\[ N = 2 \] superparticle near horizon of extreme Kerr–Newman-AdS-dS black hole

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Abstract

Conformal mechanics related to the near horizon extreme Kerr–Newman-AdS-dS black hole is studied. A unique \( N = 2 \) supersymmetric extension of the conformal mechanics is constructed.

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1. Introduction

A large class of extreme black holes in four and five dimensions exhibits $SO(2, 1)$ symmetry in the near horizon limit [1, 2]. The most interesting example of such a kind is the Kerr black hole in four dimensions [3]. The idea in [4] to consider excitations around the near horizon extreme Kerr geometry and to extend the corresponding $SO(2, 1) \times U(1)$ symmetry group to the Virasoro group of asymptotic symmetries has paved the way for extensive investigation of the Kerr/CFT correspondence\textsuperscript{1}.

A useful means of studying geometry of vacuum solutions of the Einstein equations is provided by particle mechanics on a curved background. Apart from the issue of geodesic completeness, the knowledge of conserved charges of a particle helps uncover symmetries of the background. For example, the discovery of a quadratic first integral for a massive particle in the Kerr space–time [6] preceded the construction of the second rank Killing tensor for the Kerr geometry [7]. Because $SO(2, 1)$ is the conformal group in one dimension, particle models derived from the near horizon geometries with the $SO(2, 1)$ isometry will automatically be conformal invariant.

Conformal mechanics associated with the near horizon extreme Kerr black hole in four dimensions has been studied recently in [8]. It was shown that in the near horizon limit the Killing tensor degenerates into a quadratic combination of Killing vectors corresponding to the $SO(2, 1) \times U(1)$ isometry group. Furthermore, because $SO(2, 1) \times U(1)$ is the bosonic subgroup of $SU(1, 1|1)$, an $\mathcal{N} = 2$ superconformal extension is feasible. While a relation between first integrals of a particle propagating on a curved background and Killing vectors characterizing a background geometry is well known, a link between supersymmetry charges and Killing spinors is poorly understood. A possibility to construct $\mathcal{N} = 2$ superparticle moving near the horizon of the extreme Kerr black hole thus raises the question about geometric interpretation of the supercharges.

The purpose of this work is to extend the analysis in [8] to the case of the near horizon extreme Kerr–Newman-AdS-dS black hole [9]. There are two motivations for this study. First, the analysis in [10, 11] indicates that the extreme Kerr–Newman-AdS black hole is supersymmetric. Then one can hope to relate supersymmetry generators of an $\mathcal{N} = 2$ superparticle to Killing spinors and to explore a limit in which the cosmological constant tends to zero and the electric and magnetic charges of the black hole vanish. This might give an answer to the question raised above. Second, in [8] an $\mathcal{N} = 2$ superparticle was constructed in an ad hoc manner following the recipe which was earlier applied to an $\mathcal{N} = 4$ superparticle in Bertotti–Robinson space [12, 13]. Below we give a proof that the $\mathcal{N} = 2$ model is essentially unique and is constructed in a purely group theoretical way.

The organization of this paper is as follows. In the next section background fields are briefly discussed. In Section 3 conformal mechanics associated with the near horizon extreme Kerr–Newman-AdS-dS black hole is studied within the framework of the Hamiltonian formalism. First integrals corresponding to the $SO(2, 1) \times U(1)$ isometry group are identified.

\textsuperscript{1}By now there is an extensive literature on the subject. For a more complete list of references see e.g. a recent work [5].
Section 4 is devoted to $\mathcal{N} = 2$ supersymmetric extension of the conformal mechanics which is done within the canonical formalism. Our analysis is fully general and shows that the $\mathcal{N} = 2$ model is unique. We summarize our results and discuss possible further developments in Section 5.

2. Background fields

The Kerr-Newman–AdS–dS black hole is a solution of the Einstein-Maxwell equations with a non–vanishing cosmological constant \[14\]. In Boyer–Lindquist–type coordinates it reads

$$ds^2 = \frac{\Delta_r}{\rho^2} \left( dt - \frac{a}{\Xi} \sin^2 \theta d\phi \right)^2 - \rho^2 \frac{d\rho^2}{\Delta_r} - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\Delta_\theta}{\rho^2} \sin^2 \theta \left( a dt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2,$$

$$A = - \frac{q_e r}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right) - \frac{q_m \cos \theta}{\rho^2} \left( a dt - \frac{r^2 + a^2}{\Xi} d\phi \right),$$

where

$$\Delta_r = (r^2 + a^2) \left( 1 + \frac{r^2}{l^2} \right) - 2Mr + q^2, \quad \Delta_\theta = 1 - \frac{a^2}{l^2} \cos^2 \theta,$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Xi = 1 - \frac{a^2}{l^2}, \quad q^2 = q_e^2 + q_m^2. \quad (2)$$

The parameters $M$, $a$, $q_e$ and $q_m$ are linked to the mass, angular momentum, electric and magnetic charges of the black hole, respectively (for explicit relations see e.g. \[9\]). $l^2$ is taken to be positive for AdS and negative for dS and is related to the cosmological constant via $\Lambda = -3/l^2$.

It is evident that (1) is invariant under the time translation and rotation around $z$–axis

$$t' = t + \alpha, \quad \phi' = \phi + \beta. \quad (3)$$

A less obvious fact is that this solution admits the second rank Killing tensor \[6, 7, 15\] which obeys

$$K_{mn} = K_{nm}, \quad \nabla_{(n} K_{mp)} = 0, \quad F_n^\; m K_{mp} + F_p^\; m K_{mn} = 0. \quad (4)$$

In Boyer–Lindquist–type coordinates it reads ($x^m = (t, r, \theta, \phi)$)

$$K_{mn} = Q_{mn} + r^2 g_{mn}, \quad (5)$$

\(^2\)We use metric with mostly minus signature and put $c = 1$, $G = 1$. In these conventions the Einstein–Maxwell equations read $R_{nm} - \frac{1}{2}g_{nm}(R + 2\Lambda) = -2(F_{ns} F_m^\; s - \frac{1}{4} g_{nm} F^2), \partial_n (\sqrt{-g} F^{nm}) = 0.$
where
\[
Q_{mn} = \begin{pmatrix}
-\Delta_r & 0 & 0 & \frac{a\Delta r}{\rho} \sin^2 \theta \\
0 & \frac{\rho}{\Delta r} & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{a\Delta r}{\rho} \sin^2 \theta & 0 & 0 & -\frac{2a\Delta r}{\rho^2} \sin^2 \theta
\end{pmatrix}.
\]

Generally, a Killing tensor is a key ingredient in establishing complete integrability of the geodesic equations. It also helps to separate variables for some important field equations in a given gravitational background, including Klein–Gordon and Dirac equations (for a review and further references see e.g. [16]). It should also be mentioned that, when considering superparticle models on a curved background which admits Killing–Yano tensors, extra supersymmetry charges can be constructed [17] whose Poisson bracket yields Killing tensors\(^3\).

The extreme solution is characterized by the condition that the inner and outer horizons of the black hole coalesce. Denoting the corresponding value of the radial coordinate by \(r_+\), from \(\Delta r(r_+) = 0\), \(\Delta' r(r_+) = 0\) one finds
\[
a^2 = \frac{r_+^2(1 + 3r_+^2/l^2) - q^2}{1 - r_+^2/l^2}, \quad M = \frac{r_+[(1 + r_+^2/l^2)^2 - q^2/l^2]}{1 - r_+^2/l^2},
\]
and
\[
\Delta r(r) = (r - r_+)^2[(r + r_+)^2 + 2r_+^2 + l^2 + a^2]/l^2.
\]

In order to describe the near horizon geometry, one redefines the coordinates [9]
\[
r \to r_+ + \varepsilon r_0 r, \quad t \to tr_0/\varepsilon, \quad \phi \to \phi + \frac{tr_0a\Xi}{\epsilon(r_+^2 + a^2)},
\]
where \(r_0\) is a constant to be fixed below and then takes the limit \(\varepsilon \to 0\). The first prescription in (8) followed by \(\varepsilon \to 0\) provides a natural definition of the near horizon region. The last two prescriptions are designed so as to make \(\lim_{r \to 0} ds^2\) nonsingular\(^4\)
\[
ds^2 = \Gamma(\theta) \left[ r^2 dt^2 - \frac{dr^2}{r^2} - \alpha(\theta)d\theta^2 \right] - \gamma(\theta)(d\phi + krdt)^2, \quad A = f(\theta)(d\phi + krdt),
\]
where
\[
\Gamma(\theta) = \frac{\rho^2 r_0^2}{r_+^2 + a^2}, \quad \alpha(\theta) = \frac{r_+^2 + a^2}{\Delta_0 r_0^2}, \quad \gamma(\theta) = \frac{\Delta(\theta)(r_+^2 + a^2)^2 \sin^2 \theta}{\rho^2 r_0^2}.
\]

\(^3\)For some recent developments see [18].

\(^4\)As to the vector potential, usually one calculates the near horizon field strength and derives the vector potential from there.
\[ \rho_+^2 = r_+^2 + a^2 \cos^2 \theta, \quad r_0^2 = \frac{(r_+^2 + a^2)(1 - r_+^2/l^2)}{1 + 6r_+^2/l^2 - 3r_+^4/l^4 - q^2/l^2}, \quad k = \frac{2ar_+ \Xi r_0^2}{(r_+^2 + a^2)^2}, \]

This is a solution of the Einstein-Maxwell equations [9] which generalizes the near horizon extreme Kerr metric constructed in [3]. It reduces to the latter when \( q_e = q_m = 0 \) and \( l^2 \to \infty \).

Near the throat the symmetry group is enhanced. In addition to (3) it includes the dilatation

\[
t' = t + \gamma t, \quad r' = r - \gamma r, \tag{11}
\]

and the special conformal transformation

\[
t' = t + (t^2 + \frac{1}{r^2})\sigma, \quad r' = r - 2tr\sigma, \quad \phi' = \phi - \frac{2k}{r}\sigma. \tag{12}
\]

Altogether they form \( SO(2,1) \times U(1) \) group [9].

Concluding this section we derive the second rank Killing tensor in the near horizon region. Implementing the same limit as above one finds that the second term in (5) reduces to the metric which gives a trivial contribution to the Killing tensor and can be discarded. The first terms yields

\[
K_{nm}dx^ndx^m = \Gamma(\theta)^2 \left[ r^2dt^2 - \frac{1}{r^2}dr^2 \right]. \tag{13}
\]

Given the background fields (9), it is straightforward to verify that (13) meets all the requirements formulated in (4). Note that in a four–dimensional space–time with four isometries the Killing tensor can not be irreducible. Reducibility of the Killing tensor is a priori guaranteed by the enhanced isometry. However, the explicit formula for the reduction of the Killing tensor into Killing vectors may be useful and is given below in Section 3. Note that up to a conformal factor (13) coincides with the \( AdS_2 \) metric in Poincaré coordinates.

3. Conformal mechanics

First it is worth briefly reminding a relation between symmetries of a background and conserved charges of a particle propagating in it. The conventional action functional and equations of motion read

\[
S = -\int (mds + eA) \Rightarrow m \left( \frac{d^2x^n}{ds^2} + \Gamma^n_m \frac{dx^m}{ds} \right) = e g^{nm} F_{mp} \frac{dx^p}{ds}, \tag{14}
\]

where \( m \) is the mass and \( e \) is the electric charge of the particle. In general, the coordinate transformation \( x^m = x^n + \xi^n(x) \) generated by a Killing vector \( \xi^n(x) \) leaves the background vector potential invariant if

\[
\xi^m F_{mn} + \partial_n(\xi^m A_m) = 0 \tag{15}
\]
holds. Then from (14) and (15) one finds that each Killing vector $\xi^n(x)$ gives rise to the integral of motion
\[ ξ^n(x)(mx_m dx^n_m + eA_n), \] (16)
which is linear in $\frac{dx_m}{ds}$. Likewise, each Killing tensor $K_{mn}$ yields the integral of motion
\[ K_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds}, \] (17)
which is quadratic in $\frac{dx_m}{ds}$. That (17) is conserved is verified with the use of the rightmost constraint in (4).

The action functional for a massive charged particle propagating near the horizon of the extreme Kerr-Newman–AdS–dS black hole is
\[ S = -\int dt \left( m\sqrt{Γ(θ) \left[ r^2 - \dot{r}^2/r^2 - \alpha(θ) \dot{θ}^2 \right] - γ(θ) \left[ 2 \dot{θ} + kr \right]^2 + e f(θ) \left[ \dot{φ} + kr \right]^2} \right), \] (18)
where the overdot denotes the derivative with respect to $t$. We choose the Hamiltonian formalism to analyze the model. Introducing momenta $(p_r, p_θ, p_φ)$ canonically conjugate to the configuration space variables $(r, θ, φ)$
\[ p_r = \frac{mΓ(θ)r}{r^2\sqrt{Γ(θ) \left[ r^2 - \dot{r}^2/r^2 - \alpha(θ) \dot{θ}^2 \right] - γ(θ) \left[ 2 \dot{θ} + kr \right]^2}}, \]
\[ p_θ = \frac{mΓ(θ)α(θ)\dot{θ}}{\sqrt{Γ(θ) \left[ r^2 - \dot{r}^2/r^2 - \alpha(θ) \dot{θ}^2 \right] - γ(θ) \left[ 2 \dot{θ} + kr \right]^2}}, \]
\[ p_φ = \frac{mγ(θ) \left[ \dot{φ} + kr \right]}{\sqrt{Γ(θ) \left[ r^2 - \dot{r}^2/r^2 - \alpha(θ) \dot{θ}^2 \right] - γ(θ) \left[ 2 \dot{θ} + kr \right]^2}} - ef(θ), \] (19)
one readily finds the identity
\[ \sqrt{m^2Γ(θ) + (rp_r)^2 + p_θ^2/α(θ) + Γ(θ)[p_φ + ef(θ)]^2/γ(θ)} = \]
\[ = \frac{mΓ(θ)r}{\sqrt{Γ(θ) \left[ r^2 - \dot{r}^2/r^2 - \alpha(θ) \dot{θ}^2 \right] - γ(θ) \left[ 2 \dot{θ} + kr \right]^2}}. \] (20)
Then (19) and (20) can be used to derive conserved charges, including the Hamiltonian $H$, the generator of special conformal transformations $K$, the generator of dilatations $D$ and
the generator $P$ of rotations around $z$–axis, directly from the Killing vector fields (3), (11), (12) and the general formula (16)

$$H = r \left( \sqrt{m^2 \Gamma(\theta) + (rp_r)^2 + p_\theta^2 / \alpha(\theta) + \Gamma(\theta)[p_\phi + ef(\theta)]^2 / \gamma(\theta) - kp_\phi} \right),$$

$$K = \frac{1}{r} \left( \sqrt{m^2 \Gamma(\theta) + (rp_r)^2 + p_\theta^2 / \alpha(\theta) + \Gamma(\theta)[p_\phi + ef(\theta)]^2 / \gamma(\theta) + kp_\phi} \right) +$$

$$+ t^2 H + 2trp_r, \quad D = tH + rp_r, \quad P = p_\phi.$$ \hspace{1cm} (21)

Under the Poisson bracket they form $so(2, 1) \oplus u(1)$ algebra

$$\{H, D\} = H, \quad \{H, K\} = 2D, \quad \{D, K\} = K.$$ \hspace{1cm} (22)

The Killing tensor (13) yields the integral of motion quadratic in momenta

$$L = m^2 \Gamma(\theta) + p_\theta^2 / \alpha(\theta) + \Gamma(\theta)[p_\phi + ef(\theta)]^2 / \gamma(\theta).$$ \hspace{1cm} (23)

This can be used to derive the explicit formula for the reduction of the Killing tensor into Killing vectors. Computing the Casimir element of $so(2, 1)$ one finds

$$HK - D^2 = L - k^2 P^2.$$ \hspace{1cm} (24)

Translating this back to the Lagrangian formalism one can verify that contributions linear in the vector potential are canceled as well as those quadratic in the vector potential. The rest yields

$$K_{nm} = \frac{1}{2} \left( \xi_n^{(1)} \xi_m^{(3)} + \xi_n^{(3)} \xi_m^{(1)} \right) - \xi_n^{(2)} \xi_m^{(2)} + k^2 \xi_n^{(4)} \xi_m^{(4)}.$$ \hspace{1cm} (25)

Here $(\xi_n^{(1)}, \xi_n^{(2)}, \xi_n^{(3)}, \xi_n^{(4)})$ denote the Killing vectors corresponding to the time translation, dilatation, special conformal transformation and rotation around $z$–axis, respectively. As usual, the index is lowered with the use of the metric. In view of the enhanced isometry similar formulae hold for the extremal Kerr throat geometry [8], for the near-horizon geometry of the extremal Kerr-NUT-AdS-dS black hole [19], and for the weakly charged extremal Kerr throat geometry [20].

Finally, we note that Hamiltonian equations of motion can be treated by analogy with the extreme Kerr throat case [8]. The dynamics of the radial pair $(r, p_r)$ is fixed from the conserved charges $H$ and $D$. The momentum $p_\phi$ is a constant of motion while $p_\theta$ is expressed algebraically via other variables with the use of $L$. The remaining equations of motion for $\theta$ and $\phi$ can be integrated by quadratures only.

4. $\mathcal{N} = 2$ superconformal mechanics

In this section we construct $\mathcal{N} = 2$ supersymmetric extension of the particle propagating near the horizon of the extreme Kerr-Newman–AdS–dS black hole. In addition to the
so(2, 1) ⊕ u(1) generators discussed in the previous section, su(1, 1|1) superalgebra involves
supersymmetry generators Q, \( \bar{Q} \), superconformal generators S, \( \bar{S} \), and u(1) \( R \)-symmetry
generator J. Here and in what follows the bar denotes complex conjugation. The structure
relations read
\[
\begin{align*}
\{Q, \bar{Q}\} &= -2iH, & \{K, Q\} &= S, & \{Q, \bar{S}\} &= 2i(D + iJ), \\
\{D, Q\} &= -\frac{1}{2}Q, & \{H, S\} &= -Q, & \{D, S\} &= \frac{1}{2}S, \\
\{S, \bar{S}\} &= -2iK, & \{J, Q\} &= -\frac{i}{2}Q, & \{J, S\} &= -\frac{i}{2}S, \\
\{H, D\} &= H, & \{H, K\} &= 2D, & \{D, K\} &= K,
\end{align*}
\]

plus complex conjugate relations.

A conventional way to accommodate \( \mathcal{N} = 2 \) supersymmetry in the bosonic model under
consideration is to introduce two fermionic degrees of freedom \( \psi \) and \( \bar{\psi} \) and impose the
canonical bracket
\[
\{\psi, \bar{\psi}\} = -i. \tag{27}
\]
The most general form of the supersymmetry charges is
\[
Q = Ae^{iB} \psi, \quad \bar{Q} = Ae^{-iB} \bar{\psi}, \tag{28}
\]
where \( A \) and \( B \) are real functions on the full phase space \( (r, \theta, \phi, p_r, p_\theta, p_\phi) \) to be fixed from
the structure relations of the superalgebra and the requirement that in the bosonic limit a
supersymmetric Hamiltonian reduces to the first line in (21). From \( \{Q, \bar{Q}\} = -2iH \) one finds
\[
H = \frac{1}{2}A^2 + \frac{1}{2}\{A^2, B\}\psi \bar{\psi}, \tag{29}
\]
while the bosonic limit gives
\[
H_0 = \frac{1}{2}A^2 = r \left( \sqrt{m^2\Gamma(\theta) + (rp_r)^2 + p_\theta^2/\alpha(\theta) + \Gamma(\theta)[p_\phi + ef(\theta)]^2/\gamma(\theta)} - kp_\phi \right). \tag{30}
\]
For a supersymmetric model the generators of dilatations and special conformal transforma-
tions can be chosen as in the preceding section
\[
D = tH + rp_r, \quad K = K_0 + t^2H + 2trp_r, \tag{31}
\]
\footnote{For a discussion of this algebra in the context of non–relativistic many–body mechanics see e.g. [21, 22]
and references therein.}
\footnote{When constructing a supersymmetric generalization, the action functional (18) is extended by the
fermionic kinetic term \( \int dt \left( \frac{i}{2}\bar{\psi}\dot{\psi} - \frac{i}{2}\dot{\bar{\psi}}\psi \right) \) as well as other contributions describing boson–fermion couplings.
Within the framework of the Hamiltonian formalism there appear fermionic second class constraints, (27)
being the Dirac bracket associated to them.}
where we denoted
\[
K_0 = \frac{1}{r} \left( \sqrt{m^2\Gamma(\theta) + (rp_r)^2 + \frac{p_\theta^2}{\alpha(\theta)} + \Gamma(\theta)[p_\phi^2 + ef(\theta)]^2/\gamma(\theta) + kp_\phi} \right),
\]
and \( H \) is now the full supersymmetric Hamiltonian (29).

Given \( Q \) and \( K \), superconformal generator \( S \) and \( R \)-symmetry generator \( J \) are uniquely fixed by the structure relations of the superalgebra (26). Thus it remains to compute all the brackets and derive restrictions on \( B \).

The bracket of \( D \) and \( Q \) yields
\[
\{rp_r, B\} = 0,
\]
which means that \( B \) is a function of \((rp_r)\). Computing the bracket of \( H \) and \( K \) one gets
\[
\{K_0, \{H_0, B\}\} = 0 \Rightarrow \{H_0, \{K_0, B\}\} = 0,
\]
which implies that \( \{K_0, B\} \) is the integral of motion of the original bosonic theory. In the last line the Jacobi identity was used.

The bracket of \( K \) and \( Q \) determines \( S \)
\[
S = \left( -t + i\{K_0, B\} - \frac{rp_r}{H_0} \right) Q,
\]
while \( \{S, \bar{S}\} = -2iK \) gives
\[
\{K_0, B\}^2 = \frac{(H_0 K_0 - (rp_r)^2)}{H_0^2} = \frac{(L - k^2 p_\theta^2)}{H_0^2},
\]
with \( L \) from (23). Note that the numerator coincides with the Casimir element of \( so(2,1) \) realized in the bosonic theory. The full fraction commutes with \( H_0 \) in agreement with what was found above.

Finally, computing the bracket of \( Q \) and \( \bar{S} \) one derives the \( R \)-symmetry generator
\[
J = H_0\{K_0, B\} + \left( 1 + \frac{K_0\{H_0, B\}}{2H_0\{K_0, B\}} \right) \psi\bar{\psi}.
\]
It turns out that the rest of the algebra does not impose further restrictions on the form of \( B \). Thus, in order to construct an \( \mathcal{N} = 2 \) superparticle moving near the horizon of the extreme Kerr-Newman–AdS–dS black hole, it is sufficient to solve (33) and (36) for \( B \).

As was mentioned above, (33) means that \( B \) depends on \( r \) and \( p_r \) in the form of the single argument \((rp_r)\). Taking into account the definition of \( K_0 \) from (36) one derives a linear inhomogeneous first order partial differential equation for \( B \). Its general solution is

\[\text{Here and in what follows it proves helpful to use the brackets } \{rp_r, H_0\} = -H_0, \{rp_r, K_0\} = K_0, \text{ and } \{K_0, H_0\} = -2rp_r \text{ which reproduce the structure relations of } so(2,1).\]
the sum of a particular solution to the inhomogeneous equation and the general solution to
the homogeneous equation

\[ B = -\arctan \frac{rp_r}{\sqrt{L - k^2p_\phi^2}} + B_{\text{hom}}. \] (38)

The latter is an arbitrary function of the integrals of the associated system of ordinary
differential equations (see e.g. [23]). It turns out that they all depend on \( \phi \). Because
rotation around \( z \)–axis is the symmetry of the bosonic model, the requirement that the
momentum \( p_\phi \) be conserved in the full supersymmetric theory rules out \( B_{\text{hom}} \).

Thus, \( \mathcal{N} = 2 \) supersymmetric extension is essentially unique. Substituting (38) in the
generators above one gets

\[
\begin{align*}
H &= H_0 - \left( K_0^{-1} \sqrt{L - k^2p_\phi^2} \right) \psi \bar{\psi}, \\
J &= \frac{1}{2} \psi \bar{\psi} + \sqrt{L - k^2p_\phi^2}, \\
D &= tH + rp_r, \\
K &= K_0 + t^2H + 2trp_r, \\
Q &= -i\left( K_0/2 \right)^{-1/2} \left( rp_r + i\sqrt{L - k^2p_\phi^2} \right) \psi, \\
S &= -tQ + i(2K_0)^{1/2} \psi.
\end{align*}
\] (39)

Taking into account the fact that \( H_0, \ K_0, \ rp_r \) obey the structure relations of \( so(2,1) \) and
\( H_0K_0 - (rp_r)^2 = L - k^2p_\phi^2 \) is the Casimir element, one concludes that the \( \mathcal{N} = 2 \) model is
constructed in a purely group theoretical way.

A few comments are in order. First, the second term in the definition of \( J \) might seem
odd. It is the square root of the Casimir element of \( so(2,1) \) which commutes with all the
generators of \( su(1,1|1) \). If desirable, instead of including it in \( J \), one could keep this term
as the central charge in \( \mathcal{N} = 2 \) superalgebra. Only the bracket of \( Q \) and \( \bar{S} \) would be altered.
In fact, it is this way how the \( R \)–symmetry generator is usually treated in the context of
non–relativistic \( \mathcal{N} = 2 \) many–body mechanics (see e.g. [21, 22]). Second, by construction \( p_\phi \)
commutes with all the generators of \( su(1,1|1) \). The full symmetry group of the superparticle
is thus \( SU(1,1|1) \times U(1) \). Third, \( \mathcal{N} = 2 \) supersymmetry does not impose any restriction
on the particle parameters. This is to be contrasted with the \( \mathcal{N} = 4 \) superparticle moving
near the horizon of the extreme Reissner–Nordström black hole where the higher symmetry
leads to a kind of BPS condition [13]. Finally, our consideration implies that the \( \mathcal{N} = 2 \)
superparticle associated with the near horizon extreme Kerr geometry [8] is unique as well.

5. Conclusion

To summarize, in this work we studied the conformal mechanics resulting from the near
horizon geometry of the extreme Kerr-Newman–AdS–dS black hole. Conserved charges of
the particle were identified. A unique \( \mathcal{N} = 2 \) supersymmetric extension of the conformal
mechanics was constructed within the framework of the Hamiltonian formalism.

Proceeding to open questions, the first to mention is a link of the supersymmetry charges
and the superconformal charges constructed in this work to Killing spinors characterizing
the near horizon extreme Kerr-Newman–AdS–dS geometry. A related issue is a Lagrangian formulation for the $\mathcal{N} = 2$ superparticle both in components and in superfields. Then it is worth constructing a canonical transformation to conventional conformal mechanics in the spirit of [13]. The resulting angular sector might be interesting in the context of recent studies in [24] (see also references therein). It is also intriguing to study a possible link of the model in this work to the matrix model for $AdS_2$ proposed in [25]. Finally, one could try to construct $\mathcal{N} > 2$ supersymmetric extensions of the conformal mechanics, albeit it is not clear what they might correspond to in geometric terms.

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