On $C^*$-algebras related to asymptotic homomorphisms

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Abstract

We study the $C^*$-algebras related to Mishchenko's version of asymptotic homomorphisms. In particular we show that their different versions are weakly homotopy equivalent but not isomorphic to each other. We give also the continuous version for these algebras.

1 Introduction

Let $M_k$, $k \in \mathbb{N}$, denote the $k \times k$ matrix $C^*$-algebra. To make notation uniform we will write $M_\infty$ for the $C^*$-algebra $K$ of compact operators. Let $\kappa = \{\kappa(1), \kappa(2), \ldots\}$ be a function from $\mathbb{N}$ to $\mathbb{N} \cup \{\infty\}$, i.e. $\kappa$ is a sequence with $\kappa(i)$, $i \in \mathbb{N}$, being either integers or infinity. We suppose that the sequences $\kappa = \{\kappa(i)\}$ are monotonely non-decreasing and either stabilize (there exists $\lim_{i \to \infty} \kappa(i)$) or approach infinity as $i \to \infty$.

Consider the quotient $C^*$-algebra

$$\prod_{i=1}^{\infty} M_{\kappa(i)}/ \bigoplus_{i=1}^{\infty} M_{\kappa(i)},$$

which is the target $C^*$-algebra for discrete asymptotic homomorphisms [1, 2], i.e. any discrete asymptotic homomorphism from a $C^*$-algebra $A$ into matrix algebras or into compacts gives a genuine $*$-homomorphism from $A$ into this target algebra. In [4] Mishchenko suggested to consider another version of discrete asymptotic homomorphisms, namely to add the following property: if $\varphi_n : A \to B$ is a discrete asymptotic homomorphism, it should satisfy also

$$\lim_{i \to \infty} \|\varphi_{i+1}(a) - \varphi_i(a)\| = 0$$

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for any \( a \in A \). The corresponding target \( C^* \)-algebras were constructed as follows. Since any sequence \( \kappa \) is non-decreasing, there is a natural inclusion \( M_{\kappa(i)} \subseteq M_{\kappa(i+1)} \), hence the right shift

\[
\alpha : (m_1, m_2, \ldots) \mapsto (0, m_1, m_2, \ldots),
\]

where \( m = (m_1, m_2, \ldots) \in \Pi_{i=1}^{\infty} M_{\nu(i)} \), is well-defined both on \( \Pi_{i=1}^{\infty} M_{\nu(i)} \) and on \( \Pi_{i=1}^{\infty} M_{\kappa(i)}/ \oplus_{i=1}^{\infty} M_{\kappa(i)} \). The map \( \alpha \) is an endomorphism of these \( C^* \)-algebras, and it is easy to see that the \( \alpha \)-invariant subset in \( \Pi_{i=1}^{\infty} M_{\kappa(i)}/ \oplus_{i=1}^{\infty} M_{\kappa(i)} \) is a \( C^* \)-algebra too. We denote this \( \alpha \)-invariant \( C^* \)-algebra by \( Q(\kappa) \).

If the sequence \( \kappa \) stabilizes, \( \lim_{i \to \infty} \kappa(i) = n \), \( n \in \mathbb{N} \cup \{ \infty \} \), we denote such \( \kappa \) by \( n \) (or by \( \infty \) if \( n = \infty \)). In this case we write \( Q(n) \) instead of \( Q(\kappa) \). The \( C^* \)-algebra \( Q(\kappa) \) can be described as a set \( B(\kappa) \) of all sequences \( (m_1, m_2, \ldots) \) of matrices \( m_i \in M_{\kappa(i)} \) such that

\[
\lim_{i \to \infty} \|m_{i+1} - m_i\| = 0
\]

modulo the sequences converging to zero. The set \( B(\kappa) \) is a \( C^* \)-algebra too, so \( Q(\kappa) \) is a quotient \( C^* \)-algebra of \( B(\kappa) \) modulo the ideal \( I(\kappa) \) of sequences converging to zero. Moreover, the ideal \( I(\kappa) \) is an essential ideal in \( B(\kappa) \), so \( B(\kappa) \) lies in the multiplier \( C^* \)-algebra \( M(I(\kappa)) \).

Note that though the \( C^* \)-algebras \( Q(\kappa) \) consist only of \( \alpha \)-invariant elements of \( \Pi_i, M_{\kappa(i)}/ \oplus_i M_{\kappa(i)} \), one should not think that \( Q(\kappa) \) are smaller than the latter algebras. For example, consider \( Q(\infty) \) and let \( r = r(n) \) be a sequence of increasing integers with

\[
\lim_{i \to \infty} (r(i+1) - r(i)) = \infty
\]

then it is easy to see that the sequences \( (m_1, m_2, \ldots) \) with \( m_{r(i)} = 0 \) form an ideal \( I_r \) of \( Q(\infty) \) and the corresponding quotient \( C^* \)-algebra \( Q(\infty)/I_r \) can be mapped to \( \Pi_i, M_{\infty}/ \oplus_i M_{\infty} \) by sending \( (m_1, m_2, \ldots) \) to \( (m_{r(1)}, m_{r(2)}, \ldots) \). Using (2) one can lift any element of \( \Pi_i, M_{\infty}/ \oplus_i M_{\infty} \) by linear interpolation, hence this map is epimorphic.

Remark that the \( C^* \)-algebra \( Q(1) \) is commutative, and its spectrum is the Higson corona, i.e. the quotient space of the Stone–Čech corona of \( \mathbb{N} \) modulo the standard action of \( \mathbb{Z} \). It is easy to see that the \( C^* \)-algebra \( Q(\kappa) \) is unital if and only if \( \kappa = n \) for some finite \( n \). In this case one has \( Q(n) = Q(1) \otimes M_n \). Otherwise \( Q(\kappa) \) contains the tensor product \( Q(1) \otimes M_{\infty} \) as a proper subalgebra. Indeed, the lifting of \( Q(1) \otimes M_{\infty} \) to \( B(\kappa) \) consists of uniformly vanishing sequences \( m = (m_1, m_2, \ldots) \in B(\kappa) \). This means that for any such \( m \) and for any \( \varepsilon > 0 \) there exists a number \( j \) such that

\[
\|(m_1, m_2, \ldots) - (p_j m_1 p_j, p_j m_2 p_j, \ldots)\| < \varepsilon,
\]

where \( p_j \in M_{\infty} \) denotes a projection onto the first \( j \) coordinates. For any \( \kappa \) approaching infinity it is easy to find an element of \( B(\kappa) \), which is not uniformly vanishing. To do so consider a sequence \( \{n_i\} \) with \( \lim_{i \to \infty} (n_{i+1} - n_i) = \infty \) and a non-zero element \( \lambda = (\lambda_1, \lambda_2, \ldots) \in B(1) \) such that \( \lambda_{n_i} = 0 \) for all \( i \in \mathbb{N} \). Let \( \chi_{[n_i, n_{i+1}]} \) be the characteristic function for the set \( [n_i, n_{i+1}] \subset \mathbb{N} \). Let \( \{k_i\} \) be another sequence approaching infinity and
let \( e_i \in M_\infty \) be the projection onto the \( i \)-th coordinate. Then the element \( \oplus_i \chi_{[n_i, n_{i+1})} \lambda e_{k_i} \) does not lie in \( B(1) \otimes M_\infty \).

2 Weak homotopic equivalence of \( Q(\kappa) \)

The \( C^* \)-algebras \( Q(\kappa) \) are quotient algebras modulo an essential ideal, so it is natural to consider the corresponding weaker notion of homotopy for \( * \)-homomorphisms into corona algebras. Remember that a function \( g = g(t) : [0, 1] \to B(\kappa) \) is called continuous with respect to the strict topology \([3]\) if the functions \( g(t)k(t) \) and \( k(t)g(t) \) are continuous in \( C([0, 1], I(\kappa)) \) for any continuous function \( k = k(t) \in C([0, 1], I(\kappa)) \). The \( C^* \)-algebra of all continuous functions with respect to the strict topology we denote by \( C_{str}([0, 1], B(\kappa)) \). The quotient maps \( C_{str}([0, 1], B(\kappa)) \to C_{str}([0, 1], B(\kappa))/C([0, 1], I(\kappa)) \) and \( B(\kappa) \to Q(\kappa) \) we denote by \( q \).

Let \( f_0, f_1 : A \to Q(\kappa) \) be two \( * \)-homomorphisms. They are called homotopic (as maps into quotient algebra) if there exists a family of maps (not necessarily \( * \)-homomorphisms) \( F_t : A \to B(\kappa) \) such that \( F_t(a) \) is continuous with respect to the strict topology for any \( a \in A \), the composition \( q \circ F_t : A \to C([0, 1], Q(\kappa)) \) is a \( * \)-homomorphism and \( q \circ F_i \) coincides with \( f_i \), \( i = 0, 1 \).

As we are dealing with non-separable \( C^* \)-algebras, we have to consider even weaker homotopy equivalence. Let \( A, B \) be \( C^* \)-algebras. By \([A, B]\) we denote the set of all \( * \)-homomorphisms from \( A \) to \( B \).

**Definition 2.1** We call \( A \) and \( B \) weakly homotopy equivalent if for any separable \( C^* \)-algebra \( C \) there are natural (with respect to \( C \)) maps

\[
\phi = \phi_C : [C, A] \to [C, B] \quad \text{and} \quad \psi = \psi_C : [C, B] \to [C, A]
\]

such that they preserve homotopy of \( * \)-homomorphisms and that for any \( f : C \to A \) and for any \( g : C \to B \) the compositions \( (\psi \circ \phi)(f) \) and \( (\phi \circ \psi)(g) \) are homotopic to \( f \) and \( g \) respectively.

**Lemma 2.2** If \( A \) and \( B \) are weakly homotopy equivalent separable \( C^* \)-algebras then they are homotopy equivalent.

**Proof.** Put \( f = \phi_A(id_A) : A \to B \) and \( g = \psi_B(id_B) : B \to A \). By \( f^* \) we denote the map \([B, X] \to [A, X]\) generated by the \( * \)-homomorphism \( f \). Then one has \( f = f^*(id_B) \) and naturality of \( \psi \) implies that

\[
\psi_A \phi_A(id_A) = \psi_A f^*(id_B) = f^* \psi_B(id_B) = f^* g = g \circ f
\]

so by definition \( g \circ f \) is homotopic to \( id_A \). By the same way one can show that \( f \circ g \) is homotopic to \( id_B \). \( \square \)

Though this weak homotopy equivalence is very weak indeed, it preserves the \( K \)-theory.
Lemma 2.3 If $C^*$-algebras $A$ and $B$ are weakly homotopy equivalent then $K_*(A) \cong K_*(B)$.

Proof. It is easy to see that weak homotopy equivalence of $C^*$-algebras $A$ and $B$ implies the same equivalence of $C^*$-algebras $M_\infty \otimes A$ and $M_\infty \otimes B$. The groups $K_0$ and $K_1$ are defined by homotopy classes of $\ast$-homomorphisms into these algebras of the separable $C^*$-algebras $C$ and $C_0(\mathbb{R})$ respectively. $\square$

The $K$-groups of the $C^*$-algebras $Q(\kappa)$ were calculated in [3].

Proposition 2.4 ([3]) For any $\kappa$ one has $K_0(Q(\kappa)) = \mathbb{Z}$ and $K_1(Q(\kappa)) = 0$. $K_0(Q(\kappa))$ is generated by the sequence $(e_1,e_1,e_1,\ldots)$, where $e_1$ is the projection onto the first coordinate. $\square$

Proposition 2.5 Let $\kappa_1$ and $\kappa_2$ be two sequences approaching infinity but $\kappa_j \neq \infty$, $j = 1,2$. Then the $C^*$-algebras $Q(\kappa_1)$ and $Q(\kappa_2)$ are homotopy equivalent.

Proof. Let $\kappa_3$ be a sequence majorizing both $\kappa_1$ and $\kappa_2$. Transitivity of homotopy equivalence shows that we can prove homotopy equivalence only for $Q(\kappa_3)$.

As $\kappa_1(i) \leq \kappa_3(i)$ for any $i \in \mathbb{N}$, there exists a natural inclusion $f : Q(\kappa_1) \to Q(\kappa_3)$. To define a map $g$ in the other direction consider an element $m = (m_1,m_2,\ldots) \in B(\kappa_3)$, define numbers $r = r(i)$ by $\kappa_1(i-1) < \kappa_3(r) \leq \kappa_1(i)$ and put $(g(m))_i = m_r$. Then $g(m) \in B(\kappa_1)$ and $g(m) \in I(\kappa_1)$ whenever $m \in I(\kappa_3)$, so the map $g : Q(\kappa_3) \to Q(\kappa_1)$ is well-defined. Then the composition $g \circ f : Q(\kappa_1) \to Q(\kappa_1)$ is reduced to renumbering: a sequence $m = (m_1,m_2,\ldots) \in B(\kappa_1)$ is mapped to a sequence

$$(m_1, \ldots, m_1, m_2, \ldots, m_2, \ldots).$$

(3)

We are going to construct the homotopy $F_t : Q(\kappa_1) \to C_{str}([0,1],B(\kappa_1))$ connecting $g \circ f$ with $id_{Q(\kappa_1)}$. For any $a \in Q(\kappa_1)$ choose a representative $m \in B(\kappa_1)$ of the form (3). For $t \in [0,1/2]$ define $F_t(a)$ by connecting (by a linear path) the sequence (3) with the sequence

$$F_{1/2}(a) = (m_1, m_2, \ldots, m_2, m_3, \ldots, m_3, \ldots).$$

(4)

For $t \in [1/2,3/4]$ we connect the sequence (4) with the sequence

$$F_{3/4}(a) = (m_1, m_2, m_3, \ldots, m_3, m_4, \ldots, m_4, \ldots),$$

$$s_1+s_2-1 \text{ times} \hspace{1cm} s_3 \text{ times}$$

$$s_1, m_2, \ldots, m_2, \ldots, m_3, \ldots, m_3, \ldots, \ldots,$$

$$s_1+s_2+s_3-1 \text{ times} \hspace{1cm} s_4 \text{ times}$$

etc. Finally we put $F_1(a) = (m_1,m_2,\ldots)$. It is easy to see that $F_t(a)$ is continuous with respect to the strict topology (one has to check this only at $t = 1$), that $q \circ F_t(a)$ does not depend on choice of a representative $m$, and that $F_t(\cdot)$ is a $\ast$-homomorphism modulo $C([0,1],I(\kappa_1))$, so it gives the necessary homotopy. A similar homotopy connects $f \circ g$ with $id_{Q(\kappa_3)}$. $\square$
Proposition 2.6 Let $A$ be a separable $C^*$-algebra and let $f : A \to Q(\infty)$ be a $*$-homomorphism. Then there exists a finite sequence $\kappa$ such that $f(A) \subset Q(\kappa) \subset Q(\infty)$.

**Proof.** Fix a dense sequence $\{a_i\}, \ i \in \mathbb{N}$, in $A$ and a monotonely decreasing sequence of numbers $\{\varepsilon_n\}, \ n \in \mathbb{N}$, such that $\lim_{n \to \infty} \varepsilon_n = 0$. Let $m^{(i)} = (m_1^{(i)}, m_2^{(i)}, \ldots) \in B(\infty)$ be liftings for $f(a_i)$. Then the numbers $\kappa(n)$ can be chosen to satisfy

$$\|m_j^{(i)} - p_{\kappa(n)} m_j^{(i)} p_{\kappa(n)}\| < \varepsilon_n$$

for all $1 \leq i, j \leq n$, where $p_{\kappa(n)} \in M_{\infty}$ are projections onto the first $\kappa(n)$ coordinates. It follows from (5) that for any $i$ the sequences

$$(m_1^{(i)}, m_2^{(i)}, \ldots) \quad \text{and} \quad (p_{\kappa(1)} m_1^{(i)} p_{\kappa(1)}, p_{\kappa(2)} m_2^{(i)} p_{\kappa(2)}, \ldots)$$

give the same element in $Q(\infty)$ and the last sequence lies in $B(\kappa)$. As $Q(\kappa) \subset Q(\infty)$ is a $C^*$-subalgebra, $f$ is continuous, and $f(a_i) \subset Q(\kappa)$, so $f(A) \subset Q(\kappa)$ too. \qed

Propositions 2.5 and 2.6 imply

**Theorem 2.7** If $\kappa$ approaches infinity then all $C^*$-algebras $Q(\kappa)$ are weakly homotopy equivalent to each other. \qed

### 3 Growth functions and $Q(\kappa)$

Now we are going to show that the $C^*$-algebras $Q(\kappa)$ can be different for different growth of $\kappa$. We say that $\kappa$ has polynomial growth if there exist numbers $0 < C_1 < C_2$ and $n \geq 1$ such that

$$C_1 i^n \leq \kappa(i) \leq C_2 i^n \quad \text{and} \quad \lim_{i \to \infty} \frac{\kappa(i+1) - \kappa(i)}{i^n} = 0.$$ 

It has exponential growth if there exists a number $a > 1$ such that for all $i$ one has

$$\kappa(i+1) \geq a \kappa(i).$$

**Theorem 3.1** Let $\kappa_1$ have polynomial growth and let $\kappa_2$ have exponential growth.

1. There exists a non-trivial $Q(1)$-valued trace on $Q(\kappa_1)$.

2. There is no non-trivial trace on $Q(\kappa_2)$.

**Proof.** 1. We define a trace on the $C^*$-algebra $B(\kappa_1)$ and then show that it vanishes on the ideal $I(\kappa_1)$. Let $m = (m_1, m_2, \ldots) \in B(\kappa_1)$, $m_i \in M_{\kappa_1(i)}$. Put

$$\tau(m) = \left( \frac{\text{tr}_{\kappa_1(1)}(m_1)}{\kappa_1(1)}, \frac{\text{tr}_{\kappa_1(2)}(m_2)}{\kappa_1(2)}, \frac{\text{tr}_{\kappa_1(3)}(m_3)}{\kappa_1(3)}, \ldots \right),$$

where $\text{tr}_{\kappa_1(i)}$ denotes the trace on $M_{\kappa_1(i)}$. It follows from the definition of the trace and the growth function that $\tau(m)$ is well-defined. To show that $\tau(m)$ vanishes on $I(\kappa_1)$, it suffices to show that for all $m_i$ the sequence $\frac{\text{tr}_{\kappa_1(i)}(m_i)}{\kappa_1(i)}$ converges to zero as $i \to \infty$. This follows from the definition of the trace and the growth function.

2. The proof is similar to the previous case but uses the fact that $\kappa_2$ has exponential growth.
where \( \text{tr}_i \) is the standard trace on \( M_i \) normalized by \( \text{tr}_i(1_i) = i \) for \( 1_i \) being the identity of \( M_i \). If \( \|m\| = 1 \) then \( \text{tr}_{\kappa_i(i)}(m_i) \leq \kappa_i(i) \), so \( \|\tau(m)\| \leq 1 \). One has also

\[
\frac{\|\text{tr}_{\kappa_i(i+1)}(m_{i+1}) - \text{tr}_{\kappa_i(i)}(m_i)\|}{\kappa_i(i+1)} \leq \frac{\|\text{tr}_{\kappa_i(i+1)}(m_{i+1}) - \text{tr}_{\kappa_i(i)}(m_i)\|}{\kappa_i(i+1)} + \frac{\|\text{tr}_{\kappa_i(i)}(m_i)\|}{\kappa_i(i+1)} \leq \|m_{i+1} - m_i\| + \frac{1}{\kappa_i(i+1)} \to 0,
\]

hence the image of \( \tau \) lies in \( B(1) \). Finally the property \( \tau(ab) = \tau(ba) \) for \( a, b \in B(\kappa_1) \) is obvious. So \( \tau \) is a \( B(1) \)-valued trace on \( B(\kappa_1) \). It is also obvious that \( \tau(I(\kappa_1)) \subset I(1) \), so \( \tau \) is well-defined on the quotient algebra \( Q(\kappa_1) \). It remains to show that \( \tau \) is non-trivial. Let

\[ m_i = \text{diag} \left\{ 1, \frac{\kappa_i(i) - 1}{\kappa_1(i)}, \ldots, \frac{1}{\kappa_1(i)} \right\} \]

be diagonal matrices in \( M_{\kappa_i(i)} \). Then

\[ \|m_{i+1} - m_i\| = \frac{\kappa_i(i+1) - \kappa_i(i)}{\kappa_i(i+1)} \leq \frac{1}{C_1} \frac{\kappa_i(i+1) - \kappa_i(i)}{i^\beta} \to 0, \]

so the sequence \( m = (m_1, m_2, \ldots) \) lies in \( B(\kappa_1) \), hence it defines an element in \( Q(\kappa_1) \) and one has

\[ \tau(m) = \left( \frac{\kappa_1(1) + 1}{2\kappa_1(1)}, \ldots, \frac{\kappa_i(i) + 1}{2\kappa_i(i)}, \ldots \right). \]

As \( \lim \frac{\kappa_i(i+1)}{2\kappa_i(i)} = \frac{1}{2} \neq 0 \), the trace \( \tau \) is non-trivial.

2. For convenience of notation we assume that \( a = 2 \) and that \( \kappa_2(i+1) \geq 2\kappa_2(i) \). We also assume that all \( M_i \) are embedded in the \( C^* \)-algebra of bounded operators on a Hilbert space \( H \) with a fixed basis \( \xi_1, \xi_2, \ldots \). Let \( u_1, u_2 \) be isometries on \( H \) such that \( u_1(\xi_k) = \xi_{2k-1}, u_2(\xi_k) = \xi_{2k}, k \in \mathbb{N} \). Let \( m = (m_1, m_2, \ldots) \in B(\kappa_2) \). Then one can define an element \( m \oplus m \in B(\kappa_2) \) as follows. Put \( (m \oplus m)_1 = 0 \) and

\[ (m \oplus m)_{i+1} = u_1m_1u_1^* + u_2m_2u_2^* \]

for \( i > 1 \). Then \( \text{dim}(m \oplus m)_{i+1} = 2\kappa_2(i) \leq \kappa_2(i+1) \), and it is easy to see that \( \|(m \oplus m)_{i+1} - (m \oplus m)_i\| \to 0 \). As \( \|(m \oplus m)_i\| \to 0 \) whenever \( m \in I(\kappa_2) \), so the map \( m \mapsto (m \oplus m) \) defines a homomorphism \( Q(\kappa_2) \to Q(\kappa_2) \). In a similar way one can define a direct sum of 4, 8, etc. summands.

Now suppose that there exists a trace \( \tau \) on \( Q(\kappa_2) \) with the property \( \tau(ab) = \tau(ba) \) for \( a, b \in Q(\kappa_2) \). Then \( \tau \) extends to a trace on \( B(\kappa_2) \) and we still denote this extension by \( \tau \). Let \( m \in B(\kappa_2) \) be a positive element with \( \|m\| = 1 \) and \( \tau(m) \neq 0 \). One has \( m \oplus m = m^{(1)} + m^{(2)} \), where

\[ m^{(1)} = (0, u_1m_1u_1^*, u_1m_2u_1^*, u_1m_3u_1^*, \ldots), \]

\[ m^{(2)} = (0, u_2m_1u_2^*, u_2m_2u_2^*, u_2m_3u_2^*, \ldots). \]
As all \( m_i \) are positive and of dimension \( \kappa(i) \), so the sequences
\[
a_j = (0, u_j(m_1)^{1/2}, u_j(m_2)^{1/2}, u_j(m_3)^{1/2}, \ldots)
\]
and
\[
b_j = (0, (m_1)^{1/2} u_j^*, (m_2)^{1/2} u_j^*, (m_3)^{1/2} u_j^*, \ldots),
\]
j = 1, 2, are well-defined as elements of the algebra \( B(\kappa_2) \). One has
\[
\tau(m(j)) = \tau(a_j b_j) = \tau(b_j a_j) = \tau(\alpha(m)),
\]
where \( \alpha \) is the shift \([1]\), so \( \tau(m(j)) = \tau(m) \), as \( m = \alpha(m) \) modulo \( I(\kappa_2) \). Therefore \( \tau(m \oplus m) = 2\tau(m) \). By the same way we can show that for any \( k \) one has
\[
\tau(\underbrace{m \oplus m \oplus \ldots \oplus m}_{2^k \text{ times}}) \in B(\kappa_2)
\]
and
\[
\tau(\underbrace{m \oplus m \oplus \ldots \oplus m}_{2^k \text{ times}}) = 2^k \tau(m). \quad (6)
\]
On the other hand it is easy to see that
\[
\|\underbrace{m \oplus m \oplus \ldots \oplus m}_{2^k \text{ times}}\| = \|m\|,
\]
hence
\[
\tau(\underbrace{m \oplus m \oplus \ldots \oplus m}_{2^k \text{ times}}) \leq 1 \quad (7)
\]
and the contradiction between \((6)\) and \((7)\) shows that \( \tau(m) = 0 \) and finishes the proof. □

Remember that a \( C^* \)-algebra \( A \) is called stable if \( A \cong A \otimes M_\infty \).

**Corollary 3.2** If \( \kappa \) is of polynomial growth (or smaller) then the \( C^* \)-algebra \( Q(\kappa) \) is not stable. If \( \kappa \) is of exponential growth then \( Q(\kappa) \) is stable. □

**Corollary 3.3** Let \( \kappa \) be of polynomial growth. Let \( \tau \) be a trace constructed in Theorem 3.1 and let \( a \in Q(1) \otimes M_\infty \subset Q(\kappa) \). Then \( \tau(a) = 0 \). □

Let \( \kappa \) be of polynomial growth. Consider a \( C^* \)-subalgebra \( I_\tau \subset Q(\kappa) \) generated by all positive elements \( a \in Q(\kappa) \) with \( \tau(a) = 0 \). As
\[
\tau(ba) = \tau(a^{1/2} ba^{1/2}) \leq \|b\| \tau(a) = 0
\]
for any \( b \in Q(\kappa) \), so \( I_\tau \) is an ideal in \( Q(\kappa) \). By Lemmas 2.3 and 3.3 the \( K \)-groups of \( I_\tau \) coincide with those of \( Q(\kappa) \) and the inclusion \( I_\tau \subset Q(\kappa) \) induces an isomorphism of their \( K \)-groups. Hence the six-term exact sequence shows that the quotient \( C^* \)-algebra \( Q(\kappa)/I_\tau \) is \( K \)-contractible.
4 Continuous version for \( Q(\kappa) \)

Let \( C_b([0, \infty), B) \) (resp. \( C_0([0, \infty), B) \)) be the \( C^* \)-algebra of bounded continuous (resp. vanishing at infinity) functions on the half-line taking values in a \( C^* \)-algebra \( B \). The quotient algebra \( Q_b(B) = C_b([0, \infty), B)/C_0([0, \infty), B) \) is usually used for describing continuous asymptotic homomorphisms. But this algebra is too big and one has often to make reparametrization in order to obtain slowly varying functions. Consider at first the case \( B = M_\infty \). There exists a natural inclusion

\[
Q(\infty) \longrightarrow Q_b(M_\infty),
\]

given by the natural inclusion \( N \subset [0, \infty) \) and by linear interpolation from sequences of compact operators to the \( M_\infty \)-valued functions. This inclusion is obviously not epimorphic.

For a function \( f(t) \in C_b([0, \infty), M_\infty) \) and for an interval \([a, b] \subset [0, \infty)\) put

\[
\text{Var}_{a,b} f = \sup_{t_1, t_2 \in [a, b]} \| f(t_2) - f(t_1) \|
\]

and consider in \( C_b([0, \infty), M_\infty) \) a subset of functions with variation vanishing at infinity

\[
C_{b}^{\text{vv}}([0, \infty), M_\infty) = \{ f(t) \in C_b([0, \infty), M_\infty) : \lim_{t \to \infty} \text{Var}_{a+t,b+t} f = 0 \}.
\]

This subset is a \( \ast \)-subalgebra and it does not depend on choice of \( a \) and \( b \). Denote by \( Q_b^{\text{vv}}(M_\infty) \) the quotient \( C^* \)-algebra \( C_{b}^{\text{vv}}([0, \infty), M_\infty)/C_0([0, \infty), M_\infty) \).

**Lemma 4.1** \( C^* \)-algebras \( Q(\infty) \) and \( Q_b^{\text{vv}}(M_\infty) \) are isomorphic.

**Proof.** The image of \( Q(\infty) \) under the map \( \text{(8)} \) obviously lies in \( Q_b^{\text{vv}}(M_\infty) \). To get the map in the other direction one has to assign to a function \( f(t) \in C_{b}^{\text{vv}}([0, \infty), M_\infty) \) representing an element of \( Q_b^{\text{vv}}(M_\infty) \) a sequence \( \{ f(n) \} \) of its values at integer points. One can directly check that these \( \ast \)-homomorphisms are inverse to each other. \( \square \)

**Proposition 4.2** The \( C^* \)-algebras \( Q_b(M_\infty) \) and \( Q_b^{\text{vv}}(M_\infty) \) are weakly homotopy equivalent.

**Proof.** It is easy to see that for any \( \ast \)-homomorphism of a separable \( C^* \)-algebra \( A \) into \( Q_b(M_\infty) \) one can find a reparametrization \( s = s(t) \) such that after passing from \( t \) to \( s(t) \) the image of \( A \) lies in \( Q_b^{\text{vv}}(M_\infty) \). \( \square \)

To imitate the continuous version for general \( \kappa \) one should consider telescopes. Let

\[
T(\kappa) = \{ f(t) \in C_b([0, \infty), M_\infty) : f(t) \in M_{\kappa(i)} \text{ for } t \in [i, i+1) \}
\]
be the telescope $C^*$-algebra. Put

$$C_0([0, \infty), \kappa) = \{ f(t) \in T(\kappa) : \lim_{t \to \infty} \| f(t) \| = 0 \}$$

and

$$C_{\text{du}}([0, \infty), \kappa) = \{ f(t) \in T(\kappa) : \lim_{t \to \infty} \var T_{t, t+1} f = 0 \}.$$

**Lemma 4.3** $C^*$-algebras $Q(\kappa)$ and $C_{\text{du}}([0, \infty), \kappa)/C_0([0, \infty), \kappa)$ are isomorphic. ☐

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