REMARKS ON NON-COMPACT COMPLETE RICCI EXPANDING SOLITONS

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Abstract. In this paper, we study gradient Ricci expanding solitons \((X, g)\) satisfying
\[ Rc = cg + D^2 f, \]
where \(Rc\) is the Ricci curvature, \(c < 0\) is a constant, and \(D^2 f\) is the Hessian of the potential function \(f\) on \(X\). We show that for a gradient expanding soliton \((X, g)\) with non-negative Ricci curvature, the scalar curvature \(R\) has at least one maximum point on \(X\), which is the only minimum point of the potential function \(f\). Furthermore, \(R > 0\) on \(X\) unless \((X, g)\) is Ricci flat. We also show that there is exponentially decay for scalar curvature for \(\epsilon\)-pinched complete non-compact expanding solitons.

1. Introduction

In this paper, we continue our study on Ricci solitons \([7]\), which are generated by one parameter family of diffeomorphisms and are special solutions to Ricci flow introduced by R.Hamilton in 1982 \([7]\). We assume in this paper that \((X, g)\) is a gradient expanding soliton. Let recall the definition of expanding soliton.

**Definition 1.** We call a Riemannian manifold \((X, g)\) an expanding soliton if there is a smooth solution \(f\) on a Riemannian manifold \((X, g)\) such that for some constant \(c < 0\), it holds the equation
\[ Rc = cg + D^2 f, \]
on \(X\), where \(D^2 f\) is the Hessian matrix of the function \(f\) and \(Rc\) is the Ricci tensor of the metric \(g\). We call the function \(f\) the potential function for the soliton \((X, g)\). If \(c > 0\) in \((7)\), \((X, g)\) is called a shrinking soliton; if \(c = 0\), \((X, g)\) is called a steady soliton.

In the study of Ricci flow, we often meet the following definition.
Definition 2. The Ricci curvature of a Riemannian manifold \((X, g)\) is called \(\epsilon\)-pinched if there is some \(\epsilon > 0\) such that the scalar curvature \(R > 0\) on \(X\) and
\[
Rc \geq \epsilon Rg
\]
on \(X\).

Throughout this paper, we shall assume that the Riemannian manifold \((X, g)\) is a complete non-compact Riemannian manifold of dimension \(n \geq 3\). We denote by \(R\) the scalar curvature of the metric \(g\).

Our main result is the following

Main Theorem. Assume that the Ricci curvature of the gradient expanding soliton \((X, g)\) is non-negative. Then the scalar curvature \(R\) has at least one maximum point on \(X\), which is the only minimum point of the potential function \(f\). Furthermore, \(R > 0\) on \(X\) unless \((X, g)\) is Ricci flat.

The proof of this Theorem will be proved in section 3.

In section four, we will prove the following result

Theorem 3. Assume that \((X, g)\) is a gradient expanding soliton with its Ricci curvature being \(\epsilon\)-pinched. Then its scalar curvature has the decay
\[
R(s) \leq R(o)e^{Cs - Cs^2}
\]
as the distance function \(s\) from a fixed point going to infinity, i.e., \(s = d(x, o) \to +\infty\).

We remark that a similar but weaker decay result is announced by L.Ni in Proposition 3.1 in [8]. We know the result for a while and a reason for the delay of this present is that we try to prove non-existence of this kind of expanding solitons. However, we have not been succeed yet.

Throughout \(C\) will denote various uniform constants in different places.

2. PRELIMINARY

We recall first some basic properties about Ricci solitons [6].

Taking the trace of both sides of (1), we have
\[
R = nc + \Delta f.
\]

Take a point \(x \in X\). In local normal coordinates \((x^i)\) of the Riemannian manifold \((X, g)\) at a point \(x\), we write the metric \(g\) as \((g_{ij})\).
The corresponding Riemannian curvature tensor and Ricci tensor are denoted by $Rm = (R_{ijkl})$ and $Rc = (R_{ij})$ respectively. Hence,

$$R_{ij} = g^{kl}R_{ikjl}$$

and

$$R = g^{ij}R_{ij}.$$  

We write the covariant derivative of a smooth function $f$ by $Df = (f_i)$, and denote the Hessian matrix of the function $f$ by $D^2f = (f_{ij})$, where $D$ the covariant derivative of $g$ on $X$. The higher order covariant derivatives are denoted by $f_{ijk}$, etc. Similarly, we use the $T_{ij,k}$ to denote the covariant derivative of the tensor $(T_{ij})$. We write $T^i_j = g^{ik}T_{jk}$. Then the Ricci soliton equation is

$$R_{ij} = f_{ij} + cg_{ij}.$$  

Taking covariant derivative, we get

$$f_{ijk} = R_{ij,k}.$$  

So we have

$$f_{ijk} - f_{ikj} = R_{ij,k} - R_{ik,j}.$$  

By the Ricci formula we have that

$$f_{ijk} - f_{ikj} = R^l_{ijk}f_l.$$  

Hence we obtain that

$$R_{ij,k} - R_{ik,j} = R^l_{ijk}f_l.$$  

Recall that the contracted Bianchi identity is

$$R_{ij,j} = \frac{1}{2}R_i.$$  

Upon taking the trace of the previous equation we get that

$$\frac{1}{2}R_i + R^k_{ij}f_k = 0,$$

i.e.,

$$R_k = -2R^i_{jk}f_j.$$  

Then at $x$,

$$D_k(|Df|^2 + R + 2cf) = 2f_j(f_{jk} - R_{jk} + 2cg_{jk}) = 0.$$  

So,

$$|Df|^2 + R + 2cf = M,$$

where $M$ is a constant.

In the remaining part of this section, we assume that $0 \leq Rc \leq C$ on the expanding soliton $(X, g)$ for some constant $C > 0$. Then we have
$|D^2 f| \leq C$ on $X$. Assume $f \geq 0$ and that $o$ is a critical point of the potential function $f$. Then using the Taylor’s expansion, we have

$$f(x) \leq C d^2(x, o).$$

We now study the behavior of the potential function along a minimizing geodesic curve on the expanding soliton. A similar work has been done by G. Perelman [9] (see also [5]) where he tries to give some uniform bounds on potential function $f$ on a shrinking soliton.

Fix a point $o \in X$. Take any minimizing geodesic curve $\gamma(s)$ connecting $x$ and the fixed point $p$, where $s$ is the arc-length parameter. Write by $r = d(x, o)$ and $X = \gamma'(s)$. Assume that $r > 2$. Let $\{Y_i\}$ $(i = 1, \ldots, n - 1)$ be an orthonormal parallel vector fields along $\gamma$. Let $Y$ be an orthogonal vector field along the curve $\gamma$ vanishing at end points. Then the second variational formula [10] (see also [1]) tells us that

$$\int_0^r (|Y|^2 - \langle R(X, Y)Y, X \rangle) ds \geq 0.$$

Take $Y$ to be $sY_i$ on $[0, 1]$, $= Y_i$ on $[1, r - r_0]$ where $1 < r_0 < r$, and $\frac{r-r}{r_0}Y_i$. Adding over $i$ gives that

$$\int_0^r Rc(X, X) \leq C_0(r_0) + \frac{n-1}{r_0} - \int_{r-r_0}^r \left( \frac{r-r_0}{r_0} \right)^2 Rc(X, X) ds,$$

which implies that for some constant $C > 0$,

$$\int_0^r Rc(X, X) \leq C. \tag{5}$$

Note that

$$\left( \int_0^r Rc(X, Y_1) ds \right)^2 \leq r \int_0^r |Rc(X, Y_1)|^2 ds \leq s \sum_i \int_0^r |Rc(X, Y_i)|^2 ds.$$

Thinking of $Rc$ as self-adjoint linear operator on $TX$ and taking a point-wise orthonormal frame $\{e_j\}$ as eigenvectors of $Rc = (\oplus \lambda_j)$, we have that

$$R = \sum_j \lambda_j$$

and for $X = \sum_j X_j e_j$,

$$\sum_i |Rc(X, Y_i)|^2 = < X, Rc^2 X > = \sum_j \lambda_j X_j^2 \leq RRc(X, X).$$

Then,

$$\left( \int_0^r Rc(X, Y_1) ds \right)^2 \leq Cs \int_0^r Rc(X, X) \leq C^2 s.$$
Hence, for any unit vector field $Y$ along $\gamma$, orthogonal to $X$, we have
\[
\int_0^r Rc(X, Y) ds \leq C(\sqrt{s} + 1).
\]
Using (11) we have
\[
\frac{d^2 f(\gamma(s))}{ds^2} = Rc(X, X) - c \geq -c,
\]
and
\[
\frac{d(Yf)(\gamma(s))}{ds} = Rc(X, Y).
\]
Then we have
\[
\frac{df(\gamma(s))}{ds} \geq \frac{df(\gamma(s))}{ds}(0) - cs \geq -cs + C
\]
and for $s > 2$,
\[
(Yf)(\gamma(s)) \leq |(Yf)(\gamma(0))| + \int_0^s |Rc(X, Y)| ds \leq C\sqrt{s}.
\]
Therefore, we can conclude that at large distance from $o$ the potential function $f$ has its gradient making small angle with the gradient of the distance function from $o$.

3. Proof of Main Theorem

Assume that $Rc \geq 0$ on $X$, and we also assume that for some constant $c < 0$ we have $Rc - cg > 0$ on $X$. By (11) we know that
\[
D^2 f = Rc - cg \geq -cg > 0, \quad \text{on} \quad X.
\]
Then the potential function $f$ is locally strictly convex. Since $(X, g)$ is a complete non-compact Riemannian manifold, we have that $f$ has at most one critical point, i.e., the point where $\nabla f = 0$. Using $D^2 f > 0$, we know that if $p \in X$ is the critical point of $f$, then it is a non-degenerate minimum point of $f$.

Note that along any minimizing geodesic curve $\gamma(s)$ connecting $x$ and the fixed point $p$, where $s$ is the arc-length parameter, we have
\[
< \nabla f, \gamma'(s) > |_0^s = \int_0^s f_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} ds
\]
\[
= \int_0^s (R_{ij} - cg_{ij}) \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} ds
\]
\[
= -cs + \int_0^s R_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} ds
\]
\[
\geq -cs > 0
\]
This implies that \( f(\gamma(s)) \) is growing at infinity at least the quadratic rate \(-c\) of the distance function. Then \( f \) has at least a minimum point in \( X \).

Assume that \( o \) is the only critical point of \( f \). Then by adding a constant, we can assume that and \( f(o) = 0 \) and \( f > 0 \) on \( X - \{o\} \).

Using (4), we know that \[
M = |Df|^2(o) + R(o) + 2cf(o) = R(o).
\]

Using (3) we know that \( o \) is also the critical point of \( R \).

Let \( x \in X - \{o\} \). Taking a minimizing geodesic curve \( \gamma(s) \) connecting \( x \) and the fixed point \( o \), where \( s \) is the arc-length parameter, we again have by using (7)

\[
\langle \nabla f, \gamma'(s) \rangle > -cs > 0.
\]

This implies that the integral curves of \( \nabla f \) in \( X - \{o\} \) emanating from the point \( o \) to infinity. Take a integral curve \( \sigma(t) \nabla f \) in \( X - \{o\} \). Then by (3) we have

\[
\frac{d}{dt} R(\sigma(t)) = R_i f_i = -2Rc(\nabla f, \nabla f) \leq 0.
\]

Hence \( R(x) \leq R(o) \) for all \( x \in X\{o\} \). So, \( o \) is a maximum point of \( R \).

By this we conclude that

**Assertion 4.** Assume that the Ricci curvature of the gradient expanding soliton \((X, g)\) is non-negative positive. Then the scalar curvature \( R \) has at least one maximum point of \( R \), which is the only critical point of the potential function \( f \).

If \( R(o) = 0 \), then \( R = 0 \) on \( X \). Hence \( Rc = 0 \) on \( X \), that is to say that \((X, g)\) is Ricci flat. So we have \( R(o) > 0 \). By the local strong maximum principle, we must have \( R > 0 \) on the whole space \( X \).

This finishes the **proof of Main Theorem**.

In the remaining part of this section, we consider the behavior of \( f \) at infinity. Since

\[
|Df|(x)^2 + 2cf(x) = R(o) - R(x) \geq 0,
\]

we get that

\[
|Df|^2 \geq -2cf = 2|c|f.
\]

Then we have

\[
|D\sqrt{f}| \geq \sqrt{|c|/2}.
\]

at where \( f \neq 0 \). Therefore, we have

\[
\sqrt{f}(s) \geq \sqrt{|c|/2}s
\]
and
\[ f(s) \geq \frac{|c|}{2} s^2 \]
along any minimizing geodesic curve \( \gamma(s) \) connecting \( x \) and the fixed point \( o \), where \( s \) is the arc-length parameter.

Note that using (2) we have
\[
|Df|^2(s) = -2cf(x) + R(o) - R(x) \leq -2cf(x) + R(o) \leq Cs^2 + R(o).
\]
Hence, for \( s >> 1 \),
\[
C_4 s \leq |Df|(s) \leq C_5 s.
\]

4. \( \epsilon \)-Pinched Solitons

We give a proof of Theorem 3 below. We try to make the proof more transparent and self-contained.

Proof of Theorem 3: Recall that the Ricci curvature of the non-shrinking soliton \((X, g)\) is \( \epsilon \)-pinched, i.e., for some \( \epsilon > 0 \) we have that \( R > 0 \) on \( X \) and
\[
Rc \geq \epsilon Rg
\]
on \( X \). Then using the maximum principle, we know that either \( R = 0 \) on \( X \) or \( R > 0 \). If \( R = 0 \) on \( X \), then by the pinching condition we know that \((X, g)\) is Ricci flat.

Assume that \( R > 0 \) on \( X \). Then as before, the potential function \( f \) is locally strictly convex. Since \((X, g)\) is a complete non-compact Riemannian manifold, we have that \( f \) has at most one critical point, i.e., the point where \( \nabla f = 0 \). Assume that we have a critical point for \( f \), saying that it is \( o \in X \). Then using (3), we know it is also a critical point of \( R \). Using (8), we know that is the maximum point for \( R \). In particular, we know that \( R \) is a bounded function on \( X \), saying that \( D > 0 \) is the upper bound.

Using (3) and the \( \epsilon \)-pinched condition, we have that
\[
-R|\nabla f|^2 \leq \langle \nabla R, \nabla f \rangle \leq -2Rc(\nabla f, \nabla f) \leq -\epsilon R|\nabla f|^2.
\]
Taking a minimizing geodesic curve \( \gamma(s) \) connecting \( x \) and a fixed point \( o \), where \( s \) is the arc-length parameter, we have
\[
\langle \nabla f, \gamma'(s) \rangle |_0^s = \int_0^s f_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} ds = \int_0^s (R_{ij} - cg_{ij}) \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} ds = -cs + \int_0^s R_{ij} \frac{d\gamma^i}{ds} \frac{d\gamma^j}{ds} ds.
\]
This implies that there is a constant $C_2$ such that

\[
< \nabla f, \gamma'(s) > \geq -cs + \int_0^s cR ds \geq -cs + \int_0^1 R ds \geq -cs + C_2 \geq C_2
\]

for $s >> 1$.

Using (5) and the pinching condition, we have that

\[
\int_0^s R ds \leq C_6.
\]

Using the pinching condition again, (10) also implies that

\[
< \nabla f, \gamma'(s) > \leq -cs + \int_0^s R ds \leq -cs + D.
\]

Therefore, the angle between $\nabla f$ and the gradient of the distance function from $o$ is almost fixed.

Then, using (3) and the $\epsilon$-pinched condition, we have for some constant $C_3 > 0$,

\[
(R^{-1})_s = -R^{-2} < \nabla R, \gamma'(s) > = 2R^{-2}Rc(\nabla f, \gamma'(s)).
\]

Using (6) and (9), we obtain that

\[
Rc(\nabla f, \gamma'(s)) = |\nabla f| Rc(\gamma', \gamma') + 0(\sqrt{s})
\geq \epsilon R|\nabla f| + 0(\sqrt{s}) \geq R(Cs - C),
\]

we have

\[
(R^{-1})_s \geq 2R^{-1}(Cs - C).
\]

This implies that

\[
(log R)_s \leq C - Cs
\]

and

\[
R(s) \leq R(o)e^{Cs-Cs^2}.
\]

This implies that $R \to 0$ exponentially as $s \to +\infty$. This completes the proof of Theorem 3.

Theorem 3 tells us that for such $(X, g)$ we have

\[
A = \limsup_{s \to \infty} R s^2 = 0.
\]

REFERENCES

[1] Th.Aubin, Non-linear Analysis on manifolds, Springer, New York, 1982.
[2] S.Bando, A.Kasue, and H.Nakajima, On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth, Invent. Math., 97(1989):313-349.
[3] R.Bryant, Gradient Kahler-Ricci solitons, ArXiv:math.DG/0407453, 2004.
[4] H.D.Cao, Existence of gradient Kahler-Ricci soliton, Elliptic and parabolic methods in geometry, Eds. B.Chow, R.Gulliver, S.Levy, J.Sullivan, A K Peters, pp.1-6, 1996.
[5] B. Kleiner, and J. Lott, *Notes on Perelman’s papers*, http://www.math.lsa.umich.edu/research/ricciflow/perelman.html
[6] R. Hamilton, *The formation of Singularities in the Ricci flow*, Surveys in Diff. Geom., Vol.2, pp7-136, 1995.
[7] Li Ma, *Remarks on Ricci solitons*, Arxiv.math.DG/0411426, 2004.
[8] L. Ni, *Ancient solutions to Kahler-Ricci flow*, Math. Research letters. 11(2005)10001-10020.
[9] G. Perelman, *Ricci flow with surgery on three manifolds*, arXiv:math.Dg/0303109
[10] R. Schoen and S. T. Yau, *Lectures on Differential Geometry*, IP, Boston, 1994.
[11] S. T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana University Math. Journal, 25(1976)659-670.

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