Abstract
Centrality measures characterize important nodes in networks. Efficiently computing such nodes has received a lot of attention. When considering the generalization of computing central groups of nodes, challenging optimization problems occur. In this work, we study two such problems, group-harmonic maximization and group-closeness maximization both from a theoretical and from an algorithm engineering perspective.

On the theoretical side, we obtain the following results. For group-harmonic maximization, unless $P=NP$, there is no polynomial-time algorithm achieving an approximation factor better than $1-1/(4e)$ (directed) and $1-1/(4e)$ (undirected), even for unweighted graphs. On the positive side, we show that a greedy algorithm achieves an approximation factor of $\lambda(1-2/e)$ (directed) and $\lambda(1-1/e)/2$ (undirected), where $\lambda$ is the ratio of minimal and maximal edge weights. For group-closeness maximization, the undirected case is NP-hard to be approximated to within a factor better than $1-1/(e+1)$ and a constant approximation factor is achieved by a local-search algorithm. For the directed case, however, we show that, for any $\varepsilon < 1/2$, the problem is NP-hard to be approximated within a factor of $4|V|^{-\varepsilon}$.

From the algorithm engineering perspective, we provide efficient implementations of the above greedy and local search algorithms. In our experimental study we show that, on small instances where an optimum solution can be computed in reasonable time, the quality of both the greedy and the local search algorithms come very close to the optimum. On larger instances, our local search algorithms yield results with superior quality compared to existing greedy and local search solutions, at the cost of additional running time. We thus advocate local search for scenarios where solution quality is of highest concern.

1 Introduction
The identification of important vertices in a graph $G=(V,E)$ is one of the most widely used analytics in network analysis. To this end, numerous centrality measures have been proposed that reflect different underlying network processes, see [9, 30]. Among the widely used ones are closeness and harmonic centrality, which are both based on shortest-path distances, see [27]. The textbook algorithm for computing a node ranking w.r.t. closeness or harmonic centrality solves $|V|$ single-source shortest path problems. Top-$k$ ranking queries can often be solved faster by suitable pruning [6]. Still, closeness is known to be expensive in the worst case: one cannot compute the most closeness-central vertex in time $O(|E|^{2-\varepsilon})$ for any $\varepsilon > 0$ (assuming SETH) [6].

Many network analysis applications do not only require a centrality ranking, but also ask for a group of $k$ nodes that is central as a group. This is an orthogonal problem since the nodes of the most central group need to cover the graph together and often differ significantly from the $k$ most highly ranked vertices. Group centrality problems arise in facility location, leader selection [11], and influence maximization [5], among many others.

Group-closeness and group-harmonic maximization are NP-hard problems (for group-closeness see [10]). In practice real-world instances of non-trivial size (say, beyond a few thousand nodes/edges) usually take too long with exact methods such as ILP solvers [7]. Thus, for group-closeness maximization, recent work has concentrated on heuristics [2, 7, 10]. Early attempts to attribute a constant-factor approximation to a popular greedy algorithm for group-closeness maximization were flawed (cf. discussion in Section 4.1), leaving the question open how and how well both problems can be approximated.

1.1 Related Work. Borgatti and Everett [17] were the first to extend the notion of centrality to groups...
of nodes, including group-degree, group-betweenness, and group-closeness. Group-degree maximization can be reduced from vertex cover (see e.g., \[1\]) and is thus \(NP\)-hard. A similar reduction also shows that the optimization of GED-Walk, a recent group centrality measure inspired by Katz centrality, is \(NP\)-hard. In the same paper, Angriman et al. \[1\] prove submodularity of GED-Walk and use a well-known greedy strategy to get a constant-factor approximation. For group betweenness maximization, sampling-based approximation algorithms have been proposed \[24, 32\]. The notion of group-closeness maximization in \[33\] differs from the original definition (we use the latter, which is widely accepted) and only serves as an estimate of the original. That is why their proofs of \(NP\)-hardness and submodularity do not necessarily carry over to the original one. In fact, standard group-closeness maximization is not submodular (cf. Section 4.1), so that we cannot simply apply submodular optimization results \[31\] in this case directly.

Chen et al. \[10\] argued that group-closeness maximization is \(NP\)-hard by relating it to the \(NP\)-hard \(k\)-means problem. Their realization of the common greedy algorithm was later improved by Bergamini et al. \[7\], who made the algorithm more memory-efficient and exploited the supermodularity of the reciprocal of group-closeness for search pruning. Exploiting supermodularity of the reciprocal also works when the graph distance is replaced by the resistance distance, which leads to the so-called group current-flow closeness – for which Li et al. \[23\] proposed approximation algorithms based on greedy strategies and random projections.

Still, even for group-closeness with the usual graph distance, the greedy algorithm can be time-consuming on large instances, which motivated new local search heuristics \[2\]. Depending on the quality level expected and the implementation, these heuristics can be significantly faster than the greedy method and often obtain a nearly comparable quality. All these works on group-closeness maximization did not provide approximation bounds (also see Section 4.1), leaving the question of approximability unsettled for both problems considered here. Yet, the close relationship between group-closeness maximization and the metric \(k\)-median problem as well as known local search algorithms with constant-factor approximation bounds for the latter \[4\] motivate us to investigate whether the \(k\)-median results can be transferred to group-closeness and to group-harmonic maximization.

### 1.2 Outline and Contribution

In this paper, we address theoretical and practical approximation aspects of the two group centrality maximization problems. In Section 3, we provide the first non-trivial approximation bounds for group-harmonic maximization. By proving that the problem is submodular, but not necessarily monotone, we can directly apply a local search algorithm \[22\] with approximation factor \(4/3\). We also prove that a greedy algorithm admits a \(\lambda(1 - \frac{e}{\epsilon})\)-approximation in directed graphs, where \(\lambda\) is the ratio of the smallest and the largest edge weight, respectively. In undirected graphs, the approximation factor improves to \(4/(1 - \epsilon)\). These results have to be seen in relation to our hardness of approximation results: we show that, unless \(P = NP\), there is no polynomial-time algorithm with approximation factor better than \(1 - 1/e\) (general) or \(1 - 1/(4e)\) (undirected). We proceed by studying group-closeness maximization in Section 4. Interestingly, for this problem we obtain a strong separation between undirected and directed graphs: we prove that the undirected case admits a constant-factor approximation (by relating it to known results on \(k\)-median). For the directed case, in turn, we provide the first inapproximability results: it is \(NP\)-hard to approximate the problem to within a factor better than \(\Theta(|V|^{-\epsilon})\) for any \(\epsilon < 1/2\). All our hardness results hold even in the unweighted case, hence the strong separation even holds in the unweighted case. We summarize our results on approximation in Table 1.

The purpose of Section 5 is to illustrate how to implement greedy and local search heuristics (that satisfy approximation guarantees, unlike previous implementations) efficiently for the respective group centrality maximization problem. Section 6 presents the results of our experimental study with exact and random restart results as baselines: where we can make such a comparison, greedy and local search are on average less than 0.5% away from the optimum. Local search is one to three orders of magnitude slower than greedy, but this is to be expected due to a high quality demand; indeed, unlike previous work on local search \[2\] by a subset of the authors, our new algorithms often cut greedy’s (empirical) gap to optimality by half or more.

Table 1: A summary of our approximation bounds:

| Group-closeness | Directed graphs | Undirected graphs |
|-----------------|----------------|------------------|
| Greedy          | \(\lambda(1 - \frac{e}{\epsilon})\) | \(4/(1 - \epsilon)\) |
| Greedy (general)| \(1 - \frac{e}{\epsilon}\) | \(1 - \frac{1}{4}e\) |
| Greedy (undirected)| \(1 - \frac{e}{\epsilon}\) | \(1 - \frac{1}{4}e\) |
| Greedy (directed)| \(1 - \frac{1}{4}e\) | \(1 - \frac{1}{4}e\) |

Some proofs are deferred to the appendix along with further experimental results.
2 Preliminaries

In all the problems we study, we are given a weighted (possibly directed) graph $G = (V, E, \ell)$, where $|V| = n$ and $\ell : E \rightarrow \mathbb{N}_{>0}$ is an edge-weight function. We do not assume that $G$ is connected, but we assume that there are no isolated nodes. For two nodes $u, v \in V$, we denote with $d(u, v)$ the length of a shortest path in $G$ from $u$ to $v$, where length is measured w.r.t. the function $\ell$. We denote by $\ell_{\min} := \min_{e \in E} \ell(e)$ and $\ell_{\max} := \max_{e \in E} \ell(e)$ the lowest and highest weights in graph $G$, respectively. We furthermore let $\lambda := \frac{\ell_{\min}}{\ell_{\max}}$ be the ratio of smallest and largest edge weight.

Centrality Measures. To measure the relative importance of a vertex in a graph, different centrality measures have been defined. Notably, two well-known centrality measures based on distances are closeness centrality and harmonic centrality. Formally, the closeness centrality $C(u)$ and harmonic centrality $H(u)$ of a vertex $u$ are defined as follows:

$$ C(u) := \frac{n}{\sum_{v \in V \setminus \{u\}} d(u, v)} \quad \text{and} \quad H(u) := \sum_{v \in V \setminus \{u\}} \frac{1}{d(u, v)}. $$

These two centrality measures differ by the order in which they apply the sum and inverse operations. Note that while closeness centrality may seem more natural than harmonic centrality, it suffers from its inability to address disconnected graphs. Indeed, note that in this case, all vertices have a centrality of zero. This finding has been one of the motivations to introduce harmonic centrality, which additionally enjoys several desirable properties from an axiomatic viewpoint [9].

In this work, we study the generalizations of these two centrality measures to groups of nodes. We start by extending the notion of distances to sets by defining $d(S, v) := \min_{u \in S} d(u, v)$. In words, $d(S, v)$ denotes the distance from the closest node in $S$ to $v$. This notation allows us to define the group-closeness and group-harmonic centrality measures.

Group Centralities. For a group $S \subset V$ of vertices in $G$, its group-closeness centrality is defined as

$$ \text{GC}(S) := \frac{n}{\sum_{v \in V \setminus S} d(S, v)}, $$

see for example [7]. A similar objective has been addressed in the literature as well, namely the farness of a set, defined by $\text{GF}(S) := \frac{1}{n} \cdot \sum_{v \in V \setminus S} d(S, v)$. We note that the farness of a set is the reciprocal of its closeness.

The group-harmonic centrality of a group $S \subset V$ of vertices in $G$ is defined as

$$ \text{GH}(S) := \sum_{v \in V \setminus S} \frac{1}{d(S, v)}, $$

where $\frac{1}{d(S, v)} = 0$ if there is no path from $S$ to $v$. While this definition provides a natural generalization to harmonic centrality, the way it handles the vertices in the set $S$ may seem questionable. Indeed, why should these vertices count as 0 while they are the closest ones to the group? On the other hand, making them count more than 0 by assigning them an arbitrary value would also be unsatisfactory. We work around this problem by always comparing the group-harmonic centrality of sets of equal cardinality. Indeed, the value assigned to vertices in the set does not impact such comparisons.

Computational Problems. In this work, we study the following two computational problems that consist of finding groups that maximize the two introduced centrality measures with respect to a budget constraint, i.e., for a given parameter $k$, we are interested in finding a group of size $k$ of large group-closeness centrality or group-harmonic centrality. Formally:

**GROUP-CLOSENESS MAXIMIZATION**
Input: Graph $G = (V, E, \ell)$, integer $k$.
Find: Set $S \subset V$ with $|S| = k$, s.t. $\text{GC}(S)$ is maximum.

**GROUP-HARMONIC MAXIMIZATION**
Input: Graph $G = (V, E, \ell)$, integer $k$.
Find: Set $S \subset V$ with $|S| = k$, s.t. $\text{GH}(S)$ is maximum.

While group-closeness maximization has already been studied in several settings [2], to the best of our knowledge, we are the first to study the group-harmonic maximization problem.

3 Group-Harmonic Maximization

3.1 Mathematical Properties. We start our study of the group-harmonic maximization problem by analyzing the mathematical properties of the set function $\text{GH}(\cdot)$. We observe that, while the function is submodular, it is not monotone.

**Lemma 3.1.** Function $\text{GH} : 2^V \rightarrow \mathbb{Q}_{\geq 0}$ is submodular.

To see that the function $\text{GH}(\cdot)$ is not necessarily monotone, consider the example of an undirected graph with two nodes $u, v$ and one edge between them, then $\text{GH}(\{u\}) = \text{GH}(\{v\}) = 1$, while $\text{GH}(\{u, v\}) = 0$.

3.2 Approximation Algorithms. As $\text{GH}(\cdot)$ is submodular, we can use the local-search algorithm due to Lee et al. [22] and obtain a $\left(\frac{1+\lambda}{\lambda}\right)$-approximation (the
exact cardinality constraint corresponds to the case of a single matroid base constraint, where the matroid is the uniform one). This algorithm was notably improved by Vondrák [31], who designed a randomized local-search method with an approximation factor of \((\frac{1}{4} - o(1))\). Another approximation algorithm candidate is the greedy algorithm (Algorithm 1) that provides an approximation factor of \(1 - \frac{1}{e}\) for maximizing a monotone and submodular function under a cardinality constraint \(|S| \leq k\).

**Algorithm 1** Greedy algorithm for maximizing a monotone submodular function \(f\)

1. \(S \leftarrow \emptyset\)
2. while \(|S| < k\) do
3. \(v \leftarrow \arg \max_{u \in V \setminus S} \{f(S \cup \{u\}) - f(S)\}\)
4. \(S \leftarrow S \cup \{v\}\)
5. return \(S\)

Unfortunately, as \(\text{GH}(\cdot)\) is not monotone, we cannot use this result directly. However, in what follows, we show that Algorithm 1 still guarantees interesting approximation bounds despite the non-monotonicity of \(\text{GH}(\cdot)\). Indeed, we obtain the following theorem.

**Theorem 3.1.** Algorithm 1 guarantees the following approximation factors for the group-harmonic maximization problem, where \(\lambda := \frac{\ell_{\text{max}}}{\ell_{\text{max}}}\) is the ratio of the minimum and maximum edge weight.

- \(\lambda(1 - \frac{1}{2}) > 0.264\lambda\) in the directed case;
- \(\lambda^2(1 - \frac{1}{2}) > 0.316\lambda\) in the undirected case.

While these approximation factors may be worse than the ones provided by Lee et al. [22] and Vondrák [31], they offer better guarantees for the unweighted version of the group-harmonic maximization problem.

We prove Theorem 3.1 by showing that the corresponding approximation factors hold in the unweighted case and then using the following lemma.

**Lemma 3.2.** An \(\alpha\)-approximation algorithm for the unweighted case of the group-harmonic maximization problem yields an \(\alpha\lambda\)-approximation algorithm for the general case, where \(\lambda := \frac{\ell_{\text{max}}}{\ell_{\text{max}}}\) is the ratio of the minimum and maximum edge weight.

We now analyze Algorithm 1 in the unweighted case. Let \(S_i\) be the set computed by Algorithm 1 at the end of iteration \(i\) and \(\Delta_i = \max_{u \in V \setminus S_{i-1}} \{f(S_{i-1} \cup \{u\}) - f(S_{i-1})\}\). We first show an approximation result in case of \(\Delta_i < 0\) (directed) or \(\Delta_i \leq 0\) (undirected), respectively.

**Lemma 3.3.** If \(G\) is directed (resp. undirected), and if at some iteration \(i \in \{1, \ldots, k\}\), \(\Delta_i < 0\) (resp. \(\Delta_i \leq 0\)), then the set returned by Algorithm 1 provides a 0.5-approximation.

### Analysis in the Directed Case
Let us consider a set \(\overline{S} \in \arg \max \{\text{GH}(S) : |S| \leq k\}\) with smallest size \(k' \leq k\). Note that \(\overline{S}\) may be of smaller size than \(k\), whereas group-harmonic maximization asks for solutions of size exactly \(k\). Observe that for each node \(v \in \overline{S}\), we have that there exists a node \(u \in V \setminus \overline{S}\) whose distance from \(\overline{S}\) is due to node \(v\), that is \(d(\overline{S}, u) = d(\overline{S}, v)\), as otherwise we can find an optimal solution with smaller size. This implies that \(\text{GH}(\overline{S}) \geq k'\).

Let us consider the function \(h'(S) := \text{GH}(S) + |S|\). First note that \(\overline{S}\) is an optimal solution of size \(k'\) for \(h'\). Secondly, note that \(h'\) is monotone, as for each \(v \in V \setminus S\), \(h'(S \cup \{v\}) \geq \text{GH}(S) - \frac{1}{e}d(S,v) + |S| + 1 \geq h'(S)\), and submodular, as it is the sum of two submodular functions. Moreover, note that the greedy algorithm shows the same behavior for \(h(\cdot)\) and \(h'(\cdot)\). Hence, we obtain that

\[
h'(S_{k'}) = \text{GH}(S_{k'}) + k' \geq \left(1 - \frac{1}{e}\right) h'(\overline{S})
\]

\[
= \left(1 - \frac{1}{e}\right) \left(\text{GH}(\overline{S}) + k'\right)
\]

where \(S_{k'}\) is the set obtained at iteration \(k'\) of Algorithm 1.

Hence, we obtain that

\[
\text{GH}(S_{k'}) \geq \left(1 - \frac{1}{e}\right) \left(\text{GH}(\overline{S}) + k'\right) - k'
\]

\[
= \left(1 - \frac{1}{e}\right) \left(\text{GH}(\overline{S}) - \frac{k'}{e}\right)
\]

\[
\geq \left(1 - \frac{1}{e}\right) \text{GH}(\overline{S}) - \frac{\text{GH}(\overline{S})}{e}
\]

\[
= \left(1 - \frac{2}{e}\right) \text{GH}(\overline{S}) \geq \left(1 - \frac{2}{e}\right) \text{GH}(O),
\]

where \(O\) is an optimal solution to the group-harmonic maximization problem. Let \(S\) be the solution returned by Algorithm 1. If for all iterations \(i\) of the algorithm \(\Delta_i\) is greater than or equal to 0, then we obtain that \(\text{GH}(S) \geq \text{GH}(S_{k'}) \geq \left(1 - \frac{2}{e}\right) \text{GH}(O)\). Otherwise, by Lemma 3.3, we obtain that \(S\) is a 0.5-approximation, which is better than the claimed approximation bound.

### Analysis in the Undirected Case
Let \(O\) be an optimal solution to group-harmonic maximization. We start with the following lemma lower bounding the increment achieved in each iteration of Algorithm 1.

**Lemma 3.4.** For each \(i = 1, \ldots, k\), it holds that \(\text{GH}(S_i) - \text{GH}(S_{i-1}) \geq \frac{1}{2} \left(\text{GH}(O) - \text{GH}(S_{i-1})\right) - 1\).

**Proof.** For a set of nodes \(T\), let us partition the set \(V \setminus T\) into sets \(R(u,T)\) for \(u \in T\), where \(v \in R(u, T)\) if \(d(u,v) = d(T,v)\); ties are broken arbitrarily in such a
way that \( \{R(u, T)\}_{u \in T} \) is a partition of \( V \setminus T \). Then:

\[
GH(O) - GH(S_{i-1}) = \sum_{v \in V \setminus (O \cup S_{i-1})} \frac{1}{d(O, v)} - \sum_{v \in V \setminus S_{i-1}} \frac{1}{d(S_{i-1}, v)}
\]

(3.1)

\[
= \sum_{v \in V \setminus (O \cup S_{i-1})} \left( \frac{1}{d(O, v)} - \frac{1}{d(S_{i-1}, v)} \right)
\]

\[
+ \sum_{v \in S_{i-1} \setminus O} \frac{1}{d(O, v)} - \sum_{v \in O \setminus S_{i-1}} \frac{1}{d(S_{i-1}, v)}
\]

(3.2)

\[
\leq \sum_{u \in O} \sum_{v \in R(u, O) \setminus S_{i-1}} \left( \frac{1}{d(u, v)} - \frac{1}{d(S_{i-1}, v)} \right)
\]

\[
+ \sum_{v \in S_{i-1} \setminus O} \frac{1}{d(O, v)} - \sum_{v \in O \setminus S_{i-1}} \frac{1}{d(S_{i-1}, v)}
\]

(3.3)

\[
\leq \sum_{u \in O} \sum_{v \in R(u, O) \setminus S_{i-1}} \left( \frac{1}{d(u, v)} - \frac{1}{d(S_{i-1}, v)} \right)
\]

\[
- \frac{1}{d(S_{i-1}, u)}
\]

(3.4)

where equality (3.1) is a reordering of the terms, equality (3.2) holds since \( \{R(u, O)\}_{u \in O} \) is a partition of \( V \setminus O \), inequality (3.3) holds because, for \( u \in O \setminus S_{i-1} \) and \( v \in R(u, O) \), we have \( d(u, v) \geq d(S_{i-1}, v) \) and then \( \frac{1}{d(u, v)} - \frac{1}{d(S_{i-1}, v)} \leq 0 \), and inequality (3.4) holds because \( \frac{1}{d(u, v)} \leq 1 \) for each \( v \in S_{i-1} \setminus O \) and \( |S_{i}| \leq k \).

Let \( \bar{v} \) be the node selected at iteration \( i \), i.e., \( S_{i} \setminus S_{i-1} = \{\bar{v}\} \); then, for each \( u \in O \setminus S_{i-1} \), we have

\[
GH(S_{i}) - GH(S_{i-1}) = \sum_{v \in V \setminus S_{i}} \frac{1}{d(S_{i}, v)} - \sum_{v \in V \setminus S_{i-1}} \frac{1}{d(S_{i-1}, v)}
\]

\[
= \sum_{v \in R(S_{i})} \left( \frac{1}{d(v, \bar{v})} - \frac{1}{d(S_{i-1}, \bar{v})} \right) \frac{1}{d(S_{i-1}, u)}
\]

(3.5)

\[
\geq \sum_{v \in R(u, S_{i-1} \cup \{u\})} \left( \frac{1}{d(u, v)} - \frac{1}{d(S_{i-1}, v)} \right) - \frac{1}{d(S_{i-1}, u)}
\]

(3.6)

\[
\geq \sum_{v \in R(u, S_{i-1} \cup \{u\}) \cap R(u, O)} \frac{1}{d(u, v)} - \frac{1}{d(S_{i-1}, v)}
\]

(3.7)

\[
+ \sum_{v \in R(u, O) \setminus (R(u, S_{i-1} \cup \{u\}) \cup S_{i-1})} \left( \frac{1}{d(u, v)} - \frac{1}{d(S_{i-1}, v)} \right)
\]

(3.8)

where inequality (3.5) holds since node \( \bar{v} \) is the one that maximizes the marginal increment and \( u \) is available at iteration \( i \), inequality (3.6) follows since for each \( v \in R(u, S_{i-1} \cup \{u\}) \), we have \( d(u, v) \leq d(S_{i-1}, v) \) and then all the terms in the sum are non-negative, while, for nodes \( v \in R(u, O) \setminus R(u, S_{i-1} \cup \{u\}) \), we have \( d(u, v) \geq d(S_{i-1}, v) \), since there is a shortest path from \( S_{i-1} \) to \( v \) that does not pass through \( u \), and hence all the terms in the second sum of (3.7) are non-positive, which implies the last inequality. Combining equations (3.4) and (3.8), we have

\[
GH(O) - GH(S_{i-1}) \leq \sum_{u \in O \setminus S_{i-1}} (GH(S_{i}) - GH(S_{i-1})) + k
\]

\[
\leq k \cdot (GH(S_{i}) - GH(S_{i-1})) + k,
\]

since \( |O| = k \), which implies the statement. \( \square \)

We can now prove that Algorithm 1 guarantees an approximation factor of \( \frac{1}{3} \left( 1 - \frac{1}{k} \right) \) in the unweighted undirected case.

**Proof.** As we saw in Lemma 3.3, Algorithm 1 provides a 0.5-approximation, which is larger than the claimed approximation ratio, if at some iteration \( \Delta_{i} \leq 0 \). Hence, we now assume that in all iterations of Algorithm 1, we have \( \Delta_{i} > 0 \). In this case, we prove by induction that

\[
GH(S_{i}) \geq \left( 1 - \left( 1 - \frac{1}{k} \right)^{i} \right) \frac{GH(O) - i}{k}
\]

(3.9)

for each iteration \( i = 1, \ldots, k \). The inductive basis is implied by Lemma 3.4, since for \( i = 1 \) we have \( GH(S_{1}) \geq \frac{GH(O)}{k} - 1 \). For \( i > 1 \), by Lemma 3.4 and the inductive hypothesis, we have

\[
GH(S_{i}) = GH(S_{i}) - GH(S_{i-1}) + GH(S_{i-1})
\]

\[
\geq \frac{1}{k} (GH(O) - GH(S_{i-1})) - 1 + GH(S_{i-1})
\]

\[
= \frac{1}{k} \left( 1 - \left( 1 - \frac{1}{k} \right) \right) \left( GH(O) - i + 1 \right) \left( 1 - \frac{i}{k} \right)
\]

\[
\geq \frac{1}{k} \left( 1 - \left( 1 - \frac{1}{k} \right) \right) \left( GH(O) - i + 1 \right) \left( 1 - \frac{i}{k} \right)
\]

\[
\geq GH(O) \left( 1 - \left( 1 - \frac{1}{k} \right)^{i} \right) - i + \frac{i - 1}{k}
\]

\[
\geq GH(O) \left( 1 - \left( 1 - \frac{1}{k} \right)^{i} \right) - i.
\]
We now show that $\text{GH}(S_i) \geq i$. This claim is due to the fact that the number of nodes at distance one from $S_i$ is greater than $i$, which we prove by induction. The claim is clear for $S_1$. Let us assume the claim to be true at iteration $i - 1$ and let $u$ be the node picked by the greedy algorithm at iteration $i$. First observe that there exists a distinct neighbor $v$ of $u$ which is at distance at least 2 from $S_{i-1}$ since we assumed that $\Delta_i > 0$. If $d(S_{i-1}, u) \geq 2$, then we are done. If $d(S_{i-1}, u) = 1$, we prove by contradiction that $|\{v \mid d(u, v) = 1 \text{ and } d(S_{i-1}, v) \geq 2\}| \geq 2$. Let this set be a singleton $\{v\}$; we show that picking $v$ would yield a higher increment than $u$, a contradiction. Indeed, let $T_u = \{w \mid d(u, w) < d(S_{i-1}, w)\}$. Note that $T_u$ contains necessarily other nodes than $v$, as otherwise $u$ would not yield a positive increment. Vertex $v$ is closer than $u$ to all vertices in $T_u \setminus \{v\}$. Lastly, let $h_u^v(S) = \sum_{w \in \{u, v\} \setminus S} 1/d(S, w)$; then $h_u^v(S_{i-1} \cup \{u\}) - h_u^v(S_{i-1}) = h_u^v(S_{i-1} \cup \{v\}) - h_u^v(S_{i-1}) = -1/2$. By this observation and Equation (3.9) it follows that

$$2 \text{GH}(S_i) \geq i + \text{GH}(S_i) \geq \left(1 - \left(1 - \frac{1}{k}\right)^i\right) \text{GH}(O).$$

By setting $i = k$, we get that $\text{GH}(S)$ is at least

$$\frac{1}{2} \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \text{GH}(O) \geq \frac{1}{2} \left(1 - \frac{1}{e}\right) \text{GH}(O),$$

which concludes the proof. \hfill \qed

### 3.3 Hardness Results

We conclude this section with two hardness of approximation results, one in the directed case and one in the undirected case. These results do not completely close the gap w.r.t. the guarantees of our approximation algorithms, but they provide first upper bounds on the approximation factors that can be achieved for group-harmonic maximization.

**Theorem 3.2.** Even in the unweighted case, there is no polynomial-time algorithm that can approximate group-harmonic maximization with a factor greater than $1 - 1/e$, unless $P = NP$.

**Proof.** We provide a simple reduction from the maximum coverage problem which is known to be hard to approximate better than $1 - 1/e$ [18]. In the maximum coverage problem, we are given a universe $U = \{x_1, \ldots, x_n\}$ of $n$ elements, a collection $C = \{S_1, \ldots, S_m\}$ of $m$ subsets of $U$ and a positive integer $k$. The goal is to select $k$ sets $\{S_1, \ldots, S_k\}$ in $C$ that maximize $|\bigcup_{j=1}^k S_j|$. Given an instance $(U, C, k)$, we create the following unweighted digraph. There exists a vertex $v_x$ for each element $x \in U$ and a vertex $v_S$ for each set $S \in C$. Moreover, there is one arc from $v_x$ to $v_y$ if $x \in S$. Then we consider the group harmonic maximization instance defined by this digraph and a budget $k$. The soundness of the reduction stems from the following two observations. (1) To maximize the group harmonic maximization problem, one should only select vertices associated to sets. (2) For a solution $T$ only compounded of vertices associated to sets, $\text{GH}(T) = |\bigcup_{v_S \in T} S|$.

**Theorem 3.3.** Even in the unweighted undirected case, there is no polynomial-time algorithm that can approximate group-harmonic maximization with a factor greater than $1 - 1/4e$, unless $P = NP$.

In order to prove the theorem, we assume that there exists a $\gamma$-approximation algorithm $A$ for the group harmonic maximization problem, where $\gamma > 1 - 1/4e$. We then show that, using $A$, we can get a logarithmic-factor approximation algorithm to the minimum dominating set problem, which is not possible unless $P = NP$ [16]. See Appendix A.4 for the proof.

### 4 Group-Closeness Maximization

#### 4.1 Preliminary Discussion

Different variants of the group-closeness maximization problem occur depending on whether the graph at hand is undirected or directed. When studying these problems from an approximation algorithm’s perspective, it is tempting to observe that the group-farness $\text{GF}(\cdot)$ is a supermodular set function. In the literature, see the paper by Chen et al. [10], this has been used to argue that $\text{GC}(\cdot)$ is submodular by falsely assuming that the reciprocal of a supermodular function was submodular. It is well-known [26] that maximizing a submodular set function with respect to a cardinality constraint can be done using the greedy algorithm within an approximation factor of $1 - 1/e$. Unfortunately, this approach is flawed and thus the approximation question can be considered as still unresolved for this problem. In Appendix B, we provide a counter-example to the submodularity of $\text{GC}(\cdot)$.

A similar, yet non-flawed, approach has been recently taken by Li et al. [23]. In their work, which deals with a different notion of group centrality, namely “current-flow closeness centrality”, they measure the approximation factor of their algorithms in a different way, allowing them to obtain constant-factor approximation results. In Appendix C, we argue that an approach similar to theirs can be applied also in our setting, yielding constant-factor approximation algorithms in their sense. We would like to stress, however, that this notion of approximation used in the work by Li et al. is a fundamentally different notion of approximation.
4.2 Approximation Algorithms. We will observe that the undirected and directed problems fundamentally differ from an approximation algorithm’s perspective when considering the standard notion of approximation factor. Indeed, while the undirected case allows for a constant-factor approximation, it is NP-hard to approximate directed group-closeness maximization to within a factor better than \(\Theta(n^{-\varepsilon})\) for any \(\varepsilon < 1/2\). We stress that this strong separation between the directed and undirected case even occurs in the unweighted case.

We start by introducing the metric k-Median problem following Arya et al. [4].

**Metric k-Median**

Input: Set of clients \(C\), set of facilities \(F\), cost matrix \(c\) with \(c_{i,j} \geq 0\) for \(i \in C, j \in F\), satisfying triangle inequality, integer \(k\).

Find: Set \(S \subseteq F\) with \(|S| \leq k\), s.t. \(c(S) := \sum_{i \in C} \min_{j \in S} c_{i,j}\) is minimum.

Arya et al. show that the local search algorithm that performs \(p\) swaps at a step leads to a solution with approximation ratio at most \(3 + 2/p\) for Metric k-Median.

The group-farness problem can be seen as a special case of the metric k-Median problem where \(C\) and \(F\) are both taken to be the vertex set and the cost matrix being obtained using the shortest path distances. Since GF(\cdot) is monotone, the result of Arya et al. carries over to the undirected group-farness maximization problem with exact cardinality constraint, yielding an approximation factor of \(\frac{p}{5p+2}\) for group-closeness maximization.

4.3 Hardness Results.

The Undirected Case. Following [13], it is NP-hard to approximate the metric k-median problem to within a factor of \(1 + 1/e\) even in the case when sets \(C\) and \(F\) are the same set. This is equivalent to the group-farness minimization problem in undirected connected graphs and hence we get that it is NP-hard to approximate this latter to within a factor \(\frac{1}{1+e} = (e+1)/e \approx 1.37\). Similarly, we get that it is NP-hard to approximate the undirected group-closeness maximization problem to within a factor of \(e/(e+1) = 1 - 1/(e+1) \approx 0.73\).

The Directed Case. In this paragraph, we prove that the directed case fundamentally differs from the undirected case from an approximation algorithm’s perspective, that is we will show the following theorem:

**Theorem 4.1.** It is NP-hard to approximate the group-closeness maximization problem within \(4 \cdot |V|^{-\varepsilon}\) for any \(\varepsilon \in (0,1/2)\), even in the case of an unweighted DAG.

To prove this result, we provide a reduction from the Set Cover problem. Let \(X = \{x_1, \ldots, x_n\}\) be a universe, \(C = \{C_1, \ldots, C_m\}\) be a collection of subsets of \(X\) and \(k\) be an integer. The Set Cover problem investigates whether there exists a subset \(C' \subseteq C\) of size at most \(k\) such that \(\bigcup_{C_i \in C'} C_i = X\). Importantly, note that the Set Cover problem is NP-hard even if \(m\) is less than or equal to \(3n\). Indeed, there is a simple reduction from Exact Cover by 3-Sets to the Set Cover problem and Exact Cover by 3-Sets remains NP-hard when each element appears in exactly three subsets [20].

**The reduction.** Let \((X, C, k)\) be a Set Cover instance and let \(\delta > 0\) be arbitrary. We construct the following instance of the group-closeness centrality problem: For each set \(C_j\), we create \(\alpha := \lceil n^{1+\delta} \rceil\) vertices \(\{q_j^\ell : \ell \in [\alpha]\}\) and connect them in the form of a path by \(\alpha - 1\) arcs \((q_j^\ell, q_j^{\ell+1}) : \ell \in [\alpha - 1]\). For each element \(x_i\), we create \(\Lambda := ma^2\) vertices \(\{p_i^t : t \in [\Lambda]\}\) and arcs \((q_j^t, q_j^t)\) for all \(t \in [\Lambda]\), if \(x_i \in C_j\). Lastly, we add a vertex \(s\) and arcs \((s, q_j^t)\) for all \(j \in [m]\). The budget is set to \(k + 1\). The reduction is illustrated in Figure 1.

The number of resulting vertices \(V\) can be bounded as \(|V| = ma + nma^2 + 1 \leq 4n^{4+2\delta} m\) using that \(m \leq 3n\) and assuming that \(n \geq 18\). Indeed,

\[
|V| = ma + nma^2 + 1 \\
\leq m(n^{1+\delta} + 1) + mn(n^{1+\delta} + 1)^2 + 1 \\
\leq mn^{1+\delta} + m + mn^{3+2\delta} + 2mn^{2+\delta} + mn + 1 \\
\leq 6mn^{2+\delta} + mn^{3+2\delta} \\
\leq 18n^{3+\delta} + 3n^{4+2\delta} \\
\leq 4n^{4+2\delta}.
\]

We start with the following observation that follows from \(d(v,s) = \infty\) for any \(v \neq s\).

**Observation 1.** For any \(S \subseteq V \setminus \{s\}\), it holds that \(\sum_{v \in V \setminus S} d(S, v) = \infty\).

We continue with further observations.
Observation 2. If there is a set cover \( \{C_{j_1}, \ldots, C_{j_k}\} \) of size \( k \), then \( S := \{s\} \cup \{q^*_r, r \in [k]\} \) satisfies
\[
(4.10) \quad \sum_{v \in V \setminus S} d(S, v) \leq m\alpha^2 + \Lambda n = \Lambda(n + 1)
\]
In the above inequality, the first summand is due to the vertices from \( \{q^*_j : j \in [m], l \in [a]\} \) that are not in \( S \), while the second summand is due to the \( \Lambda n \) vertices \( \{p^*_i : i \in [n], t \in [\Lambda]\} \) that are all at distance one from \( S \).

Conversely, we obtain the following observation.

Observation 3. Let \( S \subset \{s\} \cup \{q^*_j : j \in [m]\} \) be such that \( \{C_j \cup q^*_j : j \in [m]\} \) does not correspond to a set cover in \((X, C, k)\), then
\[
(4.11) \quad \sum_{v \in V \setminus S} d(S, v) \geq \Lambda\alpha.
\]
Note that the previous inequality has made an assumption on the elements that compose the set \( S \). The following lemma, will provide the rational behind this assumption.

Lemma 4.1. Let \( S \) be a set of vertices containing \( s \) such that \( \exists \{v_1, v_2\} \) where \( S \not\subseteq \{s\} \cup \{q^*_j : j \in [m]\} \) then either \( \{C_j \cup q^*_j : j \in [m]\} \) corresponds to a set cover in \((X, C, k)\) or we can, in polynomial time, build a set \( S' \) from \( S \) with \(|S'| = |S|\) such that \( S \cap \{q^*_j : j \in [m]\} \subseteq S' \cap \{q^*_j : j \in [m]\} \) and \( \sum_{v \in V \setminus S} d(S, v) > \sum_{v \in V \setminus S'} d(S', v) \).

Proof. Let \( S \) be a set of vertices containing \( s \) such that \( \exists \{v_1, v_2\} \) where \( S \not\subseteq \{s\} \cup \{q^*_j : j \in [m]\} \) and \( \{C_j \cup q^*_j : j \in [m]\} \) is not a set cover in \((X, C, k)\). We distinguish the following (non-exclusive) cases:

- There exists a vertex \( p^*_i \in S \) such that \( p^*_i \) is not at distance 1 from \( S \). In this case, let \( q^*_j \) be a vertex at distance 1 of \( p^*_i \) and set \( S' = S \setminus \{p^*_i\} \cup \{q^*_j\} \).
- There exists a vertex \( p^*_i \in S \) such that \( p^*_i \) is at distance 1 from \( S \). In this case, because \( \{C_j \cup q^*_j : j \in [m]\} \) is not a set cover of \( X \), there exists a vertex \( p^*_i \) which is at distance 1 from \( S \) and a vertex \( q^*_j \) which is at distance 1 of \( p^*_i \). Set \( S' = S \setminus \{p^*_i\} \cup \{q^*_j\} \).
- Assume that the two first cases do not occur. Then, there exists a vertex \( q^*_j \) in \( S \setminus \{q^*_j : j \in [m]\} \). If \( q^*_j \in S \) and there exists a vertex \( q^*_j \) such that \( q^*_j \) is closer to \( p^*_i \) than any vertex in \( S \), then set \( S' = S \setminus \{q^*_j\} \cup \{q^*_k\} \). Otherwise, because \( \{C_j \cup q^*_j : j \in [m]\} \) is not a set cover of \( X \), there exists a vertex \( p^*_i \) which is not at distance 1 from \( S \) and a vertex \( q^*_j \) which is at distance 1 of \( p^*_i \). Set \( S' = S \setminus \{q^*_j\} \cup \{q^*_j\} \).

In all cases, it is easy to see that we obtain a set \( S' \) which satisfies the conditions of the lemma. In the first two cases, we use the fact that \((A - k) > 1\). In the third case, the result is due to the fact that \( A > \delta \) and the possible loss incurred by vertices \( q^*_j \).

Using Lemma 4.1, we can now prove Theorem 4.1.

Proof of Theorem 4.1. Let us assume that there exists an algorithm \( A \) with approximation guarantee \( 4 \cdot |V|^{-\varepsilon} \) for the group-closeness centrality problem for some \( \varepsilon \in (0, 1/2) \). For a given instance of Set Cover, we use the reduction described above to obtain an instance of the group-closeness centrality problem and apply algorithm \( A \) using \( \delta = 4\varepsilon/(1 - 2\varepsilon) \). Observe that \( \varepsilon < 1/2 \) guarantees that \( \delta > 0 \). Let \( S \) be the solution returned by \( A \). Using Lemma 4.1, we can assume that either \( \{C_j \cup q^*_j : j \in [m]\} \) does not correspond to a set cover of \( S \) or \( \{S \cap \{q^*_j : j \in [m]\}\} \) is not a set cover of \( S \). Furthermore, suppose for the purpose of contradiction that the original instance of Set Cover is a YES instance. Then, denoting the optimum to the group-closeness centrality problem by \( OPT \) and using (4.10) and (4.11) yields that
\[
\frac{\text{GC}(S)}{\text{OPT}} \leq \frac{\Lambda(n + 1)}{\Lambda\alpha} < 2n^{-\delta} \leq 2\left(\frac{|V|}{4}\right)^{-4+2\delta} \leq 4|V|^{-\varepsilon},
\]
using that \( |V| \leq 4n^{4+2\delta} \) and \( \varepsilon = \delta/(4 + 2\delta) < 1/2 \). This contradicts the assumption that \( S \) is a \( 4 \cdot |V|^{-\varepsilon} \)-approximate solution. To summarize, we have shown that if \( A \) is a \( 4 \cdot |V|^{-\varepsilon} \)-approximation algorithm for the group-closeness centrality problem, then it provides a polynomial time algorithm for Set Cover.

5 Algorithm Engineering

In the following we propose several engineering techniques that accelerate the approximate maximization of group-closeness and group-harmonic in practice.

5.1 Group-Harmonic Maximization. We consider greedy and local search algorithms for group-harmonic centrality.

Greedy Algorithm. We start with the greedy algorithm; the pseudocode of this algorithm is given by Algorithm 3 in Appendix H. The first vertex that is added to the group is the vertex with highest harmonic centrality (Line 1); this vertex can be found by a top-\( k \) algorithm such as the ones from Refs. [6,8]. Afterwards, the algorithm iteratively adds the vertex with highest marginal gain \( \text{GH}(S \cup \{u\}) - \text{GH}(S) \) to the group.

Since \( \text{GH} \) is submodular, we can evaluate marginal gains lazily, i.e., the marginal gain \( \text{GH}(S, u) \) from previous iterations serves as an upper bound of the
marginal gain $\text{GH}(S \cup \{u\}) - \text{GH}(S)$ in the current operation. Since $\text{GH}(S, u) \geq \text{GH}(S \cup \{u\})$ holds after $S$ is initialized with the vertex with highest harmonic centrality, we initialize $\text{GH}(S, u)$ to $H(u)$ for each $u \in V \setminus S$ (more precisely, the top-$k$ closeness algorithm from Bisenius et al. [8] yields an upper bound on $H(u)$ that we can use in this initialization step). To determine the vertex with highest marginal gain, we use the well-known lazy strategy [25]: we evaluate the marginal gain of the vertex with highest upper bound (and adjust the upper bound to the true marginal gain) until we know the true marginal gain of the top vertex w.r.t. the upper bound (Lines 3-5 and Line 8 of the pseudocode, by using a priority queue).

To evaluate marginal gains, we run a pruned SSSP algorithm from $u$ that only visits vertices $v$ such that $d(u, v) < d(S, v)$ and updates $\text{GH}(S, u)$ after every vertex at distance $i$ from $u$ has been explored. The traversal is pruned if $\text{GH}(S, u) \leq \text{GH}(S \cup \{x\})$, where $x$ is the vertex with highest marginal gain computed so far; otherwise it returns the exact value of $\text{GH}(S \cup \{u\})$ once all that are vertices closer to $u$ than to $S$ have been visited. As for group-closeness, $\text{GH}$ is defined differently for weighted than for unweighted graphs.

**Pruning (Unweighted).** In unweighted graphs, we can exploit additional bounds to prune the SSSP algorithm earlier. Let us assume that the pruned SSSP (i.e., a BFS) has explored all vertices up to distance $i$. We denote by $D/S/u$ the set of vertices $v$ such that $d(u, v) \leq i$ and $d(u, v) < d(S, v)$. An additional upper bound on the marginal gain of $u$ is

$$
\text{GH}(S, u) = \sum_{v \in D/S/u \setminus \{u\}} \left( \frac{1}{d(u, v)} - \frac{1}{d(S, v)} \right) + r(u) - \frac{1}{d(S, u)} + \frac{n_{i+1}}{i+1}.
$$

The first term is the contribution of the explored vertices up to distance $i$ to the marginal gain. Then, let $D/S/u \subseteq D/S/u$ contain the vertices at distance exactly $i$ from $u$; in the second term we assume that $n_{i+1} \geq D/S/u$ vertices are at distance exactly $i + 1$ from $u$, where $n_{i+1}$ is defined as $\sum_{x \in D/S/u} \deg_{S, u}(x)$ for directed graphs, and $\sum_{x \in D/S/u} (\deg(x) - 1)$ for undirected graphs. In the third term we assume that all the remaining vertices reachable from $u$ are at distance $i + 2$ from $u$ (where $r(u)$ is the number of vertices reachable from $u$). Finally, we subtract the contribution of $u$ to the centrality of $S$.

As a further optimization for unweighted and undirected graphs, for every vertex $u \in V \setminus S$ we subtract from $r(u)$ all the vertices in $u$’s connected component that are at distance 1 from $S$. In this way we avoid to count them in the third term of Eq. (5.12).

**Pruning (Weighted).** Concerning weighted graphs, the SSSP is a pruned version of the Dijkstra algorithm. Let $i$ be the distance from $u$ to the last explored vertex. Upon completion of Dijkstra’s relaxation step, $\text{GH}(S, u)$ is updated as follows:

$$
\text{GH}(S, u) = \sum_{v \in D/S/u \setminus \{u\}} \left( \frac{1}{d(u, v)} - \frac{1}{d(S, v)} \right) + r(u) - \frac{1}{d(S, u)},
$$

i.e., to the contribution to $\text{GH}(S, u)$ of (i) the vertices visited by the SSSP, and (ii) the unexplored vertices assuming that they are all at distance $i$ from $u$.

**Local Search.** The local search algorithm by Lee et al. [22] needs to evaluate $O(n^2)$ swaps per iteration. Since this is already quite expensive, it is desirable to perform only few iterations. Hence, we initialize the local search with a greedy solution; this does not affect its approximation guarantee but offers a considerable acceleration in practice.

We cannot make use of lazy evaluation for local search (since we need to consider swaps and not vertex additions). However, we can still make use of the bound from Eq. (5.12).

**Parallelism.** Since both greedy and local search typically need to evaluate either the marginal gains or the objective function for many vertices before performing a single addition (or swap), it is desirable to utilize parallelism. We parallelize multiple evaluations of the objective function in a straightforward way. Each thread evaluates the marginal gain for one candidate vertex. It needs to store the state of a single SSSP; this incurs $O(n)$ additional memory per thread.

### 5.2 Group-Closeness Maximization

Since the greedy algorithm for group-closeness has been studied before [7], we only discuss local-search and engineering improvements.

**Local Search.** We consider a local search algorithm that evaluates all possible pairs of swaps. For the $k$-Median case, Arya et al. [4] minimize the cost function of an initial solution $S$; a swap is done only if $\text{cost}(S') \leq (1 - \varepsilon/Q) \cdot \text{cost}(S)$, where $S'$ is the solution after the swap, $Q$ is the number of neighboring solutions (i.e. how many different $S'$ are one swap away from $S$), and $\varepsilon > 0$. For group-closeness, the cost function is...
represented by GF(S) (minimum farness is maximum closure), and \( Q = k \cdot (n - k) \) i.e., the number of possible swaps. The algorithm has an approximation ratio of 5.

Like in the group-harmonic case, the local search algorithm is much faster in practice if we start from a good initial solution. We use the grow-shrink algorithm that was introduced by a subset of the authors \([2]\) to obtain such a solution. Grow-shrink is a heuristic algorithm; Ref. \([2]\) does not provide any bounds on its approximation guarantee; however, the paper demonstrates empirically that the algorithm performs well on real-world graphs. The lack of approximation ratio in grow-shrink is not an issue in our case, since the approximation guarantee of our local search does not depend on the initial solution.

Prioritizing Swaps. In practice, the number of swaps that need to be analyzed before a local optimum is reached is heavily affected by the sequence of swaps that are done. Algorithm 4 in Appendix H summarizes how we prioritize the swaps. Similarly to the original grow-shrink algorithm, we prioritize swaps depending on their estimated impact on GF(S). First, we sort in ascending order the vertices in S by the increase in GF due to their removal from S (i.e., GF(S \ {u}) - GF(S) for all u \in S, Lines 4-6 of the pseudocode). Afterward, we sort in descending order all the vertices v \in V \ S by \( \hat{G}F((S \cup \{v\}) \ \{u\}) \), which is an estimate of the decrease in farness (i.e., GF((S \cup \{v\}) \ \{u\}) - GF(S)). We use the same estimate based on the size of shortest path DAGs as Ref. \([2]\).

As a further optimization, we exclude swaps with vertices in V \ S with degree 1 as, in (strongly) connected graphs, they cannot result in a decrease in GF(S).

Additional Pruning. The grow-shrink algorithm \([2]\) performs pruned SSSPs to evaluate whether a swap is advantageous. We modify the algorithm to incorporate additional pruning conditions that prune the SSSP when a swap is not good enough to be considered in the local search (in contrast, Ref. \([2]\) perform all swaps that improve the objective function, regardless of the difference in the score). In particular, we maintain a lower bound \( \hat{G}F(S, u, v) \leq GF((S \cup \{v\}) \ \{u\}) \), so that we can interrupt the pruned SSSP as soon as \( GF(S, u, v) > (1 - \epsilon/(k \cdot (n - k))) \cdot GF(S) \).

\( \hat{G}F(S, u, v) \) is computed in two steps: we first compute \( GF^+(u) := GF(S \ \{u\}) - GF(S) \) exactly (Line 11) i.e., the increase in farness of S due to the removal of u. Then, during every pruned SSSP from v, we keep updating an upper bound of decrease in farness of S \ {u} due to the addition of v: \( \hat{G}F^-(v) := GF(S \ \{u\}) - \hat{G}F(S, u, v) \). Then, \( \hat{G}F(S, u, v) \) is computed as \( GF(S) + GF^+(u) - \hat{G}F^-(v) \).

To compute \( GF^+(u) \) exactly we maintain the following information for each vertex \( x \in V \ \{S\} \): \( d(S, x) \), a vertex \( x_r \in S \) such that \( d(r_x, x) = d(S, x) \), and \( d(S, x) = d(S \ \{r_x\}, x) \). In this way, \( GF^+(u) \) can be computed in \( O(n) \) time as done in the original grow-shrink algorithm:

\[
GF^+(u) = \sum_{x \in \{V \ \{S\} \text{ s.t. } d(S, x) = d(u, x)\}} d(S, x) - d(S^r, x).
\]

\( \hat{G}F^-(v) \) is computed differently in unweighted and weighted graphs. In unweighted graphs the pruned SSSP is a BFS, and we define bounds inspired by the ones used for top-k closeness centrality in \([6]\): For every distance \( i \leq \text{diam}(G) \) we maintain \( N^i_S \) i.e., the set of vertices at distance \( i \) from S, and \( \Phi^i_{S,v} \), i.e., the set of vertices v such that \( d(v, x) \leq i \) and \( d(v, x) < d(S, x) \). Once every vertex in \( \Phi^i_{S,v} \) has been visited by the pruned BFS, we know that at most \( \hat{n}_{i+1} := \min(|N^i_S| + \sum_{x \in \Phi^i_{S,v}} \deg(x) - 1) \) vertices can be at distance \( i + 1 \) from v (in undirected graphs, \( \hat{n}_{i+1} := \min(|N^i_S| + \sum_{x \in \Phi^i_{S,v}} \deg(x) - 1) \) while the remaining unexplored vertices will be at distance \( \geq i + 2 \). Thus, we update \( \hat{G}F^-(v) \) as follows:

\[
\hat{G}F^-(v) = \sum_{x \in \Phi^i_{S,v}} (d(S, x) - d(v, x)) + \sum_{x \in \Lambda} (d(S, x) - i - 1) + \sum_{x \in N^i_S \ \{i\} \ \{\Lambda\}} (d(S, x) - i - 2).
\]

The first term represents the decrease in farness due to the vertices that are already visited by the BFS. In the second term \( \Lambda \subseteq N^{i+2} \) contains the nearest \( \hat{n}_{i+1} \) vertices to S, and we assume at they are \( i + 1 \) hops away from v. Finally, in the third term we assume that all the remaining unvisited vertices at distance \( \geq i + 3 \) from S not counted in \( \Lambda \) can be reachable from v in \( i + 2 \) hops. From the third term we exclude vertices at distance \( i + 2 \) from S because, under our assumption, their distance from S would remain unchanged. At the cost of an additional \( O(\text{diam}(G)) \) memory, \( \hat{G}F^-(v) \) can be computed in \( O(\text{diam}(G)) \) time.

On weighted graphs we update \( \hat{G}F^-(v) \) by adapting the our strategy from GH to GF (see Eq. \((5.13)\)).

Parallelism. We employ the same parallelism as for group-harmonic centrality. The fact that evaluations of the objective function can be parallelized in the greedy and local search algorithm can be seen as an advantage over the grow-shrink algorithm since the latter operates inherently sequentially (i.e. in many cases, it performs
Table 2: Large complex networks. The “Type” column indicates whether the network is undirected (U) or directed (D).

| Graph                        | Type | |V| | |E|
|------------------------------|------|---|---|---|---|
| petster-hamster-household    | U    | 874 | 4,003 |
| petster-hamster-friend       | U    | 1,788 | 12,476 |
| petster-hamster              | U    | 2,000 | 16,098 |
| loc-brightkite_edges         | U    | 58,228 | 214,078 |
| douban                       | U    | 154,908 | 327,162 |
| petster-cat-household        | U    | 105,138 | 494,858 |
| loc-gowalla_edges            | U    | 196,591 | 950,327 |
| wikipedia_link_fy            | U    | 65,562 | 1,071,668 |
| wikipedia_link_ckb           | U    | 60,722 | 1,176,289 |
| petster-dog-household        | U    | 260,290 | 2,130,117 |
| livemocha                    | U    | 104,103 | 2,193,083 |
| flickrEdges                  | U    | 105,938 | 2,316,948 |
| petster-friendships-cat      | U    | 149,700 | 5,448,197 |
| wikipedia_link_mi            | D    | 7,996 | 116,457 |
| folklore                     | D    | 13,356 | 120,238 |
| wikipedia_link_so            | D    | 7,439 | 125,046 |
| wikipedia_link_lo            | D    | 3,811 | 132,837 |
| wikipedia_link_co            | D    | 8,252 | 177,420 |

Table 3: Large high-diameter networks. In the “Type” column the first letter indicates whether the network is undirected (U) or directed (D), while the second letter whether the network is unweighted (U) or weighted (W).

| Graph                      | Type | |V| | |E|
|-----------------------------|------|---|---|---|---|
| marshall-islands            | UW   | 1,080 | 2,557 |
| micronesia                 | UW   | 1,703 | 3,600 |
| kiribati                   | UW   | 1,867 | 4,412 |
| opahal-powergrid            | UW   | 4,941 | 6,594 |
| samoa                      | UW   | 6,926 | 15,217 |
| comores                    | UW   | 7,250 | 17,554 |
| marshall-islands            | DW   | 1,080 | 2,557 |
| micronesia                 | DW   | 1,703 | 3,600 |
| kiribati                   | DW   | 1,867 | 4,412 |
| DC                          | UW   | 9,522 | 14,807 |
| samoa                      | UW   | 6,926 | 15,217 |
| comores                    | UW   | 7,250 | 17,554 |

6 Experiments

We conduct experiments to evaluate our algorithms in terms of solution quality and running time.

For GH we first evaluate the quality of our greedy algorithm (Greedy-H), our local-search algorithm (Greedy-LS-H), and sets of vertices selected uniformly at random (Best-Random-H), the best of 100 randomly chosen sets) against the optimal solution on small-sized networks. Then, we measure the quality and running time performance of Greedy-H and Greedy-LS-H and we use Best-Random-H as baseline.

Regarding group closeness, we compare our local-search algorithm against the greedy algorithm by Bergamini et al. [7], the grow-shrink algorithm, and vertices selected uniformly at random (again, the best of 100 randomly chosen sets). Hereafter, these algorithms are referred to as Greedy-C, GS, and Best-Random-C respectively. Our local-search algorithm for group closeness uses either Greedy-C or GS to initialize the initial group: in the former case we label it as Greedy-LS-C, and GS-LS-C in the latter.

---

2 As in Ref. [2, Sec. III.B], we use a variant of this algorithm that achieves a reasonable time-quality trade-off i.e. with $p = 0.75$. 3 Experiments have been conducted on the instances in Tables 6 and 7 in Appendix G, while the rest of the experiments have been conducted on the instances in Tables 2 and 3.
Because algorithms for group-closeness maximization can only handle (strongly) connected graphs, we run them on the (strongly) connected components of the instances in our datasets. Detailed statistics are reported in Appendix G. For high-diameter networks we test mainly road networks because they are the most common type of networks in the aforementioned repositories. We are confident that our local-search algorithms are capable to handle other types of high-diameter networks as well without significant difference in performance. Because public repositories do not provide a reasonable amount of weighted complex networks, we omit these networks from our experiments.

6.3 Group Harmonic Maximization.

Comparison to Exact ILP Solutions. Figure 2 shows a comparison of the solution quality of our algorithms compared to exact solutions. We observe that random groups cover these unweighted graphs reasonably well; hence, Best-Random-H already yields solutions of > 70% of the optimum. This peculiarity is amplified by the fact that the networks are rather small in comparison to $k$ (at most 1000 vertices). Indeed, the quality of Best-Random-H increases with $k$ on complex networks, a behavior that no other algorithm shows. Still, Greedy-H yields substantially better solutions in all cases: it yields solutions of > 99.5% of the optimum for all group sizes. These solutions are further improved by Greedy-LS-H, which yields groups with at least 99.72% of the optimal quality.

In high-diameter networks (Figures 2b and 7 in Appendix E) Best-Random-H is not a serious competitor. It yields solutions less than 80% the optimal quality. Indeed, since high-diameter networks have a higher diameter compared to complex networks, it is expected that a random group of vertices is less likely to be central. On the other hand, Greedy-H and Greedy-LS-H yield solution qualities from 98.76% and 99.75%, respectively. For $k = 5$ in particular, solutions returned by Greedy-LS-H have > 99.99% the quality of the optimal solution.

Concerning weighted high-diameter networks, the ILP solver runs out of time or memory on nearly all instances. Tentative results on the two remaining instances suggest that Greedy-H yields solutions at are almost optimal, but due to the small size of the data set, we cannot conclude definitive results.

Quality and Running Time on Larger Instances. Figure 3 summarizes quality and running time results of Greedy-H and Greedy-LS-H (absolute running times are reported in Tables 10 and 11, Appendix I). Due to the size of these graphs, it is not feasible to obtain an ILP solution and we use Best-Random-H as baseline. In unweighted complex networks (Figures 3a and 8 in Appendix E), Greedy-H finds solutions with quality (compared to Best-Random-H) from 1.407 (with $k = 5$) to 1.525 (with $k = 50$) in directed networks, and from 1.445 to 1.504 in undirected networks. Compared to Greedy-H, Greedy-LS-H is not competitive: it improves the quality by at most 0.05% while being 5.7× to 27.3× slower.

Greedy-H achieves even better results in high-diameter networks: in weighted directed high-diameter networks (Figure 3b) Greedy-H’s quality is 2.4 to 2.6 of the quality returned by Best-Random-H, while being just 2.5× to 3.6× slower. Concerning Greedy-LS-H, it is less competitive than in complex networks: it improves
Greedy-H’s quality by at most 0.01%, while being 54.9× to 448.9× slower. Results are more promising in high-diameter unweighted networks (Figure 9 in Appendix E): here Greedy-LS-H improves Greedy-H’s quality by 0.58% to 0.69% while being 3.2× to 12.2× slower.

6.4 Group Closeness Maximization.

Comparison to Exact ILP Solutions. Figures 4 and 10 (Appendix F) summarize the quality of our local-search algorithms for group closeness and the competitors compared to the optimum.

Concerning unweighted complex networks, in the directed case, for groups of size 5 Greedy-LS-C is the only algorithm achieving optimal solutions, while for the remaining group sizes it yields solutions with the same quality as Greedy-C. In the undirected case (see Figure 10) Greedy-LS-C and GS-LS-C achieve solutions with at least 99.77% and 99.76% the optimal quality, resp.; for k = 5 and k = 100 in particular they achieve optimal solutions.

In high-diameter networks our local-search algorithms always achieve better results than Greedy-C and GS. The best results are on unweighted graphs: here Greedy-LS-C and GS-LS-C yield solutions at least 98.66% and 98.50% away from optimality, respectively.

Interestingly, the quality of Greedy-LS-C is often higher than GS-LS-C, especially in complex networks and high-diameter weighted networks; we conjecture that our local-search algorithm has a narrower improvement margin on solutions from GS compared to solutions from Greedy-C since GS is based on local-search as well.

6.5 Quality and Running Time on Larger Instances. In Figures 5 and 11 (Appendix F), we report the quality and running time results of GS-LS-C, Greedy-LS-C, Greedy-C and GS compared to Best-Random-C (absolute running times are reported in Tables 12 and 13, Appendix I). In terms of quality our local-search algorithms always reach the best results in all our experiments: in directed complex networks (Figure 5a) GS-LS-C, Greedy-LS-C, and Greedy-C yield similar quality, while GS has consistently the lowest quality. On the other hand, quality can be traded for running time: GS is the fastest algorithm (for small group sizes even faster than Best-Random-C), Greedy-C is on average 16.4× slower than Best-Random-C (average among all k), whereas GS-LS-C and Greedy-LS-C are respectively 28.33× to 233.01×, and 22.99× to 485.09× slower than Best-Random-C. Interestingly, for small group sizes Greedy-LS-C is often faster than GS-LS-C, and vice versa for larger groups. This is likely due to the difference between GS and Greedy-C solutions: Greedy-C aims to maximize the objective function regardless of the group size, while for GS the group size determines how many vertices are consecutively added and removed in a single iteration. Therefore, for larger groups GS solutions need less swaps to reach a local optimum compared to Greedy-C solutions.

In directed weighted high-diameter networks (Figure 5b) Greedy-LS-C always achieves the highest quality with less time overhead than GS-LS-C for all group sizes but 100.

6.6 Parallel Scalability. Strong scaling plots for Greedy-C, Greedy-LS-C, and GS are reported in Figure 6.
on both complex and high-diameter networks. This is not surprising: local search needs to evaluate at least \( k(n-k) \) swaps which is a highly parallel operation, and often much more expensive than running Greedy-C.

On high-diameter networks in particular, Greedy-C has a poor parallel scalability; we conjecture that, since closeness centrality distinguishes vertices in high-diameter networks better than in complex networks [27, Ch. 7], Greedy-C needs to evaluate only few vertices per iteration before finding the vertex with highest marginal gain. In that case, multiple cores do not speed this process up significantly.

7 Conclusions

This work has investigated theoretical and practical approximation aspects of two group centrality maximization problems, namely group-harmonic maximization and group-closeness maximization. These two problems aim to determine a group of \( k \) nodes in a network which is central as a whole.

For the first problem, we have provided approximation hardness results as well as interesting approximation guarantees for a local search algorithm and the greedy algorithm. For the second one, we showed that the undirected version of the problem admits a constant-factor approximation algorithm, while the directed version is \( NP \)-hard to approximate better than \( \Theta(|V|^{-\varepsilon}) \) for any \( \varepsilon < 1/2 \). We have illustrated how to implement efficiently greedy and local search heuristics for both problems and presented the results of a detailed experimental study. Our experiments show that the quality of both the greedy and the local search algorithms come very close to the optimum. This finding is consistent with our theoretical results which assess that in most cases these algorithms have good approximation guarantees.

Interestingly, the two methods also perform well on directed instances for group-closeness maximization despite the hardness of approximation result which holds on this class of instances.

Several future works are conceivable. First, one could try to close the gap on group-harmonic maximization between existing approximation guarantees and approximation hardness results. Second, for group-closeness maximization, it would be interesting to design an algorithm with an approximation ratio matching our hardness result in the directed case.

References

[1] Eugenio Angriman, Alexander van der Grinten, Aleksandar Bojchevski, Daniel Zügner, Stephan Günnemann, and Henning Meyerhenke. Group centrality maximization for large-scale graphs. In *ALENEX*, pages 56–69. SIAM, 2020.
[2] Eugenio Angriman, Alexander van der Grinten, and Henning Meyerhenke. Local search for group closeness maximization on big graphs. In *BigData*, pages 711–720. IEEE, 2019.
[3] Eugenio Angriman, Alexander van der Grinten, Moritz von Looz, Henning Meyerhenke, Martin Nöllenburg, Maria Predari, and Charilaos Tzovas. Guidelines for experimental algorithmics: A case study in network analysis. *Algorithms*, 12(7):127, 2019.
[4] Vijay Arya, Naveen Garg, Rohit Khandekar, Adam Meyerson, Kamesh Munagala, and Vinayaka Pandit. Local search heuristics for k-median and facility location problems. *SIAM J. Comput.*, 33(3):544–562, 2004.
[5] Suman Banerjee, Mamata Jenamani, and Dilip Kumar Pratihar. A survey on influence maximization in a social network. *Knowl. Inf. Syst.*, 62(9):3417–3455, 2020.
[6] Elisabetta Bergamini, Michele Borassi, Pierluigi Crescenzi, Andrea Marino, and Henning Meyerhenke.
Computing top-\(k\) closeness centrality faster in unweighted graphs. *ACM Trans. Knowl. Discov. Data*, 13(5):53:1–53:40, 2019.

[7] Elisabetta Bergamini, Tanya Gonser, and Henning Meyerhenke. Scaling up group closeness maximization. In *ALENEX*, pages 209–222. SIAM, 2018.

[8] Patrick Bisinius, Elisabetta Bergamini, Eugenio Angriman, and Henning Meyerhenke. Computing top-\(k\) closeness centrality in fully-dynamic graphs. In *ALENEX*, pages 21–35. SIAM, 2018.

[9] Paolo Boldi and Sebastiano Vigna. Axioms for centrality. *Internet Math.*, 10(3-4):222–262, 2014.

[10] Chen Chen, Wei Wang, and Xiaoyang Wang. Efficient maximum closeness centrality group identification. In *ADC*, volume 9877 of *Lecture Notes in Computer Science*, pages 43–55. Springer, 2016.

[11] Andrew Clark, Linda Bushnell, and Radha Poovendran. A supermodular optimization framework for leader selection under link noise in linear multi-agent systems. *IEEE Trans. Autom. Control.*, 59(2):283–296, 2014.

[12] Pierluigi Crescenzi, Gianlorenzo D’Angelo, Lorenzo Severini, and Yllka Velaj. Greedily improving our own closeness centrality in a network. *ACM Trans. Knowl. Discov. Data*, 11(1):9:1–9:32, 2016.

[13] Gianlorenzo D’Angelo, Daniele Diodati, Alfredo Navarra, and Cristina M. Pinotti. The minimum \(k\)-storage problem: Complexity, approximation, and experimental analysis. *IEEE Trans. Mob. Comput.*, 15(7):1797–1811, 2016.

[14] Camil Demetrescu, Andrew V Goldberg, and David S Johnson. *The Shortest Path Problem: Ninth DIMACS Implementation Challenge*, volume 74. American Mathematical Soc., 2009.

[15] Julian Dibbelt, Ben Strasser, and Dorothea Wagner. Customizable contraction hierarchies. *ACM J. Exp. Algorithmics*, 21(1):1.5:1–1.5:49, 2016.

[16] Irit Dinur and David Steurer. Analytical approach to parallel repetition. In *STOC*, pages 624–633. ACM, 2014.

[17] Martin G Everett and Stephen P Borgatti. The centrality of groups and classes. *The Journal of mathematical sociology*, 23(3):181–201, 1999.

[18] Uriel Feige. A threshold of \(\ln n\) for approximating set cover. *J. ACM*, 45(4):634–652, 1998.

[19] Gerald Gamrath, Daniel Anderson, Ksenia Bestuzheva, Wei-Kun Chen, Leon Eifler, Maxime Gasse, Patrick Geiandner, Ambros Gleixner, Leona Gottwald, Katrin Halbig, Gregor Hendel, Christopher Hojny, Thorsten Koch, Pierre Le Bodic, Stephen J. Maher, Frederic Matter, Matthias Miltenberger, Erik Mühmer, Benjamin Müller, Marc E. Pfetsch, Franziska Schlösser, Felipe Serrano, Yuji Shinano, Christine Tawfik, Stefan Vigerske, Fabian Wegscheider, Dieter Weninger, and Jakob Witzig. The SCIP Optimization Suite 7.0. ZIB-Report 20-10, Zuse Institute Berlin, March 2020.

[20] Teofilo F. Gonzalez. Clustering to minimize the maximum intercluster distance. *Theor. Comput. Sci.*, 38:293–306, 1985.

[21] Jérôme Kunegis. KONECT: the koblenz network collection. In *WWW (Companion Volume)*, pages 1343–1350. ACM, 2013.

[22] Jon Lee, Vahab S. Mirrokni, Viswanath Nagarajan, and Maxim Sviridenko. Maximizing nonmonotone submodular functions under matroid or knapsack constraints. *SIAM J. Discret. Math.*, 23(4):2053–2078, 2010.

[23] Huan Li, Richard Peng, Liren Shan, Yuhao Yi, and Zhongzhi Zhang. Current flow group closeness centrality for complex networks? In *WWW*, pages 961–971. ACM, 2019.

[24] Ahmad Mahmoody, Charalampos E. Tsourakakis, and Eli Upfal. Scalable betweenness centrality maximization via sampling. In *KDD*, pages 1765–1773. ACM, 2016.

[25] Michel Minoux. Accelerated greedy algorithms for maximizing submodular set functions. In J. Stor, editor, *Optimization Techniques*, pages 234–243, Berlin, Heidelberg, 1978. Springer.

[26] George L. Nemhauser, Laurence A. Wolsey, and Marshall L. Fisher. An analysis of approximations for maximizing submodular set functions - I. *Math. Program.*, 14(1):265–294, 1978.

[27] Mark Newman. *Networks*. Oxford university press, 2018.

[28] OpenStreetMap contributors. Planet dump retrieved from https://planet.osm.org .

[29] Christian L. Staudt, Aleksejs Sazonovs, and Henning Meyerhenke. Networkit: A tool suite for large-scale complex network analysis. *Netw. Sci.*, 4(4):508–530, 2016.

[30] Alexander van der Grinten, Eugenio Angriman, and Henning Meyerhenke. Scaling up network centrality computations – a brief overview. *It - Information Technology*, 62(3-4):189 – 204, 2020.

[31] Jan Vondrak. Symmetry and approximability of submodular maximization problems. *SIAM J. Comput.*, 42(1):265–304, 2013.

[32] Yuichi Yoshida. Almost linear-time algorithms for adaptive betweenness centrality using hypergraph sketches. In *KDD*, pages 1416–1425. ACM, 2014.

[33] Junzhou Zhao, John C.S. Lui, Don Towsley, and Henning Meyerhenke. Computing top-k closeness centrality faster in unweighted graphs. *ACM Trans. Knowl. Discov. Data*, 13(5):53:1–53:40, 2019.
A Omitted proofs

A.1 Proof of Lemma 3.1. We show that for any $S \subseteq T \subseteq V$ and $v \in V \setminus T$ the following holds:

$\text{GH}(S \cup \{v\}) - \text{GH}(S) \geq \text{GH}(T \cup \{v\}) - \text{GH}(T)$. 

The LHS is equal to

$$\sum_{u \in T \setminus S} \left( \frac{1}{d(S \cup \{v\}, u)} - \frac{1}{d(S, u)} \right) - \frac{1}{d(S, v)} \tag{A.1}$$

$$+ \sum_{u \in V \setminus (T \cup \{v\})} \left( \frac{1}{d(S \cup \{v\}, u)} - \frac{1}{d(S, u)} \right),$$

while the RHS is equal to

$$- \frac{1}{d(T, v)} + \sum_{u \in V \setminus (T \cup \{v\})} \left( \frac{1}{d(T \cup \{v\}, u)} - \frac{1}{d(T, u)} \right). \tag{A.2}$$

Term (a) in (A.1) is non-negative, term (b) is at least equal to term (b'), we show that term (c) is at least $c'$. We analyze each term of the sum, $u \in V \setminus (T \cup \{v\})$, separately, we have two cases: (1) $d(T \cup \{v\}, u) = d(T, u)$. In this case the term related to $u$ in (c) is equal to 0, while the one in (c') is non-negative. (2) $d(T \cup \{v\}, u) < d(T, u)$. If $u \in (T \cup \{v\}, u) < d(T, u)$. In this case we have $u, v) < d(T, u) \leq d(S, u)$ and $d(S \cup \{v\}, u) = d(T \cup \{v\}, u)$. It follows that $\frac{1}{d(S, u)} - \frac{1}{d(S, v)} \geq \frac{1}{d(T \cup \{v\}, u)} - \frac{1}{d(T, u)}$, which concludes the proof.

A.2 Proof of Lemma 3.2. Let $\mathcal{A}$ be an $\alpha$-approximation algorithm for the weighted case of the problem. Given an instance $I$ of the group-harmonic centrality problem, we denote by $I_{\mathcal{A}}$ its unweighted version (setting all weights to 1). We denote by $\text{GH}()$ and $\text{GH}_{\mathcal{A}}()$ the corresponding group-harmonic objective functions and let $O_{\mathcal{A}}$ and $O$ be optimal solutions in $I_{\mathcal{A}}$ and $I$, respectively. Apply algorithm $\mathcal{A}$ to $I_{\mathcal{A}}$ and let $S$ be the returned solution. We have that $\text{GH}(S) \geq \alpha \text{GH}_{\mathcal{A}}(O_{\mathcal{A}}) \geq \alpha \text{GH}(O)$. Moreover, for any set $T$, it is easy to observe that $\text{GH}(T) \times \frac{1}{\ell_{\text{min}}} \leq \text{GH}(T) \leq \text{GH}(T) \times \frac{1}{\ell_{\text{min}}}$. Hence, we obtain that $\text{GH}(S) \geq \alpha \lambda \text{GH}(O)$.

A.3 Proof of Lemma 3.3. In the directed (resp. undirected) case, let $i$ be the first iteration such that $\Delta_i < 0$ (resp. $\Delta_i \leq 0$). We show that in this case, for each $v \in V \setminus S_{i-1}$, $d(S_{i-1}, v) \leq 2$. By contradiction, let us consider a node $v$ such that $d(S_{i-1}, v) \geq 3$. We first argue that $d(S_{i-1}, v) \leq 3$. Indeed, in the directed case, if $d(S_{i-1}, v) = \infty$, then $v$ would yield a non-negative increment. Moreover, in the undirected case, if $d(S_{i-1}, v) = \infty$, then $v$ would yield a positive increment as we have assumed there are no isolated nodes in $G$. Hence, $d(S_{i-1}, v) < \infty$ and there exists a neighbor $u$ of $v$ on a shortest from $S_{i-1}$ to $v$. Then, $d(S_{i-1}, u) \geq 2$, $d(S_{i-1} \cup \{u\}, v) = 1$ and $\text{GH}(S_{i-1} \cup \{u\}) - \text{GH}(S_{i-1}) \geq -\frac{1}{d(S_{i-1}, u)} - \frac{1}{d(S_{i-1}, v)} \geq -\frac{1}{2} + 1 - \frac{1}{3} > 0$, a contradiction to $\Delta_i < 0$ (resp. $\Delta_i \leq 0$).

Let $S$ be the set returned by Algorithm 1. As $S_{i-1} \subseteq S$, it follows that the group-harmonic centrality of $S$ can be lower-bounded by $\frac{|V| - k}{2}$, while the optimum can be upper-bounded by $|V| - k$. Thus, the approximation ratio guaranteed by $S$ is at least 0.5.

Algorithm 2 Approximation algorithm for Minimum Dominating Set used in the proof of Theorem 3.3

1: we assume that there exists a $\gamma$-approximation algorithm $\mathcal{A}$ for the group harmonic maximization problem.

2: for $k = 1, \ldots, n$ do

3: $D_k \leftarrow \emptyset$

4: $V_k^1 \leftarrow V$

5: $j \leftarrow 1$

6: while $|V_k^j| \geq k + 1$ do

7: Let $n_k^j = |V_k^j|$ and assume w.l.o.g. that $V_k^j = [n_k^j]$

8: Build a graph $G_k^j = (V_k^j, E_k^j)$ from the subgraph of $G$ induced by $V_k^j$ as follows

9: $V_k^j \leftarrow V_k^j \cup \{x\} \cup Y_k^j \cup Z_k^j$, where $Y_k^j := \{y_i \mid i = 1, \ldots, k\}$ and $Z_k^j := \{z_i \mid i = 1, \ldots, n_k^j\}$

10: $E_k^j \leftarrow E(V_k^j) \cup \{(x, y_i) \mid y_i \in Y_k^j\} \cup \{(z_i, i) \mid i = 1, \ldots, n_k^j\}$

11: Let $S_k^j$ be the solution returned by algorithm $\mathcal{A}$ on $G_k^j$ with budget $k + 1$

12: $D_k \leftarrow D_k \cup (S_k^j \cap V_k^j)$

13: $V_k^{j+1} \leftarrow V_k^j \setminus (S_k^j \cup \cup_{v \in S_k^j} N_v)$

14: $j \leftarrow j + 1$

15: $D_k \leftarrow D_k \cup V_k^j$

16: $D \leftarrow \arg\min_{k=1,\ldots,n} |D_k|$

17: return $D$

A.4 Proof of Theorem 3.3. By contradiction, let us assume that there exists a $\gamma$-approximation algorithm $\mathcal{A}$ for the group harmonic maximization problem, where $\gamma > 1 - 1/4e$. We show that, using $\mathcal{A}$, Algorithm 2 is a logarithmic-factor approximation algorithm to the minimum dominating set problem, which is a contradiction, unless $P = NP$ [16].
The Minimum Dominating Set problem is defined as follows, let \( G = (V, E) \) be an undirected graph, where \( V = [n] \), find a dominating set, i.e., a set of nodes \( D \subset V \) such that \( V = D \cup \bigcup_{v \in D} N_v \), of minimum size. For any \( c \in (0, 1) \), there exist no \((c \ln n)\)-approximation algorithm, unless \( P = NP \) [16].

Let \( k \) be the size of a minimum dominating set of a graph \( G \). We can assume w.l.o.g. that \( k \geq 3 \), otherwise we can guess a minimum dominating set. We observe that \( |D| \leq |D_k| \) and therefore we can show a contradiction on \( D_k \) instead of \( D \). In the following we focus on iteration \( k \) of the while loop.

Let \( \eta \) be the last iteration of the while loop (the largest value of \( j \) such that the while condition holds). We will show that, at each iteration of the while loop, \( k \) nodes are added to \( D_k \). Moreover, since the exit condition of the while loop is \( |V_k^{\eta+1}| < k+1 \), then the size of \( V_k^{\eta+1} \) is at most \( k \). Therefore the size of \( D_k \) is at most \( \eta k + k \), which implies that the approximation ratio of Algorithm 2 is at most \( \eta + 1 \). In the following we show that \( |S_k^j \cap S_k^i| = k \), for each \( j \leq \eta \), and bound the value of \( \eta \).

We first show that, for each \( j \leq \eta \), any solution \( S_k^j \) returned by algorithm \( \mathcal{A} \) selects node \( x \), i.e. \( x \in S_k^j \). Indeed, we show that if \( x \notin S_k^j \), then we can find a node \( u \in S_k^j \) such that \( \mathrm{GH}(S_k^j) \leq ((S_k^j \cup \{x\}) \setminus \{u\}) \). We analyze three different cases.

- If \( y_i \in S_k^j \) for some \( y_i \in Y_k^j \), then \( \mathrm{GH}(S_k^j) \leq ((S_k^j \cup \{x\}) \setminus \{y_i\}) \) since \( d(S_k^j, x) = d((S_k^j \cup \{x\}) \setminus \{y_i\}, y_i) = 1 \) and any node different from \( y_i \) is closer to \( x \) than to \( y_i \).

- If \( Y_k^j \cap S_k^j = \emptyset \) and \( z_i \in S_k^j \) for some \( z_i \in Z_k^j \), then \( \mathrm{GH}(S_k^j) \leq 2 + \frac{1}{2} + h' \), where the first term is due to the two neighbors of \( z_i \), the second term is due to the nodes in \( Y \), and \( h' \) is the contribution of any other node, note that all such nodes are at distance at least 2 from \( z_i \). By swapping \( z_i \) with \( x \) we obtain \( \mathrm{GH}((S_k^j \cup \{x\}) \setminus \{z_i\}) \geq 1 + k + h'' \), where the first term is due to \( z_i \), the second term is due to the nodes in \( Y \), and \( h'' \) is the contribution of any other node, which are at distance at most 2 from \( x \), that is \( h'' \geq h' \). It follows that \( \mathrm{GH}((S_k^j \cup \{x\}) \setminus \{z_i\}) \geq \mathrm{GH}(S_k^j) \), for any \( k \geq 2 \).

- If \( (Y_k^j \cup Z_k^j) \cap S_k^j = \emptyset \), that is \( S_k^j \subseteq V_k^j \), then we show that there exists a node \( v \in S_k^j \) such that \( \mathrm{GH}(S_k^j) \leq ((S_k^j \cup \{v\}) \setminus \{v\}) \). For each \( v \in S_k^j \), let us define \( C(v) := \{w \in V_k^j \mid d(v, w) = 1 \land d(S_k^j \setminus \{v\}, w) > 1\} \), in other words, \( C(v) \) are the nodes of \( V_k^j \) that, among nodes in \( S_k^j \), are adjacent only to \( v \). Since, for \( k \geq 1 \), we have \( |S_k^j| \geq 2 \), then there exists at least a node \( v \in S_k^j \) such that \( |C(v)| \leq |n_k^j/2| \). We observe that among nodes in \( Z_k^j \) there are \( k+1 \) nodes at distance 1 from \( S_k^j \) and \( n_k^j - k - 1 \) other nodes at distance at least 2 (note that \( n_k^j \geq k + 1 \) due to the condition of the while loop). The group harmonic centrality of \( S_k^j \) is then \( \mathrm{GH}(S_k^j) \leq \frac{\delta}{2} + \frac{1}{2} + k + 1 + \frac{n_k^j - k - 1}{2} + |C(v)| + h' \), where \( \frac{\delta}{2} + \frac{1}{2} \) is the contribution of nodes in \( Y_k^j \cup \{x\} \) and \( h' \) is the contribution of nodes not in \( Y_k^j \cup Z_k^j \cup \{x\} \cup C(v) \). By swapping \( v \) with \( x \) we obtain \( \mathrm{GH}((S_k^j \cup \{x\}) \setminus \{v\}) = k + n_k^j + |C(v)| + h'' \), where the last term \( \frac{1}{2} \) is due to \( v \) and \( h'' \) is the contribution of nodes not in \( Y_k^j \cup Z_k^j \cup \{x\} \cup C(v) \), with \( h'' \geq h' \). Since \( |C(v)| \leq |n_k^j/2| \leq n_k^j \) and \( k \geq 3 \), we obtain the statement.

We can further assume that, since \( x \in S_k^j \), then \( S_k^j \) does not contain any node in \( Y_k^j \cup Z_k^j \). Indeed, if \( y_i \in S_k^j \), then we can swap \( y_i \) with any node in \( V_k^j \setminus S_k^j \). If \( z_i \in S_k^j \), then we can swap \( z_i \) with its neighbor in \( Y_k^j \), if it does not belong to \( S_k^j \) or with any other node in \( V_k^j \setminus S_k^j \) otherwise. In any case we do not decrease the value of the objective function.

Since \( x \in S_k^j \) and \( (Y_k^j \cup Z_k^j) \cap S_k^j = \emptyset \), it follows that \( |S_k^j \cap V_k^j| = k \).

We now bound the value of \( \eta \). For each \( j \leq \eta \), we have that the optimal value OPT of the harmonic maximization problem on \( G_k^j \) is at least \( 2n_k^j \). In fact, since \( k \) is the size of an optimal dominating set of \( G \), then there exists a dominating set of size \( k \) for the subgraph of \( G \) induced by \( V_k^j \). If we select the \( k \) nodes in a dominating set of this subgraph and node \( x \), we have that all nodes in \( V_k^j \) that are not selected and all nodes in \( Y_k^j \cup Z_k^j \) are at distance 1 from the nodes in the solution.

Let us consider the first iteration of the while loop (i.e. \( j = 1 \)) and let us denote as \( c \) (as “covered”) and \( u \) (as “uncovered”) the number of nodes in \( V_k^1 \) that are at distance 1 and 2 from \( S_k^1 \), respectively. Since \( x \in S_k^1 \) there is no node at distance greater than 2. We have that

\[ \mathrm{GH}(S_k^1) = c + \frac{u}{2} + n_k^1 + k \geq \gamma 2n_k^1, \]

where \( S_k^1 \) is a \( \gamma \) approximation to OPT. Moreover, we have \( n_k^1 = c + u + k \), that is \( c = n_k^1 - u - k \), which implies

\[ 2n_k^1 - \frac{u}{2} \geq \gamma 2n_k^1, \]

that is

\[ u \leq 4n_k^1(1 - \gamma). \]
Note that $u$ is the number of nodes in $V$ that are given in input to the next iteration, i.e. $u = n_k^2$. By iterating the above arguments, we obtain
\[
n_k^j \leq 4n_k^{j-1}(1 - \gamma) \leq n_k^1(4(1 - \gamma))^{j-1},
\]
for each $j = 2, \ldots, \eta$. By plugging $j = \eta$ and observing that $n_\eta \geq 1$, we obtain
\[
1 \leq n_k^\eta \leq n_k^1(4(1 - \gamma))^\eta - 1.
\]
Since $\gamma > 1 - \frac{1}{4e}$, we have $4(1 - \gamma) < 1$, and hence the above inequality can be solved as
\[
\eta - 1 \leq \log_{4(1 - \gamma)} \frac{1}{n_k^\eta} = \frac{\ln(n_k^1)}{\ln((4(1 - \gamma))^{-1})}.
\]
The approximation ratio of Algorithm 2 is at most $\eta + 1 \leq \frac{\ln(n_k^1)}{\ln((4(1 - \gamma))^{-1})} + 2$. Let us denote $\alpha := \frac{1}{\ln((4(1 - \gamma))^{-1})}$, since $\gamma > 1 - \frac{1}{4e}$, then $\alpha < 1$. For any $\beta$ such that $0 < \alpha < \beta < 1$ there exists a $n_\beta$ such that for each $n_\beta \geq n_\alpha$, $\alpha \ln(n_\beta^1) + 2 \leq \beta \ln(n_\beta^1)$, which implies that the approximation ratio of Algorithm 2 is at most $\beta \ln(n_\beta^1)$. Since for any $c \in (0, 1)$, there exist no $(c \ln n)$-approximation algorithm, unless $P = \text{NP}$ [16], we obtain a contradiction.

**B Counter-example on the submodularity of group closeness**

We provide here a simple example illustrating that $\text{GC}(\cdot)$ is not a submodular set function.\footnote{Note that another counter-example has already been pointed out in the most recent version of [7].} Consider a simple path graph composed of four nodes $v_1$, $v_2$, $v_3$, and $v_4$. The edge-weight function $\ell$ is defined as follows: $\ell(\{v_1, v_2\}) = L$ and $\ell(\{v_2, v_3\}) = \ell(\{v_3, v_4\}) = 1$. It is easy to check that $\text{GC}(\emptyset) = 0$, $\text{GC}(\{v_1\}) = 4/(3L + 3)$, $\text{GC}(\{v_2\}) = 4/(L + 3)$ and $\text{GC}(\{v_1, v_2\}) = 4/3$. Hence, $\text{GC}(\{v_1, v_2\}) - \text{GC}(\{v_2\}) = 4L/(3L + 1)$ and $\text{GC}(\{v_2\}) - \text{GC}(\emptyset) = 4/(L + 3)$. It is straightforward that for a large enough value of $L$ (more precisely for $L \geq 2$), we have $\text{GC}(\{v_1, v_2\}) - \text{GC}(\{v_1\}) > \text{GC}(\{v_2\}) - \text{GC}(\emptyset)$ which shows that $\text{GC}(\cdot)$ is not submodular.

**C Approximation for group-closeness maximization in the Sense of Li et al.**

The approach of Li et al. in fact works for minimizing a general supermodular monotone non-increasing function $f(\cdot)$ with respect to a cardinality constraint. They let $x_1^* \in \arg \max \{ f(\emptyset) - f(\{x\}) \}$ and use the greedy algorithm on the set function $\text{g}(S) := f(\{x_1^*\}) - f(\{x_1^*\} \cup S)$, which is a monotone non-decreasing submodular set function with $\text{g}(\emptyset) = 0$. Thus, the greedy algorithm maximizes the function with respect to a cardinality constraint within an approximation factor of $1 - 1/e$ [26]. However, there are two caveats. First, the greedy algorithm uses a budget of $k - 1$ instead of $k$ (as a budget of one is spent on identifying $x_1^*$) and thus Li et al. obtain an approximation factor of $1 - k/(k - 1)e$. Second and most importantly, observe that the approximation factor is obtained on the function $g(S)$ and not $f(S)$, i.e., they get a set $S$ of size $k - 1$ such that $f(\{x_1^*\}) - f(S \cup \{x_1^*\}) \geq \left(1 - \frac{k}{(k - 1)e}\right) \cdot (f(\{x_1^*\}) - f(S^* \cup \{x_1^*\}))$, where $S^*$ is an optimal set of size $k - 1$ for adding to $\{x_1^*\}$ with the goal of minimizing $f$. We remark that this set is not necessarily related to the set that minimizes $f$ with respect to the cardinality constraint. Clearly, this approach can be applied for the supermodulararness function $\text{GF}(\cdot)$ in the usual sense – and furthermore it would not be easily extendable to the closeness function $\text{GC}(\cdot)$.

**D Ground Truth via ILP.**

To evaluate the quality of the results yielded by our greedy algorithm, we develop an ILP formulation of the group harmonic closeness maximization problem similar to the one proposed in other works [7, 12] which we use later in our experiments to compute the optimal solution $S^*$ for some instances with limited size and we compare it to the one yielded by our greedy algorithm.

We define a binary variable $y_j$ for each vertex $v_j \in V$ that is 1 if $v_j \in S^*$, 0 otherwise. A vertex $v_i$ is assigned to $v_j \in S^*$ if $d(v_i, S^*) = d(v_i, v_j)$ (if multiple vertices satisfy this condition $v_i$ can be assigned arbitrarily to one of them). For every node pair $(v_i, v_j)$ we define a variable $x_{ij}$ that is 1 if $v_i$ is assigned to $v_j$, 0 otherwise. Note that maximizing the sum of all $x_{ij}/d(v_i, v_j)$ would not work because this would yield divisions by zero if a vertex is assigned to itself. Thus, we set the contribution of all $x_{ii}$ to zero by splitting the sum in two terms.

\[
\max \sum_{i=1}^{n} \left( \sum_{j=1}^{i-1} \frac{x_{ij}}{d(v_i, v_j)} + \sum_{j=i+1}^{n} \frac{x_{ij}}{d(v_i, v_j)} \right)
\]
\[\text{s.t. (i)} \sum_{j=1}^{n} x_{ij} + y_i = 1 \quad \forall i \in \{1, \ldots, n\}\]
\[(ii) \sum_{j=1}^{n} y_j = k\]
\[(iii) x_{ij} \leq y_j \quad \forall i, j \in \{1, \ldots, n\}\]
where $x_{ij}, y_j \in \{0, 1\}$

Condition (i) states that each vertex but the ones
in $S^*$ is assigned to exactly one vertex $v_j \in S^*$, (ii) that $|S^*| = k$, and (iii) a vertex can be assigned only to vertices in $S^*$.

E Additional Experimental Results for Group Harmonic Maximization

![Figure 7: Quality vs. the optimum over the small networks of Tables 4 and 5.](image1)

![Figure 8: Quality and running time relative to Best-Random-H over the large complex networks of Table 8.](image2)

![Figure 9: Quality and running time relative to Best-Random-H over the large high-diameter networks of Table 3.](image3)
F Additional Experimental Results for Group Closeness Maximization

(a) Complex networks
(b) High-diameter networks
(c) High-diameter networks
(d) High-diameter networks

Figure 10: Quality vs the optimum over the small networks of Tables 6 and 7.

(a) Complex networks
(b) High-diameter networks
(c) High-diameter networks
(d) High-diameter networks

Figure 11: Quality and running time relative to Best-Random-C over the complex networks of Table 8.
G  Instances Statistics

| Graph                    | Type | |V| | |E|  |
|--------------------------|------|----------------|-------|----------------|-------|
| convote                  | U    | 219 | 586  |
| dimacs10-football        | U    | 115 | 613  |
| wiki_talk_ht             | U    | 537 | 787  |
| moreno_innovation        | U    | 241 | 1,098|
| dimacs10-celegans_metabolic | U   | 453 | 2,025|
| arenas-meta              | U    | 453 | 2,025|
| foodweb-baywet           | U    | 128 | 2,166|
| contact                  | U    | 275 | 2,124|
| foodweb-baydry           | U    | 128 | 2,137|
| moreno_oz                | U    | 217 | 2,672|
| arenas-jazz              | U    | 198 | 2,742|
| sociopatterns-infectious | U    | 411 | 2,765|
| dimacs10-celegansneural  | U    | 297 | 4,296|
| radoslaw_email           | U    | 168 | 5,783|

| convote                  | D    | 219 | 586  |
| wiki_talk_ht             | D    | 537 | 787  |
| moreno_innovation        | D    | 241 | 1,098|
| foodweb-baywet           | D    | 128 | 2,166|
| foodweb-baydry           | D    | 128 | 2,137|
| moreno_oz                | D    | 217 | 2,672|
| dimacs10-celegansneural  | D    | 297 | 4,296|
| radoslaw_email           | D    | 168 | 5,783|

Table 4: Small complex networks used for group harmonic closeness experiments with ILP solver.

| Graph                                | Type | |V| | |E|  |
|--------------------------------------|------|----------------|-------|----------------|-------|
| dimacs10-celegans_metabolic          | U    | 453 | 2,025|
| arenas-meta                          | U    | 453 | 2,025|
| contact                              | U    | 274 | 2,124|
| arenas-jazz                          | U    | 198 | 2,742|
| sociopatterns-infectious             | U    | 410 | 2,765|
| dnc-coreipient                       | U    | 849 | 10,384|
| moreno_oz                            | D    | 214 | 2,658|
| wiki_talk_lv                         | D    | 510 | 2,783|
| wiki_talk_eu                         | D    | 617 | 2,811|
| dnc-temporalGraph                    | D    | 520 | 3,518|
| dimacs10-celegansneural              | D    | 297 | 4,296|
| wiki_talk_bn                         | D    | 700 | 4,316|
| wiki_talk_co                         | D    | 822 | 6,076|
| wiki_talk_gl                         | D    | 1,009 | 7,435|

Table 6: Small complex networks used for group closeness experiments with ILP solver.

| Graph                                | Type | |V| | |E|  |
|--------------------------------------|------|----------------|-------|----------------|-------|
| dbpedia-similar                      | IU   | 430 | 564  |
| niue                                 | IU   | 461 | 1,055|
| tuvalu                               | IU   | 436 | 1,082|
| librec-filmtrust-trust               | IU   | 874 | 1,853|
| niue                                 | WU   | 461 | 1,055|
| tuvalu                               | WU   | 436 | 1,082|
| niue                                 | DU   | 461 | 1,055|
| tuvalu                               | DU   | 436 | 1,082|
| librec-filmtrust-trust               | DU   | 874 | 1,853|
| niue                                 | DW   | 461 | 1,055|
| tuvalu                               | DW   | 436 | 1,082|

Table 5: Small high-diameter networks used for group harmonic closeness experiments with ILP solver.
| Graph          | Type | $|V|$  | $|E|$  |
|----------------|------|------|------|
| tuvalu         | UU   | 152  | 187  |
| niue           | UU   | 461  | 529  |
| nauru          | UU   | 618  | 729  |
| dimacs10-netscience | UU   | 379  | 914  |
| asoiaf         | UU   | 796  | 2,823|

| Graph          | Type | $|V|$  | $|E|$  |
|----------------|------|------|------|
| tuvalu         | UW   | 152  | 187  |
| niue           | UW   | 461  | 529  |
| nauru          | UW   | 618  | 729  |

Table 7: Small high-diameter networks used for group closeness experiments with ILP solver.

| Graph          | Type | $|V|$  | $|E|$  |
|----------------|------|------|------|
| tuvalu         | DU   | 152  | 374  |
| niue           | DU   | 461  | 1,055|
| librec-filmtrust-trust | DU   | 267  | 1,099|
| nauru          | DU   | 618  | 1,427|

| Graph          | Type | $|V|$  | $|E|$  |
|----------------|------|------|------|
| tuvalu         | DW   | 152  | 374  |
| niue           | DW   | 461  | 1,055|
| nauru          | DW   | 618  | 1,427|

Table 8: Largest (strongly) connected components of the complex networks in Table 2 used for group closeness experiments.

| Graph          | Type | $|V|$  | $|E|$  |
|----------------|------|------|------|
| seychelles     | DU   | 3,907| 4,322|
| comores        | DU   | 3,789| 4,630|
| andorra        | DU   | 4,219| 4,933|
| liechtenstein  | DU   | 6,215| 7,002|
| faroe-islands  | DU   | 12,129| 13,165|
| DC             | DU   | 9,522| 14,807|

| Graph          | Type | $|V|$  | $|E|$  |
|----------------|------|------|------|
| seychelles     | DU   | 3,907| 4,322|
| comores        | DU   | 3,789| 4,630|
| andorra        | DU   | 4,219| 4,933|
| liechtenstein  | DU   | 6,215| 7,002|
| faroe-islands  | DU   | 12,129| 13,165|
| DC             | DU   | 9,522| 14,807|

Table 9: Largest (strongly) connected components of the high-diameter networks in Table 3 used for group closeness experiments.
H  Pseudocodes

Algorithm 3 Greedy algorithm for group-harmonic closeness
1: \( v \leftarrow \text{topHarmonicCloseness}(); S \leftarrow \{v\} \)
2: while \(|S| < k\) do
3: \( PQ \leftarrow \text{max-PQ with key } \hat{GH}(S, u) \) and value \( u \)
4: for each \( u \in V \setminus S \) do
5: \( PQ\text{.push}(u) \)
6: \( x \leftarrow \text{null} \)
7: \( \hat{GH}(S \cup \{x\}) \leftarrow -\infty \)
8: repeat ☞ This loop is done in parallel.
9: \( u \leftarrow PQ\text{.extract_max}() \)
10: if \( \hat{GH}(S, u) \leq \hat{GH}(S \cup \{x\}) \) then
11: \( \text{break} \) ☞ \( x \) has the highest marginal gain.
12: \( \text{isExact, } \hat{GH}(S \cup \{u\}) \leftarrow \text{pruned SSSP}(u, \hat{GH}(S \cup \{x\})) \)
13: if \( \text{isExact and } \hat{GH}(S \cup \{u\}) > \hat{GH}(S \cup \{x\}) \) then
14: \( x \leftarrow u \)
15: until \( PQ \) is empty
16: \( S \leftarrow S \cup \{x\} \)
17: return \( S \)

Algorithm 4 Overview of the single-swap algorithm
1: \( S \leftarrow \text{grow-shrink}(G, k) \)
2: \( GF(S) \leftarrow \text{SSSP}(S) \)
3: repeat
4: \( PQ_u \leftarrow \text{min-PQ with key } (GF(S \setminus \{u\}) - GF(S)) \) and value \( u \)
5: for each \( w \in S \) do
6: \( PQ_u\text{.push}(w) \)
7: \( \text{didSwap} \leftarrow \text{false} \)
8: repeat
9: \( u \leftarrow PQ_u\text{.extract_min}() \)
10: \( \text{Compute exact farness increase} \)
11: \( GF^+(u) \leftarrow GF(S \setminus \{u\}) - GF(S) \)
12: compute \( GF((S \cup \{v\}) \setminus \{u\}) \) for all \( V \setminus S \)
13: \( PQ_v \leftarrow \text{max-PQ with key } GF((S \cup \{v\}) \setminus \{u\}) \) and value \( v \)
14: for each \( w \in V \setminus S \) do
15: \( PQ_v\text{.push}(w) \)
16: repeat ☞ This loop is done in parallel.
17: \( v \leftarrow PQ_v\text{.extract_max}() \)
18: \( \text{Compute exact farness decrement} \)
19: if \( GF((S \cup \{v\}) \setminus \{u\}) \leq (1 - \frac{\varepsilon}{k(n-k)}) GF(S) \) then
20: \( S \leftarrow (S \cup \{v\}) \setminus \{u\} \)
21: \( GF(S) \leftarrow \text{SSSP}(S) \)
22: \( \text{didSwap} \leftarrow \text{true} \)
23: \( \text{break} \)
24: until \( PQ_u \) is empty
25: if \( \text{didSwap} \) then
26: \( \text{break} \)
27: until \( PQ_u \) is empty
28: until not \( \text{didSwap} \)
29: return \( S \)
# Running Times

| Graph                        | Undirected unweighted | Directed unweighted | Undirected weighted | Directed weighted |
|------------------------------|-----------------------|---------------------|---------------------|-------------------|
|                             | Greedy-H | Greedy-LS-H | k | 5 | 10 | 50 | 5 | 10 | 50 |
| petster-hamster-household    | <0.1  <0.1 <0.1 <0.1 0.1 |                  |                  |                   |
| petster-hamster-friend       | <0.1  <0.1 <0.1 <0.1 0.1 |                  |                  |                   |
| petster-hamster              | <0.1  <0.1 <0.1 <0.1 0.1 |                  |                  |                   |
| loc-brightkite_edges         | 1.1   1.0 1.1 4.3 6.6 | 25.8              |                  |                   |
| doublan                      | 8.1    8.1 8.4 40.3 86.3 | 303.0             |                  |                   |
| petster-cat-household        | 0.1   0.2 0.3 19.5 23.8 | 106.1             |                  |                   |
| loc-gowalla_edges            | 8.9   8.4 8.7 59.8 97.3 | 1,064.5           |                  |                   |
| wikipedia_link_fy            | 3.8    3.8 4.0 13.3 15.7 | 137.9             |                  |                   |
| wikipedia_link_ckb           | 7.3    7.3 7.4 12.9 14.6 | 80.2              |                  |                   |
| petster-dog-household        | 10.3  10.4 10.7 131.9 212.3 | 843.8             |                  |                   |
| livemocha                    | 11.2   11.4 11.8 52.5 64.6 | 277.9             |                  |                   |
| flickrEdges                  | 44.2   45.4 46.4 119.5 128.4 | 217.6             |                  |                   |
| petster-friendships-cat      | 2.7    2.8 2.9 35.6 55.1 | 266.7             |                  |                   |

Table 10: Running time (s) of Greedy-H and Greedy-LS-H on the complex networks of Table 2.
| Undirected unweighted | Graph                      | GS-LS-C | Greedy-LS-C |
|-----------------------|----------------------------|---------|-------------|
|                       | $k$                        | 5       | 10          | 50          | 5  | 10  | 50  |
| loc-brightkite_edges  | 11.8                       | 22.1    | 146.4       | 11.5        | 20.8| 110.9|
| doublan               | 35.0                       | 59.4    | 222.0       | 26.3        | 43.5| 202.5|
| petster-cat-household | 32.7                       | 66.1    | 363.2       | 32.2        | 63.2| 341.5|
| wikipedia_link_fy     | 100.2                      | 102.5   | 476.9       | 27.5        | 50.3| 434.7|
| wikipedia_link_ckb    | 19.7                       | 103.2   | 767.2       | 19.4        | 34.1| 718.3|
| livemocha             | 58.1                       | 86.3    | 713.0       | 46.5        | 58.2| 604.9|

| Directed unweighted  | Graph                      | GS-LS-C | Greedy-LS-C |
|-----------------------|----------------------------|---------|-------------|
|                       | $k$                        | 5       | 10          | 50          | 5  | 10  | 50  |
| wikipedia_link_mi     | <0.1                       | 0.8     | 2.9         | 0.1         | 0.2 | 1.8  |
| foldoc                | 2.3                        | 3.5     | 0.5         | 1.7         | 2.2 | 5.7  |
| wikipedia_link_so     | 0.7                        | 1.5     | 23.7        | 0.5         | 0.9 | 3.3  |
| wikipedia_link_lo     | 0.8                        | 1.6     | 26.0        | 0.5         | 2.1 | 13.5 |
| wikipedia_link_co     | 0.8                        | 1.7     | 42.9        | 1.2         | 1.7 | 18.5 |
| soc-Epinions1         | 4.2                        | 6.9     | 30.1        | 3.6         | 6.0 | 28.2 |
| slashdot-zoo          | 4.1                        | 6.4     | 19.9        | 3.4         | 7.1 | 15.4 |
| web-NotreDame         | 14.9                       | 37.3    | 1,106.5     | 14.4        | 23.4| 388.6|
| wikipedia_link_jv     | 22.9                       | 97.1    | 30.0        | 17.7        | 14.5| 49.9 |

Table 12: Running time (s) of GS-LS-C and Greedy-LS-C on the complex networks of Table 8.

| Undirected weighted  | Graph                      | GS-LS-C | Greedy-LS-C |
|----------------------|----------------------------|---------|-------------|
|                       | $k$                        | 5       | 10          | 50          | 5  | 10  | 50  |
| opsalh-powergrid     | 0.9                        | 1.1     | 13.4        | 0.7         | 0.4 | 3.7  |
| andorra              | 3.6                        | 8.9     | 55.1        | 1.9         | 3.9 | 26.7 |
| seychelles           | 1.6                        | 5.3     | 26.7        | 0.9         | 3.4 | 25.0 |
| liechtenstein        | 10.8                       | 21.2    | 56.3        | 2.2         | 16.4| 38.2 |
| comores              | 1.2                        | 5.0     | 22.9        | 1.4         | 4.9 | 18.0 |
| faroe-islands        | 33.9                       | 77.2    | 313.5       | 25.3        | 96.4| 268.4|

| Directed weighted    | Graph                      | GS-LS-C | Greedy-LS-C |
|----------------------|----------------------------|---------|-------------|
|                       | $k$                        | 5       | 10          | 50          | 5  | 10  | 50  |
| andorra              | 20.5                       | 35.5    | 182.0       | 4.5         | 10.9| 64.3 |
| seychelles           | 2.6                        | 13.7    | 93.1        | 2.3         | 3.3 | 62.6 |
| liechtenstein        | 3.8                        | 8.6     | 230.3       | 4.1         | 27.0| 265.6|
| DC                   | 7.8                        | 18.9    | 473.2       | 9.9         | 14.0| 98.3 |
| comores              | 2.3                        | 10.1    | 140.1       | 2.3         | 9.6 | 55.3 |
| faroe-islands        | 17.1                       | 137.5   | 907.0       | 15.6        | 27.4| 411.3|

Table 13: Running time (s) of GS-LS-C and Greedy-LS-C on the high-diameter networks of Table 9.