NULL CONTROLLABILITY FOR A HEAT EQUATION WITH DYNAMIC BOUNDARY CONDITIONS AND DRIFT TERMS

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Abstract. We consider the heat equation in a bounded domain of \( \mathbb{R}^N \) with distributed control (supported on a small open subset) subject to dynamic boundary conditions of surface diffusion type and involving drift terms on the bulk and on the boundary. We prove that the system is null controllable at any time. The result is based on new Carleman estimates for this type of boundary conditions.

1. Introduction. In this paper, we consider the internal null controllability of the heat equation with dynamic boundary conditions of surface diffusion type and drift terms

\[
\begin{aligned}
\partial_t y - d \Delta y + B(x) \nabla y + c(x)y &= \mathbb{1}_\omega \nu + f & \text{in } (0, T) \times \Omega, \\
\partial_t y_T - \delta \Delta_T y_T + d \partial_\nu y + b(x) \nabla_T y_T + \ell(x) y_T &= g & \text{on } (0, T) \times \Gamma, \\
y_T(t, x) &= y_T(t, x) & \text{on } (0, T) \times \Gamma, \\
(y, y_T)|_{t=0} &= (y_0, y_0, \Gamma) & \text{in } \Omega \times \Gamma,
\end{aligned}
\]

where \( \Omega \) is a bounded open domain of \( \mathbb{R}^N \), with smooth boundary \( \Gamma = \partial \Omega \) of class \( C^2 \), \( d, \delta \) are positive real numbers, \( c \in L^\infty(\Omega) \), \( \ell \in L^\infty(\Gamma) \), \( B \in L^\infty(\Omega)^N \), \( b \in L^\infty(\Gamma)^N \), \( f \in L^2((0, T) \times \Omega) \), \( g \in L^2((0, T) \times \Gamma) \), \( \partial_\nu y := (\nabla y, \nu)|_{\Gamma} \), \( \nu \) is the outer unit normal field to \( \Omega \). Further, we denote also by \( \Omega_T := (0, T) \times \Omega \), and \( \Gamma_T := (0, T) \times \partial \Omega \), for \( T > 0 \), \( y_T := y_T | \Gamma \) the trace of a function \( y : \Omega \to \mathbb{R} \). Finally, \( \Delta \) is the Laplace operator, \( \Delta_T \) is the Laplace-Beltrami operator on the Riemannian submanifold \( \Gamma \), and \( \nabla_T \) is the tangential gradient on the Riemannian submanifold \( \Gamma \) (see \[28\] for more details), and \( y_0 \in L^2(\Omega) \), \( y_0, \Gamma \in L^2(\Gamma) \) are the initial states. The term \( d \partial_\nu y \) represents the interaction domain–boundary, while \( \delta \Delta_T y_T \) and \( \nabla_T \) stands respectively for a boundary diffusion and convection. The control \( \nu \) plays the role of a source applied on a small region \( \omega \) of the domain \( \Omega \), and our goal is to show that for all given \( T > 0 \), \( \omega \subseteq \Omega \) and initial data \( y_0 \in L^2(\Omega) \) in the bulk and \( y_0, \Gamma \in L^2(\Gamma) \) on the boundary there exists a control \( \nu \in L^2((0, T) \times \omega) \) such that the solution satisfies

\[ y(T, \cdot) = 0 \quad \text{in } \overline{\Omega}. \]

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As in the literature, by duality, to obtain this aim, we show an observability inequality to its associated backward adjoint problem

\[
\begin{aligned}
\partial_t \psi + d \Delta \psi - c(x) \psi + \text{div}(\psi B(x)) &= 0 & \text{on } \Omega_T, \\
\partial_t \psi_T + \delta \Delta_T \psi_T - d \partial_x \psi_T - \psi B, \nu - \ell(x) \psi_T + \text{div}_T(\psi_T b(x)) &= 0 & \text{on } \Gamma_T, \\
\psi_T(t, x) &= \psi_T(t, x) & \text{on } \Gamma_T, \\
(\psi; \psi_T)|_{t=T} &= (\psi_T, \psi_T, T) & \text{in } \Omega \times \Gamma,
\end{aligned}
\]

that is an estimate of type

\[
\|\psi(0, \cdot)\|^2_{L^2(\Omega)} + \|\psi_T(0, \cdot)\|^2_{L^2(\Gamma)} \leq C_T \int_{\omega_T} |\psi|^2 \, dx \, dt,
\]

where \(C_T\) is a positive constant, which will be explicitly defined later.

Note here that, as \(B\) and \(b\) are only in \(L^\infty(\Omega)^N\) and \(L^\infty(\Gamma)^N\), the terms \(\text{div}(\psi B(x))\) and \(\text{div}_T(\psi_T b(x))\) are understood in the sense of distributions.

The observability estimate (3) will be derived thanks to a suitable Carleman estimate for the following intermediate equation

\[
\begin{aligned}
\partial_t \varphi + d \Delta \varphi &= F_0 - \text{div}(F) & \text{in } \Omega_T, \\
\partial_t \varphi_T + \delta \Delta_T \varphi_T - d \partial_x \varphi_T &= F.\nu + F_{0, \Gamma} - \text{div}_T(F_T) & \text{on } \Gamma_T, \\
\varphi|_{t=T} &= \varphi_T(t, x) & \text{on } \Gamma_T, \\
(\varphi; \varphi_T)|_{t=T} &= (\varphi_T, \varphi_T, T) & \text{in } \Omega \times \Gamma
\end{aligned}
\]

for \(F_0 \in L^2(0, T; L^2(\Omega))\), \(F \in L^2(0, T; L^2(\Omega)^N)\), \(F_{0, \Gamma} \in L^2(0, T; L^2(\Gamma)^N)\), \(F_T \in L^2(0, T; L^2(\Gamma)^N)\), which will be obtained by showing another Carleman estimate fulfilled by the solution to (2). The main result of this paper will be the establishment of a Carleman estimate for the following thermodynamical principles and their physical interpretation was also given in [22].

Null controllability of parabolic equations, using Carleman estimates, is extensively studied in the case of static boundary conditions (Dirichlet, Neumann, or Fourier boundary conditions), see e.g. [4, 6, 11, 12, 13, 15, 25, 31, 33]. Recently, in the case of dynamic boundary conditions as above, Maniar et al. [32] achieved this aim in the absence of drift terms.

Dynamic surface and interface processes have attracted a lot of attention in recent years in the mathematical and applied literature in various contexts (such as, diffusion phenomena in thermodynamics, phase-transition phenomena in a material science, climate science, control theory, in chemical reactor theory, and in colloid chemistry and special flows in hydrodynamics and in a class of problems which arise when the diffusion or flow takes place between a solid and a fluid, see [30, 3, 5, 7, 10, 18, 19, 27, 20, 21, 22, 24, 34, 38, 41]. For some applications of dynamic boundary conditions for physiologically structured populations with diffusion, we refer the reader, for instance, to [9]. In the context of reaction-diffusion equations, dynamic boundary conditions have been rigorously derived in [18] based on first and second thermodynamical principles and their physical interpretation was also given in [22].

This work have focused, as a first step, on the heat equation with constant diffusion coefficients \(d\), but as in the case of static boundary conditions presumably our results also hold for general elliptic second order operators, with diffusion coefficients \(d(x)\), or even depending on time \(d(x, t)\). Recently, many results are obtained even in
the case where diffusion coefficients degenerate in the case of static boundary conditions, see [1] and the references therein. We can also consider possible extensions of the null controllability results above to nonlinear problems of the form
\[
\begin{align*}
\partial_t y - d \Delta y + F(y, \nabla y) &= I_w v + f(t, x) \quad \text{in } \Omega_T, \\
\partial_t y_{\Gamma} - \delta \Delta_{\Gamma} y_{\Gamma} + d \partial_{\nu} y + G(y_{\Gamma}, \nabla y_{\Gamma}) &= g(t, x) \quad \text{on } \Gamma_T, \\
y_{\Gamma}(t; x) &= y_{\Gamma}(t; x) \quad \text{on } \Gamma_T, \\
y(0, \cdot) &= y_0 \quad \text{in } \Omega, \\
y_{\Gamma}(0, \cdot) &= y_{0,\Gamma} \quad \text{on } \Gamma,
\end{align*}
\]
where \( F : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) and \( G : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) are locally Lipschitz-continuous functions with \( F(0) = G(0) = 0 \) as in the case of static boundary conditions (Dirichlet, Neumann boundary, or Fourier boundary conditions) see [6, 14]. The case of nonlinearities leading to a blowup will be treated in a forthcoming paper. The boundary null controllability of our system can be done by the technique of extending the domain \( \Omega \) as in [4, Theorem 2.3] and [32, Theorem 4.5].

2. The wellposedness. We refer to paper [28], where a complete study of the wellposedness and regularity properties of solutions to the inhomogeneous linear system
\[
\begin{align*}
\partial_t y - d \Delta y + B(x).\nabla y + c(x)y &= f \quad \text{in } \Omega_T, \\
\partial_t y_{\Gamma} - \delta \Delta_{\Gamma} y_{\Gamma} + d \partial_{\nu} y + b(x).\nabla y_{\Gamma} + \ell(x)y_{\Gamma} &= g \quad \text{on } \Gamma_T, \\
y_{\Gamma}(t; x) &= y_{\Gamma}(t; x) \quad \text{on } \Gamma_T, \\
y(0, \cdot) &= y_0 \quad \text{in } \Omega, \\
y_{\Gamma}(0, \cdot) &= y_{0,\Gamma} \quad \text{on } \Gamma,
\end{align*}
\]
and the inhomogeneous backward adjoint problem (in the transposition sense)
\[
\begin{align*}
- \partial_t \psi - d \Delta \psi + c(x)\psi - \text{div}(\psi B(x)) &= f \quad \text{on } \Omega_T, \\
- \partial_t \psi_{\Gamma} - \delta \Delta_{\Gamma} \psi_{\Gamma} + d \partial_{\nu} \psi_{\Gamma} + \psi B.\nu + \ell(x)\psi_{\Gamma} - \text{div}_{\Gamma}(\psi B(x)) &= g \quad \text{on } \Gamma_T, \\
\psi_{\Gamma}(t; x) &= \psi_{\Gamma}(t; x) \quad \text{on } \Gamma_T, \\
(\psi, \psi_{\Gamma})_{|t=0} &= (\psi_0, \psi_{0,\Gamma}) \quad \text{in } \Omega \times \Gamma,
\end{align*}
\]
with \( f \in L^2(0, T; (H^1(\Omega))') \) and \( g \in L^2(0, T; H^{-1}(\Gamma)) \), is established. Recall briefly the functional spaces introduced and the main wellposedness results of [28]. Define the Hilbert space \( L^2 := L^2(\Omega) \times L^2(\Gamma) \) with the scalar product
\[
(\langle U, V \rangle_{L^2} := \langle u, v \rangle_{L^2(\Omega)} + \langle u_{\Gamma}, v_{\Gamma} \rangle_{L^2(\Gamma)}), \quad U = (u, u_{\Gamma}), V = (v, v_{\Gamma}) \in L^2.
\]
Introduce also the interpolation spaces \( H^k := \{(u, u_{\Gamma}) \in H^k(\Omega) \times H^k(\Gamma) : u_{|\Gamma} = u_{\Gamma}\}, k \in \mathbb{N} \), endowed with the scalar product
\[
(\langle U, V \rangle_{H^k} := \langle u, v \rangle_{H^k(\Omega)} + \langle u_{\Gamma}, v_{\Gamma} \rangle_{H^k(\Gamma)}),
\]
and the dual space \( H^{-1} \) of \( H^1 \) with respect to the pivot space \( L^2 \), that is, the space of all continuous linear functionals on \( H^1 \), equipped with the (dual) norm \(||\Theta||_{H^{-1}} = \sup_{||U||_{H^1} \leq 1} \langle \Theta, U \rangle_{H^{-1}, H^1}||. \) Recall that \((H^{-1}, |\cdot|_{H^{-1}}) \) is a Hilbert space.

For \((\theta, \theta_{\Gamma}) \in (H^1(\Omega))' \times H^{-1}(\Gamma)\), we denote by \(\begin{bmatrix} \theta \\ \theta_{\Gamma} \end{bmatrix} \) the element of \( H^{-1} \) defined
by
\[
\left\langle \begin{bmatrix} \theta \\ \theta_T \end{bmatrix}, (v, v_T) \right\rangle_{E^{-1}, E} = \langle \theta, v \rangle_{(H^1(\Omega))^s, H^1(\Omega)} + \langle \theta_T, v_T \rangle_{H^{-1}(\Gamma), H^1(\Gamma)}. \]
We also introduce the Hilbert space of time
\[
E_1 := \{ f \in L^2(0, T; \mathbb{H}^2) : f' \in L^2(0, T; \mathbb{L}^2) \},
\]
with the scalar product given by
\[
\langle f, g \rangle = \int_0^T \langle f(t), g(t) \rangle_{\mathbb{H}^2} dt + \int_0^T \langle f', g' \rangle_{\mathbb{L}^2} dt,
\]
and its associated norm will be denoted by \( \| \cdot \|_{E_1} \). For the wellposedness of the adjoint problem, we need also the Banach space
\[
W_1 := \{ (u, u_T) \in L^2(0, T; \mathbb{H}^1) : \partial_t u(t) \in L^2(0, T; (H^1(\Omega))'), \partial_t u_T(t) \in L^2(0, T; H^{-1}(\Gamma)) \},
\]
with the norm
\[
\| U \|_{W_1} = \left( \| (u, u_T) \|^2_{L^2(0, T; \mathbb{H}^1)} + \left\| \left[ \partial_t u \right] \|^2_{L^2(0, T; \mathbb{H}^{-1})} \right\|^{1/2}.
\]
The space \( W_1 \) is continuously embedded in \( C([0, T]; \mathbb{L}^2) \), see [28].

System (4) can be written equivalently as a Cauchy initial value problem
\[
\begin{align*}
Y'(t) &= AY(t) + F(t), \quad t \in [0, T], \\
Y(0) &= Y_0 \in \mathbb{L}^2,
\end{align*}
(6)
\]
where
\[
A = \begin{pmatrix} d\Delta - B \nabla - c & 0 \\
-d\partial_\nu \delta & \delta \Gamma - b \nabla \Gamma - \ell \end{pmatrix}, \quad \mathcal{D}(A) = \mathbb{H}^2,
\]
and
\[
Y(t) = \begin{pmatrix} y(t) \\ y_T(t) \end{pmatrix}, \quad Y_0 = \begin{pmatrix} y_0 \\ y_{0_T} \end{pmatrix}, \quad F(t) = \begin{pmatrix} f(t) \\ f_T(t) \end{pmatrix}.
\]
We have shown, in [28], that the operator \( A \) generates an analytic semigroup \( \left( S(t) \right)_{t \geq 0} \) on \( \mathbb{L}^2 \). We adopt the following notion of solutions to system (4).

**Definition 2.1.** A function \( U := (u, u_T) \) is said to be a strong solution of (4) if \( U \in E_1 \) and fulfills (4) a.e., \( t \in [0, T] \). A function \( U := (u, u_T) \) is called a mild solution of (4) if \( U \in C([0, T]; \mathbb{L}^2) \) and satisfies
\[
U(t) = S(t)Y_0 + \int_0^t S(t-s)F(s)ds \quad \text{for} \quad t \in [0, T].
\]
We have established the following existence, uniqueness and regularity results.

**Proposition 1.** Let \( Y_0 := (y_0, y_{0_T}) \in \mathbb{L}^2 \) and \( F := (f, g) \in L^2(\Omega_T) \times L^2(\Gamma_T) \), then the following assertions are true.
1) The Cauchy problem (6), and hence system (4), has a unique mild solution \( U \) given by
\[
U(t) = S(t)Y_0 + \int_0^t S(t-s)F(s)ds \quad \text{for} \quad t \in [0, T].
\]
Moreover, there exists a constant \( C > 0 \) such that
\[
\| U \|_{C([0, T]; \mathbb{L}^2)} \leq C(\| Y_0 \|_{\mathbb{L}^2} + \| f \|_{L^2(\Omega_T)} + \| g \|_{L^2(\Gamma_T)}).
\]
2) If \( Y_0 \in \mathbb{H}^1 \), the Cauchy problem (6) and hence system (4) has a unique strong solution \( U \in \mathbb{E}_1 \), which is also a mild solution. Moreover, there exists a constant \( C > 0 \) such that
\[
\|U\|_{\mathbb{E}_1} \leq C(\|Y_0\|_{\mathbb{H}^1} + \|f\|_{L^2(\Omega_T)} + \|g\|_{L^2(\Gamma_T)}).
\]

We also gave in [28] the definition of solutions to the backward adjoint problem (5) and proved results of existence and uniqueness of such solutions.

**Definition 2.2.** Let \((u_T, u_{T, T}) \in L^2, f \in L^2(0,T;\langle H^1(\Omega)\rangle')\) and \(g \in L^2(0,T;H^{-1}(\Gamma))\).

1. We say that \(U = (u,u_T) \in \mathbb{W}_1\) is a weak solution of (5) if \((u,u_T)\) satisfies
\[
\begin{align*}
\int_0^T &\langle \partial_t u, v \rangle_{\langle H^1(\Omega)\rangle',\langle H^1(\Omega)\rangle} dt - d\int_{\Omega_T} \nabla u. \nabla v dx dt \\
- &\int_{\Omega_T} cu. v dx dt - \int_{\Omega_T} uB. v dx dt \\
+ &\int_0^T \langle \partial_t u_T, v_T \rangle_{L^2,\langle H^{-1}(\Gamma)\rangle'} dt - \delta \int_{\Gamma_T} \nabla u_T . \nabla v_T dx dt \\
- &\int_{\Gamma_T} \ell u_T v_T d\sigma dt - \int_{\Gamma_T} u_T b. v_T d\sigma dt \\
= &\int_0^T \langle f, v \rangle_{\langle H^1(\Omega)\rangle',\langle H^1(\Omega)\rangle} dt + \int_0^T \langle g, v_T \rangle_{\langle H(\Gamma)\rangle'\langle H(\Gamma)\rangle} dt
\end{align*}
\]
for each \((v,v_T) \in L^2(0,T;\mathbb{H}^1)\) and
\[
(u,u_T)(T,\cdot) = (u_T,u_{T,T}).
\]

2. A function \(U := (u,u_T)\) is said to be a strong solution to (5) if \(U \in L^2(0,T;\mathbb{H}^1) \cap H^1(0,T;\mathbb{H}^{-1})\) and fulfills (5) a.e., \(t \in [0,T]\) in \(\mathbb{H}^{-1}\).

We proved two useful characterizations of the weak solution to (5).

**Proposition 2.** The following conditions are equivalent.

1. \(\Phi = (\psi,\psi_T)\) is the weak solution to (5).

2. \(\Phi \in L^2(0,T;\mathbb{H}^1)\) and for all \((z,z_T) \in \mathbb{W}_1\), with \(z(0) = z_T(0) = 0\), we have
\[
\begin{align*}
\int_{\Omega_T} \psi (B. \nabla z + cz) dx dt + d\int_{\Omega_T} \nabla \psi \nabla z dx dt \\
+ \int_{\Gamma_T} \psi_T \ell z_T d\sigma dt + \delta \int_{\Gamma_T} \nabla \psi_T \nabla z_T d\sigma dt \\
+ \int_0^T \langle \partial_t z, \psi \rangle_{\langle H^1(\Omega)\rangle',\langle H^1(\Omega)\rangle} dt + \int_0^T \langle \partial_t z_T, \psi_T \rangle_{\langle H^{-1}(\Gamma)\rangle',\langle H^1(\Gamma)\rangle} dt \\
= &-\int_{\Omega} z(T,x) \psi(T,x) dx - \int_{\Gamma} z_T(T,x) \psi_T(T,x) d\sigma \\
- &\int_0^T < f(t), z(t) >_{\langle H^1(\Omega)\rangle',\langle H^1(\Omega)\rangle} dt - \int_0^T < g(t), z_T(t) >_{H^{-1}(\Gamma),\langle H^1(\Gamma)\rangle} dt.
\end{align*}
\]
Moreover, there exists a constant

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such that Carleman estimates.

3. For every \( \Phi \in L^2(0, T; \mathbb{L}^2) \) and for all \((z, z_T) \in \mathbb{E}_1\), with \( z(0) = z_T(0) = 0 \), we have

\[
\int_{\Omega_T} \psi (\partial_t z - d \Delta z + B \nabla z + cz) \, dx \, dt
\]

\[ + \int_{\Gamma_T} \psi_T (\partial_t z_T - \delta \Delta z_T + \ell z_T + d \partial_{\nu} z) \, d\sigma \, dt \]

\[ = - \int_{\Omega} z(T, x) \psi(T, x) \, dx - \int_{\Gamma} z_T(T, x) \psi_T(T, x) \, d\sigma \]

\[ - \int_0^T < f(t), z(t) >_{H^1(\Omega), (H^1(\Omega))'} \, dt - \int_0^T < g(t), z_T(t) >_{H^{-1}(\Gamma), H^1(\Gamma)} \, dt. \]  

We have also shown the wellposedness result for the backward problem (5).

Proposition 3. 1. For any \( \Phi_T := (\varphi_T, \varphi_T, T) \in \mathbb{L}^2 \) (resp. \( \Phi_T \in \mathcal{D}(A^*) \)), the backward problem (5) admits a unique weak solution \( U \in \mathbb{W}_1 \).

2. For any \( \Phi_T := (\varphi_T, \varphi_T, T) \in \mathbb{L}^2 \) (resp. classical solution ) \( \Phi \in C([0, T]; \mathbb{L}^2) \) (resp. \( \Phi \in C([0, T]; \mathcal{D}(A^*)) \cap C^1([0, T]; \mathbb{L}^2) \) given by

\[
\Phi(t, \cdot) = (S(T - t))^* \Phi_T, \quad t \in [0, T].
\]

Moreover, there exists a constant \( C = C(\Omega, \omega) > 0 \) such that for all \( \Phi_T := (\varphi_T, \varphi_T, T) \in \mathbb{L}^2 \)

\[
\| \Phi \|_{C([0, T]; \mathbb{L}^2)} \leq \exp(C T (1 + \| \varphi \|_{\infty} + \| B \|_{\infty} + \| \ell \|_{\infty} + \| b \|_{\infty})) \| \Phi_T \|_{\mathbb{L}^2}.
\]

3. Carleman estimates. To establish null controllability of the linear equation (1), we will first prove an observability inequality for the associated backward adjoint problem (2). This will be done by establishing first a Carleman estimate for the following intermediate system

\[
\begin{aligned}
\partial_t \varphi + d \Delta \varphi &= F_0 - \text{div}(F) & \quad & \text{in } \Omega_T, \\
\partial_t \varphi_T + \delta \Delta \varphi_T - d \partial_{\nu} \varphi &= F.T + F_0.T - \text{div}_{\Gamma}(F_T) & \quad & \text{on } \Gamma_T, \\
\varphi_T(t, x) &= \varphi(t, x) & \quad & \text{on } \Gamma_T, \\
(\varphi, \varphi_T)|_{t = T} &= (\varphi, \varphi_T) & \quad & \text{in } \Omega \times \Gamma
\end{aligned}
\]

for \( F_0 \in L^2(0, T; L^2(\Omega)), \quad F \in L^2(0, T; L^2(\Omega)^N), \quad F_0, T \in L^2(0, T; L^2(\Gamma)), \quad F_T \in L^2(0, T; L^2(\Gamma)^N). \)

We begin by recalling a lemma due to Fursikov-Imanuvilov in [25].

Lemma 3.1. For any nonempty open set \( \omega' \subset \Omega \), there is a function \( \eta^0 \in C^2(\Omega) \) such that

\[
\eta^0 > 0 \quad \text{in } \Omega, \quad \eta^0 = 0 \quad \text{on } \Gamma, \quad |\nabla \eta^0| > 0 \quad \text{in } \overline{\Omega \setminus \omega'}.
\]
Since \( |\nabla \eta|^2 = |\nabla r \eta|^2 + |\partial_r \eta|^2 \) on \( \Gamma \), there is a constant \( c > 0 \) such that
\[
\nabla r \eta = 0, \quad |\nabla \eta| = |\partial_r \eta|, \quad \partial_r \eta \leq -c < 0 \quad \text{on} \quad \Gamma.
\]

Given \( \omega' \subset \omega \) an nonempty open subset, we take \( \lambda, m > 1 \) and \( \eta_0 \) with respect to \( \omega' \) as in Lemma 3.1. Following [11] and [32], we define the weight functions \( \alpha \) and \( \xi \) by
\[
\beta(x) = e^{2m\lambda \|\eta\|_{\infty}} - e^{\lambda(m\|\eta\|_{\infty} + \eta(x))}, \quad \theta(t) = \frac{1}{t(T-t)}
\]
\[
\alpha(x, t) = \theta(t) \beta(x), \quad \xi(x, t) = \theta(t) e^{\lambda(m\|\eta\|_{\infty} + \eta(x))}
\]
for \( x \in \overline{\Omega} \) and \( t \in [0, T] \). Note that \( \alpha \) and \( \xi \) are \( C^2 \) and positive on \((0, T) \times \overline{\Omega} \) and blow up as \( t \to 0 \) and \( t \to T \). Observing that \( \theta'(t) = -(T-t)^{-1} \theta^2(t) \), and \( \theta''(t) = 2\theta'(t) + 2(T-2t)^{-2}\theta^2(t) \), there is a constant \( K > 0 \) such that
\[
\frac{4}{T^2} \leq \theta, \quad |\theta'| \leq K T \theta^2, \quad |\theta''| \leq K T^2 \theta^3.
\]

We collect in the following lemma some useful properties and estimates fulfilled by the above weight functions, see [2].

**Lemma 3.2.** (a) For all \( s > 0 \) and \( r \in \mathbb{R} \), the function \( e^{-2sa} \xi \) is bounded on \( \Omega_T \).
(b) If \( s \geq T^2 \) then for all \( \nu, \mu \in \mathbb{R} \)
\[
\nu \leq \mu \implies (s\xi)^\nu \leq (s\xi)^\mu.
\]
(c) For any \( \sigma > 0 \) and \( \mu \in \mathbb{R} \) there exists \( C > 0 \) (only depending on \( \Omega, \omega, \sigma \) and \( \mu \)) such that for any \( s \geq \sigma(T+T^2) \)
\[
|\partial_t(e^{-2sa}\xi\mu)| \leq Cs^2 e^{-2sa}\xi^{\mu+2}.
\]
(d) For any \( \sigma > 0 \), there exists \( C > 0 \) (only depending on \( \Omega, \omega \) and \( \sigma \)) such that for any \( s \geq \sigma(T+T^2) \)
\[
|\Delta(e^{-2sa}\xi)| \leq Cs^2 \lambda^2 e^{-2sa}\xi^3
\]
\[
|\partial_r(e^{-2sa}\xi)| = |\nabla(e^{-2sa}\xi).\nu| \leq \|\nabla(e^{-2sa}\xi)\|_{\mathbb{R}^N} \leq Cs^2 \lambda e^{-2sa}\xi^3.
\]

We recall also the following Carleman estimate from [32], needed to show our main results.

**Lemma 3.3.** For any nonempty open set \( \omega \subset \Omega \), there exist two constants constants \( C = C(\Omega, \omega) > 0 \), \( \lambda_0 > 0 \) and \( \sigma_0 = \sigma_0(\Omega, \omega) > 0 \) such that
\[
s^{-1} \int_{\Omega_T} e^{-2sa} \xi^{-1} (|\partial_t \varphi|^2 + |\Delta \varphi|^2) \, dx \, dt 
\]
\[
+ s^{-1} \int_{\Gamma} e^{-2sa} \xi^{-1} (|\partial_t \varphi|^2 + |\Delta_T \varphi|^2) \, d\sigma \, dt 
\]
\[
+ s\lambda^2 \int_{\Omega_T} e^{-2sa} \xi |\nabla \varphi|^2 \, dx \, dt + s\lambda \int_{\Gamma_T} e^{-2sa} \xi |\nabla_T \varphi|^2 \, d\sigma \, dt 
\]
\[
+ s^3 \lambda^4 \int_{\Omega} e^{-2sa} \xi^3 |\varphi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma} e^{-2sa} \xi^3 |\varphi|^2 \, d\sigma \, dt 
\]
\[
+ s\lambda \int_{\Gamma} e^{-2sa} \xi |\partial_r \varphi|^2 \, d\sigma \, dt
\]

introduce the solution of a variational problem on the space $E_1$. An intermediate Carleman estimate.

E_{\phi}

for all $\lambda \geq \lambda_0$, $s \geq s_0 = \sigma_0(T + T^2)$ and $(\phi, \varphi_T) \in E_1$.

The main result of this section is the next Carleman estimate which will be the key to prove the null controllability of equation (1).

**Theorem 3.4.** Let $F_0 \in L^2(0,T; L^2(\Omega))$, $F \in L^2(0,T; L^2(\Omega)^N)$, $F_{0,T} \in L^2(0,T; L^2(\Gamma))^N$ and $F_T \in L^2(0,T; L^2(\Gamma)^N)$. Then, for any nonempty open subset $\omega \Subset \Omega$, there exist constants $\lambda_1 = \lambda_1(\Omega, \omega) \geq 1$, $\sigma_1 = \sigma_1(\Omega, \omega) > 0$ and $C = C(\Omega, \omega) > 0$ such that for any $\lambda \geq \lambda_1$, $s \geq s_1 = \sigma_1(T + T^2)$ and any $(\varphi_T, \varphi_{T,T}) \in L^2$, the unique weak solution $\Phi = (\varphi, \varphi_T)$ to the system (9) satisfies the Carleman estimate

\[
\begin{align*}
\int_{\omega_T} e^{-2s\alpha}\zeta^3|\varphi|^2 \, dx \, dt + s^3\lambda^4 \int_{\Gamma_T} e^{-2s\alpha}\zeta^3|\partial_t \varphi|^2 \, d\sigma \, dt \\
+ s\lambda^2 \int_{\Omega_T} \xi e^{-2s\alpha}|\nabla \varphi|^2 \, dx \, dt + s\lambda^2 \int_{\Gamma_T} \xi e^{-2s\alpha}|\nabla \varphi|^2 \, d\sigma \, dt \\
\leq C(s^3\lambda^4 \int_{\omega_T} e^{-2s\alpha}\zeta^3|\varphi|^2 \, dx \, dt + \int_{\Omega_T} e^{-2s\alpha}|F_0|^2 \, dx \, dt \\
+ s^2\lambda^2 \int_{\Omega_T} e^{-2s\alpha}\zeta^2 \|F\|_{L^2}^2 \, dx \, dt + \int_{\Gamma_T} e^{-2s\alpha}|F_{0,T}|^2 \, d\sigma \, dt \\
+ s^2\lambda^2 \int_{\Gamma_T} e^{-2s\alpha}\zeta^2 \|F_T\|_{L^2}^2 \, d\sigma \, dt).
\end{align*}
\]

3.1. **An intermediate Carleman estimate.** Following the approach of [20], we introduce the solution of a variational problem on the space $E_1$ involving the solution $\varphi$ of (9) which will be used to derive the Carleman estimate (15). For $Y = (y, y_T) \in E_1$, we set

\[
L_y = \partial_t y - d\Delta y, \quad L^*_y = -\partial_t y - d\Delta y, \\
L_{y_T} = \partial_t y_T - \delta \Delta y_T, \quad L^*_{y_T} = -\partial_t y_T - \delta \Delta y_T.
\]

Let $(\varphi, \varphi_T)$ be the unique weak solution to (9), and consider the variational problem

\[
\begin{align*}
\int_{\Omega_T} e^{-2s\alpha} L^* y L^* v \, dx \, dt + \int_{\Gamma_T} e^{-2s\alpha}(L^*_{y_T} v_T + d\partial_r v_T)(L^*_{y_T} v_T + d\partial_r v_T) \, d\sigma \, dt \\
+ s^3\lambda^4 \int_{\omega_T} e^{-2s\alpha}\zeta^3 y v \, dx \, dt + \int_{\Omega_T} y(T,x) v(T,x) \, dx + \int_{\Gamma_T} y_T\varphi(T, x) v_T(T, x) \, d\sigma \\
= -s^3\lambda^3 \int_{\Omega_T} e^{-2s\alpha}\zeta^3 v \varphi \, dx \, dt - s^3\lambda^3 \int_{\Gamma_T} e^{-2s\alpha}\zeta^3 \varphi_T v \, d\sigma \, dt.
\end{align*}
\]

for all $V := (v, v_T) \in E_1$. To establish the existence and uniqueness of the solution to this variational problem, we use Lax-Milgram Theorem. For this, we need to prove that $E_1$ is a Hilbert space for a new suitable scalar product. Note that this idea was firstly used by Imanuvilov in [25] and Imanuvilov-Yamamoto in [26], by considering the unique solution of an auxiliary extremal problem and solvability of their optimal systems.
Lemma 3.5. For a real positive $s$, let

$$S_{\Omega_T} = \{ f \in L^2(\Omega_T) : e^{\alpha s} f \in L^2(\Omega_T) \}, \quad \langle f_1, f_2 \rangle_{S_{\Omega_T}} = \int_{\Omega_T} e^{\alpha s} f_1 f_2 \, dx \, dt,$$

$$S_{\Gamma_T} = \{ g \in L^2(\Gamma_T) : e^{\alpha s} g \in L^2(\Gamma_T) \}, \quad \langle g_1, g_2 \rangle_{S_{\Gamma_T}} = \int_{\Gamma_T} e^{\alpha s} g_1 g_2 \, d\sigma \, dt,$$

and

$$X = S_{\Omega_T} \times S_{\Gamma_T}, \quad ((f_1, g_1), (f_2, g_2))_{s,\lambda} = \langle f_1, f_2 \rangle_{S_{\Omega_T}} + \langle g_1, g_2 \rangle_{S_{\Gamma_T}}.$$  

Then,

(i) $(X, \langle \cdot, \cdot \rangle_{s,\lambda})$ is a Hilbert space.

(ii) The real bilinear form $\Phi$ on $E_1$ defined by

$$\Phi(U, V) := \int_{\Omega_T} e^{-\alpha s} L^* u L^* v \, dx \, dt + \int_{\Gamma_T} e^{-\alpha s} (L^*_1 u_1 + d\partial_\nu u)(L^*_1 v_1 + d\partial_\nu v) \, d\sigma \, dt$$

$$+ s^3 \lambda^4 \int_{\omega_T} e^{-\alpha s} \xi^3 u v \, dx \, dt + \langle U(T), V(T) \rangle_{\mathbb{H}^1}$$

is a scalar product in $E_1$ and $(E_1, \Phi)$ is a Hilbert space.

**Proof.** For assertion (i), it is obvious that $\langle \cdot, \cdot \rangle_{s,\lambda}$ is a scalar product on $X$. Set $\| \cdot \|_{s,\lambda}$ the associated norm to this scalar product and let $(f_n, g_n)_{n \in \mathbb{N}}$ be a Cauchy sequence of $(X, \| \cdot \|_{s,\lambda})$. Hence, $(e^{\alpha s} f_n, e^{\alpha s} g_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the space $(L^2, \| \cdot \|_{L^2})$, and thus it converges to an element $(f, g) \in L^2$. Therefore, $(e^{\alpha s} f_n, e^{\alpha s} g_n)_{n \in \mathbb{N}}$ converges to $(e^{-\alpha s} f, e^{-\alpha s} g)$ in $X$.

For (ii), note first that since $E_1 \hookrightarrow C([0,T], \mathbb{H}^1)$ (see [32, Proposition 2.2]), we have $\mathcal{U}(\cdot, \cdot) \in \mathbb{H}^1$, for all $U \in E_1$, and so $\Phi$ is well defined. On the other hand, $\Phi$ is symmetric bilinear positive definite, and so, it defines a scalar product in $E_1$. Let $\|U\|_1 = (\Phi(U, U))^{1/2}$ be the associated norm and $(U_n := (u_n, u_n, u_n, \Omega, T, \cdot))_{n \in \mathbb{N}}$ is a Cauchy sequence of $(E_1, \| \cdot \|_1)$. Hence, $(e^{-\alpha s} L^* u_n, e^{-\alpha s} (L^*_1 u_1 + \partial_\nu u_n))$ and $(U_n(T, \cdot)))$ are Cauchy sequences respectively of the Hilbert spaces $X$ and $\mathbb{H}^1$. Therefore, they converge respectively to $F := (f, g) \in X$ and $\Psi = (\psi, \psi, \cdot) \in \mathbb{H}^1$. Hence, the sequence $(e^{-\alpha s} L^* u_n, e^{-\alpha s} (L^*_1 u_n, \partial_\nu u_n))$ converges to $e^{-\alpha s} F$ in $L^2(0, T; L^2)$. Furthermore, since $\Psi \in \mathbb{H}^1$, we deduce by [32, Proposition 2.4] that there exists a unique strong solution $U = (u, u, \Omega, T, \cdot) \in E_1$ to the parabolic system with dynamic boundary conditions

$$\begin{cases}
L^* u = e^{\alpha s} f & \text{in } \Omega_T, \\
L^*_1 u_1 + \partial_\nu u = e^{\alpha s} g & \text{in } \Gamma_T, \\
u_1(t, x) = u_1(t, x) & \text{on } \Gamma_T, \\
(u, u_1)|_{t=T} = (\psi, \psi) & \text{on } \Omega \times \Gamma,
\end{cases}$$

and

$$\int_{\Omega_T} e^{-\alpha s} (L^* u_n - L^* u)^2 \, dx \, dt + \int_{\Gamma_T} e^{-\alpha s} (L^*_1 (u_n, \Gamma - u_1) + d\partial_\nu (u_n - u))^2 \, d\sigma \, dt$$

$$= \|((e^{-\alpha s} L^* u_n, e^{-\alpha s} (L^*_1 u_n, \Gamma + d\partial_\nu u_n)) - e^{-\alpha s} F)^2 |_{L^2(0, T; L^2)}.$$
Thus,
\[
\int_{\Omega_T} e^{-2s\alpha}(L^*u_n - L^*u)^2 \, dx \, dt \\
+ \int_{\Gamma_T} e^{-2s\alpha}(L^*_T(u_{n,T} - u_T) + d\partial_n(u_n - u))^2 \, d\sigma \, dt \to 0. \tag{17}
\]

On the other hand, \((U_n - U)\) is the strong solution to the system
\[
\begin{align*}
L^*y &= e^{s\alpha}f - L^*u_n \quad \text{in } \Omega_T, \\
L^*_T\psi_T + d\partial_n \psi &= e^{s\alpha}g - L^*_T u_{n,T} - d\partial_n u_n \quad \text{in } \Gamma_T, \\
y|_{\Gamma} &= y_T(t, x) \quad \text{on } \Gamma_T, \\
(y, \psi_T)|_{t=T} &= (\psi - u_n(T), \psi_T - u_{n,T}(T)) \quad \text{on } \Omega \times \Gamma,
\end{align*}
\]
and, by [32, Proposition 2.4], we have
\[
\|U - U_n\|_{E_1} \\
\leq C(\|U_n(T) - U(T)\|_{E_1} + \|e^{s\alpha}f - L^*u_n\|_{L^2} + \|e^{s\alpha}g - L^*_T u_{n,T} - d\partial_n u_n\|_{L^2}).
\]
Hence \(\|U_n - U\|_{E_1} \to 0\) and, since the function \(e^{-2s\alpha}\xi^3\) is bounded on \(\Omega_T\), there exists \(C > 0\) such that
\[
\int_{\omega_T} e^{-2s\alpha}\xi^3(u_n - u)^2 \, dx \, dt \leq \int_{\Omega_T} e^{-2s\alpha}\xi^3(u_n - u)^2 \, dx \, dt \leq C\|u_n - u\|_{L^2(0,T; L^2(\Omega))}^2.
\]
From this and (17), we conclude that \(\|U_n - U\|_1 \to 0\), and the proof is complete. \(\Box\)

Now, we can state the following existence result of solutions to the problem (16).

**Proposition 4.** For any \(\lambda \geq \lambda_0\) and any \(s \geq s_0\), the variational problem (16) possesses exactly one solution \(U \in E_1\).

**Proof.** Since \(\Phi\) is a scalar product on \(E_1\), it is continuous and coercive. On the other hand, define the linear form \(L\) on \(E_1\) by
\[
L : Y = (y, \psi_T) \mapsto -s^3\lambda^4 \int_{\Omega_T} e^{-2s\alpha}\xi^3y\varphi \, dx \, dt - s^3\lambda^3 \int_{\Gamma_T} e^{-2s\alpha}\xi^3\psi_T \varphi_T \, d\sigma \, dt.
\]
By Young and Carleman estimate (14), we have
\[
|L(U)| \\
\leq C \left[ \left( s^3\lambda^4 \int_{\Omega_T} e^{-2s\alpha}\xi^3|\varphi|^2 \, dx \, dt \right)^{1/2} + \left( s^3\lambda^3 \int_{\Gamma_T} e^{-2s\alpha}\xi^3|\varphi_T|^2 \, d\sigma \, dt \right)^{1/2} \right] \|U\|_1
\]
for all \(U \in E_1\), and this yields that \(L\) is continuous on \(E_1\). Finally, by Lax-Milgram Theorem, we deduce that the variational problem \(\Phi(U, V) = L(V)\) for all \(V \in E_1\) has a unique solution \(U \in E_1\). \(\Box\)

Now, we show an intermediate Carleman estimate.

**Lemma 3.6.** Let \(\lambda \geq \lambda_0\), \(s \geq s_0\) and \(U = (u, u_T)\) be the unique solution to the variational problem (16), and set
\[
z = e^{-2s\alpha}L^*u, \quad z_T = e^{-2s\alpha}(L^*_T u_T + d\partial_n u), \quad v = s^3\lambda^4 e^{-2s\alpha}\xi^3u.
\]
Then, the following assertions hold.

(i) $Z = (z, z_T)$ is the unique strong solution to the system

\[
\begin{align*}
\frac{\partial q}{\partial t} - d\Delta q &= s^3 \lambda^4 e^{-2s\alpha} \xi^3 \varphi + v I_w \\
\partial_t q - \delta \Delta q_T + d\partial_\nu q &= s^3 \lambda^3 e^{-2s\alpha} \xi^3 \varphi_T \\
q|_{t=0} &= q_T(t, x) \\
(q, q_T)|_{t=T} &= (0, 0)
\end{align*}
\]  

in $\Omega_T$, $\Gamma_T$, respectively.

(ii) There exist $\tilde{s} = \tilde{s}(\Omega, \omega)$, $\tilde{t} = \tilde{t}(\Omega, \omega)$ such that for all $s \geq \tilde{s}(T + T^2)$ and $\lambda \geq \tilde{\lambda}$

\[
s^{-3} \lambda^{-4} \int_{\omega_T} e^{2s\alpha} \xi^{-3} |v|^2 \, dx \, dt + \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt
\]

\[
+ \int_{\Gamma_T} e^{2s\alpha} |z_T|^2 \, d\sigma \, dt + s^{-2} \lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} |\nabla z|^2 \, dx \, dt
\]

\[
+ s^{-2} \lambda^{-2} \int_{\Gamma_T} e^{2s\alpha} \xi^{-2} |\nabla z_T|^2 \, d\sigma \, dt
\]

\[
\leq C \left( s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |\varphi_T|^2 \, d\sigma \, dt \right).
\]

Proof. To show the assertion (i), by [32, Proposition 2.5], it suffices to prove that $Z = (z, z_T)$ is a weak solution of (18), which follows from (16). The proof of (19) will be done in three steps.

**Step 1. Estimate of the three first terms.** As in [11], multiplying the first equation of (18) by $u$ and integrating on $\Omega_T$, we obtain

\[
\int_{\Omega_T} e^{-2s\alpha} |L^* u|^2 \, dx \, dt + s^3 \lambda^4 \int_{\Omega_T} \xi^3 e^{-2s\alpha} |u|^2 \, dx \, dt
\]

\[- d \int_{\Gamma_T} \partial_\nu z u_T \, d\sigma \, dt + d \int_{\Gamma_T} \partial_\nu z_T u_T \, d\sigma \, dt = -s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 \varphi u \, dx \, dt.
\]

Similarly, multiplying the second equation in (18) by $u_T$ and integrating by parts, we obtain

\[
\int_{\Gamma_T} e^{-2s\alpha} |L^* u_T|^2 \, dx \, dt + d \int_{\Gamma_T} \partial_\nu u_T \, d\sigma \, dt + d \int_{\Gamma_T} \partial_\nu u_T \, d\sigma \, dt = -s^3 \lambda^4 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 \varphi_T u_T \, d\sigma \, dt.
\]

Adding (20) and (21), we obtain

\[
\Phi(U, U) = -s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 \varphi u \, dx \, dt - s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 \varphi_T u_T \, d\sigma \, dt.
\]

Using Young inequality, for all $\epsilon > 0$, we obtain

\[
|\Phi(U, U)| \leq \frac{s^3 \lambda^4}{2} \left( \int_{\Omega_T} e^{-1} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt + \int_{\Omega_T} e e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt \right)
\]

\[
+ \frac{s^3 \lambda^3}{2} \left( \int_{\Gamma_T} e^{-1} e^{-2s\alpha} \xi^3 |\varphi_T|^2 \, d\sigma \, dt + \int_{\Gamma_T} e e^{-2s\alpha} \xi^3 |u_T|^2 \, d\sigma \, dt \right).
\]
On the other hand, we have

\[
\begin{align*}
& s^{-3} \lambda^{-4} \int_{\omega_T} e^{2s\alpha} \xi^{-3} |u|^2 \, dx \, dt + \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt + \int_{\Gamma_T} e^{2s\alpha} |z|^2 \, d\sigma \, dt \\
= & s^3 \lambda^4 \int_{\omega_T} e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt + \int_{\Omega_T} e^{-2s\alpha} |L^* u|^2 \, dx \, dt \\
& + \int_{\Gamma_T} e^{-2s\alpha} |L^* u| + d\partial_v u|^2 \, d\sigma \, dt \\
= & \Phi(U, U),
\end{align*}
\]

(23)

Using Carleman estimate (14), (22) and (23), for all \( s \geq s_0 := \sigma_0(T + T^2), \lambda \geq \lambda_0 \) and \( \epsilon > 0 \), one has

\[
\begin{align*}
& s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |u_T|^2 \, d\sigma \, dt \\
\leq & C s^3 \lambda^4 \int_{\omega_T} e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt + \int_{\Omega_T} e^{-2s\alpha} |L^* u|^2 \, dx \, dt \\
& + \int_{\Gamma_T} e^{-2s\alpha} |L^* u_T + d\partial_v u|^2 \, d\sigma \, dt \\
\leq & C s^3 \lambda^4 \left( \int_{\Omega_T} e^{-1} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt + \int_{\Omega_T} e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt \right) \\
& + C s^3 \lambda^4 \left( \int_{\gamma_T} e^{-1} e^{-2s\alpha} \xi^3 |\varphi_T|^2 \, d\sigma \, dt + \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |u_T|^2 \, d\sigma \, dt \right).
\end{align*}
\]

(24)

Let us set

\[
\begin{align*}
A := & s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |u_T|^2 \, d\sigma \, dt, \\
B := & s^3 \lambda^4 \int_{\omega_T} e^{-2s\alpha} \xi^3 |u|^2 \, dx \, dt + \int_{\Omega_T} e^{-2s\alpha} |L^* u|^2 \, dx \, dt \\
& + \int_{\Gamma_T} e^{-2s\alpha} |L^* u_T + d\partial_v u|^2 \, d\sigma \, dt,
\end{align*}
\]

and

\[
D := s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |\varphi_T|^2 \, d\sigma \, dt.
\]

From (24), we obtain \( B \leq \frac{\epsilon^{-1}}{2} D + \frac{\epsilon A}{2} + \frac{\epsilon CB}{2} \). Hence, \( (1 - \frac{\epsilon}{2}) B \leq \frac{\epsilon^{-1}}{2} D \).

Choosing \( 0 < \epsilon < 2C^{-1} \) and setting \( C_1 := \frac{\epsilon^{-1}}{2} (1 - \frac{\epsilon C}{2})^{-1} \), we deduce that \( B \leq C_1 D \). Therefore

\[
\begin{align*}
& s^{-3} \lambda^{-4} \int_{\omega_T} e^{2s\alpha} \xi^{-3} |u|^2 \, dx \, dt + \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt + \int_{\Gamma_T} e^{2s\alpha} |z|^2 \, d\sigma \, dt \\
\leq & C_1 \left( s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |\varphi_T|^2 \, d\sigma \, dt \right)
\end{align*}
\]

(25)

for all \( s \geq s_0 \) and \( \lambda \geq \lambda_0 \).

**Step 2. Estimates of the first order interior term.** To estimate the first order term \( s^{-2} \lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} |\nabla z|^2 \, dx \, dt \), we multiply the equation (18) by \( s^{-2} \lambda^{-2} e^{2s\alpha} \).
\( \xi^{-2}z \) and integrate by parts with respect to the space variable. So, we obtain

\[
\begin{align*}
&\quad s^{-2}\lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} \partial_t z^2 \, dx \, dt + ds^{-2}\lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} |\nabla z|^2 \, dx \, dt \\
&\quad - 2ds^{-1}\lambda^{-1} \int_{\Omega_T} e^{2s\alpha} \xi^{-1} \nabla \eta^0 \cdot \nabla z \, dx \, dt \\
&\quad - 2ds^{-2}\lambda^{-1} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} \nabla \eta^0 \nabla z \, dx \, dt + B_1 \\
&= s\lambda^2 \int_{\Omega_T} \xi \varphi z \, dx \, dt + s^{-2}\lambda^{-2} \int_{\omega_T} e^{2s\alpha} \xi^{-2} v \, dx \, dt,
\end{align*}
\]

with \( B_1 := -ds^{-2}\lambda^{-2} \int_{\Gamma_T} \partial_\nu (e^{2s\alpha} \xi^{-2} z) \, d\sigma \, dt \). Integrating by parts in the first term with respect to the time variable, and using the inequality (11), there exists a constant \( \sigma_1 > 0 \) such that, for all \( s \geq s_1 = \sigma_1 (T + T^2) \) and \( \lambda \geq \lambda_0 \), we have

\[
s^{-2}\lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} z^2 \, dx \, dt \leq C_2 \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt.
\]

The other terms can be estimated, by Young inequality, as follows. Using that \( \nabla \eta^0 \) is a bounded function on \( \Omega \) and setting \( K := \frac{1}{2} \sup_{x \in \Omega} |\nabla \eta^0(x)| \), we have

\[
2s^{-1}\lambda^{-1} \int_{\Omega_T} e^{2s\alpha} \xi^{-1} \nabla \eta^0 \cdot \nabla z \, dx \, dt + 2s^{-2}\lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} |\nabla z|^2 \, dx \, dt \\
\leq K \left( \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt + 2s^{-2}\lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} |\nabla z|^2 \, dx \, dt \right) \\
+ K \left( \epsilon^{-1} \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt + s^{-4}\lambda^{-2} \epsilon \int_{\Omega_T} e^{2s\alpha} \xi^{-4} |\nabla z|^2 \, dx \, dt \right) \\
\leq (1 + \epsilon^{-1})K \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt + \epsilon K \lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} |\nabla z|^2 \, dx \, dt
\]

for all \( \epsilon > 0 \), \( s \geq s_1 \) and \( \lambda \geq \lambda_0 \). Hence,

\[
s\lambda^2 \int_{\Omega_T} \xi \varphi z \, dx \, dt \leq \frac{1}{2} \left( s^{-1} \int_{\Omega_T} e^{2s\alpha} \xi^{-1} |z|^2 \, dx \, dt + s^3\lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt \right) \\
\leq \frac{1}{2} \left( \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt + s^3\lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt \right)
\]

for all \( s \geq s_2 = \sigma_2 (T + T^2) \) and \( \lambda \geq \lambda_0 \). Similarly, estimate (10) yields

\[
s^{-2}\lambda^{-2} \int_{\omega_T} e^{2s\alpha} \xi^{-2} v \, dx \, dt \leq \frac{1}{2} \left( \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt + s^{-4}\lambda^{-4} \int_{\omega_T} e^{2s\alpha} \xi^{-4} |v|^2 \, dx \, dt \right) \\
\leq \frac{1}{2} \left( \int_{\Omega_T} e^{2s\alpha} |z|^2 \, dx \, dt + s^{-3}\lambda^{-4} \int_{\omega_T} e^{2s\alpha} \xi^{-3} |v|^2 \, dx \, dt \right).
\]

Now, (26) yields that

\[
ds^{-2}\lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} |\nabla z|^2 \, dx \, dt + B_1 \\
= -s^{-2}\lambda^{-2} \int_{\Omega_T} e^{2s\alpha} \xi^{-2} z_{\partial_t} z \, dx \, dt + 2ds^{-1}\lambda^{-1} \int_{\Omega_T} e^{2s\alpha} \xi^{-1} \nabla \eta^0 \cdot \nabla z \, dx \, dt
\]
+ 2ds^{-2}\lambda^{-1} \int_{\Omega_T} e^{2z\alpha} \xi^{-2} \nabla \eta \cdot \nabla z \ dx \ dt + s\lambda^2 \int_{\Omega_T} \xi \varphi \ dx \ dt \\
+ s^{-2}\lambda^{-2} \int_{\omega_T} e^{2z\alpha} \xi^{-2} vz \ dx \ dt \\
\leq (dK^{-1} + C_2) \int_{\Omega_T} e^{2z\alpha} |\varphi|^2 \ dx \ dt + \frac{1}{2} s^{-3}\lambda^{-4} \int_{\omega_T} e^{2z\alpha} \xi^{-3} |v|^2 \ dx \ dt \\
+ \frac{1}{2} s^3 \lambda^4 \int_{\Omega_T} e^{-2z\alpha} \xi^3 |\varphi|^2 \ dx \ dt + KK^{-2} \int_{\Omega_T} e^{2z\alpha} \xi^{-2} |\nabla z|^2 \ dx \ dt.

Setting \( C_3 := C_1 \max((dK^{-1} + C_2, \frac{1}{2})) \), we have, for \( s \geq s_2 \) and \( \lambda \geq \lambda_0 \),

\[ (d - \epsilon K) s^{-2}\lambda^{-2} \int_{\Omega_T} e^{2z\alpha} \xi^{-2} |\nabla z|^2 \ dx \ dt + B_1 \]

\[ \leq C_3 \left( s^3 \lambda^4 \int_{\Omega_T} e^{-2z\alpha} \xi^3 |\varphi|^2 \ dx \ dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2z\alpha} \xi^3 |\varphi\Gamma|^2 \ d\sigma \ dt \right). \tag{26} \]

Therefore, choosing \( 0 < \epsilon < \frac{dK^{-1}}{2} \), for all \( s \geq s_2 \), and \( C_4 := \max_{1 \leq i \leq 3} C_i \), one has

\[ \frac{d}{2} s^{-2}\lambda^{-2} \int_{\Omega_T} e^{2z\alpha} \xi^{-2} |\nabla z|^2 \ dx \ dt + B_1 \]

\[ \leq C_4 \left( s^3 \lambda^4 \int_{\Omega_T} e^{-2z\alpha} \xi^3 |\varphi|^2 \ dx \ dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2z\alpha} \xi^3 |\varphi\Gamma|^2 \ d\sigma \ dt \right). \]

**Step 3. Estimate of the first order boundary term.** To estimate the first order boundary term, we multiply by \( s^{-2}\lambda^{-2} e^{2z\alpha} \xi^{-2} \partial_{\Gamma} z \) the second equation (18), verified by \( z \) on \( \Gamma_T \) and integrate by parts with respect to the space variable, and since \( \nabla \Gamma \alpha = \nabla \Gamma \xi = 0 \), we obtain

\[ s^{-2}\lambda^{-2} \int_{\Gamma_T} e^{2z\alpha} \xi^{-2} \partial_{\Gamma} z \xi \ d\sigma \ dt + \delta s^{-2}\lambda^{-2} \int_{\Gamma_T} e^{2z\alpha} \xi^{-2} |\nabla \Gamma z|^2 \ d\sigma \ dt + B_2 \]

\[ = s\lambda \int_{\Gamma_T} \xi \varphi \Gamma z \ d\sigma \ dt, \]

where \( B_2 := ds^{-2}\lambda^{-2} \int_{\Gamma_T} e^{2z\alpha} \xi^{-2} \partial_{\sigma} z \ d\sigma \ dt \). An integration by parts in the first term with respect to the time variable and the inequality (11) yield the existence of constants \( C_5 > 0 \) and \( \sigma_3 > \sigma_2 \) such that, for \( s \geq s_3 = \sigma_3 (T + T^2) \) and \( \lambda \geq \lambda_0 \), using (25), one has

\[ -s^{-2}\lambda^{-2} \int_{\Gamma_T} e^{2z\alpha} \xi^{-2} \partial_{\sigma} z \ d\sigma \ dt \leq C_5 \int_{\Gamma_T} e^{2z\alpha} |z\Gamma|^2 \ d\sigma \ dt \]

\[ \leq C_5 C_1 \left( s^3 \lambda^4 \int_{\Omega_T} e^{-2z\alpha} \xi^3 |\varphi|^2 \ dx \ dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2z\alpha} \xi^3 |\varphi\Gamma|^2 \ d\sigma \ dt \right). \]

On the other hand, by Young inequality and (25), we have

\[ s\lambda \int_{\Gamma_T} \xi \varphi \Gamma z \ d\sigma \ dt \leq \frac{1}{2} (\int_{\Gamma_T} e^{2z\alpha} |z\Gamma|^2 \ d\sigma \ dt + s^2 \lambda^2 \int_{\Gamma_T} e^{-2z\alpha} \xi^2 |\varphi\Gamma|^2 \ d\sigma \ dt) \]

\[ \leq \frac{1}{2} (\int_{\Gamma_T} e^{2z\alpha} |z\Gamma|^2 \ d\sigma \ dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2z\alpha} \xi^3 |\varphi\Gamma|^2 \ d\sigma \ dt) \]
Hence, for \( s \geq s_4 \), \( \lambda \geq \lambda_0 \) and \( C_6 := C_5 + \frac{14C_1}{2} \), we have
\[
d\lambda^{-2} \int \varphi^2 \zeta \mathcal{P} d\sigma dt + B_2
= \lambda \int \zeta \varphi \zeta \mathcal{P} d\sigma dt - \lambda^{-2} \int \varphi^2 \zeta \partial_z z \mathcal{P} d\sigma dt
\leq C_6 \left( s^3 \lambda^4 \int e^{-2s \alpha} \varphi^2 d\sigma dt + s^3 \lambda^3 \int e^{-2s \alpha} \zeta^2 \mathcal{P} d\sigma dt \right). \tag{27}
\]

Now, summing (25), (26) and (27), and setting \( C_7 := C_1 + C_4 + C_6 \), we obtain
\[
s^{-3} \lambda^{-4} \int \varphi^2 d\sigma dt + s^3 \lambda^3 \int e^{-2s \alpha} \zeta^2 \mathcal{P} d\sigma dt
+ \frac{d}{2} s^{-2} \lambda^{-2} \int \varphi^2 \zeta \mathcal{P} d\sigma dt + \delta s^{-2} \lambda^{-2} \int \varphi^2 \zeta \mathcal{P} d\sigma dt + B_1 + B_2
\leq C_7 \left( s^3 \lambda^4 \int e^{-2s \alpha} \varphi^2 d\sigma dt + s^3 \lambda^3 \int e^{-2s \alpha} \zeta^2 \mathcal{P} d\sigma dt \right) \tag{28}
\]
for all \( s \geq s_3 \) and \( \lambda \geq \lambda_0 \). On the other hand,
\[
- B_1 - B_2 = ds^{-2} \lambda^{-2} \left( \int \partial_z (e^{2s \alpha} \zeta) z \mathcal{P} d\sigma dt - \int \partial_z (e^{2s \alpha} \zeta^2) z \mathcal{P} d\sigma dt \right)
= ds^{-2} \lambda^{-2} \int \partial_z (e^{2s \alpha} \zeta^2) z \mathcal{P} d\sigma dt
= 2ds^{-1} \lambda^{-1} \int e^{2s \alpha} \zeta \partial_z \eta^0 \zeta \mathcal{P} d\sigma dt - 2ds^{-2} \lambda^{-1} \int e^{2s \alpha} \zeta \partial_z \eta^0 \zeta \mathcal{P} d\sigma dt
= 2ds^{-1} \lambda^{-1} \int e^{2s \alpha} \zeta \partial_z \eta^0 \zeta \mathcal{P} d\sigma dt - 2ds^{-2} \lambda^{-1} \int e^{2s \alpha} \zeta \partial_z \eta^0 \zeta \mathcal{P} d\sigma dt.
\]
Now since \( \partial_z \eta^0 \zeta \mathcal{P} < 0 \), we see that the first term in the right-hand side of the above equality is negative and using that \( |\partial_z \eta^0| = |\nabla \eta^0| \), it follows that
\[
- B_1 - B_2 \leq -2ds^{-2} \lambda^{-1} \int e^{2s \alpha} \zeta \partial_z \eta^0 \zeta \mathcal{P} d\sigma dt
\leq s^{-2} dKT^4 \int e^{2s \alpha} \zeta \mathcal{P} d\sigma dt
\]
for all \( s \geq s_4 \) and \( \lambda \geq \lambda_0 \).

So, the estimate (28) implies that
\[
s^{-3} \lambda^{-4} \int \varphi^2 d\sigma dt + s^3 \lambda^3 \int e^{-2s \alpha} \zeta^2 \mathcal{P} d\sigma dt
+ (1 - s^{-2} dKT^4) \int \varphi^2 \zeta \mathcal{P} d\sigma dt + \frac{d}{2} s^{-2} \lambda^{-2} \int e^{2s \alpha} \zeta \mathcal{P} d\sigma dt
+ \delta s^{-2} \lambda^{-2} \int e^{2s \alpha} \zeta \mathcal{P} d\sigma dt
\]
\[ s^{-3} \lambda^{-4} \int_{\omega_T} e^{2s\alpha \xi^3} |v|^2 \, dx \, dt + \int_{\Omega_T} e^{2s\alpha \xi^3} |z|^2 \, dx \, dt \]
\[ + \int_{\Gamma_T} e^{2s\alpha \xi^3} |z|^2 \, d\sigma \, dt + s^{-2} \lambda^{-2} \int_{\Omega_T} e^{2s\alpha \xi^3} |\nabla z|^2 \, dx \, dt \]
\[ + s^{-2} \lambda^{-2} \int_{\Gamma_T} e^{2s\alpha \xi^3} |\nabla z|^2 \, d\sigma \, dt + B_1 + B_2 \]
\[ \leq C_7 \left( s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha \xi^3} |\varphi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha \xi^3} |\varphi_{\Gamma}|^2 \, d\sigma \, dt \right). \]

Furthermore, for \( \tilde{\sigma} = \max(\sigma_3, \sqrt{2dK}) \), \( s \geq \tilde{s} := \tilde{\sigma}(T + T^2) \) and \( \lambda \geq \lambda_0 \), we have \( (1 - s^{-2}dKT^4) > \frac{1}{2} \), and then

\[ s^{-3} \lambda^{-4} \int_{\omega_T} e^{2s\alpha \xi^3} |v|^2 \, dx \, dt + \int_{\Omega_T} e^{2s\alpha \xi^3} |z|^2 \, dx \, dt \]
\[ + \frac{d}{2} s^{-2} \lambda^{-2} \int_{\Omega_T} e^{2s\alpha \xi^3} |\nabla z|^2 \, dx \, dt + \delta s^{-2} \lambda^{-2} \int_{\Gamma_T} e^{2s\alpha \xi^3} |\nabla z|^2 \, d\sigma \, dt \]
\[ \leq C_7 \left( s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha \xi^3} |\varphi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha \xi^3} |\varphi_{\Gamma}|^2 \, d\sigma \, dt \right), \]

where \( C = C_7 (\min(\delta, \frac{d}{2}, \frac{1}{2})^{-1} \). This achieves the proof. \( \square \)

3.2. Proof of Theorem 3.4. By the preparations of the above section, we are now able to show the main Carleman estimate (15). The proof will be done in three steps.

**Proof.** Step 1. Estimate of the two first terms of (15). In view of (8) written with \( Z = (z, z_T) \) solution of (18), we have

\[ s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha \xi^3} |\varphi|^2 \, dx \, dt + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha \xi^3} |\varphi_{\Gamma}|^2 \, d\sigma \, dt \]
\[ = - \int_{\omega_T} \varphi v \, dx \, dt + \int_0^T (-F_0(t) + \text{div}(F(t), z)) \, dt \]
\[ + \int_0^T (-F_{\Omega} - F_{\Gamma,0}(t) + \text{div}_{\Gamma}(F_{\Gamma}(t), z_T)) \, dt \]
\[ = - \int_{\omega_T} \varphi v \, dx \, dt - \int_0^T (F_0(t), z) \, dt - \int_0^T F(t, \nabla z) \, dx \, dt \]
\[ - \int_0^T F_{\Gamma,0}(t) z_T \, d\sigma \, dt - \int_0^T F_{\Gamma}(t, \nabla z_T) \, d\sigma \, dt. \]

Therefore, by Young inequality, for any \( \epsilon > 0 \), we obtain

\[ \frac{\epsilon^{-1} s^3 \lambda^4}{2} \int_{\omega_T} e^{-2s\alpha \xi^3} \varphi^2 \, dx \, dt + \frac{\epsilon s^3 \lambda^3}{2} \int_{\Gamma_T} e^{-2s\alpha \xi^3} |\varphi_{\Gamma}|^2 \, d\sigma \, dt \]
\[ \leq \frac{\epsilon^{-1} s^3 \lambda^4}{2} \int_{\omega_T} e^{-2s\alpha \xi^3} \varphi^2 \, dx \, dt + \frac{\epsilon s^3 \lambda^3}{2} \int_{\omega_T} e^{2s\alpha \xi^3} |v|^2 \, dx \, dt \]
\[ + \frac{\epsilon^{-1}}{2} \int_{\Omega_T} e^{-2s\alpha |F_0|^2} \, dx \, dt + \frac{\epsilon}{2} \int_{\Omega_T} e^{2s\alpha |z|^2} \, dx \, dt \]
for all $L_s \lambda \Phi$ is a weak solution to (9) and that $C_2\epsilon''(0) = \int_\Omega \xi^2 e^{2\sigma_0} |\nabla z|^2 dx dt$

$+ \frac{\epsilon^2}{2} s^2 \lambda^2 \int_\Omega \xi^2 |F|^2_{L^2} dx dt + \frac{\epsilon s^2 \lambda^2}{2} \int_\Gamma \xi^2 e^{2\sigma_0} |\nabla z|^2 dx dt$

$+ \frac{\epsilon^2}{2} s^2 \lambda^2 \int_\Gamma \xi^2 |F|^2_{L^2} dx dt + \frac{\epsilon s^2 \lambda^2}{2} \int_\Gamma \xi^2 e^{2\sigma_0} |\nabla z|^2 dx dt$

Now, estimate (19) provides that

$\left(1 - \frac{\epsilon C}{2}\right) \left(s^3 \lambda^4 \int_\Omega e^{-2\sigma_0} \xi^3 |\varphi|^2 dx dt + s^3 \lambda^3 \int_\Gamma e^{-2\sigma_0} \xi^3 |\varphi|^2 dx dt\right)$

$\leq \frac{\epsilon^2}{2} (s^3 \lambda^4 \int_\Omega e^{-2\sigma_0} \xi^3 |\varphi|^2 dx dt + \int_\Omega e^{-2\sigma_0} |F_0|^2 dx dt$

$+ s^2 \lambda^2 \int_\Omega e^{-2\sigma_0} \xi^2 |F|^2_{L^2} dx dt + \int_\Gamma e^{-2\sigma_0} |F_0|^2 dx dt$

$+ s^2 \lambda^2 \int_\Gamma e^{-2\sigma_0} \xi^2 |F|^2_{L^2} dx dt$)

for all $s \geq \tilde{s} = \tilde{\sigma}(T + T^2)$ and $\lambda \geq \tilde{\lambda}$. Choosing $0 < \epsilon < 2C^{-1}$ and setting $C' = \frac{1}{2} (1 - \epsilon C)^{-1}$, we deduce that

$s^3 \lambda^4 \int_\Omega e^{-2\sigma_0} \xi^3 |\varphi|^2 dx dt + s^3 \lambda^3 \int_\Gamma e^{-2\sigma_0} \xi^3 |\varphi|^2 dx dt$

$\leq C'(s^3 \lambda^4 \int_\Omega e^{-2\sigma_0} \xi^3 |\varphi|^2 dx dt + \int_\Omega e^{-2\sigma_0} |F_0|^2 dx dt$

$+ s^2 \lambda^2 \int_\Omega e^{-2\sigma_0} \xi^2 |F|^2_{L^2} dx dt + \int_\Gamma e^{-2\sigma_0} |F_0|^2 dx dt$

$+ s^2 \lambda^2 \int_\Gamma e^{-2\sigma_0} \xi^2 |F|^2_{L^2} dx dt$)

Step 2. Estimate of the first order terms of (15). We use the fact that $\Phi$ is a weak solution to (9) and that $s \lambda^2 e^{-2\sigma_0} \xi \Phi = (s \lambda^2 e^{-2\sigma_0} \xi \varphi, s \lambda^2 e^{-2\sigma_0} \xi \varphi_\Gamma) \in L^2(0, T; H^1)$ to obtain from (7), with $Z = s \lambda^2 e^{-2\sigma_0} \xi \Phi$, that

$ds^2 \int_\Omega \nabla \varphi \nabla (e^{-2\sigma_0} \xi) dx dt + ds^2 \int_\Gamma \nabla \varphi_\Gamma \nabla (e^{-2\sigma_0} \xi \varphi_\Gamma) dx dt$

$+ \int_0^T \langle \partial_t (e^{-2\sigma_0} \xi \varphi), \varphi \rangle_{H^1(\Omega)} dt + s \lambda^2 \int_0^T \langle \partial_t (e^{-2\sigma_0} \xi \varphi_\Gamma), \varphi_\Gamma \rangle_{H^{-1}(\Gamma)} dt$

$= - \int_0^T < F_0 - \text{div} F, z(t) >_{H^1(\Omega)} dt - \int_0^T < F_0, z(t) >_{H^1(\Gamma)} dt >_{H^{-1}(\Gamma)} dt$

Integrating by parts in time and space and using the equality

$\frac{d}{dt} \|\varphi(t, \varphi_\Gamma(t, \cdot))\|_{L^2} = 2 \langle (\varphi'(t), \varphi_\Gamma(t, \cdot)), (\varphi(t, \cdot), \varphi_\Gamma(t, \cdot)) \rangle _{H^{-1}, H^1}$

for a.e., $t \in [0, T]$ (see [8, Section 5.9, Theorem 3]), we obtain

$ds^2 \int_\Omega e^{-2\sigma_0} \xi |\nabla \varphi|^2 dx dt + \frac{s \lambda^2}{2} \int_\Gamma (\partial_t (e^{-2\sigma_0} \xi) - d\Delta (e^{-2\sigma_0} \xi)) |\varphi|^2 dx dt$

$+ B_3 + B_4 + \delta s^2 \int_\Omega e^{-2\sigma_0} \xi |\nabla \varphi|^2 dx$
By Cauchy-Schwarz and Young inequalities, and using (10) and (13), for all
\begin{align*}
&+ s \frac{\lambda^2}{2} \int_{\Gamma_T} \left( \partial_t (e^{-2s_\alpha} \xi) - \delta \Delta \gamma (e^{-2s_\alpha} \xi) \right) |\varphi|^2 d\sigma dt - B_4 \\
&= - s \lambda^2 \int_{\Omega_T} e^{-2s_\alpha} (\xi F_0 \varphi - \xi F_0 \nabla \varphi) dx dt + s \lambda^2 \int_{\Omega_T} F_0 \nabla (e^{-2s_\alpha} \xi) \varphi dx dt \\
&- s \lambda^2 \int_{\Gamma_T} e^{-2s_\alpha} (\xi F_{0, \Gamma} \varphi - \xi F_0 \nabla \varphi) d\sigma dt \\
&+ s \lambda^2 \int_{\Gamma_T} F_0 \nabla (e^{-2s_\alpha} \xi) \varphi d\sigma dt,
\end{align*}
where we have set
\begin{align*}
B_3 := - \frac{ds \lambda^2}{2} \int_{\Gamma_T} \partial_t (e^{-2s_\alpha} \xi) \varphi^2 d\sigma dt, \quad B_4 := ds \lambda^2 \int_{\Gamma_T} e^{-2s_\alpha} \xi \varphi \partial_t \varphi d\sigma dt.
\end{align*}
On the other hand, by Young inequality, we have
\begin{align*}
s \lambda^2 \left| \int_{\Omega_T} e^{-2s_\alpha} \xi F_0 \varphi dx dt \right| \\
&\leq C \left( s \lambda^4 \int_{\Omega_T} e^{-2s_\alpha} \xi^3 |\varphi|^2 dx dt + s^{-1} \int_{\Omega_T} e^{-2s_\alpha} \xi^{-1} |F_0|^2 dx dt \right),
\end{align*}
\begin{align*}
s \lambda^2 \left| \int_{\Omega_T} e^{-2s_\alpha} \xi F_0 \nabla \varphi dx dt \right| \\
&\leq \frac{s \lambda^2}{2} \left( \int_{\Omega_T} e^{-2s_\alpha} \xi |\varphi|^2 dx dt + \int_{\Omega_T} e^{-2s_\alpha} \xi |\nabla \varphi|^2 dx dt \right),
\end{align*}
\begin{align*}
s \lambda^2 \left| \int_{\Gamma_T} e^{-2s_\alpha} \xi F_{0, \Gamma} \varphi d\sigma dt \right| \\
&\leq C \left( s \lambda^4 \int_{\Gamma_T} e^{-2s_\alpha} \xi^3 |\varphi \Gamma|^2 d\sigma dt + s^{-1} \int_{\Gamma_T} e^{-2s_\alpha} \xi^{-1} |F_{0, \Gamma}|^2 d\sigma dt \right),
\end{align*}
and
\begin{align*}
s \lambda^2 \left| \int_{\Gamma_T} e^{-2s_\alpha} \xi F_0 \nabla \varphi d\sigma dt \right| \\
&\leq \frac{s \lambda^2}{2} \left( \int_{\Gamma_T} e^{-2s_\alpha} \xi |F_\gamma|^2 d\sigma dt + \int_{\Gamma_T} e^{-2s_\alpha} \xi |\nabla \varphi \gamma|^2 d\sigma dt \right).
\end{align*}
By Cauchy-Schwarz and Young inequalities, and using (10) and (13), for all $s \geq C(T + T^2)$, we obtain
\begin{align*}
s \lambda^2 \int_{\Omega_T} F_0 \nabla (e^{-2s_\alpha} \xi) \varphi dx dt &\leq s \lambda^2 \int_{\Omega_T} \|F\|_{\mathbb{R}^N} \|\nabla (e^{-2s_\alpha} \xi)\|_{\mathbb{R}^N} |\varphi| dx dt \\
&\leq s \lambda^2 \int_{\Omega_T} \xi \|F\|_{\mathbb{R}^N}^2 dx dt + s \lambda^2 \int_{\Omega_T} \|\nabla (e^{-2s_\alpha} \xi)\|_{\mathbb{R}^N}^2 |\varphi|^2 dx dt \\
&\leq C \left( s \lambda^2 \int_{\Omega_T} \xi \|F\|_{\mathbb{R}^N}^2 dx dt + s^3 \lambda^4 \int_{\Omega_T} e^{-2s_\alpha} \xi^3 |\varphi|^2 dx dt \right) \\
&\leq C \left( s^2 \lambda^2 \int_{\Omega_T} \xi^2 \|F\|_{\mathbb{R}^N}^2 dx dt + s^3 \lambda^4 \int_{\Omega_T} e^{-2s_\alpha} \xi^3 |\varphi|^2 dx dt \right),
\end{align*}
Thus, by this and estimate (29), we obtain estimate (15).

Similarly, we deduce from (13) that

$$-B_3 = \frac{ds \lambda^2}{2} \int_{\Gamma_T} \partial_\nu (e^{-2s_\alpha} \xi_\nu \varphi_T) \, d\sigma \, dt \leq C s^3 \lambda^3 \int_{\Gamma_T} e^{-2s_\alpha} \xi^3 |\varphi_T|^2 \, d\sigma \, dt.$$

Finally, (10) and (30) yield

$$s \lambda^2 \int_{\Omega_T} e^{-2s_\alpha} \xi_\nu |\nabla \varphi|^2 \, dx \, dt + s \lambda^2 \int_{\Gamma_T} e^{-2s_\alpha} \xi_\nu |\nabla \varphi_T|^2 \, d\sigma \, dt$$

$$\leq C \left( s^2 \lambda^2 \int_{\Omega_T} \xi_\nu^2 \|F\|_\mathbb{R}^N \, dx \, dt + s^3 \lambda^4 \int_{\Omega_T} e^{-2s_\alpha} \xi^3 |\varphi|^2 \, dx \, dt$$

$$+ s^{-1} \int_{\Omega_T} e^{-2s_\alpha} \xi^{-1} |F_0|^2 \, dx \, dt + s \lambda^2 \int_{\Gamma_T} \xi_\nu \|F_T\|_\mathbb{R}^N \, d\sigma \, dt$$

$$+ s^3 \lambda^4 \int_{\Gamma_T} e^{-2s_\alpha} \xi \varphi_T^3 \, d\sigma \, dt + s^{-1} \int_{\Gamma_T} e^{-2s_\alpha} \xi \varphi_T^{-1} |F_{0,T}|^2 \, d\sigma \, dt \right).$$

Thus, by this and estimate (29), we obtain estimate (15).

Now, we deduce, from Carleman estimate (15), important estimates for solutions of (2), which will be the key to establish its observability inequality.

**Corollary 1.** Let $T > 0$ be fixed. Then the following holds.

(i) Fixing $\lambda = \lambda_1$, there exist two constants, $\sigma_2 = \sigma_2(\Omega, \omega) > 0$ and $C_1 = C_1(\Omega, \omega) > 0$ such that for any $s \geq s_2 := \sigma_2(T_0 + T)^2 + T(\|c\|_{L^\infty}^2 + \|\ell\|_{L^\infty}^2 + \|b\|_{L^\infty}^2)$, and every $(\psi_T, \psi_{T,T}) \in L^2$, the unique mild solution to the backward problem (2) satisfies the estimate

$$\int_{\Omega_T} e^{-2s_\alpha} \xi_\nu |\psi|^2 \, dx \, dt + \int_{\Gamma_T} e^{-2s_\alpha} \xi_\nu |\psi_T|^2 \, d\sigma \, dt \leq C_1 \int_{\omega_T} e^{-2s_\alpha} \xi_\nu |\psi|^2 \, dx \, dt. \quad (31)$$

(ii) There exists a constant $C_2 = C_2(\Omega, \omega) > 0$ such that for every $(\psi_T, \psi_{T,T}) \in L^2$, the unique mild solution to the backward problem (2) satisfies the estimate

$$\int_{(T/4,3T/4) \times \Omega} |\psi|^2 \, dx \, dt + \int_{(T/4,3T/4) \times \Gamma} |\psi_T|^2 \, d\sigma \, dt$$

$$\leq \exp(C_2(1 \frac{1}{T} + |c|_{L^\infty}^2 + \|\ell\|_{L^\infty}^2 + \|F\|_{L^\infty}^2 + \|b\|_{L^\infty}^2)) \int_{\omega_T} |\psi|^2 \, dx \, dt. \quad (32)$$

**Proof.** For the first assertion, let $\Phi$ be the unique weak solution to the adjoint backward problem (2) with final data $\Phi_T$. Then, by Proposition 3, $\Phi$ is the weak solution to the problem (9), with $F_0 = c\psi$, $F = \psi B$, $F_{0,T} = \ell \psi_T$, $F_T = \psi_T b$. Therefore, Fixing $\lambda = \lambda_1$ and since $\lambda_1 \geq 1$, Carleman estimate (15) provides that
there exists \( C = C(\Omega, \omega) > 0 \) such that for all \( s \geq s_1 \)

\[
\begin{align*}
&\quad s^3 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\psi|^2 \, dx \, dt + s^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |\psi_T|^2 \, d\sigma \, dt \\
&\quad + s \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\nabla \psi|^2 \, dx \, dt + s \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |\nabla_T \psi_T|^2 \, d\sigma \, dt \\
&\quad \leq C \left( s^3 \int_{\omega_T} e^{-2s\alpha} \xi^3 |\psi|^2 \, dx \, dt + \int_{\Omega_T} e^{-2s\alpha} |\psi|^2 \, dx \, dt \\
&\quad + s^2 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |\psi_T|^2 \, d\sigma \, dt \right) .
\end{align*}
\]

Furthermore, using the inequalities

\[ \theta \leq \xi \leq M \theta, \text{ on } \overline{\Omega_T}, \text{ with } M = \max_{x \in \Omega} e^{3\lambda_1(m\|\eta^7\|_{L^\infty} + \eta^9(x))}, \theta^{-1} \leq 2^{-2} T^2, \text{ on } [0, T], \]

and setting \( \tilde{C} = M.C. \), we obtain

\[
\begin{align*}
&\quad s^3 \int_{\Omega_T} e^{-2s\alpha} \theta^3 |\psi|^2 \, dx \, dt + s^3 \int_{\Gamma_T} e^{-2s\alpha} \theta^3 |\psi_T|^2 \, d\sigma \, dt \\
&\quad + s \int_{\Omega_T} e^{-2s\alpha} \theta^3 |\nabla \psi|^2 \, dx \, dt + s \int_{\Gamma_T} e^{-2s\alpha} \theta^3 |\nabla_T \psi_T|^2 \, d\sigma \, dt \\
&\quad \leq \tilde{C} \left( s^3 \int_{\omega_T} e^{-2s\alpha} \theta^3 |\psi|^2 \, dx \, dt + \int_{\Omega_T} e^{-2s\alpha} |\psi|^2 \, dx \, dt \\
&\quad + s^2 \|B\|_{L^\infty} \int_{\Omega_T} e^{-2s\alpha} \theta^2 |\psi|^2 \, dx \, dt + \|\ell\|_{L^\infty} \int_{\Gamma_T} e^{-2s\alpha} |\psi|^2 \, d\sigma \, dt \\
&\quad + s^2 \|b\|_{L^\infty} \int_{\Gamma_T} e^{-2s\alpha} \theta^2 |\psi|^2 \, d\sigma \, dt \right) \\
&\quad \leq \tilde{C} \left( s^3 \int_{\omega_T} e^{-2s\alpha} \theta^3 |\psi|^2 \, dx \, dt + (2^{-2} T^2 \|c\|_{L^\infty}^{2/3})^3 \int_{\Omega_T} e^{-2s\alpha} \theta^3 |\psi|^2 \, dx \, dt \\
&\quad + 2^{-2} T^2 s^2 \|B\|_{L^\infty} \int_{\Omega_T} e^{-2s\alpha} \theta^3 |\psi|^2 \, dx \, dt + (2^{-2} T^2 \|\ell\|_{L^\infty}^{2/3})^3 \int_{\Gamma_T} e^{-2s\alpha} \theta^3 |\psi_T|^2 \, d\sigma \, dt \\
&\quad + 2^{-2} T^2 s^2 \|b\|_{L^\infty}^2 \int_{\Gamma_T} e^{-2s\alpha} \theta^3 |\psi_T|^2 \, d\sigma \, dt \right) .
\end{align*}
\]

Thus, we deduce that

\[
\begin{align*}
&\quad \left( 1 - (2^{-2} \tilde{C}^{1/3} s^{-1} T^2 \|c\|_{L^\infty}^{2/3})^3 - 2^{-2} T^2 s^{-1} \tilde{C} \|B\|_{L^\infty}^2 \right) \int_{\Omega_T} e^{-2s\alpha} \theta^3 |\psi|^2 \, dx \, dt \\
&\quad + \left( 1 - (2^{-2} \tilde{C}^{1/3} s^{-1} T^2 \|\ell\|_{L^\infty}^{2/3})^3 - 2^{-2} T^2 s^{-1} \tilde{C} \|b\|_{L^\infty}^2 \right) \int_{\Gamma_T} e^{-2s\alpha} \theta^3 |\psi_T|^2 \, d\sigma \, dt \\
&\quad \leq \tilde{C} \int_{\omega_T} e^{-2s\alpha} \theta^3 |\psi|^2 \, dx \, dt .
\end{align*}
\]
Hence, taking \( \sigma_2 = \max(\overline{C}, 2^{-4/3}\overline{C}^{1/3}, \sigma_1) \) yields that for any \( s \geq s_2 = \sigma_2(T + T^2 + T^2(\|c\|_{\infty}^{2/3} + \|\ell\|_{\infty}^{2/3} + \|B\|_{\infty}^2 + \|b\|_{\infty}^2)) \) we have

\[
\frac{1}{2} \leq \left( 1 - (2^{-2}\overline{C}^{1/3}s^{-1}T^2\|c\|_{\infty}^{2/3}) - 2^{-2}T^2s^{-1}\overline{C}\|B\|_{\infty}^2 \right)
\]

and thus,

\[
\int_{\Omega_T} e^{-2s_\alpha \theta^3} |\psi|^2 \, dx \, dt + \int_{\Gamma_T} e^{-2s_\alpha \theta^3} |\psi_T|^2 \, d\sigma \, dt \leq C_1 \int_{\omega_T} e^{-2s_\alpha \theta^3} |\psi|^2 \, dx \, dt, \tag{33}
\]

with \( C_1 = 2\overline{C} \). For the second assertion (ii), the solution to the backward adjoint problem (2) with \( \Phi_T = (\psi_T, \psi_T, \tau) \) satisfies (33). Hence, setting \( M_0 = \max(\beta(x)) \) and \( m_0 = \min(\beta(x)) \), we obtain that

\[
\int_{\Omega_T} e^{-2sM_0 \theta^3} |\psi|^2 \, dx \, dt + \int_{\Gamma_T} e^{-2sM_0 \theta^3} |\psi_T|^2 \, d\sigma \, dt \leq C_1 \int_{\omega_T} e^{-2sM_0 \theta^3} |\psi|^2 \, dx \, dt.
\]

On the other hand, we have

\[
\max_{T/4 \leq t \leq 3T/4} \theta(t) = \frac{16}{3T^2}, \quad \min_{T/4 \leq t \leq 3T/4} \theta(t) = \frac{4}{T^2}.
\]

Therefore, it can be easily verified that

\[
\theta^3 e^{-2s_\alpha \theta^3} \leq \theta^3 e^{-2sM_0 \theta^3} \leq \sup_{t' \in \mathbb{R}^+} \theta^3 e^{-2sM_0 t'} = \left( \frac{3}{2s_\alpha m_0} \right)^3 \quad \text{in } \Omega_T.
\]

Thus, taking \( s = s_2 \) and \( \lambda = \lambda_1 \) yields

\[
\frac{4^3}{T^6} \exp\left( \frac{-2^5 s_2 M_0}{3T^2} \right) \left( \int_{(T/4,3T/4) \times \Omega} |\psi|^2 \, dx \, dt + \int_{(T/4,3T/4) \times \Gamma} |\psi_T|^2 \, d\sigma \, dt \right)
\]

\[
\leq C_1 \int_{\omega_T} \theta^3 e^{-2s_\alpha \theta^3} |\psi|^2 \, dx \, dt \leq \left( \frac{3}{2s_\alpha m_0} \right)^3 \frac{C_1}{T^6} \int_{\omega_T} |\psi|^2 \, dx \, dt.
\]

Hence

\[
\int_{(T/4,3T/4) \times \Omega} |\psi|^2 \, dx \, dt + \int_{(T/4,3T/4) \times \Gamma} |\psi_T|^2 \, d\sigma \, dt
\]

\[
\leq K \exp\left( \frac{2^5 M_0}{3} (1 + 1/T + \|c\|_{\infty}^{2/3} + \|\ell\|_{\infty}^{2/3} + \|B\|_{\infty}^2 + \|b\|_{\infty}^2) \right) \int_{\omega_T} |\psi|^2 \, dx \, dt,
\]

with \( K = \left( \frac{3}{8s_\alpha m_0} \right)^3 C_1 \). Finally

\[
\int_{(T/4,3T/4) \times \Omega} |\psi|^2 \, dx \, dt + \int_{(T/4,3T/4) \times \Gamma} |\psi_T|^2 \, d\sigma \, dt
\]

\[
\leq \exp(C_2(1 + 1/T + \|c\|_{\infty}^{2/3} + \|\ell\|_{\infty}^{2/3} + \|B\|_{\infty}^2 + \|b\|_{\infty}^2)) \int_{\omega_T} |\psi|^2 \, dx \, dt,
\]

with \( C_2 = \max(\log(K), \frac{2^5 M_0}{3}) \), and the proof is complete. \( \square \)
4. Null controllability. In this section, we apply Carleman estimate (31) to show null controllability for (1). In the following proposition, we show first the observability inequality for the backward adjoint problem (2).

**Proposition 5.** For any fixed $T > 0$, there exists a constant $C = C(\Omega, \omega) > 0$ such that for every $\Phi_T := (\psi_T, \psi_{T, t}) \in L^2$, the unique weak solution $\Phi = (\psi, \psi_t)$ to the backward adjoint problem (2) satisfies the observability inequality

$$
\|\psi(0, \cdot)\|_{L^2(\Omega)}^2 + \|\psi_T(0, \cdot)\|_{L^2(\Gamma)}^2 \leq \exp (CK) \int_{\omega_T} |\psi|^2 \, dx \, dt,
$$

with

$$
K = K(T, \|c\|_\infty, \|B\|_\infty, \|\ell\|_\infty, \|b\|_\infty)
$$

$$
= 1 + \frac{1}{T} + \|c\|^{2/3}_\infty + \|\ell\|^{2/3}_\infty + T(\|B\|_\infty^2 + \|b\|_\infty^2 + \|c\|_\infty + \|B\|_\infty + \|\ell\|_\infty + \|b\|_\infty).
$$

(34)

**Proof.** **Step 1.** First, assume that $\Phi_T \in D(A^*)$ and let $\Phi$ be the unique strict solution to the backward adjoint problem (2) with final data $\Phi_T$. By Proposition (3), the function $\Phi$ is the unique weak solution to the backward adjoint problem (9), with $F_0 = c\psi$, $F = b\psi$, $F_0, F = \ell\psi_T$, $F_T = \psi_T b$. The version of Proposition 3 on the time interval $[0, t]$ yields there exists $C_0 = C_0(\Omega, \omega) > 0$ such that

$$
\|\psi(0, t)\|_{L^2(\Omega)}^2 + \|\psi_T(t)\|_{L^2(\Gamma)}^2 \leq \tilde{C} \|\psi(t, t)\|_{L^2}^2 \quad \text{for all } t \in [0, T],
$$

with $\tilde{C} = \exp (C_0 T(\|c\|_\infty + \|B\|_\infty + \|\ell\|_\infty + \|b\|_\infty))$. Integrating this inequality over $(T, \frac{3T}{4})$ and (32) provide that

$$
\|\psi(0)\|_{L^2(\Omega)}^2 + \|\psi_T(0)\|_{L^2(\Gamma)}^2 \leq 2\tilde{C} \int_{\frac{3T}{4}}^{2T} \|\psi(t, T)\|_{L^2}^2 dt
$$

$$
= \frac{2\tilde{C}}{T} \left( \int_{(T/4, 3T/4) \times \Omega} |\psi(t, x)|^2 \, dx \, dt + \int_{(T/4, 3T/4) \times \Gamma} |\psi_T(t, x)|^2 \, dx \, dt \right)
$$

$$
\leq \exp (CK) \int_{\omega_T} |\psi|^2 \, dx \, dt,
$$

where $C = \max(C_0, C_2, 2)$ and the constant $K$ is given by (35).

**Step 2.** Let $(\psi_T, \psi_{T, t}) \in L^2$ and $\Phi = (\psi, \psi_t)$ be the unique mild solution to the backward problem (2) with final data $\Phi_T$. By density of $D(A^*)$ in $L^2$, there exists a sequence $(\Phi_{n, T})_{n \in \mathbb{N}} := ((\psi_{n, T}, \psi_{n, T, t}), n \in \mathbb{N}) in D(A^*)$ such that $(\Phi_{n, T})$ converges to $\Phi_T$ in $L^2$. For any $n \in \mathbb{N}$, let $\Phi_n := (\psi_n, \psi_{n, T})$ be the weak solution to the backward adjoint problem (2) with final data $\Phi_{n, T}$. Then, by the first step, we have

$$
\|\psi_n(0, \cdot)\|_{L^2(\Omega)}^2 + \|\psi_{n, T}(0, \cdot)\|_{L^2(\Gamma)}^2 \leq \exp K \int_{\omega_T} e^{-2\alpha x} \xi^3 |\psi_n|^2 \, dx \, dt.
$$

(36)

Since the sequence $(\Phi_n)_{n \in \mathbb{N}}$ converges in $C([0, T]; L^2)$ to the mild solution $\Phi$ with final data $(\psi_T, \psi_{T, t})$, taking the limit in (36), we obtain (34).

We now establish the null controllability of the linear equation (1), where we allow for inhomogeneities with exponential decay at $t = 0$ and $t = T$, following the approach used in [32, Theorem 4.2]. For the constants $s_2$ and $\lambda_1$ defined in Corollary 1, let us introduce the weighted $L^2$-spaces, for $s \geq s_1$ and $\lambda = \lambda_1$,

$$
Z_\Omega = \{ f \in L^2(\Omega_T)/e^{s_2 x} \xi^{-3/2} f \in L^2(\Omega_T) \}, \quad \langle f_1, f_2 \rangle = \int_{\Omega_T} e^{2s_2 x} \xi^{-3} \int_{T/4}^{3T/4} f_1 f_2 \, dx \, dt,
$$

$$
\langle f_1, f_2 \rangle = \int_{\Omega_T} e^{2s_2 x} \xi^{-3} \int_{T/4}^{3T/4} f_1 f_2 \, dx \, dt.
$$
Thus the proof is complete.

Finally, we can state and show the aim result of this work.

**Theorem 4.1.** For $T > 0$, every nonempty open set $\omega \Subset \Omega$, all data $y_0 \in L^2(\Omega)$, $y_{0,T} \in L^2(\Gamma)$ and all $f \in Z_{\Omega T}$ and $g \in Z_T$, there is a control $v \in L^2(\omega_T)$ such that the unique mild solution $y$ to (1) satisfies $y(T, \cdot) = 0$ on $\Omega$ and $y_T(T, \cdot) = 0$ on $\Gamma$.

**Proof.** The null controllability of the system (1) will follow by the observability inequality (34) and a classical duality argument as in [32]. Define the bounded linear operator $T : L^2(\omega_T) \rightarrow \mathbb{L}^2$ by

$$T v := \int_0^T S(T - s)(1_{\omega} v, 0) \, ds.$$  

Using the continuous embedding of $Z_{\Omega T} \times Z_T$ in $L^2(\Omega_T) \times L^2(\Gamma_T)$, we also introduce the bounded linear operator $\mathcal{R} : \mathbb{L}^2 \times Z_{\Omega T} \times Z_T \rightarrow \mathbb{L}^2$ given by

$$\mathcal{R}(f, g) = S(T)f_0 + \int_0^T S(T - s)(f(s), g(s)) \, ds.$$

It is clear that

$$\mathcal{R}(f, g) + T v = (y(T, \cdot), y_T(T, \cdot)),$$

where $(y, y_T)$ is the unique mild solution to the system

$$\left\{ \begin{array}{ll}
\partial_t y - d\Delta y + B(x) \nabla y + c(x)y = 1_{\omega} v + f & \text{in } \Omega_T, \\
\partial_t y_T - \delta\Delta y_T + d\partial_\nu y_T + b(x) \nabla y_T + \ell(x)y_T = g & \text{on } \Gamma_T, \\
y_T(0, t) = y_T(t, x) & \text{on } \Gamma_T, \\
(y, y_T)|_{t=0} = (y_0, y_{0,T}) & \text{in } \Omega \times \Gamma.
\end{array} \right.$$  

Furthermore, we observe that the adjoint operator of $T$ is

$$T^* : \mathbb{L}^2 \rightarrow L^2(\omega_T), \quad T^*(\psi_T, \psi_{T,T}) = 1_{\omega} \psi,$$

where $(\psi, \psi_T)$ is the mild solution of the homogeneous backward problem (2) with final data $(\psi_T, \psi_{T,T})$, and the adjoint operator of $\mathcal{R}$ is

$$\mathcal{R}^* : \mathbb{L}^2 \rightarrow \mathbb{L}^2 \times Z_{\Omega T} \times Z_T$$

given, for $\Phi_T := (\psi_T, \psi_{T,T})$, by $\mathcal{R}^* \Phi_T = (\psi(0, \cdot), \psi_T(0, \cdot), e^{-2\alpha_0 \xi^3} \psi, e^{-2\alpha_0 \xi^3} \psi_T)$.

Using the observability inequality (34) and the Carleman estimate (31), we obtain that

$$\|\mathcal{R}^* \Phi_T\|_{\mathbb{L}^2 \times Z_{\Omega T} \times Z_T}^2 = \|\psi(0, \cdot), \psi_T(0, \cdot))\|^2_{\mathbb{L}^2} + \int_{\Omega_T} e^{-2\alpha_0 \xi^3} \psi^2 \, dx \, dt + \int_{\Gamma_T} e^{-2\alpha_0 \xi^3} \psi_T^2 \, ds \, dt$$

$$\leq C \int_{\omega_T} \psi^2 \, dx \, dt = C \|T^*(\psi_T, \psi_{T,T})\|_{L^2(\omega_T)}^2.$$  

(37)

Now, by Theorem IV.2.2 of [42], the above estimate (37) implies that $\text{Im}(\mathcal{R}) \subset \text{Im}(T)$ which means that for all $Y_0 \in \mathbb{L}^2$, $f \in Z_{\Omega T}$ and $g \in Z_T$ there is a control $v \in L^2(\omega_T)$ such that $\mathcal{R}(Y_0, f, g) = -Tv$. That is, $(y(T, \cdot), y_T(T, \cdot)) = (0, 0)$, and thus the proof is complete. \qed
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