HOMOLOGICAL STABILITY FOR FAMILIES OF COXETER GROUPS

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Abstract. We prove that certain families of Coxeter groups and inclusions $W_1 \hookrightarrow W_2 \hookrightarrow \cdots$ satisfy homological stability, meaning that in each degree the homology $H_\ast(BW_n)$ is eventually independent of $n$. This gives a uniform treatment of homological stability for the families of Coxeter groups of type $A_n$, $B_n$ and $D_n$, recovering existing results in the first two cases, and giving a new result in the third. The key step in our proof is to show that a certain simplicial complex with $W_n$-action is highly connected. To do this we show that the barycentric subdivision is an instance of the ‘basic construction’, and then use Davis’s description of the basic construction as an increasing union of chambers to deduce the required connectivity.

1. Introduction

A family of groups $G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots$ is said to satisfy homological stability if the induced maps $H_\ast(BG_{n-1}) \to H_\ast(BG_n)$ are isomorphisms when $n$ is sufficiently large relative to $\ast$. Homological stability is known for many families of groups, including symmetric groups [13], general linear groups [14], mapping class groups of surfaces [7] and 3-manifolds [11], diffeomorphism groups of highly connected manifolds [6], and automorphism groups of free groups [8], [10]. Coxeter groups are abstract reflection groups, appearing in many areas of mathematics, such as root systems and Lie theory, geometric group theory, and combinatorics. See [2], [5], and [1] for introductions to Coxeter groups from each of these three viewpoints. In this paper we will show that homological stability holds for certain families of Coxeter groups.

Recall that a Coxeter matrix on a set $S$ is an $S \times S$ symmetric matrix $M$, with values in $\mathbb{N} \cup \{\infty\}$, satisfying $m_{st} = 1$ if $s = t$ and $m_{st} \geq 2$ otherwise. The corresponding Coxeter group is the group generated by the elements of $S$, subject to the relations $(st)^{m_{st}} = e$ for $s, t \in S$. (When $m_{st} = \infty$ no relation is imposed.) It is common to represent a Coxeter matrix by the equivalent Coxeter diagram. This is the graph with vertices $S$ and edges $\{s, t\}$ for $m_{st} \geq 3$. The edge $\{s, t\}$ is labelled $m_{st}$ if $m_{st} \geq 4$.

Now consider a sequence of finite Coxeter diagrams $(\Gamma_n)_{n \geq 1}$ of the form
where every diagram has a preferred vertex, and each diagram is obtained from its predecessor by attaching a new preferred vertex to the old one by an unlabelled edge. Writing $W_n$ for the Coxeter group determined by $\Gamma_n$, the inclusion $\Gamma_{n-1} \hookrightarrow \Gamma_n$ induces an inclusion $W_{n-1} \hookrightarrow W_n$, and our main result states that the family

$$W_1 \hookrightarrow W_2 \hookrightarrow W_3 \hookrightarrow W_4 \hookrightarrow \cdots$$

satisfies homological stability.

**Main Theorem.** The map $H_*(BW_{n-1}) \to H_*(BW_n)$ is an isomorphism in degrees $2* \leq n$. Here homology is taken with arbitrary constant coefficients.

Observe that while the diagrams $\Gamma_n$ are assumed to be finite, it is not necessary for the groups $W_n$ to be finite.

**Homological stability for Coxeter groups of type $A_n$, $B_n$ and $D_n$.** The main theorem gives a uniform treatment of homological stability for the families of Coxeter groups of type $A_n$, $B_n$ and $D_n$. Recall that these are the Coxeter groups corresponding to the following diagrams, in which $n$ always denotes the total number of vertices.

![Diagram](attachment:image.png)

These families have an important place in the theory of Coxeter groups, since the classification of finite Coxeter groups states that a finite irreducible Coxeter group has type $A_n$, $B_n$ or $D_n$, or is dihedral, or is one of six exceptional examples. (See Appendix C of [5].) The sequences $\left(A_n\right)_{n \geq 1}$, $\left(B_{n+1}\right)_{n \geq 1}$ and $\left(D_{n+2}\right)_{n \geq 1}$ all have the form $(\Gamma_n)_{n \geq 1}$ described above, with the rightmost vertex taken as the preferred vertex, and therefore we may apply the main theorem to each one.

For the sequence of diagrams $(A_n)_{n \geq 1}$, the corresponding sequence of Coxeter groups is

$$\Sigma_2 \hookrightarrow \Sigma_3 \hookrightarrow \Sigma_4 \hookrightarrow \Sigma_5 \hookrightarrow \cdots$$

where $\Sigma_n$ is the symmetric group on $n$ letters and the inclusions are given by extending permutations by the identity. Applying the main theorem, we recover the following classical result.

**Corollary** (Nakaoka [13]). The map $H_*(B\Sigma_n) \to H_*(B\Sigma_{n+1})$ is an isomorphism in degrees $2* \leq n$.

For the sequence of diagrams $(B_{n+1})_{n \geq 1}$, the corresponding sequence of Coxeter groups

$$C_2 \wr \Sigma_2 \hookrightarrow C_2 \wr \Sigma_3 \hookrightarrow C_2 \wr \Sigma_4 \hookrightarrow C_2 \wr \Sigma_5 \hookrightarrow \cdots$$

consists of the wreath products of the symmetric groups with the group $C_2$ of order 2, and the inclusions are again given by extending permutations by the identity. Applying the main theorem gives the following special case of Hatcher and Wahl’s result on homological stability for wreath products. (See Proposition 1.6 of [11] and the discussion that follows it.) It also follows from Randal-Williams’s result on homological stability for unordered configuration spaces. (See Theorem A of [15] with $M = \mathbb{R}^\infty$ and $X = BC_2$.)
Corollary (Hatcher-Wahl [11]). The map $H_*(B(C_2 \wr \Sigma_n)) \to H_*(B(C_2 \wr \Sigma_{n+1}))$ is an isomorphism in degrees $2* \leq n$.

For the sequence of diagrams $(D_{n+2})_{n \geq 1}$, the corresponding sequence of Coxeter groups is

$$H_3 \hookrightarrow H_4 \hookrightarrow H_5 \hookrightarrow H_6 \hookrightarrow \cdots$$

where $H_n$ denotes the kernel of the homomorphism $C_2 \wr \Sigma_n \to C_2$ that takes the sum of the $C_2$-components. The main theorem gives the following result, which we believe to be new.

Corollary. Let $H_n$ denote the Coxeter group of type $D_n$. Then the inclusion $H_{n+1} \hookrightarrow H_{n+2}$ induces an isomorphism $H_*(BH_{n+1}) \to H_*(BH_{n+2})$ in degrees $2* \leq n$.

The concrete descriptions of the groups of type $A_n$, $B_n$ and $D_n$ that we used here can all be found in section 6.7 of [5].

Families of hyperbolic Coxeter groups. The main theorem applies to interesting families besides those of type $A_n$, $B_n$ and $D_n$ already considered. For example, if we fix an integer $m \geq 7$, then the main theorem shows that homological stability holds for the family of Coxeter groups associated to the sequence of diagrams $(\Gamma_n)_{n \geq 1}$

$$\bullet \cdots \bullet$$

$$\Gamma_n$$

in which $\Gamma_n$ has a total of $(n + 1)$ vertices, the rightmost one preferred. This family has the feature that the first group is finite, while the rest are all infinite hyperbolic. (Hyperbolicity is verified using Moussong’s condition. See Corollary 12.6.3 of [5].) It is not difficult to create other sequences of infinite hyperbolic groups to which the main theorem applies.

On the other hand, these examples show that it is possible for the first group in one of our families to be finite while the rest are infinite. Similarly, it is possible for the first group to be infinite hyperbolic while the rest are not, for example if $\Gamma_1 = \bullet \cdots \bullet$ with any preferred vertex. (The claims about hyperbolicity again follow from Moussong’s condition.)

Overview of the proof. The proof of the main theorem proceeds as follows.

First, we construct a simplicial complex $\mathcal{C}^{n}$ with an action of $W_{n}$, and prove that it is weakly Cohen-Macaulay of dimension $n$, meaning that it is $(n-1)$-connected in a certain ‘homogeneous’ way.

Second, we form the semisimplicial set $\mathcal{D}^{n}$ whose simplices are simplices of $\mathcal{C}^{n}$ with an ordering of their vertices. We show that $W_{n}$ acts transitively on the simplices in each dimension, with stabilisers the subgroups $W_{m} \subset W_{n}$ for $m < n$. From the weakly Cohen-Macaulay property of $\mathcal{C}^{n}$ we deduce that the geometric realisation $\|\mathcal{D}^{n}\|$ is $(n-1)$-connected.

Third, we study the Borel construction $EW_{n} \times_{W_{n}} \|\mathcal{D}^{n}\|$. From properties of $\mathcal{D}^{n}$ we deduce that the homology of the Borel construction matches that of $BW_{n}$ in a range of degrees, and that it can be computed by a spectral sequence whose
$E^1$-term consists of the homology groups $H_\ast(BW_m)$ for $m < n$. An argument involving this spectral sequence completes the proof.

Let us explain in more detail how we show that $C_n$ is $(n - 1)$-connected, since this is by far the hardest step in the proof. We make use of the ‘basic construction’, a technique from the topology of Coxeter groups (see chapters 5 and 8 of Davis’s book [5]). The basic construction takes a Coxeter system $(W, S)$ and a ‘mirrored space’ $X$ over $S$, and produces a topological space $U(W, X)$ with $W$-action. Now $U(W, X)$ can be expressed as an ‘increasing union of chambers’, i.e. copies of $X$, each copy of $X$ being attached to the preceding ones in a controlled way. To prove that $C_n$ is $(n - 1)$-connected, we show that the realisation of the barycentric subdivision of $C_n$ has the form $U(W_n, |\Delta|)$, where $|\Delta|$ is a topological $n$-simplex. By studying the attachments in the increasing union of chambers, we are able to deduce the required connectivity.

**Outline of the paper.** In section 2 we establish some notation, and in section 3 we prove some elementary algebraic facts about the groups $W_n$. In section 4 we define the simplicial complex $C_n$. Then we examine $C_n$ in detail: in section 5 we study the links of its simplices, in section 6 we study its action by $W_n$, and in section 7 we show that it is $(n - 1)$-connected. Then in section 8 we study the semisimplicial set of ordered simplices in $C_n$. The proof of the main theorem is completed in section 9.

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### 2. Notation

**Definition 1.** Let $(\Gamma_n)_{n \geq 1}$ be a sequence of the kind described in the introduction. We extend this sequence to the left by two terms as follows. Define $\Gamma_0$ to be the diagram obtained from $\Gamma_1$ by deleting the preferred vertex. And define $\Gamma_{-1}$ to be the diagram obtained from $\Gamma_1$ by deleting the preferred vertex and all vertices that shared an edge with it.

**Example 2** (Coxeter groups of type $A_n$, $B_n$ and $D_n$). For the sequence $(A_n)_{n \geq 1}$, the diagrams $A_0$ and $A_{-1}$ are both empty. For $(B_{n+1})_{n \geq 1}$, the diagram $B_0$ consists of a single vertex and $B_{-1+1}$ is empty. And for $(D_{n+2})_{n \geq 1}$, the diagram $D_0$ consists of two vertices with no edge and $D_{-1+2}$ is empty.

**Definition 3.** Let $(\Gamma_n)_{n \geq 1}$ be a sequence of the kind described in the introduction, and let $(\Gamma_n)_{n \geq -1}$ be the extension described above. Then for $n \geq -1$ we define $S_n$ to be the set of vertices of $\Gamma_n$, and we define $W_n$ to be the Coxeter group associated to $\Gamma_n$. Thus $(W_n, S_n)$ is a Coxeter system.

**Definition 4.** For $n \geq 1$ we define $s_n \in S_n$ to be the preferred vertex of $\Gamma_n$.

Thus $S_n = S_0 \cup \{s_1, \ldots, s_n\}$, and the parabolic subgroup of $W_n$ generated by $s_1, \ldots, s_n$ is an isomorphic copy of $\Sigma_{n+1}$, with $s_i$ acting as the transposition of $i$ and $(i + 1)$. 
3. Algebraic preliminaries

From this point onwards, unless stated otherwise we fix the sequence \((\Gamma_n)_{n \geq 1}\) and the integer \(n \geq 1\).

In several places we will consider the symbol \(s_1 \cdots s_n W_{n-1}\) for \(i\) in the range \(1 \leq i \leq n + 1\). In the case \(i = n + 1\) we take it to mean \(W_{n-1}\).

**Proposition 5.** Let \(i\) lie in the range \(1 \leq i \leq n\). Then left multiplication by the element \(s_i\) fixes the set

\[
\{ s_1 \cdots s_n W_{n-1}, s_2 \cdots s_n W_{n-1}, \ldots, s_n W_{n-1}, W_{n-1}\}.
\]

It acts on the set by transposing \(s_i \cdots s_n W_{n-1}\) and \(s_{i+1} \cdots s_n W_{n-1}\), and fixing the remaining elements.

**Proof.** The identities

\[
s_i(s_j \cdots s_n) = (s_j \cdots s_n)s_i \text{ for } i < j - 1
\]
\[
s_i(s_{i+1} \cdots s_n) = s_i \cdots s_n
\]
\[
s_i(s_i \cdots s_n) = s_{i+1} \cdots s_n
\]
\[
s_i(s_j \cdots s_n) = (s_j \cdots s_n)s_{i-1} \text{ for } i > j
\]

are simple to verify, and the claim follows immediately. \(\square\)

**Proposition 6.** \(W_{i-1} \cap (s_1 \cdots s_n W_{n-1} s_n \cdots s_i) = W_{i-2}\) for \(1 \leq i \leq n\).

**Proof.** The word \(s_1 \cdots s_n\) is \((W_{i-1}, W_{n-1})\)-reduced, meaning that it is reduced and has no reduced expression beginning with a generator of \(W_{i-1}\) or ending with a generator of \(W_{n-1}\). A result of Kilmoyer, Solomon and Tits (see Lemma 2 of [16] and the remarks that precede it) then shows that \(W_{i-1} \cap s_i \cdots s_n W_{n-1} s_1 \cdots s_i\) is the subgroup generated by \(T = S_{i-1} \cap (s_1 \cdots s_n S_{n-1} s_1 \cdots s_i)\).

So it will be enough to show that \(T = S_{i-2}\). It is immediate that \(S_{i-2} \subset T\), so suppose that \(t \in T \setminus S_{i-2}\). Thus \(t \in S_{i-1} \setminus S_{i-2}\) and \(s_n \cdots s_i t s_i \cdots s_n \in S_{n-1}\). By the first condition we have \(m(s_i, t) \geq 3\). By the second condition the word \(s_n \cdots s_i t s_i \cdots s_n\) is not reduced, so that we must be able to apply an M-move to it (see section 3.4 of [5]), and this is only possible if \(m(s_i, t)\) is exactly 3. But in this case \(s_n \cdots s_i t s_i \cdots s_n\) is already reduced, contradicting the first condition. \(\square\)

**Proposition 7.** Let \(i\) lie in the range \(1 \leq i \leq n\). If \(\sigma, \tau \in W_{n}\) satisfy

\[
\sigma s_j \cdots s_n W_{n-1} = \tau s_j \cdots s_n W_{n-1} \text{ for } j = i, \ldots, n + 1
\]

then \(\sigma^{-1} \tau \in W_{i-2}\).

**Proof.** The proposition is equivalent to the claim that

\[
W_{n-1} \cap (s_i W_{n-1} s_n) \cap \cdots \cap (s_i \cdots s_n W_{n-1} s_n \cdots s_i) = W_{i-2},
\]

which is proved by downward induction on \(i\). The initial case \(i = n + 1\) is immediate, and the induction step follows from Proposition 6. \(\square\)

**Proposition 8.** For \(c \in W_{n}\) the cosets

\[
c(s_1 \cdots s_n) W_{n-1}, \ldots, cs_n W_{n-1}, c W_{n-1}
\]

are pairwise distinct.
Proof. If \( cs_j \cdots s_n W_{n-1} = cs_k \cdots s_n W_{n-1} \) with \( j < k \), then \( (s_n \cdots s_j)(s_k \cdots s_n) \in W_{n-1} \). But \( (s_n \cdots s_j)(s_k \cdots s_n) = (s_{k-1} \cdots s_n \cdots s_{k-1})(s_{k-2} \cdots s_j) \), implying that \( s_n \in W_{n-1} \), which is a contradiction. \( \Box \)

4. The simplicial complex \( \mathcal{C}^n \)

Now we introduce the simplicial complex that will be central to our proof of the main theorem, and we prove that it is weakly Cohen-Macaulay of dimension \( n \). The proof relies on propositions that will be established in the following three sections.

Definition 9 (The simplicial complex \( \mathcal{C}^n \)). Given \( n \geq 0 \) we let \( \mathcal{C}^n \) denote the \( n \)-dimensional abstract simplicial complex with vertex set \( W_n/W_{n-1} \) and with \( k \)-simplices given by the subsets

\[
C = \{c(s_{n-k+1} \cdots s_n)W_{n-1}, \ldots, cs_nW_{n-1}, cW_{n-1}\}
\]

for \( 0 \leq k \leq n \) and \( c \in W_n \). (Proposition 8 shows that \( C \) does indeed have cardinality \( (k+1) \).) In this situation we call \( c \) a lift of the simplex \( C \).

Remark 10. We chose the name “lift” in the previous definition to emphasis the formal similarity with the concept of the same name that appears in Definition 2.1 of Wahl’s paper [17].

A given simplex can have many lifts. Choosing a lift for a simplex induces an ordering of its vertices, and all orderings occur in this way. For if \( c \) lifts a \( k \)-simplex \( C \) then so does \( cs_{n-k+1} \cdots s_n \), and the induced orderings differ by transposition of the \( i \)-th and \( (i+1) \)-st vertices (see Proposition 5). This makes it simple to verify that \( \mathcal{C}^n \) is indeed a simplicial complex, for if \( C \) is a simplex of \( \mathcal{C}^n \) and \( D \subset C \) is a nonempty subset, then we may choose a lift \( c \) of \( C \) such that \( D \) is a terminal segment in the induced ordering. Then \( c \) is also a lift of \( D \).

The natural action of \( W_n \) on \( W_n/W_{n-1} \) extends to an action on \( \mathcal{C}^n \). For if \( C \) is a simplex of \( \mathcal{C}^n \) with lift \( c \), and if \( w \in W_n \), then \( wC \) is a simplex of \( \mathcal{C}^n \) with lift \( wc \).

We now give a concrete description of \( \mathcal{C}^n \) for the families of Coxeter groups of type \( A_n \), \( B_n \) and \( D_n \) that were discussed in the introduction. Again, see section 6.7 of [5] for the concrete descriptions of these groups.

Example 11 (The family \( A_n \)). Let the sequence of diagrams \( (\Gamma_n)_{n \geq 1} \) be \( (A_n)_{n \geq 1} \), so that \( W_n = \Sigma_{n+1} \) is the symmetric group on \((n+1)\) letters. Then \( \mathcal{C}^n \) is the \( n \)-dimensional simplex \( \Delta^n \) with the action of \( \Sigma_{n+1} \) that permutes the vertices. For the vertex set of \( \mathcal{C}^n \) is \( \Sigma_{n+1}/\Sigma_n \), which is isomorphic to \( \{1, \ldots, n+1\} \) via the map that sends \( \sigma \Sigma_n \) to \( \sigma(n+1) \). Under this isomorphism, an element \( \sigma \in \Sigma_{n+1} \) is a lift of the \( k \)-simplex

\[
C = \{\sigma(n-k+1), \ldots, \sigma(n+1)\},
\]

and every subset of \( \{1, \ldots, n+1\} \) arises in this way.

Example 12 (The family \( B_n \)). Let the sequence of diagrams \( (\Gamma_n)_{n \geq 1} \) be given by \( (B_{n+1})_{n \geq 1} \), so that \( W_n = C_2 \wr \Sigma_{n+1} \). In this case \( \mathcal{C}^n \) is isomorphic to the hyperoctahedron of dimension \( n \), which is the simplicial complex whose vertex set is \( \{\pm 1, \ldots, \pm(n+1)\} \) and whose simplices are the subsets containing at
most one element from each pair \( \{i, -i\} \). In particular, its realisation is homeomorphic to the \( n \)-sphere. The action of \( C_2 \wr \Sigma_{n+1} \) on the hyperoctahedron is the one in which \( \Sigma_{n+1} \) permutes the pairs \( \pm i \) while preserving their signs, and in which the \( i \)-th copy of \( C_2 \) transposes \( i \) and \(-i\). To obtain this description, observe that the vertex set of \( \mathcal{C}^n \) is \( (C_2 \wr \Sigma_{n+1})/(C_2 \wr \Sigma_n) \), which is isomorphic to \( \{\pm 1, \ldots, \pm(n+1)\} \) via the map that sends the coset of \((\epsilon_1, \ldots, \epsilon_{n+1}, \sigma)\) to \( \epsilon_{n+1} \sigma(n+1) \). (Here we are taking \( C_2 = \{\pm 1\} \).) And under this isomorphism an element \((\epsilon_1, \ldots, \epsilon_{n+1}, \sigma)\) lifts the \( k \)-simplex
\[
C = \{\epsilon_{n-k+1} \sigma(n-k+1), \ldots, \epsilon_{n+1} \sigma(n+1)\},
\]
so that a subset of \( \{\pm 1, \ldots, \pm(n+1)\} \) spans a simplex of \( \mathcal{C}^n \) if and only if it does not contain any element and its negative.

**Example 13** (The family \( D_n \)). Let the sequence of diagrams \((\Gamma_n)_{n \geq 1}\) be given by \((D_{n+2})_{n \geq 1}\), so that \( W_n = H_{n+2} \) is the kernel of the homomorphism \( C_2 \wr \Sigma_{n+2} \to C_2 \) that takes the sum of the \( C_2 \)-components. In this case \( \mathcal{C}^n \) is the \( n \)-skeleton of the \((n+1)\)-dimensional hyperoctahedron, with the action inherited from the action of \( C_2 \wr \Sigma_{n+2} \). (See Example 12.) In particular, the realisation of \( \mathcal{C}^n \) has the homotopy type of the wedge of \((2^n - 1)\) copies of the \( n \)-dimensional sphere. To obtain this description observe that the vertex set \( H_{n+2}/H_{n+1} = (C_2 \wr \Sigma_{n+2})/(C_2 \wr \Sigma_{n+1}) \) can be identified with \( \{\pm 1, \ldots, \pm(n+2)\} \) via the map sending the coset of \((\epsilon_1, \ldots, \epsilon_{n+2}, \sigma)\) to \( \epsilon_{n+2} \sigma(n+2) \), and that under this identification the \( k \)-simplex with lift \((\epsilon_1, \ldots, \epsilon_{n+2}, \sigma)\) is
\[
C = \{\epsilon_{n-k+2} \sigma(n-k+2), \ldots, \epsilon_{n+2} \sigma(n+2)\},
\]
so that a subset of \( \{\pm 1, \ldots, \pm(n+2)\} \) spans a simplex if and only if it does not contain any element and its negative.

Recall from Definition 3.4 of [11] that a simplicial complex is called *weakly Cohen-Macaulay of dimension* \( n \) if it is \((n-1)\)-connected and the link of each \( p \)-simplex is \((n-p-2)\)-connected. In each of the three examples above, \( \mathcal{C}^n \) has the homotopy type of a wedge of \( n \)-dimensional spheres, and so is \((n-1)\)-connected. In fact, it is not hard to see that in these examples \( \mathcal{C}^n \) is weakly Cohen-Macaulay of dimension \( n \). This is an instance of the following general fact.

**Theorem 14.** \( \mathcal{C}^n \) is weakly Cohen-Macaulay of dimension \( n \).

The proof of this theorem is assembled from Propositions [15][21] and [22] which are proved over the course of the next three sections.

**Proof.** By Proposition [15] if \( C \) is a \( p \)-simplex of \( \mathcal{C}^n \) then \( \text{lk}_{\mathcal{C}^n}(C) \cong \mathcal{C}^{n-p-1} \). It therefore suffices to show that \( \mathcal{C}^n \) is \((n-1)\)-connected for all \( n \), or equivalently that the barycentric subdivision \( \text{sd} \mathcal{C}^n \) is \((n-1)\)-connected for all \( n \). Now Proposition [21] shows that \( \text{sd} \mathcal{C}^n \) is homeomorphic to the basic construction \( \mathcal{U}(W_n, |\Delta|) \), while Proposition [22] shows that \( \mathcal{U}(W_n, |\Delta|) \) is \((n-1)\)-connected. This completes the proof. \( \square \)

5. Links of simplices of \( \mathcal{C}^n \)

**Proposition 15.** Let \( C \) be a \( p \)-simplex of \( \mathcal{C}^n \). Then \( \text{lk}_{\mathcal{C}^n}(C) \cong \mathcal{C}^{n-p-1} \).
Proof. Choose a lift $c$ of $C$. Define

$$
\phi: W_{n-p-1}/W_{n-p-2} \longrightarrow W_n/W_{n-1}
$$

by $\phi(dW_{n-p-2}) = cds_{n-p} \cdots s_n W_{n-1}$ for $d \in W_{n-p-1}$. This is well defined since every generator of $W_{n-p-2}$ commutes with $s_{n-p}, \ldots, s_n$. Observe that the domain and range of $\phi$ are the vertex sets of $\mathbb{C}^{n-p-1}$ and $\mathbb{C}^n$ respectively.

**Claim 1.** $\phi$ is an injection.

To prove this claim let $d, d' \in W_{n-p-1}$ satisfy

$$
\phi d(s_{n-k} \cdots s_n)W_{n-1} = \phi d'(s_{n-k} \cdots s_n)W_{n-1}.
$$

Then

$$
d^{-1}d' \in W_{n-p-1} \cap (s_{n-p} \cdots s_n)W_{n-1}(s_{n-1} \cdots s_{n-p}) = W_{n-p-2},
$$

the latter equation by Proposition 6. Thus $d'W_{n-p-2} = dW_{n-p-2}$.

**Claim 2.** $\phi$ sends simplices of $\mathbb{C}^{n-p-1}$ to simplices of $\text{lk}_{\mathbb{C}^n}(C)$.

To prove this, suppose that $D$ is an $i$-simplex of $\mathbb{C}^{n-p-1}$. Let $d \in W_{n-p-1}$ be a lift of $D$. Then

$$
\phi D = \{cds_{n-p-i} \cdots s_n W_{n-1}, \ldots, cds_{n-p} \cdots s_n W_{n-1}\}
$$

while

$$
C = \{cs_{n-p+1} \cdots s_n W_{n-1}, \ldots, cs_n W_{n-1}, cW_{n-1}\}
$$

$$
= \{cds_{n-p+1} \cdots s_n W_{n-1}, \ldots, cd_s W_{n-1}, cd W_{n-1}\}.
$$

Thus $\phi D \cap C = \emptyset$ by Proposition 8 and $\phi D \cup C$ is a simplex of $\mathbb{C}^n$ with lift $cd$, so that $\phi D$ is a simplex of $\text{lk}_{\mathbb{C}^n}(C)$ as claimed.

**Claim 3.** Every simplex of $\text{lk}_{\mathbb{C}^n}(C)$ has the form $\phi D$ for some simplex $D$ of $\mathbb{C}^{n-p-1}$.

To prove this, suppose that $\tilde{D}$ is an $i$-simplex of $\text{lk}_{\mathbb{C}^n}(C)$. Then $\tilde{D} \cap C = \emptyset$ and $\tilde{D} \cup C$ is a simplex of $\mathbb{C}^n$. Let $c'$ be a lift of $\tilde{D} \cup C$, and assume without loss that the ordering it induces on $\tilde{D} \cup C$ contains $\tilde{D}$ as an initial segment and $C$ as a terminal segment with the ordering induced by $c$. Thus

$$
\tilde{D} = \{c'(s_{n-p-i} \cdots s_n)W_{n-1}, \ldots, c'(s_{n-p} \cdots s_n)W_{n-1}\}
$$

and

$$
c'(s_{n-p+j} \cdots s_n)W_{n-1} = c(s_{n-p+j} \cdots s_n)W_{n-1}
$$

for $j = 1, \ldots, p+1$. The latter gives $c^{-1}c' \in W_{n-p-1}$ by Proposition 7 so that $c' = cd$ for some $d \in W_{n-p-1}$. Then $\tilde{D} = \phi D$, where $D$ is the $i$-simplex of $\mathbb{C}^{n-p-1}$ with lift $d$.

We can now prove the proposition. Combining the first claim with the third in the case of 0-simplices, we see that $\phi$ is an isomorphism between the vertex sets of $\mathbb{C}^{n-p-1}$ and $\text{lk}_{\mathbb{C}^n}(C)$. The second and third claims then show that $\phi$ induces an isomorphism of simplicial complexes from $\mathbb{C}^{n-p-1}$ to $\text{lk}_{\mathbb{C}^n}(C)$. □
6. The action of $W_n$ on $\mathcal{C}^n$ and $\text{sd} \mathcal{C}^n$

Recall that if $X$ is an abstract simplicial complex, then its barycentric subdi-
vision $\text{sd} X$ is the abstract simplicial complex whose vertices are the simplices
of $X$, and in which $\{C_0, \ldots, C_k\}$ is a simplex if, after possibly reordering the
elements, we have $C_0 \subset \cdots \subset C_k$.

Here we will study the barycentric subdivision $\text{sd} \mathcal{C}^n$. The action of $W_n$ on
$\mathcal{C}^n$ induces an action of $W_n$ on $\text{sd} \mathcal{C}^n$. This new action automatically has the
property that if $w \in W_n$ and $C$ is a simplex of $\text{sd} \mathcal{C}^n$, then $w$ fixes every vertex
of $C \cap wC$. (See p.115 of [3].) In particular any element of the stabilizer of $C$
must fix $C$ pointwise.

**Lemma 16.** The action of $W_n$ on $\mathcal{C}^n$ is transitive on the set of $k$-simplices
for each $k$. For a $k$-simplex $C$ of $\mathcal{C}^n$, any permutation of the vertices of $C$
is realised by some element of $W_n$ that fixes $C$. Every simplex of $\mathcal{C}^n$ is a face of
an $n$-simplex.

**Proof.** For the first claim, let $C$ and $D$ be $k$-simplices of $\mathcal{C}^n$ with respective lifts
$c$ and $d$. Then $(dc^{-1})C = D$. For the second claim, let $C$ be a simplex of $\mathcal{C}^n$
with lift $c$, and let $\phi$ be a permutation of the vertices of $C$. Let $c'$ be the lift of $C$
whose induced ordering is obtained from the ordering induced by $c$ by applying
$\phi$. Then $c'c^{-1}$ sends each vertex of $C$ to its image under $\phi$. The final claim
follows because a $k$-simplex with lift $c$ is a face of the $n$-simplex with lift $c$. $\Box$

**Lemma 17.** $W_n$ acts transitively on the $n$-simplices of $\text{sd} \mathcal{C}^n$. Every simplex of
$\text{sd} \mathcal{C}^n$ is a face of an $n$-simplex.

**Proof.** Since $\mathcal{C}^n$ has dimension $n$, the set of $n$-simplices of $\text{sd} \mathcal{C}^n$ is in natural
bijection with the set of $n$-simplices of $\mathcal{C}^n$ equipped with an ordering of their
vertices. Lemma 16 shows that the action of $W_n$ on this set is transitive, proving
the first claim. The same lemma shows that every simplex of $\mathcal{C}^n$ is a face of an
$n$-simplex, and the second claim follows immediately. $\Box$

To understand the action of $W_n$ on $\text{sd} \mathcal{C}^n$ we may now concentrate on a single
$n$-simplex.

**Definition 18.** Let $\Delta = \{a_0, \ldots, a_n\}$ denote the $n$-simplex of $\text{sd} \mathcal{C}^n$ with
vertices $a_i = \{(s_{i+1} \cdots s_n)W_{n-1}, \ldots, s_nW_{n-1}, eW_{n-1}\}$ for $i = 1, \ldots, n$. For each
$s \in S_n$ define a face $\Delta_s$ of $\Delta$ as follows. If $s \in S_1 \setminus S_{i-1}$ for $i = 1, \ldots, n$ then
$$\Delta_s = \{a_0, \ldots, \hat{a}_i, \ldots, a_n\},$$
and if $s \in S_{-1}$ then $\Delta_s = \Delta$.

**Example 19** (The case $n = 2$). Let $n = 2$. The diagram on the left shows the $2$-simplex $\{s_1s_2W_1, s_2W_1, W_1\}$ of $\mathcal{C}^2$ with lift the identity element, and the
diagram on the right shows $\Delta$ as a face of the subdivision of this simplex.
The next diagrams show $\Delta_{s_2}$, $\Delta_{s_1}$, $\Delta_t$ and $\Delta_u$ for $t \in S_0 \setminus S_{-1}$, and $u \in S_{-1}$.

Observe that $s_2$ acts on $\{s_1s_2W_1, s_2W_1, W_1\}$ as the reflection that fixes $s_1s_2W_1$ and interchanges $W_1$ and $s_2W_1$, and $\Delta_{s_2}$ is exactly the part of $\Delta$ fixed by this reflection. Similarly, $s_1$ acts as the reflection that fixes $W_1$ and interchanges $s_1s_2W_1$ and $s_2W_1$, and $\Delta_{s_1}$ is precisely the part of $\Delta$ fixed by this reflection. Next, $t$ can be thought of as “reflection in the edge $\{W_1, s_2W_1\}$”, and $\Delta_t$ is precisely the part of $\Delta$ that it fixes. Finally, $u$ acts trivially on $\Delta$ since it lies in $W_1$ and commutes with $s_1$ and $s_2$, and indeed $\Delta_u = \Delta$ is the part of $\Delta$ fixed by $u$.

**Lemma 20.** Let $F$ be a face of $\Delta$. Then the stabilizer of $F$ under the action of $W_n$ is the subgroup generated by those $s \in S_n$ for which $F \subset \Delta_s$.

**Proof.** For the purposes of this proof, given $i \geq 0$ we write $S_{=i}$ for the difference $S_i \setminus S_{i-1}$. So for $i \geq 1$ we have $S_{=i} = \{s_i\}$, while $S_{=0}$ is the set of elements of $S_0$ that do not commute with $s_1$.

For $i \geq 0$, the stabilizer of $a_i$ is the subgroup of $W_n$ generated by $S_n \setminus S_{=i}$. For Proposition 5 shows that $s_{i+1}, \ldots, s_n$ fix $a_i = \{s_{i+1}, \ldots, s_nW_{n-1}, \ldots, s_nW_{n-1}, W_{n-1}\}$ and generate all permutations of its elements, while Proposition 7 shows that the subgroup consisting of elements that fix every element of $a_i$ is $W_{i-1}$.

Let $F = \{a_1, \ldots, a_i\}$. Then the stabilizer of $F$ is the intersection of the stabilizers of the $a_i$. By the last paragraph this is the intersection of the subgroups generated by the sets $S_n \setminus S_{=i}$, and by a general result (see Theorem 4.1.6 of [5]) this is the subgroup generated by $\bigcap (S_n \setminus S_{=i}) = S_n \setminus \bigcup S_{=i}$.

Now $F \subset \Delta_s$ for all $s \in S_{-1}$, and $F \subset \Delta_s$ for $s \in S_{=i}$ if and only if $a_i \notin F$. Thus the set of $s$ such that $F \subset \Delta_s$ is $S \setminus \bigcup S_{=i}$. Thus the subgroup generated by the $s$ such that $F \subset \Delta_s$ is precisely the pointwise stabilizer of $F$.  

7. **The barycentric subdivision of $\mathcal{C}^n$ and the basic construction**

The ‘basic construction’ is a method for building and studying certain spaces with group action. It can be used, for example, to study the topology of the Coxeter complex and Davis complex of a Coxeter group. In this section we will show that $|s \Delta \mathcal{C}^n|$ is an instance of the basic construction, and we will use this to show that it is $(n-1)$-connected.

To begin we recall the relevant notions from section 5.1 of [5]. (These are tailored to the case of Coxeter groups. For an approach to the basic construction that applies to more general groups see chapter II.12 of [1] ). Let $(W, S)$ be a Coxeter system. A mirrored space over $S$ is a space $X$ together with subspaces $X_s \subset X$, called mirrors, one for each $s \in S$. We assume that $X$ is a CW-complex and that the mirrors are subcomplexes. The basic construction is the
space

\[ \mathcal{U}(W, X) = (W \times X) / \sim \]

where \((v, x) \sim (w, y)\) if and only if \(x = y\) and \(v^{-1}w\) belongs to the subgroup generated by the \(s \in S\) for which \(x \in X_s\). The basic construction is equipped with the action of \(W\) by left translation, and we identify \(X\) with the image of \(\{e\} \times X\) in \(\mathcal{U}(W, X)\). Observe that \(\mathcal{U}(W, X)\) has the structure of a CW-complex in which each translate \(wX\) is a subcomplex.

For us the most important feature of the basic construction is that it can be described as an increasing union of chambers, meaning copies of \(X\), as we now recall from section 8.1 of [5]. Given \(w \in W\), let \(\text{In}(w) = \{ s \in S \mid \ell(ws) < \ell(w) \}\) denote the set of letters with which a reduced expression for \(w\) can end, and let \(X^{\text{In}(w)} = \bigcup_{s \in \text{In}(w)} X_s\) denote the corresponding union of mirrors. Now order the elements of \(W\) as \(w_0, w_1, w_2, \ldots\) where \(w_0 = e\) and \(\ell(w_m) \leq \ell(w_{m+1})\) for \(m \geq 0\). Define

\[ P_m = \bigcup_{i=0}^m w_iX, \]

so that \(\mathcal{U}(W, X)\) is the increasing union of the subcomplexes \(P_m\). Then

\[ P_m = P_{m-1} \cup w_mX \quad \text{and} \quad P_{m-1} \cap w_mX = w_mX^{\text{In}(w_m)}. \]

The latter equation is by Lemma 8.1.1 of [5], and it will be useful to us since it specifies exactly how each chamber is attached to its predecessor.

Now let us show that \(|\text{sd } \mathbb{C}^n|\) is an instance of the basic construction. Fix the Coxeter system \((W_n, S_n)\). Recall the simplex \(\Delta\) and the faces \(\Delta_s\) introduced in Definition 18. We make \(|\Delta|\) into a mirrored space over \(S_n\) by defining the mirror \(|\Delta|_s\) to be the subspace \(|\Delta_s|\) of \(|\Delta|\). We may therefore form the basic construction \(\mathcal{U}(W_n, |\Delta|)\). The inclusion \(|\Delta| \hookrightarrow |\text{sd } \mathbb{C}^n|\) sends \(|\Delta_s|\) into the subset fixed by \(s\), and so extends uniquely to a \(W_n\)-equivariant map \(\mathcal{U}(W_n, |\Delta|) \rightarrow |\text{sd } \mathbb{C}^n|\).

**Proposition 21.** The map \(\mathcal{U}(W_n, |\Delta|) \rightarrow |\text{sd } \mathbb{C}^n|\) is a homeomorphism.

**Proof.** The map is surjective because any point of \(|\text{sd } \mathbb{C}^n|\) is in a translate of \(|\Delta|\). This follows from Lemma 17, which shows that every simplex of \(\text{sd } \mathbb{C}^n\) is a face of a translate of \(\Delta\).

The map is injective because if \(x \in |\Delta|\), then the image of \(x\) in \(\mathcal{U}(W_n, |\Delta|)\) has stabilizer generated by those \(s\) for which \(x \in |\Delta|_s\). To see this, let \(F\) denote the unique face of \(\Delta\) for which \(x\) lies in the interior of \(|F|\). Then \(x \in |\Delta|_s = |\Delta_s|\) if and only if \(F \subset \Delta_s\), and the stabilizer of \(x\) is precisely the stabilizer of \(F\). The result then follows from Lemma 20.

The map is a homeomorphism because \(|\text{sd } \mathbb{C}^n|\) has the weak topology with respect to the realizations of its simplices. By Lemma 17 this coincides with the weak topology with respect to the realizations of its \(n\)-simplices. This is exactly the topology of \(\mathcal{U}(W_n, |\Delta|)\). \(\square\)

**Proposition 22.** \(\mathcal{U}(W_n, |\Delta|)\) is \((n - 1)\)-connected.

This relies on the following two lemmas.

**Lemma 23.** For \(w \in W_n\), \(w \neq e\), the space \(|\Delta|^{\text{In}(w)}\) is \((n - 2)\)-connected.
Proof. The set \( \text{In}(w) \) is nonempty since \( w \neq e \). Thus \( |\Delta|^{\text{In}(w)} \) is either \( |\Delta| \), or it is a nonempty union of facets of \( |\Delta| \). In the first case it is contractible, and in the second case it is either contractible (if not all facets are in the union) or it is \( \partial |\Delta| \cong S^{n-1} \) (if all facets are in the union). In all cases it is \((n-2)\)-connected. \( \square \)

Lemma 24. Let \( n \geq 1 \). Suppose that \((X; A, B)\) is a CW-triad in which \( A \) and \( B \) are \((n-1)\)-connected and \( C = A \cap B \) is \((n-2)\)-connected. Then \( X \) is \((n-1)\)-connected.

Proof. For \( n = 1 \) this is immediate since the union of two path-connected spaces with nonempty intersection is path-connected. So we assume that \( n \geq 2 \). The pairs \((A, C)\) and \((B, C)\) are \((n-1)\)-connected, and \( C \) is path-connected, so that Theorem 4.23 of [9] can be applied to show that \( \pi_i(A, C) \to \pi_i(X, B) \) is an isomorphism for \( i < 2n-2 \), and in particular for \( i \leq (n-1) \). Thus \((X, B)\) is \((n-1)\)-connected, and the same then follows for \( X \) itself. \( \square \)

Proof of Proposition 22. If \( n = 0 \) then the claim is that \( \mathcal{U}(W_n, |\Delta|) \) is nonempty, which holds vacuously. So we may assume that \( n \geq 1 \).

As in the discussion at the start of the section, order the elements of \( W_n \) as \( w_0, w_1, w_2, \ldots \) starting with the identity and respecting the length. Then \( \mathcal{U}(W_n, |\Delta|) \) is the union of subcomplexes \( P_0 \subset P_1 \subset P_2 \subset \cdots \) where \( P_0 = |\Delta| \) and

\[
P_m = P_{m-1} \cup w_m|\Delta| \quad \text{with} \quad P_{m-1} \cap w_m|\Delta| = w_m|\Delta|^{\text{In}(w_m)}.
\]

It will suffice to show that each \( P_m \) is \((n-1)\)-connected. We do this by induction on \( m \).

In the initial case \( m = 0 \) we have \( P_0 = e|\Delta| \), which is contractible and so the claim holds. For the induction step we take \( m \geq 1 \) and assume that \( P_{m-1} \) is \((n-1)\)-connected. Then \( P_m = P_{m-1} \cup w_m|\Delta| \) is the union of the subcomplexes \( P_{m-1} \) and \( w_m|\Delta| \), and their intersection \( w_m|\Delta|^{\text{In}(w_m)} \) is \((n-2)\)-connected by Lemma 23. Thus \((P_m; P_{m-1}, w_m|\Delta|)\) is a CW-triad in which the subspaces \( P_{m-1} \) and \( w_m|\Delta| \) are \((n-1)\)-connected and their intersection is \((n-2)\)-connected. It now follows from Lemma 24 that \( P_m \) is \((n-1)\)-connected as required. \( \square \)

8. The ordered simplices of \( C^n \)

In this section we introduce a semisimplicial set \( D^n \) and identify it as the semisimplicial set of ordered simplices in \( C^n \). We then use the fact that \( C^n \) is weakly Cohen-Macaulay of dimension \( n \) to deduce that the geometric realisation \( |D^n| \) is \((n-1)\)-connected, an approach we learned from Wahl’s paper [17] (see in particular Proposition 7.9 of [17], which is due to Randal-Williams). In this section and the next we will use semisimplicial spaces and their realisations. The background material we require can be found in section 2 of [15].

Definition 25. Let \( D^n \) denote the semisimplicial set with \( k \)-simplices.

\[
D_k^n = \begin{cases} W_n/W_{n-k-1} & k \leq n \\ \emptyset & k > n \end{cases}
\]

and with face maps

\[
d_i : W_n/W_{n-k-1} \to W_n/W_{n-k}
\]
defined by
\[ d_i(cW_{n-k-1}) = c(s_{n-k+i} \cdots s_{n-k+1})W_{n-k} \]
for \( i = 0, \ldots, k \).

It is a simple exercise to verify that the face maps \( d_i \) satisfy the relations \( d_i \circ d_j = d_{i+1} \circ d_i \) for \( i < j \). Alternatively, it is a consequence of the proof of Proposition 27 below.

**Definition 26.** Let \( X \) be a simplicial complex. By an ordered simplex of \( X \), we mean a simplex of \( X \) equipped with an ordering of its vertices. The semisimplicial set of ordered simplices in \( X \), denoted \( X^{\text{ord}} \), has for its \( k \)-simplices the ordered \( k \)-simplices in \( X \), with face maps \( d_i \) given by forgetting the \( i \)-th vertex of an ordered simplex.

**Proposition 27.** \( D^n \) is isomorphic to \( e^{n,\text{ord}} \).

**Proof.** We define \( \phi_k : D^n_k \rightarrow e^{n,\text{ord}}_k \) by
\[ \phi_k(cW_{n-k-1}) = \{c(s_{n-k+1} \cdots s_n)W_{n-1}, \ldots, cs_nW_{n-1}, cW_{n-1}\} \]
for \( cW_{n-k-1} \in W_n/W_{n-k-1} \). In other words, \( \phi_k(cW_{n-k-1}) \) is the \( k \)-simplex with lift \( c \), equipped with the ordering induced by \( c \). The map \( \phi_k \) is well-defined because the generators of \( W_{n-k-1} \) all commute with \( s_{n-k+1}, \ldots, s_n \). It is surjective because by definition every simplex admits a lift, and any ordering of a simplex is afforded by some lift (see the paragraph following Definition 9). It is injective because if \( \phi_k(cW_{n-k-1}) = \phi_k(c'W_{n-k-1}) \) then \( cs_i \cdots s_nW_{n-1} = c's_i \cdots s_nW_{n-1} \) for \( i = n-k+1, \ldots, n+1 \), so that \( cW_{n-k-1} = c'W_{n-k-1} \) by Proposition 7.

To complete the proof we must show that the face maps in \( e^{n,\text{ord}} \) and \( D^n \) are compatible under the \( \phi_k \). In other words, given \( 0 \leq i \leq k \leq n \), we must show that
\[ \phi_{k-1} \circ d_i = d_i \circ \phi_k. \]

Observe from the definition of \( d_i \) in \( D^n \) that for \( i \geq 1 \) we have \( d_i(cW_{n-k-1}) = d_{i-1}(cs_{n-k+i}W_{n-k+1}) \). On the other hand, Proposition 5 shows that \( \phi_k(cW_{n-k-1}) \) and \( \phi_k(cs_{n-k+i}W_{n-k-1}) \) differ only by the transposition of their \( (i-1) \)-st and \( i \)-th vertices, so that \( d_i(\phi_k(cW_{n-k-1})) = d_{i-1}(\phi_k(cs_{n-k+i}W_{n-k-1})) \). Thus the claim will follow by induction on \( i \) so long as we can show that
\[ \phi_{k-1} \circ d_0 = d_0 \circ \phi_k. \]

This follows by inspection. \( \square \)

**Corollary 28.** \( \|D^n\| \) is \((n-1)\)-connected.

**Proof.** Theorem 14 shows that \( e^n \) is weakly Cohen-Macaulay of dimension \( n \). Proposition 7.9 of [17] shows that if a simplicial complex \( X \) is weakly Cohen-Macaulay of dimension \( n \), then \( \|X^{\text{ord}}\| \) is \((n-1)\)-connected. Consequently \( \|e^{n,\text{ord}}\| \) is \((n-1)\)-connected, and by Proposition 27 the same holds for \( \|D^n\| \). \( \square \)
9. Completion of the proof

We now complete the proof of the main theorem. This section is modelled closely on section 5 of [15], from which there is little essential difference. It is also similar to the proof of Theorem 2 of [12].

We regard $D^n$ as a simplicial space by equipping its constituent sets with the discrete topology. Then we form a semisimplicial space

$$EW_n \times_{W_n} D^n$$

by setting $(EW_n \times_{W_n} D^n)_k = EW_n \times_{W_n} D^n_k$ and using the face maps obtained from those of $D^n$.

Lemma 29. The projection $EW_n \times_{W_n} D^n_0 \to BW_n$ makes $EW_n \times_{W_n} D^n$ into an augmented simplicial space over $BW_n$, and the induced map $\|EW_n \times_{W_n} D^n\| \to BW_n$ is $(n - 1)$-connected.

Proof. The composites of the projection with $d_0$ and $d_1$ coincide, so that the projection is indeed an augmentation. Since $EW_n \to BW_n$ is a locally trivial principal $W_n$-bundle, it follows that $\|EW_n \times_{W_n} D^n\| \to BW_n$ is a locally trivial bundle with fibre $\|W_n \times_{W_n} D^n\| \cong \|D^n\|$, which is $(n - 1)$-connected by Corollary [28], so that the map itself is $(n - 1)$-connected.

Lemma 30. There are homotopy equivalences $EW_n \times_{W_n} D^n_k \simeq BW_{n-k-1}$ under which the face maps $d_i : EW_n \times_{W_n} D^n_k \to EW_n \times_{W_n} D^n_{k-1}$ are all homotopic to the stabilization map $BW_{n-k-1} \to BW_{n-k}$, and under which the composite $EW_n \times_{W_n} D^n_0 \to \|EW_n \times_{W_n} D^n\| \to BW_n$ is homotopic to the stabilization map $BW_{n-1} \to BW_n$.

Proof. There is an isomorphism

$$EW_n \times_{W_n} D^n_k = EW_n \times_{W_n} (W_n/W_{n-k-1}) \xrightarrow{\cong} EW_n/W_{n-k-1}$$

sending the orbit of $(x, cW_{n-k-1})$ to the orbit of $c^{-1}x$. This identifies $d_i$ with the map

$$EW_n/W_{n-k-1} \to EW_n/W_{n-k}$$

sending the $W_{n-k-1}$-orbit of $x$ to the $W_{n-k}$-orbit of $(s_{n-k+1} \cdots s_{n-k+i})x$. Since $(s_{n-k+1} \cdots s_{n-k+i})$ commutes with every element of $W_{n-k-1}$, this map is homotopic to the one sending the $W_{n-k-1}$-orbit of $x$ to the $W_{n-k}$-orbit of $x$. Now the equivariant homotopy equivalences

$$EW_{n-k-1} \to EW_n, \quad EW_{n-k} \to EW_n$$

induce homotopy equivalences

$$BW_{n-k-1} \to EW_n/W_{n-k-1}, \quad BW_{n-k} \to EW_n/W_{n-k}$$

under which the map $EW_n/W_{n-k-1} \to EW_n/W_{n-k}$ just described becomes the stabilization map.

The skeletal filtration of $\|EW_n \times_{W_n} D^n\|$ leads to a first-quadrant spectral sequence

$$E^{1}_{k,l} = H_l(EW_n \times_{W_n} D^n_k) \Rightarrow H_{k+l}(\|EW_n \times_{W_n} D^n\|)$$
in which the differential $d^1$ is given by the alternating sum $\sum_{i=0}^{k}(-1)^i(d_i)_*$ of the maps induced by the face maps. Lemma 30 allows us to identify the $E^1$-term of this spectral sequence: $E_{k,l}^1 = H_l(BW_{n-k-1})$, and $d^1: E_{k,l}^1 \to E_{k-1,l}^1$ is the stabilization map $H_l(BW_{n-k-1}) \to H_l(BW_n)$ if $k$ is even, and is zero if $k$ is odd.

**Lemma 31.** Assume that for all $m < n$ the stabilization map $H_l(BW_{m-1}) \to H_l(BW_n)$ is an isomorphism in degrees $2l \leq m$. Then the spectral sequence has the following properties:

1. $E_{0,l}^\infty = \cdots = E_{0,l}^2 = E_{0,l}^1$ for $2l \leq n$.
2. $E_{k,l}^\infty = 0$ for $k > 0$ and $2(k+l) \leq n$.

**Proof.** The assumption allows us to deduce that $E_{k,l}^2 = 0$ when $k \geq 1$ is odd and $2l + k + 1 \leq n$, and that $E_{k,l}^2 = 0$ when $k \geq 2$ is even and $2l + k \leq n$.

For in the first case $d^1: E_{k+1,l}^1 \to E_{k,l}^1$ is the stabilization map $H_l(BW_{n-k-2}) \to H_l(BW_{n-k-1})$, and in the second case $d^1: E_{k,l}^1 \to E_{k-1,l}^1$ is the stabilization map $H_l(BW_{n-k-1}) \to H_l(BW_{n-k})$, and our assumption means that both are isomorphisms in the given range.

To prove the first claim, observe that $E_{0,l}^2 = E_{0,l}^1$ since $d^1: E_{1,l}^1 \to E_{0,l}^1$ is zero. The remaining differentials with target in bidegree $(0,l)$ are $d^k: E_{k,l-k+1}^k \to E_{0,l}^k$ with $k \geq 2$, and these have domain zero since $2(l-k+1)+k \leq 2(l-k+1)+k+1 = 2l - k + 2 \leq 2l \leq n$ so that $E_{k,l-k+1}^2 = 0$. To prove the second claim, observe that if $2(k+l) \leq n$ and $k > 0$, then certainly $2l + k < 2(l+k) \leq 2l + k + 1 \leq 2(k+l) \leq n$, so that $E_{k,l}^2 = 0$.

We can now complete the proof of the main theorem, showing by induction on $n \geq 2$ that $H_l(BW_{n-1}) \to H_l(BW_n)$ is an isomorphism for $2l \leq n$. For $n = 2$ the claim is that $H_l(BW_1) \to H_l(BW_2)$ is an isomorphism for $l = 0, 1$. For $l = 0$ this is trivial since both $BW_1$ and $BW_2$ are connected. For $l = 1$ this follows from the well-known fact that if $(W,S)$ is a Coxeter system then $H_l(BW)$ is isomorphic to the elementary abelian 2-group generated by the elements of $S$, subject to the relation that identifies $s,t \in S$ if $m_{st}$ is odd. Take $n > 2$ and suppose that the theorem holds for all smaller integers.

Lemma 31 shows that the composite

$$H_l(W_{n-1}) = E_{0,l}^1 \to E_{0,l}^\infty \to H_l(\|EW_n \times W_n \mathcal{D}^n\|) \to H_l(BW_n)$$

is the stabilization map, and we must show that this is an isomorphism for $2l \leq n$. Lemma 31 shows that the first two arrows are isomorphisms in this range, while Lemma 29 shows that the last map is an isomorphism for $l \leq n-1$, which holds since $2l \leq n$ and $n \geq 3$.

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