Optimistic posterior sampling for reinforcement learning: worst-case regret bounds

Shipra Agrawal
Columbia University
sa3305@columbia.edu

Randy Jia
Columbia University
rqj2000@columbia.edu

Abstract

We present an algorithm based on posterior sampling (aka Thompson sampling) that achieves near-optimal worst-case regret bounds when the underlying Markov Decision Process (MDP) is communicating with a finite, though unknown, diameter. Our main result is a high probability regret upper bound of $\tilde{O}(D\sqrt{SAT})$ for any communicating MDP with $S$ states, $A$ actions and diameter $D$, when $T \geq S^5A$. Here, regret compares the total reward achieved by the algorithm to the total expected reward of an optimal infinite-horizon undiscounted average reward policy, in time horizon $T$. This result improves over the best previously known upper bound of $\tilde{O}(DS\sqrt{AT})$ achieved by any algorithm in this setting, and matches the dependence on $S$ in the established lower bound of $\Omega(\sqrt{DSAT})$ for this problem. Our techniques involve proving some novel results about the anti-concentration of Dirichlet distribution, which may be of independent interest.

1 Introduction

Reinforcement Learning (RL) refers to the problem of learning and planning in sequential decision making systems when the underlying system dynamics are unknown, and may need to be learned by trying out different options and observing their outcomes. A typical model for the sequential decision making problem is a Markov Decision Process (MDP), which proceeds in discrete time steps. At each time step, the system is in some state $s$, and the decision maker may take any available action $a$ to obtain a (possibly stochastic) reward. The system then transitions to the next state according to a fixed state transition distribution. The reward and the next state depend on the current state $s$ and the action $a$, but are independent of all the previous states and actions. In the reinforcement learning problem, the underlying state transition distributions and/or reward distributions are unknown, and need to be learned using the observed rewards and state transitions, while aiming to maximize the cumulative reward. This requires the algorithm to manage the tradeoff between exploration vs. exploitation, i.e., exploring different actions in different states in order to learn the model more accurately vs. taking actions that currently seem to be reward maximizing.

Exploration-exploitation tradeoff has been studied extensively in the context of stochastic multi-armed bandit (MAB) problems, which are essentially MDPs with a single state. The performance of MAB algorithms is typically measured through regret, which compares the total reward obtained by the algorithm to the total expected reward of an optimal action. Optimal regret bounds have been established for many variations of MAB (see Bubeck et al. [2012] for a survey), with a large majority of results obtained using the Upper Confidence Bound (UCB) algorithm, or more generally, the optimism in the face of uncertainty principle. Under this principle, the learning algorithm maintains tight over-estimates (or optimistic estimates) of the expected rewards for individual actions, and at any given step, picks the action with the highest optimistic estimate. More recently, posterior sampling, aka Thompson Sampling [Thompson[1933], has emerged as another popular algorithm design principle in MAB, owing its popularity to a simple and extendible algorithmic structure, an
We should also compare our result with the very recent result of Azar et al. [2017], which provides a posterior sampling based algorithm with high probability worst-case regret upper bound of $\hat{O}(D\sqrt{S}\sqrt{T})$ for this problem. A similar bound was achieved by Bartlett and Tewari [2009], though assuming the knowledge of the diameter $D$. Jaksch et al. [2010] also established a worst-case lower bound of $\Omega(\sqrt{DSAT})$ on the regret of any algorithm for this problem.

Our main contribution is a posterior sampling based algorithm with a high probability worst-case regret upper bound of $\hat{O}(D\sqrt{S}\sqrt{T} + DS^{1/2}A^{3/4}T^{1/4})$, which is $\hat{O}(D\sqrt{S}\sqrt{T})$ when $T \geq S^2A$. This improves the previously best known upper bound for this problem by a factor of $\sqrt{S}$, and matches the dependence on $S$ in the lower bound, for large enough $T$.

Our algorithm uses an ‘optimistic version’ of the posterior sampling heuristic, while utilizing several ideas from the algorithm design structure in Jaksch et al. [2010], such as an epoch based execution and the extended MDP construction. The algorithm proceeds in epochs, where in the beginning of every epoch, it generates $\psi = \hat{O}(S)$ samples transition probability vectors from a posterior distribution for every state and action, and solves an extended MDP with $\psi$A actions and $S$ states formed using these samples. The optimal policy computed for this extended MDP is used throughout the epoch. Posterior Sampling for Reinforcement Learning (PSRL) approach has been used previously in Osband et al. [2013], Abbasi-Yadkori and Szepesvari [2014], Osband and Van Roy [2016], but in a Bayesian regret framework. Bayesian regret is defined as the expected regret over a known prior on the transition probability matrix. Osband and Van Roy [2016] demonstrate an $\hat{O}(H\sqrt{SAT})$ bound on the expected Bayesian regret for PSRL in finite-horizon episodic Markov decision processes, when the episode length is $H$. In this paper, we consider the stronger notion of worst-case regret, aka minimax regret, which requires bounding the maximum regret for any instance of the problem.

Further, we consider a non-episodic communicating MDP setting, and produce a comparable bound of $\hat{O}(D\sqrt{S}\sqrt{T})$ for large $T$, where $D$ is the unknown diameter of the communicating MDP. In comparison to a single sample from the posterior in PSRL, our algorithm is slightly inefficient as it uses multiple ($\hat{O}(S)$) samples. It is not entirely clear if the extra samples are only an artifact of the analysis. In an empirical study of a multiple sample version of posterior sampling for RL, Fonteneau et al. [2013] show that multiple samples can potentially improve the performance of posterior sampling in terms of probability of taking the optimal decision. Our analysis utilizes some ideas from the Bayesian regret analysis, most importantly the technique of stochastic optimism from Osband et al. [2014] for deriving tighter deviation bounds. However, bounding the worst-case regret requires several new technical ideas, in particular, for proving ‘optimism’ of the gain of the sampled MDP. Further discussion is provided in Section 4.

We should also compare our result with the very recent result of Azar et al. [2017], which provides an optimistic version of value-iteration algorithm with a minimax (i.e., worst-case) regret bound of

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1Worst-case regret is a strictly stronger notion of regret in case the reward distribution function is known and only the transition probability distribution is unknown, as we will assume here for the most part. In case of unknown reward distribution, extending our worst-case regret bounds would require an assumption of bounded rewards, where as the Bayesian regret bounds in the above-mentioned literature allow more general (known) priors on the reward distributions with possibly unbounded support. Bayesian regret bounds in those more general settings are incomparable to the worst-case regret bounds presented here.
We consider communicating MDPs with finite diameter (see Bartlett and Tewari [2009]). Among other related work, Burnetas and Katehakis [1997] and Tewari and Bartlett [2008] present optimistic linear programming approaches that achieve logarithmic regret bounds with problem dependent constants. Strong PAC bounds have been provided in Kearns and Singh [1999], Brafman and Tennenholtz [2002], Kakade et al. [2003], Asmuth et al. [2009], Dann and Brunskill [2015]. There, the aim is to bound the performance of the policy learned at the end of the learning horizon, and not the performance during learning as quantified by regret. Strehl and Littman [2005], Strehl and Littman [2008] provide an optimistic algorithm for bounding regret in a discounted reward setting, but the definition of regret is slightly different in that it measures the difference between the rewards of an optimal policy and the rewards of the learning algorithm along the trajectory taken by the learning algorithm.

2 Preliminaries and Problem Definition

2.1 Markov Decision Process (MDP)

We consider a Markov Decision Process $\mathcal{M}$ defined by tuple $\{ S, A, P, r, s_1 \}$, where $S$ is a finite state-space of size $S$, $A$ is a finite action-space of size $A$, $P : S \times A \to \Delta^S$ is the transition model, $r : S \times A \to [0, 1]$ is the reward function, and $s_1$ is the starting state. When an action $a \in A$ is taken in a state $s \in S$, a reward $r_{s,a}$ is generated and the system transitions to the next state $s' \in S$ with probability $P_{s,a}(s')$, where $\sum_{s' \in S} P_{s,a}(s') = 1$.

We consider ‘communicating’ MDPs with finite ‘diameter’ (see Bartlett and Tewari [2009] for an in-depth discussion). Below we define communicating MDPs, and recall some useful known results for such MDPs.

Definition 1 (Policy). A deterministic policy $\pi : S \to A$ is a mapping from state space to action space.

Definition 2 (Diameter $D(\mathcal{M})$). Diameter $D(\mathcal{M})$ of an MDP $\mathcal{M}$ is defined as the minimum time required to go from one state to another in the MDP using some deterministic policy:

$$D(\mathcal{M}) = \max_{s \neq s', s, s' \in S} \min_{\pi : S \to A} T^\pi_{s \to s'},$$

where $T^\pi_{s \to s'}$ is the expected number of steps it takes to reach state $s'$ when starting from state $s$ and using policy $\pi$.

Definition 3 (Communicating MDP). An MDP $\mathcal{M}$ is communicating if and only if it has a finite diameter. That is, for any two states $s \neq s'$, there exists a policy $\pi$ such that the expected number of steps to reach $s'$ from $s$, $T^\pi_{s \to s'}$, is at most $D$, for some finite $D \geq 0$.

Definition 4 (Gain of a policy). The gain of a policy $\pi$, from starting state $s_1 = s$, is defined as the infinite horizon undiscounted average reward, given by

$$\lambda^\pi(s) = \mathbb{E}[\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} r_{s_t, \pi(s_t)} | s_1 = s].$$

where $s_1$ is the state reached at time $t$.

Lemma 2.1 (Optimal gain for communicating MDPs). For a communicating MDP $\mathcal{M}$ with diameter $D$:

(a) [Puterman 2014] Theorem 8.1.2, Theorem 8.3.2) The optimal (maximum) gain $\lambda^*$ is state independent and is achieved by a deterministic stationary policy $\pi^*$, i.e., there exists a
We present an algorithm for the learning agent with a near-optimal upper bound on the regret \( \lambda \) where
\[
\lambda := \max_{s' \in S} \max_{\pi} \lambda^{\pi}(s') = \lambda^{\pi^*}(s), \forall s \in S.
\]

Here, \( \pi^* \) is referred to as an optimal policy for MDP \( \mathcal{M} \).

An epoch is a group of consecutive rounds. The rounds \( t \) where the next state was \( s' \) is
\[
\sum_{i=1}^{N^t_{s,a}} a_i h - h_s = \max_a r_{s,a} + P^T_{s,a} h^* - h_s, \forall s
\]
where \( h^* \), referred to as the bias vector of MDP \( \mathcal{M} \), satisfies:
\[
\max_s h^*_s - \min_s h^*_s \leq D.
\]

Given the above definitions and results, we can now define the reinforcement learning problem studied in this paper.

### 2.2 The reinforcement learning problem

The reinforcement learning problem proceeds in rounds \( t = 1, \ldots, T \). The learning agent starts from a state \( s_1 \) at round \( t = 1 \). In the beginning of every round \( t \), the agent takes an action \( a_t \in A \) and observes the reward \( r_{s_t,a_t} \) as well as the next state \( s_{t+1} \sim P_{s_t,a_t} \), where \( r \) and \( P \) are the reward function and the transition model, respectively, for a communicating MDP \( \mathcal{M} \) with diameter \( D \).

The learning agent knows the state-space \( S \), the action space \( A \), as well as the rewards \( r_{s,a} \), \( \forall s \in S, a \in A \), for the underlying MDP, but not the transition model \( P \) or the diameter \( D \). (The assumption of known and deterministic rewards has been made here only for simplicity of exposition, since the unknown transition model is the main source of difficulty in this problem. Our algorithm and results can be extended to bounded stochastic rewards with unknown distributions using standard Thompson Sampling for MAB, e.g., using the techniques in [Agrawal and Goyal, 2013].)

The agent can use the past observations to learn the underlying MDP model and decide future actions. The goal is to maximize the total reward \( \sum_{t=1}^{T} r_{s_t,a_t} \), or equivalently, minimize the total regret over a time horizon \( T \), defined as
\[
R(T, \mathcal{M}) := T \lambda^* - \sum_{t=1}^{T} r_{s_t,a_t}
\]
where \( \lambda^* \) is the optimal gain of MDP \( \mathcal{M} \).

We present an algorithm for the learning agent with a near-optimal upper bound on the regret \( R(T, \mathcal{M}) \) for any communicating MDP \( \mathcal{M} \) with diameter \( D \), thus bounding the worst-case regret over this class of MDPs.

### 3 Algorithm Description

Our algorithm combines the ideas of Posterior sampling (aka Thompson Sampling) with the extended MDP construction used in [Jaksch et al., 2010]. Below we describe the main components of our algorithm.

#### Some notations:
\( N^t_{s,a} \) denotes the total number of times the algorithm visited state \( s \) and played action \( a \) until before time \( t \), and \( N^t_{s,a}(i) \) denotes the number of time steps among these \( N^t_{s,a} \) steps where the next state was \( i \), i.e., a transition from state \( s \) to \( i \) was observed. We index the states from 1 to \( S \), so that \( \sum_{i=1}^{S} N^t_{s,a}(i) = N^t_{s,a} \) for any \( t \). We use the symbol \( 1 \) to denote the vector of all 1s, and \( 1_i \) to denote the vector with 1 at the \( i^{th} \) coordinate and 0 elsewhere.

#### Doubling epochs:
Our algorithm uses the epoch based execution framework of [Jaksch et al., 2010]. An epoch is a group of consecutive rounds. The rounds \( t = 1, \ldots, T \) are broken into consecutive epochs as follows: the \( k^{th} \) epoch begins at the round \( \tau_k \) immediately after the end of \( (k-1)^{th} \) epoch and ends at the first round \( \tau \) such that for some state-action pair \( s,a \), \( N^T_{s,a} \geq 2N^T_{s,a} \). The algorithm computes a new policy \( \pi_k \) at the beginning of every epoch \( k \), and uses that policy through all the rounds in that epoch. It is easy to observe that irrespective of how the policy \( \pi_k \) is computed, the number of epochs in \( T \) rounds is bounded by \( SA \log(T) \).
We prove the following bound on the regret of Algorithm 1 for the reinforcement learning problem.

**Theorem 1.** For any communicating MDP $\mathcal{M}$, the regret of Algorithm 1 in time $T$ is bounded as:

$$\mathcal{R}(T, \mathcal{M}) \leq \tilde{O} \left( D \sqrt{SA^4T} + DS^{7/4}A^{3/4}T^{1/4} + DS^{5/2}A \right)$$

where $C$ is an absolute constant. For $T \geq S^5A$, this implies a regret bound of

$$\mathcal{R}(T, \mathcal{M}) \leq \tilde{O} \left( D \sqrt{SAT} \right).$$

Here $\tilde{O}$ hides logarithmic factors in $S, A, T, \rho$ and absolute constants.

The rest of this section is devoted to proving the above theorem. Here, we provide a sketch of the proof and discuss some of the key lemmas, all missing details are provided in the supplementary material.
For each \((s, a)\), \(\pi_k\) is the optimal gain policy for extended MDP \(M_k\). As defined in Section 2, regret is given by \(R_k := (\tau_k + 1 - \tau_k)\lambda^* - \sum_{t=1}^{\tau_k} r_{s_t, a_t}\), where \(\lambda^*\) is the optimal gain of MDP \(M\), \(a_t\) is the action taken and \(s_t\) is the state reached by the algorithm at time \(t\). Algorithm 1 proceeds in epochs \(k = 1, 2, \ldots, K\), where \(K \leq SA\log(T)\). To bound its regret in time \(T\), we first analyze the regret in each epoch \(k\), namely,

\[
R_k := (\tau_k + 1 - \tau_k)\lambda^* - \sum_{t=1}^{\tau_k-1} r_{s_t, a_t},
\]

and bound \(R_k\) by roughly

\[
D\sum_{s, a} \frac{N_{s, a}^{\tau_k + 1} - N_{s, a}^{\tau_k}}{N_{s, a}^{\tau_k}}
\]

where, by definition, for every \(s, a\), \((N_{s, a}^{\tau_k + 1} - N_{s, a}^{\tau_k})\) is the number of times this state-action pair is visited in epoch \(k\). The proof of this bound has two main components:

(a) Optimism: The policy \(\tilde{\pi}_k\) used by the algorithm in epoch \(k\) is computed as an optimal gain policy of the extended MDP \(\hat{M}_k\). The first part of the proof is to show that with high probability, the extended MDP \(\hat{M}_k\) is (i) a communicating MDP with diameter at most \(2D\), and (ii) optimistic, i.e., has optimal gain at least (close to) \(\lambda^*\). Part (i) is stated as Lemma 4.1 with a proof provided in the supplementary material.

Algorithm 1: A posterior sampling based algorithm for the reinforcement learning problem

**Inputs:** State space \(\mathcal{S}\), Action space \(\mathcal{A}\), starting state \(s_1\), reward function \(r\), time horizon \(T\), parameters \(\rho \in (0, 1], \psi = O(S \log(SA/\rho))\), \(\omega = O(\log(T/\rho))\), \(\kappa = O(\log(T/\rho))\), \(\eta = \sqrt{\frac{T\lambda}{\Delta}} + 12\omega S^2\).

**Initialize:** \(\tau_1^1 := 1\), \(M_{s, a}^1 = \omega^1\).

for all epochs \(k = 1, 2, \ldots, do\)

**Sample transition probability vectors:** For each \(s, a\), generate \(\psi\) independent sample probability vectors \(Q_{s, a}^{j, k}, j = 1, \ldots, \psi\), as follows:

- **(Posterior sampling):** For \(s, a\) such that \(N_{s, a}^{\tau_k} \geq \eta\), use samples from the Dirichlet distribution:
  \[Q_{s, a}^{j, k} \sim \text{Dirichlet}(M_{s, a}^k),\]

- **(Simple optimistic sampling):** For remaining \(s, a\), with \(N_{s, a}^{\tau_k} < \eta\), use the following simple optimistic sampling: let
  \[P_{s, a}^{-} = \hat{P}_{s, a} - \Delta,\]

  where \(\hat{P}_{s, a}(i) = \frac{N_{s, a}^{\tau_k}(i)}{N_{s, a}^{\tau_k}}\), and \(\Delta_i = \min\left\{\frac{3\hat{P}_{s, a}(i)\log(4S)}{N_{s, a}^{\tau_k}}, 1\right\}\), and let \(z\) be a random vector picked uniformly at random from \(\{1, \ldots, 1\}\); set
  \[Q_{s, a}^{j, k} = P_{s, a}^{-} + (1 - \sum_{i=1}^{S} P_{s, a}^{-}(i))z.\]

**Compute policy \(\tilde{\pi}_k\):** as the optimal gain policy for extended MDP \(\hat{M}_k\) constructed using sample set \(\{Q_{s, a}^{j, k}, j = 1, \ldots, \psi, s \in \mathcal{S}, a \in \mathcal{A}\}\).

**Execute policy \(\tilde{\pi}_k\):**

for all time steps \(t = \tau_k, \tau_k + 1, \ldots\), until break epoch do

  Play action \(a_t = \tilde{\pi}_k(s_t)\).

  Observe the transition to the next state \(s_{t+1}\).

  Set \(N_{s, a}^{t+1}(i), M_{s, a}^{t+1}(i)\) for all \(a \in \mathcal{A}, s, i \in \mathcal{S}\) as defined (refer to Equation (3)).

  If \(N_{s, a}^{t+1} \geq 2N_{s, a}^{\tau_k}\), then set \(\tau_{k+1} = t + 1\) and break epoch.

end for

end for

4.1 Proof of Theorem 1

As defined in Section 2, regret \(R(T, M)\) is given by \(R(T, M) = T\lambda^* - \sum_{t=1}^{T} r_{s_t, a_t}\), where \(\lambda^*\) is the optimal gain of MDP \(M\), \(a_t\) is the action taken and \(s_t\) is the state reached by the algorithm at time \(t\). Algorithm 1 proceeds in epochs \(k = 1, 2, \ldots, K\), where \(K \leq SA\log(T)\). To bound its regret in time \(T\), we first analyze the regret in each epoch \(k\), namely,

\[
R_k := (\tau_k + 1 - \tau_k)\lambda^* - \sum_{t=1}^{\tau_k-1} r_{s_t, a_t},
\]

and bound \(R_k\) by roughly

\[
D\sum_{s, a} \frac{N_{s, a}^{\tau_k + 1} - N_{s, a}^{\tau_k}}{N_{s, a}^{\tau_k}}
\]

where, by definition, for every \(s, a\), \((N_{s, a}^{\tau_k + 1} - N_{s, a}^{\tau_k})\) is the number of times this state-action pair is visited in epoch \(k\). The proof of this bound has two main components:

(a) Optimism: The policy \(\tilde{\pi}_k\) used by the algorithm in epoch \(k\) is computed as an optimal gain policy of the extended MDP \(\hat{M}_k\). The first part of the proof is to show that with high probability, the extended MDP \(\hat{M}_k\) is (i) a communicating MDP with diameter at most \(2D\), and (ii) optimistic, i.e., has optimal gain at least (close to) \(\lambda^*\). Part (i) is stated as Lemma 4.1 with a proof provided in the supplementary material. Now, let \(\tilde{\lambda}_k\) be the optimal gain of the extended MDP \(\hat{M}_k\). In
Lemma 4.2, which forms one of the main novel technical components of our proof, we show that with probability $1 - \rho$,

$$\hat{\lambda}_k \geq \lambda^* - \tilde{O}(D \sqrt{\frac{SA}{T_k}}).$$

We first show that above holds if for every $s, a$, there exists a sample transition probability vector whose projection on a fixed unknown vector ($h^*$) is optimistic. Then, in Lemma 4.3, we prove this optimism by deriving a fundamental new result on the anti-concentration of any fixed projection of a Dirichlet random vector (Proposition A.1 in the supplementary material).

Substituting this upper bound on $\lambda^*$, we have the following bound on $R_k$ with probability $1 - \rho$:

$$R_k \leq \sum_{t=\tau_k}^{\tau_{k+1} - 1} \left( \hat{\lambda}_k - r_{s_t, a_t} + \tilde{O}(D \sqrt{\frac{SA}{T_k}}) \right).$$

(b) **Deviation bounds:** Optimism guarantees that with high probability, the optimal gain $\hat{\lambda}_k$ for MDP $\mathcal{M}^k$ is at least $\lambda^*$. And, by definition of $\hat{\pi}_k, \hat{\lambda}_k$ is the gain of the chosen policy $\hat{\pi}_k$ for MDP $\mathcal{M}^k$.

However, the algorithm executes this policy on the true MDP $\mathcal{M}$. The only difference between the two is the transition model: on taking an action $a^j := \pi_k(s)$ in state $s$ in MDP $\mathcal{M}^k$, the next state follows the sampled distribution

$$\tilde{P}_{s, a} := Q_{s, a}^j,$$

where as on taking the corresponding action $a$ in MDP $\mathcal{M}$, the next state follows the distribution $P_{s, a}$. The next step is to bound the difference between $\hat{\lambda}_k$ and the average reward obtained by the algorithm by bounding the deviation $(\tilde{P}_{s, a} - P_{s, a})^T \hat{h}$ for all $s, a$, to bound the first term in above. Note that $\hat{h}$ is random and can be arbitrarily correlated with $\tilde{P}$, therefore, we need to bound $\max_{h \in [0, 2D]^S} (\tilde{P}_{s, a} - P_{s, a})^T h$. (For the above term, w.l.o.g. we can assume $\hat{h} \in [0, 2D]^S$).

For $s, a$ such that $N_{s, a}^T > \eta$, $\tilde{P}_{s, a} = Q_{s, a}^j$ is a sample from the Dirichlet posterior. In Lemma 4.4, we show that with high probability,

$$\max_{h \in [0, 2D]^S} (\tilde{P}_{s, a} - P_{s, a})^T h \leq \tilde{O}(\frac{D}{\sqrt{N_{s, a}}} + \frac{DS}{N_{s, a}}).$$

This bound is an improvement by a $\sqrt{S}$ factor over the corresponding deviation bound obtainable for the optimistic estimates of $P_{s, a}$ in UCRL2. The derivation of this bound utilizes and extends the stochastic optimism technique from Osband et al. [2014]. For $s, a$ with $N_{s, a}^T \leq \eta$, $\tilde{P}_{s, a} = Q_{s, a}^j$ is a sample from the simple optimistic sampling, where we can only show the following weaker bound, but since this is used only while $N_{s, a}^T$ is small, the total contribution of this deviation will be small:

$$\max_{h \in [0, 2D]^S} (\tilde{P}_{s, a}^k - P_{s, a})^T h \leq \tilde{O}\left(D \sqrt{\frac{S}{N_{s, a}^T}} + \frac{DS}{N_{s, a}^T}\right).$$

Finally, to bound the second term in (6), we observe that $\mathbb{E}[1_{s, a}] = P_{s, a}^T \hat{h}$ and use Azuma-Hoeffding inequality to obtain with probability $(1 - \frac{SA}{2})$:

$$\sum_{t=\tau_k}^{\tau_{k+1} - 1} (P_{s, a_t} - 1_{s_t})^T \hat{h} \leq O(\sqrt{\mathbb{E}[(\tau_{k+1} - \tau_k) \log(SA/\rho)]})$$
Combining the above observations (equations (4), (9), (7), (8), (9)), we obtain the following bound on $R_k$ within logarithmic factors:

$$D(\tau_{k+1} - \tau_k) \sqrt{\frac{SA}{T}} + D \sum_{s,a} \frac{N_{s,a}^{T_k+1} - N_{s,a}^k}{\sqrt{N_{s,a}^k}} (1 (N_{s,a}^{T_k+1} > \eta) + \sqrt{S} 1 (N_{s,a}^{T_k+1} \leq \eta)) + D \sqrt{\tau_{k+1} - \tau_k}.$$  

(10)

We can finish the proof by observing that (by definition of an epoch) the number of visits of any state-action pair can at most double in an epoch, and therefore, substituting this observation in (10), we can bound (within logarithmic factors) the total regret $R(T) = \sum_{k=1}^K R_k$ as:

$$\sum_{k=1}^K \left( D(\tau_{k+1} - \tau_k) \sqrt{\frac{SA}{T}} + D \sum_{s,a:N_{s,a}^k > \eta} \sqrt{N_{s,a}^k} + D \sum_{s,a:N_{s,a}^k \leq \eta} \sqrt{SN_{s,a}^k} + D \sqrt{\tau_{k+1} - \tau_k} \right) \leq D\sqrt{SAT} + D \log(K) (\sum_{s,a} \sqrt{N_{s,a}^k}) + D \log(K) (SA \sqrt{S\eta}) + D \sqrt{KT}$$

where we used $N_{s,a}^{T_k+1} \leq 2N_{s,a}^k$ and $\sum_k (\tau_{k+1} - \tau_k) = T$. Now, we use that $K \leq SA \log(T)$, and $SA \sqrt{S\eta} = O(S^{7/4} A^{5/4} T^{1/4} + S^{5/2} A \log(T/\rho))$ (using $\eta = \sqrt{\frac{T \omega_S}{A}} + 12 \omega_S^2$). Also, since $\sum_{s,a} N_{s,a}^k \leq T$, by simple worst scenario analysis, $\sum_{s,a} \sqrt{N_{s,a}^k} \leq \sqrt{SAT}$, and we obtain:

$$R(T, M) \leq \tilde{O}(D \sqrt{SAT} + DS^{7/4} A^{5/4} T^{1/4} + DS^{5/2} A).$$

### 4.2 Main lemmas

Following lemma form the main technical components of our proof. All the missing proofs are provided in the supplementary material.

**Lemma 4.1.** Assume $T \geq CDA \log^2(T/\rho)$ for a large enough constant $C$. Then, with probability $1 - \rho$, for every epoch $k$, the diameter of MDP $\tilde{M}^k$ is bounded by $2D$.

**Lemma 4.2.** With probability $1 - \rho$, for every epoch $k$, the optimal gain $\tilde{\lambda}_k$ of the extended MDP $\tilde{M}^k$ satisfies:

$$\tilde{\lambda}_k \geq \lambda^* - O \left(D \log^2(T/\rho) \sqrt{\frac{SA}{T}} \right),$$

where $\lambda^*$ the optimal gain of MDP $M$ and $D$ is the diameter.

**Proof.** Let $h^*$ be the bias vector for an optimal policy $\pi^*$ of MDP $M$ (refer to Lemma 2.1 in the preliminaries section). Since $h^*$ is a fixed (though unknown) vector with $|h_i - h_j| \leq D$, we can apply Lemma 4.3 to obtain that with probability $1 - \rho$, for all $s, a$, there exists a sample vector $Q_{s,a}$ for some $j \in \{1, \ldots, \psi\}$ such that

$$(Q_{s,a})^T h^* \geq P^T_{s,a} h^* - \delta$$

where $\delta = O \left(D \log^2(T/\rho) \sqrt{\frac{SA}{T}} \right)$. Now, consider the policy $\pi$ for MDP $\tilde{M}^k$ which for any $s$, takes action $a^s\pi^*(s)$, and $P_{\pi^*}$ be the transition matrix whose rows are formed by the vectors $P_{s,a}^{\pi^*(s)}$. Above implies $Q_{s,a} h^* \geq P_{\pi^*} h^* - \delta 1$. We use this inequality along with the known relations between the gain and the bias of optimal policy in communicating MDPs to obtain that the gain $\tilde{\lambda}(\pi)$ of policy in $\pi$ for MDP $\tilde{M}^k$ satisfies $\tilde{\lambda}(\pi) \geq \lambda^* - \delta$ (details provided in the supplementary material), which proves the lemma statement since by optimality $\tilde{\lambda}_k \geq \tilde{\lambda}(\pi)$.

\[\square\]
Lemma 4.3. (Optimistic Sampling) Fix any vector $h \in \mathbb{R}^S$ such that $|h_i - h_{i'}| \leq D$ for any $i, i'$, and any epoch $k$. Then, for every $s, a$, with probability $1 - \frac{2}{T^\alpha}$ there exists at least one $j$ such that

$$(Q^j_{s,a})^T h \geq P^T_{s,a} h - O\left(D \log^2(T/\rho) \sqrt{\frac{SA}{T}} \right).$$

Lemma 4.4. (Deviation bound) With probability $1 - \rho$, for all epochs $k$, sample $j$, all $s, a$

$$\max_{h \in [0,2D]^S} (Q^j_{s,a} - P_{s,a})^T h \leq \begin{cases} 
O\left(D \frac{\log(SAT/\rho)}{N_{s,a}^k} + D \frac{S \log(SAT/\rho)}{N_{s,a}^k} \right), & N_{s,a}^k > \eta \\
O\left(D \frac{S \log(SAT/\rho)}{N_{s,a}^k} + D \frac{S \log(S)}{N_{s,a}^k} \right), & N_{s,a}^k \leq \eta 
\end{cases}$$

5 Conclusions

We presented an algorithm inspired by posterior sampling that achieves near-optimal worst-case regret bounds for the reinforcement learning problem with communicating MDPs in a non-episodic, undiscounted average reward setting. Our algorithm may be viewed as a more efficient randomized version of the UCRL2 algorithm of [Jaksch et al., 2010], with randomization via posterior sampling forming the key to the $\sqrt{S}$ factor improvement in the regret bound provided by our algorithm. Our analysis demonstrates that posterior sampling provides the right amount of uncertainty in the samples, so that an optimistic policy can be obtained without excess over-estimation.

While our work surmounts some important technical difficulties in obtaining worst-case regret bounds for posterior sampling based algorithms for communicating MDPs, the provided bound is tight in its dependence on $S$ and $A$ only for large $T$ (specifically, for $T \geq S^6A$). Other related results on tight worst-case regret bounds have a similar requirement of large $T$ ([Azar et al., 2017] produce an $\tilde{O}(\sqrt{HSA})$ bound when $T \geq H^3S^3A$). Obtaining a cleaner worst-case regret bound that does not require such a condition remains an open question. Other important directions of future work include reducing the number of posterior samples required in every epoch from $\tilde{O}(S)$ to constant or logarithmic in $S$, and extensions to contextual and continuous state MDPs.
References

Yasin Abbasi-Yadkori and Csaba Szepesvari. Bayesian optimal control of smoothly parameterized systems: The lazy posterior sampling algorithm. arXiv preprint arXiv:1406.3926, 2014.

Milton Abramowitz and Irene A Stegun. Handbook of mathematical functions: with formulas, graphs, and mathematical tables, volume 55. Courier Corporation, 1964.

Shipra Agrawal and Navin Goyal. Analysis of Thompson Sampling for the Multi-armed Bandit Problem. In Proceedings of the 25th Annual Conference on Learning Theory (COLT), 2012.

Shipra Agrawal and Navin Goyal. Thompson sampling for contextual bandits with linear payoffs. In Proceedings of the 30th International Conference on Machine Learning (ICML), 2013a.

Shipra Agrawal and Navin Goyal. Further Optimal Regret Bounds for Thompson Sampling. In AISTATS, pages 99–107, 2013b.

John Asmuth, Lihong Li, Michael L Littman, Ali Nouri, and David Wingate. A Bayesian sampling approach to exploration in reinforcement learning. In Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence, pages 19–26. AUAI Press, 2009.

Mohammad Gheshlaghi Azar, Ian Osband, and Rémi Munos. Minimax regret bounds for reinforcement learning. arXiv preprint arXiv:1703.05449, 2017.

Peter L Bartlett and Ambuj Tewari. REGAL: A regularization based algorithm for reinforcement learning in weakly communicating MDPs. In Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence, pages 35–42. AUAI Press, 2009.

Ronen I Brafman and Moshe Tennenholtz. R-max—a general polynomial time algorithm for near-optimal reinforcement learning. Journal of Machine Learning Research, 3(Oct):213–231, 2002.

Sébastien Bubeck and Che-Yu Liu. Prior-free and prior-dependent regret bounds for Thompson sampling. In Advances in Neural Information Processing Systems, pages 638–646, 2013.

Sébastien Bubeck, Nicolo Cesa-Bianchi, et al. Regret analysis of stochastic and nonstochastic multi-armed bandit problems. Foundations and Trends® in Machine Learning, 5(1):1–122, 2012.

Apostolos N Burnetas and Michael N Katehakis. Optimal adaptive policies for Markov decision processes. Mathematics of Operations Research, 22(1):222–255, 1997.

Olivier Chapelle and Lihong Li. An empirical evaluation of Thompson sampling. In Advances in neural information processing systems, pages 2249–2257, 2011.

Christoph Dann and Emma Brunskill. Sample complexity of episodic fixed-horizon reinforcement learning. In Advances in Neural Information Processing Systems, pages 2818–2826, 2015.

Raphaël Fonteneau, Nathan Korda, and Rémi Munos. An optimistic posterior sampling strategy for bayesian reinforcement learning. In NIPS 2013 Workshop on Bayesian Optimization (BayesOpt2013), 2013.

Charles Miller Grinstead and James Laurie Snell. Introduction to probability. American Mathematical Soc., 2012.

Thomas Jakusch, Ronald Ortner, and Peter Auer. Near-optimal regret bounds for reinforcement learning. Journal of Machine Learning Research, 11(Apr):1563–1600, 2010.

Sham Machandranath Kakade et al. On the sample complexity of reinforcement learning. PhD thesis, University of London London, England, 2003.

Emilie Kaufmann, Nathaniel Korda, and Rémi Munos. Thompson Sampling: An Optimal Finite Time Analysis. In International Conference on Algorithmic Learning Theory (ALT), 2012.

Michael J Kearns and Satinder P Singh. Finite-sample convergence rates for Q-learning and indirect algorithms. In Advances in neural information processing systems, pages 996–1002, 1999.
Robert Kleinberg, Aleksandrs Slivkins, and Eli Upfal. Multi-armed bandits in metric spaces. In *Proceedings of the fortieth annual ACM symposium on Theory of computing*, pages 681–690. ACM, 2008.

Ian Osband and Benjamin Van Roy. Why is posterior sampling better than optimism for reinforcement learning. *arXiv preprint arXiv:1607.00215*, 2016.

Ian Osband, Dan Russo, and Benjamin Van Roy. (More) efficient reinforcement learning via posterior sampling. In *Advances in Neural Information Processing Systems*, pages 3003–3011, 2013.

Ian Osband, Benjamin Van Roy, and Zheng Wen. Generalization and exploration via randomized value functions. *arXiv preprint arXiv:1402.0635*, 2014.

Martin L. Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.

Daniel Russo and Benjamin Van Roy. Learning to Optimize Via Posterior Sampling. *Mathematics of Operations Research*, 39(4):1221–1243, 2014.

Daniel Russo and Benjamin Van Roy. An Information-Theoretic Analysis of Thompson Sampling. *Journal of Machine Learning Research (to appear)*, 2015.

Yevgeny Seldin, François Laviolette, Nicolo Cesa-Bianchi, John Shawe-Taylor, and Peter Auer. PAC-Bayesian inequalities for martingales. *IEEE Transactions on Information Theory*, 58(12):7086–7093, 2012.

I. G. Shevtsova. An improvement of convergence rate estimates in the Lyapunov theorem. 82(3):862–864, 2010.

Alexander L Strehl and Michael L Littman. A theoretical analysis of model-based interval estimation. In *Proceedings of the 22nd international conference on Machine learning*, pages 856–863. ACM, 2005.

Alexander L Strehl and Michael L Littman. An analysis of model-based interval estimation for Markov decision processes. *Journal of Computer and System Sciences*, 74(8):1309–1331, 2008.

Ambuj Tewari and Peter L Bartlett. Optimistic linear programming gives logarithmic regret for irreducible MDPs. In *Advances in Neural Information Processing Systems*, pages 1505–1512, 2008.

William R Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, 25(3/4):285–294, 1933.