Non-semisimple Hopf Algebras of Dimension $p^2$

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Abstract

Let $H$ be a Hopf algebra of dimension $pq$ over an algebraically closed field of characteristic 0, where $p \leq q$ are odd primes. Suppose that $S$ is the antipode of $H$. If $H$ is not semisimple, then $S^{4p} = id_H$ and $\text{Tr}(S^{2p})$ is an integer divisible by $p^2$. In particular, if $\dim H = p^2$, we prove that $H$ is isomorphic to a Taft algebra. We then complete the classification for the Hopf algebras of dimension $p^2$.

1 Introduction

Let $p$ be a prime number and $k$ an algebraically closed field of characteristic 0. If $H$ is a semisimple Hopf algebra of dimension $p^2$, then $H$ is isomorphic to a group algebra [Mas96], namely $k[Z_p]$ or $k[Z_p \times Z_p]$. For the Hopf algebras $H$ of dimension $p^2$, the only known non-semisimple Hopf algebras of dimension $p^2$ are the Taft algebras [Taf71] (cf. [Mon98, 5]). The question whether the Taft algebras are the only non-semisimple Hopf algebras of dimension $p^2$ is open. In fact, it is also a question suggested by Susan Montgomery in several international conferences. It was proved in [AS98, Theorem A] that if both $H$ and $H^*$ have nontrivial group-like elements or the order of the antipode is $2p$, then $H$ is isomorphic to a Taft algebra provided $\dim H = p^2$. In this paper, we will give a complete answer to the question. More explicitly, we prove that for any non-semisimple Hopf algebra $H$ over $k$ of dimension $p^2$, $H$ is isomorphic to a Taft algebra. Hence, the Hopf algebras over $k$ of dimension $p^2$ can be completely classified (Theorem 6.5).

If $p \leq q$ are odd primes, whether there is a non-semisimple Hopf algebra of dimension $pq$ other than the Taft algebras is still in question. Nevertheless, we prove for any Hopf algebra of this type, the order of its antipode $S$ divides $4p$. Moreover, $\text{Tr}(S^{2p})$ is an integer divisible by $p^2$ (Theorem 6.4). The uniqueness of Taft algebras is a consequence of this result.

The article is organized as follows: In section 2, we recall some notation, general theorems and some useful statements. In section 3, we introduce the notion of the index of a Hopf algebra and we compute the index of the Taft algebras. In section 4, we consider the common eigenspaces of $S^2$ and $r(g)$ where $S$ and $r(g)$ are the antipode and the right multiplication by the distinguished group-like element $g$ of the Hopf algebra $H$. We derive some arithmetic properties of the dimensions of these eigenspaces for the Hopf algebras of odd index. We further exploit the arithmetic properties of these numbers for Hopf algebras
of odd prime index in section 3. Finally, we prove our main theorems in section 6.

2 Notation and Preliminaries

Throughout this paper \( k \) is an algebraically closed field of characteristic 0 and \( H \) is a finite-dimensional Hopf algebra over \( k \) with antipode \( S \). Its comultiplication and counit are, respectively, denoted by \( \Delta \) and \( \varepsilon \). We will use Sweedler’s notation \[\text{Swe69}\]:

\[
\Delta(x) = \sum x_{(1)} \otimes x_{(2)} .
\]

A non-zero element \( a \in H \) is called group-like if \( \Delta(a) = a \otimes a \). For the details of elementary aspects for finite-dimensional Hopf algebras, readers are referred to the references \text{Swe69} and \text{Mon93}.

The set of all group-like elements \( G(H) \) of \( H \) is a linearly independent set and forms a group under the multiplication of \( H \). The divisibility of \( \dim H \) by \( |G(H)| \) is an immediate consequence of the following generalization of Lagrange’s theorem, due to Nichols and Zoeller:

**Theorem 2.1** \text{NZ89} If \( B \) is a Hopf subalgebra of \( H \), then \( H \) is a free \( B \)-module. In particular, \( \dim B \) divides \( \dim H \).

The order of the antipode is of fundamental importance to the semisimplicity of \( H \). We recall some important results on the antipode \( S \) of finite-dimensional Hopf algebras \( H \).

**Theorem 2.2** \text{LR87}, \text{LR88} Let \( H \) be a finite-dimensional Hopf algebra over a field of characteristic 0. Then the following are equivalent:

(i) \( H \) is semisimple.

(ii) \( H^* \) is semisimple.

(iii) \( \text{Tr}(S^2) \neq 0 \).

(iv) \( S^2 = \text{id}_H \).

Let \( \lambda \) be a non-zero right integral of \( H \) and let \( \Lambda \) be a non-zero left integral of \( H^* \). There is an \( \alpha \in \text{Alg}(H, k) = G(H^*) \), independent of the choice of \( \Lambda \), such that \( \Lambda \alpha = \alpha(a) \Lambda \) for \( a \in H \). Likewise, there is a group-like element \( g \in H \), independent of the choice of \( \lambda \), such that \( \beta \lambda = \beta(g) \lambda \) for \( \beta \in H^* \). We call \( g \) the distinguished group-like element of \( H \) and \( \alpha \) the distinguished group-like element of \( H^* \). Then we have a formula for \( S^4 \) in terms of \( \alpha \) and \( g \) \text{Rad76}:

\[
S^4(h) = g(\alpha \rightarrow h \leftarrow \alpha^{-1})g^{-1} \quad \text{for } a \in H ,
\]

(2.2.1)
where \(\rightarrow\) and \(\leftarrow\) denote the natural actions Hopf algebra \(H^*\) on \(H\) described by

\[
\beta \rightarrow a = \sum a_{(1)} \beta(a_{(2)}) \quad \text{and} \quad a \leftarrow \beta = \sum \beta(a_{(1)}) a_{(2)}
\]

for \(\beta \in H^*\) and \(a \in H\). If \(\lambda\) and \(\Lambda\) are normalized, there are formulae for the trace of any linear endomorphism on \(H\).

**Theorem 2.3** [Rad94, Theorem 2] Suppose that \(\lambda(\Lambda) = 1\). Then for any \(f \in \text{End}_k(H)\),

\[
\text{Tr}(f) = \sum \lambda \left( S(\Lambda_{(2)}) f(\Lambda_{(1)}) \right) = \sum \lambda \left( S \circ f(\Lambda_{(2)}) \Lambda_{(1)} \right) = \sum \lambda \left( f \circ S(\Lambda_{(2)}) \Lambda_{(1)} \right).
\]

We shall also need the following lemma of linear algebra:

**Lemma 2.4** [AS98, Lemma 2.6] Let \(T\) be an operator on a finite dimensional vector space \(V\) over \(k\). Let \(p\) be an odd prime and let \(\omega \in k\) be a primitive \(p\)th root of unity.

(i) If \(\text{Tr}(T) = 0\) and \(T^p = id_V\), then \(\dim V_i\) is constant where \(V_i\) is eigenspace of \(T\) associated with the eigenvalue \(\omega^i\). In particular, \(p|\dim V\).

(ii) If \(\text{Tr}(T) = 0\) and \(T^{2p} = id_V\), then

\[
\text{Tr}(T^p) = pd
\]

for some integer \(d\).

### 3 Index of a Hopf algebra

The distinguished group-like element \(g\) defines a coalgebra automorphism \(r(g)\) on \(H\) as follows:

\[
r(g)(a) = ag \quad \text{for} \quad a \in H.
\]

Since \(S^2\) is an algebra automorphism on \(H\),

\[
S^2 \circ r(g) = r(g) \circ S^2.
\]

Moreover, both \(S^2\) and \(r(g)\) are of finite order. Therefore, \(S^2\) and \(r(g)\) generate a finite abelian subgroup of \(\text{Aut}_k(H)\). We will simply call the exponent of the subgroup generated by \(S^4\) and \(r(g)\) the **index** of \(H\). It is easy to see that the index of \(H\) is also the smallest positive integer \(n\) such that

\[
S^{4n} = \text{id}_H \quad \text{and} \quad g^n = 1.
\]

Obviously, \(o(g) \mid n\) and \(o(S^4) \mid n\), where \(o(g)\) and \(o(S^4)\) are the orders of \(g\) and \(S^4\) respectively. By equation (2.2.1),

\[
\text{lcm}(o(g), o(\alpha)) = n.
\]

(3.3.1)
Example 3.1

(i) If both $H$ and $H^*$ are unimodular, then $S^4 = id_H$ by (2.2.1). Therefore, the index of $H$ is 1. In particular, if $H$ is semisimple, the index of $H$ is 1.

(ii) Let $\xi \in \mathbb{k}$ be an $n$th root of unity. The Taft algebra $[Taf71]$ $T(\xi)$ over $\mathbb{k}$ is generated by $x$ and $a$, as a $\mathbb{k}$-algebra, subject to the relations

$$a^n = 1, \quad ax = \xi xa, \quad x^n = 0.$$ 

The Hopf algebra structure is given by

$$\Delta(a) = a \otimes a, \quad S(a) = a^{-1}, \quad \varepsilon(a) = 1,$$
$$\Delta(x) = x \otimes a + 1 \otimes x, \quad S(x) = -xa^{-1}, \quad \varepsilon(x) = 0.$$ 

It is known that $\{x^ia^j \mid 0 \leq i, j \leq n - 1\}$ forms a basis for $T(\xi)$. In particular, $\dim T(\xi) = n^2$. The linear functional $\lambda$, defined by

$$\lambda(x^ia^j) = \delta_{i,n}\delta_{j,0},$$

is a right integral for $T(\xi)$. One can easily see that $a$ is the distinguished group-like element of $T(\xi)$. Moreover, $S^4(x) = \xi^2x$ and $S^4(a) = a$. Therefore, the order of $S^4$ is $n / \gcd(2,n)$. Since the order of $a$ is $n$, the index of $T(\xi)$ is $n$.

Remark 3.2

(i) If the index of the Hopf algebra $H$ is greater than 1, then $H$ is not semisimple by example 3.1(i).

(ii) If $\dim H$ is odd, it follows from theorem [2.1] that the order of the distinguished group-like element $g$ of $H$ and the order of the distinguished group-like element $\alpha$ of $H^*$ are both odd. Hence, by the formula (2.2.1), the order of $S^4$ is also odd. Therefore, the index of $H$ is odd.

4 Eigenspace decompositions for Hopf algebras of odd index

In this section, we will only consider those Hopf algebras $H$ of odd index $n$. Since $r(g)^n = S^{4n} = id_H$, and $S^2$ and $r(g)$ are commuting operators on $H$, $r(g)$ and $S^2$ are simultaneously diagonalizable. Let $\omega \in \mathbb{k}$ be a primitive $n$th root of unity. Then any eigenvalue of $S^2$ is of the form $(-1)^a\omega^i$ and the eigenvalues of $r(g)$ are of the form $\omega^j$. Define

$$H^\omega_{a,i,j} = \{ u \in H \mid S^2(u) = (-1)^a\omega^iu, ug = \omega^j u \} \text{ for any } (a, i, j) \in \mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n.$$ 

We will simply write $\mathcal{K}_n$ for the group $\mathbb{Z}_2 \times \mathbb{Z}_n \times \mathbb{Z}_n$ and write $H^\omega_{(a,i,j)}$ for $H^\omega_{a,i,j}$ for convenience. We then have the decomposition

$$H = \bigoplus_{a \in \mathcal{K}_n} H^\omega_{a}.$$  

(4.4.1)
Note that $H^\omega_a$ is not necessarily non-trivial.

Since the distinguished group-like element $\alpha$ of $H^*$ is an algebra map and $g^n = 1$, we have $\alpha(g)^n = 1$. Hence, $\alpha(g)$ is an $n$th root of unity, and so $\alpha(g) = \omega^x$ for some integer $x$. Using the eigenspace decomposition of $H$ in (4.4.1), the diagonalization of the left integral of $H$ admits an interesting form.

**Lemma 4.1** Let $H$ be a Hopf algebra over the field $k$ of odd index $n$. Let $g$ and $\alpha$ be the distinguished group-like elements of $H$ and $H^*$, respectively. Suppose that $\Lambda$ is a left integral for $H$. Then

$\Delta(\Lambda) \in \sum_{a \in K_n} H^\omega_a \otimes H^\omega_{-a+x}$

where $x = (0, -x, x)$ and $\alpha(g) = \omega^x$.

**Proof.** Note that

$H \otimes H = \bigoplus_{a, b \in K_n} H^\omega_a \otimes H^\omega_b$.

In particular, we can write

$\Delta(\Lambda) = \sum_{a, b \in K_n} \left( \sum u_a \otimes v_b \right)$

where $\sum u_a \otimes v_b \in H^\omega_a \otimes H^\omega_b$. By [Rad94, Proposition 3(d)],

$S^2(\Lambda) = \alpha(g^{-1}) \Lambda = \omega^{-x} \Lambda$.

Since $S^2$ is a coalgebra automorphism on $H$, we have

$\Delta(\Lambda) = \sum_{(a, i, j), (b, s, t) \in K_n} \left( \sum u_{a, i, j} \otimes v_{b, s, t} \right)$

$= \sum_{(a, i, j), (b, s, t) \in K_n} \omega^x S^2 \otimes S^2 \left( \sum u_{a, i, j} \otimes v_{b, s, t} \right)$

$= \sum_{(a, i, j), (b, s, t) \in K_n} (-1)^{a+b} \omega^{x+i+s} \left( \sum u_{a, i, j} \otimes v_{b, s, t} \right)$.

(4.4.2)

Since $g$ is group-like and $\Lambda g = \alpha(g) \Lambda = \omega^x \Lambda$, we have

$\Delta(\Lambda) = \sum_{(a, i, j), (b, s, t) \in K_n} \left( \sum u_{a, i, j} \otimes v_{b, s, t} \right)$

$= \sum_{(a, i, j), (b, s, t) \in K_n} \omega^{-x} r(g) \otimes r(g) \left( \sum u_{a, i, j} \otimes v_{b, s, t} \right)$

$= \sum_{(a, i, j), (b, s, t) \in K_n} \omega^{-x+j+t} \left( \sum u_{a, i, j} \otimes v_{b, s, t} \right)$.

(4.4.3)

Thus, if $\sum u_{a, i, j} \otimes v_{b, s, t} \neq 0$, by equations (4.4.2) and (4.4.3),

$1 = (-1)^{a+b} \omega^{x+i+s}$ and $1 = \omega^{-x+j+t}$,
or equivalently,
\[(b, s, t) = (a, -i, -j) + (0, -x, x) = -(a, i, j) + x.\]

Thus,
\[
\Delta(\Lambda) = \sum_{a \in K_n} \left( \sum u_a \otimes v_{-a+x} \right). \tag{4.4.4}
\]

In the sequel, we will call the expression in equation (4.4.4) the normal form of \(\Delta(\Lambda)\) associated with \(\omega\). We will simply write \(u_a \otimes v_{-a+x}\) for the sum \(\sum u_a \otimes v_{-a+x}\) in the normal form of \(\Delta(\Lambda)\).

The eigenspace decomposition \(H = \bigoplus_{a \in K_n} H^\omega_a\) is associated with a unique family of projections \(E^\omega_a\) (\(a \in K_n\)) from \(H\) onto \(H^\omega_a\) such that

1. \(E^\omega_a \circ E^\omega_b = 0\) for \(a \neq b\) and
2. \(\sum_{a \in K_n} E^\omega_a = \text{id}_H\).

In particular, \(\dim H^\omega_a = \text{Tr}(E^\omega_a)\) for all \(a \in K_n\). By Lemma 4.1,
\[
\Delta(\Lambda) = \sum_{a \in K_n} \left( E^\omega_a \otimes E^\omega_{-a+x} \right) \Delta(\Lambda)
\]
and hence \((E^\omega_a \otimes E^\omega_{-a+x}) \Delta(\Lambda)\) is identical to \(\sum u_a \otimes v_{-a+x}\) in the normal form (4.4.4) of \(\Delta(\Lambda)\). Using the trace formula [Rad90, Theorem 1], we obtained the following Lemma:

**Lemma 4.2** Let \(H\) be a Hopf algebra over the field \(k\) of odd index \(n\) and let \(\omega \in k\) be a primitive \(n\)th root of unity. Suppose that \(\Lambda\) is a left integral for \(H\) and that \(\lambda\) be a right integral for \(H^*\) such that \(\lambda(\Delta(\Lambda)) = 1\). Then
\[
\dim H^\omega_a = \lambda(S(v_{-a+x})u_a) \tag{4.4.5}
\]
for all \(a \in K_n\), where \(\sum_{a \in K_n} u_a \otimes v_{-a+x}\) is the normal form of \(\Delta(\Lambda)\) associated with \(\omega\).

**Proof.** Using the normal form of \(\Delta(\Lambda)\) associated with \(\omega\) and [Rad90, Theorem 1], for any \(b \in K_n\),
\[
\dim H^\omega_b = \text{Tr}(E^\omega_b) = \sum_{a \in K_n} \lambda (S(v_{-a+x})E^\omega_b(u_a)) = \sum_{a \in K_n} \delta_{a,b} \lambda (S(v_{-a+x})u_a) = \lambda (S(v_{-b+x})u_b). \tag*{■}
\]

The family of elements \(S(v_{-a+x})u_a\) appearing in (4.4.3) are in \(H^\omega_{0,x,0}\). Moreover, if \(H\) is non-semisimple, they satisfy a system of equations.
Lemma 4.3 Let $H$ be a non-semisimple Hopf algebra over the field $k$ of odd index $n$ and let $\omega \in k$ be a primitive $n$th root of unity. Then
\[
\sum_{(a,i) \in \mathbb{Z} \times \mathbb{Z}_n} (-1)^a \omega^{-i} \dim H_{a,i,j}^\omega = 0 \quad \text{for } j \in \mathbb{Z}_n.
\]

Proof. Let $\Lambda$ be a left integral for $H$ and let $\lambda$ be a right integral for $H^*$ such that $\lambda(\Lambda) = 1$. If $H$ is not semisimple, by [Rad94, Theorem 4],
\[
\sum S^3(\Lambda_2)\Lambda_1 = 0.
\]
Hence for any integer $e$,
\[
\sum S^3(\Lambda_2)\Lambda_1 g^e = 0.
\]
Let
\[
h'_a = S^3(v_{-a+x})u_a \quad \text{for all } a \in \mathcal{K}_n
\]
where $\sum_{a \in \mathcal{K}_n} u_a \otimes v_{-a+x}$ is the normal form of $\Delta(\Lambda)$ associated with $\omega$. Then
\[
0 = \sum S^3(\Lambda_2)\Lambda_1 g^e = \sum h'_{a,i,j} g^e = \sum_{j \in \mathbb{Z}_n} \omega^{ej} \sum_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_n} h'_{a,i,j}
\]
for $e = 0, \ldots, n-1$. Since $1, \omega, \ldots, \omega^{n-1}$ are distinct elements in $k$, the Vandermonde matrix
\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & \omega & \cdots & \omega^{n-1} \\
: & : & \cdots & : \\
1 & \omega^{n-1} & \cdots & \omega^{(n-1)^2}
\end{bmatrix}
\]
is invertible. Therefore,
\[
\sum_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_n} h'_a,i,j = 0 \quad (4.4.6)
\]
for $j \in \mathbb{Z}_n$. Notice that
\[
S^3(v_{a,-i-x,-j+x}) = (-1)^a \omega^{-i-x} S(v_{a,-i-x,-j+x}).
\]
Therefore, $h'_{a,i,j} = (-1)^a \omega^{-i-x} S(v_{a,-i-x,j+x})u_{a,i,j}$ for any $(a,i,j) \in \mathcal{K}_n$. Then equation (4.4.6) becomes
\[
\sum_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_n} (-1)^a \omega^{-i} S(v_{a,-i-x,j+x})u_{a,i,j} = 0
\]
for $j \in \mathbb{Z}_n$. Applying $\lambda$ to the equation, we have
\[
\sum_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_n} (-1)^a \omega^{-i} \lambda(S(v_{a,-i-x,j+x})u_{a,i,j}) = 0
\]
for all $j \in \mathbb{Z}_n$. Then, the result follows from Lemma 4.2. ■
Lemma 4.4 Let $H$ be a non-semisimple unimodular Hopf algebra over the field $k$ of odd index $n$. Let $\omega \in k$ be a primitive $n$th root of unity. Then
\[
\sum_{i \in \mathbb{Z}_n} (-1)^a \omega^{-i} \dim H_{a,i,l-2i}^\omega = 0
\]
for $l \in \mathbb{Z}_n$.

Proof. Let $\alpha$ and $g$ be the distinguished group-like elements of $H^*$ and $H$, respectively. Since $H$ is unimodular, $\alpha = \varepsilon$ and hence $\alpha(g) = 1 = \omega^0$. Let $\Lambda$ be a left integral for $H$ and let $\lambda$ be a right integral for $H^*$ such that $\lambda(\Lambda) = 1$. It follows from Lemma 4.1 that the normal form of $\Delta(\Lambda)$ associated with $\omega$ is
\[
\sum_{a \in K_n} u_a \otimes v_{-a}.
\]
(4.4.7)

Since $H$ is not semisimple,
\[
0 = \varepsilon(\Lambda)1 = \sum \Lambda_1 S(\Lambda_2).
\]
Thus, we have
\[
0 = \sum_{a \in K_n} u_a S(v_{-a}).
\]
(4.4.8)

Note that, by equation (2.2.1) and the unimodularity of $H$,
\[
g^e a = S^{4e}(a) g^e
\]
for any integer $e$ and $a \in H$. Let $\overline{h}_a = u_a S(v_{-a})$ for $a \in K_n$. Then,
\[
g^e \overline{h}_{a,i,j} = g^e u_{a,i,j} S(v_{a,-i,-j})
= \omega^{e(2i+j)} u_{a,i,j} S(v_{a,-i,-j})
= \omega^{e(2i+j)} \overline{h}_{a,i,j}.
\]
(4.4.9)

By multiplying $g^e$ on the left in equation (4.4.8), we have
\[
0 = \sum_{(a,i,j) \in K_n} \omega^{e(2i+j)} \overline{h}_{a,i,j}
= \sum_{l \in \mathbb{Z}_n} \omega^{el} \sum_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_n} \overline{h}_{a,i,l-2i}.
\]
(4.4.10)

By the same argument used in the proof of Lemma 4.3,
\[
\sum_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_n} \overline{h}_{a,i,l-2i} = 0
\]
(4.4.11)

for $l \in \mathbb{Z}_n$. Notice that, by [Rad94, Theorem 3(a)],
\[
\lambda(\overline{h}_{a,i,j}) = \lambda(u_{a,i,j} S(v_{a,-i,-j}))
= \lambda(S^3(v_{a,-i,-j}) u_{a,i,j})
= (-1)^a \omega^{-1} \lambda(S(v_{a,-i,-j}) u_{a,i,j}).
\]
(4.4.12)
By Lemma 4.2 and equation (4.4.7),
\[ \lambda(\overline{h}_{a,j}) = (-1)^a \omega^{-i} \dim H^\omega_{a,i,j} \, . \]

Hence, we have
\[ 0 = \sum_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_n} \lambda(\overline{h}_{a,i,l}) = \sum_{i \in \mathbb{Z}_n} (-1)^a \omega^{-i} \dim H^\omega_{a,i,l-2i} \, . \]

for \( l \in \mathbb{Z}_n \). ■

5 Arithmetic properties of Hopf algebras with odd prime index

In this section, we will study the arithmetic properties for the Hopf algebras of odd prime index \( p \). Let \( \omega \in k \) be a primitive \( p \)-th root of unity. The Taft algebra \( T(\omega) \) [Taf71] is then a Hopf algebra of this type by example 3.1 (ii). The quantum double of \( T(\omega) \) is a unimodular Hopf algebra of index \( p \) (cf. [KR93]).

Lemma 5.1 Let \( H \) be a Hopf algebra of index \( p \). Then, for each \( j \in \mathbb{Z}_p \), there exists an integer \( d_j \) such that
\[ \dim H^\omega_{0,i,j} - \dim H^\omega_{1,i,j} = d_j \, . \]

for any \( i \in \mathbb{Z}_p \).

Proof. By Lemma 4.3, we have
\[ \sum_{i \in \mathbb{Z}_p} \omega^{-i}(\dim H^\omega_{0,i,j} - \dim H^\omega_{1,i,j}) = 0 \]

for any \( j \in \mathbb{Z}_p \). In particular, \( \omega^{-1} \) is a root of the integral polynomial
\[ f_j(x) = \sum_{i=0}^{p-1} (\dim H^\omega_{0,i,j} - \dim H^\omega_{1,i,j}) x^i \, . \]

Hence, \( f_j(x) = d_j \Phi_p(x) \) for some \( d_j \in \mathbb{Q} \), where \( \Phi_p(x) = 1 + x + \cdots + x^{p-1} \) is the irreducible polynomial of \( \omega^{-1} \) over \( \mathbb{Q} \). Therefore,
\[ \dim H^\omega_{0,i,j} - \dim H^\omega_{1,i,j} = d_j \, . \]

Since \( \dim H^\omega_{0,i,j} - \dim H^\omega_{1,i,j} \) is an integer, and so is \( d_j \). ■

Lemma 5.2 Let \( H \) be a Hopf algebra of index \( p \), where \( p \) is an odd prime. If \( H^* \) is not unimodular, then \( p \mid \dim H \) and
\[ \sum_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_p} \dim H^\omega_{a,i,j} = \frac{\dim H}{p} \, . \]
Proof. Since $H^*$ is not unimodular, $g \neq 1$. Then, $\text{Tr}(r(g)) = 0$ (cf. [LR95, Proposition 2.4(d)]. Moreover, $r(g)^p = id_H$. Hence, by Lemma 2.4, $p|\dim H$ and the eigenspace of $r(g)$ associated with the eigenvalue $\omega^j$ is of dimension $\frac{\dim H}{p}$ for any $j \in \mathbb{Z}_p$. Note that

$$\bigoplus_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_p} H^\omega_{a,i,j}$$

is the eigenspace of $r(g)$ associated with $\omega^j$. Therefore,

$$\frac{\dim H}{p} = \dim \left( \bigoplus_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_p} H^\omega_{a,i,j} \right) = \sum_{(a,i) \in \mathbb{Z}_2 \times \mathbb{Z}_p} \dim H^\omega_{a,i,j}.$$  

Lemma 5.3 Let $H$ be a Hopf algebra of index $p$. If $H^*$ is not unimodular and $H$ is unimodular, then:

(i) There is an integer $d$ such that

$$\dim H^\omega_{0,i,j} - \dim H^\omega_{1,i,j} = d \quad \text{for any} \quad i, j \in \mathbb{Z}_p.$$

(ii) $\text{Tr}(S^{2p}) = p^2 d$.

Proof. (i) By Lemma 4.4, for any $l \in \mathbb{Z}_p$,

$$\sum_{i \in \mathbb{Z}_p} (\dim H^\omega_{0,i,l-2i} - \dim H^\omega_{1,i,l-2i}) \omega^{-i} = 0$$

Since $\omega^{-1}$ is also a primitive $p$th root of unity in $k$, there exists an integer $c_l$ such that

$$\dim H^\omega_{0,i,l-2i} - \dim H^\omega_{1,i,l-2i} = c_l \quad (5.5.1)$$

for $i \in \mathbb{Z}_p$. By Lemma 5.1, for any $i, l \in \mathbb{Z}_p$,

$$c_l = \dim H^\omega_{0,i,l-2i} - \dim H^\omega_{1,i,l-2i} = d_{l-2i}. \quad (5.5.2)$$

Since 2 and $p$ are relative prime, $l, l-2, \ldots, l-2(p-1)$ is a complete set of representatives of $\mathbb{Z}_p$. Therefore,

$$d_j = c_l = d \quad \text{for any} \quad j, l \in \mathbb{Z}_p.$$

(ii) Since $p$ is odd,

$$\text{Tr}(S^{2p}) = \sum_{i, j \in \mathbb{Z}_p} \dim H^\omega_{0,i,j} - \dim H^\omega_{1,i,j}$$

$$\quad = \sum_{i, j \in \mathbb{Z}_p} d$$

$$\quad = p^2 d. \quad \blacksquare$$
6 Hopf algebras of dimension $pq$

In this section, we will consider the Hopf algebras $H$ of dimension $pq$ where both $p \leq q$ are odd primes. In particular, we prove that if $H$ is not semisimple and $\text{dim } H = p^2$, then $H$ is isomorphic to a Taft algebra. By [Mas96, Theorem 2], any Hopf algebra over $k$ of dimension $p^2$ is either a group algebra or a Taft algebra. We begin the section with the following lemma.

**Lemma 6.1** Let $p, q$ be two distinct prime numbers. Then there is no Hopf algebra $H$ of dimension $pq$ such that $|G(H)| = p$ and $|G(H^*)| = q$.

**Proof.** Suppose there is a Hopf algebra $H$ of dimension $pq$ such that $|G(H)| = p$ and $|G(H^*)| = q$. Let $g \in G(H)$ and $\alpha \in G(H^*)$ such that $o(g) = p$ and $o(\alpha) = q$. Then,

$$\alpha(g)^p = \alpha(g^p) = \alpha(1) = 1$$

and

$$1 = \varepsilon(g) = \alpha^q(g) = \alpha(g)^q.$$ 

Therefore, $o(\alpha(g)) = 1$ and so $\alpha(g) = 1$. Since $k[G(H^*)]$ is a Hopf subalgebra of $H^*$, $k[G(H^*)]^\perp$ is a Hopf ideal of $H$. Let $B^+$ be the augmentation ideal of $k[G(H)]$. Then $B^+H = (g-1)H$ and

$$\alpha^i((g-1)h) = (\alpha(g)^i - 1)\alpha^i(h) = 0 \text{ for } h \in H, i \in \mathbb{Z}.$$ 

Therefore,

$$B^+H \subseteq k[G(H^*)]^\perp.$$ 

It follows from [Sch92, Theorem 2.4 (2a)] that $\text{dim } H/B^+H = q$. Thus,

$$\text{dim } B^+H = pq - q = k[G(H^*)]^\perp$$

and hence,

$$B^+H = k[G(H^*)]^\perp.$$ 

Therefore, $H/B^+H$ is isomorphic to $k[G(H^*)]^*$ as Hopf algebras. In particular, $H/B^+H$ is semisimple. Let $\Lambda$ be a non-zero left integral of $H$ and $\Lambda'$ a non-zero right integral of $k[G(H)]$. Since $\text{char } k = 0$, $\varepsilon(\Lambda') \neq 0$ and hence, $\Lambda'\Lambda = \varepsilon(\Lambda')\Lambda \neq 0$. Therefore, $\Lambda \notin B^+H$ and so $\Lambda + B^+H$ is a non-zero left integral in $H/B^+H$. Since $H/B^+H$ is semisimple, $\varepsilon(\Lambda) = \varepsilon(\Lambda + B^+H) \neq 0$. Hence, $H$ is semisimple. By [EG98], $H$ is trivial and so $|G(H)| = pq$, a contradiction. \openbullet

**Proposition 6.2** Let $H$ be a non-semisimple Hopf algebra of dimension $pq$ where $p \leq q$ are odd primes. Then

(i) the order of $S^4$ is $p$ and

(ii) $H$ is of index $p$. {11}
Proof. (i) Since \( H \) is not semisimple and \( \dim H \) is odd, by [LR95, Theorem 2.1] or [AS98, Lemma 2.5], \( S^4 \neq \text{id}_H \) and \( H, H^* \) cannot both be unimodular. Let \( g \) be the distinguished group-like element of \( H \) and let \( \alpha \) the distinguished group-like element of \( H^* \). Then, \( o(\alpha) < pq \) and \( o(\alpha) < pq \), for otherwise, \( H \) is isomorphic to a group algebra which is semisimple. By Lemma 6.1,

\[
\text{lcm}(o(g), o(\alpha)) = p \text{ or } q.
\]

By the equation (2.2.1) and (6.6.1), the order of \( S^4 \) is either \( p \) or \( q \). If \( p = q \), order of \( S^4 \) and the index of \( H \) are obviously equal to \( p \). We now assume \( q > p \). We consider the following cases:

Case (a): \( H^* \) is not unimodular. Suppose that the order of \( S^4 \) is \( q \). By equation (2.2.1), \( q \mid \text{lcm}(o(g), o(\alpha)) \). Therefore, \( \text{lcm}(o(g), o(\alpha)) = q \) and hence \( o(g) = 1 \) or \( q \). Thus, the index of \( H \) is also \( q \). Let \( \omega \in k \) be a \( q \)th primitive root of unity. By Lemma 5.1, for each \( j \in \mathbb{Z}_q \) there is an integer \( d_j \) such that

\[
\dim H^\omega_{0,i,j} - \dim H^\omega_{1,i,j} = d_j \quad \text{for all } i \in \mathbb{Z}_q.
\]

Let \( X_{i,j} = \text{min}(\dim H^\omega_{0,i,j}, \dim H^\omega_{1,i,j}) \). Then,

\[
\dim H^\omega_{0,i,j} + \dim H^\omega_{1,i,j} = 2X_{i,j} + |d_j|
\]

and so

\[
\sum_{(a,i)\in\mathbb{Z}_2\times\mathbb{Z}_q} H^\omega_{a,i,j} = \sum_{i\in\mathbb{Z}_q} 2X_{i,j} + q|d_j|
\]

for each \( j \in \mathbb{Z}_q \). It follows from Lemma 5.2 that

\[
\sum_{i\in\mathbb{Z}_q} 2X_{i,j} + q|d_j| = p.
\]

Since \( p \) odd, by (6.6.4), \( |d_j| \) must be odd. However, the left hand side of (6.6.4) is then strictly greater than \( p \), a contradiction! Therefore, \( o(S^4) = p \).

Case (b): \( H^* \) is unimodular. Then \( H^{**} \cong H \) is not unimodular. By Theorem 2.2, \( H^* \) is not semisimple and \( \dim H^* = pq \). It follows from Case (a) that the order of \( S^{v_4} \) is \( p \). Since \( o(S^4) = o(S^{v_4}) \). Therefore, \( o(S^4) = p \).

(ii) Let \( n \) be the index of \( H \). Then, by (3.3.1), \( n \mid \text{lcm}(o(g), o(\alpha)) \) and \( o(S^4) \mid n \). Since \( o(S^4) = p \) and \( \text{lcm}(o(g), o(\alpha)) = p \text{ or } q \), we have \( n = p \). ■

Lemma 6.3 Let \( H \) be a Hopf algebra over \( k \) such that both the distinguished group-like elements of \( H \) and \( H^* \) are of order \( p \) where \( p \) is an odd prime. Then, \( \text{Tr}(S^{2p}) = p^2d \) for some integer \( d \).

Proof. Let \( g \) and \( \alpha \) be the distinguished group-like elements of \( H \) and \( H^* \) respectively. Let \( B \) be the group algebra \( k[g] \). It follows from the arguments in the proof of [AS98, Theorem
A] that there is an Hopf algebra map $\pi : H \to B$ such that $\pi \gamma = \text{id}_B$ where $\gamma : B \to H$ is the inclusion map. Therefore, $H$ isomorphic to the biproduct $R \times B$ as Hopf algebras where

$$R = H^{coB} = \{ h \in H | (\text{id} \otimes \pi) \Delta(h) = h \otimes 1 \}$$

(cf. [Rad85]). It is shown in [AS98, section 4] that $R$ is invariant under $S_2$. Moreover, in the identification $H \cong R \otimes H$ given by multiplication, one has $S_2 = T \otimes \text{id}_B$.

(6.6.5)

Since $H$ is not unimodular, $H$ is not semisimple and hence $\text{Tr}(S^2) = 0$. By equation (6.6.4), $\text{Tr}(S^2) = \text{Tr}(T)p$. Therefore, $\text{Tr}(T) = 0$. Moreover, $T^{2p} = \text{id}_R$ as $S_4^p = \text{id}_H$ by equation (2.2.1). Hence, by Lemma 2.4, $\text{Tr}(T^p) = pd$ for some integer $d$. Since $S^2 = T^p \otimes \text{id}_B$, we have

$\text{Tr}(S^{2p}) = \text{Tr}(T^p)\text{Tr}(\text{id}_B) = p^2d$. $\blacksquare$

**Theorem 6.4** Let $H$ be a non-semisimple Hopf algebra of dimension $pq$ where $p \leq q$ are odd primes. Then $\text{Tr}(S^{2p}) = p^2d$ for some odd integer $d$.

**Proof.** By Proposition 6.2, $S_4^p = \text{id}_H$. Let

$$H_\pm = \{ h \in H | S^{2p}(h) = \pm h \}.$$ 

Then,

$$\dim H_+ - \dim H_- = \text{Tr}(S^{2p})$$

and

$$\dim H_+ + \dim H_- = pq.$$ 

Since $pq$ is odd, $\text{Tr}(S^{2p})$ is also an odd integer. Thus, if $\text{Tr}(S^{2p}) = p^2d$, then $d$ must be an odd integer. Therefore, it suffices to show that $\text{Tr}(S^{2p}) = p^2d$ for some integer $d$. Since $H$ is not semisimple, by Theorem 2.3, $H^*$ is also not semisimple. By Proposition 5.2, the indexes of $H$ and $H^*$ are both $p$. Since $\dim H$ is odd, by [LR93, Theorem 2.2], not both of $H$ and $H^*$ are unimodular. We then have the following three cases:

(i) If $H$ is unimodular and $H^*$ is not unimodular, the result follows from Lemma 5.3.

(ii) If $H$ is not unimodular and $H^*$ is unimodular, by Lemma 6.3, $\text{Tr}(S^{2p}) = p^2d$ for some odd integer $d$. The result follows from $\text{Tr}(S^{2p}) = \text{Tr}(S^{2p})$.

(iii) If both $H$ and $H^*$ are not unimodular, by Lemma 6.1 and Proposition 5.2, the orders of the distinguished group-like elements of $H$ and $H^*$ are both equal to $p$. Thus, by Lemma 6.3, $\text{Tr}(S^{2p}) = p^2d$. $\blacksquare$

As a consequence of the above theorem, we prove that any Hopf algebra of dimension $p^2$ is either a group algebra or a Taft algebra (see example 3.1(ii)).

**Theorem 6.5** Let $H$ be a Hopf algebra over $k$ of dimension $p^2$ where $p$ is any prime number. Then, $H$ is isomorphic to one of the following Hopf algebras:

- (a) $k[Z_{p^2}]$;
(b) $k[Z_p \times Z_p]$;

(c) $T(\omega)$, $\omega \in k$ a primitive $p$th of unity.

Proof. If $H$ is semisimple, it follows from [Mas96, Theorem 2] that $H$ isomorphic to $k[Z_p^2]$ or $k[Z_p \times Z_p]$. It is also shown in [Kap75] that if $H$ is a non-semisimple Hopf algebra of dimension 4, then $H$ isomorphic to the Taft algebra $T(1)$ or $T(-1)$. We may now assume $H$ is not semisimple and $p$ is odd. Let $S$ be the antipode of $H$. By Proposition 6.2, $S^{2p} = id_H$ and so $S^{2p}$ is diagonalizable and the possible eigenvalues of $S^{2p}$ are $\pm 1$. Suppose $S^{2p} \neq id_H$. Then, $\text{Tr}(S^{2p})$ is an integer such that

$$-p^2 \leq \text{Tr}(S^{2p}) < p^2.$$  

By Theorem 6.4,

$$\text{Tr}(S^{2p}) = p^2d$$

for some odd integer $d$. Therefore, $\text{Tr}(S^{2p}) = -p^2$ and hence $S^{2p} = -id_H$. However, this is not possible since $S^{2p}(1_H) = 1_H$. Therefore, $S^{2p} = id_H$. By Proposition 6.2, the order of $S^4$ is $p$, and so is the order $S^2$. It follows from [AS98, Theorem A(ii)] that $H$ is isomorphic to a Taft algebra of dimension $p^2$. Hence, $H \cong T(\omega)$ for some primitive $p$th root of unity, $\omega \in k$. 

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References

[AS98] Nicolás Andruskiewitsch and Hans-Jürgen Schneider, *Hopf algebras of order $p^2$ and braided Hopf algebras of order $p$*, J. Algebra **199** (1998), no. 2, 430–454. MR 99c:16033

[EG98] Pavel Etingof and Shlomo Gelaki, *Semisimple Hopf algebras of dimension $pq$ are trivial*, J. Algebra **210** (1998), no. 2, 664–669. MR 99k:16079

[Kap75] Irving Kaplansky, *Bialgebras*, Department of Mathematics, University of Chicago, Chicago, Ill., 1975, Lecture Notes in Mathematics.

[KR93] Louis H. Kauffman and David E. Radford, *A necessary and sufficient condition for a finite-dimensional Drinfel’d double to be a ribbon Hopf algebra*, J. Algebra **159** (1993), no. 1, 98–114. MR 94d:16037

[LR87] Richard G. Larson and David E. Radford, *Semisimple cosemisimple Hopf algebras*, Amer. J. Math. **109** (1987), no. 1, 187–195. MR 89a:16011

[LR88] , *Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple*, J. Algebra **117** (1988), no. 2, 267–289. MR 89k:16016
[LR95] _____, *Semisimple Hopf algebras*, J. Algebra **171** (1995), no. 1, 5–35. MR 96a:16040

[Mas96] Akira Masuoka, *The $p^n$ theorem for semisimple Hopf algebras*, Proc. Amer. Math. Soc. **124** (1996), no. 3, 735–737. MR 96f:16046

[Mon93] Susan Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

[Mon98] _____, *Classifying finite-dimensional semisimple Hopf algebras*, Trends in the representation theory of finite-dimensional algebras (Seattle, WA, 1997), Amer. Math. Soc., Providence, RI, 1998, pp. 265–279. MR 99k:16084

[NZ89] Warren D. Nichols and M. Bettina Zoeller, *A Hopf algebra freeness theorem*, Amer. J. Math. **111** (1989), no. 2, 381–385.

[Rad76] David E. Radford, *The order of the antipode of a finite dimensional Hopf algebra is finite*, Amer. J. Math. **98** (1976), no. 2, 333–355. MR 53 #10852

[Rad85] _____, *The structure of Hopf algebras with a projection*, J. Algebra **92** (1985), no. 2, 322–347. MR 86k:16004

[Rad90] _____, *The group of automorphisms of a semisimple Hopf algebra over a field of characteristic 0 is finite*, Amer. J. Math. **112** (1990), no. 2, 331–357. MR 91b:16048

[Rad94] _____, *The trace function and Hopf algebras*, J. Algebra **163** (1994), no. 3, 583–622. MR 95e:16039

[Sch92] Hans-Jürgen Schneider, *Normal basis and transitivity of crossed products for Hopf algebras*, J. Algebra **152** (1992), no. 2, 289–312.

[Swe69] Moss E. Sweedler, *Hopf algebras*, W. A. Benjamin, Inc., New York, 1969, Mathematics Lecture Note Series.

[Taf71] Earl J. Taft, *The order of the antipode of finite-dimensional Hopf algebra*, Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 2631–2633. MR 44 #4075

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