OPTIMAL ASYMPTOTIC BOUNDS FOR DESIGNS ON MANIFOLDS

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Abstract. We extend to the case of a $d$-dimensional compact connected oriented Riemannian manifold $M$ the theorem of A. Bondarenko, D. Radchenko and M. Viazovska [2] on the existence of $L$-designs consisting of $N$ nodes, for any $N \geq C_M L^d$. For this, we need to prove a version of the Marcinkiewicz-Zygmund inequality for the gradient of diffusion polynomials.

1. Introduction

Let $M$ be a connected compact orientable $d$-dimensional Riemannian manifold without boundary with normalized Riemannian measure $d\mu$, such that $\mu(M) = 1$. We shall denote the Riemannian distance between $x$ and $y$ by $|x - y|$. Let $\{\varphi_k\}_{k=0}^{+\infty}$ be the eigenfunctions of the (positive) Laplace-Beltrami operator, with eigenvalues $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \ldots$, $\Delta \varphi_k = \lambda_k^2 \varphi_k$.

The space of diffusion polynomials of bandwidth $L \geq 0$ is

$$\Pi_L = \text{span} \{\varphi_k : \lambda_k \leq L\}.$$ 

We say that a set of points $\{x_j\}_{j=1}^{N} \subset M$ is an $L$-design if

$$\int_M P(x) d\mu(x) = \sum_{j=1}^{N} \frac{1}{N} P(x_j), \quad \text{for all } P \in \Pi_L.$$ 

Observe that since the above identity is trivially satisfied by constant functions, and since by orthogonality of the eigenfunctions $\varphi_k$,

$$\int_M \varphi_k(x) d\mu(x) = 0, \quad \text{for all } k \geq 1,$$

then $\{x_j\}_{j=1}^{N} \subset M$ is an $L$-design if and only if

$$\sum_{j=1}^{N} \frac{1}{N} P(x_j) = 0, \quad \text{for all } P \in \Pi_L^0,$$

where $\Pi_L^0$ is the subspace of $\Pi_L \subset L^2(M, d\mu)$ orthogonal to the constant functions, that is $\Pi_L^0 = \text{span}\{\varphi_k : 0 < \lambda_k \leq L\}$.

By Weyl’s estimates on the spectrum of an elliptic operator [13, Theorem 17.5.3], $\dim(\Pi_L) \sim L^d$. For each $L \geq 0$ denote with $N(L)$ the minimal number of points in an $L$-design in $M$.

Proposition 1. There exists a positive constant $c_M$ such that $N(L) \geq c_M L^d$ for every $L \geq 0$.

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Proof. Assume \( \{x_j\}_{j=1}^{N(L)} \) is an \( L \)-design with exactly \( N \) (\( L \)) nodes. By Theorem 2.12 in [3] there exists a constant \( \beta > 0 \) such that for every \( f \) in the Sobolev space \( W^{\alpha,1}(\mathcal{M}) \) with \( \alpha > d \) one has

\[
\left| \int_{\mathcal{M}} f(x) \, d\mu(x) - \sum_{j=1}^{N(L)} \frac{1}{N(L)} f(x_j) \right| \leq \beta L^{-\alpha} \|f\|_{\alpha,1}.
\]

On the other hand, by Theorem 2.16 in [3], there exists a constant \( \gamma > 0 \) such that for every \( L \) there exists a function \( f_L \in W^{\alpha,1}(\mathcal{M}) \) with

\[
\left| \int_{\mathcal{M}} f_L(x) \, d\mu(x) - \sum_{j=1}^{N(L)} \frac{1}{N(L)} f_L(x_j) \right| \geq \gamma N(L)^{-\alpha/d} \|f_L\|_{\alpha,1}.
\]

This gives

\[
N(L) \geq \gamma^{d/\alpha} \beta^{-d/\alpha} L^d.
\]

Korevaar and Meyers [15] conjectured that when \( \mathcal{M} \) is the \( d \)-dimensional sphere, there is a constant \( C_d \) such that \( N(L) \leq C_d L^d \) for any positive \( L \). Bondarenko, Radchenko and Viazovska [3] show an even stronger version of Korevaar and Meyer’s conjecture, namely they show that there is a constant \( C_d \) such that for every \( N \geq C_d L^d \) there exists an \( L \)-design in the \( d \)-dimensional sphere with exactly \( N \) nodes. Later, Etayo, Marzo and Ortega-Cerdà [8] by means of the same techniques as in [2], generalize the result of Bondarenko, Radchenko and Viazovska to the case of an affine algebraic manifold. In particular the main ingredients in these proofs are a result from the Brouwer degree theory, a partition of the ambient space \( \mathcal{M} \) containing a geodesic ball \( Y_j \) of radius \( c_1 N^{-1/d} \) and contains a geodesic ball \( Y_j \) of radius \( c_1 N^{-1/d} \).

The Marcinkiewicz-Zygmund inequality for diffusion polynomials on manifolds (and much more general spaces) has been proved in a series of papers by Maggioni, Mhaskar and Filbir [6, 10, 17].

Theorem 3. [18, Theorem 1.2.9.] Let \( H \) be a finite dimensional Hilbert space with inner product \( \langle \cdot, \cdot \rangle \). Let \( f : H \to H \) be a continuous mapping and \( \Omega \) an open bounded subset with boundary \( \partial \Omega \) such that \( 0 \in \Omega \subset H \). If \( \langle x, f(x) \rangle > 0 \) for all \( x \in \partial \Omega \), then there exists \( x \in \Omega \) satisfying \( f(x) = 0 \).

The second result has been proved recently in [11] for Ahlfors regular metric measure spaces, and therefore holds for compact Riemannian manifolds as well.

Theorem 4. [10, Theorem 5.1] Assume that \( c_1 \) and \( c_2 \) are constants for which Theorem 3 holds. Then there exists a constant \( C > 0 \) such that for all integers \( N \geq 1 \), for all partitions \( \mathcal{R} = \{R_j\}_{j=1}^{N} \) with constants \( c_1 \) and \( c_2 \) as in Theorem 3 for all \( x_j \in R_j \), for all \( L \leq N^{1/d} \) and for all \( P \in \Pi_L \)

\[
\left| \int_{\mathcal{M}} |P(x)| \, d\mu(x) - \sum_{j=1}^{N} \frac{1}{N} |P(x_j)| \right| \leq C LN^{-1/d} \int_{\mathcal{M}} |P(x)| \, d\mu(x).
\]
When $\mathcal{M}$ is the sphere and the diffusion polynomials are restrictions to $\mathcal{M}$ of polynomials in $d+1$ real variables of degree at most $L$, then the gradient of a polynomial is again a polynomial, and therefore the Marcinkiewicz-Zygmund inequality for gradients follows easily from the above Theorem 4. In the case of algebraic manifolds, ad hoc arguments that use the complexification of the variety $\mathcal{M}$ can be applied (see [8]). In the general case of Riemannian manifolds, the above types of arguments fail. Here we show a Marcinkiewicz-Zygmund inequality for gradients of diffusion polynomials and consequently prove the Korevaar and Meyer’s conjecture in the case of Riemannian manifolds.

**Theorem 5.** Assume that $c_1$ and $c_2$ are constants for which Theorem 3 holds. Then there exists a constant $C_3 = C_3(c_1, c_2) > 0$ such that for all integers $N \geq 1$, for all partitions $\mathcal{R} = \{R_j\}_{j=1}^N$ with constants $c_1$ and $c_2$ as in Theorem 3 for all $x_j \in R_j$, for all $L \leq N^{1/d}$ and for all $P \in \Pi^2_N$,

$$\left| \int_{\mathcal{M}} \|\nabla P(x)\| \, d\mu(x) - \frac{1}{N} \sum_{j=1}^N \|\nabla P(x_j)\| \right| \leq C_3 LN^{-1/d} \int_{\mathcal{M}} \|\nabla P(x)\| \, d\mu(x).$$

**Theorem 6.** There exists a constant $C_M$ such that for each $N \geq C_M L^d$ there exists an $L$-design in $\mathcal{M}$ with $N$ nodes.

In the proof of Proposition 1 we mentioned Theorem 2.12 in [3]. This is a result on numerical integration for functions in Sobolev spaces and it says that if $\{x_j\}_{j=1}^N$ is an $L$-design on a compact Riemannian manifold $\mathcal{M}$, then for every $1 \leq p \leq +\infty$ and for every $\alpha > d/p$ there exists a constant $\beta > 0$ such that for every $f$ in the Sobolev space $W^{\alpha,p}(\mathcal{M})$

$$\left| \int_{\mathcal{M}} f(x) \, d\mu(x) - \frac{1}{N} \sum_{j=1}^N f(x_j) \right| \leq \beta L^{-\alpha} \|f\|_{\alpha,p}.$$

By the above Theorem 6 if $\mathcal{M}$ is a connected compact oriented $d$-dimensional Riemannian manifold, for every positive integer $N$, setting $L = (N/C_M)^{1/d}$ there indeed exists an $L$-design on $\mathcal{M}$ consisting of $N$ nodes, and this immediately gives the following result on the worst case error in numerical integration

**Corollary 7.** For every $1 \leq p \leq +\infty$ and for every $\alpha > d/p$ there exists a constant $\beta > 0$ such that for every $N \geq 1$ there exists a collection of points $\{x_j\}_{j=1}^N$ such that for every $f \in W^{\alpha,p}(\mathcal{M})$

$$\left| \int_{\mathcal{M}} f(x) \, d\mu(x) - \frac{1}{N} \sum_{j=1}^N f(x_j) \right| \leq \beta C_M^{\alpha/d} N^{-\alpha/d} \|f\|_{\alpha,p}.$$

By Theorem 2.16 in [3], the exponent $-\alpha/d$ is best possible. This result should be compared with Corollary 6.3, Corollary 6.4 and Example 6.5 in [4], where the authors prove that if $1 < p \leq +\infty$ and $d/p < \alpha < d$, then a random choice of nodes $x_j \in R_j$ gives the desired decay rate $N^{-\alpha/d}$ for the worst case error in numerical integration if and only if $\alpha < d/2 + 1$. See also [5, 6] for previous results in the case of the sphere.

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2. Introduction to Riemannian manifolds

The following are well known facts about manifolds. The interested reader can find all details in [1, 7].
A differentiable manifold of dimension $d$ is a set $\mathcal{M}$ and a family of injective maps $x_\alpha : U_\alpha \subset \mathbb{R}^d \to \mathcal{M}$ such that

1. The $U_\alpha$’s are open and $\bigcup_\alpha x_\alpha(U_\alpha) = \mathcal{M}$.
2. For any pair $\alpha, \beta$ with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open and the maps $x_\beta^{-1} \circ x_\alpha$ are $C^\infty$.
3. The family $\{(U_\alpha, x_\alpha)\}$ is maximal relative to the above conditions.

Each $(U_\alpha, x_\alpha)$ is called local chart, and a family $\{(U_\alpha, x_\alpha)\}$ satisfying (1) and (2) is called differentiable structure.

This induces a natural topology on $\mathcal{M}$. A set $A$ is open in $\mathcal{M}$ if and only if $x_\alpha^{-1}(A \cap x_\alpha(U_\alpha))$ is open in $\mathbb{R}^d$ for all $\alpha$. We will assume that with this topology $\mathcal{M}$ is a Hausdorff space with a countable basis.

We also say that a differentiable manifold $\mathcal{M}$ is orientable if

4. For every pair $\alpha, \beta$ with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$ the differential of the change of coordinates $x_\beta^{-1} \circ x_\alpha$ has positive determinant.

A map $f : \mathcal{N} \to \mathcal{M}$ between two differentiable manifolds is called differentiable in $p \in \mathcal{N}$ if for every local chart $(V, y)$ at $f(p)$ there exists a local chart $(U, x)$ at $p$ such that $f(x(U)) \subset y(V)$ and the map $y^{-1} \circ f \circ x$ is $C^\infty$. We say that $f$ is differentiable in an open set of $\mathcal{N}$ if it is differentiable at all points of this open set.

A differentiable map $\alpha$ from the interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ to $\mathcal{M}$ will be called a (differentiable) curve in $\mathcal{M}$. Suppose that $\alpha(0) = p$. Call $\mathcal{D}(\mathcal{M})$ the set of functions on $\mathcal{M}$ differentiable in $p$. The tangent vector to $\alpha$ at $t = 0$ is the function $\alpha'(0) : \mathcal{D}(\mathcal{M}) \to \mathbb{R}$ given by

$$\alpha'(0)f = \frac{d(f \circ \alpha)}{dt} \bigg|_{t=0}, \quad f \in \mathcal{D}(\mathcal{M}).$$

A tangent vector at $p$ is the tangent vector at $t = 0$ of a1 curve like the one above. The set of all tangent vectors to $\mathcal{M}$ at $p$ will be indicated with $T_p\mathcal{M}$. This is a vector space of dimension $d$ called tangent space of $\mathcal{M}$ at $p$, and the choice of a local chart $(U, x)$ around $p$ determines a natural basis of $T_p\mathcal{M}$ given by

$$\left\{ \left( \frac{\partial}{\partial x_i} \right)_p \right\}_{i=1}^d$$

where setting $x^0 = x^{-1}(p)$, $(\partial/\partial x_i)_p$ is the tangent vector to the curve $x(t), \ldots, x(t) + t, \ldots, x(t)$ at $t = 0$. The set $T\mathcal{M} = \{(p, v) : p \in \mathcal{M}, v \in T_p\mathcal{M}\}$ can be given a differentiable structure that makes it a differentiable manifold of dimension $2d$ called tangent bundle. The local charts are $(U_\alpha \times \mathbb{R}^d, y_\alpha)$, where

$$y_\alpha(x^0_1, \ldots, x^0_d, u_1, \ldots, u_d) = (x_\alpha(x^0_1, \ldots, x^0_d), \sum_{i=1}^d u_i \frac{\partial}{\partial x^0_i}).$$

A vector field $X$ on a differentiable manifold $\mathcal{M}$ is a map from the tangent bundle $T \mathcal{M}$ such that $X(p) = (p, v)$ for some $v \in T_p\mathcal{M}$. We call $\mathcal{X}(\mathcal{M})$ the space of differentiable vector fields on $\mathcal{M}$.

A Riemannian metric on a differentiable manifold $\mathcal{M}$ is a correspondence which associates to each point $p \in \mathcal{M}$ an inner product $(\cdot, \cdot)_p$ (that is, a symmetric, bilinear, positive definite form) on the tangent space $T_p\mathcal{M}$ which varies differentiably in the sense that if $(U, x)$ is a local chart around $p$ and if $q = x(x_1, \ldots, x_d)$ then

$$g_{ij}(x_1, \ldots, x_d) = \left\langle \left( \frac{\partial}{\partial x_i} \right)_q, \left( \frac{\partial}{\partial x_j} \right)_q \right\rangle_q$$

is a differentiable function on $U$. A differentiable manifold with a given Riemannian metric will be called a Riemannian manifold. If $v$ is a tangent vector to $\mathcal{M}$ at $p$,
we set \( \|v\|_p^2 = \langle v, v \rangle_p \). The length of a differentiable curve \( \alpha \) from the interval \([a, b]\) to \( M \) is defined as

\[
\int_a^b \|\alpha'(t)\|_{\alpha(t)} \, dt.
\]

We define the distance \( |p - q| \) between two points in a Riemannian manifold, \( p, q \in M \), as the infimum of the lengths of all the differentiable curves joining \( p \) and \( q \). This is indeed a distance, and it turns \( M \) into a metric space that has the same topology as the manifold's natural topology.

If \( M \) is a compact Riemannian manifold, then for any two points \( p \) and \( q \) in \( M \) there exists at least one differentiable curve \( \alpha \) joining \( p \) and \( q \) that realizes the infimum of the lengths of all the differentiable curves joining \( p \) and \( q \). Furthermore, the covariant derivative of \( \alpha' \) along \( \alpha \) equals zero (curves that satisfy this property are called geodesics). We refer the reader to [7] for the precise definition of covariant derivative, here it suffices to recall that if \( \alpha \) is a geodesic then \( \|\alpha'(t)\|_{\alpha(t)} \) is constant, and one can normalize \( \alpha \) in such a way that \( \|\alpha'(t)\|_{\alpha(t)} = 1 \).

Let \( (U, x) \) be a local chart where the metric has local representation given by \( g_{ij}(x_1, \ldots, x_d) \). The positive measure \( du_{(U, x)} = \sqrt{\det(g_{ij})} \, dx_1 \cdots dx_d \) on \( U \) induces a positive measure \( d\mu_{(U, x)} \) on \( x(U) \) given by

\[
\int_{x(U)} f \, d\mu_{(U, x)} = \int_U f \circ x \, du_{(U, x)}.
\]

If \( (U, x) \) and \( (V, y) \) are two local charts with \( U \cap V = W \neq \emptyset \), then one can show that \( d\mu_{(U, x)} \) coincides with \( d\mu_{(V, y)} \) on \( W \). By a standard partition of unit argument, there exists a unique measure \( d\mu \) on \( M \) that coincides with \( d\mu_{(U, x)} \) on \( U \) for all local charts \( (U, x) \). This measure is called canonical measure on \( M \).

It can be shown that there exist two positive constants \( c_4 \) and \( c_5 \) such that for any point \( x \in M \) and for any radius \( r \leq \text{diam}(M) \), the measure of the ball \( B(x, r) = \{ y \in M : |x - y| < r \} \) satisfies the inequalities

\[
c_4 r^d \leq \mu(B(x, r)) \leq c_5 r^d. \tag{1}
\]

It follows easily that there exists a positive constant \( c_6 \) such that if \( f : [0, +\infty) \to [0, +\infty) \) is a decreasing function and if \( x \in M \) then

\[
\int_M f(|x - y|) \, d\mu(y) \leq c_6 \int_0^{+\infty} f(t) t^{d-1} \, dt. \tag{2}
\]

For any \( f \in \mathcal{D}(M) \) we define the gradient of \( f \) as a vector field \( \nabla f \) on the Riemannian manifold \( M \) given by

\[
\langle \nabla f(p), v \rangle_p = v f, \quad p \in M, \ v \in T_p M.
\]

In local coordinates, the gradient is given by the formula

\[
\nabla f(p) = \sum_{j=1}^d \sum_{i=1}^d g^{ij} \left( \frac{\partial}{\partial x_i} f \right) \frac{\partial}{\partial x_j}.
\]

where \( g^{ij} \) are the entries of the inverse matrix of \( g_{ij} \). It follows from the definition that

\[
\|\nabla f\|_p = \sup X f
\]

where the supremum is taken over all the differentiable vector fields \( X \) with norm \( \|X\|_p \leq 1 \).

If \( f \in \mathcal{D}(M) \) and \( \alpha : [0, |p - q|] \to M \) is a normalized geodesic joining \( p \) and \( q \), then

\[
|f(p) - f(q)| = \left| \int_0^{|p - q|} \frac{d}{dt} (f(\alpha(t))) \, dt \right| = \left| \int_0^{|p - q|} \langle \nabla f(\alpha(t)), \alpha'(t) \rangle \, dt \right|
\]
on $\ell, m$ there exist two positive constants $c$ and $\|X\|$. See also [1, Chapter 3.E] for details on the Minakshisundaram-Pleijel asymptotic development of the heat kernel. See [12, Theorem 1.4.3 and the following remarks], and [14, Theorem 5.5] for an extension to general elliptic operators on manifolds of bounded geometry. See also [1] Chapter 3.E] for details on the Minakshisundaram-Pleijel asymptotic development of the heat kernel.

**Theorem 8.** Let $M$ be a compact Riemannian manifold of dimension $d$. Let $X_1, \ldots, X_\ell$ and $Y_1, \ldots, Y_m$ be differentiable vector fields on $M$ such that $\|X_j\|_x \leq 1$ and $\|Y_i\|_x \leq 1$ for all $x \in M$, for all $j = 1, \ldots, \ell$ and for all $i = 1, \ldots, m$. Then there exist two positive constants $c_\ell = c_\ell(\ell, m)$ and $c_m = c_m(\ell, m)$ depending only on $\ell, m$ (and on $M$) such that for all $t \in [0, 1]$ and for all $x, y \in M$,

$$\sum_{k=0}^{+\infty} \exp(-\lambda_k^m t) X_1 \cdots X_\ell \varphi_k(x) Y_1 \cdots Y_m \varphi_k(y) \leq c_\ell t^{-\frac{\ell+1}{2}} \exp \left(-c_m \frac{|x-y|^2}{t} \right).$$
3. Estimates on summability kernels

Here we recall certain definitions and results concerning general summability kernels for Bessel systems, following [9].

Definition 9. A system \( \{\phi_k\}_{k=0}^{+\infty} \subset L^2(\mathcal{M}) \) will be called a generalized Bessel system if for any \( g \in \mathcal{D}(\mathcal{M}) \),

\[
\mathcal{N}(g) := \sum_{k=0}^{+\infty} \left| \int_{\mathcal{M}} g \phi_k d\mu \right|^2 < +\infty.
\]

Any Bessel system, that is a system \( \{\phi_k\}_{k=0}^{+\infty} \) such that for all \( f \in L^2(\mathcal{M}) \) one has \( \sum_{k=0}^{+\infty} \left| \int_{\mathcal{M}} f \phi_k d\mu \right|^2 \leq c \|f\|^2 \), is clearly a generalized Bessel system according to the above definition. In particular, any orthonormal system is a generalized Bessel system. We will also use the following type of generalized Bessel systems. Let \( X \) be a differentiable vector field on \( \mathcal{M} \) and set \( \phi_k = X \varphi_k \), where \( \{\varphi_k\}_{k=0}^{+\infty} \) are the eigenfunctions of the Laplace-Beltrami operator described before. By Green’s formula \( [4] \), for any \( g \in \mathcal{D}(\mathcal{M}) \)

\[
\sum_{k=0}^{+\infty} \left| \int_{\mathcal{M}} g(X \varphi_k) d\mu \right|^2 = \sum_{k=0}^{+\infty} \left| -\int_{\mathcal{M}} \text{div}(gX) \varphi_k d\mu \right|^2 = \|\text{div}(gX)\|_2^2 < +\infty.
\]

Similarly, if \( X_1, X_2 \) are differentiable vector fields on \( \mathcal{M} \) and we set \( \phi_k = X_1X_2 \varphi_k \), then

\[
\sum_{k=0}^{+\infty} \left| \int_{\mathcal{M}} g(X_1X_2 \varphi_k) d\mu \right|^2 = \|\text{div}(\text{div}(gX_1)X_2)\|_2^2 < +\infty,
\]

and so on for any number of vector fields.

Here is the main result of this section.

Theorem 10 (Theorem 2.1 in [9]). Let \( \{\phi_k\}, \{\psi_k\} \) be generalized Bessel systems composed by continuous functions, and assume that there exist positive constants \( \kappa_1, \kappa_2, \kappa_3, \kappa_4, A_1, A_2, A_3 \) such that for all \( t \in (0, 1] \) and \( x, y \in \mathcal{M} \)

\[
\sum_{k=0}^{+\infty} \exp(-\lambda_k^2 t) |\phi_k(x)|^2 \leq \kappa_1 t^{-A_1/2},
\]

\[
\sum_{k=0}^{+\infty} \exp(-\lambda_k^2 t) |\psi_k(x)|^2 \leq \kappa_2 t^{-A_2/2},
\]

\[
\left| \sum_{k=0}^{+\infty} \exp(-\lambda_k^2 t) \phi_k(x) \psi_k(y) \right| \leq \kappa_3 t^{-A_3} \exp(-\kappa_4 |x-y|^2/t).
\]

Let \( K = (A_1 + A_2)/2 \) and \( S > \max\{K, d\} \) integer. Let \( H : \mathbb{R} \to \mathbb{R} \) be an even function supported on \([-1, 1]\) with continuous derivatives up to order \( S \). Then there exists a positive constant \( C_9 \) such that for all \( x, y \in \mathcal{M} \) and for all \( L > 0 \),

\[
\left| \sum_{k=0}^{+\infty} H(\lambda_k/L) \phi_k(x) \psi_k(y) \right| \leq C_9 \frac{L^K}{(1 + L|x-y|)^S}.
\]

The constant \( C_9 \) does not depend on the specific Bessel systems \( \{\phi_k\} \) and \( \{\psi_k\} \) nor on the manifold \( \mathcal{M} \), but only on the constants \( \kappa_1, \kappa_2, \kappa_3, \kappa_4, A_1, A_2, A_3 \), and on the function \( H \).

In particular, assume that the generalized Bessel systems are given by

\[
\phi_k = X_1 \ldots X_l \varphi_k, \quad \psi_k = Y_1 \ldots Y_m \varphi_k
\]

where \( X_1, \ldots, X_l \) and \( Y_1, \ldots, Y_m \) are differentiable vector fields on \( \mathcal{M} \) such that \( \|X_j\|_x \leq 1 \) and \( \|Y_i\|_x \leq 1 \) for all \( x \in \mathcal{M} \), for all \( j = 1, \ldots, l \) and for all \( i = 1, \ldots, m \).
Then, by Theorem \[5\] the above Theorem \[10\] applies with \(\kappa_1 = c_7(\ell, \ell)\), \(\kappa_2 = c_7(m, m)\), \(\kappa_3 = c_7(\ell, m)\), \(\kappa_4 = c_8(\ell, m)\), \(A_1 = d + 2\ell\), \(A_2 = d + 2m\), and \(A_3 = (d + \ell + m)/2\).

Theorem \[10\] follows from a series of results, the first being Theorem 4.1 again in \[9\]. The latter is a nice elementary result on holomorphic functions, which is a simplified version of a result of A. Sikora \[19\] Theorem 2. Unfortunately, the proof of Theorem 4.1 presented in \[9\] is incomplete, as inequality (4.12) is not properly justified. Actually one needs to use, as A. Sikora does in his original proof, some version of the Phragmén-Lindelöf theorem. We give here a complete proof of this result.

**Theorem 11** (Theorem 4.1 in \[9\]). Let \(r > 0\), \(\{\alpha_k\}\) be an absolutely summable sequence of complex numbers, \(\{\ell_k\}\) be a sequence of nonnegative, nondecreasing numbers with \(\ell_k \to \infty\), and

\[
K(t) = \sum_{k=0}^{\infty} \exp(-\ell_k t) \alpha_k, \quad W(t) = \sum_{k=0}^{\infty} \cos(\ell_k t) \alpha_k.
\]

Then

\[
|K(t)| \leq \alpha t^{-\beta} \exp(-r^2/t) \sum_{k=0}^{\infty} |\alpha_k|, \quad t \in (0, 1]
\]

if and only if \(W(t) = 0\) for \(0 \leq t \leq 2r\).

**Proof.** Assume without loss of generality that \(\sum_{k=0}^{\infty} |\alpha_k| = 1\). By the well known formula on the Fourier transform of the Gaussian function, for all \(t > 0\) and for all \(k = 0, 1, 2, \ldots\)

\[
\exp(-\ell_k^2 t) = \frac{1}{\sqrt{\pi t}} \int_{0}^{+\infty} \exp(-u^2/(4t)) \cos(\ell_k u) \, du
\]

and, by the absolute and uniform convergence of the series defining \(W(t)\),

\[
K(t) = \frac{1}{\sqrt{\pi}} \int_{0}^{+\infty} \exp(-u^2/(4t)) W(u) \, du \quad (5)
\]

Assume now \(W(u) = 0\) for all \(0 \leq u \leq 2r\). If \(t \geq r^2\) then

\[
|K(t)| \leq 1 \leq \exp(1 - r^2/t) = e \exp(-r^2/t).
\]

If \(t < r^2\) then, by \(5\),

\[
|K(t)| \leq \frac{1}{\sqrt{\pi t}} \int_{2r}^{+\infty} \exp(-u^2/(4t)) W(u) \, du
\]

\[
\leq \frac{1}{\sqrt{\pi t}} \int_{2r}^{+\infty} \exp(-u^2/(4t)) \, du
\]

\[
= \frac{1}{\sqrt{\pi}} \int_{r^2/t}^{+\infty} u^{-1/2} \exp(-u) \, du
\]

\[
\leq \frac{1}{\sqrt{\pi}} \left( \frac{t}{r^2} \right)^{1/2} \exp\left(\frac{-r^2}{t}\right) \leq \frac{1}{\sqrt{\pi}} \exp\left(\frac{-r^2}{t}\right),
\]

and the thesis follows with \(\alpha = e\) and \(\beta = 0\).

Assume now that

\[
|K(t)| \leq \alpha t^{-\beta} \exp(-r^2/t), \quad t \in (0, 1].
\]

Then for any \(\varepsilon > 0\) there exists a positive constant \(\gamma\) such that

\[
|K(t)| \leq \gamma \exp\left(-\varepsilon (r - \varepsilon)^2/t\right), \quad t \in (0, 1]. \quad (6)
\]
For any complex $t = \tau + i\xi$ with $\tau \geq 0$, define the function

$$F(t) = \begin{cases} \frac{t}{1 + t} \exp \left( 4 (r - \varepsilon)^2 \tau \right) \sum_{k=0}^{+\infty} \exp \left( -\ell_k^2 \tau / (4t) \right) a_k & t \neq 0, \tau \geq 0 \\ 0 & t = 0. \end{cases}$$

It is easy to show that $F$ satisfies the hypotheses of the Phragmén-Lindelöf theorem (see [20, Theorem 3.4, page 124]) in $\{ -\frac{\pi}{2} < \arg(t) < 0 \}$ and in $\{ 0 < \arg(t) < \frac{\pi}{2} \}$. In particular, by [20], for all $\tau > 0$

$$|F(\tau)| \leq \begin{cases} \frac{\tau}{1 + \tau} \exp \left( 4 (r - \varepsilon)^2 \tau \right) \gamma \exp \left( - (r - \varepsilon)^2 4 \tau \right) & \text{if } 1/(4\tau) \leq 1 \\ \frac{\tau}{1 + \tau} \exp \left( 4 (r - \varepsilon)^2 \tau \right) & \text{if } 1/(4\tau) \geq 1 \\ \gamma & \text{if } \tau \geq 1/4 \\ \frac{1}{5} \exp \left( (r - \varepsilon)^2 \right) & \text{if } \tau \leq 1/4 \\ \leq \max \left\{ \gamma, \frac{1}{5} \exp \left( r^2 \right), 1 \right\} =: \eta. \end{cases}$$

Also, for all $\xi \in \mathbb{R} \setminus \{0\}$,

$$|F(i\xi)| \leq \frac{|\xi|}{|1 + i\xi|} \sum_{k=0}^{+\infty} |a_k| \leq 1 \leq \eta.$$

The function $F$ is continuous on $\{ \Re(t) \geq 0 \}$. The continuity outside 0 follows by the absolute and uniform convergence of the series defining $F$ in any compact subset of $\{ \Re(t) \geq 0 \} \setminus \{ 0 \}$. The continuity in 0 follows by the estimate, for $\Re(t) \geq 0$ and $0 < |t| < \delta$,

$$|F(t)| \leq |t| \exp \left( 4 (r - \varepsilon)^2 \frac{\delta}{2} \right) \sum_{k=0}^{+\infty} \exp \left( -\ell_k^2 \tau / \left( 4 |t|^2 \right) \right) |a_k| \leq |t| \exp \left( 4 (r - \varepsilon)^2 \delta \right).$$

Again by the absolute and uniform convergence of the series defining $F$, it follows that $F$ is holomorphic in $\{ \Re(t) > 0 \}$. Finally, for all $\Re(t) > 0$,

$$|F(t)| \leq \exp \left( 4 (r - \varepsilon)^2 \tau \right) \sum_{k=0}^{+\infty} \exp \left( -\ell_k^2 \tau / \left( 4 (r^2 + \xi^2) \right) \right) |a_k| \leq \exp \left( 4 (r - \varepsilon)^2 \tau \right) \leq \exp \left( 4 (r - \varepsilon)^2 |t| \right).$$

It therefore follows by the Phragmén-Lindelöf theorem that

$$|F(t)| \leq \eta, \quad \forall t : \Re(t) \geq 0.$$

The proof now follows as in [9]. Changing variables in [53] we obtain

$$\frac{1}{\sqrt{4\pi}} K \left( \frac{1}{4\tau} \right) = \int_{-\infty}^{+\infty} \exp \left( -ur \right) g(u) du$$

where

$$g(u) = \begin{cases} \frac{1}{\sqrt{4\pi u}} W(\sqrt{u}) & \text{if } u > 0 \\ 0 & \text{if } u \leq 0. \end{cases}$$
By the definition of $F$ we have for $\tau > 0$

$$\begin{aligned}
\frac{1}{\sqrt{4\pi}} K \left( \frac{1}{4\tau} \right) &= \frac{1}{\sqrt{4\pi}} \frac{1 + \tau}{\tau} \exp \left( -4 (r - e)^2 \tau \right) F(\tau),
\end{aligned}$$

which can be extended analytically to $\{\text{Re}(t) > 0\}$. Also $\int_{-\infty}^{+\infty} \exp(-ut) g(u) du$ can be extended analytically to $\{\text{Re}(t) > 0\}$, and the identity

$$\begin{aligned}
\frac{1}{(4t)^{1/2}} \frac{1 + t}{t} \exp \left( -4 (r - e)^2 t \right) F(t) &= \int_{-\infty}^{+\infty} \exp(-ut) g(u) du
\end{aligned} \quad (7)$$

holds in $\{\text{Re}(t) > 0\}$. Let now $\phi \in C^\infty(\mathbb{R})$ with support in $[0, b]$. Then

$$\begin{aligned}
\hat{\phi}(t) &= \int_{-\infty}^{+\infty} \phi(u) e^{-iut} du
\end{aligned}$$

is entire and a repeated integration by parts gives

$$\begin{aligned}
\hat{\phi}(-\xi + i\tau) &= \int_{-\infty}^{+\infty} \phi(u) e^{-iu(-\xi + i\tau)} du \\
&= \frac{(-1)^R}{(-i(\xi + i\tau))^R} \int_{-\infty}^{+\infty} \phi(R)(u) e^{-iu(-\xi + i\tau)} du \\
&= \frac{(-1)^R}{(\tau + i\xi)^R} \int_{-\infty}^{+\infty} \phi(R)(u) e^{(\tau + i\xi)u} du,
\end{aligned}$$

so that

$$\begin{aligned}
\left|\hat{\phi}(-\xi + i\tau)\right| &\leq \max \left( 1, \frac{e^{\tau b}}{(\tau^2 + \xi^2)^{R/2}} \right) \int_{-\infty}^{+\infty} \left|\phi(R)(u)\right| du.
\end{aligned} \quad (8)$$

Thus, for any $\tau > 0$,

$$\begin{aligned}
\int_{\mathbb{R}} g(u) \phi(u) du &= \int_{\mathbb{R}} e^{-ut} g(u) \{ e^{ut} \phi(u) \} du \\
&= \int_{\mathbb{R}} e^{-ut} g(u) \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi u} \left( \frac{e^{-i\xi u}}{\xi + it} \right) d\xi \right\} du \\
&= \int_{\mathbb{R}} e^{-ut} g(u) \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi u} \int_{\mathbb{R}} e^{it\xi} \phi(s) e^{-is\xi} d\xi ds du \\
&= \int_{\mathbb{R}} e^{-ut} g(u) \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi u} \int_{\mathbb{R}} \phi(s) e^{-is(\xi + i\tau)} d\xi ds du \\
&= \int_{\mathbb{R}} e^{-ut} g(u) \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} \hat{\phi}(\xi + i\tau) d\xi du.
\end{aligned}$$

One can apply Fubini’s theorem, since

$$\begin{aligned}
\left| e^{-ut} g(u) e^{-i\xi u} \hat{\phi}(\xi + i\tau) \right| &\leq e^{-ut} \left| g(u) \right| \frac{e^{\tau b}}{(\tau^2 + \xi^2)^{R/2}} \int_{-\infty}^{+\infty} \left|\phi(R)(u)\right| du,
\end{aligned}$$

so that by $(7)$

$$\begin{aligned}
\int_{\mathbb{R}} g(u) \phi(u) du &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}(\xi + i\tau) \int_{\mathbb{R}} g(u) e^{-u(\tau + i\xi)} du d\xi.
\end{aligned}$$
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\phi}(-\xi + i\tau) \frac{1}{(4(\tau + i\xi))^1/2} \frac{1 + \tau + i\xi}{\tau + i\xi} e^{-4(r-\varepsilon)^2(\tau + i\xi)} F(\tau + i\xi) d\xi.
\]

It follows from this and (8) that

\[
\left| \int_{\mathbb{R}} g(u) \phi(u) du \right| \leq \frac{\eta}{4\pi} e^{-4(r-\varepsilon)^2\tau} \int_{\mathbb{R}} \left| \widehat{\phi}(-\xi + i\tau) \right| \left| \frac{1 + \tau + i\xi}{\tau + i\xi} \right| d\xi
\]

\[
\leq \frac{\eta}{4\pi} e^{-4(r-\varepsilon)^2\tau} \int_{-\infty}^{+\infty} \left| \phi(R)(u) \right| du \int_{\mathbb{R}} \left( (1 + \tau)^2 + \xi^2 \right)^{1/2} \left( \tau^2 + \xi^2 \right)^{-R/2 - 3/4} d\xi
\]

which goes to 0 as \( \tau \to +\infty \) if \( b \leq 4(r-\varepsilon)^2 \). By the arbitrarity of \( \phi \) it follows that \( g(u) = 0 \) for \( 0 \leq u \leq 4(r-\varepsilon)^2 \), and by the arbitrarity of \( \varepsilon > 0 \), \( g(u) = 0 \) for \( 0 \leq u \leq 4r^2 \), so that \( W(t) = 0 \) for \( 0 \leq t \leq 2r \).

The next step in the proof of Theorem 11, again following [9], is

**Corollary 12** (Corollary 4.1 in [9]). Let \( G : \mathbb{R} \to \mathbb{R} \) be an even, bounded, integrable function such that the Fourier transform \( \widehat{G} \) is also integrable and supported on \([-2r, 2r]\). In the assumptions of Theorem 11, if

\[
|K(t)| \leq a t^{-\beta} \exp\left( -r^2/t \right) \sum_{k=0}^{\infty} |a_k|, \quad t \in (0, 1]
\]

then

\[
\sum_{k=0}^{\infty} G(\ell_k) a_k = 0.
\]

**Proof.** By the Fourier inversion formula,

\[
G(u) = \frac{1}{\pi} \int_{0}^{+\infty} \widehat{G}(t) \cos(tu) dt.
\]

Thus,

\[
\sum_{k=0}^{\infty} G(\ell_k) a_k = \frac{1}{\pi} \int_{0}^{+\infty} \widehat{G}(t) W(t) dt = 0
\]

by Theorem 11. \( \square \)

Let now \( V : \mathbb{R} \to \mathbb{R} \) be an even function such that \( \widehat{V} \) is infinitely differentiable with \( \widehat{V}(u) = 1 \) when \( |u| \leq 1/2 \) and \( \widehat{V}(u) = 0 \) when \( |u| \geq 1 \). Then for any \( Y > 0 \) let \( H_Y \) be defined by

\[
\widehat{H}_Y(u) = \widehat{H}(u) \widehat{V}(u/Y).
\]

**Lemma 13** (Lemma 4.3 in [9], Lemma 6.1 in [17]). In the assumptions of Theorem 11, there is a constant \( c_{10} > 0 \) (depending on \( \kappa_1, \kappa_2, \kappa_3, A_1, A_2, A_3 \) and on the function \( H \)) such that for all \( Y \geq 1/2 \) and for all \( x, y \in \mathcal{M} \)

\[
\left| \sum_{k=0}^{\infty} (H(\lambda_k/L) - H_Y(\lambda_k/L)) \phi_k(x) \psi_k(y) \right| \leq c_{10} L^K Y^{-\alpha}.
\]

The interested reader can find the proof in the above mentioned references.

**Proof of Theorem 11.** Again, we follow Filbir and Mhaskar [9, Proof of Theorem 2.1 page 646]. By the hypotheses, for all \( L \geq 1 \) we have

\[
\sum_{\lambda_k \leq L} |\phi_k(x)|^2 \leq \sum_{k=0}^{\infty} \exp(1 - \lambda_k^2/L^2) |\phi_k(x)|^2 \leq c_{14} L^{A_1},
\]

where \( c_{14} \) depends on \( \mu, \kappa_1, \kappa_2, \kappa_3, A_1, A_2, A_3 \). Then

\[
\sum_{\lambda_k \geq L} |\phi_k(x)|^2 \leq \sum_{k=0}^{\infty} \exp(1 - \lambda_k^2/L^2) |\phi_k(x)|^2 \leq c_{14} L^{A_1}.
\]

By the hypothesis, for all \( L \geq 1 \) we have

\[
\sum_{\lambda_k \geq L} |\phi_k(x)|^2 \leq \sum_{k=0}^{\infty} \exp(1 - \lambda_k^2/L^2) |\phi_k(x)|^2 \leq c_{14} L^{A_1},
\]
and similarly,

$$\sum_{\lambda_k \leq L} |\psi_k(x)|^2 \leq e^{\kappa_2} L^{A_2}.$$ 

Hence, by the Cauchy-Schwarz inequality

$$\sum_{k=0}^{+\infty} H(\lambda_k/L) \left| \phi_k(x) \psi_k(y) \right| \leq \max_{t \in \mathbb{R}} |H(t)| \left( \sum_{\lambda_k \leq L} |\phi_k(x)|^2 \right)^{1/2} \left( \sum_{\lambda_k \leq L} |\psi_k(y)|^2 \right)^{1/2}$$

$$\leq e^{(\kappa_1 \kappa_2)^{1/2}} \max_{t \in \mathbb{R}} |H(t)| L^K. \quad (9)$$

This proves the theorem when $L|x-y| \leq 1$. Assume now $L|x-y| \geq 1$ and let $Y := \sqrt{\kappa_2} L|x-y|$. Let $f_1, f_2 \in L^1(M)$ with $\|f_1\|_1 = \|f_2\|_1 = 1$ be supported in the balls centered at $x$ and $y$ respectively, and with radii $|x-y|/8$. For any $\varepsilon > 0$ there exist two functions $g_1, g_2 \in \mathcal{C}^\infty(M)$ supported in the balls centered at $x$ and $y$ respectively, and with radii $|x-y|/4$ such that $\|f_1 - g_1\|_1 < \varepsilon$ and $\|f_2 - g_2\|_1 < \varepsilon$. Therefore, by (9) and Lemma 13

$$\left| \sum_{k=0}^{+\infty} H(\lambda_k/L) \int_M \int_M \phi_k(w) \psi_k(z) f_1(w) f_2(z) d\mu(w) d\mu(z) \right|$$

$$\leq \sum_{k=0}^{+\infty} H(\lambda_k/L) \int_M \int_M \phi_k(w) \psi_k(z) f_1(w) f_2(z) - g_2(z) d\mu(w) d\mu(z)$$

$$+ \sum_{k=0}^{+\infty} H(\lambda_k/L) \int_M \int_M \phi_k(w) \psi_k(z) (f_1(w) - g_1(w)) g_2(z) d\mu(w) d\mu(z)$$

$$+ \sum_{k=0}^{+\infty} H(\lambda_k/L) \int_M \int_M \phi_k(w) \psi_k(z) g_1(w) g_2(z) d\mu(w) d\mu(z)$$

$$\leq \sum_{k=0}^{+\infty} H(\lambda_k/L) \int_M \int_M \phi_k(w) \psi_k(z) g_1(w) g_2(z) d\mu(w) d\mu(z)$$

$$+ e^{(\kappa_1 \kappa_2)^{1/2}} \varepsilon \max_{t \in \mathbb{R}} |H(t)| (\|f_1\|_1 + \|g_2\|_1) L^K$$

$$\leq \sum_{k=0}^{+\infty} H_Y(\lambda_k/L) \int_M \int_M \phi_k(w) \psi_k(z) g_1(w) g_2(z) d\mu(w) d\mu(z)$$

$$+ c_{10} L^K Y^{-S} \|g_1\|_1 \|g_2\|_1 + e^{(\kappa_1 \kappa_2)^{1/2}} \varepsilon \max_{t \in \mathbb{R}} |H(t)| (\|f_1\|_1 + \|g_2\|_1) L^K.$$

The distance between the supports of $g_1$ and $g_2$ exceeds $|x-y|/2$ and therefore for all $t \in (0, 1]$

$$\left| \sum_{k=0}^{+\infty} \exp(-\lambda_k^2 t) \int_M \int_M \phi_k(w) \psi_k(z) g_1(w) g_2(z) d\mu(w) d\mu(z) \right|$$

$$\leq \kappa_3 t^{-A_3} \exp(-\kappa_4 |x-y|^2/(4t)) \|g_1\|_1 \|g_2\|_1. \quad (10)$$

Observe that since $\{\phi_k\}$ and $\{\psi_k\}$ are generalized Bessel systems, then

$$\sum_{k=0}^{+\infty} \left| \int_M \int_M \phi_k(w) \psi_k(z) g_1(w) g_2(z) d\mu(w) d\mu(z) \right| \leq \left( \mathcal{N}_{\phi_k} (g_1) \mathcal{N}_{\psi_k} (g_2) \right)^{1/2}.$$

We may therefore apply Corollary 12 with $r = \sqrt{\kappa_4} |x-y|/2$, $G(u) = H_Y(u/L)$ and

$$a_k = \int_M \int_M \phi_k(w) \psi_k(z) g_1(w) g_2(z) d\mu(w) d\mu(z).$$
By (10),
\[ \sum_{k=0}^{+\infty} H_{\lambda_k} \int_{M} \int_{M} \phi_k(w) \psi_k(z) g_1(w) g_2(z) d\mu(w) d\mu(z) = 0. \]

Therefore
\[ \left| \sum_{k=0}^{+\infty} H_{\lambda_k} \int_{M} \int_{M} \phi_k(w) \psi_k(z) f_1(w) f_2(z) d\mu(w) d\mu(z) \right| \]
\[ \leq c_{10} L^{K} Y^{-S} \| g_1 \|_1 \| g_2 \|_1 + \epsilon (\kappa_1 \kappa_2)^{1/2} \max_{t \in \mathbb{R}} |H(t)| (\| f_1 \|_1 + \| g_2 \|_1) L^K \]
\[ \leq c_{10} (1 + \epsilon)^2 L^{K} Y^{-S} + \epsilon (\kappa_1 \kappa_2)^{1/2} (2 + \epsilon) \max_{t \in \mathbb{R}} |H(t)| L^K. \]

By the arbitrarity of \( \epsilon > 0 \) this gives
\[ \sum_{k=0}^{+\infty} H_{\lambda_k} \int_{M} \int_{M} \phi_k(w) \psi_k(z) f_1(w) f_2(z) d\mu(w) d\mu(z) \leq c_{10} L^{K} Y^{-S}, \]
and by the arbitrarity of \( f_1 \) and \( f_2 \) and the continuity of \( \phi_k \) and \( \psi_k \),
\[ \sum_{k=0}^{+\infty} H_{\lambda_k} \phi_k(x) \psi_k(y) \leq C_{2} L^{K} Y^{-S}. \]

\( \square \)

4. Proof of Theorem [5]

This proof follows the lines of the corresponding proof of Theorem [4] by Filbir and Mhaskar as found in [10], properly modified to treat the case of the gradients.

Fix \( \epsilon > 0 \), and let \( v_\epsilon : [0, +\infty) \to \mathbb{R} \) be a \( C^\infty \) function such that \( v_\epsilon (u) = u \) for \( u \geq \epsilon \), \( v_\epsilon (u) = \epsilon/2 \) for \( u < \epsilon/4 \) and \( v_\epsilon (u) \geq u \) for all \( u \geq 0 \). Let \( P \in \Pi^0_T \) and define the differentiable vector field
\[ T(x) := \frac{\nabla P(x)}{v_\epsilon (\| \nabla P(x) \|)}. \]

Then
\[ TP(x) = \left\langle \nabla P(x), \frac{\nabla P(x)}{v_\epsilon (\| \nabla P(x) \|)} \right\rangle = \frac{\| \nabla P(x) \|^2}{v_\epsilon (\| \nabla P(x) \|)} \leq \| \nabla P(x) \| \]
and
\[ \left| \int_{M} \| \nabla P(x) \| d\mu(x) - \sum_{j=1}^{N} \frac{1}{N} \| \nabla P(x_j) \| \right| \]
\[ \leq \left| \int_{M} (\| \nabla P(x) \| - TP(x)) d\mu(x) \right| + \left| \int_{M} TP(x) d\mu(x) - \sum_{j=1}^{N} \frac{1}{N} TP(x_j) \right| \]
\[ + \sum_{j=1}^{N} \frac{1}{N} (TP(x_j) - \| \nabla P(x_j) \|) \]
\[ \leq 2\epsilon + \left| \int_{M} TP(x) d\mu(x) - \sum_{j=1}^{N} \frac{1}{N} TP(x_j) \right|. \]
Let us now call \( \delta \) the maximum diameter of the balls \( X_j \), so that \( \delta \leq 2c_2 N^{-1/d} \).

Now
\[
\left| \int_M TP(x) \ d\mu(x) - \sum_{j=1}^N \frac{1}{N} TP(x) \right| \leq \sum_{j=1}^N \int_{R_j} |TP(x) - TP(x)| \ d\mu(x) \\
\leq \sum_{j=1}^N \frac{1}{N} \sup_{x,z \in R_j} |TP(x) - TP(z)|
\]

By (3), the last term can be bounded above by
\[
\sum_{j=1}^N \frac{1}{N} \sup_{x,z \in R_j} \sup_{t \in [0,|x-z|]} \| \nabla TP(\alpha(t)) \| \ |x-z|.
\]

where \( \alpha \) is a normalized geodesic joining \( x \) and \( z \). Since \( R_j \) is contained in the ball \( X_j \), the geodesic \( \alpha \) is contained in the ball \( 2X_j \) with the same center as \( X_j \) and radius twice the radius of \( X_j \). It follows that the last term is bounded above by
\[
\delta \sum_{j=1}^N \frac{1}{N} \sup_{x \in 2X_j} \| \nabla TP(x) \|.
\]

Defining now the vector field
\[
S(x) := \frac{\nabla TP(x)}{v_\varepsilon (\| \nabla TP(x) \|)}
\]

we have as before
\[
STP(x) = \frac{1}{v_\varepsilon (\| \nabla TP(x) \|)} \leq \| \nabla TP(x) \|
\]

and therefore
\[
\delta \sum_{j=1}^N \frac{1}{N} \sup_{x \in 2X_j} \| \nabla TP(x) \|
\leq \delta \sum_{j=1}^N \frac{1}{N} \sup_{x \in 2X_j} \| \nabla TP(x) \| - STP(x) | + \delta \sum_{j=1}^N \frac{1}{N} \sup_{x \in 2X_j} |STP(x)|
\leq \delta \varepsilon + \delta \sum_{j=1}^N \frac{1}{N} \sup_{x \in 2X_j} |STP(x)|.
\]

So far we have obtained the inequality
\[
\left| \int_M \| \nabla P(x) \| \ d\mu(x) - \sum_{j=1}^N \frac{1}{N} \| \nabla P(x) \| \right| \leq (2 + \delta) \varepsilon + \delta \sum_{j=1}^N \frac{1}{N} \sup_{x \in 2X_j} |STP(x)|.
\]

Let \( h \) be a \( C^\infty \) even function such that \( h(u) \) equals 1 for \( u \in [-1,1] \) and \( h(u) \) equals 0 for \( |u| \geq 2 \). For any \( L \geq 0 \), define the kernels
\[
W_L(x,y) = \sum_{0 < \lambda_k} \frac{1}{\lambda_k^2} h \left( \frac{\lambda_k}{L} \right) \varphi_k(x) \varphi_k(y)
\]
\[
\Psi_L(x,y) = \Delta \varphi W_L(x,y) = \sum_{0 < \lambda_k} h \left( \frac{\lambda_k}{L} \right) \varphi_k(x) \varphi_k(y).
\]

Since \( \Psi_L(x,y) \) is a reproducing kernel for \( \Pi_{L}^0 \), we have by Green’s formula (4)
\[
P(x) = \int_M P(y) \Psi_L(x,y) \ d\mu(y) = \int_M P(y) \Delta \varphi W_L(x,y) \ d\mu(y)
\]
Thus
\[ \text{STP}(x) = \int_{\mathcal{M}} \langle \nabla_y P(y), \nabla_y S_x T_x W_L(x, y) \rangle \, d\mu(y) \]
and
\[ |\text{STP}(x)| = \left| \int_{\mathcal{M}} \langle \nabla_y P(y), \nabla_y S_x T_x W_L(x, y) \rangle \, d\mu(y) \right| \leq \int_{\mathcal{M}} \|\nabla_y P(y)\| \|\nabla_y S_x T_x W_L(x, y)\| \, d\mu(y). \]

We will show in a moment that
\[ \|\nabla_y S_x T_x W_L(x, y)\| \leq \kappa L^{d+1} (1 + L |x - y|)^{-d - 1}. \quad (11) \]

This inequality implies that
\[
\begin{align*}
\delta \sum_{j=1}^N \frac{1}{N} \sup_{x \in 2X_j} |\text{STP}(x)| &\leq \delta \sum_{j=1}^N \frac{1}{N} \sup_{x \in 2X_j} \int_{\mathcal{M}} \|\nabla_y P(y)\| \|\nabla_y S_x T_x W_L(x, y)\| \, d\mu(y) \\
&\leq \kappa \delta \sum_{j=1}^N \frac{1}{N} \sup_{x \in 2X_j} \int_{\mathcal{M}} \|\nabla_y P(y)\| L^{d+1} (1 + L |x - y|)^{-d - 1} \, d\mu(y) \\
&\leq \kappa \delta \int_{\mathcal{M}} \|\nabla_y P(y)\| \left\{ \sum_{j=1}^N \frac{L^{d+1}}{N} \sup_{x \in 2X_j} (1 + L |x - y|)^{-d - 1} \right\} \, d\mu(y).
\end{align*}
\]

For any fixed \( y \), let now \( J = \{ j : \text{dist}(2X_j, y) \geq 2\delta \} \) and \( J' \) its complement. It is easy to see that, calling \( q_j \) the point in \( 2X_j \) that is closest to \( y \), and \( p_j \) the point in \( 2X_j \) that is farthest from \( y \), then if \( j \in J \)
\[
1 + \frac{L}{2} |p_j - y| \leq 1 + \frac{L}{2} (|q_j - y| + 2\delta) \leq 1 + L |q_j - y|
\]
and therefore by (2)
\[
\begin{align*}
\sum_{j \in J} \frac{L^{d+1}}{N} \sup_{x \in 2X_j} (1 + L |x - y|)^{-d - 1} &\leq \sum_{j \in J} \frac{L^{d+1}}{N} \left( 1 + \frac{L}{2} |p_j - y| \right)^{-d - 1} \\
&\leq \sum_{j \in J} \int_{R_j} L^{d+1} \left( 1 + \frac{L}{2} |p_j - y| \right)^{-d - 1} \, d\mu(x) \\
&\leq \sum_{j \in J} \int_{R_j} L^{d+1} \left( 1 + \frac{L}{2} |x - y| \right)^{-d - 1} \, d\mu(x) \\
&\leq \int_{\mathcal{M}} L^{d+1} \left( 1 + \frac{L}{2} |x - y| \right)^{-d - 1} \, d\mu(x) \\
&\leq c_0 L^{d+1} \int_0^{+\infty} \left( 1 + \frac{L}{2} s \right)^{-d - 1} s^{d-1} ds.
\end{align*}
\]
\[
\leq c_6 L^{d+1} \left( \int_0^{1/L} s^{d-1} ds + \left( \frac{2}{L} \right)^{d+1} \int_{1/L}^{+\infty} s^{-2} ds \right)
\]
\[
\leq (d^{-1} + 2^{d+1}) c_6 L,
\]
Observe that the cardinality of \( J' \) is uniformly bounded with respect to \( y \) and \( N \). Indeed, the cardinality of \( J' \) is bounded above by the number of inner balls \( Y_j \) that are contained in the ball \( B(y, 4\delta) \), and this number is bounded above by the ratio
\[
\frac{\mu(B(y, 4\delta))}{\min_{j=1, \ldots, N} \mu(Y_j)} \leq \frac{c_5 (8c_2 N^{-1/d})^d}{c_4 (c_1 N^{-1/d})^d} = \frac{c_5 (8c_2)^d}{c_4 c_1^d}.
\]
Therefore, since \( L \leq N^{1/d} \),
\[
\sum_{j \in J'} \frac{L_{d+1}}{N} \sup_{x \in 2X_j} (1 + L |x - y|)^{-d-1} \leq \sum_{j \in J'} \frac{L_{d+1}}{N} \leq \frac{c_5 (8c_2)^d L_{d+1}}{c_4 c_1^d} \leq \frac{8^d c_5 c_2^d}{c_4 c_1^d} L.
\]
Overall
\[
\left| \int_\mathcal{M} \| \nabla P(x) \| \, d\mu(x) - \sum_{j=1}^N \frac{1}{N} \| \nabla P(x_j) \| \right| \leq (2 + \delta) \varepsilon + \kappa \left( (d^{-1} + 2^{d+1}) c_6 + \frac{8^d c_5 c_2^d}{c_4 c_1^d} \right) \delta L \int_\mathcal{M} \| \nabla P(y) \| \, d\mu(y).
\]
Taking
\[
\varepsilon = \frac{\kappa \left( (d^{-1} + 2^{d+1}) c_6 + \frac{8^d c_5 c_2^d}{c_4 c_1^d} \right) \delta L}{2 + \delta} \int_\mathcal{M} \| \nabla P(y) \| \, d\mu(y)
\]
we obtain
\[
\left| \int_\mathcal{M} \| \nabla P(x) \| \, d\mu(x) - \sum_{j=1}^N \frac{1}{N} \| \nabla P(x_j) \| \right| \leq C_3 LN^{-1/d} \int_\mathcal{M} \| \nabla P(y) \| \, d\mu(y),
\]
where
\[
C_3 = 4c_2 \kappa \left( (d^{-1} + 2^{d+1}) c_6 + \frac{8^d c_5 c_2^d}{c_4 c_1^d} \right).
\]
It remains to show (11). Since
\[
\nabla_y S_x T_x W_L (x, y) = \sum_{0 < \lambda_k \leq 2L} \frac{1}{\lambda_k^2} h \left( \frac{\lambda_k}{L} \right) ST \varphi_k (x) \nabla \varphi_k (y),
\]
it is enough to estimate
\[
\sum_{0 < \lambda_k \leq 2L} \frac{1}{\lambda_k^2} h \left( \frac{\lambda_k}{L} \right) ST \varphi_k (x) U \varphi_k (y)
\]
for a generic vector field \( U \) with \( \| U(x) \| = 1 \) for all \( x \in \mathcal{M} \). Since for all \( u > 0 \)
\[
h(u) = \sum_{j=0}^{+\infty} h(2^j u) - h(2^{j+1} u),
\]
we have
\[
\sum_{0 < \lambda_k} \frac{1}{\lambda_k} h \left( \frac{\lambda_k}{L} \right) ST \varphi_k (x) U \varphi_k (y)
\]
\[
= \sum_{j=0}^{+\infty} \left( \sum_{0 < \lambda_k} \frac{1}{\lambda_k} \left( h \left( 2^j \lambda_k \frac{\lambda_k}{L} \right) - h \left( 2^{j+1} \lambda_k \frac{\lambda_k}{L} \right) \right) \right) ST \varphi_k (x) U \varphi_k (y).
\]
Now apply Theorem 11 with \( \phi_k = ST \varphi_k, \psi_k = U \varphi_k, \kappa_1 = c_7 (2, 2), \kappa_2 = c_7 (1, 1), \kappa_3 = c_7 (2, 1), \kappa_4 = c_8 (2, 1), A_1 = d + 4, A_2 = d + 2, A_3 = (d + 3)/2, S > d + 3,$
$H(u) = u^{-2} (h(2u) - h(4u)) \in C^3$ and is supported in $\{1/4 \leq |u| \leq 1\}$. Finally replace $L$ in Theorem 10 with $2L/2^j$. Thus for $x, y \in \mathcal{M}$,

$$\sum_{0 < \lambda_k} \frac{1}{\lambda_k} \left( h \left( \frac{2j\lambda_k}{L} \right) - h \left( \frac{2j+1\lambda_k}{L} \right) \right) \sum \varphi_k(x) U \varphi_k(y)$$

$$= \left( \frac{2L}{2^j} \right)^{-2} \sum_{0 < \lambda_k} \left( \frac{2L}{2^j \lambda_k} \right)^2 \left( h \left( \frac{2j\lambda_k}{L} \right) - h \left( \frac{2j+1\lambda_k}{L} \right) \right) \sum \varphi_k(x) U \varphi_k(y)$$

$$= \left( \frac{2L}{2^j} \right)^{-2} \sum_{0 < \lambda_k} H \left( \frac{2j\lambda_k}{2L} \right) \sum \varphi_k(x) U \varphi_k(y)$$

$$\leq C_9 \left( \frac{2L}{2^j} \right)^{-2} \frac{(2L/2^j)^{d+3}}{(1 + 2L2^{-j}|x-y|)^S} = C_9 \frac{(2L2^{-j})^{d+1}}{(1 + 2L2^{-j}|x-y|)^S}.$$ 

Adding up in $j$, 

$$\sum_{j=0}^{+\infty} \left( \sum_{0 < \lambda_k} \frac{1}{\lambda_k} \left( h \left( \frac{2j\lambda_k}{L} \right) - h \left( \frac{2j+1\lambda_k}{L} \right) \right) \sum \varphi_k(x) U \varphi_k(y) \right) \leq C_9 \sum_{j=0}^{+\infty} (2L2^{-j})^{d+1} \leq 2^d + 2L^{d+1},$$

while if $L|x - y| \geq 1$ then 

$$\sum_{j=0}^{+\infty} \frac{(2L2^{-j})^{d+1}}{(1 + 2L2^{-j}|x-y|)^S} \leq \sum_{1 \leq 2^j \leq L|x-y|} \frac{(2L2^{-j})^{d+1}}{(1 + 2L2^{-j}|x-y|)^S} + \sum_{L|x-y| \leq 2^j} \frac{(2L2^{-j})^{d+1}}{(1 + 2L2^{-j}|x-y|)^S}$$

$$\leq \sum_{1 \leq 2^j \leq L|x-y|} \frac{(2L2^{-j})^{d+1}}{(2L2^{-j}|x-y|)^S} + \sum_{L|x-y| \leq 2^j} (2L2^{-j})^{d+1}$$

$$\leq 2^{d+2-S} L^{d+1-S} |x-y|^S |x-y|^{-S-d+1} + 2^{d+2} L^{d+1} L^{-d-1} |x-y|^{-d-1} \leq 2^{d+3} |x-y|^{-d-1}.$$ 

Overall 

$$\sum_{j=0}^{+\infty} \frac{(2L2^{-j})^{d+1}}{(1 + 2L2^{-j}|x-y|)^S} \leq 2^{d+3} L^{d+1} (1 + L |x-y|)^{-d-1}.$$ 

5. Proof of Theorem 13 

Let $\Omega$ be the open subset of the vector space $\Pi^0 L \subset L^2(\mathcal{M}, d\mu)$ 

$$\Omega = \left\{ P \in \Pi^0 L : \int_{\mathcal{M}} \|\nabla P(x)\|d\mu(x) < 1 \right\}.$$
Since \( \int_{\mathcal{M}} \|\nabla P(x)\| d\mu(x) \) is a norm in the finite dimensional space \( \Pi^0_L \), it is equivalent to the \( L^2 \) norm in \( \Pi^0_L \), and it follows that \( \Omega \) is bounded in \( \Pi^0_L \subset L^2(M) \), and the map from \( \Pi^0_L \subset L^2(M) \) to \( \mathbb{R} \) given by

\[
P \mapsto \int_{\mathcal{M}} \|\nabla P(x)\| d\mu(x),
\]

is continuous, so that \( \Omega \) is open.

**Lemma 14.** There exists a continuous map \( F : \Pi^0_L \to \mathcal{M}^N \) such that for every \( P \in \partial \Omega \)

\[
\sum_{j=1}^{N} P(x_j(P)) > 0,
\]

where \( F(P) = (x_1(P), \ldots, x_N(P)) \).

Let us first show that this lemma readily implies Theorem 3. By the Riesz representation theorem, for each point \( x \in \mathcal{M} \) there exists a unique polynomial \( G_x \in \Pi^0_L \) such that

\[
\langle G_x, P \rangle = P(x) \text{ for all } P \in \Pi^0_L.
\]

Then a set of points \( x_1, \ldots, x_N \in \mathcal{M} \) forms an \( L \)-design if and only if

\[
G_{x_1} + \cdots + G_{x_N} = 0.
\]

Now let \( Z : \mathcal{M}^N \to \Pi^0_L \) be the continuous map defined by

\[
Z(x_1, \ldots, x_N) = G_{x_1} + \cdots + G_{x_N},
\]

and call \( f = Z \circ F : \Pi^0_L \to \Pi^0_L \). Clearly for every \( P \in \partial \Omega \) we have

\[
\langle P, f(P) \rangle = \sum_{j=1}^{N} P(x_j(P)) > 0
\]

by the lemma, and by Theorem 3 it follows that there exists \( Q \in \Omega \) such that \( Z(F(Q)) = 0 \), that is such that \( G_{x_1(Q)} + \cdots + G_{x_N(Q)} = 0 \) which implies that \( \{x_1(Q), \ldots, x_N(Q)\} \) is an \( L \)-design.

**Proof of Lemma 14** Take a partition of \( \mathcal{M} \) as in Theorem 3 with constants \( c_1 \) and \( c_2 \) and let \( C_M \geq \max\{1, 2^d c_1 c_2, 2^d c_2 c_1, 13 c_2^d\} \), where the constants \( C_2(c, \ldots) \) are as in Theorem 3. For each \( j = 1, \ldots, N \) choose an arbitrary \( x_j \in B_j \).

Now fix \( \varepsilon < 1/4 \) and let as before \( v_\varepsilon : [0, +\infty) \to \mathbb{R} \) be a \( C^\infty \) function such that \( v_\varepsilon(u) = u \) for \( u \geq \varepsilon \), \( v_\varepsilon(u) = \varepsilon/2 \) for \( u < \varepsilon/4 \) and \( v_\varepsilon(u) \geq u \) for all \( u \geq 0 \).

Take the mapping \( U : \Pi^0_L \to \mathcal{X}(\mathcal{M}) \) defined by

\[
U(P)(y) = -\frac{\nabla P(y)}{v_\varepsilon(\|\nabla P(y)\|)}, \quad y \in \mathcal{M}.
\]

For each \( j = 1, \ldots, N \) let \( y_j : \Pi^0_L \times [0, +\infty) \to \mathcal{M} \) be the map satisfying the differential equation

\[
\frac{dy_j(P,t)}{dt} = U(P)(y_j(P,t))
\]

with the initial condition

\[
y_j(P,0) = x_j
\]

for each \( P \in \Pi^0_L \). Since the mapping \( U(p,y) \) is Lipschitz continuous in both \( P \) and \( y \), each \( y_j \) is well defined and continuous in \( P \) and \( t \). Now set

\[
F(P) = (x_1(P), \ldots, x_N(P)) := \left( y_1 \left( P, 12c_2 N^{-1/d} \right), \ldots, y_N \left( P, 12c_2 N^{-1/d} \right) \right),
\]

for each \( P \in \Pi^0_L \).
which is continuous on $\Pi_0^1$ by definition. Let now $P \in \partial \Omega$, that is
\[
\int_{\mathcal{M}} \|\nabla P(x)\|d\mu(x) = 1.
\]
We have
\[
\frac{1}{N} \sum_{j=1}^{N} P(x_j(P)) = \frac{1}{N} \sum_{j=1}^{N} P(\gamma_j(P, 12c_2N^{-1/d}))
\]
\[
= \frac{1}{N} \sum_{j=1}^{N} P(x_j) + \int_0^{12c_2N^{-1/d}} \frac{d}{dt} \left( \frac{1}{N} \sum_{j=1}^{N} P(\gamma_j(P, t)) \right) dt.
\]
Now,
\[
\left| \frac{1}{N} \sum_{j=1}^{N} P(x_j) \right| = \sum_{j=1}^{N} \int_{R_j} (P(x_j) - P(x))d\mu(x) \leq \sum_{j=1}^{N} \int_{R_j} |P(x_j) - P(x)|d\mu(x)
\]
\[
\leq \frac{1}{N} \sum_{j=1}^{N} \text{diam}(R_j) \max_{z \in 2X_j} \|\nabla P(z)\| \leq \frac{2c_2}{N^{1+1/d}} \sum_{j=1}^{N} \|\nabla P(z_j)\|
\]
where $z_j$ is the point that realizes the maximum. Observe that the partition $\mathcal{R}' = \{R'_1, \ldots, R'_N\}$ defined by $R'_j = R_j \cup \{z_j\}$ satisfies Theorem 5 with constants $c_1$ and $2c_2$. Therefore, by Theorem 5 applied to $P$ and the partition $\mathcal{R}'$, since $C_3(c_1, 2c_2)LN^{-1/d} \leq (C_ML^dN^{-1})^{1/d}/2 \leq 1/2$, we have
\[
\left| \frac{1}{N} \sum_{j=1}^{N} P(x_j) \right| \leq \frac{2c_2}{N^{1+1/d}} \sum_{j=1}^{N} \|\nabla P(z_j)\|
\]
\[
\leq \frac{2c_2}{N^{1/d}} \sum_{j=1}^{N} \frac{1}{N} \|\nabla P(z_j)\| - \int_{\mathcal{M}} \|\nabla P(z)\|d\mu(z)\right| + \frac{2c_2}{N^{1/d}} \int_{\mathcal{M}} \|\nabla P(z)\|d\mu(z)
\]
\[
\leq \frac{3c_2}{N^{1/d}} \int_{\mathcal{M}} \|\nabla P(z)\|d\mu(z) = \frac{3c_2}{N^{1/d}}
\]
for any $P \in \partial \Omega$. On the other hand, for $t \in [0, 12c_2N^{-1/d}]$
\[
\frac{d}{dt} \left( \frac{1}{N} \sum_{j=1}^{N} P(\gamma_j(P, t)) \right) = \frac{1}{N} \sum_{j=1}^{N} \|\nabla P(\gamma_j(P, t))\|^2
\]
\[
\geq \frac{1}{N} \sum_{j=1}^{N} \|\nabla P(\gamma_j(P, t))\| \geq \frac{1}{N} \sum_{j=1}^{N} \|\nabla P(\gamma_j(P, t))\| - \varepsilon.
\]
Since clearly $|\gamma_j(P, t) - x_j| \leq t$, the partition $\mathcal{R}'' = \{R''_1, \ldots, R''_N\}$ defined by $R''_j = R_j \cup \{\gamma_j(P, t)\}$ satisfies Theorem 5 with the constants $c_1$ and $13c_2$. Therefore by Theorem 5 applied to $P$ and the partition $\mathcal{R}''$, since $C_3(c_1, 13c_2)LN^{-1/d} \leq (C_ML^dN^{-1})^{1/d}/2 \leq 1/2$, we have
\[
\frac{d}{dt} \left( \frac{1}{N} \sum_{j=1}^{N} P(\gamma_j(P, t)) \right)
\]
\[
\frac{1}{N} \sum_{j=1}^{N} \| \nabla P(y_j(P,t)) \| - \varepsilon \\
\geq \int_{M} \| \nabla P(y) \| d\mu(y) - \left| \int_{M} \| \nabla P(y) \| d\mu(y) - \frac{1}{N} \sum_{j=1}^{N} \| \nabla P(y_j(P,t)) \| \right| - \varepsilon \\
\geq \frac{1}{2} \int_{M} \| \nabla P(y) \| d\mu(y) - \varepsilon = \frac{1}{2} - \varepsilon
\]

for each \( P \in \partial \Omega \) and \( t \in [0, \frac{12c_2 N^{-1/d}}{}]. \) Finally,
\[
\frac{1}{N} \sum_{j=1}^{N} P(x_j(P)) = \frac{1}{N} \sum_{j=1}^{N} P(x_j) + \int_{0}^{\frac{12c_2 N^{-1/d}}{}} \frac{d}{dt} \left( \frac{1}{N} \sum_{j=1}^{N} P(y_j(P,t)) \right) dt \\
\geq \frac{12c_2}{N^{1/d}} \left( \frac{1}{2} - \varepsilon \right) - \frac{3c_2}{N^{1/d}} = (3 - 12\varepsilon) \frac{c_2}{N^{1/d}} > 0.
\]

\[\square\]

References

[1] M. Berger, P. Gauduchon, E. Mazet, Le spectre d’une variété riemannienne, Lecture Notes in Mathematics, 194, Springer-Verlag, Berlin-New York, 1971.

[2] A. Bondarenko, D. Radchenko, M. Viazovska, Optimal asymptotic bounds for spherical designs, Ann. Math. 178, 443–452 (2013).

[3] L. Brandolini, C. Choirat, L. Colzani, G. Gigante, R. Seri, G. Travaglini, Quadrature rules and distribution of points on manifolds, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5) XIII, 889–923 (2014).

[4] L. Brandolini, W. W. L. Chen, L. Colzani, G. Gigante, G. Travaglini, Discrepancy and Numerical Integration on Metric Measure Spaces, J. Geom. Anal. (2018), https://doi.org/10.1007/s12220-018-9993-6.

[5] J. S. Brauchart, J. Dick, E. B. Saff, I. H. Sloan, Y. G. Wang, R. S. Womersley, Covering of spheres by spherical caps and worst-case error for equal weight cubature in Sobolev spaces, J. Math. Anal. Appl. 431, 782–811 (2015).

[6] J. S. Brauchart, E. B. Saff, I. H. Sloan, R. S. Womersley, QMC designs: optimal order quasi Monte Carlo integration schemes on the sphere, Math. Comput. 83, 2821–2851 (2014).

[7] M. P. do Carmo, Riemannian Geometry, Translated from the second Portuguese edition by Francis Flaherty. Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992.

[8] U. Etayo, J. Marzo, J. Ortega-Cerdà, Asymptotically optimal designs on compact algebraic manifolds, J. Monatsh. Math. 186, 235–248 (2018).

[9] F. Filbir, H. N. Mhaskar, A Quadrature Formula for Diffusion Polynomials Corresponding to a Generalized Heat Kernel, J. Fourier Anal. Appl. 16, 629–657 (2010).

[10] F. Filbir, H. N. Mhaskar, Marcinkiewicz-Zygmund measures on manifolds, J. Complexity 27, 568–596 (2011).

[11] G. Gigante, P. Leopardi, Diameter bounded equal measure partitions of Ahlfors regular metric measure spaces, Discret. Comput. Geom. 57, 419–430 (2017).

[12] P. Greiner, An asymptotic expansion for the heat equation, Arch. Rational Mech. Anal. 41, 163–218 (1971).

[13] L. Hörmander, The analysis of linear partial differential operators, I II III IV, Springer Verlag, 1983-1985.

[14] Yu. A. Kordyukov, \(L^p\) theory of elliptic differential operators on manifolds of bounded geometry, Acta Appl. Math. 23, 223–260 (1991).

[15] J. Korevaar, J. L. H. Meyers, Spherical Faraday cage for the case of equal point charges and Chebyshev-type quadrature on the sphere, Integral Transform. Spec. Funct. 1, 105–117 (1993).

[16] J. M. Lee, Introduction to smooth manifolds, Second edition, Graduate Texts in Mathematics, 218, Springer, New York, 2013.

[17] M. Maggioni, H. N. Mhaskar, Diffusion polynomial frames on metric measure spaces, Appl. Comput. Harmon. Anal. 24, 329–353 (2008).
[18] D. O'Regan, Y. J. Cho, Y.-Q. Chen, Topological Degree Theory and Applications, Ser. Math. Anal. Appl., 10, Chapman & Hall / CRC, Boca Raton, FL, 2006.

[19] A. Sikora, Riesz transform, Gaussian bounds and the method of wave equation. Math. Z. 247, 643–662 (2004).

[20] E. M. Stein, R. Shakarchi, Complex analysis. Princeton Lectures in Analysis, 2. Princeton University Press, Princeton, NJ, 2003.

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