Consequences of Deformation of the Heisenberg Algebra

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Abstract
In this paper we will demonstrate that like the existence of a minimum measurable length, the existence of a maximum measurable momentum, also influence all quantum mechanical systems. Beyond the simple one dimensional case, the existence of a maximum momentum will induce non-local corrections to the first quantized Hamiltonian. However, these non-local corrections can be effectively treated as local corrections by using the theory of harmonic extensions of functions. We will also analyses the second quantization of this deformed first quantized theory. Finally, we will analyses the gauge symmetry corresponding to this deformed theory.

1 Introduction
The classical picture of spacetime gets modified in all most all approaches to quantum gravity. One of the most common modification that occurs in most theories of quantum gravity is that the the continuum picture of spacetime breaks down at Planck scale. So, the idea of the existence of a minimum length is naturally incorporated in most approaches to quantum gravity. A minimum measurable length naturally occurs in string theory [1]-[5]. Even in loop quantum gravity the existence of minimum length turns big bang into a big bounce [6].
In fact, there are strong indication from black hole physics that there minimum length of the order of the Planck length should arise in any theory of quantum gravity [7]-[8]. The existence of a minimum measurable length deforms all quantum mechanical Hamiltonians by \( H\psi = H_0\psi + H_1\psi \), where \( H_0 = p^2/2m + V(x) \) is the original Hamiltonian, \( p_i = -i\partial_i \), is the momentum operator corresponding to the undeformed Heisenberg algebra, and \( H_1 = \beta p_i^4/m \) is the term that occurs due to the existence of a minimum length.
According to the Heisenberg uncertainty principle there is no limit to the accuracy with which one can measure the momentum or the position of a particle separately. In other words, according to the Heisenberg uncertainty principle minimum observable length is actually zero. Thus, the Heisenberg uncertainty principle has to be modified in order to incorporate the idea of minimum length. This has been done and the resultant uncertainty principle is called the Generalized Uncertainty Principle [9]-[23]. In this picture the commutation relations between position and momentum operators in the Hilbert space are deformed.
to contain momentum dependent factors.

A further deformation of this algebra occurs in doubly special relativity \[24]-\[26]. In doubly special special relativity theories, both the velocity of light and the Planck energy are invariant quantities. This modified algebra naturally incorporates the existence of a maximal momentum for any particle. In fact, General Relativity has also been modified to keep both velocity of light and the Planck energy as invariant quantities. The resultant theory is called Gravity’s Rainbow \[27]-\[28]. Both these deformations have been combined into a single deformation of the Heisenberg algebra \[29]-\[31]. The modification to transition rate of ultra cold neutrons in gravitational field has been studied in this deformed algebra \[32]. In fact, the modification to the Lamb shift and Landau levels have also been analysed in this deformed algebra \[33]. However, both these calculations are only done for the simple one dimensional case. In this paper we will also analyse the second quantization of this deformed first quantized algebra \[33].

2 Deformed Heisenberg Algebra

The deformation of the Heisenberg algebra consistent with the existence of a minimum length is \([x^i, p_j] = i[\delta^i_j + \beta p^2 \delta^j_i + 2\beta p^ip_j]\) and the deformation of the Heisenberg algebra consistent with the existence of a maximum momentum is \([x^i, p_j] = i[1 - \beta^0 p^i \delta^j_i + \beta p^jp_j]\), with \(\beta = \beta_0 \ell_{Pl}/\hbar\), and \(\beta = \ell_{Pl}\), where \(\ell_{Pl} \approx 10^{-35}\) m is the Planck length, and \(\beta_0\) is a constant normally assumed to be of order unity. Both these deformations of Heisenberg algebra can be combined into a single deformation of the Heisenberg algebra as follows \[29]-\[31]\:

\[
[x^i, p_j] = i \left[ \delta^i_j - \alpha |p| \delta^i_j + \alpha |p|^{-1} p^i p_j + \alpha^2 p^2 \delta^i_j + 3\alpha^2 p^i p_j \right],
\]

(1)

with \(\alpha = \alpha_0/M_{Pl}c = \alpha_0 \ell_{Pl}/\hbar\), where \(M_{Pl}\) is the Planck mass, \(\ell_{Pl} \approx 10^{-35}\) m is the Planck length, and \(M_{Pl}c^2 \approx 10^{19}\) GeV is the Planck energy. In the one dimensional case this corresponds to the uncertainty relation given by \(\Delta x \Delta p = [1 - 2\alpha < p > + 4\alpha^2 < p^2 >]\). These imply the existence of a minimum length

\[
\Delta x \geq \Delta x_{min} \geq \alpha_0 \ell_{Pl}.
\]

(2)

It also implies the existence of a maximum momentum

\[
\Delta p \leq \Delta p_{max} \leq \alpha_0^{-1} M_{Pl}c.
\]

(3)

In fact, this algebra is satisfied if we chose the following representation for it, \(x_i = \tilde{x}_i\) and \(p_i = \tilde{p}_i(1 - \alpha |\tilde{p}| + 2\alpha^2 \tilde{p}^2)\), such that \([\tilde{x}^i, \tilde{p}_j] = i\delta^i_j\). Thus, \(\tilde{p}_i\) can be interpreted as the momentum at low energies. Now can use the standard representation for \(\tilde{p}\) in one dimension and obtain the following expression for \(p\),

\[
p = -i \left( 1 + i\alpha \frac{d}{dx} - 2\alpha^2 \frac{d^2}{dx^2} \right) \frac{d}{dx}.
\]

(4)
Thus, for the one dimensional quantum mechanical systems with both minimum length and maximum momentum the deformed Hamiltonian becomes \[ \frac{1}{2m} \frac{d^2 \psi}{dx^2} - \frac{i}{m} \frac{d^2 \psi}{dx^2} + \frac{5\alpha^2}{2m} \frac{d^4 \psi}{dx^4} + V(x)\psi = E\psi. \] (5)

In higher dimensions, we can also use the standard representation for \( \tilde{p}_i = -i\partial_i \) and obtain an expression for \( p_i \),

\[ p_i = -i \left( 1 + \alpha \sqrt{-\partial_i \partial_j - 2\alpha^2 \partial_i \partial_j} \right) \partial_i. \] (6)

Like the deformation corresponding to the existence of a minimum length, this deformation also influence all quantum mechanical Hamiltonians. Thus, in any dimension greater than one this Hamiltonian becomes non-local

\[ -\frac{1}{2m} \partial_i \partial_i \psi - \frac{\alpha}{m} \sqrt{-\partial_j \partial_i \partial_i \partial_i \psi + \frac{5\alpha^2}{2m} \partial_i \partial_j \partial_i \partial_i \psi + V(x)\psi = E\psi}. \] (7)

Thus, the term proportional to \( \alpha \) is a non-local term as it contains this non-local differential operator \( \sqrt{-\partial_j \partial_i} \). So, the existence of a maximum momentum will being induce non-locality in all quantum mechanical systems. However, we can still effectively treat this non-local correction as a local term by using the theory of harmonic extensions of functions from \( R^n \) to \( R^n \times (0, \infty) \) [34]-[35]. So, we define \( \sqrt{-\partial_j \partial_i} \) on functions \( \psi : R^n \to R \), such that the harmonic extension \( u : R^n \times (0, \infty) \to R \) satisfies, \( \sqrt{-\partial_j \partial_i} u(x, y) = -\partial_y u(x, y) \). If the restriction of a harmonic function \( u : R^n \times (0, \infty) \to R \) to \( R^n \), coincides with a function \( \psi : R^n \to R \), then given any function \( \psi \), it is possible to find \( u \) by solving the Dirichlet problem defined by \( u(x, 0) = \psi(x) \) and \( \partial_y^2 u(x, y) = 0 \), where \( \partial_y^2 \) is the Laplacian in \( R^{n+1} \), for \( x \in R^n \) and \( y \in R \). Thus, for a smooth function \( C_0^\infty(R^n) \) there is a unique harmonic extension \( u \in C^\infty(R^n \times (0, \infty)) \). If \( u \) is a harmonic extension of \( \psi \), then \( u_y(x, y) \) is the harmonic extension of \( \sqrt{-\partial_j \partial_i} \psi(x) \) to \( R^n \times (0, \infty) \). So, the following result can be obtained, \( \sqrt{-\partial_j \partial_i}^2 \psi(x) = \partial_y^2 u(x, y) \big|_{y=0} = -\partial_y^2 \psi(x) \). As \( \sqrt{-\partial_j \partial_i} \psi(x) = -\partial_y^2 \psi(x) \), we can consistently define \( \sqrt{-\partial_j \partial_i} = \sqrt{-\partial_y^2} \).

Now if we take \( \psi(x) = \cos \tilde{p}x \), then the bounded harmonic extension of \( \psi \) will be given by \( u(x, y) = \exp[-\tilde{p}y \cos \tilde{p}x], \) where \( y \in (0, \infty), \partial_y^2 u(x, y) + \partial_y^2 u(x, y) = 0 \). The action of \( \sqrt{-\partial_j \partial_i} \) on \( \psi(x) \) will be given by \( \sqrt{-\partial_j \partial_i} \cos \tilde{p}x = -u_y(x, y) \big|_{y=0} \) and so, we have \( \sqrt{-\partial_j \partial_i} \cos \tilde{p}x = |\tilde{p}| \cos \tilde{p}x \). Similarly, we take \( \psi(x) = \sin \tilde{p}x \), and its the bounded harmonic as \( u(x, y) = \exp[-|\tilde{p}|y \sin \tilde{p}x] \). So, again we have \( \sqrt{-\partial_j \partial_i} \sin \tilde{p}x = |\tilde{p}| \sin \tilde{p}x \). Now we can write the action of \( \sqrt{-\partial_j \partial_i} \) as \( \sqrt{-\partial_j \partial_i} \exp i\tilde{p}x = \sqrt{-\partial_j \partial_i} (\cos \tilde{p}x + i \sin \tilde{p}x) \). Thus, we have \( \sqrt{-\partial_j \partial_i} \exp i\tilde{p}x = |\tilde{p}| \exp i\tilde{p}x \). So, \( \sqrt{-\partial_j \partial_i} \) can be effectively used as a local derivative. It may be noted that if \( \psi(x) \) admits a harmonic extension \( u(x, y), \) such that \( \sqrt{-\partial_j \partial_i} \psi(x) = -u_y(x, 0) \), then the harmonic extension of \( \partial_i \psi(x) \) will be \( \partial_i u(x, y) \). Furthermore, if \( u \in C^2(R \times (0, \infty)) \), then \( \sqrt{-\partial_j \partial_i} \partial_i \psi(x) = -\partial_i u_y(x, y) \big|_{y=0} \). So, this non-local operator commutes with a derivative, \( \sqrt{-\partial_j \partial_i} \partial_i \psi(x) = \partial_i \sqrt{-\partial_j \partial_i} \psi(x) \).
3 Quantum Field Theory

It may be noted the first quantized theory with minimum length has also been second quantized [39]-[41]. In order to achieve this the deformed Heisenberg algebra has also been extended to also include a minimum time. Thus, the temporal part of the first quantized theory is also deformed like its spatial part. So, it is a natural to complete this algebra to include a maximum energy. Thus, we propose the following algebra,

\[ [x^\mu, p_\nu] = i \left[ \delta^\mu_\nu - \alpha |p^\mu p_\nu|^{-1/2} \delta^\mu_\nu + \alpha |p^\mu p_\nu|^{-1/2} p^\mu p_\nu + \alpha^2 p^2 \delta^\mu_\nu + 3\alpha^2 p^\mu p_\nu \right], \quad (8) \]

where \( p^\mu \) stands for the full four momentum of a particle. This algebra can be represented by the following deforming of the momentum operator, 

\[ p^\mu = \tilde{p}^\mu (1 - \alpha |\tilde{p}^\rho \tilde{p}_\rho|^{-1/2} - 2\alpha^2 \tilde{p}^\rho \tilde{p}_\rho). \]

We will now work in Euclidean spacetime and thus take \( \tau = it \) as the time variable. We can obtain the following expression for \( p^\mu \) in coordinate representation,

\[ p^\mu = -i(1 + \alpha \sqrt{-\partial^\nu \partial_\nu} - 2\alpha^2 \partial^\nu \partial_\nu) \partial^\mu. \quad (9) \]

Thus, we can write the deformed Klein–Gordon equation as follows,

\[ \left( 1 - 2\alpha \sqrt{-\partial^\nu \partial_\nu} + 5\alpha^2 \partial^\nu \partial_\nu \right) \partial^\mu \partial_\mu \phi + m^2 \phi = 0. \quad (10) \]

We can write the Lagrangian corresponding to it as follows,

\[ \mathcal{L} = \phi \left[ \left( 1 - 2\alpha \sqrt{-\partial^\nu \partial_\nu} + 5\alpha^2 \partial^\nu \partial_\nu \right) \partial^\mu \partial_\mu + m^2 \right] \phi. \quad (11) \]

In order to derive the deformed Klein–Gordon equation, from this Lagrangian, we needed to shift the non-local operator \( \sqrt{-\partial^\nu \partial_\nu} \) from one field to the next. So, we the harmonic extensions of fields \( \phi_1 \) and \( \phi_2 \) on \( C = R \times (0, \infty) \) be \( u_1 \) and \( u_2 \), respectively. Also let these harmonic extensions vanish for \( |x| \rightarrow \infty \) and \( |y| \rightarrow \infty \). Now we can write

\[ \int_C u_1(x, y) \partial_{n+1}^2 u_2(x, y) dxdy - \int_C u_2(x, y) \partial_{n+1}^2 u_1(x, y) dxdy = 0. \quad (12) \]

So, we can write

\[ \int_{R^n} \left( u_1(x, y) \frac{\partial}{\partial y} u_2(x, y) - u_2(x, y) \frac{\partial}{\partial x} u_1(x, y) \right) |_{y=0} dx = 0. \quad (13) \]

From this we get

\[ \int_{R^n} \left( \phi_1(x) \frac{\partial}{\partial y} \phi_2(x) - \phi_2(x) \frac{\partial}{\partial x} \phi_1(x) \right) dx = 0. \quad (14) \]

Thus, we can write

\[ \int_{R^n} \phi_1(x) \sqrt{-\partial^\nu \partial_\nu} \phi_2(x) dx = \int_{R^n} \phi_2(x) \sqrt{-\partial^\nu \partial_\nu} \phi_1(x) dx \quad (15) \]

The Euclidean Green’s function corresponding to this deformed theory can be calculated by noting that the existence of a minimum length and maximum
momentum, will deform the Euclidean Green’s function. This will occur because the momentum in the exponent will get deformed to,

\[ p_\mu = \tilde{p}_\mu (1 - \alpha |\tilde{p}| - 2\alpha^2 \tilde{p}_\mu \tilde{p}_\nu). \]

Thus, we can write,

\[ G(\tilde{p}) = \left[ J(\tilde{p}) \right]^{-1}/(\tilde{p}^2 + m^2), \] where \( J(\tilde{p}) = \text{det}[\partial p/\partial \tilde{p}] \).

Similarly, the propagator for any physical field can be obtained by taking the Jacobian determinant of the transformation between \( p \) and \( \tilde{p} \). It may be noted that a similar result has been obtained for theories with a minimum measurable length [39].

The action of the non-local operator \( T_\theta \) on the fermionic fields can also be effectively defined by using the theory of harmonic extensions of functions. Thus, we can write the Dirac equation as follows,

\[ i\gamma^\mu \left( 1 + \alpha \sqrt{-\partial^\nu \partial_\nu} - 2\alpha^2 \partial^\nu \partial_\nu \right) \partial_\mu \psi - m\psi = 0. \] (16)

Now we can obtain the following result,

\[ \int_{R^n} \bar{\psi}(x) \sqrt{-\partial^\nu \partial_\nu} \psi(x) dx = \int_{R^n} \sqrt{-\partial^\nu \partial_\nu} \bar{\psi}(x) \psi(x) dx, \] (17)

by repeating the argument used in the derivation of a similar equation in the bosonic case. So, we can write the Lagrangian for the deformed Dirac’s equation as,

\[ L = \bar{\psi} \left[ i\gamma^\mu \left( 1 + \alpha \sqrt{-\partial^\nu \partial_\nu} - 2\alpha^2 \partial^\nu \partial_\nu \right) \partial_\mu - m \right] \psi. \] (18)

Now the propagator corresponding to this equation will be given by, \( G(\tilde{p}) = i[J(\tilde{p})]^{-1}/(\gamma^\mu \tilde{p}_\mu - m) \). This deformation of the propagator of physical fields, is also a universal feature of all second quantized theories, whose first quantized equations have been deformed. It is important to stress that this is true only for physical fields, this is because in the next section, we will observe that the propagator for ghost fields does not get deformed in these theories.

4 Gauge Symmetry

It is known that the usual undeformed Dirac’s equation is invariant under a global phase transformation. This global symmetry can be promoted to a local gauge symmetry by replacing all the derivatives by covariant derivatives. Here the covariant derivatives are defined by introducing a gauge field which cancels the extra terms generated by action of the derivative on the gauge parameter. Furthermore, by making this gauge parameter a matrix valued function, this abelian gauge symmetry can be promoted to an non-abelian gauge symmetry. Finally, a kinetic term for any gauge theory can be obtained by taking a commutator of two covariant derivatives. Thus, we could proceed this way for the deformed Dirac’s theory also. So, we could define a gauge field that would cancel the extra terms generated by the action of the deformed derivatives on the gauge parameter. The problem with this approach is that, the deformed derivatives contain a non-local part. Thus, there are effectively infinity many terms in this theory and it is not possible to obtain a compact form for the non-local gauge transformations that a gauge field has to transform under, so that a covariant derivative can be constructed.

However, if we make all the derivatives covariant, then the deformed express will also be covariant, \( D_\mu = (1 + \alpha \sqrt{-D^\nu D_\nu} - 2\alpha^2 D^\nu D_\nu) D_\mu \) and \( D_\mu = \partial_\mu - igA_\mu \).
In fact, we can directly incorporate non-abelian gauge symmetry at this stage by deforming the derivatives as follows, 

\[ D_\mu = (1 + \alpha \sqrt{-D^\tau D_\tau} - 2\alpha^2 D_\mu D_\mu) D_\mu \]

and 

\[ \partial_\mu - igA^A_\mu T_A, \]

such that \( [T_A, T_B] = i f^{C}_{AB} T_C \). We can thus write a Lagrangian which will be invariant under gauge transformations as follows,

\[ \mathcal{L} = \bar{\psi} i\gamma^\mu D_\mu \psi - \frac{1}{4} \mathcal{F}^{\mu
u} \mathcal{F}_{\mu\nu}, \quad (19) \]

Here the deformed field strength \( \mathcal{F}_{\mu\nu} = \mathcal{F}^A_{\mu\nu} T_A \), used in this theory is obtained by taking commutator of two deformed covariant derivatives,

\[
\mathcal{F}_{\mu\nu} = -i[D_\mu, D_\nu] \\
= -i[D_\mu, D_\nu] + \alpha[\sqrt{-D^\tau D_\tau} D_\mu, D_\nu] + \alpha[D_\mu, \sqrt{-D^\tau D_\tau} D_\nu] \\
+ \alpha^2[\sqrt{-D^\tau D_\tau} D_\mu, \sqrt{-D^\tau D_\tau} D_\nu] - 2\alpha^2[D_\tau D_\mu, D_\nu] \\
- 2\alpha^2[D_\mu, D^\tau D_\nu]. \quad (20)
\]

Thus, we get, \( \mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu} + O(\alpha) \), where \( \mathcal{F}_{\mu\nu} \) is the convectional Yang-Mills field tensor. It may be noted that, if under gauge transformations, \( \mathcal{F}_{\mu\nu} \rightarrow UF_{\mu\nu}U^{-1} \), then under the same gauge transformations, \( \mathcal{F}_{\mu\nu} \rightarrow UF_{\mu\nu}U^{-1} \). Thus, even though \( \mathcal{F}_{\mu\nu} \) is a non-local non-abelian field strength, all the derivatives in it are covariant derivatives. So, they all transform covariant and the net result is that \( \mathcal{F}_{\mu\nu} \) transforms under local gauge transformations like \( \mathcal{F}_{\mu\nu} \).

By the same argument, the whole Lagrangian for this deformed Yang-Mills theory coupled to matter fields, is invariant under local gauge transformations. It may be noted that even a deformed abelian gauge theory with contain infinite number of interaction terms due to the non-local operator.

It is not clear how we can quantize this theory, as it is not clear if the derivative in the ghost term will also get deformed. However, we can use the fact that this theory is invariant under usual gauge transformations, and thus the structure of the BRST and the anti-BRST transformations will not change. We can now use a particular non-linear gauge, where we incorporate into the gauge-fixing Lagrangian a quartic ghost interaction \[12\]. The advantage of using this gauge is that the sum of the gauge fixing term and the ghost term contains no derivatives. However, the usual BRST and the usual anti-BRST transformations get deformed in this gauge,

\[
\begin{align*}
sc^A &= b^A - \frac{1}{2}(\tilde{c} \times c)^A, \\
sb^A &= -\frac{1}{2}(c \times b)^A - \frac{1}{8}((c \times c) \times \tilde{c})^A, \\
sA^A_\mu &= (D_\mu c)^A, \\
s\bar{c}^A &= -b^A - \frac{1}{2}(\tilde{c} \times c)^A, \\
s\bar{c}^A &= \frac{1}{2}(\tilde{c} \times c)^A - \frac{1}{2}(\tilde{c} \times c)^A - \frac{1}{8}((\tilde{c} \times e) \times c)^A, \\
s\bar{c}^A &= \frac{1}{2}(\tilde{c} \times c)^A \\
s\bar{c}^A &= \frac{1}{2}(\tilde{c} \times c)^A, \\
s\bar{c}^A &= \frac{1}{2}(\tilde{c} \times e)^A, \\
s\bar{c}^A &= (D_\mu \tilde{c})^A.
\end{align*}
\]

Now we can write the sum of the gauge fixing term \( \mathcal{L}_{gf} \) and the ghost term \( \mathcal{L}_{gh} \) as follows \[12\]

\[
\mathcal{L}_{gf} + \mathcal{L}_{gh} = \frac{i}{2} s\bar{s}(A^A_\mu A_\mu - i\xi \tilde{c}c) \\
= ib\partial^A_\mu A_\mu + \frac{\xi}{3} b^2 + \frac{i}{2} \tilde{c} \partial^A_\mu D_\mu c + \frac{1}{8} \xi (\tilde{c} \times c)^2. \quad (22)
\]
Thus, at least in this non-linear gauge the ghost propagators do not get de-
formed. This is because they depend on the gauge transformations, and this
non-local theory is invariant under regular local gauge transformations.

5 Conclusion

In this paper, we analysed a universal correction that will occur in all quantum
mechanical Hamiltonians, due to the existence of both a minimum length and
maximum momentum. These corrections occur due to the deformation of the
Heisenberg algebra. This in turn deforms the coordinate representation of the
momentum operator. This deformed momentum operator contains a non-local
term, which reduces to a local term only in one dimension. The existence of a
minimum momentum scale will induce non-locality in all quantum mechanical
processes at high energy scales. However, this non-local term can be effec-
tively treated as a local term by using the theory of harmonic extensions of
functions from $\mathbb{R}^n$ to $\mathbb{R}^n \times (0, \infty)$. We have also analysed the implications of
this deformation for quantum field theory. Thus, we were able to construct a
non-local Lagrangian for the Dirac equation. Then we coupled this Lagrangian
to non-abelian gauge fields. We observed that even though the field strength
in the theory contains non-local terms, it transformed under local gauge transformations, as a regular undeformed field
strength. Thus, by the same argument, it was argued that the whole Lagrangian
for the deformed Yang-Mills coupled to matter fields, is invariant under local
gauge transformations. The quantization of this theory was also discussed.

It may be noted that like Yang-Mills theories, it is possible to describe the
gravity in the framework of gauge theories. Thus, if we deform a covariant
derivative in curved spacetime, then the dynamics of the gravitational field it-
self will change. In this case, the Einstein Hilbert action will be built from a
generalized Riemann tensor which in turn will be defined though the commu-
tator of these deformed covariant derivatives. This generalized Einstein Hilbert
action will also contain non-local terms and the variation with respect to the
tetrad field yields the non-local Einstein equations. The first part of these equa-
tions will be the usual local Einstein equations. However, it is expected that we
will also obtain non-local parts proportional to $\alpha$. This means that the dynam-
ics of the metric structure of spacetime depend will be considerably deformed.
Usually the deformation of the Heisenberg algebra seen as a consequence of some
quantum gravitational effect. However, it will be also interesting to reverse this
argument and analyses the effects a deformation of the Heisenberg algebra can
have for the quantum theory of gravity.

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