Pfaffian form of the Grammian determinant solutions of the BKP hierarchy

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Abstract

The Grammian determinant type solutions of the KP hierarchy, obtained through the vectorial binary Darboux transformation, are reduced, imposing suitable differential constraint on the transformation data, to Pfaffian solutions of the BKP hierarchy.

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1. Binary Darboux transformations for the KP hierarchy \[7\] after iteration give Grammian determinant expressions for new potentials and wave functions. For the BKP hierarchy \[2\] instead one finds Pfaffian expressions, see \[3\] for a bilinear approach, dressing the zero background, and \[6\] for direct Darboux transformation approach (in both papers only the BKP equation was considered, not the hierarchy); in fact the appearance of Pfaffians was described in \[2\].

As the BKP hierarchy is a reduction of the KP hierarchy \[2\], it is natural to expect a relation between Grammian and Pfaffian expressions. That is, the Grammian solutions of the KP hierarchy reduces to Pfaffian solutions of the BKP hierarchy, when suitable constraints are imposed in the transformation data. In this short note we show that this correspondence holds.

The letter is organized as follows. Next, in \(\S\) 2 we remind the reader about some basic facts regarding the KP hierarchy and the vectorial binary Darboux transformation \[4\], and also about the BKP hierarchy and how to reduce the mentioned vectorial transformation to it. Section 3 is devoted to show that these Grammian expressions reduce to Pfaffians.

2. The KP hierarchy can be formulated as the compatibility of the following linear system:

\[
\frac{\partial \psi}{\partial t_n} = B_n(\psi), \quad n \geq 1, \tag{1}
\]

where

\[
B_n := \partial^n + \sum_{m=0}^{n-2} u_{n,m} \partial^m,
\]

with \(\partial := \frac{\partial}{\partial x}, \quad x = t_1\). There is also an adjoint linear system

\[
\frac{\partial \psi^*}{\partial t_n} = -B_n^*(\psi^*), \quad n \geq 1, \tag{2}
\]

where

\[
B_n^* := (-1)^n \partial^n + \sum_{m=0}^{n-2} (-1)^m \partial^m u_{n,m}.
\]

The functions \(u_{n,m}\) are differential polynomials in \(u_{2,0} =: u\).
The vectorial binary Darboux transformation is constructed in terms of the potential defined by the following compatible equations:

$$\frac{\partial \Omega(\psi, \psi^*)}{\partial t_n} = - \text{res}(\partial^{-1} \psi \otimes B_n^* \psi^* \partial^{-1}), \quad n \geq 1,$$

where we are working in the ring of pseudodifferential operators in $\partial$, see for example [7], and assuming that $\psi, \psi^*$ take values in a vectorial spaces $V$ and $W^*$ (the dual of $W$), respectively. The first equation is

$$\partial(\Omega(\psi, \psi^*)) = \psi \otimes \psi^*,$$

that for $V = V^* = \mathbb{C}^N$ has the form of a Grammian matrix when $\psi^* = \psi^\dagger$.

**Vectorial Binary Darboux Transformation** [7, 4]. For any pair of vectorial wave and adjoint wave functions, $\xi \in V = \mathbb{C}^N, \xi^* \in V^*$, respectively, we construct a new potential $\tilde{u}$ and wave function $\tilde{\psi}$,

$$\tilde{u} = u - \partial^2 \ln \det \Omega(\xi, \xi^*),$$

$$\tilde{\psi} = \psi - \Omega(\psi, \xi^*)\Omega(\xi, \xi^*)^{-1} \xi,$$

$$= \frac{1}{|\Omega(\xi, \xi^*)|} \left[ \begin{array}{c} \psi \\ \Omega(\psi, \xi^*) \end{array} \right].$$

The BKP hierarchy is the compatibility of

$$\frac{\partial \psi}{\partial t_{2n+1}} = B_{2n+1}(\psi), \quad n \geq 0,$$

where

$$B_{2n+1} := \partial^{2n+1} + \sum_{m=0}^{n-2} v_{n,m} \partial^{2m+1}.$$ 

Being $v_{n,m}$ differential polynomials in $v_{3,1}/3 =: v$. Observe that if in the KP hierarchy one has the constraint

$$B_{2m+1}^* \partial + \partial B_{2m+1} = 0, \quad m \in \mathbb{N},$$

then the odd flows correspond to the BKP hierarchy. Thus, given a BKP $\Psi(x, t_3, \ldots)$ wave function one can construct wave and adjoint wave functions for the odd flows restriction of the KP hierarchy:

$$\psi(x, 0, t_3, 0, t_5, \ldots) = \Psi(x, t_3, \ldots),$$

$$\psi^*(x, 0, t_3, 0, t_5, \ldots) = \partial \Psi(x, t_3, \ldots).$$
Moreover, we should have the identity
\[ u(x, 0, t_3, 0, t_5, \ldots) = v(x, t_3, \ldots). \]

It can be shown that given transformation data \((\xi, \xi^*)\) such that
\[ \xi^*(x, 0, t_3, 0, t_5, \ldots) = \partial \xi^t(x, 0, t_3, 0, t_5, \ldots) \]
and choosing the potential satisfying the consistent constraint
\[ \Omega(\xi, \xi^*) + \Omega(\xi, \xi^*)^t = \xi \otimes \xi^t, \]
\[ \Omega(\psi, \xi^*) + \Omega(\xi, \psi^*)^t = \psi \xi^t, \]
the corresponding vectorial binary Darboux transformation, when applied to a KP wave function \(\psi\) coming from BKP one, is such that \(\tilde{\psi}(x, t_3, \ldots) := \psi(x, 0, t_3, 0, t_5, \ldots)\) is a wave function of the BKP hierarchy.

Hence, the vectorial binary Darboux transformation can be constrained such that it preserves the BKP reduction. Thus, we have Grammian type determinant solutions of the BKP hierarchy.

Observe that the potential matrices \(\Omega := \Omega(\xi, \xi^*)\) and \(\Omega(\psi, \xi^*)\) split into its symmetric and skew-symmetric parts as follows
\[ \Omega = \frac{1}{2} [\xi \otimes \xi^t + S], \quad (3) \]
\[ \Omega(\psi, \xi^*) = \frac{1}{2} [\psi \xi^t + S(\psi, \xi^*)^t], \quad (4) \]
where \(S\) and \(S(\psi, \xi)\) are skew-symmetric potentials and for example satisfy
\[ \partial S = \xi \otimes \partial \xi^t - (\partial \xi) \otimes \xi^t, \]
\[ \partial S(\psi, \xi) = \psi \partial \xi - (\partial \psi) \xi. \]

3. Our aim now is to show that these Grammian expressions can be written, as a consequence of the constraints, in terms of Pfaffians. First we shall evaluate the determinant of \(\Omega\), from \(3\) we have
\[ |\Omega| = \frac{1}{2^N} |\xi \otimes \xi^t + S|. \]
Observe that the \(j\)-th column of \(\xi \otimes \xi^t + S\) is \(\xi_j \xi + S_j\), with
\[ S_j = (S_{1,j}, \ldots, S_{j-1,j}, 0, S_{j+1,j}, \ldots, S_{N,j})^t. \]
Hence,

$$|\Omega| = \frac{1}{2^N} \left[ \sum_{j=1}^{N} \xi_j |S(j)| + |S| \right]$$

with the matrix $S(j)$ obtained from $S$ by replacing the $j$-th column by $\xi$. Thus, we have

$$|\Omega| = \frac{1}{2^N} \left[ \begin{vmatrix} 0 & -\xi^t \\ \xi & S \end{vmatrix} + |S| \right],$$

and recalling that the determinant of an odd skew-symmetric matrix vanishes we conclude

$$|\Omega| = \begin{cases} \frac{1}{2^N} |S|, & \text{for } N \text{ even,} \\ \frac{1}{2^N} \left| \begin{vmatrix} 0 & -\xi^t \\ \xi & S \end{vmatrix} \right|, & \text{for } N \text{ odd.} \end{cases}$$

Using the fact that the determinant of an even skew-symmetric matrix is the square of a Pfaffian [5, 1]

$$|\Omega| = \begin{cases} \frac{1}{2^N} \text{Pf}(S)^2 & \text{for } N \text{ even,} \\ \frac{1}{2^N} \left( \text{Pf} \left( \begin{vmatrix} 0 & -\xi^t \\ \xi & S \end{vmatrix} \right) \right)^2 & \text{for } N \text{ odd.} \end{cases}$$

Now, we will compute the Pfaffian expressions for the wave function. As before, we will distinguish between the even and odd cases.

For even $N = 2k$ we use the first representation for the wave function:

$$\tilde{\psi} = \psi - \Omega(\psi, \xi^*) \Omega(\xi, \xi^*)^{-1} \xi.$$ 

Computing the inverse through the Cramer rule we get

$$\tilde{\psi} = \psi - \Omega(\psi, \xi^*) \cdot \frac{\Omega}{|\Omega|}$$

where

$$\Omega = \begin{pmatrix} |\Omega(1)| \\ \vdots \\ |\Omega(2)| \end{pmatrix}$$
with $\Omega(j)$ obtained from $\Omega$ by replacing the $j$-th column by $\xi$; an alternative formula is

$$\Omega = \frac{1}{2^{N-1}} \begin{pmatrix} |S(1)| \\ \vdots \\ |S(N)| \end{pmatrix} =: \frac{1}{2^{N-1}} S.$$ 

Using (4) we find

$$\tilde{\psi} = \psi - \Omega(\psi, \xi^*) \cdot \frac{\psi \xi^t + S(\psi, \xi)}{2|\Omega|},$$

and noting that

$$\xi \cdot \Omega = \frac{1}{2^{N-1}} \begin{vmatrix} 0 & -\xi^t \\ \xi & S \end{vmatrix} = 0,$$

we find out

$$\tilde{\psi} = \psi - S(\psi, \xi) \cdot \frac{S}{|S|}.$$ 

It can be shown [1]

$$|S(i)| = \text{Pf}(S) \text{Pf}(S[i])$$

where $S[i]$ is obtained from $S$ by replacing their $i$-th row and $i$-th column with $-\hat{\xi}^t$ and $\hat{\xi}$, respectively, where

$$\hat{\xi} := \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{i-1} \\ 0 \\ \xi_{i+1} \\ \vdots \\ \xi_N \end{pmatrix}.$$ 

Thus,

$$\tilde{\psi} = \psi - S(\psi, \xi) \frac{(\text{Pf}(S[1]), \ldots, \text{Pf}(S[N]))^t}{\text{Pf}(S)} = \frac{\text{Pf} \left( \begin{pmatrix} 0 & \psi & S(\psi, \xi)^t \\ -\psi & 0 & -\xi^t \\ -S(\psi, \xi) & \xi & S \end{pmatrix} \right)}{\text{Pf}(S)},$$

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where we have used the expansion rule for Pfaffians. This completes the even case and we proceed with odd case. Now, we use the second representation for the wave function, namely,

$$
\tilde{\psi} = \begin{vmatrix} \psi & \Omega(\psi, \xi^*) \\ \xi & \Omega \\ \end{vmatrix}/|\Omega|.
$$

Notice that

$$
\begin{vmatrix} \psi & \Omega(\psi, \xi^*) \\ \xi & \Omega \\ \end{vmatrix} = \frac{1}{2^{N-1}} \begin{vmatrix} \psi & S(\psi, \xi)^t \\ \xi & S \\ \end{vmatrix},
$$

where we used (3) and (4).

As in the even case, we have

$$
\begin{vmatrix} \psi & S(\psi, \xi)^t \\ \xi & S \\ \end{vmatrix} = \text{Pf} \left( \begin{array}{cc} 0 & S(\psi, \xi)^t \\ -S(\psi, \xi) & S \end{array} \right) \text{Pf} \left( \begin{array}{cc} 0 & -\xi^t \\ \xi & S \end{array} \right).
$$

Using the above formula, we arrive

$$
\tilde{\psi} = \frac{\text{Pf} \left( \begin{array}{cc} 0 & S(\psi, \xi)^t \\ -S(\psi, \xi) & S \end{array} \right)}{\text{Pf} \left( \begin{array}{cc} 0 & -\xi^t \\ \xi & S \end{array} \right)}.
$$

We can summarize the above results as follows

**Pfaffians and the Reduced Vectorial Binary Darboux Transformation.**

The reduction of the Grammian determinant type solutions of the KP hierarchy to the BKP hierarchy gives the following Pfaffian expressions:

1. For $N$ even

$$
\tilde{v} = v - 2\partial^2(\ln \text{Pf}(S)),
\quad
\tilde{\Psi} = \frac{\text{Pf} \left( \begin{array}{cccc} 0 & \Psi & S(\Psi, \xi)^t \\ -\Psi & 0 & -\xi^t \\ -S(\Psi, \xi) & \xi & S \end{array} \right)}{\text{Pf}(S)}.
$$
2. For $N$ odd

$$\tilde{\nu} = v - 2\partial^2 \left( \ln \text{Pf} \begin{pmatrix} 0 & -\xi^t \\ \xi & S \end{pmatrix} \right),$$

$$\tilde{\Psi} = \frac{\text{Pf} \begin{pmatrix} 0 & S(\Psi, \xi)^t \\ -S(\Psi, \xi) & S \end{pmatrix}}{\text{Pf} \begin{pmatrix} 0 & -\xi^t \\ \xi & S \end{pmatrix}}.$$  

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