Mobile Relays for Smart Cities: Mathematical Proofs

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1. Proofs for $S_M$ under non-stopping fmBS scenario

Fig. 1. Time of arrivals of fmAPs and $S_c$ operation for a given T2 connection round.

We first present the following two lemmas:

Lemma 1. The expectation of service time $\tau_1^{(S_c)}$ is

$$
\mathbb{E} \left[ \tau_1^{(S_c)} | M^{(S_c)} \geq 1 \right] = \frac{\lambda T_M - (1 - e^{-\lambda T_M})}{\lambda (1 - e^{-\lambda T_M})}.
$$

Lemma 2. The expectation of $t_1$ is

$$
\mathbb{E} \left[ t_1 | M^{(S_c)} \geq 1 \right] = \frac{\lambda}{e^{\lambda T_M} - 1} \int_0^{T_M} \frac{s_1 e^{\lambda s_1}}{1 - e^{\lambda s_1}} ds_1 - \frac{1}{\lambda}
$$

$$
= \frac{1}{\lambda (e^{\lambda T_M} - 1)} \left( \text{Li}_2 \left( e^{\lambda T_M} \right) - \frac{\pi^2}{6} + 1 \right)
$$

$$
+ e^{\lambda T_M} (\lambda T_M - 1) + \lambda T_M \log \left( 1 - e^{-\lambda T_M} \right) - \frac{1}{\lambda},
$$

where $\text{Li}_2 \left( x \right) \triangleq \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ is the polylogarithm function of order 2. The proofs of these two lemmas are given consecutively.
A. Proof of Lemma 1

Let us start with the conditional pdf \( f_{t_1(s_c)|M(s_c)\geq 1}(s) \). For \( M(s_c) = 0 \), we have \( \tau_1^{(s_c)} = 0 \), which is a trivial result. In the case \( M(s_c) \geq 1 \), we have \( K \geq 1 \) fmBS arrivals in the interval \( (0, T_M] \) by taking the arrival time of the initial vehicle \( C_0 \) as our time origin. Furthermore, according to the uniform property of the Poisson arrivals, the exact arrival times of these fmBSs, namely \( A_j, j \in \{1, \ldots, K\} \), are independent and distributed uniformly in \( (0, T_M] \). Therefore, we have \( \tau_1^{(s_c)} = \max_{j \in \{1, \ldots, K\}} \{A_j\} \). It is known that the pdf of the maximum of \( k \) independent uniform random variables defined in \( (0, T_M] \) satisfies

\[
\begin{align*}
f_{\tau_1^{(s_c)}|M(s_c)\geq 1, K=k}(s) &= \frac{ks^{k-1}}{T^k} , \quad s \in (0, T_M] . \tag{3}
\end{align*}
\]

Since \( K \) is a Poisson random variable with parameter \( \lambda T_M \) and we are given that \( K \geq 1 \), the expectation sum over (3) yields

\[
\begin{align*}
f_{t_1|M(s_c)\geq j+1, K=k, K\geq 1, \tau_1^{(s_c)}=s_j}(r) &= \frac{kr^{k-1}}{s_j^k} , \quad r \in (0, s_j] . \tag{5}
\end{align*}
\]

Taking the expectation over the pmf of \( K \) with the condition \( K \geq 1 \), we obtain

\[
\begin{align*}
f_{t_1|M(s_c)\geq j+1, \tau_1^{(s_c)}=s_j}(r) &= \frac{\lambda \exp(\lambda r)}{\exp(\lambda s_j) - 1} , \quad r \in (0, s_j] . \tag{6}
\end{align*}
\]

The expected value of \( t_j \), conditioned on \( \tau_1^{(s_c)} = s_j \), follows from an expectation integral of (6) over \( r \in (0, s_j] \).

\[
\begin{align*}
E \left[ t_j \mid M(s_c) \geq j+1, \tau_1^{(s_c)} = s_j \right] &= \int_0^{s_j} \frac{\lambda \exp(\lambda r)}{\exp(\lambda s_j) - 1} dr \\
&= \frac{\lambda s_j - (1 - \exp(-\lambda s_j))}{\lambda (1 - \exp(-\lambda s_j))} . \tag{7}
\end{align*}
\]

In order to remove the condition on \( \tau_1^{(s_c)} \) for the \( j = 1 \) case and obtain \( E \left[ t_1 \mid M(s_c) \geq 1 \right] \), we first multiply of the expressions in (7) and (4), to further integrate the resulting expression over \( s_1 \in (0, T_M] \), which finalizes the proof.

**Theorem 1.** The expected effective ratio of time in T2 with strategy \( S_c \) can be approximated by

\[
R_2^{(s_c)} \simeq (1 - P_V) - \frac{2 (1 - P_V) T_H}{E \left[ \tau_1^{(s_c)} \mid M(s_c) \geq 1 \right] + \frac{2}{\lambda}} , \tag{8}
\]

where \( E \left[ \tau_1^{(s_c)} \mid M(s_c) \geq 1 \right] \) follows from (7) and \( E \left[ t_1 \mid M(s_c) \geq 1 \right] \) from (2).
C. Proof of Theorem 1

For a given value \( M^{(S_c)} = n_c \), the time intervals in which new fmBS arrivals occur are disjoint time intervals of length \( t_1^{(S_c)}, t_1, t_2, \ldots, t_{n-1} \), as exemplified in Fig. 1. With an abuse of notation, we drop the superscript \( (S_c) \) in \( t_1^{(S_c)} \) since for no strategy other than \( S_M \), we deal with \( t_1^{(S)} \) in the scope of this proof. Therefore, the number of unserved fmBSs, \( U \), in a T2 round satisfies

\[
E[U|M^{(S_c)} = n] = E[U_{\tau_1} + \sum_{j=1}^{n-1} U_{t_j}]
\]

where \( U_{\tau_1} \) and \( U_{t_j} \) are the number of unserved fmBSs in the time intervals \( \tau_1 \) and \( t_j \), respectively. Based on the iterated expectations over these random service times, we obtain

\[
E[U|M^{(S_c)} = n] = E[E_{U_{\tau_1}, t_j, M^{(S_c)} = n} \left[ U_{\tau_1} + \sum_{j=1}^{n-1} U_{t_j} \right]]
\]

\[
= \lambda \left( E[\tau_1|M^{(S_c)} = n] + \sum_{j=1}^{n-1} E[t_j|M^{(S_c)} = n] \right),
\]

where we use the fact that \( E[U_{\tau_1}|\tau_1 = s_1] = \lambda s_1 \), and a similar argument is valid for \( t_j \) as well.

We should evaluate \( E[\tau_1|M^{(S_c)} = n] \) and \( E[t_j|M^{(S_c)} = n] \) in order to remove the conditions on these random variables in (9). However, it is possible to approximate the term \( E[U|M^{(S_c)} = n] \) by evaluating \( E[\tau_1|M^{(S_c)} \geq 1] \), and for only a few values of \( E[t_j|M^{(S_c)} \geq j + 1] \). Therefore, we use

\[
E[U|M^{(S_c)} = n] \approx n \lambda \frac{E[\tau_1|M^{(S_c)} \geq 1] + E[t_1|M^{(S_c)} \geq 1]}{2},
\]

where the term \( (E[\tau_1|M^{(S_c)} \geq 1] + E[t_1|M^{(S_c)} \geq 1]) / 2 \) is an approximation for the average number of unserved fmBSs between two consecutive horizontal handoffs.

On the other hand, the number of handoffs for \( S_m \) and \( S_c \) in the whole T2 round satisfy

\[
E[M^{(S_m)}] = E_{M^{(S_c)}} \left[ E[U + M^{(S_c)}|M^{(S_c)}] \right]
\]

\[
\approx E[M^{(S_c)}] \left( \frac{E[\tau_1|M^{(S_c)} \geq 1] + E[t_1|M^{(S_c)} \geq 1]}{2} + 1 \right),
\]

where we use (10). Solving it for \( E[M^{(S_c)}] \) in (11) we obtain

\[
E[M^{(S_c)}] \approx \frac{2 E[M^{(S_m)}]}{\lambda \left( E[\tau_1^{(S_c)}] + t_1|M^{(S_c)} \geq 1 \right) + 2},
\]

where the denominator follows from the results of Lemmas 1 and 2. For \( S_c \), the expected effective ratio of time in T2 is

\[
R^{(S_c)}_2 = \frac{E[T_2] - E[M^{(S_c)}] T_H}{E[T_1] + E[T_2]}
\]

where \( E[T_1] = \lambda^{-1} \) and \( E[T_2] = \frac{1 - PV}{\lambda PV} \) as for \( S_m \). The proof is completed by plugging all these known expressions and (12) into (13).
II. PROOFS FOR $S_m$ UNDER STOPPING FMBS SCENARIO

Lemma 3.

$$E[M^{(S_m)}] = \sum_{m=0}^{\infty} \prod_{j=1}^{m+1} \left(1 - \hat{P}_V^{(j)}\right),$$

(14)

where $\hat{P}_V^{(j)}$ is the probability of a vertical handoff at the end of a service time of the $(j-1)$th non-stopping one, is $\hat{P}_V^{(j)}$ and $\Delta$ is expressed as

$$\hat{P}_V^{(j)} = e^{-\lambda T_M} \left[1 - P_S(1 - e^{-P_S\lambda T_M}) \frac{1 - \Delta^j}{1 - \Delta}\right]$$

(15)

and $\Delta \triangleq (P'_S - P_S)$.

A. Proof of Lemma 3

The conditional probability that the $j$th non-stopping fmBS is a stopping one, given that the $(j-1)$th non-stopping one is $P'_S \triangleq P \{j$th non-stopping fmBS has stopped$\}$. It can be found that $P'_S = P_S \sum_{k=0}^{j-1} (P'_S - P_S)^k$ for $j = 1, 2, \ldots$. Therefore, following several steps of derivation, one reaches the result that

$$\hat{P}_V^{(j)} = P''_S^{(j-1)} \mathbb{P} \{N_{ns}(0, T_M) + N_s(0, T_M + T_S) = 0\}$$

$$+ (1 - P''_S^{(j-1)}) \mathbb{P} \{N_{ns}(0, T_M) + N_s(0, T_M) = 0\}$$

$$= P''_S^{(j-1)} \mathbb{P} \{N_{ns}(0, T_M) = 0\} \mathbb{P} \{N_s(0, T_M + T_S) = 0\}$$

$$+ (1 - P''_S^{(j-1)}) \mathbb{P} \{N_{ns}(0, T_M) = 0\} \mathbb{P} \{N_s(0, T_M) = 0\}$$

$$= e^{-\lambda T_M} \left[1 - P_S(1 - e^{-P_S\lambda T_M}) \frac{1 - \Delta^j}{1 - \Delta}\right],$$

(17)

where $N_s(,\ldots)$ and $N_{ns}(,\ldots)$ are the independent counting processes corresponding to the Poisson processes of stopping and the non-stopping fmBS arrivals as defined in (1), and $\Delta \triangleq (P'_S - P_S)$. Therefore, one can evaluate the expected value for the T2 handoffs for the stopping fmBS case utilizing $S_m$ as

$$E[M^{(S_m)}] = \sum_{m=0}^{\infty} \prod_{j=1}^{m+1} \hat{P}_V^{(m+1)} \left(1 - \hat{P}_V^{(j)}\right),$$

$$= \sum_{m=0}^{\infty} \prod_{j=1}^{m+1} \left(1 - \hat{P}_V^{(j)}\right),$$

(18)

where we used the fact that for any non-negative random variable $M$, $E[M] = \sum_{m=0}^{\infty} \mathbb{P}(M > m)$. Since the expression in (18) is an infinite summation, one can either directly utilize it to approximate $E[M^{(S_m)}]$ by truncating the summation at a finite $m$ value, or can model random variable $M^{(S_m)}$ using a new random variable $M'$ which has the following success probabilities

$$P_{M'}^{(1)} = \hat{P}_V^{(1)} = e^{-\lambda T_M} \left[1 - P_S(1 - e^{-P_S\lambda T_M})\right]$$

$$P_{M'}^{(m)} = \hat{P}_V^{(2)} = e^{-\lambda T_M} \left[1 - P_S(1 - e^{-P_S\lambda T_M})(1 + P'_S - P_S)\right],$$

$$m = 1, 2, \ldots$$
where \( m \geq 2 \). Using this model we obtain
\[
E[M^{(S_m)}] \approx E[M'] = \frac{1 - P_{M'}^{1}}{P_{M'}^{2}}
\]
\[
= \frac{1 - e^{-\lambda T_M} \left[ 1 - P_S \left( 1 - e^{-P_S \lambda T_S} \right) \right]}{e^{-\lambda T_M} \left[ 1 - P_S \left( 1 - e^{-P_S \lambda T_S} \right) \right]} \left( 1 + P_{M'} - P_S \right) .
\]

(19)

**Theorem 2.** The expected effective T2 ratio for \( S_m \) with stopping fmAPs can be approximated by
\[
\hat{R}_2^{(S_m)} \approx \frac{\hat{A}_2 - T_H E[M^{(S_m)}]}{E[\hat{T}_1] + \hat{A}_2},
\]
where \( E[\hat{T}_1] = 1/\lambda \), \( E[M^{(S_m)}] \) is given in (19), and we approximate \( E[\hat{T}_2] \approx \hat{A}_2 \), which is defined as
\[
\hat{A}_2 = T_M + P_S T_S + E[M^{(S_m)}] \frac{E[\tau_1^{(S_m)}] + E[\tau_2^{(S_m)}]}{2}.
\]

(20)

**B. Proof of Thm. 2**

The expected service time \( E[\tau_j^{(S_m)}] \) can be evaluated by using 4 possible combinations of the stopping property for the \((j - 1)^{st}\) and the \( j^{th}\) fmAPs and it simplifies to
\[
E[\tau_j^{(S_m)}] = \left( 1 - P_S^{(j-1)} \right) \frac{1 - e^{-x (1 + x)}}{\lambda (1 - e^{-x})} \bigg|_{x = P_S \lambda T_M}
\]
\[
+ \left( 1 - P_S^{(j-1)} \frac{P_{M'} - P_S}{1 - P_S} \right) \frac{1 - e^{-x (1 + x)}}{\lambda (1 - e^{-x})} \bigg|_{x = (1 - P_S) \lambda T_M}
\]
\[
+ P_S^{(j-1)} \frac{P_S}{1 - e^{-x (1 + x)}} \bigg|_{x = P_S \lambda (T_M + T_S)},
\]
\[
(22)
\]
where \( P_S' \triangleq \left( 1 - e^{-P_S \lambda T_S} \right) + e^{-P_S \lambda T_S} \left( 1 - e^{-\lambda T_M} \right) P_S \) is the probability that an fmAP is a stopping one given that the previous one has stopped and \( P_S^{(j)} \triangleq P_S \sum_{k=0}^{j-1} (P_{M'} - P_S)^k \) is the probability that the \((j - 1)^{st}\) fmAP is a stopping one.

Similar to the approximation in the proof of Thm. 1 we can approximate the expected time in T2 by using only a few of the \( E[\tau_j^{(S_m)}] \) terms. As an example, only by utilizing \( \tau_1^{(S_m)} \) and \( \tau_2^{(S_m)} \) we reach
\[
E[\hat{T}_2] \approx T_M + P_S T_S + E[M^{(S_m)}] \frac{E[\tau_1^{(S_m)}] + E[\tau_2^{(S_m)}]}{2},
\]
\[
(23)
\]
where \( E[\tau_1^{(S_m)}] \) follows from (22) with \( j = 1 \). The average time spent in T1 is not affected by the stopping fmAPs since a handoff from T1 to T2 occurs only when the next arrival is observed. Hence \( E[\hat{T}_1] = 1/\lambda \). The proof is completed when we replace \( E[\hat{T}_2] \) in both the numerator and the denominator of (20) with the approximation for \( E[\hat{T}_2] \) in (23) and use the result on \( E[M^{(S_m)}] \) from Lemma 3.

**References**

[1] T. Aktas, G. Quer, T. Javidi, and R. R. Rao, “From connected vehicles to mobile relays: Enhanced wireless infrastructure for smarter cities,” in IEEE Global Communications Conference, Dec. 2016.