The effective field theory on a finite boundary of the Bruhat-Tits tree

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Abstract

Based on bulk reconstruction from the finite boundary of the Bruhat-Tits tree, the effective field theory is obtained after integrating out fields outside this boundary. According to the $p$-adic version of Anti-de Sitter/Conformal Field Theory duality, two-point functions of dual theory living on the finite boundary are read out from the effective action. It can be regarded as a two-point function of a deformed CFT over $p$-adic numbers.

1 Introduction

It is proposed that physics should be invariant under the change of number fields [1]. For example, we should be able to use either real numbers($\mathbb{R}$) or $p$-adic numbers($\mathbb{Q}_p$) [2–4] to set up spacetime coordinates and write down the same physical laws. Such number fields should include the set of rational numbers $\mathbb{Q}$ since all measurement results in physics are rational numbers. Considering that $\mathbb{Q}_p$ and $\mathbb{R}$ are the only two candidates satisfying certain restrictions, it is necessary to study physics over $\mathbb{Q}_p$ as investigations to the above proposal. Another motivation to study physics over $\mathbb{Q}_p$ comes from the possibility that spacetime is non-Archimedean at small scales [1, 5, 6], and it is very convenient to construct such spacetime using $\mathbb{Q}_p$. String theories over $\mathbb{Q}_p$ ($p$-adic string) begin with [5, 7, 8], and the Bruhat-Tits tree($T_p$) is regarded as the $p$-adic string world-sheet in [9]. Spinors, gravity and blackholes on $T_p$ are studied in [10–15]. Relations between $T_p$ and tensor network are studied in [16–18]. The $p$-adic version of the Anti-de Sitter/Conformal Field Theory duality [19–21] is proposed in [10, 22] ($p$-adic AdS/CFT), which are followed by lots of works, such as [23–33].

Among all these references, [9] and [22] are the most important to this paper. Besides indentifying $T_p$ as the $p$-adic string world-sheet, [9] also calculates the effective field theory on the infinite boundary of $T_p$ which is obtained by integrating out fields in the bulk. “Effective” comes from the integration of fields. One key technique is the use of bulk-boundary propagators. But propagators seem useless when one wants to calculate the effective field theory on the finite(cutoff) boundary, where bulk reconstruction from the finite boundary is required. “Cutoff” usually means ignoring the outside of the boundary. “Finite boundary” is preferred to “cutoff boundary” in this paper because both the inside and outside of the boundary are handled carefully, and none of them is dropped directly.

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Fig. 1: $T_{p=2}$ and its vertical coordinate $z$.

One motivation of this paper is to extend [9]'s work which is to calculate the effective field theory on a finite boundary. Bulk reconstruction from the finite boundary is solved in section 3, and the effective field theory is calculated in section 4. Another motivation is to find some results of $p$-adic AdS/CFT which are parallel to those of AdS/CFT over $\mathbb{R}$ with a cutoff AdS boundary. Identifying $T_p$ as the $p$-adic version of AdS spacetime [22], two-point functions of a deformed CFT over $\mathbb{Q}_p$ are calculated in section 5, where the deformation comes from the “cutoff” of $T_p$, or in other words, comes from the finite boundary. Section 2 provides some basic knowledge of $T_p$ and points out the field space used in this paper. The last section is summary and discussion. In this paper, the measure $\mu$, $dx$, the $p$-adic absolute value $|\cdot|_p$ and the edge length $L$ have the dimension of length while $p$-adic numbers are dimensionless.

2 The Bruhat-Tits tree and field spaces

Referring to Fig. 1, $T_p$ is an infinite tree with $p + 1$ edges incident on each vertex, where $p$ is a prime number. Distance $d(\cdot, \cdot)$ between vertices can be defined as the number of edges between them. Letting $z(\cdot)$ denote the vertical coordinate of a vertex, there is a particular one-to-one correspondence between the upper boundary of $T_p$ and $\mathbb{Q}_p$ such that $|x - y|_p = |z(a_{xy})|_p$, where $a_{xy}$ is the lowest vertex on the line connecting $x$ and $y$ on the upper boundary ($x, y \in \mathbb{Q}_p$). $|x - y|_p$ also defines the distance between $x$ and $y$, and it is actually the regularization of $d(a, b)$ when $a \to x$ and $b \to y$. There is only one single point on the lower boundary of $T_p$, which is noted as $\infty$. Each vertex can be regarded as a subset(ball) of $\mathbb{Q}_p$ containing points on the upper boundary which are connected to this vertex from above. There is an additive measure $\mu$ of vertex $a$ which equals to $|z(a)|_p$. Several examples are provided in Fig. 1 such as

$$d(a, e) = d(b, e) = d(a, b) = 2, \quad d(a, c) = d(b, c) = d(e, c) = 1,$$

$$|x - y|_p = |z(a_{xy})|_p = |z(c)|_p = |p^n|_p = p^{-n}, \quad |x - u|_p = |y - u|_p = |z(e)|_p = p^{1-n},$$

$$x \in a, \quad y \in b, \quad u \in e, \quad a \cup b = c \subset e.$$
\[ p^2 \mu(a) = p^2 \mu(b) = p\mu(c) = \mu(e) = |z(e)|_p = p^{1-n}. \]  

Be aware that \( u \notin c \) since edge \( ec \) is attached to \( c \) from below but not from above.

Consider the action and equation of motion of a real-valued massless scalar field on \( T_p \):

\[ S = \frac{1}{2} \sum_{(ab)} \frac{(\phi_a - \phi_b)^2}{L^2}, \]

\[ \square \phi_a = 0, \quad \square f_a := \sum_{b \in \partial a} (f_a - f_b), \]

where \( (ab) \) is the edge connecting the neighboring vertices \( a \) and \( b \). The constant \( L \) is the length of edges. \( b \in \partial a \) means \( b \) is a neighboring vertex of \( a \) and the sum \( \sum_{b \in \partial a} \) is over all the neighboring vertices of \( a \). This action can be rewritten as a sum over vertices, which is

\[ 4L^2 S = \sum_a \sum_{b \in \partial a} (\phi_a - \phi_b)^2 = 2 \sum_{z(a) \leq p^N} \phi_a \square \phi_a + F_N(\phi, \phi) + R_N(\phi, \phi), \]

\[ F_N(f, g) := \sum_{z(a) = p^N} \sum_{b \in \partial a} (f_a + f_b)(g_b - g_a), \quad R_N(f, g) := \sum_{z(a) > p^N} \sum_{b \in \partial a} (f_a - f_b)(g_a - g_b). \]

\( R_N \) comes from the separation \( \sum_a = \sum_{z(a) \leq p^N} + \sum_{z(a) > p^N} \) and \( F_N \) comes from the identity

\[ \sum_{z(a) \leq p^N} \sum_{b \in \partial a} (f_a - f_b)(g_a - g_b) = 2 \sum_{z(a) \leq p^N} f_a \square g_a + F_N(f, g) = 2 \sum_{z(a) \leq p^N} g_a \square f_a + F_N(g, f). \]

It is convenient to consider a field space where \( R_N \) and \( F_N \) vanish. For the field space \( \mathcal{H} \) in this paper, we demand that

\[ \forall f, g \in \mathcal{H}, \quad \lim_{N \to \infty} F_N(f, g) = \lim_{N \to \infty} R_N(f, g) = 0. \]

### 3 Bulk reconstruction from the finite boundary

With the help of on-shell conditions in the bulk, fields there can be reconstructed from those on the boundary. In Fig. 2 there are four subgraphs of \( T_{p=2} \). From left to right their bulks and boundaries are

\[
\text{bulk} = \{1\}, \quad \text{boundary} = \{0_1, 0_2, 2\}, \\
\text{bulk} = \{1, 2, \cdots\}, \quad \text{boundary} = \{0_1, 0_2, 0_3, 0_4, 3\}, \\
\text{bulk} = \{1, 2, 3, \cdots\}, \quad \text{boundary} = \{0_1, 0_2, 0_3, 0_4, 0_5, 0_6, 0_7, 0_8, 4\}, \\
\text{bulk} = \{1, \cdots, n, \cdots\}, \quad \text{boundary} = \{0_1, \cdots, 0_p^n, n + 1\}. 
\]

“\( \cdots \)” represents vertices which are not marked out with numbers. Refer to the first subgraph on the right. \( \phi_n \) , whose location is one edge above the lower boundary \( \{n + 1\} \), can be reconstructed from \( \phi_{n+1} \) (the field on the lower boundary) and \( \phi_0 \)'s (fields on the upper boundary). After solving the cases of \( n = 1, 2, 3 \) (three subgraphs on the left), the following ansatz can be proposed:

\[ a_{n+1} \phi_n = a_n \phi_{n+1} + a_1 \sum_n \phi_0, \quad a_n := p^n - 1, \quad \sum_n \phi_0 := \frac{p^n}{\sum_i \phi_0}, \quad n \geq 1. \]
Fig. 2: Four subgraphs of $T_{p=2}$. Vertices on the upper boundary are noted as $0_i$'s. The lower boundary of each subgraph contains only one vertex.

It can be proved by mathematical induction.

What is useful in this paper is the reconstruction of $\phi_1$, whose location is one edge below the upper boundary. Letting $n = 1, 2, 3$ in (12), we have

$$a_2 \phi_1 = a_1 \phi_2 + a_1 \sum \phi_0$$
$$a_3 \phi_2 = a_2 \phi_3 + a_1 \sum \phi_0$$
$$a_4 \phi_3 = a_3 \phi_4 + a_1 \sum \phi_0$$

From these, we get

$$\frac{1}{a_1^2} \phi_1 = \frac{1}{a_1 a_2} \sum \phi_0 + \frac{1}{a_2 a_3} \sum \phi_0 + \frac{1}{a_3 a_4} \sum \phi_0 + \frac{1}{a_1 a_4} \phi_4 .$$

(13)

Referring to the third subgraph on the left in Fig. 2 [13] is the reconstruction of $\phi_1$ from $\phi_4$ and $\phi_0$'s. Therefore, the ansatz for the reconstruction of $\phi_1$ from $\phi_{n+1}$ and $\phi_0$'s can be proposed as

$$\frac{1}{a_1^2} \phi_1 = \sum_{i=1}^{n} \frac{1}{a_1 a_{i+1}} \sum \phi_0 + \frac{1}{a_1 a_{n+1}} \phi_{n+1} ,$$

(14)

which can also be proved by mathematical induction.

Let’s consider a simple case of the boundary condition on the lower boundary, which is $\phi_{a \to \infty} = 0$. Remember that $\infty$ is the lower boundary of $T_p$ (Fig. 1). Letting $\phi_{n+1} = 0$ and $n \to \infty$, the reconstruction of $\phi_1$ writes

$$\frac{1}{a_1^2} \phi_1 = \sum_{i=1}^{\infty} \frac{1}{a_1 a_{i+1}} \sum \phi_0 .$$

(15)

It can be rearranged into a more useful form. Taking the third subgraph on the left in Fig. 2 as an example, we can write

$$\sum_{i=1}^{2^3} \phi_0 = \sum_{i=1}^{2^1} \phi_0_i + \sum_{i=2^1+1}^{2^2} \phi_0_i + \sum_{i=2^2+1}^{2^3} \phi_0_i \equiv \sum_{i=1}^{1} \phi_0 + \sum_{i=2^1+1}^{1} \phi_0 + \sum_{i=2^2+1}^{1} \phi_0 ,$$

(16)

where “≡” means that we introduce new symbols on the right-hand side to denote the left-hand side. Remembering that each vertex is a ball in $\mathbb{Q}_p$, $\sum_{(i+1)/i}$ means the sum is over all vertices $0_i$'s (vertices
on the upper boundary) included in vertex $i + 1$ but not included in vertex $i$. It can be found that there are $p$ terms in $\sum_{i \neq j}$ and $p^{i+1} - p^i$ ($i \geq 1$) terms in $\sum_{(i+1) \backslash i}$. Now the reconstruction of $\phi_1$ can be rewritten as

$$
\frac{1}{a_1^i} \phi_1 = \frac{1}{a_1a_2} \sum_{1} \phi_0 + \frac{1}{a_2a_3} \sum_{2} \phi_0 + \frac{1}{a_3a_4} \sum_{3} \phi_0 + \frac{1}{a_4a_5} \sum_{4} \phi_0 + \cdots
$$

$$
= \frac{1}{a_1a_2} \sum_{1} \phi_0 + \frac{1}{a_2a_3} \left( \sum_{1} \phi_0 + \sum_{2\setminus 1} \phi_0 \right)
+ \frac{1}{a_3a_4} \left( \sum_{1} \phi_0 + \sum_{2\setminus 1} \phi_0 + \sum_{3\setminus 2} \phi_0 \right)
+ \frac{1}{a_4a_5} \left( \sum_{1} \phi_0 + \sum_{2\setminus 1} \phi_0 + \sum_{3\setminus 2} \phi_0 + \sum_{4\setminus 3} \phi_0 \right) + \cdots
$$

$$
= A_1 \sum_{1} \phi_0 + A_2 \sum_{2\setminus 1} \phi_0 + A_3 \sum_{3\setminus 2} \phi_0 + A_4 \sum_{4\setminus 3} \phi_0 + \cdots, A_k = \sum_{i=k}^{\infty} \frac{1}{a_1a_{i+1}}, k \geq 1.
$$

The distance between any vertex $0_j \subset (i+1) \setminus i$ and vertex 1 is a constant which only depends on $i$. Taking the third subgraph on the left in Fig. 2 as an example, we have

$$
0_1 \cup 0_2 = 1, d(0_1, 1) = d(0_2, 1) = 1 = 2 \cdot 1 - 1, 0_3 \cup 0_4 = 2 \setminus 1, d(0_3, 1) = d(0_4, 1) = 3 = 2 \cdot 2 - 1, 0_5 \cup 0_6 \cup 0_7 \cup 0_8 = 3 \setminus 2, d(0_5, 1) = d(0_6, 1) = d(0_7, 1) = d(0_8, 1) = 5 = 2 \cdot 3 - 1.
$$

Therefore, under the boundary condition $\phi_{a \to \infty} = 0$, the reconstruction of $\phi_1$ from $\phi_0$’s (17) also writes

$$
\frac{1}{a_1} \phi_1 = \sum_{n=1}^{\infty} A_n \sum_{d(1,0) = 2n-1} \phi_0 \equiv \sum_{0 \in \text{bdy}} A_{1+d(1,0)/2} \phi_0,
$$

where $\sum_{d(1,0) = 2n-1}$ means the sum is over fields on the upper boundary ( $\phi_0$’s) whose locations are $2n - 1$ edges away from vertex 1. $\sum_{0} \sum_{d(1,0)} \equiv \sum_{0 \in \text{bdy}}$ is the sum over all fields on the upper boundary. The weight coefficient $A_{(1+d)/2}$ only depends on the distance between $\phi_0$’s location and vertex 1.

4 The effective field theory on the finite boundary

Consider the partition function with sources only living on a finite boundary $E_M$. We can write

$$
Z_M[J] = \frac{\int_{T_p} D\phi e^{-S + \sum_{a \in E_M} \phi_a J_a}}{\int_{T_p} D\phi e^{-S}}, \quad (20)
$$

$$
S = \frac{1}{2} \sum_{(ab)} \left( \phi_a - \phi_b \right)^2 = \frac{1}{2L^2} \sum_a \phi_a \Box \phi_a, \quad E_M := \{ a | z(a) = p^M \}, \quad J_a_{\not\in E_M} = 0.
$$

5
\[
\Phi_a = \Phi_{b_2} = \Phi_{b_3} = \cdots \quad \Phi_{a'} = \Phi'_{b'_2} = \Phi'_{b'_3} = \cdots
\]

Fig. 3: The configuration of \( \Phi_a \) when \( z(a) > p^M \). Take \( a \in E_M \) as an example. \( b_1 \equiv a^- \), \( b_2 \) and \( b_3 \) are \( p + 1 = 3 \) neighboring vertices of \( a \), which satisfy \( z(b_1) = z(a^-) < z(a) \) and \( z(b_2) = z(b_3) > z(a) \). \( \Phi 's \) on vertices included in \( a \) (vertices of the red subgraph) equal to \( \Phi_a \). \( \Phi 's \) on vertices of the blue subgraph equal to \( \Phi_{a'} \), and so on.

where \( \int_{T_p} D\phi \) means \( \phi \) fluctuates on the entire \( T_p \). Decompose \( \phi \) into \( \Phi \) and \( \phi' \) which satisfy

\[
\phi_a = \Phi_a + \phi'_a , \quad \Box \Phi_a \notin E_M = 0 , \quad \phi'_a \in E_M = 0 .
\]

(22)

\( \Phi \) is on-shell outside \( E_M \) and \( \phi' \) vanishes on \( E_M \). The action then writes

\[
2L^2 S = \sum_{a \in E_M} \Phi_a \Box \Phi_a + S' = \sum_{a \in E_M} \Phi_a(\Phi_a - \Phi_a^-) + \sum_{a \in E_M} \Phi_a(p\Phi_a - \sum_{b \in \partial a} \Phi_b) + S' ,
\]

(23)

\[
S' = \sum_{a} \phi'_a \Box \phi'_a ,
\]

(24)

where (9) and (22) are used. Among \( p + 1 \) neighboring vertices of \( a \), there is only one satisfying \( z(b) < z(a) \) (noted as \( a^- \) ) and the rest satisfying \( z(b) > z(a) \) . When choosing a particular on-shell configuration of \( \Phi_a \) above \( E_M \) ( \( z(a) > p^M \) ), the second term in the action vanishes. Referring to Fig. 3 we have

\[
p\Phi_a - \sum_{b \in \partial a \atop z(b) > z(a)} \Phi_b = p\Phi_a - \sum_{b \in \partial a \atop z(b) > z(a)} \Phi_a = 0 .
\]

(25)

According to the reconstruction of \( \phi_1 \) from \( \phi_0 's \) (19), \( \Phi_{a^-} \) can be reconstructed from \( \Phi 's \) on \( E_M \). And the action can be written as

\[
2L^2 S = \sum_{a \in E_M} \Phi_a(\Phi_a - \Phi_a^-) + S' = \sum_{a \in E_M} \Phi_a(\Phi_a - a_1^2 \sum_{b \in E_M} A_{1_{\frac{1}{2}}}(-\phi) \Phi_b) + S' .
\]

(26)
Considering that there are $p$ vertices (b’s) satisfying $d(a^-,b) = 1$ and $p^n - p^{n-1}$ vertices satisfying $d(a^-,b) = 2n - 1$ when $n \geq 2$, it can be proved that

$$ a_1^2 \sum_{b \in E_M} A_{\frac{1+d(a,b)}{2}} = (p-1)^2 \left( A_1 p + A_2 (p^2 - p) + A_3 (p^3 - p^2) + A_4 (p^4 - p^3) + \cdots \right) \\
= (p-1)^2 \left( p(A_1 - A_2) + p^2 (A_2 - A_3) + p^3 (A_3 - A_4) + \cdots \right) \\
= (p-1) \left( \frac{a_2 - a_1}{a_1 a_2} + \frac{a_3 - a_2}{a_2 a_3} + \frac{a_4 - a_3}{a_3 a_4} + \cdots \right) = (p-1) \frac{1}{a_1} = 1. $$

Hence the action also writes

$$ 2L^2 S = a_1^2 \sum_{a \in E_M} \Phi_a \left( \sum_{b \in E_M} A_{\frac{1+d(a,b)}{2}} (\Phi_a - \Phi_b) \right) + S' = a_1^2 \sum_{a \in E_M} \Phi_a \left( \sum_{b \in E_M} \frac{A_{d(a,b)}}{2} (\Phi_a - \Phi_b) \right) + S'. $$

Substituting it into the partition function, terms related to $\phi'$ cancel out. And it turns out to be a partition function of a field theory on $E_M$, which is

$$ Z_M[J] = \frac{\int_{E_M} \mathcal{D}\Phi e^{-S_M + \sum_{a \in E_M} \Phi_a J_a}}{\int_{E_M} \mathcal{D}\Phi e^{-S_M}}, \quad S_M = \frac{(p-1)^2}{2L^2} \sum_{a \in E_M} \Phi_a \left( \sum_{b \in E_M} \frac{A_{d(a,b)}}{2} (\Phi_a - \Phi_b) \right). $$

$$ \int_{E_M} \mathcal{D}\Phi = \int_{T_p} \mathcal{D}\phi \int_{T_p \setminus E_M} \mathcal{D}\phi. $$

(29) is the effective field theory on the finite boundary $E_M$. Taking the limit $M \to \infty$ leads to that on the infinite boundary. Refer to Fig. 4 Given vertices $a, b \in E_M$, select two points $x$ and $y$ on the upper boundary of $T_p$ satisfying $x \in a, y \in b$. We can write

$$ |x - y|_p = |z(a_{xy})|_p = |p^{M - \frac{d(a,b)}{2}}|_p = p^{\frac{d(a,b)}{2}} |p^M|_p. $$

(31)

The action $S_M$ can be rewritten as

$$ S_M = \frac{(p-1)^2}{2L^2} \sum_{a \in E_M} |p^M|_p \Phi_a \left( \sum_{b \in E_M \setminus b \neq a} \frac{A_{d(a,b)}}{2} (\Phi_a - \Phi_b) \right) \\
= \frac{(p-1)^2}{2L^2} \sum_{a \in E_M} |p^M|_p \Phi_a \left( \sum_{b \in E_M \setminus b \neq a} |p^M|_p A_{d(a,b)} \frac{A_{d(a,b)}}{2} |\Phi_a - \Phi_b| \right)^{\frac{1}{2}}, $$

(32)

where $x \in a, y \in b$ and $|p^M|_p$ is the measure of each vertex on the finite boundary $E_M$, which tends to $dx$ in the limit $M \to \infty$. Supposing that $a \to x$ and $b \to y$ when $M \to \infty$, we can write $\Phi_a \to \Phi_x$ and $\Phi_b \to \Phi_y$ where $\Phi_x$ or $\Phi_y$ represents a field on the upper boundary (infinite boundary) of $T_p$. As for the $A_{d/2} p^d$ term, considering that $M \to \infty \iff d(a,b) \to \infty$ according to (31) when fixing $x$ and $y$, we can write

$$ A_{d(a,b)} p^{d(a,b)} = p^d \sum_{i=d/2}^{\infty} \frac{1}{(p^i - 1)(p^{i+1} - 1)} \xrightarrow{M \to \infty} p^d \sum_{i=d/2}^{\infty} \frac{1}{p^i p^{i+1}} = p \frac{1}{p^2 - 1}. $$

(33)
Finally, in the limit \( M \to \infty \), the action \( S_M \) can be written as

\[
S_{M \to \infty} = \frac{p(p-1)}{2(p+1)L^2} \int_{x \in \mathbb{Q}_p} dx \Phi_x \int_{y \in \mathbb{Q}_p, y \neq x} dy \frac{\Phi_x - \Phi_y}{|x-y|^2_p}. 
\]

(34)

It is the effective field theory on the infinite boundary of \( T_p \), which is consistent with [9], although different \( dx \) and \(| \cdot |_p \) are used in that paper.

### 5 Relations to \( p \)-adic AdS/CFT

Consider the equation

\[
Z_M[J] = \int_{T_p \backslash E_M} D\phi e^{-S + \sum_{a \in E_M} \phi_a J_a} \int_{T_p} D\phi e^{-S} 
= \frac{1}{2} \int_{E_M} D\phi \int_{T_p \backslash E_M} D\phi e^{-S + \sum_{a \in E_M} \phi_a J_a} \int_{E_M} D\Phi e^{-S_M + \sum_{a \in E_M} \phi_a J_a} \int_{E_M} D\Phi e^{-S_M}.
\]

(35)

Ignoring denominators and setting \( J = 0 \), we can write

\[
\int_{T_p \backslash E_M} D\phi e^{-S} \sim e^{-S_M} \xrightarrow{M \to \infty} \int_{T_p} D\phi e^{-S} \sim e^{-S_M \to \infty}.
\]

(36)

Therefore, \( S_M (S_{M \to \infty}) \) can be regarded as the effective action after integrating out fields on \( T_p \backslash E_M \). Now let’s identify \( T_p \) as a \( p \)-adic version of AdS spacetime [22]. According to the spirit of AdS/CFT:

\[
\langle e^{\int dx O(\phi_0)} \rangle_{\text{CFT}} = \int_{\text{AdS}} D\phi e^{-S} |_{\phi_{\text{AdS}} = \phi_0}.
\]

(37)
where $\partial \text{AdS}$ is the boundary of AdS and the fluctuation of gravity has been ignored, $e^{-S_{M \to \infty}}$ should be directly proportional to the generating functional of some CFT over $\mathcal{Q}_p$, whose two-point function reads

$$
\frac{\delta^2 e^{-S_{M \to \infty}}}{\delta \Phi_a \delta \Phi_b}|_{\Phi=0} = \frac{p(p-1)}{(p+1)L^2} \frac{1}{|x-y|_p^2}, \ x \neq y.
$$

It is consistent with [22] if setting $\eta_p = 1$, $\Delta = n = 1$ there and $L = 1$ in (38). On the other hand, if not taking the limit $M \to \infty$, the following calculation should give a two-point function of some deformed CFT over (coarse-grained) $\mathcal{Q}_p$:

$$
\frac{\delta^2 e^{-S_M}}{\delta \Phi_a \delta \Phi_b}|_{\Phi=0} = \frac{(p-1)^2}{L^2} A_{d(a,b)} = \frac{(p-1)^2}{L^2} \sum_{n = d(a,b)}^{\infty} \frac{1}{(p^n - 1)(p^{n+1} - 1)}, \ a \neq b,
$$

where $d(a,b) = 2, 4, 6, 8, \cdots$ is a positive even number and $E_M = \{a, b, \cdots\}$ is a coarse-grained $\mathcal{Q}_p$. Remember that each element in $E_M$ is a ball in $\mathcal{Q}_p$. (39) can be regarded as a counterpart to the two-point function of a deformed CFT living on the cutoff boundary of AdS over $\mathbb{R}$.

### 6 Summary and discussion

In this paper, we manage to reconstruct fields in the bulk from those on the finite boundary of $T_p$ (19). Then with the help of calculating techniques in [9], the effective field theory is calculated by integrating out fields on the entire $T_p$ except those on the finite boundary (29). Referring to AdS/CFT, two-point functions are read out: (38) on the infinite boundary and (39) on the finite boundary. The former is a two-point function of a CFT over $\mathcal{Q}_p$ which is consistent with [22], and the latter is a two-point function of a deformed CFT which should be compared with that in AdS/CFT over $\mathbb{R}$ with a cutoff AdS boundary.

Some problems still need to be explored. For example, i) relations between field spaces discussed in section 2 and those in [13,30] are still unclear. Different field spaces or boundary conditions sometimes lead to different results; ii) it may be a hard problem to find out what “deformed CFT” is which gives a two-point function like (39). It is known that the counterpart over $\mathbb{R}$ can be regarded as a $TT$-deformed CFT [34]; iii) the same calculation on $p$AdS [25] is interesting. $p$AdS is another $p$-adic version of AdS spacetime whose finite boundary is exactly $\mathcal{Q}_p$ but not the coarse-grained one.

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