CP odd weak basis invariants in minimal see-saw model and Leptogenesis

Madan Singh*

Department of Physics, M.N.S Government College Bhiwani, Bhiwani, Haryana, 127021, India.
*singhmadan179@gmail.com

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Abstract

In this paper, we derive the relationship between the weak basis invariants (WB) related to CP violation responsible for leptogenesis and CP violation relevant at low energy. We examine all the experimental viable cases of Frampton-Glashow and Yanagida (FGY) model, in order to construct the WB invariants in terms of left handed Majorana neutrino mass matrix elements, and thus finding the necessary and sufficient condition for CP conservation. Further for all the viable FGY texture zeros, we have shown the explicit dependence of WB invariants on Dirac type and Majorana type CP violating phases. In the end, we discuss the implication of such interrelationships on leptogenesis.

1 Introduction

The origin of CP violation is one of the outstanding challenges in the fermion sector. In the Standard model (SM) [1] CP violation is related to the mixing between the flavor and mass eigen states, also known as Cabibo-Kobayashi-Maskawa mechanism (CKM) [2] in the scenario of three families of quarks and non-degenerate masses, and is well established in $K^0 - \bar{K}^0$ system. On the other hand, in the lepton sector, neutrinos are exactly massless Weyl particle and lepton flavor mixing does not exist, implying that there is no CP violation in the lepton sector. However, several neutrino oscillation experiments [3–6] provide us with very strong evidence regarding the non-zero neutrino masses as well as mixings. This, in consequence, provides the first sign to search for new physics and necessitates to look beyond the Standard model. In any extended model of SM, which incorporates neutrino masses and mixing, CP violation naturally appears in the leptonic sector. In the leptonic sector, CP violation have profound implication in cosmology, playing a pivotal role in the generation of matter-antimatter asymmetry of the universe via leptogenesis [7]. In this regard, seesaw
mechanism \[8\], is widely considered to be the most plausible candidate, which, not only, explain the smallness of neutrino masses in a natural way but also provides the origin of baryon asymmetry of the Universe. The seesaw mechanism, in fact, connects the small neutrino masses to very heavy right-handed neutrino masses. In general it contains more physical parameters than can be measured at low energies. In an attempt to reduce the number of seesaw parameters, several theoretical ideas have been proposed either by introducing the texture zeros in Yukawa coupling Dirac neutrino matrix or by reducing the right handed heavy Majorana neutrinos. Among them, the most economical is the imposition of two zeros in Dirac neutrino mass matrix in the scheme of minimal seesaw model \[9,10\], popularly known as Frampton-Glashow- Yanagida (FGY) model \[11\]. However, the introduction of zeros in any specific model are not weak basis(WB) invariants, implying that a given set of texture zeros which exist in a certain WB may not be present or may appear in different entries in another WB, while leading to the same physics. This, in turn, brings forward a question of how to recognize the same texture zero model written in different bases where symmetry (or special texture zero) is not apparent. In such a scenario, CP odd weak basis invariants (WB) is considered to be an invaluable tool, and widely followed in the literature. The WB invariants were first used in \[12\] to study the CP violation in the quark sector. Similarly, leptonic WB invariants were presented for studying the CP conditions at low energy \[13–16\]. To investigate the CP violation at high energies one requires to establish a connection between the low energy physics and physics at high energies, for instance leptogenesis \[15,16\], \[17–19\], and the imposition of texture zeros in the scenario of minimal seesaw model (MSM) may serve the purpose in this regard. This makes the study of CP-odd WB invariants relevant for the model under consideration. In addition, it is crucial to examine the interrelationships between the CP-odd invariants which are required to vanish as a necessary and sufficient condition for CP conservation.

The present paper aims to study the implication of CP odd invariants for FGY ansätze. To this end, we first of all construct the CP-odd WB invariants relevant for leptogenesis (at high energies) in terms of left handed Majorana mass matrix elements at low energies for viable ansätze, and then find the necessary and sufficient condition of CP conservation. Further we derive an analytical relations showing an explicit dependence of the CP-odd WB invariants on Dirac/Majorana CP violating phase. In the end we re-investigate the implications of such interrelationships on leptogenesis for each ansätz.

2 FGY ansätze in minimal seesaw model

In the present analysis, we take into account a most simple and economical see saw model \[9,10\], which incorporates the two heavy right handed neutrinos \(N_{1,2}\) having strong hierarchical pattern (i.e. \(M_2 > M_1\)), and keep the Lagrangian of electroweak interactions invariant under the \(SU(2)_L \times U(1)_Y\) gauge transformation \[11\]. After the spontaneous electroweak symmetry breaking, this simple but interesting model
leads to the following neutrino mass term:

\[-L_{\text{mass}} = (\nu_e, \nu_\mu, \nu_\tau) M_D \begin{pmatrix} N_1^c \\ N_2^c \end{pmatrix} + \frac{1}{2}(N_1^c, N_2^c) M \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} + h.c,\]  

(1)

where \(N_i^c \equiv CN_i^T (i = 1, 2)\) with \(C\) being the charge-conjugation operator; and \((\nu_e, \nu_\mu, \nu_\tau)\) denote the left-handed neutrinos. \(M_D\) and \(M\) denote a \(3 \times 2\) Dirac neutrino mass matrix, and \(2 \times 2\) symmetric Majorana neutrino mass matrix, respectively. The scale of \(M_D\) is characterized by the electroweak scale \(v = 174\) GeV. In contrast, the scale of \(M\) can be much higher than \(v\), because \(N_1^c\) and \(N_2^c\) are \(SU(2)_L\) singlets and their corresponding mass term is not subject to the scale of gauge symmetry breaking. Then one may obtain the effective (light and left-handed) neutrino mass matrix \(M_\nu\) via the well-known seesaw mechanism [8]

\[M_\nu \approx M_D M^{-1} M_D^T.\]  

(2)

Without loss of generality, both heavy right-handed Majorana neutrino mass matrix \(M\), and the charged lepton mass matrix \(M_l\) are assumed to be diagonal, real and positive; i.e.,

\[M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_l = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix},\]  

(3)

where \(M_{1,2}\) denotes the masses of two heavy Majorana neutrinos. The choice of this specific basis implies that one of the light (left-handed) Majorana neutrinos must be zero. On the other hand, \(M_D\) is a complex \(3 \times 2\) rectangular matrix, and can be given as

\[M_D = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix},\]  

(4)

where, \(a_1, a_2, b_1, b_2, c_1, c_2\) denote the complex entries.

The minimal seesaw model itself has no restriction on the structure of \(M_D\). Framp-ton, Glashow and Yanagida [11] first introduce the two zeros, with a aim to restrict the structure of \(M_D\), whose origin comes from an underlying horizontal flavor symmetry. Such ansätze have been investigated by many authors, while taking into account both strongly hierarchical (i.e. \(M_1 << M_2\)) [20,23] as well as nearly degenerate (i.e. \(M_1 \approx M_2\)) [24,25] neutrino spectrum of heavy right-handed Majorana neutrinos. Among the fifteen different possibilities of Eq.(4), only four are found to be compatible with neutrino oscillation data for inverted mass ordering, while same are ruled out for normal mass ordering [24]. The four viable FGY ansätze are given below:

Type1: \(M_D = \begin{pmatrix} a_1 & 0 \\ b_1 & b_2 \\ 0 & c_2 \end{pmatrix}\),  

Type2: \(M_D = \begin{pmatrix} a_1 & 0 \\ 0 & b_2 \\ c_1 & c_2 \end{pmatrix}\).
Type3 : \( M_{\text{D}} = \begin{pmatrix} 0 & a_2 \\ b_1 & 0 \\ c_1 & c_2 \end{pmatrix} \),  
Type4 : \( M_{\text{D}} = \begin{pmatrix} 0 & a_2 \\ b_1 & b_2 \\ c_1 & 0 \end{pmatrix} \).  

It is worthwhile to note here that in the MSM, the low-energy phenomenological implications are driven by \( M_{\nu} \), while cosmological baryon number asymmetry is associated with \( M_D \) via the leptogenesis mechanism.

### 3 Parameterization of lepton mass matrices in MSM

Before proceeding further, we briefly go through the different parameterizations used for effective Majorana neutrino mass matrix \( (M_{\nu}) \) and Yukawa coupling Dirac neutrino mass matrix \( (M_D) \), respectively. These may be useful for deriving the relationship between CP odd invariants related to CP violation at high energies and CP violation at low energies. As mentioned earlier, the lightest neutrino in the MSM must be massless, therefore we are then left with either \( m_1 = 0 \) (normal mass ordering) or \( m_3 = 0 \) (inverted mass ordering). Since normal mass ordering is ruled out for all the FGY ansätze, therefore we restrict our analysis for inverted mass ordering. In the basis of diagonal \( M_l \), \( M_\nu \) can be parameterized as follows

\[
M_\nu \equiv \begin{pmatrix} m_{ee} & m_{e\mu} & m_{e\tau} \\ m_{e\mu} & m_{\mu\mu} & m_{\mu\tau} \\ m_{e\tau} & m_{\mu\tau} & m_{\tau\tau} \end{pmatrix} = V \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T, \tag{6}
\]

where \[26\]

\[
V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -c_{12}s_{23}s_{13} - s_{12}c_{23}e^{-i\delta} & -s_{12}s_{23}s_{13} + c_{12}c_{23}e^{-i\delta} & s_{23}c_{13} \\ -c_{12}s_{23}s_{13} + s_{12}c_{23}e^{-i\delta} & -s_{12}s_{23}s_{13} - c_{12}c_{23}e^{-i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}
\]

Here, \( c_{ij} = \cos \theta_{ij}, \ s_{ij} = \sin \theta_{ij} \) for \( i, j = 1, 2, 3 \), and \( \delta, \sigma \) denote the Dirac and Majorana CP violating phase, respectively. From Eqs.(6) and (7), it is obvious that \( M_\nu \) depends on seven low energy physical parameters: two neutrino masses \( (m_1, m_2) \), three mixing angles \( (\theta_{12}, \theta_{23}, \theta_{13}) \), two CP violating phases \( (\delta, \sigma) \), therefore, one can trivially derive each element of \( M_\nu \) in terms of these parameters. The number of available parameters here, is lesser than that found in \( M_D \), which reduces to nine after eliminating the three trivial phases by rephasing the charged-lepton field in the chosen basis. To account this difference, Casas-Ibarra-Ross \[9,10\] introduce a orthogonal complex matrix \( R \)

\[
R = \begin{pmatrix} 0 & 0 \\ \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}, \tag{8}
\]
for $m_1 = 0$, and for $m_3 = 0$ is

$$R = \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (9)$$

The complex parameter $z$ encodes the two hidden parameters \textit{viz.} a real parameter and one phase, which are required to match the total number of parameters at high energies and low energies in the MSM model. Using Eq.(2,8, 9), one can now parameterize the $M_D$ in terms of $V$, $M_\nu$, $M$, and $z$ as

$$M_D = iV\sqrt{m}R\sqrt{M}.$$  \hspace{1cm} (10)$$

Using Eq.(10), for $m_3 = 0$, one obtain

$$a_1 = i\sqrt{M_1}(V_{e1}\sqrt{m_1}c_z + V_{e2}\sqrt{m_2}s_z),$$  \hspace{1cm} (11)$$

$$a_2 = i\sqrt{M_1}(V_{e2}\sqrt{m_1}s_z - V_{e3}\sqrt{m_2}c_z),$$  \hspace{1cm} (12)$$

where, $V_{e1}, V_{e2}, V_{e3}$ denote the first row elements of neutrino mixing matrix given in Eq.(7). The remaining elements of $M_D$ can be expressed in the same manner following the generic relations used in [9,10].

4 Weak basis invariant(WB) for leptogenesis

In the seesaw mechanism, lepton number asymmetry can be generated through the decays of the heavy Majorana neutrinos $M_1, M_2$. This is called leptogenesis mechanism [7,27] and requires CP violation at high energies. Taking into account the general seesaw mechanism, it is not possible to establish a connection between leptonic CP violation at low energies and CP violation at high energies. Such a relation can only be establish in the context of flavor theory. Using the single flavor approximation for leptogenesis (i.e. in the case when wash out effects are not sensitive to the different flavors of the charged leptons into which the heavy neutrino decays), the leptogenesis can be probed using the CP odd invariants [17].

In the weak basis (WB), where $M$ and $M_l$ are real and diagonal, there are six physical phases in $M_D$, which can be used to characterize the CP violation in the leptonic sector. This corresponds to six possible CP-odd WB invariants relevant for leptogenesis [18]. For instance,

$$I_1 \equiv \text{Im} Tr[M_D^\dagger M_D(M_1 M_D^\dagger M_D)^* M].$$  \hspace{1cm} (13)$$

The non-zero value of $I_1$ signals the CP violation in leptonic sector. Since WB invariants are basis independent. Therefore, in the chosen basis, one can express $I_1$ as

$$I_1 = M_1 M_2 (M_2^2 - M_1^2) \text{Im}[k_{12}^2] + M_1 M_3 (M_3^2 - M_1^2) \text{Im}[k_{13}^2] + M_2 M_3 (M_1^3 - M_2^2) \text{Im}[k_{23}^2],$$  \hspace{1cm} (14)$$
where, \( k = M_D^\dagger M_D \) denotes the \( 3 \times 3 \) hermitian mass matrix. Clearly, \( I_1 = 0 \) implies CP conservation in leptonic sector. This condition holds for either degenerate right-handed neutrino masses or diminishing imaginary part of \( k_{ij}^2 \) (\( i \neq j, \ i=1, 2, 3 \)) or both. The interest in \( I_1 \) stems from the dependence on the term \( \text{Im} (k_{ij}^2) \), which eventually determines the strength of leptogenesis. Hence one can say that \( I_1 \) is sensitive to the CP violating phases which appear in the leptogenesis.

Following the similar WB as above, CP-odd invariants \( I_2 \) and \( I_3 \) can be expressed as

\[
I_2 \equiv \text{ImTr}[M_D^\dagger M_D (M^\dagger M)^2 M^\ast (M_D^\dagger M_D)^\ast M],
\]

\[
= M_1 M_2 (M_2^4 - M_1^4) \text{Im}[k_{12}^2] + M_1 M_3 (M_3^4 - M_1^4) \text{Im}[k_{13}^2] 
+ M_2 M_3 (M_3^4 - M_2^4) \text{Im}[k_{23}^2].
\]

and,

\[
I_3 \equiv \text{ImTr}[M_D^\dagger M_D (M^\dagger M)^2 M^\ast (M_D^\dagger M_D)^\ast M (M^\dagger M)],
\]

\[
= M_1^3 M_2^3 (M_2^2 - M_1^2) \text{Im}[k_{12}^2] + M_1^3 M_3^3 (M_3^2 - M_1^2) \text{Im}[k_{13}^2] 
+ M_2^3 M_3^3 (M_3^2 - M_2^2) \text{Im}[k_{23}^2].
\]

In the MSM, one of the diagonal elements of \( M \) (\( i.e. \ M_3 = 0 \)) is zero. Therefore, CP odd invariants in Eqs. (14), (15) and (16) are reduced to

\[
I_1 = M_1 M_2 (M_2^2 - M_1^2) \text{Im} k_{12}^2, \quad (17)
\]

\[
I_2 = M_1 M_2 (M_2^4 - M_1^4) \text{Im} k_{12}^2, \quad (18)
\]

\[
I_3 = M_1^3 M_2^3 (M_2^2 - M_1^2) \text{Im} k_{12}^2. \quad (19)
\]

It must be noted that Eqs. (14), (15) and (16) hold for the general case of seesaw model, where all the three heavy Majorana neutrino masses \( (M_1, M_2, M_3) \) are real, diagonal and non-zero, and \( M_D \) is \( 3 \times 3 \) complex matrix. Hence \( k \) turns out to be \( 3 \times 3 \) hermitian matrix, while in minimal seesaw model, \( M \) is a \( 2 \times 2 \) real and diagonal matrix. This implies that \( M_D \) is necessarily \( 3 \times 2 \) complex matrix following the see-saw mechanism in Eq. (2). Therefore, \( k \) is reduced to \( 2 \times 2 \) hermitian matrix

\[
k_{11} = |a_1|^2 + |b_1|^2 + |c_1|^2,
\]

\[
k_{12} = a_1^* a_2 + b_1^* b_2 + c_1^* c_2,
\]

\[
k_{21} = a_2^* a_1 + b_2^* b_1 + c_2^* c_1,
\]

\[
k_{22} = |a_2|^2 + |b_2|^2 + |c_2|^2. \quad (20)
\]

The remaining three CP odd invariants \( I_4, I_5 \) and \( I_6 \) can be written in a similar manner by simply substituting \( M_D^\dagger M_D \) with \( M_D^\dagger M_i M_i^\dagger M_D \)

\[
I_4 \equiv \text{ImTr}[M_D^\dagger M_i M_i^\dagger M_D (M^\dagger M) M^\ast (M_D^\dagger M_i M_i^\dagger M_D)^\ast M],
\]
Eqs. (21), (22) and (23) can be deduced in MSM model as

\[
I_5 \equiv \text{ImTr}[M_D^\dagger M_l M_l^\dagger M_D M^\dagger M^\dagger M_D] \equiv M_1 M_2 (M_2^2 - M_1^2) \text{Im}[K_{12}^2] + M_1 M_3 (M_3^2 - M_1^2) \text{Im}[K_{13}^2] \\
+ M_2 M_3 (M_3^2 - M_2^2) \text{Im}[K_{23}^2].
\]

(21)

\[
I_6 \equiv \text{ImTr}[M_D^\dagger M_l M_l^\dagger M_D (M^\dagger M)^2 M^\ast (M_D^\dagger M_l M_l^\dagger M_D)^\ast M(M^\dagger M)],
\]

\[
= M_1 M_2 (M_2^4 - M_1^4) \text{Im}[K_{12}^2] + M_1 M_3 (M_3^4 - M_1^4) \text{Im}[K_{13}^2] \\
+ M_2 M_3 (M_3^4 - M_2^4) \text{Im}[K_{23}^2].
\]

(22)

\[
I_6 \equiv \text{ImTr}[M_D^\dagger M_l M_l^\dagger M_D (M^\dagger M)^2 M^\ast (M_D^\dagger M_l M_l^\dagger M_D)^\ast M(M^\dagger M)],
\]

\[
= M_1^2 M_2^3 (M_2^2 - M_1^2) \text{Im}[K_{12}^2] + M_1^2 M_3^3 (M_3^2 - M_1^2) \text{Im}[K_{13}^2] \\
+ M_2^2 M_3^3 (M_3^2 - M_2^2) \text{Im}[K_{23}^2].
\]

(23)

Eqs. (21), (22) and (23) can be deduced in MSM model as

\[
I_4 = M_1 M_2 (M_2^2 - M_1^2) \text{Im}K_{12}^2,
\]

(24)

\[
I_5 = M_1 M_2 (M_2^4 - M_1^4) \text{Im}K_{12}^2,
\]

(25)

\[
I_6 = M_1^2 M_2^3 (M_2^2 - M_1^2) \text{Im}K_{12}^2.
\]

(26)

where, K is 2 × 2 hermitian matrix and its elements are given below:

\[
K_{11} = m_e^2 |a_1|^2 + m_\mu^2 |b_1|^2 + m_\tau^2 |c_1|^2,
\]

\[
K_{12} = m_e^2 a_1^* a_2 + m_\mu^2 b_1^* b_2 + m_\tau^2 c_1^* c_2,
\]

\[
K_{21} = m_e^2 a_2^* a_1 + m_\mu^2 b_2^* b_1 + m_\tau^2 c_2^* c_1,
\]

\[
K_{22} = m_e^2 |a_2|^2 + m_\mu^2 |b_2|^2 + m_\tau^2 |c_2|^2.
\]

(27)

where, \(m_e\), \(m_\mu\) and \(m_\tau\) denote the electron, muon and tau neutrinos, respectively.

In the following section, we shall discuss the implications of six CP odd invariants for FGY ansätze.

## 5 Implication of CP-odd WB invariants for FGY ansätze

As discussed in section 3, \(M_\nu\) consists of seven physical parameters. Since \(M_\nu\) is related to \(M_D\) and \(M\) through the seesaw relation, given in Eq. (2), therefore, the parameters of \(M_\nu\) are depend on \(M_D\) and \(M\). In principle, the light Majorana neutrino masses, flavor mixing angles and CP-violating phases can all be calculated at low energies. Hence it is possible to reconstruct \(M_D\) by the means of two heavy Majorana neutrino masses (\(M_1, M_2\)) and the complex elements of \(M_\nu\). In the following discussion, we derive the CP-odd WB invariants in terms of \(M_1, M_2\), and the complex elements of \(M_\nu\) for FGY ansätze.
5.1 Type 1

Using the seesaw mechanism in Eq. (2) and Eq. (5), one can write the expression for $M_\nu$ for type 1 as

$$M_\nu = \begin{pmatrix} a_1^2 M_1 & a_1 b_1 M_1 & 0 \\ b_1^2 M_1 + b_2^2 M_2 & b_2 c_2 M_2 & c_2^2 M_2 \\ a_2^2 M_1 + b_2^2 M_2 & b_2 c_2 M_2 & c_2^2 M_2 \end{pmatrix},$$

(28)
in terms of Dirac neutrino matrix elements $a_1, b_1, b_2, c_2$. On comparing Eqs. (6) and (28), one can trivially find $a_1, b_1, b_2, c_2$ in terms of the elements of Eq. (28), $a_1^2 = M_1 m_{ee}$, $b_1^2 = M_1 \frac{(m_{ee})^2}{m_{ee}}$, $c_2^2 = M_2 m_{\tau\tau}$, $b_2^2 = \frac{M_2 (m_{\mu\tau})^2}{m_{ee} m_{\tau\tau}}$. Since $I_i$ ($i = 1, 2, 3$) is directly proportional to $\text{Im} k_{12}^2$. Therefore it is sufficient to evaluate $I_1$ for each FGY ansatz.

Using Eq. (17), one can write $I_1$, for type 1,

$$I_1 = M_1^2 M_2^2 (M_2^2 - M_1^2) \text{Im} \left[ \frac{(m_{\mu\mu})^2}{m_{ee}^2} \right],$$

(29)

where, $k_{12}^2 = (b_1^2 b_2^2)^2$. The CP violation depends on the phase i.e. $\arg \left( \frac{(m_{ee})^2}{m_{ee} m_{\tau\tau}} \right)$. The vanishing of this phase implies CP conservation, and leads to following phase relation

$$\arg (m_{ee}) + 2 \arg (m_{ee}) = \arg (m_{ee}) + 2 \arg (m_{ee}).$$

(30)

From the above equation, one can say that CP violation is brought about by the mismatch among the phases of elements $m_{ee}^2, m_{\mu\tau}^2, m_{ee}$ and $m_{\tau\tau}$, while phase of the elements $m_{ee}$ or $m_{\mu\mu}$ does not have any contribution for CP violation and can be rephased away.

5.2 Type 2

For type 2, using Eqs. (2) and (5), we get

$$M_\nu = \begin{pmatrix} a_1^2 M_1 & 0 & a_1 c_1 M_1 \\ b_1^2 M_1 + b_2^2 M_2 & c_2^2 M_2 \\ a_2^2 M_1 + b_2^2 M_2 & b_2 c_2 M_2 \end{pmatrix}. $$

(31)

Again, using Eq. (31), one can easily find the following relations, $a_1^2 = M_1 m_{ee}$, $c_2^2 = M_1 \frac{(m_{ee})^2}{m_{ee}}$, $b_1^2 = M_2 m_{\mu\tau}$, $b_2^2 = M_2 m_{\mu\mu}$.

Using these relations and Eq. (17), one can derive $I_1$ for type 2,

$$I_1 = M_1^2 M_2^2 (M_2^2 - M_1^2) \text{Im} \left[ \frac{(m_{ee})^2}{m_{ee}^2} \right],$$

(32)

where, $k_{12}^2 = (a_1^2 c_2^2)^2$, and CP violation explicitly depends on physical phase i.e.
The necessary and sufficient condition for CP conservation for type 2 is given as

\[ \arg(m_{ee}) + 2\arg(m_{\mu\tau}) = \arg(m_{\mu\mu}) + 2\arg(m_{e\tau}). \]  

(33)

The type 1 and type 2 are phenomenologically related to each other via \( \mu - \tau \) exchange symmetry.

## 5.3 Type 3

Like type 2, type 3 also leads to \( m_{e\mu} = 0 \), and using Eqs. (2) and (5), one gets

\[ M_\nu = \begin{pmatrix} \frac{a^2}{M_2} & 0 & \frac{a^2 c^2}{M_2} \\ \frac{b^2}{M_1} & \frac{b c}{M_2} & \frac{c^2}{M_2} \\ \frac{c^2}{M_1} + \frac{a^2 c^2}{M_2} \end{pmatrix}. \]

(34)

Using Eqs. (6) and (34), we obtain the following mathematical relations for the elements of \( M_D \): \( b_1^2 = M_1 m_{\mu\mu} \), \( c_1^2 = M_1 \frac{(m_{\mu\tau})^2}{m_{\mu\mu}} \), \( c_2^2 = M_2 m_{ee}^2 \), \( a_2^2 = 2 M_2 m_{ee} \).

Using these relations, one can find

\[ I_1 = M_1^2 M_2^2 (M_2^2 - M_1^2) \text{Im} \left[ \frac{(m_{\mu\tau}^*)^2 m_{e\tau}^2}{m_{\mu\mu} m_{ee}} \right], \]

(35)

where, \( k_1^2 = (c_1^* c_2)^2 \), and CP violation for type 3 depends on the physical phase \( \arg \left[ \frac{(m_{\mu\tau}^*)^2 m_{e\tau}^2}{m_{\mu\mu} m_{ee}} \right] \), and its vanishing value leads to the following phase relation

\[ \arg(m_{\mu\mu}) + 2\arg(m_{e\tau}) = \arg(m_{ee}) + 2\arg(m_{\mu\tau}). \]  

(36)

The results obtained here are just the complex conjugate of the results obtained in case of type 2.

## 5.4 Type 4

For type 4, using Eq. (2) and (5), one can write

\[ M_\nu = \begin{pmatrix} \frac{a^2}{M_2} & \frac{a^2 b}{M_2} & 0 \\ \frac{b^2}{M_1} + \frac{b^2}{M_2} & \frac{b c}{M_2} & \frac{b c}{M_2} \\ \frac{b c}{M_1} + \frac{c^2}{M_2} & \frac{c^2}{M_2} & \frac{c^2}{M_2} \end{pmatrix}. \]

(37)

Similar to type 1, type 4 also leads to \( m_{e\tau} = 0 \). Using Eq. (37), we arrive at the following relations, \( b_1^2 = M_1 \frac{(m_{\mu\tau})^2}{m_{ee}} \), \( c_1^2 = M_1 m_{\tau\tau} \), \( b_2^2 = M_2 m_{ee}^2 \), \( a_2^2 = M_2 m_{ee} \), and consequently, we obtain

\[ I_1 = M_1^2 M_2^2 (M_2^2 - M_1^2) \text{Im} \left[ \frac{(m_{\mu\tau}^*)^2 m_{e\tau}^2}{m_{\mu\mu} m_{ee}} \right], \]

(38)
where, \( k_{12}^2 = (b_1^*b_2)^2 \), and, for CP conservation, we require
\[
\arg(m_{\tau\tau}) + 2\arg(m_{e\mu}) = \arg(m_{ee}) + 2\arg(m_{\mu\tau}).
\] (39)

Like in type 1 and type 2, we find that type 3 and type 4 are also related via \( \mu - \tau \) exchange symmetry. In addition, the results obtained in type 3 are simply a complex conjugate to that in type 1, and the same is true for type 2 and type 4 texture zeros. Similarly, one can derive the relations for \( I_2 \) and \( I_3 \) in terms of Majorana mass matrix elements using Eqs. (18, 19). The CP violating phase remain similar to \( I_1 \), while the coefficients dependence in terms of heavy right handed neutrinos are different as shown in Eqs.(17, 18,19).

On the other hand, the remaining CP-odd invariants (\( I_4, I_5, I_6 \)) depend on \( \text{Im} K_{12}^2 \). For illustration, we shall only evaluate \( I_4 \) for type 1.

Using Eq.(24), and elements of \( M_D \) provided in subsection 5.1, it is trivial to find the expression for \( I_4 \)
\[
I_4 = m_\mu^2 M_1^2 M_2^2 (M_2^2 - M_1^2) \text{Im} \left[ \frac{(m_{e\mu}^*)^2 m_{\mu\tau}^2}{m_{ee} m_{\tau\tau}} \right],
\] (40)
where, \( K_{12}^2 = m_\mu^2 (b_1^*b_2)^2 \). The above relation is similar to Eq. (29) except that \( I_4 \) depends on additional charged lepton parameter \( m_\mu \). The CP invariance condition obtained here is similar to \( I_1 \) for type 1. For the sake of completion, we have tabulated all the CP-odd invariants for all the viable FGY ansätze along with the necessary and sufficient CP invariance condition in Table[1]. The conditions on phases can be visualized as fine tuning required to have CP conservation at high energies.

From the above discussion, it is trivial to find that \( M_D \) with three or more zeros leads to CP invariance in the leptonic sector. In addition, it is found that that all the CP-odd invariants strongly depend on the effective neutrino mass term \( |m_{ee}| \) \( i.e. I_i \propto \frac{1}{|m_{ee}|} \), where \( i = 1, 2, 3, 4, 5, 6 \). If \( |m_{ee}| = 0 \), \( I_i \) simply blows up. Therefore the measurement of \( |m_{ee}| \) in neutrinoless double beta decay experiments could have serious implications on these WB invariants.

6 CP-odd WB invariants and low energy CP violating phases

In this section, we discuss how the CP odd invariants depend on CP violating phases \( (\delta, \sigma) \) in an explicit manner. Using Eqs.(11) and (12), we get the following relations
\[
tanz = R^{1/2} t_{12} e^{i\sigma},
\] (41)
\[
cotz = -R^{1/2} t_{12} e^{i\sigma},
\] (42)
for \( a_1 = 0 \) and \( a_2 = 0 \), respectively. The symbols \( R = \frac{m_2}{m_1} \), and \( t_{12} = c_{\delta_{12}} \).
Using Eq. (41), it is trivial to find the imaginary part as
\[
\text{Im}[c^2_z] = \frac{Rt^2_{12}\sin 2\sigma}{(1 + Rt^2_{12}\cos 2\sigma)^2 + (Rt^2_{12}\sin 2\sigma)^2}.
\]
(43)

where \( c_z = \cos(z) \).

For Type 1 and Type 4, we obtain, \( m_{e\tau} = 0 \) as evident from Eqs. (28) and (37).

Using this constraint, we arrive at the following relation between \( \delta \) and \( \sigma \)
\[
s_{13}\sin\delta = -\frac{t_{12}t_{23}}{c^2_{12}} \times \frac{R\sin 2\sigma}{(1 + Rt^2_{12}\cos 2\sigma)^2 + (Rt^2_{12}\sin 2\sigma)^2}.
\]
(44)

Following the same procedure as in Eq. (43), we get, using Eq. (42),
\[
\text{Im}[s^2_z] = \frac{-Rt^2_{12}\sin 2\sigma}{(1 + Rt^2_{12}\cos 2\sigma)^2 + (Rt^2_{12}\sin 2\sigma)^2}.
\]
(45)

For Type 2 and Type 3, we obtain, \( m_{e\mu} = 0 \). Using this condition, one can easily obtain the relation
\[
s_{13}\sin\delta = \frac{t_{12}}{t_{23}c^2_{12}} \times \frac{R\sin 2\sigma}{(1 + Rt^2_{12}\cos 2\sigma)^2 + (Rt^2_{12}\sin 2\sigma)^2}.
\]
(46)

On comparing Eqs. (43) and (44), we get
\[
\text{Im}[c^2_z] = +t_{12}t_{23}^{-1} c^2_{12} s_{13}\sin\delta,
\]
(47)

and,
\[
\text{Im}[s^2_z] = +t_{12}t_{23}^{-1} c^2_{12} s_{13}\sin\delta,
\]
(48)

for type 1 and type 4, respectively.

Similarly, on comparing Eqs. (45) and (46), we get
\[
\text{Im}[c^2_z] = -t_{12}t_{23} c^2_{12} s_{13}\sin\delta,
\]
(49)

and,
\[
\text{Im}[s^2_z] = -t_{12}t_{23} c^2_{12} s_{13}\sin\delta,
\]
(50)

for type 2 and type 3, respectively.

To evaluate the CP-odd invariants in terms of Dirac CP violating phase \( \delta \), we need to calculate the \( 2 \times 2 \) hermitian matrix \( k \). Using Eq. (10), we have, \( k = \sqrt{M^TR^RmR}\sqrt{M} \). On solving it, we obtain [23],
\[
k = \begin{pmatrix}
M_1(m_1|c_z|^2 + m_2|s_z|^2) & \sqrt{M_1M_2}(-m_1c_z^*s_z + m_2s_z^*c_z) \\
\sqrt{M_1M_2}(-m_1c_zs_z^* + m_2s_zc_z^*) & M_2(m_1|s_z|^2 + m_2|c_z|^2)
\end{pmatrix}.
\]
(51)

After squaring the above matrix, one can extract the term
\[
k^2_{12} = \sqrt{M_1M_2}[(M_1m_1 + M_2m_2)|c_z|^2 + (M_1m_2 + M_2m_1)|s_z|^2](-m_1c_z^*s_z + m_2s_z^*c_z).
\]
(52)
Since we know that imaginary part of pure imaginary number is again imaginary. Therefore, one can write $I_1$ using Eq.(52)

$$I_1 \simeq C \text{Im}(s^*_z c_z),$$

(53)

where, $C \simeq 2M_1^{5/2}M_2^{5/2}(M_2^2 - M_1^2)(M_1 + M_2)m^2$, is the coefficient of $I_1$, and we have used the approximation, $m_1 \simeq m_2 \simeq m$. Clearly, $I_1$ depends on the neutrino mass $m$ and heavy right-handed Majorana neutrino masses $M_1$ and $M_2$. The CP violation depends on the phase of complex term $s^*_zc_z$. On evaluating further using Eq.(41), we obtain

$$I_1 \simeq C t_{12} R^{1/2} |c_z|^2 \sin \sigma.$$  

(54)

The relation holds for type1 and type2. For Type3 and Type4, using Eq.(42), $I_1$ is given as

$$I_1 \simeq \frac{C}{t_{12} R^{1/2}} |c_z|^2 \sin \sigma.$$  

(55)

Similarly, we can easily derive the expressions for $I_2$, $I_3$, $I_4$, $I_5$ and $I_6$ in terms of $\sin \sigma$ using Eqs. (18,19, 24, 25, 26) for each ansatz. From Eqs. (54) and (55), we conclude that $\sin \sigma=0$ leads to CP conservation in leptonic sector. Taking into account the analytical relation between $\delta$ and $\sigma$ in Eqs.(44),(45), one find that CP conservation holds for $\delta, \sigma = \pm n\pi$, where $n$ is a integer.

7 Relationship between the thermal leptogenesis and left handed Majorana neutrino mass matrix

In the thermal leptogenesis in the MSM, seesaw mechanism with only two right handed neutrinos succeeds in reproducing the observed baryon asymmetry of universe for a nearly degenerate heavy neutrino mass spectrum. In [28], seesaw mechanism with thermal leptogenesis is also tested in the context of gravitational waves. D. Croon et.al [29] have studied how the observed baryon asymmetry is realized after high scale reheating into the lightest sterile neutrino in the framework of MSM.

In this choosen framework, the decays of two heavy right-handed Majorana neutrinos, $N_i \rightarrow l + H$ and $N_i \rightarrow \bar{l} + H^*$ (for $i = 1, 2$), are both lepton-number-violating and CP-violating [27]. The CP asymmetry $\epsilon_i$ originates from the interference between the tree-level and one-loop decay amplitudes. If $N_1$ and $N_2$ have a hierarchical mass spectrum ($M_1 \ll M_2$), the interactions involving $N_1$ can be in thermal equilibrium when $N_2$ decays. The asymmetry term $\epsilon_2$ is erased before $N_1$ decays. The CP-violating asymmetry $\epsilon_1$, which is produced by the out-of-equilibrium decay of $N_1$, in the choosen basis where $M_l$ and $M$ are both diagonal, can be given as

$$\epsilon_1 \equiv \frac{\Gamma(N_1 \rightarrow l + H) - \Gamma(N_1 \rightarrow \bar{l} + H^*)}{\Gamma(N_1 \rightarrow l + H) + \Gamma(N_1 \rightarrow \bar{l} + H^*)},$$
\[ \approx -\frac{3}{16\pi v^2} \times \frac{M_1}{M_2} \times \frac{Im(k\dagger k)_{12}^2}{(k\dagger k)_{11}}, \]  

(56)

In this section, we discuss the implications of FGY ansatz on leptogenesis. To this end, we find the relationship between \( \epsilon_1 \) and \( M_\nu \) for each ansatz.

Using Eq. (56) and \( a_1, b_1, b_2, c_2 \) in subsection 5.1, one can easily arrive at,

\[ \epsilon_1 = -\frac{3}{16\pi v^2} \times \frac{M_1}{|m_{ee}|} \frac{|m_{\mu\tau}|^2 (|m_{ee}|^2 + |m_{\mu\tau}|^2)}{M_{\mu\tau}} \times Im[(m_{\mu\tau})^2 m_{\mu\tau}^* m_{ee}^*], \]  

(57)

where, \( (k\dagger k)_{11} = |a_1|^2 + |c_1|^2 \). From Eq. (57), \( \epsilon_1 \) depends on physical phase i.e. \( \arg[(m_{\mu\tau})^2 m_{\mu\tau}^* m_{ee}^*] \). For type 2, one can obtain \( \epsilon_1 \), simply by the exchange of \( \mu \leftrightarrow \tau \).

Similarly, with the help of Eq. 56 and \( b_1, c_1, c_2, a_2 \) in subsection 5.3, \( \epsilon_1 \) can be expressed as

\[ \epsilon_1 = -\frac{3}{16\pi v^2} \times \frac{M_1}{|m_{ee}|} \frac{|m_{\mu\tau}|^2 (|m_{ee}|^2 + |m_{\mu\tau}|^2)}{M_{\mu\tau}} \times Im[(m_{\mu\tau})^2 m_{\mu\tau}^* m_{ee}^*], \]  

(58)

where, \( (k\dagger k)_{11} = |b_1|^2 + |c_1|^2 \). In case of type 3, \( \epsilon_1 \) depends on physical phase i.e. \( \arg[(m_{e\tau})^2 m_{e\tau}^* m_{\mu\tau}^*] \). The result for type 4 can simply be obtained through \( \mu - \tau \) exchange symmetry. From Eqs. (57) and (58), we conclude that CP-violating asymmetry \( \epsilon_1 \) requires the mismatch among the phases associated with \( m_{ee}, m_{\mu\tau}, m_{ee} \) and \( m_{\mu\tau} \) pertaining to \( M_\nu \) for type 1 ansatz, while, for type 3, same holds true for the phases associated with \( m_{e\tau}, m_{\mu\tau}, m_{ee} \) and \( m_{\mu\tau} \) this, in turn, lead to net lepton number asymmetry, \( Y_L \equiv \frac{n_L}{s} = \frac{d}{g^*} \), where \( g^* = 106.75 \) corresponds to an effective number featuring the relativistic degree of freedom which contribute to the entropy \( s \), and \( d \) is the dilution effects induced by the lepton-number-violating wash-out processes [27]. The lepton number asymmetry \( Y_L \) is finally converted into a net baryon number asymmetry \( Y_B \) through the nonperturbative sphaleron processes [30]:

\[ Y_B \equiv \frac{n_B}{s} \approx 0.5Y_L. \]

In addition to the phase dependence, \( \epsilon_1 \) depends only on \( M_1 \), for \( M_2 >> M_1 \). Another careful observation reveal that \( \epsilon_1 \) for all the FGY ansätze depends inversely on \( |m_{ee}| \). Therefore, the measurement of \( |m_{ee}| \) through various neutrinoless double beta decay experiments is important for calculating the baryon asymmetry of Universe. In the following discussion, we shall see how \( \epsilon_1 \) depends explicitly on the CP violating phases related to low energy.

With the help of Eqs. (51) and (56) we can arrive at following relations

\[ \epsilon_1 = -\frac{3}{16\pi v^2} M_1 \Delta m_{12}^2 \frac{Im[c_{12}^2]}{m_1}, \]  

(59)

or

\[ \epsilon_1 = +\frac{3}{16\pi v^2} M_1 \Delta m_{12}^2 \frac{Im[s_{12}^2]}{m_1}, \]  

(60)

where, \( m_1 = v(m_1|c_{12}|^2 + m_2|s_{12}|^2) \).

Using Eqs. (47 48) and (59 60), \( \epsilon_1 \) is given as

\[ \epsilon_1 = \frac{3}{16\pi v^2} M_1 \Delta m_{12}^2 t_{12} t_{23}^{-1} c_{12}^2 s_{13}^2 \sin\delta, \]  

(61)
for type 1 (minus) and type 4 (plus), respectively. Similarly, using Eqs. (49, 50) and (59, 60), \( \epsilon_1 \) is given as

\[
\epsilon_1 = \pm \frac{3}{16\pi v^2} \frac{M_1 \Delta m^2_{12} t_{23} c^2_{12} s_{13}}{m_1} \sin \delta, \tag{62}
\]

for type 2 (plus) and type 3 (minus), respectively. These relations show the explicit dependence of lepton asymmetry on \( \delta \). From Eqs. (44) and (46), it is clear that \( \sin \delta \) is directly proportional to \( \sin 2\sigma \), implying that \( \epsilon_1 \propto \sin 2\sigma \). Therefore lepton asymmetry depends on the Majorana CP-violating phase \( \sigma \). It is worthwhile to note that this phase parameter does not affect CP violation in neutrino oscillation, but it can be instrumental in the scenarios of leptogenesis due to the lepton number violating and CP violating decays of the two heavy right handed Majorana neutrinos. The discussion also remain consistent with Ref. [23].

## 8 Summary and Conclusion

In summary, we have considered the minimal seesaw model (MSM) augmented with two zero in the Dirac neutrino mass matrix. Taking into account the four experimentally viable ansätze with inverted mass ordering, we construct the weak basis invariants (WB) relevant for leptogenesis in terms of low energy effective neutrino mass matrix elements, and then find the necessary and sufficient conditions of CP conservation. It is shown that textures having three or more zeros lead to CP conservation. The CP violation at high energies for these ansätze requires that phases among the low energy effective Majorana mass matrix elements are not fine tuned and, in addition, the right handed Majorana neutrino masses \( M_1 \) and \( M_2 \) are non-degenerate. To extend our analysis further, we have explicitly shown the dependence of these CP odd invariants on Majorana CP violating phase \( \sigma \) for each ansatz, and find that \( \delta, \sigma = \pm n\pi \), where \( n \) is a integer, holds for CP invariance in leptonic sector at high energy scale.

In the end we re-examine the implications of these interrelationships on leptogenesis. In this regard, we have shown the relations for CP violating asymmetry in terms of left handed Majorana neutrino mass matrix for all ansätze. Further, it is shown that it’s non-zero value depends on the mismatch among the phases associated with the elements of \( M_\nu \). In addition, for all ansätze, CP violating asymmetry depends on effective neutrino mass, \( |m_{ee}| \), related to neutrinoless double beta decay.

In future long baseline experiments and neutrinoless double beta decay experiments, the precise determination of low energy parameters e.g. CP violating phases(\( \delta, \sigma \)), octant of \( \theta_{23} \), is critical to rule in or rule out the FGY ansätze.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.
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References

[1] S. L. Glashow, Nucl. Phys. 22, 597 (1961); S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam in Elementary Particle Theory, Ed. N. Swartholm (Almquist and Wiksells, Stockholm) 1969. For excellent reviews on Standard Model see, J. F. Donoghue, E. Golowich and B.R. Holstein, Dynamics of the Standard Model, Cambridge University Press (1992); H. Fritzsch, M. GellMann and H. Leutwyler, Phys. Lett. B 47, 365 (1973).

[2] N. Cabbibo, Phys. Rev. Lett. 10, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).

[3] SNO Collaboration, Q. R. Ahmad et al., Phys. Rev. Lett. 89 (2002) 011301.

[4] For a review, see: C. K. Jung et al., Ann. Rev. Nucl. Part. Sci. 51 (2001) 451.

[5] KamLAND Collaboration, K. Eguchi et al., Phys. Rev. Lett. 90 (2003) 021802; October 31, 2018; CHOOZ Collaboration, M. Apollonio et al., Phys. Lett. B 420 (1998) 397; Palo Verde Collaboration, F. Boehm et al., Phys. Rev. Lett. 84 (2000) 3764.

[6] K2K Collaboration, M. H. Ahn et al., Phys. Rev. Lett. 90 (2003) 041801.

[7] M. Fukugita and T. Yanagida, Phys. Lett. B 174, 45 (1986).

[8] T. Yanagida, in Proceedings of the Workshop on Unified Theory and the Baryon Number of the Universe, edited by O. Sawada and A. Sugamoto (KEK, 1979); M. Gell-Mann, P. Ramond, and R. Slansky, in Supergravity, edited by F. van Nieuwenhuizen and D. Freedman (North Holland, Amsterdam, 1979); R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. 44, 912 (1980).

[9] A. Ibarra, G. G. Ross, Phys. Lett. B 591(2004) 285, hep-ph/0312138.

[10] J. A. Casas and A. Ibarra, Nucl. Phys. B 618, 171 (2001), hep-ph/0103065.

[11] P. H. Frampton, S. L. Glashow and T. Yanagida, Phys. Lett. B 548, 119 (2002), hep-ph/0208157.

[12] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985).

[13] G. C. Branco, L. Lavoura and M. N. Rebelo, Phys. Lett. B 180 (1986) 264.

[14] G. C. Branco, M. N. Rebelo, J. I. Silva-Marcos, Phys. Rev. Lett. 82 (1999) 683.
[15] Herbi K. Dreiner, Jong Soo Kim, Oleg Lebedev and Marc Thormeier, Phys. Rev. D 76, 015006 (2007), hep-ph/0703074.

[16] G. C. Branco, M. N. Rebelo, J. J Silva-Marcos, JHEP 11, 001 (2017), arXiv: 1705.07758[hep-ph].

[17] A. Pilaftsis, Phys. Rev. D 56(1997), 5431-5451, arXiv: hep-ph/9707235.

[18] G. C. Branco, T. Morozumi, B. M. Nobre, M. N. Rebelo, Nucl. Phys. B 617, p-475-492 (2001), arXiv: hep-ph/0107164.

[19] G. C. Branco and M. N. Rebelo, New Journal of Physics 7, 86 (2005).

[20] W. l. Guo and Z.Z. Xing, Phys. Lett. B 583, p-163-172 (2004), arXiv: hep-ph/0310326.

[21] W. l. Guo, Z. Z. Xing, S. Zhou, Int. J. Mod. Phys. E 16, 1 (2007), arXiv: hep-ph/0612033.

[22] B. Brahmachari and N. Okada, Phys. Lett. B 660, 508(2008), arXiv: hep-ph/0612079 (2008).

[23] K. Harigaya, M. Ibe, and Tsutomu T. Yanagida, Phys. Rev. D 86, 013002, arXiv: 1205.2198 [hep-ph].

[24] J. Zhang, S. Zhou, JHEP 09, 65 (2015), arXiv: 1505.04858 [hep-ph].

[25] Thomas Rink and K. Schmitz, JHEP 03, 158 (2017), arXiv: 1611.05857 [hep-ph].

[26] H. Fritzsch, Z. Z. Xing, Phys. Lett. B 517 (2001) 363-368, arXiv: hep-ph/0103242.

[27] For recent reviews of leptogenesis with extensive references, see: W. Buchmuller and M. Plumacher, Int. J. Mod. Phys. A 15, 5047 (2000); G. F. Giudice, A. Notari, M. Raidal, A. Riotto, and A. Strumia, hep-ph/0310123.

[28] Jeff A. Dror, T. Hiramatsu, K. Kohri, H. Murayama, G. White, Phys. Rev. Lett. 124, no. 4, 041804(2020), arXiv: 1908.03227 [hep-ph].

[29] D. Croon, N. Fernandez, D. Mckeen, G. White, JHEP 06, 098 (2019), arXiv: 1903.08658[hep-ph].

[30] V. A. Kuzmin, V. A. Rubakov, and M. E. Shaposhnikov, Phys. Lett. B 155, 36 (1985).
| Cases of $M_2$ | $M_\nu$ | CP-odd invariants | Phase Relationship for CP conservation |
|----------------|---------|-------------------|----------------------------------------|
| Type 1 | \( \begin{pmatrix} \frac{s^2}{M_1^2} & \frac{a_{11}k_2}{M_1^2} & \frac{s_2^2}{M_1^2} + \frac{s_2^2}{M_2^2} & 0 \\ \frac{k_{21}k_2}{M_1^2} & \frac{s_2^2}{M_1^2} + \frac{s_2^2}{M_2^2} & \frac{k_{21}^*}{M_1^2} & \frac{k_{22}^*}{M_1^2} \end{pmatrix} \) | \( I_1 = M_1^2 M_2^2 (M_3^2 - M_4^2) \text{Im} \left( \frac{m_{\nu e}^* m_{\nu e} M_{\nu e}^2}{m_{\nu e}^2 m_{\nu e} M_{\nu e}^2} \right) \) | arg($m_{\nu e}$) + 2arg($m_{\nu e}$) = arg($M_{\nu e}$) + 2arg($m_{\nu e}$) |
| Type 2 | \( \begin{pmatrix} \frac{s^2}{M_1^2} & 0 & \frac{a_{11}k_2}{M_1^2} & \frac{s_2^2}{M_1^2} + \frac{s_2^2}{M_2^2} \\ \frac{s_2^2}{M_1^2} & 0 & \frac{k_{21}k_2}{M_1^2} & \frac{s_2^2}{M_1^2} + \frac{s_2^2}{M_2^2} \\ \frac{k_{21}^*}{M_1^2} & \frac{k_{22}^*}{M_1^2} & \frac{s^2}{M_2^2} & \frac{a_{11}k_2}{M_2^2} \\ \frac{k_{21}^*}{M_1^2} & \frac{k_{22}^*}{M_1^2} & \frac{s_2^2}{M_2^2} + \frac{s_2^2}{M_2^2} & \frac{s^2}{M_2^2} \end{pmatrix} \) | \( I_1 = M_1^2 M_2^2 (M_3^2 - M_4^2) \text{Im} \left( \frac{m_{\nu e}^* m_{\nu e} M_{\nu e}^2}{m_{\nu e}^2 m_{\nu e} M_{\nu e}^2} \right) \) | arg($m_{\nu e}$) + 2arg($m_{\nu e}$) = arg($M_{\nu e}$) + 2arg($m_{\nu e}$) |
| Type 3 | \( \begin{pmatrix} \frac{s^2}{M_1^2} & 0 & \frac{a_{11}k_2}{M_1^2} & \frac{s_2^2}{M_1^2} + \frac{s_2^2}{M_2^2} \\ \frac{s_2^2}{M_1^2} & 0 & \frac{k_{21}k_2}{M_1^2} & \frac{s_2^2}{M_1^2} + \frac{s_2^2}{M_2^2} \\ \frac{k_{21}^*}{M_1^2} & \frac{k_{22}^*}{M_1^2} & \frac{s^2}{M_2^2} & \frac{a_{11}k_2}{M_2^2} \\ \frac{k_{21}^*}{M_1^2} & \frac{k_{22}^*}{M_1^2} & \frac{s_2^2}{M_2^2} + \frac{s_2^2}{M_2^2} & \frac{s^2}{M_2^2} \end{pmatrix} \) | \( I_1 = M_1^2 M_2^2 (M_3^2 - M_4^2) \text{Im} \left( \frac{m_{\nu e}^* m_{\nu e} M_{\nu e}^2}{m_{\nu e}^2 m_{\nu e} M_{\nu e}^2} \right) \) | arg($m_{\nu e}$) + 2arg($m_{\nu e}$) = arg($M_{\nu e}$) + 2arg($m_{\nu e}$) |
| Type 4 | \( \begin{pmatrix} \frac{s^2}{M_1^2} & \frac{a_{11}k_2}{M_1^2} & \frac{s_2^2}{M_1^2} + \frac{s_2^2}{M_2^2} & 0 \\ \frac{k_{21}k_2}{M_1^2} & \frac{s_2^2}{M_1^2} + \frac{s_2^2}{M_2^2} & \frac{k_{21}^*}{M_1^2} & \frac{k_{22}^*}{M_1^2} \\ \frac{k_{21}^*}{M_1^2} & \frac{k_{22}^*}{M_1^2} & \frac{s^2}{M_2^2} & \frac{a_{11}k_2}{M_2^2} \\ \frac{k_{21}^*}{M_1^2} & \frac{k_{22}^*}{M_1^2} & \frac{s_2^2}{M_2^2} + \frac{s_2^2}{M_2^2} & \frac{s^2}{M_2^2} \end{pmatrix} \) | \( I_1 = M_1^2 M_2^2 (M_3^2 - M_4^2) \text{Im} \left( \frac{m_{\nu e}^* m_{\nu e} M_{\nu e}^2}{m_{\nu e}^2 m_{\nu e} M_{\nu e}^2} \right) \) | arg($m_{\nu e}$) + 2arg($m_{\nu e}$) = arg($M_{\nu e}$) + 2arg($m_{\nu e}$) |

Table 1: The structure of effective Majorana mass term ($M_\nu$), and the rephasing invariants $I_{1,2,3,4,5,6}$ as well as the necessary and sufficient conditions for CP invariance is given corresponding to each FGY ansätz.