Statements of the boundary value problems in mathematical simulation of a quasistationary electric field in the atmosphere and ionosphere

Valery Denisenko
Institute of Computational Modelling, SB RAS, Academgorodok, Krasnoyarsk, 660036, Russia
E-mail: denisen@icm.krasn.ru

Abstract. Boundary-value problems which have to be solved in the frame of the mathematical models of quasi-stationary electric fields and currents in the global conductor, that consists of the Earth’s ionosphere and the atmosphere are formulated. The approach based on the domain decomposition is used. A quadratic energy functional is constructed. It allows us to reduce the solution of the boundary value problem for a three-dimensional elliptic differential equation with asymmetric coefficients to minimizing of the functional. Estimates of the quadratic form in comparison with the Dirichlet principle for the Poisson equation are obtained. These estimates allow us to assess the condition number of the matrix of the system of linear algebraic equations, arising in the numerical solution to the problem.

1. Introduction
The problem of the penetration of large-scale electric fields from the atmosphere into the ionosphere is reduced to solving a quasi-stationary problem of electrical conductivity and occurs, mainly, in studying of the two phenomena. The first is the Global Electric Circuit (GEC) [1]. Thunderstorms are its generator. Electric charges separated in a thundercloud due to the external currents form the electric fields and conductivity currents, which provide closing the external currents necessary for quasi-stationary process. Although the external currents appear in the atmosphere below 15 km, the conductivity currents partially carry charges into the ionosphere, along which they are spreading, and globally flow down to the ground. Then they return to the thunderstorm areas by the well-conducting ground or salt water in the oceans. This electrical circuit is called the GEC. Naturally, one of the most important issues in studying its formation is what share of the current from the vicinity of a thundercloud goes into the ionosphere.

The total current of the GEC is well known, 1 – 2 kA. For its closure with the fair weather currents, a potential difference between the ground and the ionosphere of about 300 kV is formed [1]. The ionospheric electric fields, providing the global redistribution of currents, are too weak to be measured. They are also difficult for mathematical simulation [2]. The results of the modern model [2] differ by an order of magnitude from the results of [3], in which excessive simplifications of the conductivity problem were used.

The second important problem arose when analyzing the possibility of earthquake forecasting based on satellite measurements of the electric field [4]. Disturbances in the electric field in
the atmosphere are observed before earthquakes occur. Naturally, the question arose of the penetration of such fields into the ionosphere, where they, with a sufficiently high intensity, could be detected. This would make it possible to replace a dense network of the ground-based geophysical observatories with a satellite constellation. Our mathematical models of such a lithosphere-ionosphere connection are presented in [5, 6] with the analysis of the models by other authors.

The present paper is devoted to the formulation of the boundary value problems, arising for mathematical simulation of the global conductor, consisting of the ionosphere and atmosphere.

2. The Electric Conductivity Equation

In the atmosphere, due to the isotropy of the air, conductivity is scalar, while in the ionosphere, the magnetic field defines a special direction, and the conductivity becomes a gyrotropic tensor

\[ \hat{\sigma} = \begin{pmatrix} \sigma_p & -\sigma_H & 0 \\ \sigma_H & \sigma_p & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \]  

(1)

with components called field-aligned (\(\sigma_\parallel\)), the Pedersen (\(\sigma_p\)) and the Hall (\(\sigma_H\)) conductivities [7]. Their typical altitude distributions are shown in figure 1 for mid latitudes at night at a low solar activity. For long-term processes, the effective Pedersen and Hall conductivities must be used [8]. They are presented by the dashed lines. The tensor \(\hat{\sigma}\) has the form of (1) in the Cartesian coordinates \(x, y, z\) with the axis \(z\) directed along the magnetic induction vector \(B\).

We also use Cowling conductivity

\[ \sigma_C = (\sigma_p^2 + \sigma_H^2)/\sigma_p. \]  

(2)

By virtue of Ohm’s law, conductivity determines the relationship between the electric current density \(j\) and the electric field strength \(E\)

\[ j = \hat{\sigma}E. \]  

(3)

An electric current is accompanied by dissipation of the electrical energy with the density \(j^*E\), where * means transposition. So, the symmetric part of \(\hat{\sigma}\) is positive definite. For a tensor of the form of (1), this means that the diagonal elements are positive. Excluding ideal conductors and insulators from consideration, we assume positive definiteness in the domain \(\Omega\). It is convenient to write down these conditions as:

\[ \sigma_1 \leq \sigma_p, \quad \sigma_C \leq \sigma_2, \quad \sigma_1 \leq \sigma_\parallel \leq \sigma_2, \]  

(4)

where \(\sigma_1, \sigma_2\) are some positive constants. The ratio

\[ H = \sigma_2/\sigma_1 \]  

(5)

describes the heterogeneity of the conductor. So we call it the heterogeneity number.

For the processes whose characteristic time is much longer than the charge relaxation time in the conductive environment \(\varepsilon_0/\sigma\), where \(\varepsilon_0\) is the free space permittivity, one can use the quasi-stationary approximation [9]. As can be seen in figure 1, \(\sigma > 10^{-14}\) S/m, and, therefore, this characteristic time in the Earth’s atmosphere does not exceed 15 minutes.

In the stationary case, Faraday’s law and the charge conservation law have the form

\[ \text{curl} \, E = 0, \quad \text{div} \, j = Q, \]  

(6)
where \( Q \neq 0 \) only if there are external currents with the density \( j_{ext} \). Then \( Q = -\text{div} j_{ext} \).

The density of external currents is determined by other physical processes, and we consider \( j_{ext} \) to be a given vector function in this model. The sum \( j + j_{ext} \) is the total current.

By virtue of the first equation (6), the electric potential \( V \) can be introduced such that

\[
E = -\text{grad} V.
\] (7)

Then the system of equations (3), (6) is reduced to the electric conductivity equation

\[
-\text{div} (\hat{\sigma} \text{grad} V) = Q.
\] (8)

It is of the elliptic type due to the positive definiteness of the symmetric part of \( \hat{\sigma} \), since in the considered domain \( \Omega \) the components \( \sigma_P, \sigma_\parallel \) satisfy (4).

3. The boundary condition at the Earth’s surface

Because the conductivity of the sea water (3.2 S/m), the clay (10^{-2} S/m), and even many rocky substances (10^{-7} S/m), [10] exceed the air conductivity near the ground (10^{-14} S/m) by many orders of magnitude, the substances below the Earth’s surface can be considered as an ideal conductor with a good accuracy. The corresponding boundary condition

\[
V |_{r_c} = -V_0,
\] (9)
where $V_0$ is some unknown constant. The surface $\Gamma_G$ is given by a well-known function $h_s(\theta, \varphi)$, i.e. the height of the Earth’s surface, ($h_s = 0$ for oceans), $\theta, \varphi$ are spherical coordinates, $h$ is the altitude above the mean sea level.

4. Condition on the upper ionosphere boundary

The upper boundary of the ionosphere $\Gamma_I$, beyond which the magnetosphere begins, is conditionally determined. For large-scale problems of electrical conductivity, it can be located at an altitude of 500 km in the sense that the conductivity across the magnetic field can be neglected $\sigma_p = \sigma_H = 0$ (except for the two parts of the magnetosphere discussed below) and $\sigma_\parallel = \infty$. The last condition means that an arbitrary current can flow along the magnetic field line due to a negligibly small electric field $E_\parallel$ meaning the equipotentiality of each magnetic field line.

According to the properties of the magnetospheric conductor, the upper boundary of the ionosphere $\Gamma_I$ is divided into 6 parts: northern ($\Gamma_{CN}$) and southern ($\Gamma_{CS}$) polar caps, surrounding rings ($\Gamma_{AN}$ and $\Gamma_{AS}$), and the two parts of the mid-latitude ionosphere, including low latitudes, one of which ($\Gamma_{N}$) is located to the north from the geomagnetic equator, the other ($\Gamma_{S}$) – to the south.

The condition for the smallness of $\sigma_p, \sigma_H$ in comparison with their values in the ionosphere is not satisfied in the plasma sheet of the magnetospheric tail and near the outer boundary of the magnetosphere. It was shown in [11] that this allows one to use the equipotentiality approximation in the union of these regions and in two sections of the upper boundary of the ionosphere, connected with them by the magnetic field lines

\[ V |_{\Gamma_{AN}} = 0, \quad V |_{\Gamma_{AS}} = 0. \]  

(10)

These ring regions, usually located at latitudes around $\pm 70^\circ$, are called auroral zones, since the polar lights often occur there.

The magnetic field lines extend far from the polar caps into the magnetospheric tail, and there are no currents between such lines. Therefore, there are no currents through their lower ends, $j_\parallel = 0$. By virtue of $\sigma_p = \sigma_H = 0$ we have $j_\perp = 0$, and so the whole vector $\mathbf{j} = 0$. In particular, a component of $\mathbf{j}$ normal to the boundary $\Gamma_I$ is equal to zero. By virtue of the charge conservation law, it does not change when crossing the boundary. This gives the boundary condition for the currents in the ionosphere:

\[ j_n |_{\Gamma_{CN}} = 0, \quad j_n |_{\Gamma_{CS}} = 0. \]  

(11)

The indices $n$ and $\tau$ mark the normal and the tangential components of vectors.

At mid and low latitudes, each magnetic field line has one end north of the geomagnetic equator, the other – south. Due to the equipotentiality of the line in the magnetosphere, for each such pair of the conjugate points:

\[ V |_{\Gamma_{N}} = V |_{\Gamma_{S}}. \]  

(12)

By virtue of the charge conservation law, the current flowing into a magnetic field tube through its northern section at the boundary with the ionosphere should flow out through the southern one. Denoting through $dS_N$ and $dS_S$ the areas of these sections and using the continuity of the normal component of $\mathbf{j}$ at any surface, for the ionospheric currents we obtain the boundary condition connecting the normal components of $\mathbf{j}$ at the conjugate points:

\[ j_n |_{\Gamma_{N}} = \frac{dS_S}{dS_N} j_n |_{\Gamma_{S}}. \]  

(13)

In the approximation of a dipole magnetic field, this condition is simplified, since $dS_S = dS_N$.

Thus, in the layer between the Earth’s surface $\Gamma_G$ and the conditional upper boundary of the ionosphere $\Gamma_I$, the elliptic boundary value problem (8) - (13) is set.
5. Domain decomposition

In the numerical solution of a differential equation, a system of linear algebraic equations appears. The condition number of this system is important for the precision of an approximate solution as well as for the effectiveness of iterative methods. It could be shown that the condition number is approximately proportional to the heterogeneity number $H$ (5). In a simple case, when the conductivity $\hat{\sigma}$ is a unit tensor, equation (8) becomes Poisson equation and $H = 1$. The main advantage of the domain decomposition is due to separation of the domain $\Omega$ into a few more homogeneous subdomains. In other words it is necessary to decrease the heterogeneity number $H$ in each subdomain.

For the typical values in the domain $0 < h < 500$ km shown in figure 1 $H \approx 10^{16}$, that is, an additional multiplier for the condition numbers of similar problems for the Poisson equation. It is impossible to solve such a problem, even using the double precision numbers.

In the model [2], this domain is divided into two parts by the boundary at $h = 80$ km. Then, in the lower part, $H$ is decreased to $10^6$, and in the upper - to $10^{12}$. In both subdomains, the three-dimensional problem was not directly solved. The approximations whose errors are greatest in the layer $60$ km $< h <$ 80 km, the central part of the $D$-layer of the ionosphere were used. These errors can be eliminated only by solving the three-dimensional problem in the $D$-layer.

The height distributions of the conductivities shown in figure 1 demonstrate that it seems appropriate to divide the atmospheric-ionospheric conductor $\Omega$ into the three layers: $\Omega_A$ is the atmosphere, $\Omega_D$ is the ionospheric $D$-layer, $\Omega_I$ is the ionosphere, more precisely, its $E$- and $F$-layers. It is natural to take the two internal boundaries $\Gamma_A$ and $\Gamma_D$ at the heights of 50 km and 90 km, respectively, though they can be somewhat shifted when optimizing the algorithm.

In the atmosphere, the problem is greatly simplified due to the scalarity of $\hat{\sigma}$.

In the ionosphere, the two-dimensional model using $\sigma_P \ll \sigma_\parallel$ is obtained.

In the $D$-layer, one has to solve a three-dimensional problem, but with the heterogeneity number $H$ reduced to $10^6$ instead of $10^{16}$.

The decomposition method consists of iterations, in each one the problems in the subdomains are solved using the boundary values of the unknown functions obtained at the previous iteration in the adjacent subdomains.

We formulate such boundary conditions for subdomains. Taking into account an increase in conductivity with height, we use the previously obtained potential values $V$ at additionally arising upper boundaries, and the normal components of the current density $j_n$ at the lower ones.

5.1. Atmosphere

We use the values of the potential obtained in the $D$-layer at the upper boundary of the atmosphere

$$V|_{\Gamma_A} = V_D|_{\Gamma_A}. \quad (14)$$

We obtain a mixed boundary value problem (8), (9), (14) in the domain $\Omega_A$, which would be a spherical layer if we neglect the relief and consider the Earth not as an ellipsoid of revolution, but as a ball.

This problem is quite simple due to the scalar value $\hat{\sigma}$. Therefore, the operator of the boundary value problem is symmetric, and the heterogeneity number $H \approx 10^4$ is not too large. It was shown in [12] that in solving this problem, one can use an additional horizontal decomposition with subdomains like 500 km * 500 km. In the model [2], it turned out that it is acceptable to use the one-dimensional approximation, which reduces the problem to the calculation of height integrals.
5.2. Ionosphere
During simulation of the large-scale electric fields and currents in the ionosphere, $E_\parallel$ can be neglected due to $\sigma_\parallel \gg \sigma_P$. It makes each magnetic field line to be an equipotential one. Therefore, the potential $V$ becomes a function of two variables. This approach is well known [7], our version is presented in [11]. Simplified constructions for a dipole magnetic field were performed in [13]. The energy method for such a problem and the multigrid method of numerical solution based on it is presented in [14].

5.3. The ionospheric D-layer
In the domain of $50 \text{ km} < h < 90 \text{ km}$, the conductivity $\hat{\sigma}$ is no longer scalar as in the atmosphere, but there is no condition $\sigma_\parallel^2 \gg \sigma_P^2 + \sigma_H^2$, that would allow us to use a two-dimensional model. Therefore, it is necessary to solve a three-dimensional system of equations (3), (6) in this subdomain $\Omega_D$ with the boundary conditions set during the decomposition of the entire atmospheric-ionospheric conductor:

$$V |_{\Gamma_I} = V_{I | \Gamma_I}, \quad j_n |_{\Gamma_A} = j^A_n |_{\Gamma_A},$$

(15)

where the right-hand sides are determined with the solutions in the neighboring subdomains.

In the next section we present our approach to solving this boundary value problem. It is created in the frame of the energy method [15] which can be considered as a generalization of the Dirichlete principle for the Poisson equation.

6. The energy method for the 3-D problem
Here we consider a more general case as compared to equations (6). If the magnetic field varies with time according to a given law, then the first equation (6) has a non-zero right-hand side, denoted by $G$

$$\text{curl} E = G, \quad \text{div} j = Q.$$

(16)

The presentation of (7) is impossible when $G \neq 0$, so the first boundary condition (15) must be written down in the form that is original from the physical point of view:

$$E_\tau |_{\Gamma_I} = E^I_\tau |_{\Gamma_I}, \quad j_n |_{\Gamma_A} = j^A_n |_{\Gamma_A}.$$

(17)

It is possible to calculate the normal to the boundary component of the curl of both sides of the first boundary condition (17):

$$\text{curl}_n E_\tau |_{\Gamma_I} = \text{curl}_n E^I_\tau |_{\Gamma_I}.$$

The left-hand side also satisfies the normal to the boundary component of the first equation (16). Therefore we see the necessary condition for the given functions:

$$G_n |_{\Gamma_I} = \text{curl}_n E^I_\tau |_{\Gamma_I},$$

otherwise problem (16), (17) is not solvable. This condition is automatically satisfied in the statement of (8), (15) for the potential $V$ since its both sides are equal identically to zero in such a case.

Due to the asymmetry of $\hat{\sigma}$, the operator of the boundary value problem (16), (17) is asymmetric, complicating the solution of the problem. A new formulation of a similar problem with a uniform condition at the entire boundary was proposed in [16]. Here we give the statement for conditions (17). We believe that the proof can be carried out by analogy with [16].
We consider a pair of new unknown functions $F, P$ (scalar and vector, respectively) which satisfy the boundary conditions

$$F \big|_{\Gamma_1} = 0, \quad P_n \big|_{\Gamma_1} = 0, \quad P_r \big|_{\Gamma_2} = 0. \quad (18)$$

The energy scalar product is defined for the smooth functions satisfying these conditions

$$\left( \begin{array}{c} u \\ v \end{array} \right), \left( \begin{array}{c} F \\ P \end{array} \right) = \int \left( \begin{array}{c} \nabla u \\ \nabla \times v \end{array} \right)^* \left( \begin{array}{cc} \frac{1}{\sigma_0} \hat{\sigma} \hat{S} \hat{\sigma}^* & -\hat{\sigma} \hat{S} \\ -\hat{S} \hat{\sigma}^* & \sigma_0 \hat{S} \end{array} \right) \left( \begin{array}{c} \nabla F \\ \nabla \times P \end{array} \right) + \text{div} v \cdot \text{div} P \right) d\Omega, \quad (19)$$

where $\sigma_0$ is an arbitrary positive constant, $\hat{S}$ is a sufficiently arbitrary symmetric positive definite matrix, whose components vary in $\Omega_D$.

The matrix appearing in (19) is a symmetric one. So we can analyze the quadratic form instead of the bilinear form. The matrix is degenerate, since its upper blocks are obtained from the lower ones by multiplying by $-\hat{\sigma}/\sigma_0$.

Consider the auxiliary integral

$$\int (\nabla F)^* \nabla \times P \right) d\Omega. \quad (20)$$

We transform the integrand:

$$\int (\text{div}(F \nabla \times P) - F \text{div} \nabla \times P) d\Omega.$$

The second term is identically equal to zero. Convert the remaining integral using the Gauss-Ostrogradsky theorem:

$$\int_{\Gamma_1} F \nabla \times n P \ d\Gamma + \int_{\Gamma_2} F \nabla \times n P \ d\Gamma.$$

These integrals are equal to zero due to the first and the third conditions (18), respectively.

Therefore, integral (20) is equal to zero, and it can be added with any coefficient to the quadratic form (19), without changing its value. When adding the doubled integral (20), the matrix of the integrand quadratic form becomes equal to

$$K = \left( \begin{array}{cccccc} \frac{1}{\sigma_0} \hat{\sigma} \hat{S} \hat{\sigma}^* & 0 & 0 & 0 & \frac{\sigma_\mu}{\sigma_p} \hat{\sigma}^* & 0 \\ 0 & \frac{\sigma_\mu}{\sigma_p} & 0 & -\frac{\sigma_\mu}{\sigma_p} & 0 & 0 \\ 0 & 0 & \frac{\sigma_\mu}{\sigma_p} & 0 & 0 & 0 \\ 0 & -\frac{\sigma_\mu}{\sigma_p} & 0 & \frac{\sigma_\mu}{\sigma_p} & 0 & 0 \\ \frac{\sigma_\mu}{\sigma_p} & 0 & 0 & 0 & \frac{\sigma_\mu}{\sigma_p} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sigma_\mu}{\sigma_p} \end{array} \right),$$

where $\hat{I}$ is the identity matrix. Let us denote the maximal and minimal eigenvalues of $K$ as $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$, which are real due to the symmetry of $K$.

To obtain more precise estimates than in [16], we use the form $\hat{\sigma}$ (1) with restrictions (4) and set

$$\sigma_0 = \sqrt{\sigma_1 \sigma_2}, \quad \hat{S}^{-1} = (\hat{\sigma} + \hat{\sigma}^*)/2. \quad (21)$$

For $\hat{\sigma}$ (1), the matrix $\hat{S}$ (21) is a diagonal matrix, and the symmetrical matrix $K$ takes the form
where $\sigma_c$ is the Cowling conductivity (2).

The eigenvalues of the matrix do not change as a result of the simultaneous rearrangement of rows and columns, so we can bring the matrix $K$ to the block diagonal form with the blocks
\[
\begin{pmatrix}
\sigma_c & \sigma_H \\
\sigma_H & \sigma_p \\
\end{pmatrix}, \quad \begin{pmatrix}
\sigma_c & -\sigma_H \\
-\sigma_H & \sigma_p \\
\end{pmatrix}, \quad \frac{\sigma_0}{\sigma_0}, \frac{\sigma_\parallel}{\sigma_\parallel}, \frac{\sigma_\perp}{\sigma_\perp},
\]
(22)

The eigenvalues of the two matrices are equal. The larger eigenvalue does not exceed the trace of the matrix $\sigma_c/\sigma_0 + \sigma_p/\sigma_p \leq 2\sqrt{\sigma_2/\sigma_1}$, where the inequality is valid in view of (4).

Since $\sigma_0$ is set by (21), the last two numbers in (22) do not exceed $\sqrt{\sigma_2/\sigma_1}$. Therefore, all the eigenvalues of blocks (22) and $\lambda_{\text{max}}$ of the whole matrix $K$ do not exceed $2\sqrt{\sigma_2/\sigma_1}$ in the domain $\Omega_D$

\[
\lambda_{\text{max}} \leq 2\sqrt{\sigma_2/\sigma_1} = 2H.
\]
(23)

This explains the introduction of the heterogeneity number $H$ (5).

Since the determinants of blocks (22) are equal to 1 as well as the the product of two numbers in (22), three products of the pairs of the eigenvalues of $K$ equal 1. So, the smaller eigenvalue $\lambda_{\text{min}} = 1/\lambda_{\text{max}}$ and is estimated from below with the value inverse to (23). Simultaneously we have proved that the determinant of the matrix $K$ equals 1 and have obtained the estimate of the condition number of the matrix $K$ for any point in the domain $\Omega_D$: $\lambda_{\text{max}}/\lambda_{\text{min}} \leq 4H^2$.

These considerations give the estimates of the quadratic form of (19) from below and above:

\[
[\left( \begin{array}{c} F \\ P \end{array} \right), \left( \begin{array}{c} F \\ P \end{array} \right)] \geq \frac{1}{2H} \int ((\text{grad } F)^2 + (\text{curl } P)^2 + (\text{div } P)^2) \, d\Omega,
\]

\[
[\left( \begin{array}{c} F \\ P \end{array} \right), \left( \begin{array}{c} F \\ P \end{array} \right)] \leq 2H \int ((\text{grad } F)^2 + (\text{curl } P)^2 + (\text{div } P)^2) \, d\Omega.
\]

These estimates mean that the heterogeneity number $H$ defines important features of the studied quadratic form in comparison with the form in the simple case of the unit $\delta$ when the original equation (8) is the Poisson equation, $K$ is the unit matrix and $H = 1$.

We define the energy functional

\[
W(F, P) = \frac{1}{2} \left[ \left( \begin{array}{c} F \\ P \end{array} \right), \left( \begin{array}{c} F \\ P \end{array} \right) \right] - \int (FQ/\sigma_0 + P^* G) \, d\Omega + \frac{1}{\sigma_0} \int_{\Gamma_A} F j_n^A \, d\Gamma - \int_{\Gamma_I} P^* e_I \, d\Gamma.
\]
(24)

Let us suppose that a minimum of the energy functional is reached with the functions $F_{\text{min}}$, $P_{\text{min}}$ and denote

\[
E = -\frac{1}{\sigma_0} \text{grad } F_{\text{min}} + S \text{curl } P_{\text{min}}, \quad j = \delta E.
\]
(25)

The condition of a minimum of the energy functional with notations (25) can be written down as an identity valid for arbitrary smooth functions $u, v$, satisfying conditions (18):

\[
\int ((\text{grad } u)^* j/\sigma_0 + (\text{curl } v)^* E + \text{div } v \text{ div } P_{\text{min}} - uQ/\sigma_0 - v^* G) \, d\Omega
\]

\[
+ \int_{\Gamma_A} u j_n^A/\sigma_0 \, d\Gamma - \int_{\Gamma_I} v^* e_I \, d\Gamma = 0.
\]
(26)

If we additionally assume that all functions are smooth, the integrations by parts convert this identity to the form:

\[
\frac{1}{\sigma_0} \int u(\text{div } j - Q) \, d\Omega + \int v^* (\text{rot } E - G) \, d\Omega + \int \text{div } v \text{ div } P_{\text{min}} \, d\Omega
\]

\[
+ \frac{1}{\sigma_0} \int_{\Gamma_A} u(-j_n + j_n^A) \, d\Gamma + \int_{\Gamma_I} v^* (E_x - E_x^I) \, d\Gamma = 0,
\]
(27)
where two integrals over $\Gamma_A$ and $\Gamma_I$ are omitted since they equal zero in view of (18).

For $u \equiv 0$ and $v = \text{grad} U$ with $U = 0$ at $\Gamma_A$ and $\Gamma_I$, all but the third integrals in identity (27) equal zero, hence the third integral also equals zero:

$$\int \text{div} (\text{div} U) \text{div} P_{\text{min}} d\Omega = 0.$$  

This identity allows us to prove in the usual way that $\text{div} P_{\text{min}} = 0$, henceforth excluding the third integral in (27). Using the arbitrariness of the functions $u, v$ it is easy to prove that all multipliers for $u, v$ in (27) equal zero, so equations (16) and the boundary conditions of the original boundary value problem (17) are satisfied. Details of these proofs can be found in [16].

For the solution of the original problem $\text{div} P_{\text{min}}$ is of no significance, but it equals zero as a result of minimization. This condition is added to the system of the equations for $F, P$ to exclude multiple solutions.

The pair $F_{\text{min}}, P_{\text{min}}$ obtained by minimizing is a generalized solution. To satisfy the equations in the classical sense, the smoothness of the functions $F_{\text{min}}, P_{\text{min}}$ is additionally required.

Note that the quadratic part of (24) equals the total Joule dissipation, which explains the term ”energy functional”.

We emphasize that this energy principle has not been proved yet. Existence of its minimum was only supposed above. Difficulty arises in proving the inequality

$$\int ((\text{rot} P)^2 + (\text{div} P)^2) d\Omega \geq C_0 \int |\text{grad} P|^2 d\Omega,$$  

where $|\text{grad} P|^2$ is the sum of the squared gradients of the Cartesian components of $P$ and the constant $C_0$ is common for all the functions $P$ satisfying (18). Similar inequality was proved in [17] under simpler conditions than we need here.

Using (28) it would be possible to repeat the proofs [16] of the following statements: the energy norm is equivalent to the sum of the norms of $F$ and the Cartesian components of $P$ as the elements of $W_2^1(\Omega_D)$; the energy functional has a unique minimum $F_{\text{min}}, P_{\text{min}}$. It has been already proved that these functions give the generalized solution to the original boundary value problem.

7. Numerical methods

Using the principle of a quadratic energy functional minimum formulated in the previous section it is rather simple to design a finite element method. It is sufficient to divide the domain $\Omega_D$ into tetrahedrons and to use piece-wise linear approximating functions for each of four unknown functions, which are $F$ and the Cartesian components of $P$. Then the energy functional $W(F, P)$ becomes a quadratic form of the values of these functions in the grid nodes. The conditions of $W(F, P)$ reaching a minimum, together with the main boundary conditions (18) produce a system of linear algebraic equations. A multigrid method can be used to solve this system.

In a particular case when the conductivity $\hat{\sigma}$ is scalar $\sigma$, the energy functional $W(F, P)$ is split to independent functionals $W(F)$ and $W(P)$. The unknown function $F$ can be made equal to the electric potential $V$ since $G = 0$. It is sufficient to use the boundary conditions of the Dirichlet problem (9), (14) as the main boundary conditions for $F$ instead of (18) and put $P \equiv 0$. Then only rather a simple functional

$$W(F) = \frac{1}{2\sigma_0} \int \sigma (\text{grad} F)^2 d\Omega - \frac{1}{\sigma_0} \int FQ d\Omega$$

must be minimized. Our numerical method for this atmospheric problem is described in [12].

For the third problem arising in the ionospheric domain $\Omega_I$ after the decomposition of $\Omega$, our approach is briefly described in Section 5.2.
8. Conclusion

Thus, for the quasi-stationary electric conductivity problem in the global conductor, consisting of the ionosphere and the atmosphere of the Earth, a method of solution based on the domain decomposition is proposed.

A parameter describing the heterogeneity of a gyrotropic conductor that can be used for a preliminary estimation of the effectiveness of the decomposition is found.

The common conductor is split to three subdomains in each one the boundary value problem is formulated, and the quadratic energy functional is constructed. For two problems the energy principles have been proved, and effective numerical methods have been already designed and used. In the third case, one necessary inequality has not been proved yet.

Acknowledgements
This work was financially supported by the Russian Foundation for Basic Research (project code 18-05-00195).

References

[1] Mareev E A 2010 Achievements and prospects of the global electric circuit Phys Usp 53 504-534
[2] Denisenko V V, Rycroft M J, and Harrison R G 2019 Mathematical Simulation of the Ionospheric Electric Field as a Part of the Global Electric Circuit Surveys in Geophysics 40(1) 1-35
[3] Roble R G and Hays P B 1979 A quasi-static model of global atmospheric electricity. 2. Electric coupling between the upper and lower atmosphere J Geophys Res 84(A12) 7247-7256
[4] Kim V P, Hegaj V V and Illich-Switych P V 1994 On the possibility of a metallic ion layer forming in the E-region of the night midlatitude ionosphere before great earthquakes Geomagnetism and Aeronomy 33 658-662
[5] Denisenko V V, Nesterov S A, Boudjada M Y and Lammer H 2018 A mathematical model of quasi-stationary electric field penetration from ground to the ionosphere with inclined magnetic field Journal of Atmospheric and Solar-Terrestrial Physics 179 527-537
[6] Ampferer M, Denisenko V V, Hauleitner W, Krauss S, Stangl G, Boudjada M Y and Biernat H K 2010 Decrease of the electric field penetration into the ionosphere due to low conductivity at the near ground atmospheric layer Annales Geophysiscae 28(3) 779-787
[7] Hargreaves J K 1979 The Upper Atmosphere and Solar-terrestrial Relations (New York: Van Nostrand Reinold)
[8] Denisenko V V, Biernat H K, Mezentsev A V, Shaidurov V A and Zamay S S 2008 Modification of conductivity due to acceleration of the ionospheric medium Annales Geophysicscae 26 2111-2130
[9] Molchanov O and Hayakawa M 2008 Seismo-electromagnetics and related phenomena: history and latest results (Tokyo: TERRAPUB)
[10] Rycroft M J, Harrison G R, Nicoll K A and Mareev E A 2008 An overview of Earth’s global electric circuit and atmospheric conductivity Space Sci Rev 137 83-105
[11] Denisenko V V 2018 2-D model of the global ionospheric conductor connected with the magnetospheric conductors Preprint http://arxiv.org/abs/1802.07955
[12] Denisenko V V and Pomozov E V 2010 Global electric fields in the Earth’s atmosphere calculation Journal of Computational Technologies 15(5) 34-50
[13] Denisenko V V and Zamay S S 1992 Electric field in the equatorial ionosphere Planetary and Space Science 40(7) 941-952
[14] Denissenko V V 1995 Energy methods for elliptic equations with asymmetric coefficients (Novosibirsk: Russian Academy of Sciences Siberian Branch)
[15] Mikhlin S G 1957 Variational methods in mathematical physics (Moscow: Gostekhizdat)
[16] Denisenko V V 1997 The energy method for three dimensional elliptical equations with asymmetric tensor coefficients Siberian Mathematical Journal 38(6) 1099-1111
[17] Bykhovsky E B and Smirnov N V 1960 On the orthogonal decomposition of the space of vector functions that are quadratically summable over a given domain, and vector analysis operators Trudy MIAN im. V.A. Steklov 59 5-36