A Classical Version of the Non-Abelian Gauge Anomaly

MICHAEL STONE
University of Illinois,
Department of Physics
1110 W. Green St.
Urbana, IL 61801 USA
E-mail: m-stone5@illinois.edu

VATSAL DWIVEDI
University of Illinois,
Department of Physics
1110 W. Green St.
Urbana, IL 61801 USA
E-mail: vdwived2@illinois.edu

Abstract

We show that a version of the covariant gauge anomaly for a 3+1 dimensional chiral fermion interacting with a non-Abelian gauge field can be obtained from the classical Hamiltonian flow of its probability distribution in phase space. The only quantum input needed is the Berry phase that arises from the direction of the spin being slaved to the particle’s momentum.

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I. INTRODUCTION

There has been much recent interest on the influence of Berry phases on the electronic property of solids [1], and a number of these effects provide fruitful analogies for relativistic field theories. A particular example occurs when there is a net flux of Berry curvature through a disconnected part of the Fermi surface. In this case an analogue of the Abelian axial anomaly appears, manifesting itself as non-conservation of conduction-band particle number in the presence of external electric and magnetic fields [2, 3]. A net Berry flux through the Fermi-surface implies the existence of a Dirac-cone band-touching point somewhere within the surface, and the lower (valence) band is the source of the new particles. In a bulk crystal the Nielsen-Ninomiya theorem [4] requires that Dirac-cone degeneracies come in pairs with opposite-sign anomalies. Consequently, while the number of particles in each disconnected Fermi sea will change, the total number of particles in the conduction and valance band is conserved. On the surface of a topological insulator, however, we can have domain-wall fermions [5, 6] with single Dirac points, and in that case the additional particles flow into the surface-state valance band from the bulk via the Callan-Harvey effect [7].

The axial anomaly is usually derived via sophisticated quantum calculations, so it is perhaps surprising that Stephanov and Yin were able to obtain the result of [2, 3] from purely classical Hamiltonian phase-space dynamics [8]. Their argument works because near the Fermi surface, and well away from the Dirac point, an adiabatic classical approximation becomes accurate and the influx of extra particles to the Fermi surface can be seen and counted reliably. The only quantum input is the Berry phase, which subtly alters the classical canonical structure so that $p$ and $x$ are no longer conjugate variables.

In addition to the Abelian axial anomaly, chiral fermions may also be subject to a non-Abelian gauge anomaly, in which the failure of a current to be covariantly conserved signals a quantum breakdown of the formal gauge invariance. Since a covariant conservation law does not imply that any net charge is time independent, its failure is not necessarily due to an influx of discrete particles. Instead, it means that the fermion determinant is no longer a function but has become a section of a twisted line bundle over the space of gauge-equivalent fields [9]. It might not be expected, therefore, that the gauge anomaly can be accounted for as simply as the Abelian axial anomaly. This raises the question of what the
analogous phase-space calculation reveals for particles coupled to non-Abelian gauge fields. The purpose of this paper is to show that classical phase space dynamics does in fact lead to a version of the gauge anomaly.

In section II we provide a brief review of the argument in [8]. We do so because the Abelian calculation provides a guide for the slightly more intricate non-Abelian dynamics. In section III we review the classical-quantum correspondence for Lie group representations. In section IV we derive Liouville’s theorem for the Hamiltonian flow in the combined gauge and space-time phase space, and show how it leads to a classical analogue of the non-Abelian gauge anomaly. A last section provides a brief discussion.

II. LIOUVILLE’S THEOREM AND THE ABELIAN ANOMALY

In [8] the authors show that the adiabatic motion of a 3+1 dimensional positive-energy, positive-helicity Weyl particle may be described by the action functional

\[ S[x, p] = \int dt \left( A \cdot \dot{x} - \phi(x) + p \cdot \dot{x} - |p| - a \cdot \dot{p} \right). \]  (1)

Here \( A \) and \( \phi \) are the usual Maxwell vector and scalar potentials. The vector potential \( a(p) \) is the real-valued momentum-space Berry connection that arises because the Weyl Hamiltonian

\[ H_{\text{Weyl}} = \sigma \cdot p \]  (2)

slaves the the spin of an energy \( E = +|p| \) particle to the direction of its momentum. The Berry connection is singular:

\[ b = \nabla_p \times a = \frac{\dot{p}}{2|p|^2}, \quad \nabla \cdot b = 2\pi\delta^3(p). \]  (3)

The Dirac point \( p = 0 \) is a Berry-curvature Dirac monopole.

An adiabatic approximation has been made in [11] that subsumes all the of the effects of the particle’s spin into the Berry phase. This approximation breaks down completely in the neighbourhood of the Dirac point, but becomes better and better as we move to higher energies. In the following calculations the Dirac point may seem to be exactly where the anomaly calculation needs the approximation to hold. This is not the case however. As explained in [8], all that is important is that the approximation and its resulting classical flow equations be reliable at a positive-energy Fermi surface.
From the action we obtain the classical equations of motion.

\[ \dot{p} = E + \dot{x} \times B \]
\[ \dot{x} = \dot{p} + \dot{p} \times b \]  

(4)

The first equation is the usual Lorentz force. The second contains the expected \( \dot{p} = \nabla_p|p| \) group velocity, but in addition there is an anomalous velocity term \( \dot{p} \times b \). This term was first identified by Karplus and Luttinger \[10\], who argued that it was responsible for the anomalous Hall effect in ferromagnetic solids \[11\]. They were writing thirty years before the wide-ranging importance of the adiabatic phase was made clear by Berry, and their claim was not understood and accepted until relatively recently \[12–15\].

The equations (4) can be solved for \( \dot{x} \), \( \dot{p} \) in terms of \( x \) and \( p \), to give.

\[ (1 + b \cdot B)\dot{x} = \dot{p} + E \times b + (b \cdot \dot{p})B, \]
\[ (1 + b \cdot B)\dot{p} = E + \dot{p} \times B + (E \cdot B)b. \]

(5)

The phase space \((\dot{x}, \dot{p})\) flow is Hamiltonian, albeit with an unconventional symplectic structure \[16\] in which \( p \) is no longer the canonical conjugate of \( x \). We can therefore find a version of Liouville’s theorem for the conservation of phase-space volume.

We set \( \sqrt{G} = 1 + b \cdot B \), and use the homogeneous Maxwell equations \( \nabla_x \cdot B = 0 \) and \( \nabla_x \times E + \dot{B} = 0 \), to evaluate

\[ \frac{\partial \sqrt{G}}{\partial t} + \frac{\partial \sqrt{G}\dot{x}_i}{\partial x_i} + \frac{\partial \sqrt{G}\dot{p}_i}{\partial p_i} = b \cdot \dot{B} + b \cdot (\nabla \times E) + (b \cdot \dot{p})\nabla_x \cdot B + (E \cdot B)\nabla_p \cdot b. \]
\[ = (E \cdot B)\nabla_p \cdot b. \]  

(6)

For a non-singular Berry connection \( \nabla_p \cdot b = 0 \), and (6) shows that the conserved phase-space measure is

\[ \mu = \sqrt{G} \left( \frac{dpdx}{2\pi} \right)^3. \]

(7)

This measure with its \( \sqrt{G} \) modification was originally regarded as “non-canonical” \[17\], but it is precisely the canonical phase-space volume associated with the unconventional symplectic structure \[18\].

If we introduce a phase space density \( f(x, p, t) \) and define

\[ \rho = (1 + (b \cdot B))f \]
\[ \mathbf{j}_x = (1 + (\mathbf{b} \cdot \mathbf{B})) \dot{\mathbf{x}} \]
\[ \mathbf{j}_p = (1 + (\mathbf{b} \cdot \mathbf{B})) \dot{\mathbf{p}}, \] (8)

then, again in the non-singular case, we have
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}_x + \nabla \cdot \mathbf{j}_p = (1 + (\mathbf{b} \cdot \mathbf{B})) \left( \frac{\partial}{\partial t} + \dot{\mathbf{x}} \cdot \nabla \mathbf{x} + \dot{\mathbf{p}} \cdot \nabla \mathbf{p} \right) f. \] (9)

This shows that the phase-space probability density \( \rho = \sqrt{G} f \) is conserved when \( f \) is advected with the flow. As a consequence of the probability conservation the particle number 4-current
\[ J^0(x, t) = \int f(x, p, t) \sqrt{G} \frac{d^3p}{(2\pi)^3}, \]
\[ J^i(x, t) = \int f(x, p, t) \dot{x}^i \sqrt{G} \frac{d^3p}{(2\pi)^3}, \] (10)
is also conserved. In the singular case, with its Dirac monopole, we instead find
\[ \partial_\mu J^\mu = \frac{1}{(2\pi)^2} (\mathbf{E} \cdot \mathbf{B}) f(x, 0, t). \] (11)

If the negative-energy Dirac sea is completely filled, and in addition some of the positive energy states are filled up a Fermi energy, then we will have \( f(x, 0, t) = 1 \), and equation (11) becomes the 3+1 dimensional axial anomaly for positive-chirality particles. (For a negative chirality particle, the Berry phase and anomaly have opposite sign.)

III. CLASSICAL MECHANICS OF GROUP REPRESENTATIONS

We will generalize the Abelian calculation by making use of the Wong equations \[19\] for particle interacting with a Yang-Mills field. This requires us to appreciate that the “charge” of a particle interacting with a non-Abelian gauge field is the representation \( \Lambda \) of the gauge group \( G \) in which the particle lives. To obtain a classical version of the particle’s motion in physical space, the internal colour space must also be described classically. To do this the finite-dimensional representation space \( \Lambda \) should be replaced by a suitable finite-volume phase space \( \mathcal{O}_\Lambda \) \[20\].

The correspondence between Lie group representations, classical phase space and quantization has been explored in great generality by Kirillov, Kostant and Souriau \[21\]. We
will, however, restrict ourselves to the specific case of a compact simple group. For such a group a unitary irreducible representation is completely characterized by its highest-weight vector $|\Lambda\rangle$. The general theory in [21] then shows that the appropriate phase space is the co-adjoint orbit of the function $F(X) = \langle \Lambda|X|\Lambda \rangle$ under the map $F(X) \to F(g^{-1}Xg)$. Here $g^{-1}Xg$ denotes the adjoint action of $g^{-1}$ on the Lie algebra element $X$. Now a simple group possesses an invertible Killing-form metric tensor that we can take to be

$$\gamma_{ab} = \text{tr}\{\lambda_a \lambda_b\}, \quad (12)$$

where the trace is taken is some fixed faithful representation (usually the defining representation) of $G$, and the $\lambda_a$ are a hermitian basis for the Lie algebra obeying $[\lambda_a, \lambda_b] = if_{ab}^c \lambda_c$. Making use of this metric allows us to write

$$\langle \Lambda|g^{-1}Xg|\Lambda \rangle = \text{tr}\{\alpha_A g^{-1}Xg\} = \text{tr}\{g \alpha_A g^{-1}X\}, \quad (13)$$

where

$$\alpha_A = \alpha_A^a \lambda_a, \quad \alpha_A^a = \gamma^{ab}\langle \Lambda|\tilde{\lambda}_b|\Lambda \rangle. \quad (14)$$

Here $\tilde{\lambda}_a$ is the matrix representing the generator $\lambda_a$ in the representation $\Lambda$. In the compact simple case therefore, the second equality in (13) shows that the co-adjoint orbit of $F$ can be identified with the adjoint orbit of $\alpha_A$.

Consider now the Hamiltonian action functional

$$S[g] = \int dt \left( i \text{tr}\left\{\alpha g^{-1}\frac{dg}{dt}\right\} - i \mathcal{H}(g) \right), \quad (15)$$

where $\mathcal{H}(g) = \text{tr}\{\alpha g^{-1}Xg\}$ with $X$ an element of the Lie algebra. The equation of motion that comes from from varying $g$ is

$$[\alpha, g^{-1}(\partial_t - X)g] = 0. \quad (16)$$

This equation is equivalent to

$$g^{-1}(\partial_t - X)g + h(t) = 0, \quad (17)$$

where $h(t)$ is an arbitrary time dependent function such that $h \in \mathfrak{g} \equiv \text{Lie}(G)$ commutes with $\alpha$. The solution to the equation of motion is therefore

$$g(t) = \mathcal{T}\exp\left\{\int_0^t X dt\right\} H(t), \quad (18)$$
where $\dot{H}(t) = h(t)$ is now an arbitrary element of the subgroup $H \subseteq G$ that commutes with $\alpha$. The group element $g(t)$ is thus only well defined as an element of the coset $G/H$. The Lie-algebra valued expression

$$Q = g\alpha g^{-1} = Q^a\lambda_a$$

(19)

is insensitive to the $H(t)$ ambiguity, and its (co)-adjoint orbit can be identified with the coset $G/H$. We can define a Poisson bracket on functions on $G/H$ by setting

$$\{\mathcal{H}_1, \mathcal{H}_2\} \equiv \left. \frac{d\mathcal{H}_2}{dt} \right|_{\mathcal{H}_1}.$$  

(20)

In particular, $Q_a = \gamma_{ab}Q^b = \text{tr}\{Q\lambda_a\} = \text{tr}\{\alpha g^{-1}\lambda_ag\}$ is a function on $G/H$, and we find that

$$\{Q_a, Q_b\} = if_{abc}Q^c.$$  

(21)

This Poisson-bracket version of the Lie algebra exists for any $\alpha \in \mathfrak{g}$, but only when $\alpha$ arises as an $\alpha_\Lambda$ from equation (14) can the classical motion be consistently quantized. When we do so, we recover the representation $\Lambda$. In this case the classical-variable $\rightarrow$ quantum-operator correspondence will assign $Q_a \rightarrow \hat{\lambda}_a$, where $\hat{\lambda}_a$ is the matrix representing $\lambda_a$ in the representation $\Lambda$.

For future reference we note that the canonical volume element on the phase-space $G/H$ is given by

$$\mu_\alpha = \mathcal{N}(-\text{tr}\{\alpha(\omega_L)^2\})^N,$$  

(22)

where $\mathcal{N}$ is a normalization constant that is introduced to make

$$\int \mu_\alpha = \text{tr}_\Lambda\{I\} = \text{dim}(\Lambda).$$  

(23)

The symbol $\omega_L = \omega_L^a\lambda_a = g^{-1}dg$ denotes the left-invariant Maurer-Cartan form on the Lie algebra, and $N$ is the number of pairs of generators that fail to commute with $\alpha$. For $G = \text{SU}(2)$, for example, we can take $\alpha = \sigma_3$ and then $N = 1$. Using Euler angles to parameterize the group element

$$g = \exp\{-i\phi\sigma_3/2\} \exp\{-i\theta\sigma_2/2\} \exp\{-i\psi\sigma_3/2\},$$  

(24)

and setting $g^{-1}dg = \omega_L^a\sigma_a$, we have

$$\omega_L^1 = \frac{i}{2}(\sin \psi \, d\theta - \sin \theta \cos \psi \, d\phi),$$
\[
\omega^2_L = -\frac{i}{2}(\cos \psi \, d\theta + \sin \theta \sin \psi \, d\phi),
\]
\[
\omega^3_L = -\frac{i}{2}(d\psi + \cos \theta \, d\phi).
\]
This gives
\[
- \text{tr} \{\sigma_3(\omega_L)^2\} = -2(\omega^1_L \omega^2_L - \omega^2_L \omega^1_L)
\]
\[
= -4 \omega^1_L \omega^2_L
\]
\[
= \sin \theta \, d\theta \, d\phi,
\]
which is the element of area on the two-sphere. In this case the group manifold of \(G\) is \(S^3\) and the projection \(G \to G/H\) is the Hopf map \(S^3 \to S^2\).

This normalization of the volume \((23)\) makes the integral over the phase space of a polynomial in \(Q_a\) into a classical approximation to the symmetrized trace of the operators \(\hat{\lambda}_a\).

**IV. LIOUVILLE’S THEOREM AND THE GAUGE ANOMALY**

We now couple the internal group dynamics to the motion of our 3+1 dimensional Weyl fermion. We take as action functional
\[
S[x, p, g] = \int dt \left( i \text{tr} \left\{ \alpha g^{-1} \left( \frac{d}{dt} - i(\dot{x} \cdot A + A_0) \right) g \right\} + p \cdot \dot{x} - |p| - a \cdot \dot{p} \right).
\]
(26)
Here the factors of \(i\) have been inserted so that the non-Abelian gauge fields \(A_0 = A^a_0 \lambda_a\), \(A = A^a \lambda_a\) are hermitian. Recall that for Abelian electromagnetism we have \(A_0 = -\phi\). In the Abelian case, therefore, the action reduces to \((1)\). The functional \((26)\) is invariant under the gauge transformation
\[
- iA^h_\mu \rightarrow - h^{-1}(-iA^h_\mu)h + h^{-1}\partial_\mu h,
\]
\[
g(t) \rightarrow h^{-1}(x(t), t)g(t).
\]
(27)

From \((26)\) we obtain the equation of motion for \(g\):
\[
[\alpha, g^{-1}(\partial_t - i\dot{x} \cdot A - iA_0)g] = 0.
\]
(28)
As before, \(g(t)\) is only defined as an element of \(G/H\). For constant \(C\), however, we have the unambiguous result
\[
\frac{\partial}{\partial t} \text{tr} \{QC\} = \text{tr} \{[Q, -iA_0 - i\dot{x} \cdot A]C\}
\]

= \text{tr}\{Q[-iA_0 - i\dot{x} \cdot A, C]\} \quad (29)

We use this result when we vary $x$ to get the equation of motion

$$\dot{p}_i = -\frac{\partial}{\partial t}\text{tr}\{Q A_i\} - \dot{x}_j \frac{\partial}{\partial x_i}\text{tr}\{Q A_i\} + \frac{\partial}{\partial x_i}\text{tr}\{Q A_0\}$$

$$= \text{tr}\left\{Q \left(\left(\frac{\partial A_0}{\partial x_i} - \frac{\partial A_i}{\partial t} - i[A_i, A_0]\right) + \dot{x}_j \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} - i[A_i, A_j]\right)\right)\right\}.$$  

We have obtained the equation derived empirically by Wong \cite{19}

$$\dot{p} = \text{tr}\{Q(E + \dot{x} \times B)\}.$$  

Here $E = E^a\lambda_a$ and $B = B^a\lambda_a$ are the Lie-algebra-valued non-Abelian analogues of the electric and magnetic fields whose $i = 1, 2, 3$ components are

$$B_i = \frac{1}{2}\epsilon_{ijk}\{\partial_j A_k - \partial_k A_j - i[A_j, A_k]\},$$

$$E_i = \partial_i A_0 - \partial_0 A_i - i[A_i, A_0]. \quad (30)$$

By varying $p$ we again get

$$\dot{x} = \dot{p} + \dot{p} \times b.$$  

The full set of equations determining the motion is therefore

$$\dot{Q} = -i[Q, A_0 + \dot{x} \cdot A],$$

$$\dot{p} = \text{tr}\{Q(E + \dot{x} \times B)\},$$

$$\dot{x} = \dot{p} + \dot{p} \times b.$$  

These may again be solved for $\dot{x}$ and $\dot{p}$ in terms of $x$, $p$ and $Q$, as

$$(1 + b \cdot \text{tr}\{Q B\})\dot{x} = \dot{p} + \text{tr}\{Q E\} \times b + (b \cdot \dot{p})\text{tr}\{Q B\}.$$  

$$(1 + b \cdot \text{tr}\{Q B\})\dot{p} = \text{tr}\{Q E\} + \dot{p} \times \text{tr}\{Q B\} + (\text{tr}\{Q E\} \cdot \text{tr}\{Q B\})b. \quad (33)$$

A reasonable conjecture is that the non-Abelian generalization of the phase-space measure involves $\sqrt{G} = (1 + b \cdot \text{tr}\{Q B\})$. A slightly tedious computation of with the symplectic form confirms that this conjecture is correct, and the measure is

$$\mu = (1 + b \cdot \text{tr}\{Q B\})\mu_0 \left(\frac{dp dx}{2\pi}\right)^3.$$  

$$\mu = (1 + b \cdot \text{tr}\{Q B\})\mu_0 \left(\frac{dp dx}{2\pi}\right)^3.$$  

(34)
To obtain the non-Abelian version of Liouville’s theorem we need the analogues
\[
\nabla \cdot \mathbf{B} - i(\mathbf{A} \cdot \mathbf{B} - \mathbf{B} \cdot \mathbf{A}) = 0,
\]
\[
\dot{\mathbf{B}} - i[A_0, \mathbf{B}] + \nabla \times \mathbf{E} - i(\mathbf{A} \times \mathbf{E} + \mathbf{E} \times \mathbf{A}) = 0,
\]
(35)
of the homogeneous Maxwell equations. These homogeneous equations lead to the sum of
the three terms
\[
\left( \frac{\partial \sqrt{G}}{\partial t} \right)_Q + f_{ab}^c Q^a A_0^b \left( \frac{\partial \sqrt{G}}{\partial Q^c} \right)_{t,x,p} = \mathbf{b} \cdot \text{tr} \{ Q(\dot{\mathbf{B}} - i[A_0, \mathbf{B}]) \},
\]
\[
\left( \frac{\partial \sqrt{G} \dot{x}^i}{\partial x^i} \right)_Q + f_{ab}^c Q^a A_i^b \left( \frac{\partial \sqrt{G} \dot{x}^i}{\partial Q^c} \right)_{t,x,p} = \epsilon_{ijk} \text{tr} \{ Q(\partial_i E_j - i[A_i, E_j]) \} b_k,
\]
\[
+ (\dot{\mathbf{p}} \cdot \mathbf{b}) \text{tr} \{ Q(\partial_i B_i - i[A_i, B_i]) \},
\]
(36)
being zero — modulo the singular contribution from \( \nabla \cdot \mathbf{b} \) in the last line. This is our non-Abelian Liouville theorem. The theorem can also be derived with a bit more effort by computing the Lie derivative of the top power of the symplectic form. A tricky point here is the computation of the Lie derivative of \( \mu_a \). This derivative is not zero, and is responsible for moving an \( \dot{x}^i \) in the second line of (36) away from its natural companion \( A_i^b \) to its location inside the \( Q^c \) derivative.

To understand why this rather complicated looking relation is an expression of phase-space conservation we observe that the convective constancy of a phase-space distribution \( f(Q^a, x, p, t) \) is expressed by
\[
\left( \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + \dot{Q}^a \frac{\partial}{\partial Q^a} + \dot{p}_i \frac{\partial}{\partial p_i} \right) f = 0.
\]
(37)
Now
\[
\dot{Q}^a = -f_{bc}^a (A_0^b + \dot{x}^i A_i^b) Q^c,
\]
(38)
so we can group the \( \dot{Q} \) with the \( t \) and \( x \) derivatives together to make two \( A \)-dependent “covariant derivatives” \([22]\). These are
\[
\left( \frac{\partial f}{\partial t} \right)_Q + f_{ab}^c Q^a A_0^b \left( \frac{\partial f}{\partial Q^c} \right)_{t,x,p}
\]
(39)
and
\[
\dot{x}^i \left( \frac{\partial f}{\partial x^i} \right)_Q + f_{ab}^c Q^a A_i^b \left( \frac{\partial f}{\partial Q^c} \right)_{t,x,p}.
\]
(40)
We see that we have the same combination of terms that we had in (36). Let us verify that these combinations are naturally gauge covariant. Under a transformation $Q \rightarrow Q' = gQg^{-1}$ we have

$$Q^a \rightarrow Q'^a = G^a_b Q^b$$

(41)

where $G^a_b \equiv [\text{Ad}(g)]^a_b$ is the matrix corresponding to $g$ in the adjoint representation of $G$. Since $g(x,t)$ depends on space and time, this transformation mixes up the $Q$ and $x$ derivatives. The density $f$ is invariant,

$$f(Q, x, p, t) = f'(Q', x, p, t),$$

(42)

but

$$\left( \frac{\partial f}{\partial t} \right)_Q = \left( \frac{\partial f'}{\partial t} \right)_{Q'} + \left( \frac{\partial Q^b}{\partial t} \right)_Q \left( \frac{\partial f'}{\partial Q'^b} \right)_t,$$

(43)

and

$$\left( \frac{\partial f}{\partial Q^a} \right)_t = \left( \frac{\partial Q'^b}{\partial Q^a} \right)_t \left( \frac{\partial f'}{\partial Q'^b} \right)_t.$$

(44)

So we have

$$\left( \frac{\partial f}{\partial t} \right)_Q + f_{bc}^a A^e_0 Q^b \left( \frac{\partial f}{\partial Q^a} \right)_t$$

$$= \left( \frac{\partial f'}{\partial t} \right)_{Q'} + \left\{ (\dot{G}G^{-1})^b_d + G^b_ef_{bc}^e A^c_0 (G^{-1})^c_d \right\} Q'^d \left( \frac{\partial f'}{\partial Q'^b} \right)_t.$$

(45)

We therefore have covariance under

$$Q^a \rightarrow Q'^a = G^a_b Q^b$$

$$f_{bc}^a A^c_\mu \rightarrow f_{bc}^a A^c_\mu = (\dot{G}G^{-1})^a_b + G^a_d f_{ec}^d A^e_\mu (G^{-1})^c_b.$$ 

(46)

As the matrix representing $\lambda_c$ in the adjoint representation is $[\text{ad}(\lambda_c)]^a_b = -if_{bc}^a$, this is indeed the correct gauge transformation.

Combining Liouville’s theorem with (37) shows that $\sqrt{G}f$ is the conserved (modulo the singular contribution) phase-space probability:

$$\frac{\partial \sqrt{G}f}{\partial t} + \frac{\partial \sqrt{G}f\dot{x}^i}{\partial x^i} + \frac{\partial \sqrt{G}f\dot{p}^i}{\partial p^i} = f(Q, x, p, t) \text{tr} \{ Q \mathbf{E} \} \cdot \text{tr} \{ Q \mathbf{B} \} \nabla \cdot \mathbf{b}. $$

(47)

Now we define the gauge 4-current

$$J^0_a(t, x) = \int Q_a f(Q, x, p) \sqrt{G} \mu_\alpha \frac{d^3 p}{(2\pi)^3},$$

$$J^i_a(t, x) = \int Q_a \dot{x}^i f(Q, x, p) \sqrt{G} \mu_\alpha \frac{d^3 p}{(2\pi)^3}. $$

(48)

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and combine Liouville’s theorem, the convective constancy of the phase-space density 
\[ f(Q^a, x, p, t), \]
with
\[ \dot{Q}_a = f_{ba}^c (A_0^b + \dot{x}^i A_i^b) Q_c \]
(49)
to see that
\[ \partial_{\mu} J_{\mu}^a - f_{ba}^c A_\mu^b J_{\mu}^c = f(0) \frac{1}{(2\pi)^2} \int Q_a \text{tr} \{ QB \} \text{tr} \{ QE \} \mu_\alpha. \]
(50)

Since \( \text{tr} \{ QB \} = Q_a B^a, \text{tr} \{ QE \} = Q_a E^a \) and the integration of the three factors of \( Q_a \) over the phase-space \( G/H \) is the classical version of the symmetrized trace \( \frac{1}{2} \text{tr} A(\hat{\lambda}_a \{ \hat{\lambda}_b, \hat{\lambda}_c \}) \), this expression becomes a classical version of the “covariant” (as opposed to “consistent”) gauge anomaly \[ 23 \]
\[ \nabla_{\mu} J_{\mu}^a = \frac{1}{32\pi^2} \epsilon^{\alpha\beta\gamma\delta} \text{tr} A(\hat{\lambda}_a F_{\alpha\beta} F_{\gamma\delta}), \]
\[ = \frac{1}{(2\pi)^2} \frac{1}{2} \text{tr} A(\hat{\lambda}_a \{ \hat{\lambda}_b, \hat{\lambda}_c \}) E^b \cdot B^c. \]
(51)

V. DISCUSSION

We have considered the classical phase space Hamiltonian flow for spin-\( \frac{1}{2} \) particles interacting with a non-Abelian gauge field. We used Liouville’s theorem to identify the phase-space volume form and found that this volume form fails to be conserved in the vicinity of the Berry-phase monopole. The failure then leads to a classical version of the covariant form of the non-abelian gauge anomaly.

It is perhaps not too surprising that we obtain the “covariant” gauge anomaly rather than the “consistent” gauge anomaly. Although the Hamiltonian formalism only makes manifest the canonical structure, gauge invariance is being tacitly maintained at all points of the calculation. Also, when an anomalous chiral gauge theory makes physical sense, the Weyl particles will be domain-wall fermions residing on the boundary of some higher dimensional space. The anomaly is then accounted for by the inflow of gauge current from the bulk, and this inflowing current can obtained by functionally differentiating a bulk Chern-Simons theory. The boundary variation of the Chern-Simons term is then precisely the Bardeen-Zumino polynomial \[ 23 \] that converts the consistent gauge current to the covariant current (see for example \[ 24 \]). A similar argument shows that in an anomalous theory the current that appears in the Lorentz-force contribution to the energy-momentum conservation law is the covariant current \[ 25, 26 \].
There have been several recent works on the effects of anomalies on fluid dynamics [27, 28]. It is interesting to explore the relation between these papers, which take the anomalies as given and explore their consequences, and the present analysis that uses fluid-like kinetics to deduce their existence.

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