THE SCHRÖDINGER EQUATION WITH A MOVING POINT INTERACTION IN THREE DIMENSIONS

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(Communicated by David S. Tartakoff)

ABSTRACT. In the case of a single point interaction we improve, by using different methods, the existence theorem for the unitary evolution generated by a Schrödinger operator with moving point interactions obtained by Dell’Antonio, Figari and Teta.

1. Introduction

Let us denote by $L^2(\mathbb{R}^3)$, with the usual scalar product $\langle \cdot, \cdot \rangle_2$ and corresponding norm $\| \cdot \|_2$, the Hilbert space of square integrable measurable functions on $\mathbb{R}^3$. By $H^1(\mathbb{R}^3)$ and by $H^2(\mathbb{R}^3)$ we denote the usual Sobolev-Hilbert spaces

\[ H^1(\mathbb{R}^3) := \{ \psi \in L^2(\mathbb{R}^3) : \nabla \psi \in L^2(\mathbb{R}^3) \}, \]

\[ H^2(\mathbb{R}^3) := \{ \psi \in H^1(\mathbb{R}^3) : \Delta \psi \in L^2(\mathbb{R}^3) \}. \]

Let

\[ H \equiv -\Delta : H^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3) \]

be the self-adjoint operator giving the Hamiltonian of a free quantum particle in $\mathbb{R}^3$. For any $y \in \mathbb{R}^3$ let us consider the symmetric operator $H^0_y$ obtained by restricting $H$ to the subspace

\[ \{ \psi \in H^2(\mathbb{R}^3) : \psi(y) = 0 \}. \]

Such a symmetric operator has defect indices $(1, 1)$. Any of its self-adjoint extensions different from $H$ itself describes a point interaction centered at $y$. One has the following (see [1], section I.1.1 as regards $H^\alpha_y$, and see [9] as regards $F^\alpha_y$).

Theorem 1.1. Any self-adjoint extension of $H^0_y$ different from $H$ itself is given by

\[ H_{\alpha,y} : D(H_{\alpha,y}) \rightarrow L^2(\mathbb{R}^3), \]

\[ D(H_{\alpha,y}) := \{ \psi \in L^2(\mathbb{R}^3) : \psi(x) = \psi_\lambda(x) + \Gamma_\alpha(\lambda)^{-1} \psi_\lambda(y) G_\lambda(x-y), \psi_\lambda \in H^2(\mathbb{R}^3) \}, \]

\[ (H_{\alpha,y} + \lambda)\psi := (H + \lambda)\psi_\lambda, \]

Received by the editors February 3, 2006.

2000 Mathematics Subject Classification. Primary 47B25, 47D08; Secondary 47D06, 81Q10.

Keywords and phrases. Point interactions, singular perturbations, unitary propagators.

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the definition being $\lambda$-independent. Here $\alpha \in \mathbb{R}$,
\[ \Gamma_{\alpha}(\lambda) = \alpha + \frac{\sqrt{\lambda}}{4\pi}, \quad \mathcal{G}_{\lambda}(x) = \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}, \]
and $\lambda > 0$ is chosen in such a way that $\Gamma_{\alpha}(\lambda) \neq 0$. The kernel of the resolvent of $H_{\alpha,y}$ is given by
\[ (H_{\alpha,y} + \lambda)^{-1}(x_1, x_2) = \mathcal{G}_{\lambda}(x_1 - x_2) + \Gamma_{\alpha}(\lambda)^{-1}\mathcal{G}_{\lambda}(x_1 - y)\mathcal{G}_{\lambda}(x_2 - y). \]
The quadratic form associated with $H_{\alpha,y}$ is
\[ F_{\alpha,y} : D(F_y) \times D(F_y) \to \mathbb{R}, \]
\[ \{ \psi \in H^1(\mathbb{R}^3) : \psi(x) = \psi_{\lambda}(x) + q_{\phi}\mathcal{G}_{\lambda}(x - y), \psi_{\lambda} \in H^1(\mathbb{R}^3), q_{\phi} \in \mathbb{C} \}, \]
\[ (F_{\alpha,y} + \lambda)(\psi, \phi) = \langle \nabla \psi_{\lambda}, \nabla \phi_{\lambda} \rangle_2 + \lambda \langle \psi_{\lambda}, \phi_{\lambda} \rangle_2 + \Gamma_{\alpha}(\lambda) q_{\phi} q_{\phi}. \]
Moreover the essential spectrum of $H_{\alpha,y}$ is purely absolutely continuous,
\[ \sigma_{ess}(H_{\alpha,y}) = \sigma_{ac}(H_{\alpha,y}) = [0, \infty), \]
$\alpha < 0 \Rightarrow \sigma_{pp}(H_{\alpha,y}) = -(4\pi\alpha)^2$;
\[ \alpha \geq 0 \Rightarrow \sigma_{pp}(H_{\alpha,y}) = \emptyset. \]
Suppose now that the point $y$ is not fixed but describes a curve $y : \mathbb{R} \to \mathbb{R}^3$, thus producing the family of self-adjoint operators $H_{\alpha,y}(t) = H_{\alpha,y(t)}$. Then one is interested in showing that the time-dependent Hamiltonian $H_{\alpha,y}(t)$ generates a strongly continuous unitary propagator $U_{t,s}$. Note that both the operator and the form domain of $H_{\alpha,y}(t)$ are strongly time dependent. This renders inapplicable the known general theorems (see [3], [6]), and such a generation problem is not trivial.

By exploiting the explicit form of $H_{\alpha,y}(t)$ and in the case of several moving point interactions, Dell’Antonio, Figari and Teta obtained in [2] the following (here we state their results in the simpler case of a single point interaction).

**Theorem 1.2.** Suppose that
\[ y \in C^3(\mathbb{R}; \mathbb{R}^3), \quad \psi \in C^0(\mathbb{R}^3), \quad \psi(y(s)) = 0. \]
Then there exists a unique strongly continuous unitary propagator
\[ U_{t,s} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \]
such that
\[ \left. \left( i \frac{d}{dt} U_{t,s} \psi, \phi \right) \right|_t = F_{\alpha,y}(t)(U_{t,s} \psi, \phi) \]
for all $\phi \in D(F_y(t))$.

Here $(\cdot, \cdot)_t$ denotes the duality between $D(F_y(t))$ and its strong dual. Moreover the solution $\psi(t) := U_{t,s} \psi$ has a natural representation (see [2], (14)-(21) for the details).

In the introduction of [2] the authors conjectured that $U_{t,s}$ defines a flow on $D(F_y(t))$ which is continuous with respect to the Banach topology induced by the quadratic form $F_{\alpha,y}(t)$.

Here we show, by using different methods, that if $y \in C^2(\mathbb{R}; \mathbb{R}^3)$, then this is indeed the case, and Theorem 1.2 above holds for any $\psi \in D(F_y(s))$ (see Theorem 3.1 for the precise statements).
Our proof proceeds in the following conceptually simple way. Noting that the unitary map
\[ T_t \psi(x) := \psi(x + y(t)) \]
transforms the equation
\[ i \frac{d\psi}{dt}(t) = H\psi(t) \]
into the nonautonomous equation
\[ i \frac{d\psi}{dt}(t) = H_\psi(t)\psi(t) \equiv (H + iv(t) \cdot \nabla) \psi(t), \quad \nu(t) \equiv \frac{dy}{dt}(t), \]
we consider the point perturbations (at \( y = 0 \)) of \( H_\psi \), where \( \nu \) is a given, time-independent vector in \( \mathbb{R}^3 \). We realize (see Theorem 2.3) that the form domains \( D(F_v) \) of such singular perturbations \( H_{\psi,\alpha} \) of \( H_\psi \) are \( \nu \)-independent. Indeed one has \( D(F_v) \equiv D(F_0) \), where \( D(F_0) \) is the form domain of \( H_{\alpha,y}, y = 0 \). This allows, in the case the vector \( \nu \) is time-dependent, the application of Kisyński’s theorem (see the Appendix), thus obtaining a strongly continuous unitary propagator \( \tilde{U}_{t,s} \) which is also a strongly continuous propagator on \( D(F_0) \) with respect to the Banach topology induced by the quadratic form associated with the natural extension of \( i v \cdot \nabla \) to \( D(F_0) \) (see Remark 2.4). This allows us to show (see Theorem 3.1) that
\[ F_{\psi,\alpha} = F_{\alpha,0} + Q_\psi, \]
where \( Q_\psi \) is the quadratic form associated with the natural extension of \( iv \cdot \nabla \) to \( D(F_0) \) (see Remark 2.4). This allows us to show (see Theorem 3.1) that
\[ U_{t,s} := T_t^{-1} \tilde{U}_{t,s} T_s \]
satisfies (1.1) for any \( \psi \in D(F_{\alpha,y}(s)) \) and is a continuous flow from \( D(F_{\alpha,y}(s)) \) onto \( D(F_{\alpha,y}(t)) \). In the case \( y \in C^3(\mathbb{R}; \mathbb{R}^3) \) we also show that \( U_{t,s} \) maps \( \tilde{D}(H_{\alpha,y}(s)) \) onto \( \tilde{D}(H_{\alpha,y}(t)) \), where
\[ \tilde{D}(H_{\alpha,y}(t)) := V_t D(H_{\alpha,y}(t)), \quad V_t \psi(x) := e^{iv(t) \cdot x/2} \psi(x). \]

2. The operator \( -\Delta + iL_\psi \) with a point interaction

Let us consider the linear operator \( -\Delta + iL_\psi \), where
\[ L_\psi \psi := \sum_{k=1}^3 v_k \partial_k \psi, \quad \nu \equiv (v_1, v_2, v_3) \in \mathbb{R}^3. \]
Since, for any \( \epsilon > 0 \),
\[ \|L_\psi \psi\|_2^2 \leq \sum_{1 \leq k,j \leq 3} \|v_k v_j (\partial_{k,j}^2 \psi, \psi)\|_2 \leq 3 \|\nu\|^2 \langle \Delta \psi, \psi\rangle_2 \]
\[ \leq \frac{3}{2} \|\nu\|^2 \left( \epsilon \|\Delta \psi\|_2^2 + \frac{1}{\epsilon} \|\psi\|_2^2 \right), \]
by Kato-Rellich’s theorem one has that
\[ H_\psi := -\Delta + iL_\psi : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \]
is self-adjoint. Moreover, since, for any \( \epsilon > 0 \),
\[ \langle L_\psi \psi, \psi\rangle_2 \leq \|\nu\|_2 \|\nabla \psi\|_2 \|\psi\|_2 \leq \frac{\|\nu\|_2}{2} \left( \epsilon \|\nabla \psi\|_2^2 + \frac{1}{\epsilon} \|\psi\|_2^2 \right), \]
$H_v$ has lower bound $-|\nu|^2/4$.

Now we look for the self-adjoint extensions of the symmetric operator $H_v^0$ obtained by restricting $H_v$ to the kernel of the continuous and surjective linear map

$$\tau : H^2(\mathbb{R}^3) \to \mathbb{C}, \quad \tau \psi := \psi(0).$$

**Theorem 2.1.** Any self-adjoint extension of $H_v^0$ different from $H_v$ itself is given by

$$H_{v,\alpha} : D(H_{v,\alpha}) \to L^2(\mathbb{R}^3),$$

$$D(H_{v,\alpha}) := \{ \psi \in L^2(\mathbb{R}^3) : \psi = \psi_\lambda + \Gamma_{v,\alpha}(\lambda)^{-1}\psi_\lambda(0) G_{\lambda}, \psi_\lambda \in H^2(\mathbb{R}^3) \},$$

$$(H_{v,\alpha} + \lambda)\psi := (H_v + \lambda)\psi,$$

the definition being $\lambda$-independent. Here $\alpha \in \mathbb{R}$,

$$\Gamma_{v,\alpha}(\lambda) = \alpha + \frac{\sqrt{\lambda - |\nu|^2/4}}{4\pi}, \quad G_{\lambda}(x) = \frac{e^{-\sqrt{\lambda - |\nu|^2/4}|x|}}{4\pi|x|} e^{i\nu x/2},$$

and $\lambda > |\nu|^2/4$ is chosen in such a way that $\Gamma_{v,\alpha}(\lambda) \neq 0$. The kernel of the resolvent of $H_{v,\alpha}$ is given by

$$(H_{v,\alpha} + \lambda)^{-1}(x_1, x_2) = G_{\lambda}^v \psi(x_1 - x_2) + \Gamma_{v,\alpha}(\lambda)^{-1} G_{\lambda} \psi(x_1) G_{\lambda}^{-v}(x_2).$$

Moreover the spectrum of $H_{v,\alpha}$ is purely absolutely continuous,

$$\sigma_{ess}(H_{v,\alpha}) = \sigma_{ac}(H_{v,\alpha}) = [ -|\nu|^2/4, \infty ),$$

$$\alpha < 0 \Rightarrow \sigma_{pp}(H_{v,\alpha}) = -(4\pi\alpha)^2 - |\nu|^2/4,$$

$$\alpha \geq 0 \Rightarrow \sigma_{pp}(H_{v,\alpha}) = 0.$$

**Proof.** Let us define the bounded linear operators

$$\hat{G}(\lambda) : L^2(\mathbb{R}^3) \to \mathbb{C}, \quad \hat{G}(\lambda) := \tau (H_v + \lambda)^{-1},$$

and

$$G(\lambda) : \mathbb{C} \to L^2(\mathbb{R}^3), \quad G(\lambda) := \hat{G}(\lambda)^*.$$

Since

$$G_{\lambda}^v(x) = \frac{e^{-\sqrt{\lambda - |\nu|^2/4}|x|}}{4\pi|x|} e^{i\nu x/2}$$

is the Green function of $H_v + \lambda$ (see e.g. [5]), one obtains

$$\hat{G}(\lambda)\psi = \langle G_{\lambda}^v, \psi \rangle_2, \quad G(\lambda)q = q G_{\lambda}^v.$$

Since (see [7], Lemma 2.1)

$$(\mu - \lambda)(H_v + \lambda)^{-1}G(\mu) = G(\lambda) - G(\mu),$$

one obtains (see [7], Lemma 2.2)

$$(\mu - \lambda)\hat{G}(\lambda)G(\mu)$$

$$= \tau(G(\lambda) - G(\mu)) = \tau(G(\nu) - G(\mu)) - \tau(G(\nu) - G(\lambda))$$

$$= \frac{\sqrt{\mu - |\nu|^2/4}}{4\pi} - \frac{\sqrt{\nu - |\nu|^2/4}}{4\pi} - \left( \frac{\sqrt{\lambda - |\nu|^2/4}}{4\pi} - \frac{\sqrt{\nu - |\nu|^2/4}}{4\pi} \right)$$

$$= \frac{\sqrt{\mu - |\nu|^2/4}}{4\pi} - \frac{\sqrt{\lambda - |\nu|^2/4}}{4\pi}. $$
The theorem about $H_{v,\alpha}$ and its resolvent then follows from Theorem 2.1 in [7]. As regards the spectral properties of $H_{v,\alpha}$ one proceeds as in [1], Theorem 1.1.4. □

**Remark 2.2.** Note that, as expected, $H_{v,\alpha}$ converges in norm resolvent sense to $H_{\alpha,0}$ as $|\nu| \downarrow 0$.

**Theorem 2.3.** The quadratic form associated with $H_{v,\alpha}$ is

$$F_{v,\alpha} : D(F_0) \times D(F_0) \to \mathbb{R}, \quad F_{v,\alpha} = F_{\alpha,0} + Q_v,$$

where $D(F_0)$ is the domain of the quadratic form $F_{\alpha,0}$ associated with $H_{\alpha,y}$, $y = 0$, (see Theorem 1.1) and

$$Q_v : D(F_0) \times D(F_0) \to \mathbb{R},$$

$Q_v(\psi, \phi) := \langle iL_v\psi_\lambda, \phi_\lambda \rangle_2 + \bar{q}_\psi \langle G_\lambda, iL_v\phi_\lambda \rangle_2 + q_\phi \langle iL_v\psi_\lambda, \Gamma_\lambda \rangle_2.$

**Proof.** Given $\psi$ and $\phi$ in $D(H_{v,\alpha})$ put

$$q_\psi := \Gamma_{v,\alpha}(\lambda)^{-1}\psi(0), \quad q_\phi := \Gamma_{v,\alpha}(\lambda)^{-1}\phi(0).$$

Then

$$\langle (H_{v,\alpha} + \lambda)\psi, \phi \rangle_2 = \langle (H_v + \lambda)\psi_\lambda, \phi_\lambda \rangle_2 + q_\phi \langle (H_v + \lambda)\psi_\lambda, G^*_\lambda \rangle_2$$

$$= \langle (H_v + \lambda)\psi_\lambda, \phi_\lambda \rangle_2 + \Gamma_{v,\alpha}(\lambda) \bar{q}_\psi q_\phi.$$

Thus one is lead to define the quadratic form

$$F_{v,\alpha} : D(F_v) \times D(F_v) \to \mathbb{R}$$

by

$$D(F_v) := \{ \psi \in H^1(\mathbb{R}^3) : \psi = \psi_\lambda + q_\psi \mathcal{G}^*_\lambda, \quad \psi_\lambda \in H^1(\mathbb{R}^3), \ q_\psi \in \mathbb{C} \},$$

$$\langle (F_{v,\alpha} + \lambda)(\psi, \phi) \rangle := \langle (-\Delta + iL_v + \lambda)^{1/2}\psi_\lambda, (-\Delta + iL_v + \lambda)^{1/2}\phi_\lambda \rangle_2 + \Gamma_{v,\alpha}(\lambda) \bar{q}_\psi q_\phi.$$

It is then straightforward to check that $F_{v,\alpha}$ is closed and bounded from below. Thus $F_{v,\alpha}$ is the quadratic form associated with $H_{v,\alpha}$. Since

$$\mathcal{G}^*_\lambda - \mathcal{G}_\lambda \in H^1(\mathbb{R}^3),$$

one obtains $D(F_v) \equiv D(F_0)$. Re-writing the quadratic form above by using the decomposition entering in the definition of $D(F_0)$ and noting that

$$\forall \psi \in H^1(\mathbb{R}^3) \quad (F_v + \lambda)(\mathcal{G}^*_\lambda - \mathcal{G}_\lambda, \psi) = \langle \mathcal{G}_\lambda, iL_v\psi \rangle_2,$$

one obtains

$$\langle (F_{v,\alpha} + \lambda)(\psi, \phi) \rangle = \langle \nabla \psi_\lambda, \nabla \phi_\lambda \rangle_2 + \lambda \langle \psi_\lambda, \phi_\lambda \rangle_2 + Q_v(\psi, \phi)$$

$$+ \langle \Gamma_{v,\alpha}(\lambda) + (F_{v,\alpha} + \lambda)(\mathcal{G}^*_\lambda - \mathcal{G}_\lambda, \mathcal{G}^*_\lambda - \mathcal{G}_\lambda) \rangle \bar{q}_\psi q_\phi.$$

Since $L_v$ is skew-adjoint one has $\langle L_v\psi, \psi \rangle_2 = 0$ for any real-valued $\psi \in H^1(\mathbb{R}^3)$. Thus, by taking a real-valued $J_\epsilon \in C_0^\infty(\mathbb{R}^3)$ such that $J_\epsilon$ weakly converges to the
Dirac mass at the origin as $\epsilon \downarrow 0$, one obtains (here $*$ denotes convolution)
\[
(F_\alpha + \lambda) (G_\Lambda^\prime - G_\Lambda) = (G_\Lambda, iL\psi(G_\Lambda^\prime - G_\Lambda))_2
\]
\[
= \lim_{\epsilon \downarrow 0} \langle iL\psi, \epsilon \psi \rangle_{2} = \lim_{\epsilon \downarrow 0} \langle iL\psi, \epsilon \psi \rangle_{2} = \lim_{\epsilon \downarrow 0} \langle iL\psi, \epsilon \psi \rangle_{2} = \lim_{\epsilon \downarrow 0} \langle G_\Lambda^\prime - G_\Lambda, \epsilon J_\epsilon \rangle_{2}
\]
\[
= \lim_{\epsilon \downarrow 0} \langle G_\Lambda^\prime - G_\Lambda, (H_\psi + \lambda)G_\Lambda \ast J_\epsilon \rangle_{2} = \lim_{\epsilon \downarrow 0} \langle G_\Lambda^\prime - G_\Lambda, J_\epsilon \rangle_{2}
\]
\[
= \frac{\sqrt{\lambda} - |\psi|^2/4}{4\pi} = \Gamma_\alpha(\lambda) - \Gamma_{\psi,\alpha}(\lambda),
\]
and the proof is done. \qed

Remark 2.4. Let $J_\epsilon$ be a real-valued, compactly supported smooth function weakly converging to the Dirac mass a zero as $\epsilon \downarrow 0$. For any $\psi = \psi_\lambda + q_\psi G_\Lambda$ and $\phi = \phi_\lambda + q_\phi G_\Lambda$, let us define $\psi_\epsilon := \psi_\lambda + q_\psi G_\Lambda \ast J_\epsilon$ and $\phi_\epsilon := \phi_\lambda + q_\phi G_\Lambda \ast J_\epsilon$. Then, since $L\nu$ is skew-adjoint, one has
\[
\lim_{\epsilon \downarrow 0} \langle iL\psi_\epsilon, \phi_\epsilon \rangle_{2} = \lim_{\epsilon \downarrow 0} \langle iL\psi_\epsilon, \phi \rangle_{2} + q_\psi \langle G_\Lambda \ast J_\epsilon, iL\phi \rangle_{2}
\]
\[
+ q_\phi \langle iL\psi_\epsilon, G_\Lambda \ast J_\epsilon \rangle_{2} - i\bar{q}_\psi q_\phi \langle L\psi G_\Lambda \ast J_\epsilon, G_\Lambda \ast J_\epsilon \rangle_{2}
\]
\[
= \lim_{\epsilon \downarrow 0} Q_\nu(\psi_\epsilon, \phi_\epsilon) = Q_\nu(\psi, \phi).
\]
Thus $Q_\nu$ is the natural extension to $D(F_0)$ of the quadratic form associated with $iL\nu$.

3. The Schrödinger equation with a moving point interaction

Let us now consider a differentiable curve $y: \mathbb{R} \to \mathbb{R}^3$ and put $v(t) \equiv \frac{dy}{dt}(t)$. Thus one has the families of self-adjoint operators and associated quadratic forms
\[
H_{\alpha,y}(t) : D(H_{\alpha,y}(t)) \to L^2(\mathbb{R}^3),
\]
\[
F_{\alpha,y}(t) : D(F_{\alpha,y}(t)) \to \mathbb{R},
\]
\[
H_{\alpha,y}(t) : D(H_{\alpha,y}(t)) \to L^2(\mathbb{R}^3),
\]
\[
F_{\alpha,y}(t) : D(F_{\alpha,y}(t)) \to \mathbb{R}.
\]

Now we can state our main result.

Theorem 3.1. Let $y \in C^2(\mathbb{R}; \mathbb{R}^3)$. Then there is a unique strongly continuous unitary propagator
\[
U_{t,s} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad (t,s) \in \mathbb{R}^2,
\]
such that
1) $U_{t,s}D(F_{\alpha,y}(s)) = D(F_{\alpha,y}(t))$;
2) each $U_{t,s}$ is strongly continuous as a map from $D(F_{\alpha,y}(s))$ onto $D(F_{\alpha,y}(t))$ with respect to the Banach topologies induced by the bounded from below closed quadratic forms $F_{\alpha,y}(s)$ and $F_{\alpha,y}(t)$ respectively;
3) $\forall \psi \in D(F_{\alpha,y}(s)), \quad t \mapsto F_{\alpha,y}(t)(U_{t,s}^*\psi, U_{t,s}^*\psi)$ is in $C(\mathbb{R}; \mathbb{R})$;
4) $\forall \psi \in D(F_{\alpha,y}(s)), \quad t \mapsto U_{t,s}^*\psi$ is in $C^1(\mathbb{R}; D(F_{\alpha,y}^*(\cdot)))$.  

where $D(F_y(t))^*$ denotes the dual of $D(F_y(t))$ with respect to the $L^2(\mathbb{R}^3)$ scalar product;

5)

$$\forall \psi \in D(F_y(s)), \forall \phi \in D(F_y(t)), \left( i \frac{d}{dt} U_{t,s} \psi, \phi \right)_t = F_{\alpha,y}(t)(U_{t,s} \psi, \phi),$$

where $(\cdot, \cdot)$ denotes the duality between $D(F_y(t))$ and $D(F_y(t))^*$.

If $y \in C^3(\mathbb{R}; \mathbb{R}^3)$, then

6)

$$U_{t,s} \tilde{D}(H_{\alpha,y}(s)) = \tilde{D}(H_{\alpha,y}(t)), $$

where

$$\tilde{D}(H_{\alpha,y}(t)) := V_t D(H_{\alpha,y}(t)), \quad V_t \psi(x) := e^{i v(t) \cdot x/2} \psi(x).$$

Proof. By Theorem 2.3 we have that $y \in C^2(\mathbb{R}; \mathbb{R}^3)$ implies that

$$\forall \psi, \phi \in D(F_0), \quad t \mapsto F_{\psi,\phi}(t)(\psi, \phi) \text{ is in } C^1(\mathbb{R}).$$

Let $T > 0$. By Kisyński’s theorem (see the Appendix) applied to the family of strictly positive self-adjoint operators

$$H_{\psi,\phi}(t) + \lambda, \quad t \in [-T, T], \quad \lambda > (4\pi \min(0, \alpha))^2 + \frac{1}{4} \sup_{t \in [-T, T]} |v(t)|,$$

one knows that $H_{\psi,\phi}(t), t \in [-T, T]$, generates a strongly continuous unitary propagator $\tilde{U}_{t,s}^T, (s, t) \in [-T, T]^2$. By unicity if $T' > T$, then $\tilde{U}_{s,t}^T = \tilde{U}_{t,s}^{T'}$ for any $(s, t) \in [-T, T]^2 \subset [-T', T']^2$. Thus we obtain a unique strongly continuous unitary propagator $\tilde{U}_{t,s}, (s, t) \in \mathbb{R}^2$, generated by the family $H_{\psi,\phi}(t), t \in \mathbb{R}$. Such a propagator is also a strongly continuous propagator on $D(F_0)$ with respect to the Banach topology induced by the bounded from below closed quadratic form $F_{\alpha,0}$.

Considering the unitary map

$$T_t : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3), \quad T_t \psi(x) := \psi(x + y(t)),$$

we then define the strongly continuous unitary propagator

$$U_{t,s} := T_{t}^{-1} \tilde{U}_{t,s} T_s.$$

Since $T_t$ is a bounded operator from $D(F_y(t))$ onto $D(F_0)$, one has that $U_{t,s}$ is a bounded operator from $D(F_y(s))$ onto $D(F_y(t))$ with respect to the Banach topologies induced by the bounded from below closed quadratic forms $F_{\alpha,y}(s)$ and $F_{\alpha,y}(t)$, respectively. Moreover, for all $\psi \in D(F_y(s))$, the map

$$t \mapsto F_{\alpha,y}(t)(U_{t,s} \psi, \psi) = F_{\alpha,0}(\tilde{U}_{t,s} T_s \psi, \tilde{U}_{t,s} T_s \psi)$$

is continuous. Let us now show that, for all $\psi \in D(F_y(s))$ and for all $\phi \in D(F_y(t))$, one has

$$\left( i \frac{d}{dt} U_{t,s} \psi, \phi \right)_t = F_{\alpha,y}(t)(U_{t,s} \psi, \phi) .$$

For any $\psi \in D(F_y(s))$ and $\phi \in D(F_y(t))$ there exist $\tilde{\psi}$ and $\tilde{\phi}$ in $D(F_0)$ such that $T_s^{-1} \tilde{\psi} = \psi$ and $T_t^{-1} \tilde{\phi} = \phi$. Thus equivalently we need to show that

$$\left( i \frac{d}{dt} T_t^{-1} \tilde{U}_{t,s} \tilde{\psi}, T_t^{-1} \tilde{\phi} \right)_t = F_{\alpha,y}(t)(T_t^{-1} \tilde{U}_{t,s} \tilde{\psi}, T_t^{-1} \tilde{\phi}) \equiv F_{\alpha,0}(\tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi}) .$$
Since
\[ \left( i \frac{d}{dt} T_{t}^{-1} \tilde{U}_{t,s} \tilde{\psi}, T_{t}^{-1} \tilde{\phi} \right) = \left( i T_{t} \frac{d}{dt} T_{t}^{-1} \tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi} \right) \]
\[ = \left( i T_{t} \left( \frac{d}{dt} T_{t}^{-1} \right) \tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi} \right) + \left( i \frac{d}{dt} \tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi} \right) \]
\[ = \left( i T_{t} \left( \frac{d}{dt} T_{t}^{-1} \right) \tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi} \right) + F_{v,\alpha}(t)(\tilde{U}_{t,s} \tilde{\psi}, \tilde{\phi}), \]
by Theorem 2.3 we need to show that
\[ \left( i T_{t} \frac{d}{dt} T_{t}^{-1} \tilde{\psi}, \tilde{\phi} \right) = -Q_{v}(\tilde{\psi}, \tilde{\phi}). \]
This is obviously true in the case where either \( \tilde{\psi} \) or \( \tilde{\phi} \) is in \( H^{1}(\mathbb{R}^{3}) \) and, by taking \( J_{\epsilon} \) as in Remark 2.4,
\[ \left( T_{t} \frac{d}{dt} T_{t}^{-1} \tilde{\psi}, \tilde{\phi} \right) = \lim_{\epsilon \downarrow 0} \left( T_{t} \frac{d}{dt} T_{t}^{-1} \tilde{\psi}, \tilde{\phi} \right) = 0. \]
Thus point 5) is proven. Point 6) follows from Kisynski’s theorem again by noting that if \( y \in C^{3}(\mathbb{R}^{3}) \), then \( \tilde{U}_{t,s} \) maps \( D(H_{\nu,\alpha}(t)) \) onto \( D(H_{\nu,\alpha}(s)) \) and that
\[ D(H_{\nu,\alpha}(t)) = T_{t}V_{t}D(H_{\nu,\alpha}(s)). \]

4. APPENDIX: Kisyński’s theorem

For the reader’s convenience in this Appendix we recall Kisynski’s theorem. For the proof we refer to Kisynski’s original paper [4] (see in particular [4], section 8; see also [3], section II.7).

Let us recall that the double family \( U_{t,s}, (t, s) \in [T_{1}, T_{2}]^{2} \), is said to be a strongly continuous propagator on the Hilbert space \( \mathcal{H} \) if each \( U_{t,s} \) is a bounded operator on \( \mathcal{H} \), the map \( (t, s) \mapsto U_{t,s} \) is strongly continuous, \( U_{s,s} = 1 \) and the Chapman-Kolmogorov equation
\[ U_{t,r}U_{r,s} = U_{t,s} \]
holds. Such a propagator is then said to be unitary if each \( U_{t,s} \) is unitary.

**Theorem 4.1.** Let \( A(t), t \in [T_{1}, T_{2}] \), be a family of strictly positive self-adjoint operators on the Hilbert space \( \mathcal{H}(\mathcal{H}, \langle \cdot, \cdot \rangle) \) with time-independent form domain \( \mathcal{H}^{+} \). Suppose that \( \forall \psi, \phi \in \mathcal{H}^{+}, t \mapsto F(t)(\psi, \phi) \) is in \( C^{k}([T_{1}, T_{2}]; \mathbb{R}) \), where \( F(t) \) denotes the quadratic form associated with \( A(t) \).

If \( k = 1 \), then there is a unique strongly continuous unitary propagator
\[ U_{t,s} : \mathcal{H} \rightarrow \mathcal{H}, \quad (s, t) \in [T_{1}, T_{2}]^{2}, \]
such that
1) \( U_{t,s} \mathcal{H}^{+} = \mathcal{H}^{+} \).
2) \( U_{t,s} \) is a strongly continuous propagator on \((\mathcal{H}, \langle \cdot, \cdot \rangle_{+})\), where \( \langle \cdot, \cdot \rangle_{+} \) denotes any of the equivalent scalar products
\[
\langle \psi, \phi \rangle_{t,+} := F(t)\langle \psi, \phi \rangle;
\]
3) \( \forall \psi \in \mathcal{H}, \ t \mapsto U_{t,s}\psi \) is in \( C^{1}([T_{1}, T_{2}]; \mathcal{H}) \), where \((\mathcal{H}, \langle \cdot, \cdot \rangle_{-})\) is the completion of \( \mathcal{H} \) endowed with any of the equivalent scalar products
\[
\langle \psi, \phi \rangle_{t,-} := \langle A(t)^{-1/2} \psi, A(t)^{-1/2} \phi \rangle;
\]
4) \( \forall \psi, \phi \in \mathcal{H}_{+}, \left( i \frac{d}{dt} U_{t,s} \psi, \phi \right) = F(t)\langle U_{t,s}\psi, \phi \rangle, \)
where \( \langle \cdot, \cdot \rangle \) denotes the duality between \( \mathcal{H}_{+} \) and \( \mathcal{H}_{-} \). If \( k = 2 \), then
5) \( U_{t,s} D(A(s)) = D(A(t)) \),
6) \( \forall \psi \in D(A(s)), \ t \mapsto U_{t,s}\psi \) is in \( C^{1}([T_{1}, T_{2}]; \mathcal{H}) \cap C([T_{1}, T_{2}]; D(A(\cdot))) \),
7) \( i \frac{d}{dt} U_{t,s} \psi = A(t)U_{t,s}\psi. \)

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