Zinbiel algebras and bialgebras: main properties and related algebraic structures

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Abstract. This work provides a characterization of left and right Zinbiel algebras. Basic identities are established and discussed, showing that Zinbiel algebras are centersymmetric, and therefore Lie-admissible algebras. Their bimodules are given, and used to build a Zinbiel algebra structure on the direct sum of the underlying vector space and a finite-dimensional vector space. In addition, their matched pair is built, and related to the matched pair of their sub-adjacent Lie algebras. Besides, Zinbiel coalgebras are introduced, and linked to their underlying Lie coalgebras and coassociative coalgebras. Moreover, the related Manin triple is defined, and used to characterize Zinbiel bialgebras, and their equivalence to the associated matched pair.

1. Introduction

One of the important subclasses of nonassociative algebras is that of Lie algebras. Their properties lead to some generalizations of associative algebras such as Lie-admissible algebras [1], left-symmetric algebras [2], flexible algebras, alternative algebras, Malcev algebras, Leibniz algebras, etc.

Leibniz algebra is an algebra $\langle A, [,] \rangle$ such that the following identity is satisfied:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \forall x, y, z \in A.$$  \hspace{1cm} (1.0.1)

The category of Zinbiel algebras is Koszul dual to the category of Leibniz algebras in the sense of J-L. Loday [3], where Zinbiel algebras were called Leibniz algebras. They appear in various domains in mathematics and physics, e.g., in the theory of nonlinear geometric control [4] characterized by the following system:

$$\begin{align*}
\dot{x}_1 &= u_1, \quad |u_1| \leq 1 \\
\dot{x}_2 &= u_2, \quad |u_2| \leq 1 \\
\dot{x}_3 &= x_1 u_2 - u_2 x_1.
\end{align*}$$  \hspace{1cm} (1.0.2)
This system is completely controllable, i.e. every point can be reached from every other point, since the distribution spanned by the vector fields $f_1 = \partial_{x_1} - x_2 \partial_{x_3}$ and $f_2 = \partial_{x_1} - x_3 \partial_{x_3}$ is nonintegrable due to the nonvanishing commutator $[f_1, f_2] = 2 \partial_{x_3}$.

Performing the global coordinate change $y_1 = x_1, y_2 = x_2, y_3 = \frac{1}{2} (x_3 + x_1 x_2)$ transforms the initial system into a less symmetric but simpler form:

$$\begin{cases}
\dot{y}_1 = u_1, & |u_1| \leq 1 \\
\dot{y}_2 = u_2, & |u_2| \leq 1 \\
\dot{y}_3 = y_1 u_2.
\end{cases} \quad (1.0.3)$$

Now setting

$$u_3(t) = (u_1 * u_2)(t) := \left( \int_0^t u_1(s) ds \right) u_2(t) \quad (1.0.4)$$

yields the Zinbiel identity

$$(u_1 \ast (u_2 \ast u_3))(t) = ((u_1 \ast u_2 + u_2 \ast u_1) \ast u_3)(t). \quad (1.0.5)$$

Besides, Zinbiel algebras under q-commutator given by $x \circ_q y = x \circ y + q y \circ x$, where $q \in \mathbb{C}$, were investigated in [5].

In this work, we provide a characterization of left and right Zinbiel algebras. We establish and discussed their basic identities, showing that Zinbiel algebras are centersymmetric, and, therefore, Lie-admissible algebras. Furthermore, the bimodules are defined and used to build a Zinbiel algebra structure on the direct sum of the underlying vector space and a finite-dimensional vector space. In addition, their matched pair is constructed and related to the matched pair of their sub-adjacent Lie algebras. Besides, Zinbiel coalgebras are introduced, and linked to their underlying Lie coalgebras and coassociative algebras. Moreover, the related Manin triple is defined and used to characterize Zinbiel bialgebras, and their equivalence to the associated matched pair.

2. Zinbiel algebras: basic properties and consequences

2.1. Zinbiel algebras

**Definition 2.1** A left (resp. right) Zinbiel algebra is a vector space $\mathcal{A}$ endowed with a bilinear product $\ast$ satisfying, for all $x, y, z \in \mathcal{A}$,

$$(x \ast y) \ast z = x \ast (y \ast z) + x \ast (z \ast y), \quad (2.1.1)$$

$$(\text{resp. } x \ast (y \ast z) = (x \ast y) \ast z + (y \ast x) \ast z), \quad (2.1.2)$$

or, equivalently,

$$(x, y, z) = x \ast (z \ast y) \quad (2.1.3)$$

$$(\text{resp. } (x, y, z) = -(y \ast x) \ast z), \quad (2.1.4)$$

where, $\forall x, y, z \in \mathcal{A}, (x, y, x) = (x \ast y) \ast z - x \ast (y \ast z)$ is the associator associated to $\ast$. 

**Proposition 2.2** Let $\mathcal{A}$ be a vector space, $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(x, y) \mapsto \mu(x, y) = x \circ y$, a bilinear product, $\tau$ an exchange map defined on $\mathcal{A} \otimes \mathcal{A}$, and $\text{id}$ the identity map defined on $\mathcal{A}$. Then, if $(\mathcal{A}, \circ)$ is a left Zinbiel algebra, the relation
\[ x \circ (y \circ z) = y \circ (x \circ z), \quad \forall x, y, z \in \mathcal{A}, \] (2.1.5)
equivalent to the two identities:
\[ \mu \circ (\text{id} \otimes \mu) = \mu \circ (\text{id} \otimes \mu) \circ (\tau \otimes \text{id}), \] (2.1.6)
\text{and}
\[ \mu \circ (\text{id} \otimes \mu) = (\mu \circ \tau) \circ (\mu \otimes \text{id}) \circ (\text{id} \otimes \tau), \] (2.1.7)
holds.
Similarly, if $(\mathcal{A}, \circ)$ is a right Zinbiel algebra, the relation
\[ (x \circ y) \circ z = (x \circ z) \circ y, \quad \forall x, y, z \in \mathcal{A}, \] (2.1.8)
equivalent to the two identities:
\[ \mu \circ (\mu \otimes \text{id}) = \mu \circ (\mu \otimes \text{id}) \circ (\text{id} \otimes \tau), \] (2.1.9)
\text{and}
\[ \mu \circ (\mu \otimes \text{id}) = (\mu \circ \tau) \circ (\mu \otimes \mu) \circ (\tau \otimes \text{id}). \] (2.1.10)
is satisfied.

**2.2. Basic properties**

**Definition 2.3** The opposite algebra of the algebra $(\mathcal{A}, \circ)$ is the algebra denoted by $\mathcal{A}^{\text{opp}} = (\mathcal{A}, \circ_{\text{opp}})$ whose product is given by $x \circ_{\text{opp}} y = y \circ x$, for all $x, y \in \mathcal{A}$.

**Remark 2.4** (i) If $(\mathcal{A}, \circ)$ is a commutative algebra, then $\mathcal{A}^{\text{opp}} = (\mathcal{A}, \circ)$. Conversely, if $\mathcal{A}^{\text{opp}} = (\mathcal{A}, \circ)$, then $(\mathcal{A}, \circ)$ is a commutative algebra.

(ii) The opposite algebra of the left (resp. right) Zinbiel algebra is a right (resp. left) Zinbiel algebra under the same underlying vector space.

In the sequel, both the left and right Zinbiel algebras are simply called Zinbiel algebras.

**Proposition 2.5** [6] Let $(\mathcal{A}, \circ)$ be a Zinbiel algebra. Then $(\mathcal{A}, \{, \}_\circ)$, such that, for all $x, y \in \mathcal{A}$, $\{x, y\}_\circ = x \circ y + y \circ x$, is a commutative associative algebra.

**Proposition 2.6** Consider a Zinbiel algebra $(\mathcal{A}, \mu)$, $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(x, y) \mapsto \mu(x, y) = x \circ y$, and the bilinear map $\tau : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ given by $\tau(x \otimes y) = y \otimes x$ for all $x, y \in \mathcal{A}$. Then, the following identities hold:
\[ \mu \circ (\text{id} \otimes (\mu \circ \tau)) = \mu \circ (\mu \otimes \text{id}) \circ (\text{id} \otimes \tau) + (\mu \circ \tau) \circ (\text{id} \otimes (\mu \circ \tau)) \circ (\tau \otimes \text{id}) \] (2.2.1)
\[ (\mu \circ \tau) \circ (\mu \otimes \text{id}) = \mu \circ (\mu \otimes \text{id}) \circ (\text{id} \otimes \tau) + (\mu \circ \tau) \circ (\text{id} \otimes (\mu \circ \tau)) \circ (\tau \otimes \text{id}) \] (2.2.2)
\[ (\mu \circ \tau) \circ ((\mu \circ \tau) \otimes \text{id}) = (\mu \circ \tau) \circ (\text{id} \otimes \mu) + (\mu \circ \tau) \circ (\text{id} \otimes (\mu \circ \tau)) \] (2.2.3)
\[ \mu \circ (\text{id} \otimes \mu) = \mu \circ (\text{id} \otimes \mu) \circ (\tau \otimes \text{id}) \] (2.2.4)
Theorem 2.7 Let \( (A, \ast) \) be a Zinbiel algebra. Then \( (A, [[\cdot, \cdot]]_\ast) \) is a Lie algebra, where, for all \( x, y \in A \), \( [x, y]_\ast = x \ast y - y \ast x \).

Proof. Consider a Zinbiel algebra \( (A, \mu) \), where for all \( x, y \in A \), \( \mu(x, y) = x \ast y \). By its definition, the commutator of a bilinear product is bilinear and skew symmetric. Thus, proving that \( (A, [[\cdot, \cdot]]_\ast) := \mu - \mu \circ \tau \) is a Lie algebra, it only remains to prove that \( [[\cdot, \cdot]]_\ast := \mu - \mu \circ \tau \) satisfies the Jacobi identity. Indeed, we have, for all \( x, y, z \in A \),

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = [x, y \ast z - z \ast y] + [y, z \ast x - x \ast z] + [z, x \ast y - x \ast y] = \{x \ast (y \ast z) - y \ast (x \ast z)\} + \{y \ast (z \ast x) - z \ast (y \ast x)\} + \{(z \ast x) \ast y - (x \ast y) \ast z\} = \{\mu \circ (id \otimes \mu \circ \tau)\} - \{\mu \circ (\mu \circ \tau) \otimes id\} \circ (\tau \otimes id)\}(x, y, z) - \{\mu \circ (\mu \circ \tau) \otimes id\} \circ (\tau \otimes id)\}(z, x, y) + \{\mu \circ (\mu \circ \tau) \circ (id \otimes (\mu \circ \tau))\} - \{\mu \circ (\mu \circ \tau) \circ (id \otimes (\mu \circ \tau))\} \circ (\tau \otimes id)\}(x, y, z).
\]

Using the relations (2.1.6), (2.1.7), (2.1.10) and (2.1.9), the right hand side of the last equality of this equation vanishes. Therefore, for all \( x, y, z \in A \),

\[
[x, [y, z]]_\ast + [y, [z, x]]_\ast + [z, [x, y]]_\ast = 0,
\]

i.e. the Jacobi identity associated to the bilinear product \([\cdot, \cdot]_\ast\) holds. Hence, \( (A, [[\cdot, \cdot]]_\ast) \) is a Lie algebra.

From the definition of a center-symmetric algebra given in [7] by, for all \( x, y, z \in A \), \((x, y, z)_o := (z, y, x)_o\), where \((x, y, z)_o := (x \circ y) \circ z - x \circ (y \circ z)\) is the associator of the bilinear product \( \circ \), we have:

Proposition 2.8 Any Zinbiel algebra is a center-symmetric algebra.

Example 2.9 Consider a vector space \( A \) given by \( A = \mathbb{C}[X] := \left\{ a = \sum_{k \in \mathbb{N}} a_k X^k, a_k \in \mathbb{C} \right\} \). On \( A \), we define the bilinear products given by, for all \( a, b \in A \),

\[
a \ast b := b \int_0^X a(t)dt \quad \text{and} \quad a \circ b := \int_0^X b(t) \partial_t(a(t))dt,
\]

where \( \partial_t := \frac{d}{dt} \). Then, the associators of these products are given, for any \( a, b, c \in A \), by

\[
(a, b, c)_\ast = -(b \ast a) \ast c, \quad (a, b, c)_o = a \circ (c \circ b).
\]

Therefore, \( (\mathbb{C}[X], \ast) \) and \( (\mathbb{C}[X], \circ) \) are right and left Zinbiel algebras, respectively.
2.3. Bimodule

**Definition 2.10** A bimodule of a Zinbiel algebra is a triple \((l, r, \mathcal{A})\), where \(\mathcal{A}\) is a vector space endowed with a Zinbiel algebra structure, \(V\) is a vector space, and \(l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)\) are two linear maps satisfying the following relations, for all \(x, y \in \mathcal{A}\),

\[
l_{xy} = l_{yx} + l_{y,x}, \quad (2.3.1)
\]

\[
l_{xy} = r_{yx} + r_{y,x}. \quad (2.3.2)
\]

**Proposition 2.11** Let \((\mathcal{A}, \cdot)\) be a Zinbiel algebra. Consider a vector space \(V\) over a field \(K\) and two linear maps \(l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)\). The triple \((l, r, V)\) is a bimodule of \(\mathcal{A}\) if and only if there is a Zinbiel algebra structure on the vector space \(\mathcal{A} \oplus V\) given by, \(\forall x, y \in \mathcal{A}\) and all \(u, v \in V\),

\[
(x + u) \star (y + v) = x \cdot y + (l_x v + r_y u). \quad (2.3.3)
\]

**Proposition 2.12** Let \((l, r, V)\) be a bimodule of a Zinbiel algebra \((\mathcal{A}, \cdot)\), where \(V\) is a vector space and \(l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)\) are two linear maps. Then,

(i) the following conditions are satisfied, for all \(x, y \in \mathcal{A}\),

\[
l_{xy} = r_{yx}, \quad (2.3.4)
\]

\[
r_{xy} = r_{yx}. \quad (2.3.5)
\]

(ii) the linear map \(l - r : \mathcal{A} \rightarrow \mathfrak{gl}(V), x \mapsto l_x - r_x\) is a representation of the sub-adjacent Lie algebra \(\mathcal{G}(\mathcal{A}) := (\mathcal{A}, [[\cdot, \cdot]])\) associated to \((\mathcal{A}, \cdot)\).

2.4. Matched pair

**Definition 2.13** [8] Let \((\mathcal{G}, \cdot)\) and \((\mathcal{H}, \circ)\) be two commutative associative algebras, and \(\rho : \mathcal{H} \rightarrow \mathfrak{gl}(\mathcal{G})\) and \(\mu : \mathcal{G} \rightarrow \mathfrak{gl}(\mathcal{H})\) be two \(K\)-linear maps which are representations of \(\mathcal{H}\) and \(\mathcal{G}\), respectively, satisfying the following relations: for all \(x, y \in \mathcal{G}\) and all \(a, b \in \mathcal{H}\),

\[
\mu(x)(a \circ b) = (\mu(x)a) \circ b + \mu(\rho(a)x)b, \quad (2.4.1)
\]

\[
\rho(a)(x \cdot y) = (\rho(a)x) \cdot b + \rho(\mu(x)a)y. \quad (2.4.2)
\]

Then, \((\mathcal{G}, \mathcal{H}, \rho, \mu)\) is called a matched pair of the commutative associative algebras \(\mathcal{G}\) and \(\mathcal{H}\), denoted by \(\mathcal{G} \bowtie^p \mathcal{H}\).

In this case, \((\mathcal{G} \oplus \mathcal{H}, \ast)\) defines a commutative associative algebra with respect to the product \(\ast\), given, for all \(x, y \in \mathcal{G}\) and all \(a, b \in \mathcal{H}\), by

\[
(x + a) \ast (y + b) = x \cdot y + \mu(a)y + \mu(b)x + a \circ b + \rho(x)b + \rho(y)a.
\]

**Definition 2.14** [9] Let \((\mathcal{G}, [\cdot, \cdot]_\mathcal{G})\) and \((\mathcal{H}, [\cdot, \cdot]_\mathcal{H})\) be two Lie algebras and let \(\mu : \mathcal{H} \rightarrow \mathfrak{gl}(\mathcal{G})\) and \(\rho : \mathcal{G} \rightarrow \mathfrak{gl}(\mathcal{H})\) be two Lie algebra representations satisfying the following relations, for all \(x, y \in \mathcal{G}\) and all \(a, b \in \mathcal{H}\),

\[
\rho(x)[a, b]_\mathcal{H} - [\rho(x)a, b]_\mathcal{H} - [a, \rho(x)b]_\mathcal{H} + \rho(\mu(a)x)b - \rho(\mu(b)x)a = 0, \quad (2.4.3)
\]

\[
\mu(a)[x, y]_\mathcal{G} - [\mu(a)x, y]_\mathcal{G} - [x, \mu(a)y]_\mathcal{G} + \mu(\rho(x)a)y - \mu(\rho(y)a)x = 0. \quad (2.4.4)
\]

Then, \((\mathcal{G}, \mathcal{H}, \rho, \mu)\) is called a matched pair of the Lie algebras \(\mathcal{G}\) and \(\mathcal{H}\), denoted by \(\mathcal{G} \bowtie^\Delta \mathcal{H}\).

In this case, \((\mathcal{G} \oplus \mathcal{H}, [\cdot, \cdot]_{\mathcal{G} \oplus \mathcal{H}})\) is a Lie algebra with respect to the product \([\cdot, \cdot]_{\mathcal{G} \oplus \mathcal{H}}\) defined on the direct sum vector space \(\mathcal{G} \oplus \mathcal{H}\) by, \(\forall x, y \in \mathcal{G}\) and all \(a, b \in \mathcal{H}\),

\[
[(x + a), (y + b)]_{\mathcal{G} \oplus \mathcal{H}} = [x, y]_\mathcal{G} + \mu(a)y - \mu(b)x + [a, b]_\mathcal{H} + \rho(x)b - \rho(y)a. \quad (2.4.5)
\]
Theorem 2.15 Let \((A, \cdot)\) and \((B, \circ)\) be two Zinbiel algebras. Consider the four linear maps defined as \(l_A, r_A : A \to \mathfrak{gl}(B)\) and \(l_B, r_B : B \to \mathfrak{gl}(A)\). Suppose that \((l_A, r_A, B)\) and \((l_B, r_B, A)\) are bimodules of \(A\) and \(B\), respectively, obeying the relations, for all \(x, y \in A\) and all \(a, b \in B\),

\[
\begin{align*}
r_B(a)(x \cdot y + y \cdot x) - x \cdot (r_B(a)y) - r_B(l_A(y)a)x &= 0, \quad (2.4.6) \\
r_A(x)(a \circ b + b \circ a) - a \circ (r_A(x)b) - r_A(l_B(b)x)a &= 0, \quad (2.4.7) \\
l_B(a)(x \cdot y) &= ((l_B + r_B(a))x) \cdot y + l_B((l_A + r_A(x))a)y \quad (2.4.8) \\
l_A(x)(a \circ b) &= l_A((l_B + r_B(a))x)b + ((l_A + r_A(x))a) \circ b \quad (2.4.9)
\end{align*}
\]

Then, there is a Zinbiel algebra structure on \(A \oplus B\) given by:

\[
(x + a) \ast (y + b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \circ b + l_A(x)b + r_A(y)a). \quad (2.4.10)
\]

Proof. Suppose \(x, y, z \in A\) and \(a, b, c \in B\). Then, using the bilinear product \(\ast\) defined on \(A \oplus B\) by \((x + a) \ast (y + b) = \{x \cdot y + l_B(a)y + r_B(b)x\} + \{a \circ b + l_A(x)b + r_A(y)a\}\), we have:

\[
\begin{align*}
\{(x + a) \ast (y + b) \ast (z + c) &= (x \cdot y) \cdot z + ((l_B(a)y) \cdot z + l_B(r_A(y)az)) + l_B(a \circ b)z \\
+r_B(c)(x \cdot y) + ((r_B(b)x) \cdot z + l_B(l_A(x)b)z) &+ r_B(c)(l_B(a)y) + r_B(c)(r_B(b)x) \\
+(a \circ b) \circ c + ((l_A(x)b) \circ c + l_A(r_B(b)x)c) &+ l_A(x \cdot y)c + r_A(z)(a \circ b) \\
+r_A(z)(l_A(x)b) + ((r_A(y)a) \circ c + l_A(l_B(a)y)c) &+ r_A(z)(r_A(y)a), \quad (2.4.11)
\end{align*}
\]

\[
\begin{align*}
\{(y + b) \ast (x + a) \ast (z + c) &= (y \cdot x) \cdot z + ((l_B(b)x) \cdot z + l_B(r_A(x)bz)) + l_B(b \circ a)z \\
+r_B(c)(y \cdot x) + ((r_B(a)y) \cdot z + l_B(l_A(y)a)z) &+ r_B(c)(r_B(b)y) \\
+(b \circ a) \circ c + ((l_A(y)a) \circ c + l_A(r_B(b)y)c) &+ l_A(y \cdot x)c + r_A(z)(b \circ a) \\
+(r_A(x)b) \circ c + l_A(l_B(b)x)c &+ r_A(z)(l_A(y)a) + r_A(z)(r_A(x)b), \quad (2.4.12)
\end{align*}
\]

Summing (2.4.11) and (2.4.12) gives (2.4.13), what is equivalent to say that \((l_A, r_A, B)\) and \((l_B, r_B, A)\) are bimodules of the Zinbiel algebras \((A, \cdot)\) and \((B, \circ)\), respectively, and the four linear maps \(l_A, r_A, l_B, r_B\) satisfy the equations (2.4.6), (2.4.7), (2.4.8), and (2.4.9).

Corollary 2.16 Let \((A, B, l_A, r_A, l_B, r_B)\) be a matched pair of the Zinbiel algebras \((A, \cdot)\) and \((B, \circ)\). Then,

(i) \((G_{ass}(A), G_{ass}(B), l_A + r_A, l_B + r_B)\) is a matched pair of the commutative associative algebras \(G_{ass}(A) := (A, [\cdot, \cdot])\) and \(G_{ass}(B) := (B, [\cdot, \cdot])\).

(ii) \((G(A), G(B), l_A - r_A, l_B - r_B)\) is a matched pair of the Lie algebras \(G(A) := (A, [\cdot, \cdot])\) and \(G(B) := (B, [\cdot, \cdot])\).
3. Zinbiel coalgebras, Manin triple and bialgebra

The structures of nonassociative coalgebras and their basic properties are investigated and discussed in [10] and references therein. This section is devoted to the construction of Zinbiel bialgebra using their associated coalgebras and Manin triple.

3.1. Zinbiel coalgebras

**Definition 3.1** Let $A$ be a vector space equipped with the linear map $\Delta : A \to A \otimes A$. The couple $(A, \Delta)$ is a right Zinbiel coalgebra if $\Delta$ satisfies the following identity

$$(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta + ((\tau \circ \Delta) \otimes \text{id}) \circ \Delta,$$

(3.1.1)

illustrated by the following commutative diagram:

\[ \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow \Delta & & \downarrow \text{id} \otimes \Delta \\ A \otimes A & \xrightarrow{(\Delta + (\tau \circ \Delta)) \otimes \text{id}} & A \otimes A \otimes A \end{array} \]  

(3.1.2)

where $\text{id}$ is the identity map on $A$, and $\tau$ is the exchange map defined on $A \otimes A$.

**Definition 3.2** Let $A$ be a vector space endowed with the linear map $\Delta : A \to A \otimes A$. The couple $(A, \Delta)$ is a left Zinbiel coalgebra if $\Delta$ satisfies the following identity:

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta + (\text{id} \otimes (\tau \circ \Delta)) \circ \Delta,$$

(3.1.3)

illustrated by the following commutative diagram:

\[ \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\ A \otimes A & \xrightarrow{\text{id} \otimes (\Delta + (\tau \circ \Delta))} & A \otimes A \otimes A \end{array} \]  

(3.1.4)

where $\text{id}$ is the identity map on $A$, and $\tau$ is the exchange map defined on $A \otimes A$.

**Proposition 3.3** Let $A$ be a vector space equipped with a linear map $\Delta : A \to A \otimes A$, $\tau$ be the exchange map defined on $A \otimes A$, and $\text{id}$ be the identity map defined on $A$.

(i) If $(A, \Delta)$ is a right Zinbiel coalgebra, then the relation

$$(\text{id} \otimes \Delta) \circ \Delta = (\tau \otimes \text{id}) \circ (\text{id} \otimes \Delta) \circ \Delta = (\tau \otimes \text{id}) \circ (\Delta \otimes \text{id}) \circ (\tau \circ \Delta)$$

(3.1.5)

is satisfied.

(ii) If $(A, \Delta)$ is a left Zinbiel coalgebra, then the relation

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \tau) \circ (\text{id} \otimes \Delta) \circ (\tau \circ \Delta)$$

(3.1.6)

holds.

**Definition 3.4** A commutative infinitesimal bialgebra $(A, \cdot, \Delta)$ is an infinitesimal bialgebra $(A, \cdot, \Delta)$ such that $(A, \cdot)$ is a commutative algebra and $(A, \Delta)$ is a cocommutative coalgebra.
Definition 3.5 The opposite coalgebra of the coalgebra \((\mathcal{A}, \Delta)\) is a coalgebra \((\mathcal{A}, \tau \circ \Delta)\), where the coproduct \(\tau \circ \Delta\) is defined by: \(\tau \circ \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\).

Proposition 3.6 The opposite coalgebra of a right Zinbiel coalgebra is a left Zinbiel coalgebra. Conversely, the opposite coalgebra of a left Zinbiel coalgebra is a right Zinbiel coalgebra.

Proposition 3.7 Consider a right Zinbiel coalgebra \((\mathcal{A}, \Delta)\), and \(\tau\) the exchange map given by \(\tau : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\), for all \(x, y \in \mathcal{A}\), \(\tau(x \otimes y) = y \otimes x\). Then, the following equations hold:

\[
(id \otimes (\tau \circ \Delta)) \circ \Delta = (id \otimes \tau) \circ (\Delta \otimes id) \circ \Delta + (\tau \otimes id) \circ (id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta), \quad (3.1.7)
\]

\[
(\Delta \otimes id) \circ (\tau \circ \Delta) = (id \otimes \tau) \circ (\Delta \otimes id) \circ \Delta + (\tau \otimes id) \circ (id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta), \quad (3.1.8)
\]

\[
((\tau \circ \Delta) \otimes id) \circ (\tau \circ \Delta) = (id \otimes \Delta) \circ (\tau \circ \Delta) + (id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta). \quad (3.1.9)
\]

Proof.
Consider a right Zinbiel coalgebra \((\mathcal{A}, \Delta)\) and the exchange map \(\tau\) on \(\mathcal{A} \otimes \mathcal{A}\). We have:

\[
(id \otimes (\tau \circ \Delta)) \circ \Delta = (id \otimes \tau) \circ (\Delta \otimes id) \circ \Delta = (id \otimes \tau) \circ ((\Delta \otimes id) \circ \Delta) + (id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta).
\]

Hence, the relation (3.1.7) holds.

Besides,

\[
(id \otimes (\tau \circ \Delta)) \circ \Delta = ((\tau \circ \Delta) \otimes id) \circ (\tau \circ \Delta) = (\tau \otimes id) \circ (\Delta \otimes id) \circ (\tau \circ \Delta)
\]

\[
= (id \otimes (\tau \circ \Delta)) \circ (\Delta \otimes id) \circ (\tau \circ \Delta),
\]

which is the relation (3.1.8).

Using the following identities,

\[
((\tau \circ \Delta) \otimes id) \circ (\tau \circ \Delta) = (id \otimes (\tau \circ \Delta)) \circ \Delta,
\]

\[
(id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta) = (\Delta \otimes id) \circ (\tau \circ \Delta) = (\tau \otimes id) \circ ((\tau \circ \Delta) \otimes id) \circ \Delta
\]

\[
(id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta) = (\tau \otimes id) \circ (id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta),
\]

and

\[
(id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta) = (id \otimes \tau) \circ (id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta) = (id \otimes \tau) \circ (\Delta \otimes id) \circ \Delta,
\]

we obtain that the relations (3.1.7) and (3.1.9) are equivalent. \(\square\)

Definition 3.8 A cocommutative coassociative coalgebra is the couple \((\mathcal{A}, \Delta)\), where \(\mathcal{A}\) is a vector space and \(\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\) a linear map such that the following relations hold:

\[
\Delta = \tau \circ \Delta, \quad (3.1.10)
\]

\[
(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta. \quad (3.1.11)
\]
Proposition 3.9 Consider a vector space $A$ and $\Delta$ a linear map given by $\Delta : A \to A \otimes A$. The couple $(A, \Delta)$ is a Zinbiel coalgebra if $(A, \Delta_{(1)})$ is a cocommutative coassociative coalgebra, where $\Delta_{(1)} : A \to A \otimes A$ is given by $\Delta_{(1)} := \Delta + \tau \circ \Delta$, and $\tau$ is the exchange map defined on $A \otimes A$.

Definition 3.10 Let $A$ be a vector space equipped with the linear map $\Delta : A \to A \otimes A$. Then, the couple $(A, \Delta)$ is a Lie coalgebra if the following relations are satisfied:

$$\Delta = -\tau \circ \Delta,$$

$$\quad (\text{id} \otimes \Delta) \circ \Delta + (\text{id} \otimes \tau) \circ (\Delta \otimes \text{id}) \circ \Delta - (\Delta \otimes \text{id}) \circ \Delta = 0.$$  (3.1.12, 3.1.13)

Proposition 3.11 Consider a vector space $A$ equipped with a linear map $\Delta : A \to A \otimes A$. Then, the couple $(A, \Delta)$ is a Zinbiel coalgebra if the linear map $\Delta_{(1)} : A \to A \otimes A$ given by $\Delta_{(1)} := \Delta - \tau \circ \Delta$ satisfies the relations (3.1.12) and (3.1.13).

Proof. Let $(A, \mu)$ be a Zinbiel algebra and $(A, \Delta)$ be a Zinbiel coalgebra. From the definition of the exchange map, we have:

$$\Delta_{(1)} = \text{id} \circ \Delta_{(1)} = \tau \circ (\tau \circ \Delta_{(1)}) = \tau \circ (\tau \circ (\Delta - \tau \circ \Delta))$$

$$\quad = \tau \circ (\tau \circ \Delta - \Delta) = -\tau \circ (\Delta - \tau \circ \Delta),$$

and, hence, $\Delta_{(1)} = -\tau \circ \Delta_{(1)}$ which is the relation (3.1.12).

Besides,

$$(\text{id} \otimes \Delta_{(1)}) \circ \Delta_{(1)} = (\text{id} \otimes (\Delta - \tau \circ \Delta)) \circ (\Delta - \tau \circ \Delta)$$

$$\quad = (\text{id} \otimes \Delta) \circ \Delta - (\text{id} \otimes \Delta) \circ (\tau \circ \Delta) - (\text{id} \otimes (\tau \circ \Delta)) \circ \Delta$$

$$\quad + (\text{id} \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta).$$

Then, the relation

$$(\text{id} \otimes \Delta_{(1)}) \circ \Delta_{(1)} = (\text{id} \otimes \Delta) \circ \Delta - (\text{id} \otimes \Delta) \circ (\tau \circ \Delta) - (\text{id} \otimes (\tau \circ \Delta)) \circ \Delta$$

$$\quad + (\text{id} \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta)$$

holds. In addition,

$$(\Delta_{(1)} \otimes \text{id}) \circ \Delta_{(1)} = ((\Delta - \tau \circ \Delta) \otimes \text{id}) \circ (\Delta - \tau \circ \Delta)$$

$$\quad = (\Delta \otimes \text{id}) \circ (\Delta - \tau \circ \Delta) - ((\tau \circ \Delta) \otimes \text{id}) \circ (\Delta - \tau \circ \Delta)$$

$$\quad = (\Delta \otimes \text{id}) \circ (\Delta - (\Delta \otimes \text{id}) \circ (\tau \circ \Delta) - ((\tau \circ \Delta) \otimes \text{id}) \circ \Delta$$

$$\quad + ((\tau \circ \Delta) \otimes \text{id}) \circ (\tau \circ \Delta),$$

and the relation

$$(\Delta_{(1)} \otimes \text{id}) \circ \Delta_{(1)} = (\Delta \otimes \text{id}) \circ (\Delta - (\Delta \otimes \text{id}) \circ (\tau \circ \Delta) - ((\tau \circ \Delta) \otimes \text{id}) \circ \Delta$$

$$\quad + ((\tau \circ \Delta) \otimes \text{id}) \circ (\tau \circ \Delta)$$

is satisfied. Furthermore, we have

$$(\text{id} \otimes \tau) \circ (\Delta_{(1)} \otimes \text{id}) \circ \Delta_{(1)} = (\text{id} \otimes \tau) \circ ((\tau \circ \Delta) \otimes \text{id}) \circ (\tau \circ \Delta)$$

$$\quad - (\text{id} \otimes \tau) \circ (\Delta \otimes \text{id}) \circ (\tau \circ \Delta)$$

$$\quad - (\text{id} \otimes \tau) \circ ((\tau \circ \Delta) \otimes \text{id}) \circ \Delta$$

$$\quad + (\text{id} \otimes \tau) \circ (\Delta \otimes \text{id}) \circ \Delta.$$  (3.1.16)

In addition, by considering the right hand side of the relations (3.1.14), (3.1.15), and (3.1.16), and using the relations (3.1.7), (3.1.8), and (3.1.9), we obtain the identity (3.1.13).
3.2. Manin triple and Zinbiel bialgebras

Definition 3.12 Consider a nonassociative algebra \((A, \cdot)\) and \(B\) a bilinear form on \(A\).

- \(B\) is symmetric if \(B(x, y) = B(y, x)\), \(\forall x, y \in A\).
- \(B\) is invariant if \(B(x \cdot y, z) = B(x, y \cdot z)\), \(\forall x, y, z \in A\).
- \(B\) is nondegenerate if the set \(\{x \in A, B(x, y) = 0, \forall y \in A\}\) is reduce to \(\{0\}\).

Definition 3.13 A Manin triple of Zinbiel algebras is a triple \((A, A^+, A^-)\) endowed with a nondegenerate symmetric bilinear form \(B(\cdot, \cdot)\), invariant on \(A\), and satisfying:

(i) \(A = A^+ \oplus A^-\) as \(K\)-vector space;
(ii) \(A^+\) and \(A^-\) are Zinbiel subalgebras of \(A\);
(iii) \(A^+\) and \(A^-\) are isotropic with respect to \(B(\cdot, \cdot)\), i.e., \(B(A^+, A^+) = 0 = B(A^-, A^-)\).

Theorem 3.14 Let \((A, \cdot)\) and \((A^*, \circ)\) be two Zinbiel algebras. Then, the six-tuple \((A \oplus A^*, R^*, L^*, R^*_a, L^*_a)\) is a matched pair of Zinbiel algebras \(A\) and \(A^*\) if and only if \((A \oplus A^*, A, A^*)\) is a Manin triple endowed with the bilinear form \(B(x + a^*, y + b^*) = <x, b^*> + <y, a^*>, \forall x, y \in A\), \(\forall a^*, b^* \in A^*\), where \(<, >\) is the natural pairing between \(A\) and \(A^*\).

Proposition 3.15 Let \((A, \cdot)\) be a Zinbiel algebra and \((A^*, \circ)\) be a Zinbiel algebra on its dual space \(A^*\). Then the following conditions are equivalent:

(i) \((A \oplus A^*, A, A^*)\) is the standard Manin triple of considered Zinbiel algebra;
(ii) \((G(A), G(A^*), -ad^*, -ad^*_a)\) is a matched pair of sub-adjacent Lie algebras;
(iii) \((A, A^*, R^*, L^*, R^*_a, L^*_a)\) is a matched pair of Zinbiel algebras;
(iv) \((A, A^*)\) is a Zinbiel bialgebra.

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