Efficient quantum tensor product expanders and unitary $t$-designs via the zigzag product

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Abstract

A classical $t$-tensor product expander is a natural way of formalising correlated walks of $t$ particles on a regular expander graph. A quantum $t$-tensor product expander is a completely positive trace preserving map that is a straightforward analogue of a classical $t$-tensor product expander. Interest in these maps arises from the fact that iterating a quantum $t$-tensor product expander gives us a unitary $t$-design, which has many applications to quantum computation and information. We show that the zigzag product of a high dimensional quantum expander (i.e. $t = 1$) of moderate degree with a moderate dimensional quantum $t$-tensor product expander of low degree gives us a high dimensional quantum $t$-tensor product expander of low degree. Previously such a result was known only for quantum expanders i.e. $t = 1$.

Using the zigzag product we give efficient constructions of quantum $t$-tensor product expanders in dimension $D$ where $t = \text{polylog}(D)$. We then show how replacing the zigzag product by the generalised zigzag product leads to almost-Ramanujan quantum tensor product expanders i.e. having near-optimal tradeoff between the degree $s$ and the second largest singular value $s^{-\frac{1}{2}+O(\sqrt{\log s})}$. Both the products give better tradeoffs between the degree and second largest singular value than what was previously known for efficient constructions.

1 Introduction

Expander graphs are graphs of small degree and high connectivity, and have had many applications to combinatorics and computer science (see e.g. the survey paper [HLW06] and the references therein). One way of formalising the expansion property of an infinite family of directed graphs with out-degree and in-degree $d$ is via the requirement that the second largest singular value in absolute value of the normalised adjacency matrix be at most $1 - \Omega(1)$. In this paper, we will use this notion of an algebraic expander. An equivalent way to state the algebraic definition is as follows: A $d$-regular expander on $D$ vertices with second singular value $\lambda$ is a linear transformation $G : \mathbb{C}^D \to \mathbb{C}^D$ that can be expressed as $G(v) = \frac{1}{d} \sum_{i=1}^{d} P_i(v)$, for any $v \in \mathbb{C}^D$, where $\{P_i\}_{i=1}^{d}$ are $D \times D$ permutation matrices, such that

$$\left\| G(v) - \frac{\vec{1}^T v}{D} \vec{1} \right\|_2 \leq \lambda \|v\|_2,$$

$\vec{1}$ being the all ones $D$-tuple. The above condition is equivalent to saying that $\|G - \vec{1}\|_\infty \leq \lambda$, where $\|M\|_\infty$ is the largest singular value of $M$ aka Schatten $\ell_\infty$-norm of $M$ aka spectral norm of

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$M$, and $I$ is the ‘ideal’ linear map defined by $I := \frac{1}{D} \sum_{P \in S_D} P$, the average being over all $D \times D$ permutation matrices $P$.

The algebraic definition is directly used for many applications of expander graphs. Ben-Aroya, Schwartz and Ta-Shma [BST10] generalised the algebraic definition in a natural fashion to the quantum setting. A $D$-dimensional quantum expander $G$ is a so-called completely positive trace preserving superoperator, called CPTP map for short, which maps $D \times D$ matrices to $D \times D$ matrices. If the input $D \times D$ matrix is a density matrix, i.e. a Hermitian positive semidefinite matrix with trace one, then the output is a density matrix also. The quantum expander $G$ is said to have degree $d$ if it can be expressed as $G(M) = \frac{1}{d} \sum_{i=1}^{d} U_i M U_i \dagger$, where $M$ is the $D \times D$ input matrix and $\{U_i\}_{i=1}^{d}$ are $D \times D$ unitary matrices. Since matrices are the quantum analogue of vectors, and density matrices are the quantum analogue of probability distributions, a $D$-dimensional quantum expander of degree $d$ can be thought of a quantum analogue of a $d$-regular classical expander on $D$ vertices. The quantum expander $G$ is said to have second singular value at most $\lambda$ if

$$\left\| G(M) - \frac{\text{Tr} [M]}{D} \mathbb{I} \right\|_2 \leq \lambda \|M\|_2,$$

for all matrices $M \in \mathbb{C}^{D \times D}$, where $\mathbb{I}$ denotes the $D \times D$ identity matrix and $\|M\|_2$ denotes the Frobenius norm or the Schatten $\ell_2$-norm of $M$ which is nothing but the $\ell_2$-norm of the $D^2$-tuple obtained by rearranging the entries of matrix $M$. Equivalently, we can say that $\|G - I\|_{\infty} \leq \lambda$, where $I$ is the superoperator whose action on a $D \times D$ matrix $M$ is defined by $I(M) := \int_{U \in U(D)} U M U^\dagger$, the integration being over the Haar probability measure on the unitary group $U(D)$.

The concept of random walk of a single particle on a $d$-regular expander graph was naturally extended by Hastings and Harrow [HH09] to that of a correlated random walk of $t$ particles. This walk can be informally described as follows: Toss a fair $d$-faced coin, if it comes up with $i$ for some $1 \leq i \leq d$, then each of the $t$ particles takes the $i$th outgoing edge from their respective vertices. Algebraically, a classical $t$-tensor product expander ($t$-TPE) on $D$ vertices of degree $d$ can be defined as a linear transformation $G : (\mathbb{C}^D)^{\otimes t} \rightarrow (\mathbb{C}^D)^{\otimes t}$ that can be expressed as $G(v) = \frac{1}{d} \sum_{i=1}^{d} (P_i)^{\otimes t}(v)$, for any $v \in (\mathbb{C}^D)^{\otimes t}$, where $\{P_i\}_{i=1}^{d}$ are $D \times D$ permutation matrices. The $t$-TPE product expander is said to have second singular value $\lambda$ if $\|G - I\|_{\infty} \leq \lambda$, where $I$ is the ‘ideal’ linear map defined by $I := \frac{1}{D^t} \sum_{P \in S_D} P^{\otimes t}$, the average being over all $D \times D$ permutation matrices $P$.

In a similar fashion, Hastings and Harrow [HH09] extended the notion of a quantum expander to that of a quantum $t$-tensor product expander ($t$-qTPE). A quantum $t$-tensor product expander $G : (\mathbb{C}^{D \times D})^{\otimes t} \rightarrow (\mathbb{C}^{D \times D})^{\otimes t}$ that can be expressed as $G(M) = \frac{1}{d} \sum_{i=1}^{d} (U_i)^{\otimes t} M (U_i^\dagger)^{\otimes t}$, for any matrix $M \in (\mathbb{C}^{D \times D})^{\otimes t}$, where $\{U_i\}_{i=1}^{d}$ are $D \times D$ unitary matrices. The qTPE is said to have second singular value $\lambda$ if $\|G - I\|_{\infty} \leq \lambda$, where $I$ is the ‘ideal’ CPTP superoperator defined by its action on a matrix $M$ by $I(M) := \int_{U \in U(D)} U^{\otimes t} M (U^\dagger)^{\otimes t}$, the integration being over the Haar probability measure on the unitary group $U(D)$. A quantum expander defined above is thus a $1$-qTPE.

Quantum expanders have already found several applications in quantum algorithms and complexity e.g. [AS03, Has07, BST10]. Quantum tensor product expanders are also of great interest primarily because sequentially iterating a $t$-qTPE gives us an approximate unitary $t$-design. Unitary $t$-designs have many applications to quantum computation and information e.g. [DCEL09, AE07, Low09]. Many protocols in quantum information theory use Haar random unitaries e.g.
Sometimes, these Haar random unitaries can be replaced by approximate unitary $t$-designs which require less random bits to describe e.g. \cite{SDTR13}. Thus, unitary $t$-designs are a useful notion of pseudo-random unitaries. More precisely, they serve as the quantum analogue \cite{Low09} of $t$-wise independent random variables used in classical derandomisation applications (see e.g. \cite{BH94}). Moreover, Hastings and Harrow \cite{HH09} have shown that $D^3$-qTPEs in dimension $D$ allow one to obtain better approximations to an arbitrary $D \times D$ unitary matrix than what the Solovay-Kitaev theorem provides.

Obtaining efficient constructions of $t$-qTPEs is thus an important problem. By an efficient construction, we mean that the qTPE superoperator in dimension $D$ can be realised by a quantum algorithm running in polylog($D$) time. An efficient construction of a $t$-qTPE will automatically give us an efficient construction of an approximate unitary $t$-design simply by sequential iteration. However, the converse is not known to be true.

Ben-Aroya, Schwartz and Ta-Shma \cite{BST10} showed that the zigzag product of expander graphs first defined by Reingold, Vadhan and Wigderson \cite{RVW02} can be appropriately extended to the quantum setting to give an efficient construction of a 1-qTPE in arbitrarily large dimension with constant degree and constant singular value gap. This leaves open the case when $t > 1$. Efficient constructions of approximate unitary $t$-designs for $t = \text{polylog}(D)$ were known before this work \cite{HL09,BHH16}, but viewed as expanders, they have polylog($D$) degree. Moreover, the tradeoff between their degree and singular value gap is far from optimal. This is unsatisfactory for some applications where we want constant degree and constant singular value gap. One such application is a quantum protocol for private information retrieval via the quantum Johnson Lindenstrauss transform \cite{Sen18}.

Hastings and Harrow \cite{HH09} posed an open question asking whether the quantum expander constructions of Ben-Aroya, Schwartz and Ta-Shma \cite{BST10} can lead to quantum tensor product expanders also. In this work, we answer their question in the affirmative by showing that the zigzag product of a high dimensional quantum expander i.e. 1-qTPE with a low dimensional quantum tensor product expander i.e. $t$-qTPE gives rise to a high dimensional $t$-qTPE. Combined with Hastings and Harrow’s \cite{HH09} existential result that random unitaries form a $t$-qTPE, we obtain the first efficient construction of constant degree, constant singular value gap $t$-qTPEs in arbitrarily large dimension. Our method achieves the best known (in fact third power) tradeoff between degree and singular value gap amongst efficient constructions of approximate unitary $t$-designs.

Proving that the zigzag product gives a $t$-qTPE is similar to Ben-Aroya, Schwartz and Ta-Shma’s proof \cite{BST10} that the zigzag product gives a 1-qTPE. However, we have to take care of some technical geometric issues involving the eigenspace of the qTPE superoperator for eigenvalue one. Unlike the $t = 1$ setting, this eigenspace has dimension larger than one which introduces several complications. To address these complications, we define a subspace that is ‘close’ to the eigenspace in a certain precise sense. The proof of closeness uses some combinatorial properties of permutations. We then prove our main result by ‘switching back and forth’ between these two spaces in a slightly tricky fashion. The formal proof can be found in Section 3.

We then go further and achieve a near-optimal tradeoff between degree $s$ and second largest singular value $s^{-\frac{1}{2}+O(\sqrt{\log s})}$ of a qTPE. The optimal tradeoff of degree $s$ versus second largest singular value of $2s^{-1/2}$ is known as the Ramanujan bound. To approach the Ramanujan bound, we need to use the generalised zigzag product of graphs defined by Ben-Aroya and Ta-Shma \cite{BT11}. We can extend the generalised zigzag product to qTPEs in a natural fashion. To show that the generalised zigzag product gives an almost-Ramanujan qTPE we follow the outline of Ben-
Aroya and Ta-Shma’s proof, combined with repeated applications of the ‘back and forth’ technique explained above. We also need to exploit a version of the Johnson-Lindenstrauss property for an independent sequence of Haar random unitaries (see e.g. [Sen18]). This technical property can be viewed as the quantum generalisation of the so-called $\epsilon$-good property of a sequence of independent uniformly random permutations analysed in Ben-Aroya and Ta-Shma’s paper [BT11, Lemma 19]. Though the quantum $\epsilon$-good property is fundamentally different from the classical version in a certain sense, nevertheless it allows us to prove that the generalised zigzag product gives an almost-Ramanujan qTPE akin to the classical expander setting.

2 Preliminaries

In this paper all vector spaces are over the field of complex numbers $\mathbb{C}$, are finite dimensional and equipped with inner products. Often we will be dealing with vector spaces whose elements are matrices i.e. the elements are themselves linear operators from a Hilbert space $A$ to a Hilbert space $B$. Such vector spaces will be equipped with the Hilbert-Schmidt inner product $\langle M, N \rangle := \text{Tr} [M^\dagger N]$. This inner product is nothing but the usual dot product of ‘long’ vectors obtained by rearranging the entries of matrices as tuples. We will also consider linear maps between vector spaces that are themselves spaces of matrices. We will call such linear maps as superoperators.

For vector space $\mathbb{C}^d$, let $e_i, i \in [d]$ denote the $i$th standard basis vector which consists of a one in position $i$ and zeroes everywhere else. Let $\|v\|_2 := \sqrt{\sum_{i=1}^d |v_i|^2}$ denote the $\ell_2$-norm of a vector $v \in \mathbb{C}^d$. The dot product of two vectors is given by $\langle v, w \rangle := \sum_{i=1}^d v_i^* w_i$. For a matrix $M \in \mathbb{C}^d \times \mathbb{C}^d$, let $\|M\|_p$ denote its Schatten $p$-norm i.e. the $\ell_p$-norm of the vector of singular values of $M$. We will only be interested in Schatten $p$-norms with $p = 1, 2, \infty$. The Schatten 2-norm of $M$ turns out to be nothing but the $\ell_2$-norm of the long vector obtained by rearranging the entries of the matrix $M$. In other words, it is the norm arising from the Hilbert-Schmidt inner product.

We will denote the vector space of $d \times d$ matrices, or equivalently the vector space of linear maps $\mathbb{C}^d \to \mathbb{C}^d$, by $\mathbb{C}^{d \times d}$. In several places we will interchangeably use $\mathbb{C}^d$ in place of the tensor product $(\mathbb{C}^d)^\otimes t$. In line with this abuse of notation, we will sometimes use $\mathbb{C}^{(Dd)^t \times (Dd)^t}$ to denote the vector space of linear maps $(\mathbb{C}^D \otimes \mathbb{C}^d)^{\otimes t} \to (\mathbb{C}^D \otimes \mathbb{C}^d)^{\otimes t}$.

For a Hilbert space $V$ and subspace $W \leq V$, we define the orthogonal complement of $W$ in $V$, denoted by $V \setminus W$, to be the span of all vectors in $V$ orthogonal to $W$. When the ambient space $V$ is clear from the context, we shall denote $V \setminus W$ by the shorter notation $W^\perp$.

2.1 On permutations

Let $t, d$ be positive integers. The following lemma is easy to prove.

**Lemma 1.** Suppose $d \geq t$. Define the falling factorial $(d)_t := d(d-1) \cdots (d-t+1)$. Then

$1 - \frac{(d)_t}{d!} \leq \frac{t(t-1)}{2d}.$

**Proof.**

$$1 - \frac{(d)_t}{d!} = 1 - \frac{1}{d}(1 - \frac{2}{d}) \cdots (1 - \frac{t-1}{d}) \leq \frac{1}{d} + \frac{2}{d} + \cdots + \frac{t-1}{d} = \frac{t(t-1)}{2d}.$$

$\square$
The number of permutations of \([t]\) with \(k\) cycles, called \((unsigned)\) Stirling number of the first kind, is denoted by \([t^k]\). The unsigned Stirling numbers of the first kind satisfy the recurrence equation \([t^k + 1^k] = [t^k] + \binom{t}{k-1}\). From this, we can show by induction for \(1 \leq k \leq t - 1\) that \([t^k] \leq \binom{t}{k}\). This upper bound on \([t^k]\) is almost tight by a result of Arratia and DeSalvo [AD17, Theorem 3.2]. Using this upper bound, we prove the following lemma.

**Lemma 2.** Let \(d\) be a positive integer larger than \(t^2\). Let \(M\) be a \(t! \times t!\)-matrix whose rows and columns are indexed by the permutations of \([t]\), defined as follows:

\[
M_{\sigma \sigma'} := \begin{cases} 
\frac{d^{t_{\sigma^{-1} \sigma'} - t}}{2} & \sigma \neq \sigma' \\
0 & \sigma = \sigma',
\end{cases}
\]

where \(t_{\sigma^{-1} \sigma'}\) is the number of cycles in the permutation \(\sigma^{-1} \sigma'\). Then, \(M\) is a real symmetric matrix and \(\|M\|_\infty \leq \frac{t(t-1)}{d}\). Moreover, the eigenvalues of the real symmetric matrix \(I + M\) lie between \(1 - \frac{t(t-1)}{d}\) and \(1 + \frac{t(t-1)}{d}\).

**Proof.** Observe that since \(M\) is a real symmetric matrix, \(\|M\|_\infty\) is nothing but the largest eigenvalue of \(M\) in absolute value. By Gershgorin’s theorem and the permutation symmetry of \(M\), we conclude that

\[
\|M\|_\infty \leq \sum_{\sigma \neq() \atop \sigma \neq() } d^{t_{\sigma^{-1} \sigma'} - t} = \sum_{k=1}^{t-1} \binom{t}{k} d^{-k} \leq \sum_{k=1}^{\binom{t}{2} -1} \binom{t}{k} d^{-k} \leq \frac{t}{d} \sum_{k=0}^\infty 2^{-k} = \frac{t(t-1)}{2d}.
\]

This completes the proof.

We will also need the following lemma.

**Lemma 3.** Fix \(0 < \epsilon < \frac{1}{2t}\). Let \(N\) be a \(t! \times t!\)-matrix whose rows and columns are indexed by the permutations of \([t]\), defined as follows:

\[
N_{\sigma \sigma'} := \epsilon^{t_{\sigma^{-1} \sigma'} - 1} \quad \sigma \neq \sigma' \\
0 \quad \sigma = \sigma',
\]

where \(t_{\sigma^{-1} \sigma'}\) is the number of fixed points in the permutation \(\sigma^{-1} \sigma'\). Then, \(N\) is a real symmetric matrix and \(\|N\|_\infty \leq 2\epsilon^2 t^2\).

**Proof.** Observe that since \(N\) is a real symmetric matrix, \(\|N\|_\infty\) is nothing but the largest eigenvalue of \(N\) in absolute value. By Gershgorin’s theorem and the permutation symmetry of \(N\), we conclude that

\[
\|N\|_\infty \leq \sum_{\sigma \neq() } \epsilon^{t_{\sigma^{-1} \sigma'} - 1} = \sum_{k=1}^{t-2} \epsilon^{t-k} \frac{t}{k} (t-k)! (1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^t}{t!}) \leq \sum_{k=1}^{t-2} \epsilon^{t-k} t^{t-k} \leq 2(\epsilon t)^2,
\]

where () denotes the identity permutation. This completes the proof.
2.2 On Haar random unitaries

In this subsection, we single out a Johnson-Lindenstrauss type of property of Haar random unitaries that will be used in the proof that the generalised zigzag product gives qTPEs. Let $0 < \epsilon < 1$. Let $V, V'$ be vector spaces of dimensions $d, d'$. Let $x$ be a unit length vector in $V \otimes V'$. Let $U$ be a unitary matrix on $V \otimes V'$. Let $v$ be a computational basis vector for $V$. The unitary $U$ is said to be $\epsilon$-good for $x$ given $v_1$ if the probability of observing the outcome $v_1$ when the system $V$ of the state $Ux$ is measured in its computational basis is $\frac{1 + 3\epsilon}{d}$. The unitary $U$ is said to be $\epsilon$-good for $x$ if for all computational basis vectors $v_1 \in V$, $U$ is $\epsilon$-good for $x$ given $v_1$. The notation $Ux|v_1$ denotes the normalised vector in $V \otimes V'$ obtained by computing $Ux$, measuring only $V$ in its computational basis, and observing the result $v_1$. Suppose $X$ is a set of unit length orthogonal vectors in $V \otimes V'$. The unitary $U$ is said to be $\epsilon$-good for $X$ if it is $\epsilon$-good for each $x \in X$, and for every computational basis vector $v_1 \in V$ and $x, x' \in X$, $\langle x, x' \rangle = 0$, $|\langle Ux|v_1, Ux'|v_1 \rangle| \leq 8\epsilon$.

Let $(U_k, \ldots, U_1)$ be a $k$-tuple of unitaries on $V \otimes V'$. Let $x_0 := e_{i_0} \otimes e'_{j_0}$ be a computational basis vector of $V \otimes V'$. Let $(e_{i_k}, \ldots, e_{i_1})$ be a $k$-tuple of computational basis vectors of $V$. By induction on $j$, we define $x_j := U_{j-1}|e_{i_j}$. We say that $U_j$ is $\epsilon$-good given $(U_{j-1}, \ldots, U_1), (e_{i_{j-1}}, \ldots, e_{i_1})$, $x_0$ if $U_j$ is $\epsilon$-good for $x_{j-1}$. The unitary $U_j$ is said to be $\epsilon$-good if it is $\epsilon$-good for the set of all computational basis vectors $x_0$ of $V \otimes V'$. We declare $U_j$ to be $\epsilon$-good given $(U_{j-1}, \ldots, U_1)$ if $U_j$ is $\epsilon$-good given $(U_{j-1}, \ldots, U_1), (e_{i_{j-1}}, \ldots, e_{i_1})$, $x_0$ for the set of all possible $(j - 1)$-tuples of computational basis vectors $(e_{i_{j-1}}, \ldots, e_{i_1})$ of $V$ and all possible computational basis vectors $x_0$ of $V \otimes V'$. By induction on $j$, we say that the $j$-tuple $(U_j, \ldots, U_1)$ is $\epsilon$-good if $(U_{j-1}, \ldots, U_1)$ is $\epsilon$-good and $U_j$ is $\epsilon$-good given $(U_{j-1}, \ldots, U_1)$.

Let $\mathcal{H}_j, 1 \leq j \leq k$ be sets of unitaries on $V \otimes V'$, each set $\mathcal{H}_j$ being of size $s$. We use the shorthand $\mathcal{H}$ to denote the $k$-tuple of sets $(\mathcal{H}_k, \ldots, \mathcal{H}_1)$. Then, $\mathcal{H}$ is said to be $\epsilon$-good if all $k$-tuples of unitaries $(U_k, \ldots, U_1)$, $U_j \in \mathcal{H}_j$ are $\epsilon$-good.

The following standard result is a version of the Johnson-Lindenstrauss lemma for Haar random unitaries. It can be proved using Fact 2 and the method of Theorem 1 in [Sen18].

**Fact 1.** Let $k, s$ be positive integers. Independently choose $sk$ Haar random unitaries on $V \otimes V'$, and group them into $k$ sets $\mathcal{H}_j, 1 \leq j \leq k$, each set $\mathcal{H}_j$ being of size $s$. Then, the probability of $\mathcal{H}$ not being $\epsilon$-good is at most $4(s^k + 1d^{k+2}d')^2 \exp(-2^{-4}\epsilon^2d')$.

**Remark:** The above fact can be thought of as a quantum analogue of Lemma 19 in [BT11], which analysed a similar property about an independent sequence of uniformly random permutations. However, there is an important difference between the $\epsilon$-good properties analysed in the two statements. The closeness to the uniform distribution in the definition of the classical $\epsilon$-good property arises from the choice of a uniformly random computational basis vector of $V \otimes V'$. In contrast, Fact 1 does not require us to choose a uniformly random computational basis vector of $V \otimes V'$. In fact, Fact 1 works for any fixed computational basis vector of $V \otimes V'$, whereas Lemma 19 of [BT11] would give a deterministic result (which is the farthest possible from the uniform distribution) if one were to take a fixed computational basis vector of $V \otimes V'$. This difference arises from the inherently quantum effect of measurement being probabilistic. Thus, the two statements are fundamentally different.
2.3 Quantum tensor product expanders

We recall the definition of quantum tensor product expanders first defined by Hastings and Harrow [HDH09].

**Definition 1 (Quantum tensor product expander).** A \((d, s, \lambda, t)\)-quantum tensor product expander (qTPE) is a set of \(d \times d\) unitaries \(\{U_i\}_{i=1}^s\) such that

\[
\left\| \text{Design}_{U} \left[ U^\otimes t M(U^\dagger)^\otimes t \right] - \text{Haar}_{U} \left[ U^\otimes t M(U^\dagger)^\otimes t \right] \right\|_2 \leq \lambda \|M\|_2, 
\]

for all linear operators \(M : (\mathbb{C}^d)^\otimes t \to (\mathbb{C}^d)^\otimes t\). The notation

\[
\text{Design}_{U} [U^\otimes t M(U^\dagger)^\otimes t] := s^{-1} \sum_{i=1}^s U_i^\otimes t M(U_i^\dagger)^\otimes t
\]

denotes the expectation under the choice of a uniformly random unitary from the design. The notation

\[
\text{Haar}_{U} [U^\otimes t M(U^\dagger)^\otimes t] := \int_{U(d)} U^\otimes t M(U^\dagger)^\otimes t \, d\mu
\]

denotes the expectation under the choice of a unitary \(U\) picked from the Haar measure \(\mu\) (formalisation of uniform measure) on the group of \(d \times d\) unitary matrices \(U(d)\).

The quantity \(s\) is referred to as the degree and \(1 - \lambda\) as the singular value gap of the qTPE.

In this paper, we shall often consider Hermitian qTPEs. These are qTPEs where the corresponding linear map on matrices, aka superoperator \(M \mapsto \text{Design}_{U} [U^\otimes t M(U^\dagger)^\otimes t]\) is Hermitian under the Hilbert-Schmidt inner product of matrices. In fact, the Hermitian qTPEs constructed in this paper will be explicitly Hermitian i.e. there will be a bijective involution ‘\(-\)’ on the index set \([s]\) satisfying \(U_{-i} = U_i^{-1}\) for all \(i \in [s]\). For a Hermitian qTPE, the singular value gap is also called the eigenvalue gap as the singular values of the associated Hermitian linear map are nothing but the absolute values of its eigenvalues.

Since the second superoperator in the definition of qTPE is the Haar average over a representation of the compact group \(U(d)\), it is equal to the orthogonal projection onto the fixed space \(W\) of the representation. Let \(\sigma\) be a permutation of \([t]\). Define the matrix

\[
\alpha_\sigma := d^{-t/2} \sum_{(i_1, \ldots, i_t) \in [d]^t} (e_{i_1} \otimes \cdots \otimes e_{i_t})(e_{i_{\sigma(1)}}^\dagger \otimes \cdots \otimes e_{i_{\sigma(t)}}^\dagger).
\]

Thus \(\alpha_\sigma\) is the operator obtained by first applying the Schatten \(\ell_2\)-normalised identity matrix in \(\mathbb{C}^d^t\) followed by shuffling the registers according to the permutation \(\sigma\) i.e. register number \(a\) goes to register number \(\sigma(a)\). Let this ‘shuffling’ operator be denoted by \(\Sigma(\mathbb{C}^d)^\otimes t\). Note that \(\Sigma(\mathbb{C}^d)^\otimes t\) is a unitary matrix. Thus we have, \(\alpha_\sigma = \Sigma(\mathbb{C}^d)^\otimes t \frac{1}{d^{t/2}} 1_{(\mathbb{C}^d)^\otimes t}\). Observe that for any \(\sigma \in S_t, U \in U(d), U^\otimes t \alpha_\sigma = \alpha_\sigma U^\otimes t\). Thus it is clear that \(\alpha_\sigma\) lies in the fixed space \(W\). It turns out that \(W\) is spanned by the matrices \(\alpha_\sigma\) as \(\sigma\) ranges over all permutations of \([t]\) i.e. \(W = \text{span}_\sigma \alpha_\sigma\). This non-trivial statement follows from Schur-Weyl duality [GW98 Theorem 3.3.8]. For \(t \leq d\), it is easy to see that the matrices \(\alpha_\sigma, \sigma \in S_t\) are linearly independent and so \(\dim W = t!\).
We thus see that for the superoperator $E_U^{\text{Design}}[U^\otimes t \cdot (U^\dagger)^\otimes t] - E_U^{\text{Haar}}[U^\otimes t \cdot (U^\dagger)^\otimes t]$, the matrices $\{\alpha_\sigma\}_\sigma$ are left and right singular vectors with singular value zero. Thus, the $\{\alpha_\sigma\}_\sigma$ are eigenvectors with eigenvalue zero. This is true even if the design superoperator is not Hermitian. The bound on the Schatten $\ell_\infty$-norm of the above superoperator required by Definition 1 translates to the requirement that the other singular vectors have singular values at most $\lambda$. If the design superoperator is Hermitian, this is equivalent to the requirement that the other eigenvectors have eigenvalues between $-\lambda$ and $\lambda$.

Hastings and Harrow [HH09, Theorem 6] showed that independent Haar random choices of unitary matrices give rise to Hermitian TPEs with good tradeoff between degree and eigenvalue gap.

**Fact 2.** (Random qTPE). Let $s \geq 4$ be an even integer. Let $d \geq 100$ be an integer. Let $t$ be a positive integer satisfying $t \leq \frac{d^{1/6}}{10 \log d}$. Choose $d \times d$ unitary matrices $\{U_i\}_{i=1}^s$ independently from the Haar measure on the unitary group $\mathbb{U}(d)$. For all $1 \leq i \leq s/2$, set $U_{i+\frac{s}{2}} := U_i^\dagger$. Then $\{U_i\}_{i=1}^s$ form an explicitly Hermitian $(d, s, \lambda, t)$-qTPE, where $\lambda < \frac{s}{s^2 + 2}$ with probability at least $1 - d^{-2}$, and the involution $'-'$ is defined is $-i := i + \frac{s}{2}$ if $1 \leq i \leq s/2$, and $-i := i - \frac{s}{2}$ otherwise.

Combining the above fact with Fact 1 we get

**Lemma 4.** Let $s \geq 4$ be an even integer. Let $d \geq 100$ be an integer. Let $k \leq \log s$ be an integer. Let $0 < \epsilon < 10^{-2}$. Let $d' \geq 30 \log s (\log s + \log d) d^{k+1} \epsilon^{-2}$. Let $t$ be a positive integer satisfying $t \leq \frac{(dd')^{1/6}}{10 \log (dd')}$. Choose $(dd') \times (dd')$ unitary matrices $\{U_i(j)\}, 1 \leq i \leq s/2, 1 \leq j \leq k$ independently from the Haar measure on the unitary group $\mathbb{U}(dd')$. For all $1 \leq i \leq s/2, 1 \leq j \leq k$ set $U_{i+\frac{s}{2}}(j) := U_i^\dagger(j)$. Then, with probability at least $3/4$, for all $j \in [k], \mathcal{H}_j := \{U_i(j)\}_{i=1}^s$ is an explicitly Hermitian $(d, s, \lambda, t)$-qTPE, where $\lambda < \frac{s}{s^2 + 2}$, the involution $'-'$ being defined as $-i := i + \frac{s}{2}$ if $1 \leq i \leq s/2$, and $-i := i - \frac{s}{2}$ otherwise, and the $k$-tuple of expanders $\mathcal{H}$ is $\frac{8dk}{8dk}$-good.

We now recall the definition of an approximate unitary $t$-design according to Low [Low09].

**Definition 2 (Unitary $t$-design).** Consider $d^2$ formal variables $\{u_{ij}\}_{i,j=1}^d$. A monomial $M$ in these formal variables is said to be balanced of degree $t$ if it is a product of exactly $t$ of the formal variables and exactly $t$ of complex conjugates of the formal variables (the sets of un conjugated and conjugates variables bear no relation amongst them). For a $d \times d$ unitary matrix $U$, let $M(U)$ denote the value of the monomial $M$ obtained by evaluating it at the entries $U_{ij}$ of $U$. A balanced polynomial of degree $t$ is a linear combination of balanced monomials of degree $t$.

A unitary $(d, s, \alpha, t)$-design is a set of $d \times d$ unitaries $\{U_i\}_{i=1}^s$ such that

$$\left| \mathbb{E}_U [M(U)] - \mathbb{E}_U^{\text{Haar}} [M(U)] \right| \leq \frac{\alpha}{d^t},$$

for all balanced monomials $M$ of degree $t$.

The following facts are easy to prove from the definition of a qTPE and hold even if the qTPE is not Hermitian.

**Fact 3.** A $(d, s, \lambda, t)$-qTPE is also a $(d, s, \lambda, t')$-qTPE for $t' \leq t$. 

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Fact 4. Suppose $G := \{U_i\}_{i=1}^s$ is a $(d, s, \lambda, t)$-qTPE. Then $G' := \{U_i\}_{i=1}^s \cup \{U_i^\dagger\}_{i=1}^s$ is an explicitly Hermitian $(d, 2s, \lambda, t)$-qTPE with involution $\cdot^\dagger$ defined by $-i := i + s$ if $1 \leq i \leq s$, and $-i := i - s$ otherwise.

Fact 5. Let $H = \{U_j\}_{j=1}^s$ be a $(d, s, \lambda, t)$-qTPE. Sequentially iterating $H$ twice means applying the superoperator corresponding to $H$ twice in succession. Then, $H \circ H$ is a $(d, s^2, \lambda^2, t)$-qTPE where the $s^2$ unitaries are of the form $U_i \otimes U_j$, $1 \leq i, j \leq s$.

For a balanced monomial $M = (u_{ii_1} \cdots u_{ij_t})(u_{ii_1} \cdots u_{ij_t})^*$ of degree $t$, let $M'$ be the matrix with a one in the position $((j_1, \ldots, j_t), (j_1, \ldots, j_t))$ and zeroes elsewhere. Plugging $M'$ into the definition of a $(d, s, \lambda, t)$-qTPE, we see that sequentially iterating the qTPE $O(\frac{\log d + \log \alpha^{-1}}{\log \lambda^{-1}})$ times gives us an $\alpha$-approximate unitary $t$-design.

Fact 6. Let $H = \{U_j\}_{j=1}^s$ be a $(d, s, \lambda, 1)$-qTPE. The tensor product of $H$ with $H$ is defined as the tensor product of the corresponding superoperators. Then, $H \otimes H$ is a $(d^2, s^2, \lambda, 1)$-qTPE where the $s^2$ unitaries are of the form $U_i \otimes U_j$, $1 \leq i, j \leq s$.

3 Zigzag product gives a qTPE

Inspired by the definition of zigzag product for quantum expanders, i.e. 1-qTPEs, in [BST10], we define the zigzag product of a 1-qTPE and a $t$-qTPE as follows. Let $G = \{U_i\}_{i=1}^d$ be a $(D, d, \lambda_1, 1)$-qTPE and $H = \{V_j\}_{j=1}^s$ be a $(d, s, \lambda_2, t)$-qTPE. We will also use $G$, $H$ to denote the corresponding superoperators $C^D \times D \to C^D \times D$, $(C^d \otimes D)^{\otimes t} \to (C^d \otimes D)^{\otimes t}$. If $G$ is explicitly Hermitian, let $\cdot^{-1}$ denote the corresponding involution on $[d]$; if not, let $\cdot^{-1}$ denote the identity function on $[d]$. Define the unitary superoperator $\hat{G}$ on the vector space $C^{Dd} \cong C^D \otimes C^d$ by

$$e_a \otimes e_b \mapsto \hat{G}(U_b e_a) \otimes e_{-b},$$

where $e_a$, $e_b$ denote computational basis vectors of $C^D$, $C^d$ respectively. If $G$ is explicitly Hermitian, then $\hat{G}$ is an involution i.e. $\hat{G}^2 = \mathbb{1}_{C^{Dd}}$. In this case, $\hat{G}$ is both unitary and Hermitian. Let $\mathbb{1}_{C^D}$ denote the identity operator on $C^D$. We define the unitary superoperator $\hat{G}$ on $C^{(Dd) \times (Dd)}$ as

$$M \mapsto \hat{G} M \hat{G}^{-1}.$$

If $G$ is explicitly Hermitian, then $\hat{G}$ is Hermitian also.

Definition 3 (Zigzag product of qTPEs). The zigzag product of qTPEs $G$ and $H$, denoted by $G \otimes H$, is defined as the following set of $s^2$ unitary matrices on $C^{Dd}$:

$$G \otimes H := \{(\mathbb{1}_{C^D} \otimes V_i) \hat{G}(\mathbb{1}_{C^d} \otimes V_j) : i, j \in [s]\}.$$

Let $\mathbb{1}_{C^{D^t \times D^t}}$ denote the identity superoperator on $C^{D^t \times D^t}$. Viewed as a superoperator on $(C^D \otimes C^d)^{\otimes t} \cong C^{D^t \times D^t} \otimes C^{d^t \times d^t} \cong (C^{Dd})^{\otimes t} \cong (C^{Dd})^{\otimes t}$, the zigzag product $G \otimes H$ is nothing but

$$G \otimes H := (\mathbb{1}_{C^{D^t \times D^t}} \otimes H) \circ \hat{G}^{\otimes t} \circ (\mathbb{1}_{C^{D^t \times D^t}} \otimes H).$$

If $G$ and $H$ are explicitly Hermitian, then $G \otimes H$ is also explicitly Hermitian with involution $-(i, j) := (-j, -i)$ on $[s^2] \cong [s] \times [s]$. 

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Suppose $t \leq d \leq D \leq Dd$. Then, the eigenspace $W$ of the superoperator $\mathcal{G} \otimes \mathcal{H}$ for eigenvalue $1$ is spanned by the linearly independent matrices $\{\alpha_\sigma\}_{\sigma \in S_t}$ where

\[
\mathbb{C}^{(Dd)^t \times (Dd)^t} \ni \alpha_\sigma := \sum (\mathcal{C}^D \otimes \mathcal{C}^D)^{\otimes t} \frac{1}{(Dd)^{t/2}} = (\alpha_1)^t \otimes (\alpha_2)^t,
\]

\[
(\alpha_1)^t := \sum (\mathcal{C}^D)^{\otimes t} \frac{1}{D^{t/2}} \in \mathbb{C}^{D^t \times D^t}, \quad (\alpha_2)^t := \sum (\mathcal{C}^D)^{\otimes t} \frac{1}{d^{t/2}} \in \mathbb{C}^{d^t \times d^t}.
\]

Define the matrices

\[
\mathbb{C}^{D^t \times D^t} \ni \alpha_1 := (\alpha_1)^t = D^{-t/2} \mathbb{1}^{(\mathcal{C}^D)^{\otimes t}}, \\
\mathbb{C}^{d^t \times d^t} \ni \alpha_2 := (\alpha_2)^t = d^{-t/2} \mathbb{1}^{(\mathcal{C}^D)^{\otimes t}}, \\
\mathbb{C}^{d^t \times d^t} \ni \alpha_2' := d^{-t/2} \sum_{(j_1, \ldots, j_t) \in [d]^t, \text{distinct}} (e_{j_1} \otimes \cdots \otimes e_{j_t}) (e_{j_1}^\dagger \otimes \cdots \otimes e_{j_t}^\dagger), \\
\mathbb{C}^{(Dd)^t \times (Dd)^t} \ni \alpha' := \alpha_1 \otimes \alpha_2'.
\]

For $\sigma \in S_t$, define

\[
\mathbb{C}^{d^t \times d^t} \ni (\alpha_2')^t := \sum (\mathcal{C}^D)^{\otimes t} \alpha_2', \quad \mathbb{C}^{(Dd)^t \times (Dd)^t} \ni \alpha'' := (\alpha_1)^t \otimes (\alpha_2')^t.
\]

Observe that the respective sets of matrices $\{(\alpha_\sigma)^t\}_{\sigma \in S_t}$, $\{(\alpha_2')^t\}_{\sigma \in S_t}$ are orthogonal under the Hilbert-Schmidt inner product.

Define the vector spaces

\[
W_1 := \text{span} \{ (\alpha_1)^t \}_{\sigma \in S_t}, \quad W_1^\perp := \mathbb{C}^{(Dd)^t \times (Dd)^t} \setminus W, \\
W_2 := \text{span} \{ (\alpha_2)^t \}_{\sigma \in S_t}, \quad (W')^\perp := \mathbb{C}^{(Dd)^t \times (Dd)^t} \setminus W', \\
W_2' := \text{span} \{ (\alpha_2')^t \}_{\sigma \in S_t}, \quad (W_1')^\perp := \mathbb{C}^{D^t \times D^t} \setminus W_1, \\
W' := \text{span} \{ (\alpha_2')^t \}_{\sigma \in S_t}, \quad (W_2')^\perp := \mathbb{C}^{d^t \times d^t} \setminus W_2'.
\]

Then

\[
(W')^\perp = (\mathbb{C}^{D^t \times D^t} \otimes (W_2')^\perp) \oplus ((\mathbb{C}^{D^t \times D^t} \otimes W_2') \cap (W')^\perp).
\]

We now define a geometric property called everywhere close capturing that a subspace $W$ is ‘close’ in a certain sense to a subspace $W'$.

**Definition 4.** Let $W, W'$ be subspaces of a vector space $V$. Let $\epsilon > 0$. We say that $W$ is everywhere close to $W'$ within $\epsilon$ if for all $w \in W$, $\|w\|_2 = 1$, there is a $w' \in W'$ such that $\|w - w'\|_2 \leq \epsilon$. If $W$ is everywhere close to $W'$ within $\epsilon$ and $W'$ is everywhere close to $W$ within $\epsilon$, then we say that subspaces $W, W'$ are everywhere close within $\epsilon$.

We next prove an important property about the subspaces $W, W', \mathbb{C}^{D^t \times D^t} \otimes W_2, \mathbb{C}^{D^t \times D^t} \otimes W_2'$ and their orthogonal spaces defined in Equation 1 above.

**Lemma 5.** For the definitions of the subspaces above, the following claims are true.

1. The subspaces $W, W'$ are everywhere close to within $2 \sqrt{\frac{(t-1)}{d}}$. 

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2. For any \( \beta' = \sum_{\sigma} b_{\sigma}(\beta'_{1})_{\sigma} \otimes (\alpha'_{2})_{\sigma} \in \mathbb{C}^{D^{t} \times D^{t}} \otimes W_{2} \), \( \|\beta'\|_{2} = 1 \), define \( \beta := \sum_{\sigma} b_{\sigma}(\beta'_{1})_{\sigma} \otimes (\alpha_{2})_{\sigma} \in \mathbb{C}^{D^{t} \times D^{t}} \otimes W_{2} \). Then \( \|\beta - \beta'\|_{2} \leq 2 \frac{t(t-1)}{d} \). In particular, the subspace \( \mathbb{C}^{D^{t} \times D^{t}} \otimes W_{2} \) is everywhere close to the subspace \( \mathbb{C}^{D^{t} \times D^{t}} \otimes W_{2} \) to within \( 2 \sqrt{\frac{t(t-1)}{d}} \).

3. The subspaces \( W^\perp \), \( (W')^\perp \) are everywhere close to within \( 2 \sqrt{\frac{t(t-1)}{d}} \).

4. The subspace \( \mathbb{C}^{D^{t} \times D^{t}} \otimes (W_{2})^\perp \) is everywhere close to the subspace \( \mathbb{C}^{D^{t} \times D^{t}} \otimes (W_{2})^\perp \) to within \( 2 \sqrt{\frac{t(t-1)}{d}} \).

**Proof.** Observe that for permutations \( \sigma, \sigma' \in S_{t} \),
\[
\langle (\alpha_{2})_{\sigma'}, (\alpha_{2})_{\sigma} \rangle = d^{-t} \text{Tr} \left[ ((\Sigma^{d})^{\otimes t})^\dagger \Sigma^{(\otimes t)} \right] = d^{t(\sigma')_{1} \sigma^{-t}},
\]
where \( t(\sigma')_{1} \sigma \) is the number of cycles in the permutation \( (\sigma')_{1} \sigma \). Similarly, \( \langle (\alpha_{1})_{\sigma'}, (\alpha_{1})_{\sigma} \rangle = D^{t(\sigma')_{1} \sigma^{-t}} \).

Let \( \beta \in W \), \( \|\beta\|_{2} = 1 \). Express \( \beta \) as a linear combination \( \beta = \sum_{\sigma} a_{\sigma}(\alpha_{1})_{\sigma} \otimes (\alpha_{2})_{\sigma} \), where \( a_{\sigma} \in \mathbb{C} \). We have,
\[
1 = \langle \beta, \beta \rangle = \sum_{\sigma',\sigma} a_{\sigma'}^{*} a_{\sigma} \langle (\alpha_{1})_{\sigma'} \otimes (\alpha_{2})_{\sigma'} \rangle \langle (\alpha_{1})_{\sigma} \otimes (\alpha_{2})_{\sigma} \rangle
\]
\[
= \sum_{\sigma',\sigma} a_{\sigma'}^{*} a_{\sigma} \langle (\alpha_{1})_{\sigma'} \rangle \langle (\alpha_{2})_{\sigma} \rangle = \sum_{\sigma',\sigma} a_{\sigma'}^{*} a_{\sigma} (Dd)^{t(\sigma')_{1} \sigma^{-t}}
\]
\[
= a^{t}(I + M)a \in \|a\|_{2}^{2} \left( 1 \pm \frac{t(t-1)}{Dd} \right),
\]
where \( a \) is a \( t! \)-tuple whose \( \sigma \)-th entry is \( a_{\sigma} \), and \( M \) is the \( t! \times t! \)-matrix defined in Lemma 2 with \( Dd \) replacing \( d \). We thus conclude that \( \sum_{\sigma} |a_{\sigma}|^{2} \leq \frac{1}{1 \pm \frac{t(t-1)}{Dd}} \).

For \( \beta \in W \), \( \|\beta\|_{2} = 1 \) as defined above, let \( \beta' := \sum_{\sigma} a_{\sigma}(\alpha_{1})_{\sigma} \otimes (\alpha_{2})_{\sigma} \in W' \). Then, \( \|\beta'\|_{2}^{2} = \sum_{\sigma} |a_{\sigma}|^{2} \| (\alpha_{1})_{\sigma} \|_{2}^{2} \| (\alpha_{2})_{\sigma} \|_{2}^{2} = \sum_{\sigma} |a_{\sigma}|^{2} \| (\alpha_{2})_{\sigma} \|_{2}^{2} \). By Lemma 1 and the previous paragraph, we get
\[
1 - \frac{t(t-1)}{2d} \leq \|\beta'\|_{2}^{2} \leq \frac{1}{1 - \frac{t(t-1)}{Dd}}.
\]

Observe that if \( \sigma \neq \sigma' \), \( \langle (\alpha'_{2})_{\sigma'} \rangle = 0 \). Moreover, \( \langle (\alpha'_{2})_{\sigma} \rangle \geq 1 - \frac{t(t-1)}{2d} \). Thus,
\[
\langle \beta', \beta \rangle = \sum_{\sigma',\sigma} a_{\sigma'}^{*} a_{\sigma} \langle (\alpha_{1})_{\sigma'} \otimes (\alpha_{2})_{\sigma'} \rangle \langle (\alpha_{1})_{\sigma} \otimes (\alpha_{2})_{\sigma} \rangle
\]
\[
= \sum_{\sigma} |a_{\sigma}|^{2} \langle (\alpha_{1})_{\sigma} \rangle \langle (\alpha'_{2})_{\sigma} \rangle \langle (\alpha_{2})_{\sigma} \rangle \geq 1 - \frac{t(t-1)}{d}.
\]

Hence,
\[
\|\beta - \beta'\|_{2}^{2} = \|\beta\|_{2}^{2} + \|\beta'\|_{2}^{2} - 2 \Re \langle \beta', \beta \rangle \leq 1 + \frac{1}{1 - \frac{t(t-1)}{Dd}} - 2 \left( 1 - \frac{t(t-1)}{d} \right) \leq \frac{4t(t-1)}{d}.
\]

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This shows that for any $\beta \in W$, $\|\beta\|_2 = 1$, there exists a $\beta' \in W'$ such that $\|\beta - \beta'\|_2 \leq 2\sqrt{\frac{t(t-1)}{d}}$.

Similarly, we can show that for any $\beta' \in C^{D \times D'} \otimes W_2$, $\|\beta'\|_2 = 1$, there exists a $\beta \in C^{D \times D'} \otimes W_2$ such that $\|\beta - \beta'\|_2 \leq 2\sqrt{\frac{t(t-1)}{d}}$. Moreover, if $\beta' \in W'$, then the resulting $\beta \in W$. We proceed as follows. Suppose $\beta' := \sum_\sigma b_\sigma (\beta_1)_\sigma \otimes (\alpha_2)_\sigma$, where $b_\sigma \in C$, $(\beta_1)_\sigma \in C^{D \times D'}$, $\| (\beta_1)_\sigma \|_2 = 1$. Then by Lemma 1,

$$1 = \|\beta'\|_2^2 = \sum_\sigma |b_\sigma|^2 \| (\alpha_2)_\sigma \|_2^2 \geq \left( 1 - \frac{t(t-1)}{2d} \right) \sum_\sigma |b_\sigma|^2,$$

which implies that $\sum_\sigma |b_\sigma|^2 \leq \frac{1}{1 - \frac{t(t-1)}{2d}}$. Similarly, we can argue that $\sum_\sigma |b_\sigma|^2 \geq 1$. Define $\beta := \sum_\sigma b_\sigma (\beta_1)_\sigma \otimes (\alpha_2)_\sigma$. Then,

$$\| \beta \|_2^2 = \sum_{\sigma', \sigma} |b_{\sigma'}^* b_\sigma ( (\beta_1)_{\sigma'} \otimes (\alpha_2)_{\sigma'} ) ( (\beta_1)_\sigma \otimes (\alpha_2)_\sigma ) |$$

$$= \sum_{\sigma', \sigma} |b_{\sigma'}^* b_\sigma ( (\beta_1)_{\sigma'} , (\beta_1)_\sigma ) \cdot ( (\alpha_2)_{\sigma'} , (\alpha_2)_\sigma ) |$$

$$\leq \sum_{\sigma', \sigma} |b_{\sigma'}| |b_\sigma| |( (\beta_1)_{\sigma'} , (\beta_1)_\sigma ) | \cdot |( (\alpha_2)_{\sigma'} , (\alpha_2)_\sigma ) |$$

$$\leq \sum_{\sigma', \sigma} |b_{\sigma'}| |b_\sigma| |d^{t(\alpha_{\sigma'}) - t} \gamma | = b^\dagger (1 + M) b \leq \|b\|_2^2 \|1 + M\|_\infty \leq \frac{1 + t(t-1)}{1 - \frac{t(t-1)}{2d}},$$

where $b$ is a $t!$-tuple whose $\sigma$th entry is $|b_\sigma|$, and $M$ is the $t! \times t!$-matrix defined in Lemma 2. Hence,

$$\langle \beta', \beta \rangle = \sum_{\sigma', \sigma} b_{\sigma'}^* b_\sigma ( (\beta_1)_{\sigma'} \otimes (\alpha_2)_{\sigma'} , (\beta_1)_\sigma \otimes (\alpha_2)_\sigma )$$

$$= \sum_{\sigma} |b_\sigma|^2 \langle (\beta_1)_\sigma , (\beta_1)_\sigma \rangle \langle (\alpha_2)_\sigma , (\alpha_2)_\sigma \rangle$$

$$\geq 1 - \frac{t(t-1)}{2d}.$$

Hence,

$$\| \beta - \beta' \|_2^2 = \| \beta \|_2^2 + \| \beta' \|_2^2 - 2 \Re (\langle \beta', \beta \rangle) \leq \frac{1 + \frac{t(t-1)}{d}}{1 - \frac{t(t-1)}{2d}} + 1 - 2 \left( 1 - \frac{t(t-1)}{2d} \right) \leq \frac{3t(t-1)}{d}.$$

This proves the first two claims of the lemma.

Using the first claim of the lemma, we now argue that $(W')^\perp$ is everywhere close to $W^\perp$ to within $2\sqrt{\frac{t(t-1)}{d}}$. Let $\gamma' \in (W')^\perp$, $\| \gamma' \|_2 = 1$. Let $\beta \in W$ be the projection of $\gamma'$ onto $W$. We claim that $\| \beta \|_2 \leq 2\sqrt{\frac{t(t-1)}{d}}$. Suppose not. Then $\| \gamma' - \beta \|_2 \leq \sqrt{1 - 4\sqrt{\frac{t(t-1)}{d}}}$. Since $W'$ and $W$ are everywhere close, there exists a $\beta' \in W'$ such that

$$\| \beta - \beta' \|_2 \leq \| \beta \|_2 \sqrt{\frac{t(t-1)}{d}} \leq 2\sqrt{\frac{t(t-1)}{d}}.$$
Hence, $\|\gamma' - \beta'\|_2 \leq \sqrt{1 - 4t(l-1)/d} + 2\sqrt{(l-1)/d} < 1$, leading to a contradiction because the orthogonality of $\gamma'$ and $\beta'$ would imply $\|\gamma' - \beta'\|_2 \geq 1$. Define $\gamma := \gamma' - \beta$. Then $\gamma \in W^\perp$ and $\|\gamma' - \beta\|_2 = 2\sqrt{(l-1)/d}$. This shows that $(W')^\perp$ is everywhere close to $W^\perp$ to within $2\sqrt{(l-1)/d}$. The argument can be reversed to also show that $W^\perp$ is everywhere close to $(W')^\perp$ to within $2\sqrt{(l-1)/d}$. We have thus finished showing that $W^\perp$ and $(W')^\perp$ are everywhere close to within $2\sqrt{(l-1)/d}$.

By a similar argument, starting from the second claim of the lemma, we can prove the fourth claim.

We now prove the following theorem.

**Theorem 1.** Let $G = \{U_i\}_{i=1}^d$ be a $(D, d, \lambda_1, 1)$-qTPE and $H = \{V_j\}_{j=1}^e$ be a $(d, s, \lambda_2, t)$-qTPE, where $D \geq d \geq 10t^2$. Then $G \otimes H$ is a $(Dd, s^2, \lambda, t)$-qTPE where

$$\lambda := \lambda_1 + \lambda_2 + 24\sqrt{t(l-1)/d}.$$ 

**Proof.** In order to prove the theorem, we need to show that for any matrix $\gamma \in W^\perp$ with $\|\gamma\|_2 = 1$, $\| (G \otimes H)(\gamma)\|_2 \leq \lambda$. By Lemma 5 $W^\perp$ and $(W')^\perp$ are everywhere close. Since the Schatten $\ell_\infty$-norm of the superoperator $G \otimes H$ is one, it suffices to show that for any matrix $\gamma' \in (W')^\perp$ with $\|\gamma'\|_2 = 1$, $\| (G \otimes H)(\gamma')\|_2 \leq \lambda - 2\sqrt{t(l-1)/d}$. This is equivalent to showing that

$$\langle \delta', (G \otimes H)(\gamma') \rangle \leq \lambda - 2\sqrt{t(l-1)/d}$$

for any matrices $\gamma', \delta' \in (W')^\perp$ with $\|\gamma'\|_2 = \|\delta'\|_2 = 1$.

Let us write $\gamma' = a_1 \gamma'_1 + a_2 \gamma'_2$, $\delta' = b_1 \delta'_1 + b_2 \delta'_2$, where

$$\gamma'_1, \delta'_1 \in (C^{D' \times D'} \otimes (W'_2)) \cap (W')^\perp, \quad \gamma'_2, \delta'_2 \in C^{D' \times D'} \otimes (W'_2)^\perp,$$

$a_1, b_1, a_2, b_2 \in \mathbb{C}$, $\|\gamma'_1\|_2 = \|\delta'_1\|_2 = \|\gamma'_2\|_2 = \|\delta'_2\|_2 = 1$. Since $\gamma'_1$ is orthogonal to $\gamma'_2$ and $\delta'_1$ is orthogonal to $\delta'_2$, $|a_1|^2 + |a_2|^2 = \|\gamma'_1\|_2^2 = 1$ and $|b_1|^2 + |b_2|^2 = \|\delta'_1\|_2^2 = 1$. Thus,

$$\langle \delta', (G \otimes H)(\gamma') \rangle = b_1^* a_1 \langle \delta'_1, (G \otimes H)(\gamma'_1) \rangle + b_1^* a_2 \langle \delta'_1, (G \otimes H)(\gamma'_2) \rangle + b_2^* a_1 \langle \delta'_2, (G \otimes H)(\gamma'_1) \rangle + b_2^* a_2 \langle \delta'_2, (G \otimes H)(\gamma'_2) \rangle.$$ 

We will now bound each of the four inner products in the above equation.

We bound the fourth inner product first. By Lemma 5 there are matrices $\gamma_2, \delta_2 \in C^{D' \times D'} \otimes (W_2)^\perp$ such that

$$\|\gamma_2 - \gamma_2\|_2, \|\delta_2 - \delta_2\|_2 \leq 2\sqrt{t(l-1)/d}, \quad \|\gamma_2\|_2 \leq 1, \|\delta_2\|_2 \leq 1.$$
Hence, 

\[ \| (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_2') \|_2 \leq \| (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_2) \|_2 + \| \gamma_2' - \gamma_2 \|_2 \]

\[ \leq \lambda_2 \| \gamma_2 \|_2 + 2 \sqrt{\frac{t(t-1)}{d}} \leq \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}}. \]

Similarly,

\[ \| (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger)(\delta_2') \|_2 \leq \| (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger)(\delta_2) \|_2 + \| \delta_2' - \delta_2 \|_2 \]

\[ \leq \lambda_2 \| \delta_2 \|_2 + 2 \sqrt{\frac{t(t-1)}{d}} \leq \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}}, \]

where, in the second inequality above, we used the fact that the right singular vectors of \( \mathcal{H}^\dagger \) are the left singular vectors of \( \mathcal{H} \) and vice versa, and the fact that \( (W_2)^\perp \) is the span of the left as well as the right singular vectors of \( \mathcal{H} \) with singular value at most \( \lambda_2 \). Thus,

\[ |\langle \delta_2', (G \otimes \mathcal{H})(\gamma_2') \rangle| \]

\[ = |\langle \delta_2', (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}) \circ \tilde{G}^\otimes (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_2') \rangle| \]

\[ = |\langle (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger)(\delta_2), \tilde{G}^\otimes ((I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_2'))(\tilde{G}^\dagger)^\otimes \rangle| \]

\[ \leq \| (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger)(\delta_2) \|_2 \| \tilde{G}^\otimes ((I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_2'))(\tilde{G}^\dagger)^\otimes \|_2 \]

\[ = \| (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger)(\delta_2) \|_2 \| ((I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_2')) \|_2 \]

\[ \leq \left( \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}} \right)^2. \]

We bound the second and third inner products similarly.

\[ |\langle \delta_1', (G \otimes \mathcal{H})(\gamma_2') \rangle| \]

\[ = |\langle (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger)(\delta_1), \tilde{G}^\otimes ((I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_2'))(\tilde{G}^\dagger)^\otimes \rangle| \]

\[ \leq \| (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger)(\delta_1) \|_2 \| ((I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_2')) \|_2 \]

\[ \leq \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}}. \]

where we used the fact that the Schatten \( \ell_\infty \)-norm of the superoperator \( I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger \) is one in the inequality above. Analogously,

\[ |\langle \delta_2', (G \otimes \mathcal{H})(\gamma_1') \rangle| \]

\[ = |\langle (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger)(\delta_2), \tilde{G}^\otimes ((I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_1'))(\tilde{G}^\dagger)^\otimes \rangle| \]

\[ \leq \| (I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H}^\dagger)(\delta_2) \|_2 \| ((I_{\mathbb{C}^{D_t \times D_t}} \otimes \mathcal{H})(\gamma_1')) \|_2 \]

\[ \leq \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}}. \]

Finally, we bound the first inner product as follows. Let \( (\mathbb{C}^{D_t \times D_t} \otimes W_2') \cap (W')^\perp \ni \gamma_1' = \sum_\sigma b_\sigma (\beta_1)_\sigma \otimes (\alpha_2)_\sigma \), where \( \| \gamma_1' \|_2 = 1, \| (\beta_1)_\sigma \|_2 = 1, b_\sigma \in \mathbb{C} \). Similarly, let \( (\mathbb{C}^{D_t \times D_t} \otimes W_2') \cap (W')^\perp \ni \)}
\[ \delta_1' = \sum_{\sigma} c_{\sigma}(\theta_1)_{\sigma} \otimes (\alpha_2')_{\sigma}, \text{ where } ||\delta_1'||_2 = 1, ||(\theta_1)_{\sigma}||_2 = 1, c_{\sigma} \in \mathbb{C}. \] Let \( b \) be the \( t! \)-tuple whose \( \sigma \)th entry is the complex number \( b_{\sigma} \). Let \( c \) be the \( t! \)-tuple whose \( \sigma \)th entry is the complex number \( c_{\sigma} \).

By Lemma 1
\[ 1 = \|\gamma_1'\|^2 = \sum_{\sigma} |b_{\sigma}|^2 \| (\alpha_2')_{\sigma} \|^2 \geq \| b \|^2 \left( 1 - \frac{t(t-1)}{2d} \right), \]
which gives \( \| b \|^2 \leq \frac{1}{1 - \frac{t(t-1)}{2d}}. \) Similarly, \( \| c \|^2 \leq \frac{1}{1 - \frac{t(t-1)}{2d}}. \) Define the matrices
\[ \gamma_1 := \sum_{\sigma} b_{\sigma}(\beta_1)_{\sigma} \otimes (\alpha_2)_{\sigma}, \quad \delta_1 := \sum_{\sigma} c_{\sigma}(\theta_1)_{\sigma} \otimes (\alpha_2)_{\sigma}, \quad \gamma_1, \delta_1 \in \mathbb{C}^{D_t \times D_t} \otimes W_2. \]

By Lemma 5, \( \|\gamma_1' - \gamma_1\|_2, \|\delta_1' - \delta_1\|_2 \leq 2\sqrt{\frac{t(t-1)}{d}}. \) Recalling the fact that the Schatten \( \ell_\infty \)-norm of \( G \otimes H \) is one, we get
\[ \|\langle \delta_1', (G \otimes H)(\gamma_1') \rangle - \langle \delta_1, (G \otimes H)(\gamma_1) \rangle \|
\leq \|\langle \delta_1', (G \otimes H)(\gamma_1') \rangle - \langle \delta_1', (G \otimes H)(\gamma_1) \rangle \| + \|\langle \delta_1', (G \otimes H)(\gamma_1) \rangle - \langle \delta_1, (G \otimes H)(\gamma_1) \rangle \|
\leq 2\sqrt{\frac{t(t-1)}{d}} + 2\sqrt{\frac{t(t-1)}{d}} \left( 1 + 2\sqrt{\frac{t(t-1)}{d}} \right) \leq 9\sqrt{\frac{t(t-1)}{d}}. \]

Moreover, as \( \gamma_1' \in (W^t)\perp, \langle (\beta_1)_{\sigma}, (\alpha_1)_{\sigma} \rangle = 0 \) for all \( \sigma \in S_t \). We now evaluate
\[ \langle \delta_1, (G \otimes H)(\gamma_1) \rangle
= \langle \delta_1, (G \otimes H)^\dagger \rangle \circ (G \otimes H) \circ (G \otimes H)^\dagger (\gamma_1)
= (G \otimes H)^\dagger \gamma_1 (G \otimes H)^\dagger
= \sum_{\sigma' \neq \sigma} c_{\sigma'}^* b_{\sigma} \langle (\theta_1)_{\sigma'} \otimes (\alpha_2)_{\sigma'} \rangle, \gamma_1 (G \otimes H)^\dagger (G \otimes H)^\dagger
= \sum_{\sigma' \neq \sigma} c_{\sigma'}^* b_{\sigma} \langle (\theta_1)_{\sigma'} \otimes (\alpha_2)_{\sigma'} \rangle, \gamma_1 (G \otimes H)^\dagger (G \otimes H)^\dagger
+ \sum_{\sigma} c_{\sigma'}^* b_{\sigma} \langle (\theta_1)_{\sigma} \otimes (\alpha_2)_{\sigma} \rangle, \gamma_1 (G \otimes H)^\dagger (G \otimes H)^\dagger. \]

Fix \( \sigma, \sigma' \in S_t, \sigma \neq \sigma' \). Then,
\[ \|\langle (\theta_1)_{\sigma'} \otimes (\alpha_2)_{\sigma'} \rangle, \gamma_1 (G \otimes H)^\dagger (G \otimes H)^\dagger \|
= d^{-t} \left| \sum_{i_1, \ldots, i_t, j_1, \ldots, j_t \in [d]} \langle (\theta_1)_{\sigma'} \otimes (e_{i_1} \otimes \cdots \otimes e_{i_t}) (e_{i_{\sigma'(1)}}^\dagger \otimes \cdots \otimes e_{i_{\sigma'(t)}}^\dagger),
(\langle U_{j_1} \otimes \cdots \otimes U_{j_t}(\beta_1)_{\sigma} (U_{j_{\sigma(1)}}^\dagger \otimes \cdots \otimes U_{j_{\sigma(t)}}^\dagger)) \otimes (e_{-j_1} \otimes \cdots \otimes e_{-j_t})(e_{-j_{\sigma(1)}}^\dagger \otimes \cdots \otimes e_{-j_{\sigma(t)}}^\dagger) \rangle \right| \]
\[ \leq d^{-t} \left| \sum_{i_1, \ldots, i_t, j_1, \ldots, j_t \in [d]} \|\langle (\theta_1)_{\sigma'}, (U_{j_1} \otimes \cdots \otimes U_{j_t})(\beta_1)_{\sigma} (U_{j_{\sigma(1)}}^\dagger \otimes \cdots \otimes U_{j_{\sigma(t)}}^\dagger) \rangle \| \right| \]
\[ \left| \left( (e_{i_1} \otimes \cdots \otimes e_{i_t}) (e_{i'_{\sigma(1)}}^t \otimes \cdots \otimes e_{i'_{\sigma(t)}}^t), (e_{-j_1} \otimes \cdots \otimes e_{-j_t}^t) (e_{-j_{\sigma(1)}}^t \otimes \cdots \otimes e_{-j_{\sigma(t)}}^t) \right) \right| \leq d^{-t} \sum_{i_1, \ldots, i_t} \delta_{i'_{\sigma(1)}, i_{\sigma(1)}} \cdots \delta_{i'_{\sigma(t)}, i_{\sigma(t)}} = d^{(t')_t} t^{-t}. \]

Hence,

\[
\sum_{\sigma' \neq \sigma} c^*_{\sigma'} b_{\sigma} \langle (\theta_1)_{\sigma'} \otimes (\alpha_2)_{\sigma'}, \hat{G}^{\otimes t} ((\beta_1)_{\sigma} \otimes (\alpha_2)_{\sigma})(\hat{G}^t)^{\otimes t} \rangle 
\leq \sum_{\sigma' \neq \sigma} |c_{\sigma'}| |b_{\sigma}| \langle (\theta_1)_{\sigma'} \otimes (\alpha_2)_{\sigma'}, \hat{G}^{\otimes t} ((\beta_1)_{\sigma} \otimes (\alpha_2)_{\sigma})(\hat{G}^t)^{\otimes t} \rangle 
\leq \sum_{\sigma' \neq \sigma} |c_{\sigma'}| |b_{\sigma}| d^{(t')_t} t^{-t} = |c|^t M |b| \leq \|c\|_2 \|b\|_2 \|M\|_\infty 
\leq \frac{t(t-1)}{d} \frac{1}{1 - \frac{t(t-1)}{2d}} \leq \frac{2t(t-1)}{d},
\]

where $|c|$, $|b|$ denote the $t!$-tuples whose $\sigma$th entries are $|c_{\sigma}|$, $|b_{\sigma}|$, and Lemma [2] is used in the next to last inequality. Now fix $\sigma \in S_t$. We have,

\[
\langle (\theta_1)_{\sigma} \otimes (\alpha_2)_{\sigma}, \hat{G}^{\otimes t} ((\beta_1)_{\sigma} \otimes (\alpha_2)_{\sigma})(\hat{G}^t)^{\otimes t} \rangle 
= d^{-t} \sum_{i_1, \ldots, i_t, j_1, \ldots, j_t \in [d]} \langle (\theta_1)_{\sigma} \otimes ((e_{i_1} \otimes \cdots \otimes e_{i_t})(e^t_{i'_{\sigma(1)}} \otimes \cdots \otimes e^t_{i'_{\sigma(t)}})),

(U_{j_1} \otimes \cdots \otimes U_{j_t})(b_{\sigma(1)}^t \otimes \cdots \otimes b_{\sigma(t)}^t) \rangle 
\otimes ((e_{-j_1} \otimes \cdots \otimes e_{-j_t})(e^t_{-j_{\sigma(1)}} \otimes \cdots \otimes e^t_{-j_{\sigma(t)}})) \rangle 
= d^{-t} \sum_{i_1, \ldots, i_t \in [d]} \langle (\theta_1)_{\sigma}, (U_{-i_1} \otimes \cdots \otimes U_{-i_t})(b_{\sigma(1)}^t \otimes \cdots \otimes b_{\sigma(t)}^t) \rangle 
\otimes ((e_{-j_1} \otimes \cdots \otimes e_{-j_t})(e^t_{-j_{\sigma(1)}} \otimes \cdots \otimes e^t_{-j_{\sigma(t)}})) \rangle 
\leq \langle (\theta_1)_{\sigma} \|_2 \left\| d^{-t} \sum_{i_1, \ldots, i_t \in [d]} (U_{-i_1} \otimes \cdots \otimes U_{-i_t})(b_{\sigma(1)}^t \otimes \cdots \otimes b_{\sigma(t)}^t) \right\|_2 
= \left\| d^{-t} \sum_{i_1, \ldots, i_t \in [d]} (U_{i_1} \otimes \cdots \otimes U_{i_t})(b_{\sigma(1)}^t \otimes \cdots \otimes b_{\sigma(t)}^t) \right\|_2.
\]

Let us now express $(\beta_1)_{\sigma}$ as $\Sigma^{(C^D)^{\otimes t}} (\hat{\beta}_1)_{\sigma}$ for some $(\hat{\beta}_1)_{\sigma} \in C^{D_t \times D^t}$; $\| (\hat{\beta}_1)_{\sigma} \|_2 = 1$. Observe that

\[ 0 = \langle (\beta_1)_{\sigma}, (\alpha_1)_{\sigma} \rangle = \langle \Sigma^{(C^D)^{\otimes t}} (\hat{\beta}_1)_{\sigma}, \Sigma^{(C^D)^{\otimes t}} (\hat{\alpha}_1)_{\sigma} \rangle = \langle (\hat{\beta}_1)_{\sigma}, (\hat{\alpha}_1)_{\sigma} \rangle. \]
We can now write

$$\langle \langle \theta_1, \sigma \otimes (\alpha_2, \sigma), \hat{G}^\otimes ((\beta_1, \sigma) \otimes (\alpha_2, \sigma))| (\hat{G}^\dagger)^\otimes \rangle \rangle$$

\[
\leq \left\| d^{-t} \sum_{i_1, \ldots, i_t \in [d]} (U_{i_1} \otimes \cdots \otimes U_{i_t}) \Sigma^{(D)^\otimes}((\hat{\beta}_1, \sigma)(U_{i_{t(1)}}^\dagger \otimes \cdots \otimes U_{i_{t(t)}}^\dagger) \right\|_2 \\
= \left\| d^{-t} \sum_{i_1, \ldots, i_t \in [d]} \Sigma^{(D)^\otimes}(U_{i_1} \otimes \cdots \otimes U_{i_t})(\hat{\beta}_1, \sigma)(U_{i_{t(1)}}^\dagger \otimes \cdots \otimes U_{i_{t(t)}}^\dagger) \right\|_2 \\
= \left\| \Sigma^{(D)^\otimes} G^\otimes ((\hat{\beta}_1, \sigma)) \right\|_2 = \left\| G^\otimes ((\hat{\beta}_1, \sigma)) \right\|_2.
\]

Since $G$ is a quantum expander, i.e. a $(D, d, \lambda_1, 1)$-qTPE, $G^\otimes$ is also a quantum expander, i.e. a $(D^t, d^t, \lambda_1, 1)$-qTPE by Fact 6. Since $\langle \langle \hat{\beta}_1, \sigma, \alpha_1 \rangle \rangle = 0$, we see that $\left\| G^\otimes ((\hat{\beta}_1, \sigma)) \right\|_2 \leq \lambda_1$. Thus,

$$\langle \langle \theta_1, \sigma \otimes (\alpha_2, \sigma), \hat{G}((\beta_1, \sigma) \otimes (\alpha_2, \sigma))| (\hat{G}^\dagger)^\otimes \rangle \rangle \leq \lambda_1.$$

Hence,

$$\left| \sum_{\sigma} c_\sigma b_\sigma \langle \langle \theta_1, \sigma \otimes (\alpha_2, \sigma), \hat{G}^\otimes ((\beta_1, \sigma) \otimes (\alpha_2, \sigma))| (\hat{G}^\dagger)^\otimes \rangle \rangle \right|$$

\[
\leq \sum_{\sigma} |c_\sigma| |b_\sigma| \langle \langle \theta_1, \sigma \otimes (\alpha_2, \sigma), \hat{G}^\otimes ((\beta_1, \sigma) \otimes (\alpha_2, \sigma))| (\hat{G}^\dagger)^\otimes \rangle \rangle \leq \lambda_1 \|c\|_2 \|b\|_2 \\
\leq \frac{\lambda_1}{1 - \frac{t(t-1)}{2d}} \leq \lambda_1 \left( 1 + \frac{t(t-1)}{d} \right).
\]

This implies that

$$\langle \langle \delta_1, (G \otimes H)(\gamma_1) \rangle \rangle \leq \lambda_1 \left( 1 + \frac{t(t-1)}{d} \right) + \frac{2t(t-1)}{d},$$

which further leads to

$$\langle \langle \delta_1', (G \otimes H)(\gamma'_1) \rangle \rangle \leq \lambda_1 \left( 1 + \frac{t(t-1)}{d} \right) + \frac{2t(t-1)}{d} + 9 \sqrt{\frac{t(t-1)}{d}}$$

\[
\leq \lambda_1 + 12 \sqrt{\frac{t(t-1)}{d}}.
\]

Putting the bounds on the four inner products together, we get

\[
\langle \langle \delta', (G \otimes H)(\gamma') \rangle \rangle \leq |b_1| |a_1| \left( \lambda_1 + 12 \sqrt{\frac{t(t-1)}{d}} \right) + |b_1| |a_2| \left( \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}} \right) \\
+ |b_2| |a_1| \left( \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}} \right) + |b_2| |a_2| \left( \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}} \right)^2 \\
\leq \left( \lambda_1 + 12 \sqrt{\frac{t(t-1)}{d}} \right) + \left( \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}} \right) + \left( \lambda_2 + 2 \sqrt{\frac{t(t-1)}{d}} \right)^2,
\]

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As in [RVW02, Theorem 6.2], one can similarly define a ‘derandomised’ zigzag product as follows:

\[ \langle G \circ \mathcal{H} \rangle (\gamma) \]  
\[ \leq \left( \lambda_1 + 12 \sqrt{\frac{t(t-1)}{d}} \right) + \left( \lambda_2 + 2^{\frac{4}{\sqrt{\lambda}} t(t-1) d} \right) + \left( \lambda_2 + 2^{\frac{4}{\sqrt{\lambda}} t(t-1) d} \right)^2 + 2^{\frac{4}{\sqrt{\lambda}} t(t-1) d} \]

finishing the proof of the theorem.

Remarks:
1. Setting \( t = 1 \) recovers the eigenvalue bound on the zigzag product of quantum expanders, i.e. 1-qTPEs, proved in [BST10, Theorem 4.8].
2. For \( t = \text{polylog}(D) \), taking an efficient construction (e.g. via the zigzag product) of a \((D, d, \lambda_1, 1)\)-qTPE, \( d = (10s \log t)^6 \), \( \lambda_1 = 100d^{-1/4} \) as in [BST10], and combining it via the zigzag product with a \((d, s, \lambda_2, t)\)-qTPE, \( \lambda_2 = 8s^{-1/2} \) obtained from the random construction of Fact 2, gives us a \((Dd, s^2, \lambda, t)\)-qTPE, \( \lambda := \lambda_1 + 2\lambda_2 + O(\sqrt{\frac{t}{d}}) \) which is efficiently computable. This gives rise to a fourth power tradeoff between degree \( s^2 \) and second largest singular value \( 10s^{-1/2} \). This tradeoff is the same as in the standard zigzag product for classical [RVW02] and quantum [BST10] expanders.
3. The reader may wonder why we went from the subspace \( W^\perp \) to \( (W^\perp)^\perp \) and back in the above proof. The reason behind this seemingly unnatural strategy is because we want to ensure that in the proof of the bound on the first inner product, \((\beta_1)_{\sigma} \) is perfectly orthogonal to \((\alpha_1)_{\sigma} \). Approximate orthogonality in this step seems to give additive losses of \( \text{poly}(\frac{t}{d}) \) in the expression for \( \lambda \), which would require \( d \geq t! \), leading to the construction of efficient \( t \)-qTPEs in dimension \( N \) only for \( t \leq \frac{\log \log N}{\log \log \log N} \). This is too small for many applications. Going from \( W^\perp \) to \( (W^\perp)^\perp \) allows us to use the fact that \((\alpha'_2)_{\sigma'} \) is orthogonal to \((\alpha_2)_{\sigma} \), \( \sigma' \neq \sigma \) which finally ensures that \((\beta_1)_{\sigma} \) is indeed perfectly orthogonal to \((\alpha_1)_{\sigma} \). But then the second eigenvalue bounds on \( G \) and \( H \) are in terms of \( W^\perp \) and so we have to go back to \( W^\perp \) from \((W^\perp)^\perp \) in order to use them in the proof. By adopting this back and forth strategy, we only get additive losses of \( \text{poly}(\frac{t}{d}) \), which would require \( d \geq \text{poly}(t) \), leading to the construction of efficient \( t \)-qTPEs in dimension \( N \) for \( t = \text{polylog}(N) \).
4. An improved analysis of \( \lambda \) in the above theorem along the lines of [RVW02, Theorem 4.3] can be done, giving us the bound

\[ \lambda := \frac{1}{2}(1 - \mu_2^2)\mu_1 + \frac{1}{2}\sqrt{(1 - \mu_2^2)\mu_1^2 + 4\mu_2^2 + 2^{\frac{4}{\sqrt{\lambda}} t(t-1) d}}, \]

where

\[ \mu_1 := \lambda_1 + 9\sqrt{\frac{t(t-1)}{d}}, \quad \mu_2 := \lambda_2 + 2^{\frac{4}{\sqrt{\lambda}} t(t-1) d}. \]

This bound has several nice properties e.g. it is always less than \( \mu_1 + \mu_2 + 2^{\frac{4}{\sqrt{\lambda}} t(t-1) d} \), it is always less than \( 1 + 2^{\frac{4}{\sqrt{\lambda}} t(t-1) d} \) if \( \mu_1, \mu_2 < 1 \) etc.
5. As in [RVW02, Theorem 6.2], one can similarly define a ‘derandomised’ zigzag product as follows:
Definition 5 (Derandomised zigzag product of qTPEs). The derandomised zigzag product of explicitly Hermitian qTPEs $G$ and $H$, denoted by $G \circledast H$, is defined as the following set of $s^3$ unitary matrices on $\mathbb{C}^{Dd}$:

$$G \circledast H := \{ (\mathbb{1}^{\bar{C}D} \otimes V_i)(\mathbb{1}^{\bar{C}D} \otimes V_j^\dagger)\hat{G}(\mathbb{1}^{\bar{C}D} \otimes V_j)(\mathbb{1}^{\bar{C}D} \otimes V_k) : i, j, k \in [s] \}.$$ 

With this definition, one can similarly show that the second eigenvalue $\lambda$ of $G \circledast H$ satisfies the bound

$$\lambda := \mu_1 + 2\mu_2^2 + 2\sqrt{\frac{t(t-1)}{d}},$$

where $\mu_1, \mu_2$ are defined in the previous remark. For $t = \text{polylog}(D)$, using the derandomised zigzag product for constructing a quantum expander i.e. $(D, d, \lambda_1)$-qTPE, $d = (10st \log t)^6$, $\lambda_1 = 100d^{-1/3}$, and combining it via the derandomised zigzag product with a $(d, s, \lambda_2, t)$-qTPE, $\lambda_2 = 8s^{-1/2}$ obtained from Fact 2, gives us a $(Dd, s^3, \lambda, t)$-qTPE, $\lambda := \lambda_1 + 2\lambda_2^2 + O(\sqrt{\frac{t}{d}})$ which is efficiently computable. This gives rise to a third power tradeoff between degree $s^3$ and second largest singular value $130s^{-1}$. This tradeoff is the same as in the derandomised zigzag product for classical expanders [RVW02].

4 Generalised zigzag product gives almost Ramanujan qTPE

Inspired by the definition of generalised zigzag product for classical expanders, i.e. 1-cTPEs, in [BT11], we define the zigzag product of a 1-qTPE and a $t$-qTPE as follows.

Definition 6 (Generalised zigzag product of qTPEs). Let $G = \{U_i\}_{i=1}^d$ be a $(D, d, \lambda_1)$-qTPE. For $1 \leq j \leq k$, let $H_j = \{V_i(j)\}_{i=1}^s$ be a $(dd', s, \lambda_2, t)$-qTPE. Let $H_k := (H_k, \ldots, H_1)$. Define the unitary matrix $\hat{G}$ on the vector space $\mathbb{C}^{Ddd'} := \mathbb{C}^D \otimes (\mathbb{C}^d \otimes \mathbb{C}^{d'})$ by

$$e_a \otimes (e_b \otimes e_{b'}) \mapsto (U_b e_a) \otimes (e_b \otimes e_{b'}),$$

where $e_a, e_b, e_{b'}$ denote computational basis vectors of $\mathbb{C}^D, \mathbb{C}^d, \mathbb{C}^{d'}$ respectively. The zigzag product of qTPEs $G$ and $H$, denoted by $G \circledast H$, is defined as the following set of $s^k$ unitary matrices on $\mathbb{C}^{Ddd'}$:

$$G \circledast H := \{ (\mathbb{1}^{\bar{C}D} \otimes V_{i_1}(k))\hat{G} \cdots \hat{G}(\mathbb{1}^{\bar{C}D} \otimes V_{i_1}(1)) : i_1, \ldots, i_1 \in [s] \}.$$ 

Remarks:

1. The generalised zigzag product of Hermitian qTPEs will in general not be Hermitian because the qTPEs $H_k, \ldots, H_1$ in general have no relation amongst them. That is why we dispense with the involution ‘$-$’ in defining the unitary $\hat{G}$ and the generalised zigzag product. Note that any qTPE can be made explicitly Hermitian by doubling its degree according to Fact 4.
2. Viewed as a superoperator on $\mathbb{C}^{Ddd'}^t \times (Ddd')^t$, the generalised zigzag product $G \circledast H$ is nothing but

$$G \circledast H := (\mathbb{1}^{\bar{C}D^t \times D^t} \otimes H_k) \circ \hat{G} \otimes \cdots \otimes \hat{G} \otimes (\mathbb{1}^{\bar{C}D^t \times D^t} \otimes H_1).$$
Suppose $t \leq dd' \leq D \leq Ddd'$. Define the subspaces

$$W \subseteq \mathbb{C}^{(Ddd')^t \times (Ddd')^t}, W_1 \subseteq \mathbb{C}^{D^t \times D^t}, W_2 \subseteq \mathbb{C}^{(dd')^t \times (dd')^t}, W_2' \subseteq \mathbb{C}^{(dd')^t \times (dd')^t}, W' \subseteq \mathbb{C}^{(Ddd')^t \times (Ddd')^t}, W_1' \subseteq \mathbb{C}^{(dd')^t \times (dd')^t}, (W')^\perp \subseteq \mathbb{C}^{(Ddd')^t \times (Ddd')^t}, (W_1')^\perp \subseteq \mathbb{C}^{(dd')^t \times (dd')^t},$$

and matrices

$$\alpha_\sigma \in \mathbb{C}^{(Ddd')^t \times (Ddd')^t}, (\alpha_1)_\sigma \in \mathbb{C}^{D^t \times D^t}, (\alpha_2)_\sigma \in \mathbb{C}^{(dd')^t \times (dd')^t},$$

for a $\sigma \in S_t$ in similar fashion as before. Then

$$(W')^\perp = (\mathbb{C}^{D^t \times D^t} \otimes (W_2')^\perp) \oplus ((\mathbb{C}^{D^t \times D^t} \otimes (W_2') \cap (W')^\perp)$$

as before and Lemma 5 on everywhere closeness holds with $d$ replaced by $dd'$. For a matrix $\gamma \in W^\perp$, define $\gamma'$ to be the matrix in $(W')^\perp$ such that

$$\|\gamma' - \gamma\|_2 \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \|\gamma\|_2, \quad \|\gamma'\|_2 \leq \|\gamma\|_2,$$

whose existence is guaranteed by Lemma 5. For a matrix $\gamma' \in (W')^\perp$, define $(\gamma')^\parallel$ to be its projection onto $(\mathbb{C}^{D^t \times D^t} \otimes (W_2') \cap (W')^\perp$ and $(\gamma')^\perp$ to be its projection onto $\mathbb{C}^{D^t \times D^t} \otimes (W_2')^\perp$. Define $(\gamma')^\parallel$ to be the matrix in $\mathbb{C}^{D^t \times D^t} \otimes W_2$ such that

$$\| (\gamma')^\parallel - (\gamma')^\parallel' \|_2 \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \| (\gamma')^\parallel' \|_2,$$

whose existence is guaranteed by Lemma 5. Define $(\gamma')^\perp$ to be the matrix in $\mathbb{C}^{D^t \times D^t} \otimes (W_2')^\perp$ such that

$$\| (\gamma')^\perp - (\gamma')^\perp' \|_2 \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \| (\gamma')^\perp' \|_2, \quad \| (\gamma')^\perp \|_2 \leq \| (\gamma')^\perp' \|_2,$$

whose existence is guaranteed by Lemma 5. We now define by induction two sequences of matrices starting with $\gamma_0, \delta_0 \in W^\perp$, $\|\gamma_0\|_2 = \|\delta_0\|_2 = 1$. Let $\gamma_i', \delta_i'$ be the matrices in $(W')^\perp$ that are $2 \sqrt{\frac{t(t-1)}{dd'}}$ close to $\gamma_0$, $\delta_0$, whose existence has been shown above. For $1 \leq i \leq k-1$, define

$$\gamma_i := (\mathcal{G}^{\otimes t} \circ (\mathbb{I} \otimes \mathcal{H}_i))(\gamma_{i-1})^\perp, \quad \delta_i := (\mathcal{G}_i^{\otimes t} \circ (\mathbb{I} \otimes \mathcal{H}_{k-i+1}))((\delta_{i-1})^\perp.$$  

Observe that $\gamma_i, \delta_i$ are in $W^\perp$, so we can define $\gamma_i', \delta_i' \in (W')^\perp$ accordingly. This implies that $(\gamma_i')^\parallel, (\delta_i')^\parallel \in (\mathbb{C}^{D^t \times D^t} \cap W_2') \cap (W')^\perp$. We will assume that our parameters are such that $\lambda_2 < 1/2$. Using induction on $i$, it is easy to see that

$$\|\gamma_i\|_2 \leq \|\gamma_i\|_2 \leq \lambda_2 \|\gamma_{i-1}\|^\perp_2 \leq \lambda_2 \|\gamma_{i-1}\|^\parallel_2 \leq \lambda_2 \|\gamma_{i-1}\|^\parallel_2 \leq \lambda_2.$$

Similarly,

$$\|\delta_i\|_2 \leq \|\delta_i\|_2 \leq \lambda_2 \|\delta_{i-1}\|^\perp_2 \leq \lambda_2 \|\delta_{i-1}\|^\parallel_2 \leq \lambda_2 \|\delta_{i-1}\|^\parallel_2 \leq \lambda_2.$$
By construction, we have the orthogonal decompositions $\gamma_i = (\gamma_i)^\parallel + (\gamma_i)^\perp$, $\delta'_i = (\delta'_i)^\parallel + (\delta'_i)^\perp$. Thus, $\|\gamma_i\|^2 = \|\gamma_i\|^\parallel^2 + \|\gamma_i\|^\perp^2$, $\|\delta'_i\|^2 = \|\delta'_i\|^\parallel^2 + \|\delta'_i\|^\perp^2$. By induction, we observe that

$$\sum_{0 \leq i \leq k-1} \|\gamma_i\|^\parallel^2 \leq \|\gamma_0\|^\parallel^2 + \|\gamma_1\|^\parallel^2 + \lambda_2^2 \|\gamma_0\|^\perp^2 \leq \|\gamma_0\|^\parallel^2 + \|\gamma_0\|^\perp^2$$

Similarly,

$$\sum_{0 \leq i \leq k-1} \|\gamma_0\|^\parallel^2 \leq 1.$$

On the other hand,

$$\sum_{0 \leq i \leq k-1} \|\delta'_i\|^\parallel^2 \leq \sum_{0 \leq i \leq k-1} \|\delta'_i\|^\perp^2$$

Similarly,

$$\sum_{0 \leq i \leq k-1} \|\gamma_i\|^\parallel^2 \leq 2.$$

For $0 \leq i < j \leq k$, define

$$e_i := \langle \delta_0, ((\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_k) \circ \tilde{G}^\otimes \circ \cdots \circ \tilde{G}^\otimes (\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_1)(\gamma_i - \gamma_i^\parallel),$$

$$f_i := \langle \delta_0, ((\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_k) \circ \tilde{G}^\otimes \circ \cdots \circ \tilde{G}^\otimes (\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_1)((\gamma_i)^\perp - (\gamma_i)^\parallel),$$

$$d_i := \langle (\delta'_{k-i-1})^\perp, ((\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_{k+1})((\gamma_i)^\parallel - (\gamma_i)^\parallel),$$

$$l_i := \langle (\delta'_{k-i-1})^\parallel, ((\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_{i+1})((\gamma_i)^\parallel - (\gamma_i)^\parallel),$$

$$g_{ji} := \langle \delta'_{k-j} - \delta'_{k-j}, ((\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_k) \circ \tilde{G}^\otimes \circ \cdots \circ \tilde{G}^\otimes (\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_{i+1})(\gamma_j^\parallel),$$

$$h_{ji} := \langle (\delta'_{k-j})^\perp - (\delta'_{k-j})^\perp, ((\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_k) \circ \tilde{G}^\otimes \circ \cdots \circ \tilde{G}^\otimes (\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_{i+1})(\gamma_j^\parallel),$$

$$m_{ji} := \langle (\delta'_{k-j})^\parallel - (\delta'_{k-j})^\parallel, ((\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_j) \circ \tilde{G}^\otimes \circ \cdots \circ \tilde{G}^\otimes (\mathbb{C}^{d \times d})^\ast \otimes \mathcal{H}_{i+1})(\gamma_j^\parallel).$$

Then,

$$|e_i| \leq \|\gamma_i - \gamma_i^\parallel\| \leq 2\sqrt{\frac{t(t-1)}{dd'}} \|\gamma_i\|,$$

$$|f_i| \leq \|\gamma_i^\perp - (\gamma_i)^\parallel\| \leq 2\sqrt{\frac{t(t-1)}{dd'}} \|\gamma_i^\perp\| \leq 2\sqrt{\frac{t(t-1)}{dd'}} \|\gamma_i\|,$$

$$|d_i| \leq \|\gamma_i^\parallel - (\gamma_i)^\parallel\| \cdot \|\delta'_{k-i-1})^\perp\| \leq 2\sqrt{\frac{t(t-1)}{dd'}} \|\gamma_i^\parallel\| \cdot \|\delta'_{k-i-1})^\perp\|$$

$$\leq 2\sqrt{\frac{t(t-1)}{dd'}} \left(\|\gamma_i^\parallel\|^2 + \|\delta'_{k-i-1})^\perp\|^2\right).$$
\[ |l_i| \leq \left\| (\gamma_i')'' - (\gamma_i')'' \right\|_2 \cdot \left\| (\delta_k'_{i-1})'' \right\|_2 \\
\leq 2 \sqrt{\frac{t(t-1)}{dd'}} \left( 1 + 2 \sqrt{\frac{t(t-1)}{dd'}} \right) \left\| (\gamma_i')'' \right\|_2 \cdot \left\| (\delta_k'_{i-1})'' \right\|_2 \\
\leq 2 \sqrt{\frac{t(t-1)}{dd'}} \left( \left\| (\gamma_i')'' \right\|_2^2 + \left\| (\delta_k'_{i-1})'' \right\|_2^2 \right) \\
\]

\[ |g_{ji}| \leq \left\| \delta_k - \delta_k'_{j-1} \right\|_2 \cdot \left\| (\gamma_i')'' \right\|_2 \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \left\| \delta_k - \delta_k'_{j-1} \right\|_2 \cdot \left\| (\gamma_i')'' \right\|_2 \\
\]

\[ |h_{ji}| \leq \left\| (\delta_k'_{j-1})''' - (\delta_k'_{j-1})''' \right\|_2 \cdot \left\| (\gamma_i')'' \right\|_2 \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \left\| (\delta_k'_{j-1})''' \right\|_2 \cdot \left\| (\gamma_i')'' \right\|_2 \\
\leq 2 \sqrt{\frac{t(t-1)}{dd'}} \left\| (\delta_k')''' \right\|_2 \cdot \left\| (\gamma_i')'' \right\|_2 \\
\]

\[ |m_{ji}| \leq \left\| (\delta_k'_{j-1})'''' - (\delta_k'_{j-1})'''' \right\|_2 \cdot \left\| (\gamma_i')'' \right\|_2 \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \left\| (\delta_k'_{j-1})'''' \right\|_2 \cdot \left\| (\gamma_i')'' \right\|_2 \\
\leq 2 \sqrt{\frac{t(t-1)}{dd'}} \left\| (\delta_k')'''' \right\|_2 \cdot \left\| (\gamma_i')'' \right\|_2 \\
\]

We can now write

\[ \langle \delta_0, (\mathcal{G} \otimes \mathcal{H})(\gamma_0) \rangle = \langle \delta_0, ((\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_k) \circ \mathcal{G}^{\otimes t} \circ \cdots \circ \mathcal{G}^{\otimes t}(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_1))(\gamma_0) \rangle = \langle \delta_0, ((\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_k) \circ \mathcal{G}^{\otimes t} \circ \cdots \circ \mathcal{G}^{\otimes t}(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_1))(\gamma_0) \rangle + e_0 \\
\]

\[ = \langle \delta_0, ((\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_k) \circ \mathcal{G}^{\otimes t} \circ \cdots \circ \mathcal{G}^{\otimes t}(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_1))(\gamma_0) \rangle + f_0 \\
\]

\[ = \langle \delta_0, ((\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_k) \circ \mathcal{G}^{\otimes t} \circ \cdots \circ \mathcal{G}^{\otimes t}(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_2))(\gamma_1) \rangle + \langle \delta_0, ((\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_k) \circ \mathcal{G}^{\otimes t} \circ \cdots \circ \mathcal{G}^{\otimes t}(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_1))(\gamma_0) \rangle + e_0 + f_0 \\
\]

\[ = \langle \delta_0, ((\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_k))(\gamma_0) \rangle + \sum_{0 \leq i \leq k-1} \langle \delta_0, ((\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_k) \circ \mathcal{G}^{\otimes t} \circ \cdots \circ \mathcal{G}^{\otimes t}(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_{i+1}))(\gamma_0) \rangle + e_i + f_i \\
\]

\[ = \langle \delta_0, ((\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_k))(\gamma_0) \rangle + \sum_{0 \leq i < j \leq k} \langle (\delta_k'_{j-1})''''(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_j) \circ \mathcal{G}^{\otimes t} \circ \cdots \circ \mathcal{G}^{\otimes t}(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_{i+1}))(\gamma_0) \rangle + \sum_{0 \leq j < k} \langle (\delta_k''_{j-1})''''(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_j) \circ \mathcal{G}^{\otimes t} \circ \cdots \circ \mathcal{G}^{\otimes t}(\mathcal{L}^{D_1 \times D_2} \otimes \mathcal{H}_{i+1}))(\gamma_0) \rangle \\
\]
We will now bound each of the six terms in the last equality.

Hence,

Similarly,

Next,

\[ 0 + \sum_{0 \leq i < k} (e_i + f_i + d_i + l_i) + \sum_{0 \leq i < j < k} (g_{ji} + h_{ji} + m_{ji}) \]

We start by bounding the fifth term as follows.

\[
\sum_{0 \leq i < k} |e_i| \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \sum_{0 \leq i < k} \| \gamma_i \|_2 \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \sum_{0 \leq i < k} \lambda_i \leq 4 \sqrt{\frac{t(t-1)}{dd'}}.
\]

Similarly,

\[
\sum_{0 \leq i < k} |f_i| \leq 4 \sqrt{\frac{t(t-1)}{dd'}}.
\]

Next,

\[
\sum_{0 \leq i < k} |d_i| \leq \sqrt{\frac{t(t-1)}{dd'}} \sum_{0 \leq i < k} \left( \| (\gamma_i)\|_2^2 + \| (\delta_{k-1})_{i-1}^{\perp}\|_2^2 \right) \leq 3 \sqrt{\frac{t(t-1)}{dd'}}.
\]

Similarly,

\[
\sum_{0 \leq i < k} |l_i| \leq 4 \sqrt{\frac{t(t-1)}{dd'}}.
\]

Hence,

\[
\sum_{0 \leq i < k} (|e_i| + |f_i| + |d_i| + |l_i|) \leq 15 \sqrt{\frac{t(t-1)}{dd'}}.
\]

We now bound the sixth term as follows.

\[
\sum_{0 \leq i < j < k} |g_{ji}| \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \sum_{0 \leq i < j < k} \| \delta_{k-j} \|_2 \cdot \| \gamma_i \|_2 \leq 4 \sqrt{\frac{t(t-1)}{dd'}} \sum_{0 \leq i < k} \| \gamma_i \|_2 \leq 8 \sqrt{\frac{t(t-1)}{dd'}}.
\]

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Similarly,
\[ \sum_{0 \leq i < j \leq k} |h_{ji}| \leq 8 \sqrt{\frac{t(t-1)}{dd'}} \]
\[ \sum_{0 \leq i < j \leq k} |m_{ji}| \leq 8 \sqrt{\frac{t(t-1)}{dd'}}. \]

Hence,
\[ \sum_{0 \leq i < j \leq k} (|g_{ji}| + |h_{ji}| + |m_{ji}|) \leq 24 \sqrt{\frac{t(t-1)}{dd'}}. \]

Next we bound the first term as follows. Observe that
\[ |\langle \delta_0, (E_{D'}D') \otimes H_k)((\gamma_{k-1})^\perp)\rangle| \]
\[ \leq \| (E_{D'}D') \otimes H_k)((\gamma_{k-1})^\perp) \|_2 \leq \lambda_2 \| (\gamma_{k-1})^\perp \|_2 \leq \lambda_2 \| (\gamma_{k-1})^\perp \|_2 \leq \lambda_2 \| \gamma_{k-1} \|_2 \leq \lambda_2^\perp. \]

We now bound the third term as follows. The proof is very similar to that of [BT11, Lemma 16]. We give it below for completeness.
\[ \sum_{0 \leq i \leq k-1} \| (\delta_{k-i}^\perp), (\gamma_i^\perp) \| \]
\[ \leq \sum_{0 \leq i \leq k-1} \| (\delta_{k-i}^\perp) \|_2 \cdot \| (\gamma_i^\perp) \|_2 \]
\[ = \lambda_2^{k-1} \sum_{0 \leq i \leq k-1} \lambda_2^{-(k-i-1)} \| (\delta_{k-i-1}^\perp) \|_2 \cdot \lambda_2^{-i} \| (\gamma_i^\perp) \|_2 \]
\[ \leq \lambda_2^{k-1} \left( \sum_{0 \leq i \leq k-1} \lambda_2^{-2i} \| (\delta_i^\perp) \|_2^2 + \sum_{0 \leq i \leq k-1} \lambda_2^{-2i} \| (\gamma_i^\perp) \|_2^2 \right). \]

Now,
\[ \sum_{0 \leq i \leq k-1} \lambda_2^{-2i} \| (\delta_i^\perp) \|_2^2 \leq \sum_{0 \leq i \leq k-1} \lambda_2^{-2i} \| (\delta_i^\perp) \|_2^2 \left( 1 + 2 \sqrt{\frac{t(t-1)}{dd'}} \right)^2. \]

By induction, we observe that
\[ \sum_{0 \leq i \leq k-1} \lambda_2^{-2i} \| (\delta_i^\perp) \|_2^2 \]
\[ \leq \sum_{0 \leq i \leq k-1} \lambda_2^{-2i} \| (\delta_i^\perp) \|_2^2 + \lambda_2^{-2(k-1)} \| (\delta_{k-1}^\perp) \|_2^2 = \sum_{0 \leq i \leq k-2} \lambda_2^{-2i} \| (\delta_i^\perp) \|_2^2 + \lambda_2^{-2(k-1)} \| \delta_{k-1}^\perp \|_2^2 \]
\[ \leq \sum_{0 \leq i \leq k-2} \lambda_2^{-2i} \| (\delta_i^\perp) \|_2^2 + \lambda_2^{-2(k-2)} \| (\delta_{k-2}^\perp) \|_2^2 \leq \| \delta_0 \|_2^2 \leq \| \delta_0 \|_2^2 = 1. \]

Thus,
\[ \sum_{0 \leq i \leq k-1} \lambda_2^{-2i} \| (\delta_i^\perp) \|_2^2 \leq \left( 1 + 2 \sqrt{\frac{t(t-1)}{dd'}} \right)^2. \]
Similarly,
\[
\sum_{0 \leq i \leq k-1} \lambda_i^{-2i} \left\| (\gamma_i') \right\|_2^2 \leq \left( 1 + 2 \sqrt{\frac{t(t-1)}{dd'}} \right)^2.
\]
Thus,
\[
\sum_{0 \leq i \leq k-1} \langle (\delta_{k-i-1}'), (\gamma_i') \rangle \leq \lambda_i^{k-1} \left( 1 + 2 \sqrt{\frac{t(t-1)}{dd'}} \right)^2.
\]

We now bound the second term as follows: Fix
\[
(C^{D' \times D'} \otimes W'_2) \cap (W'^\perp) \ni (\gamma_i')' = \sum_{\sigma} b_{\sigma}(\Sigma^{(C^D)\otimes t} \otimes \Sigma^{(C^{dd'})\otimes t})(\beta_1)_{\sigma} \otimes (\alpha_2'),
\]
\[
(C^{D' \times D'} \otimes W'_2) \cap (W'^\perp) \ni (\delta_{k-j})' = \sum_{\sigma} c_{\sigma}(\Sigma^{(C^D)\otimes t} \otimes \Sigma^{(C^{dd'})\otimes t})(\theta_1)_{\sigma} \otimes (\alpha_2'),
\]
where \( \left\| (\beta_1)_{\sigma} \right\|_2 = 1, \left\| (\theta_1)_{\sigma} \right\|_2 = 1, b_{\sigma}, c_{\sigma} \in \mathbb{C} \). Let \( b \) be the \( t! \)-tuple whose \( \sigma \)th entry is \( |b_{\sigma}|. \) Let \( c \) be the \( t! \)-tuple whose \( \sigma \)th entry is \( |c_{\sigma}|. \) By Lemma 1
\[
\left\| (\gamma_i')' \right\|_2^2 = \sum_{\sigma} |b_{\sigma}|^2 \left\| (\alpha_2')_\sigma \right\|_2^2 \geq \left\| b \right\|_2^2 \left( 1 - \frac{t(t-1)}{2dd'} \right),
\]
which gives \( \left\| b \right\|_2^2 \leq \frac{\left\| (\gamma_i')' \right\|_2^2}{1 - \frac{t(t-1)}{2dd'}}. \) Similarly, \( \left\| c \right\|_2^2 \leq \frac{\left\| (\delta_{k-j})' \right\|_2^2}{1 - \frac{t(t-1)}{2dd'}}. \) Define the matrix
\[
C^{D' \times D'} \otimes W_2 \ni (\delta_{k-j})' := \sum_{\sigma} c_{\sigma}(\Sigma^{(C^D)\otimes t} \otimes \Sigma^{(C^{dd'})\otimes t})(\theta_1)_{\sigma} \otimes (\alpha_2').
\]
By Lemma 5
\[
\left\| (\delta_{k-j})' - (\delta_{k-j})' \right\|_2 \leq 2 \sqrt{\frac{t(t-1)}{dd'}} \left\| (\delta_{k-j})' \right\|_2.
\]
Moreover, as \( (\gamma_i')' \in (W'^\perp), \)
\[
0 = \langle (\gamma_i')', (\Sigma^{(C^D)\otimes t} \otimes \Sigma^{(C^{dd'})\otimes t})(\alpha_1 \otimes \alpha_2') \rangle = b_{\sigma}' \langle (\beta_1)_\sigma, \alpha_1 \rangle \cdot \langle (\alpha_2'), (\alpha_2') \rangle,
\]
implying that \( \langle (\beta_1)_\sigma, \alpha_1 \rangle = 0 \) for all \( \sigma \in S_t. \)

Observe that
\[
\langle (\delta_{k-j})', (\hat{G}^{\otimes t} \circ (\Pi^{C^D \times D'} \otimes \mathcal{H}_{j-1}) \circ \ldots \circ \hat{G}^{\otimes t} \circ (\Pi^{C^D \times D'} \otimes \mathcal{H}_{i+1}))(\gamma_i')' \rangle
\]
\[
= \sum_{\sigma \neq \sigma'} c_{\sigma} b_{\sigma}' \langle (\Sigma^{(C^D)\otimes t} \otimes (\Sigma')^{(C^{dd'})\otimes t})(\theta_1)_{\sigma'} \otimes (\alpha_2'), \rangle
\]
\[
+ \sum_{\sigma} c_{\sigma} b_{\sigma}' \langle (\Sigma^{(C^D)\otimes t} \otimes \Sigma^{(C^{dd'})\otimes t})(\theta_1)_{\sigma} \otimes (\alpha_2'), \rangle
\]
\[
+ \sum_{\sigma} c_{\sigma} b_{\sigma}' \langle (\Sigma^{(C^D)\otimes t} \otimes \Sigma^{(C^{dd'})\otimes t})(\theta_1)_{\sigma} \otimes (\alpha_2'), \rangle
\]
\[
+ \sum_{\sigma} c_{\sigma} b_{\sigma}' \langle (\Sigma^{(C^D)\otimes t} \otimes \Sigma^{(C^{dd'})\otimes t})(\theta_1)_{\sigma} \otimes (\alpha_2'), \rangle
\]
\[
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\]
\[
\sum_{\sigma \neq \sigma'} |c_{\sigma'}||b_{\sigma}| \left| \langle ((\Sigma')^{(C^d)^\otimes t} \otimes (\Sigma')^{(C^{dd'})^\otimes t})((\theta_1)_{\sigma'} \otimes \alpha_2), \\
(\Sigma^{(C^d)^\otimes t} \otimes \Sigma^{(C^{dd'})^\otimes t})((\tilde{\varphi}^{\otimes t} \circ (T^{C^d \times D^t} \otimes H_{j-1}) \circ \cdots \circ \tilde{\varphi}^{\otimes t} \circ (T^{C^d \times D^t} \otimes H_{j+1}))((\beta_1)_{\sigma} \otimes \alpha_2') \rangle \\
+ \sum_{\sigma} |c_{\sigma}||b_{\sigma}| \left| \langle ((\Sigma^{(C^d)^\otimes t}) \otimes \Sigma^{(C^{dd'})^\otimes t})((\theta_1)_{\sigma} \otimes \alpha_2), \\
(\Sigma^{(C^d)^\otimes t} \otimes \Sigma^{(C^{dd'})^\otimes t})((\tilde{\varphi}^{\otimes t} \circ (T^{C^d \times D^t} \otimes H_{j-1}) \circ \cdots \circ \tilde{\varphi}^{\otimes t} \circ (T^{C^d \times D^t} \otimes H_{j+1}))((\beta_1)_{\sigma} \otimes \alpha_2') \rangle \right| .
\]

Fix a \( \sigma \in S_t \). Let \( 1 \leq l \leq k - 1 \) be a positive integer. Fix a \( \frac{1}{2\delta}\geq \)-good sequence of unitary maps \( (U_1, \ldots, U_l) \) on \( C^d \otimes C^{dd'} \cong C^{dd'} \). Let the corresponding unitary superoperators tensored with the identity superoperator on \( C^{D \times D} \) be denoted by \( \tilde{U}_1, \ldots, \tilde{U}_l \). Let \( x_0 := e_0^\perp \otimes e_{j_0}^\perp \) denote a computational basis vector of \( (C^d)^\otimes t \otimes (C^{dd'})^\otimes t \cong C^{(dd')^t} \). We now calculate the matrix
\[
(\tilde{\varphi}^{\otimes t} \circ \tilde{U}_l^{\otimes t} \circ \cdots \circ \tilde{\varphi}^{\otimes t} \circ \tilde{U}_1^{\otimes t})((\Sigma^{(C^d)^\otimes t} \otimes \Sigma^{(C^{dd'})^\otimes t})((\beta_1)_{\sigma} \otimes (x_0x_0^\dag))) = (\Sigma^{(C^d)^\otimes t} \otimes \Sigma^{(C^{dd'})^\otimes t})((\tilde{\varphi}^{\otimes t} \circ \tilde{U}_l^{\otimes t} \circ \cdots \circ \tilde{\varphi}^{\otimes t} \circ \tilde{U}_1^{\otimes t})((\beta_1)_{\sigma} \otimes (x_0x_0^\dag)))
\]
Fix two sequences of computational basis vectors \((e^{\dag}_{i_1}, \ldots, e^{\dag}_{i_k}), \ (e^{\dag}_{i'_1}, \ldots, e^{\dag}_{i'_l})\) of \( C^d \). Starting from computational basis vector \( x_0 \) of \( C^{(dd')^t} \), define the two sequences of vectors \((x_1, \ldots, x_{l_1}), (x'_1, \ldots, x'_l)\) of \( C^{(dd')^t} \) accordingly, as in Section 2.2. Let \( p(i_1, \ldots, i_1) \), \( p(i'_1, \ldots, i'_l) \) denote the probabilities of obtaining the corresponding sequences of outcomes on measuring \( C^d \) in its computational basis. For a computational basis vector \( \tilde{t} \) of \( C^{dd'} \), let \( V^{\otimes \tilde{t}} \) be the corresponding tensor product of unitary operators on \( C^D \) arising from the 1-qTPE \( \mathcal{G} \) (note that the unitary operators of \( \mathcal{G} \) are indexed by the computational basis vectors of \( C^d \)). Let \( (V^{\otimes \tilde{i}_1}, \ldots, V^{\otimes \tilde{i}_1}, V^{\otimes \tilde{i}_1}), (V^{\dag}\otimes \tilde{i}_1, \ldots, (V^{\dag}\otimes \tilde{i}_1), (V^{\dag}\otimes \tilde{i}_1), \) be corresponding sequences of unitary maps on \( (C^d)^\otimes t \). Then
\[
(\tilde{\varphi}^{\otimes t} \circ \tilde{U}_l^{\otimes t} \circ \cdots \circ \tilde{\varphi}^{\otimes t} \circ \tilde{U}_1^{\otimes t})((\beta_1)_{\sigma} \otimes (x_0x_0^\dag)) = \sum_{(i_1, \ldots, i_1)} \sqrt{p(i_1, \ldots, i_1)p(i'_1, \ldots, i'_l)}
\]
\[
((V^{\otimes \tilde{i}_1} \cdots V^{\otimes \tilde{i}_1})(V^{\dag}\otimes \tilde{i}_1 \cdots (V^{\dag}\otimes \tilde{i}_1))) \otimes (x_1(x_1)^\dag).
\]
Let \( \sigma' \in S_t \), \( \sigma' \neq \sigma \). Suppose all the \( t \) entries of \((i_1, \ldots, i_1)\) are distinct. Then
\[
|\langle ((\Sigma')^{(C^d)^\otimes t} \otimes (\Sigma')^{(C^{dd'})^\otimes t})((\theta_1)_{\sigma'} \otimes \alpha_2), \\
(\Sigma^{(C^d)^\otimes t} \otimes \Sigma^{(C^{dd'})^\otimes t})((\tilde{\varphi}^{\otimes t} \circ \tilde{U}_l^{\otimes t} \circ \cdots \circ \tilde{\varphi}^{\otimes t} \circ \tilde{U}_1^{\otimes t})((\beta_1)_{\sigma} \otimes (x_0x_0^\dag)) \rangle | \leq \sum_{(i_1, \ldots, i_1)} \sqrt{p(i_1, \ldots, i_1)p(i'_1, \ldots, i'_l)}
\]
\[
|\langle (\theta_1)_{\sigma'}, ((\Sigma')^{-1} \Sigma)^{(C^d)^\otimes t}((V^{\otimes \tilde{i}_1} \cdots V^{\otimes \tilde{i}_1})(\beta_1)_{\sigma}((V^{\dag}\otimes \tilde{i}_1 \cdots (V^{\dag}\otimes \tilde{i}_1))) | \langle \alpha_2, ((\Sigma')^{-1} \Sigma)^{(C^{dd'})^\otimes t}(x_1(x_1)^\dag) \rangle |
\]
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$$\leq \sum_{\langle \vec{i}, \ldots, \vec{i} \rangle} \sqrt{p(\vec{r}, \ldots, \vec{r} \rangle p(\vec{s}, \ldots, \vec{s} \rangle \langle \alpha_2, ((\Sigma')^{-1})^{(C_{\text{dd}'}^{\text{dd}})} \langle x_1(x_1') \rangle \rangle |}$$

$$= (dd')^{-t/2} \sum_{\langle \vec{i}, \ldots, \vec{i} \rangle} \sqrt{p(\vec{r}, \ldots, \vec{r} \rangle p(\vec{s}, \ldots, \vec{s} \rangle \langle x_1', ((\Sigma')^{-1})^{(C_{\text{dd}'}^{\text{dd}})} \langle x_1 \rangle \rangle |}.$$
\[
\begin{align*}
&= \sum_{(i_1, \ldots, i_1)} \sum_{(i_1', \ldots, i_1')} \sqrt{p(i_1, \ldots, i_1)p(i_1', \ldots, i_1')} \\
&\quad \langle (\theta_1)_{\sigma}, (V^{\otimes i_1} \cdots V^{\otimes i_1}) (\beta_1)_{\sigma} ((V^+)^{\otimes i_1'} \cdots (V^+)^{\otimes i_1'}) \rangle \langle \alpha_2, x_l(x_l')^t \rangle \\
&= (dd')^{-t/2} \sum_{(i_1, \ldots, i_1)} \sum_{(i_1', \ldots, i_1')} \sqrt{p(i_1, \ldots, i_1)p(i_1', \ldots, i_1')} \\
&\quad \langle (\theta_1)_{\sigma}, (V^{\otimes i_1} \cdots V^{\otimes i_1}) (\beta_1)_{\sigma} ((V^+)^{\otimes i_1'} \cdots (V^+)^{\otimes i_1'}) \rangle \langle x_l', x_l \rangle \\
&\leq (dd')^{-t/2} \sum_{(i_1, \ldots, i_1)} p(i_1, \ldots, i_1) \\
&\quad \langle (\theta_1)_{\sigma}, (V^{\otimes i_1} \cdots V^{\otimes i_1}) (\beta_1)_{\sigma} ((V^+)^{\otimes i_1'} \cdots (V^+)^{\otimes i_1'}) \rangle \langle x_l, x_l \rangle \\
&\quad + (dd')^{-t/2} \sum_{T': T' \neq [t]} \sum_{(i_1, \ldots, i_1)} \sum_{(i_1', \ldots, i_1')} \sum_{(\tilde{i}_1, \ldots, \tilde{i}_1') \in (\tilde{i}_1', \ldots, \tilde{i}_1')} p((\tilde{i}_1, \ldots, \tilde{i}_1)_T') \sqrt{p((\tilde{i}_1, \ldots, \tilde{i}_1)_{[t] \setminus T'}) p((\tilde{i}_1', \ldots, \tilde{i}_1')_{[t] \setminus T'})} \\
&\quad \langle x_l', x_l \rangle |\langle x_l', x_l \rangle| \\
&\leq (dd')^{-t/2} \sum_{(i_1, \ldots, i_1)} p(i_1, \ldots, i_1) \\
&\quad \langle (\theta_1)_{\sigma}, (V^{\otimes i_1} \cdots V^{\otimes i_1}) (\beta_1)_{\sigma} ((V^+)^{\otimes i_1'} \cdots (V^+)^{\otimes i_1'}) \rangle \\
&\quad + (dd')^{-t/2} \sum_{T': T' \neq [t]} \sum_{(i_1, \ldots, i_1)} \sum_{(i_1', \ldots, i_1')} p((\tilde{i}_1, \ldots, \tilde{i}_1)_{T'}) d^{|\theta| \setminus T'} \left( \frac{c}{tdk} \right)^{|\theta| \setminus T'} \\
&\leq (dd')^{-t/2} \sum_{(i_1, \ldots, i_1)} d^{-lt} \langle (\theta_1)_{\sigma}, (V^{\otimes i_1} \cdots V^{\otimes i_1}) (\beta_1)_{\sigma} ((V^+)^{\otimes i_1'} \cdots (V^+)^{\otimes i_1'}) \rangle \\
&\quad + (dd')^{-t/2} \sum_{(i_1', \ldots, i_1')} |p(i_1, \ldots, i_1') - d^{-lt}| \\
&\quad + (dd')^{-t/2} \sum_{T': T' \neq [t]} (t^{-1} |\epsilon| |[t] \setminus T'|) \\
&\leq (dd')^{-t/2} \langle (\theta_1)_{\sigma}, (G^{\otimes t})^l ((\beta_1)_{\sigma}) \rangle + (dd')^{-t/2} \frac{3c}{8tdk} lt + (dd')^{-t/2} ((1 + t^{-1} |\epsilon|)^t - 1) \end{align*}
\]
\[
\leq (dd')^{-t/2} \left\| (\mathcal{G}^{\otimes t})^1( (\beta_1)_2) \right\|_2 + (dd')^{-t/2} \frac{3k\epsilon}{8d^k} + (dd')^{-t/2} (e^\epsilon - 1)
\]
\[
\leq (dd')^{-t/2} \lambda_1^t + 3(dd')^{-t/2} \epsilon.
\]

Above, we used the fact that the sequence \((U_1, \ldots, U_1)\) is \(\frac{\epsilon}{8d^k}\)-good in the second and fourth inequalities which implies that orthogonal states get mapped to almost orthogonal states and the measurement are almost uniform. We also used the triangle inequality for the \(\ell_1\)-distance in the fourth inequality. For the sixth inequality, we used the observation that \(\mathcal{G}^{\otimes t}\) is a 1-qTPE by Fact \(\text{[F]}\) Fact \(\text{[G]}\) and the fact that \(\langle (\beta_1)_2, \alpha_1 \rangle = 0\) which was proved earlier.

We now evaluate
\[
\sum_{\sigma \neq \sigma'} |c_{\sigma'}| |b_\sigma| \left\| \langle (\Sigma')^{(C^D)\otimes (\Sigma')^{(C^{dd'})\otimes t}} \rangle( (\theta_1)_{\sigma'} \otimes \alpha_2) \right\|
\]
\[
\leq \sum_{\sigma \neq \sigma'} |c_{\sigma'}| |b_\sigma| 2(t^{-1}e)^{t-f(\sigma')^{-1}}
\]
\[
= 2e^t N b \leq 2 \|N\|_\infty \|c\|_2 \|b\|_2
\]
\[
\leq 8e^2 \left\| \langle (\gamma'_1)'' \rangle \right\|_2 \left\| \langle (\delta'_{k-j})'' \rangle \right\|_2
\]
where we used Lemma \(\text{[F]}\) in the second inequality. Similarly,
\[
\sum_{\sigma} |c_{\sigma}| |b_\sigma| \left\| \langle (\Sigma')^{(C^D)\otimes (\Sigma')^{(C^{dd'})\otimes t}} \rangle( (\theta_1)_{\sigma} \otimes \alpha_2) \right\|
\]
\[
\leq \sum_{\sigma} |c_{\sigma}| |b_\sigma| (\lambda_1^t + 3\epsilon)
\]
\[
\leq (\lambda_1^t + 3\epsilon) \|c\|_2 \|b\|_2
\]
\[
\leq 2(\lambda_1^t + 3\epsilon) \left\| \langle (\gamma'_1)'' \rangle \right\|_2 \left\| \langle (\delta'_{k-j})'' \rangle \right\|_2
\]
We now let \((U_1, \ldots, U_1)\) range over unitaries from the sequence of qTPEs \((\mathcal{H}_{i+1}, \ldots, \mathcal{H}_{i+1})\) which is assumed to be \(\frac{\epsilon}{8d^k}\)-good by Lemma \(\text{[F]}\). This gives us
\[
\left\| \langle (\delta'_{k-j})'' \rangle, (\mathcal{G}^{\otimes t} \circ (\mathbb{I}^{C^{D^t} \times D_t} \otimes \mathcal{H}_{j-1}) \circ \cdots \circ \mathcal{G}^{\otimes t} \circ (\mathbb{I}^{C^{D^t} \times D_t} \otimes \mathcal{H}_{i+1})) \rangle( (\gamma'_i)'' \rangle \right\|
\]
\[
\leq 8e^2 \left\| \langle (\gamma'_i)'' \rangle \right\|_2 \left\| \langle (\delta'_{k-j})'' \rangle \right\|_2 + 2(\lambda_1^t)^{i-1} + 3\epsilon) \left\| \langle (\gamma'_i)'' \rangle \right\|_2 \left\| \langle (\delta'_{k-j})'' \rangle \right\|_2.
\]
We can now finally bound the second term by
\[
\sum_{0 \leq i < i+1 < j \leq k} \left\| \langle (\delta'_{k-j})'' \rangle, (\mathcal{G}^{\otimes t} \circ (\mathbb{I}^{C^{D^t} \times D_t} \otimes \mathcal{H}_{j-1}) \circ \cdots \circ \mathcal{G}^{\otimes t} \circ (\mathbb{I}^{C^{D^t} \times D_t} \otimes \mathcal{H}_{i+1})) \rangle( (\gamma'_i)'' \rangle \right\|
\]
\[
\leq 8e^2 \sum_{0 \leq i < i+1 < j \leq k} \left\| \langle (\gamma'_i)'' \rangle \right\|_2 \left\| \langle (\delta'_{k-j})'' \rangle \right\|_2
\]
\[
+ \sum_{0 \leq i < i+1 < j \leq k} 2(\lambda_1^t)^{i-1} + 3\epsilon) \left\| \langle (\gamma'_i)'' \rangle \right\|_2 \left\| \langle (\delta'_{k-j})'' \rangle \right\|_2
\]
\[
\leq (8\varepsilon^2 + 2(\lambda_1 + 3\varepsilon)) \sum_{0 \leq i < j \leq k} \left\| (\gamma_i')^T \right\|_2 \left\| (\delta_{k-j}')^T \right\|_2
\]
\[
\leq (2\lambda_1 + 14\varepsilon) \sum_{0 \leq i < j \leq k} \left\| \gamma_i' \right\|_2 \left\| \delta_{k-j}' \right\|_2
\]
\[
\leq 2(\lambda_1 + 7\varepsilon) \sum_{0 \leq i \leq k-2} \left\| \gamma_i' \right\|_2 \left( \sum_{j=i+2}^k \lambda_2^{k-j} \right)
\]
\[
\leq 4(\lambda_1 + 7\varepsilon) \sum_{0 \leq i \leq k-2} \left\| \gamma_i' \right\|_2 \leq 4(\lambda_1 + 7\varepsilon) \sum_{0 \leq i \leq k-2} \lambda_2^i \leq 8(\lambda_1 + 7\varepsilon).
\]

Putting everything together, we have finally shown that
\[
|\langle \delta_0, (\mathcal{G} \boxtimes \tilde{\mathcal{H}})(\gamma_0) \rangle| \leq \lambda_2^k + 8(\lambda_1 + 7\varepsilon) + \lambda_2^{k-1} + 8 \sqrt{\frac{t(t-1)}{dd'}} + 15 \sqrt{\frac{t(t-1)}{dd'}} + 24 \sqrt{\frac{t(t-1)}{dd'}}
\]
\[
\leq 8(\lambda_1 + 7\varepsilon) + \lambda_2^{k-1} + \lambda_2^k + 47 \sqrt{\frac{t(t-1)}{dd'}}.
\]

We have thus shown the following theorem.

**Theorem 2.** Let \( s \geq 4, d, d' \geq 100 \) be integers. Let \( k \leq \log s \) be an integer. Let \( t \) be an integer such that \( D \geq dd' \geq 10t^2 \). Let \( 0 < \varepsilon < 10^{-2} \). Let \( \mathcal{G} = \{ V_i \}_{i=1}^d \) be a \((D, d, \lambda_1, 1)\)-qTPE. Let \( \mathcal{H}_j = \{ U_i(j) \}_{i=1}^s, 1 \leq j \leq k \) be \((dd', s, \lambda_2, t)\)-qTPEs such that the sequence \( \mathcal{H} := (\mathcal{H}_k, \ldots, \mathcal{H}_1) \) is \( \frac{s}{8dd'} \)-good. Then \( \mathcal{G} \boxtimes \mathcal{H} \) is a \((dd'', s^k, \lambda, t)\)-qTPE where
\[
\lambda := 8(\lambda_1 + 7\varepsilon) + \lambda_2^{k-1} + \lambda_2^k + 47 \sqrt{\frac{t(t-1)}{dd'}}.
\]

Moreover for \( d' \geq 30 \log s(\log s + \log d)d^{2k+1} \epsilon^{-2} \), such a sequence \( \mathcal{H} \) exists with \( \lambda_2 < 8s^{-1/2} \).

**Remarks:**

1. For \( t = 1 \), we get the bound \( \lambda = 8(\lambda_1 + 7\varepsilon) + \lambda_2^{k-1} + \lambda_2^k \) which is the same as the bound in [BT11] except for the constants involving the \( \lambda_1 \) term. However, this does not affect the parameters of the iterative construction of almost Ramanujan expanders given in that paper. We thus get an infinite family of almost Ramanujan quantum expanders i.e. \((D^n, d, \lambda, 1)\)-qTPEs, \( n \geq 1 \) where \( d = 2^{s \log s} \), \( s \geq 4 \) is an even integer, \( D = (D_0)^{3 \log s} \), \( D_0 \) sufficiently large integer, and \( \lambda = d^{-\frac{1}{2} + O\left(\frac{1}{s \log s}\right)} \).

2. For integer \( t \) satisfying \( 10t^2 \leq d \), we thus get an infinite family of almost Ramanujan qTPEs i.e. \((D^n, d, \lambda_1, t)\)-qTPEs where the constraints on the parameters are the same as in above remark.

5 Conclusion

In this paper, we have shown that the famous zigzag product first defined for classical expander graphs by Reingold, Vadhan and Wigderson [RVW02] is amazingly powerful: it generalises to quantum tensor product expanders, and furthermore, it can be refined via the ideas of Ben-Aroya and Ta-Shma [BT11] to give efficient constructions of almost Ramanujan \( t \)-qTPEs for \( t \) polynomial
in the number of qubits. This leads to efficient constructions for unitary \( t \)-designs for \( t \) polynomial in the number of qubits. The only efficient construction known earlier for such large \( t \) was the local random circuit construction of Brandão, Harrow and Horodecki \(^{[BHH16]} \). For both zigzag and generalised zigzag products, our construction has the advantage of much better tradeoff between the degree and the singular value gap than what was proved by Brandão, Harrow and Horodecki. For the generalised zigzag product, our tradeoff is almost optimal by virtue of being almost Ramanujan. Achieving efficient constructions of perfectly Ramanujan \( q \)TPEs remains an open problem, even for \( t = 1 \).

Strangely, the zigzag product construction does not seem to work for classical tensor product expanders. Finding an efficient combinatorial construction of \( t \)-cTPEs for \( t > 1 \) is an important open problem. The only efficient constructions known for \( t > 1 \) are algebraic, involving Cayley graphs on the symmetric group \(^{[Kas07]} \).

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