A note on the best invariant estimation of continuous probability distributions under mean square loss

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We consider the nonparametric estimation problem of continuous probability distribution functions. For the integrated mean square error we provide the statistic corresponding to the best invariant estimator proposed by Aggarwal [9] and Ferguson [10]. The table of critical values is computed and a numerical power comparison of the statistic with the traditional Cramér-von Mises statistic is done for several representative distributions.

Keywords: Cramér-von Mises statistic; Goodness of fit criteria; Best invariant estimates; Empirical distribution function; Statistical power

In 1933, Kolmogorov [1] formally defined the empirical distribution function $F_n(x)$, and then considered how close this would be to the true distribution function $F(x)$ when it is continuous. This contribution introduced the use of $F_n(x)$ as an estimator of $F(x)$, to be followed by its use in testing a given $F(x)$. Modifying the statistics proposed earlier by Cramér [2] and von Mises [3], Smirnov [4][5][6] compared the hypothesis $F(x)$ with $F_n(x)$ by means of the quadratic loss function

$$L = \int_{-\infty}^{\infty} (F(x) - F_n(x))^2 w(F(x)) \, dF(x)$$  \hspace{1cm} (1)

where $w$ is some preassigned positive weight function. Let $x_1, ..., x_n$ be a random sample drawn from the continuous probability distribution function $F(x)$ with density function $f(x)$ and let $x_1 < x_2 < ... < x_n$ be obtained by ordering each realization $x_1, ..., x_n$. For $w = 1$, the expression (1) can be written equivalently as

$$\omega^2 = \frac{1}{12n^2} + \frac{1}{n} \sum_{i=1}^{n} \left[ F_i - \frac{2i-1}{2n} \right]^2 \hspace{1cm} (2)$$

with $F_i = F(x_i)$. The term $n\omega^2$ is commonly named as the Cramér-von Mises statistic. Smirnov [4][5][6] showed that the probability distribution of the latter is independent of $F$ for any $n$ and he obtained an asymptotic expression of its probability distribution for $n \to \infty$. For general weight functions, Anderson and Darling [7][8] presented a general method for obtaining the asymptotic distribution of (1) for $n \to \infty$.

Later on, Aggarwal [9] considered a class of invariant loss functions and obtained the best invariant estimators which are also step functions like $F_n(x)$. The canonical representation of any invariant estimator is given by

$$\hat{F}(x) = \sum_{i=0}^{n} \hat{F}_i 1_{\nu_i}(x) \hspace{1cm} (3)$$

with real valued factors $0 \leq \hat{F}_i \leq 1$, $i = 0, ..., n$, and $1_{\nu_i}(x)$ is the indicator function of the set $\nu_i$, which is defined by

$$\nu_0 = (-\infty, x_1^*) \hspace{1cm} \nu_i = [x_i^*, x_{i+1}^*) \hspace{1cm} i = 1, ..., n - 1, \hspace{1cm} (4)$$

$$\nu_n = [x_n^*, \infty).$$

Note that for the empirical distribution, each set $\nu_i$ is uniquely corresponding to the particular probability estimate $i/n$, for $i = 0, ..., n$.

For the risk function

$$R = E \int (F - \hat{F})^2 w(F) \, dF$$  \hspace{1cm} (5)

it is known [2][10] that

$$\hat{F}(x) = F_n(x)$$  \hspace{1cm} (6)

is the best invariant estimate if the weight function is $w(t) = 1/t(1-t)$, while the corresponding statistic (1) is given by

$$A^2 = -1 - \frac{1}{n} \sum_{i=1}^{n} \frac{2i-1}{n} \left[ \log F_i + \log(1 - F_{n-i+1}) \right]. \hspace{1cm} (7)$$

The expression $nA^2$ is the commonly denoted Anderson-Darling statistic. On the other hand, it is also known that the estimator

$$\hat{F}(x) = \frac{nF_n(x) + 1}{n + 2}$$  \hspace{1cm} (8)

is best invariant for the ordinary mean square error with $w(t) = 1$ in (1). Although the latter can be considered as an improvement of the Cramér-von Mises statistic, it appears that there is no explicit expression of the corresponding statistic given in literature.

Read [11] pointed out that (6) and (8) can be improved by an estimator which is stochastically smaller than the Kolmogorov statistic. However, the corresponding estimator is not a step function and is not invariant under the full group of strictly increasing transformations as are (6) and (8). Another discussion concerns the admissibility of the estimators (6) and (8). For
instance, Yu [13][14][15] has shown that (6) is admissible with respect to the weight \( w(t) = 1/t(1 - t) \) only for samples of size \( n = 1, 2 \) but inadmissible for samples of size \( n \geq 3 \). Similarly, for the estimator (5), Brown [12] has proven inadmissibility for all sample sizes \( n \geq 1 \).

Moreover, we considered the test for uniformity given in Table 3 of Stephens study [16]. When \( F(x) \) is completely specified then \( F_i \) should be uniformly distributed between 0 and 1. The power study has therefore been confined to a test of this hypothesis when \( F_i \) is drawn from alternative distributions. If the variance of the hypothesized \( F(x) \) is correct but the mean is wrong, the \( F_i \) points will tend to move toward 0 of 1; if the mean is correct but the variance is wrong, the points will move to each end, or will move to 1/2. Accordingly, in the Stephens study [16], three variants A, B and C have been defined. Case A gives random points closer to zero than expected on the hypothesis of uniformity (\( H_0 \); B gives points near 1/2; and C gives two clusters close to the boundary 0 and 1.

First, we verified the table of powers in [16] for the Cramér-von Mises statistic and subsequently computed the power of (9) for \( n = 1, \ldots, 40 \). For case A, we found that both have very similar power. In case B, the Cramér-von Mises statistic is slightly improved by (9). Actually for case C, there is a significant improvement for all sample sizes and all levels of confidence. The latter is shown in Figure 3. Here, we have qualitatively the same picture as in Figure 2. The difference of the power of both statistics is positive and reaches up to 18 percentage points.

It should be mentioned here that for case C, in the Stephens study the Anderson-Darling statistic is superior compared to the traditional Cramér and von Mises statistic. If we compare the new statistic (9) with the Anderson-Darling statistic for the same case C, then we find that the power of the former is significantly higher. This might encourage further investigations of (9).

\[ \hat{\omega}^2 = \frac{n + 8}{12(n + 2)^3} + \frac{1}{n + 2} \sum_{i=1}^{n} \left( F_i - \frac{i + \frac{1}{2}}{n + 2} \right)^2. \]
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FIG. 1: Table of critical values for the test statistic $\hat{\omega}^2$ provided in Proposition 1. The quantiles are defined by $P(\hat{\omega}^2 > Q_\alpha) = \alpha$. The asymptotic values for $n > 35$ are given by the percentage points in the last line of the table for every confidence level. The latter are to be compared with $(n + 2)\hat{\omega}^2$. The table is computed by Monte Carlo simulations of $5 \times 10^7$ respective samples.

| Sample Size $n$ | Level of Significance for $\hat{\omega}^2$-Statistic |
|-----------------|-----------------------------------------------------|
|                 | 20% | 15% | 10% | 5%  | 1%  |
| 1               | 0.0811 | 0.0880 | 0.0953 | 0.1030 | 0.1094 |
| 2               | 0.0576 | 0.0645 | 0.0743 | 0.0914 | 0.1194 |
| 3               | 0.0474 | 0.0536 | 0.0617 | 0.0752 | 0.1084 |
| 4               | 0.0396 | 0.0449 | 0.0525 | 0.0658 | 0.0954 |
| 5               | 0.0341 | 0.0389 | 0.0458 | 0.0577 | 0.0856 |
| 6               | 0.0299 | 0.0342 | 0.0404 | 0.0514 | 0.0771 |
| 7               | 0.0266 | 0.0306 | 0.0363 | 0.0463 | 0.0702 |
| 8               | 0.0240 | 0.0276 | 0.0329 | 0.0422 | 0.0643 |
| 9               | 0.0218 | 0.0252 | 0.0300 | 0.0387 | 0.0593 |
| 10              | 0.0200 | 0.0231 | 0.0277 | 0.0357 | 0.0550 |
| 11              | 0.0185 | 0.0214 | 0.0256 | 0.0331 | 0.0512 |
| 12              | 0.0172 | 0.0199 | 0.0239 | 0.0310 | 0.0480 |
| 13              | 0.0160 | 0.0186 | 0.0223 | 0.0290 | 0.0451 |
| 14              | 0.0150 | 0.0174 | 0.0210 | 0.0273 | 0.0425 |
| 15              | 0.0141 | 0.0164 | 0.0198 | 0.0258 | 0.0402 |
| 16              | 0.0134 | 0.0155 | 0.0187 | 0.0244 | 0.0382 |
| 17              | 0.0127 | 0.0147 | 0.0178 | 0.0232 | 0.0364 |
| 18              | 0.0120 | 0.0140 | 0.0169 | 0.0221 | 0.0347 |
| 19              | 0.0115 | 0.0134 | 0.0161 | 0.0211 | 0.0331 |
| 20              | 0.0109 | 0.0128 | 0.0154 | 0.0201 | 0.0317 |
| 25              | 0.0089 | 0.0104 | 0.0126 | 0.0165 | 0.0262 |
| 30              | 0.0075 | 0.0088 | 0.0107 | 0.0140 | 0.0223 |
| 35              | 0.0065 | 0.0076 | 0.0093 | 0.0122 | 0.0194 |
| over 35 $(n + 2)\hat{\omega}^2$ | 0.24124 | 0.28406 | 0.34730 | 0.46136 | 0.74346 |
FIG. 2: Difference $\Delta P$ for the power of the new statistic \((9)\) and the traditional Cramér-von Mises statistic \((2)\). The test is for the normal distribution \((H_0)\) against uniform distribution \((H_1)\). Confidence levels are 20, 15, 10, 5, 1 percentage points (from left to right). The Monte Carlo simulation is based on $10^7$ samples. The erratic behavior for very small samples is of systematical nature and not caused by insufficient sampling.

FIG. 3: The same as in Figure 2 except that the test is for uniformity (see text) corresponding to Case C of [16]. The confidence levels are 20, 15, 10, 5, 1 percentage points (from left to right). The Monte Carlo simulation is based on $10^7$ samples.