Our study of the evolution of transmission eigenvalues, due to changes in various physical parameters in a disordered region of arbitrary dimensions, results in a generalization of the celebrated DMPK equation. The evolution is shown to be governed by a single complexity parameter which implies a deep level of universality of transport phenomena through a wide range of disordered regions. We also find that the interaction among eigenvalues is of many body type that has important consequences for the statistical behavior of transport properties.

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I. INTRODUCTION

Recent advances in nanotechnology and quantum information theory have motivated an extensive research on the topic of electronic transport through disordered regions [1–3]. The random distribution of impurities in such systems give rise to fluctuations in transport properties from sample to sample. The fluctuations can also be observed in a single disordered sample under an external perturbation or a slight variation of a system parameter [4,5]. As a result, the information about the statistical behavior of transport properties is of great importance.

A variety of transport properties can be formulated in terms of the eigenvalues of transmission matrix of the region. The knowledge of the statistical behavior of transmission eigenvalues is therefore very useful in the statistical analysis of transport properties. This motivates us to study the joint probability distribution of transmission eigenvalues. Previous attempts in this direction have resulted in the well-known DMPK equation which describes the statistical evolution of transmission eigenvalues with respect to changing length of the medium [2,6]. Various assumptions made in its derivation, however, restrict its applicability to quasi one dimensional systems or under specific scattering conditions [1,2,7]. This being the only analytical tool available so far to describe the full distribution of transport properties, a generalization of DMPK equation for higher dimensions and under generic scattering conditions is required. (The other analytical method based on nonlinear sigma model provides information only about the moments of the transport properties). Further the transport properties are also sensitive to changes in other system parameters besides length e.g., boundary conditions, disorder strength and dimensionality. It would also be desirable if the generalized equation contains explicit information about the effect of various system parameters on the distribution. Our study in this paper is aimed at such a generalization.

The derivation of DMPK equation is based on the transfer matrix approach, applied to a conductor placed between two perfect leads of finite width. A transfer matrix relates the wave amplitudes on the right of the region to the left. The scattering of waves by randomly distributed impurities leads to a randomization of transferred amplitudes. The transfer matrix can therefore be modeled by a random matrix, that is, a matrix with all or some of its elements as randomly distributed [1,2,8,9]. The idea to use random matrix approach originates from the universality of the conductance fluctuations, observed in the metallic regime of different disordered systems. The universality suggested the possibility of formulating a theory which is system-independent; the suggestion motivated the use of standard random matrix ensembles [1,2] as models for the ensembles of transfer matrices in the metallic regime. These ensembles are obtained by using maximum entropy hypothesis under a single constraint of fixed eigenvalue density; their matrix elements are of almost same strength and the statistical behavior, being governed only by underlying symmetry, is universal in nature [1,2]. These ensemble can therefore serve as good models for the transfer matrix in a metallic regime or for quasi one dimensional conductors where the flux incident on one channel can be assumed to be transmitted with the same probability into all outgoing channels (known as the isotropic assumption). The DMPK equation was also derived by using the standard random matrix model for the transfer matrix of a small length of the disordered region. As expected, the equation has been very successful in predicting the behavior of transport properties in metallic regime or for quasi one dimensional conductors.

The standard random matrix ensembles are not appropriate tools to model the transfer matrix of a disordered region under generic scattering conditions. This is confirmed by the observed failure of DMPK equation beyond metallic
regime, in conductors of dimensions higher than one, or under inelastic scattering conditions. The generic scattering conditions may cause a preferential as well as multiple interactions among channels which would lead to varying degree of correlations among transfer matrix elements. The equation governing the evolution of the transmission eigenvalues therefore should be based on a model of transfer matrices free of isotropy constraints. In this paper, we consider such a model and study the evolution of the transmission eigenvalues due to change of various system parameters. As shown later, the resulting evolution equation indicates, beyond metallic regime, the presence of eigenvalue correlations stronger than those suggested by the DMPK equation. This would lead to new theoretical predictions of the statistical behavior of transport properties. However, in the metallic regime, the bulk eigenvalue correlations suggested by our evolution equation are essentially the same as those given by DMPK equation.

The paper is organized as follows. In section II, we derive the diffusion equation for the elements of the transfer matrix of a disordered region under generic scattering conditions. The equation is then used to obtain the statistics of transmission eigenvalues in section III. To maintain the flow of the discussion, only relevant steps are given in sections II, III; the details of the steps can be found in the appendices. The section IV deals with a derivation of the "Hamiltonian" formulation of the dynamics of transmission eigenvalues. The usefulness of the Hamiltonian representation is that it clearly reveals the hidden three body eigenvalue-interactions which are found to be absent in the case of DMPK equation. This is followed, in section V, by a brief discussion of the averages of the moments of transmission eigenvalues; the information is relevant for predictions of the distribution of the transport properties. The section VI contains a discussion about the scaling behavior of the transport properties. We conclude in section VII by summarizing our main results.

**II. STATISTICS OF TRANSFER MATRIX ELEMENTS**

We consider a disordered region of length $L$ and width $W$ connected to two electron reservoirs by ideal (perfectly conducting) leads. The scattering in the region as well as reservoirs is assumed to be elastic. The finite width in the transverse direction leads to the quantization of energy of the transverse part of the wavefunction. As a result, the scattering states at the Fermi energy satisfy the relation $k_F^2 = k_n^2 + \epsilon_n$ with $k_F$ as the Fermi momentum, $k_n$ the longitudinal momentum ($k_n > 0$) and $\epsilon_n$ as the transverse quantized eigenvalue. The various $k_n$ for $n = 1, 2, ..., N$ define the $N$ propagating channels. As each channel can carry two waves traveling in opposite directions, the wavefunction on either side of the disordered region is specified by a $2N$ components (corresponding to the amplitudes of $N$ waves propagating to the right and $N$ waves to the left). The normalization of the wavefunction is chosen such that it carries unit current. Various length-scales associated with the wavefunction divide the transport properties into three main regimes: (i) ballistic limit described by $l > L$ with $l$ as the mean free path, (ii) diffusive limit given by $l < L < \zeta$ with $\zeta = NL$ as the localization length and (iii) insulator limit with $L > \zeta$.

The scattering properties of the disordered region are completely characterized by a $2N \times 2N$ transfer matrix $M$ with $M_{kl} \equiv \sum_{s=1}^{2N} (i)^{s-1} M_{kl,s}$; the subscript $s$ refers to the real $(s = 1)$ and imaginary component $(s = 2)$ of the element. The transfer matrix has a multiplicative property: if the disordered system is described as a sequence of $n$ "scattering units" (thin slices), with transfer matrices $M_1, M_2, ..., M_n$, respectively, the transfer matrix of the disordered system is

$$M^{(n)} = M_n...M_2M_1$$  \hspace{1cm} (1)$$

The current conservation imposes a "pseudo-unitarity" constraint on $M$:

$$\Sigma_z M^{-1} \Sigma_z = M^\dagger$$  \hspace{1cm} (2)$$

where $\Sigma_z$ is a $2N \times 2N$ matrix: $\Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with 1 and 0 as $N \times N$ unit matrix and null matrix, respectively. The presence of time-reversal symmetry in the disordered region subjects $M$ to an additional requirement:

$$\Sigma_x M \Sigma_x = M^*$  \hspace{1cm} (3)$$

with $\Sigma_x$ as a $2N \times 2N$ matrix: $\Sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The current conservation condition on $M$ leads to correlations between its matrix elements. As a result, the number of independent elements of $M$ is $N_2 = (2N)^2$. The time-reversal symmetry along with current conservation further reduces the number of independent elements to $N_1 = N(2N + 1)$ [6,2]. In this section, we denote an independent
matrix element of $M$ by $M_{\mu}$ with $\mu$ as a single index, running from $1 \rightarrow N_\beta$, and replacing three indices $(k,l,s)$, that is, $\mu \equiv \{kl; s\}$; the subscript $(kl; s)$ will be used only if required for clarification. Here $N_\beta = N(2\beta N + 2 - \beta)$ with $\beta = 1$ in presence of time-reversal symmetry and $\beta = 2$ in its absence.

Our objective in this paper is to study the statistics of transmission eigenvalues. In principle, the eigenvalues $T_n$ of the transmission matrix $T$ of a region are known as transmission eigenvalues. However, due to a simple one-to-one relationship with $T_n$, the doubly degenerate eigenvalues $\lambda_n$ of the matrix $B = [A + A^{-1} - 2]/4$, with $A = M.M^\dagger$, are also referred as transmission eigenvalues: $T_n = \lambda_n$, $n = 1 \rightarrow N$ [1]. The $\lambda_n$ can further be expressed in terms of $2N$ eigenvalues of the matrix $A$ which exist in inverse pairs and can be denoted by $x_n$ ($n = 1,2,...,2N$) [1]: $\lambda_n = (x_n + x_n^{-1} - 2)/4$. The statistics of transmission eigenvalues can then be determined by a knowledge of the statistics of eigenvalues $x_n$ which in turn is related to the statistics of transfer matrices.

**A. Probability density of transfer matrices and its dependence on various system parameters**

For statistical analysis of transfer matrices, we consider an ensemble of matrices $M$, representing a collection of disordered conductors of length $L$ and defined by a differential probability $d\rho(M) = \rho(M)d\mu(M)$ [2,6,10]. Here $\rho(M)$ and $d\mu(M)$ are the probability density and the invariant measure associated with the transfer matrix space (see [10] for the details about $d\mu(M)$). The behavior of $\rho(M)$ depends on the physical properties of the disordered region. This can be explained as follows: The elements of $M$ describe the overlap between various channel states on two sides of the region. The presence of disorder and deterministic uncertainty due to complexity of region leads to randomization of the elements $M_{\mu}$. Based on complex nature of the disordered region, the randomness associated with $M_{\mu}$ can be of various types. Further the generic scattering conditions can cause multiple channel interactions, resulting in varying degree of correlations between matrix elements. The multiplicative property (eq.(1)) of the transfer matrices also imposes constraints on the behavior of $\rho(M)$: if $\rho_L(M)$ and $\rho_L(M_0)$ be the densities of transfer matrices $M, M_0$ of a region with lengths $L, L_0$, respectively, the density $\rho_L(M')$ of the matrix $M' = MM_0$ corresponding to the length $L' = L + L_0$ should be reproducible under convolutions of $\rho_L(M)$ and $\rho_L(M_0)$,

$$\rho_L(M') = \int \rho_L(M'M_0^{-1})\rho_L(M_0)d\mu(M_0)$$ (4)

The above formulation was used as a basis to derive the DMPK equation, by applying it to obtain the probability density for a length $L' = L + \delta L$ with $L_0 = \delta L$ as a small length increment. The $\rho_M(M_0)$ in this case was assumed to be a maximum entropy distribution obtained under isotropy constraints: $\rho_M(M_0) = \exp(-\mu - \nu TrM^\dagger M)$ with $\mu$ and $\nu$ as Lagrange Multipliers [6,10]. The assumption implies that, in DMPK case, the information about matrix-element correlations is contained only in the invariant measure $d\mu(M)$; the latter, however, contains only those correlations which are required to preserve the current conservation [10].

The generic scattering conditions in a disordered region result in the matrix element correlations beyond those due to current conservation condition. We therefore need to consider a probability density $\rho(M)$, free of isotropy constraints, with multi-channel correlations and dependent on various other system parameters besides length; (now onwards, we suppress the subscript length of $\rho$). In absence of any further information about the disordered region, the simplest and least biased hypotheses is that the system is described by the distribution $\rho(M)$ that maximizes Shannon’s information entropy

$$S[\rho(M)] = -\int \rho(M) \ln \rho(M) \ d\mu(M)$$ (5)

under the constraints (i) $\rho(M)$ is normalized, (ii) the mean $<M_{\mu}>$ and correlations $<M_{\mu}M_{\mu'}>$ are fixed:

$$<M_{\mu}> = \frac{\beta}{2} \frac{\partial \rho C}{\partial a_{\mu}}, \ <M_{\mu}M_{\mu'}> = \beta \frac{\partial \rho C}{\partial b_{\mu\mu'}}$$, with $C$ as the normalization constant and the parameters $a_{\mu}$, $b_{\mu\mu'}$ given by the system conditions. The entropy $\hat{S}$ subject to above constraints can be maximized by considering the functional $\hat{S}[\rho] = S[\rho] - (1/\beta) \sum_{\mu,\mu'} b_{\mu\mu'} <M_{\mu}M_{\mu'}> - (2/\beta) \sum_{\mu} a_{\mu} <M_{\mu}> - C$ and putting its functional derivative to zero:

$$\delta \hat{S} = -\int d\mu(M) \delta \rho(M) \left[ 1 + \ln \rho(M) - \frac{1}{\beta} \sum_{\mu,\mu'} b_{\mu\mu'} M_{\mu}M_{\mu'} - \frac{2}{\beta} \sum_{\mu} a_{\mu} M_{\mu} \right] = 0.$$ (6)

The above implies

3
\[ \rho(M, a, b) = C \exp \left[ -(1/\beta) \sum_{\mu, \mu'} b_{\mu \mu'} M_{\mu} M_{\mu'} - (2/\beta) \sum_{\mu} a_{\mu} M_{\mu} \right]. \]  

(7)

with \( C \) as a normalization constant and \( a, b \) as the matrices of distribution parameters \( a_{\mu} \) and \( b_{\mu \mu'} \), respectively. Note the symbol \( \sum \) implies a summation over different matrix elements only.

The Gaussian form of \( \rho \) in eq.(7) results due to consideration of constraints only up to second order moments of the matrix elements. The availability of information about higher order moments may lead to a non-Gaussian behavior of \( \rho \). The Gaussian assumption, however, can also be justified by the same intuitive reasoning which led to the Gaussian distribution of \( \rho(M) \) for a small length increment \( \delta L \) in the DMPK case [10]. The reasoning is based on the multiplicative property of \( M \): Dividing the length into small segments, the transfer matrix of the region can be written as a sum over matrices corresponding to the length segments [11]. Assuming the matrices in the sum to be independent of each other, a central limit theorem for the statistics of transfer matrices was proved in [11] in the weak-scattering limit. For a length macroscopically small but containing many scattering units, as in the DMPK case, the central limit theorem justifies the assumption of independent Gaussian distributions for various \( M_{\mu} \). For a macroscopically large length, however, the correlations among \( M_{\mu} \)'s should be taken into account which intuitively leads us to eq.(7).

The distribution parameters \( a_{\mu} \) and \( b_{\mu \mu'} \), being measures of the averages of the matrix elements and their correlations, are influenced by various system conditions. For example, the increase of disorder in the region leads to localization of waves and reduced channel-channel interaction which in turn affects the distribution of each \( M_{\mu} \). Similarly the increasing dimensionality \( d \) of the region increases the total number \( N \) of existing channels, \( N \approx (k_{B}W)^{d-1} \), as well as the probability of interacting channels. The latter increases the number of finite \( b \) parameters. The boundary conditions or topology of the region also affect the distribution parameters, due to their influence on the interactions among channels close to boundaries. As an example, consider the case of interaction between nearest-neighbors channels only. This gives only \( 2N(t + 1) \) finite \( b \)'s, rest of them being infinite; here \( t \), the number of nearest neighbors depends on the dimensionality as well as boundary conditions of the region. Further the strength of finite \( b \) parameters depends on the localization length \( \zeta \) of the region which, in turn, is quite sensitive to the dimensionality.

Another important system parameter affecting the sets \( a, b \), is the system length \( L \). The dependence of \( a, b \) on \( L \) can be derived by using the convolution property, given by eq.(4), as a condition. For example, the \( L \)-dependence of \( b \)'s can be seen by a simple case with \( a_{\mu} \) length-independent. The eq.(4) gives, by using eq.(7) for \( \rho(M') \) as well as \( \rho(M) \) and writing \( b(L + \delta L) \approx b(L) + \frac{\partial b}{\partial L} \delta L \),

\[ C_{L+\delta L} e^{-(1/\beta)\delta L \sum_{\mu, \mu'} \frac{\partial b}{\partial L} M_{\mu} M_{\mu'}} = C_{L} \int e^{-(1/\beta)\sum_{\mu, \mu'} b_{\mu \mu'}[M_{\mu} M_{\mu'}-M_{\mu} M_{\mu'}]} \rho_{\delta L}(M_{0}) \, d\mu(M_{0}) \]  

(8)

where \( M = M' M^{-1} = M' \Sigma_{z} M_{0} \Sigma_{z} \). The \( \frac{\partial b}{\partial L} \) can now be obtained by multiplying eq.(8) by \( \prod_{\mu} dM_{\mu}' \), followed by an integration:

\[ \text{Det} \left[ \frac{\partial b}{\partial L} \right] = \frac{C_{L+\delta L}^{2}}{C_{L}^{2} f^{2}} \]  

(9)

where \( f \) as a function of \( b \), \( f(b) = \int f_{0}(M_{0}) \rho_{\delta L}(M_{0}) \, d\mu(M_{0}) \), and \( f_{0} \) as a function of \( M_{0} \) and \( b \): \( f_{0}(M_{0}, b) = \int e^{-(1/\beta)\sum_{\mu, \mu'} b_{\mu \mu'}[M_{\mu} M_{\mu'}-M_{\mu} M_{\mu'}]} \, d\mu(M_{0}) \). The eqs.(8,9) indicate that the variation of \( b \) with respect to \( L \) depends on the scattering properties of the small length increment.

The physical properties such as current conservation and presence of time-reversal symmetry in the region also put restrictions on the strengths of the parameters \( a, b \). The current conservation condition (eq.(2)) implies \( \sum_{i} < M_{kl} M_{kl} > = \sum_{i} < M_{kl} M_{kl} > - < M_{kl} M_{kl} > = \delta_{k} s_{k} c_{k} \) (with \( c_{k} = 1 \) for all \( k \leq N \) and \( c_{k} = -1 \) for \( k > N \) which in turn connects various \( b \)'s (as \( < M_{ij} s_{i} M_{kl} s_{k} > = \beta \delta_{ij} \delta_{kl} \rho C_{ijkl} \)). Similarly the time-reversal symmetry along with current conservation results in equality \( M_{kl} = M_{k+N,l+N} \) [6] which implies \( b_{kl;ij;ss'} = (1)^{s-1}b_{k+N,l+N;ij;ss'} = (1)^{s'-1}b_{kl,i+N,j+N;ss'} = (1)^{s-s'-2}b_{k+N,l+N;ij;ss'} \) and \( a_{kl;ss'} = (1)^{-1}a_{k+N,l+N;ss'} \).

In general, the different scattering conditions can give rise to different sets of distribution parameters \( a, b \). For example, let us consider the situation in three main regimes:

(i) Ballistic Regime: The transfer matrix in this regime is almost an unity matrix. The regime can therefore be modeled by an ensemble of matrices \( M \) with a probability density given by eq.(7) where

\[ a_{\mu} \rightarrow (l/2L) \delta_{\mu \mu'}, \quad b_{\mu \mu'} \rightarrow (l/L) \delta_{\mu \mu'}; \]  

(10)
for all $\mu, \mu', \mu_d$. Here $\mu_d \equiv (kk; s)$ with $M_{\mu_d} = M_{kk,s}$ as a diagonal element. Note, for the case $L << l$, the above parametric-strengths correspond to an ensemble of matrices with almost non-random elements; it approaches the ensemble of diagonal matrices in the limit $L/l \to 0$.

(ii) Metallic Regime: For this case, (a) the overlapping between various channels is almost of the same strength, (b) the flow between two channels is not affected by the presence of other channels. The ensemble of matrices $M$ can then be modeled by eq.(7) with distribution parameters

$$a_\mu \to 0, \quad b_{\mu\mu'} \to (\zeta/L)b_{\mu\mu'}$$

(11)

for all $\mu, \mu', \mu_d$ and with $l < L < \zeta$. The above set of parameters result in an isotropic density $\rho(M) \propto \exp[-(\zeta/2L)\text{Tr}(M,M^\dagger)]$ which corresponds to a statistical behavior independent of system-details. The latter is in agreement with the observed behavior in the metallic regime.

(iii) Insulator Regime: This regime corresponds to a zero net current flow across the disordered region which implies an almost zero overlap between different channels states. The insulator state can therefore be modeled by the limit

$$a_\mu \to 0, \quad b_{\mu\mu'} \to (L/\zeta)$$

(12)

for almost all $\mu, \mu'$, and, with $L >> \zeta$. In large $L$ limit, above parameters result in an ensemble of transfer matrices with all matrix elements going to zero.

B. Single Parametric Evolution of $\rho(M)$

The distribution of the transmission eigenvalues $\lambda_n$ can be obtained, in principle, from eq.(7) by using eq.(2.11) of [10] which relates $M$ with a matrix $\tau(\lambda)$, with functions of $\lambda$ as its elements, and the unitary matrices $U,V$: $M = U\tau(\lambda)V$. An integration over matrices $U,V$, if possible, can then give the probability density for $\lambda$. The non-isotropic form of the distribution $\rho(M)$, however, makes the integration route very complicated. This motivates us to seek an alternative route. As discussed below, we reduce the technical complications by formulating the non-isotropic problem in the same form as that of the isotropic problem. This also helps in identifying a single parameter which governs the evolution of multi-parametric $\rho(M)$.

We proceed as follows. A perturbation of the disordered region due to a change in impurity structure or other system parameters perturbs the matrix elements $M_{\mu}$ and, consequently, the probability density $\rho(M,a,b)$. Due to its Gaussian form, a change in $\rho$ due to variation of $M_{\mu}$ can be well-mimicked by the change due to the distribution parameters $a, b$ (see appendix A for the derivation):

$$L \rho = T \rho$$

(13)

where $L$ and $T$ are the operators in $M$-space and parametric space, respectively,

$$L = \sum_{\mu} \frac{\partial}{\partial M_{\mu}} \left[ \frac{\beta}{2} \frac{\partial}{\partial M_{\mu}} + \gamma M_{\mu} \right]$$

(14)

$$T = \sum_{\mu,\mu'} f_{\mu\mu'} \frac{\partial}{\partial b_{\mu\mu'}} + \sum_{\mu} b_{\mu\nu} \frac{\partial}{\partial a_{\mu}}$$

(15)

with

$$f_{\mu} = \gamma a_{\mu} - 2 \sum_{\mu'} a_{\mu'} b_{\mu\mu'}$$

(16)

$$f_{\mu\mu'} = G_{\mu\mu'} \left[ \gamma b_{\mu\mu'} - \sum_{\mu''} b_{\mu''\nu} b_{\mu\mu'} \right]$$

(17)

Here $G_{\mu\mu'} = 1 + \delta_{kk} \delta_{\mu} \delta_{\mu'}$ if $\mu \equiv (kl; s)$, $\mu' \equiv (ij; s')$, and, $\delta_{xy} = 1$ if $x = y$ and $\delta_{xy} = 0$ for $x \neq y$. The normalization constant $C$ is chosen such that $TC = \sum_{\mu} (\gamma + (2\beta^{-1}a_{\mu} - b_{\mu\mu}))C$ (see appendix A for details). The parameter $\gamma$ is arbitrary and marks the end of the transition. Note, in eq.(13), the derivatives of $\rho$ with respect to different matrix elements are independent of each other but the parametric derivatives are correlated. By considering the above combination of derivatives, therefore, we transfer the information regarding the correlations between elements.
of matrix $M$ to the elements of matrix $b$. This helps in reducing the $M$-space operator $L$ (eq.(14)), in the same form as the one in the isotropic case; as mentioned above, this reduction is useful for technical reasons as well as for comparisons with earlier studies.

Contrary to the single parametric evolution of $\rho$ in the isotropic case, the evolution of $\rho$ in eq.(13) is governed by a large number of parameters. However, as discussed in appendix B, seemingly multi-parametric diffusion can be reduced to a single parametric evolution by considering a transformation $(a, b) \rightarrow y$, with $y$ as a set of parameters $y_j$, $j = 1, 2, \ldots, N$, which maps the parametric space operator $T$ (eq.(15)) as $T[a, b] \rightarrow T[y(a, b)] \equiv \sum_{\mu \mu'}$; Here $N$ is the number of $(a, b)$ parameters with finite strengths. As a result, eq.(13) can be written as

$$
\sum_{\mu} \frac{\partial}{\partial M_{\mu}} \left[ \frac{\beta}{2} \frac{\partial \rho}{\partial M_{\mu}} + \gamma M_{\mu} \rho \right] = \frac{\partial \rho}{\partial Y}
$$

where $Y = y_1$. The eq.(18) is in a form of a standard Fokker-Planck equation with "particles" $M_{\mu}$ undergoing a Brownian motion in "time" $Y$. The evolution approaches a steady state in limit $\frac{\partial \rho}{\partial Y} \rightarrow 0$ or $Y \rightarrow \infty$ which occurs when $f_{\mu}, f_{\mu'} \rightarrow 0$ or, equivalently, $a_{\mu} \rightarrow 0$, $b_{\mu\mu'} \rightarrow \gamma \delta_{\mu\mu'}$ and, therefore, $\rho \rightarrow e^{-(\gamma/2)YMM^{T}}$ (from eq.(7)). The latter is also the solution of eq.(18) in limit $\frac{\partial \rho}{\partial Y} \rightarrow \infty$; the steady state limits of eq.(7) and (18) are, therefore, consistent with each other.

The parameter $Y$, appearing in eq.(18), is a function of various parameters $a_{\mu}$ and $b_{\mu\mu'}$ which are governed by underlying complexity of the system; $Y$ can therefore be termed as the complexity parameter,

$$
Y = \sum_{\mu} \int da_{\mu} z_{\mu} X + \sum_{\mu, \mu'} \int db_{\mu\mu'} z_{\mu\mu'} X + \text{constant}
$$

where summation is implied over the distribution parameters with finite values only, and, $X = [\sum_{\mu} z_{\mu} f_{\mu} + \sum_{\mu, \mu'} z_{\mu\mu'} f_{\mu\mu'}]^{-1}$. Here the functions $z_{\mu}, z_{\mu\mu'}$ are arbitrarily chosen such that the ratio

$$
\left[ \sum_{\mu} z_{\mu} da_{\mu} + \sum_{\mu, \mu'} z_{\mu\mu'} db_{\mu\mu'} \right] X
$$

is a complete differential; the details of the eqs.(19,20) are given in appendix B.

It is worth emphasizing here that the system information in eq.(18) enters only through the parameter $Y$. The distribution parameters $a, b$ being system-dependent, $Y$ in turn is a function of the system parameters e.g. length, disorder, dimensionality and boundary conditions etc. A variation of any one of the system parameters e.g. length $L$ can therefore change $Y$ but, note, in general $Y \neq L$. (In other words, even if $L$ is the only system parameter subjected to change, the parameter governing the evolution is $Y$, a function of other system parameters besides $L$). However the case $Y \propto L$ can occur if the distribution parameters are assumed to depend only on the length of the system (ignoring the dependence on other system parameters). For example, consider the case with

$$
a_{\mu} \rightarrow 0, b_{\mu\mu'} \rightarrow (qL)^{-1} \delta_{\mu\mu'}
$$

for all $\mu, \mu'$ with $q$ as a constant. Using the values in eqs.(16, 17) gives $f_{\mu} = 0, f_{\mu\mu'} = 2(qL)^{-2}(\gamma qL - 1)$ and therefore $X^{-1} = 2N_{\beta}(qL)^{-2}[\gamma qL - 1]$; here we have chosen $z_{\mu\mu'} = 1$ as that makes the ratio (20) a complete differential. Note the variation of length $L$ is this case changes only $N_{\beta}$ parameters $b_{\mu\mu'}$, leaving all other parameters unchanged. This, along with substitution of $X$, in eq.(19) gives

$$
Y = q \int \frac{dL}{2(\gamma qL - 1)} = -\frac{1}{2\gamma} \log |1 - \gamma qL| + \text{constant}.
$$

As obvious, the system length $L \sim (\gamma q)^{-1}$ gives $Y \rightarrow \infty$ and therefore steady state of the evolution: $\rho \rightarrow e^{(-\gamma/2)YMM^{T}}$.

The dimensional sensitivity of $Y$ can also be explained by a simple example. Let us consider a disordered region with interactions between nearest-neighbor channels only, that is, by choosing

$$
a_{\mu} = \delta_{\mu\mu_d}, \quad b_{\mu\mu'} \rightarrow r_{\mu\mu'} \delta_{\mu\mu'},
$$

where $c_{\mu} = 1$ if $M_{\mu}$ corresponds to an interaction between same channels (i.e. for $\mu = \mu_d \equiv (kk; s)$ or nearest neighbors $k, l$ ($\mu \equiv (kl; s)$), $c_{\mu} = 0$ for all other $\mu$ values, (thus $\sum_{\mu} c_{\mu} = 2N(t + 1)$ with $t$ as the number of nearest
neighbors). As channel-channel interaction depends on the localization length $\zeta$, it is reasonable to consider $r = \zeta/L$. For the above $a, b$-values, eqs.(16,17) give $f_\mu = (\gamma - 2r)\delta_{\mu\mu}$, $f_{\mu\mu} = 2rc_p(\gamma - rc_p)$. Choosing again $z_\mu = z_{\mu\mu} = 1$, we get $X^{-1} = 2N[\gamma + 2\gamma r - 2(t + 1)^2]$. Substituting the above in eq.(19) gives

$$Y = 2\beta N \int dr X$$

$$= \frac{\beta N}{2\gamma(t+1)} \ln \frac{2(t+1)\zeta + q_+ L}{2(t+1)\zeta - q_- L} + \text{constant}$$

(23)

with $q_\pm = \sqrt{\gamma t^2 + (t+1)\gamma} \pm \gamma t$. The parameter $t$, being a function of dimensionality as well as boundary conditions of the region, its presence in eq.(23) implies the dependence of $Y$ on the conditions. However the most significant dimension-dependence of $Y$ comes through the localization length $\zeta$ which varies with dimension of the region (see [12]). Further, as steady state of the evolution ($Y \to \infty$) occurs at length scales $\zeta \sim \gamma L$, the approach to equilibrium is sensitive to system-dimension.

The parametric space transformation $(a, b) \to y$ maps the probability density $\rho(M, a, b)$ to $\rho(M, y(a, b))$. As a result, $\rho$ depends on various parameters $y_j$, $j = 1 \to N$ with $N$ as the number of non-zero elements in set $(a, b)$. However the diffusion, generated by the operator $L$ in the matrix-space $M$, is governed by $Y \equiv y_1$ only; the rest of them, namely, $y_j$, $j > 1$ remain constant during the evolution. Note it is always possible to define a transformation from the set $(a, b) \to y$ with $y_j$, $j > 1$ as the constants of dynamics generated by $L$. This can be explained as follows. A matrix element, say $M_{\mu}$, describes how a basis state $\psi_i$ interacts with state $\psi_j$ through $M$. This results in dependence of the matrix element correlations and, thereby, of the set $(a, b)$, on the basis parameters e.g. basis indices. As the basis is kept fixed during the evolution, the suitable functions of basis parameters can be chosen to play the role of $y_j$, $j > 1$. (Note a similar transformation has been used to obtain a single parametric evolution of multi-parametric Gaussian ensembles of Hermitian matrices; see [13] for details).

The eq.(18) forms the basis of our results obtained in next few sections. As clear, it is based on the Gaussian assumption for the probability density of the transfer matrices. The complexity of the region may however give rise to non-negligible higher order matrix element correlations which leads to non-Gaussian behavior of the $\rho(M)$. In this context, it is useful to note that the evolution of the density $\rho(H)$ of the Hermitian matrices $H$ is also known to be governed by an equation similar to eq.(18) (with $M$ replaced by $H$ and a different form of $Y$); the $\rho(H)$ can be Gaussian or non-Gaussian (see [18] for the proof). Following similar steps as used for the non-Gaussian $\rho(H)$, the eq.(18) can also be proved for a non-Gaussian density of transfer matrices e.g. for $\rho(M) \propto e^{f(M)}$, with $f(M)$ now containing products of $r$ matrix elements, $r = 1 \to n$.

### III. EVOLUTION OF TRANSMISSION EIGENVALUES DUE TO CHANGING COMPLEXITY

The single parametric distribution $P(\lambda, Y)$ of the transmission eigenvalues can, in principle, be obtained by substituting $M = UT(\lambda)V$ in the solution $\rho(M, Y)$ of eq.(18), followed by an integration over matrices $U, V$. However, again, as in the case of eq.(7), the integration route offers many technical difficulties. This again motivates us to use the differentiation option which has, as discussed later, other advantages too.

In this paper, we use the evolution of $M$, given by eq.(18), to show a single parametric diffusion of the transmission eigenvalues. The steps are given as follows. The eq.(18) gives the first and second moments of the matrix element components $M_{klt} \equiv M_{\mu}$ as (appendix C)

$$\delta M_{klt} = -\gamma M_{klt}\delta Y,$$

$$\delta M_{klt} \delta M_{mnst} = \beta \delta_{km} \delta_l \delta_{ns} \delta Y$$

(24)

with $\delta$ implying an average of $f$ over random noise. (For clarification, we again refer a matrix element component as $M_{klt}$. The above moments can further be used to determine the moments of matrix elements $A_{mn} = \sum_{k=1}^{2N} M_{nk} M_{nk}^*$ (see appendix D for details).

$$\delta A_{mn} = -2\gamma A_{mn}\delta Y$$

$$\frac{\delta A_{mn} \delta A_{mn}^*}{(\delta A_{mn}^* - 2[\beta + (2-\beta)\delta_{mn} + (4-\beta)\delta_{mn+n}])} (A_{mn} + A_{nn}) \delta Y$$

(25)

with $\beta = 1$ in presence of the time-reversal symmetry and $\beta = 2$ in its absence.

As mentioned before, the matrix $A$ has $2N$ eigenvalues $x_p$, $1 \leq n \leq 2N$ which form inverse pairs. For clarification, let us label the eigenvalues such that $x_{n+N} \equiv x_n^{-1}$. Using $2^{nd}$ order perturbation theory for Hermitian matrices, the change $\delta x_n$ of the eigenvalues $x_n$ due to a small perturbation $\delta A$ of the matrix $A$ can be given as
\[
\delta x_n = \delta A_{nn} + \sum_{m \neq n} \frac{|\delta A_{mn}|^2}{x_n - x_m} + O((\delta A_{mn})^3)
\]

The above equation can then be used to obtain the first and second moments of the eigenvalues \(x_n, n = 1 \rightarrow N\)

\[
\begin{aligned}
\delta x_n &= F_n(x) \delta Y, \\
\delta x_n \delta x_m &= 8 x_n \delta nm \delta Y
\end{aligned}
\]

with \(F_n = -2 \gamma x_n + 2 \sum_{m \neq n} [(\beta + (4 - \beta) \delta_{nm,n+N})(x_n + x_m)(x_n - x_m)^{-1}\) (appendix E). The eq.(26) along with the relations \(\lambda_n = \frac{(x_n-1)^2}{4x_n}, \delta \lambda_n = \frac{(x_n^2-1)}{4x_n} \delta x_n\) gives the first and second moments of the transmission eigenvalues \(\lambda_n\) (appendix E).

\[
\begin{aligned}
\frac{\delta \lambda_n \delta \lambda_m}{\delta nm \delta Y} &= 8 \frac{\lambda_n(\lambda_n + 1)}{x_n} \delta \lambda_n, \\
\frac{\delta \lambda_n}{\delta \lambda_n} &= -4E_n(\lambda_n) \delta Y
\end{aligned}
\]

where

\[
E_n(\lambda_n) = \frac{1}{x_n^2} \sum_{n=1}^N \frac{1}{\lambda_n^2} \left[ \frac{1}{2} \frac{\partial (\delta \lambda_n \delta \lambda_m)}{\partial \lambda_n} P + (-\delta \lambda_n) \right] P
\]

\[
= 4 \sum_{n=1}^N \frac{\partial}{\partial \lambda_n} \left[ \frac{\partial (x_n^{-1} \lambda_n(1 + \lambda_n) P)}{\partial \lambda_n} + E_n P \right]
\]

Equation (29) describes the evolution of the transmission eigenvalues for a disordered region with respect to its changing complexity [15]. The reasons for a change in complexity can be manifold, for example, due to change in length or disorder of the region, the scattering and boundary conditions. As a result, eq.(29) is not only valid for higher dimensions and beyond weak scattering limits, it can also describe the influence of various system parameters on transport; the information about all of them enters in the probability distribution through a single parameter \(Y\). Note \(Y\) can also be expressed as a function of the localization length \(\zeta\) measured in units of system length \(L\). As a result, eq.(29) can be explained as follows. A single parametric formulation of the diffusion in the matrix space \(M\) implies a same parametric dependence of the evolutions in the eigenvalue as well as eigenfunction space. This allows a complexity parameter formulation of the eigenfunction correlations e.g. \(<\psi_n(r)\psi_n(r')>= f(r,r'; Y)\) (with \(\psi_n(r)\) as \(n^{th}\) eigenstate of the transfer matrix at space point \(r\) of the disordered region). As the localization length \(\zeta\) can be expressed in terms of eigenfunction correlation \(f(r,r'; Y)\), this results in \(Y\) dependence of \(\zeta\).

It is worth comparing eq.(29) with DMPK equation. The latter describes the evolution of transmission eigenvalues due to changing length of the region in the diffusive regime and for quasi one-dimensional systems. The DMPK equation has been obtained by assuming the elements of the transfer matrix of a small length \(L\) of the region as independently distributed: \(\rho_{\text{DMPK}}(M) \propto e^{-(a_l/2L)} \rho M \rho M^\dagger\) with \(a = \beta N + 2 - \beta\). Note that \(\rho_{\text{DMPK}}\) is a special limit of eq.(7), equivalent to a choice of all \(a_\mu \rightarrow 0, b_{\mu\mu'} \rightarrow (a_l/L) \delta_{\mu\mu'}\). Consequently the complexity parameter \(Y\) for \(\rho_{\text{DMPK}}\) turns out to be same as eq.(21) with \(q = (a_l)^{-1}\). In the limit \(L < Nl, Y\) reduces in a form similar to the evolution parameter of DMPK equation: \(Y \rightarrow L/2a_l\). However the right side of eq.(29) is still slightly different from that of the DMPK equation. By further considering the strongly transmitting channels limit, namely, \(\lambda_n \ll 1\) or \(x_n \approx 1\), eq.(29) reduces to DMPK equation

\[
\begin{aligned}
\frac{\partial P}{\partial Y} &= 4 \sum_{n=1}^N \frac{\partial}{\partial \lambda_n} \left[ \frac{\partial (\lambda_n(1 + \lambda_n) P)}{\partial \lambda_n} - (2\lambda_n + 1) P - \beta \sum_{m \neq n} \frac{\lambda_n(1 + \lambda_n) P}{\lambda_n - \lambda_m} \right]
\end{aligned}
\]

Thus, for strongly transmitting channels and with length as the only changing parameter, eq.(29) describes the same statistical properties of the transmission eigenvalues as the DMPK equation. However, as eq.(30) indicates, if the
system parameters other than length are subjected to change, eq.(29) has a different evolution parameter even in strongly transmitting channels limit although its λ-dependent terms remain same as in the DMPK equation. The latter similarity is useful for the following reason. As many results for the length-dependence of transport properties e.g. conductance using DMPK equation have already been derived, a replacement of length by Y in the known results can then give us the dependence of properties on other system parameters too (in limit λn << 1 only).

IV. HAMILTONIAN FORMULATION OF THE EIGENVALUE-DYNAMICS

The distribution of transmission eigenvalues and their correlations can be used to determine the statistical behavior of various transport properties. For example, the conductance G is given by

\[ G = \sum_{n=1}^{N} \frac{1}{1 + \lambda_n} \] (in units of dimensionless conductance G)

\[ G \] can then give us the dependence of properties on other system parameters too (in limit λn << 1 only).

A knowledge of the solution of eq.(29) is therefore very desirable. Although the solution is not known so far, still we can gain a deeper insight about eigenvalue correlations by writing eq.(29) in a Schrodinger equation form. The required steps are as follows. A transformation of the eigenvalues λn = sinh2(λµn) reduces eq.(29) to a Fokker-Planck equation with constant diffusion coefficient,

\[ \frac{\partial P}{\partial Y} = \sum_{n=1}^{N} \left( \frac{\partial P}{\partial \mu_n} + \beta \frac{\partial \Omega(\mu_n)}{\partial \mu_n} \right) \] (32)

where \( \Omega(\mu_n) = -\sum_{n>m} \log|\Delta(r_n, r_m)| - \beta^{-1} \sum_{n} \log(\sinh2r_n - \gamma \mu_n + r_n) + (N(N-1)ln2)/2 \) with \( \Delta(r_n, r_m) = (\cosh2r_n - \cosh2r_m) \) and \( r_n = \log\mu_n \). By using \( P(\{\mu\}, Y) = e^{-\beta\Omega/2}\psi(\{\mu\}, Y) \), eq.(32) can be mapped to a Schrodinger equation

\[ \frac{\partial \psi}{\partial Y} = \hat{H}\psi \]

(33)

with \( \hat{H} \) as the "Hamiltonian" governing the eigenvalue dynamics:

\[ \hat{H}(\{\mu\}) = \sum_n \frac{\partial^2}{\partial \mu_n^2} + V(\{\mu\}) \] (34)

\[ V(\{\mu\}) = -\frac{\beta}{2} \sum_n \left( \frac{\beta}{2} \left( \frac{\partial \Omega}{\partial \mu_n} \right)^2 - \frac{\partial^2 \Omega}{\partial \mu_n^2} \right) \]

\[ = \sum_n \frac{1}{\mu_n^2} \left[ \frac{1}{4} (2\gamma \mu_n^2 - \gamma \mu_n^4 - 3) + \gamma \mu_n^2 \coth2r_n + \frac{1}{\sinh2r_n} + \sum_m U_{mn} \right] , \]

\[ U_{mn} = \left[ (4 + \gamma \mu_n^2) \frac{\sinh2r_n}{\Delta(r_n, r_m)} - \beta (4 - \gamma \mu_n^2) \frac{\sinh2r_n}{\Delta^2(r_n, r_m)} - \beta^2 \sum_{j,j\neq m,n} \frac{\sinh2r_n}{\Delta(r_n, r_m)\Delta(r_n, r_j)} \right] \] (35)

The Schrodinger equation formulation of eq.(29) reveals an important aspect of the eigenvalues, namely, the presence of a three body interaction terms; (note it could have been converted into a two body term in absence of the prefactor \( \mu_n^2 \)). It is well-known that the "Hamiltonian" appearing in the Schrodinger equation formulation of the DMPK equation contains only two body interaction terms [1,16]. Thus it appears that the presence of significant multiple channel interactions leads to three body eigenvalue correlations while the isotropic channel limit (used in DMPK case) restricts it to pairs of eigenvalues. The different "Hamiltonian" in the two cases result in a significant difference in their states \( \psi \) and therefore \( P(\{\mu\}) \). As a result, the theoretical prediction of the physical properties under multiple channel interactions are expected to differ significantly from the predictions based on isotropic channel considerations; the suggestion is in accordance with the already known failure of DMPK equation beyond metallic regime or for two and higher dimensions.
V. EVOLUTION OF AVERAGES

The presence of many body interactions among eigenvalues makes it difficult to calculate $P(\{\lambda\})$ under general scattering conditions. In the metallic regime, however, the fluctuations of transport properties are much weaker than the average behavior and a knowledge of their averages (instead of full distribution) is sufficient. The evolution of transmission eigenvalues can then be used to study the effect of changing complexity on the average behavior of their various functions. Consider a function $F(\{\lambda\})$ not explicitly dependent on the parameter $Y$. The multiplication of $F$ on both sides of eq.(29), followed by partial integration, leads to following behavior of $< F >= \int F(\{\lambda\})P(\{\lambda\})d\{\lambda\}$ with $< . >$ implying an ensemble average:

$$\frac{\partial < F >}{\partial Y} = 4 \left( \sum_{n=1}^{N} \left[ \lambda_n (1 + \lambda_n) x_n^{-1} \frac{\partial^2 F}{\partial \lambda_n^2} - E_n \frac{\partial F}{\partial \lambda_n} \right] \right)$$  \hspace{1cm} (36)

For example, evolution of the moment $T_1^p = (\sum_n (1 + \lambda_n)^{-1})^p$, due to changing complexity in the strongly transmitting channel limit, can be obtained by substituting $F = T_1^p$ in eq.(36) and using the limit $\lambda_n << 1$:

$$\frac{\partial < T_1^p >}{\partial Y} = -2p \left[ \beta < T_1^{p+1} > - (\beta - 2) < T_1^{p-1} T_2 > - 2(p-1) < (T_2 - T_3) T_1^{p-2} > \right]$$ \hspace{1cm} (37)

where notation $T_1^p$ implies $y^{th}$ power of the moment $T_x = \sum_n (1 + \lambda_n)^{-x}$. A substitution of $p = 1, 2$ in eq.(37) then gives the complexity dependence of the average conductance $< G > = < T_1 >$ and its variance $\sigma^2 = < T_1^2 > - < T_1 >^2$, respectively,

$$\frac{\partial < G >}{\partial Y} = -2 \left[ \beta < G^2 > - (\beta - 2) < T_2 > \right]$$ \hspace{1cm} (38)

$$\frac{\partial \sigma^2}{\partial Y} = -4 \left[ \beta (< G^3 > - < G > < G^2 >) - (\beta - 2) (< M_2 > < G > + < M_2 G >) - 2 < (T_2 - T_3) > \right]$$ \hspace{1cm} (39)

In general, the studies of fluctuations in transport properties about their average behavior require a knowledge of the averages of various combinations of moments e.g. $< T_1^p T_2^q >$. A substitution of $F = T_1^p T_2^q$ in eq.(36) leads to a hierarchy of coupled set of equations for the average behavior of the moments. The details of these calculations with DMPK equation as a basis are given in appendix B of [10]; the replacement of the length parameter by complexity parameter in eq.(B8) of [10], gives us the evolution of $< T_1^p T_2^q >$ with respect to changing complexity of a system in strongly transmitting channel limit.

The coupled form of eqs.(38,39) or eq.(B8) of [10] makes it difficult to obtain the exact behavior of the moments, even in $\lambda_n << 1$ limit. However, by assuming the number $N$ of the channels very large, the equations can be solved by using $N^{-1}$ expansion of the moments, see [10] for details. For example, the evolution of $T_1^p$ reduces to, in the leading order of $N$,

$$\frac{\partial < T_1^p >}{\partial Y} = -2p \left[ \beta < T_1^{p+1} > \right] + o(N^p)$$ \hspace{1cm} (40)

In ballistic limit, the eigenvalues $\lambda_n \rightarrow 0$ (for $n = 1 \rightarrow N$) which gives $< T_1^p > = N^p$. For ballistic case as the initial state of the disordered region, that is, $\lim Y \rightarrow Y_0 < T_1^p > = N^p$, the solution of eq.(40) can be given as

$$< T_1^p > = N^p (1 + \Lambda)^{-p} + o(N^{p-1})$$ \hspace{1cm} (41)

with $\Lambda = 2\beta N|Y - Y_0|$. Following eq.(21), $\Lambda = qL/2$ for almost same strength of interactions between any two channels and length as the only changing parameter (for $\gamma qL < 1$). The behavior of $< T_1^p >$ with respect to the above $\Lambda$ coincides with one obtained by using DMPK equation for which $q^{-1} = (\beta N + \beta - 2)l$ (see eq.(3.31) of [1]). The substitution of $p = 1$ in eq.(41) gives $Y$ dependence of the average conductance $< G > = < T_1 > \approx NA^{-1} + o(N^0)$ (in large $N$ limit). The average conductance of a wire therefore decreases linearly with increasing length which is in agreement with the already known results [1].

Equations (22,23) can be used to study the effect of changing complexity on the conductance fluctuations in the diffusive regime. For example, the weak localization correction $\delta G = < G > - (1 + \Lambda)^{-1}$ to the average conductance as well as the variance $\sigma^2 = < G^2 > - < G >^2$ can be obtained by expanding the moments to order $N^{-1}$:

$$\delta G = (1 - 2/\beta) \Lambda^3/3(1+\Lambda)^3 + o(N^{-1})$$ \hspace{1cm} (42)

$$\text{Var } G = (2/15\beta)[1 - (1 + 6\Lambda)/(1 + \Lambda)^6] + o(N^{-1}).$$ \hspace{1cm} (43)
(Above results were calculated in [10] for length variation. Due to similarity of eq.(30) with DMPK equation in limit \( \lambda_0 \ll 1 \), same results can be used to describe the complexity dependence of conductance fluctuations just by replacing \( L/l \) by \( \Lambda \). The diffusive limit \( \Lambda \to \infty \) of the above results give \( \delta G = (1 - 2/\beta)/3 \) and \( \text{Var} G = (2/15\beta) \) which is in agreement with expected universal behavior of the conductance fluctuations in this limit (see [1] for the latter).

As mentioned above, eqs.(37-43) are valid only in the limit \( \lambda_n \ll 1 \). A detailed analysis of the dependence of transport properties on \( \Lambda \), beyond strongly transmitting channel limit, is still under investigation; the results will be given elsewhere.

VI. SINGLE PARAMETER SCALING OF TRANSPORT PROPERTIES

A distribution \( P(X) \) of a physical quantity \( X \) that depends on system size \( N \) and a set of system parameters \( \{alpha_n\} \) obeys one parameter scaling if it is a function of only \( X \) and one scale dependent parameter: \( P(X; L; \{alpha\}) = f(X; Y) \) [21]. Thus the system-information appears in the distribution only through the scaling parameter \( Y \equiv Y(N, \{alpha\}) \).

Following the discussion given in sections II, III, it is clear that the distribution of transmission eigenvalues satisfies the above requirement, that is, \( P(\{lambda\}; N; a, b) = P(\{lambda\}; Y) \), with complexity parameter \( Y = Y(N, a, b) \) playing the role of scaling parameter. As a consequence, the distribution \( P_F(F) \) of any transport property \( F(\{lambda\}) \), would also follow a single parameter scaling: \( P_F(F; Y) = \int \delta(F - F(\{lambda\})))P(\{lambda\}; Y) \prod_{j=1}^{N} d\lambda_j \)

The \( Y \) governed evolution of the distribution \( P_F(F; Y) \) can be obtained with the help of eq.(29). The evolution as a function of \( Y \), however, is abrupt in large \( N \)-limit (see, for example, eqs.(40)); a smooth crossover can be seen only in terms of a rescaled parameter \( \Lambda \) where

\[
\Lambda = \frac{Y - Y_0}{D}
\]

with \( D \) as the local mean level spacing of transmission eigenvalues \( \lambda_n \). (Note the relation \( \Lambda = 2\beta N[Y - Y_0], \) given in section V, is valid only in the strongly transmitting channel regime).

The parameter \( \Lambda \) governs the statistical behavior of the transport properties. As discussed in section I, the dimensional-sensitivity of the interaction between various channels can influence both \( Y \) and \( D \). This in turn leads to dimensional-dependence of \( \Lambda \). The statistical behavior of transport properties e.g. conductance is therefore expected to be different for higher dimensions. The size-dependence of transport properties and influence of scattering conditions on them can also be explained by \( Y \) and therefore \( \Lambda \)-dependence on these system conditions.

The one-parameter scaling behavior of the distribution \( P(\{lambda\}) \) implies the existence of a universal distribution \( P^*(\{lambda\}) = \lim_{N \to \infty} P(\{lambda\}, \Lambda) \) at a critical point which is fixed by the critical value \( \Lambda^* = \lim_{N \to \infty} \Lambda(N) \). Thus the size-dependence of \( \Lambda \) plays a crucial role in locating the critical point of statistics. It is possible that various system-conditions in a disordered region may result in different \( N \)-dependence of \( |Y - Y_0| \) and \( D \), say, \( |Y - Y_0| \propto N^\alpha \) and \( D \propto N^\eta \) which gives \( \Lambda \propto N^{\alpha-\eta} \). In regions connected with leads of finite width (implying finite \( N \) ), a variation of size \( N \), therefore, leads to a smooth crossover of statistics between an initial state (\( \Lambda \to 0 \)) and the equilibrium (\( \Lambda \to \infty \)); the intermediate statistics belongs to an infinite family of ensembles, parameterized by \( \Lambda \) and given by the solution of eq.(29). However, for system-conditions leading to \( \alpha = \eta \), the statistics becomes universal for all sizes, \( \Lambda \) being \( N \)-independent; the corresponding system conditions can then be referred as the critical conditions.

As clear from above, a variation of system conditions (i.e. changing \( \alpha, \eta \)) for a given finite system size, would also lead to a crossover of statistics. In large \( N \) limit, however, the variation causes a transition of statistics. Based on relative values of \( \alpha \) and \( \eta \), the statistics for large system sizes can be divided into three different types

Case \( \alpha < \eta \): This corresponds to \( \Lambda \to 0 \) in limit \( N \to \infty \). The system under variation of such conditions rapidly approaches its initial state.

Case \( \alpha > \eta \): This gives \( \Lambda \to \infty \) for infinite system sizes; as a result, even a small perturbation of the system will cause an abrupt transition from the initial state to the equilibrium.

Case \( \alpha = \beta \): Here \( \Lambda \) being size-independent, the case corresponds to the critical point of evolution. Further, due to finite and non-zero \( \Lambda \), the eigenvalue statistics for this case is different, even in limit \( N \to \infty \), from both, the initial state as well as the equilibrium. Note, however, the statistics is same as the one at the critical point of finite systems.

As clear from preceding sections (III-V), the \( \Lambda \)-formulation of transport properties (e.g. eqs.(29-43)) is based on eq.(18). Although this paper contains the derivation of eq.(18) only for Gaussian \( \rho(M) \), but, for the reasons mentioned in section II, we expect it to be valid for non-Gaussian cases too. This suggests the application of our results for a wide range of disordered regions. Also note that eq.(18) and eq.(29) are valid under inelastic scattering conditions too.
However the connection between the transport properties and $\lambda$’s needs to be revised. For example, the Landauer’s formula $G = \sum \alpha_n (1 + \lambda_n)^{-1}$ is not applicable in the inelastic regime; the results for Conductance fluctuations given in section V are therefore not valid for that regime.

The validity of eqs.(18,29) for non-Gaussian cases as well as inelastic conditions suggests the existence of a $\Lambda$-formulation of transport properties for regions with generic scattering and disorder conditions. Although the functional form of $\Lambda$ is expected to be different under generic conditions, the critical statistics would still occur at system conditions satisfying the critical criteria $\alpha = \beta$, thus leading to an $N$-independent $\Lambda$. It should be stressed, however, that the critical criteria may not be fulfilled for all disordered systems; such systems will not show any transition in the statistics of transmission eigenvalues.

VII. CONCLUSION

In summary, we find that, under generic scattering conditions, the eigenvalues are subjected to many body interactions besides logarithmic repulsion. As a consequence, the correlations among eigenvalues and thereby the fluctuations of transport properties beyond metallic regime are significantly different from those within the regime. An explicit formulation of the fluctuations can be obtained by the knowledge of the solution of eq.(29) (or eq.(33)). Although a complete solution is not known so far, nonetheless eq.(29) reveals, for the first time, a very important characteristic of transport phenomena through disordered regions, namely, their dependence on a single complexity parameter $Y$. This implies that the disordered regions with different system parameters e.g., boundary conditions, scattering conditions etc. will show same fluctuations of transport properties if their parameters $\Lambda$ correspond to a same value. This implies a deeper level of universality lying underneath the transport phenomena, even beyond metallic regime (where it was known to exist so far). A knowledge of universality is useful from a practical viewpoint, besides other reasons. This is because the information about the transport properties of a simple disordered region (obtained by some other technique) can now be used to determine those for a complicated region, just by comparing their complexity parameters. Further the dependence of distribution of the transmission eigenvalues on various system parameters through a single parameter $Y$ also indicates the validity of one parameter scaling theory of localization for statistical behavior of the conductance (also for those transport properties which can be formulated as the functions of transmission eigenvalues).

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APPENDIX A: PROOF OF EQUATION (13)

Using eq.(7), a derivative of $\rho_0 = \rho/C$ with respect to $M_\mu \equiv M_{kl;s}$ can be written as

$$\frac{\partial \rho_0}{\partial M_\mu} = -\frac{2}{\beta} \left( \sum_{\mu'} b_{\mu\mu'} M_{\mu'} + a_\mu \right) \rho_0$$  \hspace{1cm} (A1)

where $b_{\mu\mu'} = b_{\mu'\mu}$. Further

$$G_{\mu\mu'} \frac{\partial \rho_0}{\partial b_{\mu\mu'}} = -\frac{2}{\beta} M_{\mu} M_{\mu'} \rho_0,$$  \hspace{1cm} (A2)

$$\frac{\partial \rho_0}{\partial a_\mu} = -\frac{2}{\beta} M_{\mu} \rho_0,$$  \hspace{1cm} (A3)

where $G_{\mu\mu'} = 1 + \delta_{ij} \delta_{kl}$ if $\mu \equiv \{kl;s\}$, $\mu' \equiv \{ij;s'\}$. The eqs.(A1,A2,A3) can be combined to give

$$\sum_{\mu} M_{\mu} \frac{\partial \rho_0}{\partial M_{\mu}} = \sum_{\mu,\mu'} G_{\mu\mu'} b_{\mu\mu'} \frac{\partial \rho_0}{\partial b_{\mu\mu'}} + \sum_{\mu} a_{\mu} \frac{\partial \rho_0}{\partial a_{\mu}}.$$  \hspace{1cm} (A4)

Similarly
\[ \frac{\partial^2 \rho_0}{\partial M_\mu^2} = \left( \frac{2}{\beta} \right) (2\beta^{-1}a_\mu^2 - b_{\mu \mu}) \rho_0 + \frac{4}{\beta^2} \sum_{\mu'} b_{\mu' \mu'} M_{\mu'} \left( 2a_{\mu'} + \sum_{\mu''} b_{\mu'' \mu'} M_{\mu''} \right) \rho_0 \]
\[ = \left( \frac{2}{\beta} \right) (2\beta^{-1}a_\mu^2 - b_{\mu \mu}) \rho_0 - \frac{2}{\beta} \left( \sum G_{\mu' \mu''} b_{\mu' \mu''} \frac{\partial \rho_0}{\partial b_{\mu' \mu''}} + 2 \sum a_{\mu} b_{\mu' \mu'} \frac{\partial \rho_0}{\partial a_{\mu'}} \right). \tag{A5} \]

Using above equalities,
\[ \sum_{\mu} \frac{\partial}{\partial M_\mu} \left[ \frac{\beta}{2} \frac{\partial \rho_0}{\partial M_\mu} + \gamma M_\mu \rho_0 \right] = \sum_{\mu, \mu'} f_{\mu \mu'} \frac{\partial \rho_0}{\partial b_{\mu \mu'}} + \sum_{\mu} f_{\mu} \frac{\partial \rho_0}{\partial a_{\mu}} + C_\beta \rho_0 \tag{A6} \]

where \( C_\beta = \sum_{\mu} \left[ \gamma + (2\beta^{-1}a_\mu^2 - b_{\mu \mu}) \right]. \) Now interchanging the indices in the terms inside the square brackets in eq.(A10) gives
\[ \sum_{\mu} \frac{\partial}{\partial M_\mu} \left[ \frac{\beta}{2} \frac{\partial \rho_0}{\partial M_\mu} + \gamma M_\mu \rho_0 \right] = \sum_{\mu, \mu'} f_{\mu \mu'} \left[ \frac{\partial \rho_0}{\partial b_{\mu \mu'}} + \frac{\partial \rho_0}{\partial b_{\mu' \mu}} \right] + \sum_{\mu} f_{\mu} \frac{\partial \rho_0}{\partial a_{\mu}} + C_\beta \rho_0 \tag{A7} \]

where \( f_{\mu} \) and \( f_{\mu \mu'} \) are given by eqs.(16,17). Multiplying eq.(A9) by \( C \) and using \( C \rho_0 = \rho \), we get
\[ \sum_{\mu} \frac{\partial}{\partial M_\mu} \left[ \frac{\beta}{2} \frac{\partial \rho}{\partial M_\mu} + \gamma M_\mu \rho \right] = \sum_{\mu, \mu'} f_{\mu \mu'} \left[ \frac{\partial \rho}{\partial b_{\mu \mu'}} + \frac{\partial \rho}{\partial b_{\mu' \mu}} \right] + \sum_{\mu} f_{\mu} \frac{\partial \rho}{\partial a_{\mu}} + C_\beta \rho \tag{A8} \]

The eq.(13) can now be obtained by choosing the normalization condition on \( \rho \) such that \( C \) satisfies the relation \( \sum_{\mu, \mu'} f_{\mu \mu'} \frac{\partial C}{\partial b_{\mu \mu'}} + \sum_{\mu} f_{\mu} \frac{\partial C}{\partial a_{\mu}} = C_\beta C. \)

**APPENDIX B: SINGLE PARAMETRIC FORM OF T: DERIVATION OF EQ.(18)**

Consider a transformation from the \( \tilde{N} \)-dimensional \( \{a, b\} \)-space to another parametric space, say \( y \)-space, consisting of variables \( y_i, i = 1 \to \tilde{N} \) which reduces \( T(a, b) \) (eq.(15)) into
\[ T(y|a, b)|\rho \equiv \frac{\partial \rho}{\partial y_i} \bigg|_{y_1 \to \tilde{N}} \tag{B1} \]

. The conditions to determine desired transformation can be obtained as follows, By using partial differentiation, \( T(b) \) given by eq.(15) can be transformed in terms of the derivatives with respect to \( y \),
\[ T \rho = \sum_k t_k \frac{\partial \rho}{\partial y_k} \tag{B2} \]

where
\[ t_k \equiv \sum_{\mu} f_{\mu} \frac{\partial y_k}{\partial a_{\mu}} + \sum_{\mu, \mu'} f_{\mu \mu'} \frac{\partial y_k}{\partial b_{\mu \mu'}}. \tag{B3} \]

The eq.(B2) can be reduced in the desired form of eq.(B1), if the transformation \( b \to y \) satisfies following condition:
\[ t_k = \delta_{k1} \quad \text{for} \quad k = 1 \to \tilde{N} \tag{B4} \]

According to the theory of partial differential equations (PDE) [20], the general solution of a linear PDE
\[ \sum_{i=1}^{\tilde{N}} P_i(x_1, x_2, \ldots, x_{\tilde{N}}) \frac{\partial Z}{\partial x_i} = R \tag{B5} \]
is $F(u_1, u_2, ..., u_n) = 0$ where $F$ is an arbitrary function and $u_i(x_1, x_2, ..., x_n, Z) = c_i$ (a constant), $i = 1, 2, ..., n$ are independent solutions of the following equation

$$
\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_k}{P_k} = \ldots = \frac{dx_N}{P_N} = \frac{dZ}{R} \tag{B6}
$$

Note the function $F$ being arbitrary, it can also be chosen as

$$F \equiv \sum_j (u_j - c_j) = 0 \tag{B7}$$

The equations for various $y_j$ in the set of eq.(B4) are of the same form as eq.(B5) and, therefore, can be solved as described above. Let us first consider the equation for $y_1$; its general solution can be given by a relation $F(u_1, u_2, ..., u_N) = 0$ where function $F$ is arbitrary and $u_j$ are the functions of $N$ parameters of set $\{a, b\}$ such that $u_j(\{a, b\}, y_1) = c_j$ (with $c_j$’s as constants). The solutions $u_j$ are the independent solutions of the equation

$$
\frac{db_{\mu\mu'}}{f_{\mu\mu'}} = \ldots = \frac{da_\mu}{f_\mu} = \ldots = dy_1 \tag{B8}
$$

where the equality between ratios is implied for all possible indices $\mu, \mu'$ (with the total number of ratios as $N + 1$). It is easy to see that each of the above ratios is equal to $\sum_\mu z_\mu f_\mu + \sum_\mu z_\mu f_\mu'$, where $z_\mu', z_{\mu\mu'}$ are arbitrary functions. The eq.(B8) can therefore be rewritten as

$$
y_1 = \sum_\mu z_\mu f_\mu + \sum_\mu z_\mu f_\mu' + \sum_\mu z_{\mu\mu'} \tag{B9}
$$

A solution, say $u_1$ of eq.(B9), or alternatively eq.(B8), can now be obtained by choosing the functions in the set $z$ such that the right side of the above equation becomes an exact differential:

$$u_1 \equiv y_1 - \sum_\mu \int da_\mu z_\mu X + \sum_\mu \int db_{\mu\mu'} z_{\mu\mu'} X = constant \tag{B10}
$$

where $X = [\sum_\mu z_\mu f_\mu + \sum_\mu z_{\mu\mu'} f_{\mu\mu'}]^{-1}$. The general solution for $y_1$ can therefore be given by a combination of all possible functions $u$ obtained by using arbitrary set of $z$-functions. It can be shown that each such solution differs from the other only by a constant: $u_j = u_i + constant$ (this is due to equality of the two ratios obtained by choosing two different sets $z$ of the functions). The $y_1$ can therefore be written as follows,

$$y_1 = \sum_\mu \int da_\mu z_\mu X + \sum_\mu \int db_{\mu\mu'} z_{\mu\mu'} X + constant \tag{B11}
$$

which gives eq.(19) for $Y \equiv y_1$.

The set of equations (B4) can similarly be solved for other $y_j$ ($j > 2$). For example, the solution for $y_k$ can be given by the function $F_k(v_1, ..v_M) = 0$ where $v_j(\{a, b\}, y_k) = constant$ are the independent solutions of following equality

$$
\frac{db_{\mu\mu'}}{f_{\mu\mu'}} = \ldots = \frac{da_\mu}{f_\mu} = \ldots = \frac{dy_k}{0} \tag{B12}
$$

A solution, say $v_1$, of eq.(B12) can now be given as

$$v_1 \equiv y_k - \sum_\mu \int db_{\mu\mu'} z_{\mu\mu'} - \sum_\mu \int da_\mu z_\mu = constant \tag{B13}
$$

where the set $z$ is a set of arbitrarily chosen functions which satisfy the condition

$$\sum_\mu z_{\mu\mu'} f_{\mu\mu'} + \sum_\mu z_\mu f_\mu = 0 \tag{B14}
$$

As obvious, one possible choice for $z$ functions satisfying the above condition is $z_{\mu\mu'} = 0, z_\mu = 0$ for all $\mu, \mu'$ which gives $y_k = constant$. As each solution of eq.(B12) is different from the other only by a constant, the general solution for $y_k, k > 1$, can now be given as

$$y_k = \sum_\mu \int db_{\mu\mu'} z_{\mu\mu'} + \sum_\mu \int da_\mu z_\mu + constant. \tag{B15}$$
APPENDIX C: PROOF OF EQUATION (7)

The eq.(7) can be obtained from eq.(18), by following standard routes of calculation of the moments from a Fokker-Planck equation; see [17,14] for details. For completeness sake, we describe here one such route.

Let \( \rho(M - \delta M; Y) \) be the joint probability density that the elements of transfer matrix will be at the positions \( M - \delta M = \{ M_\mu - \delta M_\mu \} \) at parameter value \( Y \). At \( Y + \delta Y \), let the positions of the elements change to \( M = \{ M_\mu \} \). The changes \( \delta M_\mu \), for all \( \mu \) values, are random variables and different, in general, for every member of the ensemble. Expanding \( \rho(M - \delta M; Y) \) on both sides of eq.(C1) in a power series of \( \delta M \) and \( \delta Y \) around \( \rho(M, Y) \), one obtains eq.(24). The latter can also be derived by a direct integration of eq.(C3).

\[
\rho(M; Y + \delta Y) = \int d(\delta M) \psi(M - \delta M, Y|M, Y + \delta Y) \rho(M - \delta M; Y)
\]  

(C1)

where \( \psi \) is the probability that the position of the transfer matrix will change from \( M - \delta M \) to \( M \) in a parametric interval \( \delta Y \). Expanding \( \rho \) on both sides of eq.(C1) in a power series of \( \delta M \) and \( \delta Y \) around \( \rho(M, Y) \), one gets

\[
\rho(M, Y) + \frac{\partial \rho}{\partial Y} \delta Y + o((\delta Y)^2) = \int d(\delta M) \psi(M, Y) \rho(M, Y)
\]

(C2)

Now by using the definition \( \overline{\delta M} = \int (\delta M)^n \psi(M - \delta M, Y|M, Y + \delta Y) d(\delta M) \) and the equality \( \int \psi(M - \delta M, Y|M, Y + \delta Y) d(\delta M) = 1 \), the eq.(C2) can be reduced to

\[
\frac{\partial \rho}{\partial Y} = \frac{1}{2} \sum_{\mu, \mu'} \left[ (\delta M_\mu \delta M_{\mu'}) + \frac{\partial^2 \rho}{\partial M_\mu \partial M_{\mu'}} \right] - \sum_{\mu} \frac{\partial \rho}{\partial M_\mu}
\]

(C3)

By comparing eq.(C3) with eq.(18) one obtains eq.(24). The latter can also be derived by a direct integration of eq.(18); see [17] for details.

APPENDIX D: PROOF OF EQUATION (25)

The matrix \( A \) is defined by \( A = M.M^+ \) with its elements \( A_{mn} = \sum_{k=1}^{2N} M_{mk}^* M_{nk} \). The average of a small change \( \delta A \) over random noise can then be expressed in terms of \( \delta M \):

\[
\overline{\delta A_{mn}} = \sum_{k=1}^{2N} \left[ M_{mk} \delta M_{nk}^* + M_{nk}^* \delta M_{mk} \right]
\]

(D1)

\[
= -\gamma \sum_{k=1}^{2N} \left[ M_{mk} M_{nk}^* + M_{nk}^* M_{mk} \right] = -2\gamma A_{mn} \delta Y.
\]

(D2)

Here eq.(D2) is obtained from eq.(D1) by using the relation (24).

The current conservation condition on the transfer matrix \( M \), given by eq.(2), introduces various relations between its matrix elements:

\[
\sum_{k=1}^{N} \left[ M_{nk} M_{mk}^* - M_{nk+n} M_{m,k+N}^* \right] = g \delta_{nm}
\]

(D3)

\[
\sum_{k=1}^{N} \left[ M_{nk}^* M_{mk+n} - M_{nk+n} M_{m,k+N}^* \right] = 0
\]

(D4)

with \( g = 1 \) if \( n \leq m \) and \( g = -1 \) for \( n > m \). The presence of time-reversal symmetry alongwith current conservation, further implies that \( M_{mk} = M_{mk+n}^* |k+N| \) (\( |x| = x \) if \( x \leq 2N \), \( |x| = x-2N \) if \( x > 2N \)). As a result, the second moment of \( \delta A_{mn} \) depends on the presence or absence of time-reversal symmetry in the disordered region and is different for the cases \( m = n \), \( m \neq n \) and \( m = n + N \).
1. Case $m \neq n, m \neq n + N$

$$\delta A_{mn}\delta A_{nm} = 2^N \sum_{k,l=1}^{2N} \{M_{mk}\delta M_{nk}^* + M_{nk}\delta M_{mk}^*\} \{M_{ml}\delta M_{nl} + M_{nl}\delta M_{ml}^*\}$$

$$= 2^N \left[ M_{mk}M_{ml}(\delta M_{nk}^*\delta M_{nl}) + M_{nk}M_{nl}(\delta M_{mk}\delta M_{ml}^*) + M_{mk}M_{ml}(\delta M_{nk}^*\delta M_{nl}) + M_{nk}M_{ml}^*(\delta M_{mk}\delta M_{ml}) \right]$$

(D5)

Following eq.(24), we have

$$\langle \delta M_{jk}\delta M_{rl} \rangle = \sum_{s_1,s_2} i^{s_1+s_2-2}(\delta M_{jk};s_1\delta M_{rl};s_2) = 0$$

$$\langle \delta M_{jk}\delta M_{rl} \rangle = \sum_{s_1,s_2} (-1)^{s_1-i^{s_1+s_2-2}(\delta M_{jk};s_1\delta M_{rl};s_2) = 2\beta\delta_{jr}\delta_{kl}\delta Y$$

(D6)

Consequently the last two terms inside the square bracket $\{\}$ in eq.(D6) do not contribute in this case and, with help of eq.(24), we get

$$\delta A_{mn}\delta A_{nm} = 2\beta\delta Y \sum_{k,l} \{\delta_{kl}M_{mk}M_{ml}^* + \delta_{kl}M_{nk}^*M_{ml}\}$$

$$= 2\beta[A_{mm} + A_{nn}]\delta Y$$

(D7)

2. Case $m = n + N$

Using current conservation condition given by eq.(D4), we can write

$$A_{nn+N} = 2\sum_{k=1}^{N} M_{nk}M_{n+Nk}^*$$

(D10)

The above equation gives

$$\delta A_{n,n+N}\delta A_{n,n+N}^* = 4 \sum_{k,l=1}^{N} \left[ M_{n+Nk}\delta M_{n+Nl}^*(\delta M_{mk}\delta M_{nl}) + M_{nk}M_{nl}(\delta M_{n+Nk}^*\delta M_{n+Nl}) \right.$$  

$$\left. + M_{n+Nl}\delta M_{mk}\delta M_{n+Nk}^* + M_{n+Nl}^*M_{n+Nk}(\delta M_{mk}\delta M_{nl}) \right]$$

(D11)

In absence of time-reversal symmetry, the contribution from the last two terms inside the bracket is zero (see eq.(D7)). However the time-reversal symmetry along with current conservation condition in a region implies $M_{mk} = M_{m+n+N||k+N}^*$. As a result, the last two terms inside the bracket can be rewritten as follows

$$M_{mk}M_{n+Nl}(\delta M_{mk}\delta M_{nl}) = M_{n+Nl}|k+N|M_{n+Nl}\langle\delta M_{mk}\delta M_{nl}\rangle\delta_{l1}\delta_{1l}$$

$$M_{n+Nl}^*M_{n+Nk}\langle\delta M_{mk}\delta M_{nl}\rangle = M_{n,Nl}|k+N|M_{n,Nl}\langle\delta M_{mk}\delta M_{nl}\rangle\delta_{1l}\delta_{l1}$$

(D12)

The presence of time-reversal symmetry as well its absence therefore results in

$$\langle\delta A_{n,n+N}\rangle^2 = 8\beta\delta Y \sum_{k,l=1}^{N} \delta_{kl} \left[ (M_{n+Nk}\delta M_{n+Nl}^* + M_{n+Nk}^*\delta M_{n+Nl})(1 + \delta_{l1}\delta_{1l}) \right]$$

(D13)

The above equation can further be simplified by using eq.(D3) which gives $A_{jj} = 2\sum_{k=1}^{N} M_{jk}M_{jk}^* - g$. Using the equality in eq.(), we get

$$\langle\delta A_{n,n+N}\rangle^2 = 8(A_{nn} + A_{n+n+N})\delta Y$$

(D14)
3. Case \( m = n \)

For a clear understanding, here we consider the cases with and without time-reversal symmetry separately. As shown below, \( |\delta A_{nn}|^2 \) turn out to be same in both the cases.

**Region without Time-Reversal Symmetry, \( (\beta = 2) \):**

As \( A_{nn} = \sum_{s=1}^{2} \sum_{k=1}^{2N} M^{2}_{nk; s} = \sum_{r=0, N}^{2} \sum_{s=1}^{N} \sum_{k=1}^{N} M^{2}_{nk+r; s} \), we get

\[
\delta A_{nn} \delta A_{nn}^{*} = 4 \sum_{r_1, r_2, s_1, s_2} \sum_{k, l=1}^{N} M_{n+k+r; s_1} M_{n+l+r; s_2} (\delta M_{n+k+r; s_1} \delta M_{n+l+r; s_2})
\]

\( (D15) \)

\[
= 8\delta Y \sum_{r_1, r_2, s_1, s_2} \sum_{k, l=1}^{N} M_{n+k+r_1; s_1} M_{n+l+r_2; s_2} \delta r_1 \delta r_2 \delta s_1 \delta s_2
\]

\( (D16) \)

\[
= 8\delta Y \sum_{r} \sum_{s} \sum_{k=1}^{N} M^{2}_{n+k+r; s} = 8A_{nn} \delta Y
\]

\( (D17) \)

Here eq.(D16) is obtained from eq.(D5) by using eq.(24) (or alternatively eq.(D7)).

**Region with Time-Reversal Symmetry**

As \( M_{nk; s} = (-1)^{s-1} M_{n+N, |k+N|; s} \) under time-reversal symmetry, \( A_{nn} \) can be rewritten as \( A_{nn} = \sum_{r=0, N}^{2} \sum_{s=1}^{N} \sum_{k=1}^{N} M^{2}_{n+r+k; s} \). This gives

\[
\delta A_{nn} \delta A_{nn}^{*} = 4 \sum_{r_1, r_2, s_1, s_2} \sum_{k, l=1}^{N} M_{n+r_1, k; s_1} M_{n+r_2, l; s_2} (\delta M_{n+r_1, k; s_1} \delta M_{n+r_2, l; s_2})
\]

\( (D19) \)

\[
= 4\delta Y \sum_{r_1, r_2, s_1, s_2} \sum_{k, l=1}^{N} M_{n+r_1, k; s_1} M_{n+r_2, l; s_2} [\delta r_1 \delta r_2 \delta s_1 \delta s_2 + (-1)^{s_2-1} \delta r_1 \delta r_2 + N \delta r_2 + N \delta s_1 \delta s_2] + (-1)^{s_2-1} M_{n+r+k+N; N; s} M_{n+r+k; s}
\]

\( (D20) \)

\[
= 4\delta Y \sum_{r} \sum_{s} \sum_{k=1}^{N} [M^{2}_{n+r+k; s} + (-1)^{s-1} M_{n+r+N+k+N; s} M_{n+r+k; s}]
\]

\( (D21) \)

\[
= 8\delta Y \sum_{r} \sum_{s} \sum_{k=1}^{N} M^{2}_{n+r+k; s} = 8A_{nn} \delta Y
\]

\( (D22) \)

Here eq.(D22) is obtained from eq.(D21) by using the relation \( M_{n+r+N+k+N; s} = (-1)^{s-1} M_{n+r+k; s} \).

The eqs.(D9), *(D14) and (D23) can be combined together to give eq.(25).

**APPENDIX E: PROOF OF EQUATION (10)**

The matrix \( A \) is Hermitian in nature. Let the inverse pair eigenvalues of \( A \), at value \( Y \) of the complexity parameter, be given by \( x_n \) and \( x_n^{-1} \), \( n = 1 \rightarrow N \) at parametric values \( Y \). A small change \( \delta Y \) in parameter \( Y \) changes \( A \) and its eigenvalues. By considering matrix \( A + \delta A \) in the diagonal representation of matrix \( A \), a small change \( \delta x_n \) in the eigenvalues can be given as

\[
\delta x_n = \delta A_{nn} + \sum_{m \neq n} \frac{|\delta A_{mn}|^2}{x_n - x_m} + o((\delta A_{nn})^3)
\]

\( (E1) \)

where \( A_{mn} = x_n \delta_{mn} \) at value \( Y \) of complexity parameter. This further gives,
\[ \delta x_n = \delta A_{nn} + \sum_{m=1, m\neq n}^{2N} \frac{|\delta A_{mn}|^2}{x_n - x_m} \]  
(E2)

\[ = \left[ -2\gamma A_{nn} + 2 \sum_{m=1, m\neq n}^{2N} (\beta + (4 - \beta)\delta_{m,n+N}) \frac{A_{nn} + A_{mm}}{x_n - x_m} \right] \delta Y \]  
(E3)

\[ = \left[ -2\gamma x_n + 2 \sum_{m=1, m\neq n}^{2N} (\beta + (4 - \beta)\delta_{m,n+N}) \frac{x_n + x_m}{x_n - x_m} \right] \delta Y \]  
(E4)

\[ = F_n(x) \delta Y \]  
(E5)

Here eq.(E3) in the above has been obtained from eq.(E2) by using eq.(25). Similarly, up to first order of \( \delta Y \),

\[ \overline{\delta x_n \delta x_m} = \overline{\delta A_{nn} \delta A_{mm}} = 8A_{nn} \delta_{nm} \delta Y = 8 x_n \delta_{nm} \delta Y \]  
(E6)

The eq.(E5) can now be used to obtain first moment of eigenvalues \( \lambda_n = \frac{(x_n-1)^2}{x_n} \) of the matrix \( B \).

\[ \overline{\delta \lambda_n} = \frac{(x_n^2 - 1)}{4x_n^2} \delta x_n = \frac{\sqrt{\lambda_n(1 + \lambda_n)}}{x_n} F_n(\lambda) \delta Y \]  
(E7)

where

\[ F_n = \left[ -2\gamma x_n + 2\beta \sum_{m=1, m\neq n}^{N} \frac{(x_n + x_m)(x_n - x_m) + (x_n - x_m)(x_n + x_m^{-1})}{(x_n - x_m)(x_n - x_m^{-1})} + 8 \frac{x_n + x_m^{-1}}{x_n - x_m^{-1}} \right] \]

\[ = \frac{-1}{\sqrt{\lambda_n(1 + \lambda_n)}} \left[ 2\gamma x_n \sqrt{\lambda_n(1 + \lambda_n)} - 4(2\lambda_n + 1) - 4\beta \sum_{m \neq n} \frac{\lambda_n(1 + \lambda_n)}{\lambda_n - \lambda_m} \right] \]

\[ = -4 \frac{x_n}{\sqrt{\lambda_n(1 + \lambda_n)}} E_n \]  
(E8)

where \( E_n \) is given by eq.(28). The average of the second moment \( \overline{(\delta \lambda_n)^2} \) can similarly be calculated,

\[ \overline{(\delta \lambda_n)^2} = \frac{(x_n^2 - 1)(x_n^{-2} - 1)}{4x_n^2} \overline{\delta x_n \delta x_m} = \frac{8 \lambda_n(1 + \lambda_n)}{x_n} \delta_{nm} \delta Y \]  
(E9)

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Note, although the diffusion coefficient in eq. (29) is position dependent, however the Ito-Stratanovich ambiguity of Brownian motion does not arise here. This is because eq. (27) explicitly relates the change in the variables $\lambda_n$ to their values prior to the change.

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