K-IN Variant Hilbert Modules and Singular Vector Bundles on Bounded Symmetric Domains

Harald Upmeier

Dedicated to the Memory of Ottmar Loos

Abstract. We show that the "eigenbundle" (localization bundle) of certain Hilbert modules over bounded symmetric domains of rank \( r \) is a "singular" vector bundle (linearly fibrered complex analytic space) which decomposes as a stratified sum of homogeneous vector bundles along a canonical stratification of length \( r + 1 \). The fibres are realized in terms of representation theory on the normal space of the strata.

0. Introduction

Let \( \mathcal{P}_E \approx \mathbb{C}[z_1, \ldots, z_d] \) be the algebra of all polynomials on a vector space \( E \approx \mathbb{C}^d \). Let \( \mathcal{M}_\zeta \) be the maximal ideal at \( \zeta \in E \). For any ideal \( I \subset \mathcal{P}_E \) the quotient module

\[ L_\zeta := I/\mathcal{M}_\zeta I \]

has finite dimension, and the disjoint union

\[ L := \bigcup_{\zeta \in E} L_\zeta \]

has the structure of a "linearly fibrered complex analytic space" [14], also called a "singular vector bundle." We call \( L \) the "localization bundle" of the ideal \( I \). For example, if \( I \) is a prime ideal whose vanishing locus \( X \) consists only of smooth points, then a result of Duan-Guo [10] states that the localization bundle has rank 1 on the regular set \( E \setminus X \) whereas on \( X \) the bundle is isomorphic to the (dual) normal bundle to the submanifold \( X \). Thus we have a "stratification" of length 2. In this paper we consider ideals over bounded symmetric domains of arbitrary rank \( r \) and obtain localization bundles which are stratified of length \( r + 1 \).

Besides their interest in complex and algebraic geometry, as the dual objects of coherent analytic module sheaves [5, 14], these bundles play a fundamental role in multi-variable operator theory for commuting tuples \((T_1, \ldots, T_d)\) of non-selfadjoint operators acting on a Hilbert space \( H \) of holomorphic functions on a bounded domain \( D \subset \mathbb{C}^d \). Here a central concept is the so-called "eigenbundle"

\[ H_\zeta := \{ \phi \in H : T_j^* \phi = \zeta_j \cdot \phi \quad \forall \ 1 \leq j \leq d \} \]

of \( H \) viewed as a Hilbert module [7]. The main idea is that the differential-geometric properties (Chern connection, curvature, etc.) of the eigenbundle \( H \), viewed as a (singular) hermitian holomorphic vector bundle over \( D \), should characterize the underlying operator tuple up to unitary equivalence. In this sense the complex-analytic properties of \( H \) are analogous to the spectral theorem in the self-adjoint case. If \( H = \mathcal{T} \) is the Hilbert closure of a polynomial ideal \( I \) then there is a natural isomorphism \( H \approx L|_D \).

2020 Mathematics Subject Classification. Primary 32M15, 46E22; Secondary 14M12, 17C36, 47B35.

Key words and phrases. bounded symmetric domain, Hilbert module, complex-analytic fibre space, Jordan triple.
In the original approach by Cowen-Douglas [7] this program was fully realized for a certain class of operator tuples where the eigenbundle is a genuine vector bundle without singularities. In general, for example in the situation of the Duan-Guo theorem, the eigenbundle is not a smooth vector bundle anymore and its fibre dimension varies over different strata within the domain $D$. If $D = G/K$ is a bounded symmetric domain of arbitrary rank $r$ there is a canonical stratification into "Kepler varieties" defined by a rank condition, and our main result characterizes the localization bundle $J^\Lambda$ for certain polynomial ideals $J^\Lambda$ determined by any partition $\lambda$ of length $\leq r$. Only the "fundamental" partitions $\lambda = (1, \ldots, 1, 0, \ldots, 0)$ give rise to prime ideals.

We show that the associated eigenbundle is stratified of length $r + 1$ (including the open stratum) and the fibre at a point $\zeta$ is described in terms of certain polynomials (instead of linear forms as for prime ideals and smooth points) on the normal space at $\zeta$. The realization is given by an explicit geometric construction, taking a projection of a polynomial to the normal space, and the main challenge is to identify the kernel of this projection map. The theory of Jordan algebras and Jordan triples [6, 13] is used to carry out the general discussion in a uniform way without using the classification of bounded symmetric domains.

The results of this paper have numerous consequences, further developed in [26]. For example, for any Hilbert closure $H = \mathring{J}$, with reproducing kernel $K(z, \zeta)$, the localization bundle is explicitly identified with the subbundle $H \subset D \times H$, by taking certain "normal derivatives" of kernel functions constructed from $K$. This is important for introducing the hermitian metric on the singular vector bundle. Moreover, in the spirit of the Borel-Weil-Bott theorem, the fibres $J^\Lambda_{\zeta}$ can be described by holomorphic sections of line bundles over flag manifolds. It is also shown how to extend the analysis to arbitrary $K$-invariant ideals, in particular the (Jordan) determinantal ideals which are defined by vanishing conditions on the underlying stratification.

## 1. Hilbert modules and their eigenbundles

Let $D$ be a bounded domain in a finite dimensional complex vector space $E \approx \mathbb{C}^d$. Denote by $\mathcal{P}_E \approx \mathbb{C}[z_1, \ldots, z_d]$ the algebra of all polynomials on $E$. A Hilbert space $H$ of holomorphic functions $f$ on $D$ (supposed to be scalar-valued) is called a Hilbert module if for any polynomial $p \in \mathcal{P}_E$ the multiplication operator $T_p := pf$ leaves $H$ invariant and is bounded. Using the adjoint operators $T_p^*$, the closed linear subspace

$$H^\Lambda_{\zeta} := \{ f \in H : T_p^* f = \overline{p(\zeta)} f \ \forall \ p \in \mathcal{P}_E \}$$

is called the joint eigenspace at $\zeta \in D$. Since $T_pT_q = T_{pq}$ for polynomials $p, q$ it suffices to consider linear functionals or just the coordinate functions. The disjoint union

$$H = \bigcup_{\zeta \in D} H^\Lambda_{\zeta}$$

becomes a subbundle of the trivial vector bundle $D \times H$, which is called the eigenbundle of $H$, although it is not locally trivial in general. One also requires that the fibres have finite (non-constant) dimension and their union is total in $H$. For more details, cf. [17, 5].

The eigenbundle is closely related to the concept of localization. For a given point $\zeta \in D$, the complex numbers become a $\mathcal{P}_E$-module, denoted by $\mathcal{C}_\zeta$, under the action $p \cdot \xi := p(\zeta) \xi$. Denote by

$$\mathcal{M}_\zeta = \{ f \in \mathcal{P}_E : f(\zeta) = 0 \} \subset \mathcal{P}_E$$

(1.1)
the maximal ideal at $\zeta \in E$. Define the module tensor product
\[ H \otimes_{\mathcal{P}_E} C_\zeta = H/\mathcal{M}_\zeta \mathcal{H} \quad (1.2) \]
where $\mathcal{M}_\zeta \mathcal{H}$ is the closed submodule generated by $T_p \psi - p(\zeta) \psi$, for all $p \in \mathcal{P}_E$ and $\psi \in H$.

**Lemma 1.1.** The map $\phi \mapsto [\phi]$ from $H_\zeta$ to $H/\mathcal{M}_\zeta \mathcal{H}$ is a Hilbert space isomorphism, with inverse given by
\[ H/\mathcal{M}_\zeta \mathcal{H} \rightarrow H_\zeta, \quad f + \mathcal{M}_\zeta \mathcal{H} \mapsto \pi_\zeta f, \]
where $\pi_\zeta : H \rightarrow H_\zeta$ is the orthogonal projection. Thus $H_\zeta$ is the "quotient module" for the submodule $\mathcal{M}_\zeta \mathcal{H}$.

**Proof.** Let $z_i, 1 \leq i \leq d$ denote the coordinate functions. By definition we have
\[ H_\zeta = \bigcap_{i=1}^d \ker T_{z_i-\zeta_i}^* = \bigcap_{i=1}^d (T_{z_i-\zeta_i} H)^\perp = \bigcap_{i=1}^d (T_{z_i-\zeta_i} H) = (\mathcal{M}_\zeta H)^\perp \approx H/\mathcal{M}_\zeta \mathcal{H}. \quad \square \]

Classical examples of Hilbert modules are the **Bergman space** $H^2(D)$ of square-integrable holomorphic functions, whose reproducing kernel is called the Bergman kernel, and the **Hardy space** $H^2(\partial D)$ if $D$ has a smooth boundary $\partial D$. For general Hilbert modules $H$, a **reproducing kernel function** is a sesqui-holomorphic function $K(z, \zeta)$ on $D \times D$ such that for each $\zeta \in D$ the holomorphic function
\[ K_\zeta(z) := K(z, \zeta) \]
belongs to $H$, and we have
\[ \psi(z) = (K_z|\psi)_H \]
for all $\psi \in H$ and $z \in D$. Here $(\phi|\psi)_H$ is the inner product on $H$ (anti-linear in the first variable). Thus $H$ is the closed linear span of the holomorphic functions $K_\zeta$, where $\zeta \in D$ is arbitrary. In terms of an orthonormal basis $\phi_\alpha$ of $H$, we have
\[ K(z, \zeta) = \sum_\alpha \phi_\alpha(z)\overline{\phi_\alpha(\zeta)}. \]

The identity
\[ (T_p^* K_\zeta|\psi)_H = (K_\zeta|p\psi)_H = p(\zeta)\psi(\zeta) = p(\zeta) (K_\zeta|\psi)_H = (p(\zeta) K_\zeta|\psi)_H \]
for $p \in \mathcal{P}_E$ and $\psi \in H$ shows that $T_p^* K_\zeta = p(\zeta) K_\zeta$, so that
\[ K_\zeta \in H_\zeta \]
for each $\zeta \in D$. If $K$ has no zeros (e.g., the Bergman kernel of a strongly pseudo-convex domain) then the eigenbundle $H$ is spanned by the reproducing kernel functions $K_\zeta$ and hence becomes a **hermitian holomorphic line bundle**.

In general, Hilbert modules have reproducing kernel function which vanish along certain analytic subvarieties of $D$. In this case the eigenbundle is not locally trivial, its fibre dimension can jump along the varieties and we obtain a **singular vector bundle** on $D$, also called a "linearly fibered complex analytic space". Such singular vector bundles are important in Several Complex Variables since, by [14], they are in duality with the category of **coherent analytic module sheaves**, whereas (regular) vector bundles correspond to locally free sheaves.
An important class of Hilbert modules is given by the Hilbert closure $H = \overline{I}$ of a polynomial ideal $I \subset \mathcal{P}_E$. In this case we define, analogous to (1.2), the localization

$I_\zeta := I/M_\zeta I$

at $\zeta \in E$, and call the disjoint union

$I := \bigcup_{\zeta \in E} I_\zeta$

the localization bundle over $E$. We first show that this has finite rank.

**Proposition 1.2.** Let $p_1, \ldots, p_t$ be generators of $I$. Then for any $\zeta \in E$ the linear map

$\mathbb{C}^t \to I_\zeta, \ (a_1, \ldots, a_t) \mapsto \mathcal{M}_\zeta I + \sum_{i=1}^{t} a_i \ p_i$

is surjective, and hence $\dim I/M_\zeta I \leq t$.

**Proof.** Write $f \in I$ as $f = \sum_{i=1}^{t} h_i \ p_i$ with $h_i \in \mathcal{P}_E$. Then

$f - \sum_{i} h_i(\zeta) p_i = \sum_{i}(h_i - h_i(\zeta)) p_i \in \mathcal{M}_\zeta I$.

Putting $a_i = h_i(\zeta)$, the assertion follows. □

For any ideal $I \subset \mathcal{P}_E$ consider the vanishing locus

$\mathcal{V}^I = \{ \zeta \in E : p(\zeta) = 0 \ \forall \ p \in I \}$

and the "regular set"

$\tilde{E} = E \setminus \mathcal{V}^I$.

**Proposition 1.3.** For $\zeta \in \tilde{E}$ there is an isomorphism

$I_\zeta \to \mathbb{C}, \ f + \mathcal{M}_\zeta I \mapsto f(\zeta)$.

Thus the localization bundle has rank 1 on the regular set $E \setminus \mathcal{V}^I$.

**Proof.** If $f = gh$, with $g \in \mathcal{M}_\zeta$ and $h \in I$, then $g(\zeta) = 0$ and hence $f(\zeta) = g(\zeta)h(\zeta) = 0$. Thus the map (1.3) is well-defined. Since $\zeta \in \tilde{E}$ there exists $p \in I$ such that $p(\zeta) \neq 0$. Thus the map (1.3) is non-zero and hence surjective. To show injectivity, we may assume $p(\zeta) = 1$. If $f \in I$ satisfies $f(\zeta) = 0$, then

$f = (1-p)f + fp \in \mathcal{M}_\zeta I$

since both $1-p$ and $f$ belong to $\mathcal{M}_\zeta$. □

Passing to a Hilbert module completion $H = \overline{I}$ it is shown in [10] that $\dim I/\mathcal{M}_\zeta I = \dim H/\mathcal{M}_\zeta H < +\infty$ and hence the map

$I_\zeta \to H_\zeta, \ f + \mathcal{M}_\zeta I \mapsto \pi_\zeta f$

is an isomorphism. The difference between $I$ and $H$ is that $H$ carries an additional hermitian fibre metric, being embedded in $D \times H$. Also, $I$ is defined on all of $E$, whereas $H$ is defined only on $D$. In this sense we have

$H = I|_D$, \hspace{1cm} (1.4)

but equipped with a holomorphic hermitian metric. As shown in [5] the analytic module sheaf associated with $H$ is coherent, so that these bundles become "complex-analytic linear fibre spaces" in the sense of [14]. We prefer the term singular vector bundle.
Let us recall the following \cite[Lemma 2.3]{5}:

**Lemma 1.4.** Let \( p_1, \ldots, p_t \) be a finite set of generators of \( I \). Then \( f \in H^\sim_\zeta \) satisfies

\[
p_i(\zeta)f = (p_i|f)_H K_\zeta
\]

for all \( i \).

**Proof.** For any \( 1 \leq i, j \leq t \) we have

\[
p_i(\zeta)(f|p_j)_H = (p_i(\zeta)f|p_j)_H = (T_{p_i}^*f|p_j)_H = (f|p_ip_j)_H = p_j(\zeta)(f|p_i)_H.
\]

Let \( q_j \in \mathcal{P}_E \) be arbitrary. Using the reproducing property it follows that

\[
p_i(\zeta)\left(f|\sum_j p_jq_j\right)_H = p_i(\zeta)\sum_j (T_{q_j}^*f|p_j)_H = p_i(\zeta)\sum_j q_j(\zeta)(f|p_j)_H = \left((p_i|f)_H K_\zeta|\sum_j p_jq_j\right)_H.
\]

Since the vector subspace \( \{\sum_j p_jq_j : q_j \in \mathcal{P}_E\} \) is dense in \( H = \hat{T} \), the assertion follows. \( \square \)

Define the "regular set"

\[
\hat{D} := D \setminus \mathcal{V}^I = \bigcup_{j=1}^t \{\zeta \in D : p_j(\zeta) \neq 0\}
\]

as a dense open subset of \( D \).

**Corollary 1.5.** For \( \zeta \in \hat{D} := D \cap \hat{E} = D \setminus \mathcal{V}^I \) we have

\[
H^\sim_\zeta = C K_\zeta.
\]

Thus the eigenbundle \( H \) restricted to the regular set is a holomorphic line bundle spanned by the reproducing kernel functions \( K_\zeta \), \( \zeta \in \hat{D} \).

**Proof.** Lemma \cite[14]{14} implies \cite[15]{15} since \( p_i(\zeta) \neq 0 \) for some \( i \). \( \square \)

The behavior of \( H \) on the singular set \( D \setminus \hat{D} \) is more complicated and has so far been studied mostly when the vanishing locus of the reproducing kernel is a smooth subvariety of \( D \), for example given as a complete intersection of a regular sequence of polynomials. The case where \( I \) is a prime ideal whose vanishing locus \( X := \mathcal{V}^I \) consists of smooth points has been studied by Duan-Guo \cite{10}. They showed that for \( \zeta \in D \setminus X \)

\[
H^\sim_\zeta = \langle K_\zeta \rangle
\]

is 1-dimensional, whereas \( H|_X \) is isomorphic to the (dual) normal bundle of the submanifold \( X \). Thus we have a stratification of length 2. We consider a more general situation for bounded symmetric domains \( D \) of arbitrary rank \( r \), where we have a stratification of length \( r + 1 \). Here the relevant algebraic varieties are not smooth and the ideal \( I \) is not prime in general.
2. Bounded symmetric domains and Jordan triples

We use the Jordan theoretic description of bounded symmetric domains \([1, 6, 13, 18, 24]\). Every (hermitian) bounded symmetric domain can be realized as the unit ball, with respect to the so-called spectral norm, in a complex vector space \(E\) endowed with a Jordan triple product. This is a ternary operation

\[ E \times E \times E \to E \quad (u, v, w) \mapsto \{uw^*v\} \]

which is symmetric bilinear in the outer variables \((u, w)\), conjugate-linear in the inner variable \(v\) and satisfies the ”Jordan triple identity”

\[ [u \Box v^*, z \Box w^*] = \{uv^*z\} \Box w^* - z \Box \{wu^*v\}^* \quad (2.1) \]

for all \(u, v, z, w \in E\). Here

\[ (u \Box v^*)z := \{uv^*z\} \quad (2.2) \]

denotes the ”triple multiplication” operator. The ”star” occurring here is a formal symbol.

A complex vector space \(E\) carrying such a structure will be called a hermitian Jordan triple (or \(J^*\)-triple) if the sesqui-linear product \((u | v) = \text{tr} \ u \Box v^*\) is hermitian and positive-definite. Geometrically, this inner product coincides with the Bergman metric at the origin 0 \(\in D\).

We consider only finite dimensional Jordan triples \(E\), although the Jordan theoretic description of symmetric manifolds carries over to the case of Banach manifolds \([23, 6]\). We also assume that \(E\) is irreducible of rank \(r\).

The primary example of a hermitian Jordan triple is the matrix space \(E = C^{r \times s}\), with \(r \leq s\), endowed with the anti-commutator triple product

\[ \{uv^*w\} = uv^*-wv^*u. \]

The associated domain is the matrix unit ball for the operator norm. We will often illustrate the general theory with this example. If \(r = s\) one can take \(v = e\) (unit matrix) and obtains the classical anti-commutator

\[ \{ue^*w\} = uw + wu \]

which is the prototype of the so-called Jordan algebras \([13]\). For rank \(r = 1\), we obtain the unit ball \(D \subset E = C^{1 \times d}\) with Jordan triple product \(\{xy^*z\} = (x|y)z + (z|y)x\). The unit disk \(D \subset E = C\) corresponds to the Jordan triple product \(\{xy^*z\} = 2x\overline{yz}\).

Irreducible hermitian Jordan triples of rank \(r\) are classified by two characteristic multiplicities \(a\) and \(b\) such that

\[ \frac{d}{r} = 1 + \frac{a}{2}(r-1) + b. \]

If \(b = 0\) the domain \(D\) is called of tube type. In this case \(E\) is a Jordan algebra. The full classification is

- **matrix triple** \(E = C^{r \times s}\), rank \(r \leq s\), \(a = 2\), \(b = s - r\) (complex case)
- **symmetric matrices** \(a = 1\), \(b = 0\) (real case)
- **anti-symmetric matrices** \(a = 4\) (quaternion case)
- **spin factor** \(E = C^d\), \(r = 2\), \(a = d - 2\), \(b = 0\)
- **exceptional Jordan triples** of dimension 27 \((r = 3, a = 8, b = 0)\) and 16 \((r = 2)\) (octonion case)
By the classification the only Jordan triple of rank 1 is the row-space \( E = C^* = C^{1 \times s} \). Its unit ball \( D \subset C^* = C^{1 \times s} \) is the only bounded symmetric domain which is strictly pseudo-convex or has a smooth boundary.

Let \( G \) be the identity component of the biholomorphic automorphism group of \( D \). Then \( D = G/K \), where the stabilizer subgroup \( K \subset G \) at the origin consists of linear Jordan triple automorphisms of \( E \). The “structure group” \( \hat{K} \), a complexification of \( K \), is a complex Lie subgroup of \( GL(E) \) endowed with an involution \( h \mapsto h^* \) such that
\[
h(u \square v^*)h^{-1} = (hu)\square(h^{-1}v)^*
\]
for all \( u, v \in E \). Let \( E \square E^* \) denote the vector space spanned by linear transformations \( u \square v^* \), defined in (2.2), for \( u, v \in E \). By the Jordan triple identity (2.1) this is a Lie algebra which coincides with the Lie algebra \( \hat{K} \) of \( \hat{K} \). We have
\[
\text{tr } u \square v^* = \text{const } (u|v)
\]
for the \( K \)-invariant inner product on \( E \) and put
\[
E_{\square}E^*: = \{ A \in E \square E^* : \text{tr } A = 0 \}.
\]
In the matrix case \( E = C^{r \times s} \) \( \hat{K} \) consists of all transformations of the form
\[
hz = azd
\]
where \( a \in GL_r(C) \), \( d \in GL_s(C) \). The adjoint is \( h^*z = a^*zd^* \). Also, \( \hat{f} = E \square E^* \) consists of all transformations
\[
z \mapsto az + zd
\]
where \( a \in C^{r \times r} \) and \( d \in C^{s \times s} \).

In a Jordan triple \( E \) the central notion of tripotent generalizes the idempotents in an algebra. An element \( c \in E \) is called a tripotent (triple idempotent) if the Jordan triple product satisfies
\[
\{cc^*c\} = 2c.
\]
For \( 0 \leq j \leq r \) the set \( S_j \) of all tripotents of rank \( j \) is a compact real-analytic manifold which is homogeneous under \( \hat{K} \). We have \( S_0 = \{0\} \), the elements of \( S_1 \) are called minimal tripotents, and the maximal tripotents \( S := S_r = S_1 \) form the Shilov boundary of \( D \) [18].

For the matrix triple \( C^{r \times s} \) the tripotents are the so-called partial isometries characterized by \( cc^*c = c \). The minimal tripotents are the rank 1 matrices \( \xi \eta^* \), where \( \xi \in C^r \) and \( \eta \in C^s \) are unit vectors. For \( r = s \), any self-adjoint projection \( (c = c^* = c^2) \) and any unitary matrix \( (cc^* = c^*c = 1) \) is a tripotent. In the rank 1 case \( E = C^d \) \( S = S_1 \) consists of all unit vectors.

A frame of \( E \) is a family \((e_1, \ldots, e_r)\) of minimal orthogonal tripotents. Every \( \zeta \in E \) has a spectral decomposition
\[
\zeta = \sum_j \zeta_j e_j
\]
(2.3)
where \( e_j \) is a frame and \( \zeta_1 \geq \zeta_2 \geq \ldots \geq \zeta_r \geq 0 \) are the singular values. For matrices this is the classical singular value decomposition under \( U(r) \times U(s) \).

Any tripotent \( c \) induces a Peirce decomposition
\[
E = E_c^2 \oplus E_c^1 \oplus E_c^0
\]
(2.4)
into eigenspaces \( E_c^\alpha = \{ z \in E : \{cc^*z\} = \alpha z \} \) for \( \alpha = 0, 1, 2 \). The Peirce spaces \( E_c^\alpha \) are Jordan subtriples of \( E \). In the matrix case the tripotent
\[
c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]
of rank $\ell$ induces the Peirce decomposition
\[
C^{r\times s} = \begin{pmatrix}
C^{\ell\times \ell} & C^{\ell\times (s-\ell)} \\
C^{(r-\ell)\times \ell} & C^{(r-\ell)\times (s-\ell)}
\end{pmatrix} = \begin{pmatrix}
E_2 & E_1 \\
E_1 & E_0
\end{pmatrix}.
\]
For the spin factor $E = \mathbb{C}^{2+\nu}$ of rank 2 consider the minimal tripotent $c = (1/2, i/2, 0, \ldots, 0)$ and put $\overline{c} = (1/2, -i/2, 0, \ldots, 0)$. Then $e = c + \overline{c} = (1, 0, \ldots, 0)$ is the unit element and we have
\[
E_c^2 = C \cdot c, \quad E_0^0 = C \cdot \overline{c} = E_1^2, \quad E_c^1 = E_0^1 = \{c, \overline{c}\}^\perp \approx \mathbb{C}^\nu.
\]
We often abbreviate $E_c := E_c^2$ and $E^c := E_c^1$.

The number of non-zero singular values $\zeta_j$ in (2.3) is called the rank of $\zeta$. For each $0 \leq \ell \leq r$ the Kepler manifold [11]
\[
\mathring{E}_\ell = \{\zeta \in E : \text{rank}(\zeta) = \ell\}
\]
is a complex-analytic manifold which is a $K$-orbit containing the compact submanifold $S_\ell$ of all tripotents of rank $\ell$. For maximal $\ell = r$ the set
\[
\mathring{E} := \mathring{E}_r
\]
is an open dense subset of $E$. The closure
\[
\mathring{E}_\ell := \{\zeta \in E : \text{rank}(\zeta) \leq \ell\} = \bigcup_{i=0}^\ell \mathring{E}_i
\]
of $\mathring{E}_\ell$ is called the Kepler variety. It is irreducible and normal [11]. The smooth points of $\mathring{E}_\ell$ coincide with $\mathring{E}_\ell$. For matrices $E = \mathbb{C}^{r\times s}$ we obtain the classical "determinantal variety" of all matrices of rank $\leq \ell$. The spin factor $E = \mathbb{C}^d$ yields the quadric $\mathring{E}_1 = \{z \in E : z \cdot z = \sum_i z_i^2 = 0\}$. The identity (2.5) is typical of a stratification
\[
E = \bigcup_{\ell=0}^r \mathring{E}_\ell, \quad \text{(disjoint union)}
\]
into $r+1$ complex analytic $K$-orbits $\mathring{E}_\ell$. The maximal stratum $\mathring{E} = \mathring{E}_r$ is open and dense, and the only closed stratum is the minimal one $\mathring{E}_0 = \mathring{E}_0 = \{0\}$.

3. $K$-invariant ideals and Hilbert modules

For a $J^*$-triple $E$ let $\mathcal{P}_E$ denote the algebra of all (holomorphic) polynomials, endowed with the "Fischer-Fock" inner product $(\phi|\psi)$ (anti-linear in the first variable). The natural action
\[
(k \cdot f)(z) := f(k^{-1}z)
\]
of $K$ on functions $f$ on $E$ induces a multiplicity-free Peter-Weyl decomposition [20][12]
\[
\mathcal{P}_E = \sum_{\lambda \in \mathbb{N}_r^+} \mathcal{P}_E^\lambda.
\]
Here $\mathbb{N}_r^+$ denotes the set of all partitions
\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)
\]
of integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$. A partition $\lambda$ is often identified with its Young diagram
\[
[\lambda] := \{(i,j) : 1 \leq i \leq r, 1 \leq j \leq \lambda_i\}
\]
(3.1)
viewed as a subset of $\mathbb{N} \times \mathbb{N}$. The polynomials in $\mathcal{P}_E^\lambda$ are homogeneous of degree $|\lambda| := \lambda_1 + \ldots + \lambda_r$.

Hence for any $n \in \mathbb{N}$ the $n$-homogeneous polynomials $\mathcal{P}_E^n$ decompose as

$$\mathcal{P}_E^n = \sum_{|\lambda| = n} \mathcal{P}_E^\lambda.$$  

As special cases $\mathcal{P}_E^{0,\ldots,0} = \mathcal{P}_E^0 = \mathbb{C}$ (constant functions) and $\mathcal{P}_E^{1,0,\ldots,0} = \mathcal{P}_E^1 = E^*$ (linear dual space).

Let $\phi_\alpha^\lambda$ be an orthonormal basis of $\mathcal{P}_E^\lambda$. The sesqui-polynomial

$$\mathcal{E}_E^\lambda(z, \zeta) = \sum_\alpha \phi_\alpha^\lambda(z) \overline{\phi_\alpha^\lambda(\zeta)} \quad (3.2)$$

is called the **Fischer-Fock reproducing kernel** for $\lambda$. For example, $\lambda = (1,0,\ldots,0)$

gives rise to the normalized $K$-invariant inner product

$$\mathcal{E}_E^{1,0,\ldots,0}(z, \zeta) = (z|\zeta).$$

**Lemma 3.1.** Let

$$\chi_\lambda(k) := \text{tr}_{\mathcal{P}_E^\lambda}(k \cdot)$$

denote the character of the representation $\mathcal{P}_E^\lambda$ and put

$$d_\lambda := \dim \mathcal{P}_E^\lambda.$$  

Then the orthogonal projection

$$\pi^\lambda : \mathcal{P}_E \to \mathcal{P}_E^\lambda, \ f \mapsto \pi^\lambda f =: f^\lambda,$$

is given by the ”character integral formula”

$$\pi^\lambda f = d_\lambda \int_K d k \ \chi_\lambda(k) \ f \circ k \quad (3.3)$$

for all $f \in \mathcal{P}_E$.

**Proof.** By definition,

$$\chi_\lambda(k) = \sum_\gamma (\phi_\gamma^\lambda | \phi_\gamma^\lambda \circ k^{-1}) = \sum_\gamma (\phi_\gamma^\lambda \circ k | \phi_\gamma^\lambda).$$

For any partition $\mu$ it follows from Schur orthogonality that

$$d_\lambda \int_K d k \ \chi_\lambda(k) \ (\phi_\alpha^\mu | \phi_\beta^\mu \circ k) = d_\lambda \int_K d k \ \sum_\gamma (\phi_\gamma^\lambda \circ k | \phi_\gamma^\lambda) \ (\phi_\alpha^\mu | \phi_\beta^\mu \circ k)$$

$$= \delta_\lambda^\mu \sum_\gamma (\phi_\gamma^\lambda | \phi_\beta^\mu)(\phi_\alpha^\mu | \phi_\gamma^\lambda) = \delta_\lambda^\mu \delta_\alpha^\beta (\phi_\alpha^\mu | \phi_\beta^\mu) = (\phi_\alpha^\lambda | \pi^\lambda \phi_\beta^\mu).$$

since $\mathcal{P}_E^\lambda$ and $\mathcal{P}_E^\mu$ are inequivalent $K$-modules if $\lambda \neq \mu$. □

Any $K$-invariant Hilbert module $H$ of holomorphic functions on $D$ carries a $K$-invariant inner product $(\phi|\psi)_H$ which is uniquely determined by the condition

$$(p|q) = a_\lambda \ (p|q)_H$$

for all $p,q \in \mathcal{P}_E^\lambda \subset H$ where $a_\lambda > 0$ are constants, and $\lambda$ runs over all partitions such that $\mathcal{P}_E^\lambda \subset H$. Taking only those partitions we thus have a Fourier type decomposition

$$H = \sum_\lambda \mathcal{P}_E^\lambda \quad (\text{Hilbert sum})$$

9
determined by the sequence \((a_\lambda)\) of coefficients. Using (3.2) it follows that \(H\) has the reproducing kernel 
\[
K(z, \zeta) = \sum_\lambda a_\lambda \mathcal{E}^\lambda(z, \zeta),
\]
summed over all partitions \(\lambda\) such that \(P_E \subset H\).

**Lemma 3.2.** Let \(H\) be a \(K\)-invariant Hilbert module. If \(f \in H\) then \(f^\mu \in H\) for all \(\mu\).

**Proof.** Using the character integration formula (3.3), this follows from the fact that \(H_0\) is a closed subspace of \(H\) which is \(K\)-invariant.

A \(K\)-invariant reproducing kernel function \(K(z, \zeta) = K_\zeta(z)\) satisfies \(K(kz, k\zeta) = K(z, \zeta)\) for all \(k \in K\). By analytic continuation this implies
\[
K_\zeta \circ h^* = K_h \zeta
\]
for all \(\zeta \in D\) and \(h \in \hat{K}\) such that \(h\zeta \in D\).

**Lemma 3.3.** Let \(H\) be a \(K\)-invariant Hilbert module and \(h \in \hat{K}\). Then
\[
H_\zeta \circ h^* = H_{h \zeta}
\]

**Proof.** For \(f, g \in H\) we have \((f \circ k^{-1}|g)_H = (f|g \circ k)_H\) for all \(k \in K\) since \(H\) is \(K\)-invariant. It follows that
\[
(f \circ h^*|g)_H = (f|g \circ h)_H
\]
(3.4)
for all \(h \in \hat{K}\), since both sides of (3.4) are holomorphic in \(h\) and (3.4) holds for \(h \in K\) where \(h^* = h^{-1}\). Now let \(g = \phi \psi\) with \(\phi \in \mathcal{M}_{h \zeta}\) and \(\psi \in H\). Then
\[
g \circ h = (\phi \circ h)(\psi \circ h)
\]
with \(\psi \circ h \in H\) and \(\phi \circ h \in P_E\) satisfies \((\phi \circ h)(\zeta) = \phi(h\zeta) = 0\). Therefore \(\phi \circ h \in \mathcal{M}_\zeta\) and \(g \in \mathcal{M}_\zeta H\). If \(f \in H_\zeta\) then \((f \circ h^*|g) = (f|g \circ h) = 0\) by Lemma 1.1. Hence (3.4) implies \(f \circ h^* \in H_{h \zeta}\). Therefore \(H_\zeta \circ h \subset H_{h \zeta}\). Passing to \(h^{-1}\) yields equality. \(\square\)

The structure group \(\hat{K}\) acts transitively on each Kepler manifold \(\hat{E}_\ell\). It follows that
\[
\hat{E}_\ell = \hat{K} / \hat{K}^c
\]
where \(\hat{K}^c = \{\gamma \in \hat{K} : \gamma c = c\}\) for some (fixed) tripotent \(c \in S_\ell\). If \(\gamma \in \hat{K}^c\) then \(H_\gamma \circ \gamma^* = H_\gamma\) by Lemma 3.3 and this defines an action of \(\hat{K}^c\) on \(H_\gamma\). As usual define the homogeneous vector bundle
\[
\hat{K} \times_{\hat{K}^c} H_\gamma := \{[h, \phi] = [h\gamma, \phi \circ \gamma^{-*}] : h \in \hat{K}, \gamma \in \hat{K}^c, \phi \in H_\gamma\},
\]
endowed with the \(\hat{K}\)-action \(h \cdot [h', \phi] := [hh', \phi]\).

**Proposition 3.4.** For a \(K\)-invariant Hilbert module \(H\) the restriction \(H|_{\hat{E}_\ell}\) of the eigenbundle \(H\) to each stratum \(\hat{E}_\ell\) is \(\hat{K}\)-isomorphic to the homogeneous vector bundle
\[
H|_{\hat{E}_\ell} \cong \hat{K} \times_{\hat{K}^c} H_\gamma.
\]
For the fibre at \(\zeta \in \hat{E}_\ell\) the isomorphism is given by \(\varepsilon_\zeta f = [h, f \circ h^{-*}]\) for \(f \in H_\gamma\), where \(h \in \hat{K}\) satisfies \(hc = \zeta\).
Proof. If \( f \in H_\zeta \), with \( \zeta = hc \), then \( f \circ h^{-s} \in H_c \) by Lemma 3.3 giving the equivalence class \( \varepsilon_\zeta f = [h, f \circ h^{-s}] \). For any \( \gamma \in \hat{K} \) the definition (3.5) yields \([h_\gamma, f \circ (h_\gamma)^{-s}] = [h_\gamma, f \circ h^{-s} \circ \gamma^{-s}] = [h, f \circ h^{-s}]\), showing that \( \varepsilon_\zeta \) is well-defined. To show equivariance let \( h' \in \hat{K} \). Then \( f \circ h'^{-s} \in H_{h'\zeta} \) by Lemma 3.3. Since \( h'\zeta = h'hc \) we have
\[
\varepsilon_{h'\zeta}(f \circ h'^{-s}) = [h', f \circ h'^{-s}] = [h', h \circ f \circ h^{-s}] = h' \cdot \varepsilon_\zeta f.
\]

Thus for \( K \)-invariant Hilbert modules \( H \) it suffices to determine the fibre \( H_r \) for a fixed tripotent \( c \in S_\ell \). The full eigenbundle will then automatically be a ”stratified union” of homogeneous vector bundles along each stratum \( E_\ell \). Note that these vector bundles are really anti-holomorphic in the fibre variable \( \zeta \), as is standard for Hilbert modules of holomorphic functions.

The basic examples of Hilbert modules on a bounded symmetric domain are the so-called weighted Bergman spaces and Hardy type spaces on the Shilov boundary and the other boundary strata \([3]\). Using the spectral decomposition \([2.3]\), there exists a unique \( K \)-invariant sesqui-polynomial \( \Delta : E \times E \to \mathbb{C} \) such that
\[
\Delta(\zeta, \zeta) = \sum_{j=1}^{r} (1 - \zeta_j^2).
\]
For \((r \times s)\)-matrices, we have
\[
\Delta(z, \zeta) = \text{det}(I_r - z\zeta^*) = \text{det}(I_s - \zeta^*z).
\]
There exists a scale of weighted Bergman spaces \( H^2_s(D) \), for a scalar parameter \( s > 1 + a(r - 1) + b \), with reproducing kernel
\[
K_s(z, \zeta) = \Delta(z, \zeta)^{-s}.
\]
For \( s = 2 + a(r - 1) + b \) we obtain the standard Bergman space. The well-known Faraut-Korányi binomial formula \([12]\)
\[
\Delta(z, \zeta)^{-s} = \sum_\lambda (s)_\lambda \mathcal{E}^\lambda(z, \zeta)
\]
expresses the kernel functions in terms of the Fischer-Fock reproducing kernels \( \mathcal{E}^\lambda \). Here \((s)_\lambda\) denotes the multivariable Pochhammer symbol (a quotient of Gindikin \( \Gamma \)-functions.) Using this formula, one can determine the analytic continuation of the scale of weighted Bergman spaces as
\[
s > \frac{a}{2}(r - 1) \quad \text{(continuous Wallach set)}
\]
for \( s = \ell \frac{a}{2} \) for \( 0 \leq \ell \leq r - 1 \) (discrete Wallach set). A deep result \([4]\) says that the corresponding Hilbert space \( H_s \) is a Hilbert module if and only if \( s \) belongs to the continuous Wallach set. For parameter
\[
s = \frac{d}{r} + (r - \ell)\frac{a}{2}
\]
with \( 1 \leq \ell \leq r \) we obtain Hardy type spaces \( H^2(\partial_\ell D) \) which are supported on the boundary \( G \)-orbits \( \partial_\ell D \). For \( \ell = r \) we obtain the Shilov boundary \( \partial_r D = S \) and the ”standard” Hardy space \( H^2(S) \).

The Hilbert modules mentioned above contain \( \mathcal{P}_E \) as a dense subspace and their reproducing kernel function does not vanish. In this paper we study submodules where this is no longer the case. For any partition \( \lambda \in \mathbb{N}_+ \) denote by \( J^\lambda \subset \mathcal{P}_E \) the ideal generated
by $\mathcal{P}_E^\Lambda$ (or any linear basis). For example, $J^{0,\ldots,0} = \mathcal{P}_E$ and $J^{1,0,\ldots,0} = \mathcal{M}_0$ is the maximal ideal at $0 \in E$. These ”partition” ideals $J^\Lambda$ are the main subject of the paper. Define the (partial) containment ordering of partitions by
\[ \mu \succeq \lambda \iff \mu_i \geq \lambda_i \ \forall \ 1 \leq i \leq r. \]
This is equivalent to inclusion $[\mu] \supset [\lambda]$ of the respective Young diagrams (3.1). Our first main result is

**Theorem 3.5.** $J^\Lambda$ has the Peter-Weyl decomposition
\[ J^\Lambda = \bigoplus_{\mu \succeq \lambda} \mathcal{P}_E^\mu. \]

For the matrix space $E = \mathbb{C}^{r \times s} (a = 2)$ this result is proved in [8, Theorem 4.1] using the theory of standard tableaux. Our proof, valid in the more general setting of $J^*$-triples, is based on harmonic analysis of spherical polynomials.

**Lemma 3.6.** $J := \bigoplus_{\mu \succeq \lambda} \mathcal{P}_E^\mu$ is an ideal in $\mathcal{P}_E$.

**Proof.** By [25, Corollary 2.10] we have for any $\ell \in E^*$ and $\mu \in \mathbb{N}_r^+$
\[ \ell \cdot \mathcal{P}_E^\mu \subset \sum_{i=1}^r \mathcal{P}_E^{\mu', \varepsilon_i}. \] (3.5)
where the sum is over all $i$ such that $\mu + \varepsilon_i$ is a partition. Since $\mu \succeq \lambda$ implies $\mu + \varepsilon_i \succeq \lambda$ it follows that $J$ is invariant under multiplication by linear forms and is therefore an ideal in $\mathcal{P}_E$. □

It follows that $J^\Lambda \subset J$. The converse inclusion requires more effort.

**Lemma 3.7.** Let $\lambda, \mu, \nu$ be partitions and suppose that
\[ \Sigma := \{(pq)^{\lambda} : p \in \mathcal{P}_E^\mu, q \in \mathcal{P}_E^\nu\} \]
contains a non-zero vector. Then $\mathcal{P}_E^{\lambda}$ is spanned by $\Sigma$.

**Proof.** Since the $\lambda$-projection satisfies
\[ (pq)^{\lambda} \circ k = ((pq) \circ k)^{\lambda} = ((p \circ k)(q \circ k))^\lambda \]
for all $k \in K$, the set $\Sigma$ is $K$-invariant. Hence its linear span $\langle \Sigma \rangle$ is a $K$-invariant subspace of $\mathcal{P}_E^{\lambda}$ which by assumption is non-zero. Irreducibility implies $\mathcal{P}_E^{\lambda} = \langle \Sigma \rangle$. □

The crucial technical result is the following:

**Lemma 3.8.** Suppose that $\mu$ and $\mu + \varepsilon_i$ are partitions. Then $\mathcal{P}_E^{\mu + \varepsilon_i}$ is spanned by terms $(\ell q)^{\mu + \varepsilon_i}$, where $\ell \in E^*$ and $q \in \mathcal{P}_E^\mu$.

**Proof.** By Lemma 3.7 it suffices to show that some term $(\ell q)^{\mu + \varepsilon_i}$ is non-zero. Suppose first that $E$ is of tube type, with unit element $e$. Denote by $\Phi^\mu \in \mathcal{P}_E^\mu$ the spherical polynomial [13]. The so-called Pieri formula [16, 4, 22] is
\[ (z|e)\Phi^\mu(z) = \sum_i \Phi^{\mu + \varepsilon_i}(z) \prod_{j \neq i} \frac{\mu'_i - \mu'_j + \frac{a}{2}}{\mu'_i - \mu'_j} \] (3.6)
where we define
\[ \mu'_i := \mu_i - \frac{a}{2}(i - 1) \]
and the sum is over all $1 \leq i \leq r$ such that $\mu + \varepsilon_i$ is a partition. We claim that the coefficient $(\mu'_i - \mu'_j + \frac{a}{r})(\mu'_i - \mu'_j)$ in (3.6) is always $> 0$. If $i < j$ then $\mu_i \geq \mu_j$ and

$$\mu'_i - \mu'_j = \mu_i - \mu_j + \frac{a}{2}(j - i) \geq \frac{a}{2}(j - i) \geq \frac{a}{2}.$$  

Hence $\mu'_i - \mu'_j + \frac{a}{r} \geq a$. If $j < i$ then $\mu_j \geq \mu_i$ and

$$\mu'_j - \mu'_i = \mu_j - \mu_i + \frac{a}{2}(i - j) \geq \frac{a}{2}.$$  

Moreover, $\mu'_j - \mu'_i - \frac{a}{r} = \mu_j - \mu_i + \frac{a}{r}(i - j - 1)$. Since both summands are non-negative, we have $\mu'_j - \mu'_i - \frac{a}{r} = 0$ only if $\mu_j = \mu_i$ and $i - 1 = j$. Thus $\mu_{i-1} = \mu_j = \mu_i$ and $\mu + \varepsilon_i$ cannot be a partition. This proves the claim. It follows that $(z|e)\Phi^\mu(\mu + \varepsilon_i) \neq 0$ whenever $\mu + \varepsilon_i$ is also a partition.

In the general case, choose a maximal tripartite $e \in E$. The Fischer-Fock kernel $E^\mu(z) = E^\mu(z, e)$ has a restriction to $E_e$ which is proportional to $\Phi^\mu$ by a strictly positive factor. As a consequence, we have

$$(z|e)E^\mu(z) = \sum_i \alpha_i E^{\mu+\varepsilon_i}(z)$$

with $\alpha_i > 0$ whenever $\mu + \varepsilon_i$ is a partition. This shows $((z|e)E^\mu(\mu + \varepsilon_i) \neq 0$.

\[\blacksquare\]

**Lemma 3.9.** Let $\mu > \lambda$ be partitions. Then there exists a partition $\nu \geq \lambda$ such that $\mu = \nu + \varepsilon_j$ for some $j \leq r$.

**Proof.** We have $\mu_i > \lambda_i$ for some $i \leq r$. Put $j := \max\{i \leq r : \mu_i > \lambda_i\}$. Then $\mu_1 \geq \ldots \geq \mu_j > \lambda_j \geq \lambda_{j+1} = \mu_{j+1} \geq \lambda_{j+2} = \mu_{j+2} \geq \ldots \geq \lambda_r = \mu_r$.

It follows that $\nu := \mu - \varepsilon_j$ is a partition with $\nu \geq \lambda$.

\[\blacksquare\]

The proof of Theorem 3.5 is now completed by showing that $J \subset J^\lambda$, i.e., $P^\mu_E \subset J^\lambda$ for all partitions $\mu \geq \lambda$. We use induction over $|\mu|$. If $\mu = \lambda$, the assertion is trivial. If $\mu > \lambda$ there exists a partition $\nu \geq \lambda$ such that $P^\mu_E$ is spanned by terms $(\ell q)^\mu$, with $\ell \in K^*$ and $q \in P^\mu_E$. Since $|\nu| = |\mu| - 1$, the induction hypothesis implies $P^\nu_E \subset J^\lambda$. Applying (3.3) to the character $\chi_\mu$ of $P^\mu_E$ yields

$$\langle Q^\mu \rangle = \int \chi_\mu(k) (\ell q) \circ k = \int \chi_\mu(k) (\ell \circ k)(q \circ k).$$

Since $q \circ k \in P^\nu_E \subset J^\lambda$ for all $k \in K$ and $J^\lambda$ is an ideal, it follows that $(\ell q)^\mu \in J^\lambda$ (the integral is actually performed in a finite-dimensional subspace of $P_E$). Therefore $P^\mu_E \subset J^\lambda$. This completes the induction step and proves $J \subset J^\lambda$.

**Corollary 3.10.** Let $\lambda, \mu$ be partitions. Then $J^\mu \subset J^\lambda$ if and only if $\mu \geq \lambda$.

**Proof.** If $\mu \geq \lambda$ then for any partition $\nu \geq \mu$ we have $\nu \geq \lambda$ and hence $P^\nu_E \subset J^\lambda$ by Theorem 3.5. Since $\nu$ is arbitrary, it follows that $J^\mu \subset J^\lambda$. Conversely, if $J^\mu \subset J^\lambda$ then $P^\mu_E \subset J^\lambda$ and hence $\mu \geq \lambda$.

\[\blacksquare\]

As a first application of Theorem 3.5 we show that for any partition $\lambda$ the ideal $J^\lambda$ can be written as an intersection of ideals defined by “rectangular” partitions. For $n \in \mathbb{N}$, put

$$n^{(m)} = (n, \ldots, n, 0, \ldots, 0),$$

with $n$ repeated $m$ times. Thus the Young diagram $[n^{(m)}] = [1, m] \times [1, n]$. Any partition can be written in the form

$$\lambda = (n_1^{(\ell_1)}, n_2^{(\ell_2-\ell_1)}, \ldots, n_r^{(\ell_r-\ell_{r-1})}, 0^{(r-\ell_r)}),$$

(3.7)
where $1 \leq \ell_1 < \ldots < \ell_t \leq r$, $n_1 > n_2 > \ldots > n_t > 0$. In other words,

$$\lambda_1 = \ldots = \lambda_{\ell_1} = n_1 > \lambda_{\ell_1+\ell_2} = \ldots = \lambda_{\ell_t} = n_2 > \ldots$$

Thus we have $n_s = \lambda_j$ for $\ell_{s-1} < j \leq \ell_s$. In particular, $n_s = \lambda_{\ell_s}$. The Young diagram

$$[\lambda] = \bigcup_{s=1}^{t} [n_{s}^{(\ell_{s})}] = \bigcup_{s=1}^{t} [1, \ell_{s}] \times [1, n_{s}]$$

is a (non-disjoint) union of rectangular diagrams.

**Proposition 3.11.** Writing $\lambda$ in the form (3.7) for "rectangular" partitions $n_{s}^{(\ell_{s})}$ we have

$$J^\lambda = \bigcap_{s=1}^{t} J^ {n_{s}^{(\ell_{s})}}.$$ 

**Proof.** This follows from Corollary 3.10 and (3.8). \hfill $\square$

**Proposition 3.12.** The "maximal fibre" at $\zeta = 0$ is given by

$$H_0 = \mathcal{P}_E^\lambda.$$ 

**Proof.** We first show the easy inclusion $\mathcal{P}_E^\lambda \subset H_0$. Let $p \in \mathcal{P}_E^\lambda$ and $\mu \geq \lambda$. Then for all $q \in \mathcal{P}_E^\mu$ and $\ell \in E^*$ we have

$$(T^*_\ell p|q)_H = (p|\ell q)_H = \sum_{i} (p|((\ell q)^{\mu+\varepsilon})_H = 0$$

since $\mu + \varepsilon_i > \mu \geq \lambda$ is different from $\lambda$. Since $T^*_\ell p \in H$ and $\mu \geq \lambda$ is arbitrary, it follows that $T^*_\ell p = 0$.

For the converse, let $f = \sum_{\mu \geq \lambda} f^\mu \in H_0$. For any partition $\mu > \lambda$ there exists a partition $\nu \geq \lambda$ such that $\mathcal{P}_E^\mu$ is spanned by terms $(\ell q)^\mu$, where $\ell \in E^*$ and $q \in \mathcal{P}_E^\nu$. Then

$$(f^\mu|((\ell q)^\mu)_H = (f^\mu|\ell q)_H = (T^*_\ell f^\mu|q)_H = 0$$

since $q \in J^\lambda$ and $f^\mu \in H_0$ by Lemma 3.2. It follows that $f^\mu$ is orthogonal to $\mathcal{P}_E^\mu$ and hence vanishes. Therefore $f = f^\lambda \in \mathcal{P}_E^\lambda$. \hfill $\square$

4. Normal projections

An irreducible Jordan algebra $E$ with unit element $e$ has a unique determinant polynomial $\Delta_e : E \to \mathbb{C}$ normalized by $\Delta_e(e) = 1$ [13, 19]. For the matrix algebra $E = \mathbb{C}^{r \times r}$ and the symmetric matrices $E = \mathbb{C}_{sym}^{r \times r}$ this is the usual determinant. For the antisymmetric matrices $E = \mathbb{C}_{asym}^{2r \times 2r}$ we obtain the Pfaffian determinant instead. For the spin factor (of rank 2) we have $\Delta_e(z) = z \cdot z = \sum z_i^2$.

The determinant polynomial $\Delta_e$ has the semi-invariance property

$$\Delta_e(kz) = \Delta_e(k e) \Delta_e(z)$$

for all $k \in K$ and $z \in E$. The map $\chi : K \to T$ defined by

$$\chi(k) := \Delta_e(k e)$$

is a character of $K$. It follows that for any $k \in K$

$$\Delta_{ke}(z) := \Delta_e(k^{-1}z)$$

is a Jordan determinant normalized at $ke$. 

14
Now consider a frame $e_1, \ldots, e_r$ and for $1 \leq m \leq r$ put $[m] := \{1, \ldots, m\}$ and 
\[ e_{[m]} := e_1 + \ldots + e_m \in S_m. \]

Let $\Delta_{[m]}$ denote the Jordan determinant of the Peirce 2-space $E_{[m]} = E_{e_{[m]}}$ and define the $m$-th Jordan theoretic minor $N_m$ by 
\[ N_m(z) := \Delta_{[m]}(P_{[m]}z) \]
where $P_{[m]}$ is the Peirce projection onto $E_{[m]}$. For any partition $\lambda \in \mathbb{N}^*_+$ define the conical polynomial 
\[ N^\lambda := N_1^{\lambda_1 - \lambda_2} \cdot N_2^{\lambda_2 - \lambda_3} \cdots N_r^{\lambda_r}. \]

The irreducible $K$-module $P^\lambda_E$ is the linear span of such polynomials (for various frames) since the highest weight vector is of this form [24].

As a crucial step towards identifying the eigenbundle of the ideals $J^\lambda$ (or a Hilbert completion $\mathcal{T}^\lambda$) we consider certain projection mappings. Let $0 \leq \ell \leq r$ and consider a rank $\ell$ tripotent $c \in S_\ell$. Its Peirce decomposition will be denoted by 
\[ E = E^2_c \oplus E^1_c \oplus E^0_c = U \oplus V \oplus W. \]

In the matrix case $E = \mathbb{C}^{r \times s}$, with $c = \begin{pmatrix} 1_\ell & 0 \\ 0 & 0 \end{pmatrix}$, this corresponds to the decomposition 
\[ z = \begin{pmatrix} u & v_0 \\ v_0 & 0 \end{pmatrix} \in \begin{pmatrix} U & V \\ V & W \end{pmatrix} \]

of $z \in E$ as a block-matrix with $u$ of size $\ell \times \ell$. The Kepler manifold $\tilde{E}_\ell$ has the tangent space 
\[ T_c(\tilde{E}_\ell) = E^2_c \oplus E^1_c = U \oplus V \]
and hence the normal space 
\[ T_c^\perp(\tilde{E}_\ell) = E^0_c = W. \]

Define the normal projection 
\[ \pi_c : \mathcal{P}_E \rightarrow \mathcal{P}_W, \quad \pi_c f(w) := f(c + w) \]
for $f \in \mathcal{P}_E$ and $w \in W$. If $\ell = 0$ then $\pi_0$ is the identity map.

**Lemma 4.1.** Let $\lambda \in \mathbb{N}^*_+$ and $f \in \mathcal{P}^\lambda_E$. Then $f|_W = 0$ if $\lambda_{r-\ell+1} > 0$, and $f|_W \in \mathcal{P}_W^{\lambda_1 \cdots \lambda_{r-\ell}}$ if $\lambda_{r-\ell} = 0$.

**Proof.** We may assume that $f(z) = E^\lambda(z, b)$ for some $b \in E$. If $w \in W$ then 
\[ f(w) = E^\lambda(w, b) = E^\lambda(P_W w, b) = E^\lambda(w, P_W b), \]
where $P_W$ is the Peirce 0-projection onto $W = E^0_c$. If $\lambda_{r-\ell+1} > 0$ then $E^\lambda(w, P_W b) = 0$ since $W$ has rank $r - \ell$. If $\lambda_{r-\ell+1} = 0$ then $\lambda = (\lambda_1, \ldots, \lambda_{r-\ell}, 0^{(\ell)}) \equiv (\lambda_1, \ldots, \lambda_{r-\ell}) \in \mathbb{N}^*_+^{r-\ell}$ and $E^\lambda(w, P_W b) \in \mathcal{P}_W^{\lambda_1 \cdots \lambda_{r-\ell}}$. \[ \square \]

Since $W = E^c$ is an irreducible $J^*$-triple of rank $r - \ell$, the polynomial algebra $\mathcal{P}_W$ has its own Peter-Weyl decomposition 
\[ \mathcal{P}_W = \sum_{\alpha \in \mathbb{N}^*_+^{r-\ell}} \mathcal{P}_W^\alpha \]
with respect to the Jordan automorphism group $K_W$ of $W$, ranging over all partitions 
\[ \alpha = (\alpha_{\ell+1}, \ldots, \alpha_r) \in \mathbb{N}^*_+^{r-\ell} \]
of length $r - \ell$. We can therefore consider the ideal

$$J^\lambda_W := \sum_{a \in \mathcal{N}_+^{r-\ell}, \; a \geq \lambda^*} \mathcal{P}_W^a$$

generated by the "truncated" partition

$$\lambda^* := (\lambda_{\ell+1}, \ldots, \lambda_r)$$

of length $r - \ell$. The main result of this section is

**Theorem 4.2.** Let $\lambda \in \mathbb{N}_+^r$ be a partition. Then for any tripotent $c \in S_\ell$ with Peirce 0-space $W$ the normal projection (4.3) satisfies

$$f(z) = N_c(z)^n = \Delta_c(P_z z)^n$$

where $e \in E$ is a tripotent of rank $m$ and $\Delta_c$ is the Jordan determinant of the Peirce 2-space $E_c$, with Peirce 2-projection $P_c : E \to E_c$. Regarding $E_c$ as a Jordan algebra with unit element $e$ it has been shown in [19, Theorem 1] that for a minimal projection $e_1 \in E_c$

$$\Delta_c(\xi e_1 + u) = \Delta_c(u) + \xi \Delta_{e-e_1}(P_{e-e_1} u)$$

for all $u \in E_c$ and $\xi \in \mathbb{C}$. Since $c$ is a minimal tripotent, $P_c$ has rank $\leq 1$ and hence there exists a minimal tripotent $c_1 \in E_c$ such that $P_c c = \xi c_1$ where $\xi \in \mathbb{C}$. Choose $k \in K$ commuting with $P_c$ such that $ke_1 = c_1$. Then $\kappa := k|_{E_c} \in K_{E_c}$. For $z \in E$ we have

$$P_c(c + z) = \xi c_1 + P_c z = \kappa(\xi e_1 + \kappa^{-1} P_c z) = \kappa(\xi e_1 + P_c k^{-1} z)$$

and semi-invariance of $\Delta_c$ implies

$$\frac{1}{\Delta_c(k e)} N_c(c + z) = \frac{1}{\Delta_c(k e)} \Delta_c(P_c(c + z)) = \frac{1}{\Delta_c(k e)} \Delta_c(\kappa(\xi e_1 + P_c k^{-1} z))$$

$$= \Delta_c(\xi e_1 + P_c k^{-1} z) = \Delta_c(P_c k^{-1} z) + \xi \Delta_{e-e_1}(P_{e-e_1} P_c k^{-1} z)$$

$$= \Delta_c(P_c k^{-1} z) + \xi \Delta_{e-e_1}(P_{e-e_1} k^{-1} z) = N_c(k^{-1} z) + \xi N_{e-e_1}(k^{-1} z)$$

since $P_{e-e_1} P_c = P_{e-e_1}$. Taking the $n$-th power it follows that $f(c + z) = N_c(c + z)^n$ is a linear combination of polynomials $(N_c^{n-h} N_{e-e_1}) \circ k^{-1}$, where $0 \leq h \leq n$. These have signature $(n^{(m-1)}, h, 0^{(r-m)})$. By Lemma 3.1 the restriction to $W = E^e$ if not zero, has signature

$$(n^{(m-1)}, h, 0^{(r-m-1)}) \geq (n^{(m-1)}, 0, 0^{(r-m-1)}) = (n^{(m-1)}, 0^{(r-m)}) = (\lambda_2, \ldots, \lambda_r).$$

It follows that $\pi_c f \in J^\lambda_W$ for all $f \in \mathcal{P}_W^e$ and, a fortiori, for all $f \in J^\lambda$. Thus the assertion holds for minimal tripotents $c$ and rectangular partitions $\lambda$. We can also write the conclusion in the form

$$J^\lambda W \xrightarrow{\pi_c} J^\lambda W^m.$$  (4.8)

In the **second step** assume only that $c$ is a minimal tripotent but $\lambda \in \mathbb{N}_+^r$ is arbitrary. Using the representation (3.5) we have (since $c$ has rank 1)

$$[\lambda^*] = [(\lambda_2, \ldots, \lambda_r)] = \bigcup_{s=1}^t [n_s^{(l_s-1)}].$$
By step 1, we have \( J_{n_s}^{(t_s)} \xrightarrow{\pi_c} J_{W_{n_s}}^{(t_s-1)} \) for each \( s \leq t \). Proposition 3.3 implies

\[
J^\lambda = \bigcap_{s=1}^{t} J_{n_s}^{(t_s)} \xrightarrow{\pi_c} \bigcap_{s=1}^{t} J_{W_{n_s}}^{(t_s-1)} = J_{W}^{\lambda^*}.
\]

For the general case suppose, by induction, that the assertion is true for tripotents \( c' \) of rank \( \ell - 1 \). Let \( c \) be a minimal tripotent orthogonal to \( c' \). Then

\[
E^{c+c'} = (E^c)^{c'}
\]

and there is a commuting diagram

\[
\begin{array}{ccc}
\mathcal{P}_E & \xrightarrow{\pi_c} & \mathcal{P}_{E^c} & \xrightarrow{\pi_{c+c'}} & \mathcal{P}_{E^{c+c'}} \\
\pi_{c+c'} & & & & \\
\end{array}
\]

where \( \pi_{c+c'}^{E^c} \) denotes the \( c' \)-projection relative to \( E^c \). In fact, if \( w \in (E^c)^{c'} \), then \( c' + w \in E^c \) and

\[
\pi_{c+c'}^{E^c}(\pi_c f)(w) = (\pi_c f)(c' + w) = f((c + (c' + w)) = f((c + c') + w) = (\pi_{c+c'} f)(w).
\]

By the second part of the proof we have

\[
J^\lambda \xrightarrow{\pi_c} J_{E^c}^{\lambda_2, \ldots, \lambda_r}.
\]

The induction hypothesis applied to \( E^c \) (of rank \( r - 1 \)) and the partition \( (\lambda_2, \ldots, \lambda_r) \) yields

\[
J_{E^c}^{\lambda_2, \ldots, \lambda_r} \xrightarrow{\pi_{E^c}} J_{(E^c)^{c'}}^{\lambda_{t+1}, \ldots, \lambda_r} = J_{E^{c+c'}}^{\lambda_{t+1}, \ldots, \lambda_r}.
\]

With (4.9) we obtain \( J^\lambda \xrightarrow{\pi_{E^c}} J_{E^{c+c'}}^{\lambda_{t+1}, \ldots, \lambda_r} \). Thus the assertion holds for the tripotent \( c + c' \) of rank \( \ell \). An induction argument finishes the proof.

As an example, consider the fundamental partition \( \lambda = 1^{(m)} \). If \( c \in S_\ell \) with \( \ell \leq m \), then \( \lambda^* = (\lambda_{\ell+1}, \ldots, \lambda_r) = 1^{(m-\ell)} \) is again a fundamental partition relative to \( W = E^c \). In general, let

\[
\mathcal{M}_X := \{ \pi_c : \pi_c |_X = 0 \} = \bigcap_{\zeta \in X} \mathcal{M}_\zeta
\]

denote the ideal associated with a variety \( X \subset E \). Then Theorem 4.2 says

\[
J_{\hat{W}_{m-\ell-1}}^{(m-\ell)} = \mathcal{M}_X \xrightarrow{\pi_c} J_{\hat{W}_{m-\ell-1}}^{(m-\ell)} = \mathcal{M}_{\hat{W}_{m-\ell-1}}^{(m-\ell)}.
\]

This is clear since \( f|_{\hat{W}_{m-\ell-1}} = 0 \) implies \( (\pi_c f)(w) = f(c + w) = 0 \) for \( w \in \hat{W}_{m-\ell-1} \), because

\[
\text{rank}(c + w) = \text{rank}(c) + \text{rank}(w) = \ell + \text{rank}(w) \leq \ell + (m - \ell - 1) = m - 1.
\]

5. The Main Theorem

Let \( c \) be a tripotent of rank \( \ell \), with Peirce 0-space \( W = E^c \). We combine the normal projection map \( \pi_c \) with the Peter-Weyl projection \( \pi_W^\lambda : \mathcal{P}_W \rightarrow \mathcal{P}_W^{\lambda^*} \) onto the lowest \( K_W \)-type (4.7). This yields a map

\[
J^\lambda \xrightarrow{\pi_W^\lambda} \mathcal{P}_W^{\lambda^*}, \quad \pi_W^\lambda f := \pi_W^\lambda(\pi_c f) = (\pi_c f)^{\lambda^*}.
\]
Proposition 5.1. For any tripotent $c \in S_\ell$ we have
\[ \mathcal{M}_c J^\lambda \subset \ker(\pi^\lambda_c). \]
Thus we have an induced mapping
\[ J^\lambda_c = J^\lambda / \mathcal{M}_c J^\lambda \xrightarrow{\pi^\lambda_c} \mathcal{P}^\lambda_W, \quad f + \mathcal{M}_c J^\lambda \mapsto \pi^\lambda_c f. \]

Proof. Let $g \in \mathcal{M}_c$ and $h \in J^\lambda$. By Theorem 4.2, $\pi_c h$ has only $K_W$-components of type $\mu \geq \lambda^*$. Since $g(c) = 0$ we have $\pi_c g \in \mathcal{M}_{W,0}$. Therefore (3.5) implies that $\pi_c(gh) = (\pi_c g)(\pi_c h)$ has only $K_W$-components of type $\mu > \lambda^*$. Therefore $\pi^\lambda_c (gh) = 0$. \hfill \Box

The main result of this paper is

Theorem 5.2. For each partition $\lambda$ and each tripotent $c$, with Peirce 0-space $W = E^0_c$, the map (5.1), mapping $f \in J^\lambda$ to the lowest $K_W$-type of its normal projection $\pi_c f$, is surjective and has the kernel $\mathcal{M}_c J^\lambda$. Thus it induces an isomorphism
\[ J^\lambda_c \xrightarrow{\pi^\lambda_c} \mathcal{P}^\lambda_W. \quad (5.2) \]

In view of Proposition 5.3 [3,4] Theorem 5.2 yields a realization of the full localization bundle $J^\lambda$ on $E$ as a stratified sum of homogeneous vector bundles supported on the Kepler manifolds $\tilde{E}_\ell$, for $0 \leq \ell \leq r$. A description in terms of reproducing kernels will be given in [26]. The fibre dimension of $J^\lambda$ on the $\ell$-th stratum $\tilde{E}_\ell$ can be computed using the known dimension formula [23] for $\dim \mathcal{P}_E^\lambda$.

It is instructive to check the isomorphism (5.2) in the extremal cases $\ell = 0$ (maximal fibre) and $\ell \geq \ell(\lambda)$ (minimal fibre).

Proposition 5.3. For $\ell = 0$ we have $c = 0$, $W = E^c = E$ and $\lambda^* = \lambda$. In this case there is an isomorphism
\[ J^\lambda_0 \xrightarrow{\pi^\lambda} \mathcal{P}_E^\lambda, \quad f + \mathcal{M}_0 J^\lambda \mapsto f^\lambda. \]

Proof. If $\ell \in E^\ast$ and $q \in J^\lambda$, then $q$ has only components of type $\mu \geq \lambda$ and $\ell q$ has components of type $\mu + \varepsilon_i$ with $\mu \geq \lambda$. It follows that $f \in \mathcal{M}_0 J^\lambda$ has vanishing component $f^\lambda = 0$. Thus the map (4.3) is well-defined and surjective since $\mathcal{P}_E^\lambda \subset J^\lambda$. To show injectivity we use an argument analogous to the proof of Theorem 3.3. If $f \in J^\lambda$ satisfies $f^\lambda = 0$ then $f$ has only components $f^\mu$, where $\mu > \lambda$. For each such $\mu$ there exists a partition $\nu \geq \lambda$ such that $\mu = \nu + \varepsilon_i$ for some $i \leq r$. By Lemma 4.3 $\mathcal{P}_E^\mu$ is spanned by terms $(q)^\mu$, where $\ell \in E^\ast$ and $q \in \mathcal{P}_E^\mu$. Since $\ell \circ k \in \mathcal{M}_0$ and $q \circ k \in \mathcal{P}_E^\mu$ for all $k \in K$, the character integration formula (3.3) yields $f^\mu \in \mathcal{M}_0 \mathcal{P}_E^\mu \subset \mathcal{M}_0 J^\lambda$ and therefore $f \in \mathcal{M}_0 J^\lambda$. \hfill \Box

In view of Proposition 5.3 [5.2] is equivalent to
\[ J^\lambda_c \xrightarrow{\pi_c} J^\lambda_W \xrightarrow{\pi^\lambda_W} \mathcal{P}^\lambda_W. \]
The right hand side corresponds to the maximal fibre of the eigenbundle relative to the normal space $W = E^c$. This formulation may be valid in more general situations, for stratified varieties which are compatible under passing to the normal space.

At the other extreme, for the minimal (1-dimensional) fibres, suppose that $c$ has maximal rank $r$ or, more generally,
\[ \ell = \text{rank}(c) \geq \ell(\lambda). \]
Then $\lambda^* = (\lambda_{r+1}, \ldots, \lambda_r) = 0^{(r-\ell)}$ and hence $\mathcal{P}_{\ell^*}^c W = \mathbb{C}$ (constant functions). This holds even in case $\ell = r$ where $W = \{0\}$. Thus (5.2) amounts to an isomorphism

$$\mathcal{J}_{\ell^*}^c \simeq_{c} 0^{(r-\ell)} \approx \mathcal{P}_{\ell^*}^0 W \approx \mathbb{C}.$$ 

This follows from Corollary [1,3] since $c$ is a regular point for $J^\lambda$.

A third case that is easily checked is the ”Duan-Guo situation” where we have a prime ideal evaluated at a smooth point. In our case this corresponds to fundamental partitions $J^{(\ell+1)} = \mathcal{M}_{\ell^*}$ and a tripotent $c \in \hat{E}_\ell$ of rank $\ell$. In this case $\lambda^* = (1, 0, \ldots, 0)$ and hence $\mathcal{P}_{\ell^*}^c W = W^*$ (linear dual space). Thus (5.2) amounts to

$$J_{\ell}^{(\ell+1)} \approx_{c} \mathcal{P}_{\ell^*}^{1,0,0} W = W^*.$$ 

Since $W = E^c$ is the normal space to the variety $\hat{E}_\ell$ at the smooth point $c$, this is exactly the result proved (in a general context) in [10]. Note that Theorem 5.2 includes the singular points of $\hat{E}_\ell$ having rank $< \ell$. The corresponding fibres involve non-linear functions on $W$.

The proof of Theorem 5.2 will occupy the rest of this section. Fix a frame $e_1, \ldots, e_r$ of minimal orthogonal tripotents $e_i$. Let $1 \leq \ell \leq r$ and put

$$c = e_{[\ell]} = e_1 + \ldots + e_\ell,$$

noting that the case $c = 0$ is covered by Proposition 5.3. As a replacement of the usual matrix coordinates, any irreducible $J^*$-triple $E$ has a joint Peirce decomposition

$$E = \sum_{0 \leq i,j \leq r} E_{ij}$$

into subspaces $E_{ij} = E_{ji}$, satisfying the ”Peirce multiplication rules” [18]

$$\{E_{ij} E_{mn} E_{mn}^* \} \subset E_{mn}.$$ 

Moreover, such a triple product vanishes if there is no possible ”matching” of the indices.

For any $1 \leq m \leq r$ we put $[m] := \{1, \ldots, m\}$. The Peirce decomposition (4.3) under $e_{[m]}$ has the form

$$E_{[m]} = E_{e_{[m]}} = \sum_{i,j \in [m]} E_{ij}, \quad E_{e_{[m]}}^1 = \sum_{i \in [m], j \notin [m]} E_{ij}, \quad E_{e_{[m]}}^c = \sum_{i,j \notin [m]} E_{ij}.$$ 

This notation applies also for domains not of tube type, where the index 0 occurs. By (4.2) the $m$-th minor $N_m$ has the derivative

$$N_m'(z) = \Delta_{[m]}(P_m z) P_m z$$

for $z \in E$, where $\Delta_{[m]}$ is the Jordan determinant of $E_{[m]}$ with unit element $e_{[m]}$. For $A \in \hat{E}$ this implies

$$(A^0 N_m)(z) = N_m'(z) Az = \Delta_{[m]}(P_m z) P_m z Az$$

(5.3)

**Step 1** (of the proof) constructs a ”good” spanning set of vectors in $\mathcal{P}_{\ell^*}^c E$. For related arguments, cf. [21].

**Lemma 5.4.** Let $0 \leq p, q, j \leq r$ and $p, q, j$ distinct. Then the following identities hold for $z \in E_{pq}$, $x \in E_{pj}$ and $y \in E_{jq}$:

$$z \square e_q^* = e_p \square \{e_p z e_q\}^*,$$

$$e_p \square \{y e_q\}^* = x \square y^* = \{e_q y^* x\} \square e_q^*.$$ 

Here an index $i \neq 0$ if the identity involves $e_i$. 


Proof. Since $p \neq q$ we have $z \in E_{ep}^1$ and hence $z = \{e_pe_p^*z\}$. Moreover, $\{e_qe_p^*e_p\} = 0 = \{ze_qe_p\}$. Therefore the Jordan triple identity implies
\[
\begin{align*}
  z \Box e_q^* &= \{e_pe_p^*\} \Box e_q^* = \{e_pe_p^*z\} \Box e_q^* - z \Box \{e_qe_p^*e_p\}^* = [e_p \Box e_q^*, z \Box e_q^*] \\
  &= -[ze_qe_p, e_pe_p^*] = -\{ze_qe_p\} \Box e_q^* + e_p \Box \{e_pz^*e_q\}^* = e_p \Box \{e_pz^*e_q\}^*.
\end{align*}
\]
This yields the first assertion.

Since $p \neq j$ we have $x \in E_{ep}^1$ and hence $x = \{e_pe_j^*x\}$. Here $j = 0$ is allowed. Moreover, $\{e_py^*x\} = \{e_qy^*y\} = 0$ since $p$ does not match $j$ or $q$. Therefore the Jordan triple identity implies
\[
\begin{align*}
x \Box y^* - e_p \Box \{yx^*e_p\}^* &= \{xe_q^*e_p\} \Box y^* - e_p \Box \{yx^*e_p\}^* \\
  &= [x \Box e_q^*, e_pe_p^*y] = -[e_p \Box y^*, x \Box e_q^*] = -\{e_py^*x\} \Box e_q^* + x \Box \{e_pe_q^*y\} = 0.
\end{align*}
\]
Similarly,
\[
\begin{align*}
  \{e_qy^*x\} \Box e_q^* - x \Box y^* &= \{e_qy^*x\} \Box e_q^* - x \Box \{e_qy^*y\}^* \\
  &= [e_q \Box y^*, x \Box e_q^*] = -[x \Box e_q^*, e_qy^*] = -\{xe_q^*e_q\} \Box y^* + e_q \Box \{yx^*e_q\} = 0
\end{align*}
\]
since $\{e_qy^*y\} = y$ and $\{xe_q^*e_q\} = 0 = \{yx^*e_q\}$. This yields the second assertion. \qed

We first consider the ”annihilation operators” in $\hat{\mathfrak{f}}$.

**Proposition 5.5.** Let $j \neq 0$ and $p \notin [q]$. Then
\[
(E_{pj} \Box E_{jq}^*)^0 N_m = 0
\]
for $0 \leq j \leq r$ and $1 \leq m \leq r$.

**Proof.** Let $x \in E_{pj}$ and $y \in E_{jq}$. Lemma 5.4 implies $A := x \Box y^* = z \Box e_q^*$, where $z := \{xy^*e_q\} \in E_{pq}$. If $p \notin [m]$ then $AE = \{ze_q^*e_q\} \subset E_{[m]}^+ \subset E_{[m]}^+ \subset E_{[m]}^+$ (since $p$ does not match $q$). Therefore (5.4) follows from (5.3). If $p \in [m]$ then $p \neq 0$ and $p \notin [q]$ means $q < p$. Therefore $q \in [m]$ and $z \in E_{pq} \subset E_{[m]}$. It follows that $A \in E_{[m]} \Box E_{[m]}$. Since
\[
\text{tr}_{E_{[m]}} A = \text{const} \cdot (z \Box e_q) = 0
\]
it follows that $A$ is a commutator sum in $E_{[m]} \Box E_{[m]}$. Now (5.4) follows from semivariance of $\Delta_{[m]}$. \qed

Now consider the ”creation operators” in $\hat{\mathfrak{f}}$.

**Lemma 5.6.** Let $p \neq 0$ and $q \notin [p]$. Then $E_{pj} \Box E_{jq}^*$ is spanned by linear transformations in $U \Box U^*$, $U \Box V^*$ and $W \Box W^*$ for the Peirce decomposition (1.3) under $c$ (the subscript 0 means vanishing trace).

**Proof.** Let $A = x \Box y^*$ with $x \in E_{pj}$ and $y \in E_{jq}$. Since $p \neq 0$ Lemma 5.4 implies $A := x \Box y^* = e_p \Box z^*$, where $z := \{yx^*e_p\} \in E_{pq}$. There are three cases: If both $p, q \in [\ell]$ then $e_p \in U$ and $z \in U$. Therefore $A \in U \Box U^*$. If both $p, q \notin [\ell]$ then $e_p \in W$ and $z \in W$. Therefore $A \in W \Box W^*$. In both cases the relation $(e_p|z) = 0$ implies vanishing trace. If $p \in [\ell]$ and $q \notin [\ell]$ then $e_p \in U$ and $z \in V$. Therefore $A \in U \Box V^*$. Finally, if $p \notin [\ell]$ and $q \in [\ell]$ then $1 \leq q \leq \ell < p$ since $p \neq 0$. Thus $q \in [p]$, showing that this case cannot occur. \qed

**Lemma 5.7.** The following commutation relations hold:
\[
[U \Box U^*, U \Box V^*] \subset U \Box V^*, \quad [W \Box W^*, U \Box V^*] \subset U \Box V^*, \quad [U \Box U^*, W \Box W^*] = 0.
\]


Proposition 5.8. \( P_E^\lambda \) is spanned by terms
\[
Y_1^\beta \cdots Y_s^\beta X_1^\beta \cdots X_t^\beta Z_1^\beta \cdots Z_r^\beta N^\lambda,
\]
where \( s, t, r \geq 0 \) and \( Y_i \in U \Box V^* \), \( X_j \in U \Box_0 U^* \), \( Z_k \in W \Box_0 W^* \).

Proof. There exists a Cartan subalgebra \( \mathfrak{h} \subset \hat{\mathfrak{k}} = E \Box E^* \) containing
\[
\mathfrak{h}_- := \mathbb{C}\langle e_j \Box e_j^* : 1 \leq j \leq r \rangle
\]
such that the root decomposition
\[
\hat{\mathfrak{k}} = \mathfrak{h} \oplus \sum_\alpha \hat{\mathfrak{k}}_\alpha
\]
has \( N^\lambda \) as the highest weight vector [24]. Then \( P_E^\lambda \) is spanned by terms \( A_1^\beta \cdots A_n^\beta N^\lambda \),
where \( n \geq 0 \) and \( A_i \in \hat{\mathfrak{k}}_\alpha \) for roots \( \alpha > 0 \). By [24] Theorem 1.7
\[
\mathcal{A} := \sum_{p \neq 0, q \notin [p]} \sum_j E_{pj} \Box E_{jq} = \sum_{\alpha > 0, a_{\mathfrak{h}-} \neq 0} \hat{\mathfrak{k}}_\alpha
\]
and
\[
\mathcal{D} := \sum_{p=0}^r \sum_j E_{pj} \Box E_{jp} = \mathfrak{h} \oplus \sum_{a_{\mathfrak{h}-} = 0} \hat{\mathfrak{k}}_\alpha.
\]
Thus the positive root spaces \( \hat{\mathfrak{k}}_\alpha \) belong to \( \mathcal{A} \oplus \mathcal{D} \). We claim that
\[
[\mathcal{A}, \mathcal{D}] \subset \mathcal{A}.
\]
In fact, let \( x \in E_{pj}, y \in E_{jq}, u \in E_{ab}, v \in E_{ba} \). Then \( x \Box y^* = e_p \Box z^* \) for some \( z \in E_{pq} \).
Hence
\[
[x \Box y^*, u \Box v^*] = [e_p \Box z^*, u \Box v^*] = \{e_p \Box z^* u\} \Box v^* - u \Box \{v e_p^* z\}^*.
\]
If \( \{e_p z^* u\} \neq 0 \) then \( q \) must match \( a \) or \( b \). Assume \( q = a \). Then \( \{e_p z^* u\} \Box v^* \in E_{pb} \Box E_{ba} \subset \mathcal{A} \).
Similarly, if \( \{v e_p^* z\} \neq 0 \) then \( p \) must match \( a \) or \( b \). Assume \( p = a \). Then \( u \Box \{v e_p^* z\}^* \in E_{pb} \Box E_{ba} \subset \mathcal{A} \).
This proves the claim.
Combining [24] Theorem 1.7 and Lemma 3.4] one obtains
\[
\mathcal{D}^\beta N^\lambda \subset \mathcal{C} N^\lambda.
\]
It follows that \( P_E^\lambda \) is spanned by terms \( A_1^\beta \cdots A_n^\beta N^\lambda \), where \( n \geq 0 \) and \( A_i \in \mathcal{A} \). By Lemma 5.7 \( \mathcal{A} \) is spanned by transformations of the form \( X \in U \Box U^* \), \( Y \in U \Box V^* \) and \( Z \in W \Box W^* \).
Now the required ordering (5.5) follows from Lemma 5.7 since
\[
Z^\beta X^\beta = X^\beta Z^\beta + [Z^\beta, X^\beta] = X^\beta Z^\beta + [Z, X]^\beta = X^\beta Z^\beta,
\]
\[
Z^\beta Y^\beta = Y^\beta Z^\beta + [Z^\beta, Y^\beta] = Y^\beta Z^\beta + [Z, Y]^\beta,
\]
\[
X^\beta Y^\beta = Y^\beta X^\beta + [X^\beta, Y^\beta] = Y^\beta X^\beta + [X, Y]^\beta
\]
with \( [Z, X] = 0 \), \( [Z, Y] \in U \Box V^* \) and \( [X, Y] \in U \Box V^* \). □
Let $X' := (\lambda_1 - \lambda_{\ell+1}, \ldots, \lambda_\ell - \lambda_{\ell+1}, 0^{(r-\ell)})$

and $\hat{\lambda}' = (\lambda'_{\ell+1}, \lambda_{\ell+1}, \ldots, \lambda_r)$. Then

$$N^\lambda = N^{X'} N^{\hat{\lambda}'}.$$ 

**Lemma 5.9.** Let $X \in U\square U^*$ and $f \in \mathcal{P}_U$. Then

$$X^\theta(f \circ P_U) = (X|_U^\theta f) \circ P_U.$$ 

More generally, for $X_1, \ldots, X_t \in U\square U^*$,

$$X_1^\theta \cdots X_t^\theta(f \circ P_U) = (X_1|_U^\theta \cdots X_t|_U^\theta f) \circ P_U.$$ 

**Proof.** Since $X$ preserves the Peirce decomposition \([4.3]\) it follows that $P_U X z = P_U X P_U z = X|_U P_U z$. Therefore

$$X^\theta(f \circ P_U) z = (f \circ P_U)'(z) X z = f'(P_U z) P_U X z = f'(P_U z) X|_U P_U z = (X|_U^\theta f) P_U z.$$

□

Consider the subgroup $\hat{K}_U := < \exp U \square U^* > \subset \hat{K}$. In the matrix case \([4.4]\) $\hat{K}_U$ consists of the transformations

$$k \begin{pmatrix} u & v_0 \\ v^0 & w \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u & v_0 \\ v^0 & w \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} au + \alpha v_0 & \alpha v \\ v^0 & w \end{pmatrix}. $$

**Lemma 5.10.**

$$(J^\lambda_U \circ P_U) N^{\hat{\lambda}'} \subset J^\lambda.$$ 

**Proof.** Let $h \in J^\lambda_U$. We may assume that $h \in \mathcal{P}_U^\lambda$ and moreover that $h = N^\lambda_U \circ k|_U$ for some $k \in \hat{K}_U$. Then $N^\lambda_U \circ P_U = N^\lambda$ and $N^{\hat{\lambda}'} \circ k = N^{\hat{\lambda}'}$. Therefore

$$(h \circ P_U) N^{\hat{\lambda}'} = ((N^\lambda_U \circ k|_U) \circ P_U) N^{\hat{\lambda}'} = ((N^\lambda_U \circ P_U) \circ k) (N^{\hat{\lambda}'} \circ k)$$

$$= (N^\lambda \circ k) (N^{\hat{\lambda}'} \circ k) = (N^\lambda N^{\hat{\lambda}'}) \circ k = N^\lambda \circ k \in J^\lambda.$$ 

□

**Proposition 5.11.** Let $X_1, \ldots, X_t \in U\square U^*$. Then there exists a constant $C$ such that

$$X_1^\theta \cdots X_t^\theta N^\lambda - CN^\lambda \in \mathcal{M}_e J^\lambda.$$ 

**Proof.** Since $\text{rank}(c) = \ell = \text{rank}(U)$ it follows that $c$ is a regular point for $J^\lambda_U$. Therefore the maximal ideal $\mathcal{M}_{U,e} \subset \mathcal{P}_U$ at $c \in U$ satisfies $J^\lambda_U + \mathcal{M}_{U,e} = \mathcal{P}_U$. Thus for every $f \in J^\lambda_U$ we have

$$f - f(c)N^\lambda_U \in J^\lambda_U \cap \mathcal{M}_{U,e} = \mathcal{M}_{U,e} J^\lambda_U,$$

using \([2]\) p. 6 in the last equation. In particular there exist a constant $C$ and $h \in \mathcal{M}_{U,e} J^\lambda_U$ such that

$$X_1|_U^\theta \cdots X_t|_U^\theta N^\lambda_U = C \ N^\lambda_U + h.$$ 

Applying Lemma 5.9 we obtain

$$X_1^\theta \cdots X_t^\theta N^{\lambda'} = X_1^\theta \cdots X_t^\theta (N^\lambda_U \circ P_U) = (X_1|_U^\theta \cdots X_t|_U^\theta N^\lambda_U) \circ P_U$$

$$= (C \ N^\lambda_U + h) \circ P_U = C \ N^{\lambda'} + h \circ P_U.$$ 

Since $\text{tr}_U X_i = 0$ it follows that $X_i$ is a commutator sum in $U\square U^*$ and therefore also in $E_{[m] \square E_{[m]}}$ for all $m > \ell$. Therefore $X_1^\theta N_m = 0$. It follows that

$$X_1^\theta \cdots X_t^\theta N^\lambda = X_1^\theta \cdots X_t^\theta (N^{\lambda'} N^{\hat{\lambda}'}) = (X_1^\theta \cdots X_t^\theta N^{\lambda'}) N^{\hat{\lambda}'}.$$
of "Cramer’s rule" in a Jordan algebra setting. For any Jordan triple $E$
and consider the Bergman endomorphism $B$. Since $P \in E$
In the matrix case $E = \mathbb{C}^{r \times s}$ this has the form
Let $(z|z) = \Delta^2(z, y)z = z - \{xy^*z\} + Q_{x}Q_{y}z$.
For a unital $J^*$-triple $E$ with unit element $e$ and involution $Q_{e}z = z^*$ the linear transformation
is called the quadratic representation of $z \in E$. An element $z \in E$ is called invertible if $P_{z}$ is invertible. In this case, the inverse $z^{-1}$ is defined by $z^{-1} := P_{z}^{-1}z$. For square matrices $E = \mathbb{C}^{r \times r}$ we have $P_{z}w = zwz$ and $P_{z}^{-1}z = z^{-1}zz^{-1} = z^{-1}$ is the usual inverse.
The Bergman endomorphism $B(x, y)$ defined in (5.6) belongs to $\hat{K}$ if $\Delta(x, y) \neq 0$. Since $B(x, y)^* = B(y, x)$ and $(x\Box y)^* = y\Box x^*$ it follows that $(Q_{x}Q_{y})^* = Q_{y}Q_{x}$. In particular,
\[ P_{z}^* = (Q_{z}Q_{e})^* = Q_{e}Q_{z} = Q_{e}Q_{z}Q_{e}Q_{e} = Q_{Q_{e}Q_{e}}Q_{e} = Q_{Q_{e}Q_{e}}Q_{e} = P_{z^*}. \]
Let $(z|\zeta)$ denote the $K$-invariant inner product normalized by $(e|e) = r$. At the unit element $e$ the Jordan determinant $\Delta_{e}$ has the derivative
\[ \Delta'_{e}(e)u = (u|e). \]
For square matrices, the right hand side is the trace of $u$. If $z \in E$ has strictly positive real part then $P_{z}^{1/2}$ belongs to the structure group $\hat{K}$. It follows that
\[ \Delta_{e} \circ P_{z}^{1/2} = \Delta_{e}(z)\Delta_{e} \]
since $\Delta_{e}(z)\Delta_{e}(u) = \Delta_{e}(P_{z}^{1/2}e)\Delta_{e}(u) = \Delta_{e}(P_{z}^{1/2}u) = (\Delta_{e} \circ P_{z}^{1/2})(u)$ for all $u$. Taking the derivative at $e$ yields
\[ \Delta'_{e}(z)(P_{z}^{1/2}u) = (\Delta_{e} \circ P_{z}^{1/2})'(e)u = \Delta_{e}(z)\Delta'_{e}(e)u = \Delta_{e}(z)(u|e). \]
By analytic continuation this implies "Cramer’s rule"
\[ \Delta'_{e}(z)v = \Delta_{e}(z)(P_{z}^{-1/2}v|e) = \Delta_{e}(z)(v|P_{z}^{-1/2}e) = \Delta_{e}(z)(v|z^{-*}) \]
whenever $z$ is invertible.
In the following we fix $1 \leq \ell < n \leq r$ and put
\[ n^+ := \begin{cases} n + 1 & n < r \\ 0 & n = r \end{cases} \]
The second case arises only for domains not of tube type. Define
\[ E_{[m]n^+} := \sum_{i=1}^{m} E_{in^+}. \]
Proposition 5.12. Let $1 \leq m \leq n$ and $y \in E_{[m]n^+}$. Then

$$N'_m(z)\{e_{[m]}(e_{[m]}(P_{[m]}z)^*y)z\} = (z|y)\ N_m(z).$$

Proof. For $x \in E_{[m]}$ the Peirce multiplication rules imply $\{xye_{[m]}\} = 0$. Hence the Jordan triple identity yields

$$y \boxtimes x^* = \{ye_{[m]}e_{[m]}\} \boxtimes x^* - e_{[m]}\{xye_{[m]}\}^* = [y \boxtimes e_{[m]}^*, e_{[m]} \boxtimes x^*]

= -\{e_{[m]} \boxtimes x^*, ye_{[m]}\} = -\{e_{[m]}x^* \boxtimes e_{[m]}^*, y \boxtimes e_{[m]}e_{[m]}x^*\} = -\{e_{[m]}x^* \boxtimes e_{[m]}^*\} + 2y \boxtimes x^*.

Therefore

$$\{e_{[m]}x^*y\} \boxtimes e_{[m]}^* = y \boxtimes x^*. \quad (5.7)$$

Now suppose that $P_{[m]}z$ has maximal rank $m$, so that $e_{[m]}$ is a supporting tripotent of $P_{[m]}z$. Let $(P_{[m]}z)^{-*}$ denote the inverse of $(P_{[m]}z)^* = Qe_{[m]}P_{[m]}z$ in $E_{[m]}$. Putting $x = (P_{[m]}z)^*$ in (5.4) and evaluating at $(P_{[m]}z)^{-*}$ yields

$$\{\{e_{[m]}(P_{[m]}z)^*y\}e_{[m]}(P_{[m]}z)^{-*}\} = \{y(P_{[m]}z)^*(P_{[m]}z)^{-*}\} = \{ye_{[m]}e_{[m]}\} = y.$$

By density, we may assume that $z \in E$ has $P_{[m]}z \in E_{[m]}$ of maximal rank $m$. Applying Cramer’s rule to the determinant $\Delta_m$ of $E_{[m]}$ we obtain for $\xi \in E$

$$N'_m(z)\xi = \Delta_m'(P_{[m]}z)P_{[m]}\xi = \Delta_m'(P_{[m]}z) (P_{[m]}\xi|(P_{[m]}z)^{-*}) = N_m(z) \left(\xi |(P_{[m]}z)^{-*}\right).$$

Putting $\xi = \{e_{[m]}\{e_{[m]}(P_{[m]}z)^*y\}z\}$ and using associativity of $(\cdot)$ we obtain

$$N'_m(z)\{e_{[m]}(e_{[m]}(P_{[m]}z)^*y)z\} = N_m(z) \{\{e_{[m]}\{e_{[m]}(P_{[m]}z)^*y\}z\}\} \left((P_{[m]}z)^{-*}\right)

= N_m(z) \left(\{e_{[m]}(P_{[m]}z)^*y\}e_{[m]}(P_{[m]}z)^{-*}\right) = N_m(z)(z|y)$$

\[\square\]

Let $R_u$ denote the representation

$$R_u v := \{uc^*v\}$$

of $u \in U = E_{[\ell]}$ acting on $v \in E_{[\ell]n^+}$. For matrices $E = \mathbb{C}^{r \times s}$ these are the transformations

$$R \left(\begin{array}{cc} u & 0 \\ 0 & 0 \end{array}\right) \left(\begin{array}{cc} 0 & v_0 \\ v_0 & 0 \end{array}\right) = \left(\begin{array}{cc} u & 0 \\ 0 & 0 \end{array}\right) \left(\begin{array}{cc} 0 & v_0 \\ v_0 & 0 \end{array}\right) = \left(\begin{array}{cc} 0 & u^0v_0 \\ v_0 & 0 \end{array}\right).$$

Proposition 5.13. Let $\ell \leq n \leq r$ and $v \in E_{[\ell]n^+}$. Then we have for $\ell < m \leq n$

$$(z|R_u^{-*}v)\ N_m(z) = N'_m(z)\{cv^*z\} + \sum_{i=\ell+1}^n N'_m(z)\{e_i(e_i(P_{[m]}z)^*(R_u^{-*})v)z\} = (z|y)\ N_m(z). \quad (5.8)$$

Proof. Let $u = P_{[\ell]}z = P_{[\ell]}z$ and assume that $N_m(z) = \Delta_{e}(u) \neq 0$. Then $y = R_u^{-*}v \in E_{[\ell]n^+}$. If $z_{ab} \in E_{ab}$ with $a, b \in [m]$ then $\{cz_{ab}y\} \neq 0$ implies $a, b \in [\ell]$ since $a$ or $b$ cannot match $n^+$. It follows that

$$\{c(P_{[m]}z)^*y\} = \{c(P_{[\ell]}z)^*y\} = \{cu^*y\} = \{u^*c^*y\} = R_u^*y = v.$$ 

Now let $\ell < i \leq m$. If $z_{ab} \in E_{ab}$, with $a, b \in [n]$ and $\{e_i z_{ab}y\} \neq 0$, then both $a, b$ cannot match $n^+$ and hence one index $b \in [\ell]$. Since $i > \ell$ it follows that the other index $a = i$. Thus both indices $a, b \in [m]$ and hence

$$\{e_i(P_{[m]}z)^*y\} = \{e_i(P_{[m]}z)^*y\} \in E_{m^+}.$$ 

Therefore

$$\{e_{[m]}(P_{[m]}z)^*y\} = \{c(P_{[m]}z)^*y\} + \sum_{i=\ell+1}^m \{e_i(P_{[m]}z)^*y\} = v + \sum_{i=\ell+1}^m \{e_i(P_{[m]}z)^*y\}. \quad (5.10)$$
If \( \ell < i \leq m \) then \( v \in E_{[\ell]n^+} \subset E^\times_i \). Hence \( e_i \sqcup v^* = 0 \) and

\[
e_i \sqcup v^* = e_i \sqcup v^* = e \sqcup v^*. \tag{5.11}
\]

If \( j \in [m] \) with \( j \neq i \) then \( \{e_i(P_{[n]}z)y\} \in E_{m^+} \subset E^\times_j \) and hence \( e_j \sqcup \{e_i(P_{[n]}z)y\}^* = 0 \). Therefore

\[
e_{[m]} \sqcup \{e_i(P_{[n]}z)y\}^* = e_i \sqcup \{e_i(P_{[n]}z)y\}^*. \tag{5.12}
\]

Applying Proposition 5.12 together with (3.3), (5.11) and (5.12) we obtain

\[
(z | y) N_m(z) = N_m'(z)\{e_{[m]} \{e_{[m]}(P_{[n]}z)y\}^*z\} = N_m'(z)\{e_{[m]}(v + \sum_{i=\ell+1}^m \{e_i(P_{[n]}z)y\}^*)z\}
\]

\[
= N_m'(z)\{e_{[m]}v^*z\} + \sum_{i=\ell+1}^m N_m'(z)\{e_i(P_{[n]}z)y\}^*z\}
\]

\[
= N_m'(z)\{cv^*z\} + \sum_{i=\ell+1}^m N_m'(z)\{e_i(P_{[n]}z)y\}^*z\}
\]

\[
= N_m'(z)\{cv^*z\} + \sum_{i=\ell+1}^m N_m'(z)\{e_i(P_{[n]}z)y\}^*z\}.
\]

In the last step we use that for \( m < i \leq n \) we have \( \{e_i(P_{[n]}z)y\}^*z \in \sum \ E_{ik} \subset E_{[m]}^1 \) and hence \( N_m'(z)\{e_i(P_{[n]}z)y\}^*z = 0 \).

The transformation \( R_u \) on \( U \) has determinant \( \Delta_c(u) = N_c(z)^6 \), where \( u = P_c z \) and

\[
\delta := \dim \ E_{in^+} = \begin{cases} a \quad n < r \\ b \quad n = r. \end{cases}
\]

Computing \( R_u^{-1} \) with (the usual) Cramer’s rule it follows that for fixed \( v \) there exists a ”cofactor polynomial” \( \Psi_{\xi}^v(z) \) depending linearly on \( \xi \in E \) such that

\[
\Psi_{\xi}^v(z) = N_i(z)^6 \ (\xi | R_u^{-1}v)
\]  

(5.13)

for all \( z \) with \( N_i(z) \neq 0 \) (a dense open subset of \( E \)). We apply this construction to \( \xi = z \) and to \( \xi = \{v_\alpha e_i^*(P_{[n]}z)\} \), with \( \ell < i \leq n \) and \( v_\alpha \) is an orthonormal basis of \( E_{in^+} \).

**Lemma 5.14.** The ”cofactor polynomials” \( \Psi_{\xi}^z(z) \) and \( \Psi_{\{v_\alpha e_i^*(P_{[n]}z)\}}^z(z) \) vanish at \( c = e_{[\ell]} \) and hence belong to \( M_c \).

**Proof.** Since \( N_i(c) = 1 \) and \( R_c = \text{id} \) (5.13) implies

\[
\Psi_{\xi}^v(c) = N_i(c)^6 \ (\xi | R_c^{-1}v) = (\xi | v)
\]

for \( \xi \) evaluated at \( z = c \). For \( \xi = z \) evaluating at \( z = c \) yields \( (c | v) = 0 \) since \( v \in E_{[\ell]n^+} \subset U^\perp \). For \( \xi = \{v_\alpha e_i^*(P_{[n]}z)\} \), with \( \ell < i \leq n \), evaluating at \( z = c \) yields \( \xi = \{v_\alpha e_i^*(P_{[n]}c)\} = \{v_\alpha e_i^*c\} = 0 \) since \( i > \ell \).

**Lemma 5.15.** If \( Y \in U \sqcup V^* \) then \(YC = 0 \).

**Proof.** Let \( Y = uc \sqcup v^* \) with \( u \in U \) and \( v \in V \). The Peirce multiplication rules imply \( Yc = \{uv^*c\} \in \{UV^*U\} = \{0\} \).

**Proposition 5.16.** Let \( Y \in U \sqcup V^* \). Then \( Y^\Delta N^\lambda \in M_c J^\lambda \).
Proof. By Lemma 5.4 we may assume that $Y = c\Box v^*$ for some $v \in E_{[m]}$ with $\ell \leq n \leq r$. Multiplying the identity (5.8) by $N_\ell(z)^\delta$ and applying the definition of $\Psi_v^*(z)$ and $\Psi_v^*(v^*_{\ell+1}\{P_{[m]}\})(z)$ one obtains

$$N_\ell(z)^\delta (c\Box v^*)^\delta N_m = \Psi_v^*(z) N_m(z) - \sum_{i=\ell+1}^n \sum_{\alpha} \Psi_v^*(v^*_{\ell+i}\{P_{[m]}\})(z) (e_i\Box v^*)^\delta N_m$$

whenever $\ell < m \leq n$. In case $n < r$ we have $(c\Box v^*)^\delta N_m = 0$ if $\ell < n \leq m \leq r$ since $n^+ \leq m$ and hence $c, v \in E_{[m]}$ with $(c|v) = 0$ (for $n = r$ the condition is empty). Therefore (5.14) implies

$$N_\ell^\delta (c\Box v^*)^\delta N_\Lambda^\ast = N_\ell^\delta \sum_{m=\ell+1}^n (\lambda_m - \lambda_{m+1}) (c\Box v^*)^\delta N_m = N_\ell^\delta \sum_{m=\ell+1}^n (\lambda_m - \lambda_{m+1}) (e_i\Box v^*)^\delta N_m$$

$$= \sum_{m=\ell+1}^n (\lambda_m - \lambda_{m+1}) \left( \Psi_v^* - \sum_{i=\ell+1}^n \sum_{\alpha} \Psi_v^*(v^*_{\ell+i}\{P_{[m]}\}) \lambda_m \lambda_{m+1} \right) \frac{(e_i\Box v^*)^\delta N_m}{N_m}$$

$$= C \cdot \Psi_v^* - \sum_{i=\ell+1}^n \sum_{\alpha} \Psi_v^*(v^*_{\ell+i}\{P_{[m]}\}) \sum_{m=\ell+1}^n (\lambda_m - \lambda_{m+1}) \frac{(e_i\Box v^*)^\delta N_m}{N_m}.$$

Here the constant

$$C = \sum_{m=\ell+1}^n (\lambda_m - \lambda_{m+1}) = \begin{cases} \lambda_{\ell+1} - \lambda_{n+} & n < r \\ \lambda_{\ell+1} & n = r \end{cases}.$$ 

On the other hand, if $\ell < i \leq n$ then

$$\frac{(e_i\Box v^*)^\delta N_\Lambda^\ast}{N_m} = \sum_{m=\ell+1}^n (\lambda_m - \lambda_{m+1}) (e_i\Box v^*)^\delta N_m = \sum_{m=\ell+1}^n (\lambda_m - \lambda_{m+1}) (e_i\Box v^*)^\delta N_m$$

since $(e_i\Box v^*)^\delta N_m = 0$ if $i \leq n < m$. It follows that

$$N_\ell(z)^\delta (c\Box v^*)^\delta N_\Lambda^\ast = C \Psi_v^* (z) N_\Lambda^\ast - \sum_{i=\ell+1}^n \sum_{\alpha} \Psi_v^*(v^*_{\ell+i}\{P_{[m]}\})(z) (e_i\Box v^*)^\delta N_\Lambda^\ast$$

and therefore

$$N_\ell^\delta N^{2\lambda'}(Y^\partial N_\Lambda^\ast) = N^{2\lambda'}(C \Psi_v^* N_\Lambda^\ast - \sum_{i=\ell+1}^n \sum_{\alpha} \Psi_v^*(v^*_{\ell+i}\{P_{[m]}\})(z) (e_i\Box v^*)^\delta N_\Lambda^\ast)$$

$$= C \cdot N^{2\lambda'} \Psi_v^* N_\Lambda^\ast - \sum_{i=\ell+1}^n \sum_{\alpha} \Psi_v^*(v^*_{\ell+i}\{P_{[m]}\})(z) N^{2\lambda'}(e_i\Box v^*)^\delta N_\Lambda^\ast. \quad (5.15)$$

For any $A \in \widehat{\mathfrak{f}}$ we have $A^\partial N_\Lambda^\ast = A^\partial (N_\Lambda^\ast N_\Lambda^\ast) = (A^\partial N_\Lambda^\ast) N_\Lambda^\ast + N_\Lambda^\ast (A^\partial N_\Lambda^\ast)$ and therefore

$$N^{2\lambda'}(A^\partial N_\Lambda^\ast) = N^{2\lambda'}(A^\partial N_\Lambda^\ast) - (A^\partial N_\Lambda^\ast) N_\Lambda^\ast \in J^\Lambda.$$

This implies

$$N^{2\lambda'}(e_i\Box v^*)^\delta N_\Lambda^\ast \in J^\Lambda$$

and (5.23) and Lemma 5.14 imply $N_\ell^\delta N^{2\lambda'}(Y^\partial N_\Lambda^\ast) \in \mathcal{M}_c J^\Lambda$. Since $N_\ell(c) = 1$ it follows that

$$N^{2\lambda'}(Y^\partial N_\Lambda^\ast) \in \mathcal{M}_c J^\Lambda.$$
We have $Y^\partial N^{\lambda'} \in \mathcal{M}_c$ since $Yc = 0$ by Lemma 5.15. Therefore the identity

$$N^{\lambda'}(Y^\partial N^{\lambda}) = N^{\lambda'}(Y^\partial N^{\lambda'} N^{\lambda} + N^{\lambda'}(Y^\partial N^{\lambda'})) = (Y^\partial N^{\lambda'}) N^{\lambda} + N^{2\lambda'}(Y^\partial N^{\lambda'})$$

shows that $N^{\lambda'}(Y^\partial N^{\lambda}) \in \mathcal{M}_c J^{\lambda}$. Since $N^{\lambda'}(c) = 1$ this completes the proof. \hfill $\square$

**Step 4** constructs an equivariant cross-section to the (surjective) map $\omega$. We need a "compression formula" for Jordan determinants which in the matrix case amounts to the well-known relation

$$\det \begin{pmatrix} u & v_0 \\ v_0 & w \end{pmatrix} = \det(u) \det(w - v^0 u^{-1} v_0)$$

for block-matrices, with the $(\ell \times \ell)$-submatrix $u$ invertible. If $E$ is of tube type (i.e. $r = s$ in the matrix case) its Jordan algebra determinant $N(z)$ satisfies

$$N(B(x,y)z) = \Delta(x,y)^2 N(z). \quad (5.16)$$

This follows from the fact that $B(x,y)$ belongs to $\hat{K}$ whenever $\Delta(x,y) \neq 0$. Using the decomposition **[1.3]** define the open dense subset

$$\Omega := \{ z = u + v + w \in E : u \in U \text{ invertible} \}$$

and a rational map $\omega : \Omega \to W$ by

$$\omega(z) := w - Q_v u^{-*} = w - \frac{1}{2} \{ vu^{-1} v \}.$$ 

Here $u^{-1}$ denotes the inverse of $u \in U$ and $u^{-*} \in U$ denotes the inverse of $u^*$ for the involution $u^* = Q_v u$ on $U$. Consider the subgroup $\hat{K}_W := < \exp W \square W^* >$ of $\hat{K}$. In the matrix case **[4.4]** $\hat{K}_W$ consists of the transformations

$$k \begin{pmatrix} u & v_0 \\ v_0 & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} u & v_0 \\ v_0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} u & v_0 \delta \\ dv^0 & dw \delta \end{pmatrix}.$$ 

Any $k \in \hat{K}_W$ satisfies $k(u + v + w) = u + kv + kw$, with $kv \in V$ and $kw \in W$. Therefore $\hat{K}_W$ leaves $\Omega$ invariant and

$$\omega(kz) = kw - Q_{kv} u^{-*} = k(w - Q_v u^{-*}) = k|_W \omega(z).$$

**Lemma 5.17.** Let $z = u + v + w \in E$ with $u$ invertible. Then we have the "compression formula"

$$N_m(z) = \Delta_c(u) N_{m-\ell}(\omega(z)) \quad (5.17)$$

for $\ell < m \leq r$. Here $N_{m-\ell} = N_{e_{m-\ell}} = N_{e_{\ell+1}+\ldots+e_m}$ denotes the $(m - \ell)$-th minor relative to $W$.

**Proof.** The Peirce multiplication rules imply $Q_v u^{-*} \in W$. Therefore

$$B(v, u^{-*})(u + v + w) = u + (w - Q_v u^{-*}) = u + \omega(z)$$

has no $V$-component. Moreover, $\Delta(v, u^{-*}) = 1$. Applying (5.16) to the unital $J^*$-triple $E_{[m]}$ we obtain

$$N_m(z) = N_m(u + v + w) = \Delta(v, u^{-*})^2 N_m(u + v + w) = N_m(B(v, u^{-*})(u + v + w)) = N_m(u + \omega(z)) = \Delta_c(u) N_{m-\ell}(\omega(z)). \hfill \square$$
For $\alpha = (\alpha_{\ell+1}, \ldots, \alpha_r) \in \mathbb{N}_+^{r-\ell}$ put
\[
\hat{\alpha} := (\alpha_{\ell+1}, \alpha_{\ell+1}, \ldots, \alpha_r) \in \mathbb{N}^r
\]
and denote by
\[
N_W^\alpha = \prod_{m=\ell+1}^r (N_{m-\ell}^W)^{\alpha_m - \alpha_{m+1}}
\]
the conical polynomial on $W$ relative to $\alpha$. If $u = P_U z$ is invertible, then (5.17) implies
\[
N^\hat{\alpha}(z) = \prod_{m=\ell+1}^r N_{m}^{\alpha_m - \alpha_{m+1}}(z) = \Delta_c(u)^{\alpha_{\ell+1}} N_W^\alpha(\omega(z)).
\]

**Corollary 5.18.** Let $z = u + v + w \in \Omega$ and $Z_1, \ldots, Z_r \in W^*$. Then
\[
(Z_1^0 \cdots Z_r^0 N^\hat{\alpha})(z) = \Delta_c(u)^{\alpha_{r+1}} (Z_1^0 \cdots Z_r^0 N_W^\alpha)(\omega(z)).
\]

**Proof.** Each $k \in \hat{K}_W$ leaves $\Omega$ invariant and satisfies $k|_U = \text{id}$. Therefore Lemma 5.17 implies
\[
N^\hat{\alpha}(kz) = \Delta_c(u)^{\alpha_{r+1}} N_W^\alpha(\omega(kz)) = \Delta_c(u)^{\alpha_{r+1}} N_W^\alpha(k|_W \omega(z)).
\]
Taking a 1-parameter group $k_t = \exp(tZ)$ with $Z \in W^* W^*$ this implies
\[
(Z_1^0 \cdots Z_r^0 N^\hat{\alpha})(z) = \frac{d}{dt} N^\hat{\alpha}(k_t z) = \Delta_c(u)^{\alpha_{r+1}} \frac{d}{dt} N_W^\alpha(k_t|_W \omega(z)) = \Delta_c(u)^{\alpha_{r+1}} (Z_1^0 \cdots Z_r^0 N_W^\alpha(\omega(z)).
\]
Iterating this relationship, the assertion follows. \hfill \square

**Proposition 5.19.** For each $\lambda \in \mathbb{N}_+^r$ there is a unique linear map $\tau_\lambda : \mathcal{P}_W^\lambda \to \mathcal{P}_E^\lambda$ with
\[
\tau_\lambda N_W^\lambda = N^\lambda,
\]
which for all $\phi \in \mathcal{P}_W^\lambda$ has the cross-section property
\[
\pi_c^\lambda(\tau_\lambda \phi) = \phi \quad (5.18)
\]
and the invariance property
\[
(\tau_\lambda \phi) \circ k = \tau_\lambda(\phi \circ k|_W) \quad (5.19)
\]
under $k \in \hat{K}_W$.

**Proof.** Let $u = P_U z$ be invertible. Any $k \in \hat{K}_W$ satisfies $P_U(kz) = P_U z = u$ and $\omega(kz) = k|_W \omega(z)$. It follows that
\[
N^\hat{\alpha}(kz) = \Delta_c(P_U(kz))^{\alpha_{r+1}} N_W^\alpha(\omega(kz)) = \Delta_c(u)^{\alpha_{r+1}} N_W^\alpha(k|_W \omega(z)).
\]
Therefore a relation $\sum_i C_i N_W^\alpha \circ k_i|_W = 0$, for constants $C_i$ and $k_i \in \hat{K}_W$, implies
\[
\sum_i C_i N^\hat{\alpha}(k_i z) = \Delta_c(u)^{\alpha_{r+1}} \sum_i C_i N_W^\alpha(k_i|_W \omega(z)) = 0
\]
on a dense open subset of $E$ and hence on all of $E$. By irreducibility under $K_W$, every $\phi \in \mathcal{P}_W^\alpha$ has the form $\phi = \sum_i C_i N_W^\alpha \circ k_i|_W$. Thus there is a well-defined linear map $\tau_\alpha : \mathcal{P}_W^\alpha \to \mathcal{P}_E^\hat{\alpha}$ given by
\[
\phi = \sum_i C_i N_W^\alpha \circ k_i|_W \mapsto \tau_\alpha \phi := \sum_i C_i N^\hat{\alpha} \circ k_i.
\]

Then
\[ \tau_\alpha N^\alpha_W = N^{\bar{\alpha}} \]
and we have the invariance property
\[ (\tau_\alpha \phi) \circ k = \tau_\alpha (\phi \circ k)|_W \] (5.20)
for all \( k \in \hat{K}_W \), as follows with
\[ \phi \circ k|_W = \left( \sum_i C_i N^\alpha_W \circ k_i|_W \right) \circ k|_W = \sum_i C_i N^\alpha_W \circ (k_i|_W k)|_W = \sum_i C_i N^\alpha_W \circ (k_i k)|_W \]
and
\[ (\tau_\alpha \phi) \circ k = \sum_i C_i (N^{\bar{\alpha}} \circ k_i) \circ k = \sum_i C_i N^{\bar{\alpha}} \circ (k_i k). \]
Since \( \pi^\alpha c N^{\bar{\alpha}} = N^\alpha_W \) this also yields the cross-section property
\[ \pi^\alpha c (\tau_\alpha \phi) = \phi \] (5.21)
for all \( \phi \in \mathcal{P}^\alpha_W \). Since \( N^\lambda \) is invariant under \( \hat{K}_W \) and \( \pi_c N^\lambda = 1 \), the required cross-section \( \tau_\lambda \) is defined by
\[ \tau_\lambda \phi := N^\lambda (\tau_\lambda \phi). \]
\[ \square \]

**Step 5** completes the proof of Theorem 5.2.

**Proposition 5.20.** \( \mathcal{P}^\lambda_E \) is spanned by terms
\[ (Y^\rho_1 \cdots Y^\rho_s X^\omega_1 \cdots X^\omega_t N^\lambda) \circ k \]
where \( s, t \geq 0, Y_i \in U \Box V^*, X_j \in U \Box U^* \) and \( k \in \hat{K}_W \).

**Proof.** By Proposition 5.18 it suffices to consider terms
\[ Y^\rho_1 \cdots Y^\rho_s X^\omega_1 \cdots X^\omega_t Z^\rho_1 \cdots Z^\rho_r N^\lambda \]
where \( s, t, r \geq 0 \) and \( Y_i \in U \Box V^*, X_j \in U \Box U^* \) and \( Z_k \in W \Box W^* \). Since \( \hat{K}_W \) acts irreducibly on \( \mathcal{P}^\alpha_W \) it follows that
\[ Z^\rho_1 \cdots Z^\rho_r N^\lambda = Z^\rho_1 \cdots Z^\rho_r (\tau_\lambda N^\lambda_W) = \tau_\lambda (Z^\rho_1 \cdots Z^\rho_r N^\lambda_W) \]
is a linear combination of \( \tau_\lambda (N^\lambda_W \circ k)|_W \) = \( (\tau_\lambda N^\lambda_W) \circ k = N^\lambda \circ k \) for \( k \in \hat{K}_W \). The proof is concluded by noting that for \( k \in \hat{K}_W \) we have
\[ X^\rho (f \circ k) = (X^\rho f) \circ k \]
if \( X \in U \Box U^* \) and
\[ Y^\rho (f \circ k) = ((kYk^{-1})^\rho f) \circ k \]
if \( Y \in U \Box V^* \), with \( kYk^{-1} \in U \Box V^* \). This follows from
\[ Y^\rho (f \circ k)(z) = (f \circ k)'(z)Yz = f'(kz)kYz = f'(kz)(kYk^{-1})kz = ((kYk^{-1})^\rho f)(kz). \]
\[ \square \]

**Lemma 5.21.**
\[ \mathcal{P}^\lambda_E \subset (N^\lambda \circ \hat{K}_W) + M_c J^\lambda = \tau_\lambda \mathcal{P}^\lambda_W + M_c J^\lambda. \]
Proof. Since $\mathcal{M}_c \circ \mathcal{K}_W = \mathcal{M}_c$ it suffices by Proposition 5.22 to show that
\begin{equation}
g := Y_1^0 \cdots Y_s^0 X_1^0 \cdots X_t^0 N^\lambda \in \langle N^\lambda \rangle + \mathcal{M}_c J^\lambda, \quad (5.22)
\end{equation}
where $s, t \geq 0$ and $Y_i \in U \vartriangleleft V^*$. By Lemma 5.11 there exist a constant $C$ and $h \in \mathcal{M}_c J^\lambda$ such that $X_1^0 \cdots X_t^0 N^\lambda = CN^\lambda + h$. This proves (5.22) if $s = 0$. Now let $s \geq 1$. Proposition 5.16 implies $Y^0 N^\lambda \in \mathcal{M}_c J^\lambda$ for all $Y \in U \vartriangleleft V^*$. With $Y^0 (pq) = (Y^0 p)q + p(Y^0 q)$ we also have
\[Y^0 (\mathcal{M}_c J^\lambda) \subset \mathcal{M}_c J^\lambda\]
since $Y c = 0$ implies $(Y^0 p)(c) = p'(c) Y c = 0$ for all $p \in \mathcal{P}_E$ and hence $Y^0 \mathcal{P}_E \subset \mathcal{M}_c$. Therefore $Y^0 h \in \mathcal{M}_c J^\lambda$ and hence $g = Y_1^0 \cdots Y_s^0 (CN^\lambda + h) \in \mathcal{M}_c J^\lambda$. □

Proposition 5.22. \(\ker(\pi^\lambda_c) \subset \mathcal{M}_c J^\lambda\).

Proof. Let $(\phi_i)_{i \in I}$ be a basis of $\mathcal{P}^\lambda_W$. Then $\tau_c \phi_i \in \mathcal{P}^\lambda_E$ satisfies $\pi^\lambda_c \tau_c \phi_i = \phi_i$ and $(\tau_c \phi_i)$ are linearly independent. Let $(p_j)_{j \in J}$ be a basis of $\ker(\pi^\lambda_c) \cap \mathcal{P}^\lambda_E$. Then $(\tau_c \phi_i)_{i \in I} \cup (p_j)_{j \in J}$ form a basis of $\mathcal{P}^\lambda_E$. Now let $f \in \ker(\pi^\lambda_c) \subset J^\lambda$. By definition of $J^\lambda$ there exist $f_i \in \mathcal{M}_c$, $a_i \in \mathbb{C}$ and $q_j \in \mathcal{P}_E$ such that
\begin{equation}
f = \sum_{i \in I} (f_i + a_i) (\tau_c \phi_i) + \sum_{j \in J} q_j p_j = \sum_{i \in I} f_i (\tau_c \phi_i) + \sum_{j \in J} q_j p_j + \sum_{i \in I} a_i (\tau_c \phi_i). \quad (5.23)
\end{equation}
By Lemma 5.21 each $p_j$ can be written as $p_j = \tau_c \psi_j + h_j$, where $\psi_j \in \mathcal{P}^\lambda_W$ and $h_j \in \mathcal{M}_c J^\lambda$. Since $p_j \in \ker(\pi^\lambda_c)$ and $\mathcal{M}_c J^\lambda \subset \ker(\pi^\lambda_c)$ by Proposition 5.3 it follows that
\[0 = \pi^\lambda_c p_j = \pi^\lambda_c (\tau_c \psi_j) + \pi^\lambda_c h_j = \pi^\lambda_c (\tau_c \psi_j) = \psi_j.\]
Therefore $p_j = h_j \in \mathcal{M}_c J^\lambda$. By Theorem 4.22 $\pi_c \tau_c \phi_i$ has only signatures $\geq \lambda^*$. Since $\pi_c f_i$ vanishes at $0 \in W$, it follows that $(\pi_c f_i)(\pi_c \tau_c \phi_i)$ has only signatures $> \lambda^*$. The same holds for $\pi_c p_j$ and hence for $(\pi_c q_j)(\pi_c p_j)$ since $p_j \in \ker(\pi^\lambda_c)$. With $\pi^\lambda_c \tau_c \phi_i = \phi_i$ it follows from (5.23) that
\[0 = \pi^\lambda_c f = \sum_{i \in I} \left( (\pi_c f_i)(\pi_c \tau_c \phi_i) \right)^\lambda + \sum_{j \in J} \left( (\pi_c q_j)(\pi_c p_j) \right)^\lambda + \sum_{i \in I} a_i (\pi^\lambda_c \tau_c \phi_i) = \sum_{i \in I} a_i \phi_i.\]
Since $(\phi_i)_{i \in I}$ are linearly independent, we have $a_i = 0$ for all $i$. With (5.23) we obtain $f \in \mathcal{M}_c J^\lambda$ since $\tau_c \phi_i \in J^\lambda$ and $p_j \in \mathcal{M}_c J^\lambda$. □

References

[1] J. Arazy: A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains.
Contemp. Math. 185 (1995), 7-65
[2] M. Atiyah, I. Macdonald: Introduction to Commutative Algebra. Addison-Wesley (1969)
[3] J. Arazy, H. Upmeier: Boundary measures for symmetric domains and integral formulas for the discrete Wallach points. Int. Equ. Op. Th. 47 (2003), 375-434
[4] J. Arazy, G. Zhang: Homogeneous multiplication operators on bounded symmetric domains. J. Funct. Anal. 202 (2003), 44-66
[5] S. Biswas, G. Misra, M. Putinar: Unitary invariants for Hilbert modules of finite rank. J. Reine Angew. Math. 662 (2012), 165-204
[6] Cho-Ho Chu: Bounded Symmetric Domains in Banach Spaces. World Scientific (2021)
[7] C. Cowen, R. Douglas: Complex geometry and operator theory. Acta Math. 141 (1978), 187-261
[8] C. deConcini, D. Eisenbud, C. Procesi: Young diagrams and determinantal varieties. Inv. Math. 56 (1980), 129-165
[9] R. Douglas, G. Misra, C. Varughese: On quotient modules - the case of arbitrary multiplicity, J. Funct. Anal. 174 (2000), 364-398
[10] Y. Duan, K. Guo: Dimension formula for localization of Hilbert modules. J. Operator Theory 62 (2009), 439-452
[11] M. Englis, H. Upmeier: Reproducing kernel functions and asymptotic expansions on Jordan-Kepler varieties, Adv. Math. 347 (2019), 780-826
[12] J. Faraut, A. Korányi, Function spaces and reproducing kernels on bounded symmetric domains. J. Funct. Anal. 88 (1990), 64-89
[13] J. Faraut, A. Korányi, Analysis on Symmetric Cones. Clarendo Press, Oxford (1994)
[14] G. Fischer: Lineare Faserräume und kohärente Modulgarben über komplexen Räumen. Arch. Math. 18 (1967), 609-617
[15] K. Guo: Algebraic reduction for Hardy submodules over polydisk algebras. J. Operator Theory 41 (1999), 127-138
[16] K. Kadell: The Selberg-Jack symmetric functions. Adv. Math. 130 (1997), 33-102
[17] A. Korányi, G. Misra: A classification of homogeneous operators in the Cowen-Douglas class, Adv. Math. 226 (2011), 5338-5360
[18] O. Loos: Bounded Symmetric Domains and Jordan Pairs. Univ. of California, Irvine (1977)
[19] E. Neher: An expansion formula for the norm function of a Jordan algebra. Arch. Math. 69 (1997), 105-111
[20] W. Schmid: Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen. Invent. Math. 9 (1969), 61-80
[21] B. Schwarz: Dissertation Marburg (2010)
[22] R. Stanley: Some combinatorial properties of Jack symmetric functions. Adv. Math. 77 (1989), 76-115
[23] H. Upmeier: Symmetric Banach Manifolds and Jordan $C^*$-Algebras. North Holland (1985)
[24] H. Upmeier: Jordan algebras and harmonic analysis on symmetric spaces. Amer. J. Math. 108 (1986), 1-25
[25] H. Upmeier: Toeplitz operators on bounded symmetric domains. Trans. Amer. Math. Soc. 280 (1983), 221-237
[26] H. Upmeier: Stratified Hilbert modules on bounded symmetric domains. Preprint (2022)

FACHBEREICH MATHEMATIK, UNIVERSITÄT MARBURG, D-35032 MARBURG, GERMANY
Email address: upmeier@mathematik.uni-marburg.de