NOTE ON THE CORTEX OF TWO-STEP NILPOTENT LIE ALGEBRAS

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ABSTRACT. In this paper, we construct an example of a family of 4d-dimensional two-step nilpotent Lie algebras \((\mathfrak{g}_d)_{d \geq 2}\) so that the cortex of the dual of each \(\mathfrak{g}_d\) is a projective algebraic set. More precisely, we show that the cortex of each dual \(\mathfrak{g}_d^*\) of \(\mathfrak{g}_d\) is the zero set of a homogeneous polynomial of degree \(d\). This example is a generalization of one given in "Irreducible representations of locally compact groups that cannot be Hausdorff separated from the identity representation" by "M.E.B. BEKKA, AND E. KANIUTH".

1. Introduction

The cortex of general locally compact group \(G\) was defined in \cite{4} as
\[
cor(G) = \{ \pi \in \hat{G}, \text{ which cannot be Hausdorff-separated from the identity representation}1_G \},
\]
where \(\hat{G}\) is the dual of \(G\) (set of unitary irreducible representations of \(G\)), that is, \(\pi \in \cor(G)\) if and only if for all neighborhood \(V\) of \(1_G\) and for each neighborhood \(U\) of \(\pi\), one has \(V \cap U\) is non-empty set. Note that \(\hat{G}\) is equipped with the topology of Fell which can be described in terms of weak containment (see \cite{9}) and, in general, is not separated. However, if \(G\) is abelian, then \(\hat{G}\) is separated and hence \(\cor(G) = \{1_G\}\). The set
\[
\Cor(\mathfrak{g}^*) = \{ \ell = \lim_{m \to \infty} \Ad_{s_m}^{-1}(\ell_m), \text{ where } \{s_m\} \subset G \text{ and } \{\ell_m\} \subset \mathfrak{g}^* \text{ such that } \lim_{m \to \infty} \ell_m = 0 \}
\]
and we have \(\pi_\ell \in \cor(G)\) if and only if \(\ell \in \Cor(\mathfrak{g}^*)\). Note that in the case of general Lie groups, the two definitions are not so easily related. With the above in mind, it comes out that the set in question can be defined on \(\mathfrak{g}^*/\Ad^*G\) as in \cite{5} and one can investigate the parametrization of \(\mathfrak{g}^*\) to determine such a set.

Motivated by this situation, the authors in \cite{5} define the cortex \(C_V(G)\) of a representation of a locally compact group \(G\) on a finite-dimensional vector space \(V\) as the set of all \(v \in V\) for which \(G.v\) and \(\{0\}\) cannot be Hausdorff-separated in the orbit-space \(V/G\). They give a precise description of \(C_V(G)\) in the case \(G = \mathbb{R}\). Moreover, they consider the subset \(IC_V(G)\) of \(V\) consisting of the common zeroes of all \(G\)-invariant polynomials \(p\) on \(V\) with \(p(0) = 0\). They show that \(IC_V(G) = C_V(G)\) when \(G\) is a nilpotent Lie group of the form \(G = \mathbb{R} \times \mathbb{R}^n\) and \(V = \mathfrak{g}^*\) the dual of the Lie algebra \(\mathfrak{g}\). This fails for a general nilpotent Lie group, even in the case of two-step nilpotent Lie group, in \cite{4}, the authors show that the cortex of any two-step nilpotent Lie algebra \(\mathfrak{g}\) is the closure of the set:
\[
\{\ad_{\ell}^*X, X \in \mathfrak{g}, \ell \in \mathfrak{g}^*\}
\]
and they give a counter-example of 8-dimensional Lie algebra \(\mathfrak{g}\) for which \(\Cor(\mathfrak{g}^*) \subsetneq ICor(\mathfrak{g}^*)\).

In \cite{4}, the author gives an explicit description of \(\Cor(\mathfrak{g}^*)\) when \(\mathfrak{g}\) is a nilpoten Lie algebra with
dim(g) ≤ 6 by the means of parametrization of the coadjoint orbits in g∗. In [4] one gives an explicit description of the cortex of ceratin class of exponential Lie algebras g by investigating the tools of parametrization of the coadjoint orbits in the dual space g∗ (see for example [1] [2]).

Fixing the class of two-step nilpotent Lie algebras, for any ℓ belonging to the dual space of a such Lie algebra, its coadjoint orbit O satisfies

\[ O = \{ \ell \} + T_{\ell}O, \]

where \( T_{\ell}O \) is the tangent space to the orbit O at ℓ. From this, we deduce that each coadjoint orbit is a flat (affine) symplectic manifold. Combining this with the property (1), we intuitively think that the cortex of such class of Lie algebras needs not to be far from affine sets however in this paper, we show that for this class of Lie algebras the cortex can be very far from affine or flat sets. Indeed, we give a generalization of the example given in [4] pp. 210, our example is a family of 4d-dimensional two-step nilpotent Lie algebras \((g_d)_{d≥2}\) such that the cortex of each \(g_d^*\) is the zero set of a homogeneous polynomial of degree d in the complement \(\mathbb{J}_d\) of the center \(\mathbb{J}_d\) of \(g_d\).

The paper is organized as follows, in Section 2, we give the definition of the main tools of cortex and we recall some essential results. In section 3, we give the main example and we conclude the paper by some conjecture about the optimality of the family \((g_d)_{d≥2}\).

2. Preliminaries

2.1. Definitions and notations. We begin by setting some notations and useful facts which will be used throughout the paper. This material is quite standard. Throughout, G will always denote an n-dimensional connected and simply connected two-step nilpotent Lie group with (real) Lie algebra g so that \([g, [g, g]] = 0\). We denote by \(\mathfrak{j}\) the center of g and \(g^\ast\) denotes the dual of g. Let us denote \(\mathfrak{j}^\perp\) to be

\[ \mathfrak{j}^\perp = \{ \ell \in g^\ast : \ell(Z) = 0 \quad \forall Z \in \mathfrak{j}\}. \]

G acts on g by the adjoint action denoted by Ad and on g∗ by the coadjoint action denoted by Ad∗. Since G is nilpotent then the exponential mapping from g onto G is a global diffeomorphism, and we can write \(Ad_{\exp}(X) = e^{adX} \) where \(adX(Y) = [X, Y]\). Since g is two-step nilpotent Lie algebra, one has \(e^{adX} = Id_g + adX\), and hence the coadjoint action of G on the dual space g∗ of g is given by \(Ad^*_{\exp}(X) = Id_{g^\ast} + ad^*X\) where \(ad^*\) is the coadjoint action of g on g∗. As a consequence, for any \(\ell \in g^\ast\), if \(O = Ad^*(G)\ell\) denotes the coadjoint orbit of \(\ell\), then

\[ O = \{ \ell \} + T_{\ell}O, \]

where \(T_{\ell}O\) is the tangent space of O at \(\ell\). Thus we see that the coadjoint orbits in two-step nilpotent Lie algebras are flat symplectic manifolds.

Following [3], we recall the following:

**Definition 2.1.** Let \(\ell \in g^\ast\). We define the cortex of \(g^\ast\) as

\[ Cor(g^\ast) = \left\{ \lim_{m \to \infty} Ad^*_s(\ell_m) | (s_m)_m \subset G, (\ell_m)_m \subset g^\ast \quad \text{with} \quad \lim_{m \to \infty} \ell_m = 0 \right\} \]

As a consequence of this definition, since g is two-step nilpotent Lie algebra, then we can write

\[ Cor(g^\ast) = \left\{ \lim_{\ell \to 0} ad^*_{X_n}(\ell_n), \quad (X_n)_n \subset g, (\ell_n)_n \subset g^\ast \right\} = \lim_{\ell \to 0} T_{\ell}O. \]

In [3] there is a study of the cortex of several locally compact groups and especially the case of connected Lie groups. In particular, when G is a connected two-step nilpotent Lie group, one has the following:

**Proposition 2.1.** Let g be a nilpotent Lie algebra of class 2 (i.e., \([g, [g, g]] = 0\)), and let \(G = \exp(g)\) be the associated Lie group. Denote by ad∗ the coadjoint representation of g on g∗. Let \(f \in g^\ast\). Then the corresponding representation \(\pi_f\) of G belongs to cor(G) if and only if \(f\) belongs to the closure of the subset \(\{ad^*_X(\ell), X \in g, \ell \in g^\ast\}\) of g∗.

In [3], the author shows the following:
Lemma 2.2. If \( g \) is a two-step nilpotent Lie algebra, and if the coadjoint orbits have codimension 0 or 1 in \( z^1 \), then
\[
\text{Cor}(g^*) = z^1.
\]

Since the coadjoint orbits in two-step nilpotent Lie algebras are affine, we naturally think that the cortex of any two-step nilpotent Lie algebra will be an affine subset of \( z^1 \). Again, in [4] it was given an interesting example of a two-step nilpotent Lie algebra \( g \) so that the corresponding cortex in \( g^* \) is a projective algebraic set given by a quadric, let us recall it.

Example 2.3. Let \( g \) be the Lie algebra of dimension 8 with basis \((X_1, \ldots, X_6, Z_1, Z_2)\) and nontrivial commutators
\[
[X_1, X_3] = [X_2, X_3] = Z_1, [X_1, X_6] = [X_2, X_4] = Z_2.
\]
Then \( g \) is nilpotent of class 2, and it is easily verified that the center \( z \) of \( g \) equals \( \mathbb{R} Z_1 + \mathbb{R} Z_2 \).
Let \((X_1^*, \ldots, X_6^*, Z_1^*, Z_2^*)\) denote the corresponding basis of \( g^* \), and let \( G = \exp(g) \). It follows by Proposition Z.3 that
\[
\text{cor}(G) = \{ \pi_f \in \hat{G}; f = \sum_{i=1}^{6} t_i X_i^* \text{ with } t_3 t_6 = t_4 t_5 \},
\]
or equivalently,
\[
\text{Cor}(g^*) = \{ f \in g^* : f = \sum_{i=1}^{6} t_i X_i^* \text{ with } t_3 t_6 = t_4 t_5 \}.
\]

In this example, it was shown that the set of invariant polynomials on \( g^* \) is
\[
\text{Pol}(g^*)^G = \mathbb{R}[z_1, z_2, z_1 x_4 - z_2 x_3, z_1 x_6 - z_2 x_5],
\]
and
\[
\text{ICor}(g^*) = \{ \ell \in g^* : P(\ell) = P(0), \forall P \in \text{Pol}(g^*)^G \} = z^1.
\]
Then \( \text{Cor}(g^*) \subsetneq \text{ICor}(g^*) \). Thus the cortex of \( G \) is strictly contained in the set of all irreducible representations of \( G \) having a trivial infinitesimal character.

Up to writing this note, I don’t see any example of two-step nilpotent Lie algebra \( g \) so that that \( \text{Cor}(g^*) \) is the zero set of polynomials whose degree is grater than 2. In the sense that we think \( \text{Cor}(g^*) \) is the zero set of polynomials whose degree is equal of less then 2.

3. Main example

Let \( d \) be an integer with \( d \geq 2 \) and \( g_d \) be the Lie algebra whose Jordan-Höder basis
\[
\mathcal{B} = (Z_1, \ldots, Z_d, Y_1, Y_2, \ldots, Y_{2d-1}, Y_{2d}, X_1, \ldots, X_d),
\]
and nontrivial brackets
\[
[X_i, Y_{2i-1}] = Z_1, i = 1, \ldots, d, [X_k, Y_{2k}] = Z_{k+1}, k = 1, \ldots, d-1, [X_{2d}, Y_{2d}] = Z_2 + \cdots + Z_d.
\]

Proposition 3.1. For each Lie algebra \( g_d \) \( (d \geq 2) \), one has
(i). The generic orbits are 2d dimensional affine manifold defined by the minimal layer
\[
\Omega_d = \{ \ell \in g_d^* : \ell(Z_1) \neq 0 \},
\]
(ii). Any coadjoint orbit \( \mathcal{O} \) of \( \ell \in \Omega_d \) is given by
\[
\mathcal{O} = G \cdot \ell = \left\{ \prod_{i=1}^{d} \text{Ad}^*_{\exp(t_i Y_{2i-1})} \text{Ad}^*_{\exp(s_i X_i)}(\ell), \ (t_1, \ldots, t_d, s_1, \ldots, s_d) \in \mathbb{R}^{2d} \right\}.
\]
(iii). The algebra of \( G \)-invariant polynomials is
\[
\text{Pol}(g_d^*)^G = \mathbb{R}[z_1, \ldots, z_d, z_1 y_2 - z_2 y_1, \ldots, z_1 y_{2d-2} - z_{d-1} y_{2d-3}, z_1 y_{2d} - (z_2 + \cdots + z_d)y_{2d-1}].
\]
Proof. (i) Fixing the Jordan Hölder basis $B_d$ and denote $B_d = (U_1, \ldots, U_{4d})$ and $B_d^* = (U^*_1, \ldots, U^*_{4d})$ its dual basis with

$$U_i = \begin{cases} 
Z_i, & \text{if } 1 \leq i \leq d, \\
Y_{i-d}, & \text{if } d + 1 \leq i \leq 3d, \\
X_{i-3d}, & \text{if } 3d + 1 \leq i \leq 4d.
\end{cases}$$

Using the methods of [1], we can see that the minimal layer in $g^*_d$ is

$$\Omega_d = \{ \ell \in g^* : \ell(U_1) = \ell(Z_1) \neq 0 \},$$

and it corresponds to the set of jump indices $\mathbf{e}_d = \mathbf{i}_d \cup \mathbf{j}_d$ with $\mathbf{i}_d = \{ d + 1 < d + 3 < \cdots < 3d - 1 \}$ and $\mathbf{j}_d = \{ 3d + 1, 3d + 2, \ldots, 4d \}$. The cross-section $\Sigma_d$ is given by

$$\Sigma_d = \left( \sum_{k \in \mathbf{e}} R U^*_k \right) \cap \Omega = \left( \sum_{k=1}^d R Z^*_k + R Y^*_2 \right) \cap \Omega_d.$$

Concerning (ii) and (iii), they can be easily showed by using also the methods of [1] in the parametrization of coadjoint orbits.

\[ \square \]

**Theorem 3.2.** Let $(Z_1^*, \ldots, Z_d^*, Y_1^*, \ldots, Y_{2d}^*, X_1^*, \ldots, X_d^*)$ be the corresponding dual basis in $g^*_d$. The cortex of $g^*_d$ is the dual of the Lie algebra $g_d$ is a projective set. More precisely, if we denote $\ell = \sum_{i=1}^d (z_i Z_i^* + x_i X_i^*) + \sum_{j=1}^{2d} y_j Y_j^* \in g^*$ by $\ell = (z_1, y_j, x_k)$ then $\text{Cor}(g^*_d)$ is the projective algebraic set given by

$$\text{Cor}(g^*_d) = \{ \ell = (z_1, y_j, x_k) : z_1 = \cdots = z_d = y_{2d-1}(\sum_{i=1}^{d-1} y_{2i} \prod_{j=1, j \neq i}^{d-1} y_{2j-1}) - y_{2d} \prod_{j=1}^{d-1} y_{2j-1} = 0 \}.$$  

**Proof.** Note that since $g_d$ is exponential Lie algebra and $\Omega_d$ is dense in $g^*_d$ (Zariski open set) then

$$\text{Cor}(g^*_d) = \{ \lim m \, \text{Ad}_{exp(X_m)}^* \ell_m, \ (\ell_m)_m \in g^*_d, \ (\ell_m)_m \in \Omega_d, \text{ and } \lim m = 0 \}.$$  

On other hand each coadjoint orbit in $\Omega$ corresponds to the set of jump indices $\mathbf{e}_d$ and satisfies

$$\mathcal{O} = G \cdot \ell = \left\{ \prod_{i=1}^d \text{Ad}_{exp(t, Y_{2i-1})}^* \text{Ad}_{exp(s, X_i)}^* \ell, \ (t_1, \cdots, t_d, s_1, \cdots, s_d) \in \mathbb{R}^{2d} \right\}.$$  

Hence we can deduce that $\text{Cor}(g^*_d)$ is the closure of the set

$$\{ ad_X^*(\ell), \ (\ell) \in \Omega_d \text{ and } X \in \text{Vect}\{ Y_{2k-1}, X_k, 1 \leq k \leq d \} \}.$$  

Now let $\ell = (z_1, y_j, x_k) \in \Omega_d$ and $\xi \in \mathcal{O}$, with $\xi = \sum_{i=1}^d \lambda_i Z_i^* + \sum_{j=1}^{2d} \gamma_j Y_j^* + \sum_{k=1}^d \beta_k X_k^*$, then one has

$$\xi = \begin{cases} 
\lambda_i &= z_i, \text{ if } i = 1, \cdots, d; \\
\gamma_{2j-1} &= y_{2j-1} - s_j z_1, \text{ if } j = 1, \cdots, d - 1; \\
\gamma_{2j} &= y_{2j} - s_j z_{j+1}, \text{ if } j = 1, \cdots, d - 1; \\
\gamma_{2d-1} &= y_{2d-1} - s_d z_1; \\
\gamma_{2d} &= y_{2d} - s_d (z_2 + \cdots + z_d); \\
\beta_k &= x_k + t_k z_1, \text{ if } k = 1, \cdots, d.
\end{cases}$$

On other hand, one can see that the tangent space to $\mathcal{O}$ at $\ell$ is

$$T_\ell \mathcal{O} = \{ ad_X^*(\ell), X \in \text{VectVect}\{ Y_{2k-1}, X_k, 1 \leq k \leq d \} \},$$

and hence any element in this space has coordinates

$$\lambda_i = 0, \text{ if } i = 1, \cdots, d; \quad \gamma_{2j-1} = -s_j z_1, \text{ if } j = 1, \cdots, d - 1; \quad \gamma_{2j} = -s_j z_{j+1}, \text{ if } j = 1, \cdots, d - 1; \quad \gamma_{2d-1} = -s_d z_1; \quad \gamma_{2d} = -s_d (z_2 + \cdots + z_d); \quad \beta_k = t_k z_1, \text{ if } k = 1, \cdots, d.$$
Then we conclude

\[ \text{Cor}(g^*_d) = \{ \ell = (z_i, y_j, x_k) \in g^*_d : z_i = 0, y_{2d-1} (\sum_{i=1}^{d-1} y_{2i} \prod_{j=1, j \neq i}^{d-1} y_{2j-1}) - y_{2d} \prod_{j=1}^{d-1} y_{2j-1} = 0 \} \].

\[ \square \]

**Corollary 3.1.** For each integer \( d \geq 2 \) let \( z_d \) denotes the center of the Lie algebra \( g_d \). Then for any \( d \geq 2 \) one has

\[ \text{Cor}(g^*_d) \subseteq \text{ICor}(g^*_d) = z^\bot. \]

### 3.1. Concluding remarks.

1. Let us remark that the cross-section mapping \( P_d : \Omega_d \to \Sigma_d \) is as follows

\[
P_d : \ell = (z_i, y_j, x_k) \mapsto P(\ell) = \sum_{i=1}^{d} z_i Z_i^* + \sum_{i=1}^{d-1} (y_{2i} - \frac{z_{i+1}}{z_i} y_{2i-1}) Y_{2i}^* + (y_{2d} - \frac{z_1 + \cdots + z_d}{z_1}) Y_{2d}^*.
\]

2. I believe that the family of Lie algebras \( (g_d)_{d \geq 2} \) is optimal in the sense that if for some \( k \)-dimensional two-step nilpotent Lie algebra \( g \) the cortex of the dual of \( g^* \) satisfies

\[ \text{Cor}(g^*) = \{ \ell \in z^\bot, P_i(\ell) = 0, \text{ with each } P_i \text{ is a homogeneous polynomial of degree } k \}, \]

then \( \dim(z) \geq k \) and \( \dim(g) \geq 4k \) where \( z \) is the center of \( g \).

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