Topologies and Measurable Structures on the Projective Hilbert Space

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Abstract
A systematic review of the various topologies that can be defined on the projective Hilbert space \( \mathcal{P}(\mathcal{H}) \), i.e., on the set of the pure quantum states, is presented. It is shown that \( \mathcal{P}(\mathcal{H}) \) carries a natural topology as well as a natural measurable structure.

Key words: Projective Hilbert space, pure quantum states, topology and Borel structure.

1 Introduction

Based on ideas of Misra [13], it was essentially the late S. Bugajski who recognized that the statistical (probabilistic) framework of quantum mechanics can be understood as a reduced classical probability theory on the projective Hilbert space \( \mathcal{P}(\mathcal{H}) \) [3,4,11,14,6]. In a forthcoming paper [7] a suggestive definition of a classical extension of quantum mechanics is given and it is proved that every such classical extension is essentially equivalent to the Misra-Bugajski scheme. Moreover, it is known that \( \mathcal{P}(\mathcal{H}) \), considered as a real differentiable manifold, carries a Riemannian as well as a symplectic structure, the latter enabling one to reformulate quantum dynamics on the (in general infinite-dimensional) phase space \( \mathcal{P}(\mathcal{H}) \) [10,12,8,9,3,2]. Thus, quantum mechanics can be considered as some reduced classical statistical mechanics.

In order to take the projective Hilbert space \( \mathcal{P}(\mathcal{H}) \) as a sample space for classical probability theory, \( \mathcal{P}(\mathcal{H}) \) must be equipped with a measurable structure. To make \( \mathcal{P}(\mathcal{H}) \) a differentiable manifold, it should be equipped with a topology first. The elements of \( \mathcal{P}(\mathcal{H}) \) can be interpreted as equivalence classes of vectors \( \varphi \in \mathcal{H}, \varphi \neq 0 \), as equivalence classes of unit vectors, as the one-dimensional subspaces of \( \mathcal{H} \), or as the one-dimensional orthogonal projections acting in \( \mathcal{H} \). So, on the one hand, two quotient topologies can be defined on \( \mathcal{P}(\mathcal{H}) \) whereas, on the other hand, all the different operator topologies induce topologies on...
In Section 2 we undertake a systematic review and comparison, already sketched out in [5], of the various topologies on $\mathcal{P}(\mathcal{H})$ and show that all the topologies are the same, thus giving a natural topology $T$ on $\mathcal{P}(\mathcal{H})$. Furthermore, in Section 3 we present a simple proof that the Borel structure generated by the topology $T$ coincides with the measurable structure generated by the transition probability functions of the pure quantum states; our proof simplifies a proof of Misra from 1974 for a corresponding statement [13].

2 The Topology of the Projective Hilbert Space

Let $\mathcal{H} \neq \{0\}$ be a nontrivial separable complex Hilbert space. Call two vectors of $\mathcal{H}^* := \mathcal{H} \setminus \{0\}$ equivalent if they differ by a complex factor, and define the *projective Hilbert space* $\mathcal{P}(\mathcal{H})$ to be the set of the corresponding equivalence classes which are often called *rays*. Instead of $\mathcal{H}^*$ one can consider only the unit sphere of $\mathcal{H}$, $S := \{\varphi \in \mathcal{H} | \|\varphi\| = 1\}$. Then two unit vectors are called equivalent if they differ by a phase factor, and the set of the corresponding equivalence classes, i.e., the set of the *unit rays*, is denoted by $S/S^1$ (in this context, $S^1$ is understood as the set of all phase factors, i.e., as the set of all complex numbers of modulus 1). Clearly, $S/S^1$ can be identified with the projective Hilbert space $\mathcal{P}(\mathcal{H})$. Furthermore, we can consider the elements of $\mathcal{P}(\mathcal{H})$ also as the one-dimensional subspaces of $\mathcal{H}$ or, equivalently, as the one-dimensional orthogonal projections $P = P_\varphi = |\varphi\rangle\langle\varphi|$, $\|\varphi\| = 1$.

The set $\mathcal{H}^*$ and the unit sphere $S$ carry the topologies induced by the metric topology of $\mathcal{H}$. Using the canonical projections $\mu : \mathcal{H}^* \to \mathcal{P}(\mathcal{H})$, $\mu(\varphi) := [\varphi]$, and $\nu : S \to S/S^1$, $\nu(\chi) := [\chi]_S$, where $[\varphi]$ is a ray and $[\chi]_S$ a unit ray, we can equip the quotient sets $\mathcal{P}(\mathcal{H})$ and $S/S^1$ with their quotient topologies $T_\mu$ and $T_\nu$. Considering $T_\nu$, a set $O \subseteq S/S^1$ is called open if $\nu^{-1}(O)$ is open.

**Theorem 1** The set $S/S^1$, equipped with the quotient topology $T_\nu$, is a second-countable Hausdorff space, and $\nu$ is an open continuous mapping.

**Proof.** By definition of $T_\nu$, $\nu$ is continuous. To show that $\nu$ is open, let $U$ be an open set of $S$. From

$$\nu^{-1}(\nu(U)) = \nu^{-1}(\{[\chi]_S | \chi \in U\}) = \bigcup_{\lambda \in S^1} \lambda U,$$

$S^1 = \{\lambda \in \mathbb{C} | \|\lambda\| = 1\}$, it follows that $\nu^{-1}(\nu(U)) \subseteq S$ is open. So $\nu(U) \subseteq S/S^1$ is open; hence, $\nu$ is open.

Next consider two different unit rays $[\varphi]_S$ and $[\psi]_S$ where $\varphi, \psi \in S$ and $|\langle\varphi|\psi\rangle| = 1 - \varepsilon$, $0 < \varepsilon \leq 1$. Since the mapping $\chi \mapsto |\langle\varphi|\chi\rangle|$, $\chi \in S$, is continuous, the sets

$$U_1 := \{\chi \in S | |\langle\varphi|\chi\rangle| > 1 - \frac{\varepsilon}{2}\}$$

and

$$U_2 := \{\chi \in S | |\langle\varphi|\chi\rangle| < 1 - \frac{\varepsilon}{2}\}$$

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are open neighborhoods of \( \varphi \) and \( \psi \), respectively. Consequently, the sets \( O_1 := \nu(U_1) \) and \( O_2 := \nu(U_2) \) are open neighborhoods of \( [\varphi]_S \) and \( [\psi]_S \), respectively. Assume \( O_1 \cap O_2 \neq \emptyset \). Let \( [\xi]_S \in O_1 \cap O_2 \), then \( [\xi]_S = \nu(\chi_1) = \nu(\chi_2) \) where \( \chi_1 \in U_1 \) and \( \chi_2 \in U_2 \). It follows that \( \chi_1 \) and \( \chi_2 \) are equivalent, so \( |\langle \varphi | \chi_1 \rangle| = |\langle \varphi | \chi_2 \rangle| \), in contradiction to \( \chi_1 \in U_1 \) and \( \chi_2 \in U_2 \). Hence, \( O_1 \) and \( O_2 \) are disjoint, and \( \mathcal{T}_\nu \) is separating.

Finally, let \( \mathcal{B} = \{ U_n \mid n \in \mathbb{N} \} \) be a countable base of the topology of \( S \) and define the open sets \( O_n := \nu(U_n) \). We show that \( \{ O_n \mid n \in \mathbb{N} \} \) is a base of \( \mathcal{T}_\nu \). For \( O \in \mathcal{T}_\nu \), we have that \( \nu^{-1}(O) \) is an open set of \( S \) and consequently \( \nu^{-1}(O) = \bigcup_{n \in M} U_n \) where \( U_n \in \mathcal{B} \) and \( M \subseteq \mathbb{N} \). Since \( \nu \) is surjective, it follows that

\[
O = \nu(\nu^{-1}(O)) = \nu \left( \bigcup_{n \in M} U_n \right) = \bigcup_{n \in M} \nu(U_n) = \bigcup_{n \in M} O_n.
\]

Hence, \( \{ O_n \mid n \in \mathbb{N} \} \) is a countable base of \( \mathcal{T}_\nu \). \( \square \)

Analogously, it can be proved that the topology \( \mathcal{T}_\mu \) on \( \mathcal{P}(H) \) is separating and second-countable and that the canonical projection \( \mu \) is open (and continuous by the definition of \( \mathcal{T}_\mu \)). Moreover, one can show that the natural bijection \( \beta : \mathcal{P}(H) \to \mathcal{S}/\mathcal{S}^1, \beta([\varphi]) := \frac{[\varphi]}{\| \varphi \|} \| \varphi \|, \beta^{-1}([\xi]|_S) = [\xi], \) is a homeomorphism. Thus, identifying \( \mathcal{P}(H) \) and \( \mathcal{S}/\mathcal{S}^1 \) by \( \beta \), the topologies \( \mathcal{T}_\mu \) and \( \mathcal{T}_\nu \) are the same.

We denote the real vector space of the self-adjoint trace-class operators by \( \mathcal{T}_s(H) \) and the real vector space of all bounded self-adjoint operators by \( \mathcal{B}_s(H) \); endowed with the trace norm and the usual operator norm, respectively, these spaces are Banach spaces. As is well known, \( \mathcal{B}_s(H) \) can be considered as the dual space \( (\mathcal{T}_s(H))' \) where the duality is given by the trace functional. Let \( \mathcal{S}(H) \) be the convex set of all positive trace-class operators of trace 1; the operators of \( \mathcal{S}(H) \) are the density operators and describe the quantum states. We recall that the extreme points of the convex set \( \mathcal{S}(H) \), i.e., the pure quantum states, are the one-dimensional orthogonal projections \( P = P_\varphi, \| \varphi \| = 1 \). We denote the set of these extreme points, i.e., the extreme boundary, by \( \partial_s \mathcal{S}(H) \).

The above definition of \( \mathcal{P}(H) \) and \( \mathcal{S}/\mathcal{S}^1 \) as well as of their quotient topologies is related to a geometrical point of view. From an operator-theoretical point of view, it is more obvious to identify \( \mathcal{P}(H) \) with \( \partial_s \mathcal{S}(H) \) and to restrict one of the various operator topologies to \( \partial_s \mathcal{S}(H) \). A further definition of a topology on \( \partial_s \mathcal{S}(H) \) is suggested by the interpretation of the one-dimensional projections \( P \in \partial_s \mathcal{S}(H) \) as the pure quantum states and by the requirement that the transition probabilities between two pure states are continuous functions. Next we consider, taking account of \( \partial_s \mathcal{S}(H) \subseteq \mathcal{S}(H) \subseteq \mathcal{T}_s(H) \subseteq \mathcal{B}_s(H) \), the metric topologies on \( \partial_s \mathcal{S}(H) \) induced by the trace-norm topology of \( \mathcal{T}_s(H) \), resp., by the norm topology of \( \mathcal{B}_s(H) \). After that we introduce the weak topology on \( \partial_s \mathcal{S}(H) \) defined by the transition-probability functions as well as the restrictions of several weak operator topologies to \( \partial_s \mathcal{S}(H) \). Finally, we shall prove the surprising result that all the many topologies on \( \mathcal{P}(H) \cong \mathcal{S}/\mathcal{S}^1 \cong \partial_s \mathcal{S}(H) \) are equivalent.

**Theorem 2** Let \( P_\varphi = |\varphi \rangle \langle \varphi| \in \partial_s \mathcal{S}(H) \) and \( P_\psi = |\psi \rangle \langle \psi| \in \partial_s \mathcal{S}(H) \) where \( \| \varphi \| = \| \psi \| = 1 \). Then
\((a)\)

\[\rho_n(P_\varphi, P_\psi) := \|P_\varphi - P_\psi\| = \sqrt{1 - |\langle \varphi | \psi \rangle|^2} = \sqrt{1 - \text{tr} P_\varphi P_\psi}\]

where the norm \(\|\cdot\|\) is the usual operator norm

\((b)\)

\[\rho_{tr}(P_\varphi, P_\psi) := \|P_\varphi - P_\psi\|_{tr} = 2 \|P_\varphi - P_\psi\|,\]

in particular, the metrics \(\rho_n\) and \(\rho_{tr}\) on \(\partial_c S(H)\) induced by the operator norm \(\|\cdot\|\) and the trace norm \(\|\cdot\|_{tr}\) are equivalent

\((c)\)

\[\|P_\varphi - P_\psi\| \leq \|\varphi - \psi\|,\]

in particular, the mapping \(\varphi \mapsto P_\varphi\) from \(S\) into \(\partial_c S(H)\) is continuous, \(\partial_c S(H)\) being equipped with \(\rho_n\) or \(\rho_{tr}\).

**Proof.** To prove (a) and (b), assume \(P_\varphi \neq P_\psi\), otherwise the statements are trivial. Then the range of \(P_\varphi - P_\psi\) is a two-dimensional subspace of \(H\) and is spanned by the two linearly independent unit vectors \(\varphi\) and \(\psi\). Since eigenvectors of \(P_\varphi - P_\psi\) belonging to eigenvalues \(\lambda \neq 0\) must lie in the range of \(P_\varphi - P_\psi\), they can be written as \(\chi = \alpha \varphi + \beta \psi\). Therefore, the eigenvalue problem \((P_\varphi - P_\psi)\chi = \lambda \chi, \chi \neq 0\), is equivalent to the two linear equations

\[
(1 - \lambda)\alpha + \langle \varphi | \psi \rangle \beta = 0 \\
-\langle \psi | \varphi \rangle \alpha - (1 + \lambda)\beta = 0
\]

where \(\alpha \neq 0\) or \(\beta \neq 0\). It follows that \(\lambda = \pm \sqrt{1 - |\langle \varphi | \psi \rangle|^2} = \lambda_{1,2}\). Hence, \(P_\varphi - P_\psi\) has the eigenvalues \(\lambda_1, 0, \lambda_2\). Now, from \(\|P_\varphi - P_\psi\| = \max\{|\lambda_1|, |\lambda_2|\}\) and \(\|P_\varphi - P_\psi\|_{tr} = |\lambda_1| + |\lambda_2|\), we obtain the statements (a) and (b).—From

\[
\|P_\varphi - P_\psi\|^2 = 1 - |\langle \varphi | \psi \rangle|^2 = \|\varphi - (\varphi | \psi \rangle \psi\|^2 = \|(I - P_\psi)\varphi\|^2 \\
\leq \|(I - P_\psi)\varphi\|^2 + \|\psi - P_\psi \varphi\|^2 \\
= \|(I - P_\psi)\varphi - (\psi - P_\psi \varphi)\|^2 \\
= \|\varphi - \psi\|^2
\]

we conclude statement (c). \(\Box\)

According to statement (b) of Theorem 2, the metrics \(\rho_n\) and \(\rho_{tr}\) give rise to the same topology \(T_n = T_{tr}\) as well as to the same uniform structures.

**Theorem 3** Equipped with either of the two metrics \(\rho_n\) and \(\rho_{tr}\), \(\partial_c S(H)\) is separable and complete.

**Proof.** As a metric subspace of the separable Hilbert space \(H\), the unit sphere \(S\) is separable. Therefore, by statement (c) of Theorem 2 the metric space \((\partial_c S(H), \rho_n)\) is separable and so is \((\partial_c S(H), \rho_{tr})\) (the latter, moreover, implies the trace-norm separability of \(T_1(H)\)). Now let \(\{P_n\}_{n \in \mathbb{N}}\) be a Cauchy sequence in \((\partial_c S(H), \rho_{tr})\). Then there exists an operator \(A \in T_1(H)\) such that
$\|P_n - A\|_\text{tr} \to 0$ as well as $\|P_n - A\| \to 0$ as $n \to \infty$ (remember that, on $\mathcal{T}_s(\mathcal{H})$, $\|\cdot\|_\text{tr}$ is stronger than $\|\cdot\|$). From

$$\|P_n - A^2\| = \|A^2 - P_n\| \leq \|A^2 - A P_n\| + \|A P_n - P_n^2\| \leq \|A\| \|A - P_n\| + \|A - A_n\| \to 0$$

as $n \to \infty$ we obtain $A = \lim_{n \to \infty} P_n = A^2$; moreover,

$$\text{tr} A = \text{tr} A I = \lim_{n \to \infty} \text{tr} P_n I = 1.$$ 

Hence, $A$ is a one-dimensional orthogonal projection, i.e., $A \in \partial_s S(\mathcal{H})$. □

Next we equip $\partial_s S(\mathcal{H})$ with the topology $\mathcal{T}_0$ generated by the functions

$$P \mapsto h_Q(P) := \text{tr} PQ = |\langle \psi | \psi \rangle|^2$$

where $P = |\psi \rangle \langle \psi | \in \partial_s S(\mathcal{H})$, $Q = |\varphi \rangle \langle \varphi | \in \partial_s S(\mathcal{H})$, and $\|\psi\| = \|\varphi\| = 1$. That is, $\mathcal{T}_0$ is the coarsest topology on $\partial_s S(\mathcal{H})$ such that all the real-valued functions $h_Q$ are continuous. Note that $\text{tr} PQ = |\langle \varphi | \psi \rangle|^2$ can be interpreted as the transition probability between the two pure states $P$ and $Q$.

**Lemma 1** The set $\partial_s S(\mathcal{H})$, equipped with the topology $\mathcal{T}_0$, is a second-countable Hausdorff space. A countable base of $\mathcal{T}_0$ is given by the finite intersections of the open sets

$$U_{klm} := h_Q^{-1} \left( \left[ q_l - \frac{1}{m}, q_l + \frac{1}{m} \right] \right)$$

where $Q_k \in \mathbb{N}$ is a sequence of one-dimensional orthogonal projections being $\rho_n$-dense in $\partial_s S(\mathcal{H})$, $\{q_l\}_{l \in \mathbb{N}}$ is a sequence of numbers being dense in $[0, 1] \subseteq \mathbb{R}$, and $m \in \mathbb{N}$.

**Proof.** Let $P_1$ and $P_2$ be any two different one-dimensional projections. Choosing $Q = P_1$ in (3), we obtain $h_{P_1}(P_1) = 1 \neq h_{P_1}(P_2) = 1 - \varepsilon$, $0 < \varepsilon \leq 1$. The sets

$$U_1 := \{ P \in \partial_s S(\mathcal{H}) \mid h_{P_1}(P) > 1 - \frac{\varepsilon}{2} \}$$

and

$$U_2 := \{ P \in \partial_s S(\mathcal{H}) \mid h_{P_1}(P) < 1 - \frac{\varepsilon}{2} \}$$

(cf. Eqs. 1 and 2) are disjoint open neighborhoods of $P_1$ and $P_2$, respectively. So $\mathcal{T}_0$ is separating.

For an open set $O \subseteq \mathbb{R}$, $h_Q^{-1}(O)$ is $\mathcal{T}_0$-open. We next prove that

$$U := h_Q^{-1}(O) = \bigcup_{U_{klm} \subseteq U} U_{klm}$$

with $U_{klm}$ according to (4). Let $P \in U$. Then there exists an $\varepsilon > 0$ such that the interval $[h_Q(P) - \varepsilon, h_Q(P) + \varepsilon[$ is contained in $O$. Choose $m_0 \in \mathbb{N}$ such that
\[ \frac{1}{m_0} < \frac{1}{\varepsilon}, \] and choose a member \( q_{l_0} \) of the sequence \( \{q_l\}_{l \in \mathbb{N}} \) and a member \( Q_{k_0} \) of \( \{Q_k\}_{k \in \mathbb{N}} \) such that \( |\text{tr} \ PQ - q_{l_0}| < \frac{1}{2m_0} \) and \( \|Q_{k_0} - Q\| < \frac{1}{2m_0} \). It follows that
\[
\begin{align*}
|\text{tr} \ PQ_{k_0} - q_{l_0}| & \leq |\text{tr} \ PQ_{k_0} - \text{tr} \ PQ| + |\text{tr} \ PQ - q_{l_0}| \\
& \leq \|Q_{k_0} - Q\| + |\text{tr} \ PQ - q_{l_0}| \\
& < \frac{1}{m_0}
\end{align*}
\]
which, by (4), means that \( P \in U_{k_0l_0m_0} \). We further have to show that \( U_{k_0l_0m_0} \subseteq U \). To that end, let \( \bar{P} \in U_{k_0l_0m_0} \). Then, from
\[
|\text{tr} \ \bar{P} Q - \text{tr} \ PQ| \leq |\text{tr} \ \bar{P} Q - \text{tr} \ PQ_{k_0}| + |\text{tr} \ PQ_{k_0} - q_{l_0}| + |q_{l_0} - \text{tr} \ PQ|
\]
where the first term on the right-hand side is again smaller than \( \|Q - Q_{k_0}\| \) and, by (4), the second term is smaller than \( \frac{1}{m_0} \), it follows that
\[
|h_Q(\bar{P}) - h_Q(P)| = |\text{tr} \ \bar{P} Q - \text{tr} \ PQ| \leq \frac{1}{2m_0} + \frac{1}{m_0} + \frac{1}{2m_0} = \frac{2}{m_0} < \varepsilon.
\]
This implies that \( h_Q(\bar{P}) \in |h_Q(P) - \varepsilon, h_Q(P) + \varepsilon[ \subseteq O \), i.e., \( \bar{P} \in h_Q^{-1}(O) = U \). Hence, \( U_{k_0l_0m_0} \subseteq U \).

Summarizing, we have shown that, for \( P \in U, P \in U_{k_0l_0m_0} \subseteq U \). Hence, \( U \subseteq \bigcup_{U_{k_0l_0m_0} \subseteq U} U_{k_0l_0m_0} \subseteq U \), and assertion (4) has been proved. The finite intersections of sets of the form \( U = h_Q^{-1}(O) \) constitute a basis of the topology \( \mathcal{T}_0 \). Since every set \( U = h_Q^{-1}(O) \) is the union of sets \( U_{k_0l_0m_0} \), the intersections of finitely many sets \( U = h_Q^{-1}(O) \) is the union of finite intersections of sets \( U_{k_0l_0m_0} \). Thus, the finite intersections of the sets \( U_{k_0l_0m_0} \) constitute a countable basis of \( \mathcal{T}_0 \). \( \square \)

Later we shall see that the topological space \( (\partial_S(H), \mathcal{T}_0) \) is homeomorphic to \( (\partial_S(H), \mathcal{T}_0) \) as well as to \( (S/S^1, \mathcal{T}_0) \). So it is also clear by Theorem 3 or Theorem 4 that \( (\partial_S(H), \mathcal{T}_0) \) is a second-countable Hausdorff space. The reason for stating Lemma 1 is that later we shall make explicit use of the particular countable base given there.

The weak operator topology on the space \( B_s(H) \) of the bounded self-adjoint operators on \( H \) is the coarsest topology such that the linear functionals
\[ A \mapsto \langle \varphi | A \psi \rangle \]
where \( A \in B_s(H) \) and \( \varphi, \psi \in H \), are continuous. It is sufficient to consider only the functionals
\[ A \mapsto \langle \varphi | A \varphi \rangle \tag{6} \]
where \( \varphi \in H \) and \( ||\varphi|| = 1 \). The topology \( \mathcal{T}_w \) induced on \( \partial_S(H) \subset B_s(H) \) by the weak operator topology is the coarsest topology on \( \partial_S(H) \) such that the restrictions of the linear functionals (6) to \( \partial_S(H) \) are continuous. Since these restrictions are given by
\[ P \mapsto \langle \varphi | P \varphi \rangle = \text{tr} \ PQ = h_Q(P) \]
where \( P \in \partial_S(H) \) and \( Q := |\varphi\rangle \langle \varphi | \in \partial_S(H) \), the topology \( \mathcal{T}_w \) on \( \partial_S(H) \) is, according to (4), just our topology \( \mathcal{T}_0 \).

Now we compare the weak topology \( \mathcal{T}_0 \) with the metric topology \( \mathcal{T}_n \).
Theorem 4 The weak topology $T_0$ on $\partial_c S(\mathcal{H})$ and the metric topology $T_n$ on $\partial_c S(\mathcal{H})$ are equal.

Proof. According to (3), a neighborhood base of $P \in \partial_c S(\mathcal{H})$ w.r.t. $T_0$ is given by the open sets

$$U(P;Q_1,\ldots,Q_n;\varepsilon) := \bigcap_{i=1}^n h_Q^{-1}(h_{Q_i}(P) - \varepsilon, h_{Q_i}(P) + \varepsilon]$$

(7)

$$= \{ \hat{P} \in \partial_c S(\mathcal{H}) \mid |h_{Q_i}(\hat{P}) - h_{Q_i}(P)| < \varepsilon \text{ for } i = 1,\ldots,n \}$$

$$= \{ \hat{P} \in \partial_c S(\mathcal{H}) \mid |\text{tr} \hat{P}Q_i - \text{tr} PQ_i| < \varepsilon \text{ for } i = 1,\ldots,n \}$$

where $Q_1,\ldots,Q_n \in \partial_c S(\mathcal{H})$ and $\varepsilon > 0$; a neighborhood base of $P$ w.r.t. $T_n$ is given by the open balls

$$K_\varepsilon(P) := \{ \hat{P} \in \partial_c S(\mathcal{H}) \mid \|\hat{P} - P\| < \varepsilon \}.$$  

(8)

If $\|\hat{P} - P\| < \varepsilon$, then

$$|\text{tr} \hat{P}Q_i - \text{tr} PQ_i| = |\text{tr} Q_i(\hat{P} - P)| \leq \|Q_i\|_{tr} \|\hat{P} - P\| = \|\hat{P} - P\| < \varepsilon;$$

hence, $K_\varepsilon(P) \subseteq U(P;Q_1,\ldots,Q_n;\varepsilon)$. To show some converse inclusion, take account of Theorem 2 part (a), and note that

$$\|\hat{P} - P\|^2 = 1 - \text{tr} \hat{P}P = |\text{tr} \hat{P}P - \text{tr} PP|.$$  

In consequence, by (7) and (8), $U(P;P;\varepsilon^2) = K_\varepsilon(P)$. Hence, $T_0 = T_n$. □

It looks surprising that the topologies $T_0$ and $T_n$ coincide. In fact, consider the sequence $\{P_\varphi\}_{\varphi \in \mathcal{H}}$ where the vectors $\varphi \in \mathcal{H}$ constitute an orthonormal system. Then, w.r.t. the weak operator topology, $P_\varphi \rightarrow 0$ as $n \rightarrow \infty$ whereas $\|P_\varphi - P_{\varphi + 1}\| = 1$ for all $\varphi \in \mathcal{H}$. However, $0 \notin \partial_c S(\mathcal{H})$; so $\{P_\varphi\}_{\varphi \in \mathcal{H}}$ is convergent neither w.r.t. $T_w = T_0$ nor w.r.t. $T_n$. Finally, like in the case of the weak operator topology, there is a natural uniform structure inducing $T_0$. The uniform structures that are canonically related to $T_0$ and $T_n$ are different: $\{P_\varphi\}_{\varphi \in \mathcal{H}}$ is a Cauchy sequence w.r.t. the uniform structure belonging to $T_0$ but not w.r.t. that belonging to $T_n$, i.e., w.r.t. the metric $\rho_n$.

We remark that besides $T_0$ and $T_w$ several further weak topologies can be defined on $\partial_c S(\mathcal{H})$. Let $C_s(\mathcal{H})$ be the Banach space of the compact self-adjoint operators and remember that $(C_s(\mathcal{H}))' = T_s(\mathcal{H})$. So the weak Banach-space topologies of $C_s(\mathcal{H})$, $T_s(\mathcal{H})$, and $B_s(\mathcal{H})$ as well as the weak*-Banach-space topologies of $T_s(\mathcal{H})$ and $B_s(\mathcal{H})$ can be restricted to $\partial_c S(\mathcal{H})$, thus giving the topologies $T_1 := \sigma(C_s(\mathcal{H}),T_s(\mathcal{H})) \cap \partial_c S(\mathcal{H})$, $T_2 := \sigma(T_s(\mathcal{H}),C_s(\mathcal{H})) \cap \partial_c S(\mathcal{H})$, $T_3 := \sigma(T_s(\mathcal{H}),B_s(\mathcal{H})) \cap \partial_c S(\mathcal{H})$, $T_4 := \sigma(T_s(\mathcal{H}),T_s(\mathcal{H})) \cap \partial_c S(\mathcal{H})$, and $T_5 := \sigma(B_s(\mathcal{H}),T_s(\mathcal{H}))' \cap \partial_c S(\mathcal{H})$. Moreover, the strong operator topology induces a topology $T_s$ on $\partial_c S(\mathcal{H})$. From the obvious inclusions

$$T_w \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq T_4 \subseteq T_5 \subseteq T_s,$$

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\[ T_1 = T_4 \subseteq T_5 = T_1, \]
and
\[ T_w \subseteq T_s \subseteq T_n \]
as well as from the shown equality
\[ T_0 = T_w = T_n = T_r \]
it follows that the topologies \( T_1, \ldots, T_5 \) and \( T_s \) also coincide with \( T_0 \).

Finally, we show that all the topologies on \( \partial_\mu S(\mathcal{H}) \) are equivalent to the quotient topologies \( T_\mu \) and \( T_\nu \) on \( \mathcal{P}(\mathcal{H}) \), resp., \( S/S^1 \).

**Theorem 5** The mapping \( F : S/S^1 \rightarrow \partial_\mu S(\mathcal{H}) \), \( F([\varphi]_S) := P\varphi \) where \( \varphi \in S \), is a homeomorphism between the topological spaces \( (S/S^1, T_\nu) \) and \( (\partial_\mu S(\mathcal{H}), T_0) \).

**Proof.** The mapping \( F \) is bijective. The map \( h_Q \circ F \circ \nu : S \rightarrow \mathbb{R} \) where \( h_Q \) is any of the functions given by Eq. (3) and \( \nu \) is the canonical projection from \( S \) onto \( S/S^1 \), reads explicitly
\[
(h_Q \circ F \circ \nu)(x) = h_Q(F(\varphi)) = h_Q(P\varphi) = \text{tr } P\varphi = |\langle \varphi | \varphi \rangle|;
\]
therefore, \( h_Q \circ F \circ \nu \) is continuous. Consequently, for an open set \( O \subseteq \mathbb{R} \),
\[
(h_Q \circ F \circ \nu)^{-1}(O) = \nu^{-1}(F^{-1}(h_Q^{-1}(O)))
\]
is an open set of \( S \). By the definition of the quotient topology \( T_\nu \), it follows that \( F^{-1}(h_Q^{-1}(O)) \) is an open set of \( S/S^1 \). Since the sets \( h_Q^{-1}(O) \), \( Q \in \partial_\mu S(\mathcal{H}) \), \( O \subseteq \mathbb{R} \) open, generate the weak topology \( T_0 \), \( F^{-1}(U) \) is open for any open set \( U \in T_0 \). Hence, \( F \) is continuous.

To show that \( F \) is an open mapping, let \( V \in T_\nu \) be an open subset of \( S/S^1 \) and let \([\varphi_0]_S \subseteq V \). Since the canonical projection \( \nu \) is continuous, there exists an \( \varepsilon > 0 \) such that
\[
\nu(K_\varepsilon(\varphi_0) \cap S) \subseteq V
\]
where \( K_\varepsilon(\varphi_0) := \{ \varphi \in \mathcal{H} | \| \varphi - \varphi_0 \| < \varepsilon \} \). Without loss of generality we assume that \( \varepsilon < 1 \).

The topology \( T_0 \) is generated by the functions \( h_Q \) according to (3): \( T_0 \) is also generated by the functions \( P \mapsto g_Q(P) := \sqrt{h_Q(P)} = \sqrt{\nu P}Q \). In consequence, the set
\[
U_\varepsilon := g_Q^{-1}\left(1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2}\right) \cap h_Q^{-1}\left(1 - \frac{\varepsilon^2}{4}, 1 + \frac{\varepsilon^2}{4}\right)
\]
where \( Q := P_{\varphi_0} \) and \( \varphi_0 \) and \( \varepsilon \) are specified in the preceding paragraph, is \( T_0 \)-open. Using the identity
\[
1 - |\langle \varphi_0 | \varphi \rangle|^2 = \| \varphi - \langle \varphi_0 | \varphi \rangle \varphi_0 \|^2
\]
where \( \varphi \in \mathcal{H} \) is also a unit vector, we obtain
\[
U_\varepsilon = \left\{ P\varphi \in \partial_\mu S(\mathcal{H}) | |g_Q(P\varphi) - 1| < \frac{\varepsilon}{2} \text{ and } |h_Q(P\varphi) - 1| < \frac{\varepsilon^2}{4} \right\} = \left\{ P\varphi \in \partial_\mu S(\mathcal{H}) | |\langle \varphi_0 | \varphi \rangle| - 1| < \frac{\varepsilon}{2} \text{ and } |\langle \varphi_0 | \varphi \rangle| - 1| < \frac{\varepsilon}{2} \right\} = \left\{ P\varphi \in \partial_\mu S(\mathcal{H}) | |\langle \varphi_0 | \varphi \rangle| - 1| < \frac{\varepsilon}{2} \text{ and } |\varphi - \langle \varphi_0 | \varphi \rangle \varphi_0 \| < \frac{\varepsilon}{2} \right\}.
\]
Now let \( P_\varphi \in U_\varepsilon \). Since \( \varepsilon < 1 \), we have that \( \langle \varphi | \varphi_0 \rangle \neq 0 \). Defining the phase factor \( \lambda := \frac{\langle \varphi | \varphi_0 \rangle}{\langle \varphi | \varphi_0 \rangle} \), it follows that
\[
\| \lambda \varphi - \varphi_0 \| = \| \lambda \varphi - \lambda \langle \varphi_0 | \varphi \rangle \varphi_0 \| + \| \lambda \langle \varphi_0 | \varphi \rangle \varphi - \varphi_0 \|
= \| \varphi - \langle \varphi_0 | \varphi \rangle \varphi_0 \| + \| \langle \varphi_0 | \varphi \rangle | \varphi_0 - \varphi_0 \| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
That is, \( P_\varphi \in U_\varepsilon \) implies that \( \lambda \varphi \in K_\varepsilon(\varphi_0) \); moreover, \( \lambda \varphi \in K_\varepsilon(\varphi_0) \cap S \).

Taking the result (\ref{eq:K}) into account, we conclude that, for \( P_\varphi \in U_\varepsilon, [\varphi]_S = [\lambda \varphi]_S = \nu(\lambda \varphi) \in V \). Consequently, \( P_\varphi = F([\varphi]_S) \in F(V) \). Hence, \( U_\varepsilon \subseteq F(V) \). Since \( U_\varepsilon \) is an open neighborhood of \( P_{\varphi_0} \), \( P_{\varphi_0} \) is an interior point of \( F(V) \). So, for every \( [\varphi_0]_S \in V \), \( F([\varphi_0]_S) = P_{\varphi_0} \) is an interior point of \( F(V) \), and \( F(V) \) is a \( T_0 \)-open set. Hence, the continuous bijective map \( F \) is open and thus a homeomorphism. \( \square \)

In the following, we identify the sets \( \mathcal{P}(\mathcal{H}), S/S^1 \), and \( \partial_c S(\mathcal{H}) \) and call the identified set the projective Hilbert space \( \mathcal{P}(\mathcal{H}) \). However, we preferably think about the elements of \( \mathcal{P}(\mathcal{H}) \) as the one-dimensional orthogonal projections \( P = P_\varphi \). On \( \mathcal{P}(\mathcal{H}) \) then the quotient topologies \( T_\mu, T_\nu, T_\omega, T_1, \ldots, T_5, T_s \), and the metric topologies \( T_n, T_\text{tr} \) coincide. So we can say that \( \mathcal{P}(\mathcal{H}) \) carries a natural topology \( T; (\mathcal{P}(\mathcal{H}), T) \) is a second-countable Hausdorff space.

For our purposes, it is suitable to represent this topology \( T \) as \( T_0, T_n, \) or \( T_\text{tr} \). As already discussed, the topologies \( T_0, T_n, \) and \( T_\text{tr} \) are canonically related to uniform structures. With respect to the uniform structure inducing \( T_0 \), \( \mathcal{P}(\mathcal{H}) \) is not complete. The uniform structures related to \( T_n \) and \( T_\text{tr} \) are the same since they are induced by the equivalent metrics \( \rho_n \) and \( \rho_\text{tr} \); \( (\mathcal{P}(\mathcal{H}), \rho_n) \) and \( (\mathcal{P}(\mathcal{H}), \rho_\text{tr}) \) are separable complete metric spaces. So \( T \) can be defined by a complete separable metric, i.e., \( (\mathcal{P}(\mathcal{H}), T) \) is a polish space.

### 3 The Measurable Structure of \( \mathcal{P}(\mathcal{H}) \)

It is almost natural to define a measurable structure on the projective Hilbert space \( \mathcal{P}(\mathcal{H}) \) by the \( \sigma \)-algebra \( \Xi = \Xi(T) \) generated by the \( T \)-open sets, i.e., \( \Xi \) is the smallest \( \sigma \)-algebra containing the open sets of the natural topology \( T \). In this way \( (\mathcal{P}(\mathcal{H}), \Xi) \) becomes a measurable space where the elements \( B \in \Xi \) are the Borel sets of \( \mathcal{P}(\mathcal{H}) \). However, since the topology \( T \) is generated by the transition-probability functions \( h_\Omega \) according to Eq. \((\ref{eq:K})\), it is also obvious to define the measurable structure of \( \mathcal{P}(\mathcal{H}) \) by the \( \sigma \)-algebra \( \Sigma \) generated by the functions \( h_\Omega \), i.e., \( \Sigma \) is the smallest \( \sigma \)-algebra such that all the functions \( h_\Omega \) are measurable. A result due to Misra (1974) \[\text{[3]}\] Lemma 3 clarifies the relation between \( \Xi \) and \( \Sigma \). Before stating that result, we recall the following simple lemma which we shall also use later.
Lemma 2 Let \((M, T)\) be any second-countable topological space, \(B \subseteq T\) a countable base, and \(\Xi = \Xi(T)\) the \(\sigma\)-algebra of the Borel sets of \(M\). Then \(\Xi = \Xi(T) = \Xi(B)\) where \(\Xi(B)\) is the \(\sigma\)-algebra generated by \(B\); \(B\) is a countable generator of \(\Xi\).

\begin{proof}\end{proof}

Theorem 6 (Misra) The \(\sigma\)-algebra \(\Xi = \Xi(T)\) of the Borel sets of the projective Hilbert space \(P(H)\) and the \(\sigma\)-algebra \(\Sigma\) generated by the transition-probability functions \(h_Q, Q \in P(H)\), are equal.

\begin{proof}\end{proof}

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