SOLUTION OF A LINEARIZED MODEL OF 
HEISENBERG’S FUNDAMENTAL EQUATION II

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Abstract. We propose to look at (a simplified version of) Heisenberg’s fundamental field equation (see [2]) as a relativistic quantum field theory with a fundamental length, as introduced in [1] and give a solution in terms of Wick power series of free fields which converge in the sense of ultrahyperfunctions but not in the sense of distributions.

The solution of this model has been prepared in [5] by calculating all $n$-point functions using path integral quantization. The functional representation derived in this part is essential for the verification of our condition of extended causality. The verification of the remaining defining conditions of a relativistic quantum field theory is much simpler through the use of Wick power series. Accordingly in this second part we use Wick power series techniques to define our basic fields and derive their properties.

Contents

1. Introduction 2
1.1. Motivation and outline of paper 2
1.2. Localization properties of tempered ultra-hyperfunctions 4
2. Verification of extended causality 5
3. Convergence of Wick power series for $\rho(x) = e^{g\phi(x)^2}$ : 12
4. Verification of the equation $\partial_{\mu}\rho(x) = 2it\ell^2 : \rho(x)\phi(x)\partial_{\mu}\phi(x) :$ 16
5. Wightman’s Axioms for general type fields 18
6. Some Consequences of the Axioms 20
7. Multiplication of $\rho(x)$ and $\psi(x)$ 24
8. Conclusion 26
References 27

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1. Introduction

1.1. Motivation and outline of paper. Heisenberg’s fundamental field equation (see [2])

\[ \gamma_\mu \frac{\partial}{\partial x_\mu} \psi(x) \pm l^2 \gamma_\mu \gamma_5 \psi(x) \gamma^\mu \gamma_5 \psi(x) := 0 \]  

(1.1)

contains a parameter \( l \) of the dimension of length and accordingly one might speculate that this parameter can play the rôle of the fundamental length of a quantum field theory with a fundamental length as introduced in [1]. Unfortunately, nobody knows to solve this equation. However there is a simplification of Heisenberg’s equation which is solvable in the sense of classical field theory, namely the system of equations

\[ \begin{cases} (\Box + m^2)\phi(x) = 0 \\ i\gamma_\mu \frac{\partial}{\partial x_\mu} - M \psi(x) = -2l^2 \gamma_\mu : \psi(x) \phi(x) \frac{\partial \phi(x)}{\partial x_\mu} \end{cases} \]

(1.2)

for a Klein-Gordon field \( \phi \) and a spinor field \( \psi \). It is this system of coupled equations which we discuss in the framework of [1]. In [3], this system with the Lagrangian density

\[ L(x) = L_{Ff}(x) + L_{Fb}(x) + L_I(x), \quad L_{Ff}(x) = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - \tilde{m})\psi(x), \]

\[ L_{Fb}(x) = \frac{1}{2}\{((\partial_\mu \phi(x))^2 - m^2\phi(x)^2) \}, \quad L_I(x) = 2l^2(\bar{\psi}(x)\gamma^\mu \psi(x))\phi(x)\partial_\mu \phi(x) \]

is quantized by the method of path integral, that is, the \( n \)-point Schwinger functions are calculated by the Euclideanized lattice approximation (with infinitesimal spacing, in the framework of nonstandard analysis) of following path integral:

\[ \int \prod_{j=1}^{n} \psi_{\alpha_j}(x_j) \exp \left\{ \int_{\mathbb{R}^4} L_I(x)dx \right\} d\mathcal{D}(\psi, \bar{\psi})d\mathcal{G}(\phi) \]

\[ \times \left\{ \int \exp i \left\{ \int_{\mathbb{R}^4} L_I(x)dx \right\} d\mathcal{D}(\psi, \bar{\psi})d\mathcal{G}(\phi) \right\}^{-1}, \]

\[ d\mathcal{G}(\phi) = \exp \left\{ \int_{\mathbb{R}^4} L_{Fb}(x)dx \right\} \prod_{x \in \mathbb{R}^4} d\phi(x) \]

\[ d\mathcal{D}(\psi, \bar{\psi}) = \exp i \left\{ \int_{\mathbb{R}^4} L_{Ff}(x)dx \right\} \prod_{x \in \mathbb{R}^4} \prod_{\alpha = 1}^{4} \psi_{\alpha}(x)\psi_{\alpha}(x), \]

where \( \psi^1 = \psi, \psi^2 = \bar{\psi} \).

After renormalization we obtain the continuous limit of the lattice Schwinger functions. Then the Wightman functions are obtained by Wick rotation of the Schwinger functions. If these Wightman functions satisfy the axioms of the relativistic quantum field theory, then, by the
reconstruction theorem, we can construct the operator valued generalized functions $\phi(x)$ and $\psi(x)$. We understand that these fields $\phi(x)$ and $\psi(x)$ are the solutions of the system defined by the Lagrangian density (1.3) according to standard interpretation of renormalization procedure.

In this paper, we try to construct the quantum fields $\phi(x)$ and $\psi(x)$ which satisfy the system of differential equations (1.2), then show that these fields satisfy the axioms of the relativistic quantum field theory. In [5], it is shown that the Wightman functions of $\psi(x)$ are not tempered distributions used in the usual Wightman axioms but tempered ultra-hyperfunctions which are used to formulate the quantum field theory with a fundamental length in [1].

In Section 2 we show that the $n$-point functionals constructed in this way satisfy the spinor version of the functional characterization of our condition of extended causality of [1]. In order to verify the remaining defining conditions of our relativistic field theory with a fundamental length we use Wick power series to define the theory. Accordingly, in this second part we construct an operator valued generalized function $\psi(x)$ satisfying (1.2). The basic idea to solve the system (1.2) is quite natural:

Take a Klein-Gordon field of mass $m$ and suppose that we can show the following three statements:

A) the Wick power series

$$\rho(x) := e^{it^2\phi(x)^2} := \sum_{n=0}^{\infty} i^n t^{2n} : \phi(x)^{2n} : / n!$$

and

$$\rho^*(x) := e^{-it^2\phi(x)^2} := \sum_{n=0}^{\infty} (-i)^n t^{2n} : \phi(x)^{2n} : / n!$$

are well-defined as an operator-valued ultra-hyperfunctions.

B) $\rho(x)$ satisfies

$$\frac{\partial}{\partial x^\mu} \rho(x) = 2it^2 : e^{it^2\phi(x)^2} \phi(x) \frac{\partial}{\partial x^\mu} \phi(x) :$$

$$= 2it^2 : \rho(x) \phi(x) \frac{\partial}{\partial x^\mu} \phi(x) :. \quad (1.4)$$

C) the free Dirac field $\psi_0(x)$ is a multiplier for the field $\rho$ and so, define the field

$$\psi(x) = \psi_0(x) \rho(x), \quad (1.5)$$
and calculate
\[
\left( i\gamma_\mu \frac{\partial}{\partial x^\mu} - M \right) \psi(x) = \left[ \left( i\gamma_\mu \frac{\partial}{\partial x^\mu} - M \right) \psi_0(x) + \gamma_\mu \psi_0(x) \frac{\partial}{\partial x^\mu} \rho(x) \right] \rho(x) + \gamma_\mu \psi_0(x) \frac{\partial}{\partial x^\mu} \phi(x) : \\
= -2l^2 \gamma_\mu \psi_0(x) : \rho(x) \phi(x) \frac{\partial}{\partial x^\mu} \phi(x) :.
\]
Thus, if A) – C) hold, the operator-valued ultra-hyperfunction \( \psi(x) \) satisfies Equation (1.2).

In [1] statement A) is shown together with the fact that the fields \( \phi(x), \rho(x) \) and \( \rho^*(x) \) satisfy the axioms of ultra-hyperfunction quantum field theory (UHFQFT). In Section 3 the convergence of the Wick power series for \( \rho(x) = e^{\phi(x)} \) is recalled from [1]. In the next section the important differential equation \( \partial_\mu \rho(x) = 2i l^2 : \rho(x) \phi(x) \partial_\mu \phi(x) : \)
is proven. Then in order to prepare the treatment of Dirac fields, in Section 5 the axioms of UHFQFT with a fundamental length \( \ell \), for general type of (in particular spinor) fields are presented. In order to show statement C), we study some properties of \( \rho(x) \) which follow from the axioms of UHFQFT in Section 5. In Section 7 it is shown that the pointwise product (1.5) of two operator-valued tempered ultra-hyperfunctions is well-defined and thus statement C) can be established; and it is shown that \( \phi(x), \psi(x) = \psi_0(x) \rho(x) \) and \( \tilde{\psi}(x) = \rho^*(x) \tilde{\psi}_0(x) = \tilde{\psi}_0(x) \rho^*(x) \) satisfy all axioms of UHFQFT for general type fields as presented in Section 4, and their Wightman functions are the same ones obtained in [5] using path integral methods.

1.2. Localization properties of tempered ultra-hyperfunctions.
As announced, in Section 2 we are going to show that the system of \( n \)-point functionals as constructed in the first part satisfy the condition of extended causality. Since this condition is based on the localization properties of tempered ultra-hyperfunctions we explain here briefly the technical realization of these localization properties. To simplify matters we use a simple one-dimensional model first.

Denote \( T(-\ell, \ell) = \mathbb{R} + i(-\ell, \ell), T[-k, k] = \mathbb{R} + i[-k, k] \subset \mathbb{C} \), and let \( \mathcal{T}(T(-\ell, \ell)) \) be the set of functions \( f \) holomorphic in \( T(-\ell, \ell) \) and rapidly decreasing in any \( T[-k, k] \subset T(-\ell, \ell) \). Then for \( |a| < \ell \), we get
\[
\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{a^n}{n!} \delta^{(n)}(x) f(x) dx = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} f^{(n)}(0) \\
= f(-a) = \int_{-\infty}^{\infty} \delta(x + a) f(x) dx.
\]
The above equality implies the following two facts.
(A) If \(|a| < \ell\) then \(\Delta_N(x) = \sum_{n=0}^{N} \frac{a^n}{n!} \delta^{(n)}(x)\) converges to \(\delta(x + a) = \delta_{-a}(x)\) in \(\mathcal{T}(\mathcal{T}(-\ell, \ell))'\) as \(N \to \infty\). Clearly, for all \(N \in \mathbb{N}\), \(\text{supp} \Delta_N = \{0\}\) while for the limit we find \(\text{supp} \delta_{-a} = \{-a\}\).

(B) If \(|a| > \ell\), \(\Delta_N(x)\) does not converge in \(\mathcal{T}(\mathcal{T}(-\ell, \ell))'\).

(A) and (B) say: Elements in \(\mathcal{T}(\mathcal{T}(-\ell, \ell))'\) do not allow to distinguish between \(\{0\}\) and \(\{-a\}\), if \(|a| < \ell\), but if \(|a| > \ell\) then elements in \(\mathcal{T}(\mathcal{T}(-\ell, \ell))'\) can be used to distinguish between the locations \(\{0\}\) and \(\{-a\}\). Such a length \(\ell\) is considered to be the fundamental length. \(\mathcal{T}(\mathcal{T}(-\infty, \infty))'\) is called the space of the tempered ultrahyperfunctions, where \(\mathcal{T}(\mathcal{T}(-\infty, \infty)) = \lim_{\infty \to \ell} \mathcal{T}(\mathcal{T}(-\ell, \ell))\) is the space of rapidly decreasing entire functions. \(\mathcal{T}(\mathcal{T}(-\ell, \ell))'\) is the space of tempered ultrahyperfunctions whose carrier are contained in \(\mathcal{T}(-\ell, \ell)\). The standard locality condition of quantum field theory in terms of Schwartz distributions is extended using the notion of carrier of analytic functionals (functionals over the test-function space of analytic functions) instead of the notion of support of Schwartz distributions.

For a field \(\phi(x)\) satisfying the standard Wightman axioms, the two-point functional \((\Phi, \phi(x)\phi(y)\Psi)\) is a functional over the test-function space \(\mathcal{S}(\mathbb{R}^{2+4})\), i.e., a tempered distribution. However, for the field \(\psi(x)\) satisfying Equation (1.2), \((\Phi, \psi(x)\psi(y)\Psi)\) is not a functional over the test-function space \(\mathcal{S}(\mathbb{R}^{2+4})\) but, as shown in sections 2 and 7 of this paper, a functional over the test-function space \(\mathcal{T}(\mathcal{T}(\mathcal{L}^{\ell}))\) for any \(\ell' > \ell = \ell_m(l) = \ell/\sqrt{2\pi} + O(l^2),\) where

\[\mathcal{T}(\mathcal{L}^{\ell'}) = \mathbb{R}^{2+4} + i\mathcal{L}^{\ell'}, \quad \mathcal{L}^{\ell'} = \{(y_1, y_2) \in \mathbb{R}^{2+4}; |y_1 - y_2| < \ell'\}.\]

Thus such a functional can distinguish two events occurring at \(x_1\) and \(x_2\) if the distance between \(x_1\) and \(x_2\) is greater than \(\ell\), and cannot distinguish them if the distance is smaller than \(\ell\). In that sense, the field \(\psi(x)\) does not define a local field but a quasi-local field with a fundamental length \(\ell\).

It is quite interesting that the parameter \(l\) with the dimension of length contained in Equations (1.2) is essentially the fundamental length \(\ell' > \ell = \ell_m(l) = \ell/\sqrt{2\pi} + O(l^2)\) in the sense of this theory.

2. **Verification of Extended Causality**

In this section we are going to prove that the system of functionals (5.7) of Part I (see [5]), i.e., the functionals on \(\mathcal{T}(\mathcal{T}(\mathbb{R}^{4n}))\)

\[W_n^\alpha(f) = \int_{\prod_{j=1}^n \Gamma_j} (\det A(z))^{-1/2} W_0^\alpha(z_1, \ldots, z_n) f(z) dz, \quad (2.1)\]

where \(A(z)\) is the \(n \times n\) symmetric matrix whose entries \(a_{j,k}\) are given by

\[a_{j,k} = a_{k,j} = 2\hbar r_j \hbar r_k l^2 D_m^{(-)}(z_j - z_k)\]
for \( r_j = \pm 1, h_{\pm 1} = e^{\pm in/4} \), \( j < k \) and \( a_{j,j} = 1 \), and where the paths \( \Gamma_j \) are \( \mathbb{R}^4 + i(y_j^0, 0, 0, 0) \) for appropriately chosen constants \( y_j^0 \), satisfies the spinor version of condition (R3) of extended locality as presented in \[1\]. For convenience we recall this condition here:

**(R3) (Condition of extended causality):** For all \( n = 2, 3, \ldots \) and all \( j = 1, \ldots, n - 1 \) denote

\[
L_j^\ell = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^{4n}; |x_j - x_{j+1}| < \ell \},
\]

\[
W_j^\ell = \{ (z_1, \ldots, z_n) \in \mathbb{C}^{4n}; z_j - z_{j+1} \in V^\ell \},
\]

where

\[
V^\ell = \{ z \in \mathbb{C}^4; \exists x \in V; |\text{Re } z - x| < \ell, |\text{Im } z_1| < \ell \}. \quad (2.2)
\]

is a complex neighborhood of light cone \( V \). Then, for any \( \ell' > \ell \),

(i) the functional on \( \mathcal{T}(T(\mathbb{R}^{4n})) \)

\[
\mathcal{T}(T(\mathbb{R}^{4n})) \ni f \rightarrow W_{\alpha}^\ell (f) \in \mathbb{C}
\]

is extended continuously to \( \mathcal{T}(T(L_j^\ell)) \), and

(ii) the functional on \( \mathcal{T}(T(\mathbb{R}^{4n})) \)

\[
f \rightarrow W_{\alpha_1, \alpha_j, \alpha_{j+1}, \ldots, \alpha_n}^{r_1, \ldots, r_j, r_{j+1}, \ldots, r_n} (f) + W_{\alpha_1, \alpha_j, \alpha_{j+1}, \ldots, \alpha_n}^{r_1, \ldots, r_j, r_{j+1}, \ldots, r_n} (f) \in \mathbb{C}
\]

is extended continuously to \( \mathcal{T}(W_j^\ell) \).

**Remark 2.1.** In our previous paper \[1\], we defined a complex neighbourhood \( V^\ell \) by

\[
V^\ell = \{ z \in \mathbb{C}^4; \exists x \in V; |\text{Re } z - x| + |\text{Im } z|_1 < \ell \}. \quad (2.3)
\]

But we found that to treat the present model, the neighbourhood \( (2.2) \) is convenient, and by this change of the \( \ell \)-neighbourhood of \( V \), our theory \[1\] is not affected.

In order to verify this condition fix \( j \in \{1, \ldots, n - 1\} \) and assume

\[
y_{j+1}^0 - y_j^0 > \ell = l/(\sqrt{2\pi}), \quad (2.4)
\]

then by estimate (5.6) of Part I, i.e., the global estimate

\[
|D_m^{(-)}(x^0 - i\epsilon, \alpha^i) - (2\pi\epsilon)^{-2} \text{ for all } x \in \mathbb{R}^4, \quad \forall \epsilon > 0, \quad (2.5)
\]

it follows

\[
|4l^4 D_m^{(-)}(z_j - z_{j+1})^2 | < 1.
\]

Introduce

\[
Q_{n,j}(a_{i,k}) = \sum_{(i,k,\ldots,l) \neq (1,2,\ldots,n)} sgn (i,k,\ldots,l) a_{1,j} a_{2,k} \cdots a_{n,l} \quad (2.6)
\]
and denote by $\sigma(j+1, j)$ the permutation $(1, \ldots, j-1, j, j+1, \ldots, n) \rightarrow (1, \ldots, j-1, j+1, \ldots, n)$. Then we have

$$P_n(a, i, k) = \operatorname{sgn}(\sigma(j+1, j))a_{1, i}a_{2, 2} \cdots a_{j-1, j-1}a_{j, j+1}a_{j+1, j+1} \cdots a_{n, n}$$

$+$

$$Q_n(a, i, k) = -a_{j, j+1}^2 + Q_n(a, i, k) = \pm 4l^2 D_m^{-}(z_j - z_{j+1})^2 + Q_n(a, i, k).$$

Hence we can rewrite (5.5) of Part I, i.e.,

$$\det A = 1 + P_n(a, j, k)$$

(2.7)

where $P_n(a, j, k)$ is the sum of homogeneous polynomials of degrees $m = 2, \ldots, n$ in the entries $a_{j, k}$, $1 \leq j < k \leq n$ with integer coefficients, as

$$\det A = 1 + P_n(a, j, k) = 1 \pm 4l^2 D_m^{-}(z_j - z_{j+1})^2 + Q_n(a, i, k).$$

It is clear from (2.7), (2.6) and the details provided about the polynomial $P_n$ that each term of $Q_n(a, i, k)$ contains products of 2-points functions $D_m^{-}$ at arguments different from $z_j - z_{j+1}$. If we choose the arguments $y_0^i - y_k^0 (i < k)$ in these 2-points functions sufficiently large, $Q_n(a, i, k)$ becomes very small; and for these points $z_j$ the determinant $(\det A(z))^{-1/2}$ is holomorphic and the function

$$(\det A(z))^{-1/2} \mathcal{W}_{0, \alpha}(z_1, \ldots, z_n)$$

defines a functional in $\mathcal{T}(\mathcal{T}(L_f^\ell))'$ for any $\ell' > \ell$ by Formula (2.1) for all $f \in \mathcal{T}(\mathcal{T}(L_f^\ell))$. In fact, for $\ell' > \ell$, we choose $\ell' > y_{j+1}^0 - y_j^0 \geq \ell$ and other $y_0^i - y_k^0$ sufficiently large so that $(\det A(z))^{-1/2}$ is a bounded function of $x$. Then the corresponding integration path $\prod_{j=1}^n \Gamma_j$ of (2.1) is contained in

$$T(L_f^\ell) = \{z = x + iy \in \mathbb{C}^n; |y_j - y_{j+1}|_1 < \ell\},$$

where $|y_1| = |y_0^0| + |y_1|$. We conclude that the functional defined by

$$(\det A(z))^{-1/2} \mathcal{W}_{0, \alpha}(z_1, \ldots, z_n)$$

satisfies Axiom (i) of (R3).

The transposition of $z_j$ and $z_{j+1}$ causes the change of $a_{j, j+1} = a_{j+1, j}$:

$D_m^{-}(z_j - z_{j+1}) \rightarrow D_m^{-}(z_{j+1} - z_j)$

and for an index $k$ with $j < k \neq j + 1$ the change

$$a_{j, k} = a_{k, j} = D_m^{-}(z_j - z_k) \rightarrow D_m^{-}(z_{j+1} - z_k) = a_{j+1, k} = a_{k, j+1},$$

$$a_{j+1, k} = a_{k, j+1} = D_m^{-}(z_{j+1} - z_k) \rightarrow D_m^{-}(z_{j+1} - z_k) = a_{j, k} = a_{k, j},$$

results while for an index $k$ with $j > k \neq j + 1$ the change is

$$a_{j, k} = a_{k, j} = D_m^{-}(z_k - z_j) \rightarrow D_m^{-}(z_k - z_{j+1}) = a_{j+1, k} = a_{k, j+1},$$

$$a_{j+1, k} = a_{k, j+1} = D_m^{-}(z_k - z_{j+1}) \rightarrow D_m^{-}(z_k - z_{j+1}) = a_{j, k} = a_{k, j}.$$

We consider the matrix $B = (b_{j, j})$ obtained from $A$ by the change of $j$-th and $j + 1$-th rows and $j$-th and $j + 1$-th columns. Then we have

$$\det A = \det B.$$
$D_m^-(z_{j+1} - z_j)$. If $x_j$ and $x_{j+1}$ are space-like separated, then $D_m^-(x_j - x_{j+1})$ is analytic (space-like points $x$ are Jost points of $D_m^-(x)$) and $D_m^-(x_j - x_{j+1}) = D_m^-(x_{j+1} - x_j)$. Therefore for space-like separated $x_j, x_{j+1}$ ($y_j^0 - y_{j+1}^0 = 0$) and other $y_k^0 - y_i^0$ sufficiently large, we have $\det A = \det C$. Note that $\mathcal{W}_{0,\alpha}^r(z_1, \ldots, z_n)$ is also expressed by the sum of products of the two-point functions of the Dirac field as in the scalar case, and for space-like separated $x_j, x_{j+1}$ ($y_j^0 - y_{j+1}^0 = 0$) and other $y_k^0 - y_i^0$ positive, we have

$$\mathcal{W}_{0,\alpha}^r(z_1, \ldots, x_j, x_{j+1}, \ldots, z_n) = -\mathcal{W}_{0,\alpha}^r(z_1, \ldots, x_{j+1}, x_j, \ldots, z_n).$$

In order to proceed we need some estimates for $D_m^-(x_{j+1} - x_j)$ which are developed below.

**Proposition 2.2.** Let $\omega(|p|) = \sqrt{|p|^2 + m^2}$ and introduce the auxiliary function

$$g_m(z, x) = \int_0^\infty e^{-i\omega(|p|)z} e^{-ip|x|} \frac{m}{|p|^2 + m^2 + |p|\omega(|p|)} dp.$$

Then we have

$$D_m^-(x^0 - i\epsilon, x) = [(2\pi)^3]^{-1} e^{-i\epsilon |x^0|} \frac{-1}{(x^0 - i\epsilon)^2 - |x|^2}$$

$$+ \frac{mi}{[(2\pi)^3]^2} \left[ -\frac{g_m(x^0 - i\epsilon, -|x|)}{x^0 - i\epsilon - |x|} + \frac{g_m(x^0 - i\epsilon, -|x|)}{x^0 - i\epsilon + |x|} \right],$$

and for $\Im z \leq 0$ and $\Im x = 0$, the estimate

$$|g_m(z, x)| \leq g_m(0, 0) \leq \frac{\sqrt{2\pi}}{4}$$

follows.

**Proof.** From the definition of $D_m^-$ we know

$$D_m^-(x^0 - i\epsilon, x) = [2\pi^3]^{-1} \int \omega(|p|)^{-1} e^{-i\omega(|p|)(x^0 - i\epsilon)} e^{ip|x|} dp$$

$$= [2\pi^3]^{-1} \int \omega(|p|)^{-1} e^{-i\omega(|p|)(x^0 - i\epsilon)} \exp(i|p||x| \cos \theta) |p|^2 \sin \theta dp |d\theta d\phi$$

$$= [2\pi^2]^{-1} \frac{1}{i|x|} \int_0^\infty \omega(|p|)^{-1} e^{-i\omega(|p|)(x^0 - i\epsilon)} [e^{i|p||x|} - e^{-i|p||x|}] |p| dp.$$
\[-[2(2\pi)^2]^{-1} \frac{1}{i |x|} \int_m^\infty e^{-it(x^0 - i\epsilon + |x|)} e^{-i(\sqrt{t^2 - m^2} - t)|x|} dt = \]

\[
\left[2(2\pi)^2 \right]^{-1} \frac{1}{i |x|} \left[ e^{-it(x^0 - i\epsilon - |x|)} e^{i(\sqrt{t^2 - m^2} - t)|x|} \right]_{t=m}^\infty \]

\[-[2(2\pi)^2]^{-1} \frac{1}{i |x|} \left[ e^{-i(x^0 - i\epsilon - |x|)} \right]_{t=m}^\infty \]

\[
+ \left[2(2\pi)^2 \right]^{-1} \frac{1}{i |x|} \left[ e^{-i(x^0 - i\epsilon + |x|)} e^{-i(\sqrt{t^2 - m^2} - t)|x|} \right]_{t=m}^\infty \]

Since

\[
\int_m^\infty e^{-it(x^0 - i\epsilon - m^2)|x|} \left[ \frac{t}{\sqrt{t^2 - m^2}} - 1 \right] dt = \int_0^\infty e^{-i\omega(|p|)(x^0 - i\epsilon)|p|} [1 - |p|\omega(|p|)] d|p| = m \int_0^\infty e^{-i\omega(|p|)(x^0 - i\epsilon)|p|} \frac{m}{|p|^2 + m^2 + |p|\omega(|p|)} d|p| = mg_m(x^0 - i\epsilon, \pm |x|)
\]

and

\[
\frac{1}{i |x|} \left[ e^{-it(x^0 - i\epsilon - |x|)} e^{i(\sqrt{t^2 - m^2} - t)|x|} \right]_{t=m}^\infty \]

\[-2 \]

\[
D^{(-)}_m(x^0 - i\epsilon, |x|) = \left[2(2\pi)^2 \right]^{-1} e^{-im(x^0 - i\epsilon)} \frac{1}{(x^0 - i\epsilon)^2 - |x|^2} \]

\[
+ \frac{m}{[2(2\pi)^2]} \left[ \frac{-g_m(x^0 - i\epsilon, |x|)}{x^0 - i\epsilon - |x|} + \frac{g_m(x^0 - i\epsilon, |x|)}{x^0 - i\epsilon + |x|} \right] .
\]

\[
|g_m(z, x)| \leq \int_0^\infty \frac{m}{|p|^2 + m^2 + |p|\omega(|p|)} d|p| = \left( \frac{\sqrt{2}}{m} \tan^{-1} \sqrt{2} \right) m \int_0^\infty \frac{\sqrt{2}}{4} = \frac{\sqrt{2}\pi}{4}.
\]
Corollary 2.3. **Introduce**

\[ \ell_m(l) = \left[ \frac{1}{(2\pi)} \right] l^2 \sqrt{2/8 + \ell \sqrt{2 + 2(m/8)^2l^2}} \]

and \( a = \min_{\pm} |x^0 - i\epsilon \pm |x||. \) Then, if \( a > \ell_m(l) \), the estimate

\[ 2l^2|D_m^{(-)}(x^0 - i\epsilon \ell(x), x)| < 1 \]

holds.

**Proof.** We have the following inequalities.

\[
|D_m(x^0 - i\epsilon, x)| \leq (2\pi)^{-2} \left( \frac{1}{(x^0 - i\epsilon - |x|)(x^0 - i\epsilon + |x|)} \right) \\
+ \left[ (2\pi)^{-2} m \frac{\sqrt{2\pi}}{4} \left( \frac{1}{x^0 - i\epsilon - |x|} + \frac{1}{x^0 - i\epsilon + |x|} \right) \right] \\
\leq (2\pi)^{-2} \frac{1}{a^2} + (2\pi)^{-2} m \frac{\sqrt{2\pi}}{4} \frac{1}{a}, \\
2l^2|D_m(x^0 - i\epsilon, x)| \leq 2l^2 \left( (2\pi)^{-2} \frac{1}{a^2} + (2\pi)^{-2} m \frac{\sqrt{2\pi}}{4} \frac{1}{a} \right).
\]

As a solution of the inequality

\[ 2l^2(2\pi)^{-2} \left( \frac{1}{a^2} + m \frac{\sqrt{2\pi}}{4} \frac{1}{a} \right) < 1, \]

we have

\[ a > \ell_m(l) = \left[ \frac{1}{(2\pi)} \right] l^2 \sqrt{2/8 + \ell \sqrt{2 + 2(m/8)^2l^2}}. \]

This completes the proof. \( \square \)

**Corollary 2.4.** Denote by \( \text{dist}(x, \bar{V}) \) the distance between \( x \) and the closed light cone \( \bar{V} = \{ x = (x^0, x) \in \mathbb{R}^4; |x^0| \geq |x| \} \), and for \( \ell > 0 \)

\[ V_\ell = \{ x \in \mathbb{R}^4; \text{dist}(x, \bar{V}) < \ell \}. \]

Define \( \epsilon_\ell(x) \) by \( \epsilon_\ell(x) = \ell \) if \( \text{dist}(x, \bar{V}) \leq \ell / \sqrt{2}, \epsilon_\ell(x) = \sqrt{2\ell^2 - 2\text{dist}(x, V)^2} \) if \( \ell / \sqrt{2} \leq \text{dist}(x, \bar{V}) \leq \ell \) and \( \epsilon_\ell(x) = 0 \) if \( \text{dist}(x, \bar{V}) \geq \ell \). Then

\[ 0 \leq \epsilon_\ell(x) \leq \ell \text{ and } \text{supp} \epsilon_\ell(x) \subset \bar{V}_{\ell}. \]

Let \( \ell = l/(\sqrt{2\pi}) \) and assume \( \sqrt{2\ell} > \ell_m(l), \) e.g., assume \( ml < 2. \) Then, if \( \ell'' > \ell \), the estimate

\[ 2l^2|D_m^{(-)}(x^0 - i\epsilon_{\ell''}(x), x)| < 1 \]

holds.

**Proof.** The support property of \( \epsilon_{\ell''}(x) \) follows immediately from the definitions, and it is easy to see that

\[ |x^0 \pm |x|| \geq \sqrt{2} \text{dist} (x, \bar{V}) \]

and we have,

\[ a(x)^2 = \min_{\pm} |x^0 - i\epsilon_{\ell''}(x) \pm |x||^2 \geq 2\text{dist} (x, \bar{V})^2 + \epsilon_{\ell''}(x)^2. \]
If \( \text{dist}(x, V) \geq \ell''/\sqrt{2} \), then \( a(x)^2 = 2\ell''^2 > 2\ell^2 > \ell_m(l)^2 \), and the estimate holds. If \( \text{dist}(x, V) \leq \ell''/\sqrt{2} \), then \( \epsilon_{\ell''}(x) = \ell'' > \ell \), and the estimate follows from the inequality \( [23] \). This completes the proof.

For any \( \ell' > \ell \), we choose \( \ell < \ell'' < \ell' \). Let \( \epsilon(x) = \epsilon_{\ell''}(x) \), and \( a_{j,j+1} = D_m^{-1}(x_j - x_{j+1} + i\epsilon(x_j - x_{j+1})) \) and for the other \( a_{i,k} \) take \( y_k^0 - y_j^0 \) sufficiently large. Then \( (\det A(x))^{-1/2} \) and \( (\det C(x))^{-1/2} \) are well-defined continuous functions of \( x \) and \( (\det A(x))^{-1/2} = (\det C(x))^{-1/2} \) if \( x_j - x_{j+1} \in \mathbb{R}^4 \setminus V'' \).

Let

\[
W_{\alpha}^{r}(z_1, \ldots, z_n) = (\det A(z))^{-1/2}W_{0,\alpha}^{r}(z_1, \ldots, z_n)
\]

and

\[
W_{\alpha}^{r,j}(z) = W_{\alpha}^{r'}(z'), z' = (z_1, \ldots, z_{j+1}, z_j, \ldots, z_n),
\]

\[
r' = (r_1, \ldots, r_{j+1}, r_j, \ldots, r_n), \quad \alpha' = (\alpha_1, \ldots, \alpha_{j+1}, \alpha_j, \ldots, \alpha_n).
\]

Then, by deforming the path \( \Gamma_j \times \Gamma_{j+1} \) into \( G_{j,j+1} \), we can write

\[
W_{\alpha}^{r}(f) + W_{\alpha}^{r,j}(f) = \int_{G_{j,j+1,\Pi_{i\neq j+1}}} W_{\alpha}^{r}(z) f(z) dz + \int_{G_{j+1,\Pi_{i\neq j+1}}} W_{\alpha}^{r,j}(z) f(z) dz,
\]

where \( y_j^0 = y_j^0 \) and

\[ G_{j,j+1} = \{(x_j^0 + iy_j^0 - i\epsilon(x_j - x_{j+1}), x_j, x_{j+1}^0 + iy_{j+1}^0, x_{j+1}); (x_j, x_{j+1}) \in \mathbb{R}^{2^4}\}, \]

\[ G_{j+1,j} = \{(x_j^0 + iy_j^0, x_j, x_{j+1}^0 + iy_{j+1}^0 - i\epsilon(x_{j+1} - x_j), x_{j+1}); (x_j, x_{j+1}) \in \mathbb{R}^{2^4}\}. \]

Since \( W_{\alpha}^{r}(z) + W_{\alpha}^{r,j}(z) = 0 \) for \( x_j - x_{j+1} \in \mathbb{R}^4 \setminus V'' \),

\[
W_{\alpha}^{r}(f) + W_{\alpha}^{r,j}(f) = \int_{G_{j,j+1,\Pi_{i\neq j+1}}} W_{\alpha}^{r}(z) f(z) dz + \int_{G_{j+1,\Pi_{i\neq j+1}}} W_{\alpha}^{r,j}(z) f(z) dz,
\]

where

\[
G_{j,j+1}'' = \{(x_j^0 + iy_j^0 - i\epsilon(x_j - x_{j+1}), x_j, x_{j+1}^0 + iy_{j+1}^0, x_{j+1}); x_j - x_{j+1} \in \mathbb{R}^4 \setminus V''\},
\]

\[
G_{j+1,j}'' = \{(x_j^0 + iy_j^0, x_j, x_{j+1}^0 + iy_{j+1}^0 - i\epsilon(x_{j+1} - x_j), x_{j+1}); x_j - x_{j+1} \in \mathbb{R}^4 \setminus V''\}. \]

Since \( G_{j,j+1,\Pi_{i\neq j+1}} \times G_{j+1,j,\Pi_{i\neq j+1}} \subset W_{j}^{r} \), this shows that

\[
T(W_{j}^{r}) \ni f \rightarrow W_{\alpha}^{r}(f) + W_{\alpha}^{r,j}(f) \in \mathbb{C},
\]

is continuous and satisfies the axiom (ii) of (R3) of \( [11] \).
3. Convergence of Wick power series for $\rho(x) = e^{g\phi(x)^2}$

Our starting point are the well-known results of Jaffe [4] on formal Wick power series of free fields. If we consider the power series of a free field $\phi$

$$\rho(i)(x) = \sum_{n=0}^{\infty} a_n^{(i)} \phi(x)^n : \frac{n!}{n!},$$  \hspace{1cm} (3.1)

then we have the following theorem.

**Theorem 3.1 (Theorem A.1 of [4]).** In the sense of formal power series the following identity holds

$$\left( \Phi_0, \rho^{(1)}(x_1) \cdots \rho^{(n)}(x_n) \Phi_0 \right) = \sum_{r_{ij}=0; 1 \leq i < j \leq n} A(R) T^R \prod_{1 \leq i < j \leq n} \left( g_n^i g_n^j \right) t_{ij}^{r_{ij}}$$  \hspace{1cm} (3.2)

where

$$r_{ij} = r_{ji}, \quad r_{ii} = 0, \quad R_i = \sum_{j=1}^{n} r_{ij}, \quad A(R) = \prod_{j=1}^{n} a_n^{(i)}_{R_j}$$

$$R! = \prod_{1 \leq i < j \leq n} (r_{ij})!, \quad T^R = \prod_{1 \leq i < j \leq n} (t_{ij})^{r_{ij}}$$

$$t_{ij} = (\Phi_0, \phi(x_i) \phi(x_j) \Phi_0) = D_m^(-)(x_i - x_j).$$

**Corollary 3.2.** In the case of

$$\sigma^{(i)}(x) = : e^{g\phi(x)} : = \sum_{n=0}^{\infty} g_n^i \phi(x)^n : \frac{n!}{n!},$$  \hspace{1cm} (3.2)

(3.2) becomes

$$\left( \Phi_0, \sigma^{(1)}(x_1) \cdots \sigma^{(n)}(x_n) \Phi_0 \right) = \exp \left\{ \sum_{1 \leq i < j \leq n} g_i g_j t_{ij} \right\}.$$

**Proof.** The chain of identities

$$\left( \prod_{1 \leq i < j \leq n} \left\{ g_i g_j \right\}^{r_{ij}} \right)^2 = \prod_{1 \leq i < j \leq n} \left\{ g_i g_j \right\}^{r_{ij}} = \prod_{i=1}^{n} \prod_{j=1}^{n} g_i^{r_{ij}} g_j^{r_{ij}}$$

$$= \prod_{i=1}^{n} \left( \sum_{j=1}^{n} r_{ij} g_j^{r_{ij}} \right) = \prod_{i=1}^{n} \prod_{j=1}^{n} g_i^{R_i} g_j^{r_{ij}} = \prod_{i=1}^{n} \prod_{j=1}^{n} g_i^{R_i} g_j^{r_{ij}} = A(R)^2$$

shows that

$$\prod_{1 \leq i < j \leq n} \left\{ g_i g_j \right\}^{r_{ij}} = A(R),$$

and thus we get

$$\exp \left\{ \sum_{1 \leq i < j \leq n} g_i g_j t_{ij} \right\} = \prod_{1 \leq i < j \leq n} \exp \left\{ g_i g_j t_{ij} \right\} = \prod_{1 \leq i < j \leq n} \sum_{r_{ij}=0}^{\infty} \left\{ g_i g_j t_{ij} \right\}^{r_{ij}}$$
\[
\sum_{r_{ij}=0,1 \leq i < j \leq n} \prod_{1 \leq i < j \leq n} \{g_i g_j \}^{r_{ij}} = \sum_{r_{ij}=0,1 \leq i < j \leq n} \prod_{1 \leq i < j \leq n} \{g_i g_j \}^{r_{ij}} \frac{T(R)}{R!}.
\]

\[
\sum_{r_{ij}=0,1 \leq i < j \leq n} A(R) \frac{T(R)}{R!} = (\Phi_0, \sigma^{(1)}(x_n) \cdots \sigma^{(n)}(x_n) \Phi_0).
\]

Assume that for some \(\sigma > 0\)

\[
\lim_{n \to \infty} \left[ \frac{|a_n^{(i)}|^2 / n!}{|n|} \right]^{1/n} = \sigma.
\]

Then Theorem 6.3 of \[1\] says that the power series (3.1) defines an ultra-hyperfunction quantum field with fundamental length \(\ell = \sqrt{\sigma / (2\pi)}\) if \(\phi\) is a massless free field. Now consider

\[
\rho(x) := e^{ig\phi(x)^2} := \sum_{n=0}^{\infty} \frac{(ig)^n \phi(x)^{2n}}{n!}.
\]

\[
= \sum_{n=0}^{\infty} \frac{(ig)^n (2n)! \phi(x)^{2n}}{n! (2n)!},
\]

\[
\rho^*(x) := e^{-ig\phi(x)^2} := \sum_{n=0}^{\infty} \frac{(-ig)^n \phi(x)^{2n}}{n!}.
\]

In this case we find for the above limit \(\sigma = 2|g|\). Suppose that the 0 < \(t_{ij}\)'s satisfy

\[
\sum_{1 \leq i < j \leq n} t_{ij} < \frac{1}{2|g|}.
\]

Then the power series

\[
\sum_{r_{ij}=0,1 \leq i < j \leq n} A(R) \frac{Z^R}{R!}
\]

of \(z_{ij} (1 \leq i < j \leq n)\) for \(\rho^{(j)}(x) = \rho(x)\) or \(\rho^{(j)}(x) = \rho^*(x)\), where \(Z^R = \prod_{1 \leq i < j \leq n} (z_{ij})^{r_{ij}}\), is absolutely convergent for \(|z_{ij}| < t_{ij} (1 \leq i < j \leq n)\). This shows the convergence of the vacuum expectation value

\[
(\Phi_0, \rho^{(1)}(x_1) \cdots \rho^{(n)}(x_n) \Phi_0)
\]

in the sense of tempered ultra-hyperfunctions, and moreover implies the strong convergence of

\[
\rho_N(f) \Phi = \sum_{n=0}^{N} \frac{(ig)^n \phi(x)^{2n} \cdot (f)}{n!} \Phi
\]

for \(N \to \infty\) (in the Fock space), where \(\Phi = \rho^{(1)}(f_1) \cdots \rho^{(m)}(f_m) \Phi_0\) for \(f_k \in \mathcal{T}(T(\mathbb{R}^4))\). For the definition and basic properties of the
test function space $\mathcal{T}(T(\mathbb{R}^4))$ of tempered ultrahyperfunctions we refer to [1].

**Proposition 3.3.** Abbreviate

\[ \rho^{(j)}(x_j) = e^{-r_j i t^2 \phi(x_j)^2}; \]

with $r_j = \pm 1$. Then the vacuum expectation values of these fields are given by

\[ (\Phi_0, \rho^{(1)}(x_1) \cdots \rho^{(n)}(x_n) \Phi_0) = (\det A)^{-1/2}, \quad (3.5) \]

where $A$ is the $n \times n$ symmetric matrix whose entries $a_{j,k}$ are given by

\[ a_{j,k} = 2 h r_j h r_k D_m^{-1}(x_j - x_k) \]

for $h = e^{\pm i \pi/4}$, $j < k$ and $a_{j,j} = 1$.

Note that the result (3.5) is the same as the corresponding result in [5].

**Proof.** The equation

\[ (2\pi)^{-1/2} \int e^{itp} e^{-t^2/2} dt = e^{-p^2/2} \]

can be considered as an equation for the following two power series of the variable $p$:

\[ (2\pi)^{-1/2} \int \sum_{n=0}^{\infty} [(itp)^n / n!] e^{-t^2/2} dt = \sum_{n=0}^{\infty} (-p^2/2)^n / n!, \]

and by inserting $p = \sqrt{2} \phi(x)$ and using Wick products we get, as a formal series

\[ (2\pi)^{-1/2} \int \sum_{n=0}^{\infty} [(it \sqrt{2} \phi(x))^n / n!] e^{-t^2/2} dt = \sum_{n=0}^{\infty} (- (h \phi(x))^2)^n / n!. \]

We write this as

\[ (2\pi)^{-1/2} \int e^{it \sqrt{2} h \phi(x)} : e^{-t^2/2} dt = e^{-(h \phi(x))^2} :. \]

Let $h = e^{ir_j \pi/4}$ and denote $\sigma^{(j)}(x) = e^{it_j \sqrt{2} h r_j \phi(x)} :$. Then Corollary 2.2 says

\[ (\Phi_0, \sigma^{(1)}(x_1) \cdots \sigma^{(n)}(x_n) \Phi_0) = \exp \left\{ \sum_{1 \leq j < k \leq n} -2 t_j t_k h r_j h r_k D_m^{-1}(x_j - x_k) \right\} \]

and thus we get

\[ (\Phi_0, \rho(\phi(x_1) \cdots \rho(\phi(x_n) \Phi_0) = 1 / (2\pi)^{n/2} \int e^{\sum_{1 \leq j < k \leq n} -2 t_j t_k h r_j h r_k D_m^{-1}(x_j - x_k) - \sum_{j=1}^{n} t_j^2 / 2} dt_1 \ldots dt_n \]

\[ = (\det A)^{-1/2}. \]

□
Note that \( \phi(x) = \rho(i)(x) \) if \( a_i^{(i)} = 1 \) and \( a_n^{(i)} = 0 \) for \( n \neq 1 \). Let \( U(a, \Lambda) \) be the unitary representation of the proper Poincaré group for the free neutral scalar field in the Fock space \( \mathcal{H} \). Then the system \( \{ \mathcal{H}, \Phi_0, U(a, \Lambda), \phi(x), \rho(x), \rho^*(x) \} \) satisfies the axioms of UHFQFT.

4. Verification of the equation
\[
\partial_{\mu} \rho(x) = 2i\ell^2 : \rho(x) \phi(x) \partial_{\mu} \phi(x) : 
\]
We begin by recalling some basic facts about Wick products of free fields which are then used to study Wick polynomials and Wick power series.

Let \( \mathcal{H} \) be the Hilbert space defined by
\[
\mathcal{H} = \oplus_{n=0}^{\infty} \mathcal{H}_n.
\]
Here, \( \mathcal{H}_n \) is the set of symmetric square-integrable functions on the direct product of the momentum space hyperboloids
\[
\xi_k^2 = m^2, \quad \xi_k^0 > 0, \quad k = 1, \ldots, n
\]
with respect to the Lorentz invariant measure
\[
d\Omega_m(\xi) = \frac{d\xi_1 d\xi_2 d\xi^3}{\sqrt{\sum_{k=1}^{3} (\xi_k^2)^2 + m^2}}.
\]
In the fundamental paper [6], we find the following quite general formula (3.44) for the definition of Wick products of a free field \( \phi \) of mass \( m \) as operators in \( \mathcal{H} \): For \( f \in \mathcal{S}(\mathbb{R}^4) \) and \( \Phi \in \mathcal{H} \) one has:
\[
\langle :D^\alpha(1) \phi D^\alpha(2) \phi \cdots D^\alpha(l) \phi : (f) \Phi \rangle_{(n)}(\xi_1, \ldots, \xi_n) = \frac{\pi^{l/2}}{(2\pi)^{2l-1}} \sum_{j=0}^{l-1} \prod_{k=1}^{l} d\Omega_m(\eta_k) \times
\]
\[
\sum_{1 \leq k_1 < k_2 < \ldots < k_{l-j} \leq n} (j!)^{-1} \prod_P P (-i\eta_1)^{\alpha(1)} \cdots (-i\eta_j)^{\alpha(j)} (i\xi_{k_1})^{\alpha(j+1)} \cdots \cdots (i\xi_{k_{l-j}})^{\alpha(l)} \tilde{f} \left( \sum_{r=1}^{j} \eta_r - \sum_{r=1}^{l-j} \xi_{k_r} \right) \Phi^{(n-l+2j)}(\eta_1, \ldots, \eta_j, \xi_1, \ldots, \xi_{k_1}, \ldots, \xi_{k_{l-j}}, \ldots, \xi_n),
\]
where in the summation \( \sum_j \), only those terms are to be retained for which \( n - l + 2j \geq 0 \), and the sum \( \sum_P \) is over all permutation of the
variables $\eta_1, \ldots, \eta_j, (-\xi_{k_1}), \ldots, (-\xi_{k_{l-j}})$. We reconsider this formula in the sense of operator-valued ultra-hyperfunctions. Let $|\beta| = 1$ and $|\alpha^{(1)}| = |\alpha^{(2)}| = \ldots = |\alpha^{(l)}| = 0$. Then we have from (3.44)

\[
(\phi_j : (-D^\beta f)\Phi)^{(n)}(\xi_1, \ldots, \xi_n)
\]

\[
= \frac{\pi^{l/2}}{(2\pi)^{2(l-1)}} \sum_{j=0}^{l-1} \left[ \frac{(n - l + 2j)!}{n!} \right]^{1/2} \int \cdots \int \left( \prod_{k=1}^{j} d\Omega_m(\eta_k) \right)
\times \sum_{1 \leq k_1 < k_2 < \ldots < k_{l-j} \leq n} (j!)^{-1}
\]

\[
\times \Phi^{(n-l+2j)}(\eta_1, \ldots, \eta_j; \xi_1, \ldots, \hat{\xi}_{k_1}, \ldots, \hat{\xi}_{k_{l-j}}, \ldots, \xi_n).
\]

\[
= \frac{\pi^{l/2}}{(2\pi)^{2(l-1)}} \sum_{j=0}^{l-1} \left[ \frac{(n - l + 2j)!}{n!} \right]^{1/2} \int \cdots \int \left( \prod_{k=1}^{j} d\Omega_m(\eta_k) \right)
\times \sum_{1 \leq k_1 < k_2 < \ldots < k_{l-j} \leq n} (j!)^{-1} \sum_{P} \left( l(i\eta_1)^\beta \hat{f} \left( \sum_{r=1}^{j} \eta_r - \sum_{r=1}^{l-j} \xi_{k_r} \right) \right)
\times \Phi^{(n-l+2j)}(\eta_1, \ldots, \eta_j; \xi_1, \ldots, \hat{\xi}_{k_1}, \ldots, \hat{\xi}_{k_{l-j}}, \ldots, \xi_n).
\]

Observe that

\[
\sum_{P} P(\eta_i) = \sum_{P} P(-\xi_{k_r})
\]

for any $i$ and $r$. This implies for $|\beta| = 1$,

\[
\sum_{P} P((\eta_i)^\beta) = \sum_{P} P((-\xi_{k_r})^\beta)
\]

and therefore

\[
\sum_{P} \left( l(i\eta_1)^\beta \hat{f} \left( \sum_{r=1}^{j} \eta_r - \sum_{r=1}^{l-j} \xi_{k_r} \right) \right)
\]

\[
= \sum_{P} \left( l(i\eta_1)^\beta \hat{f} \left( \sum_{r=1}^{j} \eta_r - \sum_{r=1}^{l-j} \xi_{k_r} \right) \right)
\]

On the other hand, we also have from (3.44), for $|\alpha^{(1)}| = 1$, and $|\alpha^{(2)}| = \ldots = |\alpha^{(l)}| = 0$

\[
(\phi : (D^\alpha)\Phi) \hat{\phi}^{-1} : (f)\Phi)^{(n)}(\xi_1, \ldots, \xi_n)
\]

\[
= \frac{\pi^{l/2}}{(2\pi)^{2(l-1)}} \sum_{j=0}^{l-1} \left[ \frac{(n - l + 2j)!}{n!} \right]^{1/2} \int \cdots \int \left( \prod_{k=1}^{j} d\Omega_m(\eta_k) \right)
\]

\[
\times \Phi^{(n-l+2j)}(\eta_1, \ldots, \eta_j; \xi_1, \ldots, \hat{\xi}_{k_1}, \ldots, \hat{\xi}_{k_{l-j}}, \ldots, \xi_n).
\]
\[ \times \sum_{1 \leq k_1 < k_2 < \ldots < k_{l-j} \leq n} P \left( (i \eta_1)_{1}^{(l-j)} f \left( \sum_{r=1}^{j} \eta_r - \sum_{r=1}^{l-j} \xi_{k_r} \right) \right) \]
\[ \times \Phi^{(n^{l-2})} (\eta_1, \ldots, \eta_{j}, \xi_1, \ldots, \xi_{k_1}, \ldots, \xi_{k_{l-j}}, \ldots, \xi_n). \]

This shows that
\[ (\varphi^{l} : (-D^{\alpha(l)} f) \Phi)_{(n)} = l : (D^{\alpha(l)} \phi) \varphi^{l-1} : (f) \Phi)_{(n)}, \quad (4.2) \]
that is,
\[ D^{\alpha(l)} : \phi(x)^l := l : (D^{\alpha(l)} \phi(x)) \phi^{l-1}(x) : . \quad (4.3) \]

Let \(D_0\) be the set generated by the vectors of the form
\[ \rho^{(1)}(f_1) \cdots \rho^{(n)}(f_n) \Phi_0, f_k \in T(T(\mathbb{R}^4)), \]
where \(\rho^{(k)}(x)\) is one of \(\phi(x), \rho(x)\) and \(\rho^*(x)\), and \(\Phi \in D_0\). Then we have seen in the previous section that
\[ \rho(-D^{\alpha(l)} f) \Phi =: e^{i g \phi^2} : (-D^{\alpha(l)} f) \Phi \]
is strongly convergent, and by \((4.2)\)
\[ : \phi^{2l} : (-D^{\alpha(l)} f) \Phi = l : (D^{\alpha(l)} \phi) \phi^{l-1} : (f) \Phi. \]

This shows that
\[ \sum_{l=0}^{\infty} \frac{(i g)^l}{l!} : \phi^{2l} : (-D^{\alpha(l)} f) \Phi \]
\[ = \sum_{l=1}^{\infty} \frac{(i g)^l}{(l-1)!} : (D^{\alpha(l)} \phi) \phi^{2(l-1)} : (f) \Phi \]
\[ = \sum_{l=0}^{\infty} 2(i g) \frac{(i g)^l}{l!} : (D^{\alpha(l)} \phi) \phi^{2l} : (f) \Phi. \]

We write the last expression as
\[ = 2(i g) : (D^{\alpha(l)} \phi) \phi \sum_{l=0}^{\infty} \frac{(i g)^l}{l!} : \phi^{2l} : (f) \Phi \]
\[ = 2 i g : (D^{\alpha(l)} \phi) \phi \rho : (f) \Phi. \]

That is, the formal expression (which is difficult to give a direct meaning)
\[ 2 i g : (D^{\alpha(l)} \phi(x)) \phi(x) : e^{i g \phi(x)^2} : \Phi \]
\[ = 2 i g : (D^{\alpha(l)} \phi(x)) \phi(x) \sum_{l=0}^{\infty} \frac{(i g)^l}{l!} : \phi^{2l}(x) : \Phi \]
can be understood as
\[
\sum_{l=0}^{\infty} 2i g \cdot (D^{\alpha(1)} \phi(x)) \cdot \phi(x) \cdot \frac{(ig)^l}{l!} \cdot \phi^{2l}(x) : \Phi
\]
\[
= \sum_{l=1}^{\infty} 2 \cdot (D^{\alpha(1)} \phi(x)) \cdot \frac{(ig)^l}{(l-1)!} \cdot \phi^{2l-1}(x) : \Phi.
\]
Then by (4.3), the above expression equals
\[
\sum_{l=1}^{\infty} \frac{(ig)^l}{l!} \cdot D^{\alpha(1)} \cdot \phi^{2l}(x) : \Phi,
\]
and this is equal to
\[
D^{\alpha(1)} \sum_{l=1}^{\infty} \frac{(ig)^l}{l!} \cdot \phi^{2l}(x) : \Phi = D^{\alpha(1)} \rho(x) \Phi
\]
in the sense of generalized functions. In the above understanding, we have
\[
D^{\alpha(1)} \rho(x) \Phi = 2i g \cdot (D^{\alpha(1)} \phi(x)) \cdot \phi(x) \cdot \rho(x) : \Phi,
\]
that is, if the Wick product
\[
(D^{\alpha(1)} \phi(x)) \cdot \phi(x) \cdot \rho(x) :
\]
is defined by the Wick power series
\[
\sum_{l=0}^{\infty} 2i g \cdot (D^{\alpha(1)} \phi(x)) \cdot \phi(x) \cdot \frac{(ig)^l}{l!} \cdot \phi^{2l}(x) :,
\]
then we have (4.4), i.e., (1.4).

5. WIGHTMAN’S AXIOMS FOR GENERAL TYPE FIELDS

In Wightman’s scheme, the concept of a relativistic quantum field \( \phi^{(\kappa)} \) of type \( \kappa \) plays a fundamental role. Such a field, for example a scalar, tensor or spinor field, has a finite number of Lorentz components \( \phi^{(\kappa)}_j \) \( (j = 1, \ldots, r_\kappa) \).

The field components \( \phi^{(\kappa)}_j(x) \) are operator-valued generalized functions, i.e.,
\[
\phi^{(\kappa)}_j(f) = \int \phi^{(\kappa)}_j(x) f(x) d^4 x
\]
are densely defined linear operators in a complex Hilbert space \( \mathcal{H} \). They are not assumed to be bounded.

Here we state Wightman’s axioms for the ultra-hyperfunction quantum field theory \[\[\]. For the neutral scalar fields, these axioms are the axioms in \[\[\].
W.I. Relativistic invariance of the state space: There is a complex Hilbert space $\mathcal{H}$ with positive metric in which a unitary representation $U(a, A)$ of the Poincaré spinor group $\mathcal{P}_0$ acts. $(a, A) \mapsto U(a, A)$ is weakly continuous.

W.II. Spectral property: The spectrum $\Sigma$ of the energy-momentum operator $P$ which generates the translations in this representation, i.e., $e^{iap} = U(a, 1)$, is contained in the closed forward light cone $\overline{V}_+ = \{p = (p^0, \ldots, p^3) \in \mathbb{R}^4; p^0 \geq |p|\}$.

W.III. Existence and uniqueness of the vacuum: In $\mathcal{H}$ there exists unit vector $\Phi_0$ (also denoted by $|0\rangle$ and called the vacuum vector) which is unique up to a phase factor and which is invariant under all space-time translations $U(a, 1), a \in \mathbb{R}^4$.

W.IV. Fields: The components $\phi_j^{(\kappa)}$ of the quantum field $\phi^{(\kappa)}$ are operator-valued generalized functions $\phi_j^{(\kappa)}(x)$ over the space $\mathcal{T}(T(\mathbb{R}^4))$ with common dense domain $\mathcal{D}$; i.e., for all $\Psi \in \mathcal{D}$ and all $\Phi \in \mathcal{H}$,

$$\mathcal{T}(T(\mathbb{R}^4)) \ni f \rightarrow (\Phi, \phi_j^{(\kappa)}(f)\Psi) \in \mathbb{C}$$

is a tempered ultrahyperfunction. It is supposed that the vacuum vector $\Phi_0$ is contained in $\mathcal{D}$ and that $\mathcal{D}$ is taken into itself under the action of the operators $\phi_j^{(\kappa)}(f)$ and $U(a, A)$, i.e.,

$$\phi_j^{(\kappa)}(f)\mathcal{D} \subset \mathcal{D}, \quad U(a, A)\mathcal{D} \subset \mathcal{D}.$$ 

Moreover it is assumed that there exist indices $\bar{\kappa}, \bar{j}$ such that $\phi_{\bar{j}}^{(\bar{\kappa})}(\bar{f}) \subset \phi_j^{(\kappa)}(f)^*$ where $^*$ indicates the Hilbert space adjoint of the operator in question.

W.V. Poincaré-covariance of the fields: According to the type of the field, there is a finite dimensional real or complex matrix representation $V^{(\kappa)}(A)$ of $SL(2, \mathbb{C})$ such that

$$U(a, A)\phi_j^{(\kappa)}(x)U(a, A)^{-1} = \sum_\ell V_{j,\ell}^{(\kappa)}(A^{-1})\phi_\ell^{(\kappa)}(A(A)x + a),$$

i.e., for any $f \in \mathcal{T}(T(\mathbb{R}^4))$ and $\Psi \in \mathcal{D}$,

$$U(a, A)\phi_j^{(\kappa)}(f)U(a, A)^{-1}\Psi = \sum_\ell V_{j,\ell}^{(\kappa)}(A^{-1})\phi_\ell^{(\kappa)}(f(a, A))\Psi,$$

where $f(a, A)(x) = f(A(A)^{-1}(x - a))$. We have $V^{(\kappa)}(-1) = \pm 1$. If $V^{(\kappa)}(-1) = 1$, then the field is called a tensor field. If $V^{(\kappa)}(-1) = -1$, then the field is called a spinor field.
Any two field components $\phi^{(\kappa)}_j(x)$ and $\phi^{(\kappa')}_i(y)$ either commute or anticommute if the distance between $x$ and $y$ is greater than $\ell$:

a) The functionals

$$\mathcal{T}(T(\mathbb{R}^4)) \times \mathcal{T}(T(\mathbb{R}^4)) \ni f \otimes g \to (\Phi, \phi^{(\kappa)}_j(f)\phi^{(\kappa')}_i(g)\Psi)$$

and

$$\mathcal{T}(T(\mathbb{R}^4)) \times \mathcal{T}(T(\mathbb{R}^4)) \ni f \otimes g \to (\Phi, \phi^{(\kappa')}_i(g)\phi^{(\kappa)}_j(f)\Psi)$$

can be extended continuously to $\mathcal{T}(T(L^\ell))$ in some Lorentz frame, for arbitrary elements $\Phi, \Psi$ in the common domain $\mathcal{D}$ of the field operators $\phi^{(\kappa)}_j(f)$, where

$$T(L^\ell) = \{(z_1, z_2) \in \mathbb{C}^{4^2}; |\text{Im} z_1 - \text{Im} z_2| < \ell\}.$$

b) The carrier of the functional

$$(f, g) \rightarrow (\Phi, [\phi^{(\kappa)}_j(f), \phi^{(\kappa')}_i(g)]_\mp \Psi)$$

on $\mathcal{T}(T(\mathbb{R}^4)) \times \mathcal{T}(T(\mathbb{R}^4))$ is contained in the set

$$W^\ell = \{(z_1, z_2) \in \mathbb{C}^{4^2}; z_1 - z_2 \in V^\ell\},$$

where

$$V^\ell = \{z \in \mathbb{C}^4; \exists x \in V, |\text{Re} z - x| + |\text{Im} z| < \ell\}$$

is a complex neighborhood of light cone $V$, i.e., this functional can be extended continuously to $\mathcal{T}(W^\ell)$.

W.VII. Cyclicity of the vacuum: The set $\mathcal{D}_0$ of finite linear combinations of vectors of the form

$$\phi^{(\kappa_1)}_{j_1}(f_1) \cdots \phi^{(\kappa_n)}_{j_n}(f_n)\Phi_0, \ f_j \in \mathcal{T}(T(\mathbb{R}^4)) \ (n = 0, 1, \ldots)$$

is dense in $\mathcal{H}$.

6. SOME CONSEQUENCES OF THE AXIOMS

A vector-valued generalized function $\Phi^{(\kappa)}_{j_1 \cdots j_n}(f)$ is defined as follows:

First, let $g(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n)$ for $f_j \in \mathcal{T}(T(\mathbb{R}^4))$, and define $\Phi^{(\kappa_1 \cdots \kappa_n)}_{j_1 \cdots j_n}(g)$ by:

$$\phi^{(\kappa_1 \cdots \kappa_n)}_{j_1 \cdots j_n}(g) = \phi^{(\kappa_1)}_{j_1}(f_1) \cdots \phi^{(\kappa_j)}_{j_j}(f_j) \cdots \phi^{(\kappa_n)}_{j_n}(f_n)\Phi_0.$$

If $\mathcal{T}(T(\mathbb{R}^4)) \otimes^n \ni g_k \rightarrow f(x_1, \ldots, x_n) \in \mathcal{T}(T(\mathbb{R}^{4n}))$ in the topology of $\mathcal{T}(T(\mathbb{R}^{4n}))$,

$$\|\phi^{(\kappa_2 \cdots \kappa_n)}_{j_2 \cdots j_n}(g_k - g_0)\|^2 = W^{(\kappa_n \cdots \kappa_1 \kappa_2 \cdots \kappa_n)}_{j_n \cdots j_1 j_2 \cdots j_n}((g_k - g_0)^* \otimes (g_k - g_0)) \rightarrow 0.$$
This shows that there exists a vector $\Phi^{(\kappa_1, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_n}(f)$ such that
\[
\Phi^{(\kappa_1, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_n}(g_k) \rightarrow \Phi^{(\kappa_1, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_n}(f) = \Phi^{(\kappa_1, \ldots, \kappa_n)}_{\mu_n}(f),
\]
and the mapping
\[
\mathcal{T}(T(\mathbb{R}^{4n})) \ni f \rightarrow \Phi^{(\kappa_1, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_n}(f) \in \mathcal{H}
\]
is continuous. The Wightman (generalized) function $\mathcal{W}^{(\kappa_1, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_n}(f)$ is defined by
\[
\mathcal{T}(T(\mathbb{R}^{4n})) \ni f \rightarrow \mathcal{W}^{(\kappa_1, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_n}(f) = (\Phi_0, \Phi^{(\kappa_1, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_n}(f)) \in \mathbb{C}.
\]

With the definition of the Fourier transform $\tilde{\Phi}_{\mu_n}$ of $\Phi^{(\kappa_n)}_{\mu_n}$ by
\[
\Phi^{(\kappa_n)}_{\mu_n}(f) = \tilde{\Phi}^{(\kappa_n)}_{\mu_n}(\tilde{f}).
\]
we find
\[
U(a, 1)\tilde{\Phi}^{(\kappa_n)}_{\mu_n}(f) = \Phi^{(\kappa_n)}_{\mu_n}(f) = \tilde{\Phi}^{(\kappa_n)}_{\mu_n}(f) \exp \left( i \left( \sum_{k=1}^{n} p_k a \right) \right).
\]

According to standard strategy we use this identity to determine support properties of the Fourier transforms of the field operators. Let $h \in \mathcal{T}(T(\mathbb{R}^4))$. Then we have
\[
(2\pi)^2 \tilde{h}(P)\tilde{\Phi}^{(\kappa_n)}_{\mu_n}(\tilde{f}) = \int_{\mathbb{R}^4} h(a)U(a, 1)da\tilde{\Phi}^{(\kappa_n)}_{\mu_n}(\tilde{f})
\]
\[
= (2\pi)^2(\tilde{\Phi}^{(\kappa_n)}_{\mu_n})(p_1, \ldots, p_n), \tilde{h}(p_1 + \cdots + p_n) \cdot \tilde{f}(p_1, \ldots, p_n).
\]
Let $\chi_n$ be a linear mapping defined by
\[(p_1, \ldots, p_n) = \chi_n(q_0, \ldots, q_{n-1}), p_k = q_{k-1} - q_k (k = 1, \ldots, n-1), p_n = q_{n-1}.
\]
The inverse mapping $\chi_n^{-1}$ is:
\[q_k = \sum_{j=k+1}^{n} p_j \quad (k = 0, \ldots, n-1).
\]

Define $\tilde{Z}^{(\kappa_n)}_{\mu_n}$ by
\[
\tilde{Z}^{(\kappa_n)}_{\mu_n}(\tilde{f} \circ \chi_n) = \Phi^{(\kappa_n)}_{\mu_n}(\tilde{f}).
\]

Then
\[
\tilde{Z}^{(\kappa_n)}_{\mu_n}(\tilde{g}) = \tilde{\Phi}^{(\kappa_n)}_{\mu_n}(\tilde{g} \circ \chi_n^{-1}).
\]
Let $\tilde{g}_2 \in H(\mathbb{R}^{4(n-1)}; \mathbb{R}^4)$ and $\tilde{g}_1 \in H(\mathbb{R}^4; \mathbb{R}^4)$. Then we have
\[
\tilde{h}(P)\tilde{Z}^{(\kappa_n)}_{\mu_n}(\tilde{g}_1 \otimes \tilde{g}_2) = \tilde{Z}^{(\kappa_n)}_{\mu_n}(\tilde{h} \cdot \tilde{g}_1 \otimes \tilde{g}_2)
\]
\[
= \tilde{g}_1(P)\tilde{Z}^{(\kappa_n)}_{\mu_n}(\tilde{h} \otimes \tilde{g}_2).
\]
These equalities show that the vector-valued generalized function
\[ H(\mathbb{R}^4; \mathbb{R}^4) \ni \tilde{g}_1 \to \tilde{Z}_{\mu_n}^{(\omega)}(\tilde{g}_1 \otimes \tilde{g}_2) \in \mathcal{H} \]
has its support contained in the spectrum \( \Sigma \) of energy-momentum operator \( P \) (see Proposition 4.5 of [1]), and
\[
\hat{h}(0)(\Phi_0, \tilde{Z}_{\mu_n}^{(\omega)}(\tilde{g}_1 \otimes \tilde{g}_2))) = (\Phi_0, \hat{h}(P) \tilde{Z}_{\mu_n}^{(\omega)}(\tilde{h} \otimes \tilde{g}_2))) = (\Phi_0, \tilde{g}_1(0)(\Phi_0, \tilde{Z}_{\mu_n}^{(\omega)}(\tilde{h} \otimes \tilde{g}_2))).
\]
This equality allows us to define a functional \( \tilde{W}_{\mu_n}^{(\omega)} \) by
\[
(2\pi)^2 \tilde{W}_{\mu_n}^{(\omega)}(\tilde{g}_2) = (\Phi_0, \tilde{Z}_{\mu_n}^{(\omega)}(\tilde{h} \otimes \tilde{g}_2))
\]
for \( \tilde{h} \in H(\mathbb{R}^4; \mathbb{R}^4) \) with \( \tilde{h}(0) = 1 \), since the right hand side of the above equality does not depend on \( \tilde{h} \in H(\mathbb{R}^4; \mathbb{R}^4) \) provided \( \tilde{h}(0) = 1 \), equivalently,
\[
\int h(x)dx = (2\pi)^2.
\]
Moreover, we have
\[
(2\pi)^2 \tilde{g}_1(0) \tilde{W}_{\mu_n}^{(\omega)}(\tilde{g}_2) = (\Phi_0, \tilde{Z}_{\mu_n}^{(\omega)}(\tilde{g}_1 \otimes \tilde{g}_2)).
\]
and this shows that
\[
\tilde{W}_{\mu_n}^{(\omega)} \circ \chi_n(q_0, q_1, \ldots, q_{n-1}) = (2\pi)^2 \delta(q_0) \tilde{W}_{\mu_n}^{(\omega)}(q_1, \ldots, q_{n-1}).
\]
Let \( \tilde{f}_j = \tilde{g}_j \circ \chi_n^{-1} \) \((j = 1, 2)\). Then
\[
\tilde{f}_1 \otimes \tilde{f}_2(p_1, \ldots, p_{m+n}) = \tilde{f}_1(-p_m, \ldots, -p_1) \tilde{f}_2(p_{m+1}, \ldots, p_{m+n}) = \tilde{g}_1(q_m - q_0, \ldots, q_1 - q_0) \tilde{g}_2(q_m, \ldots, q_{m+n-1}),
\]
and
\[
(\tilde{Z}_{\mu_{m+n}}^{(\omega)}(\tilde{g}_1), \tilde{Z}_{\mu_{m+n}}^{(\omega)}(\tilde{g}_2)) = (\tilde{f}_{1}^{(\omega)}(\tilde{f}_1^{*}), \tilde{f}_{1}^{(\omega)}(\tilde{f}_2^{*})) = (\tilde{f}_{1}^{(\omega)}(f_1), \tilde{f}_{1}^{(\omega)}(f_2)) = \tilde{W}_{\mu_{m+n}}^{(\omega)}(f_1^{*} \otimes f_2) = (2\pi)^2 \delta(q_0) \tilde{W}_{\mu_{m+n}}^{(\omega)}(f_1^{*} \otimes f_2)
\]
\[
= \langle (2\pi)^2 \delta(q_0) \tilde{W}_{\mu_{m+n}}^{(\omega)}(f_1^{*} \otimes f_2), \tilde{g}_1(q_m - q_0, \ldots, q_1 - q_0) \tilde{g}_2(q_m, \ldots, q_{m+n}) \rangle = (2\pi)^2 \langle \tilde{W}_{\mu_{m+n}}^{(\omega)}(q_1, \ldots, q_{m+n-1}), \tilde{g}_1(q_m, \ldots, q_1) \tilde{g}_2(q_m, \ldots, q_{m+n-1}) \rangle.
\]
This identity implies that the support of \( \tilde{W}_{\mu_n}^{(\omega)}(q_1, \ldots, q_{n-1}) \) is contained in \( \Sigma^{n-1} \) (see Proposition 4.6 of [1]). Moreover, the equality
\[
(\tilde{Z}_{\mu_n}^{(\omega)}(\tilde{g}), \tilde{Z}_{\mu_n}^{(\omega)}(\tilde{g})) = (2\pi)^2 \langle \tilde{W}_{\mu_n}^{(\omega)}(q_1, \ldots, q_{2n-1}), \tilde{g}(q_m, \ldots, q_1) \tilde{g}(q_m, \ldots, q_{2n-1}) \rangle
\]
shows that the support of \( \tilde{Z}_{\mu_n}^{(\omega)}(q_0, \ldots, q_{n-1}) \) is contained in \( \Sigma^n \). From this support property it follows that \( \tilde{Z}_{\mu_n}^{(\omega)}(\tilde{g}) \) exists for a much wider
class of test functions \( \tilde{g} \) than was originally considered. For example, the function
\[
\tilde{g}_\zeta(q) = (2\pi)^{-2n} e^{i\sum_{j=0}^{n-1} q_j \zeta_j}, \quad \text{Im} \zeta_j \in V_+ + \ell_j(1, 0)
\]

belongs to the class of test functions for sufficiently large \( \ell_j \). We investigate the region of holomorphy of the following function

\[
\langle \tilde{W}_{\tilde{Z}}^{(\zeta)}(q_1, \ldots, q_{2n-1}), \tilde{g}_\zeta(q_1, \ldots, q_n) \tilde{g}_\zeta(q_n, \ldots, q_{2n-1}) \rangle
\]

\[
= \frac{1}{(2\pi)^{4n}} \langle \tilde{W}_{\tilde{Z}}^{(\zeta)}(q_1, \ldots, q_{2n-1}), e^{-i\sum_{j=1}^{n+1} \tilde{c}_j q_{j-1}} e^{i\sum_{k=1}^{k+n+k-1} \zeta_{k-1}} \rangle
\]

\[
= \tilde{W}_{\tilde{Z}}^{(\zeta)}(-\tilde{c}_{n-1}, \ldots, -\tilde{c}_0 + \zeta_0, \ldots, \zeta_{n-1}).
\]

Now, we recall the following proposition.

**Proposition 6.1** (Proposition 4.7 of [1]). There exist decreasing functions \( R_{ij}(r) \) defined for \( \ell < r \) such that \( \tilde{W}_{\tilde{Z}}^{(\zeta)}(\zeta_1, \ldots, \zeta_{2n-1}) \) is holomorphic in

\[
\bigcup_{i=1}^{2n-1} \{ \zeta \in \mathbb{C}^{4(2n-1)}; \text{Im} \zeta_i \in V_+ + (\ell', 0), \text{Im} \zeta_j \in V_+ + (R_{ij}(\ell'), 0), \ell < \ell', j \neq i \}.
\]

This proposition shows that \( Z_{\tilde{Z}}^{(\zeta)}(\zeta_0, \ldots, \zeta_{n-1}) \) is holomorphic in the domain \( \text{Im} \zeta_0 \in V_+ + (\ell, 0)/2 \) and \( \text{Im} \zeta_k \in V_+ + (\ell_k, 0) \) for sufficiently large \( \ell_k \) for \( k = 1, \ldots, n - 1 \). Note that

\[
(\tilde{g}_\zeta \circ \chi_n^{-1})(p_1, \ldots, p_n) = (2\pi)^{-2n} \exp i\zeta, \chi_n^{-1} p)
\]

\[
= (2\pi)^{-2n} \exp i\chi_n^{-1T} \zeta, p = (2\pi)^{-2n} \exp i\zeta, p,
\]

where \( z = \chi_n^{-1T} \zeta \) and \( \zeta = \chi_n^T z \), that is,

\[
\zeta_0 = z_1, \quad \zeta_j = z_{j+1} - z_j \quad (j = 1, \ldots, n - 1),
\]

\[
z_1 = \zeta_0, \quad z_j = \sum_{k=0}^{j-1} \zeta_k \quad (j = 2, \ldots, n).
\]

Therefore we get

\[
Z_{\tilde{Z}}^{(\zeta)}(\zeta_0, \ldots, \zeta_{n-1}) = \tilde{Z}_{\tilde{Z}}^{(\zeta)}(\tilde{g}_\zeta)
\]

\[
= \Phi_{\tilde{Z}}^{(\zeta)}(\tilde{g}_\zeta \circ \chi_n^{-1}) = \Phi_{\tilde{Z}}^{(\zeta)}(z_1, \ldots, z_n),
\]

and

\[
\Phi_{\tilde{Z}}^{(\zeta)}(f) = \int \Phi_{\tilde{Z}}^{(\zeta)}(x_1 + i\ell_0, \ldots, x_n + i \sum_{k=1}^{n} \ell_{k-1})
\]

\[
\times f(x_1 + i\ell_0, \ldots, x_n + i \sum_{k=1}^{n} \ell_{k-1}) dx_1 \cdots dx_n,
\]

where \( \ell_0 = \ell/2 + \epsilon \) for any \( \epsilon > 0 \). Note that the Poincaré group acts on \( \tilde{g}_\zeta(q) \) as

\[
(a, A) : \tilde{g}_\zeta(q) \to \tilde{g}_\zeta(\Lambda(A)^{-1} q) e^{i\alpha q_0} = (2\pi)^{-2n} e^{i\sum_{j=0}^{n-1} \Lambda(A)^{-1} q_j \zeta_j} e^{i\alpha q_0}
\]
holomorphy of $\Phi$ implies the following simple formula of covariance in the domain of

$$U(a, A)\Phi_{\mu_1\ldots\nu_n}(f) = \sum_{\nu_1, \ldots, \nu_n, j=1}^n \mathbb{V}^{(\nu_j)}(A^{-1})\Phi_{\nu_1\ldots\nu_n}(f(a, A))$$

implies the following simple formula of covariance in the domain of holomorphy of $\Phi_{\mu_1\ldots\nu_n}(z_1, \ldots, z_n)$ in complex space:

$$U(a, A)\Phi_{\mu_1\ldots\nu_n}(z_1, \ldots, z_n) = \sum_{\nu_1, \ldots, \nu_n, j=1}^n \mathbb{V}^{(\nu_j)}(A^{-1})\Phi_{\nu_1\ldots\nu_n}(A(A)z_1 + \ldots, A(A)z_n + a).$$

7. Multiplication of $\rho(x)$ and $\psi(x)$

As stated at the end of Section 3, $\{\mathcal{H}, \Phi_0, U(a, A), \phi(x), \rho(x), \rho^*(x)\}$ satisfies the axioms of UHFQFT. Let $\rho^{(\chi)}(x) = \rho(x)$ and $\rho^{(\kappa)}(x) = \rho^*(x)$. Then, as we learned in the previous section, the vector-valued function $\rho^{(\lambda_1)}(z_1)\cdots\rho^{(\lambda_n)}(z_n)\Phi_0$ is holomorphic in

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^{4n}, \text{Im } z_1 \in V_+ + (\ell_0, 0), \text{Im } (z_{j+1} - z_j) \in V_+ + (\ell_j, 0)\}$$

for some $\ell_j > \ell > 0$ ($j = 1, \ldots, n - 1$), where $\rho^{(\lambda)}(x)$ is one of $\rho^{(\chi)}(x)$, $\rho^{(\kappa)}(x)$ and $\phi(x)$. Let $\psi^{(\chi)}(x) = \psi_{0,\alpha}(x)$ and $\psi^{(\kappa)}(x) = \psi_{0,\bar{\alpha}}(x)$ be a free Dirac fields of mass $M$. Then the system

$$\{\mathcal{K}, \Psi_0, V(a, A), \psi^{(\chi)}(x), \psi^{(\kappa)}(x)\}$$

satisfies the axioms of tempered field theory (and consequently, that of UHFQFT), and therefore $\psi^{(\lambda_1)}(z_1)\cdots\psi^{(\lambda_n)}(z_n)\Psi_0$ is holomorphic in

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^{4n}, \text{Im } z_1 \in V_+, \text{Im } (z_j - z_{j-1}) \in V_+\},$$

where $\lambda = \kappa$, $\beta = \alpha$ or $\lambda = \bar{\kappa}$, $\beta = \bar{\alpha}$. Therefore, $\rho(z)\Phi$ for $\Phi = \rho^{(\lambda_2)}(f_2)\cdots\rho^{(\lambda_n)}(f_n)\Phi_0$, $f_j \in \mathbb{T}(T(\mathbb{R}^4))$ is holomorphic in

$$\{z \in \mathbb{C}^4, \text{Im } z \in V_+ + (\ell/2, 0)\}$$

and $\psi_{0,\alpha_1}(z)\Psi$ for $\Psi = \psi^{(\lambda_2)}_{0,\beta_2}(g_2)\cdots\psi^{(\lambda_n)}_{0,\beta_n}(g_n)\Psi_0$, $g_j \in \mathcal{S}(\mathbb{R}^4)$ is holomorphic there too.

The composite system

$$\{\mathcal{H} \otimes \mathcal{K}, \Phi_0 \otimes \Psi_0, U(a, A) \otimes V(a, A), \phi(x) \otimes I_\mathcal{K}, \rho(x) \otimes I_\mathcal{K}, \rho^*(x) \otimes I_\mathcal{K}, I_\mathcal{H} \otimes \psi_{0,\alpha}(y), I_\mathcal{H} \otimes \psi_{0,\bar{\alpha}}(y)\}$$

is the tensor product of two systems and thus satisfies all the axioms of UHFQFT. Although the tensor product is well-defined, the pointwise product is not necessarily well-defined for generalized (vector-valued)
functions. In the category of distributions, the following theorem is well-known:

**Theorem 7.1** (Theorem 8.2.10 of [3]). If \( u, v \in \mathcal{D}'(X) \) then the product \( uv \) can be defined as the pullback of the tensor product \( u \otimes v \) by the diagonal map \( \delta : X \to X \times X \) unless \( (x, \xi) \in \text{WF}(u) \) and \( (x, -\xi) \in \text{WF}(v) \).

In our case, the condition that \( \rho(z)\Phi \) and \( \psi_{0,\alpha}(z)\Psi \) have the common domain of holomorphy,

\[
\{ z \in \mathbb{C}^4; \text{Im} \, z \in V_+ + (\ell/2, \mathbf{0}) \},
\]

which corresponds to the condition of the wave front sets \( \text{WF}(u) \) and \( \text{WF}(v) \) of distributions, implies that the product \( (\psi_{0,\alpha}(f)) \) is well-defined by the formula

\[
(\psi_{0,\alpha}(f)) = \int_{\Gamma_N} f(z)\psi_{0,\alpha}(z)\Psi \otimes \rho(z)\Phi dz,
\]

\[
\Gamma_N = \{ z \in \mathbb{C}^4; \, z = x + i(N, \mathbf{0}) \}
\]

for suitable \( N > 0 \). Thus the field \( \psi_0(x) \) is a multiplier of the field \( \rho(x) \). Similarly one can show that \( \partial_{\alpha x} \psi_{0,\alpha} \) is a multiplier for \( \rho(x) \) and then we calculate

\[
(\partial_{\alpha x} \psi_{0,\alpha}) (f) \Psi \otimes \Phi
\]

\[
= (\psi_{0,\alpha}(f)) \Psi \otimes \Phi = \int_{\Gamma_N} (\partial_{\alpha x} \psi_{0,\alpha}(f)) \Psi \otimes \rho(z)\Phi dz
\]

\[
= \int_{\Gamma_N} f(z) \left\{ (\partial_{\alpha x} \psi_{0,\alpha}(z)\Psi) \otimes \rho(z)\Phi + \psi_{0,\alpha}(z)\Psi \otimes \rho(z)\Phi \right\} dz
\]

\[
= (\partial_{\alpha x} \psi_{0,\alpha}) (f) \Psi \otimes \Phi + (\psi_{0,\alpha}) \partial_{\alpha x} \rho(f) \Psi \otimes \Phi.
\]

This gives

\[
\frac{\partial}{\partial x}(\psi_{0,\alpha}(x)\rho(x)) (\Psi \otimes \Phi) =
\]

\[
(\partial_{\alpha x} \psi_{0,\alpha}(x)) \rho(x) \Psi \otimes \Phi + \psi_{0,\alpha}(x) \partial_{\alpha x} \rho(x) \Psi \otimes \Phi.
\]

Let \( \psi(x) = \psi_0(x)\rho(x) \) and \( \bar{\psi}(x) = \bar{\psi}_0(x)\rho^*(x) \). We can easily see that the fields \( \psi(x), \bar{\psi}(x), \phi(x) \) satisfy the axioms of UHFQFT except for the extended causality, which is proved in Section 2. In fact, the conditions WI - WV follow from those of the systems

\[
\{ \mathcal{H}, \Phi_0, U(a, \Lambda), \phi(x), \rho(x), \rho^*(x) \} \text{ and } \{ \mathcal{K}, \Psi_0, V(a, \Lambda), \psi^{(k)}_0(x), \psi^{(\bar{k})}_0(x) \}
\]

(for WV the relation (5.1) is used). For WVII, we only have to restrict the Hilbert space \( \mathcal{H} \otimes \mathcal{K} \) to the subspace generated by

\[
\phi_{j_1}^{(k_1)}(f_1) \cdots \phi_{j_n}^{(k_n)}(f_n) \Phi_0 \otimes \Psi_0, \quad f_j \in \mathcal{T}(T(\mathbb{R}^4)) \quad (n = 0, 1, \ldots),
\]
where \( \phi^{(\kappa)}_j(x) \) is \( \psi_\alpha(x) = (I_\mathcal{H} \otimes \psi_{0,\alpha}(x)) \cdot (\rho(x) \otimes I_\mathcal{K}) = \rho(x) \otimes \psi_{0,\alpha}(x) \) or \( \tilde{\psi}_\alpha(x) = (\rho^*(x) \otimes I_\mathcal{K}) \cdot (I_\mathcal{H} \otimes \tilde{\psi}_{0,\alpha}(x)) = \rho^*(x) \otimes \tilde{\psi}_{0,\alpha}(x) \) or \( \phi(x) \otimes I_\mathcal{K} \).

At the end of this section we complete the proof of the condition of extended causality in the form of axiom WVI by showing that this axiom is equivalent to Condition (R3) for the Wightman functionals which has been verified in Section 2.

**Proposition 7.2.** Assuming the validity of the other axioms, the axiom of extended causality WVI is equivalent to the following condition

(R3) For all \( n = 2, 3, \ldots \) and all \( i = 1, \ldots, n-1 \) denote

- \( L_\ell^i = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^{4n}; |x_i - x_{i+1}| < \ell \} \),
- \( W_\ell^i = \{ x = (z_1, \ldots, z_n) \in \mathbb{C}^{4n}; z_i - z_{i+1} \in V^\ell \} \).

Then, for any \( \ell' > \ell \),

(i) the functional

\[
\mathcal{T}(T(\mathbb{R}^{4n})) \ni f \rightarrow W^{(\kappa_1, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_n}(f) \in \mathbb{C}
\]

is extended continuously to \( \mathcal{T}(T(L_\ell^i)) \), and

(ii) the functional on \( \mathcal{T}(T(\mathbb{R}^{4n})) \)

\[
f \rightarrow W^{(\kappa_1, \ldots, \kappa_j + 1, \kappa_j + 2, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_j, \mu_{j+1}, \ldots, \mu_n}(f) + W^{(\kappa_1, \ldots, \kappa_{j+1}, \kappa_{j+2}, \ldots, \kappa_n)}_{\mu_1, \ldots, \mu_{j+1}, \mu_{j+2}, \ldots, \mu_n}(f) \in \mathbb{C}
\]

is extended continuously to \( \mathcal{T}(W_\ell^i) \).

**Proof.** Since the spinor/tensor indices do not play a role in this statement the proof given in [1] for the scalar case applies (see Propositions 4.3, 4.4 and Theorem 5.1 of [1]).

From the tensor structure of the composite system of \( \rho(x) \) and \( \psi_0(x) \), the Wightman function of \( \psi(x) = \psi_0(x) \rho(x) \) is the product of the Wightman functions of \( \psi_0(x) \) and \( \rho(x) \). Then it follows from Proposition 3.3 that the Wightman functional of \( \psi(x) \) is just (2.1) for which the extended causality (R3) is proven in Section 2. Thus the axiom WVI is verified.

8. Conclusion

After the condition of extended causality had been verified in its functional version (Section 2), this second part of our study of a linearized model of Heisenberg’s fundamental equation established first the convergence of the Wick power series

\[
\rho(x) =: e^{i\phi(x)^2} = \sum_{n=0}^{\infty} \frac{(i\phi(x)^2)^n}{n!} : \phi(x)^{2n} :
\]

through Wick power series techniques. It turns out that this power series converges in the sense of tempered ultra-hyperfunctions but not in the sense of (tempered) Schwartz distributions.
Next through the use of further Wick product techniques it is shown that this field $\rho$ satisfies the differential equation (in the sense of operator-valued tempered ultra-hyperfunctions)

$$\partial_\mu \rho(x) = 2il^2 : \rho(x) \phi(x) \partial_\mu \phi(x) :$$

where we used the abbreviation $\partial_\mu = \frac{\partial}{\partial x_\mu}$.

Finally, in order to solve the system (1.2) by the ansatz

$$\psi(x) = \psi_0(x) \rho(x)$$

with $\psi_0$ being a free Dirac field two results have been established, namely

a) the concept of a relativistic quantum field with a fundamental length of general type $\kappa$ (i.e., a scalar, tensor or spinor field) generalizing the case of a scalar field presented in [1] and

b) the free Dirac field $\psi_0$ is a multiplier of the field $\rho$.

Then it follows that the field $\psi$ in (8.1) is a relativistic quantum field with a fundamental length of spinor type which satisfies the system (1.2). The interpretation and the motivation of our use of the concept of a quantum field theory with a fundamental length can also be found in the introduction to part I and in [1].

We find it a very remarkable fact that the length parameter $l$ in the linearized version of Heisenberg’s fundamental equation can be interpreted as the fundamental length in the sense of our theory of relativistic quantum field theory with a fundamental length as developed in [1].

As important physical consequences we mention that therefore the solution of the linearized version of Heisenberg’s fundamental equation falls in the class of quantum field theories for which the PCT and spin-statistic theorems hold and for which a scattering theory is available.

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