Infinitesimal objects associated to Dirac groupoids
and their homogeneous spaces

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Abstract

Let \((G \equiv P, D_G)\) be a Dirac groupoid. We show that there are natural Lie algebroid structures on the units \(\mathfrak{A}(D_G)\) and on the core \(I(D_G)\) of the multiplicative Dirac structure. In the Poisson case, the Lie algebroid \(\mathfrak{A}^*G\) is isomorphic to \(\mathfrak{A}(D_G)\) and in the case of a closed 2-form, the IM-2-form \(\mathfrak{A}\) is equivalent to the core algebroid that we find. We construct a vector bundle \(\mathfrak{B}(D_G) \to P\) associated to any (almost) Dirac structure. In the Dirac case, \(\mathfrak{B}(D_G)\) has the structure of a Courant algebroid that generalizes the Courant algebroid defined by the Lie bialgebroid of a Poisson groupoid \([17]\). This Courant algebroid structure is induced in a natural way by the ambient Courant algebroid \(T G \oplus T^* G\).

The theorems in \([7]\), \([18]\) and \([10]\) about one-one correspondence between the homogeneous spaces of a Poisson Lie group (respectively Poisson groupoid, Dirac Lie group) and suitable Lagrangian subspaces of the Lie bialgebra or Lie bialgebroid are generalized to a classification of the Dirac homogeneous spaces of a Dirac groupoid. \(D_G\)-homogeneous Dirac structures on \(G/H\) are related to suitable Dirac structures in \(\mathfrak{B}(D_G)\). In the case of almost Dirac structures, we find Lagrangian subspaces of \(\mathfrak{B}(D_G)\), that are invariant under an induced action of the bisections of \(H\) on \(\mathfrak{B}(D_G)\).

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1 Introduction

A Poisson groupoid is a Lie groupoid endowed with a Poisson structure that is compatible with the partial multiplication. Poisson Lie groups were introduced in [30], as a common generalization of the symplectic groupoids and the Poisson Lie groups [8], and studied in [31], [23] among others.

A Poisson structure on a homogeneous space of a Poisson groupoid is homogeneous if the action of the Lie groupoid on the homogeneous space is compatible with the Poisson structures on the Lie groupoid and on the homogeneous space. Poisson homogeneous spaces of a Poisson groupoid are in correspondence with suitable Dirac structures in the direct sum of the Lie algebroid with its dual [15]. Hence, the homogeneous spaces are encoded in terms of the infinitesimal data of the Poisson groupoid. We show that this correspondence result fits into a more general and natural context: the one of Dirac groupoids, which are objects generalizing Poisson groupoids and multiplicative closed 2-forms on groupoids [3]. This result gives some insight on the problem of finding the infinitesimal data of Dirac groupoids, something that is not fully understood yet.

Let \( G \rightrightarrows P \) be a Lie groupoid endowed with a Poisson bivector field \( \pi_G \in \Gamma \left( \wedge^2 T^*G \right) \). The bivector field \( \pi_G \) is multiplicative if the vector bundle map \( \pi^*_G : T^*G \rightarrow TP \) is a Lie groupoid morphism over some map \( A^*G \rightarrow TP \) [23], where \( A^*G \) is the dual of the Lie algebroid \( AG \) of \( G \rightrightarrows P \) and \( T^*G \oplus A^*G \) are endowed with the tangent and cotangent Lie groupoid structures (see [1], [28], [22]). Equivalently, the graph of \( \pi^*_G \), Graph(\( \pi^*_G \)) \( \subseteq TG \oplus T^*G \), is a subgroupoid of the Pontryagin groupoid \( (TG \oplus T^*G) \rightrightarrows (TP \oplus A^*G) \). The pair \( (G \rightrightarrows P, \pi_G) \) is then a Poisson groupoid.

A Poisson groupoid \( (G \rightrightarrows P, \pi_G) \) induces a Lie algebroid structure on the dual \( A^*G \) of \( AG \) and a Courant algebroid structure on the direct sum \( AG \oplus A^*G \). This was shown by [23] and [17], the pair \( (AG, A^*G) \) is the Lie bialgebroid associated to \( (G \rightrightarrows P, \pi_G) \). If \( G \rightrightarrows P \) is a target-simply connected Lie groupoid, and if \( (AG, A^*G) \) is a Lie bialgebroid, then there exists a unique multiplicative Poisson structure \( \pi_G \) on \( G \) such that \( (G \rightrightarrows P, \pi_G) \) is a Poisson groupoid with Lie bialgebroid \( (AG, A^*G) \) [21]. This generalizes a theorem in [6] about one-one correspondence between Lie bialgebra structures on \((g, g^*)\) over the Lie algebra \( g \) of a connected and simply connected Lie group \( G \), and multiplicative Poisson structures on \( G \) (see for instance [19]).

In the same spirit, a closed 2-form \( \omega_G \) on a Lie groupoid \( G \rightrightarrows P \) is multiplicative if \( m^* \omega_G = pr_1^* \omega_G + pr_2^* \omega_G \), where \( m : G \times_P G \rightarrow G \) is the multiplication map of the groupoid and \( pr_1, pr_2 : G \times_P G \rightarrow G \) are the projections. Equivalently, the map \( \omega_G^* : TG \rightarrow T^*G \) associated to \( \omega_G \) is a Lie groupoid morphism over a map \( \lambda : TP \rightarrow A^*G \). It has been shown in [2] and [1] that multiplicative closed 2-forms on a Lie groupoid \( G \rightrightarrows P \) are in one-one correspondence with IM-2-forms; special maps \( \sigma : AG \rightarrow T^*P \) satisfying some algebraic and differential conditions. The correspondence is given by \( \sigma = -\lambda^t \).

Dirac structures generalize simultaneously Poisson brackets and closed 2-forms in the sense that the graphs of the vector bundle homomorphisms \( \pi^z : T^*M \rightarrow TM \) and \( \omega^z : TM \rightarrow T^*M \) associated to a Poisson bivector \( \pi \) on \( M \) and a closed 2-form \( \omega \in \Omega^2(M) \) define Dirac structures on the manifold \( M \). Hence, it is natural to ask how to recover the two results above on classification of multiplicative Poisson bivectors and closed 2-forms on a Lie groupoid in terms of data on its algebroid, which are by nature very different, as special cases of a more general result about the infinitesimal data of Dirac groupoids. These objects have been defined in [27]: a Dirac groupoid is a groupoid endowed with a Dirac structure that is a subgroupoid of the Pontryagin groupoid \( (TG \oplus T^*G) \rightrightarrows (TP \oplus A^*G) \).

It has been shown in [27] that multiplicative Dirac structures on a Lie groupoid \( G \rightrightarrows P \) are in one-one correspondence with Dirac structures on the Lie algebroid \( AG \rightarrow P \), which are at the same time subalgebroids of the natural Lie algebroid \( T(AG) \oplus T^*(AG) \rightarrow TP \oplus A^*G \) defined by the Lie algebroid structure on \( AG \).

Yet, this result does not generalize the results given above in the Poisson and closed 2-form cases, but relates, modulo canonical identifications, the multiplicative Poisson bivectors and closed 2-forms to the associated Lie algebroid maps \( A(\pi^*_G) : A(TG) \rightarrow A(T^*G) \) and \( A(\omega^*_G) : A(T^*G) \rightarrow A(TG) \). In [21] and [1],
the construction of these maps from the infinitesimal data is an intermediate step in the reconstruction of \( \pi_G \) and \( \omega_G \) from the Lie bialgebroid \((A, A^*)\) and, respectively, the IM-2-form \( \sigma : AG \to T^*P \).

We show in this paper that, given a Dirac groupoid \((G \rightrightarrows P, D_G)\), there is an induced Lie algebroid structure on the units \(A(D_G) = D_G \cap (TP \oplus A^*)\) of the multiplicative Dirac structure (Theorem 3.24). This was predicted by [27] and, since \(A(D_G)\) is the graph of the anchor map of \(A^*G\) in the Poisson case, generalizes the fact that a multiplicative Poisson structure on \(G \rightrightarrows P\) defines a Lie algebroid structure on \(A^*G\). Since \(D_G\) has then the structure of an \(LA\)-groupoid, we recover the natural Lie algebroid structure on the core \(I^1(D_G) = D_G \cap (AG \oplus T^*P)\) of the multiplicative Dirac structure (Proposition 3.25), see [21].

In the case of a closed 2-form, this Lie algebroid is just the graph of the IM-2-form, hence completely equivalent to it. We show then that the Courant algebroid structure on \(TG \oplus T^*G\) defines naturally a Courant algebroid \(\mathfrak{B}(D_G)\) over \(P\), on a vector bundle that is isomorphic to \(A(D_G) \oplus \mathfrak{B}(D_G)^*\) (Theorem 3.35). In the Poisson case, we recover exactly the Courant algebroid \(AG \oplus A^*G \to P\) defined by the Lie bialgebroid \((AG, A^*G)\). This new approach shows hence how to see the Courant algebroid \(AG \oplus A^*G\) defined by the Lie bialgebroid \((AG, A^*G)\) of a Poisson groupoid as a suitable restriction of the ambient Courant algebroid structure on \(TG \oplus T^*G\).

We show also that the integrability properties of the Dirac structure are completely encoded in the pair of algebroids \((\mathfrak{A}(D_G), I^1(D_G))\) (Theorem 3.32).

In the second part of this paper, we focus on Dirac homogenous spaces of Dirac groupoids. A Poisson homogeneous space \((X, \pi_X)\) of a Poisson groupoid \((G \rightrightarrows P, \pi_G)\) is a homogeneous space \(X\) of \(G \rightrightarrows P\) endowed with a Poisson structure \(\pi_X\) that is compatible with the action of \(G \rightrightarrows P\) on \(J : X \to P\) (see [18] for more details).

It has been shown in [7] that the Poisson homogeneous spaces of a Poisson Lie group are classified by a special class of Lagrangian subalgebras of the double Lie algebra \(\mathfrak{g} \times \mathfrak{g}^*\) defined by the Lie bialgebra \((\mathfrak{g}, \mathfrak{g}^*)\) of the Poisson Lie group. This result has been generalized to Dirac homogeneous spaces of Dirac Lie groups in [10], and to Poisson homogeneous spaces of Poisson groupoids in [17]. Poisson homogeneous spaces of a Poisson groupoid are in one-one correspondence with a special class of Dirac structures in the Courant algebroid \(AG \oplus A^*G\).

Here, we prove a classification theorem for Dirac homogeneous spaces of Dirac groupoids, that generalizes the theorems in [7, 17] and [10]. Dirac homogeneous spaces of a Dirac groupoid are related to Dirac structures in \(\mathfrak{B}(D_G)\) (Theorem 4.17). In the case of almost Dirac structures, we classify the homogeneous spaces in terms of an action of the bisections of \(G \rightrightarrows P\) on the vector bundle \(\mathfrak{B}(D_G)\). This action is found in Theorem 5.11, and is already interesting independently since it generalizes the adjoint actions of a Poisson Lie group on its Lie bialgebra (17).

The geometry is more involved in the Lie groupoid setting than in the Lie group case, where the Lie bialgebra of the Dirac Lie group can be defined using the theory that is already known about Poisson Lie groups: multiplicative Dirac structures on a Lie group are only a slight generalization of the graphs of multiplicative bivector fields [23, 10]. Here, we need first to construct in the first part of the paper the object \(\mathfrak{B}(D_G)\) that will play the role of the Lie bialgebroid in this more general setting. Since we find the right object for the classification of the homogeneous spaces, our classification theorem suggests that a lot of information about the Dirac groupoid is contained in the data \((AG, \mathfrak{A}(D_G), I^1(D_G), \mathfrak{B}(D_G))\).

The natural question that arises is then in what sense these infinitesimal objects determine infinitesimal invariants of a multiplicative Dirac structure, and how to reconstruct the Dirac structure and its counterpart on \(AG\) as in [27] from these Lie algebroids. The case of multiplicative foliations on Lie groupoids, i.e., multiplicative Dirac structures \(D_G = F_G \oplus F_G^*\), with \(F_G \subseteq TG\) an involutive, multiplicative subbundle of \(TG\), has been solved in [12]. We show there that the two Lie algebroids and a partial connection related to the Courant algebroid encode completely the multiplicative foliation.

The general case is the subject of a work in progress with C. Ortiz and T. Drummond [8]. In the special case of Poisson groupoids, we will recover the result of [24]. In the case of multiplicative closed 2-forms, we will recover [3], see also [1], and in the setting of a multiplicative foliation, we will find [12] as a corollary. Hence, this paper is the first part of a series of articles showing how to understand all these
descriptions of the infinitesimal data of multiplicative structures, which are by nature very different, in a common framework.

Outline of the paper  Backgrounds about Lie groupoids and their Lie algebroids are recalled in §2.1 and generalities about Dirac manifolds are recalled in §2.2.

The definition of a Dirac groupoid is given in §3.1 together with examples. In §3.2 we give (under certain hypotheses) a generalization to this setting of a theorem in [30] about the induced Poisson structure on the units of a Poisson groupoid. We also explain shortly how the situation is more complicated in the general case than in the group case.

In §3.3 we study the set $\mathfrak{A}(D_G)$ of units of the Dirac structure, seen as a subgroupoid of $(TG \oplus T^*G) \to (TP \oplus A^*G)$. We show that there is a Lie algebroid structure on this vector bundle over $P$. The Lie algebroid structure on $I^1(D_G)$ is then a consequence and we show in §3.4 that the integrability properties of the Dirac structure are completely encoded in these two algebroids. Then, in §3.5 we define a vector bundle over $P$ that is associated to the Dirac structure $D_G$. We prove the existence of a natural Courant algebroid structure on this vector bundle. We compute this Courant bracket in three standard examples. In the case of a Poisson groupoid, we recover the Courant algebroid structure on $AG \oplus A^*G$, and in the case of a Lie groupoid endowed with a closed multiplicative 2-form, we find simply the standard Courant bracket on $TP \oplus T^*P$. In §3.6 we prove that there is an induced action of the bisections of $G\rightrightarrows P$ on the vector bundle defined in §3.5. In Section 3 each one of the main results is illustrated by the three special examples of Poisson groupoids, multiplicative closed 2-forms and pair Dirac groupoids.

Dirac homogeneous spaces of Dirac groupoids are defined in Section 4. For this, we use the fact that if $X \to P$ is a homogeneous space of a Lie groupoid $G\rightrightarrows P$, the action $\phi : G \times P X \to X$ of $G \rightrightarrows P$ on $J : X \to P$ induces an action of $(TG \oplus T^*G) \rightrightarrows (TP \oplus A^*G)$ on some momentum map $(TX \oplus T^*X) \to (TP \oplus A^*G)$ (this is proved in [9]). The pair $(X, DX)$ is then defined to be $(G \rightrightarrows P, D_G)$-homogeneous if this action restricts to an action of $D_G \rightrightarrows \mathfrak{A}(D_G)$ on $DX$. Our main theorem (Theorem 4.17) about the correspondence between (almost) Dirac homogeneous spaces of an (almost) Dirac groupoid and Lagrangian subspaces (subalgebroids) of the Courant algebroid (vector bundle) $\mathfrak{B}(D_G)$ is then proved.

Notations and conventions  Let $M$ be a smooth manifold. We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the spaces of (local) smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \to M$, the space of (local) sections of $E$ will be written $\Gamma(E)$. We will write $\text{Dom}(\sigma)$ for the open subset of the smooth manifold $M$ where the local section $\sigma \in \Gamma(E)$ is defined.

The Pontryagin bundle of $M$ is the direct sum $TM \oplus T^*M \to M$. The zero section in $TM$ will be considered as a trivial vector bundle over $M$ and written $0_M$, and the zero section in $T^*M$ will be written $0^*M$. The pullback or restriction of a vector bundle $E \to M$ to an embedded submanifold $N$ of $M$ will be written $E|_N$. In the special case of the tangent and cotangent spaces of $M$, we will write $T_NM$ and $T^*_NM$. The annihilator in $T^*_M$ of a smooth subbundle $F \subseteq TM$ will be written $F^c \subseteq T^*_M$.

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2 Review of necessary backgrounds

2.1 Generalities on Lie groupoids and Lie algebroids

The general theory of Lie groupoids and their Lie algebroids can be found in [22], [25]. We fix here first of all some notations and conventions.

A groupoid $G$ with base $P$ will be written $G\rightrightarrows P$. The set $P$ will be considered most of the time as a subset of $G$, that is, the unity $1_p$ will be identified with $p$ for all $p \in P$. A Lie groupoid is a groupoid $G$ on base $P$ together with the structures of smooth Hausdorff manifolds on $G$ and $P$ such that the source and
target maps \( s, t : G \to P \) are surjective submersions, and such that the object inclusion map \( \epsilon : P \to G \), \( p \mapsto 1_p \) and the partial multiplication \( m : G \times P \to G : \{(g, h) \in G \times G \mid s(g) = t(h)\} \to G \) are all smooth.

Since \( t \) and \( s \) are smooth surjective submersions, the kernels \( \ker(Tt) \) and \( \ker(Ts) \) are smooth subbundles of \( TG \). These two vector bundles over \( G \) are written \( T^sG := \ker(Ts) \) and \( T^kG := \ker(Tt) \).

Let \( g \in G \), then the right translation by \( g \) is

\[
R_g : s^{-1}(t(g)) \to s^{-1}(s(g)), \quad h \mapsto R_g(h) = h * g.
\]

The left translation by \( L_g : t^{-1}(s(g)) \to t^{-1}(t(g)) \) is defined in an analogous manner.

Let \( G \rightrightarrows P \) be a Lie groupoid. A right translation on \( G \) is a pair of diffeomorphisms \( \Phi : G \to G \), \( \phi : P \to P \) such that \( s \circ \Phi = \phi \circ s \), \( t \circ \Phi = t \) and, for all \( p \in P \), the map \( \Phi|_{s^{-1}(p)} : s^{-1}(p) \to s^{-1}(\phi(p)) \) is \( R_g \) for some \( g \in G \). A bisection of \( G \rightrightarrows P \) is a smooth map \( K : P \to G \) which is right-inverse to \( t : G \to P \) and is such that \( s \circ K \) diffeomorphism. The set of bisections of \( G \) is denoted by \( \mathcal{B}(G) \). If \( K : P \to G \) is a bisection of \( G \rightrightarrows P \), then the right translations by \( K \) is a right translation:

\[
R_K : G \to G, \quad g \mapsto R_K(s(g))(g) = g * K(s(g)).
\]

We will also use the left translation by \( K \),

\[
L_K : G \to G, \quad g \mapsto L_K((s \circ K)^{-1}(t(g)))(g).
\]

The set \( \mathcal{B}(G) \) of bisections of \( G \) has the structure of a group. For \( K, L \in \mathcal{B}(G) \), the product \( L \ast K \) is given by

\[
L \ast K : P \to G, \quad (L \ast K)(p) = L(p) \ast K((s \circ L)(p)) \quad \forall p \in P.
\]

The composition \( t \circ (L \ast K) \) is equal to \( \text{Id}_P \) and the composition \( s \circ (L \ast K) \) is equal to \( (s \circ K) \circ (s \circ L) \), which is a diffeomorphism of \( P \). The identity element in \( \mathcal{B}(G) \) is the identity section \( \epsilon : P \to G \). The inverse \( K^{-1} : P \to G \) of \( K \in \mathcal{B}(G) \) is given by

\[
K^{-1}(p) = (K((s \circ K)^{-1}(p)))^{-1}
\]

for all \( p \in P \). Finally, we have \( R_{L \ast K} = R_K \circ R_L \) for all \( K, L \in \mathcal{B}(G) \). Since \( R_\epsilon = \text{Id}_G \), we have then also the equality \( R_{K^{-1}} = R_K^{-1} \) for all \( K \in \mathcal{B}(G) \).

We will also consider local bisections of \( G \) in the following, without saying it always explicitly. A local bisection of \( G \rightrightarrows P \) is a map \( K : U \to G \) defined on an open set \( U \subseteq P \) such that \( t \circ K = \text{Id}_U \) and \( s \circ K \) is a diffeomorphism on its image. We will write \( \mathcal{B}_U(G) \) for the set of local bisections of \( G \rightrightarrows P \) with the domain of definition \( U \subseteq P \). The local right translation induced by the local bisection \( K : U \to G \) is the map \( R_K : s^{-1}(U) \to s^{-1}((s \circ K)(U)), g \mapsto g * K(s(g)) \).

The Lie algebroid of a Lie groupoid In this paper, the Lie algebroid of the Lie groupoid \( G \rightrightarrows P \) is \( AG := T^p_0M \), equipped with the anchor map \( T^s|_AG \) and the Lie bracket defined by the left invariant vector fields. We write \( (AG, a, [,]_{AG}) \) for the Lie algebroid of the Lie groupoid \( G \).

Note that the vector field \( X^l \), for \( X \in \Gamma(AG) \), satisfies \( X^l \sim_s a(X) \in \mathfrak{X}(P) \) since we have \( T_g s X^l(g) = T_g s(T_{s(g)} L_g X(s(g))) = T_{s(g)} s X(s(g)) \) for all \( g \in G \).

We recall here the definition of the exponential map for a Lie groupoid, see [22] (note that the Lie algebroid is defined there with the right-invariant vector fields, and the bisections of \( G \rightrightarrows P \) are defined to be inverse to the source map).

Let \( G \rightrightarrows P \) be a Lie groupoid and choose \( X \in \Gamma(AG) \). Let \( \{\phi_t^X : U \to U_t\} \) be a local flow for \( X^l \in \mathfrak{X}(G) \). Since \( T_t x^l(g) = 0 \) for all \( g \in G \), we have \( t \circ \phi_t^X(g) = t(g) \) for all \( t \in \mathbb{R} \) and \( g \in G \) where this makes sense. For each \( t \in \mathbb{R} \) where this is defined and \( p \in P \), the map \( \phi_t^X : t^{-1}(p) \to t^{-1}(p) \) restricts to \( \phi_t^X : t^{-1}(p) \to t^{-1}(p) \). Choose \( h \in G \) such that \( s(h) = p \). We have then \( L_h : t^{-1}(p) \to t^{-1}(t(h)) \) and \( L_h \circ \phi_t^X = \phi_t^X \circ L_h \) since the vector field \( X^l \) satisfies \( X^l(h * g) = T_g L_h X^l(g) \) for all \( g \in t^{-1}(p) \). Recall that \( \tilde{X} := a(X) \in \Gamma(P) \) is defined
on $s(U) := V \subseteq P$ and is such that $X^t \sim_s \tilde{X}$. Let $\tilde{\phi}_t^X$ be the flow of $\tilde{X}$. Then we have $\{\tilde{\phi}_t^X : V \to V_t\}$, where $V_t = s(U_t)$, and $\tilde{\phi}_t^X \circ s = s \circ \phi_t^X$ for all $t$ where this makes sense.

Each $\phi_t^X$ is the restriction to $U$ of a unique local right translation $R_{\text{Exp}(tX)}$ with $\text{Exp}(tX) \in \mathcal{B}_V(G)$. The local bisection $\text{Exp}(tX)$ is defined by $\text{Exp}(tX)(p) = g^{-1} \star \phi_t^X(g)$ for any $g \in U \cap s^{-1}(p)$. We have then $t \circ \text{Exp}(tX) = \text{Id}_V$ and $s \circ \text{Exp}(tX) = \phi_t^X$ is a local diffeomorphism on its image $V_t$. For any $g \in U$, we have

$$\phi_t^X(g) = g \star \text{Exp}(tX)(s(g)) = R_{\text{Exp}(tX)}(g)$$

and the flow of $X^t$ is hence the right translation by $\text{Exp}(\cdot X)$. This is summarized in Proposition 3.6.1 of [22]:

**Proposition 2.1** [22] Let $G \rightrightarrows P$ be a Lie groupoid, choose $X \in \Gamma(AG)$ and set $W = \text{Dom}(X)$. For all $p \in W$ there exists an open neighborhood $U$ of $p$ in $W$, a flow neighborhood for $X$, an $\varepsilon > 0$ and a unique smooth family of local bisections $\text{Exp}(tX) \in \mathcal{B}_U(G)$, $|t| < \varepsilon$, such that:

1. $\frac{d}{dt} \big|_{t=0} \text{Exp}(tX) = X$,
2. $\text{Exp}(0X) = \text{Id}_U$,
3. $\text{Exp}((t + s)X) = \text{Exp}(tX) \star \text{Exp}(sX)$, if $|t|, |s|, |s + t| < \varepsilon$,
4. $\text{Exp}(-tX) = (\text{Exp}(tX))^{-1}$,
5. $\{s \circ \text{Exp}(tX) : U \to U_1\}$ is a local 1-parameter group of transformations for $a(X) \in \mathcal{X}(P)$.

Let $G \rightrightarrows P$ be a Lie groupoid and let $C_p$ be the connectedness component of $p$ in $t^{-1}(p)$. Then the union

$$C(G) := \bigcup_{p \in P} C_p$$

is a wide Lie subgroupoid of $G \rightrightarrows P$ (see [22]), the identity-component subgroupoid of $G \rightrightarrows P$. The set of values $\text{Exp}(tX)(p)$, for all $X \in \Gamma(AG)$, $p \in P$ and $t \in \mathbb{R}$ where this makes sense, is the identity-component subgroupoid $C(G)$ of $G \rightrightarrows P$ (see [21],[22]). Hence, if $G \rightrightarrows P$ is t-connected, that is, if all the t-fibers of $G$ are connected, then $G = C(G)$ is the set of values of $\text{Exp}(tX)(p)$, $X \in \Gamma(AG)$, $p \in P$ and $t \in \mathbb{R}$ where defined.

Note that we can show in the same manner that the flow of a right invariant vector field $Y^r$ is the left translation by a family of bisections $\{L_t\}$ of $G$ satisfying $s \circ L_t = \text{Id}$ on their domains of definition and such that $\tau \circ L_t$ are diffeomorphisms on their images. Hence, the flow of $Y^r$ commutes with the flow of $X^l$ for any left invariant vector field $X^l$ and we get the fact that $[Y^r, X^l] = 0$ for all $Y \in \Gamma(T_P G)$ and $X \in \Gamma(AG)$.

The tangent prolongation of a Lie groupoid Let $G \rightrightarrows P$ be a Lie groupoid. Applying the tangent functor to each of the maps defining $G$ yields a Lie groupoid structure on $TG$ with base $TP$, source $Ts$, target $Tt$ (these maps will be written $s$ and $t$ in the following) and multiplication $Tm : T(G \times_P G) \to TG$. The identity at $v_p \in T_pP$ is $1_{v_p} = T_p(1_{v_p})$. This defines the tangent prolongation of $G \rightrightarrows P$ or the tangent groupoid associated to $G \rightrightarrows P$.

The cotangent Lie groupoid defined by a Lie groupoid If $G \rightrightarrows P$ is a Lie groupoid, then there is an induced Lie groupoid structure on $T^*G \rightrightarrows A^*G = (TP)^\circ$. The source map $\tilde{s} : T^*G \to A^*G$ is given by

$$\tilde{s}(\alpha_g) \in A^*_{s(g)} G$$

for all $\alpha_g \in A^*_{s(g)} G$, and the target map $\tilde{t} : T^*G \to A^*G$ is given by

$$\tilde{t}(\alpha_g) \in A^*_{t(g)} G, \quad \tilde{t}(\alpha_g)(u_{t(g)}) = \alpha_g(T_{t(g)} R_g(u_{t(g)} - T_{t(g)} \tilde{s}u_{t(g)}))$$
for all $u(t(g)) \in A_{t(g)}G$. If $\hat{s}(v_g) = \hat{t}(v_h)$, then the product $\alpha_g \ast \alpha_h$ is defined by

$$(\alpha_g \ast \alpha_h)(v_g \ast v_h) = \alpha_g(v_g) + \alpha_h(v_h)$$

for all composable pairs $(v_g, v_h) \in T_{(g,h)}(G \times P G)$.

This Lie groupoid structure was introduced in [4] and is explained in [4], [28] and [22].

**The Pontryagin groupoid defined by a Lie groupoid**  If $G \rightrightarrows P$ is a Lie groupoid, there is hence an induced Lie groupoid structure on $P_G = TG \oplus T^*G$ over $TP \oplus A^*G$. We will write $T_\ast$ for the target map $P_G \to TP \oplus A^*G$, and in the same manner $T_\ast : P_G \to TP \oplus A^*G$ for the source map.

**Lie groupoid actions** Let $G \rightrightarrows P$ be a Lie groupoid and $A \rightrightarrows P$ a set with a map $J : M \to P$. Consider the set $G \times_P M = \{(g,m) \in G \times M \mid s(g) = J(m)\}$.

A groupoid action of $G \rightrightarrows P$ on $J : M \to P$ is a map $\Phi : G \times_P M \to M$, $\Phi(g,m) = g \cdot m = gm$ such that

- $J(g \cdot m) = t(g)$ for all $(g,m) \in G \times_P M$,
- $g \cdot (h \cdot m) = (g \ast h) \cdot m$ for all $(h,m) \in G \times_P M$, and $g \in G$ such that $s(g) = t(h)$,
- $1_{J(m)} \cdot m = m$ for all $m \in M$.

**Example 2.2** Let $G \rightrightarrows P$ be a groupoid.

1. $G \rightrightarrows P$ acts on $t : G \to P$ via the multiplication.
2. $G \rightrightarrows P$ acts on $\text{Id}_P : P \to P$ via $\Phi : G \times_P P \to P$, $(g,p) \mapsto t(g \ast p) = t(g)$. $\diamond$

**Homogeneous spaces** Let $G \rightrightarrows P$ be a Lie groupoid and $H \rightrightarrows P$ a wide subgroupoid of $G$. Define the equivalence relation

$$g \sim_H g' \iff \exists h \in H \text{ such that } g \ast h = g'$$

on $G$ and $G/H := G/\sim_H = \{gH \mid g \in G\}$, where $gH = \{g \ast h \mid s(g) = t(h) \text{ and } h \in H\}$. Since $t(g \ast h) = t(g)$ for all $g \ast h \in gH$, the map $t$ factors to a map $J : G/H \to P$, $J(gH) = t(g)$ for all $gH \in G/H$. The multiplication $m : G \times_P G \to G$ factors to a groupoid action $\Phi$ of $G \rightrightarrows P$ on $J : G/H \to P$, $\Phi(g,g')H = (g \ast g')H$ for all $(g,g'H) \in G \times_P (G/H) = \{(g,g'H) \mid s(g) = J(g'H) = t(g')\}$.

**Definition 2.3** A $G$-space $X$ over $P$ is homogeneous if there is a section $\sigma$ of the moment map $J : X \to P$ which is saturating for the action in the sense that $G \ast \sigma(P) = X$. The isotropy subgroupoid of the section $\sigma$ consists of those $g \in G$ for which $g \ast \sigma(P) \subseteq \sigma(P)$.

It is shown in [18] that a $G$-space is homogeneous if and only if it is isomorphic to $G/H$ for some wide subgroupoid $H \subseteq G$.

**Example 2.4** Let $G \rightrightarrows P$ be a groupoid. The two extreme examples of homogeneous spaces of $G$ are the following.

1. In the case where the wide subgroupoid is $P$, the equivalence classes are $gP = \{g \ast p \mid p \in P, p = s(g)\} = \{g\}$ and the quotient is just $G/P = G$ with the first action of Example 2.2.
2. If the wide subgroupoid is $G$ itself, then the equivalence classes are $gG = \{g \ast h \mid h \in G, t(h) = s(g)\} = t^{-1}(t(g))$ and the quotient is $G/G = P$, with projection equal to the target map $t : G \to G/G \simeq P$ and with the second action in Example 2.2. $\diamond$
Assume that $H$ is a $t$-connected wide Lie subgroupoid of $G$ and that $G/H$ is a smooth manifold such that the projection $q : G \to G/H$ is a smooth surjective submersion.

Consider the vector bundle $AH = T^*_H H \subseteq T^*_H G \subseteq T^*_P G$ over $P$ and the subbundle $\mathcal{K} \subseteq TG$ defined as the left invariant image of $AH$, i.e., $\mathcal{K}(g) = T^*_H A_0 A(g)$ for all $g \in G$. We show that $\mathcal{K} = \ker T_q$ and $G/H$ is the leaf space of the foliation on $G$ defined by the involutive subbundle $\mathcal{K} \subseteq TG$.

The vector bundle $\mathcal{K}$ is spanned by the left invariant vector fields $X^l$, for $X \in \Gamma(\mathcal{A}H) \subseteq \Gamma(\mathcal{A}G)$. Since $H$ is an immersed submanifold of $G$, $\mathcal{K}$ is a subalgebra of $\mathcal{A}G$, and $\Gamma(\mathcal{K})$ is hence closed under the Lie bracket.

It is easy to check that $\mathcal{K} = \ker(T_q)$. Hence, if $g$ and $g'$ are in the same leaf of $\mathcal{K}$, we have $g'/H = gH$. Conversely, if $gH = g'H$, it is easy to see, using the fact that $H$ is $t$-connected, and hence $H = C(H) = \{\text{Exp}(tX) \mid t \in \mathbb{R}, X \in \Gamma(\mathcal{A}H)\}$ that $g$ and $g'$ are in the same leaf of $\mathcal{K}$.

Consider the set $\mathcal{B}(H)$ of (local) bisections $K : U \subseteq P \to H$ of $H$ such that $t \circ K = \text{Id}_U$ and $s \circ K$ is a diffeomorphism. We have $gH = \{R_K(g) \mid K \in \mathcal{B}(H)\}$ and $G/H$ is the quotient of $G$ by the right action of $\mathcal{B}(H)$ on $G$. A function $f \in C^\infty(G)$ pushes forward to the quotient $G/H$ if and only if it is invariant under $R_K$ for all bisections $K \in \mathcal{B}(H)$.

**Lie bialgebroids, associated Courant algebroids, the special case of the Pontryagin bundle**

Let $M$ be a smooth manifold and $(A \to M, a, [\cdot, \cdot])$ a Lie algebroid on $M$. Assume that the dual $A^* \to M$ of $A$ is endowed with a Lie algebroid structure $(A^* \to M, a_*, [\cdot, \cdot], *)$ such that

$\mathbf{d}^{\,[\cdot, \cdot]}_* = [\mathbf{d}^{\cdot}, \cdot]_* + [\cdot, \mathbf{d}^{\cdot}],_*$ or equivalently $\mathbf{d}_*^{\cdot, \cdot} = [\mathbf{d}_*^{\cdot}, \cdot] + [\cdot, \mathbf{d}_*^{\cdot}]$

hold for the induced maps $\mathbf{d}^{\cdot} : \Gamma(\Lambda^* A^*) \to \Gamma(\Lambda^* A)$ and $\mathbf{d}_*^{\cdot} : \Gamma(\Lambda^* A) \to \Gamma(\Lambda^* A)$ and the brackets $[\cdot, \cdot]$ (respectively $[\cdot, \cdot]_*$) induced on $\Gamma(\Lambda^* A)$ (respectively $\Gamma(\Lambda^* A^*)$), see for instance [15]. We refer to [23] for a quick review of the definitions of these objects since they will not be needed explicitly here. We just recall that if $(A \to M, a, [\cdot, \cdot])$ is a Lie algebroid, then the map $\mathbf{d}^{\cdot} : \Gamma(\Lambda^* A^*) \to \Gamma(\Lambda^* A^* + 1)$ is defined on $C^\infty(M) = \Gamma(\Lambda^0 A^*)$ by $(\mathbf{d}^{\cdot})(f)(x) = a_*(X)(f)$ for all $f \in C^\infty(M)$ and $X \in \Gamma(A)$.

For instance, if $(G \rightrightarrows P, \pi_G)$ is a Poisson groupoid, the dual $A^*_G$ of the Lie algebroid $A$ inherits the structure of a Lie algebroid such that $(AG, A^*_G)$ is a Lie bialgebroid (see [23], [22]).

The direct sum vector bundle $A \oplus A^* \to P$ endowed with the map $\rho = a \oplus a_*$, the symmetric non degenerate bilinear form $\langle \cdot, \cdot \rangle$ given by $\langle(x, \alpha_p), (y, \beta_p)\rangle = \alpha_p(y) + \beta_p(x)$ for all $(x, \alpha_p), (y, \beta_p) \in (A \oplus A^*)(p)$ and the bracket on its sections given by

$\langle([X, \alpha], (Y, \beta)) = \left([X, Y] + \mathbf{L}_\alpha Y - \mathbf{L}_\beta X - \frac{1}{2} \mathbf{d}_*(\alpha(Y) - \beta(X)), [\alpha, \beta]_* + \mathbf{L}_X \beta - \mathbf{L}_Y \alpha + \frac{1}{2} \mathbf{d}^{(\alpha(Y) - \beta(X))}\right)$

for all $(X, \alpha), (Y, \beta) \in \Gamma(A \oplus A^*)$, is then a Courant algebroid in the sense of the definition below, see [17].

A Courant algebroid over a manifold $M$ is a vector bundle $E \to M$ equipped with a fiberwise non degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a skew-symmetric bracket $[\cdot, \cdot]$ on the smooth sections $\Gamma(E)$, and a vector bundle map $\rho : E \to TM$ called the anchor, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

1. $\langle[e_1, e_2], e_3\rangle + [e_2, e_3], e_1\rangle + [e_3, e_1], e_2\rangle = \frac{1}{3} \mathbf{D} \langle\langle[e_1, e_2], e_3\rangle \rangle + \langle[e_2, e_3], e_1\rangle + \langle[e_3, e_1], e_2\rangle$,
2. $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$,
3. $\langle e_1, f   e_2\rangle = f[e_1, e_2] + (\rho(e_1)f)e_2 - (\rho(e_2)f)e_1 \mathbf{D} f$,
4. $\rho \circ \mathbf{D} = 0$, i.e., for any $f, g \in C^\infty(M)$, $\langle \mathbf{D} f, \mathbf{D} g \rangle = 0$,
5. $\rho(e_1)\langle e_2, e_3\rangle = \langle e_1, e_2 \rangle + \mathbf{D}(e_1, e_2), e_3\rangle + \langle e_2, e_1 \rangle + \mathbf{D}(e_1, e_3)$,
where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by
\[
\langle \mathcal{D} f, e \rangle = \frac{1}{2} \rho(e)(f)
\]
for all $f \in C^\infty(M)$ and $e \in \Gamma(E)$, that is, $\mathcal{D} = \frac{1}{2} \beta^{-1} \circ \rho^* \circ \mathbf{d} : C^\infty(M) \to \Gamma(E)$. Here, $\beta : E \to E^*$ is the isomorphism defined by the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$.

**Example 2.5** Consider a smooth manifold $M$, the Lie algebroid $(TM, [\cdot, \cdot], a = \text{Id}_{TM})$ and its dual, the cotangent space $T^*M$ endowed with the trivial bracket $[\cdot, \cdot] = 0$ and the trivial anchor map $a_* = 0$. The map $\mathbf{d}$ induced by $TM$ on the sections of $\bigwedge^* T^*M$ is here simply the usual de Rham derivative. The map $\mathbf{d}_*$ induced by $(T^*M, 0, 0)$ on the sections of $\bigwedge^* T^*M$ is trivial. The equation $\mathbf{d}[\cdot, \cdot]_* = [\mathbf{d}\cdot, \cdot]_* + [\cdot, \mathbf{d}\cdot]_*$ is obviously satisfied and the pair $(TM, T^*M)$ is a Lie bialgebroid.

The direct sum $P_M = TM \oplus T^*M$ endowed with the projection on $TM$ as anchor map, $\rho = \text{pr}_{TM}$, the symmetric bracket $\langle \cdot, \cdot \rangle$ given by
\[
\langle (v_m, \alpha_m), (w_m, \beta_m) \rangle = \alpha_m(w_m) + \beta_m(v_m)
\]
for all $m \in M$, $v_m, w_m \in T_mM$ and $\alpha_m, \beta_m \in T^*_mM$ and the Courant bracket given by
\[
\langle [X, \alpha], (Y, \beta) \rangle = \left( [X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} \mathbf{d}(\alpha(Y) - \beta(X)) \right)
\]
for all $(X, \alpha), (Y, \beta) \in \Gamma(P_M)$, is then a Courant algebroid. The map $\mathcal{D} : C^\infty(M) \to \Gamma(P_M)$ is given by $\mathcal{D} f = \frac{1}{2} (0, df)$.

### 2.2 Generalities on Dirac structures

As we have seen in Example 2.5, the Pontryagin bundle $P_M := TM \oplus T^*M$ of a smooth manifold $M$ is endowed with the non-degenerate symmetric fiberwise bilinear form of signature $(\dim M, \dim M)$ given by (1). An *almost Dirac structure* (see [3]) on $M$ is a Lagrangian vector subbundle $D \subset P_M$. That is, $D$ coincides with its orthogonal relative to $\mathfrak{d}$ and so its fibers are necessarily $\dim M$-dimensional.

Let $(M, D)$ be a Dirac manifold. For each $m \in M$, the Dirac structure $D$ defines two subspaces $G_0(m), G_1(m) \subset T_mM$ by
\[
G_0(m) := \{v_m \in T_mM \mid (v_m, 0) \in D(m)\}
\]
and
\[
G_1(m) := \{v_m \in T_mM \mid \exists \alpha_m \in T^*_mM : (v_m, \alpha_m) \in D(m)\},
\]
and two subspaces $P_0(m), P_1(m) \subset T^*_mM$ defined in an analogous manner. The distributions $G_0 = \cup_{m \in M} G_0(m)$ and $P_0 = \cup_{m \in M} P_0(m)$ are not necessarily smooth. The distributions $G_1 = \cup_{m \in M} G_1(m)$ (respectively $P_1 = \cup_{m \in M} P_1(m)$) are smooth since they are the projections on $TM$ (respectively $T^*M$) of $D$.

The almost Dirac structure $D$ is a *Dirac structure* if
\[
[\Gamma(D), \Gamma(D)] \subset \Gamma(D).
\]
Since $\langle (X, \alpha), (Y, \beta) \rangle = 0$ if $(X, \alpha), (Y, \beta) \in \Gamma(D)$, this integrability property of the Dirac structure is expressed relative to a non-skew-symmetric bracket that differs from (2) by eliminating in the second line the third term of the second component. This truncated expression is called the *Courant-Dorfman bracket* in the literature:
\[
[(X, \alpha), (Y, \beta)] = ([X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha)
\]
for all \((X, \alpha), (Y, \beta) \in \Gamma(D)\). The restriction of the Courant bracket to the sections of a Dirac bundle is skew-symmetric and satisfies the Jacobi identity. It satisfies also the Leibniz rule:

\[
[(X, \alpha), f(Y, \beta)] = f[(X, \alpha), (Y, \beta)] + X(f) \cdot (Y, \beta)
\]
for all \((X, \alpha), (Y, \beta) \in \Gamma(D)\) and \(f \in C^\infty(M)\).

Note that in the following, we work in the general setting of almost Dirac structures. To simplify the notation, we will simply call almost Dirac structures “Dirac structures” and always state it explicitly if the integrability condition \(\text{(3)}\) is assumed to be satisfied. We will say in this case that the Dirac structure is \textit{closed} or \textit{integrable}.\(^1\)

The class of Dirac structures given in the next example will be very important in the following.

**Example 2.6** Let \(M\) be a smooth manifold endowed with a globally defined bivector field \(\pi \in \Gamma \left( \Lambda^2 TM \right)\). Then the subdistribution \(D_\pi \subseteq P_M\) defined by

\[
D_\pi(m) = \left\{ (\pi^2(\alpha_m), \alpha_m) \mid \alpha_m \in T^*_m M \right\} \quad \text{for all } m \in M,
\]

where \(\pi^2 : T^*M \to TM\) is defined by \(\pi^2(\alpha) = \pi(\alpha, \cdot) \in \mathfrak{X}(M)\) for all \(\alpha \in \Omega^1(M)\), is a Dirac structure on \(M\). It is closed if and only if the bivector field satisfies \([\pi, \pi] = 0\), that is, if and only if \((M, \pi)\) is a Poisson manifold.

**Dirac maps and Dirac reduction** Let \((M, D_M)\) and \((N, D_N)\) be two Dirac manifolds and \(F : M \to N\) a smooth map. Then \(F\) is a \textit{forward Dirac map} if for all \(n \in N, m \in F^{-1}(n)\) and \((v_n, \alpha_n) \in D_N(n)\) there exists \((v_m, \alpha_m) \in D_M(m)\) such that \(T_m F v_m = v_n\) and \(\alpha_m = (T_m F)\ast \alpha_n\). The map \(F\) is a \textit{backward Dirac map} if for all \(m \in M, n = F(m)\) and \((v_m, \alpha_m) \in D_M(m)\) there exists \((v_n, \alpha_n) \in D_N(n)\) such that \(T_m F v_m = v_n\) and \(\alpha_m = (T_m F)\ast \alpha_n\).

Assume that \(G \Rightarrow P\) is a Lie groupoid, and that \(G\) is endowed with a Dirac structure. Let \(H\) be a t-connected, wide Lie subgroupoid of \(G\) such that \(G/H\) has a smooth manifold structure and \(q : G \to G/H\) is a smooth surjective submersion. Since \(G/H\) is the leaf space of \(\mathcal{H}\), where \(\mathcal{H}\) is the left invariant image of \(A H\) (see the previous section), we can apply the results in \(^{32}\) (see also \(^{13}\)) for Dirac reduction. Assume that the Dirac structure \(D\) on \(G\) is such that \(D \cap (T G \oplus \mathcal{H})\) has constant rank on \(G\) and

\[
\left[ \Gamma(D), \Gamma(\mathcal{H} \oplus \{0\}) \right] \subseteq \Gamma(D + (\mathcal{H} \oplus \{0\})),
\]

then \(D\) induces a Dirac structure \(q(D)\) on the quotient \(G/H\). The Dirac structure \(q(D)\) on \(G/H\) is given by

\[
\Gamma(q(D)) = \{ (X, \alpha) \in \Gamma(P_{G/H}) \mid \exists X \in \mathfrak{X}(G) \text{ such that } X \sim_q \bar{X} \text{ and } (X, q^* \alpha) \in \Gamma(D) \}.
\]

In other words, \(q(D)\) is the forward Dirac image of \(D\) under \(q : G \to G/H\). If the Dirac structure \(D\) is closed, then \(q(D)\) is closed.

If \(\mathcal{H} \oplus \{0\} \subseteq D\), then \(D = q^*(q(D))\), where for any Dirac structure \(\bar{D}\) on \(G/H\), its \textit{pullback} \(q^*(\bar{D})\) to \(G\) is the Dirac structure on \(G\) defined by

\[
q^*(\bar{D})(g) = \{ (v_g, (T_g q)^* \alpha_{gH}) \in \Gamma_{G}(g) \mid (T_g q v_g, \alpha_{gH}) \in \bar{D}(gH) \}
\]

for all \(g \in G\). (The bundle \(q^*(\bar{D})\) is the backward Dirac image of \(D\) under \(q\).)

Note that if we can verify that

\[
(R^*_K X, R^*_K \alpha) \in \Gamma(D) \quad \text{for all } (X, \alpha) \in \Gamma(D) \text{ and } K \in \mathcal{B}(H),
\]

then condition \(\text{(6)}\) is satisfied.

\(^1\)We prefer the terminology “closed” because integrability of a Dirac structure can also signify that it is integrable as a Lie algebroid, i.e., it integrates to a presymplectic groupoid as in \(^2\).
3 The geometry of Dirac groupoids.

3.1 Definition and examples

Definition 3.1 ([27]) A Dirac groupoid is a Lie groupoid $G \rightrightarrows P$ endowed with a Dirac structure $D_G$ such that $D_G \subseteq TG \oplus T^*G$ is a Lie subgroupoid. The Dirac structure $D_G$ is then said to be multiplicative.

Note that in [27], Dirac manifolds are always closed by definition.

Example 3.2 Consider a Poisson groupoid, that is, a Lie groupoid $G \rightrightarrows P$ endowed with a Poisson structure $\pi_G$ such that the graph $\Gamma \subseteq G \times G \times G$ of the multiplication map is a coisotropic submanifold of $(G \times G \times G, \pi_G \oplus \pi_G \oplus (−\pi_G))$. Poisson groupoids were introduced in [30] and studied in [30], [31], [23] among other, see also [22].

It is shown in [23] that $(G \rightrightarrows P, \pi_G)$ is a Poisson groupoid if and only if the vector bundle map $\pi_G^* : T^*G \to TG$ associated to $\pi_G$ is a morphism of Lie groupoids over some map $\alpha : A^*G \to TP$ (the restriction of $\pi_G^*$ to $A^*G$). Using this, it is easy to see that $(G \rightrightarrows P, \pi_G)$ is a Poisson groupoid if and only if $(G \rightrightarrows P, D_{\pi_G})$ is a closed Dirac groupoid (recall the definition of $D_{\pi_G}$ from Example 2.6).

Example 3.3 Let $G \rightrightarrows P$ be a Lie groupoid. A 2-form $\omega_G$ on $G$ is multiplicative if the partial multiplication map $m : G \times_P G \to G$ satisfies $m^* \omega_G = pr_1^* \omega_G + pr_2^* \omega_G$. The graph $D_{\omega_G} = \text{Graph}(\omega_G : TG \to T^*G)$ is then multiplicative, and $(G \rightrightarrows P, D_{\omega_G})$ is a Dirac groupoid, see [27], [1]. If the 2-form is closed, then the Dirac groupoid is closed.

Conversely, if a Dirac groupoid $(G \rightrightarrows P, D_G)$ is such that $G_1 = TG$, then $D_G$ is the graph of the vector bundle homomorphism $TG \to T^*G$ induced by a multiplicative 2-form. If the set of smooth sections of $D_G$ is closed under the Courant bracket, then the 2-form is closed.

Note that presymplectic groupoids have been studied in [3], [2]. These are Lie groupoids endowed with closed, multiplicative 2-forms satisfying some additional non degeneracy properties that will be recalled in Example 3.19.

Example 3.4 Let $(G, D_G)$ be a Dirac Lie group in the sense of [10]. We have seen there that it is a Dirac Lie group in the sense of [27], that is, $D_G$ is a subgroupoid of the Pontryagin groupoid $TG \oplus T^*G \rightrightarrows \{0\} \oplus g^*$. The set of units is here $D_G(e) \cap (\{0\} \oplus g^*) = \{0\} \oplus p_1$ since we know that $D_G(e)$ is equal to a direct sum $g_0 \oplus p_1 \subseteq g \oplus g^*$, with $g_0$ an ideal in $g$ and $p_1 \subseteq g^*$ its annihilator.

Example 3.5 Consider a smooth Dirac manifold $(M, D_M)$ and the pair Lie groupoid $(M \times M) \rightrightarrows M$ associated to $M$, that is, $D_M$ is a subgroupoid of the Pontryagin groupoid $T^*M \oplus T^*M \rightrightarrows \{0\} \oplus g^*$. The tangent bundle $TM \rightrightarrows M$ is an embedded submanifold of $M \times M$ via the smooth map $\iota : M \to \Delta_M, m \mapsto (m, m)$. The tangent groupoid $T(M \times M) \rightrightarrows TM$ of $M \times M$ is easily seen to be $(TM \times TM) \rightrightarrows TM$, the pair groupoid associated to $TM$.

The Lie algebroid $A(M \times M)$ of $M \times M \rightrightarrows M$ is the set $T_{\Delta_M}^*(M \times M)$. A vector $(v_m, w_m) \in T_{(m,m)}^*(M \times M)$ lies in $T_{(m,m)}^*(M \times M)$ if $0 = T_{(m,m)}(v_m, w_m) = v_m$. Hence, we have $A(M \times M) = (0_M \oplus TM)|\Delta_M$ and its dual $A^*(M \times M) \simeq (T\Delta_M)^0 \subseteq T^*(M \times M)|\Delta_M$ is given by $A^*_{(m,m)}(M \times M) = \{(-\alpha_m, \alpha_m) \mid \alpha_m \in T^*_mM\}$ for all $m \in M$. Hence, we can give the structure of the cotangent groupoid $T^*(M \times M) \rightrightarrows A^*(M \times M)$. If $(\alpha_m, \alpha_n) \in T^*_{(m,n)}(M \times M)$, then it is easy to check that $t(\alpha_m, \alpha_n) \in (\{0_m\} \times T_{m,M})^* = A^*_{(m,m)}(M \times M)$, where

$$t(\alpha_m, \alpha_n)(0_m, v_m) = -\alpha_m(v_m)$$

for all $v_m \in T_mM$, and hence $t(\alpha_m, \alpha_n) = (\alpha_m, -\alpha_m)$. In the same manner, we show that $s(\alpha_m, \alpha_n) = (-\alpha_n, \alpha_m)$. The product of $(\alpha_m, \alpha_n)$ and $(-\alpha_n, \alpha_p)$ is then given by

$$(\alpha_m, \alpha_n) \ast (-\alpha_n, \alpha_p) = (\alpha_m, \alpha_p).$$
It is easy to check that the Dirac structure $D_M \oplus D_M$, defined by

$$(D_M \oplus D_M)(m, n) = \left\{ (v_m, -v_n), (\alpha_m, \alpha_n) \in P_{M \times M}(m, n) \mid (v_m, \alpha_m) \in D_M(m) \quad \text{and} \quad (v_n, \alpha_n) \in D_M(n) \right\}$$

for all $(m, n) \in M \times M$, is a multiplicative Dirac structure on $M \times M \Rightarrow M$. This generalizes the fact that if $(M, \pi_M)$ is a Poisson manifold, then $M \times M \Rightarrow M$ endowed with $\pi_M \oplus (-\pi_M)$ is a Poisson groupoid.

We call the Dirac groupoid $(M \times M \Rightarrow M, D_M \oplus D_M)$ the pair Dirac groupoid associated to $(M, D_M)$. It is closed if and only if $(M, D_M)$ is closed.

Remark 3.6 In the Poisson case, it is known by results in [30] that any multiplicative Poisson structure on a pair groupoid is $\pi_M \oplus (-\pi_M)$ for some Poisson bivector $\pi_M$ on $M$. This is not true in general. For instance, let $M$ be a smooth manifold with a smooth free action of a Lie group $H$ with Lie algebra $\mathfrak{h}$. Then the diagonal action of $H$ on $M \times M$ is by Lie groupoid morphisms, and its vertical space $V \subseteq T(M \times M)$, $V(m, n) = \{(\xi_M(m), \xi_M(n)) \mid \xi \in \mathfrak{h}\}$ for all $m, n \in M$, is multiplicative (see for instance [11]). The Dirac structure $V \oplus V^\circ$ is then multiplicative, but cannot be written as a pair Dirac structure on $M \times M$.

3.2 General properties of Dirac groupoids

First, we study the characteristic distribution of an arbitrary Dirac groupoid. The results here illustrate how the situation in the case of Dirac groupoids is different from the case of the Dirac Lie groups.

The proof of the first proposition is straightforward.

Proposition 3.7 Let $(G \rightrightarrows P, D_G)$ be a Dirac groupoid. Then the subbundle $G_0 \subseteq TG$ is a (set) subgroupoid over $TP \cap G_0$.

In the group case, $G_0$ is automatically the bi-invariant image of an ideal in the Lie algebra $[26, 10]$, hence involutive of constant rank. In general, the distribution $G_0$ does not even need to be smooth. Since each manifold can be seen as a (trivial) groupoid over itself (i.e., with $t = s = \text{Id}_M$), any Dirac manifold can be seen as a Dirac groupoid, which will, in general not satisfy these conditions. Thus, trivial Dirac groupoids and pair Dirac groupoids yield already many examples of Dirac groupoids that do not have these properties.

If $G_0$ associated to a closed Dirac groupoid $(G \rightrightarrows P, D_G)$ is assumed to be a vector bundle on $G$, then we are in the same situation as in the group case. Yet, we know by the considerations in [11] that, even if it is regular, the quotient $G/G_0$ does not necessarily inherit a groupoid structure. If it does, we have the following result, which is shown in [11].

Theorem 3.8 Let $(G \rightrightarrows P, D_G)$ be a closed Dirac groupoid. Assume that $G_0$ is a subbundle of $TG$ and that it is complete [11]. If the leaf spaces $G/G_0$ and $P/G_0$ have smooth manifold structures such that the projections are submersions, then there is an induced multiplicative Poisson structure on the Lie groupoid $G/G_0 \rightrightarrows P/G_0$, such that the projection $\text{pr} : G \rightarrow G/G_0$ is a forward Dirac map.

Remark 3.9 In the Lie group case, the Poisson Lie group $(G/N, q(D_G))$ associated to a closed Dirac Lie group $(G, D_G)$ satisfying the necessary regularity assumptions was also a Poisson homogeneous space of the Dirac Lie group. Here, the Poisson groupoid associated to the Dirac groupoid is, in general, not a Poisson homogeneous space of the Dirac groupoid since the quotient $G/G_0$ is not a homogeneous space of the Lie groupoid $G \rightrightarrows P$.

For the sake of completeness, we show next how the result in [30] about the induced Poisson structure on the units of a Poisson groupoid can be generalized to the situation of Dirac groupoids. For that, we need to study the units of the Dirac groupoid. It is natural to ask what the set of units of $D_G$ is, when seen as a subgroupoid of $(TG \oplus T^*G) \rightrightarrows (TP \oplus A^*G)$. It is easy to see that $D_G$ is a Lie groupoid over $D_G \cap (TP \oplus A^*G)$. This intersection will be written $\mathcal{A}(D_G) := D_G \cap (TP \oplus A^*G)$. Here, we will show that it is a vector bundle over $P$. 

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**Definition 3.10** 1. Let \((G \rightrightarrows P, D_G)\) be a Dirac groupoid and \(\mathfrak{A}(D_G)\) the set of units of \(D_G\), i.e., the subdistribution \(D_G \cap (TP \oplus A^*G)\) of \(TP \oplus A^*G\). We write \(a_* : \mathfrak{A}(D_G) \to TP\) for the map defined by \(a_*(v_p, \alpha_p) = v_p\) for all \(p \in P\), \((v_p, \alpha_p) \in \mathfrak{A}_p(D_G)\).

2. We write \(\ker Ts\), respectively \(\ker Tt\) for the kernel \(T^*G \oplus (T^*G)^o\) (respectively \(T^*G \oplus (T^*G)^\circ\)) of the source map \(Ts : P_G \to TP \oplus A^*G\) (respectively the target map \(Tt : P_G \to TP \oplus A^*G\)). We denote by \(I^s(D_G)\) the restriction to \(P\) of \(D_G \cap \ker Ts\), i.e.,

\[
I^s(D_G) := D_G \cap (T_p^sG \oplus (T_p^sG)^o) = (D_G \cap \ker Ts)|_P. 
\]

In the same manner, we write \(I^t(D_G) := D_G \cap (T_p^tG \oplus (T_p^tG)^o) = (D_G \cap \ker Tt)|_P.\)

**Theorem 3.11** Let \((G \rightrightarrows P, D_G)\) be a Dirac groupoid. Then the Dirac subspaces \(D_G|_P\) splits as a direct sum

\[
D_G|_P = \mathfrak{A}(D_G) \oplus I^s(D_G)
\]

and in the same manner

\[
D_G|_P = \mathfrak{A}(D_G) \oplus I^t(D_G).
\]

The three intersections are smooth and have constant rank on \(P\).

**Proof:** Choose \(p \in P\) and \((v_p, \alpha_p) \in D_G(p)\). Then we have \(Tt(v_p, \alpha_p) \in D_G(p)\) and hence also \((v_p, \alpha_p) - Tt(v_p, \alpha_p) \in D_G(p)\). We find that \(v_p - T_p^s v_p \in T_p^s G\) and \(T_p^t v_p \in T_p P\), and in the same manner \(t(\alpha_p) \in A_p^* G = (T_p^s P)^o\), by definition, and \(\alpha_p - t(\alpha_p) \in (T_p^t G)^o\).

Since

\[
(v_p, \alpha_p) = Tt(v_p, \alpha_p) + (v_p, \alpha_p) - Tt(v_p, \alpha_p),
\]

we have shown the first equality. The second formula can be shown in the same manner, using the map \(Ts : D_G(p) \to D_G(p) \cap (T_p^s P \times A_p^* G)\).

Next, we show that the intersection of \(D_G\) with \(TP \oplus A^*G\) is smooth. Choose \(p \in P\) and \((v_p, \alpha_p) \in D_G(p) \cap (T_p^s P \times A_p^* G)\). Since \(D_G\) is a smooth vector bundle on \(G\), we find a section \((X, \alpha) \in \Gamma(D_G)\) defined on a neighborhood of \(p\) such that \((X, \alpha)(p) = (v_p, \alpha_p)\). The restriction \((X, \alpha)|_P\) is then a smooth section of \(D_G|_P\). We have \(Ts((X, \alpha)|_P) \in \Gamma(D_G \cap (TP \oplus A^*G))\) and \(Ts(X, \alpha)(p) = (T_p^s v_p, \alpha_p|_{T_p^s G}) = (v_p, \alpha_p)\) since \(v_p \in T_p P\) and \(\alpha_p \in A_p^* G = (T_p^s P)^o\).

Thus, we have found a smooth section of \(D_G \cap (TP \oplus A^*G)\) defined on a neighborhood of \(p\) in \(P\) and taking value \((v_p, \alpha_p)\) at \(p\).

Since \((D_G|_P)^\perp = D_G|_P\) and \(TP \oplus A^*G = (TP \oplus A^*G)^\perp\) are smooth subbundles of \(P_G|_P\), we get from Proposition 4.4 in [14] that \(D_G \cap (TP \oplus A^*G)\) has constant rank on \(P\). By the splittings shown above and the fact that \(D_G|_P\) has constant rank on \(P\), we find that the two other intersections have constant rank on \(P\), and are thus smooth.

In the case of a Dirac Lie group, the bundle \(I^s(D_G) \to P\) is \(g_0 \to \{e\}\), as is shown in the next example. We will see later that \(I^s(D_G)\) has a crucial role in the construction of the Courant algebroid associated to a Dirac groupoid \((G \rightrightarrows P, D_G)\). The fact that the left and right invariant images of this subspace are exactly the characteristic distribution of the Dirac structure is a very special and convenient feature in the group case, that makes the Dirac Lie groups much easier to understand than arbitrary Dirac groupoids (see [10]).

**Example 3.12** Let \((G, D_G)\) be a Dirac Lie group (Example 3.4) and set \(p_1 = P_1(e) \subseteq g^*\) and \(g_0 = G_0(e) \subseteq g\). We have \(P = \{e\}\) (the neutral element of \(G\)),

\[
D_G(e) \cap (T_e P \times (T_e P)^o) = D_G(e) \cap (\{0\} \times g^*) = \{0\} \times p_1.
\]
and

\[ D_G(e) \cap (T^*_s G \times (T^*_c G)^o) = D_G(e) \cap (\mathfrak{g} \times \{0\}) = \mathfrak{g}_0 \times \{0\}. \]

We recover hence the equality \( D_G(e) = \mathfrak{g}_0 \times p_1 \). \( \square \)

In this particular case, \( D_G \) is a Poisson structure if and only if \( D_G(e) \) is equal to the set of units of \( TG \oplus T^*G \ni \{0\} \times g^* \), i.e. \( \mathfrak{g}_0 = \{0\} \) and \( p_1 = g^* \). In the general case, this is not true. \( \diamond \)

**Lemma 3.13** Let \((G \rightrightarrows P, D_G)\) be a Dirac groupoid. For all \( g \in G \), we have

\[ D_G(g) \cap \ker Tt = (0_g, 0_g) \ast I^*_s(g)(D_G) \]

and

\[ D_G(g) \cap \ker Ts = I^*_t(g)(D_G) \ast (0_g, 0_g). \]

The intersections \( D_G \cap \ker Tt \) and \( D_G \cap \ker Ts \) have consequently constant rank on \( G \).

**Proof:** Choose \( g \in G \) and \( v_{s(g)} \in T^*_s(G), \alpha_{s(g)} \in T^*_s(P) \) such that \((v_{s(g)}, (T^*_s s) \ast \alpha_{s(g)}) \in D_G(s(g))\). Then we have \( Tt(v_{s(g)}, (T^*_s s) \ast \alpha_{s(g)}) = (0_{s(g)}, 0_{s(g)}) \) and \((0_g, 0_g) \in D_G(g) \) with \( Ts(0_g, 0_g) = (0_{s(g)}, 0_{s(g)}) \). Thus, the product

\[ (0_g, 0_g) \ast (v_{s(g)}, (T^*_s s) \ast \alpha_{s(g)}) \]

makes sense and is an element of \( D_G(g) \cap \ker Tt \). Conversely, it is easy to see that

\[ (0_g, 0_g) \ast (v_g, 0_g) \]

makes sense and is an element of \( D_G(s(g)) \cap \ker Tt = I^*_t(g)(D_G) \) for all \( v_g, 0_g \in D_G(g) \cap (T^*_s G \oplus (T^*_c G)^o) \).

Since \( (0_g, 0_g) = (0_g, 0_g)^{-1} \), this shows \( (v_g, 0_g) \in (0_g, 0_g) \ast I^*_s(g)(D_G) \).

There is hence an isomorphism

\[ D_G(s(g)) \cap \left( T^*_s G \times (T^*_c G)^o \right) \leftrightarrow D_G(g) \cap (T^*_s G \times (T^*_c G)^o). \]

As a consequence, \( D_G \cap \ker Tt \) has constant rank along \( s \)-fibers. Since \( D_G \cap \ker Tt \) has constant rank on \( P \) by Theorem 3.11, it has hence constant rank on the whole of \( G \). \( \square \)

**Example 3.14** If \((G \rightrightarrows P, \pi_G)\) is a Poisson groupoid, then \( \pi_G^0(d(s^*f)) \in \Gamma(T^i G) \) for all \( f \in C^\infty(P) \) (see [30]). The intersection \( D_{\pi_G} \cap \ker Tt \) is hence spanned by the sections \((\pi_G^0(d(s^*f)), d(s^*f))\), with \( f \in C^\infty(P) \), and has constant rank. The intersection \( D_{\pi_G} \cap \ker Ts \) is spanned by the sections \((\pi_G^0(d(t^*f)), d(t^*f))\) with \( f \in C^\infty(P) \). \( \diamond \)

Using this, we will show the next main theorem of this section. We will need the following lemma.

**Lemma 3.15** Let \( G \rightrightarrows P \) be a Lie groupoid. Choose \( g \in G \) and set \( p = t(g) \). Then, for all \( \alpha_p \in T^*_p P \), we have

\[ -(T_{g^{-1}} s)^* \alpha_p = (T^*_g t)^* \alpha_p^{-1}. \]

**Proof:** For any \( u_p \in A_p G \), one gets easily

\[ \hat{\alpha}(T^*_g t)^* \alpha_p)(u_p) = -((T^*_p s)^* \alpha_p)(u_p). \]

and

\[ \hat{s}(-(T_{g^{-1}} s)^* \alpha_p)(u_p) = -((T^*_p s)^* \alpha_p)(u_p). \]

In the same manner,

\[ \hat{t}(-(T_{g^{-1}} s)^* \alpha_p) = 0, \quad \hat{s}((T^*_g t)^* \alpha_p) = 0. \]
Hence, we can compute \((T_qg)^*\alpha_p) \ast ((T_qg)^{-1}) \ast \alpha_p) \ast (T_qg)^* \alpha_p) \ast (T_qg)^* \alpha_p). By choosing for any \(u_q \in T_qG\) two vectors \(u_{g-1} \in T_{g-1}G\) and \(u_g \in T_gG\) such that \(u_q = u_{g-1} \ast u_g\), one gets immediately

\[
\left((T_qg)^{-1}) \ast (T_qg)^* \alpha_p\right) \ast (T_qg)^* (u_g) = \left(\left((T_qg)^{-1}) \ast (T_qg)^* \alpha_p\right) \ast (T_qg)^* (w_{g-1}) + (T_qg)^* \alpha_p\right)(u_g) = 0,
\]

which shows that \((T_qg)^* \alpha_p) \ast (T_qg)^* \alpha_p) = 0 = \hat{s}((T_qg)^* \alpha_p). For any \(w_p = w_g \ast w_{g-1} \in T_pG\),

\[
((T_qg)^* \alpha_p) \ast (T_qg)^* \alpha_p) = \alpha_p(T_qg w_g) - \alpha_p(T_qg^{-1} w_{g-1}) = \alpha_p(T_qg w_g).
\]

Thus, \((T_qg)^* \alpha_p) \ast (T_qg)^* \alpha_p) = \hat{t}((T_qg)^* \alpha_p).

\[\square\]

**Remark 3.16** If \((v_p,(T_pg)^* \alpha_p)\) is such that \(T_pg v_p = 0_p\), then \(T_pg(T_pg)^* \alpha_p) = (0_p,0_p). If \(g \in G\) is such that \(s(g) = p\), then \((0_g,0_g) \ast (v_p,(T_pg)^* \alpha_p) = (T_pL_g v_p, (T_pg)^* \alpha_p)\) for all \(g \in s^{-1}(p)\).

To see this, let \(c : (-\varepsilon,\varepsilon) \to t^{-1}(p)\) be a curve such that \(c(0) = p\) and \(c(0) = v_p\). We can then compute \(g \ast v_p = T_{(g,p)}m(0_g,v_p) = \frac{d}{d\sigma} \mid_{\sigma=0} g \ast c(\sigma) = T_pL_g v_p\). If \(v_g \in T_gG\), the equality \(v_g = v_g \ast (T_g \alpha_g)\) yielding \((0_g \ast (T_pg)^* \alpha_p) (v_g) = 0_g(v_g) + ((T_pg)^* \alpha_p) (T_g \alpha_g) = \alpha_p(T_g \alpha_g) = (T_g)^* \alpha_p) (v_g).

\[\triangle\]

Now we can prove a generalization of the fact that the units of a Poisson groupoid inherit a Poisson structure such that the target map if a Poisson map and the source is anti-Poisson (see [30]).

**Theorem 3.17** Assume that \((G\Rightarrow P, D_G)\) is a Dirac groupoid such that \(TP \cap G_0\) is smooth. Define the subspace \(D_P\) of \(P_P\) by

\[
D_P(p) = \left\{(v_p,\alpha_p) \in P_P(p) \mid \exists (w_p,(T_pw_p)^* \alpha_p) \in D_G(p) \cap (T_pG \times (A_pG)^0) \text{ such that } v_p = T_pw_p\right\}
\]

for all \(p \in P\). Then \(D_P\) is a Dirac structure on \(P\).

Furthermore, if for all \(g \in G\), the restriction to \(G_0(g)\) of the target map \(T_g t : G_0(g) \to G_0(t(g)) \cap T_{t(g)} P\) is surjective, then the maps \(t : (G,D_G) \to (P,D_P)\) and \(s : (G,D_G) \to (P,-D_P)\) are forward Dirac maps, where \(-D_P\) is the Dirac structure defined on \(P\) by \(D_P(p) = \{(v_p,\alpha_p) \in P_P(p) \mid (v_p,\alpha_p) \in D_P(p)\}\).

The characteristic distribution \(G^P_0\) of \((P,D_P)\) is then equal to the intersection \(G_0 \cap TP\). Note that this theorem generalizes Theorem 4.2.3 in [30] (see also [29], [1] for the special case of symplectic groupoids), since in the Poisson case, we have \(G_0 = 0_{TG}\) and the hypotheses are consequently trivially satisfied. If all conditions are satisfied, the Dirac structure on \(P\) is just the push forward of the Dirac structure on \(G\) under the quotient map \(t : G \to G/G \simeq P\) (see Example [2.4]).

**Proof:** First note that \(P(D_G) \oplus (G_0 \cap TP) \oplus A^0_{G}} = D_G \cap (T_pG \oplus A^0_G)\). By the hypothesis on \(G_0 \cap TP\), the intersection \(D_G \cap (T_pG \oplus A^0_G)\) is hence smooth and has consequently constant rank on \(P\) by a Proposition in [14]. The space \(D_P\) is smooth since it is spanned by the smooth sections of \(G_0 \cap TP\) and the smooth sections \((T_xtX, \alpha)\) for all \((X^0, t^* \alpha) \in T(D_G \cap \ker T\Sigma)\).

The inclusion \(D_P \cap D_P^\perp\) is obvious. Conversely, if \((v_p,\alpha_p) \in D_P(p)^\perp \subseteq P_P(p)\) and \(y_p \in T_pG\) is chosen such that \(T_p t y_p = v_p\), then we have

\[
\langle (w_p, (T_pw_p)^* \beta_p), (y_p, (T_pw_p)^* \alpha_p) \rangle = \langle (T_pw_p, \beta_p), (v_p, \alpha_p) \rangle = 0
\]

for all \((w_p, (T_pw_p)^* \beta_p) \in D_G(p) \cap (T_pG \times A_pG^0)\). Hence, we get

\[
(y_p, (T_pw_p)^* \alpha_p) \in (D_G(p) \cap (T_pG \times A_pG^0))^\perp = (D_G(p) + A_pG \times \{0_p\})
\]

and consequently \((y_p, (T_pw_p)^* \alpha_p) = (y_p', (T_pw_p)^* \alpha_p) + (u_p,0)\) for some \((y_p', (T_pw_p)^* \alpha_p) \in D_G(p)\) and \(u_p \in A_pG\). But then \(T_p t y_p = T_p t y_p = v_p\) and \((v_p, \alpha_p) \in D_P(p)\).

Assume that the target map \(T_g t : G_0(g) \to G_0(t(g)) \cap T_{t(g)} P\) is surjective for all \(g \in G\). We show that \(t : (G,D_G) \to (P,D_P)\) is a forward Dirac map. Choose \(p \in P\), \(g \in t^{-1}(p)\) and \((v_p, \alpha_p) \in D_P(p)\). We have to prove that there exists \((v_g, \alpha_g) \in D_G(g)\) such that \(\alpha_g = (T_g)^* \alpha_p\) and \(T_g t v_g = v_p\). By definition of \(D_P\)
and the considerations above, there exists \( u_p \in T_p^* G \) and \( z_p \in G_0(p) \cap T_p P \) such that \( T_p u_p + z_p = v_p \) and \( (u_p, (T_p t)^* \alpha_p) \in P_p^*(D G) \). Then the pair \( (T_p R_g u_p, (T_g t)^* \alpha_p) = (u_p, (T_p t)^* \alpha_p) \ast (0_g, 0_g) \) is an element of \( D G(g) \) and by hypothesis, we find \( z_g \in G_0(g) \) such that \( T_g z_g = z_p \). The pair \( (T_p R_g u_p + z_g, (T_g t)^* \alpha_p) \) is then an element of \( D G(g) \) and \( T_g (T_p R_g u_p + z_g) = T_p u_p + z_p = v_p \).

It remains to prove that \( s : (G, D G) \to (P, -D P) \) is also a forward Dirac map. Choose \( p \in P, g \in s^{-1}(p) \) and \( (v_p, \alpha_p) \in -D P(p) \). Then \( (-v_p, \alpha_p) \in D P(p) \) and, since \( t(g^{-1}) = s(g) = p \), there exists by the considerations above \( w_g \in T_{g^{-1}} G \) such that \( T_g t w_g^{-1} = v_p \) and \( (-w_g^{-1}, (T_g t)^* \alpha_p) \in D G(g^{-1}) \). By Lemma 3.15, we have then \( (-w_g^{-1}, (T_g t)^* \alpha_p) \in D G(g) \). This leads to \( ((w_g^{-1})^{-1}, (T_g s)^* \alpha_p) \in D G(g) \) and since \( T_g s((w_g^{-1})^{-1}) = T_{g^{-1}} t w_g^{-1} = v_p \), the proof is finished.

Note that the hypotheses on the distribution \( G_0 \) in Theorem 3.17 are rather strong. The following example shows that this theorem can hold under weaker hypotheses.

**Example 3.18** Assume that \((M, D M)\) is a smooth Dirac manifold such that \( G_0 \) is a singular distribution. Then, the induced pair Dirac groupoid \((M \times M \rightrightarrows M, D M \odot D M)\) as in Example 3.15 does not satisfy the conditions for Theorem 3.17. The space \( P(D M \odot D M) \) is here given by
\[
P_{(m,m)}^s(D M \odot D M) = \{(v_m, 0_m, \alpha_m, 0_m) \mid (v_m, \alpha_m) \in D_M(m)\}
\]
for all \( m \in M \), and the space \( G_0 \cap T \Delta_M \) is given by
\[
G_0(m, m) \cap T_{(m,m)} \Delta_M = \{(v_m, v_m) \mid (v_m, 0_m) \in D_M(m)\}
\]
for all \( m \in M \). Hence, we have
\[
P_{(m,m)}^s(D M \odot D M) + (G_0(m, m) \cap T_{(m,m)} \Delta_M) \times M \times M \{0\}
\]
\[
= \{(v_m, w_m, \alpha_m, 0_m) \mid (v_m, \alpha_m) \in D_M(m), (w_m, 0_m) \in D_M(m)\}
\]
and we find that the same construction as in Theorem 3.17 defines a Dirac structure on \( M \rightrightarrows \Delta_M \), which equals the original Dirac structure \( D M \) on \( M \) since its fiber over \( m \in M \) is \( \{(v_m, \alpha_m) \mid (v_m, w_m, \alpha_m, 0_m) \in P_{(m,m)}^s(D M \odot D M) + G_0(m, m) \cap T_{(m,m)} \Delta_M\} \).

\( \Box \)

**Example 3.19** Let \( G \rightrightarrows P \) be a Lie groupoid and \( \omega_G \in \Omega^2(G) \) be a closed 2-form on \( G \). Then \((G \rightrightarrows P, \omega_G)\) is a presymplectic groupoid if \( \omega_G \) is multiplicative, \( \dim G = 2 \dim P \) and \((\ker \omega_G)(p) \cap T^*_p G \cap T^*_p G = \{0_p\}\) for all \( p \in P \).

Note that \((\ker \omega_G)(p) = G_0(p)\), if \( G_0 \) is the characteristic distribution associated to the Dirac groupoid \((G \rightrightarrows P, D_{\omega_G})\), see Example 3.15. In \( G \), a multiplicative 2-form is said to be of Dirac type if it has a property that is shown to be equivalent to our hypothesis on surjectivity of \( T_{\ker \omega_G} \). If the bundle of Dirac structures defined as in 3.14 by the multiplicative Dirac structure \( D_{\omega_G} \) associated to a multiplicative 2-form \( \omega_G \) of Dirac type is smooth, then \( D P \) is a Dirac structure on \( P \) such that the target map \( t : (G, D G) \to (P, D P) \) is a forward Dirac map. Thus, we recover here the two conditions in 3.14 since we made the hypothesis on smoothness of \( G_0 \cap TP \) to ensure the smoothness of \( D P \).

It is shown in 3.15 that a presymplectic groupoid \((G \rightrightarrows P, \omega_G)\) satisfy automatically these conditions and hence that there exists a Dirac structure \( D P \) on \( P \) such that the target map \( t : (G, D G) \to (P, D P) \) is a forward Dirac map.

\( \Box \)

**Remark 3.20** In the situation of Theorem 3.15 the multiplicative subbundle \( G_0 \) of \( TG \) has constant rank on \( G \). In particular, the intersection \( TP \cap G_0 \) is a smooth vector bundle over \( P \) and for each \( g \in G \), the restriction to \( G_0(g) \) of the target map, \( T_g t : G_0(g) \to G_0(t(g)) \cap T_{t(g)} P \), is surjective. By Theorem 3.17 there exists then a Dirac structure \( D P \) on \( P \) such that \( t : (G, D G) \to (P, D P) \) is a forward Dirac map. Since \((G/G_0 \rightrightarrows P/G_0, pr(D G))\) is a Poisson groupoid, we know also by a theorem in 3.15 that there is a Poisson structure \( \{\cdot, \cdot\}_{P/G_0} \) on \( P/G_0 \) such that \( [t]:(G/G_0, pr(D G)) \to (P/G_0, \{\cdot, \cdot\}_{P/G_0}) \) is a Poisson map. It is easy to check that the map \( pr_0 : (P, D P) \to (P/G_0, \{\cdot, \cdot\}_{P/G_0}) \) is then also a forward Dirac
map, i.e., the graph of the vector bundle homomorphism $T^*(P/G_0) \rightarrow T(P/G_0)$ defined by the Poisson structure is the forward Dirac image of $D_P$ under $pr_o$.

\[
\begin{array}{ccc}
(G,D_G) & \xrightarrow{t} & (P,D_P) \\
pr \downarrow & & \downarrow \text{pr}_o \\
(G/G_0, pr(D_G)) & \xrightarrow{\iota} & (P/G_0, \{\cdot, \cdot\}_P/G_0)
\end{array}
\]

\[\triangle\]

### 3.3 The units of a Dirac groupoid

In this section, we discuss further properties of the set of units $\mathfrak{A}(D_G)$ of a multiplicative Dirac structure.

**Proposition 3.21** Let $\tilde{\xi} = (\tilde{X}_\xi, \tilde{\theta}_\xi)$ be a section of $D_G \cap (TP \oplus A^*G) = \mathfrak{A}(D_G)$. Then there exists a smooth section $\xi = (X_\xi, \theta_\xi)$ of $D_G$ such that $\xi|_P = \tilde{\xi}$ and $T_\xi(s(g)) = \tilde{\xi}(s(g))$ for all $g \in s^{-1}(\text{Dom}(\tilde{\xi}))$.

Following [21], we say that then the section $\xi$ of $D_G$ is a $s$-star section or simply star section and we write $\xi \sim_s \tilde{\xi}$. Indeed, since $X_\xi \in \Gamma(TP)$ and $T_\xi(sX_\xi(g)) = \tilde{X}_\xi(s(g))$ for all $g \in G$ where this makes sense, the vector fields $X_\xi$ and $\tilde{X}_\xi$ are $s$-related, $X_\xi \sim_s \tilde{X}_\xi$. Note that outside of $P$, $\xi$ is defined modulo sections of $D_G \cap \ker Ts$.

**Proof:** We have shown in Lemma 3.13 that $D_G \cap \ker Ts$ is a subbundle of $D_G$. Hence, we can consider the smooth vector bundle $D_G/(D_G \cap \ker Ts)$ over $G$. Since $D_G$ is a Lie subgroupoid of $P_G = (TP \oplus A^*G)$, we can consider the restriction to $D_G$ of the source map, $Ts : D_G \rightarrow \mathfrak{A}(D_G)$. Since $D_G \cap \ker Ts$ is the kernel of this map, we have an induced smooth vector bundle homomorphism $\overline{Ts} : D_G/(D_G \cap \ker Ts) \rightarrow \mathfrak{A}(D_G)$ over the source map $s : G \rightarrow P$, that is bijective in every fiber. Hence, there exists a unique smooth section $[\xi]$ of $D_G/(D_G \cap \ker Ts)$ such that $\overline{Ts}([\xi](g)) = \tilde{\xi}(s(g))$ for all $g \in G$. If $\xi \in \Gamma(D_G)$ is a representative of $[\xi]$ such that $\xi|_P = \tilde{\xi}$, then $Ts(\xi(g)) = \tilde{\xi}(s(g))$ for all $g \in G$. □

**Lemma 3.22** Choose $\tilde{\xi}, \tilde{\eta} \in \Gamma(\mathfrak{A}(D_G))$ and star sections $\xi \sim_s \tilde{\xi}, \eta \sim_s \tilde{\eta}$ of $D_G$. Then, if $\xi = (X_\xi, \theta_\xi)$ and $\eta = (X_\eta, \theta_\eta)$, the identity

\[
\theta_\eta(L_{Z\xi}X_\xi) + (L_{Z\xi}\theta_\xi)(X_\eta) = s^\ast\left((\theta_\eta(L_{Z\xi}X_\xi) + (L_{Z\xi}\theta_\xi)(X_\eta))|_P\right)
\]

**Proof:** Choose $g \in G$ and set $p = s(g)$. For all $t \in (-\varepsilon, \varepsilon)$ for a small $\varepsilon$, we have

\[
\xi(g \ast \text{Exp}(tZ)(p)) = \left(\xi(g \ast \text{Exp}(tZ)(p))\right) \ast \left(\xi(\text{Exp}(tZ)(p))\right)^{-1} \ast \left(\xi(\text{Exp}(tZ)(p))\right).
\]

The pair

\[
(\xi(g \ast \text{Exp}(tZ)(p))) \ast (\xi(\text{Exp}(tZ)(p)))^{-1}
\]

is an element of $D_G(g)$ for all $t \in (-\varepsilon, \varepsilon)$ and will be written $\delta_t(g)$ to simplify the notation. Note that we have

\[
\delta_t(g) = \left(T_{\text{Exp}(tZ)}R_{\text{Exp}(-tZ)}X_\xi(g \ast \text{Exp}(tZ)(p)), \theta_\xi(g \ast \text{Exp}(tZ)(p)) \circ T_g R_{\text{Exp}(tZ)}\right)
\]

\[\delta_t(g) = \left(T_{\text{Exp}(tZ)}R_{\text{Exp}(-tZ)}X_\xi(\text{Exp}(tZ)(p)), \theta_\xi(\text{Exp}(tZ)(p)) \circ T_g R_{\text{Exp}(tZ)}\right).
\]
We compute
\[
(\theta_\eta(\mathcal{L}_{\bar{X}}X_\xi) + (\mathcal{L}_{\bar{X}}\theta_\xi)(X_\eta)) (g)
\]
\[
= \left\langle \eta(g), \frac{d}{dt} \bigg|_{t=0} \left( R^*_{\text{Exp}(t\bar{Z})}X_\xi \right)(g), \left( R^*_{\text{Exp}(t\bar{Z})}\theta_\xi \right)(g) \right\rangle
\]
\[
\quad \text{for all } \bar{Z} \in \Gamma(\mathfrak{A}(\mathcal{D}_G)),
\]
\[
\quad \text{since } \bar{X}_\xi \sim \bar{X}_\xi \text{ and } X_\eta \sim X_\eta \text{ on points in } \bar{Z} \in \mathfrak{X}(A).
\]
\[
= \left\langle \eta(g), \delta_t(g) \right\rangle \left( \left( R^*_{\text{Exp}(t\bar{Z})}X_\xi \right)(p), \left( R^*_{\text{Exp}(t\bar{Z})}\theta_\xi \right)(p) \right)
\]
\[
= \left\langle \eta(p), \delta_t(p) \right\rangle \left( \left( R^*_{\text{Exp}(\bar{Z})}X_\xi \right)(p), \left( R^*_{\text{Exp}(\bar{Z})}\theta_\xi \right)(p) \right)
\]
\[
= \left( \frac{d}{dt} \bigg|_{t=0} \right) + \left\langle \eta, (\mathcal{L}_{\bar{X}}X_\xi, \mathcal{L}_{\bar{X}}\theta_\xi) \right\rangle (p) = (\bar{\theta}_\eta(\mathcal{L}_{\bar{Z}}X_\xi) + (\mathcal{L}_{\bar{Z}}\theta_\xi)(X_\eta)) (s(g)).
\]
\[
\text{Proposition 3.23} \quad \text{Let } (G \rightrightarrows P, \mathcal{D}_G) \text{ be a Dirac groupoid. Choose } \bar{\xi}, \bar{\eta} \in \Gamma(\mathfrak{A}(\mathcal{D}_G)) \text{ and star sections } \xi \sim \bar{\xi}, \eta \sim \bar{\eta} \text{ of } \mathcal{D}_G, \text{ as in Proposition 3.27. Then the Courant-Dorfman bracket}
\]
\[
[\xi, \eta] = ([X_\xi, X_\eta], \mathcal{L}_{X_\xi}\theta_\eta - i_{X_\xi}d\theta_\xi)
\]
\[
is a star section and its values on } P \text{ are elements of } TP \oplus A^*G.
\]
\[
\text{Proof: Since } X_\xi \sim \bar{X}_\xi \text{ and } X_\eta \sim \bar{X}_\eta, \text{ we know that } [X_\xi, X_\eta] \sim [\bar{X}_\xi, \bar{X}_\eta]. \text{ Since } X_\xi|P = \bar{X}_\xi, X_\eta|P = \bar{X}_\eta,
\]
\[
\text{the value of } [X_\xi, X_\eta] \text{ on points in } P \text{ is equal to the value of } [\bar{X}_\xi, \bar{X}_\eta] \in \mathfrak{X}(P). \text{ We check that for all } p \in P,
\]
\[
\text{we have } \bar{s} \left( \left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right)(g) \right) = \left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right)(p) \text{ for any } g \in s^{-1}(p).
\]
\[
\text{We have for any } Z \in \Gamma(\mathfrak{A}(P)):
\]
\[
\bar{s} \left( \left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right)(g) \right) (Z(p)) = \left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right) \left( Z^*(p) \right).
\]
\[
\text{Hence, we compute with } [9]
\]
\[
\left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right)(Z^*) = \left( \mathcal{L}_{\bar{X}}\theta_\eta \right)(Z^*) + \theta_\eta(\mathcal{L}_{\bar{Z}}X_\xi) - \theta_\eta(\mathcal{L}_{\bar{Z}}\theta_\xi)
\]
\[
= \left( X_\xi(\bar{\theta}_\eta(Z)) + \bar{\theta}_\eta(\mathcal{L}_{\bar{Z}}X_\xi) - \bar{\theta}_\eta(\mathcal{L}_{\bar{Z}}\theta_\xi) \right). \quad \text{for any } p \in P:
\]
\[
\left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right) (Z(p)) = \left( \bar{X}_\xi(\bar{\theta}_\eta(Z)) + \bar{\theta}_\eta(\mathcal{L}_{\bar{Z}}X_\xi) - \bar{\theta}_\eta(\mathcal{L}_{\bar{Z}}\theta_\xi) \right), \quad \text{for any } X \in \Gamma(\mathfrak{A}(P)),
\]
\[
\text{Choose } X \in \Gamma(\mathfrak{A}(P)), \text{ then}
\]
\[
\left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right) (X(p)) = \left( \mathcal{L}_{\bar{X}}\theta_\eta \right)(X(p)) + \theta_\eta([X, \bar{X}_\xi])(p) - \theta_\eta(\bar{X}_\xi)(X(p))
\]
\[
+ \bar{X}_\xi(\theta_\eta(Z))(p) - \bar{X}_\xi(\mathcal{L}_{\bar{Z}}\theta_\xi)(p) = 0
\]
\[
since \theta_\eta, \theta_\xi \in \Gamma(\mathfrak{A}(P)) \text{ and } X, [X, \mathcal{L}_{\bar{X}}X_\xi], \bar{X}_\xi, [\bar{X}_\xi, \bar{X}_\xi] \in \Gamma(\mathfrak{A}(P)).
\]
\[
\text{Thus, we have shown that } \left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right)|_P \text{ is a section of } A^*G = TP^0 \text{ and}
\]
\[
\bar{s} \left( \left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right)(g) \right) = \left( \mathcal{L}_{\bar{X}}\theta_\eta - i_{\bar{X}}d\bar{\theta}_\xi \right) (s(g)) \quad \text{for all } g \in G.
\]
\[
\text{Theorem 3.24} \quad \text{Let } (G \rightrightarrows P, \mathcal{D}_G) \text{ be a Dirac groupoid. Then there is an induced antisymmetric bracket}
\]
\[
[\cdot, \cdot]_* \equiv \Gamma(\mathfrak{A}(\mathcal{D}_G)) \times \Gamma(\mathfrak{A}(\mathcal{D}_G)) \rightarrow \Gamma(\mathfrak{A}(\mathcal{D}_G)) \times \Gamma(\mathfrak{A}(\mathcal{D}_G)) \rightarrow \Gamma(\mathfrak{A}(\mathcal{D}_G)) \rightarrow \Gamma(\mathfrak{A}(\mathcal{D}_G))
\]
\[
defined by } [\bar{\xi}, \bar{\eta}]_* = [\xi, \eta]|_P \text{ for any choice of star sections } \xi \sim \bar{\xi}, \eta \sim \bar{\eta} \text{ of } \mathcal{D}_G. \text{ If } (G \rightrightarrows P, \mathcal{D}_G) \text{ is closed, then } (\mathfrak{A}(\mathcal{D}_G), \cdot, \cdot)_* \text{ is a Lie algebroid over } P.
\]
Proof: By Proposition 3.23 if $\xi \sim s \tilde{\xi}$, $\eta \sim s \tilde{\eta}$ then

\[ [(X_\xi, \theta_\xi), (X_\eta, \theta_\eta)] \sim s ([\tilde{X}_\xi, \tilde{X}_\eta], (\mathcal{L}_{X_\xi} \theta_\eta - i_{X_\xi} d\theta_\xi) | p). \]

Thus, we have first to show that the right-hand side of this equation does not depend on the choice of the sections $\xi$ and $\eta$. Choose a star section $\nu \sim s 0$ of $\mathbb{D}_G$, i.e., $\nu \in \Gamma(\mathbb{D}_G \cap \ker Ts)$ with $\nu|_P = 0$. For any $Z \in \Gamma(\mathbb{A}(G))$, we find as in the proof of Proposition 3.23

\[
(\mathcal{L}_{X_\nu} \theta_\xi - i_{X_\xi} d\theta_\nu) (Z(p)) = - (\mathcal{L}_{Z_\nu}(X_\xi, \theta_\xi)(p), (\tilde{X}_\nu, \tilde{\theta}_\nu)(p)) = 0.
\]

Hence, at any $p \in P$, we find

\[
(\mathcal{L}_{X_\nu} \theta_\xi - i_{X_\xi} d\theta_\nu)(p) = 0_p \text{ since we know by the previous proposition that } (\mathcal{L}_{X_\nu} \theta_\xi - i_{X_\xi} d\theta_\nu)(p) \in A^*_\nu G = T_p P^0. \text{ We get hence }
\]

\[
[(X_\nu, \theta_\nu), (X_\xi, \theta_\xi)](p) = ([\tilde{X}_\nu, \tilde{X}_\xi], \mathcal{L}_{X_\nu} \theta_\xi - i_{X_\xi} d\theta_\nu)(p) = ([0, \tilde{X}_\nu]|_p, 0_p) = (0_p, 0_p).
\]

This shows that the bracket on $\Gamma(\mathbb{A}(\mathbb{D}_G))$ is well-defined. It is antisymmetric because the Courant-Dorfman bracket on sections of $\mathbb{D}_G$ is antisymmetric.

If $\mathbb{D}_G$ is closed, then for all star sections $\xi, \eta \in \Gamma(\mathbb{D}_G)$, the bracket $[\xi, \eta]$ is also a section of $\mathbb{D}_G$ and its restriction to $P$ is a section of $\mathbb{A}(\mathbb{D}_G)$ since it is a section of $TP \oplus A^* G$. The Jacobi identity is satisfied by $[\cdot, \cdot, \cdot]$, because the Courant-Dorfman bracket on sections of $\mathbb{D}_G$ satisfies the Jacobi identity. For any $\tilde{\xi}, \tilde{\eta} \in \Gamma(\mathbb{A}(\mathbb{D}_G))$ and $f \in C^\infty(P)$, we have

\[
a_* [\tilde{\xi}, \tilde{\eta}]_* = [\tilde{X}_\xi, \tilde{X}_\eta] = [a_*(\tilde{\xi}), a_*(\tilde{\eta})]
\]

and

\[
[\tilde{\xi}, f \cdot \tilde{\eta}]_* (p) = [[X_\xi, \theta_\xi], (s^* f)(X_\eta, \theta_\eta)](p) = X_\xi (s^* f)(X_\eta, \theta_\eta)(p) + (s^* f) [[X_\xi, \theta_\xi], (X_\eta, \theta_\eta)](p) = a_*(\tilde{\xi})(f)(p) \cdot (\tilde{X}_\eta, \tilde{\theta}_\eta)(p) + f(p) \cdot [\tilde{\xi}, \tilde{\eta}]_* (p)
\]

for all $p \in P$. \[\square\]

Let $G \rightarrow P$ be a Lie groupoid, $TG \rightarrow TP$ its tangent prolongation and $(A \rightarrow P, a, [\cdot, \cdot, \cdot]_a)$ a Lie algebroid over $P$. Let $\Omega$ be a smooth manifold. The quadruple $(\Omega; G, A; P)$ is a $\mathcal{L}A$-groupoid if $\Omega$ has both a Lie groupoid structure over $A$ and a Lie algebroid structure over $G$ such that the two structures on $\Omega$ commute in the sense that the maps defining the groupoid structure are all Lie algebroid morphisms. (The bracket on sections of $\mathbb{A}(\mathbb{D}_G)$ can be defined in the same manner with the target map, and the fact that the multiplication in $T^* G \oplus TG$ is a Lie algebroid morphism is shown in 27.) The double source map $(\tilde{q}, \tilde{s}) : \Omega \rightarrow G \oplus A$ has furthermore to be a surjective submersion.

Recall from 5 that if $\mathbb{D}_G$ is closed, then $\mathbb{D}_G \rightarrow G$ has the structure of a Lie algebroid with the Courant-Dorfman bracket and the projection on $TM$ as anchor. Thus, the previous theorem shows that
the quadruple \((D_G; G, \mathfrak{A}(D_G); P)\) is a \(\mathcal{L}A\)-groupoid (see also [27]):

We recover hence our star sections of \(D_G\) as the star sections of the \(\mathcal{L}A\)-groupoid \((D_G; G, \mathfrak{A}(D_G); P)\) as in [21]. It is shown in [21] (see also [20]), that the bracket of two star sections is again a star section. Here, we have shown this fact in Proposition 3.23 and get as a consequence the fact that \(\mathfrak{A}(D_G)\) has the structure of a Lie algebroid over \(P\).

The next interesting object in [21] is the core \(K\) of \(\Omega\). It is defined as the pullback vector bundle across \(\epsilon : P \rightarrow G\) of the kernel \(\ker(\mathfrak{g} : \Omega 
abla A)\). Hence, it is here exactly the vector bundle \(P^s(D_G)\) over \(P\). It comes equipped with the vector bundle morphisms \(\delta_{\mathfrak{A}(D_G)} : P^s(D_G) \rightarrow \mathfrak{A}(D_G)\), \((v_p, \alpha_p) \mapsto \mathfrak{T}t(v_p, \alpha_p)\) and \(\delta_{\mathcal{A}G} : P^s(D_G) \rightarrow \mathcal{A}G\), \((v_p, \alpha_p) \mapsto v_p\). We have then \(\tilde{a} \circ \delta_{\mathcal{A}G} = a \circ \delta_{\mathfrak{A}(D_G)} = : k\). Furthermore, there is an induced bracket \([\cdot, \cdot]_{\mathcal{A}G}\) on sections of \(P^s(D_G)\) such that \((P^s(D_G), [\cdot, \cdot]_{\mathcal{A}G}, k)\) is a Lie algebroid over \(P\). We prove this fact for our special situation in the following proposition.

Recall that if \((v_p, \alpha_p), p \in P\), is an element of \(P^s_p(D_G)\), then \(\alpha_p\) can be written \((T_p t)^* \beta_p\) with some \(\beta_p \in T_p^p G\). Furthermore, if \(\sigma\) is a section of \(P(D_G) \subseteq (T^s G \oplus (T^s G)^\circ)|_P\), then \(\sigma'\) defined by \(\sigma'(g) = \sigma(t(g)) \ast (0_g, 0_g)\) for all \(g \in G\) is a section of \(D_G \cap \ker \mathfrak{T}s\) by Lemma 3.13 and Remark 3.15. We write \(\sigma' = (X^r, t^* \alpha_{\sigma})\) with some \(X_{\sigma} \in \Gamma(\mathcal{A}G)\) and \(\alpha_{\sigma} \in \Omega^1(P)\).

**Proposition 3.25** Let \((G \Rightarrow P, D_G)\) be a Lie groupoid. Define \([\cdot, \cdot]_{\mathcal{A}G} : \Gamma(P^s(D_G)) \times \Gamma(P^s(D_G)) \rightarrow \Gamma((\ker \mathfrak{T}s)|_P)\) by

\[
([\sigma, \tau]_{P(D_G)})^r = [\sigma', \tau']
\]

for all sections \(\sigma, \tau \in \Gamma(P^s(D_G))\), i.e.,

\[
[\sigma, \tau]_{\mathcal{A}G} = \left([X^r_{\sigma}, X^r_{\tau}]_{\mathcal{A}G}, \left(\mathfrak{T}^*(\mathcal{L}_{\mathfrak{A}G} \alpha_{\tau} - \mathfrak{i}_{\mathfrak{A}G} \mathfrak{d}\alpha_{\sigma})\right)|_P\right).
\]

If \(D_G\) is closed, this bracket has image in \(\Gamma(P^s(D_G))\) and \(P^s(D_G)\) has the structure of a Lie algebroid over \(P\) with the anchor map \(k\) defined by \(k(v_p, \alpha_p) = T_p t v_p\) for all \((v_p, \alpha_p) \in P^s_p(D_G), p \in P\).

Note that this bracket on \(P^s(D_G)\) is the restriction of \(P^s(D_G)\) of a bracket defined in the same manner on the sections of \((\ker \mathfrak{T}s)|_P\). Note also that, if \(D_G\) is closed, the space \(P^s(D_G)\) has in the same manner the structure of an algebroid over \(P\).

**Proof:** Choose \(\sigma, \tau \in \Gamma(P^s(D_G))\) and assume that \(D_G\) is closed. The bracket

\[
[\sigma', \tau'] = [(X^r_{\sigma}, t^* \alpha_{\sigma}), (X^r_{\tau}, t^* \alpha_{\tau})]
\]

is then itself a section of \(D_G\). The identity

\[
[(X^r_{\sigma}, t^* \alpha_{\sigma}), (X^r_{\tau}, t^* \alpha_{\tau})] = \left(\left([X^r_{\sigma}, X^r_{\tau}]_{\mathcal{A}G}\right)^r, \mathfrak{T}^*(\mathcal{L}_{\mathfrak{A}G} \alpha_{\tau} - \mathfrak{i}_{\mathfrak{A}G} \mathfrak{d}\alpha_{\sigma})\right)
\]

shows hence that \([\sigma', \tau'] \in \Gamma(D_G \cap \ker \mathfrak{T}s)\) is right invariant and consequently

\[
[\sigma, \tau]_{P(D_G)} = \left([X^r_{\sigma}, X^r_{\tau}]_{\mathcal{A}G}, (X^r_{\tau}, t^* \alpha_{\tau})\right)|_P \in \Gamma(P^s(D_G)).
\]

The bracket \([\cdot, \cdot]_{\mathcal{A}G}\) satisfies then the Jacobi identity because the Courant bracket on sections of \(D_G\) satisfies it. The Leibniz rule is easy to check. □
As in [21], we have thus four Lie algebroids over $P$:

$$
\begin{array}{c}
\delta_{\mathfrak{a}_G} \\
\delta_{\mathfrak{a}_G} \\
a_* \\
AG \\
TP
\end{array}
\quad
\begin{array}{c}
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G)
\end{array}
\quad
\begin{array}{c}
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G)
\end{array}
\quad
\begin{array}{c}
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G) \\
\mathfrak{a}(D_G)
\end{array}
$$

The anchors $\mathfrak{a}$, $a_*$ and the map $\delta_{\mathfrak{a}_G}$ are obviously Lie algebroid morphisms and the theory in [20], [21] yields that $\delta_{\mathfrak{a}_G}$ is also a Lie algebroid morphism.

Next, we compute the Lie algebroid $\mathfrak{a}(D_G) \to P$ for our three “standard” examples.

**Example 3.26** Let $(G := P, \Sigma_G)$ be a Poisson groupoid and $D_{\Sigma_G}$ the graph of the vector bundle homomorphism $\pi^*_G : T^* G \to TG$ associated to $\pi_G$. The pair $(G := P, D_{\Sigma_G})$ is a closed Dirac groupoid. The set of units $\mathfrak{a}(D_{\Sigma_G})$ of $D_{\Sigma_G}$ equals here $\text{Graph} \left( \pi^*_G \big|_{A^* G} : A^* G \to TP \right)$ and is hence isomorphic to $A^* G$ as a vector bundle, via the maps $\Theta := \text{pr}_{A^* G} : \mathfrak{a}(D_{\Sigma_G}) \to A^* G$ and $\Theta^{-1} = \left( \pi^*_G \big|_{A^* G}, \text{Id}_{A^* G} \right) : A^* G \to \mathfrak{a}(D_{\Sigma_G})$ over $\text{Id}_P$.

The vector bundle $A^* G$ has the structure of a Lie algebroid over $P$ with anchor map given by $A^* G \to TP$, $\alpha_p := \pi^*_G(\alpha_p) \in TP$ and with bracket the restriction to $A^* G$ of the bracket $[,]_{\Sigma_G}$ on $\Omega^1(G)$ defined by $\pi_G$: 
\[
[\alpha, \beta]_{\Sigma_G} = L_{\pi^*_G(\alpha)} \beta - L_{\pi^*_G(\beta)} \alpha - d \pi^*_G(\alpha, \beta)
\] for all $\alpha, \beta \in \Omega^1(G)$ [4]. Thus, $A^* G$ with this Lie algebroid structure and $\mathfrak{a}(D_{\Sigma_G})$ are isomorphic as Lie algebroids via $\Theta$ and $\Theta^{-1}$. \hfill \Box

**Example 3.27** Let $\omega_G$ be a multiplicative closed 2-form on a Lie groupoid $G := P$ and consider the associated multiplicative Dirac structure $D_{\omega_G}$ on $G$. The Lie algebroid $\mathfrak{a}(D_{\omega_G}) \to P$ is here equal to

$$
\mathfrak{a}(D_{\omega_G}) = \text{Graph} \left( \omega^h_G \big|_{TP} : TP \to A^* G \right)
$$

with anchor map $a_* : \mathfrak{a}(D_{\omega_G}) \to TP$ given by $a_*(v, \omega^h_G(v)) = v$. The bracket of two sections $(X, \omega^h_G(X)), (Y, \omega^h_G(Y)) \in \Gamma(\mathfrak{a}(D_{\omega_G}))$ is simply given by

$$
\left(\left[ X, \omega^h_G(X) \right], \left[ Y, \omega^h_G(Y) \right] \right) = \left( [X, Y], \omega^h_G([X, Y]) \right).
$$

The Lie algebroid $\mathfrak{a}(D_{\omega_G})$ is obviously isomorphic to the tangent Lie algebroid $TP \to P$ of $P$, via the maps $\text{pr}_{TP} : \mathfrak{a}(D_{\omega_G}) \to TP$ (the anchor map) and $\left( \text{Id}_{TP}, \omega^h_G \big|_{TP} \right) : TP \to \mathfrak{a}(D_{\omega_G})$.

Note that if $(G := P, \omega)$ is a Lie groupoid endowed with a multiplicative closed 2-form, then $\mathfrak{a}(D_\omega)$ is the graph of the dual of the map $\sigma_\omega : AG \to T^* P$ in [1]. \hfill \Box

**Example 3.28** Let $(M, D_M)$ be a smooth Dirac manifold and $(M \times M \rightrightarrows M, D_M \oplus D_M)$ the associated pair Dirac groupoid as in Example 3.3. The set $\mathfrak{a}(D_M \oplus D_M)$ is defined here by

$$
\mathfrak{a}(D_M \oplus D_M)(m, m) = \mathfrak{t}((D_M \oplus D_M)(m, m)) = \{ (v_m, v_m, \alpha_m, -\alpha_m) \mid (v_m, \alpha_m) \in D_M(m) \}
$$

for all $m \in M$. Hence, we have an isomorphism $\mathfrak{a}(D_M \oplus D_M) \to D_M$ over the map $\text{pr}_1 : \Delta_M \to M$. Sections of $\mathfrak{a}(D_M \oplus D_M)$ are exactly the sections $(X, X, \alpha, -\alpha)_{\Delta_M}$ for sections $(X, \alpha) \in D_M$. The section $(X, X, \alpha, -\alpha)$ of $D_M \oplus D_M$ defined on $M \times M$ by $(X, X, \alpha, -\alpha)(m, n) = (X(m), X(n), \alpha(m), -\alpha(n))$ for all $(m, n) \in M \times M$ is then easily shown to be star sections $s$-related to $(X, X, \alpha, -\alpha)_{\Delta_M}$. Using this, one can check that, if $(M, D_M)$ is closed, the Lie algebroid structure on $\mathfrak{a}(D_M \oplus D_M)$ corresponds to the Lie algebroid structure on $(M, D_M)$ (see [5]). \hfill \Box
3.4 Integrability criterion

The main theorem of this section shows that the integrability of a Dirac groupoid is completely encoded in its square of Lie algebroids. The proof, which is very technical, will be only summarized here. It can be found with more details in [9]. We begin by showing a derivation formula for star sections, that will also be useful later.

**Theorem 3.29** Let \((G \rightrightarrows P, \sD_G)\) be a Dirac groupoid, \(\xi \sim \bar{\xi}\) a star section of \(\sD_G\) and \(Z \in \Gamma(AG)\). Then the derivative \(\sL_{Z!}(X_t, \theta_{\xi})\) can be written as a sum

\[
\sL_{Z!}(X_t, \theta_{\xi}) = (X_{\xi \circ}, \theta_{\xi \circ}) + (Y_{\xi, Z}^l, s^* \alpha_{\xi, Z}) =: \sL_{Z!} \xi + (\sigma_{\xi, Z})^l
\]

with \(\xi, Z \in \Gamma(AG)\), \(\alpha_{\xi, Z} \in \Omega^1(P)\) and \(\sL_{Z!} \xi := (X_{\xi \circ}, \theta_{\xi \circ})\) a star section of \(\sD_G\). We have \(\sL_{Z!} \xi \sim \bar{s}\) \(\mathbb{T}_s'(\sL_{Z!}(X_t, \theta_{\xi})(s(g)))\) in the sense that

\[
\mathbb{T}_s'(\sL_{Z!}(X_t, \theta_{\xi})(g)) = \mathbb{T}_s'(\sL_{Z!}(X_t, \theta_{\xi})(s(g)))
\]

for all \(g \in G\).

In addition, if \((X_{\nu}, \nu) \sim s (0, 0)\), then \(\sL_{Z!}(X_{\nu}, \theta_{\nu}) \in \Gamma(\sD_G \cap \ker \mathbb{T}_s)\). In particular, its restriction to \(P\) is a section of \(\mathbb{P}^s(\sD_G)\).

The following lemma will be useful for the proof of this theorem. The proof is easy and shall be omitted.

**Lemma 3.30** Let \(G \rightrightarrows P\) be a Lie groupoid. Choose \((X, (t^* \alpha)|_P) \in \Gamma((\ker \mathbb{T}_s)|_P)\) and \(Z \in \Gamma(AG)\). Then we have

\[
\sL_{Z!}(X_t, t^* \alpha) = 0.
\]

**Proof (of Theorem 3.29):** Note first that, in general \(\sL_{Z!}(X_t, \theta_{\xi})\) is a section of \(\sD_G + \ker \mathbb{T}_s\): for all \(\sigma^r = (X^r_{\sigma}, t^* \alpha_{\sigma}) \in \Gamma(\sD_G + \ker \mathbb{T}_s)\), we have

\[
\langle \sL_{Z!}(X_t, \theta_{\xi}), \sigma^r \rangle = \sL_{Z!}((X_{\xi}, \theta_{\xi}) + (X^r_{\sigma}, t^* \alpha_{\sigma})) = \sL_{Z!}(X^r_{\sigma}, t^* \alpha_{\sigma}) = 0
\]

using \(\sD_G = \sD_G^\perp\) and Lemma 3.30. This leads to \(\sL_{Z!}(X_t, \theta_{\xi}) \in \Gamma((\sD_G \cap \ker \mathbb{T}_s)^\perp) = \Gamma(\sD_G + \ker \mathbb{T}_s)\).

Choose \(g \in G\). Then

\[
T_g s(\sL_{Z!}(X_t)(g)) = T_g s \left[ Z^l, X_t \right] (g) = [a(Z), X_{\xi}] \ (s(g))
\]

and for any \(W \in \Gamma(AG)\)

\[
s(\sL_{Z!}(\theta_{\xi})(W(s(g)))) = (\sL_{Z!}(\theta_{\xi}))(W^l)(g) = (Z^l(s^*(\theta_{\xi}(W)) - s^*(\theta_{\xi}([Z, W]_\text{AG}))))(g) = ([a(Z)](Z_{\xi}^l)(W) - \theta_{\xi}([Z, W]_\text{AG}))(s(g)).
\]

This shows that \(\mathbb{T}_s(\sL_{Z!}(X_t, \theta_{\xi}))(g)\) depends only on the values of \(Z, X_{\xi}, \theta_{\bar{\xi}}\) at \(s(g)\).

Set

\[
(Y_{\xi, Z}^l, s^* \alpha_{\xi, Z})(g) := (0_g, 0_g) * \left( \langle \sL_{Z!}(X_t, \sL_{Z!}(\theta_{\xi})) s(g) \rangle - \mathbb{T}_s'(\sL_{Z!}(X_t, \sL_{Z!}(\theta_{\xi}))(s(g))) \right)
\]

and

\[
(X_{\xi \circ}, \theta_{\xi \circ})(g) := (\sL_{Z!}(X_t, \sL_{Z!}(\theta_{\xi}))(g) - (Y_{\xi, Z}^l, s^* \alpha_{\xi, Z})(g))
\]

for all \(g \in G\). Then \((Y_{\xi, Z}^l, s^* \alpha_{\xi, Z})\) is a smooth section of \(\ker \mathbb{T}_s\) satisfying

\[
\mathbb{T}_s\left( \left( Y_{\xi, Z}^l, s^* \alpha_{\xi, Z} \right)(g) \right) = \mathbb{T}_s\left( \left( Y_{\xi, Z}^l, s^* \alpha_{\xi, Z} \right)(s(g)) \right)
\]
by construction for all \( g \in G \) and \( \mathcal{L}_Z \xi = (X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi}) \) is consequently a star section if we can show that
\[
\mathcal{T}_s ((X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi})(g)) = (X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi})(s(g))
\]
for all \( g \in G \). Using the computations above for \( \mathcal{T}_s (\mathcal{L}_Z \xi, \theta_{\xi}) \), it is easy to see that, for \( g \in G \), we have
\[
\mathcal{T}_s ((X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi})(g)) = \mathcal{T} \mathcal{T}_s (\mathcal{L}_Z \xi, \mathcal{L}_Z \xi)(s(g)),
\]
which, by definition, is equal to \( (X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi})(s(g)) \).

It remains hence to show that \( (X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi}) \) is a section of \( \mathcal{D}_G \). The equality
\[
\langle \sigma^*, (X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi}) \rangle = \langle \sigma^*, (\mathcal{L}_Z \xi \mathcal{X}_\xi, \mathcal{L}_Z \xi \theta_{\xi}) \rangle - \langle \sigma^*, (\mathcal{L}_Z \xi \xi^\alpha \xi, \xi) \rangle = 0 - 0
\]
holds for all \( \sigma^* \in \Gamma(\ker \mathcal{T}_s \cap \mathcal{D}_G) \), and for all star sections \( (X_\eta, \theta_\eta) \) of \( \mathcal{D}_G \), we compute
\[
\langle (X_\eta, \theta_\eta), (X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi}) \rangle (g) = \langle (X_\eta, \theta_\eta)(g), (\mathcal{L}_Z \xi \xi, \mathcal{L}_Z \xi \theta_{\xi})(g) - (\mathcal{L}_Z \xi \xi^\alpha \xi, \xi)(g) \rangle
\]
\[
= \langle (X_\eta, \theta_\eta)(g), (\mathcal{L}_Z \xi \xi, \mathcal{L}_Z \xi \theta_{\xi})(g) \rangle - \langle (X_\eta, \theta_\eta)(g) \ast (X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi})(s(g)),
\]
\[
(0, \theta_\eta) \ast ((\mathcal{L}_Z \xi \xi, \mathcal{L}_Z \xi \theta_{\xi})(s(g)) - \mathcal{T} \mathcal{T}(\mathcal{L}_Z \xi \xi, \mathcal{L}_Z \xi \theta_{\xi})(s(g)))
\]
\[
= \langle (X_\eta, \theta_\eta)(s(g)), (\mathcal{L}_Z \xi \xi, \mathcal{L}_Z \xi \theta_{\xi})(s(g)) \rangle - \langle (X_\eta, \theta_\eta)(s(g)), (\mathcal{L}_Z \xi \xi, \mathcal{L}_Z \xi \theta_{\xi})(s(g)) \rangle - \mathcal{T} \mathcal{T}(\mathcal{L}_Z \xi \xi, \mathcal{L}_Z \xi \theta_{\xi})(s(g)))
\]
\[
= \langle (X_\eta, \theta_\eta)(s(g)), (\mathcal{L}_Z \xi \xi, \mathcal{L}_Z \xi \theta_{\xi})(s(g)) \rangle = 0
\]
since \( \mathcal{T} \mathcal{P} \mathcal{L} \mathcal{A}^* \mathcal{G} = (\mathcal{T} \mathcal{P} \mathcal{L} \mathcal{A}^* \mathcal{G})^{-1} \). Thus, we have shown that \( (X_{\xi \cdot \xi}, \theta_{\xi \cdot \xi}) \in \Gamma(\ker \mathcal{T}_s \cap \mathcal{D}_G) \).

For the proof of the second statement, assume that \( (X_\nu, \theta_\nu) \) is a smooth section of \( \mathcal{D}_G \) that is a star section \( s \)-related to \( (X_\nu, \theta_\nu) \) = (0, 0). For all left invariant sections \( (Y^l, s^* \gamma) \) of \( \mathcal{K} \mathcal{T}_s \), we have
\[
\langle \mathcal{L}_Z (X_\nu, \theta_\nu), (Y^l, s^* \gamma) \rangle = \mathcal{L}_Z (\langle (X_\nu, \theta_\nu), (Y^l, s^* \gamma) \rangle - \langle (X_\nu, \theta_\nu), \mathcal{L}_Z (Y^l, s^* \gamma) \rangle
\]
\[
= \mathcal{L}_Z (s^*(\gamma(X_\nu) + \theta_\nu(Y))) - s^*(\theta_\nu([Z, Y]_{AG}) + (\mathcal{L}_s \gamma)(X_\nu)) = 0
\]
since \( X_\nu = 0 \) and \( \theta_\nu = 0 \). Choose any star section \( \xi = (X_\xi, \theta_\xi) \) of \( \mathcal{D}_G \). Then
\[
\langle \mathcal{L}_Z (X_\nu, \theta_\nu), (X_\xi, \theta_\xi) \rangle = \mathcal{L}_Z (\langle (X_\nu, \theta_\nu), (X_\xi, \theta_\xi) \rangle - \langle (X_\nu, \theta_\nu), \mathcal{L}_Z (X_\xi, \theta_\xi) \rangle = 0
\]
since \( \mathcal{L}_Z (X_\xi, \theta_\xi) \in \Gamma((\mathcal{D}_G \cap \ker \mathcal{T}_s)^{-1}) \). We have also \( \mathcal{L}_Z (X_\nu, \theta_\nu) \in \Gamma((\mathcal{D}_G \cap \ker \mathcal{T}_s)^{-1}) \) and, because the star sections of \( \mathcal{D}_G \) and the sections of \( \mathcal{D}_G \cap \ker \mathcal{T}_s \) span \( \mathcal{D}_G \), this shows that \( \mathcal{L}_Z (X_\nu, \theta_\nu) \in \Gamma((\mathcal{D}_G + \ker \mathcal{T}_s)^{-1}) = \Gamma(\mathcal{D}_G \cap \ker \mathcal{T}_s) \).

We have also for any star section \( \xi \) of \( \mathcal{D}_G \), any section \( \sigma \in \Gamma(\mathcal{P}(\mathcal{D}_G)) \) and \( Z \in \Gamma(\mathcal{AG}) \):
\[
\frac{d}{dt} \langle R_{\mathcal{L}_Z \xi \xi} \xi, \sigma^* \rangle (g) = \frac{d}{dt} \langle \xi, R_{\mathcal{L}_Z \xi \xi} \sigma^* \rangle (R_{\mathcal{L}_Z \xi \xi} \xi)(g) = \frac{d}{dt} \langle \xi, \sigma^* \rangle (R_{\mathcal{L}_Z \xi \xi} \xi)(g) = 0
\]
since \( R_K \sigma^* = \sigma^* \) for all bisections \( K \in \mathcal{B}(G) \). Hence, we get
\[
\langle R_{\mathcal{L}_Z \xi \xi} \xi, \sigma^* \rangle (g) = \langle R_{\mathcal{L}_Z \xi \xi} \xi, \sigma^* \rangle (g) = \langle \xi, \sigma^* \rangle (g) = 0
\]
for all \( g \in G, \sigma \in \Gamma(I^2(D_G)) \) and \( t \in \mathbb{R} \) where this makes sense and we find consequently \( R^*_t \Exp(tZ) \xi \in \Gamma(D_G + \ker Tt) \). If the s-fibers of \( G \Rightarrow P \) are connected, the set of bisections of \( G \) is generated by the bisections \( \Exp(tZ), t \in \mathbb{R} \) small enough and \( Z \in \Gamma(AG) \) (see [24]). We know then that \( R^*_K \xi \in \Gamma(D_G + \ker Tt) \) for any bisection \( K \in \mathcal{B}(G) \).

We denote here by \( S(D_G) \) the set of star sections of \( D_G \). Note that \( D_G \) is spanned on \( G \setminus P \) by the values of the elements of \( S(D_G) \), since \( D_G \cap \ker Ts \) is spanned there by the values of the star sections that vanish on \( P \).

Consider the vector bundle \( E := D_G/(D_G \cap \ker Tt) \simeq (D_G + \ker Tt)/\ker Tt \) over \( G \). Since the fiber \( D_G(g) \) over \( g \) of the Dirac structure is spanned for each \( g \in G \setminus P \) by the values of the elements of \( S(D_G) \) at \( g \) and, for each \( p \in P \), the vector space \( E(p) \) is spanned by the classes \( \xi(p) + I^t_p(D_G) \) for all star sections \( \xi \) of \( D_G \), we find that the vector bundle \( E \) is spanned at each point \( g \in G \) by the elements \( \xi(g) + (D_G \cap \ker Tt)(g) \) for all \( \xi \in S(D_G) \). To simplify the notation, we write \( \hat{\xi} \) for the image of the section \( \xi \in S(D_G) \) in \( E \), and \( \hat{S(D_G)} \) for the set of these special sections of \( E \). By the considerations above, for any \( K \in \mathcal{B}(G) \) and \( \xi \in S(D_G) \), we can define \( R^*_K \hat{\xi} := R^*_K \hat{\xi} \). If we set in the same manner

\[
\mathcal{L}_{Zt} \hat{\xi} = \mathcal{L}_{Zt} \hat{\xi} + \mathcal{l}_{\xi} \mathcal{Z} = \mathcal{L}_{Zt} \hat{\xi} \in \hat{S(D_G)}
\]

for all \( \xi \in S(D_G) \) and \( Z \in \Gamma(AG) \), we find for any \( g \in G \):

\[
\frac{d}{dt} \bigg|_{t=0} \left( R^*_t \Exp(tZ) \hat{\xi} \right)(g) = \frac{d}{dt} \bigg|_{t=0} R^*_t \Exp(tZ) \xi(g)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} R^*_t \Exp(tZ) \xi(g)
\]

\[
= \mathcal{L}_{Z} \hat{\xi}(g) = \mathcal{L}_{Z} \xi(g) = L^t \xi(g).
\]

Assume here that the bracket on sections of \( \mathfrak{g}(D_G) \) induced by \( D_G \) as in Theorem 3.24 has image in \( \Gamma(\mathfrak{g}(D_G)) \). Recall from [5] that the integrability of a Dirac structure is measured by the Courant 3-tensor \( T \) defined on sections of \( D_G \) by

\[
T(\xi, \eta, \zeta) = ([\xi, \eta], \zeta)
\]

for all \( \xi, \eta, \zeta \in \Gamma(D_G) \). We show that \( T \) induces a tensor \( \hat{T} \in \Gamma \left( \wedge^3 \mathfrak{e}^* \right) \). By the considerations above, we can define a 3-tensor \( \hat{T} \) by its values on the elements of \( \hat{S(D_G)} \). Set

\[
\hat{T}(\hat{\xi}, \hat{\eta}, \hat{\zeta}) = T(\xi, \eta, \zeta)
\]

for all \( \xi, \eta, \zeta \in S(D_G) \). To see that \( \hat{T} \) is well-defined, choose \( g \in G \) and \( \sigma_g \in (D_G \cap \ker Tt)(g) \). Then there exists \( \sigma \in \Gamma(I^1(D_G)) \) such that \( I^t(g) = \sigma_g \). Then, since \( [\xi, \eta] \) is a star section \( s \)-related to \( [\hat{\xi}, \hat{\eta}]^s \), we have:

\[
T(\xi(g), \eta(g), \sigma_g) = T(\xi, \eta, \sigma^t)(g) = \langle [\xi, \eta], \sigma^t \rangle(g) = \langle [\hat{\xi}, \hat{\eta}]^s, \sigma \rangle(s(g)) = 0
\]

since \( [\hat{\xi}, \hat{\eta}]^s(s(g)) \in D_G(s(g)) \) by hypothesis.

For any bisection \( K \in \mathcal{B}(G) \), we can define the 3-tensor \( R^*_K \hat{T} \) by

\[
\left( R^*_K \hat{T} \right)(\hat{\xi}, \hat{\eta}, \hat{\zeta}) = R^*_K \hat{T} \left( R^*_K \hat{\xi}, R^*_K \hat{\eta}, R^*_K \hat{\zeta} \right)
\]

for all \( \xi, \eta, \zeta \in S(D_G) \). For \( Z \in \Gamma(AG) \), we can thus define \( \mathcal{L}_{Zt} \hat{T} \) by

\[
\mathcal{L}_{Zt} \hat{T} \left( \xi, \eta, \zeta \right) = \left. \frac{d}{dt} \left( R^*_t \Exp(tZ) \hat{T} \right) \right|_{t=0}
\]
We have
\[ R^*_{\Exp(tZ)} \left( \tilde{T} \left( \tilde{\xi}, \tilde{\eta}, \tilde{\zeta} \right) \right) = \left( R^*_{\Exp(tZ)} \tilde{T} \right) \left( R^*_{\Exp(tZ)} \tilde{\xi}, R^*_{\Exp(tZ)} \tilde{\eta}, R^*_{\Exp(tZ)} \tilde{\zeta} \right) \]
for all \( \xi, \eta, \zeta \in \mathcal{S}(D_G) \), which yields easily
\[ \mathcal{L}_{Z_1} (T (\tilde{\xi}, \tilde{\eta}, \tilde{\zeta})) = \left( \mathcal{L}_{Z_1} \tilde{T} \right) \left( \tilde{\xi}, \tilde{\eta}, \tilde{\zeta} \right) + T (\mathcal{L}_{Z_1} \xi, \mathcal{L}_{Z_1} \eta, \mathcal{L}_{Z_1} \zeta) + T (\xi, \eta, \mathcal{L}_{Z_1} \zeta). \] (12)

We will need the following lemma for the proof of the main result of this subsection.

**Lemma 3.31** Let \((G \xrightarrow{\pi} P, D_G)\) be a Dirac groupoid. Consider three star sections \( \xi \sim_s \tilde{\xi}, \eta \sim_s \tilde{\eta} \) and \( \zeta \sim_s \tilde{\zeta} \) of \( D_G \). Then, if \([\tilde{\xi}, \tilde{\eta}]_s \in \Gamma(\mathfrak{A}(D_G))\), we have
\[ [\tilde{\zeta}, [\tilde{\xi}, \tilde{\eta}]_s]_s = [\zeta, [\xi, \eta]]_P \]
where the bracket on the right-hand side is the Courant-Dorfman bracket on sections of \( TG \oplus T^*G \).

**Proof:** If \( \tau := [\tilde{\xi}, \tilde{\eta}]_s \in \Gamma(\mathfrak{A}(D_G)) \), then there exists a star section \( \tau \) of \( D_G \) such that \( \tau \sim_s \tilde{\tau} \). Since \([\xi, \eta]_P = [\tilde{\xi}, \tilde{\eta}]_s = [\tilde{\xi}, \tilde{\eta}]_s = \tau_P \), and \( \mathcal{T}s([\xi, \eta](g)) = [\tilde{\xi}, \tilde{\eta}](s(g)) = \mathcal{T}s(\tau(g)) \) for all \( g \in G \), there exists then a section \( \chi \) of \( \ker \mathcal{T}s \) that is vanishing on \( P \) such that \( \tilde{\tau} - [\tilde{\xi}, \tilde{\eta}]_s = \chi \). Choose \( p \in P \). Then, on a neighborhood \( U \) of \( p \) in \( G \), the section \( \chi \) of \( \ker \mathcal{T}s \) can be written \( \chi = \sum_{i=1}^n \sigma_i \) with functions \( f_1, \ldots, f_n \in \mathcal{C}^\infty(U) \) that vanish on \( P \cap U \) and basis sections \( \sigma_1, \ldots, \sigma_n \) of \( \ker \mathcal{T}s \) on \( U \cap P \). We have then
\[ [\tilde{\zeta}, [\tilde{\xi}, \tilde{\eta}]_s]_s = [\zeta, [\xi, \eta]_P]_P = [\zeta, [\xi, \eta]_P + \chi]_P = [\zeta, [\xi, \eta]]_P + [\zeta, \chi]_P. \]

If we write \( \zeta = (X_\zeta, \omega_\zeta) \), we can compute using (5)
\[ [\zeta, \chi] = \sum_{i=1}^n (f_i [\zeta, \sigma_i^\tau] + X_\zeta(f_i) \sigma_i^\tau). \]
Since \( X_\zeta \) is tangent to \( P \) on \( P \) and \( f_1, \ldots, f_n \) vanish on \( P \), we have \( X_\zeta(f_i)|_P = 0 \). This shows that \([\zeta, \chi](p) = 0 \) for all \( p \in P \). Hence, we have \([\tilde{\zeta}, [\tilde{\xi}, \tilde{\eta}]_s](p) = [\zeta, [\xi, \eta]]_P(p) \).

Now we can state the main theorem of this section.

**Theorem 3.32** Let \((G \xrightarrow{\pi} P, D_G)\) be a Dirac groupoid. Assume that \( G \xrightarrow{\pi} P \) is t-connected. Then the Dirac structure \( D_G \) is closed if and only if:

1. the induced bracket as in Theorem 3.24 has image in \( \Gamma(\mathfrak{A}(D_G)) \) and satisfies the Jacobi identity
\[ [\tilde{\zeta}, [\tilde{\xi}, \tilde{\eta}]_s]_s + [\tilde{\eta}, [\tilde{\xi}, \tilde{\zeta}]_s]_s + [\tilde{\xi}, [\tilde{\eta}, \tilde{\zeta}]_s]_s = 0 \]
for all \( \tilde{\xi}, \tilde{\eta}, \tilde{\zeta} \in \Gamma(\mathfrak{A}(D_G)) \)

and

2. the induced bracket on sections of \( I^s(D_G) \) as in Proposition 3.25 has image in \( \Gamma(I^s(D_G)) \).

**Example 3.33**

1. In the Poisson case, the integrability is ensured by the fact that \( A^G \) is an algebroid, since it defines then a linear Poisson structure \( \pi_A \) on \( AG \) such that \( \pi_A^* : T^* A \rightarrow TA \) is a Lie algebroid morphism, that integrates modulo canonical identifications to \( \pi_C^* : T^* G \rightarrow TG \).

2. In the case of a multiplicative 2-form, closedness of the form is ensured by the condition of compatibility of the corresponding IM-2-form with the Lie algebroid bracket \([1]\). This is exactly the same as the condition on \( I^s(D_G) \) to be a Lie algebroid.

3. In the pair Dirac groupoid case, we have seen that \( I^s(D_M \oplus D_M) \) and \( \mathfrak{A}(D_M \oplus D_M) \) are both isomorphic to \( D_M \). We know already that \( D_M \oplus D_M \) is closed if and only if \( D_M \) is. \( \diamond \)
Proof (of Theorem 3.32): We have shown in Theorem 3.24 and Proposition 3.25 that the integrability of $D_G$ implies 1) and 2).

Conversely, assume that 1) and 2) hold. We will show that $D_G$ is closed. First choose $p \in P$. The fiber $D_G(p)$ of $D_G$ over $p$ is spanned by the values of the sections in $I^s(D_G)$ defined at $p$, and the values at $p$ of the star sections of $D_G$. Since the brackets on sections of $I^s(D_G)$ and $\mathfrak{A}(D_G)$ have values in $\Gamma(I^s(D_G))$, and respectively $\Gamma(\mathfrak{A}(D_G))$, we find for all $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(I^s(D_G))$ and $\xi_1, \xi_2, \xi_3 \in \mathcal{S}(D_G)$:

$$
\mathcal{T}(\sigma_1^*, \sigma_2^*, \sigma_3^*)(p) = \langle \langle \sigma_1, \sigma_2 \rangle \rangle(p) = \langle \langle \sigma_1, \sigma_2 \rangle \rangle(p) = 0
$$

Hence, $\mathcal{T}$ vanishes over points in $P$.

Consider the 3-tensor $\mathcal{T}$ induced on the sections of $E = D_G/(D_G \cap \ker \mathcal{T})$ by $T$ and choose $\xi_1, \xi_2, \xi_3 \in \mathcal{S}(D_G)$ and $Z \in \Gamma(AG)$. We show that $(L_{Z^*} \mathcal{T})(\xi_1, \xi_2, \xi_3) = 0$. A long but straightforward computation yields

$$
Z^i(\mathcal{T}(\xi_1, \xi_2, \xi_3)) = T(\mathcal{L}_{Z^*} \mathcal{T})(\xi_1, \xi_2, \xi_3) + s^*(\alpha_{\xi_1, Z}(\bar{X}_{\xi_2}, \bar{X}_{\xi_3})) + s^*(\bar{\omega}_{\xi_1}(Y_{\xi_2}, Y_{\xi_3}, Z))
$$

By [12], this leads to

$$(L_{Z^*} \mathcal{T})(\xi_1, \xi_2, \xi_3)(g) = s^*(\alpha_{\xi_1, Z}(\bar{X}_{\xi_2}, \bar{X}_{\xi_3})) + s^*(\bar{\omega}_{\xi_1}(Y_{\xi_2}, Y_{\xi_3}, Z)) + s^*(\bar{X}_{\xi_1}(\alpha_{\xi_2, Z}(\bar{X}_{\xi_3}))) + s^*(\bar{X}_{\xi_1}(\bar{\omega}_{\xi_2}(Y_{\xi_3}, Z))) + s^*(\bar{\omega}_{\xi_2}(Y_{\xi_1}, \bar{X}_{\xi_2}, \bar{X}_{\xi_3})) + c.p.
$$

for all $g \in G$. But since $\mathcal{T}$ vanishes on the units by hypothesis, we find by [12] that

$$(L_{Z^*} \mathcal{T})(\xi_1, \xi_2, \xi_3)(s(g)) = Z^i(\mathcal{T}(\xi_1, \xi_2, \xi_3))(s(g)).
$$

A quick computation shows that the cotangent part of $[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]]$ is equal to $d(\mathcal{T}(\xi_1, \xi_2, \xi_3))$, see also [5]. Using this and Lemma 3.31 we find finally

$$(L_{Z^*} \mathcal{T})(\xi_1, \xi_2, \xi_3)(g) = Z^i(\mathcal{T}(\xi_1, \xi_2, \xi_3))(s(g)) = 0$$

since by condition 1), $[\cdot, \cdot],*$ satisfies the Jacobi identity.

Hence, we have shown that $L_{Z^*} \mathcal{T} = 0$ for all $Z \in \Gamma(AG)$. This yields that $R^*_{Exp(tZ)} \mathcal{T} = \mathcal{T}$ for all $Z \in \Gamma(AG)$ and $t \in \mathbb{R}$ where this makes sense and hence, since $G$ is t-connected and $\mathcal{T}$ vanishes on the units, we find $\mathcal{T} = 0$. Thus, $T = 0$ on $G$ and the proof is finished.
Remark 3.34 For $Z \in \Gamma(AG)$, define $\nabla_Z : \Gamma(E) \to \Gamma(E)$ by $\nabla_Z \xi = \tilde{L}_Z \xi$ for all $\xi \in \mathcal{S}(D_G)$, and $\nabla_Z \left( \sum_{i=1}^{n} f_i \hat{\xi}_i \right) = \sum_{i=1}^{n} \left( \tilde{Z}(f_i) \hat{\xi}_i + f_i \nabla_Z \hat{\xi}_i \right)$ for all $f_1, \ldots, f_n \in C^\infty(G)$ and $\hat{\xi}_1, \ldots, \hat{\xi}_n \in \mathcal{S}(D_G)$. Then $\nabla_Z$ is a derivative endomorphism of $E$ over $Z^!$. The map $\Gamma(AG) \to \Gamma(D(E))$, $Z \to \nabla_Z$ is a derivative representation of $AG$ on $E$ associated to the action of $AG$ on $s : G \to P$, $Z \in \Gamma(AG) \to Z^!$ (see \cite{10}).  

3.5 The Courant algebroid associated to a closed Dirac groupoid

The dual space of $\mathfrak{A}(D_G)$ can be identified with $P_G|_P/\mathfrak{A}(D_G)^\perp$. Since

$$\mathfrak{A}(D_G)^\perp = D_G|_P + (TP \oplus A^*G) = I^!(D_G) \oplus (TP \oplus A^*G)$$

and

$$P_G|_P = (TP \oplus A^*G) + \ker T|_P,$$

we have

$$\mathfrak{A}(D_G)^\perp \cong \frac{\ker T|_P}{I^!(D_G)}.$$

Since $D_G|_P \subseteq \mathfrak{A}(D_G) \oplus \ker T|_P$, we have $I^!(D_G) \subseteq \mathfrak{A}(D_G) \oplus \ker T|_P$ and the quotient

$$\mathfrak{B}(D_G) := \frac{\mathfrak{A}(D_G) \oplus \ker T|_P}{I^!(D_G)}$$

is a smooth vector bundle over $P$. Consider the map

$$\Psi : \ker T|_P \oplus \mathfrak{A}(D_G) \to \mathfrak{B}(D_G),$$

$$\Psi (\sigma + \tilde{\xi}) = \sigma + \tilde{\xi} + I^!(D_G)$$

for all $\sigma \in \Gamma(\ker T|_P)$ and $\tilde{\xi} \in \Gamma(\mathfrak{A}(D_G))$. If $\Psi(\sigma + \tilde{\xi}) \in I^!(D_G)$, then we have $\sigma + \tilde{\xi} \in \Gamma(D_G|_P)$ and hence $\sigma \in \Gamma(I^!(D_G))$ since $\tilde{\xi} \in \Gamma(D_G|_P)$. This yields $\sigma \in \Gamma(I^!(D_G))$ and the map $\Psi$ factors to a vector bundle homomorphism

$$\tilde{\Psi} : (\mathfrak{A}(D_G))^\ast \oplus \mathfrak{A}(D_G) \to \mathfrak{B}(D_G)$$

over the identity $\text{Id}_P$.

Set $r = \text{rank} I^!(D_G)$, $n = \text{dim} G$. Then we have also $r = \text{rank} I^!(D_G)$ and we can compute $\text{rank} \mathfrak{B}(D_G) = \text{rank}(\ker T|_P) + \text{rank} \mathfrak{A}(D_G) - \text{rank} I^!(D_G) = n + (n - r) - r = 2n - 2r$. We have also rank$(\mathfrak{A}(D_G))^\ast \oplus \mathfrak{A}(D_G)) = n - r + n - r = 2n - 2r$ and since $\tilde{\Psi}$ is surjective, it is hence a vector bundle isomorphism.

Since $(\ker T|_P \oplus \mathfrak{A}(D_G))^\perp = (\ker T|_P \oplus D_G|_P)^\perp = I^!(D_G)$, the bracket $\langle \cdot, \cdot \rangle$ restricts to a non degenerate symmetric bracket on $\mathfrak{B}(D_G)$, that will also be written $\langle \cdot, \cdot \rangle$ in the following.

Recall from Example 3.26 that if $(G \rightleftharpoons P, D_{\pi_G})$ is a Poisson groupoid, the bundle $\mathfrak{A}(D_{\pi_G})$ is equal to Graph$(\pi^\sharp_G|_{A^*G}) \simeq A^*G$, $a_\ast(x) = \pi^\sharp_G(x)$ for all $x \in \Gamma(A^*G)$ and the bracket on sections of $\mathfrak{A}(D_G)$ is the bracket induced by the Poisson structure. In the same manner, we have $(\mathfrak{A}(D_G))^\ast = \ker T|_P / I^!(D_G) = \ker T|_P / \text{Graph} \left( \pi^\sharp_G|_{(\pi^\sharp_p G)^\ast} \right)$ which is isomorphic as a vector bundle to $AG$. The vector bundle $\mathfrak{B}(D_{\pi_G})$ is thus the vector bundle underlying the Courant algebroid associated to $(G \rightleftharpoons P, \pi)$ We will study this example in more detail in Example 3.30 where we will show that $\mathfrak{B}(D_{\pi_G})$ carries a natural Courant algebroid structure that makes it isomorphic as a Courant algebroid to $AG \oplus A^*G$.

We show here that if the Dirac groupoid $(G \rightleftharpoons P, D_G)$ is closed, the vector bundle $\mathfrak{B}(D_G) \to P$ always inherits the structure of a Courant algebroid from the ambient standard Courant algebroid structure of $P_G$.

Because of the special case of Poisson groupoids, we have chosen the notation $\mathfrak{B}(D_G)$: this Courant algebroid will play the role of the “Lie bialgebroid of the Dirac groupoid $(G \rightleftharpoons P, D_G)$".
Theorem 3.35 Let $(G \Rightarrow P, D_G)$ be a closed Dirac groupoid and
\[
\mathfrak{B}(D_G) = \frac{\mathfrak{B}(D_G) \oplus \ker T_P|_P}{T^*_P(D_G)} \to P
\]
the associated vector bundle over $P$. Set $b : \mathfrak{B}(D_G) \to TP$, $b(v_p, \alpha_p) = T_p s v_p$. Define
\[
[\cdot, \cdot] : \Gamma(\mathfrak{B}(D_G)) \times \Gamma(\mathfrak{B}(D_G)) \to \Gamma(\mathfrak{B}(D_G))
\]
by
\[
\left[\xi + \sigma + P^t(D_G), \eta + \tau + P^t(D_G)\right] = \left[\xi + \sigma', \eta + \tau'\right] \Big|_P + P^t(D_G)
\]
for all $\sigma, \tau \in \Gamma(\ker T_P|_P)$, $\xi, \eta \in \Gamma(\mathfrak{A}(D_G))$ and star sections $\xi \sim_s \xi$, $\eta \sim_s \eta$ of $D_G$, where the bracket on the right-hand side of this equation is the Courant bracket on sections of the Courant algebroid $P_G$. This bracket is well-defined and $(\mathfrak{B}(D_G), b, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ is a Courant algebroid.

**Proof:** This proof can be found with more detailed computations in [11]. The map $b$ is well-defined since $T_p s v_p = 0_p$ for all $(v_p, \alpha_p) \in P^t(D_G)$. We show that the bracket on sections of $\mathfrak{B}(D_G)$ is well-defined, that is, that it has image in $\Gamma(\mathfrak{B}(D_G))$ and does not depend on the choice of the sections $\xi + \sigma$ and $\eta + \tau$ representing $\xi + \sigma + P^t(D_G)$ and $\eta + \tau + P^t(D_G)$. We have, writing $\sigma^t = (X^l, s^t \alpha)$ and $\tau^t = (Y^l, s^t \beta)$,
\[
\left[\xi + \sigma + P^t(D_G), \eta + \tau + P^t(D_G)\right] = \left[\xi + \sigma', \eta + \tau'\right] \Big|_P + P^t(D_G)
\]
\[
\left[\xi + \sigma + P^t(D_G), \eta + \tau + P^t(D_G)\right] = \left[\xi + \sigma', \eta + \tau'\right] \Big|_P + P^t(D_G)
\]
\[
\left[\xi + \sigma, \eta + \tau\right] = \left[\xi + \sigma', \eta + \tau'\right] \Big|_P + P^t(D_G)
\]

By Theorems [3.24] and [3.29], the restriction of this to $P$ is a section of $\mathfrak{A}(D_G) \oplus \ker T_P|_P$ and depends on the choice of the star sections $(X^l, \theta^l), (X^l, \theta^l)$ only by sections of $P^t(D_G)$.

Choose $\sigma \in \Gamma(P^t(D_G))$. Then we have for all $(Y^l, s^t \beta) \in \Gamma(\ker T_P)$:
\[
\left[\sigma^t, (Y^l, s^t \beta)\right] = \left(0, \mathcal{L}_{X^l} (s^t \beta) - \mathcal{L}_{Y^l} (t^s \alpha) + \frac{1}{2} d (\langle t^s \alpha (Y^t) - (s^t \beta) X^l \rangle) \right) = (0, 0).
\]

We have used Lemma [3.30]. If $(X^l, \theta^l)$ is a section of $D_G$ that is a star section $s$-related to $(X^l, \theta^l) = (0, 0)$, then we have $(X^l, \theta^l) \in \Gamma(D_G \cap \ker T^s)$ and we find smooth sections $\sigma^r_1, \ldots, \sigma^r_k \in \Gamma(D_G \cap \ker T^s)$ and functions $f^1, \ldots, f^k \in C^\infty(G)$ such that $(X^l, \theta^l) = \sum_{i=1}^k f^i \sigma^r_i$. Then we get easily for all $(Y^l, s^t \beta) \in \Gamma(\ker T_P)$:
\[
\left[\sigma^t, (Y^l, s^t \beta)\right] = \sum_{i=1}^k \mathfrak{L}^t \left( (f^i \sigma^r_i) (Y^l) \right) = (0, 0).
\]

which is a section of $D_G \cap \ker T^s$. Hence, the restriction to $P$ of $\left[\left( (X^l, \theta^l) , (Y^l, s^t \beta) \right) \right]$ is a section of $P^t(D_G)$. In the same manner, for $i = 1, \ldots, k$ and any star section $(X^l, \theta^l) \sim_s (X^l, \theta^l)$,
\[
\left[\sigma^r_i, (X^l, \theta^l)\right] = - \sum_{i=1}^k \mathfrak{L}^t \left( (f^i \sigma^r_i) (Y^l) \right) = (0, 0).
\]

In the case of $i = 1$, we have $\left[\sigma^r_1, (X^l, \theta^l)\right] = (0, 0)$ and we compute for any $Y \in \Gamma(A_G)$, using the equality $(t^s \alpha (X^l) (Y^l) = - \theta^l (X^l - Y^l))$:
\[
\left( \mathfrak{L}^t X^l \theta^l - i_{X^l} \mathfrak{d} (t^s \alpha (X^l)) \right) (Y^l) = X^l \theta^l (Y^l) - \theta^l \left( \left[ X^l, Y^l \right] - X^l (t^s \alpha (X^l)) (Y^l) \right) + Y^l \left( (t^s \alpha (X^l)) (Y^l) \right) = X^l (s^t \theta^l (Y^l)) + \left( \mathfrak{L}^t (t^s \alpha (X^l)) \right) (X^l) = 0.
\]
This shows that \([X_{t}^{*}, t^{*} \alpha_{n}] = (X_{t}, \theta_{E}(\xi))\) is a section of \(\ker \mathbb{T}_{G} \cap D_{G}\) for \(i = 1, \ldots, k\). Then we get as above
\[
[(X_{t}, \theta_{E}(\xi)), (X_{t}, \theta_{E}(\xi))] = \sum_{i=1}^{k} \left( f_{i} \left( \sigma_{t}^{*}, (X_{t}, \theta_{E}(\xi)) \right) - X_{t}(f_{i})\sigma_{t}^{*} + \frac{1}{2}\left( \sigma_{t}^{*}, (X_{t}, \theta_{E}(\xi)) \right)(0, df_{i}) \right)
\]
\[
= \sum_{i=1}^{k} \left( f_{i} \left( \sigma_{t}^{*}, (X_{t}, \theta_{E}(\xi)) \right) - X_{t}(f_{i})\sigma_{t}^{*} \right),
\]
which is a section of \(D_{G} \cap \ker \mathbb{T}_{G}\) by the considerations above. Hence, its restriction to \(P\) is a section of \(I^{p}(D_{G})\). If \(\bar{\xi} + \sigma \in \Gamma(I^{p}(D_{G}))\), then as above, we find that \(\sigma \in \Gamma(I^{p}(D_{G}))\). The section \(\bar{\xi} + \sigma^{l} - (\bar{\xi} + \sigma)\) is then a section of \(D_{G}\) that is a star section \(s\)-related to 0. Since by the considerations above, we know that
\[
\left[ \bar{\xi} + \sigma^{l} - (\bar{\xi} + \sigma), \eta + \tau \right] \mid_{P} \in \Gamma(I^{p}(D_{G}))
\]
and
\[
\left[ (\bar{\xi} + \sigma)\tau, \eta + \tau \right] \mid_{P} \in \Gamma(I^{p}(D_{G}))
\]
for all star sections \(\eta \sim \bar{\eta}\) of \(D_{G}\) and \(\tau \in \Gamma((\ker \mathbb{T})_{|P})\), we have shown that the bracket does not depend on the choice of the representatives for \((X_{t}, \theta_{E})(\bar{\xi} + (X_{t}, (s\alpha)|p) + I^{p}(D_{G})\) and \((X_{t}, \theta_{E})(\bar{\eta} + ((s\beta)|p) + I^{p}(D_{G})\).

We show now that \((\mathcal{B}(D_{G}), b, [\cdot, \cdot], \langle \cdot, \cdot \rangle)\) is a Courant algebroid. The map
\[
\mathcal{D} : C^\infty(P) \to \Gamma(\mathcal{B}(D_{G}))
\]
is simply given by
\[
\mathcal{D} f = \frac{1}{2}(0, s^{*}df + I^{p}(D_{G})
\]
since
\[
\langle \mathcal{D} f, (v_{p}, \alpha_{p}) \rangle = \frac{1}{2}b \left( (v_{p}, \alpha_{p}) \right)(f) = \frac{1}{2}T_{P}sv_{p}(f)
\]
for all \((v_{p}, \alpha_{p}) \in \mathcal{B}_{P}(D_{G})\). We check all the Courant algebroid axioms. Choose
\[
(X_{t} + X_{l}, \theta_{E} + (s\alpha)|p) + I^{p}(D_{G}), \quad (X_{t} + Y_{l}, \theta_{E} + (s\beta)|p) + I^{p}(D_{G})
\]
and
\[
(X_{t} + Z, \theta_{E} + (s\gamma)|p) + I^{p}(D_{G}) \in \Gamma(\mathcal{B}(D_{G}))
\]
and let \(f\) be an arbitrary element of \(C^\infty(P)\).

1. By (13), the bracket
\[
\left[ (X_{t} + X_{l}, \theta_{E} + s\alpha), (X_{t} + Y_{l}, \theta_{E} + s\beta) \right]
\]
can be taken as the section extending
\[
\left[ (X_{t} + X_{l}, \theta_{E} + s\alpha)|p, + I^{p}(D_{G}), (X_{t} + Y_{l}, \theta_{E} + s\beta)|p) + I^{p}(D_{G}) \right]
\]
to compute its bracket with \((X_{t} + Z, \theta_{E} + s\gamma)|p) + I^{p}(D_{G})\). Since \(P_{G}\) is a Courant algebroid, we have
\[
\frac{1}{6} \left( 0, df \left( \left[ (X_{t} + X_{l}, \theta_{E} + s\alpha), (X_{t} + Y_{l}, \theta_{E} + s\beta) \right] \right) + \text{c.p.} \right)
\]
and
\[
= \frac{1}{6} \left( 0, df \left( \left[ (X_{t} + X_{l}, \theta_{E} + s\alpha), (X_{t} + Y_{l}, \theta_{E} + s\beta) \right] \right) + \text{c.p.} \right)
\]
+ c.p.
If we write \( e_{\xi,X,\alpha} \) for \((X_\xi + X, \tilde{\theta}_\xi + (s^* \alpha)|_P) + I^s(D_G)\), etc., this can be checked to restrict to
\[
\frac{2a}{3} D \left( \left( [e_{\xi,X,\alpha}, e_{\eta,Y,\beta}], e_{\tau,Z,\gamma}\right) + \left( [e_{\eta,Y,\beta}, e_{\tau,Z,\gamma}], e_{\xi,X,\alpha}\right) + \left( [e_{\tau,Z,\gamma}, e_{\xi,X,\alpha}], e_{\eta,Y,\beta}\right) \right)
\]
on \( P \).

2. We have
\[
b[e_{\xi,X,\alpha}, e_{\eta,Y,\beta}] = Ts \left[ X_\xi + X^I, X_\eta + Y^I \right]_P = \left[ Ts \left( X_\xi + X^I \right), Ts \left( X_\eta + Y^I \right) \right]
\]
\[
= [X_\xi + a(X), X_\eta + a(Y)] = b(e_{\xi,X,\alpha}, b(e_{\eta,Y,\beta})].
\]

3. We compute
\[
\frac{2a}{3} D \left( \left( [e_{\xi,X,\alpha} \cdot e_{\eta,Y,\beta}] = \left( X_\xi + X^I, \theta_\xi + s^* \alpha \right), (s^* f) \cdot (X_\eta + Y^I, \theta_\eta + s^* \beta) \right) \right)
\]
\[
= \left( (s^* f) \left( X_\xi + X^I, \theta_\xi + s^* \alpha \right), (X_\eta + Y^I, \theta_\eta + s^* \beta) \right)
\]
\[
+ \frac{2a}{3} (0, d(s^* f)) \right) \right) + I^s(D_G)
\]
\[
f = \left( [e_{\xi,X,\alpha}, e_{\eta,Y,\beta}] = b(e_{\xi,X,\alpha})(f) \cdot e_{\eta,Y,\beta}
\]
\[
- \left( (X_\xi + X, \tilde{\theta}_\xi + s^* \alpha \right), (X_\eta + Y, \tilde{\theta}_\eta + s^* \beta) \right) D f
\]
\[
f = \left( [e_{\xi,X,\alpha}, e_{\eta,Y,\beta}] = b(e_{\xi,X,\alpha})(f) \cdot e_{\eta,Y,\beta} - \left( e_{\xi,X,\alpha}, e_{\eta,Y,\beta} \right) D f.
\]

4. We have obviously \( b \circ D = 0 \).

5. Finally, since \( P_G \) is a Courant algebroid, the corresponding equality for sections of \( P_G \) yields easily, with the same computations as in the previous points
\[
b(e_{\xi,X,\alpha}, e_{\eta,Y,\beta}, e_{\tau,Z,\gamma}) = \left( [e_{\xi,X,\alpha}, e_{\eta,Y,\beta}] + D \left( e_{\xi,X,\alpha}, e_{\eta,Y,\beta}, e_{\tau,Z,\gamma} \right) \right)
\]
\[
+ \left( e_{\eta,Y,\beta} \cdot e_{\xi,X,\alpha}, e_{\tau,Z,\gamma} + D \left( e_{\xi,X,\alpha}, e_{\tau,Z,\gamma} \right) \right).
\]

**Example 3.36** We see in this example that in the special case of a Poisson groupoid \((G \rightrightarrows P, D_{P_G})\), the obtained Courant algebroid is isomorphic to the Courant algebroid defined by the Lie bialgebroid associated to \((G \rightrightarrows P, \pi_G)\), see [17], [18]. This shows how the Courant algebroid structure on \( AG \oplus A^*G \) induced by the Lie bialgebroid of the Poisson groupoid \((G \rightrightarrows P, \pi_G)\) can be related to the standard Courant algebroid structure on \( P_G = TG \oplus T^*G \).

Recall that the Courant algebroid \( E_{P_G} = AG \oplus A^*G \) associated to the Lie bialgebroid \((AG, A^*G)\) of \((G \rightrightarrows P, \pi_G)\) is endowed with the anchor \( \rho : AG \oplus A^*G \to TP \) defined by \( \rho(v_p, \alpha_p) = a(v_p) + \pi^s_G(\alpha_p) \) for all \( p \in P \) and \( (v_p, \alpha_p) \in A_pG \times A^*_pG \) and the symmetric bracket \( \langle \cdot, \cdot \rangle \) defined by \( \langle (v_p, \alpha_p), (w_p, \beta_p) \rangle = \alpha_p(w_p) + \beta_p(v_p) \) for all \( p \in P \) and \( (v_p, \alpha_p), (w_p, \beta_p) \in A_pG \times A^*_pG \). Its Courant bracket is given by
\[
\langle (X, \xi), (Y, \eta) \rangle = \left( [X, Y]_{AG} + \mathcal{L}_{\xi}Y - \mathcal{L}_{\eta}X - \frac{1}{2} d_s(\xi(Y) - \eta(X)),
\]
\[
\mathcal{L}_{\xi}Y - \mathcal{L}_{\eta}X + \frac{1}{2} d_s(\xi(Y) - \eta(X)) \right),
\]
where, if \( X_\xi := \pi^s_G(\xi) \in \Gamma(TP) \) for \( \xi \in \Gamma(A^*G) \),
\[
\mathcal{L}^s_{\xi}Y \in \Gamma(AG), \quad \tau(\mathcal{L}^s_{\xi}Y) = X_\xi(\tau(Y)) - [\xi, \tau](Y) \quad \forall \tau \in \Gamma(A^*G)
\]
\[
\mathcal{L}_{\xi}Y \in \Gamma(A^*G), \quad (\mathcal{L}_{\xi}Y)(Z) = a(X)(\eta(Z)) - [\eta([X, Z]_{AG}), \forall Z \in \Gamma(AG)
\]

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and, for any \( f \in C^\infty(P) \),
\[
\begin{align*}
(d_s f)(\tau) &= X_\tau(f) = X_\tau(s^* f)|_P = -\tau(X_{s^* f}), &\text{hence } d_s f = -X_{s^* f}|_P \\
(df)(Z) &= a(Z)(f) = Z(s^*(f)), &\text{hence } df = d(s^* f)|_{AG} = \hat{s}(d(s^* f)).
\end{align*}
\]

The isomorphism \( \Psi : AG \oplus A^*G \to \mathfrak{B}(D_{\pi_G}) \) is given by
\[
\Psi(X(p), \xi(p)) = (X + X_\xi, \xi(p)) + I_p^p(D_{\pi_G}),
\]
with inverse
\[
\Psi^{-1}(v_p, \alpha_p + I_p^p(D_{\pi_G})) = (v_p - \pi_G^\sharp(\alpha_p), \hat{s}(\alpha_p)).
\]
The verification of the equalities \( \Psi^{-1} \circ \Psi = \text{Id}_{E_{\pi_G}} \) and \( \Psi \circ \Psi^{-1} = \text{Id}_{\mathfrak{B}(D_{\pi_G})} \) is easy, in the same spirit as
the corresponding equalities in the following example. In \( \mathfrak{B} \), it is shown that
\[
\Psi^{-1}(\varphi(X, \xi), \varphi(Y, \eta)) = (\varphi(X, \xi), (Y, \eta))
\]
for \( (X, \xi), (Y, \eta) \in \Gamma(AG \oplus A^*G) \). Since the computations are long, but straightforward, we omit them here.

\[\diamondsuit\]  

**Example 3.37** Consider a Lie groupoid \( G \rightrightarrows P \) endowed with a closed multiplicative 2-form \( \omega_G \in \Omega^2(G) \). The Courant algebroid \( \mathfrak{B}(D_{\omega_G}) \) is given here by
\[
\mathfrak{B}(D_{\omega_G}) = \left( \text{Graph}(\omega_G^\flat|TP : TP \to A^*G) + \ker \mathcal{T}_p \right) / \text{Graph} \left( \omega_G^\flat|T_pG : T_p^*G \to (T_p^*G)^\flat \right).
\]
We show that it is isomorphic as a Courant algebroid to the standard Courant algebroid \( \mathcal{P}P = TP \oplus T^*P \).

For this, consider the maps
\[
\Lambda : \mathfrak{B}(D_{\omega_G}) \to TP \oplus T^*P, \quad \Lambda \left( (v_p, \alpha_p) \right) = (T_p s v_p, \beta_p),
\]
where \( (T_p s)^* \beta_p = \alpha_p - \omega_G^\flat(v_p) \), and
\[
\Lambda^{-1} : TP \oplus T^*P \to \mathfrak{B}(D_{\omega_G}), \quad \Lambda^{-1} (v_p, \alpha_p) = (\epsilon(v_p), (T_p s)^* \alpha_p + \omega_G^\flat(\epsilon(v_p))).
\]
Note that \( \Lambda \) is well-defined: if \( (v_p, \alpha_p) \in \mathfrak{A}_p(D_{\omega_G}) + \ker \mathcal{T}_p \), then \( \hat{t}(\alpha_p) = \omega_G^\flat(T_p tv_p) \) and hence \( \hat{t}(\alpha_p - \omega_G^\flat(v_p)) = \hat{t}(\alpha_p) - \omega_G^\flat(T_p tv_p) = 0 \). Thus, the covector \( \alpha_p - \omega_G^\flat(v_p) \) can be written \( (T_p s)^* \beta_p \) with some \( \beta_p \in T_p^*P \). (For simplicity, we will identify elements \( \beta_p \) of \( T_p^*P \) with \( (T_p s)^* \beta_p \in (T_p^*G)^\flat \subset T_p^*G \) and \( v_p \in T_pP \) with \( \epsilon(v_p) \in T_pP \subset T_pG \) in the following.) Furthermore, if \( u_p \in T_p^*G \), we have \( \Lambda \left( (u_p, \omega_G^\flat(u_p)) \right) = (T_p s u_p, \omega_G^\flat(u_p) - \omega_G^\flat(v_p)) = (0, 0) \). The map \( \Lambda^{-1} \) has image in \( \mathfrak{B}(D_{\omega_G}) \) because for any \( (v_p, \alpha_p) \in \mathfrak{P}_P(p), \) we have
\[
\left( v_p, (T_p s)^* \alpha_p + \omega_G^\flat(v_p) \right) = \mathcal{T}_p \left( v_p, \omega_G^\flat(v_p) \right) + (0, (T_p s)^* \alpha_p) = \mathfrak{A}_p(D_{\omega_G}) + (\ker \mathcal{T}_p)_p.
\]
Choose now \( p \in P, (v_p, \alpha_p) \in \mathfrak{A}_p(D_{\omega_G}) \) and compute
\[
\begin{align*}
(\Lambda^{-1} \circ \Lambda) \left( (v_p, \alpha_p) \right) &= \Lambda^{-1}(T_p s v_p, \alpha_p - \omega_G^\flat(v_p)) \\
&= (T_p s v_p, \alpha_p - \omega_G^\flat(v_p) + \omega_G^\flat(T_p s v_p)) \\
&= (v_p, \alpha_p) + (T_p s v_p - v_p, \omega_G^\flat(T_p s v_p - v_p)) = (v_p, \alpha_p).
\end{align*}
\]
In the same manner, if \( (v_p, \alpha_p) \in \mathfrak{P}_P(p) \), we have
\[
\left( \Lambda \circ \Lambda^{-1} \right) (v_p, \alpha_p) = \Lambda \left( (v_p, \alpha_p + \omega_G^\flat(v_p)) \right) = (T_p s v_p, \alpha_p + \omega_G^\flat(v_p) + \omega_G^\flat(v_p)) = (v_p, \alpha_p)
\]
31
since \( v_p \in T_p P \). The equality \( b \circ \Lambda^{-1} = \text{pr}_P \) is immediate.

Now if \((\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta}) \in \Gamma(P_P)\), we choose \( X, Y \in \mathfrak{X}(G) \) such that \( X \sim_s \bar{X}, X|_P = \bar{X}, Y \sim_s \bar{Y}, Y|_P = \bar{Y} \) and we compute

\[
\Lambda \left[ \Lambda^{-1}(\bar{X}, \bar{\alpha}), \Lambda^{-1}(\bar{Y}, \bar{\beta}) \right] = \Lambda \left( \begin{pmatrix} [X, Y], \omega_G^\#([X, Y]) + \mathcal{L}_X(s^* \bar{\beta}) - \mathcal{L}_Y(s^* \bar{\alpha}) + \frac{1}{2}d((s^* \bar{\alpha})(Y) - (s^* \bar{\beta})(X)) \end{pmatrix} \right)
\]

Then the Courant bracket on \( \mathcal{C}(\mathbb{R}) \). Hence, we get an isomorphism

\[
\Pi : \mathfrak{B}(\mathfrak{D}_M \otimes \mathfrak{D}_M) \rightarrow TM \times_M T^* M, \quad (v_m, w_m, \alpha_m, \beta_m) \mapsto (w_m, \beta_m)
\]

over \( \text{pr}_1 : \Delta_M \rightarrow M \), with inverse

\[
\Pi^{-1} : TM \times_M T^* M \rightarrow \mathfrak{B}(\mathfrak{D}_M \otimes \mathfrak{D}_M), \quad (w_m, \beta_m) \mapsto (0_m, w_m, 0_m, \beta_m).
\]

The Courant bracket on \( \mathfrak{B}(\mathfrak{D}_M \otimes \mathfrak{D}_M) \) is easily seen to correspond via this isomorphism to the standard Courant bracket on \( P_M = TM \times_M T^* M \) (and hence, does not depend on \( \mathfrak{D}_M \)).

\[\triangleleft\]

**Example 3.38** Consider the pair Dirac groupoid \((M \times M \to M, \mathcal{D}_M \otimes \mathcal{D}_M)\) associated to a closed Dirac manifold \((M, \mathcal{D}_M)\) (see Example 3.28). The vector bundle \( \mathfrak{B}(\mathfrak{D}_M \otimes \mathfrak{D}_M) \to \Delta_M \) is defined here by

\[
\mathfrak{B}_{(m,m)}(\mathfrak{D}_M \otimes \mathfrak{D}_M) = \mathfrak{A}_{(m,m)}(\mathfrak{D}_M \otimes \mathfrak{D}_M) = \mathfrak{A}(\mathfrak{D}_M) \oplus \mathfrak{A}(\mathfrak{D}_M) = \mathfrak{A}(\mathfrak{D}_M) \oplus (\{0\} \times \mathcal{P}_m \times \mathcal{P}_m)
\]

for all \( m \in M \) (recall that we have computed \( \mathfrak{A}(\mathfrak{D}_M \otimes \mathfrak{D}_M) \) in Example 3.28). Hence, we get an isomorphism

\[
\Pi : \mathfrak{B}(\mathfrak{D}_M \otimes \mathfrak{D}_M) \rightarrow TM \times_M T^* M, \quad (v_m, w_m, \alpha_m, \beta_m) \mapsto (w_m, \beta_m)
\]

over \( \text{pr}_1 : \Delta_M \rightarrow M \), with inverse

\[
\Pi^{-1} : TM \times_M T^* M \rightarrow \mathfrak{B}(\mathfrak{D}_M \otimes \mathfrak{D}_M), \quad (w_m, \beta_m) \mapsto (0_m, w_m, 0_m, \beta_m).
\]

**Remark 3.39** Consider a Dirac groupoid as in Theorem 3.38. Set \( N := G/G_0 \) and \( Q := P/(TP \cap G_0) \). Then the Courant algebroid \( TG \oplus T^* G \) projects under \( pr \) to the Courant algebroid \( TN \times_N T^* N \). We have a map \( Tpr : TG \oplus P_1 \rightarrow TN \times_N T^* N, (v_g, \alpha_g) \mapsto (T_g pr(v_g), \alpha_{[g]}), \) where \( \alpha_{[g]} \) is such that \( \alpha_g = (T_g pr)^* \alpha_{[g]} \). By definition of the reduced Dirac structure \( pr(D_G) = D_{\pi} \), the restriction of this map to \( \mathfrak{A}(\mathfrak{D}_G) \oplus \ker(T|_P) \cap (TG \oplus P_1) \) has image \( \mathfrak{A}(\mathfrak{D}_G) \) (see Example 3.28). Furthermore, we find that \( (v_p, (T_p pr)^* \alpha_{[p]})) \in I^*_{\mathfrak{T}^0}(\mathfrak{D}_G) + \mathfrak{G}_0(p) \times 0_p \) if and only if \( (T_p pr v_p, \alpha_{[p]}) \in I^*_{\mathfrak{T}^0}(\mathfrak{D}_G) \).

Hence, the map \( Tpr \) factors to a map \( \mathfrak{B}(\mathfrak{D}_G) \to \mathfrak{B}(\mathfrak{D}_G) \). It is straightforward to check that this is a morphism of Courant algebroids.

\[\triangle\]

### 3.6 Induced action of the group of bisections on \( \mathfrak{B}(\mathfrak{D}_G) \)

We show here how the action of \( G \) on \( \mathfrak{g}/\mathfrak{g}_0 \times \mathfrak{p}_1 \) found in [10] in the Lie group can be generalized to the setting of Dirac groupoids. In this section, the Dirac groupoids that we consider are not necessarily closed. Hence, the vector bundle \( \mathfrak{B}(\mathfrak{D}_G) \) exists, but does not necessarily have a Courant algebroid structure.

We begin with a lemma, which will also be useful in the following section about Dirac homogeneous spaces.

**Lemma 3.40** Let \((G \to P, \mathfrak{D}_G)\) be a Dirac groupoid and \((v_p, \alpha_p) \in \mathfrak{A}_p(\mathfrak{D}_G) \oplus (\ker \mathfrak{T}|_P) \) for some \( p \in P \). If \( \mathfrak{T}(v_p, \alpha_p) = (X_\xi(p), \theta_\xi(p)) \in \mathfrak{A}_p(\mathfrak{D}_G), \) then \( (v_p, \alpha_p) = (X_\xi(p), \theta_\xi(p)) + (u_p, (T_p s)^* \gamma_p) \) with some \( u_p \in A_p G \) and \( \gamma_p \in T^*_p P \) and

\[
((X_\xi, \theta_\xi)(g)) \ast (v_p, \alpha_p) = (X_\xi(g) + T_p L_g u_p, \theta_\xi(g) + (T_p s)^* \gamma_p)
\]

for any \( g \in s^{-1}(p) \) and \((X_\xi, \theta_\xi) \in \Gamma(\mathfrak{D}_G)\) such that \((X_\xi, \theta_\xi) \simeq_s (\bar{X}_\xi, \bar{\theta}_\xi)\).
Proof: If \((v_p, \alpha_p) \in \mathfrak{X}_p(D_G) \oplus (\ker \mathbb{T}t)|_p\), then \(\mathbb{T}t(v_p, \alpha_p) \in \mathfrak{X}_p(D_G)\) and hence \(\mathbb{T}t(v_p, \alpha_p) = (X_{\xi}, \tilde{\theta}_{\xi})(p)\) for some section \((X_{\xi}, \tilde{\theta}_{\xi}) \in \Gamma(\mathfrak{X}_p(D_G))\). The difference \((v_p, \alpha_p) - (X_{\xi}, \tilde{\theta}_{\xi})(p) = (v_p, \alpha_p) - \mathbb{T}t(v_p, \alpha_p)\) is then an element of \((\ker \mathbb{T}t)|_p\) and there exists \(\gamma_p \in T_p P\) such that \((v_p, \alpha_p) - (X_{\xi}, \tilde{\theta}_{\xi})(p) = (u_p, (T_p s)^* \gamma_p)\) if we set \(u_p = v_p - T_p v_p\).

Since the section \((X_{\xi}, \tilde{\theta}_{\xi}) \in \Gamma(D_G)\) is a pair that is a star section \(s\)-related to \((\tilde{X}_{\xi}, \tilde{\theta}_{\xi})\), the product \((X_{\xi}, \tilde{\theta}_{\xi})(g) \ast (v_p, \alpha_p)\) is defined for any \(g \in s^{-1}(p)\).

We compute, using a bisection \(K\) through \(g\),

\[X_{\xi}(g) \ast v_p = X_{\xi}(g) + T_p L_K v_p - T_p L_K(T_p v_p) = X_{\xi}(g) + T_p L_K(u_p) = X_{\xi}(g) + T_p L_g(u_p).\]

For the first equality, we have used the formula proved in [31], see also [22].

We have also, for any \(v_g = v_g \ast (T_g s v_g) \in T_g G\),

\[(\theta_{\xi}(g) \ast \alpha_p)(v_g) = (\theta_{\xi}(g) \ast \alpha_p)(v_g \ast (T_g s v_g)) = \theta_{\xi}(g)(v_g) + \alpha_p(T_g s v_g) = \theta_{\xi}(g)(v_g) + (\alpha_p - \tilde{\theta}_{\xi})(\mathbb{T}g s \gamma_p)(v_g) + (T_p s^* \gamma_p)(T_g s v_g) = (\theta_{\xi}(g) + (T_g s^* \gamma_p))((v_g)).\]

\[\square\]

Theorem 3.41 Let \((G \equiv P, D_G)\) be a Dirac groupoid. Choose a bisection \(K \in \mathcal{B}(G)\) and consider

\[r_K : \mathfrak{X}(D_G) \oplus \ker \mathbb{T}t|_p \to \mathcal{B}(D_G)\]

\[r_K(v_p, \alpha_p) = (T_K(p)^{-1} R_K(v_K(p)^{-1} \ast v_p), (I_{\mathbb{R}(K(p))} D_K^{-1})^* (\alpha_{K(p)^{-1}} \ast \alpha_p)) + I_{\mathbb{R}(K(p))} D_G,\]

where \((v_K(p)^{-1}, \alpha_{K(p)^{-1}}) \in D_G(K(p)^{-1})\) is such that

\[\mathbb{T}s(v_K(p)^{-1}, \alpha_{K(p)^{-1}}) = \mathbb{T}(v_p, \alpha_p).\]

The map \(r_K\) is well-defined and induces the right translation by \(K\),

\[\rho_K : \mathcal{B}(D_G) \to \mathcal{B}(D_G),\]

\[(v_p, \alpha_p) + I_{\mathbb{R}(K(p))} D_G \mapsto r_K(v_p, \alpha_p).\]

The map \(\rho : \mathbb{B}(G) \times \Gamma(\mathfrak{X}(D_G)) \to \Gamma(\mathcal{B}(D_G))\) is a right action.

For the proof of this theorem, we will need the following lemma, which proof is straightforward.

Lemma 3.42 Let \((G \equiv P, D_G)\) be a Lie groupoid. Choose \(g, h \in G\) and \(K \in \mathcal{B}(G)\). Choose \((v_h, \alpha_h) \in \mathcal{P}_G(h), (v_g, \alpha_g) \in \mathcal{P}_G(g)\) such that \(\mathbb{T}s(v_g, \alpha_g) = \mathbb{T}(v_h, \alpha_h)\). Then

\[T_{g \ast h} R_K(v_g \ast v_h) = v_g \ast (T_h R_K v_h)\]

and

\[\alpha_g \ast ((T_{R_K(h)} R_{K}^{-1})^* \alpha_h) = (T_{R_K(g \ast h)} R_{K}^{-1})^* (\alpha_g \ast \alpha_h).\]

Proof (of Theorem 3.41): First, we check that the map \(r_K\) is well-defined, that is, that it has image in \(\mathcal{B}(D_G)\) and does not depend on the choices made.

Choose \(p \in P\), \((v_p, \alpha_p) \in \mathfrak{X}_p(D_G) \oplus (\ker \mathbb{T}t)|_p\) and \(K \in \mathcal{B}(G)\). Set \(K(p) = g\). Since the map \(r_K\) is linear in every fiber of \(\mathfrak{X}(D_G) \oplus (\ker \mathbb{T}t)|_p\), it suffices to show that the image of \((0_p, 0_p)\) is \(I_{\mathbb{R}(G)} D_G\) for any choice of \((v_g, \alpha_g) \in D_G(g^{-1})\) such that \(\mathbb{T}s(v_g, \alpha_g) = (0_p, 0_p)\) to prove that it is well-defined. Using (16) and (17), we get

\[T_{g^{-1}} R_K(v_g^{-1} \ast 0_p) = v_g^{-1} \ast (T_p R_K 0_p) = v_g^{-1} \ast 0_g\]

and

\[(\alpha_{g^{-1}} \ast 0_p) \circ T_{s(g)} R_{K^{-1}} = \alpha_{g^{-1}} \ast (0_p \circ T_g R_{K^{-1}}) = \alpha_{g^{-1}} \ast 0_g.\]
Thus, we have shown that

\[ r_K(0_p, 0_p) = (v_{g-1}, \alpha_{g-1}) \star (0_g, 0_g) \in D_G(s(g)) \cap \ker T s = I_{s(g)}^*(D_G). \]

Choose next \((v_p, \alpha_p) \in \mathfrak{B}(D_G) \oplus (\ker T t)_p\) such that \((v_p, \alpha_p) \in I_{s(g)}^*(D_G)\), that is, such that \(v_p, \alpha_p = 0\) in \(\mathfrak{B}_p(D_G)\). Choose \((v_{g-1}, \alpha_{g-1}) \in D_G(g^{-1})\) such that \(T s(v_{g-1}, \alpha_{g-1}) = T t(v_p, \alpha_p)\). Then we have \(T_{g-1}^{-1} R_K(v_{g-1} \ast v_p) = T_{g-1}^{-1} R_K(v_{g-1} \ast v_p \star 0_p) = v_{g-1} \ast v_p \ast 0_g\), since \(T_p \circ \gamma = 0\). We have also \(s(\alpha_p) = 0\), and by \([17]\):

\[ (T_s(g) R_K^{-1})^*(\alpha_{g-1} \ast \alpha_p) = (T_s(g) R_K^{-1})^*(\alpha_{g-1} \ast \alpha_p \ast 0_p) = \alpha_{g-1} \ast \alpha_p \ast 0_g. \]

Thus, \(r_K(v_p, \alpha_p) = (v_{g-1}, \alpha_{g-1}) \ast (v_{g-1}, \alpha_{g-1}) \ast 0_g \in I_{s(g)}^*(D_G)\). The map \(\rho_K : \mathfrak{B}(D_G) \to \mathfrak{B}(D_G)\) is consequently well-defined.

We show now that \(\rho : \mathfrak{B}(G) \times \mathfrak{B}(D_G) \to \mathfrak{B}(D_G)\) defines an action of the group of bisections of \(G \rightrightarrows P\) on \(\Gamma(\mathfrak{B}(D_G))\).

Choose \(K, L \in \mathfrak{B}(G)\), \(p \in P\) and \((v_p, \alpha_p) \in \mathfrak{B}_p(D_G)\). Set \(K(p) = g\) and choose a pair \((v_{g-1}, \alpha_{g-1}) \in D_G(g^{-1})\) such that \(T s(v_{g-1}, \alpha_{g-1}) = T t(v_p, \alpha_p)\). Set also \(h := L(s(g))\) and choose \((w_{h^{-1}}, \beta_{h^{-1}}) \in D_G(h^{-1})\) such that \(T s(w_{h^{-1}}, \beta_{h^{-1}}) = T t(v_{g-1}, \alpha_{g-1})\). Then we have \((K \ast L)(p) = g \ast h\) and we compute, using \([18]\) and \([17]\):

\[
\rho L \left( \rho_K \left( \frac{v_p}{v_p, \alpha_p} \right) \right) = \rho L \left( \left( T_{g^{-1}}^{-1} R_K, (T_s(g) R_K^{-1})^* \right) \left( (v_{g-1}, \alpha_{g-1}) \star (v_p, \alpha_p) \right) \right) = (T_{(g \ast h)^{-1}}^{-1} R_{K \ast L}, (T_s(g \ast h) R_{K \ast L}^{-1})^*) \left( (w_{h^{-1}}, \beta_{h^{-1}}) \star (v_{g-1}, \alpha_{g-1}) \star (v_p, \alpha_p) \right) + I_{s(h)}^*(D_G) = \rho_{K \ast L} \left( \frac{v_p}{v_p, \alpha_p} \right)
\]

since \((w_{h^{-1}}, \beta_{h^{-1}}) \star (v_{g-1}, \alpha_{g-1})\) is an element of \(D_G((g \ast h)^{-1})\) satisfying

\[ T s ((w_{h^{-1}}, \beta_{h^{-1}}) \star (v_{g-1}, \alpha_{g-1})) = T s ((v_{g-1}, \alpha_{g-1})) = (v_p, \alpha_p). \]

\[ \square \]

**Example 3.43** Consider a Poisson groupoid \((G \rightrightarrows P, \pi_G)\). We will compute the action of the bisections \(\mathfrak{B}(G)\) on \(\mathfrak{B}(D_{\pi_G}) \simeq AG \oplus A^*G\).

Choose \(K \in \mathfrak{B}(G)\), \(p \in P\) and \(\Psi(X(p), \xi(p)) \in \mathfrak{B}_p(D_{\pi_G})\), \((X(p), \xi(p)) \in A_p G \times A^*_p G\) (recall that \(\Psi\) has been defined in Example \(3.30\)). If \(K(p) = g\) and \(\theta_\xi \in \Omega^1(G)\) is a star section that is \(s\)-related to \(\xi\), then \(\rho_K(\Psi(X(p), \xi(p)))\) is given by

\[
\rho_K(\Psi(X(p), \xi(p))) = \left( T_{g^{-1}}^{-1} R_K(\pi_G^* \theta_\xi(g^{-1})) + X^l(g^{-1}), (R_K^{-1})^* \theta_\xi(s(g)) \right)
\]

and corresponds to

\[
\left( T_{g^{-1}}^{-1} R_K(\pi_G^* \theta_\xi(g^{-1})), X^l(g^{-1}) \right) \mapsto \left( T_p(L_g \circ R_K) X^l(g^{-1}) + T_{g^{-1}}^{-1} R_K(\pi_G^* \theta_\xi(g^{-1})), (R_K^{-1})^* \theta_\xi(s(g)) \right)
\]

Note that by the general theorem, this does not depend on the choice of \(\theta_\xi\). In the case of a Poisson Lie group, we recover the action of \(G\) on the Lie bialgebroid, see \([7]\). In the case of a trivial Poisson groupoid, i.e., with \(s_\xi = 0\), this is simply the pair of maps on \(AG\) and \(A^*G\) generalizing \(\text{Ad}\) and \(\text{Ad}^*\) in the Lie group case.

\[ \diamond \]
Consider a Lie groupoid $G \rightrightarrows P$ endowed with a closed multiplicative 2-form $\omega_G \in \Omega^2(G)$. We will compute the action of the bisections $\mathcal{B}(G)$ on $\mathcal{B}(D_{\omega_G}) \cong TP \oplus T^*P$.

Choose a bisection $K \in \mathcal{B}(G)$, a vector $\Lambda^{-1}(v_p, \alpha_p) = (v_p, (T_p s)^* \alpha_p + \omega_G(v_p)) \in \mathcal{B}_p(D_{\omega_G})$, $(v_p, \alpha_p) \in P_p(p)$ (recall Example 3.37) and set $K(p) = g \in G$. Then we have

$$\mathcal{T}(v_p, (T_p s)^* \alpha_p + \omega_G(v_p)) = (v_p, \omega_G(v_p)) \in \mathcal{A}_p(D_{\omega_G}).$$

Hence, any vector $v_{g^{-1}} \in T_{g^{-1}}G$ such that $T_{g^{-1}} s v_{g^{-1}} = v_p$ leads to $(v_{g^{-1}}, \omega_G(v_{g^{-1}})) \in \mathcal{D}_{\omega_G}(v_{g^{-1}})$ and $\mathcal{T}(v_{g^{-1}}, \pi_G(v_{g^{-1}})) = \mathcal{T}(v_p, (T_p s)^* \alpha_p + \omega_G(v_p))$.

The vector $\rho_K(\Lambda^{-1}(v_p, \alpha_p)) \in \mathcal{B}_s(g)(\mathcal{D}_{\omega_G})$ is then given by

$$\rho_K(\Lambda^{-1}(v_p, \alpha_p)) = (T_{g^{-1}} R_K(v_{g^{-1}}) \ast v_p, (T_{s(g)} R_K^{-1})^* \left( \omega_G(v_{g^{-1}}) \ast \left( (T_p s)^* \alpha_p + \omega_G(v_p) \right) \right)) = (T_{g^{-1}} R_K v_{g^{-1}}, (T_{s(g)} R_K^{-1})^* \left( \omega_G(v_{g^{-1}}) + (T_{g^{-1}} s)^* \alpha_p \right)).$$

Thus, we get

$$(\Lambda \circ \rho_K \circ \Lambda^{-1})(v_p, \alpha_p) = (T_{g^{-1}}(s \circ R_K) v_{g^{-1}}, (T_{s(g)} R_K^{-1})^* \left( \omega_G(v_{g^{-1}}) + (T_{g^{-1}} s)^* \alpha_p \right) - \omega_G(T_{g^{-1}} R_K v_{g^{-1}}))$$

Again, by the general theorem, this does not depend on the choice of $v_{g^{-1}}$. In the trivial case $\omega_G = 0$, $\rho_K$ is simply the map $(T(s \circ K), (T(s \circ K)^{-1})^*)$ induced by the diffeomorphism $s \circ K$ on $P_p$. 

Example 3.45 Let $(M, D)$ be a smooth Dirac manifold and consider the pair Dirac groupoid $(M \times M \rightrightarrows M, D \circ D)$ associated to it. The set of bisections of $M \times M \rightrightarrows M$ is equal to $\mathcal{B}(M \times M) = \{\text{Id}_M\} \times \text{Diff}(M)$. Choose $K = (\text{Id}_M, \phi_K) \in \mathcal{B}(M \times M)$, $p := (m, m) \in \Delta_M$ and $(v_m, w_m, \alpha_m, \beta_m) \in \mathcal{B}_p(D_M \circ D_M)$. Then we have $\mathcal{T}(v_m, w_m, \alpha_m, \beta_m) = (v_m, v_m, \alpha_m, -\alpha_m) \in \mathcal{A}_p(D_M \circ D_M)$. Set $n := \phi_K(m)$. Then we have $K(p)^{-1} = (n, m)$ and $(0_n, v_m, 0_n, -\alpha_m) \in (D_M \circ D_M)(n, m)$ such that $\mathcal{T}(0_n, v_m, 0_n, -\alpha_m) = (v_m, v_m, \alpha_m, -\alpha_m) = \mathcal{T}(v_m, w_m, \alpha_m, \beta_m)$. The vector $\mathcal{R}_K \left( (v_m, w_m, \alpha_m, \beta_m) \right)$ is thus given by

$$\mathcal{R}_K \left( (v_m, w_m, \alpha_m, \beta_m) \right) = (T_{[n,m]} R_K(0_n, w_m), (T_{[n,m]} R_K^{-1})^*(0_n, \beta_m)) = (0_n, T_m \phi_K v_m, 0_n, (T_m \phi_K^{-1})^* \beta_m).$$

Recall that $\mathcal{B}(D_M \circ D_M)$ is isomorphic to $P_M$ via $[13]$. It is easy to see that the action of $\mathcal{B}(M \times M)$ on $\mathcal{B}(D_M \circ D_M)$ corresponds via this identification to the action of $\text{Diff}(M)$ on $P_M$ given by $\phi \cdot (v_m, \alpha_m) = (T_m \phi v_m, (T_m \phi)^{-1})^* \alpha_m$ for all $\phi \in \text{Diff}(M)$ and $(v_m, \alpha_m) \in P_M(m)$. 

4 Dirac homogeneous spaces and the classification

In this section, we show that the Courant algebroid found in Section 3 is the right ambient Courant algebroid for the classification of the Dirac homogeneous spaces of a Dirac groupoid.

We prove our main theorem (Theorem 4.17) about the correspondence between (closed) Dirac homogeneous spaces of a (closed) Dirac groupoid and Lagrangian subspaces (subalgebroids) of the vector bundle (Courant algebroid) $\mathcal{B}(D_G)$. This result generalizes the result of [7] about the Poisson homogeneous spaces of Poisson Lie groups, of [8] about Poisson homogeneous spaces of Poisson groupoids and the result in [10] about the Dirac homogeneous spaces of Dirac Lie groups. To be able to define the notion of a homogeneous Dirac structure on a homogeneous space of a Lie groupoid, we have to prove the following proposition. The proof is straightforward and will be omitted here. For more details, see [9].
Proposition 4.1 Let \( G \rightrightarrows P \) be a Lie groupoid acting on a smooth manifold \( M \) with momentum map \( J : M \to P \). Then there is an induced action of \( TG \rightrightarrows TP \) on \( TJ : TM \to TP \).

Assume that \( M \cong G/H \) is a smooth homogeneous space of \( G \) and let \( q : G \to G/H \) be the projection. The map \( \hat{J} : T^*(G/H) \to A^*G, \hat{J}(\alpha_{gH}) = \delta((TqH)^*\alpha_{gH}) \) for all \( gH \in G/H \) is well-defined and \( \hat{\Phi} : T^*G \times_{A^*G} T^*(G/H) \to T^*(G/H) \) given by

\[
(\hat{\Phi}(\alpha_g, \alpha_{gH}))(T(g', gH)\hat{\Phi}(v_{g'}, v_{gH})) = \alpha_{g'}(v_{g'}) + \alpha_{gH}(v_{gH})
\]

defines an action of \( T^*G \rightrightarrows A^*G \) on \( \hat{J} : T^*(G/H) \to A^*G \).

In the following, we write \( \alpha_g \cdot \alpha_{g'H} \) for \( \hat{\Phi}(\alpha_g, \alpha_{g'H}) \).

Corollary 4.2 If \( G/H \) is a smooth homogeneous space of \( G \rightrightarrows P \), there is an induced action \( T\Phi = (T\Phi, \hat{\Phi}) \) of

\[(TG \oplus T^*G) \rightrightarrows (TP \oplus A^*G)\]
on
\( \bar{TJ} := TJ \times \hat{J} : (T(G/H) \times_{G/H} T^*(G/H)) \to (TP \oplus A^*G) \).

We will show that the following definition generalizes in a natural manner the notion of Poisson homogeneous space of a Poisson groupoid.

Definition 4.3 Let \( (G \rightrightarrows P, D_G) \) be a Dirac groupoid, and \( G/H \) a smooth homogeneous space of \( G \rightrightarrows P \) endowed with a Dirac structure \( D_{G/H} \). The pair \( (G/H, D_{G/H}) \) is a Dirac homogeneous space of the Dirac groupoid \( (G \rightrightarrows P, D_G) \) if the induced action of \( (TG \oplus T^*G) \rightrightarrows (TP \oplus A^*G) \) on \( \bar{TJ} : (T(G/H) \times_{G/H} T^*(G/H)) \to (TP \oplus A^*G) \) restricts to an action of

\[D_G \rightrightarrows \mathfrak{A}(D_G)\]
on
\( \bar{TJ}|_{D_{G/H}} : D_{G/H} \to \mathfrak{A}(D_G) \).

Let \( (G, D_G) \) be a Dirac Lie group and \( H \) a closed connected Lie subgroup of \( G \). Let \( G/H = \{gH \mid g \in G\} \) be the homogeneous space defined as the quotient space by the right action of \( H \) on \( G \). Let \( q : G \to G/H \) be the quotient map. For \( g \in G \), let \( \sigma_g : G/H \to G/H \) be the map defined by \( \sigma_g(g'H) = gg'H \).

Proposition 4.4 Let \( (G, D_G) \) be a Dirac Lie group and \( H \) a closed connected Lie subgroup of \( G \). Let \( G/H \) be endowed with a Dirac structure \( D_{G/H} \). The pair \( (G/H, D_{G/H}) \) is a Dirac homogeneous space of \( (G, D_G) \) if and only if the left action

\[\sigma : G \times G/H \to G/H, \quad \sigma_g(g'H) = gg'H\]
is a forward Dirac map, where \( G \times G/H \) is endowed with the product Dirac structure \( D_G \oplus D_{G/H} \).

Example 4.5 Consider a Poisson homogeneous space \( (G/H, \pi) \) of a Poisson groupoid \( (G \rightrightarrows P, \pi_G) \), i.e., the graph \( \text{Graph}(\pi) \subseteq G \times G/H \times \overline{G/H} \) is a coisotropic submanifold (see [18]).

Consider the Dirac groupoid \( (G \rightrightarrows P, D_{\pi_G}) \) defined by \( (G \rightrightarrows P, \pi_G) \) and the Dirac manifold \( (G/H, D_{G/H}) \), defined by \( D_{G/H} = \text{Graph}(\pi^t \circ \pi : T^*(G/H) \to T(G/H)) \). The verification of the fact that \( (G/H, D_{G/H}) \) is a Dirac homogeneous space of the Dirac groupoid \( (G \rightrightarrows P, D_{\pi_G}) \).

Conversely, we show in a similar manner that if \( T\Phi \) restricts to an action of \( D_{\pi_G} \) on \( \pi^t \), then the graph of the left action of \( G \) on \( G/H \) is coisotropic.

Example 4.6 Let \( (G \rightrightarrows P, \omega_G) \) be a presymplectic groupoid and \( H \) a wide subgroupoid of \( G \rightrightarrows P \). Assume that \( G/H \) has a smooth manifold structure such that the projection \( q : G \to G/H \) is a surjective submersion. Let \( \omega \) be a closed 2-form on \( G/H \) such that the action \( \Phi : G \oplus (G/H) \to G \) is a presymplectic groupoid action, i.e., \( \Phi^*\omega = \text{pr}_{G/H}^*\omega + \text{pr}_G^*\omega_G \). Let \( D_\omega \) be the graph of the vector bundle map \( \omega^t : T(G/H) \to T^*(G/H) \) associated to \( \omega \). It is easy to check that the pair \( (G/H, D_\omega) \) is a closed Dirac homogeneous space of the closed Dirac groupoid \( (G \rightrightarrows P, D_{\omega_G}) \), see Example 3.3.

Example 4.7 Let \( (G \rightrightarrows P, D_G) \) be a Dirac groupoid. Then \( (t : G \to P, D_G) \) is a Dirac homogeneous space of \( (G \rightrightarrows P, D_G) \).
4.1 The moment map

In the Poisson case, if \((G/H, D_{G/H})\) is a Poisson homogeneous space of a Poisson groupoid \((G\rightrightarrows P, \pi_G)\), then the map \(J : G/H \rightarrow P\) is a Poisson map (see [18]). This is also true here under some regularity conditions on the characteristic distributions of the involved Dirac structures.

**Theorem 4.8** Let \((G\rightrightarrows P, D_G)\) be a Dirac groupoid such that Theorem 3.7 holds and \((G/H, D_{G/H})\) a Dirac homogeneous space of \((G\rightrightarrows P, D_G)\). Assume that the map \(J_{|G/H} : G_0^\ast \cap TP \rightarrow G_0 \cap TP\) is surjective in every fiber, where \(G_0^\ast \cap TP\) is the characteristic distribution defined on \(G/H\) by \(D_{G/H}\). Then the map \(J : (G/H, D_{G/H}) \mapsto (P, D_P)\) is a forward Dirac map.

**Proof:** Choose \((v_p, \alpha_p) \in D_P(p)\), for some \(p \in P\) and \(gH \in G/H\) such that \(t(g) = p\). Then there exists \(w_p \in T^*_p G\) and \(u_p \in G_0(p) \cap T_p P\) such that \((w_p, (T_p)^\ast \alpha_p) \in D_p^\ast(D_G)\) and \(v_p = u_p + T_p w_p\). Thus, we have shown that \((w_p, (T_p)^\ast \alpha_p) \in D_p^\ast(D_G)\).

4.2 The homogeneous Dirac structure on the classes of the units

Let \((G\rightrightarrows P, D_G)\) be a Dirac groupoid and \((G/H, D_{G/H})\) a Dirac homogeneous space of \((G\rightrightarrows P, D_G)\). Then \(\mathfrak{D} \subseteq P_G|_P\) defined as in ([18]) satisfies

\[
I^\ast(D_G) \subseteq \mathfrak{D} \subseteq \mathfrak{A}(D_G) \oplus (\ker T\mathfrak{t})|_P.
\]

Thus, the quotient \(\mathfrak{D} = \mathfrak{D} / I^\ast(D_G)\) is a smooth subbundle of \(\mathfrak{A}(D_G)\). We have by definition \(AH \oplus \{0\} \subseteq \mathfrak{D}\).

**Proof:** Choose \(p \in P\) and \((v_p, \alpha_p) \in I^\ast(D_G) = D_G(p) \cap \ker T\mathfrak{s}\). Then \(T\mathfrak{s}(v_p, \alpha_p) = (0_p, 0_p)\) and the product \((v_p, \alpha_p) \cdot (0_p, 0_p)\) makes sense. Since \((0_p, 0_p) \in D_G(pH)\), we have then \((T_p q v_p, \alpha_p \cdot 0_p) = (v_p, \alpha_p) \cdot (0_p, 0_p)\) \(\in D_G(pH)\). But \(\alpha_p \cdot 0_pH\) is such that \((T_p q)^\ast(\alpha_p \cdot 0_p) = \alpha_p \ast ((T_p q)^\ast 0_p) = \alpha_p\), and we have hence \((v_p, \alpha_p) \in \mathfrak{D}(p)\) by definition of \(\mathfrak{D}\).

The inclusion \(I^\ast(D_G) \subseteq \mathfrak{D}\) yields immediately \(\mathfrak{D} = \mathfrak{D}^\perp \subseteq (I^\ast(D_G))^\perp = D_G|_P + (\ker T\mathfrak{t})|_P = \mathfrak{A}(D_G) \oplus (\ker T\mathfrak{t})|_P\).

**Theorem 4.10** Let \((G\rightrightarrows P, D_G)\) be a Dirac groupoid and \(\mathfrak{D}\) a Dirac subspace of \(P_G|_P\) satisfying (19). Then the set \(\mathfrak{D} = D_G : \mathfrak{D} \subseteq P_G\) defined by

\[
\mathfrak{D}(g) = \left\{ (v_g, \alpha_g) \ast (v_{s(g)}, \alpha_{s(g)}) \mid (v_g, \alpha_g) \in D_G(g), \right. \\
\left. (v_{s(g)}, \alpha_{s(g)}) \in \mathfrak{D}(s(g)), \\
T\mathfrak{s}(v_g, \alpha_g) = T\mathfrak{t}(v_{s(g)}, \alpha_{s(g)}) \right\}
\]

is a Dirac structure on \(G\) and \((G, \mathfrak{D})\) is a Dirac homogeneous space of \((G\rightrightarrows P, D_G)\).
Proof: By Lemma 4.40 D is spanned by sections $\xi + \sigma^*$ such that $\xi \in \Gamma(\mathfrak{A}(D_G))$ and $\sigma \in \Gamma((\ker T)^{(p)})$ and all the sections of $D_G \cap \ker T$s. This shows that $D$ is smooth.

Choose $(v_g, \alpha_g) \ast (v_s(g), \alpha_s(g))$ and $(w_g, \beta_g) \ast (w_s(g), \beta_s(g)) \in D(g)$, that is, with $(v_g, \alpha_g), (w_g, \beta_g) \in D(g)$ and $(v_s(g), \alpha_s(g)), (w_s(g), \beta_s(g)) \in D(s(g))$. We have then

$$
\langle (v_g, \alpha_g) \ast (v_s(g), \alpha_s(g)), (w_g, \beta_g) \ast (w_s(g), \beta_s(g)) \rangle = \langle (w_g, \beta_g), (v_g, \alpha_g) \ast (v_s(g), \alpha_s(g)) \rangle + \langle \xi(g^{-1}), (v_g, \alpha_g)^{-1} \rangle = 0,
$$

since $(v_g, \alpha_g) \ast (v_s(g), \alpha_s(g)) \in D(g)$ and $(v_g, \alpha_g)^{-1} \in D(g)^{-1}$). This proves that

$$
\xi(g^{-1}) \ast (w_g, \beta_g) = (w_s(g), \beta_s(g)) \in D(s(g)),
$$

and hence, if we write $\xi(g^{-1}) = (w_g, \beta_g)^{-1} \ast (w_s(g), \beta_s(g)) \in D(g)$.

The second claim is obvious since the restriction to $D$ of the map TJ has image in $T\mathfrak{t}(D_G) = \mathfrak{A}(D_G)$ and, by construction of $D$, the map $D_G \times \mathfrak{A}(D_G) \to D \ni (v_g, \alpha_g) \mapsto (v_g, \alpha_g) \ast (v_h, \alpha_h)$ is a well-defined Lie groupoid.

Theorem 4.11 In the situation of the preceding theorem, if $D$ is the restriction to $P$ of the pullback $q^*(D_G/H)$ (as in [13]) for some Dirac homogeneous space $(G/H, D_G/H)$ of $(G \rightrightarrows P, D_G)$, then $D = q^*(D_G/H)$.

Proof: Choose $(v_g, \alpha_g) \in q^*(D_G/H)(g)$. Then $\alpha_g$ is equal to $(T_gq)^* \alpha_{gH}$ for some $\alpha_{gH} \in T_g^*H(gH)$ such that $(T_gqv_g, \alpha_{gH}) \in D_G/H(gH)$. Then $T\mathfrak{t}(T_gqv_g, \alpha_{gH}) = T\mathfrak{t}(v_g, \alpha_g) \in \mathfrak{A}(D_G)$ and there exists $(w_g, \beta_g^{-1}) \in D(g)^{-1}$ such that

$$
T\mathfrak{s}(w_g, \beta_g^{-1}) = T\mathfrak{t}(T_gqv_g, \alpha_{gH}).
$$

Set $p = s(g)$ and consider $(u_g, \gamma_g) := (w_g, \beta_g^{-1}) \ast (T_gqv_g, \alpha_{gH}) \in D_G/H(pH)$. Then we have

$$
(T_gq)^* \gamma_g \in (w_g, \beta_g^{-1}) \ast (T_gq)^* \alpha_{gH} = (w_g, \beta_g^{-1}) \ast \alpha_g
$$

by Proposition 4.4 about the action of $T^*G \rightrightarrows A^*G$ on $\widehat{\mathfrak{g}}: T^*(G/H) \to A^{\mathfrak{g}}$, and

$$
u_{gH} = w_g \ast (T_gqv_g) = T_gq(w_g \ast v_g).
$$

Thus, $(u_g, \gamma_g) := (w_g, \beta_g^{-1}) \ast (v_g, \alpha_g)$ is an element of $D(p)$, and we have $(v_g, \alpha_g) = (w_g, \beta_g^{-1})^{-1} \ast (u_g, \gamma_g)$. Since $D_G$ is multiplicative and $(w_g, \beta_g^{-1}) \in D_G(g^{-1})$, the pair $(w_g, \beta_g^{-1})^{-1}$ is an element of $D_G(g)$ and we have shown that $(v_g, \alpha_g) \in D(g)$.

Since $q^*(D_G/H) \subseteq D$ is an inclusion of Dirac structures, we have then equality.
Remark 4.12 Note that Theorem 4.11 shows that if $(G \rightrightarrows P, D_G)$ is a Dirac groupoid, a $D_G$-homogeneous Dirac structure on $G/H$ is uniquely determined by its restriction to $q(P) \subseteq G/H$. △

Example 4.13 We have seen in Example 4.7 that if $(G \rightrightarrows P, D_G)$ is a Dirac groupoid, then $(t : G \to P, D_G)$ is a Dirac homogeneous space of $(G \rightrightarrows P, D_G)$.

The space $\mathcal{D}$ is here the direct sum $P^*(D_G) \oplus \mathfrak{A}(D_G)$. The corresponding Dirac structure $\mathcal{D}$ is equal to $D_G$ by the last theorem. This can also be seen directly from the definition of $\mathcal{D}$, since $\mathcal{D}$ is spanned by the sections $(X, \theta)$ for $(X, \theta) \in \Gamma(\mathfrak{A}(D_G))$ and the sections $\sigma^*$ for all $\sigma \in \Gamma(P^*(D_G))$, which are spanning sections for $D_G$.

4.3 The Theorem of Drinfel’d

Recall that if $(G \rightrightarrows P, D_G)$ is a Dirac groupoid, then there is an induced action of the set of bisections $\mathcal{B}(G)$ of $G$ on the vector bundle $\mathfrak{B}(D_G)$ associated to $D_G$ (see Theorem 3.41). If $H$ is a wide Lie subgroupoid of $G \rightrightarrows P$, this action restricts to an action of $\mathcal{B}(H)$ on $\mathfrak{B}(D_G)$. We use this action to characterize $D_G$-homogeneous Dirac structures on $G/H$.

Theorem 4.14 Let $(G \rightrightarrows P, D_G)$ be a Dirac groupoid, $H$ a $t$-connected wide subgroupoid of $G$ such that the homogeneous space $G/H$ has a smooth manifold structure and $q : G \to G/H$ is a smooth surjective submersion. Let $\mathcal{D}$ be a Dirac subspace of $P_G|_{P}$ satisfying (19) and such that $AH \oplus \{0\} \subseteq \mathcal{D}$. Then the following are equivalent:

1. $\mathcal{D}$ is the pullback $q^*(D_{G/H})|_P$ as in (18), where $D_{G/H}$ is some $D_G$-homogeneous Dirac structure on $G/H$.
2. $\mathcal{D} = \mathcal{D}/P^*(D_G) \subseteq \mathfrak{B}(D_G)$ is invariant under the induced action of $\mathcal{B}(H)$ on $\mathfrak{B}(D_G)$.
3. The $D_G$-homogeneous Dirac structure $\mathcal{D} = D_G \cdot \mathcal{D} \subseteq P_G$ as in Theorem 4.10 pushes-forward to a $(D_G$-homogeneous) Dirac structure on the quotient $G/H$.

Note that, together with Theorem 4.11 this shows that a Dirac structure $D_{G/H}$ on $G/H$ is $D_G$-homogeneous if and only if $P^*(D_G) \subseteq (q^*D_{G/H})|_P$ and $q^*D_{G/H} = D_G \cdot (q^*D_{G/H})|_P$, that is, $(G \rightrightarrows P, D_G)$ is $(G \rightrightarrows P, D_G)$-homogeneous if and only if $(G, q^*D_{G/H})$ is.

For the proof of Theorem 4.14, we will need the following Lemma.

Lemma 4.15 In the situation of Theorem 4.14, we have $\mathcal{D} = \mathcal{D}|_P$.

Proof: Choose $p \in P$ and $(v_p, \alpha_p) \in \mathcal{D}(p)$. Then $Tt(v_p, \alpha_p) \in \mathfrak{A}_p(D_G) \subseteq D_G(p)$ and $(v_p, \alpha_p) = Tt(v_p, \alpha_p) \ast (v_p, \alpha_p) \in D(p)$. This shows $\mathcal{D} \subseteq \mathcal{D}|_P$ and we are done since both vector bundles have the same rank.

Proof (of Theorem 4.14): Assume first that $\mathcal{D} = q^*(D_{G/H})|_P$ for some $D_G$-homogeneous Dirac structure $D_{G/H}$ on $G/H$ and choose $K \in \mathcal{B}(H)$ and $(v_p, \alpha_p) \in D(p)$, $p \in P$. Then there exists $\alpha_{pH} \in T^*_{\mathcal{B}H}(G/H)$ such that $\alpha_{pH} = T_{qH}q^*H(\mathfrak{A}_p(D_G))$ such that $\alpha_{pH} = T_{qH}q^*H(D_G)$, and $(T_{qH}q^*H(D_G)) \subseteq D_G(p) \subseteq \mathfrak{B}(D_G)$, we have

$$\rho_K(v_p, \alpha_p) = (T_{qH}q^*H(v_p) \ast (\alpha_{pH} \ast \alpha_p)) + T_{qH}q^*H(D_G)$$

for any $(v_{h^{-1}}, \alpha_{h^{-1}}) \in D_G(h^{-1})$ satisfying $T\mathcal{S}(v_{h^{-1}}, \alpha_{h^{-1}}) = Tt(v_p, \alpha_p)$. Since

$$Tt(T_{qH}q^*H(p, \alpha_{pH})) = Tt(v_p, \alpha_p) = T\mathcal{S}(v_{h^{-1}}, \alpha_{h^{-1}}),$$

the product $(v_{h^{-1}}, \alpha_{h^{-1}}) \cdot (T_{qH}q^*H(p, \alpha_{pH}))$ makes sense and is an element of $D_G/(\mathcal{S}(h)H)$. Note that since $K \in \mathcal{B}(H)$, we have $q \circ R_K = q$. The pair $(T_{qH}R_N(v_{h^{-1}} \ast v_p), (T_{qH}R_N^{-1})^*(\alpha_{h^{-1}} \ast \alpha_p))$ satisfies $T_{qH}\mathcal{S}(q, v_{h^{-1}} \ast v_p) \in T\mathcal{S}(h)H(G/H)$,

$$T_{qH}q(T_{qH}R_N(v_{h^{-1}} \ast v_p)) = T_{qH}q(q \circ R_K(v_{h^{-1}} \ast v_p)) = T_{qH}q(v_{h^{-1}} \ast v_p) = v_{h^{-1}} \circ T_{qH}q(v_p)$$
and

\((T_{s(h)} R_K^{-1})^* (\alpha_{h^{-1}} \ast \alpha_p) = (T_{s(h)} R_K^{-1})^* (\alpha_{h^{-1}} \ast (T_p q)^* \alpha_{pH})\)

\[= (T_{s(h)} R_K^{-1})^* ((T_{h^{-1}} q)^* (\alpha_{h^{-1}} \cdot \alpha_{pH})) = (T_{s(h)} q)^* (\alpha_{h^{-1}} \cdot \alpha_{pH}).\]

Thus, \((T_{h^{-1}} R_K (v_{h^{-1}} \ast v_p), (T_{s(h)} R_K^{-1})^* (\alpha_{h^{-1}} \ast \alpha_p))\) is an element of \(\mathcal{D}(s(h))\) and \(\rho_K \left( (v_p, \alpha_p) \right)\) is an element of \(\mathcal{D}(s(h))\). This shows \((1) \Rightarrow (2)\).

Assume now that \(\mathcal{D}\) is invariant under the action of \(\mathcal{B}(H)\) on \(\mathcal{B}(D_G)\). Recall the backgrounds about Dirac reduction in Section 2.2. Set \(\mathcal{K} = \mathcal{H} \oplus \mathcal{T}^* G\), and hence \(\mathcal{K}^\perp = TG \oplus \mathcal{H}^0\).

We have \(AH \cap \{0\} \subseteq \mathcal{H}\) by hypothesis. By definition of \(\mathcal{D}\) and \(\mathcal{H}\), this yields immediately \(\mathcal{K} = \mathcal{H} \oplus \{0\} \subseteq \mathcal{D}\) and \(\mathcal{D} \cap \mathcal{K}^\perp = D\) has constant rank on \(G\). By (7), we have to show that \(\mathcal{D}\) is invariant under the right action of \(\mathcal{B}(H)\) on \(G\). We will use the fact that \(\mathcal{D}\) is spanned by the sections \(\sigma^r \in \Gamma(D_G \cap \ker T_s)\) for all \(\sigma \in \Gamma(P'(D_G))\) and \((X_\xi, \theta_\xi) + (X^l, s^* \alpha)\) for all sections \((X_\xi, \theta_\xi) + (X^l, s^* \alpha)|_p \in \Gamma(\mathcal{D}) \subseteq \Gamma(\mathcal{A}(D_G) \oplus (\ker T_T)|_p)\).

Choose \(K \in \mathcal{B}(H)\). It is easy to verify that

\[(R_K^* Z^r, R_K^* (t^* \gamma)) = (Z^r, t^* \gamma) \quad \text{for all} \quad (Z, (t^* \gamma)|_p) \in \Gamma(\ker T_s|_p).\]

Choose a section \((X_\xi, \theta_\xi) + (X^l, s^* \alpha)\) of \(D\). We want to show that \((R_K^* (X_\xi + X^l), R_K^* (\theta_\xi + s^* \alpha))\) is then also a section of \(D\). Choose \(g\) and \(s\) for generality set \(h = K(s(g)) \in H, p = s(h), q = t(h) = s(g)\) and \((\tilde{X}_\xi + X, \tilde{\theta}_\xi + s^* \alpha)(p) =: (u_p, \gamma_p) \in \mathcal{D}(p)\). Then \(((X_\xi, \theta_\xi) + (X^l, s^* \alpha))(g \ast h) = (X_\xi, \theta_\xi)(g \ast h) \ast (u_p, \gamma_p)\) and we can compute

\[
\left( R_K^* (X_\xi + X^l), R_K^* (\theta_\xi + s^* \alpha) \right)(g) = \left( T_{s(h)} R_K^{-1} (X_\xi + X^l)(g \ast h), (T_g R_K)^* ((\theta_\xi + s^* \alpha)(g \ast h)) \right) = \left( T_{s(h)} R_K^{-1} (X_\xi (gh) \ast u_p), (T_g R_K)^* (\theta_\xi(gh) \ast \gamma_p) \right).
\]

Choose \((v_g, \alpha_g) \in D_G(g)\) such that \(T_t(v_g, \alpha_g) = T_t(X_\xi(gh), \theta_\xi(gh))\). Then the product \((w_h, \alpha_h) := (v_g, \alpha_g)^{-1} \ast (X_\xi(gh), \theta_\xi(gh))\) is an element of \(D_G(h)\) such that \(T_s(w_h, \alpha_h) = (X_\xi, \theta_\xi)(p)\) and we have

\[
\left( R_K^* (X_\xi + X^l), R_K^* (\theta_\xi + s^* \alpha) \right)(g) = (v_g, \alpha_g) \ast \left( T_h R_K^{-1} (v_g^{-1} \ast X_\xi(gh) \ast u_p), (T_g R_K)^* (\alpha_g^{-1} \ast \theta_\xi(gh) \ast \gamma_p) \right) = (v_g, \alpha_g) \ast \left( T_h R_K^{-1} (w_h \ast u_p), (T_g R_K)^* (\beta_h \ast \gamma_p) \right).
\]

But since \(\mathcal{D}\) is invariant under the action of \(\mathcal{B}(H)\) on \(\mathcal{B}(D_G)\) and \((u_p, \gamma_p) = (u_p, \gamma_p) + I_p^* (D_G)\) is an element of \(\mathcal{D}(p)\), we have

\[
(T_h R_K^{-1} (w_h \ast u_p), (T_q R_K)^* (\beta_h \ast \gamma_p)) + I_p^* (D_G) = \rho_K^{-1} \left( (u_p, \gamma_p) \right) \in \mathcal{D}(g).
\]

Because \(I_p^* (D_G) \subseteq D_G(g)\), we have consequently

\[
(T_h R_K^{-1} (w_h \ast u_p), (T_q R_K)^* (\beta_h \ast \gamma_p) \in \mathcal{D}(q)
\]

and hence

\[
\left( R_K^* (X_\xi + X^l), R_K^* (\theta_\xi + s^* \alpha) \right)(g) = (v_g, \alpha_g) \ast \left( T_h R_K^{-1} (w_h \ast u_p), (T_q R_K)^* (\beta_h \ast \gamma_p) \right) \in D_G(g).
\]

We show then that the push-forward \(q(D)\) is a \(D_G\)-homogeneous Dirac structure on \(G/H\). By definition of \(T_J\), we have \(T_J(q(D)) = T_t(D) \subseteq T_t(D_G) = \mathfrak{a}(D_G)\). Choose \((v_{g_H}, \alpha_{g_H}) \in q(D)(gH)\) and \((w_{g'}, \beta_{g'}) \in D_G(g')\) such that \(T_s(w_{g'}, \beta_{g'}) = T_J(v_{g_H}, \alpha_{g_H})\). Then there exists \(v_g \in T_g G\) such that

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The pair \((v_g, (T_gq)^* \alpha_{gH})\) satisfies then \(T \ell(v_g, (T_gq)^* \alpha_{gH}) = T \ell(v_w, \beta_g) = T \ell(w_g, \beta_g) \in D(g) \) and since \((G, D)\) is a Dirac homogeneous space of \((G\rightrightarrows P, D_G)\), we have \( \langle w_g, \beta_g \rangle \in D(g) \) and the identities \((T_gq^*q)^* (\beta_g \cdot \alpha_{gH}) = \beta_q \cdot (T_gq)^* \alpha_{gH}\), and \(T_gq^*q\) is an element of \(q(D)(g)\) and \(q(D)\) is shown to be \(D_G\)-homogeneous. Hence, we have shown \((2) \Rightarrow (3)\).

To show that \((3)\) implies \((1)\), we have then just to show that the vector bundle \(\mathcal{D} \to P\) is the restriction to \(P\) of the pullback \(q^*(q(D))\). Since \(D|_P = \mathcal{D}\) by Lemma 4.15, we can show that \(D = q^*(q(D))\). This follows from the inclusion \(\mathcal{H} \oplus 0_{T_G} \subseteq D\). Choose \((v_g, \alpha_g) \in D(g)\). Then \(\alpha_g \in \mathcal{H}(g)\). Thus, there exists \(\alpha_g \in T^*_gH(G/H)\) such that \(\alpha_g = (T_gq)^* \alpha_{gH}\) and, by definition of \(q(D)\), the pair \((T_gq^*q)^* (\beta_q \cdot \alpha_{gH}) = \beta_q \cdot (T_gq)^* \alpha_{gH}\), and \(T_gq^*q\) is an element of \(q(D)(g)\). Thus, the pair \((w_g, \beta_g) \in D(g)\) is shown to be \(D_G\)-homogeneous. Hence, we have shown \((2) \Rightarrow (3)\).

**Theorem 4.16** Let \((G\rightrightarrows P, D_G)\) be a closed Dirac groupoid. In the situation of the previous theorem, the following are equivalent:

1. The Dirac structure \(q(D) = D_{G/H}\) is closed.
2. The Dirac structure \(D\) is closed.
3. The set of sections of \(\tilde{\mathcal{D}} \subseteq \mathcal{B}(D_G)\) is closed under the bracket on the sections of the Courant algebroid \(\mathcal{B}(D_G)\).

**Proof:** If \(D\) is closed, then \(q(D)\) is closed by a Theorem in [32] about Dirac reduction by foliations (see the generalities about Dirac reduction in Section 2.2). Conversely, assume that \(q(D)\) is closed. Since \(\mathcal{D} \subseteq T^G \oplus T^\mathcal{H}\) and by the proof of Theorem 4.14 the Dirac structure \(D\) is spanned by \(q\)-descending sections, that is, sections \((X, \alpha)\) such that \(\alpha \in \Gamma(\mathcal{H}^c)\) and \(R^G_{q}(X, \alpha) = (X, \alpha)\) for all \(K \in \mathcal{B}(H)\). Choose two descending sections \((X, \alpha), (Y, \beta)\) of \(D\). Choose \((\tilde{X}, \tilde{\alpha}), (\tilde{Y}, \tilde{\beta}) \in \Gamma(q(D))\) such that \((\tilde{X}, \tilde{\alpha} \sim q (X, \alpha)\) and \((Y, \beta) \sim q (Y, \beta)\). Then the bracket \([X, \alpha], [Y, \beta]\) descends to \([\tilde{X}, \tilde{\alpha}], [\tilde{Y}, \tilde{\beta}]\) which is a section of \(q(D)\) since \((G/H, q(D))\) is closed. But since \(\mathcal{H} \oplus 0_{T^G} \subseteq D\), we have \(D = q^*(q(D))\) (recall the proof of Theorem 4.14). Since \([X, \alpha], [Y, \beta]\) is a section of \(q^*(q(D))\), we have shown that \([X, \alpha], [Y, \beta] \in \Gamma(D)\). This proves \((1) \iff (2)\).

Assume that \((G, D)\) is closed and choose two sections \(e_{\xi, X, \alpha} = (\tilde{X}_\xi + X, \tilde{\theta}_\xi + s^* \alpha + I^* (D_G), e_{\eta, Y, \beta} = (\tilde{X}_\eta + Y, \tilde{\theta}_\eta + s^* \beta + I^* (D_G)) \subseteq \mathcal{B}(D_G)\). Then the two pairs \((X + X^t, \theta + s^* \alpha)\) and \((X + Y^t, \theta + s^* \beta)\) are smooth sections of \(D\) by construction and since \((G, D)\) is closed, we have \((X + X^t, \theta + s^* \alpha)\) and \((X + Y^t, \theta + s^* \beta)\) are smooth sections of \(D\). But since \(D = D|_P\) and \([e_{\xi, X, \alpha}, e_{\eta, Y, \beta}] = [(X + X^t, \tilde{\theta} + s^* \alpha), (X + Y^t, \tilde{\theta} + s^* \beta)]\) and \((X + Y^t, \theta + s^* \beta)\) are smooth sections of \(D\), we have \([e_{\xi, X, \alpha}, e_{\eta, Y, \beta}] = [(X + X^t, \tilde{\theta} + s^* \alpha), (X + Y^t, \tilde{\theta} + s^* \beta)]\) and \((X + Y^t, \theta + s^* \beta)\) are smooth sections of \(D\). Conversely, assume that \(\Gamma(\tilde{D})\) is closed under the Courant bracket on sections of \(\mathcal{B}(D_G)\) and choose two spanning sections \((X + X^t, \theta + s^* \alpha), (X + Y^t, \theta + s^* \beta)\) of \(D\) corresponding to \((X + X^t, \tilde{\theta} + s^* \alpha|_P)\) and \((X + Y^t, \theta + s^* \beta|_P) \in \mathcal{B}(D_G) \subseteq \Gamma(\mathcal{B}(D_G) \oplus (\ker T\ell)|_P\). Since \([e_{\xi, X, \alpha}, e_{\eta, Y, \beta}]\) is then an element of \(\mathcal{F}(\tilde{D})\) and \(I^* (D_G) \subseteq \tilde{D}\), we have

\[
[(X + X^t, \theta + s^* \alpha), (X + Y^t, \theta + s^* \beta)]|_P \in \Gamma(\tilde{D})
\]

by definition of the bracket on the sections of \(\mathcal{B}(D_G)\). By Theorem 3.29 Lemma 3.40 and [13], the value of \([[(X + X^t, \theta + s^* \alpha), (X + Y^t, \theta + s^* \beta)]|_P\) at \(g \in G\) equals

\[
\left\langle ([X + X^t, \theta + s^* \alpha], [X + Y^t, \theta + s^* \beta])|_P (s(g)) \right\rangle
\]
and we find that \([X + X^t, \theta + s^*\alpha, (X_\eta + Y^t, \theta_\eta + s^*\beta)]\) is a section of \(D\), since the first factor is an element of \(D_G(g)\), and the second an element of \(\mathcal{D}(s(g))\). Recall also that, by the proof of Theorem \ref{lemma:closedDiracSubspaces}, we know that \([\sigma_1, \sigma_2] \in \Gamma(D_G \cap \ker T_s)\) for all \(s \in \Gamma(I^s(D_G))\). Finally, since \(D_G\) is closed, we know that \([\sigma_1, \sigma_2] \in \Gamma(D_G)\) for all \(\sigma_1, \sigma_2 \in \Gamma(D_G \cap \ker T_s)\). Thus, by the Leibniz identity for the restriction to \(\Gamma(D)\) of the Courant bracket on \(P_G\), we have shown that \((G, D)\) is closed.

As a corollary of the Theorems \ref{thm:closedDiracSubspaces}, \ref{thm:DiracStructureProjective}, \ref{thm:DiracStructureQuotient} and \ref{thm:DiracStructurePullback} we get our main result, that generalizes the correspondence theorems in \cite{Hennion2015, Hennion2013} and \cite{Hennion2015}.

**Theorem 4.17** Let \((G \rightrightarrows P, D_G)\) be a Dirac groupoid. Let \(H\) be a wide Lie subgroupoid of \(G\) such that the quotient \(G/H\) is a smooth manifold and the map \(q : G \to G/H\) a smooth surjective submersion. There is a one-to-one correspondence between \(D_G\)-homogeneous Dirac structures on \(G/H\) and Dirac subspaces \(\mathcal{D}\) of \(P_G\) such that \(AH \times \{0\} + I^s(D_G) \subseteq \mathcal{D} \subseteq \mathfrak{A}(D_G) \oplus (\ker Tt)|_P\) and \(\mathcal{D} := \mathcal{D} / I^s(D_G)\) is a \(\mathcal{B}(H)\)-equivariant Dirac subspace of \(\mathcal{B}(D_G)\).

If \((G \rightrightarrows P, D_G)\) is closed, then closed \(D_G\)-homogeneous Dirac structures on \(G/H\) correspond in this way to Lagrangian subalgebroids \(\mathcal{D}\) of \(\mathcal{B}(D_G)\).

**Remark 4.18** Assume that \((G \rightrightarrows P, D_G)\) is a closed Dirac groupoid, \(\mathcal{D} \subseteq P_G|_P\) a Dirac subspace satisfying \(\ref{thm:DiracStructurePullback}\) and \(AH \times \{0\} \subseteq \mathcal{D}\) for some t-connected wide Lie subgroupoid \(H = G \rightrightarrows P\), and such that \(\mathcal{D} / I^s(D_G) \subseteq \mathcal{B}(D_G)\) is closed under the bracket on \(\mathcal{B}(D_G)\). It is easy to check (as in the proof of Theorem \ref{thm:DiracStructurePullback}) that we have then \([X^t, 0, (X, \alpha)] \in \Gamma(D_G \cdot \mathcal{D})\) for all \((X, \alpha) \in \Gamma(D_G^s \cdot \mathcal{D})\) and \(X \in \Gamma(AH)\). Since \(H\) is t-connected, we get then the fact that \(\mathcal{D} / I^s(D_G) \subseteq \mathcal{B}(D_G)\) for all bisections \(K \subseteq \mathcal{B}(H)\) and the Dirac structure \(\mathcal{D} \cdot \mathcal{D}\) projects to a Dirac structure on \(G/H\), that is \(D_G\)-homogeneous. The quotient \(\mathcal{D} / I^s(D_G)\) is then automatically invariant under the induced action of the bisections \(\mathcal{B}(H)\) on \(\mathcal{B}(D_G)\) and this shows that the condition 2 of Theorem \ref{thm:DiracStructurePullback} is always satisfied if \(D_G\) is closed, \(\mathcal{D} / I^s(D_G)\) is closed under the Courant bracket on sections of \(\mathcal{B}(D_G)\) and \(H\) is t-connected.

**Example 4.19** In \cite{Hennion2015}, it is shown that for a Poisson groupoid \((G \rightrightarrows P, \pi_G)\), there is a one to one correspondence between \(\pi_G\)-homogeneous Poisson structures on smooth homogeneous spaces \(G/H\) and regular closed Dirac structures \(L\) of the Courant algebroid \(AG \oplus A^*G\), such that \(H\) is the t-connected subgroupoid of \(G\) corresponding to the subalgebroid \(L \cap (AG \times 0_{A^*G})\). Since pullbacks to \(G\) of Poisson structures on \(G/H\) correspond to closed Dirac structures on \(G\) with characteristic distribution \(\mathcal{K}\), we recover this result as a special case of Theorem \ref{thm:DiracStructurePullback} using Remark \ref{thm:DiracStructurePullback} and the isomorphism in Example \ref{example:DiracSubspaces}.

Note that in this particular situation of a Poisson groupoid, Theorem \ref{thm:DiracStructurePullback} classifies not only the Poisson homogeneous spaces of \((G \rightrightarrows P, \pi_G)\), but all its (not necessarily closed) Dirac homogeneous spaces.

**Example 4.20** Let \((G \rightrightarrows P, \pi_G)\) be a Poisson groupoid and \(H\) a wide subgroupoid of \(G\). Assume that the Poisson structure descends to the quotient \(G/H\), i.e., that \(\pi_G\) is invariant under the action of the bisections of \(H\). Let \(\pi\) be the induced structure on \(G/H\). We show that \((G, q^*D_\pi)\) is a Dirac homogeneous space of \((G \rightrightarrows P, \pi_G)\). This is equivalent to the fact that \((G/H, \pi)\) is a Poisson homogeneous space of \((G \rightrightarrows P, \pi_G)\).

The Dirac structure \(q^*D_\pi\) is equal to \((\mathcal{H} \oplus 0_{T\pi(G)}) \oplus \text{Graph}\left(\pi_G^r \bigg|_{\mathcal{H}^o} : \mathcal{H} \to TG\right)\). Since \(\mathcal{H} \subseteq T^sG\), the inclusion \(\mathcal{Tt}(q^*D_\pi) \subseteq \mathfrak{A}(D_{\pi_G})\) is obvious. Choose \((v_g, \alpha_g) \in (q^*D_\pi)(g)\) and \(\alpha_h \in T^h_h G\) such that

\[
\mathcal{Tt}(\pi^r_G(\alpha_h), \alpha_h) = \mathcal{Tt}(v_g, \alpha_g).
\]

Then we have \((v_g, \alpha_g) = (u_g + \pi^r_G(\alpha_g), \alpha_g)\) with some \(u_g \in \mathcal{H}(g)\) and the product \((\pi^r_G(\alpha_h), \alpha_h) \ast (v_g, \alpha_g)\) is equal to

\[
(\pi^r_G(\alpha_h), \alpha_h) \ast (u_g + \pi^r_G(\alpha_g), \alpha_g) = (\pi^r_G(\alpha_h) \ast \pi^r_G(\alpha_g) + 0_h) \ast u_g, \alpha_g \ast \alpha_h)
\]

since \(\pi_G\) is multiplicative. The vector \(T_gL_h u_g\) is an element of \(\mathcal{H}\) by definition and consequently, \((\pi^r_G(\alpha_h), \alpha_h) \ast (u_g + \pi^r_G(\alpha_g), \alpha_g)\) is an element of \(q^*(D_\pi)\), which is shown to be \(\pi_G\)-homogeneous. It
corresponds to the Lagrangian subalgebroid \((AH \times 0_{T^* D}) \oplus \text{Graph}(\pi_H^* : AH^0 \to TP) + P(D_{\pi_G})\) of \(\mathfrak{B}(D_{\pi_G})\), or more simply, to the Lagrangian subalgebroid \(AH \oplus A^* H\) in the Courant algebroid \(AG \oplus A^* G\).

Thus, Theorem 4.17 together with the isomorphism in Example 3.36 shows that the multiplicative Poisson structure on \(G\) descends to \(G/H\) if and only if the Lagrangian subspace \(AH \oplus A^* H\) is a subalgebroid of the Courant algebroid \(AG \oplus A^* G\).

The Poisson homogeneous space that corresponds in this way to the Lagrangian subalgebroid \(AG \oplus 0_{A^* G}\) is the Poisson manifold \((P, \pi_P)\), where \(\pi_P\) is the Poisson structure induced on \(P\) by \(\pi_G\), see [30] and also Theorem 3.17. Note that the other trivial Dirac structure 0 of \(AG \oplus A^* G\) corresponds to \((G, \pi_G)\) seen as a Poisson homogeneous space of \((G\Rightarrow P, \pi_G)\) (see Example 2.4).

In the same manner, we can show that if a Dirac groupoid \((G\Rightarrow P, D_G)\) is invariant under the action of a wide subgroupoid \(H\), and the Dirac structure descends to the quotient \(G/H\), then \((G/H, q(D_G))\) is \((G\Rightarrow P, D_G)\)-homogeneous. For that, we use the formula \(q^*(q(D_G)) = \mathfrak{K}_H + D_G \cap \mathfrak{K}_G\). In particular, the Dirac structure on \(P\) obtained under some regularity conditions in Theorem 4.17 is \(D_G\)-homogeneous. As in the Poisson case, we find hence that the Dirac structure descends to \(G/H\) if and only if

\[
AH \oplus 0_{T^* P} \oplus \mathfrak{A}(D_G) \cap (TP \oplus A^* H) \subseteq \mathfrak{B}(D_G)
\]

is invariant under the induced action of \(\mathfrak{B}(H)\).

\(\diamondsuit\)

**Example 4.21** Let \((M, D_M)\) be a smooth Dirac manifold and \((M \times M \Rightarrow M, D_M \oplus D_M)\) the pair Dirac groupoid associated to it.

The wide Lie subgroupoids of \(M \times M \Rightarrow M\) are the equivalence relations \(R \subseteq M \times M\), and the corresponding homogeneous spaces are the products \(M \times M/R\). For instance, if \(\Phi : G \times M \to M\) is an action of a Lie group \(G\) (with Lie algebra \(\mathfrak{g}\)) on \(M\), the subset \(R_G = \{(m, \Phi_g(m)) \mid m \in M, g \in G\}\) is a wide subgroupoid of \(M \times M\), and \((M \times M)/R_G\) is easily seen to equal \(M \times M/G\). Hence, if the action is free and proper, the homogeneous space \((M \times M)/R_G\) has a smooth manifold structure such that the projection \(q : M \times M \to M \times M/G\) is a smooth surjective submersion.

In this case, the bisections of \(R_G\) are the diffeomorphisms of \(M\) that leave the orbits of \(G\) invariant. For instance, for every \(g \in G\), the map \(K_g : \Delta_M \simeq M \to M \times M \to m \mapsto (m, \Phi_g(m))\) is a bisection of \(R_G\).

Choose \((m, m) \in \Delta_M\). A vector \((v_m, w_m) \in T_{(m,m)}(M \times M)\) is an element of \(T^1_{(m,m)}R_G\) if and only if \(v_m = 0_m\) and \((0_m, w_m) \in T_{(m,m)}R_G\), that is if and only if \(v_m = 0_m\) and \(w_m \in T_m(G \cdot m)\). Thus, if \(V\) is the vertical space of the action, we find that \(\mathfrak{A}_G = \{\{0\} \oplus V\} \Delta_M\).

By Theorem 4.14 \(D_M \oplus D_M\)-homogeneous Dirac structures on \(M \times M/G\) are in one-one correspondence with Lagrangian subspaces \(\mathfrak{D}\) of \(\mathfrak{P}_G|\mathfrak{P}\) such that \(\mathfrak{P}(D_M \oplus D_M) \supseteq \mathfrak{D}\) and such that \(\mathfrak{D}/\mathfrak{P}(D_M \oplus D_M)\) is invariant under the induced action of \(\mathfrak{B}(R_G)\) on \(\mathfrak{B}(D_M \oplus D_M)\). But since \(\mathfrak{P}_{(m,m)}(D_M \oplus D_M) + A_{(m,m)}R_G = \{(v_m, \xi_M(m), \alpha_m, 0_m) \mid (v_m, \alpha_m) \in D_M(m), \xi \in \mathfrak{g}\}\), we find, using the isomorphism in Example 3.38 and the considerations in Example 3.15 that \((D_M \oplus D_M) \cdot \mathfrak{D}\) is a product of Dirac structures \(D_M \oplus D\) such that \(\mathfrak{V} \oplus 0_{\mathfrak{P}G} \subseteq \mathfrak{D}\) and \((\Phi^*_g X, \Phi^*_g \alpha) \in \Gamma(\mathfrak{D})\) for all \(g \in G\) and \((X, \alpha) \in \Gamma(\mathfrak{D})\). But Dirac structures \(\mathfrak{D}\) satisfying these conditions are exactly the pullbacks to \(M\) of Dirac structures on \(M/G\) and we find that the \(D_M \oplus D_M\)-homogeneous Dirac structures on \(M \times M/G\) are of the form \(D_M \oplus \mathfrak{D} := D_M \oplus q_G(D)\), where \(q_G : M \to M/G\) is the canonical projection.

With Example 3.38 and Theorem 4.16 we get hence that \(\mathfrak{D}\) is closed if and only if \(D\) is closed. \(\diamondsuit\)

**Example 4.22** The left invariant Dirac structures on a Lie group \(G\) are the homogeneous structures relative to the trivial Poisson bracket on \(G\) ([10]). Hence, if we consider this example in the groupoid situation, we should recover the “right” definition for left invariant Dirac structures on a Lie groupoid. We say that a Dirac structure \(\mathfrak{D}\) on a Lie groupoid \(G\Rightarrow P\) is left-invariant if the action \(T\Phi\) of \(TG \oplus T^* G\) on \(Tt : TG \oplus T^* G \to TP \oplus A^* G\) restricts to an action of \(0_{TG} \oplus T^* G\) on \(D\), i.e.,

\[
(0_{TG} \oplus T^* G) \cdot \mathfrak{D} = \mathfrak{D}.
\]
In [18], a Dirac structure on a Lie groupoid $G \rightrightarrows P$ is said to be left-invariant if it is the pullback under the map
\[
\Phi : T^* G \oplus T^* G \to AG \oplus A^* G
\]
\[
(v_g, \alpha_g) \mapsto (T_g L_{g^{-1}} v_g, \hat{s}(\alpha_g)) \in A_{s(g)} G \times A_{s^* (g)}^* G
\]
of a Dirac structure in $AG \oplus A^* G$. These two definitions are easily seen to be equivalent, the inclusion $0_{T G \oplus (T^* G)^*} \subseteq D$ is immediate and it is easy to check that $D$ is invariant under the lifted right actions of the bisections if and only if the corresponding Dirac structure in $\mathfrak{B}(T^* G)$ is invariant under the induced action of $\mathfrak{B}(G)$ on $\mathfrak{B}(T^* G)$ (compare with Proposition 6.2 in [18]).

The result in Theorem 4.16 implies that a left-invariant Dirac structure $D$ is closed if and only if the corresponding Dirac structure $\Phi(D|_P) \subseteq AG \oplus A^* G$ is a subalgebroid. ♦

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