Entanglement Relativity in the Foundations of The Open Quantum Systems Theory

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Abstract Realistic many-particle systems dynamically exchange particles with their environments. In classical physics, small variations in the number of constituent particles are commonly considered practically irrelevant. However, in the quantum mechanical context, such and similar structural variations are generically taxed due to the so-called Entanglement Relativity. In this paper we point out difficulties in deriving master equation for a subsystem of an alternative partition of the closed quantum system. We find that the Nakajima-Zwanzig projection method cannot be straightforwardly used to solve the problem. The emerging tasks and prospects for the consistent foundations are examined.

1. Introduction

Realistic many-particle systems dynamically exchange particles with their environments and with other systems. This trivial observation is as yet largely intact in the foundations of quantum theory, including quantum measurement, decoherence and the open quantum systems theory. While intuitively exchange of particles is completely clear, it sets a blurred separating line between a system and its environment. Classical physics straightforwardly tackles such situations e.g. in motion with variable mass or in the Grand Canonical Ensemble for systems in thermal equilibrium. However, systematic quantum-mechanical description of such processes is lacking.

In quantum measurement or decoherence [1,2,3] or more generally in open quantum systems [4,5,6] theory, the border line between 'system' and 'environment' is rarely analyzed in depth. We construe this fact as a symptom of a subtle and hard problem that is the subject of this paper.

The aim of this paper is to diagnose the problem that is still largely unrecognized but nevertheless of the fundamental importance in the field of open quantum systems and applications. We hope, that this first step in

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noticing and identifying the problem can help in setting outlines of further progress in the field.

In Section 2 we define the problem. In Section 3 we generalize our findings in the context of the so-called projection-method approach. Section 4 is an illustration of our considerations and their subtlety. Section 5 is brief discussion and conclusion.

2. The problem

Planet Earth is constantly bombarded from the outer space. Provided that the captured-projectile mass is much less than the mass of the Earth, the variations in the Earth’s orbit around the Sun are practically negligible. Furthermore, classical physics straightforwardly accounts for huge stochastic change in the number of particles in the many-particle systems in thermal equilibrium. The Grand Canonical Ensemble of the standard classical statistical mechanics straightforwardly describes a composite system of $N$ particles with the variations of the number of particles from the set \{1, 2, 3, ..., $N$\}. There is also no problem with description of non-equilibrium mesoscopic systems, such as Brownian particle, which is dressed by the water molecules that constantly stick to and come off the particle’s surface–the particle’s ‘hydration shell’ known also for large molecules (e.g. protein molecules and other biopolymers) in a solution.

Except for the ‘chaotic’ systems, it seems that classical physics embraces the following rule: Knowledge of dynamics of an $N$-particle classical system allows straightforward deduction of dynamical laws for such $(N - n)$-particle systems under the same physical conditions (external fields and interactions), as long as $n \ll N$. More intuitively, it seems that individuality of many-particle classical systems is not threatened by tiny changes in the system’s separation from the rest of the world–e.g. Brownian particle with $n$ attached water molecules is typically regarded as practically the same system as with $n'$ ($\neq n$) attached water molecules.

However, in the quantum mechanical context, the things stand differently. To see this, let us consider the Hamiltonian of interest in the quantum Brownian model, in which the environment $E$ monitors the particle’s center of mass position [4] (and references therein). For simplicity, we refer to one dimensional system and the environment consisting of non-interacting harmonic oscillators:
\[ \hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_{SE} \] (1)

where

\[ \hat{H}_S = \sum_{i=1}^{N_S} \frac{\hat{p}_i^2}{2m_i} + \sum_{i \neq j=1}^{N_S} V(|\hat{x}_i - \hat{x}_j|) \]

\[ \hat{H}_E = \sum_{\alpha=1}^{N_E} \left( \frac{\hat{p}_\alpha^2}{2m_\alpha} + \frac{1}{2} m_\alpha \omega_\alpha^2 \hat{x}_\alpha^2 \right) \]

\[ \hat{H}_{SE} = \hat{X}_{CM} \otimes \sum_{\alpha=1}^{N_E} \kappa_\alpha \hat{x}_\alpha \] (2)

where the Latin indices refer to the \( S \) system and the Greek indices to the \( E \) system, with the numbers \( N_S \) and \( N_E \) of particles in the system and the environment, respectively, with the pair interactions \( V \), while \( \hat{X}_{CM} = \sum_{i=1}^{N_S} m_i \hat{x}_i / M \), \( M = \sum_{i=1}^{N_S} m_i \).

Consider now that the \( i_0 \)th particle of the \( S \) system becomes a part of the environment. This is, instead of the \( S \) system and the environment \( E \) there is the new open system \( S' = S \setminus i_0 \) and the new environment \( E' = E \cup i_0 \), which symbolically reads as a structural transformation:

\[ \{S, E\} \rightarrow \{S', E'\}, \] (3)

while the composite system \( C \) as a whole remains intact by the transformation: \( S + E = C = S' + E' \).

Needless to say, the composite system’s Hamiltonian eq.(1) remains intact by the transformation, while it takes another form:

\[ \hat{H} = \hat{H}_{S'} + \hat{H}_{E'} + \hat{H}_{S'E'} \] (4)

where:
\[ \hat{H}_{S'} = \sum_{i=1}^{N_S-1} \frac{p_i^2}{2m_i} + \sum_{i \neq j=1}^{N_S-1} V(|\hat{x}_i - \hat{x}_j|) \]

\[ \hat{H}_{E'} = \hat{H}_E + \frac{\hat{p}_{i_0}^2}{2m_{i_0}} + \frac{1}{M} \hat{x}_{i_0} \otimes \sum_{a=1}^{N_E} \kappa_\alpha \hat{x}_\alpha \]

\[ \hat{H}_{S'E'} = \frac{1}{M} \sum_{i=1}^{N_S-1} m_i \hat{x}_i \otimes \sum_{a=1}^{N_E} \kappa_\alpha \hat{x}_\alpha + \sum_{j=1}^{N_S-1} V(|\hat{x}_{i_0} - \hat{x}_j|). \quad (5) \]

The following simplifications can make the two models eq.(2) and eq.(5) similar to each other: (i) for \( N_S \gg 1 \), the total mass \( M \approx M' = \sum_{i=1}^{N_S-1} m_i \), (ii) for large \( M \) (i.e. \( M' \)), one can neglect the last term in \( \hat{H}_{E'} \), or more generally to introduce the normal coordinates for the new environment \( E' \) such that it consists of mutually non-interacting quasi-particles [7,8] and (iii) the pair interactions in \( \hat{H}_{S'E'} \) to consider as a weak perturbation. Then, according to the classical intuition, the two open systems \( S \) and \( S' \) may appear essentially mutually indistinguishable.

However, formal similarity of the quantum Hamiltonians does not in general guarantee that reduced dynamics of the \( S \) system straightforwardly implies the reduced dynamics of the \( S' \) system. This subtle point requires careful examination.

Central to derivation of master equations in the density matrix theory of open quantum systems is the tensor product:

\[ \dot{\hat{\rho}}_S(t) \otimes \dot{\hat{\rho}}_E, \quad (6) \]

which is an ansatz known as Born approximation [4,5,6] and follows in a systematic way from the projection of the composite system’s state \( \dot{\hat{\rho}}(t) \) [9,10]:

\[ \mathcal{P}(\dot{\hat{\rho}}(t)) = \dot{\hat{\rho}}_S(t) \otimes \dot{\hat{\rho}}_E, \quad (7) \]

with \( \mathcal{P}^2 = \mathcal{P} \) and \( \mathcal{Q} = \mathcal{I} - \mathcal{P} \) such that \( \mathcal{Q}^2 = \mathcal{Q} \), while \( \dot{\hat{\rho}}_S(t) = tr_E \dot{\hat{\rho}}(t) \) carries all information about the open system \( S \). Linearity of \( \mathcal{P} \) excludes the choice \( \dot{\hat{\rho}}_E = tr_S \dot{\hat{\rho}}(t) \).

Let us suppose that the reduced dynamics for the \( S \) system is well described by a proper master equation, which assumes validity of eq.(6), i.e. eq.(7). The main observation of this paper is as follows:
(O) As distinct from the classical counterpart, reduced dynamics of the $S$ system cannot in general be used to derive or deduce dynamics of the $S'$ system.

In order to justify the (O), we first emphasize the formal yet substantial distinction between the classical and quantum state spaces. In classical physics, the ‘phase space’ of $N$ particles is Cartesian product of the $N$ ‘phase spaces’ for individual particles. However, in quantum mechanics, the particles state spaces are not in Cartesian but in tensor product. For the composite system $C$ decomposed (structured) as $S + E$, the Hilbert state space $\mathcal{H}$ is the tensor product of the state spaces for the subsystems:

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E. \quad (8)$$

The structural transformation eq. (3) gives rise to re-factorization:

$$\mathcal{H} = \mathcal{H}_{S'} \otimes \mathcal{H}_{E'}. \quad (9)$$

Invariants of the transformation eq. (3) are: (a) the Hilbert state space $\mathcal{H}$, (b) the composite system’s Hamiltonian $\hat{H}$ and (c) the composite system’s state in every instant of time. However, the transformation eq.(3) induces a change in the form of the Hamiltonian (e.g. eqs. (2) and (5)) as well as re-factorization eq.(8) $\rightarrow$ eq.(9). The transformation eq.(3) also leads to a change in the form of the composite system’s instantaneous state. For a pure state in an instant of time $t$, this is known as Entanglement Relativity:

$$|\phi\rangle_S \otimes |\chi\rangle_E = \sum_i c_i |i\rangle_{S'} \otimes |i\rangle_{E'}. \quad (10)$$

Entanglement Relativity (ER) is a recently established (and rediscovered) [11-16] corollary of quantum mechanics that asserts: Virtually every structural transformation that induces tensor re-factorization also induces a change in amount of quantum entanglement in the composite system $C$. If the composite system’s state is tensor-product for one structure ($S + E$) it is of the entangled form for practically all the alternative structures ($S' + E'$) of the composite system. ER regards every (pure) state in every instant in time $t$.

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$^2$ Exceptions to ER are also known, but do not alter our main point. E.g. the transformation eq.(3) does not change a tensor-product state. However, the more general structural transformations change even such states and ER applies [16].
On the basis of ER, it can be shown that also the more general non-classical correlations, quantified e.g. by 'quantum discord', are structure-dependent [17]: A tensor-product mixed state for one structure ($S + E$) acquires a quantum correlated (entangled or discordant) form for virtually arbitrary alternative structure ($S' + E'$) of the composite system, e.g.:

$$\hat{\rho}_S \otimes \hat{\rho}_E = \sum_i \lambda_i \hat{\rho}_{S'i} \otimes \hat{\rho}_{E'i}, \quad \sum_i \lambda_i = 1. \quad (11)$$

Now the classically unknown conditions eqs. (10) and (11) plausibly justify the (O) statement as follows; the more rigorous consideration is given in Section 3. The projection eq.(7), as well as the generalizations given in Section 3, do not possess any quantum correlations. Equations (6) and (7) are of exactly the same form as the l.h.s. of eqs. (10) and (11) which refer to the $S + E$ structure. The r.h.s. of eqs. (10) and (11) that refer to the $S' + E'$ structure are typically not of the form of eqs. (6) and (7). Hence derivation of master equation based on eq.(6), i.e. on eq.(7), for the $S$ system is practically never a simultaneous derivation of the master equation for the $S'$ system. Consequently, the knowledge of dynamics of the $S$ system does not suffice to conclude much about dynamics of the $S'$ system.

For the probably most relevant class of Markovian open systems, eqs. (10) and (11) reveal another layer of consideration. The tensor-product initial state $\hat{\rho}_S(t = 0) \otimes \hat{\rho}_E$ is a necessary condition for Markovian dynamics [5,6]; typically, the environment is supposed thermal, $\hat{\rho}_E = \exp(-\beta \hat{H}_E)/\text{tr}_E \exp(-\beta \hat{H}_E)$, on the inverse temperature $\beta = 1/k_B T$. Due to eqs. (10) and (11), as long as eq.(6) is valid for the $S + E$ structure, it is practically never fulfilled regarding the alternative $S' + E'$ structure. Then Markovian dynamics for the $S$ system is virtually never applicable for the alternative $S'$ system. More specifically: even if eq.(5) can be reduced to eq.(2), and even if the new environment may also be in thermal-equilibrium state, $\sum_i \lambda_i \hat{\rho}_{S'i} = \exp(-\beta' \hat{H}_{E'})/\text{tr}_{E'} \exp(-\beta' \hat{H}_{E'})$, there is initial correlation for the $S'$ and $E'$ systems. Consequently, the reduced $S'$ system’s dynamics is not Markovian [5,6] and is also possibly non-completely positive [18].

Hence, typically, derivation of the $S'$ system’s dynamics has to be started from the scratch–by setting eq.(6) for the $S' + E'$ structure in an independent derivation of master equation for the $S'$ system.

3. On the use of the Nakajima-Zwanzig projection method
One may still wonder if, somehow, dynamics of the $S$ system can be useful for drawing conclusions on the $S'$ system’s dynamics. After all, it’s just one tiny-particle difference between the two structures pertaining to eq. (2) and eq. (5). The correlations present on the r.h.s. of eqs. (10) and (11) are due only to a single particle denoted $i_0$. May it be possible to approximate the r.h.s. of eqs. (10) and (11) by some tensor-product states?

In certain special cases (e.g., when the use of the Born-Oppenheimer approximation is allowed), this may be the case. Nevertheless there still remains open the question as to whether ‘tiny correlations’ can be safely discarded from considerations in general. In this section we are not interested in such subtle and deep questions. Rather, we consider usefulness of the Nakajima-Zwanzig projection method [9,10] in regard of the ($O$) statement.

Our motivation comes from the fact that the Nakajima-Zwanzig and the related (projection-based) methods provide systematic introduction of eq.(6) and set the basis for the general considerations thus representing the up-to-date most general methodological basis of the open systems field [4,5,6]. If the ($O$) statement remains valid in the context of the projection-based methods, then it presents a serious limitation not only to our classical intuition but also to operational procedures in describing dynamics of the alternate open systems. Bearing in mind the classical intuition of Sections 1 and 2, in such a case we face yet another non-trivial task in the context of the problem of transition from quantum to classical.

Below, we completely generalize previous considerations. We are interested in arbitrary bipartitions of a composite system $C$, $S + E = C = S' + E'$, i.e. in arbitrary linear canonical transformations (LCTs) that induce the tensor-product structures (i.e. tensor re-factorization) of the composite system’s Hilbert state space.

The key idea behind the Nakajima-Zwanzig projection method [9,10] is presented by eq.(7). It consists of the introduction of a linear projection operator, $\mathcal{P}$, which acts on the operators of the state space of the composite system ‘system+environment’ ($S + E$). If $\hat{\rho}$ is the density matrix of the composite system, the projection $\mathcal{P}\hat{\rho}$ (the ‘relevant part’ of the composite density matrix) serves to represent a simplified effective description through a reduced state of the composite system. The complementary part (the ‘irrelevant part’ of the composite density matrix), $\mathcal{Q}\hat{\rho} = (I - \mathcal{P})\hat{\rho}$. For the ‘relevant part’, $\mathcal{P}\hat{\rho}(t)$, one derives closed (‘autonomous’) equations of motion in the form of integro-differential equation. The open system’s density matrix $\hat{\rho}_S(t) = tr_E\mathcal{P}\hat{\rho}(t)$ is required to carry all information about the open system.
S, equivalently $tr_E Q \hat{\rho} = 0$.

The Nakajima-Zwanzig projection method assumes a concrete, in advance chosen and fixed for all time-instants, system-environment split (a 'structure'), $S + E$. This split is uniquely defined by the associated tensor product structure of the composite system’s Hilbert space, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$. Division of the composite system into 'system' and 'environment' is practically motivated. In principle, the projection method can equally describe arbitrary system-environment split i.e. arbitrary factorization of the composite system’s Hilbert space. By definition, different factorizations introduce different projectors: $P$ for the $S + E$ structure, and $P'$ for some alternative $S' + E'$ structure of the composite system, such that $\hat{\rho}_S = tr_E P \hat{\rho}(t)$ carries all information about the $S$ system (equivalently $tr_E Q \hat{\rho}(t) = 0$), and $\hat{\rho}_{S'} = tr_{E'} P' \hat{\rho}(t)$ carries all information about the $S'$ system (equivalently $tr_{E'} Q' \hat{\rho}(t) = 0$).

The linear projections can be defined [1,19]: (i) $P \hat{\rho}(t) = (tr_E \hat{\rho}(t)) \otimes \hat{\rho}_E$ [for some $\hat{\rho}_E \neq tr_S \hat{\rho}$], which is eq.(7), (ii) $P \hat{\rho}(t) = \sum_n (tr_E \hat{P}_n \hat{\rho}(t)) \otimes \hat{\rho}_{E_n}$ [with arbitrary orthogonal supports for the $\hat{\rho}_{E_n}$], and (iii) $P \hat{\rho}(t) = \sum_i (tr_E \hat{P}_E \hat{\rho}(t)) \otimes \hat{P}_{E_i}$ [with arbitrary orthogonal projectors for the $E$ system]; by $\hat{P}$, we denote the projectors on the respective Hilbert state (factor) spaces. The physical context fixes the choice of the projection—e.g. by an assumption about the initial state. In this paper we stick to the projection (i), which is by far of the largest interest in foundations and applications of the open systems theory. As it can be easily shown, all the projections (i)-(iii) are free from the quantum correlations (entanglement or discord).

Now we provide the main results of this section that are borrowed from [20] with the proofs placed in the appendices.

**Lemma 1.** For the most part of the composite system’s dynamics, validity of

$$tr_E Q \hat{\rho}(t) = tr_E (\hat{\rho}(t) - P \hat{\rho}(t)) = tr_E (\hat{\rho}(t) - \hat{\rho}_S(t) \otimes \hat{\rho}_E) = 0, \forall t. \quad (12)$$

implies non-validity of

$$tr_{E'} Q' \hat{\rho}(t) = tr_{E'} (\hat{\rho}(t) - \rho_S(t) \otimes \rho_E) = 0, \forall t, \quad (13)$$

and vice versa.

Lemma 1 reveals that the information 'irrelevant part’ of a projected state for one structure contains some relevant information regarding an alternative structure of the composite system for the most of time instants $t$. In formal
terms: for the most part of the composite system’s dynamics, the projection \( Q \hat{\rho} (Q') \hat{\rho} \) brings some information about the open system \( S' \) at variance with the Nakajima-Zwanzig projection idea. Hence \( \partial P \hat{\rho}(t) / \partial t \) allows ‘tracing out’ regarding only one structure. If that structure is \( S + E \), then \( tr_E \partial P \hat{\rho}(t) / \partial t \neq \partial \hat{\rho}_{S'}(t) / \partial t \) [as long as \( \hat{\rho}_{S'}(t) = tr_E \hat{\rho}(t) \)]. This can be seen also from the following argument, which is not restricted to the projection-based methods. Tracing out the \( E \) system is dependent on, but not equal to, the tracing out the \( E' \) system, and vice versa. This dependence follows from the fact that the \( S \) and \( E \) degrees of freedom are intertwined with the \( S' \) and \( E' \) degrees of freedom. Intuitively: ‘\( tr_E \)’ (e.g. integrating over the \( E \)’s degrees of freedom) partly encompasses both the \( S' \) and the \( E' \) degrees of freedom.

**Lemma 2.** The two structure-adapted projectors \( P \) and \( P' \) do not mutually commute

\[
[P, P'] \hat{\rho}(t) \neq 0
\]  

for the most of the time instants \( t \).

Very much like noncomutativity of quantum observables, Lemma 2 asserts that the projection-based information contents regarding different structures of a composite system are mutually exclusive for the most of the time instants \( t \).

Formally, there is no state \( \hat{\rho}(t) \) of a composite system for which the equality \( P \hat{\rho}(t) = \rho(t) = P' \hat{\rho}(t) \) can be fulfilled for arbitrary instant of time \( t \).

Formally, \( \partial P \hat{\rho}(t) / \partial t = \partial \hat{\rho}_{S}(t) / \partial t \otimes \hat{\rho}_E \) is in conflict with \( \partial P' \hat{\rho}(t) / \partial t = \partial \hat{\rho}_{S'}(t) / \partial t \otimes \hat{\rho}_{E'} \): due to eq.(11), only one of them can be correct for arbitrary instant in time \( t \).

Lemma 1 and lemma 2 refer to all projection-based methods and exclude acquisition of information about an open system \( S' \) from the master equation known for the alternative (albeit possibly similar) open system \( S \) in the same instant (or interval) of time, and vice versa. In effect, the \((O)\) statement is justified and leads to the conclusion that derivation of master equations has to be performed for every set of the degrees of freedom (for every open system, \( S, S' \), etc.) separately, in accord with equations (10) and (11).

4. Analysis of the quantum Brownian motion

In order to illustrate subtlety of our considerations, we stick to eq.(2) as the Caldeira-Leggett model of quantum Brownian motion (QBM) [4,21].
In the Schrödinger picture the QBM master equation for the initial separable state \((\hat{\rho}(t = 0) = \hat{\rho}_S(t = 0) \otimes \hat{\rho}_E)\) with the environment on temperature \(T\):

\[
\frac{d\hat{\rho}_S(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_S, \hat{\rho}_S(t)] - \frac{2m\gamma k_B T}{\hbar^2}[\hat{x}_S, [\hat{x}_S, \hat{\rho}_S(t)]] - \frac{i\gamma}{\hbar}[\hat{\rho}_S(t), \{\hat{p}_S, \hat{\rho}_S(t)\}].
\]

(15)

The curly brackets denote the 'anticommutator', \(m\) is the mass while \(\hat{x}_S\) and \(\hat{p}_S\) are the position and momentum of the particle and \(\gamma\) is the semi-empirical friction coefficient.

Eq. (15) is not of the Lindblad form and hence by definition \([5,6]\) is not Markovian. Interestingly enough, eq. (15) applies even for initially correlated state and for arbitrary strength of interaction in the composite system as well as for arbitrary 'spectral density' (which defines the friction coefficient \(\gamma\))\(^3\).

Non-Markovianity of eq. (15) is behind its 'robustness', which is the ultimate basis of the observation of QBM effect for an alternative structure of the total system \(C\) \([7]\).

Now we emphasize the variations offered by eq. (15). First, it is known that eq. (15) can be transformed to a Lindblad form for sufficiently high temperature \(T\) \([4]\). Second, for the massive particle, the second term proportional to \(\gamma\) can be neglected. This constitutes the 'recoilless' variant of the Caldeira-Leggett model and provides the Lindblad-form master equation:

\[
\frac{d\hat{\rho}_S(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_S, \hat{\rho}_S(t)] - \frac{2m\gamma k_B T}{\hbar^2}[\hat{x}_S, [\hat{x}_S, \hat{\rho}_S(t)]]).
\]

(16)

which is similar with the scattering-decoherence master equation—see eq. (3.66) in \([1]\).

It is remarkable that already at the level of fixed structure, \(C = S + E\), we can see non-trivial variations in the form of master equation and consequently regarding the \(S\) system’s dynamics.

Now we refer to certain structural variations of the dynamics described by eq. (16). We are aiming at the cases in which the classical intuition can be justified; see e.g. comments below eq. (5). In all other cases we do not expect the classical intuition to be very useful.

Concretely, we are interested in eq. (3) as well as in the opposite case, i.e. when an environmental particle, denoted \(\alpha_\circ\), is joined the \(S\) system

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\(^3\)See e.g. eq. (4.226) in Ref. [4].
thus providing a new open system, \( S'' = S \cup \alpha \), and new environment, \( E'' = E \setminus \alpha \); needless to say, \( C = S + E = S' + E' = S'' + E'' \). This situation also describes the 'Schrödinger’s cat’—the \( \alpha \)th particle flows out of the radioactive source.

Regarding the structural transformation

\[
(S, E) \rightarrow (S'', E''),
\]

that is accompanied by tensor re-factorization, \( \mathcal{H}_S \otimes \mathcal{H}_E \rightarrow \mathcal{H}_{S''} \otimes \mathcal{H}_{E''} \), the Hamiltonian eq.(2) i.e. eq.(5) takes the form:

\[
\hat{H} = \hat{H}_{S''} + \hat{H}_{E''} + \hat{H}_{S''E''}.
\]

In eq.(18):

\[
\begin{align*}
\hat{H}_{S''} &= \hat{H}_S + \frac{p_{\alpha}^2}{2m_\alpha} + \hat{X}_{CM} \otimes \kappa_\alpha \hat{x}_\alpha \\
\hat{H}_{E''} &= \sum_{\alpha=1}^{N_E-1} \left( \frac{p_\alpha^2}{2m_\alpha} + \frac{1}{2} m_\alpha \omega_\alpha^2 \hat{x}_\alpha^2 \right) \\
\hat{H}_{S''E''} &= \hat{X}_{CM} \otimes \sum_{\alpha=1}^{N_E-1} \kappa_\alpha \hat{x}_\alpha.
\end{align*}
\]

As distinct from eq.(5), eq.(19) is already of the form of eq.(2): there only appears a new interaction in the system’s (in the \( S'' \) system’s) self Hamiltonian. Therefore, the two structure variations, eq.(3) and eq.(17), are not mutually equivalent.

Our starting model is eq.(2), for which we assume the initial tensor product state and thermal environment \( E \):

\[
\dot{\rho}(t = 0) = \dot{\rho}_S \otimes \dot{\rho}_E = \rho_S \otimes \alpha \exp(-\beta \hat{H}_\alpha) Z_\alpha,
\]

with the one-particle 'statistical sum' \( Z_\alpha = tr_\alpha \exp(-\beta \hat{H}_\alpha) \).

\[4\text{For examples of the more general structural transformations see e.g. Refs. [7,16,22,23].}
\]

\[5\text{Rigorously, the new structure (} S + \alpha \) + \( E'' \), where the} \theta \text{ particle is not in interaction with} \ E''. \text{ If we assume that the mass} \ M'' \text{ of} \ S'' \text{ is approximately} \ M, \text{ the two models eq.(2)} \text{ and eq.(19) become practically indistinguishable.} \]
The pair interactions $V$ in eq.(2) produce correlation of the $i$-th particle with some particles in the $S$ system, $\hat{\rho}_S(t = 0) = \sum_m \mu_m \hat{\rho}_m^S \otimes \hat{\rho}_m^i$, which gives correlated initial state for the $S' + E'$ structure:

$$\hat{\rho}(t = 0) = \sum_m \mu_m \hat{\rho}_m^S \otimes \hat{\rho}_m^E \equiv \sum_m \mu_m \hat{\rho}_m^S \otimes (\hat{\rho}_m^i \otimes \hat{\rho}_m^E), \quad \sum \mu_m = 1. \quad (21)$$

However, at variance with eq.(11), eq.(20) directly provides tensor product state for the $S'' + E''$ structure:

$$\hat{\rho}(t = 0) = \hat{\rho}_S \otimes \hat{\rho}_E \equiv \left( \hat{\rho}_S \otimes \frac{\exp(-\beta \hat{H}_\alpha \otimes \exp(-\beta \hat{H}_\alpha) / Z_{\alpha}}{Z_{\alpha}} \right) \otimes \frac{\exp(-\beta \hat{H}_\alpha \otimes \exp(-\beta \hat{H}_\alpha) / Z_{\alpha}}{Z_{\alpha}} \otimes. \quad (22)$$

Further we focus on the $S'' + E''$ structure. The projection adapted to the $S + E$ structure is defined:

$$\mathcal{P}\hat{\rho}(t) = \hat{\rho}_S(t) \otimes \frac{\exp(-\beta \hat{H}_\alpha)}{Z_{\alpha}} \otimes \exp(-\beta \hat{H}_\alpha) / Z_{\alpha}. \quad (23)$$

providing that $\otimes \exp(-\beta \hat{H}_\alpha) / Z_{\alpha} = tr S\hat{\rho}(t)$.

For the $S'' + E''$ structure, the projection reads:

$$\mathcal{P}''\hat{\rho}(t) = (tr_{E',\alpha \otimes \hat{\rho}(t)}) \otimes \hat{\rho}_{E''}. \quad (24)$$

In an instant of time $t > 0$, we expect correlations in the $S''$ system, e.g. $\hat{\rho}_{S''}(t) = \sum_i \lambda_i(t) \hat{\rho}_i^S(t) \otimes \hat{\rho}_i^\alpha(t)$. Hence with the use of eq.(24):

$$tr_{E''} \frac{\partial \mathcal{P}''\hat{\rho}(t)}{\partial t} = \frac{\partial}{\partial t} \sum_i \lambda_i(t) \hat{\rho}_i^S(t) \otimes \hat{\rho}_i^\alpha(t). \quad (25)$$

However, from eq.(23):

$$tr_{E''} \mathcal{P}\hat{\rho}(t) = \hat{\rho}_S(t) \otimes \frac{\exp(-\beta \hat{H}_\alpha)}{Z_{\alpha}} \otimes. \quad (26)$$

\footnote{This nicely exhibits the subtlety of 'quantum correlations relativity', eq.(10) and eq.(11): for some special states (here: tensor-product states) and for a special pair of structures (here: $S + E$ and $S'' + E''$) one should not worry about the quantum correlations relativity. However, the worry remains for almost all other kinds of re-structuring even for the initial tensor-product state regarding the starting $S + E$ structure of the total system.}
which instead of eq.(25) gives:

$$tr_{E''} \frac{\partial \hat{P} \hat{\rho}(t)}{\partial t} = \frac{\partial}{\partial t} \hat{\rho}_S(t) \otimes \frac{\exp(-\beta \hat{H}_{\alpha_0})}{Z_{\alpha_0}}. \tag{27}$$

The absence of the exact correlations appearing in eq.(25) clearly illustrates Lemma 1, i.e. implies that $tr_{E''} Q \hat{\rho}(t) \neq 0$, even in this case in which eq.(11) is not applicable.

Hence despite the classical similarity, the open systems $S$, $S'$ and $S''$ are subjected to different dynamics. While the $S$ system, as well possibly as the $S''$ system, undergoes a frictionless Markovian dynamics eq.(16), the $S'$ system may be expected to be subjected to neither Markovian [5,6] nor frictionless [4,21] and possibly non-completely-positive [18] dynamics. Exact master equations for the $S'$ and $S''$ systems can only follow from independent e.g. projection-based analysis, which comes out of the scope of the present paper.

5. Discussion

Recently, it was shown [7] that non-similar subsystems of a composite system can have similar dynamics. In contrast to the classical intuition of Sections 1 and 2, in this paper we show that classically indistinguishable many-particle systems can undergo non-equivalent dynamics. Moreover, the shortcuts in describing dynamics of the alternative open systems may not even exist. It is our conjecture that the classical intuition fits with some special structures of many-particle systems [24] that can still require certain assumptions, such as e.g. the Born-Oppenheimer adiabatic approximation. Those results set a new layer in the long standing problem of the transition from quantum to classical [1] that will be discussed elsewhere.

Perhaps not surprisingly, our findings are implied by the classically unknown Entanglement Relativity [11-16]–a not-yet-fully-appreciated rule of the universally valid quantum theory–and some other, classically surprising, findings may be expected.

Currently it appears that description of the alternative subsystems dynamics should be performed for every alternate open system separately. The classical intuition, that similar systems should bear similar dynamics, appears unreliable–which is our conclusion.

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Appendix A Proof of Lemma 1

Given eq. (12), i.e. $tr_E Q \rho(t) = 0$, $\forall t$, we investigate the conditions that should be fulfilled in order for eq. (13), i.e. $tr_{E'} Q \rho(t) = 0$, $\forall t$, to be fulfilled. The $Q$ projector refers to the $S + E$, not to the $S' + E'$ structure. Therefore, in order to calculate $tr_{E'} Q \rho(t)$, we use ER. We refer to the projection (i), Section 3, in an instant of time:

$$P \rho = (tr_E \rho) \otimes \rho_E. \quad (28)$$

A) Pure state $\rho = |\Psi\rangle\langle\Psi|$, while, due to eq. (12), $tr_E Q |\Psi\rangle\langle\Psi| = 0$.

We consider the pure state presented in its (not necessarily unique) Schmidt form

$$|\Psi\rangle = \sum_i c_i |i\rangle_S |i\rangle_E, \quad (29)$$

where $\rho_S = tr_E |\Psi\rangle\langle\Psi| = \sum_i p_i |i\rangle_S \langle i|$, $p_i = |c_i|^2$ and for arbitrary $\rho_E \neq tr_S |\Psi\rangle\langle\Psi|$. Given $\rho_E = \sum_\alpha \pi_\alpha |\alpha\rangle_E \langle \alpha|$, we decompose $|\Psi\rangle$ as:

$$|\Psi\rangle = \sum_{i,\alpha} c_i C_{i\alpha} |i\rangle_S |\alpha\rangle_E, \quad (30)$$

with the constraints:

$$\sum_i |c_i|^2 = 1 = \sum_\alpha \pi_\alpha, \sum_\alpha |C_{i\alpha}|^2 = 1, \forall i, \quad (31)$$

Then

$$Q |\Psi\rangle\langle\Psi| = |\Psi\rangle\langle\Psi| - \sum_{i,\alpha} p_i \pi_\alpha |i\rangle_S \langle i| \otimes |\alpha\rangle_E \langle \alpha|. \quad (32)$$

We use ER:

$$|i\rangle_S |\alpha\rangle_E = \sum_{m,n} D_{mn}^{i\alpha} |m\rangle_{S'} |n\rangle_{E'}, \quad (33)$$
with the constraints:

\[ \sum_{m,n} D_{mn}^\alpha D_{mn}^{\alpha'} = \delta_{\alpha \alpha'} \delta_{\alpha \alpha'} \delta_{\alpha \alpha'}. \]  \hspace{1cm} (34)

With the use of eqs. (30) and (33), eq. (32) reads:

\[ \sum_{m,m',n,n'} \left[ \sum_{i,i',\alpha,\alpha'} c_i C_i^\alpha C_i^{\alpha'} D_{mn} D_{m'n'} \right] \left[ m \rangle S \langle m' | \otimes | n \rangle E \langle n' | - \sum_{i,\alpha} p_i \pi_\alpha D_{mn}^{i\alpha} D_{m'n'}^{i\alpha'} \right] \]  \hspace{1cm} (35)

After tracing out, \( tr_E \):

\[ \sum_{m,m'} \left\{ \sum_{i,\alpha,n} \sum_{i',\alpha'} c_i C_i^\alpha C_i^{\alpha'} D_{mn} D_{m'n'} - p_i \pi_\alpha D_{mn}^{i\alpha} D_{m'n'}^{i\alpha'} \right\} \left| m \rangle S \langle m' | \right\} = 0, \forall m, m'. \]  \hspace{1cm} (37)

Introducing notation, \( \Lambda_n^m \equiv \sum_{i,\alpha} c_i C_i^\alpha \), one obtains:

\[ \sum_{m,m'} \left\{ \sum_{i,\alpha,n} \sum_{i',\alpha'} c_i C_i^\alpha C_i^{\alpha'} D_{mn} D_{m'n'} - p_i \pi_\alpha D_{mn}^{i\alpha} D_{m'n'}^{i\alpha'} \right\} \left| m \rangle S \langle m' | \right\} = 0, \forall m, m'. \]  \hspace{1cm} (38)

Notice:

\[ \sum_m A_{mm} = 0. \]  \hspace{1cm} (39)

which is equivalent to \( tr_S Q | \Psi \rangle \langle \Psi | = 0 \), see eq. (32).

B) Mixed (e.g. non-entangled) state.

\[ \hat{\rho} = \sum_i \lambda_i \hat{\rho}_S \hat{\rho}_E, \quad \hat{\rho}_S = \sum_{m,n} \rho_{im} | \chi_{im} \rangle S \langle \chi_{im} |, \quad \hat{\rho}_E = \sum_{m,n} \pi_{im} | \phi_{im} \rangle E \langle \phi_{im} |, \]  \hspace{1cm} (40)

In eq. (40), having in mind eq. (28), \( tr_E Q \hat{\rho} = 0 \), while \( tr_E \hat{\rho} = \sum_p \kappa_p | \varphi_p \rangle S \langle \varphi_p | \), and \( \hat{\rho}_E = \sum_q \omega_q | \psi_q \rangle E \langle \psi_q | \neq tr_S \hat{\rho}. \)
Constraints:
\[ \sum_i \lambda_i = 1 = \sum_p \kappa_p = \sum_q \omega_q, \quad \sum_m p_{im} = 1 = \sum_n \pi_{in}, \forall i. \] (41)

Now we make use of ER and, for comparison, we use the same basis \{\left| a \right\rangle_{S'}, \left| b \right\rangle_{E'}\}\}
\[ \left| \chi_{im} \right\rangle_S \left| \phi_{in} \right\rangle_E = \sum_{a,b} C^{imn}_{ab} \left| a \right\rangle_{S'} \left| b \right\rangle_{E'} = \sum_{a,b} D^{pq}_{ab} \left| a \right\rangle_{S'} \left| b \right\rangle_{E'}. \] (42)

Constraints:
\[ \sum_{a,b} C^m_{ab} C^m_{ab}^* = \delta_{nn'} \delta_{nn'}, \quad \sum_{a,b} D^p_{ab} D^p_{ab}^* = \delta_{pp'} \delta_{qq'}. \] (43)

So
\[ \mathcal{Q} \hat{\rho} = \hat{\rho} - (tr \hat{\rho}) \otimes \hat{\rho}_E = \sum_{a,a',b,b'} \left\{ \sum_{i,m,n} \lambda_i \pi_{in} C^m_{ab} C^m_{a'b'} - \sum_{p,q} \kappa_p \omega_q D^{pq}_{ab} D^{pq}_{a'b'} \right\} \left| a \right\rangle_{S'} \langle a' | \otimes \left| b \right\rangle_{E'} \langle b' |. \] (44)

Hence
\[ tr_{E'} \mathcal{Q} \hat{\rho} = 0 \iff \Lambda_{aa'} \equiv \sum_{i,m,n,b} \lambda_i \pi_{in} C^m_{ab} C^m_{a'b'} - \sum_{p,q,b} \kappa_p \omega_q D^{pq}_{ab} D^{pq}_{a'b'} = 0, \forall a, a'. \] (45)

Again, for \(a = a'\):
\[ \sum_a \Lambda_{aa} = 0, \] (46)
as being equivalent with \( tr \mathcal{Q} \hat{\rho} = 0 \), see eq.(44).

Validity of eq.(13) assumes validity of eq.(38) for pure and of eq.(45) for mixed states. Both eq.(38) and eq.(45) represent the sets of simultaneously satisfied equations. We do not claim non-existence of the particular solutions to eq.(38) and/or to eq.(45), e.g. for the finite-dimensional systems. Nevertheless, we want to emphasize that the number of states they might refer to is apparently negligible compared to the number of states for which this is
not the case. For instance, already for the fixed $a$ and $a'$, a small change e.g. in $\kappa$ (while bearing eq.(41) in mind) undermines equality in eq.(45).

Quantum dynamics is continuous in time. Provided eq.(12) is fulfilled, validity of eq.(13) might refer only to a special set of the time instants. So we conclude: for the most part of the open $S'$-system’s dynamics, eq.(13) is not fulfilled. By exchanging the roles of eq.(12) and eq.(13) in our analysis, we obtain the reverse conclusion, which completes the proof. Q.E.D.

Appendix B Proof of Lemma 2

The commutation condition, $[\mathcal{P}, \mathcal{P}']\hat{\rho}(t) = 0, \forall t$. With the notation $\hat{\rho}_P(t) \equiv \mathcal{P}\hat{\rho}(t)$ and $\hat{\rho}_{P'}(t) \equiv \mathcal{P}'\hat{\rho}(t), \forall t$. The commutativity reads: $\mathcal{P}\hat{\rho}_{P'}(t) = \mathcal{P}'\hat{\rho}_P(t), \forall t$. Then $\mathcal{P}\hat{\rho}_{P'}(t) = tr_E\hat{\rho}_{P'}(t) \otimes \hat{\rho}_E = \hat{\rho}_S(t) \otimes \hat{\rho}_E$, while $\mathcal{P}'\hat{\rho}_P(t) = tr_E\hat{\rho}_P(t) \otimes = \sigma_{S'}(t) \otimes \sigma_{E'}$. So, the commutativity requires the equality $\sigma_{S'}(t) \otimes \sigma_{E'} = \hat{\rho}_S(t) \otimes \hat{\rho}_E, \forall t$. However, quantum dynamics is continuous in time. Likewise in Proof of Lemma 1, quantum correlations relativity guarantees, that for the most of the time instants the equality will not be fulfilled. Q.E.D.