On self-dual double circulant codes

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Abstract

Self-dual double circulant codes of odd dimension are shown to be dihedral in even characteristic and consta-dihedral in odd characteristic. Exact counting formulae are derived for them and used to show they contain families of codes with relative distance satisfying a modified Gilbert-Varshamov bound.

Key Words: quasi-cyclic codes, dihedral group, consta-dihedral codes, Artin primitive root conjecture

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1 Introduction

It has been known for forty years that quasi-cyclic codes are good \[2\] in the asymptotic sense: the product of their rate by their relative distance does not vanish when the length goes to infinity. Thirteen years ago, it was shown that even the self-dual subclass was good \[8\]. A decade ago, it was proved that binary dihedral codes of rate one half were good \[1\], then that their self-dual doubly even subclass is also good \[10\]. The last two papers used a non-constructive probabilistic argument, where the order of 2 modulo the length is controlled but not determined. In the present article, we will consider so-called pure double circulant codes, that is 2-quasi cyclic codes with a systematic generator matrix consisting of two circulant matrices. These codes have been studied in a number of papers since the 1960’s \[2, 6, 15\]. In particular, it is known that binary extended square codes, which form one of the oldest and most studied family of self-dual codes, are double circulant in many lengths \[5, 12\]. We will show that double circulant self-dual codes over an arbitrary finite
field of order \( q \) are either dihedral or consta-dihedral depending on the parity of \( q \). (A special case of the first statement is anticipated in [12]). While the notion of dihedral codes has been considered by several authors, the notion of constadihedral codes has been introduced in [15] in terms of twisted group rings. We give an alternative definition in terms of group representations. We believe, but do not prove here that the two definitions are related.

Further, building on the Chinese Remainder Theorem (CRT) approach of [7], we will give exact counting formulae for these codes. From there, we will give an alternative proof that dihedral codes are good, with codes of lengths a prime number as in [2]. Our proof depends on Artin conjecture [11], proved under the Generalized Riemann Hypothesis (GRH) in [3]. It is however, conceptually clearer, and valid for more general alphabets than that of [11]. Also we give a new family of good long self-dual quasi-cyclic codes. They differ from that of [8] by the index, the power of the shift under which they are invariant.

The material is organized as follows. Section 2 collects together the definitions and notation that we need thereafter. Section 3 studies the automorphism group of self-dual double circulant codes, first for even then for odd characteristic. A general notion of consta-dihedral codes is introduced in the language of representation theory. Section 4 studies the asymptotics of double circulant self-dual codes by combining enumerative formulae with the expurgated random coding argument made familiar by the Gilbert-Varshamov bound.

## 2 Definitions and notation

Let \( GF(q) \) denote a finite field of characteristic \( p \). In the following, we will consider codes over \( GF(q) \) of length \( 2n \) with \( n \) odd and coprime to \( q \). Their generator matrix \( G \) will be of the form \( G = (I, A) \) where \( I \) is the identity matrix of order \( n \) and \( A \) is a circulant matrix of the same order. We will call these codes double circulant. These codes are sometimes called pure double circulant to distinguish them from bordered double circulant which are not quasi-cyclic [14].

By a dihedral group \( D_n \), we will denote the group of order \( 2n \) with two generators \( r \) and \( s \) of respective orders \( n \) and \( 2 \) and satisfying the relation \( srs = r^{-1} \). A code of length \( 2n \) is called dihedral if it is invariant under \( D_n \) acting transitively on its coordinate places.

If \( C(n) \) is a family of codes of parameters \([n, k_n, d_n]\), the rate \( R \) and relative distance \( \delta \) are defined as
\[
R = \limsup_{n \to \infty} \frac{k_n}{n},
\]
and
\[
\delta = \liminf_{n \to \infty} \frac{d_n}{n}.
\]
Both limits are finite as limits of bounded quantities. Such a family of codes is said to be good if \( R\delta \neq 0 \).
3 Symmetry

3.1 Even $q$

Let $q$ be an even prime power. Let $M_n(q)$ denote the set of all $n \times n$ matrices over $GF(q)$.

**Lemma 1** If $A$ is a circulant matrix in $M_n(q)$, then there exists an $(n \times n)$-permutation matrix $P$ such that $PAP = A^t$ where $A^t$ denotes the transpose of $A$.

**Proof.** Assume, for simplicity, that $n$ is odd. Denote by $\pi$ the permutation $(1, n)(2, n-1)\cdots \left(\frac{n-1}{2}, \frac{n+3}{2}\right)$. Permuting the columns of $A$ with respect to $\pi$ yields a symmetric back-circulant matrix. Let $P$ be the permutation matrix attached to $\pi$. The preceding explanation shows that $AP = (AP)^t = P^tA^t$, or $PAP = A^t$.

□

**Theorem 2** For $n \geq 3$ odd, and $q$ even, every self-dual double circulant code over $GF(q)$ of length $2n$ is dihedral.

**Proof.** Let $C$ be a self-dual double circulant code of length $2n$ with generator matrix $G = (I, A)$. The parity-check matrix $H = (A^t, I)$ is also a generator matrix of $C$ due to self-duality. Let $P$ be the $(n \times n)$ permutation matrix such that $PAP = A^t$. Since left multiplication by $P$ amounts to changing the positions of some rows, $PH = (PA^t, P)$ is also a generator matrix for $C$.

On the other hand, right multiplication of $PH$ by $P$ is equivalent to multiplying $PH$ by the block diagonal matrix $(P 0 0)$, yielding $PHP = (A, I)$. This right multiplication corresponds to applying the following permutation in $S_{2n}$

$$\pi = (2, n)(3, n-1)\cdots \left(\frac{n+1}{2}, \frac{n+3}{2}\right)(n+2, 2n)(n+3, 2n-1)\cdots \left(\frac{3n+1}{2}, \frac{3n+3}{2}\right).$$

i.e. $PHP = PH\pi$. Moreover, we can obtain the generator matrix $(I, A)$ from $(A, I)$ by applying the permutation

$$\sigma = (1, n+1)(2, n+2)(3, n+3)\cdots (n, 2n).$$

Hence $C$ is invariant under the following product

$$\pi\sigma = (1, n+1)(2, 2n)(3, 2n-1)\cdots (n-1, n+3)(n, n+2).$$

Furthermore, since $I$ and $A$ are circulant matrices, $C$ is invariant under also the permutation

$$\tau = (1, 2, \ldots, n)(n+1, n+2, \ldots, 2n).$$
Therefore, $C$ is invariant under the subgroup $\langle \tau, \pi \sigma \rangle$ of $S_{2n}$. Since $\tau$ is a product of $n$-cycles and $\pi \sigma$ is a product of transpositions, we have $\tau^n = 1$ and $(\pi \sigma)^2 = 1$. Observe that

$$(\pi \sigma)\tau = (1, n + 2)(2, n + 1)(3, 2n)(4, 2n - 1) \ldots (n - 1, n + 4)(n, n + 3).$$

Then we can easily obtain the following equality

$$(\pi \sigma)\tau(\pi \sigma) = (1, n, n - 1, n - 2, \ldots, 2)(n + 1, 2n, 2n - 1, \ldots, n + 2) = \tau^{-1}.$$

Therefore, $\langle \tau, \pi \sigma \rangle$ is isomorphic to the dihedral group $D_n$. □

3.2 Odd $q$

Recall that a monomial matrix over $GF(q)$ of order $g$ has exactly one nonzero element per row and per column. The monomial matrices form a group $M(g, q)$ of order $g!(q - 1)^g$ under multiplication. This group is abstractly isomorphic to the wreath product $\mathbb{Z}_{q-1} \wr S_g$.

By a monomial representation of a group $G$ over $GF(q)$ we shall mean a group morphism from $G$ into $M(g, q)$. A code of length $2n$ will be said to be consta-dihedral if it is held invariant under right multiplication by a monomial representation of the dihedral group $D_n$. An alternative, but related definition can be found in [13]. We can now state the main result of this subsection.

**Theorem 3** For $n \geq 3$ odd, and $q$ odd, every self-dual double circulant code $C$ of length $2n$ over $GF(q)$ is consta-dihedral.

**Proof.** Keep the matrix notations of Theorem 2. Let the generator matrix of $C$ be $G = (I, A)$ with $A$ circulant and $AA^t = -I$. Computing $A^tG = (A^t, -I)$ and conjugating by $P$ of Lemma 1 we get $PA^tGP = (A, -I)$. Define the antiswap involution $a$ by the rule $a(x, y) = a(y, -x)$, where $x, y$ are vectors of length $n$ over $GF(q)$. Note that $a^2 = -1$. Clearly $a \in M(2n, q)$. Thus $\pi a \in M(2n, q)$ and it preserves $C$. A monomial representation of $D_n$ is then $\langle \tau, \pi a \rangle$. Thus $C$ is consta-dihedral. □

4 Asymptotics

4.1 Enumeration

In this section we give enumerative results for self-dual double circulant codes. It is important to notice that there are 2-quasi-cyclic codes that are not double circulant. An example in length 168 is given in [5]. Thus, the formula of [7 Prop. 6.2] does not apply.
We will need the following counting formula. An alternative proof for \( q \) prime can be found in [9, Th 1.3, Th 1.3'] where the number of orthogonal circulant matrices over \( GF(q) \) for \( q \) prime is computed. Recall that \(-1\) is a square in \( GF(q) \), a field of characteristic \( p \), if one of the following conditions holds

1. \( q \) is even
2. \( p \equiv 1 \pmod{4} \)
3. \( p \equiv 3 \pmod{4} \) and \( q \) is a square.

Note that [7, Prop. 6.2] we know that 2-quasi-cyclic self-dual codes, hence a fortiori self-dual double circulant codes over \( GF(q) \) exist only if \(-1\) is a square in \( GF(q) \).

**Lemma 4** Let \( n \) denote a positive odd integer. Assume that \(-1\) is a square in \( GF(q) \). If \( x^n - 1 \) factors as a product of two irreducible polynomials over \( GF(q) \), the number of self-dual double circulant codes of length \( 2n \) is \( 2(q^2 - 1) + 1 \) if \( q \) is odd and \( (q^2 - 1) + 1 \) if \( q \) is even.

**Proof.** By the CRT approach of [7] any 2-quasi-cyclic code of length \( 2n \) over \( GF(q) \) decomposes as the 'CRT product' of a self-dual code \( C_1 \) of length 2 over \( GF(q) \) and of a hermitian self-dual code \( C_n \) of length 2 over \( GF(q^{n-1}) \). To obtain a double-circulant code we must ensure that the leftmost entry of their generator matrix \( G \) is \( G_{1,1} = 1 \).

If \( q \) is even the only possibility for \( C_1 \) is the code spanned by \([1, 1]\). If \( q \) is odd there are two codes \([1, a]\) and \([1, -a]\) where \( a^2 = -1 \).

For \( C_n \) the generator matrix is \([1, b]\) with \( b \) such that \( 1 + b^{1+r} = 0 \), with \( q^{n-1} = r^2 \). By finite field theory, this equation in \( b \) admits \( 1 + r \) roots in \( GF(r^2) \). Note that if \( q \) is even, \( b \) ranges over the elements of order dividing \( 1 + r = \frac{r^2 - 1}{r - 1} \), and that if \( q \) is odd, \( b^2 \) ranges over elements of order \( 2(1 + r) \). In both cases, we use the fact that the multiplicative group of \( GF(r^2) \) is cyclic of order \( r^2 - 1 \). \( \square \)

The following, more general, result is an analogue for double circulant codes of the Proposition [7 Prop. 6.2] for 2-quasi-cyclic codes. It is of interest in its own right, but not needed for the asymptotic bounds of this section.

**Proposition 5** Let \( n \) be an odd integer, and \( q \) a prime power coprime with \( n \). Suppose that \(-1\) is a square in \( GF(q) \). Assume that the factorization of \( x^n - 1 \) into irreducible polynomials over \( GF(q) \) is of the form

\[
x^n - 1 = \alpha (x - 1) \prod_{j=2}^s g_j(x) \prod_{j=1}^t h_j(x) h_j^*(x),
\]

with \( \alpha \) a scalar of \( GF(q) \), \( n = s + 2t \) and \( g_j \) a self-reciprocal polynomial of degree \( 2d_j \), the polynomial \( h_j \) is of degree \( e_j \) and \(* \) denotes reciprocation. For convenience, let \( g_1 = x - 1 \)
and, in case of $n$ even, let $g_2 = x + 1$. The number of self-dual 2-quasi-cyclic codes over $GF(q)$ is then

\[
4 \prod_{j=3}^{s} (1 + q^{d_j}) \prod_{j=1}^{t} (q^{e_j} - 1)
\]

if $q$ is odd and $n$ is even

\[
2 \prod_{j=2}^{s} (1 + q^{d_j}) \prod_{j=1}^{t} (q^{e_j} - 1)
\]

if $q$ is odd and $n$ is odd

\[
\prod_{j=2}^{s} (1 + q^{d_j}) \prod_{j=1}^{t} (q^{e_j} - 1)
\]

if $q$ is even and $n$ is odd.

**Proof.** (sketch). The part of the proof dealing with self-reciprocal polynomials $g_j$ is analogous to the previous lemma. In the case of reciprocal pairs $(h_j, h_j^*)$, note that the number of linear codes of length 2 over some $GF(Q)$ admitting, along with their duals, a systematic form is $Q - 1$, all of dimension 1. Indeed their generator matrix is of the form $[1, u]$ with $u$ nonzero. We conclude by letting $Q = q^{e_j}$.

### 4.2 Arithmetic

In number theory, Artin’s conjecture on primitive roots states that a given integer $q$ which is neither a perfect square nor $-1$ is a primitive root modulo infinitely many primes $\ell$ [11]. It was proved conditionally under GRH by Hooley [3]. In this case, by the correspondence between cyclotomic cosets and irreducible factors of $x^n - 1$ [4], the factorization of $x^n - 1$ into irreducible polynomials over $GF(q)$ contains exactly two factors, one of which is $x - 1$ [2].

### 4.3 Distance bound

We will need a $q$-ary version of a classical lemma from [2]. Let $a(x)$ denote a polynomial of $GF(q)[x]$ coprime with $x^n - 1$, and let $C_a$ be the double circulant code with generator matrix $(1, a)$. Assume the factorization of $x^n - 1$ into irreducible polynomials is $x^n - 1 = (x - 1)h(x)$. We call constant vectors the codewords of the cyclic code of length $n$ generated by $h$.

**Lemma 6** If $u$ is not a constant vector then there are only at most $(q - 1)$ polynomials $a$ such that $u \in C_a$.

**Proof.** Write $u = (v, w)$ with $v, w$ of length $n$. The condition $u \in C_a$ is equivalent to the equation $w = av \pmod{x^n - 1}$. If $v$ is invertible $(mod \; x^n - 1)$, then $v$ is uniquely determined by this equation. If not and if $u$ is not a constant vector the only possibility is that
both $w$ and $v$ are multiples of $(x-1)$. Letting $v = (x-1)v'$, and $w = (x-1)w'$, yields $w' = av' \pmod{h(x)}$, which gives $a \pmod{h(x)}$, since $v'$ is invertible $\pmod{h(x)}$. Now $a \pmod{(x-1)}$ can take $q-1$ nonzero values. The result follows by the CRT applied to $a$, since $a$, being of degree at most $n-1$ is completely determined by its residue $\pmod{x^n-1}$. □

Recall the $q$-ary entropy function defined for $0 < x < \frac{q-1}{q}$ by

$$H_q(x) = x \log_q(q - 1 - x \log_q(x)) - (1 - x) \log_q(1 - x).$$

We are now ready for the main result of this section.

**Theorem 7** If $q$ is not a square, then there are infinite families of self-dual double circulant codes of relative distance

$$\delta \geq H_q^{-1}(\frac{1}{4}).$$

**Proof.** Let $q$ be fixed and $n$ a prime going to infinity that satisfies the Artin conjecture for $q$. The double circulant codes containing a vector of weight $d \sim \delta n$ or less are by standard entropic estimates of [4] and Lemma 6 of the order of $(q - 1)q^{2nH_q(\delta)}$, up to subexponential terms. This number will be less than the total number of self-dual double circulant codes which is by Lemma 4 of the order of $q^{n/2}$, as soon as $\delta$ is of the order of the stated bound. □

5 Conclusion and Open problems

In this paper, we have studied the class of double circulant self-dual codes over finite fields, under the aspects of symmetry, enumeration, and asymptotic performance. The self-dual condition shows that these codes in odd dimension are held invariant by the dihedral group of order the length of the code in the even characteristic case, and by a monomial representation of that group in the odd characteristic case. It is possible that a similar phenomenon occurs for $n$ even and, more generally, for quasi-cyclic codes of higher index than two. Further, we have derived an exact enumeration formula for this family of codes. This formula can be interpreted as an enumeration of circulant orthogonal matrices over finite fields, thus generalizing a result of MacWilliams [9] in the prime field case, to general finite fields. Our approach to asymptotic bounds on the minimum distance relies on some deep number-theoretic conjectures (Artin or GRH). It would be a worthwhile task to remove this dependency by looking at lengths where the factorization of $x^n - 1$ into irreducible polynomials contains more than two elements.
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