Outliers in the spectrum of large deformed unitarily invariant models

S. T. Belinschi, H. Bercovici, M. Capitaine, M. Février

May 2, 2014

Abstract

We investigate the asymptotic behavior of the eigenvalues of the sum $A_N + U_N^* B_N U_N$, where $A_N$ and $B_N$ are deterministic $N \times N$ Hermitian matrices having respective limiting compactly supported distributions $\mu$ and $\nu$, and $U_N$ is a random $N \times N$ unitary matrix distributed according to Haar measure. We assume that $A_N$ has a fixed number of fixed eigenvalues (spikes) outside the support of $\mu$ whereas the distances between the other eigenvalues of $A_N$ and the support of $\mu$, and between the eigenvalues of $B_N$ and the support of $\nu$ uniformly go to zero as $N$ goes to infinity.

We establish that only a particular subset of the spikes will generate some eigenvalues of $A_N + U_N^* B_N U_N$ outside the support of the limiting spectral measure, called outliers. This phenomenon is fully described in terms of free probability involving the subordination function related to the free additive convolution of $\mu$ and $\nu$. Only finite rank perturbations had been considered up to now.

1 Introduction

The set of possible spectra for the sum of two deterministic matrices $A_N$ and $B_N$ depends in complicated ways on the spectra of $A_N$ and $B_N$ (see [21]). Nevertheless, if one adds some randomness to the eigenspaces and assumes them to be in generic position with respect to each other, when $N$ becomes large, free probability provides a good understanding of the global behavior of the spectrum of the sum of matrices. Indeed, if $X_N = A_N + U_N^* B_N U_N$, where $U_N$ is a Haar unitary random matrix (i.e. from the set of unitary matrices equipped with the normalized Haar measure as probability measure), if the spectral measures of $A_N$ and $B_N$ converge weakly towards respective compactly supported distributions $\mu$ and $\nu$, then building on the groundbreaking result of Voiculescu [34], Speicher proved in [32] the almost sure weak convergence of the spectral measure of $X_N$ to the free convolution $\mu \boxplus \nu$, which is known to be a compactly supported probability measure on $\mathbb{R}$. We refer the reader to [36] for an introduction to free probability theory.

In [11], the authors investigated the case where $A_N$ has a finite rank $r$ independent of $N$. More precisely, they considered a deterministic Hermitian
perturbation matrix $A_N$ having $r$ non-zero eigenvalues $\gamma_1 \geq \cdots \geq \gamma_s > 0 > \gamma_{s+1} \geq \cdots \geq \gamma_r$. Note that in that case, $\mu \equiv \delta_0$ and the global limiting behavior of the spectrum of $X_N$ is not affected by such a matrix $A_N$. Thus, the spectral measure of $X_N = A_N + U_N^* B_N U_N$ still converges to the limiting spectral measure of $B_N$. Nevertheless, in [11], the authors uncovered a phase transition phenomenon whereby the limiting value of the extreme eigenvalues of $X_N$ differs from that of $B_N$ if and only if the eigenvalues of $A_N$ are above a certain critical threshold:

**Theorem 1.1** (Theorem 2.1 in [11]). Denote by $\lambda_i(X_N) \geq \cdots \geq \lambda_N(X_N)$ the ordered eigenvalues of $X_N$. Let $a$ and $b$ be respectively the infimum and supremum of the support of $\nu$. Assume that the smallest and largest eigenvalue of $B_N$ converge almost surely to $a$ and $b$. Then, we have for each $1 \leq i \leq s$, almost surely,

$$
\lambda_i(X_N) \to_{N \to +\infty} \begin{cases} 
G^{-1}_\nu(1/\gamma_i) & \text{if } \gamma_i > 1/\lim_{z \downarrow b} G_\nu(z), \\
b & \text{otherwise},
\end{cases}
$$

while for each fixed $i > s$, almost surely, $\lambda_i(X_N) \to_{N \to +\infty} b$. Similarly, for the smallest eigenvalues, we have for each $0 \leq j < r-s$, almost surely,

$$
\lambda_{N-j}(X_N) \to_{N \to +\infty} \begin{cases} 
G^{-1}_\nu(1/\gamma_{r-j}) & \text{if } \gamma_{r-j} < 1/\lim_{z \uparrow a} G_\nu(z), \\
a & \text{otherwise},
\end{cases}
$$

while for each fixed $j \geq r-s$, almost surely, $\lambda_{N-j}(X_N) \to_{N \to +\infty} a$. Here,

$$
G_\nu : \mathbb{C} \setminus \text{supp}(\nu) \to \mathbb{C}, \quad G_\nu(z) = \int_{\mathbb{R}} \frac{d\nu(t)}{z-t},
$$

is the Cauchy-Stieltjes transform of $\nu$, $G^{-1}_\nu$ is its functional inverse.

Note that [11] lies in the lineage of recent works studying the influence of some finite rank additive or multiplicative perturbations on the extremal eigenvalues of classical random matrix models, the seminal paper being [5] where Baik, Ben Arous and Péché pointed out the so-called BBP phase transition (cf. [25, 5, 6] for sample covariance matrices, [22, 28, 19, 15, 29] for deformed Wigner models and [27] for information-plus-noise models). Such problems were first extended to non-finite rank perturbations in [30] and [3] for sample covariance matrices and in [16] for deformed Wigner models. In this last paper, the authors pointed out that the subordination function relative to the free additive convolution of a semicircular distribution with the limiting spectral distribution of the perturbation plays an important role in the fact that some eigenvalues of the deformed Wigner model separate from the bulk. Note that in [14], the author explained how the results of [30] and [3] in the sample covariance matrix setting can also be described in terms of free probability involving the subordination function related to the free multiplicative convolution of a Marchenko-Pastur distribution with the limiting spectral distribution of the multiplicative perturbation.
In this paper, we investigate the asymptotic spectrum of the model $X_N = A_N + U_N^* B_N U_N$ without the extra requirement that one of the limiting spectral measures $\mu$ and $\nu$ of the respective deterministic Hermitian matrices $A_N$ and $B_N$ is a point mass. We assume that $A_N$ has a fixed number $r$ of fixed eigenvalues (spikes) outside the support of $\mu$ whereas the distances between the other eigenvalues of $A_N$ and the support of $\mu$, and between the eigenvalues of $B_N$ and the support of $\nu$ uniformly go to zero as $N$ goes to infinity. We are interested in the following questions: are some of the eigenvalues of $X_N$ almost surely separating from the bulk, that is being located outside the support of the limiting spectral distribution $\mu \boxplus \nu$? What is the characterization of the spiked eigenvalues of $A_N$ that generate such outliers in the spectrum of $X_N$? Is there an interpretation of these phenomena in terms of the subordination functions related to the free additive convolution of $\mu$ and $\nu$?

In the particular case where $r = 0$, it is proved in [18] that, almost surely, for large enough $N \in \mathbb{N}$, all eigenvalues of $X_N$ are included in a neighborhood of the support of the limiting spectral distribution $\mu \boxplus \nu$. In the following, we will therefore assume that $r \geq 1$. Thus, this paper may be seen as an extension of [11] since it extends the framework of [11] to non-finite rank perturbations but also as an extension of [16] since it extends the free probabilistic interpretation of outliers phenomena in terms of subordination functions described in [16] for Wigner deformed models to deformed unitarily invariant models. Here, the characterization in terms of subordination functions of the spiked eigenvalues of $A_N$ that generate outliers in the spectrum of $X_N$ turns out to be more complex that the one presented in [16] for deformed Wigner models, but it is a completely natural extension as we explain in Remark 3.2. It is worth noticing that we uncover here a new phenomenon: a single spiked eigenvalue of $A_N$ may generate asymptotically a finite or countably infinite set of outliers of $X_N$. This comes from the fact that the restriction to the real line of some subordination functions may be many-to-one, unlike the subordination function related to free convolution with a semicircular distribution studied in [16].

The approach of the proof of our main result (i.e Theorem 4.1) is in the spirit of [11] and comes down to prove the almost sure convergence of a certain $r \times r$ matrix, involving the resolvent of the deformation of $U_N^* B_N U_N$ by some matrix $A'_N$ without spikes. Its almost sure convergence is proved by establishing an approximate matricial subordination equation and using a concentration argument. Then, the problem is reduced to solving an equation involving the spikes and the subordination function related to the free convolution of $\mu$ and $\nu$.

The paper is organized as follows. In Section 2, we introduce the additive deformed models we consider in this paper; we also introduce some basic notations that will be used throughout the paper. Section 3 is devoted to definitions and results concerning free convolution and subordination functions, some of them being necessary to state our main result Theorem 4.1 in Section 4. The proof of Theorem 4.1 is presented in Sections 4 and 5. More precisely, we explain in Section 4 how the proof comes down to prove the almost sure convergence of a $r \times r$ matrix and Section 5 deals with the proof of this convergence.
2 Notations and presentation of the model

Throughout this paper, we will use the following notations.

- $\mathbb{C}^+$ will denote the complex upper half-plane $\{z \in \mathbb{C}, \Im z > 0\}$. Similarly, $\mathbb{C}^-$ will stand for $\{z \in \mathbb{C}, \Im z < 0\}$.

- We will denote by $M_m(\mathbb{C})$ the set of $m \times m$ matrices with complex entries and $GL_m(\mathbb{C})$ the subset of invertible ones. $\|\|$ will denote the operator norm.

- For any matrix $M$, we will denote its kernel by $\text{Ker}(M)$.

- $E_{ij}$ stands for the matrix such that $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

- For any $N \times N$ Hermitian matrix $M$, we will denote by
  $$\lambda_1(M) \geq \ldots \geq \lambda_N(M)$$
  its ordered eigenvalues.

- For a probability measure $\tau$ on $\mathbb{R}$, we denote by $\text{supp}(\tau)$ its topological support.

- $C, C_1, C_2, C'$ denote nonnegative constants which may vary from line to line.

In this note, we consider the following model $X_N = A_N + U_N^* B_N U_N$, where:

- $A_N$ is a deterministic $N \times N$ Hermitian matrix whose spectral measure
  $\mu_{A_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(A_N)}$ weakly converges to some compactly supported probability measure $\mu$ on $\mathbb{R}$. We assume that there exists a fixed integer $r \geq 0$ (independent from $N$) such that $A_N$ has $N-r$ eigenvalues $\alpha_j^{(N)}$ satisfying
  $$\max_{1 \leq j \leq N-r} \text{dist}(\alpha_j^{(N)}, \text{supp}(\mu)) \xrightarrow{N \to \infty} 0.$$ 
  We also assume that there are $J$ fixed real numbers $\theta_1 > \ldots > \theta_J$ independent of $N$ which are outside the support of $\mu$ and such that each $\theta_j$ is an eigenvalue of $A_N$ with a fixed multiplicity $k_j$ (with $\sum_{j=1}^{J} k_j = r$). The $\theta_j$’s will be called the spikes or the spiked eigenvalues of $A_N$.

- $B_N$ is a deterministic $N \times N$ Hermitian matrix whose spectral measure
  $\mu_{B_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(B_N)}$ weakly converges to some compactly supported probability measure $\nu$ on $\mathbb{R}$. We assume that the eigenvalues $\beta_j^{(N)}$ of $B_N$ satisfy
  $$\max_{1 \leq j \leq N} \text{dist}(\beta_j^{(N)}, \text{supp}(\nu)) \xrightarrow{N \to \infty} 0.$$ 

- $U_N$ is a random $N \times N$ unitary matrix distributed according to Haar measure.
Free convolution appears as a natural analogue of the classical convolution in the context of free probability theory. Denote by $\mathcal{M}$ the set of Borel probability measures supported on the real line. For $\mu$ and $\nu$ in $\mathcal{M}$ one defines the free additive convolution $\mu \boxplus \nu$ of $\mu$ and $\nu$ as the distribution of $X + Y$ where $X$ and $Y$ are free self-adjoint random variables with distribution $\mu$ and $\nu$. We refer the reader to [36] for an introduction to free probability theory and to [33] and [10] for free convolution. In this section, we recall the analytic approach developed in [33] to calculate the free convolution of measures, we present the important subordination property and some related fundamental results we will refer to later.

### 3.1 Additive Free convolution

For any positive finite Borel measure $\tau$ on $\mathbb{R}$, the Cauchy-Stieltjes transform of $\tau$

$$G_{\tau} : \mathbb{C} \setminus \text{supp}(\tau) \to \mathbb{C}, \quad G_{\tau}(z) = \int_{\mathbb{R}} \frac{d\tau(t)}{z - t}$$

is analytic, maps the upper half-plane $\mathbb{C}^+$ into the lower half-plane $\mathbb{C}^-$ and satisfies the conditions $G_{\tau}(\overline{z}) = \overline{G_{\tau}(z)}$, $\lim_{y \to +\infty} iyG_{\tau}(iy) = \tau(\mathbb{R})$. These conditions in fact characterize functions which are Cauchy-Stieltjes transforms of positive finite measures. For probability measures $\tau$ with compact support, $G_{\tau}$ is analytic on the neighbourhood of infinity $\{z \in \mathbb{C} : |z| > \max\{|x| : x \in \text{supp}(\tau)\}\}$. The Cauchy-Stieltjes transform allows us to recover the measure $\tau$ as the weak*-limit

$$d\tau(x) = \lim_{y \to 0} \frac{-1}{\pi} \Im G_{\tau}(x + iy).$$

For the absolutely continuous part of $\tau$, the situation is better: the relation

$$\frac{d\tau}{dx} = \lim_{y \to 0} \frac{1}{\pi} \Im G_{\tau}(x + iy)$$

holds Lebesgue-a.e. as equality of functions. For some purposes, it is convenient to work with the reciprocal Cauchy-Stieltjes transform, which is the analytic self-map of the upper half-plane defined by:

$$\forall z \in \mathbb{C}^+, \quad F_{\tau}(z) = \frac{1}{G_{\tau}(z)}.$$

The Cauchy-Stieltjes transform of a compactly supported probability measure $\tau$ is invertible in the neighborhood of infinity, with functional inverse, denoted by $G_{\tau}^{-1}$, defined in a neighborhood of 0. Define then the R-transform of $\tau$ by:

$$R_{\tau}(z) = G_{\tau}^{-1}(z) - \frac{1}{z}.$$
Given two compactly supported probability measures \( \mu \) and \( \nu \), there exists a unique probability measure \( \lambda \) such that

\[
R_\lambda = R_\mu + R_\nu
\]

on a domain where these functions are defined. The probability measure \( \lambda \) is called the free additive convolution of \( \mu \) and \( \nu \) and denoted by \( \mu \boxplus \nu \). The support of the probability measure \( \mu \boxplus \nu \) being a compact set, we will denote it by

\[
K := \text{supp}(\mu \boxplus \nu).
\]

Moreover, given \( \varepsilon > 0 \), we will use the following notations:

\[
K_\varepsilon^\mathbb{R} := \{ x \in \mathbb{R} | d(x, \text{supp}(\mu \boxplus \nu)) \leq \varepsilon \},
\]

\[
K_\varepsilon^\mathbb{C} := \{ z \in \mathbb{C} | d(z, \text{supp}(\mu \boxplus \nu)) \leq \varepsilon \}.
\]

### 3.2 Free subordination phenomenon

We recall the subordination phenomenon for the Cauchy-Stieltjes transform of the free additive convolution of measures. Given Borel probability measures \( \mu \) and \( \nu \) on \( \mathbb{R} \), the Cauchy-Stieltjes transform of the free additive convolution \( \mu \boxplus \nu \) is subordinated to the Cauchy-Stieltjes transform of any of \( \mu \) or \( \nu \): there are two analytic self-maps of the upper half-plane \( \omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+ \) such that:

\[
\forall z \in \mathbb{C}^+, G_{\mu \boxplus \nu}(z) = G_\mu(\omega_1(z)) = G_\nu(\omega_2(z)).
\]

The subordination maps \( \omega_1, \omega_2 \) are also related by:

\[
\forall z \in \mathbb{C}^+, \omega_1(z) + \omega_2(z) = z + F_{\mu \boxplus \nu}(z). \quad (3.1)
\]

Moreover, for \( j \in \{1, 2\} \),

\[
\lim_{y \to +\infty} \frac{\omega_j(iy)}{iy} = 1.
\]

It then follows, using Nevanlinna representation of analytic self-maps of the upper half-plane, that:

\[
\forall z \in \mathbb{C}^+, \Re \omega_j(z) \geq \Re z;
\]

equality can occur only when one of the measures \( \mu, \nu \) is a point mass (more specifically, \( \exists z \in \mathbb{C}^+, \Re \omega_1(z) = \Re z \iff \forall z \in \mathbb{C}^+, \Re \omega_1(z) = \Re z \iff \nu \) is a point mass, and \( \exists z \in \mathbb{C}^+, \Re \omega_2(z) = \Re z \iff \forall z \in \mathbb{C}^+, \Re \omega_2(z) = \Re z \iff \mu \) is a point mass). These results, first obtained in full generality in [13], have been given a new interpretation in terms of Denjoy-Wolff points of analytic functions in [7]. We will state the upper half-plane version of the Denjoy-Wolff theorem below:

If \( f : \mathbb{C}^+ \to \mathbb{C}^+ \) is analytic and not a Möbius transformation, then only one of the following three cases can occur:
1. there exists a unique point \( \omega \in \mathbb{C}^+ \) so that \( f(\omega) = \omega \) and \( |f'(\omega)| < 1 \). Then the iterations \( f^{\circ n} \) converge uniformly on compacts to the constant function taking the value \( \omega \);

2. there exists a unique \( \omega \in \mathbb{R} \) so that \( \lim_{r \downarrow 0} f(ir + \omega) = \omega \) and

   \[
   0 < \lim_{r \downarrow 0} \frac{f(ir + \omega) - \omega}{ir} \leq 1.
   \]

Then the iterations \( f^{\circ n} \) converge uniformly on compacts to the constant function taking the value \( \omega \);

3. \( \lim_{r \uparrow +\infty} f(ir) = \infty \) and

   \[
   1 \leq \lim_{r \uparrow +\infty} \frac{f(ir)}{ir} < \infty.
   \]

Then the iterations \( f^{\circ n} \) converge uniformly on compacts to infinity.

The point \( \omega \) in situations 1. and 2. (and infinity in 3.) is called the Denjoy-Wolff point of \( f \). Note that the second limit in 2. above always exists in \((0; +\infty]\) as soon as \( \lim_{r \downarrow 0} f(ir + \omega) = \omega \), and is called the Julia-Carathéodory derivative of \( f \) at \( \omega \). Denjoy and Wolff proved that any analytic function \( f : \mathbb{C}^+ \rightarrow \mathbb{C}^+ \) as above has a Denjoy-Wolff point (see [23] for details). Using this result, a new proof of Biane’s subordination result for free convolution was given in [7], by identifying \( \omega_1(z) \) as the Denjoy-Wolff point of the function

\[
f_z(\omega) := F_\nu(F_\mu(\omega) - \omega + z) - (F_\mu(\omega) - \omega + z) + z.
\]

We will need the following lemma collecting results on extensions of the subordination maps:

**Lemma 3.1.** For \( j \in \{1, 2\} \), the function \( \omega_j \), defined on \( \mathbb{C}^+ \), has an extension (still denoted by \( \omega_j \)) to \( \mathbb{C} \) so that:

- (a) \( \omega_j \) is continuous on \( \mathbb{C}^+ \cup \mathbb{R} \);
- (b) \( \omega_1(\{\infty\} \cup \mathbb{R} \setminus \text{supp}(\mu \boxplus \nu)) \subseteq \{\infty\} \cup \mathbb{R} \setminus \text{supp}(\mu) \);
- (b') \( \omega_2(\{\infty\} \cup \mathbb{R} \setminus \text{supp}(\mu \boxplus \nu)) \subseteq \{\infty\} \cup \mathbb{R} \setminus \text{supp}(\nu) \);
- (c) \( \forall z \in \mathbb{C} \setminus \mathbb{R}, \omega_j(z) = \omega_j(\overline{z}) \);
- (d) \( \omega_j \) is meromorphic on \( \mathbb{C} \setminus \text{supp}(\mu \boxplus \nu) \).

**Proof.** As noted in [9, Theorem 3.3], \( \omega_j|_{\mathbb{C}^+} \) has a continuous extension to the real line. This proves (a). Letting \( z \) tend to \( x \in \mathbb{R} \setminus \text{supp}(\mu \boxplus \nu) \) in (3.1), one notices that \( \omega_j(x) \) necessarily belongs to the boundary of \( \mathbb{C}^+ \), because so does \( F_\nu(F_\mu(z) - \omega + z) \). To be more specific, if \( F_\nu(F_\mu(z) - \omega + z) = \infty \), then \( x \) is an isolated simple pole of \( F_\nu(F_\mu(z) - \omega + z) \), and the same (3.1) tells us that either \( \omega_1(x) = \infty \) and
ω_2(x) = -m_1(μ) + x, or ω_2(x) = ∞ and ω_1(x) = -m_1(ν) + x, and thus x is
an isolated simple pole of exactly one of the two ω_j. (Here m_1(τ) denotes
the first moment of τ.) This proves (b), (b') and (d). We then extend ω_j to ℂ^− by
requiring:

∀z ∈ ℂ^−, ω_j(z) = \overline{ω_j(z)};

as in [13], so that (c) holds. □

It should be noted that the proof from [7] shows that ω_1(z) is the Denjoy-Wolff
point of f_z only for z ∈ ℂ^+. Indeed, it is still an open problem whether this
is true for all z ∈ ℂ^+ ∪ ℜ. However, if x ∈ ℜ is so that ω_1 is analytic in x
and real on some interval around x, the Julia-Carathéodory Theorem requires
ω_1(x) ∈ (0, +∞). As noted in the above lemma, this implies that F_μ is real
meromorphic around ω_1(x) and F_μ real analytic around ω_2(x) = F_μ(ω_1(x)) −
ω_1(x) + x. Thus, f_μ will be real analytic around ω_1(x). Taking limits shows
that f_μ(ω_1(x)) = ω_1(x) and ω_1'(x) = \partial_x f_μ(ω_1(x)) [1 − \partial_ω f_μ(ω_1(x))]^{-1}. Since
\partial_ω f_μ(ω_1(x)) = F_μ'(ω_2(x)) > 0, in order for ω_1'(x) > 0 as required by the Julia-
Carathéodory Theorem, we must have \partial_ω f_μ(ω_1(x)) < 1, and hence ω → f_μ(ω)
has ω_1(x) as a fixed point in which the derivative is less than one. This is
the case 2. of the upper half-plane version of the Denjoy-Wolff theorem given
above. Thus, in this case ω_1(x) is still necessarily the unique Denjoy-Wolff point
of ω → f_μ(ω).

Let us give a slightly different formulation for the results of [7] concerning the
subordination functions, more appropriate to the needs of our paper. Assume
neither μ nor ν is a point mass and denote

h_μ(z) = F_μ(z) − z, h_ν(z) = F_ν(z) − z.

We re-write the ideas of [7], where ω_1 and ω_2 are identified as the Denjoy-Wolff
points of self-maps of ℂ^+ indexed by z (see above), but with a formulation
chosen to make the statement somehow independent of complex dynamics. It
follows from [7] that ω_1 and ω_2 are identified uniquely by the following system
of equations

\{ ω_1(z) − h_ν(ω_2(z)) = z \\
ω_2(z) − h_μ(ω_1(z)) = z .

We can look upon this system as an implicit equation for a two-variable map:
we define f(w_1, w_2, z) = (w_1 − h_ν(w_2) − z, w_2 − h_μ(w_1) − z). A straightfor-ward
application of the implicit function theorem indicates that h_ν'(w_1)h_μ'(w_2) = 1
is the only obstacle to an analytic solution z → (ω_1(z), ω_2(z)) to the equation
f(ω_1(z), ω_2(z), z) = (0, 0). The analysis in [7] shows that

∀z ∈ ℂ^+, |h_ν'(ω_1(z))h_μ'(ω_2(z))| < 1.

This is a consequence of the fact that non-trivial - i.e. not Möbius - maps have
derivatives less than one in their Denjoy-Wolff points. Since the func-
tions h and ω map the parts of ℜ which lay in their domains of analyticity
in ℵ and their derivatives on ℜ are necessarily positive, we conclude that
\[ 0 < h'_\mu(\omega_1(x))h'_\nu(\omega_2(x)) < 1 \] for any \( x \in \mathbb{R} \) in the domain of analyticity of \( \omega_1 \) and \( \omega_2 \). In particular, \( 0 < h'_\mu(\omega_1(x))h'_\nu(h_\mu(\omega_1(x)) + x) < 1 \), and the obstacle to an analytic solution \( \omega_1 \) is described by the equality \( h'_\mu(\omega_1(x))h'_\nu(h_\mu(\omega_1(x)) + x) = 1 \).

**Theorem 3.1.** Assume that neither \( \mu \) nor \( \nu \) is a point mass. Given \( \theta \in \mathbb{R} \setminus \text{supp}(\mu) \), then \( \rho \in \mathbb{R} \) is a solution of the equation

\[ \omega_1(\rho) = \theta \]

belonging to \( \mathbb{R} \setminus \text{supp}(\mu \oplus \nu) \) if and only if \( h_\mu(\theta) + \rho \) belongs to the domain of analyticity of \( h_\nu \), and \( \rho \) is a solution of:

\[ h_\nu(h_\mu(\theta) + \rho) - \theta + \rho = 0, \quad 0 < h'_\mu(\theta)h'_\nu(h_\mu(\theta) + \rho) < 1. \]  \hspace{1cm} (3.2)

**Proof:** Let \( \rho \in \mathbb{R} \setminus \text{supp}(\mu \oplus \nu) \) be a solution of the equation \( \omega_1(\rho) = \theta \). Taking the limit \( z \rightarrow \rho \) in the second equality of the system above, we obtain

\[ h_\nu(h_\mu(\theta) + \rho) = \omega_2(\rho) \]

which belongs to \( \{ \infty \} \cup \mathbb{R} \setminus \text{supp}(\nu) \) by Lemma 3.1 (b'), and therefore to the domain of analyticity of \( h_\nu \). Taking the limit \( z \rightarrow \rho \) in (3.1), one gets

\[ h_\nu(h_\mu(\theta) + \rho) = h_\nu(\omega_2(\rho)) = \omega_1(\rho) - \rho = \theta - \rho. \]

Since \( \rho \in \mathbb{R} \setminus \text{supp}(\mu \oplus \nu) \), \( \omega_1 \) is analytic at \( \rho \) (see Lemma 3.1 (d) - by hypothesis \( \omega_1(\rho) = \theta \neq \infty \)), and there must be no obstacle to the existence of an analytic solution around \( \rho \) as in the discussion above. Hence

\[ 0 < h'_\mu(\theta)h'_\nu(h_\mu(\theta) + \rho) < 1. \]

Conversely, let \( \rho \in \mathbb{R} \) be such that \( h_\mu(\theta) + \rho \) belongs to the domain of analyticity of \( h_\nu \), and \( \rho \) is a solution of (3.2). This in particular implies that \( \theta \) is the Denjoy-Wolff point of \( f_\rho(\omega) = h_\nu(h_\mu(\omega) + \rho) + \rho, \omega \in \mathbb{C}^+, \) as \( f_\rho(\theta) = \theta \) and \( f'_\rho(\theta) \in ]0, 1[. \) An application of the implicit function theorem shows that the dependence of \( \theta \) on the complex variable \( \rho \) is analytic around the given real point, and has a positive derivative. Analytic continuation, the uniqueness of the Denjoy-Wolff point and the considerations above guarantee that \( \omega_1(\rho) = \theta \).

\[ \square \]

**Remark 3.1.** We should note that the set of points \( \rho \) that satisfy the equality from (3.2), being the set of zeroes of an analytic map, is necessarily discrete, by the principle of isolated zeroes. Therefore the set of solutions of \( \omega_1(\rho) = \theta \) in \( \mathbb{R} \setminus \text{supp}(\mu \oplus \nu) \) is discrete as well. This set of solutions may be empty, finite or countably infinite, as illustrated in the next remarks.

**Remark 3.2.** If \( \nu \) is \( \boxplus \)-infinitely divisible, it is known (see [12]) that \( \omega_1 \) is a conformal bijection between \( \mathbb{C}^+ \) and a simply connected domain \( \Omega \subseteq \mathbb{C}^+ \), whose inverse is the restriction to \( \Omega \) of the map \( H \) defined on \( \mathbb{C}^+ \) by:

\[ H(z) := z + R_\nu(G_\mu(z)), \]
where $R_\nu$ is the $R$-transform of the measure $\nu$. Under this extra hypothesis, the content of Theorem 3.1 reads simply: $\rho \in \mathbb{R}$ is a solution of the equation
\[
\omega_1(\rho) = \theta
\]
belonging to $\mathbb{R} \setminus \text{supp}(\mu \boxplus \nu)$ if and only if
\[
\rho = H(\theta), \quad H'(\theta) > 0. \tag{3.3}
\]
Indeed, if $\rho \in \mathbb{R}$ satisfy (3.3), notice first that
\[
h_\mu(\theta) + \rho = F_\mu(\theta) + R_\nu(G_\mu(\theta)) = G_\nu^{-1}(G_\mu(\theta)),
\]
which is in the domain of analyticity of $h_\nu$ (recall that $\theta \notin \text{supp}(\mu)$ by hypothesis and thus $G_\nu(h_\mu(\theta) + \rho) = G_\mu(\theta)$). Then
\[
h_\nu(h_\mu(\theta) + \rho) + \rho = F_\nu(G_\nu^{-1}(G_\mu(\theta))) - h_\mu(\theta) = F_\mu(\theta) - h_\mu(\theta) = \theta.
\]
Finally, because of the relation
\[
H(\theta) = \theta - h_\nu(h_\mu(\theta) + \rho),
\]
the condition $H'(\theta) > 0$ implies the inequality in (3.2). Conversely, assume that $\rho \in \mathbb{R}$ is such that $h_\mu(\theta) + \rho$ belongs to the domain of analyticity of $h_\nu$ and satisfies (3.2). The condition (3.3) being equivalent to
\[
\theta = \rho - R_\nu(G_\mu \boxplus \nu(\rho)), \quad H'(\theta) > 0, \tag{3.4}
\]
it is possible to check that $\rho - R_\nu(G_\mu \boxplus \nu(\rho))$ is the unique Denjoy-Wolff point of $f_\rho$, which yields the equality above. The inequality follows immediately from the inequality in (3.2). Note that there is at most one solution belonging to $\mathbb{R} \setminus \text{supp}(\mu \boxplus \nu)$ which satisfies the equation
\[
\omega_1(\rho) = \theta
\]
when $\nu$ is $\boxplus$-infinitely divisible.

**Example 3.1.** We should note that there are many examples in which the restriction to the real line of subordination functions are many-to-one on their domain of analyticity. A trivial example comes from free Brownian motions, in which the spikes are associated to the matrix approximating the semicircular distribution (note the difference from the problem studied in [16]!). Consider a Bernoulli distribution $b = (\delta_{-1} + \delta_1)/2$ and a standard semicircular $\gamma_t$ of variance $t$. We know that $G_{\delta_1 \oplus \delta_1}(z) = G_{\gamma_t}(\omega_1(z)) = G_b(\omega_2(z))$, where $\omega_2$ satisfies
\[
\omega_2(z) = z - tG_b(\omega_2(z)), \quad z \in \mathbb{C}^+ \cup \mathbb{R}.
\]
We claim that for $t > 0$ small enough, there are real values taken twice by $\omega_1$. Indeed,
\[
h_b(h_{\gamma_t}(\theta) + \rho) = \frac{-1}{h_{\gamma_t}(\theta) + \rho} = \frac{-2}{-\theta + \sqrt{\theta^2 - 4t} + 2\rho},
\]
and then
\[ h_t(h_{\gamma t}(\theta) + \rho) = \theta - \rho \iff 2\rho^2 + (\sqrt{\theta^2 - 4t} - 3\theta)\rho + \theta^2 - \theta\sqrt{\theta^2 - 4t} - 2 = 0. \]

The two solutions are
\[ \rho_{1,2} = \frac{\theta - \sqrt{\theta^2 - 4t} \pm \sqrt{2\theta\sqrt{\theta^2 - 4t} + 2\theta^2 - 4t + 16 - 4t}}{4}. \]

For these solutions to be real, we must have \(|\theta| \geq 2\sqrt{t}\) and
\[ \theta^2 + 8 - 2t \geq -\theta\sqrt{\theta^2 - 4t}. \]

For our example we shall consider a positive spike, and so the requirements translate into \(\theta \geq 2\sqrt{t}\) and \(\theta^2 + 8 - 2t \geq -\theta\sqrt{\theta^2 - 4t}\). For \(t \leq 1\), this reduces to \(\theta \geq 2\sqrt{t}\). On the other hand, the condition \(h'_{\gamma t}(\theta) = \frac{\theta}{h_{\gamma t}(\theta) + \rho} = \theta - \rho\). Thus this inequality is simply \(h'_{\gamma t}(\theta)(\rho - \theta)^2 < 1\). We note that \(h'_{\gamma t}(\theta) = \frac{\theta}{2\sqrt{\theta^2 - 4t}} - \frac{1}{2} \in [0,1]\) for all \(\theta > 3\sqrt{t}\). Moreover, \(\lim_{\theta \to +\infty} \theta^2 h'_{\gamma t}(\theta) = 2t\), and, again when \(\theta \to +\infty\), the two values of \(\rho\) tend to infinity at a speed of the order of \(\theta\) and at zero with a speed of order \(\frac{\theta}{2t}\), respectively. Now we conclude easily: for \(\theta\) large enough and \(t < \frac{1}{2}\) (strict inequality!), both conditions in (3.2) are satisfied for the existence of \(\rho\), and thus we obtain two spikes explicitly computed.

**Remark 3.3.** In the case, not covered by Theorem 3.1, where \(\mu\) or \(\nu\) is a point mass, \(\omega_1\) is equal either to \(F_{\nu}\) or to a real translation. In the second case, the equation
\[ \omega_1(\rho) = \theta \]

is trivial. In the first case, \(F_{\nu}\) establishing continuously increasing bijections between each connected component of the complement of the support of \(\nu\) and some subsets of \(\mathbb{R}\), the set of solutions may be determined by real analysis: for example, if \(\nu = \sum 2^{-n}\delta_{1/n}\), then there are countably many open intervals included in \([0,1]\) which are mapped bijectively onto \(\mathbb{R}\) by \(F_{\nu}\).

In this paper we shall be concerned almost exclusively with \(\omega_1\), so for simplicity we shall adopt the

**Notation convention:** \(\omega_1 = \omega\).

**4 Main results and sketch of proof**

**4.1 Main result and examples**

**Definition 4.1.** For each \(j \in \{1, \ldots, J\}\), define \(O_j\) the set of solutions in \(\mathbb{R} \setminus \text{supp}(\mu \boxdot \nu)\) of the equation
\[ \omega(\rho) = \theta_j, \quad (4.1) \]
and
\[ O = \bigcup_{1 \leq j \leq J} O_j. \]

Recall that the sets \( O_j \) defined above may be empty, finite, or countably infinite.

**Theorem 4.1.** Denote by \( \text{sp}(X) \) the spectrum of the operator \( X \). The following results hold almost surely:

- for each \( \rho \in O_j \), for all small enough \( \varepsilon > 0 \), for all large enough \( N \),
  \[ \text{card}\{\text{sp}(X_N) \cap [\rho - \varepsilon; \rho + \varepsilon]\} = k_j; \]

- for almost all \( \eta > 0 \), for all small enough \( \varepsilon > 0 \), for large enough \( N \),
  \[ \text{sp}(X_N) \cap \mathbb{C} \setminus K^R_\eta \subset \bigcup_{\rho \in O_j \cap \mathbb{C} \setminus K^R_\eta} [\rho - \varepsilon; \rho + \varepsilon]. \]

**Remark 4.1.** The proof of this theorem, starting at the end of this section, and completed in the next one, works without any change when the spikes may depend on \( N \), more precisely if we assume that the spectrum of \( A_N \) consists of \( N - r \) eigenvalues \( \alpha_j^{(N)} \) satisfying

\[ \max_{1 \leq j \leq N - r} \text{dist}(\alpha_j^{(N)}, \text{supp}(\mu)) \to 0, \]

and \( r \) spikes \( \theta_1^N \geq \ldots \geq \theta_r^N \) such that:

\[ \forall i \in \{k_1 + \ldots + k_{j-1} + 1, \ldots, k_1 + \ldots + k_j\}, \theta_i^N \to \theta_j. \]

The only reason we chose not to write the proofs under these slightly more general hypotheses is to avoid confusion in the notations.

**Remark 4.2.** Actually, the conclusion of the preceding theorem holds for a random matrix \( X_N = A_N + \tilde{B}_N \), where \( A_N \) and \( \tilde{B}_N \) are independent random Hermitian matrices satisfying almost surely the assumptions given in the first section and in the preceding remark, under the extra assumption that the distribution of the random matrix \( \tilde{B}_N \) is invariant by conjugation by any unitary matrix in \( \mathbb{U}_N \). Note however that the quantities \( J, k_1, \ldots, k_J, \theta_1, \ldots, \theta_J \) are supposed deterministic. In particular, one recovers results from [11], [15] and [16].

It is clear that a random Hermitian matrix \( \tilde{B}_N \) whose distribution is invariant by conjugation by any unitary matrix in \( \mathbb{U}_N \) has the same distribution as \( U_N^* B_N U_N \), where \( B_N \) is a random real diagonal matrix, and \( U_N \) is a random unitary matrix distributed according to Haar measure and independent of \( B_N \). The proof would then proceed, on the almost sure event on which \( A_N \) and \( \tilde{B}_N \) satisfy all the assumptions required, by conditioning with respect to the sigma-field generated by the sequences of random matrices \( A_N, B_N \), and then applying Theorem 4.1.
Example 4.1. When the rank of $A_N$ remains finite, one recovers, using the preceding remark, results on the extremal eigenvalues from \cite{11} recalled in Theorem 1.1 (see also the Gaussian case in \cite{15}). Indeed, in this setting, $\mu \equiv \delta_0$, and the set $O$ is therefore the set of solutions $\rho \in \mathbb{R} \setminus \text{supp}(\nu)$ of equations

$$F_\nu(\rho) = \theta_j,$$

as explained in Remark 3.3. Theorem 4.1 implies the conclusion of Theorem 1.1, the condition

$$\gamma_i > 1 \lim_{z \downarrow b} G_\nu(z)$$

(resp. $\gamma_{r-j} < 1 \lim_{z \uparrow a} G_\nu(z)$)

being equivalent to the existence of elements of $O$ greater (resp. lower) than the maximum (resp. minimum) of the support of $\nu$, and the limiting points $G^{-1}_\nu(1/\gamma_i)$, (resp. $G^{-1}_\nu(1/\gamma_{r-j})$) being solutions of $F_\nu(\rho) = \gamma_i$, (resp. $F_\nu(\rho) = \gamma_{r-j}$). Notice that Theorem 4.1 deals in addition with the outliers of $X_N$ located in bounded components of the complement of the support of $\nu$.

Example 4.2. When $B_N$ is drawn from the $GUE(N, \sigma^2)$, which satisfies the assumptions stated in the preceding remark (invariance by unitary conjugation, almost sure weak convergence of the spectral measure towards the semicircular distribution \cite{2}, almost sure convergence to 0 of the distance between the eigenvalues and the semicircular support \cite{1}), one recovers a particular case of the results on the outliers from \cite{16}. Indeed, in this setting, $\nu$ is the semicircular distribution, which is $\boxplus$-infinitely divisible, and the set $O$ is therefore described by Remark 3.2. The conclusion of Theorem 4.1 is then exactly the one of the main result of \cite{16}.

Analogously, when $B_N$ is a $N \times N$ Wishart matrix, which also satisfies the assumptions stated in the preceding remark (invariance by unitary conjugation, almost sure weak convergence of the spectral measure towards the Marchenko-Pastur distribution, almost sure convergence to 0 of the distance between the eigenvalues and the Marchenko-Pastur support \cite{37}), one recovers the result on the outliers established by the two last authors of this paper and presented in \cite{20}. In this setting, $\nu$ is the Marchenko-Pastur distribution, which is $\boxplus$-infinitely divisible, and the set $O$ is therefore described by Remark 3.3. The conclusion of Theorem 4.1 is then exactly the one of the main result of Chapter 7 of \cite{20}.

4.2 Reduction of the problem to the almost sure convergence of a $r \times r$ matrix

In this section, we explain how we reduce the problem of locating outliers of $X_N$ to a problem of convergence of a certain $r \times r$ matrix in the spirit of \cite{11}. Due to the invariance of the Haar measure under multiplication by any unitary
matrix, we may assume without loss of generality that both $A_N$ and $B_N$ are real diagonal matrices:

$$A_N = \text{Diag}(\theta_1, \ldots, \theta_1, \ldots, \theta_J, \ldots, \theta_J, \alpha_1^{(N)}, \ldots, \alpha_{N-r}^{(N)}),$$

$$B_N = \text{Diag}(\beta_1^{(N)}, \ldots, \beta_N^{(N)}).$$

Moreover, from the beginning of our argument, we will make use of the following additive decomposition of $A_N$:

$$A_N = A'_N + A''_N,$$

$$A'_N = \text{Diag}(\alpha, \ldots, \alpha, \alpha_1^{(N)}, \ldots, \alpha_{N-r}^{(N)}),$$

$$A''_N = \text{Diag}(\theta_1 - \alpha, \ldots, \theta_1 - \alpha, \ldots, \theta_J - \alpha, \ldots, \theta_J - \alpha, 0, \ldots, 0),$$

where the choice of $\alpha \in \text{supp}(\mu)$ is made so that $\lim_{y \downarrow 0} G_\mu(iy + \alpha) \in \mathbb{R}+i[-\infty, 0)$.

Note that $A''_N = tP\Theta P$, where $P$ is the $r \times N$ matrix defined by

$$P = (I_r|0_{r \times (N-r)}),$$

$\Theta$ is the $r \times r$ matrix

$$\Theta = \text{Diag}(\theta_1 - \alpha, \ldots, \theta_1 - \alpha, \ldots, \theta_J - \alpha, \ldots, \theta_J - \alpha),$$

and $^tX$ denotes the transpose of the matrix $X$. Under our assumptions, the spectral measure of $A'_N$ (resp. $B_N$) weakly converges to $\mu$ (resp. $\nu$), and all eigenvalues of $A'_N$ (resp. $B_N$) belong to any given neighborhood of $\text{supp}(\mu)$ (resp. $\text{supp}(\nu)$) for large enough $N \in \mathbb{N}$. Hence, applying Corollary 3.1 of [18], one gets that, for any $k \in \mathbb{N}^*$, almost surely,

$$\exists N_k \in \mathbb{N}, \forall N \geq N_k, \text{sp}(A'_N + U_N^*B_NU_N) \subseteq K^R_k.$$

We obtain thus the almost sure existence of a sequence $(\eta_N)_{N \geq N_1}$ of positive numbers converging to 0 so that:

$$\forall N \geq N_1, \text{sp}(A'_N + U_N^*B_NU_N) \subseteq K^R_{\eta_N}.$$

(Choose for instance $\eta_N = \frac{1}{N}$, for $N_k \leq N < N_{k+1}$.) In the following, we will restrict ourselves to the almost sure event on which the sequence $(\eta_N)_{N \geq N_1}$ is well-defined. Then, for $N \geq N_1$, for any $\lambda \in \mathbb{C} \setminus K^R_{\eta_N}$, the matrix $\lambda I_N - (A'_N + U_N^*B_NU_N)$ is invertible and

$$\det(\lambda I_N - X_N) = \det(\lambda I_N - (A'_N + U_N^*B_NU_N)) \det(I_N - R_N(\lambda)^tP\Theta P),$$

where

$$R_N(\lambda) = (\lambda I_N - (A'_N + U_N^*B_NU_N))^{-1}. \quad (4.2)$$
Using that, for rectangular matrices $X \in M_{N,r}(\mathbb{C}), Y \in M_{r,N}(\mathbb{C})$, one has 
\[
\det(I_N - XY) = \det(I_r - YX),
\]
one obtains:
\[
\det(\lambda I_N - X_N)) = \det(\lambda I_N - (A'_N + U_N B_N U_N)) \det(I_r - PR_N(\lambda)^r P\Theta).
\]
Hence, for $N \geq N_1$, the eigenvalues of $X_N$ outside $K_{N_1}^\mathbb{R}$ are precisely the zeros of $\det(I_r - PR_N(\lambda)^r P\Theta)$ in that open set. We will denote by
\[
M_N := I_r - PR_N^t P\Theta
\]
the analytic function defined on $\mathbb{C} \setminus K_{N_1}^\mathbb{R}$, with values in the set of $r \times r$ complex matrices.

Then, the following fundamental lemma allows to reduce the problem to the convergence of the sequence of analytic functions $(M_N)_{N \geq N_1}$.

\textbf{Lemma 4.1.} Let $M : \mathbb{C} \setminus K \to M_r(\mathbb{C})$ be a normal-operator-valued analytic function (i.e. $M(z) \in M_r(\mathbb{C})$ is normal for each $z \in \mathbb{C} \setminus K$) so that

(a) $\forall z \in \mathbb{C} \setminus K, M(z)^* = M(\overline{z})$.

(b) $\exists z > 0 \implies \Re M(z)$ invertible.

Assume that there exists a sequence of positive numbers $\{\eta_N\}_{N \in \mathbb{N}}$ decreasing to zero and a sequence of analytic maps $M_N : \mathbb{C} \setminus K_{N_1}^\mathbb{R} \to M_r(\mathbb{C})$ so that

1. there exists $C > 0$ such that for all $z \in \mathbb{C}$ such that $|z| > C$, for any $N$,
\[M_N(z)\] is invertible,

2. for any $z \in \mathbb{C} \setminus \mathbb{R}$ we have $M_N(z) \in GL_r(\mathbb{C})$.

3. for any $\eta > 0$, $M_N$ converges to $M$, uniformly on $\mathbb{C} \setminus K_{\eta}^\mathbb{R}$.

If, for a fixed $\eta > 0$ such that the boundary points of $K_{\eta}^\mathbb{R}$ are not zeroes of $\det(M)$, $\{\rho_1, \ldots, \rho_{p(\eta)}\}$ is the set of points $z \in \mathbb{C} \setminus K_{\eta}^\mathbb{R}$ such that $M(z)$ is not invertible, then

(i) $\{\rho_1, \ldots, \rho_{p(\eta)}\} \subset \mathbb{R}$;

(ii) $\dim(\ker(M(\rho_j)))$ equals the order of zero of $\det(M(\rho_j))$;

(iii) For any $0 < \varepsilon < \frac{1}{2} \min\{|\rho_i - \rho_j|, \ d(\rho_i, K_{\eta}^\mathbb{R}) : 1 \leq i \neq j \leq p(\eta)\}$, there exists an $N_0 \in \mathbb{N}$ so that for any $N \geq N_0$ the function $M_N$ is defined on $\mathbb{C} \setminus K_{\eta}^\mathbb{R}$ has exactly $\dim(\ker(M(\rho_j)))$ zeroes in $(\rho_j - \varepsilon, \rho_j + \varepsilon)$ for any $j \in \{1, \ldots, p(\eta)\}$ and exactly $\dim(\ker(M(\rho_1))) + \cdots + \dim(\ker(M(\rho_{p(\eta)})))$ zeroes in $\mathbb{C} \setminus K_{\eta}^\mathbb{R}$, counted with multiplicity, so that
\[
\{z \in \mathbb{C} \setminus K_{\eta}^\mathbb{R} : \det(M_N(z)) = 0\} \subset \bigcup_{j=1}^{p(\eta)} (\rho_j - \varepsilon, \rho_j + \varepsilon).
\]
**Proof:** To begin with, by [24], \(M(z)M(z') = M(z')M(z)\) for all \(z, z' \in \overline{\mathbb{C}} \setminus K\). Thus, there exists a unitary matrix \(U\) so that for all \(z \in \overline{\mathbb{C}} \setminus K\)

\[
U^* M(z) U = \text{diag}(h_1(z), \ldots, h_r(z)).
\]

Pick an \(x \in \mathbb{R} \setminus K\) so that \(\det M(x) = 0\). The condition \(M(z)^* = M(x)\) implies that \(M(x)^* = M(x)\) whenever \(x \in \mathbb{R}\), and thus \(h_i(x) \in \mathbb{R}\) for any \(i \in \{1, \ldots, r\}\).

Let \(I = \{i_1, \ldots, i_j\}\) be such that \(h_l(x) = 0\) if \(l \in I\) and \(h_l(x) \neq 0\) else. Denote by \(m_k\) the multiplicity of the zero of \(h_{i_k}(z)\) at \(x\). The zero of \(\det(M(x))\) is of order equal to \(m_1 + m_2 + \cdots + m_j\). We only need now to argue that \(m_k = 1\) for all \(1 \leq k \leq j\). According to (b), \(\Im h_j(z) \neq 0\) whenever \(\Im z \neq 0\). This, in particular, implies that \(\Im z \Im h_j(z)\) has constant sign on half-planes, and by the Julia-Carathéodory Theorem, \(h_j'(x) \neq 0\) for any \(x \in \mathbb{R} \setminus K\). This proves (i) and (ii).

If we denote by \(f(z) = \det(M(z))\) and \(f_N(z) = \det(M_N(z))\), then Hurwitz’s Theorem [31] Kapitel 8.5 guarantees that for \(N\) large enough, \(f_N\) will have exactly as many zeros - multiplicity included - as \(f\) has in \(\overline{\mathbb{C}} \setminus K_N^\ell\) - and since all zeros of \(f_N\) are known to be real, in \(\overline{\mathbb{C}} \setminus K_N^R\) - and these zeros will cluster towards \(\{\rho_1, \ldots, \rho_{p\eta}\}\) in the sense that for any given \(\varepsilon > 0\) there exists an \(N_\varepsilon \in \mathbb{N}\) so that

\[
\{z \in \overline{\mathbb{C}} \setminus K_N^R: \det(M_N(z)) = 0\} \subset \bigcup_{j=1}^{p\eta} B(\rho_j, \varepsilon)
\]

whenever \(N \geq N_\varepsilon\). Moreover, for \(\varepsilon > 0\) small enough, there are exactly \(\dim(\text{Ker}(M(\rho_j)))\) zeros of \(f_N\) in \(B(\rho_j, \varepsilon)\), multiplicity included. Since by 2., \(M_N\) is invertible in the two half-planes, we must have

\[
\{z \in \overline{\mathbb{C}} \setminus K_N^R: \det(M_N(z)) = 0\} \subset \bigcup_{j=1}^{p\eta} (\rho_j - \varepsilon, \rho_j + \varepsilon).
\]

\(\square\)

5 Convergence of \(M_N\) defined by (4.3)

5.1 Preliminary results on the resolvent \(R_N\)

We begin this section by recording some facts on the resolvent \(R_N\) defined by (4.2). Recall that, if \(X\) is a selfadjoint operator on a Hilbert space with spectrum \(\sigma(X)\), then we shall denote by \(R_X(z) = (z - X)^{-1}\) its resolvent. It is known that this resolvent is analytic on \(\mathbb{C} \setminus \sigma(X)\). If \(\varphi\) is a positive unital linear functional on the unital algebra generated by \(X\), then the distribution \(\mu_{X, \varphi}\) of \(X\) with respect to \(\varphi\) can be recovered as

\[
G_{\mu_{X, \varphi}}(z) = \varphi(R_X(z)), \quad z \notin \sigma(X).
\]
Since in most cases it will be clear from the context which functional \( \varphi \) is considered, we shall suppress \( \varphi \) from the notation \( \mu_{X,\varphi} \).

A more general notion of resolvent of \( X \), which we shall use only sparingly in this paper, can be defined, following Voiculescu, as below: for an arbitrary operator \( b \) on the same Hilbert space as \( X \), we can write its decomposition

\[
b = \frac{b + b^*}{2\Re b} + i \frac{b - b^*}{2\Im b},
\]

where \( \Re b, \Im b \) are selfadjoint. We shall write \( \Im b > 0 \) if \( \Im b \geq 0 \) as operator on Hilbert space, and \( (\Im b)^{-1} \) exists and is bounded. It has been noted by Voiculescu [35] that

\[
R_X(b) = (b - X)^{-1}, \quad \Im b > 0
\]

is an analytic map so that \( \Im [R_X(b)]^{-1} = \Im b \geq \Im b \). Moreover, as noted in [8, Remark 2.5], if \( E \) is a positive unit-preserving linear map which leaves the algebra of \( b \) invariant, then

\[
\Im [E[R_X(b)]]^{-1} \geq \Im b.
\]

For all \( z \in \mathbb{C} \setminus \mathbb{R} \), \( R_N(z) \) defined by (4.2) satisfies:

\[
\forall z \in \mathbb{C} \setminus \mathbb{R}, \|R_N(z)\| \leq \frac{1}{|\Im z|}.
\]

This implies that, for any \( z \in \mathbb{C} \setminus \mathbb{R} \), the matrix \( E[R_N(z)] \) is well-defined and also satisfies:

\[
\|E[R_N(z)]\| \leq \frac{1}{|\Im z|}.
\]

**Lemma 5.1.** When \( \Im z \neq 0 \), \( E[R_N(z)] \) is an invertible matrix and

\[
\forall z \in \mathbb{C} \setminus \mathbb{R}, \|E[R_N(z)]^{-1}\| < |z| + C_1 + \frac{4C_2}{|\Im z|},
\]

where \( C_1 \) is any constant greater than \( \sup_N (\|A'_N\| + \|B_N\|) \) and \( C_2 \) any constant greater than \( \sup_N (\text{tr}_N(B_N^2) - [\text{tr}_N(B_N)]^2) \).

**Proof.** As noted in equation (5.1) above applied to \( b = z - A'_N \), \( \Im (z - A'_N - U_N^*B_NU_N)^{-1} < 0 \). Since \( E \) is both positive and faithful, it follows that for any \( z \in \mathbb{C}^+ \), \( \Im E [(z - A'_N - U_N^*B_NU_N)^{-1}] < 0 \), and thus by the same remark of Voiculescu [35], \( E [(z - A'_N - U_N^*B_NU_N)^{-1}] \) is invertible. The second statement of the lemma is equivalent to a statement about the power series expansion of \( z \mapsto E[R_N(z)]^{-1} \) around infinity. The power series expansion

\[
E[R_N(z)] = \sum_{n=0}^{\infty} \frac{E[(A'_N + U_N^*B_NU_N)^n]}{z^{n+1}}, \quad |z| > \|A'_N\| + \|B_N\|,
\]
assures us that

\[
f(z) := (z - A_N' - E[U_N' B_N U_N]) - E[R_N(z)]^{-1}
\]

\[
= z - E[A_N' + U_N^* B_N U_N] - \left[ \sum_{n=0}^{\infty} \frac{E[(A_N' + U_N^* B_N U_N)^n]}{z^{n+1}} \right] - 1
\]

\[
= \left\{ \left( z - E[A_N' + U_N^* B_N U_N] \right) \left[ \sum_{n=0}^{\infty} \frac{E[(A_N' + U_N^* B_N U_N)^n]}{z^{n+1}} \right] - 1 \right\}
\times \left[ \sum_{n=0}^{\infty} \frac{E[(A_N' + U_N^* B_N U_N)^n]}{z^{n+1}} \right]^{-1}
\]

\[
= \frac{1}{z} \left[ E[(A_N' + U_N^* B_N U_N)^2] - E[A_N' + U_N^* B_N U_N]^2 + \frac{1}{z} O(1) \right]
\times \left[ 1 + \sum_{n=1}^{\infty} \frac{E[(A_N' + U_N^* B_N U_N)^n]}{z^{n}} \right]^{-1}
\]

\[
= \frac{1}{z} \left[ E[(A_N' + U_N^* B_N U_N)^2] - E[A_N' + U_N^* B_N U_N]^2 \right] + \frac{1}{z^2} O(1).
\]

This power series expansion holds uniformly in \( N \) as long as \( \|A_N'\| + \|B_N\| \) is bounded uniformly in \( N \). In particular, we obtain

\[
\lim_{z \to \infty} zf(z) = E[(A_N' + U_N^* B_N U_N)^2] - E[(A_N' + U_N^* B_N U_N)^2] = tr_N((B - tr_N(B))^2) \cdot 1,
\]

uniform limit in \( N \).

As noted in equation (5.2) above, \( \Im f(z) < 0 \), so for any positive linear functional \( \varphi \) on \( M_N(\mathbb{C}) \), the function \( z \mapsto \varphi(f(z)) \) maps \( \mathbb{C}^+ \) into the lower half-plane, and \( \lim_{z \to \infty} z\varphi(f(z)) = \varphi(1)(tr_N(B^2) - [tr_N(B)]^2) \). Thus, \( z \mapsto \varphi(f(z)) \) is the Cauchy-Stieltjes transform of a positive measure supported on \([-\|A_N\| - \|B_N\|, \|A_N\| + \|B_N\|]\) of total mass \( \varphi(1)(tr_N(B^2) - [tr_N(B)]^2) \). It follows that

\[
|\varphi(f(z))| < \frac{\varphi(1)}{3z}(tr_N(B^2) - [tr_N(B)]^2), \quad z \in \mathbb{C}^+.
\]

Now, since positive linear functionals on von Neumann algebras reach their norm on the unit, the Jordan decomposition of linear functionals allows us to write

\[
\|f(z)\| \leq \sup_{\|\varphi\|=1} |\varphi(f(z))| \leq 4 \sup_{\varphi \geq 0, \varphi(1)=1} |\varphi(f(z))| < \frac{4}{3z}(tr_N(B^2) - [tr_N(B)]^2).
\]

Since \( E[R_N(z)]^{-1} = z - E[A_N' + U_N^* B_N U_N] - f(z) \), for any \( z \in \mathbb{C} \setminus \mathbb{R} \), we conclude that

\[
\|E[R_N(z)]^{-1}\| \leq |z| + \|A_N'\| + \|B_N\| + \|f(z)\| < |z| + C_1 + \frac{4C_2}{|3z|^4},
\]

as stated in our lemma. \( \square \)
One now states concentration results that will allow to reduce the proof of Proposition 5.1 to the convergence of a sequence of deterministic matrices and to estimate the variance of each entry of the resolvant $R_N(z)$.

**Lemma 5.2.** (i) $\forall z \in \mathbb{C} \setminus \mathbb{R}, \text{PR}_N(z)^tP - P\mathbb{E}[R_N(z)]^tP \stackrel{a.s.}{\to} N \to +\infty 0.$

(ii) $\forall z \in \mathbb{C} \setminus \mathbb{R}, \forall (k, l) \in \{1, \ldots, N\}^2, \text{Var}(R_N(z))_{kl} \leq \frac{C}{|\Im z|^4}.$

**Proof.** Fix $z \in \mathbb{C} \setminus \mathbb{R}$. It is clear from the remark that writing $P(R_N(z) - \mathbb{E}[R_N(z)])^tP$ corresponds to taking the upper left $r \times r$ corner of $R_N(z) - \mathbb{E}[R_N(z)]$, that (i) is equivalent to:

$$\forall (k, l) \in \{1, \ldots, r\}^2, (R_N(z) - \mathbb{E}[R_N(z)])_{kl} \stackrel{a.s.}{\to} N \to +\infty 0. \quad (5.5)$$

Now, for any $(k, l) \in \{1, \ldots, N\}^2$, since the function $f : U_N \mapsto R_N(z)_{kl}$ is Lipschitz on the unitary group $U_N$ with Lipschitz bound $\frac{C}{|\Im z|^2}$, by Corollary 4.4.28 of the book [1], for any $0 < \alpha < \frac{1}{2},$

$$\mathbb{P}\left(\left| (R_N(z) - \mathbb{E}[R_N(z)])_{kl} \right| > \frac{\epsilon}{N^{\frac{1}{2} - \alpha}} \right) \leq 2 \exp\left(-\frac{a}{N^{\frac{1}{2} - \alpha}}\right).$$

Hence, one gets (5.5) by a standard application of Borel-Cantelli lemma, and (ii) by the classical formula holding for a positive random variable $X$:

$$\mathbb{E}(X) = \int_0^{+\infty} \mathbb{P}(X > t)dt. \quad \square$$

### 5.2 Convergence of $M_N$

We investigate the convergence of the sequence of analytic functions $(M_N)_{N \geq N_1}$ defined by (4.3). We need the following preliminary lemma.

**Lemma 5.3.** The function $\chi: z \mapsto \frac{1}{\omega(z) - \alpha}$ is analytic on $\mathbb{C} \setminus \text{supp}(\mu \boxplus \nu)$ and satisfies $\chi(\overline{z}) = \overline{\chi(z)}$ for any $z$ in $\mathbb{C} \setminus \text{supp}(\mu \boxplus \nu)$.

**Proof.** This lemma readily follows from Lemma 3.1. \square

**Proposition 5.1.** Almost surely, for any $\eta > 0$, the sequence $(M_N)_{N \geq N_0}$, where $N_0 \geq N_1$ is such that $\forall N \geq N_0, \eta_N < \eta$, converges to

$$M = \text{diag}(1 - (\theta_1 - \alpha)\chi, \ldots, 1 - (\theta_J - \alpha)\chi),$$

uniformly on compact subsets of $\mathbb{C} \setminus K_C^\phi$. 19
Since
\[ M_N = I_r - PR_N^tP\Theta, \]
it is equivalent to prove the convergence of \( PR_N^tP \) towards \( \chi I_r \).

We first study the convergence of the sequence of analytic functions \( (E[R_N])_{N \geq 1} \):

**Proposition 5.2.**
\[ \forall z \in \mathbb{C} \setminus \mathbb{R}, \quad P_E[R_N(z)]^tP \xrightarrow{N \rightarrow +\infty} \chi(z)I_r. \]  
(5.6)

Fix \( z \in \mathbb{C}^+ \) (which is sufficient by a reflection argument). We now break the proof in three lemmas. First, a strengthening of the result from [26] stating that the matrix \( E[R_N(z)] \) is diagonal:

**Lemma 5.4.** If \( b \in M_N(\mathbb{C}) \) is so that \( b - U_N^*B_NU_N \) is invertible for each value of the Haar unitary \( U_N \), then \( E[(b - U_N^*B_NU_N)^{-1}] \in \{b\}'' \), where the bicommutant \( \{b\}'' \) is taken in \( M_N(\mathbb{C}) \).

**Proof.** Pick an arbitrary unitary \( V \) in the commutant \( \{b\}' \) of \( b \). By the invariance of the Haar measure,
\[ V^*E[(b - U_N^*B_NU_N)^{-1}]V = E[(V^*bV - V^*U_N^*B_NU_NV)^{-1}] = E[(b - U_N^*B_NU_N)^{-1}], \]
so that \( E[(b - U_N^*B_NU_N)^{-1}] \in \{V\}' \). Thus, since a von Neumann algebra equals the span of its unitaries, \( E[(b - U_N^*B_NU_N)^{-1}] \in \{b\}'' \), as claimed. \( \square \)

Recall that the first \( r \) eigenvalues of \( A_N' \) are all equal to \( \alpha \). We apply the above lemma to \( b = z - A_N' \) to conclude that the first \( r \) eigenvalues of \( E[R_N(z)] \) are all equal, and thus \( P_E[R_N(z)]^tP = \chi_N(z)I_r \) for the Cauchy-Stieltjes transform \( \chi_N \) of some probability measure depending on \( A_N', B_N \).

Our next task is to establish an approximate matricial subordination equation, namely to prove that \( E[R_N(z)] \) is asymptotically equal to \( (\omega_N(z)I_N - A_N')^{-1} \), for a certain complex number \( \omega_N(z) \). Then, we prove the uniform convergence on the compact subsets of \( \mathbb{C}^+ \) of the sequence of analytic functions \( (\omega_N)_{N \geq 1} \) towards \( \omega \).

**Lemma 5.5.** For \( z \in \mathbb{C}^+ \), one has:
\[ \|E[R_N(z)] - (\omega_N(z)I_N - A_N')^{-1}\| \xrightarrow{N \rightarrow +\infty} 0, \]
where
\[ \omega_N(z) := \frac{1}{E[R_N(z)]_{11}} + \alpha = \frac{1}{\chi_N(z)} + \alpha. \]  
(5.7)
Proof. First notice, using (5.2), that $\omega_N$ defined by (5.7), satisfies:
\begin{equation}
\forall z \in \mathbb{C}^+, \Im \omega_N(z) \geq \Im z.
\end{equation}

Fix $z \in \mathbb{C}^+$ and define
$$
\Omega_N(z) := \mathbb{E}[R_N(z)]^{-1} + A_N',
$$
which belongs, according to Lemma 5.4, to $\{A_N'\}''$. We denote its $k$-th diagonal entry
$$(\Omega_N(z))_{kk} = \frac{1}{\mathbb{E}[R_N(z)]_{kk}} + (A_N')_{kk}.$$

Note that $\omega_N(z) = (\Omega_N(z))_{11}$ and that, using (5.3) and (5.8),
$$
\| (\omega_N(z) - A_N')^{-1} \leq \frac{1}{\Im \omega_N(z)} \leq \frac{1}{\Im z}.
$$

and
$$
\| (\Omega_N(z) - A_N')^{-1} \leq \| \mathbb{E}[R_N(z)] \| \leq \frac{1}{\Im z}.
$$

If we prove that:
$$
\exists C'(z) > 0, \forall N \geq 1, \forall (k, l) \in \{1, \ldots, N\}, \| (\Omega_N(z))_{kk} - (\Omega_N(z))_{ll} \| \leq \frac{C'(z)}{N},
$$
then we may conclude:
$$
\| \mathbb{E}[R_N(z)] - (\omega_N(z)I_N - A_N')^{-1} \|
\begin{align*}
&= \| (\Omega_N(z) - A_N')^{-1} - (\omega_N(z)I_N - A_N')^{-1} \|
&= \| (\Omega_N(z) - A_N')^{-1}(\omega_N(z)I_N - \Omega_N(z))(\omega_N(z)I_N - A_N')^{-1} \|
&\leq \| (\Omega_N(z) - A_N')^{-1} \| \| (\omega_N(z)I_N - \Omega_N(z)) \| \| (\omega_N(z)I_N - A_N')^{-1} \|
&\leq \frac{1}{|\Im z|^2} \| (\Omega_N(z))_{11}I_N - \Omega_N(z) \|
&\leq \frac{C'(z)}{N|\Im z|^2}
&\rightarrow 0
\end{align*}
$$

For $(k, l) \in \{1, \ldots, N\}$, observe that
$$(\Omega_N(z))_{kk} - (\Omega_N(z))_{ll} = (\Omega_N(z))_{kk}E_{kl} - E_{kl}(\Omega_N(z))_{kl}.
$$

Define, for a given deterministic matrix $X \in M_N(\mathbb{C})$,
$$
\Delta_X := \mathbb{E}[(R_N(z) - \mathbb{E}[R_N(z)])(A_N'X - XA_N')(R_N(z) - \mathbb{E}[R_N(z)])] - \mathbb{E}[R_N(z)](A_N'X - XA_N')\mathbb{E}[R_N(z)].
$$
Noting that for any deterministic Hermitian matrix $X$,
\[
\frac{d}{dt} \mathbb{E} \left( (z - A_N' - e^{-itX}U_N B_N U_N e^{itX})^{-1} \right)_{t=0} = 0,
\]
we readily deduce that
\[
\mathbb{E}[R_N(z)(A_N'X - XA_N')R_N(z)] = \mathbb{E}[R_N(z)]X - X\mathbb{E}[R_N(z)],
\]
and then extend this identity by linearity to any matrix $X \in M_N(\mathbb{C})$. It follows
that
\[
\Omega_N(z)X - X\Omega_N(z) = -\mathbb{E}[R_N(z)]^{-1} \Delta X \mathbb{E}[R_N(z)]^{-1}.
\] (5.10)

For $X = E_{kl}$, (5.9) and (5.10) yield
\[
|\Omega_N(z)_{kk} - (\Omega_N(z))_{ll}| \leq |\mathbb{E}[R_N(z)]_{kk} - (\mathbb{E}[R_N(z)])_{ll}| \leq C \left( |z| + 1 + \frac{1}{|z|^3} \right)^2 \frac{V((R_N(z))_{kk})}{N|3z|^4}
\]
where we used (5.4) and Lemma 5.2 (ii) in the three last inequalities. And we are done. □

We now study the convergence of the sequence $(\omega_N)_{N \geq 1}$ defined by (5.7).

**Lemma 5.6.** The sequence of analytic functions $(\omega_N)_{N \geq 1}$ defined on $\mathbb{C} \setminus \mathbb{R}$ converges uniformly towards $\omega$ on the compact subsets of $\mathbb{C} \setminus \mathbb{R}$.

**Proof.** It follows from Lemma 5.3 by taking the normalized trace, and using the notation
\[
g_N(z) := \mathbb{E}[G_{\mu_{A_N' + U_N B_N U_N}}(z)]
\]
that
\[
g_N(z) - G_{\mu_{A_N'}}(\omega_N(z)) \xrightarrow{N \to +\infty} 0.
\]

Using Lemma 7.7 in [14] and (5.3), we deduce that
\[
g_N(z) - G_{\nu}(\omega_N(z)) \xrightarrow{N \to +\infty} 0. \quad (5.11)
\]

22
The sequence of analytic functions \((\omega_N)_{N \geq 1}\) is normal, and thus there exists at least one converging subsequence. For any fixed \(z \in \mathbb{C}^+\), let us consider a converging subsequence \(\omega_{\phi(N)}(z)\) of \(\omega_N(z)\) and denote by \(l(z)\) the limit. As noted above in (5.8), \(\Im \omega_N(z) \geq \Re z\). Thus, if \(\Re z\) is large enough, \(|\omega_N(z)|\) will be large, and we can then uniquely invert with respect to composition uniformly in \(N\) in (5.11) to obtain \(G_\nu^{-1}(g_N(z) - o(1)) = \omega_N(z)\). Letting \(N\) go to infinity, Voiculescu’s asymptotic freeness result guarantees that \(g_N(z) \to G_{\nu \boxplus \nu}(z)\). Thus, \(l(z) = G_\nu^{-1}(G_{\nu \boxplus \nu}(z))\), independent of the convergent subsequence \(\omega_{\phi(N)}(z)\) we chose. This implies that \(\lim_{N \to \infty} \omega_N = l\) uniformly on compact sets of \(\mathbb{C}^+\), and by analytic continuation that \(l = \omega\). Therefore, for any \(z \in \mathbb{C}^+\), \(\omega(z)\) is the unique cluster point of \(\omega_N(z)\). \(\Box\)

**Proof of Proposition 5.2** Fix \(z \in \mathbb{C}^+\) (which is sufficient by a reflection argument). We proved that

\[
P(\omega_N(z) = z) = \frac{1}{\omega_N(z) - \alpha} P_r.
\]

(5.12)

We simply conclude by using Lemma 5.6 and \(\Im \omega_N(z) \geq \Re z\), \(\Re \omega(z) \geq \Re z\). \(\Box\)

It will be important in the proof of Proposition 5.1 to have the following analytic continuation result.

**Theorem 5.1.** (lemma 7.6.5 [17]) Let \(A\) be an open nonempty subset of \(\mathbb{C}\) and \(D \subset A\) such that \(\overline{D} = A\). Let \((g_n)\) be a sequence of locally bounded holomorphic functions on \(A\) such that \(\lim_{n \to +\infty} g_n(d)\) exists for any \(d \in D\). Then, the sequence \((g_n)\) converges towards a function \(g\) which is holomorphic on \(A\), uniformly on each compact subset of \(A\).

**Proof of Proposition 5.1** Define

\[D_\eta = \{z \in \mathbb{C} \setminus K_\eta^C, \Re z \in \mathbb{Q}, \Im z \in \mathbb{Q}^+\}.\]

According to Lemma 5.2(i) and (5.5), for any \(z \in D_\eta\), almost surely, \(P \Re \omega_N(z) = z\) converges towards \(\chi(z) I_r\). Hence, almost surely, for any \(k, l \in \{1, \ldots, r\}\), for any \(\eta > 0\), \((R_N(z))_{kl}\) is a bounded sequence of holomorphic functions on \(\mathbb{C} \setminus K_\eta^C\) such that the limit of \((R_N(z))_{kl}\) exists for any \(z \in D_\eta\). Therefore, according to Theorem 5.1, we can deduce that, almost surely, \((R_N(z))_{kl}\) converges towards an holomorphic function \(\chi_{kl}\) on \(\mathbb{C} \setminus K_\eta^C\), uniformly on each compact subset of \(\mathbb{C} \setminus K_\eta^C\). Of course, \(\chi_{kl}\) coincides with \(\frac{1}{\omega(z) - \alpha}\) on \(\mathbb{C}^+\) so that \(\chi_{kl} = \delta_{kl} \chi\). The proof is complete. \(\Box\)

We will now prove Theorem 4.1 by applying Lemma 4.1 on an almost sure event on which its assumptions are satisfied.

**Proof of Theorem 4.1** We consider the almost sure event, whose existence is guaranteed by Proposition 5.1 on which there exists an integer \(N_1\), a sequence
of positive numbers converging to 0, so that
\[ \text{sp}(A'_N + U^*_N B_N U_N) \subseteq K_{\eta N}^R \]
and, for any \( \eta > 0 \), the sequence \((M_N)_{N \geq N_0}\), where \( N_0 \geq N_1 \) is such that \( \forall N \geq N_0, \, \eta_N < \eta \), converges to
\[ M = \text{diag}(1 - (\theta_1 - \alpha)\chi, \ldots, 1 - (\theta_J - \alpha)\chi), \]
uniformly on compact subsets of \( \mathbb{C} \setminus K_{\eta N}^R \). On this event, apply Lemma 4.1 to the sequence \((M_N)_{N \geq N_0}\) and its uniform limit \( M \). The function \( M : \mathbb{C} \setminus K \rightarrow M_r(\mathbb{C}) \) is indeed a normal-operator-valued analytic function satisfying trivially conditions (a) and (b) of Lemma 4.1. The sequence \((M_N)_{N \geq N_0}\) consists of (random) analytic maps on \( \mathbb{C} \setminus K_{\eta N}^R \). Condition 3. of Lemma 4.1 is guaranteed by Proposition 5.1. Condition 2. is straightforward. To check condition 1., it is sufficient to argue that for any \( z \) such that \( |z| > C_1 \),
\[ \|R_N(z)\| \leq \frac{1}{d(z, [-C_1, C_1])}, \]
where \( C_1 \) is chosen as in Lemma 6.1. Almost every \( \eta > 0 \) is such that the boundary points of \( K_{\eta N}^R \) are not zeroes of \( \det(M) \), so for such \( \eta \)’s, we get exactly the conclusion of Theorem 4.1. Indeed, as explained in Section 3.2, eigenvalues of \( X_N \) in \( \mathbb{C} \setminus K_{\eta N}^R \) are exactly zeroes of \( \det(M_N) \), and the set of points \( z \) such that \( M(z) \) is not invertible is precisely \( O \). □

References

[1] G. W. Anderson, A. Guionnet, and O. Zeitouni. An introduction to random matrices, volume 118 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2010.

[2] L. Arnold. On the asymptotic distribution of the eigenvalues of random matrices. J. Math. Anal. Appl. 20:262268, 1967.

[3] Z. D. Bai and J. Yao. On sample eigenvalues in a generalized spiked population model. J. Multivariate Anal., doi:10.1016/j.jmva.2011.10.009

[4] Z. D. Bai and Y. Q. Yin. Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix. Ann. Probab., 16: 17291741, 1988.

[5] J. Baik, G. Ben Arous, and S. Péché. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. Ann. Probab., 33(5):1643–1697, 2005.

[6] J. Baik and J. W. Silverstein. Eigenvalues of large sample covariance matrices of spiked population models. J. Multivariate Anal., 97(6):1382–1408, 2006.
[7] S. T. Belinschi and H. Bercovici. A new approach to subordination results in free probability. *J. Anal. Math.*, 101:357–365, 2007.

[8] S. T. Belinschi, M. Popa, and V. Vinnikov. Infinite divisibility and a non-commutative Boolean-to-free Bercovici-Pata bijection. *J. Funct. Anal.*, 262(1):94–123, 2012.

[9] S. Belinschi. The Lebesgue decomposition of the free additive convolution of two probability distributions. *Probab. Theory Related Fields*, 142(1-2):125–150, 2008.

[10] H. Bercovici and D. Voiculescu. Free convolution of measures with unbounded support. *Indiana Univ. Math. J.*, 42(3):733–773, 1993.

[11] F. Benaych-Georges and R. R. Nadakuditi. The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices. *Advances in Mathematics*, 227(1): 494–521, 2011.

[12] P. Biane. On the free convolution with a semi-circular distribution. *Indiana Univ. Math. J.*, 46(3):705–718, 1997.

[13] P. Biane. Processes with free increments. *Math. Z.*, 227(1):143–174, 1998.

[14] M. Capitaine. Additive/multiplicative free subordination property and limiting eigenvectors of spiked additive deformations of Wigner matrices and spiked sample covariance matrices. *Journal of Theoretical Probability*, 2012, DOI: 10.1007/s10959-012-0416-5.

[15] M. Capitaine, C. Donati-Martin, and D. Féral. The largest eigenvalues of finite rank deformation of large Wigner matrices: convergence and nonuniversality of the fluctuations. *Ann. Probab.*, 37(1):1–47, 2009.

[16] M. Capitaine, C. Donati-Martin, D. Féral and M. Février. Free convolution with a semi-circular distribution and eigenvalues of spiked deformations of Wigner matrices. *Electronic Journal of Probability*, 16: 1750–1792, 2011.

[17] S. D. Chatterji. Cours d’analyse 2 Analyse complexe Presses polytechniques et universitaires romandes, 1997.

[18] B. Collins and C. Male. The strong asymptotic freeness of Haar and deterministic matrices. *ArXiv e-prints*, May 2011.

[19] D. Féral and S. Péché. The largest eigenvalue of rank one deformation of large Wigner matrices. *Comm. Math. Phys.*, 272(1):185–228, 2007.

[20] M. Février. Infinitesimal freeness and deformed matrix models. Doctorat de l’Université Paul Sabatier 2010

[21] W. Fulton. Eigenvalues of sums of Hermitian matrices (after A. Klyachko). *Astérisque*, (252):Exp. No. 845, 5, 255–269, 1998. Séminaire Bourbaki. Vol. 1997/98.
[22] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.

[23] John B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.

[24] J. Globevnik and I. Vidav. A note on normal-operator-valued analytic functions. *Proc. Amer. Math. Soc.*, 37:619–621, 1973.

[25] I. Johnstone. On the distribution of the largest eigenvalue in principal components analysis. *Ann. Stat.*, 29:295–327, 2001.

[26] V. Kargin. Subordination of the resolvent for a sum of random matrices. *ArXiv e-prints*, September 2011.

[27] P. Loubaton and P. Vallet. Almost sure localization of the eigenvalues in a Gaussian information-plus-noise model. Application to the spiked models. 2010 Available at [http://front.math.ucdavis.edu/1009.5807](http://front.math.ucdavis.edu/1009.5807).

[28] S. Péché. The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probab. Theory Related Fields*, 134:127–173, 2006.

[29] A. Pizzo, D. Renfrew, A. Soshnikov On Finite Rank Deformations of Wigner Matrices. To appear in Annales de l’Institut Henri Poincaré (B) Probabilités et Statistiques, 2011.

[30] N. R. Rao and J. W. Silverstein. Fundamental limit of sample generalized eigenvalue based detection of signals in noise using relatively few signal-bearing and noise-only samples. *IEEE Journal of Selected Topics in Signal Processing*, 4(3): 468–480, 2010.

[31] Reinhold Remmert. *Funktionentheorie. I*, volume 5 of *Grundwissen Mathematik [Basic Knowledge in Mathematics]*. Springer-Verlag, Berlin, 1984.

[32] R. Speicher. Free convolution and the random sum of matrices. *Publ. Res. Inst. Math. Sci.*, 29(5):731–744, 1993.

[33] D. Voiculescu. Addition of Certain Non commuting Random Variables *J. Funct. Anal.*, 66:323–346, 1986.

[34] D. V. Voiculescu. Limit laws for random matrices and free products. *Invent. Math.*, 104(1):201–220, 1991.

[35] D. V. Voiculescu. The coalgebra of the free difference quotient and free probability. *Internat. Math. Res. Notices*, (2):79–106, 2000.

[36] D. V. Voiculescu, K. J. Dykema, and A. Nica. *Free random variables*, A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups. volume 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992.
[37] Y. Q. Yin, Z. D. Bai, and P. R. Krishnaiah. On the limit of the largest eigenvalue of the large-dimensional sample covariance matrix. *Probab. Theory Related Fields*, 78(4): 509-521, 1988.

S. T. Belinschi: Queen’s University and Institute of Mathematics
“Simion Stoilow” of the Romanian Academy.
Address: Department of Mathematics and Statistics,
Queen’s University, Jeffrey Hall,
Kingston, ON K7L 3N6, Canada
Email: sbelinsch@mast.queensu.ca

H. Bercovici: Indiana University.
Address: Department of Mathematics,
Indiana University, Rawles Hall,
Bloomington, IN 47405, USA.
Email: bercovic@indiana.edu

M. Capitaine: CNRS Toulouse.
Address: CNRS, Institut de Mathématiques de Toulouse,
Equipe de Statistique et Probabilités,
F-31062 Toulouse Cedex 09, France.
Email: mireille.capitaine@math.univ-toulouse.fr

M. Février: Université Paris Sud.
Address: Université Paris Sud, Laboratoire de Mathématiques,
Bât. 425 91405 Orsay Cedex, France.
Email: maxime.fevrier@math.u-psud.fr