On the stability of exact ABCs for the reaction-subdiffusion equation on unbounded domain*

Can Li1†

1Department of Applied Mathematics, School of Sciences,
Xi’an University of Technology, Xi’an, Shaanxi 710054, P.R. China.

Abstract

In this note we propose the exact artificial boundary conditions formula to the fractional reaction-subdiffusion equation on an unbounded domain. With the application of Laplace transformation, the exact artificial boundary conditions (ABCs) are derived to reformulate the original problem on the unbounded domain to an initial-boundary-value problem on the bounded computational domain. By the Kreiss theory, we prove that the reduced initial-boundary value problem is stability. Based on the properties of tempered fractional calculus, we obtain that the reduced initial-boundary value problem is long-time stability.

Keywords: Fractional reaction-subdiffusion equation, Exact artificial boundary conditions, Stability, Tempered fractional calculus, Stability.

1 Introduction

In the past two decades, the continuous time random walks (CTRWs) model are popular to describe the movement of anomalous diffusion particles on the mesoscopic level [17, 16]. For the sub-diffusion particles, with a long tailed waiting time density and with a reduction in particle concentration driven by constant per capita linear reaction dynamics, this process can be described by the following reaction-sub-diffusion equation [18, 11, 12]

\[
\frac{\partial u(x, t)}{\partial t} = \kappa \gamma_0 D_t^{1 - \gamma, \lambda}(u_{xx}) - \lambda u(x, t), x \in \mathbb{R}, t > 0,
\]

where \( \kappa \) is the positive diffusion coefficient, \( \lambda \) is a constant reaction rate and the operator \( D_t^{1 - \gamma, \lambda} \) denotes the Riemann-Liouville tempered fractional derivative [11, 12, 15]

\[
D_t^{\alpha, \lambda} u(t) = e^{-\lambda t} D_t^{\alpha} (e^{\lambda t} u(t)) = \frac{e^{-\lambda t}}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t e^{\lambda(s - t)} u(s) ds, \ 0 < \alpha < 1.
\]

Compare with the previous fractional diffusion models, model (1) behaviors many good properties, such as it preserves the positivity of solution and recovers the reaction kinetic equation for homogeneous concentrations, for more physical meaning about this model we refer the references [18, 11, 12]. The model (1) is derived in infinite domain. To get the exact solution of model (1), we usual need the assumption on boundary

\[
u \to 0, \ |x| \to \infty.
\]

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†Corresponding author. Email addresses: mathlican@xaut.edu.cn(C.Li)
However, to numerically solve this model, we need impose proper boundaries on the finite computing domain. One of powerful tools to overcome the difficulty of unboundedness of the domain is the artificial boundary methods (ABMs)\cite{9}. The main idea of ABMs is to design suitable boundary conditions to absorb the waves arriving at artificial boundaries. And it has been successfully used to solve many kinds of linear and nonlinear partial differential equations with classical derivatives, see the review papers \cite{2,10}. So far, few work is reported for the fractional diffusion equation on unbounded domain due to the non-locality of fractional operators. In the literature, the exact ABCs are constructed for the one-dimensional time fractional diffusion equation in \cite{6,7}, and the convergence rate and stability of the finite difference methods are established. Brunner et al \cite{4} obtained a series of artificial boundary conditions (ABCs) for two-dimensional time fractional diffusion-wave equation. Ghaffari and Hossein \cite{8} derive the exact ABCs for sub-diffusion equation by using circle artificial boundaries. Awotunde et al. \cite{3} obtained ABCs for a modified fraction diffusion problem. Using the similar approach given by Engquist and Majda, Dea obtains a ABCs for the two-dimensional time-fractional wave equation in his recent work \cite{5}. It is important that the boundaries are designed such that the reduced problem is well-posed. In this note, we focus on the stability of model (1) with proper artificial boundary. By the Kreiss theory, we prove that the reduced initial-boundary value problem is stability. Base on the properties of tempered fractional calculus, we obtain that the reduced initial-boundary value problem is long-time stability.

The rest of the note is organized as follows. In Section 2 we derive of the exact ABCs for model (1). The original problem on bounded domain with the boundary condition (3) is reduced an initial-boundary-value problem on a bounded computational domain. In Section 3, we investigate the stability of the reduced initial-boundary value problem.

## 2 The exact artificial boundary conditions

We first derive the exact artificial boundary conditions for the fractional reaction-subdiffusion equation (1) with the boundary condition (3). If we introduce two artificial boundaries $x_l$ and $x_r$, Then the real line $\mathbb{R}$ is divided into three parts: $\Omega_l := \{x| -\infty < x < x_l\}$, $\Omega_i := \{x|x_l < x < x_r\}$, $\Omega_r := \{x|x_r < x < \infty\}$. The constants $x_l$, $x_r$ are chosen such that sup$\{u_0(x)\} \subset \Omega_i$. Denote the exterior domain by $\Omega_e = \Omega_l \cup \Omega_r$. Restricting the solution $u(x,t)$ to $\Omega_e$, we have

\[ \frac{\partial u(x,t)}{\partial t} = \kappa_\gamma 0D_t^{1-\gamma,\lambda} (u_{xx}) - \lambda u(x,t), \quad x \in \Omega_e, t > 0, \]  \[ u(x,0) = 0, \quad x \in \Omega_e, \]  \[ u(x,t)|_{x=x_l} = u(x_l,t) \]  \[ u(x,t)|_{x=x_r} = u(x_r,t), \]  \[ u \to 0, \quad \text{as} \ |x| \to \infty. \]

To derive artificial boundary conditions, we introduce the Laplace transform of a function $f(t)$ by

\[ \mathcal{L}\{f(t); s\} = \hat{f}(s) = \int_0^{+\infty} e^{-st} f(t) dt, \quad \text{Re}(s) > 0, \]  and the inverse Laplace transform of a function $g(s)$ gives

\[ \mathcal{L}^{-1}\{g(s); t\} = g(t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{st} g(s) ds, \quad i^2 = -1. \]

**Lemma 1** For $0 < \alpha < 1$, the Laplace transform of the Riemann-Liouville tempered fractional derivative \cite{7,11,12}

\[ 0D_t^{\alpha,\lambda} f(t) = \frac{e^{-\lambda t}}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{e^{\lambda s} f(s)}{(t-s)^\alpha} ds, \]  \[ (11) \]
is given by
\[ \mathcal{L}\{_0D_t^{\alpha,\lambda}f(t); s\} = (s + \lambda)^\alpha \hat{f}(s). \]  
(12)

And the Laplace transform of the Caputo tempered fractional derivative is directly form
\[ C^\alpha_0D_t^{\alpha,\lambda}f(t) = \frac{e^{-\lambda t}}{\Gamma(1 - \alpha)} \int_0^t \frac{1}{(t - s)^\alpha} \frac{d(e^{\lambda s}f(s))}{ds} ds, \]  
(13)

is given by
\[ \mathcal{L}\{C^\alpha_0D_t^{\alpha,\lambda}f(t); s\} = (s + \lambda)^\alpha \hat{f}(s) - (s + \lambda)^{\alpha-1} f(0). \]  
(14)

**Proof.** The Laplace transform of the Caputo tempered fractional derivative is directly form
\[ (15). \]  
For the Laplace transform of the Riemann-Liouville tempered fractional derivative, from
\[ (15), \]  
we have
\[ \mathcal{L}\{_0D_t^{\alpha,\lambda}f(t); s\} = (s + \lambda)^\alpha \hat{f}(s) - (s + \lambda) \left[ _0D_t^{\alpha-1}(e^{\lambda t}f(t))\right]_{t=0}. \]  
(15)

where \(_0D_t^{\alpha-1,\lambda}\) denotes the Riemann-Liouville integral operator given as
\[ _0D_t^{\alpha-1,\lambda}f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t e^{-\lambda(t-s)}(t - s)^{-\alpha} f(s)ds. \]  
(16)

Furthermore, using the fact
\[ \lim_{t \to 0} \frac{1}{\Gamma(1 - \alpha)} \int_0^t e^{-\lambda(t-s)}(t - s)^{-\alpha} f(s)ds \]
\[ = \lim_{t \to 0} \frac{e^{-\lambda t}}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} f(s)e^{\lambda s} ds \]
\[ = \lim_{t \to 0} \frac{1}{\Gamma(\alpha)} \left[ t^{1-\alpha}e^{-\lambda t}f(0) + \int_0^t (t - s)^{1-\alpha}e^{-\lambda(t-s)}[f'(s) + \lambda f(s)]ds \right] \]
\[ = 0. \]  
(17)

and the assumption on the smoothness of \(f(t)\), we get \((12).\)

Using relation \((12),\) and notice that the initial value \((13),\) we obtain
\[ (s + \lambda)^\gamma \Hat{u}(x, s) = \kappa_\gamma \Hat{u}_{xx}(x, s), \quad x \in \Omega_e, \quad \text{Re}(s) > 0 \]
\[ \Hat{u} \to 0, \quad \text{as } \mid x \mid \to \infty. \]  
(18)

(19)

Using the boundary conditions \((19),\) the solutions of differential equation \((18)\) gives
\[ \Hat{u}(x, s) = c_1(s)e^{-\frac{(s+\lambda)^{\gamma/2}}{\sqrt{\kappa_\gamma}}x}, \quad x \in \Omega_r, \]  
(20)
\[ \Hat{u}(x, s) = c_2(s)e^{-\frac{(s+\lambda)^{\gamma/2}}{\sqrt{\kappa_\gamma}}x}, \quad x \in \Omega_l. \]  
(21)

Differentiating equations \((20)\) and \((21)\) with respect to variable \(x\), we have
\[ \Hat{u}_x(x, s) = -\frac{(s + \lambda)^{\gamma/2}}{\sqrt{\kappa_\gamma}}\Hat{u}(x, s), \quad x \in \Omega_r, \]  
(22)
\[ \Hat{u}_x(x, s) = \frac{(s + \lambda)^{\gamma/2}}{\sqrt{\kappa_\gamma}}\Hat{u}(x, s), \quad x \in \Omega_l. \]  
(23)
Lemma 3 ([20])
Suppose that the Parseval relation
Lemma 2 ([20])
given by (22)-(23), give as
where we used the Laplace transform of the Caputo fractional derivative (14) and the initial value
condition (5). If take (24)-(25) as artificial boundary conditions, then the previous problem (4)-(5)
on unbounded domain is reduced to the following initial-boundary value problem on bounded domain
\[
\frac{\partial u(x,t)}{\partial t} = \kappa_{\gamma} D_t^{1-\gamma, \lambda}(u_{xx}) - \lambda u(x,t), \quad x \in \Omega_{in}, \ t > 0, \tag{28}
\]
\[
u(x,0) = u_0(x), \quad x \in \Omega_i, \tag{29}
\]
\[
u_x(x,t) = \frac{1}{\sqrt{2\pi}} \tilde{C}_0 D_t^{2, \lambda} u(x,t), \quad x = x_l, \tag{30}
\]
\[
u_x(x,t) = -\frac{1}{\sqrt{2\pi}} \tilde{C}_0 D_t^{2, \lambda} u(x,t), \quad x = x_r. \tag{31}
\]

3 Stability analysis of the reduced problems

Next we consider the stability analysis of the reduced problem (28)-(31). The main tool is the Kreiss theory [13]. Firstly, we introduce the notations of the inner product, the classic $L^2$ norm given by
\[
(u,v) = \int_{\Omega_i} u(x)v(x)dx, \quad \|u\|^2_{L^2(\Omega_i)} = \|u\|^2 = (u,u).
\]

Lemma 2 ([20]) (Parseval relation) Suppose that $u(t), v(t)$ is well defined on $[0, +\infty)$, then for $s_0 > 0$ we have
\[
\int_{-\infty}^{+\infty} \hat{u}(s_0 + i\zeta)\hat{v}(s_0 - i\zeta)d\zeta = 2\pi \int_0^{+\infty} e^{-2s_0t}u(t)v(t)dt, t^2 = -1. \tag{32}
\]

Lemma 3 ([20]) Suppose that $v(t)$ is well defined on $[0, \infty)$, then for $s_0 > 0$ we have
\[
\int_0^{+\infty} e^{-2s_0t} \frac{dv(t)}{dt} v(t)dt \leq 0. \tag{33}
\]
then
\[
\int_0^{+\infty} e^{-2s_0t} v^2(t)dt \leq \frac{1}{2s_0} v^2(0). \tag{34}
\]

Theorem 1 The initial-boundary value problem (28)-(31) holds the prior estimate
\[
\|u\|^2_{L^2(H^2_\omega)} \leq \frac{1}{2s_0} \|u_0(x)\|^2, \tag{35}
\]
where $\|u\|_{L^2(H^2_\omega)} = \int_0^{+\infty} \|u(\cdot,t)\|^2 e^{-2s_0t}dt$. 
\[
\]
Proof. Applying the Laplace transform to (28)-(31), we have

\[
\frac{\partial \hat{u}(x, s)}{\partial t} = \kappa_\gamma (s + \lambda)^{1-\gamma} \hat{u}_{xx}(x, s) - \lambda \hat{u}(x, s), \quad x \in \Omega_in, \tag{36}
\]
\[
u(x, 0) = u_0(x), \quad x \in \Omega_i, \tag{37}
\]
\[
\hat{u}_x(x_t, s) = \frac{(s + \lambda)^{\gamma/2}}{\sqrt{\kappa_\gamma}} \hat{u}(x_t, s), \tag{38}
\]
\[
\hat{u}_x(x_r, s) = -\frac{(s + \lambda)^{\gamma/2}}{\sqrt{\kappa_\gamma}} \hat{u}(x_r, s), \tag{39}
\]
where \(\frac{\partial \hat{u}(x, s)}{\partial t} := s\hat{u}(x, s) - u_0(x)\). Multiplying equation (36) by \(\hat{u}(x, s)\) and integrate on \(\Omega_in\), we obtain

\[
\left( \frac{\partial \hat{u}}{\partial t}, \hat{u} \right) = \kappa_\gamma (s + \lambda)^{1-\gamma} \left( \hat{u}_{xx}(x, s), \hat{u}(x, s) \right) - \lambda \left( \hat{u}, \hat{u} \right). \tag{40}
\]
By integrating (40) by parts we get

\[
\left( \frac{\partial \hat{u}}{\partial t}, \hat{u} \right) = -\kappa_\gamma (s + \lambda)^{1-\gamma} \|\hat{u}_x\|^2 - \kappa_\gamma (s + \lambda)^{1-\gamma} \hat{u}_x(x, s) \overline{\hat{u}(x, s)} \bigg|_{x_l}^{x_r} - \lambda \|\hat{u}\|^2. \tag{41}
\]
In view of the boundary conditions (39), we have

\[
\left( \frac{\partial \hat{u}}{\partial t}, \hat{u} \right) = -\kappa_\gamma (s + \lambda)^{1-\gamma} \|\hat{u}_x\|^2 - \lambda \|\hat{u}\|^2 \\
- \sqrt{\kappa_\gamma} (s + \lambda)^{1-\gamma} \left[ \|\hat{u}(x_l, s)\|^2 + |\hat{u}(x_r, s)|^2 \right]. \tag{42}
\]
Hence, we have

\[
Re\left( \left( \frac{\partial \hat{u}}{\partial t}, \hat{u} \right) \right) = -\kappa_\gamma Re(\gamma (s + \lambda)^{1-\gamma} \|\hat{u}_x\|^2 - \lambda Re(\|\hat{u}\|^2) \\
- \sqrt{\kappa_\gamma} Re(\gamma (s + \lambda)^{1-\gamma} \|\hat{u}(x_l, s)\|^2 + |\hat{u}(x_r, s)|^2). \tag{43}
\]
Using the fact \(\arg(s_0 + \lambda + i\zeta) \in (-\pi/2, \pi/2)\) for \(s_0 + \lambda > 0\), we have

\[
(s_0 + \lambda + i\zeta)^\beta = \left((s_0 + \lambda + i\zeta) e^{i\arg(s_0 + \lambda + i\zeta)}\right)^\beta = \left((s_0 + \lambda)^2 + (\zeta)^2\right)^{\beta/2} e^{\beta i \arg((s_0 + \lambda + i\zeta))}. \tag{44}
\]
which means \(Re((s + \lambda)^\beta) = |(s + \lambda)|^\beta \cos(\beta \arg((s + \lambda))) > 0\), for \(\beta = 1 - \gamma\) or \(1 - \frac{\gamma}{2}, \gamma \in (0, 1)\). Above inequality implies \(Re((s + \lambda)^{1-\gamma}) > 0\) and \(Re((s + \lambda)^{1-\gamma}) > 0\), and hence \(Re\left( \left( \frac{\partial \hat{u}}{\partial t}, \hat{u} \right) \right) \leq 0\). Paseval’s relation (32) then leads

\[
\int_{-\infty}^{\infty} \left( \frac{\partial \hat{u}}{\partial t}, \hat{u} \right) (s_0 + i\zeta) d\zeta = \int_{\Omega_i} \int_{-\infty}^{\infty} \frac{\partial \hat{u}}{\partial t} (s_0 + i\zeta) \overline{\hat{u}(s_0 + i\zeta)} d\zeta dx \\
= 2\pi \int_{\Omega_i} \int_{0}^{\infty} e^{-2s_0 t} \frac{\partial u}{\partial t} u(t) dt dx. \tag{45}
\]
Furthermore, using Lemma \(3\), we have

\[
\int_{\Omega_i} \int_{0}^{\infty} e^{-2s_0 t} \frac{\partial u}{\partial t} u(x, t) dt dx \leq \frac{1}{2s_0} \int_{\Omega_i} u_0^2(x) dx, \quad (48)
\]
which is the desired inequality (35).
The model (11) can be rewritten as

\[
\frac{\partial e^{\lambda t}u(x,t)}{\partial t} = \kappa_\gamma 0D_t^{1-\gamma}(e^{\lambda t}u_{xx}).
\]  

(49)

Performing Riemann-Liouville fractional integral operator \(0D_t^{\gamma-1}\) on both side of (49), using the composite properties fractional derivative and integral [17], we arrive at

\[
0D_t^{\gamma}(e^{\lambda t}u(x,t)) = \kappa_\gamma e^{\lambda t}u_{xx},
\]  

(50)
or, equivalently,

\[
0D_t^{\gamma-\lambda}(u(x,t)) = \kappa_\gamma u_{xx},
\]  

(51)

Hence, the reduced problem (28)-(31) equivalent to the following initial-boundary value problem

\[
0D_t^{\gamma}(e^{\lambda t}u(x,t)) = \kappa_\gamma e^{\lambda t}u_{xx},
\]  

(52)

\[
u(x, 0) = u_0(x),
\]  

(53)

\[
(e^{\lambda t}u(x,t))_x = \frac{1}{\sqrt{\kappa_\gamma}} 0D_t^{\frac{\gamma}{2}}(e^{\lambda t}u(x,t)),
\]  

(54)

\[
(e^{\lambda t}u(x,t))_x = -\frac{1}{\sqrt{\kappa_\gamma}} 0D_t^{\frac{\gamma}{2}}(e^{\lambda t}u(x,t)),
\]  

(55)

For the initial-boundary value problem (52)-(55), we have the long-time stability.

**Theorem 2** The solution \(u(x,t)\) of the initial-boundary value problem (52)-(55) holds the prior estimate

\[
\|u(x,t)\|^2 + 2\kappa_\gamma 0D_t^{-\gamma,2\lambda}\|u_x(x,t)\|^2 \leq e^{-2\lambda t}\|u_0(x)\|^2.
\]  

(56)

where \(0D_t^{-\gamma,2\lambda}\) denotes the Riemann-Liouville fractional integral operator given in (16).

**Proof.** Taking \(v(x,t) = e^{\lambda t}u(x,t)\), form (52)-(55), we have

\[
0D_t^{\gamma}v(x,t) = \kappa_\gamma v_{xx}(x,t),
\]  

(57)

\[
u(x, 0) = u_0(x),
\]  

(58)

\[
v_x(x,t) = \frac{1}{\sqrt{\kappa_\gamma}} 0D_t^{\frac{\gamma}{2}}v(x,t),
\]  

(59)

\[
v_x(x,t) = -\frac{1}{\sqrt{\kappa_\gamma}} 0D_t^{\frac{\gamma}{2}}v(x,t),
\]  

(60)

We multiply equation (57) by \(v(x,t)\) and integrate on \(\Omega_i\), we get

\[
(0D_t^{\gamma}v, v) = -\kappa_\gamma \|v_x\|^2 + \kappa_\gamma v_x(x,t)v(x,t)|^{x_r}_{x_l}.
\]  

(61)

In view of the boundary conditions (59)-(60), we have

\[
(0D_t^{\gamma}v, v) = -\kappa_\gamma \|v_x\|^2 - \sqrt{\kappa_\gamma} \left[ v(x_r,t)0D_t^{\frac{\gamma}{2}}v(x_r,t) + v(x_l,t)0D_t^{\frac{\gamma}{2}}v(x_l,t) \right].
\]  

(62)

With help of the inequality [1]

\[
w(t)0D_t^{\alpha}w(t) \geq \frac{1}{2}\alpha 0D_t^{\alpha}(w^2(t)), 0 < \alpha < 1,
\]  

(63)

and the fact

\[
0D_t^{\frac{\gamma}{2}}[v^2(x_r,t) + v^2(x_l,t)] \geq 0,
\]

(64)
we get
\[ C_0 D_t^\gamma \|v\|^2 + 2\kappa_\gamma \|v_x\|^2 \leq 0. \]  
(64)

By applying the fractional integral operator \( D_t^{-\gamma} \) to both sides of inequality (64), using the fact [14]
\[ D_t^{-\gamma} D_t^\gamma(w(t)) = w(t) - w(0), \]
we obtain the estimate
\[ \|v\|^2 + 2\kappa_\gamma C_0 D_t^{-\gamma} \|v_x\|^2 \leq \|v_0(x)\|^2. \]  
(65)

Taking \( u(x,t) = e^{-\lambda t}v(x,t) \) we get the estimate (56).

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References

[1] A.A. Alikhanov, A priori estimates for solutions of boundary value problem for fractional-order equations, Diff. Eq., 46 (2010) 660-666.
[2] X. Antoine, A. Arnold, C. Besse, M. Ehrhardt, A. Schädle, A Review of transparent and artificial boundary conditions techniques for linear and nonlinear Schrödinger equations, Commun. Comput. Phys., 4(2008)729-796.
[3] A.A. Awotunde, R.A. Ghanam, N. Tatar, Artificial boundary condition for a modified fractional diffusion problem, Boundary Value Problems, 1(2015) 1-17.
[4] H. Brunner, H. Han, D. Yin, Artificial boundary conditions and finite difference approximations for a time-fractional diffusion-wave equation on a two-dimensional unbounded spatial domain, J. Comput. Phys., 276(2014) 541-562.
[5] J. R. Dea, Absorbing boundary conditions for the fractional wave equation, Appl. Math. Comput., 219 (2013) 9810-9820.
[6] G. H. Gao, Z. Z. Sun, Y. N. Zhang, A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions, J. Comput. Phys., 231 (2012) 2865-2879.
[7] G. H. Gao, Z. Z. Sun, The finite difference approximation for a class of fractional sub-diffusion equations on a space unbounded domain, J. Comput. Phys., 236 (2013) 443-460.
[8] R. Ghaffari, S. M. Hosseini, Obtaining artificial boundary conditions for fractional sub-diffusion equation on space two-dimensional unbounded domains, Comput. Math. Appl., 68 (2014) 13-26.
[9] H. Han, X. Wu, Artificial Boundary Method, Tsinghua Univ. Press, 2013.
[10] H. Han, X. Wu, A survey on artificial boundary method, Science China Mathematics, 56(2013) 2439-2488.
[11] B. I. Henry, T. A. M. Langlands, S. L. Wearne, Anomalous diffusion with linear reaction dynamics: From continuous time random walks to fractional reaction-diffusion equations, Phys. Rev. E, 74(2006) 031116.
[12] B. I. Henry, T. A. M. Langlands, S. L. Wearne, Anomalous subdiffusion with multispecies linear reaction dynamics, Phys. Rev. E, 77(2008) 021111.
[13] H. Kreiss, J. Lorenz, Initial boundary value problems and the Navier-Stokes equations, III. Series: Pure and applied mathematics, Academic Press, Vol. 136, 1989.
[14] C. P. Li, W. H. Deng, Remarks on fractional derivatives, Appl. Math. Comput. 187 (2007) 777-784.
[15] C. Li, W. H. Deng, L. J. Zhao, Well-posedness and numerical algorithm for the tempered fractional ordinary differential equations, arXiv:1501.00376v1 [math.CA] 2 Jan 2015.

[16] R. Metzler, J. Klafter, The random walk’s guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep., 339 (2000) 1-77.

[17] I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999.

[18] I. M. Sokolov, M. G. W. Schmidt, F. Sagués, Reaction-subdiffusion equations, Phys. Rev. E, 73 (2006) 031102.

[19] S. Samko, A. Kilbas, O. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, London, 1993.

[20] Y. Yan, G. Fairweather, Orthogonal spline collocation methods for some partial integrodifferential equations, SIAM J. Numer. Anal., 29 (1992), 755-768.