CONTIGUITY RELATIONS OF LAURICELLA’S $F_D$ REVISITED

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Abstract. We study contiguity relations of Lauricella’s hypergeometric function $F_D$, by using the twisted cohomology group and the intersection form. We derive contiguity relations from those in the twisted cohomology group and give the coefficients in these relations by the intersection numbers. Furthermore, we construct twisted cycles corresponding to a fundamental set of solutions to the system of differential equations satisfied by $F_D$, which are expressed as Laurent series. We also give the contiguity relations of these solutions.

1. Introduction

Lauricella’s hypergeometric series $F_D$ of $m$ variables $x_1, \ldots, x_m$ with complex parameters $a, b_1, \ldots, b_m, c$ is defined by

$$F_D(a, b, c; x) = \sum_{n_1, \ldots, n_m=0}^{\infty} \frac{(a, n_1 + \cdots + n_m)(b_1, n_1) \cdots (b_m, n_m)}{(c, n_1 + \cdots + n_m)! n_1! \cdots n_m!} x_1^{n_1} \cdots x_m^{n_m},$$

where $x = (x_1, \ldots, x_m)$, $b = (b_1, \ldots, b_m)$, $c \notin \{0, -1, -2, \ldots\}$, and $(a, n) = \Gamma(a + n)/\Gamma(a)$. This series converges in the domain $\{x \in \mathbb{C}^m \mid |x_i| < 1 \ (1 \leq i \leq m)\}$. It is known that $F_D(a, b, c; x)$ admits an Euler-type integral representation:

$$F_D(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^\infty t^{\sum_{i=1}^m b_i - c}(t - 1)^{c-a-1} \prod_{i=1}^m (t-x_i)^{-b_i} dt. \tag{1}$$

The contiguity relations of Lauricella’s $F_D$ have been studied from several points of view. In the 1970s, W. Miller Jr. [6] gave the contiguity relations of $F_D$ as a representation of a Lie algebra, and Aomoto [1] studied the contiguity relations of $F_D$ and its generalization to the hypergeometric functions of type $(k, n)$. In 1991, Sasaki [11] studied the contiguity relations in the framework of the Aomoto-Gel’fand system on the Grassmannian manifold. In 1989, an algorithmic method that used Gröbner bases to derive the contiguity relations was given by Takayama [12]. Recently, Ogawa, Takemura, and Takayama [7] have illustrated that the Pfaffian system and the contiguity relations for $F_D$ combine to give a method to evaluate the normalizing constant of the hypergeometric distribution on the 2 by $N$ contingency tables with given marginal sums. On the other hand, Matsumoto [5] recently proposed a method that utilizes the intersection numbers of twisted cohomology groups to derive Pfaffian systems. In this paper, we reconsider the problem of the contiguity relations of $F_D$, in order to produce formulas for application to statistics [7]. Matsumoto’s method can be applied to derive the contiguity relations for our purpose, and further generalizations will be possible.

We derive the contiguity relations of $F_D$ by considering the twisted cohomology groups associated with the integral representation (1). We regard the contiguity relations as those between the twisted cocycles. To obtain the coefficients in the contiguity relations, we use the intersection form of the twisted cohomology group.

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In the way, we are able to derive the contiguity relations for the basis given in [5], which was also used in [7]. An advantage of our method is that it makes it easy to systematically derive the contiguity relations for a given basis of the twisted cohomology group.

This paper is arranged as follows. In Sections 2, 3, and 4, we introduce our method for using the intersection form to derive the contiguity relations. By evaluating the intersection numbers, we obtain explicit forms for the contiguity relations. In Section 5, we introduce the system \( E_D(a, b, c) \) of differential equations satisfied by \( F_D(a, b, c; x) \), and we introduce the Laurent series solution \( f^{(k)}(a, b, c; x) \) to \( E_D(a, b, c) \) and construct a fundamental set of solutions. In Section 6, we construct the twisted cycle \( r_k \) corresponding to the solution \( f^{(k)}(a, b, c; x) \). Since our contiguity relations are obtained from those in the twisted cohomology group, the integration on \( r_k \) gives the contiguity relations of \( f^{(k)} \). In Section 7, we present an application of our formula, in which we evaluate the normalizing constant of the hypergeometric distribution on the 2 by \((m + 1)\) contingency tables; this is also explained in [7] in the context of statistics. We also explain how to apply our results when the parameters \((a, b, c)\) are integers.

Although the contiguity relations of \( F_D \) have been studied by several authors, those of the other solutions \( f^{(k)} \) that appear in applications to statistics have not been studied.

2. Twisted cohomology group and intersection pairing

We summarize some results in [2], [3], and [5] that will be used in this paper. We consider the twisted cohomology group for 
\[ T_x := \mathbb{C} - \{x_0, x_1, \ldots, x_m, x_{m+1}\} \]
and the multivalued function
\[ u_x(t) := \prod_{i=0}^{m+1} (t - x_i)^{\alpha_i}, \]
where
\[ x_0 := 0, \quad x_{m+1} := 1, \]
\[ \alpha_0 := -c + \sum_{j=1}^{m} b_j, \quad \alpha_k := -b_k \quad (1 \leq k \leq m), \quad \alpha_{m+1} := c - a, \quad \alpha_{m+2} := a. \]

Except in Section 7, we assume the condition
\[ \alpha_k \not\in \mathbb{Z} \quad (0 \leq k \leq m + 2). \]

We denote the vector space consisting of the smooth \( k \)-forms on \( T_x \) and that with compact support by \( \mathcal{E}^k(T_x) \) and \( \mathcal{E}^k_c(T_x) \), respectively. We set \( \omega := d \log u_x \) and \( \nabla_\omega := d + \omega \wedge \), where \( d \) is the exterior derivative with respect to the variable \( t \) (note that this is not with respect to \( x_1, \ldots, x_m \), which are regarded as parameters). The twisted cohomology group and that with compact support are defined as
\[ H^1(T_x, \nabla_\omega) = \text{Ker}(\nabla_\omega : \mathcal{E}^1(T_x) \to \mathcal{E}^2(T_x))/\nabla_\omega(\mathcal{E}^0(T_x)), \]
\[ H^1_c(T_x, \nabla_\omega) = \text{Ker}(\nabla_\omega : \mathcal{E}^1_c(T_x) \to \mathcal{E}^2_c(T_x))/\nabla_\omega(\mathcal{E}^0_c(T_x)), \]
respectively. The expression \([11]\) means that the integral
\[ \int_{t=1}^{\infty} u_x \varphi_0, \quad \varphi_0 := \frac{dt}{t - 1} \]
represents $F_D(a, b, c, x)$ modulo Gamma factors. By [2], $H^1(T_x, \nabla_\omega)$ has $(m+1)$-dimensions, and there is a canonical isomorphism $j : H^1(T_x, \nabla_\omega) \rightarrow H^1(T_x, \nabla_\omega)$; see also [3, Fact 6.1]. Hereafter, we identify $H^1(T_x, \nabla_\omega)$ with $H^1(T_x, \nabla_\omega)$.

The intersection form $I_c$ on the twisted cohomology groups is the pairing between $H^1(T_x, \nabla_\omega)$ and $H^1(T_x, \nabla_-\omega)$, and it is defined as follows:

$$I_c(\psi, \psi') := \int_{T_x} j(\psi) \wedge \psi', \quad \psi \in H^1(T_x, \nabla_\omega), \quad \psi' \in H^1(T_x, \nabla_-\omega).$$

We put

$$\varphi_{i,m+2} := \frac{dt}{t-x_i}, \quad \varphi_{i,j} := \varphi_{i,m+2} - \varphi_{j,m+2} = \frac{(x_i - x_j)dt}{(t-x_i)(t-x_j)},$$

$$\varphi_0 = \varphi_{m+1,m+2} = \frac{dt}{t-1}, \quad \varphi_k := \varphi_{m+1,k} = \frac{(1-x_k)dt}{(t-x_k)(t-1)},$$

where $0 \leq i, j \leq m+1$ and $1 \leq k \leq m$. The intersection numbers among these 1-forms are evaluated in [3]; see also [5, Fact 6.2].

**Fact 1** ([3]). We have

$$I_c(\varphi_{i,j}, \varphi_{p,q}) = 2\pi \sqrt{-1} \left( \frac{\delta_{i,p} - \delta_{i,q}}{\alpha_i} - \frac{\delta_{j,p} - \delta_{j,q}}{\alpha_j} \right),$$

where $i, j, p, q \in \{0, 1, \ldots, m+2\}$, and $\delta_{i,p}$ is the Kronecker delta. Thus, the intersection matrix $C(a, b, c) := (I_c(\varphi_{i,j}))_{i,j=0,\ldots,m}$ is

$$C(a, b, c) = 2\pi \sqrt{-1} \left\{ \frac{1}{\alpha_{m+1}} N + \text{diag} \left( \frac{1}{\alpha_{m+2}}, \frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_m} \right) \right\},$$

where

$$N = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$  

Under assumption [3], we have

$$\det(C(a, b, c)) = (2\pi \sqrt{-1})^{m+1} \prod_{i=1}^{m+2} \frac{-\alpha_0}{\alpha_i} \neq 0,$$

and hence $\varphi_0, \ldots, \varphi_m$ form a basis of $H^1(T_x, \nabla_\omega)$.

3. Contiguity relations

In this section, we derive the contiguity relations by using the intersection form. We define two column vectors of size $m+1$:

$$F(a, b, c; x) := \begin{pmatrix} F_D(a, b, c; x), \frac{x_1 - 1}{\alpha_1} \frac{\partial}{\partial x_1} F_D(a, b, c; x), & \cdots, & \frac{x_m - 1}{\alpha_m} \frac{\partial}{\partial x_m} F_D(a, b, c; x) \end{pmatrix},$$

$$\tilde{F}(a, b, c; x) := \frac{\Gamma(a) \Gamma(c-a)}{\Gamma(c)} F(a, b, c; x).$$

For $(v_0, \ldots, v_m) \in \mathbb{C}^{m+1}$, we regard $v_i$ as the $i$-th entry. For example, the 0-th entry of $F(a, b, c; x)$ is $F_D(a, b, c; x)$. By [3, Corollary 7.2], we have

$$\left( \int_1^\infty u_x \varphi_0, \ldots, \int_1^\infty u_x \varphi_m \right) = \tilde{F}(a, b, c; x).$$

Our main theorem states the contiguity relations of the vector-valued function $F(a, b, c; x)$. 

For $0 \leq k \leq m+1$, there exist $p^{(k)}_{i,j}(a,b,c,x)$’s such that

$$
(t - x_k) \cdot \varphi_i = \sum_{l=0}^{m} p^{(k)}_{i,l}(a,b,c,x) \cdot \varphi_l
$$

as elements in the twisted cohomology group $H^1(T_x, \nabla_\omega)$. We put $P_k(a,b,c,x) := \left( p^{(k)}_{i,j}(a,b,c,x) \right)_{i,j}$.

**Lemma 2.**

$$
\tilde{F}(a - 1, b, c; x) = P_{m+1}(a,b,c;x)\tilde{F}(a,b,c;x),
$$

$$
\tilde{F}(a - 1, b, c - 1; x) = P_0(a,b,c;x)\tilde{F}(a,b,c;x),
$$

$$
\tilde{F}(a - 1, b - e_k, c - 1; x) = P_k(a,b,c;x)\tilde{F}(a,b,c;x) \quad (1 \leq k \leq m).
$$

Here, $e_k$ is the $k$-th unit vector in $\mathbb{C}^m$.

For example, we have $b - e_1 = (b_1 - 1, b_2, \ldots, b_m)$.

**Proof.** Recall that $x_{m+1} = 1$. We consider the integration of (4) on $(1, \infty)$. By (1), we have

$$
\int_1^{\infty} u_x \cdot (t - 1)\varphi_0 = \int_1^{\infty} t\sum_{i=1}^{m} b_i c(t - 1)^{c-1} \prod_{i=1}^{m} (t - x_i)^{-b_i} dt
$$

$$
= \frac{\Gamma(a-1)\Gamma(c-a+1)}{\Gamma(c)} F_D(a - 1, b, c; x),
$$

which is the 0-th entry of $\tilde{F}(a - 1, b, c; x)$. Then, the first equality follows. The other ones are shown in an analogous way.

The following lemma is obvious.

**Lemma 3.**

$$
\tilde{F}(a - 1, b, c; x) = P_{m+1}(a,b,c;x)\tilde{F}(a,b,c;x),
$$

$$
\tilde{F}(a, b, c - 1; x) = P_0(a+1,b,c;x)P_{m+1}(a+1, b, c; x)^{-1}\tilde{F}(a,b,c;x),
$$

$$
\tilde{F}(a, b - e_k, c; x) = P_k(a+1,b,c+1;x)P_0(a+1, b, c+1; x)^{-1}\tilde{F}(a,b,c;x).
$$

To evaluate $P_k(a,b,c;x)$, we use the intersection form $I_c$. By (4), we have

$$
I_c ((t - x_k)\varphi_i, \varphi_j) = \sum_{l=0}^{m} p^{(k)}_{i,l}(a,b,c;x) \cdot I_c (\varphi_l, \varphi_j).
$$

Let $Q_k(a,b,c;x) := (I_c ((t - x_k)\varphi_i, \varphi_j))_{i,j}$. Then we obtain $Q_k(a,b,c;x) = P_k(a,b,c;x)C(a,b,c)$, that is,

$$
P_k(a,b,c;x) = Q_k(a,b,c;x)C(a,b,c)^{-1}.
$$

We can reduce Lemma 3 to the relations between the $F(a,b,c;x)$’s by using the formula $\Gamma(s + 1) = s \cdot \Gamma(s)$.

**Theorem 4** (Contiguity relations). We have

$$
F(a - 1, b, c; x) = D_a(a,b,c;x)F(a,b,c;x),
$$

$$
F(a, b, c - 1; x) = D_c(a,b,c;x)F(a,b,c;x),
$$

$$
F(a, b - e_k, c; x) = D_k(a,b,c;x)F(a,b,c;x) \quad (1 \leq k \leq m),
$$
Remark 6. We have
\[ D_0(a, b, c; x) := \frac{a - 1}{c - a} \cdot Q_{m+1}(a, b, c; x) \cdot C(a, b, c)^{-1}, \]
\[ D_c(a, b, c; x) := \frac{c - a - 1}{c - 1} \cdot Q_0(a + 1, b, c; x) \cdot Q_{m+1}(a + 1, b, c; x)^{-1}, \]
\[ D_k(a, b, c; x) := Q_k(a + 1, b, c + 1; x) \cdot Q_0(a + 1, b, c + 1; x)^{-1}. \]

Therefore, we need to evaluate the matrices \( Q_k(a, b, c; x) \) explicitly.

**Proposition 5.** We have
\[ Q_k(a, b, c; x) \]
\[ = 2\pi \sqrt{-1} \left\{ \frac{1- x_k}{\alpha_{m+1}} N + \frac{1}{\alpha_{m+2}} \text{diag} (0, 1-x_1, \ldots, 1-x_m) \cdot N \cdot \text{diag} (1, 0, \ldots, 0) \right. \]
\[ \left. - \frac{1}{1- \alpha_{m+2}} \text{diag} (1, 0, \ldots, 0) \cdot N \cdot \text{diag} (0, 1-x_1, \ldots, 1-x_m) \right. \]
\[ + \text{diag} \left( \frac{1- x_k}{\alpha_{m+2}} - \frac{1}{1- \alpha_{m+2}} \left( \sum_{p=1}^{m+1} \alpha_p x_p \right) + 1 \right) \cdot \frac{x_1-x_k}{\alpha_1}, \ldots, \frac{x_m-x_k}{\alpha_m} \}. \]

We will give the proof in the next section. Note that
\[ \text{diag}(p_0, \ldots, p_m) \cdot N \cdot \text{diag}(q_0, \ldots, q_m) = \begin{pmatrix} p_0 q_0 & p_0 q_1 & \cdots & p_0 q_m \\ p_1 q_0 & p_1 q_1 & \cdots & p_1 q_m \\ \vdots & \vdots & & \vdots \\ p_m q_0 & p_m q_1 & \cdots & p_m q_m \end{pmatrix}. \]

**Remark 6.** The determinant of \( Q_k(a, b, c; x) \) is as follows:
\[ \det Q_k(a, b, c; x) = (2\pi \sqrt{-1})^{m+1} \cdot \frac{\alpha_0 (1 + \delta_{k,0} \alpha_0)}{\prod_{j=1}^{m+2} \alpha_j \cdot (\alpha_{m+2} - 1)} \cdot \prod_{j=0}^{m+1} (x_j - x_k). \]

**Example 7.** If \( m = 2 \), the matrices \( C(a, b, c) \) and \( Q_k(a, b, c; x) \) are as follows:
\[ C(a, b, c; x) = 2\pi \sqrt{-1} \left\{ \frac{1}{\alpha_3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{\alpha_4} & 0 & 0 \\ 0 & \frac{1}{\alpha_4} & 0 \\ 0 & 0 & \frac{1}{\alpha_2} \end{pmatrix} \right\}. \]

\[ Q_k(a, b, c; x) \]
\[ = 2\pi \sqrt{-1} \left\{ \frac{1- x_k}{\alpha_3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1-x_k}{\alpha_4} & -\frac{1}{\alpha_4} & \frac{1-x_k}{\alpha_4} \\ -\frac{1}{\alpha_4} & \frac{1-x_k}{\alpha_4} & 0 \\ \frac{1-x_k}{\alpha_4} & 0 & \frac{1-x_k}{\alpha_2} \end{pmatrix} \right\}. \]

The first equality in Theorem 4 is written as
\[ \begin{pmatrix} \frac{1-x_1}{b_1} & \frac{a}{b_2} \cdot \frac{\partial}{\partial x_1} F_D(a-1, b_1, b_2; c, x_1, x_2) \\ \frac{b_2}{b_2} \cdot \frac{\partial}{\partial x_2} F_D(a-1, b_1, b_2; c, x_1, x_2) \end{pmatrix} \]
\[ = \begin{pmatrix} \frac{a c-1}{b_2 (a-c) (x_2-1)} & \frac{b_1 x_1}{(a-c) (x_2-1)} & \frac{b_2 x_2}{c-a} \\ \frac{b_1 x_1}{(a-c) (x_2-1)} & \frac{b_2 x_2}{c-a} & 0 \end{pmatrix} \begin{pmatrix} \frac{1-x_1}{b_1} & \frac{a}{b_2} \cdot \frac{\partial}{\partial x_1} F_D(a, b_1, b_2; c, x_1, x_2) \\ \frac{b_2}{b_2} \cdot \frac{\partial}{\partial x_2} F_D(a, b_1, b_2; c, x_1, x_2) \end{pmatrix}. \]

The \( 3 \times 3 \) matrix on the right-hand side is equal to \( D_0(a, b, c; x) = \frac{a}{c-a} Q_3(a, b, c; x) \cdot C(a, b, c)^{-1} \).
4. Proof of Proposition 5

In this section, we evaluate the intersection numbers that are the entries of $Q_k(a, b, c; x)$, by using Fact 4.

We denote $\varphi \sim \psi$, if $\varphi$ is $\nabla_\omega$-cohomologous to $\psi$, that is,

$$\varphi \sim \psi \iff \varphi = \psi + \nabla_\omega f \quad \text{for some } f \in E^0(T_x),$$

$$\iff \varphi \text{ and } \psi \text{ give the same element in } H^1(T_x, \nabla_\omega).$$

Lemma 8.

$$dt \sim \frac{1}{1 - \alpha_{m+2}} \sum_{p=0}^{m+1} \alpha_p x_p \varphi_{p,m+2}.\tag{$\dagger$}$$

Proof. This lemma follows from

$$0 \sim \nabla_\omega(t) = dt + \sum_{p=0}^{m+1} \alpha_p \frac{t - x_p}{t - x_p} dt = dt + \sum_{p=0}^{m+1} \alpha_p \frac{t - x_p + x_p dt}{t - x_p}$$

$$= (1 + \sum_{p=0}^{m+1} \alpha_p) dt + \sum_{p=0}^{m+1} \alpha_p x_p \frac{dt}{t - x_p} = (1 - \alpha_{m+2}) dt + \sum_{p=0}^{m+1} \alpha_p x_p \varphi_{p,m+2}.$$

Here, we use $x_0 = 0$ and $\sum_{p=0}^{m+1} \alpha_p = 0$. □

Then, we have

$$(t - x_k) \cdot \varphi_{i,m+2} = \frac{t - x_k}{t - x_k} dt = \frac{t - x_k + x_k - x_k}{t - x_k} dt$$

$$\sim (t - x_k) \varphi_{i,m+2} - \frac{1}{1 - \alpha_{m+2}} \sum_{p=0}^{m+1} \alpha_p x_p \varphi_{p,m+2}.$$

Fact $\dagger$ and a straightforward calculation show the following lemma.

Lemma 9.

$$I_c((t - x_k) \varphi_{i,m+2}, \varphi_j)$$

$$= \begin{cases} 
2 \pi \sqrt{-1} \left( (t - x_k) \left( \frac{1}{\alpha_{m+1}} + \frac{1}{\alpha_{m+2}} \right) - \frac{1}{1 - \alpha_{m+2}} \left( \frac{\sum_{p=1}^{m+1} \alpha_p x_p}{\alpha_{m+2}} + 1 \right) \right) & (j = 0), \\
2 \pi \sqrt{-1} \left( (t - x_k) \frac{\delta_{i,m+1} - \delta_{i,j}}{\alpha_{i}} - \frac{1 - x_k}{1 - \alpha_{m+2}} \right) & (1 \leq j \leq m). 
\end{cases}$$

Proof of Proposition 5 Let $Q_k(i,j)$ be the $(i,j)$ entry of $Q_k(a, b, c; x)$, that is, $Q_k(i,j) = I_c((t - x_k) \cdot \varphi_i, \varphi_j)$. For $1 \leq i, j \leq m$, we have

$$Q_k(0,0) = I_c((t - x_k) \cdot \varphi_{m+1,m+2}, \varphi_0)$$

$$= 2 \pi \sqrt{-1} \left( \frac{1 - x_k}{\alpha_{m+1}} + \frac{1 - x_k}{\alpha_{m+2}} - \frac{1}{1 - \alpha_{m+2}} \left( \frac{\sum_{p=1}^{m+1} \alpha_p x_p}{\alpha_{m+2}} + 1 \right) \right),$$

$$Q_k(0,j) = I_c((t - x_k) \cdot \varphi_{m+1,m+2}, \varphi_j)$$

$$= 2 \pi \sqrt{-1} \left( \frac{1 - x_k}{\alpha_{m+1}} - \frac{1 - x_j}{\alpha_{m+2}} \right),$$

$$Q_k(0,0) = I_c((t - x_k) \cdot \varphi_{m+1,m+2}, \varphi_0) - I_c((t - x_k) \cdot \varphi_{i,m+2}, \varphi_0)$$

$$= 2 \pi \sqrt{-1} \left( \frac{1 - x_k}{\alpha_{m+1}} + \frac{1 - x_k}{\alpha_{m+2}} \right),$$

$$Q_k(i,j) = I_c((t - x_k) \cdot \varphi_{m+1,m+2}, \varphi_j) - I_c((t - x_k) \cdot \varphi_{i,m+2}, \varphi_j)$$

$$= 2 \pi \sqrt{-1} \left( \frac{1 - x_k}{\alpha_{m+1}} + \frac{1 - x_k}{\alpha_{m+2}} \right).$$
5. Differential equations and solutions

Lauricella’s $F_D(a, b, c; x)$ satisfies the differential equations

\[ [\theta_i(\theta + c - 1) - x_i(\theta + a)(\theta_i + b_i)] f(x) = 0 \quad (1 \leq i \leq m), \]

\[ [(x_i - x_j) \partial_i \partial_j - b_j \partial_i + b_i \partial_j] f(x) = 0 \quad (1 \leq i < j \leq m), \]

where $\partial_i := \frac{\partial}{\partial x_i}$, $\theta_i := x_i \partial_i$, and $\theta := \sum_{j=1}^{m} \theta_j$. The system generated by them is called Lauricella’s hypergeometric system $E_D(a, b, c)$ of differential equations. It is known that the $A$-hypergeometric system associated with the matrix $A(\Delta_1 \times \Delta_m)$ can be transformed into the system $E_D(a, b, c)$, and combinatorial methods for constructing a fundamental set of solutions to the $A$-hypergeometric system are known \cite{4, 10}. Thus, we can use the general method for constructing series solutions to $A$-hypergeometric systems to obtain a fundamental set of solutions to $E_D(a, b, c)$ with generic parameters $(a, b, c)$.

**Fact 10** (\cite{4} Section 3.3, \cite{10} Section 1.5). For $1 \leq k \leq m$, we put

\[
f(k)(a, b, c; x) := \prod_{l=1}^{k-1} \frac{x_l^{-b_l}}{x_k^{n_l}} \cdot \sum_{n_1, \ldots, n_m=0}^{\infty} \frac{1}{\Gamma^{(k)}_{n_1, \ldots, n_m}(a, b, c)} \prod_{l=1}^{k-1} \left( \frac{x_k}{x_l} \right)^{n_l} \cdot x_k^{n_k} \cdot \prod_{l=k+1}^{m} \left( \frac{x_l}{x_k} \right)^{n_l},
\]

where

\[
\Gamma^{(k)}_{n_1, \ldots, n_m}(a, b, c) := \Gamma(c - a - n_k) \cdot \prod_{1 \leq \ell \leq m, \ell \neq k} \Gamma(1 - b_l - n_l) \cdot \prod_{l=1}^{m} \Gamma(1 + n_l)
\]

\[
\cdot \Gamma \left( - \sum_{l=1}^{k} b_l + c - \sum_{l=1}^{k} n_l + \sum_{l=k+1}^{m} n_l \right) \cdot \Gamma \left( 2 + \sum_{l=1}^{k-1} b_l - c + \sum_{l=1}^{k} n_l - \sum_{l=k+1}^{m} n_l \right).
\]

Then, each $f(k)(a, b, c; x)$ is a solution to $E_D(a, b, c)$. Moreover, the set of $F_D(a, b, c; x)$ and $f(k)(a, b, c; x)$ ($1 \leq k \leq m$) is a set of fundamental solutions to $E_D(a, b, c)$.

6. Twisted cycles corresponding to solutions

We consider the twisted homology group $H_1(T_x, u_x)$ on $T_x$ that is associated with the multivalued function $u_x(t)$. For the definition of the twisted homology groups, refer to \cite{2} and \cite{5}. By \cite{2}, $H_1(T_x, u_x)$ has $(m + 1)$ dimensions. If $(a, b, c; x)$ are generic, then the local solution space $\text{Sol}_a$ of $E_D(a, b, c)$ around $x$ can be identified with the twisted homology group $H_1(T_x, u_x)$ by the integration of $u_x \varphi_0$; see \cite{5} Proposition 4.1]. Thus, there exists a twisted cycle that corresponds to the series solution $f(k)(a, b, c; x)$. In this section, we construct such a cycle explicitly.

Let $\varepsilon$ and $\xi$ be real numbers satisfying

\[ 0 < \varepsilon < \frac{1}{2}, \quad \varepsilon < \frac{1}{1 + \varepsilon}. \]

We construct the twisted cycle $r_k$ in $T_x$ with $x$ belonging to a small neighborhood of

\[ x^{(k)} := (\xi, \xi^2, \ldots, \xi^{k-1}, e^{-\pi \sqrt{\varepsilon} \xi^k}, \xi^{k+1}, \ldots, \xi^m). \]
Once we construct the twisted cycle in $T_{x(k)}$, this cycle is uniquely continued to the twisted cycle in each $T_x$. Thus, we may assume $x = x(k)$. We put $S_x := \mathbb{C} - \left\{ \frac{x_k}{x_m}, \ldots, \frac{x_k}{x_{k+1}}, \frac{x_k}{x_{k-1}}, \ldots, \frac{x_k}{x_1}, x_k, 0, 1 \right\}$, $v_x(s) := \prod_{l=1}^{k-1} \left( \frac{s-x_k}{x_l} \right)^{\alpha_l} \cdot \left( s-x_k \right)^{\alpha_{m+1}-1} \cdot \prod_{l=k+1}^{m} \left( 1 - \frac{x_l}{x_k}s \right)^{\alpha_l} \cdot s^{\alpha_{m+2}} \cdot (1-s)^{\alpha_k+1}$. We put $\lambda_j := e^{2\pi \sqrt{-1}/\alpha_j}$ and $\tilde{r}_k := \frac{1}{\prod_{l=1}^{k-1} \lambda_l \cdot \lambda_m \cdot \lambda_{m+2}} C_0 \otimes v_x + [\varepsilon, 1-\varepsilon] \otimes v_x - \frac{1}{\lambda_k} \cdot C_1 \otimes v_x$. Here, $C_0$ (resp. $C_1$) is the circle of center 0 (resp. 1) and radius $\varepsilon$ with starting point $\varepsilon$ (resp. $1-\varepsilon$), which turns in the counterclockwise direction, and the branch of $v_x$ is obtained by the analytic continuation along $C_0$ (resp. $C_1$). Let us verify that $\tilde{r}_k$ is a twisted cycle. Let $D_i$ be the disk whose boundary is $C_i$ ($i = 0, 1$). Since $\frac{x_k}{x_{k-1}} = \xi < \varepsilon < 1 < \frac{1}{\xi} = \frac{x_k}{x_{k+1}}$, we have $D_0 \cap (\mathbb{C} - S_x) = \left\{ \frac{x_k}{x_{k-1}}, \ldots, \frac{x_k}{x_1}, x_k, 0 \right\}$, $D_1 \cap (\mathbb{C} - S_x) = \{1\}$; see Figure 1. Then, the difference between the branches of $v_x$ at the ending and starting points of the circle $C_0$ (resp. $C_1$) is $\prod_{l=1}^{k-1} \lambda_l \cdot \lambda_m \cdot \lambda_{m+2}$ (resp. $\lambda_k$), which implies that $\tilde{r}_k$ is a twisted cycle (cf. [2, Example 2.1]).

![Figure 1. $\tilde{r}_k$](image-url)
Lemma 11.

(5)

\[ \int_{\tilde{r}_k} v_x \frac{ds}{s(1-s)} = \Gamma(c-a) \cdot \prod_{l=1}^{m} \Gamma(1-b_l) \cdot \Gamma \left( \sum_{l=1}^{k-1} b_l - c \right) \cdot \Gamma \left( 1 - \sum_{l=1}^{k-1} b_l + c \right) \cdot \sum_{n_1, \ldots, n_m=0}^{\infty} \frac{1}{\Gamma(n_1) \ldots n_m (a, b, c)} \cdot \prod_{l=1}^{k-1} \left( \frac{x_k}{x_l} \right)^{n_l} \cdot \prod_{l=k+1}^{m} \left( \frac{x_l}{x_k} \right)^{n_l}. \]

Proof. Note that if \( s \) belongs to \( C_0 \cup \{ \varepsilon, 1-\varepsilon \} \cup C_1 \), it satisfies \( \varepsilon < |s| < 1 + \varepsilon \). Since

\[ \left| \frac{x_k}{x_l} \right| < \xi^{k-l} \cdot \frac{1}{\xi} < 1 \quad (1 \leq l \leq k-1), \]

\[ \left| \frac{x_k}{x_l} \right| < \xi^{k-1} \cdot \frac{1}{\xi} < 1, \]

\[ \left| \frac{x_l}{x_k} \right| < \xi^{l-k} \cdot (1+\varepsilon) < 1 \quad (k+1 \leq l \leq m), \]

the following power series expansions are uniformly and absolutely convergent on \( C_0 \cup \{ \varepsilon, 1-\varepsilon \} \cup C_1 \):

\[ \left( 1 - \frac{x_k}{x_l} \right)^{\alpha_l} = \sum_{n_l=0}^{\infty} \frac{(-\alpha_l, n_l)}{n_l!} \left( \frac{x_k}{x_l} \right)^{n_l} \quad (1 \leq l \leq k-1), \]

\[ \left( 1 - \frac{x_k}{x_l} \right)^{\alpha_{m+1-l}} = \sum_{n_k=0}^{\infty} \frac{(1-\alpha_{m+1-l}, n_k)}{n_k!} \left( \frac{x_k}{x_l} \right)^{n_k}, \]

\[ \left( 1 - \frac{x_l}{x_k} \right)^{\alpha_l} = \sum_{n_l=0}^{\infty} \frac{(-\alpha_l, n_l)}{n_l!} \left( \frac{x_l}{x_k} \right)^{n_l} \quad (k+1 \leq l \leq m). \]

We replace the power functions on the left-hand side of (5) by these expansions, and exchange the sum and the integral. Then, the coefficient of \( \prod_{l=1}^{k-1} \left( \frac{x_k}{x_l} \right)^{n_l} \cdot \frac{x_l}{x_k}^{n_l} \) is

\[ \prod_{l=k+1}^{m} \left( \frac{x_l}{x_k} \right)^{n_l} \]

(6)

\[ \frac{1-\alpha_{m+1}, n_k}{n_k!} \prod_{l \neq k} \frac{(-\alpha_l, n_l)}{n_l!} \int_{\tilde{r}_k} \frac{\sum_{l=1}^{k-1} \alpha_l + \alpha_{m+1} + \alpha_{m+2} - 1 - \sum_{l=1}^{k} n_l + \sum_{l=k+1}^{m} n_l}{s^{1-\alpha_{k+1}}} \frac{ds}{s(1-s)}. \]

By the construction of \( \tilde{r}_k \), the twisted cycle \( \tilde{r}_k \) of this integral can be identified with the usual regularization of the open interval \((0, 1)\) loaded with the multivalued function

\[ \frac{\sum_{l=1}^{k-1} \alpha_l + \alpha_{m+1} + \alpha_{m+2} - 1 - \sum_{l=1}^{k} n_l + \sum_{l=k+1}^{m} n_l}{s^{1-\alpha_{k+1}}} \]

on \( \mathbb{C} \setminus \{0,1\} \). Hence the integral in (6) is equal to

\[ \frac{\Gamma \left( \sum_{l=k-1}^{k-1} \alpha_l + \alpha_{m+1} + \alpha_{m+2} - 1 - \sum_{l=k}^{k} n_l + \sum_{l=k+1}^{m} n_l \right)}{\Gamma \left( \sum_{l=k}^{k} \alpha_l + \alpha_{m+1} + \alpha_{m+2} - \sum_{l=k}^{k} n_l + \sum_{l=k+1}^{m} n_l \right)} \cdot \Gamma(\alpha_{k+1}). \]
Proof. Note that arg(\(\frac{\Gamma (1 - c + a + n_b)}{\Gamma (1 - c + a)}\prod_{l \neq k} \frac{\Gamma (b_l + n_l)}{\Gamma (b_l)}\)) is equal to

\[
\frac{\Gamma (1 - c + a + n_b)}{\Gamma (1 - c + a)} \cdot \frac{\prod_{l \neq k} \Gamma (b_l + n_l)}{\Gamma (b_l)} \cdot \frac{1}{\Gamma (1 - \sum_{l \leq k} b_l + c - \sum_{l \leq k} n_l + \sum_{l \geq k+1} n_l) \prod_{l=1}^{m} \frac{1}{\Gamma (1 + n_l)}}.
\]

By using \(\Gamma (z)\Gamma (1 - z) = \pi / \sin (\pi z)\), we obtain, for example,

\[
\frac{\Gamma (b_l + n_l)}{\Gamma (b_l)} = \frac{1}{\Gamma (1 - b_l - n_l) \sin (\pi b_l + n_l)} = (-1)^{n_l} \pi \frac{1}{\Gamma (1 - b_l - n_l) \Gamma (b_l) \sin (\pi b_l)} = (-1)^{n_l} \frac{\Gamma (1 - b_l)}{\Gamma (1 - b_l - n_l)}
\]

In an analogous way, other Gamma functions with \(n_l\)’s in the numerator can be moved to the denominator. Thus, we obtain the lemma.

We will construct a twisted cycle standing for the series solution \(f^{(k)}(a, b, c; x)\) by the bijection

\[\iota : S_x \to T_x; \quad s \mapsto t = \frac{x_k}{s} \\] .

Let \(r_k\) be the twisted cycle defined as \(r_k := \iota_*(\tilde{r}_k)\), which gives an element in \(H_1(T_x, u_x)\).

**Theorem 12.**

\[
\int_{r_k} u_x \varphi_0 = \Gamma (c - a) \cdot \prod_{l=1}^{m} \Gamma (1 - b_l) \cdot \Gamma \left(\sum_{l=1}^{k-1} b_l - c\right) \cdot \Gamma \left(1 - \sum_{l=1}^{k-1} b_l + c\right) \cdot e^{\pi \sqrt{\left(\sum_{l=1}^{k-1} b_l - c + a\right)}}, \quad f^{(k)}(a, b, c; x).
\]

**Proof.** Note that \(\arg(x_k) = -\pi\). We have

\[
u_x(\iota(s)) \cdot \iota^* \varphi_0 = \left(\frac{x_k}{s}\right)^{\alpha_0} \cdot \left(\frac{x_k}{s} - 1\right)^{\alpha_m+1} \cdot \prod_{l=1}^{k-1} \left(\frac{x_k}{s} - x_l\right)^{\alpha_l} \cdot \left(\frac{x_k}{s} - x_k\right)^{\alpha_k} \prod_{l=k+1}^{m} \left(\frac{x_k}{s} - x_l\right)^{\alpha_l} \cdot \frac{-x_k ds}{s^2 \left(\frac{x_k}{s} - 1\right)}
\]

\[
= -\prod_{l=1}^{k-1} \frac{x_k^\alpha_l}{x_k^\alpha_l + \sum_{l=1}^{m} \alpha_l+1} \cdot \frac{s^{-\alpha_0 - \sum_{l=1}^{m} \alpha_l-1}}{s^{-\alpha_0 - \sum_{l=1}^{m} \alpha_l-1}} \cdot \left(\frac{x_k}{s} - 1\right)^{\alpha_m+1-1} \cdot \prod_{l=1}^{k-1} \left(\frac{x_k}{s} - x_l\right)^{\alpha_l} \cdot (1 - s)^{\alpha_k+1} \prod_{l=k+1}^{m} \left(1 - \frac{x_l s}{x_k}\right)^{\alpha_l} \cdot \frac{ds}{s(1 - s)}
\]

\[
= e^{-\pi \sqrt{\left(\sum_{l=1}^{k-1} \alpha_l + \alpha_m+1\right)}} \prod_{l=1}^{k-1} \frac{x_k^\alpha_l}{x_k^\alpha_l + \sum_{l=1}^{m} \alpha_l+1} \cdot \left(x_k^\alpha_k + \sum_{l=1}^{m} \alpha_l+1\right) \cdot e_x(s) \cdot \frac{ds}{s(1 - s)}.
\]
Here, we use $-\alpha_0 - \sum_{l=k}^m \alpha_m = \sum_{l=1}^{k-1} \alpha_l + \alpha_{m+1} + \alpha_{m+2}$. By Lemma 11 and the relations

\[ \alpha_l = - b_l \quad (1 \leq l \leq k - 1), \]
\[ \alpha_0 + \sum_{l=1}^m \alpha_l + 1 = - \sum_{l=1}^{k-1} \alpha_l - \alpha_{m+1} - \alpha_{m+2} + 1 = \sum_{l=1}^{k-1} b_l - c + 1, \]
\[ \sum_{l=1}^{k-1} \alpha_l + \alpha_{m+1} = - \sum_{l=1}^{k-1} b_l + c - a, \]

we obtain the identity of the theorem. \[ \square \]

By replacing the cycle $(1, \infty)$ in Section 3 with $r_k$, we can obtain the contiguity relations of $f^{(k)}$. We put

\[ F^{(k)}(a, b, c; x) := \frac{\partial}{\partial x_1} f^{(k)}(a, b, c; x), \ldots, \frac{\partial}{\partial x_m} f^{(k)}(a, b, c; x). \]

By Theorem 12 and [5], we have

\[ \bar{F}^{(k)}(a, b, c; x) := \left( \int_{x_1} u_1 \varphi_0, \ldots, \int_{x_k} u_k \varphi_m \right) = \Gamma(c - a) \prod_{l=1}^m \Gamma(1 - b_l) \cdot \Gamma \left( \sum_{l=1}^{k-1} b_l - c \right) \cdot e^{\pi \sqrt{1 - (\sum_{l=1}^{k-1} b_l - c + a)}}, \]

where

\[ D_{\bullet}^{(k)}(a, b, c; x) := \frac{1}{a - c} \cdot Q_{m+1}(a, b, c) \cdot C(a, b, c)^{-1}, \]
\[ D_{c}^{(k)}(a, b, c; x) := (c - a - 1) \cdot Q_0(a + 1, b, c) \cdot Q_{m+1}(a + 1, b, c; x)^{-1}, \]
\[ D_{l}^{(k)}(a, b, c; x) := \frac{1}{1 - b_l} \cdot Q_l(a + 1, b, c + 1; x) \cdot Q_0(a + 1, b, c + 1; x)^{-1}. \]

In fact, $D_{\bullet}^{(k)}$ is independent of $k$.}

### 7. Application—Normalizing Constant for $2 \times (m + 1)$ Contingency Tables

Contiguity relations of $F_D$ and $f^{(k)}$ are applied to the numerical evaluation of the normalizing constant of the hypergeometric distribution of the $2 \times (m + 1)$ contingency tables with fixed marginal sums. In this section, we explain how our results are applied.

We consider the $2 \times (m + 1)$ contingency table

\[ u = \begin{pmatrix} u_{10} & u_{11} & \cdots & u_{1m} \\ u_{20} & u_{21} & \cdots & u_{2m} \end{pmatrix} \in M_{2,m+1}(\mathbb{Z}_{\geq 0}) \]
with row sums $\beta_1$ and $\beta_2$ and columns sums $\gamma_0, \ldots, \gamma_m$. We put $t := \beta_1 + \beta_2 = \sum_{i=0}^m \gamma_i$. We use the multi-index notation

$$p^u = \prod_{i=1}^2 \prod_{j=0}^m \eta_{ij}^{u_{ij}}, \quad u! = \prod_{i=1}^2 \prod_{j=0}^m u_{ij}!,$$

where $p$ is the $2 \times (m+1)$ matrix variable. The polynomial

$$Z(\beta, \gamma; p) = t! \sum_u p^u / u!$$

is called the normalizing constant, where the sum is taken over all contingency tables $u$ with marginal sums $\beta = (\beta_1, \beta_2)$ and $\gamma = (\gamma_0, \ldots, \gamma_m)$. It is a fundamental problem in statistics to evaluate $Z(\beta, \gamma; p)$ numerically, where $\beta, \beta_0 \in \mathbb{Z}_{\geq 0}$ and $p_{ij} \in \mathbb{Q}_{\geq 0}$.

The normalizing constant $Z$ can be expressed by $F_D$ or $f^{(k)}$. To explain this, we will first define some notation. We put

$$B_0 := \{ (\beta_1, \beta_2, \gamma_0, \ldots, \gamma_m) \in (\mathbb{Z}_{\geq 0})^{m+3} \mid \beta_1 + \beta_2 = \sum_{i=0}^m \gamma_i, \beta_1 - \gamma_0 \leq 0 \},$$

$$B_k := \{ (\beta_1, \beta_2, \gamma_0, \ldots, \gamma_m) \in (\mathbb{Z}_{\geq 0})^{m+3} \mid \beta_1 + \beta_2 = \sum_{i=0}^m \gamma_i, \beta_1 - \gamma_0 > 0, \beta_1 - \sum_{i=0}^k \gamma_i \leq 0 \},$$

where $1 \leq k \leq m$. Then, $\{ (\beta_1, \beta_2, \gamma_0, \ldots, \gamma_m) \in (\mathbb{Z}_{\geq 0})^{m+3} \mid \beta_1 + \beta_2 = \sum_{i=0}^m \gamma_i \}$ is the disjoint union of $B_0, \ldots, B_m$. We also put

$$\ell_1 := \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix}, \quad \ell_2 := \begin{pmatrix} -1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 \end{pmatrix}, \quad \cdots,$$

$$\ell_m := \begin{pmatrix} \ell_1 & \cdots & \ell_{m-1} \end{pmatrix},$$

$$u_0 := \begin{pmatrix} \beta_1 & 0 & 0 & \cdots & 0 \\ \gamma_0 - \beta_1 & \gamma_1 & \gamma_2 & \cdots & \gamma_m \end{pmatrix},$$

$$u_k := \begin{pmatrix} \gamma_0 & \cdots & \gamma_{k-1} & \beta_1 - \sum_{i=0}^{k-1} \gamma_i & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sum_{i=0}^{k} \gamma_i - \beta_1 & \gamma_{k+1} & \cdots & \gamma_m \end{pmatrix} \quad (1 \leq k \leq m),$$

$$x_i := p^{\ell_i} = \frac{p_{11}p_{20}}{p_{10}p_{21}}.$$

If $(\beta_1, \beta_2, \gamma_0, \ldots, \gamma_m) \in B_k$, then all of the entries of $u_k$ are non-negative integers, and hence it is one of the contingency tables with marginal sums $\beta$ and $\gamma$. By straightforward calculation, we can prove the following lemma.

**Lemma 14.**

(1) If $(\beta, \gamma) \in B_0$, then

$$Z(\beta, \gamma; p) = \frac{t!}{u_0!} \cdot p^{u_0} \cdot F_D(-\beta_1, -\gamma_1, \ldots, -\gamma_m, \gamma_0 - \beta_1 + 1; x_1, \ldots, x_m).$$

(2) If $(\beta, \gamma) \in B_k$ with $1 \leq k \leq m$, then

$$Z(\beta, \gamma; p) = t! \cdot p^{u_0} \cdot f^{(k)}(-\beta_1, -\gamma_1, \ldots, -\gamma_m, \gamma_0 - \beta_1 + 1; x_1, \ldots, x_m).$$

In [7], our contiguity relations are applied for the difference holonomic gradient method, which evaluates the numerical value of the column vector $F(a, b, c; x)$ or that of $F^{(k)}(a, b, c; x)$, with $a, b, c \in \mathbb{Z}_{\geq 0}$. For example, it follows from the below discussion of contiguity relations for integer parameters $a, b, c$ that we can easily
evaluate the numerical value of \( F(a, b, c; x) \) from that of \( F(-1, b, c; x) \) by using the matrix \( D_a \) in the contiguity relation. Note that

\[
F_D(-1, b, c; x) = 1 - \sum_{i=1}^{m} \frac{b_i}{c_i} x_i.
\]

For details of the difference holonomic gradient method, see [7].

We now consider the case in which the parameters are integers. Since \( \beta_1, \beta_2, \gamma_0, \ldots, \gamma_m \) are integers, the parameters \( (a, b, c) = (-\beta_1, (-\gamma_1, \ldots, -\gamma_m), \gamma_0 - \beta_1 + 1) \) do not satisfy the condition (3). For the above application, we need to give the contiguity relations that are valid even when the parameters are integers.

**Proposition 15.** (1) If \( (\beta, \gamma) \in \mathcal{B}_0 \), then the relation

\[
F(a - 1, b, c; x) = \frac{a - 1}{c - a} \cdot P_{m+1}(a, b, c; x) \cdot F(a, b, c; x)
\]

holds when the generic parameter vector is specialized to an integral point \( (a, b, c) \rightarrow (-\beta_1, (-\gamma_1, \ldots, -\gamma_m), \gamma_0 - \beta_1 + 1) \).

(2) When \( (\beta, \gamma) \in \mathcal{B}_k \) with \( 1 \leq k \leq m \), we consider the relation

\[
F^{(k)}(a - 1, b, c - 1; x) = -P_0(a, b, c; x) \cdot F^{(k)}(a, b, c; x).
\]

If \( (\beta_1 + 1, \beta_2, \gamma_0, \ldots, \gamma_m) \in \mathcal{B}_k \), then this relation holds when the generic parameter vector is specialized to an integral point \( (a, b, c) \rightarrow (-\beta_1, (-\gamma_1, \ldots, -\gamma_m), \gamma_0 - \beta_1 + 1) \).

We put

\[
\Gamma_{n_1, \ldots, n_m}^{(a, b, c)} := \Gamma \left( 1 - a - \sum_{l=1}^{m} n_l \right) \cdot \Gamma \left( c + \sum_{l=1}^{m} n_l \right) \cdot \prod_{l=1}^{m} \Gamma (1 - b_l - n_l) \cdot \prod_{l=1}^{m} \Gamma (1 + n_l).
\]

To prove this proposition, we will use the following lemma.

**Lemma 16.** (1) Let \( \tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c} \) be integers, and assume \( \tilde{c} > 0 \). Then, there exists \( \tilde{x} \in \mathbb{C}^m \) such that the power series

\[
\sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{1}{\Gamma_{n_1, \ldots, n_m}^{(\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c})}} \prod_{l=1}^{m} x_l^{n_l}
\]

as a function in \( (a, b, c; x) \) is holomorphic on a small neighborhood of \( (\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c}) \). In particular, if \( x \in \mathbb{C}^m \) belongs to a small neighborhood of \( \tilde{x} \), then this series has a limit as \( (a, b, c) \rightarrow (\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c}) \).

(2) For \( 1 \leq k \leq m \), let \( \tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c} \) be integers that satisfy

\[
-\sum_{i \leq k} \tilde{b}_i + \tilde{c} > 0, \quad 2 \sum_{i \leq k-1} \tilde{b}_i - \tilde{c} > 0.
\]

Then, there exists \( \tilde{x} \in \mathbb{C}^m \) such that the Laurent series

\[
\sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{1}{\Gamma_{n_1, \ldots, n_m}^{(\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c})}} \cdot \prod_{l \leq k-1} \frac{x_l^{n_l}}{x_l^{n_l}} \cdot x_k^{n_k} \cdot \prod_{l \geq k+1} \frac{x_l^{n_l}}{x_l^{n_l}}
\]

as a function in \( (a, b, c; x) \) is holomorphic on a small neighborhood of \( (\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c}) \). In particular, if \( x \in \mathbb{C}^m \) belongs to a small neighborhood of \( \tilde{x} \), then this series has a limit as \( (a, b, c) \rightarrow (\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c}) \).

Further, we can differentiate these series term by term, and the partial derivatives of them also have limits as \( (a, b, c) \rightarrow (\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_m, \tilde{c}) \).
uniformly and absolutely convergent, and it defines a holomorphic function.

(1) The definition of \( F(X) \) is well-defined and invertible.\(^*\)

\[ F(a, b, c) := \frac{1}{\prod_{l=1}^{m} \Gamma(1 - b_l)} \cdot \prod_{n_1, \ldots, n_m=0}^\infty \frac{1}{\prod_{l=1}^{m} \Gamma(1 - b_l)} \cdot \prod_{n_1, \ldots, n_m=0}^\infty \prod_{l=1}^{m} x_l^{n_l}. \]

Sketch of Proof. Let \( 0 \leq j \leq m \). First, we can show that there exist \( C, \rho_1, \ldots, \rho_m > 0 \) such that the inequality

\[ \left| \frac{1}{\Gamma^{(j)}_{n_1, \ldots, n_m}(a, b, c)} \right| \leq C \rho_1^{\alpha_1} \cdots \rho_m^{\alpha_m} \]

holds on a small neighborhood of \((\tilde{a}, \tilde{b}, \tilde{c})\). Next, we put

\[ \rho := \max\{\rho_1, \ldots, \rho_m, 2\}, \quad \tilde{x}_i := \frac{1}{\rho^{x_i}}, \]

and \( \tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_m) \). We can show that there exists \( 0 < \eta < 1 \) such that the series in the lemma has the form

\[ \sum_{n_1, \ldots, n_m=0}^\infty \left( \prod_{l=1}^{m} \eta^{n_l} \right) \]

for a majorant on a small neighborhood of \((\tilde{a}, \tilde{b}, \tilde{c}, \tilde{x})\). Therefore, the series is uniformly and absolutely convergent, and it defines a holomorphic function. □

Proof of Proposition 15. If \((a, b, c) = (-\beta_1, (-\gamma_1, \ldots, -\gamma_m), \gamma_0 - \beta_1 + 1)\), then the \( \alpha_l \)'s are expressed as follows:

\[ \alpha_0 = \beta_1 - \sum_{l=0}^{m} \gamma_l - 1 = -\beta_2 - 1, \quad \alpha_k = \gamma_k \quad (1 \leq k \leq m), \]

\[ \alpha_{m+1} = \gamma_0 + 1, \quad \alpha_{m+2} = -\beta_1. \]

Since these values and \( 1 - \alpha_{m+2} = \beta_1 + 1 \) are not zero, it follows from Fact 14 Proposition 5 and Remark 6 that both of the matrices \( Q_k(a, b, c; x) \) and \( C(a, b, c) \) are well-defined and invertible.

(1) The definition of \( F_D(a, b, c; x) \) can be expressed by the Gamma function:

\[ F_D(a, b, c; x) = \Gamma(1 - a) \cdot \Gamma(c) \cdot \prod_{l=1}^{m} \Gamma(1 - b_l) \cdot \sum_{n_1, \ldots, n_m=0}^\infty \frac{1}{\prod_{l=1}^{m} \Gamma(1 - b_l)} \cdot \prod_{l=1}^{m} x_l^{n_l}. \]

By \((\beta, \gamma) \in \mathcal{B}_0\), we have \( c = \gamma_0 - \beta_1 + 1 > 0 \). Then we can apply Lemma 14(1) to \( F(a, b, c; x) \). Note that \( a - 1 = -\beta_1 - 1 \neq 0 \), and \( c - a = \gamma_0 + 1 \neq 0 \).

(2) Let \( \sigma = 0 \) or 1. \((\beta_1 + \sigma, \beta_2, \gamma_0, \ldots, \gamma_m) \in \mathcal{B}_k\) implies

\[ -k \sum_{l=1}^{k-1} b_l + (c - \sigma) = -k (\beta_1 + \sigma) + \sum_{l=0}^{k-1} \gamma_l + 1 > 0, \]

\[ 2 + \sum_{l=1}^{k-1} b_l - (c - \sigma) = (\beta_1 + \sigma) - \sum_{l=0}^{k-1} \gamma_l + 1 > 0. \]

Then we can take the limit of \( F^{(k)}(a, b, c; x) \) as \((a, b, c) \to (-\beta_1, (-\gamma_1, \ldots, -\gamma_m), \gamma_0 - \beta_1 + 1)\) by Lemma 14(2).

By the identity theorem for holomorphic functions, it is sufficient to prove the proposition on a small neighborhood of some \( x \in \mathbb{C}^m \). Therefore, the proof is completed. □
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