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PROJECTIVE LOGARITHMIC POTENTIALS

S. ASSERDA, FATIMA Z. ASSILA AND A. ZERIAHI

A tribute to Professor Ahmed Intissar

ABSTRACT. We study the projective logarithmic potential $G_\mu$ of a Probability measure $\mu$ on the complex projective space $\mathbb{P}^n$ equipped with the Fubini-Study metric $\Omega$. We prove that the Green operator $G : \mu \mapsto G_\mu$ has strong regularizing properties.

It was shown by the second author in [As17] that the range of the operator $G$ is contained in the (local) domain of definition of the complex Monge-Ampère operator on $\mathbb{P}^n$. This result extend earlier results by Carlehed [Carlehed99].

Here we will show that the complex Monge-Ampère measure $(\omega + dd^c G_\mu)^n$ of the logarithmic potential of $\mu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{P}^n$ if and only if the measure $\mu$ has no atoms. Moreover when the measure $\mu$ has a "positive dimension", we give more precise results on regularity properties of the potential $G_\mu$ in terms of the dimension of $\mu$.

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1. INTRODUCTION

In Classical Potential Theory (CPT) various potentials associated to Borel measures in the euclidean space $\mathbb{R}^N$ were introduced. They play a fundamental role in many problems (see [Carleson65, Lat72]). This is due to the fact that CPT is naturally associated to the Laplace operator which is a linear elliptic partial differential operator of second order with constant coefficients. Indeed any subharmonic function is locally equal (up to a harmonic function) to the Newton potential of its Riesz measure when $N \geq 3$, while in the case of the complex plane $\mathbb{C} \simeq \mathbb{R}^2$, we need to consider the logarithmic potential.
On the other hand, it is well known that in higher complex dimension \( n \geq 2 \), plurisubharmonic functions are rather connected to the complex Monge-Ampère operator which is a fully non linear second order differential operator. Therefore Pluripotential theory cannot be completely described by logarithmic potentials as shown by Magnus Carlehed in [Carlehed]. However the class of logarithmic potentials provides a natural class of plurisubharmonic functions which turns out to be included in the local domain of definition of the complex Monge-Ampère operator. This study was started by Carlehed ([Carlehed99], Carlehed) in the case when the measure is compactly supported on \( \mathbb{C}^n \) with a locally bounded potential.

Our main goal is to extend this study to the case of arbitrary probability measures on the complex projective space \( \mathbb{P}^n \). The motivation for this study comes from the fact that the complex Monge-Ampère operator plays an important role in Kähler geometry when dealing with the Calabi conjecture and the problem of the existence of a Kähler-Einstein metric (see [GZ17]). Indeed in geometric applications one needs to consider degenerate complex Monge-Ampère equations. To this end a large class of singular potentials on which the complex Monge-Ampère operator is well defined was introduced (see [GZ07], [BEGZ10]). This leads naturally to a general definition of the complex Monge-Ampère operator (see [CGZ08]). However the global domain of definition of the complex Monge-Ampère operator on compact Kähler manifolds is not yet well understood.

Besides this, thanks to the works of Cegrell and Blocki (see [Ce04], [Bl04], [Bl06]), the local domain of definition is characterized in terms of some local integrability conditions on approximating sequences.

Our goal here is then to study a natural class of projective logarithmic potentials and show that it is contained in the (local) domain of definition of the complex Monge-Ampère operator on the complex projective space \( \mathbb{P}^n \), giving an interesting subclass of the domain of definition of the complex Monge-Ampère operator on the complex projective space and exhibiting many different interesting behaviours.

More precisely, let \( \mu \) be a probability measure on the complex projective space \( \mathbb{P}^n \). Then its projective logarithmic potential is defined on \( \mathbb{P}^n \) as follows:

\[
G_{\mu}(\zeta) := \int_{\mathbb{P}^n} \log \frac{\|\zeta \wedge \eta\|}{\|\zeta\|\|\eta\|} \, d\mu(\eta), \quad \zeta \in \mathbb{P}^n.
\]

Our first result is a regularizing property of the operator \( G \) acting on the convex compact set of probability measures on \( \mathbb{P}^n \).

**Theorem A.** Let \( \mu \) be a probability measure on \( \mathbb{P}^n \) \((n \geq 2)\). Then

1) \( G_{\mu} \in DMA_{loc}(\mathbb{P}^n, \omega) \) and for any \( 0 < p < n \), \( G_{\mu} \in W^{2,p}(\mathbb{P}^n) \);

2) for any \( 1 \leq k \leq n - 1 \), the \( k \)-Hessian measure \( (\omega + ddG_{\mu})^k \wedge \omega^{n-k} \) is absolutely continuous with respect to the Lebesgue measure on \( (\mathbb{P}^n, \omega) \);
3) the Monge-Ampère measure \((\omega + dd^c G_\mu)^n\) is absolutely continuous with respect to the Lebesgue measure on \((\mathbb{P}^n, \omega)\) if the measure \(\mu\) has no atom in \(\mathbb{P}^n\); conversely if the measure \(\mu\) has an atom at some point \(a \in \mathbb{P}^n\), the complex Monge-Ampère measure \((\omega + dd^c G_\mu)^n\) also has an atom at the same point \(a\).

Here \(\omega := \omega_{FS}\) is the Fubini-Study metric on the projective space \(\mathbb{P}^n\) and \(\text{DMA}_{\text{loc}}(\mathbb{P}^n, \omega)\) denotes the local domain of definition of the complex Monge-Ampère operator on the Kähler manifold \((\mathbb{P}^n, \omega)\) (see Definition 2.2 below).

The fact that the projective logarithmic potential of any measure belongs to the domain of definition of the complex Monge-Ampère operator as well as the third property was proved earlier by the second named author ([AS17]).

These results generalize and improve previous results by M. Carlehed who considered the local setting (see [Carlehed, Carlehed99]).

When the measure has a “positive dimension”, we obtain a strong regularity result in terms of this dimension as defined by the formula (3.2) in section 3.2.

**Theorem B**: Let \(\mu\) be a probability measure on \(\mathbb{P}^n (n \geq 2)\) with positive dimension \(\gamma(\mu) > 0\). Then the following holds:

1) \(G_\mu \in W^{1,p}(\mathbb{P}^n)\) for any \(p\) such that \(1 \leq p < (2n - \gamma(\mu))/(1 - \gamma(\mu))\); in particular \(G_\mu\) is Hölder continuous of any exponent \(\alpha\) such that
\[
0 < \alpha < 1 - 2n \frac{(1 - \gamma(\mu))_+}{2n - \gamma(\mu)};
\]

2) \(G_\mu \in W^{2,p}(\mathbb{P}^n)\) for any
\[
1 < p < \frac{2n - \gamma(\mu)}{(2 - \gamma(\mu))_+},
\]

3) the density of the complex Monge-Ampère measure \((\omega + dd^c G_\mu)^n\) with respect to the Lebesgue measure on \(\mathbb{P}^n\) satisfies \(g_\mu \in L^q(\mathbb{P}^n)\) for any
\[
1 < q < \frac{2n - \gamma(\mu)}{n(2 - \gamma(\mu))_+}.
\]

Recall that for a real number \(s \in \mathbb{R}\), we set \(s_+ := \max\{s, 0\}\). Observe that in the last two statements, the critical exponent is \(+\infty\) when \(2 \leq \gamma(\mu) \leq 2n\).

2. Preliminaries

2.1. Lagrange identities. Let us fix some notations. For \(a = (a_0, \cdots, a_n) \in \mathbb{C}^{n+1}\) and \(b = (b_0, \cdots, b_n) \in \mathbb{C}^{n+1}\) we set
\[
a \cdot \bar{b} := \sum_{j=0}^n a_j \bar{b}_j, \quad |a|^2 := \sum_{j=0}^n |a_j|^2.
\]
We can define the wedge product $a \wedge b$ in the vector space $\bigwedge^2 \mathbb{C}^{n+1}$. The hermitian scalar product on $\mathbb{C}^{n+1}$ induces a natural hermitian scalar product on $\bigwedge^2 \mathbb{C}^{n+1}$. If $(e_j)_{0 \leq j \leq n}$ is the associated canonical orthonormal basis of $\mathbb{C}^{n+1}$, then the sequence $(e_i \wedge e_j)_{0 \leq i < j \leq n}$ is an orthonormal basis of the vector space $\bigwedge^2 \mathbb{C}^{n+1}$. Therefore we have

$$a \wedge b = \sum_{0 \leq i < j \leq n} (a_i b_j - a_j b_i) e_i \wedge e_j,$$

and

$$|a \wedge b|^2 = \sum_{0 \leq i < j \leq n} |a_i b_j - a_j b_i|^2.$$

Lemma 2.1. (Lagrange Identities)

1. For any $a, b \in \mathbb{C}^{n+1} \setminus \{0\}$,

$$|a \wedge b|^2 = |a|^2 |b|^2 - |a \cdot \bar{b}|^2.$$

2. For any $a, b \in \mathbb{C}^{n+1} \setminus \{0\}$,

$$|a \wedge b|^2 = 1 - |a \cdot \bar{b}|^2.$$

3. For any $z, w \in \mathbb{C}^n$,

$$|z - w|^2 + |z \wedge w|^2 \leq (1 + |z|^2)(1 + |w|^2) = 1 - \frac{|1 + z \cdot \bar{w}|^2}{(1 + |z|^2)(1 + |w|^2)}.$$

Proof. Here $a \cdot \bar{b} := \sum_{j=0}^n a_j \bar{b}_j$ is the euclidean hermitian product on $\mathbb{C}^{n+1}$. The first identity is the so called Lagrange identity, while the second follows immediately from the first.

The third identity follows from the second one applied to the vectors $a = (1, z)$ and $b = (1, w)$.

2.2. The complex projective space. Let $\mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}_*$ be the complex projective space of dimension $n \geq 1$ and

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n,$$

the canonical projection. which sends a point $\zeta = (\zeta_0, \cdots, \zeta_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ to the complex line $\mathbb{C}_* \cdot \zeta$ which we denote by $[\zeta] = [\zeta_0, \cdots, \zeta_n]$.

By abuse of notation we will denote by $\zeta = [\zeta_0, \cdots, \zeta_n]$ and call the $\zeta_j$’s the homogenuous coordinates of the (complex line) $\zeta$.

As a complex manifold $\mathbb{P}^n$ can be coved by a finite number of charts given by

$$\mathcal{U}_k := \{ \zeta \in \mathbb{P}^n ; \zeta_k \neq 0 \}, \quad 0 \leq k \leq n.$$

For a fixed $k = 0, \cdots, n$, the corresponding coordinate chart is defined on $\mathcal{U}_k$ by the formula

$$z^k(\zeta) = z_k := (z_j^k)_{0 \leq j \leq n, j \neq k}, \text{ where } z_j^k := \zeta_j / \zeta_k \text{ for } j \neq k.$$
The map $\gamma_k : U_k \simeq \mathbb{C}^n$ is an homeomorphism and for $k \neq \ell$ the transition functions (change of coordinates)

$$z^k \circ (z^\ell)^{-1} : z^\ell(U_\ell \cap U_k) \rightarrow z^k(U_\ell \cap U_k)$$

is given by

$$w = z^\ell \circ (z^k)^{-1}(z_1, \ldots, z_n), \quad \text{for } (z_1, \ldots, z_n) \in \mathbb{C}^n, z_\ell \neq 0,$n

where $w_i = z_i/z_\ell$ for $i \notin \{k, \ell\}$, $w_k = 1/z_\ell$ and $w_\ell = z_k/z_\ell$.

The $(1,1)$-form $dd^c \log |z|$ is smooth, $d$-closed on $\mathbb{C}^{n+1} \setminus \{0\}$ and invariant under the $\mathbb{C}^*$-action. Therefore it descends to $\mathbb{P}^n$ as a smooth closed $(1,1)$-form $\omega_{FS}$ on $\mathbb{P}^n$ so that

$$dd^c \log |z| = \pi^*(\omega_{FS}), \quad \text{in } \mathbb{C}^{n+1} \setminus \{0\}.$$n

In the local chart $(U_k, z^k)$ we have

$$\omega | U_k = \frac{1}{2} dd^c \log(1 + |z^k|^2)$$n

Observe that the Fubini-Study form $\omega$ is a Kähler form on $\mathbb{P}^n$ and the corresponding Fubini-Study volume form $dV_{FS} = \omega^n/n!$ is given in the chart $(U_k, z^k) \simeq (\mathbb{C}^n, z)$ by the formula

$$dV_{FS} | \mathbb{C}^n = c_n \frac{dV_2(z)}{(1 + |z|^2)^{n+1}},$$

where $dV_2(z) = \beta_n = \beta^n/n!$ is the euclidean volume form on $\mathbb{C}^n$, $\beta := dd^c|z|^2$ being the standard Kähler metric on $\mathbb{C}^n$.

2.3. **The complex Monge-Ampère operator.** Here we recall some definitions and give a useful characterization of the local domain of definition of the complex Monge-Ampère operator given by Z. Blocki (see [B104], [B106]).

**Definition 2.2.** Let $X$ be a complex manifold of dimension $n$ and $\eta$ a smooth closed (semi)-positive $(1,1)$-form on $X$.

1) We say that a function $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ is $\eta$-plurisubharmonic in $X$ if it is locally the sum of a plurisubharmonic function and a smooth function and $\eta + dd^c \varphi$ is a positive current on $X$. We denote by $PSH(X, \eta)$ the convex set of all of $\eta$-plurisubharmonic functions in $X$.

2) By definition, the set $DMA_{Loc}(X, \eta)$ is the set of functions $\varphi \in PSH(X, \eta)$ for which there exists a positive Borel measure $\sigma = \sigma_\varphi$ on $X$ such that for all open $U \subset \subset \Omega$ and $\forall (\varphi_j) \in PSH(U, \eta) \cap C^\infty(U)$, $\varphi_j$ in $U$, the sequence of local Monge-Ampère measures $(\eta + dd^c \varphi_j)^n$ converges weakly to $\sigma$ on $U$.

In this case, we set $(\eta + dd^c \varphi)^n = \sigma_\varphi$ and call it the complex Monge-Ampère measure of the function $\varphi$ in the manifold $(X, \eta)$.

When $\eta = 0$, we write $DMA_{Loc}(X) = DMA_{Loc}(X, 0)$.

In the case when $X \subset \mathbb{C}^n$ is an open subset and $\eta = 0$, Bedford and Taylor [BT76, BT82] extended the complex Monge-Ampère operator $(dd^c u)^n$ to plurisubharmonic functions which are locally bounded in $X$. Moreover,
they showed that this operator is continuous under decreasing sequences in $\text{PSH}(X) \cap L^\infty_{\text{Loc}}(X)$.

In [De87], Demailly extended the complex Monge-Ampère operator $\left(dd^c\right)^n$ to plurisubharmonic functions locally bounded near the boundary $\partial X$ i.e. in the complement of a compact subset of $X$.

When $X = \mathbb{P}^n$ and $\eta = \omega_{FS}$, it was proved in [CGZ08] that if $\varphi \in \text{PSH}(\mathbb{P}^n, \omega_{FS})$ is bounded in a neighbourhood of a divisor in $\mathbb{P}^n$, then $\varphi \in DMA_{\text{loc}}(\mathbb{P}^n, \omega_{FS})$.

When $\Omega \subset \mathbb{C}^n$ is a bounded hyperconvex domain, the set $DMA_{\text{loc}}(\Omega)$ coincides with the Cegrell class $\mathcal{E}(\Omega)$ (see [Ce04], [Bl06]). Moreover for any $u_1, \ldots, u_n DMA_{\text{loc}}(\Omega)$, it is possible to define the intersection current $dd^c u_1 \wedge \cdots \wedge dd^c u_n$ as a positive Borel measure on $\Omega$ ([Ce04]).

Therefore for any $\varphi_1, \ldots, \varphi_n \in DMA_{\text{loc}}(X, \eta)$, it is possible to define the intersection current

$$M(\varphi_1, \ldots, \varphi_n) := (\eta + dd^c \varphi_1) \wedge \cdots \wedge (\eta + dd^c \varphi_n)$$

as a positive Borel measure on $X$. In particular if $X$ is compact and $\eta$ is a smooth $(1,1)$–form on $X$ which is closed semi-positive and big i.e. $\int_X \eta^n > 0$, then

$$\int_X (\eta + dd^c \varphi_1) \wedge \cdots \wedge (\eta + dd^c \varphi_n) = \int_X \eta^n.$$

Moreover the operator $M$ is continuous for monotone convergence of sequences in $DMA_{\text{loc}}(X, \eta)$.

In dimension two, a simple characterization of the local domain of definition of $\left(dd^c\right)^2$ was given by [Bl04].

**Theorem 2.3.** [Bl04] Suppose $\Omega$ is an open subset of $\mathbb{C}^2$, then

$$DMA_{\text{Loc}}(\Omega) = \text{PSH}(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega),$$

where $W^{1,2}_{\text{loc}}(\Omega)$ is the usual Sobolev space.

In higher dimension, the characterization of the set $DMA_{\text{Loc}}(\Omega)$ is more complicated (see [Bl06]).

**Theorem 2.4.** Let $u$ be a negative plurisubharmonic function on $\Omega \subset \mathbb{C}^n$, $n \geq 3$. The following are equivalent:

1. $u \in DMA_{\text{Loc}}(\Omega)$,

2. $\forall z \in \Omega$, $\exists U_z \subset \Omega$ an open neighborhood of $z$ such that for any sequence $u_j \in \text{PSH} \cap C^\infty(U_z) \nabla u$ in $U_z$, the sequences

$$|u_j|^{n-p-2} du_j \wedge d^c u_j \wedge (dd^c u_j)^p \wedge \omega^{n-p-1}, \quad p = 0, 1, \ldots, n-2$$

are locally weakly bounded in $U_z$.

It is clear that these results extend to the case of $DMA_{\text{loc}}(X, \eta)$.

**Remark 2.5.** We can define the global domain of definition $DMA(X, \eta)$ by taking only bounded global approximants. Therefore $DMA_{\text{loc}}(X, \eta) \subset$
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2.4. The Lelong Class. Recall that the Lelong class \( L(\mathbb{C}^n) \) is defined as the convex set of plurisubharmonic functions in \( \mathbb{C}^n \) satisfying the following growth condition at infinity:

\[
(2.4) \quad u(z) \leq C_u + \frac{1}{2} \log(1 + |z|^2), \quad \forall z \in \mathbb{C}^n,
\]

where \( C_u \) is a constant depending on \( u \) (see [Le68]).

There are two important subclasses of \( L(\mathbb{C}^n) \). Set

\[
\mathcal{L}_*(\mathbb{C}^n) := \{ u \in L(\mathbb{C}^n); u(z) = \frac{1}{2} \log(1 + |z|^2) + O(1), \text{ as } |z| \to +\infty \}.
\]

Associated to a function \( u \in L(\mathbb{C}^n) \) we define its Robin function of \( u \) (BT88)

\[
\rho_u(\xi) := \limsup_{\xi \to \infty} (u(\lambda \xi) - \log |\lambda \xi|) = \limsup_{\xi \to \infty} (u(\lambda \xi) - \frac{1}{2} \log(1 + |\lambda \xi|^2)).
\]

When \( \rho_u^* \neq -\infty \), then it is a homogenous function of order 0 on \( \mathbb{C}^n \) such that \( \rho_u^*(z) + \log |z| \) is plurisubharmonic in \( \mathbb{C}^n \). This implies that \( \rho_u^* \) is a well defined function on the projective space \( \mathbb{P}^{n-1} \) which is actually an \( \omega_{FS} \)-psh on \( \mathbb{P}^{n-1} \), where \( \omega_{FS} \) is the Fubini-Study metric on \( \mathbb{P}^{n-1} \).

Following (BT88), we can define the subclass of logarithmic Lelong potentials as follows:

\[
\mathcal{L}_*(\mathbb{C}^n) := \{ u \in L(\mathbb{C}^n); \rho_u \neq -\infty \}.
\]

Observe that \( \mathcal{L}_*(\mathbb{C}^n) \subset \mathcal{L}_*(\mathbb{C}^n) \) and \( u \in \mathcal{L}_*(\mathbb{C}^n) \) iff \( \rho_u \in L^1(\mathbb{P}^{n-1}) \). We refer to [Zc07] for more properties of this class.

Then there is a 1-1 correspondence between the Lelong class \( \mathcal{L}(\mathbb{C}^n) \) and the set \( PSH(\mathbb{P}^n, \omega) \) of \( \omega \)-plurisubharmonic functions on \( \mathbb{P}^n \). Indeed we will write \( \mathbb{P}^n = U_0 \cup H_\infty \), where

\[
H_\infty := \{ \zeta \in \mathbb{P}^n; \zeta_0 = 0 \} \equiv \mathbb{P}^{n-1},
\]

is the hyperplane at infinity and observe that \( U_0 = \mathbb{P}^n \setminus H_\infty \).

Given \( u \in \mathcal{L}(\mathbb{C}^n) \), we associate the function \( \varphi \) defined on \( U_0 \) by

\[
\phi_u(\zeta) := u(z_1, \cdots, z_n) - \frac{1}{2} \log(1 + |z|^2), \quad \text{with } z := z^0 = (\zeta_1/\zeta_0, \cdots, \zeta_n/\zeta_0).
\]

By definition of the class \( \mathcal{L}(\mathbb{C}^n) \), the function \( \phi_u \) is locally upper bounded in \( U_0 \) near \( H_\infty \). Since \( H_\infty = \{0\} \times \mathbb{P}^{n-1} \) is a proper analytic subset (hence a pluripolar subset) of \( \mathbb{P}^n \), the function \( \phi_u \) can be extended into an \( \omega \)-plurisubharmonic function on \( \mathbb{P}^n \) by setting

\[
\phi_u(0, \zeta') = \limsup_{\xi \to (0, \zeta')} \varphi(\zeta), \quad \zeta' \in \mathbb{P}^{n-1}.
\]
Then we have a well defined "homogenization" map
\[ \mathcal{L}(\mathbb{C}^n) \ni \mapsto \phi_u \in PSH(\mathbb{P}^n, \omega_{FS}). \]
This is a bijective map and its inverse map is defined as follows: given \( \phi \in PSH(\mathbb{P}^n, \omega_{FS}), \) we can define
\[ u_\phi(z) := \phi([1, z]) + (1/2) \log(1 + |z|^2). \]
It is clear that \( u_\phi \in \mathcal{L}(\mathbb{C}^n) \) and \( H(u_\phi) = \phi. \) Observe that for \( \zeta' \in \mathbb{P}^{n-1}, \) we have
\[ \rho_u(\zeta') = \phi_u(0, \zeta'). \]

**Example 2.6.** Let \( P \in \mathbb{C}_d[z_1, \ldots, z_n] \) be a polynomial of degree \( d \geq 1. \) Then \( u := (1/d) \log |P| \in \mathcal{L}(\mathbb{C}^n) \). It is easy to see that its homogenization \( \phi = \phi_u \) is given by
\[ \phi(\zeta) := (1/d) \log |Q(\zeta)| - \log |\zeta|, \quad \zeta \in \mathbb{P}^n, \]
where \( Q \) is the unique homogeneous polynomial on \( \mathbb{C}^{n+1} \) of degree \( d \) such that \( Q(1, z) = P(z) \) for \( z \in \mathbb{C}^n. \)

**Proposition 2.7.** Let \( u_1, \ldots, u_n \in \mathcal{L}_+(\mathbb{C}^n) \) and for each \( i = 1, \ldots, n \) set \( \phi_i := \phi_{u_i} \) the homogenization of \( u_i \) on \( \mathbb{P}^n. \) Then
\[ (2.5) \quad \int_{\mathbb{C}^n} dd^c u_1 \wedge \cdots \wedge dd^c u_n = \int_{\mathbb{P}^n} (\omega + dd^c \phi_1) \wedge \cdots \wedge (\omega + dd^c \phi_n) = 1. \]

Proof. Observe that if \( u \in \mathcal{L}_+(\mathbb{C}^n), \) its homogenization \( \phi = \phi_u \in PSH(\mathbb{P}^n, \omega) \) is a bounded \( \omega \)-psh function in a neighbourhood of the hyperplane at infinity \( H_\infty. \) Hence by \([CGZ08], \mathcal{L}_+(\mathbb{C}^n) \subset DMA_{loc}(\mathbb{P}^n, \omega) \) and its complex Monge-Ampère measure is well defined and puts no mass on \( H_\infty. \) Hence
\[ 1 = \int_{\mathbb{P}^n} (\omega + dd^c \phi^n) = \int_{\mathbb{C}^n} (\omega + dd^c \phi)^n. \]
Since \( \omega + dd^c \phi = dd^c u \) in the weak sense on \( \mathbb{C}^n, \) the formula (2.5) follows. \( \square \)

### 3. Projective Logarithmic Potentials on \( \mathbb{C}^n \)

#### 3.1. The euclidean logarithmic potential in \( \mathbb{C}^n \)

Our main motivation is to study projective logarithmic potentials on \( \mathbb{P}^n. \) It turns out that when localizing these potentials in affine coordinates, we end up with a kind of projective logarithmic potential on \( \mathbb{C}^n \) which we shall study here.

We first introduce the normalized logarithmic kernel on \( \mathbb{C}^n \times \mathbb{C}^n \) defined by
\[ K(z, w) := \frac{1}{2} \log \frac{|z - w|^2}{1 + |w|^2}, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}^n. \]
Let \( \mu \) be a probability measure on \( \mathbb{C}^n. \) We define its logarithmic potential as follows, for \( z \in \mathbb{C}^n \)
\[ (3.1) \quad U_\mu(z) = \int_{\mathbb{C}^n} K(z, w) d\mu(w) = \frac{1}{2} \int_{\mathbb{C}^n} \log \frac{|z - w|^2}{1 + |w|^2} d\mu(w) \]
It is well known that for any \( w \in \mathbb{C}^n \), the function \( K_w := K(\cdot, w) \) is plurisubharmonic in \( \mathbb{C}^n \), \( K_w \in \mathcal{L}_+(\mathbb{C}^n) \) and satisfies the complex Monge-Ampère equation

\[
(dd^c K(\cdot, w))^n = \delta_w,
\]

in the sense of currents on \( \mathbb{C}^n \), where \( \delta_w \) is the unit Dirac mass at \( w \).

**Theorem 3.1.** Let \( \mu \) be a probability measure on \( \mathbb{C}^n \). Then for any \( z \in \mathbb{C}^n \),

\[
U_\mu(z) \leq \frac{1}{2} \log(1 + |z|^2).
\]

and if \( \sigma_{2n-1} \) is the normalized Lebesgue measure on the unit sphere \( \mathbb{S}^{2n-1} \),

\[
\int \left\{ |z| = 1 \right\} U_\mu(z) d\sigma_{2n-1}(z) \geq -\log \sqrt{5}.
\]

In particular \( U_\mu \in \mathcal{L}(\mathbb{C}^n) \).

If moreover \( \mu \) satisfies the following logarithmic moment condition at infinity i.e.

\[
\int_{\mathbb{C}^n} \log(1 + |w|^2) d\mu(w) < +\infty,
\]

then \( U_\mu \in \mathcal{L}_*(\mathbb{C}^n) \) and its Robin function \( \rho_\mu := \rho_{U_\mu} \) satisfies the lower bound

\[
\int_{\mathbb{P}^{n-1}} \rho_\mu(\zeta) \omega_{n-1} \geq -(1/2) \int_{\mathbb{C}^n} \log(1 + |w|^2) d\mu(w),
\]

where \( \omega_{n-1} \) is the Fubini-Study volume form on \( \mathbb{P}^{n-1} \).

**Proof.** By (2.3), for any \( z, w \in \mathbb{C}^n \),

\[
\frac{|z - w|^2}{(1 + |z|^2)(1 + |w|^2)} \leq 1 - \frac{|1 + z \cdot w|^2}{(1 + |z|^2)(1 + |w|^2)}.
\]

Then for \( z, w \in \mathbb{C}^n \),

\[
K(z, w) := \frac{1}{2} \log \frac{|z - w|^2}{1 + |w|^2} \leq \frac{1}{2} \log(1 + |z|^2).
\]

For any fixed \( w \in \mathbb{C}^n \), the function \( z \rightarrow K(z, w) \) is plurisubharmonic in \( \mathbb{C}^n \). It follows that \( U_\mu \in \mathcal{L}(\mathbb{C}^n) \) provided that we prove that \( U_\mu \not\equiv -\infty \).

To prove the last statement, write for \( z \in \mathbb{C}^n \)

\[
2U_\mu(z) = I_1(z) + I_2(z),
\]

where

\[
I_1(z) := \int_{|w| \leq 2} \log \frac{|z - w|^2}{1 + |w|^2} d\mu(w),
\]

and

\[
I_2(z) := \int_{|w| > 2} \log \frac{|z - w|^2}{1 + |w|^2} d\mu(w).
\]
Let us prove that $I_1 \not\equiv -\infty$ and $I_2 \not\equiv -\infty$. For the first integral observe that for $|z| \geq 3$, we have

$$I_1(z) \geq -\mu(\mathbb{B}(0, 2)) \log 5 > -\infty.$$ 

For the second integral, observe that for $|z| \leq 1$, \(\frac{(|w|-1)^2}{1+|w|^2} \geq \frac{1}{5}\) for $|w| \geq 2$. Hence for $|z| < 1$,

$$I_2(z) \geq \int_{|w| \geq 2} \log \left( \frac{|w|-1}{1+|w|^2} \right) d\mu(w) \geq -\log 5 > -\infty.$$ 

Therefore $I_1$ and $I_2$ belongs to $PSH(\mathbb{C}^n) \subset L^1_{\text{loc}}(\mathbb{C}^n)$ and then $U_\mu \in PSH(\mathbb{C}^n)$.

This proves that $U_\mu \not\equiv -\infty$.

To prove the last statement observe that $\rho_\mu \leq 0$ on $\mathbb{C}^n$ and

$$\lim_{|\lambda| \to +\infty} (\log |\lambda z - w|^2 - \log |\lambda z|^2) = 0.$$ 

Then apply Fatou’s lemma to obtain the conclusion. \(\square\)

### 3.2. Riesz potentials

Here we prove a technical lemma which is certainly well known but since we cannot find the right reference for it, we will give all the details needed in the sequel. Here we work in the euclidean space $\mathbb{R}^N$ with its usual scalar product and its associated euclidean norm $\| \cdot \|$.

Let $\mu$ be a probability measure with compact support on $\mathbb{R}^N$. Define its Riesz potentials by

$$J_{\mu, \alpha}(x) := \int_{\mathbb{R}^N} \frac{d\mu(w)}{|x-y|^\alpha} = \mu \ast J_\alpha(x), \ x \in \mathbb{R}^N,$$
where

\[ J_\alpha(x) := \frac{1}{|x|^\alpha}, \quad x \in \mathbb{R}^N. \]

Observe that \( J_\alpha \in L^p_{\text{loc}}(\mathbb{R}^N) \) if and only if \( 0 < p < N/\alpha \).

Given a probability measure \( \mu \) on \( \mathbb{R}^N \), there are many different notions of dimension for the measure \( \mu \). Here we use the following one (see [LMW02]). We define the Lévy concentration functions of \( \mu \) as follows:

\[ \mu(x, r) := \mu(B(x, r)), \quad Q_\mu(r) := \sup \{ \mu(x, r); x \in \text{Supp}\mu \}, \]

where \( B(x, r) \) is the euclidean (open) ball of center \( x \) and radius \( r > 0 \).

The lower concentration dimension of \( \mu \) is given by the following formula:

\[
(3.2) \quad \gamma(\mu) = \gamma_-(\mu) := \liminf_{r \to 0^+} \frac{\log Q_\mu(r)}{\log r}.
\]

We will call it for convenience the dimension of the measure \( \mu \). The following property is well known.

**Lemma 3.2.** Let \( \mu \) be a probability measure \( \mu \) with compact support on \( \mathbb{C}^n \). Then its dimension \( \gamma(\mu) \) is the supremum of all the exponents \( \gamma \geq 0 \) for which the following estimates are satisfied: there exists \( C > 0 \) such that \( \forall x \in \mathbb{R}^N, \forall r \in [0, 1[ \),

\[ \mu(B(x, r)) \leq Cr^\gamma. \]

Moreover we have \( 0 \leq \gamma(\mu) \leq N \).

**Proof.** The first statement is obvious and we have \( \gamma(\mu) \geq 0 \) since \( \mu \) has a finite mass. To prove the upper bound, observe that for any fixed \( r > 0 \), the function \( x \mapsto \mu(x, r) \) is a non negative Borel function on \( \mathbb{R}^n \). By Fubini’s Theorem, we have for any \( r > 0 \),

\[
\int_{\mathbb{R}^N} \mu(x, r) dx = \int_{\mathbb{R}^N} \left( \int_{y \in B(x, r)} d\mu(y) \right) dx
= \int_{\mathbb{R}^N} \left( \int_{B(y, r)} dx \right) d\mu(y).
\]

If we denote by \( \tau_N \) the volume of the euclidean unit ball in \( \mathbb{R}^N \), we obtain

\[
(3.3) \quad \int_{\mathbb{R}^N} \mu(x, r) dx = \tau_N r^N.
\]

Therefore for any \( r > 0 \), we have

\[ \tau_N r^N \leq Q_\mu(r) \mu(\mathbb{R}^N) = Q_\mu(r), \]

which implies immediately that \( \gamma(\mu) \leq N \). \( \square \)

**Example 3.3.** 1. Then the measure \( \mu \) has no atom in \( \mathbb{R}^N \) if and only if \( \gamma(\mu) > 0 \).

2. Let \( 0 < k \leq N \) be any real number and let \( A \subset \mathbb{R}^N \) be any Borel subset such that its \( k \)-dimensional Hausdorff measure satisfies \( 0 < \lambda_k(A) < +\infty \).
Then the restricted measure \( \lambda_{k,A} := \mathbb{1}_A \lambda_k \) is a Borel measure of dimension \( k \). In particular the \( N \)-dimensional Lebesgue measure \( \lambda_{k,A} \) has dimension \( N \).

**Lemma 3.4.** Let \( \mu \) be a probability measure with compact support on \( \mathbb{R}^N \) and \( 0 < \alpha < N \) and \( \gamma := \gamma(\mu) \) its dimension. Then

\[
J_{\mu,\alpha} \in L^p_{\text{Loc}}(\mathbb{R}^N) \quad \text{if} \quad 1 < p < \frac{N - \gamma}{(\alpha - \gamma)_+},
\]

where \( x_+ := \max\{x, 0\} \) for a real number \( x \in \mathbb{R} \).

**Proof.** We follow an idea from [P16] where the case when \( \alpha = N - 1 \) is considered (see [P16, Proof of Lemma 10.12]). For convenience, we give all the details here.

By the Cavalieri principle for any fixed \( x \in \mathbb{R}^N \), we have

\[
J_{\mu,\alpha}(x) = \alpha \int_0^{+\infty} \mu(x, r) \frac{dr}{r^{\alpha+1}}.
\]

Then by Minkowski inequality, we obtain

\[
\|\mu \ast J_{\alpha}\|_p \leq \alpha \int_0^{+\infty} \|\mu(\cdot, r)\|_p \frac{dr}{r^{\alpha+1}},
\]

here \( \| \cdot \|_p \) means the \( L^p \)-norm with respect to the Lebesgue measure on \( \mathbb{R}^N \).

Recall that \( Q_{\mu}(r) := \sup_{x} \mu(x, r) \) for \( r > 0 \). This is a bounded Borel function on \( \mathbb{R}^+ \) such that \( 0 \leq Q_{\mu}(r) \leq 1 \) since \( \mu \) is a Probability measure. For \( p > 1 \) fixed, we can write for any \( x \in \mathbb{R}^N \) and \( r > 0 \),

\[
\mu(x, r)^p = \mu(x, r)^{p-1} \mu(x, r) \leq Q_{\mu}(r)^{p-1} \mu(x, r) \leq \min\{Q_{\mu}(r), 1\}^{p-1} \mu(x, r).
\]

Then by (3.3) we get,

\[
\|\mu(\cdot, r)\|_p^p = \int_{\mathbb{R}^N} \mu(x, r)^p dx \\
\leq \min\{Q_{\mu}(r), 1\}^{p-1} \int_{\mathbb{R}^N} \mu(x, r) dx \\
\leq C \min\{Q_{\mu}(r), 1\}^{p-1} r^{2n}.
\]

Therefore for any fixed \( \kappa > 0 \), we have

\[
\|\mu \ast J_{\alpha}\|_p \leq C\alpha \int_{\kappa}^{\infty} Q_{\mu}(r)^{(p-1)/p} r^{2n/p} \frac{dr}{r^{\alpha+1}} \\
+ \int_{\kappa}^{\infty} r^{2n/p} \frac{dr}{r^{\alpha+1}}.
\]

Using the estimate on \( Q_{\mu} \) in terms of the dimension and minimizing in \( \kappa > 0 \), we get the required result. \( \square \)
3.3. **The projective logarithmic kernel.** Our main motivation is to study projective logarithmic potentials on $\mathbb{P}^n$. It turns out that when localizing these potentials in affine coordinates, we end up with a kind of projective logarithmic potential on $\mathbb{C}^n$ which we shall study first.

Recall the definition of the logarithmic kernel from the previous section,

$$K(z, w) := \frac{1}{2} \log \frac{|z - w|^2}{1 + |w|^2}.$$

The projective logarithmic kernel on $\mathbb{C}^n \times \mathbb{C}^n$ is defined by the following formula:

$$N(z, w) := \frac{1}{2} \log \frac{|z - w|^2 + |z \wedge w|^2}{1 + |w|^2}, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}^n.$$

**Lemma 3.5.** 1. The kernel $N$ is upper semi-continuous in $\mathbb{C}^n \times \mathbb{C}^n$ and smooth off the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$.

2. For any fixed $w \in \mathbb{C}^n$, the function $N_w := N(\cdot, w) : z \mapsto N(z, w)$ is plurisubharmonic in $\mathbb{C}^n$ and satisfies the following inequality

$$K(z, w) \leq N(z, w) \leq (1/2) \log(1 + |z|^2), \quad \forall (z, w) \in \mathbb{C}^n \times \mathbb{C}^n$$

hence for any $w \in \mathbb{C}^n$, $N(\cdot, w) \in \mathcal{L}_+(\mathbb{C}^n)$.

3. The kernel $N$ has a logarithmic singularity along the diagonal i.e. for any $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$,

$$0 \leq N(z, w) - K(z, w) \leq \frac{1}{2} \log(1 + \min\{|z|^2, |w|^2\}).$$

4. For any $w \in \mathbb{C}^n$,

$$(dd^c N_w)^n = \delta_w,$$

where $\delta_w$ is the unit Dirac measure on $\mathbb{C}^n$ at the point $w$.

**Proof.** The first statement is obvious. Let us prove the second one. By the formula 2.3 we have for any $z, w \in \mathbb{C}^n$,

$$|z - w|^2 + |z \wedge w|^2 \geq \left(1 + |z|^2\right)\left(1 + |w|^2\right).$$

Thus for $z, w \in \mathbb{C}^n$,

$$N(z, w) = \frac{1}{2} \log \frac{|z - w|^2 + |z \wedge w|^2}{1 + |w|^2} \leq \frac{1}{2} \log(1 + |z|^2).$$

This yields

$$N(z, w) - K(z, w) = (1/2) \log \left(1 + \frac{|z \wedge w|^2}{|z - w|^2}\right).$$

Now observe that by Lemma 2.3 we have

$$|z \wedge w|^2 = |(z - w) \wedge w|^2 \leq |z - w|^2 |w|^2.$$

By symmetry we obtain the required inequality.
To prove the third property, observe that for a fixed \( w \in \mathbb{C}^n \), the two functions
\[
u(z) := N(z, w), \quad v(z) := K(z, w),
\]
belongs to \( \mathcal{L}_+(\mathbb{C}^n) \). Hence by (2.5) they have the same total Monge-Ampère mass in \( \mathbb{C}^n \), i.e.
\[
\int_{\mathbb{C}^n} (dd^c \nu)^n = 1.
\]
On the other we know that
\[
(dd^c v)^n = \delta_w.
\]
Since \( \lim_{z \to w} \frac{\nu(z)}{v(z)} = 1 \), by the comparison theorem of Demailly [De93], they have the same residual Monge-Ampère mass at the point \( w \). Therefore the total Monge-Ampère mass of the measure \( (dd^c \nu)^n \) is concentrated at the point \( w \), which proves our statement. \( \square \)

3.4. The projective logarithmic potential. Let \( \mu \) be a probability measure with compact support on \( \mathbb{C}^n \). We define the projective logarithmic potential of \( \mu \) as follows:
\[
(3.4) \quad V_\mu(z) = \frac{1}{2} \int_{\mathbb{C}^n} \log \left( \frac{|z - w|^2 + |z \wedge w|^2}{1 + |w|^2} \right) d\mu(w),
\]
It follows that \( V_\mu \in \mathcal{L}(\mathbb{C}^n) \) provided that we prove that \( V_\mu \neq -\infty \).

**Theorem 3.6.** Let \( \mu \) be a probability measure with compact support on \( \mathbb{C}^n \). Then for any \( z \in \mathbb{P}^n \),
\[
U_\mu(z) \leq V_\mu(z) \leq \frac{1}{2} \log(1 + |z|^2).
\]
and
\[
\int_{\{|z|=1\}} V_\mu(z) d\sigma_{2n-1}(z) \geq -\log \sqrt{\pi}.
\]
Moreover \( V_\mu \in \mathcal{L}_+(\mathbb{C}^n) \) and its Robin function \( \rho_\mu := \rho_{V_\mu} = \rho_{U_\mu} \) satisfies the lower bound
\[
\int_{\mathbb{P}^{n-1}} \rho_\mu(\xi) \omega_{n-1} \geq -(1/2) \int_{\mathbb{C}^n} \log(1 + |w|^2) d\mu(w),
\]
where \( \omega_{n-1} \) is the Fubini-Study volume form on \( \mathbb{P}^{n-1} \).

**Proof.** The right hand side estimate in the first inequality follows from lemma 3.5, while the other statements follow from Theorem 3.1 since \( V_\mu \geq U_\mu \) in \( \mathbb{C}^n \). \( \square \)
4. The projective logarithmic potential on $\mathbb{P}^n$

4.1. The projective logarithmic kernel. The projective logarithmic kernel on $\mathbb{P}^n \times \mathbb{P}^n$ is defined by the following formula:

$$G(\zeta, \eta) := \frac{1}{2} \log \frac{\|\zeta \wedge \eta\|^2}{\|\zeta\|^2 \|\eta\|^2}, \quad (\zeta, \eta) \in \mathbb{P}^n \times \mathbb{P}^n.$$ 

**Lemma 4.1.**

1. The kernel $G$ is a non positive upper semi-continuous function in $\mathbb{P}^n \times \mathbb{P}^n$ and smooth off the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$.

2. For any fixed $\eta \in \mathbb{P}^n$, the function $G(\cdot, \eta) : \zeta \mapsto G(\zeta, \eta)$ is a non positive $\omega$-plurisubharmonic in $\mathbb{P}^n$ which is smooth in $\mathbb{P}^n \setminus \{\eta\}$ i.e. $G(\cdot, \eta) \in PSH(\mathbb{P}^n, \omega)$.

3. The kernel $G$ has a logarithmic singularity along the diagonal. More precisely for a fixed $\eta \in U^k$, in the coordinate chart $(U^k, z^k)$ we have

$$0 \leq G(\zeta, \eta) - \frac{1}{2} \log \frac{|z^k(\zeta) - z^k(\eta)|^2}{(1 + |z^k(\zeta)|^2)(1 + |z^k(\eta)|^2)} \leq \log(1 + \min\{|z^k(\zeta)|^2, |z^k(\eta)|^2\}).$$

4. For any $\eta \in \mathbb{P}^n$, $G(\cdot, \eta) \in DMA_{loc}(\mathbb{P}^n, \omega)$ and

$$(\omega + dd^c G(\cdot, \eta))^n = \delta_\eta,$$

where $\delta_\eta$ is the unit Dirac measure on $\mathbb{P}^n$ at the point $\eta$.

**Proof.** This lemma follows from Lemma 3.5 by observing that in each open chart $(U^k, z^k)$ we have for $(\zeta, \eta) \in U^k \times U^k$,

$$G(\zeta, \eta) = N(z, w) - \frac{1}{2} \log(1 + |z|^2),$$

where $z := z^k(\zeta)$ and $w := z^k(\eta)$. \qed

4.2. The projective logarithmic potential. Let $\text{Prob}(\mathbb{P}^n)$ be the convex compact set of probability measures on $\mathbb{P}^n$. Given $\mu \in \text{Prob}(\mathbb{P}^n)$ we define its (projective) logarithmic potential as follows

$$\mathcal{G}_\mu(\zeta) := \int_{\mathbb{P}^n} G(\zeta, \eta) d\mu(\eta),$$

$$= \int_{\mathbb{P}^n} \log \frac{\|\zeta \wedge \eta\|}{\|\zeta\| \|\eta\|} d\mu(\eta),$$

As observed in [As17] the projective kernel $\mathcal{G}$ can be expressed in terms of the geodesic distance $d$ on the Kähler manifold $(\mathbb{P}^n, \omega_{FS})$. Namely we have

$$\mathcal{G}_\mu(\zeta) = \int_{\mathbb{P}^n} \log \sin\left(\frac{d(\xi, \eta)}{\sqrt{2}}\right) d\mu(\eta).$$

Thanks to this formula, we see that if $f$ is a radial function on $\mathbb{P}^n$ i.e. $f(\zeta) := g(d(\zeta, a))$ for a fixed point $a \in \mathbb{P}^n$. Then choosing polar coordinates around $a$ we have

$$\int_{\mathbb{P}^n} f(\zeta) dV(\zeta) = \int_0^{\pi/\sqrt{2}} g(r) A(r) dr,$$
where $A(r)$ is the "area" of the sphere of center $a$ and radius $r$. The expression of $A(r)$ is given by the formula

$$A(r) = c_n \sin^{2n-2}(r/\sqrt{2}) \sin(\sqrt{2}r),$$

where $c_n$ a constant depending on the volume of the unit ball in $\mathbb{R}^n$. For more details, see [Rag71, Page 168], [AB77, Section 3] or [Hel65, Lemma 5.6].

This allows to give a simple example.

**Proposition 4.2.**

1. Let $\sigma$ be the Lebesgue measure associated to the Fubini-Study volume form $dV$. Then for any $\zeta \in \mathbb{P}^n$,

$$G_\sigma(\zeta) = -\alpha_n,$$

where $\alpha_n > 0$ is a numerical constant given by the formula (4.3) below.

2. For any $\mu \in \text{Prob}(\mathbb{P}^n)$, $G_\mu$ is a negative $\omega$-plurisubharmonic function in $\mathbb{P}^n$ such that

$$(4.2) \quad \int_{\mathbb{P}^n} G_\mu(\zeta)dV(\zeta) = -\alpha_n.$$

**Proof.** We use the same computations based on the formula (4.1) as in ([AS17]).

1. By (4.1) and the co-area formula

$$G_\sigma(\zeta) = \int_{\mathbb{P}^n} \log \sin\left(\frac{d(\zeta, \eta)}{2}\right)dV(\eta)$$

$$= \int_0^{\infty} \log \sin\left(\frac{r}{\sqrt{2}}\right)A(r)dr$$

$$= c_n \int_0^{\infty} \log \left(\sin\left(\frac{r}{\sqrt{2}}\right)\right) \sin^{2n-2}\left(\frac{r}{\sqrt{2}}\right) \sin(\sqrt{2}r)dr$$

$$= 2\sqrt{2}c_n \int_0^{\frac{\pi}{2}} (\log \sin(t)) \sin^{2n-1}(t) \cos(t)dt$$

$$= 2\sqrt{2} \int_0^{1} u^{2n-1} \log u du.$$

Therefore the statement follows if we set

$$(4.3) \quad \alpha_n := \sqrt{2} \int_0^{1} u^{2n-1} \log u du$$

2. To prove the second statement it is enough to observe the following symmetry

$$\int_{\mathbb{P}^n} G_\mu(\zeta)d\sigma(\zeta) = \int_{\mathbb{P}^n} G_\sigma(\eta)d\mu(\eta) = -\alpha_n \mu(\mathbb{P}^n) = -\alpha_n,$$

thanks to the formula (4.2).
5. The Monge-Ampère measure of the potentials

5.1. The Monge-Ampère measure of $V_\mu$ in $\mathbb{C}^n$. We begin this section by showing that $V_\mu$ belongs to the domain of definition of the complex Monge-Ampère operator for ($n \geq 3$).

**Theorem 5.1.** Let $\mu$ be a probability measure on $\mathbb{C}^n$ ($n \geq 2$) with compact support. Then

1) $V_\mu \in DMA_{\text{loc}}(\mathbb{C}^n)$.

2) $V_\mu \in W^{2,p}_{\text{loc}}(\mathbb{C}^n)$ for any $0 < p < n$. In particular, for any $1 \leq k \leq n-1$, the measure $(dd^c V_\mu)^k \wedge \omega^{n-k}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}^n$.

**Proof.** The first part of the theorem was proved in [As17] but for convenience we reproduce the proof here since we will use the same computations to prove the other statements.

1. We will use the characterization due to Blocki ([Bl06]). For $\varepsilon > 0$, set

$$V_\varepsilon^\mu(z) := \frac{1}{2} \int_{\mathbb{C}^n} \log \left( \frac{|z-w|^2 + |z \wedge w|^2 + \varepsilon^2}{1 + |w|^2} \right) d\mu(w).$$

It is clear that $V_\varepsilon^\mu \in L(\mathbb{C}^n) \cap C^\infty(\mathbb{C}^n)$ and decreases towards $V_\mu$ as $\varepsilon$ decreases to 0. Since $V_\mu$ in plurisubharmonic in $\mathbb{C}^n$, we have

$$|V_\varepsilon^\mu|^{n-p-2} \in L^r_{\text{loc}}(\mathbb{C}^n) \quad \text{for any} \quad r_1 > 0.$$

On the other hand, recall that

$$|z \wedge w|^2 = \sum_{1 \leq i < k \leq n} |z_i w_k - z_k w_i|^2,$$

and observe that

$$\frac{\partial}{\partial z_m}(|z-w|^2 + |z \wedge w|^2) = z_m - w_m + \sum_{m<j \leq n} w_j (z_m w_j - z_j w_m) - \sum_{1 \leq i < m} w_i (z_i w_m - z_m w_i).$$

Then we have for any $z \in \mathbb{C}^n$,

$$2 \frac{\partial}{\partial z_m} V_\varepsilon^\mu(z) = \int_{\mathbb{C}^n} \frac{z_m - w_m + \sum_{m<j \leq n} w_j (z_m w_j - z_j w_m) - \sum_{1 \leq i < m} w_i (z_i w_m - z_m w_i)}{|z-w|^2 + |z \wedge w|^2 + \varepsilon^2} d\mu(w).$$

Thus
\[ |\nabla V^\varepsilon_\mu(z)| \leq \frac{1}{2} \int_{\mathbb{C}^n} \frac{|z - w| + |w||z \wedge w|}{|z - w|^2 + |z \wedge w|^2} d\mu(w) \]

\[
\leq \frac{\sqrt{2}}{2} \int_{\mathbb{C}^n} \frac{(1 + |w|)\sqrt{|z - w|^2 + |z \wedge w|^2}}{|z - w|^2 + |z \wedge w|^2} d\mu(w) 
\leq \frac{\sqrt{2}}{2} \int_{\mathbb{C}^n} \frac{(1 + |w|)}{|z - w|} d\mu(w) 
\leq \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(1 + |z|) \int_{\mathbb{C}^n} \frac{d\mu(w)}{|z - w|}.
\]

In conclusion we have for \( z \in \mathbb{C}^n \),

\[
(5.1) \quad \|\nabla V^\varepsilon_\mu(z)\| \leq \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(1 + |z|)J_{\mu,1}(z).
\]

By Lemma 3.4 we conclude that

\[ |\nabla V^\varepsilon_\mu| \in L^q_{\text{Loc}}(\mathbb{C}^n), \forall q < 2n, \]

and so

\[ |\nabla V^\varepsilon_\mu(z)|^2 \in L^2_{\text{Loc}}(\mathbb{C}^n) \text{ for } r_2 < n \]

The same computation shows that the second partial derivatives satisfy

\[
|\frac{\partial^2}{\partial z_k \partial z_m} V^\varepsilon_\mu(z)| + |\frac{\partial^2}{\partial z_k \partial z_m} V^\varepsilon_\mu(z)| \leq c_n \int_{\mathbb{C}^n} \frac{(1 + |w|^2)}{|z - w|^2 + |z \wedge w|^2} d\mu(w) 
\leq c_n + c_n(1 + |z|^2)J_{\mu,2}(z).
\]

Hence recalling that \( \mu \) has compact support and using Lemma 3.4, we obtain the inequality

\[
(5.3) \quad \left|\nabla^2 V^\varepsilon_\mu(z)\right| \leq c_n + C(n, \mu)J_{\mu,2} \in L^p_{\text{Loc}}(\mathbb{C}^n),
\]

for any \( p < n \), which proves the second statement.

To prove the first statement, observe that for a fixed compact set \( K \subset \mathbb{C}^n \), we have

\[
\int_K |V^\varepsilon_\mu|^{n-p-2} dV^\varepsilon_\mu \wedge d\varepsilon V^\varepsilon_\mu \wedge (d\varepsilon V^\varepsilon_\mu)^p \wedge \omega^{n-p-1} 
\leq \int_K \|(V^\varepsilon_\mu)^{n-p-2}\|\nabla V^\varepsilon_\mu^2 \underbrace{\ldots}_{p-\text{times}} \omega^n 
\leq C \sum_{l,k=1}^n \int_K \|(V^\varepsilon_\mu)^{n-p-2}\|\nabla V^\varepsilon_\mu^2 \left|\frac{\partial^2}{\partial z_{k_1} \partial \widehat{z}_{l_1}} V^\varepsilon_\mu(z)\right| \ldots \left|\frac{\partial^2}{\partial z_{k_p} \partial \widehat{z}_{l_p}} V^\varepsilon_\mu(z)\right| \omega^n 
\leq C \sum_{l,k=1}^n \|(V^\varepsilon_\mu)^{n-p-2}\|_r \|\nabla V^\varepsilon_\mu^2\|_r \left|\frac{\partial^2}{\partial z_{k_1} \partial \widehat{z}_{l_1}} V^\varepsilon_\mu(z)\right| \ldots \left|\frac{\partial^2}{\partial z_{k_p} \partial \widehat{z}_{l_p}} V^\varepsilon_\mu(z)\right|_s,
\]

where \( \| \cdot \|_t \) denotes the norm in \( L^t(K) \).
Since $0 \leq p \leq n - 2$, in this estimates we have a product of $p + 2 \leq n$ terms such that $p + 1 \leq n - 1$ terms are in $L^k_{loc}$ with $k < n$ and one term in $L^k_{loc}$ for any $k > 0$. In order to apply Hölder inequality, we need to choose $r_1 > 1$ and $r_2, s_1, \ldots, s_n \in ]1, n]$ such that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{s_1} + \ldots + \frac{1}{s_p} = 1$$

Indeed since $p + 1 < n$, we can set $r_j = s_j = p + 1 + \epsilon < n - \epsilon$ for $j = 1, \ldots, p$ and $r_1 = \frac{n + 1 + \epsilon}{\epsilon}$ to obtain the required condition. Thus the complex Monge-Ampère measure $(dd^c V_{\mu})^n$ is well defined by Blocki’s Theorem 2.4.

(5.4) $\frac{\partial^2}{\partial z_k \partial z_m} V_\mu(z) \leq C J_{\mu, 2} \in L^p_{loc}(\mathbb{C}^n)$, for any $0 < p < n$ with a uniform constant $C > 0$.

Since $\frac{\partial^2}{\partial z_k \partial z_m} V_\mu \to \frac{\partial^2}{\partial z_k \partial z_m} V_\mu$ weakly on $\mathbb{C}^n$ as $\epsilon \to 0$, it follows from standard Sobolev space theory that $\frac{\partial^2}{\partial z_k \partial z_m} V_\mu \in L^p$ for any $0 < p < n$.

We note for later use that the kernel

$$N(z, w) := \frac{1}{2} \log \left(\frac{|z - w|^2 + |z \wedge w|^2}{1 + |w|^2}\right), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}^n.$$

can be approximated by the smooth kernels

$$N_\epsilon(z, w) := \frac{1}{2} \log \left(\frac{|z - w|^2 + |z \wedge w|^2 + \epsilon^2}{1 + |w|^2}\right), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}^n.$$

Since each function $N_\epsilon(\cdot, w_j) \in \mathcal{L}_+(\mathbb{C}^n)$, we know by Lemma 3.4 that the measures

$$dd^c N_\epsilon(z, w_1) \wedge \ldots \wedge dd^c N_\epsilon(z, w_n)$$

are well defined probability measures on $\mathbb{C}^n$. Hence we have for all $w_1, \ldots, w_n \in \mathbb{C}^n$

$$\int_{\mathbb{C}^n} dd^c N_\epsilon(z, w_1) \wedge \ldots \wedge dd^c N_\epsilon(z, w_n) = 1.$$

**Proposition 5.2.** If $\mu$ be a probability measure on $\mathbb{C}^n$ then the complex Hessian currents associated to $V_\mu$ are given by the following formula: for any $1 \leq k \leq n$,

$$(dd^c V_{\mu})^k = \int_{\mathbb{C}^n} dd^c N(\cdot, w_1) \wedge \ldots \wedge dd^c N(\cdot, w_k) d\mu(w_1) \ldots d\mu(w_k),$$

in the sense of $(k, k)$-currents on $\mathbb{C}^n$.

**Proof.** Since $V_\mu$ belongs to the domain of definition $DMA_{loc}(\mathbb{C}^n)$, the complex Monge-Ampère current $(dd^c V_{\mu})^k$ is well defined.
We assume that $2 \leq k \leq n$. Set

$$V_{\mu, \varepsilon}(z) := \frac{1}{2} \int_{\mathbb{C}^n} \frac{1}{2} \log \left( \frac{|z - w|^2 + |z \wedge w|^2 + \varepsilon^2}{1 + |w|^2} \right) \, d\mu(w)$$

$$= \int_{\mathbb{C}^n} N_\varepsilon(z, w) \, d\mu(w)$$

Let $\chi \geq 0$ be a positive smooth $(n-k, n-k)$-form with compact support in $\mathbb{C}^n$ and denote for $j = 2, \cdots, n$, by $\mu^{(j)} = \mu^{\otimes j}$ the product measure on $(\mathbb{C}^n)^j$. Then applying Fubini's theorem and integrating by parts formula, we obtain

$$A_\varepsilon := \int_{z \in \mathbb{C}^n} \chi(z) \wedge \int_{(\mathbb{C}^n)^k} d\varepsilon N_\varepsilon(z, w_1) \wedge \cdots \wedge d\varepsilon N_\varepsilon(z, w_k) \, d\mu^{(k)}(w)$$

$$= \int_{(\mathbb{C}^n)^k} \left( \int_{z \in \mathbb{C}^n} N_\varepsilon(z, w_1) d\varepsilon N_\varepsilon(z, w_2) \wedge \cdots \wedge d\varepsilon N_\varepsilon(z, w_k) \right) \, d\mu^{(k)}(w),$$

where $w := (w_1, \cdots, w_k) \in (\mathbb{C}^n)^k$.

Integrating by parts and applying again Fubini's theorem we obtain

$$A_\varepsilon = \int_{z \in \mathbb{C}^n} V_\varepsilon(z) \int_{(\mathbb{C}^n)^{k-1}} d\varepsilon N_\varepsilon(z, w_2) \wedge \cdots \wedge d\varepsilon N_\varepsilon(z, w_k) \, d\mu^{(n-1)}(w').$$

where $w' := (w_2, \cdots, w_k) \in \mathbb{C}^{k-1}$.

Using Fubini's theorem and integrating parts once again, we obtain when $k \geq 2$

$$A_\varepsilon = \int_{(\mathbb{C}^n)^{k-2}} \left( \int_{z \in \mathbb{C}^n} N_\varepsilon(z, w_2) d\varepsilon N_\varepsilon(z, w_3) \wedge \cdots \wedge d\varepsilon N_\varepsilon(z, w_k) \right) \, d\mu^{(k-2)}(w').$$

Repeating this process $k$ times we get the final equation

$$(5.5) \quad \int_{\mathbb{C}^n} \chi \wedge \int_{(\mathbb{C}^n)^k} d\varepsilon N_\varepsilon(\cdot, w_1) \wedge \cdots \wedge d\varepsilon N_\varepsilon(\cdot, w_k) \, d\mu^{(k)}(w)$$

$$= \int_{\mathbb{C}^n} \chi \wedge (d\varepsilon V_\varepsilon')^k.$$ 

Now we want to pass to the limit as $\varepsilon \searrow 0$. The first term can be written as follows:

$$\int_{\mathbb{C}^n} \chi \wedge \int_{(\mathbb{C}^n)^k} d\varepsilon N_\varepsilon(\cdot, w_1) \wedge \cdots \wedge d\varepsilon N_\varepsilon(\cdot, w_k) \, d\mu^{(k)}(w_1, \cdots, w_k)$$

$$= \int_{(\mathbb{C}^n)^k} I_\varepsilon(w_1, \cdots, w_k) \, d\mu^{(k)}(w_1, \cdots, w_k),$$

where

$$I_\varepsilon(w_1, \cdots, w_k) := \int_{\mathbb{C}^n} \chi \wedge d\varepsilon N_\varepsilon(\cdot, w_1) \wedge \cdots \wedge d\varepsilon N_\varepsilon(\cdot, w_k).$$
Observe that for any fixed \((w_1, \cdots, w_k) \in (\mathbb{C}^n)^k\)

\[
\text{dd}^c N_{\varepsilon}(\cdot, w_1) \land \cdots \land \text{dd}^c N_{\varepsilon}(\cdot, w_k) \to \text{dd}^c N(\cdot, w_1) \land \cdots \land \text{dd}^c N(\cdot, w_k)
\]

weakly in the sense of currents on \(\mathbb{C}^n\) as \(\varepsilon \to 0\). Hence the family of functions

\[
I_{\varepsilon}(w_1, \cdots, w_k)
\]

are uniformly bounded on \((\mathbb{C}^n)^n\) and by Fubini’s theorem, it converges as \(\varepsilon \to 0\) pointwise to the function

\[
I(w_1, \cdots, w_k) := \int_{\mathbb{C}^n} \chi \land \text{dd}^c N(\cdot, w_1) \land \cdots \land \text{dd}^c N(\cdot, w_k).
\]

Therefore by Lebesgue convergence theorem we conclude that

\[
\lim_{\varepsilon \to 0} \int_{(\mathbb{C}^n)^k} I_{\varepsilon}(w_1, \cdots, w_k) d\mu^{(k)} = \int_{(\mathbb{C}^n)^k} I(w_1, \cdots, w_k) d\mu^{(k)}
\]

\[= \int_{\mathbb{C}^n} \chi \land \left( \int_{(\mathbb{C}^n)^k} \text{dd}^c N(\cdot, w_1) \land \cdots \land \text{dd}^c N(\cdot, w_k) d\mu^{(k)} \right)(w_1, \cdots, w_k).
\]

For the second term in \((5.5)\), observe that since \(V_{\mu, \varepsilon} \lesssim V_{\mu}\) as \(\varepsilon \to 0\), it follows by the convergence theorem that the second term converges also and

\[
\int_{\mathbb{C}^n} \chi \land (\text{dd}^c V_{\mu, \varepsilon})^k \to \int_{\mathbb{C}^n} \chi \land (\text{dd}^c V_{\mu})^k
\]

as \(\varepsilon \to 0\).

Now passing to the limit in \((5.5)\), we obtain the required statement. \(\Box\)

We will need a more general result. Let \(\phi\) be a plurisubharmonic function in \(\mathbb{C}^n\) such that \(\phi \in DMA_{loc}(\mathbb{C}^n)\). We define the twisted potential associated to a Probability measure \(\mu\) by setting

\[
V_{\mu}^{\phi} := V_{\mu} + \phi.
\]

Then we have the following representation formula:

\[
V_{\mu}^{\phi}(z) = \int_{\mathbb{C}^n} N^{\phi}(z, w)d\mu(w),
\]

where

\[
N^{\phi}(z, w) := N(z, w) + \phi(z), \quad z \in \mathbb{C}^n.
\]

We can prove a similar representation formula for the Monge-Ampère measure of the twisted potential.

**Proposition 5.3.** If \(\mu\) be a probability measure on \(\mathbb{C}^n\) and \(\phi \in DMA_{loc}(\mathbb{C}^n)\) then the complex Monge-Ampère currents associated to \(V_{\mu}^{\phi}\) are given by the following formula: for any \(1 \leq k \leq n\),

\[
(dd^c V_{\mu}^{\phi})^k = \int_{(\mathbb{C}^n)^k} dd^c N^{\phi}(\cdot, w_1) \land \cdots \land dd^c N^{\phi}(\cdot, w_k)d\mu(w_1) \cdots d\mu(w_k),
\]

in the sense of \((k, k)\)-currents on \(\mathbb{C}^n\).

The proof is the same as above.
5.2. The regularizing property. We prove a regularizing property of the operator $V$ which generalizes and improves a result of Carlehed [Carlehed99]. Recall that a positive measure $\mu$ on $\mathbb{C}^n$ is said to have an atom at some point $a \in \mathbb{C}^n$ if $\mu(\{a\}) > 0$.

**Theorem 5.4.** Let $\mu$ be a probability measure on $\mathbb{C}^n$ and $\phi \in DMA_{loc}(\mathbb{C}^n)$ which is smooth in some domain $B \subset \mathbb{C}^n$. Then the Monge-Ampère current $(dd^c V^\phi_{\mu})^n$ are absolutely continuous with respect to the Lebesgue measure on $B$ if and only if $\mu$ has no atoms on $B$.

The proof of the “if part” of the theorem is based on the following lemma.

**Lemma 5.5.** Assume that $\phi$ is smooth in some domain $B \subset \mathbb{C}^n$ and let $w_1, \ldots, w_n \in \mathbb{C}^n$ be such that $w_1 \neq w_2$. Then the Borel measure

$$dd^c N^\phi(\cdot, w_1) \wedge \cdots \wedge dd^c N^\phi(\cdot, w_n),$$

is absolutely continuous with respect to the Lebesgue measure on $B$.

**Proof.** The proof is based on an idea of Carlehed [Carlehed99]. We are reduced to the proof of the following fact: let $a, b_1, \ldots, b_m \in \mathbb{C}^n$ with $1 \leq m \leq n - k$ such that $b_j \neq a$, for $1 \leq j \leq m$, then the following current

$$\mu_{m,k} := (dd^c N^\phi(\cdot, a))^k \wedge_{1 \leq j \leq m} dd^c N^\phi(\cdot, b_j).$$

is absolutely continuous with respect to the Lebesgue measure on $B$.

Indeed, since the current $\mu_k$ is smooth in $B \setminus \{a, b_1, \ldots, b_m\}$, it is enough to show that $\mu_{m,k}$ puts no mass at the points $a, b_1, \ldots, b_m$ that belong to $B$. Assume for simplicity that $a \in B$ and define the function

$$u(z) = \log(|z - a|^2 + |z \wedge a|^2) + \phi(z) + \sum_{j=1}^m N^\phi(z, b_j).$$

Observe that $u$ is a plurisubharmonic function in $\mathbb{C}^n$ such that $u \in DMA_{loc}(\mathbb{C}^n)$. Since $u(z) \simeq \log(|z - a|^2 + |z \wedge a|^2)$ as $z \to a$, it follows from Demailly’s comparison theorem ([De93]) that

$$\int_{\{a\}} (dd^c u)^n = \int_{\{a\}} \left(dd^c \log(|\cdot - a|^2 + |\cdot \wedge a|^2)\right)^n = 1.$$

On the other hand, performing the exterior product, we obtain

$$(dd^c u)^n \geq \left(dd^c \log(|\cdot - a|^2 + |\cdot \wedge a|^2)\right)^n + \mu_k \geq \delta_a + \mu_{m,k},$$

in the weak sense on $\mathbb{C}^n$. Therefore $\mu_{m,k}(\{a\}) = 0$, which proves the required statement and the Lemma follows. $\square$

We now prove the theorem.
Proof. Assume first that the measure $\mu$ has an atom at some point $w \in \mathbb{C}^n$ then $\mu \geq c\delta_w$, where $c := \mu(\{w\}) > 0$. Therefore from the definition we see that $\forall z \in \mathbb{C}^n$

$$V^\phi_\mu(z) \leq cN(z, w) + c\phi(z).$$

This implies that $V^\phi_\mu$ has a positive Lelong number at $w$ at least equal to $c$ and then the Monge-Ampère mass satisfies $(dd^cV^\phi_\mu)^n(\{w\}) \geq c^n > 0$ (see [Ce04]).

Assume now that $\mu$ has no atoms in $B$. We want to show that $(dd^cV^\phi_\mu)^n$ is absolutely continuous with respect to Lebesgue measure on $B$. Indeed, let $K \subset B$ be a compact set such that $\lambda_{2n}(K) = 0$ and let us prove that $(dd^cV^\phi_\mu)^n(K) = 0$. Set

$$\Delta_B := \{(w, \ldots, w) : w \in B\} \subset \mathbb{C}^n \times \cdots \times \mathbb{C}^n.$$

By Fubini theorem, denoting by $w = (w_1, \ldots, w_n) \in (\mathbb{C}^n)^n$, we get

$$\int_K (dd^cV^\phi_\mu)^n = \int_K \int_{(\mathbb{C}^n)^n} dd^cN^\phi(\cdot, w_1) \wedge \cdots \wedge dd^cN^\phi(\cdot, w_n) d\mu^{(n)}(w)$$

$$= \int_K \int_{(\mathbb{C}^n)^n \setminus \Delta_B} dd^cN^\phi(\cdot, w_1) \wedge \cdots \wedge dd^cN^\phi(\cdot, w_n) d\mu^{(n)}(w)$$

$$+ \int_K \int_{\Delta_B} dd^cN^\phi(\cdot, w_1) \wedge \cdots \wedge dd^cN^\phi(\cdot, w_n) d\mu^{(n)}(w).$$

Since $\mu$ puts no mass at any point in $B$, it follows from Fubini’s theorem that $\Delta_B$ has a zero measure with respect to the product measure $\mu^{(n)} = \mu \otimes \cdots \otimes \mu$ on $(\mathbb{C}^n)^n$. Hence we have

$$\int_K (dd^cV^\phi_\mu)^n = \int_K \left( \int_{(\mathbb{C}^n)^n \setminus \Delta_B} dd^cN^\phi(\cdot, w_1) \wedge \cdots \wedge dd^cN^\phi(\cdot, w_n) d\mu^{(n)}(w_1, \ldots, w_n) \right)$$

$$= \int_{(\mathbb{C}^n)^n \setminus \Delta_B} \int_K dd^cN^\phi(\cdot, w_1) \wedge \cdots \wedge dd^cN^\phi(\cdot, w_n) d\mu^{(n)}(w_1, \ldots, w_n).$$

We set

$$f(w_1, \ldots, w_n) = \int_K dd^cN^\phi(\cdot, w_1) \wedge \cdots \wedge dd^cN^\phi(\cdot, w_n).$$

Then using the previous lemma, we see that if $(w_1, \ldots, w_n) \notin \Delta_B$, the measure

$$dd^cN^\phi(\cdot, w_1) \wedge \cdots \wedge dd^cN^\phi(\cdot, w_n),$$

is absolutely continuous with respect to the Lebesgue measure on $B$. Hence $f(w_1, \ldots, w_n) = 0$ if $(w_1, \ldots, w_n) \notin \Delta_B$ and then

$$\int_K (dd^cV^\phi_\mu)^n = \int_{(\mathbb{C}^n)^n \setminus \Delta_B} f(w_1, \ldots, w_n) d\mu^{(n)}(w_1, \ldots, w_n) = 0$$

and the theorem is proved. \qed
6. Proofs of Theorems

6.1. Localization of the potential. To study the projective logarithmic potential we will localize it in the affine charts and use the previous results. Let \((\chi_j)_{0 \leq j \leq n}\) a fixed partition of unity subordinated to the covering \((U_j)_{0 \leq j \leq n}\). We define \(m_j := \int \chi_j d\mu\) and \(I_\mu := \{j \in \{0, \ldots, n\}; m_j \neq 0\}\). Then \(I_\mu \neq \emptyset\) and for \(j \in I_\mu\), the measure \(\mu_j := (1/m_j)\chi_j \mu\) is a probability measure on \(\mathbb{P}^n\) supported in the chart \(U_j\) and we have the following convex decomposition of \(\mu_\mu\)

\[
\mu = \sum_{j \in I} m_j \mu_j.
\]

Therefore the potential \(G_\mu(\zeta)\) can be written as

\[
G_\mu(\zeta) = \sum_{j \in I} m_j G_{\mu_j}(\zeta), \quad \zeta \in \mathbb{P}^n,
\]

so that we are reduced to the case of a compact measure supported in an affine chart.

Without loss of generality we may always assume the \(\mu\) is compactly supported in \(U_0\). Then the potential \(G_\mu\) can be written as follows

\[
G_\mu(\zeta) := \int_{U_0} G(\zeta, \eta) d\mu(\eta) = (1/2) \int_{U_0} \log \left| \frac{\zeta \land \eta}{\zeta^2 \eta^2} \right| d\mu(\eta).
\]

Since we integrate on \(U_0\) we have \(\eta_0 \neq 0\) and we can use the affine coordinates

\[
\eta := (\eta_1/\eta_0, \ldots, \eta_n/\eta_0).
\]

Therefore

\[
|\zeta \land \eta|^2 = \sum_{1 \leq j \leq n} |\zeta_0 w_j - \zeta_j|^2 + \sum_{1 \leq i < j \leq n} |\zeta_i w_j - \zeta_j w_i|^2.
\]

Since the measure \(\mu\) is supported on \(U_0\), the potential \(G_\mu(\zeta)\) is smooth outside the compact set \(\text{Supp} \mu\) and we are reduced to the study of the potential \(G_\mu(\zeta)\) on the open set \(U_0\).

The restriction of \(G(\zeta, \eta)\) to \(U_0 \times U_0\) can be expressed in the affine coordinates as follows:

Set

\[
z := (\zeta_1/\zeta_0, \ldots, \zeta_n/\zeta_0);
\]

Then the kernel can be written as

\[
(6.1) \quad G(\zeta, \eta) = (1/2) \log \left( \frac{|z - w|^2 + |z \land w|^2}{(1 + |z|^2)(1 + |w|^2)} \right)
\]

\[
(6.2) \quad = (1/2) \log \left( \frac{|z - w|^2 + |z \land w|^2}{1 + |w|^2} \right) - (1/2) \log(1 + |z|^2)
\]

\[
= N(z, w) - (1/2) \log(1 + |z|^2),
\]
where

\[ N(z, w) := (1/2) \log \frac{|z - w|^2 + |z \wedge w|^2}{1 + |w|^2} \]

is the projective logarithmic kernel on \( \mathbb{C}^n \) which was studied in the previous sections.

6.2. Proof of Theorem A. 1. From the previous localization, it follows that for each \( j \in I \), we have for \( \zeta \in U_j \)

\[ G_{\mu_j}(\zeta) = V_{\mu_j}(z) - (1/2) \log(1 + |z|^2), \]

where \( z := z^j(\zeta) \) are the affine coordinates of \( \zeta \) in \( U_j \). By Theorem 2.10 and Theorem 5.2, it follows that for each \( j \in I \), the function \( G_{\mu_j} \) is \( \omega_{FS} \)-plurisubharmonic and \( \| \nabla G_{\mu_j} \| \in L^p(\mathbb{P}^n) \) for any \( 0 < p < n \). Therefore the convex combination \( G_{\mu} = \sum_{j \in I} G_{\mu_j} \) also satisfies the same properties i.e. \( G_{\mu} \) is \( \omega_{FS} \)-plurisubharmonic and \( \| \nabla G_{\mu} \| \in L^p(\mathbb{P}^n) \) for any \( 0 < p < n \).

To study the complex Hessian \( (\omega + dd^c G_{\mu})^k \), it is enough to localize to a small ball \( B \subset U_j \simeq \mathbb{C}^n (0 \leq j \leq n) \) such that \( \mu(B) > 0 \).

Then if we set \( \mu_B := 1_B \mu \), we have

\[ \mu = s\mu_B + (1 - s)\nu, \]

where \( 0 < s \leq 1 \) is a positive number, \( \mu_B \) is a probability measure supported on \( B \) and \( \nu \) is a probability measure supported on the complement of \( B \).

Therefore we have

\[ G_{\mu} = sG_{\mu_B} + (1 - s)G_{\nu}, \]

where \( G_{\nu} \) is a smooth \( \omega \)-psh function in \( B \), since the support of \( \nu \) is contained in the complement of \( B \).

Then working on the coordinates \( z \) in the chart \( U_j \simeq \mathbb{C}^n \), and setting \( \ell(z) := (1/2) \log(1 + |z|^2) \), the local potential of \( \omega \) in \( U_j \), we conclude that

\[ G_{\mu} + \ell = sV_{\mu_B} + (1 - s)V_{\nu} = V_{s\mu_B} + \phi, \]

where \( \phi \) is a plurisubharmonic function in \( \mathbb{C}^n \) which is smooth in \( B \).

By Theorem 5.3 (5.2) it follows that \( W_B := V_{s\mu_B} + \phi \in PSH(B) \cap W^{2,p}(B) \) for any \( 0 < p < n \). Let \( 1 \leq k \leq n - 1 \). Since for any \( 1 \leq i, j \leq n \),

\[ \partial^2 W_B/\partial z_i \partial \bar{z}_j \in L^p(B), \forall 0 < p < n, \]

it follows by Hölder inequality that the coefficients of the current \( (dd^c W_B)^k \) are in \( L^r(B) \) as far as \( r \geq 1 \) and \( 1/r = k/p \) i.e. \( r = p/k \geq 1 \). This is possible by choosing \( p \) such that \( k < p < n \) since \( 1 \leq k < n \). Therefore \( (dd^c W^k) \wedge \omega^{n-k} \) is absolutely continuous w.r.t. the Lebesgue measure on \( B \) with a density in \( L^r(B) \) for any \( 1 < r < n/k \).

On the other hand

\[ G_{\mu} + \ell = V_{s\mu_B} + \phi = V_{s\mu_B}^\phi, \]

is a twisted logarithmic potential. Therefore by Theorem 5.4 it follows that

\[ (\omega + dd^c G_{\mu})^n = (dd^c V_{s\mu_B}^\phi)^n, \]
is absolutely continuous w.r.t. the Lebesgue measure on $B$. This proves Theorem A.

6.3. **Proof of Theorem B.** We will use Lemma 3.4. By the localization principle, it is enough to assume that $\mu$ is compactly supported in a chart $U_j \simeq \mathbb{C}^n$. Then in the affine coordinates we have

$$G_\mu(\zeta) = V_\mu(z) - (1/2) \log(1 + |z|^2),$$

in $\mathbb{C}^n$.

From (5.1) we see that locally $|\nabla V_\mu|$ is dominated by the Riesz potential $J_{\mu,1}$ and using the Lemma 3.4 with $\alpha = 1$ we obtain the first statement of the theorem about the gradient of $V_\mu$. The Hölder continuity of $G_\mu$ follows then by the classical lemma of Sobolev-Morrey.

The second statement is proved in the same way. Indeed we have $\omega + dd^c G_\mu = dd^c V_\mu$, and by (5.3), the coefficients of the current $dd^c V_\mu$ are locally dominated by the Riesz potential $J_{\mu,2}$. Again using Lemma 3.4 with $\alpha = 2$, we conclude that

$$dd^c V_\mu \in L^p_{2,\text{loc}}(\mathbb{C}^n), \forall p < \frac{2n - s}{(2 - s)},$$

the second statement of the theorem follows by Hölder inequality.

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