One-particle Hilbertspace of 2+1 dimensional gravity using non-commuting coordinates

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After a review of multi-particle solutions in classical 2+1 dimensional gravity we will construct a one-particle Hilbertspace. As we will use a curved momentum space, the coordinates $x^\mu$ are represented as non-commuting Hermitian operators on this Hilbertspace. Finally we will indicate how to construct a Schrödinger equation.

1. INTRODUCTION

In 1963 Staruszkiewicz [1] considered general relativity in 2+1 dimensions for the first time. He solved the gravitational field surrounding a point-particle and found that it represents a conical spacetime. The subject was revived in 1984 by Deser, Jackiw and 't Hooft [2] where it was shown that multiparticle solutions can be constructed by cutting wedges out of spacetime and identifying the boundaries according to a Poincaré transformation. This idea was worked out by 't Hooft in [3] who also proved that for closed universes there can be no closed timelike curves. It is important to notice that one will never find gravitational wave solutions in 2+1 dimensions because the gravitational field carries no degrees of freedom. So all degrees of freedom must come from either topology (handles) or from matter.

A completely different viewpoint on the subject was put forward by Achucarro and Townsend (1986) and Witten (1988). They proved that 2+1 dimensional gravity was equivalent to a Chern-Simons theory with the Poincaré group as its gauge group.

Quantization programs have mainly concentrated on matter free universes with torus or higher genus topology. In this paper we will treat the quantization of one particle states as was advocated by 't Hooft [5] and possible variations on that theme.

2. CLASSICAL MULTI-PARTICLE SOLUTIONS

If we solve the gravitational field surrounding a static point particle we find that it is a conical space. Therefore we may choose flat (Minkowskian) coordinates globally but with unconventional ranges. In polar coordinates ($r = 0$ is the position of the particle), the angle $\varphi$ runs from 0 to $2\pi(1 - 4Gm)$, where $G$ is Newton’s constant and $m$ is the mass of the particle. So we can picture space by cutting out a wedge and identifying the boundaries. To describe a moving particle we simply boost this solution. The Lorentz contraction widens the angle of the wedge that is ‘missing’ from spacetime. Also the identification rule is now a Poincaré transformation of the following form:

$$x' = a + BRB^{-1}(x - a)$$

Here $x$ and $x'$ are opposite points on the boundaries, $a$ is the position of the particle, $B$ is a boost matrix and $R$ is a rotation over an angle $8\pi Gm$. We have pictured the situation in figure [1] where one can also find the variables $\beta$ (half the deficit angle), $\eta$ (perpendicular rapidity of the boundary), $\xi$ (the rapidity of the particle: $v = \tanh(\xi)$) and $\mu = 4\pi Gm$. It is easy to deduce some relations among these variables:

$$\tan(\beta) = \cosh(\xi) \tan(\mu)$$
$$\tanh(\eta) = \sin(\beta) \tanh(\xi)$$
$$\cos(\mu) = \cos(\beta) \cosh(\eta)$$
$$\sinh(\eta) = \sin(\mu) \sinh(\xi)$$

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It is important to notice that we chose the wedge ‘behind’ the particle so that we can avoid time jumps across the wedge.

If we want to describe a multiparticle solution it is convenient to construct a Cauchy surface consisting of flat patches of Minkowski space. The patches must be glued together in such a way that the metric is continuous across the boundaries. The result of such a construction is that the boundaries as seen by observers on the neighboring patches has equal length and can only move perpendicular to itself. Moreover the velocities in both frames have the same magnitude, but not necessarily the same sign. On the spot where three edges meet we have a vertex. The angles $\alpha_i$ (see figure (2)) need not add up to $2\pi$ so that we can construct curved surfaces.

We can introduce particles on this surface by putting them inside a polygon. The ‘tailpipe’ introduced earlier connects to a neighbouring patch where it forms a vertex. Of course, besides the vertices, also particles introduce curvature on the surface. It should be noted however that the three dimensional curvature of a vertex vanishes as there is no matter present in contrast to the three-curvature of a particle which is proportional to its mass. If the system evolves in time all edges shrink or grow linearly in time. So now and then an edge appears or disappears (according to certain rules that can be derived). It is also possible that a particle hits a boundary of a patch after which it proceeds with a Lorentztransformed velocity in another polygon. We will call these events ‘transitions’. It is important that this is a completely deterministic system that evolves Cauchy surfaces in time. This is why there will be no trouble with causality. It is even possible to reformulate it as a Hamiltonian system. If we choose our Hamiltonian to be the total deficit angle of the surface (which equals the total energy contained in that surface) and the length $L_i$ as our configuration variables then we find (by solving Hamilton’s equations) that the momentum conjugated to the boundary variables is $p_i = 2\eta_i$ which is the rapidity with which the boundary moves. One may notice now that there are far too many degrees of freedom, as the only degrees of freedom are connected with the particles and there are many more boundaries. This is due to the fact that there are also some constraints in the model connected with the closure of the polygons. For instance, the angles inside a polygon (which are functions of the momenta) should add up to $(N - 2)\pi$ ($N$ is the number of edges surrounding a polygon). There are also two more constraints to ensure that the last boundary $L_N$ of a polygon, viewed a vector for a moment, bites the first boundary $L_1$ in its tail. These constraints constitute a system of first class constraints which generate ‘gauge-transformations’ in the following
sense. The closure of angles generate time translations of that particular polygon, and the closure of boundaries generate Lorentz transformations of the polygon. So we have now a constrained Hamiltonian model at our disposal for which quantization seems straightforward. The transitions however, that must be taken into account as boundary conditions on the wavefunction, are troublesome. This is the reason why we will first concentrate on the one particle quantization.

3. ONE-PARTICLE HILBERT SPACE

In this section we will follow a quantization scheme first proposed by Snyder in 1947 (!) [9] and reinvented by ’t Hooft [8] in the case of 2+1 gravity. The main idea behind Snyders paper was to introduce a curved momentum space (he used De Sitter space) that still has the maximal number of symmetries among which the full Lorentz group. In the case of De Sitter space or anti-De Sitter space one trades the translations of the Poincaré group for four more (in the case of 3+1 dimensions) Lorentz type transformations. So one cannot expect to preserve the full group of translations as an invariance group. But the amazing thing is that we can still define coordinates as Hermitian operators that act on wavefunctions living on this curved (but maximally symmetric) momentum space that transform covariantly under the full group of Lorentz transformations. In the previous section we have seen that the variables \( L_i \) and \( \eta_i \) are each others conjugate and that \( \eta_i \) is really a hyperbolic angle. For one particle there is an alternative pair of conjugate variables. If we denote the position of the particle in the two dimensional plane by \( (x, 0) \) and give a speed \( v = \tanh(\xi) \) in the \( x \)-direction than we find that the conjugate momentum is an angle \( \theta \). The ‘Schrödinger equation’ (4) then becomes

\[
\cos(H) = \cos(\mu) \cos(\theta) \tag{6}
\]

Moreover, we have seen in section 2 that the Hamiltonian was also given by an angle. The next step is to choose a curved momentum space. There are many possibilities. Denote by \( H_q \) a hyperboloid given by the following relation:

\[
Q_1^2 + ... + Q_p^2 - Q_{p+1}^2 - ... - Q_q^2 = 1 \tag{7}
\]

Inspired by the fact that the momentum variables and the Hamiltonian are given by angles ’t Hooft studied the possibilities:

a) \( H_0^2 \times H_0^2 \) \((= S^2 \times S^1)\)

b) \( H_0^2 \) \((= S^3)\)

So in the first case the momentum variables live on a sphere and the energy lives on a circle, in the second case all variables are combined in a three sphere. But there are also different possible choices which are presently being studied by the author. Interesting choices seem to be:

c) \( H_1^2 \times H_0^2 \)

d) \( H_2^2 \)

How can we do quantum mechanics on these spaces? We should of course study wavefunctions that live on these homogeneous momentum spaces. In particular we would like a complete set of orthonormal, square integrable functions to define a basis in our Hilbert space. Fortunately there is a lot of literature on this subject and one of the results is that on all this homogeneous spaces there exists such a complete, orthonormal, square integrable (with respect to a suitable measure) set of basis functions. For instance, on the sphere we have the well known spherical harmonics \( Y_{\ell m}(\theta, \varphi) \) as our basis.

Let us elaborate a bit on case a). The sphere is given by the equation:

\[
Q_1^2 + Q_2^2 + Q_3^2 = 1 \tag{8}
\]

If we define:

\[
x^k = i\ell_p (Q_3 \frac{\partial}{\partial Q_k} - Q_k \frac{\partial}{\partial Q_3}) \tag{9}
\]

\[
t = i\ell_p \frac{\partial}{\partial H} \tag{10}
\]

and

\[
\tan\left(\frac{\ell_p}{\cos(\mu)} p_k\right) \equiv \tan(\theta_k) = \frac{Q_k}{Q_3} \tag{12}
\]
where \( k = x, y \) and \( \ell_p \) is the Planck-length, then the coordinates \( x^k \) and the vector \((\tan(\theta_x), \tan(\theta_y))\) will transform covariantly under rotations generated by:

\[
L = i(Q_2 \frac{\partial}{\partial Q_1} - Q_1 \frac{\partial}{\partial Q_2})
\]  

(13)

This is of course checked by calculating the commutators:

\[
[L, x] = -y \quad [L, \tan(\theta_x)] = -\tan(\theta_y)
\]  

(14)

\[
[L, y] = x \quad [L, \tan(\theta_y)] = \tan(\theta_x)
\]  

(15)

The price that we are paying is that the usual commutation relations among the phase space variables are changed. For instance:

\[
[x, y] = i \frac{\ell_p^2}{\cos^2(\mu)} L
\]  

(16)

This implies that also the coordinates become subject to uncertainty relations. If we choose topology d) as our momentum space, also the time coordinate mixes into the non-commutative structure. In that case Lorentz transformations become simple pointtransformations.

Finally we would like to comment on the construction of a Schrödinger equation on these spaces. The basisfunctions of the momentum space are polynomials of the embedding coordinates \( Q_\nu \). For instance in the above example the basisfunctions are:

\[
\Psi_{\ell,m,t}(Q_j, H) = Y_{\ell m}(Q_j) \exp[iHt]
\]  

(17)

where \( j = 1, 2, 3 \) and \( \ell \) is the degree of this polynomial. The equation (18) can be written in terms of the \( Q \)-variables:

\[
\cos(H)\Psi_{\ell,m,t}(Q_j, H) = \cos(\mu)Q_3 \Psi_{\ell,m,t}(Q_j, H)
\]  

(18)

The action of \( \cos(H) \) on \( \Psi \) is simple: it is a shift of one time step in the positive direction minus a shift of one time step in the negative direction. Because \( Q_3 = Y_{10} \), the action of \( Q_3 \) on \( \Psi \) is just the calculation of Clebsch-Gordon coefficients for the decomposition of the tensor product of two \( \text{SO}(3) \) representations in its irreducible representations. In this case we find that the action of \( Q_3 \) is a linear combination of shifts of \( \ell \) of one step in the positive and negative direction. Because of the time steps in positive and negative direction, equation (18) is the analogue of the Klein-Gordon equation and therefore suffers from the same disease: \(|\Psi|^2\) cannot be interpreted as a probability distribution. The solution for this problem is to construct a Dirac-like equation (see 3).

When we use topology d) as our momentum space, the Hilbertspace is given by polynomials of the \( Q_\nu \) where \( \nu \) runs from 1 to 4. They are the infinite dimensional representations of \( \text{SO}(2,2) \) (which spectrum contains a continuous and a discreet part) and form an orthonormal set of square integrable basisfunctions. Although the \( Q_\nu \) transform now under the nonunitary finite dimensional representations of \( \text{SO}(2,2) \) their action on the infinite dimensional unitary representations can still be calculated using recurrence relations of the hypergeometric function. Although the Dirac-like equation is not completely satisfactory yet the advantage of this Hilbertspace seems to be that Lorentztransformations act very easy on it; they only involve nearest neighbours.

For two particles the complicated boundary conditions on the wave functions play an essential role. To formulate these boundary conditions on the wave functions it is very convenient to have a discreet spacetime. The above quantization procedure seems to provide that structure.

REFERENCES

1. A. Staruszkiewicz, Acta Phys. Polon. 24 (1963) 734.
2. S. Deser, R. Jackiw and G. ’t Hooft, Ann. Phys. 152 (1984) 220.
3. G. ’t Hooft, Class. Quant. Grav. 9 (1992) 1335.
4. A. Achucarro and P. Townsend, Phys. Lett. B180 (1986) 85.
5. E. Witten, Nucl. Phys. B331 (1988) 46.
6. M. Welling, Proceedings of the 7th Summer School: Theoretical and Mathematical Physics, Kazan, Russia. hep-th/9511211
7. S. Carlip, Canadian Gen. Rel. 1993 (1993) 215.
8. G. ’t Hooft, Class. Quant. Grav. 13 (1996) 1023.
9. H. Snyder, Physical Review 71 (1947) 38.