REACTION-ADVECTION-DIFFUSION COMPETITION MODELS
UNDER LEthal BOUNDARY CONDITIONS

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Abstract. In this study, we consider a Lotka-Volterra reaction-diffusion-advection model for two competing species under homogeneous Dirichlet boundary conditions, describing a hostile environment at the boundary. In particular, we deal with the case in which one species diffuses at a constant rate, whereas the other species has a constant rate diffusion rate with a directed movement toward a better habitat in a heterogeneous environment with a lethal boundary. By analyzing linearized eigenvalue problems from the system, we conclude that the species dispersion in the advection direction is not always beneficial, and survival may be determined by the convexity of the environment. Further, we obtain the coexistence of steady-states to the system under the instability conditions of two semi-trivial solutions and the uniqueness of the coexistence steady states, implying the global asymptotic stability of the positive steady-state.

1. Introduction. Over the past several decades, competition models of various types have been studied in ecology, ranging from homogeneous species habitats of species to nonuniform environments. In particular, studies on the asymptotic properties of two competing species, which lead to the extinction of one species or the coexistence of two species, have been performed considering various types of species dispersal under Neumann/Dirichlet boundary conditions. There have been a tremendous amount of studies for competition models for two species with constant diffusions (see [11, 12], and the references therein), and self-cross diffusion [28–30]. In ecology, it can be easily predicted that environmental heterogeneity dramatically affects the density and distribution of species living in a certain area [13]. In particular, species dispersion as a response to the environment is an essential factor in adaptation and evolution [4–6, 10, 27]. A significant amount of research

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has been conducted on the dynamics of two species with competing interactions in a bounded region under the assumption of a heterogeneous environment and the absence of species flux at the boundary [1,4–8,10,18,20–24]. It is now a well-known result that, under certain conditions, species with advection to the environment have a survival advantage over other species that only have a constant diffusion rate [4, 8, 10]. The coexistence of two competing species was previously investigated in [5, 23]. Concentration phenomena for high advection and the limiting structure of reaction–advection–diffusion systems were also investigated in [20–24]. Further, the authors of [6] considered the logarithmic form of advection and investigated the ideal free distribution. The dynamics of a model in which both species had advection-diffusion was studied in [7]. Recently, in [33, 34], the authors dealt with reaction-diffusion-advection competitive systems with general advection form, which incorporates distinct advective directions and inter-specific competition intensities. They obtained the parameter region in which all coexistence steady states are linearly stable. However, unlike cases in which there is no species flux at the boundary of the region, those in a hostile environment are different. For example, under the assumption of a hostile environment at the boundary, Belgacem and Cosner [2] discussed a species model revealing that dispersion in the advection direction together with a constant diffusion rate is not always beneficial for a species, and survival may be determined according to the convexity of the environment. However, the dynamics of the competitive interaction models including the directed movement of two species are not known for homogeneous Dirichlet boundary conditions. With regard to diffusive competition models with Dirichlet boundary conditions, there has been considerable work on the existence, uniqueness and nonexistence of stable coexistence states in homogeneous models with constant parameters (see [9,11,12,14] and the references therein) and of heterogeneous models with spatial parameters (see [15–17,26], and the references therein).

In this study, we extend the case of one species to that of two competing species, considering their interaction. To achieve this, we assume that when two species compete in a bounded region, the diffusion of one occurs linearly, whereas that of the other is also linear but is combined with advection owing to the heterogeneity of the environment. Our aim is to investigate the role of diffusion rate values and the sensitivity of directed movement on the local behavior of the competitive interaction between two species under a lethal-boundary environment as well as on the connection between local behavior to the long-term behavior of the system. Thus, we consider the following reaction–advection–diffusion model under a homogeneous Dirichlet boundary condition:

\[
\begin{align*}
  u_t &= \nabla \cdot (\mu \nabla u - \alpha u \nabla m) + u(m(x) - u - v) \\
  v_t &= \nu \Delta v + v(m(x) - u - v) \\
  u &= v = 0 \\
  u(x, 0) &= u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0
\end{align*}
\]

in \( \Omega \times (0, \infty) \),

on \( \partial \Omega \times (0, \infty) \),

in \( \Omega \).

Here \( \Omega \subseteq \mathbb{R}^n \) is a bounded region with smooth boundary \( \partial \Omega \) with \( n \geq 1 \). The unknown functions \( u(x, t) \) and \( v(x, t) \) are the population densities of two competing species. The function \( m(x) \) is a positive smooth function and represents the spatially varying resource productivity that affects both the resource rate and the carrying capacity. Further, \( \nabla \cdot (-\alpha u \nabla m) \) represents the directional movement of the species \( u \) toward \( m \) considering a sensitivity parameter \( \alpha > 0 \).
In this paper, to evaluate the asymptotic behavior of two competing species, we investigate the stability of the semi-trivial solution wherein one of the two species is absent. This stability depends on the convexity of the environment. Specifically, it is found that the stability of $(0, v)$, which indicates invasibility or survival, depends on the shape of the environment $m(x)$. Thus we investigate the three cases: (i) $\Delta m > 0$, (ii) $\Delta m < 0$, and (iii) $\Delta m$ changes its sign. Further, we obtain the unique coexistence of two competing species for a sufficiently large sensitivity of advection for the corresponding time-independent model of (1), which implies the global stability of the positive steady state.

The remainder of this paper is organized as follows. In Section 2, we present the main results regarding the stability of semi-trivial solutions $(\theta_{\alpha,\mu}, 0)$ and $(0, \theta_{\nu})$ when $\alpha$ is sufficiently small or within a proper range. Further, we give the uniqueness result of the coexistence steady state for a sufficiently large sensitivity parameter of the directed movement. Finally, we discuss the biological interpretation of the obtained results. In Section 3, we prove the results presented in Section 2.

2. Main results. We consider scalar equations

\[
\begin{aligned}
\nabla \cdot (\mu \nabla u - \alpha u \nabla m) + u(m(x) - u) &= 0 & \text{in } \Omega, \\
u = 0 & & \text{on } \partial \Omega,
\end{aligned}
\]

and

\[
\begin{aligned}
\nu \Delta v + v(m(x) - v) &= 0 & \text{in } \Omega, \\
v = 0 & & \text{on } \partial \Omega.
\end{aligned}
\]

From [2], (2) has a unique positive solution $\theta_{\alpha,\mu}$ if $\sigma_1 > 0$, where $\sigma_1$ is the principal eigenvalue of

\[
\begin{aligned}
\sigma \Phi &= \nabla \cdot (\mu \nabla \Phi - \alpha \Phi \nabla m) + m \Phi & \text{in } \Omega, \\
\Phi &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]

Let $\phi = e^{-\alpha m/\mu} \Phi$. Then the first equation of (4) becomes

\[
\mu \nabla \cdot (e^{\alpha m/\mu} \nabla \phi) + m e^{\alpha m/\mu} \phi = \sigma e^{\alpha m/\mu} \phi.
\]

Thus, we have the variational form of the principal eigenvalue as follows:

\[
\sigma_1 = \sup_{\phi \in H_0^1(\Omega) \setminus \{0\}, \int_{\Omega} e^{\alpha m/\mu} \phi^2 = 1} \int_{\Omega} -\mu e^{\alpha m/\mu} |\nabla \phi|^2 + m e^{\alpha m/\mu} \phi^2.
\]

In addition, it is well known from [3] that (3) has a unique positive solution $\theta_{\nu}$ when $\sigma_2 > 0$, where $\sigma_2$ is the principal eigenvalue of

\[
\begin{aligned}
\sigma \psi &= \nu \Delta \psi + m(x) \psi & \text{in } \Omega, \\
\psi &= 0 & \text{on } \partial \Omega.
\end{aligned}
\]

Therefore, if $\sigma_1$ and $\sigma_2$ are positive, (1) has two semi-trivial solutions: $(\theta_{\alpha,\mu}, 0)$ and $(0, \theta_{\nu})$. Based on [13] , the following result can be obtained regarding the stability of semi-trivial solutions when $\alpha = 0$ in system (1) by replacing the Neumann boundary conditions with Dirichlet boundary conditions. Thus, we omit the proof here.

**Theorem 2.1.** Assume that $\alpha = 0$ in system (1). Let $\sigma_1 > 0$ and $\sigma_2 > 0$.

(i) If $\mu < \nu$, then $(\theta_{\nu}, 0)$ is locally asymptotically stable, and $(0, \theta_{\nu})$ is unstable;

(ii) If $\mu > \nu$, then $(0, \theta_{\nu})$ is locally asymptotically stable, and $(\theta_{\nu}, 0)$ is unstable.
Theorem 2.1 implies that, for the competition model in a heterogeneous environment, the slower diffusor eventually survives as time goes to $\infty$.

We now introduce the results of this paper. We first give a result for the global asymptotic behavior of $(u,v)$ when either $\sigma_1 < 0$ or $\sigma_2 < 0$, or both are negative.

**Theorem 2.2.** Let $(u,v)$ be a positive solution of (1).
\begin{itemize}
  \item[(i)] If $\sigma_1 < 0$ and $\sigma_2 < 0$, then $(u,v)$ converges to $(0,0)$ as $t \to \infty$.
  \item[(ii)] If $\sigma_1 > 0$ and $\sigma_2 < 0$, then $(u,v)$ converges to $(\theta_{\alpha,\mu},0)$ as $t \to \infty$.
  \item[(iii)] If $\sigma_1 < 0$ and $\sigma_2 > 0$, then $(u,v)$ converges to $(0,\theta_\nu)$ as $t \to \infty$.
\end{itemize}

Next, we obtain the local stability of semi-trivial solutions of (1) given the case in which $\alpha > 0$ is sufficiently small.

**Theorem 2.3.** Let $\sigma_1 > 0$ and $\sigma_2 > 0$ and assume that $\alpha > 0$ is sufficiently small.
\begin{itemize}
  \item[(i)] $(\theta_{\alpha,\mu},0)$ is locally asymptotically stable if $\mu < \nu$, and it is unstable if $\mu > \nu$.
  \item[(ii)] $(0,\theta_\nu)$ is locally asymptotically stable if $\mu > \nu$, and it is unstable if $\mu < \nu$.
\end{itemize}

Theorem 2.3 implies that when $\alpha$ is sufficiently small, the stability of semi-trivial solutions does not change compared with the case in which $\alpha = 0$.

Subsequently, we consider large values of $\alpha$. We further consider three cases for $m(x)$: (i) $\Delta m > 0$, (ii) $\Delta m < 0$, and (iii) $\Delta m$ changes sign. The following corresponds to the case of $\Delta m > 0$.

**Theorem 2.4.** Assume that $\Delta m > 0$ on $\overline{\Omega}$ and $\sigma_2 > 0$. Then, there exists $\alpha_1 > 0$ such that $(0,\theta_\nu)$ is locally asymptotically stable if $\alpha \geq \alpha_1$. Moreover, $\alpha_1 = 0$ when $\mu > \nu$.

When compared with Theorem 2.1, Theorem 2.4 indicates that advective movement results in a survival disadvantage when the resource $m(x)$ is distributed convexly (Figure 1). Figure 1 shows that a species that can survive without directed movement may become extinct if it disperses with advective direction and linear diffusion.

To present the results for the remaining cases, we define the following notation. For a continuous function $f$, denote
\begin{align*}
A_f &= \int_{\Omega} \frac{1}{4\mu} f^2 |\nabla m|^2, \\
B_f &= \int_{\Omega} \frac{1}{2} f^2 \Delta m, \\
C_f &= \int_{\Omega} \frac{(\nu - \mu) |\nabla f|^2}{4}. 
\end{align*}

Next, we state the result corresponding to the case of $\Delta m < 0$ on $\overline{\Omega}$.

**Theorem 2.5.** Assume that $\Delta m < 0$ on $\overline{\Omega}$ and $\sigma_2 > 0$. If $B_{\theta_\nu}^2 + 4A_{\theta_\nu}C_{\theta_\nu} > 0$, there exists $\alpha_3 > \alpha_2$ such that $\alpha_3 > 0$ and $(0,\theta_\nu)$ is unstable for $\alpha \in (\max\{0,\alpha_2\},\alpha_3)$. Moreover, if $\mu < \nu$, then, $B_{\theta_\nu}^2 + 4A_{\theta_\nu}C_{\theta_\nu} > 0$ holds and $(0,\theta_\nu)$ is unstable for $\alpha \in (0,\alpha_3)$.

In Theorem 2.5, we remark that the condition $B_{\theta_\nu}^2 + 4A_{\theta_\nu}C_{\theta_\nu} > 0$ is satisfied for $\mu > \nu$, once the condition
\[ \mu - \nu < \frac{B_{\theta_\nu}^2}{A_{\theta_\nu} \int_{\Omega} |\nabla \theta_\nu|^2} \]
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Figure 1. Case $\Delta m > 0$: (a) Instabilities of $(0, \theta_\nu)$ when $\alpha = 0$, (b) Stability of $(0, \theta_\nu)$ when $\alpha = 0.1$ $(m(x) = 1 - 0.4 \sin(\pi x), \mu = 0.01, \nu = 0.02)$.

holds for a given $m$ and $\theta_\nu$. In other words, if the difference between the two distinct diffusion rates of two competing species is not sufficiently large, the species with the advective movement can survive given a suitable $\alpha$.

It follows from Theorem 2.5 that advection provides a survival advantage when $m(x)$ is a concave function (Figure 2). Figure 2 shows that even if it is possible that the species moving with linear diffusion can be extinct, it may survive if it disperses with advective direction and linear diffusion.

We now consider the last case, assuming that $\Delta m$ changes sign. In this case, subsets $\{\Omega_i^-\}$ of $\Omega$ exist for $i = 1, \cdots, N$ such that $\Omega_i^-$ is connected and

$$
\bigcup_{i=1}^N \Omega_i^- = \{x \in \Omega : \Delta m(x) < 0\}.
$$
We define the principal eigenvalue $\eta_1^i$ of the problem as

$$\begin{cases}
\eta_1^i \phi = \nu \Delta \phi + m(x) \phi & \text{in } \Omega_i^-,
\phi = 0 & \text{on } \partial \Omega_i^-.
\end{cases} \quad (7)$$

Let $k$ be such that $\eta_M := \max_{i \in [1,N]} \eta_1^i = \eta_1^k$. If $\eta_M > 0$, there exists a positive solution $v_M$ of

$$\begin{cases}
\nu \Delta v + v(m(x) - v) = 0 & \text{in } \Omega_k^-,
v = 0 & \text{on } \partial \Omega_k^-.
\end{cases} \quad (8)$$

Regarding $v_M$ as a function defined on $\Omega$ such that $v_M = 0$ for $x \in \Omega \setminus \Omega_k^-$, $A_{v_M}$, $B_{v_M}$ and $C_{v_M}$ are well defined. Thus, we obtain the following result.
Theorem 2.6. Assume that $\Delta m$ changes its sign and $\eta_M > 0$. If $B_{cM}^2 + 4A_{cM}C_{cM} > 0$, then there exists $\alpha_5 > \alpha_4$ such that $\alpha_5 > 0$ and $(0, \theta_v)$ is unstable for $\alpha \in (\max\{0, \alpha_4\}, \alpha_5)$. 

Similar to that in Theorem 2.5, the condition $B_{cM}^2 + 4A_{cM}C_{cM} > 0$ holds when the difference between the diffusion rates of two competing species is not too large or the diffusion rate of the prey is lower than that of the predators.

If we regard $m(x)$ as the species resources, it is expected that the advective movement toward $m$ will result in a survival disadvantage when $m(x)$ is distributed near the boundary because the boundary condition is lethal. However, Theorem 2.6 indicates that although $m(x)$ has its maximum on the boundary, the directional movement toward $m$ provides a survival advantage so long as $\eta_M > 0$ (Figure 3). Figure 3 shows that a species that might become extinct without directed movement can survive if it disperses in the advective direction with linear diffusion, even when $\Delta m$ changes sign.

Next, we obtain the stability of $(\theta_{\alpha, \mu}, 0)$.

Theorem 2.7. Assume that $\sigma_1 > 0$, $\sigma_2 > 0$ and the set of critical points of $m(x)$ has Lebesgue measure zero. Then, $\alpha_6 > 0$ exists such that $(\theta_{\alpha, \mu}, 0)$ is unstable if $\alpha > \alpha_6$.

It should be noted that since $\sigma_1 \geq \lambda_1(L, \alpha)$, $\theta_{\alpha, \mu}$ exists when $(0, \theta_v)$ is unstable, where $\lambda_1(L, \alpha)$ is the principal eigenvalue of the following equation (see Section 3 in detail):

$$
\begin{align*}
\mu \nabla \cdot (e^{\alpha m/\mu} \nabla \phi) + (m(x) - \theta_v(x))e^{\alpha m/\mu} \phi &= \lambda e^{\alpha m/\mu} \phi \quad \text{in } \Omega, \\
\phi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

(9)

We further note that $(0, 0)$ is unstable if $\sigma_1 > 0$ and $\sigma_2 > 0$. The coexistence of steady states follows from the fact that (1) is a strongly monotone system (see [5]) when two semi-trivial states are unstable. Therefore, using Theorems 2.5, 2.6 and 2.7, we can conclude that (1) has the coexistence when $\alpha$ is properly chosen and $\Delta m$ is negative somewhere in $\Omega$ (Figures 2, 3).

Let us denote that

$$
\Lambda_1 = \{ \alpha \in \mathbb{R}^+ : \lambda_1(L, \alpha) > 0 \},
$$

$$
\Lambda_2 = \{ \alpha \in \mathbb{R}^+ : \lambda_1(\nu \Delta + (m - \theta_{\alpha, \mu})) > 0 \}.
$$

Now, we state the following coexistence result of steady-states to system (1).

Theorem 2.8. Assume that $\sigma_1 > 0$, $\sigma_2 > 0$ and the set of critical points of $m(x)$ has Lebesgue measure zero.

(i) Suppose that $\Delta m < 0$ on $\Omega$. If $B_{cM}^2 + 4A_{cM}C_{cM} > 0$ and $\alpha_3 \geq \alpha_6$, then $\Lambda_1 \cap \Lambda_2 \neq \emptyset$, and (1) has at least one coexistence steady-state for $\alpha \in \Lambda_1 \cap \Lambda_2$;

(ii) Suppose that $\Delta m$ changes its sign and $\eta_M > 0$. If $B_{cM}^2 + 4A_{cM}C_{cM} > 0$ and $\alpha_5 \geq \alpha_6$, then $\Lambda_1 \cap \Lambda_2 \neq \emptyset$, and (1) has at least one coexistence steady-state for $\alpha \in \Lambda_1 \cap \Lambda_2$.

Furthermore, if the coexistence steady-state is unique, it is globally asymptotically stable [18]. Thus, we obtain the following result.

Theorem 2.9. Assume that $\sigma_1 > 0$ and $\sigma_2 > 0$. Suppose that $\Delta m < 0$. There exists $\alpha_7$ such that if $[\alpha_7, \infty) \cap (\Lambda_1 \cap \Lambda_2) \neq \emptyset$, then the coexistence steady state obtained in Theorem 2.8 is unique for $\alpha \in [\alpha_7, \infty) \cap (\Lambda_1 \cap \Lambda_2)$, and thus it is globally asymptotically stable.
3. Proof of theorems. Before starting this section, we consider an eigenvalue problem

$$\begin{cases}
\lambda \psi = \nu \Delta \psi + g(x)\psi & \text{in } \Omega, \\
\psi = 0 & \text{on } \partial \Omega.
\end{cases}$$

(10)

We denote the principal eigenvalue of (10) by $\lambda_1(\nu \Delta + g(x))$, and $\lambda_1(\nu \Delta + g(x))$ has the variational form:

$$\lambda_1(\nu \Delta + g(x)) = \sup_{\psi \in H_0^1(\Omega) \setminus \{0\}, \int_\Omega \psi^2 = 1} \int_\Omega -\nu |\nabla \psi|^2 + g(x)\psi^2.$$
We also consider an eigenvalue problem with advection
\[
\begin{aligned}
\lambda \Phi &= \nabla \cdot (\mu \nabla \Phi - \alpha \Phi \nabla A) + A(x)\Phi \quad \text{in } \Omega, \\
\Phi &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\] (11)

Similar to \( \sigma_1 \) in Section 2, we have the variational form of the principal eigenvalue as follows:
\[
\lambda_1 = \sup_{\phi \in H_0^1(\Omega) \setminus \{0\}, \int_\Omega e^{\alpha A/\mu} \phi^2 = 1} \int_\Omega -\mu e^{\alpha A/\mu} |\nabla \phi|^2 + Ae^{\alpha A/\mu} \phi^2.
\] (12)

We denote \( \lambda_1 = \lambda_{1,A} \). Then, we first give a proof of Theorem 2.2.

**Proof of Theorem 2.2.** By a comparison with the scalar equations of \( u \) and \( v \), it is straightforward that (i) holds. In addition, since (ii) and (iii) can be proved in a similar manner, we only give a proof of (ii) here.

Let \((u, v)\) be a solution of (1). Since \( \sigma_2 = \lambda_1 (\nu \Delta + m) < 0 \), and the comparison principle, we have the following:
\[
\lim_{t \to \infty} v(x, t) = 0.
\] (13)

For \( u \), we observe that
\[
u_t = \nabla \cdot (\mu \nabla u - \alpha u \nabla m) \leq u(m(x) - u).
\]

From [2], the scalar equation
\[
\begin{aligned}
U_t &= \nabla \cdot (\mu \nabla U - \alpha U \nabla A) + U(A(x) - U) \quad \text{in } \Omega \times (0, \infty), \\
U &= 0 \quad \text{on } \partial \Omega \times (0, \infty),
\end{aligned}
\] (14)

has a unique positive equilibrium \( \Theta[A] \), which is globally attracting among nonzero non-negative solutions if \( \lambda_{1,A} > 0 \). Otherwise, the trivial solution is globally stable.

Since \( \sigma_1 = \lambda_{1,m} > 0 \), it follows from the comparison principle that
\[
\limsup_{t \to \infty} u(x, t) \leq \Theta[m] := \theta_{\alpha,\mu}.
\] (15)

Let \( \epsilon \) be sufficiently small such that \( \sigma_1 - \epsilon > 0 \). From (13), there exists a \( T_0 \geq 0 \) such that \( v(x, t) \leq \epsilon \) for all \( x \in \Omega \), \( t > T_0 \). Then, \( u \) satisfies
\[
u_t = \nabla \cdot (\mu \nabla u - \alpha u \nabla m) \geq u(m(x) - \epsilon - u).
\]

Let \( \phi_1 \) be a principal eigenfunction with respect to \( \sigma_1 \). Since \( \int_\Omega e^{\alpha m/\mu} \phi_1^2 = 1 \), we have
\[
\sigma_1 - \epsilon = \int_\Omega -\mu e^{\alpha m/\mu} |\nabla \phi_1|^2 + (m - \epsilon)e^{\alpha m/\mu} \phi_1^2.
\]

Let \( \bar{\phi} = e^{\alpha \epsilon/2\mu} \phi_1 \). Then, \( \bar{\phi} \) satisfies \( \int_\Omega e^{\alpha (m-\epsilon)/\mu} \bar{\phi}^2 = 1 \) and
\[
\sigma_1 - \epsilon \leq \sup_{\phi \in H_0^1(\Omega) \setminus \{0\}, \int_\Omega e^{\alpha (m-\epsilon)/\mu} \phi^2 = 1} \int_\Omega -\mu e^{\alpha (m-\epsilon)/\mu} |\nabla \phi|^2 + (m - \epsilon)e^{\alpha (m-\epsilon)/\mu} \phi^2
\]
\[
= \lambda_{1,m-\epsilon}.
\]

Based on the choice of \( \epsilon \), \( \lambda_{1,m-\epsilon} \) is positive. Then by comparison, we obtain
\[
\liminf_{t \to \infty} u(x, t) \geq \Theta[m - \epsilon].
\]
By the continuity for $\epsilon$, if we let $\epsilon \to 0$, we obtain
\[
\liminf_{t \to \infty} u(x,t) \geq \theta_{\alpha,\mu}.
\] (16)

Combining (15) and (16), it can be concluded that $u(x,t) \to \theta_{\alpha,\mu}$ as $t \to \infty$. \qed

Next, we consider the linearized eigenvalue problem at $(\theta_{\alpha,\mu}, 0)$ and $(0, \theta_{\nu})$, which can be obtained by a simple calculation. At $(\theta_{\alpha,\mu}, 0)$, we have
\[
\begin{align*}
\lambda_1 &= \nabla \cdot (\mu \nabla \Phi - \alpha \Phi \nabla m) + \Phi (m(x) - 2\theta_{\alpha,\mu}) - \theta_{\alpha,\mu} \psi \\
\lambda \psi &= \nu \Delta \psi + \psi (m(x) - \theta_{\alpha,\mu}) \\
\Phi &= \psi = 0
\end{align*}
\] (17)

and at $(0, \theta_{\nu})$,
\[
\begin{align*}
\lambda_1 &= \nabla \cdot (\mu \nabla \Phi - \alpha \Phi \nabla m) + \Phi (m(x) - \theta_{\nu}) \\
\lambda \psi &= \nu \Delta \psi + \psi (m(x) - 2\theta_{\nu}) - \theta_{\nu} \Phi \\
\Phi &= \psi = 0
\end{align*}
\] (18)

Let $\phi = e^{-\alpha m/\mu} \Phi$. Then, the first equation of (18) becomes a weighted eigenvalue problem, as follows:
\[
\mu \nabla \cdot (e^{\alpha m/\mu} \nabla \phi) + (m(x) - \theta_{\nu}(x)) e^{\alpha m/\mu} \phi = \lambda e^{\alpha m/\mu} \phi,
\] (19)

and (19) can be written as $\lambda e^{\alpha m/\mu} \phi = L \phi$, where
\[
L \phi = \mu \nabla \cdot (e^{\alpha m/\mu} \nabla \phi) + (m(x) - \theta_{\nu}(x)) e^{\alpha m/\mu} \phi.
\]

Note that since $L$ is a self-adjoint operator, the principal eigenvalue has a variational form as follows:
\[
\lambda_1(L, \alpha) := \sup_{\|\phi\|_2 = 1} \int_{\Omega} -\mu e^{\alpha m/\mu} |\nabla \phi|^2 + (m(x) - \theta_{\nu}(x)) e^{\alpha m/\mu} \phi^2.
\]

Considering the stability of two semi-trivial solutions $(\theta_{\alpha,\mu}, 0)$ and $(0, \theta_{\nu})$ of system (1), the following two lemmas can be obtained from Lemma 5.5 in [7] by considering homogeneous Dirichlet boundary conditions instead of Neumann boundary conditions.

**Lemma 3.1.** Assume that $\sigma_2 > 0$. Let $\lambda_1(L, \alpha)$ be the principal eigenvalue of (9).

(i) If $\lambda_1(L, \alpha) > 0$, then $(0, \theta_{\nu})$ is unstable;

(ii) If $\lambda_1(L, \alpha) < 0$, then $(0, \theta_{\nu})$ is locally asymptotically stable.

**Lemma 3.2.** Assume that $\sigma_1 > 0$. Let $\lambda_1(\nu \Delta + (m - \theta_{\alpha,\mu}))$ be the principal eigenvalue.

(i) If $\lambda_1(\nu \Delta + (m - \theta_{\alpha,\mu})) > 0$, $(\theta_{\alpha,\mu}, 0)$ is unstable;

(ii) If $\lambda_1(\nu \Delta + (m - \theta_{\alpha,\mu})) < 0$, $(\theta_{\alpha,\mu}, 0)$ is locally asymptotically stable.

We will use Lemma 3.1 and 3.2 to prove the stability results in Section 2.

### 3.1. Stability of $(0, \theta_{\nu})$. Herein, we will prove Theorems 2.3, 2.4, 2.5, and 2.6. Recall that the principal eigenvalue of (19) can be written by
\[
\lambda_1(L, \alpha) = \sup_{\|\phi\|_{H^1_0(\Omega)} \neq 1} \int_{\Omega} -\mu e^{\alpha m/\mu} |\nabla \phi|^2 + (m(x) - \theta_{\nu}(x)) e^{\alpha m/\mu} \phi^2.
\] (20)
Consider \( \varphi = e^{\alpha m/2\mu} \phi \). Since \( \phi \in H^1_0(\Omega) \), \( \varphi \) is also in \( H^1_0(\Omega) \). Also, \( \phi \mapsto e^{\alpha m/2\mu} \phi \) is a bijection mapping. If we let \( \phi = e^{-\alpha m/2\mu} \varphi \) and take the supremum over \( \varphi \in H^1_0(\Omega) \), we can rewrite the (20) as

\[
\lambda_1(L, \alpha) = \sup_{\varphi \in H^1_0(\Omega) \setminus \{0\}, ||\varphi||_2 = 1} \int_{\Omega} -\mu |\nabla \varphi|^2 + \alpha \varphi \nabla \varphi \cdot \nabla m + \left( -\frac{\alpha^2}{4\mu} |\nabla m|^2 + m(x) - \theta_\nu(x) \right) \varphi^2.
\]

Then, by Green’s identity, (EV1) can be written as

\[
\lambda_1(L, \alpha) = \sup_{\varphi \in H^1_0(\Omega) \setminus \{0\}, ||\varphi||_2 = 1} \int_{\Omega} -\mu |\nabla \varphi|^2 + \left( -\frac{\alpha^2}{4\mu} |\nabla m|^2 + m(x) - \theta_\nu(x) \right) \varphi^2.
\]

By the uniqueness of positive principal eigenfunction and the definition of the principal eigenvalue, \( \lambda_1(L, \alpha) \) is the principal eigenvalue of

\[
\begin{cases} 
\mu \Delta \varphi + \left( -\frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2 + m(x) - \theta_\nu(x) \right) \varphi = \lambda \varphi & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Let \( \phi_1 \) be the principal eigenfunction with respect to \( \lambda_1(L, \alpha) \). From (EV2), we obtain

\[
\lambda_1(L, \alpha) = \int_{\Omega} -\mu |\nabla \varphi_1|^2 + \left( -\frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2 + (m(x) - \theta_\nu(x)) \right) \varphi_1^2,
\]

where \( \varphi_1 = e^{\alpha m/2\mu} \phi_1 \). Since the positive principal eigenfunction is unique, \( \varphi_1 \) is the principal eigenfunction with respect to \( \lambda_1 \left( \mu \Delta - \frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2 + (m(x) - \theta_\nu(x)) \right) \).

Therefore,

\[
\lambda_1(L, \alpha) = \lambda_1 \left( \mu \Delta - \frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2 + (m(x) - \theta_\nu(x)) \right).
\]

We will utilize the forms (EV1) and (EV2) to prove our results.

**Proof of Theorem 2.3.** Based on Lemma 3.1, it is necessary to check the sign of \( \lambda_1(L, \alpha) \). This can be easily shown by a standard perturbation argument with the continuity of the principal eigenvalue on the advection rate \( \alpha \) and diffusion rates of species \( \mu \) and \( \nu \).

**Proof of Theorem 2.4.** Let \( (\lambda_1(L, \alpha), \varphi_1) \) be the principal eigenpair of (18) with \( ||\varphi_1||_2 = 1 \). It suffices to check the sign of \( \lambda_1(L, \alpha) \). From (EV2), we obtain

\[
\lambda_1(L, \alpha) = \int_{\Omega} -\mu |\nabla \varphi_1|^2 + \left( -\frac{\alpha}{2} \Delta m + \frac{\alpha^2}{4\mu} |\nabla m|^2 \right) \varphi_1^2 + \frac{1}{\alpha} \varphi_1^2 (m - \theta_\nu) \\
\leq \int_{\Omega} -\mu |\nabla \varphi_1|^2 + \left( -\frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2 \right) \varphi_1^2 + \frac{1}{\alpha} ||m - \theta_\nu||_\infty \varphi_1^2 < 0,
\]

for large \( \alpha > 0 \) satisfying \( -\frac{1}{2} \min_{\Omega} \Delta m + \frac{1}{2} ||m - \theta_\nu||_\infty < 0 \). By the assumption \( \Delta m > 0 \) and (EV2), \( \lambda_1(L, \alpha) \) is monotonically decreasing with respect to \( \alpha \). Hence, \( \alpha_1 > 0 \) exists such that \( \lambda_1(L, \alpha) < 0 \) for \( \alpha \geq \alpha_1 \).
Assume that $\nu < \mu$. Since $\lambda_1(\nu \Delta + (m - \theta_\nu)) = 0$, we have
\[
\lambda_1(L, \alpha) = \int_{\Omega} -\mu |\nabla \varphi_1|^2 - \left(\frac{\alpha}{2} \Delta m + \frac{\alpha^2}{4\mu} |\nabla m|^2\right) \varphi_1^2 + \varphi_1^2 (m - \theta_\nu) \\
\leq \int_{\Omega} (\nu - \mu) |\nabla \varphi_1|^2 - \left(\frac{\alpha}{2} \Delta m + \frac{\alpha^2}{4\mu} |\nabla m|^2\right) \varphi_1^2 < 0.
\]
Thus, $\lambda_1(L, \alpha) < 0$ for all $\alpha > 0$; therefore, $\alpha_1 = 0$.

Proof of Theorem 2.5. We only need to check the sign of $\lambda_1(L, \alpha)$. Since $\theta_\nu$ satisfies $\nu \Delta \theta_\nu + \theta_\nu (m - \theta_\nu) = 0$,
\[
\int_{\Omega} \nu |\nabla \theta_\nu|^2 = \int_{\Omega} \theta_\nu^2 (m - \theta_\nu).
\]
Then, from (EV2),
\[
\lambda_1(L, \alpha) = \sup_{\varphi \in H_0^1(\Omega) \setminus \{0\}, \|\varphi\|_2 = 1} \int_{\Omega} -\mu |\nabla \varphi|^2 + \left(-\frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2 + m(x) - \theta_\nu(x)\right) \varphi^2 \\
\geq \left[\int_{\Omega} -\mu |\nabla \theta_\nu|^2 + \left(-\frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2 + m(x) - \theta_\nu(x)\right) \theta_\nu^2 \right] / \int_{\Omega} \theta_\nu^2 \\
= \left[\int_{\Omega} (\nu - \mu) |\nabla \theta_\nu|^2 + \left(-\frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2\right) \theta_\nu^2 \right] / \int_{\Omega} \theta_\nu^2.
\]
Since $B_{\theta_\nu}^2 + 4A_{\theta_\nu}C_{\theta_\nu} > 0$, there exist two real roots, $\alpha_2, \alpha_3$, of $A_{\theta_\nu}\alpha^2 + B_{\theta_\nu}\alpha - C_{\theta_\nu} = 0$, such that $\alpha_2 < \alpha_3$ and $\alpha_3 > 0$. Then, $\lambda_1(L, \alpha) > 0$ for $\alpha \in (\alpha_2, \alpha_3)$. Since $\mu \geq 0$, we have that $\lambda_1(L, \alpha) > 0$ for $\alpha \in (\max\{0, \alpha_2, \alpha_3\})$. Furthermore, If $\mu < \nu$, then $B_{\theta_\nu}^2 + 4A_{\theta_\nu}C_{\theta_\nu}$ is always positive and $\lambda_1(L, \alpha) > 0$ for $\alpha \in (0, \alpha_3)$.

Proof of Theorem 2.6. Since $\eta_M = \eta_1^k > 0$, there exists a unique positive solution $v_M$ of
\[
\begin{cases}
\nu \Delta v + v(m(x) - v) = 0 & \text{in } \Omega_k^-, \\
v = 0 & \text{on } \partial \Omega_k^-.
\end{cases}
\]
Define
\[
\tilde{\phi}(x) = \begin{cases}
0 & \text{for } x \in \Omega_k \setminus \Omega^-_k, \\
v_M & \text{for } x \in \Omega^-_k.
\end{cases}
\]
Then,
\[
\lambda_1(L, \alpha) = \sup_{\varphi \in H_0^1(\Omega) \setminus \{0\}, \|\varphi\|_2 = 1} \int_{\Omega} -\mu |\nabla \varphi|^2 + \left(-\frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2 + m(x) - \theta_\nu(x)\right) \varphi^2 \\
\geq \left[\int_{\Omega_k} -\mu |\nabla \tilde{\phi}|^2 + \left(-\frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2 + m(x) - \theta_\nu(x)\right) \tilde{\phi}^2 \right] / \int_{\Omega_k} \tilde{\phi}^2 \\
= \left[\int_{\Omega_k} (\nu - \mu) \nu |\nabla v_M|^2 + \left(-\frac{\alpha}{2} \Delta m - \frac{\alpha^2}{4\mu} |\nabla m|^2\right) v_M^2 \right] / \int_{\Omega_k} v_M^2.
\]
Similar to the proof of Theorem 2.5, there exists $\alpha_4, \alpha_5$ such that $\alpha_4 < \alpha_5$, $\alpha_5$ is positive, and $\lambda_1(L, \alpha) > 0$ for $\alpha \in (\max\{0, \alpha_4, \alpha_5\})$. Hence, $(0, \theta_\nu)$ is unstable when $\alpha \in (\max\{0, \alpha_4, \alpha_5\})$. \qed
3.2. Stability of \((\theta_{\alpha,\mu}, 0)\).

**Proof of Theorem 2.7.** It is necessary to check the sign of \(\lambda_1(\nu \Delta + (m(x) - \theta_{\alpha,\mu}))\). Let \(\psi_1\) be the principal eigenfunction corresponding to \(\sigma_2 > 0\) and satisfying \(\|\psi_1\|_2 = 1\). Then,

\[
\lambda_1(\nu \Delta + (m - \theta_{\alpha,\mu})) \geq \int_{\Omega} \nu|\nabla \psi_1|^2 + \psi_1^2(m - \theta_{\alpha,\mu}) = \sigma_2 - \int_{\Omega} \theta_{\alpha,\mu} \psi_1^2 \geq \sigma_2 - \|\theta_{\alpha,\mu}\|_2 \|\psi_1^2\|_2.
\]

Note that \(\|\psi_1\|_\infty < \infty\), based on the standard elliptic regularity. In addition, note that the principal eigenvalue with a no-flux boundary condition is greater than that with a Dirichlet boundary condition. Then, it follows from the assumption \(\sigma_1 > 0\) that the principal eigenvalue of

\[
\left\{ \begin{array}{ll}
\mu \frac{\partial \psi}{\partial m} = 0 & \text{in } \Omega, \\
\frac{\partial \psi}{\partial m} = 0 & \text{on } \partial \Omega
\end{array} \right.
\]

is positive. Thus, there exists a solution \(u_{\alpha,N}\) of

\[
\left\{ \begin{array}{ll}
\nabla \cdot (\mu \nabla u - \alpha u \nabla m) + u(m(x) - u) = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial m} = 0 & \text{on } \partial \Omega.
\end{array} \right.
\]

Since \(u_{\alpha,N}\) is the upper solution of \(\theta_{\alpha,\mu}\), based on the fact presented in [5],

\[\|u_{\alpha,N}\|_2 \to 0 \quad \text{as } \alpha \to \infty.\]

Then, it follows that

\[\|\theta_{\alpha,\mu}\|_2 \to 0 \quad \text{as } \alpha \to \infty.\]

Therefore, we have

\[\lambda_1(\nu \Delta + (m - \theta_{\alpha,\mu})) \geq \lambda_1(\nu \Delta + m) - \|\theta_{\alpha,\mu}\|_2 \|\psi_1^2\|_2 \to \lambda_1(\nu \Delta + m) > 0 \quad \text{as } \alpha \to \infty.
\]

Thus, there exists \(\alpha_6\) such that

\[\lambda_1(\nu \Delta + m) - \|\theta_{\alpha_6,\mu}\|_2 \|\psi_1^2\|_2 = 0,
\]

and \(\lambda_1 \geq 0\) for \(\alpha \geq \alpha_6\). Hence, \((\theta_{\alpha,\mu}, 0)\) is unstable if \(\alpha > \alpha_6\). \(\square\)

3.3. Coexistence of steady states. In this subsection, we assume \(\Delta m < 0\) on \(\bar{\Omega}\). We then employ the fixed-point index theory to prove the uniqueness of coexistence steady states to (1) for a large \(\alpha > 0\) satisfying the instability conditions of the semi-trivial solutions of system (1).

**Fixed point index theory.** Let \(E\) be a real Banach space and \(W \subset E\) be a closed convex set. \(W\) is said to be a total wedge if \(\alpha W \subset W\) for all \(\alpha \geq 0\) and \(\overline{W - W} = E\). A wedge is said to be a cone if \(W \cap (-W) = \{0\}\). For \(y \in W\), we define

\[W_y = \{x \in E : y + \gamma x \in W \text{ for some } \gamma > 0\} \text{ and } S_y = \{x \in \overline{W_y} : -x \in \overline{W_y}\}.
\]

Then, \(\overline{W_y}\) is a wedge containing \(W\), \(y\), and \(-y\), whereas \(S_y\) is a closed subspace of \(E\) containing \(y\).

Let \(T\) be a compact linear operator on \(E\) satisfying \(T(\overline{W_y}) \subset \overline{W_y}\). \(T\) is said to have property \(\alpha\) on \(\overline{W_y}\) if \(t \in (0, 1)\) and \(w \in \overline{W_y} \setminus S_y\) exist such that \(w - tTw \in S_y\). Let \(F : W \to W\) be a compact operator with a fixed point \(y \in W \) and \(F\) be Fréchet
differentiable at \( y \). Let \( L = F'(y) \) be the Fréchet derivative of \( F \) at \( y \). Then, \( L \) maps \( W_y \) into itself. For an open subset \( U \subset W \), we define

\[
\text{Ind}_W(F,U) = \text{Ind}(F,U,W) = \deg_W(I - F, U, 0),
\]

where \( I \) is the identity map. If \( y \) is an isolated fixed point of \( F \), then the fixed point index of \( F \) at \( y \) in \( W \) is defined by

\[
\text{Ind}_W(F, y) = \text{Ind}(F, y, W) = \text{Ind}(F, U(y), W),
\]

where \( U(y) \) is a small open neighborhood of \( y \) in \( W \).

To prove Theorem \ref{thm:main}, we first obtain a priori bounds.

**Lemma 3.3.** Let \((u,v)\) be the positive solution of system \((1)\). Then, \((u,v)\) satisfies

\[
\begin{align*}
 u(x) &\leq \alpha \|\Delta m\|_{\infty} + \max_{\overline{\Omega}} m := Q_1 \\
 v(x) &\leq \max_{\overline{\Omega}} m := Q_2
\end{align*}
\]

for all \( x \in \overline{\Omega} \).

**Proof.** By the maximum principle, we obtain \( v(x) \leq \max_{\overline{\Omega}} m \). Assume that \( u(x) \) has the maximum value at \( x_0 \in \overline{\Omega} \). Note that \( \nabla \cdot (\mu \nabla u - \alpha u \nabla m) = \mu \Delta u - \alpha \nabla u \cdot \nabla m - \alpha u \Delta m \). By applying the maximum principle to the first equation of \((1)\), we have

\[
 u(x_0) \leq -\alpha \Delta m(x_0) + m(x_0) \leq \alpha \|\Delta m\|_{\infty} + \max_{\overline{\Omega}} m,
\]

which is the desired result. \( \Box \)

Since \( u \) is in \( C^2(\overline{\Omega}) \) by the standard regularity theory of elliptic equations, there exists a positive constant \( M \) such that \( \|\nabla u\|_{C^1} \leq M \). Now we present the following notations.

\[
Q = \max\{Q_1, Q_2\} + 1; \quad C_\nu = \{ w \in C(\overline{\Omega}) : w = 0 \text{ on } \partial \Omega \}; \quad E = C_\nu \oplus C_\nu \\
D_Q = \{ w \in C_\nu : w \leq Q \}; \quad D = D_Q \oplus D_Q; \quad K = \{ w \in C_\nu : 0 \leq w(x) \} \\
W = K \oplus K; \quad D' = (\text{Int } D) \cap W.
\]

Define

\[
 F : C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})
\]

via

\[
 F(u,v) = (-\Delta + P)^{-1}(P\mathcal{I} + G)(u,v)^T,
\]

where \( P \) is sufficiently large positive constant and \( \mathcal{I} \) is an identity operator and

\[
 G(u,v) = \left( \frac{1}{\nu} [ \alpha \nabla m \cdot \nabla u - \alpha u \Delta m + u(m - u - v)] \right) \left( \frac{1}{\nu} (m - u - v) \right).
\]

Now we calculate the indices at semi-trivial solutions of \((1)\).

**Lemma 3.4.** Let \((\theta_{\alpha,\mu},0)\) and \((0,\theta_{\nu})\) be the semi-trivial solutions of \((1)\). Then,

(i) \( \text{Ind}_W(F, D') = 1 \),

(ii) If \( \sigma_1 > 0 \) and \( \sigma_2 > 0 \), \( \text{Ind}_W(F,(0,0)) = 0 \),

(iii) If \( \lambda_1(\nu \Delta + (m - \theta_{\alpha,\mu})) > 0 \), \( \text{Ind}_W(F,(\theta_{\mu},0)) = 0 \),

(iv) If \( \lambda_1(L,\alpha) > 0 \), \( \text{Ind}_W(F,(0,\theta_{\nu})) = 0 \).
We first claim that it is only necessary to show that the coexistence steady state exists. Proof of Theorem 2.9.

The indices in (i)-(iii) can be calculated similarly as in [25, 31], we calculate only \[ \text{Ind}_{W} = K \bigoplus C_{\nu} \text{ and } S(0, \theta_{e}) = \{ 0 \} \bigoplus C_{\nu}. \]

Define \[ \mathcal{L} := F'(0, \theta_{e}) \]
\[ = (-\Delta + P)^{-1} \begin{pmatrix} \frac{1}{\mu}[-\alpha \nabla m \cdot \nabla - \alpha \Delta m + m - \theta_{e}] + P & 0 \\ \frac{1}{\nu}[m - 2\theta_{e}] + P \end{pmatrix}. \]

We first claim that \( \mathcal{I} - \mathcal{L} \) is invertible. Let \( \mathcal{L}(\phi, \psi)^T = (\phi, \psi)^T \in W(0, \theta_{e}). \) Then, \( (\phi, \psi) \) satisfies
\[
\begin{cases}
-\mu \Delta \phi = -\alpha \nabla m \cdot \nabla \phi - \alpha \phi \Delta m + (m - \theta_{e}) \phi \\
-\nu \Delta \psi = -\theta_{e} \phi + (m - 2\theta_{e}) \psi
\end{cases}
\text{in } \Omega,
\phi = \psi = 0 \text{ on } \partial \Omega.
\]

For \( \phi \geq 0, \) we have
\[ \nabla \cdot [\mu \nabla \phi - \alpha \phi \nabla m] + (m(x) - \theta_{e}) \phi = 0. \]

Define \( Z = e^{-\alpha m/\mu} \phi. \) Then \( Z \geq 0 \) and
\[ \nabla \cdot [\mu e^{\alpha m/\mu} \nabla Z] + (m(x) - \theta_{e}) Z e^{\alpha m/\mu} = 0. \tag{25} \]

Let \( \Phi_{1} \) be the principal eigenfunction of \( \lambda_{1}(L, \alpha) > 0 \) satisfying \( \int_{\Omega} e^{\alpha m/\mu} \Phi_{1}^{2} = 1. \)

Multiplying \( \Phi_{1} \) and integrate over \( \Omega \) in (25), we obtain
\[ \int_{\Omega} \Phi_{1} \nabla \cdot [\mu e^{\alpha m/\mu} \nabla Z] + e^{\alpha m/\mu} \Phi_{1}(m(x) - \theta_{e}) Z = \lambda_{1}(L, \alpha) \int_{\Omega} e^{\alpha m/\mu} Z \Phi_{1} \geq 0, \]
which implies \( Z = 0. \) Hence \( \phi = 0. \) Next, we show \( \psi = 0. \) Since \( \lambda_{1}(\nu \Delta + (m - \theta_{e})) = 0 \) with the corresponding principal eigenfunction \( \theta_{e}, \lambda_{1}(\nu \Delta + (m - 2\theta_{e})) < 0 \) by the monotonicity. We multiply the second equation of (24) by \( \psi \) and integrate over \( \Omega \) to obtain
\[ 0 = \int_{\Omega} \theta_{e} \phi \psi = \int_{\Omega} -\nu |\nabla \psi|^{2} + (m - 2\theta_{e}) \psi^{2} \leq \lambda_{1}(\nu \Delta + (m - 2\theta_{e})) \int_{\Omega} \psi^{2} \leq 0, \]
which implies \( \psi = 0. \) Hence, \( \mathcal{I} - \mathcal{L} \) is invertible.

Next, we denote \( r := \text{the spectral radius of } (-\Delta + P)^{-1} \begin{pmatrix} \frac{1}{\mu}[-\alpha \nabla m \cdot \nabla - \alpha \Delta m + m - \theta_{e}] + P \\ \frac{1}{\nu}[m - 2\theta_{e}] + P \end{pmatrix}. \)

The assumption \( \lambda_{1}(L, \alpha) > 0 \) implies that the largest eigenvalue of (18) is also positive. Thus, we have \( r > 1, \) and there exists corresponding eigenvector \( \xi_{1} \in K \setminus \{ 0 \} \) such that \( (\xi_{1}, 0) \in W(0, \theta_{e}) \setminus S(0, \theta_{e}) \text{ and } (\xi_{1}, 0)^T - \frac{1}{r} \mathcal{L}(\xi_{1}, 0)^T \in S(0, \theta_{e}). \)

which implies that \( \mathcal{L} \) has property \( \alpha. \) Therefore, by Theorem 2.4 in [31], \( \text{Ind}_{W} = 0. \) \( \square \)

Now, we prove Theorem 2.9 using the above results.

Proof of Theorem 2.9. It is only necessary to show that the coexistence steady state of (1) is unique. Let \( (u_{j}, v_{j}) \) be the positive coexistence steady-state of (1) for \( 1 \leq j \leq \tau, \) where \( \tau \) is a positive integer. Then, once we can show that \( (u_{j}, v_{j}) \) is non-degenerate and linearly stable, it can easily be shown that the linearized
obtain the following eigenvalue problem: of (1) with \( \alpha \).

Suppose that \((u, v)\) be a positive steady state of (1). We then linearize (1) at \((u^*, v^*)\), and obtain the following eigenvalue problem:

\[
\begin{cases}
\lambda \phi = \nabla \cdot (\mu \nabla \phi - \alpha \phi \nabla m) + \phi(m(x) - 2u^* - v^*) - u^* \psi \\
\lambda \psi = \nu \Delta \psi + \psi(m(x) - u^* - 2v^*) - v^* \phi \\
\phi = \psi = 0
\end{cases}
\]  

in \( \Omega \), on \( \partial \Omega \).

We also set \((\hat{\phi}_i, \hat{\psi}_i)\) such that

\[
\hat{\phi}_i = \phi_i/\|\phi_i\|^2_{H^1(\Omega)} \text{ and } \hat{\psi}_i = \psi_i/\|\psi_i\|^2_{H^1(\Omega)}.
\]

Then, \((\hat{\phi}_i, \hat{\psi}_i)\) satisfies (26). For convenience, we use the notation \((\hat{\phi}_i, \hat{\psi}_i)\) instead of \((\phi_i, \psi_i)\). Let \( \varphi_i = e^{-\alpha_i m/\mu} \phi_i \). Then, (26) becomes

\[
\begin{cases}
\lambda_i \varphi_i = \nabla \cdot (\mu e^{\alpha_i m/\mu} \nabla \varphi_i) + e^{\alpha_i m/\mu} \varphi_i(m(x) - 2u_i^* - v_i^*) - u_i^* \psi \\
\lambda_i \psi_i = \nu \Delta \psi_i + \psi_i(m(x) - u_i^* - 2v_i^*) - v_i^* \varphi_i \\
\phi_i = \psi_i = 0
\end{cases}
\]  

in \( \Omega \), on \( \partial \Omega \). (27)

Since (1) is a monotone system, it follows from the Krein-Rutman theorem [19] that (27) has a principal eigenvalue \( \lambda_{i,i} \), and we can assume that the corresponding eigenfunction \((\varphi_i, \psi_i)\) satisfies \( \varphi_i < 0 < \psi_i \). Multiplying \( \psi_i \) with the second equation of (27) and integrating over \( \Omega \), we obtain

\[
\lambda_i \int_\Omega \psi_i^2 = \int_\Omega -\nu |\nabla \psi_i|^2 + \psi_i^2(m(x) - u_i^* - 2v_i^*) - v_i^* \varphi_i e^{\alpha_i m/\mu}. \tag{28}
\]

Note that \( v_i^* \) is the principal eigenfunction with a zero principal eigenvalue of

\[
\begin{cases}
\sigma \psi = \nu \Delta \psi + (m - u_i^* - v_i^*) \psi \\
\psi = 0
\end{cases}
\]  

in \( \Omega \), on \( \partial \Omega \). (29)

Then, we have from (28) that

\[
\lambda_i \int_\Omega \psi_i^2 \leq \int_\Omega -v_i^* \psi_i^2 - v_i^* \varphi_i e^{\alpha_i m/\mu}. \tag{30}
\]

Recall that, by comparison principle and Theorem 2.7, as \( i \to \infty \),

\[
u_i^* \leq \theta_{\alpha_i \mu} \to 0 \quad \text{in } L^2(\Omega).
\]

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Then, up to subsequence, $u_i^* \to 0$ almost uniformly in $x$. From this point on, for the sake of convenience, the limit of function can be up to subsequence if necessary. In addition, $v_i^* \to \theta_\nu$ in $W_0^{2,2}$. Further, since $||\phi_i||_2^2 \leq 1$ and $m > 0$ on $\Omega$,
\[
\int_{\Omega} \phi_i^2 \to 0, \quad \int_{\Omega} e^{\alpha_i m/\mu} \phi_i^2 \to 0, \quad \phi_i = e^{\alpha_i m/\mu} \phi_i \to \phi \text{ in } L^2
\]
as $i \to \infty$ for some function $\phi$. From (30) with the above convergences, $\lambda_i$ is bounded above and independent of $\alpha_i$. In addition, since $||\phi_i||_{H^1}^2 + ||\psi_i||_{H^1}^2 = 1$, we can assume that $\lambda_i \to \lambda^*$ and $(\phi_i, \psi_i) \to (\phi, \psi)$ in $H^1_0$ as $i \to \infty$. Suppose $\psi \equiv 0$. Since $\lambda^*$ and $(\phi, \psi)$ satisfy
\[
\lambda^* \psi = \nu \Delta \psi + \psi(m - 2\theta_\nu) - \theta_\nu \phi,
\]
we have $\phi \equiv 0$, which contradicts $||\psi||_{H^1}^2 + ||\phi||_{H^1}^2 = 1$. Thus, $\psi \not\equiv 0$. Further, from the second equation of (26), we can obtain
\[
\lambda_i \int_{\Omega} \phi_i^2 = \int_{\Omega} -\mu |\nabla \phi_i|^2 + \alpha_i \phi_i \nabla \phi_i \cdot \nabla m + \int_{\Omega} \phi_i^2 (m(x) - 2u_i^* - v_i^*) - u_i^* \psi_i \phi_i
\]
\[
= \int_{\Omega} -\mu |\nabla \phi_i|^2 + \alpha_i \left(\frac{1}{2} \phi_i^2\right) \cdot \nabla m + \int_{\Omega} \phi_i^2 (m(x) - 2u_i^* - v_i^*) - u_i^* \psi_i \phi_i
\]
\[
= \int_{\Omega} -\mu |\nabla \phi_i|^2 + \int_{\Omega} -\frac{\alpha_i}{2} \Delta m \phi_i^2 + \int_{\Omega} \phi_i^2 (m(x) - 2u_i^* - v_i^*) - u_i^* \psi_i \phi_i.
\]
(31)

Dividing (31) by $\alpha_i$ and taking $i \to \infty$, $\phi$ must be 0 by the assumption $\Delta m < 0$ on $\Omega$. Then, letting $i \to \infty$ on (30), $\lambda^* = 0$. Thus, $\psi$ satisfies
\[
0 = \lambda^* \psi = \nu \Delta \psi + \psi(m - 2\theta_\nu).
\]
(32)

Multiplying $\psi$ with (32) and integrating over $\Omega$, we obtain
\[
0 = \int_{\Omega} -\nu |\nabla \psi|^2 + \psi(m - \theta_\nu) - \int_{\Omega} \psi \theta_\nu \leq - \int_{\Omega} \psi \theta_\nu < 0,
\]
which is a contradiction. Hence, the positive steady-state of (1) is linearly stable for large $\alpha$.

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\textbf{References}

[1] I. Averill, K.-Y. Lam and Y. Lou, The role of advection in a two-species competition model: A bifurcation approach, Mem. Amer. Math. Soc., 245 (2017), 117 pp.

[2] F. Belgacem and C. Cosner, The effects of dispersal along environmental gradients on the dynamics of populations in heterogeneous environment, Canad. Appl. Math. Quart., 3 (1995), 379–397.

[3] R. S. Cantrell and C. Cosner, Spatial Ecology via Reaction-Diffusion Equations, John Wiley & Sons, 2003.

[4] R. S. Cantrell, C. Cosner and Y. Lou, Movement toward better environments and the evolution of rapid diffusion, Math. Biosci., 204 (2006), 199–214.
[5] R. S. Cantrell, C. Cosner and Y. Lou, Advection-mediated coexistence of competing species, *Proc. Roy. Soc. Edinburgh Sect. A*, 137 (2007), 497–518.

[6] R. S. Cantrell, C. Cosner and Y. Lou, Evolution of dispersal and the ideal free distribution, *Math. Biosci. Eng.*, 7 (2010), 17–36.

[7] X. Chen, K.-Y. Lam and Y. Lou, Dynamics of a reaction-diffusion-advection model for two competing species, *Discrete Contin. Dyn. Syst.*, 32 (2012), 3841–3859.

[8] C. Cosner, Reaction-diffusion-advection models for the effects and evolution of dispersal, *Discrete Contin. Dyn. Syst.*, 34 (2014), 1701–1745.

[9] C. Cosner and A. C. Lazer, Stable coexistence states in the Volterra-Lotka competition model with diffusion, *SIAM J. Appl. Math.*, 44 (1984), 1112–1132.

[10] C. Cosner and Y. Lou, Does movement toward better environments always benefit a population?, *J. Math. Anal. Appl.*, 277 (2003), 489–503.

[11] E. N. Dancer, On positive solutions of some pairs of differential equations, *Trans. Amer. Math. Soc.*, 284 (1984), 729–743.

[12] E. N. Dancer, On the existence and uniqueness of positive solutions for competing species models with diffusion, *Trans. Amer. Math. Soc.*, 326 (1991), 829–859.

[13] J. Dockery, V. Hutson, K. Mischaikow and M. Pernarowski, The evolution of slow dispersal rates: A reaction-diffusion model, *J. Math. Biol.*, 37 (1998), 61–83.

[14] J. C. Eilbeck, J. E. Furter and J. López-Gómez, Coexistence in the competition model with diffusion, *J. Differential Equations*, 107 (1994), 363–398.

[15] J. E. Furter and J. López-Gómez, Diffusion-mediated permanence problem for a heterogeneous Lotka–Volterra competition model, *Proc. Roy. Soc. Edinburgh Sect. A*, 127 (1997), 281–336.

[16] P. Hess, *Periodic-Parabolic Boundary Value Problems and Positivity*, Pitman, New York, 1991.

[17] M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in Banach space (in Russian), Usp. Mat. Nauk., 3 (1948), 3–95. English translation in *Amer. Math. Soc. Transl.*, 26 (1950).

[18] Y. Lou and F. Lutscher, Evolution of dispersal in open advective environments, *J. Math. Biol.*, 69 (2014), 1319–1342.

[19] Y. Lou and W.-M. Ni, Diffusion vs cross-diffusion: An elliptic approach, *Trans. Amer. Math. Soc.*, 305 (1988), 143–166.

[20] Y. Lou and W.-M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differential Equations*, 131 (1996), 79–131.

[21] Y. Lou and W.-M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differential Equations*, 131 (1996), 79–131.

[22] Y. Lou and W.-M. Ni, Diffusion vs cross-diffusion: An elliptic approach, *J. Differential Equations*, 154 (1999), 157–190.

[23] K. Ryu and I. Ahn, Positive solutions for ratio-dependent predator-prey interaction systems, *J. Differential Equations*, 218 (2005), 117–135.

[24] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2nd edition, Springer-Verlag, New York, 1994.
[33] P. Zhou, D. Tang and D. Xiao, On Lotka-Volterra competitive parabolic systems: Exclusion, coexistence and bistability, *J. Differential Equations*, **282** (2021), 596–625.

[34] P. Zhou and D. Xiao, Global dynamics of a classical Lotka-Volterra competition-diffusion-advection system, *J. Funct. Anal.*, **275** (2018), 356–380.

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