Correlation Functions in 2-Dimensional Integrable Quantum Field Theories

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Abstract

In this talk I discuss the form factor approach used to compute correlation functions of integrable models in two dimensions. The Sinh-Gordon model is our basic example. Using Watson’s and the recursive equations satisfied by matrix elements of local operators, I present the computation of the form factors of the elementary field $\phi(x)$ and the stress-energy tensor $T_{\mu\nu}(x)$ of the theory.
1 Introduction

Important progress in Quantum Field Theories has been made in recent years by studying two dimensional relativistic models. The reason essentially lays in the successful application of several non-perturbative approaches which lead to exact solutions of the quantum field dynamics. In this talk I am concerned with the determination of multi-point correlation functions of local operators for a massive field theory

\[ G_n(r_1, r_2, \ldots, r_n) = \langle O_{i_1}(r_1) O_{i_2}(r_2) \ldots O_{i_n}(r_n) \rangle. \]  

(1.1)

My basic example will the the Sinh-Gordon model which is discussed in a work made in collaboration with A. Fring and P. Simonetti [1].

For two-dimensional systems, there is usually a special limit where the computation of the \( G_n \)'s is greatly simplified. This is the ultraviolet asymptotic regime of the correlation functions, defined by

\[ |r_i - r_j| \ll \xi. \]

where the correlation length is given by \( \xi = m^{-1} \). This regime can be equivalently reached by keeping finite values of the distances but taking the limit \( \xi \rightarrow \infty \). Either ways, a scale invariant behaviour occurs in the model and the resulting 2-D infinite-dimensional conformal symmetry induces a system of linear differential equations satisfied by the correlators \( G_n \) [2]. Their solution provides an explicit expression for the correlation functions of the theory in the ultraviolet regime [2, 3, 4]. However, the behaviour of correlation functions in the cross-over region

\[ |r_i - r_j| \simeq \xi, \]

and in the infrared limit

\[ |r_i - r_j| \gg \xi, \]
is more complicated. Consider, for instance, the case of two-point correlation functions

\[ \langle \mathcal{O}_i(r) \mathcal{O}_i(0) \rangle = \frac{1}{r^{\eta_i}} \Phi_i(mr) . \]  

With this parametrization, the exponent \( \eta_i \) is usually identified with twice the anomalous dimension of the field \( \mathcal{O}_i \) whereas \( \Phi_i(x) \) is a scaling function of the variable \( x = mr \). Although conformal invariance severely restricts the possible values of \( \eta_i \) and gives quite powerful classification of the ultraviolet behaviours of two-dimensional models, the computation of the scaling functions \( \Phi_i(r) \) is, on the contrary, quite difficult.

There exists, however, a large class of relativistic models where the determination of the scaling functions can be explicitly worked out. This is the case of the massive integrable systems whose dynamics is strongly constrained by an infinite number of integrals of motion. In particular, the \( S \)-matrix of these models presents factorization and elasticity properties and can be explicitly constructed [5-11]. In order to compute multi-point correlators for massive integrable models, we may exploit their spectral representation, i.e. their decomposition into an infinite sum over intermediate multi-particle contributions (fig. 1). For instance, the two-point function of a hermitian scalar operator \( \mathcal{O}_i(x) \) in real Euclidean space can be written as

\[ \langle \mathcal{O}_i(x) \mathcal{O}_i(0) \rangle = \sum_{n=0}^{\infty} \int \frac{d\beta_1 \cdots d\beta_n}{n!(2\pi)^n} < 0|\mathcal{O}_i(x)|\beta_1, \ldots, \beta_n >_{in} < \beta_1, \ldots, \beta_n | \mathcal{O}_i(0) | 0 > 
\]

\[ = \sum_{n=0}^{\infty} \int \frac{d\beta_1 \cdots d\beta_n}{n!(2\pi)^n} | F^\mathcal{O}_i_n(\beta_1 \cdots \beta_n) |^2 \exp \left( -mr \sum_{i=1}^{n} \cosh \beta_i \right) , \]  

(1.4)

*Scaling functions can be introduced as well for the \( n \)-point correlators. We concentrate our discussion on the two-point functions for clarity and simplicity.

\[ \beta \] is the rapidity variable, related to the two-dimensional momentum by

\[ p^0 = m \cosh \beta , \quad p^1 = m \sinh \beta . \]  

(1.3)
where $r$ denotes the radial distance $r = \sqrt{x_0^2 + x_1^2}$.

Figure 1

The functions

$$F_n^{\mathcal{O}_i}(\beta_1, \beta_2, \ldots, \beta_n) = <0|\mathcal{O}_i(0)|\beta_1, \beta_2, \ldots, \beta_n>_{\text{in}}$$

are the form factors of the operator $\mathcal{O}_i$ (fig. 2).

Figure 2

Analytic properties of the form factors have been investigated by several authors and particularly important contributions can be found in [13-19]. As I will show in
the following, the form factor approach is a successful method in order to compute correlation functions. The expansion of the correlators in terms of the number of intermediate particles rather than in a perturbative series of the coupling constant presents, in fact, several advantages. First of all, the form factors take into account coupling constant dependence of correlation functions to all order in perturbation theory. Secondly, expressions like that one in eq. (1.4) are fast convergent series in the number $N$ of intermediate particles even for moderate value of the distance and therefore, for practical applications, it is often sufficient to compute the form factors of the operator $\mathcal{O}_i$ with few external particles. Moreover, this spectral representation of two-point functions formally coincides with the grand partition function of a one-dimensional gas system, as noticed in [17, 19]. Hence, one can borrow successful techniques developed for one-dimensional system in order to extract non-perturbative parameters of the theory. For instance, as shown in [17, 19], the anomalous dimension of the operator $\mathcal{O}$ in the ultraviolet limit is equal to the pressure of the corresponding one-dimensional gas at one specific value of the fugacity.

I will focus on the form factor approach for the Sinh-Gordon model, described by the Lagrangian

$$
\mathcal{L} = \frac{1}{2}(\partial_{\mu}\phi)^2 - \frac{m_0^2}{g^2}\cosh g \phi(x) .
$$

The SG model belongs to the class of Toda Field Theory constructed on the simply-laced root systems, in this case $A_1$. The reason why I will consider this model is its simplicity which allows us to understand the basic principles of the computation without being masked by algebraic complicancies. Let us discuss initially the basic features of this theory.
2 The Sinh-Gordon Theory

The SG model is the simplest example of an affine Toda Field Theories \[20\], possessing a \(Z_2\) symmetry \(\phi \rightarrow -\phi\). By an analytic continuation in \(g\), i.e \(g \rightarrow ig\), it can formally be mapped to the Sine-Gordon model.

There are numerous alternative viewpoints for the Sinh-Gordon model. First, it can be regarded either as a perturbation of the free massless conformal action by means of the relevant operator \(\cosh g\phi\). Alternatively, it can be considered as a perturbation of the conformal Liouville action

\[
S = \int d^2x \left[ \frac{1}{2} (\partial_{\mu} \phi)^2 - \lambda e^{g\phi} \right]
\]

by means of the relevant operator \(e^{-g\phi}\) or as a conformal affine \(A_1\)-Toda Theory \[21\] in which the conformal symmetry is broken by setting the free field to zero.

In a perturbative approach to the quantum field theory defined by the Lagrangian (1.6), the only ultraviolet divergences which occur in any order in \(g\) come from tadpole graphs and can be removed by a normal ordering prescription with respect to an arbitrary mass scale \(M\). All other Feynman graphs are convergent and give rise to finite wave function and mass renormalisation. The coupling constant \(g\) does not renormalise.

An essential feature of the Sinh-Gordon theory is its integrability, which in the classical case can be established by means of the inverse scattering method \[24\].

In order to obtain the expressions of the (classical) conserved currents one can use Bäcklund transformation, as shown in \[1\], and obtains an infinite set of conservation laws

\[
\partial_z T_{s+1} = \partial_x \Theta_{s-1}.
\]

\(^\dagger\) Although the anomalous dimension of this operator (computed with respect to the free conformal point), is negative, \(\Delta = -g^2/8\pi\), the resulting theory is unitary. This is due to the existence of non a nonzero vacuum expectation values of some of the fields \(\mathcal{O}_i\) in the theory. A detailed discussion of this point can be found in \[18\].
The corresponding charges $Q_s$ are given by

$$Q_s = \oint [T_s + 1 \, dz + \Theta_{s-1} \, d\tau] .$$  \hspace{1cm} (2.3)

The integer-valued index $s$ which labels the integrals of motion is the spin of the operators. Non-trivial conservation laws are obtained for odd values of $s$

$$s = 1, 3, 5, 7, \ldots$$ \hspace{1cm} (2.4)

In analogy to the Sine-Gordon theory \cite{23}, an infinite set of conserved charges $Q_s$ with spin $s$ given in (2.4) also exists for the quantized version of the Sinh-Gordon theory. They are diagonalised by the asymptotic states with eigenvalues given by

$$Q_s |\beta_1, \ldots, \beta_n> = \chi_s \sum_{i=1}^{n} e^{s\beta_i} |\beta_1, \ldots, \beta_n>,$$ \hspace{1cm} (2.5)

where $\chi_s$ is the normalization constant of the charge $Q_s$. The existence of these higher integrals of motion precludes the possibility of production processes and hence guarantees that the $n$-particle scattering amplitudes are purely elastic and factorized into $n(n-1)/2$ two-particle $S$-matrices. The exact expression for the Sinh-Gordon theory is given by \cite{10}

$$S(\beta, B) = \frac{\tanh \frac{1}{2}(\beta - i\pi B/2)}{\tanh \frac{1}{2}(\beta + i\pi B/2)} ,$$ \hspace{1cm} (2.6)

where $B$ is the following function of the coupling constant $g$

$$B(g) = \frac{2g^2}{8\pi + g^2} .$$ \hspace{1cm} (2.7)

This formula has been checked against perturbation theory in ref. \cite{10} and can also be obtained by analytic continuation of the $S$-matrix of the first breather of the Sine-Gordon theory \cite{5}. For real values of $g$ the $S$-matrix has no poles in the physical sheet and hence there are no bound states, whereas two zeros are present at the crossing symmetric positions

$$\beta = \begin{cases} 
\frac{i\pi B}{2} \\
\frac{i\pi(2-B)}{2}
\end{cases}$$ \hspace{1cm} (2.8)
An interesting feature of the S-matrix is its invariance under the map \[ B \rightarrow 2 - B \] \[ \text{(2.9)} \]
i.e. under the \textit{strong-weak} coupling constant duality
\[ g \rightarrow \frac{8\pi}{g}. \] \[ \text{(2.10)} \]
This duality is a property shared by the unperturbed conformal Liouville theory \[ (2.1) \] \[ \text{[22]} \] and it is quite remarkable that it survives even when the conformal symmetry is broken.

3 Form Factors

The form factors (FF) are matrix elements of local operators between the vacuum and \( n \)-particle in-state
\[ F_n^O(\beta_1, \beta_2, \ldots, \beta_n) = \langle 0|O(0)|\beta_1, \beta_2, \ldots, \beta_n \rangle_{\text{in}}. \] \[ \text{(3.1)} \]
For local scalar operators \( O(x) \), relativistic invariance implies that \( F_n \) are functions of the difference of the rapidities. Except for the poles corresponding to the one-particle bound states in all sub-channels, we expect the form factors \( F_n \) to be analytic inside the strip \( 0 < \text{Im}\beta_{ij} < 2\pi \).

The form factors of a hermitian local scalar operator \( O(x) \) satisfy a set of functional equations, known as Watson’s equations \[ [12] \], which for integrable systems assume a particularly simple form
\[ F_n(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n) = F_n(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n)S(\beta_i - \beta_{i+1}), \] \[ \text{(3.2)} \]
\[ F_n(\beta_1 + 2\pi i, \ldots, \beta_{n-1}, \beta_n) = F_n(\beta_2, \ldots, \beta_n, \beta_1) = \prod_{i=2}^{n} S(\beta_i - \beta_1)F_n(\beta_1, \ldots, \beta_n). \]
The first equation states that as a result of the commutation of two particles in the asymptotic state we get a scattering process whereas the second equation fixes the
discontinuity of the functions $F_n$ on the cuts $\beta_i = 2\pi i$. In the case $n = 2$, eqs. (3.2) reduce to

$$
F_2(\beta) = F_2(-\beta)S_2(\beta),
$$

$$
F_2(i\pi - \beta) = F_2(i\pi + \beta).
$$

The general solution of Watson’s equations for diagonal $S$-matrix systems can always be brought into the form

$$
F_n(\beta_1, \ldots, \beta_n) = K_n(\beta_1, \ldots, \beta_n) \prod_{i<j} F_{\min}(\beta_{ij}),
$$

where $F_{\min}(\beta)$ has the properties that it satisfies (3.3), is analytic in $0 \leq \text{Im} \beta \leq \pi$, has no zeros in $0 < \text{Im} \beta < \pi$, and converges to a constant value for large values of $\beta$. These requirements uniquely determine this function, up to a normalization. In the case of the SG model, $F_{\min}(\beta)$ is given by

$$
F_{\min}(\beta, B) = \mathcal{N} \exp \left[ 8 \int_0^\infty \frac{dx}{x} \sinh \left( \frac{x B}{4} \right) \sinh \left( \frac{x}{2} \left( 1 - \frac{B}{2} \right) \right) \sin \left( \frac{x \hat{\beta}}{2 \pi} \right) \sinh \left( \frac{x}{2} \sinh \left( \frac{x \hat{\beta}}{2 \pi} \right) \right) \right].
$$

We choose our normalization to be

$$
\mathcal{N} = \exp \left[ -4 \int_0^\infty \frac{dx}{x} \sinh \left( \frac{x B}{4} \right) \sinh \left( \frac{x}{2} \left( 1 - \frac{B}{2} \right) \right) \sinh \left( \frac{x}{2} \right) \right].
$$

The analytic structure of $F_{\min}(\beta, B)$ can be easily read from its infinite product representation in terms of $\Gamma$ functions

$$
F_{\min}(\beta, B) = \prod_{k=0}^\infty \left| \frac{\Gamma \left( k + \frac{3}{2} + i\frac{B}{2\pi} \right) \Gamma \left( k + 1 + \frac{B}{4} + i\frac{\beta}{2\pi} \right)}{\Gamma \left( k + \frac{1}{2} + i\frac{B}{2\pi} \right) \Gamma \left( k + \frac{3}{2} - \frac{B}{4} + i\frac{\beta}{2\pi} \right)} \right|^2.
$$

$F_{\min}(\beta, B)$ has a simple zero at the threshold $\beta = 0$ since $S(0) = -1$ and its asymptotic behaviour is given by

$$
\lim_{\beta \to \infty} F_{\min}(\beta, B) = 1.
$$

It satisfies the functional equation

$$
F_{\min}(i\pi + \beta, B)F_{\min}(\beta, B) = \frac{\sinh \beta}{\sinh \beta + \sinh \frac{i\pi B}{2}}.
$$
which is useful in order to find a convenient form for the recursive equations of the
form factors.

A useful expression for the numerical evaluation of \( F_{\text{min}}(\beta, B) \) is given by

\[
F_{\text{min}}(\beta, B) = \mathcal{N} \prod_{k=0}^{N-1} \left[ \left( 1 + \left( \frac{\beta/2\pi}{k + 1 - \frac{1}{2}} \right)^2 \right) \left( 1 + \left( \frac{\beta/2\pi}{k + 1 + \frac{1}{2}} \right)^2 \right) \right]^{k+1} \\
\times \exp \left[ 8 \int_0^\infty \frac{dx}{x} \frac{\sinh \left( \frac{xB}{4} \right)}{\sinh^2 x} \frac{\sinh \left( \frac{x}{2} \right)}{\sinh \left( \frac{x}{2} \right)} \left( N + 1 - Ne^{-2x} \right) e^{-2Nx} \sin^2 \left( \frac{x \hat{\beta}}{2\pi} \right) \right].
\] (3.10)

The rate of convergence of the integral may be improved substantially by increasing
the value of \( N \).

The remaining factors \( K_n \) in (3.4) then satisfy Watson’s equations with \( S_2 = 1 \),
which implies that they are completely symmetric, \( 2\pi i \)-periodic functions of the
\( \beta_i \). They must contain all the physical poles expected in the form factor under
consideration and must satisfy a correct asymptotic behaviour for large value of \( \beta_i \).
Both requirements depend on the nature of the theory and on the operator \( O \).

Taking into account the one-particle pole in the three-particle channel at \( \beta_{ij} = i\pi \), the general form factors of a scalar hermitian operator in the Sinh-Gordon model
can be parameterized as

\[
F_n(\beta_1, \ldots, \beta_n) = H_n Q_n(x_1, \ldots, x_n) \prod_{i<j} F_{\text{min}}(\beta_{ij}) \frac{1}{(x_i + x_j)},
\] (3.11)

where we have introduced the variables

\[
x_i = e^{\beta_i},
\] (3.12)

and the normalization constant \( H_n \). The functions \( Q_n(x_1, \ldots, x_n) \) are symmetric polynomials\footnote{The polynomial nature of the functions \( Q_n \) is dictated by the locality of the theory \cite{14}.} in the variables \( x_i \). They can be expressed in terms of \textit{elementary symmetric polynomial} \( \sigma_k^{(n)}(x_1, \ldots, x_n) \) which are generated by \cite{25}

\[
\prod_{i=1}^n (x + x_i) = \sum_{k=0}^n x^{n-k} \sigma_k^{(n)}(x_1, x_2, \ldots, x_n).
\] (3.13)
Relativistic invariance demands that the total degree of $Q_n$ entering a form factor of spinless operators should be $n(n - 1)/2$ (in order to match the total degree of the denominator in (3.11)). The order of the degree of $Q_n$ in each variable $x_i$ is fixed, on the other hand, by the nature and by the asymptotic behaviour of the operator $\mathcal{O}$ which is considered. We will come back to this point later, when we discuss specific operators.

### 3.1 Pole Structure and Residue Equations for the Form Factors

The pole structure of the form factors induces a set of recursive equations for the $F_n$. For the Sinh-Gordon model there is only one kind of poles that arise from kinematical poles located at $\beta_{ij} = i\pi$. They are related to the one-particle pole in a subchannel of three-particle states which, in turn, corresponds to a crossing process of the elastic $S$-matrix. The corresponding residues are computed by the LSZ reduction \cite{15,16} and give rise to a recursive equation between the $n$-particle and the $(n + 2)$-particle form factors (fig. 3)

$$-i \lim_{\beta \to \tilde{\beta}} (\tilde{\beta} - \beta) F_{n+2}(\tilde{\beta} + i\pi, \beta, \beta_1, \beta_2, \ldots, \beta_n) = \left(1 - \prod_{i=1}^{n} S(\beta - \beta_i)\right) F_n(\beta_1, \ldots, \beta_n).$$

(3.14)
This equation establishes a recursive structure between the \((n + 2)\)- and \(n\)-particle form factors.

## 4 Solution of the Recursive Equations

According to the \(Z_2\) symmetry of the SG model, we can label the operators by their parity. For operators which are \(Z_2\)-odd the only possible non-zero form factors are those involving an odd number of particles. For \(Z_2\)-even operators the only possible non-zero form factors are those involving an even number of particles. The vacuum expectation value of \(Z_2\)-even operators can in principle be different from zero.

The simplest representative of the odd sector is given by the (renormalised) field \(\phi(x)\) itself. It creates a one-particle state from the vacuum. Our normalization is fixed to be

\[
F_1^\phi(\beta) = \langle 0 \mid \phi(0) \mid \beta >_{\text{in}} = \frac{1}{\sqrt{2}}.
\]

For the even sector, an important operator is given by the energy-momentum tensor

\[
T_{\mu\nu}(x) = 2\pi (\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}(x))
\]

where \(\cdot \cdot\) denotes the usual normal ordering prescription with respect an arbitrary mass scale \(M\). Its trace \(T^{\mu}_\mu(x) = \Theta(x)\) is a spinless operator whose normalization is fixed in terms of its two-particle form factor

\[
F_2^\Theta(\beta_{12} = i\pi) = \langle \text{out} \mid \Theta(0) \mid \beta_{12} >_{\text{in}} = 2\pi m^2,
\]

where \(m\) is the physical mass.

In the following we shall compute the form factors of the operators \(\phi(x)\) and \(\Theta(x)\). Employing the parameterization (3.11), the recursive equations (3.14) take on the form

\[
(-)^n Q_{n+2}(-x, x_1, \ldots, x_n) = xD_n(x, x_1, x_2, \ldots, x_n) Q_n(x_1, x_2, \ldots, x_n)
\]

(4.4)
where we have introduced the function

\[
D_n = \frac{-i}{4 \sin(\pi B/2)} \left( \prod_{i=1}^{n} [(x + \omega x_i)(x - \omega^{-1} x_i)] - \prod_{i=1}^{n} [(x - \omega x_i)(x + \omega^{-1} x_i)] \right) \tag{4.5}
\]

with \( \omega = \exp(i\pi B/2) \). The normalization constants for the form factors of odd and even operators are conveniently chosen to be

\[
H_{2n+1} = H_1 \left( \frac{4 \sin(\pi B/2)}{F_{\text{min}}(i\pi, B)} \right)^n \tag{4.6}
\]

\[
H_{2n} = H_2 \left( \frac{4 \sin(\pi B/2)}{F_{\text{min}}(i\pi, B)} \right)^{n-1}
\]

where \( H_1 \) and \( H_2 \) are the initial conditions, fixed by the nature of the operator.

Using the generating function \( (3.13) \) of the symmetric polynomials, the function \( D_n \) can be expressed as

\[
D_n = \frac{1}{2 \sin(\pi B/2)} \sum_{l,k=0}^{n} (-1)^l \sin \left( (k - l) \frac{\pi B}{2} \right) x^{2n-l-k} \sigma_{l}^{(n)} \sigma_{k}^{(n)}. \tag{4.7}
\]

As function of \( B \), \( D_n \) is invariant under \( B \to -B \).

As shown in [1], the symmetric polynomials \( Q_{2n+1} \) entering the form factors of the elementary field \( \phi(x) \) can be factorized as

\[
Q_{2n+1}(x_1, \ldots, x_{2n+1}) = \sigma_{2n+1}^{(2n+1)} P_{2n+1}(x_1, \ldots, x_{2n+1}) \quad n > 0 , \tag{4.8}
\]

whereas the analogous polynomials entering the form factors of the trace of the stress-energy tensor can be written as

\[
Q_{2n}(x_1, \ldots, x_{2n}) = \sigma_{2n}^{(2n)} P_{2n}(x_1, \ldots, x_{2n}) \quad n > 1 . \tag{4.9}
\]

\( P_n(x_1, \ldots, x_n) \) is a symmetric polynomial of total degree \( n(n - 3)/2 \) and of degree \( n - 3 \) in each variable \( x_i \). Using the following property of the elementary symmetric polynomials

\[
\sigma_{k}^{(n+2)}(-x, x, x_1, \ldots, x_n) = \sigma_{k}^{(n)}(x_1, x_2, \ldots, x_n) - x^2 \sigma_{k-2}^{(n)}(x_1, x_2, \ldots, x_n) , \tag{4.10}
\]
the recursive equations (4.4) can then be written in terms of the $P_n$ as

$$(-)^{n+1}P_{n+2}(-x,x,x_1,\ldots,x_n) = \frac{1}{x}D_n(x,x_1,x_2,\ldots,x_n)P_n(x_1,x_2,\ldots,x_n).$$

(4.11)

Using the recursive equations (4.11) and the transformation property of the elementary symmetric polynomials (4.10), the explicit expressions of the first polynomials $P_n(x_1,\ldots,x_n)$ are given by

$$P_3(x_1,\ldots,x_3) = 1$$
$$P_4(x_1,\ldots,x_4) = \sigma_2$$
$$P_5(x_1,\ldots,x_5) = \sigma_2\sigma_3 - c_1^2\sigma_5$$
$$P_6(x_1,\ldots,x_6) = \sigma_3(\sigma_2\sigma_4 - \sigma_6) - c_1^2(\sigma_4\sigma_5 + \sigma_1\sigma_2\sigma_6 - \sigma_3\sigma_6)$$
$$P_7(x_1,\ldots,x_7) = \sigma_2\sigma_3\sigma_4\sigma_5 - c_1^2(\sigma_4\sigma_5^2 + \sigma_1\sigma_2\sigma_5\sigma_6 + \sigma_2^2\sigma_3\sigma_7 - c_1^2\sigma_2\sigma_5\sigma_7) +$$
$$-c_2(\sigma_1\sigma_2\sigma_4\sigma_7 + \sigma_3\sigma_5\sigma_6 - c_2\sigma_1\sigma_6\sigma_7) + c_1^2c_2^2\sigma_7^2$$

where $c_1 = 2\cos(\pi B/2)$ and $c_2 = 1 - c_1^2$. Expression of the higher $P_n$ are easily computed by an iterative use of eqs. (4.4). For practical application the first representatives of $P_n$ are sufficient to compute with a high degree of accuracy the correlation functions of the fields. In fact, the $n$-particle term appearing in the correlation function of the fields (1.4) behaves as $e^{-n(mr)}$ and for quite large values of $mr$ the correlator is dominated by the lowest number of particle terms. This conclusion is also confirmed by an application of the $c$-theorem which I discuss at the end of the talk. It is interesting to notice that closed expressions for $P_n$ can be found for particular values of the coupling constant.

4.0.1 The Self-Dual Point

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*The upper index of the elementary symmetric polynomials entering $P_n$ is equal to $n$ and we suppress it, in order to simplify the notation.*
The self-dual point in the coupling constant manifold has the special value

\[ B \left( \sqrt{8\pi} \right) = 1. \] (4.13)

The two zeros of the \( S \)-matrix merge together and the function \( D_n(x, x_1, x_2, \ldots, x_n) \) acquires the particularly simple form

\[
D_n(x|x_1, x_2, \ldots, x_n) = \left( \sum_{k=0}^{n} (-1)^{k+1} \sin \frac{k\pi}{2} x^{n-k} \sigma_k^{(n)} \right) \left( \sum_{l=0}^{n} (-1)^{l} \cos \frac{l\pi}{2} x^{n-l} \sigma_l^{(n)} \right).
\] (4.14)

In this case the general solution of the recursive equations (4.11) is given by

\[
P_n(x_1, x_2, \ldots, x_n) = \det \mathcal{A}(x_1, x_2, \ldots, x_n)
\] (4.15)

where \( \mathcal{A} \) is an \((n - 3) \times (n - 3)\) matrix whose entries are

\[
\mathcal{A}_{ij}(x_1, x_2, \ldots, x_n) = \sigma_{2j-i+1}^{(n)} \cos^2 \left( (i-j) \frac{\pi}{2} \right),
\] (4.16)

i.e.

\[
\mathcal{A} = \begin{pmatrix}
\sigma_2 & 0 & \sigma_6 & 0 & \cdots \\
0 & \sigma_3 & 0 & \sigma_7 & \cdots \\
1 & 0 & \sigma_4 & 0 & \cdots \\
0 & \sigma_1 & 0 & \sigma_5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\] (4.17)

This can be proved by exploiting the properties of determinants.

### 4.0.2 The “Inverse Yang-Lee” Point

A closed solution of the recursive equations (4.11) is also obtained for

\[ B \left( 2\sqrt{\pi} \right) = \frac{2}{3}. \] (4.18)

The reason is that, for this particular value of the coupling constant the \( S \)-matrix of the Sinh-Gordon theory coincides with the inverse of the \( S \)-matrix \( S_{YL}(\beta) \) of the Yang-Lee model or, equivalently

\[
S(\beta, -\frac{2}{3}) = S_{YL}(\beta).
\] (4.19)
Since the recursive equations (4.11) are invariant under $B \to -B$ (see sect. 4.2), a solution is provided by the same combination of symmetric polynomials found for the Yang-Lee model \[16, 18\], i.e.

$$P_n(x_1, x_2, \ldots, x_n) = \det B(x_1, x_2, \ldots, x_n)$$

(4.20)

with the following entries of the $(n - 3) \times (n - 3)$-matrix $B$

$$B_{ij} = \sigma_{3j-2i+1}$$

(4.21)

## 5 Form factors and c-theorem

The Sinh-Gordon model can be regarded as deformation of the free massless theory with central charge $c = 1$. This fixed point governs the ultraviolet behaviour of the model whereas the infrared behaviour corresponds to a massive field theory with central charge $c = 0$. Going from the short- to large-distances, the variation of the central charge is dictated by the $c$-theorem of Zamolodchikov [26]. An integral version of this theorem has been derived by Cardy [27] and related to the spectral representation of the two-point function of the trace of the stress-energy tensor in \[28, 29\], i.e.

$$\Delta c = \int_0^\infty d\mu \, c_1(\mu)$$

(5.1)

where $c_1(\mu)$ is given by

$$c_1(\mu) = \frac{6}{\pi^2} \frac{1}{\mu^3} \text{Im} G(p^2 = -\mu^2)$$

(5.2)

$$G(p^2) = \int d^2 x \ e^{-ip\cdot x} < 0|\Theta(x)\Theta(0)|0_{\text{conn}}$$

Inserting a complete set of in-state into (5.2), we can express the function $c_1(\mu)$ in terms of the form factors $F_{2n}^\Theta$

$$c_1(\mu) = \frac{12}{\mu^3} \sum_{n=1}^\infty \frac{1}{(2n)!} \int \frac{d\beta_1 \ldots d\beta_{2n}}{(2\pi)^{2n}} \ | F_{2n}^\Theta(\beta_1, \ldots, \beta_{2n}) |^2$$

(5.3)

$$\times \delta\left(\sum_i m \sinh \beta_i\right) \delta\left(\sum_i m \cosh \beta_i - \mu\right)$$
For the Sinh-Gordon theory $\Delta c = 1$ and it is interesting to study the convergence of this series increasing the number of intermediate particles. For the two-particle contribution, we have the following expression

$$\Delta c^{(2)} = \frac{3}{2F_{\text{min}}^2(i\pi)} \int_0^\infty \frac{d\beta}{\cosh^4 \beta} |F_{\text{min}}(2\beta)|^2 .$$

The numerical results for different values of the coupling constant $g^2/4\pi$ are listed in the table below It is evident that the sum rule is saturated by the two-particle form factor also for large values of the coupling constant. Hence, the expansion in the number of intermediate particles results in a fast convergent series, as it is confirmed by the computation of the next terms involving the form factor with four and six particles.
6 Conclusions

The computation of the Green functions is a central problem in a Quantum Field Theory. For integrable models, a promising approach to this question is given by the bootstrap principle applied to the computation of the matrix elements of local operators. It would be interesting to use this approach in order to derive differential equations satisfied by the quantum correlators and also to classify the operator content of a quantum integrable field theory.

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