Mode-Coupling Theory as a Mean-Field Description of the Glass Transition

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Mode-coupling theory (MCT) is conjectured to be a mean-field description of dynamics of the structural glass transition and the replica theory to be its thermodynamic counterpart. However, the relationship between the two theories remains controversial and quantitative comparison is lacking. In this Letter, we investigate MCT for monatomic hard sphere fluids at arbitrary dimensions above three and compare the results with replica theory. We find grave discrepancies between the predictions of two theories. While MCT describes the nonergodic parameter quantitatively better than the replica theory in three dimension, it predicts a completely different dimension dependence of the dynamical transition point. We find it to be due to the pathological behavior of the nonergodic parameters derived from MCT, which exhibit negative tails in real space at high dimensions.

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The nature of the glass transition remains elusive despite of decades of discussion. Many theories and scenarios have been proposed to explain the drastic slow down of dynamics of supercooled liquids but we still lack conclusive microscopic understanding of the phenomenon. Amongst various theories, mode-coupling theory (MCT) $^1$ and replica theory $^2$ $^3$ are arguably the only first principles theories.

On the one hand, MCT describes the slow dynamics of the mildly supercooled liquids using the static structure factor as a sole input. It quantitatively captures the onset of the two-step relaxation of correlation functions, the scaling properties at the intermediate time scale (the $\beta$ relaxation), and the algebraic increase of the structural relaxation time. MCT, however, predicts a spurious freezing transition at a lower density $\varphi_{\text{mct}}$ (or higher temperature $T_{\text{mct}}$) than the experimentally determined glass transition point $\varphi_g$ (or $T_g$). On the other hand, the replica theory is a static mean-field description of the glass transition $^2$. It predicts that the fluid undergoes a thermodynamic “ideal” glass transition at a higher density $\varphi_K$ than $\varphi_g$ (or lower temperature $T_K$ than $T_g$), characterized by the one-step replica symmetry breaking. The replica theory also predicts that the dynamical transition takes place at $\varphi_d < \varphi_K$ (or $T_d > T_K$) where the phase space or energy landscape starts splitting into numerous metastable states, or basins. MCT is conjectured to be the dynamical counterpart of the replica theory and $\varphi_{\text{mct}}$ to be identical to $\varphi_d$, because mathematical structure of MCT is equivalent to the dynamical equation of the so-called $p$-spin spherical model with $p = 3$, a mean-field model for which the relation between the dynamical and ideal glass transition is rigorously established $^4$ $^5$. According to this mean-field scenario, the absence of the dynamic transition at $\varphi_{\text{mct}}$ in mean-field systems is interpreted as the round-off of the dynamic freezing by activated processes between basins in finite dimensions $^6$.

Despite the apparent and simple parallelism with spin glasses, the relationship of MCT with replica theory and physical insights from the mean-field treatments have never been fully understood. MCT was originally derived as a generalization of kinetic theories, using the projection operator formalism with numerous uncontrolled approximations $^1$, whereas the replica theory is based on purely thermodynamic argument developed for disordered systems. In this Letter, we compare the MCT and replica theory results quantitatively, in order to clarify the relationship between the two theories developed in totally different arenas of physics communities. We especially focus on the dimension dependence of the glass transition point and the nonergodic parameter $f_\infty(q)$, the plateau height of the density correlation function.

To simplify the argument, we focus on the monatomic hard sphere system in $d$-dimension, for which the sole system parameter is the number density $\rho = N/V$ or the volume fraction $\varphi = V_d \rho$, where $V_d$ is the volume of a single hard sphere. We show that MCT is more quantitative than the replica theory at $d = 3$, which can be largely attributed to the lack of accurate approximation schemes in the replica theory to evaluate the static correlation functions of the replicated liquids. In higher dimensions where the static correlation functions become trivial, discrepancies between the two theories become catastrophic. MCT’s dynamical transition point ($\varphi_{\text{mct}}$) scales with dimension $d$ differently from the replica counterpart ($\varphi_d$). This discrepancy comes from the spurious negative tails of the van Hove correlation function, a generically positive quantity, that MCT predicts in high dimensions. This pathological negative tail is the origin of the non-Gaussian shape of $f_\infty(q)$ and thus the different $d$-dependence of $\varphi_{\text{mct}}$ from $\varphi_d$. These results shed serious doubts over the validity of MCT in higher dimensions and call for reconsideration of MCT as a dynamic theory of the mean-field scenario of the glass transition.

MCT is expressed as a set of nonlinear integro-differential equations for correlation functions such as the intermediate scattering function $F(q,t) = N^{-1} \langle \delta \rho(q,t) \delta \rho(-q,0) \rangle$, where $\delta \rho(q,t)$ is the density fluctu-
ation in reciprocal space at time $t$. The MCT equation for $F(q,t)$ in $d$-dimension is given by

$$\Omega^{-1}_q \ddot{F}(q,t) + F(q,t) + \int_0^t ds \, M(q,t-s) \ddot{F}(q,s) = 0, \quad (1)$$

where $\Omega_q = \sqrt{\frac{k_B T q^{d-1}}{mS(q)}}$ is the phonon frequency and $S(q) = F(q,t=0)$ is the static structure factor. The memory function $M(q,t)$ is given by

$$M(q,t) = \int_0^\infty dk \int_{|q-k|}^{|q+k|} dp \, V(q,k,p) F(k,t) F(p,t). \quad (2)$$

In this expression, $V(q,k,p) = \rho \, S(q) \, s_{d-1} \, J^{d-3} \, kp \{f_{+c}(k)+f_{-c}(p)\}/q^{d+2} \, (4\pi)^d$ is the vertex function, where $c(q) = \rho^{-1}(1-1/S(q))$ is the direct correlation function, $s_d$ is the surface of $d$-dimensional unit sphere, $f_{\pm} = q^{d} \pm (k^2 - p^2)$, and $J = \sqrt{4k^2p^2-(k^2-q^2+p^2)}$ is the Jacobian term. MCT predicts a nonergodic transition at $\varphi_{\text{mct}}$, beyond which the nonergodic parameter (NEP) $f_\infty(q) = F(q,\infty)/S(q)$ becomes non-zero. NEP can be calculated from the long time limit of Eq. (1),

$$\frac{f_\infty(q)}{1-f_\infty(q)} = M(q,\infty). \quad (3)$$

The self-part of the intermediate scattering function $F_s(q,t) = \langle \rho_\sigma(q,t) \rho_\sigma(-q,0) \rangle$ can be also described by an equation similar to Eq. (1). The set of MCT equations can be solved numerically using $S(q)$ as a sole input.

![FIG. 1: $f_\infty(q)$ evaluated from MCT (solid line) and the replica theory (dashed line) at $\varphi_{\text{mct}}$ and $\varphi_d$, respectively. $q$ is scaled with the diameter of a sphere $\sigma$. $S(q)$ from HNC closure is used for MCT calculation.](image)

On the other hand, the replica liquid theory for the structural glass transition has been developed by Mézard and Parisi and recently applied to hard-sphere fluids by Parisi and Zamponi. In this theory, replicated systems with a weak attractive interaction of order $\epsilon$ between them are considered. The free energy of the whole system is calculated as a function of intra- and inter-replica correlations. The relevant physical observables are evaluated by taking the limit $\epsilon \to 0$ at the end of the calculation. The theory predicts a thermodynamical transition characterized by the replica symmetry breaking at which the configurational entropy vanishes. The transition point is often identified with the Kauzmann point $\varphi_K$. The theory also predicts a dynamic transition point $\varphi_d < \varphi_K$ at which the free energy splits into numerous metastable basins. The system can not explore the whole phase space above $\varphi_d$ because of the infinite free-energy barrier which separates the basins. In the replica interpretation, the inter-replica pair density correlation function, $\tilde{g}(r)$, plays the role of order parameter and is identified with NEP $\rho h(q)/S(q) = f_\infty(q)$, where $h(q)$ is the wavevector representation of $\tilde{g}(r)-1$. The NEP is determined from variational condition of the replicated free energy. In analogy with the $p$-spin spherical model, it is believed that $\varphi_d$ should be identical to $\varphi_{\text{mct}}$ and that the NEP calculated from MCT should match that derived from the replica theory at the dynamic transition point.

In order to check the validity of this conjecture, we numerically compare the results of both theories for $\varphi_{\text{mct}}$, $\varphi_d$, and $f_\infty(q)$ for $d \geq 3$. First, we look at a $d = 3$ hard sphere system. MCT for three-dimensional hard spheres was studied by Götze et al. and On the other hand, the quantitative accuracy of the replica theory sensitively depends on the approximation scheme employed to calculate the free energy of replicated liquids. The small cage expansion technique is known to be a good approximation near the Kauzmann point, but it does not describe the dynamic transition in low dimensions. Therefore, we use another scheme, the replicated hypernetted chain (RHNC) approximation, the only method at present which captures the dynamic transition in finite dimensions. RHNC consists of a set of closure equations for both inter- and intra-pair density correlation functions, $\tilde{g}(r)$ and $g(r)$, given by

$$\begin{align*}
\ln g(r) &= \beta v(r) + \frac{1}{(2\pi)^d} \int d\vec{q} \, \rho h^2(q) / \left(1 + \rho h(q) \right), \\
\ln \tilde{g}(r) &= \frac{1}{(2\pi)^d} \int d\vec{q} \, \rho h^2(q) / \left(1 + \rho h(q) \right) - \frac{\rho h(q) - \tilde{h}(q)}{1 + \rho (h(q) - \tilde{h}(q))}.
\end{align*} \quad (4)$$

Here, $\rho h(q) = S(q) - 1$ and $v(r)$ is the interaction potential. The first equation is the HNC equation of a simple liquid and the second equation describes the inter-replica coupling. The dynamic transition point $\varphi_d$ is defined as the volume fraction beyond which $h(q)$ becomes non-zero. We solve MCT equation, Eq. (3), and the RHNC theory, Eq. (4), for the monatomic hard sphere system. For MCT calculation, we employ the HNC equation to evaluate $S(q)$ in order to make the comparison consistent. The dynamic transition points thus obtained are $\varphi_{\text{mct}} = 0.523$ (which is slightly larger than 0.515 obtained from Percus-Yevick closure) and...
TABLE I: Values of $\varphi_{mct}$ and $\varphi_{K}$ from $d = 4$ to $d = 8$. $\varphi_{K}$ is from Parisi et al. [3].

| $d$ | $4$ | $5$ | $6$ | $7$ | $8$ |
|-----|-----|-----|-----|-----|-----|
| $\varphi_{mct}$ | 0.3652 | 0.2542 | 0.1736 | 0.1159 | 0.0751 |
| $\varphi_{K}$ | 0.4319 | 0.2894 | 0.1883 | 0.1194 | 0.0739 |

$\varphi_d = 0.612$ [12]. $f_{\infty}(q)$ calculated from MCT at $\varphi_{mct}$ and the replica theory at $\varphi_d$ are shown in Figure 1. Quantitative difference between the shape of the NEP from the two theories is obvious. Since it is well established that MCT’s $f_{\infty}(q)$ agrees very well with simulation [10] and experimental results [13], this discrepancy could be mainly due to poor performance of the replica theory. However, it is not clear whether this is attributed to the inherent inconsistency of MCT with replica theory or solely to a lack of accuracy of the RHNC approximation.

In order to give the two theories more stringent test, we discuss the dimension dependence of quantities near the dynamic transition point. We start with $d = 4$ -- 8 and solve Eq. (2) to evaluate $\varphi_{mct}$ and $f_{\infty}(q)$. An algorithm by Baus and Colot [14] is used to evaluate $S(q)$. $\varphi_{mct}$ thus obtained is listed on Table I along with $\varphi_{K}$ reported in Ref. [3]. Note that $\varphi_{mct}$ is smaller than $\varphi_{K}$ in lower dimensions, but the gap narrows with increasing dimension and at $d = 8$ $\varphi_{mct}$ exceeds $\varphi_{K}$.

Next, we study the $d$-dependence in even higher dimensions, where the static properties of the liquid and the replicated liquid become insensitive to the approximation schemes. Therefore, it is possible to check the relationship between the two theories without obscuration from approximations for the static inputs. In the high $d$-limit where the diagrammatic expansions of the free energy is given by a simple function of the Mayer function $e^{-\beta v(r)} - 1$, exact analytical expressions for static correlation functions are available. For the hard sphere system, the direct correlation function $c(q)$ is simply given by $c(q) = -2(2\pi/\sigma^2)^d/J_{1/2}(\sigma q)$, where $J_l(x)$ is the $l$th Bessel function of the first kind and $\sigma$ is the diameter of a sphere. Recently, accurate replica theory calculations of the free energy in high dimensions was carried for the monatomic hard sphere system, using this $c(q)$ as an input and the cage expansion method [3]. The dynamic transition point was shown to scale with $d$ as

$$\varphi_d = 4.8 \times 2^{-d} d$$

and the Kauzmann point as $\varphi_K = 2^{-d} d \ln d$ in the high dimension limit $d \to \infty$ [3]. We solve the MCT equation with the same $c(q)$, keeping the convergence of discretization error and the numerical accuracy of the Bessel function under control. In Figure 2, $\varphi_{mct}$ is shown as a function of $d$. We find that $\varphi_{mct}$ scales as $\varphi_{mct} = 0.22 \times 2^{-d} d^2$, in stark contrast with the replica prediction for $\varphi_d$, Eq. (3). We also calculate the NEPs from MCT and find that their $q$-dependence are non-Gaussian shaped in high $d$. In low dimensions below $d = 8$, decay of $f_{\infty}(q)$ at $q\sigma > 10$ and $f_{s,\infty}(q)$ are well fitted by a Gaussian form, but in higher dimensions they decay faster than Gaussian at large $q$’s (not shown). Different $d$-dependence of $\varphi_{mct}$ from $\varphi_d$ originates from this non-Gaussianity. This can be shown by solving Eq. (3) assuming that $f_{\infty}(q)$ and $f_{s,\infty}(q)$ both have a Gaussian shape, i.e., $f_{\infty}(q)$, $f_{s,\infty}(q) \approx e^{-Bq^2/d}$. Here, we assume that $f_{\infty}(q) \approx f_{s,\infty}(q)$, the so-called Vineyard approximation [11]. Substituting this Gaussian form in Eq. (3), we obtain a self-consistent equation for $R$,

$$\frac{1}{R} = \frac{\rho s_d}{2\sigma^d} \int_0^{\infty} dq q^{d+1} c^2(q) S(q) e^{-Bq^2/d}.$$ (6)

This expression is strikingly analogous to the equation for the density field for the amorphous solid obtained using Gaussian approximation in the framework of the density functional theory [13]. Solving Eq. (6), we find that the equation has a finite solution for $R$ above a volume fraction $\varphi_{mct}^{(G)}$ that behaves in high $d$ as [22]

$$\varphi_{mct}^{(G)} = 8.27 \times 2^{-d} d.$$ (7)

Though the prefactor differs, it retrieves the same $d$-dependence as the replica theory, which ascertains the origin of discrepancies between MCT and the replica theory.

This non-Gaussian shape of the NEP that MCT predicts in high dimensions is the compelling evidence that MCT breaks down. Analyzing the van Hove correlation function $G_s(r, t)$, which is the real space representation of $F_s(q, t)$, makes this breakdown clear. $G_s(r, t)$ is a distribution function of the distance for one particle to explore during the time interval $t$, $G_s(r, t) = \langle \delta(r - |\vec{R}_i(t) - \vec{R}_i(0)|) \rangle$, where $\vec{R}_i(t)$ is the position of the $i$th particle at time $t$. By definition, the van Hove function is a non-negative quantity. In Figure 3 we plot $G_s(r, t) = G_s(r, t = \infty)$ derived from MCT for several dimensions. Following the standard convention a multiplying factor $s_d d^{d-1}$ is used. In high dimensions, it
exhibits a negative dip whose depth becomes larger as dimension increases. We checked that $f_{\infty}(q)$ retrieves the Gaussian shapes if the negative dips are absent.

These results suggest that MCT in its present form is not consistent with replica theory and that moreover MCT suffers from serious deficiencies. The validity of the dimension dependence $\varphi_{\text{mct}} \sim d^2/2^d$ that MCT predicts is suspicious because it originates from the pathologic behavior of the NEP. It is noteworthy that the non-Gaussian shape of $f_\infty(q)$ and the negative tails of $G_{s,\infty}(r)$ already appear in $d = 8$, below the upper-critical dimension $d_c = 8$ of the glass transition [16, 17]. Assuming a Gaussian shape for $f_\infty(q)$ in MCT recovers the linear dimension dependence of $2^d\varphi_{\text{mct}}$ (see Eq. (7)) but the prefactor still does not match with the replica theory prediction. This fact implies that a quick remedy is unlikely to fix the problem. In hindsight, a convincing reason to conjecture that two theories are related is lacking, except for their apparent mathematical similarity with the p-spin spherical mean-field model of spin glasses. Deceptively similar structure between Gaussian-approximated MCT, Eq. (10), and the equation for the density profile derived in the mean-field analysis of the density functional theory [15] also hints that MCT is a mean-field-like theory, but a small yet non-negligible difference between these equations leaves us a nagging suspicion over MCT’s validity. One of routes to resolve these problems is to reformulate MCT in a field theoretic languages in which parallels and differences with the mean field theory of spin glasses and the dynamic liquid theory is highlighted. Efforts in this direction have suffered from series of difficulties associated with consistencies with the fluctuation-dissipation theorem [18], the double-counting problem of the potential interactions, and the reconciliation between the dynamic and static liquid theories [19]. Although it is not clear if the replica theory and the density functional theory correctly describe the dynamic transition, at least our results clearly indicate that reconsideration and revision of MCT from the ground up are in order. We conjecture, however, that prospective revisions leave general mathematical properties of MCT equation intact. It is argued that MCT should be seen as a Landau theory in a sense that critical behavior and scaling properties that MCT describes near the dynamical transition point is universal [20]. Indeed, a recent numerical study indicates that MCT works better for critical behaviors in high dimensions [8].

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Note added.– As this manuscript was being finalized for submission, we became aware of a paper by Schmid and Schilling [21]. They have solved MCT equation for $d \geq 10$ and shown the same dimension dependence of $\varphi_{\text{mct}}$ as reported in Fig. 2 and non-Gaussian shape of $f_\infty(q)$.

![Graph](image_url)

**FIG. 3:** The van Hove function $G_{s,\infty}(r)$ evaluated using MCT in several dimensions. From left to right, $d = 4, 6, 8, 10, 15$.

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