Genericness of inflation in isotropic loop quantum cosmology

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Non-perturbative corrections from loop quantum cosmology (LQC) to the scalar matter sector is already known to imply inflation. We prove that the LQC modified scalar field generates exponential inflation in the small scale factor regime, for all positive definite potentials, independent of initial conditions and independent of ambiguity parameters. For positive semi-definite potentials it is always possible to choose, without fine tuning, a value of one of the ambiguity parameters such that exponential inflation results, provided zeros of the potential are approached at most as a power law in the scale factor. In conjunction with generic occurrence of bounce at small volumes, particle horizon is absent thus eliminating the horizon problem of the standard Big Bang model.

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The Standard Big Bang model so far is the most successful large scale description of our universe. In this description, the evolution of our universe begins from a singularity. Within the context of homogeneous and isotropic expanding space-times, the singularity is unavoidable as long as the matter satisfies the so called strong energy condition. The singularity in this context means that the scale factor (or size of the universe) vanishes a finite time ago. This vanishing size also implies that the energy density diverges at this time. Furthermore, the scale factor vanishes slower than linearly with the synchronous time making the conformal time integral finite thus implying the existence of particle horizon.

The particle horizon with respect to a space-time point is defined by the maximum proper distance a particle could have traveled since the beginning of the universe. Due to the behavior of the scale factor, this is a finite distance. It also means that any space-time point could have causal contact only with a finite patch of the space-time around it. By itself, existence of particle horizon need not be a problem. However, in conjunction with the thermal history of the universe, the finite horizon size implies that the surface of last scatter of the cosmic microwave background photons has regions which could not have been in causal contact. Yet, there is remarkable isotropy (to within few parts of thousand) in their angular distribution. This is the horizon problem of the Big Bang model.

The most popular approach to solve this puzzle (along with few other puzzles) is to introduce a phase of inflation \textsuperscript{1}. Phase of inflation generally refers to a period during which the universe goes through a rapid (generally exponential) expansion. Clearly there must be a violation of the strong energy condition during inflation. This is generally achieved by introducing a scalar field (an inflaton) with a self interaction potential. By now there are several versions of inflationary models \textsuperscript{2}. Generically these solve the horizon problem (and other traditional problems such as the flatness problem) and in addition make specific predictions about the power spectra of inhomogeneous perturbations. While these are attractive features of inflationary models, generally they need fine tuning the potential and initial conditions for the inflaton to ensure a sufficient amount of inflation with graceful exit. In a sense, the isotropic singularity in Einsteinian gravity implies existence of particle horizon which leads to the horizon problem which needs an inflationary scenario to be implanted.

The space-time singularity, however, signals breakdown of the theoretical framework of classical general relativity. It is widely expected that a quantum theory of gravity will provide a more accurate description which will hopefully be free of such breakdowns. A fully satisfactory quantum theory of gravity is not yet available. Over the past couple of decades, two promising approaches have emerged, String Theory \textsuperscript{3} and Loop Quantum Gravity (LQG) \textsuperscript{4}. In the last few years, a detailed adaptation of LQG methods to cosmological context has been developed and has come to be known as Loop Quantum Cosmology (LQC) \textsuperscript{5}. In this letter we work within the LQC framework \textsuperscript{6} and more specifically within the context of spatially flat or close isotropic models.

It has already been shown that the LQC framework is free of singularity, both in the isotropic context \textsuperscript{7} as well as more generally for homogeneous diagonal models \textsuperscript{8,9}. There are two aspects of this singularity-free property. The imposition of the Hamiltonian constraint ("Wheeler–DeWitt equation") of LQC leads to a difference equation with eigenvalues of the densitized triad variable serving as labels. These eigenvalues can take negative values corresponding to reversal of orientation. The difference equation, viewed as an evolution equation in these labels, allows solutions to evolve through the zero eigenvalue (zero size). Thus there is no breakdown of evolution equation at the classically indicated singularity at zero size. This is the first aspect of absence of singularity. The second aspect is that matter densities and curvatures remain finite at all sizes. The inverse scale factor operator that enters the definitions of these
quantities turns out to have bounded spectrum. For an explanation and details see [2, 10].

While quantum theory is well specified at the kinematical level, one still does not have a physical inner product to have a bona-fide Hilbert space of solutions of the Hamiltonian constraint (the difference equation). The issue of Dirac observables is also under-explored (but see [11]). Consequently, understanding of semi-classical behavior in terms of expectation values of observables is not yet available. To relate implications of LQC which is based on a discrete quantum geometry, to observable quantities turns out to have bounded spectrum. For an explanation and details see [6, 10].

The idea of an effective Hamiltonian has been proposed [12, 13]. This Hamiltonian contains the modifications implied by LQC to the usual classical Hamiltonian. This approach retains the kinematical framework of Robertson-Walker geometry but gives modification of the dynamics of the Einstein equations.

The effective Hamiltonian in LQC is generally derived in two steps [13, 14]. In first step, one develops a continuum approximation to the fundamental difference equation to obtain a differential equation [15]. For large volumes where one expects the manifestation of discrete geometry to be negligible, the differential equation matches with the usual Wheeler-DeWitt equation (with certain factor ordering). In the second step one looks for a WKB form for solution of the differential equation to derive the corresponding Hamilton-Jacobi equation and read-off the Hamiltonian. This is the effective Hamiltonian. The effective Hamiltonian differs from the classical Hamiltonian due to the modifications in the differential equation derived from the difference equation. There are two sources of modifications. In the matter sector, the modifications come from using the modified inverse triad operator which incorporate the small volume deviations. These involve inverse powers of the Planck length and thus are non-perturbative. One can also get modifications in the gravity sector for small volumes. These have been obtained recently [14], exploiting non-separable structure of the kinematical Hilbert space of LQC [2].

It turns out that the dynamics (evolution with respect to the synchronous time) implied by the effective Hamiltonian captures essential features of the difference equation, in particular the dynamics is non-singular. A universe beginning at some large volume will never reach zero volume when evolved backwards. Since the framework for effective dynamics is that of the usual pseudo-Riemannian geometry, the arguments leading to the singularity theorem are applicable and therefore non-singular evolution must imply violation of the strong energy condition on effective matter density and pressure. While the effective density and pressure [14] includes contributions from gravity sector alone, in this letter we concentrate on the matter sector modifications only.

The question we address is whether the modifications in the matter sector imply violation of strong energy condition. In general the strong energy condition requires

$$R_{\alpha\beta}\xi^\alpha\xi^\beta = 8\pi G(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)\xi^\alpha\xi^\beta \geq 0,$$

for all time-like vectors $\xi^\alpha$. Within the context of homogeneous and isotropic geometries, the strong energy condition applied to the congruence of isotropic observers (or four velocity of the matter perfect fluid), becomes

$$R_{00} = 4\pi G(\rho + 3P) \geq 0 \ \text{where} \ \rho \ \text{is the total energy density and} \ P \ \text{is the total pressure of the matter fluid. Defining} \ \omega := P/\rho \ \text{(with} \ \rho \ \text{assumed to be positive definite)} \ \text{as the equation of state variable, the violation of}\n
strong energy condition is conveniently stated as} \ \omega < -\frac{1}{3}.

Note that since $R_{00} = -\frac{3}{a^2}$, violation of the strong energy condition in this context also implies an accelerated evolution of the scale factor or in other words an inflationary phase.

For simplicity, let the matter sector consists of a single scalar field with a standard kinetic term and self interaction potential. While quantum theory is well specified at the kinematical level, one still does not have a physical inner product to have a bona-fide Hilbert space of solutions of the Hamiltonian constraint (the difference equation). The issue of Dirac observables is also under-explored (but see [11]). Consequently, understanding of semi-classical behavior in terms of expectation values of observables is not yet available. To relate implications of LQC which is based on a discrete quantum geometry, to observable quantities turns out to have bounded spectrum. For an explanation and details see [6, 10].

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of the Hamiltonian, this route is not available. It is possible to define the density and pressure directly in terms of the matter Hamiltonian. This has been done generally in [14] (See also [19]). The relevant definitions, in terms of the notation in [14], are \( \alpha \) denotes \( \frac{d}{a} \)

\[
\rho = \frac{32}{3} \alpha \ H, \quad P = \frac{32}{9} \alpha \ a^3 \left( 1 - \frac{\alpha}{\alpha'} \right) H - aH \tag{2}
\]

In the above \( \alpha \) is a specific function of \( a \). For large \( a^2, \alpha \) goes as \( \frac{1}{a^2} \) giving the familiar form of density as \( 8Ha^{-3} \) [14]. For the LQC modified scalar matter Hamiltonian, and for large \( a \), these definitions of density and pressure match with those of [19]. The conservation equation defined by the effective Hamiltonian and by the classical Hamiltonian. This has been done generally [14] (See also [19]). The relevant definitions, in terms of the matter Hamiltonian. This follows.

Consider the two equation of state variables, \( P/\rho \), defined by the effective Hamiltonian and by the classical Hamiltonian.

\[
\omega^{\text{eff}} = -\frac{1}{2} \left[ \frac{\bar{F}_j(\bar{a})}{a} \right]^2 \left( 1 - \frac{\alpha}{\alpha'} \right) + V(\phi) + \frac{1}{\bar{p}^2} a^{-3} \left( \frac{\bar{F}_j(\bar{a})}{a} \right)^2 + V(\phi)
\]

\[
\omega = \frac{\bar{p}^2 a^{-6} - V(\phi)}{\bar{p}^2 a^{-6} + V(\phi)} \tag{3}
\]

The second term in eq. (3) vanishes for large volumes and goes to \(-1/3\) for small volumes [14]. It is independent of the matter variables and will be suppressed below (see remarks on the scales at the end). The dynamical evolutions of these equations of state is of course governed by the corresponding Hamiltonians. It is however possible to derive qualitative behavior of \( \omega^{\text{eff}} \) for small scale factors \textit{without having to know explicit time evolution}, as follows.

The equations (3) can be thought of as two homogeneous algebraic equations for \( p^2, V(\phi) \). For non-trivial values of these, the determinant must vanish which gives a relation between the two \( \omega \)'s as,

\[
\omega^{\text{eff}} = -1 + \frac{1}{\left( 1 + \omega \right) a^3} \left[ \frac{\bar{F}_j(\bar{a})}{a} \right]^2 \left( 1 - \frac{\alpha}{\alpha'} \right) \left( 1 - \omega \right) \left( 1 + \omega \right) a^3 \left[ \frac{\bar{F}_j(\bar{a})}{a} \right]^2 + (1 - \omega) . \tag{5}
\]

Using the expression (4) it is easy to see that for the large values of the scale factor \( a \), where one expects the quantum effects to be small, \( \omega^{\text{eff}} = \omega \) and the dynamical evolution is controlled by the classical Hamiltonian. However for small values of \( a \) the \( \omega^{\text{eff}} \) differs from the classical \( \omega \) dramatically.

The numerator in the second terms of (3) vanishes as \( a^{-3+\frac{3}{1-\omega}} \). If \( 1 - \omega \) in the denominator dominates, then clearly \( \omega^{\text{eff}} \to -1 \). This would happen either because \( 1 - \omega \to 0 \) or it vanishes slower than \( a^{3+\frac{3}{1-\omega}} \). In the former case we already have violation of strong energy condition. It is possible to get constraints on the behavior of \( \omega \) as the scale factor vanishes. For instance, the conservation equation expressed in terms of the scale factor implies that if \( \omega \to 1 \) then \( \rho \sim a^{-6} \). This equation is independent of the LQC modification and applies also to effective density. Furthermore, from the definition it follows that \( 1 - \omega = \frac{2V(\phi)}{\rho} \). Thus, the \( 1 - \omega \) term in the denominator will dominate if \( V(\phi)(a) a^{-\frac{3}{2}} \) diverges as \( a \to 0 \). This dominance is ensured if either (i) the potential never vanishes during the evolution or (ii) \( V(\phi)(a) \) vanishes at the most as a power law, \( a^\xi \). In the former case, \( \omega^{\text{eff}} \to -1 \) will hold independent of the ambiguity parameter \( \ell \) while in the latter case, for any given \( \alpha \) we can always choose \( \ell > \frac{\alpha}{\alpha'} \) so that \( \omega^{\text{eff}} \to -1 \) is achieved. Note that this is not a fine tuning.

For the special case of identically vanishing potential, we get \( \omega = 1 \) and the expression for \( \omega^{\text{eff}} \) simplifies to \( \frac{a[\bar{F}_j(\bar{a})]}{2\bar{F}_j(\bar{a})} \). For small scale factor \( \omega^{\text{eff}} \to -\frac{1}{1-\gamma} \approx -1 \) and violation of strong energy condition follows. Indeed, since \( \omega^{\text{eff}} \approx -1 \) holds, one has a phase of \textit{super-inflation}.

In fact this feature corresponds to situation considered in [20-21]. However this feature is rather special because even a tiny but non-negative potential will force \( \omega^{\text{eff}} \) to take the form (5) (see figure 1).

\[ \text{FIG. 1: Plot of } \omega^{\text{eff}} \text{ as a function of } a^2 \text{ and } \omega \text{ for different constant values } \omega = 0.9, 0.33, 0.0001 \text{. The ambiguity parameters are } j = 5, l = 0.5 \text{ and } a^2 \text{ is in units of } \frac{1}{\gamma^2} \text{. For small scale factor, } \omega^{\text{eff}} \text{ always approaches } -1 \text{ from below while for larger values it approaches } \omega \text{.} \]

The spikes in the figure correspond to the non-differentiability of the \( F_j(q) \) at \( q = 1 \). These can be removed by a local smoothing of the function around \( q = 1 \) and thus have no physical significance.

In summary, we find that if the scalar field potential
satisfies $V(\phi) > 0$ then irrespective of what values we choose for the ambiguity parameters and irrespective of ‘initial conditions’ for the scalar field, there is always a violation of strong energy condition in the small volume regime and of course a corresponding inflationary phase. Furthermore since the effective equation of state variable approaches $-1$, we get to a phase of exponential inflation. If the potential has zeros which are approached as a power law for small scale factor, one can always choose a value of $\ell$ to get the same result. We emphasize that unlike the usual inflationary scenarios we do not need to invoke ‘slow roll conditions’ which constrain the potential as well as initial conditions for the scalar and effectively posit the equation of state variable to be $-1$. It is enough to have the evolution get to small volume regime to generate (exponential) inflation.

A couple of remarks are in order. Firstly, if LQC modifications from gravity sector (quantum geometry potential) are also included \[14\], then the results regarding behavior of the effective equation of state as a function of the scale factor, are unchanged. These gravitational contributions to density and pressure violate the strong energy condition by themselves. Their effective equation of state parameter is $+1$ but both the density and pressure are negative.

A second remark concerns the scales. There are two basic scales available: (i) the ‘quantum geometry scale’, $L_{qg}^2 := \frac{1}{6} \gamma_0 \mu_0 \gamma^2 = p_0 \ell_\gamma^2$ and (ii) the ‘inverse scale factor scale’ $L_j^2 := \frac{1}{6} \gamma_0 \mu_0 \ell_j^2(2j) = 2jp_0$. The former sets the scale for non-perturbative modifications in the gravitational sector while the latter does the same for the matter sector. Clearly, $L_{qg} \leq L_j$. It is easy to see \[14\] that the WKB approximation gets poorer close to $L_{qg}$. This is consistent with the physical expectation that below this scale one is in the deep quantum regime. Furthermore, the effective model, including the modifications in both gravity and matter sector, always shows a bounce i.e. a non-zero minimum scale factor at which $\dot{a}$ vanishes \[22\]. This introduces a third scale, $L_{\text{bounce}}^2$ which is smaller than $L_j^2$. Clearly, $L_{\text{bounce}}^2 > L_{qg}^2$ must hold to remain within the domain of validity of WKB approximation. In summary, $a \gg L_j$ is the classical regime, while for $a < L_{\text{bounce}}$ one is strictly in the quantum domain in the WKB sense and the effective Hamiltonian is not valid. The semi-classical regime for the purposes of this paper has the scale factor between $L_{\text{bounce}}$ and $L_j$. This implies that the suppressed term in the eq. \[22\] is vanishingly small in the semiclassical regime thus $\omega^{\text{eff}} \rightarrow -1$.

The issue of whether the effective dynamics admits particle horizon or not, is a separate issue. In view of the generic bounce in the effective model \[22\], the universe would have existed for infinite time in the past. The evolution could have been oscillatory or there could have been just one bounce in the past. In the large volume regime, we have the usual decelerating evolution (modulo $\Lambda$-term) implying that the scale factor will diverge at the most as linear power of the synchronous time. For both possibilities, the conformal time integral would be infinite implying absence of particle horizon. However, independent of the non-existence of particle horizon, inflation comes built-in with the LQC modifications.

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\[\text{[1]}\quad\text{Bojowald M, Phys. Rev. D 89, 261301 (2002);}\]
\[\text{[2]}\quad\text{Bojowald M and Vandersloot K, Phys. Rev. D 67, 124023 (2003)}\]
\[\text{[3]}\quad\text{Bojowald M, Phys. Rev. Lett. 89, 261301 (2002);}\]
\[\text{[4]}\quad\text{Bojowald M, Phys. Rev. D 70, 063521 (2004);}\]
\[\text{[5]}\quad\text{Bojowald M, Phys. Rev. Lett. 89, 261301 (2002);}\]
\[\text{[6]}\quad\text{Bojowald M and Vandersloot K, Phys. Rev. D 67, 124023 (2003)}\]
\[\text{[7]}\quad\text{Bojowald M, Class. Quantum Grav. 21, 179 (2004);}\]
\[\text{[8]}\quad\text{Bojowald M, Class. Quantum Grav. 21, 3541 (2004);}\]
\[\text{[9]}\quad\text{Bojowald M, Date G and Vandersloot K, Class. Quantum Grav. 21, 2595 (2004);}\]
\[\text{[10]}\quad\text{Bojowald M, Date G and Vandersloot K, Class. Quantum Grav. 21, 3541 (2004);}\]
\[\text{[11]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 179 (2004);}\]
\[\text{[12]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 3541 (2004);}\]
\[\text{[13]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 4941 (2004);}\]
\[\text{[14]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 121 (2004);}\]
\[\text{[15]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 121 (2004);}\]
\[\text{[16]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 121 (2004);}\]
\[\text{[17]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 5113 (2001);}\]
\[\text{[18]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 5113 (2001);}\]
\[\text{[19]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 5113 (2001);}\]
\[\text{[20]}\quad\text{Hossain G M, Class. Quantum Grav. 21, 5113 (2001);}\]