Canonical and Lie-algebraic twist deformations of Galilei algebra

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Abstract

We describe various nonrelativistic contractions of two classes of twisted Poincare algebra: canonical one ($\theta_{\mu\nu}$-deformation) and the one leading to Lie-algebraic models of noncommutative space-times. The cases of contraction-independent and contraction-dependent twist parameters are considered. We obtain five models of noncommutative nonrelativistic space-times, in particular, two new Lie-algebraic nonrelativistic deformations of space-time, respectively, with quantum time/classical space and with quantum space/classical time.
1 Introduction

In the last decade the interest in a class of theories describing kinematics different from Special Relativity is growing rapidly. The main reason for such type of considerations follows from many phenomenological suggestions, which state that relativistic space-time symmetries should be modified (deformed) at Planck scale, while the classical Poincaré invariance still remains valid at larger distances [1]-[4]. There are also several formal arguments based mainly on quantum gravity [5], [6] and string theory [7], [8] indicating that space-time at Planck-length should be noncommutative, i.e. it should have a quantum nature.

An important candidate for a modification of relativistic symmetry in ultra-high energy regime, so-called $\kappa$-deformed Poincaré algebra, has been proposed in the Hopf algebraic framework of quantum groups in [9], [10]. As a result of contraction of $q$-deformed anti-De-Sitter algebra one obtains $\kappa$-Poincaré group which leads to Lie-algebraic space-time noncommutativity, the $\kappa$-Minkowski space [11], [12], with the mass-like deformation parameter $\kappa$. Besides, it also gives a formal framework for such theoretical constructions as Double Special Relativity (see e.g. [13]-[16]), which postulates two observer-independent scales, of velocity, describing the speed of light, and of mass, which can be identify with $\kappa$-parameter and is expected to be of the order of fundamental Planck mass.

In accordance with the general classification of all possible deformations of the Poincaré group [17], there exist another two interesting modifications of relativistic symmetries. First of them corresponds to the so-called soft twisted Poincaré Hopf algebra, and leads to very popular at present, canonical commutation relations for quantum Minkowski space [18]-[23]

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} ; \quad \theta_{\mu\nu} = \text{const} .$$

The second deformation, associated with deformation as well generated by twist, is closer to the $\kappa$-deformed case as it introduces the Lie-algebraic type of space-time noncommutativity ([25]; see also [24])

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta^0_{\mu\nu} x_\rho ,$$

with particularly chosen coefficients $\theta^0_{\mu\nu}$ being constants. Nevertheless, it should be noted that such a Lie-algebraic modification of relativistic symmetry looks simpler than the $\kappa$-deformed one. Its classical r-matrix satisfies the classical Yang-Baxter equation, and it can be generated by a twist transformation of the undeformed Poincaré Hopf structure [26]. Besides, in contrast to the $\kappa$-deformed case, the explicit form of the corresponding quantum R-matrix is known.

The deformations (1) and (2) are supposed to describe the kinematics of a very fast (relativistic) object in ultra-high energy regime. However, one can ask about their non-relativistic limit, i.e. about their deformed Galilei counterparts describing much slower objects in the transplanckian region. In the case of $\kappa$-Poincaré algebra there exist two nonrelativistic contractions to Galilei group depending on the way in which we embed parameter $c$ in $\kappa$ [27]-[29]. First contraction uses substitution $\kappa = \hat{\kappa}/c$ and in the contraction limit $c \to \infty$ follows the $\kappa$-deformed Galilei algebra firstly studied in the article.
The second one ($\kappa$ is replaced by $\hat{\kappa}c$) has been investigated in the context of so-called extended deformed Galilei Hopf structure ([28], [29], see also [30]-[33]), and in the presence of additional central generator leads to the extended $\kappa$-deformed Galilei group [28]. The main consequences of these contracted algebras, for example the type of corresponding noncommutative space-times or differential calculi, have been studied in [28], [29]. Besides, it was also shown (see e.g. [34]) that the above quantum groups, similarly as their relativistic counterparts, exhibit the bicross-product structure [36].

The main aim of present paper is to consider nonrelativistic quantum contractions with $c$-dependent twist parameters in context of canonical and Lie-algebraic twisted Poincaré group [19], [25]. In such a way we find three new, deformed Galilei Hopf algebras, which can be also obtained by twisting of the classical Galilei Hopf algebras. Further, we also investigate the simplest type of relativistic contraction with $c$-independent deformation parameter [37] (see also [38]). In such a way we recover two quantum groups corresponding to the soft- and Lie-twisted Poincare algebras - the soft- and Lie-twisted Galilei algebras. Finally, we derive the corresponding noncommutative ”space-times” describing the representation spaces (Hopf modules) of our Galilean twisted quantum groups [39], [19], [20].

The paper is organized as follows. In second Section we recall basic facts concerning the soft and Lie-algebraic twisted relativistic symmetries. Sections three and four are devoted to their nonrelativistic contractions to Galilei quantum groups with contraction-independent and contraction-dependent twist parameter, respectively. In Section five we derive the corresponding nonrelativistic deformed space-times. The results are discussed and summarized in the last Section.

2 Canonical and Lie-algebraic twisted Poincaré algebra

In this section we review some known facts concerning the soft and Lie-algebraic twisted classical Poincaré algebras (see [19], [20], [25]), which we denote by $U_\theta(P)$ and $U_\kappa(P)$, respectively.

First of all, let us start with canonical twisted modification of relativistic symmetry with carrier algebra described by commuting fourmomentum generators $P_\mu$. According to [17] (see also [18]) its classical $r$-matrix looks as follows

$$r_\theta = \frac{1}{2} \theta^{\mu\nu} P_\mu \wedge P_\nu ,$$

with $a \wedge b = a \otimes b - b \otimes a$ and $\theta^{\mu\nu} = -\theta^{\nu\mu}$. Such Abelian $r$-operator satisfies trivially the classical Yang-Baxter equation (CYBE)

$$[[r_\theta, r_\theta]] = [r_{\theta 12}, r_{\theta 13} + r_{\theta 23}] + [r_{\theta 13}, r_{\theta 23}] = 0 ,$$

1This contraction is rather of mathematical nature. If we relate e.g. $\kappa$ with Planck mass, this contraction corresponds to vanishing Planck mass.
where the symbol \([\cdot,\cdot]\) denotes the Schouten bracket while \(r_{\theta 12} = \frac{1}{2}\theta^{\mu\nu} P_\mu \wedge P_\nu \wedge 1\), \(r_{\theta 13} = \frac{1}{2}\theta^{\mu\nu} P_\mu \wedge 1 \wedge P_\nu\) and \(r_{\theta 23} = \frac{1}{2}\theta^{\mu\nu} 1 \wedge P_\mu \wedge P_\nu\). We get the \(\theta^{\mu\nu}\)-deformed Poincaré algebra by twisting of the classical coproducts with use of the twist factor

\[
\mathcal{F}_\theta = \exp \frac{i}{2} (\theta^{\mu\nu} P_\mu \wedge P_\nu) ,
\]

satisfying the classical cocycle condition \([40]\)

\[
\mathcal{F}_{\theta 12} \cdot (\Delta_0 \otimes 1) \mathcal{F}_\theta = \mathcal{F}_{\theta 23} \cdot (1 \otimes \Delta_0) \mathcal{F}_\theta ,
\]

and the normalization one

\[
(\epsilon \otimes 1) \mathcal{F}_\theta = (1 \otimes \epsilon) \mathcal{F}_\theta = 1 ,
\]

with \(\mathcal{F}_{\theta 12} = \mathcal{F}_\theta \otimes 1\) and \(\mathcal{F}_{\theta 23} = 1 \otimes \mathcal{F}_\theta\). The algebraic sector of \(U_\theta(P)\) is not changed, while coproducts and antipodes transform according to

\[
\Delta_\theta(a) \rightarrow \Delta_\theta(a) = \mathcal{F}_\theta \circ \Delta_0(a) \circ \mathcal{F}_\theta^{-1} ,
\]

\[
S_\theta(a) = u(\theta) S_0(a) u^{-1}(\theta) ,
\]

where \(\Delta_0(a) = a \otimes 1 + 1 \otimes a\), \(S_0(a) = -a\) and \(u(\theta) = \sum f(1) S_0(f(2))\) (we use Sweedler’s notation \(\mathcal{F} = \sum f(1) \otimes f(2)\)). The twisted Hopf algebra \(U_\theta(P)\) has the form \((\eta_{\mu\nu} = (-,+,+,+))\)

\[
[M_{\mu\nu}, M_{\rho\sigma}] = i (\eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma}) ,
\]

\[
[M_{\mu\nu}, P_\rho] = i (\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu) , \quad [P_\mu, P_\nu] = 0 ,
\]

with coalgebra

\[
\Delta_\theta(P_\mu) = \Delta_0(P_\mu) ,
\]

\[
\Delta_\theta(M_{\mu\nu}) = \mathcal{F}_\theta \circ \Delta_0(M_{\mu\nu}) \circ \mathcal{F}_\theta^{-1}
= \Delta_0(M_{\mu\nu}) - \theta^{\rho\sigma} [(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \otimes P_\sigma
+ P_\rho \otimes (\eta_{\sigma\mu} P_\nu - \eta_{\sigma\nu} P_\mu)] ,
\]

and the classical antipodes and counits

\[
S_\theta(P_\mu) = -P_\mu , \quad S_\theta(M_{\mu\nu}) = -M_{\mu\nu} , \quad \epsilon(M_{\mu\nu}) = \epsilon(P_\mu) = 0 .
\]

Let us now turn to the Lie-algebraic twist of Poincare algebra \(U_\epsilon(P)\) \([25]\) (see also \([24]\)) with carrier algebra described by three commuting generators \(M_{\alpha\beta}\), \(P_\epsilon\) where \(\epsilon \neq \alpha, \beta\) and where the indices \(\alpha, \beta\) are fixed. The corresponding classical \(r\)-matrix has the form \([17]\)

\[
r_\epsilon = \frac{1}{2} \zeta^\lambda P_\lambda \wedge M_{\alpha\beta} ,
\]
with the vector $\zeta^\lambda$ having vanishing components $\zeta^\alpha$, $\zeta^\beta$. In particular, the $r$-matrix (13) for the special choices $\{M_{\alpha\beta} = M_{03}, \epsilon = 1, 2\}$ and $\{M_{\alpha\beta} = M_{12}, \epsilon = 0, 3\}$ has been discussed some time ago in [24] and also considered very recently in [35]. Obviously, the operator (13) satisfies the classical Yang-Baxter equation (4), and twist factor corresponding to (13) looks as follows

$$F_\zeta = \exp \frac{i}{2} (\zeta^\lambda P_\lambda \wedge M_{\alpha\beta}).$$  

(14)

Using the twist procedure one gets (besides the undeformed algebraic commutation relations (10)) the following coproducts [25]

$$\Delta_\zeta(P_\mu) = \Delta_0(P_\mu) + (-i)^\gamma \sinh(i^\gamma \zeta^\lambda P_\lambda) \wedge (\eta_{\alpha\mu} P_\beta - \eta_{\beta\mu} P_\alpha) \quad (15)$$

$$\Delta_\zeta(M_{\mu\nu}) = \Delta_0(M_{\mu\nu}) + M_{\alpha\beta} \wedge \zeta^\lambda (\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu) + i [M_{\mu\nu}, M_{\alpha\beta}] \perp (-1)^{1+\gamma}(\cosh(i^\gamma \zeta^\lambda P_\lambda) - 1)$$

$$+ M_{\alpha\beta}(-i)^\gamma \sinh(i^\gamma \zeta^\lambda P_\lambda) \perp \zeta^\lambda (\psi_\lambda P_\alpha - \chi_\lambda P_\beta)$$

$$+ \zeta^\lambda (\psi_\lambda \eta_{\alpha\alpha} P_\beta + \chi_\lambda \eta_{\beta\beta} P_\alpha) \perp M_{\alpha\beta}(-1)^{1+\gamma}(\cosh(i^\gamma \zeta^\lambda P_\lambda) - 1),$$

(16)

where, in the above formulas $a \perp b = a \otimes b + b \otimes a$,

$$\zeta^\lambda = \frac{\zeta^\lambda}{2}, \quad \psi_\lambda = \eta_{\nu\lambda} \eta_{\beta\mu} - \eta_{\mu\alpha} \eta_{\lambda\nu}, \quad \chi_\lambda = \eta_{\nu\lambda} \eta_{\alpha\mu} - \eta_{\mu\alpha} \eta_{\nu\lambda},$$

and $\gamma = 0$ when $M_{\alpha\beta}$ is a boost or $\gamma = 1$ for a space rotation. Antipodes and counits remain classical. The relations (10) and (15), (16) define the Lie-algebraic twist deformation $U_\zeta(\mathcal{P})$ of Poincaré algebra. Of course, we get the undeformed classical Poincaré Hopf structure $U_0(\mathcal{P})$ in the $\zeta \to 0$ limit.

3 Nonrelativistic contractions of twisted Poincaré algebras: contraction-independent twist parameters

In this section we present the nonrelativistic contractions of Hopf structures described in previous section, i.e. we find their nonrelativistic counterparts - the $\theta^{\mu\nu}$-Galilei algebra $U_0(\mathcal{G})$ and Lie-algebraic one $U_\zeta(\mathcal{G})$. For this purpose let us introduce the following standard redefinition of Poincaré generators [37]

$$P_0 = \frac{\Pi_0}{c}, \quad P_i = \Pi_i, \quad M_{ij} = K_{ij}, \quad M_{i0} = c V_i,$$

(17)

where parameter $c$ describes the light velocity. We begin with canonical twisted algebra $U_\zeta(\mathcal{P})$. In a first step of contraction procedure one can rewrite the algebraic relations
coproducts \( \Pi_0, \Pi_i, K_{ij}, V_i \). Next, one should take a proper nonrelativistic limit \((c \to \infty)\) of rewritten formulas. In such a way we obtain the classical \( D = 4 \) Galilean algebra

\[
[K_{ij}, K_{kl}] = i(\delta_{il}K_{jk} - \delta_{jl}K_{ik} + \delta_{jk}K_{il} - \delta_{ik}K_{jl}),
\]
\[
[K_{ij}, V_k] = i(\delta_{jk}V_i - \delta_{ik}V_j), \quad [K_{ij}, \Pi_\rho] = i(\eta_{j\rho}V_i - \eta_{i\rho}V_j),
\]

\( (18) \)

\[
[V_i, V_j] = [V_i, \Pi_j] = 0, \quad [V_i, \Pi_0] = -i\Pi_i, \quad [\Pi_\rho, \Pi_\sigma] = 0,
\]

and the following deformation of coalgebraic sector

\[
\Delta_\theta(\Pi_\rho) = \Delta_{0}(\Pi_\rho), \quad \Delta_\theta(V_i) = \Delta_{0}(V_i),
\]

\( (19) \)

\[
\Delta_\theta(K_{ij}) = \Delta_{0}(K_{ij}) - \theta^{kl}[(\delta_{ki}V_j - \delta_{kj}V_i) \otimes \Pi_l
+ \Pi_k \otimes (\delta_{li}V_j - \delta_{lj}V_i)].
\]

\( (20) \)

\( (21) \)

The antipodes and counits remain undeformed

\[
S(\Pi_\rho) = -\Pi_\rho, \quad S(K_{ij}) = -K_{ij}, \quad S(V_i) = -V_i,
\]

\( (22) \)

\[
\epsilon(\Pi_\rho) = \epsilon(K_{ij}) = \epsilon(V_i) = 0,
\]

\( (23) \)

where \( k, l = 1, 2, 3 \). The relations \( (18)-(23) \) constitute the Hopf algebra structure which we shall call the canonically deformed Galilei Hopf algebra \( U_\theta(G) \). We note that one can also perform the contraction of classical \( r \)-matrix \( (3) \) and of the twist factor \( (5) \). We get

\[
r_\theta = \frac{1}{2}\theta^{kl}\Pi_k \wedge \Pi_l,
\]

\( (24) \)

\[
K_\theta = \exp i\frac{1}{2}(\theta^{kl}\Pi_k \wedge \Pi_l),
\]

\( (25) \)

respectively. We see that the \( r \)-operator \( (24) \) satisfies the classical Yang-Baxter equation \( (1) \) while the factor \( (5) \) defines twist element, which satisfies the classical cocycle and normalization conditions \( (6), (7) \), and provides the formulas \( (19)-(21) \).

In the case of Lie-type twisted Poincaré algebra \( U_\zeta(P) \) the contraction is more complicated. For carrier algebra \( \{ M_{kl}, P_\gamma; \gamma \neq k, l, 0 \} \) one can check that after contracting the formulas \( (15) \) and \( (16) \) we get the classical Galilei algebraic sector \( (18) \) and the following deformed coproducts

\[
\Delta_\zeta(\Pi_0) = \Delta_{0}(\Pi_0), \quad
\]

\( (26) \)

\[
\Delta_\zeta(\Pi_i) = \Delta_{0}(\Pi_i) + \sin(\zeta \Pi_\gamma) \wedge (\delta_{ki}\Pi_l - \delta_{li}\Pi_k)
+ (\cos(\zeta \Pi_\gamma) - 1) \perp (\delta_{ki}\Pi_k + \delta_{li}\Pi_l),
\]

\( (27) \)
\[
\Delta_\xi(\mathcal{K}_{ij}) = \Delta_0(\mathcal{K}_{ij}) + \mathcal{K}_{kl} \wedge \xi (\delta_{i\gamma} \Pi_j - \delta_{j\gamma} \Pi_i) \\
+ \ i [\mathcal{K}_{ij}, \mathcal{K}_{kl}] \wedge \sin(\xi \Pi_j) \\
+ \ [\mathcal{K}_{ij}, \mathcal{K}_{kl}] \perp (\cos(\xi \Pi_i) - 1) \\
+ \ \mathcal{K}_{kl} \sin(\xi \Pi_j) \perp \xi (\psi_\gamma \Pi_k - \chi_\gamma \Pi_l) \\
+ \ \xi (\psi_\gamma \Pi_i + \chi_\gamma \Pi_k) \wedge \mathcal{K}_{kl} (\cos(\xi \Pi_j) - 1),
\]

(28)

\[
\Delta_\xi(V_i) = \Delta_0(V_i) + i [V_i, \mathcal{K}_{kl}] \wedge \sin(\xi \Pi_j) \\
+ \ [\mathcal{K}_{ij}, \mathcal{K}_{kl}] \perp (\cos(\xi \Pi_i) - 1),
\]

(29)

where

\[\xi = \frac{\zeta}{2}, \quad \psi_\gamma = \delta_{j\gamma} \delta_{i\iota} - \delta_{i\gamma} \delta_{j\iota}, \quad \chi_\gamma = \delta_{j\gamma} \delta_{k\iota} - \delta_{i\gamma} \delta_{k\j} .\]

The antipodes and counits remain classical (see (22), (23)). One can also derive the contraction limit for the \(\zeta\)-deformed classical \(r\)-matrix

\[r_\zeta = \frac{1}{2} \zeta \Pi_\gamma \wedge \mathcal{K}_{kl},\]

(30)

which satisfies the classical Yang-Baxter equation

\[[ [r_\zeta, r_\zeta ]] = 0 .\]

(31)

Similarly, by contracting (13) we get the following twist factor

\[\mathcal{K}_\zeta = \exp \frac{i}{2} (\zeta \Pi_\gamma \wedge \mathcal{K}_{kl}) ,\]

(32)

which can be used to get \(U_\zeta(\mathcal{G})\) by twisting the classical Galilei Hopf algebra \(U_0(\mathcal{G})\).

In the case \(\{ \mathcal{M}_{kl}, \mathcal{P}_0 (\gamma = 0) \}\) the contracted algebra is trivial, i.e. its coproducts are primitive while its classical \(r\)-matrix disappears

\[r_\zeta = 0 , \quad \Delta_\zeta(a) = \Delta_0(a) .\]

(33)

On the other side, for the carrier algebra containing boosts \(\{ \mathcal{M}_{k0}, \mathcal{P}_l ; k \neq l \}\) the contraction limit (17) does not exist. In fact, one can check that the coproducts for rotations and corresponding classical \(r\)-matrix are divergent in the contraction limit \(c \to \infty\).

4 Nonrelativistic contractions of twisted Poincaré algebras: contraction-dependent twist parameters

Let us now look at the contraction with light velocity \(c\) hidden in the deformation parameter of quantum group. As already mentioned in Introduction such a type of contraction has been investigated in the case of \(\kappa\)-Poincaré Hopf algebra [28, 29].
We begin with the canonical deformation $U_\theta(P)$ with $\theta^{\mu\nu} = \hat{\theta}^{\mu\nu}/\kappa$ ($[\kappa] = L^{-1}$). Let us introduce the parameter $\hat{\kappa}$ such that $\kappa = \hat{\kappa}/c$ ($[\hat{\kappa}] = T^{-1}$). We perform the contraction limit of co-sector (11) in two steps. Firstly, we rewrite the formula (11) in term of the operators (17) and parameter $\hat{\kappa}$. Next, we take the $c \to \infty$ limit and in such a way, in the case $\theta^{ij} = 0$ and $\theta^{0i} = \hat{\theta}^{0i}/\hat{\kappa}$, one gets

$$\lim_{c \to \infty} \Delta_{\theta^{0i}}(\Pi_\mu) = \Delta_0(\Pi_\mu),$$  \hspace{1cm} (34)

$$\lim_{c \to \infty} \Delta_{\theta^{0i}}(K_{ij}) = \Delta_0(K_{ij}) - \frac{\hat{\theta}^{0k}}{\hat{\kappa}}\Pi_0 \wedge (\delta_{k,i}\Pi_j - \delta_{k,j}\Pi_i),$$  \hspace{1cm} (35)

$$\lim_{c \to \infty} \Delta_{\theta^{0i}}(V_i) = \Delta_0(V_i) - \frac{\hat{\theta}^{0k}}{\hat{\kappa}}\Pi_i \wedge \Pi_k.$$  \hspace{1cm} (36)

The algebraic relations (10) and coproducts (34)-(36) constitute the canonical deformed Galilei Hopf algebra $U_{\hat{\kappa}}(G)$ with the corresponding classical $r$-matrix

$$r_{\hat{\kappa}} = \frac{\hat{\theta}^{0k}}{\hat{\kappa}}\Pi_0 \wedge \Pi_k,$$  \hspace{1cm} (37)

and the following twist factor

$$K_{\hat{\kappa}} = \exp \left( \frac{i\hat{\theta}^{0k}}{\hat{\kappa}}\Pi_0 \wedge \Pi_k \right).$$  \hspace{1cm} (38)

Unfortunately, one can check that for $\theta^{ij} \neq 0$ the contracted coproducts become in the limit $c \to \infty$ divergent.

If we introduce a new deformation parameter $\pi; \kappa = \pi c$ ($[\pi] = TL^{-2}$), one can repeat the above contraction procedure. In such a way, in the contraction limit, we obtain however the undeformed Galilei Hopf algebra $U_0(G)$.

In the case of Lie-algebraic space-time noncommutativity corresponding to $U_\zeta(P)$ with $\zeta = \frac{1}{L}$ the situation is more complicated. For $\kappa = \hat{\kappa}/c$ and carrier $\{M_{kl}, P_0\}$, we get the Lie-algebraic Galilei algebra $U_\hat{\kappa}(G)$ with classical algebraic sector (18). Its coproducts for momentum and rotation are given by (26)-(28) with $\gamma = 0$, $\zeta = \frac{1}{\hat{\kappa}}$, while for boost it looks as follows

$$\Delta_\zeta(V_i) = \Delta_0(V_i) + \frac{1}{2\hat{\kappa}}K_{kl} \wedge \Pi_i + i[V_i, K_{kl}] \wedge \sin\left(\frac{1}{2\hat{\kappa}}\Pi_0\right)$$

$$+ \left[[V_i, K_{kl}], K_{kl}\right] \perp (\cos\left(\frac{1}{2\hat{\kappa}}\Pi_0\right) - 1)$$

$$+ K_{kl} \sin\left(\frac{1}{2\hat{\kappa}}\Pi_0\right) \perp \frac{1}{2\hat{\kappa}}(\delta_{k,i}\Pi_l - \delta_{k,l}\Pi_i)$$

$$- \frac{1}{2\hat{\kappa}}(\delta_{k,i}\Pi_k + \delta_{k,i}\Pi_l) \wedge K_{kl}(\cos\left(\frac{1}{2\hat{\kappa}}\Pi_0\right) - 1).$$  \hspace{1cm} (39)

One can also check that its classical $r$-matrix has the form

$$r_{\hat{\kappa}}^{(k,l)} = \frac{1}{2\hat{\kappa}}\Pi_0 \wedge K_{kl}.$$  \hspace{1cm} (40)
and the twist element is given by

\[ K_{(k,l)}^\kappa = \exp \frac{i}{2\kappa} (\Pi_0 \wedge K_{kl}) \, . \]  

(41)

In the case \( \{ M_{kl}, P_\gamma ; \gamma \neq k, l, 0 \} \) it is easy to see that the coproducts for rotations do not have the contraction limits. The similar situation appears for "boost-like" carrier \( \{ M_{k0}, P_l ; k \neq l \} \)- the rotation coproduct is divergent.

For \( \kappa = \overline{\kappa}c \), by simple calculation one can check that in the case of "space-like" carrier \( \{ M_{kl}, P_\gamma ; \gamma \neq k, l \} \) we get the undeformed Galilei algebra \( \mathcal{U}_0(\mathcal{G}) \), i.e. its contracted \( r \)-matrix disappears \( (r = 0) \) and the coproducts remain primitive. For "boost-like" carrier \( \{ M_{k0}, P_l ; k \neq l \} \) situation is less-trivial - the contracted Galilei Hopf algebra \( \mathcal{U}_\kappa(\mathcal{G}) \) is given by the classical commutation relations (18) and the following coproducts

\[
\Delta_\kappa(\Pi_0) = \Delta_0(\Pi_0) + \frac{1}{2\kappa} \Pi_l \wedge \Pi_k ,
\]

(42)

\[
\Delta_\kappa(\Pi_i) = \Delta_0(\Pi_i) , \quad \Delta_\kappa(V_i) = \Delta_0(V_i) ,
\]

(43)

\[
\Delta_\kappa(K_{ij}) = \Delta_0(K_{ij}) + \frac{i}{2\kappa} [K_{ij}, V_k] \wedge \Pi_l + \frac{1}{2\kappa} V_k \wedge (\delta_{il} \Pi_j - \delta_{lj} \Pi_i) ,
\]

(44)

By straightforward calculations one can check that the above quantum group can be obtain by twist transformation of the classical Galilei algebra with use of the following factor

\[ K^\kappa_{(l,k)} = \exp \frac{i}{2\kappa} (\Pi_l \wedge V_k) . \]

(45)

5 Noncommutative Galilei space-times

In previous sections we found by nonrelativistic contractions five deformations of Galilei Hopf algebras. Two of them correspond to the simplest contraction (17) with \( c \)-independent parameter of deformation (Section 3). They are given by:

i) the deformed relations (19)-(21) for canonically deformed \( \mathcal{U}_\theta(\mathcal{G}) \),

ii) the deformed formulas (26)-(29) for Lie-algebraic type deformation with carrier \( \{ M_{kl}, P_\gamma ; \gamma \neq k, l, 0 \} - \mathcal{U}_\kappa(\mathcal{G}) \).

We also get the quantum groups corresponding to the contraction (17) with the parameter \( c \) hidden in \( \kappa = \hat{\kappa}/c \) (see Section 4). In the case of canonical deformation we obtain:

iii) the quantum Galilei group \( \mathcal{U}_\kappa(\mathcal{G}) \) equipped with the deformed coproducts (34)- (36).
For Lie-algebraic noncommutativity, in the case of carrier \( \{ M_{kl}, P_0 \} \), we get:

iv) the deformed coproducts (26)-(28) with \( \gamma = 0 \) supplemented by the formula (39).

Finally, if we perform the contraction (17) with parameter \( \kappa = \kappa_c \) we have:

v) the Galilei Hopf algebra \( U_{\kappa_c}(G) \) with deformed coproducts (42)-(44) for \( \{ M_{k0}, P_l \; ; \; k \neq l \} \).

In this section we construct nonrelativistic space-times corresponding to the Galilei quantum groups i) - v\(^2\). They are defined as the quantum representation spaces (Hopf modules) for quantum Galilei algebras, with action of the deformed symmetry generators satisfying suitably deformed Leibnitz rules [39], [19], [20]. The action of Galilei group on a Hopf module of functions depending on space-time coordinates \((t, x_i)\) is given by

\[
\Pi_0 \triangleright f(t, \mathbf{x}) = i \partial_t f(t, \mathbf{x}) \quad \Pi_i \triangleright f(t, \mathbf{x}) = i \partial_i f(t, \mathbf{x}),
\]

and

\[
K_{ij} \triangleright f(t, \mathbf{x}) = i (x_i \partial_j - x_j \partial_i) f(t, \mathbf{x}) \quad V_i \triangleright f(t, \mathbf{x}) = it \partial_i f(t, \mathbf{x}),
\]

while the \( \ast \)-multiplication of arbitrary two functions is defined as follows

\[
f(t, \mathbf{x}) \ast \cdot g(t, \mathbf{x}) := \omega \circ \left( \mathcal{K}^{-1} \triangleright f(t, \mathbf{x}) \otimes g(t, \mathbf{x}) \right). \]

In the above formula \( \mathcal{K} \) denotes twist factor corresponding to a proper Galilei group and \( \omega \circ (a \otimes b) = a \cdot b \).

Hence, we have:

i) the twist factor (25) is represented by

\[
\mathcal{K}_\theta = \exp \left( -\frac{i}{2} \theta^{kl} \partial_k \wedge \partial_l \right), \quad (49)
\]

and using the formulas (48), (49) one can check that "time-like" commutators vanish

\[
[ t, x_i ]_{\ast \theta} = t \ast \theta x_i - x_i \ast \theta t = 0 \quad (50)
\]

while "space-like" ones introduce a canonical type of noncommutativity

\[
[ x_i, x_j ]_{\ast \theta} = i \theta^{kl} (\delta_{ik} \delta_{jl} - \delta_{jk} \delta_{il}) = 2i \theta^{ij}. \quad (51)
\]

The relations (50) and (51) describe the nonrelativistic "magnetic" form of known canonical deformation of Minkowski space coordinates.

\(^2\)The similar classification of twist factors and corresponding superspaces in the case of superconformal algebra has been proposed in [43].
ii) For Lie-algebraic deformation $U_\zeta(G)$ the twist factor (32) is represented on Hopf module as follows (see (16), (17))

$$K_\zeta = \exp \left(-\frac{i}{2} \zeta \partial_\gamma \wedge (x_k \partial_l - x_l \partial_k) \right) ; \gamma \neq k, l, t . \tag{52}$$

Using (48) and (52) we obtain the following commutation relations

$$[x_i, x_j]_\zeta = i \zeta \delta_{\gamma j} (\delta_{ki} x_l - \delta_{li} x_k) + i \zeta \delta_{\gamma i} (\delta_{lj} x_k - \delta_{kj} x_l) , \quad [t, x_i]_\zeta = 0 , \tag{53}$$

which define the new model of Lie-algebraic $\zeta$-deformed space-time for Galilei algebra $U_\zeta(G)$.

iii) In this case the twist factor (38) is given by

$$K_\theta = \exp \left(-i \theta^0 \partial_t \wedge \partial_k \right) . \tag{54}$$

while corresponding space-time noncommutativity, due to (48) and (54), has the form

$$[t, x_i]_{\theta} = 2 i \frac{\theta^0}{\kappa} , \quad [x_i, x_j]_{\theta} = 0 , \tag{55}$$

and corresponds to ”electric” deformation of Minkowski space coordinates.

iv) For $U_\kappa(G)$ the twist factor (41) and corresponding space-time are given by

$$K_\kappa = \exp \left(-\frac{i}{2\kappa} \partial_t \wedge (x_k \partial_l - x_l \partial_k) \right) , \tag{56}$$

and

$$[t, x_i]_{\kappa} = \frac{i}{\kappa} (\delta_{li} x_k - \delta_{ki} x_l) , \quad [x_i, x_j]_{\kappa} = 0 , \tag{57}$$

respectively. We see that we obtained by (57) the nonrelativistic variant of $\kappa$-Minkowski space with classical three-space and quantum time.

v) Finally, for quantum group $U_\pi(G)$ we have (see (45))

$$K_\pi = \exp \left(-\frac{i}{2\pi} \partial_t \wedge t \partial_k \right) . \tag{58}$$

We also obtain the following commutation relations

$$[x_i, x_j]_{\pi} = \frac{i}{\pi} t (\delta_{li} \delta_{kj} - \delta_{ki} \delta_{lj}) , \quad [t, x_i]_{\pi} = 0 . \tag{59}$$

The deformation (59) of nonrelativistic space-time describes conceptually new type of noncommutativity. It should be interesting to consider dynamical models based on deformed space-time (59), with quantum space and classical time coordinate (see [14]).
6 Final remarks

In this paper we investigate three different nonrelativistic contractions of soft and Lie-algebraic Poincaré algebras. In such a way we get five deformations of Galilei groups - three of Lie-type and two of a soft kind. Further, we derive as well the corresponding nonrelativistic space-times representing the Hopf module of our Galilean quantum algebras.

The present studies can be extended in various ways. First of all, one can construct associated with the presented algebras suitable phase spaces derived with the help of Heisenberg double construction [36], [45], [46], [47]. Besides, we could also look for the dual Galilei quantum groups, which can be recovered with use of FRT method (see [48]) or by canonical quantization of a proper Poisson-Lie structure [41], [49]. Finally, one can ask about explicit realization of others $D = 3+1$ Galilei Hopf algebras and their deformed space-times, which are predicted by the general classification of all ”nonrelativistic” classical r-matrices [44] (see also for two-dimensional case [50], [51]). These problems are under considerations and they are postponed for further investigation.

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References

[1] S. Coleman, S.L. Glashow, Phys. Rev. D 59, 116008 (1999)
[2] R.J. Protheore, H. Meyer, Phys. Lett. B 493, 1-6 (2000)
[3] F.W. Stecker, S.L. Glashow, Astroparticle Phys. 16, 97-99 (2001)
[4] G. Amelino-Camelia, T. Piran, Phys. Lett. B 497, 265-270 (2001)
[5] S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. B 331, 39-44 (1994); Comm. Math. Phys. 172, 187-220 (1995); hep-th/0303037
[6] A. Kempf and G. Mangano, Phys. Rev. D 55, 7909-7920 (1997); hep-th/9612084
[7] N. Seiberg and E. Witten, JHEP 9909:032 (1999); hep-th/9908142
[8] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, Phys. Lett. B 574, 98-104 (2003); hep-th/0302078
[9] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B 264, 331-338 (1991)
[10] J. Lukierski, A. Nowicki and H. Ruegg, Phys. Lett. B 293, 344-352 (1992)
[11] S. Majid, H. Ruegg, Phys. Lett. B 334, 348-354 (1994); hep-th/9405107
[12] J. Lukierski, H. Ruegg, W.J. Zakrzewski, Annals Phys. 243, 90-116 (1995)
[13] G. Amelino-Camelia, Phys. Lett. B 510, 255-263 (2001); hep-th/0012238
[14] G. Amelino-Camelia, Mod. Phys. Lett. A 17, 899-922 (2002); gr-qc/0204051
[15] B. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, Phys. Lett. B 522, 133-138 (2001); hep-th/0107039
[16] J. Magueijo and L. Smolin, Phys. Rev. Lett. 88, 190403 (2002); hep-th/0112090
[17] S. Zakrzewski, "Poisson Structures on the Poincare group"; q-alg/9602001
[18] R. Oeckl, Nucl. Phys. B 581, 559-574 (2000)
[19] M. Chaichian, P.P. Kulish, K. Nishijima and A. Tureanu, Phys. Lett. B 604, 98-102 (2004); hep-th/0408069
[20] J. Wess, "Deformed coordinate spaces: Derivatives"; hep-th/0408080
[21] P. Kosinski and P. Maslanka, "Lorentz-invariant interpretation of noncommutative space-time: Global version"; hep-th/0408100
[22] F. Koch and E. Tsouchnika, Nucl. Phys. B 717, 387-403 (2005); hep-th/0409012
[23] C. Gonera, P. Kosinski, P. Maslanka and S. Giller, "Space-times symmetry of non-commutative field theory"; hep-th/0504132
[24] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, J. Phys. A 27, 2389-2400 (1994)
[25] J. Lukierski and M. Woronowicz, Phys. Lett. B 633, 116-124 (2006); hep-th/0508083
[26] N.Yu. Reshetikhin, Lett. Math. Phys. 20, 331-335 (1990)
[27] S. Giller, P. Kosinski, M. Majewski, P. Maslanka and J. Kunz, Phys. Lett. B 286, 57-62 (1992)
[28] J.A. de Azcarraga and J.C. Perez Bueno, J. Math. Phys. 36, 6879-6896 (1995)
[29] J.A. de Azcarraga and J.C. Perez Bueno, J. Phys. A 29, 6353-6362 (1996)
[30] V. Aldaya and J.A. de Azcarraga, Int. J. of Theor. Phys. 24, 141 (1985)
[31] J.A. de Azcarraga and J. M. Izquierdo, "Lie algebras, Lie groups cohomology and some applications in physics", Camb. Univ. Press (1995)
[32] J.A. de Azcarraga and J.C. Perez Bueno, "Contractions, Hopf algebra extensions and covariant differential calculus"; in "From field theory to quantum groups"; World Scientific, Singapore (1996)

[33] A. Ballesteros, E. Celeghini and F.J. Herranz, J. Phys. A 33, 3431-3444 (2000)

[34] S. Giller, C. Gonera, P. Kosinski, P. Maslanka, "The quantum Galilei group"; q-alg/9505007

[35] M. Chaichian, P.P. Kulish, A. Turenau, R.B. Zhang, X. Zhang, "Noncommutative fields and actions of twisted Poincare algebra"; arXiv: 0711.0371 [hep-th]

[36] S. Majid, H. Ruegg, Phys. Lett. B 329, 189-194 (1994)

[37] E. Inönü and E.P. Wigner, Proc. Nat. Acad. Sci. 39, 510-524 (1953)

[38] A. Barut, R. Raczka, "Theory of group representation" Moskow 1980 (in Russian)

[39] C. Blohmann, J. Math. Phys. 44, 4736-4755 (2003); q-alg/0209180

[40] V.G. Drinfeld, Soviet Math. Dokl. 32, 254-258 (1985); Algebra i Analiz (in Russian), 1, Fasc. 6, p. 114 (1989)

[41] Y. Brihaye, E. Kowalczyk, P. Maslanka, "Poisson-Lie structure on Galilei group"; math/0006167

[42] J. Lukierski and M. Woronowicz, "Twisted space-time symmetry, non-commutativity and particle dynamics", published in Tianjin 2005, "Differential geometry and physics" 333-342; hep-th/0512046

[43] M. Irisawa, Y. Kobayashi, S. Sasaki, Prog. Theor. Phys. 118, 83-96 (2007)

[44] M. Daszkiewicz, C.J. Walczyk, in preparation

[45] S. Majid, "Foundations of quantum group theory", Cambridge University Press, April 2000

[46] J. Lukierski and A. Nowicki, "Heisenberg double description of κ-Poincare algebra and κ-deformed phase-space"; XXI International Colloquium of Group-Theoretic Methods, Goslar, July 1996; q-alg/9702003

[47] P. Czerchoniak, J. Phys. A 36, 9655-9672 (2003)

[48] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtadzhyan, Leningrad Math. J. 1 (1990)

[49] L.A. Takhtajan, "Introduction to Quantum Groups"; in Clausthal Proceedings, Quantum groups 3-28 (see High Energy Physics Index 29 (1991) No. 12256)
[50] E. Kowalczyk, Acta Phys. Pol. B 28, 1893-1906 (1997)

[51] A. Opanowicz, J. Phys. A 31, 8387-8396 (1998), A. Opanowicz, J. Phys. A 33, 1941-1953 (2000)