Quantum and Classical Aspects of Deformed $c = 1$ Strings

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ABSTRACT

The quantum and classical aspects of a deformed $c = 1$ matrix model proposed by Jevicki and Yoneya are studied. String equations are formulated in the framework of the Toda lattice hierarchy. The Whittaker functions now play the role of generalized Airy functions in $c < 1$ strings. This matrix model has two distinct parameters. Identification of the string coupling constant is thereby not unique, and leads to several different perturbative interpretations of this model as a string theory. Two such possible interpretations are examined. In both cases, the classical limit of the string equations, which turns out to give a formal solution of Polchinski’s scattering equations, shows that the classical scattering amplitudes of massless tachyons are insensitive to deformations of the parameters in the matrix model.

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1 Introduction and Summary

Matrix models have been a very powerful tool for our understanding of nonperturbative string theories in two or fewer space-time dimensions [1]. If a matrix model is to be taken as a nonperturbative definition of string theory, all solutions of string theory should be described by this approach. In particular it is very interesting to provide a matrix model formulation of the two-dimensional critical string theory in the black-hole background [2], [3] since it will give us a full quantum mechanical (and possibly nonperturbative) understanding of black-hole physics. An important step in this direction has been taken by Jevicki and Yoneya [4]. They proposed that a stationary black-hole of mass $m$ with tachyon condensation is described by the large $N$ hermitian matrix quantum mechanics with potential $V(\Phi) = \text{Tr} \{ -\Phi^2/2 + m\Phi^{-2}/2N^2 \}$. This matrix model has two entirely different parameters, the Fermi energy $\mu$ and the black-hole mass $m$. They parametrize deformations of the linear dilaton solution of the critical string theory in a flat space-time background (i.e. $\mu = m = 0$). It is also argued in [4] that, whereas the string coupling constant in the standard $c = 1$ string is given by $g_{st} \sim 1/\mu$, that of the black-hole should be modified as $g_{st} \sim 1/\sqrt{m}$. These two different identifications of the string coupling constant lead us to two different perturbative string theoretical interpretations of this deformed matrix model, i.e., a deformation from a flat space-time background by the black-hole mass operator, and a deformation from the black-hole background by the Fermi level (both with tachyon condensates). Many attempts have been done towards nonperturbative understanding of this deformed $c = 1$ string model. Classical scattering amplitudes of massless tachyons in the black-hole background (i.e. $\mu = 0$) have been studied in [4], [5], [6], [7] mainly by the collective field approach to this deformed matrix model.

In this article, we present a nonperturbative treatment of this deformed matrix model. The goal is to derive string equations that encode the dynamics of deformed $c = 1$ strings. The tachyon scattering amplitudes will then be reproduced in the classical limit as the string coupling constant $g_{st} \sim \hbar \to 0$. To formulate such string equations, we need to specify an underlying integrable structure.
In the case of $c < 1$ string theory the nonperturbative partition function of $(p,q)$ string is a $\tau$ function of the Kadomtsev-Petriashivil (KP) hierarchy specified by the string equation \[ [P,Q] = 1 \] where $P$ and $Q$ are differential operators with $\deg P = p$ and $\deg Q = q$. In the particular case of $(p,1)$ string, the solution of the string equation is written in terms of the generalized Airy equation

\[(\partial^p_\lambda - \lambda)f_0(\lambda) = 0 ,\]

or equivalently its first order form

\[\lambda f_k(\lambda) = f_{k+p}(\lambda) - kf_{k-1}(\lambda) , \quad \partial_k f_k(\lambda) = f_{k+1}(\lambda) ,\]

where $0 \leq k \leq p - 1$. The partition function of this model is given by a generalized matrix Airy function \[ [10],[11] \] and several topological aspects appear through the asymptotic expansion of this function \[ [10],[12] \]. This matrix integral representation of $(p,1)$ string gives a nonperturbative formulation of $c < 1$ string theory.

Is there a counterpart of the generalized Airy functions for these deformed $c = 1$ strings? Motivated by this question we devote Section 2 to a detailed study of a nonrelativistic fermion system which describes \[ [13],[14],[15] \] the large $N$ limit of this deformed matrix model. Our fundamental standpoint is the observation that the Whittaker functions give a $c = 1$ counterpart of the generalized Airy functions in $(p,1)$ strings. We construct two sets of operators $B_n$ and $\bar{B}_n$ ($n \in N$) in this fermion system, part of which turn out to be creation operators of massless tachyons in these deformed $c = 1$ strings. (This point is discussed at the end of Section 3 in comparison with classical scattering data obtained from the collective field approach.) On the basis of the analysis of the one-body system we also introduce the asymptotic fields for these nonrelativistic fermions. Besides the $S$-matrix elements between these asymptotic fields, the asymptotic forms of $B_n$ and $\bar{B}_n$ are thus also determined. It should be stressed that these results are derived essentially from the recursion relations and connection formula of the Whittaker functions $M_{\kappa,\mu}$ ($W_{\kappa,\mu}$)

\[ x\partial_x M_{\kappa,\mu}(x) = \pm(x/2 - \kappa)M_{\kappa,\mu}(x) + (\mu \pm \kappa + 1/2)M_{\kappa,\pm1,\mu}(x) ,\]

\[ W_{\kappa,\mu}(x) = \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - \kappa)}M_{\kappa,\mu}(x) + \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - \kappa)}M_{\kappa,-\mu}(x) .\]
As we show in Section 4, the asymptotic expressions of \( B_n \) and \( \bar{B}_n \) determine string equations.

In Section 3, we consider the compactification at self-dual radius. This makes the ingredients of Section 2 more tractable. We identify the asymptotic Hilbert spaces of the nonrelativistic fermion system with that of two-dimensional relativistic fermions. The asymptotic operators \( B^{\text{in,out}}_n \) and \( \bar{B}^{\text{in,out}}_n \) are then identified with even generators of the \( U(1) \) current subalgebra in a fermionic realization of \( W_{1+\infty} \) algebra obtained from relativistic fermions. For an example, \( B^{\text{out}}_n \sim g^{-1}W^{(0)}_{2n}g \) and \( \bar{B}^{\text{out}}_n \sim W^{(0)}_{-2n} \) are the standard generators of the \( U(1) \) subalgebra, and \( g \) is an element of \( GL(\infty) \) that acts on the Hilbert space of relativistic fermions. Thus \( g \) is exactly a counterpart of the \( S \)-matrix of Section 2, now realized on the Hilbert space of relativistic fermions. Furthermore, as one can see from the aforementioned expression of \( B^{\text{in,out}}_n \) and \( \bar{B}^{\text{in,out}}_n \), \( g \) also plays the role of an intertwining operator for \( W_{1+\infty} \) algebra. We shall indeed make use of this intertwining property of \( g \) to construct \( W_\infty \) symmetry of the deformed \( c = 1 \) strings.

In the latter half of this section we consider classical limits of these intertwining equations. One can obtain a classical counterpart of the intertwining equations by identifying the Planck constant \( \hbar \) with a particular combination of the parameters \((\mu,m)\) of the system and taking \( \hbar \to 0 \) limit. In the case of \( \hbar = -1/(i\sqrt{m}) \), the classical limit becomes independent of \( \mu \), and gives a formal solution of Polchinski’s classical scattering equations

\[
\alpha_\pm(y) = \alpha_\mp(y \mp 2\ln(1 + \alpha_\pm^2(y)))
\]

which are derived in [3] by the collective field approach to this deformed matrix model. On the other hand, in the case of \( \hbar = -1/(i\mu) \), the classical limit becomes independent of \( m \), and gives a formal solution of another variation of Polchinski’s classical scattering equations

\[
\alpha_\pm(y) = \alpha_\mp(y \mp \ln(1 + \alpha_\pm(y)))
\]

which are given in [16],[17] by the collective field approach to the standard \( c = 1 \) matrix model.

In Section 4, string equations are presented for these \( c = 1 \) strings (compactified at self-dual radius). As in the case of the standard \( c = 1 \) string [20],[21],[22], these
string equations are formulated in the framework of the Toda lattice hierarchy [18], [19].

We begin Section 4 by a rather detailed overview on the Toda lattice hierarchy in three different languages, that is, difference operators, infinite matrices and free fermions. A fundamental observation here is that, though the space of solutions of the Toda lattice hierarchy has a $W_{1+\infty} \oplus W_{1+\infty}$ symmetry, an intertwining relation as described in Section 3 can induce a relation between these two $W_{1+\infty}$ symmetries. This indeed occurs if the solution is characterized by a fixed point condition under these symmetries, and string equations are nothing else than such a fixed point condition. We then return to the deformed $c = 1$ strings. String equations turn out to have the forms

\[
\begin{align*}
L^{-2} &= \frac{1}{\mu^2 + m} \left(- (ML^{-1} - i\mu L^{-1})^2 + mL^{-2} \right), \\
L^2 &= \frac{1}{\mu^2 + m} \left(- (\bar{M}\bar{L} + (1 - i\mu)\bar{L})^2 + m\bar{L}^2 \right),
\end{align*}
\]

where $L, M, \bar{L}$ and $\bar{M}$ are the Lax and Orlov-Shulman operators of the Toda lattice hierarchy. The solution of these string equations, as in the case of $m = 0$ [20], is given by the generating function of (nonperturbative) tachyon scattering amplitudes (expressed in terms of the relativistic fermions)

\[
e^{F(t, \bar{t})} = \langle 0|e^{\sum_{k \geq 1} t_k W_k^{(0)}} e^{-\sum_{k \geq 1} \bar{t}_k W_k^{(0)}} |0 \rangle.
\]

The string coupling constant $g_{st}(\sim \hbar)$ will be recovered by the rescaling of parameters, $t_k \rightarrow t_k/\hbar$ and $\bar{t}_k \rightarrow \bar{t}_k/\hbar$. The results of Section 3 for classical limit are now described in the terminology of the dispersionless Toda hierarchy. The classical string equations are

\[
\begin{align*}
\bar{L}^{-2} &= \bar{M}\bar{L}^{-2} + \bar{L}^{-2}, \\
\bar{L}^2 &= \bar{M}\bar{L}^2 + \bar{L}^2,
\end{align*}
\]

for the case of $\hbar = -1/(i\sqrt{m})$,

\[
\begin{align*}
\bar{L}^{-2} &= (\bar{M}\bar{L}^{-1} + \bar{L}^{-1})^2, \\
\bar{L}^2 &= (\bar{M}\bar{L} + \bar{L})^2,
\end{align*}
\]

for the case of $\hbar = -1/(i\mu)$. Here $\bar{L}, \bar{M},$ etc are the classical analogues of the Lax and Orlov-Shulman operators of the Toda lattice hierarchy. These classical string equations clearly show that the classical scattering amplitudes, in both the pictures, are insensitive to deformations of the parameters.
2 Nonrelativistic Fermion

The matrix model becomes, after diagonalizing and double-scaling, a theory of free non-relativistic fermions in an inverted oscillator potential deformed by $1/x^2$. The hamiltonian of second-quantized fermions is given by

$$H = \int_0^{+\infty} dx \, \Psi^*(x) \left\{ -\frac{1}{2} \partial_x^2 + V(x) - \mu \right\} \Psi(x),$$

where the potential $V(x)$ is

$$V(x) = -\frac{1}{2} x^2 + \frac{m^2}{2x^2}, \quad m > 0,$$

and $\mu$ is the Fermi energy. $m$ is a deformation parameter of this nonrelativistic fermion system. $\Psi(x)$ and $\Psi^*(x)$ are second-quantized fermion fields with $x$ the rescaled matrix eigenvalue. Their equal time anti-commutation relations are set to

$$\{ \Psi^*(x_1), \Psi(x_2) \} = \delta(x_1 - x_2),$$
$$\{ \Psi^*(x_1), \Psi^*(x_2) \} = \{ \Psi(x_1), \Psi(x_2) \} = 0.$$

Notice that the integration in the R.H.S of equation (1) is over the region $[0, +\infty)$. This is due to the appearance of an infinite wall of the potential $V(x)$ at the origin. Then wave functions of the one-body system will be defined on this region and vanish at the origin $x = 0$, which can be considered as a boundary condition for second-quantized system (1).

2.1 Algebraic properties

We shall begin by studying algebraic properties associated with hamiltonian (1), which will play an important role in the description of the underlying Euclidean $c = 1$ strings. For this purpose we shall look into the one-body hamiltonian operator $L_x = -\frac{1}{2} \partial_x^2 + V(x)$. In this quantum mechanical system it may be convenient to introduce the following second-order differential operators

$$B_x = \frac{1}{2} (\partial_x - ix)^2 - \frac{m}{2x^2},$$
$$\bar{B}_x = \frac{1}{2} (\partial_x + ix)^2 - \frac{m}{2x^2}.$$
These two operators give an analogue of spectral generating algebra for this quantum mechanical system. They satisfy the commutation relations

\[
\begin{align*}
[ L_x , B_x ] &= 2iB_x , \\
[ L_x , \bar{B}_x ] &= -2i\bar{B}_x , \\
[ B_x , \bar{B}_x ] &= -4iL_x .
\end{align*}
\] (5)

So one can construct the series of hamiltonian eigenstates by the successive operations of \( B_x(\bar{B}_x) \) to the ground-state wave function. Notice that with an appropriate rescaling of the operators these commutation relations are reduced to those of \( su(1,1) \) algebra:

\[
\begin{align*}
[h,f] &= -if , \\
[h,e] &= ie , \\
[e,f] &= -2ih .
\end{align*}
\]

Since the Casimir element of \( su(1,1) \) is given by

\[
\Omega = h^2 - (ef + fe)/2,
\]

the corresponding operator \( \Omega_x \) becomes constant

\[
\begin{align*}
\Omega_x &\equiv \frac{1}{4} \left( L_x^2 - \frac{B_x\bar{B}_x + \bar{B}_x B_x}{2} \right) , \\
&= \frac{1}{4} \left( \frac{3}{4} - m \right) .
\end{align*}
\] (6)

To summarize one may say that algebraic nature of this quantum mechanical system is governed by the envelops of \( su(1,1) \) algebra (with Casimir condition (6)).

Let us return to nonrelativistic fermion system (1). It should be remarked first that an algebra of differential operators such as (5) lifts up to the second-quantized form. In fact, for differential operators \( D^{(i)}_x(i = 1, 2) \), we can construct second-quantized operators \( \hat{D}^{(i)} \) as

\[
\hat{D}^{(i)} = \int dx \, \Psi^*(x)D^{(i)}_x\Psi(x) .
\] (7)

The commutation relation among these operators satisfies

\[
[ \hat{D}^{(1)} , \hat{D}^{(2)} ] = [\hat{D}^{(1)},\hat{D}^{(2)}] ,
\] (8)

which tells us that the original algebra is preserved in the second-quantized form. We shall introduce the second-quantized operators which correspond to \( B_x \) and \( \bar{B}_x \)

\[
\begin{align*}
B_1 &= \int_0^\infty dx \, \Psi^*(x)B_x\Psi(x) , \\
\bar{B}_1 &= \int_0^\infty dx \, \Psi^*(x)\bar{B}_x\Psi(x) .
\end{align*}
\] (9)
su(1, 1) algebra \( \mathfrak{su}(1, 1) \) lifts up in the second-quantized form

\[
\begin{align*}
[H, B_1] &= 2iB_1, \\
[H, \bar{B}_1] &= -2i\bar{B}_1, \\
[B_1, \bar{B}_1] &= -4i(H + \mu N),
\end{align*}
\]

where \( N = \int_0^\infty dx \Psi^\dagger(x)\Psi(x) \) is the fermion number operator. Other elements of the \( \mathfrak{su}(1, 1) \) enveloping algebra in the one-body system may be described by the following set of second-quantized operators

\[
O_{a, J, M}^a \equiv \int_0^\infty dx \Psi^\dagger(x)L_x^a\bar{B}_x^J\bar{B}_x^{-M}\Psi(x),
\]

where \( a, J \in \mathbb{Z}_{\geq 0} \) and \( M = -J, -J + 2, \cdots, J - 2, J \). (Because of Casimir relation (6) all these operators are not independent.) One can also say these operators as the fermionic realization of the \( W_\infty \) algebra [23] which appears in the collective field analysis [4, 24] of the deformed matrix model. In this sense, as we shall verify it in the later section, the operators

\[
B_n \equiv O_{a=0, -n}^a, \quad \bar{B}_n \equiv O_{a=0}^a_{n,n}
\]

will be identified with the creation operators of the massless tachyon states with momentum \( \pm 2n \) in the Euclidean \( c = 1 \) strings.

### 2.2 Analytic properties

Let us consider analytical properties of nonrelativistic fermion system (1).

Firstly we shall look into the effect of the boundary condition imposed by the infinite wall of the potential \( V(x) \). The eigenvalue problem for the one-body hamiltonian operator \( L_x (\equiv -\frac{1}{2} \partial_x^2 + V(x)) \)

\[
L_x u_\xi(x) = \xi u_\xi(x)
\]

will be solved by using the Whittaker function. Since the Whittaker function \( W(z)(\equiv \frac{(\xi + e^z)^{J/2}}{\sqrt{2\pi e^z}}) \)
\[ M_{\kappa,\mu}(z), W_{\kappa,\mu}(z) \] satisfies the differential equation
\[
\left\{ \partial_z^2 + \left( -\frac{1}{4} + \frac{\kappa}{z} - \frac{\mu^2 - \frac{1}{4}}{z^2} \right) \right\} W(z) = 0 ,
\]
the following can be taken as the independent solutions for this eigenvalue problem
\[
x^{-\frac{1}{2}} M_{\frac{\kappa}{2}, \pm \alpha}(ix^2) \quad \text{or} \quad x^{-\frac{1}{2}} W_{\pm \frac{\kappa}{2}, \alpha}(\pm ix^2) ,
\]
where we set
\[
\alpha = \frac{1}{4} \sqrt{1 + 4m}.
\]

The behaviors at the origin \( x = 0 \) of these eigenfunctions are
\[
x^{-\frac{1}{2}} M_{\frac{\kappa}{2}, \pm \alpha}(ix^2) = e^{\mp \frac{\xi}{2} (1 + \alpha)} e^{-\frac{\xi^2}{2} x^{2\alpha}} \left[ 1 + O(x^2) \right] ,
\]
\[
x^{-\frac{1}{2}} W_{\pm \frac{\kappa}{2}, \alpha}(\pm ix^2) = e^{\mp \xi x^{2\alpha}} \left\{ \frac{\Gamma(-2\alpha) e^{\mp \frac{\xi^2}{2} (1 + \alpha)}}{\Gamma(\frac{1}{2} - \alpha + \frac{\kappa}{2})} x^{2\alpha} + \frac{\Gamma(2\alpha) e^{\mp \frac{\xi^2}{2} (1 - \alpha)}}{\Gamma(\frac{1}{2} + \alpha + \frac{\kappa}{2})} x^{-2\alpha} \right\} \left[ 1 + O(x^2) \right] .
\]

Since \( m > 0 \) implies \( \alpha = \sqrt{1 + 4m}/4 > 1/4 \), the physical boundary condition, \( u_\xi(x)|_{x=0} = 0 \), will be satisfied only for
\[
x^{-\frac{1}{2}} M_{\frac{\kappa}{2}, \alpha}(ix^2) .
\]
Thus it is very conceivable that any wave function can be expanded by the eigenfunctions \( x^{-\frac{1}{2}} M_{\frac{\kappa}{2}, \alpha}(ix^2) \). Say, for an example, we may expand the fermion fields \( \Psi(x), \Psi^*(x) \) as
\[
\Psi(x) = \int d\xi \, \Psi_{-\xi} x^{-\frac{1}{2}} M_{\frac{\kappa}{2}, \alpha}(ix^2) ,
\]
\[
\Psi^*(x) = \int d\xi \, \Psi^*_{\xi} x^{-\frac{1}{2}} M_{\frac{\kappa}{2}, \alpha}(ix^2) .
\]

---

1 We follow the normalization in [23]

\[ M_{\kappa,\mu}(x) = x^{\mu+\frac{1}{2}} e^{-\frac{x^2}{2}} F(\mu - \kappa + 1, 2\mu + 1; x) ,
\]
\[ W_{\kappa,\mu}(x) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - \kappa)} M_{\kappa,\mu}(x) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \kappa)} M_{-\kappa,\mu}(x) ,
\]
where \( F(a, b; x) \) is a degenerate hypergeometric function.
Notice that, from the linear independence of solutions (13), the wave function \( x^{-\frac{1}{2}}M_{\frac{\xi}{2},\alpha}(ix^2) \)
can be also written as the linear combination of \( x^{-\frac{1}{2}}W_{\pm\frac{\xi}{2},\alpha}(\pm ix^2) \)

\[
x^{-\frac{1}{2}}M_{\frac{\xi}{2},\alpha}(ix^2) = \Gamma(2\alpha + 1)e^{-\frac{\pi\xi^2}{4}} \left\{ \frac{e^{\frac{\pi\xi(1+2\alpha)}{2}}}{{\Gamma}(\alpha + \frac{1}{2} + \frac{\xi}{2\Gamma})} x^{-\frac{1}{2}}W_{\frac{\xi}{2},\alpha}(ix^2) + \frac{1}{{\Gamma}(\alpha + \frac{1}{2} - \frac{\xi}{2\Gamma})} x^{-\frac{1}{2}}W_{-\frac{\xi}{2},\alpha}(-ix^2) \right\} .
\]

The asymptotic behavior of the wave function \( x^{-\frac{1}{2}}M_{\frac{\xi}{2},\alpha}(ix^2) \) can be read from those of \( x^{-\frac{1}{2}}W_{\pm\frac{\xi}{2},\alpha}(\pm ix^2) \)

\[
x^{-\frac{1}{2}}W_{\pm\frac{\xi}{2},\alpha}(\pm ix^2) \sim e^{\frac{2\pi i}{4}x} e^{-\frac{1}{2}x^2 \pm i\xi\text{ln}x}[1 + O(\frac{1}{x^2})],
\]

which is valid for \(-\frac{\pi}{4} < \text{Arg}x < \frac{\pi}{4}\).

Nextly we shall consider eigenvalue problem (13) from rather different points of view. For this purpose we will introduce unitary transforms on the space of the one-body wave functions. These transforms will be used in the construction of the asymptotic fields for \( \Psi(x) \) and \( \Psi^*(x) \). The underlying classical mechanics of the hamiltonian operator \( L_x \) is described by

\[
\frac{\partial x}{\partial \tau} = p , \quad \frac{\partial p}{\partial \tau} = -\frac{\partial V(x)}{\partial x} .
\]

Especially on the dynamical orbit with the energy \( E(= \frac{p^2}{2} + V(x)) \) equal to the Fermi energy \( \mu \), the measure \( d\tau \) has the form

\[
d\tau = \frac{dx}{\sqrt{2(\mu - V(x))}},
\]

from which one can see that the R.H.S of (23) is the 1-form invariant under the shift \( \tau \rightarrow \tau + \delta\tau \) on this orbit. The integration of (23)

\[
\tau = \int_{x_0}^{x} \frac{dy}{\sqrt{2(\mu - V(y))}}, \quad (x \geq x_0),
\]

where \( x = x_0 \) is the turning point of the dynamical orbit \( E = \mu \), maps the region \( x_0 \leq x < +\infty \) onto the region \( 0 \leq \tau < +\infty \) bijectively. Let us introduce similarity transforms
\( \mathcal{J}_\pm \) which map the wave functions \( u(x) \)\(^\mathbb{R} \) to the functions on the region \( 0 \leq \tau < +\infty \)

\[
\mathcal{J}_\pm : u(x) \mapsto \mathcal{J}_\pm u(\tau) \equiv J_\pm (x(\tau)) u(x(\tau)) ,
\]

where

\[
J_\pm (x) = \{2(\mu - V(x))\}^{1/4} e^{\pm i \int_{x_0}^x \sqrt{2(\mu - V(y))} dy} .
\]

Notice that, especially for the wave functions which support are on the region \( x > x_0 \), these similarity transforms \( \mathcal{J}_\pm \) are unitary, that is, preserve the \( L^2 \)-norm,

\[
|\mathcal{J}_\pm u\rangle^2 = \int_0^{+\infty} d\tau \overline{\mathcal{J}_\pm u(\tau)} \mathcal{J}_\pm u(\tau) ,
\]

\[
= \int_0^{+\infty} d\tau \overline{dx \left( \frac{d}{d\tau} u(x(\tau))u(x(\tau)) \right)} ,
\]

\[
= \int_0^{+\infty} dx \overline{u(x)u(x)} ,
\]

\[
\equiv |u|_x^2 .
\]

Under these transforms the one-body hamiltonian operator \( L_x \) will change into \( \tilde{L}_\tau^\pm \). Their explicit forms are given by

\[
\tilde{L}_\tau^\pm \equiv J_\pm L_x J_\pm^{-1} ,
\]

\[
= \pm i \partial_\tau + \mu - \frac{1}{2} f^{\frac{1}{2}} \partial_\tau f^{-\frac{1}{2}} \partial_\tau f^{-\frac{1}{2}} ,
\]

where \( f(\tau) = \sqrt{2(\mu - V(x(\tau)))} \). One can also consider the eigenvalue problems for these transformed operators \( \tilde{L}_\tau^\pm \),

\[
\tilde{L}_\tau^\pm v_\xi^\pm (\tau) = \xi v_\xi^\pm (\tau) .
\]

Though these eigenvalue problems are equivalent to (13) via the correspondence \( v_\xi^\pm (\tau) = \mathcal{J}_\pm u_\xi(\tau) \), we may consider them perturbatively. Namely we divide \( \tilde{L}_\tau^\pm \) into

\[
\tilde{L}_\tau^\pm = (\pm i \partial_\tau + \mu) + \tilde{L}_\tau^{\text{int}} ,
\]

\[
(\tilde{L}_\tau^{\text{int}} \equiv -\frac{1}{2} f^{-\frac{1}{2}} \partial_\tau f^{-1} \partial_\tau f^{-\frac{1}{2}})
\]

\(^2\) Strictly speaking we should restrict the class of the wave functions to those which have their support on \( x > x_0 \).
and then treat the second order differential operator $\tilde{L}^\text{int}_\tau$ as a perturbation to $\pm i\partial_\tau + \mu$.

The zero-th order approximation for eigenvalue problems (29) gets the form

$$(\pm i\partial_\tau + \mu) v^\pm_\xi (0) (\tau) = \xi v^\pm_\xi (0) (\tau),$$

from which one can see

$$v^\pm_\xi (0) (\tau) = e^{\mp i(\xi - \mu)\tau}. \quad (32)$$

Under the inverse transforms $J^{-1}_\pm$ the zero-th order solutions $v^\pm_\xi (0) (\tau)$ are mapped to

$$J^{-1}_\pm v^\pm_\xi (0) (x) = \frac{1}{2(\mu - V(x))^{\frac{1}{4}}} e^{\mp i \int_{x_0}^x \sqrt{2(\mu - V(y))} dy} e^{\mp i(\xi - \mu)\tau(x)}, \quad (33)$$

which asymptotic behaviors can be read as

$$J^{-1}_\pm v^\pm_\xi (0) (x) \sim e^{\mp i a} e^{\pm \frac{i}{4} \ln \frac{\mu^2 + m}{4}} x^{-\frac{1}{2}} e^{\mp \frac{i}{4} \xi \ln x} \left[1 + O\left(\frac{1}{x^2}\right)\right], \quad (34)$$

where $a$ is the constant independent of $\xi$ (and is irrelevant to our discussion). These asymptotic behaviors coincide (up to the leading order) with those of $x^{-\frac{1}{2}} W_{\pm,\xi,\alpha} (\pm ix^2)$ (21) by constant multiplications. So one may expect that the eigenfunctions $v^\pm_\xi (\tau)$ which are constructed from $v^\pm_\xi (0) (\tau)$ by the perturbative expansion, that is, $v^\pm_\xi (\tau) \equiv v^\pm_\xi (0) (\tau) + v^\pm_\xi (1) (\tau) + \cdots$, lead to the asymptotic expansion of $x^{-\frac{1}{2}} W_{\pm,\xi,\alpha} (\pm ix^2)$

$$e^{\mp i a - \frac{x^2}{2} + \frac{\mu}{4} \ln \frac{\mu^2 + m}{4}} x^{-\frac{1}{2}} W_{\pm,\xi,\alpha} (\pm ix^2) \sim J^{-1}_\pm v^\pm_\xi (x). \quad (35)$$

We shall proceed to the the second-quantized system. Under similarity transforms $J_\pm$ (25) second-quantized hamiltonian operator $H$ (1) will become

$$\tilde{H}_\pm = \tilde{H}_\pm^{(0)} + \tilde{H}_\pm^\text{int}, \quad (36)$$

$$\tilde{H}_\pm^{(0)} = \int_0^{+\infty} \! d\tau \, \tilde{\Psi}_\pm^*(\tau)(\pm i\partial_\tau)\tilde{\Psi}_\pm(\tau), \quad \tilde{H}_\pm^\text{int} = \int_0^{+\infty} \! d\tau \, \tilde{\Psi}_\pm^*(\tau)\tilde{L}_\tau^\text{int}\tilde{\Psi}_\pm(\tau),$$

$^3$The following estimates are useful

$$\int_{x_0}^x \! \frac{dy}{\sqrt{2(\mu - V(y))}} \sim \ln x - \frac{1}{4} \ln \frac{\mu^2 + m}{4} + O\left(\frac{1}{x^2}\right), \quad (37)$$

$$\int_{x_0}^x \! \frac{dy}{\sqrt{2(\mu - V(y))}} \sim \frac{x^2}{2} + \mu \ln x - \frac{\mu}{4} \ln \frac{\mu^2 + m}{4} + a + O\left(\frac{1}{x^2}\right). \quad (38)$$

where $a = \frac{\mu}{2} - \frac{\sqrt{m}}{2} \left(\frac{\pi}{2} + \arcsin \frac{\mu}{\sqrt{\mu^2 + m}}\right)$. 

12
where $\tilde{H}^{(0)}_\pm$ can be considered as the analogue of the hamiltonian for 1 + 1 dimensional free relativistic fermions [13, 14].

With the assumption that the contributions of $\tilde{H}^{int}_\pm$ at $\tau \gg 0$ are negligible it may be possible to introduce the asymptotic fields $\tilde{\Psi}_{in,out}(\tau)$ by

$$
\tilde{\Psi}_{in}(\tau) = \int d\xi \, \psi_{\xi}^{in}(\tau) v^{(0)}_{\xi}(\tau),
$$

$$
\tilde{\Psi}_{out}(\tau) = \int d\xi \, \psi_{\xi}^{out}(\tau) v^{(0)}_{\xi}(\tau),
$$

(37)

such that $\tilde{\Psi}_+(\tau) \rightarrow \tilde{\Psi}_{in}(\tau)$, $\tilde{\Psi}_-(\tau) \rightarrow \tilde{\Psi}_{out}(\tau)$ as $\tau \rightarrow +\infty$. It is important to note that these asymptotic fields $\tilde{\Psi}_{in,out}(\tau)$ are not independent to each other. Due to expansion (19) of $\Psi(x)$ by the Whittaker functions, each mode $\psi_{\xi}^{in,out}$ in (37) is related with the mode $\Psi_{\xi}^{-\xi}$ in (19) through asymptotic relation (35) and decomposition (20) of the one-body eigenfunction. This constraint gives us the following relation between $\psi_{\xi}^{in}$ and $\psi_{\xi}^{out}$.

$$
\psi_{\xi}^{in} = e^{i(2a+n+1+2\alpha)} e^{-i\frac{\xi}{2} \ln \frac{2^{1+2\alpha} \Gamma(\frac{1}{2} + \alpha + \frac{i}{2} \xi)}{\Gamma(\frac{1}{2} + \alpha - \frac{i}{2} \xi)}} \psi_{\xi}^{out}.
$$

(38)

We can also repeat the similar discussion for the conjugate field $\Psi^*(x)$. By introducing the asymptotic fields as

$$
\tilde{\Psi}_{in}^*(\tau) = \int d\xi \, \psi_{\xi}^{*in} v^{(0)}_{\xi}(\tau),
$$

$$
\tilde{\Psi}_{out}^*(\tau) = \int d\xi \, \psi_{\xi}^{*out} v^{(0)}_{\xi}(\tau),
$$

(39)

where the bar denotes the complex conjugation, the relation between the modes $\psi_{\xi}^{*in,out}$ can be read as

$$
\psi_{\xi}^{*in} = e^{-i(2a+n+1+2\alpha)} e^{i\frac{\xi}{2} \ln \frac{2^{1+2\alpha} \Gamma(\frac{1}{2} + \alpha - \frac{i}{2} \xi)}{\Gamma(\frac{1}{2} + \alpha + \frac{i}{2} \xi)}} \psi_{\xi}^{*out}.
$$

(40)

Notice that, from the forms of these asymptotic fields, the (non-vanishing) anti-commutation relations among these in-coming (out-going) fields can be expressed in terms of their modes.

$$
\{ \psi_{\xi_1}^{*in}, \psi_{\xi_2}^{*in} \} = \delta(\xi_1 + \xi_2)
$$

$$
\{ \psi_{\xi_1}^{*out}, \psi_{\xi_2}^{*out} \} = \delta(\xi_1 + \xi_2).
$$

(41)

Scattering relations (38) and (40) are of course consistent with these anti-commutation relations.
2.3 Euclidean strings

The scattering amplitudes of the Euclidean strings may be described by the analytic continuation of the theory to the Euclidean region. This prescription for the analytic continuation was explained in [15]. The energy $\xi$ of the fermion modes $\psi_{i\xi}^{in,out}$ and $\psi_{i\xi}^{*in,out}$ will replace $\xi - \mu \to iq$, where $q \in R$. From now on we shall write these fermion modes as $\psi_{-q}^{in,out}$ and $\psi_{q}^{*in,out}$. After this continuation, relations (38) and (40) between the in-coming and out-going fermion modes may be written as

$$
\psi_{q}^{*in} = r(q)\psi_{q}^{*out},
\psi_{-q}^{in} = r(q)^{-1}\psi_{-q}^{out},
$$

(42)

where we introduce $r(q)$,

$$
r(q) = \left(\frac{4}{\mu^2 + m}\right)^{\frac{3}{2}} \frac{\Gamma(\frac{1}{2} + \alpha - i\mu \frac{q}{2} + \frac{q}{2})}{\Gamma(\frac{1}{2} + \alpha + i\mu \frac{q}{2} - \frac{q}{2})}.
$$

(43)

In this expression we absorb the $q$–independent phases which appear in (38) and (40) into the normalizations of the asymptotic fields $\tilde{\Psi}_{in,out}^{in}(\tau)$ and $\tilde{\Psi}_{in,out}^{*}(\tau)$. The value of $r(q)$ coincides with that obtained in [13] from the asymptotic behavior of the resolvent kernel, $< x | L_{x+\mu+iq} | y >$, and, as is expected from the form of the one-body hamiltonian $L_x$, taking $m \to 0$, the value of $r(q)$ becomes equal to that for the type I theory of the inverse harmonic oscillator potential, $-x^2/2$, with an infinite wall at the origin [13].

Let us introduce the in-coming and out-going vacuums of these asymptotic fermions by the condition

$$
< in | \psi_{q}^{in} >= 0, \quad < in | \psi_{q}^{*in} >= 0 \quad \text{for} \quad q < 0,
\psi_{q}^{out} | out >= 0, \quad \psi_{q}^{*out} | out >= 0 \quad \text{for} \quad q > 0.
$$

(44)

The scattering amplitudes among the nonrelativistic fermions will be reduced to the combination of the following matrix elements

$$
< in | \prod_i \psi_{p_i}^{in} \prod_j \psi_{q_j}^{*in} \prod_k \psi_{r_k}^{*out} \prod_l \psi_{s_l}^{out} | out >= .
$$

(45)
It is important to note that these matrix elements can be also realized in terms of two-dimensional relativistic fermion [15]. In fact matrix element (45) is equal to
\[
< 0 | \prod_i \psi_{p_i} \prod_j \psi^*_{q_j} g \prod_k \psi^*_{r_k} \prod_l \psi_{s_l} | 0 > ,
\]
where \( \psi_q \) and \( \psi^*_q \) are the relativistic fermions with their (non-vanishing) anti-commutation relation,
\[
\{ \psi_q, \psi^*_q \} = \delta(q_1 + q_2) ,
\]
and the ground-state is introduced by the condition,
\[
< 0 | \psi_q = 0, \quad < 0 | \psi^*_q = 0 \quad \text{for} \quad q < 0 ,
\]
\[
\psi_q | 0 > = 0, \quad \psi^*_q | 0 > = 0 \quad \text{for} \quad q > 0 .
\]

\( g \), an element of \( GL(\infty) \), is introduced such that it transforms the relativistic fermions as
\[
g^{-1} \psi^*_q g = r(q) \psi^*_q ,
\]
\[
g^{-1} \psi_{-q} g = r(q)^{-1} \psi_{-q} ,
\]
for \( \forall q \). Notice that relation (12) is now translated to the adjoint action of \( g \) on the relativistic fermions \( \psi_q \) and \( \psi^*_q \).

Nextly let us consider the asymptotic forms of operators \( B_n \) and \( \bar{B}_n \) (12). From the definitions of these operators (11) their actions on the nonrelativistic fermions are
\[
[ B_n , \Psi(x) ] = -B^n_x \Psi(x) ,
\]
\[
[ \bar{B}_n , \Psi(x) ] = -\bar{B}^n_x \Psi(x) .
\]

The actions of the \( 2n \)-th differential operators \( B^n_x \left( \equiv \left( \frac{1}{2} (\partial_x - ix)^2 - \frac{m}{2x^2} \right)^n \right) \) and its complex conjugate \( \bar{B}^n_x \) on the nonrelativistic fermions (the R.H.Ss of (18)) can be analyzed further from their actions on one-body wave function \( x^{-\frac{1}{2}} M_{\frac{\xi}{2},\alpha}(ix^2) \) (18). Taking account of the Euclidean continuation we may expand the nonrelativistic fermion field \( \Psi(x) \) by the Whittaker function \( x^{-\frac{1}{2}} M_{-\frac{\mu}{2} + \frac{\xi}{2},\alpha}(ix^2) \)
\[
\Psi(x) = \int dq \, \Psi_{-q} x^{-\frac{1}{2}} M_{-\frac{\mu}{2} + \frac{\xi}{2},\alpha}(ix^2) .
\]
The actions of $B_x$ and $\bar{B}_x$ on the wave function $x^{-\frac{1}{2}}M_{-\frac{\mu}{2}+\frac{q}{2},\alpha}(ix^2)$ can be obtained from the recursion relations of the Whittaker function

$$B_x x^{-\frac{1}{2}}M_{-\frac{\mu}{2}+\frac{q}{2},\alpha}(ix^2) = -2i \left( \frac{1}{2} + \alpha - \frac{i\mu}{2} + \frac{q}{2} \right) x^{-\frac{1}{2}}M_{-\frac{\mu}{2}+\frac{q}{2},\alpha}(ix^2) ,$$

$$\bar{B}_x x^{-\frac{1}{2}}M_{-\frac{\mu}{2}+\frac{q}{2},\alpha}(ix^2) = 2i \left( \frac{1}{2} + \alpha + \frac{i\mu}{2} - \frac{q}{2} \right) x^{-\frac{1}{2}}M_{-\frac{\mu}{2}+\frac{q}{2},\alpha}(ix^2) .$$

(50)

Then we can evaluate the both sides of equations (48) so that we can see from (48) that $B_n$ and $\bar{B}_n$ transform the fermion modes $\Psi_{-q}$ as

$$[ B_n , \Psi_{-q} ] = -(-2i)^n \frac{\Gamma(\frac{1}{2} + \alpha - \frac{i\mu}{2} + \frac{q}{2})}{\Gamma(\frac{1}{2} + \alpha - \frac{i\mu}{2} + \frac{q+2n}{2})} \Psi_{-q+2n} ,$$

$$[ \bar{B}_n , \Psi_{-q} ] = -(-2i)^n \frac{\Gamma(\frac{1}{2} + \alpha + \frac{i\mu}{2} - \frac{q}{2})}{\Gamma(\frac{1}{2} + \alpha + \frac{i\mu}{2} - \frac{q+2n}{2})} \Psi_{-q-2n} .$$

(51)

These actions on $\Psi_{-q}$ will be handed over on the asymptotic modes $\psi^{in, out}_{-q}$. Using decomposition (20), fermionic field $\Psi(x)$ (49) can be written as the sum

$$\Psi(x) = \int dq \ s_+(q) \Psi_{-q} \left( \frac{4}{\mu^2 + m} \right)^{\frac{1}{4}} x^{-\frac{1}{2}}W_{-\frac{\mu}{2}+\frac{q}{2},\alpha}(ix^2)$$

$$+ \int dq \ s_-(q) \Psi_{-q} \left( \frac{4}{\mu^2 + m} \right)^{-\frac{1}{4}} x^{-\frac{1}{2}}W_{-\frac{\mu}{2}-\frac{q}{2},\alpha}(-ix^2) .$$

(52)

Each integration in the R.H.S of (52) will reduce to the asymptotic fields $(J_\pm)^{-1} \tilde{\psi}_{in, out}(x) = \int dq \ \psi^{in, out}_{-q} (J_\pm)^{-1} \psi^{\pm(0)}_{\mu+iq}(x)$. In fact, by using the explicit correspondence between the wave functions (35)

$$\left( \frac{4}{\mu^2 + m} \right)^{\pm\frac{1}{4}} x^{-\frac{1}{2}}W_{-\frac{\mu}{2}+\frac{q}{2},\alpha}(\pm ix^2) \sim (J_\pm)^{-1} \psi^{\pm}_{\mu+iq}(x) ,$$

(53)

---

The recursion relations are 23.

$$x \partial_x M_{\kappa,\mu}(x) = \pm \left( \frac{x}{2} - \kappa \right) M_{\kappa,\mu}(x) + (\mu \pm \kappa + \frac{1}{2}) M_{\kappa,\mu}(x).$$

su(1, 1) structure (3) of the one-body system has its origin in these recurrent relations of the Whittaker function.

---

4. We introduce the following quantities

$$s_+(q) = e^{\frac{\pi}{2}(i\mu-q+1+2\alpha)} \frac{\Gamma(1+2\alpha)}{\Gamma(\frac{1}{2}+\alpha-\frac{\mu}{2}+\frac{q}{2})} \left( \frac{4}{\mu^2 + m} \right)^{-\frac{1}{4}} ,$$

$$s_-(q) = e^{\frac{\pi}{2}(i\mu-q)} \frac{\Gamma(1+2\alpha)}{\Gamma(\frac{1}{2}+\alpha+\frac{\mu}{2}-\frac{q}{2})} \left( \frac{4}{\mu^2 + m} \right)^{\frac{1}{4}} .$$

---

5. The recursion relations are 23.

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\[x \partial_x M_{\kappa,\mu}(x) = \pm \left( \frac{x}{2} - \kappa \right) M_{\kappa,\mu}(x) + (\mu \pm \kappa + \frac{1}{2}) M_{\kappa,\mu}(x).\]
one can derive the relations between these several fermion modes
\[
\psi_{-q}^{\text{in}} = \frac{e^{-i \pi q}}{\Gamma(\frac{1}{2} + \alpha - \frac{\mu}{2} + \frac{q}{2})} \left( \frac{4}{\mu^2 + m} \right)^{-\frac{q}{2}} \Psi_{-q},
\]
\[
\psi_{-q}^{\text{out}} = \frac{e^{-i \pi q}}{\Gamma(\frac{1}{2} + \alpha + \frac{\mu}{2} - \frac{q}{2})} \left( \frac{4}{\mu^2 + m} \right)^{\frac{q}{2}} \Psi_{-q}. \tag{54}
\]
The asymptotic actions of \(B_n\) and \(\bar{B}_n\) will be obtained from transforms (51) by changing \(\Psi_q\) to \(\psi_{q}^{\text{in, out}}\) through correspondence (54)
\[
\begin{align*}
\left[ B_n^{\text{in}}, \psi_{-q}^{\text{in}} \right] &= -(2i)^n \left( \frac{\mu^2 + m}{4} \right)^{\frac{q}{2}} \psi_{-q+2n}^{\text{in}} , \\
\left[ \bar{B}_n^{\text{in}}, \psi_{-q}^{\text{in}} \right] &= -(2i)^n \left( \frac{\mu^2 + m}{4} \right)^{\frac{q}{2}} \frac{r(q + 2n)}{r(q)} \psi_{-q-2n}^{\text{in}} , \\
\left[ B_n^{\text{out}}, \psi_{-q}^{\text{out}} \right] &= -(2i)^n \left( \frac{\mu^2 + m}{4} \right)^{\frac{q}{2}} \frac{r(q)}{r(q - 2n)} \psi_{-q+2n}^{\text{out}} , \\
\left[ \bar{B}_n^{\text{out}}, \psi_{-q}^{\text{out}} \right] &= -(2i)^n \left( \frac{\mu^2 + m}{4} \right)^{\frac{q}{2}} \psi_{-q-2n}^{\text{out}} . \tag{55}
\end{align*}
\]
Since \(B_n\) and \(\bar{B}_n\) have fermion bilinear forms (11) these actions completely determine their forms. It may be also convenient to write them in terms of the relativistic fermions. Let us introduce the operators
\[
\begin{align*}
B_n &\equiv -(2i)^n \left( \frac{\mu^2 + m}{4} \right)^{\frac{q}{2}} \int dp \psi_{-p}^* \psi_{p+2n} , \\
\bar{B}_n &\equiv -(2i)^n \left( \frac{\mu^2 + m}{4} \right)^{\frac{q}{2}} \int dp \psi_{-p}^* \psi_{p-2n} , \tag{56}
\end{align*}
\]
for \(n > 0\). Then we may write \(B_n^{\text{in, out}}\) and \(\bar{B}_n^{\text{in, out}}\) as
\[
\begin{align*}
B_n^{\text{in}} &= B_n , & \bar{B}_n^{\text{in}} &= g \bar{B}_n g^{-1} , \\
B_n^{\text{out}} &= g^{-1} B_n g , & \bar{B}_n^{\text{out}} &= \bar{B}_n , \tag{57}
\end{align*}
\]
which tell us the scattering relations of \(B_n\) and \(\bar{B}_n\). From expressions (56) we can see that \(B_n\) and \(\bar{B}_n\) are essentially the (even) modes of a relativistic free boson field \(\partial \phi\) or equivalently \(B_n^{\text{in}}\) and \(\bar{B}_n^{\text{out}}\) are the (even) modes of asymptotic (free) boson fields \(\partial \phi_{\text{in, out}}\). In the next section we will show that these asymptotic boson fields can be exactly identified with those appeared in the collective field approach.
3 Compactification at Self-Dual Radius

In this section we shall study the Euclidean strings compactified at self-dual radius. The prescription for the compactification was explained in \[26, 20\]. At the compactification radius $\beta$ the Euclidean momentum $q \in \mathbb{R}$ of the relativistic fermions $\psi_q, \psi^*_q$ are discretized to $\frac{n+\frac{1}{2}}{\beta}$ ($n \in \mathbb{Z}$). In particular, at the self-dual radius $\beta = 1$, action (47) of $g$ on the relativistic fermions gets the form

$$
g^{-1}\psi^*_ng = e^{\alpha_n}\psi^*_n, \quad g^{-1}\psi_{-n}g = e^{-\alpha_n}\psi_{-n},
$$

(58)

where we set

$$
e^{\alpha_n} \equiv r(n + \frac{1}{2}) = \left(\frac{4}{\mu^2 + m}\right)^{n+\frac{1}{2}} \frac{\Gamma\left(\frac{3}{4} + \alpha - i\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{4} + \alpha + i\frac{n}{2} - \frac{1}{2}\right)}.
$$

(59)

Notice that, for the notational convenience, we shift the label of the fermion modes by 1/2. We shall summarize our usage of the relativistic fermions: $\psi(z) = \sum_n \psi_n z^{-n}$ and $\psi^*(z) = \sum_n \psi^*_n z^{-n-1}$ are relativistic fermion fields with the nontrivial operator product expansion, $\psi(z)\psi^*(w) \sim \frac{1}{z-w}$, or equivalently $\{\psi_n, \psi^*_m\} = \delta_{n+m,0}$. Their ground-state $|n\rangle$ ($n \in \mathbb{Z}$) is introduced by the condition

$$
\psi_m|n\rangle = 0 \quad m > -n ,
$$

$$
\psi^*_m|n\rangle = 0 \quad m < n .
$$

(60)

$n$ is dual to $|n\rangle$, that is, $\langle n|m\rangle = \delta_{n,m}$, and it satisfies the condition

$$
\langle n|\psi_m = 0 \quad m \leq -n ,
$$

$$
\langle n|\psi^*_m = 0 \quad m < n .
$$

(61)

3.1 $su(1, 1)$ structure

Let us begin by studying the implications of relations (57) between $B^\text{in}_n$ and $B^\text{out}_n$ (or $B^\text{out}_n$ and $B^\text{out}_n$) in these compactified strings. For this purpose it may be convenient to review

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6 Strictly speaking, under this shift of modes, $(\psi, \psi^*)$ is the ghost system with central charge $c = -2$. 

the free fermion realization of pseudo-differential operators \[28\]. We shall introduce an infinite dimensional vector space \(\mathcal{V}\) with basis \(\{e_n \, (n \in \mathbb{Z})\}\), on which \(X\) and \(\partial_X\) act as \(Xe_n = (n+1)e_{n+1}\) and \(\partial_X e_n = e_{n-1}\). \(\psi_n\) and \(\psi_n^*\) have the following realization on \(\wedge \mathcal{V}\),

\[
\psi_n = e_{-n} \wedge, \\
\psi_n^* = i_{e_n},
\]

where \(v \wedge\) and \(i_v\) denote respectively the exterior and interior products by \(v \in \mathcal{V}\). With this representation of the fermions any polynomial of \(X\), \(\partial_X\) and \(\partial_X^{-1}\) gets the fermion bilinear form.\[7\]

Especially pseudo-differential operators, \(X^k \partial_X^{n+k}\), have the form

\[
:X^k \partial_X^{n+k} : = \sum_{p=-\infty}^{+\infty} k! \left(\frac{p}{k}\right) :\psi_{-p} \psi_{p+n}^*:.
\]

The normal ordering \(:\) is prescribed relative to the ground-state \(|0\rangle \in \wedge \mathcal{V}\), and \(\left(\frac{p}{k}\right) \equiv \frac{p(p-1)\cdots(p-k+1)}{k!}\) is the binomial coefficient. These (normal ordered) actions of pseudo-differential operators on \(\wedge \mathcal{V}\) constitute \(W_{1+\infty}\) algebra\[8\], a subalgebra of \(\hat{gl}(\infty)\) algebra. Notice that the R.H.S of \((63)\) is a mode of the higher spin field \(W^{(k)}(z) \equiv: \partial_z^k \psi \psi^*: (z)\). By introducing the mode expansion for this higher spin field as

\[
W^{(k)}(z) = \sum_{n=-\infty}^{+\infty} W^{(k)}_n z^{-n-k-1},
\]

the R.H.S of \((63)\) becomes equal to \(W^{(k)}_n\). Therefore the generators of this \(W_{1+\infty}\) algebra can be also written as

\[
W^{(k)}_n = : X^k \partial_X^{n+k} :.
\]

With this brief review we shall return to our problem. We notice that \(B_n\) and \(\bar{B}_n\) \((56)\) become essentially equal to \(W^{(0)}_{2n}\) and \(W^{(0)}_{-2n}\) after this compactification. Then, from realizations \((57)\) of \(B_{2n}^{in,out}\) and \(\bar{B}_{2n}^{in,out}\), one can see that the relations between \(\bar{B}_{2n}^{in}\) and \(\bar{B}_{2n}^{out}\) (or \(B_{2n}^{in}\) and \(B_{2n}^{out}\)) are reduced to those between \(W^{(0)}_{2n}\) and \(g W^{(0)}_{-2n} g^{-1}\) (or \(W^{(0)}_{2n}\) and \(g^{-1} W^{(0)}_{2n} g\)). So it may be interesting to describe \(g W^{(0)}_{-2n} g^{-1}\) (or \(g^{-1} W^{(0)}_{2n} g\)) in terms of

---

\[7\] The action of pseudo-differential operators on \(\mathcal{V}\) lifts up on \(\wedge \mathcal{V}\) with a standard normal ordering prescription.

\[8\] With our realization \(\{W^{(k)}_n\}\) include the Virasoro algebra \(\{W^{(1)}_n\}\) with central charge equal to \(-2\).
the generators of $W_{1+\infty}$ algebra. In particular, for the case of $n = 1$, $gW_{-2}^{-1}$ gets the form

$$
g W_{-2}^{-1} = \sum_p : g\psi_p g^{-1} \cdot g\psi_p^* g^{-1} : ,
$$

$$
= \sum_p \frac{e^{\alpha_p}}{e^{\alpha_{p-2}}} : \psi_p \psi_p^* : ,
$$

$$
= \frac{1}{\mu^2 + m} \left\{ (m - i\mu(i\mu + 1)) W_{-2}^{(0)} + 2i\mu W_{-2}^{(1)} - W_{-2}^{(2)} \right\} . \tag{66}
$$

Through correspondence (65) one can also interpret this equation by pseudo-differential operators

$$
g : \partial_X^{-2} : g^{-1} = \frac{1}{\mu^2 + m} : \left\{ -(X - i\mu \partial_X^{-1})^2 + m\partial_X^{-2} \right\} : . \tag{67}
$$

As for $g^{-1}W_{2}^{(0)} g$, it has the form

$$
g^{-1} W_{2}^{(0)} g = \frac{1}{\mu^2 + m} \left\{ (m - (i\mu - 1)(1\mu - 2)) W_{2}^{(0)} + 2(i\mu - 2)W_{2}^{(1)} - W_{2}^{(2)} \right\} , \tag{68}
$$

or equivalently

$$
g^{-1} : \partial_X^2 : g = \frac{1}{\mu^2 + m} : \left\{ -(X \partial_X^2 + (1 - i\mu)\partial_X)^2 + m\partial_X^2 \right\} : . \tag{69}
$$

Notice that, as we will clarify it in the next section, these two equations (67) and (69) constitute string equations for the deformed $c = 1$ strings.

Explicit forms (66) (or (68)) of $gW_{-2}^{-1}$ or $g^{-1}W_{2}^{(0)} g$ are useful to investigate the $su(1,1)$ structure of compactified $c = 1$ strings. Let us introduce the operator $\mathcal{H}$ as

$$
\mathcal{H} \equiv \frac{-i}{\sqrt{\mu^2 + m}} \left\{ W_0^{(1)} + \left( \frac{1}{2} - i\mu \right) W_0^{(0)} - \frac{m + \mu^2}{2} \right\} . \tag{70}
$$

Then $W_{2}^{(0)} , gW_{-2}^{-1}$ and $\mathcal{H}$ forms the $su(1,1)$ algebra

$$
[ W_{2}^{(0)} , gW_{-2}^{-1} ] = \frac{-4i}{\sqrt{\mu^2 + m}} \mathcal{H} ,
$$

$$
[ \mathcal{H} , W_{2}^{(0)} ] = \frac{2i}{\sqrt{\mu^2 + m}} W_{2}^{(0)} ,
$$

$$
[ \mathcal{H} , gW_{-2}^{-1} ] = \frac{-2i}{\sqrt{\mu^2 + m}} gW_{-2}^{-1} . \tag{71}
$$
The above commutators are evaluated, after substituting the R.H.S of (66) for \( g W^{(0)}_2 g^{-1} \), by utilizing the following commutation relations of \( W_{1+\infty} \) algebra [27]

\[
\left[ W^{(k)}_m, W^{(l)}_n \right] = \sum_{r \geq 0} \left\{ \binom{k + m}{r} \binom{k + l - r}{k} - \binom{l + n}{r} \binom{k + l - r}{l} \right\} \frac{k!!}{(k + l - r)!!} \frac{W^{(k+l-r)}_{m+n}}{W^{(k+l-r)}_{m+n}}
\]

\[
+ (-1)^k k!! \binom{k + m}{k + l + 1} \delta_{m+n,0}
\]

(72)

which are obtainable from the operator product expansions among the higher spin fields \( W^{(k)}(z) \) realized by the free fermion. Since \( \mathcal{H} \) (70) is invariant under the adjoint action of \( g \), we can equivalently construct the \( su(1, 1) \) algebra by \( g^{-1} W^{(0)}_2 g, W^{(0)}_{-2} \) and \( \mathcal{H} \)

\[
\left[ g^{-1} W^{(0)}_2 g, W^{(0)}_{-2} \right] = -\frac{4i}{\sqrt{\mu^2 + m}} \mathcal{H}
\]

\[
\left[ \mathcal{H}, g^{-1} W^{(0)}_2 g \right] = \frac{2i}{\sqrt{\mu^2 + m}} g^{-1} W^{(0)}_2 g
\]

\[
\left[ \mathcal{H}, W^{(0)}_{-2} \right] = -\frac{2i}{\sqrt{\mu^2 + m}} W^{(0)}_{-2}
\]

(73)

This appearance of \( su(1, 1) \) algebra can be considered as the result of same algebraic structure (5) revealed in the study of the one-body hamiltonian operator \( L_x \). More precisely, since \( su(1, 1) \) structure (5) has its origin in the recursion relations of the Whittaker function and the element \( g \in GL(\infty) \) itself is essentially determined from the properties of this function, one can say that \( su(1, 1) \) structure (71) in these compactified \( c = 1 \) strings comes from the Whittaker function.

For the case of \( n \geq 2 \), the explicit description of \( g W^{(0)}_{-2n} g^{-1} (\equiv g : \partial^{-2n}_X : g^{-1}) \) and \( g^{-1} W^{(0)}_{2n} g (\equiv g^{-1} : \partial^{2n}_X : g) \) by the generators of \( W_{1+\infty} \) algebra is very complicated. It may be instructive to give their forms in terms of pseudo-differential operators

\[
g : \partial^{-2n}_X : g^{-1} = \left( \frac{1}{\mu^2 + m} \right)^n : \{- (X - i\mu \partial_X)^{-2} \partial^{-2}_X \}^n : ,
\]

\[
g^{-1} : \partial^{2n}_X : g = \left( \frac{1}{\mu^2 + m} \right)^n : \{- (X \partial^2_X + (1 - i\mu) \partial_X)^{2} \partial^{2}_X \}^n : .
\]

(74)

The R.H.Ss of these equations can be translated to the sums of \( W^{(l)}_{\pm 2n} \) according to (58). Taking account of these expressions of \( g W^{(0)}_{-2n} g^{-1} \) (or \( g^{-1} W^{(0)}_{2n} g \)) besides commutation
relations (71) (or (73)), one may see that these elements $gW_{-2n}^{(0)}g^{-1}$ and $W_{2m}$ (or $g^{-1}W_{2n}^{(0)}g$ and $W_{-2m}$) with their combinations are giving a representation of (central extended) enveloping algebra of $su(1,1)$. Therefore one can construct a representation of $W_{\infty}$ algebra from these elements.

In order to proceed to the next subsection it may be convenient to give some remarks related with this $W_{\infty}$ symmetry. Because the operators $X^k\partial_{X}^{n+k}$ and $g^{-1}X^k\partial_{X}^{n+k}$ will constitute $W_{1+\infty}^{(1)}$ algebras respectively, let us prepare two $W_{1+\infty}^{(1)}$ algebras. For the definitive we shall denote them as $W_{1+\infty}^{(+)}$ and $W_{1+\infty}^{(-)}$, each of which is generated by $X^k\partial_{X}^{n+k}$: or $Y^k\partial_{Y}^{n+k}$. These two $W_{1+\infty}^{(1)}$ algebras are not independent. They are related to each other by the adjoint action of $g$

\[
W_{1+\infty}^{(-)} = g^{-1}W_{1+\infty}^{(+)} g 
\]

\[
(i.e. W_{n}^{(k)}(-) = g^{-1}W_{n}^{(k)}(+) g ) \tag{75}
\]

$su(1,1)$ algebra (71) (or (73)) can be regarded as the consequence of intertwining (75) between these two $W_{1+\infty}^{(1)}$ algebras. Therefore our $W_{\infty}$ algebra is also the result of this intertwining. Notice that $W_{1+\infty}^{(1)}$ algebra can be realized in terms of free bosons, that is, the bosonization of the higher spin fields $X^k\psi\psi^*$: $(z)$ $(\equiv W_{1+\infty}^{(k)}(z))$. By introducing a free boson field $\partial_z\phi(z)$ with the operator product expansion $\partial_z\phi(z)\partial_w\phi(w) \sim \frac{1}{(z-w)^2}$, the higher spin field gets the form

\[
W_{1+\infty}^{(k)}(z) = \frac{1}{k+1} : \left\{ (\partial_z\phi)^{k+1} + \cdots \right\} : (z) \tag{76}
\]

where "\cdots" denotes the quantum corrections. Especially, in the cases of $k = 0, 1$ and 2, $W_{1+\infty}^{(k)}(z)$ have the following bosonized forms

\[
W_{1+\infty}^{(0)}(z) = \partial\phi(z) \quad , \\
W_{1+\infty}^{(1)}(z) = \frac{1}{2} : (\partial^2\phi) : (z) \quad , \\
W_{1+\infty}^{(2)}(z) = \frac{1}{3} : (\partial^3\phi) : (z) \quad . \tag{77}
\]

These two $W_{1+\infty}^{(1)}$ algebras, $W_{1+\infty}^{(\pm)}$, will be realized by boson fields $\partial\phi_+$ and $\partial\phi_-$ respectively. It is important to note that, under these bosonizations of $W_{1+\infty}^{(\pm)}$, intertwining
relation (75) will replace the scattering relations between $\partial \phi^\pm$. As we will see it in the next subsection, these scattering relations are precisely the nonperturbative analogue of Polchinski’s classical scattering equations.

### 3.2 Classical limits and Polchinski’s scattering equations

In this subsection we will consider the properties of ”classical limits” of the several equations obtained in the last subsection. Classical limit may be taken by the following substitution under $\bar{\hbar} \to 0$

\[ \bar{\hbar}\partial X \to P , \]

\[ [\bar{\hbar}\partial X, X] = \bar{\hbar} \to \{ P, Q \}_\text{p.b.} = 1 , \]

where we identify $X$ with $Q$. The Poisson bracket is given by $\{ F, G \}_\text{p.b.} = \partial F / \partial P \partial G / \partial Q - \partial G / \partial P \partial F / \partial Q$.

The problem about ”taking classical limits” of the equations in the previous subsection lies on the ambiguity in identifying the Planck constant $\bar{\hbar}$ with the parameters which appear in these equations.

Let us first consider the classical limit under the identification

\[ \bar{\hbar} \equiv -\frac{1}{i\sqrt{m}} . \]

With this identification one may rewrite equations (67) and (69) as

\[ (1 - \mu^2 \hbar^2)g : (\bar{\hbar}\partial X)^2 : g^{-1} = : \left( (X - i\mu \bar{\hbar} (\bar{\hbar}\partial_x)^{-1})^2 + (\bar{\hbar}\partial X)^2 \right) : , \]

\[ (1 - \mu^2 \hbar^2)g^{-1} : (\bar{\hbar}\partial X)^2 : g = : \left( (X (\bar{\hbar}\partial X)^2 + (1 - i\mu)\bar{\hbar} (\bar{\hbar}\partial X))^2 + (\bar{\hbar}\partial X)^2 \right) : . \]

Then the $\hbar = -1/(i\sqrt{m}) \to 0$ limits of these equations become independent of $\mu$

\[ P_+^{-2} = Q_+^2 + P_-^{-2} , \]

\[ P_-^2 = Q_+^2 P_+^4 + P_+^2 , \]

where we introduce $P_\pm$ and $Q_\pm$ as the classical counterparts of the following operators,

\[ P_+ \sim : \hbar\partial X : , \quad Q_+ \sim : X : , \]

\[ P_- \sim g^{-1} : \hbar\partial X : g , \quad Q_- \sim g^{-1} : X : g . \]
Notice that these classical limits (81) preserve the Poisson bracket \( \{ P, Q \}_{p.b.} = 1 \). This is the classical analogue of the simple fact that the adjoint actions of \( g \) on \( X \) and \( \hbar \partial_X \) preserve their commutator. So they define a symplectic diffeomorphism on two-plane, that is, on the classical phase space \((Q, P)\). On the other hand these quantities besides \( Q - P - 1 \) (or equivalently \( Q + P + 1 \)) form the \( su(1, 1) \) Poisson algebra

\[
\begin{align*}
\{ P^2, P^2 \}_{p.b.} &= 4 Q P, \\
\{ QP, P^2 \}_{p.b.} &= -2 P^2, \\
\{ QP, P^2 \}_{p.b.} &= 2 P^2,
\end{align*}
\]

which is the classical limit of (81) (83)). Equations (84) will replace under this limit

\[
P_{+2} = (Q^2 + P_{-2})^n, \quad P_{-2} = (Q^2P_+^4 + P_+^2)^n,
\]

where the former is the limit of : \( \partial_X^{2n} X : \) described in terms of \( g^{-1} : X^l \partial_X^{-2n} \) : \( g \) and the latter is that of \( g^{-1} : \partial_X^{2n} \) : \( g \) evaluated by : \( X^l \partial_X^{2n} \) :. So it follows that, under the Poisson bracket, polynomials of \( QP - P \) and \( P^2 \) constitute the enveloping algebra of \( su(1, 1) \), that is, \( w_\infty \) algebra. It is the classical limit of the \( W_\infty \) algebra discussed in the previous subsection.

The another identification will be

\[
h = -\frac{1}{i\mu}.
\]

With this identification equations (87) and (92) get the forms

\[
\begin{align*}
(1 - m\hbar^2) g : (\hbar \partial_X)^{-2} : g^{-1} &= \left\{ \left( X + (\hbar \partial_X)^{-1} \right)^2 - m\hbar^2 (\hbar \partial_X)^{-2} \right\}, \\
(1 - m\hbar^2) g^{-1} : (\hbar \partial_X)^2 : g &= \left\{ (X (\hbar \partial_X)^2 + (1 + \hbar) (\hbar \partial_X)) - m\hbar^2 (\hbar \partial_X)^2 \right\},
\end{align*}
\]

from which we can read the \( \hbar = -1/(i\mu) \to 0 \) limits of these equations as

\[
\begin{align*}
P_{+2} &= (Q_+ + P_{-1})^2, \\
P_{-2} &= (Q_+P_+^2 + P_+^2),
\end{align*}
\]

which are independent of \( m \). Notice that equations (87) define another symplectic diffeomorphism on the classical phase space. It also follows that \( P_{+2}, P_{-2} \) and \( QP - 1 \) form
the $su(1,1)$ Poisson algebra

\[
\{ P_-^2, P_+^{-2} \}_{p.b.} = 4(Q_- P_+ + 1),
\]

\[
\{ Q_- P_+ + 1, P_-^2 \}_{p.b.} = -2P_-^2,
\]

\[
\{ Q_- P_+ + 1, P_+^{-2} \}_{p.b.} = 2P_+^{-2}.
\]

(88)

This Poisson algebra is also the classical analogue of (71). Since the classical limits of equations (74) are

\[
P_-^{-2n} = (Q_- + P_-^{-1})^{2n}, \quad P_+^{2n} = (Q_+ P_+^2 + P_+)^{2n},
\]

(89)

polynomials of $P_-^{-2}, P_+^2$ and $Q_- P_-$ constitute $w_\infty$ algebra under the Poisson bracket. It is also the classical limit of the $W_\infty$ algebra obtained in the last subsection.

At this stage it may be convenient to give some remarks on these identifications of $\hbar$ from the string theoretical point of view. In string theory the genus expansion of the free energy may be considered as the asymptotic expansion by $\hbar$. The leading contribution is that of the classical string. So the above classical limits will be related with the classical (genus zero) contributions of string. The different identifications of $\hbar$ will lead to the different asymptotic expansions of the nonperturbative free energy, hence, the different string models. Our first identification (79) is based on the proposal by Jevicki and Yoneya \[4\], in which they tried to search the nonperturbative nature of string in the black-hole background. On the other hand our second choice is the standard identification in the $c = 1$ string \[20\]. As we have shown, these exists the $w_\infty$ symmetry in both classical limits. But their constituents are different from each other.

Since equations (67) and (69) are equivalent to expressions (66) and (68), it may be also interesting to study these equations from the view of the classical limit of $W_{1+\infty}$ algebra. The classical limit of $W_n^{(k)}$ will be given by

\[
w_n^{(k)} \equiv Q^k P^{n+k}.
\]

(90)

They constitute the following Poisson algebra

\[
\{ w_n^{(k)}, w_m^{(l)} \}_{p.b.} = (ln - km)w_{n+m}^{(k+l-1)}.
\]

(91)
This algebra is precisely the algebra of area-preserving (or symplectic) diffeomorphisms on two-plane, that is, \( w_{1+\infty} \) algebra. As \( W_{1+\infty} \) algebra has the realization by free bosons, its classical limit will be also written by using a classical boson field. Let us introduce a classical boson field \( \alpha(y) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{-iny} \) with the Poisson structure, \( \{ \alpha(y_1), \alpha(y_2) \} = 2\pi i \partial_y \delta(y_1 - y_2) \). Generators (90) of \( w_{1+\infty} \) algebra are given by

\[
w_n^{(k)} = \frac{1}{k+1} \int \frac{dy}{2\pi i} e^{iny} \alpha(y)^{k+1},
\]

which are the classical counterpart of the bosonic realization of \( W_n^{(k)} \).

We shall write down our two different classical equations (84) and (89) in terms of these classical bosons. Since we have two \( w_{1+\infty} \) algebras constructed respectively from the pairs \( (P_+, Q_+) \) and \( (P_-, Q_-) \), that is, the classical limits of \( W_{1+\infty}^{(\pm)} \) which are described at the end of the last subsection, it may be convenient to introduce two classical boson fields for their realizations

\[
\alpha_\pm(y) = \sum_{n=-\infty}^{+\infty} \alpha_n^{(\pm)} e^{-iny}.
\]

Let us first consider classical equations (84) which are derived from (69) and (67) under \( \hbar = -1/(i\sqrt{m}) \to 0 \) limit. Using \( \alpha_\pm(y) \), the R.H.Ss of these equations (84) can be evaluated into

\[
(Q_-^2 + P_-^{2r})^n = \sum_{r\geq 1} \frac{\Gamma(1+n)}{\Gamma(2+n-r)(r-1)!} Q_-^{2r-2} P_-^{2n-2r-2},
\]

\[
= \sum_{r\geq 1} \frac{\Gamma(1+n)}{(2r-1)\Gamma(2+n-r)(r-1)!} \int \frac{dy}{2\pi i} e^{-2iny} \alpha_-^r \delta(2n+\sum_{j=1}^{2r-1} m_j \alpha_-^{m_j}) \delta(2n+\sum_{j=1}^{2r-1} m_j \alpha_0),
\]

\[
= \sum_{r\geq 1} \frac{\Gamma(1+n)}{(2r-1)\Gamma(2+n-r)(r-1)!} \sum_{m_1,\ldots, m_{2r-1}} \delta(2n+\sum_{j=1}^{2r-1} m_j \alpha_-^{m_j} \alpha_0) \cdot \delta(2n+\sum_{j=1}^{2r-1} m_j \alpha_0) \cdot \delta(2n+\sum_{j=1}^{2r-1} m_j \alpha_0),
\]

and

\[
(Q_+^2 + P_+^{2r+2})^n = \sum_{r\geq 1} \frac{\Gamma(1+n)}{\Gamma(2+n-r)(r-1)!} Q_+^{2r-2} P_+^{2n+2r-2},
\]
\[ \sum_{r \geq 1} \frac{\Gamma(1 + n)}{(2r - 1) \Gamma(2 + n - r)(r - 1)!} \int \frac{dy}{2\pi i} e^{2iny} (y)^{2r-1}, \]

\[ = \sum_{r \geq 1} \frac{\Gamma(1 + n)}{(2r - 1) \Gamma(2 + n - r)(r - 1)!} \sum_{m_1, \ldots, m_{2r-1}} \delta_{-2n + \sum_{j=1}^{2r-1} m_j} \alpha^{(+) m_1} \alpha^{(+) m_{2r-1}}. \]

(95)

Thus we obtain the bosonic realizations for these classical equations (84)

\[ \alpha^{(-)} = \sum_{r \geq 1} \frac{\Gamma(1 + n)}{(2r - 1) \Gamma(2 + n - r)(r - 1)!} \sum_{m_1, \ldots, m_{2r-1}} \delta_{2n + \sum_{j=1}^{2r-1} m_j} \alpha^{(-) m_1} \alpha^{(-) m_{2r-1}}. \]

(96)

Though the above equations have been derived in the framework of the compactified string, it may be very plausible that the similar equations also hold for the noncompact case by simply changing the sum of momentums in the R.H.Ss of (96) to the integrations

\[ \alpha^{(\pm)} = \sum_{r \geq 1} \frac{\Gamma(1 + n)}{(2r - 1) \Gamma(2 + n - r)(r - 1)!} \int \prod_{j=1}^{2r-1} dp_j \delta(-q + \sum_{j=1}^{2r-1} p_j) \alpha^{(\mp)}_{p_1} \alpha^{(\mp)}_{p_{2r-1}}. \]

(97)

It is important to note that, by analytically continuing to the Minkowski region, these equations (97) are giving formal solutions of Polchinski’s scattering equations

\[ \alpha_{\pm}(y) = \alpha_{\mp}(y + \frac{1}{2} \ln(1 + \alpha_{\pm}^2(y))). \]

(98)

These scattering equations were derived in [4] from the asymptotic behavior of the classical collective fields \( \tilde{\alpha}_\pm(t, x) \) which satisfy the equation of motion

\[ \partial_t \tilde{\alpha}_\pm = -\partial_x V(x) - \tilde{\alpha}_\pm \partial_x \tilde{\alpha}_\pm. \]

(99)

According to the beautiful explanation given in [16] the general solution for this equation of motion is described by the deviation of the Fermi surface from static ground-states. The small deviation from the \( \mu = 0 \) static ground-state leads functional scattering equations (98).

Nextly we shall consider another classical equations (89) which are obtained from (67) and (69) under \( \hbar = -1/(i\mu) \to 0 \) limit. We can follow the same steps as for the first case.
Equations (89) get the following bosonized forms

\[ \alpha^{(+)}_{-2n} = \sum_{r \geq 1} \frac{\Gamma(1 + 2n)}{\Gamma(2 + 2n - r)r!} \sum_{m_1, \ldots, m_r} \delta_{2n + \sum_{j=1}^{r} m_j, 0} \alpha_{m_1}^{(-)} \cdots \alpha_{m_r}^{(-)} , \]

\[ \alpha^{(-)}_{2n} = \sum_{r \geq 1} \frac{\Gamma(1 + 2n)}{\Gamma(2 + 2n - r)r!} \sum_{m_1, \ldots, m_r} \delta_{-2n + \sum_{j=1}^{r} m_j, 0} \alpha_{m_1}^{(+)} \cdots \alpha_{m_r}^{(+)} . \] (100)

These equations are also expected to hold for the noncompact case with the similar modification as in (97)

\[ \alpha^{(\pm)}_{q} = \sum_{r \geq 1} \frac{\Gamma(1 \mp q)}{\Gamma(2 \mp q - r)r!} \int \prod_{j=1}^{r} dp_j \delta(-q + \sum_{j=1}^{r} p_j) \alpha_{p_1}^{(\mp)} \cdots \alpha_{p_r}^{(\mp)} . \] (101)

These are formal solutions for another variation of Polchinski’s scattering equations

\[ \alpha_{\pm}(y) = \alpha_{\pm}(y \mp \ln(1 + \alpha_{\pm}(y))) , \] (102)

which were derived in [16, 17] from the collective field study of the classical scattering in the undeformed \( c = 1 \) string model.

After the quantization of these (collective) boson fields \( \alpha_{\pm}(y) \), their modes \( \alpha_{n}^{(\pm)} \) and \( \alpha_{-n}^{(\pm)} \) \((n > 0)\), multiplied by appropriate leg factors, can be considered as the creation operators of the massless tachyons with momentum \( n \) and \(-n\) respectively [4, 17]. Since classically \( \alpha_{n}^{(\pm)} = P_{n}^{\pm} \) and \( \alpha_{-n}^{(\pm)} = P_{-n}^{\pm} \) in our context, the quantum analogue of this identification is \( \alpha_{n}^{(\pm)} = W_{n}^{(0)} \) and \( \alpha_{-n}^{(\pm)} = g^{-1}W_{-n}^{(0)}g \), that is, \( \alpha_{n}^{(\pm)} = W_{n}^{(0)(\pm)} \). Hence the free boson fields \( \partial \phi_{\pm} \) which are introduced for realizations (76) of \( W_{1+\infty}^{(\pm)} \) are identified with these quantized collective fields \( \alpha_{\pm} \). With this identification it follows that the nonperturbative analogues of formal solutions (97) and (101) of Polchinski’s classical scattering equations are equations (74) realized by \( \partial \phi_{\pm} \). Equivalently, intertwining relation (74) of \( W_{1+\infty}^{(\pm)} \) can be considered as the nonperturbative form of Polchinski’s classical scattering equations. We should also notice that, under correspondence (57), these modes of the \( U(1) \) current, \( W^{(0)(\pm)}(z) \), give the asymptotic operators \( B_{n}^{\text{in}} \) and \( g^{-1}B_{-n}^{\text{out}}g \). Therefore, taking it in reverse order, one can say that \( B_{n}^{\text{in}} \) and \( B_{-n}^{\text{out}} \), the asymptotic forms of \( B_{n} \) and \( B_{-n} \), act as the creation operators of the massless tachyons with momentum \( 2n \) and \(-2n\) respectively. In particular, the nonperturbative scatterings of the (rescaled) massless tachyons in this compactified string will be described by the matrix elements

\[ < in | \prod_{i} B_{n_i}^{\text{in}} \prod_{j} B_{m_j}^{\text{out}} | out > , \] (103)
which become equivalent to

\[
< 0 | \prod_i W_{2n_i}^{(0)} g \prod_j W_{-2m_j}^{(0)} | 0 > .
\] (104)

So the generating function of these nonperturbative tachyon scattering amplitudes, \( \mathcal{F} \), are given by

\[
e^{-\mathcal{F}(t,\bar{t})} = < 0 | e^{\sum_{k \geq 1} t_k W_k^{(0)}} g e^{-\sum_{k \geq 1} \bar{t}_k W_{-k}^{(0)}} | 0 > .
\] (105)

This partition function can be also considered as a \( \tau \) function of the Toda lattice hierarchy, which makes it possible to investigate the tachyon scattering amplitudes from the view of integrable system. It is a main topic in the next section.

The classical scattering amplitudes of the tachyons of the \( c = 1 \) string in the black-hole background were described in [4, 5, 6] as the tree-level scattering amplitudes among these collective modes \( \alpha_n^{(\pm)} \), which are obtained by utilizing the quantum analogue of relations (97). Relations (97) themself imply the data of the black-hole background. In these calculations, the Fermi energy \( \mu \) was put zero from the beginning. But classical relations (97) are applicable even for the case of \( \mu \neq 0 \) due to the fact that the contribution of \( \mu \) vanishes, by the dimensional reason, under the classical limit \( \hbar = -1/(i\sqrt{m}) \to 0 \). Therefore, at least, these classical scattering amplitudes do not alter under the shift of \( \mu \). The effect of \( \mu \) will appear at the string multi-loops, that is, the higher order contribution of the asymptotic expansion by \( \hbar = -1/(i\sqrt{m}) \). One may observe it from (80), in which \( \mu \) never arise alone, but only in such a form as \( \mu \hbar \). This explains why \( \mu \) appears only in the multi-loop amplitudes. The same reasoning does hold for the case of the classical scattering amplitudes of the tachyons in the flat background. These amplitudes was given [17] [30, 31] by the tree-level amplitudes among the collective modes which are obtainable making use of the quantum analogue of (101). These equations (101) were derived [16, 17] as the classical result for the collective field approach with the inverse harmonic oscillator potential \( (m = 0) \) but, as we have shown, they also hold for the case of \( m \neq 0 \). Thus the classical scattering amplitudes do not change under the shift of \( m \). The effect of \( m \) will appear at the higher order contribution of the asymptotic expansion by \( \hbar = -1/(i\mu) \). (see equations (86).) These features will mean that, though we have two parameters \( \mu \) and \( m \) in
our nonrelativistic fermion system (I) which defines a critical theory for nonperturbative
$c = 1$ strings, one of these parameters becomes irrelevant at the classical string level
which is defined as the leading contribution of the asymptotic expansion by the another
parameter. In this sense, though fermion system (I) itself can be easily deformed from
the type I theory of $c = 1$ string in the flat background ($m = 0$) to that in the proposed
black hole background ($\mu = 0$) on the parameter space ($\mu, m$), this deformation is very
nonperturbative from the string theoretical point of view.

4 Toda Lattice Hierarchy and Compactified $c = 1$
Strings

The goal of this section is to demonstrate that the Toda lattice hierarchy provides a clear
interpretation of our observations on compactified $c = 1$ strings. We present below a
rather detailed overview on the theory of the Toda lattice hierarchy [18] in three different
languages — difference operators, infinite matrices and free fermions. The $W_{1+\infty}$ algebra
discussed in the last section will emerge in several different forms in these three frame-
works. We then return to $c = 1$ strings, and attempt to reorganize the contents of the
last section in terms of the Toda lattice hierarchy.

Throughout this section, $t = (t_1, t_2, \ldots)$ and $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, \ldots)$ denote the two sets of
“time” variables in the Toda lattice hierarchy, and $q \in \mathbb{Z}$ the “lattice” coordinate. Thus
$q$ should not be confused with the momentum index $q$ of fermion modes in the preceding
sections. One will however notice that they actually play the same role.

4.1 Lax formalism, dressing operators and $\tau$ function

The Lax formalism of the Toda lattice hierarchy is based on the following two Lax oper-
ators $L$ and $\tilde{L}$ and the Orlov-Shulman operators $M$ and $\tilde{M}$.

\begin{align*}
L &= e^{\partial_q} + \sum_{n=0}^{\infty} u_{n+1} e^{-n\partial_q}, \\
M &= \sum_{n=1}^{\infty} nt_n L^n + q + \sum_{n=1}^{\infty} v_n L^{-n}.
\end{align*}
\[ \bar{L} = \tilde{u}_0 e^{\partial_q} + \sum_{n=0}^{\infty} \tilde{u}_{n+1} e^{(n+2)\partial_q}, \]
\[ \bar{M} = -\sum_{n=1}^{\infty} n\tilde{t}_n L^{-n} + q + \sum_{n=1}^{\infty} \tilde{v}_n L^n, \]  
(106)

where \( e^{n\partial_q} \) denotes the shift operator that acts on a function of \( q \) as \( e^{n\partial_q} f(q) = f(q + n) \).

The above operators are formal linear combinations of these shift operators, and like pseudo-differential operators in the Lax formalism of the KP hierarchy, these “pseudo-difference” operators form a well defined non-commutative algebra. The coefficients \( u_n, v_n, \tilde{u}_n \) and \( \tilde{v}_n \) are functions of \((t, \tilde{t}, q)\), \( u_n = u_n(t, \tilde{t}, q) \), etc. We shall frequently write \( u_n = u_n(q) \) etc., omitting \((t, \tilde{t})\), in order to stress the \( q \) dependence. The expansion of \( L \) and \( \bar{L} \) appears somewhat asymmetric; it is rather \( \bar{L}^{-1} \) that should be considered a counterpart of \( L \). Let \( \bar{u}_n \) denote coefficients of the expansion of \( \bar{L} \):
\[ \bar{L}^{-1} = \bar{u}_0 e^{-\partial_q} + \sum_{n=0}^{\infty} \bar{u}_{n+1} e^{n\partial_q} . \]
(107)

Those Lax-Orlov-Shulman operators are required to satisfy the twisted canonical commutation relations
\[ [ L \, , \, M ] = L \, , \quad [ \bar{L} \, , \, \bar{M} ] = \bar{L} \]
(108)

and the Lax equations
\[ \frac{\partial L}{\partial t_n} = [A_n, L], \quad \frac{\partial M}{\partial t_n} = [\bar{A}_n, M], \]
\[ \frac{\partial L}{\partial \tilde{t}_n} = [A_n, \bar{L}], \quad \frac{\partial M}{\partial \tilde{t}_n} = [\bar{A}_n, \bar{M}], \]
\[ \frac{\partial \bar{L}}{\partial t_n} = [A_n, \bar{L}], \quad \frac{\partial \bar{M}}{\partial t_n} = [\bar{A}_n, \bar{M}], \]
\[ \frac{\partial \bar{L}}{\partial \tilde{t}_n} = [A_n, \bar{L}], \quad \frac{\partial \bar{M}}{\partial \tilde{t}_n} = [\bar{A}_n, \bar{M}], \]
(109)

where \( A_n \) and \( \bar{A}_n \) are given by
\[ A_n = (L^n)_{\geq 0}, \quad \bar{A}_n = (\bar{L}^{-n})_{< 0}, \]
(110)

and \((\quad)_{\geq 0, < 0}\) denotes the projection
\[ (\sum_n a_n e^{n\partial_q})_{\geq 0} = \sum_{n \geq 0} a_n e^{n\partial_q}, \quad (\sum_n a_n e^{n\partial_q})_{< 0} = \sum_{n < 0} a_n e^{n\partial_q}. \]
(111)
We call (108) “twisted” because it is rather the twisted operators \( ML^{-1} \) and \( \bar{M} \bar{L}^{-1} \) that give canonical conjugate variable of \( L \) and \( \bar{L} \):

\[
[ L , ML^{-1} ] = 1 , \quad [ \bar{L} , \bar{M} \bar{L}^{-1} ] = 1 .
\] (112)

The same kind of twisting will be used in the derivation of string equations.

The above equations for the Lax-Orlov-Shulman operators, as in the case of the KP hierarchy, can be converted into equations for dressing operators. The Toda lattice hierarchy needs two dressing operators

\[
W = 1 + \sum_{n=1}^{\infty} w_n e^{-n \partial_t}, \quad w_n = w_n(t, \bar{t}, q),
\]

\[
\bar{W} = \bar{w}_0 + \sum_{n=1}^{\infty} \bar{w}_n e^{n \partial_t}, \quad \bar{w}_n = \bar{w}_n(t, \bar{t}, q)
\] (113)

because of the presence of two different types of Lax-Orlov-Shulman operators. Twisted canonical commutation relations (108) are automatically satisfied if \( L, M, \bar{L} \) and \( \bar{M} \) are written

\[
L = We^{\partial_t} W^{-1}, \quad M = W(q + \sum_{n=1}^{\infty} nt_n e^{n \partial_t}) W^{-1},
\]

\[
\bar{L} = \bar{W} e^{\partial_t} W^{-1}, \quad \bar{M} = \bar{W}(q - \sum_{n=1}^{\infty} n\bar{t}_n e^{-n \partial_t}) W^{-1}.
\] (114)

This does not determine \( W \) and \( \bar{W} \) uniquely. By virtue of this arbitrariness, one can single out a suitable pair of \( W \) and \( \bar{W} \) in such a way that they satisfy the equations

\[
\frac{\partial W}{\partial t_n} = A_n W - W e^{n \partial_t}, \quad \frac{\partial W}{\partial \bar{t}_n} = \bar{A}_n W,
\]

\[
\frac{\partial \bar{W}}{\partial t_n} = A_{\bar{n}} \bar{W}, \quad \frac{\partial \bar{W}}{\partial \bar{t}_n} = \bar{A}_{\bar{n}} \bar{W} - \bar{W} e^{-n \partial_t}.
\] (115)

In fact, one can rewrite \( A_n \) and \( \bar{A}_{\bar{n}} \) in terms of \( W \) and \( \bar{W} \) by inserting the above expression of \( L \) and \( \bar{L} \) into the definition of \( A_n \) and \( \bar{A}_{\bar{n}} \). Thus the above equations can also be written

\[
\frac{\partial W}{\partial t_n} = -(W e^{n \partial_t} W^{-1})_{<0} W, \quad \frac{\partial W}{\partial \bar{t}_n} = (\bar{W} e^{-n \partial_t} W^{-1})_{<0} W,
\]

\[
\frac{\partial \bar{W}}{\partial t_n} = (W e^{n \partial_t} W^{-1})_{<0} \bar{W}, \quad \frac{\partial \bar{W}}{\partial \bar{t}_n} = -(\bar{W} e^{-n \partial_t} W^{-1})_{<0} \bar{W}.
\] (116)

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Note that this gives a closed system of equations defining commuting flows in the space of dressing operators.

We can now introduce the notion of Baker-Akhiezer functions and $\tau$ function of the Toda lattice hierarchy. The Baker-Akhiezer functions are functions of $(t, \bar{t}, q)$ and $z$ (“spectral parameter”) of the form

$$\Psi = (1 + \sum_{n=1}^{\infty} w_n z^{-n}) z^q \exp\left(\sum_{n=1}^{\infty} t_n z^n\right),$$
$$\bar{\Psi} = (\bar{w}_0 + \sum_{n=1}^{\infty} \bar{w}_n z^n) z^q \exp\left(\sum_{n=1}^{\infty} \bar{t}_n z^{-n}\right),$$

(117)

and satisfy a system of linear equations, and its integrability condition is exactly the Lax equations and the twisted canonical commutation relations above. The $\tau$ function $\tau = \tau(t, \bar{t}, q)$, by definition, is a function that is connected with the Baker-Akhiezer functions as:

$$\Psi = \exp\left(-\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n}\right) \tau(t, \bar{t}, q) \tau(t, t, q) z^q \exp\left(\sum_{n=1}^{\infty} t_n z^n\right),$$
$$\bar{\Psi} = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{\partial}{\partial \bar{t}_n}\right) \tau(t, \bar{t}, q + 1) \tau(t, t, q) z^q \exp\left(\sum_{n=1}^{\infty} \bar{t}_n z^{-n}\right).$$

(118)

If expanded into Laurent series of $z$, these relations give a system of linear differential equations for $\tau$, whose integrability condition is now ensured by the linear equations of the Baker-Akhiezer functions or, equivalently, by equations (116) of commuting flows of the dressing operators.

A few historical remarks will be now in order. The Toda lattice hierarchy was first developed as a discrete (or difference) version of the KP hierarchy [19]. Fundamental tools such as the dressing operators, the Baker-Akhiezer functions, the tau function, etc., were simultaneously imported from the theory of the KP hierarchy at that stage [27, 28, 29]. (Our present notations, however, are considerably different from notations in these earlier works.) Meanwhile, the notion of the Orlov-Shulman operators appeared several years later [32], and extended to the Toda lattice hierarchy rather recently [33, 34]. Orlov and his collaborators used such an operator to describe the so called “additional symmetries” (such as Virasoro symmetries) of integrable hierarchies within the Lax formalism.
4.2 Matrix representation and semi-infinite determinant

The Toda lattice hierarchy can also be formulated in the language of infinite matrices. This is due to the following correspondence between difference operators and infinite \((Z \times Z)\) matrices:

\[
\sum_n a_n(q)e^{nq} \leftrightarrow \sum_n \text{diag}[a_n(i)]\Lambda^n \tag{119}
\]

where

\[
\text{diag}[a_n(i)] = (a_n(i)\delta_{ij}), \quad \Lambda^n = (\delta_{i,j-n}) \tag{120}
\]

and the indices \(i\) (row) and \(j\) (column) run over \(Z\). With this correspondence, algebraic operations for difference operators are mapped to the corresponding operations for infinite matrices. Similarly, the projection \((\ )_{\geq 0, < 0}\) becomes the projection onto upper triangular part (including the diagonal) and lower triangular part (excluding the diagonal) of infinite matrices:

\[
(A)_{\geq 0} = \left(\theta(j - i)a_{ij}\right), \\
(A)_{< 0} = \left((1 - \theta(j - i))a_{ij}\right). \tag{121}
\]

Accordingly, the Lax, Orlov-Shulman and dressing operators have the corresponding infinite matrices

\[
L \leftrightarrow L, \quad M \leftrightarrow M, \quad \bar{L} \leftrightarrow \bar{L}, \quad \bar{M} \leftrightarrow \bar{M}, \quad W \leftrightarrow W, \quad \bar{W} \leftrightarrow \bar{W}, \tag{122}
\]

and those matrices obey the same form of equations as we have presented in the previous subsection. Note, in particular, that dressing representation \((114)\) of the Lax and Orlov-Shulman operators turns into matrix relations of the form

\[
L = W\Lambda W^{-1}, \quad M = W(\Delta + \sum_{n=1}^{\infty} nt_n\Lambda^n)W^{-1}, \\
\bar{L} = W\bar{\Lambda}W^{-1}, \quad \bar{M} = W(\Delta - \sum_{n=1}^{\infty} n\bar{t}_n\Lambda^{-n})W^{-1}. \tag{123}
\]

Here \(\Delta\) is the infinite matrix

\[
\Delta = \left(i\delta_{ij}\right) \tag{124}
\]
that represents the multiplication operator $q$. The infinite matrices $W$ and $\bar{W}$ are lower and upper triangular matrices of the form

$$W = \begin{pmatrix}
\vdots & & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & & \ddots & w_1(-1) & 1 \\
\vdots & & & \ddots & w_1(0) & 1 \\
\vdots & & & & \ddots & \ddots
\end{pmatrix},$$

$$\bar{W} = \begin{pmatrix}
\vdots & & \bar{w}_0(-1) & \bar{w}_1(-1) & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & & \ddots & \bar{w}_0(0) & \bar{w}_1(0) & \ddots \\
\vdots & & & \ddots & 0 & \ddots \\
\vdots & & & & \ddots & \ddots
\end{pmatrix}. \quad (125)$$

Having this reformulation of the Toda lattice hierarchy, we now show that solving the Toda lattice hierarchy can be reduced to a problem of linear algebra (of $\mathbb{Z} \times \mathbb{Z}$ matrices). A key role is played by the “matrix ratio” $U(t, \bar{t})$ of $W = W(t, \bar{t})$ and $\bar{W} = \bar{W}(t, \bar{t})$:

$$U(t, \bar{t}) = W(t, \bar{t})^{-1} \bar{W}(t, \bar{t}). \quad (126)$$

It is not hard to show from the matrix counterpart of (116) that $U(t, \bar{t})$ satisfies the simple linear equations

$$\frac{\partial U(t, \bar{t})}{\partial t_n} = \Lambda^n U(t, \bar{t}), \quad \frac{\partial U(t, \bar{t})}{\partial \bar{t}_n} = -U(t, \bar{t}) \Lambda^{-n}. \quad (127)$$

In other words, the complicated nonlinearity of the Toda lattice hierarchy is “linearized” in $U(t, \bar{t})$:

$$U(t, \bar{t}) = \exp(\sum_{n=1}^{\infty} t_n \Lambda^n) U(0, 0) \exp(-\sum_{n=1}^{\infty} \bar{t}_n \Lambda^{-n}). \quad (128)$$

What is more important is that this process is reversible. Namely, if an infinite matrix $U(t, \bar{t})$ of this form is given, and if one can find such a pair of triangular infinite matrices $W(t, \bar{t})$ and $\bar{W}(t, \bar{t})$ of the form (125) that satisfy (126), then $W(t, \bar{t})$ and $\bar{W}(t, \bar{t})$ turn out to obey (116), i.e., give a solution of the Toda lattice hierarchy. To see this, differentiate the both hand sides of (126) with respect to $t$ and $\bar{t}$, rewrite the derivatives of $U(t, \bar{t})$ by
(127), and finally eliminate $U(t, \bar{t})$ using (126). One will then obtain
\[
\frac{\partial W}{\partial t} W^{-1} + W \Lambda^n W^{-1} = \frac{\partial W}{\partial t} W^{-1},
\]
\[
\frac{\partial W}{\partial \bar{t}} W^{-1} - W \Lambda^{-n} W^{-1} = \frac{\partial W}{\partial \bar{t}} W^{-1}. \tag{129}
\]
The $(\_ < 0)$ part of these equations gives
\[
\frac{\partial W}{\partial t} W^{-1} = -(W \Lambda^n W^{-1})_{< 0}, \quad \frac{\partial W}{\partial \bar{t}} W^{-1} = (W \Lambda^{-n} W^{-1})_{< 0}, \tag{130}
\]
which is the first two equations of (116). The $(\_ \geq 0)$ part, similarly, becomes
\[
\frac{\partial W}{\partial t} W^{-1} = (W \Lambda^n W^{-1})_{\geq 0}, \quad \frac{\partial W}{\partial \bar{t}} W^{-1} = -(W \Lambda^{-n} W^{-1})_{\geq 0}, \tag{131}
\]
and gives the other two of (116).

This kind of solution technique is called a “factorization method” or a “Riemann-Hilbert problem” in the terminology of nonlinear integrable systems [19]. (The KP hierarchy, too, can be treated by a similar factorization method [35].) In the present case, factorization problem (126) is a $GL(\infty)$ version of the Gauss decomposition in ordinary finite dimensional linear algebra. At least formally, therefore, one will be able to apply a standard method using the Cramer formula to obtain the two factors $W$ and $\bar{W}$ explicitly. Their matrix elements $w_n(q)$ and $\bar{w}_n(q)$ can be written as a quotient of two determinants. This calculation eventually leads to the following formula for the $\tau$ function:
\[
\tau(t, \bar{t}, q) = \det \left( u_{ij}(t, \bar{t})(i, j < q) \right), \tag{132}
\]
where $u_{ij}(t, \bar{t})$ are matrix elements of $U(t, \bar{t})$. This determinant, as well as those emerging in the determinant formulas of $w_n(q)$ and $\bar{w}_n(q)$, is a semi-infinite determinant, and needs some justification. Fortunately, these semi-infinite determinants are known to be well defined under a suitable interpretation [36].

4.3 Symmetries and constraints

Another implication of (126) is the existence of underlying $GL(\infty)$ and $W_{1+\infty}$ symmetries. One can use the language of $W_{1+\infty}$ symmetries to formulate string equations in a general form.
GL(∞) symmetries are symmetries on the space of solutions of the Toda lattice hierarchy, and induced by the left and right action of $GL(\infty)$ on $U = U(0, 0)$. Thus, more precisely, they are $GL(\infty) \times GL(\infty)$ symmetries. Since we are mostly interested in their infinitesimal form, let us consider a one-parameter family of such deformations,

$$U \to U(\epsilon) = e^{-\epsilon A} U e^{\epsilon \bar{A}},$$

(133)

where $A = (a_{ij})$ and $\bar{A} = (\bar{a}_{ij})$ are arbitrary $Z \times Z$ matrices (i.e., elements of $gl(\infty)$). By Gauss decomposition (126), this should induce a one-parameter family of deformations of the dressing operators $W$ and $\bar{W}$. As we shall show below, one can easily derive differential equations satisfied by $W$ and $\bar{W}$ with respect to the deformation parameter. These differential equations give an infinitesimal expression of the $GL(\infty)$ symmetries. (This kind of methods for constructing finite and infinitesimal symmetries of nonlinear integrable systems have been used for years, and called “Riemann-Hilbert transformations”, “dressing transformations”, etc. The formulation presented here is borrowed from a similar formulation in the self-dual Yang-Mills equations and the self-dual Einstein equation [37].)

Let $W(\epsilon) = W(\epsilon, t, \bar{t})$ and $\bar{W}(\epsilon) = \bar{W}(\epsilon, t, \bar{t})$ denote the corresponding deformations of $W$ and $\bar{W}$. They are characterized by the Gauss decomposition

$$U(\epsilon, t, \bar{t}) = W(\epsilon, t, \bar{t})^{-1} \bar{W}(\epsilon, t, \bar{t}),$$

(134)

where

$$U(\epsilon, t, \bar{t}) = \exp\left(\sum_{n=1}^{\infty} t_n \Lambda^n\right) U(\epsilon) \exp\left(-\sum_{n=1}^{\infty} \bar{t}_n \Lambda^{-n}\right).$$

(135)

One can now repeat almost the same calculations as we derived (110) from (120) to obtain the following equations with respect to the deformation parameter $\epsilon$:

$$\frac{\partial W(\epsilon)}{\partial \epsilon} = \left( W(\epsilon) A(t) W(\epsilon)^{-1} - W(\epsilon) \bar{A}(\bar{t}) \bar{W}(\epsilon) \right)_{<0} W(\epsilon),$$

$$\frac{\partial \bar{W}(\epsilon)}{\partial \epsilon} = \left( \bar{W}(\epsilon) \bar{A}(\bar{t}) \bar{W}(\epsilon) - \bar{W}(\epsilon) A(t) W(\epsilon)^{-1} \right)_{\geq 0} \bar{W}(\epsilon).$$

(136)

Here $A(t)$ and $\bar{A}(\bar{t})$ are given by

$$A(t) = \exp\left(\sum_{n=1}^{\infty} t_n \Lambda^n\right) A \exp\left(-\sum_{n=1}^{\infty} t_n \Lambda^{-n}\right).$$

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\[ \tilde{A}(\bar{t}) = \exp(\sum_{n=1}^{\infty} \bar{t}_n \Lambda^{-n}) \tilde{A} \exp(-\sum_{n=1}^{\infty} \bar{t}_n \Lambda^{-n}). \]  

(137)

We can thus obtain the following expression of an infinitesimal \( GL(\infty) \times GL(\infty) \) symmetry:

\[ \delta_{A,\bar{A}} W = \left( W A(t) W^{-1} - W \tilde{A}(\bar{t}) \tilde{W}^{-1} \right)_{<0} W, \]
\[ \delta_{A,\bar{A}} \tilde{W} = \left( \tilde{W} \bar{A}(\bar{t}) \tilde{W}^{-1} - WA(t) W^{-1} \right)_{\geq 0} \tilde{W}. \]  

(138)

Viewing \( \delta_{A,\bar{A}} \) as an abstract variational operator acting on functionals of dressing operators, one can derive the commutation relations

\[ [\delta_{A_1,A_1}, \delta_{A_2,\bar{A}_2}] = \delta_{[A_1,A_2],[A_2,\bar{A}_2]}. \]  

(139)

In particular, \( \delta_{A,0} \) and \( \delta_{0,\bar{A}} \) give two independent \( GL(\infty) \) symmetries that commute with each other. Of course this is to be expected from their origin as left and right actions of \( GL(\infty) \) on the matrix \( U \).

\( W_{1+\infty} \) symmetries (more precisely, \( W_{1+\infty} \oplus W_{1+\infty} \) symmetries) are a subset of these \( GL(\infty) \) symmetries with generators of the special form

\[ A = A(\Delta, \Lambda) = \sum_{m,\ell} a_{m\ell} \Delta^m \Lambda^\ell, \]
\[ \bar{A} = \bar{A}(\Delta, \Lambda) = \sum_{m,\ell} \bar{a}_{m\ell} \Delta^m \Lambda^\ell, \]  

(140)

where \( m \) runs over nonnegative integers and \( \ell \) over all integers. Note that these matrices are in one-to-one correspondence with difference operators,

\[ A(\Delta, \Lambda) \leftrightarrow A(q, e^{\partial_q}), \quad \bar{A}(\Delta, \Lambda) \leftrightarrow \bar{A}(q, e^{\partial_q}), \]  

(141)

and altogether form a closed Lie algebra with the fundamental commutation relation

\[ [\Delta, \Lambda] = \Delta \leftrightarrow [e^{\partial_q}, q] = e^{\partial_q}. \]  

(142)

This is essentially a (centerless) \( W_{1+\infty} \) algebra, the generators \( W^{(k)}_n \) being given by

\[ W^{(k)}_n = (\Delta \Lambda^{-1})^k \Lambda^{n+k} \sim (qe^{-\partial_q})^k e^{(n+k)\partial_q}. \]  

(143)
One can easily see that for such $A$ and $\bar{A}$, previous formula (138) becomes

$$\delta_{A,A}W = \left(A(M,L) - \bar{A}(M,\bar{L})\right)_{<0}W,$$

$$\delta_{A,A}\bar{W} = \left(\bar{A}(\bar{M},\bar{L}) - A(M,L)\right)_{\geq 0}\bar{W},$$

(144)

where

$$A(M,L) = \sum_{m,\ell} a_{m\ell} M^m L^\ell, \quad \bar{A}(\bar{M},\bar{L}) = \sum_{m,\ell} \bar{a}_{m\ell} \bar{M}^m \bar{L}^\ell.$$  

(145)

The left piece $\delta_{A,0}$ and the right piece $\delta_{0,A}$, as in the case of $GL(\infty)$ above, give two independent sets of $W_{1+\infty}$ symmetries commuting with each other, i.e., a realization of $W_{1+\infty} \oplus W_{1+\infty}$.

Furthermore, these symmetries turn out to induce the following symmetries on the Lax-Orlov-Shulman operators:

$$\delta_{A,A}L = \left[\left(A(M,L) - \bar{A}(M,\bar{L})\right)_{<0}, L\right],$$

$$\delta_{A,\bar{A}}M = \left[\left(A(M,L) - \bar{A}(M,\bar{L})\right)_{<0}, M\right],$$

$$\delta_{A,A}\bar{L} = \left[\left(\bar{A}(\bar{M},\bar{L}) - A(M,L)\right)_{\geq 0}, \bar{L}\right],$$

$$\delta_{A,\bar{A}}\bar{M} = \left[\left(\bar{A}(\bar{M},\bar{L}) - A(M,L)\right)_{\geq 0}, \bar{M}\right].$$

(146)

Note that these relations are closed within the Lax-Orlov-Shulman operators — the dressing operators have disappeared. This fact lies in the heart of our understanding of string equations.

Because of this, we can formulate a $W_{1+\infty}$ constraint as a constraint on the Lax-Orlov-Shulman operators. In general, such a $W_{1+\infty}$ constraint can be formulated as the fixed point condition $U(\epsilon) = U$, which is equivalent to

$$A(\Delta, \Lambda)U = U\bar{A}(\Delta, \Lambda).$$

(147)

In terms of the Lax-Orlov-Shulman operators, this fixed point condition can be restated as $\delta_{A,\bar{A}}L = 0$, etc., or, equivalently, as

$$A(M,L) = \bar{A}(\bar{M},\bar{L}).$$

(148)
In actual applications, several constraints of this form are considered simultaneously, frequently taking the form of a \(\{(P, Q)\}\) pair. This type of equations are what we call “string equations”.

One can further translate the above constraint into equations of the form

\[
X_A(t, \partial_t, q) \tau = \bar{X}_A(\bar{t}, \partial_{\bar{t}}, q) \tau,
\]

where \(X_A(t, \partial_t, q)\) and \(\bar{X}_A(\bar{t}, \partial_{\bar{t}}, q)\) are linear differential operators in \(t\) and \(\bar{t}\) that represent the action of \(\delta_A, 0\) and \(\delta_{0, \bar{A}}\) in terms of the \(\tau\) function. This is a general form of the so-called “\(W_{1+\infty}\)-constraints” in the literature on \(c = 1\) strings. We shall specify the origin of these differential operators \(X_A(t, \partial_t, q)\) and \(\bar{X}_A(\bar{t}, \partial_{\bar{t}}, q)\) in the next subsection.

These constraints may be thought of as imposing a relation between the (otherwise independent) left and right components of the full \(W_{1+\infty} \oplus W_{1+\infty}\) structure. We have indeed observed many aspects of such a relation in the analysis of deformed \(c = 1\) strings. This issue, too, will be discussed in detail below.

### 4.4 Free fermion formalism

We are now in a position to reformulate the contents of the preceding two subsections in terms of free fermions. Notations concerning fermions, such as the fermion operators \(\psi_n, \psi_n^*\), etc., are the same as those used in the analysis of compactified \(c = 1\) strings.

To begin with, let us recall the following correspondence between the \(gl(\infty)\) algebra (more precisely, its central extension \(\hat{gl}(\infty)\)) and fermion bilinear forms:

\[
A = (a_{ij}) \leftrightarrow A(\psi, \psi^*) = \sum_{i,j} a_{ij} : \psi_i \psi_j^* :.
\]

(150)

Combining this correspondence with the aforementioned correspondence between difference operators and infinite matrices, one can obtain a free fermion realization of difference operators:

\[
: q^m e^\ell \partial_q : = \sum_{p=-\infty}^{\infty} p^m : \psi_{-p} \psi_{p+\ell}^* :.
\]

(151)

In particular, the \(W_{1+\infty}\) generators \(W_n^{(k)}\) are now represented by

\[
W_n^{(k)} = : (qe^{-\partial_q})^k e^{(n+k)\partial_q} :.
\]

(152)
The next step is to lift up the above correspondence to the group level. A fermion bilinear form induces a linear map on the vector space spanned by $\psi_n$ and $\psi^*_n$:

\[
\begin{align*}
\left[ A(\psi, \psi^*) , \psi_{-j} \right] &= \sum_i a_{ij} \psi_{-i}, \quad \left[ A(\psi, \psi^*) , \psi^*_i \right] = - \sum_j a_{ij} \psi^*_j . \quad (153)
\end{align*}
\]

An operator of the form $g = \exp A(\psi, \psi^*)$, accordingly, induces an invertible linear transformation in the same vector space:

\[
\begin{align*}
g\psi_{-j} g^{-1} &= \sum_i u_{ij} \psi_{-i}, \quad g\psi^*_i g^{-1} = \sum_j v_{ij} \psi^*_j, \\
(v_{ij}) &= (u_{ij})^{-1}. \quad (154)
\end{align*}
\]

It is this $Z \times Z$ matrix $U = (u_{ij})$ that we now identify with the previous matrix $U = U(0,0)$ derived in the matrix formalism of the Toda lattice hierarchy.

To this end, we first note that the special matrix element $<q | g | q>$ of the operator $g$ can be rewritten as the determinant of a semi-infinite submatrix of $U$. Let us write the $q$-th ground state as the fermi sea of a half filled vacuum:

\[
\begin{align*}
|q> &= \psi_{-q+1} \psi_{-q+2} \cdots |-\infty> . \quad (155)
\end{align*}
\]

Repeated use of (154) will then give

\[
\begin{align*}
g | q> &= \left( \sum_i u_{i,q-1} \psi_{-i} \right) g | q - 1> \\
&= \left( \sum_i u_{i,q-1} \psi_{-i} \right) \left( \sum_i u_{i,q-2} \psi_{-i} \right) g | q - 2> \\
&= \ldots \\
&= \left( \sum_i u_{i,q-1} \psi_{-i} \right) \left( \sum_i u_{i,q-2} \psi_{-i} \right) \cdots |-\infty> . \quad (156)
\end{align*}
\]

The last expression can be expanded into an infinite linear combination of Fock states

\[
| \{ n_i (i < q) \} > = \psi_{-n_q+1} \psi_{-n_q+2} \cdots |-\infty> , \quad (157)
\]

and the coefficients are given by semi-infinite determinants of the form $\det \left( u_{n_i,j} (i, j < q) \right)$. Pairing with $<q | g | q>$ leaves only the term proportional to $|q>$ non-vanishing, thereby we obtain the formula

\[
<q | g | q> = \det \left( u_{ij} (i, j < q) \right) . \quad (158)
\]
The last formula gives the special value $\tau(0, 0, q)$ of the $\tau$ function. The $\tau$ function itself is given by the same semi-infinite determinant with $U$ replaced by $U(t, \bar{t})$. In the free fermion formalism, this amounts to replacing

$$g \rightarrow g(t, \bar{t}) = e^{H(t)} g e^{-\bar{H}(\bar{t})},$$

where

$$H(t) = \sum_{n=1}^{\infty} t_n J_n, \quad \bar{H}(\bar{t}) = \sum_{n=1}^{\infty} \bar{t}_n J_n,$$

$$J_n = W_n^{(0)} = \sum_{p=-\infty}^{\infty} :\psi_{-p} \psi_{p+n}^* :.$$

(159)

The reason is that $J_n$ give a fermionic representation of $e^{n\partial_t} \sim \Lambda^n$ (i.e., $J_n = e^{n\partial_t}$). One can indeed easily check the following commutation relations with $\psi_{-j}$ and $\psi_i^*$:

$$[J_n, \psi_{-j}] = \psi_{-j+n}, \quad [J_n, \psi_i^*] = \psi_{i+n}.$$

(161)

(Compare these relations with those of the previous fermion bilinear operators $B_n$ and $\bar{B}_n$.) Accordingly $e^{H(t)}$ and $e^{-\bar{H}(\bar{t})}$ give a fermionic representation of $\exp(\sum t_n \Lambda^n)$ and $\exp(\sum \bar{t}_n \Lambda^{-n})$. We thus obtain the following fermionic representation of the $\tau$ function [38, 39]:

$$\tau(t, \bar{t}, q) = \langle q | e^{H(t)} g e^{-\bar{H}(\bar{t})} | q \rangle.$$

(162)

This is actually a straightforward generalization of a similar representation of the $\tau$ function of the KP hierarchy [28]. In fact, one can readily see, from such a comparison [13], that $\tau(t, \bar{t}, q)$ may be viewed as a $\tau$ function of the KP hierarchy in two opposite ways, namely (i) $t$ as time variables and $(\bar{t}, q)$ as parameters, and (ii) $\bar{t}$ as time variables and $(t, q)$ as parameters.

This fermionic representation of the $\tau$ function also shows that the $\tau$ function is actually a generating function of fermion scattering amplitudes discussed in the previous sections. To demonstrate this, let us insert a complete set of Fock states, $1 = \sum | \{n_i(i < q)\} > < \{n_i(i < q)\}|$, between $e^{H(t)}$ and $g$, and similarly, between $g$ and $e^{-\bar{H}(\bar{t})}$. This gives the expansion

$$\tau(t, \bar{t}, q) = \sum_{\{n_i\}, \{\bar{n}_i\}} < q | e^{H(t)} | \{n_i(i < q)\} > \times$$

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\[ \times < \{ n_i(i < q) \} \mid g \mid \{ \bar{n}_i(i < q) \} < \{ \bar{n}_i(i < q) \} \mid e^{-H(t)} \mid q >. \]

(163)

In fact, the matrix elements \(< q \mid e^{H(t)} \mid \{ n_i(i < q) \} >\) and \(< \{ \bar{n}_i(i < q) \} \mid e^{-H(t)} \mid q >\) survive only for such states as \(n_i = i\) for \(i << q\), and coincide with the so called “Shur functions” \[27, 28, 29\]. The matrix elements of \(g\) are compactified analogues of the scattering amplitudes.

\(W_{1+\infty}\) symmetries take a particularly simple form in this fermionic representation of the \(\tau\) function — they are given by inserting the corresponding fermion bilinear forms to the left or right of \(g\). Furthermore, as already mentioned in the end of the last subsection, such a \(W_{1+\infty}\) symmetry can also be represented by a linear differential operator in \(t\) or \(\bar{t}\).

These facts are summarized into the following generating functional formulas \[28, 39, 33\]:

\[
< q \mid e^{H(t)} : \psi(z)\psi^*(w) : ge^{-H(t)} \mid q > = X(z, w) < q \mid e^{H(t)}ge^{-H(t)} \mid q >,
\]

\[
< q \mid e^{H(t)} g : \psi(z)\psi^*(w) : e^{-H(t)} \mid q > = \bar{X}(z, w) < q \mid e^{H(t)}ge^{-H(t)} \mid q >,
\]

(164)

where \(X(z, w)\) and \(\bar{X}(z, w)\) are given by

\[
X(z, w) = \frac{\exp\left(\sum_{n=1}^{\infty} t_n(z^n - w^n)\right)\left(\frac{z}{w}\right)^q\exp\left(-\sum_{n=1}^{\infty} \frac{z^n - w^n}{n} \frac{\partial}{\partial t_n}\right) - 1}{z - w},
\]

\[
\bar{X}(z, w) = \frac{\exp\left(\sum_{n=1}^{\infty} \bar{t}_n(z^n - w^n)\right)\left(\frac{\bar{z}}{\bar{w}}\right)^q\exp\left(-\sum_{n=1}^{\infty} \frac{z^n - w^n}{n} \frac{\partial}{\partial \bar{t}_n}\right) - 1}{z - w}.
\]

(165)

An explicit form of the differential operators \(X_A\) and \(\bar{X}_A\) mentioned in the end of the last subsection can be derived from these formulas. For instance, differential operators representing the \(W_{1+\infty}\) generators \(W^{(k)}_n\) are given by

\[
X^{(k)}_n = \oint_{\frac{dz}{2\pi i}} z^{n+k} \frac{\partial^k}{\partial z^k} \left. X(z, w) \right|_{w=z},
\]

\[
\bar{X}^{(k)}_n = \oint_{\frac{dz}{2\pi i}} z^{n+k} \frac{\partial^k}{\partial \bar{z}^k} \left. X(z, w) \right|_{w=z}.
\]

(166)
These two sets of differential operators $X_n^{(k)}$ and $\bar{X}_n^{(k)}$ give generators of the two (left and right) components in $W_{1+\infty} \oplus W_{1+\infty}$ symmetries of the Toda lattice hierarchy.

The above correspondence between the fermion bilocal field and a differential operator is just a disguised form of the well known formula

$$\psi(z)\psi^*(w) = \frac{e^{\phi(z)-\phi(w)}}{z-w}$$

in the standard bosonization of relativistic free fermions, $\psi(z) = e^{\phi(z)}$, $\psi^*(z) = e^{-\phi(z)}$.

This fact is particularly well understood in the case of the KP hierarchy [28]. In that case, the boson (or $U(1)$ current) Fourier modes $\alpha_n$ can be identified with differential operators acting on the $\tau$ function as $\alpha_n \sim \partial/\partial t_n$ and $\alpha_{-n} \sim nt_n$. The right hand side of the above bosonization formula then reproduces the operator $X(z,w)$.

A new feature in the Toda lattice hierarchy is that there are two sets of time variables $t$ and $\bar{t}$. This implies the existence of two bosonizations with two boson fields, say $\phi(z)$ and $\bar{\phi}(z)$. These boson fields correspond to the two boson fields $\phi_{\pm}(z)$ of Section 3. The differential operators $X(z,w)$ and $\bar{X}(z,w)$ may be thought of as vertex operators in these two bosonization schemes. By Taylor expansion at $z = w$ and Fourier (Laurent) expansion in $z$, one obtains two bosonic realizations $X_n^{(k)}$ and $\bar{X}_n^{(k)}$ of the $W_{1+\infty}$ algebra.

One can also see in the present framework that $g$ plays the role of an intertwining operator between the two copies of the $W_{1+\infty}$ algebra. As abstract differential operators, $X_n^{(k)}$ and $\bar{X}_n^{(k)}$ commute with each other and generate two independent $W_{1+\infty}$ algebras. As differential operators acting on the $\tau$ function, however, they are related (and this relation is nothing else but the string equations or the $W_{1+\infty}$ constraints), because the two vertex operators $X(z,w)$ and $\bar{X}(z,w)$ correspond to the same fermion bilocal field $\psi(z)\psi^*(w)$.

The only difference is whether the fermion bilocal field is inserted to the left or right of $g$ in the fermionic representation of the $\tau$ function. Thus, somewhat symbolically, the two $W_{1+\infty}$ algebras are related as

$$\{\bar{X}_n^{(k)}\} = g^{-1}\{X_n^{(k)}\}g.$$  

This is exactly what we observed throughout Section 3.

In fact, these two $W_{1+\infty}$ symmetries can also be viewed as symmetries of two copies of the KP hierarchy. In fact, as already mentioned, the $\tau$ function $\tau(t,\bar{t},q)$ is also a $\tau$
function of the KP hierarchy with respect to both $t$ and $\bar{t}$. In other words, two copies of the KP hierarchy are embedded into the Toda lattice hierarchy. Since the Lax formalism of the KP hierarchy is formulated in terms of pseudo-differential operators, one can consider two algebras of pseudo-differential operators with two different spatial variables, say $X$ and $Y$, associated with these copies of the KP hierarchy. One can naturally obtain two realizations of the $W_{1+\infty}$ algebra with generators $X^k \partial^{n+k}_X$ and $Y^k \partial^{n+k}_Y$. In fact, following the usual construction of the KP hierarchy, one can identify $X = t_1$ and $Y = \bar{t}_1$. This is a way how to justify the heuristic interpretation of the two $W_{1+\infty}$ algebras in terms of two kinds of pseudo-differential operators.

4.5 $c = 1$ strings at self-dual radius

It is now rather straightforward to translate the contents of Section 3 into the framework of the Toda lattice hierarchy. The matrix $U$ is now diagonal,

$$U = \left( e^{\alpha_i \delta_{ij}} \right),$$ (169)

and satisfies the relations

$$U \Lambda^{-2} U^{-1} = \frac{1}{\mu^2 + m} \left( - (\Delta \Lambda^{-1} - i \mu \Lambda^{-1})^2 + m \Lambda^{-2} \right),$$

$$U^{-1} \Lambda^2 U = \frac{1}{\mu^2 + m} \left( - (\Delta + (1 - i \mu) \Lambda)^2 + m \Lambda^2 \right)$$ (170)

as a consequence of relations (67) and (69) satisfied by the operator $g$. These relations turn into the relations

$$\bar{L}^{-2} = \frac{1}{\mu^2 + m} \left( - (M \bar{L}^{-1} - i \mu \bar{L}^{-1})^2 + m \bar{L}^{-2} \right),$$

$$L^2 = \frac{1}{\mu^2 + m} \left( - (\bar{M} \bar{L} + (1 - i \mu) \bar{L})^2 + m \bar{L}^2 \right)$$ (171)

among the Lax and Orlov-Shulman operators of the Toda lattice hierarchy. In other words, they are (nonperturbative) string equations of compactified $c = 1$ strings at self-dual radius.

The next issue is to give an interpretation of the classical limit discussed in Section 3. To this end, let us briefly review a standard prescription of classical limit in the Toda
lattice hierarchy \cite{17}. The first step is to reformulate the Toda lattice hierarchy in an $\hbar$-dependent way. Naively, this is achieved by rescaling the variables $(t, \bar{t}, q)$ as

$$
t_n \rightarrow \hbar^{-1} t_n, \quad \bar{t}_n \rightarrow \hbar^{-1} \bar{t}_n, \quad q \rightarrow \hbar^{-1} q.
$$

(172)

The lattice coordinate $q$ now takes values in the lattice $\hbar \mathbb{Z}$, i.e., we are considering a lattice with spacing $\hbar$. Actually, this is just a heuristic argument; we rather restart from an $\hbar$-dependent definition of the Lax-Orlov-Shulman operators,

$$
L = e^{\eta \partial_q} + \sum_{n=0}^{\infty} u_{n+1} e^{-n \hbar \partial_q},
$$

$$
M = \sum_{n=1}^{\infty} nt_n L^n + q + \sum_{n=1}^{\infty} v_n L^{-n},
$$

$$
\bar{L} = \tilde{u}_0 e^{\eta \partial_q} + \sum_{n=0}^{\infty} \tilde{u}_{n+1} e^{(n+1) \eta \partial_q},
$$

$$
\bar{M} = -\sum_{n=1}^{\infty} n \bar{t}_n \bar{L}^{-n} + q + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^{n},
$$

(173)

and consider the $\hbar$-dependent twisted canonical commutation relations

$$
[L, M] = \hbar L, \quad [\bar{L}, \bar{M}] = \hbar \bar{L},
$$

(174)

and Lax equations

$$
\hbar \frac{\partial L}{\partial t_n} = [A_n, L], \quad \hbar \frac{\partial L}{\partial \bar{t}_n} = [\bar{A}_n, L],
$$

$$
\hbar \frac{\partial M}{\partial t_n} = [A_n, M], \quad \hbar \frac{\partial M}{\partial \bar{t}_n} = [\bar{A}_n, M],
$$

$$
\hbar \frac{\partial \bar{L}}{\partial t_n} = [A_n, \bar{L}], \quad \hbar \frac{\partial \bar{L}}{\partial \bar{t}_n} = [\bar{A}_n, \bar{L}],
$$

$$
\hbar \frac{\partial \bar{M}}{\partial t_n} = [A_n, \bar{M}], \quad \hbar \frac{\partial \bar{M}}{\partial \bar{t}_n} = [\bar{A}_n, \bar{M}].
$$

(175)

$A_n$ and $\bar{A}_n$ are defined in the the same way as in the case of $\hbar = 1$. (Note that we have also rescaled $M$ and $\bar{M}$ as

$$
M \rightarrow \hbar^{-1} M, \quad \bar{M} \rightarrow \hbar^{-1} \bar{M},
$$

(176)

so that they have the above expression.) Upon this reformulation, the coefficients $u_n$, etc. can depend on $\hbar$ in an arbitrary way as $u_n = u_n(\hbar, t, \bar{t}, q)$, i.e., we do not have to assume
that they are obtained from $\hbar$-independent functions of $(t, \bar{t}, q)$ by the above rescaling. To ensure the existence of classical limit, however, we have to assume that

$$u_n(h, t, \bar{t}, q) = u_n^{(0)}(t, \bar{t}, q) + o(h), \quad v_n(h, t, \bar{t}, q) = v_n^{(0)}(t, \bar{t}, q) + o(h),$$

as $h \to 0$, and construct from the leading terms the following Laurent series of a new variable $P$:

$$L = P + \sum_{n=0}^{\infty} u_n^{(0)} P^{-n},$$

$$M = \sum_{n=1}^{\infty} nt_n L^n + q + \sum_{n=1}^{\infty} v_n^{(0)} L^{-n},$$

$$\tilde{L} = \tilde{u}_0^{(0)} P + \sum_{n=0}^{\infty} \tilde{u}_{n+1}^{(0)} P^{n+2},$$

$$\tilde{M} = -\sum_{n=1}^{\infty} nt_n \tilde{L}^{-n} + q + \sum_{n=1}^{\infty} \tilde{v}_n^{(0)} \tilde{L}^n. \quad (178)$$

These Laurent series give a classical counterpart of the Lax and Orlov-Shulman operators, and indeed turn out to obey the classical twisted canonical relations

$$\{ L , M \} = L, \quad \{ \tilde{L} , \tilde{M} \} = \tilde{L} \quad (179)$$

and the Lax equations

$$\frac{\partial L}{\partial t_n} = \{ A_n, L \}, \quad \frac{\partial L}{\partial \bar{t}_n} = \{ \tilde{A}_n, L \},$$

$$\frac{\partial M}{\partial t_n} = \{ A_n, M \}, \quad \frac{\partial M}{\partial \bar{t}_n} = \{ \tilde{A}_n, M \},$$

$$\frac{\partial \tilde{L}}{\partial t_n} = \{ A_n, \tilde{L} \}, \quad \frac{\partial \tilde{L}}{\partial \bar{t}_n} = \{ \tilde{A}_n, \tilde{L} \},$$

$$\frac{\partial \tilde{M}}{\partial t_n} = \{ A_n, \tilde{M} \}, \quad \frac{\partial \tilde{M}}{\partial \bar{t}_n} = \{ \tilde{A}_n, \tilde{M} \}. \quad (180)$$

where $\{ , \}$ denotes the new Poisson bracket

$$\{ A(P, q), B(P, q) \} = P \frac{\partial A(P, q)}{\partial P} \frac{\partial B(P, q)}{\partial q} - \frac{\partial A(P, q)}{\partial q} P \frac{\partial B(P, q)}{\partial P}. \quad (181)$$
This somewhat unusual definition is rather natural in view of the quantum-classical correspondence

\[ [ e^{\hbar \frac{\partial}{\partial \mu}} , \ q ] = \hbar e^{\hbar \frac{\partial}{\partial \mu}} \rightarrow \{ P , \ q \} = P. \quad (182) \]

In fact, if we define \( Q = qP^{-1} \), \( P \) and \( Q \) form a canonical conjugate pair,

\[ \{ P , Q \} = 1. \quad (183) \]

Thus this Poisson bracket is essentially the same as the Poisson bracket of Section 3. The \( W_{1+\infty} \) algebra of difference operators turns into a \( w_{1+\infty} \) algebra realized by this Poisson bracket. Finally, \( A_n \) and \( \bar{A}_n \) are given by

\[ A_n = (\mathcal{L}^n)_{\geq 0}, \quad \bar{A}_n = (\mathcal{L}^{-n})_{< 0}, \quad (184) \]

where \((\quad)_{\geq 0}\) and \((\quad)_{< 0}\) now stand for the projection onto nonnegative and negative powers of \( P \). This classical limit of the Toda lattice hierarchy is also called the “dispersionless Toda hierarchy” in the literature [40].

Let us return to the issue of \( c = 1 \) strings. We have derived string equations (171) in the \( \hbar = 1 \) formulation. An \( \hbar \)-dependent reformulation is obtained by rescaling

\[ q \rightarrow \hbar^{-1}q, \quad M \rightarrow \hbar^{-1}M, \quad \bar{M} \rightarrow \hbar^{-1}\bar{M} \quad (185) \]

along with similar rescaling of \( t \) and \( \bar{t} \). Actually, we have seen in Section 3 that there are two different choices of \( \hbar \) and, accordingly, two different prescriptions of classical limit.

In the first choice with \( \hbar = -1/(i\sqrt{m}) \), rescaling of the string equations gives

\[ \bar{L}^{-2} = \frac{1}{\mu^2 - \hbar^{-2}} \left( - (\hbar^{-1}ML^{-1} - i\mu L^{-1} - L^{-2}), \right. \]
\[ L^2 = \frac{1}{\mu^2 - \hbar^{-2}} \left( - (\hbar^{-1}M\bar{L} + (1-i\mu)\bar{L})^2 - \bar{L}^2 \right). \quad (186) \]

In the classical limit as \( \hbar \rightarrow 0 \), these string equations are replaced by

\[ \bar{L}^{-2} = \mathcal{M}L^{-2} + L^{-2}, \]
\[ L^2 = \mathcal{M}^2 \bar{L}^2 + \bar{L}^2. \quad (187) \]
This reproduces classical scattering relations (81) upon identifying

\[ \begin{align*}
P_+ &= \bar{L}, & Q_+ &= \bar{M}\bar{L}^{-1}, \\
P_- &= L, & Q_- &= M\bar{L}^{-1}.
\end{align*} \tag{188} \]

Note that they are indeed a canonical pair with respect to the Poisson bracket introduced above, i.e., \( \{P_\pm, Q_\pm\} = 1 \).

Similarly, in the second choice with \( \hbar = -1/(i\mu) \), we obtain the string equations

\[ \begin{align*}
\bar{L}^{-2} &= \frac{1}{m - \hbar^{-2}} \left( -(\hbar^{-1}ML^{-1} + \hbar^{-1}L^{-1})^2 + mL^{-2} \right), \\
L^2 &= \frac{1}{m - \hbar^{-2}} \left( -(\hbar^{-1}\bar{M}\bar{L} + (1 + \hbar^{-1})\bar{L})^2 + m\bar{L}^2 \right), \tag{189} \end{align*} \]

and their classical limit

\[ \begin{align*}
\bar{L}^{-2} &= (M\bar{L}^{-1} + \bar{L}^{-1})^2, \\
L^2 &= (\bar{M}\bar{L} + \bar{L})^2. \tag{190} \end{align*} \]

These string equations, too, agree with classical scattering relations (87) under the same identification of \( P_\pm \) and \( Q_\pm \).

We have thus translated the scattering relations of Section 3 into string equations of the Lax-Orlov-Shulman operators of the Toda lattice hierarchy. By identifying \( J_n \) in \( H(t) \) and \( J_{-n} \) in \( \bar{H}(t) \) with creation operators of massless tachyons, the \( \tau \) function can be interpreted as the partition function of \( c = 1 \) strings in the presence of tachyon condensates besides the black hole background. The time variables \( t \) and \( \bar{t} \) play the role of sources that turn on tachyon condensation. The string equations can be rewritten as \( W_{1+\infty} \)-constraints to the \( \tau \) function. Finding an explicit form of those constraints is now a rather straightforward exercise. This should however be a crucial step for better understanding of higher orders of multi-loop expansion.

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