Study of Loschmidt echo for two-dimensional Kitaev model

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Abstract. In this paper, we study the Loschmidt Echo (LE) of a two-dimensional Kitaev model residing on a honeycomb lattice which is chosen to be an environment that is coupled globally to a central spin. The decay of the LE is highly influenced by the quantum criticality of the environmental spin model, e.g., it shows a sharp dip close to the anisotropic quantum critical point (AQCP) of its phase diagram. The early time decay and the collapse and revival as a function of time at AQCP also exhibit interesting scaling behavior with the system size, which is verified numerically. It has also been observed that the LE stays vanishingly small throughout the gapless phase of the model. The above study has also been extended to the 1D Kitaev model, i.e. when one of the interaction terms vanishes.

Keywords: spin chains, ladders and planes (theory), quantum phase transitions (theory)
1. Introduction

A quantum phase transition is a zero temperature transition of a quantum many body system driven by a non-commuting term of the quantum Hamiltonian which is associated with a diverging length as well as a diverging time scale [1, 2]. In recent years, a plethora of studies have been carried out which attempt to bridge a connection between quantum phase transition and quantum information theory [3, 4]. For example, information theoretic measures such as entanglement, quantum fidelity [5–8], decoherence [9–12] and quantum discord [13, 14] are being studied close to the quantum critical point (QCP). These measures not only capture the singularities associated with the QCP but also show distinct scaling relations which characterizes it. There have also been numerous studies on decoherence (or loss of phase information) in a quantum critical system which is closely connected to the LE to be discussed in this work; understanding decoherence is essential for successful achievement of the quantum computation.

To study the LE in a quantum critical environment, we make resort to the central spin model [15] in which a central spin $S$ is coupled globally to an environmental spin model $E$ (which in this case is the two-dimensional Kitaev model). The LE (with $E$ in some ground state $|\psi_0\rangle$) is given by

$$L(t) = |\langle \psi_0 | e^{iH_0 t} e^{-i(H_0 + \delta H) t} | \psi_0 \rangle|^2.$$  

Here $H_0$ and $H_0 + \delta H(t)$ are the two Hamiltonians with which the ground state $|\psi_0\rangle$ evolves, where the term $\delta H(t)$ arises due to the coupling of model $E$ with $S$. It has been established that the LE shows a decay near the critical point of $E$ with a decay rate that marks the universality associated with the QCP of $E$ [15–18]. Also the LE shows collapse and revival as a function of time when $E$ is at the QCP.

The proposed work is organized in the following way: section 1 presents the model Hamiltonian, the phase diagram and a discussion about the AQCP. In section 2, we

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describe the general calculation of the LE and in the subsequent subsections we study the scaling of the short time decay close to the AQCP and its collapse and revival with time.

2. Model, phase diagram and anisotropic quantum critical point (AQCP)

The Hamiltonian of the Kitaev model on a honeycomb lattice is given by [19]

\[
H = \sum_{j+l=\text{even}} \left( J_1 \sigma_{j,l}^x \sigma_{j+1,l}^x + J_2 \sigma_{j-1,l}^y \sigma_{j,l}^y + J_3 \sigma_{j,l}^z \sigma_{j,l+1}^z \right)
\]

(1)

where \(j\) and \(l\) signify the column and row indices respectively of the honeycomb lattice while \(J_1, J_2\) and \(J_3\) are coupling parameters for the three bonds (see figure 1); \(\sigma^\alpha_{j,l}\) are the Pauli spin matrices, with \(\alpha (= x, y\) and \(z)\) denoting the spin component.

We will assume the parameters \(J_1, J_2\) and \(J_3\) are all positive and confine our analysis on the plane \(J_1 + J_2 + J_3 = 4\), since only the ratio of the coupling parameters appear in the subsequent calculations. The most exciting property of this model is that even in two dimensions it can be exactly solved using Jordan–Wigner (JW) transformation [19]–[26] in terms of Majorana fermions given by

\[
\begin{align*}
    a_{j,l} &= \left( \prod_{i=-\infty}^{j-1} \sigma_{i,l}^z \right) \sigma_{j,l}^y \quad \text{for even } j + l, \\
    a'_{j,l} &= \left( \prod_{i=-\infty}^{j-1} \sigma_{i,l}^z \right) \sigma_{j,l}^x \quad \text{for even } j + l, \\
    b_{j,l} &= \left( \prod_{i=-\infty}^{j-1} \sigma_{i,l}^z \right) \sigma_{j,l}^x \quad \text{for odd } j + l, \\
    b'_{j,l} &= \left( \prod_{i=-\infty}^{j-1} \sigma_{i,l}^z \right) \sigma_{j,l}^y \quad \text{for odd } j + l.
\end{align*}
\]

(2)
Here, $a_{j,l}$, $a'_{j,l}$, $b_{j,l}$ and $b'_{j,l}$ are all Majorana fermion operators, they obey the relations $a_{j,l}^\dagger = a_{j,l}$, $b_{j,l}^\dagger = b_{j,l}$, $\{a_{j,l}, a_{m,n}\} = \{b_{j,l}, b_{m,n}\} = 2\delta_{j,m}\delta_{l,n}$ and $\{a_{j,l}, b_{m,l}\} = 0$. One can now change the lattice site indices $(j, l)$ of the honeycomb lattice to a two-dimensional vector $\vec{n}$, where $\vec{n} = \sqrt{3}\vec{m}_1 + (\frac{\sqrt{3}}{2} + \frac{3}{2})\vec{m}_2$, which labels the midpoints of the vertical bonds of the honeycomb lattice. Here $n_1$ and $n_2$ take all integer values so that the vectors $\vec{n}$ form a triangular lattice. The Majorana fermions $a_\vec{n}$ and $b_\vec{n}$ are placed at the top and bottom sites respectively of the bond labeled by $\vec{n}$. The whole lattice is spanned by the vectors $\vec{M}_1 = \frac{\sqrt{3}}{2}\vec{i} - \frac{3}{2}\vec{j}$ and $\vec{M}_2 = \frac{\sqrt{3}}{2}\vec{i} + \frac{3}{2}\vec{j}$, see figure 1.

Under the transformation to Majorana fermions as defined in equation (2), Hamiltonian (1) takes the form

$$H = i\sum_{\vec{n}} (J_1 b_{\vec{n}} a_{\vec{n} - \vec{M}_1} + J_2 b_{\vec{n}} a_{\vec{n} + \vec{M}_2} + J_3 D_{\vec{n}} b_{\vec{n}} a_{\vec{n}}),$$

(3)

where $D_{\vec{n}} = i\vec{b}_{\vec{n}} a_{\vec{n}}'$. These $D_{\vec{n}}$ operators have eigenvalues $\pm 1$ independently for each $\vec{n}$ and commute with each other and also with $H$, which makes the Kitaev model exactly solvable. Since $D_{\vec{n}}$ is a constant of motion one can use one of the eigenvalues $\pm 1$ for each $\vec{n}$ in the Hamiltonian. The ground state of the model corresponds to $D_{\vec{n}} = 1 \forall \vec{n}$ [19]. With $D_{\vec{n}} = 1$, we can easily diagonalize the Hamiltonian (3) quadratic in Majorana fermions.

The Fourier transform of the Majorana fermions can be defined as

$$a_{\vec{n}} = \sqrt{\frac{4}{N}} \sum_{\vec{k}} (a_{\vec{k}} e^{i\vec{k} \cdot \vec{n}} + a_{\vec{k}}^\dagger e^{-i\vec{k} \cdot \vec{n}}),$$

(4)

similarly $b_{\vec{n}}$ also has same Fourier transform relation. The $a_{\vec{k}}$’s and $b_{\vec{k}}$’s are Dirac fermions which follow the fermionic anti-commutation relations. Here, $N$ is the total number of sites and $N/2$ is the number of unit cells. In the above sum given in equation (4), $\vec{k}$ is extended over half of the Brillouin zone of the hexagonal lattice due to Majorana nature of the fermions [20]. We recall that the full Brillouin zone on the reciprocal lattice represents a rhombus with vertices $(k_x, k_y) = (\pm 2\pi\sqrt{3}, 0)$ and $(0, \pm 2\pi/3)$. In the momentum space the Hamiltonian (3) takes the form $H = \sum_{\vec{k}} \psi_{\vec{k}}^\dagger H_{\vec{k}} \psi_{\vec{k}}$, where $\psi_{\vec{k}} = (a_{\vec{k}}^\dagger, b_{\vec{k}}^\dagger)$ and the reduced $2 \times 2$ Hamiltonian $H_{\vec{k}}$ can be expressed in terms of Pauli matrices as

$$H_{\vec{k}} = \alpha_{\vec{k}} \sigma^1 + \beta_{\vec{k}} \sigma^2,$$

where $\alpha_{\vec{k}} = 2[J_1 \sin(\vec{k} \cdot \vec{M}_1) - J_2 \sin(\vec{k} \cdot \vec{M}_2)]$, and $\beta_{\vec{k}} = 2[J_3 + J_1 \cos(\vec{k} \cdot \vec{M}_1) + J_2 \cos(\vec{k} \cdot \vec{M}_2)].$

(5)

The eigenenergies of the $H_{\vec{k}}$ are given by

$$E_{\vec{k}}^\pm = \pm \sqrt{\alpha_{\vec{k}}^2 + \beta_{\vec{k}}^2}.$$  

(6)

This energy spectrum corresponds to two energy bands; it is noteworthy that, for $|J_1 - J_2| \leq J_3 \leq (J_1 + J_2)$, the band gap $\Delta_{\vec{k}} = E_{\vec{k}}^+ - E_{\vec{k}}^-$ vanishes for some particular $\vec{k}$ modes, leading to the gapless phase of the Kitaev model. The phase diagram of the model is shown in an equilateral triangle satisfying the relation $J_1 + J_2 + J_3 = 4$ and $J_1, J_2, J_3 > 0$ (see figure 2); one can easily show that the whole phase is divided into three gapped phases, separated by a gapless phase (inner equilateral triangle) which is bounded by gapless critical lines $J_1 = J_2 + J_3$, $J_2 = J_3 + J_1$ and $J_3 = J_1 + J_2$. 

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Figure 2. Phase diagram of the Kitaev model, satisfying $J_1 + J_2 + J_3 = 4$. The inner equilateral triangle corresponds to the gapless phase in which the coupling parameters satisfy the relations $J_1 \leq J_2 + J_3$, $J_2 \leq J_3 + J_1$ and $J_3 \leq J_1 + J_2$. Along the three paths I, II and III $J_3$ is varied, so as to study the LE. The path I, II and III are defined by the equations $J_1 = J_2$, $J_1 = J_2 + 1$ and $J_1 + J_3 = 4$ respectively.

On the critical line $J_3 = J_1 + J_2$, the energy gap goes to zero for the four $\vec{k}$ modes given by $(k_x, k_y) = (\pm 2\pi/\sqrt{3}, 0)$ and $(\pm 2\pi/3, 0)$, which are the four corner points of the Brillouin zone. One can now expand $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ around one of the critical modes for $J_3 = J_1 + J_2$, in the form

$$
\begin{align*}
\alpha_{\vec{k}} &= \sqrt{3}(J_2 - J_1)k_x + 3(J_1 + J_2)k_y, \\
\beta_{\vec{k}} &= \frac{3}{4}(J_1 + J_2)k_x^2 + \frac{9}{4}(J_1 + J_2)k_y^2 + \frac{3\sqrt{3}}{2}(J_2 - J_1)k_xk_y,
\end{align*}
$$

where $k_x$ and $k_y$ are the deviations from the above-mentioned critical modes. We note that $\alpha_{\vec{k}}$ varies linearly and $\beta_{\vec{k}}$ varies quadratically in $k_x$ and $k_y$. The point $J_3c = 2J_1$ (where $J_1 = J_2$) denoted by A in (figure 2) needs to be checked carefully. This is an AQCP [27] with energy dispersion $E_{\vec{k}} \sim k_x^2$ along $k_x$ ($k_y = 0$) and $E_{\vec{k}} \sim k_y$ along $k_y$ ($k_x = 0$). The corresponding dynamical exponents are given by $z_\perp = 1$ and $z_\parallel = 2$, respectively.

For $J_1 \neq J_2$, $J_3c = J_1 + J_2$ is also an AQCP, as can be shown using a rotation to a new coordinate system

$$
k_1 = \sqrt{3}(J_2 - J_1)k_x + 3(J_1 + J_2)k_y, \quad k_2 = 3(J_1 + J_2)k_x - \sqrt{3}(J_2 - J_1)k_y,
$$

in which $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ take the form

$$
\alpha_{\vec{k}} = k_1, \quad \beta_{\vec{k}} = c_1k_1^2 + c_2k_2^2 + c_3k_1k_2,
$$

where $c_1 = 9(J_1 + J_2)(4J_1^2 + 4J_1J_2 + J_1J_2)$, $c_2 = 27J_1J_2(J_1 + J_2)$, and $c_3 = 18\sqrt{3}J_1J_2(J_2 - J_1)$.

Therefore, for a general AQCP ($J_1 \neq J_2$) the dispersion will vary linearly and quadratically along $\hat{k}_1$ and $\hat{k}_2$ directions, respectively, with two dynamical exponents $z_1 = 1$ and $z_2 = 2$.  

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3. Qubit coupled to the $J_3$ term of the Kitaev Hamiltonian

In this section we will provide a general calculation of the LE considering the Kitaev model on a honeycomb lattice as an environment ($E$) that is coupled to a central spin-$\frac{1}{2}$ ($S$). We shall denote the ground state and excited state of the central spin $S$ by $|g\rangle$ and $|e\rangle$ respectively. $S$ is coupled to $J_3$ term of the $E$ Hamiltonian only when the central spin is in the excited state $|e\rangle$. Therefore the composite Hamiltonian takes the form

$$H_T(J_3, \delta) = \sum_{j+l=\text{even}} (J_1\sigma^x_j\sigma^x_{j+1} + J_2\sigma^y_{j-1}\sigma^y_{j,l} + J_3\sigma^z_{j,l}\sigma^z_{j,l+1} + \delta|e\rangle\langle\sigma^z_{j,l}\sigma^z_{j,l+1}),$$

where $\delta$ is the coupling strength of $S$ to $E$. We shall work in the limit of $\delta \to 0$.

We consider that $S$ is initially in a generalized state $|\phi(0)\rangle_S = c_g|g\rangle + c_e|e\rangle$ (with the coefficients satisfying the condition $|c_g|^2 + |c_e|^2 = 1$), and $E$ is initially in the ground state $|\varphi(J_3, 0)\rangle$. The evolution of the environmental spin model splits into two branches, given by $|\varphi(J_3, t)\rangle = \exp(-iH(J_3)t)|\varphi(J_3, 0)\rangle$ and $|\varphi(J_3 + \delta, t)\rangle = \exp(-iH(J_3 + \delta)t)|\varphi(J_3, 0)\rangle$; the evolution of $|\varphi(J_3, t)\rangle$ is driven by the Hamiltonian $H(J_3) = H_T(J_3, 0)$ (when $S$ is in the ground state and hence there is no $\delta$ term present in the Hamiltonian), whereas $|\varphi(J_3 + \delta, t)\rangle$ evolves with $H(J_3 + \delta) = H_T(J_3, 0) + V_e$, where $V_e = \delta\sum_{j+l=\text{even}}\sigma^z_{j,l}\sigma^z_{j,l+1}$, is the effective potential arising due to the coupling between $S$ and $E$. The wave function of the composite system at time $t$ is given by

$$|\psi(t)\rangle = c_g|g\rangle \otimes |\varphi(J_3, t)\rangle + c_e|e\rangle \otimes |\varphi(J_3 + \delta, t)\rangle.$$

As a result the LE is given by

$$L(J_3, t) = |\langle\varphi(J_3, t)|\varphi(J_3 + \delta, t)\rangle|^2,$$

$$= |\langle\varphi(J_3, 0)| \exp(-iH(J_3 + \delta)t)|\varphi(J_3, 0)\rangle|^2. \quad (12)$$

Here, we have exploited the fact that $|\varphi(J_3, 0)\rangle$ is an eigenstate of the Hamiltonian $H(J_3)$.

Following Fourier transformation and Bogoliubov transformation the diagonalized form of the Hamiltonian (1) is given by

$$H(J_3) = \sum_{\vec{k}} [-\varepsilon_{\vec{k}}(J_3)A^\dagger_{\vec{k}}A_{\vec{k}} + \varepsilon_{\vec{k}}(J_3)B^\dagger_{\vec{k}}B_{\vec{k}}], \quad (13)$$

where the $A_{\vec{k}}$s and $B_{\vec{k}}$s are Bogoliubov fermionic operators defined as

$$A_{\vec{k}} = \frac{1}{\sqrt{2}}[a_{\vec{k}} - e^{-i\theta_{\vec{k}}b_{\vec{k}}}], \quad B_{\vec{k}} = \frac{1}{\sqrt{2}}[a_{\vec{k}} + e^{-i\theta_{\vec{k}}b_{\vec{k}}}], \quad \text{with } e^{i\theta_{\vec{k}}} = \frac{\alpha_{\vec{k}} + i\beta_{\vec{k}}}{\sqrt{\alpha^2_{\vec{k}} + \beta^2_{\vec{k}}}}. \quad (14)$$

and the energy spectrum is given by (see equations (5) and (6))

$$\varepsilon_{\vec{k}}(J_3) = \sqrt{\alpha^2_{\vec{k}} + \beta^2_{\vec{k}}} \quad \text{and} \quad \varepsilon_{\vec{k}}(J_3 + \delta) = \sqrt{\alpha^2_{\vec{k}} + \beta^2_{\vec{k}}}. \quad (15)$$

where $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ are defined in equation (5), and $\beta'_{\vec{k}}$ corresponds to the value with $J_3 + \delta$ instead of $J_3$.

The complete ground state of $H(J_3)$ can be written in the form (see [28] for details),

$$|\varphi(J_3, 0)\rangle = \prod_{\vec{k}}[|2(a_{\vec{k}}^\dagger - e^{i\theta_{\vec{k}}b_{\vec{k}}})(a_{\vec{k}}^\dagger + ib_{\vec{k}}^\dagger)|\Phi\rangle, \quad (16)$$

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where \( \vec{k} \) runs over half of the Brillouin zone of the hexagonal lattice. Following mathematical steps identical to those in [15, 18], it can be shown that equation (16) leads to the expression for the LE given by

\[
L(J_3, t) = \prod_{\vec{k}} L_{\vec{k}} = \prod_{\vec{k}} [1 - \sin^2(2\phi_{\vec{k}})\sin^2(\varepsilon_{\vec{k}}(J_3 + \delta)t)],
\]

(17)

where \( \tan \theta_{\vec{k}}(J_3 + \delta) = \alpha_{\vec{k}}/\beta_{\vec{k}}' \) and \( \phi_{\vec{k}} = [\theta_{\vec{k}}(J_3) - \theta_{\vec{k}}(J_3 + \delta)]/2 \). The expression for the LE closely resembles that of the case when the transverse Ising chain is chosen to be the

3.1. Path I: anisotropic quantum critical point \((J_1 = J_2)\)

As discussed in section 2, \( J_3 = 2J_1 \) is an AQCP with critical exponents \( \nu_\perp = z_\perp = 1 \) along the \( \hat{\mathbf{j}} \) direction and \( \nu_\parallel = 1/2, z_\parallel = 2 \) along the \( \hat{\mathbf{i}} \) direction. At this point the energy gap vanishes for the three critical modes given by \((2\pi/\sqrt{3}, 0)\) and \((0, \pm 2\pi/3)\) in half of the Brillouin zone. Now we will study the short time behavior of the LE (in equation (17)) close to the AQCP. We define a cut-off frequency \( K_c = (K_{xc}, K_{yc}) \) such that modes up to this cut-off only are considered to calculate the decay of the LE at short time close to
Figure 3. LE as a function of parameter $J_3$ ($J_3$ is varied along path ‘I’) shows a sharp dip at point A ($J_3 = 2 - \delta$) and after a small revival in the gapless phase it again decays at point B, $J_3 = 0$ (see figure 2) with $N_x = N_y = 200$, $\delta = 0.01$ and $t = 10$. Inset (a) shows the variation in the LE when the parameter $J_3$ is varied along the path II ($J_1 = J_2 + 1$) in the phase diagram for $N_x = N_y = 200$, $\delta = 0.01$ and $t = 10$, clearly showing a sharp dip at point P ($J_3 = 2 - \delta$) and a rise again at point Q ($J_3 = 1 - \delta$). Inset (b) marks the dip in the LE when $J_3$ is varied along the path III ($J_1 + J_3 = 4$); for this case $N = 400$, $\delta = 0.01$ and $t = 10$ so that $E$ realize the change in the behavior at $J_3 = 2 - \delta$. Details of these three cases are provided in sections 3.1, 3.2 and 3.3 respectively.

the AQCP. Then the LE is given by

$$L_c(J_3, t) = \prod_{k_x, k_y > 0}^{K_c} L_{k_c}.$$  \tag{19}$$

We define the quantity $S(J_3, t)$, such that $S(J_3, t) = \ln L_c \equiv -\sum_{k_x, k_y > 0}^{K_c} \ln |L_{k_c}|$. Expanding around one of the critical modes up to the cut-off, we get $\sin^2 \varepsilon_k (J_3 + \delta) t \approx 4(J_3 + \delta - 2J_1)^2 t^2$ and $\sin^2 (2\phi_k) \approx 9J_1^2 k_y^2 \delta t^2/(J_3 - 2J_1)^2 (J_3 + \delta - 2J_1)^2$, therefore we obtain

$$S(J_3, t) \approx -\frac{36\mathcal{E}(K_c) J_1^2 \delta^2 t^2}{(J_3 - 2J_1)^2},$$  \tag{20}$$

where $\mathcal{E}(K_c) = 4\pi^2 N_c (N_c + 1)(2N_c + 1)/54N_y^2$ and $N_c$ is the integer nearest to $3N_y K_c/2\pi$. We therefore find an exponential decay of the LE in the early time limit given by

$$L_c(J_3, t) \approx \exp(-\gamma t^2),$$  \tag{21}$$

where $\gamma = 36\mathcal{E}(K_c) J_1^2 \delta^2/(J_3 - 2J_1)^2$. The anisotropic nature of the quantum critical point is reflected in the fact that $\gamma$ scales as $1/N_y^2$ and is independent of $N_x$. Further, using the expression of $L_c(J_3, t)$, one can easily observe that it is invariant under the transformation $N_y \to N_y \alpha$, $\delta \to \delta/\alpha$ and $t \to t \alpha$, where $\alpha$ is some integer.

Now we fix $J_1 = J_2 = 1$ and $J_3 = 2 - \delta$ (point ‘A’ in figure 2) and observe collapse and revival of the LE with time (presented in the figure 4). The time period of collapse and revival is proportional to $N_y$, and is unaffected by the changes in $N_x$; this confirms the scaling result of the decay rate $\gamma$ for the short time limit near the AQCP discussed above.

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3.2. Path II: anisotropic quantum critical point ($J_1 \neq J_2$)

It has been shown that the point P in figure 2 ($J_1 \neq J_2, J_{3,c} = J_1 + J_2$) is an AQCP, which can be seen by choosing directions $\hat{k}_1 = \sqrt{3}(J_2 - J_1)\hat{i} + 3(J_1 + J_2)\hat{j}$ (see section 2) and $\hat{k}_2$, perpendicular to $\hat{k}_1$ [28]. The critical exponents associated with this critical point are given by $\nu_1 = z_1 = 1$, and $\nu_2 = 1/2, z_2 = 2$ along $\hat{k}_1$ and $\hat{k}_2$ directions, respectively. To calculate the early time scaling in a similar spirit as in section 3.1, we expand equation (17) near one of the critical modes up to the cut-off $K_c$ to obtain

$$\sin^2(\varepsilon_k(J_3 + \delta)t) \approx 4(J_3 + \delta - J_1 - J_2)^2t^2$$

and

$$\sin^2(2\phi_k) \approx k_1^2\delta^2/4(J_3 - J_1 - J_2)^2(J_3 + \delta - J_1 - J_2)^2.$$  

In the short time limit, the LE becomes

$$L_c(J_3, t) \approx \exp(-\gamma t^2),$$  

where $\gamma = \delta^2 \mathcal{E}(K_c)/(J_3 - J_1 - J_2)^2$, $\mathcal{E}(K_c) = 8\pi^2 J_2^2 N_c(N_c + 1)(2N_c + 1)/3N^2$ and $N_c$ is the integer nearest to $NK_c/4\pi J_2$.

In fact, comparing with the previous section 3.1, one can see that in this case $k_1$ (instead of $k_y$) appears in the expression of the LE in the short time limit. Further, from equation (22) and the expression of $\gamma$, one observes that $L_c(J_3, t)$ is invariant under the transformation $N_x = N_y = N \rightarrow N\alpha, \delta \rightarrow \delta/\alpha$ and $t \rightarrow t\alpha$, with $\alpha$ being some integer, which is also observed in the collapse and revival behavior (see figure 5).

3.3. Path III: one-dimensional quantum critical point ($J_2 = 0$)

As mentioned already, along the line $J_1 + J_3 = 4, (J_2 = 0)$, the two-dimensional spin model reduces to an equivalent one-dimensional spin chain with the energy gap vanishing at $J_1 = J_3$ for $k_c = \pi$ and the corresponding dynamical exponent being $z = 1$. We shall now expand $\sin(\varepsilon_k(J_3, \delta)t)$ and $\sin(2\phi_k)$ around the critical mode $k_c$ to analyze the short time decay of the LE, resulting in

$$\sin^2(\varepsilon_k(J_3 + \delta)t) \approx 4(J_3 + \delta - J_1)^2t^2$$

and

$$\sin^2(2\phi_k) \approx k_1^2\delta^2/4(J_3 - J_1 - J_2)^2(J_3 + \delta - J_1 - J_2)^2.$$  

In the short time limit, the LE becomes

$$L_c(J_3, t) \approx \exp(-\gamma t^2),$$  

where $\gamma = \delta^2 \mathcal{E}(K_c)/(J_3 - J_1 - J_2)^2$, $\mathcal{E}(K_c) = 8\pi^2 J_2^2 N_c(N_c + 1)(2N_c + 1)/3N^2$ and $N_c$ is the integer nearest to $NK_c/4\pi J_2$.
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Figure 5. Variation of the LE with $t$ at the AQCP P ($J_1 = 3/2, J_2 = 1/2$ and \(J_3 = 2 - \delta\)) shows collapse and revival with different $N_x = N_y = N$ and $\delta = 0.001$.

Figure 6. The LE as function of time at the QCP (point ‘R’ in figure 2) $J_1 = 2$ and $J_3 = 2 - \delta$ with different $N$ and $\delta = 0.01$, verifying the analytical scaling with $N$, $\delta$ and $t$.

\[\sin^2(2\phi_k) \approx J_1^2 k^2 \delta^2 / (J_3 - J_1)^2 (J_3 + \delta - J_1)^2.\]

The LE hence takes the from

\[L_c(J_3, t) \approx \exp(-\gamma t^2),\]

where $\gamma = 4J_1^2 \delta^2 \mathcal{E}(K_c)/(J_3 - J_1)^2$, $\mathcal{E}(K_c) = 4\pi^2 N_c(N_c + 1)(2N_c + 1)/6N^2$ and $N_c$ is the integer nearest to $NK_c/2\pi$. From equation (23), we find that the LE shows a similar scaling relation, as is expected for a one-dimensional chain with $z = 1$ [15]; this is also confirmed by studying the collapse and revival of the LE (see figure 6).

4. Conclusion

In this paper we study a variant of the central spin model in which a central spin (qubit) is globally coupled to an environment which is chosen to be a two-dimensional Kitaev model on a honeycomb lattice through the interaction term $J_3$. Using the exact solvability of the Kitaev model, we have derived an exact expression of the LE when the interaction $J_3$ is varied in a way such that the system enters the gapless phase crossing the AQCP of the phase diagram. However, the behavior of the LE as a function of $J_3$ depends on the path along which $J_3$ is varied. In the case when the AQCP, Q (with $J_1 \neq J_2$ see figure 2) is
crossed, one observes a complete revival of the echo when the system exits the gapless
phase to re-enter the gapped phase; this is in contrast to the case $J_1 = J_2$. For the case
of $J_2 = 0$ there is only one sharp dip at the critical point $J_3 = 2 - \delta$, which is associated
with the QCP of the one-dimensional Kitaev model.

The early time scaling behavior for both the paths I and II close to the AQCP bear
the signature of the fact that the gapless phase is entered crossing an AQCP with different
exponents along different spatial directions. This is also confirmed by studying the collapse
and the revival of the LE as a function of time. However, one does not observe a perfect
collapse and revival (except for the equivalent one-dimensional case); this may be because
of the proximity to a gapless phase. The quasi-period of collapse and revival in all cases
scale with the system size as $N^z$. The case with $J_2 = 0$ reflects the fact that the system
is essentially one-dimensional in this limit. It is straightforward to relate these results to
the decoherence of the central spin close to a critical point.

This study of the LE can be verified experimentally as presented by Zhang et al [16];
they measure the LE as an indicator of quantum criticality for a one-dimensional quantum
Ising model with an antiferromagnetic interaction using NMR quantum simulators. In this
experiment, they prepare the ground state of the Hamiltonian (using the gate sequences)
which need not be the true ground state but could be a state that approximates the
ground state of the system well, and then measure the LE for finite number of spins.
Similar experiments can be realized with the approximate ground state of the Kitaev
model. Also, since the Kitaev model can also be realized using an optical lattice [29,
30] (where the couplings can be separately tuned with the help of different microwave
radiations), there also exists a possibility of verifying these results in an optical lattice.

It should be noted that in a recent work, Pollmann et al [31] have studied the
problem of the LE in a transverse Ising spin chain in the presence of a longitudinal
field; more precisely they calculated the magnitude of the overlap between the final state
reached following a slow quench across the QCP and its time evolved counterpart at time
$t$ (generated following the time evolution with the final Hamiltonian). They observe a
cusp-like minimum in the echo as a function of time in the limit when the spin chain is
integrable. However, this behavior is smeared in the non-integrable case (with non-zero
longitudinal field), thus providing a probe for integrable versus non-integrable behavior.
In the present paper, however, we deal with an equilibrium situation in which the spin
chain is not quenched across the QCP, and observe the collapse and revival only at the
QCP.

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