Nonstandard homology theory for uniform spaces

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Abstract

We introduce a new homology theory of uniform spaces, provisionally called \( \mu \)-homology theory. Our homology theory is based on hyperfinite chains of microsimplices. This idea is due to McCord. We prove that \( \mu \)-homology theory satisfies the Eilenberg-Steenrod axioms. The characterization of chain-connectedness in terms of \( \mu \)-homology is provided. We also introduce the notion of S-homotopy, which is weaker than uniform homotopy. We prove that \( \mu \)-homology theory satisfies the S-homotopy axiom, and that every uniform space can be S-deformation retracted to a dense subset. It follows that for every uniform space \( X \) and any dense subset \( A \) of \( X \), \( X \) and \( A \) have the same \( \mu \)-homology. We briefly discuss the difference and similarity between \( \mu \)-homology and McCord homology.

Keywords: homology theory, nonstandard analysis, uniform space, chain-connected space, homotopy equivalence

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1. Introduction

McCord [1] developed a homology of topological spaces using nonstandard methods. McCord’s theory is based on hyperfinite chains of microsimplices. Intuitively, microsimplices are abstract simplices with infinitesimal diameters. Garavaglia [2] proved that McCord homology coincides with Čech homology for compact spaces. Živaljević [3] proved that McCord cohomology also coincides with Čech cohomology for locally contractible paracompact spaces. Korppi [4] proved that McCord homology coincides with Čech homology with compact supports for regular Hausdorff spaces.

In this paper, we introduce a new microsimplicial homology theory of uniform spaces, provisionally called \( \mu \)-homology theory. \( \mu \)-homology theory satisfies the Eilenberg-Steenrod axioms. Vanishing of the 0-th reduced \( \mu \)-homology characterizes chain-connectedness. We also introduce the notion of S-homotopy,
which is weaker than uniform homotopy. \(\mu\)-homology theory satisfies the S-

homopty axiom. Hence \(\mu\)-homology is an S-homotopy invariant. Every uni-

form space can be S-deformation retracted to a dense subset. It follows that for

every uniform space \(X\) and any dense subset \(A\) of \(X, X\) and \(A\) have the same \(\mu\-

homology. We briefly discuss the difference and similarity between \(\mu\)-homology

and McCord homology.

2. Preliminaries

The basics of nonstandard analysis are assumed. We fix a universe \(U\), the

standard universe, satisfying sufficiently many axioms of ZFC. All standard

objects we consider belong to \(U\). We also fix an elementary extension \(*U\) of

\(U\), the internal universe, that is \(|U|^+\)-saturated. The map \(x \mapsto *x\) denotes

the elementary embedding from \(U\) into \(*U\). We say “by transfer” to indicate the use

of the elementary equivalence between \(U\) and \(*U\). We say “by saturation” when

using the saturation property of \(*U\).

Let us enumerate some well-known facts of nonstandard topology. Let \(X\) be

a topological space. The monad of \(x \in X\) is \(\mu(x) = \bigcap \{U \mid x \in U \in \tau\}\), where

\(\tau\) is the topology of \(X\). A subset \(U\) of \(X\) is open if and only if \(\mu(x) \subseteq *U\) for all

\(x \in U\). A subset \(F\) of \(X\) is closed if and only if \(\mu(x) \cap *F \neq \emptyset\) implies \(x \in F\) for

all \(x \in X\). A subset \(K\) of \(X\) is compact if and only if for any \(x \in *K\) there is a

\(y \in K\) with \(x \in \mu(y)\). A map \(f : X \to Y\) of topological spaces is continuous at

\(x \in X\) if and only if for any \(y \in \mu(x)\) we have \(*f(y) \in \mu(f(x))\).

Next, let \(X\) be a uniform space. Two points \(x, y\) of \(*X\) are said to be

infinitely close, denoted by \(x \approx y\), if for any entourage \(U\) of \(X\) we have \((x, y) \in

\(*U\). \approx\) is an equivalence relation on \(*X\). The monad of \(x \in X\) is equal to

\(\mu(x) = \{y \in *X \mid x \approx y\}\). Thus, in the case of uniform spaces, one can define

the monad of \(x \in *X\). For each entourage \(U\) of \(X\), the \(U\)-neighbourhood of

\(x \in X\) is \(U[x] = \{y \in X \mid (x, y) \in U\}\). A map \(f : X \to Y\) of uniform spaces

is uniformly continuous if and only if \(x \approx y\) implies \(*f(x) \approx *f(y)\) for all

\(x, y \in *X\).

Let \(\{X_i\}_{i \in I}\) be a family of uniform spaces. Let \(P\) be the product \(\prod_{i \in I} X_i\)

of \(\{X_i\}_{i \in I}\), and let \(Q\) be the coproduct \(\coprod_{i \in I} X_i\) of \(\{X_i\}_{i \in I}\). Let \(\approx_X\) denote

the “infinitely close” relation of a uniform space \(X\). For any \(x, y \in P, x \approx_P y\) if

and only if \(x(i) \approx_X y(i)\) for all \(i \in I\). For any \(x, y \in Q, x \approx_Q y\) if and only if

there is an \(i \in I\) such that \(x, y \in X_i\) and \(x \approx_X y, y\).

3. Definition of \(\mu\)-homology theory

Let \(X\) be a uniform space and \(G\) an internal abelian group. We denote by

\(C_pX\) the internal free abelian group generated by \(*X^{p+1}\), and by \(C_p(X; G)\) the

internal abelian group of all internal homomorphisms from \(C_pX\) to \(G\). Each

member of \(C_p(X; G)\) can be represented in the form \(\sum_{i=0}^n g_i \sigma_i\), where \(\{g_i\}_{i=0}^n\)

is an internal hyperfinite sequence of members of \(G\), and \(\{\sigma_i\}_{i=0}^n\) is an internal

hyperfinite sequence of members of \(*X^{p+1}\). A member \((a_0, \ldots, a_p)\) of \(*X^{p+1}\)
is called a \( p \)-microsimplex if \( a_i \approx a_j \) for all \( i, j \leq p \), or equivalently, \( \mu (a_0) \cap \cdots \cap \mu (a_p) \neq \emptyset \). A member of \( C_p (X; G) \) is called a \( p \)-microchain if it can be represented in the form \( \sum_{i=0}^{n} g_i \sigma_i \), where \( \{ g_i \}_{i=0}^{n} \) is an internal hyperfinite sequence of members of \( G \), and \( \{ \sigma_i \}_{i=0}^{n} \) is an internal hyperfinite sequence of \( p \)-microsimplices. We denote by \( M_p (X; G) \) the subgroup of \( C_p (X; G) \) consisting of all \( p \)-microchains. The boundary map \( \partial_p : M_p (X; G) \to M_{p-1} (X; G) \) is defined by

\[
\partial_p (a_0, \ldots, a_p) = \sum_{i=0}^{p} (-1)^i (a_0, \ldots, \hat{a}_i, \ldots, a_p).
\]

More precisely, we first define an internal map \( \partial_p' : C_p (X; G) \to C_{p-1} (X; G) \) by the same equation. We see that \( \partial_p' (M_p (X; G)) \subseteq M_{p-1} (X; G) \). \( \partial_p \) is defined by the restriction of \( \partial_p' \) to \( M_p (X; G) \). Thus \( M_* (X; G) \) forms a chain complex.

Let \( f : X \to Y \) be a uniformly continuous map. By the nonstandard characterization of uniform continuity, we see that for every \( p \)-microsimplex \( (a_0, \ldots, a_p) \) on \( X \), \( (\ast f (a_0), \ldots, \ast f (a_p)) \) is a \( p \)-microsimplex on \( Y \). The induced homomorphism \( M_* (f; G) : M_* (X; G) \to M_* (Y; G) \) of \( f \) is defined by

\[
M_p (f; G) (a_0, \ldots, a_p) = (\ast f (a_0), \ldots, \ast f (a_p)).
\]

Thus we have the functor \( M_* (\cdot; G) \) from the category of uniform spaces to the category of chain complexes. \( \mu \)-homology theory is the composition of functors \( H_* (\cdot; G) = H_* M_* (\cdot; G) \), where \( H_* \) in the right hand side is the ordinary homology theory of chain complexes.

Let \( X \) be a uniform space and \( A \) be a subset of \( X \). The induced homomorphism \( M_p (i; G) : M_p (A; G) \to M_p (X; G) \) of the inclusion map \( i : A \hookrightarrow X \) is injective. Let us identify \( M_* (A; G) \) with a subchain complex of \( M_* (X; G) \) and define

\[
M_* (X, A; G) = \frac{M_* (X; G)}{M_* (A; G)}.
\]

Every uniformly continuous map \( f : (X, A) \to (Y, B) \) induces a homomorphism \( M_* (f; G) : (X, A; G) \to (Y, B; G) \). Thus \( M_* (\cdot, \cdot; G) \) is a functor from the category of pairs of uniform spaces to the category of chain complexes. Relative \( \mu \)-homology theory is the composition of functors \( H_* (\cdot, \cdot; G) = H_* M_* (\cdot, \cdot; G) \).

4. Eilenberg-Steenrod axioms

In this section, we will verify that \( \mu \)-homology theory satisfies the Eilenberg-Steenrod axioms: uniform homotopy, exactness, weak excision, dimension, and finite additivity.

Recall that two uniformly continuous maps \( f, g : X \to Y \) are said to be \textit{uniformly homotopic} if there is a uniformly continuous map \( h : X \times [0, 1] \to Y \), called a \textit{uniform homotopy} between \( f \) and \( g \), such that \( h (\cdot, 0) = f \) and \( h (\cdot, 1) = g \).
Proposition 1 (Uniform homotopy). If two uniformly continuous maps \( f, g : X \to Y \) are uniformly homotopic, then the induced homomorphisms \( M_\bullet(f; G) \) and \( M_\bullet(g; G) \) are chain homotopic. Hence \( H_\bullet(f; G) = H_\bullet(g; G) \). This also holds for relative \( \mu \)-homology.

Proof. Let \( h \) be a uniform homotopy between \( f \) and \( g \). Fix an infinite hypernatural number \( N \). Define \( h_i = * h (\cdot, i/N) \). For each hypernatural number \( i \leq N \), we define a map \( P_{i,p} : M_p(X; G) \to M_{p+1}(Y; G) \) by letting

\[
P_{i,p}(a_0, \ldots, a_p) = \sum_{j=0}^{p} (-1)^j (h_i(a_0), \ldots, h_i(a_j), h_{i+1}(a_j), \ldots, h_{i+1}(a_p)).
\]

Note that the hyperfinite sequence \( \{ P_{i,p} u \}_{i=0}^{N-1} \) is internal. Hence the hyperfinite sum \( P_p u = \sum_{i=0}^{N-1} P_{i,p} u \) exists. Thus we obtain the prism map \( P_p : M_p(X; G) \to M_{p+1}(Y; G) \). Let us verify that \( P_\bullet \) is a chain homotopy between \( M_\bullet(f; G) \) and \( M_\bullet(g; G) \).

\[
\partial_{p+1} P_{i,p}(a_0, \ldots, a_p) = \\
\sum_{k<j} (-1)^{j+k} \left( h_i(a_0), \ldots, \widehat{h_i(a_j)}, \ldots, h_i(a_j), h_{i+1}(a_j), \ldots, h_{i+1}(a_p) \right) \\
- \sum_{j<k} (-1)^{j+k} \left( h_i(a_0), \ldots, h_i(a_j), h_{i+1}(a_j), \ldots, \widehat{h_{i+1}(a_k)}, \ldots, h_{i+1}(a_p) \right) \\
+ (h_{i+1}(a_0), \ldots, h_{i+1}(a_p)) - (h_i(a_0), \ldots, h_i(a_p)),
\]

and

\[
P_{i,p-1} \partial_p (a_0, \ldots, a_p) = \\
\sum_{j<k} (-1)^{j+k} \left( h_i(a_0), \ldots, h_i(a_j), h_{i+1}(a_j), \ldots, \widehat{h_{i+1}(a_k)}, \ldots, h_{i+1}(a_p) \right) \\
- \sum_{k<j} (-1)^{j+k} \left( h_i(a_0), \ldots, \widehat{h_i(a_k)}, \ldots, h_i(a_k), h_{i+1}(a_k), \ldots, h_{i+1}(a_p) \right). 
\]

Thus we obtain

\[
(\partial_{p+1} P_{i,p} + P_{i,p-1} \partial_p) (a_0, \ldots, a_p) = (h_{i+1}(a_0), \ldots, h_{i+1}(a_p)) - (h_i(a_0), \ldots, h_i(a_p))
\]

and

\[
\partial_{p+1} P_p + P_p \partial_p = M_p(g; G) - M_p(f; G). \tag*{\Box}
\]

Proposition 2 (Exactness). Let \( X \) be a uniform space and \( A \) a subset of \( X \). The sequence

\[
0 \longrightarrow M_\bullet(A; G) \xrightarrow{i_\bullet} M_\bullet(X; G) \xrightarrow{j_\bullet} M_\bullet(X, A; G) \longrightarrow 0
\]

is exact, where \( i_\bullet \) is the inclusion map and \( j_\bullet \) is the projection map. Moreover, the above short exact sequence splits.

Proof. The first part is immediate from the definition. We will construct a right inverse \( s_\bullet \) of \( j_\bullet \). Let \( v \in M_p(X, A; G) \). Choose a representative \( u = \sum_i g_i \sigma_i \in M_p (X; G) \).
$M_p(X;G)$. Since $\sum_{i \in A} g_i \sigma_i \subset M_p(A;G)$, $u' = u - \sum_{i \in A} g_i \sigma_i$ is also a representative of $v$. $u'$ is uniquely determined by $v$ and does not depend on the choice of $u$. Define $s_p(v) = u'$. It is easy to see that $s_p$ is a right inverse of $j_\bullet$. \hfill \square

**Proposition 3** (Weak excision). Let $X$ be a uniform space. Let $A$ and $B$ be subsets of $X$ such that $X = \text{int} A \cup \text{int} B$. If either $A$ or $B$ is compact, then the inclusion map $i : (A, A \cap B) \leftrightarrow (X, B)$ induces the isomorphism $H_\bullet(i; G) : H_\bullet(A, A \cap B; G) \cong H_\bullet(X, B; G)$.

**Proof.** It suffices to show the following two inclusions:

1. $M_p(A;G) \cap M_p(B;G) \subseteq M_p(A \cap B;G)$,
2. $M_p(X;G) \subseteq M_p(A;G) + M_p(B;G)$.

The first inclusion is clear. We will only prove the second inclusion. Suppose $u = \sum_{i \in A} g_i \sigma_i \subset M_p(X;G)$. If each $\sigma_i$ is contained in either $^*A$ or $^*B$, then $u \in M_p(A;G) + M_p(B;G)$.

**Case 1.** $A$ is compact. Suppose that $\sigma_i$ is not contained in $^*B$. Then $\sigma_i$ intersects $^*A$. By the nonstandard characterization of compactness, there is an $x \in A$ such that all vertices of $\sigma_i$ are infinitely close to $x$. $x$ must belong to $\text{int} A$. Otherwise, by the nonstandard characterization of open sets, $\sigma_i \subseteq \mu(x) \subseteq ^*B$, a contradiction. Hence $\sigma_i \subseteq \mu(x) \subseteq ^*A$.

**Case 2.** $B$ is compact. Suppose that $\sigma_i$ is not contained in $^*A$. Then $\sigma_i$ intersects $^*B$. There is an $x \in B$ such that all vertices of $\sigma_i$ are infinitely close to $x$. $x$ must contained in $\text{int} B$. Otherwise, $\sigma_i \subseteq \mu(x) \subseteq ^*A$, a contradiction. Hence $\sigma_i \subseteq \mu(x) \subseteq ^*B$.

\hfill \square

**Proposition 4** (Dimension). If $X$ is the one-point space, then

$$H_p(X;G) = \begin{cases} G, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

**Proof.** Immediate by definition. \hfill \square

**Proposition 5** (Finite additivity). Let $X = \coprod_{i=0}^n X_i$ be a finite coproduct of uniform spaces. Then $M_\bullet(X;G) = \bigoplus_{i=0}^n M_\bullet(X_i;G)$. This also holds for relative $\mu$-homology.

**Proof.** Suppose $u = \sum_{i \in A} g_i \sigma_i \subset M_p(X;G)$. Any two points in different components of $^*X$ are not infinitely close. Each $\sigma_i$ is contained in one and only one of the $^*A_i$. Hence $u \in \bigoplus_{i=0}^n M_\bullet(X_i;G)$.
5. Chain-connectedness

Consider the augmented chain complex $\tilde{M}_\bullet (X; G)$:

$$
\begin{array}{ccccccc}
G & \xrightarrow{\varepsilon} & M_0 (X; G) & \xleftarrow{\partial_1} & M_1 (X; G) & \xleftarrow{\partial_2} & M_2 (X; G) & \cdots \\
\end{array}
$$

where $\varepsilon$ is the augmentation map $\varepsilon \sum g_i \sigma_i = \sum g_i$. Reduced $\mu$-homology theory is defined by $\tilde{H}_\bullet (\cdot ; G) = H_\bullet \tilde{M}_\bullet (\cdot ; G)$. For each $p > 0$, $\tilde{H}_p (X; G)$ is identical to $H_p (X; G)$. $\tilde{H}_0 (X; G)$ is a subgroup of $H_0 (X; G)$.

**Proposition 6.** Let $X$ be a nonempty uniform space. The sequence

$$
\begin{array}{c}
0 \longrightarrow \tilde{H}_0 (X; G) \xrightarrow{i} H_0 (X; G) \xrightarrow{\tilde{\varepsilon}} G \longrightarrow 0 \\
\end{array}
$$

is exact and splits, where $i$ is the inclusion map and $\tilde{\varepsilon}$ is the map induced by the augmentation map $\varepsilon$.

**Proof.** The well-definedness of $\tilde{\varepsilon}$ follows from $\text{im} \partial_1 \subseteq \ker \varepsilon$. Since $i$ is injective, the sequence is exact at $\tilde{H}_0 (X; G)$. The exactness at $H_0 (X; G)$ is clear. Let us fix a point $x$ of $X$. For any $g \in G$, we have $\varepsilon [gx] = \varepsilon (gx) = g$. Hence $\tilde{\varepsilon}$ is surjective. The exactness at $G$ is proved. Define a map $s : G \to \tilde{H}_0 (X; G)$ by letting $s (g) = [gx]$. Clearly $s$ is a right inverse of $\tilde{\varepsilon}$.

The 0-th $\mu$-homology relates to the notion of chain-connectedness of uniform spaces. Recall that a uniform space $X$ is said to be chain-connected if it is $U$-connected for all entourage $U$ of $X$. Here $X$ is said to be $U$-connected if for any $x, y \in X$ there is a finite sequence $\{ x_i \}_{i=0}^n$ of points of $X$, called a $U$-chain, such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in U$ for all $i < n$. A chain-connected space is also called a well-chained space. Every connected uniform space is chain-connected. The converse is not true, e.g., the real line without one point $\mathbb{R} \setminus \{ 0 \}$ is chain-connected but not connected. We can easily get the following characterization.

**Lemma 7.** A uniform space $X$ is chain-connected if and only if for any $x, y \in *X$ there is an internal hyperfinite sequence $\{ x_i \}_{i=0}^n$ of points of $*X$, called an infinitesimal chain, such that $x_0 = x$, $x_n = y$ and $x_i \approx x_{i+1}$ for all $i < n$.

**Proof.** Suppose first that $X$ is chain-connected. By saturation, there is an internal entourage $U$ of $*X$ with $U \subseteq (\approx)$. By transfer, for any $x, y \in *X$ there is an internal $U$-chain connecting $x$ to $y$. This is an infinitesimal chain.

Conversely, suppose that for any $x, y \in *X$ there is an infinitesimal chain connecting $x$ to $y$. Let $x, y$ be any points of $X$. Let $U$ be any entourage of $X$. There is an infinitesimal chain connecting $x$ to $y$. This is an internal $U$-chain connecting $x$ to $y$. By transfer, there is a $U$-chain connecting $x$ to $y$. □

**Theorem 8.** If $X$ is chain-connected then $\tilde{H}_0 (X; G) = 0$. 
Proof. It suffices to show that the sequence

\[ G \xleftarrow{\varepsilon} M_0(X; G) \xrightarrow{\partial_1} M_1(X; G) \]

is exact. The only nontrivial part is \( \ker \varepsilon \subseteq \im \partial_1 \). Let \( \sigma, \tau \) be 0-microsimplices. We shall identify a point \( x \) of \( ^*X \) with the 0-microsimplex \( (x) \). By Lemma 7 there is an infinitesimal chain \( \{ x_i \}_{i=0}^n \) connecting \( \sigma \) to \( \tau \). Since \( \{ (x_i, x_{i+1}) \}_{i=0}^{n-1} \) is an internal hyperfinite sequence of 1-microsimplices, the 1-microchain \( \sum_{i=0}^{n-1} (x_i, x_{i+1}) \) is well-defined, and \( \tau - \sigma = \partial_1 \sum_{i=0}^{n-1} (x_i, x_{i+1}) \in \im \partial_1 \).

Suppose \( u = \sum_{i=0}^n g_i \sigma_i \in \ker \varepsilon \). By induction, we have

\[ u = \sum_{i=0}^n \sum_{j=0}^i g_j (\sigma_i - \sigma_{i+1}) + \sum_{j=0}^n g_j \sigma_n = \sum_{i=0}^n \sum_{j=0}^i g_j (\sigma_i - \sigma_{i+1}). \]

Hence \( u \in \im \partial_1 \).

The converse is also true.

**Theorem 9.** Suppose that \( G \) is nontrivial. If \( \tilde{H}_0(X; G) = 0 \) then \( X \) is chain-connected.

Proof. The support of a \( p \)-microchain \( u \) is \( \{ \sigma \in ^*X^{p+1} \mid u(\sigma) \neq 0 \} \). We denote by \( V_u \) the set of all vertices of members of the support of \( u \). \( V_u \) is an internal hyperfinite set. Let \( u \) be a 1-microchain. We say that a vertex \( x \) is accessible to a vertex \( y \) on \( u \), denoted by \( x \leftrightarrow_u y \), if there is an internal hyperfinite sequence \( \{ x_i \}_{i=0}^n \) in \( V_u \) such that \( x, y \in \{ x_i \}_{i=0}^n \) and for each \( i < n \) either \( (x_i, x_{i+1}) \) or \( (x_{i+1}, x_i) \) belongs to the support of \( u \). Clearly \( \leftrightarrow \) is an internal equivalence relation on \( V_u \) and \( \leftrightarrow \) implies \( \approx \).

Let \( x, y \in ^*X \). We will show that \( x \) and \( y \) can be connected by an infinitesimal chain. Since \( \tilde{H}_0(X; G) = 0 \), there is a \( u = \sum g_i \sigma_i \in M_1(X; G) \) with \( \partial_1 u = x - y \). Let \( V_1, \ldots, V_n \) be all equivalence classes of \( \leftrightarrow \). Suppose for a contradiction that \( x \) and \( y \) belong to different equivalence classes \( V_\xi \) and \( V_\eta \), respectively. Then,

\[ \partial_1 u = \sum g_j \partial_1 \sigma_i \]

\[ = \sum_{j=1}^n \sum_{\sigma_i \subseteq V_j} g_j \partial_1 \sigma_i \]

\[ = \sum_{\sigma_i \subseteq V_\xi} g_j \partial_1 \sigma_i + \sum_{\sigma_i \subseteq V_\eta} g_j \partial_1 \sigma_i + \sum_{j=1}^n \sum_{\sigma_i \subseteq V_j, j \neq \xi, \eta} g_j \partial_1 \sigma_i. \]

There are \( u_\xi, u_\eta, v \in M_0(X; G) \) such that \( V_{u_\xi} \subseteq V_\xi, V_{u_\eta} \subseteq V_\eta, V_\xi \cap (V_\xi \cup V_\eta) = \emptyset \) and \( \partial_1 u = x - u_\xi + u_\eta - y + v \). By comparing the coefficients, we have \( v = 0, \angle \)
Proof. Similar to Proposition 1.

Theorem 11. Let \( H \) be a chain homotopic. Hence \( M_{\text{s-homotopic}} \), then the induced homomorphisms \( H_0(\cdot;G) \) of \( \leftrightarrow \)

Proof. The case \( n = 1 \) is immediate from Proposition 5 and Theorem 8. The general case can be reduced to this case using Proposition 5.

6. S-homotopy and S-retraction

We now introduce a weaker notion of homotopy. Let \( X \) and \( Y \) be uniform spaces. An internal map \( f : \ast X \to \ast Y \) is said to be \( S \)-continuous at \( x \in \ast X \) if \( f(y) \approx f(x) \) for all \( y \approx x \). For example, if a map \( f : X \to Y \) is uniformly continuous on \( X \), the nonstandard extension \( \ast f : \ast X \to \ast Y \) is \( S \)-continuous on \( \ast X \), and vice versa. We say that two \( S \)-continuous maps \( f, g : \ast X \to \ast Y \) are \( S \)-homotopic if there is an \( S \)-continuous map \( h : \ast X \times \ast [0,1] \to \ast Y \), called an \( S \)-homotopy between \( f \) and \( g \), such that \( h(\cdot,0) = f \) and \( h(\cdot,1) = g \). If two uniformly continuous maps \( f, g : X \to Y \) are uniformly homotopic, then the nonstandard extensions \( \ast f, \ast g : \ast X \to \ast Y \) are \( S \)-homotopic. The converse is not true, e.g., the inclusion map \( i : \{ \pm 1 \} \to \mathbb{R} \setminus \{ 0 \} \) and the constant map \( c : \{ \pm 1 \} \to \{ \pm 1 \} \subseteq \mathbb{R} \setminus \{ 0 \} \).

Every \( S \)-continuous map \( f : \ast X \to \ast Y \) induces a homomorphism \( M_\ast(f;G) : M_\ast(X;G) \to M_\ast(Y;G) \) in the usual way. Thus the domain of \( M_\ast(\cdot;G) \) and \( H_\ast(\cdot;G) \) can be extended to the category of uniform spaces with \( S \)-continuous maps. We obtain the following analogue of Proposition 1.

Theorem 11 (S-homotopy). If two \( S \)-continuous maps \( f, g : \ast X \to \ast Y \) are \( S \)-homotopic, then the induced homomorphisms \( M_\ast(f;G) \) and \( M_\ast(g;G) \) are chain homotopic. Hence \( H_\ast(f;G) = H_\ast(g;G) \). This also holds for relative \( \mu \)-homology.

Proof. Similar to Proposition 1.

Let \( X \) be a uniform space and \( A \) a subset of \( X \). Let \( i : \ast A \to \ast X \) be the inclusion map. An \( S \)-continuous map \( r : \ast X \to \ast A \) is called an \( S \)-retraction if \( ri = \text{id}_A \). An \( S \)-retraction \( r : \ast X \to \ast A \) is called an \( S \)-deformation retraction if \( ir \) is \( S \)-homotopic to \( \text{id}_A \).

Theorem 12. Let \( A \) be a dense subset of a uniform space \( X \). Then there is an \( S \)-deformation retraction \( r : \ast X \to \ast A \).

Proof. Let \( i : \ast A \to \ast X \) be the inclusion. By saturation, there is an internal entourage \( U \) of \( \ast X \) with \( U \subseteq (\approx) \). For any \( x \in \ast X \), by transfer, we have \( U[x] \cap \ast A \neq \emptyset \). The family \( \{ U[x] \cap \ast A \mid x \in \ast X \} \) of nonempty sets is internal. By transfer and the axiom of choice, there is an internal map \( r : \ast X \to \ast A \).
such that \( r(x) \in U[x] \cap \ast A \) for all \( x \in \ast X \). We can assume that \( r(a) = a \) for all \( a \in \ast A \), i.e. \( ri = \text{id}_A \). For any \( x \approx y \in \ast X \), since \( (x, r(x)) \in U \) and \( (y, r(y)) \in U \), we have that \( r(x) \approx x \approx y \approx r(y) \). Hence \( r \) is \( \mathbb{S} \)-continuous. It remains to show that \( \text{id}_X \) is \( \mathbb{S} \)-homotopic to \( \text{id}_X \). Fix a hyperreal number \( 0 < \alpha < 1 \) and define an internal map \( h : \ast X \times [0,1] \to \ast X \):

\[
h(x,t) = \begin{cases} 
ir(x), & 0 \leq t < \alpha, \\
x, & \alpha \leq t \leq 1.
\end{cases}
\]

Clearly \( h \) is an \( \mathbb{S} \)-homotopy between \( \text{id}_X \) and \( \text{id}_X \). The proof is completed. \( \square \)

The following generalizes the fact that every uniform space with a chain-connected dense subset is chain-connected.

**Corollary 13.** Let \( A \) be a dense subset of a uniform space \( X \). The \( \mu \)-homology of \( \ast A \) is isomorphic to the \( \mu \)-homology of \( \ast X \).

**Proof.** Let \( i : \ast A \hookrightarrow \ast X \) be the inclusion. By Theorem [12] there exists an \( \mathbb{S} \)-deformation retraction \( r : \ast X \to \ast A \). Then \( H_* (r;G) H_* (i;G) = H_* (\text{id}_A;G) \). By Theorem [14] \( H_* (i;G) H_* (r;G) = H_* (\text{id}_X;G) \). Hence \( H_* (i;G) \) and \( H_* (r;G) \) are isomorphisms. \( \square \)

**Corollary 14.** Let \( \bar{X} \) be a uniform completion of a uniform space \( X \). The \( \mu \)-homology of \( \bar{X} \) is isomorphic to the \( \mu \)-homology of \( X \).

**Corollary 15.** Let \( K \) be a uniform compactification of a uniform space \( X \). The \( \mu \)-homology of \( \bar{K} \) is isomorphic to the \( \mu \)-homology of \( X \).

### 7. Relation to McCord homology theory

Recall that a member \( (a_0, \ldots, a_p) \) of \( \ast X^{p+1} \) is called a \( p \)-microsimplex in the sense of McCord if there is a point \( x \) of \( X \) with \( \{a_0, \ldots, a_p\} \subseteq \mu (x) \). This is equivalent to our definition for compact uniform spaces. Hence \( \mu \)-homology theory coincides with McCord homology theory for compact uniform spaces. This does not hold for noncompact uniform spaces.

We say that a topological space \( X \) is **NS-chain-connected** if for any \( x, y \in X \) there are an internal hyperfinite sequence \( \{x_i\}_{i=0}^n \) of points of \( \ast X \) and a sequence \( \{y_i\}_{i=0}^{n-1} \) of points of \( X \) such that \( x_0 = x, x_n = y \) and \( \{x_i, x_{i+1}\} \subseteq \mu (y_i) \) for all \( i < n \). NS is the acronym of Near Standard. Every path-connected space is NS-chain-connected, and every NS-chain-connected space is connected. The converses are not true, e.g., the closed topologist’s sine curve \( \{ (x, \sin (1/x)) \mid 0 < x \leq 1 \} \cup \{ (0) \times [-1,1] \} \) is NS-chain-connected but not path-connected, and the topologist’s sine curve \( \{ (x, \sin (1/x)) \mid 0 < x \leq 1 \} \cup \{ (0,0) \} \) is connected but not NS-chain-connected. However, every compact connected space is NS-chain-connected.

**Theorem 16.** If a topological space is NS-chain-connected, then the 0-th reduced McCord homology vanishes. If the coefficient group is nontrivial, the converse is also true.
Proof. Similar to Theorem 8 and Theorem 9.

**Example 17.** Let $T$ be the closed topologist’s sine curve. Since $T$ is NS-chain-connected, the 0-th reduced McCord homology of $T$ vanishes. On the other hand, $T$ is not path-connected, so the 0-th reduced singular homology of $T$ does not vanish.

For compact uniform spaces, chain-connectedness and NS-chain-connectedness are equivalent. There are, however, chain-connected but not NS-chain-connected uniform spaces. Consequently, in general, $\mu$-homology theory does not coincide with McCord’s one.

**Example 18.** Let $X$ be the real line without one point. Since $X$ is chain-connected, the 0-th reduced $\mu$-homology of $X$ vanishes. On the other hand, $X$ is neither connected nor NS-chain-connected, so the 0-th reduced McCord homology of $X$ does not vanish.

**8. Questions**

Let $\Delta^p$ be the standard $p$-simplex and $X$ a uniform space. An S-continuous map from $\ast \Delta^p$ to $\ast X$ is called an $S$-singular $p$-simplex on $X$. One can define a homology theory based on hyperfinite chains of $S$-singular simplices. Does this homology theory coincide with $\mu$-homology theory?

One can also define a homotopy theory by using $S$-homotopy equivalence. The higher-dimensional analogue of chain-connectedness can be formulated in terms of this homotopy theory. Does the analogue of Hurewicz theorem hold for this homotopy theory and $\mu$-homology theory?

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CORRIGENDUM TO “NONSTANDARD HOMOLOGY THEORY FOR UNIFORM SPACES”

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The author regrets for the errors presented in the previous paper [1], including one critical (and only partially recoverable) error.

(1) In Abstract and Introduction, the sentence “every uniform space can be S-deformation retracted to a dense subset” is ambiguous. The correct sentence is “every uniform space can be S-deformation retracted to any dense subset”.

(2) In page 23, paragraph 4 is incorrect when the index set I is infinite. In addition, there are some omissions of the star ∗. Here is the correction:

Let \( X = \prod_{i \in I} X_i \) be a product of uniform spaces. For every \( x, y \in X \), \( x \approx y \) if and only if \( x(i) \approx y(i) \) for all \( i \in I \). This equivalence is still true when the index set I is infinite. Next, let \( Y = \coprod_{j \in J} Y_j \) be a finite sum of uniform spaces. For every \( x, y \in Y \), \( x \approx y \) if and only if \( x, y \in Y_j \) and \( x \approx y_j \) for some \( j \in J \). This equivalence is not true when the index set J is infinite.

(3) In page 24, line 5, the sentence “\( M \times (f; G) : (X, A; G) \to (Y, B; G) \)” should be “\( M \times (f; G) : M(X, A; G) \to M(Y, B; G) \)”.

(4) The proof of Proposition 5 is incomplete (and has typos). We give a correct proof: Let \( u = \sum_{j=0}^m g_j \sigma_j \in M_p(X; G) \). Since any two points in different components of \( X \) are not infinitely close, each \( \sigma_j \) is contained in one and only one of the \( X_i \)'s. It follows that

\[
u = \sum_{i=0}^n \sum_{\sigma_j \in X_i} g_j \sigma_j \in M_p(X; G)\]

and such a decomposition is unique. Hence \( M_p(X; G) = \bigoplus_{i=0}^n M_p(X_i; G) \).

(5) The proof of Theorem 8 is a hand-waving and has errors. For accuracy and to correct any mistakes, the second paragraph of this proof should be replaced with the following: Let \( u = \sum_{i=0}^n g_i \sigma_i \in \ker \varepsilon \). Since \( \varepsilon u = 0 \), we have that

\[
u = \sum_{i=0}^n g_i (\sigma_i - \sigma_0) + \sum_{i=0}^n g_i \sigma_0 = \sum_{i=0}^n g_i (\sigma_i - \sigma_0) .\]

According to the proof of Lemma 7, for each 0-microsimplices \( \sigma \) and \( \tau \), we can internally find an infinitesimal chain connecting \( \sigma \) to \( \tau \). Hence for each \( 0 \leq i \leq n \) we can internally find a 1-microchain \( v_i \) that kills \( \sigma_i - \sigma_0 \), i.e., \( \sigma_i - \sigma_0 = \partial_1 v_i \). By the internality, the hyperfinite sum \( \sum_{i=0}^n g_i v_i \) exists and kills \( u \). Hence \( u \in \im \partial_1 \).

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(6) In page 29, line 13, the sentence “the topologist’s sine curve \{(x, \sin((1/n))) | 0 < x \leq 1\} ∪ \{(0, 0)\} is not NS-chain-connected” is certainly true but nontrivial. We denote by \(T\) the topologist’s sine curve. Consider a moving point \(P\) on \(\ast T\) starting from \((1, \sin 1)\). \(P\) can do infinitesimal jumping. To jump to \((0, 0)\) from the \((1/n)\)-part, \(P\) should infinitesimally approach the \(y\)-axis, so \(P\) should cross the monad of points on the \(y\)-axis other than \((0, 0)\). These points are remote (i.e. not nearstandard) in \(\ast \mathbb{T}\). Hence \((1, \sin 1)\) and \((0, 0)\) cannot be connected by a nearstandard infinitesimal chain. To justify this intuition, we use the following lemmas:

**Lemma A.** Let \(X\) be a topological space. Let \(A\) be an internal subset of \(\ast X\). The set \(\ast X = \{ x \in X | \mu(x) \cap A \neq \emptyset \}\) is closed.

**Proof.** Let \(x\) be a closure point of \(\ast A\). For each open neighbourhood \(U\) of \(x\), there exists an \(x_U \in \ast A\) such that \(x_U \in U\). Each \(x_U\) is infinitesimally close to some \(y_U \in A\). Since \(U\) is open, we have that \(y_U \in \mu(x_U) \subseteq \ast U\). Hence \(\ast U \cap A \neq \emptyset\) holds. By the overspill principle, there is an internal (open) neighbourhood \(U\) of \(x\) such that \(U \subseteq \mu(x)\) and \(U \cap A \neq \emptyset\). It follows that \(\mu(x) \cap A \neq \emptyset\) and \(x \in \ast A\). □

**Lemma B.** Let \(X\) be a metric space. Let \(A\) be a subset of \(\ast X\). If \(A\) is \(S\)-bounded (i.e. contained in some standard ball), then \(\ast A\) is bounded.

**Proof.** For some \(x_0 \in X\) and \(r \in \mathbb{R}_+\) we have that \(A \subseteq \ast B_r(x_0)\). Let \(x \in \ast A\). There exists an \(a \in A\) such that \(d(x, a) \approx 0\). Then, \(d(x, x_0) \leq \ast d(x, a) + \ast d(a, x_0) < 1 + r\). Hence \(\ast A \subseteq B_{1+r}(x_0)\) and \(\ast A\) is bounded. □

We here say that a subset \(A\) of a topological space \(X\) is **connected in the weak sense** if there are no disjoint open sets \(U, V\) of \(X\) such that \(A \subseteq U \cup V\), \(A \cap U \neq \emptyset\) and \(A \cap V \neq \emptyset\). It is obvious that every connected subset (in the usual sense) is connected in the weak sense. The converse does not hold (e.g. the topological space \(X = \{a, b, c\}\) equipped with the topology \(\{\emptyset, \{a, c\}, \{b, c\}, \{c\}, X\}\) and the subset \(A = \{a, b\}\)).

**Lemma C.** Let \(X\) be a topological space. Let \(A\) be an internal set of nearstandard points of \(\ast X\). If \(A\) is internally connected in the weak sense, then \(\ast A\) is connected in the weak sense.

**Proof.** Suppose not, and let \(U\) and \(V\) be disjoint open sets of \(X\) such that \(\ast A \subseteq U \cup V\), \(\ast A \cap U \neq \emptyset\) and \(\ast A \cap V \neq \emptyset\). By transfer, \(\ast U\) and \(\ast V\) are disjoint internal open sets of \(\ast X\). For each \(y \in A\), there exists an \(x \in \ast A(\subseteq U \cup V)\) such that \(y \in \mu(x) \subseteq \ast U \cup \ast V\). Hence \(A \subseteq \ast U \cup \ast V\). Let \(a \in U \cap \ast A\) and \(b \in V \cap \ast A\). There exist \(a', b' \in A\) such that \(a' \in \mu(a)\) and \(b' \in \mu(b)\). Since \(U\) and \(V\) are open, \(a' \in \ast U\) and \(b' \in \ast V\) hold. Hence \(\ast U \cap A\) and \(\ast V \cap A\) are both nonempty. This is a contradiction and the lemma is proved. □

We consider \(\mathbb{T}\) as a subset of the plane \(\mathbb{R}^2\). Now, let \(\{x_i\}_{i=0}^n\) be an infinitesimal chain in \(\ast \mathbb{T}\) that connects between \((1, \sin 1)\) and \((0, 0)\). Let \(A\) be the internal polygonal path \(x_0x_1 \cdots x_n\) in \(\ast \mathbb{R}^2\). Then, \(\ast A \subseteq \ast \mathbb{T} = \text{cl}(\mathbb{T})\) holds. By the above lemmas, \(\ast A\) is closed, bounded and connected in the weak sense, so compact. However, no compact subset of \(\mathbb{T}\) containing \((1, \sin 1)\) and \((0, 0)\) is connected in the weak sense. Hence \(\ast A\) must have
some points in \( \text{cl}(T) \setminus T \) and \( \{ x_i \}_{i=0}^{n} \) must have some remote points of *T. Therefore, T is not NS-chain-connected. See also [2] Proposition 3 and Proposition 5.

(7) In page 29, line 14, the sentence “every compact connected space is NS-chain-connected” is also true but nontrivial. We first prove the uniformisable case:

**Proposition D.** Every compact, connected, uniformisable space is NS-chain-connected.

**Proof.** Let \( X \) be such a space. We equip \( X \) with a (unique) uniformity and consider \( X \) as a uniform space. Then, \( X \) is chain-connected. Let \( x, y \in X \). By [1] Lemma 7], there exists an infinitesimal chain \( \{ x_i \}_{i=0}^{n} \) that connects \( x \) and \( y \). Since \( X \) is compact, for each \( 0 < i < n \), we can find an \( a_i \in X \) such that \( x_i \in \mu(a_i) \). It follows that \( x_{i+1} \in \mu(a_i) \).

Note that this proof works in arbitrary enlargements *U of U. In other words, we only need a weak form of the saturation principle. Next, we prove the general case:

**Proposition E.** Every compact connected space is NS-chain-connected.

**Proof.** Since \( X \) is connected, the 0-th reduced Čech homology of \( X \) vanishes. By [3] Theorem 2], the reduced 0-th McCord homology of \( X \) also vanishes. By the last half of [1] Theorem 16], \( X \) is NS-chain-connected.

This proof works only in \(|U|^+\)-saturated elementary extensions *U of U, because Garavaglia’s result depends on the full saturation principle (see [3] Theorem 4)).

(8) The first half of Theorem 16 cannot be proved in the way of [1], and we do not know whether it is true or false. The same way of Theorem 8 does not work to prove the first half of Theorem 16, because we may not be able to internally find for each \( 0 \leq i \leq n \) a 1-microchain that kills \( \sigma_i - \sigma_0 \). However, under the compactness, we can prove the first half of Theorem 16 as follows.

**Lemma F.** Every NS-chain-connected space is connected (page 29, line 11).

**Proof.** Let \( X \) be an NS-chain-connected space. Suppose that \( X \) has a nontrivial clopen subset \( A \). Fix an \( a \in A \) and a \( b \in X \setminus A \). Since \( X \) is NS-chain-connected, there exists an internal hyperfinite sequence \( \{ x_i \}_{i=0}^{n} \) in *X such that \( x_0 = a, x_n = b \) and for each \( 0 \leq i < n \) we can find a \( y_i \in X \) with \( \{ x_i, x_{i+1} \} \subseteq \mu(y_i) \). Take \( i_0 \) such that \( x_{i_0} \in \mu(A) \) but \( x_{i_0+1} \notin \mu(A) \). Then, \( x_{i_0} \in \mu(y_i) \cap \mu(A) \neq \emptyset \) and \( x_{i_0+1} \notin \mu(y_{i_0}) \cap \mu(X \setminus A) \neq \emptyset \). Since \( A \) and \( X \setminus A \) are both closed, we see that \( y_{i_0} \in A \) and \( y_{i_0} \notin X \setminus A \), a contradiction.

**Proposition G.** If a topological space \( X \) is compact and NS-chain-connected, the reduced 0-th McCord homology of \( X \) vanishes.

**Proof.** By the previous lemma, \( X \) is connected. Hence the reduced 0-th Čech homology of \( X \) vanishes. By [3] Theorem 2], the reduced 0-th McCord homology of \( X \) also vanishes.
The author would like to apologise for the errors and for any confusion caused by them.

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