REGULARITY AND WELL POSEDNESS FOR THE LAPLACE OPERATOR ON POLYHEDRAL DOMAINS

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Abstract. We announce a well-posedness result for the Laplace equation in weighted Sobolev spaces on polyhedral domains in \( \mathbb{R}^n \) with Dirichlet boundary conditions. The weight is the distance to the set of singular boundary points. We give a detailed sketch of the proof in three dimensions.

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Introduction

Let \( \Omega \subset \mathbb{R}^n \) be an open set. Consider the boundary value problem

\[
\Delta u = f, \quad u|_{\partial \Omega} = g,
\]

where \( \Delta \) is the Laplace operator. For \( \Omega \) smooth and bounded, this boundary value problem has a unique solution \( u \in H^{s+2}(\Omega) \) depending continuously on \( f \in H^s(\Omega) \) and \( g \in H^{s+3/2}(\partial \Omega), s \geq -1 \). See the books of Evans [11] or Taylor [23] for a proof of this basic and well known result.

It is also well known that this result does not extend to non-smooth domains \( \Omega \). A deep analysis of the difficulties that arise for \( \partial \Omega \) Lipschitz is contained in the papers of Babuška and Guo [14], Băcuță, Bramble, and Xu [3], Jerison and Kenig [15], Kenig [16], Mitrea and Taylor [22], Verchota [24], and others (see the references in the aforementioned papers). Results specific to polyhedral domains are contained in the papers of Costabel [6], Dauge [7, 8], Elschner [9], Kondratiev [17], Mazya and Rossman [21] and others. Good references are also the monographs [12, 10, 20].

In this paper, we consider the boundary value problem (1) when \( \Omega \) is a bounded polyhedron in \( \mathbb{R}^n \), and Poisson’s equation \( \Delta u = f \) is replaced by a strongly elliptic system. Let us denote by \( \Omega^{(n-2)} \) the set of points \( p \in \partial \Omega \) such \( \partial \Omega \) is not smooth.
in a neighborhood of \( p \) and by \( \eta_n - 2(x) \) the distance from a point \( x \in \Omega \) to the set \( \Omega^{(n-2)} \subset \partial \Omega \) of non-smooth boundary points of \( \Omega \).

We shall work in the weighted Sobolev spaces

\[
K^\mu_a(\Omega) = \{ u \in L^2_{loc}(\Omega), \eta_n^{-a} \partial^\alpha u \in L^2(\Omega), \text{ for all } |\alpha| \leq \mu \}, \quad \mu \in \mathbb{Z}^+,
\]

which we endow with the induced Hilbert space norm. We shall call these spaces the Babuška–Kondratiev spaces. A similar definition (i.e., using the same weight, see Definition 3) yields the Babuška–Kondratiev (or weighted Sobolev) spaces \( K^\mu_a(\partial \Omega) \), \( \mu \in \mathbb{Z}^+ \). The spaces \( K^s_a(\partial \Omega) \), \( s \in \mathbb{R}^+ \), are defined by interpolation. The Babuška–Kondratiev spaces are closely related to weighted Sobolev spaces on non-compact manifolds. See the works of Erkip and Schrohe [10] and Grubb [13], for related results on boundary value problems on non-compact manifolds and, more generally, on the analysis on non-compact manifolds. Here is our main result for the Laplace equation on a polyhedral domain.

**Theorem 0.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded, polyhedral domain and \( \mu \in \mathbb{Z}^+ \). Then there exists \( \eta > 0 \) such that the boundary value problem (1) has a unique solution \( u \in K^{\mu+1}_a(\Omega) \) for any \( f \in K^{\mu-1}_a(\Omega) \), any \( g \in K^{\mu+1/2}_a(\partial \Omega) \), and any \( |a| < \eta \). This solution depends continuously on \( f \) and \( g \). If \( \mu = 0 \), this solutions is the solution of the associated variational problem.

The proof can be carried out, without much change, to yield the same result for strongly elliptic, strictly positive systems on curvilinear polyhedral domains. The complications are mostly of topological nature, so we shall discuss this result in [14]. The analytic part of the proof is however the same both for polyhedral domains and for curvilinear polyhedral domains, therefore the reader interested mostly in analysis will benefit from the simplified account included in this paper.

We now describe the contents of the sections of the paper in more detail. The first section introduces the weighted Sobolev spaces (also called the Babuška–Kondratiev spaces) that appear in our main result, Theorem 0.1. The second section contains a statement of three intermediate results: a Hardy–Poincaré inequality, a regularity result, and a trace theorem. A proof of proof of the Hardy–Poincaré type inequality in dimensions \( n = 3 \) is given in the third section. A sketch of the proof of our main result is given in Section 4. This proof is based on the three intermediate results mentioned above. The last two intermediate results are particular cases of some results proved in [14], provided that we show that polyhedral domains fit into the framework of Lie domains developed in that paper. This is however highly nontrivial in higher dimensions and leads to topological and geometric complications that will be treated in detail in [14] in the more general framework of curvilinear polyhedral domains.

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### 1. Sobolev spaces

We introduce in this section the Babuška–Kondratiev (or weighted Sobolev) spaces \( K^\mu_a(\Omega) \), \( K^s_a(\partial \Omega) \), \( \mu \in \mathbb{Z} \) and \( s \in \mathbb{R} \), for the case when \( \Omega \) is a straight polyhedron.
(straight polyhedra are defined below by induction). Throughout this paper \( \Omega \) will be an open set. Recall that we denoted by \( \eta_{n-2}(x) \) the distance from a point \( x \in \mathbb{R}^n \) to \( \Omega^{(n-2)} \subset \partial \Omega \), the set of singular boundary points of \( \Omega \).

Below, by an affine space we shall denote the translation of a subspace of a vector space \( V \). We define the concept of a straight polyhedron of dimension \( n \) by induction. A subset \( \Omega \) of an affine space is a straight polyhedron of dimension \( n \) if it is a finite union of open intervals (bounded or not) on a line such that \( \partial \Omega = \partial \overline{\Omega} \). An open subset \( \Omega \subset V \) with finitely many connected components of an affine space \( V \) of dimension \( n \geq 2 \) will be called a straight polyhedron of dimension \( n \) if \( \partial \Omega = \partial \overline{\Omega} \) and there exist disjoint straight polyhedra \( D_j \subset \partial \Omega \) of dimension \( n-1 \) such that \( \partial \Omega = \bigcup D_j \).

The condition \( \partial \Omega = \partial \overline{\Omega} \) is equivalent to the fact that \( \Omega \) is the interior of its closure \( \overline{\Omega} \). This condition is designed to rule out the case when \( \Omega \) lies on both sides of its boundary. To deal with this case, as well as with more general domains, we need the concept of a “curvilinear polyhedral domain,” which will be discussed in [4]. A simple example of a polyhedron is the interior of the convex hull of a finite set of points in \( \mathbb{R}^n \), provided that this set is not empty.

Let \( \Omega \) be a straight polyhedron in an affine space \( V \). For simplicity we shall take \( V = \mathbb{R}^n \). Let \( f \) be a continuous function on \( \Omega \), \( f > 0 \) on the interior of \( \Omega \). We define the \( \mu \)th Sobolev space with weight \( f \) (and index \( a \)) by

\[
K_{a,f}^\mu(\Omega) = \{ u \in L^2_{\text{loc}}(\Omega), f^{[n]} \partial^a u \in L^2(\Omega), \text{ for all } |\alpha| \leq \mu \}, \quad \mu \in \mathbb{Z}_+.
\]

The norm on \( K_{a,f}^\mu(\Omega) \) is \( \| u \|^2_{K_{a,f}^\mu(\Omega)} = \sum_{|\alpha| \leq \mu} \| f^{[n]} \partial^a u \|^2_{L^2(\Omega)} \).

**Definition 1.1.** We let \( K_a^\mu(\Omega) = K_{a,f}^\mu(\Omega) \) and \( K^\mu_a(\partial \Omega) = K_{a,f}^\mu(\partial \Omega) \), where \( f = \eta_{n-2} \) is the distance to \( \Omega^{(n-2)} \).

For example, \( K_0^\mu(\Omega) = L^2(\Omega) \). For \( \Omega \) a polygon in the plane, \( \eta_{n-2}(x) = \eta_0(x) \) is the distance from \( x \) to the vertices of \( \Omega \) and the resulting spaces \( K_{a,f}^\mu(\Omega) \) are the spaces introduced by Kondratiev [17]. Let us notice that we define both Sobolev spaces \( K^\mu_a(\Omega) \) and \( K^\mu_a(\partial \Omega) \) using the same weight function.

If \( \mu \in \mathbb{N} = \mathbb{Z}_+ \setminus \{ 0 \} \), we define \( K_{-\mu}^\circ_a(\Omega) \) to be the dual of

\[
K_{a}^\mu(\Omega) := K_{a}^\mu(\Omega) \cap \{ \partial_j u | u \partial \Omega = 0, j = 0, 1, \ldots, \mu - 1 \}
\]

with pivot \( K_{a}^\circ(\Omega) \). Since \( C^\infty_c(\Omega) \) is dense in \( K_{a}^\mu(\Omega) \) by Theorem 3.4 of [4], an equivalent definition of the space \( K_{-\mu}^\circ_a(\Omega) \) is as follows. First define for any \( u \in C^\infty(\Omega) \)

\[
\| u \|_{K_{-\mu}^\circ_a(\Omega)} = \sup_{\| v \|_{K_{a}^\mu(\Omega)} < \infty} \frac{|(u, v)|}{\| v \|_{K_{a}^\mu(\Omega)}}, \quad 0 \neq v \in C^\infty_c(\Omega).
\]

Then we let \( K_{-\mu}^\circ_a(\Omega) \) to be the completion of the space of smooth functions \( u \) on \( \Omega \) that are such that \( \| u \|_{K_{-\mu}^\circ_a(\Omega)} < \infty \). The spaces \( K^\mu_a(\partial \Omega) \), with \( s \in \mathbb{Z} \), are defined by complex interpolation.

The following result can be proved in small dimensions directly using spherical or polar coordinates. In higher dimensions, it follows using also the result of [4], and it will be dealt with in [4].

**Proposition 1.2.** Let \( P \) be a differential operator of order \( m \) on \( \Omega \) with smooth coefficients. Then \( P \) maps \( K_{a}^\mu(\Omega) \) to \( K_{a-m}^\mu(\Omega) \) continuously, for any admissible
weight \( h \) and any \( \mu \in \mathbb{Z} \). Moreover, the resulting family \( h^{-\lambda}P_{\lambda}h^{\lambda} : \mathcal{K}_a^\mu(\Omega) \to \mathcal{K}_a^{-m}(\Omega) \) of bounded operators depends continuously on \( \lambda \).

2. Three intermediate results

We now state in the three main intermediate results needed for the proof of our main result, Theorem 2.1, namely, a Hardy–Poincaré type inequality (Theorem 2.1), a regularity result for polyhedra (Theorem 2.2), and a theorem on the general properties of the trace map between weighted Sobolev spaces (Theorem 2.3).

**Theorem 2.1.** There exists a constant \( \kappa_{\Omega} > 0 \), depending only on \( \Omega \), such that

\[
\|u\|_{\mathcal{K}_1^1(\Omega)}^2 \leq \kappa_{\Omega} \int_\Omega |\nabla u(x)|^2 \, dx , \quad dx = dx_1 dx_2 \ldots dx_n ,
\]

for any function \( u \in H_{c}^{1}(\Omega) \) such that \( u|_{\partial \Omega} = 0 \).

The regularity result, stated next, is of independent interest.

**Theorem 2.2.** Assume that \( \Delta u \in \mathcal{K}_1^{-\lambda}(\Omega) \) and \( u|_{\partial \Omega} \in \mathcal{K}_1^{\mu+1/2}(\partial \Omega) \), \( \mu \in \mathbb{Z}_+ \), for some \( u \in \mathcal{K}_1^1(\Omega) \). Then \( u \in \mathcal{K}_1^{\mu+1}(\Omega) \) and

\[
\|u\|_{\mathcal{K}_1^{\mu+1}(\Omega)} \leq C \left( \|\Delta u\|_{\mathcal{K}_1^{\mu+1}(\Omega)} + \|u\|_{\mathcal{K}_1^1(\Omega)} + \|u|_{\partial \Omega}\|_{\mathcal{K}_1^{\mu+1/2}(\partial \Omega)} \right).
\]

We shall need also the following result on weighted Sobolev spaces, which generalizes the well known results on Sobolev spaces on domains with smooth boundary. Let \( \mathcal{C}_\infty(\Omega) \) be the space of compactly supported functions on the open set \( \Omega \).

**Theorem 2.3.** The restriction \( \mathcal{C}_\infty^\infty(\Omega \setminus \Omega^{(n-2)}) \ni u \to u|_{\partial \Omega} \in \mathcal{C}_\infty^\infty(\partial \Omega \setminus \Omega^{(n-2)}) \) extends to a continuous, surjective map

\[
\mathcal{K}_a^\mu(\Omega) \to \mathcal{K}_a^{\mu-1/2}(\partial \Omega), \quad \mu \geq 1 .
\]

Moreover, \( \mathcal{C}_\infty^\infty(\Omega) \) is dense in the kernel of this map if \( \mu = 1 \).

Theorems 2.2 and 2.3 will follow from Theorems 3.4 and 3.7 of \[1\], once we will have identified our weighted Sobolev spaces on \( \Omega \) with the Sobolev spaces introduced in \[1\]. This will be done, in the more general setting of curvilinear polyhedral domains in \[1\]. The proofs of the quoted results from \[1\] is to reduce to the case of a half-space using a suitable partition of unity. The construction of this partition of unity is, in turn, based on the geometric framework of Lie manifolds introduced in \[2\].

Let us give only a brief hint of the role of Lie algebras of vector fields (and Lie manifolds) in the study of weighted Sobolev spaces on a polyhedron. There exits an explicit smooth function \( r_\Omega \) on \( \Omega \) such that \( r_\Omega \) is equivalent to \( r_{n-2} \) (i.e., \( r_\Omega / r_{n-2} \) is bounded from above and bounded away from \( 0 \)) and, moreover, \( r_\Omega^f K^\mu_a(\Omega) = K^\mu_{a+f}(\Omega) \). This function is constructed as follows. Let \( g_0 \) be the Euclidean metric. We shall define the metrics \( g_{k+1} \) and the functions \( \tilde{\rho}_k \), \( k \geq 1 \), as follows. Let \( \rho_k \) be the distance to the faces of dimension \( k \) of \( \Omega \) in the metric \( g_k \) and let \( \tilde{\rho}_k \) be a smooth function that coincides with \( \rho_k \) when \( \rho_k \) is small and otherwise satisfies \( \rho_k/2 \leq \tilde{\rho}_k \leq \rho_k \). We then let \( g_{k+1} = \tilde{\rho}_k^{-2} g_k \). Finally, we define \( r_{n-2} = \rho_0 \rho_1 \ldots \rho_{n-2} \). The vector fields that we are interested are of the form \( f(x)r_\Omega X \), where \( X \) is a vector field on a neighborhood of \( \Omega \) and \( f \) is a function that is smooth in suitable generalized spherical coordinates. The set of such vector fields is closed under the Lie bracket of vector fields. See \[3\] for more details.
3. A Poincaré type inequality

The rest of this section is devoted to a proof of the Hardy–Poincaré type inequality stated in Theorem 2.1 in dimension is $n = 3$. A proof by induction for arbitrary $n$ is included in [4]. That proof requires, however, the use of curvilinear polyhedral domains on the unit sphere, which explains why it is convenient to use domains more general than the straight polyhedral ones in higher dimensions.

3.1. Proof of the Hardy–Poincaré type inequality for $n = 3$. The idea of the proof is to cover the domain $\Omega$ with open sets $C$ on which the integration simplifies and we can use the usual Poincaré inequality. Then we add the corresponding inequalities.

We shall write $dV = dx dy dz$ for the volume element. Note that $\eta_{n-2}^2 = \eta_1$, since we have fixed our dimension.

Let us consider the apparently weaker inequality

$$\|u\|_{K^2(C)}^2 := \int_C \frac{|u(x)|^2}{\eta_1(x)^2} \, dx \leq C \int_C |
abla u(x)|^2 \, dx, \quad u = 0 \text{ on } C \cap \partial \Omega.$$  \hspace{1cm} (7)

For $C = \Omega$, as in the case of smooth bounded domains, this inequality is immediately seen to be equivalent to our result. Hence we shall concentrate on proving this inequality for suitable $C$, including $C = \Omega$. In fact, the proof of our inequality for $C = \Omega$ will be obtained by adding certain analogues of the inequality (7) for suitable domains $C$.

Assume that $u$ is a smooth function on $\overline{\Omega}$ with $u|_{\partial \Omega} = 0$. We further consider two small enough positive numbers $\epsilon > \delta > 0$, depending only on $\Omega$, such that the following three properties are satisfied:

1. For any edge $e$ of $\Omega$, we consider the right cylindrical domain $Cil_e$ of radius $\delta$ whose axis of symmetry is the line containing the edge $e$ and whose bases intersect $e$ at distance $\epsilon$ from its two vertices. (These two bases are orthogonal to $e$, by assumption.) We assume that $\epsilon$ and $\delta$ were chosen small enough so that the domain $\Omega_e = \Omega \cap Cil_e$ can be characterized, in suitable cylindrical coordinates, by

$$\Omega_e = \{(r, \theta, z) \mid 0 < r < r_e, \quad 0 < \theta < \theta_e, \quad 0 < z < z_e := |e| - 2\epsilon\},$$

where $|e|$ is the length of the edge $e$, and $\eta_1 = r$ on $\Omega_e$. In these cylindrical coordinates, the edge $e$ is on the $z$-axis (in particular, $r = 0$ on $e$).

2. For any vertex $v$ and any edge $e$ containing $v$, we consider the right conical domain $Con_{v,e}$ with vertex $v$ and base the same with one of the bases of $Cil_e$ (the one which closer to the vertex $v$) and whose symmetry axis is the line containing the edge $e$. We assume that $\epsilon$ and $\delta$ were chosen small enough so that the domain $\Omega_{v,e} = \Omega \cap Con_{v,e}$ can be characterized in cylindrical coordinates by

$$\Omega_{v,e} = \{(r, \theta, z) \mid 0 < r < \frac{\delta}{\sqrt{\epsilon^2 + \delta^2}}, \quad 0 < \theta < \theta_e, \quad 0 < z < \epsilon\}$$

and $\eta_1 = r$ on $\Omega_{v,e}$. 3. Let $B(v, t)$ be the open ball of radius $t$ centered at $v$. For any vertex $v$ of $\Omega$, the domain $\Omega_v = \Omega \cap B(v, 2\epsilon)$ can be characterized in (suitable) spherical coordinates centered at $v$ by

$$\Omega_v = \{(\rho, \omega) \mid 0 < \rho < 2\epsilon, \quad \omega \in \omega_v\},$$

where $B(v, r)$ is the three dimensional ball centered at $v$ and of radius $r$, $\omega_v$ is a "polygonal region" on the unit sphere $S^2 \subset R^2$, and $\rho = 0$ corresponds to $v$. 


We shall now prove (4) for $C$ one of the domains $\Omega_e$ or $\Omega_{v,e}$. We need to stress at this point the crucial importance of the relation $\eta_1 = r$ on these domains.

Let $W_a$ be the angle $0 < r < a$ and $0 < \alpha$. Let us next prove first the inequality (7) when $C = \Omega_e$, that is, when $C$ is the cylindrical domain described in cylindrical coordinates $(r, \theta, z)$ as $C := W_a \times (0, z_e)$, with $a = a_e$ and $\alpha = \theta_e$ as above.

Let us consider first a smooth function $v$ on $W_a$ such that $v(r, 0) = v(r, a) = 0$. We then have the one-dimensional Poincaré inequality

$$\int_0^{\theta_e} |v|^2 \, d\theta \leq \frac{\pi}{\theta_e} \int_0^{\theta_e} |\partial_\theta v|^2 \, d\theta. \quad (8)$$

By integrating in polar coordinates we obtain

$$\int_{W_a} \frac{|u|^2}{r^2} \, dx \, dy = \int_{W_a} \frac{|u|^2}{r} \, dr \, d\theta \leq \frac{\pi}{\theta_e} \int_{W_a} \left( \frac{|\partial_\theta u|^2}{r} \right) \, dr \, d\theta. \quad (9)$$

Indeed, using Equation (8) and the formula for the $|\nabla u|$ in cylindrical coordinates, we get

$$\int_{C} \frac{|u|^2}{r^2} \, dV = \int_{0}^{\theta_e} \int_{W_a} \frac{|u|^2}{r} \, dr \, d\theta \, dz \leq \frac{\pi}{\theta_e} \int_{0}^{\theta_e} \int_{W_a} \left( \frac{|\partial_\theta u|^2}{r} \right) \, dr \, d\theta \, dz \leq \frac{\pi}{\theta_e} \int_{0}^{\theta_e} \int_{W_a} \left( \frac{|\partial_\theta u|^2}{r} + r |\partial_r u|^2 + r |\partial_z u|^2 \right) \, dr \, d\theta \, dz = \frac{\pi}{\theta_e} \int_{C} |\nabla u(x)|^2 \, dV. \quad (10)$$

If $C = \Omega_{v,e}$, then the proof proceeds exactly in the same way, except that we replace $W_a$ in the integrals of the last equation with $W_{z_e}$.

Now, if $C = \Omega_v$, we proceed as in Equation (8), using also the formula

$$|\nabla u|^2 = u_\rho^2 + \frac{1}{\rho^2} u_\phi^2 + \frac{1}{\rho^2 \sin^2 \phi} u_\theta^2 \quad (11)$$

and the relation

$$\int_{\omega_v} |u|^2 \, dS \leq C_v \int_{\omega_v} \left( u_\phi^2 + \frac{1}{\sin^2 \phi} u_\phi^2 \right) \, dS \leq C_v \int_{\omega_v} |\nabla (\phi, \theta, \phi)|^2 \, dS,$$

which is nothing but the usual Poincaré’s inequality on $\omega_v$ ($dS$ is the volume element on $\omega_v$). We then obtain (6) = $C_v$)

$$\int_{C} \frac{|u|^2}{\rho^2} \, dV = \int_{0}^{2\pi} \int_{\omega_v} \frac{|u|^2}{\rho^2} \, d\phi \, d\rho \leq C_v \int_{0}^{2\pi} \int_{\omega_v} \left( u_\phi^2 + \frac{1}{\sin^2 \phi} u_\phi^2 \right) \, d\phi \, d\rho \leq C_v \int_{0}^{2\pi} \int_{\omega_v} \left( u_\rho^2 + \frac{1}{\rho^2} u_\phi^2 + \frac{1}{\rho^2 \sin^2 \phi} u_\phi^2 \right) \rho^2 \, d\phi \, d\rho = C_v \int_{C} |\nabla u(x)|^2 \, dV. \quad (12)$$

Adding the inequalities (10) for all $C = \Omega_e$ and all $C = \Omega_{w,e}$, the inequalities (12) for all $C = \Omega_v$, and the usual Poincaré inequality, $\int_{\Omega} |u|^2 \, dV \leq C \int_{\Omega} |\nabla u(x)|^2 \, dV$, we obtain

$$\int_{\Omega} h|u|^2 \, dV \leq C \int_{\Omega} |\nabla u(x)|^2 \, dV,$$

where $h(x)$ is a sum of 1 and terms of the form $r^{-2}$, $\rho^{-2}$. There will be one term $r^{-2}$ each time when $x \in \Omega_e$ or $x \in \Omega_{w,e}$ and one term $\rho^{-2}$ each time when $x \in \Omega_v$. 
Therefore $h \geq Cr^{-2}$ on $\Omega_e$ and on $\Omega_{v,e}$, $h \geq Cr^{-2}$ on $\Omega_v$ and outside all of $\Omega_e \cup \Omega_{v,e}$, and, finally, $h \geq 1 \geq Cr^{-2}$ outside of $\Omega_e \cup \Omega_{v,e} \cup \Omega_v$. Therefore $h \geq Cr^{-2}$ on the whole of $\Omega$. This completes the proof of our inequality for $u$ smooth. By a standard density argument, we obtain the desired result (7) for all functions in $H_0^1(\Omega)$.

4. Proof of the main result

In this section, we prove our main result, Theorem 0.1, assuming the intermediate results stated in the previous sections. We shall follow the pattern of proof from [11]. First, let us notice that Theorem 2.3 allows us to reduce the proof to the case when $g = 0$.

Recall the function $r_\Omega$ introduced at the end of Section 2. We can check directly that $r_\Omega^2 \Delta r_\Omega^\lambda$ depends continuously on $\lambda$ and that $r_\Omega^t K_\mu(\Omega) = K_\mu^{t+1}(\Omega)$ (see [4] for details in the case of higher dimensions). This allows us to reduce the proof to the case $a = 0$, as in [5].

We shall denote by $(u, v) := \int_\Omega u(x)v(x)dx$ the (real) inner product on $L^2(\Omega)$. Let $\mathcal{H} \subset K^1_1(\Omega)$ be the subspace consisting of the functions $u \in K^1_1(\Omega)$ such that $u = 0$ on $\partial \Omega$. Thus $\mathcal{H}$ is the kernel of the trace map $K^1_1(\Omega) \to K^{1/2}_{1/2}(\partial \Omega)$. The Hardy–Poincaré inequality (Theorem 2.1) then gives the following inequality

$$B(u, u) := (\Delta u, u) = \sum_{j=1}^n (\partial_j u, \partial_j u) \geq \epsilon \|u\|^2_{K^1_1(\Omega)}$$

for any $u \in \mathcal{H}$. In particular, $B$ defines a continuous, bilinear, coercive form on $\mathcal{H}$. The assumptions of the Lax-Milgram lemma are therefore satisfied, and hence $\Delta : \mathcal{H} \to \mathcal{H}^* = K^{-1}_{-1}(\Omega)$ is an isomorphism (by this we understand that $\Delta$ is continuous with continuous inverse), by the definition of negative order Sobolev spaces on $\Omega$. This proves the result for $\mu = 0$.

We now prove the result for an arbitrary $\mu \in \mathbb{Z}_+$. Theorem 2.2 and the result we have just proved for $\mu = 0$ give that the map

$$\Delta : K^{\mu+2}_0(\Omega) \cap \{u|_{\partial \Omega} = 0\} \to K^{-\mu}_{-1}(\Omega)$$

is surjective. Since this map is also continuous (Proposition 1.2) and injective (from the case $\mu = 0$), it is an isomorphism by the open mapping theorem. The proof is now complete.

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