Fair Cuts, Approximate Isolating Cuts, and Approximate Gomory-Hu Trees in Near-Linear Time

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Abstract

In this paper, we introduce a robust notion of (1+\epsilon)-approximate (s, t)-mincuts in undirected graphs where every cut edge can be simultaneously saturated (in the same direction) to a \frac{1}{1+\epsilon} fraction by an (s, t)-flow. We call these (1 + \epsilon)-fair cuts. Unlike arbitrary approximate (s, t)-mincuts, fair cuts can be uncrossed, which is a key property of (s, t)-mincuts used in many algorithms. We also give a near-linear \tilde{O}(m/\epsilon^3)-time algorithm for computing an (s, t)-fair cut. This offers a general tool for trading off a (1 + \epsilon)-approximation for near-linear running time in mincut based algorithms.

As an application of this new concept, we obtain a near-linear time algorithm for constructing a (1 + \epsilon)-approximate Gomory-Hu tree, thereby giving a nearly optimal algorithm for the (1 + \epsilon)-approximate all-pairs max-flows (APMF) problem in undirected graphs. Our result is obtained via another intermediate tool of independent interest. We obtain a near-linear time algorithm for finding (1 + \epsilon)-approximate isolating cuts in undirected graphs, a concept that has gained wide traction over the past year.

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1 Introduction

The study of minimum cuts in undirected graphs $G = (V, E)$ is one of the central themes of combinatorial optimization. An $(s, t)$-mincut is a cut of minimum value that separates a given pair of vertices $s, t$. A GLOBAL mincut is a cut of minimum value that separates the graph into two connected components. Both $s$-$t$ and GLOBAL mincut are generalized by the STEINER mincut problem, which asks for a cut of minimum value that separates any pair of vertices in a given subset $U$. ($U = \{s, t\}$ for $(s, t)$-mincut and and $U = V$ for GLOBAL mincut.) Further generalization yields the ALL-PAIRS mincuts problem that asks for $(s, t)$-mincuts separating every pair of vertices $s, t$ in the graph. ¹ These problems have been at the forefront of graph algorithms research in the last six decades.

In 1956, Ford and Fulkerson [FF56] initiated research in this area by establishing flow-cut duality to obtain an $O(mF)$-time algorithm for the $(s, t)$ max-flow/mincut problems, where $F$ is the flow value. ² A few years later in 1961, Gomory and Hu [GH61] showed the remarkable result that the ALL-PAIRS mincuts problem can be solved using $n - 1$ max-flows. For this purpose, they introduced the notion of a cut-equivalent tree, later called a Gomory-Hu tree, which is a compact way to represent the mincuts between all pairs of vertices. Since then, although algorithms progressively became faster for $(s, t)$-mincut and GLOBAL mincut, ³ there were no improvements to the STEINER and ALL-PAIRS mincuts problems in general, weighted graphs. This is in spite of much effort, e.g., better algorithms are known in special cases such as unweighted graphs [BHKP07, KL15, AKT21a], simple graphs [AKT21b, AKT22, LPS21, Zha21], planar graphs [BSW15], surface-embedded graphs [BENW16], bounded treewidth graphs [ACZ98, AKT20], etc. (See the survey [Pan16].) Recently, the running time for STEINER mincut has been improved to that of polylog($n$) max-flows [LP20], the current record for the latter being $\tilde{O}(m + n^{3/2})$ [vdBLLN20], $\tilde{O}(m^{3 - \frac{1}{1.49}})$ [GLP21] and $\tilde{O}(m^{4/3+o(1)}U^{1/3})$ [KLS20] (where $U$ is the maximum edge weight). For ALL-PAIRS mincuts, however, the state of the art is Gomory and Hu’s 60-year old algorithm using $n - 1$ max-flows.

The holy grail remains to solve the ALL-PAIRS mincuts problem in near-linear time. This would imply that all the above problems can be solved in (the nearly optimal) $\tilde{O}(m)$ time. A major obstacle, however, is to first obtain a near-linear time algorithm even for the special case of $(s, t)$-mincut (i.e., for max-flow), which remains one of the biggest open questions in graph algorithms. However, the situation is different for $(1 + \epsilon)$-approximation algorithms. A beautiful line of work in [KLOS14, She13, Pen16] has led to an $\tilde{O}(m)$-time $(1 + \epsilon)$-approximate $(s, t)$-mincut (or $(1 - \epsilon)$-approximate max-flow) algorithm. This suggests a natural goal: can we solve $(1 + \epsilon)$-approximate ALL-PAIRS mincuts in near-linear time?

In this paper, we answer this question in the affirmative. In particular, we give an $\tilde{O}(m)$-time algorithm for constructing a $(1 + \epsilon)$-approximate Gomory-Hu tree on the vertices $V$ of the input graph with the property that for any pair of vertices $s, t \in V$, the vertex partition corresponding to the $(s, t)$-mincut in the tree is a $(1 + \epsilon)$-approximate $(s, t)$-mincut in the graph.

¹ As will be clear later, by ALL-PAIRS mincuts, we refer to computing the so-called cut-equivalent tree or Gomory-Hu tree, and not computing the values of mincuts between all pairs of vertices explicitly. Such a tree allows us to solve $s$-$t$, GLOBAL, and STEINER mincuts in linear time.

² As per convention, $m$ and $n$ respectively denote the number of edges and vertices in the graph.

³ For $(s, t)$-mincut, algorithmic improvements were obtained by, e.g., [Din70, EK72, GT88, GR98, Mad13, Mad16, LS14, LS20, KLS20], while [HO94, SW97, NI92, KS96, Kar00, MN20, GMW20, GMW21] obtained improvements for GLOBAL mincuts.
Theorem 1.1. For any $\epsilon > 0$, there is a $\tilde{O}(m \cdot \text{poly}(1/\epsilon))$-time randomized algorithm that constructs, with high probability, a $(1+\epsilon)$-approximate Gomory-Hu tree in weighted, undirected graphs.

Prior to our work, Li and Panigrahi [LP21] took a step in answering the above question when they show that a $(1+\epsilon)$-approximate Gomory-Hu tree can be computed in time roughly the same as computing the exact $(s,t)$-mincut, whose current record is a $\tilde{O}(m + n^{3/2})$-time algorithm [vdBLL+21]. Their main tool – a framework called isolating cuts [LP20] – fundamentally relies on a property of exact $(s,t)$-mincuts, called uncrossing, that $(1+\epsilon)$-approximate $(s,t)$-mincuts do not satisfy. Precisely stated, the uncrossing property says the following:

Definition 1.2. Suppose we have a $(s,t)$-mincut $X \subset V$ for any two vertices $s,t$, and let $x_1, x_2$ be any two vertices in $X$. Now, if $Y$ denotes a $(x_1,x_2)$-mincut, then the uncrossing property asserts that either $X \cap Y$ or $X \setminus Y$ (which is a set entirely contained in $X$) is also an $(x_1, x_2)$-mincut.

The uncrossing property is very useful from an algorithmic (and combinatorial) perspective since it gives a natural recursive tool – after finding an $(s,t)$-mincut, we can contract and recurse on each side of the cut without affecting the $(x_1, x_2)$-mincut values for pairs of vertices $(x_1, x_2)$ on the same side $X$. Indeed, the uncrossing property is more generally true for symmetric, submoular minimization problems and is at the heart of most of the beautiful structure displayed by undirected graph cuts and other symmetric, submodular functions. The uncrossing property, however, does not hold for $(1+\epsilon)$-approximate mincuts in general. This prevents the structure of exact mincuts from extending to approximate mincuts (which are easier to find algorithmically as described above); in our case, this is the bottleneck that prevents using the isolating cuts framework with $(1+\epsilon)$-approximate mincuts. Indeed, obtaining a reduction from $(1+\epsilon)$-approximate all-pairs mincuts to $(1+\epsilon)$-approximate $(s,t)$-mincuts remains open.

We subvert this bottleneck in this paper and obtain an $\tilde{O}(m)$-time $(1+\epsilon)$-approximate all-pairs mincuts algorithm. Instead of obtaining a more robust reduction, we introduce a more robust notion of $(1+\epsilon)$-approximate mincuts called fair cuts. Fair cuts preserve the respective algorithmic and combinatorial benefits of approximate and exact $(s,t)$-mincuts in the following sense – we give an $\tilde{O}(m)$-time algorithm to find a fair cut, and although it is not an exact $(s,t)$-mincut, we show that fair cuts can be uncrossed. Consequently, they allow us to develop a $(1+\epsilon)$-approximate isolating cuts framework, which we use in the reduction method of Li and Panigrahi [LP21] to obtain a $\tilde{O}(m)$-time $(1+\epsilon)$-approximate all-pairs algorithm.

In addition to our result, we consider the mere definition of fair cuts to be a significant contribution of this paper. The uncrossing property is at the heart of the structural richness of mincuts, and of more general concepts such as minimizers of symmetric submodular functions. But, we also know that from an algorithmic perspective, the corresponding minimization problems are conceptually much easier if we allowed a $(1+\epsilon)$-approximation. Fair cuts exactly bridge this gap – they offer the algorithmic simplicity of minimizing approximately, but preserve the structural richness of the exact minimizers. We hope that the use of fair cuts will significantly improve our understanding of the approximation-efficiency tradeoff for minimum cut problems and its generalizations such as the minimization of symmetric, submodular functions.

\footnote{For sparse graphs, this can be slightly improved to $\tilde{O}(m^{2 - \frac{\epsilon}{2\alpha}})$ [GLP21], and for graphs with small edge weights, one can obtain $\tilde{O}(m^{4/3 + o(1)}U^{1/3})$ [KLS20].}

\footnote{See also Abboud, Krauthgamer, and Trabelsi [AKT21c] for a similar framework obtained independently.}
1.1 Fair Cuts

We start with the definition of a fair cut.

Definition 1.3. Let \( G = (V,E) \) be an undirected graph with edge capacities \( c \in \mathbb{R}_{>0}^E \). Let \( s,t \) be two vertices in \( V \). For any parameter \( \alpha \geq 0 \), we say that a cut \( (S,T) \) is a \((1+\alpha)\)-fair \((s,t)\)-cut if there exists a feasible \((s,t)\)-flow \( f \) such that \( f(u,v) \geq \frac{1}{(1+\alpha)} \cdot c(u,v) \) for every \((u,v) \in E(S,T)\) where \( u \in S \) and \( v \in T \).

Note that if a cut is an \((1+\alpha)\)-fair \((s,t)\)-cut, it implies by standard flow-cut duality that it is also an \((1+\alpha)\)-approximate \((s,t)\)-mincut. But not all \((1+\alpha)\)-approximate \((s,t)\)-mincuts are \((1+\alpha)\)-fair \((s,t)\)-cuts. As a simple example, consider a path \( v-s-t \) on three vertices. Clearly, the cut \{\text{\{s}}\} contains both edges and is therefore a 2-approximate \((s,t)\)-mincut. However, there is no \((s,t)\)-flow that can saturate both edges to fraction \( \frac{1}{2} \). To motivate our choice of terminology (fair cuts), note that if an \((s,t)\)-cut is a \((1+\alpha)\)-approximate \((s,t)\)-mincut, it follows by flow-cut duality that any \((s,t)\)-maxflow will \textit{cumulatively} saturate the edges of the cut to a fraction \( \geq \frac{1}{1+\alpha} \). But, as we saw in the previous example, this saturation need not be \textit{fair} in the sense that some edges might not be saturated at all. In this context, a \((1+\alpha)\)-fair cut demands the additional property that \textit{each} edge be saturated to a fraction \( \geq \frac{1}{\alpha} \) (in the sense of “max-min” fairness).

In some situations, it would be convenient to extend the definition of fair cuts from two vertices \( s,t \) to two disjoint sets of vertices \( X,Y \). Intuitively, we do this by contracting \( X \) and \( Y \) into single vertices and using the previous definition of fair cuts on this contracted graph. We formally define this below.

Definition 1.4. Let \( G = (V,E) \) be an undirected graph with edge capacities \( c \in \mathbb{R}_{>0}^E \). Let \( X,Y \subset V \) be two disjoint sets of vertices. For any parameter \( \alpha \geq 0 \), we say that a cut \( (S,T) \) is a \((1+\alpha)\)-fair \((X,Y)\)-cut if there exists a feasible flow \( f \) from the vertices in \( X \) to the vertices in \( Y \) such that \( f(u,v) \geq \frac{1}{(1+\alpha)} \cdot c(u,v) \) for every \((u,v) \in E(S,T)\) where \( u \in S \) and \( v \in T \).

1.2 Computing Fair Cuts

The main technical tool that we introduce in this paper is a near-linear time algorithm for computing a \((1+\alpha)\)-fair \((s,t)\)-cut.

Theorem 1.5 (Fair Cut). Given a graph \( G = (V,E) \), two vertices \( s,t \in V \), and \( \alpha \in (0,1] \), we can compute with high probability a \((1+\alpha)\)-fair \((s,t)\)-cut in \( \tilde{O}(m/\alpha^3) \) time.

We note that the only reason why our algorithm is randomized is because we use the \textit{congestion approximator} by [RST14, Pen16]. This can be made deterministic based on the algorithm by [CGL+20], but the running time would be \( m^{1+o(1)}/\alpha^3 \) instead.

Moreover, we remark that although we will focus on \((1+\alpha)\)-fair \((s,t)\)-cuts, the corresponding \((s,t)\)-flow can be obtained from a fair cut in \( \tilde{O}(m/\alpha) \) time using a standard application of a \((1+\alpha)\)-approximate max-flow algorithm of Sherman [She17].

Our key subroutine for computing fair cuts is called \textsc{AlmostFair}. Here, we describe at a high-level what the \textsc{AlmostFair} subroutine does, how to use it for computing fair cuts, and finally how to obtain the \textsc{AlmostFair} subroutine itself.

Say we are given an \((s,t)\)-cut \((S,T)\) which may be far from being fair. The \textsc{AlmostFair} subroutine works on one side of the \((s,t)\)-cut, say \( T \), and returns a partition \((P_t, T')\) of \( T \) such
that $t \in T'$. We think of $P_t$ as the part that we “prune” out of $T$. Our first guarantee is that the remaining part $T'$ is “almost fair” in the following sense: each boundary edge in $E(S, T')$ not incident to the pruned set $P_t$ can simultaneously send flow of value at least $(1 - \beta)$-fraction of its capacity to $t$, for a small parameter $\beta$ that we can choose. This guarantee alone would have been weak if the pruned set $P_t$ is so big that there are few edges left in $E(S, T')$. However, the second guarantee of AlmostFair says that, if $P_t$ is big, then $(V \setminus T', T')$ is actually a much better $(s, t)$-cut than the original cut $(S, T)$ with respect to cut value. More formally, we have $\delta_G(T') \leq \delta_G(T) - \beta \cdot \delta_G(S, P_t)$ meaning that the decrease in the cut size is at least $\beta$ times the total capacity of $E(S, P_t)$.

With these two guarantees of AlmostFair, given any $(s, t)$-cut $(S, T)$, we can iteratively improve this cut to make it fair as follows. We call AlmostFair on both $S$ and $T$ and obtain $(P_s, S')$ and $(P_t, T')$. Let’s consider two extremes. If both pruned sets $P_s$ and $P_t$ are tiny, then there is an $(s, t)$-flow that almost fully saturates every edge in $E(S, T')$. This certifies $(S, T)$ is very close to being fair as $P_s$ and $P_t$ are tiny. However, if either $P_s$ or $P_t$ is very big, say $P_t$, then $(S \cup P_t, T')$ is an $(s, t)$-cut of much smaller value than the original cut $(S, T)$. Therefore, this is progress too and we can recursively work on this new cut $(S \cup P_t, T')$. To make the intuition on these two extremes work, we iteratively call AlmostFair using a parameter $\beta$ that increases slightly in each round. The full algorithm is presented in Section 4.

Now, let us sketch the AlmostFair subroutine itself. This subroutine is based on Sherman’s algorithm for computing a $(1 + \epsilon)$-approximate max-flow [She13] (for any $\epsilon > 0$), which in turn uses the multiplicative weight update (MWU) framework. Given the $t$-side $T$ of an $(s, t)$-cut, if we call Sherman’s algorithm where the demand is specified so that each boundary edge should send flow at its full capacity to sink $t$, then his algorithm would either return a flow satisfying this demand with congestion $(1 + \epsilon)$ or return a “violating” cut certifying that the demand is not feasible. In the former case, this would satisfy the guarantee of AlmostFair where $P_t = \emptyset$ after scaling down the flow by a $(1 + \epsilon)$ factor. Unfortunately, in the latter case, his algorithm does not guarantee the existence of the flow that we want. The reason behind this problem is that whenever his algorithm detects a violating cut, the algorithm is just terminated. In a more general context, this holds for most (if not all) MWU-based algorithms for solving linear programs; in each round of the MWU algorithm, whenever “the oracle” certifies that the linear program is infeasible, then we just terminate the whole algorithm.

Interestingly, we fix this issue by “insisting on continuing” the MWU algorithm. Once we detect a violating cut, we include it into the pruned set, cancel the demand inside this pruned set, and continue updating weights in the MWU algorithm. After the last round, the flow constructed via MWU indeed sends flow from each remaining boundary edge which is not pruned out, which is exactly our goal. The detailed algorithm is presented in Section 3.

1.3 From Fair Cuts to Approximate Isolating Cuts

We believe that the notion of fair cuts can be useful in many contexts since it offers a more robust alternative to approximate mincuts. In this paper, we use it to obtain an approximate isolating cuts algorithm. We define the isolating cuts problem first.

6Sherman’s original presentation in [She13] does not explicitly use the MWU framework. Although this alternative interpretation was already known to experts, our MWU-based presentation of his algorithm is arguably simpler and more intuitive.
Definition 1.6. Given a weighted, undirected graph \( G = (V, E) \) and a subset of terminals \( S = \{s_1, s_2, \ldots, s_k\} \), the goal of the isolating cuts problem is to find a set of disjoint sets \( S_1, S_2, \ldots, S_k \) such that for each \( i \), the cut \( (S_i, V \setminus S_i) \) is a mincut that separates \( s_i \in S_i \) from the remaining terminals \( S \setminus \{s_i\} \subseteq V \setminus S_i \). If \( S_i \) is a \((1 + \epsilon)\)-approximate mincut separating \( s_i \) from the remaining terminals, then the corresponding problem is called the \((1 + \epsilon)\)-approximate isolating cuts problem.

Using fair cuts, we obtain a near-linear time algorithm for approximate isolating cuts.

Theorem 1.7. There is an algorithm for finding \((1 + \epsilon)\)-approximate isolating cuts that takes \( \tilde{O}(m \cdot \text{poly}(1/\epsilon)) \) time.

Li and Panigrahi [LP20] gave an algorithm for finding exact isolating cuts using \( O(\log n) \) max-flow/mincut computations that crucially relies on the uncrossing property of mincuts. This property ensures that if we take a minimum isolating cut \( X \) containing a terminal vertex \( s \) and a crossing mincut \( Y \), then their intersection \( X \cap Y \) or difference \( X \setminus Y \) (depending on which set the terminal vertex \( s \) is in) is also a minimum isolating cut. This allows us to partition the graph by removing edges corresponding to a set of mincuts, such that each terminal is in one of the components of this partition. For each terminal, we can now find the corresponding minimum isolating cut by simply contracting the rest of the components and running a max-flow algorithm on this contracted graph. The advantage of this contraction is that the total size of all the graphs on which we are running max-flows is only a constant times the size of the overall graph.

Unfortunately, approximate mincuts don’t satisfy this uncrossing property, which renders this method unusable if we replace exact mincut subroutines with faster \((1 + \epsilon)\)-approximate mincuts. But, if we instead used fair cuts, then we can show the following: if \( X \) is a \((1 + \epsilon)\)-approximate minimum isolating cut containing terminal \( s \) and \( Y \) is a \((1 + \alpha)\)-fair cut, then either \( X \cap Y \) or \( X \setminus Y \) (whichever set contains \( s \)) is a \((1 + \epsilon)(1 + \alpha)\)-approximate minimum isolating cut. This allows us to use Li and Panigrahi’s framework. Since the number of fair cuts we remove in forming the components is only \( O(\log n) \), the multiplicative growth in the approximation factor can be offset by scaling the parameter in fair cuts by the same logarithmic factor. The advantage in using fair cuts over exact mincuts is that the running time of the former is near-linear by Theorem 1.5, which helps establish Theorem 1.7. The details of this algorithm are presented in Section 5.

1.4 Applications of Approximate Isolating Cuts

The first, immediate application of approximate isolating cuts is to the Steiner connectivity problem. Recall that in this problem, we can given a set of terminals \( S \) and the goal is to find a \((1 + \epsilon)\)-approximate minimum cut that has at least one terminal on each side of the cut. We sample the terminals at doubling probability levels (we use multiple independent samples at each level to boost the probability of success), and for each sampling level, we find all the \((1 + \epsilon)\)-approximate minimum isolating cuts. If the actual Steiner mincut has \( p \) fraction of the terminals on the smaller side, then the sampling level between \( p \) and \( 2p \) has exactly one sampled terminal from the smaller side with constant probability. In this case, the \((1 + \epsilon)\)-approximate minimum isolating cut is also a \((1 + \epsilon)\)-approximate minimum Steiner cut. So, we can simply report the cut with minimum value among all the isolating cuts reported by the different sampling levels, and this will, with high probability after appropriate boosting at each level, be a \((1 + \epsilon)\)-approximate minimum Steiner cut. The details of this algorithm are presented in Section 5.1.
Theorem 1.8. There is a randomized algorithm that finds a \((1 + \epsilon)\)-approximate minimum Steiner cut with high probability in \(\tilde{O}(m \cdot \text{poly}(1/\epsilon))\) time.

The more complicated application of approximate isolating cuts is the approximate Gomory-Hu tree problem. In the recursive algorithm of Li and Panigrahi [LP21], it is crucial to ensure that the approximation is one-sided in the sense that the “large” recursive subproblem preserves mincut values exactly. But, in general, if we use the approximate isolating cuts subroutine as a black box, this would not be the case. To alleviate this concern, we augment the approximate isolating cuts procedure using an additional fairness condition for the isolating cuts returned by the algorithm. This fairness condition ensures that although we do not have one-sided approximation, the approximation factor in the “large” subproblem can be controlled using a much finer parameter than the overall approximation factor of the algorithm, which then allows us to run the recursion correctly. The details of the Gomory-Hu tree algorithm establishing Theorem 1.1 are presented in Section 6.

2 Preliminaries

Given a undirected capacitated/weighted graph \(G = (V, E)\) with edge capacities/weights is \(c \in \mathbb{R}^E_{\geq 0}\), we let \(c(E') = \sum_{e \in E'} c(e)\) be the total capacity of \(E'\). For simplicity, we assume that the ratio between the largest and lowest edge capacities or weights are poly\(n\). For any disjoint sets \(S, T \subseteq V\), we let \(\delta_G(S) = c(E(S, V \setminus S))\) denote the cut size of \(S\) and \(\delta_G(S, T) = c(E(S, T))\) denote the total capacity of edges from \(S\) to \(T\). For any distinct vertices \(s\) and \(t\), let \(\lambda_G(s, t)\) be the minimum-weight \(s\)-\(t\) cut. We sometimes omit \(G\) when it is clear from the context.

**Flow.** A flow \(f : V \times V \rightarrow \mathbb{R}\) satisfies \(f(u, v) = -f(v, u)\) and \(f(u, v) = 0\) for \(\{u, v\} \notin E\). The notation \(f(u, v) > 0\) means that mass is routed in the direction from \(u\) to \(v\), and vice versa. The congestion of \(f\) is \(\max_{(u, v) \in E} \frac{|f(u, v)|}{c(e)}\). If the congestion is at most 1, we say that \(f\) respects the capacity or \(f\) is feasible. For each vertex \(u \in V\), the net flow out of vertex \(u\), denoted by \(f(u) = \sum_{v \in V} f(u, v)\), is the total mass going out of \(u\) minus the total mass coming into \(u\). More generally, for any vertex set \(S \subseteq V\), we can define the net flow out of \(S\) as \(f(S) = \sum_{u \in S} f(u) = \sum_{u \in S, v \in V} f(u, v)\). The new flow out from \(S\) to \(T\) is denoted by \(f(S, T) = \sum_{u \in S, v \in T} f(u, v)\). Observe that we always have \(f(V) = 0\).

A source function \(\Delta : V \rightarrow \mathbb{R}\) is a function where \(\sum_{v \in V} \Delta(v) = 0\). We say that flow \(f\) satisfies source \(\Delta\) if \(f(v) = \Delta(v)\) for all \(v \in V\). For any \(S \subseteq V\), let \(\Delta(S) = \sum_{v \in S} \Delta(v)\) be the total source on \(S\). Observe \(\Delta(V) = f(V) = 0\). By the max-flow min-cut theorem, we have the following:

**Fact 2.1.** For any \(\epsilon \geq 0\), \(|\Delta(S)| \leq \epsilon \cdot \delta(S)\) for all \(S \subseteq V\) iff there is a flow with congestion \(\epsilon\) that satisfies \(\Delta\).

For a flow \(f\) and a source function \(\Delta\), define the excess \(\Delta^f\) as \(\Delta^f(v) = \Delta(v) - f(v)\) for every \(v \in V\). We think of excess as a remaining source function. We say that \(f\) \(\epsilon\)-satisfies \(\Delta\) if \(|\Delta^f(S)| \leq \epsilon \cdot \delta(S)\) for all \(S \subseteq V\). That is, by Fact 2.1, there exists a flow \(f_{\text{aug}}\) with congestion \(\epsilon\) where \(f + f_{\text{aug}}\) satisfies \(\Delta\). Note that \(f\) \(0\)-satisfies \(\Delta\) iff \(f\) satisfies \(\Delta\).

For any two vertices \(s, t \in V\), an \((s, t)\)-cut \((S, T)\) is a cut such that \(s \in S\) and \(t \in T\). An \((s, t)\)-flow \(f\) obeys \(f(v) = 0\) for all \(v \neq s, t\). Similarly, an \((s, t)\)-source function \(\Delta\) obeys \(\Delta(v) = 0\) for all \(v \neq s, t\). That is, an \((s, t)\)-source function is satisfied only by an \((s, t)\)-flow.
Congestion Approximators. When we want to argue that flow $f$ $\epsilon$-satisfies a source function $\Delta$, it can be inconvenient to ensure that $|\Delta'(S)| \leq \epsilon \cdot \delta(S)$ for all $S \subseteq V$ because there are exponentially many sets. Surprisingly, there is a collection $S$ of linearly many sets of vertices, where if $|\Delta(S)| \leq \epsilon \cdot \delta(S)$ for each $S \in S$, then this is also true for all $S \subseteq V$ with some polylog($n$) blow-up factor. Moreover, $S$ can be computed in near-linear time.

**Theorem 2.2** (Congestion approximator [RST14, Pen16]). There is a randomized algorithm that, given a graph $G = (V, E)$ with $n$ vertices and $m$ edges, constructs in $\tilde{O}(m)$ time with high probability a laminar family $S$ of subsets of $V$ such that

1. $S$ contains at most $2n$ sets,
2. each vertex appears in $O(\log n)$ sets of $S$, and
3. for any source function $\Delta$ on $V$, if $|\Delta(S)| \leq \delta(S)$ for all $S \in S$, then $|\Delta(R)| \leq \gamma_S \delta(R)$ for all $R \subseteq V$ for a quality factor $\gamma_S = O(\log^4 n)$.

Graphs with Boundary Vertices. Given a set $U \subseteq V$, let $G\{U\}$ denote the following “induced subgraph with boundary vertices”: start with induced subgraph $G[U]$, and for each edge $e = (u, v) \in E(U, V \setminus U)$ with endpoint $u \in U$, create a new vertex $x_e$ and add the edge $(x_e, u)$ to $G\{U\}$ of the same capacity as $e$. Let $N_G\{U\}$ be the vertex set of $G\{U\}$ and define $N_G(U) = N_G\{U\} \setminus U$. We call vertices in $N_G(U)$ boundary vertices. We simply write $N\{U\}$ and $N(U)$ instead of $N_G\{U\}$ and $N_G(U)$ when the context is clear. Observe that the degree $\deg_{G\{U\}}(x_e)$ of each boundary vertex $x_e \in N(U)$ in $G\{U\}$ is simply the capacity $c(e)$ of edge $e$. We will use this notation very often in the paper.

Boundary Source Functions. In our context, the sink node $t \in U$ is usually given. The full $U$-boundary source function $\Delta_U : V(G\{U\}) \rightarrow \mathbb{R}$ is defined such that

$$
\Delta_U(v) = \begin{cases} 
\deg_{G\{U\}}(v) & \text{if } v \in N\{U\} \\
0 & \text{if } v \in U \setminus t \\
-\Delta(N\{U\}) & \text{if } v = t.
\end{cases}
$$

That is, any flow satisfying $\Delta_U$ sends flow from each boundary vertex of $G\{U\}$ at full capacity to $t$. We also write $\Delta_{U,t}$ when it is not clear from the context what $t$ is. More generally, given any source function $\Delta' : V(G\{U\}) \rightarrow \mathbb{R}$, we say that $\Delta'$ is a $U$-boundary source function if $\Delta'(v) = 0$ for all $v \in U \setminus t$, $\Delta'(t) = -\Delta(N\{U\})$. That is, $\Delta'$ is completely determined once we specify the source values on boundary vertices $N\{U\}$.

3 Almost Fair Cuts via Multiplicative Weight Updates

The key subroutine used for proving Theorem 1.5 is the algorithm below.

**Theorem 3.1** (Almost Fair Cuts). There is an algorithm $\text{AlmostFair}(G, U, t, \epsilon, \beta)$ that, given a graph $G = (V, E)$ with a sink node $t \in V$, a set $U \subseteq V$ where $t \in U$, and parameters $\beta \geq 0$ and $\epsilon > 0$, returns a partition $(P, U')$ of $U$ where $t \in U'$ with the following properties:
1. \( \delta_G(U') \leq \delta_G(U) - \beta \delta_G(P, V \setminus U) \) (equivalently, \( \delta_G(P, U') \leq (1 - \beta) \delta_G(P, V \setminus U) \)), and

2. There exists a flow \( f'_{\text{sat}} \) in \( G\{U'\} \) with congestion \((1 + \epsilon)\) satisfying a \( U' \)-boundary source function \( \Delta' \) such that
   \[
   \Delta'(v) = (1 - \beta) \deg_{G(U')}(v) \quad \text{for all old boundary vertices } v \in N(U') \cap N(U)
   \]
   \[
   |\Delta'(v)| \leq (1 + \epsilon) \deg_{G(U')}(v) \quad \text{for all new boundary vertices } v \in N(U') \setminus N(U)
   \]

The algorithm takes \( \tilde{O}(|E(G\{U\})|/\epsilon^2) \) time and is correct with high probability.\(^7\)

The rest of this section is for proving Theorem 3.1. For convenience, we write \( H = G\{U\} \) and let \( n \) and \( m \) denote the number of vertices and edges in \( H \) throughout this section. Let \( B \) be the incidence matrix of \( H \). Observe that, for any flow \( f \) on \( H \), we have \( (Bf)_v = f(v) \) is the net flow out of \( v \). We can view \( Bf \) as a vector in \( \mathbb{R}^{V(H)} \). Define
\[
\Delta = (1 - \beta)\Delta_U
\]
as the full \( U \)-boundary source function on \( G\{U\} \) after scaled down by \((1 - \beta)\) factor. For any \( U' \subseteq U \), the restriction \( \Delta|_{U'} \) of \( \Delta \) is a \( U \)-boundary source function obtained from \( \Delta \) by zeroing out the entries on \( N(U) \setminus N(U') \), i.e., the boundary vertices of \( U \) which are not boundaries of \( U' \), and then setting the entry on \( t \) so that \( \sum_{v \in V(H)} \Delta|_{U'}(v) = 0 \). Similarly, we view \( \Delta \) and also \( \Delta|_{U'} \) as vectors in \( \mathbb{R}^{V(H)} \).

3.1 Algorithm

Initialization. We start by computing a congestion approximator \( S \) of \( H \) with quality \( \gamma_S = O(\log^4 n) \) using Theorem 2.2. For a technical reason, it is more convenient if no set in \( S \) contains sink \( t \). Below, we show why this can be assumed.

Proposition 3.2. We can assume that every set \( S \) in the family \( S \) from Theorem 2.2 does not contains \( t \).

Proof. Replace each set \( S \in S \) where \( t \in S \) with its complement \( V(H) \setminus S \). Observe that \( S \) is now a larminar family on \( V(H) \setminus t \) where \( |S| \) does not change, and the number of sets containing each vertex may increase only by \( O(\log n) \). Hence, the first and second properties of Theorem 2.2 still hold. The third property still holds as well because \( |\Delta(S)| = |\Delta(V(H) \setminus S)| \) for all \( S \).

Our algorithm is based on the Multiplicative Weight Update framework and so it works in rounds. For round \( i \), we maintain weights \( w_{i,S}^{\circ} \geq 0 \) for each \( S \in S \) and \( \circ \in \{+, -\} \) and define the potential \( \phi^i_v \in \mathbb{R}^{V(H)} \) where
\[
\phi^i_v = \sum_{S \ni v} \frac{1}{\delta_H(S)} (w_{i,S,+}^v - w_{i,S,-}^v)
\]
for each vertex \( v \). As no set \( S \in S \) contains \( t \), we will always have \( \phi^i_t = 0 \) for all \( i \). Initially, we set \( w_{1,S}^{1,0} = 1 \) for all \( S \in S \), \( \circ \in \{+, -\} \).

\(^7\)Note that the guarantee that \( |\Delta'(v)| \leq (1 + \epsilon) \deg_{G(U')} (v) \) for all new boundary vertices \( v \in N(U') \setminus N(U) \) in fact follows from the guarantee that \( f'_{\text{sat}} \) has congestion \((1 + \epsilon)\). We state both guarantees explicitly for convenience.
The algorithm also maintains a decremental subset $V^i$ where $t \in V^i \subseteq V^{i-1}$ for all $i$. We initialize $V^0$ as follows. First, set $V^0 = V(H)$. While there exists $S \in \mathcal{S}$ where $\Delta|_{V^0}(S) > \delta_H(S)$, which certifies that there is no feasible flow on $H$ satisfying $\Delta|_{V^0}$ by Fact 2.1, we update $V^0 \leftarrow V^0 \setminus S$ (in particular, the function $\Delta|_{V^0}$ changes). Let $D^0$ contain all the vertices we removed from $V^0$.

Now, we are ready to state the main algorithm.

**Main Algorithm.** For round $i = 1, 2, \ldots, T$ where $T = \Theta(\log(n)/\alpha^2)$ and $\alpha = \epsilon/\gamma_S$, we do the following:

1. Define $f^i$ on $H$ such that for each edge $(u, v)$, $f^i(u, v)$ flows from high potential to low potential at maximum capacity. That is, for every edge $(u, v)$ in $H$,
   
   $f^i(u, v) = \begin{cases} 
   c(u, v) & \text{if } \phi^i_u > \phi^i_v \\
   0 & \text{if } \phi^i_u = \phi^i_v \\
   -c(u, v) & \text{if } \phi^i_u < \phi^i_v.
   \end{cases}$

2. Using Lemma 3.9, compute a deletion set $D^i \subseteq V(H) \setminus t$ and set $V^i \leftarrow V^i - 1 \setminus D^i$, where $D^i$ satisfies the following:

   if $D^i \neq \emptyset$, then $\Delta|_{V^{i-1}}(D^i) > \delta_H(D^i)$, and

   $\langle \phi^i, \Delta|_{V^i} \rangle = \langle \phi^i, \Delta|_{V^{i-1} \setminus D^i} \rangle \leq \langle \phi^i, Bf^i \rangle$.

3. For each $S \in \mathcal{S}$, let
   
   $r^i_S = \frac{(\Delta|_{V^i})^f(S)}{\delta_H(S)} = \frac{\Delta|_{V^i}(S) - f^i(S)}{\delta_H(S)}$

   be the relative total excess at $S$ compared to the cut size in round $i$.

4. Update the weights
   
   $w^{i+1}_{S,+} = w^{i}_{S,+} \cdot e^{ar^i_S}$ and $w^{i+1}_{S,-} = w^{i}_{S,-} \cdot e^{-ar^i_S}$.

After $T$ rounds, we compute the pruned set $P = \bigcup_{i=0}^{T} D^i$ and let $U' = U \setminus P$. Finally, we return the partition $(P, U')$.

### 3.2 Correctness

We prove that the partition $(P, U')$ outputted by our algorithm satisfies the requirement in Theorem 3.1. The first important thing to understand our algorithm is to formally see how it is captured by the Multiplicative Weight Update (MWU) algorithm, which we recall below:

**Theorem 3.3** (Multiplicative Weights Update [AHK12]). Let $J$ be a set of indices, and let $\alpha \leq 1$ and $\omega > 0$ be parameters. Consider the following algorithm:

1. Set $w^{(1)}_j \leftarrow 1$ for all $j \in J$

2. For $i = 1, 2, \ldots, T$ where $T = O(\omega^2 \log(|J|)/\alpha^2)$:
(a) The algorithm is given a “gain” vector \( g^i \in \mathbb{R}^J \) satisfying \( \|g^i\|_\infty \leq \omega \) and \( \langle g^i, w^i \rangle \leq 0 \).

(b) For each \( j \in J \), set \( w_j^i \leftarrow w_j^{i-1} \exp(\alpha g_j^i) = \exp(\alpha \sum_{v \in J} g_j^i) \).

At the end of the algorithm, we have \( \frac{1}{T} \sum_{i \in [T]} g_j^i \leq \alpha \) for all \( j \in J \).

To apply Theorem 3.3 into our setting, we define \( J = S \times \{+, -\} \). That is, we work with indices \((S, +)\) and \((S, -)\) for \( S \in S \). We use the same weights \( w^i \) and error parameter \( \alpha \) as the algorithm, and we set \( \omega = 2 \). For each iteration \( i \) and \( S \in S \), we define

\[
g_{S, \pm}^i = \pm r_{S}^i = \pm \frac{\Delta|V^i(S) - f^i(S)}{\delta_H(S)}.\]

Observe that the weights \( w_{S, \pm}^i \) are updated in Step 4 exactly as \( w_{S, \pm}^i \leftarrow w_{S, \pm}^{i-1} \exp(\alpha g_{S, \pm}^i) \). With this setting, we show that our gain vector \( g^i \) indeed satisfies the condition in Step 2a of Theorem 3.3.

**Lemma 3.4.** For each \( i \), we have \( \|g^i\|_\infty \leq 2 \) and \( \langle g^i, w^i \rangle \leq 0 \).

**Proof.** To show \( \|g^i\|_\infty \leq 2 \), we have

\[
|g_{S, \pm}^i| = \left| \frac{\Delta|V^i(S) - f^i(S)}{\delta_H(S)} \right| \leq \frac{\Delta|V^i(S)}{\delta_H(S)} + \frac{f^i(S)}{\delta_H(S)} \leq 1 + 1,
\]

To see why the last inequality holds, we have (1) \( \Delta|V^0(S) \leq \delta_H(S) \) for all \( S \in S \) by the initialization of \( V^0 \), (2) \( \Delta|V^i(S) \geq 0 \) for all \( i \) because \( t / S \), and (3) \( \Delta|V^i(S) \) may only decrease as \( V^i \) is a decremental set. Also, we have \( |f^i(S)| \leq \delta_H(S) \) because each \( f^i \) respects the capacity.

To show \( \langle g^i, w^i \rangle \leq 0 \), first observe that \( \langle g^i, w^i \rangle = \langle \phi^i, \Delta|V^i \rangle - \langle \phi^i, Bf^i \rangle \) exactly.

\[
\langle g^i, w^i \rangle = \sum_{S \in S} (g_{S, +}^i w_{S, +}^i + g_{S, -}^i w_{S, -}^i)
= \sum_{S \in S} (w_{S, +}^i - w_{S, -}^i) r_{S}^i
= \sum_{S \in S} \frac{w_{S, +}^i - w_{S, -}^i}{\delta_H(S)} (\Delta|V^i(S) - f^i(S))
= \sum_{S \in S} \frac{w_{S, +}^i - w_{S, -}^i}{\delta_H(S)} \sum_{v \in S} (\Delta|V^i(v) - (Bf^i)_v)
= \sum_{v \in V(H)} (\Delta|V^i(v) - (Bf^i)_v) \sum_{S \ni v} \frac{w_{S, +}^i - w_{S, -}^i}{\delta_H(S)}
= \sum_{v \in V(H)} (\Delta|V^i(v) - (Bf^i)_v) \phi^i_v
= \langle \phi^i, \Delta|V^i \rangle - \langle \phi^i, Bf^i \rangle.
\]

Since the deletion set \( D^i \) from Step 2 is designed to guarantee that \( \langle \phi^i, \Delta|V^i \rangle \leq \langle \phi^i, Bf^i \rangle \), we have that \( \langle g^i, w^i \rangle \leq 0 \).

---

\*More generally, for any value \( \text{val} \), if we have \( \langle g^i, w^i \rangle \leq \text{val} \) for all \( i \), the MWU algorithm guarantees that \( \frac{1}{T} \sum_{i \in [T]} g_j^i \leq \text{val} + \alpha \) for all \( j \). Here, we use a special case when \( \text{val} = 0 \).
From the above, we have verified that our algorithm is indeed captured by the MWU algorithm. Now, we derive the implication of this fact. Only for analysis, we define the average flow \( \bar{f} = \frac{1}{T} \sum_{i=1}^{T} f_i \in \mathbb{R}^{E(H)} \) on \( H \) and the average \( U \)-boundary source function \( \bar{\Delta} = \frac{1}{T} \sum_{i=1}^{T} \Delta_{V_i} \in \mathbb{R}^{V(H)} \) on \( H \).

**Lemma 3.5.** We have \( \bar{f} \) \( \epsilon \)-satisfies \( \bar{\Delta} \) in \( H \).

**Proof.** Define \( \bar{r} = \frac{1}{T} \sum_{i=1}^{T} r^i \in \mathbb{R}^S \). First, we prove that \( |r_S| \leq \alpha \) for all \( S \in \mathcal{S} \). This is because

\[
\pm r_S = \frac{1}{T} \sum_{i \in [T]} \pm r^i = \frac{1}{T} \sum_{i \in [T]} g^i_{S, \pm} \leq \alpha
\]

where the last inequality is precisely the guarantee of the MWU algorithm from Theorem 3.3. Next observe that the excess is

\[
\bar{\Delta}^f(S) = \bar{\Delta}(S) - \bar{f}(S) = \bar{r}_S \delta_H(S).
\]

Therefore, we have that \( |\bar{\Delta}^f(S)| \leq \alpha \delta_H(S) \) for all \( S \in \mathcal{S} \). Since \( \mathcal{S} \) is a congestion approximator, it follows by Theorem 2.2 that

\[
|\bar{\Delta}^f(S)| \leq \gamma_S \alpha \delta_H(S) = \epsilon \delta_H(S)
\]

for all \( S \subseteq V(H) \). This precisely means that \( \bar{f} \) \( \epsilon \)-satisfies \( \bar{\Delta} \). \( \square \)

Now, we are ready to prove Item 2 of Theorem 3.1. By Lemma 3.5, there exists a flow \( \bar{f}_{\text{aug}} \) in \( H \) with congestion \( \epsilon \) such that \( \bar{f}_{\text{sat}} := \bar{f} + \bar{f}_{\text{aug}} \) satisfies \( \bar{\Delta} \). We define \( f'_{\text{sat}} \) as the restriction of \( \bar{f}_{\text{sat}} \) into \( G\{U'\} \). That is, for each new boundary vertex \( x_v \in N(U') \setminus N(U) \) where \( u \) is its unique neighbor, we set \( f'_{\text{sat}}(x_v, u) = \bar{f}_{\text{sat}}(e) \). For every other edge \( e \in E(G\{U'\}) \), we set \( f'_{\text{sat}}(e) = \bar{f}_{\text{sat}}(e) \).

Let \( \Delta' \) be a \( U' \)-boundary source function where, for each \( U' \)-boundary vertex \( v \in N(U') \), we set \( \Delta'(v) = f'_{\text{sat}}(v) \) as the net flow out of \( v \) via \( f'_{\text{sat}} \).

**Lemma 3.6.** We have

1. \( f'_{\text{sat}} \) is a flow in \( G\{U'\} \) with congestion at most \( (1 + \epsilon) \) that satisfies \( \Delta' \).
2. \( \Delta' \) is a \( U' \)-boundary source function where

\[
\Delta'(v) = (1 - \beta) \deg_{G(U')}(v) \quad \text{for all old boundary vertices } v \in N(U') \cap N(U)
\]

\[
\Delta'(v) \leq (1 + \epsilon) \deg_{G(U)}(v) \quad \text{for all new boundary vertices } v \in N(U') \setminus N(U)
\]

**Proof.** (1) As \( f'_{\text{sat}} \) is a restriction of \( \bar{f}_{\text{sat}} \) into \( G\{U'\} \), then the congestion of \( f'_{\text{sat}} \) is at most that of \( \bar{f}_{\text{sat}} \) which is \((1 + \epsilon)\). To see why \( f'_{\text{sat}} \) satisfies \( \Delta' \), we have that \( \Delta'(v) = f'_{\text{sat}}(v) \) for all \( U' \)-boundary vertex \( v \in N(U') \) by construction. For non-boundary vertex \( v \in U' \setminus t \), we have \( f'_{\text{sat}}(v) = \bar{f}(v) = 0 = \Delta'(v) \). So \( f'_{\text{sat}}(v) = \Delta'(v) \) for all \( v \neq t \). This implies that \( f'_{\text{sat}}(t) = \Delta'(t) \) too and so \( f'_{\text{sat}} \) satisfies \( \Delta' \).

(2) For each new boundary vertex \( v \in N(U') \setminus N(U) \), we have \( \Delta'(v) = f'_{\text{sat}}(v) \) and so \( |\Delta'(v)| \leq (1 + \epsilon) \deg_{G(U)}(v) \) because \( f'_{\text{sat}} \) has congestion \((1 + \epsilon)\) in \( G\{U'\} \). For each old boundary vertex \( v \in N(U') \cap N(U) \), we have \( \Delta'(v) = f'_{\text{sat}}(v) = \bar{f}_{\text{sat}}(v) \). As \( \bar{f}_{\text{sat}} \) satisfies \( \bar{\Delta} \), we have \( \bar{f}_{\text{sat}}(v) = \bar{\Delta}(v) \). But \( \bar{\Delta}(v) = (1 - \beta) \deg_{G(U)}(v) \) as, for every \( i \), \( \Delta|_{V_i} = (1 - \beta) \deg_{G(U)}(v) \) for every \( v \notin P \). Therefore, \( \Delta'(v) = (1 - \beta) \deg_{G(U')}(v) \).

This proves Item 2 of Theorem 3.1. It remains to prove Item 1 of Theorem 3.1.
Lemma 3.7. \( \delta_{G(U)}(P) \leq \Delta(P) \).

Proof. First observe that \( \delta_{H}(D^0) \leq \Delta(D^0) \) because every time we remove a set \( S \) from \( V^0 \), we have \( \delta_{H}(S) < \Delta|V^0(S) \) and we can charge \( \delta_{H}(S) \) to the decrease of \( \Delta|V^0(S) \). Next, the sets \( D^i \) for \( i \geq 1 \) satisfy \( \delta_{H}(D^i) \leq \Delta|V^{i-1}(D^i) \), so

\[
\delta_{G(U)}(P) = \delta_{H}(P) \leq \sum_{i \geq 0} \delta_{H}(D^i) \leq \Delta(D^0) + \sum_{i \geq 1} \Delta|V^{i-1}(D^i) = \Delta(P).
\]

\[\square\]

Corollary 3.8. \( \delta_{G}(U') \leq \delta_{G}(U) - \beta \cdot \delta_{G}(P, V \setminus U) \).

Proof. We have \( \delta_{G(U)}(P) = \delta_{G}(P, U') \) and \( \Delta(P) = (1 - \beta)\delta_{G}(V \setminus U, P) \). By adding \( \delta_{G}(V \setminus U, U') \) into both sides of the inequality of Lemma 3.7, we have

\[
\delta_{G}(V \setminus U, U') + \delta_{G}(P, U') \leq \delta_{G}(V \setminus U, U') + \delta_{G}(V \setminus U, P) - \beta \delta_{G}(V \setminus U, P)
\]

which concludes the proof because \( \delta_{G}(U') = \delta_{G}(V \setminus U, U') + \delta_{G}(P, U') \) and \( \delta_{G}(V \setminus U, U') + \delta_{G}(V \setminus U, P) = \delta_{G}(V \setminus U) = \delta_{G}(U) \). \[\square\]

This proves the correctness of Theorem 3.1.

3.3 Running Time

Here, we explain some implementation details and analyze the total running time. Computing the congestion approximator \( S \) takes \( \tilde{O}(m) \) by Theorem 2.2. The step which ensures that no set in \( S \) contains \( t \) is at most \( O(n \log n) \) time because \( t \) was contained in at most \( O(\log n) \) sets \( S \) and the complement of \( S \) has size at most \( n \).

Next, we explain how to implement the initialization of \( V^0 \) efficiently. Observe that, for any \( S \in \mathcal{S} \), if \( \Delta|V^0(S) > \delta_{H}(S), \) then we set \( V^0' \leftarrow V^0 \setminus S \) and then we have \( \Delta|V^0(S) = 0 \). Otherwise, if \( \Delta|V^0(S) \leq \delta_{H}(S) \), then it remains so forever because \( \Delta|V^0(S) \) is monotonically decreasing when \( V^0 \) is a decremental set. In any case, for each \( S \in \mathcal{S} \), we only need to compare \( \Delta|V^0(S) \) with \( \delta_{H}(S) \) once, which takes time at most \( O(|S| + |E_H(S, V(H))|) \). So the total time is \( O(m \log n) \) because \( S \) can be partitioned into \( O(\log n) \) layers of disjoint subsets by the second property of Theorem 2.2.

In round \( i \) of the main algorithm, computing \( f^i \) takes \( O(m) \) time. Using the fact that \( \mathcal{S} \) is a laminar family and \( \mathcal{S} \) contains \( O(n) \) sets, we can compute \( r_{S}^i \) for all \( S \in \mathcal{S} \) in \( O(n) \) time, and so we can compute the weights \( w_{S_o}^{i+1} \) for all \( S \in \mathcal{S}, o \in \{+, -\} \) in \( O(n) \). The most technical step is Step 2 whose implementation details is shown at the end of the section.

Lemma 3.9. The “deletion set” \( D^i \subseteq V(H) \setminus t \) from Step 2 can be computed in \( O(m + n \log n) \) time.

In total, the running time is \( \tilde{O}(m) + T \cdot O(m + n \log n) \) time. Recall that \( m = |E(H)| = O(|E(G\{U\})|) \). So we conclude the running time analysis:

Lemma 3.10. The total running time of the algorithm for Theorem 3.1 is at most \( \tilde{O}(|E(G\{U\})|/\epsilon^2) \).
3.4 Proof of Lemma 3.9

In this section, we show how to construction $D^i \subseteq V(H) \setminus t$ where

\[
\begin{align*}
&\text{if } D^i \neq \emptyset, \text{ then } \Delta|_{V_i-1}(D^i) > \delta_H(D^i) \\
&\langle \phi^i, \Delta|_{V_i} \rangle = \langle \phi^i, \Delta|_{V_i-1} \rangle \leq \langle \phi^i, Bf^i \rangle.
\end{align*}
\]

(1)

(2)

If $\langle \phi^i, \Delta|_{V_i-1} \rangle \leq \langle \phi^i, Bf^i \rangle$, then we simply set $D^i = \emptyset$, which trivially fulfills both conditions. For the remainder of the proof, we assume that $\langle \phi^i, \Delta|_{V_i-1} \rangle > \langle \phi^i, Bf^i \rangle$.

For real number $x$, define $V_{>x} = \{ v \in V(H) : \phi^i_v > x \}$. Fix some large number $M > \max_{v \in N\{U\}} |\phi^i_v|$.

We first prove the chain of relations

\[
\int_{x=-M}^{x=M} \Delta|_{V_i-1}(V_{>x})dx = \langle \phi^i, \Delta|_{V_i-1} \rangle > \langle \phi^i, Bf^i \rangle = \int_{x=-M}^{x=M} \delta_H(V_{>x})dx.
\]

(3)

We start with

\[
\begin{align*}
\int_{x=-M}^{x=M} \Delta|_{V_i-1}(V_{>x})dx &= \int_{x=-M}^{x=M} \left( \sum_{v \in V(H)} \Delta|_{V_i-1}(v) \cdot 1\{ \phi^i_v > x \} \right) dx \\
&= \sum_{v \in V(H)} \Delta|_{V_i-1}(v) \int_{x=-M}^{x=M} 1\{ \phi^i_v > x \} dx \\
&= \sum_{v \in V(H)} \Delta|_{V_i-1}(v)(\phi^i_v - (-M)).
\end{align*}
\]

Since $\sum_{v \in V(H)} \Delta|_{V_i-1}(v) = 0$ by construction, this is equal to

\[
\sum_{v \in V(H)} \Delta|_{V_i-1}(v) \phi^i_v = \langle \phi^i, \Delta|_{V_i-1} \rangle.
\]

By definition of the flow $f^i$,

\[
\langle \phi^i, Bf^i \rangle = \sum_{(u,v) \in E(H)} c_H(u,v)|\phi^i_u - \phi^i_v|
\]

\[
= \sum_{(u,v) \in E(H)} c_H(u,v) \int_{x=-M}^{x=M} 1\{ (u,v) \in \partial_H(V_{>x}) \} dx
\]

\[
= \int_{x=-M}^{x=M} \sum_{(u,v) \in E(H)} c_H(u,v)1\{ (u,v) \in \partial_H(V_{>x}) \} dx
\]

\[
= \int_{x=-M}^{x=M} \delta_H(V_{>x}) dx.
\]

Together with the assumption $\langle \phi^i, \Delta|_{V_i-1} \rangle > \langle \phi^i, Bf^i \rangle$, we obtain (3).

Let $x^*$ be the largest value such that

\[
\int_{x=-M}^{x^*} \Delta|_{V_i-1}(V_{>x})dx = \int_{x=-M}^{x^*} \delta_H(V_{>x})dx,
\]

13
which must exist since \( x^* = -M \) works. Next, we claim that we must have
\[
\Delta |_{V_{i-1}}(V_{> x^*}) > \delta_H(V_{> x^*}). \tag{4}
\]
Otherwise, for small enough \( \epsilon > 0 \) we would have \( \int_{x=-M}^{x^*+\epsilon} \Delta |_{V_{i-1}}(V_{> x}) dx \leq \int_{x=-M}^{x^*+\epsilon} \delta_H(V_{> x}) dx \), and since \( \int_{x=-M}^{M} \Delta |_{V_{i-1}}(V_{> x}) dx > \int_{x=-M}^{M} \delta_H(V_{> x}) dx \), there is another choice of \( x^* \) between \( x^* + \epsilon \) and \( M \) that achieves equality, a contradiction.

We now claim that \( t \notin V_{> x^*} \). Otherwise, since \( \Delta |_{V_{i-1}}(V(H)) = 0 \) and \( \Delta |_{V_{i-1}}(t) \) is the only negative entry, we would have \( \Delta |_{V_{i-1}}(V_{> x^*}) \leq 0 \) which would violate (4). Since \( t \notin V_{> x^*} \) and \( \phi^i_t = 0 \), we conclude that \( x^* \geq 0 \).

Let \( \bar{\phi}^i = \min\{\phi^i, x^*\} \) as \( \phi^i \) truncated to a maximum of \( x^* \). Then, similar to (3), we obtain
\[
\langle \bar{\phi}^i, \Delta |_{V_{i-1}} \rangle = \int_{x=-M}^{x^*} \Delta |_{V_{i-1}}(V_{> x}) dx = \int_{x=-M}^{x^*} \delta_H(V_{> x}) dx = \langle \bar{\phi}^i, Bf^i \rangle. \tag{5}
\]
Define our deletion set as \( D^i \triangleq V_{> x^*} \), so \( t \notin D^i \) and Equation (1) follows from (4). We now prove the chain of relations
\[
\langle \phi^i, \Delta |_{V_{i-1} \setminus D^i} \rangle = \langle \bar{\phi}^i, \Delta |_{V_{i-1} \setminus D^i} \rangle \leq \langle \bar{\phi}^i, \Delta |_{V_{i-1}} \rangle = \langle \bar{\phi}^i, Bf^i \rangle \leq \langle \phi^i, Bf^i \rangle,
\]
which would fulfill Equation (2). For the first relation, if \( \phi^i_t \neq \bar{\phi}^i_t \) then \( v \in D^i \), which means that \( \Delta |_{V_{i-1} \setminus D^i}(v) = 0 \). For the second relation, we use \( \bar{\phi}^i_t = \phi^i_t = 0 \) to obtain
\[
\langle \bar{\phi}^i, \Delta |_{V_{i-1} \setminus D^i} \rangle = \sum_{v \in V(H) \setminus t} \bar{\phi}^i_t(v) \Delta |_{V_{i-1} \setminus D^i}(v) = \sum_{v \in V(H) \setminus t} \bar{\phi}^i_t(v) \Delta |_{V_{i-1}}(v) - x^* \Delta |_{V_{i-1}}(D^i)
\]
\[
= \langle \bar{\phi}^i, \Delta |_{V_{i-1}} \rangle - x^* \Delta |_{V_{i-1}}(D^i)
\]
which is at most \( \langle \bar{\phi}^i, \Delta |_{V_{i-1}} \rangle \) since \( x^* \geq 0 \). The third relation follows from (5). For the last relation, we have
\[
\langle \bar{\phi}^i, Bf^i \rangle = \sum_{(u, v) \in E(H)} c_H(u, v) |\bar{\phi}^i_u - \bar{\phi}^i_v| \leq \sum_{(u, v) \in E(H)} c_H(u, v) |\phi^i_u - \phi^i_v| = \langle \phi^i, Bf^i \rangle.
\]
This concludes Equation (2).

Finally, we claim the running time \( O(m + n \log n) \). The only nontrivial step in the algorithm is computing \( x^* \). We first sort the values \( \phi^i_v \) in \( O(n \log n) \) time. Then, by sweeping through the sorted list, we can compute \( \Delta |_{V_{i-1}}(V_{> x}) - \delta_H(V_{> x}) \) for all \( x \in \{\phi^i_v : v \in V(H)\} \) in \( O(m) \) time. The function \( \Delta |_{V_{i-1}}(V_{> x}) - \delta_H(V_{> x}) \) is linear between consecutive values of \( \phi^i_v \), so we can locate the largest value \( x^* \) for which the function is 0.

### 4 From Almost Fair Cuts to Fair Cuts

In this section, we prove Theorem 1.5 using the ALMOSTFAIR subroutine.
Recall that we assume \( C = \text{poly}(n) \). We also assume \( \alpha \geq \frac{1}{\text{poly}(n)} \), otherwise we could solve the problem using exact max flow algorithms.

Our algorithm runs in iterations where in iteration \( j \) we compute \((S^j, T^j, k^j, \text{def}^j)\) where \((S^j, T^j)\) is an \((s, t)\)-cut where \( s \in S^j \) and \( t \in T^j \), \( k^j \in \mathbb{Z}_{\geq 0} \), and \( \text{def}^j \in \mathbb{R}_{\geq 0} \) represents an upper bound of the deficit which will be explained in the analysis. Define \( \beta = \Theta(\alpha/\log n) \) and \( \epsilon = \beta/16 \). Initially, \((S^0, T^0)\) is an arbitrary \((s, t)\)-cut, \( \text{def}^0 = \delta(G(S^0, T^0)) \), and \( k^0 = 0 \).

While \( \text{def}^j > \beta c_{\text{min}} \), do the following starting from \( j = 0, 1, 2, \ldots \)

1. Compute

\[
(P^j_s, S^j \setminus P^j_s) = \text{AlmostFair}(G, S^j, s, \epsilon, (k^j + 1)\beta), \quad \text{and} \quad (P^j_t, S^j \setminus P^j_t) = \text{AlmostFair}(G, T^j, t, \epsilon, (k^j + 1)\beta)
\]

by calling Theorem 3.1.

2. If \( \max\{\delta_G(P^j_s, T^j), \delta_G(P^j_t, S^j)\} \leq \text{def}^j / 40 \), then we update

\[
k^{j+1} = k^j + 1, \quad \text{and} \quad \text{def}^{j+1} = \text{def}^j / 2.
\]

Then, we set \( T^{j+1} = T^j \setminus P^j_t \) and \( S^{j+1} = V \setminus T^{j+1} \).

3. Else, if \( \max\{\delta_G(P^j_s, T^j), \delta_G(P^j_t, S^j)\} > \text{def}^j / 40 \), then we update

\[
k^{j+1} = k^j, \quad \text{and} \quad \text{def}^{j+1} = (1 - \beta/80)\text{def}^j
\]

If \( \delta_G(P^j_s, T^j) > \text{def}^j / 40 \), then, we set \( S^{j+1} = S^j \setminus P^j_s \) (and \( T^{j+1} = V \setminus S^{j+1} \)). Otherwise, we set \( T^{j+1} = T^j \setminus P^j_t \) (and \( S^{j+1} = V \setminus T^{j+1} \)).

After the while loop, we return \((S^j, T^j)\) as a \((1 + \alpha)\)-fair \((s, t)\)-cut. As \( \text{def}^0 \leq c(E) \) we have that \( \text{def}^j \leq (1 - \beta/80)c(E) \) for all \( j \). So there are at most \( O(\log(C/\beta)/\beta) \) iterations before \( \text{def}^j < \beta c_{\text{min}} \). Therefore, the algorithm takes \( O(\log(C/\beta)/\beta) \times \tilde{O}(m/\epsilon^2) = \tilde{O}(m/\epsilon^3) \) total time by Theorem 3.1. It remains to show the correctness of the algorithm.

4.2 Analysis

For convenience, whenever we refer to an edge \((a, b) \in E(A, B)\), we mean \( a \in A \) and \( b \in B \). Only for the analysis, we construct a feasible flow \( f^j \) in \( G \) on each iteration \( j \), and ensure that \( f^j \) satisfies the following two properties:

\footnote{We could also symmetrically set \( S^{j+1} = S^j \setminus P^j_s \) and \( T^{j+1} = V \setminus S^{j+1} \). This choice is arbitrary.}
1. Define the deficit of flow $f^j$ as $\text{def}^j(f^j) = \sum_{(u,v) \in E(S^j,T^j)} \max\{0, (1 - k^j \beta)c(u,v) - f^j(u,v)\}$. We maintain an invariant that $\text{def}^j(f^j) \leq \text{def}^j$.

2. For all $R \subseteq V \setminus \{s,t\}$, we require that $|f^j(R)| \leq \epsilon \delta_G(R)$. Equivalently, $f^j$ $\epsilon$-satisfies an $(s,t)$-source function in $G$.

In words, each cut edge $(u,v) \in E(S^j,T^j)$ contributes to the deficit of flow $f^j$ when the flow in $f^j$ from $u$ to $v$ is less than $(1 - k^j \beta)$-fraction of its capacity. With our definition of deficit in Property 1, we have that the cut is fair whenever the deficit is very small:

**Proposition 4.1.** If $\text{def}^j < \beta c_{\text{min}}$, then $(S^j,T^j)$ is a $(1 + \alpha)$-fair $(s,t)$-cut.

**Proof.** First we claim that $k^j = O(\log n)$. This is because everytime $k^j$ increments, $\text{def}$ is halved. So at the end of the algorithm, we have $\frac{\beta c_{\text{min}}}{2} < \text{def}^j < c(E)/2^{k^j}$, which implies $k^j = O(\log(C/\beta)) = O(\log n)$. Now, by the assumption and Property 1, for all $(u,v) \in E(S^j,T^j)$, we have $1 - k^j \beta)c(u,v) - f^j(u,v) < \beta \cdot c_{\text{min}}$ and so

$$f^j(u,v) \geq (1 - (k^j + 1)\beta)c(u,v) \geq \frac{1}{1 + \alpha/2}c(u,v)$$

where the last inequality is because $k^j = O(\log n)$ and we can set the constant in $\beta = \Theta(\alpha/\log n)$ to be small enough. Since $f^j$ $\epsilon$-satisfies an $(s,t)$-source function, by the observation below Fact 2.1, there exists $f_{\text{aug}}$ with congestion $\epsilon$ such that $f^* = f^j + f_{\text{aug}}$ is an $(s,t)$-flow. Now, we have that for all $(u,v) \in E(S^j,T^j)$,

$$f^*(u,v) \geq f^j(u,v) - \epsilon c(u,v) \geq \frac{1}{1 + \alpha}c(u,v)$$

because $\epsilon = \beta/16 = \Theta(\alpha/\log n)$ and the constant in it is small enough. Therefore, $f^*$ certifies that $(S^j,T^j)$ is a $(1 + \alpha)$-fair $(s,t)$-cut. \qed

Initially, we set $f^0$ as the zero flow, which satisfies both properties since $\text{def}^0 = \delta_G(S^0,T^0)$. Property 2 will help us show the following inductive step, which would conclude the correctness of Theorem 1.5.

**Lemma 4.2.** Suppose there exists a feasible flow $f^j$ satisfying Properties 1 and 2 for $j$. Then, we can construct a feasible flow $f^{j+1}$ satisfying Properties 1 and 2 for $j + 1$.

We analyze the two cases based on $\max\{\delta_G(P^j_s,T^j), \delta_G(P^j_t,S^j)\}$ in the subsections below.

**Case 1:** $\max\{\delta_G(P^j_s,T^j), \delta_G(P^j_t,S^j)\} \leq \text{def}^j/40$

Let $S'^j = S^j \setminus P^j_s$. By the guarantees of ALMOSTFAIR($G$, $S^j$, $s$, $\epsilon$, $(k^j + 1)\beta$), let $\Delta_s$ be the $S'^j$-boundary source function satisfied by a flow $f_s$ in $G(S'^j)$ with congestion $(1 + \epsilon)$. As $k^{j+1} = k^j + 1$ in this case, by Theorem 3.1, we have $f_s(v) = \Delta_s(v) = (1 - k^{j+1} \beta)\deg_G(S^j)(v)$ for all old boundary vertices $v \in N(S'^j) \cap N(S^j)$. Let $T'^j, \Delta_t, f_t$ be defined symmetrically. From $f_s$ and $f_t$, we will construct a new flow $f^{j+1}$ in three steps.
Step 1: Concatenate. Get $\hat{f}$. Consider the “concatenation” of $f_s$ and $f_t$, denoted by $f_{st}$, where we reverse the direction of $f_s$ so that the flow is sent out of $s$. The concatenated flow $f_{st}$ is on the graph $G \{ S^{ij} \} \cup G \{ T^{ij} \}$ where the two graphs share $N(S^{ij}) \cap N(T^{ij})$ as common boundary vertices. Now, we want to define a flow $\hat{f}$ on $G$ that corresponds to $f_{st}$ in a natural way.

For each edge $e \in E(G \{ S^{ij} \} \cup E(G \{ T^{ij} \})$ in the “interior” of $S^{ij}$ or $T^{ij}$, we set $\hat{f}(e) = f_{st}(e)$. For each a common boundary vertex $x_e \in N(S^{ij}) \cap N(T^{ij})$ where $e = (u, v) \in E(S^{ij}, T^{ij})$, we have $f_{st}(u, x_e) = f_{st}(x_e, v) = (1-k^j+1) \beta c(e)$ and so we set $\hat{f}(e) = (1-k^j+1) \beta c(e)$. For each new boundary vertex $x_e \in (N(S^{ij}) \setminus N(S^{ij})) \cup (N(T^{ij}) \setminus N(T^{ij}))$ where $e = (u, v) \in E(S^{ij}, P^j_s) \cup E(T^{ij}, P^j_t)$, we set $\hat{f}(e) = f_{st}(u, x_e)$. For each edge in the “interior” of $P^j_s$ or $P^j_t$, we set $\hat{f}(e) = 0$. By construction, $\hat{f}$ satisfies some a source function $\Delta$ where $\Delta(v) = 0$ for $v \notin \{s, t\} \cup V(P^j_s) \cup V(P^j_t)$.

Step 2: Remove Flow Paths Through New Boundaries. Get $\hat{f}'$. Take a path decomposition of $\hat{f}$ in $G$, and then remove all paths starting or ending at vertices in $V(P^j_s) \cup V(P^j_t)$; let the resulting flow be $\hat{f}'$, which satisfies some source function that is only nonzero at $s, t$. That is, $\hat{f}'$ is an $(s, t)$-flow. Note that $\hat{f}'$ still has congestion at most $(1 + \epsilon)$.

Step 3: Truncate to a Feasible Flow. Get $f^{j+1}$. Finally, for any edges congested by more than 1 in $\hat{f}'$, lower the flow along that edge to congestion exactly 1. We define $f^{j+1}$ as the resulting flow.

Proving Properties of $f^{j+1}$. Since $f^{j+1}$ is obtained from the $(s, t)$-flow $\hat{f}'$ by removing a flow of congestion at most $\epsilon$, Property 2 is satisfied. Now, we prove Property 1. We write the deficit of $f^{j+1}$ as follows

$$\text{def}^{j+1}(f^{j+1}) = \sum_{e \in E(S^{j+1}, T^{j+1})} \max\{0, (1-k^{j+1} \beta c(e) - f^{j+1}(e))\}$$

$$\leq \sum_{e \in E(S^{j+1}, T^{j+1})} \left( \max\{0, (1-k^{j+1} \beta c(e) - \hat{f}(e))\} + |\hat{f}(e) - \hat{f}'(e)| + |\hat{f}'(e) - f^{j+1}(e)| \right)$$

$$= \sum_{e \in E(S^{j+1}, T^{j+1})} \max\{0, (1-k^{j+1} \beta c(e) - \hat{f}(e))\} + \sum_{e \in E(S^{j+1}, T^{j+1})} |\hat{f}(e) - \hat{f}'(e)| + \sum_{e \in E(S^{j+1}, T^{j+1})} |\hat{f}'(e) - f^{j+1}(e)|$$

Now, we bound each of the three terms above. We use the fact $T^{j+1} = T^{ij}$ and $S^{j+1} = S^{ij} \cup P^j_s \cup P^j_t$.

For the first term, we consider the concatenated flow $\tilde{f}$. We have $\tilde{f}(e) = (1-k^{j+1} \beta c(e)$ for each old boundary edge $e \in E(S^{ij}, T^{ij})$. So, the first term is bounded by

$$\sum_{e \in E(S^{ij} \cup P^j_s \cup P^j_t, T^{ij})} \max\{0, (1-k^{j+1} \beta c(e) - \tilde{f}(e))\} \leq \sum_{e \in E(P^j_s \cup P^j_t, T^{ij})} (1-k^{j+1} \beta c(e) - \tilde{f}(e))$$

$$\leq ((1-k^{j+1} \beta) + (1 + \epsilon)) \cdot \delta(P^j_s \cup P^j_t, T^{ij})$$

$$\leq (2 + \epsilon) \cdot \delta(P^j_s \cup P^j_t, T^{ij})$$

where the second inequality is because $\tilde{f}$ has $(1 + \epsilon)$ congestion.
For the second term, consider the flow $\hat{f}'$ obtained by the flow-path removal. We rewrite the second term as

$$\sum_{e \in E(P_s^j \cup P_t^j, T^j)} |\hat{f}(e) - \hat{f}'(e)| + \sum_{e \in E(S^j, T^j)} |\hat{f}(e) - \hat{f}'(e)|.$$  

Trivially, we have

$$\sum_{e \in E(P_s^j \cup P_t^j, T^j)} |\hat{f}(e) - \hat{f}'(e)| \leq (1 + \epsilon)\delta(P_s^j \cup P_t^j, T^j)$$

because the flow has congestion $(1 + \epsilon)$. Now, we claim that

$$\sum_{e \in E(S^j, T^j)} |\hat{f}(e) - \hat{f}'(e)| \leq (1 + \epsilon)\delta(P_s^j \cup P_t^j, S^j \cup T^j).$$

To see this, consider each flow-path $P$ removed from $\hat{f}$ to obtain $\hat{f}'$. Observe that $P$ cannot cross directly from $T^j$ to $S^j$ because, for every edge $e \in E(S^j, T^j)$, the flow goes from $S^j$ to $T^j$ as $f(e) = (1 - kj^{1+\beta})c(e)$. Thus, between any two consecutive times that $P$ crosses from $S^j$ to $T^j$, $P$ must have crossed from $T^j$ to $P_s^j \cup P_t^j$. Also, note that the first edge of $P$ is from $E(P_s^j \cup P_t^j, S^j \cup T^j)$. Therefore, we can charge the flow changes in edges of $E(S^j, T^j)$ to the changes in edges of $E(P_s^j \cup P_t^j, S^j \cup T^j)$. So $\sum_{e \in E(S^j, T^j)} |\hat{f}(e) - \hat{f}'(e)| \leq \sum_{e \in E(P_s^j \cup P_t^j, S^j \cup T^j)} |\hat{f}(e) - \hat{f}'(e)|$ as claimed.

Finally, for the third term, we consider the truncated flow $f^{j+1}$ with congestion at most 1 on all edges. Again, we have $\hat{f}'(e) - f^{j+1}(e) = 0$ for all $e \in E(S^j, T^j)$ because 0 $\leq \hat{f}'(e) \leq (1 - kj^{1+\beta})c(e)$. In particular, the congestion on $e$ was already less than 1. Also, we have $|\hat{f}'(e) - f^{j+1}(e)| \leq \epsilon_{c}(e)$ for any edges $e$ as $\hat{f}'$ has congestion $1 + \epsilon$. Hence, we have

$$\sum_{e \in E(S^{j+1}, T^{j+1})} |\hat{f}'(e) - f^{j+1}(e)| \leq \sum_{e \in E(P_s^j \cup P_t^j, T^j)} \epsilon_{c}(e) = \epsilon_{d}(P_s^j \cup P_t^j, T^j).$$

From the above bounds, we obtain

$$\text{def}^{j+1}(f^{j+1}) \leq ((2 + \epsilon) + (1 + \epsilon) + (1 + \epsilon) + \epsilon)\delta(P_s^j \cup P_t^j, S^j \cup T^j).$$

Now, write $\delta(P_s^j \cup P_t^j, S^j \cup T^j) = \delta(P_s^j, S^j) + \delta(P_t^j, S^j) + \delta(P_s^j, T^j) + \delta(P_t^j, T^j)$. Note that $\delta(P_s^j, T^j) \leq \delta(P_t^j, S^j)$ and $\delta(P_s^j, S^j) \leq \delta(P_t^j, T^j)$ by the guarantee of ALMOSTFAIR. Trivially, we also have $\delta(P_t^j, S^j) \leq \delta(P_s^j, S^j)$ and $\delta(P_s^j, T^j) \leq \delta(P_t^j, T^j)$. But we have $\delta(P_s^j, S^j), \delta(P_t^j, T^j) \leq \text{def}^j/40$ by the assumption of this case. So we have, as $\epsilon \leq 1/4$,

$$\text{def}^{j+1}(f^{j+1}) \leq (4 + 4\epsilon) \cdot 4 \cdot \frac{\text{def}^j}{40} \leq \frac{\text{def}^j}{2} = \text{def}^{j+1}$$

fulfilling Property 1.

**Case 2:** $\max\{\delta_G(P_s^j, T^j), \delta_G(P_t^j, S^j)\} > \text{def}^j/40$

In this case, we set $f^{j+1}$ as the same old flow $f^j$. So Property 2 of $f^{j+1}$ trivially continues to hold. For Property 1, assume without loss of generality the case $\delta_G(P_s^j, S^j) > \text{def}^j/40$, so $T^{j+1} = T^j \setminus P_t^j$. 

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(The case \(\delta_G(P^j_k, T^j) > \text{def}^j / 40\) is symmetric, so we omit it.) As \(f^{j+1} = f^j\) and \(k^{j+1} = k^j\), we have

\[
\text{def}^{j+1}(f^{j+1}) = \sum_{e \in E(S^{j+1}, T^{j+1})} \max\{0, (1 - k^j \beta) c(e) - f^j(e)\}
\]

\[
= \text{def}^j(f^j) - \sum_{e \in E(S^j, P^j)} \max\{0, (1 - k^j \beta) c(e) - f^j(e)\} + \sum_{e \in E(P^j_k, T^{j+1})} \max\{0, (1 - k^j \beta) c(e) - f^j(e)\}.
\]

For the second term (without the minus sign), we can lower bound it as

\[
\geq \sum_{e \in E(S^j, P^j)} (1 - k^j \beta) c(e) - f^j(e) = (1 - k^j \beta) \delta(S^j, P^j) - f^j(S^j, P^j).
\]

For the third term, we can upper bound it as

\[
\leq \sum_{e \in E(P^j_k, T^{j+1})} c(e) - f^j(e) = \delta(P^j_k, T^{j+1}) - f^j(P^j_k, T^{j+1}).
\]

where the first inequality is because \(0 \leq c(e) - f^j(e)\) as \(f^j\) is feasible. Putting these together, we have

\[
\text{def}^{j+1}(f^{j+1}) \leq \text{def}^j(f^j) - \left( (1 - k^j \beta) \delta(S^j, P^j) - \delta(P^j_k, T^{j+1}) \right) + \left( f^j(S^j, P^j) - f^j(P^j_k, T^{j+1}) \right).
\]

That is, the increase in deficit can be upper bounded as follows. It will decrease proportional to \((1 - k^j \beta) \delta(S^j, P^j) - \delta(P^j_k, T^{j+1})\) which is related cut size. It may increase proportional to \(f^j(S^j, P^j) - f^j(P^j_k, T^{j+1})\) which is related to flow.

For the decrease caused by cut size, \textsc{AlmostFair}(\(G, T^j, t, \epsilon, (k^j+1)\beta\)) guarantees that \(\delta(P^j_k, T^{j+1}) \leq (1 - (k^j+1)\beta) \delta(S^j, P^j)\). So the deficit must decrease by at least \((1 - k^j \beta) - (1 - (k^j + 1)\beta)\) \(\delta(S^j, P^j) \geq \beta \delta(S^j, P^j)\). For the increase caused by flow, we have that \(f^j(S^j, P^j) - f^j(P^j_k, T^{j+1}) = f^j(S^j, P^j) + f^j(T^{j+1}, P^j) = -f^j(P^j_k)\) is exactly the net flow of \(f^j\) into \(P^j_k\). As \(|f^j(P^j_k)| \leq \epsilon \delta_G(P^j_k)\) by Property 2 on \(P^j_k\), we now have

\[
\text{def}^{j+1}(f^{j+1}) \leq \text{def}^j(f^j) - \beta \delta(S^j, P^j) + \epsilon \delta_G(P^j_k).
\]

Observe that \(\delta_G(P^j_k) = \delta_G(S^j, P^j) + \delta_G(P^j_k, T^{j+1})\) but \(\delta(P^j_k, T^{j+1}) \leq \delta(S^j, P^j)\) by \textsc{AlmostFair} again. So \(\epsilon \delta_G(P^j_k) \leq 2 \epsilon \delta_G(S^j, P^j) \leq \frac{\beta}{2} \delta_G(S^j, P^j)\) because \(\epsilon \leq \beta / 4\). Therefore,

\[
\text{def}^{j+1}(f^{j+1}) \leq \text{def}^j(f^j) - \frac{\beta}{2} \delta(S^j, P^j) \leq (1 - \frac{\beta}{80}) \text{def}^j(f^j) = \text{def}^{j+1}
\]

because \(\delta_G(S^j, P^j) > \text{def}^j / 40\) by our initial assumption.

### 5 Approximate Isolating Cuts and Steiner Cut

The focus of this section is to compute approximate isolating cuts and show its application in computing the Steiner mincut. The approximate minimum isolating cuts are defined below.
**Definition 5.1.** Given an undirected graph $G = (V, E)$ with non-negative edge weights and a set of terminals $T \subseteq V$, a cut $\emptyset \subseteq S \subseteq V$ is said to be an isolating cut for a terminal $t \in T$ if $T \cap S = \{t\}$. A minimum isolating cut for $t$ is a minimum value cut among all the isolating cuts for $t$. Similarly, a $(1+\epsilon)$-approximate minimum isolating cut for $t$ is an isolating cut for $t$ whose value is at most $(1+\epsilon)$ times that of a minimum isolating cut for $t$.

Below is our main theorem. We state our result in general before plugging in the current best runtime from Theorem 1.5.

**Theorem 5.2.** Fix any $\epsilon < 1$. Given an undirected graph $G = (V, E)$ on $m$ edges and $n$ vertices with non-negative edge weights and a set of terminals $T \subseteq V$, there is an algorithm that outputs a $(1+\epsilon)$-approximate minimum isolating cut for every terminal $t \in T$ in $O(m)$ time plus a set of $(1+\gamma)$-fair $(s,t)$-cut calls on undirected graphs that collectively contain $O(m\log |T|)$ edges and $O(n \log |T|)$ vertices, where $\gamma = \frac{\epsilon}{4(1+\epsilon)}$. Moreover, the sets $S_t$ are disjoint, and for each $t \in T$, the set $S_t$ is a $(1+\gamma)$-fair $(\{t\}, T \setminus \{t\})$-cut.

Using Theorem 1.5 to compute $(1+\gamma)$-fair $(s,t)$-cuts, our algorithm for $(1+\epsilon)$-approximate minimum isolating cuts runs in $O(m/\epsilon^3)$ time.

**Algorithm 1** $(1+\epsilon)$-approximate Minimum Isolating Cuts Algorithm on terminal set $T$

1: Arbitrarily order the terminals in $T = \{t_1, t_2, \ldots, t_{|T|}\}$
2: **Phase 1:**
3: for $i = 1$ to $\lfloor \log |T| \rfloor$ do
4: $X_i \leftarrow \{v_j \in V : j^{th}$ bit in $i$ is $1\}$
5: $Y_i \leftarrow \{v_j \in V : j^{th}$ bit in $i$ is $0\}$
6: Use Theorem 1.5 to find a $(1+\gamma)$-fair $(X_i, Y_i)$-cut $S_i$
7: end for
8: **Phase 2:**
9: for every terminal $t \in T$ do
10: $S_t \leftarrow \cap_i S_t^i$, where $S_t^i = S_t$ if $t \in S_i$ and $S_t^i = V \setminus S_i$ if $t \notin S_i$. In other words, $S_t$ is the connected component containing $t$ in the graph where we delete all the edges in cuts $\delta S_i$ for all $i$.
11: $G_t$ is obtained from $G$ by contracting all vertices in $V \setminus S_t$ into a single vertex $\tilde{s}_t$. {To implement this step efficiently, we construct a new graph that is identical to $G_t$ instead of contracting $G$.}
12: Find a $(1+\beta)$-approximate minimum $(t, \tilde{s}_t)$-cut in graph $G_t$; call this cut $C_t$
13: end for
14: Return the cuts $\{C_t : t \in T\}$

To establish Theorem 5.2, we describe Algorithm 1 for finding $(1+\epsilon)$-approximate isolating cuts. First, we establish correctness of the algorithm by showing that the cut $C_t$ returned by Algorithm 1 for a terminal $t \in T$ is indeed a $(1+\epsilon)$-approximate minimum isolating cut for $T$. The following claim establishes an approximate version of the standard uncrossing property of minimum cuts, and is crucial for the correctness of our algorithm.

**Lemma 5.3.** Let $A$ be a $(1+\alpha)$-approximate minimum isolating cut for some terminal $t$ and let $B$ be a $(1+\gamma)$-fair $(X,Y)$-cut where $X \cup Y = T$, $t \in X$, and $X \subseteq B$. Then, $A \cap B$ is a $(1+\alpha)(1+\gamma)$-approximate minimum isolating cut for $t$. 
Proof. First, note that since $A$ is an isolating cut for $t$ and $t \in X, X \subseteq B$, it follows that $A \setminus B$ does not contain any terminal and $A \cap B$ contains a single terminal $t$. Now, consider the two cuts $A$ and $A \cap B$. We can write

$$E(A, V \setminus A) = E(A \cap B, V \setminus (A \cup B)) \cup E(A \cap B, B \setminus A) \cup E(A \setminus B, V \setminus A)$$

$$E(A \cap B, V \setminus (A \cup B)) = E(A \cap B, V \setminus (A \cup B)) \cup E(A \cap B, B \setminus A) \cup E(A \cap B, A \setminus B).$$

Since the first two sets are identical, we only need to compare the third sets $E(A \setminus B, V \setminus A)$ and $E(A \cap B, A \setminus B)$. Since $B$ is a $(1 + \gamma)$-fair $(X, Y)$-cut, there is a flow that saturates all the edges in $E(B, V \setminus B)$ and terminates in the vertices of set $Y$ while congesting each edge at most to a factor of $1 + \gamma$. Now, consider the flow on the edges in $E(A \cap B, A \setminus B)$. Since $Y \cap (A \setminus B) = \emptyset$, it follows that this flow must exit the set $A \setminus B$ on the edges in $E(A \setminus B, V \setminus (A \cup B))$. Thus,

$$\delta(A \cap B, A \setminus B) \leq (1 + \gamma) \cdot \delta(A \setminus B, V \setminus (A \cup B)) \leq (1 + \gamma) \cdot \delta(A \setminus B, V \setminus A).$$

It follows that $\delta(A \cap B) \leq (1 + \gamma) \cdot \delta(A)$, which proves the lemma. \hfill \square

Lemma 5.4. For $\gamma = \frac{\epsilon}{4|\lg |T||}$ and $\beta = \frac{\epsilon}{4}$, the cut $C_t$ returned by Algorithm 1 is a $(1 + \epsilon)$-approximate minimum isolating cut for every $t \in T$.

Proof. Lemma 5.3 implies that in Algorithm 1, the minimum isolating cut of $t$ in graph $G_t$, i.e., the minimum $t - \bar{s}_t$ cut, is a $(1 + \gamma) \cdot |\lg |T||$-approximate minimum isolating cut of $t$ in the input graph $G$. Since $C_t$ is a $(1 + \beta)$-approximate minimum $t - \bar{s}_t$ cut, it follows that $C_t$ is a $(1 + \gamma) \cdot |\lg |T|| \cdot (1 + \beta)$-approximate minimum isolating cut of $t$ in the input graph $G$. Using the values of $\gamma$ and $\beta$, we have

$$\left(1 + \frac{\epsilon}{4|\lg |T||}\right)^{|\lg |T||} \cdot \left(1 + \frac{\epsilon}{4}\right) \leq e^{\epsilon/4} \cdot e^{\epsilon/4} = e^{\epsilon/2} \leq 1 + \epsilon$$

since $\epsilon < 1$. \hfill \square

For the $(1 + \beta)$-approximate mincut in Step 12, we can use Theorem 1.5 to compute a $(1 + \gamma)$-fair cut, which is also a $(1 + \beta)$-approximate mincut since $\gamma \leq \beta$. This also guarantees that the cut $C_t$ is a $(1 + \gamma)$-fair $(V \setminus C_t, t)$-cut. Finally, it is clear from the algorithm that all cuts $C_t$ are disjoint.

The runtime analysis is identical to that in [LP20], so we omit it for brevity.

5.1 $(1 + \epsilon)$-approximate Minimum Steiner Cut

As an immediate application of our isolating cut result, we can solve the Steiner cut problem below efficiently.

Definition 5.5. Given an undirected graph $G = (V, E)$ with non-negative edge weights and a set of terminals $T \subseteq V$, a minimum Steiner cut is a cut of minimum value among all cuts $\emptyset \subset S \subset V$ that satisfy $\emptyset \subset S \cap T \subset T$.

Using Theorem 5.2, we give the following algorithm for finding a $(1 + \epsilon)$-approximate minimum Steiner cut.

Theorem 5.6. Given an undirected graph $G = (V, E)$ on $m$ edges and $n$ vertices and with non-negative edge weights and a set of terminals $T \subseteq V$, Algorithm 2 computes a $(1 + \epsilon)$-minimum Steiner cut for $T$ with probability at least $1 - 1/n$ in $O(m)$ time.
Algorithm 2 \((1 + \epsilon)\)-approximate minimum Steiner cut Algorithm on terminal set \(T\)

\[
\text{for } i = 1 \text{ to } \lceil \log |T| \rceil \text{ do } \\
\quad \text{for } j = 1 \text{ to } \lceil \log 8/7 n \rceil \text{ do } \\
\qquad T_{ij} \text{ is drawn i.i.d. from } T \text{ where every vertex } t \in T \text{ appears in } T_{ij} \text{ with probability } 1/2^i \\
\qquad \text{ Use Theorem 5.2 to find isolating cuts } S_{ij} = \{S_t : t \in T_{ij}\} \text{ for the terminal set } T_{ij} \\
\quad \text{end for } \\
\text{end for } \\
\text{Return } \arg \min \{\delta(S) : S \in S_{ij}, i \in \lceil \log |T| \rceil, j \in \lceil \log 8/7 n \rceil\}
\]

\textbf{Proof.} Fix a minimum Steiner cut for the terminal set \(T\) and let \(S\) denote the side of this cut such that \(|T \cap S| \leq |T \setminus S|\). Let \(i \in \lceil \log |T| \rceil\) such that \(2^{i-1} \leq |S \cap T| < 2^i\). Then, \(T_{ij}\) contains exactly one vertex in \(T \cap S\) with probability

\[
\frac{|T \cap S| \cdot 2^i}{2^i} \cdot \left(1 - \frac{1}{2^i}\right)^{|T \cap S| - 1} \geq 2^{i-1} \cdot \left(1 - \frac{1}{2^i}\right) \geq \frac{1}{2} \cdot 1 \cdot 4 = \frac{1}{8}.
\]

This implies that the probability that there is no index \(j \in \lceil \log 8/7 n \rceil\) such that \(T_{ij}\) contains exactly one terminal in \(T \cap S\) is at most \(1/n\), thereby establishing the correctness of the algorithm.

The running time bound follows from Theorem 5.2.

\(\square\)

6 Approximate Gomory-Hu Tree Algorithm

The main result in this section is the near-linear time algorithm for computing an approximate Gomory-Hu tree. In fact, our algorithm can solve a more general problem called approximate Gomory-Hu Steiner tree defined below. (The definition is copied verbatim from [LP21].)

\textbf{Definition 6.1} (Approximate Gomory-Hu Steiner tree). Given a graph \(G = (V, E)\) and a set of terminals \(U \subseteq V\), the \((1 + \epsilon)\)-approximate Gomory-Hu Steiner tree is a weighted tree \(T\) on the vertices \(U\), together with a function \(f : V \to U\), such that

- For all \(s, t \in U\), consider the minimum-weight edge \((u, v)\) on the unique \(s - t\) path in \(T\).
  - Let \(U'\) be the vertices of the connected component of \(T - (u, v)\) containing \(s\). Then, the set \(f^{-1}(U') \subseteq V\) is a \((1 + \epsilon)\)-approximate \((s, t)\)-mincut, and its value is \(w_T(u, v)\).

Our main result is stated below. Recall that we assume that the ratio between the largest and lowest edge weights are \(\text{poly}(n)\).

\textbf{Theorem 6.2.} Let \(G\) be a weighted, undirected graph. There is a randomized algorithm that w.h.p., outputs a \((1 + \epsilon)\)-approximate Gomory-Hu Steiner tree in \(\tilde{O}(m \cdot \text{poly}(1/\epsilon))\) time.

The algorithm and analysis are similar to those in [LP21], except we replace (exact) minimum isolating cuts with an approximate version, which requires overcoming a few more technical issues. For completeness, we redo all the proofs. We also restate Theorem 5.2 below in the form we precisely need.

\textbf{Theorem 6.3.} Fix any \(\epsilon < 1\). Given an undirected graph \(G = (V, E)\) on \(m\) edges and \(n\) vertices with non-negative edge weights and a set of terminals \(T \subseteq V\), there is an algorithm that outputs a \((1 + \epsilon)\)-approximate minimum isolating cut \(S_t \subseteq V\) for every terminal \(t \in T\) in \(\tilde{O}(m/\epsilon^{O(1)})\) time. Moreover, the sets \(S_t\) are disjoint, and for each \(t \in T\), the set \(S_t\) is a \((1 + \gamma)\)-fair \((\{t\}, T \setminus \{t\})\)-cut.
6.1 Cut Threshold Step Algorithm

We begin with the following “Cut Threshold Step” subroutine from [LP21], described in Algorithm 3 below. Loosely speaking, the algorithm inputs a source vertex $s$ and a threshold $W$, and aims to find a large fraction of vertices whose mincut from $s$ is approximately at most $W$.

Algorithm 3 $(1 + \gamma)$-approximate “Cut Threshold Step” on inputs $(G, U, W, s)$

1. Initialize $D \leftarrow \emptyset$
2. for independent iteration $i \in \{0, 1, 2, \ldots, \lfloor \log |U| \rfloor \}$ do
3. $R^i \leftarrow$ sample of $U$ where each vertex in $U \setminus \{s\}$ is sampled independently with probability $1/2^i$, and $s$ is sampled with probability $1$
4. Compute $(1 + \frac{\gamma}{2 \lfloor \log |U| \rfloor})$-approximate minimum isolating cuts $\{S^i_v : v \in R^i\}$ on inputs $G$ and $R^i$ with the additional guarantees of Theorem 6.3 (for large enough constant $c > 0$)
5. Let $F^i$ be the family of sets $S^i_v$ satisfying $\delta S^i_v \leq (1 + \gamma)W$, and let $D^i \leftarrow \bigcup_{S^i_v \in F^i} S^i_v \cap U$
6. end for
7. Let $i_{\text{max}}$ be the index $i$ maximizing $|D^i|$
8. Return $D \leftarrow D^{i_{\text{max}}}$, $R \leftarrow R^{i_{\text{max}}}$, and $F \leftarrow F^{i_{\text{max}}}$

Lemma 6.4. For any $i$, each set $S^i_v$ added to $D^i$ satisfies $\lambda(s, v) \leq (1 + \gamma)W$.

Proof. For each $v \in D^i$, the corresponding set $S^i_v$ on line 5 contains $v$ and not $s$, so $\lambda(s, v) \leq \delta S^i_v \leq (1 + \gamma)W$. \hfill $\square$

Lemma 6.5. Let $D^*$ be all vertices $v \in U \setminus s$ for which $\lambda(s, v) \leq W$. Then, $\mathbb{E}[|D^*|] = \Omega(|D^*|/ \log |U|)$.

Proof. We will show that

$$
\mathbb{E} \left[ \sum_{i=0}^{\lfloor \log |U| \rfloor} |D^i| \right] \geq \Omega(|D^*|), \tag{6}
$$

which is sufficient, since the largest $D^i$ will have at least $1/(\lfloor \log |U| \rfloor + 1)$ fraction of the total size. Fix a vertex $v \in D^*$. For each $0 \leq j \leq \lfloor \log |U| \rfloor$, define $C^j_v \subseteq V$ as the $(s, v)$-cut of weight at most $(1 + \frac{\gamma}{2 \lfloor \log |U| \rfloor})jW$ that minimizes $|C^j_v \cap U|$, which must exist since $v \in D^*$. By construction, $|C^j_v \cap U|$ is decreasing in $j$.

We focus on a value $j^* (0 \leq j^* < \lfloor \log |U| \rfloor)$ satisfying $|C^{j^*+1}_v \cap U| \geq |C^{j^*}_v \cap U|/2$, which is guaranteed to exist. Consider sampling iteration $i = \lfloor \log |C^{j^*}_v \cap U| \rfloor$, where each vertex in $U \setminus \{s\}$ is sampled with probability $1/2^i$. With probability $\Omega(1/|C^{j^*}_v \cap U|)$, we have $C^{j^*}_v \cap R^i = \{v\}$, i.e., we sampled $v$ and nothing else in $C^{j^*}_v \cap U$. If this occurs, then $C^{j^*}_v$ is a valid isolating cut separating $v$ from $R^i \setminus \{v\}$. Since $S^i_v$ is a $(1 + \frac{\gamma}{2 \lfloor \log |U| \rfloor})$-approximate minimum isolating cut, we have

$$
\delta S^i_v \leq \left(1 + \frac{\gamma}{2 \lfloor \log |U| \rfloor}\right) \delta C^{j^*}_v \leq \left(1 + \frac{\gamma}{2 \lfloor \log |U| \rfloor}\right)^{j^*+1} W \leq \left(1 + \frac{\gamma}{2 \lfloor \log |U| \rfloor}\right)^{\lfloor \log |U| \rfloor} W \leq e^{\gamma/2} W \leq (1 + \gamma)W,
$$

so $S^i_v \cap U$ is added to $D^i$ on line 5. By definition of $C^{j^*+1}_v$, we have $|S^i_v \cap U| \geq |C^{j^*+1}_v \cap U|$, which is at least $|C^{j^*}_v \cap U|/2$ by our choice of $j^*$. In other words, if $C^{j^*}_v \cap R^i = \{v\}$, which occurs with probability $\Omega(1/|C^{j^*}_v \cap U|)$, then $v$ is “responsible” for adding at least $|C^{j^*}_v \cap U|/2$ vertices to $D^i$. 

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Thus, each vertex \( v \in D^* \) is responsible for adding \( \Omega(1) \) vertices in expectation to some \( D^i \), which increases \( E \left[ \sum_{i=0}^{\lceil \log |U| \rceil} |D^i| \right] \) by \( \Omega(1) \) in expectation. Finally, (6) follows by linearity of expectation over all \( v \in D^* \).

For our approximate Gomory-Hu tree algorithm, we actually need a bound on \( E[|D \cap D^*|] \), not \( E[|D|] \), since we want to remove \( D \) from \( U \) and claim that the size of the new \( D^* \) drops by a large enough factor. Unfortunately, it is possible that \( D \) is largely disjoint from \( D^* \), so a bound on \( E[|D|] \) does not directly translate to a bound on \( E[|D \cap D^*|] \). Therefore, we wrap Algorithm 3 into another routine that achieves a good bound on \( E[|D \cap D^*|] \). We actually prove the stronger guarantee that \( D^* \) can be any subset of all vertices \( v \in U \setminus s \) for which \( \lambda(s,v) \leq W \), which is needed in our Gomory-Hu tree algorithm.

Algorithm 4 \((1 + \gamma)\)-approximate Gomory-Hu Steiner tree “step” on inputs \((G,U_0,W_0,s)\)

```plaintext
Initialize \( U \leftarrow U_0 \\
for O(\log^3 n) \) sequential iterations do 
    for independent iteration \( j \in \{0,1,2,\ldots,\lceil \log |U| \rceil - 1 \} \) do 
        Call Algorithm 3 on parameter \( 2^{\lceil \log |U| \rceil} \) and inputs \((G,U,(1 + 2^{\lceil \log |U| \rceil}W_0,s)\) and let \((D_j,R_j,F_j)\) be the output 
    end for 
    Update \( U \leftarrow U \setminus \bigcup_j D_j \) for the values \( D_j \) computed on this sequential iteration 
end for 
Return an output \((D,R,F)\) selected uniformly at random out of the \( O(\log^3 n \log |U|) \) calls to Algorithm 3.
```

Lemma 6.6. Each set \( S \in F \) in the output \((D,R,F)\) of Algorithm 4 satisfies \( \delta S \leq (1 + \gamma)W_0 \).

Proof. By Lemma 6.4 applied to any \( j \in \{0,1,2,\ldots,\lceil \log |U| \rceil - 1 \} \), each set \( S \in F_j \) satisfies

\[
\delta S \leq \left(1 + 2^{\lceil \log |U| \rceil} \right) \cdot \left(1 + \frac{\gamma}{2^{\lceil \log |U| \rceil}} \right)^j W_0 \leq \left(1 + \frac{\gamma}{2^{\lceil \log |U| \rceil}} \right)^{\lceil \log |U| \rceil} W_0 \leq e^{\gamma/2}W_0 \leq (1 + \gamma)W_0.
\]

So the same holds for the randomly chosen output \((D,R,F)\).

Lemma 6.7. Let \( D^* \) be an arbitrary set of vertices \( v \in U \setminus s \) satisfying \( \lambda(s,v) \leq W_0 \). The output \((D,R,F)\) satisfies \( E[D \cap D^*] \geq \Omega(|D^*|/\log^4 n) \).

Proof. We claim that after \( O(\log^3 |U|) \) iterations of the main for loop, the set \( D^* \cap U \) becomes empty. This would mean that \( D^* \) is contained in the union of all \( O(\log^4 n) \) sets \( D_j \) computed over all iterations, so a random set \( D_j \) must contain a \( \Omega(1/\log^4 n) \) fraction of \( D^* \) in expectation. For the rest of the proof, we prove this claim.

For each \( 0 \leq \gamma \leq \lceil \log |U| \rceil \), let \( D^*_j \) be all vertices \( v \in U \setminus s \) for which \( \lambda(s,v) \leq (1 + 2^{\lceil \log |U| \rceil}W_0). By construction, \( D^* \subseteq D^*_0 \subseteq D^*_1 \subseteq \cdots \subseteq D^*_{\lceil \log |U| \rceil} \). We track the sets \( D^*_j \cap U \) throughout the algorithm. Consider the set \( U \) at the beginning of one of the \( O(\log^3 |U|) \) sequential iterations. We focus on a value \( j^* \) \((0 \leq j^* < \lceil \log |U| \rceil)\) satisfying \( |D^*_{j^*} \cap U| \geq |D^*_{j^*+1}||/2 \). Consider iteration \( j^* \) of the inner for loop. By Lemma 6.4, we have \( \lambda(s,v) \leq (1 + 2^{\lceil \log |U| \rceil} \cdot (1 + 2^{\lceil \log |U| \rceil}W_0 = (1 + 2^{\lceil \log |U| \rceil})^{j^*+1}W_0 \), so in particular, \( D^*_j \subseteq D^*_{j^*+1} \). By Lemma 6.5, we have
Since the product is at most $\prod_{j=1}^{\lfloor \log |U| \rfloor} D_j^*$, we delete $\bigcup_j D_j$ at the end of this sequential iteration, the size of $D_j^*$ drops by factor $(1 - \Omega(1/\log |U|))$ in expectation.

In other words, on each sequential iteration, there exists $j^*$ (satisfying $1 \leq j^* \leq \lfloor \log |U| \rfloor$) for which the size of $D_j^* \cap U$ decreases by factor $(1 - \Omega(1/\log |U|))$ in expectation. Since the other sets $D_j^*$ can never increase in size, the product $\prod_{j=1}^{\lfloor \log |U| \rfloor} D_j^* \cap U$ decreases by factor $(1 - \Omega(1/\log |U|))$ in expectation. Since the product is at most $|U|^{\Omega(n)} \leq 2^{O(n^2)}$ initially, it follows that after $O(n^3)$ sequential iterations, the product becomes zero w.h.p. Therefore, at the end of the algorithm, there exists $j$ (satisfying $1 \leq j^* \leq \lfloor \log |U| \rfloor$) with $D_j^* \cap U = \emptyset$. Since $D^* \subseteq D_j^*$, we also get $D^* \cap U = \emptyset$, which proves the claim. 

6.2 The Algorithm for Approximating Gomory-Hu Steiner Tree

We present our approximate Gomory-Hu tree algorithm in Algorithm 5. It uses Algorithm 4 as a subroutine. See Figure 6.2 for a visual guide to the algorithm. Once again, the algorithm and analysis closely follow those in [LP21].

We require the lemma below for both running time and approximation guarantee analysis.

Algorithm 5 $(1 + \epsilon)$-approximate Gomory-Hu Steiner tree on inputs $(G_0, U)$. Assume $\epsilon < 1/100$.

1: If $|U| = 1$, then return the trivial Gomory-Hu Steiner tree $(T, f)$ where $T$ is the empty tree on the single vertex $u \in U$, and $f(v) = u$ for all vertices $v$. Otherwise, if $|U| > 1$, then do the steps below.

2: $\gamma \leftarrow \epsilon^2 / \log^6 n$

3: $\lambda \leftarrow (1 + \epsilon)$-approximate global Steiner mincut of $G$ with terminals $U$, so that the Steiner mincut is in the range $\lfloor (1 - \epsilon)\lambda, \lambda \rfloor$

4: $s \leftarrow$ uniformly random vertex in $U$

5: Construct graph $G'$ by starting with $G$ and adding an edge $(s, u)$ of weight $18\epsilon\lambda/|U|$ for each $u \in U$

6: Call Algorithm 4 on parameter $\gamma$ and inputs $(G', U, (1 + 10\epsilon)\lambda, s)$, and let $(D, R, F)$ be the output. Write $F = \{S_v : v \in R\}$ where $v \in S_v$ for all $v \in R$.

7: **Phase 1: Construct recursive graphs and apply recursion**

8: for each $v \in R$

9: Let $G_v$ be the graph $G$ with vertices $V \setminus S_v$ contracted to a single vertex $x_v$

10: Let $U_v \leftarrow S_v \cap U$

11: Recursively call $(G_v, U_v)$ to obtain output $(T_v, f_v)$

12: end for

13: Let $G_{\text{large}}$ be the graph $G$ with (disjoint) vertex sets $S_v$ contracted to single vertices $y_v$ for all $v \in R$

14: Let $U_{\text{large}} \leftarrow U \cup \bigcup_{v \in R} (S_v \cap U)$

15: Recursively call $(G_{\text{large}}, U_{\text{large}})$ to obtain $(T_{\text{large}}, f_{\text{large}})$

16: **Phase 2: Merge the recursive Gomory-Hu Steiner trees**

17: Construct $T$ by starting with the disjoint union $T_{\text{large}} \cup \bigcup_{v \in R} T_v$ and, for each $v \in R$, adding an edge between $f_v(x_v) \in U_v$ and $f_{\text{large}}(y_v) = U_{\text{large}}$ of weight $w(\partial T_v S_v)$

18: Construct $f : V \rightarrow U$ by $f(v') = f_{\text{large}}(v')$ if $v' \in U_{\text{large}}$ and $f(v') = f_v(v')$ if $v' \in U_v$ for some $v \in R$

19: Return $(T, f)$
Lemma 6.8. Each set $S \in \mathcal{F}$ satisfies $\delta_G S \leq (1 + \gamma)(1 + 10\epsilon)\lambda$ and $|S \cap U| \leq 2|U|/3$.

Proof. By Lemma 6.6 on the call to Algorithm 4 (line 6), each set $S \in \mathcal{F}$ satisfies $\delta_{G'} S \leq (1 + \gamma)(1 + 10\epsilon)\lambda$. We now prove the second statement. By construction, the cut $\partial_G S$ has $|S \cap U|/3$ edges of weight $18\epsilon\lambda/|U|$ that were added to $G'$. Since $\partial_G S$ is a valid Steiner cut in $G$ and the Steiner mincut is at least $(1 - \epsilon)\lambda$, the cut $\partial_G S$ has at least $(1 - \epsilon)\lambda$ weight of edges from $G$. So $\delta_{G'} S \geq (1 - \epsilon)\lambda + |S \cap U|/3 \cdot 18\epsilon\lambda/|U|$. Suppose for contradiction that $|S \cap U| > 2|U|/3$; then, this becomes $\delta_{G'} S > (1 - \epsilon)\lambda + 12\epsilon\lambda = (1 + 11\epsilon)\lambda$, which contradicts the earlier statement $\delta_{G'} S \leq (1 + \gamma)(1 + 10\epsilon)\lambda$.

6.3 Running Time Bound

Let $P(G, U, W)$ be the set of unordered pairs of distinct vertices whose mincut is at most $W$:

$$P(G, U, W) = \left\{ \{u, v\} \in \binom{U}{2} : \lambda_G(u, v) \leq W \right\}.$$ 

In particular, we will consider its size $|P(G, U, W)|$, and show the following expected reduction:

Lemma 6.9. For any $W$ that is at most $(1 + \epsilon)$ times the Steiner mincut of $G$, we have

$$\mathbb{E}[|P(G_{\text{large}}, U_{\text{large}}, W)|] \leq \left( 1 - \Omega \left( \frac{1}{\log^4 n} \right) \right) |P(G, U, W)|,$$
where the expectation is taken over the random selection of \(s\) and the randomness in Algorithm 4.

Before we prove Lemma 6.9, we show how it implies progress on the recursive call for \(G_{\text{large}}\).

**Corollary 6.10.** Let \(\lambda_0\) be the global Steiner mincut of \(G\). W.h.p., after \(\Omega(\log^5 n)\) recursive calls along \(G_{\text{large}}\) (replacing \(G \leftarrow G_{\text{large}}\) each time), the global Steiner mincut of \(G\) is at least \((1 + \epsilon)\lambda_0\) (where \(\lambda_0\) is still the global Steiner mincut of the initial graph).

**Proof.** Let \(W = (1 + \epsilon)\lambda_0\). Initially, we trivially have \(|\mathcal{P}(G, U, W)| \leq \binom{|U|}{2}\). The global Steiner mincut can only increase in the recursive calls, since \(G_{\text{large}}\) is always a contraction of \(G\), so \(W\) is always at most \((1 + \epsilon)\) times the current Steiner mincut of \(G\). By Lemma 6.9, the value \(|\mathcal{P}(G, U, W)|\) drops by factor \(1 - \Omega\left(\frac{1}{\log^4 n}\right)\) in expectation on each recursive call, so after \(\Omega(\log^5 n)\) calls, we have

\[
\mathbb{E}[|\mathcal{P}(G, U, W)|] \leq \left(\frac{|U|}{2}\right) \cdot \left(1 - \Omega\left(\frac{1}{\log^4 n}\right)\right)^{\Omega(\log^5 n)} \leq \frac{1}{\text{poly}(n)}.
\]

In other words, w.h.p., we have \(|\mathcal{P}(G, U, W)| = 0\) at the end, or equivalently, the Steiner mincut of \(G\) is at least \((1 + \epsilon)\lambda_0\). \(\Box\)

Combining both recursive measures of progress together, we obtain the following bound on the recursion depth:

**Lemma 6.11.** W.h.p., each path down the recursion tree of Algorithm 5 has \(O(\log n)\) calls on a graph \(G_v\), and between two consecutive such calls, there are \(O(\epsilon^{-1} \log^6 n)\) calls on the graph \(G_{\text{large}}\).

**Proof.** For any \(\Theta(\log^5 n)\) successive recursive calls down the recursion tree, either one call was on a graph \(G_v\), or all \(\Theta(\log^5 n)\) of them were on the graph \(G_{\text{large}}\). In the former case, \(|U|\) drops by a constant factor by Lemma 6.8, so it can happen \(O(\log n)\) times total. In the latter case, by Corollary 6.10, the global Steiner mincut increases by factor \((1 + \epsilon)\). Let \(w_{\text{min}}\) and \(w_{\text{max}}\) be the minimum and maximum weights in \(G\), so that \(\Delta = w_{\text{max}}/w_{\text{min}}\), which we assume to be \(\text{poly}(n)\). Note that for any recursive instance \((G', U')\) and any \(s, t \in U'\), we have \(w_{\text{min}} \leq \lambda_{G'}(s, t) \leq w(\partial(\{s\})) \leq nw_{\text{max}}\), so the global Steiner mincut of \((G', U')\) is always in the range \([w_{\text{min}}, nw_{\text{max}}]\). It follows that the global Steiner mincut can increase by factor \((1 + \epsilon)\) at most \(O(\epsilon^{-1} \log(nw_{\text{max}}/w_{\text{min}})) = O(\epsilon^{-1} \log n)\) times. Therefore, there are at most \(O(\epsilon^{-1} \log^6 n)\) consecutive calls on \(G_{\text{large}}\) before a call on some \(G_v\) must occur. \(\Box\)

**Lemma 6.12.** For an unweighted/weighted graph \(G = (V, E)\), and terminals \(U \subseteq V\), Algorithm 5 takes time \(\hat{O}(m\epsilon^{-1})\) plus calls to Theorem 6.3 with parameter \(\gamma = \epsilon^2/\log^6 n\) on unweighted/weighted instances with a total of \(\hat{O}(m\epsilon^{-1})\) vertices and \(\hat{O}(m\epsilon^{-1})\) edges.

**Proof.** For a given recursion level, consider the instances \(\{(G_i, U_i, W_i)\}\) across that level. By construction, the terminals \(U_i\) partition \(U\). Moreover, the total number of vertices over all \(G_i\) is at most \(n + 2(|U| - 1) = O(n)\) since each branch creates 2 new vertices and there are at most \(|U| - 1\) branches.

To bound the total number of edges, we consider the unweighted and weighted cases separately, starting with the unweighted case. The total number of new edges created is at most the sum of weights of the edges in the final \((1 + \epsilon)\)-approximate Gomory-Hu Steiner tree. For an unweighted graph, this is \(O(m)\) by the following well-known argument. Root the Gomory-Hu Steiner tree \(T\) at any vertex \(r \in U\); for any \(v \in U \setminus r\) with parent \(u\), the cut \(\partial\{v\}\) in \(G\) is a \((u, v)\)-cut of value \(\text{deg}(v)\),
so \( w_T(u, v) \leq (1 + \epsilon)\lambda_G(u, v) \leq (1 + \epsilon)\deg(v) \). Overall, the sum of the edge weights in \( T \) is at most 

\[
(1 + \epsilon)\sum_{v \in U} \deg(v) \leq (1 + \epsilon) \cdot 2m.
\]

For the weighted case, define a parent vertex in an instance as a vertex resulting from either (1) contracting \( V \setminus S_u \) in some previous recursive \( G_v \) call, or (2) contracting a component containing a parent vertex in some previous recursive call. There are at most \( O(\log n) \) parent vertices: at most \( O(\log n) \) can be created by (1) since each \( G_v \) call decreases \( |U| \) by a constant factor (Lemma 6.8), and (2) cannot increase the number of parent vertices. Therefore, the total number of edges adjacent to parent vertices is at most \( O(\log n) \) times the number of vertices. Since there are \( O(n) \) vertices in a given recursion level, the total number of edges adjacent to parent vertices is \( O(n \log n) \) in this level. Next, we bound the number of edges not adjacent to a parent vertex by \( m \). To do so, we first show that on each instance, the total number of these edges over all recursive calls produced by this instance is at most the total number of such edges in this instance. Let \( P \subseteq V \) be the parent vertices; then, each \( G_v \) call has exactly \( |E(G[S_u \setminus P])| \) edges not adjacent to parent vertices (in the recursive instance), and the \( G_{\text{large}} \) call has at most \( |E(G[V \setminus P])| \cup \bigcup_{v \in R} E(G[S_v \setminus P])| \), and these sum to \( |E(G[V \setminus P])| \), as promised. This implies that the total number of edges not adjacent to a parent vertex at the next level is at most the total number at the previous level. Since the total number at the first level is \( m \), the bound follows.

Therefore, there are \( O(n) \) vertices and \( \tilde{O}(m) \) edges in each recursion level. By Lemma 6.11, there are \( O(\epsilon^{-1} \log^6 n) \) levels, for a total of \( \tilde{O}(n\epsilon^{-1}) \) vertices and \( \tilde{O}(m\epsilon^{-1}) \) edges. In particular, the instances to the max-flow calls have \( \tilde{O}(n\epsilon^{-1}) \) vertices and \( \tilde{O}(m\epsilon^{-1}) \) edges in total.

Finally, we prove Lemma 6.9, restated below.

**Lemma 6.9.** For any \( W \) that is at most \((1 + \epsilon)\) times the Steiner mincut of \( G \), we have

\[
\mathbb{E}[|P(G_{\text{large}}, U_{\text{large}}, W)|] \leq \left(1 - \Omega\left(\frac{1}{\log^4 n}\right)\right)|P(G, U, W)|,
\]

where the expectation is taken over the random selection of \( s \) and the randomness in Algorithm 4.

**Proof.** Define \( D^* \) as the set of vertices \( v \in U \setminus s \) for which there exists an \((s, v)\)-cut in \( G \) of weight at most \( W \) whose side containing \( v \) has at most \(|U|/2\) vertices in \( U \). Let \( P_{\text{ordered}}(G, U, W) \) be the set of ordered pairs \((u, v): u, v \in V \) for which there exists a \((u, v)\)-mincut of weight at most \( W \) with at most \(|U|/2\) vertices in \( U \) on the side \( S(u, v) \subseteq V \) containing \( u \). We now state and prove the following four properties:

(a) For all \( u, v \in U \), \( \{u, v\} \in P(G, U, W) \) if and only if either \((u, v) \in P_{\text{ordered}}(G, U, W) \) or \((v, u) \in P_{\text{ordered}}(G, U, W) \) (or both).

(b) For each pair \((u, v) \in P_{\text{ordered}}(G, U, W) \), we have \( u \in D^* \) with probability at least \( 1/2 \),

(c) For each \( u \in D^* \), there are at least \(|U|/2\) vertices \( v \in U \) for which \((u, v) \in P_{\text{ordered}}(G, U, W) \).

(d) Over the randomness in Algorithm 3 on \((G, U, (1 + \epsilon)\lambda)\), \( \mathbb{E}[|D \cap D^*|] \geq \Omega(|D^*|/\log^4 |U|) \).

Property (a) follows by definition. Property (b) follows from the fact that \( u \in D^* \) whenever \( s \notin S(u, v) \), which happens with probability at least \( 1/2 \). Property (c) follows because any vertex \( v \in U \setminus S(u, v) \) satisfies \((u, v) \in P_{\text{ordered}}(G, U, W) \), of which there are at least \(|U|/2\). For property (d), observe by construction of \( G' \) that for each vertex \( v \in D^* \), the \((s, v)\)-mincut has weight at most
\[ W + |U|/2 \cdot 18\epsilon \lambda/|U|. \] This is at most \((1 + \epsilon)\lambda + 9\epsilon \lambda = (1 + 10\epsilon)\lambda\) since \(W\) is at most \((1 + \epsilon)\) times the Steiner mincut of \(G\) (which is at most \(\lambda\)). It follows that each \(v \in D^*\) satisfies \(\lambda_G(s, v) \leq (1 + 10\epsilon)\lambda\). Property (d) follows from Lemma 6.7 applied to input \((G, U, (1 + 10\epsilon)\lambda, s)\) and set \(D^*\).

With properties (a) to (d) in hand, we now finish the proof of Lemma 6.9. For any vertex \(u \in D\), all pairs \((u, v) \in P_{\text{ordered}}(G, U, W)\) (over all \(v \in U\)) disappear from \(P_{\text{ordered}}(G, U, W)\), which is at least \(|U|/2\) many by (c). In other words,

\[ |P_{\text{ordered}}(G, U, W) \setminus P_{\text{ordered}}(G_{\text{large}}, U_{\text{large}}, W)| \geq \frac{|U| \cdot |D|}{2}. \]

Taking expectations and applying (d),

\[ \mathbb{E}[|P_{\text{ordered}}(G, U, W) \setminus P_{\text{ordered}}(G_{\text{large}}, U_{\text{large}}, W)|] \geq \frac{|U| \cdot \mathbb{E}[|D|]}{2} \geq \Omega\left(\frac{|U| \cdot |D^*|}{\log^4 |U|}\right). \]

Moreover,

\[ |U| \cdot |D^*| \geq \mathbb{E}[|\{(u, v) : u \in D^*\}|] \geq \frac{1}{2} |P_{\text{ordered}}(G, U, W)|, \]

where the second inequality follows by (b). Putting everything together, we obtain

\[ \mathbb{E}[|P_{\text{ordered}}(G, U, W) \setminus P_{\text{ordered}}(G_{\text{large}}, U_{\text{large}}, W)|] \geq \Omega\left(\frac{|P_{\text{ordered}}(G, U, W)|}{\log^4 |U|}\right). \]

Finally, applying (a) gives

\[ \mathbb{E}[|P(G, U, W) \setminus P(G_{\text{large}}, U_{\text{large}}, W)|] \geq \Omega\left(\frac{|P(G, U, W)|}{\log^4 |U|}\right). \]

Finally, we have \(P(G_{\text{large}}, U_{\text{large}}, W) \subseteq P(G, U, W)\) since the \((u, v)\)-mincut for \(u, v \in U_{\text{large}}\) can only increase in \(G_{\text{large}}\) due to \(G_{\text{large}}\) being a contraction of \(G\). Therefore,

\[ |P(G, U, W)| - |P(G_{\text{large}}, U_{\text{large}}, W)| = |P(G, U, W) \setminus P(G_{\text{large}}, U_{\text{large}}, W)|, \]

and combining with the bound on \(\mathbb{E}[|P(G, U, W) \setminus P(G_{\text{large}}, U_{\text{large}}, W)|]\) concludes the proof.

### 6.4 Approximation

We first prove the two lemmas below before concluding the approximation guarantee.

**Lemma 6.13.** For any distinct vertices \(p, q \in U_{\text{large}}\), we have \(\lambda_G(p, q) \leq \lambda_{G_{\text{large}}}(p, q) \leq (1 + \gamma)\lambda_G(p, q)\).

**Proof.** Since \(G_{\text{large}}\) is a contraction of \(G\), we have \(\lambda_G(p, q) \leq \lambda_{G_{\text{large}}}(p, q)\). To show the other inequality, fix any \((p, q)\)-mincut \((A, B)\) in \(G\). We iteratively “uncross” the cut \((A, B)\) with each set \(S_v \in \mathcal{F}\) \((v \in R)\) as follows: if \(v \in A\), then replace \((A, B)\) with \((A \cup S_v, B \setminus S_v)\), and if \(v \in B\), then replace \((A, B)\) with \((A \setminus S_v, B \cup S_v)\). By construction, the final cut is a \((p, q)\)-cut that contains each \(S_v\) on one side of the cut, so it survives upon contraction into \(G_{\text{large}}\) and is a valid \((p, q)\)-cut in \(G_{\text{large}}\). We claim that the final cut has weight at most \((1 + \gamma)\lambda_G(p, q)\), which would prove \(\lambda_{G_{\text{large}}}(p, q) \leq (1 + \gamma)\lambda_G(p, q)\).
Let \((A, B)\) be the current cut in the iterative process, and let \(S_v\) be the next cut we wish to uncross. Since \(S_v\) is a one-sided \((1 + \gamma\))-fair \((\{v\}, R \setminus \{v\})\)-cut on \(G'\), there is a flow from \(\partial G \setminus S_v\) to \(v\) that saturates \(\partial G \setminus S_v\) and congests each edge in \(E_{G'}[S_v]\) to within a factor \((1 + \gamma)\). Decompose the flow into paths and ignore the paths originating from edges in \(G' - G\) (which are all in \(\partial G \setminus S_v\)), obtaining a flow from \(\delta G \setminus S_v\) to \(v\) that saturates \(\partial G \setminus S_v\) and congests each edge in \(E_{G}[S_v]\) to within a factor \((1 + \gamma)\).

Suppose first that \(v \in B\). Further restrict the flow paths to only those originating from edges in the subset \(E_G(A \setminus S_v, A \cap S_v)\) of \(\partial G \setminus S_v\). Each of these paths must cross \(E_G(A \cap S_v, B \cap S_v)\), and from the congestion guarantee, we conclude that \(W(E_{G'}(A \setminus S_v, A \cap S_v)) \leq (1 + \gamma)W(E_{G'}(A \cap S_v, B \cap S_v))\). In the operation that uncrosses \(S_v\), the newly cut edges are precisely \(E_G(A \setminus S_v, A \cap S_v)\), and all edges in \(E_G(A \cap S_v, B \cap S_v)\) disappear. We charge the newly cut edges \(E_G(A \setminus S_v, A \cap S_v)\) to the deleted edges \(E_G(A \cap S_v, B \cap S_v)\) at a \(1 + \gamma\) to \(1\) ratio. Finally, if \(v \in A\), then the argument is symmetric by replacing \(A\) and \(B\), and the charging is identical.

Since newly cut edges are outside any \(E_G[S_v]\) for any \(v \in R\), they are not charged to again, so each edge is either charged to or charged from, but not both. If the total weight of charged-to edges is \(W\), then the total weight of newly cut edges is at most \((1 + \gamma)W\), so the final cut has weight at most \(\lambda_G(p, q) - W + (1 + \gamma)W \leq (1 + \gamma)\lambda_G(p, q)\), as promised.

**Lemma 6.14.** For any \(v \in R\) and any distinct vertices \(p, q \in U_v\), we have \(\lambda_G(p, q) \leq \lambda_{G_v}(p, q) \leq (1 + 13\epsilon)\lambda_G(p, q)\).

**Proof.** The lower bound \(\lambda_G(p, q) \leq \lambda_{G_v}(p, q)\) holds because \(G_v\) is a contraction of \(G\), so we focus on the upper bound. Fix any \((p, q)\)-mincut in \(G\), and let \(S\) be the side of the mincut not containing \(s\) (recall that \(s \in U\) and \(s \notin S_v\)). Since \(S_v \cup S\) is a \((p, s)\)-cut (and also a \((q, s)\)-cut), it is in particular a Steiner cut for terminals \(U\), so \(\delta G(S_v \cup S) \geq (1 - \epsilon)\lambda\). Also, \(\delta G(S_v) \leq (1 + \gamma)(1 + 10\epsilon)\lambda \leq (1 + 11\epsilon)\lambda\) by Lemma 6.8. Together with the submodularity of cuts, we obtain

\[
(1 + 11\epsilon)\lambda + \delta_G S \geq \delta G S_v + \delta G S \geq \delta G(S_v \cup S) + \delta G(S_v \cap S) \geq (1 - \epsilon)\lambda + \delta G(S_v \cap S),
\]

The set \(S_v \cap S\) stays intact under the contraction from \(G\) to \(G_v\), so \(\delta G_v(S_v \cap S) = \delta G(S_v \cap S)\). Therefore,

\[
\lambda_{G_v}(p, q) \leq \delta_{G_v}(S_v \cap S) = \delta G(S_v \cap S) \leq \delta G S + 12\epsilon \lambda = \lambda_G(p, q) + 12\epsilon \lambda.
\]

Finally, \(\lambda_G(p, q)\) is at least the Steiner mincut of \(G\), which is at least \((1 - \epsilon)\lambda\), so the above is at most \(\lambda_G(p, q) + 12\epsilon \cdot \lambda_G(p, q) / (1 - \epsilon) \leq (1 + 13\epsilon)\lambda_G(p, q)\), as promised. \(\square\)

Combining the lemmas above, we can conclude the following.

**Lemma 6.15.** Algorithm 5 outputs a \(((1 + 13\epsilon)(1 + \gamma)^{O(\epsilon^{-1}\log^6 n)})^{\log_{1.5}|U|}\)-approximate Gomory-Hu Steiner tree. With \(\gamma = c^2 / \log^6 n\), the approximation factor is \((1 + \epsilon)^{O(\log |U|)}\).

**Proof.** To avoid clutter, define \(\alpha = Cc^{-1}\log^6 n\) for large enough constant \(C > 0\). Consider the path down the recursion tree leading up to the current recursive instance, and let \(k\) be the number of consecutive recursive calls of type \(G_{\text{large}}\) directly preceding the current instance. We apply induction on \(|U|\) and \(k\) to prove an \(((1 + 13\epsilon)(1 + \gamma)^{\alpha})^{\log_{1.5}|U|}(1 + \gamma)^{-k}\)-approximation factor. By Lemma 6.8, we have \(|U_v| \leq 2|U|/3\) for all \(v \in R\), so by induction, the recursive outputs \((T_v, f_v)\) are Gomory-Hu Steiner trees with approximation \(((1 + 13\epsilon)(1 + \gamma)^{\alpha})^{\log_{1.5}|U_v|} \leq ((1 + 13\epsilon)(1 + \gamma)^{\alpha})^{\log_{1.5}|U|/3-1}\). By
definition, this means that for all \( s,t \in U_v \) and the minimum-weight edge \((u,u')\) on the \( s-t \) path in \( T_v \), letting \( U'_v \subseteq U_v \) be the vertices of the connected component of \( T_v - (u,u') \) containing \( s \), we have that \( f^{-1}_v(U'_v) \) is a \(((1 + 13\epsilon)(1 + \gamma)^\alpha)\log_{1.5}|U|-1\)-approximate \((s,t)\)-mincut in \( G_v \) with value \( w_T(u,u') \). Define \( U \subseteq U_v \) as the vertices of the connected component of \( T - (u,u') \) containing \( s \).

By construction of \((T,f)\) (lines 17 and 18), the set \( f^{-1}(U') \) is simply \( f^{-1}(U'_v) \) with the vertex \( x_v \) replaced by \( V \setminus S_v \) in the case that \( x_v \in f^{-1}(U') \). Since \( G_v \) is simply \( G \) with all vertices \( V \setminus S_v \) contracted to \( x_v \), we conclude that \( \delta_G(f^{-1}(U'_v)) = \delta_G(f^{-1}(U')) \). By Lemma 6.14, the values \( \lambda_G(s,t) \) and \( \lambda_G(v,t) \) are within factor \((1 + 13\epsilon)\) of each other, so \( \lambda_G(f^{-1}(U')) \) approximates the \((s,t)\)-mincut in \( G \) to a factor \((1 + 13\epsilon) \cdot ((1 + 13\epsilon)(1 + \gamma)^\alpha)\log_{1.5}|U|-1 \), which we want to show is at most \(((1 + 13\epsilon)(1 + \gamma)^\alpha)\log_{1.5}|U|(1 + \gamma)^{-k} \). This follows by Lemma 6.11 since w.h.p., we always have \( k \leq C\epsilon^{-1}\log^6 n = \alpha \) for large enough constant \( C > 0 \). Thus, the Gomory-Hu Steiner tree condition for \((T,f)\) is satisfied for all \( s,t \in U_v \) for some \( v \in R \).

We now focus on the case \( s,t \in U_{\text{large}} \). By induction, the recursive output \((T_{\text{large}}, f_{\text{large}})\) is a Gomory-Hu Steiner tree with approximation \(((1 + 13\epsilon)(1 + \gamma)^\alpha)\log_{1.5}|U|(1 + \gamma)^{-(k+1)} \). Again, consider \( s,t \in U_{\text{large}} \) and the minimum-weight edge \((u,u')\) on the \( s-t \) path in \( T_{\text{large}} \), and let \( U'_{\text{large}} \subseteq U_{\text{large}} \) be the vertices of the connected component of \( T_{\text{large}} - (u,u') \) containing \( s \). Define \( U' \subseteq U \) as the vertices of the connected component of \( T - (u,u') \) containing \( s \). By a similar argument, we have \( \delta_{G_{\text{large}}}(f^{-1}_{\text{large}}(U'_{\text{large}})) = \delta_G(f^{-1}(U')) \). By Lemma 6.13, we also have \( \lambda_{G_{\text{large}}}(s,t) = (1 + \gamma)\lambda_G(s,t) \), so \( \delta_G(f^{-1}(U')) \) is a \(((1 + 13\epsilon)(1 + \gamma)^\alpha)\log_{1.5}|U|(1 + \gamma)^{-(k+1)} \cdot (1 + \gamma)\)-approximate \((s,t)\)-mincut in \( G \), fulfilling the Gomory-Hu Steiner tree condition for \((T,f)\) in the case \( s,t \in U_{\text{large}} \).

There are two remaining cases: \( s \in U_v \) and \( t \in U_{v'} \) for distinct \( v,v' \in R \), and \( s \in U_v \) and \( t \in U_{\text{large}} \); we treat both cases simultaneously. Since \( G \) has Steiner mincut at least \( \lambda \), each of the contracted graphs \( G_{\text{large}} \) and \( G_v \) also has Steiner mincut at least \( \lambda \). Since all edges on the approximate Gomory-Hu Steiner tree correspond to actual cuts in the graph, every edge in \( T_v \) and \( T_{\text{large}} \) has weight at least \( \lambda \). By construction, the \( s-t \) path in \( T \) has at least one edge of the form \((f_v(x_v),f_{\text{large}}(y_v))\), added on line 17; this edge has weight \( \delta_G S_v \leq (1 + \epsilon)(1 + \gamma)\lambda \) by Lemma 6.8. Therefore, the minimum-weight edge on the \( s-t \) path in \( T \) has weight at least \( \lambda \) and at most \((1 + \epsilon)(1 + \gamma)\lambda \); in particular, it is a \((1 + \epsilon)(1 + \gamma)\)-approximation of \( \lambda_G(s,t) \), which fits the bound since \(|U| \geq 2 \). If the edge is of the form \((f_v(x_v),f_{\text{large}}(y_v))\), then by construction, the relevant set \( f^{-1}(U') \) is exactly \( S_v \), which is a \((1 + \epsilon)\)-approximate \((s,t)\)-mincut in \( G \). If the edge is in \( T_{\text{large}} \) or \( T_v \) or \( T_{v'} \), then we can apply the same arguments used previously.

Finally, we can reset \( \epsilon \leftarrow \Theta(\epsilon/\log n) \) so that the \((1 + \epsilon)^{O(\log |U|)}\)-approximation becomes \((1 + \epsilon)\).

This concludes Theorem 6.2.

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