THE NEW MIXED HYPOEXPONENTIAL-G FAMILY

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ABSTRACT. Through viewing out the literature, many generated distributions took a new special form of probability density function (PDF) in which it is written as a linear combination of n other distributions. Therefore, we define in this paper a new type of distributions called "The New Mixed Distribution" form in which a distribution is written as a linear combination of n others and derive its characteristics. Second we construct "The New Mixed T-G family" a family of distributions following another new defined type "The New Mixed T-G Distribution". Third, we generate "The New Mixed Hypoexponential G-Family" out of 6 new mixed distributions with characterizing their PDFs, CDFs, hazard rate and reliability functions, MGF, and nth moment, and studying their maximum likelihood estimator and method of moments.

1. INTRODUCTION

The Hypoexponential distribution is a continuous distribution that interferes in everyday events by playing an important role in several fields such as queuing theory, telegraphic engineering, and stochastic processes. It is named Hypoexponential distribution as its coefficient is less than one and obtained by adding \( n \geq 2 \) independent Exponential random variables. As a result, the PDF of Hypoexponential distribution takes a special form that is a summation of \( n \) Exponential distributions. See [7].

In the same manner, statisticians reached the same special form of distributions when generating the PDF of ratio of 2 Hypoexponential distribution (2014), PDF of summation of Extension Exponential distribution by Kadri et al. (2022), See [3].

Based on what proceeds, in this paper, three are defined: New Mixed Distribution form, New Mixed T-G Family, and the New Mixed Hypoexponential G-Family.

This paper introduces the new type of distributions "New Mixed Distribution" in which a distribution PDF is written as a linear combination of \( n \) other known distributions. Then, we derive the general form of CDF, hazard rate and reliability functions, moment generating function, and moment of order \( k \), and generalize expressions for maximum likelihood estimator and method of moments of these distributions. Next, we construct the "New Mixed T-G Distribution" a type of distributions that follow the New Mixed Distribution form, with some specific characteristics. Out of previous, we construct a new family of distributions named the New Mixed T-G family and set its first example "The New Mixed Hypoexponential G-Family" which consists of the Mixed Hypoexponential Weibull distribution, Mixed...
Hypoexponential Frechet distribution, Mixed Hypoexponential Pareto distribution, Mixed Hypoexponential Power distribution, Mixed Hypoexponential Gumbel distribution, and Mixed Hypoexponential Extreme Value distribution. Finally, the PDF with the previously mentioned properties of these distributions are found in a direct defined manner according to the properties of the New Mixed Distribution form.

2. SOME PRELIMINARIES

2.1. Hypoexponential Distribution. Hypoexponential Distribution is a continuous distribution. If a random variable \( X \sim \text{Hypoexp}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) then \( X = \sum_{i=1}^{n} X_i \) where \( X_i \sim \text{Exp}(\lambda_i) \) for \( i = 1, 2, \ldots, n \).

**Theorem 1.** Let \( X_i \sim \text{Exp}(\alpha_i) \), \( i = 1, 2, \ldots, n \) be independent random variables with \( \alpha_i \neq \alpha_j \). Then \( S_n = \sum_{i=1}^{n} X_i \sim \text{hypoexp}(\vec{\alpha}) \), where \( \vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) has PDF

\[
f_{S_n}(t) = \sum_{i=1}^{n} \frac{f_{X_i}(t)}{P_i}, \quad t > 0
\]

where \( P_i = \prod_{j=1, j \neq i}^{n} (1 - \frac{\alpha_i}{\alpha_j}) \).

Next is the graph of PDF of Hypoexponential distribution for some values of \( k \) and \( \lambda_i \).

2.2. Definitions of some Random Variables. Next, we present 6 different random variables with their corresponding PDF, CDF, MGF, Reliability and Hazard Rate functions, and moment of order \( k \).

**Definition 1.** The Weibull distribution is a continuous probability distribution. Let \( X \sim \text{Weibull}(\lambda, k, \gamma) \), then \( X \) has the following PDF

\[
f(t) = \frac{\lambda_j}{k} \left( \frac{t - \gamma}{k} \right)^{k-1} e^{-\left( \frac{t - \gamma}{k} \right)^k}
\]

where \( f(t) > 0, \lambda > 0 \) is shape parameter, \( k > 0 \) is scale parameter, \( -\infty < \gamma < \infty \) is location parameter. The 2-parameter Weibull PDF is obtained by setting
\( \gamma = 0 \) and is given by
\[
f(t) = \frac{\lambda}{k} \left( \frac{t}{\lambda} \right)^{\lambda-1} e^{-\left( \frac{t}{\lambda} \right)^\lambda}
\]
with the following characteristics
\[
F(t) = (1 - e^{-\left( \frac{t}{\lambda} \right)^\lambda}), \quad R(t) = e^{-\left( \frac{t}{\lambda} \right)^\lambda}
\]
\[
h(t) = \frac{k}{\lambda} \left( \frac{t}{\lambda} \right)^{\lambda-1}, \quad \Phi(t) = \sum_{n=0}^{\infty} \frac{t^nk^n}{m^n} \Gamma \left( \frac{1+n}{\lambda} \right)
\]
\[
E[X^n] = k^n \Gamma(1 + \frac{k}{\lambda}).
\]

where \( \Gamma(s) \) is Gamma distribution defined as \( \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \). See [5].

**Definition 2.** *The Frechet distribution is also known as Inverse Weibull distribution. Let \( X \sim \text{Frechet}(k, \lambda, \gamma) \), then \( X \) has the following PDF
\[
f(t) = \frac{k}{\lambda} \left( \frac{t-\gamma}{\lambda} \right)^{-1-k} e^{-\left( \frac{t-\gamma}{\lambda} \right)^{-k}}
\]
where \( \lambda > 0 \) is scale parameter, \( k > 0 \) is shape parameter, \( -\infty < \gamma < \infty \) is location parameter. Taking the location parameter \( \gamma = 0 \) then the PDF of 2-parameter Frechet distribution is given by
\[
f(t) = \frac{k}{\lambda} \left( \frac{t}{\lambda} \right)^{-1-k} e^{-\left( \frac{t}{\lambda} \right)^{-k}}
\]
and the CDF, Reliability Function, MGF, and nth moment are as the following
\[
F(t) = e^{-\left( \frac{t}{\lambda} \right)^{-k}}, \quad R(t) = e^{-\left( e^{\left( \frac{t}{\lambda} \right)^{-1}} \right)^{-k}}
\]
\[
\Phi(t) = \sum_{m=0}^{\infty} \frac{k^m e^m}{m^m} \Gamma \left( \lambda - \frac{m}{\lambda} \right), \quad E[X^n] = \Gamma(1 - \frac{n}{k}).
\]
for \( \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \). See [4].

**Definition 3.** Consider the random variable \( X \) following Pareto distribution, then \( X \sim \text{pareto}(k, \lambda) \) of first kind has the following PDF
\[
f(t) = \frac{k \lambda}{k t^{k+1}}
\]
where \( k > 0 \) is shape parameter and \( \lambda > 0 \) is scale parameter. Its properties are given as
\[
F(t) = (1 - \left( \frac{k}{t} \right)^k), \quad R(t) = \left( \frac{k}{t} \right)^k \Gamma(-k, -\lambda t), \quad E[X^n] = \lambda^n \frac{k}{k-n}.
\]
where \( \Gamma(n, \theta t) \) is the Upper incomplete Gamma function defined as \( \Gamma(n, \theta t) = (n-1)! \sum_{k=0}^{n-1} \frac{(\theta t)^k}{k!} e^{-\theta t} \). See [6].

**Definition 4.** Let \( X \) follows Power distribution i.e. \( X \sim \text{power}(k, \lambda) \) where \( k \) is the domain parameter and \( \lambda \) is the shape parameter. Its PDF is given as
\[
f(t) = \lambda k^t \lambda^{-1+B}
\]
whereas its CDF, reliability function, MGF, nth moment are given as

\[
F(t) = \begin{cases} \left(\frac{t}{\lambda}\right)^\lambda & 0 < t < \frac{1}{\lambda} \\ 1 & t > \frac{1}{\lambda} \end{cases}, \quad R(t) = k^\lambda t^\lambda \text{ for } 0 < t < \frac{1}{\lambda}, \\
\Phi(t) = \frac{-k^\lambda}{\lambda} \left[\Gamma\left(\frac{t}{\lambda}, \frac{1}{\lambda}\right) - \Gamma(\lambda)\right], \quad E[X^n] = k^{-n} \frac{1}{n+\lambda},
\]

where \(\Gamma(n, \theta t)\) is the Upper incomplete Gamma function defined as \(\Gamma(n, \theta t) = \sum_{k=0}^{(n-1)!} \frac{(\theta t)^k}{k!} e^{-\theta t}\).

**Definition 5.** Consider the Gumbel distribution or Generalized Extreme Value distribution Type-1, let \(X \sim \text{Gumbel}(k, \lambda)\) then \(X\) has the following PDF

\[
f(t) = \frac{1}{\lambda} e^{-\left(\frac{t+k}{\lambda}\right)},
\]

where \(\lambda > 0\) is scale parameter and \(k \in \mathbb{R}\) is location parameter with the following properties

\[
F(t) = e^{-\left(\frac{t+k}{\lambda}\right)}, \quad R(t) = \left(-e^{-\left(\frac{t+k}{\lambda}\right)}\right), \\
\Phi(t) = \Gamma(1-\lambda t)e^{\lambda t}, \quad E[X^n] = \int e^{\left(\frac{t-k}{\lambda}\right)} - e^{-\left(\frac{t-k}{\lambda}\right)} dt.
\]

for \(\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt\). See [2].

**Definition 6.** Let \(X\) follows the Extreme Value distribution i.e. \(X \sim \text{Extreme Value Distribution}(k, \lambda)\) then it has the following PDF

\[
f(t) = e^{\left(-\frac{t-k}{\lambda}\right)},
\]

where \(\lambda > 0\) is scale parameter and \(k \in \mathbb{R}\) is location parameter. Its CDF, Reliability function, MGF, and nth moment are as follows

\[
F(t) = e^{-\left(\frac{t-k}{\lambda}\right)}, \quad R(t) = \left(1 - e^{\left(-\frac{t-k}{\lambda}\right)}\right), \\
\Phi(t) = \int e^{xt} \left(-\frac{e^{-\frac{t-k}{\lambda}}}{\lambda} + \frac{e^{-\frac{t-k}{\lambda}}}{\lambda}\right) dx, \quad E[X^n] = \int_{-\infty}^{\infty} t^n e^{\left(-\frac{t-k}{\lambda}\right)} dt.
\]

3. **The New Mixed Distribution**

In this section we propose a new type of distributions called The New Mixed Distribution in which a distribution is written as a linear combination of \(n\) others. In addition, we derive the different properties for this type such as PDF, CDF, MGF, nth moment, reliability and hazard rate functions, and also we discuss some parameter estimations.

**Definition 7.** Let \(X_i, i \in I \subset \mathbb{N}\) be vector of random variables that follow a distribution. A distribution \(Y\) is called a Mixed Distribution of \(X_i\) if it has PDF of the form

\[
f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t)
\]

where \(A_i \in \mathbb{R}\) and \(\sum_{i \in I} A_i = 1\). Whenever \(A_i \geq 0\) this distribution is the known mixture distribution of the random variables \(X_i\) where \(A_i\) is the weight probability
distribution along $X_i$. However, our new distribution represents a general expression having $A_i \in \mathbb{R}$.

3.1. Properties of New Mixed distribution. Regarding this special form of PDF we are able to generalize the CDF, the reliability and hazard rate functions, MGF and nth moment of each distribution having the New Mixed Distribution form.

**Theorem 2.** Let $X_i, i \in I \subseteq \mathbb{N}$ follows a known distribution and given a new distribution $Y$ such that

$$f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t)$$

then its CDF is given as

$$F_Y(t) = \sum_{i \in I} A_i F_{X_i}(t).$$

**Proof.** Let $f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t)$, where $Y$ is a continuous distribution then its CDF is given as

$$F_Y(t) = \int_{-\infty}^{t} f_Y(x)dx = \int_{-\infty}^{t} \sum_{i \in I} A_i f_{X_i}(x)dx = \sum_{i \in I} A_i \int_{-\infty}^{t} f_{X_i}(x)dx = \sum_{i \in I} A_i F_{X_i}(t).$$

Now, suppose that $Y$ is a discrete distribution then

$$F_Y(t) = \sum_{x \leq t} f_Y(x) = \sum_{x \leq t} \sum_{i \in I} A_i f_{X_i}(x) = \sum_{i \in I} A_i \sum_{x \leq t} f_{X_i}(x) = \sum_{i \in I} A_i F_{X_i}(t).$$

□

**Theorem 3.** Let $X_i, i \in I \subseteq \mathbb{N}$ follows a known distribution and given a new distribution $Y$ such that

$$f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t)$$

then its reliability function is given as

$$R_Y(t) = \sum_{i \in I} A_i R_{X_i}(t).$$

**Proof.** Let $f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t)$, first consider $Y$ is a continuous distribution then the reliability function of $Y$ is

$$R_Y(t) = \int_{t}^{\infty} f_Y(s)ds = \int_{t}^{\infty} \sum_{i \in I} A_i f_{X_i}(s)ds = \sum_{i \in I} A_i \int_{t}^{\infty} f_{X_i}(s)ds = \sum_{i \in I} A_i R_{X_i}(t).$$

Next, consider $Y$ as a discrete distribution then

$$R_Y(t) = \sum_{x > t} f_Y(x) = \sum_{x > t} \sum_{i \in I} A_i f_{X_i}(x) = \sum_{i \in I} A_i \sum_{x > t} f_{X_i}(x) = \sum_{i \in I} A_i R_{X_i}(t).$$

□
Corollary 1. Let \( X_i, i \in I \subset \mathbb{N} \) follows a known distribution and given a new distribution \( Y \) such that

\[
f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t)
\]

then its hazard rate function is given as

\[
h_Y(t) = \frac{\sum_{i \in I} A_i F_{X_i}(t)}{\sum_{i \in I} A_i R_{X_i}(t)}.
\]

Proof. Let \( f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t) \), then the hazard rate function of \( Y \) is

\[
h_Y(t) = \frac{f_Y(t)}{R_Y(t)} = \frac{\sum_{i \in I} A_i f_{X_i}(t)}{\sum_{i \in I} A_i R_{X_i}(t)}.
\]

\( \square \)

Theorem 4. Let \( X_i, i \in I \subset \mathbb{N} \) follows a known distribution and given a new distribution \( Y \) such that

\[
f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t)
\]

then its moment generating function MGF is given as

\[
\Phi_Y(t) = \sum_{i \in I} A_i \Phi_{X_i}(t).
\]

Proof. Let \( f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t) \), first consider \( Y \) is a continuous distribution then its MGF is

\[
\Phi_Y(t) = \int_{-\infty}^{+\infty} e^{tx} f_Y(x) dx = \int_{-\infty}^{+\infty} e^{tx} \sum_{i \in I} A_i f_{X_i}(x) dx = \sum_{i \in I} A_i \int_{-\infty}^{+\infty} e^{tx} f_{X_i}(x) dx = \sum_{i \in I} A_i \Phi_{X_i}(t)
\]

while in case \( Y \) is discrete its MGF is

\[
\Phi_Y(t) = \sum_{j \in J} e^{tx_j} f_Y(x_j) = \sum_{j \in J} e^{tx_j} \sum_{i \in I} A_i f_{X_i}(x_j) = \sum_{i \in I} A_i \sum_{j \in J} e^{tx_j} f_{X_i}(x_j) = \sum_{i \in I} A_i \Phi_{X_i}(t)
\]

\( \square \)

Theorem 5. Let \( X_i, i \in I \subset \mathbb{N} \) follows a known distribution and given a new distribution \( Y \) such that

\[
f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t)
\]

then its moment of order \( k \) is

\[
E[Y^k] = \sum_{i \in I} A_i E[X_i^k].
\]

Proof. Let \( f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t) \), where \( Y \) is a continuous distribution then its moment of order \( k \) is

\[
E[Y^k] = \int_{-\infty}^{+\infty} x^k f_Y(x) dx = \int_{-\infty}^{+\infty} x^k \sum_{i \in I} A_i f_{X_i}(x) dx = \sum_{i \in I} A_i \int_{-\infty}^{+\infty} x^k f_{X_i}(x) dx = \sum_{i \in I} A_i E[X_i^k]
\]
similarly, in case Y is discrete then
\[ E[Y^k] = \sum_{j \in J} x_j^k f_Y(x_j) = \sum_{j \in J} x_j^k \sum_{i \in I} A_i f_{X_j}(x_j) = \sum_{i \in I} A_i \sum_{j \in J} x_j^k f_{X_j}(x_j) = \sum_{i \in I} A_i E[X_j^k]. \]

Lemma 1. Consider the mixed distribution Y such that \( f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t) \) where \( i \in I \subset \mathbb{N} \) and \( X_i \) are independent random variables. Then \( \sum_{i \in I} A_i = 1 \).

Proof. Consider \( f_Y(t) = \sum_{i \in I} A_i f_{X_i}(t) \) the probability density function of Y then from Theorem 2 \( F_Y(t) = \sum_{i \in I} A_i F_{X_i}(t) \) is the CDF of Y. Then,
\[
\lim_{t \to \infty} F_Y(t) = 1 = \lim_{t \to \infty} \sum_{i \in I} A_i F_{X_i}(t) = \sum_{i \in I} A_i \lim_{t \to \infty} F_{X_i}(t) = 1
\]
as \( F_{X_i}(t) \) is the CDF of \( X_i \) then \( \lim_{t \to \infty} F_{X_i}(t) = 1 \). Therefore,
\[
\sum_{i \in I} A_i = 1.
\]

Therefore, in this part we defined the New Mixed Distribution type and generalized its CDF, reliability function, hazard rate function, MGF and nth moment.

3.2. Parameter Estimation of New Mixed distribution. In addition to what proceeds, we present the following methods to generalize the estimation of parameters of any distribution of this form.

3.2.1. Maximum Likelihood Estimation.

Theorem 6. Given a mixed distribution Y of parameter \( \theta \) such that \( f_Y(x) = \sum_{j \in J} A_j f_{X_j}(x) \) where \( X_j, j \in J \subset \mathbb{N} \) are independent random variables that follow a known distribution. Given independent observations \( x_1, x_2, ..., x_n \) then the maximum likelihood estimator \( \hat{\theta} \) is that which maximizes the likelihood function
\[
L(x_1, x_2, ..., x_n; \theta) = f_Y(x_1, \theta) f_Y(x_2, \theta) ... f_Y(x_n, \theta)
\]
is obtained by solving the equality
\[
\sum_{i=1}^{n} \sum_{j \in J} \frac{\partial}{\partial \theta} A_j f_{X_j}(x_i | \theta) = 0.
\]

Given a distribution Y of parameter \( \theta \) such that \( f_Y(x) = \sum_{j \in J} A_j f_{X_j}(x) \) where \( X_j, j \in J \subset \mathbb{N} \) be independent random variables that follow a known distribution.
Suppose that the random sample $x_1, x_2, ..., x_n$ is taken from the distribution then to find the maximum likelihood estimate of $\theta$ given that

$$L(x_1, x_2, ..., x_n; \theta) = \prod_{i=1}^{n} f_Y(x_i; \theta)$$

Proof. second,

$$\ln L(x_1, x_2, ..., x_n; \theta) = \ln \prod_{i=1}^{n} f_Y(x_i; \theta)$$

$$= \sum_{i=1}^{n} \ln f_Y(x_i; \theta)$$

$$= \sum_{i=1}^{n} \ln \sum_{j \in J} A_j f_{X_j}(x_i; \theta)$$

third for parameter $\theta$

$$\frac{\partial \ln L(x_1, x_2, ..., x_n; \theta)}{\partial \theta} = \frac{\partial}{\partial \theta} \sum_{i=1}^{n} \ln \sum_{j \in J} A_j f_{X_j}(x_i; \theta)$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial \theta} \ln \sum_{j \in J} A_j f_{X_j}(x_i; \theta)$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial \theta} \sum_{j \in J} A_j f_{X_j}(x_i; \theta)$$

$$= \sum_{i=1}^{n} \sum_{j \in J} \frac{\partial}{\partial \theta} A_j f_{X_j}(x_i; \theta)$$

fourth

$$\sum_{i=1}^{n} \frac{\partial}{\partial \theta} A_j f_{X_j}(x_i; \theta) \sum_{j \in J} A_j f_{X_j}(x_i; \theta) = 0.$$

3.2.2. Method of Moments.

**Theorem 7.** Suppose the problem is to estimate $n$ unknown parameters $\theta_1, ..., \theta_n$ characterizing the distribution

$$f_Y(x; \theta) = \sum_{i \in I} A_i f_{X_i}(x; \theta)$$

of the random variable $Y(\theta_1, ..., \theta_n)$ where $X_i, i \in I \subset \mathbb{N}$ are independent random variables that follow a known distribution. Then, the first $n$ moments are expressed
as follows

\[ \mu_1 = g_1(\theta_1, ..., \theta_n) = E[Y^1] = \sum_{i \in I} A_i E[X_i^1] \]
\[ \mu_2 = g_2(\theta_1, ..., \theta_n) = E[Y^2] = \sum_{i \in I} A_i E[X_i^2] \]
\[ \vdots \]
\[ \mu_n = g_n(\theta_1, ..., \theta_n) = E[Y^n] = \sum_{i \in I} A_i E[X_i^n] \]

Suppose a sample of size \( m \) is drawn, resulting in the values \( y_1, y_2, ..., y_m \) for \( k = 1, 2, ..., n \) let

\[ \hat{\mu}_k = \frac{1}{m} \sum_{j=1}^{m} y_j^k \]

be the \( k \)-th sample moment, an estimate of \( \mu_k \). The method of moments estimator for \( \theta_1, ..., \theta_n \) denoted by \( \hat{\theta}_1, ..., \hat{\theta}_n \) is defined as the solution to the equation

\[ \hat{\mu}_k = g_k(\hat{\theta}_1, ..., \hat{\theta}_n) \quad k = 1, 2, ..., n. \]

Finally, in this section we were able to generate a new type of distributions called "The New Mixed Distribution" and examine different properties regarding its form such as PDF, MGF, reliability and hazard rate functions, and moment of order \( k \) then generalize 2 methods of estimations which are maximum likelihood estimation and method of moments.

4. The New Mixed T-G Family

In this section, we generate from the mixed distribution "T" a new type of distributions named the New Mixed T-G Distribution which is a mixed distribution, leading to construct The New Mixed T-G Family of same mother mixed distribution T.

4.1. The New Mixed T-G Distribution. Next, we define the second New type of distributions which is "The New Mixed T-G Distribution" and generalize some of its properties.

**Theorem 8.** Given a random variable \( Z \) with a probability density function \( f_Z(t) = \sum_{i \in I} A_i f_{X_i}(t) \), where \( X_i, i \in I \subset \mathbb{N} \) be vector of random variables and \( A_i \in \mathbb{R} \). Let \( Y = g(X) \) be a 1-1 function of a random variable \( X \). Then

\[ \sum_{i \in I} A_i f_{g(X_i)}(y) \]

is a valid distribution generated from the mixed distribution of \( X_i \).

**Proof.** The aim is to prove that \( \sum_{i \in I} A_i f_{g(X_i)}(t) \) is a valid PDF. First, suppose that \( g(X_i) \) is continuous. We start by proving the expression is positive. We have

\[ f_{g(X_i)}(y) = \sum_{j=1}^{k} f_{X_i}(w_j(y)) |J_j| = \sum_{j=1}^{k} f_{X_i}(x_j) |J_j| \]
where \( J_i = w_i'(y) \) Now, \( \sum_{i \in I} A_i f_{g(X_i)}(y) = \sum_{i \in I} A_i \sum_{j=1}^{k} f_{X_i}[x_j] \) \( J_i \) = \( \sum_{j=1}^{k} f_{Z}(x_j) \) \( J_i \). Now, since \( f_{Z}(x) \) is a PDF of \( Z \), then \( f_{Z}(x) \geq 0 \) and \( |J_i| \geq 0 \). Thus \( \sum_{j=1}^{k} f_{Z}(x_j) \leq |J_i| \geq 0 \). Next, we need to prove that \( \int_{R} f_T(y)dy = 1 \). We have \( \int_{R} f_T(y)dy = \int_{R} \sum_{i \in I} A_i f_{g(X_i)}(y)dy = \sum_{i \in I} A_i f_{g(X_i)}(y)dy = \sum_{i \in I} A_i \) as \( \int_{R} f_{g(X_i)}(y)dy = 1 \). Also from Definition \( \sum_{i \in I} A_i = 1 \). Hence \( \int_{R} f_T(y)dy = 1 \). Therefore, \( \sum_{i \in I} A_i f_{g(X_i)}(t) \) is a valid PDF.

Second, we will reprove the previous but for \( g(X_i) \) is discrete random variable. First,

\[
\sum_{j=1}^{k} f_{X_i}[x_j] = \sum_{j=1}^{k} f_{X_i}[x_j]
\]

And, \( \sum_{i \in I} A_i f_{g(X_i)}(y) = \sum_{i \in I} A_i \sum_{j=1}^{k} f_{X_i}[x_j] = \sum_{j=1}^{k} A_i f_{X_i}[x_j] = \sum_{j=1}^{k} A_i f_{Z}(x_j) \).

As \( f_{Z}(x) \) is a PDF of \( Z \), then \( f_{Z}(x) \geq 0 \), thus \( \sum_{j=1}^{k} A_i f_{X_i}[x_j] = \sum_{j=1}^{k} A_i f_{Z}(x_j) \).

Still to prove that \( \sum_{y} f_T(y) = 1 \). We have \( \sum_{y} f_T(y) = \sum_{y \in I} \sum_{i \in I} A_i f_{g(X_i)}(y) = \sum_{y \in I} A_i \sum_{i \in I} f_{g(X_i)}(y) = 1 \). Also from Definition \( \sum_{i \in I} A_i = 1 \). Hence \( \sum_{y} f_T(y) = 1 \). Therefore, \( \sum_{i \in I} A_i f_{g(X_i)}(t) \) is a valid PDF.

**Definition 8.** Given our Mixed distribution of \( X_i \), having a PDF

\[
\sum_{i \in I} A_i f_{X_i}(t)
\]

that follows a distribution named "T" and suppose that \( g(X_i) \) is a known unique distribution \( g(X_i) \) named "G". We denote the Mixed T-G distribution generated from T as the distribution having a PDF

\[
\sum_{i \in I} A_i f_{g(X_i)}(y).
\]

**4.2. Properties of New Mixed T-G Distribution.** We point out that the mixed T-G distribution having a PDF \( \sum_{i \in I} A_i f_{g(X_i)}(y) \) is a mixed distribution thus we may consider that all the properties in Theorems 2, 3, 1, 4, 5 may be used for our new distribution. Thus we state some important results. The CDF of our new Mixed T-G distribution is

\[
\sum_{i \in I} A_i F_{g(X_i)}(y),
\]

the reliability function is

\[
\sum_{i \in I} A_i R_{g(X_i)}(y)
\]

the hazard rate function

\[
\frac{\sum_{i \in I} A_i f_{g(X_i)}(t)}{\sum_{i \in I} A_i R_{g(X_i)}(t)}
\]
MGF is
\[ \sum_{i \in I} A_i \Phi_g (X_i) (t) \]
and moment of order k is
\[ \sum_{i \in I} A_i E[g(X_i)^k]. \]

In this part, we generate the second type of distributions "The New Mixed T-G distribution" out of a mixed distribution T of X_i and a transformation G of X_i then it is a mixed distribution of transformation of X_i. Moreover, the CDF, MGF, reliability and hazard rate functions, and moment of order k for this type are generalized. Finally, We end this Section to point out that the New Mixed T-G Family is generated by fixing a parent mixed distribution T of X_i then substituting X_i by its transformations G to obtain different Mixed T-G Distributions belonging to the same family.

5. The New Mixed Hypoexponential-G Family

In this section we adopt the Hypoexponential distribution with different parameters to be the parent distribution T of the Mixed Hypoexponential T-G Family. This can be defined as the Hypoexponential distribution is a Mixed distribution of the exponential distribution from Definition 7. Therefore, we generate some distributions and examine deep the properties obtained for these Mixed distributions.

5.1. The Mixed Hypoexponential Weibull Distribution. In this section we introduce a new distribution denoted by the Mixed Hypoexponential Weibull distribution. This distribution is generated from the Hypoexponential distribution with different parameters with a Weibull transform distribution.

We start from the Hypoexponential distribution with different parameters \( S_n \sim hypexp(\alpha) \), where \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \) and \( \alpha_i \neq \alpha_j \). The PDF of \( S_n \) is given from Theorem 1 as
\[ f_{S_n} (t) = \sum_{i=1}^{n} \frac{f_{X_i} (t)}{P_i}, \quad t > 0 \]
where \( X_i \sim \text{Exp}(\alpha_i) \) and \( P_i = \prod_{j=1, j \neq i}^{n} \left( 1 - \frac{\alpha_i}{\alpha_j} \right) \).

On the other hand we use the transform distribution of \( X_i \sim \text{Exp}(\alpha_i) \), as \( Y_i = X_i^c \sim \text{Weibull}(\frac{1}{c}, \frac{1}{\alpha_i}) \) where \( c > 0 \). Now, suppose that \( k = \frac{1}{c}, \lambda_i = \frac{1}{\alpha_i} \), then \( \alpha = \frac{1}{X_i^c} = \frac{1}{X^k} \) and \( c = \frac{1}{k} \) and we may write \( Y_i \sim \text{Weibull}(k, \lambda_i), \ i = 1, 2, ..., n. \)

Thus we write \( P_i = \prod_{j=1, j \neq i}^{n} \left( 1 - \frac{\alpha_i}{\alpha_j} \right) = \prod_{j=1, j \neq i}^{n} \left( 1 - \frac{\lambda_j}{\lambda_i} \right) = \prod_{j=1, j \neq i}^{n} \left( 1 - \left( \frac{\lambda_j}{\lambda_i} \right)^k \right) \), we call this \( PW_i \) which is transformed from \( P_i \). Therefore, we obtain our new distribution, the Mixed Hypoexponential Weibull distribution \( Z \sim MHW(k, \lambda_1, \lambda_2, ..., \lambda_n), k, \lambda_i > 0 \) with PDF
\[ f_Z (t) = \sum_{i=1}^{n} \frac{1}{PW_i} f_{Y_i} (t) \]
where \( Y_i \sim \text{Weibull}(k, \lambda_i) \) and \( PW_i = \prod_{j=1, j \neq i}^{n} \left( 1 - \left( \frac{\lambda_i}{\lambda_j} \right)^k \right) \).
5.1.1. The Properties of Mixed Hypoexponential Weibull Distribution. Here we generalize the CDF, MGF, Reliability and Hazard Rate functions, for the Mixed Hypoexponential Weibull Distribution.

**Theorem 9.** Let $Z \sim MHW(k, \lambda_1, \lambda_2, ..., \lambda_n)$ then according to Theorems 2, 3, 4 and 5 $Z$ has the following properties

$$F_X(t) = \sum_{i=1}^{n} \frac{1}{PW_i} F_{Y_i}(t)$$

$$R_X(t) = \sum_{i=1}^{n} \frac{1}{PW_i} R_{Y_i}(t)$$

$$h_X(t) = \sum_{i=1}^{n} \frac{1}{PW_i} h_{Y_i}(t)$$

$$E[Z^h] = \sum_{i=1}^{n} \frac{1}{PW_i} E[Y_i^h]$$

where $Y_i \sim Weibull(k, \lambda_i)$ and $PW_i = \prod_{j=1, j \neq i}^{n} \left(1 - \left(\frac{\lambda_j}{\lambda_i}\right)^k\right)$.

**Corollary 2.** Let $Z \sim MHW(k, \lambda_1, \lambda_2, ..., \lambda_n)$ then according to Theorem 9 the CDF, reliability and hazard rate functions, MGF and moment of order $h$ of $Z$ are

$$F_Z(t) = \sum_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - \left(\frac{\lambda_j}{\lambda_i}\right)^k\right) e^{-\left(\frac{t}{\lambda_i}\right)^k}$$

$$R_Z(t) = \sum_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - \left(\frac{\lambda_j}{\lambda_i}\right)^k\right) e^{-\left(\frac{t}{\lambda_i}\right)^k}$$

$$h_Z(t) = \sum_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - \left(\frac{\lambda_j}{\lambda_i}\right)^k\right) \frac{e^{-\left(\frac{t}{\lambda_i}\right)^k} k\lambda_i^k}{\Gamma(1+\frac{k}{\lambda_i})}$$

$$E[Z^h] = \sum_{i=1}^{n} \prod_{j=1, j \neq i}^{n} \left(1 - \left(\frac{\lambda_j}{\lambda_i}\right)^k\right) \frac{k^h \Gamma(1+\frac{k}{\lambda_i})}{\Gamma(1+\frac{k}{\lambda_i})}$$

Next, we give some different graphs of PDF and CDF for some values of $k$ and $\lambda_i$, showing the flexibility of adding extra parameters to a new distribution.

**Figure 2.** PDF and CDF of different distributions of $MHW(k, \lambda_1, \lambda_2, ..., \lambda_n)$

5.2. The Mixed Hypoexponential Frechet Distribution. The second distribution of the New Mixed Hypoexponential- G Family.
Theorem 10. Given $T \sim \text{hypoexp}(\alpha)$, then $f_T(t) = \sum_{i=1}^{n} \frac{f_{X_i}(t)}{P_i}$, $t > 0$ where $X_i \sim \text{Exp}(\alpha_i)$. Take $Y_i = X_i^{-c} \sim \text{Frechet}(\frac{1}{c}, \alpha_i^c)$ where $\alpha_i > 0$, $c > 0$. Then the Mixed Hypoexponential Frechet distribution $Z$, i.e. $Z \sim \text{MHF}(k, \lambda_1, \lambda_2, \ldots, \lambda_n)$, $k, \lambda_i > 0$, has PDF

$$f_Z(t) = \sum_{i=1}^{n} \frac{1}{PF_i} f_{Y_i}(t)$$

where $Y_i \sim \text{Frechet}(k, \lambda_i)$ and $PF_i = \prod_{j=1, j \neq i}^{n} (1 - \left(\frac{\lambda_i}{\lambda_j}\right)^k)$. 

Corollary 3. Let $Z \sim \text{MHF}(k, \lambda_1, \lambda_2, \ldots, \lambda_n)$ then

$$f_Z(t) = \sum_{i=1}^{n} \frac{1}{PF_i} f_{Y_i}(t)$$

where $Y_i \sim \text{Frechet}(k, \lambda_i)$, $i = 1, 2, \ldots, n$ be $n$ random variables with $\lambda_i \neq \lambda_j$ and $PF_i = \prod_{j=1, j \neq i}^{n} (1 - \left(\frac{\lambda_i}{\lambda_j}\right)^k)$. Then according to Theorem 3, $Z$ has the following CDF

$$F_Z(t) = \sum_{i=1}^{n} \frac{1}{PF_i} F_{Y_i}(t) = \frac{e^{-\left(\frac{t}{\lambda_i}\right)^k}}{\prod_{j=1, j \neq i}^{n} (1 - \left(\frac{\lambda_i}{\lambda_j}\right)^k)}$$

Next, we give some different graphs of PDF and CDF for some values of $k$ and $\lambda_i$, showing the flexibility of adding extra parameters to a new distribution.

![Figure 3. PDF and CDF of different distributions of MHF(k,\lambda_1, \lambda_2, \lambda_3)](image)

5.3. The Mixed Hypoexponential Pareto Distribution.

Theorem 11. Let $T \sim \text{hypoexp}(\alpha)$ then $f_T(t) = \sum_{i=1}^{n} \frac{f_{X_i}(t)}{P_i}$, $t > 0$ where $X_i \sim \text{Exp}(\alpha_i)$ and $P_i = \prod_{j=1, j \neq i}^{n} (1 - \frac{\alpha_i}{\alpha_j})$. Consider $Y_i = ce^{X_i} \sim \text{Pareto}(c, \alpha_i)$ where $\alpha_i > 0$, $c > 0$. Then, our new distribution is the Mixed Hypoexponential Pareto distribution $Z \sim \text{MHT}(k, \lambda_1, \lambda_2, \ldots, \lambda_n)$ with PDF

$$f_Z(t) = \sum_{i=1}^{n} \frac{1}{PF_i} f_{Y_i}(t)$$
where \( Y_i \sim \text{Pareto}(k, \lambda_i) \) and \( PT_i = \prod_{j=1, j \neq i}^{n} (1 - \frac{\lambda_j}{\lambda_i}). \)

**Corollary 4.** Let \( Z \sim \text{MHT}(k, \lambda_1, \lambda_2, ..., \lambda_n) \). Then according to Theorem 2 the CDF of \( Z \) is

\[
F_Z(t) = \sum_{i=1}^{n} \frac{1}{PT_i} F_{Y_i}(t) = \sum_{i=1}^{n} \frac{(1 - (\frac{\lambda_i}{t})^k)}{\prod_{j=1, j \neq i}^{n} (1 - \frac{\lambda_j}{\lambda_i})}
\]

Here are the graphs of PDF and CDF for some values of \( k \) and \( \lambda_i \), showing the flexibility of adding extra parameters to New Mixed Hypoexponential Pareto distribution.

**Figure 4.** PDF and CDF of different distributions of MHT\((k, \lambda_1, \lambda_2, \lambda_3)\)

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5.4. The Mixed Hypoexponential Power Distribution. Fourth is the Mixed Hypoexponential Power distribution.

**Theorem 12.** Let \( T \sim \text{hypo}(\alpha) \), then \( f_T(t) = \sum_{i=1}^{n} \frac{f_{X_i}(t)}{P_i} \), \( t > 0 \) where \( X_i \sim \text{Exp}(\alpha_i) \). Consider \( Y_i = ce^{-X_i} \sim \text{Power}(\frac{1}{c}, \alpha_i) \) where \( \alpha_i > 0, c > 0, \) domain \((0, c)\). Then the Mixed Hypoexponential Power distribution \( Z \sim \text{MHP}(k, \lambda_1, \lambda_2, ..., \lambda_n) \) has PDF

\[
f_Z(t) = \sum_{i=1}^{n} \frac{1}{PP_i} f_{Y_i}(t)
\]

where \( Y_i \sim \text{Power}(k, \lambda_i) \) and \( PP_i = \prod_{j=1, j \neq i}^{n} (1 - \frac{\lambda_j}{\lambda_i}) \).

**Corollary 5.** Let \( Z \sim \text{MHP}(k, \lambda_1, \lambda_2, ..., \lambda_n) \) then CDF of \( Z \) is

\[
F_Z(t) = \sum_{i=1}^{n} \frac{1}{PP_i} F_{Y_i}(t) = \begin{cases} \sum_{i=1}^{n} \frac{(kt)^{\lambda_i}}{\prod_{j=1, j \neq i}^{n} (1 - \frac{\lambda_j}{\lambda_i})} & 0 < t < \frac{1}{k} \\ \sum_{i=1}^{n} \frac{1}{\prod_{j=1, j \neq i}^{n} (1 - \frac{\lambda_j}{\lambda_i})} & t > \frac{1}{k} \end{cases}
\]

The following are the graphs of PDF and CDF of Mixed Hypoexponential distribution for some values of \( k \) and \( \lambda_i \).
Figure 5. PDF and CDF of different distributions of MHP(k,\(\lambda_1, \lambda_2, \lambda_3\)).

5.5. The Mixed Hypoexponential Gumbel Distribution. Next is the fifth distribution of the Mixed Hypoexponential - G Family.

**Theorem 13.** Let \(T \sim \text{hypoexp}(\alpha)\), then \(f_T(t) = \sum_{i=1}^{n} \frac{f_{X_i}(t)}{P_i}, t > 0\) Consider \(Y_i = c \ln X_i \sim \text{Gumbel}(-c \ln (\alpha_i), c)\) where \(\alpha_i > 0, c > 0\), domain \(\mathbb{R}\). Then, the Mixed Hypoexponential Gumbel distribution \(Z \sim \text{MHG}(k_1, k_2, ..., k_n, \lambda)\) has PDF

\[
f_Z(t) = \sum_{i=1}^{n} \frac{1}{P_{G_i}} f_{Y_i}(t)
\]

where \(Y_i \sim \text{Gumbel}(k_i, \lambda)\) and \(P_{G_i} = \prod_{j=1, j \neq i}^{n} (1 - e^{-\frac{k_i + k_j}{\lambda}})\).

**Corollary 6.** Let \(Z \sim \text{Mixed Hypoexponential Gumbel distribution} \ \text{MHG}(k_1, k_2, ..., k_n, \lambda)\) then

\[
f_Z(t) = \sum_{i=1}^{n} \frac{1}{P_{G_i}} f_{Y_i}(t)
\]

where \(Y_i \sim \text{Gumbel}(k_i, \lambda)\), \(i = 1, 2, ..., n\) be \(n\) random variables with \(k_i \neq k_j\) and \(P_{G_i} = \prod_{j=1, j \neq i}^{n} (1 - e^{-\frac{k_i + k_j}{\lambda}})\). Then,

\[
F_Z(t) = \sum_{i=1}^{n} \frac{1}{P_{G_i}} F_{Y_i}(t) = \sum_{i=1}^{n} \frac{1}{\prod_{j=1, j \neq i}^{n} (1 - e^{-\frac{k_i + k_j}{\lambda}})} e^{-\frac{(t-k_i)}{\lambda}}
\]

Next, we give the graphs of PDF and CDF for some values of \(k_i\) and \(\lambda\) of the New Mixed Hypoexponential Gumbel distribution.

5.6. The Mixed Hypoexponential Extreme Value Distribution. Finally, the last distribution of the Mixed Hypoexponential - G Family is The Mixed Hypoexponential Extreme Value Distribution.

**Theorem 14.** Let \(T \sim \text{hypoexp}(\alpha)\), then \(f_T(t) = \sum_{i=1}^{n} \frac{f_{X_i}(t)}{P_i}, t > 0\) where \(X_i \sim \text{Exp}(\alpha_i)\). Suppose \(Y_i = c \ln X_i \sim \text{ExtremeValue}(-c \ln (\alpha_i), -c)\) where \(\alpha_i > 0, c < \)
Thus, the Mixed Hypoexponential Extreme Value distribution \( Z \sim MHE(k_1, k_2, ..., k_n, \lambda) \) is the new distribution with PDF

\[
f_Z(t) = \sum_{i=1}^{n} \frac{1}{PE_i} f_{Y_i}(t)
\]

where \( Y_i \sim ExtremeValue(k_i, \lambda) \) and \( PE_i = \prod_{j=1, j \neq i}^{n} \left(1 - e^{-\frac{k_i-k_j}{\lambda}}\right) \).

Same as previous, and referring to Theorem 2 as \( Z \sim MHE(k_1, k_2, ..., k_n, \lambda) \) then

\[
F_Z(t) = \sum_{i=1}^{n} \frac{1}{PE_i} F_{Y_i}(t) = \sum_{i=1}^{n} \frac{1}{\prod_{j=1, j \neq i}^{n} \left(1 - e^{-\frac{k_i-k_j}{\lambda}}\right)} e^{-e^{-\frac{(t-k_i)}{\lambda}}}
\]

and the graphs of the PDF and CDF for different values of \( k_i \) and \( \lambda \) are as the following

Finally, we obtain the Mixed Hypoexponential-G family out of 6 different Mixed T-G distributions. However the properties of each of these distribution that are the CDF, MGF, hazard rate and reliability functions, and moment of order \( k \) are obtained in the same manner as those of Mixed Hypoexponential Weibull Distribution.
6. Conclusion

New Mixed Hypoexponential G-Family is an example of New Mixed T-G Family in which its distributions are derived by substituting the exponential distribution in the Hypoexponential distribution by its inverse, scalar multiple, k-th power, exponential, logarithm, and other transformations. Distributions belonging to this family take a common general form of PDF which is the New Mixed distribution form where their CDF, moment generating function, reliability function hazard rate function, and their parameter estimation can be determined easily according to the properties of New Mixed distribution. substitutions in a distribution of the New Mixed form.

References

[1] Abdelkader, Y. H. (2003). Erlang distributed activity times in stochastic activity networks. Kybernetika, 39(3), 347-358.
[2] Andrade, T., Rodrigues, H., Bourguignon, M., & Cordeiro, G. (2015). The exponentiated generalized Gumbel distribution. Revista Colombiana de Estadística, 38(1), 123-143.
[3] Kadri, T., Kadri, S., Kadry, S., & Smaili, K. (2022). The New Mixed Erlang Distribution: A Flexible Distribution for Modeling Lifetime Data. Reliability: Theory & Applications, 17(1), 411-420.
[4] Mansoor, M., Tahir, M. H., Alzaatreh, A., Cordeiro, G. M., Zubair, M., & Ghazali, S. S. (2016). An extended Fréchet distribution: properties and applications. Journal of Data Science, 14(1), 167-188.
[5] Rinne, H. (2008). The Weibull distribution: a handbook. CRC press.
[6] Rytgaard, M. (1990). Estimation in the Pareto distribution. ASTIN Bulletin: The Journal of the IAA, 20(2), 201-216.
[7] Smaili, K., Kadri, T., & Kadry, S. (2013). Hypoexponential distribution with different parameters.
[8] Smaili, K., Kadri, T., & Kadry, S. (2014). The Exact Distribution of The Ratio of Two Independent Hypoexponential Random Variables. British Journal of Mathematics & Computer Science, 4(18), 2665-2675.
[9] Walpole, R. E., & Myers, R. H. (2012). Probability & statistics for engineers & scientists. Pearson Education Limited.

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