EQUATIONS OF CAMASSA–HOLM TYPE AND JACOBI ELLIPSOIDAL COORDINATES

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Abstract. We consider the integrable Camassa–Holm equation on the line with positive initial data rapidly decaying at infinity. On such phase space we construct a one parameter family of integrable hierarchies which preserves the mixed spectrum of the associated string spectral problem. This family includes the CH hierarchy. We demonstrate that the constructed flows can be interpreted as Hamiltonian flows on the space of Weyl functions of the associated string spectral problem. The corresponding Poisson bracket is the Atiyah–Hitchin bracket. Using an infinite dimensional version of the Jacobi ellipsoidal coordinates we obtain a one parameter family of canonical coordinates linearizing the flows.

1. Introduction.

1.1. Arnold’s problem. Separation of variables is the simplest and the most powerful integration method for equations of motion in classical mechanics. Of course, there is no a general rule which allows one to find such separating coordinates in the general setting. “Therefore we have to go in the opposite direction and knowing some remarkable substitution to find a problem where it can be successfully applied”, [12]. One such remarkable substitution, known as Jacobi ellipsoidal coordinates is widespread. Below is a list (incomplete) of classical problems where this substitution can be applied, [2].

• Plane motion in the field of two attracting centers (Euler 1760).
• Kepler’s problem in the homogeneous force field (Lagrange 1766).
• Geodesic motion on $n$-dimensional ellipsoid (Jacobi 1866).
• Motion of uncoupled harmonic oscillators constrained to move on the $n$-dimensional sphere by the field of a quadratic potential (Neumann 1859).

The following question$^1$ was posed by V.I. Arnold:

Problem. 1981-29. [1]. Generalize Jacobi ellipsoidal coordinates to the infinite dimensional setting. Find equations of mathematical physics integrable by this method.

The goal of the present paper is to demonstrate that the Camassa–Holm equation is an example of a PDE integrable with this technique. The Camassa–Holm equation,

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$^1$See also Problem 1985–16, [1].

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[8], is an approximation to the Euler equation describing an ideal fluid

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} R \left[ v^2 + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 \right] = 0
\]

in which \( t \geq 0 \) and \(-\infty < x < \infty\), \( v = v(x, t) \) is velocity, and \( R \) is inverse to \( L = 1 - d^2/dx^2 \) i.e.

\[
R[f](x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|x-y|} f(y) dy.
\]

Introducing the function \( m = L[v] \) one writes the equation in the form \(^2\)

\[
m\dot{} + (mD + Dm) v = 0.
\]

The CH equation is a Hamiltonian system \( m\dot{} + \{m, \mathcal{H}\} = 0 \) with Hamiltonian

\[
\mathcal{H} = \frac{1}{2} \int_{-\infty}^{+\infty} mv \, dx = \text{energy}
\]

and the bracket

\[
\{A, B\} = \int_{-\infty}^{+\infty} \frac{\delta A}{\delta m} (mD + Dm) \frac{\delta B}{\delta m} \, dx. \tag{1.1}
\]

The integration of the CH equation is based on the spectral theory of an associated Sturm–Liouville operator. In connection with this spectral problem Weyl, [28], in 1910 introduced what is now called the Weyl function. The Weyl function depends on the spectral parameter and defined such that some linear combination of fundamental solutions and its derivatives vanishes at the end of an interval. The novel point here is that the Weyl function carries a natural Poisson structure, the Atiyah–Hitchin bracket. It corresponds to the Poisson structure 1.1 on the phase space. The fact that Jacobi elliptic coordinates are separating coordinates for 1.1 is proved by contour integration.

It seems that the CH equation on the line is the simplest among all integrable PDE. The reason for this is a degeneration of its spectral curve. Such degenerate reducible rational curves appear in the compactification of the space of smooth hyperelliptic curves. It is important that the constructions of the present paper can be extended to other basic integrable models, i.e., the KdV/KP and NLS equations, [26, 27].

The rest of the introduction describes what is happening for the CH equation.

\(^2\)We use notation \( D \) for the \( x \)-derivative and \( \dot{} \) for the \( t \)-derivative. We use \( \delta \) for the Frechet derivative.
1.2. The Camassa–Holm hierarchy. We consider the CH equation with non-negative \((m \geq 0)\) initial data and such decay at infinity that:

\[
\int_{-\infty}^{+\infty} m(x)e^{|x|}dx < \infty.
\]

We denote this class of functions by \(\mathcal{M}\). For such data a solution of the initial value problem exists for all times, see [9]. An initial profile moves from the left to the right and changes its shape under the flow. We also need a subclass of functions \(\mathcal{M}_0 \subset \mathcal{M}\) which vanish far enough to the left. Evidently \(\mathcal{M}_0\) is invariant under the flow.

The CH Hamiltonian is just one of infinitely many conserved quantities of motion

\[
\mathcal{H}_1 = \int v \, dx,
\]

\[
\mathcal{H}_2 = \frac{1}{2} \int v^2 + (Dv)^2 \, dx,
\]

\[
\mathcal{H}_3 = 2 \int v \left[ v^2 + (Dv)^2 \right] \, dx,
\]

\[
\mathcal{H}_4 = \frac{1}{2} \int_0^1 v^4 + \int_0^1 v^2(Dv)^2 + 2 \int_0^1 \left[ v^2 + \frac{(Dv)^2}{2} \right] G \left[ v^2 + \frac{(Dv)^2}{2} \right] \, dx, \text{ etc.}
\]

The first integral

\[
\mathcal{H}_1 = \int_{-\infty}^{+\infty} m \, dx = \text{momentum}
\]

produces the flow of translation

\[
m\cdot \{m, \mathcal{H}_1\} = m\cdot + Dm = 0.
\]

The second is the CH Hamiltonian \(\mathcal{H}_2 = \mathcal{H}\). The integrals \(\mathcal{H}_3, \mathcal{H}_4, \text{ etc.}\), produce higher flows of the CH hierarchy.

An interesting feature of the Camassa–Holm equation is that the invariant manifold, or in other terms the spectral class specified by these integrals, is not compact. As we will see this is reflected in the non-periodicity of the angle variables which occupy the entire real line.

1.3. The spectral problem. Action–angle variables. One can associate to the CH equation an auxiliary string spectral problem, \(^3\)

\[
f''(\xi) + \lambda g(\xi)f(\xi) = 0, \quad -2 \leq \xi \leq 2.
\]

The background information for this spectral problem can be found in [13, 15, 10].

The variables \(\xi\) and \(x\) are related by

\[
x \rightarrow \xi = 2 \tanh \frac{x}{2}.
\]

\(^3\)We use prime ’ to denote \(\xi\)-derivative.
Also the potential $g(\xi)$ is related to $m(x)$ by the formula $g(\xi) = m(x) \cosh^4 \frac{x}{2}$. For initial data from $\mathcal{M}$ the total mass of the associated string is finite $\int_{-\infty}^{+\infty} g(\xi) d\xi < \infty$. Evidently, for initial data from $\mathcal{M}_0$ there is an interval of length $l$ where the potential vanishes: $g(\xi) = 0, \quad \xi \in [-2, -2 + l]$.

The solutions of the string spectral problem are continuous, but the left $f'_-(\xi)$ and right $f'_+(\xi)$ derivatives at some point $\xi$ may be different due to a concentrated mass at this point. Two solutions $\varphi(\xi, \lambda)$ and $\psi(\xi, \lambda)$ of the string spectral problem play an important role in the whole discussion. They are specified by the initial data

\[
\begin{align*}
\varphi(-2, \lambda) &= 1 & \psi(-2, \lambda) &= 0 \\
\varphi'(-2, \lambda) &= 0 & \psi'(-2, \lambda) &= 1.
\end{align*}
\]

All potentials of the string spectral problem with fixed Dirichlet spectrum

\[
\lambda_n : \quad \psi(2, \lambda_n) = 0, \quad n = 1, 2, \ldots;
\]

constitute a spectral class. The variables

\[
I_n = -\frac{1}{\lambda_n}, \quad \theta_n = \log \varphi(2, \lambda_n)
\]

are called "action–angle" variables. The angles $\theta_n$ take values in $\mathbb{R}$; they are not a cyclic variables.

It turns out that the Dirichlet spectrum is preserved under the flows of the CH hierarchy. The flows are linearized in terms of angle variables. This follows from the canonical relations

\[
\begin{align*}
\{I_k, I_n\} &= 0, \\
\{\theta_k, \theta_n\} &= 0, \\
\{\theta_k, I_n\} &= \delta^k_n;
\end{align*}
\]

and the trace formulas of McKean, [20]:

\[
\begin{align*}
\mathcal{H}_1 &= \sum_n \lambda^{-1}_n = -\sum_n I_n, \\
\mathcal{H}_2 &= \frac{1}{4} \sum_n \lambda^{-2}_n = \frac{1}{4} \sum_n I^2_n, \quad etc.
\end{align*}
\]

We note that the CH flow was linearized in terms of the so-called coupling constants by Beals, Sattinger and Smigielski, [4].
1.4. **Mixed boundary conditions.** As we noted the CH flows do not change the Dirichlet spectrum. It is natural to consider the mixed boundary condition

\[ a\varphi(2, \lambda) + b\psi(2, \lambda) = 0, \]

where \(a, b\) are real constants. The set of all boundary conditions, \(i.e.,\), the pairs \((a, b)\), constitutes \(\mathbb{RP}\), real projective space. For example, the Dirichlet boundary condition corresponds to the point \((0, 1)\). Our goal is to associate to each boundary condition (point of \(\mathbb{RP}\)) the family of Hamiltonian flows which preserve the spectrum corresponding to this boundary condition. It turns out that the space \(\mathbb{RP}\) can be covered by two charts.

The first chart corresponds to the boundary condition:

\[ \psi(2, \lambda) - C\varphi(2, \lambda) = 0, \]

in which \(C \leq l\). It can be constructed for all initial data from \(\mathcal{M}\). The family of roots \(\lambda_n = \lambda_n(C), \ n = 1, 2, \ldots\); produces generalized "action–angle" variables

\[ I_n(C) = -\frac{1}{\lambda_n(C)}; \quad \theta_n(C) = \log \varphi(2, \lambda_n(C)). \]

For each value of \(C \leq l\) one has the family of integrals. Here are the first two:

\[ H_1(C) = -\frac{C}{4-C} \int_{-\infty}^{+\infty} m(x)(1 + e^{-x})dx + \frac{4}{4-C} \int_{-\infty}^{+\infty} m(x)dx; \]

\[ 4H_2(C) = \frac{1}{(4-C)^2} \left( C \int_{-\infty}^{+\infty} m(x)(1 + e^{-x})dx - 4 \int_{-\infty}^{+\infty} m(x)dx \right)^2 + \]

\[ + \frac{2}{4-C} \left[ \frac{C - 4}{2} \left( \int_{-\infty}^{+\infty} d\xi m(\xi) \right)^2 - 2C \int_{-\infty}^{+\infty} d\xi e^{-\xi} m(\xi)D^{-1}m(\xi) + \right. \]

\[ \left. + (4-C) \int_{-\infty}^{+\infty} d\xi m(\xi)v(\xi) \right]. \]

Evidently for \(C = 0\) we have classical action-angle variables and the CH Hamiltonians. The Hamiltonians produce the flows\(^5\)

\[ m^* + \{m, H_k(C)\} = 0, \quad k = 1, 2, \ldots. \]

\(^4\)The anti-derivative \(D^{-1}\) is defined by the formula \(D^{-1}f(x) = \frac{1}{2} \left[ \int_{-\infty}^{x} d\xi f(\xi) - \int_{x}^{+\infty} d\xi f(\xi) \right].\)

\(^5\)The question of existence will be discussed at the end of the introduction.
These flows preserve \( I(C) \) and move \( \theta(C) \) linearly. This follows from the trace formulas

\[
\mathcal{H}_1(C) = - \sum_n I_n(C),
\]
\[
\mathcal{H}_2(C) = \frac{1}{4} \sum_n I_n^2(C);
\]

and the canonical relations for the variables \( I(C) \) and \( \theta(C) \).

Another chart corresponds to the boundary condition

\[
\varphi(2, \lambda) + F \psi(2, \lambda) = 0,
\]
in which \( F > -1/l \). It is constructed for initial data from \( \mathcal{M}_0 \). For \( F = 0 \) the roots \( \mu_k(0) \) form the second spectrum of the string spectral problem

\[
\mu_k : \quad \varphi(2, \mu_k) = 0 \quad k = 1, 2, \ldots.
\]

The roots \( \mu_k(F), \ k = 1, 2, \ldots \); produce \( J(F) \) and \( \tau(F) \), the second family of canonical coordinates:

\[
J_k(F) = - \frac{1}{\mu_k(F)}, \quad \tau_k(F) = \log \psi(2, \mu_k(F)).
\]

Again, for \( F > -1/l \) one can write a family of integrals. The first two are

\[
\mathcal{T}_1(F) = \frac{1}{1 + 4F} \left[ \int_{-\infty}^{+\infty} m(x)(1 + e^{-x})dx + \frac{4F}{1 + 4F} \int_{-\infty}^{+\infty} m(x)dx \right];
\]
\[
4\mathcal{T}_2(F) = \frac{1}{(1 + 4F)^2} \left( \int_{-\infty}^{+\infty} m(x)(1 + e^{-x})dx + 4F \int_{-\infty}^{+\infty} m(x)dx \right)^2 - \frac{2}{1 + 4F} \times \left[ \frac{1 + 4F}{2} \left( \int_{-\infty}^{+\infty} m(x)dx \right)^2 - 2 \int_{-\infty}^{+\infty} dx e^{-x} m(x)D^{-1}m(x) - (1 + 4F) \int_{-\infty}^{+\infty} dx m(x)v(x) \right].
\]

These Hamiltonians produce the flows

\[
m^\ast + \{m, \mathcal{T}_k(F)\} = 0, \quad k = 1, 2, 3, \ldots
\]

The trace formulas have the form

\[
\mathcal{T}_1(F) = - \sum_n J_n(F),
\]
\[
\mathcal{T}_2(F) = \frac{1}{4} \sum_n J_n^2(F), \quad \text{etc.}
\]
The formulas becomes especially simple when $F = 0$. The first integral
\[ T_1(0) = \int_{-\infty}^{+\infty} m(x)(1 + e^{-x}) \, dx, \]
produces the flow
\[ m \cdot + \{m, T_1(0)\} = m \cdot - 2me^{-x} + (1 + e^{-x})Dm = 0, \]
which is linearized in the variables $J(0)$ and $\tau(0)$. The flows corresponding to higher Hamiltonians are nonlocal.

1.5. Ellipsoidal coordinates. The Atiyah–Hitchin bracket. The key observation in the construction of the generalized action–angle variables $I$ and $\theta$ and $J$ and $\tau$ is that the spectra $\lambda_n(C)$ and $\mu_k(F)$ can be interpreted as Jacobi ellipsoidal coordinates. This allows one to avoid any use of the hierarchy of differential equations as well as the string spectral problem. As a result the dynamics can be reformulated purely in the language of meromorphic functions on $\mathbb{C}P$. First, let us explain the construction of $I(C)$.

For the string $S_1$, i.e., for the string with free left end and fixed right end, we define so-called the Weyl function
\[
\Omega_0(\lambda) = \psi(2, \lambda) \varphi(2, \lambda) = l + \sum_k \frac{\sigma_k}{\mu_k - \lambda},
\]
where $l \geq 0$ is an interval free of mass, $\sigma_k > 0$ and $\sum \frac{\sigma_k}{\mu_k} < \infty$. The roots of the equation $\Omega_0(\lambda) = C$, where $C \leq l$ are the points $\lambda_n(C)$. This is a classical way to define ellipsoidal coordinates. The construction is depicted on Figure 1. We have the sequence of points $\lambda_k = \lambda_k(C), \ k = 1, 2, \ldots$ which depend on the constant $C$. They interlace the poles of $\Omega_0(\lambda)$:
\[ \mu_1 < \lambda_1(C) < \mu_2 < \lambda_2(C) < \ldots. \]
The action variables are defined by the formula
\[ I_n(C) = -\frac{1}{\lambda_n(C)}, \quad n = 1, 2, \ldots. \]
Classical action variables correspond to the case $C = 0$.

Now we can give a similar description of the variables of $J(F)$. We assume that $m(x) \in \mathcal{M}_0$ and therefore $l > 0$. For the string $S_0$ with fixed left and right ends we have the Weyl function
\[
E_0(\lambda) = -\frac{1}{\Omega_0(\lambda)} = -\frac{\varphi(2, \lambda)}{\psi(2, \lambda)} = -\frac{1}{l} + \sum_k \frac{\rho_k}{\lambda_k - \lambda},
\]
where $\rho_k > 0$ and $\sum \frac{\rho_k}{\lambda_k} < \infty$. The points $\mu_k(F)$ appear as roots of the equation $E_0(\lambda) = F, \ F > -1/l$. This is shown on Figure 2.
We have a sequence of points $\mu_k = \mu_k(F)$ interlacing the poles of $E_0(\lambda)$:

$$\mu_1(F) < \lambda_1 < \mu_2(F) < \ldots.$$

The action variables $J(F)$ are defined by the formula

$$J_n(F) = -\frac{1}{\mu_n(F)}, \quad n = 1, 2, \ldots.$$

The second spectrum corresponds to the case $F = 0$. 
The next step is to relate the functions $\Omega_0$ and $E_0$ to Poisson geometry. First we change the spectral parameter by the rule $\lambda \rightarrow \lambda' = -1/\lambda$. Note that this transformation maps the spectrum into the corresponding action variables. The Poisson bracket 1.1 for the function $\Omega_0$ corresponding two different values of the spectral parameter is given by the formula

$$\{\Omega_0(\lambda'), \Omega_0(\mu')\} = \frac{(\Omega_0(\lambda') - \Omega_0(\mu'))^2}{\lambda' - \mu'}.$$ 

This is standard Atiyah–Hitchin bracket in the form found by Faybusovich and Gehtman, [11]. The bracket (or more precisely the symplectic form) was originally introduced for rational functions in terms of their singularities in [3]. In [11] coordinate free form of the bracket for rational functions was found. This coordinate free form is identically the same for an infinite-dimensional case which we consider here. The bracket is invariant under linear–fractional transformations. Namely, for any function $\hat{\Omega}$ defined as

$$\hat{\Omega} = \frac{a\Omega_0 + b}{c\Omega_0 + d},$$

where $a, b, c$ and $d$ are real constants, the Poisson bracket is given by the same formula, see [26]. In particular, the bracket for the function $E_0$ is given by the same formula. The canonical relations for the generalized action–angle variables $I(C)$ and $\theta(C)$ and also $J(F)$ and $\tau(F)$ are proved by contour integration.

1.6. The spectral curve and Baker–Akhiezer function. It is instructive to present an algebraic–geometrical approach to the inverse problems with pure discrete spectrum developed in [18, 25]. Such problems include finite Jacobi matrices [18, 25], the quantum mechanical oscillator [21] and the string spectral problem. The algebraic–geometrical viewpoint will provide an additional geometrical insight into the nature of the flows constructed from the Hamiltonians $\mathcal{H}(C)$ and $\mathcal{T}(F)$.

The Riemann surface $\Gamma$ (Figure 3) associated with the spectral problem consists of two components $\Gamma_-$ and $\Gamma_+$, two copies of $\mathbb{C}P$. A point on the curve is denoted by $Q = (\lambda, \pm)$, where $\lambda \in \mathbb{C}P$ and the sign $\pm$ refers to the component. An infinites of $\Gamma_\pm$ are denoted by $P_\pm$ correspondingly. The components are glued together at the points of the Dirichlet spectrum $(\lambda_k, \pm)$.

The Baker–Akhiezer function $e(\xi, Q)$ is defined on $\Gamma$ and depends on the variable $\xi, -2 \leq \xi \leq +2$ as a parameter. It is defined by the formula

$$e(\xi, Q) = \begin{cases} \psi(\xi, \lambda) : & \text{if } Q \in \Gamma_+, \lambda = \lambda(Q); \\ \frac{1}{E_0(\lambda)} \varphi(\xi, \lambda) + \psi(\xi, \lambda), & \text{if } Q \in \Gamma_-, \lambda = \lambda(Q); \end{cases}$$

where

$$E_0(\lambda) = -\frac{\varphi(2, \lambda)}{\psi(2, \lambda)}.$$
The Baker–Akhiezer function has essential singularities at the infinities $P_\pm$ and simple poles at the points of the divisor $\gamma_k = (\mu_k, -)$, $k = 1, 2, \ldots$. It is holomorphic everywhere else. The Baker–Akhiezer function satisfies the gluing condition

$$e(\xi, (\lambda_k, +)) = e(\xi, (\lambda_k, -)), \quad k = 1, 2, \ldots.$$  

When $Q \in \Gamma_-$ we have

$$E_0(\lambda) = \frac{e'(-2, Q)}{e(-2, Q)}, \quad \lambda = \lambda(Q).$$

This type of formula expressing the Weyl function through the Baker–Akhiezer function is a starting point for consideration in the case of hyperelliptic curves, [26, 27].

The flows of the Camassa–Holm hierarchy constructed from the Hamiltonians $\mathcal{H}(0)$ preserve the moduli $\lambda_k$, $k = 1, 2, \ldots$, but move points of the divisor $\gamma_k$, $k = 1, 2, \ldots$. The flows constructed from the Hamiltonians $\mathcal{T}(0)$ preserve the divisor of the poles and move the moduli. All other flows constructed from the Hamiltonians $\mathcal{H}(C)$ and $\mathcal{T}(F)$ move both the moduli and the divisor.
1.7. The spectral class. Before going into the infinite–dimensional problem of
the description of the spectral class of a string we present its finite–dimensional
counterpart. Consider a finite Jacobi matrix

$$L_0 = \begin{bmatrix} v_0 & c_0 & 0 & \cdots & 0 \\ c_0 & v_1 & c_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & c_{N-3} & v_{N-2} & c_{N-2} \\ 0 & \cdots & 0 & c_{N-2} & v_{N-1} \end{bmatrix}, \quad c_k > 0.$$ 

All matrices with the same spectrum $\lambda_0 < \ldots < \lambda_{N-1}$ as $L_0$ constitute a spectral
class of $L_0$. There are no restrictions on the spectrum in this finite–dimensional
case. The fact that the spectral class is diffeomorphic to $R^{N-1}$ was already noted
by Moser, [22]. Tomei, [24] showed that it can be compactified and becomes a
convex polyhedron. The symplectic interpretation of this result as a version of
the Atiyah–Guillemin–Sternberg convexity theorem was given by Bloch, Flaschka
and Ratiu [5]. In [25] we proved that points of the divisor can serve as natural
coordinates on the spectral class and described their range. In this paper we give
a similar description of the string spectral class in terms of the divisor.

First we note that the two spectra

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots; \quad (1.4)$$

and

$$\mu_1 < \mu_2 < \mu_3 < \ldots; \quad (1.5)$$
appear as poles of the functions $E_0$ and $\Omega_0$ respectively. Their range is described
by two theorems of Krein. The first theorem gives a description of all singular
Riemann surfaces which might appear for initial data from $\mathcal{M}$.

**Theorem 1.1.** [16] *For a given increasing sequence 1.4 to be the spectrum of a
string $S_0$ it is necessary and sufficient that there exists a finite limit

$$\lim_{n \to \infty} \frac{n}{\sqrt{\lambda_n}}$$

and

$$\sum_{n=1}^{+\infty} \frac{1}{\lambda_n^2 |D'\lambda_n|} < \infty, \quad D(\lambda) = \prod_{n=1}^{+\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right).$$

The next Theorem gives a description of all possible divisors.

**Theorem 1.2.** [16] *For a given increasing sequence 1.5 to be the spectrum of a
string $S_1$ it is necessary and sufficient that there exists a finite limit

$$\lim_{n \to \infty} \frac{n}{\sqrt{\mu_n}}$$
and
\[ \sum_{n=1}^{+\infty} \frac{1}{\mu_n^{3/2} |D'_1(\mu_n)|} < \infty, \quad D_1(\lambda) = \prod_{n=1}^{+\infty} \left( 1 - \frac{\lambda}{\mu_n} \right). \]

Now we can present a description of the spectral class which is also due to Krein.

**Theorem 1.3.** [15] For two increasing interlacing sequences of \( \lambda \)'s and \( \mu \)'s
\[ 0 < \mu_1 < \lambda_1 < \mu_2 < \ldots \]
to be the two spectra of a string with finite mass and length it is necessary and sufficient that
\[ -\sum_{k=1}^{+\infty} \frac{\mu^2_k D'_1(\mu_k) D(\mu_k)}{D'_1(\mu_k) D(\mu_k)} < \infty. \]

Now we have a complete description of the spectral curve and divisor. We already introduced the formal flows corresponding to the Hamiltonians \( \mathcal{H}(C) \) and \( \mathcal{T}(F) \). We can say that the flows exist if the corresponding sequences 1.4 and 1.5 satisfy the conditions of Theorem 1.3. To write an explicit form of the equations describing the motion of the moduli and the divisor is a routine exercise.

Finally, we note that the problem of the description of the spectral class for the quantum mechanical oscillator, [21], remains open.

1.8. **The range of ellipsoidal coordinates.** Now we want
- given \( \mu \)'s, to describe the range of variables \( \lambda(C) \) when \( C < l \);
- given \( \lambda \)'s, to describe the range of variables \( \mu(F) \) when \( F > -1/l \).

An answer to these questions is given by two recent results.

The first is the theorem of Kostuchenko–Stepanov, [17], a direct generalization of an ancient (1857) formula of Boole, [6]. In the context of the functions \( E_0 \) and \( \Omega_0 \) their result takes the following form.

**Theorem.** If \( \sum_k \rho_k < \infty \) in formula 1.3 or \( \sum_k \sigma_k < \infty \) in formula 1.2 then the corresponding sequence of \( \mu_k(F) \) or \( \lambda_k(C) \) satisfies
\[ \sum_k [\lambda_k - \mu_k(F)] < \infty, \]
or
\[ \sum_k [\mu_{k+1} - \lambda_k(C)] < \infty. \]

The conditions of the theorem are not only necessary but also sufficient for the sequence of \( \mu \)'s and \( \lambda \)'s to arise as coordinates for some summable sequence of \( \rho \)'s or \( \sigma \)'s.

Unfortunately the condition of summability for the sequence of \( \lambda \)'s and \( \mu \)'s is too restrictive for our purposes. It never holds for initial data from the phase space \( \mathcal{M} \). We need the second result which due to Nazarov and Yuditskii, [23].
Theorem. If $\sum \rho_k/\lambda_k < \infty$ or $\sum \sigma_k/\mu_k < \infty$, then

$$\sum_k \frac{\lambda_k}{\mu_k(F)} - 1 < \infty,$$

or

$$\sum_k \frac{\mu_{k+1}}{\lambda_k(C)} - 1 < \infty.$$ 

The conditions of the theorem are not only necessary but also sufficient for the sequence of $\lambda$’s and $\mu$’s to arise as coordinates for some summable sequence of $\rho/\lambda$’s or $\sigma/\mu$’s.

1.9. Organization of the paper. In Section 2 we introduce an auxiliary Sturm–Liouville eigenvalue problem. Section 3 transforms this spectral problem into the Krein string by the classical Liouville transformation. We write the Poisson bracket and the flows of the CH hierarchy in terms of the potential of the string spectral problem. Section 4 presents standard facts about the spectral theory of the string. Jacobi ellipsoidal coordinates are discussed in Section 5. The Poisson bracket for the Weyl function is computed in Section 6. Action–angle coordinates are introduced in Section 7. There also CH flows are linearized. Section 8 contains examples of the computation of the Weyl function for one and two peakon solutions. Section 9 introduces generalized action–angle coordinates which include the classical ones as a particular case. Section 10 contains another family of canonical coordinates. Both families cover all possible spectra which arise from various mixed boundary conditions. Section 11 presents the derivation of the trace formulas.

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2. The spectral problem

We consider the Camassa–Holm equation

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0 \quad p(x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|x-y|} \left[ v^2 + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 \right] dy,$$

where $t \geq 0$ and $-\infty < x < \infty$. The integral operator $R$ defined by

$$R[f](x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|x-y|} f(y) \, dy.$$
is an inverse of \( L = 1 - D^2 \). Introducing \( m = L[v] \) we specify the phase space \( \mathcal{M} \) as all smooth positive functions with such decay at infinity that

\[
\int_{-\infty}^{\infty} m(x)e^{|x|} \, dx < \infty.
\]

We also need \( \mathcal{M}_0 \) the subspace of functions which vanish far enough to the left. \( \mathcal{M} \) and \( \mathcal{M}_0 \) are invariant under the CH flow, [9].

The first four Hamiltonians, integrals of motion, are listed in 1.2. The first Hamiltonian \( \mathcal{H}_1 = \int_{-\infty}^{+\infty} v \, dx \) = momentum produces the flow of translation:

\[
0 = m^\bullet + \{ m, \mathcal{H}_1 \} = m^\bullet + Dm,
\]

where the bracket is

\[
\{ A, B \} = \int_{-\infty}^{+\infty} \frac{\delta A}{\delta m} (mD + Dm) \frac{\delta B}{\delta m} \, dx.
\]

The CH Hamiltonian \( \mathcal{H}_2 = \frac{1}{2} \int_{-\infty}^{+\infty} v^2 + Dv^2 \, dx \) = energy is the second in the infinite series of commuting integrals of motion. The equation can be written as

\[
0 = m^\bullet + \{ m, \mathcal{H}_2 \} = m^\bullet + (mD + Dm) v = \left( \frac{\partial}{\partial t} + 2(Dv) + vD \right)(1 - D^2)v.
\]

All other integrals produce higher flows of the hierarchy.

The CH equation is a compatibility condition between

\[
D^2 f - \frac{1}{4} f + \lambda m f = 0
\]

and

\[
f^\bullet = -\left( v + \frac{1}{2\lambda} \right) Df + \frac{1}{2}(Dv)f;
\]

i.e., \( (D^2 f)^\bullet = D^2 (f^\bullet) \) is the same as 2.3. The isospectrality of 2.4 is a key to the integrability of the CH dynamics.

### 3. The Liouville Correspondence

The standard Liouville’s transformation

\[
x \rightarrow \xi(x) = 2 \tanh \frac{x}{2}, \quad f(x) \rightarrow f(\xi) = \frac{f(x)}{\cosh \frac{x}{2}}
\]

converts 2.4 into the string spectral problem

\[
f'' + \lambda g f = 0,
\]

or

\[
f'' + \frac{\lambda}{g} f = 0.
\]
with \( g(\xi) = m(x) \cosh^4 \frac{x}{2} \) and \(-2 \leq \xi \leq +2\). The transformation changes the length element by the rule

\[
dx \rightarrow d\xi = dx \mathcal{J}(\xi), \quad \text{with} \quad \mathcal{J}(\xi) = 1 - \frac{\xi^2}{4}.
\]

Under the assumptions made, the phase space of the string is regular, \textit{i.e.}, its mass is finite:

\[
\int_{-2}^{+2} g(\xi) \, d\xi = \int_{-\infty}^{+\infty} m(x) \cosh^2 \frac{x}{2} \, dx < \infty.
\]

The transformation reduces the problem 2.4 with two singular ends to the regular string problem on the finite interval. It is necessary for introduction of the ellipsoidal coordinates but first we consider what is happening on the phase space.

The functions \( m(x) \) and \( v(x) \) are related by \( m = L[v] \) and \( v = R[m] \). Introducing \( k(\xi) = v(x) \cosh^4 \frac{x}{2} \), we have the relations

\[
g(\xi) = \mathcal{L}[k](\xi) = k(1 - 4\mathcal{J}'^2 - 2\mathcal{J}\mathcal{J}'' - 5k'\mathcal{J}\mathcal{J}' - k''\mathcal{J}^2),
\]

and

\[
k(\xi) = \mathcal{R}[g](\xi) = \frac{1}{2} \int_{-2}^{+2} \mathcal{R}(\xi, \eta)g(\eta) \, d\eta
\]

with

\[
\mathcal{R}(\xi, \eta) = \begin{cases} \left(\frac{1+\eta/2}{1+\xi/2}\right)^2 \frac{1}{\mathcal{J}(\xi)}, & \text{for } \eta \leq \xi; \\
\left(\frac{1-\eta/2}{1-\xi/2}\right)^2 \frac{1}{\mathcal{J}(\xi)}, & \text{for } \eta \geq \xi.
\end{cases}
\]

The functions \( k(\xi) \) and \( g(\xi) \) are new coordinates on the phase space. Best of all, the relation between the functions \( m, v \) and \( k, g \) can be expressed by the diagram:

\[
m(x) : m = L[v] \quad \longleftrightarrow \quad v(x) : v = R[m]
\]

\[
g(\xi) = m(x) \cosh^4 \frac{x}{2} \quad \downarrow \quad \downarrow \quad k(\xi) = v(x) \cosh^4 \frac{x}{2}
\]

\[
g(\xi) : g = \mathcal{L}[k] \quad \longleftrightarrow \quad k(\xi) : k = \mathcal{R}[g]
\]
Now we want to express the first two flows 2.1 and 2.3 of the CH hierarchy in terms of the new coordinate, the density function $g$. The conserved quantities are

$$
\mathcal{H}_1 = \int_{-2}^{+2} g J d\xi,
$$

$$
\mathcal{H}_2 = \frac{1}{2} \int_{-2}^{+2} g R[g] J^3 d\xi, \text{ etc.}
$$

The bracket 2.2 takes the form

$$
\{A, B\} = \int_{-2}^{+2} \frac{\delta A}{\delta g} \left( J g \partial J^{-1} + J^{-1} \partial g J \right) \frac{\delta B}{\delta g} d\xi.
$$

(3.2)

Computing the variation $\delta \mathcal{H}_1 = \int \delta g J d\xi$ we write the translation flow 2.1 as

$$
0 = g^\bullet + \{g, \mathcal{H}_1\} = g^\bullet + J^{-1} \partial (g J^2).
$$

The self-adjointness of $R$ acting on the functions $f$ and $g$ of the $\xi$–variable,

$$
\int_{-2}^{+2} R[g] f J^3 d\xi = \int_{-2}^{+2} g R[f] J^3 d\xi
$$

implies the formula for the variation of the basic Hamiltonian

$$
\delta \mathcal{H}_2 = \frac{1}{2} \int \delta g R[g] J^3 d\xi + \frac{1}{2} \int g R[\delta g] J^3 d\xi = \int \delta g R[g] J^3 d\xi.
$$

Therefore, for the CH flow 2.3, we have

$$
0 = g^\bullet + \{g, \mathcal{H}_2\} = g^\bullet + (g J \partial J^{-1} + J^{-1} \partial g J) k J^3.
$$

Evidently the Liouville’s transformation complicates the formulas for the flow.

### 4. The spectral theory of the string

In this section we introduce the so-called Weyl function, the main tool in construction of ellipsoidal coordinates. We explain its relation to the spectral theory of the string. The direct and inverse spectral theory of the string with nonnegative mass was constructed by M.G. Krein in the 1950’s and presented in [15], see also [13, 10]. To formulate the results we need some facts of function theory. These can be found in [14].

$^6$\(\partial\) stands for the derivative in $\xi$ variable.
The function of a complex variable \( F(\lambda) \) belongs to the class \((R)\) and is called an \( R \)–function if it is holomorphic on \( \mathbb{C} - \mathbb{R} \) with \( F(\lambda) = \overline{F(\lambda)} \) and \( \Im F(\lambda) \) has the same sign as \( \Im \lambda \). Any such function can be represented in the form

\[
F(\lambda) = \alpha + \beta \lambda + \int_{-\infty}^{+\infty} \left( \frac{1}{\zeta - \lambda} - \frac{\zeta}{1 + \zeta^2} \right) d\sigma(\zeta) \quad (\Im \lambda \neq 0),
\]

where \( \beta \geq 0 \) and \( \alpha \) is a real constant. The positive measure \( d\sigma \) is such that

\[
\int_{-\infty}^{+\infty} \frac{d\sigma(\zeta)}{1 + \zeta^2} < \infty.
\]

Denote by \((\tilde{R})\) the class \((R)\) adjoined with the function identically equal to infinity. Evidently, if \( F(\lambda) \in (\tilde{R}) \) then \(-1/F(\lambda) \in (\tilde{R})\).

Let \((R_1)\) be a subclass of functions which admit an absolutely convergent integral representation

\[
F(\lambda) = \alpha + \int_{-\infty}^{+\infty} \frac{d\sigma(\zeta)}{\zeta - \lambda} \quad (\Im \lambda \neq 0),
\]

where \( \alpha \) is a real constant and \( d\sigma \) is a positive measure.

We introduce two solutions \( \varphi(\xi, \lambda) \) and \( \psi(\xi, \lambda) \) of the eigenvalue problem 3.1 with the boundary conditions

\[
\varphi(-2, \lambda) = 1 \quad \psi(-2, \lambda) = 0 \\
\varphi'(-2, \lambda) = 0 \quad \psi'(-2, \lambda) = 1.
\]

The functions \( \varphi(2, \lambda) \) and \( \psi(2, \lambda) \) can be written in the form

\[
\varphi(2, \lambda) = \prod_{k=1}^{\infty} \left( 1 - \frac{\lambda}{\mu_k} \right), \quad \psi(2, \lambda) = 4 \prod_{k=1}^{\infty} \left( 1 - \frac{\lambda}{\lambda_k} \right).
\]

The roots interlace each other

\[
0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots,
\]

Given some function \( N(\lambda) \) which belongs to the function class \((\tilde{R})\). This function describes the boundary condition at the right end of the string. For example, if the right end of the string is fixed then \( N(\lambda) = 0 \) and \textit{vice-versa}. The function \( \Omega_N \) was introduced by H. Weyl, [28], in his study of the Sturm–Liouville problem. It is defined such that the corresponding Weyl solution \( \chi = -\psi + \Omega_N \varphi \) satisfies boundary condition

\[
\chi'_+(2) N(\lambda) + \chi(2) = 0 \quad \text{for all } \lambda.
\]

The Weyl function is the main ingredient of the spectral theorem.
To formulate the spectral theorem we consider the measure $dM(\xi) = g(\xi)d\xi$ where $g(\xi) \geq 0$ is the function from 3.1 and $L^2_M = L^2_M[-2,2]$ the set of $M$-measurable complex-valued functions $f(\xi)$ such that

$$||f||^2_M = \int_{-2}^{2} |f(\xi)|^2 dM(\xi) < \infty.$$ 

Evidently, $L^2_M$ is a Hilbert space. For any nondecreasing function $\sigma(\lambda)$ given on the interval $(-\infty, +\infty)$, we denote by $L^{(2)}_\sigma = L^{(2)}_\sigma(-\infty, +\infty)$ the set of $\sigma$-measurable functions $F(\lambda)$ such that

$$||F||^2_\sigma = \int_{-\infty}^{+\infty} |F(\lambda)|^2 d\sigma(\lambda) < \infty.$$ 

The string with a fixed left end is denoted by $S_0$ and the string with a free left end is denoted by $S_1$. Consider the string $S_1$ first. The nondecreasing function $\sigma$ is called a spectral function of a string $S_1$ if the mapping $U : f \rightarrow F$, where $f \in L^2_M[-2,2]$, and

$$F(\lambda) = \int_{-2}^{2} f(\xi) \varphi(\xi, \lambda) dM(\xi),$$

isometrically carries the space $L^2_M[-2,2]$ into the Hilbert space $L^{(2)}_\sigma$. Accordingly the Parseval equation

$$\int_{-\infty}^{+\infty} |F(\lambda)|^2 d\sigma(\lambda) = \int_{-2}^{2} |f(\xi)|^2 dM(\xi)$$

holds. The spectral function $\sigma$ of the string is said to be orthogonal if the mapping $U : f \rightarrow F$ maps $L^2_M$ onto the entire space $L^{(2)}_\sigma$.

We say that the string has heavy endpoints if $\xi = -2/ + 2$ are points of increase/decrease of the function $M(\xi)$.

**Theorem 4.1. The Spectral Theorem.** Suppose the string $S_1$ has heavy endpoints and does not carry a concentrated mass at the right endpoint. Then the formula

$$\Omega_N(\lambda) = \frac{\psi^*_+(2, \lambda) N(\lambda) + \psi(2, \lambda)}{\varphi^*_+(2, \lambda) N(\lambda) + \varphi(2, \lambda)}$$

($\Re \lambda \neq 0$)
defines an $R_1$-function for any choice of $N(\lambda) \in (\tilde{R})$. Any such function admits the integral representation 4.2 with $\alpha = 0$:

$$
\Omega_N(\lambda) = \int_{-\infty}^{+\infty} \frac{d\sigma_N(\zeta)}{\zeta - \lambda}.
$$

These formulas establish a one-to-one correspondence $N \leftrightarrow \sigma$ between the class $(\tilde{R})$ of functions $N(\lambda)$ and the set of all spectral functions $\sigma(\lambda)$ of the string $S_1$. The spectral function $\sigma_N(\lambda)$ will be orthogonal if and only if the function $N(\lambda)$ corresponding to it degenerates into a real constant, possibly infinity.

To state some facts about strings we need two additional subclasses of $(R)$.

We say that the function $F(\lambda) \in (R)$ belongs to the class $(S)$ if it is holomorphic on $(-\infty, 0]$ and $\geq 0$ there. All such functions have an absolutely convergent integral representation

$$
F(\lambda) = \alpha + \int_{0}^{+\infty} \frac{d\sigma(\zeta)}{\zeta - \lambda},
$$

(4.4)

where $\alpha \geq 0$ and $d\sigma$ is a positive measure. Evidently $(S) \subseteq (R_1)$. Denote by $(\tilde{S})$ the class $(S)$ adjoined with the function identically equal to infinity.

Similar, the function $F(\lambda) \in (R)$ belongs to the class $(S^{-1})$ if it is holomorphic on $(-\infty, 0]$ and $\leq 0$ there. All such functions have an integral representation

$$
F(\lambda) = \alpha + \beta \lambda + \int_{0}^{+\infty} \left( \frac{1}{\zeta - \lambda} - \frac{1}{\zeta} \right) d\rho(\zeta),
$$

(4.5)

where $\alpha \leq 0$, $\beta \geq 0$ and $d\rho$ is a positive and such that

$$
\int_{0}^{+\infty} \frac{d\rho(\zeta)}{\zeta + \zeta^2} < \infty.
$$

Denote by $(\tilde{S}^{-1})$ the class $(S^{-1})$ adjoined with the function identically equal to infinity. It can be proved that $F(\lambda) \in (\tilde{S})$ if and only if $-1/F(\lambda) \in (\tilde{S}^{-1})$.

Remark 4.2. Let the right end of the string is fixed ($N(\lambda) = 0$). For such a string $S_1$ the Weyl function $\Omega_0 \in (\tilde{S})$ and according to the general formula 4.4

$$
\Omega_0(\lambda) = \frac{\psi(2, \lambda)}{\varphi(2, \lambda)} = \sum_{k=1}^{\infty} \frac{\sigma_k}{\mu_k - \lambda}.
$$

Remark 4.3. If the left end of the string $S_1$ contains an interval free of mass then the Weyl function takes the form 4.4 with $\alpha = l$ where the constant $l$ is the length of the interval. Therefore, for the string $S_1$ with fixed right end, we have

$$
\Omega_0(\lambda) = l + \sum_{k=1}^{\infty} \frac{\sigma_k}{\mu_k - \lambda}.
$$

(4.6)
From formula 4.3 we have

$$\Omega_0(0) = \frac{\psi(2,0)}{\varphi(2,0)} = 4.$$  

Combining this with formula 4.6 we obtain

$$l + \sum_{k=1}^{\infty} \frac{\sigma_k}{\mu_k} = 4. \quad (4.7)$$

**Remark 4.4.** Given $\mu$’s which are spectra of the string $S_1$, what $l$ and what sequence of $\sigma$’s can appear in formula 4.6? For any sequence of $\sigma$ such that

$$\sum_{k=1}^{\infty} \frac{\sigma_k}{\mu_k} < \infty,$$

one has a function $\Omega_0 \in (\tilde{S})$ given by formula 4.6. If $l$ is such that $\Omega_0(0) = 4$ (what is equivalent to 4.7), then to such $\Omega_0$ corresponds a unique regular string $S_1$ with heavy right end and the interval at the left of length $l$ free of mass.

**Remark 4.5.** For the string $S_0$ with fixed left end, the set of all Weyl functions is given by the formula

$$E_N = -\frac{1}{\Omega_N}.$$  

If the right end is also fixed then $E_0(\lambda) \in (\tilde{S}^{-1})$ and according to general theory

$$E_0(\lambda) = -\frac{1}{\Omega_0(\lambda)} = -\frac{\varphi(2,\lambda)}{\psi(2,\lambda)} = -\frac{1}{4} + \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\lambda_k} \right) \rho_k. \quad (4.8)$$

The poles of $E_0(\lambda)$ are zeros of $\psi(2,\lambda)$ and as we will see they do not move under the CH flow. The sum

$$\sum_{k=1}^{\infty} \frac{\rho_k}{\lambda_k} < \infty$$

if and only if $l > 0$. For such a string $S_0$, the formula 4.8 becomes

$$E_0(\lambda) = -\frac{1}{l} + \sum_{k=1}^{+\infty} \frac{\rho_k}{\lambda_k - \lambda}. \quad (4.9)$$

Evidently

$$-\frac{1}{l} + \sum_{k=1}^{+\infty} \frac{\rho_k}{\lambda_k} = -\frac{1}{4} \quad (4.10)$$
Remark 4.6. Given the sequence of λ’s which is the spectrum of the string \(S_0\), what \(l\) and sequence of \(\rho\) can appear in formula 4.9? For any sequence of \(\rho\) such that
\[
\sum_{k=1}^{\infty} \frac{\rho_k}{\mu_k} < \infty
\]
one has a function \(E_0\) defined by formula 4.9. If \(l\) is such that \(E_0(0) = -\frac{1}{4}\) (what is equivalent to 4.10), then \(E_0(-x) < E_0(0) < 0\) for any \(x > 0\). Therefore, \(E_0 \in (\tilde{S}_1^{-1})\). The function
\[
\Omega_0 = -\frac{1}{E_0}
\]
belongs to \(\tilde{S}\). The function \(\Omega_0\) has the form 4.6 and satisfies the condition \(\Omega_0(0) = 4\). According to Remark 4 there exists a unique corresponding regular string \(S_1\) with heavy right end and the interval at the left free of mass.

5. Jacobi ellipsoidal coordinates

This section contains information about Jacobi coordinates in finite and infinite dimensions, see [12] lecture 26. The results are used to describe the range of the canonical variables constructed in the subsequent sections. The present discussion (and notations) are independent from all other sections.

First we explain the construction of Jacobi coordinates in finite dimensions. Consider the function
\[
\Upsilon(\lambda) = \sum_{k=1}^{N} \frac{\nu_k}{\gamma_k - \lambda}, \quad \nu_k > 0.
\]
Assume that the poles \(\gamma_1 < \gamma_2 < \ldots < \gamma_N\) are ordered and fixed. The residues \(\nu_1, \ldots, \nu_N\) are considered as variables and they fill up \(R_+^N\). Pick \(C \neq 0\) and consider points where \(\Upsilon(\lambda) = C\). We have \(N\) real roots \(\chi_k = \chi_k(C), \ k = 1, \ldots, N\). These are Jacobi coordinates. If \(C > 0\) then,
\[
\chi_1 < \chi_1 < \chi_2 < \ldots < \chi_N < \gamma_N;
\]
and if \(C < 0\), then
\[
\gamma_1 < \chi_1 < \chi_2 < \ldots < \gamma_N < \chi_N.
\]
The converse is also true. Pick some \(C\) and consider the points \(\chi_k, \ k = 1, \ldots, N\) which satisfy inequalities 5.1 or 5.2 correspondingly. There exist a unique sequence of \(\nu_1, \ldots, \nu_N\) from \(R_+^N\) corresponding to the sequence of \(\chi\)'s.

The following formula\(^7\) is due to Boole, [6]:
\[
C \sum_{k=1}^{N} (\gamma_k - \chi_k) = s_0, \quad C \neq 0,
\]
\(^7\)A. Volberg pointed this to me.
where
\[ s_n = \sum_k \nu_k \gamma_k^n, \quad n = 0, 1, \ldots \]

In fact Boole’s formula is the first in the infinite sequence of identities. The second that we found is
\[ C^2 \sum_{k=1}^{N} \gamma_k^2 - \chi_k^2 = 2s_1C - s_0^2, \quad C \neq 0. \]

One can continue and obtain the formulas for
\[ \sum_{k=1}^{N} \gamma_k^p - \chi_k^p \]
in terms of the moments \( s_0, \ldots, s_{p-1} \) for any integer \( p \). We present two proofs of Boole’s formula. The first is classical and can be found in [19].

The identity
\[ C - \Upsilon(\lambda) = C \prod_k (\lambda - \chi_k)/\prod_k (\lambda - \gamma_k) \]
implies
\[ C \prod_k (\lambda - \gamma_k) - \sum_k \nu_k \prod_{p \neq k} (\lambda - \gamma_p) = C \prod_k (\lambda - \chi_k). \]

The coefficients of \( \lambda^N \) of both sides match. Matching coefficients of \( \lambda^{N-1} \) we obtain 5.3.

The second proof is a little more complicated, but it can be easily generalized. Note that near infinity,
\[ \Upsilon(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \frac{s_2}{\lambda^3} - \ldots. \]

Consider the contour integral
\[ \frac{1}{2\pi i} \int \lambda \frac{\Upsilon'(\lambda)}{\Upsilon(\lambda) - C} \, d\lambda \]
over a circle of sufficiently large radius. The poles and zeros of the function \( \Upsilon - C \) produce a contribution to the integral equal to
\[ \sum \gamma_k - \chi_k. \]

From another side, near infinity
\[ \frac{\Upsilon'(\lambda)}{\Upsilon(\lambda) - C} = -\lambda^{-2}s_0/C - \lambda^{-3}(2s_1/C - s_0^2/C^2) - \ldots. \]

Thus the integral is equal to
\[ \frac{s_0}{C} \]
which is Boole’s formula.
The second and all higher identities can be proved by contour integration with $\lambda$ replaced by $\lambda^p$, $p \geq 1$.

Consider now the infinite–dimensional case. Let

$$\Upsilon(\lambda) = \sum_{k=1}^{\infty} \frac{\nu_k}{\gamma_k - \lambda},$$

where $\gamma_k \to +\infty$ as $k \to \infty$ and $s_0 = \sum \nu_k < \infty$. For any $C$ we have a sequence of $\chi$’s which interlace the sequence of $\lambda$’s similarly to 5.1 or 5.2. Pick some $C \neq 0$.

Can any sequence of $\chi$’s interlacing $\lambda$’s arise from some sequence of $\nu$’s similarly to the finite–dimensional case? In fact, there is an asymptotic condition on $\chi$’s which follows from the next result.

**Theorem 5.1.** [17] The $\chi$’s are coordinates, i.e. they arise from some $C \neq 0$ and some summable sequence of $\nu$’s if and only if

$$\sum_{k=1}^{+\infty} |\gamma_k - \chi_k| < \infty.$$

If this holds then

$$C \sum_{k=1}^{+\infty} (\gamma_k - \chi_k) = s_0.$$

This result can be obtained from the rational case in the limit when the number of poles tends to infinity.

Unfortunately the condition $s_0 < \infty$ does not hold in most cases we consider. For the string with light left end (i.e. without concentrated mass) we have only

$$s_{-1} = \sum_{k=1}^{+\infty} \frac{\nu_k}{\gamma_k} < \infty.$$

**Theorem 5.2.** [23] Pick some $C > 0$. The $\chi$’s are ellipsoidal coordinates, i.e. they arise from some sequence of $\nu$’s with $s_{-1} < \infty$ if and only if

$$\sum_{k=1}^{+\infty} \left( \frac{\gamma_k}{\chi_k} - 1 \right) < \infty.$$

If $C < 0$, then the above condition should be replaced by

$$\sum_{k=1}^{+\infty} \left( \frac{\chi_k}{\gamma_k} - 1 \right) < \infty.$$

**Proof.** Without loss of generality we assume that $C = \Upsilon(0) > 0$. Then,

$$\Upsilon(\lambda) - \Upsilon(0) = \exp F(\lambda)$$
where $F \in (R)$. Using 4.1 we have

$$\Upsilon(\lambda) - \Upsilon(0) = A \exp \left( \int f_0(\zeta) \frac{1 + \zeta \lambda}{\zeta - \lambda} \frac{d\zeta}{1 + \zeta^2} \right),$$

where

$$f_0(\lambda) = \begin{cases} 1 & \text{for } \gamma_k < \lambda < \chi_{k+1}, \quad k = 0, 1, \ldots; \\ 0 & \text{for } \chi_k < \lambda < \gamma_k, \quad k = 1, 2, \ldots. \end{cases}$$

We put formally $\gamma_0 = -\infty$. Writing $f_0 = 1 - f$, we have

$$\Upsilon(\lambda) - \Upsilon(0) = -A \exp \left( -\int_0^{+\infty} f(\zeta) \frac{1 + \zeta \lambda}{\zeta - \lambda} \frac{d\zeta}{1 + \zeta^2} \right).$$

Now pass to the limit with $\lambda = -x \rightarrow -\infty$ along the real axis. Then the limit

$$\lim_{x \rightarrow +\infty} \int_0^{+\infty} f(\zeta) \frac{\zeta x - 1}{\zeta + x} \frac{d\zeta}{1 + \zeta^2} = \int_0^{+\infty} f(\zeta) \frac{\zeta}{\zeta^2 + 1} d\zeta < \infty$$

exists. This implies

$$\sum_{k=2}^{\infty} \log \frac{\gamma_k}{\chi_k} < \infty,$$

which is equivalent to the condition of the theorem. Moreover,

$$\Upsilon(\lambda) - \Upsilon(0) = -\Upsilon(0) \exp \left( -\int_0^{+\infty} f(\zeta) d\zeta \right).$$

Conversely, pick $C > 0$ and the sequence of $\chi$’s which interlace the sequence of $\gamma$’s:

$$0 = \chi_1 < \gamma_1 < \chi_2 < \ldots,$$

and satisfy the condition of the theorem. Define the function $\Upsilon(\lambda)$ by the identity

$$\Upsilon(\lambda) = C - C \exp \left( -\int_0^{+\infty} f(\zeta) d\zeta \right),$$

where

$$f(\zeta) = \sum_{k=1}^{\infty} 1_{[\chi_k,\gamma_k]}(\zeta).$$

Evidently, $\Upsilon(\lambda)$ maps the upper half–plane into itself. It also vanishes at infinity and has poles at $\gamma_k$. Thus, applying 4.1 again

$$\Upsilon(\lambda) = \alpha + \beta \lambda + \sum_{k=1}^{\infty} \frac{1 + \gamma_k \lambda}{\gamma_k - \lambda} \frac{\nu_k}{1 + \gamma_k^2},$$
Now pass to the limit with $\lambda \to -\infty$ along the real axis. The existence of the limit implies that $\beta = 0$ and
\[
\sum_{k=1}^{\infty} \frac{\gamma_k \nu_k}{1 + \gamma_k^2} < +\infty.
\]
Therefore, since $\Upsilon(\lambda)$ vanishes at infinity,
\[
\Upsilon(\lambda) = \alpha + \sum_{k=1}^{\infty} \left( \frac{1}{\gamma_k - \lambda} - \frac{\gamma_k}{1 + \gamma_k^2} \right) \nu_k
\]
\[
= \left( \alpha - \sum_{k=1}^{\infty} \frac{\gamma_k \nu_k}{1 + \gamma_k^2} \right) + \sum_{k=1}^{\infty} \frac{\nu_k}{\gamma_k - \lambda}
\]
\[
= \sum_{k=1}^{\infty} \frac{\nu_k}{\gamma_k - \lambda}.
\]
When $C > 0$ we have $\chi_k < \gamma_k$ for all $k$, and the condition
\[
\sum_{k=1}^{+\infty} \left( \frac{\gamma_k}{\chi_k} - 1 \right) < \infty
\]
is equivalent
\[
\prod_{k=1}^{+\infty} \frac{\gamma_k}{\chi_k} < \infty.
\]
This product can be expressed in the spirit of Bool’s formula as
\[
\prod_{k=1}^{+\infty} \frac{\gamma_k}{\chi_k} = \frac{C}{C - s_{-1}}.
\]
To prove the formula we consider an algebraic situation with $N < \infty$. We assume $C > \Upsilon(0) = s_{-1}$ and this implies that $\chi_1 > 0$. The function
\[
\log(\Upsilon(\lambda) - C)
\]
is single valued on the plane cut along the segments $[\chi_k, \gamma_k]$, $k = 1, \ldots, N$. Consider the contour integral
\[
\frac{1}{2\pi i} \int \frac{1}{\lambda} \log(\Upsilon(\lambda) - C) \, d\lambda
\]
over a circle of sufficiently large radius. Evaluating this integral when the radius tends to infinity, we have
\[
\log(-C).
\]
Evaluating it in the finite part of the plane we have to account a contribution from the origin and small circles surrounding the cuts. At the origin
\[
\frac{1}{2\pi i} \int \frac{1}{\lambda} \log(\Upsilon(\lambda) - C) \, d\lambda = \log(s_{-1} - C),
\]
and over a circle surrounding \([\chi_k, \gamma_k]\)
\[
\frac{1}{2\pi i} \int \frac{1}{\lambda} \log(\Upsilon(\lambda) - C) \, d\lambda = -\frac{1}{2\pi i} \int \log \lambda \frac{\Upsilon'(\lambda)}{\Upsilon(\lambda) - C} \, d\lambda = \log \gamma_k - \log \chi_k.
\]
Thus
\[
\log(-C) = \log(s_{-1} - C) + \sum_{k=1}^{N} (\log \gamma_k - \log \chi_k).
\]
The result follows by exponentiating this equation. The formula extends to all other values of \(C\) and to functions \(\Upsilon\) with an arbitrary number of poles. We do not dwell on this.

6. Poisson bracket for the Weyl function

The goal of the present section is to compute the Poisson bracket 3.2 for the Weyl function.

**Theorem 6.1.** Let \(N(\zeta) = N_0\) be a real constant, possibly infinity. Then,
\[
\{\Omega_{N_0}(\lambda), \Omega_{N_0}(\mu)\} = \frac{\lambda \mu}{\lambda - \mu} (\Omega_{N_0}(\lambda) - \Omega_{N_0}(\mu))^2.
\]

This fact is central for all further discussion. The proof follows the steps developed in [26] for the Dirac operator. We start with two auxiliary lemmas.

**Lemma 6.2.** The gradient of \(\Omega_{N_0}(\lambda)\) is
\[
\frac{\partial \Omega_{N_0}(\lambda)}{\partial g(\xi)} = \lambda \chi^2(\xi),
\]
where \(\chi = -\psi + \Omega_{N_0}\varphi\).

**Proof.** The proof is now standard, see [26].

Let \(f = f(\xi, \lambda)\) and \(s = s(\xi, \mu)\) be arbitrary solutions of the string spectral problem
\[
f'' + \lambda g f = 0, \quad s'' + \mu g s = 0.
\]

**Lemma 6.3.** Let \(\mathcal{J} = \mathcal{J}(\xi)\) be an arbitrary function. Then
\[
s^2 g \mathcal{J} \partial s^2 \mathcal{J} - f^2 g \mathcal{J} \partial f^2 \mathcal{J} = \frac{1}{\mu - \lambda} \partial [f's' - f s]^2.
\]
This formula can be verified directly.

Proof of Theorem 6.1. Using formula 3.2 for the Poisson bracket, Lemma 6.2 and 6.3 and integration by parts, we compute

\[
\{ \Omega_{N_0}(\lambda), \Omega_{N_0}(\mu) \} = \int_{-2}^{+2} \frac{\delta \Omega_{N_0}(\lambda)}{\delta g(\xi)} \left( J g \partial J^{-1} + J^{-1} \partial g J \right) \frac{\delta \Omega_{N_0}(\mu)}{\delta g(\xi)} d\xi
\]

\[
= \lambda \mu \int_{-2}^{+2} \chi^2(\lambda) \left( J g \partial J^{-1} + J^{-1} \partial g J \right) \chi^2(\mu) d\xi
\]

\[
= -\lambda \mu \int_{-2}^{+2} \chi^2(\mu) g J \partial \frac{\chi^2(\lambda)}{J} - \chi^2(\lambda) g J \partial \frac{\chi^2(\mu)}{J} d\xi
\]

\[
= -\frac{\lambda \mu}{\mu - \lambda} [\chi'(\lambda) \chi(\mu) - \chi(\lambda) \chi'(\mu)]^2 \bigg|_{-2}^{+2}.
\]

To compute the value at the point \( \xi = 2 \) we write

\[
\chi'(\lambda) \chi(\mu) - \chi(\lambda) \chi'(\mu) = \left[ \frac{\chi'(\lambda)}{\chi(\lambda)} - \frac{\chi'(\mu)}{\chi(\mu)} \right] \chi(\lambda) \chi(\mu).
\]

Note that \( \chi'N_0 + \chi = 0 \) at \( \xi = 2 \). Therefore, at this point

\[
\frac{\chi'(\lambda)}{\chi(\lambda)} = \frac{\chi'(\mu)}{\chi(\mu)} = N_0,
\]

and the contribution from the upper limit vanishes.

At the lower limit \( \xi = -2 \), using \( \chi = -\psi + \Omega_{N_0} \varphi \) we have

\[
\chi(\lambda) = \Omega_{N_0}(\lambda), \quad \chi'(\lambda) = -1.
\]

Finally,

\[
= -\frac{\lambda \mu}{\mu - \lambda} [\chi'(\lambda) \chi(\mu) - \chi(\lambda) \chi'(\mu)]^2 \bigg|_{-2} = \frac{\lambda \mu}{\lambda - \mu} \left( \Omega_{N_0}(\lambda) - \Omega_{N_0}(\mu) \right)^2.
\]

The proof is finished.

If one changes the spectral parameter by the rule

\[
\lambda' = -\frac{1}{\lambda}, \quad \mu' = -\frac{1}{\mu},
\]

then the formula of the theorem becomes

\[
\{ \Omega_{N_0}(\lambda'), \Omega_{N_0}(\mu') \} = \frac{(\Omega_{N_0}(\lambda') - \Omega_{N_0}(\mu'))^2}{\lambda' - \mu'}. \quad (6.2)
\]

This is the standard Atiyah–Hitchin bracket, [3], in the form found by Faybusovich and Gehtman, [11]. Note that the function \( \Omega_{N_0}(\lambda') \) maps the upper half–plane into itself. Due to the invariance of the AH bracket under linear–fractional transformations, [26], the bracket for the function \( E_{N_0}(\lambda') \) is given by the same formula 6.2.
7. Action–angle variables. Dynamics of the Weyl function

The main result of this section is

**Theorem 7.1.** The variables

\[ I_n = -\frac{1}{\lambda_n}, \quad \theta_k = \log \varphi(2, \lambda_k) \]

are "action–angle" variables

\[
\{I_k, I_n\} = 0, \quad \{\theta_k, \theta_n\} = 0, \quad \{\theta_k, I_n\} = \delta^n_k.
\]

We start the proof with two auxiliary lemmas. In section 4 we introduced the representation 4.8 for the function \( E_0(\lambda) \):

\[ E_0(\lambda) = -\frac{1}{4} + \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\lambda_k} \right) \rho_k. \]

Introducing

\[ \lambda'_k = -\frac{1}{\lambda_k}, \quad \rho'_k = \frac{\rho_k}{\lambda_k^2}, \]

we have the identity

\[ \frac{\rho_k}{\lambda_k - \lambda} = \frac{\rho'_k}{\lambda'_k - \lambda'} + \frac{\rho_k}{\lambda_k}. \]

Therefore for the function \( E_0(\lambda') \) we obtain

\[ E_0(\lambda') = -\frac{1}{4} + \sum_{k=1}^{\infty} \frac{\rho'_k}{\lambda'_k - \lambda'}. \]

First, we compute the bracket between the the parameters \( \lambda'_k \) and \( \rho'_k \) entering into formula 7.3.

**Lemma 7.2.** The bracket 6.2 in \( \lambda' \) and \( \rho' \) coordinates has the form

\[
\{\rho'_k, \rho'_n\} = \frac{2\rho'_k \rho'_n}{\lambda'_n - \lambda'_k} (1 - \delta^n_k), \tag{7.4}
\]

\[
\{\rho'_k, \lambda'_n\} = \rho'_k \delta^n_k, \tag{7.5}
\]

\[
\{\lambda'_k, \lambda'_n\} = 0. \tag{7.6}
\]

The proof is identical to the proof of Theorem 2 in [25].

Now we can compute the bracket between the parameters \( \lambda_k \) and \( \rho_k \) entering into formula 4.8.
Lemma 7.3. The bracket $6.1$ in $\lambda$ and $\rho$ coordinates has the form

\[
\{\rho_k, \rho_n\} = \frac{2\lambda_k \lambda_n \rho_k \rho_n}{\lambda_k - \lambda_n}, \tag{7.7}
\]
\[
\{\rho_k, \lambda_n\} = \lambda_k^2 \rho_k \delta^n_k, \tag{7.8}
\]
\[
\{\lambda_k, \lambda_n\} = 0. \tag{7.9}
\]

The proof follows from the results of the previous lemma.

Proof of Theorem 7.1. The first identity follows trivially from 7.9. Commutativity of angles is more complicated. Using 4.3

\[
\{\theta_n, \theta_m\} = \sum_{k,k'} \{\log \left(1 - \frac{\lambda_n}{\mu_k}\right), \log \left(1 - \frac{\lambda_m}{\mu_{k'}}\right)\}
\]
\[
= \sum_{k,k'} \frac{1}{(1 - \frac{\lambda_n}{\mu_k})(1 - \frac{\lambda_m}{\mu_{k'}})} \{\frac{\lambda_n}{\mu_k}, \frac{\lambda_m}{\mu_{k'}}\}
\]
\[
= \sum_{k < k'} \frac{\lambda_n \lambda_m}{(1 - \frac{\lambda_n}{\mu_k})(1 - \frac{\lambda_m}{\mu_{k'}})} \left(\{\mu_k, \mu_{k'}\} + \{\mu_{k'}, \mu_k\}\right)
\]
\[
= 0.
\]

The last sum vanishes due to skew symmetry of the bracket.

To prove the last relation we use the identity $\varphi(2, \lambda_k) = \psi'(2, \lambda_k) \rho_k$. We have

\[
\{\theta_k, I_n\} = \{\log \psi'(\lambda_k) + \log \rho_k, -\frac{1}{\lambda_n}\}
\]
\[
= \frac{1}{\rho_k \lambda_n^2} \{\rho_k, \lambda_n\} = \frac{1}{\rho_k \lambda_n^2} \lambda^2_k \rho_k \delta^n_k = \delta^n_k. \quad \blacksquare
\]

H. McKean, [20] expressed the conserved quantities $\mathcal{H}_1$ and $\mathcal{H}_2$ in terms of the Dirichlet spectrum

\[
\mathcal{H}_1 = \int_{-\infty}^{+\infty} m \, dx = \int_{-2}^{+2} g J \, d\xi = \sum_{n=1}^{+\infty} \lambda_n^{-1},
\]
\[
\mathcal{H}_2 = \frac{1}{2} \int_{-\infty}^{+\infty} m R[m] \, dx = \frac{1}{2} \int_{-2}^{+2} g R[g] J^3 \, d\xi = \frac{1}{4} \sum_{n=1}^{+\infty} \lambda_n^{-2}.
\]

These are obtained by matching coefficients of two expansions near $\lambda = 0$ for the solution $\psi$. One expansion is obtained from formula 4.3; the another form a Neumann series\(^8\).

\(^8\)These and more general trace formulas will be obtained in Section 11.
The evolution of residues under the translation flow
\[ \rho_k^\bullet = \{ \rho_k, \mathcal{H}_1 \} = \sum_n \{ \rho_k, \lambda_n^{-1} \} = -\lambda_k^{-2} \{ \rho_k, \lambda_k \} = -\rho_k, \]
i.e.
\[ \rho_k(t) = \rho_k(0) e^{-t}. \]

Therefore, under the translation flow
\[ E_0(\lambda) = -\frac{1}{4} + \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\lambda_k} \right) \rho_k(0) e^{-t}. \]

The evolution of residues under the CH flow obeys
\[ \rho_k^\bullet = \{ \rho_k, \mathcal{H}_2 \} = \frac{1}{4} \sum_n \{ \rho_k, \lambda_n^{-2} \} = -\frac{1}{2} \lambda_k^{-3} \{ \rho_k, \lambda_k \} = -\overline{p}_k \rho_k, \]
where \( \overline{p}_k = \frac{1}{2\lambda_k} \). Therefore,
\[ \rho_k(t) = \rho_k(0) e^{-\overline{p}_k t}, \]
and
\[ E_0(\lambda) = -\frac{1}{4} + \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k - \lambda} - \frac{1}{\lambda_k} \right) \rho_k(0) e^{-\overline{p}_k t}. \]

8. **Examples. 1 and 2 peakon solutions**

A remarkable class of solutions of the CH equation are peakon–antipeakon solutions of the form \( v(x,t) = \sum_n p_n e^{-|x-q_n|} \). The parameters \( q_n \) and \( p_n \) satisfy the Hamiltonian flow
\[ q_n^\bullet = \partial H / \partial p_n, \quad p_n^\bullet = -\partial H / \partial q_n \]
with Hamiltonian \( H = \frac{1}{2} \sum p_i p_j e^{-|q_i-q_j|} \) and the classical Poisson bracket.

It is instructive to compute the evolution of Weyl functions for the simplest one and two peakons solutions. The equations of motion of two peakons were already integrated in [7], but the evolution of the Weyl function computed here is an illustration of results of the previous section. Since everything is algebraic we allow \( p_n \) to be of both signs.

**Example 8.1. One peakon.**

The solution has the form \( v(x,t) = p(t) e^{-|x-q(t)|} \), with \( p(t) = p_0 \) and \( q(t) = q_0 + pt \). The peakon \((p > 0)\) travels to the right while the anti–peakon \((p < 0)\) travels to the left.

The solutions \( \varphi \) and \( \psi \) take the form
\[ \varphi(2, \lambda) = -\lambda m_1 l_1 + 1, \]
\[ \psi(2, \lambda) = -\lambda m_1 l_0 l_1 + 4. \]
Using \( m_1 l_0 l_1 = 8 p \) for the Weyl function of the string \( S_0 \) with fixed right end \((N \equiv 0)\) we obtain

\[
E_0(\lambda) = -\frac{\varphi(2, \lambda)}{\psi(2, \lambda)} = -\frac{1}{l_0} + \frac{\rho(t)}{2p - \lambda},
\]
with

\[
\rho(t) = \frac{1}{m_1 l_0^2} = \frac{1}{8p} e^{-q} = \frac{1}{8p} e^{-q_0} e^{-pt}.
\]

**Example 8.2.** Two peakons.

The solution is of the form

\[
v(x, t) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|}.
\]

The two–particle Hamiltonian is

\[
H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + p_1 p_2 e^{-|q_1-q_2|} = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2,
\]
where \( \overline{p}_n \) is the asymptotic velocity of the \( n \)-th particle. The equations of motion

\[
q_1^\ast = \frac{\partial H}{\partial p_1} = p_1 + p_2 e^{-|q_1-q_2|}, \quad p_1^\ast = -\frac{\partial H}{\partial q_1} = p_1 p_2 \text{ sign } (q_1 - q_2)e^{-|q_1-q_2|},
\]

\[
q_2^\ast = \frac{\partial H}{\partial p_2} = p_2 + p_1 e^{-|q_1-q_2|}, \quad p_2^\ast = -\frac{\partial H}{\partial q_2} = -p_1 p_2 \text{ sign } (q_1 - q_2)e^{-|q_1-q_2|},
\]
are integrated by introducing the new variables

\[
P = p_1 + p_2, \quad Q = q_1 + q_2, \quad p = p_1 - p_2, \quad q = q_1 - q_2.
\]

This reduces the equations of motion to

\[
P^\ast = 0, \quad Q^\ast = P (1 + e^{-|q|}),
\]

\[
p^\ast = \frac{1}{2} (P^2 - p^2) \text{ sign } q e^{-|q|}, \quad q^\ast = p (1 - e^{-|q|}).
\]

Now the Hamiltonian is

\[
H = \frac{1}{4} [P^2 + p^2 + (P^2 - p^2)e^{-|q|}].
\]

Consider the case\(^9\) \( \overline{p}_1 > \overline{p}_2 > 0 \). This is the pure peakon case. The law of conservation of energy takes the form

\[
\frac{4H - P^2 - p^2}{P^2 - p^2} = e^{-|q|},
\]

\(^9\)The case \( \overline{p}_1 = \overline{p}_2 \) is excluded by the equations of motion.
with left hand side less than 1 for all times. This implies that \( q = q_1 - q_2 < 0 \) for all times. The solution is

\[
p(t) = A \frac{1 - \beta e^{At}}{1 + \beta e^{At}},
\]

\[
q(t) = \log \frac{A^2 \beta e^{At}}{(p_1 + p_2 \beta e^{At})(p_2 + p_1 \beta e^{At})},
\]

\[
Q(t) = Q(0) + Pt - \log \frac{p_1 + p_2 \beta e^{At}}{p_2 + p_1 \beta e^{At}} + \log \frac{p_1 + \beta p_2}{p_2 + \beta p_1};
\]

with

\[
A = p_1 - p_2, \quad \beta = \frac{A - p(0)}{A + p(0)} > 0.
\]

The solutions \( \varphi \) and \( \psi \) are

\[
\varphi(2, \lambda) = \lambda^2 l_1 l_2 m_1 m_2 - \lambda (m_1 l_1 + m_1 l_2 + m_2 l_2) + 1, \quad (8.2)
\]

\[
\psi(2, \lambda) = \lambda^2 m_1 m_2 l_0 l_1 l_2 - \lambda (m_1 l_0 l_2 + m_1 l_0 l_1 + m_2 l_0 l_2 + m_2 l_1 l_2) + 4. \quad (8.3)
\]

Note that

\[
m_1 m_2 l_0 l_1 l_2 = 8P^2 - 16H = 16p_1 p_2
\]

and

\[
m_1 l_0 l_2 + m_1 l_0 l_1 + m_2 l_0 l_2 + m_2 l_1 l_2 = 8P = 8(p_1 + p_2).
\]

Therefore, for the poles \( \lambda : \psi(\lambda) = 0 \) we obtain

\[
\lambda_1 = \frac{1}{2p_1}, \quad \lambda_2 = \frac{1}{2p_2}.
\]

Thus:

\[
E_0(\lambda) = -\frac{\varphi(2, \lambda)}{\psi(2, \lambda)} = -\frac{1}{l_0} + \frac{\rho_1(t)}{\varphi_1 - \lambda} + \frac{\rho_2(t)}{\varphi_2 - \lambda}
\]

with

\[
\rho_1(t) = \frac{\varphi(2, \frac{t}{\varphi_1})}{8(p_2 - p_1)} = \frac{R}{8A} e^{-\frac{9t}{\varphi_1}}, \quad (8.4)
\]

\[
\rho_2(t) = \frac{\varphi(2, \frac{t}{\varphi_2})}{8(p_1 - p_2)} = \frac{R}{8A} \beta e^{-\frac{9t}{\varphi_2}}, \quad (8.5)
\]

and

\[
R^2 = \frac{e^{-Q(0)} p_2 + \beta p_1}{\beta} + \frac{p_1 + \beta p_2}{32}. \quad (8.6)
\]
**Peakon-antipeakon.** Now to the case $\vec{p}_1 > 0 > \vec{p}_2$. The integration of the equations of motion is identical to the pure peakon case. The solution is

$$p(t) = A \frac{1 + \zeta e^{At}}{1 - \zeta e^{At}},$$

$$q(t) = \log \frac{A^2 \zeta e^{At}}{(\vec{p}_1 \zeta e^{At} - \vec{p}_2)(\vec{p}_1 - \vec{p}_2 \zeta e^{At})},$$

$$Q(t) = Q(0) + Pt - \log \frac{\vec{p}_1 - \vec{p}_2 \zeta e^{At}}{\vec{p}_1 - \vec{p}_2} + \log \frac{\vec{p}_2 - \vec{p}_1 \zeta e^{At}}{\vec{p}_2 - \vec{p}_1 \zeta};$$

where

$$\zeta = \frac{p(0) - A}{p(0) + A} > 0.$$ 

Thus,

$$E_0 = -\frac{\varphi(2, \lambda)}{\psi(2, \lambda)} = -\frac{1}{l_0} + \rho_1(t) + \rho_2(t),$$

with

$$\rho_1 = \frac{D}{8A} e^{-\vec{p}_1 t}, \quad \rho_2(t) = -\frac{D}{8A} \zeta e^{-\vec{p}_2 t},$$

and

$$D^2 = \frac{e^{-Q(0)} \vec{p}_1 \zeta - \vec{p}_2}{\zeta \vec{p}_1 - \vec{p}_2 \zeta}.$$ 

The formulas for residues are obtained in the same way as in the pure peakon case. At the moment of the collision the mass of the first particle tends to infinity and $s_1 = \rho_1(t) + \rho_2(t) = 0$.

9. **Generalized action–angle variables**

The action–angle variables introduced in section 7 are a particular case of the following construction. According to formula 4.6, the function $\Omega_0(\lambda)$ has the form

$$\Omega_0(\lambda) = \frac{\psi(2, \lambda)}{\varphi(2, \lambda)} = l + \sum_{k=1}^{\infty} \frac{\sigma_k}{\mu_k - \lambda}, \quad l \geq 0.$$ 

All possible sequences of $\mu$’s are described in Theorem 1.2.

Pick some constant $C$ which does not exceed $l$, and find the points where $\Omega_0(\lambda) = C$ (see Figure 1). We have the sequence of points $\lambda_k = \lambda_k(C), \ k = 1, 2, \ldots$ which depend on the constant $C$. They interlace the poles of $\Omega_0(\lambda)$:

$$0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots$$

(9.1)

As it was explained in section 5, these are Jacobi ellipsoidal coordinates. Section 7 describes the case when $C = 0$. Given $\mu$’s, what is the range of all possible sequences of $\lambda(C)$’s?
The Nazarov–Yuditskii theorem implies
\[
\sum_{k=1}^{\infty} \left( \frac{\lambda_k(C)}{\mu_k} - 1 \right) < \infty.
\]
Conversely, let the sequence of \(\lambda\)'s satisfy 9.1 and the above condition. Then again by the Nazarov–Yuditskii theorem there exists a function
\[
\Upsilon(\lambda) = \sum_{k=1}^{\infty} \frac{\sigma_k^0}{\mu_k - \lambda}, \quad \sum_{k=1}^{\infty} \frac{\sigma_k^0}{\mu_k} < \infty;
\]
such that \(\Upsilon(\lambda_k) = -1, \quad k = 1, 2, \ldots\) Note that \(\Upsilon(0) > 0\) depends only on the given sequence of \(\mu\)'s and \(\lambda\)'s. Pick \(l, \Phi_0 \leq l < 4\) and define
\[
\Omega_0(\lambda) = l + \frac{(4-l)\Upsilon(\lambda)}{\Upsilon(0)}.
\]
By construction
\[
\Omega_0(\lambda_k) = l - \frac{(4-l)}{\Upsilon(0)} = C, \quad k = 1, 2, \ldots
\]
the range of \(C\) being the interval \(-4/\Upsilon(0) \leq C < 4\). The function \(\Omega(\lambda)\) has the form 4.6 and satisfies condition 4.7. Therefore, according to remark 4.4, for any choice of \(l\), or equivalently \(C\), there exists a string \(S_1\) with the function \(\Omega_0\) given by formula 9.2.

**Remark 9.1.** For \(C = 0\), there exists a string \(S_1\) with \(l > 0\) and corresponding Dirichlet spectrum coinciding with the \(\lambda\)'s.

Now we want to construct variables canonically conjugate to \(\lambda(C)\).

**Theorem 9.2.** The variables
\[
I_n(C) = -\frac{1}{\lambda_n(C)}, \quad \theta_k(C) = \log \varphi(2, \lambda_k(C))
\]
are canonically conjugate variables
\[
\{I_k(C), I_n(C)\} = 0, \\
\{\theta_k(C), \theta_n(C)\} = 0, \\
\{\theta_k(C), I_n(C)\} = \delta_k^a.
\]

To prove the theorem we introduce the function \(E_0^c(\lambda)\):
\[
E_0^c = -\frac{1}{\Omega_0 - C}.
\]
It has the representation

$$E^c_0(\lambda) = -\frac{\varphi(\lambda)}{\psi(\lambda) - C\varphi(\lambda)} = -\frac{\varphi(\lambda)}{\psi^c(\lambda)}.$$  

The points $\lambda_k(C)$ are the roots of the equation $\psi^c(\lambda) = 0$. The function $\psi^c(\lambda)$ can be written as

$$\psi^c(\lambda) = (4 - C) \prod_{k=1}^{\infty} \left( 1 - \frac{\lambda}{\lambda_k(C)} \right).$$

Evidently $4 - C > 0$, since $C \leq l < 4$. The function $E^c_0(\lambda)$ maps the upper half–plane into itself and can be written as:

$$E^c_0(\lambda) = -\frac{1}{l - C} + \sum_{k=1}^{\infty} \frac{\rho_k(C)}{\lambda_k(C) - \lambda}. \quad (9.3)$$

One can change the variable $\lambda \rightarrow \lambda' = -\frac{1}{\lambda}$ and get the formula

$$E^c_0(\lambda') = -\frac{1}{4 - C} + \sum_{k=1}^{\infty} \frac{\rho_k'(C)}{\lambda'_k(C) - \lambda'}. \quad (9.4)$$

After that the proof of the theorem follows familiar steps. The Poisson bracket for the function $E^c_0(\lambda')$ is given by formula 6.2. For the variables $\rho'(C)$ and $\lambda'(C)$ from 9.4, an analog of Lemma 7.2 holds. Then similarly to Lemma 7.3 we can compute the bracket between the variables $\rho(C)$ and $\lambda(C)$ entering into formula 9.3. Finally, the proof of the canonical relations is identical to the proof of Theorem 7.1.

We will construct a family of Hamiltonian flows which will be linearized in the variables $I(C)$ and $\theta(C)$ in section 11.

10. SECOND FAMILY OF CANONICAL COORDINATES

The arguments of the previous section prompt the construction of another family of canonical coordinates $J$ and $\tau$. As it was explained in the introduction these variables are complimentary to $I$ and $\theta$. We assume that the initial data belong to the smaller phase space $\mathcal{M}_0$.

According to formula 4.8, the function $E_0(\lambda)$ has the form

$$E_0(\lambda) = -\frac{\varphi(2, \lambda)}{\psi(2, \lambda)} = -\frac{1}{l} + \sum_{k=1}^{\infty} \frac{\rho_k}{\lambda_k - \lambda}.$$  

The range of all possible sequences of $\lambda$’s is described in Theorem 1.1.

Pick some $F$ such that $-\frac{1}{l} < F$ and consider the points where $E_0(\lambda) = F$ (see Figure 2). We have the sequence of points $\mu_k = \mu_k(F)$, $k = 1, 2, \ldots$ interlacing the poles of $E_0(\lambda)$:

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots. \quad (10.1)$$
Given \( \lambda \)'s, how can the range of all possible sequences of \( \mu(F) \)'s be described?

The Nazarov–Yuditskii theorem for this case implies

\[
\sum_{k=1}^{\infty} \left( \frac{\lambda_k}{\mu_k(F)} - 1 \right) < \infty.
\]

Conversely, suppose the sequence of \( \mu \)'s satisfies 10.1 and the above condition. Then again, by the Nazarov–Yuditskii theorem, there exists a function

\[
\Upsilon(\lambda) = \sum_{k=1}^{\infty} \frac{\rho_k^0}{\lambda_k - \lambda}, \quad \sum_{k=1}^{\infty} \rho_k^0 < \infty;
\]

such that \( \Upsilon(\mu_k) = 1, \quad k = 1, 2, \ldots \). Note that \( \lambda_1 > 0 \) and \( \Upsilon(0) > 0 \) depends only on the chosen sequence of \( \mu \)'s and \( \lambda \)'s. Pick \( l, \quad 0 < l < 4 \) and define

\[
E_0(\lambda) = -\frac{1}{l} + \frac{(4 - l)\Upsilon(\lambda)}{4l \Upsilon(0)}.
\]

By construction

\[
E_0(\mu_k) = -\frac{1}{l} + \frac{(4 - l)\Upsilon(\lambda)}{4l \Upsilon(0)} = F, \quad k = 1, 2, \ldots.
\]

For the range of \( F \) we have three options.

- If \( \mu_1 > 0 \) then \( \Upsilon(0) < 1 \) and \( F \) satisfies \( -1/4 < F < +\infty \).
- If \( \mu_1 < 0 \) then \( \Upsilon(0) > 1 \) and \( F \) satisfies \( -\infty < F < -1/4 \).
- If \( \mu_1 = 0 \) then \( \Upsilon(0) = 1 \) and \( F = -1/4 \).

The function \( E_0(\lambda) \) has the form 4.9 and satisfies condition 4.10. Therefore, according to remark 4.6 for any choice of \( l \), or equivalently \( F \), there exists a string \( S_0 \) with the function \( E_0 \) given by formula 10.2.

**Remark 10.1.** Let \( \mu_1 > 0 \) and \( F = 0 \). Then there exists a string \( S_0 \) with \( l > 0 \) and corresponding spectrum \( \mu(0)'s. \)

The next theorem is an analog of Theorem 9.2. We pick \( F \) such that \( \mu_1(F) \neq 0. \)

**Theorem 10.2.** The variables

\[
J_n(F) = -\frac{1}{\mu_n(F)}; \quad \tau_k(F) = \log \psi(2, \mu_k(F))
\]

are canonically conjugate variables

\[
\{J_k(F), J_n(F)\} = 0, \quad \{\tau_k(F), \tau_n(F)\} = 0, \quad \{\tau_k(F), J_n(F)\} = \delta_k^n.
\]
To prove the theorem we introduce the function $\Omega^F_0(\lambda)$:

$$\Omega^F_0(\lambda) = -\frac{1}{E_0(\lambda) - F}.$$ 

It can be written as

$$\Omega^F_0(\lambda) = \frac{\psi}{\varphi + F\psi} = \frac{\psi}{\varphi^F}.$$ 

The points $\mu_k(F)$ are the roots of the equation $\varphi^F(\lambda) = 0$. The function $\varphi^F$ can be written as:

$$\varphi^F(\lambda) = (1 + 4F) \prod_{k=1}^{\infty} \left( 1 - \frac{\lambda}{\mu_k(F)} \right).$$ 

The function $\Omega^F_0(\lambda)$ maps the upper half–plane into itself and has the expansion

$$\Omega^F_0(\lambda) = \frac{l}{1 + Fl} + \sum_{k=1}^{\infty} \frac{\sigma_k(F)}{\mu_k(F) - \lambda}. \quad (10.3)$$ 

One can change the variable $\lambda \to \lambda' = -\frac{1}{\lambda}$ and get the formula

$$\Omega^F_0(\lambda') = \frac{4}{1 + 4F} + \sum_{k=1}^{\infty} \frac{\mu_k'(F)}{\sigma_k'(F) - \lambda'}. \quad (10.4)$$ 

After that the proof of the theorem follows familiar steps. The Poisson bracket for the function $\Omega^F_0(\lambda')$ is given by formula 6.2. For the variables $\sigma'(F)$ and $\mu'(F)$ from 10.4 an analog of Lemma 7.2 holds. Then similarly to Lemma 7.3 we can compute the bracket between the variables $\sigma(F)$ and $\mu(F)$ entering into formula 10.3. Finally, the proof of the canonical relations is identical to the proof of Theorem 7.1. 

We will construct a family of Hamiltonian flows which will be linearized in the variables $J(F)$ and $\tau(F)$ in the next section.

11. Trace formulas

Let $\lambda_k(a, b)$ be a spectrum of the boundary value problem

$$a\varphi(2, \lambda) + b\psi(2, \lambda) = 0.$$

In this section we will express the sums

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k(a, b)}, \quad \sum_{k=1}^{+\infty} \frac{1}{\lambda_k^2(a, b)}, \quad etc;$$

in terms of the potential. In order to simplify calculations we work with the Sturm–Liouville problem 2.4. We assume that the potential $m(x)$ has compact support. The solutions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ of the Sturm–Liouville problem are...
the image of the corresponding solutions of the string problem under the Liouville transformation. Note

\[ \varphi(x, 0) = \varphi_0(x) = \cosh x/2, \quad \psi(x, 0) = \psi_0(x) = 2e^{x/2}. \]

We also introduce the solution \( \theta(x, \lambda) \) which corresponds to the solution \( \theta(\xi, \lambda) \) specified by the boundary condition at the right end

\[ \theta(+2, \lambda) = 0, \quad \theta'_+(+2, \lambda) = 1. \]

Note that

\[ \theta(\xi, 0) = \xi - 2, \quad \theta(x, 0) = \theta_0(x) = 2e^{-x/2}. \]

At the points of the spectrum \( \lambda_k(a, b) \), the solution \( \Delta(x, \lambda) = a\varphi(x, \lambda) + b\psi(x, \lambda) \) tends to zero when \( x \) tends to +\( \infty \). At these points, the Wronskian

\[ W(\lambda) = \theta(x, \lambda)D\Delta(x, \lambda) - \Delta(x, \lambda)D\theta(x, \lambda), \]

vanishes.

The expansion can be obtained near \( \lambda = 0 \)

\[ W(\lambda) = I_0(a, b) + I_1(a, b)\lambda + I_2(a, b)\lambda^2 + \ldots \]

in two different ways. The first arises from the Hadamard formula

\[ W(\lambda) = (a + 4b)\prod_{k=1}^{\infty} \left( 1 - \frac{\lambda}{\lambda_k} \right), \quad \lambda_k = \lambda_k(a, b). \]

The coefficients have the form

\[ I_0(a, b) = a + 4b, \]

\[ I_1(a, b) = -(a + 4b) \sum_k \frac{1}{\lambda_k}, \]

\[ I_2(a, b) = (a + 4b) \sum_{n\neq k} \frac{1}{\lambda_k\lambda_n}. \]

Another expansion is obtained from the Neumann series for the solutions

\[ \varphi(x, \lambda) = \varphi_0(x) + \varphi_1(x)\lambda + \varphi_2(x)\lambda^2 + \ldots; \]

\[ \psi(x, \lambda) = \psi_0(x) + \psi_1(x)\lambda + \psi_2(x)\lambda^2 + \ldots; \]

\[ \theta(x, \lambda) = \theta_0(x) + \theta_1(x)\lambda + \theta_2(x)\lambda^2 + \ldots. \]

Therefore,

\[ I_0(a, b) = \theta_0 D\Delta_0 - D\theta_0\Delta_0; \quad (11.1) \]

\[ I_1(a, b) = \theta_0 D\Delta_1 + \theta_1 D\Delta_0 - D\theta_0\Delta_1 - D\theta_1\Delta_0; \quad (11.2) \]

\[ I_2(a, b) = \theta_0 D\Delta_2 + \theta_1 D\Delta_1 + \theta_2 D\Delta_0 - D\theta_0\Delta_2 - D\theta_1\Delta_1 - D\theta_2\Delta_0; \quad (11.3) \]
where $\Delta_k(x) = a\varphi_k(x) + b\psi_k(x)$. Identifying the corresponding coefficients we obtain the first two trace formulas

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k(a,b)} = -\frac{I_1(a,b)}{a + 4b},$$

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k^2(a,b)} = \left[ \frac{I_1(a,b)}{a + 4b} \right]^2 - \frac{2I_2(a,b)}{a + 4b}. \quad (11.5)$$

The coefficients $\varphi_k(x)$ and $\psi_k(x)$ are given by the formulas

$$\varphi_0(x) = \cosh \frac{x}{2},$$

$$\varphi_1(x) = -2 \int_{-\infty}^{x} d\xi \sinh \frac{x-\xi}{2} \cosh \frac{\xi}{2} m(\xi),$$

$$\varphi_2(x) = 4 \int_{-\infty}^{x} d\xi \int_{-\infty}^{\xi} d\eta \sinh \frac{x-\xi}{2} \sinh \frac{\xi-\eta}{2} \cosh \frac{\eta}{2} m(\xi)m(\eta),$$

$$\psi_0(x) = 2e^{x/2},$$

$$\psi_1(x) = -4 \int_{-\infty}^{x} d\xi \sinh \frac{x-\xi}{2} e^{\xi/2} m(\xi),$$

$$\psi_2(x) = 8 \int_{-\infty}^{x} d\xi \int_{-\infty}^{\xi} d\eta \sinh \frac{x-\xi}{2} \sinh \frac{\xi-\eta}{2} e^{\eta/2} m(\xi)m(\eta).$$

Also,

$$\theta_0 = 2e^{-x/2},$$

$$\theta_1 = 4 \int_{x}^{+\infty} d\xi \sinh \frac{x-\xi}{2} e^{-\xi/2} m(\xi),$$

$$\theta_2 = 8 \int_{x}^{+\infty} \int_{x}^{+\infty} d\xi \sinh \frac{x-\xi}{2} \sinh \frac{\xi-\eta}{2} e^{-\eta/2} m(\xi)m(\eta).$$

It is easy to check that formula 11.1 produces $I_0(a,b) = a + 4b$. Note that the coefficients $I_k(a,b)$ are linear in the parameters $a$ and $b$: $I_k(a,b) = aI_k(1,0) +$
For the first trace formula 11.4 we have
\[ \sum_{k=1}^{+\infty} \frac{1}{\lambda_k(a, b)} = -\frac{a I_1(1, 0)}{a + 4b} - \frac{b I_1(0, 1)}{a + 4b}. \]

From 11.2 with the help of formulas for the coefficients of expansions of the solutions we have
\[ I_1(1, 0) = -\int_{-\infty}^{+\infty} dx m(x)(1 + e^{-x}), \]
\[ I_1(0, 1) = -4 \int_{-\infty}^{+\infty} dx m(x). \]

For the second trace formula 11.5 we have
\[ \sum_{k=1}^{+\infty} \frac{1}{\lambda_k^2(a, b)} = \left( \frac{a I_1(1, 0) + b I_1(0, 1)}{a + 4b} \right)^2 - 2 \times \frac{a I_2(1, 0) + b I_2(0, 1)}{a + 4b}. \]

From 11.3 with the help of the formulas for the coefficients of the expansions we have
\[ I_2(1, 0) = \frac{1}{2} \left( \int_{-\infty}^{+\infty} dx m(x) \right)^2 - 2 \int_{-\infty}^{+\infty} dx e^{-x} m(x) D^{-1} m(x) - \int_{-\infty}^{+\infty} dx m(x) v(x), \]
\[ I_2(0, 1) = 2 \left( \int_{-\infty}^{+\infty} dx m(x) \right)^2 - 4 \int_{-\infty}^{+\infty} dx m(x) v(x). \]

The first series of Hamiltonians \( \mathcal{H}_k(C) \) corresponds to the case \( a = -C, \ b = 1 \). Two Hamiltonians \( \mathcal{H}_1(C) \) and \( \mathcal{H}_2(C) \) are given in 1.4. The second series of Hamiltonians \( \mathcal{T}_k(F) \) corresponds to the case \( a = 1, \ b = F \). The first few also can be found in 1.4.

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