Nilpotent quandles

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Abstract

A nilpotent quandle is a quandle whose inner automorphism group is nilpotent. Such quandles have been called reductive in previous works, but it turns out that their behaviour is in fact very close to nilpotency for groups. In particular, we show that it is easy to characterise generating sets of such quandles, and that they have the Hopf property. We also show how to construct free nilpotent quandles from free nilpotent groups. We then use the properties of nilpotent quandles to describe a simple presentation of their associated group, and we use this to recover the classification of abelian quandles by Lebed and Mortier [LM21]. We also study reduced quandles, and we show that the reduced fundamental quandle is equivalent, as an invariant of links, to the reduced peripheral system, sharpening a previous result of Hughes [Hug11]. Finally, we give a characterisation of nilpotency in terms of the associated invariants of braids.

Introduction

Quandles are algebraic structures which have been introduced independently in 1982 by Joyce [Joy82a] and Matveev [Mat82]. They introduced the fundamental quandle of a knot, and they both showed that this is a complete invariant of oriented knots, up to ambient homeomorphisms (that is, up to mirror image). The fundamental quandle is in fact a complete invariant of links up to taking mirror images of non-split components.

Quandles (and racks, who are close relatives) have been the subject of a rich literature, studying them from many different points of view, with many different motivations, ranging from the original knot-theoretic context [Eis03], or closely related contexts such as knotted surfaces [CJK+03] or set-theoretic solutions of the Yang-Baxter equation [LV19, JPZD20] to singularity theory [Bri88] or classification of Hopf algebras [AG03]. Their cohomology has also been widely studied, algebraically [Szy19] or via the construction of classifying spaces [FRS07]. The algebraic study of quandles has also flourished [Joy82b, Eis14, EGM16, BN19, BS19]. Needless to say, this brief mention of the literature on racks and quandles cannot pretend to be exhaustive, but only to give some idea of the vastness of the topic. The interested reader can find an introduction to (part of) this vast literature in the survey [Car12].

Nilpotent quandles. The present paper is an attempt at building a theory of nilpotency for quandles which would closely resemble the theory of nilpotent groups. As it turns out, the right candidates for such a theory were already present in the literature: the quandles that we call $n$-nilpotent are the ones called $n$-reductive elsewhere. The latter terminology was coming from universal algebra [PR98]: these quandles are
characterised by some equations, and algebras satisfying them were called \textit{n-reductive} in this context. We feel, however, that these quandle deserve to be called \textit{nilpotent}. Here are some reasons why:

- Eisermann’s quandle coverings \cite{Eis14} are close analogues to central extensions of groups. In the same way as nilpotent groups are obtained as iterated central extensions of the trivial group, nilpotent quandles are obtained as iterated coverings of the trivial quandle (Th. 2.12).
- For nilpotent groups, surjectivity of morphisms and generation by subsets can be tested at the level of abelianisations. A similar statement holds for nilpotent quandles (Prop. 3.5 and Cor. 3.6).
- Finitely generated nilpotent group are Hopfian; so are finitely generated nilpotent quandles (Th. 3.13).
- Nilpotent quandles are exactly the ones whose enveloping group is nilpotent (Cor. 2.5).

Using the last point, we get some answers to \cite[Pb. 3.6]{BN19}, where it was asked if one can classify quandles having a nilpotent enveloping group. In particular, we recover the classification of abelian quandles from \cite{LM21}.

Our notion of nilpotency for quandles is different from the one obtained by Bonatto and Stanovský by specialization of a general notion of nilpotency in categories of universal algebras in \cite{BS19}. Our requirement is stronger than theirs, because of \cite[Lem. 6.2]{BS19}. We show below that is strictly stronger: no non-trivial connected quandle are nilpotent in our sense (Lemma 3.7).

**Main results.** Since we are exploring a notion that has not been studied for itself before (although appearing in many guises in different contexts), the paper contains many new results. Among these, let us first quote the ones that give different characterisations of \(c\)-nilpotent quandles, which we define as the ones whose inner automorphism group is \((c - 1)\)-nipotent. Namely, a quandle \(Q\) is \(c\)-nilpotent if and only if:

- It is \(c\)-reductive (Cor. 2.10).
- The map \(Q \to \ast\) to the one-element quandle is a composition of \(c\) covering maps.
- Colourings of a welded link diagram by elements of \(Q\) are invariant under inserting a welded braids in \(\Gamma_c(wP_n)\), for any number \(n\) of strands (Th. 7.1).

As far as the author knows, only the first of the above equivalences was already known \cite[Th. 4.1]{JPZD20}.

Let us now quote what we consider to be the four main results of our paper:

- The free \(c\)-nilpotent quandle on \(x_1, \ldots, x_n\) is the union of conjugacy classes of the \(x_i\) in the free \(c\)-nilpotent group on \(x_1, \ldots, x_n\) (Prop. 2.18). A similar statement holds for free reduced quandles (Prop. 6.5).
- Nilpotent quandles of finite type satisfy the Hopf property (Th. 3.13).
- Enveloping groups of nilpotent quandles admit a nice presentation (Th. 4.10).
- The reduced fundamental quandle is equivalent to a weak version of the reduced peripheral system of links (Cor. 6.11).

The first one gives a good understanding of free objects. The second one adapts a very important property of nilpotent groups to nilpotent quandles. The third one offers a good calculation tool for computing enveloping groups of our quandles. Finally, the last one, more topological in nature, is a sharper version of the main result of \cite{Hug11}. 
Outline of the paper. After a quick reminder of the group-theoretic notions we use (§0), we recall the basics of quandle theory in §1. Next, in §2, we define nilpotent quandles and we begin their study, giving the first two characterisations quoted above, and studying universal nilpotent quotients and free nilpotent quandles. The next section (§3) is devoted to adapting some group-theoretic result to the world of nilpotent quandles. In particular, we show that nilpotent quandles are generated by representatives of their orbits (Cor. 3.6), before turning to the proof of the Hopf property for finitely generated nilpotent quandles. In §4, we revisit Joyce’s construction of quandles from group-theoretic data, a key tool used many times in the sequel. We use it to find our presentation of enveloping groups of nilpotent quandles, and to give a useful injectivity criterion. We then turn our attention to the particular case of 2-nilpotent quandles (called abelian in [LM21]), whose study is the subject of §5. Finally, the last two sections are more closely related to the theory of knots and links. Namely, §6 is devoted to the the study of reduced quandles, which are the ones used to get link-homotopy invariants of links. The main result of this section is a precise comparison between the universal reduced quotient of the fundamental quandles and the classical reduced peripheral system of links. The last section (§7) gives our characterisation of nilpotent quandles in terms of colourings. Let us finish this outline by mentioning the two appendices: the first one deals with what our notion of nilpotency becomes for racks, while in the second one, a trace argument is used to show the existence of non-tame automorphism of free nilpotent quandles.

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0 Notations and reminders: group theory

Notation 0.1. Let $G$ be a group. If $x, y \in G$, we denote by $[x, y]$ their commutator $x y^{-1} x^{-1}$, and we use the usual exponential notations $x^y = y^{-1} x y$ and $^y x = y x y^{-1}$ for conjugation in $G$. If $A, B \subseteq G$ are subsets of $G$, we denote by $[A, B]$ the subgroup generated by commutators $[a, b]$ with $a \in A$ and $b \in B$.

The lower central series $\Gamma_*(G)$ of a group $G$ is defined by $\Gamma_1(G) := G$ and $\Gamma_{i+1}(G) := [G, \Gamma_i(G)] \subseteq \Gamma_i(G)$. The group $G$ is called nilpotent (resp. residually nilpotent) if we have $\Gamma_{c+1}(G) = \{1\}$ for some $c$ (resp. $\bigcap \Gamma_i(G) = \{1\}$). The minimal such $c$ is its nilpotency class.

Let us mention that the associated graded abelian group

$$\text{gr}(\Gamma_*(G)) := \bigoplus \Gamma_i(G)/\Gamma_{i+1}(G)$$

bears the structure of a graded Lie ring, whose bracket is induced by commutators in $G$. This Lie ring is easily seen to be generated by its degree 1, its degree-one part being $\Gamma_1(G)/\Gamma_2(G) = G^{ab}$ [Dar19b, Prop. 1.9].

Let $F_n$ be the free group on $n$ generators. By a theorem of Magnus, $\text{gr}(F_n) \cong \mathbb{L}_n$ is the free Lie ring on $n$ generators, seen as a graded Lie ring via the usual degree of Lie monomials. Let $F_{n,c} := F_n/\Gamma_{c+1}$ denote the free $c$-nilpotent group. Magnus’ theorem implies that $\text{gr}(F_{n,c}) \cong \mathbb{L}_{n,c}$ is the free $c$-nilpotent Lie ring, which is the quotient of $\mathbb{L}_n$ by its elements of degree at least $c + 1$.

1 General theory of quandles

This section is a brief introduction to quandles and to the general facts we need about them. This is mostly classical material, that can be found for instance in [Joy82a, AG03, Eis14].

Definition 1.1. A quandle is a set $Q$ endowed with a binary operation $\triangleright$ satisfying the following axioms:

1. $\forall x, y, z \in Q, \ x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z),$
2. $\forall x \in Q, \ x \triangleright (-) : Q \to Q$ is a bijection,
3. $\forall x \in Q, \ x \triangleright x = x.$
Quandles and their morphisms form a category, denoted by $\text{Qdl}$. The first axiom means that the maps $x \triangleright (-)$ are endomorphisms of $(Q, \triangleright)$. The second axiom says that they are in fact automorphisms.

Remark 1.2. We prefer this convention to the more usual one, where the law, usually denoted by $\triangleleft$, is distributive on the left side. In that, like for instance the authors of [AG03] or [Szy19], we follow our habit of having groups acting on the left. The difference between the two conventions is only a matter of easy translation.

Example 1.3 (Trivial quandles). Any set $X$ is a quandle, for the law defined by $x \triangleright y = y$ for all $x, y \in X$. Such quandles are called trivial. A morphism between two trivial quandles is just a map between the underlying sets.

Notation 1.4. If $x$ is an element of the quandle $Q$, we denote by $x^{-1} \triangleright (-)$ the inverse of $x \triangleright (-)$. More generally, for any $n \in \mathbb{Z}$, we denote by $x^{n} \triangleright (-)$ the $n$-th iterate of $x \triangleright (-)$, for any $n \in \mathbb{Z}$.

Definition 1.5. Let $x$ be an element of a quandle $Q$. The automorphism $x \triangleright (-)$ of $Q$ is called the behaviour of $x$. Two elements having the same behaviour are said to behave in the same way.

Definition 1.6. Let $G$ be a group. The law $\triangleright$ defined by $g \triangleright h = ghg^{-1}$ turns $G$ into a quandle, called its conjugation quandle $c(G)$.

Convention 1.7. Conjugation is the default quandle structure on a group $G$: it is the one implied when no other structure is specified. For instance, we may speak of morphisms from a quandle $Q$ to a group $G$, meaning quandle morphisms $Q \to c(G)$.

Definition 1.8. Let $Q$ be a quandle. Its enveloping group $G(Q)$ is the group defined by the following presentation:

$$G(Q) = \langle Q \mid \forall x, y \in Q, \ x \triangleright y = xyx^{-1} \rangle.$$ 

The map $\epsilon_Q : G(Q) \to \mathbb{Z}$ induced by $q \mapsto 1$ (for $q \in Q$) is called the augmentation.

Example 1.9. If $Q$ is a trivial quandle, then the relations defining $G(Q)$ are saying exactly that the generators commute with each other. Thus, in this case, $G(Q) = \mathbb{Z}[Q]$ is the free abelian group with basis $Q$. For instance, $G(*) = \mathbb{Z}$, which allows us to interpret the augmentation $\epsilon_Q$ as $G(Q \to *)$.

The canonical map from $Q$ to $G(Q)$ is in fact (by definition of $G(Q)$) a quandle morphism $\eta_Q : Q \to c(G(Q))$, which turns out to be the unit of an adjunction. Indeed, the following proposition is easy to check:

**Proposition 1.10.** The constructions $c(-)$ and $G(-)$ define functors between the category of quandle and the category $\text{Grp}$ of groups, which are adjoint to each other:

$$G : \text{Qdl} \rightleftarrows \text{Grp} : c.$$ 

Remark that $\eta_Q$ does not need to be injective in general. In fact, it is often not injective (see for instance Example 1.21 below). It thus makes sense to give a name to quandles injecting into their enveloping group:
Definition 1.11. A quandle \( Q \) is called injective when \( \eta_Q : Q \to G(Q) \) is injective.

Remark 1.12. Injectivity is weaker than faithfulness, which asks for the action of \( Q \) on itself to be faithful, that is, for the canonical map \( Q \to \text{Inn}(Q) \) to be injective. In our context, faithfulness will be irrelevant, since the only nilpotent quandle acting faithfully on itself consists of one point. On the contrary, injectivity plays an important role in our theory of nilpotent quandles.

Definition 1.13. An action of a quandle \( Q \) on another quandle \( L \) is a morphism from \( Q \) to \( c(\text{Aut}(L)) \). Using the above adjunction, it can also be seen as an action of the group \( G(Q) \) on \( L \) by automorphisms (that is, a morphism from \( G(Q) \) to Aut(\( L \))).

Example 1.14 (Adjoint action). Any quandle \( Q \) acts on itself by \( x \mapsto x \triangle (-) \). Indeed, this defines a morphism from \( Q \) to the conjugation quandle of Aut((\( Q \)), since:
\[
(x \triangle y) \triangle z = (x \triangle y) \triangle (x \triangle (x^{-1} z)) = x \triangle (y \triangle (x^{-1} z)).
\]
This action (or the corresponding action of \( G(Q) \) on \( Q \)) is called the adjoint action.

Lemma 1.15. Let \( f : Q \to Q' \) be a quandle morphism. Then \( f \) is equivariant with respect to adjoint actions. Precisely, if \( f_* : G(Q) \to G(Q') \) is the morphism induced by \( f \), then:
\[
\forall g \in G, \forall x \in Q, \ f(g \cdot x) = f_*(g) \cdot f(x).
\]
Proof. Recall that we denote by \( \eta : Q \to G(Q) \) the canonical map. It is enough to check equivariance with respect to the generators \( g = \eta(y) \) \((y \in Q)\) of \( G(Q) \). By definition of the adjoint action, \( f(\eta(y) \cdot x) = f(y \triangle x) = f(y) \triangle f(x) = f_*(\eta(y)) \cdot f(x) \), whence our claim.

Definition 1.16. The group of inner automorphisms of a quandle \( Q \), denoted by \( \text{Inn}(Q) \), is the subgroup of \( \text{Aut}(Q) \) generated by the automorphisms \( x \triangle (-) \) (for \( x \in Q \)). Equivalently, it is the image of \( G(Q) \) in \( \text{Aut}(Q) \) by the adjoint action.

The adjoint action (or, equivalently, the action by inner automorphisms) induces a decomposition of a quandle \( Q \) into orbits. The corresponding equivalence relation is easily seen to be compatible with the quandle law, allowing us to define the quandle of orbits \( \pi_0(Q) = Q/G(Q) \). It is a trivial quandle, and any morphism from \( Q \) to a trivial quandle factors uniquely through the projection \( Q \to \pi_0(Q) \). For this reason, we call it the trivialisation of \( Q \).

Remark 1.17. The orbits of \( Q \) are often called connected components, and a quandle with one connected component is then called connected. This terminology is motivated by the study of the fondamental quandle of a link: orbits of the fondamental quandle do correspond to the connected components of the link. However, since connected components of a quandle are not themselves connected, we feel that this terminology is somewhat misleading, and must be used with some care. We will mostly avoid it, but we use the notation \( \pi_0 \). In fact, the reader should keep in mind that taking the trivialization of a quandle is very similar to taking the abelianisation of a group, and bears little resemblance to taking the group of connected components of a topological group. In this regard, see Cor. 2.17.
We can refine the decomposition into orbits by taking orbits under the action of a smaller subgroup:

**Lemma 1.18.** Let $Q$ be a quandle and $G$ be a subgroup of $\text{Aut}(Q)$ which is normalized by $\text{Inn}(Q)$. Then the decomposition of $Q$ into $G$-orbits is compatible with the quandle law. The quandle of $G$-orbits will be called the quotient of $Q$ by $G$, denoted by $Q/G$.

**Proof.** Let $x, y \in Q$, and $g, h \in G$. We need to prove that $(gx)\triangleright (hy)$ belongs to the $G$-orbit of $x \triangleright y$. Since $(gx)\triangleright (hy) = g(x\triangleright (g^{-1}hy))$, it is enough to prove that $x \triangleright G y \subseteq G(x \triangleright y)$. By taking $z = x \triangleright y$, this is equivalent to showing that $x \triangleright G(x \triangleright z) \subseteq Gz$. But $x\triangleright (−)$ is an element of $\text{Inn}(Q)$, and $G$ is normalized by $\text{Inn}(Q)$, whence the result, remarking that the same is true if we replace $\triangleright$ by $\triangleright^{-1}$.

The next two lemmas allow us to compare $G(Q)$ with $\text{Inn}(Q)$.

**Lemma 1.19.** [Eis14, Prop. 2.26] For any quandle $Q$, the projection $G(Q) \twoheadrightarrow \text{Inn}(Q)$ is a central extension.

**Proof.** The morphism $G(Q) \twoheadrightarrow \text{Inn}(Q)$ can be composed with $\varphi \mapsto \varphi_*$ from $\text{Aut}(Q)$ to $\text{Aut}(G(Q))$, to get a morphism $c : G(Q) \to \text{Aut}(G(Q))$. It is easy to see (by checking this on the generators of $G(Q)$ corresponding to elements of $Q$) that this morphism $c$ is the usual morphism sending any element $g \in G$ to the associated inner automorphism. Since $\ker(c) = Z(G(C))$, the conclusion follows.

The kernel of $G(Q) \twoheadrightarrow \text{Inn}(Q)$ (which consists of elements $g \in G(Q)$ acting trivially on $Q$) is not equal to the whole center $Z(G(Q))$ in general, but there is one important case when it is:

**Lemma 1.20.** If a quandle $Q$ is injective, then $\text{Inn}(Q) \cong G(Q)/Z$.

**Proof.** If $Q$ injects into $G(Q)$, then the action of $G(Q)$ on $Q$ identifies with the action of $G(Q)$ by conjugation on its subset $Q$. Since $Q$ generates $G(Q)$, an element $g \in G(Q)$ acts trivially on $Q$ if and only if it acts trivially on $G(Q)$, and the latter means that $g \in Z(G(Q))$.

**Example 1.21.** There are numerous examples of quandles with an abelian enveloping group which are not trivial. If $Q$ is such a quandle, then $Z(G(Q)) = G(Q)$, but $\text{Inn}(Q) \not\cong G(Q)/Z = 1$ (else $Q$ would be trivial). Such examples are characterised in §5.3. The simplest one is $Q_{1,2}$, which consists of 3 elements $a, b, c$, with $a\triangleright (−)$ exchanging $b$ and $c$, and $b\triangleright (−) = c\triangleright (−) = id$. Then $G(Q_{1,2})$ is easily seen to be free abelian on $\eta(a)$ and $\eta(b) = \eta(c)$.

**Inner automorphisms and surjections**

The construction $\text{Inn}(−)$ is not functorial, at least not in any way that would turn the canonical maps $c_Q : Q \to \text{Inn}(Q)$ into a natural transformation. indeed, if $q$ is any element of any quandle $Q$, naturality with respect to the inclusion $\{q\} \hookrightarrow Q$ would force $c_Q(q)$ to be trivial, which is impossible. However, this construction is functorial when restricted to the subcategory of surjective morphisms:
Proposition 1.22. [Eis14, Prop. 2.31] Let \( p : Q \to Q' \) be a surjective morphism. Then \( p \) induces a unique morphism \( p_\ast : \text{Inn}(Q) \to \text{Inn}(Q') \) such that the following diagram commutes:

\[
\begin{array}{ccc}
Q & \xrightarrow{c_Q} & \text{Inn}(Q) \\
\downarrow p & & \downarrow p_\ast \\
Q' & \xrightarrow{c_Q} & \text{Inn}(Q').
\end{array}
\]

This construction defines a functor from the category of surjections between quandles to the category of surjections between groups.

Remark 1.23. We use the notation \( p_\ast \) for both the induced morphisms \( G(Q) \to G(Q') \) and \( \text{Inn}(Q) \to \text{Inn}(Q') \). Since the latter is induced by the former (see the proof of the Proposition), this should not be a source of confusion. In particular, Lemma 1.15 works with both of them as \( p_\ast \). Whenever we really need to distinguish between the two, we will call the first one \( G(p) \), as for instance in the following proof.

Proof of Proposition 1.22. Unicity is obvious, since \( c_Q(Q) \) generates \( \text{Inn}(Q) \). To see that \( p_\ast \) exists, we use that \( \text{Inn}(Q) \) is a quotient of \( G(Q) \), which is a functorial construction, so that the morphism we seek must fit into the following commutative diagram:

\[
\begin{array}{ccc}
Q & \xrightarrow{c_Q} & G(Q) \\
\downarrow p & & \downarrow G(p) \\
Q' & \xrightarrow{c_Q} & G(Q') \\
& & \downarrow p_\ast \\
& & \text{Inn}(Q').
\end{array}
\]

Thus we only need to show that \( \pi \circ G(p) \) vanishes on \( \ker(\pi) \). Let \( \gamma \in G(Q) \). Suppose that \( \pi(\gamma) = \text{id}_Q \). This means that \( \gamma \) acts trivially on \( Q \). Let \( z \in Q' \), and \( t \in Q \) such that \( z = p(t) \). Then (using Lemma 1.15) \( G(p)(\gamma) \cdot z = G(p)(\gamma) \cdot p(t) = p(\gamma \cdot t) = p(t) = z \), which means that the action of \( G(p)(\gamma) \) on \( Q' \) is trivial: \( \pi \circ G(p)(\gamma) = \text{id}_{Q'} \), as announced. \( \square \)

**Quandle coverings**

The theory of quandle coverings was developed by Eisermann in [Eis14], where a Galois correspondence was introduced for classifying them. This Galois correspondence, inspired by the Galois theory of covering spaces, and similar in some ways to Kervaire’s theory of coverings of perfect groups, is in fact a particular case of Janelidze’s categorical Galois theory [Eve14]. Coverings are defined as follows:

**Definition 1.24.** A **quandle covering** is a surjective quandle morphism \( p : Q \to R \) such that all the elements of each fiber \( p^{-1}(x) \) behave in the same way.

In other words, a surjective morphism \( p : Q \to R \) is a covering if and only if the adjoint action \( Q \to \text{Inn}(Q) \) factors through \( p \). Using the universal property of the enveloping group, one sees that it is equivalent to the existence of a group morphism making the following diagram commute:

\[
\begin{array}{ccc}
Q & \xrightarrow{c_Q} & G(Q) \\
\downarrow p & & \downarrow p_\ast \\
Q' & \xrightarrow{c_Q} & G(Q') \\
& & \downarrow p_\ast \\
& & \text{Inn}(Q').
\end{array}
\]
The following result will be used later for interpreting nilpotency in terms of coverings:

**Proposition 1.25.** [Eis14, Prop. 2.49] Let \( p : Q \to R \) be a quandle covering. Then both \( p_* : G(Q) \to G(R) \) and \( p_* : \text{Inn}(Q) \to \text{Inn}(R) \) are central extensions.

**Proof.** In the diagram (1.24.1), the \( \pi \) are central extensions (Lem. 1.19). Firstly, \( \pi = qp_* \) implies that \( \ker(p_*) \subseteq \ker(\pi) \) is central, whence our first statement. Secondly, for all \( \varphi \in \text{Inn}(Q) \), there is some \( g \in G(Q') \) such that \( q(g) = \varphi \). If \( p_*(\varphi) = \text{id} \), then \( \pi(g) = \text{id} \), hence \( g \) is central in \( G(Q') \), and \( \varphi \) must be central in \( \text{Inn}(Q) \), since \( q \) is surjective. This proves our second claim. \( \square \)

## 2 Nilpotent quandles

We now introduce our notion of nilpotent quandles, and we begin their study. In particular, we recover the precise link between our definition and the classical notion of reduced quandle, we characterise nilpotent quandles in terms of coverings, and we study universal nilpotent quandles and free nilpotent ones, that we describe in group-theoretic terms.

### 2.1 Definition

**Definition 2.1.** A quandle \( Q \) is called *nilpotent of class at most c*, or *c-nilpotent*, if its group of inner automorphism is \((c - 1)\)-nilpotent.

**Example 2.2.** A quandle \( Q \) is 1-nilpotent when \( \text{Inn}(Q) \) is trivial, that is, when \( q \triangleright (-) \) is the identity for all \( q \in Q \). Thus a quandle is 1-nilpotent if and only if it is trivial.

**Example 2.3.** A quandle \( Q \) is 2-nilpotent when \( \text{Inn}(Q) \) is abelian, that is, when \( x \triangleright (y \triangleright (-)) = y \triangleright (x \triangleright (-)) \) for all \( x, y \in Q \). We call such quandles *abelian*, following [LM21], whose classification we recover in §5.3.

Since \( \text{Inn}(Q) \) is not functorial in \( Q \), it helps to have a characterization of nilpotency in terms of \( G(Q) \):

**Lemma 2.4.** A quandle \( Q \) is \( c \)-nilpotent if and only if \( \Gamma_c(G(Q)) \) acts trivially on \( Q \) via the adjoint action.

**Proof.** The morphism associated to the adjoint action is a surjection \( \pi \) from \( G(Q) \) onto \( \text{Inn}(Q) \). As a consequence, \( \pi(\Gamma_c(G(Q))) = \Gamma_c(\text{Inn}(Q)) \), whence the result. \( \square \)

**Corollary 2.5.** A quandle \( Q \) is nilpotent if and only if \( G(Q) \) is. If \( Q \) is nilpotent of class \( c \), then \( G(Q) \) is nilpotent of class \( c \) or \( c - 1 \). Moreover, if \( Q \) is injective, then \( Q \) and \( G(Q) \) have the same nilpotency class (whenever they are nilpotent).

**Proof.** Combine Lemmas 2.4 and 1.19 to get the first two statements. Lemma 1.20 implies the last one. \( \square \)

The link between the nilpotency class of \( Q \) and that of \( G(Q) \), though strong, is not an obvious one. For instance, as will see later (see §5.3), there are non-trivial quandles with abelian enveloping groups. At this point, the reader may wonder why our notion of nilpotency class should be better than the one using the nilpotency class
of $G(Q)$. The reason is in fact very simple: nilpotency in a group is described by certain identities, and identities in $\text{Inn}(Q)$ translate (by evaluating them at every element of $Q$) into identities in the quandle $Q$. Whereas identities in $G(Q)$ cannot be translated into identities in $Q$, since $Q$ does not in general embed into $G(Q)$. This explains why our notion of nilpotency defines Birkhoff subcategories of the category of quandles, whereas the other notion does not. In particular, we have the following:

**Lemma 2.6.** If $Q$ is a c-nilpotent quandle, then any quotient and any subquandle of $Q$ is c-nilpotent.

This Lemma follows directly from the above remarks, but we give a direct proof in terms of our definition.

**Proof of Lemma 2.6.** If $Q$ surjects onto a quandle $R$, then $\text{Inn}(Q)$ surjects onto $R$ (Prop. 1.22), hence $R$ is c-nilpotent when $Q$ is. For subquandles, we can use Lemma 2.4. Namely, if $R$ is a subquandle of $Q$, then the action of $G(R)$ on $R$ is obtained by restricting the action of $G(Q)$ on $Q$ through the morphism $G(R) \to G(Q)$ induced by the inclusion. This morphism sends $\Gamma_c(G(R))$ into $\Gamma_c(G(Q))$, which acts trivially on $Q$, hence also on $R$. \hfill $\Box$

**Example 2.7.** On the contrary, if $Q$ is a quandle having an abelian enveloping group, there can be subquandles of $Q$ whose enveloping group is not abelian. For instance, let us consider the quandle $Q_{n,n}$ from §5.3 (with $n \neq 1$, possibly $n = 0$). It is the disjoint union $(\mathbb{Z}/n) \sqcup (\mathbb{Z}/n)$ of two orbits, where any element acts trivially on its orbit and acts via the permutation $n \mapsto n + 1$ on the other. Let us embed $Q_{n,n}$ into the quandle $Q$, defined as the disjoint union $Q_{n,n} \cup \{z\}$, where the elements of $Q_{n,n}$ act trivially on $z$, and $z$ acts via $n \mapsto n + 1$ on both orbits of $Q_{n,n}$. One can easily check that $Q$ is a quandle with 3 orbits, generated by representatives $x$, $y$ and $z$ of its orbits (this can also be deduced from our classification results – see Example 5.3). Its enveloping group is generated by $\eta(x)$, $\eta(y)$ and $\eta(z)$. Since $x \triangleright y = y \triangleright z = z$, $\eta(x)$ and $\eta(y)$ commute with $\eta(z)$. Moreover, $x$ and $z$ both act via the same permutation on the orbit of $y$, so $x \triangleright y = z \triangleright y$, which implies that $\eta(x)\eta(y)\eta(x)^{-1} = \eta(z)\eta(y)\eta(z)^{-1} = \eta(y)$. This means that $\eta(x)$ and $\eta(y)$ also commute with each other, hence $G(Q)$ is abelian (necessarily isomorphic to $\mathbb{Z}^3$). However, $G(Q_{n,n})$ is not (§5.3).

**Characterisation by identities**

The following result displays two explicit sets of generating identities for c-nilpotency of a quandle. The first one will be used in the proof of Theorem 7.1 below. The second one can also be found in [JPZD20, Th. 4.1].

**Proposition 2.8.** For any quandle $Q$, $\Gamma_c(G(Q))$ is generated by the either one of the two families of elements, both indexed by $(q_i) \in Q^c$:

- The $[\eta(q_1), [\eta(q_2), \ldots, [\eta(q_{n-1}), \eta(q_c)] \cdots]]$,
- $\eta((q_1 \triangleright q_2) \triangleright q_3) \cdots \triangleright q_c) \cdot \eta(((q_2 \triangleright q_3) \triangleright \cdots) \triangleright q_c)^{-1}$,

where, by convention, both the above expressions denote only $\eta(q_1)$ for $c = 1$.

**Proof.** Recall that if $g_1, \ldots, g_n$ are elements of a group $G$, we denote by $[g_1, \ldots, g_n]$ their iterated commutator $[g_1, [g_2, \ldots, [g_{n-1}, g_n] \cdots]]$ (which is only $x_1$ when $n = 1$, by convention). We also introduce the notation $w_n(q_1, \ldots, q_n) = (\cdots (q_1 \triangleright q_2) \triangleright q_3) \cdots) \triangleright q_n$, for elements $q_1, \ldots, q_n$ of a quandle (which is also only $q_1$ when $n = 1$, by convention).
We show both statements by induction on \( c \geq 1 \). They are clearly true for \( c = 1 \). Let \( c \geq 2 \), and let us suppose that they hold for \( c - 1 \). Let us first recall that \( g\eta(q)g^{-1} = \eta(g \cdot q) \) (for all \( g \in G(Q) \) and all \( q \in Q^c \)) and that \( G(Q) \) acts on \( Q^c \) by automorphisms. From this, we deduce that both our candidates for generating sets of \( \Gamma_c(G(Q)) \) are stable under conjugation by elements of \( G(Q) \). As a consequence, it is enough to show that \( \Gamma_c(G(Q)) \) is normally generated by each family. Equivalently, we need to show that the above elements belong to \( \Gamma_c \), and that \( \Gamma_c = 1 \) in the quotient of \( G(Q) \) by the relations \([\eta(q_1), \ldots, \eta(q_c)] = 1 \) (resp. \( \eta(w_n(q_1, \ldots, q_n)) = \eta(w_n-1(q_2, \ldots, q_n)) \), for \((q_i)_i \in Q^c \).

The first formulas obviously define elements of \( \Gamma_c(G(Q)) \). Then, the relations \([\eta(q_1), \ldots, \eta(q_c)] = 1 \) say that generators of the group (the \( \eta(q_i) \)) must commute with the elements \([\eta(q_2), \ldots, \eta(q_c)] \) which are, by the induction hypothesis, generators of \( \Gamma_{c-1} \). This means that modulo these relations, \( \Gamma_{c-1} \) is central, hence \( \Gamma_c = 1 \), proving our first claim.

Now, remark that since \( \eta \) is a quandle morphism, we have \( \eta(w_n(q_1, \ldots, q_n)) = w_n(\eta(q_1), \ldots, \eta(q_c)) \) (where the quandle law on the right is conjugation in \( G(Q) \)). Thus, the second formula can be written:

\[
(\eta(w_{c-1}(q_1, \ldots, q_{c-1})) \triangleright \eta(q_c)) \cdot (\eta(w_{c-2}(q_2, \ldots, q_{c-1}))) \triangleright \eta(q_c))^{-1}.
\]

By induction, \( \eta(w_{c-1}(q_1, \ldots, q_{c-1})) \cdot \eta(w_{c-2}(q_2, \ldots, q_{c-1}))^{-1} \) is an element of \( \Gamma_{c-1} \). Applying the formula \((u \triangleright x)(v \triangleright x)^{-1} = v^{-1}u, x)\), we deduce that (2.8.1) define elements of \( \Gamma_c \), as expected. Moreover, killing these elements is equivalent to imposing the relations:

\[
\eta(w_{c-1}(q_1, \ldots, q_{c-1})) \triangleright \eta(q_c) = \eta(w_{c-2}(q_2, \ldots, q_{c-1})) \triangleright \eta(q_c),
\]

which mean that generators of the group (the \( \eta(q_c) \)) must commute with the elements \( \eta(w_{c-1}(q_1, \ldots, q_{c-1})) \cdot \eta(w_{c-2}(q_2, \ldots, q_{c-1}))^{-1} \). The latter are, by the induction hypothesis, generators of \( \Gamma_{c-1} \). This means that modulo these relations, \( \Gamma_{c-1} \) is central, hence \( \Gamma_c = 1 \), proving our second claim.

**Remark 2.9.** The fact that \( \eta(Q) \) is stable by conjugation is crucial here. Indeed, if \( S \) is a set of generators of a group, it is not even clear that the commutators of the form \([s_1, \ldots, s_c] \) should normally generate \( \Gamma_c(G) \) is \( G \), let alone generate it.

In particular, we recover the classical consequence:

**Corollary 2.10.** A quandle \( Q \) is \( c \)-nilpotent if and only if it is \( c \)-reductive, that is, if and only if it satisfies the following identities:

\[
\forall q_1, \ldots, q_{c+1} \in Q, \quad ((q_1 \triangleright q_2) \triangleright q_3) \cdot q_{c+1} = ((q_1 \triangleright q_2) \triangleright q_3) \cdot q_{c+1}.
\]

**Proof.** From the formula:

\[
\eta((q_1 \triangleright q_2) \triangleright q_3) \cdot q_{c+1} = ((q_1 \triangleright q_2) \triangleright q_3) \cdot q_{c+1},
\]

we deduce that the second family of Proposition 2.8 acts trivially on \( Q \) if and only if \( Q \) is \( c \)-reductive. Then the conclusion follows from Proposition 2.8 and Lemma 2.4. \( \square \)
Characterisation in terms of coverings

The next Lemma will enable us to give a characterisation of nilpotency in terms of Eisermann’s quandle coverings.

**Lemma 2.11.** In any quandle \( Q \), two elements in the same orbit under \( Z(\text{Inn}(Q)) \) behave in the same way.

**Proof.** Let \( \gamma \in Z(\text{Inn}(Q)) \), and \( x \in Q \). Then, since \( \gamma \) is an automorphism of \( Q \) which commutes with \( x \triangleright (-) \):

\[
\forall y \in Q, \quad \gamma(x) \triangleright y = \gamma(x \triangleright \gamma^{-1}(y)) = x \triangleright y.
\]

This means exactly that \( x \) and \( \gamma(x) \) have the same behaviour. \( \Box \)

**Theorem 2.12.** A quandle \( Q \) is nilpotent of class less than \( c \) if and only if there exist a chain of quandle coverings:

\[
Q \xrightarrow{p_c} \cdots \xrightarrow{p_2} \xrightarrow{p_1} *.
\]

**Proof.** As a consequence of Lemma 2.11, we see that for any quandle \( Q \) and for any subgroup \( G \) of \( Z(\text{Inn}(Q)) \), the quotient map \( Q \to Q/G \) (defined using Lemma 1.18) is a covering. In particular, when \( Q \) is \((c + 1)\)-nilpotent, the projection \( Q \to Q/\Gamma_c \) is a covering onto a \( c \)-nilpotent quandle (see §2.2 below), which proves one implication. Conversely, if \( Q \to Q' \) is a covering onto a \( c \)-nilpotent quandle, then it induces a projection \( \text{Inn}(Q) \to \text{Inn}(Q') \), which is a central extension by Proposition 1.25, thus \( Q \) is \((c + 1)\)-nilpotent. \( \Box \)

**Example 2.13.** A quandle \( Q \) is 2-nilpotent if and only if it is a covering of a trivial quandle. In fact, it is 2-nilpotent if and only if the canonical projection \( Q \to \pi_0(Q) \) is a covering (see §5).

### 2.2 Universal nilpotent quotients

Let \( Q \) be a quandle. We can turn it into a \( c \)-nilpotent quandle by taking its quotient by \( \Gamma_c(\text{Inn}Q) \) (see Lemma 1.18), which we denote, for short, by \( Q/\Gamma_c \). We now show that this is in fact the universal \( c \)-nilpotent quotient of \( Q \), in the same way as \( G/\Gamma_{c+1}G \) is the universal \( c \)-nilpotent quotient of a group \( G \).

**Proposition 2.14.** The full subactegory \( \mathcal{Q}dl_c \subset \mathcal{Q}dl \) of \( c \)-nilpotent quandles is a reflexive subcategory, and the left adjoint to its inclusion into \( \mathcal{Q}dl \) is the functor \((-)/\Gamma_c \) defined above.

**Proof.** Let \( Q \) be any quandle. The quotient map \( p : Q \to Q/\Gamma_c \) induces a surjection from \( G(Q) \) onto \( G(Q/\Gamma_c) \), and thus from \( \Gamma_c(G(Q)) \) onto \( \Gamma_c(G(Q/\Gamma_c)) \). Equivariance of \( p \) with respect to the adjoint actions (Lemma 1.15), applied to \( g \in \Gamma_c(G(Q)) \) and \( x \in Q \), gives:

\[
p_*(g) \cdot p(x) = p(g \cdot x) = p(x),
\]

by definition of \( p \). As \( x \) and \( g \) vary, \( p(x) \) and \( p_*(g) \) describe \( Q/\Gamma_c \) and \( \Gamma_c(G(Q/\Gamma_c)) \) respectively, so we deduce that the action of the latter on the former is trivial, whence the \( c \)-nilpotency of \( Q/\Gamma_c \) by Lemma 2.4.

Now, let \( f : Q \to Q' \) be a quandle morphism whose target is \( c \)-nilpotent. The subgroup \( f_*(\Gamma_c(G(Q))) \subseteq \Gamma_c(G(Q')) \) acts trivially on \( Q' \). As a consequence, \( f \) factorizes uniquely (as a set map, which must be a quandle morphism) through \( Q/\Gamma_c \). \( \Box \)
We remarked above that the enveloping group functor \( G(-) \) sends \( c \)-nilpotent quandles to \( c \)-nilpotent groups, because of Lemma 1.19. The converse is true for the functor \( c(-) \):

**Lemma 2.15.** A group \( G \) is \( c \)-nilpotent if and only if its conjugation quandle \( c(G) \) is.

**Proof.** The group \( \text{Inn}(c(G)) \) identifies with \( \text{Inn}(G) \), which is isomorphic to \( G/\mathbb{Z}(G) \). Thus, it is \((c - 1)\)-nilpotent if and only if \( G \) is \( c \)-nilpotent. \( \square \)

As a consequence, the adjunction between \( c(-) \) and \( G(-) \) (Prop. 1.10) restricts to an adjunction between the full subcategory \( Qdl_c \subset Qdl \) and the full subcategory \( Grp_c \subset Grp \) of \( c \)-nilpotent groups. This allows us to write the following square of adjunctions:

\[
\begin{array}{c}
Qdl_c & \xrightarrow{c} & Grp_c \\
\downarrow & & \downarrow \\
(-)/\Gamma_c & \xrightarrow{c} & (-)/\Gamma_{c+1}
\end{array}
\]

The square of right adjoints obviously commutes, so the square of left adjoints does too. Hence we have proved:

**Proposition 2.16.** Let \( Q \) be any quandle. Then \( G(Q/\Gamma_c) = G(Q)/\Gamma_{c+1} \).

Recall that 1-nilpotent quandles are trivial ones (Ex. 2.2), hence \( Q/\Gamma_1 \) coincides with the trivialization functor \( \pi_0 \). Thus, using the calculation of the enveloping group of a trivial quandle (Ex. 1.9), we see that for \( c = 1 \), Proposition 2.16 reduces to the following easy classical result:

**Corollary 2.17.** Let \( Q \) be any quandle. Then \( G(Q)^{ab} = \mathbb{Z}[\pi_0 Q] \).

In particular, the abelianisation of \( G(Q) \) is always a free abelian group.

### 2.3 Free nilpotent quandles

Let \( X \) be a set, and \( F[X] \) be the free group on \( X \). Recall that the free nilpotent quandle \( Q[X] \) on \( X \) is the subquandle of the conjugation quandle of \( F[X] \) given by the (disjoint) union of the conjugacy classes of the elements of the basis \( X \). It is actually not difficult to check that this quandle is indeed free on the set \( X \): if \( f : X \to Q \) is a map from \( X \) to some quandle \( Q \), then \( \eta_Q \circ f : X \to G(Q) \) extends uniquely to a group morphism \( f_\# : F[X] \to G(Q) \). Then one checks easily that the unique extension \( \tilde{f} \) of \( f \) to a quandle morphism from \( Q[X] \) to \( Q \) is defined by:

\[
\forall x \in X, \forall w \in F[X], \tilde{f}(w^x) = f_\#(w) \cdot f(x).
\]

We now show that the analogous construction works for nilpotent quandles:

**Proposition 2.18.** The free \( c \)-nilpotent quandle on a set \( X \) is the subquandle of the conjugation quandle of the free \( c \)-nilpotent group \( F[X]/\Gamma_{c+1} \) given by the (disjoint) union of the conjugacy classes of the elements of the basis \( X \).
Proof. By composing adjunctions, we know that the free $c$-nilpotent quandle on a set $X$ is given by $Q[X]/\Gamma_c$ (see Prop. 2.14). Because of Proposition 2.16, we also know that its enveloping group is the free $c$-nilpotent quotient of $G(Q[X])$. But $G(Q[X]) \cong F[X]$ (by composing adjunctions), so that $G(Q[X]/\Gamma_c)$ is the free nilpotent group $F[X]/\Gamma_{c+1}$.

We then deduce from the description of $Q[X]$ recalled above that the image of $\eta : Q[X]/\Gamma_c \rightarrow F[X]/\Gamma_{c+1}$ is exactly the quandle described in the proposition. In order to prove our claim, we need to prove that this map $\eta$ is injective.

Consider the commutative diagram of canonical maps:

$$
\begin{array}{ccc}
Q[X] & \xrightarrow{p} & F[X] \\
\downarrow & & \downarrow \\
Q[X]/\Gamma_c & \xrightarrow{\eta} & F[X]/\Gamma_{c+1}.
\end{array}
$$

Consider two elements of $Q[X]/\Gamma_c$. We can write them as $p(vx)$ and $p(wy)$, where $v, w \in F[X]$ and $x, y \in X$, so that $vx$ and $wy$ are two elements of $Q[X]$. Suppose that their images by $\eta$ are the same, that is, that $q(vx) = q(wy)$. First, we see that $x = y$, by projecting $q(vx)$ and $q(wy)$ into $(F[X]/\Gamma_{c+1})^{ab} = \mathbb{Z}[X]$. Then we can write:

$$
q(vx) = q(wy) \iff vx \equiv wy \pmod{\Gamma_{c+1}F[X]} \\
\iff w^{-1}vx \equiv x \pmod{\Gamma_{c+1}F[X]} \\
\iff [w^{-1}v, x] \in \Gamma_{c+1}F[X].
$$

By applying [Dar19b, Lem. 6.3], we deduce that there exists and integer $m$ such that $w^{-1}vx^m \in \Gamma_cF[X]$. Since we can replace $v$ by $vx^m$ without changing the element $vx$, we can assume that $w^{-1}v$ is in $\Gamma_cF[X]$. Then $\gamma := vw^{-1} = w(w^{-1}v)w^{-1} \in \Gamma_cF[X]$ satisfies $\gamma(vx) = vx$. Hence $vx$ and $wx$ are in the same orbit modulo $\Gamma_cF[X]$, which means exactly that $p(vx) = p(wx)$. Thus we have proved that $\eta$ is injective, whence the result. \hfill \square

### 2.4 Residually nilpotent quandles

Recall that a group $G$ is called residually nilpotent if any non-trivial element $x \in G$ has a non-trivial class in some nilpotent quotient of $G$. In particular, nilpotent groups are residually nilpotent. By analogy, we make the following definition:

**Definition 2.19.** A quandle $Q$ is called residually nilpotent when, for all $x, y \in Q$ satisfying $x \neq y$, there exists a morphism $f : Q \rightarrow R$ to a nilpotent quandle $R$ such that $f(x) \neq f(y)$.

Remark that it is equivalent to asking that the classes of $x$ and $y$ are different in $Q/\Gamma_c$ for some $c \geq 1$ (the morphism $f$ of the definition factors through $Q/\Gamma_c$ if $c$ is the nilpotency class of $R$).

**Proposition 2.20.** If $Q$ is a residually nilpotent quandle, then $\text{Inn}(Q)$ is a residually nilpotent group.

**Proof.** Suppose that $Q$ is residually nilpotent, and let $\varphi \in \text{Inn}(Q) \setminus \{id\}$. There is some $x \in Q$ such that $\varphi(x) \neq x$. Since $Q$ is residually nilpotent, there is some $c \geq 1$ such that $\varphi(x) \neq x [\Gamma_c(\text{Inn}(Q))]$. This implies that $\varphi \notin \Gamma_c(\text{Inn}(Q))$, whence our claim. \hfill \square
There is a universal nilpotent quotient of a quandle $Q$, which is the quotient $Q/\sim$ by the equivalence relation:

$$x \sim y \iff \forall c \geq 1, \ x \equiv y \ [\Gamma_c].$$

This relation is coarser than congruence modulo $\Gamma_\infty(\text{Inn}(Q))$, possibly strictly so. Otherwise said, it is not clear whether the converse of Proposition 2.20 should hold in general.

Remark 2.21. The relation to the residual nilpotency of the enveloping group is also not clear, despite Lemma 1.19.

For injective quandles, the situation is clearer, because of:

Lemma 2.22. Let $G$ be a residually nilpotent group. Then $c(G)$ is a residually nilpotent quandle.

Proof. Let $x, y \in G$ such that $x \neq y$. Since $G$ is residually nilpotent, there is a group morphism $f : G \to N$ such that $f(x) \neq f(y)$ and $N$ is nilpotent. By Lemma 2.15, $c(f) : c(G) \to c(N)$ is a morphism to a nilpotent quandle, whence the conclusion.

In particular, since a subquandle of a residually nilpotent quandle is clearly residually nilpotent, we get:

Proposition 2.23. If $Q$ is injective and $G(Q)$ is residually nilpotent, then $Q$ is too.

Example 2.24. Free quandles are residually nilpotent, since free groups are, and they are injective.

Example 2.25. Reduced free quandles (see §6.2) are residually nilpotent, since reduced free groups are [Dar19a, Prop. 1.3], and they are injective (Prop. 6.5). We can also deduce this directly from the fact that $RQ[X]$ is $|X|$-nilpotent if $X$ is finite (Cor. 6.6), by adapting the proof for groups. Remark that if $X$ is infinite, $RQ[X]$ cannot be nilpotent, since it surjects onto $RQ[X']$ for every finite subset $X'$ of $X$.

3 Nilpotency for groups and quandles

Nilpotent groups enjoy very nice properties, and we now investigate analogues of these properties for nilpotent quandles.

3.1 Basic properties of nilpotent quandles

Some group theory

Let us recall a few classical elementary results from the theory of nilpotent groups:

Proposition 3.1. Let $G$ be a group and let $N$ be a nilpotent group. A morphism $f : G \to N$ is surjective if and only if the induced morphism $\overline{f} : G^{ab} \to N^{ab}$ is.

Corollary 3.2. Let $S \subseteq G$ be a subset of a nilpotent group $G$. Then $S$ generates $G$ if and only if its image in $G^{ab}$ generates $G^{ab}$.

Proof. Apply Prop. 3.1 to the morphism $F[S] \to G$ induced by the inclusion $S \subseteq G$ (where $F[S]$ is the free group on $S$).
Corollary 3.3. Let $G$ be a nilpotent group whose abelianisation is cyclic. Then $G$ is cyclic.

Proof. Consider $t \in G$ such that its class $\bar{t}$ generates $G^{ab}$, and apply Cor. 3.2 to $S = \{t\}$. 

Proof of Prop. 3.1. Suppose that the induced morphism $\bar{f} : G^{ab} \to N^{ab}$ is surjective. This morphism is the degree-one part of the morphism $\text{gr}(f) : \text{gr}(\Gamma_* G) \to \text{gr}(\Gamma_* N)$ between the graded Lie rings associated to the lower central series. These Lie rings are generated in degree one \cite[Prop. 1.9]{Dar19b} (see also §0), thus $\text{gr}(f)$ is surjective. Then, by applying the Five Lemma to the diagram:

$$
\begin{array}{cccc}
\text{gr}_i(\Gamma_* G) & \longrightarrow & G/\Gamma_{i+1} & \longrightarrow & G/\Gamma_i \\
\downarrow \text{gr}_i(f) & & \downarrow f & & \downarrow f \\
\text{gr}_i(\Gamma_* N) & \longrightarrow & N/\Gamma_{i+1} & \longrightarrow & N/\Gamma_i,
\end{array}
$$

we deduce, by induction, that the induced morphism $\bar{f} : G/\Gamma_i \to N/\Gamma_i$ is surjective. Since there exists some $i$ satisfying $N = N/\Gamma_i$, we get that $f : G \to G/\Gamma_i \cong N/\Gamma_i = N$ is surjective. 

Some quandle theory

We now show that the analogues of the above group-theoretic results hold for quandles. In fact, the following result will allow us to deduce quandle-theoretic properties from group-theoretic ones:

Proposition 3.4. A quandle morphism $f : Q \to R$ is surjective if and only if the induced group morphism $f_* : G(Q) \to G(R)$ is.

Proof. The “only if” part follows readily from the fact that the image of $R$ generates $G(R)$. Let us show the converse. The identification $G(Q)^{ab} \cong \mathbb{Z}[\pi_0 Q]$ from Corollary 2.17 is natural in $Q$. In particular, the map induced by $f_*$ on the abelianisations identifies with $\mathbb{Z}[\pi_0(f)]$. Thus, if $f_*$ is surjective, $\pi_0(f)$ must be too. This means that for each $y \in R$, there exists an element $x'$ of $Q$ such that $f(x')$ is in the orbit of $y$. Let us write $y = h \cdot f(x')$ for some $h \in G(R)$. Since $f_*$ is surjective, $h$ must be equal to $f_*(g)$, for some $g \in G(Q)$. Then, using Lemma 1.15, $y = f_*(g) \cdot f(x') = f(g \cdot y) \in f(Q)$, and our claim is proved. 

Proposition 3.5. Let $Q$ and $Q'$ be quandles, such that $Q'$ is nilpotent. A morphism $f : Q \to Q'$ is surjective if and only if $\pi_0(f)$ is.

Corollary 3.6. Let $S \subseteq Q$ be a subset of a nilpotent quandle $Q$. Then $S$ generates $Q$ if and only if there is one element of $S$ in each orbit of $Q$.

Proof. Apply Prop. 3.5 to the morphism $Q[S] \to Q$ induced by the inclusion $S \subseteq Q$ (where $Q[S]$ is the free quandle on $Q$). 

Corollary 3.7. Let $Q$ be connected (that is, $|\pi_0(Q)| = 1$) and nilpotent. Then $Q \cong *$.

Proof. Apply Cor. 3.6 to $S = \{q\}$ for any $q \in Q$. 

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**Remark 3.8.** Let $Q$ be a quandle whose enveloping group is cyclic. Then (using Lemma 2.5), we can apply Cor. 3.7 to show that $Q \cong \ast$. The reader can convince himself that this is not a completely obvious result. In fact, we will see later (§5.3) that quandles with abelian enveloping groups are not necessarily trivial.

**Proof of Prop. 3.5.** Suppose that the induced morphism $\pi_0(f) : \pi_0(Q) \to \pi_0(Q')$ is surjective. We have $G(-)^{ab} \cong \mathbb{Z}[\pi_0(-)]$ (Cor. 2.17), thus the induced morphism from $G(Q)^{ab}$ to $G(Q')^{ab}$ is surjective. Then, using the fact that $G(Q')$ is nilpotent (Lem. 1.19), we can apply Proposition 3.1 to deduce that $f_* : G(Q) \to G(Q')$ is surjective. Then Proposition 3.4 gives the desired conclusion. 

### 3.2 Decomposition as an extension

One very important tool in the study of nilpotent group, that allows us to reason by induction on the nilpotency class, is the decomposition of a $c$-nilpotent group $G$ into a central extension:

$$\Gamma_c(G) \hookrightarrow G \twoheadrightarrow G/\Gamma_c(G).$$

There is a similar decomposition of a nilpotent quandle, that will play the same role. It is a decomposition into an extension of a quandle, in the following sense (which is slightly different from Eisermann’s [Eis14, Def. 4.14] – see Remark 3.10):

**Definition 3.9.** Let $R$ be a quandle and $\Lambda$ be an abelian group. A $\Lambda$-extension (or a $\Lambda$-covering) of $R$ consists in a covering $p : Q \to R$, together with a group action $\Lambda \circ \Lambda$ by quandle automorphisms, such that for each $r \in R$, the action of $\Lambda$ preserves the fibre $p^{-1}(r)$ and acts transitively on it. We call $\Lambda$ a structure group of the extension. If moreover $\Lambda$ acts freely on the fibers, then the extension is called principal (or a principal $\Lambda$-covering). Since $R$ can be recovered as the quotient $Q/\Lambda$, we will sometimes simply say that $Q$ (endowed with some action of $\Lambda$) is a $\Lambda$-extension.

The fact that $\Lambda$ preserves the fibers of the covering $p$ implies that for all $\lambda \in \Lambda$ and $x \in Q$, the elements $\lambda \cdot x$ and $x$ (which have the same image by $p$) behave in the same way. As a consequence, the condition saying that the action is by quandle morphisms reads:

$$\forall \lambda \in \Lambda, \forall x, y \in Q, \; (\lambda \cdot x \triangleright y) = (\lambda \cdot x) \triangleright (\lambda \cdot y) = x \triangleright (\lambda \cdot y). \quad (3.9.1)$$

**Remark 3.10.** Since $\Lambda$ is assumed to act transitively on the fibers, we can, instead of asking $p$ to be a covering, ask all the elements in a given $\Lambda$-orbit behave in the same way. This, together with the above, allows us to reformulate our definition in terms closer to Eisermann’s [Eis14, Def. 4.14]: a $\Lambda$-extension of $R$ (in our sense) is a surjective morphism $p : Q \to R$, together with an action of the group $\Lambda$ on the set $Q$ such that:

- For all $x, y \in Q$ and for all $\lambda \in \Lambda$, $(\lambda \cdot x) \triangleright y = x \triangleright y$ and $x \triangleright (\lambda \cdot y) = \lambda \cdot (x \triangleright y)$;
- The action preserves the fibers of $p$ and is transitive on each one.

Thus, our definition is weaker than his, which corresponds exactly to our principal extensions. However, as will become apparent after the discussion below, if $R$ is connected, every $\Lambda$-extension (in our sense) which is itself connected is a principal extension of $R$ by some quotient of $\Lambda$.

The condition (3.9.1) says that the action of $\Lambda$ commutes with the $x \triangleright (-)$, that is, with the action of $\text{Inn}(Q)$. In other words, the action of $\text{Inn}(Q)$ on $Q$ is $\Lambda$-equivariant: for all $g \in \text{Inn}(Q)$, for all $\lambda \in \Lambda$, and all $q \in Q$, we have $g \cdot (\lambda \cdot q) = \lambda \cdot (g \cdot q)$. As a consequence:
Fact 3.11. Let $Q$ be a $\Lambda$-extension. If $q$ and $q'$ are in the same orbit of $Q$, then $\Lambda \cdot q \cong \Lambda \cdot q'$ as $\Lambda$-sets, an isomorphism being induced by any $g \in \text{Inn}(Q)$ sending $q$ to $q'$.

Recall that for an abelian group $\Lambda$, isomorphism classes of transitive $\Lambda$-sets correspond bijectively to subgroups of $\Lambda$. Precisely, on the one hand, we associate to each transitive $\Lambda$-set $X$ the stabilizer $H$ of any point $x \in X$. The fact that $H$ does not depend on the choice of $x$ comes from the commutativity of $\Lambda$, since stabilizers of two points of $X$ are conjugate. On the other hand, we associate to each subgroup $H$ of $\Lambda$ the $\Lambda$-set $\Lambda/H$. These constructions are inverse to each other, an isomorphism $X \cong \Lambda/\text{Stab}(x)$ been given by $\overline{g} \mapsto g \cdot x$. Thus, the above fact implies that if $p$ is a $\Lambda$-extension, the induced map from the orbit of some $x$ to the orbit of $p(x)$ is a principal $(\Lambda/\text{Stab}(x))$-extension.

Let us get back to nilpotent quandles. We already have a projection of a $(c+1)$-nilpotent quandle $Q$ onto its universal $c$-nilpotent quotient $Q/\Gamma_c$

Proposition 3.12. Let $Q$ be a $(c+1)$-nilpotent quandle. Then $Q \rightarrow Q/\Gamma_c$ is a $\Gamma_c(\text{Inn}(Q))$-extension.

Proof. The projection $Q \rightarrow Q/\Gamma_c$ is the quotient of $Q$ by the action of $\Lambda := \Gamma_c(\text{Inn}(Q))$ (as in Lem. 1.18). This action is by quandle automorphisms and, by definition of the quotient, the fibres of this projection are transitive $\Lambda$-sets. Moreover, since $Q$ is $(c+1)$-nilpotent, we have seen (see the proof of Th. 2.12) that the projection onto $Q/\Gamma_c$ is a covering (it is a quotient by a central subgroup of $\text{Inn}(Q)$).

3.3 The Hopf property

The Hopf property is one very important feature of finitely generated (residually) nilpotent groups: any surjective endomorphism of such a group is bijective. Groups satisfying this Hopf property are also called hopfian. This can be seen as a kind of finiteness property (any finite group is obviously hopfian).

We now show that (residually) nilpotent quandles do satisfy the same property:

Theorem 3.13. Let $Q$ be a finitely generated nilpotent quandle. Then $Q$ is hopfian, that is, any surjective endomorphism of $Q$ is bijective.

Corollary 3.14. A residually nilpotent quandle of finite type is hopfian.

Proof of the Corollary. Let $Q$ be a residually nilpotent quandle and $f : Q \rightarrow Q$ be a surjective morphism. Let $x, y \in Q$ be distinct elements. Since $Q$ is residually nilpotent, their classes $\overline{x}$ and $\overline{y}$ are distinct in $Q/\Gamma_c$, for some integer $c \geq 1$. The endomorphism $f$ induces a surjective endomorphism $\overline{f}$ of the finitely generated nilpotent quandle $Q/\Gamma_c$, which must be injective by Theorem 3.13. Thus $\overline{f(x)} = \overline{f(x)} \neq \overline{f(y)} = \overline{f(y)}$, whence $f(x) \neq f(y)$.

Proof of Theorem 3.13. The proof is by induction on the nilpotency class $c$ of $Q$, relying on the decomposition of Prop. 3.12 for the induction step. The case $c = 1$ is obvious. Indeed, the only generating set of a trivial quandle is the whole quandle, so a finitely generated trivial quandle must be finite, and a surjection from a finite set onto itself is a bijection. Now, let us suppose that the result holds for $c$-nilpotent quandles of finite type, and let us show that any finitely generated $(c+1)$-nilpotent quandle is hopfian.
Let $f$ be a surjective endomorphism of a finitely generated $(c+1)$-nilpotent quandle $Q$, and let $x, y \in Q$ such that $f(x) = f(y)$. We want to show that $x = y$. Let us first consider the endomorphism $\tilde{f}$ of $Q/\Gamma_c$ induced by $f$. It is a surjective endomorphism of a finitely generated $c$-nilpotent quandle, to which we can apply our induction hypothesis: $\tilde{f}$ is an automorphism. If $\tilde{x}$ and $\tilde{y}$ denote the respective classes of $x$ and $y$ in $Q/\Gamma_c$, we have $\tilde{f}(\tilde{x}) = \tilde{f}(\tilde{x}) = \tilde{f}(\tilde{y}) = \tilde{f}(\tilde{y})$, hence $\tilde{x} = \tilde{y}$. Using the description of the universal $c$-nilpotent quotient as the quotient by $\Lambda := \Gamma_c(\text{Inn}(Q))$, we see that it means that $x$ and $y$ are in the same orbit under the action of $\Lambda$. Thus, in order to conclude, we need to show that the restriction of $f$ to each $\Lambda$-orbit is injective.

Our quandle $Q$ is finitely generated, so $\text{Inn}(Q)$ is too. Since nilpotent groups of finite type are noetherian (see the remarks at the end of §3.3), the subgroup $\Lambda$ is also finitely generated. The surjective endomorphism $f$ induces a surjective endomorphism $f_*$ of $\text{Inn}(Q)$ (see Proposition 1.22). By definition of the lower central series, the surjectivity of $f$ implies $f_*(\Lambda) = \Lambda$ (in fact, because of the Hopf property for finitely generated nilpotent groups, $f_*$ must be an automorphism, inducing an automorphism of the characteristic subgroup $\Lambda$ of $\text{Inn}(Q)$). Recall that $f$ is $f_*$-equivariant (Lemma 1.15): for all $q \in Q$ and all $g \in \text{Inn}(Q)$, $f(g \cdot q) = f_*(g) \cdot f(q)$.

Using the fact that $f_*(\Lambda) = \Lambda$ and the equivariance, we see that for every $x \in Q$, the endomorphism $f$ restricts to a surjective map $\Lambda \cdot x \to \Lambda \cdot f(x)$. Since $x$ and $f(x)$ may not be in the same orbit of $Q$, we cannot make a direct use of the comparison between $\Lambda$-orbits from Fact 3.11. However, as in the beginning of the proof, $\pi_0(Q)$ is a finitely generated trivial quandle, so it is finite, and $f$ induces a permutation $\pi_0(f)$ of this set, which must be of finite order $n$. And if we show that the surjective endomorphism $f^n$ is injective, then $f$ will be. As a consequence, by replacing $f$ with $f^n$, we can assume that $\pi_0(f)$ is the identity. Then $x$ and $f(x)$ are in the same orbit of $Q$, so Fact 3.11 says that $\Lambda \cdot x$ and $\Lambda \cdot f(x)$ are isomorphic as (transitive) $\Lambda$-sets. Thus, after identifying them, we can apply Lemma 3.17 to the $f_*$-equivariant map $\Lambda \cdot x \to \Lambda \cdot f(x)$ to get that the restriction of $f$ to each $\Lambda$-orbit is injective, whence our claim. \hfill \Box

The above proof relies on Lemma 3.17 below, whose proof uses the next two Lemmas.

**Lemma 3.15.** Let $\Lambda$ be an abelian group. Let $X$ be a transitive $\Lambda$-set. Then any equivariant map $f : X \to X$ is a bijection.

**Proof.** The commutativity of $\Lambda$ implies that for $\lambda \in \Lambda$, $\lambda \cdot (-)X \to X$ is $\Lambda$-equivariant. Let us choose $x \in X$ and (by transitivity) $\lambda_0 \in \Lambda$ such that $f(x) = \lambda_0 \cdot x$. For any $\lambda \in \Lambda$, we have $f(\lambda \cdot x) = \lambda \cdot f(x) = \lambda\lambda_0 \cdot x = \lambda_0 \cdot (\lambda \cdot x)$. Thus $f = \lambda_0 \cdot (-)$ is invertible (with inverse $\lambda_0^{-1} \cdot (-)$). \hfill \Box

**Lemma 3.16.** Let $G$ be a group, and $H$ be a normal subgroup such that $G/H$ is Hopfian. If an automorphism $\alpha$ of $G$ satisfies $\alpha(H) \subseteq H$, then $\alpha^{-1}(H) \subseteq H$, i.e. $\alpha(H) = H$.

**Proof.** If $\alpha(H) \subseteq H$, $\alpha$ induces and endomorphism $\bar{\alpha}$ of $G/H$, which is clearly surjective. Since $G/H$ is hopfian, $\bar{\alpha}$ is bijective. But $\ker(\bar{\alpha})$ is the projection of $\alpha^{-1}(H)$ in $G/H$, which is trivial if and only if $\alpha^{-1}(H) \subseteq H$. \hfill \Box

**Lemma 3.17.** Let $\Lambda$ be an finitely generated abelian group, and $\alpha \in \text{Aut}(\Lambda)$. Let $X$ be a transitive $\Lambda$-set. Then any $\alpha$-equivariant map $f : X \to X$ is a bijection.
Proof. For all \( \lambda \in \Lambda \), for all \( x \in X \), \( f(\lambda \cdot x) = \alpha(\lambda) \cdot f(x) \). Thus, if \( \lambda \in \text{Stab}(x) \) (for some \( x \in X \)), then \( \alpha(\lambda) \in \text{Stab}(f(x)) \). Since \( X \) is a transitive \( \Lambda \)-set and \( \Lambda \) is abelian, \( H := \text{Stab}(x) \) does not depend on \( x \in X \), so the previous sentence translates as: \( \alpha(H) \subseteq H \). Since \( \Lambda/H \) is abelian of finite type, it is hopfian, so we can apply Lemma 3.16 to get \( \alpha(H) = H \). Now, \( f \) can be seen as a \( \Lambda \)-equivariant morphism from \( X \) to \( \alpha^*X \), where \( \alpha^*X \) is \( X \) as a set, on which \( \lambda \in \Lambda \) acts via the action of \( \alpha(\lambda) \) on \( X \). Moreover, in the same way as \( X \cong \Lambda/H \), we have \( \alpha^*X \cong \Lambda/\alpha^{-1}(H) \), since \( \alpha^{-1}(H) \) is the common stabilizer of the elements of \( \alpha^*X \). Here, \( \alpha^{-1}(H) = H \), so \( X \cong \alpha^*X \) as \( \Lambda \)-sets. By identifying these, we can see \( f \) as an equivariant map from \( X \) to itself, which has to be an isomorphism by Lemma 3.15. \( \square \)

Noetherianity

Another important finiteness property of nilpotent groups is noetherianity. Recall that a group is called noetherian if every ascending chain of subgroups stops, which is equivalent to every subgroup being of finite type. For instance, cyclic groups are noetherian. It is easy to check that an extension of two noetherian groups is noetherian. Since nilpotent groups of finite type are polycyclic, they are noetherian.

By replacing the words “group” and “subgroup”, respectively, by “quandle” and “subquandle” in the definition above (in fact in both of them), we get the notion of noetherian quandles. One could then ask if finitely generated quandles are noetherian. It turns out that they are not, in general:

Example 3.18. In 2-nilpotent quandles, orbits are trivial subquandles. A trivial subquandle is finitely generated if and only if it is finite. And there are (many) 2-nilpotent quandles of finite type with infinite orbits, which are thus not noetherian. Such quandles will appear in §5.3; the simplest one is \( Q_{1,0} = \{a\} \sqcup \mathbb{Z} \), whose law is given by \( a \triangleright n = n + 1 \) and \( n \triangleright (-) = \text{id} \) for \( n \in \mathbb{Z} \), so that \( \{a\} \) and \( \mathbb{Z} \) are its orbits.

4 Construction of quandles

We now turn to the construction of quandles from group-theoretic data. In §4.1, we recall, with a slightly new point of view, a classical construction of any quandle from such data. We then use this construction in §4.2, to get a simple presentation of the enveloping group of a quandles generated by representatives of its orbits. Finally, in §4.3, we use it to get a characterisation of injective quandles in terms of subgroups of their enveloping groups.

4.1 Quandles from groups

We first recall a construction due to Joyce [Joy82a, §7], allowing one to construct any quandle from a group \( G \) together with some group-theoretic data. Our setting is slightly different from Joyce’s, in order to allow group actions that are not necessarily faithful. We begin with the following easy result:

Proposition 4.1. Let \( Q \) be a quandle, and let a group \( G \) act on \( Q \) by quandle automorphisms, such that the image of the corresponding morphism \( G \to \text{Aut}(Q) \) contains \( \text{Inn}(Q) \). Let us choose one element \( q_i \) in each \( G \)-orbit. Let \( z_i \in G \) act via \( q_i \triangleright (-) \), and
let $H_i$ denote $\text{Stab}(q_i)$. Then the $G$-equivariant map

$$f : \left\{ \begin{array}{ll} \bigsqcup_{i \in I} G/H_i & \longrightarrow Q \\ gH_i & \longmapsto g \cdot q_i \end{array} \right.$$ 

is a bijection. The corresponding quandle law on $\bigsqcup G/H_i$ (with respect to which $f$ becomes a quandle isomorphism) reads:

$$\forall i, j \in I, \forall x, y \in G, \quad xH_i \triangleright yH_j := x z_i x^{-1} yH_j.$$

**Proof.** The kernel of the action of $G$ on $Q$ is contained in all the $H_i$, so we can replace $G$ by its image in $\text{Aut}(Q)$ without altering the construction: we can assume that $G$ is a subgroup of $\text{Aut}(Q)$ containing $\text{Inn}(Q)$. The fact that $f$ is well-defined, $G$-equivariant and bijective comes directly from the decomposition of the $G$-set $Q$ into $G$-orbits:

$$Q = \bigsqcup G \cdot q_i.$$

We only need to check that the quandle law described in the statement is the one obtained by transport of structure along $f$:

$$\forall i, j \in I, \forall x, y \in G, \quad f(xH_i) \triangleright f(yH_j) = (x \cdot q_i) \triangleright (y \cdot q_j) = x \cdot (q_i \triangleright (x^{-1} y \cdot q_j)) = (xz_i x^{-1} y) \cdot q_j = f(xz_i x^{-1} yH_j),$$

where we have used that $x \in G$ acts via a quandle automorphism and that $z_i$ acts via $q_i \triangleright (-)$. \quad $\square$

**Example 4.2.** By applying this to $G(Q)$ acting on $Q$, we see that although one cannot recover $Q$ from $G(Q)$, one can recover $Q$ from $G(Q)$, together with elements $z_i (= \eta(q_i))$ and subgroups $H_i (= \text{Stab}(q_i))$, indexed by orbits of $Q$. Using the notation introduced in Lemma 4.5 below, we have:

$$Q = Q(G(Q), (H_i, z_i)_{i \in I}).$$

This recipe is the one used to recover the fundamental quandle from the peripheral system, and we will use it to recover the reduced fundamental quandle from the reduced peripheral system in §6.3.

**Remark 4.3.** In the sequel, we will often use this result with a free group as our acting $G$. Then, the distinction between $G$ and its image in $\text{Aut}(Q)$ will come in handy: the latter can have a much more complicated structure than the former.

**Remark 4.4.** Asking that for any inner automorphism $\varphi$, there is an element of $G$ which acts via $\varphi$ may seem too strong an hypothesis, since we have used it only for the $q_i \triangleright (-)$. However, if $z_i$ acts via $q_i \triangleright (-)$, then $g z_i g^{-1}$ acts via $(g \cdot q_i) \triangleright (-)$. Indeed, since $g$ acts via a quandle automorphism, we have:

$$\forall q \in Q, \quad g z_i g^{-1} \cdot q = g \cdot (q_i \triangleright (g^{-1} \cdot q)) = (g \cdot q_i) \triangleright q.$$  

But by definition of the $q_i$, the $g \cdot q_i$ describe all the elements of $Q$. As a consequence, if there are $z_i$ in $G$ acting via the $q_i \triangleright (-)$, then any inner automorphism of $Q$ must be realised as the action of some element of $G$.

Proposition 4.1 tells us how to recover a given quandle from some group-theoretic data. It is then natural to ask when we can construct a quandle from such data.
Lemma 4.5. Let $G$ be a group, $(H_i)_{i \in I}$ be a family of subgroup of $G$, and $(z_i)_{i \in I}$ be a family of elements of $G$ indexed by the same set $I$. Then the formulas:

$$\forall i, j \in I, \forall x, y \in G, \ xH_i \triangleright yH_j := xz_i x^{-1} yH_j$$

do define a quandle structure on $\bigsqcup G/H_i$ if and only if the following conditions hold, for all $i \in I$:

- The action of $H_i$ on $\bigsqcup G/H_j$ commutes with the action of $z_i$.
- $z_i \in H_i$.

When these conditions hold, the resulting quandle will be denoted by $Q(G, (H_i, z_i)_{i \in I})$.

Proof. The operation is well-defined if and only if, for all $i \in I$, for all $x, y \in G$, for all $h \in H_i$ and $k \in H_j$, we have $(xh)z_i(xh)^{-1}(yk)H_j = xz_i x^{-1} yH_j$. This is equivalent to $hz_i h^{-1} x^{-1} yH_j = z_i x^{-1} yH_j$, which in turn (by defining $t := x^{-1} y$ and $h' := h^{-1}$) is equivalent to $(z_i h') \cdot tH_j = (h' z_i) \cdot tH_j$ (for all $t \in G$ and all $h' \in H_i$). In other words, the action of $z_i$ on $G/H_j$ must commute with the action of $H_i$. The operation is thus well-defined if and only if the latter holds for every $j \in I$, which is exactly our first condition. It is then easy to check the first two quandle axioms, and the third is equivalent to $z_i H_i = H_i$, which holds if and only if $z_i \in H_i$. □

Remark 4.6. These conditions are obviously satisfied when the data comes from a quandle by the construction of Proposition 4.1.

Remark 4.7. The hypothesis $z_i \in H_i$ is only used in the verification of the third quandle axiom; without this assumption, one does only get a rack.

Remark 4.8. The first condition is obviously satisfied if $z_i$ centralizes $H_i$. In fact, it is equivalent to $z_i$ centralizing $H_i$ when the action of $G$ on $\bigsqcup G/H_j$ is faithful.

Remark 4.9. The first condition can also be formulated as follows: $[H_i, z_i] \subseteq \bigcap gH_j g^{-1}$ (where the intersection is over $j \in I$ and $g \in G$). Indeed, $K := \bigcap gH_j g^{-1}$ is the kernel of the action of $G$ on $\bigsqcup G/H_j$, so $hz_i h^{-1} z_i^{-1} \in K$ if and only if $h z_i \equiv z_i h [K]$, which is equivalent to the action of $z_i$ on $\bigsqcup G/H_j$ commuting with the action of $h$.

4.2 A presentation of enveloping groups

Let $Q$ be a quandle. Suppose that we can choose a generating family $(q_s)_{s \in S}$ with $S \cong \pi_0(Q)$, that is, with exactly one element in each orbit (which is possible if $Q$ is nilpotent, by Corollary 3.6, but also for instance when $Q$ is free). Then the map $x_s \mapsto q_s \triangleright (-)$ extends to a group morphism $F[S] \to \text{Inn}(Q)$, which is an action of the free group $F[S]$ on $Q$ satisfying the hypotheses of Proposition 4.1 (see Remark 4.4). As a consequence, we get:

$$Q \cong Q(F[S], (H_s, x_s)),$$

where $H_s$ is the stabilizer of $q_s$ with respect to the above action of $F[S]$ on $Q$.

It turns out that we can get a presentation of the enveloping group of $Q$ from this description of $Q$:

Theorem 4.10. Let $Q$ be a quandle having a generating family $(q_s)_{s \in S}$ with exactly one element in each orbit. Then:

$$G(Q) \cong \left\langle (x_s)_{s \in S} \ \middle| \ [H_s, x_s] = 1 \right\rangle,$$

where $H_s$ is the stabilizer of $q_s$ with respect to the action of the free group $F[S]$ on $Q$ making $x_s$ act via $q_s \triangleright (-)$. 

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We are going to show that the group defined by the presentation of the statement has the universal property of $G(Q)$, using the following characterisation of morphisms from $Q$:

**Lemma 4.11.** Let $T$ be any quandle and let $(t_s)_{s \in S}$ be a family of elements of $T$. Consider the action of the free group $F[S]$ on $T$ making $x_s$ act via $t_s \mapsto (-)$. Then, with the above notation, the map $q_s \mapsto t_s$ extends (uniquely) to a quandle morphism $f : Q \to T$ if and only if for all $s \in S$, $H_s$ acts trivially on $t_s$.

**Proof.** Suppose first that such an extension exists. Since the $q_s$ generate $Q$, it is unique. In fact, we must have $f(w \cdot q_s) = w \cdot t_s$ for all $w \in F[S]$ and all $s \in S$. Indeed, if $w = x_s^{n_1} \cdots x_s^{n_k}$, then the quandle morphism $f$ must send $w \cdot q_s = q_{s_1} \circ \cdots \circ q_{s_k} \circ q_s$ to $t_{s_1} \cdot \cdots \cdot t_{s_k} \cdot t_s = w \cdot t_s$. Thus, if $w \cdot q_s = q_s$ (that is, if $w \in H_s$), we must have $w \cdot t_s = f(w \cdot q_s) = f(q_s) = t_s$, whence the necessity of the condition.

Suppose now that for all $s \in S$, $H_s$ acts trivially on $t_s$, and let us show that $w \cdot q_s \mapsto w \cdot t_s$ defines a morphism from $f : Q \to T$. First, we need to check that $f$ is well-defined. Here we use our hypothesis on $Q$: every element of $Q$ can be written as $w \cdot q_s$ for a unique $s \in S$. Thus, $w \cdot q_s = w' \cdot q_s$ implies $s = s'$, and $q_s = w^{-1}w' \cdot q_s$, which means that $w^{-1}w' \in H_s$, hence $t_s = w^{-1}w' \cdot t_s$, that is $w \cdot t_s = w' \cdot t_s$. Finally, we need to check that $f$ is a morphism, which is easy. Indeed, for all $w, v \in F[S]$, and all $r, s \in S$, we have:

$$f((w \cdot q_s) \circ (v \cdot q_r)) = f(w \cdot (q_s \circ (w^{-1}v \cdot q_r))) = f(wxw^{-1}v \cdot q_r) = wxw^{-1}v \cdot t_r = w \cdot (t_s \circ (w^{-1}v \cdot t_r)) = (w \cdot t_s) \circ (v \cdot t_r) = f(w \cdot q_s) \circ f(v \cdot q_r),$$

where we have used the definition of $f$, the fact that $w$ acts as a quandle morphism on both $Q$ and $T$, and the fact that $x_s$ acts via $q_s \mapsto (-)$ on $Q$ and via $t_s \mapsto (-)$ on $T$. \(\square\)

**Proof of Theorem 4.10.** Let $G_0$ be the group defined by the presentation of the theorem. Remark that if $w \in H_s$ and $s \in S$, then the relation $[w, x_s] = 1$ can be written as $wxw^{-1} = x_s$. Thus the family $(x_s)_{s \in S}$ in the conjugation quandle of $G_0$ satisfies the hypothesis of Lemma 4.11 (the corresponding action of $F[S]$ on $G_0$ is induced by the conjugation action of $F[S]$ on itself). As a consequence, $q_s \mapsto x_s$ defines a quandle morphism from $Q$ to $G_0$.

Now, let us show that this morphism is the universal one from $Q$ to a group. Let $G$ be a group and $f : Q \to G$ be a quandle morphism. Consider the group morphism $\tilde{f} : F[S] \to G$ sending $x_s$ to $t_s := f(q_s)$. If $w \in H_s$, then $\tilde{f}(wxw^{-1}) = \tilde{f}(w)t_s\tilde{f}(w)^{-1} = w \cdot t_s$ (by definition of the action of $F[S]$ on $G$ associated to the $t_s$). Using Lemma 4.11, we see that $w \cdot t_s = t_s = f(x_s)$. Thus $\tilde{f}$ vanished on the relations defining $G_0$, which means that it factors to a morphism $\overline{f} : G_0 \to G$. Since $\overline{f}(x_s) = t_s = f(x_s)$, the morphism $\overline{f}$ is the required (unique) extension of $f$. \(\square\)

### 4.3 An injectivity criterion

Let $Q$ be a quandle, and let us consider the canonical morphism $\eta : Q \to G(Q)$ to its enveloping group. If $q \in Q$, we can consider its stabilizer $\text{Stab}(q) \leq G(Q)$. A closely related subgroup is the centralizer $C(\eta(q)) \leq G(Q)$ of $\eta(q)$. In fact, since $\eta$ is $G(Q)$-equivariant (with respect to the conjugation action on the target), if $g \cdot q = q$, then $g\eta(q)g^{-1} = \eta(g \cdot q) = \eta(q)$. Thus $\text{Stab}(q) \subseteq C(\eta(q))$. However, the converse inclusion
is not true in general: from $\eta(g \cdot q) = \eta(q)$, we cannot deduce that $g \cdot q = q$, except when $\eta^{-1}(\eta(q)) = \{q\}$. In particular, when $Q$ is injective (that is, when $\eta$ is injective), then for all $q \in Q$, we have $\text{Stab}(q) = C(\eta(q))$. We show that the converse holds:

**Proposition 4.12.** Let $Q$ be a quandle, and let us choose one element $q_i$ is each of its orbits (where $i \in I \cong \pi_0(Q)$). Then $Q$ is injective if and only if for all $i \in I$, the inclusion $\text{Stab}(q_i) \subseteq C(\eta(q_i))$ is an equality.

**Proof.** We apply Proposition 4.1 to the quandle $Q$ and to its image $\eta(Q)$, with respect to the action of the group $G(Q)$ on both of them. By definition of the $q_i$, there is one in each $G(Q)$-orbit, and $G(Q)$ surjects onto $\text{Inn}(Q)$ by construction; precisely, the element $\eta(q_i)$ of $G(Q)$ acts via the inner automorphism $q_i \triangleright (-)$. Proposition 4.1 then gives:

$$Q \cong \mathcal{Q}(G(Q), (\text{Stab}(q_i), \eta(q_i))_{i \in I}).$$

There is also one $\eta(q_i)$ (for a unique $i \in I$) in each $G(Q)$-orbit of $\eta(Q)$. Indeed, there is at least one, since $\eta(Q)$ is a $G(Q)$-equivariant quotient of $Q$, and if we have $g\eta(q_i)g^{-1} = \eta(q_j)$ for some $g \in G$ and some $i, j \in I$, we can project this relation in $G(Q)^{ab} \cong \mathbb{Z}[\pi_0(Q)]$, and we get $\overline{\eta}_i = \overline{\eta}_j$ in there, hence $i = j$. Moreover, the inner automorphism of $\eta(Q)$ associated to $\eta(q_i)$ (for any $q \in Q$) is precisely given by the action of $\eta(q) \in G(Q)$. Proposition 4.1 then gives:

$$\eta(Q) \cong \mathcal{Q}(G(Q), (C(\eta(q_i)), \eta(q_i))_{i \in I}).$$

Now, the morphism $\eta : Q \to \eta(Q)$ identifies as the morphism

$$\mathcal{Q}(G(Q), (\text{Stab}(q_i), \eta(q_i))_{i \in I}) \to \mathcal{Q}(G(Q), (C(\eta(q_i)), \eta(q_i))_{i \in I})$$

induced by $\text{id}_{G(Q)}$. This is the disjoint union of the canonical maps $G(Q)/\text{Stab}(q_i) \to G(Q)/C(\eta(q_i))$ induced by the inclusions $\text{Stab}(q_i) \subseteq C(\eta(q_i))$, and the former are bijections if and only if the latter are equalities. \qed

**Remark 4.13.** To show that $Q$ is injective, one needs to check the condition $\text{Stab}(q) = C(\eta(q))$ for at least one $q$ in each orbit (and then it holds for all $q \in Q$). In particular, it is enough to check it for $q$ in a given generating subset $S$ of $Q$.

## 5 2-nilpotent quandles

In this section, we apply the results of the previous section to classify 2-nilpotent quandles, and to compute their enveloping groups. 2-nilpotent quandles are the one called *abelian* in [LM21], and we recover their classification results with a slightly different point of view. In particular, we recover the calculations of [Eis14, Prop. 2.38].

### 5.1 Classification

By definition, a quandle $Q$ is 2-nilpotent exactly when its group of inner automorphisms is abelian. That $\text{Inn}(Q)$ is abelian means exactly that for all $x, y \in Q$, the automorphisms $x \triangleright (-)$ and $y \triangleright (-)$ commute, that is:

$$\forall x, y, z \in Q, \ x \triangleright (y \triangleright z) = y \triangleright (x \triangleright z).$$

Because of Lemma 2.11 (or directly from the above, putting $t = y \triangleright z$ to get $x \triangleright t = (y \triangleright x) \triangleright t$), this implies that the behaviour of an element depends only on its orbit
under $\text{Inn}(Q)$ (or $G(Q)$), that is, on its class in $\pi_0(Q)$. In other words, the action of $Q$ on itself factors through $\pi_0(Q)$. Which means that the action of $G(Q)$ on $Q$ factors through the enveloping group of the trivial quandle $\pi_0(Q)$, which is $\mathbb{Z}[\pi_0(Q)]$.

**Remark 5.1.** Each orbit of a 2-nilpotent quandle $Q$ must be a trivial subquandle:

$$\forall x \in Q, \forall y \in G(Q) \cdot x, \quad y \triangleright x = x \triangleright x = x.$$ 

In order to understand 2-nilpotent quandles, we can apply the constructions of §4.1, with acting group $G = \mathbb{Z}[\pi_0(Q)]$. In fact, we can start with any free abelian group $G = \mathbb{Z}^I$. Since this $G$ is abelian, the only relevant condition from Lemma 4.5 is the second one. As an application of Proposition 4.1 and Lemma 4.5, we get the following:

**Theorem 5.2.** Let us denote by $(e_i)_{i \in I}$ the canonical basis of the free abelian group $\mathbb{Z}^I$. For each $i \in I$, let $H_i$ be a subgroup of $\mathbb{Z}^I$ containing $e_i$. Let us define:

$$Q((H_i)_{i \in I}) := \bigcup_i \mathbb{Z}^I/H_i.$$ 

We get a 2-nilpotent quandle structure on this set via:

$$\forall i, j, \forall x \in Q_i, \forall y \in Q_j, \quad x \triangleright y := e_i + y.$$ 

Every 2-nilpotent quandle can be obtained in this way. Moreover, there is an isomorphism $Q((H_i)_{i \in I}) \cong Q((K_j)_{j \in J})$ if and only there is a bijection $\sigma : I \cong J$ such that the corresponding isomorphism $\mathbb{Z}^I \cong \mathbb{Z}^J$ sends $H_i$ to $K_{\sigma(i)}$.

**Proof.** The only part of the statement which still needs a proof is the last one. It is clear that such a $\sigma$ induces an isomorphism between $Q_1 = Q((H_i)_{i \in I})$ and $Q_2 = Q((K_j)_{j \in J})$. Conversely, remark that we can recover $I$ from $Q((H_i)_{i \in I})$ as its set of orbits, and we can recover $H_i$ as the stabilizer of any element of the $i$-th orbit. Thus, an isomorphism $f : Q_1 \cong Q_2$ induces a bijection $\sigma := \pi_0(f)$ between the sets of orbits, and the induced isomorphism $\mathbb{Z}[\pi_0(f)]$ (which is induced by $f_* : G(Q_1) \to G(Q_2)$) must send the stabilizer of an element $q \in Q_1$ to the stabilizer of $f(q)$, because of Lemma 1.15. 

**Example 5.3.** The quandle constructed in Example 2.7 is 2-nilpotent, with 3 orbits. It corresponds to $H_1 = \langle e_1, ne_2, e_2 - e_3 \rangle$, $H_2 = \langle ne_1, e_2, e_1 - e_3 \rangle$ and $H_3 = \mathbb{Z}^3$.

### 5.2 Enveloping groups

We can apply Theorem 4.10 to compute the enveloping group of a 2-nilpotent quandle $Q$. Indeed, since $Q$ is nilpotent, it satisfies the required hypothesis (see Corollary 3.6): if we choose one element $q_i$ in each orbit, these generate $Q$. We know that $G(Q)$ is 2-nilpotent (Corollary 2.5), generated by the images of the $q_i$, so it is a quotient of the free 2-nilpotent group $F_2[I]$ on $I$. Precisely, it is the quotient of $F_2[I]$ by the image of the relators in $F[I]$ from Theorem 4.10.

The group $F_2[I]$ can be seen canonically as a central extension:

$$\Lambda^2(\mathbb{Z}^I) \hookrightarrow F_2[I] \twoheadrightarrow \mathbb{Z}^I.$$ 

The quotient $\mathbb{Z}^I$ is the abelianisation of $F_2[I]$, which is free abelian on the classes $e_i$ of the generators $x_i$, and the kernel $\Lambda^2(\mathbb{Z}^I)$ identifies with $\Gamma_2(F_2[I])$ via $[x_i, x_j] \mapsto e_i \wedge e_j$. 

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Theorem 5.4. Let $Q$ be a 2-nilpotent quandle, and let $(q_i)_{i \in I}$ be a family of representatives of the orbits of $Q$ (where $I = \pi_0(Q)$). Then $G(Q)$ is the quotient of $F_2[I]$ by the following relations belonging to $\Gamma_2(F_2[I]) \cong \Lambda^2(\mathbb{Z}[I])$:

$$
e_i \wedge H_i = 0,$$

where $H_i$ is the stabilizer of $q_i$ with respect to the action of $\mathbb{Z}[I]$ on $Q$ making $e_i$ act via $q_i \triangleright (-)$.

Proof. This is a direct application of Theorem 4.10, with the simplification discussed above. Namely, the $H_i$ of our current statement are the images of the $H_i$ from Theorem 4.10 by the projection $F[I] \twoheadrightarrow \mathbb{Z}(I)$, and the $e_i \wedge H_i$ are the images of the relators from Theorem 4.10 in the free 2-nilpotent group, modulo the identification of $\Gamma_2(F_2[I])$ with $\Lambda^2(\mathbb{Z}[I])$.

Remark 5.5. The theorem also applies, with the same proof, for 3-nilpotent quandles whose enveloping group happen to be 2-nilpotent.

Remark 5.6. If $G$ is a 2-nilpotent group, a necessary condition for it to be the enveloping group of a nilpotent quandle is for its abelianisation to be free. However, this condition is not sufficient. For instance, if $|I| = 4$, the element $e_1 \wedge e_2 - e_3 \wedge e_4$ is easily seen not to be of the form $u \wedge v$, so the subgroup of $\Lambda^2(\mathbb{Z}^4)$ that it generates is not of the form $\sum \epsilon_i \wedge H_i$ for any choice of basis $(\epsilon_i)$ of $\mathbb{Z}^4$. As a consequence, $G := F_2[4]/([x_1, x_2] = [x_3, x_4])$ is not the enveloping group of a 2-nilpotent quandle. In fact, it is not the enveloping group of any quandle. Indeed, if it were of the form $G(Q)$, $Q$ would need to be nilpotent of class at most 3 (Corollary 2.5), and using Remark 5.5, $G$ would have to be of the form described in the theorem, and we just remarked that it is not.

5.3 Applications

We now apply our constructions to classify 2-nilpotent quandles, and to compute their enveloping groups, notably recovering the classification results of [LM21, §2-5] in our larger context.

2-nilpotent quandles with two orbits

Let us apply Theorem 5.2 with $n = 2$. The subgroups $H_1$ and $H_2$ of $\mathbb{Z}^2$ must be of the form $H_1 = \mathbb{Z}e_1 + m\mathbb{Z}e_2$ and $H_2 = \mathbb{Z}e_2 + n\mathbb{Z}e_1$, with $m, n \geq 0$. Hence, our two orbits are $\mathbb{Z}^2/H_1 \cong \mathbb{Z}/m$ and $\mathbb{Z}^2/H_2 \cong \mathbb{Z}/n$. Moreover, the quandle law that we get on $(\mathbb{Z}/m) \sqcup (\mathbb{Z}/n)$ is given by:

$$
\forall x, x', y, y' \in \mathbb{Z}/m, \forall y, y' \in \mathbb{Z}/n,
\begin{cases}
x \triangleright y = y + 1; \\
y \triangleright x = x + 1;
\end{cases}
\begin{cases}
x \triangleright x' = x'; \\
y \triangleright y' = y'.
\end{cases}
$$

This is exactly the quandle $Q_{m,n}$ defined in [Eis14, Ex. 1.6]. This should not come as a surprise: 2-nilpotent quandles are coverings of trivial ones (Th. 2.12), and the ones with two orbits are coverings of the trivial quandle with two elements. Remark that among them, the free 2-nilpotent quandle on two generators, which is the one with the less relations, is the universal cover $Q_{0,0}$ of the trivial quandle with two elements.
We can use Theorem 5.4 to recover [Eis14, Prop. 2.38]. Namely, the free 2-nilpotent group on 2 generators $F_2[2]$ is the Heisenberg group:

$$\mathcal{H} = \langle x_1, x_2, z \mid [x_1, x_2] = z, \ [z, x_1] = [z, x_2] = 1 \rangle,$$

which is a central extension of $\mathbb{Z}^2$ (generated by $e_1, e_2$) by $\mathbb{Z} = \Lambda^2(\mathbb{Z}^2)$ (generated by $e_1 \wedge e_2$). The group $G(Q_{m,n})$ is the quotient of $F_2[2]$ by the relators $e_1 \wedge (me_2)$ and $(ne_1) \wedge e_2$, so that:

$$G(Q_{m,n}) = \mathcal{H}/z^{\gcd(m,n)},$$

which is a central extension of $\mathbb{Z}^2$ by $\mathbb{Z}/\gcd(m,n)$. In particular, we recover the fact that the enveloping group of the free 2-nilpotent quandle $Q_{0,0}$ is the free 2-nilpotent group $\mathcal{H}$. We also remark that if $m$ and $n$ are coprime, then the enveloping group is $\mathbb{Z}^2$, which is abelian, even if $Q_{m,n}$ is not trivial. In fact, $G(Q_{m,n})$ is abelian if and only if $m$ and $n$ are coprime. Finally, we remark that the subquandle of $\mathcal{H}/z^d$ generated by $x_1$ and $x_2$ is isomorphic to $Q_{d,d}$, so the $Q_{d,d}$ are the only injective ones in this family, and if $d = \gcd(m,n)$, the image of $Q_{m,n}$ inside $G(Q_{m,n}) = \mathcal{H}/z^d$ is the quotient $Q_{d,d}$ of $Q_{m,n}$.

**Finite-type 2-nilpotent quandles**

Finite 2-nilpotent quandles were classified in [LM21, §2-5] (where 2-nilpotent quandles are called *abelian*). Thus, we should recover their results as a particular case of ours. It is indeed the case, and we now explain the precise link between our construction and theirs. In particular, we remark that their construction applies with very little change to the classification of *finitely generated* (but not necessarily finite) 2-nilpotent quandles.

Finite-type 2-nilpotent quandles are the 2-nilpotent quandles with a finite number of orbits, say $n$ (see Corollary 3.6). Theorem 5.2 classifies them: they are constructed from $n$-tuples $(H_1, ..., H_n)$ of subgroups of $\mathbb{Z}^n$, with each $H_i$ containing the $i$-th element $e_i$ of the canonical basis of $\mathbb{Z}^n$. If our quandle $Q$ is fixed, the $H_i$ are defined up to a permutation of the basis, which correspond to a permutation of the orbits of $Q$. As in [LM21, §2-5], we can fix a total ordering of the orbits of $Q$, so that the $H_i$ associated to $Q$ are uniquely defined: the $i$-th orbit of $Q$ is $\mathbb{Z}^n/H_i$, which is denoted by $G(M^{(i)})$ in [LM21].

If we want to do computations, we need a precise way of describing the $H_i$. This is the role played by parameters from [LM21]. Before introducing them, let us remark that $H_i$ must contain $e_i$, so we only need to describe its image $\overline{H_i}$ in $\mathbb{Z}^n/e_i \cong \mathbb{Z}^{n-1}$. Now, we can filter $\mathbb{Z}^n/e_i$ by

$$V_0 := \{0\} \subset V_1 := \langle e_{i+1} \rangle \subset V_2 := \langle e_{i+1}, e_{i+2} \rangle \subset V_3 := \langle e_{i+1}, e_{i+2}, e_{i+3} \rangle \subset \cdots$$

where the indices are to be understood modulo $n$. Then we can describe $H_i$ by parameters $m_k$ (for $1 \leq k \leq n-1$) and $m_{k,l}$ (for $1 \leq l < k$) as follows:

$$\begin{align*}
\overline{H_i} \cap V_1 &= \langle m_1 e_{i+1} \rangle; \\
\overline{H_i} \cap V_2 &= \langle m_1 e_{i+1}, m_2 e_{i+2} + m_{2,1} e_{i+1} \rangle; \\
\overline{H_i} \cap V_3 &= \langle m_1 e_{i+1}, m_2 e_{i+2} + m_{2,1} e_{i+1}, m_3 e_{i+3} + m_{3,2} e_{i+2} + m_{3,1} e_{i+1} \rangle; \\
\text{etc.}
\end{align*}$$
At the $k$-th step, we consider the image of $\overline{H}_i \cap V_k$ in $V_k/V_{k-1} \cong \mathbb{Z}e_k$. Is it generated by $m_ke_k$, which we lift to an element of $\mathbb{Z}^{n-1}$. We then add this lift to the list of generators of $\overline{H}_i \cap V_{k-1}$ to get a list of generators of $\overline{H}_i \cap V_k$. If we take the convention that $0$ is always lifted to $0$, the list of generators is in fact a basis, with possibly zeros added. This holds at each step, and in particular for $\overline{H}_i \cap V_{n-1} = \overline{H}_i$. Remark that $m_k \neq 0$ for all $k$ if and only if $H_i$ has rank $n$, which is equivalent to the orbit $\mathbb{Z}^n/H_i$ being finite. This explains why the $m_k$ from [LM21] were never trivial: they are concerned with finite quandles there.

In order for the above parameters to be uniquely determined by our data, we need to impose some conditions. Namely, we take $m_k \geq 0$, $0 \leq m_{k,l} < m_l$ if $m_l \neq 0$, and $m_{k,l} = 0$ if $m_k = 0$. This determines the choice of lift at each stage of the process; the fact that we can take a unique lift satisfying these conditions comes from the fact that we can add multiples of the previous generators to a given lift to perform Euclid’s algorithm on each coordinate. The last condition did not appear in [LM21], again because the $m_k$ are not zero if $Q$ is finite (and not merely finitely generated).

The above correspondence explains why our $\mathbb{Z}^n/H_i$ are the $G(M^{(i)})$ from [LM21], their $x_j \in G(M^{(i)})$, also denoted by $x_j^{(i)}$, corresponding to the class of $e_{i+j}$ in $\mathbb{Z}^n/H_i$. We also see that the quandle law from Theorem 5.2 coincides with theirs. Namely, if $a$ is in the $i$-th orbit and $b$ in the $j$-th one, $b \triangleright a$ (denoted by $a \triangleright b$ in [LM21], where a convention opposite to ours is used for the quandle law) is $a + e_j$, and $e_j \in \mathbb{Z}^n/H_i$ is $x_j^{(i)}$.

Now that we have explained why our classification results are the same as theirs for finite quandles, and how it needs to be adapted in order to include all the finitely generated 2-nilpotent quandles, let us turn to the computation of enveloping groups (called structure groups in [LM21]). Namely, [LM21, Th. 3.2] can be seen as another formulation of our Theorem 5.4. Precisely, it follows from Theorem 5.4 that $G(Q)$ is a central extension of its abelianisation $\mathbb{Z}^n = \mathbb{Z} \pi_0(Q)$ by $\Lambda^2(\mathbb{Z}^n)$ quotiented by the $e_i \wedge H_i$. The kernel of this extension, which is also the commutator group of $G(Q)$, is thus generated by the elements $g_{ij} := e_i \wedge e_j$. The element $g_{ij}$ can be obtained by taking the commutator of any element of an image of the $i$-th orbit with an element in the image of the $j$-th orbit; the fact that this does not depend on the choices made is a particular instance of the theory of Lie rings associated to lower central series of groups. Relations defining the quotient are the one defining $\Lambda^2(\mathbb{Z}^n)$ (namely, the $g_{ij}$ commute with one another, $g_{ii}$ is trivial and $g_{ji}$ is inverse to $g_{ij}$), together with a generating family for $\sum e_i \wedge H_i$. Such a generating family can easily be expressed in terms of bases of the $H_i$, whence in terms of the above parameters. This recovers the presentation of the parameter group from [LM21]; their parameter matrix is the matrix of coefficients of such a generating family with respect the basis $(e_i \wedge e_j)_{i<j}$ of $\Lambda^2(\mathbb{Z}^n)$. Moreover, we also recover [LM21, Th. 4.2]: $G(Q)$ is abelian (isomorphic to $\mathbb{Z}^n$) if and only if the $e_i \wedge H_i$ generate $\Lambda^2(\mathbb{Z}^n)$, which holds if and only if the maximal minors of the above parameter matrix are globally coprime.

6 Reduced quandles

Reduced quandles (called quasi-trivial in [Ino13]) are the ones used to get invariants of links up to link-homotopy. In fact, the group now known as the reduced free group was introduced by Milnor to this end, long before the invention of quandles: it is the link group of the trivial link from [Mil54]. However, “being reduced” is not a
property of a group, but only of a group endowed with a system of (conjugacy classes of) generators, making the group-theoretic language somewhat awkward when dealing with such objects. On the contrary, we can speak of reduced quandles.

In this section, we first define reduced quandles, and we show that finitely generated ones are nilpotent. Then we construct the universal reduced quotient $R(Q)$ of any quandle $Q$. Applied to the free quandle, this construction gives the free reduced quandle, which we study next. In particular, we show that it is injective (Prop. 6.5). Applied to the fundamental quandle of a link, $R(-)$ gives the reduced fundamental quandle of [Hug11], which is an invariant of links up to link-homotopy. One of the main results of [Hug11] was to show that this invariant was stronger than the data of the reduced fundamental group with conjugacy classes of meridians. We show that it is in fact equivalent to the data of (a weak version of) the reduced peripheral system (up to equivalence).

### 6.1 Definition

The following definition first appeared as [Hug11, Def. 4.1] (in a geometric context).

**Definition 6.1.** A quandle is called **reduced** if its orbits are trivial subquandles. In other words, $Q$ is reduced if and only if: $\forall x \in Q, \forall \varphi \in \text{Inn}(Q), \varphi(x) \triangleright x = x$.

**Remark 6.2.** Recall that $\text{Inn}(Q)$ was defined as the image of $G(Q)$ in $\text{Aut}(Q)$ by the canonical action. As a consequence, $Q$ is reduced if and only if:

$$\forall x \in Q, \forall g \in G(Q), (g \cdot x) \triangleright x = x.$$ 

The following fact is easy to check:

**Fact 6.3.** The full subcategory $R\text{Q}_{dl} \subset \text{Q}_{dl}$ is reflexive. Precisely, the universal reduced quotient of a quandle $Q$ is its quotient $R(Q)$ by the relations:

$$\forall x \in Q, \forall \varphi \in \text{Inn}(Q), \varphi(x) \triangleright x = x.$$ 

The relevance of reduced quandles to our context comes from the following:

**Proposition 6.4.** Reduced quandles of finite type are nilpotent. Precisely, if $Q$ is a reduced quandle of finite type, it is nilpotent of class at most $|\pi_0 Q|$.

**Proof.** Let $S \subseteq Q$ be a finite set of generators. The surjection $Q[S] \twoheadrightarrow Q$ from the free quandle on $S$ induces a surjection $RQ[S] \twoheadrightarrow Q$. As we will see below (Lemma 6.6), the reduced free quandle $RQ[S]$ is $|S|$-nilpotent. Thus, by Lemma 2.6, $Q$ is nilpotent of class at most $|S|$. Then Lemma 3.6 allows us to choose as a generating set $S$ in the reasoning above any set of representatives of the orbits of $Q$, whence the result. 

### 6.2 Reduced free quandles

Recall that the **reduced free group** $RF[X]$ on a set $X$ is the quotient of the free group on $X$ by the relations saying that each generator $x \in X$ commutes with all it conjugates. In a similar fashion, the **reduced free quandle** $RQ[X]$ on $X$ is the quotient of the free quandle on $X$ by the relations saying that the orbit of each generator $x \in X$ acts trivially on $x$: $RQ[X]$ is precisely $R(Q[X])$. From the latter point of view, we see that the **reduced free quandle** is indeed the free object on $X$ in the category of reduced quandles.
One question arises naturally: is $RF[X]$ the enveloping group of $RQ[X]$? One can check that it is indeed so, since $G(RQ[X])$ and $RF[X]$ satisfy the same universal property. Precisely, both of them represent the functor from groups to sets:

$$G \mapsto \{ f : X \to G \mid \forall x \in X, \forall w \in \langle f(X) \rangle, \quad [w f(x), f(x)] = 1 \}.$$  

We will now show that $RQ[X]$ injects into its enveloping group $RF[X]$. Thus, in this regard, the free reduced quandle behaves in the same way as the free quandle (as recalled at the beginning of §2.3) and the free nilpotent quandles (see Prop. 2.18).

**Proposition 6.5.** The canonical map from $RQ[X]$ to its enveloping group $RF[X]$ is injective. As a consequence, $RQ[X]$ identifies with the union of the conjugacy classes of the basis $X$ inside $RF[X]$.

When $X$ has $n$ elements, the group $RF[X]$ is $n$-nilpotent [HL90, Lem. 1.3]. Thus, we can apply Lemma 2.5 to get:

**Corollary 6.6.** The reduced free quandle on $n$ elements is $n$-nilpotent.

**Proof of Prop. 6.5.** We use the injectivity criterion (Prop. 4.12). Since $X$ generates $RQ[X]$, it is enough to show that $C(\eta(x)) \subseteq \text{Stab}(x)$ for $x \in X$ (see the remark following Prop. 4.12). Moreover, the centralizer of $x$ (that is, of $\eta(x)$) in $RF[X] = G(RQ[X])$ is exactly its normal closure $\mathcal{N}(x)$ [Dar19a, Lemma 1.15].

Now, let $g \in RF[X]$. By definition of reduced quandles, we have $(g \cdot x) \triangleright x = x$. Which means that $\eta(g \cdot x) \cdot x = x$. In other words, $\eta(g \cdot x) \in \text{Stab}(x)$. Since $\eta(g \cdot x) = g \eta(x) g^{-1}$, this means that $\text{Stab}(x)$ contains the conjugates of $\eta(x)$, that is, it contains $\mathcal{N}(x) = C(\eta(x)) \subset RF[X]$, whence our conclusion.

**Remark 6.7.** An alternative strategy of proof consists in adapting the proof of Proposition 2.18 to the present context. But either way, the computation of centralizers of elements of $X$ in $RF[X]$ would have been central in the argumentation. In fact, in the opposite direction, one can interpret the application of [Dar19b, Lem. 6.3] in the proof of Proposition 2.18 as the computation of centralizers of elements of $X$ is $F[X]/\Gamma_{n+1}$, and rewrite this proof as another application of the injectivity criterion (Prop. 4.12).

### 6.3 Reduced peripheral systems of links

The fundamental quandle is equivalent, as an invariant of links, to (a weak version of) the peripheral system. This was the main motivation for introducing the fundamental quandle: it encodes the peripheral system in a nice way, and the latter is a complete invariant of links up to taking mirror images of non-split components (which can easily be strengthened to a complete invariant of links – see Remark 6.8). In this section, we show that the reduced fundamental quandle is equivalent, in a quite straightforward way, to the (weak) reduced peripheral system. The (strong) reduced peripheral system (which is obtained from the weak version by adding a little bit of information) has recently been shown to be a complete invariant of (welded) links up to link-homotopy [AM19].

The fundamental quandle

For the convenience of the reader, let us recall how the fundamental quandle encodes the (weak) peripheral system of a link. Recall that a peripheral system of a link $L$ is
we define, for any subset $X$ of the fundamental quandle $\lambda$ of the world of peripheral systems, this is done by choosing, for each component, a particular element. One can get a complete invariant of links by adding some information to it. In the remark that once one knows that these constructions are reciprocal bijections between peripheral systems and fundamental quandles with a choice of an element $x$ of peripheral systems.

Remark 6.8. It is a well-known consequence of a celebrated theorem of Waldhausen on Haken 3-manifolds [Wal68] that the peripheral system that we have just defined is a complete invariant of links up to equivalence. Recall that a reduced peripheral system $(RG_L, (P_i, \mu_i)_{i \in I})$ is obtained from a peripheral system $(G_L, (P_i, \mu_i)_{i \in I})$ by taking a certain quotient $RG_L$ of $G_L$ and, for each $i$ the images of the $\mu_i$ in this quotient (still denoted by $\mu_i$) and the products $P_iN_i$ of the image of $P_i$ (still denoted by $P_i$) with the normal closure $N_i = N(\mu_i)$ of $\mu_i$ in $RG_L$. Precisely, $RG_L$ (which is Milnor’s link group from [Mil54]) is the quotient $R(G_L, \{\mu_i\}_i)$, where we define, for any subset $X$ of any group $G$:

$$R(G, X) := G/\langle [x, gxg^{-1}] \mid x \in X, \ g \in G \rangle.$$
Let us remark that the construction $R(G, X)$ depends only on the set of conjugacy classes of elements of $X$, since the relations defining it are exactly the ones forcing two elements of each of these classes to commute. In particular, $RG_L = R(G_L, \eta(Q_L))$.

The main result of the present section is Corollary 6.11 below. It will follow from the classical equivalence between the fundamental quandle and the peripheral system, using the following algebraic result, whose proof takes up the rest of the section:

**Theorem 6.10.** Let $Q$ be a quandle, and let $(q_i)_{i \in I}$ be a family of representatives of its orbits. Let $P_i$ denote the image in $G(R(Q))$ of the stabilizer of $q_i$ in $G(Q)$ and let $N_i$ be the normal closure of $\eta(q_i)$ in $G(R(Q)) = R(G(Q), \eta(Q))$. Then we have a canonical isomorphism:

$$R(Q) \cong Q(R(G(Q), \eta(Q)), (P_i N_i, \eta(q_i))_{i \in I}).$$

**Proof.** Apply Proposition 4.1 to the action of $G(R(Q))$ on $R(Q)$ (as in Example 4.2), using Lemma 6.14 for the description of the enveloping group, and Proposition 6.15 for the description of stabilizers. □

The reduced peripheral of link is an invariant only up to equivalence, which is defined exactly as for peripheral systems.

**Corollary 6.11.** As an invariant of links, the reduced fundamental quandle (up to isomorphism) is equivalent to the (weak) reduced peripheral system (up to equivalence).

**Proof.** Apply Theorem 6.10 to $Q = Q_L$, for a link $L$, recalling that the peripheral subgroup $P_i$ of $G_L$ is the stabilizer of $\mu_i \in Q_L$ [Joy82a, Th. 16.1]. As in the case of peripheral systems, it is easy to see that an isomorphism of quandles corresponds to an equivalence of peripheral systems. Precisely, isomorphisms preserving the choices of representatives of orbits will induce isomorphisms of reduced peripheral systems, and changing $q_i$ to $q'_i = g_i \cdot q_i$ in the same orbit will induce conjugation of $P_i N_i$ and $\eta(q_i)$ by $g_i$. □

**Remark 6.12.** This was partially obtained as [Hug11, Th. 4.5], who showed that part of the reduced peripheral system could be obtained from the reduced fundamental quandle.

**Remark 6.13.** Like for links up to isotopy (Remark 6.8), our version of the reduced peripheral system is a weak one, compared to the (strong) one shown to be a complete invariant of (welded) links up to welded link-homotopy in [AM19]. Again, this strengthening corresponds to choosing preferred longitudes. So one way to turn the result of [AM19] into a result about quandles would be to add some information to our reduced quandles (which could probably take the form of classes in its cohomology) in order to get an invariant equivalent to theirs. Alternatively, one could look for a result about the strength of the reduced fundamental quandle as an invariant of welded links. In fact, we conjecture that the reduced fundamental quandle is an complete invariant of welded links up to taking mirror images of non-split components (in the 4-dimensional sphere).

Our first step towards proving Theorem 6.10 is the following Lemma which, applied to $Q_L$, gives $RG_L = G(R(Q_L))$.

**Lemma 6.14.** For any quandle $Q$, $R(G(Q), \eta(Q)) \cong G(R(Q))$. 

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Proof. We can compare the universal properties. Namely, a group morphism from $R(G(Q), \eta(Q))$ to a group $G$ is the same thing as a morphism $f$ from $G(Q)$ to $G$ such that $f(g\eta(q)g^{-1})$ commutes with $f(\eta(q))$ for all $q \in Q$ and $g \in G$. Since $\eta(q)g^{-1} = \eta(g \cdot q)$ (for all $q \in Q$ and $g \in G$), we can rephrase the condition on $f$ as follows: $f$ must send $(g\eta(q)g^{-1})\eta(q)(g\eta(q)g^{-1})^{-1} = \eta(\eta(g \cdot q) \cdot q) = \eta((g \cdot q) \triangleright q)$ and $\eta(q)$ to the same element. But the action of $G(Q)$ on $Q$ factors through its image $\text{Inn}(Q)$ in $\text{Aut}(Q)$, so (using the universal property of $\text{Inn}(Q)$), this is the same thing as a quandle morphism from $Q$ to $G$ identifying $\varphi(q) \triangleright q$ and $q$ for all $\varphi \in \text{Inn}(Q)$ and all $q \in Q$. By definition of $R(-)$, this is the same thing as a quandle morphism from $R(Q)$ to $G$ or, equivalently, as a group morphism from $G(R(Q))$ to $G$. Thus $\text{Hom}(R(G(Q), \eta(Q)), -) = \text{Hom}(G(R(Q)), -)$, whence our claim.

For a quandle $Q$, let us consider the following commutative diagram:

$$
\begin{array}{ccc}
G(Q) & \longrightarrow & \text{Aut}(Q) \\
\downarrow & & \downarrow \\
G(R(Q)) & \longrightarrow & \text{Aut}(R(Q)).
\end{array}
$$

The following result is the second step in the proof of Theorem 6.10, allowing us to describe stabilizers in the action of $G(R(Q))$ on $R(Q)$:

**Proposition 6.15.** Let $Q$ be a quandle, and $q \in Q$. The stabilizer of $\bar{q} \in R(Q)$ under the action of $G(R(Q))$ is the product $PN$ of the image $P$ of the stabilizer of $q$ in $G(Q)$ with the normal closure $N$ of $\eta(q)$ in $G(R(Q))$.

In order to prove the proposition, we need to understand precisely how the quotient $Q \to R(Q)$ works. Namely, we need to understand the congruence generated by the relations $\varphi(x) \triangleright x \sim x$ for $x \in Q$ and $\varphi \in \text{Inn}(Q)$, or, equivalently, $(g \cdot x) \triangleright x \sim x$ for $x \in Q$ and $g \in G(Q)$. Let us first remark that the action of $G(Q)$ on the quotient $R(Q)$ factors through $G(R(Q))$, so the kernel $K$ of $G(Q) \to G(R(Q))$ acts trivially on the quotient. In other words, $Q \to R(Q)$ can be factored as $Q \to Q/K \to R(Q)$.

**Remark 6.16.** This intermediate quotient $Q/K$ is the one called “operator-reduced” in [Hug11].

Then, remark that since $(g \cdot x) \triangleright x = (g\eta(x)g^{-1}) \cdot x$, the class of $x$ must contain $N_x \cdot x$, where $N_x$ is the subgroup of $G(Q)$ normally generated by $\eta(x)$. Remark that the image of $N_x \cdot x$ in $Q/K$ is also $N_x \cdot x$, if we still denote by $x$ the class of $x$ modulo $K$ and by $N_x$ the subgroup of $G(Q/K)$ normally generated by $\eta(x)$.

**Lemma 6.17.** In $Q/K$, the $N_x \cdot x$ are the classes of a congruence relation.

**Proof.** Let $x$ and $y$ be elements of $Q/K$ such that $(N_x \cdot x) \cap (N_y \cdot y) \neq \emptyset$. This means that $g \cdot x = h \cdot y$ for some $g \in N_x$ and $h \in N_y$. Then $g\eta(x)g^{-1} = \eta(g \cdot x) = \eta(h \cdot y) = h\eta(y)h^{-1}$. In particular, $\eta(x)$ and $\eta(y)$ generate the same normal subgroup $N_x = N_y$ of $G(Q/K)$. This subgroup clearly contains $h^{-1}g$, hence:

$$N_x \cdot x = N_x \cdot h^{-1}g \cdot x = N_y \cdot y.$$

This proves that the $N_x \cdot x$ are a partition of $Q/K$. In order to show that the corresponding equivalence relation is a congruence relation, we need to show that for $x, y \in Q/K$, $g \in N_x$ and $h \in N_y$, we have $(g \cdot x) \triangleright (h \cdot y) \sim x \triangleright y$. This uses the fact that the action
of $K$ is trivial. Namely, by definition of $K$, we have $[\eta(x)^{-1}, g] \in \{\eta(x)^{-1}, N_x\} \subseteq K$, so $(\eta(x)^{-1} g \eta(x) g^{-1}) \cdot (h \cdot y) = h \cdot y$, whence:

$$(g \cdot x) \triangleright (h \cdot y) = (g \eta(x) g^{-1}) \cdot (h \cdot y) = \eta(x) \cdot (h \cdot y) = x \triangleright (h \cdot y).$$

Finally, $x \triangleright (h \cdot y) = \eta(x) h \eta(x)^{-1} \cdot (\eta(x) \cdot y) \in N_{x \triangleright y} \cdot (x \triangleright y)$, since $\eta(x) N_y \eta(x)^{-1} = N_{x \triangleright y}$ (both are normally generated by $\eta(x \triangleright y) = \eta(x) \eta(y) \eta(x)^{-1}$).

\begin{proof}[Remark 6.18] The first part of the proof works in general to show that the $N_x \cdot x$ are the classes of an equivalence relation. But this equivalence relation is not a congruence in general. In fact, the calculations in the second part of the proof show that it is a congruence relation if and only if $K$ (which is normally generated by the $[x, N_x]$ for $x \in Q$) acts trivially.

By construction, the congruence relation of Lemma 6.17 (or its preimage by $Q \to Q/K$) is the one we are interested in, namely, the one generated by the relations $\varphi(x) \triangleright x \sim x$ for $x \in Q$ and $\varphi \in \text{Inn}(Q)$. Thus, $Q/K \to R(Q)$ is well-understood: its fibers are the $N_x \cdot x$. Finally, we are ready to prove the proposition:

\begin{proof}[Proof of Prop. 6.15] First, remark that $P$ is clearly contained in $\text{Stab}(\bar{q})$, and that the latter must also contain $N$ (by the same argument as in the proof of Prop. 6.5). Since $N \triangleleft G(R(Q))$, the product $NP$ is the subgroup of $G(R(Q))$ generated by $N$ and $P$.

Each element of $G(R(Q))$ is the class $\bar{g}$ of some $g \in G(Q)$. Suppose that $\bar{g}$ fixes the element $\bar{q} \in R(Q)$, which is the class of $q \in Q$. The preimage of $\bar{g}$ by the projection $Q/K \to R(Q)$ is $N_q \cdot q$, so we have $g \cdot q \equiv n \cdot q \ [K]$ for some $n \in N_q$. In other words, $g \cdot q = k n \cdot q$ for some $n \in N_q$ and some $k \in K$. But this equation means that $(kn)^{-1} g$ lies in the stabilizer $\widetilde{P}$ of $q$ in $G(Q)$. Finally, $g \in N_q K \widetilde{P}$, whence $\bar{g} \in NP$.\end{proof}

\section{Nilpotent quandles and invariants of loop braids}

Here we give yet another characterisation of nilpotent quandles, this time in terms of braid colourings.

\subsection{Automorphisms of free quandles}

On the one hand, the group $wB_n$ of loop braids (also called welded braids) identifies with the group of basis-conjugating automorphisms of the free group $[FRR97]$. On the other hand, the free quandle $Q_n$ on $n$ generators $x_1, \ldots, x_n$ identifies with the union of the conjugacy classes of the $x_i$ in the free group $F_n$, which is its enveloping group (as recalled at the beginning of §2.3). The functor $G(\_)$ induces a map $G_* : \text{Aut}(Q_n) \to \text{Aut}(G(Q_n)) = \text{Aut}(F_n)$, extending any automorphism of $Q_n$ to an automorphism of $F_n$. Since the $x_i$ generate $F_n$, an automorphism of $F_n$ is determined by the image of the $x_i$, hence this map must be injective. Moreover, an element in the image of $G_*$ must send each generator $x_i$ into $Q_n$, whence $\text{Im}(G_*) \subseteq wB_n$. This inclusion is in fact an equality: any basis-conjugating automorphism of $F_n$ must preserve $Q_n \subseteq F_n$, so it must belong to the image of $G_*$. Thus, we have recovered the classical:

$$\text{Aut}(Q_n) \cong wB_n.$$  

Such an identification has the following consequence: for any quandle $Q$, the group $wB_n$ acts (from the right) on the set $\text{Hom}(Q_n, Q) \cong Q^n$. This is the action via Nielsen
transformation [MKS76, Chap. 3], which is similar to the action of \( \text{Aut}(F_n) \) on the set \( \text{Hom}(F_n, G) \cong G^n \) for any group \( G \). The latter has the following description: if \( \varphi \) is an automorphism of \( F_n \) sending each \( x_i \) on a word \( w_i(x_1, \ldots, x_n) \), then:

\[
\varphi \cdot (g_1, \ldots, g_n) = (w_i(g_1, \ldots, g_n))_i.
\]

It is easy to see that the same description applies to the action of \( wB_n \) on \( Q^n \): if \( \beta \in wB_n \) sends each \( x_i \) on \( w_{x_{\sigma(i)}} \) (\( \sigma \in \Sigma_n \) and \( w_i = w_i(x_1, \ldots, x_n) \in F_n \)), then:

\[
\beta \cdot (q_1, \ldots, q_n) = (w_i(q_1, \ldots, q_n) \cdot q_{\sigma(i)})_i,
\]

where \( w_i(q_1, \ldots, q_n) \in G(Q) \) is obtained from \( w_i \) by evaluating each \( x_j \) at \( \eta(q_j) \in G(Q) \).

In fact, this action may already be known to the reader, with a more geometric description: it corresponds to colouring (welded) braids by elements of our quandle. Precisely, if we colour the top of the strands of \( \beta \in wB_n \) by the colours \( q_1, \ldots, q_n \), there is a unique way to colour the rest of the braid by following the rules of Figure 1, and \( \beta \cdot (q_1, \ldots, q_n) \) is exactly the list of colours that we get at the bottom of our braid.

![Figure 1: Colouring of a braid by a quandle](image)

7.2 Nilpotency via Nielsen transformations

We have seen (Prop. 2.18) that the free nilpotent quandle \( Q_{n,c} = Q_n/\Gamma_c \) of class \( c \) on \( n \) generators embeds in the free nilpotent group \( F_{n,c} = F_n/\Gamma_{c+1} \) and that, as a consequence, it identifies with the union of the conjugacy classes of the \( x_i \) in \( F_{n,c} \). We can thus use the same reasoning as in §7.1 to describe its group of automorphisms as the group \( \text{Aut}_C(F_{n,c}) \) of basis-conjugating automorphisms of \( F_{n,c} \). The image of the canonical morphism from \( wB_n = \text{Aut}(Q_n) \) to \( \text{Aut}_C(F_{n,c}) = \text{Aut}(Q_{n,c}) \) identifies with the quotient of \( wB_n \) by the \( c \)-th Andreadakis kernel \( \mathcal{A}_C^c \), which consists of (basis-conjugating) automorphisms of \( F_n \) acting trivially on \( F_{n,c} \). This image is a proper subgroup of \( \text{Aut}(Q_{n,c}) \) if \( n \geq 2 \) and \( c \geq 3 \), called the subgroup of tame automorphisms of \( Q_{n,c} \) (see the appendix for a proof of this).

**Theorem 7.1.** Let \( c \geq 1 \) be an integer, and \( Q \) be a quandle. The following are equivalent:

- \( Q \) is \( c \)-nilpotent,
- The canonical action of \( \Gamma_c(wP_n) \) on \( Q^n \) trivial for all \( n \),
- The canonical action of \( \Gamma_c(wP_n) \) on \( Q^n \) trivial for any \( n > c \)

**Proof.** If \( Q \) is \( c \)-nilpotent, then \( Q^n = \text{Hom}(Q_n, Q) = \text{Hom}(Q_n/\Gamma_c, Q) \). Thus the action of \( wB_n \) on \( Q^n \) factorizes through \( wB_n/\mathcal{A}_c \). Since \( \Gamma_c(wB_n) \subset \mathcal{A}_c \), it acts trivially on \( Q^n \).

Suppose now that the action of \( \Gamma_c(wB_n) \) on \( Q^n \) is trivial for some fixed \( n > c \). Let \( w = [x_1, [x_2, \ldots, x_c, \ldots]] \in F_n \), and consider the automorphism \( \gamma \) of \( F_n \) fixing all \( x_i \) save
$x_{c+1}$, which it sends to $x_{c+1}$ (it is an automorphism because $x_{c+1}$ does not appear in $w$, so $x_{c+1} \mapsto x_{c+1}^w$ defines its inverse). The corresponding Nielsen transformation sends $(q_i) \in Q^n$ to $(q_1, \ldots, q_c, w(q_1, \ldots, q_c) \cdot q_{c+1}, q_{c+2}, \ldots, q_n)$, where $w(q_1, \ldots, q_c)$ is computed in $G(Q)$ (precisely, it is $w(\eta(q_1), \ldots, \eta(q_c)))$. Thus, $\gamma$ acts trivially on $Q^n$ if and only if $w(q_1, \ldots, q_c) \cdot q_{c+1} = q_{c+1}$ for all $q_1, \ldots, q_c \in Q$. Which means exactly that $w(q_1, \ldots, q_c)$ acts trivially on $Q$, for all $q_1, \ldots, q_c \in Q$. Since these elements $w(q_1, \ldots, q_c)$ generate $\Gamma_c(G(G))$ (Proposition 2.8), this means that $\Gamma_c(G(G))$ acts trivially on $Q$, hence $Q$ is $c$-nilpotent (Lemma 2.4).

\section{Appendix : Nilpotent racks}

We now review how our notion of nilpotency can be adapted to racks. Recall that \textit{racks} are defined by the first two axioms of definition 1.1. Compared to quandles, the third axiom is no longer required. Since this axiom is the one ensuring that colourings are compatible with the first Reidemeister move, racks can no longer be used to colour knots. They are, however, very suited to colouring \textit{braids}, for which the first Reidemeister move is irrelevant. In particular, they play a prominent role in the theory of set-theoretic solutions of the Yang Baxter equation, where a rack (called the \textit{derived solution}) is associated to any non-degenerate solution (see [Sol00, Th. 2.3] or [LV17, Def. 5.8]).

The inclusion of quandles into racks is that of a subvariety (in the sense of universal algebra), so it has a left adjoint $Q(-)$ (in fact, $Q(R)$ is the quotient of $R$ by the relations $x \triangleright x = x$). By composing adjunctions, we see that $\iota \circ c$ has a left adjoint $G \circ Q$, and we usually denote by $G(R)$ the group $G(Q(R))$, which is the usual enveloping group (or \textit{structure group}) of a rack. Remark that the natural map from $R$ to $G(R)$ factors through $Q(R)$, so it cannot be injective unless $R$ is actually a quandle. As a consequence, injectivity issues do not make sense for general racks, and we can disregard them in this appendix.

Looking for a notion of $c$-nilpotency having nice properties and giving back the previous one for racks that are in fact quandles, we find two different (but closely related) notions that seem worth investigating. The nicest one comes from a direct generalisation of Definition 2.1:

\textbf{Definition A.1.} A rack $R$ is called \textit{nilpotent of class at most $c$}, or \textit{$c$-nilpotent}, if its group of inner automorphism is $(c-1)$-nilpotent.

With this definition, most of our the results obtained for quandles can be generalised easily to racks, with little to no change. Precisely, here is a list of the changes required:

- The characterisation in terms of coverings (Th. 2.12) needs to be modified slightly: a rack is $c$-nilpotent if and only if it is a $(c-1)$-fold covering of a trivial rack. This is slightly stronger than being a $c$-fold covering of the one-point rack – see the discussion below.

- The free $c$-nilpotent rack is easier to understand than the free $c$-nilpotent quandle. Recall that the free rack on a set $X$ is $R[X] = F[X] \times X$, with the law $(v, x) \triangleright (w, y) = (v x v^{-1}, w, y)$. Its enveloping group is $F[X]$, acting by left multiplication on the first factor, so the free $c$-nilpotent rack is the quotient by the action of $\Gamma_c(F[X])$, which is $(F[X]/\Gamma_c) \times X$.

- In Corollary 3.7, the conclusion should be replaced by: our rack is generated by a single element. Which means that it is isomorphic $\mathbb{Z}/n$, endowed with the law $i \triangleright j = j + 1$, for some $n \geq 0$. 

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• Remark 4.7 is the key to adapting results of sections 4 and 5: the condition $z_i \in H_i$ (which becomes $e_i \in H_i$ later on) needs to be dropped. Apart from this, the results of both these sections can be adapted to racks in a straightforward way.

• It does not mean anything for a rack to be reduced (resp. injective) unless it is a quandle, so results about reduced (resp. injective) quandles do not have any meaningful generalisation to the present context.

• The results of §7 hold without change.

The first of these changes points to another notion of nilpotency for racks: let us call a rack weakly \(c\)-nilpotent if it is a \(c\)-fold covering of the one-point rack. Since any trivial rack is a covering of the one-point rack, a \(c\)-nilpotent rack is weakly \(c\)-nilpotent. Since the one-point rack is a trivial rack, a weakly \(c\)-nilpotent rack is \((c + 1)\)-nilpotent.

**Example A.2.** The free rack \(R_1\) on one generator (which is \(\mathbb{Z}\) endowed with the law \(i \triangleright j = j + 1\)) is a covering of the one-point rack, but it is not a trivial rack: \(\text{Inn}(R_1) \cong \mathbb{Z}\). Thus, it is weakly 1-nilpotent, but only 2-nilpotent, and not 1-nilpotent.

Weak nilpotency is the same as nilpotency, even if the nilpotency class may differ. Thus, all the results assuming only nilpotency that work for nilpotent racks do hold for weakly nilpotent ones: they are generated by representatives of orbits, they are Hopfian, etc. However, the results depending on the nilpotency class cannot be adapted directly. For instance, we loose the description of the universal \(c\)-nilpotent quotient as the quotient by a group action. Nevertheless, we are going to show that such a quotient does exist, by showing that the subcategory of weakly \(c\)-nilpotent racks is in fact a subvariety of the variety of racks, described by equations very similar to the ones from Corollary 2.10.

Before showing this, let us examine the reason why our two notions agree for quandles. A rack is a covering of the one-point rack if and only if it all its elements behave in the same way. For quandles, they then have to behave trivially, since \(x \triangleright y = y \triangleright y = y\), so being a covering of the one-point quandle is the same as being trivial. But for racks, these two notions may differ: the equations \(x \triangleright y = y \triangleright y = y\) define the same subvariety for quandles, but not for racks. The following proposition generalises this by exhibiting equations defining weak \(c\)-nilpotency which are equivalent to the ones from Corollary 2.10 for quandles (take \(x_1 = x_2\)), but not for racks.

**Proposition A.3.** A rack \(R\) is weakly \(c\)-nilpotent if and only if it satisfies the following identities:

\[
\forall x_1, \ldots, x_{c+1}, x'_1 \in R, \ (\cdots ((x_1 \triangleright x_2) \triangleright x_3) \triangleright \cdots \triangleright x_{c+1}) = (\cdots ((x'_1 \triangleright x_2) \triangleright x_3) \triangleright \cdots \triangleright x_{c+1}).
\]

For the proof, we need to introduce the rack \(\text{Inn}(R)\), which is the image of \(R\) in \(\text{Aut}(R)\) by the rack morphism \(p_0 : x \mapsto x \triangleright (-)\). The latter (corestricted to \(\text{Inn}(R)\)) is a covering, and a surjective quandle morphism \(p : R \to S\) is a covering if and only if there is a (necessarily unique and surjective) morphism \(f : S \to \text{Inn}(R)\) such that \(f \circ p = p_0\).

**Proof of Proposition A.3.** For \(c = 1\), this is clear. Suppose that we already know the result for \(c - 1\). Remark that this implies that a quotient of a weakly \((c - 1)\)-nilpotent rack is again weakly \((c - 1)\)-nilpotent. For short, let us write \(w_c(x_1, \ldots, x_{c+1})\) for \((\cdots ((x_1 \triangleright x_2) \triangleright x_3) \triangleright \cdots) \triangleright x_{c+1}\). That \(w_c(x_1, \ldots, x_{c+1})\) does not depend on \(x_1\) means exactly that \(w_{c-1}(x_1, \ldots, x_c) \triangleright (-)\) does not. But the latter is \(p_0(w_{c-1}(x_1, \ldots, x_c)) = \)

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which, by the induction hypothesis, does not depend on \( x_1 \) if and only if \( \text{inn}(R) \) is weakly \((c-1)\)-nilpotent. So we need to show that \( R \) is weakly \( c \)-nilpotent if and only if \( \text{inn}(R) \) is weakly \((c-1)\)-nilpotent. One implication follows clearly from the fact that \( R \) is a covering of \( \text{inn}(R) \). Conversely, if \( R \) is weakly \( c \)-nilpotent, then there is a covering \( p : R \to S \) with \( S \) weakly \((c-1)\)-nilpotent, whence also its quotient \( \text{inn}(R) \).

**Remark A.4.** The same proof allows one to get a direct link between the characterisation of \( c \)-nilpotency in terms of identities (Cor. 2.10) and the one in terms of coverings (Th. 2.12).

We can deduce from this that every rack \( R \) has a *universal weakly \( c \)-nilpotent* \( N_c(R) \), which is the quotient of \( R \) by (the congruence generated by) the relations from Proposition A.3:

**Corollary A.5.** The inclusion of the (full) subcategory of weakly \( c \)-nilpotent racks into racks has a left adjoint \( N_c \).

**Remark A.6.** Since weakly \( c \)-nilpotency is weaker than \( c \)-nilpotency, the quotient \( N_c(R) \) of \( R \) is bigger than \( R/\Gamma_c \). However, since the two sets of relations are equivalent for quandles, we can see that \( Q(N_c(R)) = Q(R/\Gamma_c) \), which is also \( Q(R)/\Gamma_c \) (in other words, \( Q \circ N_c, Q \circ (-)/\Gamma_c \) and \( Q(-)/\Gamma_c \) are all left adjoint to the same functor of inclusion of \( c \)-nilpotent quandles into racks, so they are equal). Moreover, this implies that they \( N_c(R) \) and \( R/\Gamma_c \) have the same enveloping group \( G(R)/\Gamma_{c+1} \).

### B Appendix : Non-tame automorphisms

Let \( F_n \) denote the free group on \( n \) generators and \( F_{n,c} := F_n/\Gamma_{c+1} \) be its universal \( c \)-nilpotent quotient. It is known that the canonical morphism from \( \text{Aut}(F_n) \) to \( \text{Aut}(F_{n,c}) \) is not surjective if \( c \geq 3 \) and \( n \geq 2 \). Automorphisms of \( F_{n,c} \) not coming from an automorphism of \( F_{n,c} \) are called *non-tame* [BGLM90]. The main obstruction to tameness is the Morita-Satoh trace [BGLM90, Th. 3.2]. It is also the only one for \( n \geq c+1 \) [Dar19b, §2.6].

We now investigate the similar situation for quandles, which corresponds to the restriction to basis-conjugating automorphisms. Precisely, we show that the canonical morphism from \( \text{Aut}(Q_n) \cong \text{Aut}_C(F_n) \cong wB_n \) to \( \text{Aut}(Q_{n,c}) \cong \text{Aut}_C(F_{n,c}) \) is not surjective if \( c \geq 3 \) and \( n \geq 2 \), using the Morita-Satoh trace. Here, \( \text{Aut}_C \) denotes the set of automorphisms sending each element \( x_i \) of a fixed basis on a conjugate of some \( x_j \), and the isomorphisms are obtained using the reasoning from §7.1 (which can be readily adapted to the nilpotent case, as remarked at the beginning of §7.2). We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Aut}(F_n) & \xrightarrow{\varphi} & \text{Aut}(F_{n,c}) \\
\downarrow & & \downarrow \\
\text{Aut}_C(F_n) & \xrightarrow{\varphi} & \text{Aut}_C(F_{n,c}) \\
\cong \text{Aut}(Q_n) & & \cong \text{Aut}(Q_{n,c}),
\end{array}
\]

where \( \varphi \) denotes both the canonical maps.
Disclaimer: This appendix is using techniques that are very different form the rest of the paper. It is also quite technical. However, the result are quite striking, and we feel that even the non-specialist could benefit from reading it and getting a taste of the techniques involved.

B.1 Andreadakis filtrations

Let us recall [Dar19b, §1.3] that for any group $G$, the automorphism group $\text{Aut}(G)$ is filtered by the Andreadakis filtration $\mathcal{A}_*(G)$, where $\mathcal{A}_j(G)$ is the subgroup of automorphisms acting trivially on $G/\Gamma_{j+1}$:

$$\text{Aut}(G) = \mathcal{A}_0 \subseteq \mathcal{A}_1(G) \subseteq \mathcal{A}_2(G) \subseteq \cdots.$$  

Let us define

$$\text{gr}_j(\mathcal{A}_*(G)) := \mathcal{A}_j(G)/\mathcal{A}_{j+1}(G).$$

If $j \geq 1$, then this quotient is abelian, and the direct sum of these groups bears the structure of a graded Lie ring $\text{gr}(\mathcal{A}_*(G))$, whose bracket is induced by commutators in $\text{Aut}(G)$.

Let us remark that $\ker(\varphi) = \mathcal{A}_c(F_n)$, so the previous square becomes a square of injections:

$$\begin{array}{ccc}
\text{Aut}(F_n)/\mathcal{A}_c(F_n) & \xleftarrow{\varphi} & \text{Aut}(F_{n,c}) \\
\uparrow & & \uparrow \\
\text{Aut}_C(F_n)/\mathcal{A}^C_c(F_n) & \xleftarrow{\varphi} & \text{Aut}_C(F_{n,c}),
\end{array}$$

where $\mathcal{A}^C_c(F_n) := \mathcal{A}_c(F_n) \cap \text{Aut}_C(F_n)$ is the filtration on $\text{Aut}_C(F_n)$ induced by the Andreadakis filtrations of $\text{Aut}(F_n)$. Remark that the filtrations considered on these groups are finite ones, since $\mathcal{A}_{c+1}(F_{n,c}) = \{id\}$, by construction. On the associated graded, one gets, for $j \geq c$:

$$\begin{array}{ccc}
\text{gr}_j(\mathcal{A}_*(F_n)) & \xleftarrow{\varphi_j} & \text{gr}_j(\mathcal{A}_*(F_{n,c})) \\
\uparrow & & \uparrow \\
\text{gr}_j(\mathcal{A}^C_c(F_n)) & \xleftarrow{\varphi_j} & \text{gr}_j(\mathcal{A}^C_c(F_{n,c})).
\end{array}$$

The vertical maps are injective, by definition of $\mathcal{A}^C_c(F_n)$ and $\mathcal{A}^C_c(F_{n,c}).$ The injectivity of the horizontal maps comes from $\varphi^{-1}(\mathcal{A}_{j+1}(F_{n,c})) = \mathcal{A}_{j+1}(F_n)$. We are looking to obstructions to the surjectivity of the horizontal maps, from which we will deduce that the $\varphi$ are not surjective either.

- For $j = 0$, the quotients in the square are the images of the previous automorphism groups in $\text{Aut}(F_n^{ab}) = \text{GL}_n(\mathbb{Z})$. The top groups are both identified to $\text{GL}_n(\mathbb{Z})$, the bottom ones to the subgroup of permutation matrices, and both $\varphi_0$ are isomorphisms.
- For $j \geq 1$, these squares are the homogeneous components of a commutative square of graded Lie rings, which we now investigate.
B.2 Associated Lie rings.

Recall [Dar19b, §1.4] that for any group $G$, the action of $\text{Aut}(G)$ on $G$ induces and action of $\text{gr}(\mathcal{A}_s(G))$ on $\text{gr}(G) := \text{gr}(\Gamma_s(G))$ by homogeneous derivations. This action can be encoded by a morphism of graded Lie rings:

$$\tau : \text{gr}(\mathcal{A}_s(G)) \hookrightarrow \text{Der}_s(\text{gr}(G)),$$

which is injective by definition of $\mathcal{A}_s(G)$. This morphism, called the Johnson morphism associated to $G$ (by analogy with the usual Johnson morphism for mapping class groups), is given by the explicit formula:

$$\forall \sigma \in \mathcal{A}_j, \forall x \in \Gamma_j(G), \quad \tau(\sigma) : x \mapsto \overline{\sigma}(x)x^{-1}.$$

Let us recall that derivations of the free Lie ring decompose uniquely as sums of homogeneous derivations. This holds also for the free $c$-nilpotent Lie ring, and $\text{Der}(\mathbb{L}_{n,c})$ is in fact the truncation $\text{Der}_{<c}(\mathbb{L}_n)$ of $\text{Der}(\mathbb{L}_n)$. The following result is due to Bartholdi [Bar13, Th. 5.1] (see also [Dar19b, Cor. 2.46]):

**Proposition B.1.** The Johnson morphism associated to $\mathbb{F}_{n,c}$ is an isomorphism:

$$\text{gr}(\mathcal{A}_s(\mathbb{F}_{n,c})) \cong \text{Der}(\mathbb{L}_{n,c}) \cong \text{Der}_{<c}(\mathbb{L}_n).$$

Let $\text{Der}_1(\mathbb{L}_n)$ be the Lie subalgebra of tangential derivations of $\text{Der}(\mathbb{L}_n)$, which are the ones sending each element $x_i$ of the fixed basis to $[x_i, l_i]$, for some $l_i \in \mathbb{L}_n$. It is easy to see that the Johnson morphisms sends classes of basis-conjugating automorphisms to tangential derivations. Moreover, we can adapt the previous result to this context:

**Proposition B.2.** The previous isomorphism identifies the subalgebra $\text{gr}(\mathcal{A}_s(\mathbb{F}_{n,c}))$ of $\text{gr}(\mathcal{A}_s(\mathbb{F}_{n,c}))$ with the subalgebra $\text{Der}_1(\mathbb{L}_{n,c}) \cong \text{Der}_1(\mathbb{L}_{n,c})$ of $\text{Der}_{<c}(\mathbb{L}_n)$.

**Proof.** The Johnson morphism is injective by construction, and sends classes of basis-conjugating automorphisms to tangential derivations. We need to show that every homogeneous tangential derivation is the image of such an automorphism of $\mathbb{F}_{n,c}$. Let $d$ be such a derivation, of degree $k < c$, sending $x_i$ to $[x_i, l_i]$, for some $l_i \in (\mathbb{L}_n)_k$. This $l_i$ is the class of some $w_i \in \text{gr}_k(F_{n,c})$ in $(\mathbb{L}_n)_k \cong \Gamma_k(F_n)/\Gamma_{k+1}(F_n)$. Let us define an endomorphism $f$ of $\mathbb{F}_{n,c}$ by $x_i \mapsto w_i x_i w_i^{-1}$. Since $f$ induces the identity on $F_{n,c}^{ab}$, it is an automorphism [Dar19b, Lem. 2.42]. Moreover, it is clear that $f \in \mathcal{A}_k(\mathbb{F}_{n,c})$, and that $\tau(f) = d$. \hfill \Box

The Morita-Sato trace can be described using free differential calculus [Dar19b], but we will use the following explicit description:

$$\text{Tr} : \text{Der}_k(\mathbb{L}_n) \cong V^* \otimes (\mathbb{L}_n)_{k+1} \overset{\iota}{\longrightarrow} V^* \otimes V \otimes V \overset{\Phi}{\longrightarrow} V \otimes V \longrightarrow C_k(V),$$

where $V := \mathbb{Z}^n$, $V^*$ is its dual, $\iota$ is induced the usual inclusion of the free Lie algebra into the tensor algebra, $C_k(V) = V \otimes k/(\mathbb{Z}/k)$ is the cyclic power, and $\Phi$ is the contraction with respect to the first variable:

$$\Phi(\omega \otimes v_1 \cdots v_{k+1}) = \omega(v_1)v_2 \cdots v_{k+1}.$$ 

**Proposition B.3.** [Dar19b, Prop. 2.36]. The trace vanishes on the image of the Johnson morphism $\tau : \text{gr}(\mathcal{A}_s(F_n)) \hookrightarrow \text{Der}(\mathbb{L}_n)$.
Using Propositions B.1 and B.2, we can see that the commutative square of the previous section identifies with the left square in:

\[
\begin{array}{ccc}
\text{gr}_{<c}(A_4(F_n)) & \overset{\tau}{\longrightarrow} & \text{Der}(\mathbb{L}_{n,c}) \\
\uparrow & & \uparrow \text{Tr} \\
\text{gr}_{<c}(A_4^c(F_n)) & \overset{\tau}{\longrightarrow} & \text{Der}_l(\mathbb{L}_{n,c}).
\end{array}
\]

The following formulas define an endomorphism of \( F_{2,3} \):

\[
\begin{cases}
a \mapsto [a, b]a[a, b]^{-1} \\
b \mapsto b
\end{cases}
\]

This endomorphism \( \sigma \) acts trivially on the abelianisation, so it is an automorphism [Dar19b, Lem. 2.42]: \( \sigma \in \text{Aut}_C(F_{2,3}) \). In fact, it acts trivially modulo \( \Gamma_3 \), so it is in \( A_2(F_{2,3}) \). The associated (tangential) derivation \( \tau(\bar{\sigma}) \) sends \( B \) to 0 and \( A \) to \([ [A, B], A] \).

Its trace is thus:

\[
\text{Tr}(\tau(\bar{\sigma})) = -[A, B] + BA = BA \in C_2.
\]

It is not trivial, so, by Proposition B.3, \( \tau(\bar{\sigma}) \) cannot be \( \tau(\bar{\varphi}) \) for \( \varphi \in \text{Aut}(F_2) \), which implies that \( \sigma \) cannot come from an automorphism of \( F_2 \).

**Remark B.7.** For \( c = 1 \) and \( c = 2 \), there are no non-tame automorphisms of \( Q_{n,c} \). For \( c = 1 \), this is easy to see: \( Q_{n,1} \) is trivial with \( n \) elements, so its automorphism group is just the symmetric group \( S_n \), and every permutation lifts trivially to a basis-conjugating automorphism of the free group \( F_n \) (the projection \( \text{Aut}_C(F_n) \rightarrow \text{Aut}_C(\mathbb{Z}^n) = S_n \) is in fact split). For \( c = 2 \), we need to use the tools introduced above. Namely, \( \text{Aut}_C(F_{n,2}) \) is an extension of \( S_n \) by \( A_2^c(F_{n,2}) \). Since \( A_2^c(F_{n,2}) = 1 \), Proposition B.2 says that \( A_2^c(F_{n,2}) \cong \text{Der}_l(\mathbb{L}_{n,2}) \subset \text{Hom}(\mathbb{Z}^n, A_2(\mathbb{Z}^n)) \), via the Johnson morphism \( \tau_1 \) (where \( \mathbb{L}_{n,2} \) is identified with \( A_2(\mathbb{Z}^n) \)) via \([u, v] \mapsto u \wedge v \). Then \( \varphi \) induces a morphism of extensions:

\[
\begin{array}{ccc}
P\Sigma_n & \longrightarrow & \text{Aut}_C(F_n) \\
\downarrow \tau_1 & & \downarrow \varphi \\
\text{Der}_l(\mathbb{L}_{n,2}) & \longrightarrow & \text{Aut}_C(F_{n,2}) \rightarrow S_n,
\end{array}
\]

where the kernel \( P\Sigma_n \) (the McCool group) is the group of automorphisms of the form \( x_i \mapsto w_i x_i w_i^{-1} \). A basis of \( \text{Der}_l(\mathbb{L}_{n,2}) \) is given by the \( d_{ij} : e_k \mapsto \delta_{ik} \cdot e_i \wedge e_j \) (for \( i \neq j \)). Since \( e_i \wedge e_j \) is the class of \([x_i, x_j] \), we have \( \delta_{ik} = \tau(K_{ij}) \), where \( K_{ij} \) fixes the \( x_k \) if \( k \neq i \) and \( K_{ij}(x_i) = x_j x_i x_j^{-1} \). Thus \( \tau_1 \) is surjective, and so is \( \varphi \).
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