrt{g}_{ab})_{a,b=1,\ldots,p}$. Since $\sqrt{\hat{g}}_{ab} = \sqrt{g}_{ab} - \frac{1}{2\sqrt{T}} (\hat{Z}_{a,b} + \sqrt{g}_{a2}^{a2}; \hat{v}_{a,a1} \hat{Z}_{a2}) + o_p(T^{-1/2})$ on $\Omega_T = \{\omega \in \Omega \mid g_T \hat{g}_T : \text{positive definite} \} \cap \Omega_T$, we easily get

$$
\sqrt{T} \sqrt{\hat{g}_{a1a1}}(\hat{\theta}_T - \theta_0)^{a1} = \hat{Z}_a + \frac{1}{2\sqrt{T}} \sqrt{\hat{g}_{a1a2}}; \hat{Z}_{a,a1} \hat{Z}_{a2} + o_p\left(T^{-\frac{1}{2}}\right),
$$

where $\hat{Z}_{a,a} = \sqrt{\hat{g}}_{a1a1}^{a1};$ and $\hat{v}_{a,A} = \sqrt{\hat{g}}_{aa1}^{a1};$ for an index sequence $A$. The same discussion of the proof for Theorem 2.4 leads the next corollary.

**Corollary 2.5** Under the same conditions of Theorem 2.4, there exists $\Gamma \in \mathbb{N}$ such that for any $f \in \mathcal{S}(\Gamma, L_1, L_2)$,

$$
E \left[ f(\sqrt{T} \hat{g}_T^{-\frac{1}{2}}(\hat{\theta}_T - \theta_0)) \right] - \int_{\mathbb{R}^p} f(z^{(1)}) \tilde{q}_{T,3}(z^{(1)}) dz^{(1)} = o\left(T^{-\frac{1}{2}}\right),
$$

where

$$
\tilde{q}_{T,3}(z^{(1)}) = \phi(z^{(1)}; \hat{g}_T^{-1}) + \frac{1}{\sqrt{T}} \left\{ \left( \frac{1}{6} \hat{\kappa}_T^{a1a2a3}; + \hat{\mu}_{b1b2}^{a3}; \hat{g}_{b2a2}^{a3}; h_{a1a2a3}(z^{(1)}); \hat{g}_T^{-1} \right)
+ \hat{\mu}_{b1b2}^{a1}; I_{b1b2} h_{a1}(z^{(1)}); \hat{g}_T^{-1} \right\} \phi(z^{(1)}; \hat{g}_T^{-1}).
$$

(2.16)

Remark 2.6 When we replace the coefficients in (2.16) with the ones computed from the plug-in distribution of an MLE, the order of the approximation is maintained.
3 Hawkes process with an exponential kernel

In this section, we will define the Hawkes process with an exponential kernel. Furthermore, we will discuss its properties. In the second half, we define the Hawkes core process and investigate its properties. Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}\) be a filtration that satisfies the usual conditions.

3.1 Definition

First, we define a point process and its intensity. A sequence of stopping times \(\{\tau_n\}_{n \in \mathbb{N}}\) with respect to \(\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}\) is called a point process, if it satisfies the following properties:

(i) \(\tau_1 > 0\) a.s.
(ii) \(\tau_n < \tau_{n+1}\) on \(\{\tau_n < \infty\}\) a.s.\(^2\)
(iii) \(\tau_n = \tau_{n+1}\) on \(\{\tau_n = \infty\}\) a.s.

Let \(\tau_\infty = \lim_{n \to \infty} \tau_n\). Define a stochastic process \(N_t\) by \(N_t = \sum_{n \geq 1} 1_{\{\tau_n \leq t\}} 1_{\{\tau_n > t\}}\). Then, \((N_t, \mathcal{F}_t)\) is also called a point process. Define the intensity process of \((N_t, \mathcal{F}_t)\) as a non-negative \(\mathcal{F}_t\)-progressively measurable process \(\lambda_t\) such that \(\int_0^t \lambda_s \, ds\) is the compensator of \((N_t, \mathcal{F}_t)\). The filtration \(\mathcal{F}_t\) is called the history of \(N_t\), in the sense that \(\sigma(N_s; s \leq t) \subset \mathcal{F}_t\).

Definition 3.1 (Hawkes process with an exponential kernel) A Hawkes process with an exponential kernel is a point process \((N_t^x, \mathcal{F}_t)\) with the \(\mathcal{F}_t\)-predictable intensity

\[
\lambda_t^x = \mu + xe^{-\beta t} + \int_{(0,t]} \alpha e^{-\beta(t-s)} \, dN_s^x,
\]

where \(\mu, \alpha\) and \(\beta\) are positive constants with \(\alpha / \beta < 1\) and \(x \geq 0\).

The following remarks are fundamental. Let \(\mathcal{F}_t^x = \sigma(N_s^x; s \leq t)\).

Remark 3.2 There exists a Hawkes process with an exponential kernel \(N_t^x\) with the history \(\{\mathcal{F}_t^x\}_{t \in \mathbb{R}_+}\). One may prove this existence in the same fashion as the proof of Theorem 1(a) in Brémaud and Massoulié (1996). Moreover, in this construction, the measurability of \(x \mapsto N_t^x\) and \(x \mapsto \lambda_t^x\) are brought about spontaneously.

Remark 3.3 Denote the \(n\)-th jump time of \(N^x\) by \(\tau^x_n\) and let \(\tau^x_\infty = \lim_{n \to \infty} \tau^x_n\). Then \(\tau^x_\infty = \infty\) a.s. In particular, \(N_t^x\) and \(\lambda_t^x\) have finite paths.

Proof (Proof of Remark 3.3) Fix \(t \geq 0\) arbitrarily. From the definition of \(N_t^x\), we have

\[
E\left[N_{t \wedge \tau^x_n}^x\right] = E\left[\int_{(0,t \wedge \tau^x_n]} \lambda_s^x \, ds\right] = \mu E[t \wedge \tau_n^x] + \frac{x}{\beta} E[1 - e^{-\beta(t \wedge \tau^x_n)}] + \frac{\alpha}{\beta} E\left[\int_{(0,t \wedge \tau^x_n]} (1 - e^{-\beta(t \wedge \tau^x_n - u)}) \, dN^x_u\right]
\]

\[
\leq \mu t + \frac{x}{\beta} + \frac{\alpha}{\beta} E\left[N_{t \wedge \tau^x_n}^x\right].
\]

Thus,

\[
E\left[N_{t \wedge \tau^x_n}^x\right] \leq \frac{\beta \mu t + x}{\beta - \alpha}.
\]

\(^2\) Such notation means that \(P[\{\tau_n < \tau_{n+1}\} \cap \{\tau_n < \infty\}] = P[\tau_n < \infty]\).
However, if $\mathbb{P}[\tau^x_\infty \leq t] > 0$ holds, the above inequality and the monotone convergence theorem imply that $N^t_{\tau^x_\infty} < \infty$ on $\{\tau^x_\infty \leq t\}$ a.s. and it contradict the definition of $\tau^x_\infty$. Therefore, $\tau^x_\infty > t$ a.s. Since $t$ is arbitrarily, $\tau^x_\infty = \infty$ holds almost surely. \hfill $\Box$

### 3.2 Markovian property of exponential Hawkes intensity

The main purpose of this subsection is revealing the Markovian property of the Hawkes process with an exponential kernel. This property is well-known, see Oakes (1975). However, in Oakes (1975), the way of definition of the Hawkes process is somewhat different from ours. Therefore, to strictly handle the Markovian property under our settings and to make this paper self-contained, we will give another proof via the extended generator of the intensity process $\lambda^x_t$ that is defined later. In this subsection, we deal with the outline only. The detail of proofs can be found in “Appendix”.

Before getting into the main topic, we define some symbols used throughout this and the next Subsection. Let $N^x_t$ be a Hawkes process as defined in Definition 3.1, and set $\{\mathcal{F}^x_t\}_{t \in \mathbb{R}_+}$ as the history of $N^x$. By considering a sufficiently rich $\Omega$, $\{\mathcal{F}^x_t\}_{t \in \mathbb{R}_+}$ is regarded as a right-continuous filtration, see Lemma 18.4 in Liptser and Shiryaev (2000). Moreover, we regard that $\{\mathcal{F}^x_t\}_{t \in \mathbb{R}_+}$ is augmented and use the same notation $\{\mathcal{F}^x_t\}_{t \in \mathbb{R}_+}$. Then, $\{\mathcal{F}^x_t\}_{t \in \mathbb{R}_+}$ satisfies the usual condition.

First, we see that $\lambda^x_t$ has the finiteness of its moment. To show this property, we prepare the following lemma that is shown in the proof of Proposition 4.5 in Clinet and Yoshida (2017).

**Lemma 3.4** Let $\alpha$, $\beta$ and $\mu$ be parameters of the Hawkes process $N^x_t$. For a differentiable function $f$, we define the operator $\mathcal{A}$ by

$$
\mathcal{A} f(y) = y (f(y + \alpha) - f(y)) - \beta (y - \mu) \frac{d}{dy} f(y).
$$

Then, there exist positive constants $M_1$, $K_1$ and $K_2$ such that for $V(y) = e^{M_1 y}$

$$
\mathcal{A} V(y) \leq -K_1 V(y) + K_2.
$$

**Lemma 3.4** ensures the existence of the moment-generating function of $\lambda^x_t$ on a neighborhood of the origin:

**Proposition 3.5** There exists a positive constant $M_1$ such that

$$
\sup_{t \in \mathbb{R}_+} E \left[ e^{M_1 \lambda^x_t} \right] < \infty.
$$

Second, we see the Markov property of the exponential Hawkes intensity. To show this property, we use the idea of the extended generator that is an extension of the infinitesimal generator. It can be found, for instance, by Meyn and Tweedie (1993).

**Definition 3.6** Let $(X_t, \mathcal{F}_t)$ be a $d$-dimensional adapted process. We denote by $\text{Dom}(\mathcal{A})$ the set of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ for which there exists a measurable function $U : \mathbb{R}^d \to \mathbb{R}$ such that $f(X_t) - f(X_0) - \int_{(0,t]} U(X_s) ds$ is a $\mathcal{F}_t$-martingale. Then, we write $U = \mathcal{A} f$ and call $\mathcal{A}$ as the extended generator of $(X_t, \mathcal{F}_t)$.

For a differentiable function $f : \mathbb{R} \to \mathbb{R}$, we define the operator $\mathcal{A}$ by (3.2) as one of the extended generator of $(\lambda^x_t, \mathcal{F}^x_t)$. We investigate the domain of $\mathcal{A}$. Let $\mathcal{P} = \{\text{polynomial functions on } \mathbb{R}\}$. Then, the next lemma follows.

**Lemma 3.7** $\mathcal{P} \subset \text{Dom}(\mathcal{A})$. 

\[\require{cancel}\]
From Lemma 3.7 and the definition of \( \mathcal{A}, \mathcal{A}^k p \in \text{Dom}(\mathcal{A}) \) holds for any \( p \in \mathcal{P} \) and \( k \in \mathbb{N} \). Therefore, for any \( p \in \mathcal{P} \), an inductive calculation yields

\[
E \left[ p(\lambda_s^x) - p(\lambda_s^x \mid \mathcal{F}_s^x) \right] = E \left[ \int_{(s,t]} \mathcal{A} p(\lambda_{u_1}^x) du \bigg| \mathcal{F}_s^x \right] = \int_{(s,t]} E \left[ \mathcal{A} p(\lambda_{u_1}^x) \bigg| \mathcal{F}_s^x \right] du_1 \\
= (t-s) \mathcal{A} p(\lambda_s^x) + \int_{(s,t]} E \left[ \mathcal{A} p(\lambda_{u_1}^x) - \mathcal{A} p(\lambda_s^x) \bigg| \mathcal{F}_s^x \right] du_1 \\
= (t-s) \mathcal{A} p(\lambda_s^x) + \int_{(s,t]} E \left[ \int_{(s,u_1]} \mathcal{A}^2 p(\lambda_{u_2}^x) du_2 \bigg| \mathcal{F}_s^x \right] du_1 \\
= (t-s) \mathcal{A} p(\lambda_s^x) + \int_{(s,t]} \int_{(s,u_1]} E \left[ \mathcal{A}^2 p(\lambda_{u_2}^x) \bigg| \mathcal{F}_s^x \right] du_2 du_1 \\
= \cdots = \sum_{k=1}^{n-1} \frac{(t-s)^k}{k!} \mathcal{A}^k p(\lambda_s^x) \\
+ \int_{(s,t]} \int_{(s,u_1]} \cdots \int_{(s,u_{n-1}]} E \left[ \mathcal{A}^n p(\lambda_{u_n}^x) \bigg| \mathcal{F}_s^x \right] du_n \cdots du_2 du_1,
\]

where \( \mathcal{A}^k p(\lambda_s^x) \) means \( \mathcal{A}^k p(y)|_{y=\lambda_s^x} \). Furthermore, the remainder term converges to 0 in \( L^1 \)-sense, that is,

**Lemma 3.8** For any \( p \in \mathcal{P} \),

\[
\int_{(s,t]} \int_{(s,u_1]} \cdots \int_{(s,u_{n-1}]} E \left[ \mathcal{A}^n p(\lambda_{u_n}^x) \bigg| \mathcal{F}_s^x \right] du_n \cdots du_2 du_1 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad \text{in} \quad L^1.
\]

Set the operator \( e^{(t-s)\mathcal{A}} f(\lambda_s^x) = \sum_{k=0}^{\infty} \frac{(t-s)^k}{k!} \mathcal{A}^k f(\lambda_s^x) \) for a function \( f \). From Lemma 3.8, we have \( E[p(\lambda_s^x) \mid \mathcal{F}_s^x] = e^{(t-s)\mathcal{A}} p(\lambda_s^x) \) a.s. for any \( p \in \mathcal{P} \). Then, since \( e^{(t-s)\mathcal{A}} p(\lambda_s^x) \) is \( \sigma(\lambda_s^x) \)-measurable, we immediately get the Markovian property only for \( p \in \mathcal{P} \), i.e.

\[
E[p(\lambda_s^x) \mid \mathcal{F}_s^x] = E[p(\lambda_t^x) \mid \lambda_s^x] \quad \text{a.s.} \tag{3.3}
\]

For the sake of Proposition 3.5, the Eq. (3.3) can be extended to the Markovian property for any bounded function. We summarize this statement as the following theorem.

**Theorem 3.9** For any bounded measurable function \( f \), \( E[f(\lambda_t^x) \mid \mathcal{F}_s^x] = E[f(\lambda_t^x) \mid \lambda_s^x] \) a.s.

### 3.3 Markovian property and ergodicity of Hawkes core process

To consider the asymptotic expansion for the MLE of the Hawkes process, we have to deal with the derivatives of the log-likelihood process of the Hawkes process with respect to its parameters. Moreover, these derivatives are represented as functionals of the derivatives of the Hawkes intensity process with respect to time. First, we introduce the concept of the Hawkes core process. For \( x_1 \in \mathbb{R}_+ \), let \( N_t^{x_1} \) be an exponential Hawkes process with intensity \( \lambda_t^{x_1} = \mu + x_1 e^{-\beta t} + \int_{(0,t]} \alpha e^{-\beta(t-u)} dN_u^{x_1} \) defined as in Definition 3.1.

**Definition 3.10** For \( n \in \mathbb{N} \) and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we define

\[
X_t^{x,n} = e^{-\beta t} \sum_{k=1}^{n} \left( \frac{n-1}{k-1} \right) x_k t^{n-k} + \int_{(0,t]} \alpha(t-u)^{n-1} e^{-\beta(t-u)} dN_u^{x_1}.
\]

We call \( X_t^{x,n} \) as the \( n \)-th Hawkes core process of \( N_t^{x_1} \).
Remark that \( X_t^{x,(n)} \) is obviously \( \sigma(X_u^{x_1,(1)}; u \leq t) \)-measurable for any \( n \in \mathbb{N} \). However, \( X_t^{x,(n)} - X_t^{x_0,(n)} \) is not \( \sigma(X_s^{x_1,(1)}; s < u \leq t) \)-measurable for \( 0 \leq s < t \) and \( n \geq 2 \). When we consider the second order derivative of the Hawkes intensity \( \lambda_t^{x_i} \) with respect to their parameters, the following process is essential:

\[
X_t^x = \left( X_t^{x,(1)}, X_t^{x,(2)}, X_t^{x,(3)} \right)
\]  

(3.4)

where \( x = (x_1, x_2, x_3) \in \mathbb{R}_+^3 \). The detailed reason why we consider \( X_t^x \) is explained in Sect. 4. Here, we reveal the properties of \( X_t^x \). In this subsection, we deal only with the overview. The detail of proofs can be found in “Appendix”.

We can also deduce the Markovian property of the process \( X_t^x \).

**Proposition 3.11** For any bounded measurable function \( f \), \( E[f(X_t^x)|\mathcal{F}_s^x] = E[f(X_t^x)|X_s^x] \) a.s. where \( \mathcal{F}_t^x = \sigma(X_s^x; s \leq t) \).

Second, we see the time-homogeneous Markovian property. We define the Markov kernel as below. For \( x \in \mathbb{R}_+^3 \) and \( A \in \mathcal{B}(\mathbb{R}_+^3) \),

\[
P^t(x, A) = P[X_t^x \in A].
\]

(3.5)

For this Markov kernel, the time-homogeneous property holds, i.e.;

**Proposition 3.12** For any bounded measurable function \( f \),

\[
E[f(X_t^x)|\mathcal{F}_s^x] = \int_{\mathbb{R}_+^3} f(y) P^{t-s}(X_s^x, dy) \quad \text{a.s.}
\]

We concretely give the invariant measure of \( X_t^x \) under our settings. On some probability space \((\Omega, \mathcal{F}, \bar{P})\), there exists a stationary multivariate Hawkes process \((\tilde{N} = (\tilde{N}_1, \tilde{N}_2, \tilde{N}_3), \mathcal{F}_{t}^{\tilde{N}})\) with the \( \mathcal{F}_{t}^{\tilde{N}} \)-intensity \( \tilde{\lambda}_t = (\tilde{\lambda}_t^{(1)}, \tilde{\lambda}_t^{(2)}, \tilde{\lambda}_t^{(3)}) \) such that

\[
\tilde{\lambda}_t^{(1)} = \mu + \int_{(-\infty, t)} \alpha e^{-\beta(t-s)} d\tilde{N}_1^{(1)}, \quad \tilde{\lambda}_t^{(2)} = \mu + \int_{(-\infty, t)} \alpha(t-s)e^{-\beta(t-s)} d\tilde{N}_1^{(1)},
\]

and

\[
\tilde{\lambda}_t^{(3)} = \mu + \int_{(-\infty, t)} \alpha(t-s)^2 e^{-\beta(t-s)} d\tilde{N}_1^{(1)},
\]

where \( \mathcal{F}_t^{\tilde{N}} = \bigvee_{i=1}^3 \mathcal{F}_t^{\tilde{N}_i} \) and \( \mathcal{F}_t^{\tilde{N}_i} = \sigma(\tilde{N}_i, C); C \in \mathcal{B}(\mathbb{R}), C \subset (-\infty, t) \), see Theorem 7 in Brémaud and Massoulié (1996). We write \( \tilde{N} = \tilde{N}_1^{(1)} \), allowing the abuse of the notation. Then, the following process \( \tilde{X} \) is stationary:

\[
\tilde{X}_t = \left( \int_{(-\infty, t)} \alpha e^{-\beta(t-u)} d\tilde{N}_u, \int_{(-\infty, t)} \alpha(t-u)e^{-\beta(t-u)} d\tilde{N}_u, \int_{(-\infty, t)} \alpha(t-u)^2 e^{-\beta(t-u)} d\tilde{N}_u \right).
\]

Denote the distribution of \( \tilde{X}_t \) by \( P^{\tilde{X}} \).

**Proposition 3.13** \( P^{\tilde{X}} \) is the invariant probability measure for \( X_t^x \), i.e. for any \( t \geq 0 \) and \( A \in \mathcal{B}(\mathbb{R}_+^3) \),

\[
P^{\tilde{X}}[A] = \int_{\mathbb{R}_+^3} P^t(x, A) P^{\tilde{X}}(dx).
\]

\(^3 \) \( X_t^{x,(1)} \) and \( X_t^{x,(2)} \) mean \( X_t^{x_1,(1)} \) and \( X_t^{x_1,x_2,(2)} \), respectively.

\(^4 \) An abusive use of “\( P^t \)”: \( P^{\tilde{X}}[A] = P[\tilde{X} \in A] \) for \( A \in \mathcal{B}(\mathbb{R}_+^3) \).
Furthermore, $X_t^x$ has a strong finiteness of moments same as $\lambda_t^x$.

**Proposition 3.14** There exists a positive constant vector $M = (M_1, M_2, M_3)$ such that

$$\sup_{t \in \mathbb{R}_+} E \left[ e^{M X_t^x} \right] < \infty.$$ 

Finally, similarly to Proposition 4.5 in Clinet and Yoshida (2017), it is ensured that $X_t^x$ has the geometric ergodicity in the following meaning.

**Proposition 3.15** There exist a positive constant vector $M = (M_1, M_2, M_3)$ and positive constants $B > 0$ and $r \in (0, 1)$ such that

$$\|P^t(x, \cdot) - P^x\|_M \leq B(e^{Mx} + 1)r^t.$$ 

Here, for a measurable function $V \geq 1$, $\| \cdot \|_V$ designates the $V$-variation norm, i.e. for any signed measure $\mu$ on a measurable space $(S, \mathcal{F})$,

$$\|\mu\|_V = \sup_{\psi : |\psi| \leq V} \left| \int_S \psi(x) \mu(dx) \right|.$$ 

## 4 Edgeworth expansion for functionals related to the Hawkes process

As mentioned in Introduction, both computation of the maximum likelihood estimator (MLE) and simulation methods for the one-dimensional exponential Hawkes process was revealed in Ogata (1979). Furthermore, it was proved that the quasi maximum likelihood estimator (QMLE) of the multi-dimensional exponential Hawkes process has the asymptotic normality and the convergence of moments, see Clinet and Yoshida (2017). In this section, we apply Theorem 2.4 to the MLE of the one-dimensional exponential Hawkes process and we will give the second order asymptotic expansion for the distribution of the MLE. Proofs of each statement are given in “Appendix”. First, we prepare the necessary notation and establish the conditions.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. As in Definition 3.1, let $N_t$ be an exponential Hawkes process with the $\mathcal{F}_t = \sigma(N_s; s \leq t)$-predictable intensity

$$\lambda_t = \mu_0 + \int_{(0,t)} \alpha_0 e^{-\beta_0(t-s)} dN_s.$$ 

Moreover, we define parametrized intensity process by

$$\lambda_t(\theta) = \mu + \int_{(0,t)} \alpha e^{-\beta(t-s)} dN_s, \quad \text{for } \theta = (\mu, \alpha, \beta).$$ 

Furthermore, we define $X_t(\theta)$ by referring (3.4):

$$X_t(\theta) = \left( \int_{(0,t)} \alpha e^{-\beta(u-s)} dN_u, \int_{(0,t)} \alpha(t-u) e^{-\beta(t-s)} dN_u, \int_{(0,t)} \alpha(t-u)^2 e^{-\beta(t-s)} dN_u \right)'$$

for $\theta = (\mu, \alpha, \beta)$. We consider a relatively compact and open parameter set $\Theta \subset \mathbb{R}^3_+$. Assume that $\theta_0 = (\mu_0, \alpha_0, \beta_0) \in \Theta$ is the true parameter. If there is no confusion, we often omit the true parameter, i.e. write $\lambda_t(\theta_0) = \lambda_t$, $X_t(\theta_0) = X_t$ and so on. The log-likelihood process of $\lambda_t(\theta)$ is defined by

$$l_T(\theta) = \int_0^T \log(\lambda_s(\theta)) dN_s - \int_0^T \lambda_s(\theta) ds.$$
for $\theta \in \Theta$, see Liptser and Shiryaev (2000), Theorem 5.45 in chapter III. Since Lemma A.5 in Clinet and Yoshida (2017) guarantees a verification of the permutation of the symbols $\partial_{\theta}$ and $\int_0^T$, the derivative of the log-likelihood process with respect to their parameter can be calculated as below.

$$
\partial_{\theta} l_T(\theta)|_{\theta=\theta_0} = \int_0^T \frac{\partial_{\theta} \lambda_s}{\lambda_s} d\tilde{N}_s,
$$

(4.1)

where $\tilde{N}_t = N_t - \int_0^t \lambda_s ds$. Moreover,

$$
\partial^2_{\theta} l_T(\theta)|_{\theta=\theta_0} = \int_0^T \frac{\partial^2_{\theta} \lambda_s - (\partial_{\theta} \lambda_s)^{\otimes 2}}{\lambda_s^2} d\tilde{N}_s - \int_0^T (\partial_{\theta} \lambda_s)^{\otimes 2} ds,
$$

where, for a vector $x \in \mathbb{R}^k$, $x^{\otimes 2}$ stands for the product of $x$ and its transpose, i.e. $x^{\otimes 2} = xx' \in \mathbb{R}^{k \times k}$. Let the operator $\mathcal{D}$ be as in (2.9), for example, $\mathcal{D}^2 = \frac{\partial}{\partial \sigma}$, $\mathcal{D}^{(1,2)} = \frac{\partial^2}{\partial \mu \partial \sigma}$, etc.

Note that, when we write $X_t = (X_t^{(1)}, X_t^{(2)}, X_t^{(3)})$, then $\partial_{\theta} \lambda_s$ and $\partial^2_{\theta} \lambda_s$ are computed as

$$
\partial_{\theta} \lambda_s = \begin{pmatrix} \mathcal{D}^1 \lambda_s \\ \mathcal{D}^2 \lambda_s \\ \mathcal{D}^3 \lambda_s \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha_0} X_t^{(1)} \\ -X_t^{(2)} \\ -X_t^{(3)} \end{pmatrix}
$$

and

$$
\partial^2_{\theta} \lambda_s = \begin{pmatrix} \mathcal{D}^{(i,j)} \lambda_s \end{pmatrix}_{i,j=1,2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha_0^{-1} X_t^{(2)} \\ 0 & -\alpha_0^{-1} X_t^{(2)} \\ -\alpha_0^{-1} X_t^{(2)} \\ -\alpha_0^{-1} X_t^{(2)} \\ -\alpha_0^{-1} X_t^{(2)} \end{pmatrix}
$$

respectively. Corresponding to Sect. 2.2, we introduce some notation. Let $l_{a_1 \ldots a_k}(\theta) = \mathcal{D}^{(a_1 \ldots a_k)} l_T(\theta)$ and $\nu_{a_1 \ldots a_k}(\theta) = E \left[ \frac{1}{T} l_{a_1 \ldots a_k}(\theta) \right]$ for integers $a_1, \ldots, a_k$. We have the following representations:

$$
\partial_{\theta} l_T = (l_1, l_2, l_3)' = \left( \int_0^T \frac{1}{\lambda_s} d\tilde{N}_s, \int_0^T X^{(1)}_s \alpha_0 \lambda_s d\tilde{N}_s, -\int_0^T X^{(2)}_s \lambda_s d\tilde{N}_s \right)'
$$

and

$$
\partial^2_{\theta} l_T = (l_{ij})_{i,j=1,2,3}, \text{ where } l_{ij} \text{ is symmetric with respect to } i,j \text{ and each component has the following representation:}
$$

$$
l_{11} = -\left( \int_0^T \frac{1}{\lambda_s^2} d\tilde{N}_s + \int_0^T \frac{1}{\lambda_s} ds \right), \quad l_{12} = -\frac{1}{\alpha_0} \left( \int_0^T \frac{X^{(1)}_s}{\lambda_s^2} d\tilde{N}_s + \int_0^T \frac{X^{(1)}_s}{\lambda_s} ds \right),
$$

$$
l_{13} = \int_0^T \frac{X^{(2)}_s}{\lambda_s^2} d\tilde{N}_s + \int_0^T \frac{X^{(2)}_s}{\lambda_s} ds, \quad l_{22} = -\frac{1}{\alpha_0^2} \left( \int_0^T \frac{(X^{(1)}_s)^2}{\lambda_s^2} d\tilde{N}_s + \int_0^T \frac{(X^{(1)}_s)^2}{\lambda_s} ds \right),
$$

$$
l_{23} = \frac{1}{\alpha_0} \left( \int_0^T \frac{(X^{(1)}_s - X^{(2)}_s)^2}{\lambda_s^2} d\tilde{N}_s + \int_0^T \frac{(X^{(1)}_s - X^{(2)}_s)^2}{\lambda_s} ds \right), \quad \text{and}
$$

$$
l_{33} = \int_0^T \frac{X^{(3)}_s - (X^{(2)}_s)^2}{\lambda_s^2} d\tilde{N}_s - \int_0^T \frac{(X^{(2)}_s)^2}{\lambda_s} ds.
$$

Put $\mathcal{B}_I = \bigcap_{\varepsilon>0} \sigma(X_u; u \in [s, t+\varepsilon]) \vee \mathcal{N}$ for $I = [s, t] \subset \mathbb{R}_+$, where $\mathcal{N}$ is the $\sigma$-field generated by null sets in $\mathcal{F}$. Let

$$
Z_a = \frac{1}{\sqrt{T}} l_a \quad \text{and} \quad Z_{ab} = \sqrt{T} \left( \frac{1}{T} l_{ab} - \nu_{ab} \right).
$$
Moreover, we write
\[ Z_T^{(1)} = T^{\frac{1}{2}}(Z_1, Z_2, Z_3) \quad \text{and} \quad Z_T^{(2)} = T^{\frac{1}{2}}(Z_{11}, Z_{12}, Z_{13}, Z_{21}, Z_{22}, Z_{23}, Z_{31}, Z_{32}, Z_{33}). \]

Finally, we set \( Z_T = (Z_T^{(1)}, Z_T^{(2)}) \). Then, \( Z_T - Z_s \) is \( \mathcal{B}_I \)-measurable for every \( s, t \in \mathbb{R}_+, 0 \leq s \leq t \) and \( Z_0 \in \mathcal{F} \mathcal{B}_I \). About the definition of \( \mathcal{B}_I \), note the following points.

**Remark 4.1** Obviously, \( \sigma(X^{(s)}_t; s \in I) = \sigma(\lambda_t; s \in I) \) holds for any interval \( I \).

**Remark 4.2** For any \( s \geq 0, \sigma(X_s; s \in [0, t]) \subset \sigma(\lambda_t; s \in [0, t]) \) holds. However, for a general interval \( I \) and \( s \in I, X^{(2)}_s \) and \( X^{(3)}_s \) are not always measurable with respect to \( \sigma(\lambda_t; t \in I) \). Thus, if we consider the expansion of the distribution of \( Z_T \), we have to extend \( \sigma(\lambda_t; t \in I) \). In this reason, we introduced the process \( X_t \) and defined \( \mathcal{F}_I \) as above.

**Remark 4.3** \( \sigma(X_t; t \in [u, v]) \), in particular \( \sigma(X^{(s)}_t; t \in [u, v]) \), has almost all the information of \( \sigma(N_t - N_s; s, t, \in [u, v]) \). However, the information of the jump at \( v \) is not contained in \( \sigma(X_t; t \in [u, v]) \). Therefore, we need to consider the right-continuous \( \sigma \)-fields.

A functional of the process \( X_t \) has the geometric mixing property.

**Proposition 4.4** \( \mathcal{B}_I \) satisfies the condition [A1].

From Proposition 4.4 and Theorem 2.1, we immediately obtain the asymptotic expansion for the distribution of a functional of the Hawkes core process.

**Theorem 4.5** Let \( p \in \mathbb{N} \) with \( p \geq 2 \) and \( L_1, L_2 > 0 \). Assume that a \( \mathcal{B}_I \)-adapted stochastic process \( Y_T \) satisfies the condition [A2]. Then, there exist \( D > 0 \) and \( \Gamma \in \mathbb{N} \) such that for any \( f \in \mathcal{E}(\Gamma, L_1, L_2) \),

\[ \left| E \left[ f \left( \frac{Y_T}{\sqrt{T}} \right) \right] - \int_{\mathbb{R}^d} f(z)p_{T,p,D}(z)dz \right| = o \left( T^{-(p-2)/2} \right), \]

where \( p_{T,p,D}(z) \) is defined as in (2.8) with replaced \( Z_T \) by \( Y_T \).

We may also apply Theorem 2.4. Write \( g_T = (g_{ab})_{a,b=1,2,3} = (-v_{ab}(\theta_0))_{a,b=1,2,3} \). As proved in “Appendix”, the exponential Hawkes process satisfies the condition [A3]. Thus, we can also define \( g_T^{-1} = (g_{ab}^{-1})_{a,b=1,2,3} \). The following statement is the main theorem of this article. (For the definition of each symbol, see Sect. 2.)

**Theorem 4.6** Let \( L_1, L_2 > 0 \). The conditions [A1]–[A3] and [C1] are satisfied. Moreover, there exist \( L > 1, \gamma \in (0, 1) \) and \( q_1, q_2, q_3 > 0 \) such that [B0]–[B4] with (2.15) hold. Thus, there exists \( \Gamma \in \mathbb{N} \) such that for any \( f \in \mathcal{E}(\Gamma, L_1, L_2) \),

\[ \left| E \left[ f \left( \sqrt{T}(\hat{\Theta}_T - \theta_0) \right) \right] - \int_{\mathbb{R}^3} f(z^{(1)})q_{T,3}(z^{(1)})dz^{(1)} \right| = o \left( T^{-\frac{1}{2}} \right), \]

where

\[ q_{T,3}(z^{(1)}) = \phi(z^{(1)}; g_T^{-1}) + \frac{1}{\sqrt{T}} \left\{ \frac{1}{6}z^{a_1a_2z_3} + \mu^{a_3; g_{b_1b_2}} h_{a_1a_2a_3} (z^{(1)}; g_T^{-1}) \right\} \phi(z^{(1)}; g_T^{-1}), \]

and
− \(h_{a_1a_2a_3}(z^{(1)}; g_T^{-1})\) and \(h_{a_1}(z^{(1)}; g_T^{-1})\) are the Hermite polynomials defined by (2.7);

− \(\tilde{\kappa}_T^{a_1a_2a_3} = T^{1/2} \tilde{\kappa}_T^{a_1a_2a_3}\); and \(\tilde{\lambda}_T^{a_1a_2a_3}\) is the \((a_1, a_2, a_3)\)-cumulant of \(g_T^{-1}Z_T^{(1)}\);

− \(\hat{\mu}_{a_1a_2} = (V_{a_1a_2} + V_{a_2a_1} + V_{a_1a_2})/2\) and \(V_{bc} = \text{Cov}[Z_b; Z_a]\) for an index sequence \(A\).

Corollary 2.5 leads the next result for an observed information \(\hat{g}_T = (-\frac{1}{T} l_{ab}(\hat{\theta}_T))_{a,b=1,2,3}\).

Write \(\frac{1}{\hat{g}_T} = (\sqrt{g}_{ab})_{a,b=1,2,3} = (\sqrt{g}^{ab})^{-1}_{a,b=1,2,3}\) and let \(\hat{Z}_{a,A} = \sqrt{g}_{aa_1}Z_{a_1}^{a_1}\) for an index sequence \(A\).

**Corollary 4.7** There exists \(\Gamma \in \mathbb{N}\) such that for any \(f \in \mathcal{E}(\Gamma, L_1, L_2)\),

\[
\left| E \left[ f(\sqrt{T} \frac{1}{\hat{g}_T}(\hat{\theta}_T - \theta_0)) \right] - \int_{\mathbb{R}^3} f(z^{(1)}) \tilde{q}_{T,3}(z^{(1)}) dz^{(1)} \right| = o\left(T^{-\frac{1}{2}}\right),
\]

where

\[
\tilde{q}_{T,3}(z^{(1)}) = \phi(z^{(1)}) + \frac{1}{\sqrt{T}} \left\{ \left( \frac{1}{6} \tilde{\kappa}_T^{a_1a_2a_3} + \tilde{\mu}_{a_1a_2a_3} \right) h_{a_1a_2a_3}(z^{(1)}) + \lambda_{a_1a_2a_3} b_{a_1a_2a_3}^a h_{a_1b_1b_2} l_{a_1b_1b_2} h_{a_1}(z^{(1)}) \right\} \phi(z^{(1)}),
\]

and

− \(h_{a_1a_2a_3}(z^{(1)})\) and \(h_{a_1}(z^{(1)})\) are the Hermite polynomials defined by (2.7);

− \(\tilde{\kappa}_T^{a_1a_2a_3} = T^{1/2} \tilde{\kappa}_T^{a_1a_2a_3}\); and \(\tilde{\lambda}_T^{a_1a_2a_3}\) is the \((a_1, a_2, a_3)\)-cumulant of \(g_T^{-1}Z_T^{(1)}\);

− \(\hat{\mu}_{a_1a_2a_3} = \sqrt{g}^{a_1} \text{Cov}[\hat{Z}_{a_1,b}, \hat{Z}_{a_1}]/2\).

**5 Simulation**

In this section, we show the result of a simulation for Corollary 4.7. We need to compute \(\tilde{\kappa}_T^{a_1a_2a_3}\) and \(\hat{\mu}_{a_1a_2a_3}\). However, it is difficult to get their expressions for the parameter. Here, we approximate these values numerically using the Monte Carlo method. Moreover, considering the situation that we do not know the true parameters, we compute coefficients using the plug-in distribution of an MLE, see Remark 2.6. By using the algorithm in Ogata (1981), we simulate the values of the Hawkes process for \(MC\) times. From these data, we can get \(MC\) number of values for \(Z_T\). From these data, we can get the unbiased estimator of \(\text{Var}[Z_T]\). With the help of the condition [B0] (iii), we can compute the value of \(\hat{g}_T\) from \(\text{Var}[Z_T]\). Then we get an approximated value of \(\hat{\mu}_{a_1a_2a_3}\). From the representation of cumulants by moment, \(\tilde{\kappa}_T^{a_1a_2a_3}\) is computed from means of \(\hat{Z}_{a_1}, \hat{Z}_{a_2}, \hat{Z}_{a_1}\). All experiments are done by using R. The code can be found on GitHub page https://github.com/goda235/Edgeworth_expansion_for_Hawkes_MLE.

We set an exponential Hawkes process \(N_t\) with its parameters \(\mu = 0.3, \alpha = 1.0\) and \(\beta = 1.4\), i.e. its intensity \(\lambda_t\) has the representation

\[\lambda_t = 0.3 + \int_{(0,t)} 1.0e^{-1.4(t-s)} dN_s.\]

We set an observation time \(T = 100\). For this model, we compute MLEs for 1000 times and obtain histograms of \(\sqrt{T} \frac{1}{\hat{g}_T}(\hat{\theta}_T - \theta_0)\) where \(\hat{g}_T^{1/2}\) is an observed information at \(\hat{\theta}_T\). In addition, we add the density function curves for the marginal distributions of \(\phi(z)\) and \(\tilde{q}_{T,3}(z)\). Here, \(\tilde{q}_{T,3}(z)\) is computed by the above method with \(MC = 1000\). The curve of \(\phi(z)\) is described by a broken black line, and \(\tilde{q}_{T,3}(z)\) is described by a solid red line. The simulation results are as follows (Figs. 1 and 2).
We can see that the curve of $\tilde{q}_{T,3}(z)$ fits the histogram better than the normal distribution. The bootstrap approximation, that is using an empirical distribution derived from the Monte Carlo method, is a popular method to approximate the error distribution of an estimator. In many cases, it is known that a Studentized bootstrap method also approximates the error distribution of an estimator with order $o_p(T^{-1/2})$, see Hall (1992). This evaluation is supported by an idea of Edgeworth expansion and thus it is a future problem related to our results. We compare the Edgeworth expansion with the bootstrap approximation. Here, we construct the bootstrap approximation by the 1000 times Monte Carlo computation. The next figure is Q–Q plot for each marginal distribution.
We can see that Edgeworth expansion and the bootstrap approximation are better than the normal approximation. However, the bootstrap method takes more computational costs than Edgeworth expansion because of iterative computation for MLEs.

Acknowledgements I am deeply grateful to Professor Yoshida. Without his guidance and help, this article would not have been completed. This research was supported by FMSP program of The University of Tokyo and Japan Science and Technology Agency CREST JPMJCR14D7.

6 Appendix

Hereafter, when we write as $X(T) \lesssim T^a$ for $a \in \mathbb{R}$, it means that there exist positive constants $C$ and $T'$ such that $X(T) \leq CT^a$ holds for any $T > T'$.

6.1 Proofs of Section 2.1

To prove Theorem 2.1, we should give the asymptotic expansion for the characteristic function of $S_T$. The following discussion is a rework of Götze and Hipp (1983) and Yoshida (2004) to a form allowed when the variance is non-degenerate.

First, we introduce some notations. Let $N(T) = |T| + 1$ and divide the interval $[0, T]$ into intervals $\{I_i\}_{i=0, \ldots, N(T)}$ such that $I_0 = [0, 0]$, $I_i = [i-1, i]$ for $i = 1, \ldots, N(T) - 1$ and $I_{N(T)} = [N(T) - 1, T]$. Denote $Z_{I_i}$ as $Z_i$ for any $i = 0, \ldots, N(T)$. There exists a smooth function $\phi : \mathbb{R}^d \rightarrow [0, 1]$ such that $\phi(x) = 1$ if $|x| \leq 1/2$, and $\phi(x) = 0$ if $|x| \geq 1$. Choose a positive constant $\beta \in (0, 1/2)$ and put $\phi_T(x) = x^{\phi(x/2T^\beta)}$. Then $\phi_T(x) = x$ if $|x| \leq T^\beta$ and $\phi_T(x) = 0$ if $|x| \geq 2T^\beta$. Let $S_i^* = \phi_T(Z_i) - E[\phi_T(Z_i)]$ for any $i = 0, \ldots, N(T)$ and $S_T^* = T^{-\frac{1}{2}} \sum_{i=0}^{N(T)} Z_i^*$. Write the characteristic function of $S_T^*$ by $H_T(u) = E[e^{iu^\top S_T^*}]$ for $u \in \mathbb{R}^d$. For random variables $X$ and $V$, we define $E[X](V) = E[Xe^{iV}]/E[e^{iV}]$. Define the cumulant of real-valued random variables $X_1, \ldots, X_r$ shifted by a random variable $V$ as

$$
\kappa [X_1, \ldots, X_r](V) = \frac{\partial^r}{\partial e_1 \cdots \partial e_r} \log \left( \mathbb{E} \left[ e^{i e_1 X_1 + \cdots + i e_r X_r} \right] \right)_{e_1 = \cdots = e_r = 0}
$$

and write $\kappa [X_1, \ldots, X_r] = \kappa [X_1, \ldots, X_r](0)$. In this subsection, we assume that the conditions [A1] and [A2] hold. By using the mixing property, it is possible to evaluate cumulants as follows. Write the $i$-th element of a vector $X$ as $X^{(i)}$.

Proposition 6.1 Let $\bar{L} > 0$. Set $r \in \mathbb{N}$ with $r \leq \bar{L}$ and $a_1, \ldots, a_r \in \{1, \ldots, d\}$. Then, for any $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$
1_{\{|u| < T^\delta\}}(u) \left| \kappa \left( S_T^{a_1}, \ldots, S_T^{a_r} \right) \right| \leq T^{-\frac{r}{2} + \varepsilon(r-1)} \quad \text{uniformly in } u \in \mathbb{R}^d \text{ and } \eta \in [0, 1].
$$

Proof It follows in a similar way as the proof of Lemma 5 in Yoshida (2004).

The next proposition is similar to Lemma 6 in Yoshida (2004). However, our assumption [A2] is stronger than the assumption in Yoshida (2004), thus we may take an arbitrary $L_3 > 0$ as the following.

\footnote{We must remark that the notation of $Z_i$ has a different mean in the other section.}
Proposition 6.2 For any $L_3 > 0$, $r \in \mathbb{N}$ and $a_1, \ldots, a_r \in \{1, \ldots, d\}$,

$$\kappa \left[ S^*(a_1), \ldots, S^*(a_r) \right] - \kappa \left[ S_T(a_1), \ldots, S_T(a_r) \right] \lesssim T^{-L_3 \beta}.$$ 

Proof We immediately get

$$\kappa \left[ S_T^*(a_1), \ldots, S_T^*(a_r) \right] - \kappa \left[ S_T(a_1), \ldots, S_T(a_r) \right]$$

$$\leq T^{-r/2} \sum_{0 \leq j_1, \ldots, j_r \leq N(T)} \kappa \left[ Z_{j_1}^*(a_1), \ldots, Z_{j_r}^*(a_r) \right] - \kappa \left[ Z_{j_1}(a_1), \ldots, Z_{j_r}(a_r) \right]$$

$$\leq T^{-r/2} \sum_{0 \leq j_1, \ldots, j_r \leq N(T)} \sum_{l=1}^{r} \sum_{a_i, \ldots, a_i; a_1 + \cdots + a_r = 1, \ldots, r} \left( -1 \right)^{l-1} \frac{1}{l} \prod_{m=1}^{l} E \left[ \prod_{i \in a_m} \phi_T^{(a_i)}(Z_{ji}) \right]$$

$$- \prod_{m=1}^{l} E \left[ \prod_{i \in a_m} Z_{ji}^{(a_i)} \right].$$

Moreover,

$$\left| \prod_{m=1}^{l} E \left[ \prod_{i \in a_m} \phi_T^{(a_i)}(Z_{ji}) \right] - \prod_{m=1}^{l} E \left[ \prod_{i \in a_m} Z_{ji}^{(a_i)} \right] \right|$$

$$\leq \sum_{m=1}^{l} \left( \prod_{m'=1}^{m-1} E \left[ \prod_{i \in a_{m'}} \phi_T^{(a_i)}(Z_{ji}) \right] \right) \left| E \left[ \prod_{i \in a_m} \phi_T^{(a_i)}(Z_{ji}) - \prod_{i \in a_m} Z_{ji}^{(a_i)} \right] \right|$$

$$\leq \left( \prod_{m'=m+1}^{l} E \left[ \prod_{i \in a_{m'}} Z_{ji}^{(a_i)} \right] \right),$$

and the condition [A2] yields

$$E \left[ \prod_{i \in a_m} \phi_T^{(a_i)}(Z_{ji}) - \prod_{i \in a_m} Z_{ji}^{(a_i)} \right]$$

$$= \sum_{k'}^{k} E \left[ \left( \prod_{i=1}^{k'-1} \phi_T^{(a_i)}(Z_{ji}) \right) \left( \prod_{i=k'+1}^{k} Z_{ji}^{(a_i)} \right) \left( \phi_T^{(a_{k'})}(Z_{j_{k'}}) - Z_{j_{k'}}^{(a_{k'})} \right) \prod_{i=k'+1}^{k} Z_{ji}^{(a_i)} \right]$$

$$\leq T^{-L_3 \beta} E \left[ \left( \prod_{i=1}^{k'-1} \phi_T^{(a_i)}(Z_{ji}) \right) \left( \prod_{i=k'+1}^{k} Z_{ji}^{(a_i)} \right) \right] \left( \phi_T^{(a_{k'})}(Z_{j_{k'}}) - Z_{j_{k'}}^{(a_{k'})} \right) \left| Z_{j_{k'}}^{(a_{k'})} \right| \left| Z_{j_{k'}}^{(a_{k'})} \right| \lesssim T^{-L_3 \beta}. $$

Therefore,

$$\kappa \left[ S_T^*(a_1), \ldots, S_T^*(a_r) \right] - \kappa \left[ S_T(a_1), \ldots, S_T(a_r) \right] \lesssim T^{-L_3 \beta + \tilde{c}}.$$ 

Since $L_3$ is arbitrary, we get the conclusion. \qed
From Propositions 6.1 and 6.2, we immediately get the following statement.

**Corollary 6.3** Let $\tilde{L} > 0$. Set $r \in \mathbb{N}$ with $r \leq \tilde{L}$ and $a_1, \ldots, a_r \in \{1, \ldots, d\}$. Then, for any $\varepsilon > 0$,

$$|\kappa \left[ S_T^{(a_1)}, \ldots, S_T^{(a_r)} \right] | \leq T^{-\frac{r-2}{2} + \varepsilon (r-1)}.$$  

We evaluate the gap between $H_T(u)$ and its expansion $\hat{\Psi}_{T,p,D}(u)$. Allowing for the abuse of symbols, we define $\mathbb{D}$ as the derivative with respect to $u$ in the same way as (2.9).

**Proposition 6.4** Let $\tilde{L} > 0$. There exist $D > 0$, $\delta > 0$ and $\delta_0 > d \delta$ such that

$$1_{\{|u| < T^4\}}(u) \left| \mathbb{D}^n \left( H_T(u) - \hat{\Psi}_{T,p,D}(u) \right) \right| \lesssim T^{-\frac{p^2}{2} - \delta_0}$$

uniformly in $u \in \mathbb{R}^d$ and $n \in \{1, \ldots, d\}$ with $l \leq \tilde{L}$.

**Proof** Denote $\kappa^*[u^\otimes r](V) = \kappa \left[ u^\otimes r \right](V)$, $\kappa u^\otimes r = \kappa [u^\otimes r](0)$ and $\kappa^*[u^\otimes r] = \kappa [u^\otimes r](0)$. We have

$$H_T(u) = \hat{\Psi}_{T,p}(u) + R_{p+1}(u),$$

where

$$\hat{\Psi}_{T,p}(u) = \exp \left( \chi_{T,2}(u) \right)$$

$$\left\{ 1 + \sum_{j=1}^{p} \sum_{r_1, \ldots, r_j=1}^{p-2} 1_{\{r_1+\cdots+r_j \leq p-2\}} (-1)^j \frac{k^*[u^\otimes r_1+2] \cdots k^*[u^\otimes r_j+2]}{j!(r_1+2)! \cdots (r_j+2)!} \right\},$$

$$R_{p+1}(u) = \exp \left( \chi_{T,2}(u) \right)$$

$$\left\{ \sum_{j=1}^{p} \sum_{r_1, \ldots, r_j=1}^{p-2} 1_{\{r_1+\cdots+r_j \geq p-1\}} (-1)^j \frac{k^*[u^\otimes r_1+2] \cdots k^*[u^\otimes r_j+2]}{j!(r_1+2)! \cdots (r_j+2)!} \frac{1}{j!} \frac{1}{j'} \left( \sum_{r=3}^{p} \frac{i^r}{r!} k^*[u^\otimes r] \right)^{j'} \right\}$$

$$+ \sum_{j=1}^{p} \sum_{j'=0}^{j-1} \frac{1}{j!} \left( \sum_{r=3}^{p} \frac{i^r}{r!} k^*[u^\otimes r] \right)^{j'} \left( R_{p+1}(u) \right)^{j-j'}$$

$$+ \left( \sum_{r=3}^{p} \frac{i^r}{r!} k^*[u^\otimes r] + R_{p+1}(u) \right)^{p+1} \frac{1}{p!} \int_0^1 (1-t)^p$$

$$\exp \left( t \sum_{r=3}^{p} \frac{i^r}{r!} k^*[u^\otimes r] + t R_{p+1}(u) \right) dt \right\}$$

and

$$R_{p+1}(u) = \frac{i^{p+1}}{p!} \int_0^1 (1-s)^p \kappa^*[u^\otimes p+1](su^\otimes r) ds - \frac{1}{2} \left( \kappa^*[u^\otimes 2] + \chi_{T,2}(u) \right).$$

First, we consider $\left| \mathbb{D}^n \left( \hat{\Psi}_{T,p}(u) - \hat{\Psi}_{T,p,D}(u) \right) \right|$. From the definition, we have

$$\left| \mathbb{D}^n \left( \hat{\Psi}_{T,p}(u) - \hat{\Psi}_{T,p,D}(u) \right) \right|$$

$$\lesssim \left| \mathbb{D}^n \left( e^{\chi_{T,2}(u)} - e^{-\frac{1}{2}u^\otimes r} \right) \right|$$
\[
\left\{ 1 + \sum_{j=1}^{p} \sum_{r_1, \ldots, r_j = 1}^{p-2} 1_{[r_1 + \ldots + r_j \leq p-2]} \kappa^* [u^{\otimes r_1 + 2}] \cdots \kappa^* [u^{\otimes r_j + 2}] \right\}
\]
\[+
\mathbb{D}^n \left[ e^{-\frac{1}{2} u' \Sigma_T \cdot u} \sum_{j=1}^{p} \sum_{r_1, \ldots, r_j = 1}^{p-2} 1_{[r_1 + \ldots + r_j \leq p-2]} \left( \kappa^* [u^{\otimes r_1 + 2}] \cdots \kappa^* [u^{\otimes r_j + 2}] - \kappa [u^{\otimes r_1 + 2}] \cdots \kappa [u^{\otimes r_j + 2}] \right) \right].
\]

In the following, we assume that \( u \) satisfies \( |u| \leq T^\delta \). We have
\[
\left| e^{\hat{Y}_{T,2}(u)} - e^{-\frac{1}{2} u' \Sigma_T \cdot u} \right| \leq \left| e^{\frac{1}{2} T-D} |u|^2 - 1 \right| \lesssim T^{-D+2\delta}.
\] (6.1)

For the first term, by applying Proposition 6.1, Corollary 6.3 and (6.1), we get
\[
\mathbb{D}^n \left[ \left( e^{\hat{Y}_{T,2}(u)} - e^{-\frac{1}{2} u' \Sigma_T \cdot u} \right) \left\{ 1 + \sum_{j=1}^{p} \sum_{r_1, \ldots, r_j = 1}^{p-2} 1_{[r_1 + \ldots + r_j \leq p-2]} \kappa^* [u^{\otimes r_1 + 2}] \cdots \kappa^* [u^{\otimes r_j + 2}] \right\} \right]
\leq \left( 1 + |\text{Var}[S_T]| + T^{-D} \right)^{2p} T^{-D+\delta (2+2p)}
\sum_{j=1}^{p} \sum_{r_1, \ldots, r_j = 1}^{p-2} 1_{[r_1 + \ldots + r_j \leq p-2]} T^{(r_1 + \ldots + r_j) \left( -\frac{1}{2} + \epsilon + \delta \right) + j (\epsilon + 2\delta)}
\lesssim T^{-D+\delta(p+L)+\delta(2p+2+L)}.
\]

By Proposition 6.2 and Corollary 6.3, the second term is estimated as below;
\[
\mathbb{D}^n \left[ e^{-\frac{1}{2} u' \Sigma_T \cdot u} \sum_{j=1}^{p} \sum_{r_1, \ldots, r_j = 1}^{p-2} 1_{[r_1 + \ldots + r_j \leq p-2]} \left( \kappa^* [u^{\otimes r_1 + 2}] \cdots \kappa^* [u^{\otimes r_j + 2}] - \kappa [u^{\otimes r_1 + 2}] \cdots \kappa [u^{\otimes r_j + 2}] \right) \right]
\leq \mathbb{D}^n \left[ e^{-\frac{1}{2} u' \Sigma_T \cdot u} \sum_{j=1}^{p} \sum_{r_1, \ldots, r_j = 1}^{p-2} 1_{[r_1 + \ldots + r_j \leq p-2]} \times \sum_{k=1}^{j} \kappa^* [u^{\otimes r_1 + 2}] \cdots \kappa^* [u^{\otimes r_{k-1} + 2}] \left( \kappa^* [u^{\otimes r_k + 2}] - \kappa [u^{\otimes r_k + 2}] \right) \kappa [u^{\otimes r_{k+1} + 2}] \cdots \kappa [u^{\otimes r_j + 2}] \right]
\lesssim \left( 1 + |\text{Var}[S_T]| + T^{-D} \right)^{2p} T^{-\frac{5}{2} \alpha T - L \beta + \epsilon p + \delta (3p-2)} \lesssim T^{-L \beta + \epsilon (p+L) + \delta(3p-2+L)}.
\]

Therefore, for any \( \epsilon, \delta \) and \( \delta_0 \), we can choose sufficiently large \( D \) and \( L_3 \) such that
\[
\mathbb{D}^n \left( \hat{Y}_{T,p}^* (u) - \hat{Y}_{T,p,D} (u) \right) \lesssim T^{-D-\delta_0}.
\]

Finally, we have to show that \( \mathbb{D}^n \left( H_T (u) - \hat{Y}_{T,p}^* (u) \right) \right) = \mathbb{D}^n R_{p+1}^* (u) \lesssim T^{-\frac{p+2}{2} - \delta_0} \). However, it follows by the same method as the proof of Lemma 7 in Yoshida (2004). In particular, we can choose \( \delta_0 \) and \( \delta \) with \( \delta_0 > d \delta \) in this proof. \( \Box \)

Referring to G"otze and Hipp (1978), we will prove Theorem 2.1 by using the smoothness of a function \( f \). Let \( S_T' = T^{-\frac{1}{2}} \sum_{i=0}^{N(T)} \phi_T (Z_i) \) and \( e_T = T^{-\frac{1}{2}} \sum_{i=0}^{N(T)} E[\phi_T (Z_i)] \). Note that
\[
|E[\phi_T (Z_i)]| = |E[\phi_T (Z_i) - Z_i]| \leq E[|Z_i|I_{|Z_i| > T^\beta}] \leq T^{-\beta} E[|Z_i|^n + 1]
\]
for any \( i = 0, \ldots, N(T) \) and \( n \in \mathbb{N} \). Therefore, [A2] yields
\[
|e_T| \lesssim T^{-L_4}
\]
for an arbitrarily large constant \( L_4 > 0 \).

**Proof** (Proof of Theorem 2.1) Let \( \Gamma = \lceil \frac{\log \delta}{2\delta} \rceil \) for some \( \delta \in (0, 1) \) and \( f \in \mathcal{E}(\Gamma, L_1, L_2) \).

Since
\[
\left| E[f(S_T)] - \int_{\mathbb{R}^d} f(z)p_{T,p,D}(z)dz \right|
\leq \left| E[f(S_T)] - E[f(S_T')] \right|
+ \left| E[f(S_T')] - \int_{\mathbb{R}^d} f(z + e_T)p_{T,p,D}(z)dz \right|
+ \int_{\mathbb{R}^d} f(z + e_T)p_{T,p,D}(z)dz - \int_{\mathbb{R}^d} f(z)p_{T,p,D}(z)dz =: \Delta_1 + \Delta_2 + \Delta_3,
\]
we only have to estimate \( \Delta_1, \Delta_2 \) and \( \Delta_3 \).

First, we consider \( \Delta_1 \). Let \( \eta > 0 \). We can assume that \( L_1 \) is even by retaking \( L_1 \) and \( L_2 \) that satisfy \( \sup_{|x| \leq \Gamma} |\partial^\alpha f(x)| \leq \mathcal{L}_2 (1 + |x|)^{-L_1} \) for every \( x \in \mathbb{R}^d \). We set \( A = \{ |S_T| \leq T^\eta \} \) and \( B = \{ |S_T'| \leq T^\eta \} \). Similarly to Lemma 3.3 in Götze and Hipp (1983), we get
\[
\Delta_1 \lesssim T^{nL_1} \mathbb{P}[S_T \neq S_T'] + E[|S_T|^{L_1} 1_{A^c}] + E[|S_T'|^{L_1} 1_{B^c}]
\lesssim T^{nL_1} \mathbb{P}[S_T \neq S_T'] + E[|S_T'|^{L_1} 1_{B^c}]
+ E[|S_T|^{L_1} 1_{A^c}]
\]
\[
\lesssim T^{nL_1} \mathbb{P}[S_T \neq S_T'] + E[|S_T'|^{L_1} 1_{B^c}]
+ E[|S_T|^{L_1} 1_{A^c}].
\]

Let \( L > 0 \) be an arbitrary large constant. Since \( \mathbb{P}[S_T \neq S_T'] \leq \sum_{i=0}^{N(T)} \mathbb{P}[|Z_i| > T^\beta] \lesssim T^{-\eta \beta + \frac{1}{2}} \) for any \( n \in \mathbb{N} \), \( T^{nL_1} \mathbb{P}[S_T \neq S_T'] \lesssim T^{-L} \) holds. Since \( |S_T'| \leq |S_T| + |e_T| \), we have
\[
E[|S_T'|^{L_1} 1_{B^c}] \lesssim E\left[|S_T'|^{L_1} 1_{|S_T'| > T^\eta} \right] + |e_T|^{L_1}.
\]

The moment has the representation by cumulants; for any even \( n \in \mathbb{N} \)
\[
E[|S_T'|^n] = \sum_{|\alpha|=n} \sum_{k=1}^n \sum_{\alpha_1, \ldots, \alpha_k} \frac{\alpha!}{k! \alpha_1! \cdots \alpha_k!} \prod_{m=1}^k \kappa_{\alpha_m}[S_T^*],
\]
where \( \kappa_{\alpha_m}[S_T^*] = (-i)^{|\alpha_m|} |\alpha_0^m| \log E[e^{iu\alpha^*}]_{\alpha=0} \) and \( \alpha^* = \alpha^1 \! : \! \cdots \! : \alpha^d \! : \! \) for \( \alpha = (\alpha^1, \ldots, \alpha^d) \in \mathbb{Z}_+^d \). From Proposition 6.1 and the representation of moments by cumulants, \( E[|S_T'|^n] \lesssim T^{en} \) for any even \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). Therefore,
\[
E\left[|S_T|^{L_1} 1_{|S_T| > T^\eta} \right] \leq \left( \frac{T^\eta}{2} \right)^{-n} E\left[|S_T'|^{L_1+n} \right] \lesssim T^{-\eta n + \varepsilon (L_1+n)}.\]

Equations (6.2), (6.3) and the above inequality lead \( E[|S_T'|^{L_1} 1_{B^c}] \lesssim T^{-L} \) by choosing \( \eta > \varepsilon \) and sufficiently large \( n \). Since we took \( L_1 \) as an even number, the representation of moments by cumulants leads
\[
\left| E[|S_T|^{L_1}] - E[|S_T'|^{L_1}] \right|
\lesssim T^{-L}.\]
From the definition of the cumulant, \( \kappa_{\alpha_m} [S_T] = \kappa_{\alpha_m} [S_T^*] \) for \( |\alpha_m| \geq 2 \) and \( \kappa_{\alpha_m} [S'_T] = \varepsilon T \) for \( |\alpha_m| = 1 \). Thus, Propositions 6.1, 6.2, Corollary 6.3 and (6.2) yield

\[
\prod_{m=1}^{k} \kappa_{\alpha_m} [S_T] - \prod_{m=1}^{k} \kappa_{\alpha_m} [S_T'] \leq \prod_{m=1}^{k} \kappa_{\alpha_m} [S_T] - \prod_{m=1}^{k} \kappa_{\alpha_m} [S_T^*] + T^{-L} \leq \prod_{m=1}^{k} \kappa_{\alpha_m} [S_T] - \prod_{m=1}^{k} \kappa_{\alpha_m} [S_T'] + T^{-L} \leq T^{-\frac{1}{2}(|\alpha_1| + \cdots + |\alpha_k| - 2k) + \varepsilon(|\alpha_1| + \cdots + |\alpha_k| - k) - L\beta} + T^{-L} \leq T^{-\frac{p-2}{2}}.
\]

Therefore, we get \( \Delta_1 \lesssim T^{-L} \). Since \( L \) is an arbitrary constant, we get \( \Delta_1 \lesssim T^{-\frac{p-2}{2}} \).

Second, we estimate \( \Delta_2 \). Write \( h_T(z) = h(z + \varepsilon T) \) for any function \( h \) on \( \mathbb{R}^d \). Denote the distribution of \( S_T^* \) as \( dQ_T^* \). Then, we can rewrite

\[
\Delta_2 = \left| \int_{\mathbb{R}^d} f_T(z) d\left(Q_T^* - \Psi_{T,p,D}(z)\right) \right|
\]

For \( z, u \in \mathbb{R}^d \), the Taylor’s theorem yields

\[
f_T(z) = \sum_{\alpha: |\alpha| \leq \Gamma} \frac{\partial^\alpha f_T(z + u)}{\alpha!} (-u)^\alpha + g_T^{-1}(z, u)
\]

where

\[
g_T^{-1}(z, u) = \sum_{\alpha: |\alpha|=\Gamma} (-u|^\alpha|_\Gamma) \int_0^1 \nu^\Gamma \left( \partial^\alpha f_T(z + \nu u) - \partial^\alpha f_T(z + u) \right) d\nu.
\]

Let \( \mathcal{K} \) be a probability measure on \( \mathbb{R}^d \) such that \( \int_{\mathbb{R}^d} |z|^\ell d\mathcal{K}(z) < \infty \) for sufficiently large \( \ell > 0 \) and its Fourier transformation \( \hat{\mathcal{K}}(u) \) satisfies \( \hat{\mathcal{K}}(u) = 0 \) if \( |u| > 1 \). (Such \( \mathcal{K} \) exists. See Theorem 10.1 in Bhattacharya and Rao 1976.) Moreover, let \( d\mathcal{K}_{T,u} = d\mathcal{K}(T^{-\delta} u) \) and \( d\mathcal{K}_{T,\alpha}(u) = u^\alpha d\mathcal{K}_{T}(u) \). We have

\[
\int_{\mathbb{R}^d} f_T(z) d\left(Q_T^* - \Psi_{T,p,D}(z)\right) = \sum_{\alpha: |\alpha| \leq \Gamma} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\partial^\alpha f_T(z + u)}{\alpha!} (-u)^\alpha d\left(Q_T^* - \Psi_{T,p,D}(z)\right) d\mathcal{K}_{T,u}(u)
\]

\[
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} g_T^{-1}(z, u) d\left(Q_T^* - \Psi_{T,p,D}(z)\right) d\mathcal{K}_{T,u}(u)
\]

\[
= \sum_{\alpha: |\alpha| \leq \Gamma} \frac{(-1)^\alpha}{\alpha!} \int_{\mathbb{R}^d} \partial^\alpha f_T(x) d\left(\mathcal{K}_{T,\alpha} \ast \left(Q_T^* - \Psi_{T,p,D}\right)\right)(x)
\]
\[
\sum_{\alpha:|\alpha| \leq \Gamma} (-1)^{\alpha} \int_{\mathbb{R}^d} \partial^{\alpha} f_{eT}(x) d \left( K_{\Gamma, \alpha} \ast \left( Q^*_T - \psi_{T-p, D} \right) \right)(x)
\]

We know that \( \partial^\alpha f_{eT}(x) = \partial^{\alpha} f(x + e_T) \) and \( |e_T| \) is bounded in \( T \). Then, Lemma 11.6 Bhattacharya and Rao (1976) and well-known properties of Fourier transform lead

\[
\sum_{\alpha:|\alpha| \leq \Gamma} \frac{(-1)^{\alpha}}{\alpha!} \int_{\mathbb{R}^d} \partial^{\alpha} f_{eT}(x) d \left( K_{\Gamma, \alpha} \ast \left( Q^*_T - \psi_{T-p, D} \right) \right)(x)
\]

\[
\leq \left( \sup_{|\alpha| \leq \Gamma_x \in \mathbb{R}^d} \frac{|\partial^{\alpha} f(x)|}{1 + |x|^{L_1}} \right) \sum_{\alpha:|\alpha| \leq \Gamma} \frac{(-1)^{\alpha}}{\alpha!} \int_{\mathbb{R}^d} 1 + |x + e_T|^{L_1} d \left( K_{\Gamma, \alpha} \ast \left( Q^*_T - \psi_{T-p, D} \right) \right)(x)
\]

\[
\lesssim \sum_{\alpha:|\alpha| \leq \Gamma} \max_{|\beta| \leq L_1 + d + 1} \int_{\mathbb{R}^d} \left| \partial^{\beta} \left( \hat{K}_{\Gamma, \alpha}(u) \left( H_T(u) - \hat{\psi}_{T-p, D}(u) \right) \right) \right| du.
\]

Since \( \hat{K}_{\Gamma, \alpha}(u) = i^{-|\alpha|} \partial^{\alpha} \hat{K}_T(u) \), we have supp \( \hat{K}_{T, \alpha}(u) \subset \{|u| < T^\delta\} \). Moreover, \( \left| \partial^{\beta} \hat{K}_{\Gamma, \alpha}(u) \right| \leq \int_{\mathbb{R}^d} |\alpha^{\beta_1} \hat{\psi}_{\Gamma, \alpha}(u)| \left| \partial^{\beta_2} (H_T(u) - \hat{\psi}_{T-p, D}(u)) \right| du \)

\[
\lesssim T^{-\frac{p-2}{2} - \delta_0 + d \delta} \lesssim T^{-\frac{p-2}{2}}.
\]

It means that

\[
\left| \sum_{\alpha:|\alpha| \leq \Gamma} \frac{(-1)^{\alpha}}{\alpha!} \int_{\mathbb{R}^d} \partial^{\alpha} f_{eT}(x) d \left( K_{\Gamma, \alpha} \ast \left( Q^*_T - \psi_{T-p, D} \right) \right)(x) \right| \lesssim T^{-\frac{p-2}{2}}.
\]

On the other hand, from Proposition 6.1, \( |\Sigma_{T,D}| \leq T^\varepsilon \) holds for any \( \varepsilon > 0 \). Therefore, \( \int_{\mathbb{R}^d} |z|^{L_1} p_{T-p,D}(z) dz \leq T^{L_5 \varepsilon} \) for some constant \( L_5 > 0 \) which depends on \( p \). Thus, by taking sufficiently small \( \varepsilon \),

\[
\int_{\mathbb{R}^d} \mathbb{G}_T^{-1}(z, u) d \left( Q^*_T - \psi_{T-p, D} \right)(z) dK_T(u)
\]

\[
= \left| \int_{\mathbb{R}^d} \mathbb{E} \left[ \sum_{\alpha:|\alpha| = \Gamma} (-u)^{\alpha} \int_0^1 v^F \left( \partial^{\alpha} f_{eT}(z + vu) - \partial^{\alpha} f_{eT}(z + u) \right) v^F \right] d\mathbb{Q}_T^* \right|
\]

\[
\lesssim (T^{-\delta})^F \left( \sup_{|\alpha| \leq \Gamma_x \in \mathbb{R}^d} \frac{|\partial^{\alpha} f(x)|}{1 + |x|^{L_1}} \right) \int_{\mathbb{R}^d} \int_0^1 u^{\alpha} v^F \left( 1 + |z + T^{-\delta} u + e_T|^{L_1} \right) d\mathbb{Q}_T^* d\mathbb{K}(u)
\]

\[
\lesssim T^{-\frac{p-1}{2}} \int_{\mathbb{R}^d} 1 + |z|^{L_1} d \left( Q^*_T - \psi_{T-p, D} \right)(z)
\]
\[ T^{-\frac{p-1}{2}} \left( E \left[ |S_T^n| L_1 \right] + \int_{\mathbb{R}^d} |z|^{L_1} p_{T,p,D}(z)dz \right) \lesssim T^{-\frac{p-2}{2}}. \]

In conclusion, we get \( \Delta_2 \lesssim T^{-\frac{p-2}{2}} \) from (6.4), (6.5) and the above inequality.

Finally, we consider \( \Delta_3 \). With the help of the mean value theorem, we can deduce as below; for some \( \tau \in (0, 1) \),

\[ \Delta_3 \leq \int_{\mathbb{R}^d} |f(z + e\tau) - f(z)| p_{T,p,D}(z)dz \]

\[ \leq \sum_{|\alpha|=1} \int_{\mathbb{R}^d} \left| \partial^\alpha f(z + \tau e) \right| |e\tau| p_{T,p,D}(z)dz \]

\[ \lesssim |e\tau| \left( \sup_{|\alpha| \leq 1} \left| \partial^\alpha f(x) \right| \right) \int_{\mathbb{R}^d} (1 + |e\tau|^{L_1}) p_{T,p,D}(z)dz \]

\[ \lesssim T^{-L_4 + \frac{L_1}{2}} \lesssim T^{-\frac{p-2}{2}}. \]

Therefore, we get the conclusion. \( \Box \)

### 6.2 Proofs of Section 2.2

Before prove Proposition 2.2, we consider an asymptotic expansion of \( \tilde{Z}_T \). We assume that \( Z_T \) satisfies the conditions \([A1]\) and \([A2]\). Then, from Theorem 2.1, for any \( L_1, L_2 > 0 \), there exist \( D > 0 \) and \( \Gamma \in \mathbb{N} \) such that for any \( f \in \mathcal{E}(\Gamma, L_1, L_2) \),

\[ |E \left[ f \left( \frac{Z_T}{\sqrt{T}} \right) \right] - \int_{\mathbb{R}^d} f(z) p_{T,3,D}(z)dz| = o \left( T^{-1/2} \right), \]

where

\[ p_{T,3,D}(z) = \phi(z; \Sigma_{T,D}) + \frac{1}{6\sqrt{T}} \kappa_{T}^{a_1 a_2 a_3} h_{a_1 a_2 a_3}(z; \Sigma_{T,D}) \phi(z; \Sigma_{T,D}), \]

for the modified cumulant \( \kappa_{T}^{a_1 a_2 a_3} \) and the Hermite polynomial \( h_{a_1 a_2 a_3}(z; \Sigma_{T,D}) \) defined in (2.6) and (2.7) respectively. From the concrete form of \( C_T \) and \( M_{T,D} \), they are clearly non-degenerate and \( f \circ M_{T,D} \circ C_T \in \mathcal{E}(\Gamma, L_1, L_2) \) holds for any \( f \in \mathcal{E}(\Gamma, L_1, L_2) \). Owing to the variable transformation and the multi-linearity of the cumulant, the following inequality is immediately obtained. (See Proposition 7.1 in Sakamoto and Yoshida 2004 for the proof details.)

**Lemma 6.5** Let \( L_1, L_2 > 0 \). Suppose that the conditions \([A1]\)–\([A3]\) and \([B0]\) hold. Then, there exist \( D > 0 \) and \( \Gamma \in \mathbb{N} \) such that for any \( f \in \mathcal{E}(\Gamma, L_1, L_2) \),

\[ |E \left[ f \left( \tilde{Z}_T \right) \right] - \int_{\mathbb{R}^d} f(z) \tilde{p}_{T,3,D}(z)dz| = o \left( T^{-1/2} \right), \]

where \( \tilde{\kappa}_{T}^{a_1 a_2 a_3} \) is the \((a_1, a_2, a_3)\)-cumulant of \( \tilde{Z}_T \), \( \tilde{\kappa}_{T}^{a_1 a_2 a_3} = T^{1/2} \kappa_{T}^{a_1 a_2 a_3} \); and

\[ \tilde{p}_{T,3,D}(z) = \phi(z; \tilde{\Sigma}_{T,D}) + \frac{1}{6\sqrt{T}} \tilde{\kappa}_{T}^{a_1 a_2 a_3} h_{a_1 a_2 a_3}(z; \tilde{\Sigma}_{T,D}) \phi(z; \tilde{\Sigma}_{T,D}). \]
Proof (Proof of Proposition 2.2) It is proved in the same way as the proof of Theorem 5.1 in Sakamoto and Yoshida (2004) for

$$\hat{q}_{T,3,D}(z^{(1)}) = \int_{\mathbb{R}^p^2} \phi(z; \tilde{\Sigma}_{T,D})dz^{(2)} + \frac{1}{\sqrt{T}} \left\{ \int_{\mathbb{R}^p^2} \frac{1}{6} \tilde{\kappa}_T^{a_1a_2a_3} h_{a_1a_2a_3}(z; \tilde{\Sigma}_{T,D})\phi(z; \tilde{\Sigma}_{T,D})dz^{(2)} \right\} - \sum_{a=1,...,p} \frac{\partial}{\partial z^a} \int_{\mathbb{R}^p^2} \tilde{Q}_1^{a_i}(z)\phi(z; \tilde{\Sigma}_{T,D})dz^{(2)} \right\} \tag{Proof of Proposition 2.3}$$

This proof is the almost same as Theorem 6.2. in Sakamoto and Yoshida (2004) for

$$\hat{q}_{T,3,D}(z^{(1)}) = \int_{\mathbb{R}^p^2} \phi(z; \tilde{\Sigma}_{T,D})dz^{(2)} + \frac{1}{\sqrt{T}} \left\{ \int_{\mathbb{R}^p^2} \frac{1}{6} \tilde{\kappa}_T^{a_1a_2a_3} h_{a_1a_2a_3}(z; \tilde{\Sigma}_{T,D})\phi(z; \tilde{\Sigma}_{T,D})dz^{(2)} \right\} - \sum_{a=1,...,p} \frac{\partial}{\partial z^a} \int_{\mathbb{R}^p^2} \tilde{Q}_1^{a_i}(z)\phi(z; \tilde{\Sigma}_{T,D})dz^{(2)} \right\} \tag{Proof of Proposition 2.3}$$

by using the Bhattacharya–Ghosh map and transforming asymptotic expansion in Lemma 6.5. Thus, We only need to consider the form of $\hat{q}_{T,3,D}$. Due to the orthogonalization, it immediately follows that

$$h_A(z; \tilde{\Sigma}_{T,D})\phi(z; \tilde{\Sigma}_{T,D}) = h_A^{(1)}(z^{(1)}; \tilde{\Sigma}_{T,D})\phi(z^{(1)}; \tilde{\Sigma}_{T,D}) + h_A^{(2)}(z^{(2)}; \tilde{\Sigma}_{T,D})\phi(z^{(2)}; \tilde{\Sigma}_{T,D})$$

for $A^{(1)} = A \cap \{1, \ldots, p\}$ and $A^{(2)} = A \cap \{p+1, \ldots, p+p^2\}$. We decompose the polynomial $\tilde{Q}_1^{a_i}(z)$ by the Hermite polynomials. Let

$$\tilde{Q}_1^{a_i}(z) = \pi_1^{a_i}(z^{(1)}) + \pi_1^{a_i}(z^{(1)})h^{a_1}(z^{(2)}; \tilde{\Sigma}_{T,D}),$$

where $h^{a_i}(x; \sigma) = \sigma^{a_1i}; h^{a_1}(x; \sigma)$ for $\sigma = (\sigma^{ab}); a,b=1,\ldots,p$ and $\pi_1^{a_i}(z^{(1)}), \pi_1^{a_i}(z^{(1)})$ are polynomials for $z^{(1)}$. It is well known that the orthogonality of the Hermite polynomial

$$\int h_A(z^{(2)}; \tilde{\Sigma}_{T,D})h_B(z^{(2)}; \tilde{\Sigma}_{T,D})\phi(z^{(2)}; \tilde{\Sigma}_{T,D})dz^{(2)} = \left\{ \begin{array}{ll} A! & \text{if } A = B \\ 0 & \text{otherwise} \end{array} \right.$$ 

This orthogonality gives the following representation of $\pi_1^{a_i}(z^{(1)})$,

$$\pi_1^{a_i}(z^{(1)}) = \int_{\mathbb{R}^p^2} \tilde{Q}_1^{a_i}(z)\phi(z; \tilde{\Sigma}_{T,D})dz^{(2)} = \mu_{a_1a_2}^{a_1b_1}; \tilde{z}^{a_1b_2}; h_{b_1b_2}(z^{(1)}; \tilde{\Sigma}_{T,D}) + \mu_{a_1a_2}^{a_1b_1}; \tilde{z}^{a_1b_2},$$

where we used $h_{b_1b_2}(z^{(1)}; \tilde{\Sigma}_{T,D}) = \tilde{g}_{b_1b_2} z^{a_1}; \tilde{z}^{a_2}; \tilde{g}_{b_1b_2}$. Therefore, we get

$$\sum_{a=1,...,p} \frac{\partial}{\partial z^a} \int_{\mathbb{R}^p^2} \tilde{Q}_1^{a_i}(z)\phi(z; \tilde{\Sigma}_{T,D})dz^{(2)}$$

$$= \sum_{a=1,...,p} \frac{\partial}{\partial z^a} \int_{\mathbb{R}^p^2} \left( \pi_1^{a_i}(z^{(1)}) + \pi_1^{a_i}(z^{(1)})h^{a_1}(z^{(2)}; \tilde{\Sigma}_{T,D}) \right) \phi(z^{(1)}; \tilde{\Sigma}_{T,D})\phi(z^{(2)}; \tilde{\Sigma}_{T,D})dz^{(2)}$$

$$= \left( \mu_{a_1a_2}^{a_1b_1}; \tilde{z}^{a_1b_2}; h_{b_1b_2}(z^{(1)}; \tilde{\Sigma}_{T,D}) + \mu_{a_1a_2}^{a_1b_1}; \tilde{z}^{a_1b_2}; h_{a}(z^{(1)}; \tilde{\Sigma}_{T,D}) \right) \phi(z^{(1)}; \tilde{\Sigma}_{T,D}) \phi(z^{(2)}; \tilde{\Sigma}_{T,D}).$$

Thus, we can get the desired form of $\hat{q}_{T,3,D}$. \hfill \Box

Proof (Proof of Proposition 2.3) This proof is the almost same as Theorem 6.2. in Sakamoto and Yoshida (2004). Let $\gamma' \in \left( \frac{2}{3}, \gamma - \frac{L}{q^2} \right)$ and $\gamma'' \in \left( \frac{L}{q^3}, 3\gamma - 2 \right)$. We set

$$X_{T,0} = \left\{ \omega \in \Omega : \inf_{T>0, |x|=1} x' \int_0^1 \nu_{ab} (\theta_1 + s(\theta_2 - \theta_1)) ds \right\}$$
for some constant $C’ > 0$, and
\[
\mathcal{X}_T,1 = \left\{ \omega \in \Omega \mid |T^{-1}l_{a_1a_2} - v_{a_1a_2}| < T^{-\frac{\gamma'}{2}}, |T^{-1}l_{a_1a_2a_3} - v_{a_1a_2a_3}| < T^{-\frac{\gamma'}{2}}, \sup_{\theta \in \Theta} |T^{-1}l_{a_1a_2a_3a_4}(\theta)| < T^{-\frac{\gamma''}{2}} \right\}.
\]

For appropriate $C’ > 0$ and sufficiently large $T$, it is known that there exists a unique $\hat{\theta}_T \in \hat{\Theta}$ such that $\partial_\theta l_T(\hat{\theta}_T) = 0$ and $|\hat{\theta}_T - \theta_0| < T^{-\frac{1}{2}}$ on the set $\mathcal{X}_T,0$. In particular, $\mathcal{X}_T,0 \subset \Omega_T$ holds for large $T$. Moreover, it is also proved that $P(\mathcal{X}_T,0) \lesssim T^{-\frac{1}{2}}$. Here, we used the conditions [B0], [B1], [B2] and [B3], see the proof of Theorem 6.1 in Sakamoto and Yoshida (2004) for details. On the other hand, the conditions [B2] and [B4] lead
\[
P[(\mathcal{X}_T,1)^c] \leq \left[ T^{-\frac{\gamma'}{2}} T^{-1}l_{a_1a_2} - v_{a_1a_2} \right] + P \left[ T^{-\frac{\gamma'}{2}} T^{-1}l_{a_1a_2a_3} - v_{a_1a_2a_3} \right] + P \left[ \sup_{\theta \in \Theta} |T^{-1}l_{a_1a_2a_3a_4}(\theta)| \right] \lesssim T^{-\frac{\gamma'}{2}} + T^{-\frac{\gamma''}{2}} \lesssim T^{-1}.
\]

Since $g_T^{-1}$ converge to a non-singular matrix by the condition [A3], we have $|Z_{a_1}^a| \lesssim T^{-\frac{1}{2}}$, $|Z_{a_1a_2}^a| \lesssim T^{-\frac{1}{2}}\gamma$, $|Z_{a_1a_2}^a| \lesssim T^{-\frac{1}{2}}\gamma$ and $|\bar{\theta}| \lesssim T^{-\frac{1}{2}}$ on $\mathcal{X}_T,0 \cap \mathcal{X}_T,1$. Moreover, the condition [B4] guarantees
\[
|v_{a_1a_2}^a| \leq |g^{ab}| E \left[ |T^{-1}l_{b,a_1a_2}| \right] < \infty \text{ uniformly with respect to } T. \tag{6.6}
\]

Hereafter, we consider the following inequalities on $\mathcal{X}_T,0 \cap \mathcal{X}_T,1$. Let $a \in \{1, \ldots, p\}$. First, we get
\[
\left| T^{-\frac{1}{2}} \tilde{R}_2^a \right| = \left| T^{-\frac{1}{2}} \left( \frac{1}{2} Z_{a^1a_2}^a \tilde{\theta}^a_{a_1a_2} + \frac{1}{2} \int_0^1 (1-u)^2 g^{ab} \left( \frac{1}{T} l_{b,a_1a_2a_3}(\theta_0 + u(\hat{\theta}_T - \theta_0)) \right) du \right) \tilde{\theta}^a_{a_1a_2a_3} \right| \lesssim T^{-\frac{1}{2}} - \frac{\gamma - 1}{2} + T^{-\frac{1}{2}} + T^{-\frac{1}{2}} - \frac{3(\gamma - 1)}{2} \lesssim T^{-\frac{1}{2}}
\]

for some small constant $0 < \varepsilon < \min(2\gamma' + \gamma' - 2, 3\gamma - \gamma'' - 2)$. Similarly, we have
\[
\left| \tilde{R}_1^a \right| = \left| Z_{a_1}^a \tilde{\theta}^a_{a_1a_2} + \frac{1}{2} v_{a_1a_2a_3}^a \tilde{\theta}^a_{a_1a_2a_3} + T^{-\frac{1}{2}} \tilde{R}_2^a \right| \lesssim T^{-\frac{1}{2}} - \frac{\gamma - 1}{2} + T^{-\frac{1}{2}} - (\gamma - 1) + T^{-\frac{1}{2}} \lesssim T^{-\frac{1}{2}}
\]

for a positive constant $0 < \varepsilon' < \min(\gamma + \gamma' - 2, \varepsilon)$. Finally we have
\[
\left| T^{-\frac{1}{2}} \tilde{R}_2^a \right| = \left| T^{-\frac{1}{2}} \left( Z_{a_1}^a \tilde{R}_1^a + \tilde{R}_2^a \right) \right| + T^{-1} \left( \frac{1}{2} v_{a_1a_2}^a \tilde{R}_1^a \tilde{R}_2^a \right) \lesssim T^{-\frac{1}{2}} - \frac{\gamma - 1}{2} + T^{-\frac{1}{2}} + T^{-1}\varepsilon' \lesssim T^{-\frac{1}{2}}
\]
Therefore, we get the desired conclusion
\[
P \left[ \Omega_T \cap \left\{ T^{-1} |\tilde{R}_2^a| \leq CT^{-\frac{1+\varepsilon}{2}} \right\}, \ a = 1, \ldots, p \right]\geq P[\mathcal{D}_{T,0} \cap \mathcal{D}_{T,1}] = 1 - o(T^{-\frac{1}{2}}).
\]

\[\square\]

**Proof** (Proof of Theorem 2.4) From Proposition 2.2, we see that
\[
\left\{ f(\sqrt{T}(\hat{\theta}_T - \theta_0)) - \int f(z^{(1)})q_{T,3}(z^{(1)})dz^{(1)} \right\}_{\Omega_T}\leq \left\{ f(\sqrt{T}(\hat{\theta}_T - \theta_0)) - f(\tilde{S}_T) \right\}_{\Omega_T \cap \left\{ T^{-1} |\tilde{R}_2^a| \geq CT^{-\frac{1+\varepsilon}{2}} \right\}}
\]
\[
\leq \left\{ f(\tilde{S}_T + T^{-1} \tilde{R}_2) - f(\tilde{S}_T) \right\}_{\Omega_T \cap \left\{ T^{-1} |\tilde{R}_2^a| \leq CT^{-\frac{1+\varepsilon}{2}} \right\}}
\]
\[
+ \int_{\mathbb{R}^d} f(z^{(1)})\tilde{q}_{T,3,d}(z^{(1)})dz^{(1)} - \int_{\mathbb{R}^d} f(z^{(1)})q_{T,3}(z^{(1)})dz^{(1)} + T^{-\frac{1}{2}}
\]
\[\leq \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + T^{-\frac{1}{2}}.
\]

From the definition of \(\tilde{S}_T\) and the representation of (2.13),
\[
\|\tilde{S}_T\|_{L^k(\mathcal{P})} \leq \sum_{a=1,\ldots,p} \left\| Z^{a_1+} + T^{-\frac{1}{2}} Z_a^{a_2} Z^{a_1} + \frac{1}{2} T^{-\frac{1}{2}} v_{a_1 a_2} Z^{a_1} Z^{a_2} \right\|_{L^k(\mathcal{P})}.
\]
By Corollary 6.3 and the representation of moments by cumulants, we have
\[
T^{-\frac{1}{2}} Z_T \|_{L^k(\mathcal{P})} \lesssim T^{\varepsilon k}
\]
for any \(\varepsilon > 0\) and \(k > 0\). From (6.6) and the above inequality, \(\|\tilde{S}_T\|_{L^k(\mathcal{P})} \lesssim T^{\varepsilon k}\) holds for any \(\varepsilon > 0\) and \(k > 0\). Thus, the condition [C1] yields \(T^{-1} \tilde{R}_2 \|_{L^k(\mathcal{P})} \lesssim T^{\varepsilon k}\) for any \(\varepsilon > 0\) and \(k > 0\).

We evaluate \(\Delta_1\), \(\Delta_2\), \(\Delta_3\) and \(\Delta_4\). From Proposition 2.3 and by choosing sufficiently small \(\varepsilon\), we get
\[
\Delta_1 \lesssim E \left[ \left( 1 + |\tilde{S}_T| + |\sqrt{T}(\hat{\theta}_T - \theta_0)| \right)^{L_1} 1_{\Omega_T} \right] \lesssim T^{-\frac{1}{2}}.
\]

Similarly,
\[
\Delta_2 \lesssim E \left[ \left( 1 + |\tilde{S}_T| + |T^{-1} \tilde{R}_2| \right)^{L_1} 1_{\Omega_T \cap \left\{ T^{-1} |\tilde{R}_2^a| \leq CT^{-\frac{1+\varepsilon}{2}} \right\}} \right] \lesssim T^{-\frac{1}{2}}.
\]

On the other hand, from the Taylor expansion, we have
\[
f(\tilde{S}_T + T^{-1} \tilde{R}_2) - f(\tilde{S}_T) = \sum_{|a| = 1} T^{-1} \tilde{R}_2 \int_0^1 \partial^a f(\tilde{S}_T + uT^{-1} \tilde{R}_2)du.
\]
\[\square\]
\[ \lesssim T^{-1} |\tilde{R}_2| \left( 1 + |S_T| + |T^{-1} \tilde{R}_2| \right)^{L_1}. \]

Thus, we get
\[ \Delta_3 \lesssim E \left[ T^{-1} |\tilde{R}_2| \left( 1 + |S_T| + |T^{-1} \tilde{R}_2| \right)^{L_1} \left\{ T^{-1} |\tilde{R}_2| \leq \epsilon T^{-1/2} \right\} \right] \lesssim T^{-1/2 - \epsilon} \lesssim T^{-1/2}, \]
since we can choose small \( \epsilon \) arbitrary. Finally, we only have to show that \( \Delta_4 \lesssim T^{-1/2} \). From the definition of \( f \),
\[ \Delta_4 \lesssim \int_{\mathbb{R}^d} (1 + |z^{(1)}|) L_1 \left| \tilde{q}_{T, 3, D}(z^{(1)}) - q_{T, 3}(z^{(1)}) \right| dz^{(1)} \]
\[ \lesssim \int_{\mathbb{R}^d} (1 + |z^{(1)}|) L_1 \left| \frac{\tilde{q}_{T, 3, D}(z^{(1)})}{\phi(z^{(1)}; \tilde{g}_T^{-1})} - \frac{q_{T, 3}(z^{(1)})}{\phi(z^{(1)}; g_T^{-1})} \right| \phi(z^{(1)}; g_T^{-1}) dz^{(1)} \]
\[ + \int_{\mathbb{R}^d} (1 + |z^{(1)}|) L_1 \left| \frac{q_{T, 3}(z^{(1)})}{\phi(z^{(1)}; g_T^{-1})} \left( 1 - \frac{\phi(z^{(1)}; g_T^{-1})}{\phi(z^{(1)}; \tilde{g}_T^{-1})} \right) \right| \phi(z^{(1)}; \tilde{g}_T^{-1}) dz^{(1)}. \quad (6.7) \]

We see that \( \tilde{g}_T - g_T = (I - g_T \tilde{g}_T^{-1}) \tilde{g}_T = T^{-D} g_T^{-1} \tilde{g}_T T \). Since \( g_T^{-1} \tilde{g}_T^{-1} = I + T^{-D} g_T^{-1} \) is positive definite, \( g_T^{-1} \tilde{g}_T \) is also positive definite. With the help of the conditions [A2]–[A3] and Corollary 6.3, we can choose a sufficiently large \( K > 0 \) such that
\[ \left| \frac{\tilde{q}_{T, 3, D}(z^{(1)})}{\phi(z^{(1)}; \tilde{g}_T^{-1})} - \frac{q_{T, 3}(z^{(1)})}{\phi(z^{(1)}; g_T^{-1})} \right| \]
\[ = \frac{1}{\sqrt{T}} \left| \left\{ \left( \frac{1}{6} K_T a_1 a_2 a_3; \tilde{g}_T^{-1} \right) h_{a_1 a_2 a_3} (z^{(1)}; \tilde{g}_T^{-1}) + \mu a_1; \tilde{g}_T^{-1} h_{a_1} (z^{(1)}; \tilde{g}_T^{-1}) \right\} \right| \]
\[ - \left\{ \left( \frac{1}{6} K_T a_1 a_2 a_3; g_T^{-1} \right) h_{a_1 a_2 a_3} (z^{(1)}; g_T^{-1}) + \mu a_1; g_T^{-1} h_{a_1} (z^{(1)}; g_T^{-1}) \right\} \right| \]
\[ \lesssim T^{-D} (1 + |z^{(1)}|)^K, \quad (6.8) \]

and
\[ \left| \frac{q_{T, 3}(z^{(1)})}{\phi(z^{(1)}; g_T^{-1})} \right| \lesssim (1 + |z^{(1)}|)^K. \quad (6.9) \]

On the other hand, we obtain
\[ \left| 1 - \frac{\phi(z^{(1)}; g_T^{-1})}{\phi(z^{(1)}; \tilde{g}_T^{-1})} \right| \]
\[ = 1 - \frac{|\tilde{g}_T^{-1}|}{|g_T^{-1}|} \exp \left( -\frac{1}{2} \tilde{z}^{(1)'} (g_T - \tilde{g}_T) \tilde{z}^{(1)} \right) \]
\[ \lesssim 1 - \frac{|\tilde{g}_T^{-1}|}{|g_T^{-1}|} + \frac{|\tilde{g}_T^{-1}|}{|g_T^{-1}|} \left| 1 - \exp \left( -\frac{T^{-D}}{2} \tilde{z}^{(1)'} g_T^{-1} \tilde{g}_T \tilde{z}^{(1)} \right) \right| \]
\[ \lesssim 1 - \frac{|g_T^{-1} + T^{-D} (g_T^{-1})^2|}{|g_T^{-1}|} \]
\[ + \frac{|g_T^{-1} + T^{-D} (g_T^{-1})^2|}{|\tilde{g}_T^{-1}|} \frac{2}{|g_T^{-1}|} \]
Since \( \lambda \) operator in Lemma 3.4, and Shiryaev (2000), (6.11) and (6.12) yield (Proof of Proposition 3.5) Let \( m \) with

\[
\int_0^t g(y, t) = e^{M_1} e^{K_{1t}} + K_1 e^{M_{1t}}. \]

From (6.7), (6.8), (6.9) and (6.10), we get the conclusion by taking sufficiently large \( D > 0 \).

\( \square \)

6.3 Proofs of Section 3.2

Throughout this subsection, denote the \( i \)-th jump time of \( N^x_i \) by \( \tau^x_i \).

Proof (Proof of Proposition 3.5) Let \( M_1, K_1 \) and \( K_2 \) be positive constants and \( \mathcal{A} \) be the operator in Lemma 3.4. Define \( g(y, t) \) and \( \mathcal{A} \) by \( g(y, t) = e^{M_1} e^{K_{1t}} \) and \( \mathcal{A} g(y, t) = e^{K_{1t}} (e^{M_1} + K_1 e^{M_{1t}}) \). From Lemma 3.4,

\[
\mathcal{A} g(y, t) \leq e^{K_{1t}} (-K_1 e^{M_1} + K_1 + K_1 e^{M_{1t}}) = e^{K_{1t}} K_2. \quad (6.11)
\]

Since \( \lambda^{x, \tau^x_i} \) is bounded, thus \( g(\lambda^{x, \tau^x_i}, t \wedge \tau^x_i) = e^{M_1 \lambda^{x, \tau^x_i}} e^{K_{1t}} \) is integrable. Furthermore, one may get

\[
g(\lambda^{x, \tau^x_i}, t \wedge \tau^x_i) - g(\lambda^{x, 0, 0}, 0) \leq \int_{(0,t \wedge \tau^x_i]} g(\lambda^{x, \tau^x_i}, \alpha, s) - g(\lambda^{x, \tau^x_i}, s)dN^x_s
\]

\[
+ \int_{(0,t \wedge \tau^x_i]} \frac{d}{ds} g(\lambda^{x, \tau^x_i}, s)ds
\]

\[
= \int_{(0,t \wedge \tau^x_i]} g(\lambda^{x, \tau^x_i}, \alpha, s) - g(\lambda^{x, \tau^x_i}, s)dN^x_s
\]

\[
+ \int_{(0,t \wedge \tau^x_i]} \mathcal{A} g(\lambda^{x, \tau^x_i}, s)ds. \quad (6.12)
\]

Since \( \int_{(0,t \wedge \tau^x_i]} g(\lambda^{x, \tau^x_i}, \alpha, s) - g(\lambda^{x, \tau^x_i}, s)dN^x_s \) is a \( \tau^x_i \)-local martingale, see Theorem 18.7 in Liptser and Shiryaev (2000), (6.11) and (6.12) yield

\[
E \left[ g(\lambda_{t \wedge \tau^x_i}, t \wedge \tau^x_i) \right] = g(x, 0) + E \left[ \int_{(0,t \wedge \tau^x_i]} \mathcal{A} g(\lambda^{x, \tau^x_i}, s)ds \right]
\]

\[
\leq g(x, 0) + E \left[ \int_{(0,t \wedge \tau^x_i]} e^{K_{1t}} K_2 ds \right] \leq g(x, 0) + \frac{K_2}{K_1} \left( e^{K_{1t}} - 1 \right).
\]

Then, by the Fatou’s lemma, we have

\[
E \left[ e^{M_1 \lambda} \right] \leq e^{-K_{1t}} \left( g(x, 0) + \frac{K_2}{K_1} \left( e^{K_{1t}} - 1 \right) \right).
\]

Thus, we get the conclusion.

\( \square \)

Proof (Proof of Lemma 3.7) Since \( \mathcal{A} \) is linear, we only have to prove that, for \( p(y) = y_m \) with \( m \in \mathbb{N} \), \( M^p_t = p(\lambda^x_t) - p(\lambda^x_0) - \int_{(0,t]} \mathcal{A} p(\lambda^x_s)ds \) is a \( \mathcal{F}_t \)-martingale. Take any large \( T > 0 \). In the same way as (6.12) in the proof of Proposition 3.5, one may confirm that

\[
M^p_t = \int_{[s \leq T]} \left( \lambda^x_s + \alpha \right)^m - (\lambda^x_0)^m dN^x_s \quad \text{for} \; t \leq T.
\]

Then, Theorem 18.7 in Liptser and Shiryaev (2000) and Proposition 3.5 lead the conclusion.

\( \square \)
Proof (Proof of Lemma 3.8) Let \( p(y) = a_m y^m + \cdots a_1 y + a_0 \), where \( a_0, \ldots, a_m \in \mathbb{R} \) and \( m \in \mathbb{N} \). Then, the linearity of \( \mathcal{A} \) leads

\[
E \left[ \int_{(s,t]} \int_{(s,u_1]} \cdots \int_{(s,u_{n-1}]} \int_{(s,u_{n-1}]} E \left[ \mathcal{A}^{n} p(\lambda_{u_n}^{x}) \big| \mathcal{F}_{s}^{X} \right] du_n \cdots du_1 \right] \\
\leq \int_{(s,t]} \int_{(s,u_1]} \cdots \int_{(s,u_{n-1}]} \int_{(s,u_{n-1}]} E \left[ |\mathcal{A}^{n} p(\lambda_{u_n}^{x})| \right] du_n \cdots du_1 \\
\leq \sum_{k=1}^{m} |a_k| \int_{(s,t]} \int_{(s,u_1]} \cdots \int_{(s,u_{n-1}]} \int_{(s,u_{n-1}]} E \left[ |\mathcal{A}^{n}(\lambda_{u_n}^{x})^k| \right] du_n \cdots du_1.
\]

Therefore, we only have to evaluate \( \int_{(s,t]} \int_{(s,u_1]} \cdots \int_{(s,u_{n-1}]} \int_{(s,u_{n-1}]} E \left[ |\mathcal{A}^{n}(\lambda_{u_n}^{x})^k| \right] du_n \cdots du_1 \) for any \( k \in \mathbb{N} \). In particular,

\[
\int_{(s,t]} \int_{(s,u_1]} \cdots \int_{(s,u_{n-1}]} \int_{(s,u_{n-1}]} E \left[ |\mathcal{A}^{n}(\lambda_{u_n}^{x})^k| \right] du_n \cdots du_1 \leq \frac{(t-s)^n}{n!} \sup_{u \in (s,t]} E \left[ |\mathcal{A}^{n}(\lambda_{u}^{x})^k| \right].
\]

thus it is enough to prove that the above right hand side converges to zero as \( n \to \infty \). There exist constants \( C_i, i = 0, \ldots, k \) such that \( \mathcal{A} y^k = C_k y^k + \cdots + C_1 y + C_0 \). Then, we inductively get

\[
\mathcal{A} y^k = \mathcal{A}^{n-1} (\mathcal{A} y^k) = C_k \mathcal{A}^{n-2} (\mathcal{A} y^k) + \sum_{i=1}^{k-1} C_i \mathcal{A}^{n-1} y^i \\
= C_k^2 \mathcal{A}^{n-3} (\mathcal{A} y^k) + C_k \sum_{i=1}^{k-1} C_i \mathcal{A}^{n-2} y^i + \sum_{i=1}^{k-1} C_i \mathcal{A}^{n-1} y^i \\
= \cdots = C_k^{n-1} \mathcal{A} y^k + C_k^{n-2} \sum_{i=1}^{k-1} C_i \mathcal{A} y^i + C_k^{n-3} \sum_{i=1}^{k-1} C_i \mathcal{A}^2 y^i + \cdots + \sum_{i=1}^{k-1} C_i \mathcal{A}^{n-1} y^i \\
= C_k^n y^k + \sum_{i=1}^{k-1} C_i \left( C_k^{n-1} y^i + C_k^{n-2} \mathcal{A} y^i + \cdots + \mathcal{A}^{n-1} y^i \right) + C_0 C_k^{n-1}.
\]

Hence,

\[
\sup_{u \in (s,t]} E \left[ |\mathcal{A}^{n}(\lambda_{u}^{x})^k| \right] \leq C_k^n \sup_{u \in (s,t]} E \left[ |(\lambda_{u}^{x})^k| \right] \\
+ \sum_{i=1}^{k-1} C_i \sup_{u \in (s,t]} E \left[ |C_k^{n-1}(\lambda_{u}^{x})^i + C_k^{n-2} \mathcal{A}(\lambda_{u}^{x})^i + \cdots + \mathcal{A}^{n-1}(\lambda_{u}^{x})^i| \right] \\
+ C_0 C_k^{n-1}.
\]

Furthermore, one may concretely compute as \( C_k = k(\alpha - \beta) \). Now, we introduce the following assumption.

\textbf{ASS(k) 1} For any \( i = 0, \ldots, k - 1 \) and \( C = j(\alpha - \beta) \) with \( j = k, k+1, \ldots \),

\[
\frac{(t-s)^n}{n!} \sup_{u \in (s,t]} E \left[ |C_k^{n-1}(\lambda_{u}^{x})^i + C_k^{n-2} \mathcal{A}(\lambda_{u}^{x})^i + \cdots + \mathcal{A}^{n-1}(\lambda_{u}^{x})^i| \right] \to 0 \quad \text{as} \quad n \to \infty.
\]
We prove that \( \text{ASS}(k) \) holds for any \( k \). For \( C \), Proposition 3.5 guarantees \( \sup_{u \in (s, t]} E \left[ \left( \lambda_u^x \right)^k \right] < \infty \). Thus, if \( \text{ASS}(k) \) holds, by taking \( C = k(\alpha - \beta) \) in \( \text{ASS}(k) \), we have

\[
\frac{(t-s)^n}{n!} \sup_{u \in (s, t]} E \left[ \left( \lambda_u^x \right)^k \right] \to 0.
\]

We prove that \( \text{ASS}(k) \) holds for any \( k \in \mathbb{N} \) by induction. In the case of \( k = 1 \), this assumption is obvious. Assume that \( \text{ASS}(k) \) holds. Again we denote \( \mathcal{A} y^k = C_k y^k + \cdots + C_1 y + C_0 \). For \( C = j(\alpha - \beta) \) with \( j = k + 1, k + 2, \ldots \), by using the Eq. (6.13), we have

\[
C^{n-1} y^k + C^{n-2} \mathcal{A} y^k + \cdots + \mathcal{A}^{n-1} y^k
\]

\[
= C^{n-1} y^k + C^{n-2} \left\{ C_k y^k + \sum_{i=1}^{k-1} C_i y^i + C_0 \right\}
\]

\[
+ C^{n-3} \left\{ C_k^2 y^k + \sum_{i=1}^{k-1} C_i (C_k y^i + \mathcal{A} y^i) + C_0 C_k \right\}
\]

\[
+ C^{n-4} \left\{ C_k^3 y^k + \sum_{i=1}^{k-1} C_i (C_k^2 y^i + C_k \mathcal{A} y^i + \mathcal{A}^2 y^i) + C_0 C_k^2 \right\}
\]

\[
+ \cdots + \left\{ C_k^{n-1} y^k + \sum_{i=1}^{k-1} C_i \left( C_k^{n-2} y^i + C_k^{n-3} \mathcal{A} y^i + \cdots + \mathcal{A}^{n-2} y^i \right) + C_0 C_k^{n-2} \right\}
\]

\[
= \sum_{i=0}^{n-1} C_i C_k^{n-1-i} y^k + C_{k-1} \sum_{j=0}^{n-2} \left( \sum_{i=0}^{n-2-j} C_i C_k^{n-2-i-j} \right) \mathcal{A}^{j} y^{k-1}
\]

\[
+ C_{k-2} \sum_{j=0}^{n-2-j} \left( \sum_{i=0}^{n-2-j} C_i C_k^{n-2-i-j} \right) \mathcal{A}^{j} y^{k-2}
\]

\[
+ \cdots + C_1 \sum_{j=0}^{n-2-j} \left( \sum_{i=0}^{n-2-j} C_i C_k^{n-2-i-j} \right) \mathcal{A}^{j} y + C_0 \left( \sum_{i=0}^{n-2-j} C_i C_k^{n-2-i-j} \right)
\]

\[
= \frac{C^n - C_k^n y^k}{C - C_k} + C_{k-1} \sum_{j=0}^{n-2} \frac{C^{n-1-j} - C_k^{n-1-j}}{C - C_k} \mathcal{A}^{j} y^{k-1}
\]

\[
+ C_{k-2} \sum_{j=0}^{n-2} \frac{C^{n-2-j} - C_k^{n-2-j}}{C - C_k} \mathcal{A}^{j} y^{k-2}
\]

\[
+ \cdots + C_1 \sum_{j=0}^{n-2} \frac{C^{n-2-j} - C_k^{n-2-j}}{C - C_k} \mathcal{A}^{j} y + C_0 \frac{C^n - C_k^n}{C - C_k}
\]

\[
= \frac{1}{C - C_k} \left\{ (C^n - C_k^n y^k) + C_{k-1} \left\{ \sum_{i=0}^{n-1} C_i \mathcal{A}^{n-1-i} y^{k-1} - \sum_{i=0}^{n-1} C_k^i \mathcal{A}^{n-1-i} y^{k-1} \right\} \right\}
\]

\[
+ C_{k-2} \left\{ \sum_{i=0}^{n-1} C_i \mathcal{A}^{n-1-i} y^{k-2} - \sum_{i=0}^{n-1} C_k^i \mathcal{A}^{n-1-i} y^{k-2} \right\} + \cdots
\]
\[ + C_1 \left\{ \sum_{i=0}^{n-1} C_i \partial_y^{n-1-i} y - \sum_{i=0}^{n-1} C_i \partial_y^{n-1-i} y \right\} + C_0 (C^{n-1} - C_k^{n-1}) \].

Therefore,
\[
\frac{(t - s)^n}{n!} \sup_{u \in (s, t)} \left[ C^n - C_k^n \right] \sup_{u \in (s, t)} \left( C^{n-k} + C^{n-k-1} \partial_y^{n-k-1} + \cdots + C^{n-k-1} \partial_y^{n-k-1} \right) \\
\leq \frac{(t - s)^n}{n!} \left[ C^n - C_k^n \right] \left\{ \sum_{i=0}^{n-1} C_i \partial_y^{n-1-i} \right\} + C_k^{-1} \left\{ \sum_{i=0}^{n-1} C_i \partial_y^{n-1-i} \right\} \\
+ C_k^{-2} \left\{ \sum_{i=0}^{n-1} C_i \partial_y^{n-1-i} \right\} + \cdots + \left\{ \sum_{i=0}^{n-1} C_i \partial_y^{n-1-i} \right\} \\
+ C_0 (C^{n-1} - C_k^{n-1}) \right\}.
\]

Hence, ASS(k) leads ASS(k + 1) and we have completed the proof. \( \square \)

To prove Theorem 3.9, we prepare the following lemmas.

**Lemma 6.6** For any \( u \in \mathbb{R} \) and \( t \geq s \geq 0 \), \( E[e^{iu\lambda_s^x} | \mathcal{F}_{s}^x] = E[e^{iu\lambda_s^x}] \ a.s. \)

**Proof** Fix \( s \) and \( t \) with \( t \geq s \geq 0 \). It is sufficient to show that for any bounded \( \mathcal{F}_{s}^x \)-measurable function \( g : \Omega \rightarrow \mathbb{R} \),
\[
E[e^{iu\lambda_s^x} g] = E \left[ E[e^{iu\lambda_s^x} | \lambda_s^x] g \right].
\]

Note that
\[
E \left[ E[e^{iu\lambda_s^x} | \lambda_s^x] g \right] = E \left[ E[e^{iu\lambda_s^x} | \lambda_s^x] E[g | \lambda_s^x] \right] = E \left[ e^{iu\lambda_s^x} E[g | \lambda_s^x] \right].
\]

Let \( D = \{ z \in \mathbb{C}; \ Re z < \frac{M_1}{2} \} \), where \( M_1 \) is the positive constant chosen in Proposition 3.5. First, we will prove that \( f(z) = E[e^{z\lambda_s^x} g] \) is holomorphic on \( D \) for any \( \mathcal{F}_{s}^x \)-measurable function \( g \). Let \( z = a + ib \), where \( a, b \in \mathbb{R} \) with \( a < \frac{M_1}{2} \). Then, we have
\[
|f(z)| \leq \|g\| \|E[e^{z\lambda_s^x}]\| < \infty,
\]
and thus \( f(z) = E[e^{z\lambda_s^x} g] \) is defined on \( D \). Define \( u(a, b) \) and \( v(a, b) \) as the real part and the imaginary part of \( f(z) \) respectively, namely,
\[
f(z) = E[\cos(b\lambda_s^x) e^{a\lambda_s^x}] + i E[\sin(b\lambda_s^x) e^{a\lambda_s^x}] = u(a, b) + i v(a, b).
\]

Write \( \partial_a = \frac{\partial}{\partial a} \) and \( \partial_b = \frac{\partial}{\partial b} \), \( |\partial_a (\cos(b\lambda_s^x) e^{a\lambda_s^x})|, |\partial_b (\cos(b\lambda_s^x) e^{a\lambda_s^x})|, |\partial_a (\sin(b\lambda_s^x) e^{a\lambda_s^x})| \) and \( |\partial_b (\sin(b\lambda_s^x) e^{a\lambda_s^x})| \) are dominated by an integrable random variable \( |\lambda_s^x e^{z \lambda_s^x}| \) on \( D \). Hence, the permutation of differential and integral is permitted, and thus we have
\[
\partial_a u(a, b) = \partial_b v(a, b) = E[\lambda_s^x \cos(b\lambda_s^x) e^{a\lambda_s^x}] \quad \text{and} \quad \partial_b u(a, b) = -\partial_a v(a, b) = -E[\lambda_s^x \sin(b\lambda_s^x) e^{a\lambda_s^x}].
\]

\( \square \) Springer
The Lebesgue’s theorem guarantees that \( \partial_{a}u(a, b), \partial_{b}u(a, b), \partial_{a}v(a, b) \) and \( \partial_{b}v(a, b) \) are continuous with respect to \( a \) and \( b \). In particular, they are totally differentiable. Then, the Cauchy–Riemann relations lead that \( f(z) \) is holomorphic on \( D \). Completely similarly, we can prove that \( z \mapsto E \left[ e^{z \mu_{x}^{i}} E[g \mu_{x}^{i}] \right] \) is also holomorphic on \( D \).

Second, we will confirm that \( E[e^{z \mu_{x}^{i}}] = E \left[ e^{z \mu_{x}^{i}} E[g \mu_{x}^{i}] \right] \) for \( z \in \left( -\frac{M_{1}}{2}, \frac{M_{1}}{2} \right) \). Let

\[
p_{N}(x) = \sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!}.
\]

Then, (3.3) leads

\[
E \left[ \sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!} g \right] = E \left[ \sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!} \mathcal{F}_{x}^{i} \right] g = E \left[ \sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!} \lambda_{x}^{i} \right] g = E \left[ \sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!} E[g \lambda_{x}^{i}] \right].
\]

On the other hand, since \( \sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!} \to e^{z \lambda_{x}^{i}} \) as \( N \to \infty \) and \( |\sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!}| \leq |\sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!}| \leq e^{z \lambda_{x}^{i}} \) hold for every \( z \in \left( -\frac{M_{1}}{2}, \frac{M_{1}}{2} \right) \), we have

\[
\sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!} \to e^{z \lambda_{x}^{i}} \quad \text{as} \quad N \to \infty \quad \text{in} \quad L^{1} \text{-sense},
\]

and

\[
\sum_{n=0}^{N} \frac{(z \lambda_{x}^{i})^{n}}{n!} E[g \lambda_{x}^{i}] \to e^{z \lambda_{x}^{i}} E[g \lambda_{x}^{i}] \quad \text{as} \quad N \to \infty \quad \text{in} \quad L^{1} \text{-sense}
\]

by the Lebesgue’s theorem. Therefore, we get the desired equation

\[
E \left[ e^{z \mu_{x}^{i}} \right] = E \left[ e^{z \mu_{x}^{i}} E[g \mu_{x}^{i}] \right] \quad \text{for} \quad z \in \left( -\frac{M_{1}}{2}, \frac{M_{1}}{2} \right).
\]

Now, the identity theorem guarantees the conclusion. \( \Box \)

**Proof** (Proof of Theorem 3.9) For almost every \( a, b \in \mathbb{R} \) with \( a < b \), we will prove that

\[
P_{\mathcal{F}_{x}^{i}} \left[ \lambda_{x}^{i} \in (a, b) \right] = \lim_{x \to \infty} \frac{1}{2\pi} \int_{-\pi}^{x} \frac{e^{-iua} - e^{-iub}}{iu} E\left[ e^{iu \lambda_{x}^{i}} \right] \mathcal{F}_{x}^{i} du \quad \text{a.s.} \quad (6.14)
\]

and

\[
P_{\mathcal{F}_{x}^{i}} \left[ \lambda_{x}^{i} \in (a, b) \right] = \lim_{x \to \infty} \frac{1}{2\pi} \int_{-\pi}^{x} \frac{e^{-iua} - e^{-iub}}{iu} E\left[ e^{iu \lambda_{x}^{i}} \right] \lambda_{x}^{i} du \quad \text{a.s.} \quad (6.15)
\]

However, by considering a probability measure \( P_{F}(d\omega) = P[d\omega \cap F] / P[F] \), the Lévy’s inversion formula gives

\[
P_{\mathcal{F}_{x}^{i}} \left[ \lambda_{x}^{i} \in (a, b) \right] \cap F = \lim_{x \to \infty} \frac{1}{2\pi} \int_{-\pi}^{x} \frac{e^{-iua} - e^{-iub}}{iu} E\left[ e^{iu \lambda_{x}^{i}} \right] 1_{F} du
\]

for any set \( F \in \mathcal{F}_{x}^{i} \). Moreover, we know

\[
\frac{1}{2\pi} \int_{-\pi}^{x} \frac{e^{-iua} - e^{-iub}}{iu} E\left[ e^{iu \lambda_{x}^{i}} \right] 1_{F} du = E \left[ \frac{1}{\pi} D_{A}(\lambda_{x}^{i} - a) - D_{A}(\lambda_{x}^{i} - b) \right] \left| \mathcal{F}_{x}^{i} \right] 1_{F}
\]
where $D_A$ is the Dirichlet integral, i.e.

$$D_A(\alpha) = \int_{0}^{A} \frac{\sin(ua)}{u} du.$$ 

Then, as is well known, we can apply the Lebesgue’s theorem and get

$$\lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{-iua} - e^{-iub}}{iu} E \left[ e^{iu\lambda_{s}^{\nu}} | \mathcal{F}_{s}^{x} \right] du = E \left[ \left( \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{-iua} - e^{-iub}}{iu} E \left[ e^{iu\lambda_{s}^{\nu}} | \mathcal{F}_{s}^{x} \right] du \right) 1_{F} \right].$$

Thus, (6.14) holds. In the same way, (6.15) also holds. Then, from Lemma 6.6, we get $P[\lambda_{s}^{x} \in (a, b) | \mathcal{F}_{s}^{x}] = P[\lambda_{t}^{x} \in (a, b) | \lambda_{s}^{x}]$ a.s. for almost every $a, b \in \mathbb{R}$ with $a < b$. With the help of the monotone class theorem, $E[f(\lambda_{s}^{x}) | \mathcal{F}_{s}^{x}] = E[f(\lambda_{t}^{x}) | \lambda_{s}^{x}]$ a.s. holds for any bounded measurable function $f$. \hfill \Box

### 6.4 Proofs of Section 3.3

#### 6.4.1 Markovian property

**Proof** (Proof of Proposition 3.11) From Theorem 3.9, we immediately get, for any $t \geq s \geq 0$ and bounded measurable function $f$,

$$E \left[ f(X_{t}^{x,(1)}) | \mathcal{F}_{s}^{x} \right] = E \left[ f \left( X_{t}^{x,(1)} \right) \big| X_{s}^{x,(1)} \right] \text{ a.s.} \quad (6.16)$$

$X_{t}^{x}$ has the following relation. For any $t \geq s \geq 0$,

$$X_{t}^{x,(2)} = (x_{1}t + x_{2}) e^{-\beta t} + \int_{(0,s)} \alpha(t-u)e^{-\beta(t-u)} dN_{u}^{x_{1}} + \int_{[s,t]} \alpha(t-u)e^{-\beta(t-u)} dN_{u}^{x_{1}}$$

$$= \left\{ (t-s) \left( x_{1} e^{-\beta s} + \int_{(0,s)} \alpha e^{-\beta(s-u)} dN_{u}^{x_{1}} \right) \right\} e^{-\beta(t-s)}$$

$$+ \left\{ (x_{1}s + x_{2}) e^{-\beta s} + \int_{(0,s)} \alpha(s-u)e^{-\beta(s-u)} dN_{u}^{x_{1}} \right\} e^{-\beta(t-s)}$$

$$+ \int_{[s,t]} \alpha(t-u)e^{-\beta(t-u)} dN_{u}^{x_{1}}$$

$$= X_{s}^{x,(1)}(t-s) + X_{s}^{x,(2)} e^{-\beta(t-s)} + \int_{[s,t]} \alpha(t-u)e^{-\beta(t-u)} dN_{u}^{x_{1}},$$

and similarly,

$$X_{t}^{x,(3)} = \left( X_{s}^{x,(1)}(t-s)^{2} + 2X_{s}^{x,(2)}(t-s) + X_{s}^{x,(3)} \right) e^{-\beta(t-s)}$$

$$+ \int_{[s,t]} \alpha(t-u)^{2} e^{-\beta(t-u)} dN_{u}^{x_{1}}.$$

Therefore, $X_{t}^{x}$ is $\sigma(X_{s}^{x,(1)}, X_{s}^{x,(2)}, X_{s}^{x,(3)}; u \in [s, t])$-measurable for any $t \geq s \geq 0$.

Let $g_{1}$, $g_{2}$ and $g_{3}$ be $\sigma(X_{u}^{x,(1)}, u \in [s, t]), \sigma(X_{s}^{x,(2)})$ and $\sigma(X_{s}^{x,(3)})$ measurable bounded functions respectively. Then, (6.16) and the monotone class theorem lead

$$E[g_{1}|X_{s}^{x}] = E[g_{1}|\mathcal{F}_{s}^{x}|X_{s}^{x}] = E[g_{1}|X_{t}^{x,(1)}]X_{s}^{x} = E[g_{1}|X_{s}^{x,(1)}] = E[g_{1}|\mathcal{F}_{s}^{x}] \text{ a.s.}$$
Thus,
\[ E[g_1 g_2 g_3 | \mathcal{F}^X_s] = g_2 g_3 E[g_1 | \mathcal{F}^X_s] = g_2 g_3 E[g_1 | X^X_s] = E[g_1 g_2 g_3 | X^X_s] \text{ a.s.} \]

By the monotone class theorem, we get the conclusion.

Before we prove the homogeneous Markov property of process \( X \), we prepare the following technical lemma.

**Lemma 6.7** For any bounded function \( f \) defined on the path space of \( X^{x,(1)}_u \), \( u \in [s, t] \),
\[ E\left[f\left(X^{x,(1)}_u; u \in [s, t]\right) | \mathcal{F}^X_s\right](\omega) = E\left[f\left(X^{y,(1)}_{u-s}; u \in [s, t]\right)\right]_{y=X^X_s(\omega)} \text{ a.s. } \omega. \]

**Proof** From Theorem 3.9 and the monotone class theorem, \( E\left[f\left(X^{x,(1)}_u; u \in [s, t]\right) | \mathcal{F}^X_s\right] = E\left[f\left(X^{x,(1)}_u; u \in [s, t]\right) X^{x,(1)}_s\right] \text{ a.s.} \) holds. Therefore, we only have to prove the statement replaced \( \mathcal{F}^X_s \) by \( \sigma\left(X^{x,(1)}_s\right) \). For any \( p \in \mathcal{P} \), we know that for almost every \( \omega \in \Omega \) and \( s \leq t \),
\[
E\left[p\left(X^{x,(1)}_t \bigg| X^{x,(1)}_s\right)\right](\omega) = E\left[p\left(\lambda^{x}_t - \mu\right) \bigg| \lambda^{x}_s\right](\omega)
\]
\[
= e^{(t-s)\sigma\mu} p\left(\lambda^{x}_t(\omega) - \mu\right)
\]
\[
= e^{(t-s)\sigma\mu} p\left(\lambda^{y}_{s-\mu}(\omega) - \mu\right)
\]
\[
= E\left[p\left(\lambda^{y}_{t-s} - \mu\right)\right]_{y=X^X_s(\omega)}
\]
\[
= E\left[p\left(X^{y,(1)}_{t-s}\right)\right]_{y=X^X_s(\omega)},
\]
where the operator \( e^{(t-s)\sigma\mu} \) is defined in Sect. 3.2. In particular, for \( u \in \mathbb{R} \) with \( |u| \leq M_1 \) and \( p_N(y) = \sum_{n=0}^{N} \left(\frac{M_1^2}{n!}\right)^n \in \mathcal{P} \),
\[
E\left[p_N\left(X^{x,(1)}_t \bigg| X^{x,(1)}_s\right)\right] = E\left[p_N\left(X^{y,(1)}_{t-s}\right)\right]_{y=X^X_s(\omega)} \text{ a.s.}
\]
Moreover, Proposition 3.5 and the Lebesgue’s theorem give
\[
E\left[e^{iuX^{x,(1)}_t \bigg| X^{x,(1)}_s}\right] = E\left[e^{iuX^{y,(1)}_{t-s}}\right]_{y=X^X_s(\omega)} \text{ a.s.}
\]
In the same way of the proof of Lemma 6.6, one may confirm that Proposition 3.5 and the identity theorem guarantee that for general \( u \in \mathbb{R} \),
\[
E\left[e^{iuX^{x,(1)}_t \bigg| X^{x,(1)}_s}\right] = E\left[e^{iuX^{y,(1)}_{t-s}}\right]_{y=X^X_s(\omega)} \text{ a.s.}
\]
Finally, it is proved in the same way as Theorem 3.9 that
\[
E\left[f\left(X^{x,(1)}_t \bigg| X^{x,(1)}_s\right)\right] = E\left[f\left(X^{y,(1)}_{t-s}\right)\right]_{y=X^X_s(\omega)} \text{ a.s.}
\]
for any bounded measurable function \( f \). Thus, for any \( s \leq u_1 \leq u_2 \leq t \) and bounded functions \( g_1, g_2 \),
\[
E\left[g_1\left(X^{x,(1)}_{u_1}\right) g_2\left(X^{x,(1)}_{u_2}\right) \bigg| X^{x,(1)}_s\right] = E\left[g_1\left(X^{x,(1)}_{u_1}\right) E\left[g_2\left(X^{x,(1)}_{u_2}\right) \bigg| \mathcal{F}^X_{u_1}\right]\right]_{X^X_s(\omega)}
\]
\[
= E\left[g_1\left(X^{x,(1)}_{u_1}\right) E\left[g_2\left(X^{x,(1)}_{u_2}\right) \bigg| \mathcal{F}^X_{u_1}\right]\right]_{X^X_s(\omega)},
\]
\( \square \) Springer
By considering cylinder sets in \(\sigma\) and the monotone class theorem yields that

\[
E \left[ g_1(X_{u_1}^{x,(1)}) E[g_2(X_{u_2-u_1}^{y,(1)})] \right]_{y_1 = X_{u_1}^{x,(1)}} = X_{s}^{x,(1)}
\]

\[
E \left[ g_1(X_{u_1-s}^{y,(1)}) E[g_2(X_{u_2-u_1}^{y,(1)})] \right]_{y_1 = X_{u_1-s}^{y,(1)}} = X_{s}^{y,(1)}
\]

\[
E \left[ g_1(X_{u_2-s}^{y,(1)}) E[g_2(X_{u_2-u_1}^{y,(1)})] \right]_{y_1 = X_{u_2-s}^{y,(1)}} = X_{s}^{y,(1)}
\]

Inductively, we also get for any \(k \in \mathbb{N}, s \leq u_1 \leq \cdots \leq u_k \leq t\) and bounded functions \(g_1, \ldots, g_k\),

\[
E \left[ g_1(X_{u_1}^{x,(1)}) \cdots g_k(X_{u_k}^{x,(1)}) \right]_{y_1 = X_{s}^{y,(1)}} = E \left[ g_1(X_{u_1-s}^{y,(1)}) \cdots g_k(X_{u_k-s}^{y,(1)}) \right]_{y_1 = X_{s}^{y,(1)}}
\]

By considering cylinder sets in \(\sigma(X_{u}^{x,(1)}; u \in [s, t])\), the monotone class theorem gives the conclusion.

\[\square\]

**Proof** (Proof of Proposition 3.12) Let \(h_1\) be a bounded function defined on the path space of \(X_{u}^{x,(1)}, u \in [s, t]\). Moreover, let \(h_2\) and \(h_3\) be bounded functions on \(\mathbb{R}\). Lemma 6.7 leads

\[
E \left[ h_1(X_{u}^{x,(1)}; u \in [s, t]) h_2(X_{s}^{x,(2)}) h_3(X_{s}^{x,(3)}) \right]_{X_{s}^{x}}
\]

\[
= h_2(X_{s}^{x,(2)}) h_3(X_{s}^{x,(3)}) E \left[ h_1(X_{u}^{x,(1)}; u \in [s, t]) \right]_{X_{s}^{x}}
\]

\[
= h_2(X_{s}^{x,(2)}) h_3(X_{s}^{x,(3)}) E \left[ h_1(X_{u}^{x,(1)}; u \in [s, t]) \right]_{X_{s}^{x}}
\]

\[
= h_2(X_{s}^{x,(2)}) h_3(X_{s}^{x,(3)}) E \left[ h_1(X_{u_k-s}^{y,(1)}; u \in [s, t]) \right]_{y_1 = X_{s}^{y,(1)}}
\]

Therefore, the monotone class theorem yields that

\[
E \left[ h(X_{u}^{x,(1)}, X_{s}^{x,(2)}, X_{s}^{x,(3)}; u \in [s, t]) \right]_{X_{s}^{x}}
\]

\[
= E \left[ h(X_{u-s}^{y,(1)}, y_2, y_3; u \in [s, t]) \right]_{(y_1, y_2, y_3) = X_{s}^{x}}
\]

for any bounded function \(h\). Note that \(X_{u}^{x,(1)}(\omega), u \in [s, t]\) completely determines the jumps of \(N_{u}^{x,(1)}(\omega), u \in [s, t]\). Thus, for any bounded measurable function \(f\), we can conclude

\[
E[f(X_{s}^{x})]_{X_{s}^{x}}
\]

\[
= E \left[ f \left( \left( X_{s}^{y,(1)}(t-s) + X_{s}^{y,(2)} \right) e^{-\beta(t-s)} + \int_{[t,s]} \alpha(t-u)e^{-\beta(t-u)}dN_{u}^{y,(1)} \right) \right]_{X_{s}^{x}}
\]

\[
= E \left[ f \left( \left( y_1(t-s) + y_2 \right) e^{-\beta(t-s)} + \int_{[t,s]} \alpha(t-u)e^{-\beta(t-u)}dN_{u}^{y,(1)} \right) \right]_{(y_1, y_2) = X_{s}^{x}}
\]
Thus, similarly as the proofs of Lemma 6.7 and Proposition 3.12, we get the conclusion.

Lemma 6.8 For any $0 \leq s < t$ and a bounded measurable function $f$, 

$$E \left[ f(X^X_{t-s}) \right]_{X=X_t} = \tilde{E} \left[ f(\tilde{X}_t) \right] (\tilde{\omega}) \quad \text{a.s.} \quad \tilde{\omega}$$

Proof We again set $g(x, t) = e^{M_{1|x}} e^{K_{1|t}}$ and the operator $\mathcal{A}$ same as Proposition 3.5. Denote the $i$-th jump time of $\tilde{N}_t$ from time zero by $\tau_i$, i.e. $\tau_i = \inf\{t \geq 0 \mid \tilde{N}((0, t]) = i\}$. Then, $f(\tau_t) g(\tilde{\lambda}_s^{(1)}, \alpha, s) - g(\tilde{\lambda}_s^{(1)}, \alpha) ds$ is a $\tau_i$-local martingale, see Theorem 18.7 in Liptser and Shiryaev (2000). In the same way as the proof of Proposition 3.5, we get,

$$g(\tilde{\lambda}_s^{(1)} \wedge \tau_t, t \wedge \tau_t) - g(\tilde{\lambda}_0^{(1)}, 0) = \int_{(0, t \wedge \tau_t]} g(\tilde{\lambda}_s^{(1)} + \alpha, s) - g(\tilde{\lambda}_s^{(1)}, \alpha) ds(\tilde{N}_s - \tilde{\lambda}_s^{(1)}) ds$$

Thus, we have

$$\tilde{E} \left[ g(\tilde{\lambda}_s^{(1)} \wedge \tau_t, t \wedge \tau_t) - g(\tilde{\lambda}_0^{(1)}, 0) \right] = \left. \frac{K_2}{K_1} \right( e^{K_{1|t}} - 1 \right).$$

From the Fatou’s lemma, we have

$$\tilde{E} \left[ e^{M_{1|\tilde{\lambda}_0^{(1)}}} e^{K_{1|t}} \right] \left( \tilde{\lambda}_0^{(1)} = x \right) - e^{M_{1|x}} \leq \frac{K_2}{K_1} \left( e^{K_{1|t}} - 1 \right).$$

Then, from the stationarity of $\tilde{\lambda}_0^{(1)}$, we also get the finiteness of moments of $\tilde{\lambda}_t^{(1)}$ by

$$\tilde{E} \left[ e^{M_{1|\tilde{\lambda}_t^{(1)}}} \right] \leq \frac{K_2}{K_1}.$$

For the operator $\mathcal{A}$ same as (3.2), $\tilde{E} \left[ p(\tilde{\lambda}_s^{(1)}) \tilde{\lambda}_s^{(1)} \right] = e^{(t-s)\mathcal{A}} p(\tilde{\lambda}_s^{(1)})$ a.s. holds for any $p \in \mathfrak{P}$ in the same way of the proofs for Lemmas 3.7, 3.8 and (3.3). These properties lead the Markovian property of $\tilde{\lambda}_t^{(1)}$ as in the proof of Theorem 3.9. Furthermore, for almost every $\tilde{\omega}$,

$$\tilde{E} \left[ p(\tilde{\lambda}_s^{(1)}) \tilde{\lambda}_s^{(1)} \right] (\tilde{\omega}) = e^{(t-s)\mathcal{A}} p \left( \tilde{\lambda}_s^{(1)} (\tilde{\omega}) - \mu \right)$$

$$= e^{(t-s)\mathcal{A}} p \left( \tilde{\lambda}_0^{(1)} - \mu \right) = \tilde{E} \left[ p(\tilde{\lambda}_s^{(1)}) \right] \left|_{X=X_t^{(1)}} \right. (\tilde{\omega}).$$

Thus, similarly as the proofs of Lemma 6.7 and Proposition 3.12, we get the conclusion. □

Proof (Proof of Proposition 3.13) Let $t \geq 0$ and $A \in \mathfrak{B}(\mathbb{R}^2)$. By taking $s = 0$, $f(x) = 1_A(x)$ and integrating both sides of the equation of Lemma 6.8,

$$\int_{\Omega \times \Omega} \left( X^X_t(\omega) \right)_{X=X_0(\tilde{\omega})} dP(\omega) d\tilde{P}(\tilde{\omega}) = \tilde{P} \left[ X_t \in A \right] = \mathcal{P}^X [A].$$
where we used the stationarity of $\tilde{X}$. The above left hand side equals
\[
\int_{\Omega \times \Omega} 1_{A} \left( X_{t}^{\tau} (\omega) \big|_{X = \tilde{X}_{0}(\omega)} \right) dP (\omega) d\tilde{P} (\tilde{\omega}) = \int_{\mathbb{R}^{3}} \int_{\Omega} 1_{A} \left( X_{t}^{\tau} (\omega) \right) dP (\omega) dP^{\tilde{X}} (x)
\]
\[
= \int_{\mathbb{R}^{3}} P^{\tau} (x, A) dP^{\tilde{X}} (x),
\]
and then we are done. \hfill \Box

### 6.4.2 Ergodicity

The $V$-geometric ergodicity has been proved for the process $X^{(1)}$, see Proposition 4.5 in Clinet and Yoshida (2017). For the Hawkes core process $X = (X^{(1)}, X^{(2)}, X^{(3)})$, we can also prove it in a similar way. That is, we apply Theorem 6.1 in Meyn and Tweedie (1993). First, we again consider the extended generator and the drift criterion. The following lemma is proved by the same method as Proof of Proposition 4.5. in Clinet and Yoshida (2017).

**Lemma 6.9** Let $\alpha$, $\beta$ and $\mu$ be the parameters of the Hawkes process $N^{X_{1}}$. For a differentiable function $f : \mathbb{R}^{3} \to \mathbb{R}$, we define the operator $\mathcal{A}_{X}$ by
\[
\mathcal{A}_{X} f (y) = (\mu + y_{1}) \left\{ f \left( y + \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \right) - f (y) \right\} + \left( \partial_{y} f (y) \right)^{T} \begin{pmatrix} 0 \\ y_{1} \\ 2 y_{2} \end{pmatrix}, \quad y = \begin{pmatrix} y_{1} \\ y_{2} \\ y_{3} \end{pmatrix} \in \mathbb{R}^{3}.
\]

Then, there exist a positive constant vector $M = (M_{1}, M_{2}, M_{3})$ and positive constants $K_{1}, K_{2}$ such that for $V (y) = e^{M y}$,
\[
\mathcal{A}_{X} V (y) \leq -K_{1} V (y) + K_{2}.
\]

Then, we can prove Proposition 3.14 with the help of this operator $\mathcal{A}_{X}$.

**Proof** (Proof of Proposition 3.14) Now, it is proved in the completely same way as the proof of Proposition 3.5 replaced $g (x, t) = e^{M_{1} x_{t}} e^{K_{1} t}$ and $\mathcal{A}$ by $g_{X} (x, t) = e^{M_{X} x} e^{K_{1} t}$ and $\mathcal{A}_{X}$ satisfying $\mathcal{A}_{X} g_{X} (x, t) = e^{K_{1} t} (\mathcal{A}_{X} e^{M_{X} x} + K_{1} e^{M_{X}})$ respectively. \hfill \Box

Second, we need to show that every compact set is petite for some skeleton chain, i.e. there exists $\delta > 0$ such that for any compact set $C \in \mathscr{B} (\mathbb{R}^{3})$, we can choose a probability measure $a$ on $\mathbb{Z}_{+}$ and a non-trivial measure $\phi_{a}$ on $\mathbb{R}^{3}$ such that
\[
\sum_{n \in \mathbb{Z}_{+}} P^{\delta n} (x, A) a [n] \geq \phi_{a} [A] \quad \text{for all } x \in C \quad \text{and} \quad A \in \mathscr{B} (\mathbb{R}^{3}).
\]

The following concepts are closely related to petite sets. We call $\{ X_{\delta n}^{x} \}_{n \in \mathbb{Z}_{+}}$ is an irreducible, if there exists a finite measure $\phi$ on $\mathscr{B} (\mathbb{R}^{3})$ such that if $\phi [A] > 0$ then
\[
\sum_{n=1}^{\infty} P^{\delta n} (x, A) > 0 \quad \text{for any } x \in \mathbb{R}_{+}^{3}.
\]

Moreover, we call $\{ X_{\delta n}^{x} \}_{n \in \mathbb{Z}_{+}}$ is a $T$-chain, if there exist $k \in \mathbb{Z}_{+}$ and non-trivial kernel $T$ such that
Lemma 6.10 Suppose that \( \{X^x_{\delta n}\}_{n \in \mathbb{Z}_+} \) is an irreducible \( T \)-chain. Then, every compact set is petite.

Furthermore, we call \( x^* \in \mathbb{R}^3 \) is reachable, if for any open set \( G \in \mathcal{B}(\mathbb{R}^3) \) with \( x^* \in G \),

\[
\sum_{n=0}^{\infty} p^{\delta n}(y, G) > 0 \quad \text{for any} \quad y \in \mathbb{R}^3_+.
\]

**Proof** (Proof of Proposition 3.15) We only have to prove that there exists \( \delta > 0 \) such that \( \{X^x_{\delta n}\}_{n \in \mathbb{Z}_+} \) is an irreducible \( T \)-chain. First, we check the \( T \)-chain property.

Denote the \( i \)-th jump time of \( N^x_i \) by \( \tau^x_i \). Let \( \Delta \tau^x_i \) be the interval time between the \( (i-1) \)-th and \( i \)-th jump of \( N^x_i \), i.e., \( \Delta \tau^x_i = \tau^x_i - \tau^x_{i-1} \). As mentioned in Lemma A.4 of Clinet and Yoshida (2017), \( \Delta \tau^x_i \) has the conditional probability density (with respect to Lebesgue measure)

\[
f_{\Delta \tau^x_i}(t | X^x_{\tau^{x}_{i-1}} = y) = \left( \mu + y_1 e^{-\beta t} \right) \exp \left( \int_0^t \mu + y_1 e^{-\beta s} ds \right),
\]

where \( y = (y_1, y_2, y_3)^t \in \mathbb{R}^3_+ \). Moreover, it is known that

\[
f_{\max(\Delta \tau^x_i, \ldots, \Delta \tau^x_j)}(t_1, \ldots, t_i | y) = f_{\Delta \tau^x_i}(t_i | X^x_{\tau^{x}_{i-1}} = X(t_1, \ldots, t_{i-1} | y)) \times f_{\Delta \tau^x_j}(t_j | X^x_{\tau^{x}_{j-1}} = X(t_1, \ldots, t_{j-1} | y)) \times \ldots \times f_{\Delta \tau^x_j}(t_1 | X^x_{\tau^{x}_{1}} = y),
\]

where denote \( \sum_{k=i}^j t_k \) by \( T(i,j) \) and

\[
X(t_1, \ldots, t_j | y) = \begin{pmatrix} X(1)(t_1, \ldots, t_j | y) \\ X(2)(t_1, \ldots, t_j | y) \\ X(3)(t_1, \ldots, t_j | y) \end{pmatrix} = \begin{pmatrix} y_1 e^{-\beta T(1,j)} + \sum_{l=1}^j \alpha e^{-\beta T(l+1,j)} \\ (y_1 T(1,j)) + y_2 e^{-\beta T(1,j)} + \sum_{l=1}^j \alpha T(l+1,j) e^{-\beta T(l+1,j)} \\ (y_1 T(1,j) + 2y_2 T(1,j) + y_3) e^{-\beta T(1,j)} + \sum_{l=1}^j \alpha T(l+1,j) e^{-\beta T(l+1,j)} \end{pmatrix}.
\]

Note that \( f_{\max(\Delta \tau^x_i, \ldots, \Delta \tau^x_j)}(t_1, \ldots, t_i | y) \) is obviously smooth in \( y \). Then, for any \( \delta > 0 \) and \( A \in \mathcal{B}(\mathbb{R}^3) \),

\[
P^\delta(x, A) = P \left[ X^x_{\delta} \in A \right] \\ \geq P \left[ X^x_{\delta} \in A, \exists (j | \tau^x_j < \delta) = 3 \right] \\ = \int_{\mathbb{R}_+^4} 1_{(x(\delta,t_1,t_2,t_3,t_4) \in A)} |(t_{1,3} < \delta) \cap (t_{1,4} \geq \delta)| f_{\max(\Delta \tau^x_i, \ldots, \Delta \tau^x_j)}(t_1, t_2, t_3, t_4 | x) dt_1 dt_2 dt_3 dt_4.
\]
where
\[ \tilde{X}(\delta; t_1, t_2, t_3|x) = \begin{pmatrix} x_1 e^{-\beta \delta} + \sum_{l=1}^{3} \alpha l e^{-\beta (\delta - T(1,l))} \\
(x_1 \delta + x_2) e^{-\beta \delta} + \sum_{l=1}^{3} \alpha (\delta - T(1,l)) e^{-\beta (\delta - T(1,l))} \\
(x_1 \delta^2 + 2x_2 \delta + x_3) e^{-\beta \delta} + \sum_{l=1}^{3} \alpha (\delta - T(1,l))^2 e^{-\beta (\delta - T(1,l))} \end{pmatrix} \]
and it is obviously smooth in \( x \). However, the indicator function \( 1_{\tilde{X}(\delta; t_1, t_2, t_3|x) \in A} \) is not always lower semi-continuous in \( x \). Thus, we consider a change of variable for the map \( H_x,\delta : (t_1, t_2, t_3) \mapsto \tilde{X}(\delta; t_1, t_2, t_3|x) \), as in Proof of Lemma A.3 of (Clinet and Yoshida 2017). Denote the Jacobian matrix of \( H_x,\delta \) at \((t_1, t_2, t_3)\) by \( J_\delta(t_1, t_2, t_3) \). Then, completely elementary calculations leads
\[ J_\delta(t_1, t_2, t_3) = (J_{i,j})_{i,j=1,2,3}, \]
where for \( j = 1, 2, 3 \)
\[ J_{1,j} = \sum_{l=j}^{3} \alpha l e^{-\beta (\delta - T(1,l))}; \quad J_{2,j} = \sum_{l=j}^{3} \alpha \{ \beta (\delta - T(1,l)) - 1 \} e^{-\beta (\delta - T(1,l))}, \]
and \( J_{3,j} = \sum_{l=j}^{3} \alpha \{ \beta (\delta - T(1,l))^2 - 2(\delta - T(1,l)) \} e^{-\beta (\delta - T(1,l))} \).

The determinant of the Jacobian matrix has the following representation.
\[ |J_\delta(t_1, t_2, t_3)| = \prod_{l=1}^{3} \alpha l e^{-\beta (\delta - T(1,l))} \times \begin{vmatrix} 1 \\ (\delta - T(1,1)) \end{vmatrix} \begin{vmatrix} 1 \\ (\delta - T(1,2)) \end{vmatrix} \begin{vmatrix} 1 \\ (\delta - T(1,3)) \end{vmatrix}. \]
It is a Vandermonde determinant and thus not zero if \((t_1, t_2, t_3) = (\tau, \tau, \tau)\) for \( \tau \in (0, \delta/3) \).
We consider a neighborhood at the such point \((t_1, t_2, t_3) = (\tau, \tau, \tau)\). Set \( B(t_1, t_2, t_3, t_4) = \{ T(1,3) < \delta \} \cap \{ T(1,4) > \delta \} \cap \{ (t_1, t_2, t_3) \in (\tau - \varepsilon, \tau + \varepsilon)^3 \} \) for sufficient small \( \varepsilon > 0 \). Then, we get a non-trivial component \( T(x, A) \) as below.
\[ P^\delta(x, A) \geq \int_{[0,\delta]_+} 1_{\tilde{X}(\delta; t_1, t_2, t_3|x) \in A} |J_\delta(T(1,3) \leq \delta) \cap \{ T(1,4) \geq \delta \}| f^{(\Delta T_1^+, \Delta T_2^+, \Delta T_3^+, \Delta T_4^+)}(t_1, t_2, t_3, t_4|x) dt_1 dt_2 dt_3 dt_4 \]
\[ \geq \int_{[0,\delta]_+} 1_{\{y_1, y_2, y_3\} \in A} |J_\delta(H_{x,\delta}^{-1}(y_1, y_2, y_3)| f^{(\Delta T_1^+, \Delta T_2^+, \Delta T_3^+, \Delta T_4^+)}(H_{x,\delta}^{-1}(y_1, y_2, y_3), t_4|x) \]
\[ \left|J_\delta(H_{x,\delta}^{-1}(y_1, y_2, y_3))\right|^{-1} dy_1 dy_2 dy_3 dt_4 =: T(x, A). \]
Since \( B(t_1, t_2, t_3, t_4) \) is a countable union of open intervals, continuity of \( H_{x,\delta}^{-1} \) in \( x \) leads that \( x \mapsto T(x, A) \) is lower semi-continuous. Thus, \( \{ X_{5n}^x \}_{n \in \mathbb{Z}_+} \) is a \( T \)-chain.

Finally, we prove that \( \{ X_{5n}^x \}_{n \in \mathbb{Z}_+} \) is irreducible. Since \( \{ X_{5n}^x \}_{n \in \mathbb{Z}_+} \) is a \( T \)-chain, we only have show that there exists a reachable point \( x^* \in \mathbb{R}^3_+ \), i.e. for any open set \( O \in \mathcal{B}(\mathbb{R}^3) \) containing \( x^* \),
\[ \sum_{n=0}^{\infty} P_{\delta n}(y, O) > 0 \text{ for any } y \in \mathbb{R}^3_+, \]
see Proposition 6.2.1 in Meyn and Tweedie (1993). However, we can easily show that \((0, 0, 0)\) is a reachable point. Indeed, if a jump will never occur, for any neighborhood \( O \) of \((0, 0, 0)\),
On the other hand, we have

\[ X_{\delta n}^x \in O \] for sufficient large \( n \in \mathbb{N} \). By the form of \( f^{A_{\tau_i}}(t|x) \), the probability there is no jump on \([0, \delta n]\) is positive. Thus, we get the conclusion.

\[ \square \]

### 6.5 Proofs of Section 4

In this subsection, we will prove Theorem 4.6. For this purpose, it is enough to confirm that there exist some constants satisfying (2.15) and the conditions [A1]–[A3], [B0]–[B4], [C1] hold. We explain each condition separately by dividing each small section.

### 6.5.1 Proof of Proposition 4.4 (condition [A1])

In Markovian framework, as mentioned in Kusuoka and Yoshida (2000) and Yoshida (2004), the mixing property is derived from the ergodicity. Concretely, the geometric mixing property is reduced to the following property;

[A1'] There exists a positive constant \( a \) such that

\[
\sup_{f \in \mathcal{F}_{[1, \infty)}, \|f\|_\infty \leq 1} \|E[f] - E[f]X_s\|_{L^1(P)} < a^{-1}e^{-a(t-s)} \quad \text{for any} \quad t > s > 0.
\]

**Proposition 6.11** The Markovian property in Proposition 3.11 and [A1'] lead [A1].

**Proof** For any \( f \in \mathcal{F}_{[0, s]} \) and \( g \in \mathcal{F}_{[t, \infty)} \) with \( \|f\|_\infty \leq 1 \) and \( \|g\|_\infty \leq 1 \),

\[
|E[fg] - E[f]E[g]| = |E[f(g - E[g])]| = |E[fE[g - E[g]|\mathcal{F}_{[0, s]}]||
\]

\[
\leq \|E[g - E[g]|\mathcal{F}_{[0, s]}]\|_{L^1(P)} = \|E[g|X_s] - E[g]\|_{L^1(P)}
\]

\[
\leq a^{-1}e^{-a(t-s)}.
\]

\[ \square \]

**Proof** (Proof of Proposition 4.4) We confirm that [A1'] follows from Proposition 3.15. Let \( s \leq t \) and \( f \in \mathcal{F}_{[t, \infty)} \) with \( \|f\|_\infty \leq 1 \). From the Markovian property, we have

\[
E[f|X_s] = E[E[f|\mathcal{F}_{[0, s]}]|X_s] = E[f|X_t]|X_s].
\]

There exists a measurable function \( g \) such that \( E[f|X_t] = g(X_t) \) and \( \|g\|_\infty \leq 1 \). From Proposition 3.12, we get

\[
E[f|X_s] = E[g(X_t)|X_s] = \int_{\mathbb{R}^3} g(y)P^{t-s}(X_s, dy).
\]

On the other hand, we have

\[
E[f] = E[g(X_t)] = \int_{\mathbb{R}^3} g(y)P^t(X_0, dy).
\]

Therefore, by using Proposition 3.15,

\[
\sup_{f \in \mathcal{F}_{[1, \infty)}, \|f\|_\infty \leq 1} \|E[f|X_s] - E[f]\|_{L^1(P)}
\]

\[
\leq \sup_{g: \|g\|_\infty \leq 1} \left\| \int_{\mathbb{R}^3} g(y)P^{t-s}(X_s, dy) - \int_{\mathbb{R}^3} g(y)P^t(X_0, dy) \right\|_{L^1(P)}.
\]
Finally, from Proposition 3.14, we may choose sufficient small $a > 0$ that satisfies [A1']. □

### 6.5.2 Condition [A2]

$Z_0 \in \bigcap_{p \geq 1} L^p(P)$ and $P[Z_0] = 0$ are obvious. We can write each component of $Z_{t+h}^I$ as

$$
\int_t^{t+h} \frac{p_1(X_s)}{\lambda_s^2} d\bar{N}_s + \int_t^{t+h} \frac{p_2(X_s)}{\lambda_s^2} ds - E \left[ \int_t^{t+h} \frac{p_2(X_s)}{\lambda_s^2} ds \right]
$$

where $p_1$ and $p_2$ are 3-variable polynomial functions. From Proposition 3.14, we have $\sup_t |X_t|_{L^p(P)} < \infty$ for any $p > 1$. By considering $t \in [0, T]$ for an arbitrary $T > 0$, $\int_0^t p_1(X_s)/\lambda_s^2 d\bar{N}_s$ is a square integrable martingale, see Theorem 18.8 in Liptser and Shiryaev (2000). Thus, we immediately get $E[Z_{t+h}^I] = 0$ for any $\Delta > 0$ and $t > 0$.

The rest of the proof is $\sup_{r \in \mathbb{R}, 0 \leq h \leq \Delta} \|Z_{t+h}^I\|_{L^p(P)} < \infty$. When we consider the $L^p$ boundedness, it is enough to consider the form of $p = 2^k$ for $k \in \mathbb{N}$. We get

$$
\left\| E \left[ \int_t^{t+h} \frac{p_2(X_s)}{\lambda_s^2} ds \right] \right\|_{L^p(P)} < \sup_s E[|p_2(X_s)|] h.
$$

Moreover, since $h^{-1} ds$ is a probability measure on $[t, t+h]$, by the Jensen’s inequality,

$$
\left\| \int_t^{t+h} \frac{p_2(X_s)}{\lambda_s^2} ds \right\|_{L^p(P)} \leq \left( E \left[ \int_t^{t+h} \left( \frac{p_2(X_s)}{\lambda_s^2} h^{-1} ds \right)^p \right] \right)^{1/p} \leq \sup_s \|p_2(X_s)\|_{L^p(P)} h.
$$

On the other hand,

$$\mathcal{M}_h = \int_t^{t+h} \frac{p_1(X_s)}{\lambda_s^2} d\bar{N}_s
$$

is also a square integrable martingale. Then, the Burkholder–Davis–Gundy inequality leads that there exists a positive constant $C_k$ (take again new $C_k$ in the last step) such that

$$
E \left[ |\mathcal{M}_h|^{2k} \right] \leq C_k E \left[ |\mathcal{M}_h|^{2k-1} \right] = C_k E \left[ \int_t^{t+h} \left( \frac{p_1(X_s)}{\lambda_s^2} \right)^2 d\bar{N}_s \right]^{2k-1}.
$$
\[ \leq C_k \left( E \left[ \left| \int_t^{t+h} \left( \frac{p_1(X_s)}{\lambda_s^2} \right)^2 d\tilde{N}_s \right|^{2k-1} \right] \right. \\
+ \left. E \left[ \left| \int_t^{t+h} \left( \frac{p_1(X_s)}{\lambda_s^2} \right)^2 \lambda_s d\tilde{N}_s \right|^{2k-1} \right] \right), \]

where \([\mathcal{M}]_h\) represents the quadratic variation of \(\mathcal{M}_h\). We used the Jensen’s inequality in the last estimation. By induction, one gets some constant \(Q_k\) (take again new \(Q_k\) in the last step) such that

\[ E \left[ \left| \mathcal{M}_h \right|^{2^k} \right] \leq Q_k \sum_{j=1}^{k} E \left[ \left| \int_t^{t+h} \left( \frac{p_1(X_s)}{\lambda_s^2} \right)^{2^j} \lambda_s d\tilde{N}_s \right|^{2^{k-j}} \right] \]

\[ \leq Q_k \sum_{j=1}^{k} E \left[ \int_t^{t+h} \left( \frac{p_1(X_s)}{\lambda_s^2} \right)^{2^k} \lambda_s^{2^k-j} h^{2^k-j} h^{-1} d\tilde{N}_s \right] \]

\[ \leq Q_k (h + 1)^{2^{k-1}}. \]  \hspace{1cm} (6.17)

Therefore, for any \(\Delta > 0\) and \(p > 0\), \(\sup_{t \in \mathbb{R}_+, 0 \leq h \leq \Delta} \| Z_{t+h}^{\ell} \|_{L^p(P)} < \infty \) holds. Then, the condition [A2] is verified.

### 6.5.3 Condition [A3]

The condition [A3] follows from Lemma 3.15. and the proof of Lemma A.7. in Clinet and Yoshida (2017).

### 6.5.4 Condition [B0]

(i), (ii) and (iv) are obvious. (iii) immediately follows a square integrable martingale property:

\[ Cov \left[ \int_0^t \frac{p_1(X_s)}{\lambda_s} d\tilde{N}_s, \int_0^t \frac{p_2(X_s)}{\lambda_s} d\tilde{N}_s \right] = E \left[ \int_0^t \frac{p_1(X_s) p_2(X_s)}{\lambda_s^2} d[\tilde{N}_s] \right] \]

\[ = E \left[ \int_0^t \frac{p_1(X_s) p_2(X_s)}{\lambda_s} d\tilde{N}_s \right] \]

for any 3-variable polynomial functions \(p_1\) and \(p_2\).

### 6.5.5 Condition [B1]

We take any constant \(L > 1\). From (4.1) and (6.17), we immediately deduce that for any \(k \in \mathbb{N}\)

\[ E \left[ \left| T^{-\frac{1}{2}} l_a(\theta_0) \right|^{2^k} \right] \leq Q_k (1 + T^{-1})^{2^{k-1}}. \]

Therefore, the condition [B1] holds for any \(q_1 > 1\). In particular, we can choose \(q_1\) satisfying \(q_1 > 3L\).
6.5.6 Condition [B2]

Let $L$, $q_1$ and $q_3$ be positive constants with $L > 1$, $q_1 > 3L$ and $q_3 > \frac{q_1 L}{q_1 - 3L}$. We arbitrarily set a positive constant $q_2$ with $q_2 > \max \left(3, \frac{3q_1 L}{q_1 - 3L}\right)$ for given constants $L$ and $q_1$. Let

$$Y_t(\theta) = \left( X_t^{(1)}(\theta_0), X_t^{(1)}(\theta), X_t^{(2)}(\theta), X_t^{(3)}(\theta), X_t^{(4)}(\theta) \right)$$

for $\theta = (\mu, \alpha, \beta)$. From the relation

$$\partial_\theta X_t^n(\theta) = \begin{pmatrix} 0 \\ \alpha^{-1} X_t^n(\theta) \\ -X_t^{n+1}(\theta) \end{pmatrix}$$

and a verification of the permutation rule of the symbol $\partial_\theta$ and $\int_0^T$, we can write, for both of the case $k = 2$ and $k = 3$,

$$T^\frac{\gamma}{2} \left( T^{-1} l_{a_1 \ldots a_k}(\theta) - \nu_{a_1 \ldots a_k}(\theta) \right) = T^\frac{\gamma}{2} - 1 \int_0^T \frac{p_1(Y_s(\theta))}{\lambda_s^4(\theta)} d\tilde{N}_s + T^\frac{\gamma}{2} \left\{ \frac{1}{T} \int_0^T \frac{p_2(Y_s(\theta))}{\lambda_s^4(\theta)} ds - E \left[ \frac{1}{T} \int_0^T \frac{p_2(Y_s(\theta))}{\lambda_s^4(\theta)} ds \right] \right\}$$

with some polynomial functions $p_1$ and $p_2$. Lemma A.5. in Clinet and Yoshida (2017) guarantees

$$\sup_t \sum_{i=0}^4 \| \sup_{\theta \in \Theta} \partial_\theta^i \lambda_t(\theta) \|_{L^p(P)} < \infty$$

(6.18)

for any $p > 1$. Thus, we have $\sup_t \| \sup_{\theta \in \Theta} Y_t(\theta) \|_{L^p(P)} < \infty$ for any $p > 1$. Moreover, $Y_t(\theta)$ is $\sigma(N_s; s \leq t)$-predictable. From the restriction of (2.15), $\frac{\gamma}{2} - 1 < -\frac{1}{2}$ holds. Then, in the same method of the proof of the condition [A2], we can see that

$$\sup_{T > 0, \theta \in \Theta} E \left[ \frac{1}{T} \int_0^T \frac{p_1(Y_s(\theta))}{\lambda_s^4(\theta)} d\tilde{N}_s \right] < \infty$$

for any $k \in \mathbb{N}$.

The later term is estimated by using the ergodicity of $X_t^{(1)}(\theta_0)$. Let

$$\tilde{Y}(s, t, \theta) = \left( X_t^{(1)}(\theta_0), \int_{(s,t)} \alpha e^{-\beta(t-u)} dN_u^{x_1}, \int_{(s,t)} \alpha(t-u) e^{-\beta(t-u)} dN_u^{x_1}, \int_{(s,t)} \alpha(t-u)^2 e^{-\beta(t-u)} dN_u^{x_1}, \int_{(s,t)} \alpha(t-u)^3 e^{-\beta(t-u)} dN_u^{x_1} \right).$$

Denote $D^\dagger_+ (\mathbb{R}^5_+, \mathbb{R})$ as the set of functions $\psi : \mathbb{R}^5_+ \to \mathbb{R}$ that satisfy:

- $\psi$ are of class $C^1(\mathbb{R}^5_+)$.
- $\psi$ and $| \nabla \psi |$ are polynomial growth.

By replacing $X^\alpha(t, \theta)$ by $Y_t(\theta)$ and $\tilde{X}^\alpha(s, t, \theta)$ by $\tilde{Y}(s, t, \theta)$ in the proof of Lemma A.6. and using Lemma 3.16. in Clinet and Yoshida (2017), we can get the following ergodicity property: There exist a mapping $\pi : D^\dagger_+ (\mathbb{R}^5_+, \mathbb{R}) \times \Theta \to \mathbb{R}$ and a constant $\gamma' \in (0, \frac{1}{2})$ such that for any $\psi \in D^\dagger_+ (\mathbb{R}^5_+, \mathbb{R})$ and for any $p > 1$,

$$\sup_{\theta \in \Theta} T^{\gamma'} \left\| \frac{1}{T} \int_0^T \psi(Y_s(\theta)) ds - \pi(\psi, \theta) \right\|_{L^p(P)} \to 0 \text{ as } T \to \infty.$$
However, in the case of the exponential Hawkes process, we can choose $\gamma' \in (0, \frac{1}{2})$ arbitrarily. This arbitrariness follows from the fact $\|Y_t(\theta) - \bar{Y}_t(\theta)\|_{L_1(\mathcal{P})}$ is exponentially decreasing uniformly in $\theta$ for some stationary process $\bar{Y}_t(\theta)$, see the proof of the stability condition part in Lemma A.6. of Clinet and Yoshida (2017). Therefore, by taking $\gamma' \in (0, \frac{1}{2})$ and $\gamma = 2\gamma'$ satisfying $\frac{2}{3} + \max\left(\frac{L}{q_2}, \frac{L}{3q_3}\right) < \gamma < 1 - \frac{L}{q_1}$, we get

$$
\sup_{T \in \Theta, \theta, |x|=1} T^2 \left\{ \frac{1}{T} \int_0^T \frac{p_2(Y_s(\theta))}{\lambda_s^2(\theta)} ds - \mathbb{E} \left[ \frac{1}{T} \int_0^T \frac{p_2(Y_s(\theta))}{\lambda_s^2(\theta)} ds \right] \right\} \to 0 \quad \text{as} \quad T \to \infty
$$

for any $p > 1$. It means that the condition [B2] holds for any $q_2 > \max\left(3, \frac{3q_1L}{q_1 - 3L}\right)$ and some $\gamma$ with $\frac{2}{3} + \max\left(\frac{L}{q_2}, \frac{L}{3q_3}\right) < \gamma < 1 - \frac{L}{q_1}$.

### 6.5.7 Condition [B3]

We only have to show that there exist an open set $\tilde{\Theta}$ including $\theta_0$ and a positive constant $T_0$ such that

$$
\inf_{T > T_0, \theta \in \Theta, |x|=1} \left| x' v_{ab}(\theta) \right| > 0. \quad (6.19)
$$

Because, if (6.19) holds, continuity of $v_{ab}(\theta)$ and $x' v_{ab}(\theta) \neq 0$ lead

$$
\left| \int_0^1 x' v_{ab}(\theta_1 + s(\theta_2 - \theta_1)) ds \right| > \inf_{\theta \in \Theta} \left| x' v_{ab}(\theta) \right|
$$

for any $\theta_1, \theta_2 \in \tilde{\Theta}$, $T > T_0$ and $x$ with $|x| = 1$. Therefore, we consider to prove (6.19). We can write

$$
v_{ab}(\theta) = -\mathbb{E} \left[ \frac{1}{T} \int_0^T \frac{(\partial_\theta \lambda_s(\theta))^\otimes 2}{\lambda_s^2(\theta)} \lambda_s(\theta_0) ds \right] + \mathbb{E} \left[ \frac{1}{T} \int_0^T \frac{\partial_\theta^2 \lambda_s(\theta)}{\lambda_s(\theta)} \lambda_s(\theta_0) ds \right].
$$

With the help of (6.18), for the first term, we have

$$
\left| g_T - \mathbb{E} \left[ \frac{1}{T} \int_0^T \frac{(\partial_\theta \lambda_s(\theta))^\otimes 2}{\lambda_s^2(\theta)} \lambda_s(\theta_0) ds \right] \right| \
\leq \frac{\sup_{\theta \in \Theta} \left| \theta_0 - \theta \right|}{T} \mathbb{E} \left[ \int_0^T \sup_{\theta \in \Theta} \left| \partial_\theta \lambda_s(\theta) \right| ds \right] \lambda_s(\theta_0) ds \right| \leq C_{\Theta, 1} |\theta_0 - \theta|,
$$

where $C_{\Theta, 1}$ is a positive constant that does not depend on $T$. For the second term, we also get

$$
\left| \mathbb{E} \left[ \frac{1}{T} \int_0^T \frac{\partial_\theta^2 \lambda_s(\theta)}{\lambda_s(\theta)} (\lambda_s(\theta_0) - \lambda_s(\theta)) ds \right] \right| \
\leq \frac{\sup_{\theta \in \Theta} \left| \theta_0 - \theta \right|}{T} \mathbb{E} \left[ \int_0^T \sup_{\theta \in \Theta} \left| \partial_\theta \lambda_s(\theta) \right| ds \right] \sup_{\theta \in \Theta} \left| \partial_\theta \lambda_s(\theta) \right| ds \right| \leq C_{\Theta, 2} |\theta_0 - \theta|,
$$

where $C_{\Theta, 2}$ is a positive constant independent of $T$. Since we have assumed $g_T$ is non-singular for large $T$ in the condition [A3], we may choose $\tilde{\Theta}$ and $T_0 > 0$ such that

$$
\inf_{T > T_0, \theta \in \tilde{\Theta}, |x|=1} \left| x' v_{ab}(\theta) \right| \geq \inf_{T > T_0, |x|=1} \left| x' g_T \right|
$$

 Springer
− \sup_{T > T_0, \theta \in \Theta} g_T - E \left[ \frac{1}{T} \int_{0}^{T} \left( \frac{1}{\lambda^2_s(\theta)} \lambda_s(\theta_0) \right) ds \right] \\
− \sup_{T > T_0, \theta \in \Theta} E \left[ \frac{1}{T} \int_{0}^{T} \left( \frac{\partial^2 \lambda_s(\theta)}{\lambda_s(\theta)} (\lambda_s(\theta_0) - \lambda_s(\theta)) \right) ds \right] > 0.

6.5.8 Condition [B4]

Let $L$ and $q_1$ be positive constants with $L > 1$ and $q_1 > 3L$. We apply Sobolev’s inequality (see Theorem 4.12 of Adams and Fournier 2003). We take any integer $q_3 > \max \left( 3, \frac{q_1 L}{q_1 - 3L} \right)$ and some constant $K(\Theta, q_3)$ such that

$$E \left[ \sup_{\theta \in \Theta} \left| \frac{1}{T} \int_{0}^{T} p_1(Y'_s(\theta)) \lambda^8_s(\theta) d\tilde{N}_s \right|^{q_3} \right] \leq K(\Theta, q_3) \left\{ \int_{\Theta} E \left[ \left| \frac{1}{T} l_{a_1 \ldots a_4}(\theta) \right|^{q_3} \right] d\theta \right. \\
+ \int_{\Theta} E \left[ \left| \frac{1}{T} \theta l_{a_1 \ldots a_4}(\theta) \right|^{P} \right] d\theta \left. \right\} \lesssim \sum_{k=1}^{4} \sup_{\theta \in \Theta} E \left[ \left| \frac{1}{T} l_{a_1 \ldots a_4}(\theta) \right|^{q_3} \right].$$

Let $Y'_t(\theta) = (X^{(1)}_t(\theta_0), X^{(2)}_t(\theta), X^{(3)}_t(\theta), X^{(4)}_t(\theta), X^{(5)}_t(\theta))$. We may easily confirm that

$$\frac{1}{T} l_{a_1 \ldots a_4}(\theta) = \frac{1}{T} \int_{0}^{T} \frac{p_1(Y'_s(\theta))}{\lambda^8_s(\theta)} d\tilde{N}_s + \frac{1}{T} \int_{0}^{T} \frac{p_2(Y'_s(\theta))}{\lambda^8_s(\theta)} ds$$

with some polynomial functions $p_1$ and $p_2$. In a similar way as Lemma A.5 in Clinet and Yoshida (2017), we can prove that $\sup_{T > 0, \theta \in \Theta} \left\| Y'_t(\theta) \right\|_{L^p(P)} < \infty$ for any $p > 1$. Then, like (6.17), we have

$$\sup_{T > 0, \theta \in \Theta} E \left[ \left| \frac{1}{T} \int_{0}^{T} \frac{p_1(Y'_s(\theta))}{\lambda^8_s(\theta)} d\tilde{N}_s \right|^{2k} \right] < \infty$$

for any $k \in \mathbb{N}$. On the other hand, by the Jensen’s inequality,

$$\sup_{T > 0, \theta \in \Theta} E \left[ \left| \frac{1}{T} \int_{0}^{T} \frac{p_2(Y'_s(\theta))}{\lambda^8_s(\theta)} ds \right|^{P} \right] \leq \sup_{T > 0} \frac{1}{T} \int_{0}^{T} E \left[ \sup_{\theta \in \Theta} \left| \frac{p_2(Y'_s(\theta))}{\lambda^8_s(\theta)} \right|^{P} \right] ds < \infty$$

for any $p > 1$. Therefore, the condition [B4] holds for any constant $q_3 > \max \left( 3, \frac{q_1 L}{q_1 - 3L} \right)$.

6.5.9 Condition [C1]

In Theorem 4.6 of Clinet and Yoshida (2017), the convergence of moments is proved for $\sqrt{T} (\hat{\theta}_T - \theta_0)$. The condition [C1] directly follows from this statement.
References

Abergel F, Anane M, Chakraborti A, Toke IM (2016) Limit order books, 1st edn. Cambridge University Press, Cambridge. ISBN: 978-1-107-16398-0

Adams RA, Fournier JJF (2003) Sobolev Spaces, 2nd edn. Academic Press, London. ISBN: 978-0-1204-4143-3

Bhattacharya RN, Rao RR (1976) Normal approximation and asymptotic expansions, 1st edn. Wiley, London. ISBN: 978-0-470-07201-0

Brémaud P, Massoulié L (1996) Stability of nonlinear Hawkes processes. Ann Probab 24(3):1563–1588

Clinet S, Yoshida N (2017) Statistical inference for ergodic point processes and application to limit order book. Stoch Process Appl 127(6):1800–1839

Götze F, Hipp C (1978) Asymptotic expansions in the central limit theorem under moment conditions. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 42(1):67–87

Götze F, Hipp C (1983) Asymptotic expansions for sums of weakly dependent random vectors. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 64(2):211–239

Hall (1992) The bootstrap and Edgeworth expansion, 1st edn. Springer Series in Statistics. ISBN: 978-1-4612-4384-7

Hawkes AG (1971) Spectra of some self-exciting and mutually exciting point processes. R Stat Soc Publ 58(1):83–90

Kusuoka S, Yoshida N (2000) Malliavin calculus, geometric mixing, and expansion of diffusion functionals. Probab Theory Relat Fields 116(4):457–484

Liptser RS, Shiryaev AN (2000) Statistics of random processes: II. Applications, 2nd rev. and exp. ed. edn. Springer, Berlin. ISBN: 978-3-642-08365-5

Meyn SP, Tweedie RL (1992) Stability of Markovian processes I: criteria for discrete-time chains. Adv Appl Probab 24(3):542–574

Meyn SP, Tweedie RL (1993) Markov chains and stochastic stability, 1st edn. Springer, Berlin. ISBN: 978-1-4471-3267-7

Meyn SP, Tweedie RL (1993) Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. Adv Appl Probab 25(3):518–548

Oakes D (1975) The Markovian self-exciting process. J Appl Probab 12(1):69–77

Ogata Y (1979) Maximum likelihood estimation of Hawkes’ self-exciting point processes. Ann Inst Stat Math 31(1):145–155

Ogata Y (1981) On Lewis’ simulation method for point processes. IEEE Trans Inf Theory 27(1):23–31

Ogata Y (1988) Statistical models for earthquake occurrences and residual analysis for point processes. J Am Stat Assoc 83(401):9–27

Rizoiu MA, Lee Y, Mishra S, Xie L (2017) Hawkes processes for events in social media. Front Multimed Res 191–218

Sakamoto Y, Yoshida N (2004) Asymptotic expansion formulas for functionals of ε-Markov processes with a mixing property. Ann Inst Stat Math 56(3):545–597

Vladimir Filimonov DS (2012) Quantifying reflexivity in financial markets: towards a prediction of flash crashes. Phys Rev E 85(5)

Yoshida N (2004) Partial mixing and Edgeworth expansion. Probab Theory Relat Fields 129(4):559–624

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.