A Note on Lie-Lorentz Derivatives

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Abstract

The definition of “Lie derivative” of spinors with respect to Killing vectors is extended to all kinds of Lorentz tensors. This Lie-Lorentz derivative appears naturally in the commutator of two supersymmetry transformations generated by Killing spinors and vanishes for Vielbeins. It can be identified as the generator of the action of isometries on supergravity fields and its use for the calculation of supersymmetry algebras is revised and extended.
Introduction

Spinors are defined by their transformation properties under $SO(n_+, n_-)$ ("Lorentz") transformations and, thus, they can only be introduced in the tangent space of curved spaces using the formalism introduced by Weyl in Ref. [1]. This formalism makes use of Vielbeins $\{e_a^\mu\}$ which form an orthonormal basis in tangent space

$$ e_a^\mu e_b^\nu g_{\mu\nu} = \eta_{ab}, \quad (\eta_{ab}) = \text{diag}(+ \cdots +, - \cdots -),$$

and it is covariant w.r.t. local transformations that preserve this orthonormality: local $SO(n_+, n_-)$ transformations. These are the only transformations that act non-trivially on any Lorentz tensor

$$ T'(x') = \Gamma_r[g(x)]T(x),$$

where $\Gamma_r[g(x)]$ is the representation of the position-dependent group element $g(x)$ that can be constructed by exponentiation

$$ \Gamma_r[g(x)] = \exp[\frac{i}{2} \sigma^{ab}(x) \Gamma_r(M_{ab})]$$

where $\Gamma_r(M_{ab})$ are the $SO(n_+, n_-)$ generators in the representation $r$. For the contravariant vector and spinor representations

$$ \Gamma_v(M_{ab})^c_d = 2\eta^{[c} \eta_{d]}, \quad \Gamma_s(M_{ab}) = \frac{i}{2} \Gamma_{[a} \Gamma_{b]}, \quad \{\Gamma_a, \Gamma_b\} = 2\eta_{ab}.$$ (4)

Local Lorentz covariance is required in order to be able to gauge away the additional degrees of freedom that the Vielbein ($d^2$) has, as compared with the metric ($d(d+1)/2$), and it is achieved by introducing a covariant derivative $D_\mu$ which acts on $T$ according to

$$ D_\mu T \equiv (\partial_\mu - \omega_{r\mu}) T, \quad \omega_{r\mu} \equiv \frac{1}{2} \omega^{ab}_{\mu} \Gamma_r(M_{ab}) .$$ (5)

$\omega^{ab}_{\mu}$ is the $SO(n_+, n_-)$ (spin) connection, i.e. it is antisymmetric $\omega^{ab}_{\mu} = -\omega^{ba}_{\mu}$, which implies

$$ D_\mu \eta_{ab} = 0 .$$ (6)

and transforms according to

$$ \omega'_{r\mu} = \Gamma_r[g(x)]\omega_{r\mu}\Gamma^{-1}_r[g(x)] + \left(\partial_\mu \Gamma_r[g(x)]\right)\Gamma^{-1}_r[g(x)].$$ (7)

In order to have only one connection the spin connection is related to the affine connection $\Gamma_{\mu\nu}^\rho$ by the Vielbein postulate

$$ \nabla_\mu e_a^\nu = \partial_\mu e_a^\nu + \Gamma_{\mu\rho}^\nu e_a^\rho - e_b^\nu \omega_{\mu a}^b = 0,$$ (8)

We will call Lorentz tensor any object transforming in some finite-dimensional representation of the Lorentz group like vectors and spinors.
\( \nabla_\mu \) denotes the total (general and Lorentz) covariant derivative) which implies

\[
\omega_{\mu a}^b = \Gamma_{\mu a}^b - e_a^\nu \partial_\mu e_\nu^b. \tag{9}
\]

The Vielbein postulate together with Eq. (3) imply that the affine connection is metric compatible and the spin connection is completely determined by the Vielbein, their inverses, and the contorsion tensor \( K_{abc} \):

\[
\omega_{abc} = -\Omega_{abc} + \Omega_{bca} - \Omega_{cab} + K_{abc}, \quad \Omega_{abc} = e_a^\mu e_b^\nu \partial_{[\mu} e_{c]\nu]. \tag{10}
\]

In Weyl’s formalism Lorentz tensors (in particular spinors) behave, then, as scalars under general coordinate transformations (g.c.t.’s). However, we can use Weyl’s formalism just to work in curvilinear coordinates in Minkowski spacetime and we would find that a standard Lorentz transformation becomes a g.c.t. that does not act on the spinorial indices. This looks strange, but it is unavoidable in Weyl’s formalism.

On the other hand, Lorentz transformations and g.c.t.’s do not commute and the result of a g.c.t. on a Lorentz tensor is strongly frame-dependent. This shows up in the standard Lie derivative (an infinitesimal g.c.t.) which does not transform Lorentz tensors into Lorentz tensors. Thus, while the Lie derivative of the metric \( g_{\mu\nu} \) with respect to a Killing vector field vanishes (by definition) the Lie derivative of the inverse Vielbein \( e_a^\mu \) associated to the same metric with respect to the same Killing vector in general does not.

For spinors \( \psi \) this situation was solved in Ref. [3] where a spinorial Lie derivative \( \mathbb{L}_k \) with respect to Killing vector fields \( k \) was defined by

\[
\mathbb{L}_k \psi \equiv k^\mu D_\mu \psi + \frac{1}{2} D_{[a} k_{b]} \Gamma^{ab} \psi. \tag{11}
\]

This derivative is a derivation that transforms spinors into spinors (it is Lorentz-covariant) and satisfies the property

\[
[\mathbb{L}_{k_1}, \mathbb{L}_{k_2}] \psi = \mathbb{L}_{[k_1, k_2]} \psi, \tag{12}
\]

but, most importantly, the second term defines an action of certain g.c.t.’s (the isometries) on the spinorial indices: an infinitesimal Lorentz transformation with parameter \( \frac{1}{2} D_{[a} k_{b]} \). This is precisely what one expects on physical grounds. Let us consider an example: Minkowski spacetime \( g_{\mu\nu} = \eta_{\mu\nu} \) with the obvious Vielbein \( e_a^\mu = \delta_a^\mu \) and the infinitesimal g.c.t. generated by the Killing vector

\[
k^\mu = \sigma^\mu _\nu x^\nu, \tag{13}
\]

where \( \sigma^{\mu\nu} = -\sigma^{\nu\mu} \) and constant. This is a standard infinitesimal Lorentz transformation and, indeed, we find

\[2\text{For a different approach an further references, see, e.g. } [4].\]

\[3\text{We use the symbol } \mathbb{L}_k \text{ to avoid confusion with the standard Lie derivative that we keep denoting by } \mathcal{L}_k. \text{ On spinors, then, } \mathcal{L}_k \psi = k^\mu \partial_\mu \psi.\]
\[ \mathbb{L}_k = k^\mu \partial_\mu \psi + \frac{1}{4} \sigma_{ab} \Gamma^{ab} \psi, \]  
which is the result that one would obtain in the standard spinor formalism.

The spinorial Lie derivative can be seen as a Lorentz covariantization of the standard Lie derivative (first term) supplemented by an infinitesimal local Lorentz transformation that trivializes the holonomy. It is clear that these ideas can be extended to other Lorentz tensors and we can define a Lie-lorentz derivative which is first a Lorentz covariantization of the standard Lie derivative supplemented by an infinitesimal Lorentz transformation that trivializes the holonomy. The parameter of this transformation has to be exactly the same as in the spinorial case and, thus, we arrive to the following definition.

## 1 Definition and Properties of the Lie-Lorentz Derivative

On pure Lorentz tensors \( T \) we define the Lie-Lorentz derivative with respect to the Killing vector \( k \) by

\[ \mathbb{L}_k T \equiv k^\rho \nabla_\rho T + \frac{1}{2} \nabla_{[a} k_{b]} \Gamma^r_{\, (M^{ab})} T. \]  

On tensors that also have world indices \( T_{\mu_1 \cdots \mu_m}^{\nu_1 \cdots \nu_n} \)

\[ \mathbb{L}_k T_{\mu_1 \cdots \mu_m}^{\nu_1 \cdots \nu_n} \equiv k^\rho \nabla_\rho T_{\mu_1 \cdots \mu_m}^{\nu_1 \cdots \nu_n} - \nabla_\rho k^{\nu_1} T_{\mu_1 \cdots \mu_m}^{\rho \nu_2 \cdots \nu_n} - \ldots \]

\[ + \nabla_{\mu_1} k^\rho T_{\rho \mu_2 \cdots \mu_m}^{\nu_1 \cdots \nu_n} + \ldots + \frac{1}{2} \nabla_{[a} k_{b]} \Gamma^r_{\, (M^{ab})} T_{\mu_1 \cdots \mu_m}^{\nu_1 \cdots \nu_n}. \]  

In all the cases that we are going to consider \( \nabla_\mu \) is the full (affine plus Lorentz) torsionless covariant derivative satisfying the Vielbein postulate.

In the following \( T_1, T_2 \) will be two mixed tensors of any kind, \( k_1, k_2 \) any two conformal Killing vector fields and \( a^1, a^2 \) two arbitrary constants. The Lie-Lorentz derivative has the following basic properties:

1. Leibnitz rule:

\[ \mathbb{L}_k (T_1 T_2) = \mathbb{L}_k (T_1) T_2 + T_1 \mathbb{L}_k T_2. \]

Thus, it is a derivation.

2. The commutator of two Lie-Lorentz derivatives

\[ [\mathbb{L}_{k_1}, \mathbb{L}_{k_2}] T = \mathbb{L}_{[k_1, k_2]} T, \]  

where \([k_1, k_2]\) is their Lie bracket.
3. The Lie-Lorentz derivative is linear in the vector fields

\[ L_{a}^{\mu}k_{1} + a^{2}k_{2} T = a^{1} L_{k_{1}} T + a^{2} L_{k_{2}} T , \] (19)

and, thus, the Lie-Lorentz derivative with respect to the conformal killing vector fields forms a representation of the Lie algebra of conformal isometries of the manifold.

Some immediate consequences of the definition and basic properties are:

1. The Lie-Lorentz derivative of the Vielbein is

\[ L_{k} e_{\mu}^{a} = \frac{1}{d} \nabla_{\rho} k_{\rho} e_{\mu}^{a} , \] (20)

and vanishes when \( k \) is a Killing vector. In this case, we have the desirable property

\[ L_{k} \xi^{a} = e_{\mu}^{a} L_{k} \xi^{\mu} . \] (21)

2. The Lie-Lorentz derivative of gamma matrices is zero.

\[ L_{k} \gamma^{a} = 0 . \] (22)

3. As a consequence of Eqs. (17), (20) and (22), the Lie-Lorentz derivative with respect to Killing vectors preserves the Clifford action of vectors \( v \) on spinors \( \psi \):

\[ [L_{k}, v] \psi = [k, v] \cdot \psi . \] (23)

4. Also for Killing vectors \( k \) it preserves the covariant derivative

\[ [L_{k}, \nabla_{v}] T = \nabla_{[k,v]} T . \] (24)

5. All these properties imply that the Lie-Lorentz derivative with respect to Killing vectors preserves the supercovariant derivative of supergravity theories, at least in the simplest cases that we are going to examine next.

We stress that the properties 3, 4, 5 are only valid for Killing (not just conformal Killing) vectors.
2 The Lie-Lorentz Derivative and Supersymmetry

The Lie-Lorentz derivative occurs naturally in supergravity theories. To start with, let us consider the local on-shell supersymmetry algebra, in particular the commutator of two infinitesimal, local supersymmetry transformations $\delta_Q(\epsilon)$ in $N = 1, d = 4$ supergravity

$$\delta_Q(\epsilon) e^a_{\mu} = -i \bar{\epsilon} \gamma^a \psi_{\mu}, \quad \delta_Q(\epsilon) \psi_{\mu} = D_{\mu} \epsilon.$$  \hfill (25)

which is usually written in the form

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{\text{gct}}(\xi) + \delta_{\text{LL}}(\sigma) + \delta_Q(\epsilon),$$  \hfill (26)

where $\delta_{\text{gct}}(\xi)$ is an infinitesimal general coordinate transformation with parameter

$$\xi^\mu = -i \bar{\epsilon}_1 \gamma^\mu \epsilon_2.$$  \hfill (27)

and is given by $\delta_{\text{gct}}(\xi) = -\mathcal{L}_\xi$, $\delta_{\text{LL}}(\sigma)$ is an infinitesimal local Lorentz transformation with parameter

$$\sigma^{ab} = \xi^\nu \omega^{ab}_\nu,$$  \hfill (28)

and where

$$\epsilon = \xi^\mu \psi_{\mu}.$$  \hfill (29)

We are interested in obtaining the global superalgebra from the commutators of all the symmetry transformations of the theory. There is no unique superalgebra associated to a given supergravity theory, rather there are superalgebras associated to given bosonic solutions of the supergravity theory and their (super) symmetries. Solutions with a high degree of (super) symmetry are usually considered vacua of the supergravity theory and their associated superalgebras are of special interest.

Let us, then, consider a given vacuum solution of the $N = 1, d = 4$ supergravity equations of motion admitting Killing spinors $\varepsilon$ and Killing vectors $k$

$$D_{\mu} \varepsilon = 0, \quad \nabla_{(\mu} k_{\nu)} = 0.$$  \hfill (30)

On this vacuum, the commutator Eq. (26) should reduce to the commutator of two supercharges $Q_{1,2} = \varepsilon_{1,2} Q$ which should be proportional to translations $\sim k^a P_a$ where $k^a = -i \bar{\varepsilon}_1 \gamma^a \varepsilon_2$ is a Killing vector if, as we have assumed, $\varepsilon_{1,2}$ are Killing spinors. However, the commutator Eq. (26) does not give that result if we naively interpret $\delta_{\text{gct}}(k)$ as an infinitesimal translation since there is another term $\delta_{\text{LL}}$ whose meaning is unclear.

Observing that, actually, all the Killing vectors $k^a = -i \bar{\varepsilon}_1 \gamma^a \varepsilon_2$ are covariantly constant, we can write instead

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_P(k), \quad k^a = -i \bar{\varepsilon}_1 \gamma^a \varepsilon_2$$  \hfill (31)

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where now we have identified the generator of the vacuum isometries acting on the supergravity fields

$$\delta_P(k) \equiv -\mathbb{L}_k.$$  \hfill (32)

This commutator has an immediate interpretation in terms of the global superalgebra. On the other hand, in this form, on account of Eq. (20), it is evident the the commutator of two supersymmetry transformations generated by two Killing spinors leaves invariant the Vierbein, as it should be.

To check that these definitions and interpretations actually make sense, let us consider a more complicated case: gauged $N = 2, d = 4$ supergravity, whose supersymmetry transformation rules are

$$\delta_Q(\epsilon^a) e^a_\mu = -i\bar{\epsilon}\gamma^a \psi_\mu, \quad \delta_Q(\epsilon) A_\mu = -i\bar{\epsilon}\sigma^2 \psi_\mu, \quad \delta_Q(\epsilon) \psi_\mu = \tilde{D}_\mu \epsilon,$$  \hfill (33)

where

$$\tilde{\nabla}_\mu = \hat{D}_\mu + igA_\mu \sigma^2 + \frac{1}{4} \tilde{F}\gamma_\mu \sigma^2,$$  \hfill (34)

is the supercovariant derivative and

$$\hat{D}_\mu = D_\mu - ig\frac{1}{2} \gamma_\mu,$$  \hfill (35)

is the $AdS_4 \sim SO(2, 3)$ covariant derivative. The commutator of two $N = 2, d = 4$ supersymmetry transformations is usually written in this form

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{gct}(\xi) + \delta_{LL}(\sigma) + \delta_e(\Lambda) + \delta_Q(\epsilon),$$  \hfill (36)

where the parameter of the infinitesimal local Lorentz transformations is now

$$\sigma^{ab} = \xi^\mu \omega^{ab}_\mu - g \bar{\epsilon}_2 \gamma^{ab} \epsilon_1 - i\bar{\epsilon}_2 \left( \tilde{F}^{ab} + i\gamma_5 \tilde{F}^{ab} \right) \sigma^2 \epsilon_1,$$  \hfill (37)

and where $\delta_e(\Lambda)$ are $U(1)$ gauge transformations of the vector field and charged gravitino with parameter

$$\Lambda = -i\bar{\epsilon}_2 \sigma^2 \epsilon_1 + \xi^\nu A_\nu.$$  \hfill (38)

It is easy to see that on a vacuum solution admitting Killing spinors $\varepsilon$ and Killing vectors $k$

$$\tilde{\nabla}_\mu \varepsilon = 0, \quad \nabla(\mu k_\nu) = 0.$$  \hfill (39)

the commutator Eq. (36) can be written in the form

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_P(k) + \delta_e(\chi), \quad k^a = -i\bar{\varepsilon}_1 \gamma^a \varepsilon_2, \quad \chi = -i\bar{\varepsilon}_2 \sigma^2 \varepsilon_1 + k^\nu A_\nu.$$  \hfill (40)
The interpretation is again immediate. This commutator should vanish on all the fields of the theory. It clearly does on the Vierbein. On the vector and gravitino fields, it tells us that the Lie derivative vanishes up to a gauge transformation.

Let us consider now the remaining commutators.

Since we are identifying the bosonic generators of the the bosonic subalgebra of the supersymmetry algebra of the vacuum with the Lie-Lorentz derivative with respect to the Killing vector fields of the solution, we are going to get as bosonic subalgebra the Lie algebra of isometries of the vacuum, on account of Eq. (18):

\[
[\delta_P(k_1), \delta_P(k_2)] = \delta_P([k_1, k_2]).
\] (41)

The commutator of local supersymmetry transformations and g.c.t.’s. these are difficult to compute, as a matter of principle, since the standard Lie derivative of Lorentz tensors is not a Lorentz tensor and then it does not make sense to perform on the transformed tensor a further supersymmetry transformation. Further, while we have a prescription to calculate the effect of an infinitesimal g.c.t. on any geometrical object, we do not know how to calculate the effect of an infinitesimal supersymmetry transformation of geometrical objects which are not fields of our theory (for instance, the vector that generates the infinitesimal g.c.t.).

Thus, it is necessary to give a prescription to calculate these commutators. In Ref. [7] the following prescription was proposed:

\[
[\delta_Q(\varepsilon), \delta_P(k)] = \delta_Q(\mathcal{L}_k \varepsilon).
\] (42)

This prescription is based on the property (checked for certain geometrical Killing spinors in that reference) that the Lie-Lorentz derivative preserves the supercovariant derivative and, therefore, transforms Killing spinors into Killing spinors.

This property also holds in the simple theories we are considering here: on account of Eqs. (23),(24), we get

\[
[\mathcal{L}_k, \hat{D}_v] \varepsilon = \hat{D}_{[k,v]} \varepsilon,
\] (43)

(so the \textit{AdS}_4 covariant derivative is preserved and the Lie-Lorentz derivative transforms \( N = 1, \textit{AdS}_4 \) Killing spinors into Killing spinors [7]) and, if

\[
\mathcal{L}_k A_a = 0,
\] (44)

we find that the \( N = 2, \textit{AdS}_4 \) supercovariant derivative is also preserved

\[
[\mathcal{L}_k, \hat{D}_v] \varepsilon = \hat{D}_{[k,v]} \varepsilon,
\] (45)

and the Lie-Lorentz derivative transforms again Killing spinors into Killing spinors.

The condition Eq. (44) is satisfied in most cases in which \( k \) is a Killing vector (up to a gauge transformation). If it was not satisfied in any gauge, the Killing vector \( k \) would be an isometry but not a symmetry of the complete supergravity background and would not be a generator of the vacuum supersymmetry algebra. Thus, it must always be satisfied.
It is clear that these results can be generalized to higher dimensions and supersymmetries.

Acknowledgments
I would like to thank E. Bergshoeff and specially P. Meessen for interesting conversations, the Institute for Theoretical Physics of the University of Groningen for its hospitality and financial support and M.M. Fernández for her continuous support. This work has been partially supported by the Spanish grant FPA2000-1584.

References
[1] H. Weyl, Z. Phys. 330 56 (1929). Translated in Ref. [2]
[2] L. O’Raifeartaigh, Princeton University Press, Princeton, New Jersey (1997).
[3] Y. Kosmann, Annali di Mat. Pura Appl. (IV) 91 (1972) 317-395.
[4] D. J. Hurley and M. A. Vandyck, J. Phys. A 27 (1994) 4569.
[5] M. A. Vandyck, Gen. Rel. Grav. 20 (1988) 261.
[6] M. A. Vandyck, Gen. Rel. Grav. 20 (1988) 905.
[7] J. M. Figueroa-O’Farrill, Class. Quant. Grav. 16 (1999) 2043 [arXiv:hep-th/9902066].