Scattering theory for nonlinear Schrödinger equation with inverse square potential

Jiqiang Zheng

Université Nice Sophia-Antipolis

Based on joint work with: Changxing Miao (IAPCM) and Junyong Zhang (BIT)

February 2-6, 2015 Saint-Étienne-de-Tinée
Let $d \geq 3$, $1 + \frac{4}{d} < p \leq 1 + \frac{4}{d-2}$, consider

\[
\begin{cases}
(i\partial_t - P_a)u = |u|^{p-1}u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d \\
u(0, x) = u_0(x),
\end{cases}
\]  

(1.1)

where $u(t, x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$, and we denote $P_a$ by the Friedrichs extension of $-\Delta + \frac{a}{|x|^2}$ with $a > -\frac{(d-2)^2}{4}$. The elliptic operator $P_a$ is the self-adjoint and

\[
\sigma(P_a) = \sigma_{ac}(P_a) = [0, +\infty), \quad \|P_a^{\frac{1}{2}} f\|_{L^2_{\mathbb{R}^d}} \approx \|(-\Delta)^{\frac{1}{2}} f\|_{L^2(\mathbb{R}^d)}.
\]

The Schrödinger flows for the elliptic operator $P_a$ is of interest in quantum mechanics (see [4, 5]).
The solution to (1.1) is left invariant by the scaling

\[ u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) \triangleq u_\lambda(t, x), \quad \lambda > 0. \] (1.2)

Furthermore,

\[ \| u_\lambda(t, \cdot) \|_{\dot{H}^{s_c}_x(\mathbb{R}^d)} = \| u(\lambda^2 t, \cdot) \|_{\dot{H}^{s_c}_x(\mathbb{R}^d)}, \quad s_c = \frac{d}{2} - \frac{2}{p - 1}. \] (1.3)
The solution to (1.1) is left invariant by the scaling

\[ u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) \equiv u_\lambda(t, x), \quad \lambda > 0. \quad (1.2) \]

Furthermore,

\[ \|u_\lambda(t, \cdot)\|_{\dot{H}^s_x(\mathbb{R}^d)} = \|u(\lambda^2 t, \cdot)\|_{\dot{H}^s_x(\mathbb{R}^d)}, \quad s_c = \frac{d}{2} - \frac{2}{p-1}. \quad (1.3) \]

**Mass** \[ M[u(t)] = \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx = M(u_0); \]

**Energy** \[ E[u(t)] = \int \left( \frac{1}{2} |\nabla u(t, x)|^2 + \frac{a}{2} \frac{|u|^2}{|x|^2} + \frac{1}{p+1} |u(t, x)|^{p+1} \right) \, dx = E(u_0). \]

When \( p = 1 + \frac{4}{d-2} \), i.e. \( s_c = 1 \), (1.1) is called **energy-critical**; \( s_c < 1 \), **energy-subcritical**; \( p = 1 + \frac{4}{d} \), mass-critical.
Strichartz estimate

(1) $a = 0$.

$$\|e^{it\Delta} f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \leq C \|f\|_{L_x^2},$$

where $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q, r, d) \neq (2, \infty, 2)$. Dispersive estimate

$$\|e^{it\Delta} f\|_{L_x^\infty} \leq C |t|^{-\frac{d}{2}} \|f\|_{L_x^1}.$$

(2) $H = -\Delta + V(x)$, $V(x)$ is less singular than the inverse square potential at the origin, for instance, when it belongs to the Kato class; see D’Ancona-Fanelli-Vega-Visciglia[8], Schlag[15], Schlag-Soffer-Staubach[16, 17], etc.

(3) We do not have the dispersive estimate for $e^{itP_a}$. When $a < 0$, the classical dispersive estimate does not hold for the wave equation with inverse square potential.
\[ \| e^{-itP} f \|_{L^q_t L^r_x} \leq C \| f \|_{L^2_x}. \] (1.4)

Since \( u(t, x) := e^{-itP} f \) solves

\[ i \partial_t u + \Delta u = \frac{a}{|x|^2} u, \quad u(0, x) = f(x), \]

we have

\[ u(t, x) = e^{it\Delta} f + ia \int_0^t e^{i(t-s)\Delta} (|x|^{-2} u)(s) ds. \]
\[ \|e^{-itP_a}f\|_{L_t^q L_x^r} \leq C\|f\|_{L_x^2}. \] (1.4)

Since \( u(t, x) := e^{-itP_a}f \) solves

\[ i\partial_t u + \Delta u = \frac{a}{|x|^2} u, \quad u(0, x) = f(x), \]

we have

\[ u(t, x) = e^{it\Delta}f + ia \int_0^t e^{i(t-s)\Delta}(|x|^{-2}u)(s)ds. \]

Then,

\[ \left\| \int_0^t e^{i(t-s)\Delta}(|x|^{-2}u)(s)ds \right\|_{L_t^{2d}\frac{2d}{d-2} L_x^{\frac{2d}{d-2}}} \leq \left\| \int_0^t e^{i(t-s)\Delta}(|x|^{-2}u)(s)ds \right\|_{L_t^2 L_x^{\frac{2d}{d-2},2}} \leq C\||x|^{-2}u\|_{L_t^2 L_x^{\frac{2d}{d-2},2}} \leq C\||x|^{-1}\|_{L_t^{d,\infty}}\||x|^{-1}u\|_{L_t^2 L_x^{1,\infty}} \leq C\|f\|_{L_x^2}, \]

where we use local smoothing estimate \((d \geq 3)\)

\[ \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{|P_a^{\frac{1}{4} - \alpha} u|^2}{|x|^{1+4\alpha}} \, dxdt \leq C\|f\|_{L_x^2}^2, \quad 0 < \alpha < \frac{1}{4} + \frac{1}{2} \sqrt{\frac{(d-2)^2}{4}} + a. \]
Background for Energy-subcritical NLS with $a = 0$

\[
\begin{aligned}
    i\partial_t u + \Delta u &= |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
    u(0, x) &= u_0(x) \in H^1(\mathbb{R}^d),
\end{aligned}
\]

(1.5)

with $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$ and $d \geq 3$.

\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} u_\pm\|_{H^1_x(\mathbb{R}^d)} = 0.
\]

(1) Ginibre-Velo[9], 1985, Morawetz estimate

\[
\iint_{\mathbb{R} \times \mathbb{R}^d} \frac{|u(t, x)|^{p+1}}{|x|} \, dt \, dx \lesssim \|u\|^2_{L^\infty_t H^\frac{1}{2}_x} \lesssim C(M(u_0), E(u_0)).
\]

(2) There is another simple proof by using the following interaction Morawetz estimate (Colliander-Keel-Staffilani-Takaoka-Tao[6], 2003)

\[
\| |\nabla|^{-\frac{d-3}{4}} u\|^4_{L^4_t L^\infty_x(\mathbb{R}^d)} \lesssim C\|u_0\|^3_{L^2_x} \|u\|^2_{L^\infty_t L^1_x(\mathbb{R}^d)}, \quad d \geq 3.
\]
Interaction Morawetz estimate

Proposition 1 (Morawetz-type estimates)

Let $u$ be an $H^{\frac{1}{2}}$-solution to (1.1) on the spacetime slab $I \times \mathbb{R}^d$, the dimension $d \geq 3$ and $a > \frac{1}{4} - \frac{(d-2)^2}{4}$, then we have

$$\left\| |\nabla|^{\frac{3-d}{4}} u \right\|^2_{L^4_{t,x}(I \times \mathbb{R}^d)} \leq C \left\| u(t_0) \right\|_{L^2} \sup_{t \in I} \left\| u(t) \right\|_{\dot{H}^{\frac{1}{2}}}. \quad (1.6)$$

Proof: Consider the quadratic Morawetz quantity $M(t) := \partial_t \langle |u|^2, |x| \ast |u|^2 \rangle$.

$$\left\| |\nabla|^{\frac{3-d}{4}} u \right\|^4_{L^4_{t,x}(I \times \mathbb{R}^d)} \leq C \left\| u(t_0) \right\|_{L^2}^2 \left( \sup_{t \in I} \left\| u(t) \right\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_I \int_{\mathbb{R}^d} \frac{|u(t,x)|^2}{|x|^3} dx dt \right).$$

And consider the Virial quantity

$$V(t) = \text{Im} \int_{\mathbb{R}^d} \bar{u} \frac{x}{|x|} \cdot \nabla u \, dx,$$

we deduce that

$$\int_I \int_{\mathbb{R}^d} \frac{|u(t,x)|^2}{|x|^3} dx dt \lesssim \sup_{t \in I} \left\| u(t) \right\|_{\dot{H}^{\frac{1}{2}}}^2. \quad (1.7)$$
Sobolev norm equivalence

From Hardy's inequality \( \left\| |x|^{-1} f \right\|_{L_x^2(\mathbb{R}^d)} \leq \frac{2}{d-2} \left\| f \right\|_{\dot{H}^1(\mathbb{R}^d)} \) and interpolation, it is easy to see that for \( a > -\frac{(d-2)^2}{4} \)

\[
\left\| P_a^s f \right\|_{L_x^2(\mathbb{R}^d)} \approx \left\| f \right\|_{\dot{H}^s(\mathbb{R}^d)}, \quad -1 \leq s \leq 1.
\]

Theorem 1 (Equivalence of Sobolev norm)

Let \( a > -\frac{(d-2)^2}{4} \), \( \sigma = \frac{d-2}{2} - \sqrt{\frac{(d-2)^2}{4} + a} \). Let \( 0 \leq s < 2 \) and \( \max \{1, \frac{d}{d-\sigma}\} < p < \infty \), then, we have for \((s + \sigma)p < d\)

\[
\left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L_p(\mathbb{R}^d)} \lesssim d, p, s \left\| P_a^s f \right\|_{L_p(\mathbb{R}^d)}, \quad \forall f \in C_c^\infty(\mathbb{R}^d),
\]

(1.8)

and for \((s + \sigma)p < d \) and \( sp < d \)

\[
\left\| P_a^s f \right\|_{L_p(\mathbb{R}^d)} \lesssim d, p, s \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L_p(\mathbb{R}^d)}, \quad \forall f \in C_c^\infty(\mathbb{R}^d),
\]

(1.9)

(1.8) with \( s = 1 \) corresponds to the boundedness of Riesz transform in Hassell-Lin [10].
Main result for energy-subcritical NLS

Theorem 2 (Zhang-Z, 14, JFA)

Let $d \geq 3$, $\lambda_d = \frac{(d-2)^2}{4}$ and let $p \in \left(1 + \frac{4}{d}, 1 + \frac{4}{d-2}\right)$. Assume that $a > -\frac{4p}{(p+1)^2} \lambda_d$ and $u_0 \in H^1(\mathbb{R}^d)$. Then the solution $u$ to (1.1) is global. Moreover, if $a \geq \frac{4}{(p+1)^2} - \lambda_d$ for $d \geq 4$, and $a \geq 0$ for $d = 3$, then, the solution $u$ scatters in the sense that there exists a unique $u_\pm \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t \to \pm \infty} \|u(t) - e^{-itP_a} u_\pm\|_{H^1_x(\mathbb{R}^d)} = 0.$$

Remark 1

This result allows some negative inverse-square potential when $d \geq 4$. 
Energy-critical with $a = 0, u_0 \in \dot{H}^1(\mathbb{R}^d) \Rightarrow u$ scatters.

The main difficulty in energy-critical case stems from the fact that none of the known monotonicity formulas (i.e. Morawetz estimates) for NLS scale like the energy ($\dot{H}^1_\times(\mathbb{R}^d)$).

| $d$ | radial | non-radial |
|-----|--------|------------|
| 3   | Bourgain [1] | I-term[7], Killip-Visan [11] |
| 4   | Tao [18] | Ryckman-Visan [14], Visan [21] |
| $\geq 5$ | Tao [18] | |

1. (Bourgain, 1999): ‘induction on energy’ technique paved the way for how to proceed in such a scenario: by finding a bubble of concentration inside a solution, one can introduce a characteristic length scale into the problem, bringing the available Morawetz estimates back into play (despite their non-critical scaling).

   $\textbf{space localized Morawetz : } \int \int_I |x| \leq C \| u \|^{6}_{L^6} \left( \frac{|u(t,x)|^2}{|x|} \right) dtdx \leq \| u \|^{\frac{1}{2}} \| u \|^{\frac{1}{2}} \| \hat{u} \|^{\frac{1}{2}} \| \hat{u} \|^{\frac{1}{2}} C(\| u \|_{\dot{H}^1_\times} C(\| u \|_{L^\infty \dot{H}^1_\times}$.)

2. (Colliander–Keel–Staffilani–Takaoka–Tao, 2004): Induction on energy $\oplus$

   $\textbf{frequency-localized interaction Morawetz: } \int \int |P_{\geq N} u(t,x)|^2 |P_{\geq N} u(t,y)|^2 \frac{1}{|x-y|^3} dxdy.$

Jiqiang Zheng

NLS with inverse square potential
Main result for energy-critical NLS

\[ \begin{cases} (i \partial_t - P_a)u = |u|^4 u, & (t, x) \in \mathbb{R} \times \mathbb{R}^3 \\ u(0, x) = u_0(x) \in \dot{H}^1(\mathbb{R}^3). \end{cases} \] (1.10)

Theorem 3 (Miao-Zhang-Z, 14)

Let \( a \geq -\frac{7 \sqrt{7} - 10}{54} > -\frac{1}{4} \). Given \( u_0 \in \dot{H}^1(\mathbb{R}^3) \), and assume in addition that \( u_0 \) is radial when \( a < 0 \), then there exists a unique global solution \( u \) to (1.10) satisfying

\[ \int_{\mathbb{R}} \int_{\mathbb{R}^3} |u(t, x)|^{10} \, dx \, dt < +\infty. \] (1.11)

Thus, the solution \( u \) scatters.

We will show by the concentration-compactness approach to induction on energy.
Main difficulty: Gaussian bound of the heat kernel $e^{-tP_a}$ fail to hold when $a$ is negative.

**Lemma 1 (Heat kernel boundedness, Liskevich-Sobol[12], Milman-Semenov[13])**

Assume $a > -\frac{(d-2)^2}{4}$ and $\sigma = \frac{d-2}{2} - \sqrt{\frac{(d-2)^2}{4}} + a$, let $H(t, x, y)$ be the kernel of the operator $e^{-tP_a}$. Then, for all $t > 0$ and all $x, y \in \mathbb{R}^d \setminus \{0\}$

$$C_1\left(\frac{\sqrt{t}}{|x|} \vee 1\right)^{\sigma} \left(\frac{\sqrt{t}}{|y|} \vee 1\right)^{\sigma} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{c_1t}} \leq H(t, x, y) \leq C_2\left(\frac{\sqrt{t}}{|x|} \vee 1\right)^{\sigma} \left(\frac{\sqrt{t}}{|y|} \vee 1\right)^{\sigma} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{c_2t}}.$$

**Lemma 2 (Riesz kernels)**

Let $s \in (0, d)$ and let $\mathcal{L}_{a}^{-\frac{s}{2}}(x, y)$ be the kernel of the operator $P_a^{-\frac{s}{2}}$. Then, we have for $x, y \in \mathbb{R}^d \setminus \{0\}$

$$\mathcal{L}_{a}^{-\frac{s}{2}}(x, y) = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-tP_a}(x, y) t^{\frac{s}{2}} \frac{dt}{t} \approx |x - y|^{s-d} \left(\frac{4|x|}{|x-y|} \wedge \frac{4|y|}{|x-y|} \wedge 1\right)^{-\sigma}, \quad (2.1)$$

when $d - s - 2\sigma \notin \{0, -2, -4, \cdots\}$. Here $A \wedge B := \min\{A, B\}$, and $A \vee B := \max\{A, B\}$. 
Lemma 3 (Hardy inequality for $P_a$)

Let $p > \max\{1, \frac{d}{d-\sigma}\}$, $(\sigma + s)p < d$ and $0 < s < d$. Then there exists a constant $C$ such that for all $f$ such that $P_{\frac{s}{2}}f \in L^p(\mathbb{R}^d)$,

$$\left\| x^{-s} f(x) \right\|_{L^p(\mathbb{R}^d)} \leq C \left\| P_{\frac{s}{2}} f \right\|_{L^p(\mathbb{R}^d)}.$$  \hspace{1cm} (2.2)

The estimate (2.2) is sharp in sense that it fails for $(s + \sigma)p \geq d$ or $p \leq \frac{d}{d-\sigma}$ with $\sigma > 0$. 

Jiqiang Zheng

NLS with inverse square potential
Lemma 3 (Hardy inequality for $P_a$)

Let $p > \max\{1, \frac{d}{d-\sigma}\}$, $(\sigma + s)p < d$ and $0 < s < d$. Then there exists a constant $C$ such that for all $f$ such that $P_{\frac{s}{a}}^s f \in L^p(\mathbb{R}^d)$,

$$\left\| |x|^{-s} f(x) \right\|_{L^p(\mathbb{R}^d)} \leq C \left\| P_{\frac{s}{a}}^s f \right\|_{L^p(\mathbb{R}^d)}.$$  \hspace{1cm} (2.2)

The estimate (2.2) is sharp in sense that it fails for $(s + \sigma)p \geq d$ or $p \leq \frac{d}{d-\sigma}$ with $\sigma > 0$.

Theorem 4 (Multiplier theorem)

Let $\sigma$ be as in Lemma 1 and let $m : [0, \infty) \to \mathbb{C}$ satisfy

$$|\partial^k \lambda m(\lambda)| \leq \lambda^{-k}, \quad \forall \ 0 \leq k \leq \left[\frac{d}{2}\right] + 1.$$  \hspace{1cm} (2.3)

Then $m(\sqrt{P_a})$, which we define via the $L^2$ functional calculus, extends uniquely from $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to a bounded operator on $L^p(\mathbb{R}^d)$, for all $1 < p < \infty$ when $\sigma \leq 0$, and for all $r_0 < p < r_0' := \frac{d}{\sigma}$ when $0 < \sigma < \nu := \left[\frac{d}{2}\right] - \frac{d}{2} + 1$. 

Jiqiang Zheng  
NLS with inverse square potential
Let $\phi : [0, \infty) \to [0, 1]$ be a smooth positive function obeying

$$\phi(\lambda) = 1 \quad \text{for} \quad 0 \leq \lambda \leq 1 \quad \text{and} \quad \phi(\lambda) = 0 \quad \text{for} \lambda \geq 2.$$

For each dyadic number $N \in 2^\mathbb{Z}$, we define

$$\phi_N(\lambda) := \phi(\lambda/N) \quad \text{and} \quad \psi_N(\lambda) := \phi_N(\lambda) - \phi_{N/2}(\lambda);$$

It is clear that $\{\psi_N(\lambda)\}_{N \in \mathbb{Z}}$ is a partition of unity for $(0, \infty)$. Define the Littlewood–Paley projections by

$$P_{a \leq N} := \phi_N(\sqrt{P_a}), \quad P_N := \psi_N(\sqrt{P_a}), \quad \text{and} \quad P_{a > N} := I - P_{a \leq N}.$$ 

We define another family of Littlewood–Paley projections as follows

$$\tilde{P}_{a \leq N} := e^{-P_a/N^2}, \quad \tilde{P}_N := e^{-P_a/N^2} - e^{-4P_a/N^2}, \quad \text{and} \quad \tilde{P}_{a > N} := I - \tilde{P}_{a \leq N}.$$
Lemma 4 (Bernstein estimates)

For \( s \in \mathbb{R}, 1 < p \leq q \leq \infty \) when \( a \geq 0 \) and \( r_0 < p < q < r_0' \) when \( a < 0 \). Then,

1. All operators \( P_a^N, P_N^a, \tilde{P}_N^a \) and \( \tilde{P}_a^N \) are bounded in \( L^p \);
2. \( P_a^N, P_N^a, \tilde{P}_N^a \) and \( \tilde{P}_a^N \) are bounded from \( L^q \) to \( L^p \) with \( O(N^d(\frac{1}{p} - \frac{1}{q})) \);
3. \( N^s \| P_N^a f \|_{L^p(\mathbb{R}^d)} \sim \left\| (P_a)^{\frac{s}{2}} P_N^a f \right\|_{L^p(\mathbb{R}^d)}, \forall f \in C_c^\infty(\mathbb{R}^d). \)

Theorem 5 (Square function estimates)

Fix \( s \geq 0, 1 < p < \infty \) when \( a \geq 0 \) and \( r_0 < p < r_0' \) when \( a < 0 \). Then for any \( f \in C_c^\infty(\mathbb{R}^d) \),

\[
\left\| \left( \sum_{N \in 2\mathbb{Z}} N^{2s} |P_N^a f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \sim \left\| P_a^{\frac{s}{2}} f \right\|_{L^p(\mathbb{R}^d)} \sim \left\| \left( \sum_{N \in 2\mathbb{Z}} N^{2s} |\tilde{P}_N^a|^k f^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)}, k \geq 1, 2k > s.
\]

\[
\left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^p_x} - \left\| P_a^{\frac{s}{2}} f \right\|_{L^p_x} \lesssim \left\| \left( \sum_{N \in 2\mathbb{Z}} N^{2s} |\tilde{P}_N^a|^2 f^2 \right)^{\frac{1}{2}} - \left( \sum_{N \in 2\mathbb{Z}} N^{2s} |\tilde{P}_N^a|^2 f^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \lesssim |||x|^{-s} f(x)|||_{L^p(\mathbb{R}^d)}. 
\]
Lemma 5 (Modified Morawetz inequality)

For $a \geq -\frac{7\sqrt{7} - 10}{54}$. Let $I$ be a time interval and let $u$ be a $\dot{H}^1_x(\mathbb{R}^3)$-solution to (1.10) on $I$. Then for any $C \geq 1$, we have

$$
\int_I \int_{|x| \leq C|I|^1/2} \frac{|u(t, x)|^6}{|x|^{1-\gamma}} \, dx \, dt \lesssim C^{1+\gamma}|I|^{1+\gamma/2} \sup_{t \in I} \|u(t)\|_{\dot{H}^1_x}, \quad \gamma = \frac{\sqrt{7} - 1}{3},
$$

where the implicit constant depends only on the energy of $u$.

**Proof:** Consider the Morawetz quantity

$$
M(t) := \text{Im} \int_{\mathbb{R}^3} \overline{v} |x|^\gamma \frac{x}{|x|} \cdot \nabla v dx, \quad v(t, x) = u(t, x) \chi_R(x).
$$

Error term

$$
\left[2a + \gamma - \frac{\gamma^2(1+\gamma)}{2}\right] \int_{\mathbb{R}^3} \frac{|v|^2}{r^{3-\gamma}} dx \geq 0.
$$

$$
\max_{\gamma \geq 0} \left[\frac{\gamma}{2} - \frac{\gamma^2(1+\gamma)}{4}\right] = \frac{7\sqrt{7} - 10}{54}.
$$
Local smoothing estimate

Lemma 6 (Local smoothing)

For $-\frac{1}{4} < a < 0$. Let $u = e^{-itP_a} u_0$. Then

$$
\int_{-T}^{T} \int_{|x| \leq R} |\nabla u(t, x)|^2 \, dx \, dt \leq R \|u_0\|_{L_x^2} \|\nabla u_0\|_{L_x^2} + \|u_0\|_{L_x^2}^2, \quad (2.5)
$$

uniformly for $T > 0$ and $R > 0$. For $a \geq 0$,

$$
\int_{-T}^{T} \int_{|x-z| \leq R} |\nabla u(t, x)|^2 \, dx \, dt \leq R \|u_0\|_{L_x^2} \|\nabla u_0\|_{L_x^2}, \quad \forall \ z \in \mathbb{R}^3. \quad (2.6)
$$

Error term can be controlled by local smoothing estimate

$$
\int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} \, dx \, dt \leq C \|u_0\|_{L_x^2(\mathbb{R}^3)}^2, \quad a > -\frac{1}{4}.
$$

Remark: The local smoothing estimate does guarantee local energy decay, it falls short of fulfilling the role of a dispersive estimate.
Outline of energy-critical: concentration-compactness

For $0 \leq E < +\infty$, define

$$L(E) := \sup \{ S_I(u) : u : I \times \mathbb{R}^3 \rightarrow \mathbb{C} \text{ s.t } E(u_0) \leq E \},$$

\[ S_I(u) = \|u\|_{L^{10}_{t,x}(I \times \mathbb{R}^3)}. \]

By the small data theory, we know that $L(E)$ is finite for $E$ sufficiently small.

Define

$$E_c = \sup \{ E_0, L(E_0) < +\infty \},$$

then, Theorem 3 is equivalent to $E_c = +\infty$. 
For $0 \leq E < +\infty$, define

$$L(E) := \sup \left\{ S_I(u) : u : I \times \mathbb{R}^3 \to \mathbb{C} \text{ s.t } E(u_0) \leq E \right\},$$

where $S_I(u) = \|u\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)}$.

By the small data theory, we know that $L(E)$ is finite for $E$ sufficiently small.

Define

$$E_c = \sup \{ E_0, L(E_0) < +\infty \},$$

then, Theorem 3 is equivalent to $E_c = +\infty$.

We argue by contradiction. If $E_c < +\infty$, then we know that there exist a sequence $\{u_n(t)\}$ such that as $n \to \infty$

$$E(u_n) \to E_c, \quad \text{and} \quad S_{I_n}(u_n) = \infty.$$
Theorem 6 ($\dot{H}^1_a(\mathbb{R}^3)$ linear profile decomposition for $a < 0$)

Assume $a < 0$. Let $\{f_n\}$ be a bounded radial sequence in $\dot{H}^1_a(\mathbb{R}^3)$. After passing to a subsequence, there exist $J^* \in \{0, 1, 2, \ldots, \infty\}$, $\{\phi^j\}_{j=1}^{J^*} \subset \dot{H}^1_a(\mathbb{R}^3)$, $\{\lambda^j_n\}_{j=1}^{J^*} \subset (0, \infty)$, and $\{t^j_n\}_{j=1}^{J^*} \subset \mathbb{R}$ such that for each finite $0 \leq J \leq J^*$, we have the decomposition

$$f_n = \sum_{j=1}^J \phi^j_n + w^j_n, \quad \phi^j_n(x) = G^j_n[e^{-it^j_nP_a}\phi^j](x),$$

with $[G^j_n f](x) := (\lambda^j_n)^{-\frac{1}{2}}f\left(\frac{x}{\lambda^j_n}\right)$ and $w^j_n \in \dot{H}^1_a(\mathbb{R}^3)$ satisfying

$$\lim_{J \to J^*} \lim_{n \to \infty} \|e^{-it^j_nP_a}w^j_n\|_{L_{t,x}^{10}(\mathbb{R} \times \mathbb{R}^3)} = 0, \quad \lim_{n \to \infty} \left\{\|f_n\|_{\dot{H}^1_a(\mathbb{R}^3)}^2 - \sum_{j=1}^J \|\phi^j_n\|_{\dot{H}^1_a(\mathbb{R}^3)}^2 - \|w^j_n\|_{\dot{H}^1_a(\mathbb{R}^3)}^2\right\} = 0,$$

asymptotic orthogonality:

$$\lim_{n \to \infty} \log \frac{\lambda^j_n}{\lambda^k_n} + \frac{|t^j_n(\lambda^j_n)^2 - t^k_n(\lambda^k_n)^2|}{\lambda^j_n \lambda^k_n} = \infty, \quad j \neq k.$$
Lemma 7 (Refined Strichartz estimate for \( a < 0 \))

Assume \(-\frac{1}{4} < a < 0\), \( 6 < r < r' = \frac{3}{\sigma} = \frac{6}{1-\sqrt{1+4a}} \) and \( \frac{2}{q} + \frac{3}{r} = \frac{1}{2} \). Let \( f \in \dot{H}^1_a(\mathbb{R}^3) \). Then we have

\[
\| e^{-itP_a} f \|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^3)} \lesssim \| f \|_{\dot{H}^1} \sup_{N \in 2\mathbb{Z}} \| e^{-itP_a} P_N f \|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^3)}^{\frac{1}{2}}.
\]

Lemma 8 (Inverse Sobolev embedding)

Assume \(-\frac{1}{4} < a < 0\) and the function \( f \) is radial. Let \( 6 < r < \frac{3}{\sigma} \) and

\[
\| P_N f \|_{L^r_x} \gtrsim \eta N^{\frac{1}{2}-\frac{1}{r}}, \quad \text{and} \quad \| f \|_{\dot{H}^1} \lesssim A. \tag{2.7}
\]

Then, we have

\[
\int_{|x| \leq \frac{1}{N}} |f(x)|^6 dx \gtrsim \eta \left( \frac{\eta}{A} \right)^{\frac{36}{r-6}}. \tag{2.8}
\]
Proposition 2 (critical element)

Assume that $E_c < +\infty$. Then, there exists a radial maximal life-span solution $u_c : I \times \mathbb{R}^3 \to \mathbb{C}$ satisfying

$$E(u_c) = E_c, \quad \text{and} \quad S_I(u_c) = +\infty.$$  \hspace{1cm} (2.9)

Moreover, there exist $N(t) \geq 1$ such that the set

$$K := \left\{ \frac{1}{N(t)^{\frac{1}{2}}} u(t, \frac{x}{N(t)}) \mid t \in I \right\}$$  \hspace{1cm} (2.10)

is precompact in $\dot{H}^1_x(\mathbb{R}^3)$.

Our goal is to show that the above critical element does not exist.
Remark 2

The Arzelà–Ascoli theorem tells us that a family of functions $\mathcal{F}$ is precompact in $L^2_x(\mathbb{R}^3)$ if and only if it is norm-bounded and there exists a compactness modulus function $C(\eta)$ such that

$$\int_{|x| \geq C(\eta)} |f(x)|^2 \, dx + \int_{|\xi| \geq C(\eta)} |\hat{f}(\xi)|^2 \, d\xi \leq \eta$$

uniformly for $f \in \mathcal{F}$. Thus, from the fact that the set $K := \left\{ \frac{1}{N(t)^{1/2}} u(t, \frac{x}{N(t)}) \mid t \in I \right\}$ is precompact in $\dot{H}^1_x(\mathbb{R}^3)$, there exists a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{|x| \geq \frac{C(\eta)}{N(t)}} |\nabla u(t, x)|^2 \, dx + \int_{|\xi| \geq C(\eta)N(t)} |\xi|^2 |\hat{u}(t, \xi)|^2 \, d\xi \leq \eta \quad (2.11)$$

for all $t \in I$. By the Sobolev embedding $\dot{H}^1_x(\mathbb{R}^3) \hookrightarrow L^6_x(\mathbb{R}^3)$ and $\inf N(t) \geq 1$,

$$\inf_{t \in I} \int_{|x| \leq C(u)} |u(t, x)|^6 \, dx \gtrsim u \quad 1. \quad (2.12)$$
Extinction of the global critical solutions

From the above remark, we have $R \gg 1$

\[ \int_{|x| \leq R} |u(t, x)|^6 \, dx \geq 1, \text{ uniformly for } t \in \mathbb{R}. \] (2.13)

Integrating over a time interval of length $|I| \geq 1$, we obtain

\[ |I| \lesssim R^{1-\gamma} \int_{I} \int_{|x| \leq R} \frac{|u(t, x)|^6}{|x|^{1-\gamma}} \, dx \, dt \leq R^{1-\gamma} \int_{I} \int_{|x| \leq R} \frac{|u(t, x)|^6}{|x|^{1-\gamma}} \, dx \, dt. \]

The Morawetz inequality (Lemma 5) gives

\[ \int_{I} \int_{|x| \leq R} \frac{|u(t, x)|^6}{|x|^{1-\gamma}} \, dx \, dt \lesssim R^{1+\gamma} |I|^{\frac{1+\gamma}{2}}. \]

\[ |I| \lesssim R^2 |I|^{\frac{1+\gamma}{2}}, \quad \gamma = \frac{\sqrt{7}-1}{3} < 1. \]

Taking $|I|$ sufficiently large depending on $R$ and $E_c$, we derive a contradiction.

Jiqiang Zheng

NLS with inverse square potential
Extinction of finite time blowup solution

We argue by contradiction. Assume that \( T_{\text{max}} := \sup I < +\infty \).

\[
\liminf_{t \uparrow T_{\text{max}}} N(t) = \infty, \quad \limsup_{t \uparrow T_{\text{max}}} \int_{|x| \leq R} |u(t, x)|^2 \, dx = 0, \quad \forall \ R > 0. \tag{2.14}
\]

For \( t \in I \), define

\[
M_R(t) = \int_{|x| \leq R} \phi \left( \frac{|x|}{R} \right) |u(t, x)|^2 \, dx, \quad \phi(r) = \begin{cases} 1, & r \leq 1, \\ 0, & r \geq 2, \end{cases} \tag{2.15}
\]

We derive

\[
\|u_0\|_2^2 \leq |T_{\text{max}} - t_1|.
\]

Letting \( t_1 \nearrow T_{\text{max}} \), we obtain \( u_0 \equiv 0 \). Therefore, \( u(t) \equiv 0 \), which contradicts with

\[
\|u\|_{L^{10}_{t,x} (\mathbb{R}^3)} = \infty.
\]
Outlook: Focusing energy-critical NLS

Let \( d \geq 3 \), we consider

\[
\begin{aligned}
(i \partial_t - P_a)u &= -|u|^{\frac{4}{d-2}} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d \\
\begin{cases}
u(0, x) = u_0(x) & \in \dot{H}^1(\mathbb{R}^d).
\end{cases}
\end{aligned}
\] (3.1)

It has been proved that in the defocusing case, globally well-posed and scattering for any \( u_0 \in \dot{H}^1 \).

In the focusing case, there are known counterexamples to global well-posedness and scattering for (3.1). Let \( W \) be the positive solution to the elliptic equation (S. Terracini [19])

\[
-\Delta W + \frac{a}{|x|^2} W = |W|^{\frac{4}{d-2}} W.
\]

Then \( u(t, x) = W(x) \) is a solution to (3.1) that blows up both forward and backward in time.

What is the threshold for blowup and GWP?
J. Bourgain, *Global well-posedness of defocusing 3D critical NLS in the radial case*. J. Amer. Math. Soc., 12 (1999), 145–171. MR1626257.

N. Burq, F. Planchon, J. Stalker and A. S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*. J. Funct. Anal., 203(2003), 519-549.

N. Burq, F. Planchon, J. Stalker and S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay*. Indiana Univ. Math. J., 53 (2004), 1665-1680.

H. E. Camblong, L. N. Epele, H. Fanchiotti and C. A. Garcia Canal, *Quantum anomaly in molecular physics*. Phys. Rev. Lett., 87(2001), 220402(4).

K. M. Case, *Singular potentials*. Physical Rev.(2), 80: 797-806, 1950.

J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equations on \( \mathbb{R}^3 \)*. Comm. Pure. Appl. Math., 57(2004), 987-1014.

J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in \( \mathbb{R}^3 \)*. Ann. Math., 167 (2008), 767–865.
Reference II

- P. D'Ancona, L. Fanelli, L. Vega and N. Visciglia, *Endpoint Strichartz estimates for the magnetic Schrödinger equation*. J. Funct. Anal., 258(2010), 3227-3240.

- J. Ginibre and G. Velo, *Scattering theory in energy space for a class nonlinear Schrödinger equations*. J. Math. Pure Appl., 64(1985), 363-401.

- A. Hassell and P. Lin, *The Riesz transform for homogeneous Schrödinger operators on metric cones*. Revista Mat. Iberoamericana, 30 (2014), 477-522.

- R. Killip and M. Visan, *Global well-posedness and scattering for the defocusing quintic NLS in three dimensions*. Analysis and PDE 5 (2012), 855–885.

- V. Liskevich and Z. Sobol, *Estimates of integral kernels for semigroups associated with second order elliptic operators with singular coefficients*. Potential Anal., 18(2003), 359-390.

- P. D. Milman and Yu. A. Semenov, *Global heat kernel bounds via desingularizing weights*. J. Funct. Anal., 212(2004), 373-398.

- E. Ryckman and M. Visan, *Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$*. Amer. J. Math. 129 (2007), 1-60. MR2288737
Reference III

W. Schlag, *Dispersive estimates for Schrödinger operators: a survey.* Ann. of Math., 163(2007), 255-285.

W. Schlag, A. Soffer and W. Staubach, *Decay for the wave and Schrödinger evolutions on manifolds with conical ends, I.* Trans. Amer. Math. Soc., 362(2010), 19-52.

W. Schlag, A. Soffer and W. Staubach, *Decay for the wave and Schrödinger evolutions on manifolds with conical ends, II.* Trans. Amer. Math. Soc., 362(2010), 289-318.

T. Tao, *Global well-posedness and scattering for higher-dimensional energy-critical non-linear Schrödinger equation for radial data.* New York J. of Math. 11 (2005), 57–80. MR2154347

S. Terracini, *On positive entire solutions to a class of equations with a singular coefficient and critical exponent.* Advances in Differential Equations, 1(1996), 241-264.

J. L. Vazquez and E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential.* J. Funct. Anal., 173(2000), 103-153.

M. Visan, *The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions.* Duke Math. J., 138 (2007) 281–374. MR2318286.
Thanks for your attention!