EXISTENCE AND ITERATIVE APPROXIMATION METHOD
FOR SOLVING MIXED EQUILIBRIUM PROBLEM UNDER
GENERALIZED MONOTONICITY IN BANACH SPACES

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Abstract. We study a new class of mixed equilibrium problem, in short MEP, under weakly relaxed $\alpha$–monotonicity in Banach spaces. This class of problems extends and generalizes some related fundamental results such as mixed variational-like inequalities, variational inequalities, and classical equilibrium problems as special cases. Existence and uniqueness of the solution to the problem is established. Auxiliary principle technique is used to obtain an iterative algorithm. Solvability of the auxiliary problem is established in the paper and finally the convergence of the iterates to the exact solution is proved. As applications of the approach developed in this paper, we study the existence and algorithmic approach for a general class of nonlinear mixed variational-like inequalities. The results obtained in this paper are interesting and improve considerably many existing results in literature.

1. Introduction. Study of equilibrium problems has been an interesting field of research as it includes and inter relates various fundamental areas like variational inequalities and its generalizations, such as, quasivariational and variational-like inequalities, optimization problems, minimax problems, and Game theory. The equilibrium problem is defined as follows:

(EP) Find $\bar{u} \in K$ such that $\phi(\bar{u}, v) \geq 0$, for all $v \in K$,

where $K$ is a nonempty subset of a topological vector space and $\phi : K \times K \to \mathbb{R}$ is a real-valued equilibrium bifunction, i.e. $\phi(u, u) = 0$ for all $u \in K$. This formulation was first used by Nikaido-Isoda [19] in 1955 to characterize the Nash equilibrium. After that in 1972, Ky Fan [8] re-initiated the study of the problem (EP) and established some existence results for its solution under the strong assumption that

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K is a compact convex subset of a real Hausdorff topological vector space. Since then, several people have weakened the compactness of K using some suitable form of coercivity condition on $\phi$. In 1975 and later in 1979, Mosco [18] and Joly-Mosco [12] introduced a general formulation of equilibrium problem, called implicit variational problem, which includes equilibrium problems, fixed point problems, variational and quasi-variational inequalities, and Nash equilibria as special cases. This general formulation of equilibrium problem is described by the sum of two bifunctions. Later, it is called mixed equilibrium problem. Inspired by the notion of monotonicity for operator in the sense of Minty, they introduced the notion of monotonicity for bifunctions and obtained some existence results for a solution of this kind of equilibrium problem under some weaker assumption than those in the results of Fan [8].

The appellation equilibrium problems appeared in the seminal paper by Blum-Oettli [3] where they provided the evidence of unifying aspect of (EP) and gave some fundamental notions and results. Later on these problems were studied extensively by various authors in the context of both Hilbert and Banach spaces under different kinds of generalized monotonicities [2, 13, 14, 20, 25]. All these papers are based on bifunction equilibrium problems.

Our aim in this paper is to study a general form of mixed equilibrium problem which reduces to bifunction equilibrium problem, mixed variational-like inequalities and variational inequalities under some special cases. We study this problem under weakly relaxed $\alpha$-monotonicity which is a weaker condition than monotonicity or relaxed $\alpha$-monotonicity, studied earlier, in [14]. The problem considered is as the following:

Find $\bar{w} \in K$ such that $\phi(y, \bar{w}, v) + b(\bar{w}, v) - b(\bar{w}, \bar{w}) \geq 0$, for all $y, v \in K$, \hspace{1cm} (1)

where $K$ be a nonempty closed and convex subset of a real Banach space $E$ with dual space $E^*$, $\phi : K \times K \times K \rightarrow \mathbb{R}$ is a function and $b : E \times E \rightarrow \mathbb{R}$ is a bifunction.

As applications, we study the existence and algorithmic approach of the following nonlinear mixed variational-like inequality: Find $\bar{w} \in K$ such that

$$\langle N(\bar{w}, y), \eta(v, \bar{w}) \rangle + b(\bar{w}, v) - b(\bar{w}, \bar{w}) \geq 0,$$

for all $y, v \in K$, \hspace{1cm} (2)

where $N : E \times E \rightarrow E^*$ is a nonlinear operator, $b : E \times E \rightarrow \mathbb{R}$ is a bifunction and $\eta : K \times K \rightarrow E$. Particular forms of the problem (2) have been considered by Fang-Huang [10] and Preda-Beldiman-Bătătorescu [24] by using the Kakutani-Fan-Glisberg fixed-point theorem and the notion of relaxed $\eta - \alpha$ semimonotone mapping, as an extension of the notion of semimonotone mapping introduced by Chen [5].

In a first steep, we study the existence of solutions for the mixed equilibrium problem (1) in a general setting of Hausdorff topological vector spaces $E$ under some appropriate conditions. In a second steep and in the setting of Banach spaces, we give an iterative algorithm for finding approximate solutions of the mixed equilibrium problem (1). Our approach is based on an auxiliary principle technique. The auxiliary principle technique is a method initially proposed by Glowinski, et al. [26] to study approximating solutions for variational inequalities, and which is not depending on the projection. In literature, projection type methods, proximal-type methods, or resolvent operator type methods were mainly used to find the equilibria after the iterative algorithms are formulated [11, 17, 27]. But for finding solutions of mixed equilibrium problems involving variational-like inequalities,
projection type methods can not be applied, due to the nature of these problems. To overcome such difficulties, auxiliary principle technique was proposed in a substantial number of papers on existence and uniqueness, formulation of iterative methods and convergence analysis for bifunction equilibrium problems and variational inequalities, see for instance [4, 7, 15, 20, 22, 25, 27, 16, 21, 23] and the references therein.

To approximate the solutions of the mixed equilibrium problem (1), we proceed first by formulating the auxiliary problem and study its solvability. The auxiliary problem is afterward used to generate a sequence of approximating solutions that converges strongly to a solution of the problem (1) under some appropriate conditions. Those conditions are relaxed when \( E \) is a reflexive Banach space. As an application, we obtain the existence and iterative method for the nonlinear mixed variational-like inequality (2).

The results obtained in this paper are interesting in the sense that they improve considerably the ones obtained in [10] and [24] by avoiding the use of the semimonotonicity notion, which is a strong assumption requiring that the operator \( N \) in the problem (2) is completely continuous with respect to the first argument, see [10, Definition 3.1]. Furthermore, the problems studied in [10] and [24] are particular forms of (1) and (2). Many concepts introduced and used in [24] can be avoided by the approach developed in this paper.

The paper is organized as follows. In section 2, we give some definitions and preliminary concepts. In section 2, we study the existence of solutions for the problem (1) in the setting of Hausdorff topological vector space. The uniqueness of the solution of problem (1) is also discussed in the framework of Banach spaces. Section 4 is devoted to study the approximation of solutions for the problem (1) by an auxiliary principle technique. In section 5 and as applications of the approach developed in this paper, we study the existence and algorithmic approach for a general class of nonlinear mixed variational-like inequalities. Finally, we end the paper by a conclusion in which we present the interest of the problem studied in this paper in connection with previous similar problems studied in literature.

2. Preliminaries. In this section, we give some definitions and results that will be required in the sequel. Here \( E \) denotes the real Banach space with dual space \( E^* \), \( K \) is a nonempty closed and convex subset of \( E \). We shall denote by \( \text{co}(A) \) the convex hull of a subset \( A \) of \( E \). The norm in \( E \) will be denoted by \( \|·\| \) and we shall denote by \( \langle·,·\rangle \) the duality pairing between \( E^* \) and \( E \).

**Definition 2.1.** If \( T : K \to E^* \) and \( \eta : K \times K \to E \), then \( T \) is said to be

(i) \( \eta \)-upper hemicontinuous provided \( f(t) = \langle T(w + t(v - w)), \eta(v, w + t(v - w)) \rangle \) is upper semicontinuous at 0 for any \( v, w \in K \), where \( f : [0, 1] \to (-\infty, +\infty) \);
(ii) \( \delta \)-strongly positive if and only if there exists \( \delta > 0 \) such that \( \langle T(w), w \rangle \geq \delta \|w\|^2 \), for all \( w \in E \);
(iii) monotone if and only if \( \langle T(w) - T(v), w - v \rangle \geq 0 \), for all \( w, v \in K \);
(iv) \( \gamma \)-strongly monotone if and only if there exists \( \gamma > 0 \) such that \( \langle T(w) - T(v), w - v \rangle \geq \gamma \|w - v\|^2 \), for all \( w, v \in K \).

**Remark 1.** If \( T : E \to E^* \) is linear, bounded and \( \delta \)-strongly positive, then it is \( \delta \)-strongly monotone and \( ||T|| \)-Lipschitz continuous, i.e. \( ||T(w) - T(v)|| \leq ||T|| \|w - v\| \) where \( ||T|| \) is the norm of \( T \).

**Definition 2.2.** A bifunction \( \psi : K \times K \to \mathbb{R} \) is said to be
(i) monotone if and only if $\psi(w, v) + \psi(v, w) \leq 0$, for all $y, v, w \in K$;
(ii) upper hemicontinuous if and only if for all $v, w \in K$ the function $h : [0, 1] \to \mathbb{R}$ defined by $h(t) = \psi(tv + (1 - t)w, v)$ is upper semicontinuous at $t = 0$.

**Definition 2.3.** A bifunction $b : K \times K \to \mathbb{R}$ is said to be skew-symmetric if and only if for all $v, w \in K$, $b(w, w) - b(w, v) - b(v, w) + b(v, v) \geq 0$.

**Remark 2.** Skew-symmetric bifunctions have certain properties, see [1], which can be regarded as analogs of the conditions governing the gradient monotonicity and nonnegativity of the second derivative of convex functions.

**Definition 2.4.** A trifunction $\phi : K \times K \times K \to \mathbb{R}$ is said to be,

(i) monotone if and only if $\phi(y, w, v) + \phi(y, v, w) \leq 0$, for all $y, v, w \in K$;
(ii) relaxed $\alpha$-monotone if and only if there exists $\alpha : E \to \mathbb{R}^+$, with $\alpha(tz) = t^p\alpha(z)$, for all $t > 0$, where $p > 1$ and $z \in E$, such that for all $y \in K$,

\[ \phi(y, w, v) + \phi(y, v, w) \leq \alpha(v - w); \]

(iii) weakly relaxed $\alpha$-monotone if and only if

\[ \phi(y, w, v) + \phi(y, v, w) \leq \alpha(v - w), \quad \text{for all } y \in K, \]

with $\lim_{t \to 0} \alpha(tz) = 0$ and $\lim_{t \to 0} \frac{d}{dt} \alpha(tz) = 0$;
(iv) $\gamma$-strongly monotone (here $E$ is supposed to be a normed space) if and only if there exists a positive constant $\gamma$, such that,

\[ \phi(y, w, v) + \phi(y, v, w) \leq -\gamma\|w - v\|^2, \quad \text{for all } y, v, w \in K; \]
(v) upper hemicontinuous if and only if for all $y, v, w \in K$ the function $g : [0, 1] \to \mathbb{R}$ defined by $g(t) = \phi(y, tv + (1 - t)w, v)$ is upper semicontinuous at $t = 0$.

**Remark 3.** It is clear from the above definitions that monotonicity $\implies$ relaxed $\alpha$-monotonicity $\implies$ weakly relaxed $\alpha$-monotonicity. $\alpha$-strongly monotonicity is a special case of relaxed $\alpha$-monotonicity, but weakly relaxed $\alpha$-monotonicity does not always imply relaxed $\alpha$-monotonicity, as shown in the following example.

**Example**
Let $K = [0, 1]$ and $\phi : K \times K \times K \to \mathbb{R}$ defined by $\phi(y, w, v) = y \sin 2\pi w - y \sin 2\pi v$. Then $\phi(y, v, w) = y \sin 2\pi v - y \sin 2\pi w$. Let $\alpha : E \to \mathbb{R}$ be given by $\alpha(z) = \sin^2 z$. So we have,

\[ \phi(y, w, v) + \phi(y, v, w) \leq \alpha(v - w) \]

and $\lim_{t \to 0} \alpha(tz) = 0$ and $\lim_{t \to 0} \frac{d}{dt} \alpha(tz) = 0$. But $\alpha(tz) \neq t^p\alpha(z)$. Here $\phi(\cdot, \cdot, \cdot)$ is weakly relaxed $\alpha$-monotone, but not relaxed $\alpha$-monotone as $\alpha(tz) \neq t^p\alpha(z)$.

**Definition 2.5.** Let $N : E \times E \to E^*$ be a nonlinear operator and $\eta : E \times E \to E$. The operator $N$ is said to be

(i) $\eta$-monotone if and only if $(N(w, y) - N(v, y), \eta(w, v)) \geq 0$ for all $w, v, y \in K$;
(ii) $(\eta, \delta)$-strongly monotone if and only if there exists $\delta > 0$ such that

\[ (N(w, y) - N(v, y), \eta(w, v)) \geq \delta\|w - v\|^2 \quad \text{for all } w, v, y \in K; \]
(iii) weakly relaxed $(\eta, -\alpha)$-monotone if and only if there exists $\alpha : E \to \mathbb{R}$ such that

\[ (N(w, y) - N(v, y), \eta(w, v)) \geq -\alpha(w - v) \quad \text{for all } w, v, y \in K, \]

with $\lim_{t \to 0} \alpha(tz) = 0$ and $\lim_{t \to 0} \frac{d}{dt} \alpha(tz) = 0$. 

Remark 4. Let \( N : E \times E \to E^* \) be a nonlinear operator and \( \eta : E \times E \to E \) such that \( \eta(w, v) = -\eta(v, w) \). If \( N \) is weakly relaxed \((\eta, -\alpha)\)-monotone, then the trifunction \( \phi : E \times E \times E \to \mathbb{R} \) defined by \( \phi(y, w, v) = \langle N(w, y), \eta(v, w) \rangle \) is weakly relaxed \( \alpha \)-monotone.

**Definition 2.6.** A trifunction \( \phi : K \times K \times K \to \mathbb{R} \) is said to be \( \theta \)-**diagonally convex**, if for each finite subsets \( \{y_1, \cdots, y_n\} \) and \( \{v_1, \cdots, v_n\} \) of \( K \), and any \( w_0 = \sum_{i=1}^{n} \lambda_i v_i \)

\[
(\lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1), \text{ we have } \sum_{i=1}^{n} \lambda_i \phi(y_i, w_0, v_i) \geq 0.
\]

**Remark 5.** If \( \phi : K \times K \times K \to \mathbb{R} \) is \( 0 \)-diagonally convex, then \( \phi(y, w, w) \geq 0 \) for any \( y, w \in K \).

**Definition 2.7.** Let \( N : E \times E \to E^* \) be a nonlinear operator and \( \eta : E \times E \to E \) such that \( \eta(x, y) = -\eta(y, x) \). Let \( \nu : E \to \mathbb{R} \) be a real-valued bifunction such that

\[
\lim \inf_{x \to x_0} \nu(x, y, w) \leq 0, \quad \text{for all } x \in K \text{ and compact for some } x_0 \in K, \text{ then } \bigcap_{x \in K} N(x_0, y, w) = \{0\}.
\]

**Definition 2.8.** \( F : K \to E^* \) is a KKM mapping if, for any \( \{x_1, \cdots, x_n\} \subset K \), \( \text{co}\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i) \).

**Definition 2.9.** \( f : K \to (-\infty, +\infty) \) is lower semicontinuous at \( x_0 \), if \( f(x_0) \leq \lim \inf_{x \to x_0} f(x) \).

The following Lemma plays an important role in proving the existence result for our problem.

**Lemma 2.10** ([9]). If \( M \) is a nonempty subset of a Hausdorff topological vector space \( X \), \( F : M \to 2^X \) is a KKM mapping, \( F(x) \) is closed in \( X \), for all \( x \in K \) and compact for some \( x \in K \), then \( \bigcap_{x \in M} F(x) \neq \emptyset \).

3. Existence results. We first establish an equivalence between the problems (3) and (4) given below.

**Lemma 3.1.** Let \( K \) be a nonempty closed convex subset of a topological vector space \( E \). Let \( \phi : K \times K \times K \to \mathbb{R} \) be a real-valued trifunction and \( b : E \times E \to \mathbb{R} \) be a real-valued bifunction such that

(i) \( \phi \) weakly relaxed \( \alpha \)-monotone, \( \theta \)-diagonally convex and upper hemicontinuous;
(ii) \( b \) is convex and lower semicontinuous with respect to the second argument.

Then the following problems are equivalent:

\[
w \in K, \quad \phi(y, w, v) + b(w, v) - b(w, w) \geq 0, \quad \text{for all } y, v \in K, \tag{3}
\]

\[
w \in K, \quad \phi(y, v, w) + b(w, w) - b(w, v) \leq \alpha(v - w), \quad \text{for all } y, v \in K. \tag{4}
\]

**Proof.** Let \( \bar{w} \in K \) be a solution of (3). By the weak relaxed \( \alpha \)-monotonicity of \( \phi \), we have,

\[
\phi(y, v, \bar{w}) + b(\bar{w}, v) - b(\bar{w}, \bar{w}) \leq \alpha(v - \bar{w}) - \phi(y, \bar{w}, \bar{w}) + b(\bar{w}, \bar{w}) - b(\bar{w}, v) \\
\leq \alpha(v - \bar{w}), \quad \text{for all } y, v \in K.
\]
So \( \tilde{w} \) is a solution of (4). Conversely, to prove \( (4) \implies (3) \), we consider, \( w_t = tv + (1 - t)\tilde{w}, \ t \in (0, 1) \). So \( w_t \in K \). Now replacing \( v \) by \( w_t \) in (4), we have,

\[
\phi(y, w_t, \tilde{w}) \leq \alpha(w_t - \tilde{w}) + b(\tilde{w}, w_t) - b(\tilde{w}, \tilde{w}). \tag{5}
\]

By the 0-diagonal convexity of \( \phi \), we have

\[
t\phi(y, w_t, v) + (1 - t)\phi(y, w_t, \tilde{w}) \geq 0 \tag{6}
\]

Using relation (5) and the convexity of \( b(\tilde{w}, \cdot) \), we have from realtion (6),

\[
0 \leq t\phi(y, w_t, v) + (1 - t)[\alpha(w_t - \tilde{w}) + tb(\tilde{w}, v) + (1 - t)b(\tilde{w}, \tilde{w}) - b(\tilde{w}, \tilde{w})] \\
\leq t \left[ \phi(y, w_t, v) + (1 - t)b(\tilde{w}, v) - (1 - t)b(\tilde{w}, \tilde{w}) + (1 - t)\frac{\alpha(t(v - \tilde{w}))}{t} \right].
\]

Hence,

\[
\phi(y, w_t, v) + (1 - t)b(\tilde{w}, v) - (1 - t)b(\tilde{w}, \tilde{w}) + (1 - t)\frac{\alpha(t(v - \tilde{w}))}{t} \geq 0. \tag{7}
\]

Since \( \phi \) is upper hemicontinuous, it follows, by considering the upper limit when \( t \to 0^+ \) in (7) and applying L’Hospital’s rule along with condition \( \lim_{t \to 0^+} \frac{\alpha(tz)}{t} = 0 \), that

\[
\phi(y, \tilde{w}, v) + b(\tilde{w}, v) - b(\tilde{w}, \tilde{w}) \geq 0.
\]

So \( \tilde{w} \) satisfies (3). This completes the proof. \( \square \)

**Theorem 3.2.** Let \( K \) be a nonempty closed and convex subset of a Hausdorff topological vector space \( E \). Let \( \phi : K \times K \times K \to \mathbb{R} \) be a real-valued trifunction and \( b : E \times E \to \mathbb{R} \) be a real-valued bifunction. Suppose that

(i) \( \phi \) is weakly relaxed \( \alpha \)-monotone, with \( \limsup \alpha(z_{\lambda}) \leq \alpha(z) \) for any net \( z_{\lambda} \) converging to \( z \);

(ii) \( \phi \) is 0-diagonally convex and \( b \) is convex with respect to the second argument;

(iii) \( \phi \) is upper hemicontinuous, and the function \( (y, w) \in K \times K \mapsto \phi(y, v, w) \) is lower semicontinuous for each fixed \( v \in K \);

(iv) \( b \) is lower semicontinuous and upper semicontinuous with respect to the first argument;

(v) (Coercivity) There exists a nonempty convex and compact subset \( D \) of \( K \) such that for all \( w \in K \setminus D \), there exists \( v \in D \) and \( y \in D \) satisfying

\[
\phi(y, w, v) + b(w, v) - b(w, w) < 0.
\]

Then problem (1) has at least one solution.

**Proof.** Let \( A = \{v_1, v_2, \ldots, v_n\} \) and \( B = \{y_1, y_2, \ldots, y_p\} \) be two finite subsets of \( K \). Let us consider \( C := \text{co}(A \cup D) \) and \( \bar{C} := \text{co}(B \cup D) \), which are compact and convex subsets of \( K \). For \( v \in C \), let us consider the following set

\[
F(v) = \bigcap_{y \in \bar{C}} \{w \in C : \phi(y, w, v) + b(w, v) - b(w, w) \geq 0\}.
\]

From (ii), we have that \( \phi(y, v, v) \geq 0 \) for all \( y, v \in K \), it follows that \( F(v) \neq \emptyset \) since it contains \( v \). On the other hand, the weakly relaxed \( \alpha \)-monotonicity assumption of \( \phi \) permits to have

\[
F(v) \subset H(v), \tag{8}
\]
where $H(v) = \bigcap_{y \in C} \{ w \in C : \phi(y, v, w) + b(w, w) - b(w, v) \leq \alpha(v - w) \}$. Now, let us verify that
\[
\bigcap_{v \in C} H(v) \subset \bigcap_{v \in C} F(v).
\] (9)

Indeed, let $w \in \bigcap_{v \in C} H(v)$. For $v \in C$, let $v_t = tv + (1 - t)w \in C$, $t \in [0, 1]$. Since $w \in H(v_t)$, we have
\[
\phi(y, v_t, w) + b(w, w) - b(w, v_t) \leq \alpha(v_t - w), \quad \text{for all } y \in \tilde{C}.
\] (10)

By taking account of (ii) and relation (10), we obtain
\[
0 \leq t\phi(y, v_t, v) + (1 - t)\phi(y, v_t, w) \\
\leq t\phi(y, v_t, v) + (1 - t)[\alpha(v_t - w) + b(w, v_t) - b(w, w)] \\
\leq t\phi(y, v_t, v) + (1 - t)[\alpha(v_t - w) + tb(w, v) + (1 - t)b(w, w) - b(w, w)] \\
\leq t \left[ \phi(y, v_t, v) + (1 - t)b(w, v) - (1 - t)b(w, w) + (1 - t)\frac{\alpha(t(v - w))}{t} \right].
\]

Hence,
\[
\phi(y, v_t, v) + (1 - t)b(w, v) - (1 - t)b(w, w) + (1 - t)\frac{\alpha(t(v - w))}{t} \geq 0, \quad \forall y \in \tilde{C}. \tag{11}
\]

Since $\phi$ is upper hemicontinuous, it follows, by considering the upper limit when $t \to 0^+$ in (11) and applying L’Hospital’s rule along with condition $\lim_{t \to 0^+} \frac{\alpha(tz)}{t} = 0$, that
\[
\phi(y, w, v) + b(w, v) - b(w, w) \geq 0, \quad \text{for all } y \in \tilde{C}.
\]

Since $v$ is an arbitrary element in $C$, we deduce that $w \in \bigcap_{v \in C} F(v)$. Consequently, by taking account of (8), we get
\[
\bigcap_{v \in C} F(v) = \bigcap_{v \in C} H(v). \tag{12}
\]

We claim that $\bigcap_{v \in C} H(v) \neq \emptyset$. To verify this affirmation, we apply the Ky Fan Lemma (Lemma 2.10). To this aim, first we verify that the family of sets $\{H(v)\}_{v \in C}$ has the KKM-property. Suppose by contradiction that there exists a finite subset $\{z_1, z_2, \cdots, z_m\}$ of $C$ such that $\text{col}(\{z_1, z_2, \cdots, z_m\})$ is not contained in $\bigcap_{i=1}^m H(z_i)$. Thus, there exists $\tilde{w} = \sum_{i=1}^m \lambda_i z_i$ with $\lambda_i \geq 0$ for $i = 1, \cdots, m$ and $\sum_{i=1}^m \lambda_i = 1$ such that $\tilde{w} \notin H(z_i)$ for all $i = 1, \cdots, m$. Hence, for each $i \in \{1, \cdots, m\}$ there exists $y_i \in C$ such that
\[
\phi(y_i, z_i, \tilde{w}) + b(\tilde{w}, \tilde{w}) - b(\tilde{w}, z_i) > \alpha(z_i - \tilde{w}).
\]

Since $\phi$ is weakly relaxed $\alpha$-monotone, it follows that
\[
\phi(y_i, \tilde{w}, z_i) < b(\tilde{w}, \tilde{w}) - b(\tilde{w}, z_i), \quad \text{for all } i = 1, \cdots, m.
\]

Hence,
\[
\sum_{i=1}^m \lambda_i \phi(y_i, \tilde{w}, z_i) < b(\tilde{w}, \tilde{w}) - \sum_{i=1}^m \lambda_i b(\tilde{w}, z_i) \leq b(\tilde{w}, \tilde{w}) - b(\tilde{w}, \tilde{w}) = 0,
\]

which is in contradiction with (ii). On the other hand, from conditions (iii)-(iv) and the condition on $\alpha$ given in (i), we have that $H(v)$ is a closed subset of $C$ for each $v \in C$. Consequently, by Lemma 2.10, we deduce that $\bigcap_{v \in C} H(v) \neq \emptyset$, and hence $\bigcap_{v \in C} F(v) \neq \emptyset$. Now, let us verify that $\bigcap_{v \in C} F(v) \subset D$. Suppose by contradiction
that there exists \( w \in \bigcap_{v \in C} F(v) \) such that \( w \notin D \), then \( w \in K \setminus D \). From the coercivity assumption (v), we deduce that there exists \( v_0 \in D \) and \( y_0 \in D \) such that
\[
\phi(y_0, w, v_0) + b(w, v_0) - b(w, w) < 0,
\]
which contradicts the fact that \( w \in F(v_0) \). Therefore, we have
\[
\emptyset \neq \bigcap_{v \in C} H(v) = \bigcap_{v \in C} F(v) \subset D.
\]
(13)
Thus, there exists \( w \in D \) such that \( w \in H(v) \) for all \( v \in C \). Since \( C := \text{co}(\{v_1, v_2, \ldots, v_n\} \cup D) \), we deduce that \( w \in \bigcap_{i=1}^{n} H(v_i) \). Consequently, we have proved that for any finite subset \( \{v_1, v_2, \ldots, v_n\} \) of \( K \), there exists \( w \in D \) such that
\[
\phi(y, v_i, w) + b(w, w) - b(w, v_i) \leq \alpha(v_i - w), \text{ for all } y \in \tilde{C} \text{ and all } i = 1, \ldots, n.
\]
Hence, for any finite subset \( \{v_1, v_2, \ldots, v_n\} \) of \( K \), we have
\[
\bigcap_{i=1}^{n} \{w \in D : \phi(y, v_i, w) + b(w, w) - b(w, v_i) \leq \alpha(v_i - w), \forall y \in \tilde{C}\} \neq \emptyset. \tag{14}
\]
For \( v \in K \), let
\[
K(v) := \{w \in D : \phi(y, v, w) + b(w, w) - b(w, v) \leq \alpha(v - w), \forall y \in \tilde{C}\},
\]
which is a closed subset of \( D \). From relation (14), we have that the family \( \{K(v)\}_{v \in K} \) of closed subsets of \( D \) has the finite intersection property. It follows that
\[
\bigcap_{v \in K} K(v) \neq \emptyset.
\]
Then, we have the existence of \( w \in D \) such that
\[
\phi(y, v, w) + b(w, w) - b(w, v) \leq \alpha(v - w), \forall v \in K, \forall y \in \tilde{C}. \tag{15}
\]
Therefore, we have shown that for each finite subset \( \{y_1, y_2, \ldots, y_p\} \) of \( K \), there exists \( w \in D \) satisfying
\[
\phi(y, v, w) + b(w, w) - b(w, v) \leq \alpha(v - w), \forall v \in K, \forall y \in \text{co}(\{y_1, y_2, \ldots, y_p\} \cup D). \tag{16}
\]
Hence,
\[
\phi(y_i, v, w) + b(w, w) - b(w, v) \leq \alpha(v - w), \forall v \in K, \forall i = 1, \ldots, p.
\]
It follows that for each finite subset \( \{y_1, y_2, \ldots, y_p\} \) of \( K \), we have
\[
\bigcap_{i=1}^{p} \{w \in D : \phi(y_i, v, w) + b(w, w) - b(w, v) \leq \alpha(v - w), \forall v \in K\} \neq \emptyset. \tag{17}
\]
For \( y \in K \), let us set
\[
G(y) := \{w \in D : \phi(y, v, w) + b(w, w) - b(w, v) \leq \alpha(v - w), \forall v \in K\},
\]
which is a closed subset of \( D \). From (17), the family \( \{G(y)\}_{y \in K} \) has the finite intersection property. It follows that
\[
\bigcap_{y \in K} G(y) \neq \emptyset.
\]
Consequently, we have the existence of \( w \in D \) such that
\[
\phi(y, v, w) + b(w, w) - b(w, v) \leq \alpha(v - w), \forall v \in K, \forall y \in K.
\]
From Lemma 3.1, we deduce that
\[
\phi(y, w, v) + b(w, v) - b(w, w) \geq 0, \forall v \in K, \forall y \in K,
\]
which completes the proof.

Remark 6. If $E$ is a reflexive Banach space endowed with the weak topology $\sigma(X,X^*)$ and the trifunction $\phi : K \times K \times K \to \mathbb{R}$ is monotone and convex with respect to the third argument with $\phi(y,v,v) = 0$ for all $y, v \in K$, then the coercivity assumption (v) in Theorem 3.2 is satisfied if we suppose that there exists $v_0 \in K$ such that

$$b(w,v_0) - b(w,w) \geq -\infty \quad \text{when} \quad \|w - v_0\| \to +\infty. \quad (18)$$

Indeed, let $r > 0$ and let $\bar{B}(v_0,r) := \{v \in K : \|v - v_0\| \leq r\}$. We have that $\bar{B}(v_0,r)$ is a convex and $\sigma(X,X^*)$-compact subset of $E$. Since the function $\phi(\cdot, v_0, \cdot)$ is lower semicontinuous, it follows that there exists $k_0 \in \mathbb{R}$ such that

$$\phi(y,v_0,v) > k_0, \quad \text{for all } y, v \in \bar{B}(v_0,r). \quad (19)$$

Let $w \in K \setminus \bar{B}(v_0,r)$ and let us set

$$\bar{w} = \frac{r}{\|w - v_0\|} w + (1 - \frac{r}{\|w - v_0\|})v_0,$$

note that $\bar{w} \in \bar{B}(v_0,r)$. Let $y_0$ be an arbitrary element in $\bar{B}(v_0,r)$, we have from (19) that $\phi(y_0,v_0,\bar{w}) > k_0$. Since $\phi(y_0,\cdot,\cdot)$ is a convex function, it follows that

$$\phi(y_0,v_0,w) + (1 - \frac{r}{\|w - v_0\|})\phi(y_0,v_0,v) > k_0.$$

Hence,

$$\phi(y_0,v_0,w) > \frac{k_0}{r}\|w - v_0\|, \quad \text{for all } w \in K \setminus \bar{B}(v_0,r). \quad (20)$$

By using the monotonicity of $\phi$ and relation (20), we obtain

$$\phi(y_0,w,v_0) + b(w,v_0) - b(w,w) < -\frac{k_0}{r}\|w - v_0\| + b(w,v_0) - b(w,w), \quad \forall w \in K \setminus \bar{B}(v_0,r). \quad (21)$$

From (18) and (21), we deduce that there exists $\bar{r} > 0$ such that

$$\phi(y_0,w,v_0) + b(w,v_0) - b(w,w) < 0, \quad \text{for all } w \in K \setminus \bar{B}(v_0,\bar{r}),$$

which completes the proof of the assertion. Note that the result still true if we suppose that $\phi$ is weakly relaxed $\alpha$-monotone with $\phi$ defined by $\alpha(z) = \|z\|^p$ ($p \leq 1$).

The assumptions on the bifunction $b$ in the previous theorem can be relaxed by assuming that $b$ is skew symmetric. We obtain the following result.

Theorem 3.3. Let $K$ be a nonempty closed and convex subset of a Hausdorff topological vector space $E$. Let $\phi : K \times K \times K \to \mathbb{R}$ be a real-valued trifunction and $b : E \times E \to \mathbb{R}$ be a real-valued bifunction. Suppose that

(i) $\phi$ is weakly relaxed $\alpha$-monotone, with $\lim sup \alpha(z_\lambda) \leq \alpha(z)$ for any net $z_\lambda$ converging to $z$;

(ii) $\phi$ is $\theta$-diagonally convex and $b$ is convex with respect to the second argument;

(iii) $\phi$ is upper hemicontinuous, and the function $(y,w) \in K \times K \mapsto \phi(y,v,w)$ is lower semicontinuous for each fixed $v \in K$;

(iv) $b$ is skew symmetric, lower semicontinuous with respect to the second argument and the bifunction $\varphi : E \times E \to \mathbb{R}$ defined by $\varphi(w,v) = b(w,v) - b(w,w)$ is upper hemicontinuous;
(v) (Coercivity) There exists a nonempty convex and compact subset $D$ of $K$ such that for all $w \in K \setminus D$, there exists $v \in D$ and $y \in D$ satisfying

$$
\phi(y, w, v) + b(w, v) - b(w, w) < 0.
$$

Then problem (1) has at least one solution.

Proof. Let us consider the trifunction $\psi : K \times K \times K \to \mathbb{R}$ defined by $\psi(y, w, v) = \phi(y, w, v) + b(w, v) - b(w, w)$. We can easily verify that the trifunction $\psi$ satisfies all the assumptions of Theorem 3.2. □

In the following results, we give some conditions assuring the existence of unique solution for the mixed equilibrium problem (1).

**Theorem 3.4.** Suppose that $E$ is a Banach space and that the conditions (ii), (iii) and (v) of Theorem 3.2 are satisfied. If conditions (i) and (iv) of Theorem 3.2 are replaced with

(i)' $\phi$ is strongly $\gamma$-monotone,

(iv)' $b(\cdot, v)$ is linear, $b(u, v) - b(u, w) \leq b(u, v - w)$, and there exists a positive constant $\mu$, such that, $b(u, v) \leq \mu\|u\|\|v\|$, for all $u, v, w \in K$,

(iv)” $\frac{\alpha}{\mu} > 1$, where $\gamma$ is the constant of strong monotonicity,

then the mixed equilibrium problem (1) has a unique solution.

Proof. The existence of solutions follows from Theorem 3.2. We need only to verify the uniqueness of the solution. Suppose that there exist two arbitrary solutions, say, $w_1$ and $w_2$, for the MEP (1). So we have,

$$
\phi(y, w_1, v) + b(w_1, v) - b(w_1, w_1) \geq 0, \quad (22)
$$

$$
\phi(y, w_2, v) + b(w_2, v) - b(w_2, w_2) \geq 0, \quad (23)
$$

Adding the inequalities obtained after replacing $v$ by $w_2$ in (22) and by $w_1$ in (23) and then using the conditions (i)' and (iv)', we get,

$$
-\alpha\|w_1 - w_2\|^2 \geq \phi(y, w_1, w_2) + \phi(y, w_2, w_1) \geq -\mu\|w_1 - w_2\|^2.
$$

This implies, $\frac{\alpha}{\mu} \leq 1$, which contradicts condition (iv)”. Hence the solution must be unique. □

**Theorem 3.5.** Suppose that $E$ is a Banach space and that the conditions (ii)-(v) of Theorem 3.2 are satisfied. If conditions (i) of Theorem 3.2 is replaced with

(i)' $\phi$ is strongly $\gamma$-monotone,

then the mixed equilibrium problem (1) has a unique solution.

Proof. We need only to verify the uniqueness of the solution since the existence is assured by Theorem 3.2. Suppose that there exists two solutions $w_1$ and $w_2$, for the MEP (1). It follows that for any $y \in K$,

$$
\phi(y, w_1, w_2) + b(w_1, w_2) - b(w_1, w_1) \geq 0, \quad (24)
$$

$$
\phi(y, w_2, w_1) + b(w_2, w_1) - b(w_2, w_2) \geq 0. \quad (25)
$$

By adding the inequalities (24) and (25), and using the skew symmetry assumption of $b$ and condition (i)', we obtain

$$
-\gamma\|w_1 - w_2\|^2 \geq \phi(y, w_1, w_2) + \phi(y, w_2, w_1) \geq 0.
$$

Hence, $w_1 = w_2$, which completes the proof. □
4. Approximations using auxiliary principle technique. In this section, we obtain an iterative algorithm for finding approximate solutions to the mixed equilibrium problem studied in this paper, using the auxiliary principle technique. After formulating the auxiliary variational inequality, we establish the solvability result and obtain an iterative algorithm for strong convergence of the iterates to the exact solution.

We suppose that $E$ is a Banach space and $K$ is a nonempty closed and convex subset of $E$. We consider the following mixed equilibrium problem:

\[\text{(MEP)} \begin{cases} \text{Find } \bar{w} \in K, \text{ such that} \\ \psi(y, \bar{w}, v) + \varphi(\bar{w}, v) - \varphi(\bar{w}, \bar{w}) \geq 0, \text{ for all } y, v \in K, \end{cases} \tag{26}\]

where $\psi : K \times K \times K \to \mathbb{R}$ is a trifunction and $\varphi : K \times K \to \mathbb{R}$ is a bifunction.

In order to approximate the solutions of the problem (26), we consider the following auxiliary problem: For $u \in K$ and $\rho > 0$,

\[\text{(AuxEP)} \begin{cases} \text{Find } \bar{w} \in K, \text{ such that for all } y, v \in K, \\ \rho \left[ \psi(y, \bar{w}, v) + \varphi(\bar{w}, v) - \varphi(\bar{w}, \bar{w}) \right] + \langle T(v) - T(\bar{w}), \bar{w} - u \rangle \geq 0 \tag{27}\end{cases}\]

where $T : E \to E^*$ is a given linear operator with some appropriate conditions.

First, we give some results on the existence of solution for the auxiliary problem (27).

**Theorem 4.1.** Let $K$ be a nonempty closed and convex subset of a Banach space $E$. Let $\psi : K \times K \times K \to \mathbb{R}$ be a real-valued trifunction, $\varphi : E \times E \to \mathbb{R}$ be a real-valued bifunction and $T : E \to E^*$ be a $\delta$-strongly positive bounded linear operator. Suppose that

(i) $\psi$ is monotone and 0-diagonally convex;
(ii) $\varphi$ is skew symmetric and convex with respect to the second argument;
(iii) $\psi$ is upper hemicontinuous, and the function $(y, w) \in K \times K \mapsto \psi(y, v, w)$ is lower semicontinuous for each fixed $v \in K$;
(iv) $\varphi$ is lower semicontinuous and upper semicontinuous with respect to the first argument;
(v) (Coercivity) For each $u \in K$, there exists a nonempty convex and compact subset $D_u$ of $K$ such that for all $w \in K \setminus D_u$, there exists $v \in D_u$ and $y \in D_u$ satisfying

\[\rho \left[ \psi(y, w, v) + \varphi(\bar{w}, v) - \varphi(\bar{w}, \bar{w}) \right] + \langle T(v) - T(\bar{w}), \bar{w} - u \rangle < 0.\]

Then for each $u \in K$, the auxiliary problem (27) has a unique solution.

**Proof.** The existence of solution for the auxiliary problem (27) follows from Theorem 3.2 by considering the trifunction $\phi : K \times K \times K \to \mathbb{R}$ and the bifunction $b : E \times E \to \mathbb{R}$ defined by

\[\phi(y, w, v) = \rho \left[ \psi(y, w, v) + \varphi(w, v) - \varphi(w, w) \right], \text{ and } b(w, v) = \langle T(v) - T(w), w - u \rangle.\]

We need only to verify the uniqueness of the solution. To this aim, suppose that the problem 3.2 has two solutions $w_1$ and $w_2$ in $K$. Then, for all $y, v \in K$, we have

\[\rho \left[ \psi(y, w_1, v) + \varphi(w_1, v) - \varphi(w_1, w_1) \right] + \langle T(v) - T(w_1), w_1 - u \rangle \geq 0, \tag{28}\]

\[\rho \left[ \psi(y, w_2, v) + \varphi(w_2, v) - \varphi(w_2, w_2) \right] + \langle T(v) - T(w_2), w_2 - u \rangle \geq 0. \tag{29}\]
By considering \( v = w_2 \) in (28) and \( v = w_1 \) in (29), and adding the two obtained inequalities, we get
\[
\langle T(w_2 - w_1), w_2 - w_1 \rangle \leq \rho \psi(y, w_1, w_2) + \psi(y, w_2, w_1) + \varphi(w_2, w_1) + \varphi(w_1, w_2) - \varphi(w_1, w_1) - \varphi(w_2, w_2).
\]
Since \( \psi \) is monotone, \( \varphi \) is skew symmetric and \( T \) is \( \delta \)-strongly positive, it follows that
\[
0 \geq \langle T(w_2 - w_1), w_2 - w_1 \rangle \geq -\delta \|w_1 - w_2\|^2.
\]
Therefore \( w_1 = w_2 \), which completes the proof.

In the following result, we show that in the setting of reflexive Banach spaces the result obtained in Theorem 4.1 can be improved considerably since the coercivity condition (v) can be dropped.

**Theorem 4.2.** Let \( K \) be a nonempty closed and convex subset of a Banach space \( E \). Let \( \psi : K \times K \times K \to \mathbb{R} \) be a real-valued trifunction, \( \varphi : E \times E \to \mathbb{R} \) be a real-valued bifunction and \( T : E \to E^* \) be a \( \delta \)-strongly positive bounded linear operator. Suppose that

(i) \( \psi \) is monotone and 0-diagonally convex;
(ii) \( \varphi \) is skew symmetric and convex with respect to the second argument;
(iii) \( \psi \) is upper hemicontinuous, and the function \( (y, w) \in K \times K \mapsto \psi(y, v, w) \) is lower semicontinuous and upper semicontinuous for each fixed \( v \in K \);
(iv) \( \varphi \) is lower semicontinuous and upper semicontinuous with respect to the first argument.

Then for each \( u \in K \), the auxiliary problem (27) has a unique solution.

**Proof.** For \( u \in K \) and \( \rho > 0 \), let us consider the trifunction \( \phi : K \times K \times K \to \mathbb{R} \) and the bifunction \( b : K \times K \to \mathbb{R} \) defined by
\[
\phi(y, w, v) = \rho \psi(y, v, w) + \varphi(w, v) - \varphi(w, w) \quad \text{and} \quad b(w, v) = \langle T(v) - T(w), w - u \rangle.
\]
One can easily verify that the conditions (i)-(iv) above imply the conditions (i)-(iv) of Theorem 3.2. We need only to show that the coercivity condition (v) of Theorem 3.2 is satisfied. To this aim, taking into account of Remark 6, we need to show that for some \( v_0 \in K \) one has \( \frac{b(w, v_0) - b(w, w)}{\|w - v_0\|} \to -\infty \) when \( \|w - v_0\| \to +\infty \). Let \( v_0 \in K \), then
\[
b(w, v_0) - b(w, w) = \langle T(v_0) - T(w), w - u \rangle
= \langle T(v_0 - w), v_0 - u \rangle + \langle T(v_0 - w), v_0 - w \rangle + \langle T(v_0 - w), v_0 - u \rangle
\leq -\delta \|v_0 - w\|^2 + \|T\| \|v_0 - w\| \|v_0 - u\|.
\]
Therefore
\[
\frac{b(w, v_0) - b(w, w)}{\|w - v_0\|} \leq -\delta \|v_0 - w\| + \|T\| \|v_0 - u\|.
\]
It follows that \( \frac{b(w, v_0) - b(w, w)}{\|w - v_0\|} \to -\infty \) when \( \|w - v_0\| \to +\infty \).
Algorithm (A):

Step 0: Take \( \{\rho_n\}_{n \in \mathbb{N}} \subset [0, +\infty[ \) such that \( \rho_n \to +\infty \), choose \( w_0 \in K \) arbitrary, and let \( n := 0 \).

Step 1: Given \( w_n \in K \) compute \( w_{n+1} \in K \) such that for all \( y, v \in K \),

\[
\rho_{n+1} [\psi(y, w_{n+1}, v) + \varphi(w_{n+1}, v) - \varphi(w_{n+1}, w_{n+1})] + (T(v) - T(w_{n+1}), w_{n+1} - w_n) \geq 0. \tag{31}
\]

Update \( n := n + 1 \) and go to Step 1.

The analysis of the convergence of the algorithm above is given in the following theorem.

Theorem 4.3. Let \( E \) be a Banach space and \( K \) be a nonempty closed and convex subset of \( E \). Let \( \psi : K \times K \times K \to \mathbb{R} \) be a real-valued trifunction, \( \varphi : E \times E \to \mathbb{R} \) be a real-valued bifunction and \( T : E \to E^* \) be a \( \delta \)-strongly positive bounded linear operator. Suppose that

(i) \( \psi \) is \( \gamma \)-strongly monotone and \( 0 \)-diagonally convex;
(ii) \( \varphi \) skew symmetric and convex with respect to the second argument;
(iii) \( \psi \) is upper hemicontinuous, and the function \( (y, w) \in K \times K \mapsto \psi(y, v, w) \) is lower semicontinuous for each fixed \( v \in K \);
(iv) \( \varphi \) is lower semicontinuous and upper semicontinuous with respect to the first argument;
(v) (Coercivity) For each \( u \in K \), there exists a nonempty convex and compact subset \( D_u \) of \( K \) such that for all \( w \in K \setminus D_u \), there exists \( v \in D_u \) and \( y \in D_u \) satisfying

\[
\rho_n [\psi(y, w, v) + \varphi(w, v) - \varphi(w, w)] + (T(v) - T(w), w - u) < 0.
\]

Furthermore, we assume that the following condition hold:

\[
(C) \exists k \in [0, 1[, \text{ such that } \frac{\rho_{n+1} \|T\|}{\rho_n (\delta + \gamma \rho_{n+1})} < k.
\]

Then the iterative sequence \( \{w_n\}_{n \in \mathbb{N}} \) generated by the Algorithm (A) converges strongly to a solution \( \overline{w} \in K \) of the mixed equilibrium problem (26).

Proof. Let us consider the relation (31) in the Algorithm (A) for the iterations \( n+1 \) and \( n \). Then, for all \( y, v \in K \),

\[
\rho_{n+1} [\psi(y, w_{n+1}, v) + \varphi(w_{n+1}, v) - \varphi(w_{n+1}, w_{n+1})] + (T(v) - T(w_{n+1}), w_{n+1} - w_n) \geq 0, \tag{32}
\]

and

\[
\rho_n [\psi(y, w_n, v) + \varphi(w_n, v) - \varphi(w_n, w_n)] + (T(v) - T(w_n), w_n - w_{n-1}) \geq 0. \tag{33}
\]
Therefore, the sequence \( \{ \psi_n \} \) is a Cauchy sequence and hence converges strongly to a point \( w \in K \).

Now, let us verify that \( w \) is a solution of the mixed equilibrium problem (32).

To this aim, from relation (32) we have for all \( y, v \in K \):

\[
\langle y, \gamma \rangle - \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) \geq \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right)
\]

Hence, by taking into account of the condition (C), we deduce

\[
\| \psi_{n+1} - \psi \| \leq \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right)
\]

Since \( \psi \) is \( \delta \)-strongly positive and bounded, it follows from (36) that

\[
\| \psi_{n+1} - \psi \| \leq 2 \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right)
\]

Since \( \psi \) is \( \gamma \)-strongly monotone and \( \phi \) is skew-symmetric, it follows from relation (34) that

\[
\| \psi_{n+1} - \psi \| \leq 2 \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right) + \frac{1}{\rho_n} \left( \frac{\|T(v) - T(w_n) - v\|^2}{\|T(v) - T(w_n)\|} \right)
\]

By taking \( v = w_n \) in relation (32) and \( v = w_n \), in relation (33), we obtain after dividing by \( \rho_n \), for all and adding the two obtained inequalities.
By considering the lower limit when $n \to +\infty$ in the previous inequality, and taking account of the conditions (iii) and (iv), we obtain
\[ \varphi(\bar{w}, v) - \varphi(\bar{w}, \bar{v}) \geq \psi(y, v, \bar{w}) \text{ for all } y, v \in K. \]

We complete the proof by using Lemma 3.1 with $\phi = \psi$ and $b(w, v) = \varphi(w, v) - \varphi(w, \bar{w})$. \hfill $\Box$

In the case where $E$ is a reflexive Banach space, then the coercivity condition (v) in the previous theorem can be dropped. We obtain the following result.

**Theorem 4.4.** Let $E$ be a reflexive Banach space and $K$ be a nonempty closed and convex subset of $E$. Let $\psi : K \times K \times K \to \mathbb{R}$ be a real-valued trifunction, $\varphi : E \times E \to \mathbb{R}$ be a real-valued bifunction and $T : E \to E^*$ be a $\delta$-strongly positive bounded linear operator. Suppose that

(i) $\psi$ is $\gamma$-strongly monotone and $0$-diagonally convex;
(ii) $\varphi$ skew symmetric and convex with respect to the second argument;
(iii) $\psi$ is upper hemicontinuous, and the function $(y, w) \in K \times K \mapsto \psi(y, v, w)$ is lower semicontinuous and upper semicontinuous for each fixed $v \in K$;
(iv) $\varphi$ is lower semicontinuous and upper semicontinuous with respect to the first argument.

Furthermore, we assume that the following condition hold:

\[ (C) \exists k \in ]0, 1[, \text{ such that } \frac{\rho_{n+1}}{\rho_n} \frac{\|T\|}{\delta + \gamma \rho_{n+1}} < k. \]

Then the iterative sequence $\{w_n\}_{n \in \mathbb{N}}$ generated by the Algorithm (A) converges strongly to a solution $\bar{w} \in K$ of the mixed equilibrium problem (26).

**Remark 7.** The condition $(C)$ is satisfied if we consider for instance $\rho_n = n^p$ with $p > 1$. If we choose $p$ great, then the algorithm (A) is fast.

5. **Application to mixed variational inequalities.** In this section, we consider the following nonlinear mixed variational-like inequality, in short NMVLIP: Find $w \in K$ such that

\[ \langle N(w, y), \eta(v, w) \rangle + b(w, v) - b(w, w) \geq 0, \text{ for all } y, v \in K, \]

where $N : E \times E \to E^*$ is a nonlinear operator, $b : E \times E \to \mathbb{R}$ is a bifunction and $\eta : K \times K \to E$.

Let $\psi : K \times K \times K \to \mathbb{R}$ be denoted as $\psi(y, w, v) = \langle N(w, y), \eta(v, w) \rangle$. Hence the mixed variational-like inequality (40) is now equivalent to mixed equilibrium problem and consequently results obtained for the latter can be applied to the NMVLIP with mild modifications. Here we just state the results as corollaries.

**Corollary 1.** Let $K$ be a nonempty closed and convex subset of a Banach space $E$. Let $N : K \times K \to E^*$ be a nonlinear operator, $\varphi : E \times E \to \mathbb{R}$ be a real-valued bifunction, and $\eta : E \times E \to E$ be a functional such that $\eta(u, v) = -\eta(v, u)$ for all $u, v \in E$. Suppose that

(i) $N$ is weakly relaxed $(\eta, -\alpha)$-monotone, with $\limsup \alpha(z_n) \leq \alpha(z)$ for any sequence $\{z_n\}_{n \in \mathbb{N}}$ converging to $z$;
(ii) $N$ is $(\eta, 0)$-diagonally convex and $\varphi$ is convex with respect to the second argument;
(iii) $N(y, \cdot)$ is $\eta$-upper hemicontinuous, and the function $(y, w) \in K \times K \mapsto \langle N(y, v), \eta(w, v) \rangle$ is lower semicontinuous for each fixed $v \in K$;
Algorithm \((\mathcal{B})\):

**Step 0:** Take \(\{\rho_n\}_{n \in \mathbb{N}} \subset ]0, +\infty[\) such that \(\rho_n \to +\infty\), choose \(w_0 \in K\) arbitrary, and let \(n := 0\).

**Step 1:** Given \(w_n \in K\) compute \(w_{n+1} \in K\) such that for all \(y, v \in K\),

\[
\rho_{n+1} \left[ \langle N(y, w_{n+1}), \eta(v, w_{n+1}) \rangle + \varphi(w_{n+1}, v) - \varphi(w_{n+1}, w_{n+1}) \right] \\
+ \langle T(v) - T(w_{n+1}), w_{n+1} - w_n \rangle \geq 0.
\]

(41)

Update \(n := n + 1\) and go to Step 1.

Here \(T : E \to E^*\) is a \(\delta\)-strongly positive bounded linear and bounded operator.

The convergence analysis of the algorithm \((\mathcal{B})\) is discussed in the following corollary, which is a direct consequence of Theorem 4.3.

**Corollary 2.** Let \(K\) be a nonempty closed and convex subset of a Banach space \(E\). Let \(N : K \times K \to E^*\) be a nonlinear operator, \(\varphi : E \times E \to \mathbb{R}\) be a real-valued bifunction, \(\eta : E \times E \to E\) be a functional such that \(\eta(u, v) = -\eta(v, u)\) for all \(u, v \in E\), and \(T : E \to E^*\) be a \(\delta\)-strongly positive bounded linear operator. Suppose that

(i) \(N\) is \((\eta, \gamma)\)-strongly monotone;

(ii) \(N\) is \((\eta, 0)\)-diagonally monotone and \(\varphi\) is convex with respect to the second argument;

(iii) \(N(y, \cdot)\) is \(\eta\)-upper hemicontinuous, and the function \((y, w) \in K \times K \mapsto \langle N(y, v), \eta(w, v) \rangle\) is lower semicontinuous for each fixed \(v \in K\);

(iv) \(\varphi\) is skew symmetric, lower semicontinuous and upper semicontinuous with respect to the first argument;

(v) (Coercivity) There exists a nonempty convex and compact subset \(D\) of \(K\) such that for all \(w \in K \setminus D\), there exists \(v \in D\) and \(y \in D\) satisfying

\[
\langle N(y, w), \eta(v, w) \rangle + \varphi(w, v) - \varphi(w, w) < 0.
\]

Furthermore, we assume that the following condition hold:

\[
(C) \ \exists k \in ]0, 1[ , \text{ such that } \frac{\rho_{n+1}}{\rho_n} \frac{||T||}{|\delta + \gamma \rho_{n+1}|} < k.
\]

Then the iterative sequence \(\{w_n\}_{n \in \mathbb{N}}\) generated by the Algorithm \((\mathcal{B})\) converges strongly to a solution \(\bar{w} \in K\) of the nonlinear mixed variational-like inequality (40).

**Remark 8.** In the case where \(E\) is a reflexive Banach space, the coercivity condition (v) in Corollaries 1 and 2 can be dropped.
6. **Concluding Remarks.** In this paper, we have studied both the existence, uniqueness and approximation of the solution of a general class of mixed equilibrium problem. The results are obtained in the framework of generalized monotonicity, which is, weakly relaxed $\alpha$–monotonicity. Existence result is established using KKM technique and an iterative algorithm is obtained using auxiliary principle technique. We have also shown that the iterates converge to the exact solution strongly. The interest of the problem considered in this paper is that it encompasses several important type of problems as special cases. In comparison with existing results in the literature, the approach developed in this paper improves considerably the results obtained in [10] and [24] in the following sense:

- We have avoided the use of the semimonotonicity notion, which is a strong assumption requiring that the operator $N$ in the problem (2) is completely continuous with respect to the first argument, see [10, Definition 3.1];
- The problems studied in [10] and [24] are particular forms of the problems considered in this paper;
- Many concepts introduced and used in [24] can be avoided by the approach developed in this paper.

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