SPECTRAL INCLUSIONS BETWEEN $C_0$-QUASI-SEMIGROUPS AND THEIR GENERATORS

A. TAJMOUATI, Y. ZAHOUAN AND M.A. OULD MOHAMED BABA

Abstract

In this paper, we show a spectral inclusion of a different spectra of a $C_0$-quasi-semigroup and its generator and precisely for ordinary, point, approximate point, residual, essential and regular spectra.

keywords: $C_0$-quasi-semigroup, $C_0$-semigroup, semi-regular, ascent, descent, spectrum, point spectrum, essential, regular spectra.

1. Introduction and preliminaries

Let $X$ be a complex Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators on $X$. We denote by $D(T)$, $Rg(T)$, $Rg^\infty(T) := \cap_{n \geq 1} Rg(T^n)$, $N(T)$, $\rho(T)$, $\sigma(T)$, and $\sigma_p(T)$ respectively the domain, the range, the hyper range, the kernel, the resolvent and the spectrum of $T$, where $\sigma(T) = \{ \lambda \in \mathbb{C} \mid \lambda - T \text{ is not bijective} \}$. The point, the approximate point, the residual and regular spectra are defined by

- $\sigma_p(T) = \{ \lambda \in \mathbb{C} \mid \lambda - T \text{ is not injective} \}$
- $\sigma_a(T) = \{ \lambda \in \mathbb{C} \mid \lambda - T \text{ is not injective or } Rg(\lambda - T) \text{ is not closed in } X \}$
- $\sigma_r(T) = \{ \lambda \in \mathbb{C} \mid Rg(\lambda - T) \text{ is not dense in } X \}$
- $\sigma_{\gamma}(T) = \{ \lambda \in \mathbb{C} \mid \lambda - T \text{ is not semi regular} \}$, i.e. $\lambda \in \sigma_{\gamma}(T)$ if $Rg(\lambda - T)$ is not closed or $N(\lambda - T) \not\subset Rg^\infty(\lambda - T)$.

An operator $T \in \mathcal{B}(X)$ is called Fredholm operator, in symbol $T \in \Phi(X)$, if $\alpha(T) = \dim N(T)$ and $\beta(T) = \text{codim} Rg(T)$ are finite, and the essential spectrum is defined by,

$$\sigma_e(T) = \{ \lambda \in \mathbb{C} : \lambda - T \notin \Phi(X) \}.$$ 

The family $(T(t))_{t \geq 0} \subseteq \mathcal{B}(X)$ is a $C_0$-semigroup if it has the following properties:

1. $T(0) = I$
2. $T(t)T(s) = T(t+s)$;
3. The map $t \to T(t)x$ from $[0, +\infty[$ into $X$ is continuous for all $x \in X$;

In this case, its generator $A$ is defined by

$$\mathcal{D}(A) = \{ x \in X / \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \},$$
with

\[ Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} . \]

The theory of quasi-semigroups of bounded linear operators, as a generalization of semigroups of operators, was introduced by Leiva and Barcenas [3], [4], [5].

**Definition 1.1.** [3] Let \( X \) be a complex Banach. The family \( \{ R(t, s) \} \) for every \( t, s, r \geq 0 \) and \( x \in X \),

1. \( R(t, 0) = I \), the identity operator on \( X \),
2. \( R(t, s + r) = R(t + r, s)R(t, r) \),
3. \( \lim_{s \to 0^+} ||R(t, s)x - x|| = 0 \),
4. There exists a continuous increasing mapping \( M : [0; +\infty[ \to [1; +\infty[ \) such that,

\[ ||R(t, s)|| \leq M(t + s) \]

**Definition 1.2.** [3] For a \( C_0 \)-quasi-semigroup \( \{ R(t, s) \} \) on a Banach space \( X \), let \( D \) be the set of all \( x \in X \) for which the following limits exist,

\[ \lim_{s \to 0^+} \frac{R(t,s)x - x}{s} \quad \text{and} \quad \lim_{s \to 0^+} \frac{R(t,s)x - x}{s} = \lim_{s \to 0^+} \frac{R(t,s)x - x}{s} , \quad t > 0 \]

For \( t \geq 0 \) we define an operator \( A(t) \) on \( D \) as \( A(t)x = \lim_{s \to 0^+} \frac{R(t,s)x - x}{s} \).

The family \( \{ A(t) \} \) is called infinitesimal generator of the \( C_0 \)-quasi-semigroups \( \{ R(t, s) \} \).

Throughout this paper we denote \( T(t) \) and \( R(t, s) \) as \( C_0 \)-semigroups \( \{ T(t) \} \) and \( C_0 \)-quasi-semigroup \( \{ R(t, s) \} \) respectively. We also denote \( D \) as domain for \( A(t) \), \( t \geq 0 \).

**Remark 1.1.** [12] Examples 2.3 and 3.3 In the semigroups theory, if \( A \) is an infinitesimal generator of \( C_0 \)-semigroup with domain \( D(A) \), then \( A \) is a closed operator and \( D(A) \) is dense in \( X \). These are not always true for any \( C_0 \)-quasi-semigroups.

**Remark 1.2.** If \( T(t) \) be a \( C_0 \)-semigroup on a Banach space \( X \) with its generator \( A \) then \( R(t, s) \) with \( R(t, s) = T(s), t, s \geq 0 \) defines a \( C_0 \)-quasi-semigroup on \( X \) with generator \( A(t) = A, t \geq 0 \) and \( D = D(A) \).

**Examples 1.** Let \( T(t) \) be a \( C_0 \)-semigroup on a Banach space \( X \) with its generator \( A \).

For \( t, s \geq 0 \), \( R(t, s) = T(g(t + s) - g(t)), \) where \( g(t) = \int_0^t a(u)du \) and \( a \in C([0; \infty[) \) with \( a(t) > 0 \)

The \( R(t, s) \) is \( C_0 \)-quasi-semigroup on \( X \) with generator \( A(t) = a(t)A \).

**Proof.**

(1) \( R(t, 0) = T(g(t) - g(t)) = T(0) = I \).
\[
R(t, s + r) = T(g(t + s + r) - g(t + r) + g(t + r) - g(t)) \\
= T(g(t + s + r) - g(t + r)T(g(t + r) - g(t)) \\
= R(t + r, s)R(t, r),
\]

(2) 

\[
\text{lim}_{s \to 0} |R(t, s)x - x| = \text{lim}_{s \to 0} |T(g(t + s) - g(t))x - x| = 0, \text{ since } g \text{ is continuous.}
\]

(3) 

\[
\text{lim}_{s \to 0} |R(t, s)x - x| = \text{lim}_{s \to 0} |T(g(t + s) - g(t))x - x| = 0, \text{ since } g \text{ is continuous.}
\]

(4) 

\[
\text{Since } T(t) \text{ is strongly continuous on } X, \text{ there exists } \omega \text{ and } M_\omega > 0 \text{ such that,}
\]

\[
||T(t)|| \leq M_\omega e^{\omega t}
\]

Therefore, \( ||R(t, s)|| \leq M(t + s), \) where \( M(t + s) = M_\omega e^{\omega t}\).

Moreover,

\[
A(t)x = \lim_{s \to 0^+} \frac{R(t, s)x - x}{s} \\
= \lim_{s \to 0^+} \frac{T(g(t + s) - g(t))x - x}{s} \\
= g(t + s) \frac{d}{ds} [T(g(t + s) - g(t))x] \bigg|_{s=0} \\
= a(t)Ax
\]

Thus, \( R(t, s) \) is \( C_0 \)-quasi-semigroup on \( X \) with generator \( A(t) = a(t)A \).

The following results are obtained recently by Sutrima, Ch. Rini Indrati and others \[12\], there are show some relations between a \( C_0 \)-quasi-semigroup and its generator.

**Theorem 1.1.** \[12\] Let \( R(t, s) \) be a \( C_0 \)-quasi-semigroup on \( X \) with generator \( A(t) \) then,

1. For each \( t \geq 0 \), \( R(t, \cdot) \) is strongly continuous on \([0; +\infty[.\)
2. For each \( t \geq 0 \) and \( x \in X \),

\[
\lim_{s \to 0^+} \frac{1}{s} \int_0^s R(t, h)xdh = x
\]

3. If \( x \in \mathcal{D}, \ t \geq 0 \) and \( t_0, s_0 \geq 0 \) then, \( R(t_0, s_0)x \in \mathcal{D} \) and

\[
R(t_0, s_0)A(t)x = A(t)R(t_0, s_0)x
\]

4. For each \( s > 0 \), \( \frac{d}{dt} R(t, s)x = A(t + s)R(t, s)x = R(t, s)A(t + s)x; \ x \in \mathcal{D}. \)
5. If \( A(\cdot) \) is locally integrable, then for every \( x \in \mathcal{D} \) and \( s \geq 0 \),

\[
R(t, s)x = x + \int_0^s A(t + h)R(t, h)xdh.
\]

6. If \( f : [0; +\infty[ \to X \) is a continuous, then for every \( t \in [0; +\infty[\)

\[
\lim_{r \to 0^+} \int_s^{s+r} R(t, h)f(h)xdh = R(t, s)f(s)
\]

In this work, we show that the spectral inclusion of different spectra of \( C_0 \)-semigroups valid for \( C_0 \)-quasi-semigroups.
2. Main results

For later use, we introduce the following operator acting on $X$ and depending on the parameters $\lambda \in \mathbb{C}$ and $t, s \geq 0$ :

$$D_\lambda(t, s)x = \int_0^s e^{\lambda(s-h)} R(t, h)xdh \quad \text{for all } x \in X.$$ $D_\lambda(t, s)$ is a bounded linear operator on $X$

**Theorem 2.1.** Let $A(t)$ be the generator of the $C_0$-quasi-semigroup $\{R(t, s)\}_{t, s \geq 0}$. Then for all $\lambda \in \mathbb{C}$ and all $t, s \geq 0$ we have

1. For all $x \in X$,
   $$ (\lambda - A(t)) D_\lambda(t, s)x = [e^{\lambda s} - R(t, s)]x.$$

2. For all $x \in D$,
   $$ D_\lambda(t, s) (\lambda - A(t)) x = [e^{\lambda s} - R(t, s)]x.$$

**Proof.** (1) For all $x \in X$ we have

$$ R(0, r) D_\lambda(t, s)x = R(0, r) \int_0^s e^{\lambda(s-h)} R(t, h)xdh $$

$$ = \int_0^s e^{\lambda(s-h)} R(0, r) R(t, h)xdh $$

And we obtain,

$$ \lim_{r \to 0^+} \frac{R(0, r) D_\lambda(t, s)x - D_\lambda(t, s)x}{r} = \lim_{r \to 0^+} \frac{\int_0^s e^{\lambda(s-h)} R(0, r) R(t, h)xdh - \int_0^s e^{\lambda(s-h)} R(t, h)xdh}{r} $$

$$ = \frac{\partial}{\partial r} \left[ \int_0^s e^{\lambda(s-h)} R(0, r) R(t, h)xdh \right]_{r=0} $$

Then $\lim_{r \to 0^+} \frac{R(0, r) D_\lambda(t, s)x - D_\lambda(t, s)x}{r}$ exists.

And,
\[ \lim_{r \to 0^+} \frac{R(t, r)D_\lambda(t, s)x - D_\lambda(t, s)x}{r} = \lim_{r \to 0^+} \frac{\int_0^s e^{\lambda(s-h)} R(t, r)R(t, h)x dh - \int_0^s e^{\lambda(s-h)} R(t, h)x dh}{r} \]

\[ = \frac{\partial}{\partial r} \left[ \int_0^s e^{\lambda(s-h)} R(t, r)R(t, h)x dh \right]_{r=0} \]

\[ = \left[ \int_0^s e^{\lambda(s-h)} \frac{\partial}{\partial r}(R(t, r))R(t, h)x dh \right]_{r=0} \]

\[ = \left[ \int_0^s e^{\lambda(s-h)} A(t)R(t, r)R(t, h)x dh \right]_{r=0} \]

\[ = \int_0^s e^{\lambda(s-h)} A(t)R(t, h)x dh \]

Moreover,

\[ \lim_{r \to 0^+} \frac{R(t-r, r)D_\lambda(t, s)x - D_\lambda(t, s)x}{r} = \lim_{r \to 0^+} \frac{\int_0^s e^{\lambda(s-h)} R(t-r, r)R(t, h)x dh - \int_0^s e^{\lambda(s-h)} R(t, h)x dh}{r} \]

\[ = \frac{\partial}{\partial r} \left[ \int_0^s e^{\lambda(s-h)} R(t-r, r)R(t, h)x dh \right]_{r=0} \]

\[ = \left[ \int_0^s e^{\lambda(s-h)} \frac{\partial}{\partial r}(R(t-r, r))R(t, h)x dh \right]_{r=0} \]

\[ = \left[ \int_0^s e^{\lambda(s-h)} A(t)R(t-r, r)R(t, h)x dh \right]_{r=0} \]

\[ = \int_0^s e^{\lambda(s-h)} A(t)R(t, h)x dh \]

Thus, \( \lim_{r \to 0^+} \frac{R(t,r)x}{r} = \lim_{r \to 0^+} \frac{R(t-r,r)x}{r} \)

Hence, we deduce that \( D_\lambda(t, s)x \in \mathcal{D} \) And,

\[ A(t)D_\lambda(t, s)x = \int_0^s e^{\lambda(s-h)} A(t)R(t, h)x dh \]

\[ = \int_0^s e^{\lambda(s-h)} A(t+h)R(t, h)x dh, \text{ because } A(t+h) = A(t) \text{ for all } t, h \geq 0. \]

\[ = \int_0^s e^{\lambda(s-h)} \frac{\partial}{\partial h}(R(t, h))x dh \]

\[ = \left[ e^{\lambda(s-h)} R(t, h) \right]_0^s + \lambda \int_0^s e^{\lambda(s-h)} R(t, h)x dh \]

\[ = R(t, s)x - e^{\lambda s}x + \lambda D_\lambda(t, s)x \]

Finally, \( (\lambda - A(t))D_\lambda(t, s)x = [e^{\lambda s} - R(t, s)]x \) for all \( x \in X. \)
(2) For all \( x \in D \) and all \( t, s \geq 0 \) we have,
\[
D_\lambda(t, s)A(t)x = \int_0^s e^{\lambda(s-h)}R(t,h)A(t)xdh \\
= \int_0^s e^{\lambda(s-h)}R(t,h)A(t+h)xdh \\
= \int_0^s e^{\lambda(s-h)}\frac{\partial}{\partial r}(R(t,h))xdh \\
= \left[e^{\lambda(s-h)}R(t,h)\right]_0^s + \lambda \int_0^s e^{\lambda(s-h)}R(t,h)xdh \\
= R(t,s)x - e^{\lambda s}x + \lambda D_\lambda(t,s)x
\]
Thus, we deduce for all \( x \in D(A) \)
\[
D_\lambda(t, s)(\lambda - A(t))x = [e^{\lambda s} - R(t, s)]x.
\]

Corollary 2.1. In the case of \( C_0 \)-semigroup \( T(s) = R(t, s) \), we retrieve the equality \([1]\),
\[
(\lambda - A)D_\lambda(s)x = [e^{\lambda s} - T(s)]x
\]
With \( D_\lambda(s)x = \int_0^s e^{\lambda(s-h)}T(h)xdh \) for all \( x \in X \) and \( s \geq 0 \)

Corollary 2.2. Let \( A(t) \) be the generator of a \( C_0 \)-quasi-semigroup \((R(t, s))_{t,s \geq 0}\). Then for all \( \lambda \in C \), \( t, s \geq 0 \) and \( n \in \mathbb{N} \),

(1) For all \( x \in X \),
\[
(\lambda - A(t))^n[D_\lambda(t,s)]^nx = [e^{\lambda s} - R(t, s)]^nx.
\]

(2) For all \( x \in D^n \) (Domain of \( A(t)^n \)),
\[
[D_\lambda(t,s)]^n(\lambda - A(t))^nx = [e^{\lambda s} - R(t, s)]^nx.
\]

(3) \( N[\lambda - A(t)] \subseteq N[e^{\lambda s} - R(t, s)] \).

(4) \( Rg[e^{\lambda s} - R(t, s)] \subseteq Rg[\lambda - A(t)] \).

(5) \( N[\lambda - A(t)]^n \subseteq N[e^{\lambda s} - R(t, s)]^n \).

(6) \( Rg[e^{\lambda s} - R(t, s)]^n \subseteq Rg[\lambda - A(t)]^n \).

(7) \( Rg^\infty[e^{\lambda s} - R(t, s)] \subseteq Rg^\infty[\lambda - A(t)] \).

Proof. follow easily from,
\[
e^{\lambda s}x - R(t,s)x = (\lambda - A(t))D_\lambda(t,s)x
\]
\[
= D_\lambda(t,s)(\lambda - A(t))x
\]

The following theorem characterizes the ordinary, point, approximate point, essential and residual spectra of a \( C_0 \)-quasi-semigroup.

Theorem 2.2. For the generator \( A(t) \) of a \( C_0 \)-quasi-semigroup \((R(t, s))_{t,s \geq 0}\) there exist the spectral inclusions
Proof.

(1) Let $\lambda \in \mathbb{C}$ such that for all $t \geq 0$

$$e^{\lambda s} \notin \sigma(R(t, s)),$$

then the operator $e^{\lambda s} - R(t, s)$ is invertible where $F_\lambda(t, s)$ its inverse.

By Theorem 2.1 we obtain for every $x \in \mathcal{D}$

$$x = F_\lambda(t, s)[e^{\lambda s} - R(t, s)]x$$

$$= F_\lambda(t, s)[D_\lambda(t, s)(\lambda - A(t))]x$$

$$= [F_\lambda(t, s)D_\lambda(t, s)](\lambda - A(t))x.$$

On the other hand, also from Theorem 2.1 we obtain for every $x \in X$

$$x = [e^{\lambda s} - R(t, s)]F_\lambda(t, s)x;$$

$$= [(\lambda - A(t))D_\lambda(s)]F_\lambda(t, s)x;$$

$$= (\lambda - A(t))[D_\lambda(t, s)F_\lambda(t, s)]x.$$

Since we know that $R(t, s)F_\lambda(t, s) = F_\lambda(t, s)R(t, s)$, then

$$F_\lambda(t, s)D_\lambda(t, s) = D_\lambda(t, s)F_\lambda(t, s).$$

Finally, we conclude that $\lambda - A(t)$ is invertible and hence $\lambda \notin \sigma(A(t)).$

(2) Let $\lambda \in \sigma_p(A(t))$, then there exists $x \neq 0$ such that $x \in N(\lambda - A(t))$. From Corollary 2.2 we get $x \in N[e^{\lambda s} - R(t, s)]$. Therefore, we conclude that $e^{\lambda t} \in \sigma_p(R(t, s)).$

(3) Let $\lambda \in \sigma_n(A(t))$ and a corresponding approximate eigenvector $(x_n)_{n \in \mathbb{N}} \subset \mathcal{D}$, we define the sequence $(y_n)$ by $y_n := e^{\lambda s}x_n - R(t, s)x_n$

By theorem 2.1 we have

$$y_n := \int_0^s e^{\lambda(s-h)}R(t, h)(\lambda - A(t))x_n dh.$$ 

So there is a constant $c$ ) 0 such that

$$\|y_n\| = \int_0^s \|e^{\lambda(s-h)}R(t, h)(\lambda - A(t))x_n\|dh \leq c\|(\lambda - A(t))x_n\| \to 0 \text{ as } n \to \infty.$$

Hence, $e^{\lambda s}$ is an approximate eigenvalue of $R(t, s)$, and $(x_n)_{n \in \mathbb{N}}$ serves as the same approximate eigenvector for all $t, s \geq 0$. 

(1) $e^{\sigma(A(t))s} \subset \sigma(R(t, s))$

(2) $e^{\sigma_p(A(t))s} \subset \sigma_p(R(t, s))$

(3) $e^{\sigma_n(A(t))s} \subset \sigma_n(R(t, s))$. 

(4) $e^{\sigma_e(A(t))s} \subset \sigma_e(R(t, s))$. 

(5) $e^{\sigma_r(A(t))s} \subset \sigma_r(R(t, s))$. 


In consequence, the operator \( \lambda \) with generator \( \mathcal{A} \) and from theorem 2.2 (3) \( \lambda \not\in \sigma_e(R(t, s)) \).

Then we have \( \alpha[e^{\lambda t} - R(t, s)] < +\infty \) and \( \beta[e^{\lambda t} - R(t, s)] < +\infty \). Therefore, by Corollary 2.2, we conclude that \( \alpha[\lambda - A(t)] < +\infty \) and \( \beta[\lambda - A(t)] < +\infty \), and hence \( \lambda \not\in \sigma_e(A) \).

(5) Let \( \lambda \in \sigma_e(A(t)) \), then \( Rg[\lambda - A(t)] \) is not dense, now we use the corollary 2.2, we obtain

\[ Rg[e^{\lambda s} - R(t, s)] \subseteq Rg[\lambda - A(t)]. \]

Thus \( Rg[e^{\lambda s} - R(t, s)] \) is not dense and finally \( e^{\lambda s} \in \sigma_e(R(t, s)) \).

\[ \square \]

In the next theorem, we will prove that the spectral inclusion of \( C_0 \)-quasi-semigroups remains true for the regular spectrum.

**Theorem 2.3.** For the generator \( (A(t), D) \) of a \( C_0 \)-quasi-semigroup \( (R(t, s))_{t, s \geq 0} \) on a Banach space \( X \), we have the inclusion :

\[ e^{\sigma_e(A(t)) s} \subseteq \sigma_e(R(t, s)) \]

To prove this result, we need the following proposition and lemma.

**Proposition 2.1.** [9 Corollary 1.5] Let \( T \) be a closed operator. If \( T \) is semi-regular then \( Rg^\infty(T) \) is closed.

**Lemma 2.1.** [8 Lemma 1] Let \( T \in B(X) \), If \( T \) is semi-regular, then the operator

\[ \hat{T} : X/Rg^\infty(T) \to X/Rg^\infty(T) \]

induced by \( T \) is bounded below.

**Proof.** Let \( \lambda \in \mathbb{C} \) and \( s_0 > 0 \) be fixed such that \( e^{\lambda s_0} \notin \sigma_e(R(t, s_0)) \) for all \( t \geq 0 \), then \( e^{\lambda s_0} - R(t, s_0) \) is semi-regular. We show that \( \lambda - A(t) \) is semi-regular.

For this, consider the closed \( (R(t, s))_{t, s \geq 0} \)-invariant subspace \( M := Rg^\infty(e^{\lambda s_0} - R(t, s_0)) \) of \( X \) and the quotient \( C_0 \)-quasi-semigroup \( (\widetilde{R(t, s)})_{t, s \geq 0} \) defined on \( X/M \) by

\[ \widetilde{R(t, s)}\hat{x} := \hat{R(t, s)}x, \quad \text{for} \ \hat{x} \in X/M \]

with generator \( \widetilde{A(t)} \) defined by

\[ \widetilde{D} := \{ \hat{x}, \ x \in D \}, \quad \widetilde{A(t)}\hat{x} := \widetilde{A(t)}x, \quad \text{for} \ \hat{x} \in \widetilde{D} \]

From Lemma 2.1, it follows that the operator \( e^{\lambda s_0} - R(t, s_0) \) is bounded below. Thus, \( e^\lambda \notin \sigma_e(R(t, s_0)) \) and from theorem 2.2 (3) \( \lambda \notin \sigma_e(\widetilde{A(t)}) \).

In consequence, the operator \( \lambda - \widetilde{A(t)} \) is injective and the closed range and from corollary 2.2 (7) Then,

\[ N(\lambda - \widetilde{A(t)}) \subset Rg^\infty((\lambda - \widetilde{A(t)})). \]
Now, we show that $Rg(\lambda - A(t))$ is closed.

To do this, consider a sequence $(u_n)_{n \geq 0}$ of elements of $Rg(\lambda - A(t))$, which converges to $u$. Then, there exists a sequence $(v_n)_{n \geq 0}$ of elements of $\mathcal{D}$ such that $(\lambda - A(t))u_n = v_n \rightarrow u$. Since $Rg(\lambda - \hat{A}(t))$ is closed, there exists $\hat{w} \in \hat{\mathcal{D}}$ such that $\hat{u} = (\lambda - A(t))\hat{w}$.

Hence, $u - (\lambda - A(t))w \in Rg^\infty(e^{\lambda s_0} - R(t, s_0)) \subset Rg^\infty(\lambda - A(t)) \subset Rg(\lambda - A(t))$. Therefore, $u \in Rg(\lambda - A(t))$

Consequently, the operator $\lambda - A(t)$ is semi-regular. $\square$

In the next work we will try to demonstrate the equality of the spectra or give counter-examples in the case of a strict inclusion.

REFERENCES

[1] P. Aiena, Fredholm and Local Spectral Theory with Applications to Multipliers, Kluwer. Acad. Press, 2004.
[2] W. Arendt, Vector-valued Laplace Transforms and Cauchy Problems, Israel J. Math, 59 (3) (1987), 327-352.
[3] D. Barcenas and H. L eiva, Quasisemigroups, Evolutions Equation and Controllability, Notas de Matematicas no. 109, Universidad de Los Andes, Merida, Venezuela, 1991.
[4] D. Barcenas and H. L eiva, Quasisemigroups and evolution equations, International Journal of Evolution Equations , vol. 1, no. 2, pp. 161-177, 2005.
[5] D.Barcenas, H.Leiva and Moya , The Dual Quasi-Semigroup and Controllability of Evolution Equations, Journal of Mathematical Analysis and Applications , vol. 320, no. 2, pp. 691-702, 2006.
[6] A. Elkoutri and M. A. Taoudi, Spectral Inclusions and stability results for strongly continuous semigroups, Int. J. of Math. and Mathematical Sciences, 37 (2003), 2379-2387.
[7] K. J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.
[8] V. Kordula , V. Mülller, The distance from the Apostol spectrum, Proc. Amer. Math. Soc. 124 (1996) 3055-3061.
[9] M. Mbekhta, On the generalized resolvent in Banach spaces, J. Math. Anal. Appl. 189 (1995) 362-377.
[10] V. Mülller, Spectral theory of linear operators and spectral systems in Banach algebras 2nd edition, Oper.Theo.Adva.Appl, 139 (2007).
[11] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, Springer-Verlag, New York 1983.
[12] Sutrima, Ch. Rini Indrati, Lina Aryati, and Mardiyan, The fundamental properties of quasi-semigroups, Journal of Physics: Conf. Series 855 (2017) 012052.
[13] A.E. Taylar and D.C. Lay, Introduction to Functional Analysis, 2nd ed. New York: John Wiley and Sons, 1980.

A. Tajmouati, Y. Zahouan and M.A. Ould Mohamed Baba
Sidi Mohamed Ben Abdellah Univeristy, Faculty of Sciences Dhar Al Mahraz, Fez, Morocco.

E-mail address: abdelaziz.tajmouati@usmba.ac.ma
E-mail address: zahouanyouness1@gmail.com
E-mail address: bbaba2012@gmail.com