Channel Leakage and Fundamental Limits of Privacy Leakage for Streaming Data

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Abstract—In this paper, we first introduce the notion of channel leakage as the minimum mutual information between the channel input and channel output. As its name indicates, channel leakage quantifies the (minimum) information leakage to the malicious receiver. In a broad sense, it can be viewed as a dual concept of channel capacity, which characterizes the (maximum) information transmission to the targeted receiver. We obtain explicit formulas of channel leakage for the white Gaussian case and colored Gaussian case. We also study the implications of channel leakage in characterizing the fundamental limitations of privacy leakage for streaming data.

I. INTRODUCTION

The topic of privacy for streaming data or dynamic data (see, e.g., [1]–[7] and the references therein) is attracting more and more attention in recent years. One important characteristic of streaming data or dynamic data is that information is not only contained in the samples at each time instant but also in the correlation over time [8]. On the other hand, information-theoretic privacy (see, e.g., [1], [2], [4], [7], [9]–[17] and the references therein) features a fundamental privacy concept, and the most commonly used information-theoretic measure of privacy leakage is mutual information (see, e.g., [1], [2], [4], [7], [9]–[17] and the references therein). What we are discussing in this paper will be in the general scope of information-theoretic privacy for streaming data (see, e.g., [1], [2], [7] and the references therein).

Particularly, in this paper we first formally introduce the concept of channel leakage, which is defined as the minimum mutual information between the channel input and channel output of a dynamic channel, supposing that the density function of the channel input is given while that of the channel noise can be designed, oftentimes subject to certain power constraints. For a given information source (as channel input) while subject to constraints on the information mask (as channel noise), channel leakage characterizes the minimum information leakage to a malicious receiver who has access to the masked version of the information source, i.e., the information source added with information mask (as channel output). We examine particularly the white Gaussian case and the colored Gaussian case, obtaining analytical formulas for the channel leakage as well as the power spectrum of the noise, indicating explicitly how to design the optimal information mask. It is also worth mentioning that, when designing the optimal information mask, channel leakage leads to “fire-extinguishing” power allocation policies, which are fundamentally different from the “water-filling” policy for channel capacity as well as the “reverse water-filling” policy for rate distortion.

Naturally, this notion of channel leakage may be employed to characterize the fundamental limits of privacy leakage of streaming data. Specifically, consider the scenario in which a privacy mask is to be added to a given data stream, resulting in a masked data stream that an eavesdropper may have access to. The information-theoretic privacy leakage (on average) would then be defined as the mutual information rate between the original data stream (as a stochastic process) and the masked data stream (as another stochastic process), that is, how much information can be extracted from the latter about the former on average. Accordingly, we may ask two questions. The first question is: Given a certain power constraint (on the privacy mask or on the masked data stream), what would be the minimum average privacy leakage in the long run, and how to design the privacy mask to achieve this lower bound? Or equivalently (in a dual manner): Given a certain requirement on the privacy level in terms of average privacy leakage, what would be the minimum average power needed on the privacy mask or on the masked data, and how to design the privacy mask to achieve this lower bound? It turns out the channel leakage and the “fire-extinguishing” power allocation provide mathematically explicit and physically intuitive solutions to all these problems.

The rest of the paper is organized as follows. Section II introduces the technical preliminaries. In Section III, we introduce the notion of channel leakage and discuss its properties. Section IV presents the fundamental limits of privacy leakage for streaming data based upon Section III. Conclusions are given in Section V.

II. PRELIMINARIES

Throughout the paper, we consider real-valued continuous random variables and random vectors, as well as discrete-time stochastic processes. All random variables, random vectors, and stochastic processes will be assumed to be zero-mean. We represent random variables and random vectors using boldface letters. Given a stochastic process \(\{x_k\}\), we denote the sequence \(x_0, \ldots, x_k\) by \(x_0^{k}\) for simplicity. The logarithm is with base 2. A stochastic process \(\{x_k\}\), \(x_k \in \mathbb{R}\) is said to be stationary if \(R_x(i, k) = E[x_ix_{i+k}]\) depends only on \(k\), and can thus be denoted as \(R_x(k)\) for simplicity. The power
spectrum of a stationary process \( \{ x_k \}, x_k \in \mathbb{R} \) is defined as 
\[
S_x(\omega) = \sum_{k=-\infty}^{\infty} R_x(k) e^{-i\omega k}.
\]

Moreover, the variance of \( \{ x_k \} \) is given by 
\[
\sigma^2_x = \lim_{k \to \infty} E[x_k^2] = R_x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_x(\omega) \, d\omega.
\]

Entropy and mutual information are the most basic notions in information theory \([8]\), which we introduce below.

**Definition 1:** The differential entropy of a random vector \( x \in \mathbb{R}^m \) with density \( p_x(x) \) is defined as
\[
h(x) = -\int p_x(x) \log p_x(x) \, dx.
\]

The conditional differential entropy of random vector \( x \in \mathbb{R}^m \) given random vector \( y \in \mathbb{R}^n \) with joint density \( p_{x,y}(x,y) \) and conditional density \( p_{x|y}(x|y) \) is defined as
\[
h(x|y) = -\int p_{x,y}(x,y) \log p_{x|y}(x|y) \, dx \, dy.
\]

The mutual information between random vectors \( x \in \mathbb{R}^m, y \in \mathbb{R}^n \) with densities \( p_x(x), p_y(y) \) and joint density \( p_{x,y}(x,y) \) is defined as
\[
I(x;y) = \int p_{x,y}(x,y) \log \frac{p_{x,y}(x,y)}{p_x(x) p_y(y)} \, dx \, dy.
\]

The entropy rate of a stochastic process \( \{ x_k \}, x_k \in \mathbb{R}^m \) is defined as
\[
h_\infty(x) = \lim_{k \to \infty} \sup \frac{h(x_0, \ldots, k)}{k+1}.
\]

The mutual information rate between two stochastic processes \( \{ x_k \}, x_k \in \mathbb{R}^m \) and \( \{ y_k \}, y_k \in \mathbb{R}^n \) is defined as
\[
I_\infty(x;y) = \lim_{k \to \infty} \sup \frac{I(x_0, \ldots, k; y_0, \ldots, k)}{k+1}.
\]

Properties of these notions can be found in, e.g., \([8],[13],[18],[19]\). On the other hand, consider an additive noise channel
\[
y_k = x_k + z_k,
\]
where \( \{ x_k \}, x_k \in \mathbb{R}^m \) denotes the channel input, \( \{ y_k \}, y_k \in \mathbb{R}^n \) denotes the channel output, and \( \{ z_k \}, z_k \in \mathbb{R}^m \) denotes the additive noise. For such a channel, the channel capacity is defined as follows \([8]\).

**Definition 2:** The channel capacity \( C \) of the communication channel given in (1) is defined as
\[
C = \sup_{P_x} \limsup_{k \to \infty} \frac{I(x_0, \ldots, k; y_0, \ldots, k)}{k+1},
\]
where the supremum is taken over all possible densities \( p_x \) of the input process allowed for the channel.

### III. Channel Leakage

Note first that channel leakage can be defined for classes of communication channels broader than additive noise channels. In this paper, however, we focus on additive channels for simplicity.

**Definition 3:** The channel leakage \( L \) of the communication channel given in (1) is defined as
\[
L = \inf_{p_x} I_\infty(x;y) = \inf_{p_x} \limsup_{k \to \infty} \frac{I(x_0, \ldots, k; y_0, \ldots, k)}{k+1},
\]
where the infimum is taken over all possible densities \( p_x \) of the noise process allowed for the channel.

As its name indicates, channel leakage quantifies the (minimum) information leakage to a malicious receiver. In comparison, channel capacity characterizes the (maximum) information transmission to a targeted receiver. In a broad sense, channel leakage may be viewed as a dual notion of channel capacity. Particularly, channel leakage is defined as the minimum mutual information rate between the channel input and channel output, with the channel input given; meanwhile, channel capacity is defined as the maximum mutual information rate between the channel input and channel output, with the channel noise given.

On the other hand, the following relationship between channel leakage and channel capacity may be established in general.

**Proposition 1:** Denote the channel leakage with (given) input density \( p_x \) and noise power constraint \( E[z_k^2] \leq N \) as
\[
L(p_x, N) = \inf_{E[z_k^2] \leq N} I_\infty(x;y),
\]
and denote the channel capacity with (given) noise density \( p_z \) and input power constraint \( E[z_k^2] \leq P \) as
\[
C(P, p_z) = \sup_{E[z_k^2] \leq P} I_\infty(x;y).
\]

If
\[
\sigma^2_x = \int_{-\infty}^{\infty} x^2 p_x(x) \, dx = P,
\]
and
\[
\sigma^2_z = \int_{-\infty}^{\infty} z^2 p_z(z) \, dz = N,
\]
then
\[
L(p_x, N) \leq C(P, p_z).
\]

**Proof:** Since
\[
\sigma^2_x = \int_{-\infty}^{\infty} x^2 p_x(x) \, dx = P,
\]
we have \( \sigma^2_z \leq P \); similarly, since
\[
\sigma^2_z = \int_{-\infty}^{\infty} z^2 p_z(z) \, dz = N,
\]
we have \( \sigma_z^2 \leq N \). As a result,
\[
L(p_x, N) = \inf_{E[|x|^2] \leq N} I_{\infty}(x; y) = I_{\infty}(x; y) \mid_{p_x, p_y}
\]
\[
\leq \sup_{E[|x|^2] \leq P} I_{\infty}(x; y) = C(P, p_x),
\]
which concludes the proof.

Let us now consider some special classes of communication channels. We shall start with the white Gaussian case.

**Theorem 1:** Consider a scalar channel of \( m = 1 \) and suppose that the channel input \( \{x_k\} \) is stationary white Gaussian with variance \( \sigma_x^2 = E[x_k^2] \). Suppose also that \( \{z_k\} \) is independent of \( \{x_k\} \). Then, the channel leakage with noise power constraint \( E[z_k^2] \leq N \) is given by
\[
L = \frac{1}{2} \log \left(1 + \frac{\sigma_x^2}{N}\right).
\]
\[\tag{9}\]

**Proof:** See Appendix A.

It is known [8] that the channel capacity of a scalar additive white Gaussian noise channel, where the channel noise \( \{z_k\} \) is stationary white Gaussian with variance \( \sigma_z^2 = E[z_k^2] \) and the input power constraint is \( E[z_k^2] \leq P \), is given by
\[
C = \frac{1}{2} \log \left(1 + \frac{P}{\sigma_z^2}\right).
\]
\[\tag{10}\]

Note that the distribution of a zero-mean stationary white Gaussian process is fully determined by its variance (second moment) [20]. As such, if \( \sigma_x^2 = P \) and \( \sigma_z^2 = N \), then it holds for this pair that
\[
L(p_x, N) = L(\sigma_x^2, N) = \frac{1}{2} \log \left(1 + \frac{P}{N}\right)
\]
\[\tag{11}\]
\[\tag{12}\]
\[\tag{13}\]

\[
N(\omega) = \frac{\zeta}{2 \left[1 + \sqrt{1 + \frac{\zeta}{S_x(\omega)}}\right]},
\]

and \( \zeta \geq 0 \) satisfies
\[
\frac{1}{2\pi} \int_0^{2\pi} N(\omega) d\omega = \frac{1}{2\pi} \int_0^{2\pi} \frac{\zeta}{2 \left[1 + \sqrt{1 + \frac{\zeta}{S_x(\omega)}}\right]} d\omega = N.
\]

**Proof:** See Appendix B.

The power allocation policy in (12) may be viewed as a “fire-extinguishing” policy, referring to a policy that delivers more power to frequencies that are with larger (input) spectra, which is opposite to the “water-filling” policy for channel capacity [8]. Particularly, the channel capacity of a scalar additive colored Gaussian noise channel, where the channel noise \( \{z_k\} \) is stationary colored Gaussian with power spectrum \( S_x(\omega) \) and the input power constraint is \( E[z_k^2] \leq P \), is given by [8]
\[
C = \frac{1}{2\pi} \int_0^{2\pi} \log \left[1 + \frac{P}{S_x(\omega)}\right] d\omega,
\]

where
\[
P(\omega) = \max\{0, \zeta - S_x(\omega)\},
\]

and \( \zeta \geq 0 \) satisfies
\[
\frac{1}{2\pi} \int_0^{2\pi} P(\omega) d\omega = \frac{1}{2\pi} \int_0^{2\pi} \max\{0, \zeta - S_x(\omega)\} d\omega = P.
\]
\[\tag{14}\]
\[\tag{15}\]

Note that the power allocation policy given in (14) and (15) is also known as “water-filling” (in the spectral domain) [8].

On the other hand, the distribution of a zero-mean stationary colored Gaussian process is fully determined by its power spectrum (essentially second moments) [20]. As such, if
\[
\sigma_x^2 = \frac{1}{2\pi} \int_0^{2\pi} S_x(\omega) d\omega = P,
\]

and
\[
\sigma_z^2 = \frac{1}{2\pi} \int_0^{2\pi} S_z(\omega) d\omega = N,
\]

then it holds for this pair that
\[
L(p_x, N) = L(S_x(\omega), N)
\]
\[\tag{16}\]
\[\tag{17}\]
\[\tag{18}\]
\[\tag{19}\]
\[\tag{20}\]
where \( N(\omega) \) and \( P(\omega) \) are given by (12) and (14), respectively. In fact, it may be further verified that
\[
N(\omega) = \max\{0, \zeta - S_x(\omega)\}
\]
\[
P(\omega) = \max\{0, \zeta - S_x(\omega)\}
\]
and \( \zeta \geq 0 \) satisfies
\[
\frac{1}{2\pi} \int_0^{2\pi} N(\omega) d\omega = \frac{1}{2\pi} \int_0^{2\pi} \max\{0, \zeta - S_x(\omega)\} d\omega = N.
\]

**Corollary 1:** Consider \( m \) parallel (independent) channels with
\[
y = x + z,
\]
where \( x, y, z \in \mathbb{R}^m \), and \( z \) is independent of \( x \). In addition, \( x \) is Gaussian with covariance
\[
\Sigma_x = \text{diag} \left( \sigma_x^2(1), \ldots, \sigma_x^2(m) \right),
\]
where \( x(i), i = 1, \ldots, m \), denotes the \( i \)-th element of \( x \), and \( \sigma_x^2(i) \) denotes its variance. Suppose that the noise power constraint is given by
\[
\text{tr} \left( \Sigma_x \right) = \mathbb{E} \left[ \sum_{i=1}^{m} x^2(i) \right] \leq N,
\]
where \( x(i) \) denotes the \( i \)-th element of \( z \). Then, the channel leakage is given by
\[
L = \sum_{i=1}^{m} \frac{1}{2} \log \left[ 1 + \frac{\sigma_x^2(i)}{N_i} \right],
\]
where
\[
N_i = \frac{\zeta}{2 \left[ \sqrt{1 + \frac{\zeta}{\sigma_x^2(i)}} + 1 \right]},
\]
with \( \zeta \geq 0 \) satisfying
\[
\sum_{i=1}^{m} N_i = \sum_{i=1}^{m} \frac{\zeta}{2 \left[ \sqrt{1 + \frac{\zeta}{\sigma_x^2(i)}} + 1 \right]} = N.
\]

The aim that we single out this result is to have a direct comparison between the “fire-extinguishing” policy for channel leakage and the “reverse water-filling” policy for rate distortion [8]; on the other hand, we already showed the difference between the “fire-extinguishing” policy for channel leakage and the “water-filling” policy for channel capacity in the discussions after Theorem 2. As such, channel leakage is seen to be essentially different from rate distortion as well (in addition to channel capacity).

Particularly, consider a parallel Gaussian source with \( m \) independent Gaussian random variables \( x_1, \ldots, x_m \). Suppose that the variances of \( x_1, \ldots, x_m \) are \( \sigma_1^2, \ldots, \sigma_m^2 \), respectively, and the distortion measure is \( \sum_{i=1}^{m} (\bar{x}_i - x_i)^2 \). Then, the rate distortion function is
\[
R(D) = \sum_{i=1}^{m} \frac{1}{2} \log \frac{\sigma_i^2}{D_i},
\]
where the distortion \( D_i \) for \( x_i \) is given by
\[
D_i = \begin{cases} 
\zeta, & \text{if } \zeta < \sigma_i^2, \\
\sigma_i^2, & \text{if } \zeta \geq \sigma_i^2,
\end{cases}
\]
and \( \zeta \) satisfies
\[
\sum_{i=1}^{m} D_i = D.
\]

On the other hand, the variance of \( \bar{x}_i \) is given by
\[
\tilde{\sigma}_i^2 = \sigma_i^2 - D_i = \begin{cases} 
\sigma_i^2 - \zeta, & \text{if } \zeta < \sigma_i^2, \\
0, & \text{if } \zeta \geq \sigma_i^2.
\end{cases}
\]

This allocation policy is also known as the “reverse water-filling” [8].

In parallel, the power constraint might be imposed on the channel output. In this case, we present the following result for the colored Gaussian case.

**Theorem 3:** Consider a scalar channel of \( m = 1 \) and suppose that the channel input \( \{x_k\} \) is stationary colored Gaussian with power spectrum \( S_x(\omega) \). Suppose also that \( \{z_k\} \) is independent of \( \{x_k\} \). Then, the channel leakage with output power constraint \( \mathbb{E} \left[ y_k^2 \right] \leq Y \) is given by
\[
L = \frac{1}{2\pi} \int_{0}^{2\pi} \log \sqrt{1 + \frac{S_x(\omega)}{N(\omega)}} \, d\omega,
\]
where
\[
N(\omega) = \frac{\zeta}{2 \left[ 1 + \sqrt{1 + \frac{\zeta}{S_x(\omega)}} \right]},
\]
and \( \zeta \geq 0 \) satisfies
\[
\frac{1}{2\pi} \int_{0}^{2\pi} N(\omega) \, d\omega = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\zeta}{2 \left[ 1 + \sqrt{1 + \frac{\zeta}{S_x(\omega)}} \right]} \, d\omega = Y - \frac{1}{2\pi} \int_{0}^{2\pi} S_x(\omega) \, d\omega.
\]

Note that Theorem 3 is essentially equivalent to the channel leakage with noise power constraint
\[
\mathbb{E} \left[ x_k^2 \right] \leq Y - \frac{1}{2\pi} \int_{0}^{2\pi} S_x(\omega) \, d\omega.
\]

Specifically, since \( \{z_k\} \) is independent of \( \{x_k\} \), we have
\[
S_y(\omega) = S_{x+z}(\omega) = S_x(\omega) + S_z(\omega),
\]
and thus
\[
\mathbb{E} \left[ y_k^2 \right] = \frac{1}{2\pi} \int_{0}^{2\pi} N(\omega) \, d\omega = \frac{1}{2\pi} \int_{0}^{2\pi} S_y(\omega) \, d\omega - \frac{1}{2\pi} \int_{0}^{2\pi} S_x(\omega) \, d\omega
\]
\[
= \mathbb{E} \left[ y_k^2 \right] - \frac{1}{2\pi} \int_{0}^{2\pi} S_x(\omega) \, d\omega \leq Y - \frac{1}{2\pi} \int_{0}^{2\pi} S_x(\omega) \, d\omega.
\]

**IV. FUNDAMENTAL LIMITS OF PRIVACY LEAKAGE FOR STREAMING DATA**

In this section, we present the fundamental lower bounds on the information leakage rate of streaming data. Specifically, consider the scenario in which a privacy mask is to be added to a given data stream, resulting in a masked data stream that an eavesdropper may have access to. The information-theoretic privacy leakage (on average) would then be defined as the mutual information rate between the original data stream (as a stochastic process) and the masked data stream (as another stochastic process), that is, how much information can be extracted from the latter about the former on average. Accordingly, we may ask two questions. The first question is: Given a certain power constraint (on the privacy mask
or on the masked data stream), what would be the minimum average privacy leakage in the long run, and how to design the privacy mask to achieve this lower bound? Or equivalently (in a dual manner): Given a certain requirement on the privacy level in terms of average privacy leakage, what would be the minimum average power needed on the privacy mask or on the masked data, and how to design the privacy mask to achieve this bound? We shall address these questions one by one.

We first consider the case of noise power constraint.

**Theorem 4:** Consider a data stream \( \{ x_k \} , x_k \in \mathbb{R} \). Suppose that \( \{ x_k \} \) is stationary colored Gaussian with power spectrum \( S_x(\omega) \). For the sake of privacy, a noise \( \{ n_k \}, n_k \in \mathbb{R} \) that is independent of \( \{ x_k \} \) is to be added to \( \{ x_k \} \) as its privacy mask, resulting in a masked streaming data \( \{ \mathbf{x}_k \} \), \( \mathbf{x}_k = x_k + n_k \), whereas the properties of \( \{ n_k \} \) can be designed subject to a power constraint \( \mathbb{E} [ n_k^2 ] \leq N \). Then, in order to minimize the information leakage rate

\[
I_\infty (x; \mathbf{x}) ,
\]

(31)

the noise \( \{ n_k \} \) should be chosen as a stationary Gaussian process. In addition, the power spectrum of \( \{ n_k \} \) should be chosen as

\[
N(\omega) = \frac{\zeta}{2 \left[ 1 + \sqrt{1 + \frac{\zeta}{S_x(\omega)}} \right]} ,
\]

(32)

where \( \zeta \geq 0 \) satisfies

\[
\frac{1}{2\pi} \int_0^{2\pi} N(\omega) \, d\omega = \frac{1}{2\pi} \int_0^{2\pi} \frac{\zeta}{2 \left[ 1 + \sqrt{1 + \frac{\zeta}{S_x(\omega)}} \right]} \, d\omega = N ,
\]

(33)

and the minimum information leakage rate is given by

\[
\inf_{\mathbb{E}[n_k^2] \leq N} I_\infty (x; \mathbf{x}) = L (S_x(\omega), N) = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + \frac{S_x(\omega)}{N(\omega)}} \, d\omega .
\]

(34)

**Proof:** The proof resembles that of Theorem 2 by viewing \( n_k \) and \( \mathbf{x}_k \) as \( z_k \) and \( y_k \) therein, respectively. That is to say, \( \{ n_k \} \) should be stationary Gaussian with power spectrum \( \{ n_k \} \) (32), while the minimum information leakage rate is given by

(33).

Note the lower bound is essentially given by the channel leakage of the virtual channel

\[
\mathbf{x}_k = x_k + n_k .
\]

(35)

Note also that

\[
I_\infty (x; \mathbf{x}) = h_\infty (x) - h_\infty (x|\mathbf{x}) ,
\]

(36)

and hence

\[
h_\infty (x|\mathbf{x}) = h_\infty (x) - I_\infty (x; \mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{2\pi e S_x(\omega)} \, d\omega - I_\infty (x; \mathbf{x}) .
\]

(37)

Since \( S_x(\omega) \) is pre-given, minimizing \( I_\infty (x; \mathbf{x}) \) is in fact equivalent to maximizing \( h_\infty (x|\mathbf{x}) \), which is another privacy measure that is oftentimes employed in estimation problems (see, e.g., [8, 21]). Particularly,

\[
\inf_{\mathbb{E}[n_k^2] \leq N} h_\infty (x|\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{2\pi e S_x(\omega)} \, d\omega - \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + \frac{S_x(\omega)}{N(\omega)}} \, d\omega = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{2\pi e S_x(\omega) N(\omega)} \, d\omega ,
\]

(38)

where \( N(\omega) \) is given by (32).

On the other hand, a dual problem to that of Theorem 4 would be: Given a certain privacy level, what is the minimum power of the noise to be added? The following corollary answers this question.

**Corollary 2:** Consider a data stream \( \{ x_k \} , x_k \in \mathbb{R} \). Suppose that \( \{ x_k \} \) is stationary colored Gaussian with power spectrum \( S_x(\omega) \). For the sake of privacy, a noise \( \{ n_k \}, n_k \in \mathbb{R} \) that is independent of \( \{ x_k \} \) is to be added to \( \{ x_k \} \) as its privacy mask, resulting in a masked streaming data \( \{ \mathbf{x}_k \} \), \( \mathbf{x}_k = x_k + n_k \), whereas the properties of \( \{ n_k \} \) can be designed. Then, in order to make sure that the information leakage is upper bounded by a constant \( R > 0 \) as

\[
I_\infty (x; \mathbf{x}) \leq R ,
\]

(39)

the minimum power of the noise \( \{ n_k \} \) to be added is given by

\[
\inf_{I_\infty (x; \mathbf{x}) \leq R} \mathbb{E} [ n_k^2 ] = \frac{1}{2\pi} \int_0^{2\pi} \frac{\zeta}{2 \left[ 1 + \sqrt{1 + \frac{\zeta}{S_x(\omega)}} \right]} \, d\omega ,
\]

(40)

where \( \zeta \geq 0 \) satisfies

\[
\frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + \frac{S_x(\omega)}{N(\omega)}} \, d\omega = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + \frac{S_x(\omega)}{N(\omega)}} \, d\omega = R .
\]

(41)

Consider next the case of output power constraint.

**Theorem 5:** Consider a data stream \( \{ x_k \} , x_k \in \mathbb{R} \). Suppose that \( \{ x_k \} \) is stationary colored Gaussian with power spectrum \( S_x(\omega) \). For the sake of privacy, a noise \( \{ n_k \}, n_k \in \mathbb{R} \) that is independent of \( \{ x_k \} \) is to be added to \( \{ x_k \} \) as its privacy mask, resulting in a masked streaming data \( \{ \mathbf{x}_k \} \), \( \mathbf{x}_k = x_k + n_k \), whereas the properties of \( \{ n_k \} \) can be designed subject to a power constraint on \( \mathbf{x}_k \) as \( \mathbb{E} [ \mathbf{x}_k^2 ] \leq X \). Then, in order to minimize the information leakage rate

\[
I_\infty (x; \mathbf{x}) ,
\]

(42)
the noise \( \{ n_k \} \) should be chosen as a stationary Gaussian process. In addition, the power spectrum of \( \{ n_k \} \) should be chosen as
\[
N(\omega) = \frac{\zeta}{2 \left[ 1 + \sqrt{1 + \frac{\zeta}{S_x(\omega)}} \right]},
\]
where \( \zeta \geq 0 \) satisfies
\[
\frac{1}{2\pi} \int_0^{2\pi} N(\omega) \, d\omega = \frac{1}{2\pi} \int_0^{2\pi} \frac{\zeta}{2 \left[ 1 + \sqrt{1 + \frac{\zeta}{S_x(\omega)}} \right]} \, d\omega
\]
and the minimum information leakage rate is given by
\[
\inf_{E[\mathbf{x}|\mathbf{x}]} I_{\infty}(\mathbf{x}; \mathbf{x}) = L \left( S_k(\omega), \frac{1}{2\pi} \int_0^{2\pi} S_k(\omega) \, d\omega \right)
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \log \left[ 1 + \frac{S_k(\omega)}{N(\omega)} \right] d\omega.
\]  

**Proof:** Since \( \{ n_k \} \) is independent of \( \{ x_k \} \), the power constraint reduces to
\[
E[n_k^2] \leq \overline{X} - \frac{1}{2\pi} \int_0^{2\pi} S_k(\omega) \, d\omega.
\]

Then, Theorem 5 follows by invoking Theorem 4.

**Corollary 3:** Consider a data stream \( \{ x_k \} \), \( x_k \in \mathbb{R} \). Suppose that \( \{ x_k \} \) is stationary colored Gaussian with power spectrum \( S_k(\omega) \). For the sake of privacy, a noise \( \{ n_k \} \), \( n_k \in \mathbb{R} \) that is independent of \( \{ x_k \} \) is to be added to \( \{ x_k \} \) as its privacy mask, resulting in a masked streaming data \( \{ \tilde{x}_k \} \), \( \tilde{x}_k = x_k + n_k \), whereas the properties of \( \{ n_k \} \) can be designed. Then, in order to make sure that the information leakage is upper bounded by a constant \( R > 0 \) as
\[
I_{\infty}(\mathbf{x}; \mathbf{x}) \leq R,
\]
the minimum power of the masked data \( \{ \tilde{x}_k \} \) is given by
\[
\inf_{I_{\infty}(\mathbf{x}; \mathbf{x}) \leq R} E[\tilde{x}_k^2] = \frac{1}{2\pi} \int_0^{2\pi} \frac{\zeta}{2 \left[ 1 + \sqrt{1 + \frac{\zeta}{S_k(\omega)}} \right]} \, d\omega
\]
\[
+ \frac{1}{2\pi} \int_0^{2\pi} S_k(\omega) \, d\omega,
\]  

where \( \zeta \geq 0 \) satisfies
\[
\frac{1}{2\pi} \int_0^{2\pi} \log \left[ 1 + \frac{2}{\zeta} \left[ 1 + \sqrt{1 + \frac{\zeta}{S_k(\omega)}} \right] S_k(\omega) \right] d\omega = R.
\]  

**V. Conclusion**

In this paper, we have formally introduced the notion of channel leakage as the minimum mutual information rate between the channel input and channel output, which characterizes the (minimum) information leakage rate to the malicious receiver. We have obtained explicit formulas of channel leakage for the white Gaussian case and colored Gaussian case. We have also investigated the implications of channel leakage in characterizing the fundamental limits of privacy leakage for streaming data. Potential future research directions include the investigation of non-Gaussian cases.

**APPENDIX**

**A. Proof of Theorem 7**

Since \( \{ x_k \} \) is white, we have
\[
I(x_0, \ldots, x_k; y_0, \ldots, k) = h(x_0, \ldots, x_k) - h(x_0, \ldots, x_k | y_0, \ldots, k)
\]
\[
= \sum_{i=0}^{k} h(x_i) - h(z_0, \ldots, k | y_0, \ldots, k)
\]
\[
= \sum_{i=0}^{k} h(x_i) - \sum_{i=0}^{k} h(z_i | y_0, \ldots, k, z_0, \ldots, i-1)
\]
\[
\geq \sum_{i=0}^{k} h(x_i) - \sum_{i=0}^{k} h(z_i | y_i) = \sum_{i=0}^{k} h(x_i) - \sum_{i=0}^{k} h(x_i | y_i)
\]
\[
= \sum_{i=0}^{k} I(x_i; y_i),
\]
and
\[
I(x_0, \ldots, x_k; y_0, \ldots, k) = \sum_{i=0}^{k} I(x_i; y_i)
\]

if \( \{ z_k \} \) is white. On the other hand, since \( x_i \) and \( z_i \) are independent, we have
\[
I(x_i; y_i) = h(y_i) - h(y_i | x_i) = h(y_i) - h(z_i)
\]
\[
i.e. \quad h(y_i) - h(z_i),
\]
and
\[
I(z_i; y_i) = h(y_i) - h(y_i | z_i) = h(y_i) - h(x_i | z_i)
\]
\[
i.e. \quad h(y_i) - h(x_i).
\]

Then, according to the entropy power inequality [8], we have
\[
2^{2h(y_i)} \geq 2^{2h(x_i)} + 2^{2h(z_i)}
\]
and hence
\[
2^{2[h(z_i) - h(y_i)]} + 2^{2[h(x_i) - h(y_i)]} \leq 1.
\]

Consequently,
\[
I(x_i; y_i) = h(y_i) - h(z_i) \geq -\frac{1}{2} \log \left\{ 1 - 2^{2[h(x_i) - h(y_i)]} \right\}
\]
\[
= -\frac{1}{2} \log \left[ 1 - 2^{-2I(z_i; y_i)} \right].
\]

On the other hand, it can be verified [8] that \( I(z_i; y_i) \) reaches its maximum
\[
\frac{1}{2} \log \left( 1 + \frac{N}{\sigma_z^2} \right)
\]  

when \( z_i \) is Gaussian and \( E[z_i^2] = N \). Note also that if \( z_i \) is Gaussian, then
\[
2^{2h(y_i)} = 2^{2h(z_i)} + 2^{2h(x_i)}.
\]
and thus

\[ I(\mathbf{x}_i; \mathbf{y}_i) = -\frac{1}{2} \log \left[ 1 - 2^{-2I(\mathbf{z}_i; \mathbf{y}_i)} \right]. \]

That is to say, \( I(\mathbf{x}_i; \mathbf{y}_i) \) reaches its minimum if \( \mathbf{z}_i \) is Gaussian and \( \mathbb{E}[\mathbf{z}_i^2] = N \), and the minimum is given by

\[ \min_{\mathbb{E}[\mathbf{z}_i^2] \leq N} I(\mathbf{x}_i; \mathbf{y}_i) = -\frac{1}{2} \log \left( 1 - 2^{-2 \log \left( 1 + \frac{\sigma^2}{N} \right)} \right) = \frac{1}{2} \log \left( 1 + \frac{\sigma^2}{N} \right). \]

As such, as \( k \to \infty \), \( \{\mathbf{x}_k\}, \{\mathbf{z}_k\} \), and \( \{\mathbf{y}_k\} \) are stationary white (cf. the proof of Theorem 2), and

\[ \min_{\mathbb{E}[\mathbf{z}_i^2] \leq N} I(\mathbf{x}_i; \mathbf{y}_i), \forall i \in \mathbb{N} \]

\[ = \inf_{\mathbb{E}[\mathbf{z}_i^2] \leq N} \lim_{k \to \infty} \frac{\sum_{i=0}^{k} I(\mathbf{x}_i; \mathbf{y}_i)}{k + 1} \]

\[ = \inf_{\mathbb{E}[\mathbf{z}_i^2] \leq N} \lim_{k \to \infty} \frac{\sum_{i=0}^{k} I(\mathbf{x}_i; \mathbf{y}_i)}{k + 1} \]

\[ = \inf_{\mathbb{E}[\mathbf{z}_i^2] \leq N} \lim_{k \to \infty} \frac{n}{k + 1} \]

\[ = \inf_{\mathbb{E}[\mathbf{z}_i^2] \leq N} I_{\infty}(\mathbf{x}; \mathbf{y}) = L. \]

In other words,

\[ L = \frac{1}{2} \log \left( 1 + \frac{\sigma^2}{N} \right), \]

which is achieved when \( \{\mathbf{z}_i\} \) is stationary white Gaussian with variance \( \mathbb{E}[\mathbf{z}_i^2] = N \). Note that although mathematically this proof is in the same spirit as that for channel blurredness in [19] in general, the implications are totally different. To prevent possible confusion and for the sake of completeness, in this paper we still present a step-by-step proof. This statement applies to the proof of Theorem 2 as well.

**B. Proof of Theorem 2**

We first consider the case of a finite number of parallel (dependent) channels with

\[ \mathbf{y} = \mathbf{x} + \mathbf{z}, \]

where \( \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^m \), and \( \mathbf{z} \) is independent of \( \mathbf{x} \). In addition, \( \mathbf{x} \) is Gaussian with covariance \( \Sigma_x \), and the noise power constraint is given by

\[ \text{tr} (\Sigma_\mathbf{z}) = \mathbb{E} \left[ \sum_{i=1}^{m} \mathbf{z}^2(i) \right] \leq N, \]

where \( \mathbf{z}(i) \) denotes the \( i \)-th element of \( \mathbf{z} \). (Note that the case of parallel independent channels, as discussed in Corollary 11 is a special case of that of dependent channels for when \( \Sigma_x \) is diagonal.) In addition, since \( \mathbf{x} \) and \( \mathbf{z} \) are independent, we have

\[ I(\mathbf{x}; \mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y}|\mathbf{x}) = h(\mathbf{y}) - h(\mathbf{z}|\mathbf{x}) = h(\mathbf{y}) - h(\mathbf{z}), \]

and

\[ \Sigma_y = \Sigma_{\mathbf{x} + \mathbf{z}} = \Sigma_x + \Sigma_z. \]

On the other hand, the minimum of \( I(\mathbf{x}; \mathbf{y}) \) is achieved if \( \mathbf{z} \) is Gaussian (see Section 11.9 of [18]), whereas when \( \mathbf{z} \) is Gaussian, we have

\[ I(\mathbf{x}; \mathbf{y}) = h(\mathbf{y}) - h(\mathbf{z}) \]

\[ = \frac{1}{2} \log \left[ (2\pi e)^m \det \Sigma_y \right] - \frac{1}{2} \log \left[ (2\pi e)^m \det \Sigma_z \right] \]

\[ = \frac{1}{2} \log \frac{\det \Sigma_y}{\det \Sigma_z} = \frac{1}{2} \log \frac{\det (\Sigma_x + \Sigma_z)}{\det \Sigma_z} \]

\[ = \frac{1}{2} \log \frac{\det (\Sigma_x + \Sigma_z)}{\det \Sigma_z} = \frac{1}{2} \log \left( \frac{\det (\Sigma_x + \Lambda_x)}{\det \Sigma_z} \right), \]

where \( \Sigma_x = \Sigma_x \mathbf{U} \mathbf{U}^T \) is the eigen-decomposition of \( \Sigma_x \) with

\[ \Lambda_x = \text{diag} (\lambda_1, \ldots, \lambda_m), \]

while \( \Sigma_z = \Sigma_z \mathbf{U} \mathbf{U}^T \mathbf{U} \mathbf{U}^T \). (Note that for a diagonal \( \Sigma_x \), we have \( \lambda_i = \sigma^2_x(i) \), where \( \mathbf{x}(i) \) denotes the \( i \)-th element of \( \mathbf{x} \), and \( \sigma^2_x(i) \) denotes its variance.) Hence,

\[ \text{tr} (\Sigma_z) = \text{tr} (\mathbf{U}^T \Sigma_z \mathbf{U} \mathbf{U}^T) = \text{tr} (\mathbf{U} \Sigma_x \mathbf{U}^T) \]

\[ = \text{tr} (\Sigma_x) = \mathbb{E} \left[ \sum_{i=1}^{m} \sigma^2_x(i) \right] \leq N. \]

It is known (see Lemma 3.2 of [19]) that

\[ \frac{1}{2} \log \frac{\det (\Sigma_x + \Lambda_x)}{\det \Sigma_z} \geq \frac{1}{2} \log \prod_{i=1}^{m} \left[ \frac{\sigma^2_x(i) + \lambda_i}{\sigma^2_x(i)} \right], \]

where \( \sigma^2_x(i), i = 1, \ldots, m \), are the diagonal terms of \( \Sigma_x \), and the equality holds if \( \Sigma_x \) is diagonal, whereas when \( \Sigma_x \) is diagonal, we denote

\[ \Sigma_x = \text{diag} (\sigma^2_x(1), \ldots, \sigma^2_x(m)) = \text{diag} (N_1, \ldots, N_m) \]

for simplicity. Then, the problem reduces to that of choosing \( N_1, \ldots, N_m \) to minimize

\[ \frac{1}{2} \log \prod_{i=1}^{m} \left( \frac{N_i + \lambda_i}{N_i} \right) = \sum_{i=1}^{m} \frac{1}{2} \log \left( 1 + \frac{\lambda_i}{N_i} \right) \]

subject to the constraint that

\[ \sum_{i=1}^{m} N_i = \text{tr} (\Sigma_\mathbf{z}) = N. \]

Define the Lagrange function by

\[ \sum_{i=1}^{m} \frac{1}{2} \log \left( 1 + \frac{\lambda_i}{N_i} \right) + \eta \left( \sum_{i=1}^{m} N_i - N \right), \]

and differentiate it with respect to \( N_i \), then we have

\[ \log e \left( \frac{1}{N_i + \lambda_i} - \frac{1}{N_i} \right) + \eta = 0, \]
or equivalently,
\[ N_i = \frac{\sqrt{X_i^2 + \zeta x_i} - \lambda_i}{2} = \frac{\zeta}{2 \left( \sqrt{1 + \frac{x_i}{X_i}} + 1 \right)}, \]
where \( \zeta = 2 \log e / \eta \geq 0 \) satisfies
\[ \sum_{i=1}^{m} N_i = \sum_{i=1}^{m} \frac{\zeta}{2 \left( \sqrt{1 + \frac{x_i}{X_i}} + 1 \right)} = N. \]
Consider now a scalar (dynamic) channel
\[ y_k = x_k + z_k, \]
where \( x_k, y_k, z_k \in \mathbb{R} \), and \( \{ z_k \} \) is independent of \( \{ x_k \} \).
In addition, \( \{ x_k \} \) is stationary colored Gaussian with power spectrum \( S_m(\omega) \), and the noise power constraint is given by \( \mathbb{E} \left[ z_i^2 \right] \leq N \). We may then consider a block of consecutive uses of this channel (from time 0 to \( k \)) as \( k+1 \) channels in parallel with dependent noise \( [8] \).

Particularly, let the eigen-decomposition of \( \Sigma_{x_0, \ldots, k} \) be given by
\[ \Sigma_{x_0, \ldots, k} = U_{x_0, \ldots, k} \Lambda_{x_0, \ldots, k} U_{x_0, \ldots, k}^T, \]
where \( \Lambda_{x_0, \ldots, k} = \text{diag}(\lambda_0, \ldots, \lambda_k) \).
Then, we have
\[ \min_{p_{x_0, \ldots, k} : \Sigma_{x_0, \ldots, k}} \mathbb{E} \left[ \left| z_i \right|^2 \leq (k+1)N \right] \frac{I(x_0, \ldots, k; y_{0, \ldots, k})}{k+1} \]
\[ = \frac{1}{k+1} \sum_{i=0}^{k} \frac{1}{2} \log \left( 1 + \frac{\lambda_i}{N_i} \right), \]
where \( N_i = \frac{\zeta}{2 \left( \sqrt{1 + \frac{x_i}{X_i}} + 1 \right)}, \ i = 0, \ldots, k. \)
Herein, \( \zeta \geq 0 \) satisfies
\[ \sum_{i=0}^{k} N_i = \sum_{i=0}^{k} \frac{\zeta}{2 \left( \sqrt{1 + \frac{x_i}{X_i}} + 1 \right)} = (k+1)N, \]
or equivalently,
\[ \frac{1}{k+1} \sum_{i=0}^{k} N_i = \frac{1}{k+1} \sum_{i=0}^{k} \frac{\zeta}{2 \left( \sqrt{1 + \frac{x_i}{X_i}} + 1 \right)} = N. \]
In addition, as \( k \to \infty \), the processes \( \{ x_k \}, \{ z_k \} \), and \( \{ y_k \} \) are stationary, and
\[ \lim_{k \to \infty} p_{x_0, \ldots, k} : \min_{p_{x_0, \ldots, k} : \Sigma_{x_0, \ldots, k}} \mathbb{E} \left[ \left| z_i \right|^2 \leq (k+1)N \right] \frac{I(x_0, \ldots, k; y_{0, \ldots, k})}{k+1} \]
\[ = \inf_{p_x} \lim_{k \to \infty} \frac{I(x_0, \ldots, k; y_{0, \ldots, k})}{k+1} \]
\[ = \inf_{p_x} \limsup_{k \to \infty} \frac{I(x_0, \ldots, k; y_{0, \ldots, k})}{k+1} = \inf_{p_x} I_{\infty}(x; y) = L. \]
On the other hand, since the processes are stationary, the covariance matrices are Toeplitz \([22]\), and their eigenvalues approach their limits as \( k \to \infty \). Moreover, the densities of eigenvalues on the real line tend to the power spectra of the processes \([23]\).
Accordingly,
\[ L = \lim_{k \to \infty} p_{x_0, \ldots, k} : \min_{p_{x_0, \ldots, k} : \Sigma_{x_0, \ldots, k}} \mathbb{E} \left[ \left| z_i \right|^2 \leq (k+1)N \right] \frac{I(x_0, \ldots, k; y_{0, \ldots, k})}{k+1} \]
\[ = \lim_{k \to \infty} \frac{1}{k+1} \sum_{i=0}^{k} \frac{1}{2} \log \left( 1 + \frac{\lambda_i}{N_i} \right) \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \log \left( 1 + \frac{S_k(\omega)}{N(\omega)} \right) d \omega \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left( 1 + \frac{S_k(\omega)}{N(\omega)} \right) d \omega, \]
where
\[ N(\omega) = \frac{\zeta}{2 \left( \sqrt{1 + \frac{\zeta}{S_k(\omega)} + 1} \right)}. \]
and \( \zeta \geq 0 \) satisfies
\[ \lim_{k \to \infty} \frac{1}{k+1} \sum_{i=0}^{k} N_i = \frac{1}{2\pi} \int_{-\pi}^{\pi} N(\omega) d \omega = N. \]
This concludes the proof.

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