Construction of Banach frames and atomic decompositions of anisotropic Besov spaces

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Abstract

We construct generalised shift-invariant systems of functions of several real variables for anisotropic Besov spaces that can be generated by the decomposition method using any given expansive matrix and establish the conditions on those systems under which they will constitute Banach frames or sets of atoms for the anisotropic homo- or inhomogeneous Besov spaces.

Keywords Anisotropic Besov spaces · Anisotropic wavelets · Banach frames · Atomic decompositions

Mathematics Subject Classification 42B35 · 46E35 · 42C15 · 42C40

1 Introduction

Besov spaces, originally constructed by the approximation method [1], play an extremely important role in the theory of differentiable functions of several real variables as they, on the one hand, constitute a closed system with respect to embedding theorems and are, on the other hand, closely related to Sobolev spaces [13]. Along with Sobolev, Besov spaces are an integral part of the embedding theory, which studies connexions between differential properties of functions in different metrics [2, 12, 13]. Harmonic analysis uses either bases or frames and sets of atoms to decompose functions of a function space into basic building blocks or synthesise them from those blocks. The isotropic Besov spaces are known [14] to have orthonormal bases made of wavelets [7]. Herein, we shall use the innovative approach reported in [15, 16] to construct Banach frames and sets of atoms for the anisotropic homo- and inhomogeneous Besov spaces introduced in [4] as decomposition spaces [8]. Similar results, although formulated and achieved rather differently, were reported in [4] and [3].

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The work [4] was, in its turn, an ingenious generalisation of the ideas developed in [9] and [10].

This report is structured as follows: the elements of the decomposition method essential to the present work are outlined at the beginning of Sect. 2. The definitions of the anisotropic homo- and inhomogeneous Besov spaces viewed as decompositions spaces follow in Sects. 2.1 and 2.2. The notion of the Banach frame and that of the set of atoms for the decomposition space are reminded at the beginning of Sect. 3. In the same section, we give the statements of the two theorems that we shall use to construct Banach frames and atomic decompositions of anisotropic Besov spaces. In Sect. 4.1 of Sect. 4, the set of anisotropic homogeneous Besov wavelets is defined as a generalised shift-invariant system and the conditions are established under which this set will form a Banach frame or a set of atoms for the anisotropic homogeneous Besov space. Finally, anisotropic inhomogeneous wavelets are defined in Sect. 4.2, along with the conditions under which they will be a Banach frame or a set of atoms for the anisotropic inhomogeneous Besov space.

Throughout this report, the information is provided as it is first needed. All definitions and propositions borrowed from other works are supplied with corresponding references. All the new propositions are followed by their proofs.

2 Construction of anisotropic Besov spaces by the decomposition method

In this study, we shall concern ourselves with the anisotropic Besov spaces as they were defined in [4] by the decomposition method, initially reported in [8] and further developed in [3]. This definition involves three basic building blocks, namely an almost structured cover \( Q \) of the open subset of the frequency space, a regular partition of unity on the subset subordinate to the covering \( Q \) and a \( Q \)-moderate weight. Here are the definitions of these three notions and those of the decomposition space and its reservoir.

Definition 2.1 The set \( Q = \{ Q_i \}_{i \in I} \) is called an almost structured cover of the open subset \( O \) of \( \mathbb{R}^d \) where \( d \in \mathbb{N} \), if

1. \( Q \) is admissible, i.e. the number of elements in the sets \( \{ i' \in I : Q_{i'} \cap Q_i \neq \emptyset \} \) is uniformly bounded for all \( i \in I \);
2. There is a set \( \{ T_{i' \cdot + b_i} \}_{i \in I} \) of invertible affine-linear maps and finite sets \( \{ Q_{n_i} \}_{n=1}^N \) and \( \{ P_{n_i} \}_{n=1}^N \) of non-empty open and bounded subsets of \( \mathbb{R}^d \) such that

   (a) \( P_n \subset Q'_n \) for all \( 1 \leq n \leq N \);
   (b) For each \( i \in I \) there is such an \( n_i \in \{ 1, ..., N \} \) that \( Q_i = T_i Q_{n_i} + b_i \);
   (c) There is such a constant \( C > 0 \) that \( \| T_{i'}^{-1} T_i \| \leq C \) for all such \( i \) and \( i' \in I \) that \( Q_i \cap Q_{i'} \neq \emptyset \); and
   (d) \( O \subset \bigcup_{i \in I} (T_i P_{n_i} + b_i) \).

Definition 2.2 Let \( Q = \{ Q_i \}_{i \in I} \) and \( \{ T_{i' \cdot + b_i} \}_{i \in I} \) be an almost structured cover of the open subset \( O \) of \( \mathbb{R}^d \) and the set of invertible affine-linear maps associated with it, respectively. The set of functions \( \Phi = \{ \phi_i \}_{i \in I} \) is called regular partition of unity subordinate to \( Q \), if
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1. \( \phi_i \in C^\infty_c(O) \) with \( \text{supp} \phi_i \subset Q_i \) for all \( i \in I \);
2. \( \sum_{i \in I} \phi_i \equiv 1 \) on \( O \); and
3. \( \sup_{i \in I} \| \partial^\alpha \phi_i^\# \|_{L^\infty} < \infty \) for all \( \alpha \in \mathbb{N}_0^d \), where \( \phi_i^\# : \mathbb{R}^d \to \mathbb{C} \), \( \delta \mapsto \phi_i(T_i \delta + b_i) \).

**Definition 2.3** The sequence \( w = \{w_i\}_{i \in I} \) of positive numbers is called weight. The weight is called \( Q \)-moderate where \( Q = \{Q_i\}_{i \in I} \) stands for an almost structured cover of the open subset \( O \) of \( \mathbb{R}^d \), if there is such a positive number \( C \) that \( w_i \leq C \cdot w_{i'} \) for all such \( i \) and \( i' \in I \) that \( Q_i \cap Q_{i'} \neq \emptyset \).

**Definition 2.4** The topological dual \( Z' \) of the space \( Z := \mathcal{F}(C^\infty_c(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d) \) equipped with the unique topology that makes the Fourier transform \( \mathcal{F} : C^\infty_c(\mathbb{R}^d) \to Z \) into a homeomorphism will be referred to as reservoir.

Such a definition of the reservoir ensures that the decomposition space that we shall construct with its aid will be complete [5].

**Definition 2.5** Let \( Q = \{Q_i\}_{i \in I} \), \( \Phi = \{\phi_i\}_{i \in I} \), \( w = \{w_i\}_{i \in I} \) and \( Z' \) be an almost structured cover of the open subset \( O \) of \( \mathbb{R}^d \), a regular partition of unity on \( O \subset \mathbb{R}^d \) subordinate to \( Q \) and a \( Q \)-moderate weight and the reservoir, respectively, and let \( p \) and \( q \in (0, \infty] \). The set

\[
\mathcal{D}(Q, L^p, \ell^q_w) := \{ g \in Z' : \| g \|_{\mathcal{D}(Q, L^p, \ell^q_w)} < \infty \}
\]

equipped with the quasi-norm

\[
\| g \|_{\mathcal{D}(Q, L^p, \ell^q_w)} := \left\| \left( w_i \cdot \| \mathcal{F}^{-1}(\phi_i \cdot \mathcal{F}g) \|_{L^p} \right)_{i \in I} \right\|_{\ell^q} \in [0, \infty]
\]

is called the decomposition space.

We shall now specify covers, regular partitions of unity subordinate to them and moderate weights for the anisotropic homogeneous Besov spaces.

### 2.1 Construction of anisotropic homogeneous Besov spaces

**Definition 2.6** The matrix \( A \) whose elements are real numbers and whose spectrum \( \sigma(A) \) is such that

\[
\min_{\lambda \in \sigma(A)} |\lambda| > 1
\]

is called expansive.

**Definition 2.7** Let \( Q_0 \) be such a compact subset of \([-1, 1]^d \setminus \{0\} \) and \( A \) such a \( d \times d \) expansive matrix that \( \bigcup_{i \in \mathbb{Z}} Q_i = \mathbb{R}^d \setminus \{0\} \)

where \( Q_i := A^i Q_0 \), then the set \( \hat{Q}_B := \{Q_i\}_{i \in \mathbb{Z}} \) will be referred to as anisotropic homogeneous cover of \( \mathbb{R}^d \setminus \{0\} \).
The proof of existence of such a set \( Q_B := \{ Q_i \}_{i \in \mathbb{Z}} \) and it being indeed an almost structured cover of \( \mathbb{R}^d \setminus \{ 0 \} \) can be found in Lemma 5.2 in [6].

**Definition 2.8** Let \( \phi \in C^\infty(\mathbb{R}^d \setminus \{ 0 \}) \) and \( \text{supp} \phi \subset Q_0 \) as in Definition 2.7, then the set \( \Phi_B := \{ \phi_i \}_{i \in \mathbb{Z}} \) where

\[
\phi_i(\xi) := \phi(A^{-i} \xi)
\]

as \( \xi \in \mathbb{R}^d \) and \( i \in \mathbb{Z} \) will be referred to as anisotropic homogeneous partition of unity.

From Remark 2.3 in [6], it can be inferred that \( \Phi_B \) is a regular partition of unity on \( \mathbb{R}^d \setminus \{ 0 \} \) subordinate to \( Q_B \).

**Definition 2.9** Let \( s \in \mathbb{Z} \). Then, the set \( \hat{w}_B := \{ w_i \}_{i \in \mathbb{Z}} \) where

\[
 w_i := | \det A |^{s \Delta i}
\]

for any \( i \in \mathbb{Z} \) will be referred to as anisotropic homogeneous weight.

**Lemma 2.10** The weight \( \hat{w}_B \) is \( Q_B \)-moderate.

**Proof.** In Lemma 5.2 of [6], it was proved that \( Q_i \cap Q_{i'} = \emptyset \) where \( Q_i \) and \( Q_{i'} \in \hat{Q}_B \) if \( |i - i'| \) exceeds a certain finite integer \( \Delta i \). Therefore,

\[
\frac{w_i}{w_{i'}} = | \det A |^{s(i - i')} \leq | \det A |^{s|\Delta i|}
\]

as \( Q_i \cap Q_{i'} \neq \emptyset \) and \( | \det A |^{s|\Delta i|} \) can be taken as the positive number \( C \) mentioned in Definition 2.3 of the moderate weight. \( \square \)

**Definition 2.11** The decomposition space as defined by (1) with the almost structured cover \( Q_B \), partition of unity \( \Phi_B \) subordinate to \( Q_B \) and \( Q_B \)-moderate weight \( \hat{w}_B \) will be referred to as anisotropic homogeneous Besov space and denoted by \( \hat{B}^{a}_{p,q} \).

From this definition, one can deduce that the fact that a function \( g \) belongs to the space \( \hat{B}^{a}_{p,q} \) does not, generally speaking, imply that the composition of functions \( g \circ R \) where \( R \) stands for a rotation matrix will also belong to it, which justifies the attributive anisotropic in the name of the space. Only if we choose to use a scalar expansive matrix \( A \) for generating the space \( \hat{B}^{a}_{p,q} \), the latter will be isotropic. This remark, as it will transpire later, also applies to anisotropic inhomogeneous Besov spaces to whose definition we now turn.

### 2.2 Construction of anisotropic inhomogeneous Besov spaces

**Definition 2.12** Let \( Q_0 \) be such a subset of \( [-1, 1]^d \), \( Q_1 \) such a compact subset of \( [-1, 1]^d \setminus \{ 0 \} \) and \( A \) such a \( d \times d \) expansive matrix that

\[
\bigcup_{i \in \mathbb{N}_0} Q_i = \mathbb{R}^d
\]
where $Q_i := A^{i-1}Q_1$ as $i \in \mathbb{N}$, then the set $Q_B := \{Q_i\}_{i \in \mathbb{N}_0}$ will be referred to as anisotropic inhomogeneous cover of $\mathbb{R}^d$.

The proof of existence of such a set $Q_B := \{Q_i\}_{i \in \mathbb{N}_0}$ and it being indeed an almost structured cover of $\mathbb{R}^d$ can be found in Lemma 5.2 in [6].

**Definition 2.13** Let $\phi_0$ and $\phi_1 \in C^\infty(\mathbb{R}^d \setminus \{0\})$, supp$\phi_0 \subset Q_0$ and supp$\phi_1 \subset Q_1$ with $Q_0$ and $Q_1$ as in Definition 2.12, then the set $\Phi_B := \{\phi_i\}_{i \in \mathbb{N}_0}$ where

$$\phi_i(\xi) := \phi_1(A^{-(i-1)}\xi)$$

as $\xi \in \mathbb{R}^d$ and $i \in \mathbb{N}$ will be referred to as anisotropic inhomogeneous partition of unity.

From Remark 2.3 in [6], it can be inferred that $\Phi_B$ is a regular partition of unity on $\mathbb{R}^d$ subordinate to $Q_B$.

**Definition 2.14** Let $s \in \mathbb{Z}$. Then, the set $\omega_B := \{w_i\}_{i \in \mathbb{N}_0}$ where $w_0 := 1$ and

$$w_i := |\det A|^s(i-1)$$

for any $i \in \mathbb{N}$ will be referred to as anisotropic inhomogeneous weight.

**Lemma 2.15** The weight $\omega_B$ is $Q_B$-moderate.

The proof of this lemma is almost identical to that of 2.10.

**Definition 2.16** The decomposition space as defined by (1) with the almost structured cover $Q_B$, partition of unity $\Phi_B$ subordinate to $Q_B$ and $Q_B$-moderate weight $\omega_B$ will be referred to as anisotropic inhomogeneous Besov space and denoted by $B^a_{p,q}$.

## 3 Construction of Banach frames for and atomic decompositions of the decomposition spaces

The decomposition space is a example of the quasi-Banach space, i. e. the complete quasi-normed vector space. Banach frames and sets of atoms provide the quasi-Banach space with those basic building blocks into which any of the element of the space can be decomposed or from which an element of the space can be synthesised. Here are the definitions of these two notions [11], along with one axillary definition.

**Definition 3.1** Let $A \subset C^I$ where $C^I$ stands for the quasi-Banach space consisting of sequences of complex numbers indexed by $i \in I$, and let $a = \{a_i\}_{i \in I} \in A$. If $|a_i| \leq |a_i|$ for all $i \in I$ implies that $a' = \{a'_i\}_{i \in I} \in A$ and $||a'||_C \leq ||a||_C$, then $A$ is called solid.

**Definition 3.2** A set $\{\psi_i\}_{i \in I}$ in the dual space $X'$ of a quasi-Banach space $X$ is called Banach frame for $X$ if there is a well-defined bounded map, called analysis operator, $A : X \rightarrow x,f \mapsto \{\langle \psi_i,f\rangle\}_{i \in I}$ where $x := \{\langle \psi_i,f\rangle\}_{i \in I} : f \in X$ is a solid quasi-Banach subspace of $C^I$ and there is such a bounded linear map $A^*_i : x \rightarrow X$ that $A^*_i \circ A = I_X$ where $I_X$ stands for an identity operator on $X$. 

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Definition 3.3 A set \( \{ \phi_i \}_{i \in I} \) in a quasi-Banach space \( X \) is called set of atoms in \( X \) if there is a well-defined bounded map, called synthesis operator, \( S: x \rightarrow X, \{ c_i \}_{i \in I} \rightarrow \sum_{i \in I} c_i \phi_i \) where the coefficient space \( x := \{ c_i \}_{i \in I} \) associated with \( \{ \phi_i \}_{i \in I} \) is a solid subspace of \( C^I \) and there is such a bounded linear map \( S_r^{-1}: X \rightarrow x \) that \( S \circ S_r^{-1} = I_X \) where \( I_X \) stands for an identity operator on \( X \). The series expansion \( g = \sum_{i \in I} c_i \phi_i \) of a given function \( g \in \hat{X} \) where \( \{ \phi_i \}_{i \in I} \) is a set of atoms is called atomic decomposition of \( g \).

To construct Banach frames and sets of atoms for the anisotropic Besov spaces \( \hat{B}^a_{p,q} \) and \( B^p_{q,p,q} \) we shall use the following two concepts and two theorems.

Definition 3.4 Let \( \delta > 0, \{ T_i \cdot + b_i \}_{i \in I} \) be a denumerable set of invertible affine-linear maps on \( \mathbb{R}^d \) and \( \{ \phi_n \}_{n=1}^N \) a finite set of square integrable functions on \( \mathbb{R}^d \). Then,

\[
\Psi := \{ L_{\delta T_i^{-1}} \psi_i (t) \}_{i \in I, \ k \in \mathbb{R}^d}
\]

where

\[
L_{\delta}f(z) := f(z - x) \quad \text{and} \quad \psi_i (t) := | \det T_i |^{1/2} M_{\psi} \phi_n (T_i t)
\]

where \( M_{\psi}f(z) := e^{2\pi i z} f(z) \) and \( \phi_n \in \{ \phi_n \}_{n=1}^N \) is called generalised shift-invariant system.

Definition 3.5 Let \( p \) and \( q \in (0, \infty] \) and \( w = \{ w_i \}_{i \in I} \) be a weight. Then,

\[
\mathcal{C}^p_w := \left\{ (c^{(i)}_k)_{k \in \mathbb{Z}^d} \in \mathbb{C}^{I \times \mathbb{Z}^d} : \| c \|_{\mathcal{C}^p_w} := \left\| \left( | \det T_i |^{\frac{1}{q}} \cdot w_i \cdot \| (c^{(i)}_k)_{k \in \mathbb{Z}^d} \|_{\ell^Q_{r,p}} \right)_{i \in I} \right\|_{\ell_q} < \infty \right\}
\]

is called coefficient space associated with \( \Psi \).

The functions \( \{ \phi_n \}_{n=1}^N \) in Definition 3.4 can be regarded as prototypes of all the functions in \( \Psi \). The next two theorems specify the conditions on the prototypes under which \( \Psi \) will constitute a Banach frame or a set of atoms for the decomposition space \( D(Q, L^p, C^p_w) \), respectively.

Theorem 3.6 Let \( \epsilon, p_0 \) and \( q_0 \in (0, 1], \ p \in [p_0, \infty], \ q \in [q_0, \infty] \) and \( \Phi = \{ \phi_i \}_{i \in I} \) and \( w = (w_i)_{i \in I} \) be, respectively, a regular partition of unity subordinate to \( Q \) and a \( Q \)-moderate weight where \( Q \) stands for the almost structured cover of an open subset \( O \) of \( \mathbb{R}^d \) whose elements \( \{ Q_i \}_{i \in I} \) are generated by having the invertible affine-linear transformations \( \{ T_i \cdot + b_i \}_{i \in I} \) acted on elements of the finite set \( \{ Q'_i \}_{i=n=1}^N \) of non-empty open and bounded subsets \( Q'_i \) of \( \mathbb{R}^d \). Cf Definitions 2.1, 2.2 and 2.3. Furthermore, let all the elements of the finite set \( \{ \phi_n \}_{n=1}^N \) of square integrable functions on \( \mathbb{R}^d \), introduced in Definition 3.4, satisfy the following conditions:

1. \( \hat{\phi}_n \in C^\infty (\mathbb{R}^d) \);
2. \( \phi_n \) and all its partial derivatives are of polynomial growth at most;
3. \( \hat{\phi}_n (\xi) \neq 0 \) as \( \xi \in \mathbb{R}^d \);
4. \( \phi_n \in \mathcal{C}^1 (\mathbb{R}^d) \) and \( \nabla \phi_n \in L^1 (\mathbb{R}^d) \cap L^\infty (\mathbb{R}^d) \); and

\[
S_1 := \sup_{i \in I} \sum_{j \in \mathbb{N}} N_{ij}^1 < \infty \quad \text{and} \quad S_2 := \sup_{j \in \mathbb{N}} \sum_{i \in I} N_{ij}^1 < \infty
\]

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where

\[ N_{ij}^1 := \left( \frac{w_i}{w_j} \right)^{\tau} (1 + \| T_i^{-1} T_j \|)^{\sigma} \]

\[ \cdot \max_{|\rho| \leq 1} \left| \det T_j \right|^{-1} \int_{Q_j} \max_{|\eta| \leq N} \left| \partial^\rho \overline{\partial}^\eta \phi_n \right| \left| \left( T_i^{-1} (\xi - b_i) \right) \right| d\xi \right)^{\tau}, \]

\[ \tau := \min\{1, p, q\}, \]

\[ \sigma := \tau \cdot \left( N + \frac{d}{\min\{1, p\}} \right) \quad \text{and} \quad N := \left[ \frac{d + \epsilon}{\min\{1, p\}} \right]. \]

Then, there is such a \( C = C(\epsilon, p_0, q_0, d, Q, w, \{\phi_n\}_{n=1}^N) > 0 \) that the generalised shift-invariant system

\[ \tilde{\Psi} := \{\tilde{\psi}_i(t)\}_{i \in I, k \in \mathbb{R}^d} := \{L_{\delta T_i^{-1}} \Psi_i(-t)\}_{i \in I, k \in \mathbb{R}^d} \]

where \( \psi_i(t) \) as defined by (9) with the coefficient space \( C_w^{p,q} \) constitutes a Banach frame for the decomposition space \( D(Q, L^p, \ell_w^q) \) as long as \( \delta \in (0, \delta_0) \) where

\[ \delta_0 = 1 \left[ 1 + C \cdot C_{Q,w}^{1/\tau} \left( S_1^{1/\tau} + S_2^{1/\tau} \right)^2 \right], \]

and, in particular,

1. The analysis operator

\[ A_{\tilde{\Psi}} : D(Q, L^p, \ell_w^q) \rightarrow C_w^{p,q}, f \mapsto \{[\tilde{\psi}_i * f](\delta T_i^{-1} k)\}_{i \in I, k \in \mathbb{Z}^d}, \]

where the convolution \( \tilde{\psi}_i * f \) is defined by

\[ \left( \tilde{\psi}_i * f \right)(t) = \sum_{j \in I} \mathcal{F}^{-1} \left( \hat{\tilde{\psi}}_i \cdot \hat{\phi}_j \cdot \hat{f} \right)(t) \]

\[ \text{(14)} \]

is well-defined and bounded as long as \( \delta \in (0, 1] \) and the series in (14) converges normally in \( L^\infty(\mathbb{R}^d) \). Moreover, if \( f \in L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{Z}' \), the convolution defined by (14) agrees with its usual definition and

\[ A_{\tilde{\Psi}} f = \{f, L_{\delta T_i^{-1}} \tilde{\Psi}_i\}_{i \in I, k \in \mathbb{Z}^d} \]

\[ \text{(15)} \]

for any \( f \in L^2(\mathbb{R}^d) \cap D(Q, L^p, \ell_w^q) \).

2. There is such a map \( A_{\tilde{\Psi}}^{-1} : C_w^{p,q} \rightarrow D(Q, L^p, \ell_w^q) \) that \( A_{\tilde{\Psi}}^{-1} \circ A_{\Psi} = id_{D(Q, L^p, \ell_w^q)} \) as long as \( \delta \in (0, \delta_0) \).

This is a slightly reformulated statement of Theorem 2.11 in [5]. The proof of this theorem in its more general and simplified formulations can be found in [15] and [16], respectively.
Theorem 3.7  Let $\epsilon$, $p_0$ and $q_0 \in (0, 1]$, $p \in [p_0, \infty]$, $q \in [q_0, \infty]$ and $w = \{ w_i \}_{i \in I}$ be $Q$-moderate weight where $Q$ stands for the almost structured cover of an open subset $O$ of $\mathbb{R}^d$ whose elements $\{ Q_i \}_{i \in I}$ are generated by having the invertible affine-linear transformations $\{ T_i \cdot + b_i \}_{i \in I}$ acted on elements of the finite set $\{ Q'_n \}_{n=1}^N$ of non-empty open and bounded subsets $Q'_n$ of $\mathbb{R}^d$. Cf Definitions 2.1 and 2.3. Furthermore, let all the elements of the finite set $\{ \phi_n \}_{n=1}^N$ of square integrable functions on $\mathbb{R}^d$, introduced in Definition 3.4, satisfy the following conditions:

1. $\hat{\phi}_n \in C^\infty(\mathbb{R}^d)$ where $\hat{\phi}_n$ stands for the Fourier transform of $\phi_n$;

2. $\hat{\phi}_n$ and all its partial derivatives are of polynomial growth at most;

3. $\hat{\phi}_n(\xi) \neq 0$ as $\xi \in Q'_n$;

4. $\sup_{\xi \in \mathbb{R}^d} \left| (1 + |\xi|)^\Lambda \cdot |\phi_n(\xi)| \right| < \infty$ where $\Lambda : = 1 + d / \min \{ 1, p \}$; and

5. There is such a set $\{ \rho_n \}_{n=1}^N$ of non-negative and absolutely integrable on $\mathbb{R}^d$ functions that

$$|\partial^\alpha \hat{\phi}_n(\xi)| \leq \rho_n(\xi) \cdot (1 + |\xi|)^{-(d+1+\epsilon)}$$

as $\xi \in \mathbb{R}^d$ and for all such $\alpha \in \mathbb{N}^d_0$ that $|\alpha| \leq N$ with $N$ as defined in Theorem 3.6;

and

$$S_3 := \sup_{i \in I} \sum_{j \in I} N_{ij}^2 < \infty \text{ and } S_4 := \sup_{i \in I} \sum_{j \in I} N_{ij}^2 < \infty$$

where

$$N_{ij}^2 : = \left( \frac{w_i}{w_j} \cdot \left( | \det T_j | / | \det T_i | \right)^\theta \right) \cdot (1 + || T_j^{-1} T_i ||)^\tau$$

$$\cdot \left( | \det T_j |^{-1} \int_{Q'_j} \rho_n(T_j^{-1}(\xi - b_j)) d\xi \right)^\tau$$

$$\theta : = (p^{-1} - 1)_+, \tau \text{ as defined in Theorem 3.6 and}$$

$$\sigma : = \begin{cases} \tau \cdot (d + 1) & \text{if } p \in [1, \infty] \\ \tau \cdot (p^{-1} \cdot d + \lfloor p^{-1} \cdot (d + \epsilon) \rfloor) & \text{if } p \in (0, 1) \end{cases}$$

Then, there is such a $C = C(\epsilon, p_0, q_0, d, Q, w, \{ \phi_n \}_{n=1}^N) > 0$ that the generalised shift-invariant system $\Psi$ with the coefficient space $C^{w,d}_w$ constitutes a set of atoms for the decomposition space $D(Q, L^p, \ell^d_w)$ as long as $\delta \in (0, \delta_0)$ where

$$\delta_0 = \min \left\{ 1, \left[ C \cdot (S_3^{1/\tau} + S_4^{1/\tau}) \right]^{-1} \right\},$$

and, in particular,
1. The synthesis operator

\[ S_\Psi : C^p,q_w \to D(Q, L^p, \epsilon^q_w), \quad \{c^{(i)}_k\}_{i \in I, k \in \mathbb{Z}^d} \mapsto \sum_{i \in I} \sum_{k \in \mathbb{Z}^d} \left[ c^{(i)}_k L_\delta_{\tau_k} \Psi_i \right] \]  

is well-defined and bounded as long as \( \delta \in (0, 1) \), i.e. the sum over the index \( k \) in (18) converges absolutely for any index \( i \in I \) to a function in \( L^1_S(\mathbb{R}^d) \cap S'((\mathbb{R}^d) \cap \mathbb{Z}^d) \) and the sum of such functions over the index \( i \) converges unconditionally in the weak sense in \( Z' \); and

2. There is such a map \( S_\Psi^{-1} : D(Q, L^p, \epsilon^q_w) \to C^p,q_w \) that \( S_\Psi S_\Psi^{-1} = \text{id}(Q, L^p, \epsilon^q_w) \) as long as \( \delta \in (0, \delta_0] \) and the action of \( S_\Psi^{-1} \) on any given \( f \in D(Q, L^p, \epsilon^q_w) \) does not depend on \( p, q \) and \( w \).

This is a slightly reformulated statement of Theorem 2.10 in [5]. The proof of this theorem in its more general and simplified formulations can be found in [15] and [16], respectively.

4 Construction of Banach frames and atomic decompositions of the anisotropic Besov spaces

4.1 Banach frames and atomic decompositions of anisotropic homogeneous Besov spaces

First of all, we define shift-invariant systems for the anisotropic homogeneous spaces and coefficient spaces associated with them. Here are these definitions.

**Definition 4.1** Let \( \delta > 0 \), \( I_d \) be \( d \times d \) identity matrix, \( A_i := A^i \), \( \psi(t) \in L^1(\mathbb{R}^d) \) and

\[ \psi_i(t) := |\det A_i|^{1/2} \psi(A_i^t) \]  

where \( i \in \mathbb{Z} \), then the set

\[ \Psi_B := \{L_{\delta A_i^{-1}} \psi_i(t)\}_{i \in \mathbb{Z}, k \in \mathbb{Z}^d} = \{ |\det A_i|^{1/2} \psi(A_i^t - \delta k)\}_{i \in \mathbb{Z}, k \in \mathbb{R}^d} \]  

will be referred to as anisotropic homogeneous Besov wavelets.

**Definition 4.2** Let \( p \) and \( q \in (0, \infty] \), then

\[ \dot{C}^p,q_{Bs} := \left\{ (c^{(i)}_k)_{i \in \mathbb{Z}, k \in \mathbb{Z}^d} \in C^{I \times \mathbb{Z}^d} : \|c\|_{\dot{C}^p,q_{Bs}} := \left\| \left( |\det A_i|^{1/(1-\frac{1}{p})} \cdot w_i \cdot \|c^{(i)}_k\|_{L^p} \right)_{i \in \mathbb{Z}} \right\|_{L^q} < \infty \right\} \]

where \( w_i \in \dot{w}_B \) will be referred to as coefficient space associated with \( \dot{\Psi}_B \).

To prove the two theorems that establish the conditions on \( \Psi_B \) under which it will be a Banach frame or a set of atoms for \( \dot{B}^u_{p,q} \), we shall make use of an auxiliary lemma.
Here is this lemma of ours as well as the lemma 2.2 from [4] that we shall use to state and prove it.

**Lemma 4.3** Let $\lambda_-\$ and $\lambda_+$ be such real numbers that

$$1 < \lambda_- < \min_{\lambda \in \sigma(A)} |\lambda| \leq \max_{\lambda \in \sigma(A)} |\lambda| < \lambda_+$$

(21)

where $\sigma(A)$ stands for the spectrum of a $d \times d$ expansive matrix $A$, then there is such a number $b > 0$ that

$$\frac{1}{b} \cdot \lambda_-^j |\xi| \leq |A^j \xi| \leq b \cdot \lambda_+^j |\xi|$$

(22)

and

$$\frac{1}{b} \cdot \lambda_+^{-j} |\xi| \leq |A^{-j} \xi| \leq b \cdot \lambda_-^{-j} |\xi|$$

(23)

where $\xi \in \mathbb{R}^d$ and $j \in \mathbb{N}_0$.

**Lemma 4.4** Let $Q_0$ and $A$ be, respectively, such an open bounded subset of $\mathbb{R}^d \setminus \{0\}$ and such a $d \times d$ expansive matrix that

$$\bigcup_{n \in \mathbb{Z}} Q_n = \mathbb{R}^d \setminus \{0\}$$

(24)

where

$$\{Q_n := A^n Q_0 \}_{n \in \mathbb{Z}}.$$  

(25)

Furthermore, let $a, \tau$ and $\sigma > 0$,

$$L > \log \lambda_- a,$$

(26)

$$N > \log \lambda_- \left( \frac{\lambda_+^{\tau/a}}{a} \right)$$

(27)

with $\lambda_-$ and $\lambda_+$ defined by (21) and

$$|\hat{\psi}(\xi)| \leq C \cdot \min \{ 1, |\xi|^L \} (1 + |\xi|)^{-N}$$

(28)

where $C > 0$ and $\xi \in Q_0$. Then,

$$\sup_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} M_{mn} \leq S < \infty$$

(29)

and

$$\sup_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} M_{mn} \leq S < \infty$$

(30)

where
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\[ M_{mn} := a^{r(m-n)} \cdot (1 + \|A^{n-m}\|) \gamma \cdot \left[ \frac{1}{|Q_n|} \int_{Q_n} |\hat{\psi}(A^{-m} \xi)| \, d\xi \right]^r \]  

(31)

and

\[ S = C^r \cdot (1 + b) \sigma \cdot \left( b \max \left\{ \frac{1}{r}, R \right\} \right)^{\max\{L,N\} r} \cdot \left[ \frac{1}{1 - \left( \frac{a}{\lambda^2} \right)^r} + \frac{2}{1 - \left( \frac{a}{(a \lambda^2)} \right)^r} \right] \]

(32)

with \( b \) defined by (22) and (23) and where \( r \) and \( R \) are such that \( 0 < r \leq |\xi| \leq R < \infty \) as \( \xi \in Q_0 \).

**Proof** First of all, we use (25) to change the set over which the integration in (31) is done from \( Q_n \) to \( Q_0 \) and obtain

\[ M_{mn} = a^{r(m-n)} \cdot (1 + \|A^{n-m}\|) \gamma \cdot \left[ \frac{1}{|A^n Q_0|} \int_{Q_0} |\hat{\psi}(A^{-m} A^n \xi)| |A^n| \, d\xi \right]^r \]

(33)

and then combine it with (28) to estimate \( M_{mn} \) from the above, namely

\[ M_{mn} \leq a^{r(m-n)} \cdot (1 + \|A^{n-m}\|) \gamma \cdot \left[ \frac{C}{|Q_1|} \int_{Q_1} \min\{1, |A^{n-m} \xi|^{L} \} \right]^r \cdot \left[ \left( 1 + \frac{b^{n-m} \xi}{1 + \frac{b^{n-m} \xi}{b}} \right)^N \right] \]

(34)

Now, we use (22) and (23) and distinguish two cases. If \( n - m \geq 0 \), then combining (34) with (22) results in

\[ M_{mn} \leq a^{r(m-n)} \cdot (1 + b \lambda^{n-m}) \gamma \cdot \left[ \frac{C}{|Q_1|} \int_{Q_1} \min\{1, \left( b \lambda^{n-m} \xi \right)^{L} \} \right]^r \cdot \left( 1 + \frac{b^{n-m} \xi}{b} \right)^N \]

(35)

Otherwise, if \( n - m \leq 0 \), then combining (34) with (23) results in
To prove that the series (29) converges, we shall divide it into several series and prove that each of them converges. In doing so, we distinguish two further cases. If \( n \geq 0 \), then we divide the series (29) into two, namely

\[
M_{mn} \leq a^{\tau(m-n)} \cdot (1 + b \lambda_{-m}^n)^{\sigma} \cdot \left[ \frac{C}{|Q_1|} \int_{Q_1} \min\{1, (b \lambda_{-m}^n|\xi|)^L\} \frac{1}{(1 + \frac{\lambda_{-m}^n|\xi|}{b})^N} \right]^\tau \]

(36)

\[
\leq a^{\tau(m-n)} \cdot (1 + b \lambda_{-m}^n)^{\sigma} \cdot \left[ \frac{\min\{1, (b \lambda_{-m}^n R)^L\}}{(1 + \frac{\lambda_{-m}^n R}{b})^N} \right]^\tau.
\]

Using the substitution \( m' := n - m \) in the former series in (37) and changing the order of summation result in

\[
\sum_{m=-\infty}^{\infty} M_{mn} \leq \sum_{m=-\infty}^{\infty} M_{m'n}.
\]

(39)

where

\[
M_{m'} := (1 + b)^{\sigma} \cdot C^\tau \cdot \left( \frac{b}{r} \right)^N \cdot \left[ \frac{\lambda_{+}^n}{(a \lambda_{+}^n)^\tau} \right]^{m'} \geq a^{-r m'} \cdot (1 + b \lambda_{+}^{m'})^{\sigma} \cdot \left[ \frac{C}{(1 + \frac{\lambda_{+}^{m'} R}{b})^N} \right]^\tau
\]

\[
\geq a^{-r m'} \cdot (1 + b \lambda_{+}^{m'})^{\sigma} \cdot \left[ \frac{\min\{1, (b \lambda_{+}^{m'} R)^L\}}{(1 + \frac{\lambda_{+}^{m'} R}{b})^N} \right]^\tau = M_{mm'}
\]

(40)

The geometrical series on the right-hand side of (38) would converge, should its general term \( M_{m'} \) satisfy the criterion

\[
\lim_{m' \to \infty} \sup \frac{M_{m'+1}}{M_{m'}} < 1
\]

(41)

or, in other words,

\[
\lim_{m' \to \infty} a^{-r} \frac{\lambda_{+}^n}{\lambda_{-m}^n} < 1.
\]

This holds if, as assumed in this lemma, (27) does. Under this condition,
Using the substitution \( m' := m - n \) in the latter series in (37) results in
\[
\sum_{m=n+1}^{\infty} M_{mn} \leq \sum_{m'=1}^{\infty} M_{m'n} \leq \sum_{m'=0}^{\infty} M_{m'n}
\] (43)

where
\[
M_{m'} := (1 + b)^\sigma \cdot C \cdot (bR)^{Lx} \cdot \left( \left( \frac{a}{\lambda^L} \right)^{m'} \right) \geq C \cdot (1 + b)^\sigma \cdot a^{m'} \cdot (b \lambda^{-m'} R)^{Lx}
\]
\[
\geq a^{m'} \cdot (1 + b \lambda^{-m'})^\sigma \cdot \left[ \frac{\min\{1, (b \lambda^{-m'} R)^{Lx}\}}{\left( 1 + \frac{\lambda^{-m'} R}{b} \right)^N} \right]^x = M_{mn}.
\] (44)

The geometrical series on the right-hand side of (43) would converge, should its general term \( M_{m'} \) satisfy the criterion (40), namely
\[
\left( \frac{a}{\lambda^L} \right)^x < 1.
\] (45)

This holds if, as assumed in this lemma, (26) does. Under this condition,
\[
\sum_{m=n+1}^{\infty} M_{mn} \leq \sum_{m'=0}^{\infty} M_{m'} = \frac{(1 + b)^\sigma C (bR)^{Lx}}{1 - \left( \frac{a}{\lambda^L} \right)^x}.
\] (46)

If \( n < 0 \), then we divide the series (29) into three, namely
\[
\sum_{m=-\infty}^{n} M_{mn} = \sum_{m=-\infty}^{n} M_{mn} + \sum_{m=n+1}^{-(n+1)} M_{mn} + \sum_{m=-\infty}^{-n} M_{mn},
\] (47)

so that \( n - m \geq 0 \) and therefore \( M_{mn} \) can be estimated as in (35) in the first and third series and \( n - m \leq 0 \) and therefore \( M_{mn} \) can be estimated as in (36) in the second series. Using the substitution \( m' := n - m \) in the first series in (47) and changing the order of summation result in the estimate identical to (38), where the geometrical series on the right-hand side will, as we already know, converge as long as (27) holds. Under this condition, the sum of the first series in (47) was estimated in (42). Since \( n \leq 0 \), using the substitution \( m' := n - m \) in the second series in (47) results in
\[
\sum_{m=n+1}^{-\infty} M_{mn} \leq \sum_{m=n}^{-\infty} M_{mn} \leq \sum_{m'=0}^{-\infty} M_{m'} \leq \sum_{m'=0}^{\infty} M_{m'}
\] (48)

where the geometrical series on the right-hand side is identical to that in (43). Thus, the second series in (47) will converge if (26) holds. Under this condition, the sum of the second series in (47) was estimated in (46). Using the substitution \( m' := n - m \) in the third series in (47) results in
\[
\sum_{m=-n}^{\infty} M_{mn} = \sum_{m'=2n}^{\infty} M_{m'} \leq \sum_{m'=0}^{\infty} M_{m'}
\]

(49)

where the geometrical series on the right-hand side is identical to that in (38). Thus, the third series in (47) will converge if (27) holds. Under this condition, its sum was already estimated in (42). This completes the proof of convergence of the series (29) with any \(n \in \mathbb{Z}\). Combining (37) and (47) with the estimates (42) and (46) results in (32).

Changing the order of summation in the series (30) converts it into series (29). Therefore, the series (30) converges if, as supposed in the lemma, \(N\) and \(L\) satisfy (27) and (26), respectively. The estimate of its sum is given in (32).

The next theorem establishes the conditions on \(\Psi_B\) under which it will be a Banach frame for \(B_{p,q}^a\).

**Theorem 4.5** Let \(\varepsilon, p_0\) and \(q_0 \in (0, 1]\). Moreover, let \(\phi \in L^1(\mathbb{R}^d)\) satisfy the following conditions:

1. \(\hat{\phi} \in C^\infty(\mathbb{R}^d)\);
2. \(\hat{\phi}\) and all its partial derivatives are of polynomial growth at most;
3. \(\hat{\phi}(\xi) \neq 0\) as \(\xi \in Q_0\), where \(Q_0\) is as in Definition 2.7;
4. \(\phi \in C^1(\mathbb{R}^d)\) and \(N\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\); and

5. \[\left| \partial^\alpha \partial^\beta \hat{\phi}(\xi) \right| = \hat{\gamma}_1(\xi) \leq C \min\{1, |\xi|^{L_1}\}(1 + |\xi|)^{-N_1}\]

where \(C\) stands for a constant,

\[L_1 > s \log_+ (|\det A|)\]

(50)

and

\[N_1 > \log_+ \left( \frac{s^\sigma/\tau}{|\det A|^{\beta}} \right)\]

(51)

(52)

as \(\xi \in \mathbb{R}^d\) and for all such \(\alpha\) and \(\beta \in \mathbb{N}_0^d\) that \(|\alpha| \leq N\) and \(|\beta| \leq 1\).

Then, there is such a \(\delta_0 = \delta_0(\varepsilon, p_0, q_0, d, A, \phi) > 0\) that the anisotropic homogeneous Besov wavelets \(\Psi_B\) with the coefficient space \(C_{B_0}^{p,q}\) constitutes a Banach frame for the anisotropic homogeneous Besov space \(B_{p,q}^a(A)\) as long as \(\delta \in (0, \delta_0]\).

**Proof** The four assumptions of this theorem are nothing, but those of Theorem 3.6 formulated for the generalised shift-invariant system \(\Psi_B\) with the coefficient space \(C_{B_0}^{p,q}\) pertaining to the cover \(Q_B\) of the set \(O := \mathbb{R}^d \setminus \{0\}\) and weight \(w_B\) that form the space \(B_{p,q}^a\). Furthermore, we recollect that \(\{T_i\}_{i \in \mathbb{Z}} = \{A_i\}_{i \in \mathbb{Z}}\) and \(\{b_i\}_{i \in \mathbb{Z}} = 0\) as \(\Psi = \Psi_B\) and that \(\{w_i\}_{i \in \mathbb{Z}} = \{|\det A|[\alpha]\}_{i \in \mathbb{Z}}\) with \(s \in \mathbb{Z}\) as \(C_{B_0}^{p,q} = C_{B_0}^{p,q}\). Therefore, \(N_1\) defined by (12) becomes

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\[ N_{ij}^1 = (|\det A|^\tau)^{\tau(i-j)} \cdot (1 + \|A^{j-i}\|)^\sigma \cdot \max_{|\beta| \leq 1} \left( |\det A|^{-i} \int_{Q_j} \max_{|a| \leq N} \left| \partial^\beta \hat{\phi} (A^{-i} \xi) \right| \, d\xi \right)^\tau \]

\[ = (|\det A|^\tau)^{\tau(i-j)} \cdot (1 + \|A^{j-i}\|)^\sigma \cdot \max_{|\beta| \leq 1} \left( \frac{2^d}{|Q_j|} \int_{Q_j} \max_{|a| \leq N} \left| \partial^\beta \hat{\phi} (A^{-i} \xi) \right| \, d\xi \right)^\tau \]

(53)

where we also noted that \(|Q_j| = |A^j Q| = |\det A|^j \cdot |Q| = |\det A|^j \cdot 2^d\). The last expression in (53) clearly equates to \(2^d \cdot M_{mn}^2\), with \(M_{mn}\) defined by (31) if \(m = i, n = j\) and \(a = |\det A|^j\) and as long as \(L_i\) and \(N_i\) in (50) are not smaller than \(L\) and \(N\) in (28), respectively. According to Lemma 4.4, the series (29) and (30) converge on the assumptions (26) and (27). Therefore, the series (11) converges on the assumptions (51) and (52). In other words, the fifth assumption of Theorem 3.6 follows from the fifth assumption of the present theorem.

\[ \square \]

The next theorem establishes the conditions on \(\hat{\Psi}_B\) under which it will be a set of atoms for \(\hat{B}_{p,q}^n\).

**Theorem 4.6** Let \(\varepsilon\), \(p_0\) and \(q_0\) \(\in (0, 1]\). Moreover, let \(\phi \in L^1(\mathbb{R}^d)\) satisfy the following conditions:

1. \(\hat{\phi} \in C^\infty(\mathbb{R}^d)\);
2. \(\hat{\phi}\) and all its partial derivatives are of polynomial growth at most;
3. \(\hat{\phi}(\xi) \neq 0\) as \(\xi \in Q_0\), where \(Q_0\) is as in Definition 2.7;
   \[
   \sup_{t \in \mathbb{R}^d} \left[ (1 + |t|)^\lambda \cdot |\phi(t)| \right] < \infty
   \]
4. \(\lambda := 1 + d/p_0\); and
   \[
   \left| \partial^\alpha \hat{\phi}(\xi) \right| \leq \rho(\xi) \cdot (1 + |\xi|)^{-(d+1+\varepsilon)}
   \]
5. where

\[
\rho : \mathbb{R}^d \to (0, \infty), \xi \mapsto C \min\{1, |\xi|^{L_2}\}(1 + |\xi|)^{-N_2},
\]

\(C\) stands for a constant,

\[
L_2 > (\theta - s) \log \lambda (|\det A|) \tag{54}
\]

and

\[
N_2 > \log \lambda \left( \frac{\ell^\sigma/\tau}{|\det A|^{\sigma-s}} \right) \tag{55}
\]

as \(\xi \in \mathbb{R}^d\) and for all such \(\alpha \in \mathbb{N}^d_0\) that \(|\alpha| \leq N\) where
\[ N := \left[ \frac{d + e}{p_0} \right]. \]

Then, there is such a \( \delta_0 = \delta_0(\epsilon, p_0, q_0, d, A, \phi > 0 \) that the anisotropic homogeneous Besov wavelets \( \Psi_B \) with the coefficient space \( \tilde{C}^{\phi,q}_{B_\theta} \) constitutes a set of atoms for the anisotropic homogeneous Besov space \( B_{p,q}^\alpha(A) \) as long as \( \delta \in (0, \delta_0] \).

**Proof** The first four assumptions of this theorem are nothing, but those of Theorem 3.7 formulated for the generalised shift-invariant system \( \Psi_B \) with the coefficient space \( \tilde{C}^{\phi,q}_{B_\theta} \) pertaining to the cover \( \hat{Q}_B \) of the set \( O := \mathbb{R}^d \setminus \{0\} \) and weight \( \hat{w}_B \) that form the space \( B_{p,q}^\alpha(A) \).

Furthermore, given that \( \{T_i\}_{i \in \mathbb{Z}} = \{A^i\}_{i \in \mathbb{Z}} \) and \( \{b_i\}_{i \in \mathbb{Z}} = 0 \) as \( \Psi = \tilde{\Psi}_B \) and that \( \{w_i\}_{i \in \mathbb{Z}} = \{|\det A|^{\sigma}\}_{i \in \mathbb{Z}} \) with \( s \in \mathbb{N} \) as \( C_{B_s}^{p,q} = \tilde{C}^{\phi,q}_{B_\theta} \), \( N_{ij}^2 \) defined by (17) becomes

\[
N_{ij}^2 = \left( |\det A|^{\sigma(i-j)} \cdot |\det A|^{\theta(j-i)} \right)^{\frac{\tau}{2}} \left( 1 + \|A^{i-j}\| \|A^{j-i}\| \right)^{\sigma - \sigma} \left( \int_{\hat{Q}_i} \rho(A^{-j}\xi) \, d\xi \right)^{\tau/\sigma}, \quad (56)
\]

The last expression in (56) equates to \( 2^d \cdot M_{mn} \) with \( M_{mn} \) defined by (31) if \( m = -i, n = -j \) and \( a = |\det A|^{\sigma - \sigma} \) and as long as \( L_1 \) and \( N_1 \) in (50) are not smaller than \( L \) and \( N \) in (28), respectively. According to Lemma 4.4, the series (29) and (30) converge on the assumptions (26) and (27). Therefore, the series (16) converges on the assumptions (54) and (55). In other words, the fifth assumption of Theorem 3.7 follows from the fifth assumption of the present theorem.

**4.2 Banach frames and atomic decompositions of anisotropic inhomogeneous Besov spaces**

We now define shift-invariant systems for the anisotropic inhomogeneous spaces and coefficient spaces associated with them. Here are these definitions.

**Definition 4.7** Let \( \delta > 0, I_d \) be \( d \times d \) identity matrix, \( A_i := A', \psi' \) and \( \psi(t) \in L^1(\mathbb{R}^d) \) and

\[ \psi_i(t) = |\det A_{i-1}|^{1/2} \psi(A_{i-1}^{-1} t) \quad (57) \]

where \( i \in \mathbb{N} \), then the set

\[
\Psi_B := \{L_{\delta I_d k} \psi'(t)\}_{k \in \mathbb{Z}^d} \cup \{L_{\delta A_i^{-1} k} \psi_i(t)\}_{i \in \mathbb{N}_+, k \in \mathbb{Z}^d} = \{|\det A_i|^{1/2} \psi(A_{i-1}^{-1} t - \delta k)\}_{i \in \mathbb{N}_+, k \in \mathbb{R}^d} \quad (58)
\]

will be referred to as **anisotropic inhomogeneous Besov wavelets**.

**Definition 4.8** Let \( p \) and \( q \in (0, \infty] \), then
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\[ C_{B^q} := \left\{ (c_k)_{k\in \mathbb{Z}^d} \in C^{\mathbb{Z}^d} : \| (c_k)_{k\in \mathbb{Z}^d} \|_{\ell^q} < \infty \right\} \]

\[ \cup \left\{ (c_k^{(i)})_{i\in \mathbb{N}_0, k\in \mathbb{Z}^d} \in C^{\mathbb{Z}^d} : \| c \|_{C_{B^q}} := \left\| \left( |\det A|^i \cdot w_i \cdot \| (c_k^{(i)})_{k\in \mathbb{Z}^d} \|_{\ell^q} \right)_{i\in \mathbb{N}_0} \right\|_{\ell^q} < \infty \right\} \]

where \( w_i \in w_B \) will be referred to as \textit{coefficient space associated with} \( \Psi_B \), respectively.

To prove the two theorems that establish the conditions on \( \Psi_B \) under which it will be a Banach frame or a set of atoms for \( B^p_{p,q} \), we shall use the following lemma.

**Lemma 4.9** Let \( Q_1 \) be an open bounded subset of \( \mathbb{R}^d \) that does not include its origin and the matrix \( A \in \mathbb{R}^{d \times d} \) with eigenvalues \( \{ \lambda_i \in \mathbb{C} : 1 < \lambda^- < |\lambda_i| < \lambda^+ \}_{i=1}^d \) be such that the denumerable set

\[ \{ Q_n := A^{n-1} Q_1 \}_{n \in \mathbb{N}} \tag{59} \]

covers \( \mathbb{R}^d \backslash Q_0 \) where \( Q_0 \) is an open bounded set that includes the origin of the \( \mathbb{R}^d \). Furthermore, let \( a, \tau, \sigma > 0, L, N \) and \( \tilde{\phi} \) be as defined in Lemma 4.4,

\[ K \geq \log \lambda^- \left( \frac{\lambda^+/a}{\lambda^-} \right) \tag{60} \]

and

\[ |\tilde{\phi}(\xi)| \leq C(1 + |\xi|)^{-K} \tag{61} \]

where \( C > 0 \) and \( \xi \in Q_1 \). Then

\[ \sup_{n \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} M_{mn} \leq S_1 < \infty \tag{62} \]

and

\[ \sup_{m \in \mathbb{N}_0} \sum_{n \in \mathbb{N}_0} M_{mn} \leq S_2 < \infty \tag{63} \]

where

\[ M_{mn} := \begin{cases} 2^\sigma \cdot \left[ \frac{1}{|Q_0|} \int_{Q_0} |\tilde{\phi}(\xi)| \, d\xi \right]^r, & (m = n = 0) \\
\alpha^{rn} \cdot \left( 1 + \| A^{-(m-1)} \| \right)^\sigma \cdot \left[ \frac{1}{|Q_0|} \int_{Q_0} |\tilde{\psi}(A^{-m} \xi)| \, d\xi \right]^r, & (m \neq 0, n = 0) \\
\alpha^{-rn} \cdot \left( 1 + \| A^{m-1} \| \right)^\sigma \cdot \left[ \frac{1}{|Q_n|} \int_{Q_n} |\tilde{\phi}(\xi)| \, d\xi \right]^r, & (m = 0, n \neq 0) \\
\alpha^{r(m-n)} \cdot \left( 1 + \| A^{m-n} \| \right)^\sigma \cdot \left[ \frac{1}{|Q_n|} \int_{Q_n} |\tilde{\psi}(A^{-m} \xi)| \, d\xi \right]^r, & (m \neq 0, n \neq 0) 
\end{cases} \tag{64} \]
Therefore, the series in (62) would converge, should

\[ S_1 := \begin{cases} 
2^\sigma \cdot C^r + \frac{(1+b)^r (b\lambda_\max R_0)^L}{1-(\frac{\lambda_\min}{\lambda_\max})^r}, (n = 0) \\
C^r \cdot (1 + b)^\sigma \cdot \left( 1 + \frac{b\lambda_-}{\min\{1, r\}} \right)^{\max\{N, K\}} r \\
\cdot (1 + bR)^L r \cdot \left[ \frac{1}{1 - \frac{\lambda_+^r}{(a\lambda)^r}} + \frac{1}{1 - \frac{\lambda_-^r}{(a\lambda)^r}} + \frac{1}{1 - \frac{a}{\lambda^r}} \right], (n \neq 0) 
\end{cases} \]  

(65)

and

\[ S_2 := \begin{cases} 
2^\sigma \cdot C^r + C^r \cdot \left( \frac{1 + b}{\lambda_+^r} \right)^\sigma \cdot \left( \frac{b\lambda_-}{\lambda_+^r} \right)^K r \\
C^r \cdot (1 + b)^\sigma \cdot (b\lambda_\max (R, R_0))^L r \\
\cdot \left( \frac{1 + b}{\min\{1, r\}} \right)^N r \cdot \left[ \frac{1}{1 - \frac{\lambda_+^r}{(a\lambda)^r}} + \frac{2}{1 - \frac{a}{\lambda^r}} \right], (m \neq 0) 
\end{cases} \]  

(66)

with \( b \) defined by (22) and (23) and where \( R_0, r \) and \( R \) are such that 0 \( \leq \) \( \xi \) \( < \) \( R_0 \) as \( \xi \) \( \in \) \( Q_0 \) and 0 \( \leq \) \( r \) \( \leq \) \( \xi \) \( < \) \( R < \infty \) as \( \xi \) \( \in \) \( Q_1 \).

**Proof** First of all, we deal with the series (62) and consider the case where \( n = 0 \). To do so, we rewrite it as

\[ \sum_{m=0}^\infty M_{mn} = M_{00} + \sum_{m=1}^\infty M_{m0}. \]  

(67)

Using (61) into (64), we estimate \( M_{00} \) from the above, namely

\[ M_{00} \leq 2^\sigma \cdot C^r. \]  

(68)

Thus, \( M_{00} \) is bounded independently of \( K \). Now, we use (22) and (23) to estimate the general term of the series in (67) from the above, namely

\[ M_{m0} \leq a^{r m} \cdot (1 + b\lambda_\max - m)^\sigma \cdot \left[ \frac{C}{|Q_0|} \int_{Q_0} \min\{1, (b\lambda_\min |\xi|)^L\} \, d\xi \right]^r \]

\[ \leq a^{r m} \cdot (1 + b\lambda_\max - m)^\sigma \cdot \left[ \frac{C}{|Q_0|} \int_{Q_0} \min\{1, (b\lambda_\min R_0)^L\} \, d\xi \right]^r. \]  

(69)

Therefore, the series in (62) would converge, should \( M_m \) satisfy the criterion (40), namely
\[
\lim_{m \to \infty} \sup_{n} \frac{a^{\varepsilon(m+1)} \cdot (1 + b)^\varphi \cdot \left( b \lambda_\varepsilon^{1-m} R_0 \right)^{L \tau} \cdot C^\tau}{a^{\varepsilon(m)} \cdot (1 + b)^\varphi \cdot \left( b \lambda_\varepsilon^{1-m} R_0 \right)^{L \tau} \cdot C^\tau} \leq \left( \frac{a}{\lambda_\varepsilon} \right)^{\varepsilon} < 1. \tag{70}
\]

This holds if, as assumed in this lemma, (26) does. Under this condition,

\[
\sum_{m=1}^{\infty} M_{m0} \leq (1 + b)^\varphi \cdot \left( b \lambda_\varepsilon R_0 \right)^{L \tau} \cdot C^\tau \cdot \sum_{m=0}^{\infty} \left[ \left( \frac{a}{\lambda_\varepsilon} \right)^{\varepsilon} \right] = \frac{(1 + b)^\varphi \cdot \left( b \lambda_\varepsilon R_0 \right)^{L \tau} \cdot C^\tau}{1 - \left( \frac{a}{\lambda_\varepsilon} \right)^{\varepsilon}}. \tag{71}
\]

Combining (68) and (71) results in (65) as \( n = 0 \).

Now, we investigate the series (62) as \( n \neq 0 \) and to do so divide it into three parts, namely

\[
\sum_{n=0}^{\infty} M_{mn} = M_{0n} + \sum_{n=1}^{n} M_{mn} + \sum_{n=n+1}^{\infty} M_{mn}, \tag{72}
\]

and deal with them separately. We use (59) to change the set over which the integration in \( M_{0n} \) is done from \( Q_n \) to \( Q_1 \) and obtain

\[
M_{0n} = a^{-\varepsilon n} \left( 1 + \| A^{n-1} \| \right)^\varphi \left[ \frac{1}{|Q_1|} \int_{Q_1} |\hat{\phi}(A^{n-1} \xi)| \, d\xi \right]^\tau
\]

and then combine it with (61) and (22) to obtain

\[
M_{0n} \leq a^{-\varepsilon n} \cdot \left( 1 + \| A^{n-1} \| \right)^\varphi \left[ \frac{C}{|Q_1|} \int_{Q_1} \frac{d\xi}{(1 + \| A^{n-1} \xi \|)^K} \right]^\tau
\]

\[
\leq a^{-\varepsilon n} \cdot \left( 1 + b \lambda_\varepsilon^{n-1} \right)^\varphi \left[ \frac{C}{1 + \left( \frac{\lambda_\varepsilon^{n-1} \xi}{b} \right)^K} \right]^\tau
\]

\[
\leq a^{-\varepsilon n} \cdot \left( 1 + b \lambda_\varepsilon^{n-1} \right)^\varphi \left[ \frac{C}{1 + \left( \frac{\lambda_\varepsilon^{n-1} \xi}{b} \right)^K} \right]^\tau
\]

\[
\leq \left( \frac{1 + b}{\lambda_\varepsilon} \right)^\varphi \cdot C^\tau \left( \frac{b \lambda_\varepsilon}{r} \right)^{K \tau} \cdot \left[ \frac{\lambda_\varepsilon^\varphi}{(a \lambda_\varepsilon^K)^\tau} \right]^n =: M_n \tag{73}
\]

The \( M_n \) function in (73) is bounded for any finite \( K \) and \( n \). Therefore, we only have to find the condition on \( K \) under which \( M_n \) stays bounded as \( n \) grows unlimitedly. We shall achieve this by finding such a \( K \) that

\[
\lim_{n \to \infty} M_n = 0. \tag{74}
\]

This holds if, as assumed in this lemma, (60) does. Under this condition,
Now, we deal with the rest of the series (72). Changing the set over which the integration in $M_{mn}$ is done from $Q_n$ to $Q_1$ results in expression identical to (33). Combining it with (28) gives the estimate $M_{mn}$ from the above identical to that in (34). Using (22), the general term in the second part of (72), in which $n \geq m$, results in its estimate identical to that in (35). Making the substitution $m' := n - m$ and changing the order of summation lead to

$$
M_{0n} \leq \sum_{n=1}^{\infty} M_{0n} \leq \sum_{n=1}^{\infty} M_n = \left( \frac{1 + b \lambda_+}{\lambda_+} \right)^\sigma \cdot C^\tau \cdot \left( \frac{b \lambda_+}{r} \right)^{k\tau} \cdot \frac{1}{1 - \frac{\lambda_+^n}{(a \lambda_0^k)^\tau}}. \quad (75)
$$

where $M_{0n}$ is defined in (39). Therefore, the series in the second term of (72) will converge if, as assumed in this lemma, (27) holds. Its sum was already estimated in (42). Using (23), the general term in the third part of (72), in which $n \leq m$, results in its estimate identical to that in (36). Making the substitution $m' := m - n$ leads to

$$
\sum_{m=1}^{n} M_{mn} \leq \sum_{m'=0}^{n-1} M_{m'} \leq \sum_{m'=0}^{\infty} M_{m'} \quad (76)
$$

where $M_{m'}$ is defined in (44). Therefore, the series in the third term of (72) will converge if, as assumed in this lemma, (26) holds. Its sum was already estimated in (46). Combining (75), (42) and (46) leads to (65) as the upper bound of the whole series in (72) as $n \neq 0$.

Now, we investigate the series (63). If $m = 0$, we divide it into two parts, i.e.

$$
\sum_{n=0}^{\infty} M_{0n} = M_{00} + \sum_{n=1}^{\infty} M_{0n} \quad (78)
$$

where $M_{00}$ and $M_{0n}$ were already estimated in (68) and (73), respectively. Therefore, the series (78) will converge if, as assumed in this lemma, (60) holds. Under this condition, combining (68) and (73) results in (66) as $m = 0$.

If $m \neq 0$, we divide the series (63) into three parts

$$
\sum_{n=0}^{\infty} M_{mn} = M_{m0} + \sum_{n=1}^{m} M_{mn} + \sum_{n=m+1}^{\infty} M_{mn} \quad (79)
$$

The term $M_{m0}$ was already estimated in (71) and will be bounded as long as the condition on $L$ in (26) holds. The second term in (79) was estimated in (36) and can be converted into a finite sum over $m'$ whose general term is identical to that of the series in (77) by using the substitution $m' := m - n$. The finite sum being smaller than the sum of the corresponding series leads us again to the condition on $L$ identical to that in (26). Thus, the upper bound of the second term in (79) is given in (46). Similarly, the third part in (79) can be converted into the series whose general term is identical to that in (39) by using the substitution $m' := n - m$. This leads to the condition on $N$ identical to that in (27). The upper bound of the third part in (79) is given in (42). Combining (69), (46) and (42) results in (66) as $m \neq 0$.

The next theorem establishes the conditions on $\Psi_B$ under which it will be a Banach frame for $B^a_{p,q}$. 

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Theorem 4.10  Let $e, p_0$ and $q_0 \in (0, 1]$. Moreover, let $\phi_1$ and $\phi_2 \in L^1(\mathbb{R}^d)$ satisfy the following conditions:

1. $\hat{\phi}_1$ and $\hat{\phi}_2 \in C^\infty(\mathbb{R}^d)$;
2. $\phi_1$ and $\phi_2$ and all its partial derivatives are of polynomial growth at most;
3. $\hat{\phi}_1(\xi) \neq 0$ as $\xi \in Q_0$ and $\hat{\phi}_2(\xi) \neq 0$ as $\xi \in Q_1$ where $Q_0$ and $Q_1$ are as in Definition 2.12;
4. $\phi \in C^1(\mathbb{R}^d)$ and $\nabla \phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$; and

\[
\left| \partial^\alpha \partial^\beta \phi_1(\xi) \right| \leq C(1 + |\xi|)^{-K_4} \tag{80}
\]

with $C$ standing for a constant and

\[
K_4 > \log_+ \left( \frac{x^{\sigma/\tau}}{|\det A|^{1-\theta}} \right) \tag{81}
\]

and

\[
\left| \partial^\alpha \partial^\beta \phi_2(\xi) \right| \leq C \min \{1, |\xi|^{L_4}\}(1 + |\xi|)^{-N_4} \tag{82}
\]

with $C$ standing for a constant,

\[
L_4 > (s - \theta) \log_+ (|\det A|) \tag{83}
\]

and

\[
N_4 > \log_+ \left( \frac{x^{\sigma/\tau}}{|\det A|^{1-\theta}} \right) \tag{84}
\]

as $\xi \in \mathbb{R}^d$ and $l \in \{1, 2\}$ and for all such $\alpha$ and $\beta \in \mathbb{N}_0^d$ that $|\alpha| \leq N$ and $|\beta| \leq 1$.

Then, there is such a $\delta_0 = b_0(e, p_0, q_0, d, A, \phi) > 0$ that the anisotropic heterogeneous Besov wavelets $\Psi_B^\phi$ with the coefficient space $C_{B^\phi}^{x,\mu}$ constitutes a Banach frame for the anisotropic heterogeneous Besov space $B_{p,q}^x(\mathbb{R}^d)$ as long as $\delta \in (0, \delta_0]$.

**Proof** The first four assumptions of this theorem are nothing, but those of Theorem 3.6 formulated for the generalised shift-invariant system $\Psi_B^\phi$ with the coefficient space $C_{B^\phi}^{x,\mu}$ pertaining to the cover $Q_B$ of $\mathbb{R}^d$ and weight $w_B$ that form the space $B_{p,q}^x(\mathbb{R}^d)$. Furthermore, we recollect that $T_0 = I_d$, $\{T_i\}_{i \in \mathbb{N}} = \{A^{i-1}\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}} = 0$ as $\Psi = \Psi_B^\phi$ and that $w_0 = 1$ and $\{w_i\}_{i \in \mathbb{N}} = \{|\det A|^{(i-1)s}\}_{i \in \mathbb{N}}$ with $s \in \mathbb{L}$ as $C_{w}^{x,q} = C_{B^\phi}^{x,q}$. Therefore, $N_4$ defined by (12) becomes
\[
2^\sigma \cdot \max_{|\beta| \leq 1} \left( \frac{2^d}{|Q_0|} \int_{Q_0} \max_{|\alpha| \leq N} \left| \partial^\alpha \hat{\phi}_1 \right| (\xi) d\xi \right)^\tau, \quad (i = j = 0)
\]

\[
(\det A)^{r(i-j)} \cdot (1 + \|A^{i-1}\|)^\sigma 
\cdot \max_{|\beta| \leq 1} \left( \frac{2^d}{|Q_0|} \int_{Q_0} \max_{|\alpha| \leq N} \left| \partial^\alpha \hat{\phi}_2 (A^{i-1}) \xi \right| d\xi \right)^\tau, \quad (i \neq 0, j = 0)
\]

\[
N^1_{ij} \leq 
(\det A)^{r(1-j)} \cdot (1 + \|A^{j-1}\|)^\sigma 
\cdot \max_{|\beta| \leq 1} \left( \frac{2^d}{|Q_0|} \int_{Q_0} \max_{|\alpha| \leq N} \left| \partial^\alpha \hat{\phi}_1 \right| (\xi) d\xi \right)^\tau, \quad (i = 0, j \neq 0)
\]

\[
(\det A)^{r(i-j)} \cdot (1 + \|A^{j-1}\|)^\sigma 
\cdot \max_{|\beta| \leq 1} \left( \frac{2^d}{|Q_0|} \int_{Q_0} \max_{|\alpha| \leq N} \left| \partial^\alpha \hat{\phi}_2 (A^{j-1}) \xi \right| d\xi \right)^\tau, \quad (i \neq 0, j \neq 0)
\]

(85)

where we also noted that $|Q_0| \leq 2^d$ and $|Q_0| = |A^{i-1} Q_1| = |\det A^{i-1}| \cdot |Q_1| \leq |\det A^{j-1}| \cdot 2^d$ as $j \in \mathbb{N}$. The expressions on the right hand of (85) can be further estimated to conclude that, for any $i$ and $j \in \mathbb{N}_0$,

\[
N^1_{ij} \leq 2^d \cdot |\det A|^{\sigma} \cdot M_{mn}
\]

(86)

with $M_{mn}$ defined by (64) if $m = i$, $n = j$ and $a = |\det A|^\sigma$ and as long as $K_3$ in (87) and $L_3$ and $N_3$ in (89) are not smaller than $K$ in (61) and $L$ and $N$ in (28), respectively. According to Lemma 4.9, the series (62) and (63) converge on the assumption (28) about $\hat{\psi}(\xi)$ with $L$ defined by (26), $N$ defined by (27) and $K$ defined by (60). Therefore, the series (11) converge on the assumption (87) about $\rho_1(\xi)$ with $K_3$ defined by (88) and the assumption (89) about $\rho_2(\xi)$ with $L_3$ defined by (90) and $N_3$ defined by (91). In other words, the fifth assumption of Theorem 3.6 follows from the fifth assumption of the present theorem.

\[\square\]

The next theorem establishes the conditions on $\Psi_B$ under which it will be a set of atoms for $B^\alpha_{p,q}$.

**Theorem 4.11** Let $\epsilon$, $p_0$ and $q_0 \in (0, 1]$. Moreover, let $\phi_1$ and $\phi_2 \in L^1(\mathbb{R}^d)$ satisfy the following conditions:

1. $\hat{\phi}_1$ and $\hat{\phi}_2 \in C^\infty(\mathbb{R}^d)$;
2. $\hat{\phi}_1$ and $\hat{\phi}_2$ and all its partial derivatives are of polynomial growth at most;
3. $\phi_1(\xi) \neq 0$ as $\xi \in Q_0$ and $\phi_2(\xi) \neq 0$ as $\xi \in \overline{Q_1}$ where $Q_0$ and $Q_1$ are as in Definition 2.12;
4. For each $l \in \{1, 2\}$
where $\Lambda := 1 + d/p_0$; and

$$
\left| \partial^\alpha \hat{\phi}_l(\xi) \right| \leq \rho_1(\xi) \cdot (1 + |\xi|)^{-(d+1+\varepsilon)}
$$

where

$$
\rho_1 : \mathbb{R}^d \to (0, \infty), \xi \mapsto C(1 + |\xi|)^{-K_3}
$$

with $C$ standing for a constant and

$$
K_3 > \log_{\Lambda_-} \left( \frac{\lambda_{+}^{\sigma/\tau}}{|\det A|^s} \right)
$$

and

$$
\rho_2 : \mathbb{R}^d \to (0, \infty), \xi \mapsto C \min\{1, |\xi|^{L_3}\}(1 + |\xi|)^{-N_3}
$$

with $C$ standing for a constant,

$$
L_3 > s \log_{\Lambda_-} (|\det A|)
$$

and

$$
N_3 > \log_{\Lambda_-} \left( \frac{\lambda_{+}^{\sigma/\tau}}{|\det A|^s} \right)
$$

as $\xi \in \mathbb{R}^d$ and for all such $\alpha \in \mathbb{N}_0^d$ that $|\alpha| \leq N$ where

$$
N := \left\lceil \frac{d + \varepsilon}{p_0} \right\rceil.
$$

Then, there is such a $\delta_0 = \delta_0(\varepsilon, p_0, q_0, d, A, \phi) > 0$ that the anisotropic heterogeneous Besov wavelets $\Psi_B$ with the coefficient space $C^{p,q}_{B \delta}$ constitutes a set of atoms for the anisotropic heterogeneous Besov space $B^{p,q}_{\delta}(A)$ as long as $\delta \in (0, \delta_0)$.

**Proof** The first four assumptions of this theorem are nothing, but those of Theorem 3.7 formulated for the generalised shift-invariant system $\Psi_B$ with the coefficient space $C^{p,q}_{B \delta}$ pertaining to the cover $Q_B$ of $\mathbb{R}^d$ and weight $w_B$ that form the space $B^{p,q}_{\delta}$. Furthermore, we recollect that $T_0 = I_d$, $\{T_i\}_{i \in \mathbb{N}} = \{A_i^{-1}\}_{i \in \mathbb{N}}$ and $\{b_i\}_{i \in \mathbb{N}} = 0$ as $\Psi = \Psi_B$ and that $w_0 = 1$ and $\{w_i\}_{i \in \mathbb{N}} = \{|\det A|^{(i-1)s}\}_{i \in \mathbb{N}}$ with $s \in \mathbb{Z}$ as $C^{p,q}_w = C^{p,q}_{B \delta}$. Therefore, $N^{ij}_j$ defined by (17) becomes
\[
\begin{align*}
N^2_{ij} & \leq \left\{ 
\begin{array}{ll}
2^\sigma \cdot \left( \frac{2^d}{|Q_0|} \int_{Q_0} \rho_1(\xi) \, d\xi \right)^\tau, & (i = j = 0) \\
(\det A)^{|i-j|} \cdot (1 + \|A^{-1}\|)^\sigma \cdot \left( \frac{2^d}{|Q_1|} \int_{Q_1} \rho_1(\xi) \, d\xi \right)^\tau, & (i \neq 0, j = 0) \\
(\det A)^{|i-j|} \cdot (1 + \|A^{-1}\|)^\sigma \cdot \left( \frac{2^d}{|Q_1|} \int_{Q_1} \rho_2(A^{-1} \xi) \, d\xi \right)^\tau, & (i = 0, j \neq 0) \\
(\det A)^{|i-j|} \cdot (1 + \|A^{-1}\|)^\sigma \cdot \left( \frac{2^d}{|Q_1|} \int_{Q_1} \rho_2(A^{-1} \xi) \, d\xi \right)^\tau, & (i \neq 0, j \neq 0)
\end{array}
\right.
\end{align*}
\]

where we also noted that \(|Q_0| \leq 2^d\) and \(|Q_1| = |A^{-1} Q_1| = |\det A^{-1}| \cdot |Q_1| \leq |\det A|^{j-1} \cdot 2^d\) as \(j \in \mathbb{N}\). The expressions on the right hand of (92) can be further estimated to conclude that, for any \(i\) and \(j\) in \(\mathbb{N}_0\),
\[
N^1_{ij} \leq 2^d \cdot |\det A|^{|i-j|\tau} \cdot M_{mn}
\]
(93)

with \(M_{mn}\) defined by (64) if \(m = j, n = i\) and \(a = |\det A|^{s-\theta}\) and as long as \(K_4\) in (80) and \(L_4\) and \(N_4\) in (82) are not smaller than \(K\) in (61) and \(L\) and \(N\) in (28), respectively. According to Lemma 4.9, the series (62) and (63) converge on the assumption (28) about \(\hat{\psi}(\xi)\) with \(L\) defined by (26), \(N\) defined by (27) and \(K\) defined by (60). Therefore, the series (11) converges on the assumption (80) about \(\phi_1(\xi)\) with \(K_4\) defined by (81) and the assumption (82) about \(\phi_2(\xi)\) with \(L_4\) defined by (83) and \(N_4\) defined by (84). In other words, the fifth assumption of Theorem 3.7 follows from the fifth assumption of the present theorem.

\[\square\]

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