NONDEGENERACY FOR STABLE SOLUTIONS TO THE ONE-PHASE FREE BOUNDARY PROBLEM

NIKOLA KAMBUROV∗ AND KELEI WANG†

Abstract. We prove the nondegeneracy condition for stable solutions to the one-phase free boundary problem. The proof is by a De Giorgi iteration, where we need the Sobolev inequality of Michael and Simon and, consequently, an integral estimate for the mean curvature of the free boundary. We then apply the nondegeneracy estimate to obtain local curvature bounds for stable free boundaries in dimension \( n \), provided the Bernstein-type theorem for stable, entire solutions in the same dimension is valid. In particular, we obtain this curvature estimate in \( n = 2 \) dimensions.

1. Introduction

In this article we study stable solutions to the one-phase free boundary problem

\[
\begin{align*}
& u \geq 0 \quad \text{in } D, \\
& \Delta u = 0 \quad \text{in } D^+ := \{ x \in D : u(x) > 0 \}, \\
& |\nabla u| = 1 \quad \text{on } F(u) := \partial D^+ \cap D.
\end{align*}
\]

Here, \( D \subset \mathbb{R}^n \) is a domain, \( D^+ \) denotes the positive phase of \( u \) in \( D \) and \( F(u) \) is its free boundary. We will be mainly interested in classical solutions, that is: \( u \) is continuous in \( D \), the free boundary \( F(u) \) is a \( C^\infty \) hypersurface with \( u = 0 \) on one side, \( u > 0 \) on the other, and \( u \in C^\infty(D^+(u)) \) satisfies the gradient condition \( |\nabla u| = 1 \) in pointwise sense, evaluated from the positive side.

The one-phase free boundary problem (FBP) arises as the Euler-Lagrange equations for the energy functional

\[
J(v, D) = \int_D (|\nabla v|^2 + 1_{\{v > 0\}}) \, dx, \quad v : D \to [0, \infty),
\]

and appears in various interface models in fluid mechanics and materials science. There is vast literature on it – see for example the books \[6\] and \[15\]. A seminal role in its study is played by the 1981 paper \[3\] of Alt and Caffarelli on minimizers of \( J \), in which the authors pioneered the use of blow-up limits to investigate the regularity of the free boundary. In the context of the one-phase FBP, the blow-up analysis is rooted in two basic estimates. For the sake of simplicity, we shall state them for solutions of (1) defined in the unit ball \( D = B_1(0) \). The first fundamental

\[2020\] Mathematics Subject Classification. 35R35; 35B36.

Key words and phrases. one-phase free boundary problem; stable solutions.

N. Kamburov was partially supported by Proyecto Fondecyt Regular No. 1201087. K. Wang was supported by the National Natural Science Foundation of China (No. 11871381 and No. 12131017).
estimate is the uniform Lipschitz bound that any solution to \((1)\) in \(B_1\) satisfies: if \(0 \in F(u)\), then
\[
|\nabla u(x)| \leq C \quad \text{for all } x \in B_{1/2}^+(u),
\]
for some positive constant \(C\) that depends only on the dimension \(n\). The second one is the uniform nondegeneracy bound:
\[
\int_{\partial B_r(p)} u \, d\mathcal{H}^{n-1} \geq cr^n \quad \text{for all } p \in F(u) \cap B_{1/4} \quad \text{and all } r \in (0, 1/4),
\]
for some dimensional constant \(c > 0\). Though valid for one-phase energy minimizers \(\text{(3)}\), the nondegeneracy condition \(\text{(4)}\) does not hold for all solutions, as exhibited by the family of solutions \(\{u_\varepsilon\}_{\varepsilon \in (0, 1)}\) in \(B_1\),
\[
u_\varepsilon(x) = \begin{cases} 
\varepsilon \left( \log \frac{|x|}{\varepsilon} \right)^+, & \text{if } n = 2, \\
\varepsilon^{-n-2} \left[ 1 - \left( \frac{|x|}{\varepsilon} \right)^{2-n} \right]^+, & \text{if } n \geq 3,
\end{cases}
\]
when \(\varepsilon > 0\) is small enough.

In their paper \(\text{[3]}\) Alt and Caffarelli used a convenient measure-theoretic reformulation of \(\text{(3)}\) and \(\text{(4)}\). Denote
\[
\mu := \mathcal{H}^{n-1}|_{F(u)}.
\]
The Lipschitz bound \(\text{(3)}\) means that for some dimensional constant \(C > 0\),
\[
\mu(B_r(x)) \leq Cr^{n-1} \quad \text{for all } x \in F(u) \cap B_{1/4}, \quad \text{and all } r \in (0, 1/4).
\]
This upper measure estimate can be easily deduced by applying the Divergence Theorem to \(\Delta u\) in \(B_{1/2}^+(u) \cap B_r(x)\), and using \(\text{(3)}\) as well as the free boundary condition, \(|\nabla u| = 1\) on \(F\). In turn, the lower estimate
\[
\mu(B_r(x)) \geq cr^{n-1} \quad \text{for all } x \in F(u) \cap B_{1/4}, \quad \text{and all } r \in (0, 1/4),
\]
where \(c > 0\) is a dimensional constant, corresponds to the nondegeneracy condition \(\text{(4)}\). The equivalence of the bounds \(\text{(3)}\)-\(\text{(4)}\) to their measure-theoretic counterparts \(\text{(5)}\)-\(\text{(6)}\) is the statement of \(\text{[3, Theorem 4.3]}\).

We will be interested in solutions \(u\) of \(\text{(1)}\) which are stable critical points of the functional \(J\) with respect to compactly supported domain deformations:
\[
\frac{d^2}{dt^2} \bigg|_{t=0} J(u(x + t\Phi(x)), D) \geq 0 \quad \text{for all vector fields } \Phi \in C_0^\infty(D, \mathbb{R}^n).
\]
For classical solutions, this condition takes the form of the so called stability inequality (see \(\text{[4]}\) for its derivation):
\[
\int_{F(u)} H \phi^2 \, d\mu \leq \int_{\Omega} |\nabla \phi|^2 \, dx, \quad \text{for all } \phi \in C_0^\infty(D),
\]
where \(H\) denotes the mean curvature of the free boundary with respect to the inner unit normal vector to the positive phase \(\Omega := D^+(u)\). The principal goal of this paper is to show that the nondegeneracy condition \(\text{(6)}\) (equivalently, \(\text{(4)}\)) holds, more generally, for stable solutions.
Theorem 1.1. There exists a constant $\varepsilon(n)$ such that if $u$ is a stable classical solution of (1) in $B_1$, then for any $x \in F(u) \cap B_{1/4}$ and $r \in (0, 1/4)$,

$$
\mu(B_r(x)) \geq \varepsilon(n) r^{n-1}.
$$

Alt and Caffarelli [3] showed that the nondegeneracy condition (4) is valid for energy minimizing solutions to the one-phase FBP. Soon thereafter, it was also established for energy minimizers within the theory of two-phase FBP [2]. More recently, non-degeneracy bounds have underlied the studies of the regularity theory for energy minimizing solutions to the “thin” one-phase FBPs ([1, 5, 12, 13]); for “almost minimizers” of the one-phase functional $J$ (see [9, 10, 14]), as well as for energy minimizing solutions to a vectorial analogue of (1) (see [7, 22, 23]). In all of these, the estimate is obtained via energy comparison methods. In works on free boundary solutions that are higher critical points of their underlying energy functionals, the nondegeneracy bound has been achieved either by viewing the solution as a constrained minimizer, as in [18, 21], and employing energy comparisons again, or by utilizing certain given topological constraints on the free boundary ([17]).

None of these methods are available in our setting. We obtain the result of Theorem 1.1 by performing a De Giorgi iteration that takes place on the free boundary surface itself. The stability inequality is used to obtain an $L^1$ estimate of the mean curvature $H$ in terms of the area of the free boundary, contained in a slightly larger scale (Lemma 2.2). We then employ the Sobolev inequality of Michael and Simon [24] to bound the area in a smaller scale and close the iteration loop. The proof of the key mean curvature estimate, Lemma 2.2, is delicate and involves plugging in a novel test function based on the gradient of $v = h - u$, where $h$ is the harmonic replacement of the solution $u$ in $B_1$ – rather than the more commonly used, natural test functions based on $|\nabla u|$.

The fact that stable solutions enjoy the nondegeneracy estimate opens up the toolbox of blow-up techniques, with which to pry the geometry of the free boundary. The second result of our paper involves obtaining interior curvature bounds for stable one-phase free boundaries, provided that the global problem in the same dimension is rigid. More precisely, we show that if one knows that the Bernstein type statement

(8) if $U$ is a classical stable solution of (1) in $\mathbb{R}^n \implies F(U)$ is flat

holds in some dimension $n$, then stable free boundaries in $B_1 \subset \mathbb{R}^n$ have uniformly bounded curvature on a smaller scale. This is expressed in the following theorem.

Theorem 1.2. Assume that (8) is true for some $n \geq 2$, and let $u$ be a stable classical solution to the one-phase FBP (1) in $B_1 \subset \mathbb{R}^n$, with $0 \in F(u)$. Then there exists a constant $C = C(n)$ such that the second fundamental form $A$ of the free boundary $F(u)$ satisfies

(9) $|A(p)| \text{dist}(p, \partial B_{1/4}) \leq C$ for all $p \in F(u) \cap B_{1/4}$.

In particular, the curvature $|A|$ of $F(u) \cap B_{1/8}$ is bounded by a dimensional constant.

We remark that the free boundary curvature bound (9) means that for some dimensional constant $c > 0$, the connected component of $B^+_c(u)$, whose boundary
contains the origin, is the supergraph of a function \( f \) with bounded \( C^2 \)-norm. By the classical regularity theory for the one-phase FBP [20], one then gets higher order derivative estimates for \( f \).

Theorem 1.2 is the free boundary analogue of a well known relation in minimal surface theory: between Bernstein-type results for complete, stable minimal hypersurfaces on the one hand, and local curvature estimates for stable minimal hypersurfaces on the other (see [27, Chapter 3]). As such, its proof employs a similar compactness argument, which in our setting is made possible because of the non-degeneracy estimate of Theorem 1.1.

The problem of finding the dimensions \( n \), for which the rigidity statement (8) is true, is open and akin to the so called “Stable Bernstein Problem” for minimal hypersurfaces in \( \mathbb{R}^n \) (see [8]). The recent construction by De Silva, Jerison and Shalgholian [11] of entire, energy minimizing, classical solutions of (1) that are asymptotic to the (non-flat) energy minimizing cone of De Silva-Jerison [25] in \( \mathbb{R}^7 \), implies that \( n \leq 6 \). Note that the stable Bernstein problem is a little different from the energy minimizing case (see Caffarelli-Jerison-Kenig [4] and Jerison-Savin [19]), because now the blowing down limit of entire solutions could be the wedge solution \(|x_n|\). Entire solutions with such an asymptotic behavior have been constructed in Hauswirth-Hélein-Pacard [16] (in dimension 2) and Liu-Wang-Wei [21] (in higher dimensions), whose free boundaries are of catenoid type. Although these known examples are unstable, it is not clear in general how to exclude such a possibility by only using the stability condition.

Here we show the veracity of (8) in dimension \( n = 2 \) (see Theorem 3.1 of Section 3), using the logarithmic cut-off trick. As a corollary, stable free boundaries, defined in disks of \( \mathbb{R}^2 \), enjoy interior curvature estimates.

**Corollary 1.3.** Let \( u \) be a stable classical solution to the one-phase FBP (1) in \( B_1 \subset \mathbb{R}^2 \), and assume \( 0 \in F(u) \). Then there exists an absolute constant \( C \) such that the curvature \( \kappa \) of the free boundary \( F(u) \) satisfies

\[
|\kappa(p)|\text{dist}(p, \partial B_{1/4}) \leq C \quad \text{for all } p \in F(u) \cap B_{1/4}.
\]

In particular, the curvature \( \kappa \) of \( F(u) \cap B_{1/8} \) is bounded by an absolute constant.

In the next section we prove Theorem 1.1. In Section 3 we establish Theorem 1.2 as well as the Bernstein type Theorem 3.1 concerning entire stable solutions of (1) in \( \mathbb{R}^2 \). In the Appendix we provide the necessary technical results for the proof of Theorem 1.2.

**2. Proof of Theorem 1.1**

In what follows the letters \( C, c \) (possibly with indices and primes) will denote positive constants which depend only on the dimension \( n \), and which may change from line to line. The ball of radius \( r \), centered at \( x \in \mathbb{R}^n \), is denoted by \( B_r(x) \) and \( B_r := B_r(0) \). By \( \mathcal{H}^k \) we will denote the Hausdorff measure of dimension \( k \). For ease of notation, we will often write \( F := F(u) \) and \( \Omega := B_1^+(u) \).

We start with the following auxiliary result about harmonic functions.
Lemma 2.1. Let \( v \) be a harmonic function in a domain \( D \subseteq \mathbb{R}^n \). Then
\[
|\nabla^2 v|^2 - |\nabla|\nabla v|^2 \leq c|\nabla v|^2 \leq |\nabla|\nabla v||^2 \leq |\nabla^2 v|^2\quad \text{a.e. in } D,
\]
for the constant \( c = (2(n - 1))^{-1} \).

Proof. If \( v = \text{const} \), then (11) obviously holds. Assume that \( v \) is a nontrivial harmonic function in \( D \). Then \( \nabla v(p) \neq 0 \) a.e. \( p \in D \). Fix such a point \( p \) and choose a Euclidean coordinate system \((x_1, \ldots, x_n)\) so that the unit vector along \( x_n \), \( e_n = \nabla v(p)/|\nabla v(p)| \). In this way, \( v_i(p) = 0 \) for \( i \in S := \{1, 2, \ldots, (n - 1)\} \) and \( v_n(p) = |\nabla v(p)| \). We then compute at \( p \) that
\[
|\nabla|\nabla v|^2 = 4v_n^2 \sum_j v_{nj}^2,
\]
thereby
\[
|\nabla^2 v|^2 - |\nabla|\nabla v||^2 = \sum_{i,j} v_{ij}^2 - \sum_j v_{nj}^2 = \frac{1}{2} |\nabla^2 v|^2 + \frac{1}{2} \sum_{i,j \in S} v_{ij}^2 - \frac{1}{2} v_{nn}^2.
\]
Now, since \( v \) is harmonic, we have \( v_{nn} = -\sum_{i \in S} v_{ii} \) and the AM-GM inequality yields
\[
\frac{v_{nn}^2}{n - 1} \leq \sum_{i \in S} v_{ii}^2.
\]
Combining it with (12), we conclude that at \( p \),
\[
|\nabla^2 v|^2 - |\nabla|\nabla v||^2 \geq \frac{1}{2} \left( |\nabla^2 v|^2 + \sum_{i \in S} v_{ii}^2 - v_{nn}^2 \right) \geq \frac{1}{2} \left( |\nabla^2 v|^2 - \left( 1 - \frac{1}{n - 1} \right) v_{nn}^2 \right) \geq \frac{1}{2} \frac{1}{2(n - 1)} |\nabla^2 v|^2.
\]

We now formulate and prove a key integral estimate for the mean curvature of stable free boundaries.

Lemma 2.2. Let \( u \) be a stable classical solution of (1) in \( B_1 \subset \mathbb{R}^n \), \( n \geq 3 \), which satisfies \( |\nabla u| \leq C \) in \( B_1^+ (u) \). Assume that \( 0 \in F \). There exist two universal constants \( \varepsilon_1 \) (small) and \( C_1 \) (large) such that if
\[
\mu(B_1) = \varepsilon \leq \varepsilon_1,
\]
then for any \( r \in (C_1 \frac{1}{\varepsilon^{n(n-1)}}, 1) \),
\[
\int_{F \cap B_1 - r} |H| \, d\mu \leq C_1 r^{-2} \varepsilon^{\frac{n}{n-1}}.
\]

Proof. We divide the proof into four steps.

Step 1. A decomposition of \( u \).
Write \( u = h - v \), where \( h \) is the harmonic extension of the boundary values of \( u \) on \( \partial B_1 \) to \( B_1 \), while \( v \) solves
\[
\begin{cases} 
-\Delta v = \mu & \text{in } B_1, \\
v = 0 & \text{on } \partial B_1.
\end{cases}
\]
Since \( u \) is Lipschitz continuous on \( \partial B_1 \), standard elliptic regularity theory implies that \( h \in C^{1/2}(\overline{B_1}) \cap C^\infty(B_1) \). Thus, \( v \in C^{1/2}(\overline{B_1}) \) and is locally Lipschitz continuous.
in $B_1$. In fact, $v$ is smooth in $\overline{\Omega}$ and $\overline{B_1 \setminus \Omega}$, but its gradient jumps when crossing the free boundary $F$.

We claim that

\begin{equation}
\sup_{B_1} v \lesssim \varepsilon^{\frac{1}{n-1}}. \tag{15}
\end{equation}

\textbf{Proof of (15).} Let $v^*$ be the solution of the Poisson equation

$$-\Delta v^* = \mu|_{B_1}, \quad \text{in } \mathbb{R}^n$$

given by the Newtonian potential

$$v^*(x) = c_n \int_{B_1} |x - y|^{2-n} d\mu(y), \quad \forall x \in \mathbb{R}^n.$$ 

Take $\rho > 0$, and divide the above integral into two parts

$$v^*(x) = \int_{B_1 \cap B_\rho(x)} + \int_{B_1 \setminus B_\rho(x)} =: I + II.$$ 

For the first one, we get by (5) that

$$I \lesssim \sum_{k=0}^{+\infty} (2^{-k}\rho)^{2-n} \mu(B_{2^{-k}\rho}(x)) \lesssim \sum_{k=0}^{+\infty} 2^{-k}\rho \lesssim \rho.$$ 

In the second part we have $|x - y| > \rho$, so

II $\lesssim \rho^{2-n}\mu(B_1).$

Combining these two estimates by choosing $\rho := \mu(B_1)^{1/(n-1)}$, we get

$$\sup_{\mathbb{R}^n} v^* \lesssim \mu(B_1)^{\frac{1}{n-1}}.$$ 

By the maximum principle, $v \leq v^*$ in $B_1$, and (15) follows. \hfill \Box

The first consequence of (15) is that

\begin{equation}
\int_{B_1} |\nabla v|^2 \, dx = -\int_{B_1} v \Delta v \leq C\varepsilon^{\frac{n}{n-1}}, \tag{17}
\end{equation}

after an integration by parts using (14) and the bound (15).

The second consequence is that the derivatives of $h$ on the free boundary $F$ are small. Indeed, since $u = 0$ on $F$, we have

$$h(p) = v(p) \leq C\varepsilon^{\frac{1}{n-1}} \quad \text{for all } p \in F.$$ 

Furthermore, since $h$ is positive and harmonic in $B_r(p) \subset B_1$, for $r = d(p, \partial B_1)$, Harnack’s inequality tells us that

\begin{equation}
h(x) \leq ch(p) \leq c_1\varepsilon^{1/(n-1)} \quad \text{for all } x \in B_{r/2}(p). \tag{18}
\end{equation}
Thus, by interior derivative estimates for harmonic functions, we get

$$\nabla h(p) \leq \frac{C\varepsilon^{\frac{1}{n-1}}}{r}, \quad |\nabla^2 h(p)| \leq \frac{C\varepsilon^{\frac{1}{n-1}}}{r^2} \quad \text{for } p \in F \text{ and } r = d(p, \partial B_1).$$

As $|\nabla u| = 1$ on $F$, (19) implies that,

$$\inf_{F \cap B_{1-r/2}} |\nabla v| \geq 1 - 2Cr^{-1}\varepsilon^{\frac{1}{n-1}} \geq 3/4,$$

$$\sup_{F \cap B_{1-r/2}} |\nabla v| \leq 1 + 2Cr^{-1}\varepsilon^{\frac{1}{n-1}} \leq 5/4,$$

whenever $r \geq C_1\varepsilon^{1/(n-1)}$, where $C_1$ is sufficiently large and $\varepsilon \leq \varepsilon_1$ sufficiently small.

**Step 2. Integral estimate on $H^-$**

Let us decompose the mean curvature $H$ of the free boundary $F$ into its positive and negative parts:

$$H = H^+ - H^-.$$

In this second step we will obtain the following integral estimate for $H^-$:

$$\int_{F \cap B_{1-r}} H^- \, d\mu \leq C\varepsilon^{n/(n-1)}r^{-2}, \quad \text{provided } r \geq (C_1\varepsilon^{1/(n-1)})^{1/n}.$$

For the purpose, observe that if $\nu$ denotes the outer unit normal to the positive phase $\Omega$ and

$$w := \frac{1}{2}(|\nabla u|^2 - 1),$$

then $H = w_{\nu}$ on $F$ and

$$-H^- = -\partial_{\nu}[w^+] \quad \text{on } F.$$

Take $\eta \in C^\infty_0(B_{1-r/2})$ to be a standard cut-off function, such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_{1-r}$ and $|\nabla \eta| \leq 4/r$. Applying the Divergence Theorem to $\eta^2\nabla[\rho_\varepsilon(w)]$ in $\Omega$, where $\rho_\varepsilon(t)$ is a standard, convex regularization of $t^+$, we get after taking the $\lim_{\varepsilon \to 0}$:

$$\int_F H^-\eta^2 \, d\mu \leq -\int_{\{\nabla u > 1\}} \eta^2 \Delta w \, dx - \int_{\{\nabla u > 1\}} \nabla \eta^2 \cdot \nabla w \, dx$$

$$= -\int_{\{\nabla u > 1\}} |\nabla^2 u|^2 \eta^2 \, dx - \int_{\{\nabla u > 1\}} 2|\nabla u|\eta (\nabla |\nabla u| \cdot \nabla \eta) \, dx$$

$$\leq -\int_{\{\nabla u > 1\}} (|\nabla^2 u|^2 - |\nabla |\nabla u||^2) \eta^2 \, dx + \int_{\{\nabla u > 1\}} |\nabla u|^2 |\nabla \eta|^2 \, dx,$$

$$\leq C \int_{\{\nabla u > 1\}} |\nabla \eta|^2 \, dx \leq Cr^{-2}|\{\nabla u > 1\} \cap B_{1-r/2}|$$

$$\leq Cr^{-2}|\{\nabla v > 1 - |\nabla h|\} \cap B_{1-r/2}|.$$

In the computation above, we have used the facts that: $\Delta |\nabla u|^2 = 2|\nabla^2 u|^2$ in $\Omega$ on account of $u$ being harmonic in $\Omega$, that $(|\nabla^2 u|^2 - |\nabla |\nabla u||^2) \geq 0$, and that $|\nabla u| \leq c.$
Now, since \( u(0) = 0 \), we have \( h(0) = v(0) \leq c \varepsilon^{1/(n-1)} \), so that Harnack’s inequality yields

\[
0 \leq h \leq c_0 h(0) r^{1-n} \leq c \varepsilon^{1/(n-1)} r^{1-n} \quad \text{in } B_{1-r/4}.
\]

Hence, by interior derivative estimates

\[
|\nabla h| \leq C \varepsilon^{1/(n-1)} r^n \leq \frac{1}{2} \quad \text{in } B_{1-r/2} \quad \text{as long as } r \geq (C \varepsilon^{1/(n-1)})^{1/n},
\]

for some large absolute constant \( C_1 \). Combining (24) with the estimate in (23), we obtain by the Chebyshev inequality the desired bound:

\[
\int_{F \cap B_{1-r}} H^{-} \, d\mu \leq C r^{-2} \{ |\nabla v| > 1 - |\nabla h| \} \cap B_{1-r/2} \leq C r^{-2} \{ |\nabla v| > 1/2 \} \cap B_{1-r/2} \leq 4 C r^{-2} \int_{B_1} |\nabla v|^2 \leq C' \varepsilon^{n/(n-1)} r^{-2}, \quad \text{provided } r \geq (C \varepsilon^{1/(n-1)})^{1/n},
\]

where the last inequality above follows from (17).

**Step 3. Using the stability condition.**

Fix \( r \in (C \varepsilon^{1/(n-1)}, 1) \) and take the same cut-off function \( \eta \in C^\infty_0(B_{1-r/2}) \) from Step 2. Plugging \( |\nabla v| \eta \) into the stability inequality (7) and integrating by parts, we obtain

\[
\int_{F \cap B_1} H |\nabla v|^2 \eta^2 \, d\mu \leq \int_{\Omega \cap B_1} |\nabla (|\nabla v| \eta)|^2 \, dx
\]

\[
= \int_{\Omega \cap B_1} |\nabla v|^2 |\nabla \eta|^2 + |\nabla |\nabla v||^2 \eta^2 + 2 |\nabla v| \eta |\nabla v| \cdot \nabla \eta \, dx
\]

\[
= \int_{\Omega \cap B_1} |\nabla v|^2 |\nabla \eta|^2 + |\nabla |\nabla v||^2 \eta^2 - \frac{1}{2} |\nabla |\nabla v||^2 \eta^2 \, dx
\]

\[
+ \frac{1}{2} \int_{F \cap B_1} \partial_\nu |\nabla v|^2 \eta^2 \, d\mu
\]

\[
= \int_{\Omega \cap B_1} |\nabla v|^2 |\nabla \eta|^2 - (|\nabla v|^2 - |\nabla |\nabla v||^2) \eta^2 \, dx
\]

\[
+ \frac{1}{2} \int_{F \cap B_1} \partial_\nu |\nabla v|^2 \eta^2 \, d\mu.
\]

To compute \( \partial_\nu |\nabla v|^2 \) along \( F \), we fix a point \( p \) on the free boundary. In suitable Euclidean coordinates \( (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \) near \( p = (0', 0) \), the free boundary \( F \) is locally given by the graph

\[
\{ x_n = g(x') \}, \quad x' \in B^{n-1}_\rho(0'),
\]

where \( g(0') = 0 \) and \( \nabla' g(0') = 0 \). Here we assume \( u > 0 \) in \( B_\rho \cap \{ x_n > g(x') \} \).

Hence \( u_\eta(0) = 1, \nabla' u(0) = 0 \) and \( \nu(0) = -\nabla u(0) = -e_n \). Then by differentiating the free boundary condition and utilizing the fact that \( \Delta u = 0 \), we get

\[
u_{in}(0) = 0, \quad u_{nn}(0) = -\Delta' u(0) = -H(0).
\]
In the following, we use summation convention over repeated indices, where \( i \in \{1, \ldots, n\} \). Under these assumptions, at the origin it holds that

\[
\frac{1}{2} \partial_v |\nabla v|^2 = -\nabla^2 v(\nabla v, \nabla u) = -v_{ii} v_i \\
= -(h_{ii} - u_{ii}) (h_i - u_i) \\
= -h_{ii} h_i - u_{ii} u_i + h_{ii} u_i + u_{ii} h_i \\
= -u_{nn} + u_{nn} h_n - h_{in} h_i + h_{nn} = u_{nn} v_n - h_{in} h_i + h_{nn} \\
= -H \nabla v \cdot \nabla u + O \left( r^{-3 \varepsilon^{2/n-1}} + r^{-2 \varepsilon^{1/n-1}} \right) \\
= -H \nabla v \cdot \nabla u + O \left( r^{-2 \varepsilon^{1/n-1}} \right) \text{ on } F \cap B_{1-r/2}, \text{ for } r \geq C_1 \varepsilon^{1/(n-1)},
\]

where the penultimate line follows from (26) and the interior estimates (19). Plugging this formula back into (25), we get

\[
\int_{\Omega \cap B_1} \left( |\nabla^2 v|^2 - |\nabla |\nabla v||^2 \right) \eta^2 \, dx \\
\leq \int_{\Omega} |\nabla v|^2 |\eta|^2 \, dx - \int_{F} H \left( |\nabla v|^2 + \nabla v \cdot \nabla u \right) \eta^2 \, dx + O \left( r^{-2 \varepsilon^{1/n-1}} \right) \int_{F} \eta^2 \, d\mu.
\]

Using (19), we obtain for \( r \geq C_1 \varepsilon^{1/(n-1)}, \)

\[
|\nabla v|^2 + \nabla v \cdot \nabla u = \nabla v \cdot \nabla h = (-\nabla u + \nabla h) \cdot \nabla h = O \left( r^{-1 \varepsilon^{1/n-1}} \right) \text{ on } F \cap B_{1-r/2}.
\]

Therefore

\[
\int_{\Omega \cap B_1} \left( |\nabla^2 v|^2 - |\nabla |\nabla v||^2 \right) \eta^2 \, dx \\
\leq \int_{\Omega \cap B_1} |\nabla v|^2 |\eta|^2 \, dx + C r^{-2 \varepsilon^{1/n-1}} \int_{F \cap B_1} (1 + |H| r) \eta^2 \, d\mu.
\]

Now the result of Lemma 2.1 allows us to bound the Hessian of \( v \):

\[
(27) \int_{\Omega \cap B_1} |\nabla^2 v|^2 \eta^2 \, dx \leq c \int_{\Omega \cap B_1} |\nabla v|^2 |\eta|^2 \, dx + C r^{-2 \varepsilon^{1/n-1}} \int_{F \cap B_1} (1 + |H| r) \eta^2 \, d\mu.
\]

**Step 4. Using the stability condition again.** Now we start from the second line in (25) and then use (27) to get

\[
\int_{F \cap B_1} H |\nabla v|^2 \eta^2 \, d\mu \leq C \int_{\Omega \cap B_1} |\nabla v|^2 |\eta|^2 + |\nabla^2 v|^2 \eta^2 \, dx \\
\leq C \int_{\Omega \cap B_1} |\nabla v|^2 |\eta|^2 \, dx + C r^{-2 \varepsilon^{1/n-1}} \int_{F \cap B_1} (1 + |H| r) \eta^2 \, d\mu.
\]

Adding \( 2 \int_{F} H |\nabla v|^2 \eta^2 \, d\mu \) to both sides of (28), we obtain

\[
(29) \int_{F} H |\nabla v|^2 \eta^2 \, d\mu \leq C \int_{\Omega} |\nabla v|^2 |\eta|^2 \, dx + C r^{-2 \varepsilon^{1/n-1}} \int_{F} (1 + |H| r) \eta^2 \, d\mu \\
+ 2 \int_{F} H^{-1} |\nabla v|^2 \eta^2 \, d\mu.
\]
If \( r \geq (C_1 \varepsilon^{1/(n-1)})^{1/n} \geq C_1 \varepsilon^{1/(n-1)} \), where \( C_1 \) is sufficiently large, then the gradient bounds \((20)-(21)\) imply

\[
\frac{1}{2} \leq |\nabla v|^2 - C r^{-1} \varepsilon^{1/(n-1)} \leq 2 \quad \text{on } F \cap B_{1-r/2}.
\]

Thus, collecting all the integral terms involving \(|H|\) on the left-hand side, \((29)\) becomes

\[
\int_F |H| \eta^2 \, d\mu \leq C \int_\Omega |\nabla v|^2 |\nabla \eta|^2 \, dx + C r^{-2} \varepsilon^{\frac{1}{n-1}} \int_F \eta^2 \, d\mu + C \int_F H - \eta^2.
\]

Now, putting together the bounds \((17)\) of Step 1 and \((22)\) of Step 2, we finally obtain the integral bound \((13)\). 

\[\square\]

**Proof of Theorem 1.1.** We will only prove the \( n \geq 3 \) case of Theorem 1.1. If \( n = 2 \), the solution of \((1)\) in \( B_1 \subset \mathbb{R}^2 \) can be viewed as a solution in \( B_1 \times \mathbb{R} \subset \mathbb{R}^3 \) by adding a redundant variable, which preserves the stability condition.

Fix \( p \in F \cap B_{1/4} \) and \( r \in (0,1/4) \). After recentering and rescaling,

\[ u \to \tilde{u}(x) := u(p + rx)/r \]

we may assume that we are dealing with a Lipschitz continuous, stable classical solution \( u \), defined in \( B_1 \), with \( 0 \in F \). We use a De Giorgi type iteration to prove that if \( \varepsilon := \mu(B_1) \) is small enough (to be chosen later in the proof), then \( \mu(B_{1/2}) = 0 \), leading to a contradiction.

Rewriting the estimate \((13)\) for scales \( 1/4 < R_1 < R_2 \), where \( 1 - r = R_1/R_2 \), we get

\[
\int_{F \cap B_{R_1}} |H| \, d\mu \leq \frac{C_1 \mu(B_{R_2})^{\frac{n}{n-1}}}{(R_2 - R_1)^2},
\]

provided

\[
R_2 - R_1 \geq C_1 R_2^{1 - \frac{1}{n}} \mu(B_{R_2})^{\frac{1}{n(n-1)}} \geq (C_1/4) \mu(B_{R_2})^{\frac{1}{n(n-1)}}.
\]

Set

\[
r_0 = 1, \quad a_0 = \mu(B_1)
\]

and for any \( m \geq 1 \),

\[
\rho_m = \max\{C_1 a_m^{\frac{1}{n(n-1)}}, 2^{-m-1}\}, \quad r_m = r_{m-1} - \rho_m, \quad a_m = \mu(B_{r_m}).
\]

Take \( R_1 = (r_m + r_{m-1})/2 \) and \( R_2 = r_{m-1} \), so that

\[
R_2 - R_1 = \rho_m/2 \geq (C_1/2) a_m^{\frac{1}{n(n-1)}} \geq (C_1/4) \mu(B_{R_2})^{\frac{1}{n(n-1)}}.
\]

Thus, we may apply \((30)\), obtaining

\[
\mu(B_{(r_m + r_{m-1})/2}) \rho_m^{-1} + \int_{F \cap B_{(r_m + r_{m-1})/2}} |H| \, d\mu \leq a_m^{-1} \rho_m^{-1} + C_1 (\rho_m/2)^{-2} a_m^{n/(n-1)}
\]

\[
\leq a_m^{-1} \rho_m^{-1} + (\rho_m^{-1} a_m^{-1})(4C_1 a_m^{1/(n-1)} \rho_m^{-1}) \leq c a_m^{-1} \rho_m^{-1}
\]

\[
\leq C a_m^{-1} 2^m,
\]

\[
(31)
\]
since \( 4C_1a_m^{-1/(n-1)} \rho_m^{-1} \leq 4a_m^{1/n} \leq 1 \) when \( a_m-1 \leq \varepsilon \) is small enough. Let \( \phi \in C^1_c(B_{r_m r_{m-1}/2}) \) be a cut-off function, such that \( 0 \leq \phi \leq 1 \), \( \phi = 1 \) in \( B_{r_m} \) and \( |\nabla \phi| \leq c \rho_m^{-1} \) in \( B_1 \). Plugging it in the Michael-Simon inequality ([24]),

\[
\left( \int_F \phi^{\frac{n-2}{n-1}} d\mu \right)^{\frac{n-2}{n-1}} \leq C_0 \int_F (|\nabla_F \phi| + |H| \phi) \, d\mu
\]

and using (31), we get

\[
a_m \leq \int_F \phi^{\frac{n-1}{n-2}} d\mu \leq C \left( \mu(B_{r_m r_{m-1}/2}) \rho_m^{-1} + \int_{F \cap B_{B_{r_m r_{m-1}/2}}} |H| \, d\mu \right)^{\frac{n-1}{n-2}} \leq c \left( a_m^{-\varepsilon^2 m} \right)^{\frac{n-1}{n-2}}.
\]

(32)

We now argue by induction that

\[
a_m \leq \varepsilon \gamma^{-m}
\]

for \( \gamma = 2^{n-1} > 1 \) and \( \varepsilon \leq \varepsilon_2(n) \) small enough. Set \( \kappa := \frac{n-1}{n-2} \). We see that (33) holds for \( m = 0 \) and assume it is true for \( m - 1 \). Using the recurrence inequality (32), we obtain

\[
a_m \leq c 2^m c a_m^{-\varepsilon \kappa \gamma^{-(-m+1)}} = c \varepsilon \gamma^{-m} (c \varepsilon \kappa \gamma^{-(-m)})^m \leq c \varepsilon \gamma^{-m},
\]

if we choose \( \gamma = 2^{\kappa/(\kappa-1)} = 2^{n-1} > 1 \) and \( \varepsilon \leq \varepsilon_2(n) := (c \gamma^{-1/k})^{-1/(\kappa-1)} \). Employing (33), we now estimate that for all \( m \in \mathbb{N} \)

\[
C_1 a_m^{-(\kappa-1)} \leq C_1 \varepsilon^{\kappa (n-1) 2^{n-1}} \frac{2^{n-1}}{n-1},
\]

whence

\[
\rho_m \leq \max\left\{2^{-m-1}, \varepsilon^{\frac{1}{n(n-1) 2^{n-1}}} \frac{2^{n-1}}{n-1}\right\} \quad \text{and} \quad r_m = 1 - \sum_{k=1}^{m} \rho_k \geq 1/2,
\]

whenever \( \varepsilon \leq \varepsilon_3(n) \) is small enough. Therefore, if we choose \( \varepsilon(n) := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} \), we obtain the desired

\[
\mu(B_{1/2}) \leq \lim_{m \to +\infty} \mu(B_{r_m}) = \lim_{m \to +\infty} a_m = 0. \quad \square
\]

3. Proof of Theorem 1.2

We will start this section by establishing the Bernstein type statement (8) for \( n = 2 \): the only entire stable classical solutions to the one-phase FBP in the plane are, up to rigid motion, the one-plane \( U(x_1, x_2) = x_1^+ \) and the two-plane solutions \( U(x_1, x_2) = x_2^+ + (x_2 + a)^- \), \( a > 0 \).

**Theorem 3.1.** Let \( U : \mathbb{R}^2 \to [0, \infty) \) be an entire stable classical solution to the one-phase FBP (1). Then \( U \) is a one-plane or a two-plane solution.
Proof. For any \( \eta \in C_0^\infty(\mathbb{R}^n) \) we plug in the test function \( \phi = |\nabla U| \eta \) in the stability inequality, obtaining
\[
\int_F H \eta^2 \, d\mu \leq \int_{\Omega} \left( |\nabla|\nabla U|^2 \eta^2 + |\nabla U|^2 |\nabla \eta|^2 + \frac{1}{2} \nabla |\nabla U|^2 \cdot \nabla \eta^2 \right) \, dx.
\]
Since \( \frac{1}{2} (|\nabla U|^2)_{\nu} = H \) on the free boundary \( F \) while \( \Delta |\nabla U|^2 = 2 |\nabla^2 U|^2 \) in the positive phase \( \Omega = \{ U > 0 \} \), an integration by parts yields
\[
\int_{\Omega} (|\nabla^2 U|^2 - |\nabla |\nabla U||^2) \eta^2 \, dx \leq \int_{\Omega} |\nabla U|^2 |\nabla \eta|^2 \, dx.
\]
Combining the fact (3) that \( U \) is Lipschitz continuous (see also Proposition A.5),
\[ |\nabla U| \leq C \text{ in } \mathbb{R}^2, \]
with the result of Proposition 2.1, gives us the estimate on the Hessian
\[ \int_{\Omega} |\nabla^2 U|^2 \eta^2 \, dx \leq C \int_{\Omega} |\nabla \eta|^2 \, dx. \tag{34} \]
By plugging in (34) a standard logarithmic cut-off function \( \eta = \eta_R \)
\[ \eta_R(x) := \begin{cases} 
1 & \text{for } |x| \leq R, \\
2 - (\log |x|)/\log R & \text{for } R < |x| \leq R^2, \\
0 & \text{for } |x| > R^2, 
\end{cases} \]
we now obtain
\[
\int_{\Omega \cap B_R} |\nabla^2 U|^2 \, dx \leq \frac{C}{\log^2 R} \int_{\Omega \cap (B_{R^2} \setminus B_R)} \frac{1}{|x|^2} \, dx \leq \frac{C'}{\log R}.
\]
Taking \( R \to \infty \), we conclude that \( |\nabla^2 U|^2 = 0 \) in \( \Omega \) so that \( u \) is a linear function in each connected component of \( \Omega \). There are only two possibilities, up to Euclidean isometry: \( U = x_2^+ \), or \( U = x_2^+ + (x_2 + a)^- \), for some \( a > 0 \). \( \square \)

We now borrow a strategy from minimal surface theory (see [27, Chapter 3]) to prove that the rigidity of the Bernstein type problem for (1) in dimension \( n \) entails interior bounds for the curvature of the free boundary in the local problem.

Proof of Theorem 1.2. Assume that the statement of the theorem is false. Then there exists a sequence of counterexamples \( \{ u_k \}_{k \in \mathbb{N}} \) defined in \( B_1 \), whose free boundaries \( F(u_k) \) have second fundamental forms \( A_k \) that satisfy:
\[ \rho_k := \max_{p \in F(u_k) \cap B_{1/4}} |A_k(p)| \text{dist}(p, \partial B_{1/4}) \not\to \infty, \quad \text{as } k \to \infty. \]
Let \( p_k \in F(u_k) \cap B_{1/4} \) be a point where the maximum \( \rho_k \) is attained and define the sequence of rescaled solutions
\[ \tilde{u}_k(x) := |A_k(p_k)| u_k(p_k + x/|A_k(p_k)|) \quad \text{for } x \in (B_{1/4} - p_k)|A_k(p_k)| \supseteq B_{p_k}, \]

We now borrow a strategy from minimal surface theory (see [27, Chapter 3]) to prove that the rigidity of the Bernstein type problem for (1) in dimension \( n \) entails interior bounds for the curvature of the free boundary in the local problem.

Proof of Theorem 1.2. Assume that the statement of the theorem is false. Then there exists a sequence of counterexamples \( \{ u_k \}_{k \in \mathbb{N}} \) defined in \( B_1 \), whose free boundaries \( F(u_k) \) have second fundamental forms \( A_k \) that satisfy:
\[ \rho_k := \max_{p \in F(u_k) \cap B_{1/4}} |A_k(p)| \text{dist}(p, \partial B_{1/4}) \not\to \infty, \quad \text{as } k \to \infty. \]
Let \( p_k \in F(u_k) \cap B_{1/4} \) be a point where the maximum \( \rho_k \) is attained and define the sequence of rescaled solutions
\[ \tilde{u}_k(x) := |A_k(p_k)| u_k(p_k + x/|A_k(p_k)|) \quad \text{for } x \in (B_{1/4} - p_k)|A_k(p_k)| \supseteq B_{p_k}, \]

We now borrow a strategy from minimal surface theory (see [27, Chapter 3]) to prove that the rigidity of the Bernstein type problem for (1) in dimension \( n \) entails interior bounds for the curvature of the free boundary in the local problem.

Proof of Theorem 1.2. Assume that the statement of the theorem is false. Then there exists a sequence of counterexamples \( \{ u_k \}_{k \in \mathbb{N}} \) defined in \( B_1 \), whose free boundaries \( F(u_k) \) have second fundamental forms \( A_k \) that satisfy:
\[ \rho_k := \max_{p \in F(u_k) \cap B_{1/4}} |A_k(p)| \text{dist}(p, \partial B_{1/4}) \not\to \infty, \quad \text{as } k \to \infty. \]
Let \( p_k \in F(u_k) \cap B_{1/4} \) be a point where the maximum \( \rho_k \) is attained and define the sequence of rescaled solutions
\[ \tilde{u}_k(x) := |A_k(p_k)| u_k(p_k + x/|A_k(p_k)|) \quad \text{for } x \in (B_{1/4} - p_k)|A_k(p_k)| \supseteq B_{p_k}, \]

We now borrow a strategy from minimal surface theory (see [27, Chapter 3]) to prove that the rigidity of the Bernstein type problem for (1) in dimension \( n \) entails interior bounds for the curvature of the free boundary in the local problem.

Proof of Theorem 1.2. Assume that the statement of the theorem is false. Then there exists a sequence of counterexamples \( \{ u_k \}_{k \in \mathbb{N}} \) defined in \( B_1 \), whose free boundaries \( F(u_k) \) have second fundamental forms \( A_k \) that satisfy:
\[ \rho_k := \max_{p \in F(u_k) \cap B_{1/4}} |A_k(p)| \text{dist}(p, \partial B_{1/4}) \not\to \infty, \quad \text{as } k \to \infty. \]
Let \( p_k \in F(u_k) \cap B_{1/4} \) be a point where the maximum \( \rho_k \) is attained and define the sequence of rescaled solutions
\[ \tilde{u}_k(x) := |A_k(p_k)| u_k(p_k + x/|A_k(p_k)|) \quad \text{for } x \in (B_{1/4} - p_k)|A_k(p_k)| \supseteq B_{p_k}, \]
which are uniformly Lipschitz continuous and uniformly nondegenerate in $B_{\rho_k}$, the latter on account of Theorem 1.1. Furthermore, the free boundary $F(\bar{u}_k)$ of $\bar{u}_k$ has second fundamental form $\tilde{A}_k$ that satisfies
\begin{equation}
|\tilde{A}_k(0)| = 1 \quad \text{and} \quad |\tilde{A}_k(p)|\text{dist}(p, \partial B_{\rho_k}) \leq \rho_k \quad \text{for all } p \in F(\bar{u}_k) \cap B_{\rho_k}.
\end{equation}
This means that for each fixed $r > 0$
\begin{equation}
\sup_{p \in F(\bar{u}_k) \cap B_r} |\tilde{A}_k(p)| \leq \frac{\rho_k}{\rho_k - r} \to 1 \quad \text{as } k \to \infty.
\end{equation}
Now, according to Proposition A.4, up to taking a subsequence, $\{\bar{u}_k\}$ converges uniformly on compacts to a globally defined *viscosity* solution $U$, while both
\begin{equation}
F(\bar{u}_k) \to F(U) \quad \text{and} \quad \{|\bar{u}_k > 0\} \to \{U > 0\}
\end{equation}
converge locally in the Hausdorff distance. Furthermore, by Proposition A.5, we know that
\begin{equation}
|\nabla U| \leq 1 \quad \text{in} \quad \{U > 0\}.
\end{equation}
We will argue that $U$ is, in fact, an entire classical stable solution of (1).

Let $q \in F(U)$. We aim to show that near $q$ the free boundary $F(U)$ is the graph of a smooth function, separating positive from zero phase. Let $q_k \in F(\bar{u}_k)$ be a sequence of points that converge to $q$ as $k \to \infty$, and let $\nu_k$ be the inner unit normal vector to $\{\bar{u}_k > 0\}$ at $q_k$. By possibly taking a further subsequence and rotating the coordinate axes, we may assume that $\nu_k$ converge to $e_n$. Now, because of the uniformly bounded (on compacts) curvature (36) of the free boundaries $F(\bar{u}_k)$, there is a small absolute constant $0 < c < 1$ such that for all large $k$, the connected component $C_k$ of $[B_c(q)]^+(\bar{u}_k)$, whose boundary contains $q_k$, is a supergraph
\begin{equation}
C_k = \{y \in B_c(q) : y = q + (x', x_n), \ x_n > f_k(x')\},
\end{equation}
of a function $f_k : B'_c \subset \mathbb{R}^{n-1} \to \mathbb{R}$ with uniformly bounded $C^2$ norm (Lemma A.6):
\begin{equation}
\|f_k\|_{C^2(B'_c)} \leq 1/c \quad \text{and} \quad \lim_{k \to \infty} f_k(0') = 0.
\end{equation}
By Arzela-Ascoli, the sequence $\{f_k\}$ subconverges in $C^{1,\alpha}(B'_c)$ to some $f \in C^2(B'_c)$ with $f(0') = 0$, so that by (37), the set
\begin{equation}
C := \{y \in B_c(q) : y = q + (x', x_n), \ x_n > f(x')\},
\end{equation}
is a connected component of $[B_c(q)]^+(U)$. By the classical regularity theory for the one-phase FBP [20], $f$ is in fact smooth,
\begin{equation}
f_k \to f \quad \text{in} \quad C^m(B'_c) \quad \text{for any } m \in \mathbb{N},
\end{equation}
and $U_+ := U_{1e}$ is a classical solution of (1) in $B_c(q)$. In particular, when $q = 0$ is the origin, the normalized curvature condition (35) implies that $F(U_+)$ has second fundamental form $\tilde{A}$ at 0 of unit magnitude:
\begin{equation}
|\tilde{A}(0)| = 1.
\end{equation}
To finish the proof that $U$ itself is a classical solution in $B_c(q)$, it remains to verify that there are no two connected components of $[B_c(q)]^+(U)$ touching at $q$. Assume, to the contrary, that there is a separate connected component $C$ of $[B_c(q)]^+(U)$, such that $q \in \partial C$. By the argument from the previous paragraph, $U_- := U1_{\hat{C}}$ is a classical solution of (1) in $B_c(q)$, as well. Denote by $H_+$ and $H_-$ the mean curvature of $F(U_+)$ and $F(U_-)$ with respect to the inner unit normal $\nu_+$ to $C$ and $\nu_- to \hat{C}$, respectively. Observe that in at least one of $C$ or $\hat{C}$ we have the strict inequality $|\nabla U_\pm| < 1$, for otherwise, Proposition A.5 would suggest that $U(x) = |(x - q)_n|$, contradicting the nontrivial curvature condition (42) at 0. Let us therefore assume that

$$|\nabla U_-| < 1 \text{ in } \hat{C}.$$  

Now, because of (38), we have

$$H_\pm = \text{div} \left( \frac{\nabla U_\pm}{|\nabla U_\pm|} \right) = -\frac{\partial_{\nu_\pm}|\nabla U_\pm|^2}{2|\nabla U_\pm|^2} = -\frac{1}{2}\partial_{\nu_\pm}|\nabla U_\pm|^2 \geq 0. \tag{44}$$

Moreover, via the Hopf Lemma, (43) entails that

$$H_- > 0. \tag{45}$$

On the other hand, since $F(U_+)$ and $F(U_-)$ touch at $q$ with normal unit vectors $\nu_+(q) = e_n = -\nu_-(q)$, we have the comparison

$$-H_-(q) \geq H_+(q). \tag{46}$$

Combining (44), (45) and (46) yields the impossible

$$0 > -H_-(q) \geq H_+(q) \geq 0. \tag{47}$$

We conclude that $U$ is an entire classical solution of the one-phase FBP. By the smooth local convergence of the free boundaries (41), we also see that for any fixed $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{F(\tilde{u}_k)} H_k \phi^2 d\mathcal{H}^{n-1} \to \int_{F(U)} H \phi^2 d\mathcal{H}^{n-1} \text{ as } k \to \infty,$$

where $H_k$ and $H$ denote the mean curvature of $F(\tilde{u}_k)$ and $F(U)$, respectively. Since by Proposition A.4,

$$1_{\{\tilde{u}_k > 0\}} \to 1_{\{U > 0\}} \text{ in } L^1_{\text{loc}}, \text{ as } k \to \infty,$$

we also have that

$$\int_{\{\tilde{u}_k > 0\}} |\nabla \phi|^2 \to \int_{\{U > 0\}} |\nabla \phi|^2 \text{ as } k \to \infty.$$

By taking the limit as $k \to \infty$, we see then that the stability of $\tilde{u}_k$ entails the stability of $U$.

However, the rigidity statement (8) now says that $F(U)$ is flat, which contradicts the fact (42) that $F(U)$ has nontrivial curvature at the origin. The proof is complete. \hfill \Box
APPENDIX A. Auxiliary results

We recall the notion of a viscosity solution of (1) (see [6]). First we define viscosity super- and subsolutions.

**Definition A.1.** A viscosity supersolution of (1) in a domain \( D \subseteq \mathbb{R}^n \) is a non-negative function \( w \in C(D) \) such that \( \Delta w \leq 0 \) in \( D^+(w) \) and for every \( x_0 \in F(w) \) with a tangent ball \( B \) from the positive side \((x_0 \in \partial B \text{ and } B \subset D^+(w))\), there is \( \alpha \leq 1 \) such that
\[
u \rightarrow x_0 \non-tangentially \text{ in } B, \text{ with } \nu \text{ the inner normal to } \partial B \text{ at } x_0.
\]

**Definition A.2.** A viscosity subsolution of (1) in a domain \( D \subseteq \mathbb{R}^n \) is a non-negative function \( w \in C(D) \) such that \( \Delta w \geq 0 \) in \( D^+(w) \) and for every \( x_0 \in F(w) \) with a tangent ball \( B \) in the zero set \((x_0 \in \partial B \text{ and } B \subset \{w = 0\})\), there is \( \alpha \geq 1 \) such that
\[
u \rightarrow x_0 \non-tangentially \text{ in } B^c, \text{ with } \nu \text{ the outer normal to } \partial B \text{ at } x_0.
\]

A viscosity solution in \( D \) is a function that is both a supersolution and a subsolution in the sense above.

The class of viscosity solutions is particularly well suited for taking uniform limits. We quote the following result

**Lemma A.3.** ([17, Lemma 4.4]) Let \( u_k \in C(D) \) be a sequence of viscosity solutions of (1) in \( D \) such that \( u_k \rightarrow u \) uniformly and \( u \) is Lipschitz continuous. Then \( u \) is also a viscosity supersolution of (1) in \( D \). If, in addition, \( D^+(u_k) \rightarrow D^+(u) \) locally in the Hausdorff distance, then \( u \) is a viscosity subsolution, as well.

As a corollary, we have the following well known result (see [6, Lemma 1.21] or [17, Proposition 4.2] for the proof) describing uniform limits of Lipschitz continuous and non-degenerate viscosity solutions.

**Proposition A.4.** Let \( D \subseteq \mathbb{R}^n \) be a domain and \( \{u_k\} \subset C(D) \) be a sequence of viscosity solutions of (1) in \( D \) which satisfies
\[
\begin{align*}
\bullet & \text{ (Uniform Lipschitz continuity) There exists a constant } C, \text{ such that } \\
\|\nabla u_k\|_{L^\infty(D)} & \leq C;
\end{align*}
\]
\[
\begin{align*}
\bullet & \text{ (Uniform non-degeneracy) There exists a constant } c, \text{ such that } \\
\int_{\partial B_r(x)} u_k d\mathcal{H}^{n-1} & \geq cr
\end{align*}
\]
for every \( B_r(x) \subseteq D \), centered at a free boundary point \( x \in F(u_k) \).

Then any limit \( u \in C(\overline{D}) \) of a uniformly convergent on compacts subsequence \( u_k \rightarrow u \) satisfies
\[
\begin{align*}
(1) \ \overline{D^+(u_k)} & \rightarrow \overline{D^+(u)} \text{ and } F(u_k) \rightarrow F(u) \text{ locally in the Hausdorff distance; } \\
(2) \ \{u_k > 0\} & \rightarrow \{u > 0\} \text{ in } L^1_{\text{loc}}(D);
\end{align*}
\]
Moreover, \( u \) is a Lipschitz continuous, non-degenerate viscosity solution of (1).
In the next proposition we establish the fact that globally defined viscosity solutions of the one-phase FBP have a gradient bounded by 1.

**Proposition A.5.** Let \( u : \mathbb{R}^n \to [0, \infty) \) be an entire viscosity solution of the one-phase FBP (1), with \( F(u) \neq \emptyset \), and let \( \Omega := \{ x \in \mathbb{R}^n : u(x) > 0 \} \) denote the positive phase of \( u \). Then
\[
|\nabla u(x)| \leq 1 \quad \text{for all } x \in \Omega.
\]
Furthermore, if \( |\nabla u(x_0)| = 1 \) for some \( x_0 \in \Omega \), then \( |\nabla u(x)| \equiv 1 \) in the connected component \( \mathcal{C} \) of \( \Omega \), containing \( x_0 \), so that \( u_{|\mathcal{C}}(x) = (x - p) \cdot e \) for some \( p \in \mathbb{R}^n \) and unit vector \( e \in \mathbb{R}^n \).

**Proof.** We sketch out the argument from [26, Proposition 2.1], written for a closely related problem.

First, we show that there is a dimensional constant \( C > 0 \) such that
\[
|\nabla u| \leq C \quad \text{in } \Omega.
\]
Pick any \( x \in \Omega \). By rescaling and recentering, we may assume that \( x = 0 \) and \( d(x, F(u)) = 1 \). By applying the Harnack inequality plus gradient estimates to \( u \) in \( B_1 \), it suffices to show that \( u(0) \leq C_0 \). Let \( p \in F(u) \cap \partial B_1 \) and note that \( B_1 \) is a ball touching \( F(u) \) from the positive side.

By the Harnack inequality, we know that \( u \geq cu(0) \) in \( B_{1/2} \). Consider the harmonic function \( h \) in the annulus \( B_1 \setminus B_{1/2} \), having boundary values \( h = cu(0) \) on \( \partial B_{1/2} \) and \( h = 0 \) on \( \partial B_1 \). Then \( h_v(p) = c_0u(0) \), where \( \nu \) denotes the inner unit normal to \( \partial B \) at \( p \). On the other hand, the maximum principle implies \( h \leq u \) in \( B_1 \setminus B_{1/2} \), so that the Hopf Lemma in conjunction with the viscosity supersolution property yield the desired bound
\[
1 \geq u_v(p) \geq h_v(p) = c_0u(0).
\]

Therefore, the supremum \( L := \sup_{\Omega} |\nabla u| \leq C \) is positive and finite. Let \( x_k \in \Omega \) be a sequence of points such that \( |\nabla u(x_k)| \rightarrow L \) as \( k \rightarrow \infty \). Define the following rescales of \( u \)
\[
v_k(x) := d_k^{-1}u(x_k + d_kx), \quad \text{where } d_k := d(x_k, F(u)).
\]
Then the \( v_k \) satisfy: \( |\nabla v_k| \leq L \), the distance \( d(0, F(v_k)) = 1 \) and \( |\nabla v_k(0)| = |\nabla u(x_k)| \rightarrow L \) as \( k \rightarrow \infty \). Thus, the sequence of uniformly Lipschitz continuous, viscosity solutions \( \{v_k\} \) subconverges uniformly on compact subsets of \( \mathbb{R}^n \) to the globally defined, Lipschitz continuous function \( v \), which is a harmonic in its positive phase \( \tilde{\Omega} := \{ v > 0 \} \), satisfies \( |\nabla v| \leq L \), and which is a viscosity supersolution on account of Lemma A.3. Furthermore, since \( v_k \rightarrow v \) uniformly on \( \overline{B_1} \) and \( v_k - v \) is harmonic in \( B_1 \), we have
\[
|\nabla v(0) - \nabla v_k(0)| \leq C|v_k - v|_{C(B_1)} \rightarrow 0 \quad \text{as } k \rightarrow \infty,
\]
so that \( |\nabla v(0)| = L \).

However, since \( \Delta |\nabla v|^2 = 2|\nabla^2 v|^2 \geq 0 \) in \( \tilde{\Omega} \), we know that \( v \) is subharmonic in \( \tilde{\Omega} \), so that the strict maximum principle implies that \( |\nabla v| \equiv L \) in the connected component \( \mathcal{C} \) of \( \tilde{\Omega} \) containing \( 0 \). Therefore, \( 2|\nabla^2 v|^2 = \Delta |\nabla v|^2 = 0 \) in \( \mathcal{C} \), so that \( v \)
is a linear function in $C$ with slope $L > 0$. This means that the component $C$ is actually a half-space and for some $p \in \mathbb{R}^n$, and unit vector $e \in \mathbb{R}^n$, we have

$$v(x) = L(x - p) \cdot e \quad \text{for all } x \in C.$$ 

We now infer that $L \leq 1$ from the fact that $v$ is a viscosity supersolution to (1).

To establish the remaining part of the proposition (regarding the possibility of \(|\nabla u(x_0)| = 1\)), we use the verbatim argument from the paragraph above. \hfill $\square$

We end the appendix with the following lemma.

**Lemma A.6.** Let $u$ be a classical solution to (1) in $B_1 \subset \mathbb{R}^n$ which is Lipschitz continuous and nondegenerate (with universal constants) in $B_1$. Assume that $0 \in F(u)$ and that the second fundamental form $A$ of $F(u)$ is bounded

$$|A(p)| \leq C \quad \text{for all } p \in F(u),$$

by some absolute constant $C > 0$. Then there exists a constant $c \in (0, 1)$ such that, in a suitable Euclidean coordinate system, the connected component $C$ of $B_1^+(u)$, whose boundary contains 0, is the supergraph:

$$C = \{x = (x', x_n) \in B_c : x_n > f(x')\},$$

for some $f : B'_c \to \mathbb{R}$ with $\|f\|_{C^2(B'_c)} \leq 1/c$.

**Proof.** Since the curvature of $F(u)$ is bounded by an absolute constant, there is a small absolute constant $c > 0$, such that the component $F$ of $F(u) \cap B_c$ containing 0, is given by a graph

$$F = \{(x', x_n) \in B_c : x_n = f(x')\},$$

with $\|f\|_{C^2(B'_c)} \leq 1/c$. Let us show that, by possibly reducing the constant $c$, the connected component $C$ of $B_1^+(u)$ bordering the origin, is the supergraph (47).

Assume that this last statement is false. Then there exist a sequence $c_k \to 0$ and a sequence of counterexamples $\{u_k\}$ such that the component $C_k$ of $B_{c_k}^+(u_k)$ bordering 0 has at least two free boundary connected components: the connected component $F_k$ of $0 \in F(u_k) \cap B_{c_k}$, and another component $\tilde{F}_k$. Choose $\tilde{F}_k$ to be the closest such component to $F_k$ and set $d_k := \text{dist}(F_k, \tilde{F}_k) \leq c_k$. Consider now the rescaled solutions

$$v_k(x) := d_k^{-1}[u_k C_k](d_k x) \quad \text{in } B_{d_k^{-1} c_k} \supseteq B_1,$$

and denote by $G_k := d_k^{-1} F_k$, $\tilde{G}_k := d_k^{-1} \tilde{F}_k$. We have that

$$\text{dist}(G_k, \tilde{G}_k) = 1,$$

while the curvature

$$|A_{G_k}| \leq d_k \sup |A_{F(u_k)}| \leq c_k C \to 0.$$ 

Now, according to Proposition A.4 the uniformly Lipschitz continuous and nondegenerate $v_k$ converge to the limit $v = x_n^+$ uniformly on compacts, with $G_k$ converging to a subset $G \subseteq \{x_n = 0\}$. On the other hand, by the Hausdorff convergence of the free boundaries, (48) suggests that $F(v)$ also has a component sitting at a unit distance away from $G$. This yields a contradiction and the proof of the lemma is complete. \hfill $\square$
References

[1] Mark Allen and Arshak Petrosyan. A two-phase problem with a lower-dimensional free boundary. *Interfaces and Free Boundaries*, 14(3):307–342, 2012.

[2] Hans W. Alt, Luis A. Caffarelli, and Avner Friedman. Variational problems with two phases and their free boundaries. *Transactions of the American Mathematical Society*, 282(2):431–461, 1984.

[3] Luis A. Caffarelli and Hans W. Alt. Existence and regularity for a minimum problem with free boundary. *Journal für die Reine und Angewandte Mathematik*, 325:105–144, 1981.

[4] Luis A. Caffarelli, David Jerison, and Carlos E. Kenig. Global energy minimizers for free boundary problems and full regularity in three dimensions. In *Noncompact problems at the intersection of geometry, analysis, and topology*, volume 350 of *Contemp. Math.*, pages 83–97. Amer. Math. Soc., Providence, RI, 2004.

[5] Luis A Caffarelli, Jean-Michel Roquejoffre, and Yannick Sire. Variational problems with free boundaries for the fractional Laplacian. *Journal of the European Mathematical Society*, 325:105–144, 2010.

[6] Luis A. Caffarelli, David Jerison, and Carlos E. Kenig. Global energy minimizers for free boundary problems and full regularity in three dimensions. In *Noncompact problems at the intersection of geometry, analysis, and topology*, volume 350 of *Contemp. Math.*, pages 83–97. Amer. Math. Soc., Providence, RI, 2004.

[7] Daniela De Silva, David Jerison, and Henrik Shahgholian. Inhomogeneous global minimizers to the one-phase free boundary problem. *Communications in Partial Differential Equations*, pages 1–24, 2022.

[8] Laurent Hauswirth, Frédéric Helein, and Frank Pacard. On an overdetermined elliptic problem. *Pacific J. Math.*, 250(2):319–334, 2011.

[9] David Jerison and Nikola Kamburov. Structure of one-phase free boundaries in the plane. *International Mathematics Research Notices*, 2016(19):5922–5987, 2016.

[10] David Jerison and Kanishka Perera. Higher critical points in an elliptic free boundary problem. *The Journal of Geometric Analysis*, 28(2):1258–1294, 2018.

[11] David Jerison and Ovidiu Savin. Almost minimizers of the one-phase free boundary problem. *Communications in Partial Differential Equations*, 45(8):913–930, 2020.

[12] Avner Friedman. *Variational Principles and Free-boundary Problems*. R.E. Krieger Publishing Company, 1988.

[13] Yong Liu, Kelei Wang, and Juncheng Wei. On smooth solutions to one phase-free boundary problem in $\mathbb{R}^n$. *International Mathematics Research Notices*, 2021(20):15682–15732, 2021.
[22] Dario Mazzoleni, Susanna Terracini, and Bozhidar Velichkov. Regularity of the optimal sets for some spectral functionals. *Geometric And Functional Analysis*, 27(2):373–426, 2017.

[23] Dario Mazzoleni, Susanna Terracini, and Bozhidar Velichkov. Regularity of the free boundary for the vectorial Bernoulli problem. *Analysis & PDE*, 13(3):741–764, 2020.

[24] James H. Michael and Leon M. Simon. Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$. *Communications on Pure and Applied Mathematics*, 26(3):361–379, 1973.

[25] Daniela De Silva and David Jerison. A singular energy minimizing free boundary. *Journal für die Reine und Angewandte Mathematik. [Crelle’s Journal]*, 2009(635):1–21, 2009.

[26] Kelei Wang. The structure of finite Morse index solutions to two free boundary problems in $\mathbb{R}^2$. *arXiv preprint arXiv:1506.00491*, 2015.

[27] Brian White. Lectures on minimal surface theory. *arXiv preprint arXiv:1308.3325*, 2013.

*Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Santiago 7820436, Chile
Email address: nikamburov@mat.uc.cl

†School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
Email address: wangkelei@whu.edu.cn