Statistics of Reduced Words in Locally Free and Braid Groups: Abstract Studies and Applications to Ballistic Growth Model

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Abstract

We study numerically and analytically the average length of reduced (primitive) words in the so-called locally free (LF$_n(d)$) and braid (B$_n$) groups. We consider the situations when the letters in the initial words are drawn either without or with correlations. In latter case, we show that the average length of the reduced word can be increased or lowered depending on the type of correlation. The ideas developed are used for analytical computation of the average number of peaks of the surface appearing in some specific ballistic growth model.

Key words: random walks, noncommutative groups, word statistics, ballistic aggregation.

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Introduction

The paper is devoted to the elaboration of a common new method of analysis of stable probability distributions in statistical systems of completely different physical nature, such as: vortices in superconductors, entangled polymer bunches and open surfaces of growing media. Our paper pursues two main goals:

(i) To construct a mathematical apparatus to adequately describe the topological properties of the above physical systems;

(ii) To apply evolved methods to the detailed analysis of specific physical phenomena.

The development of mathematical methods implies the construction of the statistical theory of random walks on nonabelian groups (Refs.[1]–[10]); while the application of elaborated methods in physics is aimed to answer the following question ([11]): how does the change in the topological state of the system affects its physical properties?
Although the general concepts of the noncommutative probability theory have been well elaborated in the field-theoretic context, their application in the related areas of mathematics and physics, such as, for instance, statistical physics of chain-like objects is highly limited. This state of affairs can be accounted for by two facts: (a) there is a communication problem, i.e. the languages used by specialists in topological field theory and probability theory are completely different at first glance; (b) physical systems give no evidence how these ideas are reflected in simple geometrical examples.

So, the present paper is mainly concerned with the probabilistic methods which allow us to solve the basic problems dealing with the limit distributions of random walks on some simplest noncommutative groups. To be more specific, our main goal of the work is as follows: we consider analytically and numerically the limit behavior of the Markov chains where the states are randomly taken from some noncommutative finite discrete group. In particular, we restrict ourselves with the so-called braid \( B_{n+1} \) and locally free \( LF_{n+1} \) groups—see the definitions in the section 1.2. The preliminary results concerning the words statistics in the locally free groups appeared in recent works [4, 10].

The reason of our investigations is forced by the following real physical motivations:

(a) The nematic-type ordering in bunches of entangled polymers as well as the consideration of thermodynamic properties of uncrossible vortex lines immediately turn us to studying of a statistics of chain-like objects with nonabelian topology. The main reason of investigation concerns the construction of the basis of the mean–field–like theory of fluctuating entangled chains in 1+1 dimensions on the basis of our knowledges about statistics of Markov chains on braid and locally free groups. A forthcoming publication will be devoted to examination of the influence of topological constraints in the standard nematic–like phase transition in the bunch of ”braided polymers” [11].

(b) It has been realized that the ballistic–type growth of some amount of deposit in a box and the investigation of the shape of the surface can be easily translated into the language of random walks over the elements of some noncommutative group. The corresponding model is considered in the section 4 of the present work.
1 Basic Definitions

We begin with the investigation of the probabilistic properties of Markov chains on simplest noncommutative groups. In the most general way the problem can be formulated as follows (see also [4]).

Take a discrete group $G_{n+1}$ constructed by the finite number of generators $\{g_1, \ldots, g_n\}$. Any arbitrary sequence of generators we call the initial word. The length, $N$, of this word is the total number of used generators ("letters"), whereas the length of the reduced (or primitive) word, $\mu$, is the shortest noncontractible length of the word which remains after applying of all possible group relations.

Later on we mainly use the rescaled variables $N' \equiv N/n$ and $\mu' \equiv \mu/n$ instead of $N$ and $\mu$ and consider the situation $n \gg 1$ neglecting the "edge effects".

The most attention in the following is paid to the computation of the mean length, $\langle \mu'(N') \rangle$, averaged over various distributions of initial words belonging to the group $G_{n+1}$ ($G_{n+1}$ is either "locally free" or braid group).

1.1 Random Walks Over Group Elements

Take the group $G_{n+1}$. Let $p$ be some distribution on the set $\{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$. For convenience we call $h_j \equiv g_i$ for $j = i$ and $h_j \equiv g_i^{-1}$ for $j = i + n$. We construct the (right-hand) random walk (the random word) on $G_{n+1}$ with a transition measure, $p$, i.e. we add with the probability $p$ the element $h_{\alpha_1}h_{\alpha_2} \ldots h_{\alpha_N}$ from the right-hand side$^1$.

The random word $W$ formed by $N$ letters taken with the probability distribution $p$ from the set $\{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$ is called the initial word of the length $N$ on the group $G_{n+1}$.

We distinguish below between the following three situations:

1. DRAWING WORDS WITHOUT ANY CORRELATIONS ("STANDARD CASE").

The probability distribution $p$ is uniform, i.e. $p = \frac{1}{2n}$ on the set $\{g_1, \ldots, g_n, g_1^{-1}, \ldots, g_n^{-1}\}$.

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$^1$Analogously we can construct the left-hand side random walk on the group $G_{n+1}$.  

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Besides this standard case, we consider also two extreme situations of words construction, hereafter referred as "weak (strong) correlations".

2. Drawing words with weak correlations (regime "A").

Suppose that in the initial word, $W_N$, we have for the last letter $h_{\alpha N} = g_k$ or $g_k^{-1}$. Then we add the next $(N+1)$th letter $h_{\alpha N+1}$ with the following probabilities:

$$h_{\alpha N+1} = \begin{cases} g_k^{\pm 1} & \text{with the probability } q_A \\ \text{any other letter} & \text{with the probability } p_A \end{cases}$$  (1)

The normalisation reads:

$$2q_A + 2(n-1)p_A = 1$$  (2)

3. Drawing words with strong correlations (regime "B").

Suppose again that in the initial word, $W_N$, we have for the last letter $h_{\alpha N} = g_k$ or $g_k^{-1}$. Then we add the next $(N+1)$th letter $h_{\alpha N+1}$ with the following probabilities:

$$h_{\alpha N+1} = \begin{cases} g_k^{\pm 1} & \text{with the probability } q_B \\ \text{any other letter} & \text{with the probability } p_B \end{cases}$$  (3)

The normalisation in this case reads:

$$4q_B + 2(n-2)p_B = 1$$  (4)

In particular, we show below that in the case "A" the length of the reduced (primitive) word decreases when $q_A$ is increased, while in the case "B" the length of the reduced word increases when $q_B$ is increased.

The absence of correlations, in both cases "A" and "B", means setting $q_{A,B} = p_{A,B} = \frac{1}{2n}$. Thus, in the limit $n \gg 1$, the "standard case" is recovered, formally, by setting $q_{A,B} = 0$ in the equations.

The investigation of such correlations is necessary in view of future physical applications [11], especially when we are dealing with the polymers entanglements. Indeed, if we think of $G_{n+1}$ as of a braid group, the weak ("A") and strong ("B") correlation regimes will correspond, respectively, to the weak and strong "entanglement regimes".
1.2 Braid and ”Locally Free” Groups

We are aimed to study the asymptotics of the limit distributions of Markov chains on the braid group $B_{n+1}$. For the case $n = 2$ the problem has been solved in [4], where the limit probability distribution as well as the conditional limit probability distribution of ”brownian bridges” on the group $B_3$ has been derived. For $n > 2$ this problem is unsolved yet. However we can extract some reliable estimations for the limit behavior of Markov chains on $B_{n+1}$ considering the random walks on so-called ”locally free groups” [2, 4, 10].

**Braid Group.** The braid group $B_{n+1}$ of $n+1$ strings has $n$ generators $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ with the following relations:

\[
\begin{align*}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\
(1 \leq i < n) \\
\sigma_i\sigma_j &= \sigma_j\sigma_i \\
(|i - j| \geq 2) \\
\sigma_i\sigma_i^{-1} &= \sigma_i^{-1}\sigma_i = \hat{e}
\end{align*}
\]

Let us mention that:

- The word written in terms of ”letters”—generators from the set $\{\sigma_1, \ldots, \sigma_n, \sigma_1^{-1}, \ldots, \sigma_n^{-1}\}$ gives a particular braid. Schematically, the generators $\sigma_i$ and $\sigma_i^{-1}$ could be represented as follows:

\[
\begin{array}{cccccccc}
& & & \ldots & \times & \ldots & & \\
1 & 2 & \cdots & i & i+1 & \cdots & n & n+1 \\
\end{array}
\]

\[
= \sigma_i
\]

\[
\begin{array}{cccccccc}
& & & \ldots & \times & \ldots & & \\
1 & 2 & \cdots & i & i+1 & \cdots & n & n+1 \\
\end{array}
\]

\[
= \sigma_i^{-1}
\]

- The *length*, $N$, of the braid is the total number of used letters while the *minimal irreducible length*, $\mu$, is the shortest noncontractible length of a particular braid remaining after all possible group relations Eq.(5) are applied. Diagramatically, the braid can be
represented as a set of crossed strings going from the top to the bottom after "gluing" the braid generators.

- The closed braid appears after gluing the "upper" and the "lower" free ends of the braid on the cylinder.

- Any braid corresponds to some knot or link. So, there is a principal possibility to use the braid group representation for the construction of topological invariants of knots and links, but the correspondence of braids and knots is not mutually single valued and each knot or link can be represented by an infinite series of different braids.

**Locally Free Group.** The group $\mathcal{LF}_{n+1}(d)$ is called *locally free* if the generators, $\{\sigma_1, \ldots, \sigma_n\}$ obey the following commutation relations:

(a) Each pair $(\sigma_j, \sigma_k)$ generates the free subgroup of the group $\mathcal{LF}_{n+1}$ if $|j - k| < d$;

(b) $\sigma_j \sigma_k = \sigma_k \sigma_j$ for $|j - k| \geq d$

We will be concerned mostly with the case $d = 2$ for which we define $\mathcal{LF}_{n+1}(2) \equiv \mathcal{LF}_{n+1}$.

The graphical representation of generators $\sigma_i$ and $\sigma_i^{-1}$ is rather similar to that of braid group:

\[
\begin{array}{cccccccc}
& & & \cdots & & & \cdots & \\
& & & 1 & 2 & \cdots & i & i+1 & \cdots & n & n+1 \\
\sigma_i & = & & & \cdots & \overset{i}{\bigcup} & \cdots & \\
& & & \cdots & & & \cdots &
\end{array}
\]

\[
\begin{array}{cccccccc}
& & & \cdots & & & \cdots & \\
& & & 1 & 2 & \cdots & i & i+1 & \cdots & n & n+1 \\
\sigma_i^{-1} & = & & & \cdots & \overset{i}{\bigcup} & \cdots & \\
& & & \cdots & & & \cdots &
\end{array}
\]

It is easy to understand that the following geometrical identity is valid:
hence, it is unnecessary to distinguish between ”left” and ”right” operators $\sigma_i$.

It can be seen that the only difference between the braid and locally free groups consists in elimination of the Yang-Baxter relations (first line in Eq.(5)).

2 Random Walks without Correlations on Locally Free and Braid groups

It has been shown in papers [1], [6]–[8] that for the free group (i.e. for the group without any commutation relations among generators) the problem of the limit distribution of Markov chains can be mapped to the investigation of statistics of random walks on a simply connected tree. In the case of locally free groups or braid groups the more complicated structure does not allow us to use this simple geometrical image directly.

2.1 The Locally Free Group $\mathcal{LF}_{n+1}(2)$

Let us begin with the following example:

Example 1.

Suppose that the $N$-letter initial word leads to the following reduced word:

$$\sigma_1^{-1} \sigma_2 \sigma_1 \sigma_4 \sigma_2^{-1} \sigma_7 \sigma_3 \sigma_5^{-1} \sigma_3 \sigma_8^{-1}$$

Now, if we add randomly a new letter from the right hand side, it is easy to see that only $\sigma_3, \sigma_5^{-1}$ or $\sigma_8^{-1}$ can be reduced (for instance, $\sigma_7$ cannot be reduced even if, by chance, we add $\sigma_7^{-1}$ because this generator cannot pass through $\sigma_8^{-1}, \ldots$).
**Definition 1** The set of letters which we can reduce in the given primitive word by adding one extra letter from the right-hand side we call the **set of reducible letters**, $I$.

The number of letters belonging to $I$ we denote as $\eta$.

In the above example $I = \{\sigma_3, \sigma_5^-1, \sigma_8^-1\}$ and $\eta = 3$. Generally speaking, $\eta' \equiv \eta/n$ is a random variable, the probability distribution of which *a priori* depends both on $N' \equiv N/n$ and $n$.

It is noteworthy to mention the following basic properties of the set $I$:

(i) If $\sigma_i^{\pm1}$ belongs to $I$ then $\sigma_i^{\mp1}$ does not belong to $I$;

(ii) If $\sigma_i^{\pm1}$ belongs to $I$ then $\sigma_{i+1}^{\pm1}, \sigma_{i-1}^{\mp1}$ and $\sigma_{i-1}^{\pm1}$ do not belong to $I$, i.e. all the elements of $I$ must commute.

On the basis of (i) and (ii) we can easily deduce that $0 \leq \eta \leq n/2$ ($\eta = 0$ corresponds to a completely reduced word, i.e. $\mu = 0$).

The set $I$ allows the following very useful geometrical interpretation. Take $n$ boxes (labelled as $k = 1, \ldots, n$) as displayed below:

```
  i 2 1 2 ...
 k 1 2 3 ...
```

The box $k$ is empty except if $\sigma_k^{\pm1}$ belongs to $I$. In the given example only boxes 3, 5, 8 are occupied. From the properties (i)–(ii) of the set $I$ we deduce that two neighboring boxes cannot be occupied.

Generally, $I$ is described by occupied boxes separated by a sequence of $i$ ($i \geq 1$) empty boxes. Let $n_i$ being the number of such sequences of length $i$. Neglecting the edge effects (i.e. for $n \gg 1$), we get the following rules:

\[
\sum_{i \geq 1} n_i = \eta \tag{6}
\]
and

\[ \sum_{i \geq 1} i n_i = n - \eta \quad (7) \]

Consider now the evolutions of the reduced word (length \( \mu \)) and of the set \( I \) (length \( \eta \)) when we add randomly a letter from the right hand side: \( N \to N+1 \) (i.e. \( N' \to N'+1/n \)). (Apparently the evolutions of the reduced word and of the set \( I \) are correlated). Two possibilities can occur:

\[
\begin{align*}
\Delta \mu' &= +\frac{1}{n} \quad \text{for the "increase" process (} \mu \to \mu + 1) \\
\Delta \mu' &= -\frac{1}{n} \quad \text{for the "decrease" process (} \mu \to \mu - 1)
\end{align*}
\]

where \( \Delta \mu' \) stands for the increment \( \mu' (N'+1/n) - \mu' (N') \).

We consider the "increase" and "decrease" processes separately.

1. The "increase" process.

It is easy to see that the added letter will necessarily belong to the new set \( I \). However, it does not mean at all that, in this case, we automatically have \( \eta' \to \eta' + 1/n \). Actually, \( \eta' \) can stay unchanged or can be changed by \( \pm 1/n \).

The latter point becomes more clear if we come back to Example 1 where \( I = \{ \sigma_3, \sigma_5^{-1}, \sigma_8^{-1} \} \). We have the following choice:

- If we add \( \sigma_3 \) (or \( \sigma_5^{-1} \) or \( \sigma_8^{-1} \)), then the set \( I \) (and, hence, \( \eta' \)) remains to be unchanged.

- If we add \( \sigma_6 \), then \( \eta' \) is still unchanged: \( I \) becomes the new set \( \{ \sigma_3, \sigma_6, \sigma_8^{-1} \} \), the letter \( \sigma_6 \) has replaced \( \sigma_5^{-1} \) in the set \( I \). The same occurs for \( \eta' \) if, instead of \( \sigma_6 \), we add \( \sigma_6^{-1} \) or \( \sigma_7^{\pm 1} \), and so on...

- If we add \( \sigma_4 \), then \( I \) becomes the new set \( \{ \sigma_4, \sigma_8^{-1} \} \) (\( \sigma_4 \) erases \( \sigma_3 \) and \( \sigma_5^{-1} \) from \( I \)) and, consequently, \( \eta' \to \eta' - 1/n \) (same change occurs for \( \eta' \) if we add \( \sigma_4^{-1} \)).
• If we add $\sigma_{10}$, then $I$ becomes $\{\sigma_3, \sigma_5^{-1}, \sigma_8^{-1}, \sigma_{10}\}$ and $\eta' \to \eta' + 1/n$ (the same is happened if we add $\sigma_1^{\pm 1}$ or $\sigma_{10}^{-1}$, and so on...).

These considerations can be generalized and careful inspection leads to the following rules for the increasing process ($\Delta N' = +1/n$, $\Delta \mu' = +1/n$):

\[
\begin{cases}
\Delta \eta' = 0 & \text{occurs with probability } \Pi_0 = \frac{1}{2n} \left( \eta + 4 \sum_{i \geq 2} n_i \right) \\
\Delta \eta' = -\frac{1}{n} & \text{occurs with probability } \Pi_- = \frac{1}{2n} 2n_1 \\
\Delta \eta' = +\frac{1}{n} & \text{occurs with probability } \Pi_+ = \frac{1}{2n} 2 \sum_{i \geq 3} n_i(i - 2)
\end{cases}
\] (8)

With the help of Eqs. (8) we get as expected, for the total probability, $\Pi_1$, of the increasing process:

\[
\Pi_1 = \Pi_0 + \Pi_- + \Pi_+ = 1 - \frac{\eta'}{2}
\] (9)

From the inequality $0 \leq \eta' \leq 1/2$ derived above immediately follows that $3/4 \leq \Pi_1 \leq 1$. For the corresponding average change of $\eta'$ we have:

\[
\frac{\langle \Delta_1 \eta' \rangle}{\Delta N'} = -\frac{n_1}{n} + \frac{1}{n} \sum_{i \geq 3} n_i(i - 2) = \frac{1}{n} \sum_{i \geq 1} n_i(i - 2) = \frac{n - 3\eta'}{n}
\]

so, we arrive at the following equation

\[
\langle \Delta_1 \eta' \rangle = (1 - 3 \langle \eta' \rangle) \Delta N'
\] (10)

where $\langle \ldots \rangle$ stands for averaging over the set of all initial words with $N'$ fixed.

2. The “decrease” process.

Now we compute the change $\langle \Delta_2 \eta' \rangle$ for the reducing process, i.e. when $N' \to N'+1/n$ and $\mu' \to \mu' - 1/n$. It occurs with the probability

\[
\Pi_2 \equiv 1 - \Pi_1 = \frac{\eta'}{2}
\]

In this operation, a letter of the set $I$ is erased and, again, we have $\Delta \eta' = 0$ or $\pm 1/n$. Recall that all the elements of $I$ commute. So, the erased letter (here $\sigma_k$) can always been considered as the last one:
From this point of view, the decrease process \((N \rightarrow N + 1, \mu \rightarrow \mu - 1)\) is rigorously the inverse of the increase one \((N - 1 \rightarrow N, \mu - 1 \rightarrow \mu)\). Thus, weighting each process with its actual probability, we get the equation:

\[
\langle \Delta_2 \eta' \rangle = -\langle \Delta_1 \eta' \frac{\Pi_2}{\Pi_1} \rangle
\] (11)

where corrections of order \(1/n\) are neglected.

Collecting the "increase" and "decrease" processes together, we obtain:

\[
\frac{\langle \Delta \eta' \rangle}{\Delta N'} = \left\langle (1 - 3\eta') \left(1 - \frac{\Pi_2}{\Pi_1}\right) \right\rangle
\]

and

\[
\frac{\langle \Delta \mu' \rangle}{\Delta N'} = \Pi_1 - \Pi_2
\]

We arrive in the limit \(N \gg 1\); \(\mu \gg 1\) at the following differential equations:

\[
\frac{d \langle \eta' \rangle}{d N'} = \left\langle (1 - 3\eta') \frac{1 - \eta'}{1 - \eta'/2} \right\rangle
\] (12)

and

\[
\frac{d \langle \mu' \rangle}{d N'} = 1 - \langle \eta' \rangle
\] (13)

It should be stressed that Eq.(13) together with the inequality \(\eta' \leq 1/2\) imply that \(\langle \mu \rangle / N \geq 1/2\).

We can get rid of the brackets in Eqs.(12)–(13) when \(n \gg 1\). To show that, let us compute the probability distribution of \(\eta'\), \(P(N', \eta')\). The function \(P(N', \eta')\) satisfies the following recursion relation

\[
\begin{cases}
    P(N' + 1/n, \eta') = P_0 P(N', \eta') + P_1 P(N', \eta' - 1/n) + P_2 P(N', \eta' + 1/n) \\
    P(0, \eta') = \delta(\eta')
\end{cases}
\] (14)

where

\[
P_0 = \Pi_0 \left(1 + \frac{\Pi_2}{\Pi_1}\right); \quad P_1 = \Pi_+ + \Pi_- \frac{\Pi_2}{\Pi_1}; \quad P_2 = \Pi_- + \Pi_+ \frac{\Pi_2}{\Pi_1}
\]
are transition rates. Expanding Eq. (14) to the lowest order in $1/n$ we get

$$\frac{\partial P}{\partial N'} = (P_2 - P_1) \frac{\partial P}{\partial \eta'} + \frac{1}{2n} (P_1 + P_2) \frac{\partial^2 P}{\partial \eta'^2} + O\left(\frac{1}{n}\right)$$  \hspace{1cm} (15)$$

When $n \to \infty$, the diffusion term becomes negligible and the equation becomes deterministic. Then, the distribution function $P$ acquires zero’s width, hence $\eta'$ is peaked at its average value. The same would be true for $\mu'$ but not for variables $\eta$ and $\mu$ (for which a non-vanishing width is expected).

From now on, as far as only $\eta'$ and $\mu'$ are concerned, we systematically omit the brackets. Solving Eq. (12) we get:

$$\frac{1 - \eta'}{(1 - 3\eta')^{5/3}} = e^{4N'}$$  \hspace{1cm} (16)$$

Using Eq. (13), we get $\mu'$ as a function of $N'$.

The comparison with the numerical simulations is displayed in the upper part of Fig. 1 (the full curve: Eqs. (13), (16); the points: simulations with $n = 100$).

We observe at small $N'$ that $\langle \mu \rangle \simeq N$, i.e. practically no reduction occurs because the words are too short compared to the set of available letters and we have only little chance to draw, in the same word, a given generator and its inverse. On the other hand, taking the limit $N' \gg 1$ in Eqs. (13)–(16) we arrive at 10:

$$\eta' = \frac{1}{3} \quad \text{and} \quad \frac{\langle \mu \rangle}{N} = \frac{2}{3}$$  \hspace{1cm} (17)$$

2.2 The Locally Free Group $\mathcal{LF}_{n+1}(d)$ for $d \geq 2$

The ideas developed above can be extended to the general case—the group $\mathcal{LF}_{n+1}(d)$ with $d \geq 2$. It is just worthwhile to mention the following simple fact. The generator $\sigma_k$ erases all $\sigma_j$’s with $j = k - (d - 1), \ldots, k - 1, k + 1, \ldots, k + (d - 1)$ from the set $I$. In other words, $\sigma_k$ ”screens” all the generators in a zone of extension $2(d - 1)$ around itself. This point of view is especially useful when we treat the correlations ("B").
In the case of the group $\mathcal{LF}_{n+1}(d)$ Eqs.(8) become:

$$\begin{cases}
\Delta \eta' = 0 \text{ occurs with } \Pi_0 = \frac{\eta'}{2} + \sum_{i=d-1}^{2(d-1)} \frac{4n_i(i - (d - 1))}{2n} + \sum_{i>2(d-1)} \frac{4n_i(d - 1)}{2n} \\
\Delta \eta' = +\frac{1}{n} \text{ occurs with } \Pi_+ = \sum_{i>2(d-1)} \frac{2n_i(i - 2(d - 1))}{2n} \\
\Delta \eta' = -\frac{1}{n} \text{ occurs with } \Pi_- = \sum_{i=d-1}^{2(d-1)} \frac{2n_i(2(d - 1) - i)}{2n}
\end{cases}
$$

while Eq.(13) remains to be unchanged.

The solution of Eq.(12) reads now

$$\frac{1 - \eta'}{(1 - (2d - 1)\eta')^{\frac{d-1}{2d-1}}} = e^{4(d-1)N'}$$

(compare to Eq.(16)). Asymptotically we get:

$$\eta' = \frac{1}{2d - 1} \text{ and } \langle \mu \rangle = \frac{2d - 2}{2d - 1} \quad (20)$$

It is easy to check that Eq.(17) is recovered for $d = 2$ (see also [10]).

### 2.3 The Braid Group $B_{n+1}$

Comparing the groups $B_{n+1}$ and $\mathcal{LF}_{n+1}(2)$ we could see that $\eta'$ in Eq.(13) has to be replaced by some $\eta''$ ($> \eta'$) in order to take into account the additional braiding relations Eq.(5).

For each $\sigma_k$ belonging to $I$, we can get additionnal reduction if and only if we can build at the end of the reduced word a sequence of letters like $\sigma_k\sigma_{k+1}\sigma_k$. Then the braiding relation Eq.(8) implies that $\sigma_{k+1}$ becomes reducible.

Let us compute the probability $Q$ of the event to find such sequence. Suppose that generators $\sigma_k$ and $\sigma_{k+1}$ emerge elsewhere in the reduced word. We have to push them to the right until they meet the generator $\sigma_k$ already belonging to the set $I$—see the figure below.
We proceed in two subsequent steps:

1. We push the generator $\sigma_k$ located at the point $A$ until it meets the generator $\sigma_{k+1}$ located at the point $B$. The local transition probability of such process is $p_1$, where

$$p_1 = \frac{2n - 6}{2n - 1} \quad \text{(21)}$$

It easy to understand that $p_1$ is the probability to commute a given generator inside the reduced word with its right neighbour.

2. Completing the first process we push the pair $\sigma_k\sigma_{k+1}$ until it meets the generator $\sigma_k$ located at the point $C$. The local transition probability of such process is $(p_1p_2)$, where

$$p_2 = \frac{2n - 2}{2n - 1} \quad \text{(22)}$$

$p_2$ is the conditional probability to commute $\sigma_k$ under the condition that $\sigma_{k+1}$ commutes as well.

We arrive finally at the equation for $Q(\mu')$:

$$Q(\mu') = \left(\frac{1}{2n}\right)^2 \left( \sum_{m,m'\leq \mu} p_1^m (p_1 p_2)^{m'} \right) \quad \text{(23)}$$

The answer for $Q(\mu')$ in the limit $n \gg 1$ reads:

$$Q(\mu') = \frac{1}{30} - \frac{1}{5} e^{-\frac{2}{3} \mu'} + \frac{1}{6} e^{-3\mu'} \quad \text{(24)}$$

Moreover, for given $\sigma_k \in I$, not only the sequence $\sigma_k\sigma_{k+1}\sigma_k$ can be used for braiding relations but also 5 other sequences (namely $\sigma_k^{-1}\sigma_{k+1}\sigma_k$, $\sigma_k^{\pm 1}\sigma_{k-1}\sigma_k$, $\sigma_k^{-1}\sigma_{k+1}^{-1}\sigma_k$, $\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_k$).

Thus, in Eq.(13), $\eta'$ has to be replaced by:

$$\eta''(\mu') = \eta'(\mu')(1 + 6Q(\mu')) \quad \text{(25)}$$
while Eq. (16) remains unchanged. The results are shown in the lowest part of Fig. 1 (the points are the numerical simulations for the $B_{n+1}$, $n = 100$; the full curve corresponds to the Eqs. (25), (16)).

At the end of this section let us mention two important facts:

- For small $N'$ (typically, for $N' < 1$ i.e. $N < n$), we get
  \[
  \left( \frac{\langle \mu \rangle}{N} \right)_{\mathcal{LF}_{n+1}} \approx \left( \frac{\langle \mu \rangle}{N} \right)_{B_{n+1}}
  \]
  i.e. the ”braiding” plays practically no role because the words are too short to produce sequences such as $\sigma_k \sigma_{k+1} \sigma_k$.

- In the asymptotic regime $N' \to \infty$ and $\mu' \to \infty$ we get
  \[
  \eta' = \frac{1}{3}, \quad \eta'' = 0.4, \quad \frac{\langle \mu \rangle}{N} = 0.6
  \]
  (27)
  We can now appreciate the impact of the braiding relations. The reductions are increased by about 20% (from 1/3 to 0.4)—see Eq. (27) and, simultaneously, $\langle \mu \rangle / N$ is decreased by about 10%. So, in that regime, the groups $\mathcal{LF}_{n+1}$ and $B_{n+1}$ do not coincide (eventhough they give the same order of magnitude for the quantity $\langle \mu \rangle / N$).
  This is consistent with our conjecture expressed in the paper [10] where we introduced the concept of the locally free group with ”errors” in commutation relations, $\mathcal{LF}_{n+1}^{err}$. Let us remind that in [10] the coincidence between the limit behavior of the irreducible words in $\mathcal{LF}_{n+1}^{err}$ and $B_{n+1}$ has been reached if we allowed 20% of errors in commutation relations.

3 Random Walks with Correlations on Locally Free and Braid Groups

We come back to the locally free group $\mathcal{LF}_{n+1}(2)$ and suppose, now, that the letters are drawn according to the rules described in the section 1.1.

The weak correlations (the case ”A”).

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The effect of correlations "A" amounts to a change of Eqs. (12)–(13) into:

\[ \frac{d\eta'}{dN'} = 2\alpha (1 - 3\eta') \frac{\beta - \alpha\eta'}{1 + \beta - \alpha\eta'} \]  
(28)

and

\[ \frac{d\mu'}{dN'} = \beta - \alpha\eta' \]  
(29)

where

\[ \alpha = 1 - 2q_A; \quad \beta = \sqrt{1 - 2q_A} \frac{1}{1 + 2q_A} \]  
(30)

Let us explain now how these equations come from.

Using the same line of thought as in the section 2.1 and taking into account the normalisation condition (2), we get:

\[ \frac{d\eta'}{dN'} = (1 - 3\eta')(1 - 2q_A) \left(1 - \frac{\Pi_2}{\Pi_1}\right) \]  
(31)

and

\[ \frac{d\mu'}{dN'} = \Pi_1 - \Pi_2 \]  
(32)

However, the probabilities, \( \Pi_1 \) and \( \Pi_2 \) corresponding to the "increase" and "decrease" processes \( \Pi_1 + \Pi_2 = 1 \) must be computed again to take into account the correlations. This is done as follows.

1. The case "A" means that we mainly take care of the situation when the next added letter is the same (with the probability \( q_A \)) as the previous added one.

Suppose that, at some time, we draw the letter \( \sigma_k \). At the next time step we can add \( \sigma_k \) or \( \sigma_k^{-1} \) with the probability \( q_A \) \( q_A \) independent with each others.

\[ W_{1D}^A : \{\sigma_k \sigma_k^{-1} \sigma_k \sigma_k^{-1} \sigma_k^{-1} \ldots\} \]

with a mean "lifetime" \( \tau_A = 1 + 2q_A + (2q_A)^2 + \ldots = 1/(1 - 2q_A) \). This implies the rescaling of the "time":

\[ N' \rightarrow (1 - 2q_A)N' \]

as it can be seen in Eq.(31).

\footnote{We draw \( \sigma_k \) with the probability \( q_A \) and \( \sigma_k^{-1} \) with the probability \( q_A \) independent with each others.}
Another contribution to Eqs.(31)–(32) is connected with the "mean length", \( \langle l \rangle \), of the random chain \( W_{1D}^A \) discussed just above. To clarify what is \( \langle l \rangle \) let us consider the following example:

| number of steps | 1D chain | the length \( l \) | probability                  |
|-----------------|----------|---------------------|------------------------------|
| 1               | \( \sigma_k \) | 1                   | \( (1 - 2q_A) \)             |
| 2               | \( \sigma_k \sigma_k^{-1} \) | 0                   | \( q_A(1 - 2q_A) \)          |
| 2               | \( \sigma_k \sigma_k \) | 2                   | \( q_A(1 - 2q_A) \)          |
| 3               | \( \sigma_k \sigma_k^{-1} \sigma_k = \sigma_k \) | 1                   | \( q_A^3 (1 - 2q_A) \)       |
| 3               | \( \sigma_k \sigma_k^{-1} \sigma_k^{-1} = \sigma_k^{-1} \) | 1                   | \( q_A^3 (1 - 2q_A) \)       |
| 3               | \( \sigma_k \sigma_k \sigma_k = \sigma_k^3 \) | 3                   | \( q_A^3 (1 - 2q_A) \)       |
| 3               | \( \sigma_k \sigma_k \sigma_k^{-1} = \sigma_k \) | 1                   | \( q_A^3 (1 - 2q_A) \)       |
| \( \ldots \)    | \( \ldots \) | \( \ldots \)        | \( \ldots \)                |

The calculation of \( \langle l \rangle \) for given \( q_A \) and infinite long random chain \( W_{1D}^A \) leads to the equation

\[
\langle l \rangle = \sum_{k=0}^{\infty} a_k q_A^k (1 - 2q_A)
\]

where \( a_k \) obeys the recursion relations

\[
a_{2k+1} = 2 a_{2k} \\
a_{2k} - 2 a_{2k-1} = C_{2k}^k
\]

The final answer for \( \langle l \rangle \) is:

\[
\langle l \rangle = \sum_{k=0}^{\infty} C_{2k}^k q_A^{2k} = \frac{1}{\sqrt{1 - 4q_A^2}}
\]

\( \langle l \rangle \) is produced during the “lifetime” \( \tau_A \). Thus, Eq.(13) should be rewritten as:

\[
\frac{1}{1 - 2q_A} \frac{d\mu'}{dN'} = \langle l \rangle - \eta'
\]

This equation enables us to extract the expressions for \( \Pi_1 \) (and, hence, for \( \Pi_2 = 1 - \Pi_1 \))—see Eq.(32) and substitute these values in Eq.(31). Now Eqs.(28)–(29) follow directly from Eqs.(31)–(32).
Let us stress that naive "time rescaling" in Eq.(13), i.e. the equation
\[
\frac{1}{1 - 2q_A} \frac{d\mu'}{dN'} = 1 - \eta'
\]
(i.e., when \(\langle l \rangle = 1\)) leads to a wrong result.

The comparison with numerical simulations is shown in Fig.2. For the group \(B_{n+1}\) — the lower part of the Fig.2—we used the same recipe as in the section 2.3 to get the analytic results (full curve). In the limit \(N' \to \infty\) our computations get the answer (for the group \(\mathcal{LF}_{n+1}\)):
\[
\eta'_{\infty} = \frac{1}{3}, \quad \frac{\langle \mu \rangle}{N}_{\infty} = \sqrt{\frac{1 - 2q_A}{1 + 2q_A}} - \frac{1}{3}(1 - 2q_A) \tag{35}
\]

The value \(\frac{\langle \mu \rangle}{N}_{\infty}\) is a monotonically decreasing function of \(q_A\) when \(q_A\) increases from 0 \((\langle \mu \rangle = \frac{2}{3})\) till its maximal value \(1/2\) \((\langle \mu \rangle = 0)\). Clearly, the correlations "A" enhance the reductions.

Let us pay attention to the fact that
\[
\frac{\langle \mu \rangle}{N} \to \frac{\langle \mu \rangle}{N'} \to 0 \sqrt{\frac{1 - 2q_A}{1 + 2q_A}} \leq 1 \quad \text{if} \ q_A \neq 0 \tag{36}
\]
where \(N' \to 0\) means that \(n \gg N \gg 1\). This fact is clearly depicted in Fig.3. One can see that the agreement between Eq.(36) (full curve) and numerical simulations for the groups \(\mathcal{LF}_{n+1}\) and \(B_{n+1}\) (points) is perfect.

**The strong correlations (the case "B").**

The correlations "B" are dealing with the situation when the next added letter to the initial word is \(\sigma_{k+1}^{\pm}\) (with the probability \(q_B\)) if the previous added one is \(\sigma_k^{\pm}\)—see the section 1.1 for the definition and the normalisation of probabilities.

In the spirit of discussion of the case "A" we can describe our process of successive letters drawing as developing of the 1D Markov chain
\[
W_{1D}^B : \{\sigma_k \sigma_{k+1}^{-1} \sigma_k^{-1} \sigma_{k-1}^{-1} \sigma_{k-2}^{-1} \sigma_{k-3}^{-1} \sigma_{k-2} \ldots\}
\]

The corresponding "lifetime", \(\tau_B\), is \(\tau_B = \frac{1}{(1 - 4q_B)}\).
The chain \( W_{1D}^B \) can be viewed as a 1D random walk in the "label space"

\[
k \rightarrow (k + 1) \rightarrow k \rightarrow (k - 1) \rightarrow (k - 2) \rightarrow (k - 3) \rightarrow (k - 2) \ldots
\]

with an extension around \( k \) of order of \( 2 \sqrt{r_B} = \frac{2}{\sqrt{1 - 4q_B}} \).

Now, if we apply the evolution mechanism of the set \( I \), we immediately realize that all the generators in a zone of extension \( \frac{2}{\sqrt{1 - 4q_B}} \) are erased. (In our example: \( \sigma_{k+1} \) erases \( \sigma_k \), \( \sigma_k^{-1} \) erases \( \sigma_{k+1} \), \( \sigma_{k-1}^{-1} \) erases \( \sigma_k^{-1} \), ...).

Comparing with the group \( LF_{n+1}(d) \) and following the remark at the beginning of section 2.2, we can define the new effective \( d = d_{\text{eff}} \) by the equation:

\[
2 (d_{\text{eff}} - 1) = \frac{2}{\sqrt{1 - 4q_B}} \quad (37)
\]

Moreover, it is easy to see that the probability \( \Pi_2 \) (reduction process) is equal to

\[
\Pi_2 = p_B \eta = n p_B \eta' = \left( \frac{1 - 4q_B}{2} \right) \eta' \quad (38)
\]

We have used the normalisation (4) and supposed that \( n \gg 1 \).

So, we get:

\[
\frac{d\mu'}{dN'} = \Pi_1 - \Pi_2 = 1 - 2\Pi_2 = 1 - (1 - 4q_B)\eta' \quad (39)
\]

For the evolution of \( \eta' \), we obtain, after the time rescaling \( N' \rightarrow (1 - 4q_B)N' \), the equation

\[
\frac{d\eta'}{dN'} = (1 - \eta' (2d_{\text{eff}} - 1)) (1 - 4q_B) \left( 1 - \frac{\Pi_2}{\Pi_1} \right) \quad (40)
\]

that is easily solved. In the limit \( N' \rightarrow \infty \) we arrive at the following equations

\[
\eta'_{\infty} = \frac{1}{2d_{\text{eff}} - 1} = \frac{1}{\sqrt{1 - 4q_B}} + 1 \quad (41)
\]

and

\[
\frac{\langle \mu \rangle}{N} \bigg|_{\infty} = \frac{2 + 4q_B \sqrt{1 - 4q_B}}{2 + \sqrt{1 - 4q_B}} \quad (42)
\]

We can see that \( \frac{\langle \mu \rangle}{N} \bigg|_{\infty} \) is a monotonically increasing function of \( q_B \) from \( 2/3 \) (for \( q_B = 0 \)) till \( 1 \) (for the maximal value \( q_B = 1/4 \)).

The correlations in the case "B" increase the length of reduced words in case of the locally free group \( LF_{n+1} \). The same behaviour is seen numerically for the braid group \( B_{n+1} \).
In the Fig.4 we compare the results of numerical simulations for the group $\mathcal{L}\mathcal{F}_{n+1}(2)$ at $q_B = 0.05$ (dots) with our analytic computations (full line—solutions of Eqs.(39)–(40)).

Our numerical computations of the normalised reduced word length, $\mu/N$, as a function of normalised initial word length, $N/n$, are summarized in the Fig.5. This plot shows the dependence $\mu(N')$ for locally free and braid groups for both kinds of correlations (“A” and “B”). The corresponding analytic results are available in all cases except for the braid group when words are drawn with correlations “B”.

4 A Ballistic Growth Model

We apply the ideas developed above to the investigation of some statistical properties of a ballistic growth process in 1+1 dimensions.

The standard ballistic deposition can be defined in the following way [12]. Take $n$ columns, of unit width each. A particle, of unit width and height, is dropped vertically in a randomly chosen column and sticks upon first contact with the evolving deposit.

Let $h(k, N)$ be the height of the column with the number $k$ ($k \in 1, \ldots, n$) after dropping of $N$ particles. The surface of the pile is determined by the function $h(k, N)$. The change of $h(k, N)$ when one extra particle is dropped in column $k$ satisfies the following rule:

$$h(k, N + 1) = \max\{h(k - 1, N), h(k, N) + 1, h(k + 1, N)\}$$

(43)

Schematically this rule corresponds to the following process.
The "active" box sides (i.e. the sides which can attract the new particles) are shown in boldface.

Let us slightly modify the rule \([13]\) and suppose that:

\[
h(k, N + 1) = \max\{h(k - 1, N), \, h(k, N), \, h(k + 1, N)\} + 1
\]

(44)

The prescription (44) corresponds to the situation shown below.
It represents the ballistic growth of the pile of unpenetrable particles still of unit height but of width slightly larger than one: two particles dropped in neighbouring columns cannot ”pass through” each other.

In course of numeric computations we get for the average height of the pile, \( h(N') \), the asymptotic value
\[
\frac{h(N')}{N'} \approx 4.05
\]
for \( N' \equiv N/n \gg 1 \); while for the standard ballistic model one has
\[
\frac{h(N')}{N'} \approx 2.13
\]
Thus, the compactness of the pile in our model is about twice smaller.

The collection of peaks and valleys in our model forms a highly rough surface, which develops in course of particles dropping. We suppose that each ”time step” corresponds to adding one extra particle to the system. Recall that \( k \) is a peak at some time \( N \) if \( h(k, N) > \max\{h(k - 1, N), h(k + 1, N)\} \). In what follows we are mainly interested in computing the average number of peaks, \( \eta(N) \). As before, we define \( \eta' \equiv \eta/n \) and \( N' \equiv N/n \).

According to the rule (44), two peaks cannot appear in neighbouring columns and we can easily establish the connection with the ideas developed above: a particle dropped in the column \( k \) can be viewed as a letter \( \sigma_k \) drawn with the probability \( 1/n \) over the set \( \{\sigma_1, \ldots, \sigma_n\} \) generating the group \( \mathcal{LF}_{n+1}(2) \). The ”hardcore” constraint implies the condition \( \sigma_i \sigma_j = \sigma_j \sigma_i \) if and only if \( i \neq j \pm 1 \). Note that we deal in this case with the ”semigroup” \( \mathcal{LF}_{n+1}^+ \) because we do not use the inverse generators \( \sigma_i^{-1} \) and do not consider the reducing process. From this point of view, our analysis, though analogous, is simpler than for the whole group \( \mathcal{LF}_{n+1} \).

Thus, we can easily deduce that the set of peaks is analogous to the set of reducible letters \( I \) and is reminiscent of the enumeration of ”partially commutative monoids” known in combinatorics [13].

Suppose that two neighbouring peaks are separated by the horizontal interval of length \( i \geq 1 \) and \( n_i \) is the number of such intervals. Now we are in position to write the
recursion relations for the process $N' \rightarrow N' + 1/n$:

\[
\begin{cases}
\eta' \rightarrow \eta' \quad \text{occurs with probability } \Pi'_0 = \frac{1}{n} \left( \eta + 2 \sum_{i \geq 2} n_i \right) \\
\eta' \rightarrow \eta' - \frac{1}{n} \quad \text{occurs with probability } \Pi'_- = \frac{n_1}{n} \\
\eta' \rightarrow \eta' + \frac{1}{n} \quad \text{occurs with probability } \Pi'_+ = \frac{1}{n} \left( \sum_{i \geq 3} n_i (i - 2) \right)
\end{cases}
\]  

(45)

where the conservation condition implies that

\[ \Pi'_0 + \Pi'_- + \Pi'_+ = 1 \]

The sum rules (6)–(7) remain to be unchanged. Comparing Eqs.(45) with Eqs.(8) we find that only $\Pi'_0$ differs from $\Pi_0$.

In terms of $\eta'$ and $N'$, we get the simple ordinary differential equation for the mean value

\[
\frac{d\eta'}{dN'} = 1 - 3\eta'
\]

(46)

The solution of Eq.(46) reads

\[
\eta' = \frac{1}{3} \left( 1 - e^{-3N'} \right)
\]

(47)

So, asymptotically ($N' \rightarrow \infty$), we get that $1/3$ of the columns are peaks.

Let us extend our consideration to the case of unpenetrable particles of widths slightly larger than $(d - 1) \times$ (unit particle width). It means that now the "hardcore" condition forces us to consider the generators with the following commutation relations: $\sigma_i \sigma_j = \sigma_j \sigma_i$ if and only if $|i - j| \geq d$ (where $d \geq 2$). This situation is shown schematically below:

It is a simple matter to see how Eqs.(45)–(47) are changed. We quote the final result only:

\[
\eta' = \frac{1}{2d - 1} \left( 1 - e^{-(2d-1)N'} \right)
\]

(48)
The case \( d = 2 \) corresponds to Eq.(17).

As it has been done for the locally free group \( \mathcal{LF}_{n+1}(2) \), we are looking at the changes in \( \eta' \) (Eq.(17)) when we allow the correlations between the subsequently dropped particles (see the section 3 for details).

**THE WEAK CORRELATIONS (THE CASE ”A”).**

If we draw \( \sigma_k \) at some moment of time, \( N \), then at the next moment of time, \( (N+1) \), we have the following situation:

\[
\begin{aligned}
&\text{The generator } \sigma_k \text{ appears with the probability } q_A \\
&\text{Any generator } \sigma_l (l \neq k) \text{ appears with the probability } p_A
\end{aligned}
\]

Due to the absence of inverse generators, the normalisation reads now:

\[ q_A + (n - 1) p_A = 1 \] (49)

(compare to Eq.(3)).

The recursion rules for the process \( N' \to N' + 1/n \) are now:

\[
\begin{aligned}
&\Delta \eta' = 0 \quad \text{occurs with probability } \Pi_0' = q_A + (\eta - 1)p_A + 2 \sum_{i \geq 2} n_i p_A \\
&\Delta \eta' = -\frac{1}{n} \quad \text{occurs with probability } \Pi_-' = n_1 p_A \\
&\Delta \eta' = +\frac{1}{n} \quad \text{occurs with probability } \Pi_+' = \sum_{i \geq 2} n_i (i - 2) p_A
\end{aligned}
\]

(50)

So, we get

\[
\frac{d\eta'}{dN'} = (1 - q_A)(1 - 3\eta') \] (51)

and

\[
\eta' = \frac{1}{3} \left(1 - e^{-3(1-q_A)N'}\right) \] (52)

Asymptotically, again 1/3 of the columns are peaks and the effect of correlations leads here only to the time rescaling \( N' \to (1 - q_A)N' \).

**THE STRONG CORRELATIONS (THE CASE ”B”).**

If we draw \( \sigma_k \) at some moment of time, \( N \), then at the next moment of time, \( (N+1) \), we have the situation:

\[
\begin{aligned}
&\text{The generator } \sigma_{k \pm 1} \text{ appears with the probability } q_B \\
&\text{Any generator } \sigma_l (l \neq k \pm 1) \text{ appears with the probability } p_B
\end{aligned}
\]
The normalisation is:

\[ 2q_B + (n - 2)p_B = 1 \]  \hspace{1cm} (53)

As it has been shown already in section 3, the effect of such correlations leads to replacement of \( d = 2 \) by some effective \( d_{\text{eff}} \). According to Eq.(53), we get:

\[ d_{\text{eff}} = \frac{1}{\sqrt{1 - 2q_B}} + 1 \]  \hspace{1cm} (54)

In addition, the time has to be rescaled as \( N' \rightarrow (1 - 2q_B)N' \). Finally we arrive at the following linear differential equation

\[ \frac{d\eta'}{dN'} = (1 - 2q_B) \left( 1 - (2d_{\text{eff}} - 1)\eta' \right) \]  \hspace{1cm} (55)

which has the solution

\[ \eta' = \frac{1}{2d_{\text{eff}} - 1} \left( 1 - e^{-(2d_{\text{eff}} - 1)(1 - 2q_B)N'} \right) \]  \hspace{1cm} (56)

It is worthwhile to notice that, in addition to the time rescaling, the correlations “B” also lead to a change of the asymptotic value of \( \eta' \).

Comparison of Eqs.(52)–(56) with numerical simulations \( (n = 1000) \) shows quite good agreement—see Fig.6 for correlations of type ”A”: (a) \( q_A = 0 \).2, (b) \( q_A = 0 \).5 and Fig.7 for correlations of type ”B”: (a) \( q_B = 0 \).1, (b) \( q_B = 0 \).2.

## 5 Conclusion

The investigation of statistical properties of random walks on braid and locally free groups is undertaken due to the following reasons:

1. On the basis of performed investigation we are going to construct the simple mean–field Flory–type theory of interacting braided random walks (bunches of ”directed polymers”) with nonabelian topology in 1+1 dimensions.

2. The minimal irreducible length of the braid (i.e. the reduced word) can be served as a well defined characteristic of the ”complexity” of knots constructed on the basis of braids. Thus, our study could be regarded as a basis for investigation of the limit behavior of knot and link topological invariants when the length of the corresponding braid tends to infinity, i.e., when the braid ”grows”.

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3. We believe that the application of the locally free group in the theory of ballistic aggregation could be used: (i) in the consideration of statistical and relaxational properties of "sandpile models" exhibiting SOC–behavior\(^3\) (ii) in the microscopic description of the surface growth phenomena.

Of course, our investigation is far from being complete. For instance, the scaling–like considerations of correlations "A" in locally free and braid group should be justified from the point of view of standard renormalisation group technique; special care should be taken for analytic consideration of correlations "B" in braid group and so on.

However we would like to finish the paper on the optimistic note. Let us express the hope that the problem of discovering the integrable models associated with the proposed locally free groups and developing the corresponding conformal field theory could help to establish the bridge between statistics of random walks on the noncommutative groups, spectral theory on multiconnected Riemann surfaces, topological field theory and statistics of rough surfaces in models of ballistic aggregation and sandpile growth.

\(^3\)SOC is the abbreviation of "self–organised criticality".
References

[1] H. Kesten, Trans. Amer. Math. Soc., 92 (1959), 336
[2] A.M. Vershik, in *Topics in Algebra*, 26, pt.2 (1990), 467, (Banach Center Publica-
tion, PWN Publ., Warszawa); Proc. Am. Math. Soc., 148 (1991), 1
[3] S. Nechaev, Ya.G. Sinai, Bol. Soc. Bras. Mat., 21 (1991), 121
[4] S.K.Nechaev, A.Yu. Grosberg, A.M.Vershik, J. Phys. (A): Math. Gen., 29 (1996),
2411
[5] P. Chassaing, G. Letac, M. Mora, in *Probability Measures on Groups*, Lect. Not.
Math., 1064 (1983)
[6] L. Koralov, S. Nechaev, Ya. Sinai, Prob. Theor. Appl. 38 (1993), 331 (in Russian)
[7] A. Khokhlov, S. Nechaev, Phys. Lett. A112 (1985), 156
[8] S. Nechaev, A. Semenov, M. Koleva, Physica, A–140 (1987), 506
[9] S. Nechaev, A. Vershik, J. Phys. (A): Math. Gen., 27 (1994), 2289
[10] J. Desbois, S. Nechaev, J. Stat. Phys., 88 (1997), 201
[11] J. Desbois, S. Nechaev, in preparation
[12] T. Halpin-Healy, Y.C. Zhang, Phys. Reports, 254 (1995), 215
[13] G.X. Viennot, Ann. N.Y. Acad. Sci., 576 (1989), 542
Figure Captions

Fig.1. The normalised length, $< \mu > /N$, of the reduced word as a function of the length of the initial word, $N/n$, for locally free and braid groups. Words are drawn without any correlations.

Fig.2. The same as Fig.1 except that words are drawn with correlations ”A” ($q_A = 0.1$).

Fig.3. The limit of $< \mu > /N$ when $0 < N/n \ll 1$ is plotted as a function of the probability $q_A$, for locally free and braid groups. Words are drawn with correlations ”A”.

Fig.4. The same as Fig.1 except that words are drawn with correlations”B” ($q_B = 0.05$). (Analytic computations for the group $B_n$ are absent).

Fig.5. The plot shows the dependence $\mu(N')$ for locally free and braid groups for both kinds of correlations (”A” and ”B”). The corresponding analytic results are available in all cases except for the braid group when words are drawn with correlations ”B”.

Fig.6. Dependence of normalised amount of the surface peaks, $\eta'$, on the normalised number of ”pile volume”, $N'$, in the ballistic aggregation model. Particles are dropped with correlations ”A”.

Fig.7. Dependence of normalised amount of the surface peaks, $\eta'$, on the normalised number of ”pile volume”, $N'$, in the ballistic aggregation model. Particles are dropped with correlations ”B”.