Magneto-polarisability of mesoscopic systems

Y. Noat 1,2, R. Deblock 1, B. Reulet 1, H. Bouchiat 1

1 Laboratoire de Physique des Solides, Associé au CNRS, Bât 510, Université Paris–Sud, 91405, Orsay, France.
2 Kamerlingh Onnes Laboratory, Niels Bohrweg 5, Leiden, Netherland.

In order to understand how screening is modified by electronic interferences in a mesoscopic isolated system, we have computed both analytically and numerically the average thermodynamic and time dependent polarisabilities of two dimensional mesoscopic samples in the presence of an Aharonov-Bohm flux. Two geometries have been considered: rings and squares. Mesoscopic correction to screening are taken into account in a self consistent way, using the response function formalism.

The role of the statistical ensemble (canonical and grand canonical), disorder and frequency have been investigated. We have also computed first order corrections to the polarisability due to electron-electron interactions.

Our main results concern the diffusive regime. In the canonical ensemble, there is no flux dependence polarisability when the frequency is smaller than the level spacing. On the other hand, in the grand canonical ensemble for frequencies larger than the mean broadening of the energy levels (but still small compared to the level spacing), the polarisability oscillates with flux, with the periodicity $h/2e$. The order of magnitude of the effect is given by $\delta \alpha / \alpha \propto (\lambda_s / W g)$, where $\lambda_s$ is the Thomas Fermi screening length, $W$ the width of the rings or the size of the squares and $g$ their average dimensionless conductance. This magnetopolarisability of Aharonov-Bohm rings has been recently measured experimentally and is in good agreement with our grand canonical result. A preliminary account of this work was given in [9].

I. INTRODUCTION

A. Motivation of the paper

Transport and thermodynamic properties of mesoscopic metallic samples are known for a long time to be quite sensitive to the quantum phase coherence of the electronic wave functions at low temperature. In particular, it was shown that a mesoscopic ring threaded by a magnetic field exhibits a persistent current periodic with the flux and with a period of one flux quantum $\phi_0 = h/e$. This current was measured in several experiments, corresponding to different situations and materials. More recently, the response of a mesoscopic system to a time dependent magnetic field was studied both theoretically and experimentally.

The purpose of the present paper is to investigate how screening is influenced by electronic interferences in a phase coherent sample. In this spirit, we discuss how the electrical polarisability $\alpha$, i.e. the response of a metallic sample to an electrostatic field, a quantity which is directly related to screening, is also sensitive to mesoscopic effects. $\alpha$ can be experimentally measured by coupling the mesoscopic samples to a capacitor. For small voltages, corresponding to the linear regime, the change in the capacitance is directly proportional to the average polarisability $< \alpha >$ of the particles.

When the typical size $L$ of the particle is much larger than the Thomas Fermi screening length $\lambda_s$, $\alpha$ is mostly determined by the geometrical shape of the particle. For a spherical particle, it is proportional to the volume $V$ of the particle, with a small correction of the order of $\lambda_s / L$.

Now, what happens in the case of a quantum coherent sample, (i.e when $L$ is the order of the phase coherence length)? Is there a mesoscopic correction to the polarisability, due to contribution from electronic interferences?

In order to answer this question, we have calculated the flux dependent correction to the average polarisability, which we call “mesoscopic correction”, for two dimensional mesoscopic samples in the presence of an Aharonov-Bohm flux. Two different geometries have been considered, namely a two dimensional ring or square in an in-plane electric field. The role of statistical ensemble, disorder, and finite frequency have been investigated. In the case of a time dependent electric field, we distinguish two different limits, whether the frequency is smaller or larger than the inverse relaxation time, or mean level broadening $\gamma$. The case $\omega \ll \gamma$ corresponds to the thermodynamic limit whereas $\omega \gg \gamma$ corresponds to the finite frequency limit.

B. Scope of the paper

This paper is organized as follows: the definition of the electrical polarisability and the classical result for a macroscopic sample are given in section I

In section II, we derive the expression of the electrical polarisability of a system of non-interacting electrons, in the spirit of the early work of Gorkov and Eliashberg (GE). A giant unphysical polarisability is obtained. However it is possible to discuss in this very simple model the flux dependence of two quantities which contribute to the polarisability, namely the matrix element of the position (or screened potential) operator and the contribution of energy levels. Then, we examine in section III how screening can be taken into account. In particular, we show how to incorporate contributions of electronic interferences. Using those results, a general expression for the polarisability and its mesoscopic correction for a time varying electric field are derived in section IV.
We study then separately the case of the grand canonical and canonical statistical ensemble. The grand canonical average polarisability is found to depend only on the flux dependent matrix elements of the screened potential whereas the Canonical average depends also on the energy levels. As a result when the frequency is smaller than the level spacing the magneto-polarisability is found much smaller in the canonical ensemble compared to the grand canonical case. Finally, we present results on the Hartree-Fock correction to the zero frequency polarisability. Section VI is devoted to a comparison of these results with recent experiments.

II. BASIC CONCEPTS

A. Screening and polarisability

When an external field \( \mathbf{E}_{\text{ext}} = -\nabla \phi_{\text{ext}} \) is applied on a conductor, the charge density adjusts in such a way that the effective field inside the conductor cancels. This phenomenon is called screening. In principle the computation of the potential inside the sample requires to solve a complex N body hamiltonian taking into account the long range Coulomb interaction between electrons. However it is possible in general to treat the interactions in a mean field approximation. The screened potential inside the conductor is then given by:

\[
\phi(r_1) = \phi_{\text{ext}}(r_1) + \int \delta \rho(r_2) U(r_1, r_2) dr_2 \quad (1)
\]

Where \( \delta \rho(r_2) \) is the charge density induced at the point \( r_2 \) and \( U(r_1, r_2) \) is the coulomb interaction potential. In the linear response limit, \( \delta \rho(r_2) \) can be described by the electric response function: \( \chi(r_1, r_2) \), which relates the variation of charge in the system at \( r_2 \) to a local perturbation at \( r_1: \delta \rho(r_1) \).

\[
\delta \rho(r_2) = \int \chi(r_1, r_2) \phi(r_1) dr_1 \quad (2)
\]

Relation \( \delta \rho(r_2) \) can then be written more conveniently in a matrix form:

\[
\phi = (1 - U \chi)^{-1} \phi_{\text{ext}} \quad (3)
\]

In a clean infinite system, the induced charge density is simply related to the screened potential \( \phi \) in Fourier space: \( \delta \rho(q) = \chi(q) \phi(q) \), where \( \phi(q) = \int e^{iq \cdot r} \phi(r) dr \). The problem is more complex in a disordered system, which does not have the translational invariance. In any case for a finite system, it is possible to compute the response function from the eigenstates \( | \alpha > \) of the system in the absence of the external field, associated to the eigenvalues \( \epsilon_\alpha \):

\[
\chi(r_1, r_2) = \sum_{\alpha=1}^{N} \sum_{\beta<\alpha} \frac{\psi_\alpha^*(r_1) \psi_\beta(r_2) \psi_\beta(r_1) \psi_\alpha^*(r_2)}{\epsilon_\beta - \epsilon_\alpha} + \text{c.c.} \quad (4)
\]

In most cases we will identify these eigenstates as the solution of the formulation of the system without electron interactions which will be taken into account only by considering the screening on the applied field. The effect of interactions on the eigenstates will be however discussed in section VII within the Hartree Fock approximation.

The application of an electric field on a finite metallic system, results in an induced dipolar moment \( \mathbf{P} \) which, for a sufficiently small field \( \mathbf{E} \), is proportional to the applied field:

\[
\mathbf{P} = \int \rho(r) r dr = \alpha \mathbf{E} \quad (5)
\]

where \( \alpha \) is by definition the polarisability of the conductor, which can be expressed as a function of the response function:

\[
\alpha = \frac{1}{\mathbf{E}^2} \int \chi(r_1, r_2) \mathbf{E} \cdot r_2 \phi(r_1) dr_1 dr_2 \quad (6)
\]

B. Classical polarisability

1. Plate with perpendicular electric field

The polarisability of a macroscopic piece of metal is described by classical electrostatics. In this approximation, the dipolar moment can be calculated by solving Poisson equation, \( \nabla^2 V + \frac{\rho_0}{\varepsilon_0} = 0 \), in all space and imposing as well that the conductor is at a constant potential. The problem is solvable analytically for a sphere of volume \( V \): \( \alpha = \varepsilon_0 (4\pi/3) V \). It is also possible to estimate the polarisability of a metallic plate of thickness \( a \) in a constant electric field perpendicular to the plate. The induced charge density is such that the electric field cancels inside the plate. Neglecting border effects, we have:

\[
\rho(r) = \frac{\varepsilon_0 \mathbf{E}_{\text{ext}}}{2} \left[ \delta \left( x - \frac{a}{2} \right) - \delta \left( x + \frac{a}{2} \right) \right] \quad (7)
\]

Integrating equation \( \delta \rho(r) \) one finds the classical Polarisability:

\[
\alpha = \varepsilon_0 \alpha^3 = \varepsilon_0 V \quad (8)
\]

\(^1\)This point will be discussed later on
FIG. 1. Induced charge density of a conducting strip in an in-plane uniform electric field: the function diverges logarithmically as a function of the cutoff $\lambda/a$.

2. 2d and 1d systems in longitudinal electric fields

We now examine the case of 2d conductors, and more specifically a strip of length $L$ and width $W$, with $L \gg W$, in an in-plane electric field perpendicular to its long axis. In order to determine the induced charge density, the effective electric field inside the strip is imposed to be zero. In the limit of an infinite strip, this condition leads to the following equation:

$$
\int \frac{1}{2\pi\epsilon_0} \frac{\rho(x')}{x-x'} dx' - E = 0
$$

(9)

We have calculated numerically the solution of (9) in the case of an infinite strip (fig. 2). In contrast with the 3 dimensional case, the charge distribution spreads over the full width of the strip in order to cancel the external field. The numerical solution is very close to a logarithm. Assuming that the density is of the form $\rho(x) \approx EW \log \left( \frac{W}{x} - \frac{W}{x} \right) / \log \left( \frac{W}{x} + x \right)$, the integral of the previous function gives $\alpha \approx \epsilon_0 W^2 L$.

This result extrapolates to a 2 dimensional sample of typical size $W$, for which the in-plane polarisability scales like $\alpha \sim \epsilon_0 W^3$. For instance the polarisability of a disk of radius $R$ is $\alpha = \epsilon_0 \frac{\pi}{16} R^3$ [10].

In the case of a one dimensional wire, whose length $L$ is much longer than its diameter $D$, in an electric field along its axis, the polarisability has been shown to be $\epsilon_0 L^3 \ln(D/L)$ [10]. Similarly as in the 2d case, the charge density has to spread over the entire wire in order to screen the external potential.

The case of a 1D ring is also exactly solvable. In order to take screening into account, we first assume that the induced charge density in the ring is of the form $\rho(\theta) = \lambda \cos(\theta) \delta(z) \delta(\rho - R)$, from which we deduce the induced potential in the ring:

$$
\phi_{ind}(\theta) = \int_0^{2\pi} \frac{1}{8\pi\epsilon_0} \frac{\rho(\theta')}{\sin \left( \frac{\lambda x'}{2} \right)} d\theta'
$$

(10)

The integral is logarythmically diverging and it is necessary to introduce a cutoff $\theta_c = W/R$, related to the finite width $W$ of the ring: $\phi_{ind}(\theta) \approx -\frac{\lambda \cos(\theta) \ln(\theta_c)}{4\pi\epsilon_0}$.

The screened potential reads $\phi(\theta) = \cos(\theta) (E + \lambda J)$, with $J = \frac{1}{4\pi\epsilon_0} \ln \left( \frac{1}{\theta_c} \right)$. The polarisability is obtained either from the potential calculated above, either from the expression of the charge. This self-consistent relation sets the value of the parameter $\lambda$, from which we deduce the polarisability of the ring.

$$
\alpha = \frac{\alpha_0}{1 + \frac{2\pi\alpha_0}{\lambda}}
$$

(11)

where $\alpha_0 = \epsilon_0 \pi^2 R^3$ which reads in the limit $J \gg 1$:

$$
\alpha \approx \frac{\epsilon_0 \pi^2 R^3}{\ln \left( \frac{R}{W} \right)}
$$

(12)

As a result both longitudinal polarisabilities of a ring and a disk scale like the cube of their radius, just like the polarisability of a sphere.

C. A mesoscopic correction to the polarisability?

In the following, we define the quantity we wish to calculate. As we have just seen, the classical polarisability of a metallic sample is determined by geometrical factors. However, due to the Pauli principle, charges can not strictly accumulate on the border of the sample, but the charge density distribution rather extends on the screening length $\lambda_s$. This effect reduces the polarisability compared to its classical value by a quantity $\delta\alpha_{TF}$ of the order of $\lambda_s/L$, where $L$ is the typical size of the conductor along the electric field. In a coherent sample, electronic interferences might also give rise to a contribution $\delta\alpha_Q$ to the polarisability. The total polarisability can thus be written: $\alpha = \alpha_c + \delta\alpha_{TF} + \delta\alpha_Q$. In order to show evidence of this correction, a magnetic flux can be applied through the sample in order to introduce a flux dependent part in the phase difference of electronic trajectories. We will call “mesoscopic correction” the flux dependent part of the polarisability $\delta\alpha(\Phi)$. We will focus
on the ensemble average of this quantity \( \overline{\delta \alpha(\Phi)} \), which is \( \Phi_0/2 \) periodic. We will characterize the flux dependence of \( \overline{\delta \alpha} \) by \( \delta \Phi(\alpha) = \overline{\delta \alpha(\Phi_0/4)} - \overline{\delta \alpha(0)} \).

D. Model used for the numerical simulations

Part of the analysis we have done rely on computer simulations. The two dimensional rings or squares are modelled by the Anderson model. Each atomic site is coupled to its nearest neighbour by a hopping term \( t \). Disorder is introduced by on-site energies randomly distributed in the interval \([-w, w]\). The Anderson hamiltonian can be written in second quantization:

\[
H = \sum_k \epsilon_k c_k^\dagger c_k + t e^{i\phi_k} c_k^\dagger c_{k+1} + t e^{-i\phi_k} c_{k+1}^\dagger c_k
\] (13)

where \( c_k^\dagger \) is the creation operator associated with site \( k \). The magnetic flux is introduced in this hamiltonian through the phase factor of the hoping element with \( \phi_k = 2\pi \Phi/\Phi_0 (x_k - x_{k+1})/L \) for a ring of size \( L \). In the case of squares we have taken into account the penetration of the magnetic field in the sample: the phase factor appearing in the hopping matrix elements is computed from the integral of the potential vector between concerned neighbors. In two dimensions, an estimation of the elastic mean free path is given by \( l_c \approx 30 \left( \frac{w}{L} \right)^2 \) allowing us to choose values of \( t/w \) so that the system is in the diffusive regime. In order to average quantities we take advantage of ergodicity properties in the diffusive regime and average over the number of electrons in the system between one quarter and three quarter filling.

III. QUANTUM POLARISABILITY OF A SYSTEM OF NON-INTERACTING ELECTRONS

We now examine the electrical response of a quantum system of non-interacting electrons, which was first calculated by GE [4] for a metallic grain both in the diffusive and ballistic limit. Their conclusion is quite surprising: in the diffusive regime, \( \alpha \) is bigger than the classical polarisability by a factor \( (a/a_0)^2 \) where \( a \) is the typical size of the grain and \( a_0 = h^2/(me^2) \) is the Bohr radius. Later on, Rice [11] stressed the lack of screening in the calculation of GE and showed that when screening is taken into account, one recovers the classical polarisability. We will see in the following that it is useful to consider the flux dependence of the polarisability within this crude approximation, since many results remain qualitatively true in the presence of screening.

A. Position of the problem

The polarisability of a system of non-interacting electrons can be understood as the sensitivity of the energy spectrum to an external electric field, just like persistent currents measure the sensitivity of the spectrum to an Aharonov-Bohm flux. As a matter of fact, the eigenstates \( |\alpha> \) of the system and the eigenvalues \( \epsilon_\alpha \), are modified by an external electric field \( E \):

\[
\epsilon'_\alpha = \epsilon_\alpha - <\alpha|eE.r|\alpha> + \sum_{\beta \neq \alpha} \frac{|<\alpha|eE.r|\beta>|^2}{\epsilon_\beta - \epsilon_\alpha} + ...
\] (14)

\[
|\alpha> \approx |\alpha> + \sum_{\beta \neq \alpha} \frac{<\beta|eE.r|\alpha>}{\epsilon_\beta - \epsilon_\alpha} |\beta> + ...
\]

Consequently, it exists an induced charge density \( \delta \rho(r) = e \sum_{\alpha=1}^N |\psi'_\alpha(r)|^2 - |\psi_\alpha(r)|^2 \), associated with the asymmetry of each wave function generated by \( E \). Using the definition \( \overline{\delta \alpha} \) of \( \alpha \), the expression of the polarisability of a system of non-interacting electrons at zero temperature can be found:

\[
\alpha = \frac{2e^2}{E^2} \sum_{\alpha=1}^N \sum_{\beta \neq \alpha} \frac{|<\alpha|E.r|\beta>|^2}{\epsilon_\beta - \epsilon_\alpha}
\] (15)

This expression depends on the eigen-energies of the unperturbed system and its eigen-functions through the matrix elements of the position operator. Due to the energy denominator, the polarisability will in particular be very sensitive to the electric field induced coupling between the last occupied levels and the first non occupied ones. Note also that this expression of the polarisability is also from expression [12] identical to the second derivative of the total energy of the system with respect to electric field.

\[
\alpha = - \sum_{\alpha=1}^N \frac{\partial^2 \epsilon_\alpha}{\partial E^2}
\]

These results present strong similarities with the paramagnetic contribution of orbital susceptibility of a mesoscopic ring pierced by a flux line which is identical to expression [13] where the \( E.r \) is replaced by \( eA.p \) where \( A \) is the potential vector and \( P \) is the kinetic momentum operator. The total magnetic susceptibility contains also a constant diamagnetic term \( -Ne^2/m \) [12], [15] which does not exist in the electric response.

B. 1 dimensional Aharonov-Bohm ring

Using expression [13] it is possible to calculate exactly the induced dipolar moment in a 1d non-disordered ring in an in plane electric field.
Wavefunctions only depend on the angle $\theta$, indicating the position in the ring, and satisfy the 1d Shrödinger equation: $-\hbar^2/2mL^2 \frac{\partial^2}{\partial \theta^2} \psi(\theta) = E \psi(\theta)$. Furthermore, the magnetic flux, associated with the periodicity in the ring, imposes the following boundary condition $\psi(\theta + 2\pi) = e^{i2\pi w} \psi(\theta)$. This equation can be solved. Using the parity and periodicity of the spectrum, we then order the eigenvalues and the corresponding wavefunctions in ascending order in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$:

$$
\begin{align*}
\psi_{2p} &= \frac{\hbar^2}{2mL^2} (p + \frac{\phi}{\pi})^2 \psi_{2p}(\theta) = \frac{1}{2\pi} e^{i\theta(p + \frac{\phi}{\pi})} \\
\psi_{2p+1} &= \frac{\hbar^2}{2mL^2} (p - \frac{\phi}{\pi})^2 \psi_{2p+1}(\theta) = \frac{1}{2\pi} e^{i\theta(-p + \frac{\phi}{\pi})}
\end{align*}
$$

(16)

As a consequence of the particular geometry of the system, the electric field only couples certain states. In particular, the matrix element between adjacent states cancels unless the quantum numbers $p$ and $q$ are such that $p - q = \pm 2$, in this case: $<p|X|q> = \frac{\hbar}{2\pi}$. The polarisability depends on the parity of the number of electrons $N$ and reads:

$$\alpha_N(\Phi) = \frac{e^2 R^2}{8\pi^2} \left( \frac{1}{\epsilon_{N+2}(\Phi) - \epsilon_N(\Phi)} - \frac{1}{\epsilon_{N+1}(\Phi) - \epsilon_{N-1}(\Phi)} \right)$$

(17)

At zero flux and for $N \gg 1$: $\alpha_N = \frac{1}{8\pi^2} \frac{\hbar^4}{\phi_0^2} \frac{1}{N^2}$. It is possible to evaluate $\delta_\Phi \alpha$ by computing $(\alpha_{N+1} - \alpha_N)/2$. As a result:

$$\frac{\delta_\Phi \alpha}{\alpha} \approx -\frac{1}{N^2}.$$  

(18)

It is a very small effect, since it decreases rapidly with the number of electrons in the ring. This result remains true for a multichannel ring. We attribute this effect to selection rules inherent to the square lattice which are responsible for the cancellation of the matrix elements of operator $X$ between eigenstates close to a level crossing. In particular it has been shown [9], [14], [15] using semiclassical arguments that the average square matrix element, $|<\alpha|X|\beta>|^2$ depends mainly of the energy difference $\epsilon = |\epsilon_\beta - \epsilon_\alpha|$ and the Thouless energy $E_c = hD/L^2$. For $\epsilon < E_c$, it is of the order of $a/g$ where $g = E_c/\Delta$ is the ratio between the Thouless energy and the mean level spacing $\Delta$. At higher energy it decreases like $1/\epsilon^2$. This behavior is illustrated in fig.3 showing numerical results on a sample for different values of disorder where these two regimes can be clearly distinguished.

C. Diffusive system

In the following we discuss a diffusive system characterized by a diffusion coefficient $D = (1/d)v_F l_e$ where $l_e$ is the elastic mean free path which is assumed to be shorter than the system size $a$ along the electric field. It has been shown [9], [14], [13] using semiclassical arguments that the average square matrix element, $|<\alpha|X|\beta>|^2$ depends mainly of the energy difference $\epsilon = |\epsilon_\beta - \epsilon_\alpha|$ and the Thouless energy $E_c = hD/L^2$. For $\epsilon < E_c$, it is of the order of $a/g$ where $g = E_c/\Delta$ is the ratio between the Thouless energy and the mean level spacing $\Delta$. At higher energy it decreases like $1/\epsilon^2$. This behavior is illustrated in fig.3 showing numerical results on a sample for different values of disorder where these two regimes can be clearly distinguished.

It is then easy to deduce the order of magnitude of the polarisability noting that the summation in expression (13) can be restricted to $|\alpha - \beta| < g$ as a result:

$$\alpha \sim e^2 a^2/\Delta$$

(19)

The GE result is then recovered assuming that the Fermi wave length is of the order of the Bohr radius. We will see in the next section that this result is modified by screening. It is however worth continuing this analysis in
This oversimplified picture of non interacting electrons in the case of an Aharonov-Bohm ring. From expression (13) the polarisability is expected to exhibit flux dependence both from the matrix elements and energy denominators. We discuss in the following separately these two contributions which is justified in the context of random matrix theory where eigenfunctions and eigenenergies constitute two sets of independent random variables [16].

1. Energy denominator

We focus on the flux dependence of the quantity $K = \langle \sum_{\alpha < \beta} \frac{1}{\epsilon_\beta - \epsilon_\alpha} \rangle$. This quantity depends only on the energy differences $(\epsilon_\alpha - \epsilon_\beta)$, and can be expressed as a function of $R(\epsilon)$, the two levels correlation function:

$$K = \int_0^{E_{\text{max}}} d\epsilon \int_0^\epsilon \frac{R(u)}{u} \, du$$  \hspace{1cm} (20)

In the diffusive regime, $R(s)$ is well described by random matrix theory [14] and the average probability to find two degenerate adjacent levels is zero. This property is characteristic of level repulsion in the spectrum of a random matrix, which is stronger in a system where reversal symmetry is broken compared to a system where it is not. As a consequence, $K$ decreases as a function of the flux at low magnetic flux and increases back in the vicinity of $\phi_0/2$ where time reversal symmetry is recovered.

We have calculated numerically $K$ in a disordered ring of length $L = 80$ and width $a = 8$ for several value of the on-site disorder $w$ corresponding to the diffusive regime. Figure [3] illustrate the effect described above. In the absence of other contributions, this term would give rise to a negative magnetopolarisability.

2. Matrix element

On figure [4] are plotted the flux dependance of the average square of the diagonal and non-diagonal matrix elements for a diffusive ring. It can be noted that they have opposite flux dependences, resulting from the fact that $Tr(X^2) = \sum_\alpha X^2_{\alpha,\alpha} + \sum_{\alpha \neq \beta} X^2_{\alpha,\beta}$ is flux independent. The diagonal elements decrease as a function of the flux, whereas the non-diagonal elements increases.

The time reversal property of the operator $X$ implies that its diagonal matrix elements are real even function of flux and can be developed in successive powers of $\cos(2\pi \phi/\phi_0)$. $< X^2_{\alpha,\alpha} >$ is maximum for multiple values of $\phi_0/2$. It is possible to evaluate analytically this flux dependence of the diagonal matrix elements from random matrix theory, using the relation between $X_{\alpha,\alpha}$ and the sensitivity of the energies to an electric field: $X_{\alpha,\alpha} = (\frac{\partial E}{\partial \phi_0})_{E=0}$. Since electric field preserves time-reversal symmetry, the typical value of the derivative of the energy levels with respect to the elec-
magnetopolarisability

The magnetopolarisability thus related to the insensitivity of persistent currents to an electrostatic field and can be qualitatively understood using a semi-classical argument. The effect of the electrostatic potential slowly varying at the scale of the Fermi wavelength can be included into the phase of the electronic wavefunctions. Persistent currents can be computed from the integral of the classical action on a circular orbit. The contribution of the static electrostatic potential on such an orbit: \((e/\hbar v_f) \int V(r) ds\) where \(V(r) = Er\cos(\theta)\) and \(ds = r d\theta\) obviously cancels out. This argument is helpful to understand why the polarisability is independent of the flux in this very low frequency regime.

IV. SCREENING

A. The Thomas-Fermi approximation

As mentioned above the mean field approximation reduces the complicated many body problem of interacting electrons to a much simpler one, in which electrons move in an effective potential \(\phi(r)\), resulting from the screening of the applied potential by other electrons. Therefore, it goes beyond a simple electrostatic calculation by taking into account the kinetic energy of electrons as well as their fermionic character [17].

In the linear regime, i.e. for a small external potential, the induced charge density is simply proportional to the effective potential [13, 13]:

\[
\delta \rho(r) = -e^2 n(E_F) \phi(r) = -\epsilon_0 k_s^2 \phi(r)
\]

at 3D with \(k_s = \sqrt{\frac{e^2 n(E_F)}{\epsilon_0}}\) and

\[
\delta \rho(r) = -e^2 n(E_F) \phi(r) = -\epsilon_0 k_s \phi(r) \delta(z)
\]

for a conducting plane at \(z=0\) with \(k_s = \frac{e^2 n(E_F)}{\epsilon_0}\).

The resolution of the self-consistent equation [13] within this Thomas Fermi approximation for a 3D sample in the presence of a uniform external applied field, gives rise to a screened potential whose value is of the order of \(E/k_s\) which is confined to the border of the sample within \(\lambda_s = 1/k_s\). This result is in principle not true any more in a 2D system where classical screening already involves charge displacement in the whole system. However since the charge distribution is always singular on the edge of the sample it is possible to approximate the screened potential for a disk of radius R by: \(\Phi(r, \theta) = (E/k_s) F((R - r)\cos(\theta))\) where F is a peaked function centered on zero of width \(1/k_s\). When screening is taken into account the expression of the quantum polarisability at \(T=0\) reads:

\[
\alpha = \frac{2e^2}{E^2} \sum_{\alpha \neq \beta} \frac{\langle \beta | \phi_{T,F}(r) | \alpha \rangle \langle \alpha | \mathbf{E}(r) | \beta \rangle}{\epsilon_\alpha - \epsilon_\beta}
\]

Within this approximation the GE result for the polarisability is modified by a factor \(1/(a k_s)^2\) and becomes identical to the classical result.
B. Beyond The Thomas-Fermi approximation

As we have seen, the Thomas-Fermi approximation does not take into account contribution of the screening due to electronic interferences. They indeed give rise to a quantum correction for the response function $\chi$, and consequently to the effective potential $\phi$ and the polarisability $\alpha$:

$$
\begin{align*}
\chi &= \chi_{T,F} + \delta \chi \\
\phi &= \phi_{T,F} + \delta \phi \\
\alpha &= \alpha_{T,F} + \delta \alpha
\end{align*}
$$

where $\chi$ is the response to the local field. Assuming that those corrections are small compared to the Thomas-Fermi value, it is possible to show, in agreement with Efetov [20] (see appendix A for the detailed calculation), that $\delta \alpha$ can be simply expressed as a function of the mesoscopic correction to the one electron response function $\delta \chi$ and the Thomas-Fermi potential:

$$
\delta \alpha \approx \frac{1}{E^2} \text{Tr} (\phi_{T,F} \delta \chi \phi_{T,F})
$$

In the following we discuss the response to a time dependent electric field. We will see that dynamical polarisability can be very different from the static one.

V. EXPRESSION OF THE POLARISABILITY

A. Response to time-dependent potential

The application of a time-dependent external potential $V(t) = e \mathbf{E} \cdot \mathbf{r} \exp (i\omega t)$ raises the problem of the relaxation of the system towards equilibrium. This process is made by inelastic processes characterized by a typical time scale, related to inelastic collisions.

In the limit of a weak coupling, this process can be described by a master equation on the density matrix [42].

$$
i\hbar \frac{\partial \rho}{\partial t} = [H_0 + V(t), \rho] - i \gamma (\rho - \rho_{eq})
$$

The parameter $\gamma$ represents the broadening of the energy levels and characterizes dissipation processes, allowing the relaxation of the system towards equilibrium. The density matrix $\rho_{eq}$ satisfies the condition $[H, \rho_{eq}] = 0$, $H = H_0 + e \mathbf{E} \cdot \mathbf{r} \exp (i\omega t) + \phi_{ind}(\mathbf{r}, \gamma, t)$ being the hamiltonian of the system in the mean field approximation, in an AC electric field at frequency $\omega$. Two limits are to be considered. When the frequency of the time dependent potential is small compared to $\gamma$, the system follows the potential and stays at every moment at equilibrium. On the contrary, at high frequency compared to $\gamma$, the system is always out of equilibrium.

In addition, we assume that the effective potential is always in phase with the external one. This is justified if the dissipative part of the polarisability is very small compared to the non-dissipative part ($\alpha'' \ll \alpha'$), this last quantity being discussed in [21].

The expression of the density matrix is then obtained by solving the master equation. From it, we deduce the response function and later on the polarisability and its quantum correction $\delta \alpha$, which can be expressed as a function of the eigenstates and eigenvalues of the hamiltonian $H_0$.

$$
\alpha = \frac{2e^2}{E^2} \text{Re} \left( \sum_{\alpha \neq \beta} \frac{f_\alpha - f_\beta}{\epsilon_\alpha - \epsilon_\beta - i \gamma} < \beta | \phi_{T,F} | \alpha > < \beta | \mathbf{E} \cdot \mathbf{r} | \alpha > \right)
$$

$$
\delta \phi \alpha = \frac{2e^2}{E^2} \delta \phi \text{Re} \left( \sum_{\alpha \neq \beta} \frac{f_\alpha - f_\beta}{\epsilon_\alpha - \epsilon_\beta - i \gamma} < \beta | \phi_{T,F} | \alpha > < \beta | \mathbf{E} \cdot \mathbf{r} | \alpha > \right)
$$

B. Quantum correction to the polarisability in the diffusive regime for the different statistical ensemble

In this section, we examine the differences between the canonical and grand canonical ensembles. The canonical ensemble corresponds to the situation for which the number of electrons in the system is fixed. It is in particular the case for electrically isolated systems. The chemical potential is determined self-consistently by the condition $N = \sum_i f_0 (\epsilon_i - \mu)$, and therefore depends on the flux through the energy levels. On the other hand, in the grand canonical ensemble, the system can exchange electrons with the thermodynamic reservoir, which imposes the value of the chemical potential. A physical realization consists in connecting the ring to a large metallic pad. At finite temperature, the occupation of the energy levels is spread over an energy interval of the order of the temperature. As a result, the sensitivity to the flux dependence of the chemical potential is suppressed when $T \gg \Delta$, where $\Delta$ is the mean level spacing. Therefore the differences between CE and GCE are expected to disappear with increasing temperature and frequency. We first discuss the polarisability in the GCE.
show that one can obtain the disorder average is equivalent to the average over the whole spectrum, one can show that the flux dependence of the trace of the screened potential: 

$$\delta \Phi^2 = \frac{2e^2}{\hbar} k_0 \Phi \left( \sum_{\alpha} < \beta | \phi_{T,F} | \alpha > \right)^2$$  

(28)

Two frequency regimes can be distinguished, depending whether the frequency is smaller or larger than the relaxation energy \( \gamma \). In the limit \( \omega \ll \gamma \), which corresponds to a static electric field, the polarisability is simply equal to the flux dependence of the trace of the screened potential:

$$\delta \Phi \tilde{\alpha} = \frac{2e^2}{\hbar} k_0 \left( \sum_{\alpha} < \beta | \phi_{T,F} | \alpha > \right)^2$$  

(27)

Due to the ergodicity property in the diffusive regime, the disorder average is equivalent to the average over the number of electrons. This property simplifies the calculation. By averaging over the whole spectrum, one can show that 

$$\langle \frac{f_\alpha - f_\beta}{\epsilon_\alpha - \epsilon_\beta} \rangle = \frac{1}{\omega_p} < \frac{1}{\omega_p} $$

where < >\( \equiv \frac{1}{2\pi} \int_{-\mu + \delta \mu/2}^{\mu - \delta \mu/2} \). Considering only regimes where \( \omega \ll \Delta \) and \( \gamma \ll \Delta \), formula(29) greatly simplifies:

$$\delta \Phi = \frac{2e^2}{\hbar} k_0 \left( \sum_{\alpha,\beta} < \beta | \phi_{T,F} | \alpha > \right)^2$$

$$+ \frac{\gamma}{i\omega} \sum_{\alpha} \delta (\epsilon_\alpha - \mu) \left| < \alpha | \phi_{T,F} | \alpha > \right|^2$$  

(29)

Therefore, at low frequency, there is no mesoscopic correction to the polarisability in the GCE as shown on figure 7.

We now turn to the dynamical limit, i.e. for \( \omega \gg \gamma \). As we have seen, the relaxation term cancels and only remains the term:

$$\delta \Phi^2 = \frac{2e^2}{\hbar} k_0 \delta \Phi \left( \sum_{\alpha > \beta} < \beta | \phi_{T,F} | \alpha > \right)^2$$

(29)

In order to determine the relevant spatial frequencies of the screened potential, this latter is then decomposed in Fourier serie: 

$$\phi_{T,F} = \sum_{n=0}^{\frac{M}{2}} A_n e^{i\theta_n} + e^{i\theta_0}$$

where \( A_n \), \( \theta_n \), and \( \theta_0 \) are random variables. At low energy, the main contribution to the matrix elements of the screened potential is dominated by the terms with the smallest wavevector corresponding to \( n = 0 \), \( q_{min} = \frac{\pi}{L} \), for which:

$$\phi^2_{n,\beta} \approx \frac{1}{2\pi} \left( \frac{8\pi}{\lambda_s} \right)^3$$

(27)

where \( \lambda_s \) is the screening length, \( \Delta = \frac{hD_{min}}{2\pi} \) is the Thouless energy and \( \lambda \) the mean level spacing between energy levels.

We can then estimate the flux dependence of the matrix elements \( |\phi_{n,\beta}|^2 \) as was done in section III for the unscreened potential:

$$\delta \phi < |\phi_{\alpha\beta}|^2 > = -\delta \phi < |\phi_{\alpha\alpha}|^2 >$$

$$= < |\phi_{\alpha\alpha}|^2 >_{GUE} - < |\phi_{\alpha\alpha}|^2 >_{GOE}$$

(31)

Thus, the mesoscopic correction to the polarisability reads:

$$\frac{\delta \alpha GCE}{\alpha_0} = \left( \frac{8}{3\pi^3} \right) \frac{\lambda_s}{W} g$$

(32)

where \( \alpha_0 \) is the classical value of the polarisability, \( g = \frac{E_c}{D} \) the dimensionless conductance. This result is in good agreement with the calculation of Blanter and Mirlin [24] using super-symmetry techiques.

We took the 2d limit corresponding to a disk for which \( \alpha_0 = \epsilon_0 \frac{16}{\pi} R^2 \). For a quasi-1d ring, \( \alpha_0 = \epsilon_0 \frac{4}{\pi} W \). (see section II B). Only the numerical factor in formula (32) is modified if one consider this last value.
The magnetopolarisability increases when a magnetic flux is applied, corresponding to a positive magnetopolarisability. A noticeable result is that this effect is inversely proportional to the conductance and thus increases with the disorder in the localized regime when wave functions and eigen energies become insensitive to the Aharonov Bohm flux.

From relation 26, we deduce that at zero temperature and zero frequency the quantum correction to the polarisability reads in this ensemble:

$$\delta \alpha = \frac{2e^2}{E^2} \delta \phi \left( \sum_{\alpha=1}^{N} \sum_{\beta=N+1}^{N_e} \frac{|\langle \alpha | \phi_{T,F}(r) | \beta \rangle|^2}{\epsilon_{\beta} - \epsilon_{\alpha}} \right)$$

(33)

Where \( N_e \) is the total number of states. Note that this expression is very similar to the Gorkov Eliashberg one where matrix elements of the \( X \) operator are replaced by the matrix elements of the screened potential. Just as previously discussed in the absence of screening, the flux dependence of the energy denominators compensates exactly the one of the matrix element. However this compensation does not exist anymore, as it has been recently pointed out by Blanter et al. [24] when the frequency is of the order of the level spacing due to the disappearance of the contribution coming from the energy denominators. A similar effect is obtained when increasing the temperature as shown precisely in [25].

C. Effect of interactions in the canonical ensemble

We have shown that in the canonical ensemble in the diffusive regime, there is no mesoscopic correction to the polarisability. However, we have neglected in this calculation electron-electron interactions.

In order to investigate a possible effect of electron-electron interaction a first and simple approach consists in taking the interaction potential as a perturbation. Rather than a Coulomb potential, we use an on-site interaction \( U \delta(r-r') \). This approximation is justified for high electronic density, since the interaction between two electrons is then strongly screened by other electrons. In first order in \( U \), the variation in total energy can be expressed in terms of the local electronic density \( n(r) = \sum_{\alpha=1}^{N} |\langle \alpha | r >|^2 \) [26]:

$$\delta E_{\text{tot}} = U \int n(r)^2 dr$$

(34)

The correction to the canonical polarisability is obtained then by the formula:

$$\delta \alpha_{\text{int}} = -\frac{\partial^2 (\delta E_{\text{tot}})}{\partial^2 E}$$

(35)

On figure [1] numerical calculations of this quantity \( \delta \alpha_{\text{int}}/\alpha \) is shown for a ring for various values of disorder.

Interactions give rise to a negative magnetopolarisability. The effect is more important in the ballistic regime than in the diffusive regime, in which it does not seem to depend on disorder. It gives rise to a magnetopolarisability which is quite enhanced compared to its value in the absence of interactions. Nevertheless, the order of magnitude of this effect remains small compared to the dynamical polarisability.
VI. COMPARISON WITH THE EXPERIMENT

The magnetopolarisability of two dimensional rings has been recently measured by Deblock et al [1]. The sample consists in an array of rings fabricated by electronic lithography in an heterojunction GaAs/GaAlAs. The authors use a resonant technique in which the rings are coupled to the capacitive part of a high frequency superconducting resonator. The polarisability exhibits oscillations as a function of the flux, with a periodicity corresponding to $\phi_0/2$ through a ring. The order of magnitude as well as the sign of the effect is in good agreement with our grand canonical results. In addition, it decreases as a function of electronic density, according to the $1/g$ dependence of formula (3.2).

Taking into account the fact that the experiments were done at a frequency which is of the order of $1/3$ of the level spacing, where differences between canonical and grand canonical results are strongly reduced, this result is in good agreement with theoretical predictions [2].

VII. CONCLUSION

We have calculated the average polarisability of mesoscopic rings and squares. In the grand canonical ensemble, we found that there exists a positive magnetopolarisability for frequencies larger than the typical broadening of the energy levels $\gamma$. The relative effect $\delta\alpha/\alpha$ scales like $1/g$, where $g$ is the dimensionless conductance. In the canonical ensemble at zero frequency and zero temperature, the magnetopolarisability cancels in the diffusive, whereas in the ballistic regime a small negative effect is found. Differences between canonical and grand canonical ensemble disappear at frequencies or temperatures larger than the level spacing. Our results are in good agreement with recent experiments [1].

A. Summary of the results

In the following table, are summarized the results obtained for $\delta\alpha/\alpha$ in the ballistic and diffusive regime, for the canonical and grand canonical ensemble.

|               | Ballistic regime | Diffusive regime |
|---------------|------------------|-----------------|
| $\omega \ll \gamma$ | $\Delta \gg \omega \gg \gamma$ | $\Delta \gg \omega \gg \gamma$ |
| C             | $\frac{1}{2\gamma^2}$ | $\frac{1}{4\gamma}$ |
| GC            | 0                 | 0               |
|               | $\omega$          | $\gamma$        |
| C             | 0                 | 0               |
| GC            | $0$               | $\frac{8}{(3\pi^2)} \frac{\Delta}{W_0}$ |

APPENDIX A: DETAILS ON THE CALCULATION OF THE MESOSCOPIC CORRECTION TO THE POLARISABILITY

In this appendix, we derive in more details the expression of the mesoscopic correction to the polarisability. In the following, we will note $\Xi = \Xi_{TF} + \delta\Xi$ and $\xi = \xi_{TF} + \delta\xi$, the response to the external and internal field respectively, defined in the matrix form as:

$$\delta\rho = \chi\phi = \Xi\phi_{ext}$$ (A1)

The relation which relate the induced charge density to the response function can be written in the matrix form:

$$\delta\rho = \chi(1 - U\chi)^{-1}\phi_{ext}$$ (A2)

The effective potential is also related to the response function by (A1). The polarisability in the Thomas-Fermi approximation reads:

$$\alpha_{TF} = \frac{1}{E} Tr(\chi_{TF}\phi) = \frac{1}{E} Tr(\chi_{TF}(1 - U\chi_{TF})^{-1}\phi_{ext})$$

$$= \frac{1}{E} Tr(\Xi_{cl}\phi_{ext})$$ (A3)

with: $\Xi_{cl} = \chi_{TF}(1 - U\chi_{TF})^{-1}$. The next step is to find the quantum correction $\delta\Xi$ to the screened response function. We set by definition:

$$\Xi = \chi(1 - U\chi)^{-1} = \Xi_{cl} + \delta\Xi$$ (A4)

Assuming that quantum corrections are small compared to the Thomas-Fermi value, second order terms can be neglected:

$$(1 - U\chi_{TF})\Xi - U\delta\chi_{cl} = \chi_{TF} + \delta\chi$$

$$\delta\Xi = \delta\chi(1 - U\chi)^{-1}(1 + U\Xi_{cl})$$
FIG. 10. Total energy of the electric field calculated in the Anderson model in a ring $40 \times 5$, and for a disorder $w = 2$.

Using the fact that $1 + U \Xi_{cl} = \chi_{TF}^{-1} \Xi_{cl}$, we show that the quantum correction to the response function can be expressed:

$$\delta \Xi = \Xi_{cl} \chi_{TF}^{-1} \delta \chi \chi_{TF}^{-1} \Xi_{cl}$$

(A5)

The quantum correction to the polarisability:

$$\delta \alpha = tr(x\delta \Xi \phi_{ext})$$

can then be written:

$$\delta \alpha = tr(\phi_{TF} \delta \chi \phi_{TF})$$

where we have used that $\chi_{TF}^{-1} \Xi_{cl} \phi_{ext} = \phi_{TF}$

APPENDIX B: DISCUSSION OF THE LINEAR RESPONSE APPROXIMATION

In order to estimate the validity of the linear response, we have calculated the total energy of a ring as a function of the electric field (screening is not here considered). It can be expanded as:

$$U(E) = U_0 - d_0 E - \alpha E^2 + ...$$

The coefficient $d_0$ correspond to the spontaneous dipolar moment of the ring, resulting from fluctuations of the charge in the presence of disorder. The ensemble average of $d_0$ is zero. The energy average varies like $E^2$ at small field. As it is shown on figure FIG. 10, a deviation from the linear behaviour is hardly observed before a critical field $E_{max}$ such that:

$$eE_{max}R \sim E_F$$

(B1)

where $R$ is the radius of the ring.

This critical value is less restrictive than the criteria $[9]$ and justifies the use of the linear response up to very high fields. For instance, in a 2D electron gas, obtained in a semiconductor heterojunction GaAs/GaAlAs for which $E_F \sim 30K$, our criteria allows to use linear response up to fields such that $ER \sim 2mV$.

[1] R. Deblock, Y. Noat, H. Bouchiat, B. Reulet and D. Mailly, Phys. Rev. Lett. 84(9), 3579, (2000).
[2] Y. Noat, B. Reulet and H. Bouchiat, Europhys. Lett. 36, 701, (1996).
[3] Y. Imry "Introduction to mesoscopic physics" Eds. Oxford University Press (1997).
[4] L. P. Lévy, G. Delan, J. Dunsmuir, and H. Bouchiat, Phys. Rev. Lett. 64, 2074 (1990).
[5] V. Chandrasekhar et al., Phys. Rev. Lett. 67, 3578 (1991).
[6] D. Mailly, C. Chapelier, and A. Benoit, Phys. Rev. Lett. 70, 2020 (1993).
[7] Reulet B. and Bouchiat H., Phys. Rev. B 50, 2259 (1994).
[8] Kamenev A., Reulet B., Bouchiat H., and Gefen Y., Europhysics Letters 28, 391 (1994).
[9] L.P. Gor’kov and G.M. Eliashberg, Sov. Phys.-JETP 21, 940(1965).
[10] L. Landau et E. Lifshitz, Electrodynamics of continuous media, Mir publishers, (1969).
[11] S. Strässler, M.J. Rice and P. Wyder, Phys. Rev.B 6 2575 (1972), M.J. Rice , W.R. Schneider and S. Strässler, Phys. Rev.B 8 474 (1973).
[12] N. Trivedi and D. A. Browne, Phys. Rev. B 38 , 9581 (1988).
[13] P. Fulde and A.A. Ovchinnikov Eur.Phys. J.B 17, 623 (2000)
[14] O. Entin-Wohlman and B. Mühlhslegel, Z.Phys.B 80, 385 (1990).
[15] K. Frahm, B. Mühlhslegel and R. Németh, Z. Phys.B 78, 91 (1990).
[16] L.M. Metha "Random Matrices" Academic press (1991).
[17] N. W. Ashcroft and N. D. Mermin “Solid State physics”, Ed. W.B. Saunders company, (1976).
[18] R. Berkovits, B.L. Altshuler, Phys. Rev. B 46 .12526 (1992).
[19] R.Berkovits and B. Altshuler, Europhysics Letters. 19, 115 (1992).
[20] K.B. Efetov, Phys. Rev. Lett., 76.1908 (1996).
[21] Y. Noat, B. Reulet, H. Bouchiat and D.Maillery, Superlattices and Microstructures 23.621 (1998).
[22] W.L. McMillan Phys. Rev.B24,2739 (1981).
[23] Y. Blanter and A. Mirlin, Phys. Rev. B 57, 4566,(1998).
[24] Y. Blanter and A. Mirlin , Phys. Rev. B 63, 113315,(2001).
[25] M. Ramin, B. Reulet and H. Bouchiat, Phys. Rev. B 51, 5582 (1995).