Relation between quantum speed limits and metrics on $U(n)$

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Abstract
Recently, Chau (2011 Quantum Inform. Comp. 11 721) found a family of metrics and pseudo-metrics on $n$-dimensional unitary operators that can be interpreted as the minimum resources (given by certain tight quantum speed limit bounds) needed to transform one unitary operator to another. This result is closely related to the weighted $\ell^1$-norm on $\mathbb{R}^n$. Here we generalize this finding by showing that every weighted $\ell^p$-norm on $\mathbb{R}^n$ with $1 \leq p \leq \pi/2$ induces a metric and a pseudo-metric on $n$-dimensional unitary operators with quantum information-theoretic meanings related to certain tight quantum speed limit bounds. Besides, we investigate how far the correspondence between the existence of metrics and pseudo-metrics of this type and the quantum speed limits can go.

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1. Introduction

Distinguishing two quantum operations and characterizing the resources needed to carry out a quantum operation are two meaningful problems in quantum information science. Various authors have studied the former problem. For instance, the problem of unambiguously distinguishing two quantum operators has been extensively studied [1–3]. Whereas one way to attack the latter problem is through the so-called quantum speed limits (QSLs), which put lower bounds on the evolution time needed to perform a unitary operation [4–7].

Given a Hamiltonian and an initial state, the evolution time $\tau$ needed to perform a unitary operation generated by the Hamiltonian is fixed. However, no explicit analytical expression for $\tau$ is known to date. The study of QSL makes use of a simple compromise to the above problem by asking what $\tau$ could be if a partial description of the quantum system, such as the energy
standard deviation [8], the energy of the system above its ground state [4–6] and the average absolute deviation from the median of the energy of the state [7], is given. Surely, the partial information given is not sufficient to deduce $\tau$. Yet surprisingly, non-trivial constraints in the form of explicit evolution time lower bounds (called QSL bounds) can be deduced. Moreover, these bounds are tight in the sense that for each of the above QSL bounds, we can find an initial state and a Hamiltonian generating the unitary operation such that the required evolution time is equal to the lower bound [4–8]. Interestingly, these bounds are mutually complimentary in the sense that none of them always gives a better evolution time lower bound than the others [7]. And this is not unexpected, for each of these QSL bounds are deduced using different partial information describing the quantum system.

Note that a few QSLs have geometric meanings. For instance, the well-known time–energy uncertainty relation [8] comes from the Bures metric on the group of unitary operators [9]. And the recently discovered families of metrics and pseudo-metrics on the group of $n$-dimensional unitary matrices $U(n)$ by Chau [10] are closely related to a QSL involving the average absolute deviation from the median energy. Actually, for any $U, V \in U(n)$, these metrics and pseudo-metrics can be written as certain weighted sums of the absolute value of the argument of eigenvalue of the unitary matrix $UV^{-1}$; hence, they are related to certain weighted average of the absolute value of the energy eigenvalues of the Hermitian operator generating $UV^{-1}$. Lately, Chau et al went further to show that these families of metrics and pseudo-metrics can be induced by the symmetric weighted $\ell^p$-norm on $\mathbb{R}^n$ [11]. (We will define symmetric weighted $\ell^p$-norm for $p \geq 1$ in section 2.) More importantly, they [11] interpreted these metrics and pseudo-metrics as the consequence of certain ‘reasonable’ cost functions to implement a unitary operation given by the tight QSL bound reported in [7]. (We will clarify what we mean by a ‘reasonable’ cost function in section 4.)

It is instructive to study how close the relation between the implementation cost in terms of say, certain QSLs and the existence of metrics or pseudo-metrics on $U(n)$ induced by such cost functions. Here we extend the findings by Chau and his co-workers [7, 10, 11] by proving the following results. First, for $p \geq 1$, there are metrics and pseudo-metrics on $U(n)$ that are functions of $|\theta_j|^p$ where $e^{\theta_j}$ are the eigenvalues of the unitary matrix $UV^{-1}$ with $\theta_j \in (-\pi, \pi]$ for all $j$. In fact, these metrics and pseudo-metrics are induced by certain symmetric weighted $\ell^p$-norms on $\mathbb{R}^n$. Second, for every $p > 0$, there are two QSL bounds. One involves $\langle |E|^p \rangle^{1/p}$, the $p$th root of the $p$th moment of the absolute value of energy of the system; and the other involves $\langle |E|^p \rangle^{1/p}$ by exploiting the freedom of choosing the reference energy level. Most importantly, these bounds are tight for $p \leq \pi/2$. Thus, for $1 \leq p \leq \pi/2$, the metrics and pseudo-metrics reported here can be interpreted as the minimum resources needed (through the tight QSL bounds involving $\langle |E|^p \rangle$ and $\langle |E|^p \rangle$, respectively) to convert one unitary operator to another. Nevertheless, our findings imply that for $p < 1$ or $p \geq \pi/2$, this close relation between the metric/pseudo-metric and the QSL breaks down because either the induced metric/pseudo-metric no longer exits or the QSL is no longer a tight bound. This work is a refinement and improvement of the research reported in the Master’s thesis of the first author [12].

2. Metrics and pseudo-metrics induced by weighted $\ell^p$-Norms

We say that a function $g : \mathbb{R}^n \to [0, \infty)$ is a symmetric norm on $\mathbb{R}^n$ if $g$ is a norm on $\mathbb{R}^n$ satisfying $g(v) = g(vP)$ for any $v \in \mathbb{R}^n$, and any permutation matrix or diagonal orthogonal matrix $P$. 

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Recall that for any fixed $p \geq 1$, a weighted $\ell^p$-seminorm on $\mathbb{R}^n$ is a function $h : \mathbb{R}^n \to [0, \infty]$ in the form $h(\mathbf{v}) = h(v_1, v_2, \ldots, v_n) = (\sum_{j=1}^n \mu_j |v_j|^p)^{1/p}$ for some $\mu_j \geq 0$ for all $j$. Surely, a weighted $\ell^p$-seminorm is indeed a seminorm on $\mathbb{R}^n$.

For any weighted $\ell^p$-seminorm $h$, we may define

$$g(\mathbf{v}) = \max \{ h(v_{P(1)}, v_{P(2)}, \ldots, v_{P(n)}) \} = \left[ \sum_{j=1}^n \mu_j \left( |v_j|^p \right) \right]^{\frac{1}{p}},$$

where the maximum is over all permutations $P$ of $\{1, 2, \ldots, n\}$. Besides, $\mu_j$ and $|v_j|^p$ denote the $j$th largest number in the sequences $(\mu_1, \mu_2, \ldots, \mu_n)$ and $(|v_1|, |v_2|, \ldots, |v_n|)$, respectively. It is straightforward to check that $g$ is a symmetric norm on $\mathbb{R}^n$ provided that not all $\mu_j$ are 0; we call this particular type of symmetric norm the symmetric weighted $\ell^p$-norm. (By taking the limit $p \to +\infty$, we have a symmetric weighted $\ell^\infty$-norm. This symmetric weighted $\ell^\infty$-norm is a special case of the symmetric weighted $\ell^1$-norm in which all but one of the weights $\mu_j$ are 0. So, we will not pay particular attention to the symmetric weighted $\ell^\infty$-norm any further.)

For any symmetric weighted $\ell^p$-norm on $\mathbb{R}^n$, we may apply the following result by Chau et al in [11] to induce a metric and a pseudo-metric on $U(n)$:

**Proposition 1 (Chau et al).** For any given symmetric norm $g : \mathbb{R}^n \to [0, \infty)$, we may define a metric $d_g$ and a pseudo-metric $d_g^\|^$ on $U(n)$ by

$$d_g(U, V) = g(|\phi|^1(UV^{-1}), \ldots, |\phi|^n(UV^{-1}))$$

and

$$d_g^\|^ (U, V) = \min_{x \in \mathbb{R}} g(|\phi|^1(e^{i\theta}UV^{-1}), \ldots, |\phi|^n(e^{i\theta}UV^{-1})).$$

Here $|\phi|^j(UV^{-1})$ denotes the $j$th largest number in the sequence $(|\phi_1|, |\phi_2|, \ldots, |\phi_n|)$ with $e^{i\theta_j}$ being the eigenvalues of $UV^{-1}$ obeying $-\pi < \theta_j \leq \pi$.

**Corollary 1.** Suppose $p \geq 1$. Then,

$$d_{p,\mu} (U, V) = \left\{ \sum_{j=1}^n \mu_j^j [ |\phi|^j(UV^{-1}) ]^p \right\}^{\frac{1}{p}}$$

$$= \left( \sum_{j=1}^n \mu_j \right)^{\frac{1}{p}} \min_{H: \exp(-\theta H)/\hbar = UV^{-1}} \max_{|\phi|\in C(H,\mu)} \left( (|E|^p)(H, |\phi|) \right)^{\frac{1}{p}}$$

and

$$d_{p,\mu}^\|^ (U, V) = \min_{x \in \mathbb{R}} d_{p,\mu} (e^{ix}U, V) = \min_{x \in \mathbb{R}} \left\{ \sum_{j=1}^n \mu_j^j [ |\phi|^j(e^{ix}UV^{-1}) ]^p \right\}^{\frac{1}{p}}$$

$$= \min_{x \in \mathbb{R}} \left( \sum_{j=1}^n \mu_j \right)^{\frac{1}{p}} \min_{H: \exp(-\theta H)/\hbar = e^{ix}UV^{-1}} \max_{|\phi|\in C(H,\mu)} \mathcal{D}_{\mu} E(H, |\phi|)$$

are metric and pseudo-metric on $U(n)$, respectively. Here $C(H, \mu)$ is the set of all (normalized) state kets in the form $\sum_{j=1}^n \alpha_j |E_j\rangle$, $|E_j\rangle$ is the energy eigenstate of $H$ with energy $E_j$, $|\alpha|^2 = \mu_j / \sum \mu_k$ and $P$ is a permutation of $\{1, 2, \ldots, n\}$. Also,

$$\langle |E|^p \rangle = \langle |E|^p \rangle(H, |\phi|) = \text{Tr}(|H|^p|\phi\rangle\langle\phi|) = \langle \phi | |H|^p |\phi\rangle$$

(6)
is the $p$th moment of the absolute value of energy of the system and
\[ D_p E \equiv D_p E (H, |\phi\rangle) = \min_{x \in \mathbb{R}} [(|E|^p)(H - xI, |\phi\rangle)]^{1/p} \] (7)
is the $p$th root of the $p$th moment of the absolute value of energy of the system minimized over the reference energy level.

Proof. Applying proposition 1 to the symmetric weighted $\ell^p$-norm in equation (1) gives the first equality in equation (4) as well as the first line of equation (5). More importantly, it implies that $d_{p, \mu}$ and $d_{p, \mu}^\downarrow$ are metric and pseudo-metric, respectively.

To show the second equality in equation (4), we adopt the strategy used in the proof of theorem 1 in [10]. Note that $\langle |E|^p (H, \sum_j |\alpha_j\rangle |E_j\rangle \rangle = \sum_j |\alpha_j|^2 |E_j|^p$. So, the rhs of equation (4) becomes $\min \{ \sum_j \mu_j^\downarrow (|E|^p)^{1/p} \}$, where $|E_j|^p$ denotes the $j$th largest element in the sequence $(|E_1|, |E_2|, \ldots, |E_n|)$. Among those $Ht$ that satisfy $\exp(-iHt/\hbar) = UV^{-1}$, the one that minimizes $\{ \sum_j \mu_j^\downarrow (|E|^p)^{1/p} \}$ can always be picked in such a way that its eigenvalues all lie in $(-\pi, \pi)$ [10]. Hence, the rhs of equation (4) is reduced to $\{ \sum_j \mu_j^\downarrow (|E|^p)^{1/p} \}$.

We omit the proof of the last line of equation (5) for it is essentially the same as that of the second inequality in equation (4).

\[ \square \]

Remark 1. From the above proof, we know that equations (4) and (5) hold irrespective of whether $d_{p, \mu}$ is a metric or not. We note further that the special cases of $d_{1, \mu}$ and $d_{1, \mu}^\downarrow$ are the metric and pseudo-metric reported originally by Chau in [10]. Moreover, it is possible to use an elementary method involving the Minkowski inequality to show that $d_{p, \mu}$ and $d_{p, \mu}^\downarrow$ are metric and pseudo-metric, respectively. Details can be found in the Master’s thesis of the first author [12].

3. Quantum speed limits via $\langle |E|^p \rangle^{1/p}$ or $D_p E$

We extend the proof concept used by Chau in [7] to find these QSLs. And we begin with the following lemma.

Lemma 1. Let $0 < p \leq 2$. Further let $f(x) = (1 - \cos x)/x^p$ for $x > 0$ and $f(0) = \lim_{x \rightarrow 0^+} f(x)$. Then $A_p = \sup \{ (1 - \cos x)/x^p : x > 0 \}$ exists and is equal to $\max_{x \in [0, \pi]} f(x) > 0$. In fact the maximum is attained by a unique $x_c \in [0, \pi]$. And this unique $x_c$ is a decreasing function of $p$ with $x_c > 0$ for $p < 2$ and $x_c = 0$ when $p = 2$. Thus,

\[ \cos x \geq 1 - A_p |x|^p \] (8)

for all $x \in \mathbb{R}$ with equality hold if and only if $x = 0, \pm x_c$.

Let us talk about the geometric meaning of the lemma before proving it. The lemma means that the curve $y = \cos x$ is always above the curve $y = 1 - A|x|^p$ provided that $A$ is a sufficiently large positive number. Besides, $A_p$ is the least possible value of $A$ for this to happen.

Proof. Since $p \leq 2$, $f(x)$ is well defined and continuous in $[0, \infty)$. Moreover, $f(x) > (1 - \cos x)/(x + 2\pi)^p = f(x + 2\pi)$ for all $x > 0$ and $f(x) > f(2\pi - x)$ for $0 \leq x < \pi$ because $p > 0$. Hence, $A_p = \max_{x \in [0, \pi]} f(x)$ is attained by a certain $x_c \in [0, \pi]$.

Surely, $df/dx|_{x=x_c} = 0$ which can be simplified to

\[ p \tan \frac{x_c}{2} = x_c. \] (9)
Note that the slope of the curve $y = \tan(x/2)$ is strictly increasing for $x \in [0, \pi)$ and is equal to $1/2$ at $x = \pi$. Hence, for $p = 2$, the only solution of equation (9) in the domain $[0, \pi]$ is $x_c = 0$. Whereas for $p \in (0,2)$, $f(0) = 0$. So, $x_c > 0$ as $A_p > 0$. Now consider the continuous function $\tilde{f}(x) = \tan(x/2)/x$ for $x > 0$, with $f([0,\pi]) = (1/2, \infty)$. This function is strictly increasing in $(0, \pi)$ for $d\tilde{f}(x)/dx = (x - \sin x) \sec^2(x/2)/2x^2 > 0$ for $0 < x < \pi$. Thus, the equation $f(x) = 1/p$ has a unique solution in the domain $(0, \pi)$ for all $p \in (0, 2)$. Clearly, this unique solution is the required $x_c$ that maximizes $f(x)$. More importantly, since $\tilde{f}$ is strictly increasing in $(0, \pi)$, $x_c$ decreases as $p$ increases.

Since the lhs and rhs of equation (8) are even functions, we only need to prove its validity for $x > 0$. The case of $x > 0$ is a consequence of $A_p \geq (1 - \cos x)/x^p$ for all $x > 0$; whereas the case of $x = 0$ is trivial. Finally, the if and only if condition for equation (8) to be an equality follows from the fact that $f(x)$ is maximized by a unique $x = x_c$.

**Corollary 2.** Let $p > 0$, and $H = \sum_{j=1}^n E_j|E_j\rangle\langle E_j|$ be a time-independent Hamiltonian acting on an n-dimensional Hilbert space. Then the time $\tau$ needed to evolve a pure state $|\Phi(0)\rangle$ to $|\Phi(\tau)\rangle$ under the action of $H$ is lower-bounded by

$$\tau \geq \tau_{c_1} \equiv h\left(1 - \sqrt{\frac{e}{A_p|E|^p}}\right)^{\frac{1}{2}}$$

(10)

Here $e = F(|\Phi(0)\rangle, |\Phi(\tau)\rangle) = |\langle\Phi(0)|\Phi(\tau)\rangle|^2$ is the fidelity between the two states and $\langle|E|^p\rangle$ is the $p$th moment of the absolute value of energy of the system defined by equation (6). Also, $A_p$ is given by lemma 1 if $p \leq 2$ and $A_p$ is defined to be $A_2$ otherwise. Actually, we can slightly optimize the bound in equation (10) to

$$\tau \geq \tau_{c_2} \equiv h\left(1 - \sqrt{\frac{e}{A_p}}\right)^{\frac{1}{2}}$$

(11)

where $\sqrt[p]{E}$ is the $p$th root of the $p$th moment of the absolute value of energy of the system minimized over the reference energy level as defined by equation (7). More importantly, these two bounds are tight for all $e \in [0,1]$ if $p \leq \pi/2$.

**Proof.** We first prove equation (10) for the case of $p \leq 2$. The initial quantum state $|\Phi(0)\rangle$ can be written in the form $\sum_{j=1}^n |\alpha_j\rangle |E_j\rangle$ with $\sum_{j=1}^n |\alpha_j|^2 = 1$. From lemma 1, the time $\tau > 0$ needed to evolve to a state $|\Phi(\tau)\rangle$ with $F(|\Phi(0)\rangle, |\Phi(\tau)\rangle) = e$ must obey

$$\sqrt{e} = |\langle\Phi(0)|\Phi(\tau)\rangle| = \left|\sum_{j=1}^n |\alpha_j|^2 e^{-iE_j\tau/h}\right| = \left|\sum_{j=1}^n |\alpha_j|^2 \cos \left(\frac{E_j\tau}{h}\right)\right| \geq \left|\sum_{j=1}^n |\alpha_j|^2 \cos \left(\frac{E_j\tau}{h}\right)\right|$$

$$\geq \sum_{j=1}^n |\alpha_j|^2 \left(1 - A_p \right) \frac{E_j\tau}{h}^{p} \frac{1}{h^p} \sum_{j=1}^n |\alpha_j|^2 |E_j|^p = 1 - A_p \tau^{p} (|E|^p) \frac{1}{h^p}.$$ (12)

Hence, the QSL in equation (10) is valid whenever $0 < p \leq 2$.

To prove the validity of this QSL for the case of $p > 2$, we only need to combine equation (10) for $p = 2$ and $(|E|^p)^{1/p} \geq (|E|^2)^{1/2}$ for all $p > 2$, which is a special case of the Lyapunov inequality [13].

Now, just having established the truth of the QSL in equation (10), the other QSL given by equation (11) follows from the fact that the reference energy level has no physical meaning. So from equation (10), we can obtain a more ‘optimized’ bound by varying the reference energy level $x$ so as to minimize $(|E|^p)(H - xI, |\Phi(0)\rangle)$ [7]. Therefore, equation (11) follows from equations (7) and (10).
we claim that the following state saturates the QSL stated in equation (10):

\[ |\Phi(0)\rangle \text{ as defined in equation (13) for } \beta = 1/A_x. \]

To show that the two QSLs are tight bounds for all fidelity \( \epsilon \in [0, 1) \), we only need to give an example of initial state that saturates the bound for all fidelity \( \epsilon \in [0, 1) \). And since the bound in equation (11) is in general more stringent than the bound in equation (10), we only need to show that the example we give saturates the former bound. Note further that there is no need to check for the case of \( \epsilon = 1 \) because the QSLs reduce to \( \tau \geq 0 \) which is trivially true. Now we claim that the following state saturates the QSL stated in equation (10):

\[ |\Phi(0)\rangle = \sqrt{1 - \beta} |0\rangle + \sqrt{\beta/2} (|+\rangle + |\rangle), \tag{13} \]

where \( \beta = (1 - \sqrt{\epsilon})/A_x p_x \) with \( x_c \) being the maximum point defined in lemma 1 so that \( \cos x_c = 1 - A_x p_x \). (Note that \( \beta \) is well defined as lemma 1 demands \( A_x > 0 \).) Since \( p \leq \pi/2 < 2 \), lemma 1 implies \( x_c > 0 \). Thus, \( \beta \geq 0 \) for all \( \epsilon \in [0, 1) \). As \( x_c \) is a decreasing function of \( p \) obeying equation (9), \( A_x p_x \geq 1 - \cos x_c \geq 1 \) (and hence \( \beta \leq 1 \) for all \( \epsilon \in [0, 1) \)) whenever \( 0 < p \leq p_c \), where \( p_c \) is the critical value of \( p \) in (0, 2] such that \( \cos x_c = 0 \) and hence \( x_c = \pi/2 \). From equation (9), \( p_c = \pi/2. \) In conclusion, equation (13) is a valid quantum state if \( 0 < p \leq \pi/2. \) Since \( A_x, x_c > 0 \) and \( \epsilon < 1 \), it is easy to check that for this particular quantum state, \( \langle E|p\rangle = \beta E p \) and \( \langle \Phi(0)|\Phi(t_c)\rangle = 1 - \beta + \beta \cos x_c = \sqrt{\epsilon}. \) So equation (10) is tight for all \( \epsilon \in [0, 1) \) provided that \( p \in (0, \pi/2) \).

**Remark 2.** From the above proof, for \( \pi/2 < p < 2 \), equations (10) and (11) are tight for some but not all \( \epsilon \in [0, 1) \) because equation (13) is a valid quantum state for \( \epsilon \) sufficiently close to 1. Note further that for \( p = 1 \), corollary 2 reduces to an earlier result obtained by Chau in [7]. Actually, the QSLs reported here also apply to the case of mixed state through the use of the purification argument by Giovannetti et al in [6]. Hence, these two QSLs can be regarded as the fundamental limit on the minimum time needed to evolve a density matrix or alternatively as a fundamental limit on the maximum possible information processing rate of a system [4–6].

Table 1 shows the actual evolution time \( \tau \) and the QSL bounds \( \tau_{c2} \) for different values of \( p \) for a few selected states in which \( \tau \) can be calculated exactly. The larger the value of \( \tau_{c2}/\tau \), the better the estimation of the lower bound. The table shows that while \( \tau_{c2} \) are different for different choices of \( p \), in general they all give reasonably good estimates on the actual \( \tau \). This is true even for the case of \( p = 2 \), which is not a tight bound. To understand why it is so, we start from lemma 1 and equation (11). They imply
for all $\epsilon \in [0, 1]$. (Interestingly, similar conclusions can be deduced for other QSL bounds including the time–energy uncertainty bound \cite{8} and the Margolus–Levitin bound \cite{4, 5}. Their proofs are straightforward and are left to the reader.) Let us use the notation used in corollary 2 to express the initial state $|\Phi(0)\rangle$ as $\sum_j \alpha_j |E_j\rangle$. Provided that $\vartheta_j \equiv \tau |E_j|/\hbar \leq \pi$ for all $j$ (that is, the phase angle $\vartheta_j$ rotated for each eigen-energy mode in the time evolution is at most $\pi$), we have $\tau \mathcal{D}_p/E/\hbar \lesssim \pi/2$. Hence, $\tau \approx \tau_{2\epsilon}$. So, our QSL bounds $\tau_{2\epsilon}$s, generally give reasonably good estimates to the actual evolution times $\tau$s, for all the initial states listed in table 1 because these states are picked so that $\vartheta_j \leq \pi$ for all $j$. Readers will find in section 4 that this discussion is essential to understand why our QSL bounds can be used to study the resources needed to carry out certain unitary operations.

4. Connection between the metrics, pseudo-metrics and the quantum speed limit bounds

By comparing $d_{p, \mathbb{R}}$ in equations (4) and $d^\varphi_{p, \mathbb{R}}$ in equation (5) of corollary 1 with the QSLs involving $\langle |E|^p \rangle$ or $\mathcal{D}_p E$ in corollary 2, we may interpret $d_{p, \mathbb{R}}$ and $d^\varphi_{p, \mathbb{R}}$ as cost functions describing the minimum amount of resources needed to convert $V$ from $U$. In the first case, the resources refer to the product of evolution time $\tau$ and the $p$th root of the $p$th moment of absolute value of energy of the system $\langle |E|^p \rangle^{1/p}$ required to carry out the conversion. And in the second case, the resources refer to the product of $\tau$ and $\mathcal{D}_p E$ (which is an ‘optimized’ version of $\langle |E|^p \rangle^{1/p}$) \cite{10}. Note that out of the Hamiltonians that generate a given unitary operation, we can always pick one so that $\vartheta_j \leq \pi$ for all $j$. Thus, the discussion in the final paragraph of section 3 implies that our cost functions are reasonably good estimates of the actual amount of resources required to covert $U$ from $V$.

Three remarks are in place. First, since this connection is done via QSL bounds, it works best when the bounds are tight for all $\epsilon$. For otherwise, the cost functions always overestimate the actual resources required. So, from corollary 2, this connection begins to lose its significance when $p \gg \pi/2$.

Second, from remark 1, we know that this interpretation works whenever $p > 0$—that is, even in the case when $d_{p, \mathbb{R}}$ is not a metric. However, Chau et al \cite{11} argued that any ‘reasonable’ cost functions $d_{p, \mathbb{R}}$ and $d^\varphi_{p, \mathbb{R}}$ should be a metric and a pseudo-metric on $U(n)$, respectively. Part of the reasons is that one way to transform $V$ to $U$ is first transforming $V$ to $W$ and then from $W$ to $U$. So, $d_{p, \mathbb{R}}$ and $d^\varphi_{p, \mathbb{R}}$ must satisfy the triangle inequality if the cost of transformation is additive—a rather modest additional requirement indeed. In this regard, the cost functions $d_{p, \mathbb{R}}$ and $d^\varphi_{p, \mathbb{R}}$ in equations (4) and (5) are ‘reasonable’ provided that $p \geq 1$.

Finally, since the overall phase of a unitary operator has no physical significance, the cost function $d^\varphi_{p, \mathbb{R}}(U, V)$ is more meaningful than $d_{p, \mathbb{R}}(U, V)$ in characterizing the resources needed to transform $V$ to $U$. Nonetheless, $d_{p, \mathbb{R}}$ is important in its own right for it gives rise to a characterization on the degree of non-commutativity between the two unitary operators $U$ and $V$ through $d_{p, \mathbb{R}}(UV, VU)$ \cite{10}.

To summarize, we have shown that any symmetric weighted $\ell^p$-norm on $\mathbb{R}^n$ induces a metric and a pseudo-metric on $U(n)$ for $p \geq 1$. These metrics and pseudo-metrics can be interpreted as ‘reasonable’ cost functions described by tight QSL bounds involving $\langle |E|^p \rangle$ and $\mathcal{D}_p E$, respectively, provided that $p \in [1, \pi/2]$. There is an open problem along this line of study. Our numerical study strongly suggests that $d_{p, \mathbb{R}}(U, V) = \sum_{j=1}^n \mu_j^p |\langle \theta_j | (U^V)^{-1} \rangle|^p$ is a metric on $U(n)$ for $0 < p < 1$ whenever $\mu_j > 0$. It is instructive to prove this conjecture and to relate it to a tight QSL bound.
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