Supersymmetric solutions of $N = 2$ $D = 4$ SUGRA:
the whole ungauged shebang

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Abstract

In this article we complete the classification of the supersymmetric solutions of $N = 2$ $D = 4$ ungauged supergravity coupled to an arbitrary number of vector- and hypermultiplets.

We find that in the timelike case the hypermultiplets cause the constant-time hypersurfaces to be curved and have $SU(2)$ holonomy identical to that of the hyperscalar manifold. The solutions have the same structure as without hypermultiplets but now depend on functions which are harmonic in the curved 3-dimensional space. We discuss an example obtained from a hyper-less solution via the c-map.

In the null case we find that the hyperscalars can only depend on the null coordinate and the solutions are essentially those of the hyper-less case.
1 Introduction

The classification of all the supersymmetric configurations of $N = 2, d = 4$ ungauged supergravity coupled to vector multiplets has recently been achieved in Ref. [1] and it is only natural to try to extend those results to more general couplings of $N = 2, d = 4$ supergravities [2, 3] since, after all, generic Calabi-Yau compactifications yield theories with more than just vector multiplets. The simplest extension, which just happens to be the one we are going to consider in this paper, is the inclusion in the theory of an arbitrary number of hypermultiplets.

This is a problem that has, so far, largely been ignored in the literature on the grounds that hypermultiplets do not couple to the vector multiplets at low energies and, therefore, their presence was irrelevant to study, for instance, black-hole solutions.\footnote{See, however, Ref. [4] and references therein.} Their generic presence is, however, to be expected, and, in general, hypermultiplets will be excited and their non-triviality will certainly modify the known solutions since they couple to gravity.

What are we to expect? In order to answer this question it is worthwhile to have a look at the c-map of the general cosmic string solution found in Ref. [1, (5.93)]:

\[
\begin{aligned}
    ds^2 &= 2\, du\, dv - 2e^{-K} \, dz\, dz^*, \\
    Z^i &= Z^i(z), \\
    F^A &= 0, \\
    q^a &= \text{const.},
\end{aligned}
\]  

(1.1)

This solution is especially suited for our purposes since it has an extremely simple form, is 1/2-BPS, and the corresponding Killing spinor is constant, thus ensuring that the dual solution is at least 1/2-BPS. Using the formulae in Appendix B we can dualize the above solution along the spacelike direction $u - v$, to another solution in minimal supergravity coupled to a certain number of hypermultiplets; the resulting spacetime metric is the one above, the graviphoton field strength still vanishes, and some of the hyperscalars have a (anti-)holomorphic spacetime dependency (the details are spelled out in Sec. 4.4.). Comparing this to the general timelike solution in Tod’s classification of supersymmetric solutions in minimal $N = 2, d = 4$ supergravity [5], we reach the conclusion that having non-trivial hyperscalars must lead to a non-trivial metric on the constant-time hypersurfaces.

The reason for this occurrence is to be found in the Killing spinor: the relevant gravitino variation equation for vanishing graviphoton field strength reads schematically $\mathcal{D}\epsilon = 0$, where $\mathcal{D}$ not only contains the spin connection but also an $\mathfrak{su}(2)$ connection which is constructed out of hyperscalars. Therefore, if we want BPS solutions with non-trivial scalars, we need a non-trivial spin connection in order to attain $\text{Hol}(\mathcal{D}) = 0$, or said differently: we need to embed one connection into the other.

The embedding of the gauge connection into the spin connection (or the other way around) was proposed originally in Refs. [6, 7] and used to achieve anomaly cancellation or absence of higher-order corrections in the context of the Heterotic String in Refs. [8, 9, 10, 11, 12]. As we are going to see, in this case this mechanism leads to unbroken supersymmetry through an exact cancellation of the $SU(2)$ and spin connections in the
gravitino supersymmetry transformation, generalizing the cancellation between \(U(1)\) gauge and 2-dimensional spin connection used in Ref. [13].

This embedding turns out to be possible in the timelike case, but not in the null case; further, it is only in the timelike case that the presence of excited hyperscalars has important consequences.

Let us summarize our results:

1. In the timelike case supersymmetric the configurations are completely determined by

   (a) A 3-dimensional space metric

   \[
   \gamma_{mn}dx^m dx^n, \quad m, n = 1, 2, 3, \tag{1.2}
   \]

   and a mapping \(q^u(x)\) from it to the quaternionic hyperscalar manifold such that the 3-dimensional spin connection\(^2\) \(\omega_x^y\) is related to the pullback of the quaternionic \(SU(2)\) connection \(A^x\) by

   \[
   \omega_m^{xy} = \varepsilon^{xyz}A_z^u \partial_m q^u, \tag{1.3}
   \]

   and such that

   \[
   U^{J_x} (\sigma_x)_J = 0, \quad U^{J_x} \equiv V^m A^z_q \partial_m q^u, \tag{1.4}
   \]

   where \(U^{J_x}\) is the Quadbein defined in Appendix A.

   (b) A choice of a symplectic vector \(I \equiv \Im(V/X)\) whose components are real harmonic functions with respect to the above 3-dimensional metric:

   \[
   \nabla_m q^m I = 0. \tag{1.5}
   \]

   Given \(I, \mathcal{R} \equiv \Re(V/X)\) can in principle be found by solving the generalized stabilization equations and then the metric is given by

   \[
   ds^2 = |M|^2(dt + \omega)^2 - |M|^{-2}\gamma_{mn}dx^m dx^n, \tag{1.6}
   \]

   where

   \[
   |M|^{-2} = \langle \mathcal{R} | I \rangle, \tag{1.7}
   \]

   \[
   (d\omega)_{xy} = 2\varepsilon_{xyz}\langle I | \partial^z I \rangle. \tag{1.8}
   \]

\(^2\)In this paper we use \(x, y, z = 1, 2, 3\) as tangent-space indices or as \(SU(2)\) indices.
The second equation implicitly contains the Dreibein of the 3-dimensional metric $\gamma$ and its integrability condition is

$$\langle \mathcal{I} \ | \ \nabla_m \partial^n \mathcal{I} \rangle = 0. \quad (1.9)$$

As is discussed in e.g. Refs. [14, 15], this condition will lead to non-trivial constraints. The vector field strengths are given by

$$F = -\frac{1}{\sqrt{2}} \left\{ d \|M\|^2 \mathcal{R}(dt + \omega) - * \|M\|^2 d\mathcal{I} \wedge (dt + \omega) \right\}, \quad (1.10)$$

and the scalar fields $Z^i$ can be computed by taking the quotients

$$Z^i = (\mathcal{V}/X)^i/\mathcal{V}/X^0. \quad (1.11)$$

The hyperscalars $q^u(x)$ are just the mapping whose existence we assumed from the onset.

These solutions can therefore be seen as deformations of those devoid of hypers, originally found in Ref. [16].

As for the number of unbroken supersymmetries, the presence of non-trivial hyperscalars breaks $1/2$ or $1/4$ of the supersymmetries of the related solution without hypers, which may have all or $1/2$ of the original supersymmetries. Therefore, we will have solutions with $1/2$, $1/4$ and $1/8$ of the original supersymmetries. The Killing spinors take the form

$$\epsilon_I = X^{1/2} \epsilon_{I0}, \quad \partial_\mu \epsilon_{I0} = 0, \quad \epsilon_{I0} + i\gamma_0 \varepsilon_{IJ} \epsilon^J_0 = 0, \quad \Pi^x_{IJ} \epsilon_{I0} = 0, \quad (1.12)$$

where the first constraint is imposed only if there are non-trivial vector multiplets and each of the other three constraints is imposed for each non-vanishing component of the $SU(2)$ connection. Each constraint breaks $1/2$ of the supersymmetries independently, but the third constraint $\Pi^x_{IJ} \epsilon_{I0} = 0$ is implied by the first two. Finally, the meaning of these last three constraints is that they enforce the embedding of the gauge connection into the gauge connection since they are in different representations.

2. In the null case the hyperscalars can only depend on the null coordinate $u$ and the solutions take essentially the same form as in the case without hypermultiplets (See Ref. [1]).

The plan of this article is as follows: in Section 2 we will discuss the theory we are dealing with and especially the supersymmetry transformations. This is followed in Sec. 3 by a short discussion of the Killing spinor identities and their implications. Secs. 4 and 5 then deal with the explicit solutions in the two possible cases that, according to the KSIs can, occur. Finally, Appendix A is devoted to quaternionic Kähler geometry and

\footnote{Our conventions, including those for the special Kähler geometry, are those of Ref. [1].}
Appendix B spells out the details for the c-map alluded to in the introduction.

2 Matter-coupled $N = 2 \ d = 4$ ungauged supergravity

The theory we are working with is an extension of the one studied in Ref. [1], the extension consisting in the additional coupling of $m$ hypermultiplets. We refer the reader to [1] for all conventions and notations except for those involving the $m$ hypermultiplets which we explain next. These are essentially those of Ref. [17] with the minor changes introduced in Ref. [1]. Each hypermultiplet consists of 4 real scalars $q$ (hyperscalars) and 2 Weyl spinors $ζ$ called hyperinos. The $4m$ hyperscalars are collectively denoted by $q^u$, $u = 1, \cdots, 4m$ and the $2m$ hyperinos are collectively denoted by $ζ^α$, $α = 1, \cdots, 2m$. The $4m$ hyperscalars parametrize a quaternionic Kähler manifold (defined and studied in Appendix A) with metric $H_{uv}(q)$.

The action of the bosonic fields of the theory is

$$S = \int d^4x \sqrt{|g|} \left[ R + 2G_{ij}^* \partial_\mu Z^i \partial^\mu Z^j + 2H_{uv} \partial_\mu q^u \partial^{\mu} q^v \right. \right]
   + 23m\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\mu\nu}F^{\Sigma}_{\mu\nu} - 2\Re\mathcal{N}_{\Lambda\Sigma}\tilde{F}^{\Lambda\mu\nu}F_{\mu\nu} \right],$$

(2.1)

For vanishing fermions, the supersymmetry transformation rules of the fermions are

$$\delta_\epsilon \psi_{I\mu} = \mathcal{D}_\mu \epsilon_I + \varepsilon_{IJ} T^{+\mu}_{\nu} \gamma^\nu \epsilon^J,$$  

(2.2)

$$\delta_\epsilon \lambda^I = i \bar{\phi} Z^I \epsilon^I + \varepsilon_{IJ} \bar{\mathcal{G}}^I \epsilon^J.$$  

(2.3)

$$\delta_\epsilon \zeta^\alpha = -i C_{\alpha\beta} U^\beta q^u \varepsilon_{IJ} \bar{\phi} q^u \epsilon^J,$$  

(2.4)

Here $\mathcal{D}$ is the Lorentz and Kähler-covariant derivative of Ref. [1] supplemented by (the pullback of) an $SU(2)$ connection $A_I^J$ described in Appendix A acting on objects with $SU(2)$ indices $I, J$ and, in particular, on $\epsilon_I$ as:

$$\mathcal{D}_\mu \epsilon_I = (\nabla_\mu + \frac{i}{2} Q_\mu) \epsilon_I + A_{\mu I}^J \epsilon_J.$$  

(2.5)

This is the only place in which the hyperscalars appear in the supersymmetry transformation rules of the gravitinos and gauginos. $U^\beta u$ is a Quadbein, i.e. a quaternionic Vielbein, and $C_{\alpha\beta}$ the $Sp(m)$-invariant metric, both of which are described in Appendix A.

The supersymmetry transformations of the bosons are the same as in the previous case plus that of the hyperscalars:

$$\delta_\epsilon e^a_\mu = -\frac{i}{4}(\bar{\psi}_{I\mu} \gamma_a \epsilon^I + \bar{\psi}^J_{I\mu} \gamma_a \epsilon^I),$$  

(2.6)
\[ \delta_\epsilon A^A_{\mu} = \frac{1}{4} (\mathcal{L}^A \epsilon^I J \bar{\psi}_{1\mu} \epsilon_J + \mathcal{L}^A \epsilon_{I J} \bar{\psi}^I \epsilon^J ) \]

\[ + \frac{i}{8} (f^A \epsilon_{I J} \bar{\lambda}^I \gamma_\mu \epsilon^J + f^A \epsilon^I J \bar{\lambda}^I \gamma_\mu \epsilon_J) , \quad (2.7) \]

\[ \delta_\epsilon Z^i = \frac{1}{4} \bar{\lambda}^I \epsilon_I , \quad (2.8) \]

\[ \delta_\epsilon q^u = U_{\alpha I} (\bar{\zeta}^\alpha \epsilon^I + C^{\alpha\beta} \epsilon^{I J} \bar{\zeta}^\beta \epsilon_J) . \quad (2.9) \]

Observe that the fields of the hypermultiplet and the fields of the gravity and vector multiplets do not mix in any of these supersymmetry transformation rules. This means that the KSIs [18, 19] associated to the gravitinos and gauginos will have the same form as in Ref. [1] and in the KSIs associated to the hyperinos only the hyperscalars equations of motion will appear.

For convenience, we denote the bosonic equations of motion by

\[ \mathcal{E}_a^\mu = - \frac{1}{2 \sqrt{|g|} \delta S} \delta e_a^\mu \], \[ \mathcal{E}_i = - \frac{1}{2 \sqrt{|g|} \delta S} \delta Z^i \], \[ \mathcal{E}_A^\mu = \frac{1}{8 \sqrt{|g|} \delta S} \delta A^A_{\mu} \], \[ \mathcal{E}^u = - \frac{1}{4 \sqrt{|g|} \delta S} \delta q^u . \quad (2.10) \]

and the Bianchi identities for the vector field strengths by

\[ B^A_{\mu} \equiv \nabla_\nu \epsilon^A_{\nu \mu} \]. \quad (2.11) \]

Then, using the action Eq. (2.1), we find that all the equations of motion of the bosonic fields of the gravity and vector supermultiplets take the same form as if there were no hypermultiplets, as in Ref. [1], except for the Einstein equation, which obviously is supplemented by the energy-momentum tensor of the hyperscalars

\[ \mathcal{E}_{uv} = \mathcal{E}_{uv} (q = 0) + 2 \mathcal{H}_{uv} \left[ \partial_\mu q^u \partial_\nu q^v - \frac{1}{2} g_{uv} \partial_\rho q^u \partial_\rho q^v \right] . \quad (2.12) \]

Furthermore, the equation of motion for the hyperscalars reads

\[ \mathcal{E}^u = \nabla_\mu \partial^\mu q^u = \nabla_\mu \partial^\mu q^u + \Gamma_{vw}^u \partial^\mu q^v \partial_\rho q^w \], \quad (2.13) \]

where \( \Gamma_{vw}^u \) are the Christoffel symbols of the 2\textsuperscript{nd} kind for the metric \( \mathcal{H}_{uv} \).

The symmetries of this set of equations of motion are the isometries of the Kähler manifold parametrized by the \( \bar{n} - 1 \) complex scalars \( Z^i \)'s embedded in \( Sp(2\bar{n}, \mathbb{R}) \) and those of the quaternionic manifold parametrized by the \( 4m \) real scalars \( q^u \).

### 3 Supersymmetric configurations: generalities

As we mentioned in Section 2, the supersymmetry transformation rules of the bosonic fields indicate that the KSIs associated to the gravitinos and gauginos are going to have the
same form as in absence of hypermultiplets. This is indeed the case, and the integrability conditions of the KSEs $\delta_\epsilon \psi_I \mu = 0$ and $\delta_\epsilon \lambda^I = 0$ confirm the results. Of course, now the Einstein equation includes an additional term: the hyperscalars energy-momentum tensor. In the KSI approach the origin of this term is clear. In the integrability conditions it appears through the curvature of the $SU(2)$ connection and Eq. (A.20). The results coincide for $\lambda = -1$.

There is one more set of KSIs associated to the hyperinos which take the form

$$ E^u U^{\alpha I} u \epsilon_I = 0, \quad (3.1) $$

and which can be obtained from the integrability condition $\mathcal{D}_\epsilon \zeta_\alpha = 0$ using the covariant constancy of the Quadbein, Eq. (A.17).

The KSIs involving the equations of motion of the bosonic fields of the gravity and vector multiplets take, of course, the same form as in absence of hypermultiplets. Acting with $\bar{\epsilon}^J$ from the left on the new KSI Eq. (3.1) we get

$$ X E^u U^{\alpha I} u = 0, \quad (3.2) $$

which implies, in the timelike $X \neq 0$ case, that all the supersymmetric configurations satisfy the hyperscalars equations of motion automatically:

$$ E^u = 0. \quad (3.3) $$

In the null case, parametrizing the Killing spinors by $\epsilon_I = \phi_I \epsilon$, we get just

$$ E^u U^{\alpha I} u \phi_I \epsilon = 0. \quad (3.4) $$

As usual, there are two separate cases to be considered: the one in which the vector bilinear $V^\mu \equiv i \bar{\epsilon}^J \gamma^\mu \epsilon_I$, which is always going to be Killing, is timelike (Section 4) and the one in which it is null (Section 5). The procedure we are going to follow is almost identical to the one we followed in Ref. [1].

## 4 The timelike case

As mentioned before, the presence of hypermultiplets only introduces an $SU(2)$ connection in the covariant derivative $\mathcal{D}_\mu \epsilon_I$ in $\delta_\epsilon \psi_I \mu = 0$ and has no effect on the KSE $\delta_\epsilon \lambda^I = 0$. Following the same steps as in Ref. [1], by way of the gravitino supersymmetry transformation rule Eq. (2.22), we arrive at

$$ \mathcal{D}_\mu X = -i T^{+ \mu} V^\nu, \quad (4.1) $$

$$ \mathcal{D}_\mu V^I_{\nu} = i \delta^I_J (X T^{+ \mu \nu} - X^* T^{+ \mu \nu}) - i (\epsilon^{JK} T^{+ - \mu \nu} \Phi_{KJ} \rho^\nu - \epsilon_{JK} T^{+ - \mu \rho} \Phi^{JK} \rho^\nu). \quad (4.2) $$
The $SU(2)$ connection does not occur in the first equation, simply because $X = \frac{1}{2} \epsilon^{IJ} M_{IJ}$ is an $SU(2)$ scalar, but it does occur in the second, although not in its trace. This means that $V^\mu$ is, once again, a Killing vector and the 1-form $\hat{V} = V_\mu dx^\mu$ satisfies the equation

$$d\hat{V} = 4i(X T^* - X^* T^+) .$$

The remaining 3 independent 1-forms

$$\hat{V}^x \equiv \frac{1}{\sqrt{2}} (\sigma_x)_{IJ} V^J_\mu \, dx^\mu ,$$

however, are only $SU(2)$-covariantly exact

$$d\hat{V}^x + \epsilon^{xyz} A^y \wedge \hat{V}^z = 0 .$$

From $\delta_\epsilon \lambda^I = 0$ we get exactly the same equations as in absence of hypermultiplets. In particular

$$V^\mu \partial_\mu Z^i = 0 ,$$

$$2iX^* \partial_\mu Z^i + 4iG^i + \mu \nu V^\nu = 0 .$$

Combine Eqs. (4.1) and (4.7), we get

$$V^\nu F^{\Lambda^+}_{\nu \mu} = \mathcal{L}^\Lambda_{\mu} X + X^* f_\Lambda^{\Lambda} i \partial_\mu Z^i = \mathcal{L}^\Lambda_{\mu} X + X^* D_\mu \mathcal{L}^\Lambda ,$$

which, in the timelike case at hand, is enough to completely determine through the identity

$$C^{\Lambda^+}_{\mu \nu} \equiv V^\nu F^{\Lambda^+}_{\nu \mu} \Rightarrow F^{\Lambda^+} = V^{-2} [\hat{V} \wedge \hat{C}^{\Lambda^+} + i \ast (\hat{V} \wedge \hat{C}^{\Lambda^+})] .$$

Observe that this equation does not involve the hyperscalars in any explicit way, as was to be expected due to the absence of couplings between the vector fields and the hyperscalars.

Let us now consider the new equation $\delta_\epsilon \zeta^I = 0$. Acting on it from the left with $\bar{\epsilon}^K$ and $\bar{\epsilon}^K \gamma_\mu$ we get, respectively

$$U^a_{\mu} \epsilon_{IJ} V^J K \nu \partial_\mu q^\nu = 0 ,$$

$$X^* U^a_{\nu} \partial_\mu q^\nu + U^a_{\nu} \epsilon_{IJ} \Phi^{IJ K} \nu \partial_\mu q^\nu = 0 .$$

Using $\epsilon_{IJ} V^J K = \epsilon_{JK} V^J I + \epsilon_{IK} V$ in the first equation we get

$$U^a_{\nu} V^J I \nu \partial_\mu q^\nu - U^a_{\nu} V^\nu \partial_\mu q^\nu = 0 .$$

$\sigma_{x J^f}$, $(x = 1, 2, 3)$ are the Pauli matrices satisfying Eq. (A.11).
It is not difficult to see that the second equation can be derived from this one using the Fierz identities that the bilinears satisfy in the timelike case (see Ref. [20]), whence the only equations to be solved are (4.12).

4.1 The metric

If we define the time coordinate $t$ by

$$V^\mu \partial_\mu \equiv \sqrt{2} \partial_t,$$  \hfill (4.13)

then $V^2 = 4|X|^2$ implies that $\hat{V}$ must take the form

$$\hat{V} = 2\sqrt{2}|X|^2(dt + \omega),$$  \hfill (4.14)

where $\omega$ is a 1-form to be determined later.

Since the $\hat{V}^x$s are not exact, we cannot simply define coordinates by putting $\hat{V}^x \equiv dx^x$. We can, however, still use them to construct the metric: using

$$g_{\mu\nu} = 2V^{-2}[V_\mu V_\nu - V_{J I}^\mu V_{I J}^\nu],$$  \hfill (4.15)

and the decomposition

$$V_{J I}^\mu = \frac{1}{2} V_\mu \delta_{J I} + \frac{1}{\sqrt{2}} (\sigma_x)_{J I}^T V_\mu,$$  \hfill (4.16)

we find that the metric can be written in the form

$$ds^2 = \frac{1}{4|X|^2} \hat{V} \otimes \hat{V} - \frac{1}{2|X|^2} \delta_{xy} \hat{V}^x \otimes \hat{V}^y.$$  \hfill (4.17)

The $\hat{V}^x$s are mutually orthogonal and also orthogonal to $\hat{V}$, which means that they can be used as a Dreibein for a 3-dimensional Euclidean metric

$$\delta_{xy} \hat{V}^x \otimes \hat{V}^y \equiv \gamma_{mn} dx^m dx^n,$$  \hfill (4.18)

and the 4-dimensional metric takes the form

$$ds^2 = 2|X|^2(dt + \omega)^2 - \frac{1}{2|X|^2} \gamma_{mn} dx^m dx^n.$$  \hfill (4.19)

The presence of a non-trivial Dreibein and the corresponding 3D metric $\gamma_{mn}$ is the main (and only) novelty brought about by the hyperscalars!

In what follows we will use the Vierbein basis

$$e^0 = \frac{1}{2|X|} \hat{V}, \quad e^x = \frac{1}{\sqrt{2}|X|} \hat{V}^x,$$  \hfill (4.20)

that is
\[
(e^a_\mu) = \begin{pmatrix} \sqrt{2}|X| & \sqrt{2}|X|\omega_m \\ 0 & \frac{1}{\sqrt{2}|X|}V^m \end{pmatrix}, \quad (e^\mu_a) = \begin{pmatrix} \frac{1}{\sqrt{2}|X|} & -\sqrt{2}|X|\omega_x \\ 0 & \sqrt{2}|X|V^m \end{pmatrix},
\]

(4.21)

where \(V^m_x\) is the inverse Dreibein \(V^m_xV^y_m = \delta^y_x\) and \(\omega_x = V^m_x\omega_m\). We shall also adopt the convention that all objects with flat or curved 3-dimensional indices refer to the above Dreibein and the corresponding metric.

Our choice of time coordinate Eq. (4.6) means that the scalars \(Z^i\) are time-independent, whence \(\iota_V Q = 0\). Contracting Eq. (4.1) with \(V^\mu\) we get

\[V^\mu \partial_\mu X = 0, \quad \Rightarrow \quad V^\mu \partial_{\mu}X = 0, \]

(4.22)

so that also \(X\) is time-independent.

We know the \(\hat{V}^x\)s to have no time components. If we choose the gauge for the pullback of the \(SU(2)\) connection \(A^x_t = 0\), then the \(SU(2)\)-covariant constancy of the \(\hat{V}^x\) (Eq. (4.5)) states that the pullback of \(A^x\), the \(\hat{V}^x\)s and, therefore, the 3-dimensional metric \(\gamma_{mn}\) are also time-independent. Eq. (4.5) can then be interpreted as Cartan’s first structure equation for a torsionless connection \(\omega\) in 3-dimensional space

\[d\hat{V}^x - \omega^{xy} \land \hat{V}^y = 0, \]

(4.23)

which means that the 3-dimensional spin connection 1-form \(\omega^{xy}_x\) is related to the pullback of the \(SU(2)\) connection \(A^x\) by

\[\omega^{xy}_m = \varepsilon^{xyz}A^z_u \partial_u q^u, \]

(4.24)

implying the embedding of the internal group \(SU(2)\) into the Lorentz group of the 3-dimensional space as discussed in the introduction.

The \(su(2)\) curvature will also be time-independent and Eq. (A.20) implies that the pullback of the Quadbein is also time-independent and its time component vanishes:

\[U^{\alpha I}_u V^\mu \partial_{\mu}q^\mu = 0. \]

(4.25)

Let us then consider the 1-form \(\omega\): following the same steps as in Ref. [1], we arrive at

\[(d\omega)_{xy} = -\frac{i}{2|X|^4} \varepsilon_{xyz}(X^*D^zX - XD^zX^*). \]

(4.26)

This equation has the same form as in the case without hypermultiplets, but now the Dreibein is non-trivial and, in curved indices, it takes the form

\[(d\omega)_{mn} = -\frac{i}{2|X|^4\sqrt{\gamma}} \varepsilon_{mnp}(X^*D^pX - XD^pX^*). \]

(4.27)

Introducing the real symplectic sections \(I\) and \(R\).
\[ R \equiv \Re(V/X), \quad I \equiv \Im(V/X), \]  

where \( V \) is the symplectic section 

\[ V = \left( \mathcal{L}^\Lambda \mathcal{M}_\Sigma \right), \quad \langle V | V^* \rangle \equiv \mathcal{L}^*\mathcal{M}_\Lambda - \mathcal{L}^\Lambda \mathcal{M}_\Sigma^* = -i, \]  

we can rewrite the equation for \( \omega \) to the alternative form 

\[ (d\omega)_{xy} = 2\varepsilon_{xyz}\langle I | \partial^z I \rangle, \]  

whose integrability condition is 

\[ \langle I | \nabla_m \partial^m I \rangle = 0, \]  

and will be satisfied by harmonic functions on the 3-dimensional space, \( i.e. \) by those real symplectic sections satisfying \( \nabla_m \partial^m I = 0 \). In general the harmonic functions will have singularities leading to non-trivial constraints like those studied in Refs. [14, 15].

### 4.2 Solving the Killing spinor equations

We are now going to see that it is always possible to solve the KSEs for field configurations with metric of the form (4.19) where the 1-form \( \omega \) satisfies Eq. (4.26) and the 3-dimensional metric has spin connection related to the \( SU(2) \) connection by Eq. (4.24), vector fields of the form (4.8) and (4.9), time-independent scalars \( Z^i \) and, most importantly, hyperscalars satisfying 

\[ U^a_J (\sigma_x)_J = 0, \quad U^{aJ}_x \equiv V^m_x \partial_m q^u U^{aJ}_u, \]  

which results from Eqs. (4.12), (4.25) and (4.16).

Let us consider first the \( \delta_\xi \zeta_\alpha = 0 \) equation. Using the Vierbein Eq. (4.21) and multiplying by \( \gamma^0 \) it can be rewritten in the form 

\[ U_{aI} x \gamma^{0x} \epsilon_I = 0, \]  

which can be solved using Eq. (4.32) if the spinors satisfy a constraint 

\[ \Pi^{xI} J \epsilon_J = 0, \quad \Pi^{xI} J (\sigma_x)_J = \frac{1}{2} [ \delta^I J - \gamma_i^{0(x)} (\sigma_{(x)})_I ] (\sigma_{(x)})_J \]  

(no sum over \( x \)),

for each non-vanishing \( U_{aI} x \). These three operators are projectors, \( i.e. \) they satisfy \( \Pi^x \Pi^x = \Pi^x \), and commute with each other. From \( (\sigma_x)_I K \Pi^{(x)K} J \epsilon_J = 0 \) we find 

\[ (\sigma_{(x)})_I J \epsilon_J = \gamma_i^{0(x)} \epsilon_I, \]  

which solves \( \delta_\xi \zeta_\alpha = 0 \) together with Eq. (4.32) and tells us that the embedding of the \( SU(2) \) connection in the Lorentz group requires the action of the generators of \( \mathfrak{su}(2) \) to
be identical to the action of the three Lorentz generators $\frac{1}{2} \gamma^0 \gamma^x$ on the spinors. When we impose these constraints on the spinors, each of the first two reduces by a factor of $1/2$ the number of independent spinors, but the third condition is implied by the first two and does not reduce any further the number of independent spinors.

Observe that

$$\Pi^x_{I,J} \equiv (\Pi^x_{I,J})^* = -\varepsilon^{IK} \Pi^x_{K,L} \varepsilon_{L,J}.$$  \hspace{1cm} (4.36)

Let us now consider the equation $\delta_\epsilon \lambda^i = 0$. It takes little to no time to realize that it reduces to the same form as in absence of hypermultiplets

$$\delta_\epsilon \lambda^i = i \partial Z^i (\epsilon^I + i \gamma_0 e^{-i \alpha} \varepsilon^I \epsilon^j) = 0,$$  \hspace{1cm} (4.37)

the only difference being in the implicit presence of the non-trivial Dreibein in $\partial Z^i$. Therefore, as before, this equation is solved by imposing the constraint

$$\epsilon^I + i \gamma_0 e^{-i \alpha} \varepsilon^I \epsilon^j = 0,$$  \hspace{1cm} (4.38)

which can be seen to commute with the projections $\Pi^x$ since, by virtue of Eq. (4.36),

$$\Pi^x_{K,I} (\epsilon^I + i \gamma_0 e^{-i \alpha} \varepsilon^I \epsilon^j) = (\Pi^x_{K,I} \epsilon^I) + i \gamma_0 e^{-i \alpha} \varepsilon^K \varepsilon^{J,L} (\Pi^x_{L,J} \epsilon^L).$$  \hspace{1cm} (4.39)

Let us finally consider the equation $\delta_\epsilon \lambda^i = 0$: in the $SU(2)$ gauge $A^x = 0$ the 0th component of the equation is automatically solved by time-independent Killing spinors using the above constraint. Again, the equation takes the same form as without hypermultiplets but with a non-trivial Dreibein. In the same gauge, the spatial (flat) components of the $\delta_\epsilon \lambda^i = 0$ equation can be written, upon use of the above constraint and the relation Eq. (4.24) between the $SU(2)$ and spatial spin connection, in the form

$$X^{1/2} \partial_y (X^{-1/2} \epsilon^I) + \frac{i}{2} A^x_y [(\sigma_x)^I_J \epsilon^J - \gamma^0 x \epsilon^I] = 0,$$  \hspace{1cm} (4.40)

which is solved by

$$\epsilon_I = X^{1/2} \epsilon_{I0}, \quad \partial_y \epsilon_{I0} = 0, \quad \epsilon_{I0} + i \gamma_0 \epsilon_{I} \epsilon_{J0} = 0, \quad \Pi^x_{I,J} \epsilon_{J0} = 0,$$  \hspace{1cm} (4.41)

where the constraints Eq. (4.34) are imposed for each non-vanishing component of the $SU(2)$ connection.

### 4.3 Equations of motion

According to the KSIs, all the equations of motion of the supersymmetric solutions will be satisfied if the Maxwell equations and Bianchi identities of the vector fields are satisfied. Before studying these equations it is important to notice that supersymmetry requires
Eqs. (4.32) to be satisfied. We will assume here that this has been done and we will study in the next section possible solutions to these equations.

Using Eqs. (4.8) and (4.9) we can write the symplectic vector of 2-forms in the form

\[ F = \frac{1}{2|X|^2} \{ \hat{V} \wedge d[|X|^2 \mathcal{R}] - \ast[\hat{V} \wedge \Im(\mathcal{V}^* \mathcal{D}X + X^* \mathcal{D}\mathcal{V})] \}, \]  

(4.42)

which can be rewritten in the form

\[ F = -\frac{1}{2} \{ d[\mathcal{R}\hat{V}] + \ast[\hat{V} \wedge d\mathcal{I}] \}. \]  

(4.43)

The Maxwell equations and Bianchi identities \( dF = 0 \) are, therefore, satisfied if

\[ d\ast[\hat{V} \wedge d\mathcal{I}] = 0, \quad \Rightarrow \nabla_m \partial_m \mathcal{I} = 0, \]  

(4.44)

i.e. if the \( 2n \) components of \( \mathcal{I} \) are as many real harmonic functions in the 3-dimensional space with metric \( \gamma_{mn} \).

Summarizing, the timelike supersymmetric solutions are determined by a choice of Dreibein and hyperscalars such that Eq. (4.32) is satisfied and a choice of \( 2n \) real harmonic functions in the 3-dimensional metric space determined by our choice of Dreibein \( \mathcal{I} \). This choice determines the 1-form \( \omega \). The full \( \mathcal{V}/\mathcal{X} \) is determined in terms of \( \mathcal{I} \) by solving the stabilization equations and with \( \mathcal{V}/\mathcal{X} \) one constructs the remaining elements of the solution as explained in Ref. [1].

### 4.4 The cosmic string scrutinized

It is always convenient to have an example that shows that we are not dealing with an empty set of solutions. As mentioned in the introduction we can find relatively simple non-trivial examples using the c-map on known supersymmetric solutions with only fields in the vector multiplets excited. A convenient solution is the cosmic string for the case \( n = 1 \) with scalar manifold \( SL(2,\mathbb{R})/U(1) \) and prepotential \( \mathcal{F} = -\frac{1}{4} \mathcal{X}^0 \mathcal{X}^1 \). Parametrizing the scalars as \( \mathcal{X}^0 = 1 \) and \( \mathcal{X}^1 = -i\tau \), we find from the formulae in appendix (B) that the only non-trivial fields of the c-dual solution are the spacetime metric

\[ ds^2 = 2du \, dv - 2 \Im(\tau) \, dzd\tau^*, \]  

(4.45)

with \( \tau = \tau(z) \), and the pull-back of the Quadbein is given by

\[ \Psi^{\alpha I} = [2\Im(\tau)]^{-3/2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \partial_\tau \gamma^{\tau z} & 0 \\ 0 & \partial_\tau \ast \gamma^{z \tau^*} \end{pmatrix}. \]  

(4.46)

From this form, then, it should be clear that the hyperscalar equation (2.4) is satisfied by

\[ \gamma^z \epsilon^2 = \gamma^{z \ast} \epsilon^1 = 0 \quad \Rightarrow \quad \gamma^z \epsilon_1 = \gamma^{z \ast} \epsilon_2 = 0, \]  

(4.47)
so that we have to face the fact that this solution can be at most 1/2-BPS.

Since we are dealing with a situation without vector multiplets and with a vanishing graviphoton, the gravitino variation (2.2) reduces to

\[ 0 = \nabla \epsilon_I + A_I^J \epsilon_J . \]  

(4.48)

For the c-mapped cosmic string, we have from Eq. (B.11), that \( A_I^J = \frac{i}{2} Q \sigma_3 I^J \). Also, for the metric at hand, the 4-d spin connection is readily calculated to be \( \frac{1}{2} \omega_{ab} \gamma^{ab} = i Q \gamma^{zz} \) (See e.g. [20]).

Due to the constraint (4.47), however, one can see that \( \gamma^{zz} \epsilon_I = \sigma_3 I^J \epsilon_J \), which, when mixed with the rest of the ingredients, leads to, dropping the \( I \)-indices,

\[ \text{Eq. (4.48)} = d\epsilon - \frac{i}{4} \omega_{ab} \gamma^{ab} \epsilon + \frac{i}{2} Q \sigma_3 \epsilon = d\epsilon, \]  

(4.49)

so that the c-mapped cosmic string is a 1/2-BPS solution with, as was to be expected, a constant Killing spinor.

5 The null case

In the null case\(^{5}\) the two spinors \( \epsilon_I \) are proportional: \( \epsilon_I = \phi_I \epsilon \). The complex functions \( \phi_I \), normalized such that \( \phi^I \phi_I = 1 \) and satisfying \( \phi^*_I = \phi^I \), carry a -1 \( U(1) \) charge w.r.t. the imaginary connection

\[ \zeta \equiv \phi^I \mathcal{D} \phi_I \rightarrow \zeta^* = -\zeta, \]  

(5.1)

opposite to that of the spinor \( \epsilon \), whence \( \epsilon_I \) is neutral. On the other hand, the \( \phi_I \)s are neutral with respect to the Kähler connection, and the Kähler weight of the spinor \( \epsilon \) is the same as that of the spinor \( \epsilon_I \), i.e. 1/2. The \( SU(2) \)-action is the one implied by the \( I \)-index structure.

The substitution of the null-case spinor condition into the KSEs (2.2, 2.4) immediately yields

\[ \mathcal{D}_\mu \phi_I \epsilon + \phi_I \mathcal{D}_\mu \epsilon + \epsilon_{IJ} \phi^J T^i_{\mu
u} \gamma^{\nu} \epsilon^* = 0, \]  

(5.2)

\[ \phi^I \mathcal{D} Z^i \epsilon^* + \epsilon^{IJ} \phi_J \mathcal{D}^i \epsilon = 0, \]  

(5.3)

\[ C_{\alpha\beta} U^{3I} u^I \epsilon_{IJ} \mathcal{D} \phi^J \epsilon^* = 0. \]  

(5.4)

Contracting Eq. (5.2) with \( \phi^I \) results in

\(^{5}\)The details concerning the normalization of the spinors and the construction of the bilinears in this case are explained in the Appendix of Ref. [20], which you are strongly urged to consult at this point.
$$\mathcal{D}_\mu \epsilon = -\phi^j \mathcal{D}_\mu \phi_I \epsilon \quad \iff \quad \tilde{\mathcal{D}}_\mu \epsilon \equiv (\mathcal{D}_\mu + \zeta_\mu)\epsilon = 0, \quad (5.5)$$

which is the only differential equation for \(\epsilon\). Substituting Eq. (5.5) into Eq. (5.2) as to eliminate the \(\mathcal{D}_\mu \epsilon\) term, we obtain

$$\left(\tilde{\mathcal{D}}_\mu \phi_I\right) \epsilon + \varepsilon_{IJ} \phi^j T^{+\mu\nu} \gamma^\nu \epsilon^* = 0, \quad \tilde{\mathcal{D}}_\mu \phi_I \equiv (\mathcal{D}_\mu - \zeta_\mu)\phi_I, \quad (5.6)$$

which is a differential equation for \(\phi_I\) and, at the same time, an algebraic constraint for \(\epsilon\). Two further algebraic constraints can be found by acting with \(\phi^I\) on Eq. (5.3):

$$\hat{\phi} Z^i \epsilon^* = G^i t^+ \epsilon = 0. \quad (5.7)$$

Finally, we add to the set-up an auxiliary spinor \(\eta\), with the same chirality as \(\epsilon\) but with all \(U(1)\) charges reversed, and impose the normalization condition

$$\bar{\epsilon} \eta = \frac{1}{2}. \quad (5.8)$$

This normalization condition will be preserved if and only if \(\eta\) satisfies the differential equation

$$\tilde{\mathcal{D}}_\mu \eta + a_\mu \epsilon = 0, \quad (5.9)$$

for some \(a\) with \(U(1)\) charges \(-2\) times those of \(\epsilon\), i.e.

$$\tilde{\mathcal{D}}_\mu a_\nu = (\nabla_\mu - 2\zeta_\mu - iQ_\mu) a_\nu. \quad (5.10)$$

\(a\) is to be determined by the requirement that the integrability conditions of the above differential equation be compatible with those for \(\epsilon\).

5.1 Killing equations for the vector bilinears and first consequences

We are now ready to derive equations involving the bilinears, in particular the vector bilinears which we construct with \(\epsilon\) and the auxiliary spinor \(\eta\) introduced above. First we deal with the equations that do not involve derivative of the spinors. Acting with \(\bar{\epsilon}\) on Eq. (5.6) and with \(\hat{\phi} \gamma^\mu\) on Eq. (5.7) we find

$$T^{+\mu\nu} l^\nu = G^{i+\mu\nu} l^\nu = 0 \quad \longrightarrow \quad F^{A+\mu\nu} l^\nu = 0, \quad (5.11)$$

which implies

$$F^{A+} = \frac{1}{2} \varphi^A \hat{l} \wedge \hat{m}^*, \quad (5.12)$$

for some complex functions \(\varphi^A\). Acting with \(\bar{\eta}\) on Eq. (5.6) we get
\[ \tilde{\nabla}_\mu \phi_I + i \sqrt{2} \varepsilon_{IJ} \phi^J T_{\mu \nu}^+ m^\nu = 0, \]  
\text{and substituting Eq. (5.12) into it, we arrive at}

\[ \tilde{\nabla}_\mu \phi_I - \frac{i}{\sqrt{2}} \varepsilon_{IJ} \phi^J T_{\Lambda} \varphi^\Lambda l_\mu = 0. \]  

Finally, acting with \( \bar{\epsilon} \) and \( \bar{\eta} \) on Eq. (5.14) we get

\[ l^\mu \partial_\mu Z^i = m^\mu \partial_\mu Z^i = 0 \rightarrow dZ^i = A^i \hat{l} + B^i \hat{m}, \]  

for some functions \( A^i \) and \( B^i \).

The relevant differential equations specifying the possible spacetime dependencies for the tetrad follow from Eqs. (5.5) and (5.9).

\[ \nabla_\mu l_\nu = 0, \]  

\[ \tilde{\nabla}_\mu n_\nu \equiv \nabla_\mu n_\nu = -a^*_\mu m_\nu - a_\mu m^*_\nu, \]  

\[ \tilde{\nabla}_\mu m_\nu \equiv (\nabla_\mu - 2 \zeta_\mu - i Q_\mu) m_\nu = -a_\mu l_\nu. \]

### 5.2 Equations of motion and integrability constraints

As was discussed in Sec. (3), the KSIs in the case at hand don’t vary a great deal, with respect to the ones derived in [1], and so we can be brief: the only equations of motion that are automatically satisfied are the ones for the graviphoton and the ones for the scalars from the vector multiplets. As one can see from Eq. (3.4), the same thing cannot be said about the equation of motion for the hyperscalar, but as we shall see in a few pages, it is anyhow identically satisfied. The, at the moment, relevant KSI is

\[ (\mathcal{E}_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \mathcal{E}_\sigma \sigma) l^\nu = (\mathcal{E}_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \mathcal{E}_\sigma \sigma) m^\nu = 0, \]  

where the relation of the equation of motion with and without hypermultiplets is given in Eq. (2.12).

Substituting the expressions (5.15) and (5.12) into the above KSIs we find the two conditions

\[ 0 = \left[ R_{\mu \nu} + 2 H_{\mu \nu} \partial_\mu q^a \partial_\nu q^a \right] l^\nu, \]  

\[ 0 = \left[ R_{\mu \nu} + 2 H_{\mu \nu} \partial_\mu q^a \partial_\nu q^a \right] m^\nu - \mathcal{G}_{ij} \left( A^i l_\mu + B^i m_\mu \right) B^j j^\nu. \]  

Comparable equations can be found from the integrability conditions of Eq. (5.5), i.e.
\[ 0 = \left[ R_{\mu\nu} + 2(d\zeta)_{\mu\nu} \right] l^\nu, \quad (5.22) \]
\[ 0 = \left[ R_{\mu\nu} + 2(d\zeta)_{\mu\nu} \right] m^{*\nu} - G_{ij^*} B^i (A^{*j^*} l_\mu + B^{*j^*} m_\mu^*), \quad (5.23) \]
and those of Eq. (5.14)
\[ 0 = \left[ R_{\mu\nu} - 2(d\zeta)_{\mu\nu} \right] n^{\nu} + 2(\tilde{\mathcal{D}} a)_{\mu\nu} m^{*\nu}. \quad (5.25) \]

In the derivation of these last identities use has been made of the formulae
\[ (dQ)_{\mu\nu} m^{*\nu} = iG_{ij^*} B^i B^{*j^*} m_\mu^*, \quad (dQ)_{\mu\nu} l^{\nu} = (dQ)_{\mu\nu} n^{\nu} = 0, \quad (5.26) \]
which follow from the definition of the Kähler connection and from Eq. (5.15).

Comparing these three sets of equations, we find that they are compatible if
\[ (d\zeta)_{\mu\nu} l^{\nu} = H_{uv} \partial_\mu q^u l^v \partial_\nu q^v, \quad (5.27) \]
\[ (d\zeta)_{\mu\nu} m^{*\nu} = H_{uv} \partial_\mu q^u m^{*v} \partial_\nu q^v, \quad (5.28) \]
and
\[ (\tilde{\mathcal{D}} a)_{\mu\nu} l^{\nu} = 0. \quad (5.29) \]
Please observe that, due to the positive definiteness of \( H \), Eq. (5.27) implies \( l^v \partial_\nu q^v = 0 \), but that Eq. (5.28) need not imply \( m^{*v} \partial_\nu q^v = 0 \).

### 5.3 A coordinate system, some more consistency and an anticlimax

In order to advance in our quest, it is useful to introduce a coordinate representation for the tetrad and hence also for the metric. Since \( \hat{l} \) is a covariantly constant vector, we can introduce coordinates \( u \) and \( v \) through \( l^\mu \partial_\mu = \partial_v \) and \( l_\mu dx^\mu = du \). We can also define a complex coordinates \( z \) and \( z^* \) by
\[ \hat{m} = e^U dz, \quad \hat{m}^* = e^U dz^*, \quad (5.30) \]
where \( U \) may depend on \( z \), \( z^* \) and \( u \), but not \( v \). Eq. (5.13) then implies that the scalars \( Z^i \) are just functions of \( z \) and \( u \):
\[ Z^i = Z^i(z, u), \quad (5.31) \]
wherefore the functions $A^i$ and $B^i$ defined in Eq. (5.15) are

$$A^i = \partial_{z^i} Z^i, \quad e^U B^i = \partial_{\zeta^i} (e^U B^i) = 0.$$  

(5.32)

Finally, the most general form that $\hat{n}$ can take in this case is

$$\hat{n} = dv + Hdu + \hat{\omega}, \quad \hat{\omega} = \omega_2 dz + \omega_3 dz^*, \quad (5.33)$$

where all the functions in the metric are independent of $v$. The above form of the null tetrad components leads to a Brinkmann pp-wave metric \[21\]

$$ds^2 = 2du (dv + Hdu + \hat{\omega}) - 2e^{2U} dz dz^*. \quad (5.34)$$

As we now have a coordinate representation at our disposal, we can start checking out the consistency conditions in this representation: Let us expand the connection $\zeta$ as

$$\zeta = i\zeta_n \hat{n} + i\zeta_l \hat{l} + \zeta_m \hat{m} - \zeta_m^* \hat{m}^*, \quad (5.35)$$

where $\zeta_l$ and $\zeta_n$ are real functions, whereas $\zeta_m$ is complex. Likewise expand

$$\hat{a} = a_t \hat{l} + a_m \hat{m} + a_{m^*} \hat{m}^* + a_n \hat{n}, \quad (5.36)$$

and

$$Q = Q_l \hat{l} + Q_m \hat{m} + Q_{m^*} \hat{m}^* + Q_n \hat{n}, \quad (5.37)$$

where, due to the reality of $Q$, $(Q_m)^* = Q_{m^*}$. Let us now consider the tetrad integrability equations (5.16)-(5.18): Eq. (5.16) is by construction identically satisfied. Eq. (5.18), with our choice of coordinate $z$ Eq. (5.30), implies

$$0 = e^{-U} \partial_{\zeta^*} U + 2\zeta_{m^*} - iQ_{m^*}, \quad (5.38)$$

$$0 = -2i\zeta_n - iQ_n, \quad (5.39)$$

and

$$\hat{a} = [\dot{U} - 2i\zeta_l - iQ_l] \hat{m} + a_t \hat{l}, \quad (5.40)$$

where $a_t = a_t(z, z^*, u)$ is a functions to be determined and dots indicate partial derivation w.r.t. the coordinate $u$. Eq. (5.31) implies that $\zeta_n = Q_n = 0$ and from Eq. (5.38) we obtain

$$\partial_{z^*} (U + \frac{1}{2}K) = -2\zeta_{z^*}. \quad (5.41)$$

This last equation states that $\zeta_{m^*}$, whence also $\zeta_m$, can be eliminated by a gauge transformation, after which we are left with

$$\hat{\zeta} = i\zeta_l \hat{l}. \quad (5.42)$$

\[6\] The components of the connection and the Ricci tensor of this metric can be found in the Appendix of Ref. \[20\].

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At this point it is wise to return to Eq. (5.28) and to deduce

$$H_{uv} \partial_\mu q^u m^* \nu \partial_\nu q^v = (d\zeta)_{\mu\nu} m^* \nu = 2e^{-U}(\partial_\zeta m_{\mu} l_\nu) + \partial_\zeta m_{\nu} (d\zeta)_{\mu\nu}m^* \nu = e^{-U}\partial_\zeta l_\mu.$$ (5.43)

This equation implies that $dq^u \sim \hat{1}$, and we are therefore obliged to accept the fact that in the null case, the hyperscalars can only depend on the spacetime coordinate $u$!

Had we been hoping for the hyperscalars to exhibit some interesting spacetime dependency, then this result would have been a bit of an anti-climax. But then, the fact that the hyperscalars can only depend on $u$, means that we can eliminate the connection $A$ from the initial set-up, which means that as far as solutions to the Killing Spinor equations is concerned, the problem splits into two disjoint parts: one is the solution to the KSEs in the null case of $N = 2$ $d = 4$ supergravity, which are to be found in [5, 1], and the solutions to Eq. (2.4).

In the case at hand Eq. (2.4) reduces to

$$0 = U_{\nu I} \varepsilon_{IJ} \partial_u q^v \gamma^u \epsilon^J,$$ (5.44)

so that either we take the hyperscalars to be constant or impose the condition $\gamma^u \epsilon^J = 0$. This last condition is however always satisfied by any non-maximally supersymmetric solution of the null case, to wit Minkowski space and the 4D Kowalski-Glikman wave. It is however obvious that these solutions are incompatible with $u$-dependent hyperscalars, and its reason takes us to the last point in this exposition: the equations of motion.

As far as the equations of motion are concerned, it is clear that, since we are dealing with a pp-wave metric, the hyperscalar equation of motion is identically satisfied. As the only coupling between vector multiplets and hypermultiplets is through the gravitational interaction, see Eq. (2.12), the only equation of motion that changes is the one in the $uu$-direction. More to the point, its sole effect is to change the differential equation [1, (5.91)] determining the wave profile $H$ in (5.34).

A fitting example of a solution demonstrating just this, consider the deformation of the cosmic string (1.1):

$$ds^2 = 2 du (dv + H(q) \mid z \mid^2) - 2 e^{-K} dz dz^*, \quad Z^i = Z^i(z), \quad F^\Lambda = 0,$$

$$q^w = q^w(u),$$ (5.45)

which is a 1/2-BPS solution.

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A Quaternionic Kähler Geometry

A quaternionic Kähler manifold is a real $4m$-dimensional Riemannian manifold $\mathcal{M}$ endowed with a triplet of complex structures $J^x : T(\mathcal{M}) \rightarrow T(\mathcal{M})$, $(x = 1, 2, 3)$ that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} + \varepsilon^{xyz} J^z,$$

and with respect to which the metric, denoted by $H$, is Hermitean:

$$H( J^x X, J^y Y ) = H( X, Y ), \quad \forall X, Y \in T(\mathcal{M}).$$

This implies the existence of a triplet of 2-forms $K^x(X, Y) \equiv H( J^x X, Y )$ globally known as the $\mathfrak{su}(2)$-valued hyperKähler 2-forms.

The structure of quaternionic Kähler manifold requires an $SU(2)$ bundle to be constructed over $\mathcal{M}$ with connection 1-form $A^x$ with respect to which the hyperKähler 2-form is covariantly closed, i.e.

$$\mathcal{D}K^x \equiv dK^x + \varepsilon^{xyz} A^y \wedge K^z = 0.$$  

Then, depending on whether the curvature of this bundle

$$F^x \equiv dA^x + \frac{1}{2} \varepsilon^{xyz} A^y \wedge A^z,$$

is zero or is proportional to the hyperKähler 2-form

$$F^x = \lambda K^x, \quad \lambda \in \mathbb{R}/\{0\},$$

the manifold is a hyperKähler manifold or a quaternionic Kähler manifold, respectively.

The $SU(2)$ connection acts on objects with vectorial $SU(2)$ indices, such as the chiral spinors in this article, as follows:

$$\mathcal{D}\xi^I \equiv d\xi^I + A^I_J \xi^J,$$

$$\mathcal{D}\chi^J \equiv d\chi^J + A^I_J \chi^I.$$

Consistency with the raising and lowering of vector $SU(2)$ indices via complex conjugation requires

$$A^I_J = (A^I_J)^*.$$

If we, following Ref. [17], put
\[ A^I_J = \frac{i}{2} A^x (\sigma_x)_I^J, \quad (A.8) \]

we get

\[ A^I_J = \frac{i}{2} A^x (\varepsilon \sigma_x \varepsilon^{-1})_I^J = -\frac{i}{2} A^x \varepsilon^{IK} (\sigma_x)_K^L \varepsilon_{LJ}. \quad (A.9) \]

Consistency between the above definitions of \(SU(2)\)-covariant derivatives, \(A^I_J\) and \(SU(2)\) curvature requires that the 3 matrices \((\sigma_x)_I^J\) satisfy

\[ [\sigma_x, \sigma_y]_I^J = -2i\varepsilon_{xyz} (\sigma_z)_I^J, \quad (A.10) \]

whence we can take them to be the (Hermitean, traceless) Pauli matrices satisfying

\[ (\sigma_x \sigma_y)_I^J = \delta_{xy} \delta_{IJ} - i\varepsilon_{xyz} (\sigma_z)_I^J. \quad (A.11) \]

It is convenient to use a Vielbein on \(HM\) having as “flat” indices a pair \(\alpha I\) consisting of one \(SU(2)\)-index \(I\) and one \(Sp(m)\)-index \(\alpha = 1, \ldots, 2m\)

\[ U^\alpha_I = U^\alpha_I u dq^u, \quad (A.12) \]

where \(u = 1, \ldots, 4m\) and from now on we shall refer to this object as the Quadbein. This Quadbein is related to the metric \(H_{uv}\) by

\[ H_{uv} = U^\alpha_I u U^\beta_J v \varepsilon_{IJ} C_{\alpha\beta}, \quad (A.13) \]

and, further, it is required that

\[ 2 U^\alpha_I(u) U^\beta_J(v) C_{\alpha\beta} = H_{uv} \varepsilon_{IJ}, \]
\[ 2m U^\alpha_I(u) U^\beta_J(v) \varepsilon_{IJ} = H_{uv} C^{\alpha}, \quad (A.14) \]

\[ U_{\alpha I u} \equiv (U^\alpha_I u)^* = \varepsilon_{IJ} C_{\alpha\beta} U^{\beta J u}. \]

The inverse Quadbein \(U^u_{\alpha I}\) satisfies

\[ U_{\alpha I u} U^\alpha_I v = \delta^u_v, \quad (A.15) \]

and, therefore,

\[ U_{\alpha I u} = H^{uv} \varepsilon_{IJ} C_{\alpha\beta} U^{\beta J v}. \quad (A.16) \]

The Quadbein satisfies a Vielbein postulate, i.e. they are covariantly constant with respect to the standard Levi-Civit\`a connection \(\Gamma_{vw}^u\), the \(SU(2)\) connection \(A_{u I J}\) and the \(Sp(m)\) connection \(\Delta_{u \alpha\beta}\).

\[ ^7\text{Of course, } F_{IJ} \equiv \frac{i}{2} F^z (\sigma_z)_I^J. \]
\[ D_u \ U^{\alpha I}_v = \partial_u U^{\alpha I}_v - \Gamma_{uvw}^{\alpha I} U^{\alpha}_w + A_u^{\alpha I} U^{\alpha}_v + \Delta_u^\alpha \beta U^{\gamma I}_v C_{\beta \gamma} = 0. \] (A.17)

This postulate relates the three connections and the respective curvatures, leading to the statement that the holonomy of a quaternionic Kähler manifold is contained in \( Sp(1) \cdot Sp(m) \), i.e.

\[ R_{ts}^{\ uv} \ U^{\alpha I}_u \ U^{\beta J}_v + \varepsilon^{IK} F_{ts}K^J C^{\alpha \beta} - 2 \overline{R}_{ts}^{\ \alpha \beta} \varepsilon^{IJ} = 0, \]

where

\[ \overline{R}_{ts}^{\ \alpha \beta} = 2 \partial_t \Delta_s^{\alpha \beta} + 2 \Delta_t^{\alpha \gamma} \Delta_s^{\delta \beta} C^{\gamma \delta}, \]

is the curvature of the \( Sp(m) \) connection.

A useful relation is

\[ F_{\mu \nu I} = 2 \lambda U_{I \alpha} U_{\nu J} \partial_{[\mu} q^u \partial_{\nu]} q_v. \]

(B.18)

B C-map and dual quaternionic manifolds

The c-map is a manifestation of the T-duality between the type IIA and IIB theories, compactified on the same Calabi-Yau 3-fold. Since T-duality in supergravity theories is implemented by dimensional reduction, to be told that the c-map is derived by dimensionally reducing an \( N = 2 \) \( d = 4 \) SUGRA coupled to \( n \) vector- and \( m \) hypermultiplets to \( d = 3 \), and dualizing every vector field into a scalar field, should not come as too big a surprise.

In order to derive the c-map, consider the, rather standard, KK-Ansatz

\[ \hat{e}^a = e^{-\phi} e^a; \quad \hat{e}^u = e^\phi (dy + A), \]

\[ \hat{A}^\Lambda = B^\Lambda + C^\Lambda (dy + A) \quad \rightarrow \quad \hat{F}^\Lambda = F^\Lambda + dC^\Lambda \wedge (dy + A), \]

\[ F^\Lambda = dB^\Lambda + C^\Lambda F, \quad F = dA, \]

and use it on the action (2.1); the resulting action reads

\[ S_{(3)} = \int d^3 \sqrt{g} \left[ \frac{1}{2} R + \frac{d \phi^2}{e^{-2 \phi}} \text{Im}(N)_{\Lambda \Sigma} dC^\Lambda dC^\Sigma + G_{ij} dZ^i d(Z^j)^* + H_{uv} dq^u dq^v \right] \]

\[ + \int \left( \frac{1}{2} \mathbf{3}^T M \wedge \ast \mathbf{3} + \mathbf{3}^T \wedge Q d\mathbf{c} \right), \]

(B.2)

where we have defined the \((\bar{n}+1)\)-vectors \( \mathbf{3}^T = (dB^\Lambda, dA) \) and \( \mathbf{c}^T = (C^\Lambda, 0) \). Furthermore the \((\bar{n}+1) \times (\bar{n}+1)\)-matrices \( M \) and \( Q \) are given by

\[ M = 2e^{2\phi} \left( \begin{array}{cc} \text{Im}(N) & \text{Im}(N) \cdot C \\ C^T \cdot \text{Im}(N) & C^T \cdot \text{Im}(N) \cdot C - \frac{e^{2\phi}}{4} \end{array} \right); \quad Q = 2 \left( \begin{array}{cc} \text{Re}(N) & 0 \\ C^T \cdot \text{Re}(N) & 0 \end{array} \right). \]

(B.3)
The field strengths can then be integrated out by adding to the above action a Lagrange multiplier term \( \mathcal{F}^T \wedge d\mathcal{L} \), imposing the Bianchi identity \( d\mathcal{F} = 0 \). \( \mathcal{F} \) can then be integrated out by using its equation of motion \( \ast \mathcal{F} = M^{-1}(d\mathcal{L} + Qd\mathcal{C}) \), resulting in 3d gravity coupled to a sigma model describing two disconnected quaternionic manifolds, one with metric \( H_{uv}dq^udq^v \), and the other one coming from the gravity- and vector multiplets. Taking \( \mathcal{L}^T = (T_\Lambda, \theta) \) we can write the metric of this 4\( \bar{n} \)-dimensional quaternionic manifold as

\[
ds_{DQ}^2 = d\phi^2 - e^{-2\phi} \text{Im}(\mathcal{N})_{\Lambda\Sigma} dC^\Lambda dC^\Sigma + e^{-4\phi} (d\theta - C^\Lambda dT_\Lambda)^2 + G_{ij}^* dZ^i d(Z^j)^* - \frac{1}{4} e^{-2\phi} \text{Im}(\mathcal{N})^{-1|\Lambda\Sigma} \left( dT_\Lambda + 2\text{Re}(\mathcal{N})_{\Lambda\bar{\Lambda}} dC_{\bar{\Sigma}} \right) \left( dT_\Sigma + 2\text{Re}(\mathcal{N})_{\Sigma\bar{\Sigma}} dC_{\bar{\Lambda}} \right) .
\]

The fact that this metric is indeed quaternionic was proven in [22]. This kind of quaternionic manifolds is, for an obvious reason, called dual quaternionic manifolds, and is generically characterized by the existence of at least 2(\( \bar{n} \) + 1)-translational isometries [23], generated by the following Killing vectors

\[
U = \partial_\phi + T_\Lambda \partial_{T_\Lambda} + C^\Lambda \partial_{C^\Lambda} + 2\theta \partial_\theta ; \quad V = \partial_\theta ,
\]

\[
X^\Lambda = \partial_{T_\Lambda} , \quad Y_\Lambda = \partial_{C^\Lambda} + T_\Lambda \partial_\theta .
\]

These vector fields satisfy the commutation relation of a Heisenberg algebra, i.e.

\[
[U, X^\Lambda] = -X^\Lambda , \quad [U, Y_\Lambda] = -Y_\Lambda ,
\]

\[
[U, V] = -2 V , \quad [X^\Lambda, Y_\Sigma] = \delta^\Lambda_\Sigma V .
\]

The automorphism group of this Heisenberg algebra is \( Sp(\bar{n}, \mathbb{R}) \), and one can find a nice \( Sp(\bar{n}, \mathbb{R}) \)-adapted coordinate system by doing the coordinate transformation \( T_\Lambda \rightarrow -2T_\Lambda \) and \( \theta \rightarrow \theta - C^\Lambda T_\Lambda \); this transformation allows us to write the metric, introducing the real symplectic vector \( S^T = (C^\Lambda, T_\Lambda) \), as

\[
ds_{DQ}^2 = d\phi^2 + G_{ij}^* dZ^i d(Z^j)^* + e^{-4\phi} (d\theta - \langle S|dS\rangle)^2 + ds^T \mathcal{M} d\mathcal{S} ,
\]

where \( \mathcal{M} \) is the 2\( \bar{n} \) \times 2\( \bar{n} \)-matrix

\[
\mathcal{M} = -\begin{pmatrix}
\text{Im}(\mathcal{N}) + \text{Re}(\mathcal{N})\text{Im}(\mathcal{N})^{-1}\text{Re}(\mathcal{N}) & -\text{Re}(\mathcal{N})\text{Im}(\mathcal{N})^{-1} \\
-\text{Im}(\mathcal{N})^{-1}\text{Re}(\mathcal{N}) & \text{Im}(\mathcal{N})^{-1}
\end{pmatrix},
\]

\[
= 2\Omega \text{Re} \left( V \nu^i + U_i G_{ij}^* U_j^i \right) \Omega^T ,
\]

where \( \Omega \) is the inner product left invariant by \( Sp(\bar{n}, \mathbb{R}) \). Moreover, \( \mathcal{M} \) is positive definite and has the correct and obvious properties [24] to make the metric \( Sp(\bar{n}, \mathbb{R}) \)-covariant.

In order to discuss the Quadbein, it is convenient to split the \( \alpha = 1 \ldots 2\bar{n} \) index as \( \alpha \rightarrow (\Lambda \tilde{\alpha}) \) where the new \( \tilde{\alpha} = 1, 2 \) and as usual \( \Lambda = 0, \ldots, n \). This means that we split
$\mathbb{C}_{\alpha\beta} = \delta_{\Lambda\Sigma} \varepsilon_{\bar{\alpha}\beta}$ and a base for the matrices satisfying Eq. \ref{eq:16} can be found with great ease, but since it will not be needed, we shall abstain from presenting them here.

It is likewise convenient to introduce the objects $\left( a = 1, \ldots, n_V \right)$

$$E^\Lambda = (E^0, E^a) \ ; \ \sqrt{2} E^0 = \ d\phi + i \ e^{-2\phi} \left[ d\theta - \langle S \mid dS \rangle \right],$$

$$E^a E^\bar{a} = \frac{1}{2} G_{ij} \cdot dZ^i d(Z^j)^* \tag{B.9}$$

$$U^\Lambda = (V, U^a) \ ; \ U^a = U_i G^{ij} E_j^{\Lambda}.$$}

With these definitions we can write the Quadbein compactly and manifestly $Sp(\pi, \mathbb{R})$-covariant as

$$U^{(\Lambda\bar{\alpha})} \ i = \begin{pmatrix} E^\Lambda & e^{-\phi} \langle dS \mid U^{\bar{\Lambda}} \rangle \\ -e^{-\phi} \langle dS \mid U^{\bar{\Lambda}} \rangle & (E^{\bar{\Lambda}})^* \end{pmatrix}, \tag{B.10}$$

In this parametrization, the $sp(1)$ connection can be seen to be

$$A_I^J = \frac{i}{2} \begin{pmatrix} Q - \sqrt{2} \text{Im}(E^0) & -2\sqrt{2} i \ e^{-\phi} \langle dS \mid V \rangle \\ 2\sqrt{2} i \ e^{-\phi} \langle dS \mid V \rangle & \sqrt{2} \text{Im}(E^0) - Q \end{pmatrix}. \tag{B.11}$$

Let us close this appendix with some comments: An interesting quaternionic manifold is the so-called universal quaternionic manifold, which is the manifold that arises from applying the c-map on minimal $N = 2 \ d = 4$ SUGRA: it is therefore given by the formulae in this section for $n = 0$. From the parent discussion it is then also paramount that we are dealing with a homogeneous space; It is admittedly less paramount that the universal quaternionic manifold is the symmetric space $SU(1,2)/U(2)$, but a quite standard calculation shows this to be the case.

We derived the c-map through dimensional reduction over a spacelike circle. Similarly one can dimensionally reduce the action over a timelike circle, resulting in a space of signature $(2\pi, 2\pi)$ and whose holonomy is contained in $Sp(1, \mathbb{R}) \cdot Sp(\pi)$. In the rigid limit, i.e. when $\lambda = 0$, one recovers the $(1,2)/\text{para-hyperKähler}$ structure discussed in e.g. \ref{25, 26} The para-universal para-quaternionic manifold, i.e. the manifold one obtains by the timelike c-map from minimal $N = 2 \ d = 4$ SUGRA, can be seen to be $SU(1,2)/U(1,1)$.

**References**

[1] P. Meessen and T. Ortín, “The supersymmetric configurations of $N = 2, d = 4$ supergravity coupled to vector supermultiplets”, to be published in Nucl. Phys B. [arXiv:hep-th/0603099].

[2] B. de Wit and A. van Proeyen, Nucl. Phys. B 245 (1984) 89.

[3] B. de Wit, P.G. Lauwers and A. van Proeyen, Nucl. Phys. B 255 (1985) 569.
[4] A. Celi, “Toward the classification of BPS solutions of $N = 2$, $d = 5$ gauged supergravity with matter couplings”, arXiv:hep-th/0405283.

[5] K.P. Tod, Phys. Lett. B 121 (1983) 241; Class. Quant. Grav. 12 (1995), 1801.

[6] F. Wilczek, in: Quark confinement and field theory, ed. Stump and Weingarte (Wiley-Interscience, New York, 1977)

[7] J.M. Charap and M.J. Duff, Phys. Lett. B 69 (1977) 445.

[8] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B 258 (1985) 46.

[9] R.R. Khuri, Nucl. Phys. B 387 (1992) 315 [arXiv:hep-th/9205081].

[10] J.P. Gauntlett, J.A. Harvey and J.T. Liu, Nucl. Phys. B 409 (1993) 363 [arXiv:hep-th/9211056].

[11] M.J. Duff, R.R. Khuri, R. Minasian and J. Rahmfeld, Nucl. Phys. B 418 (1994) 195 [arXiv:hep-th/9311120].

[12] R. Kallosh and T. Ortín, Phys. Rev. D 50 (1994) 7123 [arXiv:hep-th/9409060].

[13] J.M. Maldacena and C. Núñez, Int. J. Mod. Phys. A 16 (2001) 822 [arXiv:hep-th/0007018].

[14] F. Denef, JHEP 0008 (2000) 050 [arXiv:hep-th/0005049]; B. Bates and F. Denef, “Exact solutions for supersymmetric stationary black hole composites”, arXiv:hep-th/0304094.

[15] J. Bellorín, P. Meessen and T. Ortín, “Supersymmetry, attractors and cosmic censorship”, arXiv:hep-th/0606201.

[16] K. Behrndt, D. Lüst and W.A. Sabra, Nucl. Phys. B 510 (1998) 264 [arXiv:hep-th/9705169].

[17] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré and T. Magri, J. Geom. Phys. 23 (1997) 111 [arXiv:hep-th/9605032].

[18] R. Kallosh and T. Ortín, “Killing spinor identities”, arXiv:hep-th/9306085.

[19] J. Bellorín and T. Ortín, Phys. Lett. B 616 (2005) 118 [arXiv:hep-th/0501246].

[20] J. Bellorín and T. Ortín, Nucl. Phys. B 726 (2005) 171 [arXiv:hep-th/0506056].

[21] H.W. Brinkmann, Proc. Natl. Acad. Sci. U.S. 9 (1923) 1; Math. Annal. 94 (1925) 119.

[22] S. Ferrara and S. Sabharwal, Nucl. Phys. B 332 (1990) 317.
[23] B. de Wit and A. van Proeyen, Phys. Lett. B 252 (1990) 221.

[24] S. Ferrara and R. Kallosh, Phys. Rev. D 54 (1996) 1514 [arXiv:hep-th/9602136].

[25] I. Kath, “Killing spinors on pseudo-Riemannian manifolds”, Habilitationsschrift an der Humboldt Universität Berlin, 1999.

[26] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, JHEP 0506 (2005) 025 [arXiv:hep-th/0503094].