SHARP UPPER BOUNDS ON THE MINIMAL NUMBER OF ELEMENTS REQUIRED TO GENERATE A TRANSITIVE PERMUTATION GROUP

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Abstract. The purpose of this paper is to prove that if $G$ is a transitive permutation group of degree $n \geq 2$, then $G$ can be generated by $\lfloor cn/\sqrt{\log n} \rfloor$ elements, where $c := \sqrt{3}/2$. Owing to the transitive group $D_8 \circ D_8$ of degree 8, this upper bound is best possible. Our new result improves a 2015 paper by the author, and makes use of the recent classification of transitive groups of degree 48.

1. Introduction

For an arbitrary group $G$, let $d(G)$ be the minimal size of a generating set of $G$. In [13], the problem of finding numerical upper bounds for $d(G)$ for an arbitrary transitive permutation group $G$ of degree $n$ is considered. It had already been proved in [9] that $d(G)$ is at most $\frac{cn}{\sqrt{\log n}}$ in this case, where $c$ is an absolute constant. This bound is shown to be asymptotically best possible in [7] (that is, there exists constants $c_1$, $c_2$, and an infinite family $(G_n)_{i=1}^\infty$ of transitive groups of degree $n_i$, with $c_1 \leq \frac{n_i}{d(G_n)\sqrt{\log n_i}} \leq c_2$ for all $i$).

In [13] it is proved that, apart from a finite list of possible exceptions, the bound $d(G) \leq \frac{cn}{\sqrt{\log n}}$ holds, where $c := \frac{\sqrt{3}}{2}$. This bound is best possible in the sense that $d(G) = \frac{\sqrt{3}n}{\sqrt{\log n}}$ when $G = D_8 \circ D_8 < S_8$ and $n = 8$.

In this paper, we remove the possible exceptions listed in [13, Theorem 1.1]. More precisely, we prove:

**Theorem 1.1.** Let $G$ be a transitive permutation group of degree $n$. Then

$$d(G) \leq \left\lfloor \frac{cn}{\sqrt{\log n}} \right\rfloor$$

where $c := \frac{\sqrt{3}}{2}$.

There are a number of steps in the proof of Theorem 1.1. For this reason, we will spend the next few paragraphs outlining our general strategy for the proof.

First, by [13, Theorem 5.3], we only need to prove Theorem 1.1 when $G$ is imprimitive with minimal block size 2, and $n$ has the form $n = 2^x3^y5$ with either $y = 0$ and $17 \leq x \leq 26$; or $y = 1$ and $15 \leq x \leq 35$. Thus, in particular, $G$ may be viewed as a subgroup in a wreath product $2 \wr G^2$, where $\Sigma$ is a set of blocks for $G$ of size 2. It follows that $d(G) \leq d_{G^2}(M) + d(G^2)$, where $M$ is the intersection of $G$ with the base group of the wreath product, and $d_{G^2}(M)$ is the minimal number of elements required to generate $M$ as a $G^2$-module.

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Of course, \( d(G^\Sigma) \) can be bounded using induction. The bulk of this section will therefore be taken up with finding upper bounds on \( d_G^\Sigma(M) \). Since \( d_G^\Sigma(M) \leq d_H(M) \) for any subgroup \( H \) of \( G^\Sigma \), the strategy of the third author in [13] in this case involved replacing \( G^\Sigma \) by a convenient subgroup \( H \) of \( G^\Sigma \), and then deriving upper bounds on \( d_H(M) \), usually in terms of the lengths of the \( H \)-orbits in \( \Sigma \). This approach turns out to be particularly fruitful when \( H \) is chosen to be a soluble transitive subgroup of \( G^\Sigma \), whenever such a subgroup exists. When \( G^\Sigma \) does not contain a soluble transitive subgroup, however, the analysis becomes much more complicated. This leads to less sharp bounds, and ultimately, the omitted cases in [13, Theorem 1.1].

Our approach in this section involves a careful analysis of the orbit lengths of soluble subgroups in a minimal transitive insoluble subgroup of \( G \), building on the work in [13] in the case \( n = 2^r3 \).

The layout of the paper is as follows: First, Section 2 contains a necessary discussion of subdirect products of finite groups. Then, in Section 3 we prove a general theorem concerning minimal transitive groups of degree \( 2^r3^y5^z \), with \( 0 \leq y, z \leq 1 \). Finally, this information will allow us to prove Theorem 1.1 and we do so in Section 4.

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2. Subdirect Products of Finite Groups

As mentioned above, we begin preparations toward the proof of Theorem 1.1 with some straightforward but necessary lemmas concerning subdirect products of finite groups. First, we make the following remark concerning our notation and terminology in direct products of finite groups:

**Remark 2.1.** For a group \( T \) we will write \( T^e \) for the direct product of \( e \) copies of \( T \). More generally, let \( G_1, \ldots, G_r \) be finite groups, and consider the direct product \( G := G_1^{e_1} \times \cdots \times G_r^{e_r} \).

Fix subgroups \( H_i \leq G_i \), for \( 1 \leq i \leq r \), and set \( e := \sum_{i=1}^r k_i \). Then we write \( H_1^{e_1} \times \cdots \times H_r^{e_r} \) for the subgroup \( \{(x_1, \ldots, x_r) : x_j \in H_i \text{ for } f_{i-1} + 1 \leq j \leq f_i\} \) of \( G \), where \( f_0 := 0 \), and \( f_i := \sum_{k=1}^i e_k \) for \( i > 0 \).

Also, the subgroup \( S_{e_1} \times \cdots \times S_{e_r} \) of \( S_e \) (in its natural intransitive action) acts via automorphisms on \( G \) (by permutation of coordinates). Thus we can, and do, speak of \( (S_{e_1} \times \cdots \times S_{e_r}) \)-conjugates of subgroups of \( G \). This will be very useful for avoiding cumbersome notation.

Finally, for a group \( T \) we call a subgroup \( G \) of \( T^e \) a diagonal subgroup of \( T^e \) if there exists automorphisms \( \alpha_i \) of \( T \) such that \( G = \{(t^{a_{e_1}}, t^{a_{e_2}}, \ldots, t^{a_{e_r}}) : t \in T\} \).

Suppose now that \( G \) is a subgroup in a direct product \( G_1 \times \cdots \times G_e \) of groups \( G_i \), and write \( \pi_i : G \rightarrow G_i \) for the \( i \)th coordinate projections. Then \( G \) is a subdirect product in \( G_1 \times \cdots \times G_e \) if \( G \pi_i = G_i \) for all \( i \). We also introduce the following non-standard definition:

**Definition 2.2.** Let \( G_i \) be as above, and let \( n \) be a positive integer. We say that \( G \) is a subdirect product of the form \( G = \frac{1}{n}(G_1 \times \cdots \times G_e) \) if each of the following holds:

(a) \( G \) is a subdirect product in \( G_1 \times \cdots \times G_e \);
(b) \( G \) contains \([G_1, G_1] \times \cdots \times [G_e, G_e] \); and
(c) $G$ has index $n$ in $G_1 \times \ldots \times G_e$.

The following is a collection of easy facts concerning subdirect products in direct products of finite groups:

**Lemma 2.3.** Let $G_1, \ldots, G_e$ be finite groups, and $\pi_i : G_1 \times \ldots \times G_e \to G_i$ be as above, and let $K_i$ be the intersection of $G_i$ with the $i$th coordinate subgroup of $G_1 \times \ldots \times G_e$. Suppose that $G$ is a subdirect product in $G_1 \times \ldots \times G_e$. Then each of the following holds:

(i) $K_i \cong G_i$ for all $i$.

(ii) If $e = 2$ and $G_1$ and $G_2$ have no common nontrivial homomorphic images, then $G = G_1 \times G_2$.

(iii) Let $e = \sum_{i=1}^r e_i$ be a positive partition of $e$, and define $f_0 := 0$, and $f_i := \sum_{k=1}^{i} e_k$ for $1 \leq i \leq r$. Suppose that for each $1 \leq i \leq r$, there exists a finite simple group $T_i$ such that $G_i$ is isomorphic to $T_i$ for all $f_{i-1} + 1 \leq j \leq f_i$. Then for each $1 \leq i \leq r$ there exists a partition $\sum_{k=1}^{f_i} d_{i,k}$ of $e_i$ and diagonal subgroups $D_{i,k}$ of $T_i^{d_{i,k}}$ such that $G$ is $(S_{e_1} \times \ldots \times S_{e_r})$-conjugate to $(T_{1,1} \times \ldots \times T_{1,1}) \times \ldots \times (T_{r,1} \times \ldots \times T_{r,t_r})$.

(iv) Let $T$ be a nonabelian simple group, and suppose that $|G_i| < |T|$ for all $i$. Then $G$ has no homomorphic image isomorphic to $T$.

(v) Fix $1 < f < e$. Suppose that $G_i$ is nonabelian simple for $1 \leq i \leq f$, and that $|G_j| < |G_i|$ for all $1 \leq i \leq f$ and $f + 1 \leq j \leq e$. Assume also that $|G_i|$ does not divide the index of $G$ in $G_1 \times \ldots \times G_e$ for any $1 \leq i \leq f$. Then $G = G_1 \times \ldots \times G_f \times L$, where $L$ is a subdirect product in $G_{f+1} \times \ldots \times G_e$.

**Proof.** Part (i) is clear, while part (ii) follows from the fact that if $e = 2$, then

$$\frac{G_1}{K_1 \pi_1} \cong \frac{G_2}{K_2 \pi_2}.$$  

Part (iii) is well-known (for example, see [11, Theorem 4.16(iii)]), so we just need to prove (iv) and (v).

We begin with (iv). So assume that $T$ is a nonabelian finite simple group, and that $|G_i| < |T|$ for all $i$. We prove the claim in (iv) by induction on $e$, with the case $e = 1$ being trivial. So assume that $e > 2$, and write $\hat{\pi}$ for the projection $\hat{\pi} : G \to G_2 \times \ldots \times G_e$. Suppose that $N$ is a normal subgroup of $G$ with $G/N \cong T$. Then since $G\hat{\pi}/N\hat{\pi}$ is both a homomorphic image of $G/N \cong T$, and the subdirect product $G\hat{\pi} \leq G_2 \times \ldots \times G_e$, the inductive hypothesis implies that $N\hat{\pi} = G\hat{\pi}$. Then since $K_1$ is the kernel of $\hat{\pi}$, we have $|G| = |K_1||G\hat{\pi}|$ and $|N| = |K_1 \cap N||G\hat{\pi}|$. Hence, $|G_1| \geq |K_1| \geq |K_1/K_1 \cap N| = |T|$ -- a contradiction. This proves (iv).

Finally, assume that the hypotheses in (v) hold. Then $G$ may be viewed as a subdirect product in $R \times L$, where $R$ is a subdirect product in $G_1 \times \ldots \times G_f$, and $L$ is a subdirect product in $G_{f+1} \times \ldots \times G_e$. It then follows from (ii), (iii), and (iv) that $G = R \times L$. Moreover, since $|G_1 \times \ldots \times G_f : R|$ divides $|G_1 \times \ldots \times G_e : G|$, we must have $R = G_1 \times \ldots \times G_f$, by (iii). This completes the proof of (v), whence the lemma.

$$\square$$

3. **Minimal Transitive Groups of Degree $2^e 3^y 5^z$**

In this subsection, we investigate the structure of minimal transitive groups of degree $2^e 3^y 5^z$, with $0 \leq y, z \leq 1$. Our main result reads as follows:
Theorem 3.1. Let $G$ be an insoluble minimal transitive permutation group of degree $n = 2^t 3^u 5^v$, with $0 \leq y, z \leq 1$. Then:

(i) $G$ has at most $y + z$ nonabelian chief factors;
(ii) each nonabelian chief factor of $G$ is isomorphic to a direct product of copies of one of $A_5; A_6; S_3(4); O_8^+(2); A_8; A_{16};$ or $L_2(q)$, where $q$ has the form $q = 2^{t_1} 3^{v_1} 5^{z_1} - 1$ with $0 \leq y_1, z_1 \leq 1$;
(iii) if $G$ has two nonabelian chief factors, then they are isomorphic to $T_1^{t_1}$ and $L_2(p)^{v_2}$, where $T_1 \in \{A_5, A_6\}$, and $p$ is a Mersenne prime; and
(iv) $G$ has a soluble subgroup $F$ whose orbit lengths are given in Table 1.

Now, Table 1 requires some explanation: In Theorem 3.1, we prove that a minimal transitive permutation group of degree $2^t 3^u 5^v$ with $0 \leq y, z \leq 1$, has at most two nonabelian chief factors, which have to be isomorphic to direct powers of one of $A_5; A_6; A_8; A_{16};$ or $L_2(q)$, where $q$ has the form $q = 2^{t_1} 3^{u_1} 5^{v_1} - 1$. The first column of Table 1 will be a group $T^e$, where $T$ is one of these simple groups. This means that if $G$ is the minimal transitive group under consideration, then $G/R(G)$ has a minimal normal subgroup $L/R(G)$ which is isomorphic to a direct product of $e$ copies of $T$ (recall that $R(G)$ is the soluble radical of $G$). The second column gives the degree of $G$ as a permutation group, while the third column gives the orbit lengths of a soluble subgroup $F$ of $G$. Moreover, these orbit lengths will be given in one of the following ways:

**Rows 23–30:** In these cases, the orbit lengths are given in the form of a list with entries $\sum_{i=1}^f k_i \times l_i$, with the entry $\sum_{i=1}^f k_i \times l_i$ meaning that $F$ has $k_i$ orbits of size $l_i$ for each $1 \leq i \leq f$.

**Rows 1–11 and 13–21:** In these cases, the orbit lengths are given in the form $\alpha X_i, \beta Y_j$. Here, $\{X_i\}_i$ and $\{Y_j\}_j$ are positive unordered partitions whose sum is given in the fourth column of the table. For example, if $F$ is as in row 1, then $\{X_i\}_i$ is a positive unordered partition of 4 (there are five such partitions) and $\{Y_j\}_j$ is a positive unordered partition of 2 (there are two such partitions). Thus, there are ten possibilities for the orbit lengths of $F$. For instance, the orbit sizes (40, 100a, 100a) correspond to the partitions $4 = 4 + 2 = 1 + 1 + 4 + 2$, respectively.

**Rows 12 and 22:** The orbits lengths in these cases are given in the form $\sum_{i=0}^{e_2} \sum_{j,k} C_{j}^{(i)} \times 2^{b_3} p_5 X_k^{(i,j)}, 0 \leq i \leq e_2$. By this notation we mean that $(C_{j}^{(i)})_j$ is a positive ordered partition of a particular number $C$ for each $i$: $(X_k^{(i,j)})_k$ is a positive ordered partition of a particular number $X$ for each $i, j$; and $F$ has $C_{j}^{(i)}$ orbits of size $2^{b_3} p_5 X_k^{(i,j)}$ for each $i, j, k$. The numbers $C$, $X$, $b$, and $p$ are given in the fourth column of the table (in fact, $C := (e_2^i)$ in each case).

As an example, assume that $G$ is a minimal transitive group of degree $n = 2^{15} 15$. If a minimal normal subgroup of $G$ is isomorphic to a direct product of copies of $A_5$, then the first section of Table 1 gives us the orbit lengths of a soluble subgroup $F$ of $G$. For example, if $F$ is as in the first line of the table, then $F$ has orbit lengths $2^{12} 5 X_i$, $2^{13} 3 5 Y_j$, where $(X_i)_i \in \{(4), (3, 1), (2, 2), (1, 1, 1, 1)\}$, and $(Y_j)_j \in \{(2), (1, 1)\}$. Thus, the possible $F$-orbit lengths in this case are: $(2^{14}5, 2^{14}25)$; or $(2^{14}5, 2^{13}25, 2^{13}25)$; or $(2^{12}5, 2^{12}5, 2^{14}25)$; or $(2^{12}5, 2^{12}5, 2^{13}25, 2^{13}25)$; or $(2^{13}5, 2^{13}5, 2^{14}25)$; or $(2^{13}5, 2^{13}5, 2^{13}25, 2^{13}25)$; or $(2^{12}5, 2^{12}5, 2^{12}5, 2^{14}25)$; or $(2^{12}5, 2^{12}5, 2^{12}5, 2^{14}25)$; or
(2^{12}5, 2^{12}5, 2^{12}5, 2^{12}5, 2^{13}25, 2^{13}25). The rest of the possibilities for the orbit lengths of \( F \) can be computed in an entirely analogous way. (Although these computations seem tedious, they can be done very quickly using a computer).

**Remark 3.2.** Let \( G \) be as in the statement of Theorem 3.1 and suppose that we have proved the theorem for all degrees less than \( n \). Let \( K \) be the kernel of the action of \( G \) on a block system \( \Sigma = \{ \Delta^g : g \in G \} \) for \( G \) in \( O \). Then \( G^\Sigma \cong G/K \) is minimal transitive. Thus, we may choose a soluble subgroup \( F/K \) of \( G/K \cong G^\Sigma \) satisfying Theorem 3.1(iv) with \( G \) replaced by the minimal transitive group \( G^\Sigma \).

If \( K \) is soluble and acts transitively on \( \Delta \), then \( F \) is soluble and the \( F \)-orbits have lengths \( |\Delta||\Sigma_i| \), as \( \Sigma_i \) runs over the \( F^\Sigma \)-orbits. Suppose that the \( \Sigma_i \) are as in row \( k \) in Table 1. Then since \( |\Delta| \) divides \( n \), it is easy to see that by replacing \( a \) by \( |\Delta|a \), \( F \leq G \) has the required orbit lengths in order to fulfill the conclusion of (iv). (For example, if \( F/K \) is as in row 2 of Table 1 then \( G \) has degree \( 120|\Delta|a \), and \( F \) is soluble with orbit sizes \( 5|\Delta|aX_1, 50|\Delta|aY_j \).) This is a useful inductive tool which we will use throughout the section.

We begin preparations toward the proof of Theorem 3.1 with a reduction lemma:

**Lemma 3.3.** Let \( G \) be a counterexample to Theorem 3.1 of minimal degree. Then all minimal normal subgroups of \( G \) are nonabelian. Moreover, if \( L \) is a minimal normal subgroup of \( G \) and \( K \) denotes the kernel of the action of \( G \) on the set of \( L \)-orbits, then \( K/L \) is soluble.

**Proof.** Let \( L \) be a minimal normal subgroup of \( G \). Also, let \( \Delta \) be an \( L \)-orbit, and let \( \Sigma := \{ \Delta^g : g \in G \} \) be the set of \( L \)-orbits in \([n]\), so that \( G^\Sigma \cong G/K \). Then \( n = |\Delta||\Sigma| \), and \( G^\Sigma \) is minimal transitive. The minimality of \( G \) as a counterexample then implies that \( G^\Sigma \) satisfies the conclusion of Theorem 3.1.

Let \( E/L \) be a subgroup of \( G/L \) which is minimal with the property that \( (E/L)(K/L) = G/L \). Then \( (E/L) \cap (K/L) \leq \Phi(E/L) \), so \( (E/L) \cap (K/L) \) is soluble. But also, \( EK = G \). Since \( E \) contains \( L \) and \( E^\Sigma = G^\Sigma \) is transitive, we deduce that \( E \) is transitive, whence \( E = G \), since \( G \) is minimal transitive. Thus, \( K/L \) is soluble. This proves the second part of the lemma.

Finally, assume that \( L \) is abelian. Then \( K \) is soluble. It follows that the set of nonabelian chief factors of \( G \) is same as the set of nonabelian chief factors of \( G/K \). Thus, \( G = G^O \) satisfies (i), (ii), and (iii) in Theorem 3.1. Moreover, since the theorem holds in \( G^\Sigma \cong G/K \), we may choose a soluble subgroup \( F/K \) of \( G/K \) satisfying Theorem 3.1(iv). Then \( F \) is soluble, since \( K \) is soluble, and the \( F \)-orbits in \([n]\) have lengths \( |\Delta||\Sigma_i| \), as \( \Sigma_i \) runs over the \( F^\Sigma \)-orbits in \( \Sigma \). Then \( F \) is a subgroup of \( G \) satisfying the conclusion of Theorem 3.1(iv) (see Remark 3.2). Thus, (i)–(iv) in Theorem 3.1 hold in \( G \), contradicting the assumption that \( G \) is a counterexample. \( \square \)

By Lemma 3.3, a minimal normal subgroup \( L \) of a minimal counterexample to Theorem 3.1 must be a direct product of nonabelian simple groups. Since the subnormal subgroups of a transitive permutation group of degree \( n \) have orbit lengths dividing \( n \), this motivates us to study the pairs \( (T, H) \) where \( T \) is a nonabelian simple group, \( H \) is a proper subgroup of \( T \), and \( [T : H] \) has the form \( 2^y3^z5^z \), for some \( 0 \leq y, z \leq 1 \).

Our main tool in this regard is a paper of Liebeck, Praeger, and Saxl [8], where the finite simple groups \( T \) with a maximal subgroup \( M \) satisfying \( \pi(|M|) = \pi(|T|) \) are classified. In order
| $\mathcal{T}$ | degree of $G$ | orbit lengths of a soluble subgroup $F$ | notes |
|---|---|---|---|
| $\mathcal{A}_5^e$ | 240a | $10aX_i$, 100a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (4, 2)$ |
| | 120a | $5aX_i$, 50a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (4, 2)$ |
| | 120a | $10aX_i$, 50a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (2, 2)$ |
| | 60a | $5aX_i$, 25a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (2, 2)$ |
| | 30a | $5aX_i$, 25a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (1, 1)$ |
| | $2^b10a$ | $2^b5aX_i$ | $\sum_i X_i = 2$; $b \in \{0, 1\}$; $G/L$ soluble |
| | 60a | 10a, 50a | |
| | 30a | $5aX_i$, 10a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (2, 2)$ |
| | 120a | $5aX_i$, 25a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (4, 4)$ |
| | 60a | $5aX_i$, 25a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (2, 2)$ |
| | $2^b(p + 1)^{e_2}30a$ | $\sum_{i=0}^{e_2} \sum_{j,k} C_{j}^{(i)} \times 2^b3p^j5aX_k^{(i,j)}$ | $\sum_k C_k^{(i)} = (e_i^2)$ and $\sum_j X_k^{(i,j)} = 2$ for each $i, j$; $b \in \{0, 1\}$; $G/L$ insoluble; $p$ a Mersenne prime; $1 \leq e_2 \leq e$ |
| $\mathcal{A}_6^e$ | 240a | $10aX_i$, 50a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (4, 4)$ |
| | 120a | $5aX_i$, 25a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (4, 4)$ |
| | 60a | $5aX_i$, 25a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (2, 2)$ |
| | 120a | $10aX_i$ | $\sum_i X_i = 12$ |
| | 15a | $5aX_i$ | $\sum_i X_i = 3$ |
| | 40a | $10aX_i$ | $\sum_i X_i = 4$; $G/L$ soluble |
| | $2^b5a$ | $5aX_i$ | $\sum_i X_i = 2^b$; $b \in \{1, 2\}$; $G/L$ soluble |
| | 60a | $5aX_i$, 10a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (4, 4)$ |
| | 30a | $5aX_i$, 10a$Y_j$ | $(\sum_i X_i, \sum_j Y_j) = (2, 2)$ |
| | $2^bX(p + 1)^{e_2}15a$ | $\sum_{i=0}^{e_2} \sum_{j,k} C_{j}^{(i)} \times 2^b3p^j5aX_k^{(i,j)}$ | $\sum_j C_j^{(i)} = (e_i^2); \sum_k X_k^{(i,j)} = X$ for each $i, j$; $(X, b) \in \{(4, 1), (4, 0), (2, 0)\}$; $G/L$ insoluble; $p$ a Mersenne prime; $1 \leq e_2 \leq e$ |
| $T^e$ | $2^b120a$ | $2^b24a + 2^b96a$ | $b \in \{0, 3\}$, $T \in \{S_4(4), O_7^+(2)\}$ |
| $\mathcal{A}_8^e$ | $8^{e_2}960a$ | $\sum_{j=1}^{3} \sum_{i=0}^{e_2} a_{j}^{e_2} \times 7^i l_j$ | $0 \leq e_2 \leq e$; $1 \leq j \leq 3$, where $(s_1, l_1) = (1, 1), (s_2, l_2) = (11, 7)$, and $(s_3, l_3) = (42, 21)$. |
| | $8^{e_2}120a$ | $\sum_{j=3}^{2} \sum_{i=0}^{e_2} a_{j}^{e_2} \times 7^i l_j$ | $0 \leq e_2 \leq e$; $1 \leq j \leq 3$, where $(s_1, l_1) = (1, 1), (s_2, l_2) = (5, 7)$, and $(s_3, l_3) = (4, 21)$. |
| | $8^{e_2}15a$ | $\sum_{j=2}^{2} \sum_{i=0}^{e_2} a_{j}^{e_2} \times 7^i l_j$ | $0 \leq e_2 \leq e$; $1 \leq j \leq 2$, where $(s_1, l_1) = (1, 1)$ and $(s_2, l_2) = (2, 7)$. |
| $A_{16}^e$ | $15^{16^{e_2}+1} a_{2^b}$ | $\sum_{i=0}^{e_2} a_{i}^{e_2} \times 15^{i+1}$ | $b \in \{0, 1\}$, $0 \leq e_2 \leq e$ |
| $L_2(q)^{e_2}$ | $15a(q + 1)^{e_2+2}$ | $\sum_{i=0}^{e_2} a_{i}^{e_2} \times 15q^i + 2a_{i}^{e_2} \times 15q^i+1$ | $q$ a Mersenne prime, $0 \leq e_2 \leq e$ |
| | $3^{30^{e_2}a(q + 1)^{e_2}}$ | $\sum_{i=0}^{e_2} a_{i}^{e_2} \times 3^{30^{e_2}q^i}$ | $q$ a Mersenne prime, $1 \leq e_2 \leq e$ |
| | $2^{e|x}15a$ | $aq3^{1-y^5}5^{1-z^t} + a3^{1-y^5}5^{1-z^t}$ | $q = 2^{x}3^{y^5}5^{z^t} - 1$, $\{y^t, z^t\} = \{1, 0\}$ |

Table 1. Orbit lengths of a certain soluble subgroup of a minimal transitive group $G$ with $\mathcal{T} := L/R(G)$ a minimal normal subgroup of $G/R(G)$. 
to use this result, we will first need to determine the finite simple groups $T$ with a subgroup $H$ of index $2^e 3^p 5^q$ as above, such that $\pi(|M|) \neq \pi(|T|)$ for some maximal subgroup $M < T$ containing $H$.

**Proposition 3.4.** Let $T$ be a nonabelian finite simple group, and suppose that $T$ has a subgroup $H$ of index $2^e 3^p 5^q$, with $0 \leq y, z \leq 1$. Let $M$ be a maximal subgroup of $T$ containing $H$, and assume that either $\pi(|T|) \neq \pi(|M|)$, or that $T$ is an alternating group of degree less than 10. Then one of the following holds:

(i) $T = A_n$ is an alternating group and either
   - $n = 5$ and $H$ is any subgroup of $T$;
   - $n = 6$ and $|H| = 2^3 i$, for some $i \leq 3$;
   - $n = 7$, and either $H = \text{GL}_3(2)$ (two classes) or $H = 7 : 3 < \text{GL}_3(2)$;
   - $n = 8$ and either $H = 2^3 : \text{GL}_3(2)$ (two classes); or $H = 2^3 : (7 : 3) < 2^3 : \text{GL}_3(2)$ (two classes); or $H \cong \text{GL}_3(2) < 3^2 : \text{GL}_3(2)$ (three classes); or $H \cong 7 : 3 < 2^3 : \text{GL}_3(2)$; or
   - $n = 9$ and $H = L_2(8).3$ (two classes).

(ii) $T = L_n(q)$ and either:
   - $n = 2$, $q$ has the form $q = 2^{2a} 3^{y_1} 5^{z_1} - 1$, and $H$ is a subgroup of a maximal parabolic subgroup $M$ of $T$, with $|M : H| = 3^y 9^{z-1}$; or
   - $(n, q) = (4, 3)$, and $H = 3^3 : L_3(3)$ (two classes).

(iii) $T = S_4(3)$ with $|T : H| = 40$ (two classes); or $T = S_6(2)$, and $|T : H| \in \{120, 960\}$.

**Proof.** Assume first that $T = A_n$. We claim that in order for $|T : M|$ to have the form $|T : M| = 2^{e_1} 3^{y_1} 5^{z_1}$ with $0 \leq y_1, z_1 \leq 1$ and $\pi(|T|) \neq \pi(|M|)$, we must have $n \leq 9$. To prove this, we first list the the possibilities for $M$:

- $M = (A_k \times A_{n-k}).2$, for some $k \leq \frac{n}{2}$.
- $M = A_n \cap (S_r \cap S_s)$, where $r, s > 1$ and $rs = n$.
- $M = A_n \cap X$, with $X \leq S_n$ primitive in its natural action.

In particular, if $|M|$ is odd, then $M$ must be as in the third listed case with $n$ an odd prime, $\frac{n-1}{2}$ odd, and $M = C_n \times C_{\frac{n-1}{2}}$. However, in this case $|T : M| = (n-2)!$, which has the form $|T : M| = 2^{e_1} 3^{y_1} 5^{z_1}$ with $0 \leq y_1, z_1 \leq 1$ if and only if $n \leq 7$. So we may assume that $|M|$ is even. But then either $|T|_3 = 3$ or $|T|_5 = 5$, since $\pi(|T|) \neq \pi(|M|)$. Thus $n \leq 9$, as claimed. We can now use direct computation to quickly determine the possibilities for $H$.

So we may assume that $T$ is not an alternating group. If $T \in \{L_2(8), L_3(3), U_3(3), S_4(8)\}$ then direct computation quickly shows that $T$ has no maximal subgroups $M$ with index of the form $|T : M| = 2^{e_1} 3^{y_1} 5^{z_1}$ with $0 \leq y_1, z_1 \leq 1$. So we may assume that $T \notin \{L_2(8), L_3(3), U_3(3), S_4(8)\}$.

Suppose next that $T = L_2(p)$ with $p = 2^e - 1$ a Mersenne prime. Then clearly $M$ must contain a Sylow $p$-subgroup of $T$. Hence, $M$ is a maximal parabolic subgroup of $T$; $|T : M| = p + 1 = 2^e$ and $|M : H| = 3^p 5^q$ (since $|M|$ is odd). So assume also that $T$ is not of the form $T = L_2(p)$ with $p$ a Mersenne prime. Then by [8 Corollary 6], there exists a set $\Pi$ of odd prime divisors of $|T|$ such that $\Pi$ intersects $\pi(|T|) \setminus \pi(|M|) \subseteq \{2, 3, 5\}$ non-trivially. Moreover, the proof of [8 Corollary 6] shows that either $T$ is in [8 Table 10.1] and $\Pi$ is as in the third column of [8 Table
Since \( L_{n, q} \) in [8 Table 10.3] and \( \Pi \) is as in the third column of [8 Table 10.3]; or \( T \) is in [8 Table 10.4] and \( \Pi \) is as in the second column of [8 Table 10.4] (except when \( T = U_5(2) \), where \( \Pi := \{3, 5, 11\} \)); or \( T \) is in [8 Table 10.5] and \( \Pi \) is as in the second column of [8 Table 10.5] (except when \( T = G_2(q) \), where \( \Pi := \{p, 7, 13\}, \{5, 7, 13\}, \) or \( \{7, 13\} \) according to whether \( q \geq 5 \), \( q = 4 \), or \( q = 3 \), respectively); or \( T \) is in [8 Table 10.6] (except when \( T = M_{11}, M_{12}, \) or \( M_{24} \), where \( \Pi := \{3, 11\}, \{3, 11\}, \) or \( \{7, 23\} \), respectively). Fix \( r \) to be a prime in \( \Pi \cap (\pi(|T|) \setminus \pi(|M|)) \), so that \( r \in \{3, 5\} \). This implies in particular that \(|T|_r = r\).

By [8 Corollary 6] and inspection of simple group orders, the case \( r = 3 \) occurs only if \( T = L_2(p) \) with \( p \) of the form \( p = 2^{2a}3^{y_1}1 \), with \( 0 \leq y_1 \leq 1 \). In this case, one can see from [2 Table 8.1] that \( M \) is a maximal parabolic subgroup of \( T \). Since \( 2^{x_1} \) is the largest power of 2 dividing \(|T|\) in this case, we deduce that \( x = x_1 \) and that \( H \) must have index dividing \( 3^{y_1}5^{z_1} \) in \( M \).

So we may assume, for the remainder of the proof, that \( r = 5 \). In what follows, for a prime power \( q \) and a positive integer \( n \) we will write \( q_n \) for an arbitrary primitive prime divisor of \( q^n - 1 \). By Zsigmondy’s theorem these exists for \( n > 1 \) and \( (q, n) \neq (2, 6) \). By Fermat’s Little Theorem we have

\[(3.1) \quad n \mid q_n - 1.\]

Suppose first that \( G \) is a finite simple classical group, and write \( T = X_n(q) \), where \( X \in \{L, U, S, O^\pm, O\} \), and \( q = p^f \) with \( p \) prime. Then using (3.1) we can deduce from [8 Tables 10.1–10.6] that \( T \in \{L_n(q) \mid n \leq 5\}, U_n(q) \mid n \leq 6\}, S_n(q) \mid n \leq 8\}, O_n^\pm(q) \mid n \leq 10\}, O_n(q) \mid n \leq 11\}. \)

Below, we examine these remaining cases.

Suppose first that \( T = L_n(q) \), with \( n \leq 5 \). If \( n \in \{4, 5\} \), then (3.1) and [8 Tables 10.1 and 10.3] imply that either 5 is a primitive prime divisor of \( q^4 - 1 \) or \( p = 5 \). If \( n \in \{2, 3\} \), then 5 is a primitive prime divisor of \( q^2 - 1 \). Suppose first that \( M \) is parabolic. Then we see from [2 Tables 8.2, 8.3, 8.8, and 8.18] that \(|T : M|\) has the form \( \frac{q^n - 1}{(q^a - 1)} \) or \( \frac{(q^n - 1)(q^{n-1} - 1)}{(q^a - 1)(q^a - 1)} \). A routine exercise then shows that only the former can take the form \( 2^{x_1}3^{y_1}5^{z_1} \) with \( 0 \leq y_1, z_1 \leq 1 \), and this can only happen if \( n \in \{2, 4\} \) and \( q \) has the form \( q = 2a3^{b5^c}5^1 - 1 \). It is then easy to see that the only possibilities are either \((n, q) = (4, 3) \) or \((n, q) = (2, 2^{x_1}3^{y_1}5^{z_1} - 1) \) with \( 0 \leq y_1, z_1 \leq 1 \).

If \((n, q) = (4, 3) \), then the only possibility is \( M = H = 3^3 : L_3(3) \). If \((n, q) = (2, 2^{x_1}3^{y_1}5^{z_1} - 1) \) then \( 2^{x_1} \) is the largest power of 2 dividing \(|T|\), so \( x = x_1 \) and \( H \) has index dividing \( 3^{y_1}5^{z_1} \) in \( M \). So we may assume that \( M \) is not parabolic. Then \( p \) divides \(|T : M|\), so \( p \in \{2, 3, 5\} \). Also, as mentioned above, \( r = 5 \) does not divide \(|M|\). We can then quickly see from [2 Tables 8.2, 8.3, 8.8, and 8.18] that the only possibility is \((n, q) = (3, 2) \) with \( H \) having order 7 or 21. Since \( L_3(2) \cong L_2(7) \), this case is already listed in (ii)(a). This completes the proof in the case \( T = L_n(q) \).

The other classical cases are entirely similar: Suppose that \( T = U_n(q) \), with \( 3 \leq n \leq 6 \). If \( M \) is parabolic, then \(|T : M|\) is divisible by either \( \frac{(q^n - 1)(q^{n-1}(-1)^{n-1})}{q^2 - 1} \) or \( \prod_{i=1}^{2} (q^{2i+1} + 1) \). It is easy to see, however, that neither \( q^3 + 1 \) nor \( q^5 + 1 \) can ever have the form \( 2^{x_1}3^{y_1}5^{z_1} \) with \( 0 \leq y_1, z_1 \leq 1 \). So \( M \) must be non-parabolic. Then \( p \) divides \(|T : M|\), so \( p \in \{2, 3, 5\} \).
Also, 5 does not divide $|M|$. We can then use the tables in [2, Chapter 8] to quickly find that the only case of a maximal non-parabolic subgroups $M$ of $T$ with $|T : M| = 2^2 \cdot 3^3 \cdot 5^{z_1}$ and $0 \leq y_1, z_1 \leq 1$ occurs when $(n, q) = (4, 2)$. Since $U_4(2) \cong S_4(3)$, this case is accounted for in (iii). This completes the proof in the case $T = U_n(q)$.

The arguments in the remaining classical cases are almost identical, so we now move on to the case where $G$ is an exceptional group of Lie type. Then $r = 5$ is one of the primes occurring in the second column of the row for $T$ in [8, Table 10.5]. By using (3.1), we see that the only possibilities are $T \in \{2F_4(q)', 2B_2(q)\}$, and 5 is a primitive prime divisor of $q^4 - 1$. In each of these cases, the maximal subgroups of $T$ are known: see [10] for $T = 2F_4(q)'$ and [12] for $T = 2B_2(q)$. In each case, we can quickly check that no maximal subgroup $M$ of $T$ with $\pi(|T|) \neq \pi(|M|)$ can have $|T : M|$ of the form $2^x \cdot 3^y \cdot 5^{z_1}$, for $0 \leq y_1, z_1 \leq 1$.

Finally, assume that $G$ is a sporadic simple group. Then $T = J_2$, by [8, Table 10.6]. However, a quick check of the Web Atlas [14] shows that no maximal subgroup $M$ of $J_2$ with $\pi(|T|) \neq \pi(|M|)$ can have $|T : M|$ of the form $2^x \cdot 3^y \cdot 5^z$, for $0 \leq y, z \leq 1$. This completes the proof.

We are now ready to determine the possibilities for the pairs $(T, H)$, with $T$ simple, $|T : H|$ dividing $2^a 15$, and $H$ contained in a maximal subgroup $M$ of $T$ with $\pi(|T|) = \pi(|M|)$.

**Proposition 3.5.** Let $T$ be a nonabelian finite simple group, and suppose that $T$ has a subgroup $H$ of index $n = 2^a 3^b 5^c$, with $0 \leq y, z \leq 1$. Suppose also that $T$ is not an alternating group of degree less than 10. Let $M$ be a maximal subgroup of $T$ containing $H$, and assume that $\pi(|T|) = \pi(|M|)$. Then one of the following holds:

(i) $T = A_n$, and either $H = A_{n-1}$, or $n = 16$ and $A_{14} \leq H \leq A_{14} \cdot 2$.

(ii) $T = S_{2m}(2^f)$ with $(m, f) \in \{(4, 1), (2, 2)\}$, and $O_{2m}(2^f) \leq H$.

(iii) $T = S_4(q)$, with $q \in \{2, 4\}$, and $S_2(q^2) \leq H$.

(iv) $T = M_{11}$ and $H = L_2(11)$.

(v) $T = O^+_8(2)$ and $H = A_9$ (three classes, fused by a triality automorphism); or

(vi) $T = M_{12}$ or $M_{24}$, and $H = M_{11}$ or $H = M_{23}$, respectively.

**Proof.** We can quickly check using direct computation that $T$ can have no maximal subgroup $M$ of the required index when $T$ is $L_2(8)$, $L_3(3)$, $U_3(3)$ or $S_4(8)$. So we may assume that $T$ is not one of these groups. Suppose first that $T \neq L_2(p)$ for a Mersenne prime $p$. Then using [8, Corollary 6 and Table 10.7], the possibilities for $T$ and $M$ are as follows.

1. $T = A_n$ and $M = A_k \cap (S_k \times S_{n-k})$ for some $k \leq n - 1$ with $k \geq p$ for all primes $p \leq n$.

Then $|A_n : A_n \cap (S_k \times S_{n-k})| = \binom{n}{k}$ divides $|T : H| = 2^a 3^b 5^c$. By a well-known theorem of Sylvester and Schur (see [4]), $\binom{n}{k}$ has a prime divisor exceeding minimal $\{k, n - k\}$. Thus $k \in \{n - 1, n - 2, n - 3, n - 4\}$, since $k \geq 10$. We then quickly see that the only possibilities are $H = A_{n-1}$; or $n = 16$ and $k = 14$, which gives us what we need.

2. $T = S_{2m}(q)$ (m, q even) or $O_{2m+1}(q)$ (m even, q odd), and $M = N_T(O^-_{2m}(q))$. Now, $|N_T(O^-_{2m}(q)) : O^-_{2m}(q)| \leq 2$ using [3, Corollary 2.10.4(i) and Table 2.1.D]. Hence, $|T : O^-_{2m}(q)|$ divides $2^{a+1} 3^b 5^c$. Also, for each of the two choices of $T$ we get $|T : O^-_{2m}(q)| = q^m(q^m - 1)$. But $q^m(q^m - 1)$ has the form $2^{a+1} 3^b 5^{z_1}$ for $0 \leq y_1, z_1 \leq 1$ if and only if
$q = 2^f$ with $fm \in \{2, 4\}$, since $m$ is even. Since $S_4(2) \cong S_6$ is not simple, we deduce that
$(m, f) \in \{(4, 1), (2, 2)\}$.

(3) $T = O_{2m}^\perp(q)$ ($m$ even, $q$ odd) and $M = N_T(O_{2m-1}(q))$. By \cite{8} Corollary 2.10.4 Part (i) and Table 2.1.D, we have $|N_T(O_{2m-1}(q)) : O_{2m-1}(q)| \leq 2$. It follows that $\frac{1}{2}q^{m-1}(q^m - 1) = |T : O_{2m-1}(q)|$ divides $2^{x+1}3^y5^z$. However, since $q$ is odd, $m \geq 4$, and $0 \leq y, z \leq 1$, this cannot be the case.

(4) $T = S_4(q)$ and $M = N_T(S_2(q^2))$. Then \cite{8} Corollary 2.10.4 Part (i) and Table 2.1.D gives $|N_T(S_2(q^2)) : S_2(q^2)| \leq 2$. It follows that $q^2(q^2 - 1) = |T : S_2(q^2)|$ divides $2^{x+1}3^y5^z$. Thus, $q \in \{2, 4\}$. We then use direct computation to see that the only possibility is $S_2(q^2) \leq H$.

(5) In each of the remaining cases (see \cite{8} Table 10.7), we are given a pair $(T, H)$, where $T$ is $L_2(8), L_3(3), L_6(2), U_3(3), U_3(5), U_4(2), U_4(3), U_5(2), U_6(2), S_6(7), S_4(8), S_6(2), O_8^+(2), G_2(3), 2F_4(2)', M_{11}, M_{12}, M_{24}, HS, McL, Co_2$ or $Co_3$, and $Y$ is a subgroup of $T$ containing $H$. Apart from when $T = O_8^+(2), M_{11}, M_{12}$, and $M_{24}$, we find that $|T : Y|$ does not divide $2^{x}3^y5^z$, so we get a contradiction in all other cases. When $T = O_8^+(2)$, the only possibilities are $H = A_9$ of index $2^615$ (three $T$-conjugacy classes fused under the triality automorphism). When $T = M_{11}, M_{12},$ or $M_{24}$, the only possibilities for $H$ are $H = L_2(11) \leq M_{11}$ (of index 12), $H = M_{11} \leq M_{12}$ (of index 12), or $H = M_{23} \leq M_{24}$ (of index 24).

Finally, assume that $T = L_2(p)$, for some Mersenne prime $p$, and let $M$ be a maximal subgroup of $T$ containing $H$. Then, since $|T : M|$ divides $|T : H| = 2^x3^y5^z$ with $0 \leq y, z \leq 1$, $M$ must be parabolic (see \cite{2} Table 8.1]). But then $\pi(|T|) \neq \pi(|M|)$, since $p + 1$ is the highest power of 2 dividing $|T| - a$ contradiction. This completes the proof. \hfill $\square$

This completes our analysis of the pairs $(T, H)$ with $T$ a nonabelian simple group and $H$ a subgroup of $T$ of index $|T : H| = 2^x3^y5^z$, with $0 \leq y, z \leq 1$. The possibilities are listed in Proposition \cite{3} (i)–(iii) and Proposition \cite{3} (i)–(vi).

In particular, if $G$ is a counterexample to Theorem \cite{8} of minimal degree, $L = T^e$ is a minimal normal subgroup of $G$, and $H$ is a point stabilizer in $L$, then we know the possibilities for the group $T$ and the intersections of $H$ with the $i$th coordinate subgroups in $L$. Our next task is to use this information to find out more about the structure of $H$.

One of our main tools for doing this is the Frattini argument. The proof is well-known, but we couldn’t find a reference so we include a proof here.

**Lemma 3.6.** Let $G$ be a finite group, and let $L$ be a normal subgroup of $G$. Let $A \leq \text{Aut}(L)$ be the image of the induced action of $G$ on $L$ (by conjugation). Suppose that $H$ is a subgroup of $L$ with the property that $H$ and $H^\alpha$ are $L$-conjugate for each $\alpha \in A$. Then $G = N_G(H)L$.

**Proof.** Let $g \in G$. Then $H^g = H^l$ for some $l \in L$, by hypothesis. Hence, $gl^{-1} \in N_G(H)$, so $g \in N_G(H)L$, and this completes the proof. \hfill $\square$

With the Frattini argument in mind, the next lemma will be crucial.

**Lemma 3.7.** Let $T$ be a nonabelian finite simple group, and suppose that $T$ has a subgroup $H$ of index $n := 2^x3^y5^z$, with $0 \leq y, z \leq 1$. Assume also that:
If $T = A_5$, then $H$ is either transitive in its natural action, or $H = A_4$;

if $T = A_6$, then $H$ is either transitive in its natural action, or $H = A_5$ is a natural point stabilizer in $T$;

if $T = L_2(q)$, with $q$ of the form $q = 2^{2k}3^55^z - 1$, then $y = y_1$ and $z = z_1$;

if $T = S_4(4)$, then $H \neq S_2(16)$ and $H \neq S_2(16).2$;

if $T = O^-_5(2)$, then $H \neq A_9$ (three classes);

if $T = A_16$, then $H \neq A_{14}$ and $H \neq A_{14}.2$; and

$(T, H)$ is not $(A_8, H)$, with $H$ of shape $7:3$, $2^3 : (7:3)$, $GL_3(2)$, or $2^3 : GL_3(2)$.

Denote by $\Sigma$ the set of right cosets of $H$ in $T$. Then there exists a proper subgroup $S_0$ of $T$ with the following properties:

(i) $S_0$ and $S_0'$ are conjugate in $T$ for each $\alpha \in \text{Aut}(T)$;

(ii) $N_T(S_0)^\Sigma$ is transitive;

(iii) $N_T(S_0)$ acts transitively on the set of cosets of any subgroup of $T$ of 2-power index.

Proof. Note first that if $T = A_n$ and $K$ is a subgroup of $T$ of 2-power index, then $K = A_{n-1}$ and $n$ is a power of 2, by Proposition 3.4 and 3.5. Thus, if $T = A_n$, with $n$ not a power of 2, and $H$ is transitive, then setting $S_0$ to be a natural point stabilizer $S_0 := A_{n-1}$ suffices. Also, if $|T : H|$ is a power of 2, then setting $S_0$ to be a Sylow 2-subgroup of $T$ gives us what we need.

So we may assume that we are in neither of these cases.

By Proposition 3.4 and 3.5, the remaining possibilities for the pair $(T, H)$ are as follows:

(1) $(T, H) = (A_n, A_{n-1})$, with $n = 2^33^55^z$ and $y + z \neq 0$. Set $S_0 = \langle (1, \ldots, 3^55^z), (3^55^z + 1, \ldots, 2 \times 3^55^z), \ldots, (n - 3^55^z + 1, \ldots, n) \rangle$. Then $N_T(S_0)^\Sigma$ is transitive, and $S_0$ and $S_0'$ are $T$-conjugate for all $\alpha \in \text{Aut}(T)$ (this includes the case $n = 6$, when $\text{Out}(A_6)$ has order 4). Thus, (i) and (ii) are satisfied. Property (iii) is vacuously satisfied since $T$ has no proper subgroup of 2-power index in this case, as mentioned at the beginning of the proof.

(2) $T = S_8(2)$, and $O^-_5(2) \leq H$. Note that property (iii) is vacuously satisfied in each case, since $T$ has no proper subgroup of 2-power index. Moreover, $T$ has a maximal parabolic subgroup $S_0$ of shape $2^{10}.GL_4(2)$ satisfying (i) and (ii).

(3) $T = L_4(3)$, $H = 3^3 : L_4(3)$ (two classes). In this case, take $S_0$ to be the unique maximal subgroup of $T$ of shape $L_2(9).4.2$. Then it is easily seen that (i) and (ii) hold, for either of the two non-conjugate choices for $H$. Property (iii) holds since $T$ has no proper subgroups of 2-power index.

(4) $T = S_4(3)$, and $|T : H| = 40$ (two classes). Here, the unique maximal subgroup $S_0$ of $T$ of index 27 has properties (i) and (ii) for each of two non-conjugate choices for $H$. Property (iii) again holds since $T$ has no proper subgroups of 2-power index.

(5) $T = S_6(2)$, and $|T : H| \in \{120, 960\}$. Taking $S_0$ to be the unique maximal subgroup of $T$ of index 36 gives us (i) and (ii) in either of the cases $|T : H| = 120$ and $|T : H| = 960$. Property (iii) holds since $T$ has no proper subgroups of 2-power index.

(6) $(T, H) = (M_{12}, M_{11})$ or $(M_{24}, M_{23})$: In each case, let $S_0$ be a subgroup of $T$ generated by a fixed point free element of order 3. When $T = M_{12}$, $N_T(S_0) \cong A_4 \times S_3$ (see [14]) is a maximal subgroup of $T$, and acts transitively on the cosets of $H$. Thus, (ii) holds. Also, outer automorphisms of $M_{12}$ fix the set of $T$-conjugates of $S_0$, so $S_0$ and $S_0'$ are $T$-conjugate.
for all $\alpha \in \text{Aut}(T)$. Hence, (i) holds. When $T = M_{24}$, $N_T(S_0)$ has order 1008, and acts transitively on the cosets of $H$ (using MAGMA [1], for example). Also, Out$(T)$ is trivial. Thus, (i) and (ii) are again satisfied. Property (iii) is vacuously satisfied in each case, since $T$ has no proper subgroup of 2-power index.

$(7)$ $T = L_2(q)$, with $q$ of the form $q = 2^{2i}3^{y_1}5^{z_1} - 1$, $0 \leq y_1, z_1 \leq 1$. Since we are assuming that $y = y_1$ and $z = z_1$, $H$ must be a maximal parabolic subgroup of $T$ (see [2] Table 8.1). Let $S_0$ be a $D_{q+1}$ maximal subgroup of $T$. Then $S_0$ and $S_0^\alpha$ are $T$-conjugate for all $\alpha \in \text{Aut}(T)$, so (i) holds. Clearly, (ii) also holds. The group $T$ only has a proper subgroup of 2-power index if $y_1 = z_1 = 0$, and in this case the only such subgroup is $H$. Thus, property (iii) is also satisfied.

Corollary 3.8. Let $G$ be a minimal transitive group of degree $n = 2^x3^y5^z$, where $0 \leq y, z \leq 1$. Suppose that $G$ has a nonabelian minimal normal subgroup $L = T^e$, and let $H$ be the intersection of $L$ with a point stabilizer in $G$. Write $|L : H| = 2^{2x}3^{y_0}5^{z_0}$, where $x_0 \leq x$, $y_0 \leq y$, $z_0 \leq z$. Then:

(i) Either $y_0$ or $z_0$ is non-zero;

(ii) Suppose that $H = H\pi_1 \times \ldots \times H\pi_e$, and all but one of the groups $H\pi_i$, say $H\pi_e$, has 2-power index in $T$. Then $(T,H\pi_e)$ cannot satisfy the hypothesis of Lemma 3.7.

Proof. Suppose that either $|L : H|$ is a power of 2; or $H = H\pi_1 \times \ldots \times H\pi_e$ where all but one of the groups $H\pi_i$, say $H\pi_e$, has 2-power index in $T$, and $(T,H\pi_e)$ satisfies the hypothesis of Lemma 3.7. Then we claim that there exists a proper subgroup $S$ of $L$ with $S$ acting transitively on the cosets of $H$ in $L$, and $LN_G(S) = G$ ($\ast$). In particular, $N_G(S)$ is a proper subgroup of $G$, since $L$ is a minimal normal subgroup of $G$. But then $N_G(S) < G$ acts transitively both on an $L$-orbit, and on the set of $L$-orbits, so $N_G(S)$ is transitive. This contradicts the minimal transitivity of $G$. Thus, it will suffice to prove the existence of such a subgroup $S$.

Now, if $|L : H|$ is a power of 2, then $S \in \text{Syl}_2(L)$ satisfies ($\ast$), by Lemma 3.6 and the fact that $(|L : H|, |L : S|) = 1$. So assume that $H = H\pi_1 \times \ldots \times H\pi_e$, and all but one of the groups $H\pi_i$, say $H\pi_e$, has 2-power index in $T$. Assume that the pair $(T,H\pi_e)$ satisfies the hypothesis of Lemma 3.7, and let $S_0$ be the subgroup of $T$ exhibited therein. Set $S := S_0^\alpha \leq L$. Then $S$ acts transitively on the cosets of $H$ in $L$. Moreover, since $\text{Aut}(L) \cong \text{Aut}(T) \wr \text{Sym}_e$, Lemma 3.7(i) implies that $S$ and $S^\alpha$ are $L$-conjugate for all $\alpha \in \text{Aut}(L)$. Thus, $LN_G(S) = G$ by Lemma 3.6, so $S$ satisfies ($\ast$), and this completes the proof.

In what follows, recall from Definition 2.2 that a subdirect product of the form $G = \bigoplus_n (G_1 \times \ldots \times G_e)$ is a subdirect product of index $n$ in $G_1 \times \ldots \times G_e$ containing $[G_1,G_1] \times \ldots \times [G_e,G_e]$.

Corollary 3.9. Let $G$ be a minimal transitive group of degree $n = 2^x3^y5^z$, where $0 \leq y, z \leq 1$. Suppose that $G$ has a nonabelian minimal normal subgroup $L = T^e$, and write $\pi_i : L \to T$ for the $i$th coordinate projections. Let $H$ be the intersection of $L$ with a point stabilizer in $G$. Write $|L : H| = 2^{2x_0}3^{y_0}5^{z_0}$, where $x_0 \leq x$, $y_0 \leq y$, $z_0 \leq z$. Then one of the following holds:

(1) $T = \Lambda_n$ with $n \in \{5,6\}$ and one of the following holds:
(a) $H$ is $\text{Sym}_e$-conjugate to $T^{e-2} \times H_{e-1} \times H_e$, where $H_{e-1}$ and $H_e$ are subgroups of $T$
with $[T : H_{e-1}] = 2^{e-3}$, $[T : H_e] = 2^{e-5}$, and $x_0 = x_{e-1} + x_e$;
(b) $H$ is $\text{Sym}_e$-conjugate to $T^{e-1} \times H_e$, where $H_e \leq T$ is not transitive in its natural action;
$H_e$ is not $T$-conjugate to a natural point stabilizer $A_{n-1}$; and $[T : H_e] = |L : H|$;
(c) $T = A_5$ and $H$ is $\text{Sym}_e$-conjugate to $T^{e-2} \times J$, where $J < T^2$ is a direct product of
the form $J = \frac{1}{4}(S_3 \times D_{10}) < T^2$; or
(d) $T = A_5$ and $H$ is $\text{Sym}_e$-conjugate to $T^{e-2} \times J$, where $J$ is a diagonal subgroup of
$T^2$.
(2) $T = A_8$ and $H$ is $\text{Sym}_e$-conjugate to $T^{e_1} \times H_{e_1+1} \times \cdots \times H_{e_e-1} \times H_e$, where $H_e$ is one of the
subgroups of shape $7 : 3$, $2^3 : (7 : 3)$, $\text{GL}_3(2)$, or $2^5 : \text{GL}_3(2)$ in $T$; $H_i$ is $T$-conjugate to $A_7$
for all $e_i + 1 \leq i \leq e - 1$; and $0 \leq e_i \leq e - 1$.
(3) $T = A_{16}$ and $H$ is $\text{Sym}_e$-conjugate to $T^{e_1} \times H_{e_1+1} \times \cdots \times H_{e_e-1} \times H_e$, where $H_e$ is one of the
subgroups of shape $A_{14}$ or $A_{14} : 2$ in $T$; $H_i$ is $T$-conjugate to $A_{15}$ for all $e_i + 1 \leq i \leq e - 1$;
and $0 \leq e_i \leq e - 1$.
(4) $T = L_2(q)$, with $q$ of the form $q = 2^i 3^j 5^r$, $0 \leq y', z' \leq 1$, and one of the following holds:
(a) $y' = z' = 0$; $5 \mid q - 1$; $H$ is $\text{Sym}_e$-conjugate to $T^{e_1} \times P_1 \times \cdots \times P_{e_2} \times H_{e_e-1} \times H_e$, where
each $P_i$ is a maximal parabolic subgroup of $T$; $H_{e_i}$ [respectively $H_e$] is a subgroup of
index $3$ [resp. $5$] in a maximal parabolic subgroup of $T$; and $e_1 + e_2 = e - 2$;
(b) $y' = z' = 0$ and $H$ is $\text{Sym}_e$-conjugate to $T^{e_1} \times L_1$, where $L_1$ is a subdirect product of the
form $L_1 = \frac{1}{s}(L_2 \times H_{e_i})$; $L_2$ is a subdirect product of the form $L_2 = \frac{1}{s}(P_1 \times \cdots \times P_{e_2})$;
each $P_i$ is a maximal parabolic subgroup of $T$; $H_{e_i}$ is a subgroup of $T$ with $r = 3^5 5^{n_0}$; and $e_1 + e_2 = e - 1$; or
(c) $y' + z' = 1$, and $H$ is $\text{Sym}_e$-conjugate to $T^{e-1} \times \left(q : \frac{q - 1}{2^i 3^j 5^r}\right)$, where $r > 1$ divides $q - 1$, and
$3^j 5^r 7^{n_0} = 3^5 5^{n_0}$.
(5) $T = S_4(4)$ and $H$ is $\text{Sym}_e$-conjugate to $T^{e-1} \times H_e$, where $H_e \in \{S_2(16) \text{ (two } T\text{-classes)}$
$S_2(16).2 \text{ (two } T\text{-classes)}\}$.
(6) $T = O_5^+(2)$ and $H$ is $\text{Sym}_e$-conjugate to $T^{e-1} \times H_e$, where $H_e = A_9$ (three $T$-classes, fused
under a triality automorphism).

Proof. By Corollary 3.8, either $y_0$ or $z_0$ is non-zero. Now, set $H_i := (T_i \cap H)\pi_i$, for $1 \leq i \leq e$.
Then $H_i \leq H\pi_i$, and $[T : H_i]$ has the form $2^{e_i} 3^j 5^s$ for each $i$. Note also that since $H$ is
a subgroup of $H\pi_1 \times \cdots \times H\pi_e \leq L$, we have that $\prod_{i=1}^{e} [T : H\pi_i]$ divides $|L : H|$. Thus,
$[T : H\pi_i]$ has the form $2^{e_i} 3^j 5^s$ for each $i$, and $y_i$ [respectively $z_i$] is 1 for at most $y_0$ [resp.$z_0$] values of $i$. It follows that $[T : H\pi_i]$ is a power of 2 for all but at most two values of $i$.
Let $\Lambda := \{i : [T : H\pi_i]$ is a power of 2$\}$, so that $|\Lambda| \geq t - 2$. For a subset $E$ of $\{1, \ldots, e\}$, let
$\pi_E : L \rightarrow \prod_{i \in E} T_i$ be the natural projection homomorphism onto the ‘$E$-part’ of $L$. Then since
$H$ is a subgroup of $H\pi_\Lambda \times \prod_{i \in \Lambda} H\pi_i$ (up to permutation of coordinates), we have that

$$[T^{[\Lambda]} : H\pi_\Lambda] \prod_{i \in \Lambda} [T : H\pi_i] \text{ divides } |L : H|.$$ 

Furthermore, we can determine the subgroups of 2-power index in the simple groups $T$ from
Propositions 3.4 and 3.5; we see that either

(1) $H\pi_i = T$ for all $i \in \Lambda$
(II) $T = A_{2^e}$ and $H_{\pi_i}$ is $T$-conjugate to either $T$ or $A_{2^e-1}$ for all $i \in \Lambda$; or

(III) $T = L_2(p)$, with $p = 2^{e_i} - 1$, and $H_{\pi_i}$ is either $T$ or a maximal parabolic subgroup of $T$, for all $i \in \Lambda$.

Assume first that $|\Lambda| = t - 2$, and without loss of generality, write $[i] \setminus \Lambda = \{e - 1, e\}$. Then we may assume, again without of loss of generality, that $|T : H_{\pi_{e-1}}| = 2^{e-1}3$ and $|T : H_{\pi_e}| = 2^{e}5$. It then follows from (3.2) that

$$|T^{e-2} : H_{\pi_\Lambda}|$$

is a power of 2.

Next, from Propositions 3.4 and 3.5 we can determine all finite simple groups $T$ which contain both a subgroup of index $2^{e-1}3$ and a subgroup of index $2^e5$: we see that $T \in \{L_2(p), A_5, A_6 : p \text{ a Mersenne prime}, 15 \mid p - 1\}$. Suppose first that $T = A_n$, for $n \in \{5, 6\}$. Then since $T$ is the only subgroup of $T$ of 2-power index, we have $H_{\pi_i} = T$ for all $i \in \Lambda$. Thus, $H$ is Sym$_{\pi_i}$-conjugate to $T^{e-2} \times J$, where $J$ is a subdirect subgroup of $H_{\pi_{e-1}} \times H_{\pi_e}$, by Lemma 2.3 and (3.3). We can now quickly determine, using Magma [1] for example, all subgroups $J$ of $T^2$ where $J_{\pi_{e-1}}, J_{\pi_e} \subset T$, and $|T^2 : J|$ has the form $2^43^55^C$. This yields cases (1)(a) and (1)(c) in the statement of the corollary.

Suppose next that $T = L_2(p)$, where $p$ is a Mersenne prime. Then the only subgroups of $T$ of 2-power index are $T$ itself, together with the maximal parabolic subgroup $P < T$, of index $p + 1$. It follows from (3.3) and Lemma 2.3(v) that $H_{\pi_\Lambda} = T^{e_1} \times L_2$ (up to permutation of coordinates), where $L_2$ is a subdirect product in $P_1 \times \ldots \times P_{e_2}$, each $P_i$ is $T$-conjugate to $P$, and $e_1 + e_2 = e$. Also, since $P$ has odd order, (3.3) implies that that $L_2 = P_1 \times \ldots \times P_{e_2}$. Hence, $H_{\pi_\Lambda} = T^{e_1} \times P_1 \times \ldots \times P_{e_2}$ (up to permutation of coordinates). Thus, $H$ is a subdirect product in $T^{e_1} \times P_1 \times \ldots \times P_{e_2} \times J$, where $J := H_{\pi_{\{e-1, e\}}} \leq H_{\pi_{e-1}} \times H_{\pi_e}$. Now, the only subgroup of $T$ (up to $T$-conjugacy) with index $2^{e-1}3$ is the unique subgroup $Y_2$ of $P$ with $|P : Y| = 3$, $|T : Y| = 3(p + 1)$; while the only subgroup of $T$ (up to $T$-conjugacy) with index $2^e5$ is the unique subgroup $Y_3$ of $P$ with $|P : Y| = 5$, $|T : Y| = 5(p + 1)$. Thus, $J$ is a subdirect product in $H_{\pi_{e-1}} \times H_{\pi_e} \cong p : \frac{e_1 - 1}{6} \times p : \frac{e_2 - 1}{10}$. Since $|T^2 : J|$ has the form $2^43^55^C$ and $|J|$ is odd, we must have $J = H_{\pi_{e-1}} \times H_{\pi_e} \cong p : \frac{e_1 - 1}{6} \times p : \frac{e_2 - 1}{10}$. Moreover, since $|T_{e-1} \times T_e : H \cap (T_{e-1} \times T_e)|$ has the form $2^43^55^C$, we must have $J = H_{\pi_{e-1}} \times H_{\pi_e} = H \cap (T_{e-1} \times T_e)$. It follows that $H$ is Sym$_e$-conjugate to $T^{e_1} \times P_1 \times \ldots \times P_{e_2} \times H_{\pi_{e-1}} \times H_{\pi_e}$, and this yields case (4)(a) in the statement of the corollary.

Suppose next that $|\Lambda| = t - 1$. Without loss of generality, we may assume that $[i] \setminus \Lambda = \{e\}$. Hence, $|T : H_{\pi_e}| = 2^{e-r}$, with $r \in \{3, 5, 15\}$. It then follows from (3.2) that

$$|T^{e-1} : H_{\pi_\Lambda}| |2^{e-r}$$

divides $|L : H|$ for some $r \in \{3, 5, 15\}$. In particular, $|T|$ does not divide $|T^{e-1} : H_{\pi_\Lambda}|$. Suppose that case (I) occurs (recall that (I) is defined at the end of the first paragraph of this proof). Then $H$ is Sym$_e$-conjugate to $T^{e-1} \times H_e$ by (3.1) and Lemma 2.3(v). Suppose now that we are in case (II). Then $H_{\pi_\Lambda}$ is a subdirect product in $T^{e_1} \times H_{e_1+1} \times \ldots \times H_{e-1}$, where $0 \leq e_1 \leq e - 1$, and each $H_i$ is $T$-conjugate to $A_{2^{e_i}-1}$. The analogous argument to the above then shows that $H$ is Sym$_e$-conjugate to $T^{e_1} \times H_{e_1+1} \times \ldots \times H_e$, where $0 \leq e_1 \leq e - 1$. By Corollary, 3.8 the pair $(T, H_e)$ cannot satisfy
the hypothesis of Lemma 3.3, in either of the cases (I), (II). Since $|T : H_e|$ has the form $2^43^8$, $2^45$, or $2^415$, Propositions 3.4 and 3.5 then imply that one of the following holds:

- $T = A_n$ with $n \in \{5, 6\}$, and $H_e$ is both intransitive and not conjugate to $A_4$;
- $T = A_8$, and $H_e$ has shape $7 : 3$, $2^3 : (7 : 3)$, $GL_3(2)$, or $2^3 : GL_3(2)$;
- $T = A_{16}$, and $H_e \in \{A_{14}, A_{14}'\}$;
- $T = S_4(4)$, and either $H_e \cong S_2(16)$ or $H_e \cong S_2(16)\cdot 2$;
- $T = O_8^+(2)$, and $H \cong A_9$; or
- $T = L_2(q)$ with $q = 2^{y+3y'5^{z'}} - 1$; $|L : H| = r(q + 1)$ for some $r \in \{3, 5\}$ (so either $y'$ or $z'$ is 0), and $H_e \cong q : \frac{2^{y'+1}}{2^r}$.

The first five of these are cases (1)(b), (2), (3), (5), and (6), respectively, in the statement of the corollary. The last is case (4)(b) (with $s = t = 1$) if $y' = z' = 0$, and case (4)(c) otherwise. Finally, assume that case (III) holds. Then using (3.4) and Lemma 2.2 again, we see that $H$ is Sym$_c$-conjugate to $T^{e_1} \times L_1$, where $L_1$ is a subdirect product in $L_2 \times H\pi_e$; $L_2$ is a subdirect product in $P_1 \times \ldots \times P_{e_2}$; each $P_i$ is a maximal parabolic subgroup of $T$; and $e_1 + e_2 = e - 1$. Moreover, $H\pi_A = T^{e_1} \times L_2$ (up to permutation of coordinates). In particular, $|H| = |T^{e_1}||L_2||H_e|$. Thus, $|L : H| = |T^{e_2} : L_2||T : H\pi_e||H\pi_e : H_e|$. Let $s$ and $t$ be the odd parts of $|T^{e_2} : L_2|$ and $|H\pi_e : H_e|$, respectively. Then $s$ is the index of $L_2$ in $P_1 \times \ldots \times P_{e_2}$, $rst = 3^{s0}5^{z0}$, and one of the following must hold:

- $s = t = 1$ and $H$ is Sym$_c$-conjugate to $T^{e_1} \times L_2 \times H\pi_e$, $L_2 = P_1 \times \ldots \times P_{e_2}$.
- $s \in \{3, 5\}$, $t = 1$, and $H$ is Sym$_c$-conjugate to $T^{e_1} \times L_2 \times H\pi_e$, where $L_2$ has the form $L_2 = \frac{1}{t}(P_1 \times \ldots \times P_{e_2})$.
- $t \in \{3, 5\}$, $s = 1$, and $H$ is Sym$_c$-conjugate to $T^{e_1} \times L_1$, where $L_1$ has the form $L_1 = \frac{1}{t}(P_1 \times \ldots \times P_{e_2} \times p : \frac{2^{y'+1}}{2^r})$.

These are the cases in (4)(b) in the statement of the corollary.

Finally, assume that $|A| = t$. Suppose first that case (I) holds (see the first paragraph above), so that $H$ is a subdirect product in $L = T^e$. Then $|T|$ divides $|L : H|$ by Lemma 2.2(iii), since $H \neq L$. Thus, we must have $T = A_5$, since $|L : H|$ divides $2^515$. Moreover, $H$ is Sym$_c$-conjugate to $T^{e_2} \times J$, where $J$ is a diagonal subgroup of $T^2$, since $|T|^2$ cannot divide $|L : H|$. This is case (1)(d) in the statement of the corollary. Suppose next that case (II) holds. Then since neither $|A_{2e_2}|$ nor $|A_{2e_1-1}|$ can divide $|L : H|$ in this case, another application of Lemma 2.2 yields that $H$ is Sym$_c$-conjugate to $T^{e_1} \times Y_1 \times \ldots \times Y_{e_2}$, where $Y_i$ is $T$-conjugate to $A_{2e_i-1}$ for each $i$, and $e = e_1 + e_2$. But then $|L : H|$ is a power of 2 – a contradiction. Suppose next that case (III) holds. Then arguing as in the $|A| = t - 1$ case above we get $H$ is Sym$_c$-conjugate to $T^{e_1} \times L_2$, where $L_2$ is a subdirect product of the form $L_2 = \frac{1}{t}(P_1 \times \ldots \times P_{e_2})$, with $P_i$ a maximal parabolic subgroup of $T$ for all $1 \leq i \leq e_2$, and $e_1 + e_2 = e$. If (III) holds with $3^{s0}5^{z0} = 1$, then $|L : H|$ is a power of 2 – a contradiction. Thus, if case (III) holds then $3^{s0}5^{z0} \neq 1$. This is case (4)(b) (with $r = 1$) in the statement of the corollary. The proof is complete. □
Corollary 3.10 gives us information about the possible minimal normal subgroups $L$ (and their actions) in a minimal counterexample to Theorem 3.1. We would now like to analyse the orbit lengths of certain soluble subgroups of $L$.

Since a point stabilizer $H$ in $L$ in this case is a direct product of subgroups $X$ which are themselves subdirect products of the form $X = \frac{1}{n}(X_1 \times \ldots \times X_f)$ (see Definition 2.22), our analysis will have two parts: First, we will show how to determine the orbit lengths of certain subgroups $F < L$ acting on the cosets of a subdirect product $H$ of the form $H = \frac{1}{n}(P_1 \times \ldots \times P_e)$, where each $P_i$ is a subgroup of $T$. Then, we will show how to determine the orbit lengths of certain subgroups $F < L$ acting on the cosets of a subgroup $H < L$ of the form $H = H_1 \times \ldots \times H_r$, where $H_i \leq T^{e_i}$, and $\sum_{i=1}^r e_i = e$ (i.e. when $L$ has ‘product action’ type). These two bits of information, together with Corollary 3.9, will then allow us to prove Theorem 3.1(iv). We begin with the first part:

**Lemma 3.10.** Let $L = G_1 \times \ldots \times G_e$, where each $G_i$ is a finite group, and fix proper subgroups $P_i$, $S_i$ of $G_i$. Suppose that $H \leq P_1 \times \ldots \times P_e$ is a subdirect product of the form $H = \frac{1}{n}(P_1 \times \ldots \times P_e)$, and that $H$ is core-free in $L$. Assume that $S := S_1 \times \ldots \times S_e$ has orbit lengths $m_1, \ldots, m_r$ in its action on the cosets of $P := P_1 \times \ldots \times P_e$ in $L$, where $\sum_i m_i = [L : P]$. Assume also that $(S_i \cap P_i^{x_i})[P_i^{x_i}, P_i^{x_i}] = P_i^{x_i}$ for all $x_i \in G_i$. Then $S$ has $r$ orbits in its action on the cosets of $H$ in $L$, with lengths $nm_1, \ldots, nm_r$.

**Proof.** Note first that since $H$ is core-free in $L$, $L$ may be viewed as a transitive permutation group acting on the set $\Omega$ of cosets of $H$ in $L$. Moreover, $H < P$, so the set of cosets of $P$ in $L$ is permutation isomorphic to a set $\Sigma$ of blocks for $P$ in $\Omega$. Note that the blocks in $\Sigma$ have size $n$, $H \leq P$, and $P/H$ is abelian of order $n$. It follows that $L$ is isomorphic to a subgroup of the wreath product $n \wr \Sigma$.

Now, write $\Sigma_i$, $1 \leq i \leq r$, for the $S$-orbits in $\Sigma$, where $|\Sigma_i| = m_i$. Also, fix a block $\Delta_i \in \Sigma_i$, for each $i$. If $\text{Stab}_S(\Delta_i)^{\Delta_i}$ is transitive for each $i$, then it follows that the $S^\Omega$-orbits have sizes $|\Delta_i| |\Sigma_i| = nm_i$, for each $i$. Thus, we just need to prove that

\[ (3.5) \quad \text{Stab}_S(\Delta_i)^{\Delta_i} \text{ is transitive for each } i. \]

To this end, fix $1 \leq i \leq e$. Then $\text{Stab}_S(\Delta_i)^{\Delta_i} = S \cap P_i^{\alpha_i}$ for some $\alpha_i \in L$. To show that $\text{Stab}_S(\Delta_i)^{\Delta_i}$ is transitive, we just need to show that $H^{\alpha_i}(S \cap P_i^{\alpha_i}) = P_i^{\alpha_i}$. But $\alpha_i = (x_1, \ldots, x_e)$, for some $x_j \in G_j$. Thus, $S \cap P_i^{\alpha_i} = (S_1 \cap P_1^{x_1}) \times \ldots \times (S_e \cap P_e^{x_e})$. Since $H^{\alpha_i}$ contains $[P_1^{x_1}, P_1^{x_1}] \times \ldots \times [P_e^{x_e}, P_e^{x_e}]$, we have $H^{\alpha_i}(S \cap P_i^{\alpha_i}) = P_i^{\alpha_i}$ by hypothesis. This proves (3.5), whence the lemma.

Next, we prove a general lemma concerning the orbits of subgroups in a transitive permutation group with ‘product action’.

**Lemma 3.11.** Let $L = G_1 \times \ldots \times G_e$, where each $G_i$ is a finite group, and fix proper subgroups $H_i$, $S_i$ of $G_i$. Set $S = S_1 \times \ldots \times S_e$ and $H = H_1 \times \ldots \times H_e$. Then:

(i) Suppose that $S_i$ acts transitively on the cosets of $H_i$ in $G_i$ for $1 \leq i \leq e-1$, and that $S_e$ has $r$ orbits, of lengths $l_1, \ldots, l_r$ say, in its action on the cosets of $H_e$ in $G_e$. Then $S$ has $r$ orbits in its action on the cosets of $H$ in $L$, of lengths $l_1m, \ldots, l_rm$, where $m := \prod_{i=1}^{e-1} |G_i : H_i|$. 

(ii) Assume that there exists $0 \leq e_1 \leq e - 1$ such that $S_i$ acts transitively on the cosets of $H_i$ in $G_i$ for $i \leq e_1$; that $S_i$ has 2 orbits of lengths $k_1$ and $k_2$ in each of the cosets spaces $H_i \setminus G_i$, for $e_1 + 1 \leq i \leq e - 1$ (in particular, $|G_i : H_i| = |G_j : H_j| = k_1 + k_2$ for all $e_1 + 1 \leq i, j \leq e - 1$ in this case); and that $S_e$ has $r$ orbits, of lengths $l_1, \ldots, l_r$ say, in its action on the cosets of $H_e$ in $G_e$. Then $S$ has $2^{e-e_1-1} \times r$ orbits in its action on the cosets of $H$ in $G$, with $(e-e_1-1)$ orbits of length $l_j k_1^e k_2^{e-e_1-i}$ for each $0 \leq i \leq e - e_1 - 1$, and each $1 \leq j \leq r$.

Proof. The proof here is routine: if $S_i$ is a group acting on a set $\Omega_i$, then the orbits of $S_1 \times \ldots \times S_e$ in its product action on $\Omega_1 \times \ldots \times \Omega_e$ are precisely the sets $\Delta_{i_1,1} \times \ldots \times \Delta_{i_e,f}$, where the $\Delta_{i_1,j}, \ldots, \Delta_{i_e,j}$ are the orbits of $S_j^{\Omega_j}$, $1 \leq j \leq e$.

We will also need the following corollary, which is notationally heavy, but routine.

Corollary 3.12. Let $L$, $H$, and $S$ be as in Lemma 3.11(i), and assume that $G_i = T$ for $1 \leq i \leq e - 1$, and $G_e = T^k$ where $T$ is a fixed nonabelian finite simple group. Assume also that $S_i = S_0$ for all $1 \leq i \leq e - 1$, and $S_e = S_0^L$ where $S_0$ is a fixed subgroup of $T$. Write $n_1, \ldots, n_u$ for the distinct orbit lengths of $S_e$ acting on the cosets of $H_e$ in $T^k$, and suppose that $S_e$ has $f(n_j)$ orbits of length $n_j$. Let $G \leq \text{Sym}(\Omega)$ be a finite transitive permutation group in which $L$ is a minimal normal subgroup. Suppose that $H$ is the intersection of $L$ with a point stabilizer in $G$, and let $K$ be the kernel of the action of $G$ on the set of $L$-orbits. Let $R/K$ be a soluble subgroup of $G/K$ with orbits of lengths $a_1, \ldots, a_t$ on the set of $L$-orbits. Assume also that:

(i) $S_0$ is $T$-conjugate to $S_0^\alpha$ for all automorphisms $\alpha$ of $T$;
(ii) $N_L(S)$ is soluble (equivalently, $N_T(S_0)$ is soluble).

Then $F$ has orbit lengths $a_i X_k^{(i,j)} n_j m$ for some positive partitions $f(n_j) = \sum r_{i,j} X_k^{(i,j)}$ of $f(n_j)$, where $1 \leq j \leq u$, $1 \leq i \leq t$.

Proof. Let $\omega \in \Omega$ such that $H = L \cap \text{Stab}_G(\omega)$, and let $\Delta$ be the $H$-orbit containing the point $\omega$. Let $\Sigma := \{ \Delta^g : g \in G \}$ be the set of $L$-orbits. Fix a subgroup $B$ of $G$, and assume that $B^\Sigma$ has orbits $\Sigma_1, \ldots, \Sigma_r$, of lengths $b_1, \ldots, b_r$, respectively. Let $\Delta_1, \ldots, \Delta_r \in \Sigma$ be representatives of these orbits. Then $B$ acts on each of the sets $\bigcup_{\alpha \in B} \Delta_\alpha^\Sigma$. Suppose that $\text{Stab}_B(\Delta_\alpha)^\Sigma$ has orbits lengths $x_{i_1,1}, \ldots, x_{i_1,k_1}$. Then $B^\Sigma$ has orbit lengths $b_i x_{i_1,1}, \ldots, b_i x_{i_1,k_1}$.

Now, as in the proof of Corollary 3.8, $S_0$ being $T$-conjugate to $S_0^\alpha$ for all automorphisms $\alpha$ of $T$ implies that $S = S_0^\alpha$ is $T$-conjugate to $S_0^\alpha$ for all automorphisms $\alpha \in \text{Aut}(T) \cap S_e \cong \text{Aut}(L)$. Hence, $L N_G(S) = G$, by Lemma 3.11. Thus, we may choose a subgroup $F$ of $N_G(S)$ containing $N_L(S)$ with $LF/L = R/L$. It follows that $F^\Sigma$ has orbits $\Sigma_1, \ldots, \Sigma_t$ of lengths $a_1, \ldots, a_t$, respectively. Fix $\Delta_\alpha \in \Sigma$. Then since $F$ normalizes $S$ and $S$ acts trivially on $\Sigma$, each $\text{Stab}_F(\Delta_\alpha)^{\Delta_\alpha}$-orbit is a union of $S$-orbits of the same length. Thus, by Lemma 3.11 the orbit lengths of $\text{Stab}_F(\Delta_\alpha)^{\Delta_\alpha}$ are $X_1^{(i,j)} n_j m, \ldots, X_\alpha^{(i,j)} n_j m$, where $\sum_{k=1}^{r_i} X^{(i,j)} = f(n_j)$, for each $i, j$. Using the first paragraph above, we deduce that $F^\Sigma$ has orbits of lengths $a_i X_k^{(i,j)} n_j m$, for $1 \leq k \leq r_i$, $1 \leq j \leq u$, $1 \leq i \leq t$.

Finally, since both $FL/L = R/L$ and $F \cap L = N_L(S)$ are soluble, the group $F$ is soluble. This completes the proof. \qed
Lemma 3.13. Let $G$ be a minimal counterexample to Theorem 3.1, and let $L = T^e$ be a minimal normal subgroup of $G$, where $T = A_5$. Let $H$ be the intersection of $L$ with a point stabilizer for $G$, and let $a$ be the number of $L$-orbits. Then $G$ has a soluble subgroup $F$ such that one of the following holds:

(i) If $H$ is as in case (1)(a) of Corollary 3.11, then $F$ has orbit lengths $2^{b_1}5aX_i, 2^{b_2}50aY_j$, where

$$\left(\sum_i X_i, \sum_j Y_j, b_1, b_2\right) \in \{(4, 2, 0, 0), (4, 2, 1, 1), (2, 2, 1, 0), (2, 2, 0, -1), (2, 1, 0, 0), (1, 1, 0, -1)\}.$$ 

(ii) If $H$ is as in case (1)(b) of Corollary 3.11, and $G/L$ is soluble, then $F$ either has orbit lengths $2^{b_i}5aX_i$, where

$$\left(\sum_i X_i, b\right) \in \{(2, 1), (2, 0)\},$$

or $5aX_i, 10aY_j$, where $(\sum_i X_i, \sum_j Y_j) = (2, 2)$.

(iii) If $H$ is as in case (1)(c) of Corollary 3.11, then $F$ has orbit lengths $5aX_i, 25aY_i$, where

$$\left(\sum_i X_i, \sum_j Y_j\right) = (4, 4).$$

(iv) If $H$ is as in case (1)(d) of Corollary 3.11, then $F$ has precisely two orbits, of lengths $10a$ and $50a$.

Proof. Set $S_0 := D_{10}$, $S := S_0^e$, and $F := N_G(S)$. We note first that $N_T(S_0)$ is soluble, and $D_{10}$ is $A_5$-conjugate to $S_0^\alpha$ for all $\alpha \in S_5$. Thus, (i) and (ii) in Corollary 3.12 hold. We will also show that $G$ modulo the kernel $K$ of the action of $G$ on the set of $L$-orbits is soluble in each case, so we can take $R = G$ in Corollary 3.12. Since either $H$ is $\text{Sym}_e$-conjugate to $T^{e-2} \times J$ with $J \leq T^2$, or $H$ is $\text{Sym}_e$-conjugate to $T^{e-1} \times J$ with $J \leq T$, we therefore just need to compute the orbit lengths for $S_0^2$ [respectively $S_0$] in the case $e = 2$ [resp. $e = 1$] and then apply Lemma 3.11 and Corollary 3.12.

Suppose first that we are in case (i). Then either $|T_1 : H\pi_{e-1}| = 2^{2+3}$, and $|T_1 : H\pi_e| = 2^{2+5}$, or vice versa, where $x_1 + x_2 = x$. Thus, $G/K$, being a minimal transitive group of 2-power degree, is soluble (so we can indeed take $G = R$ in Corollary 3.12). Moreover, $H_{e-1} \in \{C_5, D_{10}\}$ and $H_e \in \{A_3, S_3, A_4\}$. The orbit lengths of $S_0$ acting on the cosets of $H_{e-1}$ and $H_e$ are then given in Table 2. If $S_0$ has orbit lengths $c_1, \ldots, c_k$ on the cosets of $H_{e-1}$, and $d_1, \ldots, d_l$ on the
cosets of $H_e$, then $S_0^e$ has orbit lengths $c_i d_j$, $1 \leq i \leq k$, $1 \leq j \leq l$, on the cosets of $H_{e-1} \times H_e$. Thus, we can use the table above and Lemma 3.11 and Corollary 3.12 to deduce the orbit lengths of the soluble group $F = N_G(S)$. For example, if $H_{e-1} = A_3$ and $H_e = C_5$, then $S_0^e$ has four orbits of length 10, and two orbits of length 100 in its action on the cosets of $H_{e-1} \times H_e$ in $T^2$. By Lemma 3.11(ii), $S_0^e$ then has orbits of the same lengths in its action on the cosets of $H$ in $L$ ($m = 1$ in this case). Thus, by Corollary 3.12 $F$ has orbit lengths 10$aX_i$, 100$aY_j$, where $X_i, Y_j \in \mathbb{N}$, and $\sum X_i = 4$, $\sum Y_j = 2$.

We now move on to cases (ii), (iii) and (iv). In case (ii), $K \geq L$, so $G/K$ is soluble by hypothesis. In cases (iii) and (iv), $|L : H| = 2^{r}15$, so $G/K$ is a minimal transitive group of 2-power degree, whence soluble. Thus, we can take again $G = R$ (and hence $t = 1$, $a = a_1$) in Corollary 3.12.

For part (ii), the computations of the possible orbit lengths is completely analogous to the computations in case (i) above (except that the cases $H_e \in \{A_4, C_5, D_{10}\}$ cannot occur, and we also need to compute the orbit lengths of the subgroups of $A_5$ of orders 1, 2, and 4). For part (iii), one can use Magma for example, to see that $S_0^e$ has four orbits of length 5 and four orbits of length 25, in its action on the cosets of $J = \frac{1}{2}(S_3 \times D_{10})$ in $T^2$. For part (iv), $S_0^e$ has one orbit of length 10 and one orbit of length 50 in its action on the cosets of any diagonal subgroup in $T^2$. We now apply Corollary 3.12 in each case to complete the proof. 

Lemma 3.14. Let $G$ be a minimal counterexample to Theorem 3.7 and let $L = T^e$ be a minimal normal subgroup of $G$, where $T = A_6$. Let $H$ be the intersection of $L$ with a point stabilizer for $G$, and let $a$ be the number of $L$-orbits. Then $G$ has a soluble subgroup $F$ such that one of the following holds:

(i) If $H$ is as in case (1)(a) of Corollary 3.7, then $F$ has orbit lengths $2^b5aX_i$, $2^b50aY_j$, where

$$\left(\sum X_i, \sum Y_j, b\right) \in \{(4, 4, 1), (4, 4, 0), (2, 2, 0)\}.$$  

(ii) If $H$ is as in case (1)(b) of Corollary 3.7, and $G/L$ is soluble, then $F$ either has orbit lengths $2^b5aX_i$, where

$$\left(\sum X_i, b\right) \in \{(4, 1), (12, 1), (4, 0), (3, 0), (2, 0)\},$$  

or $5aX_i$, $10aY_j$, where $(\sum X_i, \sum Y_j) \in \{(4, 4), (2, 2)\}$.

Proof. Set $S_0 := D_{10} < A_5 < A_6$, $S := S_0^e$, and $F := N_G(S)$. Then $S_0$ has soluble normalizer in $A_6$, and $D_{10}$ is $A_6$-conjugate to $S_0^e$ for all $\alpha \in \text{Aut}(A_6)$. Thus, (i) and (ii) in Corollary 3.12 hold. Note also that, as in the proof of Lemma 3.13, $G$ modulo the kernel $K$ of the action of $G$ on the set of $L$-orbits is soluble in each of (i) and (ii). Thus, we can take $R = G$ in Corollary 3.12 (and hence $t = 1$, $a = a_1$). Since either $H$ is $\text{Sym}_e$-conjugate to $T^{e-2} \times J$ with $J = H_{e-1} \times H_e \leq T^2$, or $H$ is $\text{Sym}_e$-conjugate to $T^{e-1} \times J$ with $J = H_e \leq T$, we therefore just need to compute the orbit lengths for $S_0^e$ [respectively $S_0$] in the case $e = 2$ [respectively $e = 1$] and then apply Lemma 3.11 and Corollary 3.12.

Suppose first that we are in case (i). Then, without loss of generality, we may assume that $|T_i : H\pi_{e-1}| = 2^{x_1}3$ and $|T_i : H\pi_e| = 2^{x_2}5$, where $x_1 + x_2 = x$. Hence, $H_{e-1}$ is in one of the
two conjugacy classes of $A_5$ in $A_6$, while $H_e \in \{A_3 \times A_3, (A_3 \times A_3) \cdot 2, (A_3 \times A_3) \cdot 2 \cdot 2\}$. The orbit lengths of $S_0$ acting on the cosets of $H_{e-1}$ and $H_e$ are then given in Table 3. The computations of the possible orbit lengths are then completely analogous to the computations in the proof of Lemma 3.13.

If we are in case (ii), then $H_e$ is either one of the groups $J$ in Table 3 with $J$ intransitive, or $|H_e| \in \{3, 6, 12, 24\}$. We can quickly compute the orbit lengths of $S_0$ in these latter cases, and part (ii) follows as in the proof of Lemma 3.13.

Lemma 3.15. Let $G$ be a minimal counterexample to Theorem 5.1, and let $L = T^e$ be a minimal normal subgroup of $G$, where $T \in \{S_4(4), O_5^+(2)^\alpha\}$. Let $H$ be the intersection of $L$ with a point stabilizer for $G$, and let $a$ be the number of $L$-orbits. Then $G$ has a soluble subgroup $F$ such that either $n = 120a$ and $F$ has two orbits of lengths $24a$ and $96a$; or $n = 960a$ and $F$ has two orbits of lengths $192a$ and $768a$.

Proof. In these cases, $H$ is $\text{Sym}_e$-conjugate to $T^{e-1} \times H_e$, where $(T, H_e)$ is one of the pairs $(S_4(4), S_2(16))$ (2classes), $(O_6^+(2), A_9)$ (3 classes), or $(O_6^+(2), O_7(2))$ (3 classes). Thus, $|T : H|$ is 120, 960, or 120, respectively. If $T = S_4(4)$, then let $S_0$ be the maximal parabolic subgroup of $T$ of index 425. If $T = O_6^+(2)$, then let $S_0$ be the maximal parabolic subgroup of $T$ of index 1575. Then in each case, $N_T(S_0)$ is soluble, and $S_0$ and $S_0^\alpha$ are $T$-conjugate for all $\alpha$ in $\text{Aut}(T)$. That is, (i) and (ii) in Corollary 3.12 hold. Moreover, $S_0$ has two orbits in its action on the cosets of $H_e$ in $T$, of lengths 24 and 96 when $|T : H| = 120$, and lengths 192 and 768 when $|T : H| = 960$.

Let $\Sigma$ be the set of $L$-orbits. Then since 15 divides $|L : H|$ in each case, $G^\Sigma$ is a 2-group. Thus, as in the proof of Lemmas 3.13 and 3.14, we see that either $n = 120a$ and $F$ has two orbits of lengths $24a$ and $96a$, or $n = 960a$ and $F$ has two orbits of length $192a$ and $768a$.

In the next three lemmas we deal with the cases $T \in \{A_8, A_{16}, L_2(q) : q = 2^e 3^y 5^z - 1\}$. Apart from the case $q = 2^e 3^y 5^z - 1$ with $\{y', z'\} = \{1, 0\}$, we will find a soluble subgroup $S$ of $L$ with convenient orbit lengths, and then simply set $F := S$ (we do not need to trouble to find a soluble subgroup with fewer orbits, as we did in the $T \in \{A_5, A_6, S_4(4), O_6^+(2)\}$ cases).

Lemma 3.16. Let $G$ be a minimal counterexample to Theorem 5.1, and let $L = T^e$ be a minimal normal subgroup of $G$, where $T = A_8$. Let $H$ be the intersection of $L$ with a point stabilizer for $G$, so that $H$ is of type (2) in Corollary 3.12. That is, $H$ is $\text{Sym}_e$-conjugate to $T^{e_1} \times Y_1 \times \ldots \times Y_{e_2} \times H_e$, where $e_1 + e_2 = e - 1$; $Y_i$ is $T$-conjugate to $A_7$ for each $i$; and

| $J$ | Orbit lengths for $S = D_{10} < A_6$ acting on the cosets of $J < A_6$ |
|-----|---------------------------------------------------------------|
| $A_3 \times A_3$ | four of length 10 |
| $(A_3 \times A_3) \cdot 2$ | four of length 5 |
| $(A_3 \times A_3) \cdot 2 \cdot 2$ | two of length 5 |
| $A_5$ (transitive) | one of length 1, one of length 5 |
| $A_5$ (intransitive) | one of length 1, one of length 5 |

Table 3. Orbit lengths for $S = D_{10} < A_6$ acting on the cosets of a subgroup $J < A_6$.
$H_e$ has shape $7 : 3$, $2^3 : (7 : 3)$, $\text{GL}_3(2)$, or $2^3 : \text{GL}_3(2)$. Let $a$ be the number of $L$-orbits, let $S_0 := 7 : 3 < T$, and set $F := S_0^e < L$. Then

(i) Suppose that $H_e = 7 : 3$, and define $(s_1, l_1) := (1, 1)$; $(s_2, l_2) := (11, 7)$; and $(s_3, l_3) := (42, 21)$. Then $F$ has $a(s_j^{(e)})$ orbits of size $7^i l_j$, for each $1 \leq i \leq e_2$, $1 \leq j \leq 3$.

(ii) Suppose that $H_e \cong 2^3 : (7 : 3)$ or $K \cong \text{GL}_3(2)$, and define $(s_1, l_1) := (1, 1)$; $(s_2, l_2) := (5, 7)$; and $(s_3, l_3) := (4, 21)$. Then $F$ has $a(s_j^{(e)})$ orbits of size $7^i l_j$, for each $1 \leq i \leq e_2$, $1 \leq j \leq 3$.

(iii) Suppose that $H_e = 2^3 : \text{GL}_3(2)$, and define $(s_1, l_1) := (1, 1)$; and $(s_2, l_2) := (2, 7)$. Then $F$ has $a(s_j^{(e)})$ orbits of size $7^i l_j$, for each $1 \leq i \leq e_2$, $1 \leq j \leq 2$.

Proof. Note that $S_0$ has orbits of lengths 7 and 1 in its action on the cosets of $A_7$ in $T$, while $S_0$ has $s_i$ orbits of length $l_i$ in its action on the cosets of $H_1$ in $T$, where $s_i$ and $l_i$ are as defined in each of the listed cases. We then apply Lemma 3.11(ii) to find the orbit lengths of $F$ in its action on the cosets of $H$ in $L$: we see that $F$ has $s_j^{(e)}$ orbits of size $7^i l_j$ in each case.

Finally, since this holds for each $G$-conjugate of $H$ in $L$, and $F$ fixes each $L$-orbit, we deduce that $F < G$ has $a(s_j^{(e)})$ orbits of size $7^i l_j$ in $[n]$, as required. \qed

Lemma 3.17. Let $G$ be a minimal counterexample to Theorem 3.1, and let $L = T^e$ be a minimal normal subgroup of $G$, where $T = A_{16}$. Let $H$ be the intersection of $L$ with a point stabilizer for $G$, so that $H$ is of type (3) in Corollary 3.9. That is, $H$ is $\text{Sym}_e$-conjugate to $T^{e_1} \times Y_1 \times \ldots \times Y_{e_2} \times H_e$, where $e_1 + e_2 = e - 1$; $Y_i$ is $T$-conjugate to $A_{15}$ for each $i$; and $H_e$ has shape $A_{14+2b}$, for some $b \in \{0, 1\}$. Let $a$ be the number of $L$-orbits, let $S_0 := C_{15} < T$, and set $F := S_0^e < L$. Then $F$ has $\frac{16}{29}(e^2)a$ orbits of size $15^{i+1}$, for each $0 \leq i \leq e_2$.

Proof. The proof is identical to the $A_8$ case above: $S_0$ has orbits of lengths 15 and 1 in its action on the cosets of $A_{15}$ in $T$, while $S_0$ has $\frac{16}{29}$ orbits of length 15 in its action on the cosets of $H_e$ in $T$. We then apply Lemma 3.11 and argue as in the proof of Lemma 3.16 to complete the proof. \qed

Lemma 3.18. Let $G$ be a minimal counterexample to Theorem 3.1, and let $L = T^e$ be a minimal normal subgroup of $G$, where $T = L_2(q)$, with $q$ an odd prime power of the form $q = 2^e 3^{y'5^z} - 1$, $0 \leq y', z' \leq 1$. Let $H$ be the intersection of $L$ with a point stabilizer for $G$, so that $H$ is of type (4) in Corollary 3.9. Let $a$ be the number of $L$-orbits, and set $F := S_0^e < L$, where $S_0$ is a maximal parabolic subgroup of $T$.

(i) Suppose that $H$ has type $(4)(a)$ from Corollary 3.9, so that $q = p$ is a Mersenne prime with 15 dividing $p - 1$, and $H$ is $\text{Sym}_e$-conjugate to $T^{e_1} \times Y_1 \times \ldots \times Y_{e_2} \times (p : \frac{p-1}{9}) \times (p : \frac{p-1}{19})$, where each $Y_i$ is a maximal parabolic subgroup of $T$. Then $F := S$ has $a(s_j^{e_2})$ orbits of size $15^p i$; $2a(s_j^{e_2})$ orbits of size $15 p^{i+1}$; and $a(s_j^{e_2})$ orbits of size $15 p^{i+2}$, for each $0 \leq i \leq e_2$.

(ii) Suppose that $H$ has type $(4)(b)$ from Corollary 3.9, so that $q = p$ is a Mersenne prime and $H$ is $\text{Sym}_e$-conjugate to $T^{e_1} \times L_1$, where $L_1$ is a subdirect product of the form $L_1 = \frac{1}{7}(L_2 \times H\pi_e)$; $L_2$ is a subdirect product of the form $L_2 = \frac{1}{8}(P_1 \times \ldots \times P_{e_2})$; each $P_i$ is a maximal parabolic subgroup of $T$; $H\pi_e$ is a subgroup of index $r$ in a maximal parabolic subgroup of $T$; $rst = 3^{y_0}5^{z_0}$ is the odd part of $|L : H|$; and $e_1 + e_2 = e - 1$. Then $F := S$ has $a(s_j^{e_2+1})$ orbits of size $3^{y_0}5^{z_0} p^i$ for each $0 \leq i \leq e_2 + 1$. 

Thus, by the arguments above, together with Lemma 3.10, we deduce that
\[ r \text{ of sizes } \] of length \( T \) in \( G \).

We need to determine the orbit lengths of \( X \in \{ 1, 5, 15 \} \) that \( S \) determine the orbit lengths of \( X \) in its action of the cosets of the subgroup \( p : q \times (p : q) < T^2 \) for part (i).

We first consider the orbits of the action of \( S \) on the cosets of itself in \( T \). This is equivalent to the action of \( S \) on the set of one dimensional subspaces in \( F_2 ^\ast \). It is well known that \( S \) has one orbit of size 1 and one orbit of size \( q \) in this action. It follows easily that if \( X \) has shape \( q : 4 \times 1 \) with \( r \) a divisor of \( 4 \), then \( S \) has one orbit of size \( r \), and one orbit of size \( r q \). Next, when \( p = q \), \( S \) has four orbits on the cosets of \( p : q \times (p : q) \), of lengths 15, 15q, 15q, and 15q^2. Lemma 3.11(ii) then implies that \( F \) has \( \binom{q}{2} \) orbits of size 15p^i; 2a \( \binom{q}{3} \) orbits of size 15p^{i+1}; and \( \binom{q}{2} \) orbits of size 15p^{i+2} in its action on the coset of \( H \) in \( L \), for each \( 0 \leq i \leq e_2 \). Since our arguments hold with \( H \) replaced by any \( G \)-conjugate of \( H \), we deduce that \( F \) has \( \binom{q}{2} \) orbits of size 15p^i; 2a \( \binom{q}{3} \) orbits of size 15p^{i+1}; and \( \binom{q}{2} \) orbits of size 15p^{i+2} in \( [n] \), for each \( 0 \leq i \leq e_2 \).

Suppose now that we are in case (ii). Note that
\begin{equation}
(3.6)
(S \cap P_i ^{2})[P_i ^{2}, P_i ^{2}] = P_i ^{2} \text{ for all } x_i \text{ in } T.
\end{equation}
Thus, by the arguments above, together with Lemma 3.10, we deduce that \( S \) has \( \binom{q}{2} \) orbits of length \( sq^i \) on the cosets of \( L_2 \) in \( T^{e_2} \), for each \( 0 \leq i \leq e_2 \). Hence, since \( S \) has two orbits, of sizes \( r \) and \( q r \), on the cosets of \( H\pi_e \) in \( T \), we deduce from Lemma 3.11 that \( S \) has \( \binom{q}{2} \) orbits of length \( s r q^i \) and \( \binom{q}{2} \) orbits of length \( s r q^{i+1} \) in its action on the cosets of \( L_2 \times H\pi_e \) in \( T^{e_2+1} \). By (3.6), we can then use Lemma 3.10 again: we get that \( S \) has \( \binom{q}{2} \) orbits of length \( r s t q^i \) and \( \binom{q}{2} \) orbits of length \( r s t q^{i+1} \) in its action on the cosets of \( L_1 \) in \( T^{e_2+1} \). Note that \( r s t = 3^{q_0} 5^p \) and \( \binom{q}{2} \) for \( i > 0 \). We can then deduce from Lemma 3.11(ii) (with \( m = 1 \)) that \( F \) has \( \binom{q}{3} \) orbits of size \( 3^{q_0} 5^p q^i \) in its action on the cosets of \( H \) in \( L \), for each \( 0 \leq i \leq e_2 + 1 \). Since our arguments are independent of the choice of \( G \)-conjugate of \( H \) in \( L \), we see as above that \( F \) has \( \binom{q}{3} \) orbits of size \( 3^{q_0} 5^p q^j \) in \( [n] \), for each \( 0 \leq i \leq e_2 + 1 \).

Finally, the proof in case (iii) is identical to the proof of Lemma 3.15. □

**Lemma 3.19.** Let \( G \) be a minimal counterexample to Theorem 3.1, and let \( L = T^e \) be a minimal normal subgroup of \( G \), where \( T = A_m \) with \( m \in \{ 5, 6 \} \), and \( G / L \) is insoluble. Let \( H \) be the intersection of \( L \) with a point stabilizer for \( G \), and let \( a \) be the number of \( L \)-orbits. Then

(i) \( H \) is as in case (1)(b) of Corollary 3.10
(ii) \( |T : H| = 2^{3+1} 5^3 \) and \( a = 2^{3+2} 3^2 \);
(iii) \( G / L \) has a unique nonabelian chief factor \( L' \) such that \( L' \cong L_2(p)^j \), where \( p \) is a Mersenne prime; and
(iv) There exist positive integers \( X, b, \) and \( e_2 \leq e; \) positive partitions \( \binom{q}{2} = \sum_j C_j ^{(i)} \) and \( X = \sum_k X_k ^{(i,j)} \) of \( \binom{q}{2} \) and \( X \) (for each \( i, j \)); and a soluble subgroup \( F \) of \( G \), such that one of the following holds:
• $T = A_5$, $(X, b) \in \{(2, 1), (2, 0)\}$, and $F$ has $C_j^{(i)}$ orbits of length $2^k3^p5X_k^{(i,j)}$, for $0 \leq i \leq e_2$, and each $j, k$.

• $T = A_6$, $(X, b) \in \{(4, 1), (4, 0), (2, 0)\}$, and $F$ has $C_j^{(i)}$ orbits of length $2^k3^p5X_k^{(i,j)}$, for $0 \leq i \leq e_2$, and each $j, k$.

Proof. Let $K$ be the kernel of the action of $G$ on the set $\Sigma$ of $L$-orbits, and fix an $L$-orbit $\Delta$. Then since $G/K$ is an insoluble minimal transitive group, its degree $a$ cannot be a power of 2. Hence, either 3 or 5 must divide $a$. Thus, the odd part of $|\Delta| = |L : H|$ is either 3 or 5, and it follows from Corollary 3.9, and by inspection of the subgroups of $A_5$ of $A_6$, that the only possibility is that $H$ is as in case (1)(b) of Corollary 3.9. Part (i) follows.

Now, by Corollary 3.9(1)(b), $H_e$ is not transitive, and not equal to a natural point stabilizer $A_{m-1} < T = A_m$. Thus, going through the subgroups of $A_5$ and $A_6$, we see that $|T : H_e|$ must be of the form $5$, $2^45$. Hence, $G/K$ is an insoluble minimal transitive group of degree $2^e3^2$. This proves (ii). Part (iii) then follows immediately from [13, Theorem 1.8].

We now prove (iv). Set $S_0 := D_{10} < T$, and $S := S_0$. Then as shown in Lemmas 3.13 and 3.14, the hypotheses in Corollary 3.12 hold with this choice of $S_0$. Thus we can, and do, apply Corollary 3.12 with $R/K$ chosen to be the soluble subgroup of $G/K$ exhibited in Lemma 3.18(ii), with $p = 2^e1 - 1$ and $rst = 3$. We can then deduce (iv) by taking the possible orbit lengths of $S_0$ acting on the cosets of $H_e$ from Tables 2 and 3 (noting again that $H_e$ is intransitive, and that $H_e \neq A_4$ in the $A_5$ case). \hfill $\square$

Finally, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $G$ be a counterexample to the theorem of minimal degree, and let $L \cong T^e$ be a minimal normal subgroup of $G$, with $T$ simple. Then $T$ is nonabelian, by Lemma 3.3. Let $H$ be the intersection of $L$ with a point stabilizer in $G$, and write $|L : H| = 2^{x_0}3^{y_0}5^{z_0}$, where $x_0 \leq x$, $y_0 \leq y$, and $z_0 \leq z$. Thus, if $K$ denotes the kernel of the action of $G$ on the set $\Sigma$ of $L$-orbits, then $G^\Sigma \cong G/K$ is minimal transitive of degree $2^{x-x_0}3^{y-y_0}5^{z-z_0}$. Also, either $y_0$ or $z_0$ is non-zero, by Corollary 3.8(1). Thus, since $K/L$ is soluble by Lemma 3.3 the minimality of $G$ as a counterexample implies that $G$ has at most $1 + y - y_0 + z - z_0 \leq y + z$ nonabelian chief factors. That is, (i) holds.

Now, $T$ is one of the groups in part (ii) of Theorem 3.1, by Corollary 3.9, so (ii) also holds. Next, we prove (iii). So assume that $G$ has two nonabelian chief factors, and let $L' \cong T^{e'}$ be a nonabelian chief factor of $G/L$. Then the minimal transitive group $G^\Sigma \cong G/K$ has $L'$ as a chief factor, since $K/L$ is soluble. Since minimal transitive groups of 2-power degree are 2-groups, we deduce that $G^\Sigma \cong G/K$ is minimal transitive of degree either (1) $2^{x-x_0}5$ or (2) $2^{x-x_0}3$. The size of an $L$-orbit in these cases is $2^{x_0}3$ or $2^{x_0}5$, respectively. Now, by Corollary 3.9, the only possibility for $T$ when the size of an $L$-orbit is $2^{x_0}3$ is $T = L_2(p)$ with $p$ a Mersenne prime; while the only possibility for $T$ when the size of an $L$-orbit is $2^{x_0}5$ is $T \in \{A_5, A_6, L_2(p) : p$ a Mersenne prime}. This gives us what we need. Moreover, inspection of Table 1 and the inductive hypothesis imply that in case (1), we must have $T' \in \{A_5, A_6, L_2(p') : p$ a Mersenne prime$\}$; while in case (2), we have $T' = L_2(p')$ with $p'$ a Mersenne prime. This proves (iii).
Finally, in Lemmas 5.13, 8.14, 9.15, 9.16, 8.17, 8.18 and Lemma 8.19 we exhibit a soluble subgroup $F$ of $G$ with particular orbit lengths. These orbit lengths are as listed in Table 1 and this completes the proof.

4. The proof of Theorem 1.1

In this section, we prove Theorem 1.1. First, we require the following definitions and lemma from [13].

**Definition 4.1.** For a positive integer $s$ with prime factorisation $s = p_1^{r_1} p_2^{r_2} \ldots p_t^{r_t}$, set

$$\omega(s) := \sum r_i, \omega_1(s) := \sum r_i p_i, K(s) := \omega_1(s) - \omega(s) = \sum r_i (p_i - 1)$$

and

$$\tilde{\omega}(s) = \frac{s}{2K(s)} \left[ \frac{K(s)}{2} \right].$$

For a prime $p$, write $s_p$ for the $p$-part of $s$.

**Definition 4.2.** Let $p$ be prime, and let $s$ be a positive integer. We define

$$E_{\text{sol}}(s, p) := \min \{ \tilde{\omega}(s), s_p \}.$$

**Lemma 4.3.** [13] Corollary 4.25(ii)(b)] Let $G$ be a transitive permutation group of degree $n$, and suppose that $G$ is imprimitive with a minimal block of size 2. Let $\Sigma$ be the associated system of blocks of size $\frac{n}{2}$, so that $G$ may be viewed as a subgroup of the wreath product $2 \wr G_{\Sigma}$. Let $F$ be a soluble subgroup of $G_{\Sigma}$, and suppose that $F$ has orbit lengths $s_1, \ldots, s_r$. Then

$$d(G) \leq \sum_{i=1}^r E_{\text{sol}}(s_i, 2) + d(G_{\Sigma}).$$

**Corollary 4.4.** Let $G$ be a transitive permutation group of degree $n$, and suppose that $G$ is imprimitive with a minimal block of size 2. Let $\Sigma$ be the associated system of blocks of size $\frac{n}{2}$, so that $G$ may be viewed as a subgroup of the wreath product $2 \wr G_{\Sigma}$. Suppose that $\frac{n}{2}$ has the form $\frac{n}{2} = 2^3 3^y 5^z$, with $0 \leq y, z \leq 1$, and that $G_{\Sigma}$ contains no soluble transitive subgroups. Let $F$ be a soluble subgroup of $G_{\Sigma}$ with orbit lengths as exhibited in Table 1.

1. If $F$ is as in row 12 of Table 1 then $d(G) \leq 2^{e_2 + b + 1} a + d(G_{\Sigma}).$
2. If $F$ is as in row 22 of Table 1 then $d(G) \leq 2^{e_2 + b + 2} a + d(G_{\Sigma}).$
3. If $F$ is as in row 24, 25, or 26 of Table 1 then $d(G) \leq 2^{e_2} 5^a + d(G_{\Sigma}); d(G) \leq 2^{e_2} 10a + d(G_{\Sigma});$ or $d(G) \leq 2^{e_2} 3a + d(G_{\Sigma})$, respectively.
4. If $F$ is as in row 27 of Table 1 then $d(G) \leq 2^{e_2 + b - 1} a + d(G_{\Sigma}).$
5. If $F$ is as in row 28 of Table 1 then $d(G) \leq 2^{e_2} 5a + d(G_{\Sigma}).$
6. If $F$ is as in row 29 of Table 1 then $d(G) \leq 2^{e_2} 10a + d(G_{\Sigma}).$
7. If $F$ is as in row 30 of Table 1 then $d(G) \leq 2a + d(G_{\Sigma})$.

**Proof.** Write $s_1, \ldots, s_r$ for the $F$-orbit lengths. Then by Lemma 4.3 and the definition of $E_{\text{sol}}$, we have

$$d(G) \leq \sum_{i=1}^r (s_i)_2 + d(G_{\Sigma}).$$
Parts (3), (4), (5), and (6) now follow immediately from (4.1) and inspection of the relevant row in Table II.

Suppose now that $F$ is as in row 12 or 22 of Table II. Then (4.1) implies that

$$d(G) \leq 2^b \sum_{i,j,k} C_j^{(i)}(X_k^{(i,j)})_2 + d(G^\Sigma).$$

Since $(X_k^{(i,j)})$ is an ordered positive partition of either 2 or 4, the result now follows easily by going through each of the possibilities for $(X_k^{(i,j)})$, and noting that $\sum_{i,j} C_j^{(i)} = \sum_i (c_i^2) = 2^{e^2}$.

We are now ready to prove Theorem I.1.

**Proof of Theorem I.1.** By [13, Theorem 1.1], we may assume that $n$ has the form $n = 2^x3^y5^z$, where $y \in \{0, 1\}$. Furthermore, we may assume that $17 \leq x \leq 26$ if $y = 0$, and $15 \leq x \leq 35$ if $y = 1$; that $G$ has a block system $\Sigma$ consisting of blocks of size 2; and that $G^\Sigma$ contains no soluble transitive subgroup.

It then follows from Theorem 3.1 that $G^\Sigma$ has a soluble subgroup $F$ whose orbit lengths occur in one of the rows of Table II. We can then use this fact, together with Lemma 4.3, to prove the theorem. Indeed, suppose that we have proved the theorem for all degrees properly dividing $n$, and that the possible orbit lengths for the soluble subgroup $F$ are $s_1, s_2, \ldots, s_r$, for $1 \leq j \leq t$. We then compute the maximum value of $\sum_{i=1}^r E_{sol}(s_i, 2)$ over $1 \leq j \leq t$, and add it to our previously obtained bound for $d(G^\Sigma)$. By Lemma 4.3 this gives a bound for $d(G)$.

Let us illustrate this strategy with some examples. Suppose first that $n = 2^{17}5$. From Table II we see that $G^\Sigma \leq S_{2165}$ has a soluble subgroup whose orbit lengths lie in one of the rows 7, 18, 19, or 29 (with $y_0 = 0$, $z_0 = 1$ in the last case). Computing as in the $n = 2^{15}15$ example given at the beginning of the section, we see that the lengths of the $F$-orbits are one of the following:

- 2 orbits of equal lengths, $i \in \{1, 2\}$;
- 2 orbits of length $2^{14}5$ and one orbit of length $2^{14}5$;
- 1 orbit of length $2^{14}5$ and 1 orbit of length $2^{14}15$; or
- $\binom{v}{2}$ or $\binom{v}{a}$ orbits of size $5^i$ and $\binom{v}{a}$ orbits of size $5^i$, for each $0 \leq i \leq e_2$, where the possibilities for $e_2$ and $p$ are $(e_2, p) \in \{(A, 2^5 - 1), (1, 2^{13} - 1) : 1 \leq A \leq 2\}$, and $\frac{p}{2} = 5a(p + 1)^e_2$.

We make some remarks about the last point above. Since $(p + 1)^e_2$ must divide $\frac{n}{7}$, we only need to consider the Mersenne primes $p = 2^u - 1$ such that $p + 1 = 2^u < 2^k := \left(\frac{k}{2}\right)_2$, and those $e_2$ with $1 \leq e_2 \leq \left\lfloor \frac{k}{u} \right\rfloor - 1$. And since we are assuming, throughout this proof, that $n_2 \leq 2^{35}$, the only Mersenne primes under consideration are $2^u - 1$, for $u$ a prime less than or equal to 31. In the case $n = 2^{17}5$ above, we must also have that 5 divides $p - 1$. So only the cases $u \in \{5, 13\}$ can occur.

We now proceed to compute a bound for $d(G)$ (for $n = 2^{17}5$) in each case, by using Table II and Corollary 4.3 as explained above. By [13, Table A.1], we have $d(G^\Sigma) \leq 65538$. If $F$ has 2 orbits of length $2^{15}5$ then we get $d(G) \leq 2E_{sol}(2^{15}5, 2) + 65538 = 123274$. Similarly, if $F$ has 4 orbits of length $2^{14}5$ then we get $d(G) \leq 126313$; if $F$ has 2 orbits of length $2^{14}5$ and
1 orbit of length $2^{15}5$ then $d(G) \leq 124793$; and if $F$ has 1 orbit of length $2^{14}15$ and 1 orbit of length $2^{14}15$ then $d(G) \leq 97115$. Suppose now that $F$ has $(\ell_i^1) d$ orbits of size $5p^i$ and $(\ell_i^2) d$ orbits of size $5p^i+1$, for each $0 \leq i \leq e_2$, where $(e_2, p) \in \{(1, 2^{5} - 1), (2, 2^{5} - 1), (1, 2^{13} - 1)\}$ and $a = \frac{n}{\log p + 12}$. If $(e_2, p) = (1, 2^{3} - 1)$, then $F$ has $2^{11}$ orbits of size 5, and $2^{11}$ orbits of size $5 \times 31 = 155$. Therefore, we get $d(G) \leq 2^{11}E_{sol}(5, 2) + 2^{11}E_{sol}(155, 2) + 65538 = 69634$. The other cases are entirely similar, and we see that $d(G)$ is always bounded above by 126313. Since

$$\left(\frac{2^{17}5}{\sqrt{\log 2^{17}5}}\right) = 129117$$

this proves Theorem 1.1 in the case $n = 2^{17}5$.

Next, we will give an example to show how the required upper bounds can be derived in the cases $n = 2^{x}15$. More precisely, we will go through the necessary computations in the case $n := 2^{15}15$. In order to do this, we will first need an upper bound on $d(G^\Sigma)$. Since $G^\Sigma$ is a transitive permutation group of degree 2$^{14}15$, we can deduce from [13, Table A.1] that $d(G^\Sigma) \leq 49156$. Using the classification of the transitive groups of degree 48 from [5], however, we can improve this bound. In order to avoid repetition of details, we will simply refer to [13, proof of Lemma 5.12, case 2, page 40], where the proofs of the bounds in [13, Table A.1] are given. The author uses the classification of the transitive permutation groups of degree up to 32 together with the bounds in Lemma 4.3 above, to derive the upper bound on $d(G^\Sigma) \leq 49156$. Within this computation, the estimate $d(X) \leq 16$ for transitive groups $X \leq S_{48}$ is used. We can now use the exact bound $d(X) \leq 10$ for transitive $X \leq S_{48}$ from [5, Section 5]. Performing the computations in [13, proof of Lemma 5.12, case 2, page 40] again with this new bound, we quickly deduce that

$$(4.2) \quad d(G^\Sigma) \leq 49150 \text{ whenever } G^\Sigma \leq S_{2^{14}15} \text{ is transitive.}$$

Using (4.2) and Lemma 1.3, we can now determine an upper bound for $d(G)$ when $n = 2^{15}15$. At the beginning of the section, we listed the possible $F$-orbit lengths in the case when $G$ has a unique nonabelian chief factor, and this chief factor is a direct product of copies of $A_5$ (these cases correspond to rows 1–11 in Table I). Entirely analogous calculations give us the possible $F$-orbit lengths when $G$ has a unique nonabelian chief factor, and this chief factor is a direct product of copies of $A_6$ (these cases correspond to rows 13–21 in Table I) or $S_4(4)$ or $O_{25}^+(2)$ (row 23). When $F$ has orbit lengths as in rows 12, 22, and 24–30 of I, we use Corollary 4.4 to bound $d(G)$. Taking the maximum of these bounds for $d(G)$, we get $d(G) \leq 97401$ in each case. This proves the theorem in this case, since

$$\left(\frac{2^{15}15}{\sqrt{\log 2^{15}15}}\right) = 97895.$$ 

The remaining cases with $n$ of the form $n = 2^{x}15$, and $16 \leq x \leq 35$, are entirely similar, except that we use the bound $d(G^\Sigma) \leq \left(\frac{2^{x-1}15}{\sqrt{\log 2^{x-1}15}}\right)$ for $d(G^\Sigma)$ (which holds by the inductive hypothesis). We note in particular that we get $d(G^\Sigma) \leq 189053$ when $x = 16$ (this will be important). The result now follows in each case, except when $x = 17$. So assume that $x = 17$. By [13, Theorem 1.1], we may assume that $G^\Sigma$ is imprimitive with a minimal block of size 2. Let $K_1$ be the kernel of the action of $G^\Sigma$ on a set $\Sigma_1$ of blocks of size 2 in $\Sigma$. Assume first that $G^\Sigma \cap K_1 \neq 1$. If $(G^\Sigma)_{\Sigma_1} \cong G^\Sigma / K_1$ contains a soluble transitive subgroup $S/K_1$, then $S$ is a soluble transitive subgroup of $G^\Sigma$ – a contradiction. Thus, $(G^\Sigma)_{\Sigma_1}$ contains no soluble transitive subgroups. Because of this, we can use our previously obtained bound $d(G^\Sigma) \leq 189053$ in this
case. If $G^\Sigma \cap K_1 = 1$, then $d(G^\Sigma) = d(G^\Sigma_1) \leq \left\lfloor \frac{c^{2^{15}15}}{\sqrt{\log 2^{1^{2^{15}15}}}} \right\rfloor = 98547$, so in either case, we get $d(G^\Sigma) \leq 189053$. By computing the upper bound for $d(G)$ as we did in the case $n = 2^{15}15$ above, we now get $d(G) \leq 371369$. This gives us what we need, since $\left\lfloor \frac{c^{2^{17}15}}{\sqrt{\log 2^{1^{2^{15}15}}}} \right\rfloor = 372380$. □

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