SUPERPARTNER STATES IN QUANTUM MECHANICS OF COLORED PARTICLE

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Abstract

Superpartner correspondence of states of colored particle in external chromomagnetic field given by non-commuting axial vector potentials is determined. Squared Dirac equation for this particle is solved exactly and explicit expressions of superpartner states are found. The wave functions of states with definite energy are found. Supersymmetry algebra and superpartner states in three dimensional motion case are considered.

Supersymmetry and superpartner states in abelian quantum mechanics is well studied [1-3]. For example, electron moving in homogenous magnetic field has superpartner states differing from each other by projection of spin and main quantum number [3, 11, 12].

Supersymmetry in quantum mechanics of colored particle in chromomagnetic field given by non-commuting vector potentials was studied in paper [8]. It was constructed supercharge operators \( Q_\pm \) and was shown, that these operators form closed algebra together with so-called Hamiltonian - the square of Dirac operator in given field. Of course, there are superpartner states - states with same energy and different quantum numbers of colored particle in such external field. But energy spectrum in this case is continuous and bosonic operators of creation and annihilation are different from that one’s in abelian quantum mechanics. So, it has sense to find superpartner states in this case as well. The supersymmetry in Dirac equation shows up the spin diagonal form of Hamiltonian for this particle [8]. This means the equations corresponding to different projections of quark’s spin are separated and can be solved independently. In this way Dirac equation can be solved and superpartner states could be found.

Let us define external chromomagnetic field by constant vector potentials [5]. Within SU(3) color symmetry group they look as

\[
A^a_1 = (0, \tau^1, 0, 0), \quad A^a_2 = (0, 0, \tau^2, 0), \quad A^a_3 = 0, \quad A^a_0 = 0 \tag{1}
\]

where \( a = 1, 8 \) is a color index and \( \tau_1, 2 \) is a constant. The field (1) is directed along third axes of ordinary and color spaces:

\[
F^{3}_{12} = H^3 = g\tau^1 \tau^2, \quad \text{other} \quad F^a_{\mu \nu} = 0. \tag{2}
\]

Here \( g \) is color interaction constant.

The Dirac equation for a colored particle in the external color field has a form

\[
(\gamma^\mu P_\mu - m) \psi = 0, \tag{3}
\]

where \( P_\mu = p_\mu + gA_\mu = p_\mu + gA^a_\mu \lambda^a_2, \lambda^a \) is Gell-Mann matrices describing particle’s color spin. The equation (3) is divided into two equations for Maiorana spinors \( \phi \) and \( \chi \), \( \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \)

\[
(\sigma^i P_i) \chi = \left( i \frac{\partial}{\partial t} - m \right) \phi
\]

\[
(\sigma^i P_i) \phi = \left( i \frac{\partial}{\partial t} + m \right) \chi, \tag{4}
\]
where Pauli matrices $\sigma^i$ describe particle’s spin. The spinors $\phi$ and $\chi$ has two components corresponding to two spin states of a particle

$$
\phi = \begin{pmatrix}
\phi(\sigma_3 = +1) \\
\phi(\sigma_3 = -1)
\end{pmatrix} = \begin{pmatrix}
\phi_+ \\
\phi_-
\end{pmatrix}, \quad \chi = \begin{pmatrix}
\chi(\sigma_3 = +1) \\
\chi(\sigma_3 = -1)
\end{pmatrix} = \begin{pmatrix}
\chi_+ \\
\chi_-
\end{pmatrix}.
$$

Each component of $\phi$ and $\chi$ transforms under fundamental representation of color group SU(3) and has three color components describing color states of a particle and corresponding to three eigenvalues of color spin $\lambda^3$

$$
\phi_\pm = \begin{pmatrix}
\phi_+(\lambda^3 = +1) \\
\phi_+(\lambda^3 = -1) \\
\phi_+(\lambda^3 = 0)
\end{pmatrix} = \begin{pmatrix}
\phi_+^{(1)} \\
\phi_+^{(2)} \\
\phi_+^{(3)}
\end{pmatrix}, \quad \chi_\pm = \begin{pmatrix}
\chi_+(\lambda^3 = +1) \\
\chi_+(\lambda^3 = -1) \\
\chi_+(\lambda^3 = 0)
\end{pmatrix} = \begin{pmatrix}
\chi_+^{(1)} \\
\chi_+^{(2)} \\
\chi_+^{(3)}
\end{pmatrix}.
$$

Acting on equations (4) by the operator $(\sigma^i P_i)$ and taking them into account once more, we get the same equations for spinors $\phi$ and $\chi$

$$
H \psi = (\sigma^i P_i)^2 \psi = -\left(\frac{\partial^2}{\partial t^2} + m^2\right) \psi. \quad (5)
$$

For the particle with $p_3 = 0$ the Hamiltonian $H$ in (5) and the supercharge operators $Q_\pm = P_\pm a_\pm$ form supersymmetry algebra $\{Q_+, Q_-\} = H^1$. Here the operators of bosonic and fermionic states are defined by formulas

$$
P_\pm = P_1 \pm iP_2, \quad a_\pm = \frac{1}{2} (\sigma_1 \pm i\sigma_2)
$$

and obey next relations

$$
\left[ P_+, P_- \right] = \lambda^3 g H^3, \quad \{a_+, a_-\} = 1. \quad (6)
$$

As we know, the superpartner states are received in the result of supercharge operators $Q_\pm$ action. The action of fermionic operators of creation and annihilation is turning over ordinary spin of particle

$$
a_+ \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \psi_- \\ 0 \end{pmatrix},
$$

$$
a_- \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_+ \end{pmatrix}. \quad (6)
$$

It is seen is from (6) that bosonic operator of creation $P_+$ act only at $\psi_+$ and annihilation operator $P_-$ only at $\psi_-$. These operators mix color states $\psi^{(1)}$ and $\psi^{(2)}$

$$
P_+ \begin{pmatrix} \psi^{(1)}_+ \\ \psi^{(2)}_+ \\ \psi^{(3)}_+ \end{pmatrix} = \begin{pmatrix} (p_1 + ip_2) \psi^{(1)}_+ + \frac{g}{2} \left(\tau^3_1 + \tau^3_2\right) \psi^{(2)}_+ \\ \frac{g}{2} \left(\tau^3_1 - \tau^3_2\right) \psi^{(1)}_+ + (p_1 + ip_2) \psi^{(2)}_+ \\ (p_1 + ip_2) \psi^{(3)}_+ \end{pmatrix} = \begin{pmatrix} \Psi^{(1)}_+ \\ \Psi^{(2)}_+ \\ \Psi^{(3)}_+ \end{pmatrix},
$$

$$
P_- \begin{pmatrix} \psi^{(1)}_+ \\ \psi^{(2)}_+ \\ \psi^{(3)}_+ \end{pmatrix} = \begin{pmatrix} (p_1 - ip_2) \psi^{(1)}_+ + \frac{g}{2} \left(\tau^3_1 - \tau^3_2\right) \psi^{(2)}_+ \\ \frac{g}{2} \left(\tau^3_1 + \tau^3_2\right) \psi^{(1)}_+ + (p_1 - ip_2) \psi^{(2)}_+ \\ (p_1 - ip_2) \psi^{(3)}_+ \end{pmatrix} = \begin{pmatrix} \Psi^{(1)}_- \\ \Psi^{(2)}_- \\ \Psi^{(3)}_- \end{pmatrix}. \quad (7)
$$

1for more detail see [8]
In the result of this mixing the superpartner states $\psi_{\pm}^{(1,2)}$ of pure color states $\psi_{\pm}^{(1)}$ and $\psi_{\pm}^{(2)}$ will be mixed color states.

By use of supercharge operators it is easy to observe the spin diagonal form of Hamiltonian $H$

$$H \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} P_- P_+ & 0 \\ 0 & P_+ P_- \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \quad (8)$$

So, due to spin diagonal form of $H$ the equation (5) is splitting into two non-mixed spin indicies + and - equations for components $\psi_+$ and $\psi_-$

$$P_- P_+ \psi_+ = - \left( \frac{\partial^2}{\partial t^2} + m^2 \right) \psi_+$$
$$P_+ P_- \psi_- = - \left( \frac{\partial^2}{\partial t^2} + m^2 \right) \psi_- \quad (9)$$

Taking into account the fact that the states $\psi_{\pm}$ are stationary ( the field (1) is time independent ) $\frac{\partial \psi_{\pm}}{\partial t} = iE \psi_{\pm}$, the explicit form of equations (9) will be

$$\begin{pmatrix} p_1^2 + p_2^2 + \frac{g^2}{4} (\tau_1 + \tau_2) I_2 + g \left( p_1 \tau_1 \frac{1}{4} \lambda^1 + p_2 \tau_2 \frac{1}{4} \lambda^2 + \frac{H_3}{2} \lambda^3 \right) \end{pmatrix} \psi_{\pm} = \left( E^2 - m^2 \right) \psi_{\pm} \quad (10)$$

Here $I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is color matrix.

The energy spectrum $E$ in the field (1) was found in [3] and equal to ( on $p_z = 0$ )

$$E_{1,2}^2 = p_1^2 + m^2 + \frac{g^2}{4} (\tau_1 + \tau_2) \pm \sqrt{\tau_1 p_1^2 + \tau_2 p_2^2 + \frac{(H_3)^2}{4}}$$
$$E_3^2 = p_1^2 + m^2, \quad (p_1^2 = p_1^2 + p_2^2). \quad (11)$$

The energy spectrum $E_3$ corresponds to colorless particle state ($\lambda^3 = 0$). So, the degeneration of $E_3$ is a spin degeneration. The spectra $E_{1,2}^2$ are received from (10) solving it for $E$ in momentum representation. The sign at last term in the expression of $E_{1,2}^2$ is defined by the operator

$$(I_{\pm}^a \lambda^a) = g \left( p_1 \tau_1 \frac{1}{4} \lambda^1 + p_2 \tau_2 \frac{1}{4} \lambda^2 + \frac{H_3}{2} \lambda^3 \right) \quad (12)$$

square of which is same for spin + and - indices

$$(I_{\pm}^a \lambda^a)^2 = g^2 \left( p_1^2 \tau_1 + p_2^2 \tau_2 + \frac{(H_3)^2}{4} \right) \quad (13)$$

and has eigenvalues $\pm \sqrt{p_1^2 \tau_1 + p_2^2 \tau_2 + \frac{(H_3)^2}{4}}$. The operator $\left( I_{\pm}^a \lambda^a \right)$ is different for $\psi_+$ and $\psi_-$, and the superpartner Hamiltonians in (10) are different for them as well. But solving characteristic equation gives same energy spectrum branches (11) for both of them. So, both energy branches $E_{1,2}$ are degenerated by ordinary spin [7]. It becomes clear that energy spectrum $E_{1,2}$ is not defined neither by spin nor by color quantum numbers \(^3\). Energy branches $E_1$ and $E_2$ are defined only by eigenvalues of $\left( I_{\pm}^a \lambda^a \right)$ operator. So, both energy branch are fourfold degenerated. This means that both spin and color states $\psi_{\pm}^{(1)}$ and $\psi_{\pm}^{(2)}$ may be in the energy branch $E_1$ or $E_2$ and the equations

\(^2\text{Sign before the root shouldn’t be confused with the spin + and - indices.}\)

\(^3\text{The quantum nimber } \lambda^3 \text{ is not a conserved quantity in the field (1) too } [H, \lambda^3] \neq 0.\)
(10) for given energy $E_1$ or $E_2$ should be solved for all spin and color states. The operator $(I^a_{\pm} \lambda^a)$ mixes the color states $\psi^{(1)}$ and $\psi^{(2)}$. But the solutions of equation (10) could be received for pure color states because of color diagonal form of (13).

In order to solve equations (10) it is reasonable to use polar coordinates $x = r \cos \theta, y = r \sin \theta$. We have two variables $(r, \theta)$ and one constraint due to the conservation of quantity $(\sigma^i P_i)$ ($[H, (\sigma^i P_i)] = 0, \text{ see } [8]\text{ as well}$. In reference frame, where $\theta$ is angle between $\vec{\sigma}$ and $\vec{P}$, $\cos \theta = \frac{(\vec{\sigma}, \vec{P})}{|\vec{\sigma}| |\vec{P}|}$ is a constant. Consequently $\theta$ is constant too and $\frac{\partial \psi^{(1,2)}_+}{\partial \theta} = 0$. Then equations for $\psi_+$ look as

\[
\begin{aligned}
\left\{ \begin{array}{l}
\left( \frac{d^2}{dr^2} + A_1 \right) \psi^{(1)}_+ + B_1 \frac{d \psi^{(2)}_+}{dr} = 0 \\
\left( \frac{d^2}{dr^2} + A_2 \right) \psi^{(2)}_+ + B_2 \frac{d \psi^{(1)}_+}{dr} = 0 \\
\left( \frac{d^2}{dr^2} + (E_3 - m^2) \right) \psi^{(3)}_+ = 0
\end{array} \right.
\end{aligned}
\]  

\[A_{1,2} = E^2 - m^2 - q^2 \left( \frac{1}{r_1^2} + \frac{1}{r_2^2} \right)^2, \quad B_{1,2} = \pm g \left( \frac{1}{r_1^2} \cos \theta \mp i \frac{1}{r_2^2} \sin \theta \right)^2 \]

from which we get equation for a pure state $\psi^{(2)}_+$

\[
\frac{d^4}{dr^4} \psi^{(2)}_+ + (A_1 + A_2 - B_1 B_2) \frac{d^2}{dr^2} \psi^{(2)}_+ + A_1 A_2 \psi^{(2)}_+ = 0.
\]

The solution of (15) will be in the form $\psi^{(2)}_+(r) \sim e^{\alpha r}$. Putting this solution in equation (15) we find next expressions for $\alpha$

\[
\alpha_{1,2} = \pm i p_\perp, \quad \alpha_{3,4} = \pm i \sqrt{p_\perp^2 + (\pm)2g \sqrt{p_1^2 (\tau_1 \cos^2 \theta + \tau_2 \sin^2 \theta) + \frac{(H_3^2)^2}{4}} + g^2 (\tau_1 \cos^2 \theta + \tau_2 \sin^2 \theta)}.
\]

The sign in the bracket is the sign in the energy spectrum. The solution $\psi^{(2)}_+$ has a form

\[\psi^{(2)}_+ = \sum C_i e^{\alpha_i r}.\]  

(16)

Here $C_i$ is arbitrary constants. The wave function $\psi^{(1)}_+$ can be found from (12) using solution $\psi^{(2)}_+$ as well

\[\psi^{(1)}_+ = -\frac{1}{B_2} \sum C_i \left( \alpha_i + \frac{A_2}{\alpha_i} \right) e^{\alpha_i r}.\]  

(17)

By the same way the solutions $\psi^{(2)}_-$ and $\psi^{(1)}_-$ could be found

\[\psi^{(2)}_- = \sum C'_i e^{\alpha_i r},\]  

(18)

\[\psi^{(1)}_- = -\frac{1}{B_2} \sum C'_i \left( \alpha_i + \frac{A_1}{\alpha_i} \right) e^{\alpha_i r}.\]  

(19)

And the simplest solutions $\psi^{(3)}_\pm$ are well known plane waves

\[\psi^{(3)}_\pm = C'_1 e^{i p_\perp + C'_2 e^{-i p_\perp}}.\]  

(20)

It is seen from (16) that for $E_2$ spectrum the expression under the root in $\alpha_{3,4}$ is negative at values $p_\perp^2 < g^2 (\tau_1 \cos^2 \theta + \tau_2 \sin^2 \theta) + g H_z^2$. So, the term $e^{\alpha r}$ in solutions $\psi^{(1,2)}_\pm$ is infinite at $r \to \infty$.  

4
This infinity is connected with that the movement of a colored particle in the field (1) is infinite in space volume, since the energy spectrum of this particle (11) is continuous [9]. So, we shouldn’t throw away this term for normalizibility of the solutions.

In order to find wave functions of states $\psi_1$ and $\psi_2$ with definite energy $E_1$ and $E_2$ we have to solve next equations for them

$$
\begin{bmatrix}
p_1^2 + p_2^2 + \frac{\tau}{4} (\tau_1 + \tau_2) I_2 + g \sqrt{\tau_1 p_1^2 + \tau_2 p_2^2 + \frac{(H_2)^2}{4}} \\
p_1^2 + p_2^2 + \frac{\tau}{4} (\tau_1 + \tau_2) I_2 - g \sqrt{\tau_1 p_1^2 + \tau_2 p_2^2 + \frac{(H_2)^2}{4}}
\end{bmatrix}
\psi_1 = (E_1^2 - m^2) \psi_1,
$$

$$
\begin{bmatrix}
p_1^2 + p_2^2 + \frac{\tau}{4} (\tau_1 + \tau_2) I_2 - g \sqrt{\tau_1 p_1^2 + \tau_2 p_2^2 + \frac{(H_2)^2}{4}} \\
p_1^2 + p_2^2 + \frac{\tau}{4} (\tau_1 + \tau_2) I_2 + g \sqrt{\tau_1 p_1^2 + \tau_2 p_2^2 + \frac{(H_2)^2}{4}}
\end{bmatrix}
\psi_2 = (E_2^2 - m^2) \psi_2.
$$

(21)

Here we have differential operators under the root. Solutions of these equations are found in appendix and are equal to following expressions

$$
\psi_{1,2} = -\frac{4\Delta_{1,2}}{a^2_{1,2}(c - a_{1,2})} \sin^2 \frac{\sqrt{a_{1,2}}}{2} r \mp \frac{8b\sqrt{-\Delta_{1,2}}}{3a^2_{1,2}(c - a_{1,2})} \sin \frac{\sqrt{a_{1,2}}}{2} r + \frac{4b^2(c - a_{1,2}) - b^2}{4(c - a_{1,2})}.
$$

Thus we can find following superpartner correspondence between states from same energy branch by use of formulae (7) and solutions (16) - (20):

$$
\psi_1^{(1)} \rightarrow \Psi_1^{(1)} = \sum C_i \left[ \frac{ie^{-i\theta}}{B_2} \left( \alpha_i^2 + A_1 \right) + \frac{g}{2} \left( \tau_1^\frac{1}{2} - \tau_2^\frac{1}{2} \right) \right] e^{\alpha_i r}
$$

$$
\psi_1^{(2)} \rightarrow \Psi_1^{(2)} = -\sum C_i \left[ \frac{g}{2B_2} \left( \tau_1^\frac{1}{2} + \tau_2^\frac{1}{2} \right) \left( \alpha_i + \frac{A_1}{\alpha_i} \right) + e^{-i\theta} \alpha_i \right] e^{\alpha_i r}
$$

$$
\psi_1^{(3)} \rightarrow \Psi_1^{(3)} = -ip e^{-i\theta} \left( C_1 e^{ip r} - C_2 e^{-ip r} \right)
$$

$$
\psi_2^{(1)} \rightarrow \Psi_2^{(1)} = \sum C_i \left[ \frac{ie^{i\theta}}{B_2} \left( \alpha_i^2 + A_2 \right) + \frac{g}{2} \left( \tau_1^\frac{1}{2} + \tau_2^\frac{1}{2} \right) \right] e^{\alpha_i r}
$$

$$
\psi_2^{(2)} \rightarrow \Psi_2^{(2)} = -\sum C_i \left[ \frac{g}{2B_2} \left( \tau_1^\frac{1}{2} - \tau_2^\frac{1}{2} \right) \left( \alpha_i + \frac{A_2}{\alpha_i} \right) + e^{i\theta} \alpha_i \right] e^{\alpha_i r}
$$

$$
\psi_2^{(3)} \rightarrow \Psi_2^{(3)} = -ip e^{i\theta} \left( C_1 e^{ip r} - C_2 e^{-ip r} \right).
$$

(22)

At now we can consider particle with $p_3 \neq 0$. Hamiltonian in (5) for this case is defined by

$$
H' = \left( \sigma^1 P_1 + \sigma^2 P_2 + \sigma^3 p_3 \right)^2
$$

and keep its spin diagonal form

$$
H' \begin{pmatrix} \psi_+ \\ \psi_-
\end{pmatrix} = \begin{pmatrix} P_+ P_+ + p_3^2 & 0 \\ 0 & P_+ P_- + p_3^2
\end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_-
\end{pmatrix}.
$$

(23)

This means operators $a_\pm$ also keep their anticommutation property and could be choosen as a fermionic operators. Solutions of (23) equations are received from (16)-(20) only by multiplying on $e^{\pm ip_3 z}$. Supercharge operators defined in the form

$$
Q'_1 = Q_1 + \sigma^3 p_3
$$

$$
Q'_2 = Q_2 + \sigma^3 p_3
$$

(24)

obey following relations
\[(Q')^2 = (Q')^2 = H', \quad [Q'_1, H'] = [Q'_2, H'] = 0, \quad \{Q'_1, Q'_2\} = 2p^2_3 \neq 0,\]  

last one of which is differ from ordinary supersymmetry algebra. Mutually hermitian conjugate operators \(Q'_\pm\) constructed in ordinary way has the form

\[Q'_\pm = \frac{1}{2} (Q'_1 \pm iQ'_2) = Q'_\pm + \frac{1}{2} \sigma^3 p_3\]  

and obey relations

\[\{Q'_+, Q'_-\} = H', \quad (Q'_\pm)^2 = \pm \frac{i}{2} p^2_3,\]  

If we define \(Q'_2 = Q_2\), we get ordinary anticommutation \(\{Q'_1, Q'_2\} = 0\) and commutation \([Q'_1, H'] = [Q'_2, H'] = 0\) supersymmetry relations. But for this definition we have \((Q'_1)^2 = H', \quad (Q'_2)^2 = H' - p^2_3\).

In addition to this definition, if we define \(Q'_\pm\) operators in following way

\[Q'_\pm = \frac{1}{2} (Q'_1 \pm iQ'_2) + \frac{i}{2} \sigma^3 p_3\]

we get for them expressions in (26) with ordinary anticommutation relation (27). The operators \(Q'_\pm\) contain \(\sigma^3\) matrix. So, superpartner states in this case are received mixed spin states

\[Q'_\pm \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right) = e^{ip_3 z} \left( \begin{array}{c} \frac{\Psi_+ + \frac{1+i}{2} p_3 \psi_-}{\sqrt{\frac{H^3_3}{4}}} \\ -\frac{1+i}{2} p_3 \psi_- \end{array} \right) + e^{-ip_3 z} \left( \begin{array}{c} \frac{\Psi_+ - \frac{1-i}{2} p_3 \psi_-}{\sqrt{\frac{H^3_3}{4}}} \\ \frac{1+i}{2} p_3 \psi_- \end{array} \right).\]

That means in this three dimensional motion case superpartner states are not states with definite projection of ordinary spin.

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1 Appendix

Let us demonstrate solution of first equation in (21), which in polar coordinates has got form

\[\frac{d^2 \psi_1}{dr^2} + b \sqrt{\frac{d^2 \psi_1}{dr^2} + c \psi_1 + a_1 \psi_1} = 0\]  

(A.1)

where we have made designations

\[a_{1,2} = p^2_\perp \pm \sqrt{-b^2 p^2_\perp + \frac{(H^3_3)^2}{4}}, \quad b = g \sqrt{-(\tau_1 \cos^2 \theta + \tau_2 \sin^2 \theta)}, \quad c = \frac{(H^3_3)^2}{4b^2}.\]

Equation (A.1) could be written in the form

\[\left( \sqrt{\frac{d^2 \psi_1}{dr^2} + c \psi_1 + \frac{b}{2}} \right)^2 - \frac{b^2}{4} + (a_1 - c) \psi_1 = 0\]

square root from which gives

\[\sqrt{\frac{d^2 \psi_1}{dr^2} + c \psi_1} = \sqrt{\frac{(c - a_1) \psi_1}{4} + \frac{b}{2}}.\]  

(A.2)
Squaring equation (A.2) again and doing replacement \( \psi_1 = \frac{\varphi_1^2}{c-a_1} - \frac{b^2}{4(c-a_1)} \) we get next equation for \( \varphi_1 \)

\[
2\varphi_1 \frac{d^2 \varphi_1}{dr^2} + 2 \left( \frac{d\varphi_1}{dr} \right)^2 + a_1 \varphi_1^2 + b(c-a_1) \varphi_1 - \frac{1}{4} b^2 (2c-a_1) = 0. \tag{A.3}
\]

Considering \( \frac{d\varphi_1}{dr} = f(\varphi_1) \) and taking into account \( \frac{d^2 \varphi_1}{dr^2} = \frac{df}{d\varphi_1} \frac{d\varphi_1}{dr} = f'f \) in equation (A.3) we get first order non-linear differential equation for \( f(\varphi_1) \)

\[
2\varphi_1 f' + 2f^2 = -a_1 \varphi_1^2 - b(c-a_1) \varphi_1 + \frac{1}{4} b^2 (2c-a_1) \tag{A.4}
\]

Equation (A.4) should be solved without right hand side at first

\[
2\varphi_1 f' + 2f^2 = 0,
\]

which implies for \( f \neq 0 \)

\[
\varphi_1 f' + f = 0. \tag{A.5}
\]

Equation (A.5) gives \( \frac{df}{f} = -\frac{d\varphi_1}{\varphi_1} \), which means

\[
f \cdot \varphi_1 = K(\varphi_1).
\]

Taking into account in equation (A.4) last constraint for \( f \), we get equation for \( K(\varphi_1) \) solution of which is

\[
K(\varphi_1) = \sqrt{-\frac{a_1}{4} \varphi_1^4 - \frac{b}{3} \varphi_1^3 + \frac{b^2}{8} \frac{2c-a_1}{3} \varphi_1^2 + K_1},
\]

where \( K_1 \) is integrate constant. We shall seek solution \( \varphi_1 \) for \( K_1 = 0 \). This means to demand function \( f(\varphi_1) \) to be finite at some finite \( r_n \) points, for which \( \varphi_1(r_n) = 0 \). Using explicit expression of \( K(\varphi_1) \) we find

\[
\frac{d\varphi_1}{dr} = \sqrt{-\frac{a_1}{4} \varphi_1^2 - \frac{b}{3} \frac{2c-a_1}{3} \varphi_1 + \frac{b^2}{8} \frac{2c-a_1}{3} \varphi_1^2}.
\tag{A.6}
\]

Integration of (A.6) is carry out for \( \Delta_1 < 0 \) and \( \Delta_1 > 0 \) values of \( \Delta_1 = -\frac{a_1 b^2}{8} (2c-a_1) - \frac{b^2}{9} (c-a_1) \) and gives same result for \( \varphi_1 \)

\[
\varphi_1 = \frac{2\sqrt{-\Delta_1}}{a_1} \sin \frac{\sqrt{a_1}}{2} r - \frac{2b(c-a_1)}{3a_1} + \text{Const}.
\]

Only solution with \( \text{Const.} = 0 \) obey initial equation. Same calculation for second equation of (21) gives

\[
\varphi_2 = \frac{2\sqrt{-\Delta_2}}{a_2} \sin \frac{\sqrt{a_2}}{2} r + \frac{2b(c-a_2)}{3a_2}.
\]

From these expressions we can find \( r_n \) points of \( \varphi_1, \varphi_2 \)

\[
r_n = \frac{2}{\sqrt{a_{1,2}}} \left( \arcsin \pm \frac{c-a_1}{3\sqrt{-\Delta_{1,2}}} + 2\pi n \right).
\]
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