MARGIN-CLOSED VECTOR AUTOREGRESSIVE TIME SERIES MODELS

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Conditions are obtained for a Gaussian vector autoregressive time series of order \( k \), \( \text{VAR}(k) \), to have univariate margins that are autoregressive of order \( k \) or lower-dimensional margins that are also \( \text{VAR}(k) \). This can lead to \( d \)-dimensional \( \text{VAR}(k) \) models that are closed with respect to a given partition \( \{S_1, \ldots, S_n\} \) of \( \{1, \ldots, d\} \) by specifying marginal serial dependence and some cross-sectional dependence parameters. The special closure property allows one to fit the subprocesses of multi-variate time series before assembling them by fitting the dependence structure between the subprocesses. We revisit the use of the Gaussian copula of the stationary joint distribution of observations in the \( \text{VAR}(k) \) process with non-Gaussian univariate margins but under the constraint of closure under margins. This construction allows more flexibility in handling higher-dimensional time series and a multi-stage estimation procedure can be used. The proposed class of models is applied to a macro-economic data set and compared with the relevant benchmark models.

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1. INTRODUCTION

The stationary Gaussian vector autoregressive (\( \text{VAR} \)) model of order \( k \) is a common model for multi-variate time series; see Lütkepohl (2005) and Wei (2006). It is a Markov model of order \( k \), but it does not imply that its univariate marginal time series are Markov of order \( k \) or autoregressive with finite order. In the copula and other multi-variate literature, it is common to start with univariate models and then add dependence to model the multi-variate relations. This is important because there are many diagnostics that help in selecting univariate models.

Despite extensive literature on \( \text{VAR} \) models, the conditions for when \( d \)-variate stationary Gaussian \( \text{VAR}(k) \) time series have univariate \( \text{AR}(k) \) margins or lower-dimensional \( \text{VAR}(k) \) margins have not been previously studied. It is clear that the marginal closure property is satisfied if the \( k \) coefficient matrices are diagonal, and the dependence comes only from the innovations for the \( d \) univariate time series. More generally, one partial univariate margin of the \( \text{VAR}(k) \) time series is \( \text{AR}(k) \) if the corresponding rows of the coefficient matrices are all zero except the diagonal element. The main goal of this article is to obtain conditions for margins to be \( \text{AR}(k) \) or \( \text{VAR}(k) \), even if the coefficient matrices are not diagonal and do not have any row with only one non-zero element. The conditions are obtained via some conditional independence relations that have to be satisfied for the marginal closure property.

Our motivation comes from multi-variate time series with variables that do not have Gaussian distributions. Assuming stationarity, univariate models can be chosen and then fitted, followed by transformation to standard normal. If the times series on the transformed scale have diagnostics that suggest \( \text{AR}(k) \), with \( k \) as the largest order over \( d \) univariate time series, then a simple idea of building dependence between \( d \) univariate time series is to try Gaussian \( \text{VAR}(k) \) dependence as a copula for the \( d \) univariate time series. In this case, we can make use of
the conditions for closure under margins to have a copula model with fewer parameters than VAR(k) without the closure property. To use other non-Gaussian copulas for the \( d \) univariate time series, more research is needed to meet the conditions for both closure under margins and strict stationarity.

The use of copulas from Gaussian VAR to represent dependence has been considered in Biller and Nelson (2003) and Biller (2009), which leads to multi-variate time series with non-Gaussian margins. However, these authors do not consider parsimonious forms resulting from closure under margins restrictions.

The special property of closure under margins enables the new framework of modeling high-dimensional time series by modeling its low-dimensional subprocesses and then assembling them. The framework makes it more flexible in analyzing marginal behavior of the multi-variate time series, and including additional marginal components. It also enables a multi-stage estimation procedure.

The rest of this article is organized as follows. Section 2 introduces notation for different parameterizations of VAR models and gives an overview of stationary multi-variate time series models with multi-variate Gaussian copulas. Section 3 discusses the property of closure under margins, gives a sufficient condition for it to hold for VAR models, and provides details of parameterizing the margin-closed multi-variate time series models. Section 4 discusses estimation of the VAR models assuming closure under margins. Section 5 presents some numerical examples and the results of an application of the model. Section 6 concludes the article. The Appendix include the proofs of the theorems in Section 3, as well as some details of solving linear systems to get the property of closure under margins.

2. GAUSSIAN VAR PROCESS

The stationary Gaussian VAR(k) process is reviewed, and it is shown how its stationary joint distribution can be used as a copula.

We start with a strictly stationary \( d \)-variate time series \( \{Z_t\}_{t \geq 0} \) where \( Z_t = (Z_{1,t}, \ldots, Z_{d,t})^T \). Assuming that the variables have been centered to have zero means, the VAR(k) time series model can be expressed as:

\[
Z_t = \Phi_1 Z_{t-1} + \cdots + \Phi_k Z_{t-k} + \epsilon_t, \quad \epsilon_t i.i.d. \sim \mathcal{N}_d(0, \Sigma_c),
\]

where \( \Phi_1, \ldots, \Phi_k \) are coefficient matrices and \( \epsilon_t \) is the innovation vector whose covariance matrix is \( \Sigma_c \). See Lutkepohl (2005) for details and properties. If the coefficient matrices and innovation covariance matrix are such that the process in equation (1) is strictly stationary and Markov of order \( k \), the model can be characterized by its stationary joint distribution of \( (k + 1) \) consecutive observations.

To consider the use of the VAR(k) model via the copula of \( Z_{t+1-t-k} = (Z_t^T, \ldots, Z_{t+k}^T)^T \) for \( i > k \), we suppose that \( Z_t \) is a vector of dependent \( \mathcal{N}(0, 1) \) random variables with

\[
Z_t = (\Phi^{-1}(F_1(X_{1,t})), \ldots, \Phi^{-1}(F_d(X_{d,t})))^T,
\]

where for \( i = 1, \ldots, d \), \( \{X_{i,t}\}_{t>0} \) is a strictly stationary univariate time series with \( X_{i,t} \sim F_i \).

Then the cumulative distribution function (CDF) of the stationary joint distribution of \( (X_t^T, \ldots, X_{t+k}^T)^T \), which is denoted by \( F_{X_{(t-k)}} \), can be represented as

\[
F_{X_{(t-k)}}(x_t, \ldots, x_{t+k}; \eta_1, \ldots, \eta_d, \delta) = C_{Z_{(t-k)}}(u_{1,t}, \ldots, u_{d,t}, \ldots, u_{1,t-k}, \ldots, u_{d,t-k}; \delta),
\]

where \( C_{Z_{(t-k)}} \) is the copula of the stationary joint distribution of \( (Z_t^T, \ldots, Z_{t+k}^T)^T \), \( \delta \) is the set of the dependence parameters, \( x_{t-i} = (x_{1,t-i}, \ldots, x_{d,t-i})^T \) with \( 0 \leq l \leq k \) is a realization of \( X_{t-i} \), and \( u_{i,l} = F_i(x_{i,t-l}; \eta_i) \) for \( 1 \leq i \leq d \).

Using standard results for copulas, such as in Joe (2014), the copula of \( X_{(t-k)} \) is:

\[
C_{X_{(t-k)}}(v; \delta) = \Phi_{d+1,d}(\Phi^{-1}(v_1), \ldots, \Phi^{-1}(v_{d+1}); R),
\]
where \( \mathbf{v} = (v_1, \ldots, v_{(k+1)d})^T \in [0, 1]^{(k+1)d} \) and \( \Phi_{(k+1)d}(.; \cdot; \mathbf{R}) \) is the joint CDF of the multi-variate Gaussian distribution with mean zero and correlation matrix \( \mathbf{R} \), and \( \mathbf{\delta} \) is the vector of the VAR model parameters. Note that \( C_{X,\nu,d}(.; \cdot; \mathbf{\delta}) \) is the same function as \( C_{X,\nu,d}(.; \cdot; \mathbf{\delta}) \) because copula is invariant with respect to strictly monotone transformations. The block Toeplitz correlation matrix \( \mathbf{R} \) of dimension \((k + 1)d \times (k + 1)d \) can be parameterized by VAR model parameter vector \( \mathbf{\delta} \) of dimension \( d(d+1)/2 + kd \). Correlation matrices of multi-variate Gaussian distributions are restricted to be positive-definite.

A disadvantage of the stationary joint distribution specified in equation (2) is the dimension of the parameter set. For multi-variate Gaussian copulas, a parsimonious correlation matrix can be adopted especially in the case of high-dimensional time series, high Markov order, or moderate sample size. In subsequent sections, we discuss a new approach to building multi-variate time series models with parsimonious dependence structures by exploring the marginal models of Gaussian VAR processes.

### 3. Closure under margins for Gaussian VAR model

Section 3.1 explains the benefits of closure under margins, and has a simple bivariate stationary VAR(1) example to show that margins may not be univariate AR(1). Section 3.2 specifies conditional independence relations in order for a stationary VAR(\(k\)) process to have marginal processes that are also autoregressive of order \(k\) or less. Section 3.3 applies the results of Section 3.2 to get conditions on the correlation matrix of \( k+1 \) consecutive vectors and on the coefficient matrices of equation (1) to have closure under margins. The ideas are illustrated through several examples.

#### 3.1. Property of closure under margins

For a multi-variate model, being closed under margins refers to a property that any subvector or subprocess follows the same type of model as the original multi-variate random vector or process. A typical example of the property is the class of multi-variate Gaussian distributions, where any subvector has univariate or lower-dimensional multi-variate Gaussian distribution. For VAR models, there are several advantages of the subclass with the closure under margins property. First, the marginal models of any subprocesses of a multi-variate time series can easily be obtained by extracting the corresponding parameters of the VAR model that the original multi-variate time series follows. For fitting VAR models, if all univariate components of the multi-variate time series follow AR models with the same Markov order, the AR models of all univariate components can be fitted first, followed by estimating the cross-correlation parameters that are contemporaneous or lagged. Even though Lutkepohl (1984) indicated that the univariate components of VAR models should follow ARMA models, this mathematical result is not of much practical use, because the bounds on the AR and MA orders can be wide.

To illustrate, let \( \{(Y_{1,t}, Y_{2,t}) \}_{t \geq 0} \) follows a bivariate stationary VAR(1) process:

\[
\begin{pmatrix}
Y_{1,t} \\
Y_{2,t}
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} \\
2a_{21} & a_{22}
\end{pmatrix} \begin{pmatrix}
Y_{1,t-1} \\
Y_{2,t-1}
\end{pmatrix} + \begin{pmatrix}
\epsilon_{1,t} \\
\epsilon_{2,t}
\end{pmatrix},
\]

where \( \{Y_{1,t}\}_{t \geq 0} \) and \( \{Y_{2,t}\}_{t \geq 0} \) follow AR(1) models. In the case that \( a_{11} = a_{21} = 0, -1 < a_{12} < 1, \) and \( -1 < a_{22} < 1 \), then \( \text{Var}(Y_{2,t}) = (1 - a_{22}^2)^{-1} \) and the random vector \( (Y_{1,t}, Y_{1,t-1}, Y_{1,t-2})^T \) follows a trivariate Gaussian distribution with zero mean and covariance matrix

\[
\frac{1}{1 - a_{22}^2} \begin{pmatrix}
1 + a_{12}^2 - a_{22}^2 & a_{12}^2 a_{22} & a_{12}^2 a_{22}^2 \\
2a_{12} a_{22} & 1 + a_{22}^2 - a_{22}^2 & a_{22}^2 a_{22} \\
a_{12}^2 a_{22} & a_{22}^2 a_{22} & 1 + a_{12}^2 - a_{22}^2
\end{pmatrix}.
\]
and the partial autocorrelation between $Y_{1,t}$ and $Y_{1,t-2}$ given $Y_{1,t-1}$ is
\[
\frac{a_{11}^2 a_{22}^2 [1 - a_{22}^2]}{(1 + a_{12}^2 - a_{22}^2)^2 - a_{12}^2 a_{22}^2}.
\]

Therefore, unless $a_{12} = 0$ or $a_{22} = 0$, the subprocess $\{Y_{1,t}\}_{t \geq 0}$ does not follow an AR(1) model, and equation (4) is not closed under margins. Indeed, as suggested in section 11.6 in Lutkepohl (2005), $\{Y_{1,t}\}_{t \geq 0}$ might not admit a finite Markov order VAR representation. This shows that in general VAR models are not closed under margins. However, the existence of the margin-closed VAR models can be verified in the special case when $a_{12} = a_{21} = 0$, $-1 < a_{11} < 1$ and $-1 < a_{22} < 1$ In this case, both subprocesses $\{Y_{1,t}\}_{t \geq 0}$ and $\{Y_{2,t}\}_{t \geq 0}$ follow AR(1) models, and the coefficient matrix is diagonal.

3.2. Conditional independence relations for closure under margins

A VAR($k$) process is closed under margins if all its coefficient matrices are diagonal. However, a margin-closed model with non-diagonal coefficient matrices may give simpler interpretations in practice. For example, non-zero values in the off-diagonal entries of coefficient matrices are meaningful when Granger–Causality (Granger, 1969) is investigated. Here, a sufficient condition under which a VAR model is closed under margins is derived. Suppose the $d$ variables have been split into a partition of 2 to $d$ components, each with cardinality of at least 1. Then, a VAR($k$) process is closed under margins with respect to this partition if and only if every subprocess in the partition is also a VAR($k$) process. For example, a three-variate VAR($k$) process $\{(Y_{1,t}, Y_{2,t}, Y_{3,t})\}_{t \geq 0}$ is closed under margins with respect to partition \{\{1, 2\}, \{3\}\} if and only if subprocesses $\{(Y_{1,t}, Y_{2,t})\}_{t \geq 0}$ and $\{(Y_{3,t})\}_{t \geq 0}$ follow a bivariate VAR($k$) model and an AR($k$) model, respectively.

With the notation of Section 2, let $\{Z_t\}_{t \geq 0}$ be a Gaussian multi-variate VAR($k$) process, where $Z_t = (Z_{1,t}, \ldots, Z_{d,t})^T$, and let $\{S_1, \ldots, S_n\}$ be a partition of $\{1, \ldots, d\}$. For simplicity, let $Z_{S,l}$ denote the subvector of $Z_t$ so that the dimension indices of the elements of $Z_{S,l}$ are in $S_l$, i.e. $Z_{S_l} = (Z_{i_1,l}, \ldots, Z_{i_m,l})^T$ if $S_l = \{i_1, \ldots, i_m\}$, where the cardinality of $S_l$ is $|S_l| = m_l$. Since $\{Z_t\}_{t \geq 0}$ is a VAR process, the random vector $(Z_{t+1}^T, \ldots, Z_{t+k}^T)^T$ follows a multi-variate Gaussian distribution for any $l < t$. As a consequence, $(Z_{S_l,l+1}^T, \ldots, Z_{S_l,t+k-1}^T)^T$ is also a multi-variate Gaussian random vector and we only need to ensure the process $\{Z_{S_l,t}\}_{t \geq 0}$ has Markov order $k$ so that it follow a VAR($k$) process. It means that $\{Z_{S_l,t}\}_{t \geq 0}$ is a VAR($k$) process if and only if for all $t > 0$, $Z_{S_l,t}$ is independent of $Z_{S_l,t-1} \ldots Z_{S_l,t-k+1}$ for any $l < t$. We write this condition as

\[
[Z_{S_l,t} \perp Z_{S_l,t-1} \perp \ldots \perp Z_{S_l,t-k+1} \mid Z_{S_l,t-k} \ldots Z_{S_l,t-k+1} \text{ for } k < l < t.]
\]

The following lemma is helpful in deriving a sufficient condition for $\{Z_{S_l,t}\}_{t \geq 0}$ to have Markov order $k$.

**Lemma 3.1.** Let $A$, $B$, $V$, and $W$ be four random vectors whose joint distribution is multi-variate Gaussian. Suppose that $A$ and $B$ are independent given $V$ and $W$. Then $A$ and $B$ are independent given $V$ if $A$ is independent of $W$ given $V$ or $B$ is independent of $W$ given $V$.

**Proof.** The proof is based on the result for the conditional distribution of multi-variate Gaussian random vectors. Let $\Sigma_{A|V}$, $\Sigma_{A|W}$ and $\Sigma_{B|W|V}$ denote the cross-covariance matrices between $A$ and $B$, $A$ and $W$, and $B$ and $W$ respectively, conditional on $V$, e.g. the $i$th row and $j$th column of matrix $\Sigma_{A|B|V}$ is the covariance between the $i$th element of $A$ and $j$th element of $B$ conditional on $V$. Then

\[
\Sigma_{A|B|V|W} = \Sigma_{A|B|V} - \Sigma_{A|W} \Sigma_{W|V}^{-1} \Sigma_{B|W|V},
\]
where $\Sigma_{A,B;V,W}$ is the covariance matrix between $A$ and $B$ given $V$ and $W$, and $\Sigma_{W|V}$ is the covariance matrix of $W$ given $V$. Since $A$ and $B$ are independent given $V$ and $W$, we have $\Sigma_{A,B;V,W} = 0$ and it follows that

$$\Sigma_{A,B|V} = \Sigma_{A,W|V} \Sigma_{W|V}^{-1} \Sigma_{B,W|V}.$$

and this is 0 if $\Sigma_{A,W|V} = 0$ or $\Sigma_{B,W|V} = 0$.

Note that the condition in Lemma 3.1 is only sufficient as $\Sigma_{A,W|V} \Sigma_{W|V}^{-1} \Sigma_{B,W|V} = 0$ does not lead to either $\Sigma_{A,W|V} = 0$ or $\Sigma_{B,W|V} = 0$.

Since $\{Z_i\}_{t=0}^\infty$ is a VAR($k$) process and hence Markov of order $k$, $Z_{S,t}$ is independent of $Z_{S,t-l}$ conditional on $(Z_{t-1}^T, \ldots, Z_{t-l+1}^T)^T$ for any $l$ with $k < l < t$. Let $Z_{-S,t}$ denote the subvector of $Z_t$ with components in index set $\{1, \ldots, d\} \setminus S_t$ and set

$$A = Z_{S,t}, \quad B = Z_{S,t-l},$$

$$V = (Z_{S,t-l}^T, \ldots, Z_{S,t-l+1}^T)^T, \quad W = (Z_{-S,t-l}^T, \ldots, Z_{-S,t-l+1}^T)^T.$$

Then using Lemma 3.1, we have $\{Z_{S,t} \perp Z_{S,t-l}\}|Z_{S,t-l+1}, \ldots, Z_{S,t+l+1}$ if

$$[Z_{S,t} \perp (Z_{S,t-l}^T, \ldots, Z_{S,t-l+1}^T)^T]|Z_{S,t-l+1}, \ldots, Z_{S,t+l+1} \quad \text{or}$$

$$[Z_{S,t-l} \perp (Z_{S,t-l}^T, \ldots, Z_{S,t-l+1}^T)^T]|Z_{S,t-l+1}, \ldots, Z_{S,t+l+1}.$$

The random vectors $A$ and $B$ are time lag $l$ apart (for subvector with indices in $S_t$), $V$ consists of the intermediate observations in this subprocess, and $W$ consists of the intermediate observations in the other subprocess. It follows that $\{Z_{S,t}\}_{t=0}^\infty$ is a VAR($k$) process if the condition in equation (5) holds for any $l$ with $k < l < t$. Moreover, it can be proved that we only need to guarantee the condition at $l = k + 1$ to make $\{Z_{S,t}\}_{t=0}^\infty$ follow a VAR($k$) process. This is formulated in the next theorem, with proof given in the Appendix A.

**Theorem 3.2.** If $\{Z_t\}_{t=0}^\infty$ is a multi-variate VAR($k$) process where $Z_t = (Z_{1,t}, \ldots, Z_{d,t})^T$, and $\{S_1, \ldots, S_n\}$ is a partition of $\{1, \ldots, d\}$. Let $-S_t$ denote the difference between two sets $\{1, \ldots, d\}$ and $S_t$, and let $Z_{S,t}$ denote the subvector of $Z_t$ with components in index set $S_t$. Then the subprocess $\{Z_{S,t}\}_{t=0}^\infty$ follows a VAR($k$) model if one of the following two conditions is satisfied:

1. $[Z_{S,t} \perp (Z_{S,t-l}^T, \ldots, Z_{S,t-k}^T)^T]|Z_{S,t-l+1}, \ldots, Z_{S,t-k},$

2. $[Z_{S,t-k} \perp (Z_{S,t-l}^T, \ldots, Z_{S,t-k}^T)^T]|Z_{S,t-l+1}, \ldots, Z_{S,t-k}.$

For illustration, consider the VAR(1) model in equation (4). When $a_{12} = a_{21} = 0$, $-1 < a_{11} < 1$, and $-1 < a_{22} < 1$, both $\{Y_{1,t}\}_{t=0}^\infty$ and $\{Y_{2,t}\}_{t=0}^\infty$ follow AR(1) models and the VAR(1) process in equation (4) is closed under margins with respect to partition $\{1\}, \{2\}\}$. It is easy to verify that $[Y_{1,t} \perp Y_{2,t-1}]|Y_{1,t-1} \text{ and } [Y_{2,t} \perp Y_{1,t-1}]|Y_{2,t-1}$ in this case; these correspond to the first conditions in Theorem 3.2 with $S_t = \{1\}$ and $S_t = \{2\}$ respectively.

The condition in Theorem 3.2 is only sufficient because Lemma 3.1 specifies a sufficient condition. A necessary and sufficient condition for the closure under margins would require investigating a necessary and sufficient condition under which $\Sigma_{A,W|V} \Sigma_{W|V}^{-1} \Sigma_{B,W|V} = 0$. We do not see a simple formulation for such a condition at the moment and hence the question is left for future research. However, we do note that sufficient conditions are adequate for model fitting.

### 3.3. Conditions on parameters for closure under margins

Here, conditions for conditional independence in Theorem 3.2 are expressed in terms of model parameters.
3.3.1. Partitions with two subprocesses

We consider the case with partition \( \{S_1, S_2\} \) and there are only two subprocesses \( \{Z_{S_1,t}\}_{t \geq 0} \) and \( \{Z_{S_2,t}\}_{t \geq 0} \). Let the dimensions of the subvectors be \( d_1 \) and \( d_2 \) respectively, and let \( R \) be the correlation matrix of the random vector \( (Z_{S_1,t}^T, \ldots, Z_{S_{t-k}}^T) \). To better present the structure of the subprocesses, we reorder the columns and rows of \( R \) to get the correlation matrix of the random vector \( (Z_{S_1,t}^T, \ldots, Z_{S_{t-k}}^T) \). Denote the reordered matrix by \( R_{\{S_1, S_2\}} \) with notation:

\[
R_{\{S_1, S_2\}} = \begin{pmatrix}
R_{S_1} & 0 \\
0 & R_{S_2}
\end{pmatrix}
\]

where the block entry \( \Sigma_{i,j} \) is the correlation matrix between \( Z_{i,t} \) and \( Z_{j,t-1} \) for \( i, j \in \{1, 2\} \) and \( l = 0, 1, \ldots, k \). These are treated as parameters of \( R_{\{S_1, S_2\}} \). Under the constraint that \( \{Z_{i,t}\}_{t \geq 0} \) is closed under margins with respect to partition \( \{S_1, S_2\} \), the submatrix \( R_{S_1} \), which is the correlation matrix of \( (Z_{S_1,t}^T, \ldots, Z_{S_{t-k}}^T) \), specifies the VAR\((k)\) subprocess \( \{Z_{S_1,t}\}_{t \geq 0} \). More specifically, \( \Sigma_{1,0} \) specifies the contemporaneous dependence and \( \Sigma_{1,1}, \ldots, \Sigma_{1,k} \) specify the serial dependence at different lags of \( \{Z_{S_1,t}\}_{t \geq 0} \). Similarly, the submatrix \( R_{S_2} \) characterizes the second VAR\((k)\) subprocess \( \{Z_{S_2,t}\}_{t \geq 0} \). As for the other block entries in the off-diagonal block \( R_{S_1, S_2} \), they measure the dependence between two subprocesses: \( \Sigma_{12,0} \) indicates the contemporaneous dependence between \( Z_{S_1,t} \) and \( Z_{S_2,t} \), while \( \Sigma_{12,-k}, \ldots, \Sigma_{12,1}, \Sigma_{12,1}, \ldots, \Sigma_{12,-k} \) summarize the cross-sectional dependence between two subprocesses at lags \(-k\) to \( k \). It is natural to consider closure under margins as a property describing the dependence structure between the subprocesses. Therefore, when imposing closure under margins for \( \{Z_{i,t}\}_{t \geq 0} \) with respect to the partition including the two subprocesses, we hold the entries modeling the individual VAR\((k)\) subprocesses fixed and investigate the constraints on the entries for the dependence between the subprocesses, i.e., the constraints on parameters \( \Sigma_{12,-k}, \ldots, \Sigma_{12,k} \) with parameters \( \Sigma_{11,0}, \Sigma_{22,0}, \ldots, \Sigma_{11,k}, \Sigma_{22,k} \) fixed.

To reformulate the conditions in Theorem 3.2 as the constraints on entries of \( R_{\{S_1, S_2\}} \) in equation (6), we first point out that \( S_1 = -S_2 \) and \( -S_1 = S_2 \) in this case of only two subprocesses. Then the conditions in Theorem 3.2 can be expressed as

\[
\begin{align*}
1. & \quad [Z_{S_1,t} \perp (Z_{S_1,t-1}^T, \ldots, Z_{S_1,t-k}^T)] | Z_{S_1,t-1}, \ldots, Z_{S_1,t-k}, \\
2. & \quad [Z_{S_1,t-k+1} \perp (Z_{S_1,t-1}^T, \ldots, Z_{S_1,t-k}^T)] | Z_{S_1,t-1}, \ldots, Z_{S_1,t-k}.
\end{align*}
\]

The reader may want to look ahead at Examples 3.3 and 3.4 to get an idea of what is involved when the dependence parameters of the two subprocesses are fixed and some cross-dependence parameters are fixed, to construct \( d \)-dimensional VAR\((k)\) processes that have two marginal VAR\((k)\) subprocesses.

Let

\[
D_1 = (\Sigma_{12,-k}^T, \ldots, \Sigma_{12,1}^T, \Sigma_{12,0}^T, \Sigma_{12,1}^T, \ldots, \Sigma_{12,k}^T)^T,
\]
Note that block coefficient matrices of dimension $d 	imes d$. The two conditions in equation (7) for $\{Z_{S_{i,t}}\}_{t \geq 0}$ can be rewritten as two linear systems:

\[
G_1 = \begin{pmatrix}
0 & \Phi_{1,k} & \cdots & \Phi_{1,2} & \Phi_{1,1} & -I_{d_1} & 0 & \cdots & 0 \\
0 & 0 & \Phi_{1,k} & \cdots & \Phi_{1,2} & \Phi_{1,1} & -I_{d_1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \Phi_{1,k} & \cdots & \Phi_{1,2} & \Phi_{1,1} & -I_{d_1}
\end{pmatrix}
\]

(8)

and

\[
H_1 = \begin{pmatrix}
-I_{d_1} & \Psi_{1,k} & \cdots & \Psi_{1,2} & \Psi_{1,1} & 0 & 0 & \cdots & 0 \\
0 & -I_{d_1} & \Psi_{1,k} & \cdots & \Psi_{1,2} & \Psi_{1,1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -I_{d_1} & \Psi_{1,k} & \cdots & \Psi_{1,2} & \Psi_{1,1} & 0
\end{pmatrix}
\]

(9)

Then the two conditions in equation (7) for $\{Z_{S_{i,t}}\}_{t \geq 0}$ can be rewritten as two linear systems:

Condition 1: $G_1 D_1 = 0$,
Condition 2: $H_1 D_1 = 0$.

(10)

Note that block coefficient matrices $G_1$ and $H_1$, defined in equation (8) and equation (9), have $k \times (2k + 1)$ blocks of dimension $d_1 \times d_1$. Matrix $D_1$ has $(2k + 1) \times 1$ blocks of dimension $d_1 \times d_2$. Moreover, one can see that $\Phi_{1,1}, \ldots, \Phi_{1,k}$ are the coefficient matrices of the VAR($k$) process $\{Z_{S_{i,t}}\}_{t \geq 0}$ when $\{Z_{t_i}\}_{t \geq 0}$ is closed under margins with respect to partition $\{S_{1}, S_{2}\}$, and so these coefficient matrices come from the conditional mean of $Z_{S_{1,t-1}}, \ldots, Z_{S_{i,t-1}}$ written as $\Phi_{1,1} Z_{S_{1,t-1}} + \cdots + \Phi_{1,k} Z_{S_{i,t-1}}$. The $\Psi_{1,j}$ coefficient matrices come from the conditional mean of $Z_{S_{1,t-k}} \cdots Z_{S_{i,t-k}}$.

Similarly, let

\[
D_2 = \begin{pmatrix}
\Sigma_{21,\ldots,k}^T & \cdots & \Sigma_{21,\ldots,k}^T \\
\Sigma_{21,0}^T & \cdots & \Sigma_{21,0}^T \\
\Sigma_{21,\ldots,k}^T & \cdots & \Sigma_{21,\ldots,k}^T
\end{pmatrix}^T
\]

(11)
The block coefficient matrices $G$ and $H$, defined in equations (11) and (12), have $k \times (2k+1)$ blocks of dimension $d_2 \times d_2$, and $D_1$ has $(2k+1) \times 1$ blocks of dimension $d_1 \times d_1$. Note that $\Phi_{2,1}, \ldots, \Phi_{2,k}$ are the coefficient matrices of the VAR($k$) process $\{Z_{S,t}\}_{t=0}^\infty$ when $\{Z_t\}_{t=0}^\infty$ is closed under margins with respect to partition $\{S_1, S_2\}$. Also, $\Psi_{2,1}, \ldots, \Psi_{2,k}$ come from the conditional mean of $Z_{S_{2,t-4}}$ given $Z_{S_{1,t-1}}, \ldots, Z_{S_{2,t-4}}$. The derivations of equations (10) and (13) are given in Appendix B.

To make the model closed under margins with respect to partition $\{S_1, S_2\}$, we only need to take one system from equation (10), and another system from equation (13), so there are 4 different combinations to consider. Moreover, $G_1, G_2, H_1, H_2$ have $2k+1$ blocks in each row and $k$ blocks in each column, so for each combination we have only $2k$ linear equations for $2k+1$ block matrices $\Sigma_{12,-4}, \ldots, \Sigma_{12,-1}, \Sigma_{12,0}, \Sigma_{12,1}, \ldots, \Sigma_{12,k}$. Thus, we must
choose a block matrix among them and fix it when solving a system of 2\(k\) linear equations in 2\(k\) block matrices of dimensions \(d_i \times d_j\).

It can be concluded that when parameters \(\Sigma_{11,0}, \Sigma_{22,0}, \ldots, \Sigma_{11,k}, \Sigma_{22,k}\) modeling two individual VAR\((k)\) subprocesses \(\{Z_{S,i}\}_{t>0}\) and \(\{Z_{S,j}\}_{t>0}\) are fixed and the chosen block matrix is given, all parameters modeling cross-sectional dependence between the two subprocesses can be uniquely solved, leading to a correlation matrix \(R_{\{S_i,S_j\}}\) for \((Z_t^T, \ldots, Z_{t+\ell}^T)^T\). If this \(R_{\{S_i,S_j\}}\) is positive definite, then the parameters of the two VAR\((k)\) subprocesses and the chosen block matrix are compatible for a margin-closed \(d\)-dimensional VAR\((k)\) process.

Next we consider four cases based on conditions in equations (10) and (13) and give two examples to illustrate the details:

**Case 1.** \(\{Z_{S,i}\}_{t>0}\) and \(\{Z_{S,j}\}_{t>0}\) satisfy Condition 1 in equations (10) and (13) respectively.

**Case 2.** \(\{Z_{S,i}\}_{t>0}\) and \(\{Z_{S,j}\}_{t>0}\) satisfy Condition 2 in equations (10) and (13) respectively;

**Case 3.** \(\{Z_{S,i}\}_{t>0}\) satisfies Condition 1 in equation (10) and \(\{Z_{S,j}\}_{t>0}\) satisfies Condition 2 in equation (13);

**Case 4.** \(\{Z_{S,i}\}_{t>0}\) satisfies Condition 2 in equation (10) and \(\{Z_{S,j}\}_{t>0}\) satisfies Condition 1 in equation (13).

In Cases 1 and 2, it can be seen that when the parameters of two individual VAR\((k)\) subprocesses are held fixed, the dependence between \(\{Z_{S,i}\}_{t>0}\) and \(\{Z_{S,j}\}_{t>0}\) can also be specified when their contemporaneous dependence parameter \(\Sigma_{12,0}\) is fixed. The cross-sectional dependence parameters \(\Sigma_{12,-k}, \ldots, \Sigma_{12,1}, \Sigma_{12,1}, \ldots, \Sigma_{12,k}\) can be uniquely determined.

Things will be different in Case 3. Since all block entries in the first column of \(G_1\) and \(H_2L_2\) are 0, \(\Sigma_{12,-k}\) cannot be solved and should be selected as the fixed parameter. Moreover, we can see that \(\Sigma_{12,1-k} = \cdots = \Sigma_{12,k} = 0\) as there are non-zero entries in each of the last 2\(k\) block columns of \(G_1\) and \(H_2L_2\). It follows that when the parameters of two individual VAR\((k)\) subprocesses \(\{Z_{S,i}\}_{t>0}\) and \(\{Z_{S,j}\}_{t>0}\) are held fixed, the cross-sectional dependence between the two subprocesses at lag \(-k\) is the fixed parameter based on which the dependence between the two subprocesses can be determined.

In Case 4, \(\Sigma_{12,-k}, \ldots, \Sigma_{12,1}\) are all 0 while \(\Sigma_{12,k}\), which is the cross-sectional dependence between \(\{Z_{S,i}\}_{t>0}\) and \(\{Z_{S,j}\}_{t>0}\) at lag \(k\), should be treated as the fixed parameter after fixing the parameters of two individual VAR\((k)\) subprocesses.

To summarize, the correlation matrix \(R_{\{S_i,S_j\}}\) can be specified by three parts: the conditions that two subprocesses take, the entries of \(R_{S_i}\) and \(R_{S_j}\), and the fixed cross-dependence parameter according to the conditions of two subprocesses. Note that the fixed parameter is \(\Sigma_{12,0}\) in Cases 1 and 2, is \(\Sigma_{12,-k}\) in Case 3, and is \(\Sigma_{12,k}\) in Case 4. The extra constraint that \(R_{\{S_i,S_j\}}\) is positive definite should hold so that the fixed parameters are compatible for a margin-closed \(d\)-dimensional VAR\((k)\) process.

The next two examples illustrate four cases above for partitions with two subprocesses. Example 3.3 is the extension of the bivariate VAR\((1)\) model in equation (4). Example 3.4 is more complex as it deals with a bivariate subprocess. General formulae for solving the linear equations of Case 1 and 2 are derived in Appendix C.1.

**Example 3.3.** (Two-dimensional VAR\((1)\) process). Consider a two-dimensional VAR\((1)\) model \(\{Z_{1,i}, Z_{2,i}\}_{t>0}\) with simplest partition \(\{\{1\}, \{2\}\}\). Following the same assumption in Section 2, \(Z_{i,t}\) has standard Gaussian margin for all \(i \in \{1, 2\}\) and \(t > 0\). Let \(\rho_{ij}\) denote the correlation coefficient between \(Z_{i,t}\) and \(Z_{j,t-1}\) for \(i, j \in \{1, 2\}\). The stationary joint distribution of \(\{Z_{1,i}, Z_{1,i-1}, Z_{2,i}, Z_{2,i-1}\}\) has correlation matrix

\[
R_{\{11,12\}} = \left(\begin{array}{ll}
R_{\{11,11\}} & R_{\{11,12\}} \\
R_{\{12,11\}} & R_{\{12,12\}}
\end{array}\right) = \left(\begin{array}{cccc}
1 & \rho_{11,1} & \rho_{11,2} & \rho_{12,1} \\
\rho_{11,1} & 1 & \rho_{12,1} & \rho_{12,2} \\
\rho_{11,2} & \rho_{12,1} & 1 & \rho_{22,1} \\
\rho_{12,1} & \rho_{12,2} & \rho_{22,1} & 1
\end{array}\right) .
\] (14)
If $R_{\{11,12\}}$ in equation (14) is positive definite, then from the conditional expectation formula for multi-variate Gaussian distributions, the coefficient matrix of the two-dimensional VAR(1) process above is
\[
\Phi = \begin{pmatrix}
\rho_{11,1} & \rho_{12,1} \\
\rho_{12,1} - \rho_{12,2} & \rho_{22,1}
\end{pmatrix}
\begin{pmatrix}
1 & \rho_{12,0}
\rho_{12,0} & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
\frac{\rho_{11,1} - \rho_{12,1} \rho_{12,0}}{1 - \rho_{12,0}^2} & \frac{\rho_{12,0} \rho_{22,1} - \rho_{11,1} \rho_{12,1}}{1 - \rho_{12,0}^2} \\
\frac{-\rho_{12,1} \rho_{12,0}}{1 - \rho_{12,0}^2} & \frac{1 - \rho_{22,1} \rho_{12,1}}{1 - \rho_{12,0}^2}
\end{pmatrix}.
\] (15)

If \{$(Z_{1,t}, Z_{2,t})$\}$_{t>0}$ is closed under margins with respect to partition \{(1, 1), (2, 2)\}, the entries of submatrix $R_{\{11\}}$ can characterize the VAR(1) subprocess $\{Z_{1,t}\}_{t>0}$. Specifically, we should have
\[
Z_{1,t} = \rho_{11,1} Z_{1,t-1} + \epsilon_{1,t}^i, \quad \epsilon_{1,t}^i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1 - \rho_{11,1}^2).
\]
Similarly, the VAR(1) subprocess $\{Z_{2,t}\}_{t>0}$ is specified by the entries of $R_{\{2\}}$:
\[
Z_{2,t} = \rho_{22,1} Z_{2,t-1} + \epsilon_{2,t}^i, \quad \epsilon_{2,t}^i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1 - \rho_{22,1}^2).
\]

Then $\rho_{11,1}$ and $\rho_{22,1}$ are considered fixed and one cross-correlation of lag 0, 1, or $-1$ must be fixed in order for \{$(Z_t)$\}$_{t>0}$ to satisfy the margin-closure property. As $\Phi_{1,1} = \Psi_{1,1} = (\rho_{11,1})$ and $\Phi_{2,1} = \Psi_{2,1} = (\rho_{22,1})$, in equation (8) to equation (12), the equations involving $G_1, H_1, G_2, H_2$ all involve scalars:
\[
G_1 D_1 = 0 \Rightarrow \Phi_{1,1} \rho_{12,0} - \rho_{12,1} = \rho_{11,1} \rho_{12,0} - \rho_{12,1} = 0,
\]
\[
H_1 D_1 = 0 \Rightarrow -\rho_{12,-1} + \rho_{11,1} \rho_{12,0} = \rho_{12,-1} + \rho_{11,1} \rho_{12,0} = 0.
\]
\[
(G_2 L_2)(L_2) = 0 \Rightarrow -\rho_{12,-1} + \Phi_{2,1} \rho_{12,0} = \rho_{12,-1} + \rho_{22,1} \rho_{12,0} = 0,
\]
\[
(H_2 L_2)(L_2) = 0 \Rightarrow \Psi_{2,1} \rho_{12,0} - \rho_{12,1} = \rho_{22,1} \rho_{12,0} - \rho_{12,1} = 0.
\]

In Case 1, based on $G_1$ and $G_2$,
\[
\rho_{12,-1} = \rho_{22,1} \rho_{12,0}, \quad \rho_{12,1} = \rho_{11,1} \rho_{12,0}.
\]

If $\rho_{12,0}$ is such that $R_{\{11,12\}}$ is positive definite, the coefficient matrix in equation (15) is
\[
\Phi = \begin{pmatrix}
\rho_{11,1} & 0 \\
0 & \rho_{22,1}
\end{pmatrix}
\]
and this is diagonal, which means it corresponds to the case of a diagonal coefficient matrix of \{$(Z_t)$\}$_{t>0}$.

In Case 2 based on $H_1$ and $H_2$, the contemporaneous dependence $\rho_{12,0}$ is the fixed parameter, and
\[
\rho_{12,-1} = \rho_{11,1} \rho_{12,0}, \quad \rho_{12,1} = \rho_{22,1} \rho_{12,0}.
\] (16)

If $\rho_{11,1}, \rho_{22,1}, \rho_{12,0}$ in equation (16) are such that $R_{\{11,12\}}$ in equation (14) is positive definite, the coefficient matrix in equation (15) is
\[
\Phi = (1 - \rho_{12,0}^2)^{-1}
\begin{pmatrix}
\rho_{11,1} - \rho_{12,0}^2 \rho_{12,1} & \rho_{12,0} \rho_{22,1} - \rho_{11,1} \\
\rho_{12,0} \rho_{11,1} - \rho_{22,1} & \rho_{22,1} - \rho_{12,0}^2 \rho_{11,1}
\end{pmatrix}.
\]

This is the most interesting case as this coefficient matrix is non-diagonal if $\rho_{12,0} \neq 0$ and $\rho_{11,1} \neq \rho_{22,1}$. Even though all entries of the coefficient matrix are non-zero, the two univariate subprocesses are AR(1).

In Case 3 based on $G_1$ and $H_2$, $\rho_{12,0} = \rho_{12,1} = 0$ if $\rho_{11,1} \neq \rho_{22,1}$. If $\rho_{11,1} \rho_{22,1} = \rho_{12,-1}$ with $\rho_{12,0} = \rho_{12,1} = 0$ are such that $R_{\{11,12\}}$ in equation (14) is positive definite, the coefficient matrix in equation (15) is
\[
\begin{pmatrix}
\rho_{11,1} & 0 \\
\rho_{22,1} & \rho_{22,1}
\end{pmatrix}.
\]
This is non-diagonal in general, but clearly \{$(Z_t)$\}$_{t>0}$ is AR(1).
In Case 4 based on $G_2$ and $H_1$, $\rho_{12,0} = \rho_{12,-1} = 0$ if $\rho_{11,1} \neq \rho_{22,1}$. If $\rho_{11,1}, \rho_{22,1}, \rho_{12,1}$ with $\rho_{12,0} = \rho_{12,-1} = 0$ are such that $R_{(11),(21)}$ in equation (14) is positive definite, the coefficient matrix in equation (15) is $\begin{pmatrix} \rho_{11,1} & \rho_{12,1} \\ 0 & \rho_{22,1} \end{pmatrix}$. This is non-diagonal in general, but clearly $\{Z_t\}_{t \geq 0}$ is AR(1).

Note that in Case 2, the range of $\rho_{12,0}$ making $R_{(11),(21)}$ in equation (14) positive definite is affected by the values of $\rho_{11,1}$ and $\rho_{22,1}$. To illustrate, numerical calculations show that when $\rho_{11,1} = \rho_{22,1} = 0.9$, $R_{(11),(21)}$ is positive definite for any $\rho_{12,0} \in (-1, 1)$; but when $\rho_{11,1} = 0.9$ and $\rho_{22,1} = -0.9$, $R_{(11),(21)}$ is not positive definite for $\rho_{12,0} \in (0.15, 1)$.

For examples where $d_1 > 1$, the following matrix inverse identity for a $2 \times 2$ block-partitioned positive definite matrix can be used:

$$
\begin{pmatrix}
\Omega_{11} & \Omega_{12} \\
\Omega_{21} & \Omega_{22}
\end{pmatrix}^{-1} = 
\begin{pmatrix}
A^{-1} & -A^{-1}\Omega_{12} \Omega_{22}^{-1} \\
-\Omega_{22}^{-1}\Omega_{21}A^{-1} \Omega_{12} \Omega_{22}^{-1} + \Omega_{22}^{-1}\Omega_{21}A^{-1} \Omega_{12} \Omega_{22}^{-1}
\end{pmatrix},
$$

(17)

where $A = \Omega_{11,2} = \Omega_{11} - \Omega_{12} \Omega_{22}^{-1} \Omega_{21}$.

**Example 3.4.** (three-dimensional VAR(1) process). Let $\{Z_t\}_{t \geq 0}$ follow a three-dimensional VAR(1) model with partition $\{(1, 2), (3)\}$, i.e., $Z_{S,t} = (Z_{1,t}, Z_{2,t})^T$ and $Z_{S,t} = (Z_{3,t})$. The correlation matrix of random vector $(Z_{1,t}, Z_{2,t}, Z_{1,t-1}, Z_{2,t-1}, Z_{3,t}, Z_{3,t-1})^T$ is

$$
R_{(12,),(3)} = \begin{pmatrix}
R_{(12,),(2)} & R_{(12,),(2)},(3) & R_{(12,),(3)} \\
R_{(12,),(2)},(3) & R_{(3)}
\end{pmatrix} = 
\begin{pmatrix}
1 & \rho_{12,0} & \rho_{11,1} & \rho_{12,1} & \rho_{13,0} & \rho_{13,1} \\
\rho_{12,0} & 1 & \rho_{12,-1} & \rho_{22,1} & \rho_{23,0} & \rho_{23,1} \\
\rho_{11,1} & \rho_{12,-1} & 1 & \rho_{22,1} & \rho_{23,1} & \rho_{23,0} \\
\rho_{12,1} & \rho_{22,1} & \rho_{12,-1} & 1 & \rho_{23,1} & \rho_{23,0} \\
\rho_{13,0} & \rho_{23,0} & \rho_{13,-1} & \rho_{23,1} & 1 & \rho_{33,1} \\
\rho_{13,1} & \rho_{23,1} & \rho_{13,0} & \rho_{23,0} & \rho_{33,1} & 1
\end{pmatrix}.
$$

(18)

Under the constraint that $\{Z_t\}_{t \geq 0}$ is closed under margins with respect to $\{(1, 2), (3)\}$, VAR(1) subprocess $\{Z_{S,t}\}_{t \geq 0}$ can be specified by the following positive definite matrices:

$$
R_{(12)} = \begin{pmatrix}
\Sigma_{11,0} & \Sigma_{11,1} \\
\Sigma_{11,1} & \Sigma_{11,0}
\end{pmatrix} = 
\begin{pmatrix}
1 & \rho_{12,0} & \rho_{11,1} & \rho_{12,1} \\
\rho_{12,0} & 1 & \rho_{12,-1} & \rho_{22,1} \\
\rho_{11,1} & \rho_{12,-1} & 1 & \rho_{22,1} \\
\rho_{12,1} & \rho_{22,1} & \rho_{12,-1} & 1
\end{pmatrix},
$$

and VAR(1) subprocess $\{Z_{S,t}\}_{t \geq 0}$ can be specified by

$$
R_{(3)} = \begin{pmatrix}
\Sigma_{22,0} & \Sigma_{22,1} \\
\Sigma_{22,1} & \Sigma_{22,0}
\end{pmatrix} = 
\begin{pmatrix}
1 & \rho_{33,1} \\
\rho_{33,1} & 1
\end{pmatrix}.
$$

For other parameters, $\Sigma_{12,0} = \begin{pmatrix} \rho_{13,0} \\ \rho_{23,0} \end{pmatrix}$ measures the contemporaneous dependence between $\{Z_{S,t}\}_{t \geq 0}$ and $\{Z_{S,t}\}_{t \geq 0}$, $\Sigma_{12,-1} = \begin{pmatrix} \rho_{13,-1} \\ \rho_{23,-1} \end{pmatrix}$ and $\Sigma_{12} = \begin{pmatrix} \rho_{13,1} \\ \rho_{23,1} \end{pmatrix}$ measures the cross-sectional dependence between $\{Z_{S,t}\}_{t \geq 0}$ and $\{Z_{S,t}\}_{t \geq 0}$ at lag $-1$ and 1 respectively. If matrix $R_{(12,),(3)}$ in equation (18) is positive definite, then from the conditional expectation formula for multi-variate Gaussian distributions, the coefficient matrix of three-dimensional
VAR(1) process \( \{Z_{1t}, Z_{2t}, Z_{3t}\}_{t \geq 0} \) is

\[
\Phi = \begin{pmatrix} \Sigma_{11,1} & \Sigma_{12,1} \\ \Sigma_{12,1}^T & \rho_{33,1} \end{pmatrix} \begin{pmatrix} \Sigma_{11,0} & \Sigma_{12,0} \\ \Sigma_{12,0}^T & 1 \end{pmatrix}^{-1} \\
= \begin{pmatrix} \Sigma_{11,1} & \Sigma_{12,1} \\ \Sigma_{12,1}^T & \rho_{33,1} \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}\Sigma_{12,0} \\ -\Sigma_{12,0}^TA^{-1} 1 + \Sigma_{12,0}A^{-1}\Sigma_{12,0} \end{pmatrix},
\]

where \( A = \Sigma_{11,0} - \Sigma_{12,0}\Sigma_{12,0}^T \). In the above, equation (17) is applied.

With \( R_{1,21} \) and \( R_{1,3} \) fixed, conditions on \( \Sigma_{12,1}, \Sigma_{12,0}, \) \( \Sigma_{12,1} \) can lead to closure under margins with respect to the specified partition. The rule of conditional expectation for multi-variate Gaussian distributions gives

\[
\Phi_{1,1} = \Psi_{1,1} = \Sigma_{11,1}^{-1}\Sigma_{11,0} \text{ and } \Phi_{2,1} = \Psi_{2,1} = \Sigma_{22,1}\Sigma_{22,0}^{-1} = \rho_{33,1}.
\]

Note that \( \Phi_{1,1} \) here is the same as in equation (15). The conditions in equations (8)–(12) become

\[
\begin{align*}
G_1D_1 &= 0 \implies \Phi_{1,1}\Sigma_{12,0} - \Sigma_{12,1} = 0, \\
H_1D_1 &= 0 \implies -\Sigma_{12,-1} + \Psi_{1,1}\Sigma_{12,0} = 0.
\end{align*}
\]

In Case 1, based on \( G_1 \) and \( G_2 \), \( \Sigma_{12,1} = \Phi_{1,1}\Sigma_{12,0} = \Sigma_{11,1}\Sigma_{11,0}\Sigma_{12,0} \) and \( \Sigma_{12,-1} = \Sigma_{12,0}\rho_{33,1} \). If equation (18) is positive definite, then Example 3.4 becomes

\[
\Phi = \begin{pmatrix} \Sigma_{11,1} & \Sigma_{11,1}^{-1}\Sigma_{11,0}\Sigma_{12,0} \\ \rho_{33,1} \Sigma_{12,0}^T & \rho_{33,1} \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}\Sigma_{12,0} \\ -\Sigma_{12,0}^TA^{-1} 1 + \Sigma_{12,0}A^{-1}\Sigma_{12,0} \end{pmatrix},
\]

after some algebraic simplifications. This leads to a diagonal coefficient matrix so that the two subprocesses are marginal VAR(1) and AR(1) respectively with dependence coming from the innovation vector.

In Case 3, based on \( G_1 \) and \( H_2 \), the general solution has \( \Sigma_{12,0} = \Sigma_{12,1} = 0 \) with \( \Sigma_{12,-1} \) fixed. If matrix \( R_{1,12,1,3} \) in equation (18) is positive definite, then the coefficient matrix in Example 3.4 becomes

\[
\Phi = \begin{pmatrix} \Sigma_{11,1} & 0 \\ \Sigma_{12,-1}^T & \rho_{33,1} \end{pmatrix} \begin{pmatrix} \Sigma_{11,0} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \Sigma_{11,1}\Sigma_{11,0}^{-1} & 0 \\ \Sigma_{12,-1}^T\Sigma_{11,0}^{-1} & \rho_{33,1} \end{pmatrix},
\]

and clearly \( \{Z_{1t}, Z_{2t}\}_{t \geq 0} \) is VAR(1).

In Case 4, based on \( H_1 \) and \( G_2 \), the general solution has \( \Sigma_{12,0} = \Sigma_{12,1} = 0 \) with \( \Sigma_{12,-1} \) fixed. If matrix \( R_{1,12,1,3} \) in equation (18) is positive definite, then

\[
\Phi = \begin{pmatrix} \Sigma_{11,1} & \Sigma_{12,1} \\ 0 & \rho_{33,1} \end{pmatrix} \begin{pmatrix} \Sigma_{11,0} & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \Sigma_{11,1}\Sigma_{11,0}^{-1} & \Sigma_{12,1} \\ \Sigma_{11,1}\Sigma_{11,0}^{-1} & \rho_{33,1} \end{pmatrix},
\]

and clearly \( \{Z_{3t}\}_{t \geq 0} \) is AR(1).
The interesting case is Case 2 based on \( H_1 \) and \( H_2 \), with \( \Sigma_{12,1} = \Phi_{11} \Sigma_{12,0} \) and \( \Sigma_{12,1} = \Sigma_{12,0} \rho_{33,1} \), as it leads to a non-diagonal \( \Phi \) coefficient matrix in general. As an illustration, setting \( \Sigma_{11,0} = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix} \), \( \Sigma_{22,0} = (0.9) \), \( \Sigma_{11,1} = (0.8 \ 0.7) \), and \( \Sigma_{12,0} = (0.5 \ 0.5) \), makes \( R_{\{1\}, \{1\}} \) positive definite and \( \Phi \) non-diagonal.

The two examples above also indicate the fact that some non-diagonal blocks of the coefficient matrix would be \( 0 \) if at least a subprocess satisfies Condition 1. Indeed, a general conclusion about the model can be drawn: if a subprocess satisfies Condition 1, then, in the VAR representation of the original time series, the regression coefficients of the subprocess on the other subprocesses will be 0. Therefore, Case 4 when both subprocesses satisfy Condition 1 corresponds to the special situation of diagonal coefficient matrices in blocks. To show this, the coefficient matrix \( \Phi_i \) for \( 1 \leq i \leq k \) is blocked as

\[
\Phi_i = \begin{pmatrix}
\Phi_{i,S_1} & \Phi_{i,S_1,S_2} \\
\Phi_{i,S_1,S_2} & \Phi_{i,S_1}
\end{pmatrix},
\]

where the dimension of \( \Phi_{i,S_1} \) is \( d_i \times d_1 \). Then \( Z_{S_1,t} \) can be represented as

\[
Z_{S_1,t} = \sum_{i=1}^{k} \Phi_{i,S_1} Z_{S_1,t-i} + \sum_{i=1}^{k} \Phi_{i,S_1,S_2} Z_{S_1,t-i} + \epsilon_{S_1,t},
\]

where \( \epsilon_{S_1,t} \) is the subvector consisting of the first \( d_i \) elements of \( \epsilon_t \). Furthermore, \( \Phi_{i,S_1,S_2} = \cdots = \Phi_{k,S_1,S_2} = 0 \) is equivalent to

\[
\left[ Z_{S_1,t-1}, \ldots, Z_{S_1,t-k}^T \right] Z_{S_1,t-1}, \ldots, Z_{S_1,t-k},
\]

which is exactly Condition 1 for \( \{Z_{S_1,t}\}_{t>0} \). Similarly, Condition 1 for \( \{Z_{S_2,t}\}_{t>0} \) is equivalent to \( \Phi_{i,S_2,S_1} = \cdots = \Phi_{k,S_2,S_1} = 0 \). And \( \Phi_i \) would be diagonal in blocks in the case that both subprocesses satisfy Condition 1 since \( \Phi_{i,S_1,S_2} = \Phi_{i,S_1,S_1} = 0 \) for \( 1 \leq i \leq k \).

### 3.3.2. Partitions with multiple subprocesses

For partitions with multiple subprocesses, let \( \{S_1, \ldots, S_4\} \) be a partition of \( \{1, \ldots, d\} \) and let \( d_i \geq 1 \) be the cardinality of \( S_i \). Reorder the row and columns of \( R \) to get the correlation matrix of the random vector \( \left[ Z_{S_1,i}^T, \ldots, Z_{S_1,i-k}^T, \ldots, Z_{S_4,i}^T, \ldots, Z_{S_4,i-k}^T \right]^T \):

\[
R_{\{S_1, \ldots, S_4\}} = \begin{pmatrix}
R_{S_1} & R_{S_1,S_2} & \cdots & R_{S_1,S_4} \\
R_{S_1,S_2} & R_{S_2} & \cdots & R_{S_2,S_3} \\
\vdots & \vdots & \ddots & \vdots \\
R_{S_1,S_4} & R_{S_2,S_3} & \cdots & R_{S_4}
\end{pmatrix},
\]

where

\[
R_{S_i} = \begin{pmatrix}
\Sigma_{i,0} & \Sigma_{i,1} & \cdots & \Sigma_{i,k} \\
\Sigma_{i,1} & \Sigma_{i,0} & \cdots & \Sigma_{i,k-1} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{i,k} & \Sigma_{i,k-1} & \cdots & \Sigma_{i,0}
\end{pmatrix},
\]

\[
R_{S_i,S_j} = \begin{pmatrix}
\Sigma_{i,j,0} & \Sigma_{i,j,1} & \cdots & \Sigma_{i,j,k-1} \\
\Sigma_{i,j,1} & \Sigma_{i,j,0} & \cdots & \Sigma_{i,j,k-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{i,j,k-1} & \Sigma_{i,j,k-2} & \cdots & \Sigma_{i,j,0}
\end{pmatrix},
\]
Here $\Sigma_{ij}$ for $1 \leq i, j \leq n$, which denote the covariance matrices between $Z_{S_{i,t}}$ and $Z_{S_{j,t-l}}$, are treated as the parameter entries of $R_{S_{i,t}}$. Notice that $R_{S_{i,t}}$ is the correlation matrix of $(Z_{S_{i,t}}, \ldots, Z_{S_{i,t-k}})^T$ and $R_{S_{j,t}}$ is the correlation matrix between $(Z_{S_{j,t}}, \ldots, Z_{S_{j,t-k}})^T$ and $(Z_{S_{j,t}}, \ldots, Z_{S_{j,t-k}})^T$. It means that when $\{Z_{i,t}\}_{t=0}^T$ is closed under margins with respect to partition $\{S_1, \ldots, S_p\}$, $R_{S_{i,t}}$ actually specifies the VAR($k$) subprocess $\{Z_{S_{i,t}}\}_{t=0}^T$ with contemporaneous correlation $\Sigma_{i0}$ and serial correlation $\Sigma_{ii1}, \ldots, \Sigma_{iik}$. And $R_{S_{j,t}}$ models the dependence between $\{Z_{S_{j,t}}\}_{t=0}^T$ and $\{Z_{S_{j,t}}\}_{t=0}^T$ with contemporaneous correlation $\Sigma_{j0}$ and cross-sectional correlations $\Sigma_{jj1}, \ldots, \Sigma_{jik}$. Therefore, the parameter entries of $R_{S_{i,j,t}}$ can be further divided into two parts given the partition $\{S_1, \ldots, S_p\}$: the entries of $R_{S_{i,t}}$, $\ldots$, $R_{S_{n,t}}$ that model all individual VAR($k$) subprocesses, and the entries of $R_{S_{i,j,t}}$ for $i \neq j$ that model the dependence structure between the subprocesses. To make $\{Z_{i,t}\}_{t=0}^T$ closed under margins with respect to $\{S_1, \ldots, S_p\}$, we still hold the parameters of $R_{S_{i,t}}$, $\ldots$, $R_{S_{n,t}}$ fixed while investigating the constraints on the parameters of $R_{S_{i,j,t}}$ for $i \neq j$. Note that we only need to consider the case of $i < j$ since $\Sigma_{p,k} = \Sigma_{k,p}^T$.

We then deal with the case of multiple subprocesses by considering all pairs of distinct subprocesses with the approach in Section 3.3.1 above. Indeed, the two conditions for the process $\{Z_{S_{i,t}}\}_{t=0}^T$ in Theorem 3.2 can be written as

1. $[Z_{S_{i,t}} \perp (Z_{S_{j,t-1}}, \ldots, Z_{S_{j,t-k}})^T] | Z_{S_{j,t-1}}, \ldots, Z_{S_{j,t-k}}$ for $j \neq i$,
2. $[Z_{S_{j,t-k-1}} \perp (Z_{S_{j,t-1}}, \ldots, Z_{S_{j,t-k}})^T] | Z_{S_{j,t-1}}, \ldots, Z_{S_{j,t-k}}$ for $j \neq i$.

It indicates that Condition 1 or 2 holds for subprocess $\{Z_{S_{i,t}}\}_{t=0}^T$ if and only if the same condition as in equation (7) holds for any pairs of subprocesses $\{Z_{S_{i,t}}\}_{t=0}^T$ and $\{Z_{S_{j,t}}\}_{t=0}^T$ for $1 \leq i \leq n$. For simplicity, let $c_i \in \{1, 2\}$ label the index of the above condition that $\{Z_{S_{i,t}}\}_{t=0}^T$ satisfies. Then the sufficient condition for pair $(i, j) \in \{1, \ldots, n\}^2$ with $i \neq j$ is similar to the results of partitions with two subprocesses in Section 3.3.1. In particular, the sufficient condition for $(i, j)$ can be expressed as the constraints on the dependence parameters between two subprocesses $\{Z_{S_{i,t}}\}_{t=0}^T$ and $\{Z_{S_{j,t}}\}_{t=0}^T$. For $1 \leq p \leq n$, the coefficient matrices $G_p$ and $H_p$ are defined as

$$G_p = \begin{pmatrix}
0 & \Phi_{p,2} & \Phi_{p,1} & -I_{dp} & 0 & \cdots & 0 \\
0 & 0 & \Phi_{p,k} & \Phi_{p,2} & \Phi_{p,1} & -I_{dp} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \Phi_{p,k} & \cdots & \Phi_{p,2} & \Phi_{p,1} & -I_{dp}
\end{pmatrix},$$

where

$$\begin{pmatrix}
\Phi_{p,1} \\
\Phi_{p,2} \\
\vdots \\
\Phi_{p,k}
\end{pmatrix}^T = \begin{pmatrix}
\Sigma_{pp,0} & \Sigma_{pp,1} & \cdots & \Sigma_{pp,k-1} \\
\Sigma_{pp,k} & \Sigma_{pp,0} & \cdots & \Sigma_{pp,k-2} \\
\vdots & \vdots & \ddots & \ddots \\
\Sigma_{pp,k-1} & \Sigma_{pp,k-1} & \cdots & \Sigma_{pp,0}
\end{pmatrix}^{-1}.$$

and

$$H_p = \begin{pmatrix}
-I_{dp} & \Psi_{p,k} & \cdots & \Psi_{p,2} & \Psi_{p,1} & 0 & 0 & \cdots & 0 \\
0 & -I_{dp} & \Psi_{p,k} & \cdots & \Psi_{p,2} & \Psi_{p,1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -I_{dp} & \Psi_{p,k} & \cdots & \Psi_{p,2} & \Psi_{p,1} & 0
\end{pmatrix},$$
where

\[
\begin{pmatrix}
\mathbf{\Psi}_1^T \\
\mathbf{\Psi}_2^T \\
\vdots \\
\mathbf{\Psi}_p^T
\end{pmatrix} = 
\begin{pmatrix}
\Sigma_{pp}^{T} \\
\Sigma_{pp,1}^{T} \\
\vdots \\
\Sigma_{pp,k}^{T}
\end{pmatrix}^T 
\begin{pmatrix}
\mathbf{\Sigma}_{pp,0}^{T} & \mathbf{\Sigma}_{pp,1} & \cdots & \mathbf{\Sigma}_{pp,k-1} \\
\mathbf{\Sigma}_{pp,1}^T & \mathbf{\Sigma}_{pp,0} & \cdots & \mathbf{\Sigma}_{pp,k-2} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{\Sigma}_{pp,k-1}^T & \mathbf{\Sigma}_{pp,k-2} & \cdots & \mathbf{\Sigma}_{pp,0}
\end{pmatrix}^{-1}
\]

Since \( R_S \) is the correlation matrix of \[ \{ \mathbf{Z}_{S_{i,j}}^T, \ldots, \mathbf{Z}_{S_{i,j-k}}^T \}^T \), it can also be checked that \( \Phi_{p,1}, \ldots, \Phi_{p,k} \) are the coefficient matrices of \( \text{VAR}(k) \) process \( \{ \mathbf{Z}_{S_{i,j}} \}_{i=0} \) when \( \{ \mathbf{Z}_{i} \}_{i=0} \) is closed under margins with respect to partition \( \{ S_1, \ldots, S_n \} \). It follows that for pair \((i,j)\), all combinations of Condition 1 and 2 for the two subprocesses of \( S_i \) and \( S_j \) can be written as

\[
(c_i, c_j) = (1, 1) : G_iD_{ij} = 0 \quad \text{and} \quad (G_iL_{ij})(L_{ij}D_{ij}) = 0, \\
(c_i, c_j) = (1, 2) : G_iD_{ij} = 0 \quad \text{and} \quad (H_iL_{ij})(L_{ij}D_{ij}) = 0, \\
(c_i, c_j) = (2, 1) : H_iD_{ij} = 0 \quad \text{and} \quad (G_iL_{ij})(L_{ij}D_{ij}) = 0, \\
(c_i, c_j) = (2, 2) : H_iD_{ij} = 0 \quad \text{and} \quad (H_iL_{ij})(L_{ij}D_{ij}) = 0.
\]

where \( D_{ij} = (\Sigma_{j,k}^{T}, \ldots, \Sigma_{j,b}^{T}, \Sigma_{j,0}^{T}, \Sigma_{j,1}^{T}, \ldots, \Sigma_{j,k}^{T})^T \),

\[
D_{jj} = (\Sigma_{j,k}^{T}, \ldots, \Sigma_{j,b}^{T}, \Sigma_{j,0}^{T}, \Sigma_{j,1}^{T}, \ldots, \Sigma_{j,k}^{T})^T = (\Sigma_{j,k}, \ldots, \Sigma_{j,1}, \Sigma_{j,0}, \Sigma_{j,-1}, \ldots, \Sigma_{j,-k})^T,
\]

and \( L_i = J_{2k+1} \otimes I_{\left[ S_i \right]} \) where \( J_{2k+1} \) is the \((2k + 1)\)-dimensional exchange matrix whose elements in the anti-diagonal are 1 and all other elements are zero.

It follows that given the condition labels \( c_1, \ldots, c_n \) and \( R_{S_1}, \ldots, R_{S_n} \) that model all individual \( \text{VAR}(k) \) subprocesses, \( \Sigma_{j,k}, \ldots, \Sigma_{j,b} \) for all pairs \((i,j)\) can be parameterized by the given or fixed parameters among them according to Cases 1–4 in Appendix C.0.2. Therefore the correlation matrix \( R_{\{S_1,\ldots,S_n\}} \) can be characterized by three groups of parameters: the condition labels \( c_1, \ldots, c_n \), the entries of \( R_{S_1}, \ldots, R_{S_n} \) that model all individual \( \text{VAR}(k) \) subprocesses, and the corresponding fixed parameters. Note that the fixed parameter is \( \Sigma_{j,0} \) if \( c_i = c_j = 1 \) or \( c_i = c_j = 2 \), \( \Sigma_{ij} \) if \( c_i = 1, c_j = 2 \), and \( \Sigma_{ij} \) if \( c_i = 2, c_j = 1 \) for each pair \((i,j)\) of \( 1 \leq i < j \leq n \). Moreover, an extra necessary constraint that \( R_{\{S_1,\ldots,S_n\}} \) is positive definite should always be guaranteed. Then correlation matrix \( R \) in equation (3) can be obtained by reordering the rows and columns of \( R_{\{S_1,\ldots,S_n\}} \).

Note that for the case of multiple subprocesses, we can still draw the general conclusion that if subprocess \( \{ \mathbf{Z}_{S_{i,j}} \}_{i=0} \) satisfies Condition 1, then in the \( \text{VAR} \) representation of the original time series, the regression coefficients of \( \mathbf{Z}_{S_{ij},j} \) on \( \mathbf{Z}_{S_{i,j-1}}, \ldots, \mathbf{Z}_{S_{i,j-k}} \) for \( i \neq j \) will be 0. It means that all coefficient matrices of the original \( \text{VAR} \) process will be diagonal in blocks if all subprocesses fulfill Condition 1, and all subprocesses should satisfy Condition 2 if all non-diagonal blocks of the coefficient matrices of the original \( \text{VAR} \) process are required to be non-zero.

The next example shows how to deal with a partition with three subprocesses based on the results of partitions with two subprocesses, and general formulae for solving the linear equations of Cases 1 and 2 are derived in Appendix C.0.2.

**Example 3.5.** (three-dimensional \( \text{VAR}(1) \) process). Let \( \{ \mathbf{Z}_{i} \}_{i=0} \) follow a three-dimensional \( \text{VAR}(2) \) model and the considered partition is \( \{ \{1\}, \{2\}, \{3\} \} \), i.e., \( \mathbf{Z}_{S_{1,j}} = Z_{1,j}, \mathbf{Z}_{S_{2,j}} = Z_{2,j}, \) and \( \mathbf{Z}_{S_{3,j}} = Z_{3,j} \). The correlation matrix of
$$(Z_{1,t}, Z_{2,t-1}, Z_{2,t}, Z_{2}, Z_{3}, Z_{3,2-1})^T$$ is

$$\begin{pmatrix} R_{[1]} & R_{[1],[2]} & R_{[1],[3]} \\ R_{[1],[2]}^T & R_{[2]} & R_{[2],[3]} \\ R_{[1],[3]}^T & R_{[2],[3]}^T & R_{[3]} \end{pmatrix} = \begin{pmatrix} 1 & \rho_{11,1} & \rho_{12,0} & \rho_{12,1} & \rho_{13,0} & \rho_{13,1} \\ \rho_{11,1} & 1 & \rho_{12,0} & \rho_{12,1} & \rho_{13,0} & \rho_{13,1} \\ \rho_{12,0} & \rho_{12,0} & 1 & \rho_{22,1} & \rho_{23,0} & \rho_{23,1} \\ \rho_{12,1} & \rho_{12,0} & \rho_{22,1} & 1 & \rho_{23,1} & \rho_{23,0} \\ \rho_{13,0} & \rho_{13,1} & \rho_{23,0} & \rho_{23,1} & 1 & \rho_{33,1} \\ \rho_{13,1} & \rho_{13,0} & \rho_{23,1} & \rho_{23,0} & \rho_{33,1} & 1 \end{pmatrix}.$$

It can be verified that given $\{Z_i\}_{i=0}^t$ is closed under margins with respect to $\{\{1\}, \{2\}, \{3\}\}$, VAR(1) subprocesses $\{Z_{i,j}\}_{i=0}^\infty$, $\{Z_{i,j}\}_{i=0}^\infty$, $\{Z_{i,j}\}_{i=0}^\infty$ can be specified by $R_{[1]}$, $R_{[2]}$, and $R_{[3]}$ respectively. Then for each pair $(i, j)$ for $1 \leq i < j \leq 3$, we fix $R_{[i]}$, $R_{[j]}$ and impose the constraint on $R_{[i,j]}$. Indeed, for two subprocesses $\{Z_{i,j}\}_{i=0}^\infty$ and $\{Z_{j,k}\}_{j=0}^\infty$ of pair $(i, j)$, according to Example 3.3, the conditions can be written as

1: $(c_1, c_2)$ = $(1, 1)$: set $\rho_{0,0}$ as the parameter and $\rho_{1,1} = \rho_{2,2} = \rho_{0,0}$; $\rho_{1,1} = \rho_{0,0} = 0$;

2: $(c_1, c_2)$ = $(1, 2)$: set $\rho_{0,0}$ as the parameter and let $\rho_{0,0} = 0$;

3: $(c_1, c_2)$ = $(2, 1)$: set $\rho_{0,0}$ as the parameter and let $\rho_{0,0} = 0$;

4: $(c_1, c_2)$ = $(2, 2)$: set $\rho_{0,0}$ as the parameter and $\rho_{0,0} = 0$.

Combining all results above, there are conditions on $R_{[11,2]} = \begin{pmatrix} \rho_{12,0} & \rho_{12,1} \\ \rho_{12,1} & \rho_{12,0} \end{pmatrix}$, $R_{[11,3]} = \begin{pmatrix} \rho_{13,0} & \rho_{13,1} \\ \rho_{13,1} & \rho_{13,0} \end{pmatrix}$, and $R_{[23,0]} = \begin{pmatrix} \rho_{23,0} \\ \rho_{23,0} \end{pmatrix}$ for any given condition label. For instance, if the condition label $(c_1, c_2, c_3)$ is $(1, 2, 2)$, then the above results for three pairs $(c_1, c_2) = (1, 2)$, $(c_1, c_3) = (1, 2)$, and $(c_2, c_3) = (2, 2)$ should be picked, i.e., $\rho_{12,0}, \rho_{13,0}, \rho_{23,0}$ should be set as the fixed parameters and other dependence parameters between the subprocesses can be solved using $\rho_{12,0} = \rho_{13,0} = 0$. However when $(c_1, c_2, c_3)$ is $(2, 2, 2)$ so that all coefficient matrices are non-diagonal in general, the dependence parameters should satisfy

$$\begin{align*}
\rho_{12,0} &= \rho_{12,1} \rho_{12,0}, \\
\rho_{12,1} &= \rho_{22,1} \rho_{12,0}, \\
\rho_{13,0} &= \rho_{13,1} \rho_{13,0}, \\
\rho_{13,1} &= \rho_{33,1} \rho_{13,0}, \\
\rho_{23,0} &= \rho_{22,1} \rho_{23,0}, \\
\rho_{23,1} &= \rho_{33,1} \rho_{23,0}.
\end{align*}$$

A compatible numerical example in this case is $\rho_{11,1} = 0.6$, $\rho_{22,1} = 0.7$, $\rho_{33,1} = 0.8$, and $\rho_{12,0} = \rho_{13,0} = \rho_{23,0} = 0.5$.

An interesting case is when the number of subprocesses in the partition is exactly $d$, and so all the subprocesses are univariate. Then, margins of any dimension and any univariate components of the $d$-dimensional VAR($k$) model are also VAR($k$) processes. Moreover, according to our discussion about the blocks of coefficient matrices at the end of Section 3.3.1, the coefficient matrices are all diagonal in this case if all univariate subprocesses satisfy Condition 1. There will be at least one coefficient matrix with non-zero non-diagonal entries if two conditions are met: at least one univariate subprocess satisfies Condition 2, and at least one dependence parameter between a subprocess satisfying Condition 2 and the other subprocesses is non-zero.

### 4. PARAMETER ESTIMATION

Details of the maximum likelihood estimation of the margin-closed stationary multi-variate time series model are given. If the stationary joint distributions of VAR($k$) Gaussian model, with possible margin closure under subprocesses, is used as a multi-variate copula, then univariate margins can be first fitted with parametric families, before probability integral transforms to standard Gaussian margins to estimate the dependence parameters of the VAR($k$) model. Note that all latent VAR($k$) models are parameterized by the block Toeplitz correlation matrices of $k + 1$ consecutive observations in our fitting procedure below. To get the VAR representations of the models, the Durbin-Levinson algorithm can be applied, see section 11.4 in Brockwell et al. (2009).
The joint probability density function (PDF) of consecutive \( k + 1 \) observations in the time series is

\[
f_{X_{t-1}, \ldots, x_{t-k}, \eta_1, \ldots, \eta_d, R} = c_{\lambda_{t-1}, \ldots, \lambda_{t-k}, R} \prod_{i=1}^{d} f(x_{t-i}; \eta_i), \tag{20}
\]

where \( x_{t-i} = (x_{t-i}, \ldots, x_{t-d})^T \) is the realization of \( X_{t-i} \) for \( 0 \leq l \leq k, f(x; \eta_i) \) and \( F(x; \eta_i) \) are the PDF and CDF of the univariate marginal component \( X_{t-i} \) for \( 1 \leq i \leq d \), and \( u_{t-i} = F(x_{t-i}; \eta_i) \). Then, the log-likelihood of the given realization of the time series \( x_1, \ldots, x_T \) is

\[
\ell(\eta_1, \ldots, \eta_d, R|x_1, \ldots, x_T) = \sum_{t=k+1}^{T} \log f_{X_{t-1}, \ldots, x_{t-k}, \eta_1, \ldots, \eta_d, R}(x_t | x_{t-1}, \ldots, x_{t-k}, \eta_1, \ldots, \eta_d, R) + \sum_{i=1}^{k} \log f_{X_{t-1}, \ldots, x_{t-k}, \eta_1, \ldots, \eta_d, R}(x_{t-i} | x_{t-i-1}, \ldots, x_{t-1}, \eta_1, \ldots, \eta_d, R), \tag{21}
\]

where \( f_{X_{t-1}, \ldots, x_{t-k}, \eta_1, \ldots, \eta_d, R} \) is the conditional density of \( X_t \) given \( X_{t-1}, \ldots, x_{t-k} \); this can be analytically derived based on equation (20) and the conditional distributions of multi-variate Gaussian random vectors.

Note that in equation (21), \( \eta_1, \ldots, \eta_d \) are parameters of the univariate margins, and the correlation matrix \( R \) should be parameterized by following the method in Section 3.3.2. The partition and condition labels are treated as hyperparameters to fit the model. Then entries of \( R \) can be divided into two groups: the parameters in \( R_{S_1}, \ldots, R_{S_d} \) that model all individual VAR(\( k \)) subprocesses, and the parameters in \( R_{S_i} \) for \( i < j \) that model the dependence structure between the subprocesses. More precisely, the log-likelihood of a subprocess \( \{X_{S_i}\}_{i=1}^{\infty} \) and realization \( x_{S_i,j} = (x_{S_i,j}, \ldots, x_{S_i,\eta_d})^T \) in \( 1 \leq t \leq T \) can actually be specified by \( \eta_1, \ldots, \eta_d \) and \( R_{S_i} \):

\[
\ell_{S_i}(\eta_1, \ldots, \eta_d, R_{S_i}|x_1, \ldots, x_T) = \sum_{t=k+1}^{T} \log f_{X_{S_i,j-1}, \ldots, x_{S_i,j-k}, \eta_1, \ldots, \eta_d, R_{S_i}}(x_{S_i,j} | x_{S_i,j-1}, \ldots, x_{S_i,j-k}, \eta_1, \ldots, \eta_d, R_{S_i}) + \sum_{i=1}^{k} \log f_{X_{S_i,j-1}, \ldots, x_{S_i,j-k}, \eta_1, \ldots, \eta_d, R_{S_i}}(x_{S_i,j}| x_{S_i,j-1}, \ldots, x_{S_i,j-i-1}, \eta_1, \ldots, \eta_d, R_{S_i}), \tag{22}
\]

where the conditional density \( f_{X_{S_i,j-1}, \ldots, x_{S_i,j-k}, \eta_1, \ldots, \eta_d, R_{S_i}} \) can be derived from the joint density

\[
f_{X_{S_i,j-1}, \ldots, x_{S_i,j-k}, \eta_1, \ldots, \eta_d, R_{S_i}} = c_{X_{S_i,j}, \ldots, x_{S_i,j-i}, \eta_1, \ldots, \eta_d, R_{S_i}} \prod_{i=1}^{d} f(x_{S_i,j-i}; \eta_i),
\]

by using the properties of the conditional distribution for multi-variate Gaussian random vectors. Based on the division of the parameter set as well as the consistency and asymptotic normality of the quasi maximum likelihood estimators based on log-likelihood of marginal densities proved by Francq and Zakoian (2013), a multiple-stage procedure of estimation can be applied.

**Step 1.** Estimate the univariate margin parameters \( \eta_1, \ldots, \eta_d \) by maximizing the quasi-likelihood of margins, i.e.,

\[
\hat{\eta}_i = \arg \max_{\eta_i} \sum_{t=1}^{T} \log f(x_{S_i,j}; \eta_i) \text{ for } i = 1, \ldots, d.
\]
Step 2. For each subprocess $i \in \{1, \ldots, n\}$ in the partition, hold $\eta_i, \ldots, \eta_d$ fixed of their estimates obtained in step 1, and estimate $R_{S_i}$ individually through maximizing the objective function in equation (22).

Step 3. Hold $\eta_1, \ldots, \eta_d$ and $R_{S_1}, \ldots, R_{S_n}$ fixed of their estimates obtained in steps 1 and 2, estimate $R_{S_{i,j}}$ for $i < j$ simultaneously according to condition labels by maximizing the log-likelihood in equation (21), through the approach of parameterization in equation (3.3.2). Note that the positive definiteness of $R$ needs to be guaranteed.

Step 4. If necessary, hold only $\eta_1, \ldots, \eta_d$ fixed of their estimates obtained in step 1, and use the estimates of $R_{S_1}, \ldots, R_{S_n}$ and $R_{S_{i,j}}$ for $i < j$ in steps 2 and 3 as a starting point, update the estimates of $R_{S_1}, \ldots, R_{S_n}$ and $R_{S_{i,j}}$ for $i < j$ simultaneously by maximizing equation (21), under the constraint that $R$ is positive definite.

The key step is to estimate the correlation submatrices $R_{S_1}, \ldots, R_{S_n}$ for all subprocesses before estimating the matrices $R_{S_{i,j}}$ that measure the dependence structure between the subprocesses. The method is based on the property that all subprocesses in the partition follow VAR($k$) models under the constraint of closure under margins. Otherwise, if a subprocess is not Markov or has Markov order not equal to $k$, the estimation of the correlation matrix of $k + 1$ consecutive observations of the subprocess may be biased. The idea of closure under margins would be helpful especially in the situation of a high-dimensional data sets with inadequate sample size for unconstrained VAR models. In this case, the set of parameters of $R_{S_{i,j}}$ for $i < j$ will be reduced significantly if the original time series are partitioned into subprocesses with much lower dimensions. More importantly, by fitting each subprocess individually before estimating the whole correlation matrix $R$, the problem of maximizing the likelihood with a high-dimension input is divided into several lower-dimensional optimization problems, and this reduces computational complexity.

5. A NUMERICAL EXAMPLE AND AN EMPIRICAL STUDY

Here a numerical example and an application are given. Section 5.1 has a numerical example that includes comparisons of coefficient matrices and covariance matrices of the innovation vector for the different cases in Section 3. Section 5.2 has an application to a macro-economic data set.

5.1. Numerical example of a bivariate VAR(2) model

We give a numerical example of a bivariate margin-closed VAR(2) model to discuss the behavior of the coefficient matrices under different condition labels. Consider a two-dimensional VAR(2) process $\{Z_{1,t}, Z_{2,t}\}_{t=0}^{\infty}$ with the partition $\{\{1\}, \{2\}\}$. Suppose the $Z_{1,t}$ and $Z_{2,t}$ have standard Gaussian margins, and the correlation matrices of $(Z_{1,t}, Z_{1,t-1}, Z_{1,t-2})^T$ and $(Z_{2,t}, Z_{2,t-1}, Z_{2,t-2})^T$ are

$$R_{(1)} = \begin{pmatrix} 1 & -0.8 & 0.6 \\ -0.8 & 1 & -0.8 \\ 0.6 & -0.8 & 1 \end{pmatrix}$$
and $R_{(2)} = \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.6 & 1 & 0.6 \\ 0.5 & 0.6 & 1 \end{pmatrix},$

respectively. Note that both $R_{(1)}$ and $R_{(2)}$ are Toeplitz matrices so that the two univariate subprocesses are stationary AR(2). The two univariate representations under the constraint that $\{(Z_{1,t}, Z_{2,t})\}_{t=0}^{\infty}$ is closed under margins with respect to $\{\{1\}, \{2\}\}$ are as follows:

$$Z_{1,t} = -0.889Z_{1,t-1} - 0.111Z_{1,t-2} + \epsilon_{Z_{1,t}}, \quad \epsilon_{Z_{1,t}} \overset{i.i.d.}{\sim} N(0, 0.356);$$
$$Z_{2,t} = 0.469Z_{2,t-1} + 0.219Z_{2,t-2} + \epsilon_{Z_{2,t}}, \quad \epsilon_{Z_{2,t}} \overset{i.i.d.}{\sim} N(0, 0.609).$$

To see the behavior of the coefficient matrices under different condition labels, suppose Corr($Z_{1,t}, Z_{2,t}$) = 0.35 in the cases of condition labels $(1, 1)$ and $(2, 2)$, Corr($Z_{1,t-2}, Z_{2,t-2}$) = 0.35 in the case of condition labels $(1, 2)$, and
two subprocesses for condition labels
between AR(2) subprocesses
condition labels
innovation vectors under the constraint of closure under margins. The corresponding coefficient matrices, as well
Condition labels
will be diagonal in blocks if all subprocesses fulfill Condition 1, and all subprocesses should satisfy Condition 2
Condition 1. It corresponds to our previous conclusion that all coefficient matrices of the original VAR process
cesssatisfiesCondition1, and thenon-diagonalelementsinthesecondroware0ifthesecondsubprocesssatisfies
threecases. However, thenon-diagonalelementsinthefirstrowofthecoefficientmatricesare0ifthefirstsubpro-
if both subprocesses satisfy Condition 1 while the non-diagonal coefficient matrices can be obtained in the other
matrices

Table I. The coefficient matrices and covariance matrix of the innovation vector of VAR(2) process \( \{Z_{1,t}, Z_{2,t}\}_{t \geq 0} \) with
cross-correlation set as 0.35, under different sufficient conditions of closure under margins with respect to partition \( \{1, 2\} \)

| Condition labels | \( \Phi_1 \) | \( \Phi_2 \) | \( \Sigma_e \) |
|------------------|-------------|-------------|-------------|
| (1, 1)           | \[
-0.889 & 0 \\
0 & 0.469
\] | \[
-0.111 & 0 \\
0 & 0.219
\] | \[
0.356 & 0.447 \\
0.447 & 0.609
\] |
| (1, 2)           | \[
-0.889 & 0 \\
0.778 & 0.469
\] | \[
-0.111 & 0 \\
0.972 & 0.219
\] | \[
0.356 & 0.039 \\
0.039 & 0.269
\] |
| (2, 1)           | \[
-0.889 & -0.328 \\
0 & 0.469
\] | \[
-0.111 & 0.547 \\
0 & 0.219
\] | \[
0.164 & -0.077 \\
-0.077 & 0.609
\] |
| (2, 2)           | \[
-0.716 & 0.656 \\
-1.842 & 0.296
\] | \[
0.353 & -0.330 \\
-0.863 & 0.736
\] | \[
0.194 & -0.196 \\
-0.196 & 0.287
\] |

Table II. The values of the fixed parameters of VAR(2) process \( \{Z_{1,t}, Z_{2,t}\}_{t \geq 0} \) under different sufficient conditions for closure
under margins with respect to partition \( \{1, 2\} \). The values were chosen to get similar correlations between components of the
innovation vector

| Condition labels | Fixed parameter | Value |
|------------------|----------------|-------|
| (1, 1)           | Corr\((Z_{1,t}, Z_{1,0})\) | 0.292 |
| (1, 2)           | Corr\((Z_{1,t-2}, Z_{1,t})\) | 0.464 |
| (2, 1)           | Corr\((Z_{1,t}, Z_{2,t-1})\) | -0.459 |
| (2, 2)           | Corr\((Z_{1,t}, Z_{2,t})\) | -0.346 |

Corr\((Z_{1,t}, Z_{2,t-2}) = 0.35\) in the case of condition labels (2, 1). Then the other cross-sectional dependence parameters
between AR(2) subprocesses \( \{Z_{1,t}\}_{t \geq 0} \) and \( \{Z_{2,t}\}_{t \geq 0} \) can be derived following the procedure in Section 3. It is easy
to verify the positive definiteness of the correlation matrix \( R \) in all cases. Table I shows the derived coefficient
matrices \( \Phi_1, \Phi_2 \) and covariance matrix \( \Sigma_e \) of the innovation vectors of VAR(2) process \( \{Z_{1,t}, Z_{2,t}\}_{t \geq 0} \).

Similar to the results of Example 3.3, the coefficient matrices of the VAR process \( \{Z_{1,t}, Z_{2,t}\}_{t \geq 0} \) are all diagonal
if both subprocesses satisfy Condition 1 while the non-diagonal coefficient matrices can be obtained in the other
three cases. However, the non-diagonal elements in the first row of the coefficient matrices are 0 if the first subpro-
cess satisfies Condition 1, and the non-diagonal elements in the second row are 0 if the second subprocess satisfies
Condition 1. It corresponds to our previous conclusion that all coefficient matrices of the original VAR process
will be diagonal in blocks if all subprocesses fulfill Condition 1, and all subprocesses should satisfy Condition 2
if all non-diagonal blocks of the coefficient matrices of the original VAR process are required to be non-zero.

Table I also shows different covariance matrices of the innovation vectors under different condition labels. \( \epsilon_{Z_{1,t}} \) and
\( \epsilon_{Z_{2,t}} \) are positively correlated for condition labels (1, 1) and (1, 2), while they are negatively correlated for
condition labels (2, 1) and (2, 2). The differences between the covariance matrices result from the same value
of cross-correlation of 0.35 but different interpretations of the fixed parameters under different condition labels.
Condition labels (1, 1) and (2, 2) refer to the fixed parameter as the contemporaneous cross-sectional dependence
between two subprocesses while the fixed parameter is the cross-sectional dependence at lag \(-2\) and \(-2\) between
two subprocesses for condition labels (1, 2) and (2, 1) respectively.

The results from the fixed parameter in Table II are helpful to further understand the correlation structure of the
innovation vectors under the constraint of closure under margins. The corresponding coefficient matrices, as well
as the correlation matrices of the innovation vectors are presented in Table III.

As shown in Table III, the values of the fixed parameters in Table II can lead to similar correlations between
\( \epsilon_{Z_{1,t}} \) and \( \epsilon_{Z_{2,t}} \) even though the non-diagonal entries in their coefficient matrices are distinct. Therefore, by choosing
appropriate values, even though the interpretation of the fixed parameter will vary with the condition label, they
Table III. The coefficient and covariance matrices of the innovation vector of VAR(2) process \((Z_{1,t}, Z_{2,t})\) with values of fixed parameters in Table II, under different sufficient conditions of closure under margins with respect to partition \([\{1\}, \{2\}]\)

| Condition labels | \(\Phi_1\) | \(\Phi_2\) | Correlation matrix of \(\epsilon_t\) |
|------------------|-----------|-----------|---------------------------------|
| (1, 1)           | \((-0.889, 0\)) | \((-0.111, 0\)) | \((1, 0.801)\) |
|                  | \((0, 0.469)\)   | \((0, 0.219)\)   | \((0.801, 1)\)   |
| (1, 2)           | \((-0.889, 0\)) | \((-0.111, 0\)) | \((1, 0.812)\) |
|                  | \((1.031, 0.469)\) | \((1.289, 0.219)\) | \((0.812, 1)\) |
| (2, 1)           | \((-0.889, 0.430)\) | \((-0.111, -0.717)\) | \((1, 0.792)\) |
|                  | \((0, 0.469)\)   | \((0, 0.219)\)   | \((0.792, 1)\)   |
| (2, 2)           | \((-0.787, -0.590)\) | \((0.243, 0.246)\) | \((1, 0.797)\) |
|                  | \((1.080, 0.367)\) | \((0.721, 0.630)\) | \((0.797, 1)\) |

indeed can result in similar correlation matrices of the innovation vectors while give different types of coefficient matrices. It shows the flexibility of the margin-closed VAR models.

5.2. Application

This section illustrates the margin-closed VAR model on a trivariate multi-variate time series from the FRED monthly database (McCracken and Ng, 2016). The database contains many macroeconomic variables with monthly frequency. All variables have been transformed, possibly through differencing, so that the assumption of stationarity in the resulting time series may be reasonable.

5.2.1. Three variables from FRED monthly database

For an illustration, three variables are used: the total consumer loans and leases outstanding (CLL), the real personal consumption expenditures (PCE), and the consumer price index (CPI). The three variables were transformed to the second, the first, and the second difference of natural log respectively. All values are presented on the scale of percentages. Based on plots, approximate stationarity seems acceptable for the period from March 1989 to August 2001, inclusive; this corresponds to 150 consecutive months of date. Figure 1 gives the plots of the transformed trivariate time series.

5.2.2. Model comparisons

We employ the dependence structure in margin-closed VAR model and the unrestricted VAR model as the multi-variate copula and compare their performances in the above dataset. To fit the model, we first determine the marginal distribution of each univariate component. As the univariate margins may have heavy tails and skewness, we fit them with both the Gaussian distribution and the extended skew t-distribution (Jones and Faddy, 2003). The Akaike information criterion (AIC) values presented in Table IV indicate that the extended skew t-distribution is preferred for the variables CLL and CPI. The quantile-quantile plots (not included here) confirm a good agreement between the model and the data, and show that the extended skew t-distribution better captures the tails for these two variables.

The extended skew t-distribution has two extra parameters \(a\) and \(b\), in addition to location and scale parameters to control the left and right tailweights. Estimated univariate marginal parameters are shown in Table V. The fitted values indicate heavy tails, relative to Gaussian, in both sides of the univariate densities of CLL and PCE. This also indicates the need for non-Gaussian margins for the stationary joint distribution in equation (2).

For the dependence structure in the stationary joint distribution in equation (2), the multi-variate Gaussian copula is a good choice because of the limited sample size. It is equivalent to a latent VAR process in the multi-variate time series model. We fit the margin-closed VAR model and the unrestricted VAR model with the pseudo-observations
Figure 1. The plots of transformed time series of the total consumer loans and leases outstanding (CLL), the real personal consumption expenditures (PCE), and the consumer price index (CPI), from March 1989 to August 2001

Table IV. The AIC values of extended skew $t$-distribution and Gaussian distribution, fitting the univariate margins of each univariate component of the trivariate time series

| Component | Extended skew $t$-distribution | Gaussian distribution |
|-----------|-------------------------------|-----------------------|
| CLL       | 403                           | 404                   |
| PCE       | 128                           | 126                   |
| CPI       | $-40$                         | $-38$                 |

Table V. The estimates of the model parameters for each univariate component of the trivariate time series

| Component | Location parameter | Scale parameter | Left tailweight parameter | Right tailweight parameter |
|-----------|--------------------|-----------------|---------------------------|---------------------------|
| CLL       | 0.850              | 0.791           | 5.739                     | 9.344                     |
| PCE       | 0.281              | 0.363           | -                         | -                         |
| CPI       | -0.032             | 0.172           | 3.053                     | 2.738                     |

Note: The CLL and CPI series are fitted using the extended skew $t$-distribution, while the PCE series is fitted using the Gaussian distribution.

of the latent VAR process obtained by transforming all univariate components to be standard normal, then compare the margin-closed and unrestricted time series models that include fitting the univariate margins. To fit the margin-closed model, other hyperparameters should be specified including the partitions and the condition labels. Since there are only three univariate components and the simpler interpretations of non-diagonal coefficient matrices are of our main interests, we consider the simple case of the partition $\{1\}, \{2\}, \{3\}$ and the condition label $(2, 2, 2)$. To determine the Markov order $k$ of the model, we compare the AIC values of two time series models with different Markov orders. Table VI shows the number of parameters and AIC values of the two models with Markov order from 1 to 5.

Because of the constraint of closure under margins, the margin-closed model has significantly fewer number of parameters than the unrestricted model, and the reduction in the parameter set would be even larger in the case of higher dimension of data sets and higher Markov orders. It can be noticed that the AIC values of margin-closed model are similar to that of the unrestricted model in the case of Markov order 1, and smaller in all case of higher Markov orders. As Markov order 2 leads to the minimum AIC values for the
The process can be AR(2) with a matrix of observation at lag. For the latent margin-closed VAR(2) model, the ACF values and multi-variate Portmanteau (Ljung–Box) test of residuals also indicate the adequacy of the model. For the latent margin-closed VAR(2) model, especially the entries in the diagonals of all fitted matrices. As for the covariance matrices of the innovation vectors, except for the closeness of the diagonals, the sign of most of the correlation coefficients in non-diagonal parts are the same except for the correlation coefficient between the innovation terms of the second and third univariate subprocesses. That may be due to the fact that its estimates in both models are close to 0. Moreover, the Frobenius norm of the difference between estimated correlation matrices of the innovation vectors of the two models is only 0.105; this verifies the closeness of the dependence structure of the innovation vectors of the two models. The sufficient conditions for closure under margins mean that all subprocesses of a given partition of a VAR(2) process are AR(2) with non-diagonal coefficient matrices of the VAR(2) process. By employing margin-closed and unrestricted models, the margin-closed model with Markov order 2 is preferred in view of model parsimony. The fitted parameters of the latent margin-closed and unrestricted VAR(2) models are presented in the forms of the coefficient matrices and the covariance matrices of the innovation vectors in Table VII. It is seen that most of the estimated parameters of the margin-closed model are near to the corresponding estimates of the unrestricted model, especially the entries in the diagonals of all fitted matrices. As for the covariance matrices of the innovation vectors, except for the closeness of the diagonals, the sign of most of the correlation coefficients in non-diagonal parts are the same except for the correlation coefficient between the innovation terms of the second and third univariate subprocesses. That may be due to the fact that its estimates in both models are close to 0. Moreover, the Frobenius norm of the difference between estimated correlation matrices of the innovation vectors of the two models is only 0.105; this verifies the closeness of the dependence structure of the innovation vectors of the two models. The sufficient conditions for closure under margins mean that all subprocesses of a given partition of a VAR(2) process are AR(2) with non-diagonal coefficient matrices of the VAR(2) process. By employing

6. DISCUSSION

The sufficient conditions for closure under margins mean that all subprocesses of a given partition of a VAR(2) process are AR(2) with non-diagonal coefficient matrices of the VAR(2) process. By employing
non-Gaussian univariate margins for each univariate component and multi-variate Gaussian copulas of stationary joint distributions of margin-closed VAR models, the margin-closed time series models have great flexibility in fitting the high-dimensional time series.

The numerical examples of the margin-closed model show its capacity to give the non-diagonal coefficient matrices without changing the correlation structure of the innovation vectors. The margin-closed model is also applied to a trivariate macro-economic data set, and the results indicate better performance of the margin-closed model compared with the unrestricted model.

Code will be made available for checking conditions for positive definiteness in cases similar to Example 3.3 to Example 3.5. This will also help with the estimation steps in Section 4.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from Federal Reserve Economic Data (FRED) database. Restrictions apply to the availability of these data, which were used under license for this study. Data are available from https://research.stlouisfed.org/econ/mccracken/fred-databases/ with the permission of Federal Reserve Economic Data (FRED) database.

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APPENDIX A: PROOFS

The following lemma is needed for proving Theorem 3.2.

**Lemma A.1.** Let $A_1, A_2, B,$ and $W$ be four multi-variate Gaussian distributed random vectors. If $B$ is independent of $(A_1^T, A_2^T)^T$ given $W$, then $B$ is independent of $A_2$ given $A_1$ and $W$.

**Proof of Lemma A.1.** Let $\Sigma_{A_1|W}, \Sigma_{A_2|W}, \Sigma_{A_1A_2|W}$ be covariance matrices between $A_1$ and $A_2$, between $A_1$ and $B$, and between $A_2$ and $B$ respectively, conditional on $W$. Also let $\Sigma_{A_1|W}$ be covariance matrix of $A_1$ given $W$. Then the
block diagonal form of the covariance matrix of \((A^T, A^T, B^T)^T\) given \(W\) implies that \(\Sigma_{A_1,B|W} = 0\) and \(\Sigma_{A_2,B|W} = 0\). From the conditional covariance of multi-variate Gaussian random vectors,
\[
\Sigma_{A_2,B|A_1,W} = \Sigma_{A_2,B|W} = \Sigma_{A_2,A_1|W}\Sigma^{-1}_{A_1|W}\Sigma_{A_1,B|W}
\]
\[
= 0 - \Sigma_{A_2,A_1|W}\Sigma^{-1}_{A_1|W} \times 0 = 0.
\]

which indicates the independence between \(A_2\) and \(B\) given \(A_1\) and \(W\).

The proof of Theorem 3.2 is given next.

**Proof of Theorem 3.2.** The main idea is to use the induction to show that the required condition in equation (5) is satisfied for all \(l \geq k + 1\) once it is satisfied for \(l = k + 1\). Let

\[
V_{1|t,l} = Z_{S_1,t}, \quad V_{2|t,l} = (Z_{S_2,t-1}, \ldots, Z_{S_2,t-l+1})^T, \\
V_{3|t,l} = Z_{S_3,t}, \quad V_{4|t,l} = (Z_{S_4,t-1}, \ldots, Z_{S_4,t-l+1})^T.
\]

Let \(\Gamma_{i,j,l}\) denote the covariance matrix between \(V_{i|t,l}\) and \(V_{j|t,l}\) given \(V_{4|t,l}\) for \(1 \leq i, j \leq 3\). The covariance matrix of \((V_{1|t,l}, V_{2|t,l}, V_{3|t,l})^T\) given \(V_{4|t,l}\) can be written as

\[
\begin{pmatrix}
\Gamma_{1,1,l} & \Gamma_{1,2,l} & \Gamma_{1,3,l} \\
\Gamma_{1,2,l} & \Gamma_{2,2,l} & \Gamma_{2,3,l} \\
\Gamma_{1,3,l} & \Gamma_{2,3,l} & \Gamma_{3,3,l}
\end{pmatrix}
\]

(A1)

Suppose equation (5) is satisfied for a fixed \(l \geq k + 1\), so that \(\Gamma_{1,2,l} = 0\) or \(\Gamma_{2,3,l} = 0\). To verify the induction hypothesis, it is necessary to show that \(\Gamma_{1,2,l+1} = 0\) or \(\Gamma_{2,3,l+1} = 0\). Since \(\{Z_t\}_{t \geq 0}\) is Markov of order \(k\), \([V_{1|t,l}V_{3|t,l}]V_{2|t,l}, V_{4|t,l}\). By Lemma 3.1, \(\Gamma_{1,2,l} = 0\) or \(\Gamma_{2,3,l} = 0\) implies \(\Gamma_{1,3,l} = 0\). We discuss Case (i) of \(\Gamma_{1,2,l} = 0\) and Case (ii) of \(\Gamma_{2,3,l} = 0\) separately.

**Case (i).** In this case \(\Gamma_{1,2,l} = 0\) and \(\Gamma_{2,3,l+1} = (V_{4|t,l}^TV_{3|t,l}^T)^T\). According to the definition, it follows that

\[
\text{Cov}(V_{1|t,l}, V_{2|t,l}|V_{4|t,l+1}) = \text{Cov}(V_{1|t,l}, V_{2|t,l}|V_{4|t,l}, V_{3|t,l})
\]

\[
= \Sigma_{V_{1|t,l}V_{2|t,l}|V_{4|t,l}} - \Sigma_{V_{1|t,l}V_{3|t,l}|V_{4|t,l}}\Sigma^{-1}_{V_{3|t,l}V_{3|t,l}}\Sigma_{V_{3|t,l}V_{2|t,l}|V_{4|t,l}}
\]

\[
= \Gamma_{1,2,l} - \Gamma_{1,3,l}\Gamma^{-1}_{3,3,l}\Gamma_{2,3,l} = 0. \quad (A2)
\]

Since \(\{Z_t\}_{t \geq 0}\) has Markov order \(k\), then \(V_{1|t,l} = Z_{S_1,t}\) is independent of \(Z_{t-1} = (V_{3|t,l}^TV_{3|t,l}^T)^T\) given \((V_{2|t,l}, V_{4|t,l})\). By Lemma A.1,

\[
\text{Cov}(V_{1|t,l}, Z_{S_2,t-1}, V_{2|t,l}, V_{4|t,l}, V_{3|t,l}) = 0. \quad (A3)
\]

Combining equations (A2) and (A3) leads to

\[
\text{Cov}(V_{1|t,l}, Z_{S_3,t}, V_{4|t,l+1}) = \Sigma_{V_{1|t,l}Z_{S_3,t-1}|V_{4|t,l}}
\]

\[
= \Sigma_{V_{1|t,l}Z_{S_3,t-1}|V_{2|t,l}, V_{3|t,l}} - \Sigma_{V_{1|t,l}V_{2|t,l}|V_{4|t,l}}\Sigma^{-1}_{V_{2|t,l}V_{3|t,l}}\Sigma_{V_{3|t,l}Z_{S_3,t-1}|V_{4|t,l}}
\]

\[
= 0.
\]
Since \( V_{1|t+1} = V_{1|t} \) and \( \Gamma_{1m2|t} = 0 \),

\[
\begin{align*}
\Gamma_{1,2|t+1} & = \text{Cov}(V_{1|t+1}, V_{2|t+1} | V_{3|t+1}) \\
& = \text{Cov}(V_{1|t+1}, (V_{2|t+1}^T Z_{t}^T)^T | V_{3|t+1}) = 0.
\end{align*}
\]

**Case (ii).** In this case \( \Gamma_{2,3|t} = 0 \) and \( V_{4|t+1} = (V_{1|t+1}^T, V_{2|t+1}^T, V_{3|t+1}^T)^T \). The stationarity of \( \{Z_t\}_{t=0}^\infty \) implies that the matrix in equation (A1) is also the covariance matrix of \( (V_{1|t}^T, V_{2|t}^T, V_{3|t}^T)^T \) given \( V_{4|t-1} \). Therefore,

\[
\text{Cov}(V_{2|t-1}, V_{3|t-1}, V_{4|t}) = \text{Cov}(V_{2|t-1}, V_{3|t-1}, V_{4|t})
\]

\[
= \sum v_{2|t-1} v_{3|t-1} v_{4|t-1} - \sum v_{2|t-1} v_{3|t-1} v_{4|t-1} \Sigma^{-1} v_{2|t-1} v_{3|t-1} v_{4|t-1} \Sigma
\]

\[
= \Gamma_{2,3|t} - \Gamma_{2,3|t} \Sigma^{-1} \Gamma_{2,3|t} = 0.
\]

(A4)

Since \( \{Z_t\}_{t=0}^\infty \) has Markov order \( k \), \( V_{3|t-1,l} = Z_{S,t-1} \) is independent of \( Z_{t-1} = (V_{1|t}^T, Z_{t}^T)^T \) given \( V_{2|t-1}, V_{4|t-1} \). By Lemma A.1,

\[
\text{Cov}(Z_{S,t-1}, V_{3|t-1,l} | V_{2|t-1}, V_{4|t-1}, V_{1|t-1,l}) = 0.
\]

(A5)

Combining equations (A4) and (A5) leads to

\[
\text{Cov}(Z_{S,t-1}, V_{3|t-1,l} | V_{2|t-1}, V_{4|t-1}, V_{1|t-1,l}) = 0.
\]

It leads to

\[
\Gamma_{2,3|t+1} = \text{Cov}(V_{2|t+1}, V_{3|t+1} | V_{4|t+1})
\]

\[
= \text{Cov}((Z_{S,t}^T)^T, V_{3|t+1} | V_{4|t+1}) = 0.
\]

Combining cases (i) and (ii), \( \Gamma_{1,2|t+1} = 0 \) or \( \Gamma_{2,3|t+1} = 0 \) and the induction hypothesis is verified. Hence \( \Gamma_{1,2|l} = 0 \) or \( \Gamma_{2,3|t} = 0 \) for all \( l \geq k + 1 \), which means that \( \{Z_{S,t}\}_{t=0}^\infty \) follows a VAR(\( k \)) model.

**APPENDIX B: DERIVING LINEAR SYSTEMS**

Let \( A = Z_{S,t} \), \( B = (Z_{S,t-1}^T, \ldots, Z_{S,0}^T)^T \), \( V = (Z_{S,t-1}^T, \ldots, Z_{S,0}^T)^T \). According to the conditional variance of Gaussian random vectors, Condition 1 in equation (7) for \( \{Z_{S,t}\}_{t=0}^\infty \) can be expressed as:

\[
\Sigma_{A,B|V} = \Sigma_{A,B} - \Sigma_{A,V} \Sigma_{V}^{-1} \Sigma_{V,B} = 0.
\]
or

\[
\begin{pmatrix}
\Sigma_{12,1} & \Sigma_{12,2} & \cdots & \Sigma_{12,k}
\end{pmatrix} - \begin{pmatrix}
\Phi_{1,1} & \Phi_{1,2} & \cdots & \Phi_{1,k}
\end{pmatrix} \begin{pmatrix}
\Sigma_{12,0} & \Sigma_{12,1} & \cdots & \Sigma_{12,k-1} \\
\Sigma_{12,-1} & \Sigma_{12,0} & \cdots & \Sigma_{12,k-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{12,-k} & \Sigma_{12,-k+1} & \cdots & \Sigma_{12,0}
\end{pmatrix} = 0,
\]

where \( \Phi_{1,1}, \ldots, \Phi_{1,k} \) are defined as

\[
\begin{pmatrix}
\Phi_{1,1}^T \\
\Phi_{1,2}^T \\
\vdots \\
\Phi_{1,k}^T
\end{pmatrix} = \begin{pmatrix}
\Sigma_{11,0} & \Sigma_{11,1} & \cdots & \Sigma_{11,k-1} \\
\Sigma_{11,-1} & \Sigma_{11,0} & \cdots & \Sigma_{11,k-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{11,-k+1} & \Sigma_{11,-k} & \cdots & \Sigma_{11,0}
\end{pmatrix}^{-1}
\begin{pmatrix}
\Phi_{1,1} \\
\Phi_{1,2} \\
\vdots \\
\Phi_{1,k}
\end{pmatrix}.
\]

Following the same idea, now with \( A = Z_{s_i,-k-1} \) and \( B \) and \( V \) as before, Condition 2 is equivalent to:

\[
\begin{pmatrix}
\Sigma_{12,-k} & \cdots & \Sigma_{12,-1}
\end{pmatrix} - \begin{pmatrix}
\Psi_{1,1} & \cdots & \Psi_{1,k}
\end{pmatrix} \begin{pmatrix}
\Sigma_{12,0} & \Sigma_{12,1} & \cdots & \Sigma_{12,k-1} \\
\Sigma_{12,-1} & \Sigma_{12,0} & \cdots & \Sigma_{12,k-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{12,-k} & \Sigma_{12,-k+1} & \cdots & \Sigma_{12,0}
\end{pmatrix} = 0,
\]

where \( \Psi_{1,1}, \ldots, \Psi_{1,k} \) are defined as

\[
\begin{pmatrix}
\Psi_{1,1}^T \\
\Psi_{1,2}^T \\
\vdots \\
\Psi_{1,k}^T
\end{pmatrix} = \begin{pmatrix}
\Sigma_{11,0} & \Sigma_{11,1} & \cdots & \Sigma_{11,k-1} \\
\Sigma_{11,-1} & \Sigma_{11,0} & \cdots & \Sigma_{11,k-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{11,-k+1} & \Sigma_{11,-k} & \cdots & \Sigma_{11,0}
\end{pmatrix}^{-1}
\begin{pmatrix}
\Psi_{1,1} \\
\Psi_{1,2} \\
\vdots \\
\Psi_{1,k}
\end{pmatrix}.
\]

Let \( D_1 = (\Sigma_{12,-k}^T, \ldots, \Sigma_{12,1}^T, \Sigma_{12,0}^T, \Sigma_{12,1}^T, \ldots, \Sigma_{12,k}^T)^T \). Then the two conditions in equation (7) can be rewritten as two linear systems. Condition 1 for \( \{Z_{s_i,t}\}_{t>0} \) is:

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & \Phi_{1,k} & \cdots & \Phi_{1,2} & \Phi_{1,1} & -I_{d_1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \Phi_{1,k} & \cdots & \Phi_{1,2} & \Phi_{1,1} & -I_{d_1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \Phi_{1,k} & \cdots & \Phi_{1,2} & \Phi_{1,1} & -I_{d_1} & 0 & \cdots & 0
\end{pmatrix} D_1 = G_1 D_1 = 0.
\]

Condition 2 for \( \{Z_{s_i,t}\}_{t>0} \) is:

\[
\begin{pmatrix}
-I_{d_1} & \Psi_{1,k} & \cdots & \Psi_{1,2} & \Psi_{1,1} & 0 & 0 & \cdots & 0 \\
0 & -I_{d_1} & \Psi_{1,k} & \cdots & \Psi_{1,2} & \Psi_{1,1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -I_{d_1} & \Psi_{1,k} & \cdots & \Psi_{1,2} & \Psi_{1,1} & 0
\end{pmatrix} D_1 = H_1 D_1 = 0.
\]
It gives equation (10). The reader can verify these for \( k = 1 \) and \( k = 2 \), and then generalize.

Similarly, let

\[
D_2 = (\Sigma_{21,-k}^T, \ldots, \Sigma_{21,-1}^T, \Sigma_{21,0}^T, \Sigma_{21,1}^T, \ldots, \Sigma_{21,k}^T)^T = (\Sigma_{12,k}, \ldots, \Sigma_{12,2}, \Sigma_{12,1}, \Sigma_{12,0}, \Sigma_{12,-1}, \ldots, \Sigma_{12,-k})^T.
\]

Then, Conditions 1 and 2 for \( \{Z_{t,j}\}_{t=0} \) can be rewritten as the following two linear systems. Condition 1 for \( \{Z_{t,j}\}_{t=0} \) is:

\[
\begin{pmatrix}
0 & \Phi_{2,k} & \ldots & \Phi_{2,2} & \Phi_{2,1} & -I_d & 0 & \cdots & 0 \\
0 & 0 & \Phi_{2,k} & \ldots & \Phi_{2,2} & \Phi_{2,1} & -I_d & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \Phi_{2,k} & \ldots & \Phi_{2,2} & \Phi_{2,1} & -I_d & \\
\end{pmatrix} D_2 = G_2 D_2 = 0. \tag{B1}
\]

Condition 2 for \( \{Z_{t,j}\}_{t=0} \) is:

\[
\begin{pmatrix}
-I_d & \Psi_{2,k} & \ldots & \Psi_{2,2} & \Psi_{2,1} & 0 & 0 & \cdots & 0 \\
0 & -I_d & \Psi_{2,k} & \ldots & \Psi_{2,2} & \Psi_{2,1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -I_d & \Psi_{2,k} & \ldots & \Psi_{2,2} & \Psi_{2,1} & 0 \\
\end{pmatrix} D_2 = H_2 D_2 = 0. \tag{B2}
\]

where

\[
\begin{pmatrix}
\Phi_{2,k}^T \\
\Phi_{2,2}^T \\
\vdots \\
\Phi_{2,1}^T
\end{pmatrix}^T = \begin{pmatrix}
\Sigma_{22,2}^T \\
\Sigma_{22,1}^T \\
\vdots \\
\Sigma_{22,1}^T
\end{pmatrix}^T \begin{pmatrix}
\Sigma_{22,0} & \Sigma_{22,1} & \cdots & \Sigma_{22,k-1} \\
\Sigma_{22,1} & \Sigma_{22,0} & \cdots & \Sigma_{22,k-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{22,0} & \Sigma_{22,1} & \cdots & \Sigma_{22,k-1}
\end{pmatrix}^{-1}
\]

and

\[
\begin{pmatrix}
\Psi_{2,k}^T \\
\Psi_{2,2}^T \\
\vdots \\
\Psi_{2,1}^T
\end{pmatrix}^T = \begin{pmatrix}
\Sigma_{22,-k}^T \\
\Sigma_{22,-1}^T \\
\vdots \\
\Sigma_{22,-1}^T
\end{pmatrix}^T \begin{pmatrix}
\Sigma_{22,0} & \Sigma_{22,1} & \cdots & \Sigma_{22,k-1} \\
\Sigma_{22,1} & \Sigma_{22,0} & \cdots & \Sigma_{22,k-2} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{22,0} & \Sigma_{22,1} & \cdots & \Sigma_{22,k-1}
\end{pmatrix}^{-1}.
\]

Let \( J_{2k+1} \) is the \((2k+1)\)-dimensional exchange matrix whose elements in the anti-diagonal (or opposite diagonal) are 1 and all other elements are zero, and set \( L_2 = J_{2k+1} \otimes I_{d} \). It follows that

\[
L_2 D_2 = L_2^{-1} D_2 = (\Sigma_{12,-k}, \ldots, \Sigma_{12,-1}, \Sigma_{12,1}, \Sigma_{12,0}, \Sigma_{12,1}, \ldots, \Sigma_{12,k})^T.
\]

Then equation (B1) and equation (B2) can be written as

**Condition 1:** \( (G_2 L_2)(L_2 D_2) = 0 \);

**Condition 2:** \( (H_2 L_2)(L_2 D_2) = 0 \).
APPENDIX C: SOLVING EQUATIONS IN VEC NOTATION

C.1. Two subprocesses

The vec(·) is useful to solve equation (10) to equation (13) for \(d_1, d_2 \geq 2\) and \(k \geq 2\), especially when the components of \(G_1, H_1, G_2, H_2\) are not scalars. In this Appendix, some details are shown.

Let \(G_{1,i}\) denote the submatrix of the \(i\)th block column of \(G_1\) and \(G_{1,-i}\) denote the submatrix obtained by removing the \(i\)th block column of \(G_1\). Similarly, define \(D_{1,-i}, \ldots, D_{1-k}\) as the submatrix obtained by removing the \(i\)th block row of \(D_1\). Because the blocks in \(D_1\) and \(L_2D_2\) are corresponding transposes, the combination of the two systems can be solved (numerically) by vectorizing the elements of \(\Sigma_{12,j}\) for \(j = -k, \ldots, k\).

**Case 1.** When both \(\{Z_{S,t}\}_{t>0}\) and \(\{Z_{S,t}\}_{t>0}\) adopt Condition 1, let \(k_1 = k + 1\) be the chosen column to fix for \(\Sigma_{12,0}\). It follows that

\[
G_{1,-k_1}D_{1,-k_1} = -G_{1,k_1} \Sigma_{12,0} \text{ and } (G_2L_2)_{-k_1}(L_2D_2)_{-k_1} = - (G_2L_2)_{k_1} \Sigma_{12,0} = -(G_2D_2)_{k_1} \Sigma_{12,0}^T,
\]

which means \(\Sigma_{12,0}\) is selected as the fixed parameter when we solve the equations.

In the above, \(\Sigma_{12-k}, \ldots, \Sigma_{12-1}, \Sigma_{12,1}, \ldots, \Sigma_{12,k}\) are stacked in \(D_{1,-k_1}\), and \(\Sigma_{12-1}, \Sigma_{12,-1}, \Sigma_{12,1}, \ldots, \Sigma_{12,k}\) are stacked in \((L_2D_2)_{-k_1}\).

By applying the vec(·) and using the formula that vec\((AB) = (A \otimes I_p)\)vec\((B^T)\) where \(A\) is \(m \times p\) and \(B\) is \(p \times n\), the above equations lead to

\[
(G_{1,-k_1} \otimes I_d) \text{vec}(D_{1,-k_1}^T) = -(G_{1,k_1} \otimes I_d) \text{vec}(\Sigma_{12,0}^T)
\]

\[
((G_2L_2)_{-k_1} \otimes I_d) \text{vec}((L_2D_2)_{-k_1}^T) = -((G_2L_2)_{k_1} \otimes I_d) \text{vec}(\Sigma_{12,0}),
\]

since

\[
\text{vec}(\Sigma_{12,i}) = K_{d_id_i} \text{vec}(\Sigma_{12,i}),
\]

for \(-k \leq i \leq k\) where the above defines for commutation matrix \(K_{d_id_i}\) as the permutation matrix to convert from vectorization by rows vs. vectorization by columns. It can be checked that

\[
\text{vec}(D_{1,-k_1}^T) = (I_{2k} \otimes K_{d_id_i}) \text{vec}((L_2D_2)_{-k_1}^T).
\]

Plugging equations (C3) and (C2) into equation (C1) leads to

\[
\begin{pmatrix}
(G_{1,-k_1} \otimes I_d) (I_{2k} \otimes K_{d_id_i}) \\
(G_2L_2)_{-k_1} \otimes I_d
\end{pmatrix}
\text{vec}([\Sigma_{12,-k}, \ldots, \Sigma_{12,-1}, \Sigma_{12,1}, \ldots, \Sigma_{12,k}])

= - \begin{pmatrix}
(G_{1,k_1} \otimes I_d) K_{d_id_i} \\
G_{2,k_1} \otimes I_d
\end{pmatrix}
\text{vec}(\Sigma_{12,0}).
\]

Therefore, given \(G_1, G_2, D_1, D_2\) and with the fact that the first coefficient matrix of vec\((D_{2,-k_1})\) is non-singular in nearly all situations, \(\Sigma_{12-k}, \ldots, \Sigma_{12-1}, \Sigma_{12,1}, \ldots, \Sigma_{12,k}\) can be uniquely solved when \(\Sigma_{12,0}\) is fixed.

**Case 2.** Submatrices of \(H\) are defined similar to submatrices of \(G\). The linear systems can be written as:

\[
\begin{pmatrix}
(H_{1,-k_1} \otimes I_d) (I_{2k} \otimes K_{d_id_i}) \\
(H_2L_2)_{-k_1} \otimes I_d
\end{pmatrix}
\text{vec}([\Sigma_{12,-k}, \ldots, \Sigma_{12,-1}, \Sigma_{12,1}, \ldots, \Sigma_{12,k}])
\]
\[ \begin{align*}
&= - \left( \begin{array}{c}
(H_{1,1} \otimes I_{d_1})K_{d_1d_2} \\
H_{2,1} \otimes I_{d_1}
\end{array} \right) \text{vec}(\Sigma_{12,0}),
\end{align*} \tag{C5} \]
and is similar to equation (C4).

C.2. Multiple subprocesses

The vec(\(\Sigma_{12,0}\)) form of the linear systems are given in this subsection for the case of 2 or more subprocesses. Similar to equations (C4) and (C5), given \(\Sigma_{ii,0}, \Sigma_{ij,k}, \ldots, \Sigma_{ii,k}, \Sigma_{jj,k}\), the equations of four different cases of \((c_i, c_j)\) can be concluded as following:

1. \((c_i, c_j) = (1, 1)\): set \(\Sigma_{ij,0}\) as fixed parameter and \(\Sigma_{ij,-k}, \ldots, \Sigma_{ij,-1}, \Sigma_{ij,1}, \ldots, \Sigma_{ij,k}\) can be obtained by solving the linear equation

\[ \text{vec}[(\Sigma_{ij,-k}, \ldots, \Sigma_{ij,-1}, \Sigma_{ij,1}, \ldots, \Sigma_{ij,k})] = - \left( \begin{array}{c}
(G_{i,1} \otimes I_{d_j})(I_{d_2} \otimes K_{d_1d_j}) \\
(G_{j,1} \otimes I_{d_i})(H_{j,1} \otimes I_{d_i})
\end{array} \right) \text{vec}(\Sigma_{ij,0}); \]

2. \((c_i, c_j) = (1, 2)\): set \(\Sigma_{ij,-k}\) as fixed parameter and \(\Sigma_{ij,-k} = \ldots = \Sigma_{ij} = 0\);

3. \((c_i, c_j) = (2, 1)\): set \(\Sigma_{ij,k}\) as fixed parameter and \(\Sigma_{ij,k} = \ldots = \Sigma_{ij,k-1} = 0\);

4. \((c_i, c_j) = (2, 2)\): set \(\Sigma_{ij,0}\) as fixed parameter and \(\Sigma_{ij,-k}, \ldots, \Sigma_{ij,-1}, \Sigma_{ij,1}, \ldots, \Sigma_{ij,k}\) can be obtained by solving the linear equation

\[ \text{vec}[(\Sigma_{ij,-k}, \ldots, \Sigma_{ij,-1}, \Sigma_{ij,1}, \ldots, \Sigma_{ij,k})] = - \left( \begin{array}{c}
(H_{i,1} \otimes I_{d_j})(I_{d_2} \otimes K_{d_1d_j}) \\
(H_{j,1} \otimes I_{d_i})
\end{array} \right) \text{vec}(\Sigma_{ij,0}). \]