The structure of dual Schubert union codes

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November 13, 2014

Abstract

In this article we study the dual codes of Schubert union codes which include Schubert codes and Grassmann codes. We prove that these codes are generated by their minimum weight codewords. In addition we prove that Schubert union codes are Tanner codes from a graph related to the Schubert varieties in an elementary, nontrivial way; and that this gives a systematic encoder for Schubert union codes from a minimal irreducible 2-conversion set of the Schubert union graph.

Introduction

The Grassmannian is a mathematical object relevant in Algebra, Geometry and Combinatorics. Several of these properties are reflected in the Grassmann code. This allows the use of coding theory techniques to study the Grassmannian and related objects. For example the dimension, minimum distance, weight distribution and generalized Hamming weights are concepts from coding theory which are relevant to the geometrical properties of the Grassmannian.

In this article we use codes coming from some particular subvarieties of the Grassmannian, known as Schubert unions, and a code construction known as Tanner codes, to prove that dual Schubert union codes are generated by their minimum weight codewords. We use a novel combinatorial approach derived from the incidence geometry of the Schubert union. This geometrical fact also gives some conditions on the generalized Hamming weights of dual Schubert union codes.

1 Codes and Puncturings

A code is a linear subspace of $F_q^A$, where $A$ is a finite set of $n$ elements. Usually one picks $A = \{1,2,\ldots,n\}$, thereby numbering the positions of the vectors. In this article we should find more useful to index the positions of a code with an arbitrary finite set. Puncturing codes is a fundamental operation in coding theory to make short codes from longer codes. We present their well–known definition.
Definition 1.1. Let $C$ be a code of $\mathbb{F}_q^A$. Let $B \subseteq A$. The projection of $C$ at $B$ is the code $C_B \subseteq \mathbb{F}_q^B$ obtained by projecting $C$ onto the coordinates given by $B$. (or equivalently discarding the coordinates which are not in $B$).

$$C_B := \{(c_i)_{i \in B} \mid (c_i)_{i \in A} \in C\}.$$  

Note that $C_B$ is a code of length $\#B$. Usually when $C_B$ has the same dimension as $C$ the code $C_B$ is also known as the puncturing of $C$ at $A \setminus B$. However in this article we interchange the terms projection and puncturing of a code.

2 Grassmann, Schubert and Schubert union varieties

In this section we state the definition of a well known variety known as the Grassmannian and some important subvarieties of the Grassmannian known as Schubert varieties and Schubert unions.

Definition 2.1. The Grassmannian $G_{\ell,m}$ is the set of all $\ell$ dimensional $\mathbb{F}_q$-linear subspaces of $\mathbb{F}_q^m$.

Definition 2.2. Let $X = (X_{i,j})$ be a $\ell \times m$ matrix on the $\ell m$ indeterminates $X_{i,j}$, where $\ell \leq m$. For $I \subseteq \{1, 2, \ldots, m\}$, where $\#I = \ell$, let $X_I$ denote the submatrix of $X$ obtained by taking the columns specified by $I$. The $\ell$-minor determined by $I$ is $\det(X_I)$.

Definition 2.3. Let $m$ be an integer. Suppose $\ell \leq m$. For each $W \in G_{\ell,m}$ pick an $\ell \times m$ matrix whose rowspace is $W$. Denote this matrix by $M_W$. Let $I_1, I_2, \ldots, I_{\binom{m}{\ell}}$ denote the $\binom{m}{\ell}$ subsets of $\{1, 2, \ldots, m\}$. The map

$$ev : G_{\ell,m} \to \mathbb{P}^{\binom{m}{\ell}-1}(\mathbb{F}_q)$$

$$ev(W) = (\det(X_{I_1}) : \det(X_{I_2}) : \cdots : \det(X_{I_{\binom{m}{\ell}}}))$$

is known as the Plücker embedding.

The Plücker embedding is a non degenerate embedding of $G_{\ell,m}$ into $\mathbb{P}^{\binom{m}{\ell}-1}(\mathbb{F}_q)$. This means we may also consider $G_{\ell,m}$ as a set of projective points. Now we present some special subsets of the Grassmannian, called Schubert varieties. We start with the notion of a flag.

Definition 2.4. Let $\mathbb{F}_q^m = \text{Span}_{\mathbb{F}_q}([a_1, a_2, \ldots, a_m])$. We define

$$A_i = \text{Span}_{\mathbb{F}_q}([a_1, a_2, \ldots, a_i])$$

for $i = 1, 2, \ldots, m$. The sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m = \mathbb{F}_q^m$ is called a flag.

Any sequence of basis elements of $\mathbb{F}_q^m$ determines a flag. Throughout this article we will work with only one fixed flag.
Definition 2.5. We define

\[ I(m, \ell) := \{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell \mid 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell \leq m \}. \]

The elements of \( I(m, \ell) \) are sets of \( \ell \) elements in increasing order. We may consider any \( \alpha \in I(m, \ell) \) as an unordered set as well. Thus we may consider \( I(m, \ell) \) as the set of subsets of \( \{1, 2, \ldots, m\} \) of size \( \ell \). One can define a partial order on \( I(m, \ell) \) as follows:

Definition 2.6. Let \( \alpha, \beta \in I(m, \ell) \). The Bruhat order is

\[ \alpha \leq \beta \text{ if and only if } \forall i = 1, 2, \ldots, \ell \quad \alpha_i \leq \beta_i. \]

Now we can give the definition of some particular subvarieties of the Grassmannian.

Definition 2.7. Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in I(m, \ell) \). The Schubert variety is

\[ \Omega_\alpha := \{ W \in G_{\ell,m} \mid \dim(W \cap A_{\alpha_i}) \geq i \}. \]

Definition 2.8. Let \( S \subseteq I(m, \ell) \). The Schubert union is

\[ \Omega_S := \bigcup_{\alpha \in S} \Omega_\alpha. \]

In case \( S = \{ \alpha \} \) it turns out that \( \Omega_S = \Omega_\alpha \). Therefore Schubert varieties are also Schubert unions. When \( S = \{ \alpha \} \) we denote \( \Omega_S \) by \( \Omega_\alpha \).

3 Grassmann, Schubert and Schubert union codes

Definition 3.1. \cite{9}

Let \( m \) be an integer. Suppose \( \ell \leq m \). For each \( W \in G_{\ell,m} \) pick an \( \ell \times m \) matrix whose rowspace is \( W \). Denote this matrix by \( M_W \). Define \( G \) as the following \( \#I(m, \ell) \times \#G_{\ell,m} \) matrix:

\[ G := (G_{I,W} = \det((M_W)_{I \in I(m, \ell), W \in G_{\ell,m}}). \]

We define the Grassmann code \( C(\ell, m) \) as the rowspace of \( G \).

Definition 3.1 is the linear code from the projective system of the points given by the embedding of \( G_{\ell,m} \) into \( \mathbb{P}^{(\ell)}(F_q) \). Although different matrices \( M_W \) representing the same \( W \in G_{\ell,m} \) give the same point in \( \mathbb{P}^{(\ell)}(F_q) \), they make a slight difference in the Grassmann code. Different choices of matrices result in different codes, but all the codes from this construction are monomially equivalent. These codes were introduced in \cite{9, 10} in the binary case and in \cite{7} for general \( q \). The parameters of the Grassmann code are \( [G_{\ell,m}, (\ell), q^{\ell(m-\ell)}] \).

Definition 3.2. \cite{8}

We define a Schubert code as the projection of \( C(\ell, m) \) onto \( \Omega_\alpha \) that is:

\[ C^\alpha(\ell, m) := C(\ell, m)\Omega_\alpha. \]
The usual definition of a Schubert code is to embed $\Omega_\alpha$ into a projective space by evaluating all the minors which do not vanish completely on $\Omega_\alpha$ as in the definition of the Grassmann codes. The definition we give in this article is equivalent as it gives exactly the same code. We prefer to define Schubert codes via code projections (or puncturings) to focus on the coding theoretical aspects of the Grassmannian and its subvarieties. The Grassmannian is also the Schubert variety $\Omega_\alpha$ where $\alpha$ is the maximal element of $I(\ell, m)$, $\alpha = (m - \ell + 1, m - \ell + 2, \ldots, m)$. Naturally the Grassmann code is also a Schubert code. Therefore the arguments presented here also apply to Grassmann codes. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in I(m, \ell)$ the parameters of $C_\alpha(\ell, m)$ are known. The dimension is $\#\{\beta \in I(m, \ell) \mid \beta \leq \alpha\}$. The dimension and length were determined in [3].

For $\delta_\alpha := \sum (\alpha_i - i)$ the minimum distance of the Schubert code was proven to be $q^{\delta_\alpha}$ in [5].

**Definition 3.3.** [6]
Let $S \subseteq I(m, \ell)$. A Schubert union code is defined as the projection of $C(\ell, m)$ on $\Omega_S$. That is

$$C^S(\ell, m) := C(\ell, m)_{\Omega_S}.$$ 

We also know the dimension and length of Schubert Union codes. From [8] we know that if $S \subseteq I(m, \ell)$ satisfies $\beta \leq \alpha$ and $\alpha \in S$ imply $\beta \in S$ then the dimension of $C^S(\ell, m)$ is $\#\{\beta \in I(m, \ell) \mid \exists \alpha \in S : \beta \leq \alpha\}$.

### 4 The geometry of Schubert unions

In this section we prove some theorems about the geometry of Schubert union. The geometrical features of Schubert unions are also reflected in Schubert union codes. These features will be used in order to determine the structure of Schubert union codes and their duals. We need some preliminary definitions and theorems first.

For an $\ell$ space $W$ we can keep track of the dimensions of the intersections with the different flag elements $A_i$.

**Definition 4.1.** Let $W \in G_{\ell, m}$ we define

$$[W] := (\alpha_1, \alpha_2, \ldots, \alpha_\ell) : \dim(W \cap A_{\alpha_i}) = i, \dim(W \cap A_{\alpha_{i-1}}) = i - 1.$$ 

Note that $[W] \in I(\ell, m)$. We may also use $[W]$ in a restatement of the definition of Schubert varieties, namely:

**Theorem 4.2.** The Schubert variety $\Omega_\alpha$ is equal to $\{W \in G_{\ell, m} \mid [W] \leq \alpha\}$.

**Definition 4.3.** [8] A line of $G_{\ell, m}$ is a set

$$\pi_Z^{Z'} := \{V \in G_{\ell, m} \mid Z \subseteq V \subseteq Z'\}$$

where $Z \in G_{\ell-1, m}$ and $Z' \in G_{\ell+1, m}$.
Any line of the Grassmannian is isomorphic to the projective line \( \mathbb{P}^1(F_q) \). Each line contains \( q + 1 \) elements of the Grassmannian. The Plücker embedding maps the lines of the Grassmannian to lines of \( \mathbb{P}^{(n)}(\mathbb{F}_q) \). In fact, the images of the lines of the Plücker embedding are exactly those lines of \( \mathbb{P}^{(n)}(\mathbb{F}_q) \) contained in the image of \( G_{\ell,m} \). The lines of the Grassmannian are also important for Grassmann codes.

**Theorem 4.4.** Let \( U, V, W \in G_{\ell,m} \) such that \( U, V, \) and \( W \) belong to the same line. Then \( U, V, \) and \( W \) are the support of a minimum weight codeword in \( C(\ell,m) \).

**Proof.** Suppose \( U, V, \) and \( W \) belong to \( \pi_{\mathbb{Z}}^{\ell} \). Then there exist \( x \) and \( y \) in \( \mathbb{F}_q^m \) such that \( U = \text{Span}(Z \cup \{x\}) \), \( V = \text{Span}(Z \cup \{y\}) \) and \( W = \text{Span}(Z \cup \{x + y\}) \). Moreover \( Z' = \text{Span}(Z \{x, y\}) \). The multilinearity of the determinant implies the conclusion. \( \square \)

**Lemma 4.5.** Let \( V, W \) be two linear spaces in \( \pi_{\mathbb{Z}}^{\ell} \cap \Omega_3 \). Then \( \pi_{\mathbb{Z}}^{\ell} \subseteq \Omega_3 \).

**Proof.** Suppose \( V \) and \( W \) belong to \( \pi_{\mathbb{Z}}^{\ell} \cap \Omega_3 \). Then there exist \( x \) and \( y \) in \( \mathbb{F}_q^m \) such that \( V = \text{Span}(Z \cup \{x\}) \) and \( W = \text{Span}(Z \cup \{y\}) \). Therefore all the points of the line are subspaces of \( Z' = \text{Span}(Z \cup \{x, y\}) \). Moreover \( \text{dim}(V \cap A_{\alpha_i}) \geq i \) and \( \text{dim}(W \cap A_{\alpha_i}) \geq i \). We need to prove that \( \text{dim}(\text{Span}_{\mathbb{F}_q}(U \cup \{x + \gamma y\}) \cap A_{\alpha_i}) \geq i \) for any \( \gamma \in \mathbb{F}_q^* \). Note that since \( \text{dim} Z = \ell - 1 \) and \( Z \subseteq V,W \) we have that then \( \text{dim} Z \cap A_{\alpha_i} \geq i - 1 \). In case \( \text{dim} Z \cap A_{\alpha_i} = i - 1 \) we have that \( \text{dim} \text{Span}_{\mathbb{F}_q}(Z \cup \{x + \gamma y\}) \cap A_{\alpha_i} \geq i \). Now we may assume \( \text{dim} Z \cap A_{\alpha_i} = i - 1 \). Since both \( \text{dim} (Z \cap A_{\alpha_i}) \geq i \) and \( \text{dim} (W \cap A_{\alpha_i}) \geq i \) we have that \( x,y \in A_{\alpha_i} \), therefore \( x + \gamma y \in A_{\alpha_i} \) and \( \text{dim}(\text{Span}_{\mathbb{F}_q}(Z \cup \{x + \gamma y\}) \cap A_{\alpha_i}) \geq i \). \( \square \)

**Lemma 4.6.** Let \( \alpha, \beta \in I(m, \ell) \) if \( \#(\alpha \cap \beta) = \ell - 1 \) then \( \alpha < \beta \) or \( \beta < \alpha \).

**Proof.** Let \( \gamma = \alpha \cap \beta = (s_1 < s_2 < \ldots < s_{\ell-1}) \). Without loss of generality we may suppose \( \alpha = \gamma \cup x \) and \( \beta = \gamma \cup y \) where \( x < y \). We will prove that \( \alpha < \beta \). If there exists \( s_i, s_{i+1} \) such that \( s_i < x < y < s_{i+1} \) then the conclusion is true. The conclusion is also true for \( x < y < s_1 \) and for \( s_{\ell-1} < x < y \). Now we may suppose \( x < s_i \leq s_j < y \). In this case

\[
\alpha = (s_1 < s_2 < \ldots < s_{i-1} < x < s_i < \ldots < s_j < s_{j+1} < \ldots s_{\ell-1})
\]

\[
\beta = (s_1 < s_2 < \ldots < s_{i-1} < s_i < \ldots < s_j < y < s_{j+1} < \ldots s_{\ell-1})
\]

\( \square \)

The fact that \( \#(\alpha \cap \beta) = \ell - 1 \) was referred to as \( \alpha \) and \( \beta \) are close in \([4]\). Subsets of \( I(m, \ell) \) where all elements are close are relevant to the Grassmann codes. We will relate close \( \ell \) sets to the lines of the Grassmannian. Later we will see how this is related to the Grassmann codes.

**Theorem 4.7.** Let \( \pi_{\mathbb{Z}}^{\ell} \) be a line of \( G_{\ell,m} \). There exists \( \alpha, \beta \in I(\ell,m) \) such that

- \( \#(\alpha \cap \beta) = \ell - 1 \)
- \( \pi_{\mathbb{Z}}^{\ell} \subseteq \Omega_3 \)
• \( \#(\pi_Z^{\prime} \cap \Omega \alpha) = 1 \)

• \( \alpha < \beta \)

Proof. From the definition of \( \pi_Z' \) we know \( Z' = \text{Span}(\{Z \cup \{x, y\}\}) \). The \( q + 1 \) elements of \( \pi_Z' \) are \( \text{Span}(\{Z \cup \{x\}\}) \) and \( \text{Span}(\{Z \cup \{\gamma x + y\}\}) \) for each \( \gamma \in \mathbb{F}_q \). Without loss of generality we may assume \( x \in A_i \setminus A_{i-1} \) and \( \gamma x + y \in A_j \setminus A_{j-1} \) for \( \gamma \in \mathbb{F}_q \) where \( i < j \).

Now we will prove that \( \text{Span}(\{Z \cup \{x\}\}) \) and \( \text{Span}(\{Z \cup \{\gamma x + y\}\}) \) satisfy the conditions of the theorem. Consider \( Z \). We may pick \( x \) and \( y \) such that \( i, j \notin [Z] \). The proof of Lemma 4.5 implies \( \text{Span}(\{Z \cup \{x\}\}) \) is \( Z \cup \{i\} \) and \( \text{Span}(\{Z \cup \{\gamma x + y\}\}) \) is \( [Z] \cup \{j\} \).

5 Tanner codes and forcing sets

Now we define Tanner codes. Essentially, a Tanner code is a long code made from shorter codes. We will make this definition precise in this section.

**Definition 5.1.** A bipartite graph \( G \) is a triple \((V_1(G), V_2(G), E(G))\) where \( V_1(G) \) and \( V_2(G) \) are finite sets and \( E(G) \subseteq V_1(G) \times V_2(G) \). The elements of \( V_1(G) \) and \( V_2(G) \) are called vertices. Two vertices \( v \) and \( u \) are adjacent if and only if \( (v, u) \in E(G) \). We say either \( v \) or \( u \) are incident to the edge \((v, u)\). The vertices \( v \) and \( u \) are the endpoints of the edge \((v, u)\).

**Definition 5.2.** Let \( G \) be a bipartite graph. For a vertex \( u \in V_2(G) \) we define the neighborhood of \( u \) as \( \{v \in V_1(G) \mid (v, u) \in E(G)\} \). The neighborhood of a vertex \( u \) is denoted by \( \mathcal{N}(u) \).

Note that for a bipartite graph \( G = (V_1(G), V_2(G), E(G)) \) we may consider \( V_2(G) \) as a collection of subsets of \( V_1(G) \) and the edge set \( E(G) \) defined by inclusion. In order to simplify the graph based code construction we impose a right regularity condition. We define right regularity as follows.

**Definition 5.3.** Let \( G \) be a bipartite graph. If \( \forall u \in V_2(G) : \#\mathcal{N}(u) = n_2 \) then \( G \) is a \( n_2 \)-right regular bipartite graph.

Note that a \( (n_2) \)-regular bipartite graph has \( \#V_2(G)n_2 \) edges.

**Definition 5.4.** Suppose \( G \) is an \( n_2 \)-right regular bipartite graph. Let \( C \) be a code of length \( n_2 \) over \( \mathbb{F}_q \). A Tanner code \((G, C)\) is a code which satisfies for any \( u \in V_2(G) \)

\[ (G, C)^{\mathcal{N}(u)} \]

is contained in a monomially equivalent subcode of \( C \).
The code \((G, C)\) is a code of length \(#V_1(G)\). We prefer to identify the positions of \((G, C)\) with the vertices of \(V_1(G)\) since the vertices of \(V_1(G)\) contain the symbols of the codewords. The vertices in \(V_1(G)\) are also known as the variable nodes. The vertices of \(V_2(G)\) are called the constraint nodes because they represent the parity check equations \((G, C)\) must satisfy. Tanner codes are also known as generalized LDPC codes.

Tanner code are also a natural method to make a long code from a short code \(C\) and an incidence structure given by the bipartite graph \(G\). For each \(u \in V_2(G)\), the vertices adjacent to \(u\) are a subset of \(V_1(G)\). The code \((G, C)\) is precisely the code such that for any \(u \in V_2(G)\) the punctured code of \((G, C)\) at \(N(u)\) is monomially and permutation equivalent to the component code \(C\).

**Definition 5.5.** Let \(G\) be an \(n_2\)-right regular graph. Let \(k\) be an integer satisfying \(k \leq n_2\). Let \(T \subseteq V_1(G)\). We say \(T\) is \(k\)-closed if \(T\) satisfies:

\[
\forall u \in V_2(G) \ (\#(T \cap N(u)) \geq k \rightarrow N(u) \subseteq T).
\]

That is if there are at least \(k\) vertices adjacent to \(u \in V_2(G)\) contained in \(T\) then all \(n_2\) vertices adjacent to \(u\) are also contained in \(T\).

In addition the notion of a \(k\)-closed set of vertices is closely related to the notion of irreducible \(k\)-threshold processes introduced in [2].

**Theorem 5.6.** Let \(G\) be an \(n_2\)-right regular graph. Let \(k\) be an integer satisfying \(k \leq n_2\). Let \(S \subseteq V_1(G)\). There exists an unique smallest \(k\)-closed set containing \(S\).

**Proof.** We define \(Z\) as follows \(Z = S \cup N(u_1) \cup N(u_2) \cup \ldots \cup N(u_a)\), where the subset \(Z_i := S \cup N(u_1) \cup N(u_2) \cup \ldots \cup N(u_i)\) satisfies \(Z_i \cap N(u_{i+1}) \geq k\) and \(Z\) is \(k\)-closed.

We claim that if \(Z'\) is another \(k\)-closed containing \(S\) then it must also contain \(Z\). Suppose \(Z'\) is \(k\)-closed and \(Z\) contains \(S\). Suppose \(Z \not\subseteq Z'\) then there exists \(Z_i\) such that \(Z_i \subseteq Z'\) but \(Z_{i+1} \not\subseteq Z'\). But since \(Z_i \cap N(z_{i+1}) \geq k\) it follows that \(Z_{i+1} \subseteq Z'\). Therefore \(Z \subseteq Z'\).

**Definition 5.7.** Suppose \(G\) is an \(n_2\)-right regular bipartite graph. Let \(S\) be a subset of \(V_1(G)\). We define the \(k\)-closure of \(S\) as the smallest \(k\)-closed set containing \(S\). We denote the \(k\)-closure of \(S\) by \(\text{cl}_{G,k}(S)\). If the \(k\)-closure \(\text{cl}_{G,k}(S)\) is \(V_1(G)\) then \(S\) is a \(k\)-forcing set.

This notion of \(k\)-closure is important because it gives us an upper bound on the dimension of the Tanner code \((G, C)\) provided the component code is MDS. This is also related to the original encoding algorithm Tanner proposed for Tanner codes. Furthermore the \(k\) forcing sets defined here are also related to the \(k\)-forcing sets introduced in [1] and the smallest irreversible \(k\)-conversion set in [2]. The next theorem relates the dimension of a Tanner code \((G, C)\) with the size of the forcing set.

**Theorem 5.8.** Let \(G\) be an \((n_2)\)-right regular bipartite graph. Let \(C\) be an MDS code of length \(n_2\) and dimension \(k\). Let \(S\) be a \(k\)-forcing set. Then \((G, C)\) is linearly isomorphic to the projection \((G, C)|^S\).
Proof. Consider $(G, C)$ and $(G, C)^S$, the puncturing of $(G, C)$ at $S$. There is a linear map from $(G, C)$ to $(G, C)^S$ where we map $c = (c_i)_{i \in V_1(G)}$ to $c_S = (c_i)_{i \in S}$. We will prove the kernel is zero dimensional. Let $c$ be a codeword of $(G, C)$ which is projected to the zero codeword in $(G, C)^S$. Therefore $c_i = 0$ for $i \in S$. For each $u \in V_2(G)$ once we know $c_i = 0$ for $k$ neighbors of $u$ have the zero symbol, we know $c_i = 0$ for all the neighbors of $u$. Therefore all the entries of $c$ indexed by an element of $cl_{G,k}(S)$ are zero. Since $S$ was assumed to be a $k$-forcing set, then $cl_{G,k}(S) = G$. Since only the zero codeword is mapped to zero, projection onto the coordinates given by $S$ is an embedding of $(G, C)$ into $(G, C)^S$.

As a corollary to Theorem 5.8 we also obtain a dimension bound and an iterative encoding algorithm for the Tanner code $(G, C)$.

Corollary 5.9. Let $G$ be an $(n_2)$-right regular bipartite graph. Let $C$ be an MDS code of length $n_2$ and dimension $k$. Let $S$ be a $k$-forcing set. Then $\dim(G, C) \leq \#S$.

Corollary 5.10. Let $G$ be an $(n_2)$-right regular bipartite graph. Let $C$ be an MDS code of length $n_2$ and dimension $k$. Let $S$ be a $k$-forcing set. Then a codeword $c' \in (G, C)^S$ may be extended uniquely to a codeword $c \in (G, C)$.

Proof. Let $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_m$ be a sequence of subsets of $V_1(G)$ which satisfying: $S_0 = S$, $S_m = cl_{G,k}(S)$ and for $i = 1, 2, \ldots m$ the set $S_i = S_{i-1} \cup \mathcal{N}(u_i)$ where $S_{i-1}$ satisfies $k \leq \#(S_{i-1} \cup \mathcal{N}(u_i)) < n_2$ and $u_i \in V_2(G)$. For a codeword $c \in (G, C)$ we will denote its projection onto the set $S_i$ by $\phi_{S_i}$. Theorem 5.8 implies that when $cl_{G,k}(S_i) = V_1(G)$ then $\phi_{S_i}$ is a linear isomorphism between $(G, C)$ and $(G, C)^{S_i}$. A codeword $c' \in (G, C)^{S_i}$ may be extended uniquely to a codeword in $c'' \in (G, C)^{S_{i+1}}$ by $c'' = \phi_{S_{i+1}}^{-1}(\phi_{S_i}^{-1}(c'))$. From the conditions on $S_{i+1}$ this is the same as specifying all the entries indexed by $\mathcal{N}(u_{i+1})$ provided we know the entries indexed by $S_i$. This is possible because we know at least $k$ entries in $\mathcal{N}(u_{i+1})$ and they must correspond to an MDS code on those positions.

6 Schubert union codes as Tanner codes

In this article we extend this result about the Grassmann code to Schubert union codes. Now we can determine the structure of dual Schubert union codes.

Definition 6.1. A line of $\Omega_S$ is a line of $\mathcal{G}_{\ell,m}$ which is also a subset of $\Omega_S$. We denote the set of all such lines by $\mathcal{L}(\Omega_S)$.

We remark that $\mathcal{G}_{1,2} = \Omega_{1,2,\ldots,\ell-1,\ell+1}$ is the line $\pi_{A_{\ell-1}}^{A_{\ell+1}}$ and $\mathcal{C}(1, 2)$ is the doubly extended Reed–Solomon code of dimension 2. It is a $[q + 1, 2, q]_q$ code. Note that the puncturing of $\mathcal{C}(\ell, m)$ or $\mathcal{C}(\ell, m)$ at any line is $\mathcal{C}(1, 2)$. Now we characterize the intersection of the lines of the Grassmannian with Schubert varieties.

Note that Theorem 4.7 implies that $\mathcal{L}(\Omega_S) = \bigcup_{\alpha \in S'} \mathcal{L}(\Omega_{\alpha'})$ where $S'$ is a set of maximal elements of $S$.
Lemma 6.3. \( \mathcal{C}^S(\ell, m) \subseteq (\Gamma_S, \mathcal{C}(1, 2)) \)

Proof. This is a restatement of the fact that \( \mathcal{C}^S(\ell, m^{\prime}) \equiv \mathcal{C}(1, 2) \) for any line of \( \Omega_S \). \( \square \)

The code \( \mathcal{C}(1, 2) \) is generated by its weight 3 codewords since it is the doubly extended Reed–Solomon code of minimum distance 3. Note that \( (\Gamma_S, \mathcal{C}(1, 2)) \) is generated by the parity checks of \( \mathcal{C}(1, 2) \) at the coordinates specified by \( \mathcal{N}(u) \) for each \( u \in V_2(\Gamma_S) \). Moreover since \( \mathcal{C}(1, 2) \) is generated by its weight 3 codewords the Tanner code \( (\Gamma_S, \mathcal{C}(1, 2)) \) is generated by the weight 3 codewords whose support is contained in a line of the Schubert variety. We will use 2-forcing sets to prove equality between the Schubert code and the Tanner code which implies the dual Schubert code is generated by its weight 3 codewords.

Definition 6.4. For \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in I(m, \ell) \) we define

\[ J_\alpha := \{ \beta \in I(m, \ell) \mid \beta \leq \alpha \}. \]

For \( J_\alpha \subseteq I(m, \ell) \) we define

\[ \mathcal{G}_{\ell, m}(J_\alpha) := \{ W \in \mathcal{G}_{\ell, m} \mid W = \text{Span}_{\mathbb{F}_q}(\{ a_i \}_{i \in I}) \ I \in J_\alpha \}. \]

We remark that \( \mathcal{G}_{\ell, m}(J_\alpha) \subset \Omega_\alpha \) and that \( \# J_\alpha = \dim \mathcal{C}^\alpha(\ell, m) \). \( \square \)

Theorem 6.5. The \# \( J_\alpha \) spaces in \( \mathcal{G}_{\ell, m}(J_\alpha) \) are a 2-forcing set for \( \Omega_\alpha \).

Proof. We will prove it by induction on \( m, \ell \) and the Bruhat order on \( I(m, \ell) \) given in Definition 2.6.

For \( \alpha = (1, 2, \ldots, \ell) \) the equality \( \mathcal{G}_{\ell, m}(J_\alpha) = \Omega_\alpha = A_\ell \) holds. In this case, the theorem is vacuously true.

For \( \alpha = (1, 2, \ldots, \ell-1, \ell+1) \), the set \( \Omega_\alpha \) is equal to \( \pi A_{\ell-1}^{\ell+1} \). Therefore \( \Omega_\alpha \) is isomorphic to \( \mathcal{G}_{1, 2} \). But \( J_\alpha \) is \( \{ (1, 2, \ldots, \ell-1, \ell), (1, 2, \ldots, \ell-1, \ell+1) \} \). We know that \( J_\alpha \) is a 2-forcing set for \( \Omega_\alpha \) since \( \Omega_\alpha \) is a line.

Now let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in I(m, \ell) \). By the induction hypothesis we assume the theorem holds for any \( \alpha' \in I(m, \ell) \) such that \( \alpha' < \alpha \) and we also assume the theorem holds for \( \alpha'' = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}) \in I(m, \ell-1) \).

Let \( U \in \Omega_\alpha \). If \( \dim(U \cap A_{\alpha_{\ell-1}}) = \ell \) then \( U \in \Omega_\beta \) where \( \beta_i \) is defined as the integer such that \( \dim U \cap A_{\beta_i} = i \) and \( \dim U \cap A_{\beta_i} = i \) and clearly \( \beta \leq \alpha \). Therefore \( U \) is in \( A_{\ell-1}(\mathcal{G}_{\ell, m}(J_\beta)) \).

Now we may assume \( \dim(U \cap A_{\alpha_{\ell-1}}) = \ell-1 \). Now we consider the highest consecutive entries of \( \alpha \). That is \( \alpha_{\ell-\delta} = \alpha_\ell - \delta \) for \( \delta = 0, 1, 2, \ldots, j \) but \( \alpha_{\ell-\jmath-1} < \alpha_\ell - j - 1 \). Now we shift the highest consecutive entries of \( \alpha \) by \(-1\), that is we define \( \alpha' = \)
(α₁, α₂, ..., α₇, α₈, α₉, α₁₀). If U ∈ Ωₙ, then U is in the 2-closure of $G_{ℓ,m}(J/α)$. If U ∉ Ωₙ, then U = $Span_{Fₚ}(U'' \cup \{x + a_α\})$, where $U'' \in Ωₙ \setminus \{α\} \subseteq G_{m-1,ℓ-1}$. If x ∈ U then U belongs to the closure of $G_{ℓ,m}$. Otherwise U is in the line given by $Span_{Fₚ}(U'' \cup \{a_α\})$ and $Span_{Fₚ}(U'' \cup \{x\})$. Since $Span_{Fₚ}(U'' \cup \{a_α\})$ is in the 2-closure of $G_{ℓ,m}(J/β)$ and $Span_{Fₚ}(U'' \cup \{x\})$ is in the 2-closure of $G_{ℓ,m}(J/α)$, the conclusion is true.

Therefore the results on Schubert codes and the 2-closure also extend naturally to Schubert union codes as follows.

**Theorem 6.6.** Let $S \subseteq I(ℓ,m)$. Then

$$cl_{G_{ℓ,m}}(Ω) = ∪_{α \in S} cl_{G_{ℓ,m}}(Ω).$$

**Proof.** Let S′ be the set of the maximal elements of S. Since α ≤ β implies Ωₙ ≤ Ωₚ we have that $∪_{α \in S} cl_{G_{ℓ,m}}(Ω) = ∪_{α \in S'} cl_{G_{ℓ,m}}(Ω)$. Now clearly $∪_{α \in S} cl_{G_{ℓ,m}}(Ω) \subseteq cl_{G_{ℓ,m}}(Ω)$. Theorem 6.7 implies that for two spaces in $Ω S$ there is a line between them if and only if they belong to the same $Ω α$. Therefore there are no other lines in $cl_{G_{ℓ,m}}(Ω)$ except those in $∪_{α \in S'} cl_{G_{ℓ,m}}(Ω)$. Therefore $cl_{G_{ℓ,m}}(Ω) = ∪_{α \in S} cl_{G_{ℓ,m}}(Ω).$

**Theorem 6.7.** Let $S \subseteq I(ℓ,m)$ such that if α ∈ S and β ≤ α then β ∈ S. Then the set $G_{ℓ,m}(S) := ∪_{α \in S} G_{ℓ,m}^{J/α}$ is a 2-forcing set.

**Proof.** Theorem 6.6 implies the 2-closure of $G_{ℓ,m}^{J/α}$ is $Ω α$. Let S′ be the set of maximal elements of S. Since Theorem 6.6 implies $cl_{G_{ℓ,m}}(Ω) = ∪_{α \in S} cl_{G_{ℓ,m}}(Ω)$. The closure of the Schubert union is $cl_{G_{ℓ,m}}(Ω) = ∪_{α \in S} cl_{G_{ℓ,m}}(Ω) = ∪_{α \in S} cl_{G_{ℓ,m}}(Ω)$. The conclusion is true.

**Lemma 6.8.**

$$C^S(ℓ,m) = (Γ_S, C(1,2))$$

**Corollary 6.9.** The code $C^S(ℓ,m)$ is generated by its minimum weight codewords.

Now we can easily prove the following theorems about the generalized Hamming weights of Schubert union codes.

**Theorem 6.10.** Let D be a proper, nonempty subcode of $C^S(ℓ,m)$. Then there exists $x \in C^S(ℓ,m)$ such that $Span_{Fₚ}(D \cup \{x\})$ has support either #supp(D) + 1 or #supp(D) + 2.

**Proof.** Since D is a proper, nonempty subcode of $C^S(ℓ,m)$, {∅} ≠ supp(D) ≠ $Ω S$. Therefore there exist $U \in supp(D)$ and W ∉ supp(D). Since we can find $V_1, V_2, ..., V_n$ in $Ω S$ such that $V_i \in Ω S$, $V_1 = U$, $V_n = W$ and dim $V_i \cap V_{i-1} = ℓ - 1$. Since in the sequence $V_1, V_2, ..., V_n$ there are two consecutive elements such that one is in supp(D) and the other one is not, we may assume dim $U \cap W = ℓ - 1$. If x is a codeword of weight 3 of $C^S(ℓ,m)$ which has support in $U$ and W then supp(x) = {U, V, W}, and supp($Span_{Fₚ}(D \cup \{x\})$) = supp(D) ∪ supp(x) which finishes the proof.
Corollary 6.11. The generalized Hamming weights of $C^S(\ell, m)\perp$ satisfy:

$$d_{i+1} - d_i \in \{1, 2\}$$

as long as $d_i \neq 0, \#\Omega_S$.

Proof. It follows from non degeneracy conditions introduced in [12] and Theorem 6.10

This also lends credence to the conjecture in [6] that the generalized Hamming weights of Grassmann codes are realized by Schubert unions.

conclusion

We have proven that the dual Schubert union codes are generated by their minimum weight codewords. From this we can prove that Schubert union codes (including Grassmann codes and Schubert codes) can be constructed from the simplest Grassmann codes, $C(1, 2)$, from the geometry of the Grassmannian. We also proved that the dual codes have a low increase in their generalized Hamming weights. We hope this will be an important step to find the generalized Hamming weight distribution of the Grassmannian.

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