Is the universe ill-posed?

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August 13, 2020

Abstract

We discuss an unusual consequence of the behaviour of general relativistic cosmological models when their initial value problem is not well-posed because of the lack of the local Lipschitz condition. A new type of 'zero universe' arises with vanishing scale factor, and its first time derivative, at all times. We discuss briefly some of the questions this raises about creation out of nothing.

1 Introduction

The initial value problem for the Einstein equations of general relativity proved to be a challenging mathematical problem due to the self-interacting non-linearity of the system of partial differential equations coupled to the space-time geometry, general coordinate covariance, and, like Maxwell’s equations, the presence of a set of constraint equations which govern initial data. Key early contributions to a rigorous analysis were made in classic work by Lichnerowicz [1]; for a comprehensive review of the development, see Isenberg [2]. The aim of this ongoing work was to establish that Einstein’s equations, in vacuum and with well defined matter sources are well posed: that good initial data determines the future evolution of the space-time uniquely and completely. Great focus was laid on these issues by the development of numerical relativity and by the Cosmic Censorship hypothesis of Penrose [3], as to whether naked singularities can evolve from well-posed data; for recent results see [4]. New extensions of general relativity, like Horndeski’s theories, have also revealed that a well-posed initial value problem can be a significant constraint on these theories [5, 6].

In cosmology, the uniqueness of the evolution from well-posed initial value was used by Collins and Stewart [7] to undermine the logic of the chaotic cosmology proposal of Misner [8, 9, 10], that dissipative processes like neutrino viscosity could explain why almost all anisotropic initial conditions could evolve to become as isotropic as the newly discovered temperature isotropy of the cosmic microwave background (then $\Delta T/T < 10^{-3}$) after 10 billion years. Simply
pick a universe that meets present isotropy limits, and evolve it uniquely and continuously backwards in time to find the required initial conditions. There will always be a set. However, when looked at from a more physical perspective this does not really work. The initial conditions required may be physically quite unreasonable: for example, by requiring radiation or gravitational wave energies to exceed Planck density massively at the Planck time and so quantum gravitational processes would intervene to equilibrate them [11], or they will have produced unacceptable levels of entropy in the universe due to the dissipation of large anisotropies at very early times [12].

These cosmological applications are of interest to us here because they make essential use of the local Lipschitz condition as a constraint on the cosmological evolution equations.

In the next section of this short letter we look at a simple application to Friedman universes that has an unusual consequence and then discuss the issues raised by our results in the closing section.

2 Indeterminate Friedman universes

Consider the Friedman metric,

\[ ds^2 = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2[d\theta^2 + \sin^2 \theta d\phi^2] \right\}, \]

where \( k \) is the curvature parameter, \( t \) is comoving proper time, \( r, \theta, \phi \) are spherical polars, and \( a(t) \) is the expansion scale factor determined by the Einstein equations for (1) after specifying an equation of state. Consider first a spatially flat universe with \( k = 0 \). We suppose that the scale factor has simple power-law time dependence, with

\[ a = t^n, \dot{a} = nt^{n-1}, \ddot{a} = n(n-1)t^{n-2} = n(n-1)a \frac{n-2}{n}, \]

\[ \frac{d^3a}{dt^3} = n(n-1)(n-2)...(n-q+1)t^{n-q} \propto a \frac{n-q}{n}, \]

where overdot denotes \( d/dt \). If we choose \( n > 1 \), then we can choose initial conditions,

\[ a(0) = \dot{a}(0) = 0. \]

We see that we have two very different solutions that both satisfy these initial conditions but diverge to the future:

Footnote: Formally, a function \( f : A \subset R^n \rightarrow R^m \) is locally Lipschitz at \( x_0 \in A \) if there exist constants \( > 0 \) and \( M \in R \) such that

\[ ||x - x_0|| < \Rightarrow ||f(x) - f(x_0)|| \leq M||x - x_0||. \] Informally, it requires trajectories that start close to stay close so requires \( f'(x) \) to be bounded.
\[ a(t) = 0, \quad \forall t \] (5)
\[ a(t) = t^n \] (6)

This non-uniqueness phenomenon arises because the choice of \( a = t^n \) with \( n > 1 \) violates the local Lipschitz condition (as \( \dot{a} \) is unbounded to the future). This behaviour is not restricted to our power-law ansatz, as the choice \( a = f(t) \), with \( f(0) = f'(0) = 0 \), for example with \( f(t) = \sinh^n(t) \), gives rise to the same non-unique behaviour if \( f' > 0 \). It has the same evolutionary behaviour as \( t^n \) for \( t \to 0 \) but is de Sitter-like (inflationary) when \( t \to \infty \).

Our power-law example is not unusual as high power-law indices \( (n > 1) \) are needed for power-law inflation. The upside-down oscillator is an instructive case and we can appreciate how inflation with a metastable equilibrium could leave the scalar field there forever (\( \dot{\varphi} = 0 \)) or roll down the potential. Both behaviours have the same initial conditions in the metastable equilibrium but very different future evolution, like in eqs. (5)-(6).

A simple Newtonian and relativistic cosmological example is the universe with scale factor \( a = t^n \) that expands at constant power, \( P \propto \ddot{a} \propto t^{2n-3} \). the power is constant when \( n = 3/2 \). Universes exerting constant force or acceleration, \( F \propto \dot{a} \propto t^{n-2} \), occur when \( n = 2 \).

The simplest example we have is a flat Friedman universe with a perfect fluid equation of state linking pressure, \( p \), and density, \( \rho \) by

\[ p = (\gamma - 1)\rho, \quad \gamma \text{ constant.} \] (7)

The solution for the Friedman model is then \( a(t) = t^{2/3\gamma} \), and so we see that we get the behaviour leading to eq. (4), corresponding to \( n > 1, \dot{a} > 0 \) as \( t \to \infty \), when \( \gamma < 2/3 \). This Is by no means an extreme or unusual state for matter (\( \gamma = 0 \) is the vacuum, \( p = -\rho \), state).

If we have \( a = t^n \) in the flat Friedman model with no equation of state linking \( p \) and \( \rho \), then the essential Einstein equations are \((8\pi G = c = 1)\),

\[
3 \left( \frac{\dot{a}^2}{a^2} \right) = \frac{3n^2}{t^2} = \rho \] (8)
\[
\frac{\dot{a}}{a} = \frac{n(n-1)}{t^2} = -\frac{(\rho + 3p)}{6} = -\frac{1}{2} \left[ \frac{n^2}{t^2} + p \right] \] (9)

and so

\[ p = -\frac{n}{t^2} [3n - 2], \] (10)

and

\[ \rho + p = \frac{2n}{t^2} \] (11)

We note that \( p = 0 \) for \( n = 2/3 \) and \( p = \rho/3 \) for \( n = 1/2 \), as expected. For the solution with \( a = 0 \) for all \( t \) we have \( \rho = p = n = 0 \), a type of ’nothing’, or zero cosmology, where part of the metric still exists.
3 Discussion

We have shown that under certain easily realised (classical) conditions, the Friedman equations has solutions like eq. (5) and (6) that have identical initial conditions, \(a(0) = \dot{a}(0) = 0\), which lead to very different future evolutions for \(a(t)\). In particular, if we consider the traditional description of ‘creation out of nothing’ in Friedman cosmology then we assume the existence of spacetime with a metric like eq. (1) and regard "\(t = 0\)”, where the density is infinity, as the beginning of the universe. Expansion follows for \(t > 0\). No explanation for why the universe comes into being, or starts expanding after \(t = 0\) is offered; nor could it be, as this question is metaphysical and any answer to it lies outside the encompass of relativistic Friedman cosmology. However, we have displayed a new ingredient to this issue here. There are solutions of the Friedman equation for isotropically expanding universes which share initial conditions with solutions like (5), which never expand but just remain forever in a unchanging state of zero expansion in which the space portion of the metric vanishes while the time portion remains. Creation out of nothing may create something that does not turn into a universe as we understand it. What does this mean? Which evolutionary path does the universe take from its \(a = 0 = \dot{a}\) initial state and with what probability? And are such zero-universe solutions stable?

Acknowledgement. The author is supported by the Science and Technology Facilities Council (STFC) of the United Kingdom.

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