SYMMEtRY Via LIE ALGEBRA COHOMOLOGY

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Abstract. The Killing operator on a Riemannian manifold is a
linear differential operator on vector fields whose kernel provides
the infinitesimal Riemannian symmetries. The Killing operator is
best understood in terms of its prolongation, which entails some
simple tensor identities. These simple identities can be viewed as
arising from the identification of certain Lie algebra cohomologies.
The point is that this case provides a model for more complicated
operators similarly concerned with symmetry.

1. Disclaimer

The results in this article are not widely known but are implicitly
already contained in [2, 3, 4], for example. The object of this short
exposition is to introduce the method, by means of familiar examples,
to a wider audience.

2. Notation

The notation in this article follows the standard index conventions of
differential geometry. Precisely, we shall follow Penrose’s abstract index
notation [8] in which tensors are systematically adorned with indices
to specify their type. For example, vector fields are denoted with an
upper index \( X^a \) whilst 2-forms have 2 lower indices \( \omega_{ab} \). The natural
contraction between them is denoted by repeating an index \( X^a \omega_{ab} \) in
accordance with the Einstein summation convention. Round brackets
are used to denote symmetrisation over the indices they enclose whilst
square brackets are used to denote skewing, e.g.

\[
\psi_{[abc]d} = \frac{1}{6} (\psi^d_{abcd} + \psi^d_{bcad} + \psi^d_{cabd} - \psi^d_{bacd} - \psi^d_{acbd} - \psi^d_{cbad}).
\]

3. The Levi-Civita Connection

Suppose \( g_{ab} \) is a Riemannian metric. The Levi-Civita connection
\( \nabla_a \) associated with \( g_{ab} \) is characterised by the following well-known
properties

- \( \nabla_a \) is torsion-free,
- \( \nabla_ag_{bc} = 0. \)

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Its existence and uniqueness boils down to a tensor identity as follows. Choose $D_a$, any torsion-free connection. Any other must be of the form

$$\nabla_a \phi_b = D_a \phi_b - \Gamma^c_{ab} \phi_c$$

for some tensor $\Gamma^c_{ab} = \Gamma^c_{(ab)}$. Then $\nabla_a g_{bc} = 0$ if and only if

$$0 = D_a g_{bc} - \Gamma^d_{ab} g_{dc} - \Gamma^d_{ac} g_{bd} = D_a g_{bc} - \Gamma_{abc} - \Gamma_{acb},$$

where we are using the metric $g_{ab}$ to ‘lower indices’ in the usual fashion. These are two conditions on $\Gamma_{abc}$, namely

$$\Gamma_{[ab]c} = 0 \quad \text{and} \quad \Gamma_{a(bc)} = \frac{1}{2} D_a g_{bc}$$

that always have a unique solution. To see this, note that the general solution of the second equation has the form

$$\Gamma_{abc} = \frac{1}{2} D_a g_{bc} - K_{abc}, \quad \text{where} \quad K_{abc} = K_{[ab]c}.$$ 

Having done this, the first equation reads

$$K_{[ab]c} = \frac{1}{2} D_a g_{bc},$$

which always has a unique solution owing to the tensor isomorphism

$$\Lambda^1 \otimes \Lambda^2 \xrightarrow{\sim} \Lambda^2 \otimes \Lambda^1$$

$$K_{abc} = K_{a[bc]} \mapsto K_{[ab]c},$$

where $\Lambda^p$ denotes the bundle of $p$-forms. This isomorphism is typical of the tensor identities to be explained in this article by means of Lie algebra cohomology.

4. The Killing operator

A vector field $X^a$ on a Riemannian manifold with metric $g_{ab}$ is said to be a Killing field if and only if $\mathcal{L}_X g_{ab} = 0$, where $\mathcal{L}_X$ is the Lie derivative along $X^a$. The geometric interpretation of Lie derivative means that the flow of $X^a$ is an isometry. Thus, a Killing field is an infinitesimal symmetry in the context of Riemannian geometry.

It is useful to regard the Killing equation $\mathcal{L}_X g_{ab} = 0$ as a linear partial differential equation on the vector field $X^a$ as follows. For any torsion-free connection $\nabla_a$,

$$\mathcal{L}_X \phi_b = X^a \nabla_a \phi_b + \phi_a \nabla_b X^a$$

so, if we use the Levi-Civita connection for $g_{ab}$, then

$$\mathcal{L}_X g_{bc} = X^a \nabla_a g_{bc} + g_{ac} \nabla_b X^a + g_{ba} \nabla_c X^a$$

$$= \nabla_b X_c + \nabla_c X_b.$$
Hence, the Killing fields $X^a$ make up the kernel of the Killing operator:

\[
\begin{align*}
\text{Tangent bundle} & \; \xrightarrow{\cong} \; \Lambda^1 & \xrightarrow{\ominus} \; \bigwedge^2\Lambda^1 \\
X^a & \; \xrightarrow{} \; X_a & \xrightarrow{} \; \nabla_{(a}X_{b)}.
\end{align*}
\]

5. PROLONGATION OF THE KILLING OPERATOR

For any torsion-free connection $\nabla_a$, the equation $\nabla_{(a}X_{b)} = 0$ may be understood as follows. Certainly, we may rewrite it as

\[
(2) \quad \nabla_aX_b = K_{ab}, \quad \text{where } K_{ab} \text{ is skew.}
\]

In this case $\nabla_{[a}K_{bc]} = 0$, a condition which we may rewrite as

\[
\nabla_aK_{bc} = \nabla_cK_{ba} - \nabla_bK_{ca}
\]

and substitute from (2) to conclude, as a differential consequence, that

\[
\nabla_aK_{bc} = \nabla_c\nabla_bX_a - \nabla_b\nabla_cX_a = R_{bc}^{\quad d}aX_d,
\]

where $R_{abc}^d$ is the curvature of $\nabla_a$ characterised by

\[
[\nabla_a\nabla_b - \nabla_b\nabla_a]X^c = R_{abc}^dX^d.
\]

Therefore,

\[
\nabla_{(a}X_{b)} = 0 \iff \begin{cases} 
\nabla_aX_b = K_{ab} \\
\nabla_aK_{bc} = R_{bc}^{\quad d}aX_d
\end{cases}
\]

In other words, Killing fields are in 1–1 correspondence with covariant constant sections of the vector bundle $\mathcal{T} = \Lambda^1 \oplus \Lambda^2$ equipped with the connection

\[
(3) \quad \mathcal{T} \ni \left[ \begin{array}{c} X_b \\ K_{bc} \end{array} \right] \xrightarrow{\nabla_a} \left[ \begin{array}{c} \nabla_aX_b - K_{ab} \\ \nabla_aK_{bc} - R_{bc}^{\quad d}aX_d \end{array} \right] \in \Lambda^1 \otimes \mathcal{T}.
\]

At this point, we may use the standard theory of vector bundles with connection to investigate Killing fields. In particular, it is immediately clear that the Killing fields on a connected manifold form a vector space whose dimension is bounded by the rank of $\mathcal{T}$, namely $n(n+1)/2$.

6. THE KILLING OPERATOR IN FLAT SPACE

Be that as it may, suppose ask only about the Killing operator on flat space. It is easily verified in this case that the connection (3) is flat (and, in fact, the same is true on any constant curvature space). Therefore, we may couple the de Rham sequence with (3) to obtain a locally exact complex

\[
\mathcal{T} \xrightarrow{\nabla} \Lambda^1 \otimes \mathcal{T} \xrightarrow{\nabla} \Lambda^2 \otimes \mathcal{T} \xrightarrow{\nabla} \Lambda^3 \otimes \mathcal{T} \xrightarrow{\nabla} \cdots
\]
and, at this point, the isomorphism \( \mathbf{1} \) re-emerges! Specifically, in the absence of the curvature term \( \mathbf{3} \) may be written as
\[
\begin{bmatrix}
X_b \\
K_{bc}
\end{bmatrix} \nabla_a \begin{bmatrix}
\nabla_a X_b \\
\nabla a K_{bc}
\end{bmatrix} - \partial \begin{bmatrix}
X_b \\
K_{bc}
\end{bmatrix},
\]
where \( \partial \begin{bmatrix}
X_b \\
K_{bc}
\end{bmatrix} = \begin{bmatrix}
K_{ab} \\
0
\end{bmatrix} \).

The homomorphism \( \partial : T \to \Lambda^1 \otimes T \) induces \( \partial : \Lambda^p \otimes T \to \Lambda^{p+1} \otimes T \) by \( \partial(\omega \otimes X) = \omega \wedge \partial X \) and we obtain a complex
\[
0 \to T \overset{\partial}{\to} \Lambda^1 \otimes T \overset{\partial}{\to} \Lambda^2 \otimes T \overset{\partial}{\to} \Lambda^3 \otimes T \overset{\partial}{\to} \cdots
\]
recognising that each of these bundles is an irreducible tensor bundle, which we may write as Young diagrams \( \mathbf{5} \).

Readers may notice that \( H^2(\Lambda^* \otimes T, \partial) \) is the natural location for the Riemann curvature tensor and that \( H^3(\Lambda^* \otimes T, \partial) \) is the natural location for the Bianchi identity. These observations are more fully explained in \( \mathbf{6} \). Here, suffice it to observe that a simple diagram chase on \( \mathbf{4} \) reveals a locally exact complex
\[
\begin{bmatrix}
\nabla \\
\nabla^{(2)}
\end{bmatrix} \nabla \begin{bmatrix}
\nabla a X_b \\
\nabla b \omega_{bc}
\end{bmatrix} \nabla a \begin{bmatrix}
\nabla a X_b \\
\nabla b \omega_{bc}
\end{bmatrix} + \nabla b \nabla d \omega_{ac} = 0.
\]

Theorem Suppose \( U \) is an open subset of \( \mathbb{R}^n \) with \( H^1(U, \mathbb{R}) = 0 \). Then a symmetric tensor \( \omega_{ab} \) on \( U \) is of the form \( \nabla (a X_b) \) for some \( X_a \) on \( U \) if and only if

\[
\nabla a \nabla c \omega_{bd} - \nabla b \nabla c \omega_{ad} - \nabla a \nabla d \omega_{bc} + \nabla b \nabla d \omega_{ac} = 0.
\]
7. Higher Killing operators

So far, we have not seen any Lie algebra cohomology, although it is lurking in the background. The identifications (5) can be obtained by elementary means. As soon as we consider more complicated operators, however, then the corresponding identifications are not so obvious. A Killing tensor of valence $\ell$ is a symmetric tensor field $X_{bc...de}$ with $\ell$ indices annihilated by the higher Killing operator

$$X_{bc...de} \mapsto \nabla(\alpha X_{bc...de}).$$

Killing tensors induce conserved quantities along geodesics and arise naturally in the theory of separation of variables. The higher Killing operators may be prolonged along the lines explained in §5. The details are more complicated and this is where Lie algebra cohomology comes to the fore. Without going into details, the prolonged bundle

$$\mathcal{T} = \Lambda^1 \oplus \Lambda^2 = □ ⊕ □$$

that we saw in §5 should be replaced by

$$\mathcal{T}^\ell = \mathcal{T}_0^\ell \oplus \cdots \oplus \mathcal{T}_\ell^\ell = \begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array} + \cdots + \begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array},$$

realised as

$$\begin{bmatrix}
X_{bc...de} = X_{(bc...de)} \\
K_{pbc...de} = K_{p(bc...de)} \text{ s.t. } K_{(pbc...de)} = 0 \\
K'_{pqbc...de} = K'_{(pq)(bc...de)} \text{ s.t. } K'_{pq(bc...de)} = 0 \\
\vdots \\
K''_{pq...rbc...de} = K''_{(pq...r)(bc...de)} \text{ s.t. } K''_{pq...r(bc...de)} = 0 \\
\end{bmatrix}$$

with $\partial : \mathcal{T}^\ell \to \Lambda^1 \otimes \mathcal{T}^\ell$ defined by

$$\partial = \begin{bmatrix}
X_{bc...de} \\
K_{pbc...de} \\
K'_{pqbc...de} \\
\vdots \\
K''_{pq...rbc...de}
\end{bmatrix} = \begin{bmatrix}
K_{abc...de} \\
K'_{apbc...de} \\
K''_{apqbc...de} \\
\vdots \\
K''''_{ap...qbc...de}
\end{bmatrix},$$

The identifications generalising (5) are as follows.

(7) $H^0(\Lambda^* \otimes \mathcal{T}^\ell, \partial) = \begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}$ $H^1(\Lambda^* \otimes \mathcal{T}^\ell, \partial) = \begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}$ and $H^2(\Lambda^* \otimes \mathcal{T}^\ell, \partial) = \begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}$ $H^p(\Lambda^* \otimes \mathcal{T}^\ell, \partial) = \begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}$ for $p \geq 2$.

(8)
The locally exact complex generalising (6) is

\[ \nabla - \to \cdots \to \nabla^{(\ell+1)} - \to \cdots \to \nabla \]

where the first operator is the higher Killing operator. It is a special case of the Bernstein-Gelfand-Gelfand resolution [3, 4].

8. Tensor identities

Be that as it may, the identifications of \( H^p(\Lambda^\bullet \otimes \mathbb{T}^\ell, \partial) \) claimed in the previous section are not so easy and entail some tricky tensor identities. The natural generalisation of (1), for example, follows by writing out the complex

\[ \cdots \to \mathbb{T}^{\ell} \to \Lambda^1 \otimes \mathbb{T}^{\ell} \to \Lambda^2 \otimes \mathbb{T}^{\ell} \to \Lambda^3 \otimes \mathbb{T}^{\ell} \to \cdots \to \mathbb{T}^{\ell} \to \Lambda^{\ell} \to \Lambda^{\ell+1} \to 0 \]

as in (4) and pinning down the locations of the cohomologies

simply by the number of boxes involved to deduce that

\[ \cdots \to \Lambda^{\ell} \to \mathbb{T}^{\ell} \to \Lambda^{\ell+1} \to 0 \]

is exact. Already the injectivity of the first homomorphism gives useful information regarding the higher Killing operator. Specifically it says that

\[ K_{apq\cdots rsbc\cdots de} = K_{a(pq\cdots rs)(bc\cdots de)} \]

\[ \Rightarrow K_{bpq\cdots rsbc\cdots de} = 0. \]
In the flat case, if $X_{bc\ldots de}$ is a Killing tensor of valence $\ell$, it follows immediately from the Killing equation $\nabla_{(a}X_{bc\ldots de)} = 0$, that

$$K_{apq\ldots rsbd\ldots de} = \nabla_a \nabla_p \nabla_q \cdots \nabla_r \nabla_s X_{bc\ldots de}$$

satisfies exactly these symmetries and hence vanishes. In other words, the Killing tensors of valence $\ell$ on $\mathbb{R}^n$ are polynomial of degree at most $\ell$. More generally, prolongation in the curved case implies that the Killing tensors of valence $\ell$ near any point are determined by their $\ell$-jet at that point.

### 9. Lie algebra cohomology

It remains to explain where (7) and (8) come from and the answer is a special case of Kostant’s generalised Bott-Borel-Weil Theorem [7], which we now explain. The special case we need involves only the cohomology of an Abelian Lie algebra but for Kostant’s results to apply it is important that this Abelian Lie algebra be contained inside a semisimple Lie algebra in a particular way. Specifically, let

$$g = \mathfrak{sl}(n+1, \mathbb{R}) = \{(n+1) \times (n+1) \text{ matrices } X \text{ s.t. } \text{trace}(X) = 0\}$$

and write $g = g_{-1} \oplus g_0 \oplus g_1$, comprising matrices of the form

$$\begin{pmatrix}
  0 & 0 & \cdots & 0 \\
  * & \ddots & & 0 \\
  \vdots & \ddots & \ddots & \ddots \\
  * & & \ddots & 0
\end{pmatrix},$$

$$\begin{pmatrix}
  0 & 0 & \cdots & 0 \\
  \vdots & \ddots & & \ast \\
  \ast & \ddots & \ddots & \ast \\
  0 & & \ddots & 0
\end{pmatrix},$$

respectively. Suppose $\mathcal{V}$ is an irreducible tensor representation of $g$. It restricts to a representation of the Abelian subalgebra $g_{-1}$. Kostant’s theorem computes the Lie algebra cohomology $H^p(g_{-1}, \mathcal{V})$. Explicitly, this means that the cohomology of the complex of $g_0$-modules

$$0 \to \mathcal{V} \xrightarrow{\partial} (g_{-1})^* \otimes \mathcal{V} \xrightarrow{\partial} \Lambda^2(g_{-1})^* \otimes \mathcal{V} \xrightarrow{\partial} \Lambda^3(g_{-1})^* \otimes \mathcal{V} \xrightarrow{\partial} \cdots$$

is computed as a $g_0$-module, where $\partial : \mathcal{V} \to (g_{-1})^* \otimes \mathcal{V}$ is defined by the action of $g_{-1}$ on $\mathcal{V}$. To state the result, we need a notation for the irreducible representations of $\mathfrak{sl}(n+1, \mathbb{R})$ and for this we follow [1] writing, for example,

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 1 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}$$

for the defining representation $\mathbb{R}^{n+1}$ and its dual $(\mathbb{R}^{n+1})^*$, respectively. In particular, Kostant’s theorem yields

$$H^0(g_{-1}, \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 1 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}) = \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}$$
where, again, we are following the [1] to denote $\mathfrak{g}_0$ and its irreducible representations. More generally,

\[
H^1(\mathfrak{g}_{-1}, \begin{array}{cccccccc}
0 & \ell & 0 & 0 & \cdots & 0 & 0
\end{array}) = -2 \ell + 1 \begin{array}{cccccccc}
0 & 0 & \cdots & 0 & 0
\end{array}
\]

\[
H^2(\mathfrak{g}_{-1}, \begin{array}{cccccccc}
0 & \ell & 0 & 0 & \cdots & 0 & 0
\end{array}) = -\ell - 3 \begin{array}{cccccccc}
0 & \ell + 1 & 0 & \cdots & 0 & 0
\end{array}
\]

\[
H^3(\mathfrak{g}_{-1}, \begin{array}{cccccccc}
0 & \ell & 0 & 0 & \cdots & 0 & 0
\end{array}) = -\ell - 4 \begin{array}{cccccccc}
0 & \ell & 1 & \cdots & 0 & 0
\end{array}
\]

\[\vdots \quad \vdots \quad \vdots \]

\[
H^{n-1}(\mathfrak{g}_{-1}, \begin{array}{cccccccc}
0 & \ell & 0 & 0 & \cdots & 0 & 0
\end{array}) = -\ell - n \begin{array}{cccccccc}
0 & \ell & 0 & \cdots & 0 & 1
\end{array}
\]

\[
H^n(\mathfrak{g}_{-1}, \begin{array}{cccccccc}
0 & \ell & 0 & 0 & \cdots & 0 & 0
\end{array}) = -\ell - n - 1 \begin{array}{cccccccc}
0 & \ell & 0 & \cdots & 0 & 0
\end{array}
\]

where the right hand side follows the affine action of the Weyl group as explained in [1]. For our purposes, the crossed node can be dropped, viewing the results as irreducible tensor representations of $\mathfrak{sl}(n, \mathbb{R})$. As tensor identities for $\mathfrak{sl}(n, \mathbb{R})$, they are exactly what we need induce \(\text{(7)}\) and \(\text{(8)}\) on a manifold.

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