On N=2 strings and classical scattering solutions of self-dual Yang-Mills in (2,2) spacetime

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ABSTRACT

Ooguri and Vafa have shown that the open N=2 string corresponds to self-dual Yang-Mills (SDYM) and also that, in perturbation theory, it has has a vanishing four particle scattering amplitude. We discuss how the dynamics of the three particle scattering implies that on shell states can only scatter if their momenta lie in the same self-dual plane and then investigate classical SDYM with the aim of comparing exact solutions with the tree level perturbation theory predictions. In particular for the gauge group SL(2,C) with a plane wave Hirota ansatz SDYM reduces to a complicated set of algebraic relations due to de Vega. Here we solve these conditions and the solutions are shown to correspond to collisions of plane wave kinks. The main result is that for a class of kinks the resulting phase shifts are non-zero, the solution as a whole is not pure gauge and so the scattering seems non-trivial. However the stress energy and Lagrangian density are confined to string like regions in the space time and in particular are zero for the incoming/outgoing kinks so the solution does not correspond to physical four point scattering.
1. Introduction

Recently Ooguri and Vafa [1,2] have revived interest in string theories with two local worldsheet supersymmetries. Such string theories were investigated early in the days of dual models [3], and were found to have a critical dimension of two. However only comparatively recently was it realised that these are two complex dimensions [4]. Thus, they naturally live in a real space-time with a (2,2) signature. A string in (1,1) real dimensions cannot have any transverse oscillations and so it is entirely natural that it does not give rise to the usual infinite tower of massive states. For a string in (2,2) dimensions it might be expected that transverse modes do occur, but for the N=2 string these turn out to correspond to local N=2 supersymmetry transformations and so do not give rise to physical particles, and so the spectrum contains only a finite number of particles. It turns out that they are all massless, bosonic and scalar (see [4,5, 6,1,2] and references therein for more details).

Ooguri and Vafa investigated the string scattering amplitudes in order to identify the field theory corresponding to the particles in the spectrum. From the closed N=2 string three particle scattering they found that the scalar can be interpreted simply as the Kähler potential of a Ricci flat complex manifold. Such manifolds can equivalently be described as self-dual gravity (SDG). This is in line with the fact that if one tries to couple the world sheet action to the geometry of the space-time then the only way to do this and preserve the N=2 supersymmetry is to couple it only via the Kähler potential. It was also shown that open strings with Chan-Paton group factors attached at their edges gave rise to self-dual Yang-Mills (SDYM) theories.

The four particle string amplitude usually consists of products of gamma functions and so there will be a infinite number of values for the Mandelstam s, t and u variables that will will lead to a pole in the scattering amplitude. For N=0 and N=1 strings these poles just correspond to the massive particles in the spectrum. However for N=2 strings there is no such infinite tower of massive states and so
there is a potential loss of unitarity. The solution is simply that the kinematic factor in front of the four particle amplitude vanishes whenever the external particles are on shell. This fact relies on a special identity concerning any three null vectors in a (2,2) space-time [2, but see also 7]. Similarly it is believed that all the higher point connected on-shell amplitudes also vanish. This gives an almost trivial solution to the problem of finding amplitudes that are dual with respect to the Mandelstam variables. This is certainly a very remarkable property for a quantum field theory to have and this paper is inspired by the hope of shedding some light on this property.

Since there are no higher mass states to integrate out it follows that the field theory (SDYM or SDG) from the N=2 string theory is not just a low energy theory and so at tree level the theories should be equivalent. When loops are included then they differ because the scalars act like Liouville modes with non-standard rules for path integrals and also because modular invariance changes regions of integration in the loops. Even at tree level the string approach has the advantage that the perturbation theory combines all connected field theory Feynman diagram into a single string diagram. However given the difficulty of a non-perturbative approach to string theory it is much easier to deal with exact solutions for the field theory. Hence in this paper we look again at classical SDYM in light of the results from the string theory.

It is well known that classical SDYM is an integrable system. For the quantum version of integrable systems in (1,1) dimensions we would usually expect to have an infinite number of conserved charges that prohibit any particle production and also for the S matrix to factorise into products of S matrices for $2 \to 2$ scattering. However, the string theory seems to predict that SDYM in (2,2) has almost the exactly opposite properties; that there is no $(2\to 2)$ scattering and that there is particle production from the $(1\to 2)$ S-matrix contribution. Of course, we will ignore the infrared problems that would occur in a careful discussion of scattering of massless states.
We observe that the kinematical properties of the string theory results of [1] suggest that the system exhibits self-dual null plane decoupling, meaning that physical particles only interact, at the S-matrix level, if their momenta lie in the same self-dual null plane. Specifically for a massless particle with momentum $k$ in coordinates $(yz\bar{y}\bar{z})$ we define $\omega = k_z/k_y$ and this parametrises the self-dual null plane in which it lies. Particles with different $\omega$ values would not see each other at the S-matrix level, suggesting that the theory is not fully four dimensional. If valid in all cases then this would presumably have strong effects on the allowed S-matrices in these theories. This is rather reminiscent of the decoupling of left and right movers in two-dimensional conformal field theory. Thus, it is of interest to investigate whether the stringy results of [1] are of general application.

One obvious proviso is that the string theory predictions are only a perturbative result and so could be misleading. Furthermore, even the perturbation theory prediction of the lack of connected $2 \rightarrow 2$ scattering has the potential problem that the calculation only makes sense when it is non-singular, that is, when $stu \neq 0$. Hence the calculation does not directly rule out contributions of the form $\delta(s)$ or similar. Normally we would exclude such a possibility by analyticity of the S-matrix, but it is not so clear that this is also true for integrable theories or for $(2,2)$ dimensions.

In particular for an integrable theory we would expect there to be solutions where the scattering does not induce a change of momentum but only a phase shift. Thus, if the momenta of the external particles are $k_i$ with $i = 1, 2, 3, 4$ and $k_i^2 = 0$, $\Sigma_i k_i = 0$ then we will be particularly interested in the case for which $k_3 = -k_1$ and $k_4 = -k_2$. This has $(k_1 + k_3)^2 = 0$ and so corresponds to a singular set of Feynman diagrams (or a string theory calculation at the boundary of moduli space where two of the vertex insertions coincide on the world sheet) and so these may not be reliable. Classically we expect this to correspond to the case in which the fields only depend on two of the four coordinates. For example it is known that under certain conditions SDYM is equivalent to the two-dimensional KdV equation [8], and this certainly has solutions that display non-trivial soliton scattering.
relevance of such “phase shift only” solutions is that \( k_1 \) and \( k_2 \) will generally not lie in the same self-dual null plane. If such scattering is truly relevant to SDYM then the decoupling of these planes would not occur, thus contradicting the perturbative results. Hence in this paper we take a (very restricted) look for classical solutions of SDYM that correspond to \( 2 \rightarrow 2 \) scattering or to \( 1 \rightarrow 2 \) scattering and attempt to clarify the situation with respect to classical scattering.

We emphasise that when talking of the triviality, or otherwise, of the “classical S-matrix” it is important to specify the type of waves that are colliding and whether they are to be considered as physical. As pointed out in [7], based on the example of the KP system, it can well be that finite size packets do not scatter but that plane waves might still suffer a non-trivial phase shift. In an integrable theory the infinite number of conserved charges might seem to restrict scattering but for a plane wave the infinite extent of the wave front could lead to infinite values for these charges thus invalidating the conservation laws.

Dimensional reduction of SDYM is of interest in its own right because many of the usual integrable systems can be generated in this fashion (for example [8,9]). One can hope that SDYM is then a master integrable system for many others. However the emphasis here is not to generate new reductions but to see what the reductions say about SDYM itself.

A slightly discordant note on the N=2 string is the redundancy of having a plethora of fermions on the world-sheet, and yet the theory contains no spacetime fermions. Also, since we have a (1,1) worldsheet embedded into a (2,2) space-time then the transverse modes must be pure gauge in order to explain the lack of higher mass states. Hence we need to identify the nilpotent translations obtained by a supersymmetry transformation on the worldsheet with a non-nilpotent translation in the space-time coordinates and it seems unnecessary to have to face this problem. This reinforces the suggestion in [2]) that the theory has a formulation in terms of some purely bosonic extended object.

In chapter 2 we discuss the peculiarities of kinematics in a (2,2) space-time. In
chapter 3 we review the J-formulation of SDYM, in which the Yang-Mills fields are written as derivatives of a pre-potential[10, 11, 12]. The pre-potential J is a scalar and seems to be the natural outcome of the N=2 string. The J-formulation is also a natural generalisation of the WZNW model in two dimensions. For the reason given above we are particularly interested in the case in which J depends on only two coordinates $\eta_1, \eta_2$ for which it is trivial to see that the system reduces to chiral models with Wess-Zumino term in (1,1) dimensions. In certain cases the system has obvious exact generic solutions and we relate these to the string predictions for three particle scattering.

In chapter 4 we consider the special case in which the gauge group is SL(2,C). This group has the advantage that a Gauss decomposition for the group element leads to Yangs equations which are derivable from an action. Using the tree level Feynman rules we verify that the connected four point contributions sum to zero for generic external null momenta. Also Yangs equations are homogeneous in the fields and de Vega exploited this to search for scattering solutions. With a fractional Hirota ansatz he reduced Yangs equations to a large set of algebraic equations, and was able to produce some scattering solutions. Here we show that with a good change of variables these equations have a reasonably simple general solution and so we obtain a wide class of classical scattering solutions. The solutions we obtain correspond to kink-kink collisions. They are non-trivial in the sense that the collision does indeed generate a phase shift for the kinks, but in order to get the phase shift we are forced to impose conditions on the asymptotic states which mean precisely that they have have zero value of the relevant stress energy tensor. In chapter 5 we conclude with a discussion of our results and their implications along with a few extra observations.
2. Kinematics in (2,2) space-time

Consider a complex four dimensional space with coordinates \( x^\alpha = (x^\mu, x^{\bar{\mu}}) \) where \( x^\mu = (y, z) \), \( x^{\bar{\mu}} = (\bar{y}, \bar{z}) \), and with a complex metric

\[
ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = 2g_{\mu\bar{\mu}}dx^\mu dx^{\bar{\mu}} = 2(dy d\bar{y} - dz d\bar{z})
\] (2.1)

There are two simple ways to recover a real (2,2) space-time. If we impose \((x^\mu)^* = x^{\bar{\mu}}\) then we still have complex coordinates and so call this a \(C^{1,1}\) slice; the metric has a manifest holomorphic \(U(1,1)\) symmetry. Alternatively we can impose that \(x^\mu\) and \(x^{\bar{\mu}}\) are independent real coordinates, we call this a \(R^{2,2}\) slice. For the sake of comparison with a real (3,1) space time we could also consider an \(R^{3,1}\) slice in which \(\bar{y} = y^*\) but \(z \in \mathbb{R}\) and \(\bar{z} \in \mathbb{R}\). Given momenta \(k_\alpha = (k_\mu, k_{\bar{\mu}}) = (k_y, k_z, k_{\bar{y}}, k_{\bar{z}})\), we use the non-symmetric scalar product

\[
\langle p\vert k \rangle := g^{\mu\bar{\mu}}p_\mu k_{\bar{\mu}} = p_y k_{\bar{y}} - p_z k_{\bar{z}}
\] (2.2)

The symmetric scalar product is \(p \cdot k := g^{\alpha\beta}p_\alpha k_\beta = \langle p\vert k \rangle + \langle k\vert p \rangle\). We will later consider vectors \(k_i\) and it is convenient to define \(k_{ij} := \langle k_i\vert k_j \rangle\), \(s_{ij} := k_{ij} + k_{ji}\) and \(c_{ij} := k_{ij} - k_{ji}\).

It turns out that all the interactions in the string theories or in SDYM or SDG are simple functions of the \(k_{ij}\); even one-loop corrections do not seem to affect this conclusion because the kinematic factors are similar to the tree level [13]. So, before proceeding to the specific case of SDYM, we can first discuss some general kinematical properties of the on-shell amplitudes.

For any 2x2 matrix \(M\) it is trivially true that

\[
M_\mu [\bar{\mu} M_\nu \bar{\nu} M_\sigma \bar{\sigma}] = 0
\] (2.3)

where the square brackets mean antisymmetrisation. So if we take \(M_\mu \bar{\mu} = g_{\mu\bar{\mu}}\) and contract the free indices with three arbitrary vectors \(k_i\) we obtain a cubic identity.
in $k_{ij}$. For the particular case that the vectors are null we obtain

$$k_{12}k_{23}k_{31} + k_{21}k_{32}k_{13} = 0 \quad (2.4)$$

This “three nulls identity” is valid in the four complex dimensional space and so is also valid for any restriction to a real four-dimensional space time. If the real slice has a (4,0) signature then there are no non-zero null vectors and so (2.4) is empty. If the real slice is (3,1) then this is essentially the same as the identity found in [7] and shown to be relevant to the consistency of multisoliton scattering in SDYM with SL(2,C) gauge group. When the real slice has signature (2,2) then as shown in [1] it is responsible for the vanishing of the kinematic factor in the four particle on-shell string amplitude.

If we consider on-shell scattering of three massless particles with momenta $k_i$, then conservation of momentum implies that the external momenta span a null plane. The bivector $k_{1\alpha}k_{2\beta} - k_{1\beta}k_{2\alpha}$ that defines the null plane can be either anti-self-dual or self-dual. The anti-self-dual case means that $k_i = (p_i, p_i\Omega, \bar{p}_i, \bar{p}_i\bar{\Omega})$ with $\Omega\bar{\Omega} = 1$, whilst for a self-dual plane we have $k_i = (p_i, \bar{p}_i\omega, \bar{p}_i, p_i\bar{\omega})$ with $\omega\bar{\omega} = 1$. However for the case of the anti-self-dual plane we see that $k_{ij} = 0$ and so there is no interaction in the theories under consideration.

In other words, if we write the momenta of all particles in the form $(p_i, \bar{p}_i\omega_i, \bar{p}_i, p_i\bar{\omega}_i)$ then particles can only interact with each other via the on-shell three leg vertex if they have the same $\omega_i$ value and so lie in the same self-dual null plane. If we convert the momenta to operators on some field $\phi$ we get $(\partial_x - \omega\partial_y)\phi = (\partial_{\bar{z}} - \bar{\omega}\partial_y)\phi = 0$ and this corresponds to what we shall call an “$\omega$-ansatz”

$$\phi = \phi(y + \bar{\omega}\bar{z}, \bar{y} + \omega z) \quad (2.5)$$

In a $\mathbb{R}^{3,1}$ slice there are no null planes but only null lines and so cannot deal with scattering in this way.
The three nulls identity (2.4) gives rise to the following identities, valid for any momenta \( k_i \) \( i = 1, \ldots, 4 \) with \( k_{ii} = 0 \) and \( \Sigma_i k_i = 0 \)

\[
\frac{k_{13}k_{42}}{s_{13}} + \frac{k_{12}k_{43}}{s_{12}} = 0 \quad \frac{k_{13}k_{24}}{s_{13}} + \frac{k_{12}k_{34}}{s_{12}} + k_{14} = 0 \quad (2.6)
\]

\[
\frac{c_{13}c_{24}}{s_{13}} + \frac{c_{12}c_{34}}{s_{12}} + s_{14} = 0 \quad (2.7)
\]

and these will be used to compute the four particle scattering.

Finally consider the kinematics of the four massless particle on-shell amplitude. For this paragraph only, let us suppose that we have a real diagonal metric \((-,+,+,\pm)\), and that \( s_{12} \neq 0 \). Then after rescaling and rotations of momenta the generic situation is that \( k_1 = (1, 1, 0, 0) \), \( k_2 = (1, -1, 0, 0) \), \( k_3 = (-1, p_1, p_2, 0) \) and \( k_4 = (-1, -p_1, -p_2, 0) \) where \( p_1^2 + p_2^2 = 1 \) and so \( s_{12} = -4 \), \( s_{13} = 2 + 2p_1 \), and \( s_{14} = 2 - 2p_1 \). The sign of \( g_{33} \) is irrelevant and so we don’t see the difference in this case between a (2,2) and a (3,1) signature. The point here is simply that in a (2,2) signature there exist reasonable, non-parallel, sets of physical momenta for the on-shell three and four particle amplitudes and so these amplitudes can have direct physical significance.

3. SDYM and the J formulation

In order to set up SDYM in a complex space we take a complex gauge group \( G_C \), and take \( F_{\alpha\beta} = [\nabla_\alpha, \nabla_\beta] \) with \( \nabla_\alpha = \partial_\alpha + A_\alpha \). The (anti)-self-duality condition is that \( F_{\alpha\beta} = -\frac{1}{2} \det(g_{\mu\bar{\mu}}) \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta} \) where \( \epsilon_{y\bar{z}y\bar{z}} = +1 \). In the above metric this reduces to

\[
F_{\mu\nu} = F_{\bar{\mu}\bar{\nu}} = 0 \quad g^{\mu\bar{\mu}} F_{\mu\bar{\mu}} = 0 \quad (3.1)
\]

Hence there exist independent group elements \( D \) and \( \bar{D} \) such that

\[
A_\mu = D^{-1} \partial_\mu D \quad A_{\bar{\mu}} = \bar{D}^{-1} \partial_{\bar{\mu}} \bar{D} \quad (3.2)
\]

Yang-Mills gauge transformations now correspond to \( D \rightarrow Dg \) and \( \bar{D} \rightarrow \bar{D}g \) with \( g \in G_C \) and so the gauge invariant quantities live on \((G_C \otimes G_C)/G_C\). A natural
representative for each orbit is the gauge invariant quantity

\[ J := D \bar{D}^{-1} \]  

(3.3)

and the self duality condition reduces to

\[ \partial \bar{y}(J^{-1} \partial_y J) - \partial \bar{z}(J^{-1} \partial_z J) = g^{\mu \bar{\mu}} \partial_{\bar{\mu}}(J^{-1} \partial_{\mu} J) = 0 \]  

(3.4)

In particular this implies that \( g^{\mu \bar{\mu}} \partial_{\bar{\mu}}(\ln \det J) = 0 \) so \( \det J \) is simply a free field and is of no dynamical interest. In the process of making the above gauge choices to go to the \( J \) formulation we have reduced the manifest Lorentz invariance, although of course gauge invariant quantities still must be fully Lorentz invariant. However we do gain the semi-local \( G_C \otimes G_C \) symmetry

\[ J \rightarrow g_L(x^\bar{\mu}) J g_R(x^\mu) \]  

(3.5)

By definition \( J \) is gauge invariant, however in order to recover any Yang-Mills structures from it we need to split first into \( D \) and \( \bar{D} \) and the arbitrariness of this split is the Yang-Mills gauge symmetry. Thus, gauge potentials and field strengths are gauge dependent functions of the gauge invariant \( J \). The symmetry (3.5) clearly leaves the Yang-Mills potentials invariant and so \( J \) is not uniquely given by the field strengths. Also, given the field strengths it is a non-local process to find a representative \( J \), however the \( J \) formulation could be considered as just as fundamental as the Yang-Mills formulation.

The classical integrability of the system is revealed by fact that (3.4) is simply the compatibility condition for the linear system \( \epsilon^{\mu \nu} J \partial_\nu \Psi + \lambda g^{\mu \bar{\mu}} \partial_{\bar{\mu}}(J \Psi) \equiv 0 \) for all \( \mu \) and all \( \lambda \). From this an infinite number of conservation laws can be derived in the standard fashion (see [12] and references therein).

Ultimately we need to impose reality conditions on the coordinates and if we also wish to finish with a real gauge group \( G_R \) then we need to impose reality
restrictions on $D$ and $\bar{D}$ and so on $J$. For the $\mathbb{R}^{2,2}$ slice we can trivially take $D, \bar{D}, J \in G_R$ so that $A_\alpha \dagger = -A_\alpha$. For the $\mathbb{C}^{1,1}$ case we need $A_\mu \dagger = -A_{\bar{\mu}}$ and so must take $\bar{D} = (D \dagger)^{-1}$ and this reality condition is preserved by gauge transformations satisfying $g \dagger g = 1$; hence the gauge invariants in this case live on $G_C/(G_C)_U$ where $(G_C)_U$ is the subgroup of unitary elements of $G_C$. In this case $J = DD \dagger$ becomes a positive hermitian element of $G_C$. The fundamental example is to take $G_R = \text{U}(N)$ and $G_C$ to be its complexification $\text{GL}(N,\mathbb{C})$, and then $J$ is a positive hermitian $N \times N$ matrix. Even though $\mathbb{C}^{1,1}$ and $\mathbb{R}^{2,2}$ are of course trivially related by coordinate changes the corresponding $J$ fields are not the same, or even in the same space in general.

The equation of motion (3.4) shows that the $J$ formulation is a dimensional generalisation or complexification of the Wess-Zumino-Novikov-Witten (WZNW) model in $(1,1)$ space-time dimensions and so one cannot expect directly to have a manifestly local and group invariant action. An action is feasible if we pick a particular parametrisation for the group (see the next chapter) or if one includes extra coordinates [14]. However without an action we can obtain some Feynman rules from (3.4). Presumably the properties of the $N=2$ string are true in any background that solves the equations of motion, but here we will only consider expansions around trivial flat backgrounds. Expanding about $J = 1$ gives a $1/p^2$ propagator and vertices with arbitrary numbers of external legs but which are all proportional to $c_{12}$.

If one now looks at four particle on-shell tree level scattering then the three diagrams have intermediate (squared) momenta $s_{12}, s_{13}$ and $s_{23}$ and so we would normally expect their differing momentum dependencies to be forbid any cancellation. However, due to the identity (2.7) the Feynman diagram with an apparent $s_{13}$ pole can be changed into an apparent $s_{12}$ channel pole and so it is now reasonable that the three diagrams combine into one term linear in $k_{ij}$ which can than cancel the contribution from the four point vertex. It is then straightforward to check that the connected four point tree level diagrams do indeed sum to zero when the external legs are all on shell. This of course matches the expectation from the
N=2 string theory [1,2]; but it is interesting to note that we do not need to use
the reality conditions at any time and so this is actually true in $\mathbb{C}^4$.

For reasons given in the introduction we now dimensionally reduce SDYM, in
the same fashion as [7], by considering generalised null plane waves.

$$J = J[\eta_1, \eta_2]$$

(3.6)

where $\eta_i = k_{i\alpha} x^\alpha = k_{i\mu} x^\mu + k_{i\bar{\mu}} x^{\bar{\mu}}$ and the $k_i$ are linearly independent and null.

Using $\partial^i = \partial/\partial \eta_i$ and $c_{ij} = \epsilon_{ij} c_{12}$ then the self-duality condition (3.4) gives

$$s_{12} [\partial^1(J^{-1}\partial^2 J) + \partial^2(J^{-1}\partial^1 J)] + c_{12} \epsilon_{ij} \partial^i (J^{-1}\partial^j J) = 0$$

(3.7)

The case of $k_2 = 0$ is trivial: if $k_1^2 = 0$ then $J[\eta]$ is arbitrary. Hence any null plane
wave solves both the full non-linear equations and their linearised versions which
is unusual for a non-linear system. If we were to attempt a classical version of
S-matrix theory perturbation theory then we would usually want to switch off the
interaction at infinity for the in and out states. This switching off of the interaction
is not very pleasing, but does not seem to be necessary for SDYM (or for SDG).

If neither $k_i$ vanishes then (3.7) just the chiral model with Wess-Zumino term
(CMWZ) model (1,1) space-time. To us the most interesting case is when the mo-
menta lie in the same null plane and so can correspond to on-shell three particle
scattering. This means that $s_{12} = 0$ and we see that the corresponding reduction
of SDYM is not an evolution equation. This is related to the previous observation
that any null wave is a solution. To see this heuristically recall that an evolution
equation will have an initial surface on which the field and some finite number of
derivatives are specified, but one can (at least locally) pick a direction perpendicu-
lar to this and since this direction is also null the solution can depend arbitrarily on
it and hence cannot be specified by a finite number of initial values. More specifi-
cally we have two cases: If the $k_i$ lie in an anti-self-dual plane then $c_{12} = 0$ and so
$J$ is totally arbitrary compared to a string prediction that there is no scattering. Whilst if the $k_i$ form a self-dual plane then we have
\[ \epsilon_{ij} \partial^j (J^{-1} \partial^i J) = 0 \] (3.8)
which corresponds to a two dimensional (topological) theory with only the WZ term and no kinetic term. The generic solution of this “pure WZ” system is that $J = J[f(x^\alpha)]$ where $J \in G_C$ and $f \in \mathbb{R}$ are arbitrary. The fact that we do not obtain evolution equations for the cases which correspond to the kinematics of the string or SDYM derived three point vertex suggests that we need to be very careful when talking of the implications for scattering in the theory. In particular it is obviously possible, but not very meaningful, to write down functions $J$ and $f$ that look like any number of solitons “scattering” into any number of others; scattering of two into one is allowed but not at all special.

In light of the above relation between pure WZ models and self-dual systems we briefly mention some work of Park [15]. With some rearrangement we can rewrite [15] in the notation of this paper and in a more symmetric fashion. Consider a tensor field with components $G_{\mu\bar{\nu}}(x^\alpha)$ and $\bar{G}_{\bar{\mu}\mu}(x^\alpha)$ and define the vector fields
\[ A_\mu := G_{\mu\bar{\nu}} \epsilon^{\bar{\nu}\bar{\rho}} \frac{\partial}{\partial x^\rho} \]
\[ \bar{A}_{\bar{\mu}} := \bar{G}_{\bar{\mu}\mu} \epsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} \] (3.9)
So each $A_\mu$ generates diffeomorphisms (not necessarily area preserving) on the self-dual null plane $x^\mu = \text{constant}$. Similarly each $\bar{A}_{\bar{\mu}}$ gives diffeomorphisms on the self-dual null plane $x^{\bar{\mu}} = \text{constant}$. Introduce the derivative $\nabla_\alpha = \partial_\alpha + e A_\alpha$ where $e$ is a coupling constant and the field strength operator $F_{\alpha\beta} = [\nabla_\alpha, \nabla_\beta]$. We then enforce the operator condition that
\[ F_{\mu\nu} = F_{\bar{\mu}\bar{\nu}} = 0 \quad \text{for all} \quad e \] (3.10)
The $O(e)$ term gives
\[ \epsilon^{\bar{\mu}\bar{\nu}} \partial_{\bar{\rho}} G_{\rho\bar{\nu}} = 0 \quad \epsilon^{\mu\nu} \partial_\rho \bar{G}_{\bar{\rho}\nu} = 0 \quad \forall \rho, \bar{\rho} \] (3.11)
with solution that $G_{\mu\bar{\nu}} = \partial_{\bar{\nu}} F_\mu$ and $\bar{G}_{\bar{\mu}\mu} = \partial_\mu \bar{F}_{\bar{\mu}}$ for some functions $F$, $\bar{F}$. In
particular this forces the diffeomorphisms to be area preserving on the corresponding planes (here our emphasis differs from [15]). To match Park we now force $G_{\mu\bar{\nu}} = \bar{G}_{\bar{\mu}\mu}$ so that we can regard it as a metric on the space-time (this stage seems ad hoc and perhaps instead we should allow coupling to an antisymmetric tensor). Then (3.11) gives $G_{\mu\bar{\nu}} = \partial_{\mu}\partial_{\bar{\nu}}\Omega$ and so this metric is Kähler. With these conditions on $G_{\mu\bar{\nu}}$ the $O(e^2)$ terms of (3.10) have two consequences. Firstly

$$G_{\bar{\mu}\mu}F_{\mu\bar{\nu}} = 0$$

for all $e$ \hspace{1cm} (3.12)

so the field strength $F$ is “anti-self-dual for all $e$” with respect to the metric $G_{\mu\bar{\nu}}$. Secondly $\det G_{\mu\bar{\nu}}$ is forced to be a constant and so $G_{\mu\bar{\nu}}$ defines a Ricci flat Kähler metric on the spacetime, meaning that we have SDG. Hence SDG in $(2,2)$ spacetime is almost SDYM with a gauge group which consists of diffeomorphisms acting on self-dual null planes embedded in the spacetime, but with the extra conditions that the self-duality holds true even under constant rescalings of the gauge fields and that the resulting metric is symmetric.

The relation to pure WZ models is simply that $F_{\mu\nu} = 0$ for all $e$, has the solution $A_{\mu} = D^{-1}\partial_{\mu}D$ with $\epsilon^{\mu\nu}\partial_{\mu}(D^{-1}\partial_{\nu}D) = 0$ which is a pure WZ model on the $x^\mu$-plane for a fixed point in $x^{\bar{\mu}}$ (and similarly for the barred quantities).

As discussed in [2] the $N=2$ string theories seem to be almost topological theories. It is also well known that the Yang-Mills stress energy tensor vanishes for SDYM and this would be a sign that the theory is topological were it not for the fact that the self-duality condition itself uses a metric. So it is interesting to note that the system discussed above does not any reference to a metric on the space-time but only assumes a complex structure; that is, we only needed the conditions $G_{\mu\bar{\nu}} = \bar{G}_{\bar{\mu}\mu}$ and the tensors $\epsilon^{\mu\nu}$ and $\epsilon^{\bar{\mu}\bar{\nu}}$ for (3.10). Instead the metric arises from the parameters of a gauge transformation. Also, it is reminiscent of Wittens work on (2,1) gravity [16] that the usual Yang-Mills perturbation theory around $A_{\mu} = \bar{A}_{\bar{\mu}} = 0$ corresponds to expanding about $G_{\mu\bar{\nu}} = 0$ and not around the usual flat metric. (However, since the conditions are to be enforced for all $e$ the standard perturbation theory will not apply.)
For completeness, we note that for SDG expanding around a flat metric gives the Plebanski equation

$$g^{\mu\bar{\mu}} \partial_\mu \partial_{\bar{\mu}} \phi + \epsilon^{\mu\nu} \epsilon^{\bar{\mu}\bar{\nu}} (\partial_\mu \partial_{\bar{\mu}} \phi) \partial_\nu \partial_{\bar{\nu}} \phi = 0$$

(3.13)

If we then dimensionally reduce as we did for SDYM by imposing \( \phi = \phi(\eta_1, \eta_2) \) with \( k_{11} = k_{22} = 0 \) we get

$$s_{12} \partial_1 \partial_2 \phi + k_{12} k_{21} [ (\partial_1^2 \phi) \partial_2^2 \phi - (\partial_1 \partial_2 \phi)^2 ] = 0$$

(3.14)

Then \( s_{12} = 0, k_{12} \neq 0 \) forces

$$(\partial_1^2 \phi) \partial_2^2 \phi - (\partial_1 \partial_2 \phi)^2 = 0$$

(3.15)

which has the generic solution that either \( \phi = \phi(a \eta_1 + b \eta_2) \) where \( a \) and \( b \) are arbitrary constants, or that \( \phi = (\eta_2 - \tilde{\eta}_2) \Phi[(\eta_1 - \tilde{\eta}_1)/(\eta_2 - \tilde{\eta}_2)] + b \) where \( \tilde{\eta}_1, b \) are arbitrary constants and \( \Phi \) is an arbitrary function. Again these do not behave as scattering solutions.

4. SDYM with gauge group SL(2,C)

This has been extensively studied previously; the motivation was either to impose reality conditions and so obtain self-dual SU(2) solutions [17,11], or to exploit the fact that it is a complex group and so can have self-dual solutions in a (3,1) metric [7]. We study it simply because it is the simplest non-trivial case. The most straightforward approach to SL(2,C) is based on the Gauss decomposition

$$J = \exp g \rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \exp g S \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \exp g \bar{\rho} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

= \begin{pmatrix} 1 & 0 \\ g \rho & 1 \end{pmatrix} \begin{pmatrix} e^{g S} & 0 \\ 0 & e^{-g S} \end{pmatrix} \begin{pmatrix} 1 \\ g \bar{\rho} \end{pmatrix}$$

(4.1)

where \( S, \rho, \) and \( \bar{\rho} \) are independent complex fields. The equations of motion from
which follow from the Lagrangian [10]

\[ L = g^{\mu\bar{\nu}} \left( \partial_\mu S \partial_{\bar{\nu}} S + e^{2gS} \partial_\mu \rho \partial_{\bar{\nu}} \bar{\rho} \right) \]

To put this in context we note that dimensional reduction, as in the previous chapter, would give a manifestly local action for the WZNW model in two dimensions, but of course manifest group invariance has been lost. From the Lagrangian we obtain a stress-energy tensor

\[ T_{\mu\nu} = \partial_\mu S \partial_{\nu} S + e^{2gS} \partial_\mu \rho \partial_{\nu} \bar{\rho} \]

\[ T_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\mu}} S \partial_{\bar{\nu}} S + e^{2gS} \partial_{\bar{\mu}} \rho \partial_{\bar{\nu}} \bar{\rho} \]

\[ T_{\bar{\mu}\mu} = T_{\mu\bar{\mu}} = \partial_\mu S \partial_{\mu} S + e^{2gS} \partial_\mu \rho \partial_{\mu} \bar{\rho} \]

the equations of motion (4.2) imply that \( \partial^\alpha T_{\alpha\beta} = 0 \). We see that \( T_{\alpha\beta} \) is symmetric, but not covariant because of the way in which \( \rho \) and \( \bar{\rho} \) appear. This \( T \) is of course different from the stress tensor from the Yang-Mills action.

If we only wish to work at tree level then we can ignore any ghosts that arise from the gauge fixing of the action from (4.3), and so it is easy to obtain the Feynman rules. In particular, after dropping total derivatives, the quadratic part of the Lagrangian is simply \( L_0 = -\frac{1}{2} g^{\alpha\beta} (\partial_\alpha S \partial_\beta S + \partial_\alpha \rho \partial_\beta \bar{\rho}) \) giving \( 1/p^2 \) propagators. The vertex with a \( \rho \) of momentum \( k_1 \), a \( \bar{\rho} \) of momentum \( k_2 \) and \( n \) legs of \( S \) comes with a factor \( (2g)^n k_{12} \). It is now straightforward to sum the connected tree-level diagrams with four on-shell external legs. Then

\[ < \rho(k_1) \bar{\rho}(k_2) \bar{\rho}(k_3) \rho(k_4) > \propto (2g)^2 \left( \frac{k_{13}k_{42}}{s_{13}} + \frac{k_{12}k_{43}}{s_{12}} \right) \]

\[ < \rho(k_1) S(k_2) S(k_3) \bar{\rho}(k_4) > \propto (2g)^2 \left( \frac{k_{13}k_{24}}{s_{13}} + \frac{k_{12}k_{34}}{s_{12}} + k_{14} \right) \]

Putting \( k_{ii} = 0 \) and \( \Sigma_i k_i = 0 \) and using (2.6) shows that in any case for which the calculation is non-singular then the answer is zero. It is interesting that this
happens despite the fact that the map from the variables $\lambda_a$ of the previous chapter to $S, \rho, \bar{\rho}$ is very non-linear. Non-linear changes of field variables in this case do not affect the conclusion that the connected four particle $S$ matrix is zero. We also note that $\rho$ and $\bar{\rho}$ appear only quadratically in the action so that in principle we could integrate them out, however this does not seem to be directly useful.

We now want to exhibit a class of exact solutions by extending the work of de Vega [7]. Without loss of generality we set $g = 1$ and put $S = -\ln \phi$ so that now

$$J = \frac{1}{\phi} \left( \frac{1}{\rho} \frac{\bar{\rho}}{\phi^2 + \rho \bar{\rho}} \right)$$

and the self-duality equations reduce to Yangs equations

$$g^{\mu \bar{\mu}}(\phi \partial_\mu \partial_{\bar{\mu}} \phi - \partial_\mu \phi \partial_{\bar{\mu}} \phi + \partial_\mu \rho \partial_{\bar{\mu}} \bar{\rho}) = 0$$

$$g^{\mu \bar{\mu}}(\phi \partial_\mu \partial_{\bar{\mu}} \rho - 2 \partial_\mu \rho \partial_{\bar{\mu}} \phi) = 0$$

$$g^{\mu \bar{\mu}}(\phi \partial_\mu \partial_{\bar{\mu}} \bar{\rho} - 2 \partial_\mu \phi \partial_{\bar{\mu}} \bar{\rho}) = 0$$

We shall immediately impose dimensional reduction, as in the last chapter, by insisting that we have functions of $\eta_1$ and $\eta_2$ only. Then one trivial solution, extending a solution of de Vega, is that $\phi = 1$, $\rho = \rho_1(\eta_1) + \rho_2(\eta_2)$ and $\bar{\rho} = c(k_{121} \rho_1 - k_{211} \rho_2) + b$ where $b, c$ are arbitrary constants and $\rho_1, \rho_2$ are arbitrary functions. To obtain non-trivial solutions we exploit the bilinearity of the equations by making the Hirota style ansatz

$$\phi = \frac{F}{\Delta} \quad \rho = \frac{N}{\Delta} \quad \bar{\rho} = \frac{\bar{N}}{\Delta}$$

where

$$\Delta = \Delta_0 + \Delta_1 e^{\eta_1} + \Delta_2 e^{\eta_2} + \Delta_{12} e^{\eta_1+\eta_2} \quad F = F_0 + F_1 e^{\eta_1} + F_2 e^{\eta_2} + F_{12} e^{\eta_1+\eta_2}$$

$$N = N_0 + N_1 e^{\eta_1} + N_2 e^{\eta_2} + N_{12} e^{\eta_1+\eta_2} \quad \bar{N} = \bar{N}_0 + \bar{N}_1 e^{\eta_1} + \bar{N}_2 e^{\eta_2} + \bar{N}_{12} e^{\eta_1+\eta_2}$$

Substituting this ansatz into Yangs equations and equating to zero the coefficients of the different powers of $e^{\eta_1}$ and $e^{\eta_2}$ gives a set of 15 complicated equations
in the 16 unknown coefficients. Hence, the system will indeed have solutions and
de Vega produced a solution for the special case in which \(F_{12} = 0\). Here we point
out that the equations resulting from the above ansatz can be solved directly for
rather general conditions. In order to do this we use a change of variables suggested
by the asymptotic properties of the ansatz. Thus,

\[
\eta_2 \to -\infty \quad \Rightarrow \quad \phi(\eta_1, \eta_2) \to \phi_-(\eta_1) = \frac{F_0 + F_1 e^{\eta_1}}{\Delta_0 + \Delta_1 e^{\eta_1}} \quad (4.10)
\]

and so the \(\phi\) field is a kink of height \(f_1 = F_1/\Delta_1 - F_0/\Delta_0\). However it is \(1/\phi\) and
not \(\phi\) that occurs in \(\chi\) and so we also look at

\[
\frac{1}{\phi_-(\eta_1)} = \frac{\Delta_0 + \Delta_1 e^{\eta_1}}{F_0 + F_1 e^{\eta_1}} \equiv \left(\frac{\Delta_1}{F_1} - \frac{\Delta_0}{F_0}\right) g(\eta - \ln\frac{F_0}{F_1}) + \frac{\Delta_0}{F_0} \quad (4.11)
\]

where \(g(\eta) := (1 + e^{\eta})^{-1}\) describes the shape (since it is independent of the coef-
ficients we only really get one type of wave in this ansatz) and so \(1/\phi_-\) is a kink
with phase \(\ln(F_0/F_1)\). Similarly,

\[
\eta_2 \to \infty \quad \Rightarrow \quad \phi(\eta_1, \eta_2) \to \phi_+(\eta_1) = \frac{F_2 + F_{12} e^{\eta_1}}{\Delta_2 + \Delta_{12} e^{\eta_1}} \quad (4.12)
\]

and so \(1/\phi_+\) has phase \(\ln(F_2/F_{12})\). Hence the phase shift between these two limits
is \(\ln(F_0 F_{12}/F_1 F_2)\). Furthermore the phase depends only on the denominator and so
only on \(F\). In particular this means that \(1/\phi = \Delta/F, \rho/\phi = N/F \) and \(\bar{\rho}/\phi = \bar{N}/F \)
all suffer the same phase shift. From this it follows that it is natural to introduce
the variable \(f_{12} := F_{12}F_0 - F_1F_2\). The phase shift is zero iff \(f_{12} = 0\). Thus we
change to variables that describe the heights of the kinks for \(\phi, \rho\) and \(\bar{\rho}\) by setting
for \(i=1,2\)

\[
F_i = \Delta_i \left(\frac{F_0}{\Delta_0} + f_i\right) \quad N_i = \Delta_i \left(\frac{N_0}{\Delta_0} + n_i\right) \quad \bar{N}_i = \Delta_i \left(\frac{\bar{N}_0}{\Delta_0} + \bar{n}_i\right) \quad (4.13)
\]

and to variables suggested by the form of the phase shift for \(1/\phi\) by

\[
F_{12} = \frac{F_1 F_2 + f_{12}}{F_0} \quad \Delta_{12} = \frac{\Delta_1 \Delta_2 + \delta_{12}}{\Delta_0} \quad N_{12} = \frac{N_1 N_2 + n_{12}}{N_0} \quad \bar{N}_{12} = \frac{\bar{N}_1 \bar{N}_2 + \bar{n}_{12}}{\bar{N}_0} \quad (4.14)
\]

18
We assume that \( \Delta_0 \Delta_1 \Delta_2 F_0 N_0 \bar{N}_0 \neq 0 \), so that this change of variables is non-singular. If \( k_{12} = -k_{21} \) or \( k_{12} = 0 \) or \( k_{21} = 0 \) then we have one of the cases considered in the previous chapter, so we now also assume that \( k_{12} k_{21} (k_{12} + k_{21}) \neq 0 \). Then the \( e^{n+m} \) terms of Yangs equations immediately yield

\[
\begin{align*}
\delta_{12} &= \frac{\Delta_0^2}{F_0^2} f_{12} + \Delta_1 \Delta_2 \frac{\Delta_0^2}{F_0^2} \left( \frac{n_1 k_{12} \bar{n}_2 + n_2 k_{21} \bar{n}_1}{k_{12} + k_{21}} \right) \\
\eta_{12} &= \frac{N_0^2}{\Delta_0^2} \delta_{12} + 2 \Delta_1 \Delta_2 \frac{N_0}{F_0} \left( \frac{n_1 k_{12} f_2 + n_2 k_{21} f_1}{k_{12} + k_{21}} \right) - \Delta_1 \Delta_2 n_1 n_2 \\
\bar{n}_{12} &= \frac{\bar{N}_0^2}{\Delta_0^2} \delta_{12} + 2 \Delta_1 \Delta_2 \frac{\bar{N}_0}{F_0} \left( \frac{f_1 k_{12} \bar{n}_2 + f_2 k_{21} \bar{n}_1}{k_{12} + k_{21}} \right) - \Delta_1 \Delta_2 \bar{n}_1 \bar{n}_2
\end{align*}
\]

The \( \exp(\eta_1 + 2\eta_2) \) and \( \exp(2\eta_1 + \eta_2) \) terms will give equations for \( f_{12} \) and the full general solution then depends on whether any of \( f_1, \ldots, \bar{n}_2 \) are zero. However, in order to obtain a solution with \( f_{12} \neq 0 \) we find that we must impose the constraints

\[
f_1^2 + n_1 \bar{n}_1 = 0 \quad f_2^2 + n_2 \bar{n}_2 = 0
\]

and we then obtain a solution of Yangs equations as long as

\[
f_{12} = -\Delta_1 \Delta_2 (2 f_1 f_2 + n_2 \bar{n}_1 + n_1 \bar{n}_2) \frac{k_{12} k_{21}}{(k_{12} + k_{21})^2}
\]

In this generic solution, besides the \( k_{ij} \) we have the 12 parameters \( \Delta_0, \Delta_1, \Delta_2, F_0, N_0, \bar{N}_0, f_1, f_2, n_1, n_2, \bar{n}_1 \) and \( \bar{n}_2 \) subject to the two constraints (4.16). However we can trivially set \( \Delta_0 = 1 \), and by shifting the coordinate origin we could set \( \Delta_1 = \Delta_2 = 1 \) and so there are really only 12-2-3=7 true parameters to this general solution. Also the overall scale leaves \( A_{YM} \) unchanged (see expressions for A given by de Vega) and so might not be considered a true parameter.

With this solution it turns out that \( (\phi^2 + \rho \bar{\rho})/\phi \) is also of the form of the Hirota ansatz and so effectively we have simply imposed such an ansatz directly on the components of \( J \).
In order to interpret the solution we can look at the behaviour of the stress energy tensor of (4.4). We find that for $\eta_2 \to -\infty$ we have that $T$ becomes proportional to the constraints (4.16) and so for this solution generically $T$ will become zero at $\eta_i = \pm \infty$. Hence in the above solution to Yangs equations the stress energy generically vanishes at long distance from the region where the kinks are colliding. In this respect is is more like an instanton solution of the underlying 2 dimensional chiral model. The particular solution that de Vega found is the special case $F_{12} = 0$ and this is a degenerate case in which the stress tensor spreads out infinitely in one particular direction.

If we impose reality conditions to try to get an SU(2) solution then we inevitably find we are forced to trivialise the solution. For example, in the $C^{1,1}$ case we must impose $\rho^* = \bar{\rho}$ and $\phi^* = \bar{\phi}$; whence also $n_i^* = \bar{n}_i$ and $f_i^* = f_i$ and then (4.16) forces $f_i = n_i = \bar{n}_i = 0$ and in particular $f_{12} = 0$. The reality conditions are homogeneous and so it is not unreasonable to have tried the Hirota ansatz in this case. There is no problem if we want SL(2,R) or SU(1,1) as we simply take a real solution or solve (4.16) with $n_i^* = -\bar{n}_i$. Perhaps one point worth mentioning with respect to SU(2) solutions is that in a (2,2) signature the Bäcklund transformations in [18] now preserve the reality conditions and so take SU(2) solutions to SU(2) solutions. They no longer alternate between SU(1,1) and SU(2) solutions as happens in a (4,0) signature.

An obvious question is whether this can be repeated for other gauge groups. The main step is the use of an Hirota ansatz and this will only be useful for homogeneous equations. Since (3.4) are homogeneous in the matrix elements of J for any G then we expect that GL(n) could be treated in this way. However, as we just saw, this will not usually work for subgroups because the extra conditions are usually not homogeneous and so not suitable for the Hirota ansatz. Note these solutions are just scattering in a general chiral model, which is integrable, and so in principle all solutions are known, but in practise are not trivial.
5. Conclusion and open questions

In this paper we briefly reviewed the kinematics in (complexified) space-time with a (2,2) signature. We found a simple way to obtain the three-nulls identity that is vital to calculations of on-shell Feynman diagrams. We then briefly covered the J formulation of SDYM, its perturbation theory and dimensional reduction to a two dimensional chiral model with Wess-Zumino term.

As pointed out in [1] a dominant feature of the perturbation theory is that the on-shell connected amplitudes vanish for four or more external legs. This leaves only three particle scattering and then the momentum dependence of the vertices has the implication that two particles (physical states) can only interact if their momenta are in the same self-dual null plane. This only applies to the S-matrix but not to Greens functions but does seem to imply that the effective dimensionality has been reduced by one. It would be interesting to classify all possible actions for which the S-matrix has the same properties. For example we might observe that the action (4.3) is a non-symmetric non-linear sigma model, and so instead we could generalise to

\[ L = G_{ab}[\phi] \partial_{\mu} \phi^a \partial_{\bar{\mu}} \phi^b g^{\mu \bar{\mu}} \]  (5.1)

Enforcing the vanishing of the appropriate Feynman diagrams will presumably then have some geometric meaning for the non-symmetric metric on the space parametrised by the $\phi^a$. Presumably including fermions will be related to SDYM with supergroups or to some supersymmetric extension of the self-duality relation: It should be noted that a tree level diagram with all bosonic external legs cannot have any internal fermion lines and so the bosonic sector of any such general theory should already possess the property that only the three point function is non-trivial. The S-matrix predicted by the perturbation theory is very simple, and if this structure is still present non-perturbatively then it would be very interesting to find the consequences for the S-matrix (compare factorisation of the S-matrix in two dimensions).
A classical equivalent of particle momenta being in the same self-dual plane is that the fields depend only on the two coordinates of the plane (or more generally to be non-linear superpositions of such fields on different planes). In this case SDYM reduces to the equation of motion from a chiral model in (1,1) spacetime with the Wess-Zumino term only. Such a “pure WZ” model is classically exactly solvable and topological. It is also not an evolution equation and so this leads us to question the validity of the perturbation theory prediction for the scattering in the theory. (This also means that SDYM is on the edge of being topological [2] and so we might expect some of the physical observables to be non-local and measured by appropriate Wilson lines.)

In order to see whether these properties hold non-perturbatively in the coupling constant (but still at tree level) we were lead to consider exact classical solutions of SDYM. In particular there are certainly some exact solutions which correspond to 2 to 2 scattering with no momentum exchange but a non-zero phase shift. For example the theory can be reduced to the KdV equation or a (1,1) chiral model. In this case the momenta would not necessarily lie on the same self-dual plane and so the factorisation considered above would be destroyed (and the perturbation theory would have been very misleading). To clarify this we extended some work of de Vega [7] on the scattering of plane waves in SL(2,C).

Using the Hirota ansatz and Gaussian decomposition of the SL(2,C) field we were able to solve the SDYM equations for the case of a generalised plane wave. The solution described the collision of kinks in the J field, and consisted of non-trivial scattering in the sense that the kinks suffered a non-zero phase shift from the collision. However the restrictions necessary to obtain such a solution were also precisely sufficient to force the in and out waves to have vanishing stress-energy tensor and Yang-Mills field strengths. That is, the stress tensor (meaning the one from the action for Yangs equations not the one from the Yang-Mills action) is non-zero only in the region where the kinks are colliding. On the positive side, this does not seem to correspond to 2 to 2 scattering of physical states and so will not affect the above S-matrix properties. The negative side is that there is still the
possibility that there is no scattering of physical states.

In our SL(2,C) solution we have a two dimensional surface of non-zero energy embedded in the spacetime which is reminiscent of a string theory again. It would be interesting to investigate whether such solutions have a stringlike behaviour in which strings beget strings along the lines suggested in [19].

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REFERENCES

1. H. Ooguri and C. Vafa,, Mod. Phys. lett. A5 (1990), 1389.
2. H. Ooguri and C. Vafa,, Nucl. Phys. B369 (1991), 469.
3. M. Ademello et al, Nucl. Phys. B111 (1976), 77.
4. S.D. Mathur and S. Mukhi, Phys. Rev. D36 (1987), 465.
5. S.D. Mathur and S. Mukhi, Nucl. Phys. B302 (1988), 130.
6. M.B. Green, J.H. Schwarz, and E. Witten, Superstring Theory, (Cambridge University press, Cambridge, 1986) Vol I, p 237.
7. H.J. de Vega, Commun. Math. Phys. 116 (1988), 659.
8. L.J. Mason and G.A.J. Sparling, Phys. Lett. A137 (1989), 29.
9. R.S. Ward, “Multi-dimensional integrable systems” in ‘Field theory, quantum gravity and strings’ Vol 2 eds H. de Vega and N. Sanchez (1987).
10. Y. Brihaye, D.B. Fairlie, J. Nuyts and R.G. Yates, J. Math. Phys. 19 (1978), 2528
11. K. Pohlmeyer, Commun. Math. Phys. 72 (1980), 37.
12. L.-L. Chau, “Geometrical integrability and equations of motion in physics: A unifying view”, in ‘Integrable Systems’ edited by X.C. Song.
13. M. Bonini, E. Gava and R. Iengo, *Mod. Phys. Lett.* **A6** (1991), 795.

14. S. Kalitzin and E. Sokatchev, *Phys. Lett.* **B262** (1991), 444.

15. Q-Han Park, *Phys. Lett.* **B238** (1990), 287.

16. E. Witten, *Nucl. Phys.* **B311** (1988), 46.

17. C.N. Yang, *Phys. Rev. Lett.* **38** (1977), 1377.

18. E.F. Corrigan, D.B. Fairlie, R.G. Yates and P. Goddard, *Commun. Math. Phys.* **58** (1978), 223.

19. M.B. Green, *Nucl. Phys.* **B293** (1987), 593.