This article is a continuation of [HK06]. Given a strongly conformal SUSY vertex algebra $V$ and a supercurve $X$, we construct a vector bundle $\mathcal{V}_X$ on $X$, the fiber of which, is isomorphic to $V$. Moreover, the state-field correspondence of $V$ canonically gives rise to (local) sections of these vector bundles. We also define chiral algebras on any supercurve $X$, and show that the vector bundle $\mathcal{V}_X$, corresponding to a SUSY vertex algebra, carries the structure of a chiral algebra.

1. Introduction

1.1. Vertex algebras were introduced about 20 years ago by Borcherds [Bor86]. They provide a rigorous definition of the chiral part of 2-dimensional conformal field theory, intensively studied by physicists. Since then, they have had important applications to string theory and conformal field theory, and to mathematics, by providing tools to study the most interesting representations of infinite dimensional Lie algebras. Since their appearance, they have been extensively studied in many papers and books (for the latter we refer to [FLM88], [FHL93], [Kac96], [Hua97], [FBZ01], [BD04]).

Vertex algebras also appeared in algebraic geometry as Factorization Algebras on complex curves [BD04], [FBZ01]. In the last five years, numerous applications of this deep connection between factorization algebras and vertex algebras have been exploited, notably in the study of the moduli spaces (of curves, vector bundles, principal bundles, etc) arising in algebraic geometry. There are also connections between...
the theory of vertex algebras and the geometric Langlands conjecture [FBZ01, ch. 17]. Vertex algebras have also given new invariants of manifolds [KV04], [MSV99] and applications to mirror symmetry [Bor01].

Even though these approaches have been successful in formalizing 2-dimensional conformal field theories, it has been known for some time to physicists, that in order to describe supersymmetric theories, similar objects should be defined on supercurves instead of simply curves (cf. [DRS90], [Coh87], [BDFM88]). With this motivation, mathematicians have studied in detail the supergeometry of manifolds, and in particular supercurves (cf. [DM99], [Man91], [Man97], [Vai90] among others).

The purpose of this article is to generalize the above objects to describe chiral algebras over supercurves. To accomplish this, we will use the supersymmetric (SUSY) vertex algebras defined in [HK06]. Roughly speaking, SUSY vertex algebras are vertex algebras with a state-field correspondence that includes the odd coordinates of a supercurve as formal parameters, that is, to any vector $a$ in a SUSY vertex algebra, we associate a superfield

$$Y(a, z, \theta^1, \ldots, \theta^N),$$

such that structural properties, similar to those of ordinary vertex algebras, hold.

Given a SUSY vertex algebra $V$ and a supercurve $X$, we want to assign a vector bundle $\mathcal{V}$ over $X$ in such a way that the fiber at a point $x \in X$ is identified with $V$. Moreover, we would like $Y$ to canonically define sections of this vector bundle (more precisely, its restricted dual). Here we find the first difference with the classical theory, namely, supercurves come in different flavors: general $1|n$ dimensional supercurves and superconformal $1|n$ supercurves. The latter are to the former what holomorphic curves are to compact connected 2-manifolds. In [HK06], two different versions of SUSY vertex algebras are defined, one which localizes to give vector bundles on a general $1|n$-dimensional supercurve (called $N_W = n$ SUSY vertex algebras) and another which gives vector bundles on superconformal supercurves (called $N_K = n$ SUSY vertex algebras).

There are several relations between these different SUSY vertex algebras. As a basic example, let us consider the cases with low odd dimensions. Roughly speaking, a general $N = 1$ supercurve is the data of a curve $X$ and a line bundle $\mathcal{L}$ over it, sections of this line bundle are considered to be the values of a coordinate in the odd direction. Similarly, an (oriented) superconformal $N = 2$ supercurve consists of a curve $X$ and two line bundles $\mathcal{L}$ and $\mathcal{H}$ over it such that $\mathcal{L} \otimes \mathcal{H}$ is the canonical bundle $\omega$ of $X$. It follows that an $N = 1$ supercurve gives rise canonically to another $N = 1$ supercurve (interchanging $\mathcal{L}$ with $\omega \otimes \mathcal{L}^{-1}$) and to a superconformal $N = 2$ supercurve (by taking $\mathcal{H} = \omega \otimes \mathcal{L}^{-1}$). On the algebraic side, any (conformal) $N_W = 1$ SUSY vertex algebra gives rise to a (conformal) $N_K = 2$ SUSY vertex algebra (this corresponds to the isomorphism between the superconformal Lie algebras $K(1|2)$ and $W(1|2)$) and both of them correspond to vertex algebras with $N = 2$ superconformal structure. It follows that any such vertex algebra gives vector bundles in both $N = 1$ supercurves and in the corresponding superconformal $N = 2$ supercurve. These three vector bundles are intimately related as we will see in section 3.3.

As in the ordinary vertex algebra case, the vector bundles we construct (more precisely quotients of them) are extensions of (powers of) the Berezinian bundle of $X$ (a super analog of the canonical bundle). The algebraic properties of $V$ reflect
...in geometric properties of $\mathcal{V}$ as in the ordinary vertex algebra case. We obtain thus superprojective structures, affine structures, global differential operators, etc. as splittings of these extensions. In particular, the state-field correspondence itself gives such splittings (locally).

1.2. After constructing these vector bundles, it is natural to ask if they carry the structure of a chiral algebra on a supercurve. It is shown that the usual definitions carry over to the super case with minor difficulties, and that the vector bundles obtained from $V$ are indeed chiral algebras. This allows us to define the coinvariants and conformal blocks of a SUSY vertex algebra in a coordinate independent way as in [FBZ01].

1.3. The organization of this article is as follows: In section 2 we recall some well known notions about vertex algebras and supercurves. We also summarize here some results on the structure theory of SUSY vertex algebras.

In section 3 we construct a vector bundle with a flat connection associated to an $\mathcal{N}=n$ SUSY vertex algebra, over any $\mathcal{N}=n$ supercurve. We also construct vector bundles associated to $\mathcal{N}=n$ SUSY vertex algebras over oriented super-conformal $\mathcal{N}=n$ supercurves. In this section we follow closely [FBZ01] since most proofs are straightforward adaptations of the those in the ordinary vertex algebra case. The new phenomena can be found in section 3.3 where we analyze in detail examples with supersymmetry.

In section 3.1 we define the groups $\text{Aut}\mathcal{O}$ of changes of coordinates and the $\text{Aut}\mathcal{O}$-torsor $\text{Aut}_X$ for a supercurve. In section 3.2 we construct the vector bundles themselves and their sections. In particular we show that the state-field correspondence for a SUSY vertex algebra is a section of the dual of the corresponding vector bundle. In section 3.3 we compute explicitly some examples of vector bundles over supercurves of low odd dimension.

In section 4 we define chiral algebras over supercurves and we prove that the vector bundles constructed from SUSY vertex algebras are examples of chiral algebras. We also define the spaces of coinvariants in a coordinate independent way.

In appendix A we give a brief description of a family of representations of the Lie algebra $\mathfrak{gl}(1|1)$ and their realizations as fibers of certain natural vector bundles over $\mathcal{N}=1$ supercurves.

Acknowledgments. The author would like to thank Victor G. Kac for reading the manuscript, encouraging him, and many useful discussions. He would also like to thank David Ben-Zvi, for very useful discussions.

2. Preliminaries

2.1. SUSY vertex algebras. In this section we collect some results and examples of SUSY vertex algebras from [HK06]. For the general theory of vertex algebras, the reader is referred to the book [Kac96] and [DSK05] for an excellent exposition.

2.1.1. Let $N$ be a non-negative integer. We will denote $Z = (z, \theta^1, \ldots, \theta^N)$ where $z$ is an even indeterminate, $\theta^i$’s are odd anticommuting indeterminates commuting with $z$. For $J = (j_1, \ldots, j_s)$ ordered subset of $\{1, \ldots, N\}$, and $j \in \mathbb{Z}$, we will denote

$$\theta^J = \theta^{j_1} \cdots \theta^{j_s}, \quad Z^{\lvert J \rvert} = \theta^J z^J.$$
and we will denote by $N \setminus J$ the ordered complement of $J$ in $\{1, \ldots, N\}$. For two disjoint subsets $I, J \subset \{1, \ldots, N\}$ define $\sigma(I, J) = \pm 1$ by $\theta^I \theta^J = \sigma(I, J) \theta^{I+J}$, and $\sigma(J) = \sigma(J, N \setminus J)$. Finally, define $e_i = \{i\}$.

Given a vector superspace $V$, we will denote by $V[[Z]]$ (resp. $V((Z))$) the space of formal power series (resp. formal Laurent series) in $Z$ with values in $V$, namely, formal sums of the form

$$\sum_{j \geq 0, J} Z^{|J|} v_{j|J} \quad \left(\text{resp.} \sum_{j \geq N_0, J} Z^{|J|} v_{j|J}\right),$$

where $N_0$ is some integer number and $v_{j|J} \in V$. Finally, we will denote by $V[Z, Z^{-1}]$ the space of Laurent polynomials with coefficients in $V$, namely, elements of $V((Z))$ which are finite sums.

Let $\mathcal{H}_W$ (resp. $\mathcal{H}_K$) be the associative superalgebra generated by an even element $T$ and $N$ odd elements $S^i$ subject to the relations

$$[T, S^i] = 0, \quad [S^i, S^j] = 0 \quad \text{(resp. 2}\delta_{i,j}T).$$

An $N_W = N$ (resp. $N_K = N$) SUSY vertex algebra $(V, [0], Y)$ is the data of a $\mathcal{H}_W$-module (resp. $\mathcal{H}_K$-module) $V$ (the space of states), an even vector $[0] \in V$ (the vacuum vector) and a parity preserving $\mathbb{C}$-bilinear product with values in Laurent series over $V$:

$$V \otimes V \to V((Z)), \quad a \otimes b \mapsto Y(a, Z)b = \sum_{j \in \mathbb{Z}, J} Z^{-1-j|N\setminus J} a_{(j|J)} b,$$

subject to the following axioms $(a, b \in V)$:

- (vacuum axioms) $Y(a, Z)[0]_{Z=0} = a, T[0] = S^i[0] = 0, \text{ for } i = 1, \ldots, N$,
- (translation invariance) $[T, Y(a, z)] = \partial_z Y(a, Z)$, $[S^i, Y(a, Z)] = \partial_{\theta^i} Y(a, Z)$ (resp. $(\partial_{\theta^i} - \theta^i \partial_z) Y(a, Z))$,
- (locality) $(z - w)^n [Y(a, Z), Y(b, W)] = 0$ for some $n \in \mathbb{Z}_+$.

**Remark 2.1.2.** Note that when $N = 0$, this definition agrees with the usual definition of vertex algebra as in [Kac96]

2.1.3. Denote $\Lambda = (\lambda, \chi^1, \ldots, \chi^N)$, where $\lambda$ is an even indeterminate and $\chi^i$’s are odd indeterminates, subject to the relations:

$$[\lambda, \chi^i] = 0, \quad [\chi^i, \chi^j] = 0 \quad \text{(resp.} -2\delta_{i,j}\lambda),$$

and write $ZA = z\lambda + \sum_{i=1}^N \theta^i \chi^i$.

Let $V$ be an $N_W = N$ (resp. $N_K = N$) SUSY vertex algebra. For $a, b \in V$ we define

$$[a, b]_\Lambda = \text{res}_Z e^{ZA} Y(a, Z)b,$$

where $\text{res}_Z$ stands for the coefficient to the right of $Z^{-1|N}$. We note that the right hand side of (2.1.3.1) is a finite sum of monomials in $\Lambda$ times elements of $V$ (cf. [HK06]). This operation is called the $\Lambda$-bracket. As in the usual vertex algebra case, it encodes the singular part of the *operator product expansion* (OPE) in $V$.

Define the *normally ordered product* $\cdot$ as a $\mathbb{C}$-bilinear product on $V$:

$$V \otimes V \to V, \quad a \otimes b \mapsto: ab := a_{(-1|N)} b.$$
The action of $\mathcal{H}_W$ (resp. $\mathcal{H}_K$) on $V$ is by derivations of both, the $\Lambda$-bracket, and the normally ordered product. As proved in [HK06], these two operations encode all the structure of the SUSY vertex algebra $V$.

**Example 2.1.4.** (Virasoro) This is an ordinary vertex algebra generated by one even field $L$ satisfying

\[(2.1.4.1) \quad [L_\lambda L] = (T + 2\lambda)L + \frac{\lambda^3}{12}c,\]

where $c \in \mathbb{C}$ is the **central charge**. Expanding this field as

\[L(z) = \sum_{n \in \mathbb{Z}} z^{-2-n}L_n,\]

we obtain that the operators $L_n$ satisfy the commutation relations of the Virasoro algebra of central charge $c$, namely:

\[[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12}c.\]

A field $L$ in an ordinary vertex algebra $V$ satisfying (2.1.4.1) will be called a **Virasoro field** of central charge $c$.

**Definition 2.1.5.** Let $V$ be an ordinary vertex algebra with a Virasoro field $L$. We will say that a vector $a \in V$ that satisfies $[L_\lambda a] = (T + \Delta \lambda)a + O(\lambda^2)$ has **conformal weight** $\Delta$. If moreover, $a$ satisfies $[L_\lambda a] = (T + \Delta \lambda)a$, we will say that $a$ is **primary**.

**Example 2.1.6.** (Neveu-Schwarz) This vertex algebra is generated by a Virasoro field as in Example 2.1.4 and an odd field $G$, primary of conformal weight $3/2$. The remaining $\Lambda$-bracket is given by:

\[[G_\lambda G] = 2L + \frac{\lambda^2}{3}c.\]

If we expand the corresponding fields as:

\[L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-2-n}, \quad G(z) = \sum_{n \in 1/2+\mathbb{Z}} G_n z^{-3/2-n},\]

then the coefficients of such expansions satisfy the following commutation relations:

\[(2.1.6.1) \quad [L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12}c, \quad [G_m, L_n] = \left(m - \frac{n}{2}\right)G_{m+n}, \quad [G_m, G_n] = 2L_{m+n} + \delta_{m,-n} \frac{m^2 - 1/4}{3}c.\]

**Remark 2.1.7.** Let $V$ be a vertex algebra with an $N = 1$ superconformal vector $\tau$ (cf. [Kac96, definition 5.9]). Namely, the Fourier modes of the fields

\[G(z) = Y(\tau, z) \sum_{n \in 1/2+\mathbb{Z}} G_n z^{-n-3/2}, \quad L(z) = \frac{1}{2} Y(G_{-1/2}\tau, z) = \sum_{n \in \mathbb{Z}} L_n z^{-2-n},\]

satisfy the relations (2.1.6.1) of a Neveu-Schwarz algebra for some $c \in \mathbb{C}$, $L_{-1}(= G_{-1/2}^2) = T$ and the operator $L_0$ is diagonalizable with eigenvalues bounded below.
Then $V$ carries a structure of an $N_K = 1$ SUSY vertex algebra with $S = G_{-1/2}$ and the superfields are defined as:

\[(2.1.7.1) \quad Y(a, z, \theta) = Y(a, z) + \theta Y(G_{-1/2}a, z).\]

**Example 2.1.8.** [Kac96, ex. 5.9a] Let $B_1$ be the vertex algebra generated by an even vector (free boson) $\alpha$ and an odd vector (free fermion) $\varphi$, namely:

\[ [\alpha_\lambda \alpha] = \lambda, \quad [\varphi_\lambda \varphi] = 1, \quad [\alpha_\lambda \varphi] = 0. \]

Then $B_1$ is a (simple) vertex algebra with a family of $N = 1$ superconformal vectors

\[ \tau = (\alpha(-1)\varphi(-1) + m\varphi(-2))[0], \quad m \in \mathbb{C}, \]

of central charge $c = \frac{3}{2} - 3m^2$.

**Example 2.1.9.** [Kac96, Thm 5.10] The $N = 2$ vertex algebra is generated by a Virasoro field $L$ of central charge $c$, an even field $J$, primary of conformal weight 1, and two odd fields $G^\pm$, primary of conformal weight 3/2. The remaining $\lambda$-brackets are:

\[ [J_\lambda J] = \frac{c}{3} \lambda, \quad [G^\pm_\lambda G^\pm] = 0, \quad [J_\lambda G^\pm] = \pm G^\pm, \]

\[ [G^\pm_\lambda G^-] = L + \frac{1}{2} \partial J + \lambda J + \frac{c}{6} \lambda^2. \]

This vertex algebra contains an $N = 1$ superconformal vector:

\[ \tau = G^+_(-1)[0] + G^-_(-1)[0]. \]

Also, this vertex algebra admits a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{C}^*$ family of automorphisms. The generator of $\mathbb{Z}/2\mathbb{Z}$ is given by $L \mapsto L, J \mapsto -J$ and $G^\pm \mapsto G^\mp$. The $\mathbb{C}^*$ family is given by $G^+ \mapsto \mu G^+$ and $G^- \mapsto \mu^{-1} G^-$. Applying these automorphisms, we get a family of $N = 1$ superconformal structures.

By expanding the corresponding fields:

\[ L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-2-n}, \]

\[ G^\pm(z) = \sum_{n \in 1/2 + \mathbb{Z}} G^\pm_n z^{-3/2-n}, \quad J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-1-n}, \]

we get the commutation relations of the Virasoro operators $L_n$, and the following remaining commutation relations

\[ [J_m, J_n] = \frac{m}{3} \delta_{m,-n} c, \quad [J_m, G^\pm_n] = \pm G^\pm_{m+n}, \]

\[ [G^\pm_m, L_n] = (m - \frac{n}{2}) G^\pm_{m+n}, \quad [L_m, J_n] = -n J_{m+n}, \]

\[ [G^+_m, G^-_n] = L_{m+n} + \frac{m-n}{2} J_{m+n} + \frac{c}{6} \left( m^2 - \frac{1}{4} \right) \delta_{m,-n}. \]

Sometimes it is convenient to introduce a different set of generating fields for this vertex algebra. We define $\tilde{L}(z) = L(z) - 1/2 \partial J(z)$. This is a Virasoro field with central charge zero, namely:

\[ [\tilde{L}_\lambda \tilde{L}] = (\partial + 2\lambda) \tilde{L}. \]
With respect to this Virasoro element, \( G^+ \) is primary of conformal weight 2 and \( G^- \) is primary of conformal weight 1; \( J \) has conformal weight 1 but is no longer a primary field. To summarize the commutation relations, we write:

\[
Q(z) = G^+(z) = \sum_{n \in \mathbb{Z}} Q_n z^{-2-n}, \tag{2.1.9.1}
\]

\[
H(z) = G^-(z) = \sum_{n \in \mathbb{Z}} H_n z^{-1-n}, \quad \tilde{L}(z) = \sum_{n \in \mathbb{Z}} T_n z^{-2-n}.
\]

The corresponding \( \lambda \)-brackets of these fields are given by:

\[
[\tilde{L}_\lambda \tilde{L}] = (\partial + 2\lambda)\tilde{L}, \quad [\tilde{L}_\lambda J] = (\partial + \lambda)J - \frac{\lambda^2 c}{6}, \tag{2.1.9.2}
\]

\[
[\tilde{L}_\lambda Q] = (\partial + 2\lambda)Q, \quad [\tilde{L}_\lambda H] = (\partial + \lambda)H,
\]

\[
[H_\lambda Q] = \tilde{L} - \lambda J + \frac{c \lambda^2}{6}.
\]

The commutation relations of the coefficients in (2.1.9.1) are:

\[
[T_m, T_n] = (m - n)T_{m+n}, \quad [Q_m, Q_n] = [H_m, H_n] = 0, \tag{2.1.9.3}
\]

\[
[T_m, H_n] = -nH_{m+n}, \quad [T_m, J_n] = -nJ_{m+n} - m(m+1)\frac{c}{12} \delta_{m,-n},
\]

\[
[T_m, Q_n] = (m - n)Q_{m+n}, \quad [H_m, Q_n] = T_{m+n} - mJ_{m+n} + m(m-1)\frac{c}{6} \delta_{m,-n}
\]

Finally, defining \( G^{(1)} = G^+ + G^- \) and \( G^{(2)} = i(G^+ - G^-) \), we obtain another set of generators for this vertex algebra. We note that, with respect to \( L \), the fields \( G^{(i)} \) are primary of conformal weight 3/2, and \( J \) is primary of conformal weight 1.

The other commutation relations between the generating fields \( L, J, G^{(i)} \) are

\[
[G^{(i)}_\lambda G^{(i)}] = 2L + \frac{c \lambda^2}{3}, \quad [G^{(1)}_\lambda G^{(2)}] = -i(\partial + 2\lambda)J,
\]

\[
[J_\lambda G^{(1)}] = -iG^{(2)}, \quad [J_\lambda G^{(2)}] = iG^{(1)},
\]

or, equivalently:

\[
[G^{(i)}_m, G^{(i)}_n] = 2L_{m+n} + \left(m^2 - \frac{1}{4}\right)\frac{c}{3} \delta_{m,-n}, \tag{2.1.9.4}
\]

\[
[G^{(1)}_m, G^{(2)}_n] = i(n - m)J_{m+n},
\]

\[
[J_m, G^{(1)}_n] = -iG^{(2)}_{m+n}, \quad [J_m, G^{(2)}_n] = iG^{(1)}_{m+n}.
\]

**Remark 2.1.10.** As in the \( N = 1 \) case, given an \( N = 2 \) superconformal vertex algebra, namely a vertex algebra with a vector \( j \) and two operators \( S^1, S^2 \) satisfying

\[
[T, S^i] = 0, \quad [S^i, S^j] = 2\delta_{ij}T,
\]

and such that the corresponding fields:

\[
J(z) = -iY(j, z), \quad L(z) = \frac{1}{2}Y(S^2 S^1, j, z),
\]

\[
G^{(1)}(z) \equiv G^+(z) + G^-(z) = -Y(S^2 j, z),
\]

\[
G^{(2)}(z) \equiv i\left(G^+(z) - G^-(z)\right) = Y(S^1 j, z),
\]


satisfy the \(\lambda\)-brackets of Example 2.1.9, \(L_{-1} = T,\ G^{(i)}_{-1/2} = S^i,\) and \(L_0\) is diagonalizable with eigenvalues bounded below, we obtain an \(\mathcal{N}_K = 2\) SUSY vertex algebra by letting \(Y(a, Z) = Y(a, Z) + \theta Y(S^i a, z) + \theta^2 Y(S^2 a, z) + \theta^2 \theta Y(S^i S^2 a, z)\).

Similarly, given a vertex algebra with two vectors \(h,\ h\) and an odd operator \(S\) such that \([T, S] = 0,\ S^2 = 0\) and the associated fields:

\[
J(z) = -Y(j, z), \quad H(z) = Y(h, z), \\
Q(z) = Y(Sj, z), \quad \bar{L}(z) = Y(Sh, z) - \partial_z J(z),
\]
satisfy the commutation relations (2.1.9.2), \(T_{-1} = T,\ Q_{-1} = S,\ J_0\) is diagonalizable, and \(T_0\) is diagonalizable with eigenvalues bounded below, we obtain an \(\mathcal{N}_W = 1\) SUSY vertex algebra by letting: \(Y(a, Z) = Y(a, z) + \theta Y(Sa, z)\).

Example 2.1.11. [Kac96, ex. 5.9d] Consider the vertex algebra generated by a pair of free charged bosons \(\alpha^\pm\) and a pair of free charged fermions \(\varphi^\pm\) where the only non-trivial commutation relations are:

\[
[\alpha^\pm, \alpha^\mp] = \lambda, \quad [\varphi^\pm, \varphi^\mp] = 1.
\]

This vertex algebra contains the following family of \(N = 2\) vertex subalgebras:

\[
G^\pm =: \alpha^\pm \varphi^\pm : \pm m \partial \varphi^\pm, \quad J =: \varphi^+ \varphi^- : -m (\alpha^+ + \alpha^-), \quad m \in \mathbb{C}, \\
L =: \alpha^+ \alpha^- : + \frac{1}{2} : \partial \varphi^+ \varphi^- : + \frac{1}{2} : \partial \varphi^- \varphi^+ : - \frac{m}{2} \partial (\alpha^+ - \alpha^-).
\]

The vector \(j = iJ_{-1}[0]\) provides this vertex algebra with the structure of an \(\mathcal{N}_K = 2\) SUSY vertex algebra, by letting \(T = L_{-1}\) and \(S^i = G^{(i)}_{-1/2}\), where \(G^{(1)} = G^+ + G^-\) and \(G^{(2)} = i(G^+ - G^-)\).

Example 2.1.12. (\(W_N\) series) Now we define an \(\mathcal{N}_W = N\) SUSY vertex algebra for each non-negative integer \(N\). When \(N = 0,\ W_0\) is the Virasoro vertex algebra of central charge \(c\).

\(W_1\) is the \(\mathcal{N}_W = 1\) SUSY vertex algebra generated\(^1\) by an odd superfield \(L\) and an even superfield \(G\) satisfying:

\[
[L \Lambda L] = (T + 2 \lambda)L, \quad [Q \Lambda Q] = SQ + \frac{\lambda \chi}{3} c, \\
[L \Lambda Q] = (T + \lambda)Q - \chi L + \frac{\lambda^2}{6} c,
\]

where \(c \in \mathbb{C}\) is the central charge. Expanding these superfields as:

\[
Q(Z) = -J(z) + \theta G^+(z), \quad L(Z) = G^-(z) + \theta \left( L(z) + \frac{1}{2} \partial_z J(z) \right),
\]

we find that the fields \(L, J, G^\pm\) generate an \(N = 2\) vertex algebra of central charge \(c\) as in example 2.1.9.

\(W_2\) is the \(\mathcal{N}_W = 2\) SUSY vertex algebra generated by an even superfield \(L\) and two odd superfields \(Q^1\) and \(Q^2\) satisfying:

\[
[L \Lambda L] = (T + 2 \lambda)L, \quad [Q^i \Lambda Q^j] = S^i Q^j, \\
[Q^1 \Lambda Q^2] = (S^1 + \chi^1)Q^2 - \chi^2 Q^1 + \frac{\lambda}{6} c, \quad [L \Lambda Q^i] = (T + \lambda)Q^i + \chi^i L.
\]

\(^1\)See [HK06] for the definition of generating fields of a SUSY vertex algebra.
Finally, for $N \geq 3$ we let $W_N$ be the $N_W = N$ SUSY vertex algebra generated by a superfield $L$ of parity $N \mod 2$ and $N$ superfields $Q^i$, $i = 1, \ldots, N$ of parity $N + 1 \mod 2$, satisfying:

\[
\begin{align*}
[L_A L] &= (T + 2\lambda)L, \quad [Q^i_A Q^j] = (S^i + \chi^i)Q^j - \chi^j Q^i, \\
[L_A Q^i] &= (T + \lambda)Q^i + (-1)^N \chi^i L.
\end{align*}
\]

(2.1.12.3)

It is proved in [HK06] that the Lie superalgebra $W(1|N)$ of derivations on $\mathbb{C}[Z, Z^{-1}]$ acts on $W_N$. We let $W(1|N)_-$ (resp. $W(1|N)_+$) be the Lie subalgebras of regular vector fields (resp. regular vector fields vanishing at the origin).

**Definition 2.1.13.** An $N_W = N$ SUSY vertex algebra $V$ is called *conformal* if there exists $N + 1$ vectors $\nu, \tau^1, \ldots, \tau^N$ in $V$ such that their associated superfields $L(Z) = Y(\nu, Z)$ and $Q^i(Z) = Y(\tau^i, Z)$ satisfy (2.1.12.3) for $N \geq 3$, (2.1.12.2) for $N = 2$, (2.1.12.1) for $N = 1$ or (2.1.4.1) for $N = 0$, and moreover:

- $\nu|_{(0|0)} = T$, $\tau^i|_{(0|0)} = S^i$.
- The operator $\nu|_{(1|0)}$ acts diagonally with eigenvalues bounded below and with finite dimensional eigenspaces.

If moreover, the action of $W(1|N)_-$ on $V$ can be exponentiated to the group of automorphisms of the $1|N$ dimensional superdisk $D^{1|N}$ (see 2.2.10 for a definition), we will say that $V$ is *strongly conformal*. This amounts to the following extra condition:

- The operators $\nu|_{(1|0)}$ and $\sum_{i=1}^N \sigma(e_i)\tau^i|_{(0|e_i)}$ have integer eigenvalues.

**Example 2.1.14** ($K_N$ series). For $N \leq 3$, let $K_N$ be the $N_K = N$ SUSY vertex algebra generated by one superfield $G$ of parity $N \mod 2$ satisfying:

\[
\begin{align*}
[G_A G] &= \left(2T + (4 - N)\lambda + \sum_{i=1}^N \chi^i S^i\right) G + \frac{\lambda^{3-N} \chi^N}{3} c,
\end{align*}
\]

(2.1.14.1)

where $c \in \mathbb{C}$ is called the *central charge*. Let $K_4$ be the $N_K = 4$ SUSY vertex algebra generated by an even superfield $G$ satisfying:

\[
\begin{align*}
[G_A G] &= \left(2T + \sum_{i=1}^4 S^i \chi^i\right) G + \lambda c.
\end{align*}
\]

(2.1.14.2)

In the case $N = 1$, if we expand the corresponding superfield as

$G(z, \theta) = G(z) + 2\theta L(z),$

we find that the fields $G(z)$ and $L(z)$ generate a Neveu-Schwarz vertex algebra of central charge $c$ as in Example 2.1.6.

When $N = 2$ expanding the corresponding superfield as:

$G(z, \theta^1, \theta^2) = \sqrt{-1}J(z) + \theta^1 G^{(2)}(z) + \theta^2 G^{(1)}(z) + 2\theta^1 \theta^2 L(z),$

where $G^{(1)} = G^+ + G^-$ and $G^{(2)} = i(G^+ - G^-)$, we find that the corresponding fields $J, L, G^\pm$ satisfy the commutation relations of the $N = 2$ vertex algebra as in Example 2.1.9.

It was proved in [HK06] that the Lie superalgebra $K(1|N)$ of vector fields on the $1|N$-dimensional superdisk $D^{1|N}$ preserving the differential 1-form

$\omega = dz + \sum_{i=1}^N \theta^i d\theta^i,$
up to multiplication by a function, acts on $K_N$. We let $K(1|N)$ (resp. $K(1|N)_\prec$) be the Lie subalgebra of regular vector fields (resp. regular vector fields vanishing at the origin).

**Definition 2.1.15.** Let $N \leq 4$, an $N_K = N$ SUSY vertex algebra $V$ is called **conformal** if there exists a vector $\tau \in V$ (called the conformal vector) such that the corresponding field $G(Z) = Y(\tau, Z)$ satisfies (2.1.14.1) for $N \leq 3$ or (2.1.14.2) for $N = 4$, and moreover

- $\tau_{(0|0)} = 2T$, $\tau_{(0|e_i)} = \sigma(N \setminus e_i) S_i$,
- the operator $\tau_{(1|0)}$ acts diagonally with eigenvalues bounded below and finite dimensional eigenspaces.

If moreover, the representation of $K(1|N)_\prec$ can be exponentiated to the group of automorphisms of the disk $D^{1|N}$ preserving the differential form $\omega$ up to multiplication by a function, we will say that $V$ is **strongly conformal**. This amounts to the extra condition

- the operator $\tau_{(1|0)}$ has integer eigenvalues, and if $N = 2$, the operator $\sqrt{-1} \tau_{(0|N)}$ has integer eigenvalues.

**Example 2.1.16.** (boson-fermion system) Let $B_1$ be the strongly conformal $N_K = 1$ SUSY vertex algebra generated by one odd superfield $\Psi$ satisfying:

$$[\Psi \Lambda \Psi] = \chi.$$  

Expanding the corresponding superfield as:

$$\Psi(Z) = \varphi(z) + \theta \alpha(z),$$

we find that the ordinary fields $\varphi$ and $\alpha$ generate the well known **boson-fermion system** as in Example 2.1.8.

2.1.17. Now we summarize some basic results in the structure theory of SUSY vertex algebras. The proofs can be found in [HK06]. We will denote $\nabla = (T, S^1, \ldots, S^N)$, and for each $(j|J)$ as above, we define:

$$\nabla_{j|J} = T_j S^J, \quad \nabla^{(j|J)} = \frac{(-1)^{j(j+1)}}{j!} \nabla_{j|J}.$$  

Let $Z = (z, \theta^1, \ldots, \theta^N)$ be as before, and $W = (w, \zeta^1, \ldots, \zeta^N)$ be such that $w$ is an even indeterminate, commuting with $z, \theta^i$'s and $\zeta^i$'s, and $\zeta^i$ are odd anticommuting indeterminates, commuting with $z, w$ and anticommuting with $\theta^i$'s. In the $N_W = N$ case we will write:

(2.1.17.1)  
$$Z - W = (z - w, \theta^1 - \zeta^1, \ldots, \theta^N - \zeta^N),$$

and in the $N_K = N$ case:

(2.1.17.2)  
$$Z - W = \left(z - w - \sum_{i=1}^{N} \theta^i \zeta^i, \theta^1 - \zeta^1, \ldots, \theta^N - \zeta^N \right).$$

We define the formal super delta-function to be:

$$\delta(Z, W) = (i_{z, w} - i_{w, z}) (Z - W)^{-1|N},$$

where $i_{z, w}$ denotes the expansion in the domain $|z| > |w|$. We note that this definition is independent of the definition of $Z - W$ as (2.1.17.1) or (2.1.17.2).
Put $Z \nabla = z T + \sum_i \theta^i S^i$. Similarly, in the $N_W = N$ (resp. $N_K = N$) case, we put $D_W = (\partial_w, \partial_{\chi^i})$ (resp. $D_W = (\partial_w, \partial_{\chi^i} + \zeta^i \partial_\omega)$).

**Proposition 2.1.18.** Let $V$ be a $N_W = N$ or $N_K = N$ SUSY vertex algebra. Then

$$[a_{(n|I)}, Y(b, W)] = \sum_{(j|J), J \geq 0} (-1)^{jN + I N + I J} \sigma(J) \sigma(I) \times$$

$$\times (D_W^{(j|J)} W^{n|J}) Y (a_{(j|J)} b, W).$$

If, moreover, $n \geq 0$, this becomes:

$$[a_{(n|I)}, Y(b, W)] = Y(e^{-W \nabla} a_{(n|I)}) e^{W \nabla} b, W).$$

**Theorem 2.1.19.** In an $N_W = N$ (resp. $N_K = N$) SUSY vertex algebra the following identities hold (see [HK06] for the definition of $(j|J)$-th product of superfields)

1. $Y(a_{(j|J)}, b)(Z) = \sigma(J)(D_W^{(j|J)} \delta(Z, W)) Y(a_{(j|J)} b, W)$ (the $(j|J)$-th product identity),
2. $Y(a_{(-1|N)} b, Z) = Y(a, Z) Y(b, Z)$ ;
3. $Y(T a, Z) = \partial_Z Y(a, Z)$,
4. $Y(S^i a, Z) = \partial_{\chi^i} Y(a, Z)$ (resp. $(\partial_{\chi^i} + \zeta^i \partial_\omega) Y(a, Z)$),
5. we have the following OPE formula:

$$(2.19.1) [Y(a, z), Y(b, w)] = \sum_{(j|J), J \geq 0} \sigma(J) (D_W^{(j|J)} \delta(Z, W)) Y(a_{(j|J)} b, W)$$

$$= \sum_{(j|J), J \geq 0} (i_{z, w} - i_{w, z}) (Z - W)^{-1 - j|\mathcal{N}|\mathcal{J}} Y(a_{(j|J)} b, W),$$

where the sum is finite and the operator $i_{z, w}$ denotes the expansion in the domain $|z| > |w|$.

**Lemma 2.1.20.** The following identity is true (note that $D_W$ and $Z - W$ here have different meanings in the $N_W = N$ and $N_K = N$ case):

$$(2.20.1) D_W^{(j|J)} \delta(Z, W) = \sigma(J, N \setminus J)(i_{z, w} - i_{w, z}) (Z - W)^{-1 - j|\mathcal{N}|\mathcal{J}}.$$

**Theorem 2.1.21** (Skew-symmetry). In a SUSY vertex algebra the following identity, called skew-symmetry, holds

$$(2.1.21.1) Y(a, Z)b = (-1)^{ab} e^{Z \nabla} Y(b, -Z)a$$

**Theorem 2.1.22** (Cousin property). For any SUSY vertex algebra $V$ and vectors $a, b, c \in V$, the three expressions:

$$Y(a, Z) Y(b, W) c \in V((Z))((W))$$
$$(-1)^{ab} Y(b, W) Y(a, Z) c \in V((W))((Z))$$
$$Y(Y(a, Z - W) b, W) c \in V((W))((Z - W))$$

are the expansions, in the domains $|z| > |w|$, $|w| > |z|$ and $|w| > |w - z|$ respectively, of the same element of $V[[Z, W]][z^{-1}, w^{-1}, (z - w)^{-1}]$. 


2.1.23. Let \( V \) be an \( N_W = N \) (resp. \( N_K = N \)) SUSY vertex algebra. It was proved in [HK06] that \( \text{Lie}(V) = \bar{V}/\bar{\nabla}V \) is naturally a Lie algebra\(^2\), where \( \bar{V} = V \otimes_{\mathbb{C}} \mathbb{C}[Z,Z^{-1}] \) and \( \bar{\nabla}V \) is the space spanned by vectors of the form:

\[
Ta \otimes f(Z) + a \otimes \partial_z f(Z),
\]

(2.1.23.1) \[ S^i a \otimes f(Z) + (-1)^{a N} a \otimes \partial_{\theta^i} f(Z), \]

(resp. \( S^i a \otimes f(Z) + (-1)^{a N} a \otimes (\partial_{\theta^i} + \theta^i \partial_z) f(Z) \)),

for \( a \in V, f(Z) \in \mathbb{C}[Z,Z^{-1}] \). Let \( \varphi : \text{Lie}(V) \to \text{End}(V) \) be the linear map defined by

\[
a_{<n|I>} = a \otimes Z^{n|I} \mapsto (-1)^{n I} \sigma(I) a_{(n|I)}, \quad a \in V.
\]

Similarly, we construct \( V \otimes_{\mathbb{C}} \mathbb{C}((Z)) \) and consider its quotient \( \text{Lie}'(V) \) by the vector space generated by vectors of the form (2.1.23.1). Then (2.1.23.2) defines a map \( \varphi' : \text{Lie}'(V) \to \text{End}(V) \).

**Theorem 2.1.24.** The maps \( \varphi \), and \( \varphi' \) are Lie algebra homomorphisms.

2.2. **Supercurves.**

2.2.1. For a general introduction to the theory of supermanifolds and superschemes, the reader should refer to [Man97]. We will follow [BR99] for the theory of supercurves over a Grassmann algebra \( \Lambda \). The deformation theory of superspaces and sheaves over them can be found in [Vai90]. The relations between superconformal Lie algebras and the moduli spaces of supercurves was stated in [Vai95]. The reader may also find useful the notes [DM99].

**Definition 2.2.2.** A *superspace* is a locally ringed space \((X, \mathcal{O}_X)\) where \( X \) is a topological space and \( \mathcal{O}_X \) is a sheaf of supercommutative rings. A *morphism* of superspaces is a graded morphism of locally ringed spaces. We will use \( X \) to denote such a superspace when no confusion should arise. A *superscheme* is a superspace such that \((X, \mathcal{O}_{X,0})\) is a scheme, where from now on \( \mathcal{O}_{X,i} \) denotes the i-th graded part of \( \mathcal{O}_X \), \( i = 0, 1 \).

2.2.3. Given a superspace \((X, \mathcal{O}_X)\) define \( \mathcal{J} = \mathcal{O}_{X,1} + \mathcal{O}_{X,-1}^2 \). \( \mathcal{J} \) is a sheaf of ideals in \((X, \mathcal{O}_X)\), the corresponding subspace \((X, \mathcal{O}_X/\mathcal{J})\) will be denoted \((X_{\mathcal{J}}, \mathcal{O}_{X_{\mathcal{J}}})\).

**Example 2.2.4.** Let \( R \) be a supercommutative ring, and let \( J = R_{1} + R_{-1}^2 \) be the ideal generated by \( R_1 \) as above, then \((\text{Spec} R, R)\) is a superscheme. Note that as topological spaces \( \text{Spec} R = \text{Spec} R/J \) since every element in \( J \) is nilpotent (we consider only homogeneous ideals with respect to the \( \mathbb{Z}/2\mathbb{Z} \)-grading).

**Definition 2.2.5** (cf. [Man97]). A *supermanifold* is a superspace \((X, \mathcal{O}_X)\) such that for every point \( x \in X \) there exists an open neighborhood \( U \) of \( x \) and a locally free sheaf \( \mathcal{E} \) of \( \mathcal{O}_{X_{\mathcal{J}}} \)-modules, of (purely odd) rank \( 0|q \) such that \((U, \mathcal{O}_X|_U)\) is isomorphic to \((U_{\mathcal{J}}, S\mathcal{O}_{X_{\mathcal{J}}}|_U(\mathcal{E})|_U)\). Here \( S(\mathcal{E}) \) denotes the symmetric algebra of a vector bundle.

2.2.6. An open sub-supermanifold of \((X, \mathcal{O}_X)\) consist of an open subset \( U \subset X \) and the restriction of the structure sheaf, namely \((U, \mathcal{O}_X|_U)\).

\(^2\)Here we need to change the parity of \( \bar{V}/\bar{\nabla}V \) if \( N \) is odd, see [HK06] for details.
2.2.7. In the analytic setting, the situation is easier to describe. The supermanifold $\mathbb{C}^{p|q}$ is the topological space $\mathbb{C}^{p}$ endowed with the sheaf of supercommutative algebras $\mathcal{O}^{p|q}$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on $\mathbb{C}^{p}$ and $\theta_{i}$ are odd anticommuting variables. A complex supermanifold is a topological space $|X|$ with a sheaf of supercommutative algebras $\mathcal{O}_{X}$ locally isomorphic to $\mathbb{C}^{p|q}$. Morphisms of supermanifolds are continuous maps $\sigma : |X| \to |Y|$ together with morphisms of sheaves $\sigma^{*} : \mathcal{O}_{Y} \to \mathcal{O}_{X}$.

Let $\Lambda = \mathbb{C}[\alpha_{1}, \ldots, \alpha_{n}]$ be a Grassmann algebra. The $0|n$-dimensional superscheme $\text{Spec} \Lambda$ has as underlying topological space a single point. We will work in the category of superschemes over $\Lambda$, namely super schemes $\text{Spec} \Lambda$ has as underlying topological space a single point. We will work in the category of superschemes over $\Lambda$, namely super schemes $\text{Spec} \Lambda$ has as underlying topological space a single point. We will work in the category of superschemes over $\Lambda$, namely super schemes $\text{Spec} \Lambda$ has as underlying topological space a single point.

More explicitly (cf. [BR99]), a smooth compact connected complex supercurve over $\Lambda$ of dimension 1 is a pair $(X, \mathcal{O}_{X})$, where $X$ is a topological space and $\mathcal{O}_{X}$ is a sheaf of supercommutative $\Lambda$-algebras over $X$ such that:

1. $(X, \mathcal{O}_{X}^{\text{red}})$ is a smooth compact connected algebraic curve. Here $\mathcal{O}_{X}^{\text{red}}$ is the reduced sheaf of $\mathcal{O}$-algebras on $X$ obtained by quotienting out the nilpotents in $\mathcal{O}_{X}$.

2. For some open sets $U_{\alpha} \subset X$ and some linearly independent odd elements $\theta_{i}^{\alpha}$ of $\mathcal{O}_{X}(U_{\alpha})$ we have $\mathcal{O}_{X}(U_{\alpha}) = \mathcal{O}_{X}^{\text{red}} \otimes \Lambda[\theta_{1}^{\alpha}, \ldots, \theta_{n}^{\alpha}]$.

The $U_{\alpha}$ above are called coordinate neighborhoods of $(X, \mathcal{O}_{X})$ and $Z_{\alpha} = (z_{\alpha}, \theta_{1}^{\alpha}, \ldots, \theta_{n}^{\alpha})$ are called local coordinates for $(X, \mathcal{O}_{X})$ if $z_{\alpha}$ (mod nilptents) are local coordinates for $(X, \mathcal{O}_{X}^{\text{red}})$. On overlaps $U_{\alpha} \cap U_{\beta}$ we have:

$$(2.2.8.1) \quad z_{\beta} = F_{\beta\alpha}(z_{\alpha}, \theta_{1}^{\alpha}), \quad \theta_{j}^{\beta} = \Psi_{\beta\alpha}(z_{\alpha}, \theta_{j}^{\alpha}),$$

where $F_{\beta\alpha}$ are even and $\Psi_{\beta\alpha}$ are odd. We will write such a change of coordinates as $Z_{\beta} = \rho_{\beta\alpha}(Z_{\alpha})$ with $\rho = (F, \Psi)$ where no confusion should arise.

2.2.9. A $\Lambda$-point of a supercurve $(X, \mathcal{O}_{X})$ is a morphism $\varphi : \text{Spec} \Lambda \to (X, \mathcal{O}_{X})$ over $\Lambda$, namely the composition of $\varphi$ with the structure morphism $(X, \mathcal{O}_{X}) \to \text{Spec} \Lambda$ is the identity. Locally, a $\Lambda$ point is given by specifying the images of the local coordinates under the even $\Lambda$-homomorphism $\varphi^{*} : \mathcal{O}_{X}(U_{\alpha}) \to \Lambda$. These local parameters $(p_{\alpha} = \varphi^{*}(z_{\alpha}), \pi_{i}^{\alpha} = \varphi^{*}(\theta_{i}^{\alpha}))$ transform as the coordinates do in (2.2.8.1).

2.2.10. The $N = q$ formal superdisk is an ind-superscheme as in the non-super situation, namely, let $R = \mathbb{C}[t, \theta^{1}, \ldots, \theta^{q}]$ and let $m$ be the maximal ideal generated by $(t, \theta^{1}, \ldots, \theta^{q})$. We define the superschemes $D^{(n)} = \text{Spec} R/m^{n+1}$ and we have embeddings $D^{(n)} \hookrightarrow D^{(n+1)}$. The formal disk is then

$$D = \lim_{n \to \infty} D^{(n)}.$$ 

If we want to emphasize the dimensions of these disks we will denote them by $D^{1/q}$.

2.2.11. Vector bundles of rank $(p|q)$ over a supermanifold $(X, \mathcal{O}_{X})$ are locally free sheaves $\mathcal{E}$ of $\mathcal{O}_{X}$-modules over $X$, of rank $p|q$. That is, locally, $\mathcal{E}$ is isomorphic to $\mathcal{O}_{X}^{\oplus (\Pi \mathcal{O}_{X})^{\theta}}$ where $\Pi$ is the parity change functor.

An example is the tangent bundle to a $p|q$-dimensional supermanifold $(X, \mathcal{O}_{X})$; it is a rank $p|q$ vector bundle. Its fiber at the point $x \in X$ is given as in the non-super case as the subset of morphisms in Hom($D^{(1)}, X$) mapping the closed point in $D^{(1)}$ to $x$. The cotangent bundle $\Omega_{X}^{1}$ of $(X, \mathcal{O}_{X})$ is the dual of the tangent bundle.
Another example is the Berezinian bundle of a supermanifold \((X, \mathcal{O}_X)\). We will define this bundle by giving local trivializations. Recall [DM99, §1.10] that given a free module \(L\) of finite type over a supercommutative algebra \(A\), the superdeterminant is a homomorphism
\[
\text{sdet} : \GL(L) \to \GL(1|0) = A^0_0,
\]
defined in coordinates as follows: for a parity preserving automorphism \(T\) of \(A^p|q\) with matrix \((K L)_{MN}\) we put:
\[
\text{sdet}(T) = \det(K - LN^{-1}M) \det(N)^{-1}.
\]
With this definition we can now define the Berezinian of the module \(L\) as the following \(A\)-module denoted \(\text{Ber}(L)\). Let \(\{e_1, \ldots, e_{p+q}\}\) be a basis of \(L\) where the first \(p\) elements are even and the last \(q\) are odd. This basis defines a one-element basis of \(\text{Ber}(L)\) denoted by \([e_1 \ldots e_{p+q}]\) of parity \(q \mod 2\). Given an automorphism \(T\) of \(L\) we put:
\[
[T e_1 \ldots T e_{p+q}] = \text{sdet}(T)[e_1 \ldots e_{p+q}].
\]
This makes \(\text{Ber}(L)\) a well defined rank \(1|0\) \(A\)-module when \(q\) is even and a rank \(0|1\) \(A\)-module when \(q\) is odd. Now we can define the Berezinian bundle of \((X, \mathcal{O}_X)\) as \(\text{Ber}_X = \text{Ber}(\Omega^1_X)\).

2.2.12. Given an \(N = 1\) supercurve \((X, \mathcal{O}_X)\) and an extension of \(\mathcal{O}_X\) by an invertible sheaf \(\mathcal{E}\):
\[
0 \to \mathcal{O}_X \to \hat{\mathcal{E}} \to \mathcal{E} \to 0,
\]
we can construct an \(N = 2\) supercurve \((Y, \mathcal{O}_Y)\) canonically. Its local coordinates are given by \((z_\alpha, \theta_\alpha, \rho_\alpha)\), where \((z_\alpha, \theta_\alpha)\) are local coordinates of \(X\) and \(\rho_\alpha\) are local sections of \(\mathcal{E}\). In each coordinate patch \(U_\alpha\) we can construct the form \(dz_\alpha - d\theta_\alpha \rho_\alpha\).

We say that the \(N = 2\) supercurve \((Y, \mathcal{O}_Y)\) is superconformal if this form is globally defined up to multiplication by a function.

This happens if on overlaps \(U_\alpha \cap U_\beta\) we have (see (2.2.8.1)):
\[
\rho_\beta = \text{sdet} \left( \begin{array}{cc} \partial_z F & \partial_\Psi \\ \partial_\theta F & \partial_\Psi \end{array} \right) \rho_\alpha + \frac{\partial F}{\partial_\Psi} \frac{\partial_\Psi}{\partial_\Psi},
\]
where \(\text{sdet}\) is the superdeterminant of an automorphism defined above, which can be written as
\[
\text{sdet} \left( \begin{array}{cc} \partial_z F & \partial_\Psi \\ \partial_\theta F & \partial_\Psi \end{array} \right) = D \left( \frac{DF}{D\Psi} \right),
\]
where \(D = \partial_\theta + \theta \partial_z\). Conversely, if (2.2.12.2) is satisfied on overlaps, the cocycle condition is satisfied and we have an extension as in (2.2.12.1).

Therefore to each \(N = 1\) supercurve \((X, \mathcal{O}_X)\), we canonically associate a \(N = 2\) superconformal curve \((Y, \mathcal{O}_Y)\).

From (2.2.12.2) we see that we have an exact sequence of sheaves on \(Y\):
\[
0 \to \mathcal{O}_X \to \mathcal{O}_Y \to \text{Ber}_X \to 0,
\]
where \(\text{Ber}_X\) is the Berezinian bundle on \((X, \mathcal{O}_X)\). The last map \(\hat{D} : \mathcal{O}_Y \to \text{Ber}_X\) is given in the above local coordinates, by the differential operator \(\partial_{\rho_\alpha}\).
Introducing new coordinates
\[ \hat{z}_\alpha = z_\alpha - \theta_\alpha \rho_\alpha, \quad \hat{\rho}_\alpha = \theta_\alpha, \quad \hat{\rho}_\alpha = \rho_\alpha, \]
we obtain on overlaps \( U_\alpha \cap U_\beta \):
\[
\begin{align*}
\hat{z}_\beta &= F(\hat{z}_\alpha, \hat{\rho}_\alpha) + \frac{DF(\hat{z}_\alpha, \hat{\rho}_\alpha)}{D\Psi(\hat{z}_\alpha, \hat{\rho}_\alpha)} \Psi(\hat{z}_\alpha, \hat{\rho}_\alpha), \\
\hat{\rho}_\beta &= \frac{DF(\hat{z}_\alpha, \hat{\rho}_\alpha)}{D\Psi(\hat{z}_\alpha, \hat{\rho}_\alpha)},
\end{align*}
\]
where \( D = \partial_\theta + \theta \partial_\zeta \) in local coordinates \((\zeta, \theta, \rho)\) as above.

We see from \((2.2.12.3)\) that \( \partial_Y \) contains the structure sheaf of another \( N = 1 \) supercurve \((X, \partial_X)\), whose local coordinates are \((\hat{z}_\alpha, \hat{\rho}_\alpha)\). We call \((\hat{X}, \partial_{\hat{X}})\) the dual curve of \((X, \partial_X)\).

Finally, we define an \( N = 1 \) superconformal curve as an \( N = 1 \) supercurve \((X, \partial_X)\) which is self-dual. We see from \((2.2.12.3)\) that the transition functions \( F, \Psi \) must satisfy:
\[
\begin{align*}
DF &= \Psi D\Psi, \\
(2.2.12.4)
\end{align*}
\]
for \((X, \partial_X)\) to be superconformal. In this case the operator \( D_\alpha = \partial_{\theta_\alpha} + \theta_\alpha \partial_{z_\alpha} \)
transforms as:
\[
(2.2.12.5)
\]
hence in this situation the supercurve \((X, \partial_X)\) carries a 0|1-dimensional distribution \( D \) such that \( D^2 \) is nowhere vanishing (since \( D^2 = \partial_\zeta \) in local coordinates).

**Remark 2.2.13.** An equivalent definition of \( N = 1 \) and \( N = 2 \) superconformal curves was given by Manin \([\text{Man}91]\) (under the name SUSY curves). Let \( X \) be a complex supermanifold of dimension \( 1|N \) \((N = 1 \text{ or } 2)\). When \( N = 1 \) we say that a locally free direct subsheaf \( \mathcal{T}^1 \subset \mathcal{T}_X \) \( (\mathcal{T}_X \text{ is the tangent sheaf of } X) \) of rank \( 0|1 \) for which the Frobenius form
\[
(\mathcal{T}^1) \otimes \mathcal{T}^1 \to \mathcal{T}^0 := \mathcal{T}_X / \mathcal{T}, \quad t_1 \otimes t_2 \mapsto [t_1, t_2] \mod \mathcal{T},
\]
is an isomorphism, is a SUSY structure on \( X \).

When \( N = 2 \), a SUSY structure consists of two locally free direct subsheaves \( \mathcal{T}'^1, \mathcal{T}'^2 \) of \( \mathcal{T}_X \) of rank \( 0|1 \) whose sum in \( \mathcal{T}_X \) is direct, they are integrable distributions and the Frobenius form
\[
\mathcal{T}'^1 \otimes \mathcal{T}'^1 \to \mathcal{T}_X / (\mathcal{T}'^1 \oplus \mathcal{T}'^2), \quad t_1 \otimes t_2 \mapsto [t_1, t_2] \mod (\mathcal{T}'^1 \oplus \mathcal{T}'^2),
\]
is an isomorphism.

Let \((X, \partial_X)\) be an \( N = 1 \) supercurve and \( D_\alpha \) be a family of vector fields in \( U_\alpha \), such that \( D_\alpha \) and \( D_\alpha^2 \) form a basis for \( \mathcal{T}_X \) on \( U_\alpha \) and \( D_\alpha = G_{\alpha\beta} D_\beta \) on \( U_\alpha \cap U_\beta \), where \( G_{\alpha\beta} \) is a family of invertible even functions. The sheaf defined by \( \mathcal{T}^1 |_{U_\alpha} = \partial_X D_\alpha \) is a SUSY structure in \((X, \partial_X)\) \([\text{Man}91]\). In local coordinates as above, the vector fields \( D_\alpha = \partial_{\theta_\alpha} + \theta_\alpha \partial_{z_\alpha} \)
satisfy these conditions when \( X \) is an \( N = 1 \) superconformal curve (see \((2.2.12.5)\)).

The \( N = 2 \) case is similar. Let \((X, \partial_X)\) be an \( N = 2 \) supercurve and \( \{D_\alpha^1, D_\alpha^2\} \) be a family of vector fields such that \( D_\alpha^1, [D_\alpha^1, D_\alpha^2] \) generate \( \mathcal{T}_X \) in \( U_\alpha \) and, moreover, we have:
\[
\begin{align*}
(D_\alpha^1)^2 &= f_{\alpha}^1 D_\alpha^1, \\
(D_\alpha^2)^2 &= f_{\alpha}^2 D_\alpha^2, \\
D_\alpha^1 &= F_{\alpha\beta} D_\beta^1, \\
D_\alpha^2 &= F_{\alpha\beta} D_\beta^2 \text{ on } U_\alpha \cap U_\beta,
\end{align*}
\]
where $f^i_\gamma$ and $F^i_{\alpha \beta}$ are even functions. Putting $\mathcal{F}'|_{U_\alpha} = \mathcal{O}_X D^1_\alpha$ and $\mathcal{F}''|_{U_\alpha} = \mathcal{O}_X D^2_\alpha$ we obtain an $N = 2$ superconformal structure on $(X, \mathcal{O}_X)$. If the two distributions $\mathcal{F}'$ and $\mathcal{F}''$ can be distinguished globally, the $N = 2$ superconformal curve is called orientable and a choice of one of these distributions is called its orientation.

It is clear that the construction given in 2.2.12 gives an oriented $N = 2$ superconformal curve; conversely, given such a curve, we can consider the functor $X \to X/\mathcal{F}'$ (recall that $\mathcal{F}'$ is integrable therefore this quotient makes sense). The duality that was explained in 2.2.12 corresponds to the duality $X/\mathcal{F}' \leftrightarrow X/\mathcal{F}''$.

2.2.14. Recall [BR99] that a $\Lambda$-point of an $N = 1$ supercurve $X$ transforms as an irreducible divisor of the dual curve $\hat{X}$. Indeed, an irreducible divisor of $X$ is given in local coordinates $(z_\alpha, \theta_\alpha)$ by expressions of the form $P_\alpha = z_\alpha - \hat{z}_\alpha - \theta_\alpha \rho_\alpha$. Two divisors $P_\alpha$ and $P_\beta$ are said to correspond to each other in the intersection $U_\alpha \cap U_\beta$ if in this intersection we have

$$P_\beta(z_\beta, \theta_\beta) = P_\alpha(z_\alpha, \theta_\alpha) g(z_\alpha, \theta_\alpha)$$

for some even invertible function $g(z_\alpha, \theta_\alpha)$ (we consider Cartier divisors). It is easy to see that the parameters $z_\alpha, \rho_\alpha$ transform as in (2.2.12.3), namely as the parameters of a $\Lambda$-point of $\hat{X}$.

2.2.15. We can define a theory of contour integration on an $N = 1$ superconformal curve as in [Fri86], [McA88], [Rog88]. We describe briefly a generalization to arbitrary $N = 1$ supercurves due to Bergvelt and Rabin (cf. [BR99]). For simplicity, we will work in the analytic category. Let us define a super contour to be a triple $\Gamma = (\gamma, P, Q)$ consisting of an ordinary contour $\gamma$ on the reduction $|X|$ and two Cartier divisors as in 2.2.14 such that their reductions to $|X|$ are the endpoints of $\gamma$. If in local coordinates

$$P = z - \hat{p} - \theta \hat{\pi}, \quad Q = z - \hat{q} - \theta \hat{\xi},$$

then the corresponding $\Lambda$-points of the dual curve $\hat{X}$ are given by $(\hat{p}, \hat{\pi})$ and $(\hat{q}, \hat{\xi})$. Let $z = \rhozd$ and $z = \theta zd$ be the equations for the reductions of these points, i.e. the endpoints for $\gamma$. We define the integral of a section $\omega_\alpha = D \hat{f}_\alpha$ of the Berezinian sheaf of $X$ (Recall that $D : \mathcal{O}_X \to \text{Ber} X$) along $\Gamma$ by:

$$\int_P^Q \omega = \int_P^Q D \hat{f} = \int_P^Q \hat{f}(\hat{q}, \hat{\xi}) - \hat{f}(\hat{p}, \hat{\pi}),$$

where we assume that the contour connecting $P$ and $Q$ lies in a single simply connected open set $U_\alpha$. If the contour traverses several open sets then we need to choose intermediate divisors on each overlap and we have to prove that the resulting integral is independent of these divisors. In what follows we will only need the integration in a sufficiently “small” open set $U_\alpha$ (the formal disk around a point).

Dually, we can integrate sections of $\text{Ber} \hat{X}$ along contours in $X$. Indeed, let $\gamma$ be a path in the topological space $|X|$ and two $\Lambda$-points $P, Q$ of $X$ whose reduced parts are the end-points of $\gamma$. Let $\hat{\omega} \in \text{Ber} \hat{X}(U_\alpha)$ and suppose that $\gamma$ lies in a simply connected open $U_\alpha$. Then $\hat{\omega} = \hat{D} f$ for some function $f \in \mathcal{O}_X(U_\alpha)$, and we put

$$\int_P^Q \hat{\omega} = f(Q) - f(P).$$
As it is shown in [BR99], this theory of integration can be understood in terms of a theory of contour integration on the corresponding $N = 2$ superconformal curve (cf. [Coh87]). For this let $X$ and $\hat{X}$ be an $N = 1$ supercurve and its dual, and let $Y$ be the corresponding $N = 2$ superconformal curve. We have two short exact sequences of sheaves in $Y$:

$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \overset{D^-}{\longrightarrow} \text{Ber} \hat{X} \to 0,$$

$$0 \to \mathcal{O}_{\hat{X}} \to \mathcal{O}_Y \overset{D^+}{\longrightarrow} \text{Ber} X \to 0.$$

We can define a sheaf operator on $\mathcal{O}^2 Y$ by the component-wise action of the differential operators $(D^-, D^+)$. It is shown in [BR99] that for $U$ a simply connected open in $|Y| = |X|$ and $(f, g)$ a section of $\mathcal{O}^2 Y(U)$ such that $(D^-, D^+) (f, g) = 0$, there exists a section $H \in \mathcal{O}_Y(U)$, unique up to an additive constant, such that $(f, g) = (D^- H, D^+ H)$. Let $\mathcal{M}$ be the subsheaf of $\mathcal{O}^2 Y$ consisting of closed sections $(f, g)$ as above. It follows that $\mathcal{M} = \text{Ber} X \oplus \text{Ber} \hat{X}$. A super contour in $Y$ consists of a triple $(\gamma, P, Q)$ where $P$ and $Q$ are $\Lambda$-points of $Y$ such that their reduced points are the endpoints of $\gamma$. If $\gamma$ is supported on a simply connected open set $U$, then any section $\omega \in \mathcal{M}(U)$ can be written as $(D^- H, D^+ H)$ and we put:

$$\int^Q_P \omega = H(Q) - H(P).$$

The extension to contours not lying in a single simply connected $U$ is straightforward but we will not need it.

2.2.16. We will define in general a superconformal $N = n$ supercurve to be a supercurve such that in some coordinate system $Z_\alpha = (z_\alpha, \theta^i_\alpha)$, the differential form

$$(2.2.16.1) \quad \omega = dz_\alpha + \sum_i \theta^i_\alpha d\theta^i_\alpha$$

is well defined up to multiplication by a function. It is easy to show that this definition agrees with the definition above in the $N = 1$ and $N = 2$ cases (cf. §3.1.4 and §3.1.5).

A set of coordinates $Z = (z, \theta^i)$ such that the form $\omega$ has the form (2.2.16.1) (up to multiplication by a function) will be called SUSY coordinates (or coordinates compatible with the superconformal structure).

Let $(z, \theta)$ and $(z', \theta')$ be two local coordinates compatible with a (local) superconformal structure on an $N = 1$ supercurve $(X, \mathcal{O}_X)$. Denote $D = \partial_\theta + \theta \partial_z$ and $D' = \partial_{\theta'} + \theta' \partial_{z'}$. Let $G$ be the invertible function such that $D = GD'$ (cf. (2.2.12.5)). We define the Schwarzian derivative of $(z', \theta')$ with respect to $(z, \theta)$ to be the (odd) function

$$(2.2.16.2) \quad \sigma(G) = \frac{D^3 G}{G^2} - 2 \frac{DG D^2 G}{G^2}.$$

**Definition 2.2.17.** A superprojective structure on an $N = 1$ superconformal curve over $\Lambda$ is a (maximal) atlas consisting of coordinates $(z_\alpha, \theta_\alpha)$ compatible with the superconformal structure, and such that its transition functions are fractional linear transformations, that is, changes of coordinates of the form:

$$z' = \frac{az + b + \alpha \theta}{cz + d + \beta \theta}, \quad \theta' = \frac{\gamma z + \delta + \epsilon \theta}{cz + d + \beta \theta}.$$
for some even constants $a, b, c, d$ and $e \in \Lambda$ and some odd constants $\alpha, \beta, \gamma$ and $\delta \in \Lambda$, such that

$$\text{sdet} \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & e \end{pmatrix} = 1$$

**Proposition 2.2.18** ([Man97] Proposition 4.7). Let $(z, \theta)$ and $(z', \theta')$ be two local coordinates on $(X, \mathcal{O}_X)$. The following statements are equivalent:

1. $(z, \theta)$ and $(z', \theta')$ are compatible with a common superconformal structure and $\sigma = 0$.
2. $(z, \theta)$ and $(z', \theta')$ define the same superprojective structure.

3. The associated vector bundles

3.1. **The groups $\text{Aut}\mathcal{O}$**.

3.1.1. We start this section by describing the groups of changes of coordinates in the formal superdisk $D^{1|N}$. We analyze in detail their corresponding Lie superalgebras in the cases $N = 1$ and $N = 2$. We then define principal bundles for these groups over any smooth supercurve.

In this section, we let $\Lambda$ be a Grassman algebra over $\mathbb{C}$. We will work in the category of superschemes over $\Lambda$ unless explicitly stated. When we work with a supergroup $G$, we will be interested in its $\Lambda$-points.

Let $\text{SSch}/k$ be the category of superschemes over a field $k$ and let $\text{Set}$ be the category of sets. Fix a non negative integer $N$ and a separated superscheme $X$ of finite type over $k$ (cf. 2.2.2). Let $D^{(m)}$ be as in 2.2.10 and $D^{1|N}$ be the formal superdisk. Define a family of contravariant functors $F_m : \text{SSch}/k \to \text{Set}$

$$F_m(Y) = \text{Hom}_k(Y \times_k D^{(m)}, X).$$

The proof of the following proposition is standard:

**Proposition 3.1.2.** The functors $F_m$ are representable by superschemes $X_m$.

Note in particular that $X_0 = X$, and when $N = 1$ we see that $X_1$ is the total tangent space of $X$.

The embeddings $D^{(m)} \hookrightarrow D^{(m+1)}$ induce projections $X_{m+1} \to X_m$ and we define the *Jet superscheme* of $X$ as

$$JX = \lim_{m \to \infty} X_m$$

3.1.3. Let us analyze first the case $N = 1$. Consider the group of continuous (even) automorphisms of the topological commutative superalgebra $\Lambda[[Z]]$, where $Z = (z, \theta)$ are topological generators. Such an automorphism is given by a pair of power series

$$z \mapsto a_{1,0}z + a_{0,1}\theta + a_{1,1}z\theta + \ldots$$
$$\theta \mapsto b_{1,0}z + b_{0,1}\theta + b_{1,1}z\theta + \ldots,$$

where the matrix $(a_{1,0} a_{0,1})$ is in $GL(1|1)^3$. Denote this supergroup by $\text{Aut}\mathcal{O}^{1|1}$. In what follows we will analyze its $\mathbb{C}$-points.

---

3Here and further, $GL(p|q)$ is the group of even automorphisms of a $p|q$ dimensional module over $\Lambda$. 
This supergroup is a semidirect product of $GL(1|1)$ and a pro-unipotent super group, namely, the subgroup $\text{Aut}_+\mathcal{O}^{1|1}$ of automorphisms where $\mathcal{O}^{1|1} = \text{Id}$. In fact,
\[
\text{Aut}_+\mathcal{O}^{1|1} = \lim_{n \to \infty} \text{Spec} \mathbb{C}[a_{1,1}, b_{1,1}, a_{2,0}, b_{2,0}, \ldots, a_{n,1}, b_{n,1}].
\]

Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{C}[Z]$ generated by $(z, \theta)$. We have
\[
\text{Aut}_+\mathcal{O}^{1|1} = \lim_{n \to \infty} \text{Aut}(\mathbb{C}[Z]/\mathfrak{m}^n).
\]
Similarly for its Lie superalgebra $\text{Der}_+\mathcal{O}^{1|1}$, we have
\[
\text{Der}_+\mathcal{O}^{1|1} = \lim_{n \to \infty} \text{Der}(\mathbb{C}[Z]/\mathfrak{m}^n),
\]
where for each $\mathbb{C}$-superalgebra $R$, we denote $\text{Der}(R)$ the Lie superalgebra of derivations of $R$. The exponential map is an isomorphism at each step, giving an isomorphism $\exp : \text{Der}_+\mathcal{O}^{1|1} \to \text{Aut}_+\mathcal{O}^{1|1}$.

The linearly compact Lie superalgebra $\text{Der}_0\mathcal{O}^{1|1} = \text{Lie}(\text{Aut}\mathcal{O}^{1|1})$ has the following topological basis:
\[
z^n\partial_z (n \geq 1), \quad z^n\partial_\theta (n \geq 1),
\]
\[
z^n\partial_\theta (n \geq 0), \quad z^n\partial_\theta (n \geq 0),
\]
or the following one $(n \geq 0)$:
\[
T_n = -z^{n+1}\partial_z - (n+1)z^n\partial_\theta, \quad J_n = -z^n\partial_\theta, \quad H_n = z^n\partial_z.
\]
These elements satisfy the commutation relations of the $N = 2$ algebra (2.1.9.3) for $n \geq 0$. In particular, we see that $\text{Der}_0\mathcal{O}^{1|1}$ is the formal completion of the Lie algebra $W(1|1)$. The Lie subalgebra $\text{Der}_+\mathcal{O}$ is topologically generated by the same vectors with $n \geq 1$.

3.1.4. We now turn our attention to the superconformal $N = 1$ case. Consider the differential form $\omega = dz + \theta d\theta$ on the formal superdisk $D^{1|1}$, and the supergroup $\text{Aut}^{\omega}\mathcal{O}^{1|1}$ of automorphisms of $D^{1|1}$ preserving this form, up to multiplication by a function. This is a subgroup of $\text{Aut}\mathcal{O}^{1|1}$ whose Lie superalgebra $\text{Der}_0^{\omega}\mathcal{O}^{1|1}$ consist of derivations $X$ in $\text{Der}_0\mathcal{O}^{1|1}$ such that $L_X\omega = f\omega$ for some formal power series $f$ (here $L_X$ denotes the Lie derivative). More explicitly, the linearly compact Lie superalgebra $\text{Der}_0^{\omega}\mathcal{O}^{1|1}$ is topologically generated by
\[
L_n = -\frac{n+1}{2}z^n\theta_\partial_\theta - z^{n+1}\partial_z, \quad n \in \mathbb{Z}_+
\]
\[
G_n = -z^{n+1/2}(\partial_\theta - \theta_\partial_\theta), \quad n \in \frac{1}{2} + \mathbb{Z}_+.
\]
These generators satisfy the commutation relation of the Neveu-Schwarz algebra as defined in (2.1.6.1). In particular, we see that $\text{Der}_0^{\omega}\mathcal{O}^{1|1}$ is the formal completion of the Lie superalgebra $K(1|1)$.

An automorphism of the formal superdisk is determined by two power series $F(Z), \Psi(Z)$ which are the images of the generators $Z = (z, \theta)$. Under this transformation we have (recall $\partial_\theta$ is an odd derivation)
\[
dz + \theta d\theta \mapsto \partial_z F dz - \partial_\theta F d\theta + \Psi(\partial_z \Psi dz + \partial_\theta \Psi d\theta)
\]
\[
= (\partial_z F + \Psi \partial_z \Psi) dz - (\partial_\theta F - \Psi \partial_\theta \Psi) d\theta,
\]
therefore we get that, in order for \( \omega \) to be preserved up to multiplication by a function, we need \((\partial_y F - \Psi \partial_y \Psi) = -\theta(\partial_z F + \Psi \partial_z \Psi)\), and this is equivalent to (2.2.12.4).

3.1.5. Finally we turn our attention to the (oriented) superconformal \( N = 2 \) case. For this we consider the differential form \( \omega = dz + \theta^1d\theta^1 + \theta^2d\theta^2 \) on the formal superdisk \( D^{1|2} \). We want to analyze the group of automorphisms of \( D^{1|2} \) preserving this form in the sense of the previous paragraph 3.1.4. Such an automorphism is determined by an even power series \( F(Z) \) and two odd power series \( \Psi^1(Z) \) and \( \Psi^2(Z) \), where \( Z = (z, \theta^1, \theta^2) \) are the coordinates on \( D^{1|2} \). Under such a change of coordinates, the differential form \( \omega \) changes to:

\[
(3.1.5.1) \quad \partial_z F dz - \partial_{\theta^1} F d\theta^1 - \partial_{\theta^2} F d\theta^2 + \Psi^1 \left( \partial_z \Psi^1 dz + \partial_{\theta^1} \Psi^1 d\theta^1 + \partial_{\theta^2} \Psi^1 d\theta^2 \right) + \\
+ \Psi^2 \left( \partial_z \Psi^2 dz + \partial_{\theta^1} \Psi^2 d\theta^1 + \partial_{\theta^2} \Psi^2 d\theta^2 \right) = \\
= \left( \partial_z F + \Psi^1 \partial_z \Psi^1 + \Psi^2 \partial_z \Psi^2 \right) dz + (-\partial_{\theta^1} F + \Psi^1 \partial_{\theta^1} \Psi^1 + \Psi^2 \partial_{\theta^1} \Psi^2) d\theta^1 + \\
+ (-\partial_{\theta^2} F + \Psi^1 \partial_{\theta^2} \Psi^1 + \Psi^2 \partial_{\theta^2} \Psi^2) d\theta^2.
\]

Collecting terms, imposing that the form \( \omega \) is preserved up to multiplication by a function, and defining the differential operators \( D^i = \partial_{\theta^i} + \theta^i \partial_z \) we obtain that the automorphisms we are considering satisfy the equations:

\[
(3.1.5.2) \quad D^i F = \Psi^1 D^i \Psi^1 + \Psi^2 D^i \Psi^2, \quad i = 1, 2.
\]

Note also that a particular case of (3.1.5.1) when \( F = z - \frac{1}{2} \theta^1 \theta^2, \quad \Psi^1 = \frac{1}{2}(\theta^2 - \theta^1) \) and \( \Psi^2 = \frac{1}{2}(\theta^1 + \theta^2) \) transforms the form

\[
\omega \rightarrow dz + \theta^2d\theta^1 - d\theta^1\theta^2,
\]

and the supergroup of automorphisms of \( D^{1|2} \) preserving the latter form is the supergroup of changes of coordinates preserving an \( N = 2 \) superconformal structure as in 2.2.12.

The linearly compact Lie superalgebra \( \text{Der}_0^\omega \mathcal{O}^{1|2} = \text{Lie}(\text{Aut}^\omega \mathcal{O}^{1|2}) \) is topologically generated by:

\[
L_n = -z^{n+1}\partial_z - \frac{n+1}{2} z^n \left( \theta^1 \partial_{\theta^1} + \theta^2 \partial_{\theta^2} \right), \quad n \in \mathbb{Z}_+
\]

\[
G_n^{(2)} = z^{n+1/2} \left( \theta^2 \partial_{\theta^2} - \partial_{\theta^1} \right) - \left( n + \frac{1}{2} \right) z^{n-1/2} \theta^1 \theta^2 \partial_{\theta^1}, \quad n \in \frac{1}{2} + \mathbb{Z}_+
\]

\[
G_n^{(1)} = z^{n+1/2} \left( \theta^1 \partial_{\theta^1} - \partial_{\theta^2} \right) + \left( n + \frac{1}{2} \right) z^{n-1/2} \theta^1 \theta^2 \partial_{\theta^2}, \quad n \in \frac{1}{2} + \mathbb{Z}_+
\]

\[
J_n = -iz^n \left( \theta^2 \partial_{\theta^1} - \theta^1 \partial_{\theta^2} \right) \quad n \in \mathbb{Z}_+.
\]

These operators satisfy the commutation relations of the \( N = 2 \) generators as in (2.1.9.4) for \( n \geq 0 \). We see that the Lie superalgebra \( \text{Der}_0^\omega \mathcal{O}^{1|2} \) is the formal completion of the Lie superalgebra \( K(1|2) \).

It is useful to consider complex coordinates \( \theta^\pm = \theta^1 \pm i\theta^2 \), and derivations \( D^\pm = \frac{1}{2}(D^1 \pm iD^2) \). In the coordinates \((z, \theta^+, \theta^-)\), these derivations are expressed as:

\[
D^\pm = \partial_{\theta^\pm} + \frac{1}{2} \theta^\pm \partial_z.
\]
If we change coordinates by \( \rho = (F, \Psi^+, \Psi^-) \), with \( \Psi^\pm = \Psi^1 \pm i\Psi^2 \), the superconformal condition (3.1.5.2) reads

\[
D^\pm F = \frac{1}{2} \Psi^+ D^\pm \Psi^- + \frac{1}{2} \Psi^- D^\pm \Psi^+.
\]

Therefore, under a change of coordinates \( (z_\alpha, \theta^\alpha_\beta) \mapsto (z_\beta, \theta^\beta_\beta) \), the operators \( D^\pm \) transform as

\[
D^\pm_\beta = (D^\pm_\beta \Psi^-_\alpha, \alpha) D^\pm_\alpha + (D^\pm_\beta \Psi^+_\alpha, \alpha) D^-_\beta.
\]

In the following sections, we will consider only oriented superconformal \( N = 2 \) supercurves (cf. remark 2.2.13), namely those for which there exists a coordinate atlas \( (U_\alpha, z_\alpha, \theta^\alpha_\alpha) \) such that on overlaps we have \cite{Coh87}:

\[
D^\pm_\alpha \Psi^\pm_\beta, \alpha = 0.
\]

In these coordinates, the topological generators of the Lie superalgebra \( \text{Der}_0 \mathcal{O}^{1|2} \) are expressed as:

\[
L_n = -z^{n+1} \theta^- \partial_z - \frac{n+1}{2} z^n (\theta^+ \partial_{\theta^+} + \theta^- \partial_{\theta^-}), \quad n \in \mathbb{Z}_+
\]

\[
J_n = -z^n (\theta^+ \partial_{\theta^+} - \theta^- \partial_{\theta^-}), \quad n \in \mathbb{Z}_+
\]

\[
G_n^\pm = -\frac{1}{2} z^{n+1/2} \partial_{\theta^\pm} - \frac{n+1/2}{2} z^{n-1/2} \theta^\mp \partial_{\theta^\mp}, \quad n \in \left\{ \frac{1}{2} + \mathbb{Z}_+ \right\}
\]

where as before we have \( G^\pm = \frac{1}{2}(G^{(1)} \mp iG^{(2)}) \).

Recall from 2.2.12 that an oriented superconformal \( N = 2 \) supercurve \((Y, \mathcal{O}_Y)\) projects onto two \( N = 1 \) supercurves \( X \) and its dual \( \hat{X} \). Defining new coordinates \((u, \theta^+, \theta^-)\), where \( u = z - \frac{1}{2} \theta^+ \theta^- \), we see that equations (3.1.5.4), for a change of coordinates \( \rho = (G = F + \frac{1}{2} \Psi^+ \Psi^-, \Psi^+, \Psi^-) \) are expressed in these coordinates as:

\[
D^- G = \Psi^- D^- \Psi^+, \quad D^+ G = 0.
\]

Moreover, the operators \( D^\pm \) are expressed as:

\[
D^+ = \partial_{\theta^-}, \quad D^- = \partial_{\theta^+} + \theta^- \partial_u.
\]

Note that the coordinate \( \theta^- \) does not appear in the transition functions for \( u, \theta^+ \), therefore these coordinates give the topological space \(|Y|\) the structure of an \( N = 1 \) supercurve. Let us call this curve \( X \). Similarly, if we define \( u' = z - \frac{1}{2} \theta^+ \theta^- \) we obtain that \( u', \theta^- \) defines the dual curve \((\hat{X}, \mathcal{O}_{\hat{X}})\).

It follows from the above discussion, that given a change of coordinates \( \rho = (G, \Psi^+) \in \text{Aut}\mathcal{O}^{1|1} \), we obtain uniquely a change of coordinates \( \rho = (G, \Psi^+, \Psi^-) \in \text{Aut}\mathcal{O}^{1|2} \), where \( \Psi^- = D^- G / D^- \Psi^+ \). This map induces an isomorphism of supergroups from \( \text{Aut}\mathcal{O}^{1|1} \) to the identity component of \( \text{Aut}\mathcal{O}^{1|2} \). This isomorphism corresponds to the isomorphism of Lie superalgebras \( K(1|2) \equiv W(1|1) \) (cf. \cite{KvdL89}), and has a geometric counterpart (cf. \cite{Vai95}) relating the moduli space of (oriented) superconformal \( N = 2 \) supercurves and the moduli space of \( N = 1 \) supercurves.

**Remark 3.1.6.** Let \( X \) be a superconformal \( N = n \) supercurve. Then for some coordinate atlas \( Z_\alpha = (z_\alpha, \theta^1_\alpha, \ldots, \theta^n_\alpha) \), the form \( \omega = d_z + \sum_{i=1}^n \theta^i d\theta^i \) is globally
defined up to multiplication by a function. Let $\omega$ be that form on the superdisk $D^{1|N}$ and on $D^{(m)}$ as well. Define the functors $F_m^\omega : \SSch/k \to \Set$ by:

$$F_m(Y) = \Hom^{\omega}_k(Y \times_k D^{(m)}, X),$$

where $\Hom^\omega$ denotes the set of morphisms preserving the form $\omega$ (up to multiplication by a function). It follows in the same way as in Proposition 3.1.2 that the functors $F_m$ are representable by superschemes $X_m^\omega$. This allows us to define the superscheme

$$JX^\omega = \varprojlim_{m \to \infty} X_m^\omega,$$

parametrizing maps $D \to X$ preserving the superconformal structure.

3.1.7. Let $X$ be an $N = n$ supercurve and let $x \in X$. If $Z = (z, \theta^1, \ldots, \theta^n)$ are local coordinates at $x$ and $\Theta_x$ denotes the completion of the local ring at $x$, we have an isomorphism

$$\Theta_x = \mathbb{C}[[Z]],$$

where we should replace $\mathbb{C}$ by $\Lambda$ if $X$ is defined over $\Lambda$. For the purposes of this section it is enough to consider curves over $\mathbb{C}$, the relative case follows easily. Let $\text{Aut}_x$ denote the set of local coordinates $Z = (z, \theta^i)$ at $x$. In the algebraic setting we mean by coordinates an étale map $Z : X \to \mathbb{A}^{1|n}$. The set $\text{Aut}_x$ is a torsor for the group $\text{Aut}^{\Theta^1|n}$. The torsors $\text{Aut}_x$ glue to form an $\text{Aut}^{\Theta^1|n}$-torsor $\text{Aut}_X$. Indeed $\text{Aut}_X$ consists of pairs $(x, Z)$ where $x$ is a point in $X$ and $Z = (z, \theta^i)$ is a local coordinate at $x$. The action of $\text{Aut}^{\Theta^1|n}$ on the fibers is by changes of coordinates. The torsor $\text{Aut}_X$ may be described as an open subscheme of $JX$ consisting of jets of maps $D^{1|n} \to X$ such that their 1-jet is in $GL(1|n)$.

Since we can cover $X$ by Zariski open subschemes $U_a$ and étale maps $f_a : U_a \to \mathbb{A}^{1|n}$ we see that the $\text{Aut}^{\Theta^1|n}$-torsor $\text{Aut}_X$ is locally trivial in the Zariski topology (cf. [FBZ01, §5.4.2]).

3.1.8. Similarly, let $X$ be a (oriented) superconformal $N = n$ supercurve and $x \in X$. Let $\text{Aut}_x^\omega$ be the set of SUSY coordinates $Z$ at $x$ (that is, compatible with the superconformal structure). It follows that this set is an $\text{Aut}^{\omega\Theta^1|n}$-torsor. Moreover these torsors glue to form an $\text{Aut}^{\omega\Theta^1|n}$-torsor $\text{Aut}_X^\omega \to X$. As in the previous paragraph, $\text{Aut}_X^\omega$ is an open sub superscheme of $JX^\omega$ (cf. 3.1.6) consisting of jets of maps $D^{1|n} \to X$ compatible with the superconformal structure and with invertible 1-jet.

**Remark 3.1.9.** Let $V$ a finite rank $\text{Aut}\Theta$-module (resp. a finite rank $\text{Aut}^{\omega\Theta}$-module), and let $X$ be an $N = n$ supercurve (resp. a superconformal $N = n$ supercurve). We define a vector bundle on $X$ by

$$\gamma_X = \text{Aut}_X^{\omega\Theta} \times V \quad \text{(resp. } \text{Aut}_X^{\omega\Theta} \times V\text{),}$$

consisting of pairs $(\tilde{x}, v)$ with $\tilde{x}$ in $\text{Aut}_X$ (resp. $\text{Aut}_X^{\omega\Theta}$) and $v \in V$ with the identification $(\tilde{x}, g, v) \sim (\tilde{x}, g \cdot v)$ for $g \in \text{Aut}\Theta$ (resp. $g \in \text{Aut}^{\omega\Theta}$). We call $\gamma_X$ the $\text{Aut}_X$ (resp. $\text{Aut}_X^{\omega\Theta}$) twist of $V$.

3.2. **Vector bundles, sections and connections.**

3.2.1. In this section we construct vector bundles on supercurves associated with SUSY vertex algebras following [FBZ01]. Briefly, a strongly conformal $N = n$ SUSY vertex algebra is a module for the Harish-Chandra pair $(\text{Der}^{\Theta^1|n}, \text{Aut}^{\Theta^1|n})$, therefore we can apply the Beilinson-Bernstein localization construction [BB93] to
get a vector bundle with a flat connection over any $N = n$ supercurve. Similarly, a strongly conformal $N_k = n$ SUSY vertex algebra ($n \leq 4$), is a module for the Harish-Chandra pair $(\text{Der}_\Lambda \mathfrak{g}_{1|n}, \text{Aut}_\Lambda \mathfrak{g}_{1|n})$, therefore we can construct vector bundles with flat connections over any oriented superconformal $N = n$ curve.

As in [FBZ01], it turns out that the state-field correspondence in all these cases can be seen as a (local) section of the corresponding bundles. The corresponding change of coordinates formula (a generalization of Huang’s formula [Hua97]) is proved in this section.

3.2.2. Let $V$ be a strongly conformal $N_W = n$ SUSY vertex algebra. Therefore we have $N + 1$ vectors $\nu$ and $\tau^1, \ldots, \tau^N$ such that their Fourier modes $\nu_{(m, l)}$ and $\tau^j_{(m, l)}$ with $m \geq 0$ generate a Lie superalgebra isomorphic to $\text{Der}_{\Lambda} \mathfrak{g}_{1|n}$. The derivation $\partial_z$ (corresponding to $\nu_{(0, 0)}$) cannot be exponentiated to the group $\text{Aut}_{\Lambda} \mathfrak{g}_{1|n}$ and the Lie superalgebra spanned by $\nu_{(m, l)}$ and $\tau^j_{(m, l)}$ for $m \geq 1$ if $I \neq 0$ is isomorphic to $\text{Der}_0 \mathfrak{g}_{1|n}$.

In order to exponentiate the representation $V$ of $\text{Der}_0 \mathfrak{g}_{1|n}$ to a representation of the group $\text{Aut}_{\Lambda} \mathfrak{g}_{1|n}$ we note as before that this Lie algebra is a semidirect product of $\mathfrak{gl}(1|n)$ with the pro-nilpotent Lie subalgebra $\text{Der}_z \mathfrak{g}_{1|n}$. Namely, the subalgebra spanned by $z\partial_z$, $\theta^i \partial_{\theta^j}$, $\bar{z}\partial_{\bar{z}}$ and $\bar{\theta}^i \partial_{\bar{\theta}^j}$ is isomorphic to $\mathfrak{gl}(1|n)$. It follows from the definition of strongly conformal $N_W = N$ SUSY vertex algebras in 2.1.13, that we can exponentiate this representation of $\mathfrak{gl}(1|n)$ (the fact that the nilpotent part of the Lie superalgebra exponentiates follows easily from the OPE formula and the locality axiom).

3.2.3. Let $X$ be an $N = n$ supercurve over a Grassman algebra $\Lambda$, let $x \in X$ and $\mathcal{O}_x$ be the completion of the local super-ring at $x$. Let $Z = (z, \theta^i)$ be local coordinates at $x$ (recall that in the formal setting, $Z$ is an étale map $X \to \Lambda[[Z]]$). With such a choice of coordinates we get an isomorphism $\mathcal{O}_x \cong \Lambda[[Z]]$, and the set of coordinates at $x$, $\text{Aut}_{\Lambda}$, is an $\text{Aut}_{\Lambda} \mathfrak{g}_{1|n}$-torsor. Let us work in the analytic setting first for the sake of simplicity as in [FBZ01]. Let $D_x$ be a small disk around $x$. Let $p$ be a $\Lambda$-point given in the local coordinates $Z = (z, \theta^i)$ by $Y = (y, \alpha^i)$. The coordinates $Z$ induce coordinates $Z - Y = (z - y, \theta^i - \alpha^i)$ at $p$. Now let $\rho \in \text{Aut}_{\Lambda} \mathfrak{g}_{1|n}$ be a change of coordinates. Recall that this change of coordinates is given by power series $(F(Z), \Psi^i(Z))$, where $F(Z) \in \Lambda[[Z]]$ is even and $\Psi^i \in \Lambda[[Z]]$ are odd. This change of coordinates induce new coordinates at $p$, given by:

\begin{equation}
(3.2.3.1) \quad \rho(Z) - \rho(Y) = (F(Z) - F(Y), \Psi^i(Z) - \Psi^i(Y)).
\end{equation}

The coordinates $Z - Y = (z - y, \theta^i - \alpha^i)$ and (3.2.3.1) at $p$ are related by a change of coordinates $\rho_Y = (F_Y, \Psi^i_Y)$ satisfying:

\begin{equation}
(3.2.3.2) \quad \rho_Y(Z - Y) = \rho(Z) - \rho(Y).
\end{equation}

Therefore, letting $W = (w, \zeta^i) = Z - Y$, we get:

Therefore, letting $W = (w, \zeta^i) = Z - Y$, we get:

\begin{equation}
(3.2.3.2) \quad \rho_Y(W) = \rho(W + Y) - \rho(Y).
\end{equation}

In the formal setting we can not consider a small disk, but given a point $x$ and coordinates $Z$ at $x$, we can still define $\rhoZ \in \text{Aut}_{\Lambda} \mathfrak{g}_{1|n}$ for any $\rho \in \text{Aut}_{\Lambda} \mathfrak{g}_{1|n}$ by formula (3.2.3.2) with $Y$ replaced by $Z$.

---

\(^4\)From now on, we will abuse notation and denote by $\text{Aut}_{\Lambda} \mathfrak{g}_{1|n}$ its identity component
Let $V$ be a strongly conformal $N_W = n$ SUSY vertex algebra, so that $V$ is an $\text{Aut}\mathcal{O}^{1|n}$-module. We will call this representation $R$.

**Theorem 3.2.4.** Let $V$ be a strongly conformal $N_W = n$ SUSY vertex algebra. Let $\rho = (F, \Psi^j) \in \text{Aut}\mathcal{O}^{1|n}$ and $a \in V$. The following change of coordinates formula is true:

$$Y(a, Z) = R(\rho)Y(R(\rho_Z)^{-1}a, \rho(Z))R(\rho)^{-1}$$

where by $\rho(Z)$ we understand the images of $z, \theta^i$ under $\rho$, namely $F(z, \theta^i), \Psi^j(z, \theta^i)$.

**Proof.** The proof is similar to the analogous formula in the ordinary vertex algebra case. Namely, the state-field correspondence $Y(\cdot, Z)$ is an element in the vector space $\text{Hom}(V, \mathcal{F}(V))$, where $\mathcal{F}(V)$ is the space of all $\text{End}(V)$-valued superfields. For each $\rho \in \text{Aut}\mathcal{O}^{1|n}$ consider the linear operator in $\text{Hom}(V, \mathcal{F}(V))$ given by

$$(T_\rho X)(a, Z) = R(\rho)X(R(\rho_Z)^{-1}a, \rho(Z))R(\rho)^{-1}.$$ 

It is easy to check that $T_\rho X \in \text{Hom}(V, \mathcal{F}(V))$. Moreover, this action defines a representation of $\text{Aut}\mathcal{O}^{1|n}$ in $\text{Hom}(V, \mathcal{F}(V))$. Recall that the group structure in $\text{Aut}\mathcal{O}^{1|n}$ is given by composition, namely, if $\rho = (F, \Psi^j)$ and $\tau = (G, \Theta^j)$ then $\rho \star \tau$ is given by $H, \Sigma^j$ where

$$H(z, \theta^i) = G(F(z, \theta^i), \Psi^k(z, \theta^j), \Sigma^j(z, \theta^i) = \Theta^j(F(z, \theta^i), \Psi^k(z, \theta^j)).$$

It follows that $\rho Z \star \tau_{\rho(Z)}(z, \theta^i) = (\rho \star \tau)z$. Indeed, the LHS, when evaluated in $W$ is given by

$$\tau_{\rho(Z)}(\rho(W + Z) - \rho(Z)) = \tau(W + Z) - \rho(Z) - \rho (Z) - \tau (\rho(Z)),$$

which is the RHS.

It follows from this formula that $\rho \mapsto T_\rho$ defines a representation of $\text{Aut}\mathcal{O}^{1|n}$. In fact, we have:

$$(T_{\rho \star \tau}X)(a, Z) = R(\rho \star \tau)X(R((\rho \star \tau)_Z)^{-1}a, \tau(\rho(Z))R(\rho \star \tau)^{-1} =$$

$$R(\rho)R(\tau)X(R(\rho_Z \star \tau_{\rho(Z)})^{-1}a, \tau(\rho(Z)))R(\tau)^{-1}R(\rho)^{-1} =$$

$$R(\rho)[R(\tau)X(R(\tau_{\rho(Z)})^{-1}a, \tau(\rho(Z)))R(\tau)^{-1}]R(\rho)^{-1} = [T_\rho(T_\tau X)](a, Z).$$

We have reduced the proof of the theorem to show that $Y(\cdot, Z)$ is fixed under this action. Since the exponential map $\text{exp}: \text{Der}_0\mathcal{O}^{1|n} \to \text{Aut}\mathcal{O}^{1|n}$ is surjective, we need only to show that $Y(\cdot, Z)$ is stable under the induced infinitesimal action of $\text{Der}_0\mathcal{O}^{1|n}$. For this we let $\rho = \text{exp}(\varepsilon \psi)$, where $\psi = \nu(Z)\partial_Z \in \text{Der}_0\mathcal{O}^{1|n}$, $\nu(Z) = (f(Z), g^1(Z), \ldots, g^n(Z))$ with $f(Z)$ an even function and $g^i(Z)$ odd functions of $Z$. As before, $\partial_Z = (\partial_z, \partial_{\theta^1}, \ldots, \partial_{\theta^n})$ and the product $\nu(Z)\partial_Z$ denotes the scalar product $f(Z)\partial_z + \sum_{i=1}^n g^i(Z)\partial_{\theta^i}$. We want to compute $\rho Z$. For this we put $\rho Z = \text{exp}(\varepsilon \psi)$. Expanding $\rho Z(W)$ in powers of $\varepsilon$, we get

$$(\mathbf{3.2.4.2}) \quad \mathbf{u} = \nu(Z + W)\partial_W - \nu(Z)\partial_W = (e^{Z\partial_W} \nu(W))\partial_W - \nu(Z)\partial_W.$$ 

Noting that the operators corresponding to $\partial_W = (\partial_w, \partial_{\theta^1}, \ldots, \partial_{\theta^n})$ are $-\nabla = (-T_1, -S_1, \ldots, -S_n)$, we obtain:

$$(3.2.4.3) \quad R(\mathbf{u}) = e^{-Z\nabla}R(\nu)e^{Z\nabla} + \nu(Z)\nabla.$$ 

The (infinitesimal) action of $T_\rho$ on $Y(a, Z)$ is given by $Y(a, Z)$ plus the linear term in $\varepsilon$, which in turn is:

$$[R(\nu), Y(a, Z)] = Y(R(\mathbf{u})a, Z) + \nu(Z)\nabla Y(a, Z).$$
The first term comes from the adjoint action of \( R(\rho) \), the second term is the \( \varepsilon \)-linear term in \( R(\rho_Z)^{-1} \), and the last term comes from the Taylor expansion of the change of coordinates. The result follows from (3.2.4.2), (3.2.4.3), Proposition 2.1.18 and Theorem 2.1.19.

3.2.5. Now we can define a vector bundle associated to an \( N_W = n \) SUSY vertex algebra over any \( N = n \) supercurve. Moreover, we will define a canonical section of this bundle and a flat connection on it. First recall that from any finite dimensional \( \text{Aut}\mathcal{O}^{1|n} \)-module we can construct a vector bundle over an \( N = n \) supercurve \( X \) by twisting this \( \text{Aut}\mathcal{O}^{1|n} \)-module by the \( \text{Aut}\mathcal{O}^{1|n} \)-torsor \( \text{Aut}_X \) (see Remark 3.1.9).

Given a strongly conformal \( N_W = n \) SUSY vertex algebra \( V \), we have a filtration \( V_{\leq i} \) by finite dimensional submodules, namely, \( V_{\leq i} \) is the span of fields of conformal weight less or equal than \( i \). By our assumptions, these are finite dimensional \( \text{Aut}\mathcal{O} \)-submodules of \( V \). Let \( \mathcal{V}_{\leq i} \) be the corresponding \( \text{Aut}_X \) twist. These vector bundles come equipped with embeddings \( \mathcal{V}_{\leq i} \hookrightarrow \mathcal{V}_{\leq i+1} \). The limit of this directed system is a \( \mathcal{O}_X \)-module \( \mathcal{V}_X \):

\[
\mathcal{V}_X = \lim_{i \to \infty} \mathcal{V}_{\leq i}.
\]

This \( \mathcal{O}_X \)-module is quasi-coherent by definition.

On the other hand, the dual modules \( V^*_{\leq i} \) come equipped with surjections \( V^*_{\leq i} \twoheadrightarrow V^*_{\leq i+1} \) therefore we get a projective system of \( \mathcal{O}_X \)-modules \( \mathcal{V}^*_{\leq i} \twoheadrightarrow \mathcal{V}^*_{\leq i+1} \). The inverse limit of this system is by definition \( \mathcal{V}^*_X \), namely:

\[
\mathcal{V}^*_X = \lim_{i \to \infty} V^*_{\leq i}.
\]

Thus, we have defined \( \mathcal{O}_X \)-modules associated with the SUSY vertex algebra \( V \). We will call these modules the SUSY vertex algebra bundle and its dual. By construction, the fiber of the bundle \( \mathcal{V} \) at a point \( x \in X \) is isomorphic as a vector space, to \( V \).

Similar constructions can be applied when \( X \) is replaced by a formal superdisk near a point \( x \in X \). Namely, let \( D_x \) be such a formal superdisk, we have as before an \( \text{Aut}\mathcal{O}^{1|n} \)-torsor \( \text{Aut}_{D_x} \) over \( D_x \). Then \( \mathcal{V}_{D_x} \) is the twist of \( V \) by this torsor. It is easy to see that in this case we get \( \mathcal{V}_X|_{D_x} = \mathcal{V}_{D_x} \).

Let \( \text{Aut}_x \) be the torsor of coordinates at \( x \) as before. Then the fiber of \( \mathcal{V} \) at \( x \) is given by:

\[
\mathcal{V}_x = \text{Aut}_x \times^\text{Aut}\mathcal{O} V.
\]

Let \( D_x^\times \) be the punctured disk at \( x \), that is the formal completion:

\[
D_x^\times = \lim_{i \to \infty} \text{Spec}(\hat{\mathcal{O}}_x/\mathfrak{m}^{i+1}),
\]

where \( \hat{\mathcal{O}}_x \) is the ring of fractions of the local ring at \( x \) and \( \mathfrak{m} \) is the maximal ideal defining \( x \). If \( Z = (z, \theta) \) are coordinates at \( x \), this is isomorphic to the formal spectrum of \( \Lambda((Z)) \).

We will define an \( \text{End}\mathcal{V}_x \)-valued section of \( \mathcal{V}^* \) on \( D_x^\times \). In order to define such a section it is enough to give its matrix coefficients, namely, for each \( \varphi \in \mathcal{V}_x^*, v \in \mathcal{V}_x \) and \( s \) a section of \( \mathcal{V}|_{D_x} \), we assign a function on \( D_x^\times \), that is an element of \( \hat{\mathcal{O}}_x \), the ring of fractions of \( \mathcal{O}_x \). This assignment is denoted by:

\[
\varphi, v, s \mapsto <\varphi, \hat{\mathcal{O}}_x(s) \cdot v >.
\]

\(^5\text{When there is no possible confusion, we will denote this bundle simply by } \mathcal{V}.\)
and should be linear in \( v \) and \( \varphi \) and \( \mathcal{O}_x \) linear in \( s \). Let \( Z = (z, \theta^1) \) be coordinates at \( x \). We obtain a trivialization \( i_Z : V[[Z]] \rightarrow \Gamma(D_x, \mathcal{Y}) \), of \( \mathcal{Y}|_{D_x} \). This induces isomorphisms \( V \cong \mathcal{Y}_x \) and \( V^* \cong \mathcal{Y}_x^* \), where \( V^* \) is the restricted dual of \( V \). Let \( v \in V \) and \( \varphi \in V^* \). Denote their images in \( \mathcal{Y}_x \) and \( \mathcal{Y}_x^* \), under these isomorphisms by \((Z, v)\) and \((Z, \varphi)\) respectively. Let \( s \in V[[Z]] \), its image under the isomorphism \( i_Z \) is a regular section of \( \mathcal{Y}_x \). By \( \mathcal{O}_x \) linearity, we may assume that \( s = a \in V \).

To this data, we assign the function:

\[
(3.2.5.1) \quad < (Z, \varphi), \mathcal{Y}_x(i_Z(a)) \cdot (Z, v) >= < \varphi, Y(a, Z)v > .
\]

**Theorem 3.2.6.** The assignment \((3.2.5.1)\) is independent of the coordinates \( Z = (z, \theta^1, \ldots, \theta^n) \) chosen, i.e. \( \mathcal{Y}_x \) is a well defined \( \text{End}(\mathcal{Y}_x) \)-valued section of \( \mathcal{Y}^* \) on \( D_x \).

**Proof.** The proof follows the lines of the ordinary vertex algebra case in [FBZ01]. To reduce the problem to prove:

\[
< \varphi, R(\rho)Y(R(\rho)^{-1}a, W)R(\rho)^{-1}v >= < \varphi, Y(a, Z)v > ,
\]

we need to show that this assignment coincides with \( \mathcal{Y} \). By the definition of the bundle \( \mathcal{Y} \) we have

\[
(Z, v) = (\rho^{-1}(W), v) = (W, R(\rho)^{-1}v),
\]

where \( R(\cdot) \) is the representation of \( \text{Aut} \mathcal{O}^{1|n} \) in \( V \). Similarly \( (Z, \varphi) = (W, \varphi R(\rho)) \).

Recall from 3.2.3 that in the analytic setting, if we trivialize \( \mathcal{Y}|_{D_x} \) with the coordinates \( Z \), we can use the coordinates \((Z - Y) = (z - y, \theta^i - \alpha^i)\) at \( Y = (y, \alpha^i) \) to identify \( \mathcal{Y}_y \) with \( V \). We obtain:

\[
(3.2.6.1) \quad (Z - Y, a) = (W - \rho(Y), R(\rho_Y)^{-1}a),
\]

therefore the section \( i_Z(a) \) is \( i_W(R(\rho_Z)^{-1}a) \) in the \( W \)-trivialization.

In the formal setting, we can replace the coordinates by their \( n \)-jets, but these in turn can be extended by definition to a small Zariski open neighborhood of \( x \), in this case, the formula \((3.2.6.1)\) is true as we have shown.

We have reduced the problem to prove:

\[
< \varphi, R(\rho)Y(R(\rho_Z)^{-1}a, W)R(\rho)^{-1}v >= < \varphi, Y(a, Z)v > ,
\]

thus, the theorem follows from Theorem 3.2.4. \( \square \)

**3.2.7.** In the superconformal case, the situation is slightly more complicated. Roughly, the only changes that we have to make in the above prescription are the induced coordinates at a \( \Lambda \)-point and consequently the definition of \( \rho_Z \).

Like in the \( N_K = n \) SUSY vertex algebra situation, given two set of coordinates \( Z = (z, \theta^1, \ldots, \theta^n) \) and \( W = (w, \xi^1, \ldots, \xi^n) \) we will write

\[
Z - W = \left( z - w - \sum_{i=1}^{n} \theta^i \xi^i, \theta^1 - \xi^1, \ldots, \theta^n - \xi^n \right).
\]

Let \( V \) be a strongly conformal \( N_K = n \) SUSY vertex algebra \((n \leq 4)\), hence \( V \) is an \( \text{Aut} \mathcal{O}^{1|n} \)-module. Moreover, \( V \) has a filtration by finite dimensional submodules \( V_{\leq i} \) given by conformal weight as above. Let \( X \) be an oriented superconformal
3.1.8). As above we can define the vertex algebra bundles $\mathcal{V}_x$ and $\mathcal{V}^*_x$. Similarly, we can define the $N_K = n$ SUSY vertex algebra bundles over the superconformal disks $D^*_x$. The fibers $\mathcal{V}_x$ of these bundles are the $\text{Aut}_x^n$-twists of $V$, where $\text{Aut}_x^n$ is the torsor of coordinates at $x$, compatible with the superconformal structure (see Remark 3.1.9). We define and $\text{End}(\mathcal{V}_x)$-valued section $\mathcal{B}_x$ of $\mathcal{V}^*$ on the punctured disk $D^*_x$ by formula (3.2.5.1).

**Theorem 3.2.8.** The assignment $\mathcal{B}_x$ is independent of the coordinates $Z = (z, \theta^i)$ chosen as long as they are compatible with the superconformal structure on $X$.

**Proof.** Let us first work in the analytic setting. If $p$ is a $\Lambda$-point in $D_x$ (now a small analytic disk near $x \in X$) given by local parameters $Y = (y, \alpha^i)$, then $Z$ induces local coordinates at $T = (t, \eta^i) = Z - Y$ near $p$. The coordinates $T$ are compatible with the superconformal structure. Indeed, we have:

\[ dt = dz + \sum_{i=1}^n \alpha^i d\theta^i, \quad d\eta^i = d\theta^i, \]

therefore:

\[ dt + \sum_{i=1}^n \eta^i d\eta^i = dz + \sum_{i=1}^n \alpha^i d\theta^i + (\theta^i + \alpha^i)d\theta^i = dz + \sum_{i=1}^n \theta^i d\theta^i. \]

If $W = (w, \zeta^i) = \rho(Z)$ is another set of coordinates compatible with the superconformal structure at $x$, with $\rho = (F, \Psi^i) \in \text{Aut}^\omega O^{1|n}$, then $W$ induces another set of coordinates at $p$, namely

\[ \rho(Z) - \rho(Y) = \left( F(z, \theta) - F(y, \alpha) - \sum_{i=1}^n \Psi^i(z, \theta)\Psi^i(y, \alpha), \Psi^i(z, \theta) - \Psi^i(y, \alpha) \right). \]

These are related with the coordinates $T$ by a change of coordinates

\[ \rho_Y = (F_Y, \Psi_Y) \in \text{Aut}^\omega O^{1|n}. \]

We have:

\[ \rho_Y(T) = \rho(T + Y) - \rho(Y), \]

where, as in the $N_K = n$ SUSY vertex algebra case, we write $T + Y = T - (Y)$.

The theorem will follow if we prove formula (3.2.4.1) for $\rho \in \text{Aut}^\omega O^{1|n}$. This is achieved as in the proof of Theorem 3.2.4 by first showing that the action $T_\rho$ is a representation of $\text{Aut}^\omega O^{1|n}$ in $\text{Hom}(V, \mathcal{F}(V))$. For this we first note that $(\rho \ast \tau)Z = \rho Z \ast \tau_{\rho(Z)}$ in exactly the same way as in the $N_W = n$ case. Again we just have to prove that $Y$ is fixed under this action, and we check this at the level of Lie algebras. Denote $D_W = (\partial_v, D^1_W, \ldots, D^n_W)$, where $D^1_W = \partial_{\zeta^i} + \zeta^i \partial_w$. Similarly, denote $\bar{D}_W = (\partial_v, \bar{D}^1_W, \ldots, \bar{D}^n_W)$, where $\bar{D}^1_W = \partial_{\bar{\zeta}^i} - \bar{\zeta}^i \partial_w$. Let $\rho = \exp(\varepsilon v)$ where $v = v(W) \bar{D}_W \in \text{Der}_\mathcal{O}^\omega O^{1|n}$, and put $\rho_Z = \exp(\varepsilon u)$. Expanding $\rho_Z$ in powers of $\varepsilon$ we find:

\[ u = v(W + Z) \bar{D}_W - v(Z) \bar{D}_W. \]

Note that in this context we have two different Taylor expansions:

\[ e^{Z_{D_W}} f(W) = f(Z + W), \quad e^{Z_{\bar{D}_W}} f(W) = f(W + Z), \]

\[ ^{\text{Note that}} Z + W \neq W + Z. \]
using the second, we see that
\[ u = \left( e^{ZD_W} v(W) \right) D_W - v(Z) D_W. \]
From this and the fact that the operators corresponding to \( D_W \) are:
\[ -\nabla = (-T, -S_1, \ldots, -S_n), \]
we obtain:
\[ R(u) = e^{-Z\nabla} R(v)e^{Z\nabla} + v(Z) \nabla. \]
The theorem now follows as in the \( N_W = n \) case. \( \Box \)

Now we construct connections on the vector bundles \( \mathcal{V} \) from the previous paragraphs.

**Theorem 3.2.9.** Let \( X \) be a \((1|N)\) dimensional supercurve. Let \( U \subset X \) be open and \( Z \) be coordinates in \( U \) defining the vector fields \( \partial_z \) and \( \partial_{\theta^i} \). Let \( V \) be a strongly conformal \( N_W = N \) SUSY vertex algebra and \( \mathcal{V} \) the associated bundle. Define the connection operators \( \nabla_{\chi} : \mathcal{V}|_U \rightarrow \mathcal{V}|_U \) for each vector field \( \chi \) in \( U \) by
\[ \nabla_{\partial_z} = \partial_z + T, \quad \nabla_{\partial_{\theta^i}} = \partial_{\theta^i} + S^i. \]
Then \( \nabla \) is a well defined (left) connection on \( \mathcal{V} \) (independent of the coordinates chosen). Moreover, this connection is flat.

**Proof.** The proof is verbatim the proof of the analogous statement in [FBZ01, 16.1]. Indeed, strongly conformal SUSY vertex algebras are modules for the Harish Chandra pair \((\text{Der}O^{1|N}, \text{Aut}O^{1|N})\) and this in turn acts simply transitively on the torsor \( \text{Aut}_X \rightarrow X \). The localization procedure of formal geometry applies without difficulties. \( \Box \)

**Remark 3.2.10.** Note that this connection endows \( \mathcal{V} \) with a structure of a left \( \mathcal{D}_X \)-module for any supercurve \( X \) and any strongly conformal \( N_W = N \) SUSY vertex algebra \( V \).

Let \( V \) be a strongly conformal \( N_K = N \) SUSY vertex algebra, and let \( \mathcal{V} \) be the associated vector bundle over the oriented superconformal curve \( X \). For an open \( U \) as before, and superconformal coordinates \( Z \) in \( U \) we will define the superconformal differential operators \( \mathcal{D}_X(U) \) to be the super ring of differential operators generated by all the \( D_Z^i \). This defines a sheaf of algebras of superconformal differential operators \( \mathcal{D}_X \) over any (oriented) superconformal curve \( X \). The assignment
\[
(3.2.10.1) \quad D^i_Z \cdot f(Z)a = (D^i_Z f(Z))a + (-1)^f f(Z) S^i a
\]
gives \( \mathcal{V} \) the structure of a left \( \mathcal{D}_X \)-module.

### 3.3. Examples.

3.3.1. In this section we give the first non-trivial examples of the super vector bundles that arise with the construction of the previous sections. To simplify the notation, we will use the ordinary description of the involved vertex algebras. For example, when we analyze the boson-fermion system (cf. example 3.3.2) we will work with the fermion \( \varphi \) and the boson \( \alpha \) instead of the superfields \( \Psi \) and \( S \Psi \). Note that the Grassman algebra \( \Lambda \) is a SUSY vertex algebra (either \( N_W = N \) or \( N_K = N \)) with \( T = S^i = 0 \) and \( |0\rangle = 1 \). In this section, given a SUSY vertex algebra \( V \), we will consider the tensor product \( W = \Lambda \otimes V \) (either of \( N_W = N \)
or \( N_K = N \) SUSY vertex algebras), therefore we can view \( W \) as a SUSY vertex algebra over \( \Lambda \), namely, \( W \) is a \( \Lambda \)-module and the vertex operators are \( \Lambda \)-linear.

Let us start with \( N_K = 1 \) bundles. For this let \( X \) be a super conformal \( N = 1 \) supercurve over \( \Lambda \). Let \( U_\alpha \) and \( U_\beta \) be open in \( X \) and \( p = (t, \zeta) \) a \( \Lambda \)-point in the intersection. Let \( V \) be a strongly conformal \( N_K = 1 \) SUSY vertex algebra, so that \( V \) carries a representation of \( \text{Der}_0^{\omega, \Theta^{1|1}} \) that exponentiates to a representation of \( \text{Aut}^{\omega, \Theta^{1|1}} \). Suppose we have coordinates \((z_\alpha, \theta_\alpha)\) in \( U_\alpha \) and \((z_\beta, \theta_\beta)\) in \( U_\beta \) that are compatible with the superconformal structure. They are related by a change of coordinates \( \rho_{\beta \alpha} = (F(z_\alpha, \theta_\alpha), \Psi(z_\alpha, \theta_\alpha)) \) satisfying \( DF = \Psi D\Psi \) where \( D = \partial_{\theta_\alpha} + \theta_\alpha \partial_{z_\alpha} \). These coordinates define coordinates at the point \( p \), therefore we obtain different trivializations of the bundle \( \mathcal{V} \). The transition functions for the structure sheaf give us transition functions for \( \mathcal{V} \), in particular, they act in the fiber at the point \( p \) as \( R(\rho_p)^{-1} \) (cf. 3.2.6.1).

In order to compute \( R(\rho_p) \) we need to look only at the odd coordinate, namely, expand in Taylor series:

\[
\Psi_{z,\theta}(t, \zeta) = \Psi(t + z + \zeta \theta, \zeta + \theta) - \Psi(z, \theta) = \zeta D\Psi + t D^2\Psi + \zeta t D^3\Psi + \frac{t^2}{2} D^4\Psi + \ldots
\]

(3.3.1.1)

\[
= \exp \left( -\sum_{i \geq 1} (v_i L_i + w_i G_i) \right) A^{-2L_0} \cdot \zeta,
\]

where as in (3.1.4.1) we have

\[
L_n = -\frac{n+1}{2} t^n \zeta \partial_\zeta - t^{n+1} \partial_t, \quad G_{(n+1/2)} = G_n = -t^{n+1/2} (\partial_\zeta - \zeta \partial_t),
\]

and \( v_i = v_i(z, \theta) \) are even functions and \( w_i = w_i(z, \theta) \) are odd functions. Truncating the series in (3.3.1.1) at order 2 we have:

\[
\Psi_{z,\theta}(t, \zeta) = A \left( \zeta + tw_1 + \zeta tv_1 + t^2 (w_2 + v_1 w_1) + \ldots \right),
\]

from where we get the equations:

\[
A = D\Psi, \quad w_1 A = D^2\Psi, \quad v_1 A = D^3\Psi, \quad (w_2 + v_1 w_1) A = \frac{1}{2} D^4\Psi.
\]

(3.3.1.2)

We can solve this system to get:

\[
v_1 = \frac{D^3\Psi}{D\Psi}, \quad w_1 = \frac{D^2\Psi}{D\Psi}, \quad w_2 = \frac{1}{2} \left( \frac{D^4\Psi}{D\Psi} - 2 \frac{D^3\Psi D^2\Psi}{(D\Psi)^2} \right) = \frac{1}{2} \sigma(D\Psi),
\]

(3.3.1.3)

where \( \sigma \) is the \( N = 1 \) super-schwarzian defined in (2.2.16.2).

**Example 3.3.2** (Free Fields). Recall the strongly conformal \( N_K = 1 \) SUSY vertex algebra \( B_1 \) defined in Example 2.1.8. We will denote this vertex algebra as \( B(1) \). As an ordinary vertex algebra, it is graded with respect to conformal weight. The fermion \( \varphi \) is primary of conformal weight \( 1/2 \) and the boson \( \alpha \) has conformal weight \( 1 \) but it is not primary unless \( m = 0 \). It follows easily from the non-commutative
Wick formula, that the only non-trivial relations with the fermion \( \varphi \) are given by:

\[
G_{(1)} \varphi = -m|0\rangle, \quad L_0 \varphi = \frac{1}{2} \varphi,
\]

therefore the subspace \( B(1) \subseteq \mathbb{1}/2 \) of \( B(1) \) spanned by \( \{0\}, \varphi \) is an \( \text{Aut}^\omega \mathcal{O}^{11} \) sub-module. For a given change of coordinates \( \rho = (F, \Psi) \) we can compute the action of \( R(\rho(z, \theta))^{-1} \). For this we write in the basis \( \{0\}, \varphi \}:

\[
R(\rho(z, \theta))^{-1} = A^{2L_0} \cdot \exp \left( \sum_{i \geq 1} (v_i L_i + w_i G_{(i)}) \right)
\]

\[
= \begin{pmatrix} 1 & -mw_1 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & -mD^2 \Psi \\ 0 & D\Psi \end{pmatrix}.
\]

Hence, if \( \mathcal{B}(1) \) is the vector bundle associated to the \( \text{Aut}^\omega \mathcal{O}^{11} \) module \( B(1) \) and \( \mathcal{B}(1) \subseteq \mathbb{1}/2 \) is the vector bundle corresponding to \( B(1) \subseteq \mathbb{1}/2 \) we see that the transition functions that define \( \mathcal{B}(1) \subseteq \mathbb{1}/2 \) are given on the intersections \( U_\alpha \cap U_\beta \) by the functions (3.3.2.1).

Dually, sections of the bundle \( \mathcal{B}(1) \subseteq \mathbb{1}/2 \) transform by (note that we use the super-transpose instead of the transpose, as defined in [Man97, § 3.1]):

\[
(3.3.2.2) \quad \begin{pmatrix} 1 & 0 \\ mD^2 \Psi & D\Psi \end{pmatrix}.
\]

Note that we have a section \( \mathcal{Y} \) of \( \mathcal{B}(1) \) which projects to a section of \( \mathcal{B}(1) \subseteq \mathbb{1}/2 \). In the basis \( \{0\}, \varphi \} \) this section is given by:

\[
\begin{pmatrix} \text{Id} \\ \varphi(z, \theta) \end{pmatrix},
\]

where, according to (2.1.7.1), we have

\[
\varphi(z, \theta) = Y(\varphi(-1)|0\rangle, z) + \theta Y(G_{-1/2}|\varphi(-1)|0\rangle, z).
\]

According to (3.3.2.2) and Theorem 3.2.8 we see that the field \( \varphi(z, \theta) \) transforms as:

\[
(3.3.2.3) \quad \varphi(z, \theta) = R(\rho) \varphi(\rho(z, \theta)) R(\rho)^{-1} D\Psi + mD^2 \Psi \text{Id},
\]

where \( \rho = (F, \Psi) \). In particular, since \( X \) is a superconformal \( N = 1 \) curve we have:

\[
D\Psi = D \left( \frac{DF}{D\Psi} \right) = \text{sdet} \begin{pmatrix} F_z & \Psi_z \\ F_\theta & \Psi_\theta \end{pmatrix}.
\]

Therefore when \( m = 0 \), \( \varphi(z, \theta)[dzd\theta] \) transforms as an \( \text{End} \mathcal{B}(1)_p \)-valued section of the Berezian bundle of \( X \) on the punctured disk \( D_\rho^\times \) for any \( \Lambda \)-point \( p \in X \). When \( m \neq 0 \) this bundle is not split and \( \varphi(z, \theta) \) gives rise to an \( \text{End} \mathcal{B}(1)_p \)-valued section of \( \mathcal{B}(1) \subseteq \mathbb{1}/2 \) that projects onto the section \( 1 \otimes \text{Id} \) of the quotient \( \mathcal{O}_X \otimes \text{End} \mathcal{B}(1)_p \) and transforms according to (3.3.2.3) with changes of coordinates. In other words, the bundle \( \mathcal{B}(1) \subseteq \mathbb{1}/2 \) is an extension:

\[
(3.3.2.4) \quad 0 \to \text{Ber}_X \to \mathcal{B}(1) \to \mathcal{O}_X \to 0
\]

which is non-split unless \( m = 0 \). In the case when \( m \neq 0 \) the section \( \mathcal{Y} \) projects into the constant section \( 1 \) of \( \mathcal{O}_X \).

\footnote{From now on, we abuse notation and forget the fact that the section \( \mathcal{Y} \) is \( \text{End}(\mathcal{Y}_p) \)-valued.}
In analogy to [FBZ01] we want to understand the geometric meaning of these sections. Equivalently, we want to find the set of splittings of the extension (3.3.2.4). This set, if non-empty, is a torsor over the space of even sections of $\text{Ber}_X$. Recall also that the operator $D = \partial_b + \theta \partial_z$ takes values in $\text{Ber}_X$ for a superconformal $N = 1$ curve.

**Theorem 3.3.3.** The superfield $\varphi(z, \theta)$ transforms as an odd differential operator $\nabla : \text{Ber}_X \to \text{Ber}^{\text{even}}_X$ locally of the form $\nabla = -mD_\alpha + g_\alpha(z_\alpha, \theta_\alpha)$, where on the open subset $U_\alpha$ with coordinates $(z_\alpha, \theta_\alpha)$ we have $D_\alpha = \partial_{b_\alpha} + \theta_\alpha \partial_{z_\alpha}$ and $g_\alpha$ is an odd function.

**Proof.** Recall that in a superconformal $N = 1$ curve the generators $[dz_\alpha d\theta_\alpha]$ of the Berezinian bundle transform as:

$$[dz_\beta d\theta_\beta] = (D_\alpha \Psi_{\beta \alpha})[dz_\alpha d\theta_\alpha],$$

where the change of coordinates is $\theta_\beta = \Psi_{\beta \alpha}(z_\alpha, \theta_\alpha)$. Since $\nabla : \text{Ber}_X \to \text{Ber}^{\text{even}}_X$ we have:

$$\nabla f_\alpha = (D_\alpha \Psi_{\beta \alpha})^2 \nabla_\beta \left((D_\alpha \Psi_{\beta \alpha})^{-1} f_\alpha\right).$$

Therefore we get:

$$\nabla f_\alpha = -mD_\alpha f_\alpha + g_\alpha f_\alpha = -m(D_\alpha \Psi_{\beta \alpha})D_\beta f_\alpha + g_\alpha f_\alpha = \left(D_\alpha \Psi_{\beta \alpha}\right)^2 \left(-m(D_\beta(D_\alpha \Psi_{\beta \alpha})^{-1} f_\alpha) + g_\beta(D_\alpha \Psi_{\beta \alpha})^{-1} f_\alpha\right)$$

$$= -m(D_\alpha \Psi_{\beta \alpha})D_\beta f_\alpha - m(D_\alpha \Psi_{\beta \alpha})^2 D_\beta \left((D_\alpha \Psi_{\beta \alpha})^{-1}\right) f_\alpha + g_\beta(D_\alpha \Psi_{\beta \alpha}) f_\alpha$$

$$= -m(D_\alpha \Psi_{\beta \alpha})D_\beta f_\alpha + m(D_\alpha \Psi_{\beta \alpha})^{-1} D_\alpha^2 \Psi_{\beta \alpha} f_\alpha + g_\beta(D_\alpha \Psi_{\beta \alpha}) f_\alpha$$

$$g_\alpha = (D_\alpha \Psi_{\beta \alpha}) g_\beta + m(D_\alpha \Psi_{\beta \alpha})^{-1} D_\alpha^2 \Psi_{\beta \alpha} = m(D_\alpha \Psi_{\beta \alpha})^{-1} D_\alpha^2 \Psi_{\beta \alpha}$$

Hence we find:

$$\left(\begin{array}{c} 1 \\ g_\alpha \end{array}\right) = \left(\begin{array}{cc} 1 & 0 \\ m(D_\alpha \Psi_{\beta \alpha})^{-1} D_\alpha^2 \Psi_{\beta \alpha} & D_\beta \end{array}\right) \left(\begin{array}{c} 1 \\ g_\beta \end{array}\right)$$

thus proving the theorem. \(\square\)

3.3.4. Given that we can integrate a section of $\text{Ber}_X$ along a super contour as in 2.2.15, we can state [FBZ01, 7.1.9] in this situation. We define an affine structure on a superconformal $N = 1$ curve to be a (equivalence class of) coordinate atlas $U_\alpha$ with coordinates $(z_\alpha, \theta_\alpha)$ such that the transition functions on overlaps satisfy:\(^8\)

\begin{align*}
    z_\beta &= F_{\beta \alpha}(z_\alpha, \theta_\alpha) = a^2 z_\alpha + \theta_\alpha \xi a + b, \\
    \theta_\beta &= \Psi_{\beta \alpha}(z_\alpha, \theta_\alpha) = \theta_\alpha a + \xi,
\end{align*}

where $a, b$ are even constants with $a$ invertible and $\xi$ is an odd constant (these constants may change with $\alpha$ and $\beta$). Given such an atlas, we can define $\nabla_\alpha = -mD_\alpha$ and we get from:

$$-mD_\alpha = -mD_\alpha \Psi_{\beta \alpha} D_\beta,$$

and the fact that $D^2 \Psi = 0$ for these transition functions, that $\nabla_\alpha$ is a well defined operator as in Theorem 3.3.3.

\(^8\)These are SUSY changes of coordinates where the odd coordinate changes by affine transformations.
On the other hand, suppose we have such a differential operator \( \nabla_\alpha = -mD_\alpha + g_\alpha \). Let \( f_\alpha dz_\alpha d\theta_\alpha \) be a section of Ber_\( X \) in \( U_\alpha \) such that \( f_\alpha \) is an even function and \( \nabla_\alpha \cdot f_\alpha = 0 \). Choose a \( \Lambda \)-point \( P = (x, \pi) \) of \( U_\alpha \) and, for any other point \( Q \) in \( U_\alpha \), we define the function \( \xi_\alpha \) to be

\[
\xi_\alpha(Q) = \int_Q^P f_\alpha.
\]

From the definition of this integral we see that \( \xi \) is an odd function, indeed, to compute this integral we need to solve \( D\omega = f \) and then this integral becomes \( \omega(Q) - \omega(P) \). By shrinking if necessary the open cover \( U_\alpha \) we may assume that \( f_\alpha \) does not vanish everywhere (it is an even function), it follows that \( D\xi \) is invertible everywhere. We now solve the differential equation \( Dw = \xi D\xi \) (we may need to shrink \( U_\alpha \) even more) and obtain thus a coordinate atlas \( U_\alpha \) with new coordinates \((w_\alpha, \xi_\alpha)\). We claim that this atlas is indeed an affine structure on \( X \). We have made some choices. One is the reference point \( P \) which shifts the function \( \xi_\alpha \) by an odd constant. The other choice was the solution \( \xi \), which is unique up to an invertible even multiple (for this we can apply a version of Cauchy’s theorem in super geometry). Therefore \( \xi \) is well defined up to affine transformations of the form \( \xi \mapsto a\xi + \zeta \). This forces \( w \) to change to \( \tilde{w} \) with

\[
D\tilde{w} = (a\xi + \zeta)D(a\xi + \zeta) = a^2\xi D\xi + a\zeta D\xi,
\]

hence \( \tilde{w} = a^2w + w' \) with \( Dw' = Da\zeta\xi \). Finally we see that \( w' = a\zeta\xi + w'' \) where \( Dw'' = 0 \), namely the choices made combine into changes of the form:

\[
\xi \mapsto a\xi + \zeta, \quad w \mapsto a^2w + a\zeta\xi + b,
\]

where \( a, b \) are even constants \((a \text{ is invertible})\) and \( \zeta \) is odd. Since these changes of coordinates are of the form (3.3.4.1), we have proved:

**Theorem 3.3.5.** Let \( X \) be an \( N = 1 \) superconformal curve. For every \( m \neq 0 \) the set of differential operators \( \nabla : \text{Ber}_X \rightarrow \text{Ber}_X \otimes \mathbb{C}^2 \) locally defined as \( \nabla_\alpha = -mD_\alpha + g_\alpha \), for odd functions \( g_\alpha \), are in one to one correspondence with the set of affine structures on the curve \( X \). These in turn are in one to one correspondence with the set of splittings of the extension (3.3.2.4).

**Example 3.3.6.** The Neveu-Schwarz vertex algebra
Recall the strongly conformal \( N_K = 1 \) SUSY vertex algebra \( K \) defined in Example 2.1.6 (see also Example 2.1.14). Denote this vertex algebra by \( K(1) \). Note that the sub-vector space spanned by the primary elements of conformal weight less or equal to 3/2, namely the vacuum vector and the \( N = 1 \) superconformal vector \( \tau \), is \( \text{Aut}^\omega \mathcal{O}^{1|1} \)-invariant. In order to compute the transition functions we use the relevant relations in this case:

\[
L_{(1)}\tau = \frac{3}{2}\tau, \quad G_{(2)}\tau = \frac{2}{3}c.
\]

Therefore we can compute in the basis \( \{ |0 \rangle, \tau \} \):

\[
R(\rho(z, \theta))^{-1} |0 \rangle = A^2L_0 \exp \left( \sum_{i \geq 1} v_i L_i + w_i G_{(i)} \right) |0 \rangle = \begin{pmatrix} 1 & \frac{2}{3}cw_2 \\ 0 & A^3 \end{pmatrix} \begin{pmatrix} |0 \rangle \\ \tau \end{pmatrix}.
\]
It follows from (3.3.1.3) and (3.3.1.2) that the transition functions for the corresponding bundle \( \mathcal{K}_{3/2}(1) \) are given by:

\[
R(\rho(z, \theta))^{-1} = \begin{pmatrix} 1 & \frac{c}{3} \sigma(D\Psi) \\ 0 & (D\Psi)^3 \end{pmatrix},
\]

where, as before, \( \sigma(D\Psi) \) is the super shwarzian derivative. Dualizing, we obtain an extension:

\[
0 \to \text{Ber}^{33} \to \mathcal{K}_{3/2}(1) \to \mathcal{O}_X \to 0.
\]

This extension is not split if \( c \neq 0 \) and, as for the free fields, we see that the section \( \mathcal{W} \) of \( \mathcal{K}_{3/2}(1)^* \) projects onto the section \( 1 \in \mathcal{O}_X \) in this case. Denote by \( \tau(z, \theta) = G(z) + 2\theta L(z) \) the superfield \( Y(\tau, z, \theta) \). By taking the super transpose of (3.3.6.1) we find that \( \tau(z, \theta) \) transforms as:

\[
\tau(z, \theta) = R(\rho)\tau(\rho(z, \theta))R(\rho)^{-1}(D\Psi)^3 - \frac{c}{3} \sigma(D\Psi)
\]

which in turn implies, according to Proposition 2.2.18 the following:

**Theorem 3.3.7.** The set of splittings of (3.3.6.2) is in one to one correspondence with the set of superprojective structures in \( X \).

3.3.8. Now we turn our attention to the oriented superconformal \( N = 2 \) case. We will use the coordinates \( (z, \theta^\pm = \theta^1 \pm i\theta^2) \) and the change of coordinates \( \rho = (F, \Psi^\pm = \Psi^1 \pm i\Psi^2) \) (cf. 3.1.5). It follows that:

\[
\Psi_{(z, \theta^+), (z, \theta^-)}(t, \zeta^+, \zeta^-) = \Psi^\pm \left( t + z + \frac{1}{2}(\zeta^+ \theta^- + \zeta^- \theta^+), \zeta^+ + \theta^+, \zeta^- + \theta^- \right) - \Psi^\pm (z, \theta^+, \theta^-),
\]

which we want to expand in Taylor series (here \( \Psi \) denotes either \( \Psi^+ \) or \( \Psi^- \)):

\[
\Psi_{(z, \theta^z)} = \left( 1 + \frac{1}{2}(\zeta^+ \theta^- + \zeta^- \theta^+) \partial_z + \frac{1}{8}(\zeta^+ \theta^- + \zeta^- \theta^+)^2 \partial_z^2 \right) \cdot \Psi(t + z, \zeta^+ + \theta^+, \zeta^- + \theta^-) - \Psi(z, \theta^+, \theta^-)
\]

\[
= \left[ \zeta^+ \partial_{\theta^+} + \frac{1}{2} \theta^- \partial_z \right] + \zeta^- \left( \partial_{\theta^-} + \frac{1}{2} \theta^+ \partial_z \right) + t \partial_z + \zeta^+ t \left( \partial_{\theta^+} + \frac{1}{2} \theta^- \partial_z \right) \partial_z + \zeta^- t \left( \partial_{\theta^-} + \frac{1}{2} \theta^+ \partial_z \right) \partial_z + \zeta^+ \zeta^- \left( \partial_{\theta^-, \theta^+} + \frac{1}{2} \theta^- \partial_{z, \theta^+} + \frac{1}{2} \theta^+ \partial_{z, \theta^-} + \frac{1}{4} \theta^+ \theta^- \partial_z^2 \right) \partial_z + \ldots
\]

\[
= \left( \zeta^+ D^- + \zeta^- D^+ + t \partial_z + \zeta^+ t D^- \partial_z + \zeta^- t D^+ \partial_z + \zeta^+ \zeta^- (D^+ D^- \partial_z + \frac{1}{2} \partial_z + \frac{1}{2} \partial_z^2 \partial_z^2) \right) \Psi + \ldots,
\]
where \( D^\pm = \partial x^\pm + \frac{1}{2} \theta^\pm \partial_\theta \). Since the curve is oriented (cf. (3.1.5.6)), this reduces to:

\[
\Psi_{(z, \theta^\pm)}^+ = \left( \zeta^+ D^- + t D^+ D^- + \zeta^+ t D^- D^+ \right) + \zeta^+ \frac{1}{2} (D^+ D^-) + \frac{1}{2} t^2 (D^+ D^-)^2 \Psi^+ + \ldots
\]

\[
\Psi_{(z, \theta^\pm)}^- = \left( \zeta^- D^- + t D^- D^+ + \zeta^- t D^- D^+ \right) + \zeta^- \frac{1}{2} (D^- D^+) + \frac{1}{2} t^2 (D^- D^+)^2 \Psi^- + \ldots
\]

We want to express these as the exponential of a vector field. For this we compute:

\[
\exp \left( - \sum_{i \geq 1} v_i L_i + u_i J_i + w^\pm G^\pm_{(i)} \right) B^{-J_0} A^{-2L_0} \cdot \zeta^\pm = B^\pm A \left[ \zeta^\pm + tw^\pm + \ldots \right]
\]

where we have used (3.1.5.7). We get the equations:

\[
B^\pm A = D^\mp \Psi^\pm, \quad w^\pm = \frac{D^\pm D^\mp \Psi^\pm}{D^\mp \Psi^\pm} = \frac{\Psi^\pm}{D^\mp \Psi^\pm},
\]

\[
v_1 + u_1 + \frac{1}{2} w^+_1 w^-_1 = \frac{D^+_1 D^+_1 \Psi^+_1}{D^+_1 \Psi^+_1}, \quad \frac{1}{2} w^+_1 (2v_1 \pm u_1) = \frac{1}{2} \frac{(D^+_1 D^+_1)^2 \Psi^+_1}{D^+_1 \Psi^+_1}.
\]

We can solve this system to get:

\[
v_1 = \frac{1}{2} \left( \frac{D^- \Psi^+_1}{D^- \Psi^+_1} + \frac{D^+ \Psi^-_1}{D^+ \Psi^-_1} \right), \quad w^+_1 = \frac{1}{2D^+_1 \Psi^+_1} \left( \frac{\Psi^+_1}{\Psi^+_1} - 1 \right), \quad \frac{3}{2} D^+_1 \Psi^+_1 + \frac{1}{2D^+_1 \Psi^+_1} = -\sigma_2 (\Psi^+, \Psi^-),
\]

where \( \sigma_2 \) is the \( N = 2 \) Schwarzian derivative (cf. [Coh87]).

**Example 3.3.9. Free Fields.** With the results of the previous sections we can compute now explicitly some vector bundles over oriented superconformal \( N = 2 \) curves. Let \( Y \) be such a curve and let \( B(2) \) be the strongly conformal \( N_K = 2 \) SUSY vertex algebra described in Example 2.1.11. Let \( \mathcal{B}(2) \) be the associated vector bundle over \( Y \). The vector subspace spanned by the vacuum vectors and the two fermions (namely the fields with conformal weight less or equal to \( 1/2 \)) is an Aut\( \theta^{1\dagger}_1 \)-submodule. Let us denote these vectors, as in 2.1.11, by \( \{ 0 \}, \varphi^\pm \) respectively, and let \( \mathcal{B}_{0,1/2}(2) \) be the associated rank \( 1/2 \) vector bundle over \( Y \). In order to compute its transition functions explicitly we note that the only nontrivial relations (for our purposes) are:

\[
G^\pm_{(1)} \varphi^\pm = \mp m |0\rangle, \quad J_0 \varphi^\pm = \pm \varphi^\pm, \quad L_0 \varphi^\pm = \frac{1}{2} \varphi^\pm.
\]
We compute the transition functions as:

\[
R(\rho)^{-1} \begin{pmatrix} [0] \\ \varphi^+ \\ \varphi^- \end{pmatrix} = A^{2L_0} B^{J_0} \exp \left( \sum_{i \geq 1} v_i L_i + u_i J_i + w_i^+ G_i^+ \right) \begin{pmatrix} [0] \\ \varphi^+ \\ \varphi^- \end{pmatrix}
\]

(3.3.9.1)

\[
= \begin{pmatrix} 1 & m w_1^- & -m w_1^+ \\ 0 & B A & 0 \\ 0 & 0 & B^{-1} A \end{pmatrix} \begin{pmatrix} [0] \\ \varphi^+ \\ \varphi^- \end{pmatrix}
\]

Recall now that an oriented superconformal \( N = 2 \) curve projects onto two \( N = 1 \) supercurves: \( X \) and its dual \( \hat{X} \) (cf. 2.2.12). Using the coordinates (cf. 3.1.5):

\[
u = z + \frac{1}{2} \theta^+ \theta^-, \theta^+, \theta^-
\]

we obtain from (3.1.5.9) and (3.1.5.8) that

(3.3.9.2)

\[
D^+ \Psi^- = D^+ \left( \frac{D^- G}{D^- \Psi^+} \right) = D^+ \left( \frac{1}{(\Psi^+_{\theta^+})^2} (\Psi^+_u - \theta^- \Psi^+_u)(G_{\theta^+} + \theta^- G_u) \right) = \frac{\Psi^+_{\theta^+} G_u - \Psi^+_u G_{\theta^+}}{(\Psi^+_{\theta^-})^2} = \text{sdet} \begin{pmatrix} G_u & \Psi^+_u \\ G_{\theta^+} & \Psi^+_{\theta^+} \end{pmatrix},
\]

where \( G = F + \frac{1}{2} \Psi^+ \Psi^- \) as in 3.1.5. Similarly, we find

(3.3.9.3)

\[
D^- \Psi^+ = \text{sdet} \begin{pmatrix} G'_{\theta^-} & \Psi^-_{\theta^-} \\ G'_{\theta^+} & \Psi^+_{\theta^+} \end{pmatrix}.
\]

Let us call \( \pi \) and \( \hat{\pi} \) the projections from \( Y \) onto \( X \) and \( \hat{X} \) respectively. We see from (3.3.9.3) and (3.3.9.2) that taking the super-transpose in (3.3.9.1) we obtain an extension (of sheaves of \( \mathcal{O}_Y \)-modules):

(3.3.9.4) \[ 0 \to \pi^* \text{Ber}_X \oplus \hat{\pi}^* \text{Ber}_{\hat{X}} \to \mathcal{B}(2)_{\leq 1/2} \to \mathcal{O}_Y \to 0. \]

As in the \( B(1) \) case, this extension is not split unless \( m \) vanishes. It follows in the same way as in the \( N = 1 \) case that the set of splittings of this extension corresponds to affine structures on the \( N = 2 \) superconformal curve \( Y \). Indeed, we see in the same way as in Theorem 3.3.3, that the pair of fields \( (\varphi^+, \varphi^-) \) transforms as a differential operator \( V : \text{Ber}_X \oplus \text{Ber}_{\hat{X}} \to \text{Ber}_X \otimes \text{Ber}_{\hat{X}} \otimes \text{Ber}_{\hat{X}} \) which is locally of the form \( (mD^+ + g^+)(-mD^- + g^-) \) for \( g^\pm \) odd functions of \( (u, \theta^+) \) and \( (u', \theta^-) \) respectively. We note that according to 2.2.15 sections of \( \text{Ber}_X \oplus \text{Ber}_{\hat{X}} \) can be integrated in \( Y \) up to an additive constant. The argument in the proof of Theorem 3.3.5 generalizes to this setting without difficulty.

We will return to this example below (cf. 3.3.12).

**Example 3.3.10.** The \( N = 2 \) vertex algebra. Let \( K(2) := K_2 \) be the strongly conformal \( N_8 = 2 \) SUSY vertex algebra described in Example 2.1.9 (see also Example 2.1.14), and let \( \mathcal{K}(2) \) be the associated vector bundle over an oriented superconformal \( N = 2 \) curve \( Y \). The vector subspace spanned by primary fields of
conformal weight 0 and 1 is an $\text{Aut}^{\omega}O^{1/2}$ submodule. Let us denote these vectors as above by $\{0\}, J$ respectively, and let $\mathcal{K}(2)_{\leq 1}$ be the associated rank 2 vector bundle over $Y$. To compute the transition functions we note that the only non-trivial relations we need are:

\[(3.3.10.1) \quad L_0 J = J, \quad J_1 J = \frac{c}{3} \langle 0 \rangle .\]

Therefore the transition functions are given by:

\[
R(\rho)^{-1} \begin{pmatrix} 0 \\ J \end{pmatrix} = \begin{pmatrix} 1 & \frac{\xi}{u} u_1 \\ 0 & A^2 \end{pmatrix} \begin{pmatrix} 0 \\ J \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\xi}{u} \sigma_2(\Psi^+, \Psi^-) \\ 0 & D^+ \Psi - D^- \Psi^+ \end{pmatrix} \begin{pmatrix} 0 \\ J \end{pmatrix} .
\]

It follows as before, by taking the super-transpose, that when $c = 0$, the superfield $J(z, \theta^+, \theta^-)$ transforms as a section of $\pi^* \text{Ber}_{\mathcal{K}} \otimes \tilde{\pi}^* \text{Ber}_{\mathcal{K}}$, namely in this case we get an extension:

\[(3.3.10.2) \quad 0 \rightarrow \pi^* \text{Ber}_{\mathcal{K}} \otimes \tilde{\pi}^* \text{Ber}_{\mathcal{K}} \rightarrow \mathcal{K}(2)_{\leq 1} \rightarrow \mathcal{O}_Y \rightarrow 0 ,
\]

which is split if and only if $c = 0$. When $c \neq 0$, the superfield $J(z, \theta^+, \theta^-)$ transforms as:

\[J(z, \theta^+, \theta^-) = (D^+ \Psi^-)(D^- \Psi^+) J(\rho(z, \theta^+, \theta^-)) + \frac{c}{3} \sigma_2(\Psi^+, \Psi^-) .
\]

We see that the section $\mathcal{O}$ is an even section projecting onto $1 \in \mathcal{O}_Y$, therefore giving a splitting of (3.3.10.2). The set of such splittings, if non-empty, is a torsor for the even part of $\pi^* \text{Ber}_{\mathcal{K}} \otimes \tilde{\pi}^* \text{Ber}_{\mathcal{K}}$.

Analyzing this algebra further, we can consider the space $K(2)_{\leq 3/2}$ spanned by vectors of conformal weight less than or equal to $3/2$. This space is spanned by $\{0, J, G^p\}$. In addition to (3.3.10.1) we have the following relations:

\[L_0 G^- = \frac{3}{2} G^-, \quad J_0 G^- = -G^-, \quad G^+ G^+ = J, \quad G^+_1 G^- = \frac{c}{3} \langle 0 \rangle ,
\]

\[L_0 G^+ = \frac{3}{2} G^+, \quad J_0 G^+ = G^+, \quad G^+ G^+ = -J, \quad G^+_1 G^+ = \frac{c}{3} \langle 0 \rangle .
\]

With these we can compute the transition functions in the basis $\{0, J, G^p, G^-\}$ explicitly:

\[(3.3.10.3) \quad R(\rho)^{-1} = \begin{pmatrix} 1 & \frac{\xi}{u} u_1 & \frac{\xi}{u} w_1^+ & \frac{\xi}{u} w_1^- \\ 0 & A^2 & A^2 w_1^+ & -A^2 w_1^- \\ 0 & 0 & A^3 B & 0 \\ 0 & 0 & 0 & A^3 B \end{pmatrix},
\]

the first three by three block being:

\[
\begin{pmatrix} 1 & -\frac{\xi}{u} \sigma_2(\Psi^+, \Psi^-) \\ 0 & (D^+ \Psi^-)(D^- \Psi^+) \langle \Psi^+, z \rangle - \frac{1}{2} \left( \frac{D^+ \Psi^-}{D^+ \Psi^+} + 3 \frac{D^- \Psi^+}{D^- \Psi^-} \right) \\ 0 & 0 & (D^+ \Psi^-)(D^- \Psi^-) \langle \Psi^+, z \rangle - \frac{1}{2} \left( \frac{D^+ \Psi^-}{D^+ \Psi^+} + 3 \frac{D^- \Psi^+}{D^- \Psi^-} \right) \\ 0 & 0 & 0 & (D^+ \Psi^-)(D^- \Psi^-) \langle \Psi^+, z \rangle - \frac{1}{2} \left( \frac{D^+ \Psi^-}{D^+ \Psi^+} + 3 \frac{D^- \Psi^+}{D^- \Psi^-} \right) \end{pmatrix} ,
\]

and the 4,4 entry in (3.3.10.3) is $(D^+ \Psi^-)(D^- \Psi^-)^2$. Taking the super-transpose of (3.3.10.3) it follows that $\mathcal{K}(2)_{Y, \leq 3/2}^*$ fits in a short exact sequence of the form:

\[0 \rightarrow \pi^* \text{Ber}_{\mathcal{K}} \otimes (\tilde{\pi}^* \text{Ber}_{\mathcal{K}})^{\otimes 2} \rightarrow \mathcal{K}(2)_{Y, \leq 3/2}^* \rightarrow \mathcal{K}^* \rightarrow 0 .\]
The bundle \( \mathcal{N} \) in turn fits in the exact sequence:

\[
0 \to (\pi^* \text{Ber}_X)^{\otimes 2} \otimes \hat{\pi}^* \text{Ber}_X \to \mathcal{N}^* \to \mathcal{H}(2)^*_{Y \leq 1} \to 0.
\]

In a more “symmetric” fashion, if we look at the lower two by two block in (3.3.10.3), we see that we have an extension:

\[
0 \to \pi^* \text{Ber}_X \otimes (\hat{\pi}^* \text{Ber}_X)^{\otimes 2} \otimes (\pi^* \text{Ber}_X)^{\otimes 2} \otimes \hat{\pi}^* \text{Ber}_X \to \mathcal{H}(2)^*_{Y \leq 3/2} \to \mathcal{H}(2)^*_{Y \leq 1} \to 0.
\]

3.3.11. We turn our attention now to the \( N_W = 1 \) case. For this let \( X \) be a general \( N = 1 \) supercurve. As before, given a change of coordinates \( \rho = (F, \Psi) \), we expand in Taylor series:

\[
F(z, \theta)(t, \zeta) = F(t + z, \zeta + \theta) - F(z, \theta) \]
\[
= tF_z + \zeta F_\theta + \zeta tF_{\theta, z} + \frac{t^2}{2} F_{z, z} + \ldots
\]
\[
\Psi(z, \theta)(t, \zeta) = \Psi(t + z, \zeta + \theta) - \Psi(z, \theta) \]
\[
= t\Psi_z + \zeta \Psi_\theta + \zeta t\Psi_{z, \theta} + \frac{t^2}{2} \Psi_{z, z}.
\]

We need to express these as:

\[
(3.3.11.1) \quad \left( \begin{array}{c}
F(z, \theta) \\
\Psi(z, \theta)
\end{array} \right) = \exp \left( -\sum_{i \geq 1} v_i T_i + u_i J_i + q_i Q_i + h_i H_i \right) \times
\]
\[
\times \exp(-q_0 Q_0) \exp(-h_0 H_0) B^{-J_0} A^{-T_0} \left( \begin{array}{c}
\frac{t}{\zeta}
\end{array} \right),
\]

where, as in (3.1.3.1), we have:

\[
T_n = -t^{n+1} \partial_t - (n + 1)t^n \zeta \partial_\zeta, \quad J_n = -t^n \zeta \partial_\zeta, \quad Q_n = -t^{n+1} \partial_\zeta, \quad H_n = t^n \zeta \partial_\zeta.
\]

Expanding (3.3.11.1) up to second order, we find:

\[
F(z, \theta) = tA(1 + q_0 h_0) + \zeta Ah_0 + t^2 (v_1 (A + Aq_0 h_0) + Aq_1 h_0) +
\]
\[
+ \zeta t(A(1 + q_0 h_0)h_1 + 2Av_1 h_0 + Ah_1 h_0) + \ldots
\]
\[
\Psi(z, \theta) = \zeta BA + t q_0 BA + t \zeta BA(2v_1 + u_1 + h_1 q_0) + t^2 BA(q_1 + v_1 q_0) + \ldots,
\]

and we get the equations:

\[
A(1 + q_0 h_0) = F_z, \quad BA = \Psi_\theta, \quad q_0 BA = \Psi_z,
\]
\[
Ah_0 = F_\theta, \quad v_1 F_z + q_1 F_\theta = \frac{1}{2} F_{z, z}, \quad h_1 \Psi_z + (2v_1 + u_1) \Psi_\theta = \Psi_{\theta, z},
\]
\[
h_1 F_z + (2v_1 + u_1) F_\theta = F_{z, \theta}, \quad v_1 \Psi_z + q_1 \Psi_\theta = \frac{1}{2} \Psi_{z, z}.
\]
From this we find:

\[
A = \frac{F_z \Psi_\theta - \Psi_z F_\theta}{\Psi_\theta}, \quad B = \frac{\Psi_\theta^2}{F_z \Psi_\theta - \Psi_z F_\theta},
\]

(3.3.11.3)

\[
h_0 = \frac{F_z \Psi_\theta - \Psi_z F_\theta}{F_\theta \Psi_\theta}, \quad q_0 = \frac{\Psi_\theta}{\Psi_\theta},
\]

\[
v_1 = \frac{1}{2} \frac{F_z \Psi_\theta - \Psi_z F_\theta}{F_\theta \Psi_\theta - \Psi_z F_\theta}, \quad q_1 = \frac{1}{2} \frac{F_z \Psi_\theta - \Psi_z F_\theta}{F_\theta \Psi_\theta - \Psi_z F_\theta},
\]

\[
h_1 = \frac{F_z \Psi_\theta - \Psi_z F_\theta}{F_z \Psi_\theta - \Psi_z F_\theta}, \quad u_1 = \frac{F_z \Psi_\theta - \Psi_z F_\theta}{F_\theta \Psi_\theta - \Psi_z F_\theta} + \frac{\Psi_\theta}{F_\theta \Psi_\theta - \Psi_z F_\theta}.
\]

Example 3.3.12. Free Fields Consider the vertex algebra \(B(2)\) as in example 3.3.9 but as a \(N_W = 1\) SUSY vertex algebra. As such, for each \(N = 1\) supercurve \(X\) we obtain a vector bundle \(\mathcal{B}(2)_X\). Recall that with respect to the Virasoro field \(L\), the vector \(\varphi^-\) has conformal weight 0. Therefore the vector space spanned by \(|0\rangle\) and \(\varphi^-\) is an \(\text{Aut}\mathcal{O}^{1|1}\) submodule. We obtain then a rank 11 vector bundle over \(X\), to be denoted \(\mathcal{B}(2)_{X, \leq 0}\). Let us compute explicitly the transition functions for this bundle. The relevant relations are in this case:

\[
J_0 \varphi^- = -\varphi^-, \quad Q_0 \varphi^- = -m|0\rangle.
\]

Hence we obtain:

\[
R(\rho)^{-1} \begin{pmatrix} |0\rangle \\ \varphi^- \end{pmatrix} = A^0 B^{h_0} \exp(h_0 H_0) \exp(q_0 Q_0) \begin{pmatrix} |0\rangle \\ \varphi^- \end{pmatrix} = \begin{pmatrix} 0 & -mQ_0 \\ 1 & B^{-1} \end{pmatrix} \begin{pmatrix} |0\rangle \\ \varphi^- \end{pmatrix},
\]

which implies:

(3.3.12.1)

\[
R(\rho)^{-1} = \begin{pmatrix} 0 & -mQ_0 \\ 1 & B^{-1} \end{pmatrix}.
\]

Noting that

\[
\text{sdet} \left( \begin{array}{cc} F_z & \Psi_z \\ F_\theta & \Psi_\theta \end{array} \right) = \frac{F_z \Psi_\theta - \Psi_z F_\theta}{\Psi_\theta^2},
\]

we see that by taking the super-transpose in (3.3.12.1) we obtain an extension

(3.3.12.2)

\[
0 \to \text{Ber}_X \to \mathcal{B}(2)_{X, \leq 0} \to \mathcal{O}_X \to 0.
\]

This short exact sequence is split if and only if \(m = 0\). In that case, we see that \(\varphi^- (z, \theta)|dz d\theta\rangle\) transforms as a section of \(\text{Ber}_X\). On the other hand, when \(m \neq 0\), (3.3.12.2) is not split and the section \(\mathcal{O}\) projects into 1 \(\in \mathcal{O}_X\), giving a splitting of (3.3.12.2). In order to analyze the splittings of these sequences, recall from 2.2.1 that to the \(N = 1\) supercurve \(X\) we have associated another “dual” curve \(\hat{X}\) and an oriented superconformal \(N = 2\) curve \(Y\). Introduce maps of sheaves on \(Y\), \(\nabla: \text{Ber}_Y \to \text{Ber}_X \otimes \text{Ber}_X\) which are locally of the form \(\nabla_\alpha = -mD^+_\alpha + g_\alpha\), for an odd function \(g_\alpha = g_\alpha(u, \theta^+)\). Here we consider \(X\) with coordinates \(u, \theta^+\) and \(\hat{X}\) with coordinates \(u', \theta^-\) as in 3.1.5. We will write \(f\) to denote a function of \(u', \theta^-\). It follows from (3.3.9.3), (3.3.9.4) and the fact that \(\nabla\) maps \(\text{Ber}_\hat{X} \to \text{Ber}_X \otimes \text{Ber}_X\) that on overlaps we must have:

\[
\nabla_\alpha f_\alpha = (D^+ \Psi^-)(D^- \Psi^+)\nabla_\beta \left((D^- \Psi^+)^{-1} f_\alpha\right).
\]
Replacing $\nabla$ in both sides by its local form and using (3.1.5.5) and (3.1.5.6), we get:

$$- mD^+_\alpha \hat{f}_\alpha + g_\alpha \hat{f}_\alpha = - mD^+ \alpha \hat{f}_\alpha + m(D^- \Psi^+)^{-1} D^+_\alpha \Psi^+ \hat{f}_\alpha + (D^+ \Psi^-) g_\beta \hat{f}_\alpha.$$  

Now noting that $D^+ D^- \Psi^+ = \Psi^+_u$ and that:

$$\frac{\Psi^+_u}{D^- \Psi^+} = \frac{\Psi^+_u}{\Psi^-_{\theta^+} + \theta^- \Psi^+_u} = \frac{\Psi^+_u}{\Psi^+_{\theta^+}},$$

we get

(3.3.12.3)  

$$g_\alpha = \text{sdet} \left( \begin{array}{cc} G_u & \Psi^+_u \\ G_{\theta^+} & \Psi^+_{\theta^+} \end{array} \right) g_\beta + m \frac{\Psi^+_u}{\Psi^+_{\theta^+}}$$

therefore proving the following

**Theorem.** The set of splittings of (3.3.12.2) for $m \neq 0$ is in one to one correspondence with operators $\nabla : \text{Ber}_X \rightarrow \text{Ber}_X \otimes \text{Ber}_X$ locally of the form $- mD^+ \alpha + g_\alpha$.

Let $\nabla$ be such an operator, and let $0 \neq \psi_\alpha \in \text{Ber}_X(U_\alpha)$ be a flat even section, namely $\nabla_\alpha \psi_\alpha = 0$. As a section of $\text{Ber}_X$, it can be integrated along any contour in $X$ (cf. 2.2.15), namely, let $P$ be a reference $\Lambda$-point in $U_\alpha$, then for any other $\Lambda$-point in $U_\alpha$ we put:

$$\zeta_\alpha(Q) = \int_Q^P \psi_\alpha.$$  

The solution $\zeta_\alpha$ is unique up to an even multiplicative constant, while changing the reference point $P$ changes $\zeta_\alpha$ by an additive odd constant, shrinking $U_\alpha$ we may assume that $D_\alpha \zeta$ is invertible. Choosing any other even function $t_\alpha$ with invertible differential, we obtain charts $U_\alpha, (t_\alpha, \zeta_\alpha)$. The transition functions between these charts are clearly affine functions for the odd coordinates, namely $\zeta_\beta = a_{\beta, \alpha} \zeta_\alpha + \varepsilon_{\beta, \alpha}$ for some even constants $a$ and odd constants $\varepsilon$. Conversely, given such a covering of $X$, we define $\nabla_\alpha = - mD^+_\alpha$, where we take $\zeta$ instead of $\theta^+$ and $t$ instead of $u$ in the definition of $D^+$. It follows from (3.3.12.3) that $\nabla$ is well defined globally since the second term in the right hand side of (3.3.12.3) vanishes.

Combining the above paragraph with the previous theorem we have

**Theorem.** The set of splittings of (3.3.12.2) for $m \neq 0$ is in one to one correspondence with (equivalence classes of) atlases $U_\alpha, z_\alpha, \theta_\alpha$, such that the transition functions are affine in the odd coordinate, namely $\theta_\beta = a \theta_\alpha + \varepsilon$ for some even constant $a$ and some odd constant $\varepsilon$.

Note that from (3.3.12.1) and (3.3.9.1) it follows that the following sequences of $\mathcal{O}_Y$-modules are exact:

$$0 \rightarrow \pi^* \text{Ber}_X \rightarrow \mathcal{B}(2)_{Y, \leq 1/2}^* \rightarrow \pi^* \mathcal{B}(2)_{X, \leq 0} \rightarrow 0,$$

$$0 \rightarrow \pi^* \text{Ber}_X \rightarrow \mathcal{B}(2)_{Y, \leq 1/2}^* \rightarrow \pi^* \mathcal{B}(2)_{X, \leq 0} \rightarrow 0.$$

The bundle $\mathcal{B}(2)_Y$ is the corresponding bundle constructed in Example 3.3.9 from this vertex algebra, but viewed as an $N_K = 2$ SUSY vertex algebra. These two extensions show how the different vector bundles constructed from the same vertex algebras in these three different curves ($X$, $\hat{X}$ and $Y$) are related.

**Example 3.3.13.** The $N = 2$ vertex algebra. Let, as before $K(2)$ be the $N = 2$ super vertex algebra defined in Example 2.1.9, but considered as an $N_W = 1$ SUSY vertex algebra. Let $X$ be an $N = 1$ supercurve. The vector space spanned by the vacuum vector, the current $J$, and the fermion $H$, is an Aut$\mathfrak{g}^{111}$-submodule.
Indeed, with respect to the Virasoro field $\tilde{L}$, the fermion $H$ has conformal weight 1. Denote the corresponding rank 2|1 vector bundle over $X$ by $\mathcal{K}(2)_{X, \leq 1}$. It follows from the general considerations in appendix A, that the dual of this vector bundle fits in a short exact sequence of the form:

$$(3.3.13.1) \quad 0 \to \Omega^1_X \otimes \text{Ber}_X \to \mathcal{K}(2)_{X, \leq 1} \to \mathcal{O}_X \to 0.$$ 

Indeed, the relevant relations are in this case:

$$T_0 J = J, \quad T_1 J = - \frac{e}{3} |0\rangle, \quad J_1 J = - \frac{e}{3} |0\rangle, \quad H_0 J = H,$$

$$T_0 H = H, \quad Q_0 H = J, \quad J_0 H = - H, \quad Q_1 H = H - \frac{e}{3} |0\rangle,$$

therefore the vector space $K(2)_1$ spanned by $\{J, H\}$ is isomorphic (as a $\mathfrak{gl}(1|1)$-module) to $\pi_+(1,0)$ (cf. Appendix A), and its dual module is then $\pi_-(-1,0) \equiv \pi_-(1,0) \otimes \pi_-(1)$. Also we know that the $\text{Aut}^{\mathfrak{h}^{1|1}}$-twist of $\pi_-(-1,0)$ (resp. $\pi_-(1)$) is $\Omega^1_X$ (resp. $\text{Ber}_X$).

We can actually compute these transition functions explicitly as before by exponentiating vector fields:

$$R(\rho)^{-1} \begin{pmatrix} |0\rangle \\ J \\ H \end{pmatrix} = A_{T_0} B_{J_0} \exp(h_0 H_0) \exp(q_0 Q_0) \times$$

$$\times \exp \left( \sum_{i \geq 1} v_i L_i + u_i J_i + q_i Q_i + h_i H_i \right) \cdot \begin{pmatrix} |0\rangle \\ J \\ H \end{pmatrix}$$

$$= A_{T_0} B_{J_0} \exp(h_0 H_0) \exp(q_0 Q_0) \cdot \begin{pmatrix} |0\rangle \\ (J + (u_1 - v_1) \frac{e}{3} |0\rangle) \\ H + q_1 \frac{e}{3} |0\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{e}{3}(v_1 - u_1) & \frac{e}{3} q_1 \\ 0 & A & A_{q_0} \\ 0 & B^{-1} Ah_0 & B^{-1} A(1 - h_0 q_0) \end{pmatrix} \begin{pmatrix} |0\rangle \\ J \\ H \end{pmatrix},$$

which, according to (3.3.11.3), implies:

$$(3.3.13.2) \quad R(\rho)^{-1} = \begin{pmatrix} 1 & \frac{e}{3} \left( \frac{F_5 \Psi + \Psi \Psi F_5}{F_5 \Psi + \Psi \Psi F_5} + \frac{3}{2} \frac{F_5 \Psi F_0 - F_5 \Psi F_0}{F_5 \Psi F_0 - F_5 \Psi F_0} \right) \frac{e}{6} \frac{F_5 \Psi - \Psi \Psi F_5}{F_5 \Psi - \Psi \Psi F_5} \\ 0 & \frac{F_5 \Psi}{\Psi} & \frac{F_5 \Psi}{\Psi} \\ 0 & \frac{F_5 \Psi}{\Psi} & \frac{F_5 \Psi}{\Psi} \end{pmatrix}.$$ 

Taking the super-transpose of the lower two by two block we easily see that this block corresponds to the transition functions in $\text{Ber}_X \otimes \Omega^1_X$, proving thus that $\mathcal{K}(2)_{X, \leq 1}$ is given by an extension as in (3.3.13.1). This extension is non-split unless $c = 0$, in which case the pair of fields $\{J(z, \theta), H(z, \theta)\}$ transforms as a section of $\text{Ber}_X \otimes \Omega^1_X$. In order to study the splittings of this extension we need to understand the differential operators appearing in the first row of (3.3.13.2). We leave this to the reader.

4. Chiral algebras on supercurves

4.0.1. In this section we follow closely the treatment in chapter 18 of [FBZ01]. We note that most definitions carry over to the “super” case with minor technical
changes. In particular we give a sheaf theoretical interpretation of the OPE formula (2.1.19.1). We define the superconformal blocks in section 4.2.

We will restrict our analysis to the $(1|1)$ dimensional case for simplicity. All the results in this section can be generalized to arbitrary odd dimensions without difficulty.

For the definitions of chiral algebras over ordinary curves, the reader is referred to [BD04] and [FBZ01]. For the theory of $\mathcal{D}$-modules, we refer to [Ber], and [Pen83] in the supermanifold case.

4.1. Chiral algebras.

4.1.1. When trying to define chiral algebras on supercurves the first problem that we encounter is that given a $(1|N)$ dimensional supercurve $X$ over $S$, the diagonal embedding $\Delta \hookrightarrow X \times_S X$ has relative codimension $(1|N)$. In particular, the diagonal is not a divisor in $X \times_S X$ unless $N = 0$.

The situation is much simpler in the superconformal case (corresponding to $N_K = N$ SUSY vertex algebras). In this case, we can define canonically a divisor in $X \times_S X$. Basically, all the arguments in the classical case work without change in the superconformal case, given that we have replaced the diagonal by a super diagonal.

Since we can carry explicitly the computations in the $N = 1$ case, without introducing extra notation, we will assume that this is the case in the following.

**Lemma 4.1.2** (6.3 [Man91]). (cf. 4.1.7 below) Let $X$ be a superconformal $N = 1$ supercurve. Let $J$ be the ideal defining the diagonal $i : \Delta \hookrightarrow X \times_S X$. In local coordinates $J$ is defined by $(z - w, \theta - \zeta)$. Let $\Delta^{(1)}$ be defined by $J^2$. Let $I$ be the kernel of the natural map $\Omega^1_{X/S} \to \text{Ber}_{X/S}$. Finally we define $\Delta^s$ by:

$$O_{\Delta^s} = O_{\Delta^{(1)}/i_*(I)}.$$ 

Then $\Delta^s$ is a $(1|0)$ codimensional divisor in $X \times_S X$, locally defined by the equation $0 = z - w - \theta \zeta$.

This divisor will be called the super diagonal and we will simply call it the diagonal when no confusion should arise.

4.1.3. Given an $\mathcal{O}_X$-module $\mathcal{M}$, we define two extensions of $\mathcal{M}$ along the super diagonal: extension by principal parts in the transversal direction and extension by delta functions in the transversal direction. The former is given by

$$\Delta^s_* \mathcal{M} := \mathcal{O} \boxtimes \mathcal{M}(\infty \Delta^s)/\mathcal{O} \boxtimes \mathcal{M},$$

and the latter by

$$\Delta^s_* \mathcal{M} := \omega \boxtimes \mathcal{M}(\infty \Delta^s)/\omega \boxtimes \mathcal{M},$$

where $\omega$ is the Berezinian bundle of $X$ defined in 2.2.11.

4.1.4. As in the non-super case, we have a sheaf-theoretical interpretation of the OPE formula. For this we let $X$ be a superconformal $N = 1$ curve over $\Lambda$. Let $V$ be a strongly conformal $N_K = 1$ SUSY vertex algebra over $\Lambda$ and let $\mathcal{V}$ be the associated vector bundle over $X$ (cf. 3.2.7). Recall that, given any $\Lambda$-point $x$ in $X$, we have defined a local section $\mathcal{Y}_x$ (cf. 3.2.8). Choose local coordinates...
Z at x compatible with the superconformal structure. Using this coordinates we trivialize the bundle \( Y \) in the formal superdisk \( D_x \) around x, namely we have an isomorphism \( i_x : V[[Z]] \rightarrow \Gamma(D_x, Y) \). Let \( W \) be another copy of \( Z \), so that \( D^2_x \) is identified with Spec \( \Lambda[Z,W] \). The bundle \( Y \otimes Y(\infty \Delta^*) \), when restricted to \( D^2_x \), is the sheaf associated to the \( \Lambda[[Z,W]] \)-module \( V \otimes V[[Z,W]]((z - w - \theta \zeta)^{-1}) \). Similarly, the restriction of the sheaf \( \Delta^*_s Y \) to \( D^2_x \) is associated to the \( \Lambda[[Z,W]] \)-module \( V[[Z,W]]((z - w - \theta \zeta))/V[[Z,W]] \).

**Theorem 4.1.5.** Define a map of \( \mathcal{O}_{D^2_x} \)-modules \( \mathcal{N}_{2,x} : Y \otimes Y(\infty \Delta^*) \rightarrow \Delta^*_s Y \) by the formula

\[
\mathcal{N}_{2,x}(f(Z,W)a \boxtimes b) = f(Z,W)Y(a, Z - W)b \quad \text{mod } V[[Z,W]].
\]

Then \( \mathcal{N}_{2,x} \) is independent of the choice of the coordinates \( Z \) as long as they are compatible with the superconformal structure induced in \( D_x \) from that of \( X \).

**Proof.** Exactly as in the non-super case, we reduce the proof of this theorem to the identity:

\[
Y(a, Z - W) = R(\mu_W)Y(\mu_Z)^{-1}a, \mu(Z) - \mu(W)) R(\mu_W)^{-1}, \quad a \in V,
\]

for any \( \mu \in \text{Aut}^{\sigma} \mathcal{O}^{11} \). This identity is equivalent to (3.2.4.1) by substituting \( Z - W \) instead of \( Z \) and \( \mu_W(Z - W) = \mu(Z) - \mu(W) \) instead of \( \rho(Z) \). Recall that in this case we have

\[
Z - W = (z - w - \theta \zeta, \theta - \zeta).
\]

\[\square\]

**Remark 4.1.6.** In order to prove a similar statement for a general \( N = 1 \) supercurve \( X \) over \( \Lambda \), we could define a “super-diagonal” as follows. Recall that any such curve \( X \) gives rise to an oriented superconformal \( N = 2 \) super curve \( Y \) (cf. 2.2.12). Recall also that the curve \( Y \) comes equipped with two maps \( \pi : Y \rightarrow X \) and \( \hat{\pi} : Y \rightarrow \hat{X} \), where \( \hat{X} \) is the dual curve. In local coordinates these maps are described by (cf. 3.3.10)

\[
(z, \theta^+, \theta^-) \xrightarrow{\pi} \left( z + \frac{1}{2} \theta^+ \theta^-, \theta^+ \right),
\]

\[
(z, \theta^+, \theta^-) \xrightarrow{\hat{\pi}} \left( z - \frac{1}{2} \theta^+ \theta^-, \theta^- \right)
\]

It is easy to show that \( Y \) embeds as a \((1,0)\) codimensional divisor in \( X \times \Lambda \hat{X} \). Indeed, for a \( \Lambda \)-point \( x \) in \( X \) given by local parameters \( Z = (z, \theta) \) the preimage in \( Y \) is given by local parameters \( (z - \frac{1}{2} \theta \zeta, \theta, \zeta) \). Similarly, for a point \( W = (w, \zeta) \) in \( \hat{X} \) we have its preimage in \( Y \) given by local parameters \( (w + \frac{1}{2} \theta \zeta, \theta, \zeta) \). Then the point \( (Z,W) \) in \( X \times \Lambda \hat{X} \) is in the image of \( Y \) if and only if \( z - w - \theta \zeta = 0 \). Note in particular that when \( X \) is superconformal, namely \( X \equiv \hat{X} \) this “diagonal” \( Y \rightarrow X \times \Lambda \hat{X} \) agrees with Manin’s super-diagonal given in Lemma 4.1.2.

We could try to repeat the argument given above for superconformal curves, but the operation \( \mathcal{N}_2 \) turns out to be coordinate-dependent.\(^9\)

\(^9\)It will be nice to find a way of describing the vertex algebra multiplication as an expression when a point \( x \in X \) “collides” with a point \( \hat{x} \in \hat{X} \) along the “diagonal” \( \Delta^s \subset X \times X \).
4.1.7. Instead of using the approach in the previous remark, note that we can define the push-forwards $\Delta_+$ and $\Delta_!$ even when $\Delta$ is not a divisor. In our case these are easy to describe. Let $\Delta$ be the diagonal $\Delta \hookrightarrow X \times_X X$. Even though $\Delta$ is not a divisor in $X \times_X X$, its reduction $|\Delta|$ is a divisor in $|X| \times_{|S|} |X|$. We have then an open immersion $j : X \times X \setminus \Delta \hookrightarrow X \times X$, where $X \times X \setminus \Delta$ is $U = |X| \times |X| \setminus |\Delta|$ as a topological space and the structure sheaf is the restriction of $\mathcal{O}_X$ to $U$. We can now define the corresponding push-forwards of an $\mathcal{O}_X$-module $\mathcal{M}$ as:

\[
\Delta_+ \mathcal{M} = j_* j^*(\mathcal{O}_X \boxtimes \mathcal{M}), \quad \Delta_! \mathcal{M} = j_* j^*(\omega \boxtimes \mathcal{M}).
\]

When no confusion can arise, for any sheaf $\mathcal{F}$, we will denote by $\mathcal{F}(\infty \Delta)$ the sheaf $j_* j^* \mathcal{F}$.

**Remark 4.1.8.** As in the non-super case, these pushforwards are in fact the push forward of left (resp. right) $\mathcal{D}_X$-modules along the diagonal, where in the superconformal case we understand for a $\mathcal{D}_X$ module, a module over the ring of superconformal differential operators as in Remark 3.2.10 (see also Remark 4.1.13).

4.1.9. We construct now a morphism of $\mathcal{O}_{D^2}$-modules $\mathcal{B}_{2,x} : j_* j^* (\mathcal{Y}_X \boxtimes \mathcal{Y}_X) \rightarrow \Delta_+ \mathcal{Y}_X$ by the formula:

\[(4.1.9.1) \quad \mathcal{B}_{2,x}(f(Z,W)a \boxtimes b) = f(Z,W) Y(a,Z-W)b \mod V[[Z,W]].\]

As in 4.1.5 we have

**Theorem 4.1.10.** The map $\mathcal{B}_{2,x}$ defined by (4.1.9.1) does not depend on the coordinates $Z$ chosen.

4.1.11. We can now generalize all the results in [FBZ01, chapter 18] on chiral algebras without difficulty. For simplicity let us assume that $X$ is a general 1|1|N-dimensional supercurve. Suppose that the sheaf $\mathcal{M}$ on $X$ carries a (left) action of the sheaf of differential operators $\mathcal{D}_X$. Let $\sigma_{12} : X^2 \rightarrow X^2$ be the transposition of the two factors. We obtain a canonical isomorphism of sheaves $\Delta_+ \mathcal{M} \simeq \sigma_{12}^* \Delta_+ \mathcal{M}$, given in local coordinates by the formula:

\[
\frac{1 \otimes \psi}{(Z-W)^{k|R}} \mapsto e^{(Z-W)\nabla} \cdot \frac{\psi \otimes 1}{(Z-W)^{k|R}} \mod \mathcal{M} \boxtimes \mathcal{O}_X,
\]

where $\psi$ is a local section of $\mathcal{M}$ and $\nabla$ is the connection that we obtain from the $\mathcal{D}$-module structure in $\mathcal{M}$. When $\mathcal{M}$ carries a right action of $\mathcal{D}_X$, we obtain similarly an isomorphism $\Delta_! \mathcal{M} \simeq \sigma_{12}^* \Delta_! \mathcal{M}$. Note that the Berezinian bundle is of rank (0|1) if $N$ is odd, hence in the above formula we need to multiply by $(-1)^{k N}$ in this case.

Similarly, if $X$ is a superconformal curve and $\mathcal{M}$ carries a (left) action of the sheaf of superconformal differential operators $\mathcal{D}_X$ (cf. 3.2.10), the above formula defines isomorphisms as in the general case.

4.1.12. The Berezinian bundle $\omega_X$ is a right $\mathcal{D}_X$-module, the action given by the Lie derivative [DM99]. Therefore for any left $\mathcal{D}_X$-module $\mathcal{F}$ we obtain a right $\mathcal{D}_X$-module $\mathcal{F}^\tau := \omega \otimes \mathcal{F}$. This operation establishes an equivalence of categories between left and right $\mathcal{D}_X$-modules [Pen83]. The same results hold for $\mathcal{D}_X$-modules over superconformal curves in the sense of 3.2.10.
Let $X$ be a supercurve, the sheaf $\omega_X \boxtimes \omega_X$ on $X^2$ is isomorphic to $\omega_{X^2}$. The natural map is expressed in local coordinates as:

\[(4.1.12.1)\]

$$dz \boxtimes dW \mapsto [dZdW],$$

where as before $dZ$ denotes the section $[dzd\theta \ldots d\theta^N]$ of $\omega_X$ and $[dZdW]$ denotes the section $[dzdwd\theta^1d\zeta^1 \ldots d\theta^N d\zeta^N]$ of $\omega_{X^2}$. We note the skew-symmetry in $(4.1.12.1)$ since (recall the definition of the Berezinian in 2.2.11)

\[(4.1.12.2)\]

$$dZ \boxtimes dW \mapsto -(-1)^N [dWdZ].$$

We obtain thus $\Delta \omega_X \simeq \omega_{X^2}((\infty) / \omega_{X^2}$. Let $\mu_\omega$ denote the composition of the identification $\omega \boxtimes \omega((\infty) \simeq \omega_{X^2}((\infty)$ with the projection onto $\Delta \omega_X$. This map is a morphism of right $\mathcal{D}_{X^2}$-modules satisfying the skew-symmetry condition:

\[(4.1.12.3)\]

$$\mu_\omega \circ \sigma_{12} = -\mu_\omega.$$

Note that this formula differs from $(4.1.12.2)$ by a factor $(-1)^N$. Indeed this factor appears when applying $\sigma_{12}$, namely the composition in the LHS of $(4.1.12.3)$ is given by:

$$dZ \boxtimes dW \xrightarrow{\sigma_{12}} (-1)^N dW \boxtimes dZ \xrightarrow{\Delta} (-1)^N [dWdZ] = -[dZdW] = -\mu_\omega dZ \boxtimes dW.$$

**Remark 4.1.13.** Let $X$ be a supercurve and $Z \mapsto X$ a closed embedding. We define the functor $\Gamma_Z$ from the category of sheaves on $X$ to itself by letting sections of $\Gamma_Z(\mathcal{F})$ be sections of $\mathcal{F}$ supported on $Z$. This functor is left exact. Let $\mathcal{H}_Z$ be the higher derived functors. In this sense the basic definitions of local cohomologies in [Har66] extend in a straightforward way to the super case. Similarly we can define the relative local cohomologies as the higher derived functors of $\Gamma_Z/\Gamma_Z/\Gamma$ where $Z' \mapsto Z$ is another closed embedding and $\Gamma_Z/\Gamma/\Gamma$ is defined in the usual way as the quotient of sections supported in $Z$ modulo those supported in $Z'$ [Har66]. From the exact sequence

$$0 \rightarrow \Gamma_Z(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow j_*(\mathcal{F}|_U) \rightarrow \mathcal{H}_Z^1(\mathcal{F}) \rightarrow 0,$$

where $U = X \setminus Z$ and $j : U \mapsto X$ is the open immersion, we obtain:

$$\Delta \omega_X = \mathcal{H}_Z^1(\omega_{X^2}).$$

This identification of sheaves extended by delta functions on the diagonals with local cohomology sheaves shows that indeed these are push-forwards of $\mathcal{D}_{X^2}$-modules.

**4.1.14.** We have also a dictionary between $\mathcal{D}_X$-modules and delta functions. The space $\mathbb{C}[\mathcal{Z}[Z^\pm, W^\pm]]$ carries a structure of a module over the algebra of differential operators $\mathbb{C}[\mathcal{Z}[\nabla, \mathcal{N}]]$ (here $\nabla_Z = (\partial_z, \partial_{\theta^N})$ in the general case and $\nabla_Z = (\partial_z, D_Z^J)$ in the superconformal case). The formal delta-function $\delta(Z, W)$ satisfies the relations:

$$(Z - W)^{1/0} \delta(Z, W) = 0, \quad (Z - W)^{0/1} \delta(Z, W) = 0, \quad (\nabla_Z + \nabla_W) \cdot \delta(Z, W) = 0.$$  

Therefore the $\mathbb{C}[\mathcal{Z}[\nabla, \mathcal{N}]]$-module of $\mathbb{C}[\mathcal{Z}[Z^\pm, W^\pm]]$ generated by $\delta(Z, W)$ is spanned by $\nabla_Z^j \delta(Z, W)$ with $j \geq 0$. This module gives rise to a $\mathcal{D}$-module on the disk $D^2 = \text{Spec} \mathbb{C}[\mathcal{Z}[W]]$ supported on $z = w$ (note that this is also the case in the superconformal case, where the poles are in $z - w - \sum \theta^N \zeta^J)$. The assignment

$$(Z - W)^{-1-j/0} \delta(Z, W) \mapsto \sigma(j) \delta^i_{W^N} \delta(Z, W),$$
induces an isomorphism of left $\mathcal{D}$-modules on $D^2$ between $\Delta_+ \omega$ and the left $\mathcal{D}$-module generated by $\delta(Z,W)$. Similarly, tensoring with $\omega$ we obtain an isomorphism of right $\mathcal{D}$-modules. In the superconformal case the situation is analogous. From where we obtain:

$D_{phism}$ of right $\mathcal{D}$-

4.1.15. Recall that from Theorem 3.2.9 and (3.2.10.1), we have a natural (left) action of differential operators on $\mathcal{V}$. It follows then that the push-forward $\Delta_+ \mathcal{V}$ is also a (left) $\mathcal{D}$-module. Indeed, the action of vector fields locally is given by $(a \in V)$:

$$\partial_z : f(Z,W)a \to (\partial_z f(Z,W))a,$$

$$\partial_w : f(Z,W)a \to (\partial_w f(Z,W))a + f(Z,W)(Ta)$$

and similarly in the superconformal case, using $D^2_Z$ resp. $D^2_W$ instead of $\partial_{\bar{z}}$ resp. $\partial_{\bar{w}}$. Also, we obtain a $\mathcal{D}$-module structure on the sheaves $\mathcal{V} \boxtimes \mathcal{V}(\infty \Delta)$ where $\partial_{\bar{z}}$ acts as $\partial_{\bar{z}} + S^1$ and $\partial_{\bar{w}}$ acts as $\partial_{\bar{w}} + S^1$. Similarly, in the superconformal case, $D^2_Z$ acts as $D^2_Z + S^1$ and $D^2_W$ acts as $D^2_W + S^1$.

Proposition 4.1.16. The map $\mathcal{B}_{2,x}$ commutes with the action of differential operators on $D^2_x$, making this map a morphism of $\mathcal{D}$-modules.

Proof. For a general supercurve $X$ the proof is the same as in the non-super case. We sketch the proof in the superconformal case where a subtlety arises. Let $X = (x, \eta^1, \ldots, \eta^N)$. The identity

$$Y(S^i a, Z - W)b = D^i_X Y(a, X)b|_{X = Z - W} = D^i_Z Y(a, Z - W)b$$

translates into:

$$\mathcal{B}_{2,x}(D^i_Z \cdot f(Z,W)a \boxtimes b) = D^i_Z \cdot \mathcal{B}_{2,x}(f(Z,W)a \boxtimes b).$$

On the other hand, consider translation invariance:

$$[S^i, Y(a, Z - W)]b = (\partial_{\bar{z}} - \eta^i \partial_z)Y(a, X)b|_{X = Z - W} =$$

$$= (-\partial_{\bar{z}} + \eta^i \partial_z - \eta^i \partial_z Y(a, Z - W)b|_{X = Z - W}$$

$$= (-\partial_{\bar{z}} - \eta^i \partial_z)Y(a, Z - W)b = -D^i_W Y(a, Z - W)b.$$

From where we obtain:

$$Y(a, Z - W)S^i b = (-1)^a S^i Y(a, Z - W)b + (-1)^a D^i_W Y(a, Z - W)b,$$

and this translates into:

$$\mathcal{B}_{2,x}(D^i_W \cdot f(Z,W)a \boxtimes b) = D^i_W \cdot \mathcal{B}_{2,x}(f(Z,W)a \boxtimes b).$$

Remark 4.1.17. Since $\Delta_+ \mathcal{V}$ is supported on the diagonal, we obtain a global version $\mathcal{V}^2$ of $\mathcal{B}_{2,x}$ by gluing these morphisms in the diagonal with the zero morphism outside of the diagonal. By the previous proposition, this morphism is a map of $\mathcal{D}$-modules on $X^2$.

Proposition 4.1.18. The map $\mathcal{V}^2 : \mathcal{V} \boxtimes \mathcal{V}(\infty \Delta) \to \Delta_+ \mathcal{V}$ satisfies $\mathcal{V}^2 = \sigma_{12} \circ \mathcal{V}^2$ under the canonical identification $\Delta_+ \simeq \sigma_{12} \Delta_+ \mathcal{V}$. 

Proof. From the skew-symmetry property of SUSY vertex algebras (2.1.21.1) it follows:

\[ Y(a, Z - W)b = (-1)^{ab}e^{(Z - W)\nabla}Y(b, W - Z)a. \]

The sign cancels when applying \( \sigma_{12} \) and the exponential \( e^{(Z - W)\nabla} \) is the coordinate expression for the parallel translation, using the \( \mathcal{D} \)-module structure on \( \mathcal{V} \), from \( W \) to \( Z \) (see 4.1.11).

4.1.19. In order to define chiral algebras over supercurves, we need to understand the composition of morphisms like \( \mathcal{V}^2 \). For this we need to understand \( \Delta_{123!}\mathcal{A} \) for any right \( \mathcal{D} \)-module \( \mathcal{A} \) over \( X \), where \( \Delta_{123} \) is the small diagonal in \( X^3 \) where the three points collide. As in the non-super case, we can write this as a composition (4.1.19.1)

\[ \Delta_{123!}\mathcal{A} \simeq \Delta_{23!}\Delta_{1!}\mathcal{A}. \]

This identity follows from the fact that the push-forward of right \( \mathcal{D} \)-modules is exact for closed embeddings (cf. [Ber]).

Now let \( \mu : \mathcal{A} \boxtimes \mathcal{A}(\infty \Delta) \to \Delta_1!\mathcal{A} \) be a morphism of \( \mathcal{D} \)-modules on \( X^2 \). We define a composition of \( \mu \):

\[ \mu_{1\{23\}} : j_* \mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}|_U \to \Delta_{123!}\mathcal{A}, \]

where \( U = X^3 \setminus \Delta_{1,j} \) and \( j : U \to X^3 \) is the open immersion. In order to define such a composition we first apply \( \mu \) to the second and third argument, and then we apply \( \mu \) to the first argument and the result (cf. [FBZ01, 18.3.1]). We define other compositions of \( \mu \) by changing the order in which we group the points. As in [FBZ01] we denote these compositions in the following way: given local sections \( a, b \) and \( c \) of \( \mathcal{A} \) and a meromorphic function \( f(X,Y,Z) \) with poles along the diagonals, we have:

\[
\begin{align*}
\mu_{1\{23\}}(f(X,Y,Z) a \boxtimes b \boxtimes c) &= \mu(f(X,Y,Z)a \boxtimes \mu(b \boxtimes c)) \\
\mu_{\{12\}3}(f(X,Y,Z) a \boxtimes b \boxtimes c) &= \mu(\mu(f(X,Y,Z)a \boxtimes b) \boxtimes c) \\
\mu_{2\{13\}}(f(X,Y,Z) a \boxtimes b \boxtimes c) &= \sigma_{12} \circ \mu(f(X,Y,Z)b \boxtimes \mu(a \boxtimes c)).
\end{align*}
\]

With these compositions defined, we can now define a chiral algebra in the usual way:

**Definition 4.1.20.** A chiral algebra on a 1|\( N \) dimensional supercurve \( X \) is a right \( \mathcal{D} \)-module \( \mathcal{A} \) equipped with a morphism of \( \mathcal{D} \)-modules: \( \mu : \mathcal{A} \boxtimes \mathcal{A}(\infty \Delta) \to \Delta_1!\mathcal{A} \), satisfying the following conditions:

- (skew-symmetry) \( \mu = -\mu \circ \sigma_{12} \).
- (Jacobi identity) \( \mu_{1\{23\}} = \mu_{\{12\}3} + \mu_{2\{13\}} \).
- (Unit) We are given a canonical embedding \( \omega_X : \mathcal{A} \) of the Berezinian bundle compatible with the homomorphism \( \mu_\omega \) defined in 4.1.12.

**Remark 4.1.21.** Note that this definition is exactly the same as in the non-super case, namely, the signs appearing when anticommuting odd-elements are taken care by the symmetric structure of the category of modules over super-rings. That is, given a super-ring \( R \) and two \( R \)-modules \( M \) and \( N \), the isomorphism \( \sigma : M \otimes N \simeq N \otimes M \) is given by:

\[ \sigma : m \otimes n \mapsto (-1)^{mn}n \otimes m. \]

Indeed the only difference with the non-super case is the fact that the unit \( \omega \) is a rank (0|1)-bundle when \( N \) is odd. From the SUSY vertex algebra point of view, this is translated into the fact that the A-bracket has parity \( N \mod 2 \).
In the superconformal case there is a subtlety. We note that the intersection of two different diagonals in the sense of 4.1.2 depends on the diagonals chosen, namely:

\[ \Delta^*_{12} \cap \Delta^*_{23} \neq \Delta^*_{13} \cap \Delta^*_{23}. \]

But despite this fact, the pushforward \( \Delta_{123}! \) is still well defined, independent of the composition chosen as in (4.1.19.1).

Using the equivalence between left \( D \)-modules and right \( D \)-modules, we obtain a right \( D \)-module \( \mathcal{V}^r = \omega_X \otimes \mathcal{V} \) from any strongly conformal SUSY vertex algebra. Similarly, this sheaf carries a multiplication \( \mu = (\mathcal{Y}^2)^r \) obtained from \( \mathcal{Y}^2 \).

**Theorem 4.1.22.** The pair \( (\mathcal{V}^r, \mu) \) carries a structure of a chiral algebra over \( X \).

**Proof.** The proof of this fact is the same as the proof in the non-super case [FBZ01, Thm 18.3.3]. This follows by considering the Cousin resolution of the Berezinian bundle in \( X^3 \) and the corresponding Cousin property of SUSY vertex algebras 2.1.22 proved in [HK06]. \( \square \)

4.2. **Conformal blocks.** In this section we define the sheaves of coinvariants of SUSY vertex algebras. The treatment follows [FBZ01]. In fact, most results carry over without change to our situation. We only mention the major differences.

4.2.1. Recall that the polar part of a SUSY vertex algebra is naturally a SUSY Lie conformal algebra (cf. [HK06]). We can consider then the operator \( \mathcal{Y}_x \), which is the polar part of \( \mathcal{Y}^2 \). The notion of Lie\(^*\) algebra over a super curve is generalized in a straightforward manner from the non-super case.

4.2.2. Let \( \mathcal{A} \) be a right \( D \)-module, the de Rham sequence of \( \mathcal{A} \) is the sequence:

\[ 0 \to \mathcal{A} \otimes \mathcal{T} \to \mathcal{A} \to 0 \]

placed in cohomological degrees 0 and \(-1\), where \( \mathcal{T} \) is the tangent sheaf of \( X \). In the superconformal case, we do not have an action of the entire tangent sheaf, but we can act by the subsheaf \( \mathcal{T}_{s} \) generated by the derivations \( D_i \) (i.e. the subsheaf \( \mathcal{T}_1 \) of remark 2.2.13 in the 1|1 dimensional case, and the sheaf \( \mathcal{T}_{1}' \oplus \mathcal{T}_{1}'' \) in the 1|2 dimensional case). We define then the de Rham sheaf \( h(\mathcal{A}) \) of \( \mathcal{A} \) as

\[ h(\mathcal{A}) = \mathcal{A} / (\mathcal{A} \cdot \mathcal{T}_{s}) \]

whereas in the superconformal case we put \( h(\mathcal{A}) = \mathcal{A} / (\mathcal{A} \cdot \mathcal{T}^s) \).

**Proposition 4.2.3.** Let \( (\mathcal{A}, \mu) \) be a chiral algebra. Then

1. \( h(\mathcal{A})(D^*_x) \) and \( h(\mathcal{A})(\Sigma) \), for any open \( x \notin \Sigma \subset X \) are Lie superalgebras, and there is a natural homomorphism of Lie superalgebras \( h(\mathcal{A})(\Sigma) \to h(\mathcal{A})(D^*_x) \).
2. \( h(\mathcal{A})(D^*_x) \) acts on the fiber \( \mathcal{A}_x \).
3. If \( (\mathcal{A}, \mu) \) is associated to a SUSY vertex algebra \( V \), then there is a canonical isomorphism \( h(\mathcal{A})(D^*_x) \simeq \text{Lie}^e(V) \) (see Theorem 2.1.24 for the definition of \( \text{Lie}^e(V) \)).

**Proof.** We can think of \( \mathcal{A} \simeq \omega \otimes \mathcal{A}^l \), where \( \mathcal{A}^l \) is a left \( D \)-module. Since we can integrate sections of the Berezinian bundle, we see immediately that we have \( h(\Delta;\mathcal{A}) = \Delta_* h(\mathcal{A}) \). On the other hand the map \( \mu : \mathcal{A} \boxtimes \mathcal{A}(\infty \Delta) \to \Delta_1 \mathcal{A} \) induces

\[ h(\mu) : h(\mathcal{A}) \boxtimes h(\mathcal{A})(\infty \Delta) \to h(\Delta_1 \mathcal{A}) \]

where \( \Delta_1 \mathcal{A} \) is the one-dimensional subbundle of \( \Delta \mathcal{A} \) spanned by \( \Delta \mathcal{A} \). Therefore, we can define a linear map \( h(\mu) : h(\mathcal{A})(\infty \Delta) \to h(\Delta_1 \mathcal{A}) \) as:

\[ h(\mu)(\mathcal{A}(\infty \Delta)) = \Delta_* h(\mathcal{A})(\infty \Delta) \]
Restricting to regular sections and pulling back along the diagonal we obtain:

\[ [ , ] : h(\mathcal{A}) \otimes h(\mathcal{A}) \to h(\mathcal{A}). \]

The fact that \([ , ]\) satisfies the axioms of a Lie superalgebra follows from the skewsymmetry and Jacobi identity of chiral algebras. The rest of the theorem is proved in the same way as [FBZ01, prop 18.4.12].

(3) follows from the definitions, in formulas (2.1.23.1). Indeed, these formulas are the equivalent of the corresponding formulas for the action of vector fields on \(\mathcal{A}^I\) as defined in Theorem 3.2.9 and in (3.2.10.1).

\[\square\]

Remark 4.2.4. As in the non-super case, for a strongly conformal SUSY vertex algebra \(V\), we have a natural map

\[ \mathcal{B}^\Lambda_x : \mathcal{V}^r(D_x^N) \to \text{End}(\mathcal{V}_x) \simeq \mathcal{V}^r_x, \]

on \(D_x^N\). Namely, given a section \(s \in \mathcal{V}^r(D_x^N)\) we obtain the endomorphism \(\mathcal{B}^\Lambda_x(s) = \text{res}_x < \mathcal{B}_x, s > \) on \(\mathcal{V}_x\). If \(s\) is a total derivative, this residue vanishes and the map \(\mathcal{B}^\Lambda_x\) factors through \(h(\mathcal{V}^r)(D_x^N)\). The resulting Lie superalgebra homomorphism \(h(\mathcal{V}^r)(D_x^N) \to \text{End}(\mathcal{V}_x^r)\) coincides with the homomorphism of Proposition 4.2.3 (2) and with the homomorphism \(\varphi^'\) of Theorem 2.1.24.

4.2.5. We can now define the spaces of coinvariants for a SUSY vertex algebra. For this let \(X\) be a supercurve and \(x \in X\) a point. We have a Lie superalgebra \(U_\Sigma = h(\mathcal{V}^r)(\Sigma)\), where \(\Sigma = X \setminus \{x\}\) and this Lie superalgebra acts in \(\mathcal{V}_x\).

Definition 4.2.6. The space of coinvariants associated to \((V, X, x)\) is

\[ H(V, X, x) = \mathcal{V}_x/(U_\Sigma \cdot \mathcal{V}_x). \]

Remark 4.2.7. The extension of this definition to the multiple point case with arbitrary module insertions is straightforward and we leave it for the reader.

Fix \(N \geq 0\). Let \(g\) be the Lie superalgebra of vector fields on the \(1|N\) dimensional punctured superdisk \(D^N\), namely \(g\) is the completion of the Lie superalgebra \(\mathcal{W}(1|N)\). Let \(g^\omega\) be the Lie subalgebra of \(g\) consisting of vector fields preserving the form \(\omega = dt + \sum \zeta^i d\zeta^i\), namely \(g^\omega\) is the completion of the Lie superalgebra \(K(1|N)\). Let \(\mathcal{M}_{g,1}\) be the moduli space of smooth \(1|N\) dimensional genus \(g\), pointed supercurves (here the genus of a supercurve \(X\) is the genus of \(X_{\text{un}}\)). Let \(\mathcal{M}_{g,1}\) be the moduli space of triples \((X, x, Z)\), where \((X, x) \in \mathcal{M}_{g,1}\) and \(Z\) is a coordinate system at \(x\). Let \(\mathcal{M}^{\omega}_{g,1}\) and \(\mathcal{M}^{\omega}_{g,1}\) be the superconformal analogous.

Theorem 4.2.8 ([Vai95]). The Lie algebra \(g\) (resp. \(g^\omega\)) acts (infinitesimally) transitively on \(\mathcal{M}_{g,1}\) (resp. \(\mathcal{M}^{\omega}_{g,1}\). This action preserves the fibers of the projection \(\mathcal{M}_{g,1} \to \mathcal{M}^{\omega}_{g,1}\) (resp. \(\mathcal{M}^{\omega}_{g,1} \to \mathcal{M}^{\omega}_{g,1}\)).

It follows from this theorem, by repeating the localization construction in [FBZ01, ch. 16] that, given a strongly conformal \(N_W = n\) SUSY vertex algebra (resp. a strongly conformal \(N_K = n\) SUSY vertex algebra) \(V\), we obtain a left \(\mathcal{D}\)-module \(\Delta(V)\) on \(\mathcal{M}_{g,1}\) (resp. \(\mathcal{M}^{\omega}_{g,1}\)), whose fiber at \((X, x)\) is the space of coinvariants \(H(X, x, V)\).
Appendix A. Representations of \( g\ell(1|1) \)

Let us pick a basis of \( g\ell(1|1) \) such that

\[
T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Then the irreducible representations such that \( T \) and \( J \) act diagonally are classified by

- 1|0 or 0|1 dimensional: these are representations on \( \mathbb{C}^{1|0} \) or \( \mathbb{C}^{0|1} \) generated by an even (resp. odd) vector \( \ell \in \mathbb{C} \) such that in this basis we have \( T = Q = H = 0 \) and \( J = j \) we call these representations \( \pi_{\pm}(j) \).

- 1|1 dimensional: for each numbers \( t, j \in \mathbb{C} \) there are two irreducible representations of dimension 1|1. These are either of highest weight \( \pi_{\pm}(t, j) \), or lowest weight \( \pi_{-}(t, j) \):

\[
T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \quad J = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \quad J = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.
\]

We note that by taking minus the super transpose we get that the duals of these representations are given (in the dual basis \( \{v^*, \omega^*\} \)) by

\[
T = \begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix}, \quad J = \begin{pmatrix} -j & 0 \\ 0 & -j \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

\[
T = \begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix}, \quad J = \begin{pmatrix} -j & 0 \\ 0 & -j \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix},
\]

which in the basis \( \{-t^{-1}v, \omega\} \) show that \( \pi_{\pm}(t, j)^{\vee} = \pi_{\mp}(-t, -j) \).

Finally we note that the parity changed modules are \( \Pi \pi_{\pm}(t, j) = \pi_{\mp}(t, j \mp 1) \).

On the formal 1|1 dimensional superdisk with coordinates \( (z, \theta) \) we have the following realization of these representations. Consider the basis for this Lie algebra

\(-T = z\partial_{z} + \theta\partial_{\theta}, J = -\theta\partial_{\theta}, Q = -z\partial_{z} \) and \( H = \theta\partial_{z} \) acting on sections of a vector bundle by the Lie derivative. By analyzing the action of these derivations on the fibers of the corresponding bundles we obtain:

\[
\Lambda^m \Omega^1 = \text{Aut}_D^{\text{Aut}^{\theta}} \times \pi_{\pm}(-m, -m + 1) \quad m \equiv 1(2),
\]

\[
\Lambda^m \Omega^1 = \text{Aut}_D^{\text{Aut}^{\theta}} \times \pi_{-}(-m, -m) \quad m \equiv 0(2),
\]

\[
S^m \Omega^1 = \text{Aut}_D^{\text{Aut}^{\theta}} \times \pi_{\pm}(-m, 0)
\]

\[
\text{Ber}_D = \text{Aut}_D^{\text{Aut}^{\theta}} \times \pi_{-}(1).
\]

References

[BB93] A. Beilinson and J. Bernstein. A proof of Jantzen conjectures. *Advances in Soviet Mathematics*, 16:1–50, 1993.

[BD04] A. Beilinson and V. Drinfeld. *Chiral Algebras*. AMS Colloquium Publications v 51. Providence, RI, 2004.

[BDFM88] T. Banks, L. Dixon, D. Friedan, and E. Martinec. Phenomenology and conformal field theory or can string theory predict the weak mixing angle? *Nucl. Phys. B.*, 299(3):613–626, 1988.

[Ber] J. Bernstein. Algebraic theory of D-modules. available online.
R. Borchers, Vertex algebras, Kac-Moody algebras and the Monster. *Proc. Nat. Acad. Sci. USA*, 83(10):3068–3071, 1986.

L. A. Borisov. Vertex algebras and mirror symmetry. *Comm. in Math. Phys.*, 215(3):517–557, 2001.

M. J. Bergvelt and J. M. Rabin. Supercurves, their Jacobians, and super KP equations. *Duke Math. Journal*, 98(1), 1999.

J. D. Cohn. \( N = 2 \) super-Riemann surfaces. *Nuclear Physics*, B284:349–364, 1987.

P. Deligne and J. W. Morgan. Notes on supersymmetry. In *Quantum fields and strings: A course for mathematicians v 1.*, AMS, 1999.

S. N. Dolgikh, A. A. Rosly, and A. S. Schwarz. Supermoduli spaces. *Communications in Mathematical Physics*, 135(1):91–100, 1990.

A. De Sole and V. G. Kac. Finite vs affine \( W \)-algebras. *Japanese J. Math. (to appear) math-ph/0511055*, 2005.

E. Frenkel and D. Ben-Zvi. *Vertex algebras and algebraic curves*. Mathematical surveys and monographs v 88. AMS, Providence, RI, 2001.

I. B. Frenkel, Y. Huang, and J. Lepowsky. On axiomatic approaches to vertex operators algebras and modules. *Mem. Amer. Math. Soc.*, v 104(494), 1993.

I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex operator algebras and the Monster*. Pure and applied Mathematics v 134. Academic Press Inc., 1988.

D. Friedan. Notes on string theory and two dimensional conformal field theory. *Proc. Workshop on Unified String Theories*, 1986.

R. Hartshorne. *Residues and duality*. Number 20 in Lecture notes in mathematics. Springer-Verlag, 1966.

R. Heluani and V.G. Kac. Supersymmetric vertex algebras. *preprint. math.QA/0603633*, 2006.

Y.Z. Huang. *Two dimensional conformal geometry and vertex operator algebras*. Progress in Mathematics v 148. Birkhäuser Boston Inc., Boston, MA, 1997.

V. G. Kac. *Vertex algebras for beginners*. University Lecture series v 10. AMS, 1996. Second edition 1998.

M. Kapranov and E. Vasserot. Vertex algebras and the formal loop space. *Publ. Math. Inst. Hautes. Études Sci.*, (100):209–269, 2004.

V. G. Kac and J. van de Leur. On classification of superconformal algebras. In *Strings-88*, pp. 77–106, 1989.

Yu. I. Manin. *Topics in noncommutative geometry*. Princeton University Press, 1991.

Yu. I. Manin. *Gauge field theory and complex geometry*. Springer, 1997.

I. N. MacArthur. Line integrals on super Riemann surfaces. *Phys Lett.*, B206:221–226, 1988.

A. Malikov, V. Schechtman, and A. Vaintrob. Chiral de Rham complex. *Comm. Math. Phys.*, 204(2):439–473, 1999.

I. B. Penkov. \( D \)-modules on super manifolds. *Inventiones mathematicae*, 71(3):501–512, 1983.

A. Rogers. Contour integration on super Riemann surfaces. *Phys. Lett.*, B213(1):37–40, 1988.

A. Yu. Vaintrob. Deformation of complex superspaces and coherent sheaves on them. *Journal of Soviet Math.*, 51(1), 1990.

A. Yu Vaintrob. Conformal Lie superalgebras and moduli spaces. *Journal of Geometry and Physics*, 15(2), 1995.

Department of Mathematics, MIT, Cambridge, MA 02139, USA

E-mail address: heluani@math.mit.edu