ASYMPTOTIC SYMMETRIES IN THE BV-BFV FORMALISM

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Abstract. We show how to derive asymptotic charges for field theories on manifolds with "asymptotic" boundary, using the BV-BFV construction. We also prove that the conservation of these charges follows naturally from the vanishing of the boundary action obtained from the BV-BFV procedure. We analyse in detail the case of electrodynamics and the massless scalar field. In particular, we address the controversy present in the literature, whether asymptotic charges really come from large gauge symmetries, or are they simply a consequence of how solutions to the wave equation behave asymptotically. Since the BV formalism describes symmetries and equations of motion in a unified way, the new interpretation that we propose has features of both these viewpoints.

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Introduction

Asymptotic symmetries for field theories in the presence of “boundaries at infinity” have received great attention recently, after they have been shown to be related to Weinberg soft theorems [We63]. The asymptotic structure of quantum electrodynamics (QED) and general relativity (GR) has also been studied in a number
of earlier works (see for example [Ash81a, Ash81c, AS81, Ash81b, Ash87] for GR and quantum gravity and [Her95, Her96, Her98, Sta98, Sta99, Sta02, Her12, Sta13, Her17] for QED). A great scientific effort has been devoted to this topic in the last decade, showing how asymptotic charges are expected to arise in a host of scenarios, including the crucial examples of general relativity [CL14, HLMS15], electrodynamics [HMPS14, CL15, KPS17] and even scalar field theories [CCM18, CC18, CFHS19]. More abstractly, the question of whether a gauge symmetry can become global, and hence present observable charges, is a relevant one for both theoretical modelling and experimental probing of fundamental theories.

One could also ask whether the conservation laws (asymptotic charges) in question really arise from symmetries of the theory, or if the transformations that generate them (often called large gauge transformations) are in fact maps between different sectors of the theory. The latter concern has been raised in [Her17], where the asymptotic charge of QED is shown to be generated by a transformation that, classically, relates QED theories in different gauges. Moreover, in [DW19] it was shown that quantum theories in different gauges could be unitarily inequivalent, so this would mean that a transformation that changes the gauge is not actually a symmetry of the quantum theory. This leads to the interpretational conundrum: are the large gauge transformations (LGT) symmetries of the theory?

In this paper, we employ a framework called BV-BFV — due to Cattaneo, Mnev and Reshetikhin [CMR14] — which allows one to pose the above questions within a mathematically rigorous setting. For a manifold with boundary, the BV-BFV framework is a combination of the Batalin–Vilkovisky (BV) approach to quantising a Lagrangian field theory associated to the “bulk” of the said manifold, and the Batalin–Fradkin–Vilkovisky approach to quantising the theory naturally associated to the boundary and formulated in the Hamiltonian framework \[1\] [BV77, BV81, BF83].

We adapt the BV-BFV framework to the case of “boundary at infinity”, to which we associate the asymptotic scaling limit of a theory assigned to the boundary of a scaled finite region. From such extended BV-BFV data we extract information on asymptotic symmetries and charges. At the classical level, we show that, in agreement with [Her17], large gauge transformations indeed have to relate theories in different gauges.

The advantage of using the BV-BFV setting to discuss the interpretation of LGTs is the model-independence and flexibility of the framework. Our approach allows us to reproduce the formulas from the literature on both sides of the controversy and point out where the interpretational discrepancies stem from. This is not surprising, since the BV-BFV data carry information about both the symmetries and the behaviour of solutions to equations of motion. Thus, after identifying the asymptotic charge with the BV-BFV boundary action at infinity (see below), we can interpret it both from the point of view of symmetries (the interpretation favoured e.g. by [KPS17, CL15, CE17]) and from the viewpoint of field equations (relating to the interpretation of [Her17]). This interplay of symmetries and equations of motion is central to the BV philosophy and in the case at hand it allows us to propose a new interpretation for the asymptotic charges, which avoids the problems that led to the controversy. We discuss our new interpretation of conserved asymptotic charges of QED in Section 2.7, where we also compare to the literature and resolve the interpretational conundrum.

We show that, assuming appropriate fall-off conditions for the fields, one can easily read off the correct expressions for asymptotic charges from the BV-BFV

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1BFV provides a resolution of the reduced phase space of the theory, i.e. of the locus defined by canonical constraints modulo symmetries.
data traditionally associated to a theory on a manifold with boundary, and prove
their conservation. Our results agree with the literature in examples of electrodynamics
and the massless scalar field. In particular, we compare the results on electromagnetic
asymptotic charges presented in [Her95, Her96, Her98] with the investigations of
[KPS17, CL17, CE17].

Using the same procedure, we derive the soft charges for the scalar field, compare
them with those derived in [Her95, CCM18, CC18], and show their conservation.
Our result is similar in spirit to the analysis of [CFHS19], but we disagree on
the issue of large gauge transformations (in their notation, shifts by closed-but-
not-exact forms). We derive scalar asymptotic charges and their conservation by
implementing the (reducible) symmetry of a model “dual” to scalar field theory in
the BV-BFV formalism. Then, we show in Section 3.6 that the notion of large gauge
transformations employed by [CFHS19] does not yield a well-defined BV structure,
making their interpretation and relevance harder to interpret.

In a broader context, our long-term goal is building a bridge between a systematic
approach to the quantisation of gauge theories in the presence of boundary\footnote{In this case the “boundary” is at infinity.}, such as
the BV-BFV formalism and asymptotic quantisation. The latter is an idea dating
back to [Ash81a] to address infrared problems in QED and in quantum gravity
related to the masslessness of the photon and graviton, and the long-range nature
of the interaction) by analysing the structure of asymptotic observables at null
infinity.

The long-range character of the electromagnetic interaction manifests itself in
the classical theory via Gauss’ law. In quantum theory, implementability of Gauss’
law, together with the assumption that observables should be local, leads to the
conclusion that the electric flux at space-like infinity is superselected (i.e., different
configurations of the flux label different unitarily-inequivalent representations of the
net of local algebras [Buc86]). Alternatively, one can implement Gauss’ law in the
quantum algebra, where the fluxes are not superselected, paying the price of giving
up the locality [Her98]. Other phenomena related to the long-range character of
electrodynamics include breaking of the Lorentz group and the infraparticle problem
[MS86]. The latter means that the electron’s spectrum is not point-like, since
the electron has to be considered together with the cloud of low energy (infrared)
photons accompanying it. This fact, in different guises, can be understood as
a necessity for “dressing” charged particles, as discussed, for example, in [DF16,
Dyb17] and references therein.

This paper is the first step towards developing a unified framework for quantisa-
tion of theories with boundaries and theories with long-range degrees of freedom,
in the spirit of perturbative algebraic quantum field theory [FR12b, FR12a]. The
framework we develop in this paper for the construction of classical asymptotic
charges is general enough to treat a broad spectrum of theories. One only needs
to specify the dynamics, the boundary/appropriate “infinity”, and the behaviour
of fields at this boundary/infinity. Then our extended BFV machinery returns the
correct conserved charge.

Although we use mainly the language of [CMR14], the translation to the frame-
work of [FR12b] is straightforward. Quantisation, and the precise relation to
[FR12a] is still work in progress. However, we would like to stress that the classical
BV-BFV construction we use here feeds into the quantisation scheme of [CMR18],
and we expect to be able to phrase soft theorems in this language, interpreting
the quantum master equation in the presence of boundaries in relation to Ward
identities.
The application of our procedure to other scenarios like nonabelian Yang–Mills, Chern–Simons and BF theories is expected to be straightforward (but still worth doing explicitly). The case of General Relativity (GR) in the Einstein–Hilbert (EH) formalism, whose BV-BFV structure for finite boundaries was investigated in [Sch16, CS16], will be studied in a further publication. In space-time dimension 3 the BV-BFV construction of GR in vielbein variables — often called Palatini–Cartan formalism — was presented in [CST19], while for its 4-dimensional analogue a crucial obstruction was found in [Sch16, CS17]. On the other hand, a BFV structure has been recently worked out from the reduced phase space of Palatini–Cartan theory in dimension $n \geq 3$ [CS19b, CCS20], independently from the BV theory in the bulk. We do expect asymptotic symmetries in this formulation to be easier to compute than their Einstein–Hilbert counterpart.

In Section 1 we review the basics of the BV-BFV approach to field theories on manifolds with boundary, and state the necessary geometric conventions for the remainder of the paper. We also introduce two descriptions of classical asymptotic data: the one in Section 1.2.1 is based on the approach of Herdegen (see e.g. [Her17]) and the other one, introduced in Section 1.2.2, is used by [CFHS19, CCM18, CC18].

Section 2 concerns the asymptotic symmetries of electrodynamics (ED): firstly without matter fields and next in the presence of scalar matter. We show how one obtains the asymptotic charges from the BFV data associated to ED, seen as abelian Yang–Mills theory, once appropriate fall-off conditions on fields are imposed. This agrees with [Her17, CE17, CL15].

In Section 3 we apply the same procedure to the (free) two-form model, dual to a (free) scalar field on-shell, and recover the soft asymptotic charges for scalar field theory, through the BV-BFV analysis of its associated dual model.

1. Preliminaries

1.1. BV-BFV formalism. A classical field theory on a manifold with boundary phrased in the BV-BFV formalism [CMR14, CMR18] is described by two sets of data, one assigned to the bulk manifold and one to the boundary, together with an appropriate map between the two. To the bulk manifold $M$ one associates BV data $$(\mathcal{F}, \Omega, S, Q)$$ composed of

1. A $(-1)$-symplectic graded manifold $(\mathcal{F}, \Omega)$
2. A degree-0 action functional $S$, a local functional on $\mathcal{F}$,
3. An odd vector field $Q$ on $\mathcal{F}$ of degree 1 with the cohomological property $[Q, Q] = 0$.

On the other hand, to a boundary $\partial M$ one assigns (exact) BFV data $$(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$$ similarly composed of

1. An exact $(0)$-symplectic graded manifold $(\mathcal{F}^\partial, \Omega^\partial = \delta \alpha^\partial)$, where $\delta$ denotes the de Rham differential on the space of local forms,
2. A degree-1 local action functional $S^\partial$ on $\mathcal{F}^\partial$.

$^3$This is the space of classical fields, supplemented with ghost fields in positive degrees, and antifields in negative degrees. In simplest cases (e.g. Yang-Mills theory), $\mathcal{F}$ is the odd cotangent bundle of some graded manifold (containing the classical fields in degree zero and the ghosts in degree one), where antifields are antisymmetric functionals on the cotangent space (i.e. can be interpreted as polyvector fields). The antifield number corresponds to the polynomial degree of these polyvector fields.

$^4$Here local means that $S$ has the form of an integral over a density that depends on fields and a finite number of derivatives (jets).
An odd vector field $Q$ on $F^0$ of degree 1 with the property: $[Q^0, Q^0] = 0$.

The BV-BFV construction connects the BV data associated with the bulk to the BFV data associated with the boundary by means of a map $\pi: F \to F^\partial$, and the following relations hold

\begin{align*}
t_Q \Omega &= \delta S + \pi^* \alpha^0 \\
\frac{1}{2} t_Q t_Q \Omega &= \pi^* S^0 \\
t_Q \delta \Omega &= \delta S^0 \\
\frac{1}{2} t_Q \delta t_Q \delta \Omega &= 0
\end{align*}

Remark 1. Observe that if $M$ has no boundary one defines BV data such that Equations (1a) and (1b) hold without the corrections coming from the boundary. In particular, in that case, $Q$ is the Hamiltonian vector field of $S$ and equation (1b) becomes the Classical Master Equation.

If the BV theory is constructed from the data of a classical field theory with (gauge) symmetries, the degree-zero part of $F$ and $S$ coincide with the classical data $(F_{cl}, S_{cl})$, the space of classical fields and the classical action functional. The BV-complex, given by $\mathfrak{BVF}^\bullet := (C^\infty(F), Q)$ is a combination of the Koszul–Tate resolution of the critical locus of $S_{cl}$ and of the Chevalley–Eilenberg complex for Lie algebra actions. In this language, the space of on-shell invariant functionals is given by the zeroth cohomology group $H^0(\mathfrak{BVF})$.

The BFV data represents the reduced phase-space of the system, as defined by the associated canonical constraints, i.e. functions $\{\phi_i\}$ in involution\footnote{The vanishing ideal of the constraints forms a Poisson subalgebra (the constraints are first-class). In our construction the contraints are functions on the space of degree-zero boundary fields.} with respect to the Poisson structure induced by $\Omega^0$. It can be seen as a cohomological resolution of the quotient of the vanishing set $C := \{\phi_i = 0\}$ with respect to the action of symmetries, in the sense that the space of invariant functions on the locus defined by $C$ is the degree-zero cohomology of the BFV complex $\mathfrak{BFV}^\bullet := (C^\infty(F^\partial), Q^\partial)$.

Remark 2. When a theory can be given a BV-BFV description, one can discuss its quantisation in this language \cite{CMR18}. The crucial piece of data in that case is the map $\pi: F \to F^\partial$ and the relations (1), connecting the BV and BFV data together. In this paper we are interested in purely classical considerations (concerning the nature of the asymptotic symmetries arising when boundaries at infinity are taken into account), and could in principle directly work with some given BFV data.

Remark 3. The information contained in $S^0$ is at least twofold. On one hand, it is analogous to a charge, as it generates gauge transformations via its Hamiltonian vector fields. Simultaneously, one recovers the canonical constraints by treating ghost fields $c^\alpha$ as Lagrange multipliers\footnote{For the application of this point of view to the nontrivial cases of General Relativity in the Einstein–Hilbert and Palatini–Cartan formalisms see \cite{CS16, CCS20}.} defining the constraints $\delta c^\alpha = 0$.

Remark 4. In discussing physical symmetries, a useful interpretation of degree-1 fields (ghosts) in the BV formalism is that of functionals on the space of generators whose evaluation returns a degree zero element, tautologically given by the generator itself. In particular, the evaluation of (the ghost-linear part of) $S^0$ over gauge generators $\Lambda \in C^\infty(M, g)$ is a degree-zero functional $S^0[\Lambda]$, which we interpret...
as the classical charge (on shell). At least for gauge theories, this is exactly the Maurer–Cartan form on a principal bundle \[BCR83\] \[BCRRSS88\].

**Remark 5.** The standard definition of charges that follows from Noether’s approach to symmetries would prescribe a charge to be given by the contraction of the symmetry generator (a vector field on phase space) with Noether’s one-form. In the BFV language this is very close to the contraction \[\iota_Q \alpha^\beta = S^\beta\], holds, but this is not always the case, as was discussed in detail in \[MSW19\]. This observation lead to the definition of the BV-BFV difference \[D := S^\beta - \iota_Q \alpha^\beta\], which is linked to gauge anomalies, descent equations and holography. All the examples that we consider in this paper show a coincidence between the boundary action \(S^\beta\) and the BFV version of Noether’s charge \(\iota_Q \alpha^\beta\) (i.e. \(D = 0\)). Further investigation of this will be deferred to future work.

1.2. **Geometric conventions.** When not stated otherwise, in this paper we work with the Minkowski spacetime with signature \((1, -1, -1, -1)\). However, our constructions can be adapted to the case of asymptotically-flat spacetimes.

For our construction of asymptotic charges, we begin with identifying a sufficiently large, precompact region \(W_R\) inside our space-time, bounded by a piecewise-null and piecewise spacelike boundary \(\partial W_R := \mathcal{I}_+^r \cup \mathcal{I}_-^r \cup \mathcal{H}_+^\tau \cup \mathcal{H}_-^\tau\), with \(R > 0\), as shown in Figure 1. Later on we will take a limit, where this region is enlarged “to infinity”.

![Region \(W_R\) inside our space-time, bounded by a piecewise-null and piecewise spacelike boundary \(\partial W_R := \mathcal{I}_+^r \cup \mathcal{I}_-^r \cup \mathcal{H}_+^\tau \cup \mathcal{H}_-^\tau\).](image)

1.2.1. **R, s, l variables.** One way to describe null asymptotics of fields is to use a set of variables introduced by Herdegen (see e.g. \[Her95\] \[Her17\]), which we refer to as \(R, s, l\) variables in this work. Let \(l\) is a future-pointing null vector, \(t\) a future-pointing timelike vector\(^8\) and \(R, s \in \mathbb{R}, \ R \geq 0\). In \[Her17\] (and previous works), one uses these parameters to define a space-time point \(x\) according to:

\[
x = Rl + s \frac{t}{l \cdot l},
\]

\(^7\)One can think of Noether’s prescription as the “degree-zero part” of \(\iota_Q \alpha^\beta\).

\(^8\)In \[Her17\] the author uses \(l\) and \(t\) rather than \(l\) and \(t\), but we want to avoid confusion with the notation for the time coordinate.
More about variables $R, s, l$ can be found in [Her17] Appendix B. In particular, a differentiable field $B$ on Minkowski spacetime, defines $\beta(R, s, l) = B(x)$ with the scaling property: $b(R/\lambda, \lambda s, \lambda l) = b(R, s, l), \lambda > 0$. Denote

$$L_{ab} = l_a \frac{\partial}{\partial l^b} - l_b \frac{\partial}{\partial l^a}.$$  

One can show that:

$$\frac{\partial}{\partial x^b} B(x) = l_b \beta(R, s, l) + \frac{r^a}{R \cdot l} L_{ab} \beta(R, s, l),$$  

where the dot denotes the derivative with respect to $s$ and

$$L_{ab} \beta(R, s, l) = R \left( l_a \frac{\partial}{\partial x^b} B(x) - l_b \frac{\partial}{\partial x^a} B(x) \right)$$

Very often we will use integration over the set of null directions. Let

$$C_+ = \{ l \cdot t = 0, l^0 > 0 \}.$$  

and for a fixed $t$, define $C^t_+$ as the intersection of $C_+$ with the $t \cdot l = 0$ plane. $C^t_+$ is a unit sphere in this plane and hence can be equipped with the usual metric whose line elements is denoted by $d\Omega^2$. Following [Her17], let $f(t)$ be a measurable function on $C_+$, homogeneous of degree $-2$. The integral defined by

$$\int f(t) \, d^2l \doteq \int_{C^t_+} f(t) \, d\Omega^2$$

does not depend on the choice of the vector $t$. In the present paper, we will often make use of this fact and identify the integral over the null directions on the left-hand side with the integral over a concrete unit sphere determined by the choice of $t$. In a fixed coordinate system, the natural choice is: $t = (1, 0, 0, 0)$. We come back to this at the end of the next section.

1.2.2. Retarded light-cone coordinates. Another convenient way to describe the null asymptotic of smooth fields on Minkowski spacetime makes use of retarded coordinates. We start with the standard set of coordinates $(t, x, y, z)$ and introduce space-like spherical coordinates $(r, x^A)$, with $x^A, A = 1, 2$. Next, we define retarded (resp. advanced) light-cone coordinates as $u_\pm = t \mp r$.

In coordinates $(u_{\pm}, r, x^1, x^2)$, a line element in Minkowski metric reads:

$$ds^2_\pm = +du_{\pm}^2 \pm 2drdu_{\pm} - r^2 d\Omega^2,$$

where $d\Omega^2$ is the line element for the unit 2-sphere. The matrix representation of Minkowski metric is

$$g_\pm = \begin{pmatrix}
+1 & \pm 1 & 0 \\
\pm 1 & 0 & 0 \\
0 & 0 & -r^2 g_{S^2}
\end{pmatrix},$$  

with determinant $\det(g_\pm) = -r^4 \det(g_{S^2})$, and the inverse

$$g^{-1}_\pm = \begin{pmatrix}
0 & \pm 1 & 0 \\
\pm 1 & -1 & 0 \\
0 & 0 & -r^2 g^{-1}_{S^2}
\end{pmatrix}.$$  

Let $\{x^A\}_{A=1,2}$ be coordinates of a point on the unit two-sphere embedded in three-dimensional Euclidean space. We denote the corresponding point of this
three-dimensional space by \( \hat{x} \). In this parametrisation, a spacetime point can be written as:

\[
x = \left( \frac{u_+ + r}{r \hat{x}} \right).
\]

To relate this particular coordinatisation to the formulation using \((R, s, l)\) variables, choose \( t = (1, 0, 0, 0) \) and consider null vectors of the form \( l = (1, \hat{x}) \), where \( \hat{x} \) is the unit three-vector determined by a sphere point \( x^A \). Then identify \( R \) with the radial coordinate \( r \), so that:

\[
x = r \left( \frac{1}{\hat{x}} \right) + s \left( \frac{1}{0} \right) = r \left( \frac{1}{\hat{x}} \right) + u_+ \left( \frac{1}{0} \right)
\]

where we used the fact that \( l \cdot t = 1 \) and we identified Herdegen’s variable \( s \) with the retarded time \( u_+ \), since \( s + r = t \) for our choice of \( l \) and \( t \). Observe that, in particular, in the \((r, u_+)\) coordinates we have \( l^r = 1 \) and \( l^{u_+} = 0 \).

1.3. Parametrisation of the boundary. In this paper we are concerned with symmetries and associated charges that appear on asymptotic boundary components and corners (i.e. boundaries of boundary components). For example, we will consider surfaces at constant coordinate \( R \) and then take the limit for \( R \to \infty \) (see Figure 1).

We denote by \( I^{\pm} \) the copy of \( S^2 \times \mathbb{R} \) obtained after taking the \( r \to +\infty \) (or \( R \to +\infty \) in the other description) limit while keeping \( u_\pm \) constant. We treat the limits \( I^{\pm} \) as boundaries at infinity, and call them future/past null infinity. From \( I^+ \), one gets two connected components of \( \partial I^+ \), denoted \( I^\pm_\pm \) and topologically homeomorphic to two-dimensional spheres, obtained by taking the limits \( u_\pm \to \pm \infty \), respectively. Similar considerations apply to \( I^- \).

1.3.1. Hyperbolic coordinates. Hyperbolic coordinates are used to analyse the behavior of smooth fields in Minkowski spacetime at timelike infinity. They are defined by

\[
\tau = \sqrt{t^2 - r^2}, \quad \rho = \frac{r}{\sqrt{t^2 - r^2}},
\]

so that

\[
t = \tau \sqrt{1 + \rho^2}, \quad r = \rho \tau.
\]

In these coordinates, the line element reads:

\[
ds^2 = d\tau^2 - \tau^2 \left( (1 + \rho^2)^{-1} d\rho^2 + \rho^2 d\Omega^2 \right),
\]

and future timelike infinity \( i^+ \) is obtained by taking the limit \( \tau \to \infty \). A spacetime point can then be written as

\[
x = \left( \frac{\tau \sqrt{1 + \rho^2}}{\tau \rho \hat{x}} \right).
\]

Finally, a point on the unit hyperboloid \( H^+ \) (i.e. \( \tau = 1 \)) is written as

\[
Y^\mu = (\sqrt{1 + \rho^2}, \rho \hat{x}).
\]

1.4. Differential forms conventions. A differential \( k \)-form is written in a local coordinate chart as

\[
\alpha = \frac{1}{k!} \alpha_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}
\]

with \( \alpha_{\mu_1 \ldots \mu_k} \) totally antisymmetric in the indices. The operation of taking the Hodge dual on a generating set of \( k \)-forms over an \( N \)-dimensional (pseudo)-Riemannian manifold \((M, \eta)\) is given by

\[
(*dx^{\rho_1} \ldots dx^{\rho_k}) = \frac{\sqrt{|\eta|}}{(n-k)!} \eta_{\mu_1 \rho_1} \cdots \eta_{\mu_k \rho_k} \epsilon_{\rho_1 \ldots \rho_k \mu_{k+1} \ldots \mu_{N-k}} dx^{\mu_{k+1}} \cdots dx^{\mu_N},
\]
which, for \(\alpha, \beta \in \Omega^k(M)\) yields
\[
\alpha \wedge \ast \beta = \frac{1}{k!} \alpha_{\mu_1 \ldots \mu_k} \beta^{\mu_1 \ldots \mu_k} d\text{Vol}_g.
\]
The indices are raised with the inverse metric, e.g., \(\beta^\mu = g^{\mu \nu} X_\nu\) denotes the components of the vector \(\beta = (g^\nu)^{-1}(X)\), for \(g^\nu : TM \to T^*M\).

The Laplace–Beltrami operator on a Lorentzian manifold of signature \((1, -1, -1, -1)\) is \(\Box = -(d + d^*d)\), with the codifferential defined by
\[
d^* \equiv \ast \cdot : \Omega^k(M) \to \Omega^{k-1}(M).
\]
Its restriction to co-closed forms, i.e., forms in the Lorenz gauge \(d^*A = 0\), is \(\Box|_{\text{closed}} = -d^*d\).

2. Electrodynamic

In this section we consider electrodynamics \(^{11}\) phrased in the Batalin–Vilkovisky language, to show how we can recover asymptotic symmetries from the appropriate manipulation of the BFV data.

2.1. BV-BFV approach to electrodynamics. Electrodynamics is formulated in terms of a \(U(1)\) Yang–Mills field theory coupled to matter. For simplicity, we model matter as a complex scalar field, but the same analysis can also be performed for Fermions. For a principal \(U(1)\) bundle \(P \to M\) on a Lorentzian spacetime \((M, g)\), possibly with boundary\(^{12}\) and given an associated \(C^2\) bundle\(^{13}\) \(\mathcal{V} \to M\), the extended space of field configurations is
\[
\mathcal{F} = T^*[1] (A_P \times \Omega^0(M, \mathcal{V}) \times \Omega^0[1](M))
\]
where \(A_P\) is the space of electromagnetic potentials, \(\Omega^0(M, \mathcal{V})\) is the space of complex scalar fields \((\varphi, \overline{\varphi})\), and \(\Omega^0[1](M)\) contains the ghost fields.

There is a canonical shifted symplectic structure \(\Omega_{YM}\) on \(\mathcal{F}\), formally given by
\[
\Omega_{YM} = \int_M \delta A \delta A^\dagger + \delta \varphi \delta \varphi^\dagger + \delta \overline{\varphi} \delta \overline{\varphi}^\dagger + \delta \overline{\varphi} \delta \overline{\varphi}^\dagger,
\]
where the fields in the cotangent fiber (also called anti-fields) are denoted by \(A^\dagger \in \Omega^3[-1](M)\), \(\varphi^\dagger \in \Omega^{\text{top}}[-1](M, \mathcal{V}^*)\) and \(c^\dagger \in \Omega^{\text{top}}[-2](M)\). Covariant derivatives for the fields \(\varphi, \overline{\varphi}\) are defined by \(A_\varphi = d\varphi + iqA \varphi \) and \(A_{\overline{\varphi}} = d\overline{\varphi} - igA \overline{\varphi}\), with \(q \in \mathbb{R}\) a coupling constant. The BV-extended action functional is then given by
\[
S = \int_M \left( -\frac{1}{8\pi} F_A \ast F_A + \frac{1}{2} \left( dA_{\overline{\varphi}} \wedge * dA_\varphi + m^2 A_{\overline{\varphi}} \right) + A_A \right) \wedge dA + \varphi c - \overline{\varphi} \overline{c}
\]
where \(*\) is the Hodge operator induced by the Lorentzian structure on \((M, g)\), and the BV operator \(Q\) is given by
\[
\begin{align*}
QA &= d_A c \\
QA^\dagger &= -\frac{1}{8\pi} d_A \ast F_A - i q \overline{\varphi} \ast d_A \varphi + i q \ast d_A \overline{\varphi} \varphi \\
Q \varphi &= c_\varphi \\
Q \varphi^\dagger &= (-d_A \ast d_A + m^2) \overline{\varphi} + \varphi^* c \\
Q \overline{\varphi} &= -c_\overline{\varphi} \\
Q \overline{\varphi}^\dagger &= (-d_A \ast d_A + m^2) \varphi - \overline{\varphi}^* c \\
Q c &= 0 \\
Q c^\dagger &= 0
\end{align*}
\]
where we used that \(d_A \overline{\varphi} \wedge \ast d_A \varphi = -\ast d_A \overline{\varphi} \wedge d_A \varphi\).

\(^{11}\)Throughout, we use the standard formulation of electrodynamics as a second-order field theory, i.e., where equations of motion are second order. Other literature prefer to employ the first order formulation instead, whose BV-BFV description can be found in \text{CMR14}.

\(^{12}\)We will later restrict to a situation where \((M, g)\) is a region in Minkowski spacetime.

\(^{13}\)On Minkowski spacetime the \(C^2\)-bundle is trivial.
Denote the matter current by $J := -i q \varphi \ast d_A \varphi + i q \ast d_A \varphi$. The classical equations of motion (the degree-zero sector of the condition $Q = 0$) are given by:

$$d_A \ast F_A = J$$
$$(-d_A \ast d_A + m^2)\varphi = 0$$
$$(-d_A \ast d_A + m^2)\varphi = 0$$

Note that $d_A J = 0$ on shell, since $[F_A, \ast F_A] = 0$, i.e. the current is conserved.

Since Yang–Mills theory satisfies the BV-BFV axioms stated in equation (1) (see e.g. [CMR14, MSW19] for details), we obtain — on a manifold with boundary — the following BFV data $(\mathcal{F}^\partial, S^\partial, Q^\partial, \Omega^\partial)$:

- The space of fields is

$$\mathcal{F}^\partial := T^* (A_{\ast} P \times \Omega^0(\partial M, \nu) \times \Omega^0_1(\partial M)),$$

where we denoted by $A_{\ast} P$ the space of connections on the induced principal bundle $\iota^* P$ on $\partial M$.

- The boundary action is given by

$$S^\partial = \frac{1}{2} \iota_0 Q \iota_0 \Omega_{YM} = \int_M d\left(\frac{1}{4\pi} d_A \ast F_A + J\right),$$

which is a degree-1 functional on $\mathcal{F}^\partial$.

- $\mathcal{F}^\partial$ is equipped with the symplectic form $\Omega^\partial$ given by

$$\Omega^\partial = \int_{\partial M} \frac{1}{4\pi} \delta A \delta[\ast F_A]_{\partial M} + \delta c \delta A^\dagger + \delta \varphi \delta[\ast d_A \varphi]_{\partial M} + \delta \varphi \delta[\ast d_A \varphi]_{\partial M},$$

- Completing the BFV data is $Q^\partial$, the Hamiltonian vector field of $S^\partial$, that is $\iota_0 Q^\partial = \delta S^\partial$.

The projection map $\pi: \mathcal{F} \rightarrow \mathcal{F}^\partial$ is simply the restriction of fields and normal jets to the boundary, composed with a redefinition of fields sending a normal jet of $A$ (resp. $\varphi$) - restricted to the boundary - to $[\ast F_A]_{\partial M}$ (resp. $[\ast d_A \varphi]_{\partial M}$). A careful analysis of the symplectic manifold of boundary fields for the scalar case was given in [CM14].

**Remark 6.** To obtain (6) we could make the following alternative choice (recall $d_A c$ has even parity):

$$\int_M d_A \varphi \wedge d_A \ast F_A = \int_M d(c \ast F_A) = \int_M d(d_A \ast F_A)$$

This choice yields the same result, but is better suited to recover Herdegen’s formulas for soft charges [Her17], while (6), which has the advantage of manifestly vanishing on shell, will be useful to reproduce formulas in [KPS17].

More on the BV-BFV structure of Yang-Mills theory in the second order formalism and its relation with extended phase spaces and edge modes can be found in [MSW19], while the first order formulation has been discussed in [CMR14].

**Remark 7.** Note that the BV data presented above has been historically associated to fields on compact manifolds, or equipped with vanishing boundary conditions, or given for compactly supported fields. As such, it was never set up to interact with a choice of fall-off conditions on fields. However, said conditions can be introduced once one extends the BV-BFV construction to noncompact manifolds, by defining fields to be sections of bundles supplemented with the appropriate falloff conditions. This is what we will do in the next section.
2.2. Asymptotic fields. In this work, instead of considering $S^\partial$ at a finite boundary, we consider $S^\partial$ at infinity. To make this precise, we need to impose some fall-off conditions on the variables $A$ and $\varphi$, to ensure that their asymptotes are well defined. Note that the natural limit for the electromagnetic potential is the null infinity $I = I^+ \cup I^-$, while the matter current $J$ propagates to time-like infinity $H = H^+ \cup H^-$. This translates to the requirement that

$$\lim_{\tau \to \infty} A_{|H^\pm} = \lim_{r \to R} J_{|I^\pm} = 0.$$  

(7)

2.2.1. Free electromagnetic field. In the notation of Her95, Her96, Her98, Her05, Her17, the asymptotic electromagnetic potentials are defined as follows:

$$\lim_{R \to \infty} R A(x + Rl) = V(x \cdot l, l),$$

$$\lim_{R \to \infty} R A(x - Rl) = V'(x \cdot l, l).$$

Using instead the $(r, u_+, z, \bar{z})$ coordinates, one expands $A$ as:

$$A = \sum_{k=1}^{1} \frac{1}{r^k} A^{(k)},$$

(8)

so that

$$V(s, l) = A^{(1)}(u_+, \hat{x}),$$

where $s = u_+$, $l = (1, \hat{x})$ and $\hat{x}$ is a point on unit 2-sphere embedded in 3-dimensional Euclidean space, as explained in section 1.2.2.

Without external currents ($J = 0$) and assuming Lorenz gauge, $A$ satisfies the wave equation

$$\Box A = 0$$

and the asymptotic fields have the “vanishing property”:

$$V(+\infty, l) = 0 = V'(-\infty, l),$$

(10)

i.e. these vanish at time-like infinity. In Her95, Her96, Her98, Her05, Her17, these also satisfy the following fall-off conditions:

$$|V_0(s, l)| < \frac{const.}{s^\epsilon},$$

(11)

$$|\dot{V}_0(s, l)| < \frac{const.}{s^{1+\epsilon}},$$

(12)

similarly for $V'$, but with the role of $-\infty$ and $+\infty$ exchanged.

2.2.2. Fields with sources. Now let us consider the equation with sources:

$$\Box A(x) = 4\pi J(x).$$

For this equation we know that the retarded and advanced Green functions $\Delta^{R/A}$ exist. We want the current $J$ to describe incoming and outgoing matter fields in a scattering experiment and the free radiation field

$$A^{rad} = A^R - A^A$$

should satisfy the fall-off conditions (11) and (12). The Pauli-Jordan function is defined by $\Delta = \Delta^R - \Delta^A$. We have, in relative coordinates,

$$\Delta(x) = \frac{1}{2\pi} \text{sgn}(x^0) \delta(x^2).$$

In Her95, $\Delta$ is represented as

$$\Delta(x) = -\frac{1}{8\pi^2} \int \delta'(x \cdot l) d^2 l,$$
In this case (see [Her17]), assume that for the current \( J \) coming from the definition of retarded and advanced solutions. So the radiation field

\[
A^{\text{rad}}(x) = 4\pi \int \Delta(x - y)J(y)dy = -\frac{1}{2\pi} \int dy \int d^2l \delta'(x - y \cdot l)J(y) = -\frac{1}{2\pi} \int \dot{V}_J(x \cdot l, l)d^2l,
\]

where

\[
V_j(s, l) = \int dy \delta(s - y \cdot l)J(y)
\]

and we have

\[
\lim_{R \to \infty} R A^R(x - RL) = V_j(-\infty, l),
\]

\[
\lim_{R \to \infty} R A^A(x + RL) = V_j(+\infty, l),
\]

from the definition of retarded and advanced solutions.

Assume that \( V_j \) is well-defined and that \( \dot{V}_j \) satisfies (12). For physical reasons (see [Her17]), assume that for \( v \) on the unit future hyperboloid (i.e. \( v \in \mathcal{H}_+ \)), the current \( J \) behaves as

\[
J \sim \tau^{-3} v \rho_{\pm}(v), \quad \tau \to \pm \infty.
\]

In this case

\[
V_j(\pm \infty, l) = \int_{\mathcal{H}_\pm} \frac{v \rho_{\pm}(v)}{v \cdot l}d\mu(v),
\]

so \( V_j(+\infty, l) \) need not vanish! (in contrast to the asymptote of the free field, see (13)).

The total field decomposes as

\[
A = A^R + A^\text{in} = A^A + A^\text{out}.
\]

Clearly, \( A^{\text{in/out}} \) have to solve the homogeneous equation (9), so they are free fields and (assuming that incoming and outgoing fields satisfy the fall-off conditions (11) and (12)), we have the following identities for the asymptotes:

\[
V(s, l) = V_j(s, l) + V^\text{in}(s, l) = V_j(+\infty, l) + V^\text{out}(s, l)
\]

\[
V'(s, l) = V_j(-\infty, l) + V^\text{in'}(s, l) = V_j(s, l) + V^\text{out'}(s, l)
\]

Hence

\[
V(+\infty, l) = V_j(+\infty, l), \quad V'(-\infty, l) = V_j(-\infty, l),
\]

which means that \( V(+\infty, l) \) and \( V'(-\infty, l) \) come entirely from matter and contribute to the hard part of the charge. We also have

\[
V(-\infty, l) = V_j(+\infty, l) + V^\text{out}(-\infty, l),
\]

\[
V'(+\infty, l) = V_j(-\infty, l) + V^\text{in'}(+\infty, l),
\]

so both soft and hard components contribute to the matching property that reads:

\[
V'(+\infty, l) = V(-\infty, l).
\]

2.3. Changing the gauge. In [Her17] Chapter 7, one considers a change of gauge:

\[
\hat{\Lambda} = A + d\Lambda,
\]

where \( A \) is a Lorentz potential, but \( \hat{\Lambda} \) is not. The asymptotic field corresponding to \( \hat{\Lambda} \) is defined by:

\[
\hat{V}_b(s, l) = \lim_{R \to \infty} R \hat{\Lambda}_b(\hat{R}l + st/t \cdot l).
\]
For this limit (and also the limit of $A$) to exist, $\Lambda$ has to be of the form (formula (36) of Her17, also confirmed by CL15):

$$\Lambda(rl + st/t \cdot l) = \varepsilon^+(l) + \frac{\beta(s, l)}{R} + o(R^{-1}),$$

which in $u, r, \hat{x}$ variables amounts to:

$$\Lambda(x) = \lambda(\hat{x}) + O(r^{-1}),$$

where $\lambda(\hat{x}) = \varepsilon^+(1, \hat{x})$. The expansion at time-like infinity takes the form (following CL15):

$$\Lambda(x) = \lambda_H(\rho, \hat{x}) + O(\tau^{-\varepsilon})$$

(16)

The resulting contribution to the potential can be computed in the $R, s, l$ variables, using the rule (2):

$$\frac{\partial}{\partial l'}\Lambda(st/t \cdot l + RL) = \frac{1}{R} \left( V^+_{\varepsilon^+}(l) + l_b \left( \hat{\beta}(s, l) - t \cdot V^+_{\varepsilon^+} / l \cdot l \right) \right)$$

(17)

where $V^+_{\varepsilon^+}$ is a vector-valued function such that

$$L_{abc}e^+_{\varepsilon^+}(l) = l_a V^+_{\varepsilon^+}(l) - l_b V^+_{\varepsilon^+}(l),$$

and it has properties (see appendix C to Her17):

$$V^+_{\varepsilon^+}(\Lambda l) = \lambda^{-1} V^+_{\varepsilon^+}(l), \quad l \cdot V^+_{\varepsilon^+}(l) = 0, \quad L_{[ab}V^+_{\varepsilon^+}(l) = 0.$$  

(18)

Crucially:

$$\varepsilon^+(l) = \frac{1}{4\pi} \int \frac{l \cdot V^+_{\varepsilon^+}(l')}{l \cdot l'} d^2 l'.$$

(19)

We also have:

$$\int \frac{\varepsilon^+(l)}{(l \cdot l)^2} d^2 l = \int \frac{l \cdot V^+_{\varepsilon^+}(l)}{l \cdot l} d^2 l.$$

(20)

2.4. Green’s function. Following CL15 CE17, we consider $\Lambda$ such that $\lambda_H$ satisfies Laplace equation on the hyperboloid. It is then given in terms of the corner data as

$$\lambda_H(y) = \int G(y; \hat{x}')\lambda(\hat{x}') d^2 \hat{x}'$$

where $y$ is the variable at the hyperboloid, $y = (\rho, \hat{x})$ and $G$ is the Green function discussed in Cam15 with the property that

$$\lim_{\rho \to \infty} G(y; \hat{x}') = \delta(\hat{x} - \hat{x}').$$

Remark 8. Observe that we do not require that $\Lambda$ satisfies the wave equation on the whole of $M$. Indeed, this would be incompatible with the observations in Her17, reported below in Section 2.5.

It is also shown in CL15 CE17 that

$$G(y; \hat{x}') = (4\pi)^{-1}(\sqrt{1 + \rho^2} - \rho \hat{x} \cdot \hat{x}')^{-2},$$

while in Herdegen’s notation:

$$\hat{G}(v; l') = (4\pi)^{-1}(l' \cdot v)^{-2}.$$

Here $\hat{G}$ is obtained from $G$, after we set $y = (\rho, \hat{x}), l' = (1, \hat{x}')$ and $v = (\sqrt{1 + \rho^2}, \rho \hat{x})$ (i.e. Herdegen’s $v$ is $Y$ from CL15 CE17, compare with Section 1.2). Thus, both references use the same Green’s function. Let $\Lambda_{\hat{G}}(v) = \lambda_H(y)$. This allows us to write:

$$\Lambda_{\hat{G}}(v) = \int \hat{G}(v; l')\varepsilon^+(l) d^2 l = \frac{1}{4\pi} \int \frac{\varepsilon^+(l)}{(v \cdot l)^2} d^2 l.$$

(21)
Using formula (20), we obtain
\[ \Lambda_H(v) = \frac{1}{4\pi} \int \frac{v \cdot V^+(l')}{v \cdot l'} d^2l'. \]

As a consistency check, consider the limit
\[ \lim_{\rho \to \infty} \int \frac{v \cdot V^+(l)}{v \cdot l} d^2l = \int \frac{l \cdot V^+(l')}{l' \cdot l'} d^2l', \]
where \( l = (1, \hat{x}) \). Using (19), we obtain
\[ \lim_{\rho \to \infty} \frac{1}{4\pi} \int \frac{v \cdot V^+(l)}{v \cdot l} d^2l = \varepsilon^+(l), \]
as expected.

2.5. Lorenz vs. other gauges. If both \( A \) and \( \hat{A} \) are in the Lorenz gauge, then (following [Her17]):
\[ \hat{V}(s,l) = V(s,l) + l \alpha(s,l). \]
Assuming \( \hat{V}(+\infty,l) = V(+\infty,l) = 0 \), it also follows that \( \alpha(+\infty,l) = 0 \). In [Her17] Section 3.2, it is shown that this implies that
\[ \Lambda(x) = -\frac{1}{2\pi} \int \alpha(x \cdot l', l') d^2l' + \gamma^+, \]
where \( \gamma^+ \) is a constant\(^{14} \). The null asymptotics are:
\[ \varepsilon^\pm = \lim_{R \to \infty} \Lambda(st \pm RL) = \gamma^\pm, \]
where
\[ \gamma^- = \gamma^+ - \frac{1}{2\pi} \int \alpha(-\infty, l') d^2l', \]
so the matching requirement \( \varepsilon^+(l) = \varepsilon^-(l) \) (see Remark 11) cannot be met!

Note that in equation (17), the contribution from the Lorenz gauge enters the term proportional to \( l_b \). More generally, the whole term proportional to \( l_b \) can be absorbed into a residual Lorenz gauge transformation of \( V \) and redefinition of \( \varepsilon^+(l) \). The non-trivial change of the asymptotics is therefore described fully by \( V^+(l) \) and, following [Her17], we interpret the resulting transformation (identified as the large gauge transformation of [KPS17]) as
\[ \hat{V}(s,l) = V(s,l) + V^+(l). \]
The gauge parameter \( \Lambda \) used to construct \( V^+(l) \) does not satisfy the wave equation in the bulk (since \( \hat{A} = A + d\Lambda \) cannot be in the Lorenz gauge), but we require that it satisfies it at time-like infinity. We assume \( \Lambda \) to be of the form:
\[ \Lambda(x) = \lambda_H(\rho, \hat{x}) + f(\tau), \]
with \( f \) vanishing for \( \tau \to \infty \).

\(^{14}\)In [Her17], \( \gamma^\pm \) is denoted \( \epsilon^\pm \). We adopt this notation to avoid confusion with \( \varepsilon^\pm \), the asymptote of the gauge generator \( \Lambda \).
2.6. Calculation of the charge. We now want to show how a correct specification of boundary fall-off conditions on the fields, and an explicit choice of coordinates around a lightlike boundary $\partial M$ (with corners!) allows us to reproduce known results on asymptotic symmetries in electrodynamics.

The leading idea behind our analysis is that the total asymptotic charge (in the literature [HMPS14, KPS17] it is derived from large gauge symmetries\textsuperscript{15}) is identified as the boundary action $S^0$ in the BFV formalism, and the “charge conservation” is the consequence of the fact that $S^0$ vanishes on-shell. Although there might be deviations from this paradigm, as discussed in Remark\textsuperscript{5} for the cases at hand we show that the BFV boundary action is the correct functional to consider.

Let us consider a region $\mathcal{W}_R$, whose boundary consists of 4 pieces: $\partial \mathcal{W}_R \equiv I^+_r \cup I^-_r \cup H^+_\tau \cup H^-\tau$ (cf. Section 1.2, Figure 1). Formula (6) takes the form
\[
S^0_{\partial \mathcal{W}_R} = \int_{\partial \mathcal{W}_R} c \left( -\frac{1}{4\pi} d_A \star F_A + J \right)
\]
and we immediately observe that, by virtue of Equation (5),
\[
S^0_{\partial \mathcal{W}_R} \approx 0
\]
for all $R$, where $\approx$ means on-shell, i.e. imposing the equations of motion.

**Remark 9.** One should notice that in general the boundary action does not vanish only by imposing the classical equations of motion, especially if it depends on ghosts and antifields of higher order (see (28) below). What we really mean with $\approx$ is taking the degree-zero cohomology of the Koszul-Tate part (the lowest antifield number\textsuperscript{16}) of the BV differential. Practically this is achieved by setting to zero antifields and ghosts for ghosts and quotienting out the EOMs. This fact should be interpreted as the BFV data being a container of a large amount of information, which needs to be extracted in the appropriate way.

We are interested in the limit
\[
S^0 = \lim_{R \to \infty} S^0_{\partial \mathcal{W}_R}.
\]
Clearly, also $S^0 \approx 0$ and $S^0$ naturally splits into two terms (corresponding to null and time-like asymptotics):
\[
S^0 = -\frac{1}{4\pi} \lim_{R \to \infty} \int_{\partial \mathcal{W}_R} c d_A \star F_A + \lim_{R \to \infty} \int_{\partial \mathcal{W}_R} c J ,
\]
\[
= \lim_{r \to \infty} S^0_{\partial, \text{soft}} + \lim_{r \to \infty} S^0_{\partial, \text{hard}},
\]
where
\[
S^0_{\partial, \text{soft}} = \int_{I^+_r \cup I^-_r} c d_A \star F_A ,
\]
and
\[
S^0_{\partial, \text{hard}} = \int_{H^+_\tau \cup H^-\tau} c J.
\]
As noted before (Remark\textsuperscript{5}), the soft term can also be written as
\[
S^0_{\partial, \text{soft}} = \int_{I^+_r \cup I^-_r} d_A c \star F_A .
\]

\textsuperscript{15}The notion of large gauge transformation is somewhat ambiguous, as different authors use the same terminology to denote different concepts. Here, it is intended as gauge transformations that do not vanish at infinity.

\textsuperscript{16}See Footnote\textsuperscript{3} for the definition.
2.6.1. Soft charge in $(R, s, l)$ variables. We start with the first term in formula (23). Our assumptions on $A$ imply that the limit is well-defined and we can re-write this term as

\[
S^{\partial, \text{soft}}_{I^+} = \lim_{R \to +\infty} S^{\partial, \text{soft}}_{I^+} = -\frac{1}{4\pi} \int_{I^+} \lim_{R \to +\infty} (R^2 dA \wedge \ast FA(x))
\]

\[
= -\frac{1}{4\pi} \int_{I^+} \lim_{R \to +\infty} (R^2 dA \wedge \ast FA(x)) - \frac{1}{4\pi} \int_{I^-} \lim_{R \to +\infty} (R^2 dA \wedge \ast FA(x'))
\]

where $x = RL + s \frac{\epsilon}{l}$ and $x' = -RL + s \frac{\epsilon}{l}$. Let’s focus on the first terms and evaluate the ghost at the gauge parameter $\Lambda$, as discussed in Remark 4. We use the fact that

\[
\lim_{R \to +\infty} RF_{ab} \left( s \frac{\epsilon}{l} t \cdot l + Rl \right) \approx t_a \dot{V}_b(s, l) - l_b \dot{V}_a(s, l),
\]

to find the limit of $FA$ and the expansion (17) to find the limit of $dA$.

Treating $V^{\epsilon^+}, l, V$ as one forms with index lowered using the metric (in the sense that e.g. $t^b = g(t, \cdot)$ is simply written as $l$), we obtain:

\[
-4\pi S^{\partial, \text{soft}}_{I^+}[A] = \int_{I^+} \lim_{R \to +\infty} (R^2 dA \Lambda \wedge \ast FA) \approx \int_{I^+} V^{\epsilon^+} (l) \wedge \ast (l \wedge \dot{V}(s, l))
\]

\[
= \frac{1}{2} \int_{I^+} \epsilon_{abfg} \eta^{gm} \eta^{fn} t_m \dot{V}_n dx^a \wedge dx^b \wedge dx^d = \frac{1}{2} \int_{I^+} \text{dVol}_{I^+} \epsilon_{abfg} \eta^{gm} \eta^{fn} \dot{V}_n
\]

\[
= \int_{-\infty}^{+\infty} ds \int_{S^2} d^2 l \left( \delta^d_1 \delta^d_2 - \delta^d_1 \delta^d_3 \right) \eta^{gm} \eta^{fn} t_m \dot{V}_n = \int_{-\infty}^{+\infty} ds \int_{S^2} d^2 l \left( V^d \epsilon \cdot V - l \cdot V^{\epsilon^+} V^r \right),
\]

where we used that $V = 1$ (see Section 12.2.2), $dx^a \wedge dx^b \wedge dx^d = \epsilon_{abfg} |\text{det}(h)|^{-\frac{1}{2}} \text{dVol}_{I^+}$ and $h$ is the induced metric on $I^+$, the determinant of which is 1. Observe that $\epsilon_{abfg} \epsilon_{abfg} = 2(\delta^d_1 \delta^d_2 - \delta^d_1 \delta^d_3)$. Since $l \cdot V^{\epsilon^+} = 0$ (see Eq. (18)), and neither $V^{\epsilon^+}$ nor $l'$ depend on $s$, we obtain finally:

\[
S^{\partial, \text{soft}}_{I^+}[A] = -\frac{1}{4\pi} \int_{I^+} \lim_{R \to +\infty} (R^2 dA \Lambda \ast FA) \approx \int_{S^2} d^2 l V^{\epsilon^+} (l) V^{\text{out}} (-\infty, l) \equiv Q^{\text{soft}+}_{\epsilon^+},
\]

and similarly, the contribution from $I^-$ gives:

\[
S^{\partial, \text{soft}}_{I^-}[A] \approx -\frac{1}{4\pi} \int_{S^2} d^2 l V^{\epsilon^-} (l) V^{\text{in}} (+\infty, l) \equiv -Q^{\text{soft}^-}_{\epsilon^-}.
\]

So, assuming the matching requirement

\[
\epsilon^+(l) = \epsilon^-(l) \equiv \epsilon(l), \quad (25)
\]

the “soft” contribution to the boundary action takes the form:

\[
S^{\partial, \text{soft}}_{I^+ \cup I^-}[A] \approx Q^{\text{soft}+}_{\epsilon} - Q^{\text{soft}^-}_{\epsilon}.
\]

Remark 10. Here we have defined charges from the evaluation of the boundary action on different boundary components following the convention that along the past infinity boundary, the sign gets reversed: $S^{\partial, \text{soft}}_{I^+} = \pm Q^{\text{soft}+}_{\epsilon}$. The on-shell vanishing of the boundary action is then directly linked to on-shell conservation of charges.
Remark 11. Note that the matching property \[15\] of the asymptotic potential \[Her95\] (see (2.26) and the following discussion) is a consequence of equations of motion and the fall-off condition. In contrast to that, the matching requirement \[25\] is an extra condition imposed on gauge parameters. It is not a priori clear if this condition can be fulfilled. In fact, it was shown in \[Her17\] that for \( A \) in the Lorentz gauge and \( \Lambda \) satisfying the wave equation (both with appropriate fall-off condition), this requirement cannot be met.

2.6.2. Soft charge in retarded coordinates. To relate this to the results of \[KPS17\], we use the retarded coordinates, as described in Section 2.2.3. From Equation (24), we compute the “soft” contribution to the boundary action:

\[
S_{\partial \mathcal{W}_R}^{\text{soft}} = -\frac{1}{4\pi} \int_{\mathcal{W}_R} d_A \wedge d_A \ast F_A = -\frac{1}{4\pi} \int_M \text{dVol}_M \text{div}(\mathcal{X})
\]

where \( \mathcal{X}_\mu = \sqrt{g} \partial_\mu \left( \sqrt{g} F^{\mu \tau} \right) \).

More directly, we get (recall that for an abelian group \( d_A \ast F_A = d \ast F_A \))

\[
S_{\partial \mathcal{W}_R}^{\text{soft}} = -\frac{1}{4\pi} \int_{\partial \mathcal{W}_R} d(c \ast F_A) = -\frac{1}{4\pi} \int_{\partial \mathcal{W}_R} \text{dVol}_\partial \left[ \frac{c}{\sqrt{g}} \partial_\lambda \left( \sqrt{g} g^{\sigma \mu} g^{\lambda \nu} F_{\mu \nu} \right) \right].
\]

Observe that the integral over \( \partial \mathcal{W}_R \) splits into a null boundary part and a hyperboloid part; however, because of the fall-off conditions on the field \( A \), in the limit for \( \tau \to \infty \) the Hyperboloid contribution vanishes (cf. with Equation (9)).

Assume \( \Lambda \in \Omega^0(M) \) is a gauge parameter as the one introduced in Section 2.3 i.e. \( \Lambda(x) = \Lambda(\hat{x}) + O(r^{-1}) \). Note that, when restricted to \( \mathcal{I} \simeq \mathbb{R} \times S^2 \), \( \Lambda \) is constant along the \( \mathbb{R} \) direction.

Using retarded/advanced light-cone coordinates and complex coordinates on the 2 dimensional sphere we get

\[
d\text{Vol} = 2r^2 dr du_{\pm} (1 + z \bar{z})^{-2} dz d\bar{z},
\]

from which, recalling the explicit expression of formulas (9) and (4) for Minkowski metric, we obtain

\[
-4\pi S_{\mathcal{I}^+ + \mathcal{I}^-}^{\text{soft}} = -4\pi \lim_{R \to \infty} S_{\mathcal{I}^+ + \mathcal{I}^-}^{\text{soft}} = \lim_{R \to \infty} \frac{1}{2\pi} \int_{\mathcal{I}^+ + \mathcal{I}^-} dt \left[ r^2 c(z, \bar{z}) \left( \partial_u F_{u \pm r} + \frac{1}{\sqrt{g}} \partial_\lambda \left( \sqrt{g} F^{\mu \nu} g^{\lambda \chi} F_{\mu \nu} \right) \right) \right]
\]

\[
= \lim_{R \to \infty} \frac{1}{2\pi} \int_{\mathcal{I}^+ + \mathcal{I}^-} dt \left[ r^2 c(z, \bar{z}) \left( \partial_u F^{(2)}_{u \pm r} + \partial_\lambda \left( \gamma^{zz} F^{(0)}_{r z} \mp \gamma^{zz} F^{(0)}_{u \pm z} \right) \right) \right]
\]

\[
\approx -\int_{\mathcal{I}^+} dt_+ 2\gamma_{zz} d\bar{z} c(z, \bar{z}) F^{(2)}_{u \mp r} + \int_{\mathcal{I}^-} dt_- 2\gamma_{zz} d\bar{z} c(z, \bar{z}) F^{(2)}_{u \pm r}
\]

where we used that \( F^{(0)}_{r z} = 0 \) due to the assumed fall-off conditions (Equation (8)), as well as the vanishing property \( F^{(2)}_{u r} (+\infty, z, \bar{z}) = F^{(2)}_{u r} (-\infty, z, \bar{z}) = 0 \) (cf. 17 Recall that \( \hat{z} \) is a coordinate parametrising \( S^2 \).
Equation (11). Observe that the combinations $g^{u+}g^{u+}$, $g^{r+}g^{z+}$ and $g^{r+}g^{z+}$ have the same sign regardless of the choice of signature for Minkowski metric.

The matching property (see Equation 15) $F^{(2)}_{u^+}(-\infty, z; \bar{z}) = F^{(2)}_{u^-}(+\infty, z; \bar{z})$ implies that

\[
0 \approx -4\pi \mathcal{S}^{\partial, \text{soft}}_{I^+\cup I^-}[\Lambda] \\
\approx -\int I^+ du_+ \gamma_{zz} dz d\bar{z} \Lambda_{zz}(\gamma_{zz} F^{(0)}_{u^+}) + \int I^- du_- \gamma_{zz} dz d\bar{z} \Lambda_{zz}(\gamma_{zz} F^{(0)}_{u^-}) + \text{c.c.} \\
\equiv -4\pi \left[ Q^{+\text{soft}}_\varepsilon - Q^{-\text{soft}}_\varepsilon \right],
\]

which proves the conservation of the charge and is tantamount to the calculations presented in [KPS17].

2.6.3. Soft charge from the BV-BFV perspective. To summarise, in the previous two subsections we have shown that, in the absence of matter, the vanishing of condition (25), we obtain:

\[
\text{Evaluating the ghost at } \Lambda \text{ implies that }
\text{the same sign regardless of the choice of signature for Minkowski metric.}
\]

Assuming a gain the matching corresponding contribution to the boundary action is given by:

\[
\text{boundary action on shell (Equation (22)) implies the conservation of the soft charge:}
\]

\[
Q^{+\text{soft}}_\varepsilon \approx Q^{-\text{soft}}_\varepsilon.
\]

This result then is independent of the coordinates chosen and the parametrisation of the asymptotic fields.

2.6.4. Hard charge in $(R, s, l)$ coordinates. For the computation of the “hard” charge we assume that the current $J$ has asymptotic behaviour determined by [13]. The corresponding contribution to the boundary action is given by:

\[
S^{\partial, \text{hard}}_{H^+\cup H^-} = \lim_{R \to \infty} \int_{\partial W_R} cJ.
\]

Evaluating the ghost at $\Lambda$, we obtain

\[
S^{\partial, \text{hard}}_{H^+\cup H^-}[\Lambda] = \int_{H^+} \lim_{\tau \to \infty} (\Lambda(v) \rho_+(v) dv - \int_{H^-} \lim_{\tau \to \infty} (\Lambda(v) \rho_-(v) dv,
\]

where the relative sign comes from the parametrisation of $H^-$. Using the asymptotic expansion of $\Lambda$ at time-like infinity (Equation (10)), this becomes:

\[
S^{\partial, \text{hard}}_{H^+\cup H^-}[\Lambda] = \int_{H^+} \Lambda_{H^+}(v) \rho_+(v) dv - \int_{H^-} \Lambda_{H^-}(v) \rho_-(v) dv,
\]

so applying (21), we obtain:

\[
S^{\partial, \text{hard}}_{H^+\cup H^-}[\Lambda] = \frac{1}{4\pi} \int \frac{\varepsilon^+(l)}{(v \cdot l)^2} \rho_+(v) d^2l - \frac{1}{4\pi} \int \frac{\varepsilon^-(l)}{(v \cdot l)^2} \rho_-(v) d^2l
\]
\[
= \frac{1}{4\pi} \int \frac{v \cdot V^+(l)}{v \cdot l} \rho_+(v) d^2l - \frac{1}{4\pi} \int \frac{v \cdot V^-(l)}{v \cdot l} \rho_-(v) d^2l,
\]

where in the second step we used the identity (20). Assuming again the matching condition (25), we obtain:

\[
S^{\partial, \text{hard}}_{H^+\cup H^-}[\Lambda] = Q^{\text{hard}+}_\varepsilon - Q^{\text{hard}^-}_\varepsilon
\]

where

\[
Q^{\text{hard}+}_\varepsilon = \frac{1}{4\pi} \int \frac{v \cdot V^+(l)}{v \cdot l} \rho_+(v) d^2l = \frac{1}{4\pi} \int V^+(l) \cdot V_J(+\infty, l) d^2l,
\]

where we inserted the expression for $V_J$ given by equation (14). Similarly:

\[
Q^{\text{hard}^-}_\varepsilon = \frac{1}{4\pi} \int \frac{v \cdot V^-(l)}{v \cdot l} \rho_-(v) d^2l = \frac{1}{4\pi} \int V^-(l) \cdot V_J(-\infty, l) d^2l.
\]
asympotic symmetries in the BV-BFV formalism

2.6.5. Hard charge in retarded coordinates. We use a coordinate system adapted to the \( \tau \)-hyperboloid part of the boundary of \( W_R \) (see Section 1.3.1), and then take a limit for \( \tau \to \infty \). Observe that the fall-off conditions for the current \( J \) imply that the terms on \( I^\pm_R \) vanish in the limit \( R \to \infty \), so we can discard them from the outset. We compute

\[
S_{\mathcal{H}^+ \cup \mathcal{H}^-}^{\partial, \text{hard}}[\Lambda] = \lim_{\tau \to \infty} S_{\mathcal{H}^+ \cup \mathcal{H}^-}^{\partial, \text{hard}}[\Lambda] = \lim_{\tau \to \infty} \int_{\mathcal{H}^+ \cup \mathcal{H}^-} d\text{Vol}_{\mathcal{H}^\pm} \Lambda|_{\mathcal{H}^\pm} J
\]

\[
= \lim_{\tau \to \infty} \int_{\mathcal{H}^+ \cup \mathcal{H}^-} d\text{Vol}_{\mathcal{H}^\pm} \tau^3 J \int d^2 \hat{x}' G(y; \hat{x}') \varepsilon(\hat{x}')
\]

\[
= \frac{1}{4\pi} \int d^2 \hat{x}' \int d\text{Vol}_{\mathcal{H}^\pm} \frac{Y \varepsilon(\hat{x}')}{(Y \cdot \hat{x}')^2} \rho_{\pm}(Y)
\]

\[
= \frac{1}{4\pi} \int d^2 \hat{x}' \int d\text{Vol}_{\mathcal{H}^\pm} \frac{Y \cdot V_{\varepsilon}}{Y \cdot \hat{x'}} \rho_{\pm}(Y)
\]

\[
= Q_{\varepsilon}^{\text{hard}+} - Q_{\varepsilon}^{\text{hard}-},
\]

where we used Equation (13) to rewrite \( J^{(3)} \) and Equation (20) between lines 4 and 5.

2.6.6. Total charge from the BV-BFV perspective. The total charges are given by:

\[
Q_\varepsilon^+ = Q_{\varepsilon}^{\text{soft}+} + Q_{\varepsilon}^{\text{hard}+} \approx \frac{1}{4\pi} \int d^2 IV_{\varepsilon}(l)(V_f(+\infty, l) + V^\text{out}(-\infty, l))
\]

\[
= \frac{1}{4\pi} \int d^2 IV_{\varepsilon}(l)V(-\infty, l)
\]

and

\[
Q_\varepsilon^- = Q_{\varepsilon}^{\text{soft}+} + Q_{\varepsilon}^{\text{hard}+} \approx \frac{1}{4\pi} \int d^2 IV_{\varepsilon}(l)(V_f(-\infty, l) + V^\text{in}(-\infty, l))
\]

\[
= \frac{1}{4\pi} \int d^2 IV_{\varepsilon}(l)V'(+\infty, l),
\]

so vanishing of the boundary action in the BV-BFV formalism implies the charge conservation

\[
Q_\varepsilon^+ \approx Q_\varepsilon^-.
\]

2.7. Summary. Using the classical BV-BFV formalism [CMR14] to induce compatible BFV data for electrodynamics, we have shown how the on-shell vanishing of the boundary action leads to the existence of a conserved quantity \( Q_\varepsilon \), which in the recent literature (see e.g. [KPS17]) is suggested to be the Noether charge of a large gauge symmetry. On the other hand, [Her17] argues that the transformation in question is not a symmetry, but rather a map into a different sector of the theory.

In our analysis, we have shown how the formulas for \( Q_\varepsilon \) presented in [KPS17] and [Her17] both follow from the, more abstract, BFV formalism, upon choosing the appropriate coordinate system and asymptotic variables. We agree with the claim of [Her17] according to which, in order to derive the conservation of \( Q_\varepsilon \), one
needs to consider transformations of the gauge potential that lead away from the Lorentz gauge. We also show that, by appropriate choice of such transformation, one can reproduce the formulas derived in [CL15, KPS17, CE17].

On the other hand, we have shown that $Q_ε$ is related to the boundary action $S^\partial$, which in the BFV language is, morally speaking, the contraction $i_\eta \alpha ^\partial$, analogous to the contraction of the symmetry operator with Noether’s one-form (see Remark 3). Hence our interpretation is also close to the one of [KPS17].

Important implications for the quantum theory follow from the fact that LGTs of [CL15, KPS17, CE17] are transformations between theories in different gauges. Indeed the results of [DW19] suggest that different gauges lead to unitarily inequivalent theories, i.e. representing different sectors, as also stated in [Her17]. We plan to say more about the behaviour of the quantised theory upon the choice of gauge in our future work.

We conclude by observing that, after phrasing charge conservation in the BV setting with boundary, one expects to recover the related quantum statement once the BV-quantisation with boundary, as developed in [CMR18], is appropriately implemented.

3. The Scalar Field Theory

In this section we discuss the case of asymptotic symmetries in the scalar field. Inspired by the work in [CC18, CFHS19], we will show how the correct asymptotic symmetries for the scalar field can be obtained from the BV-BFV approach to the dual two-form model. In doing so, we provide an alternative to the analysis presented in [CFHS19], which instead considered symmetries of the dual model to be given by elements of the cohomology group $H^2(M)$. In Section 3.5 we show that this type of symmetry (akin to constant shifts of a scalar field) does not admit a BV description.

Throughout, we consider $(M, g)$ to be a closed, 4-dimensional Lorentzian manifold with boundary, and the space of classical fields $F_{cl} := C^\infty(M, \mathbb{R}) \ni \phi$. The classical action functional is given by

$$S_{cl} = \int_M \star_g d\phi \wedge d\phi$$

where $\star_g$ is the Hodge-dual operator defined by the pseudo-Riemannian metric $g$.

Scalar field theory is classically equivalent to a theory of 2-forms $B \in \Omega^2(M)$

$$S_{\text{dual}} = \int_M dB \star_g dB,$$

meaning that, on-shell, we can set $dB := \star_g d\phi$ and — up to symmetries — we obtain a diffeomorphism of the spaces of solutions of the Euler Lagrange equations of the two models. In what follows we will drop the subscript $g$ and denote $\star_g \equiv \star$.

Remark 12. To be precise, note that the equation of motion $d \star d\phi = 0$ is the statement that the three-form $H_\phi := \star d\phi$ is closed. Assuming that $H^3(M) = 0$, we can then find $B \in \Omega^2(M)$ such that $H_\phi = dB$, as done above. This is clearly the case on Minkowski space $M = \mathbb{R}^4$, but it is in general not true, and the condition $H^3(M) = 0$ needs to be checked.

Remark 13. Clearly the definition of $B$ is not unique, and the theory enjoys a symmetry $B \to B + d\gamma$. Notice that $\gamma$ is also defined up to an exact form, and thus enjoys an additional symmetry $\gamma \to \gamma + d\tau$. This is an example of a reducible symmetry, which can be easily treated in the BV formalism.

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18One could as well consider complex-valued scalars.
In what follows we will analyse the dual field theory in the BV formalism, and then recover the “soft” scalar charges of \([CCM18]\). We assume that \(H^3(M) = 0\).

3.1. **BV-BFV analysis of the dual model.** The dual model has a built-in symmetry given by rescaling \(B\) by a closed form \(\beta\). The standard way to proceed here would be considering the “gauge” symmetry of \(B\) in terms of exact forms \(B \rightarrow B + d\gamma\), and extend the dual model to the BV setting.

A different point of view was proposed in \([CFHS19]\), which relies on symmetries \(B \rightarrow B + \beta\), generated by \(\beta \in H^2(M)\) \(dR\), i.e. a closed but not exact form\(^{19}\) on \(M\). We will turn to this latter possibility — and the complications that arise — in Section 3.6 after we have analysed the standard case.

Notice that in \(B \rightarrow B + d\gamma\), the form \(\gamma\) also enjoys a symmetry, as we can freely map \(\gamma \rightarrow \gamma + d\tau\). The BV formalism produces the extended action functional

\[
S_{BV}^{\text{dual}} = \int_M d\beta \star d\beta + B^\dagger d\gamma + \gamma^\dagger d\tau,
\]

and the BV operator \(\iota_Q \Omega_{dual} = \delta S_{BV}^{\text{dual}}\)

\[
QB = d\gamma \quad Q\gamma = d\tau \quad Q\tau = 0
\]

\(Q B^\dagger = d \star dB \quad Q\gamma^\dagger = d B^\dagger \quad Q\tau^\dagger = d\gamma^\dagger\)

that satisfies \([Q, Q] = 2Q^2 = 0\), on the space of fields:

\[
\mathcal{F}_{dual} := T^*[-1] \{ \Omega^0(M)[2] \times \Omega^1(M)[1] \times \Omega^2(M) \} \ni (\tau^\dagger, \gamma^\dagger, B, B^\dagger, \gamma, \tau).
\]

If \(M\) has a non-empty boundary, it is easy to check that the boundary one-form

\[
\alpha^B_{dual} := \int_{\partial M} \delta BH_B + B^\dagger \delta\gamma + \gamma^\dagger \delta\tau
\]

with \(H_B := \star dB|_{\partial M}\), and the boundary action

\[
S_{dual}^0 := \int_{\partial M} d\gamma H_B + d\tau B^\dagger
\]

(28)

satisfy the BV-BFV axioms, namely

\[
\iota_Q \Omega_{dual} = \delta S_{dual}^{BV} + \pi^* \alpha_{dual}^B
\]

and

\[
\frac{1}{2} \iota_Q \iota_Q \Omega_{dual} = \pi^* S_{dual}^0.
\]

For simplicity of notation, we will drop the subscript “dual” in what follows.

3.2. **Asymptotic symmetries of the dual model.** We would like now to revert to the scalar field description, and use the boundary action found so far as a generator for our asymptotic charges. In what follows, we will reconstruct asymptotic symmetries from the BV-BFV formulas obtained so far, after choosing appropriate fall-off conditions. Observe that the on-shell condition \(\star dB = d\phi\) restricts to the boundary

\[
H_B \equiv \star dB|_{\partial M} \approx d\phi|_{\partial M}
\]

and if the boundary component has a boundary of its own, for example a sphere at the corner of a lightlike boundary, we get

\[
S^0 \approx \int_{\partial M} d\gamma d\phi|_{\partial M} = \int_{\bigcup_i \partial(M)^i} d\gamma \phi,
\]

\(^{19}\)One way to go around the nontrivial topology requirement is to work with cohomologies relative to the boundary. That, to the best of our knowledge, does not seem to be mentioned in the literature on this subject.
where the symbol \( \approx \) means that we enforced the equations of motion and set anti-fields to zero, and \((\partial M)^i\) denotes the \(i\)-th connected component of the boundary (with appropriate orientation).

Evaluating the boundary action on a specific gauge parameter \( \Gamma \in \Omega^1(M) \) (see Remark 4), we can extract the volume form of \( S^2 \) from the two-form \( d\Gamma \), defining a function \( \lambda \in C^\infty(S^2) \) such that

\[
d\Gamma(x) = \lambda(x)dS^2,
\]

so that the corner term becomes

\[
S^\partial[\Gamma] = \int_{\bigcup_i \partial(\partial M)^i} \lambda(x)\phi dS^2. \tag{29}
\]

3.3. Asymptotic symmetries of the free scalar field. We would like to discuss now how formula (29) produces the correct asymptotic charge \( s \) for scalar fields. From now on, we restrict our discussion to asymptotically flat Lorentzian manifolds \( M \) and, for simplicity, one can consider Minkowski spacetime \( ^{20} \).

In retarded light-cone coordinates we define a boundary at infinity \( \mathcal{I} \) by the condition \( r = R \to \infty \). We consider scalar fields with the following radial dependence \( ^{21} \):

\[
\phi = \sum_{k=1}^{\infty} \phi^{(k)} r^{-k},
\]

with \( \phi^{(k)} \) independent of \( r \), so that

\[
d\phi = \frac{1}{r} d\phi^{(1)} + \sum_{k=2}^{\infty} \frac{1}{r^k} d\phi^{(k)} - dr \sum_{k=1}^{\infty} \frac{1}{r^{k+1}} \phi^{(k)}.
\]

Observe that the \( dr \) part is obviously not present in the restriction \( d\phi|_{\partial M} \). From the definition of \( \mathcal{H}_\phi = \star d\phi \) we get

\[
\mathcal{H}_\phi = \sum_{k=1}^{\infty} \frac{1}{r^k} \star d\phi^{(k)} - \star dr \sum_{k=1}^{\infty} \frac{1}{r^{k+1}} \phi^{(k)},
\]

so, requiring that \( \mathcal{H}_\phi = dB \), we are lead to

\[
B = \sum_{k=1}^{\infty} \frac{1}{r^k} B^{(k)},
\]

and we set the fall-off condition for \( \gamma \) to be such that \( \gamma = \sum_{k=1}^{\infty} \frac{1}{r^k} \gamma^{(k)} \).

As in the case of QED, we can express the above results in terms of \((R, s, l)\) variables. Following [Her95], we define

\[
\lim_{R \to \infty} R\varphi(x + Rl) = \chi(x \cdot l, l),
\]

and identify

\[
\chi(s, l) = \phi^{(1)}_+(u_+, \hat{x}).
\]

We assume the fall-off conditions

\[
|\chi(s, l)| < \frac{\text{const.}}{s^\epsilon}, \tag{30}
\]

\[
|\dot{\chi}(s, l)| < \frac{\text{const.}}{s^{1+\epsilon}}, \tag{31}
\]

\(^{20}\)Possible global effects will not be discussed here.

\(^{21}\)We consider this radial expansion in order to match with [CFHS19].
so that \(\chi(\pm\infty, l) = 0\) (i.e. \(\phi^{(1)}_+(\infty, \hat{x}) \equiv 0\)), and recall that in the absence of external currents we have

\(\Box \phi = 0\).

Under these assumptions, it was shown in [Her95] that

\[
\phi(x) = -\frac{1}{2\pi} \int \chi(x \cdot l, l) d^2 l.
\]

(32)

Now consider the past null asymptotics. Take a homogeneous function \(\chi'\) satisfying (30) and (31) with \(l\) replaced by past-pointing null directions \(-l\). We then have

\[
\lim_{R \to \infty} R \phi(x - R l) = \chi'(x \cdot l, l).
\]

and identify \(\chi'(s, l) = \phi^{(1)}(u_-, \hat{x})\). Fall-off conditions analogous to (30) and (31) imply that \(\chi'(\infty, l) \equiv 0\) (i.e. \(\phi^{(1)}(\infty) = 0\)).

The field can now be also expressed as:

\[
\phi(x) = -\frac{1}{2\pi} \int \chi'(x \cdot l, l) d^2 l.
\]

(33)

Comparing (32) and (33) we obtain:

\[
\int (\dot{\chi}(s, l) + \dot{\chi}'(s, l)) d^2 l = 0.
\]

It was shown in [Her95] that this in fact implies

\[
\dot{\chi}(s, l) + \dot{\chi}'(s, l) = 0.
\]

We obtain the existence of the limits \(\chi(-\infty, l)\) and \(\chi'(\infty, l)\) as well as

\[
\chi(s, l) + \chi'(s, l) = \chi(-\infty, l) = \chi'(\infty, l) \quad (34)
\]

This is again the matching property, analogous to (15).

We have seen that, as a consequence of the interplay between field equations and fall-off conditions, in the absence of external currents, the asymptotes of the scalar field at \(\mathcal{I}^+\) and \(\mathcal{I}^-\) and at \(i^\pm\) vanish:

\[
\phi^{(1)}_+(\infty) = \phi^{(1)}_-(\infty) = 0,
\]

\[
\lim_{\tau \to \infty} \phi|_{\mathcal{H}^\pm} = 0.
\]

Hence, with the area form on \(S^2\) being proportional to \(r^2\) in retarded coordinates, one shows that the corner term (29) is given by

\[
S^{\partial, \text{soft}}[\Gamma] \approx -\int_{\mathcal{I}^+} \lambda^{(1)} \phi^{(1)}_+ d^2 \Omega + \int_{\mathcal{I}^-} \lambda^{(1)} \phi^{(1)}_- d^2 \Omega \approx 0,
\]

(35)

which coincides with the conservation of the (smeared) asymptotic charge, as analysed in [CC18, CFHS19], with \(\lambda^{(1)}\) an arbitrary function on the two dimensional celestial sphere.

In the notation of [Her95], we can write this as:

\[
-\int \lambda^{(1)}(l) \chi(-\infty, l) d^2 l + \int \lambda^{(1)}(l) \chi'(\infty, l) d^2 l \approx 0,
\]

which is the smeared version of the matching property (34) and we identify:

\[
Q^{\text{soft}+}_{\chi^{(1)}} \equiv -\int_{\mathcal{I}^+} \lambda^{(1)} \chi(-\infty, l) d^2 l, \quad Q^{\text{soft}-}_{\chi^{(1)}} \equiv -\int_{\mathcal{I}^-} \lambda^{(1)} \chi'(\infty, l) d^2 l.
\]
Hence (35) is the on-shell charge conservation:

\[ Q_{\Lambda(1)}^{\text{soft}+} \approx Q_{\Lambda(1)}^{\text{soft}−} \ . \]

### 3.4. Soft charge from the Fourier transform.

Another way to interpret formula (35) uses the Fourier representation of the field, so can be applied only on Minkowski spacetime. In [Her95] one writes the Fourier representation of the field \( \phi(x) \) (denoted by \( A(x) \) in the original) as

\[
\phi(x) = \frac{1}{\pi} \int a'(p) \delta(p^2)e^{ip^0x}d^4p = \frac{1}{\pi} \int \frac{d^3\vec{p}}{2E_p} a'(E_p, \vec{p})e^{-ix\cdot(E_p-\vec{p})d^4p}
\]

Let

\[
a(\vec{p}) \equiv a'(|\vec{p}|, \vec{p}) ,
\]

and we define (analogously to [CCM18]) the unsmeared soft charge as:

\[
Q_{\Lambda(1)}^{\text{soft}+} := \lim_{\omega \to 0} \omega^2 (a(\omega \hat{x}) + a^\dagger(\omega \hat{x})) .
\] (36)

Next, we note that

\[
a'(\omega l) = -\tilde{\chi}(\omega, l)/\omega ,
\]

and use the following formula proven in [Her95]:

\[
\tilde{\chi}(0, l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(s, l)ds = \frac{1}{2\pi} \chi(-\infty, l) .
\]

It is now easy to see that fields with non-vanishing \( \chi(-\infty, l) \) are the infrared singular ones (1/\omega behavior around 0). Inserting this into (36), and identifying \( \chi(-\infty, l) \) in retarded coordinates with \( \phi_{\Lambda(1)}^{(1)}(-\infty) \), we obtain:

\[
Q_{\Lambda(1)}^{\text{soft}+} \sim \lim_{\omega \to 0} (\omega \tilde{\chi}(\omega, l)/\omega + c.c.) \sim \phi_{\Lambda(1)}^{(1)}(-\infty, \hat{x}) ,
\]

so smearing with an arbitrary function \( \lambda(\xi) \) on the two-dimensional celestial sphere, we obtain

\[
Q_{\Lambda(1)}^{\text{soft}+} \sim \int_{\mathbb{S}^2} \lambda(\xi) \phi_{\Lambda(1)}^{(1)} d^2\Omega ,
\]

as expected.

#### 3.5. Hard charges for scalar fields.

The calculation of hard charges for scalar fields (see [CCM18]) requires the introduction of source terms. Unfortunately, this breaks the duality outlined in Remark 12 in a nontrivial way, and finding an alternative local field-theoretic model relating scalar and 2-form field dynamics is a difficult problem.

The approach of [CFHS19] to hard scalar asymptotic charges, and its link to a non-local dual model with sources, do not fit directly into our language, so we were not able to simply adapt the argument used in their work.

It would be interesting to understand how the two approaches fit together and we plan to address this question in our future work.

#### 3.6. Shift symmetries by zero modes, global gauge transformations.

In this section we would like to analyse a particular class of transformations that arise from considering either shifting a scalar field by a constant or, dually, the \( \mathcal{B} \) field by an element of \( H^2(M) \). In [CFHS19] these are called large gauge transformations, because in their work they are interpreted as ultimately being the same. We prefer to resort to the more standard nomenclature and refer to them as global gauge transformations.

The action functional for a scalar field does not admit local gauge symmetries, but it admits shifts by constant maps

\[
\phi \mapsto \phi + \alpha
\]
where $\alpha$ is a constant function on $M$, i.e. $d\alpha(x) = 0$ or $\alpha \in H^0(M)$.

Similarly, we have a symmetry for $B$ generated by closed-but-not-exact forms $\beta \in H^2(M)$, i.e. $d\beta = 0$ but $\beta \neq d\gamma$. We observe, *en-passant*, that this is not possible on Minkowski space, since $H^2(M)$ = 0.

Note that both these transformations are to be considered symmetries of the zero modes (more than a symmetry of the fields), i.e. elements of the kernel of the kinetic operator (that is the quadratic part of the Lagrangian density). We will see shortly that these transformations do not really admit a BV description in the usual sense.

Let us try to construct the BV-data for this field redefinition. The space of fields is now ($\alpha$ is promoted to ghost-number 1)

$$F = T^*[−1](F_{cl} \times H^0[1](M))$$

and the extended BV action reads

$$S_{BV, large} = \int_M \ast d\phi \wedge d\phi + \phi^\dagger \alpha,$$

with $\phi^\dagger$ the cotangent field for $\phi$.

The $−1$-shifted symplectic BV-form is

$$\Omega = \int \delta \phi \delta \phi^\dagger + \delta \alpha \delta \alpha^\dagger,$$

and the action of the BV operator $Q$ on fields is

$$Q\phi = \alpha; \quad Q\phi^\dagger = d \ast d\phi; \quad Q\alpha = 0; \quad Q\alpha^\dagger = \phi^\dagger$$

so that $\iota_Q \Omega = \delta S + \pi^* \alpha \beta$. On the other hand, the BV extension for large symmetries in the case of the dual model reads

$$S_{BV, dual, large} = \int_M dB \ast dB + B^\dagger \beta$$

and the associated BV operator

$$QB = \beta \quad QB^\dagger = 0$$

$$QB^\dagger = d \ast dB \quad Q\beta^\dagger = B^\dagger$$

The problem with the above naïve construction is that these operators are not coboundaries, i.e. $Q^2 \neq 0$. In fact, we compute

$$Q^2 \alpha^\dagger = Q(Q\alpha^\dagger) = Q\phi^\dagger = d \ast d\phi \neq 0,$$

and similarly for the dual model:

$$Q^2 \beta^\dagger = d \ast dB \neq 0.$$

both of which only vanish on shell. Hence this construction (for symmetries given by constants and, dually, elements of the second cohomology group) does not yield a BV data.

Ignoring this and pushing through with the calculations, for the shift $B \rightarrow B + \beta$ with $\beta \in H^2(M)$, one gets a "boundary action"

$$S_{\partial \mathcal{W}_n}[\beta] \approx \int_{\partial M} \beta d\phi = \int_{\partial M} d(\beta \phi)$$

(37)

which is a corner term. In the limit $R \rightarrow \infty$, assuming the same fall-off $\beta = \sum_{k=1}^{\infty} r^{-k} \beta^{(k)}$ we get that $d\beta = 0$ implies that $\beta^{(k)} = d\beta^{(k+1)}$. In particular
\[ \beta^{(0)} = 0 = d\beta^{(1)} \text{ and } \beta^{(1)} = d\beta^{(2)}. \]

Then, equation (37) becomes

\[
S^\beta_{I}\left[\beta\right] = \lim_{R \to \infty} S^\beta_{W_\mathcal{R}}[\beta] = \int_{\mathcal{I}} d(\beta^{(1)} \phi^{(1)}) = \int_{S^2} d(\beta^{(2)} \phi^{(1)}) = \int_{S^2} d\text{Vol}_{S^2} \tilde{\lambda} \phi^{(1)}|_{-\infty}^{+\infty}.
\]

This is the same conclusion as the one reached in [CFHS19]. It is evident, though, that on spaces with trivial second cohomology, the procedure used in [CFHS19] needs to be better understood. One possibility might be to phrase this in terms of relative cohomology (see, e.g., [BT13]). We note, however, that the construction presented in Sections 3.1 and 3.2 avoids this problem, while still reproducing the correct asymptotic behaviour. Thus, we believe it provides a neat description of how soft charges emerge from the symmetries of the dual model.

One possible way to overcome the difficulty above might be to think of the global transformations for the scalar/dual model as shifts in zero modes, rather than proper symmetries. One can extend the BV construction in order to consider infinitesimal shifts in the space of zero modes, a framework that is related to formal geometry. In a nutshell, introducing a differential \( d \) on the space of solutions to \( d \ast dB = 0 \), which we can think of being a field-version of de Rham differential, we obtain that, for \( \beta \in \Omega^2(M)_{\text{coclosed}} \),

\[
S_{\text{formal}} = \int_{M} d(B + \beta) \ast d(B + \beta) + B^i d\beta
\]

satisfies the differential modified classical master equation:

\[ \{S_{\text{formal}}, S_{\text{formal}}\} = dS_{\text{formal}}^\beta \]

reconstructing a formal version of the BV-BFV construction. We refer to [BCM12, Section 3] and [CMW19, Section 3.3.2] for an introduction of this technique in relation to the Poisson sigma model and, more generally, AKSZ theories, and defer its analysis for the case at hand to a subsequent work.\(^{22}\)

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