THE ACCESSIBILITY RANK OF WEAK EQUivalences

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Abstract. We study the accessibility properties of trivial cofibrations and weak equivalences in a combinatorial model category and give an estimate for the accessibility rank of weak equivalences in certain Cisinski model categories. In particular, we show that the class of weak equivalences between simplicial sets is finitely accessible.

1. Introduction

A well-known and useful property of combinatorial model categories is that their classes of weak equivalences are accessible. Although each of the various proofs of this result can also give some estimate for the accessibility rank, these estimates are generally not the best possible.

The purpose of this paper is to address the issue of determining better estimates for the accessibility rank of weak equivalences in cases of interest. We prove the following theorem.

Theorem A. The full subcategory $W$ of $SSet^\to$ spanned by the class of weak equivalences is finitely accessible.

The finite accessibility of $W$ in $SSet^\to$ means that given a weak equivalence of simplicial sets $f : X \to Y$ and a morphism $g : K \to L$ between finite simplicial sets, then every commutative square

\[
\begin{array}{ccc}
K & \to & X \\
\downarrow g & & \downarrow f \\
L & \to & Y
\end{array}
\]

admits a factorization

\[
\begin{array}{ccc}
K & \to & K' & \to & X \\
\downarrow g & & \downarrow h & & \downarrow f \\
L & \to & L' & \to & Y
\end{array}
\]

where $h : K' \to L'$ is weak equivalence between finite simplicial sets.
The proofs of the accessibility of the weak equivalences in a general combinatorial model category make essential use of the Pseudopullback Theorem according to which pseudopullbacks of accessible categories and functors are again accessible categories. The Pseudopullback Theorem, which will be recalled in Section 2, also gives an estimate for the accessibility rank of the pseudopullback and thus estimates for the accessibility ranks of weak equivalences can also be obtained. Using this method, it is not difficult to show that $W \subseteq SSet^\rightarrow$ is $\aleph_1$-accessible. More generally, applying this method to get an estimate for the accessibility rank of the weak equivalences in a $\kappa$-combinatorial model category will generally produce a regular cardinal strictly greater than $\kappa$.

Our strategy for proving Theorem A uses different methods which we think are also of independent interest. An important ingredient is the fat small-object argument which was introduced in [14]. Based on this, we show in Section 3 that every trivial cofibration in a $\kappa$-combinatorial model category is a $\kappa$-directed colimit of trivial cofibrations between $\kappa$-presentable objects.

In Section 4, we prove the main result of the paper which is the following generalization of Theorem A to more general combinatorial model categories.

**Theorem B.** Let $\mathcal{M}$ be a Cisinski model category whose class of weak equivalences is closed under filtered colimits. Let $\kappa$ be a regular cardinal such that

(I) $\mathcal{M}$ is $\kappa$-combinatorial and there is a cylinder functor $\text{Cyl} : \mathcal{M} \to \mathcal{M}$ which preserves $\kappa$-presentable objects,

(II) subobjects of $\kappa$-presentable objects are $\kappa$-presentable.

Then the full subcategory $\mathcal{W}$ of $\mathcal{M}^\rightarrow$ spanned by the class of weak equivalences is $\kappa$-accessible.

Some background material on Cisinski model categories will be recalled in Section 2. In the case where $\mathcal{M} = SSet$, with the usual model structure, the class of weak equivalences is closed under filtered colimits and $\aleph_0$ satisfies properties (I) and (II), so Theorem A is a consequence of Theorem B. We emphasize that a regular cardinal $\kappa$ satisfying (I) and (II) always exists.

We end in Section 5 with some additional remarks. First, we discuss the related problem of estimating the rank of a strong generator in $\mathcal{W}$. Finally, we comment on the relation with the results of [14] and [9] on the accessibility properties of acyclic objects.
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2. Background material and preliminaries

2.1. Combinatorial model categories. Combinatorial model categories were defined by J. H. Smith to be model categories $\mathcal{M}$ which are locally presentable and cofibrantly generated. The latter means that both cofibrations and trivial cofibrations are cofibrantly generated by a set of morphisms. There is always a regular cardinal $\kappa$ such that $\mathcal{M}$ is locally $\kappa$-presentable and the generating cofibrations and trivial cofibrations are morphisms between $\kappa$-presentable objects. In this case we say that $\mathcal{M}$ is $\kappa$-combinatorial. For instance, $\mathbb{S}Set$ is finitely combinatorial ($= \aleph_0$-combinatorial).

In a $\kappa$-combinatorial model category $\mathcal{M}$, the standard ‘replacement-by-fibration’ functor $R_{\text{fib}} : \mathcal{M}^{\to} \to \mathcal{M}^{\to}$ that arises from the (trivial cofibration, fibration)-factorization, as given by the small-object argument, preserves $\kappa$-filtered colimits but it does not preserve $\kappa$-presentable objects in general. By the Uniformization Theorem (see [11, 2.4.9], or [2, 2.19]), there is a regular cardinal $\lambda \geq \kappa$ such that $R_{\text{fib}}$ preserves $\lambda$-presentable objects. This cardinal is important because it makes the category of fibrations $\lambda$-accessible - any fibration is a $\lambda$-filtered colimit in $\mathcal{M}^{\to}$ of fibrations between $\lambda$-presentable objects. To see this, consider a fibration $p : X \to Y$ in $\mathcal{M}$ and express it as a $\lambda$-filtered colimit of morphisms $f_i : X_i \to Y_i$ between $\lambda$-presentable objects. Then $R_{\text{fib}}(p)$ is a $\lambda$-filtered colimit of fibrations between $\lambda$-presentable objects. Since $p$ is a retract of $R_{\text{fib}}(p)$, it follows that $p$ can also be expressed in this way (see the proof of [11, 2.3.11]). Thus we are interested in the smallest possible $\lambda$.

Example 2.1. Let $\mathcal{M} = \mathbb{S}Set$ and $R_{\text{fib}} : \mathbb{S}Set^{\to} \to \mathbb{S}Set^{\to}$ the standard replacement by a fibration as given by the small-object argument. The functor $R_{\text{fib}}$ preserves filtered colimits and sends finitely presentable objects to $\aleph_1$-presentable ones. It follows that $R_{\text{fib}}$ preserves $\aleph_1$-presentable objects since every $\aleph_1$-presentable object is an $\aleph_1$-small filtered colimit of finitely presentable objects (see [11, 2.3.11], [2, 2.15]).

Analogously, one can consider instead the ‘replacement-by-trivial fibration’ functor $R_{\text{wfib}} : \mathcal{M}^{\to} \to \mathcal{M}^{\to}$ that comes from the standard (cofibration, trivial fibration)-factorization in $\mathcal{M}$ and ask for the smallest regular cardinal $\lambda$ such that $R_{\text{wfib}}$ preserves $\lambda$-filtered colimits and
\(\lambda\)-presentable objects. In \(\mathcal{SSet}\), the smallest such \(\lambda\) is again \(\aleph_1\), which then shows that the category \(\mathcal{F} \cap \mathcal{W}\) of trivial fibrations in \(\mathcal{SSet}^\to\) is \(\aleph_1\)-accessible.

Then the accessibility of \(\mathcal{W}\) and an estimate for its accessibility rank can be obtained by the following theorem.

**Pseudopullback Theorem.** Let \(\lambda\) be a regular cardinal and \(\mathcal{K}, \mathcal{L}\) and \(\mathcal{M}\) \(\lambda\)-accessible categories which admit \(\kappa\)-filtered colimits for some \(\kappa < \lambda\). Let \(F : \mathcal{K} \to \mathcal{L}\) and \(G : \mathcal{M} \to \mathcal{L}\) be functors which preserve \(\kappa\)-filtered colimits and \(\lambda\)-presentable objects. Consider the pseudopullback

\[
\begin{array}{ccc}
\mathcal{P} & \rightarrow & \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{K} & \rightarrow & \mathcal{L}
\end{array}
\]

Then \(\mathcal{P}\) is \(\lambda\)-accessible and has \(\kappa\)-filtered colimits.

**Proof.** This theorem is essentially shown in [6, 3.1]. Although the statement of [6, 3.1] includes stronger accessibility properties, exactly the same proof applies here too. \(\square\)

The category of weak equivalences can be expressed as a pullback

\[
\begin{array}{ccc}
\mathcal{W} & \rightarrow & \mathcal{F} \cap \mathcal{W} \\
\downarrow & & \downarrow \\
\mathcal{M} & \rightarrow & \mathcal{M}^\to
\end{array}
\]

where \(G\) is the full embedding of the trivial fibrations. Since \(G\) is transportable, this pullback is equivalent to a pseudopullback (see [11, 5.1.1]). Then the Pseudopullback Theorem applied to this pseudopullback gives the following estimate for the accessibility rank of \(\mathcal{W}\). Let us add that the accessibility of \(\mathcal{W}\) was claimed by J. H. Smith. (Proofs can be found in [10] and [16].)

**Proposition 2.2.** Let \(\mathcal{M}\) be a \(\kappa\)-combinatorial model category and \(\lambda > \kappa\) a regular cardinal such that both \(R_{\text{fib}}\) and \(R_{\text{wfib}}\) preserve \(\lambda\)-presentable objects. Then the category \(\mathcal{W}\) of weak equivalences, as a full subcategory of \(\mathcal{M}^\to\), is \(\lambda\)-accessible and admits \(\kappa\)-filtered colimits.

**Example 2.3.** The discussion above and the last proposition shows that the full subcategory \(\mathcal{W}\) of weak equivalences in \(\mathcal{SSet}^\to\) is \(\aleph_1\)-accessible.
This means that the estimate that we can get with this method for the accessibility rank of \( W \) in a \( \kappa \)-combinatorial model category \( \mathcal{M} \) is going to be strictly greater than \( \kappa \). A way of improving this estimate was opened in [14] where the idea of good colimits from [10] was used to prove that every cofibrant object in a \( \kappa \)-combinatorial model category is a \( \kappa \)-directed colimit of \( \kappa \)-presentable cofibrant objects. In Section 3, we will show how this can be used to prove that every trivial cofibration in a \( \kappa \)-combinatorial model category is a \( \kappa \)-directed colimit of trivial cofibrations between \( \kappa \)-presentable objects. This is the first step of our proof of Theorem B, but the rest of the proof requires more assumptions on \( \mathcal{M} \).

2.2. Cisinski model categories. A Cisinski model category is a combinatorial model category whose underlying category is a Grothendieck topos and whose cofibrations are the monomorphisms. A systematic study of these model structures can be found in [7].

Every Grothendieck topos \( \mathcal{K} \) is locally presentable and its class of monomorphisms is cofibrantly generated by a set of morphisms (see [7, 1.29], [5, 1.12]). This implies the existence of functorial (cofibration, trivial fibration)-factorizations, see [7, 1.30]. In particular, for every \( K \in \mathcal{K} \), we obtain a functorial cylinder object \( \text{Cyl}(K) \) from the (cofibration, trivial fibration)-factorization of the codiagonal morphism

\[
\nabla : K \coprod K \xrightarrow{j} \text{Cyl}(K) \xrightarrow{s} K
\]

where \( j \) is a cofibration (= monomorphism) and \( s \) is a trivial fibration, i.e. it has the right lifting property with respect to all monomorphisms. We write

\[
j_0 = j \cdot i_0 \quad \text{and} \quad j_1 = j \cdot i_1
\]

where \( i_0, i_1 : K \to K \coprod K \) are the two inclusions to the coproduct.

Following [7, 2.3], we require, throughout this paper, that a cylinder functor \( \text{Cyl} : \mathcal{K} \to \mathcal{K} \) in a Grothendieck topos also satisfies the following properties:

1. \( \text{Cyl} : \mathcal{K} \to \mathcal{K} \) preserves monomorphisms and colimits.
2. For every monomorphism \( f : K \to L \), the commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{f} & L \\
\downarrow{j_k} & & \downarrow{j_k} \\
\text{Cyl}(K) & \xrightarrow{\text{Cyl}(f)} & \text{Cyl}(L)
\end{array}
\]

is a pullback, for \( k = 0, 1 \).
A cylinder functor satisfying these properties is called "donné homotopique élémentaire" in [7, 2.3]. Such cylinder functors are not unique but, following [7, 2.5], there is a canonical choice given by defining
\[ \text{Cyl}(K) = K \times \Omega \]
where \( \Omega \) is the subobject classifier, \( s : K \times \Omega \to K \) is the first projection, and \( j_0 = (\text{id}_K, 1) \) and \( j_1 = (\text{id}_K, \emptyset) \) where \( 1, \emptyset : 1 \to \Omega \) classify the subobjects \( \text{id}_1 : 1 \to 1 \) and \( 0 \to 1 \) of the terminal object 1.

**Example 2.4.** The standard cylinder functor in the category of simplicial sets, given by \( K \mapsto \text{Cyl}(K) := K \times \Delta^1 \), satisfies properties (1) and (2) above. Moreover, it preserves finitely presentable objects in \( \text{SSet} \), thus satisfying also assumption (I) of Theorem B.

We will make use of the following elementary properties of cylinder functors in a Grothendieck topos.

**Lemma 2.5.** Let \( \mathcal{K} \) be a Grothendieck topos and \( \text{Cyl} : \mathcal{K} \to \mathcal{K} \) a cylinder functor. Then:

(a) for every epimorphism \( f : K \to L \), the morphism \( \text{Cyl}(f) : \text{Cyl}(K) \to \text{Cyl}(L) \) is again an epimorphism and the square

\[
\begin{array}{ccc}
\text{Cyl}(K) & \xrightarrow{\text{Cyl}(f)} & \text{Cyl}(L) \\
\downarrow s & & \downarrow s \\
K & \xrightarrow{f} & L
\end{array}
\]

is a pushout.

(b) the cylinder functor sends pullback squares,

\[
\begin{array}{ccc}
M & \xrightarrow{g} & K \\
\downarrow f & & \downarrow g \\
L & \xrightarrow{f} & X
\end{array}
\]

where all morphisms are monomorphisms, to pullback squares.

**Proof.** (a) Since \( \text{Cyl} \) preserves colimits, it preserves epimorphisms. Consider \( u : \text{Cyl}(L) \to X \) and \( v : K \to X \) such that \( u \text{Cyl}(f) = vs \). Then
\[ u(j_0 f) = (u \text{Cyl}(f)) j_0 = v(s j_0) = v \]
and
\[ u j_0 (s \text{Cyl}(f)) = (uj_0 f)s = vs = u \text{Cyl}(f). \]
Since Cyl\( (f) \) is an epimorphism, it follows that \( u_j s = u \). Thus \( u_j : L \rightarrow X \) is the desired morphism which is also unique because \( f \) is an epimorphism.

(b) Consider the union of subobjects of \( X \),

\[
\begin{align*}
  M & \overset{f}{\longrightarrow} K \\
  \downarrow & \downarrow \\
  L & \longrightarrow K \cup_M L
\end{align*}
\]

and the canonical monomorphism \( h : K \cup_M L \rightarrow X \). Since Cyl : \( K \rightarrow \mathcal{K} \) preserves pushouts and monomorphisms, the square

\[
\begin{array}{ccc}
  \text{Cyl}(M) & \xrightarrow{\text{Cyl}(f)} & \text{Cyl}(K) \\
  \text{Cyl}(g) & \downarrow & \downarrow \text{Cyl}(g) \\
  \text{Cyl}(L) & \longrightarrow & \text{Cyl}(K \cup_M L)
\end{array}
\]

is a pushout and the morphisms Cyl(\( f \)) and Cyl(\( g \)) are monomorphisms. Therefore it is also a pullback square. (This uses that equivalence relations in a Grothendieck topos are effective.) Then composition with the monomorphism Cyl(h) : Cyl(\( K \cup_M L \)) \( \rightarrow \text{Cyl}(X) \),

\[
\begin{array}{ccc}
  \text{Cyl}(M) & \xrightarrow{\text{Cyl}(f)} & \text{Cyl}(K) \\
  \text{Cyl}(g) & \downarrow & \downarrow \text{Cyl}(g) \\
  \text{Cyl}(L) & \longrightarrow & \text{Cyl}(X)
\end{array}
\]

gives again a pullback square, as required. \( \square \)

**Proposition 2.6.** Every Cisinski model category satisfies conditions (I) and (II) of Theorem B for some regular cardinal \( \kappa \).

**Proof.** Let \( \mathcal{M} \) be a Cisinski model category. There is a regular cardinal \( \lambda \) such that \( \mathcal{M} \) is \( \lambda \)-combinatorial. The functor Cyl preserves colimits, therefore by the Uniformization Theorem, there is a regular cardinal \( \kappa_1 \geq \lambda \) such that Cyl preserves \( \kappa_1 \)-presentable objects. Moreover, \( \mathcal{M} \) satisfies (I) also for every other regular cardinal \( \kappa \) such that \( \kappa \geq \kappa_1 \) (see [2, 2.20]).

Let \( \mathcal{M}^{\rightarrow \text{mono}} \) denote the full subcategory of \( \mathcal{M}^{\rightarrow} \) spanned by the monomorphisms. Following [2, 1.60], \( \mathcal{M}^{\rightarrow \text{mono}} \) is closed under \( \lambda \)-filtered colimits in \( \mathcal{M}^{\rightarrow} \). Moreover, \( \mathcal{M}^{\rightarrow \text{mono}} \) is reflective in \( \mathcal{M}^{\rightarrow} \) because the full inclusion \( G : \mathcal{M}^{\rightarrow \text{mono}} \rightarrow \mathcal{M}^{\rightarrow} \) admits a left adjoint \( F : \mathcal{M}^{\rightarrow} \rightarrow \mathcal{M}^{\rightarrow \text{mono}} \) which sends \( u : X \rightarrow Y \) to the associated monomorphism
$F(u) : \text{im}(u) \to Y$. Since $G$ preserves $\lambda$-filtered colimits, $F$ preserves $\lambda$-presentable objects and $\mathcal{M}^{\to}_{\text{mono}}$ is $\lambda$-accessible and $\lambda$-accessibly embedded in $\mathcal{M}^{\to}$. By the Uniformization Theorem, there is a regular cardinal $\kappa_2$ such that the inclusion functor $G$ preserves $\kappa_2$-presentable objects. This means that if $i : Z \to Y$ is a $\kappa_2$-presentable object of $\mathcal{M}^{\to}_{\text{mono}}$, which in particular means that $Y$ is $\kappa_2$-presentable in $\mathcal{M}$, then $i$ is $\kappa_2$-presentable in $\mathcal{M}^{\to}$, which means that $Z$ is also $\kappa_2$-presentable in $\mathcal{M}$. Therefore $\mathcal{M}$ satisfies (II) for every regular cardinal $\kappa$ such that $\kappa \geq \kappa_2$.

Then the result follows by choosing $\kappa \geq \kappa_1, \kappa_2$. \qed

3. **Trivial cofibrations**

Combinatorial categories, introduced in [13], are locally presentable categories $\mathcal{K}$ equipped with a class of morphisms $\text{cof}(\mathcal{K})$, called cofibrations, which is cofibrantly generated by a set of morphisms. This basic categorical structure is inspired by combinatorial model categories, which can be viewed as combinatorial categories in (at least) two different ways, and the aim was to analyze through this generalization the general properties of cofibrant generation by focusing on a single cofibrantly generated class of morphisms.

In analogy with combinatorial model categories, a combinatorial category $(\mathcal{K}, \text{cof}(\mathcal{K}))$ is called $\kappa$-combinatorial if it is locally $\kappa$-presentable and there is a set of generating cofibrations $I$ between $\kappa$-presentable objects. In what follows, $\mathcal{K}_\kappa$ will denote the full subcategory of $\mathcal{K}$ consisting of $\kappa$-presentable objects. Every locally presentable category carries the trivial combinatorial structure where every morphism is a cofibration. The following result is [13, 2.4] but we will recall the proof.

**Lemma 3.1.** Let $\mathcal{K}$ be a locally $\kappa$-presentable category. Then the trivial combinatorial structure on $\mathcal{K}$ is $\kappa$-combinatorial.

**Proof.** If a morphism $g$ has the right lifting property with respect to all morphisms between $\kappa$-presentable objects then $g$ is both a $\kappa$-pure monomorphism and a $\kappa$-pure epimorphism. Thus $g$ is both a regular monomorphism and an epimorphism (see [1], [3]), which means that $g$ is an isomorphism. Hence any morphism has the left lifting property with respect to $g$. \qed

We say that an object $K$ of a combinatorial category is cofibrant if the unique morphism $0 \to K$ from an initial object is a cofibration. One of the main results of [14] shows that *every cofibrant object in a $\kappa$-combinatorial category is a $\kappa$-directed colimit of $\kappa$-presentable cofibrant objects* [14, 5.1]. A consequence of this is the next theorem.
**Theorem 3.2.** Let \( \mathcal{K} \) be a \( \kappa \)-combinatorial category. Then every cofibration is a \( \kappa \)-directed colimit of cofibrations between \( \kappa \)-presentable objects.

**Proof.** Let \( \mathcal{I} \subseteq \mathcal{K}_\kappa^\to \) be a generating set of cofibrations. Consider the following sets of morphisms in \( \mathcal{K}_\kappa^\to \):

\[
\chi_1 = \{(id : A \to A) \xrightarrow{(i,i)} (id : B \to B) : i \in \mathcal{K}_\kappa^\to\}
\]

and

\[
\chi_2 = \{(id : A \to A) \xrightarrow{(id,j)} (j : A \to B) : j \in \mathcal{I}\}.
\]

Then the set \( \chi_1 \cup \chi_2 \) generates a \( \kappa \)-combinatorial structure on \( \mathcal{K}_\kappa^\to \). Following \([14, 5.1]\), it suffices to show that its cofibrant objects are precisely the cofibrations in \( \mathcal{K} \). It is easy to see that every cofibrant object in \( \mathcal{K}_\kappa^\to \) is a cofibration in \( \mathcal{K} \). Conversely, let \( f : X \to Y \) be a cofibration in \( \mathcal{K} \). An immediate consequence of Lemma 3.1 is that the object \((id : X \to X)\) is cofibrant with respect to \( \chi_1 \). Moreover, the morphism in \( \mathcal{K}_\kappa^\to \):

\[
(id, f) : (id : X \to X) \longrightarrow (f : X \to Y)
\]

is a cofibration with respect to \( \chi_2 \). Therefore \( f \) is a cofibrant object with respect to \( \chi_1 \cup \chi_2 \) and the result follows. \( \square \)

**Corollary 3.3.** Let \( \mathcal{M} \) be a \( \kappa \)-combinatorial model category. Then every trivial cofibration in \( \mathcal{M} \) is a \( \kappa \)-directed colimit of trivial cofibrations between \( \kappa \)-presentable objects.

**Proof.** It suffices to apply Theorem 3.2 to the \( \kappa \)-combinatorial category defined by the underlying category of \( \mathcal{M} \) together with the class of trivial cofibrations. \( \square \)

**Remark 3.4.** Of course, the proof of Corollary 3.3 does not require that there exists a generating set of cofibrations between \( \kappa \)-presentable objects. An alternative argument that uses this assumption is as follows. Let \( \mathcal{I} \) denote a generating set of cofibrations between \( \kappa \)-presentable objects and replace \( \chi_1 \) with

\[
\chi'_1 = \{(id : A \to A) \xrightarrow{(i,i)} (id : B \to B) : i \in \mathcal{I}\}
\]

Then \((id : X \to X)\) is \( \chi'_1 \)-cofibrant if \( X \) is cofibrant in \( \mathcal{M} \). The same argument as above then shows that a trivial cofibration between cofibrant objects is a \( \kappa \)-directed colimit of a diagram \( F : P \to \mathcal{M}_\kappa^\to \) whose values are trivial cofibrations between \( \kappa \)-presentable cofibrant objects. Another advantage of this argument is that if \( X \) and \( f \) are cellular or \( \kappa \) is uncountable, then \( F \) can be chosen so that for all \( p \in P \), the morphism \( F(p) \to f \) in \( \mathcal{M}_\kappa^\to \) is given by cofibrations (see \([14]\), esp.
the proofs of 5.1 and 5.2). We recall that a morphism in a combinatorial
category, cofibrantly generated by a set $\mathcal{X}$, is called *cellular* if it can be
obtained from $\mathcal{X}$ by transfinite compositions of pushouts. Cofibrations
are then retracts of cellular morphisms.

**Remark 3.5.** Let $\mathcal{M}$ be a $\kappa$-combinatorial model category and

\[
\begin{array}{ccc}
K & \xrightarrow{u} & X \\
\downarrow{g} & & \downarrow{f} \\
L & \xrightarrow{v} & Y
\end{array}
\]

a commutative square where $g$ is a cofibration between $\kappa$-presentable
objects and $f$ is a weak equivalence. Then there is a filling

\[
\begin{array}{ccc}
K & \xrightarrow{u_1} & K' \xrightarrow{u_2} X \\
\downarrow{g} & & \downarrow{f} \\
L & \xrightarrow{v_1} & L' \xrightarrow{v_2} Y
\end{array}
\]

where $h$ is a trivial cofibration between $\kappa$-presentable objects. Indeed
let

\[
f : X \xrightarrow{f_1} Z \xrightarrow{f_2} Y
\]

be a (trivial cofibration, trivial fibration)-factorization of $f$. Since $g$
is a cofibration, there is a morphism $t : L \to Z$ such that $f_2t = v$
and $tg = f_1u$. Thus we get a morphism $(u, t) : g \to f_1$ in $\mathcal{M}^{\to}$. By
Corollary 3.3 there is a filling

\[
\begin{array}{ccc}
K & \xrightarrow{u_1} & K' \xrightarrow{u_2} X \\
\downarrow{g} & & \downarrow{f_1} \\
L & \xrightarrow{t_1} & L' \xrightarrow{t_2} Z
\end{array}
\]

with $u = u_2u_1$ and $t = t_2t_1$. Thus

\[
\begin{array}{ccc}
K & \xrightarrow{u_1} & K' \xrightarrow{u_2} X \\
\downarrow{g} & & \downarrow{f} \\
L & \xrightarrow{t_1} & L' \xrightarrow{f_2t_2} Y
\end{array}
\]

is the required filling.

Trivial cofibrations in a $\kappa$-combinatorial model category $\mathcal{M}$ are not
closed under $\kappa$-filtered colimits in general. However, weak equivalences are,
as follows from the arguments of Section 2 and Proposition 2.2. These arguments required that both cofibrations and trivial
cofibrations are cofibrantly generated by sets of morphisms between $\kappa$-
presentable objects. We will give another proof which only needs this
for cofibrations. In fact, the proof that follows does not even require that \( \mathcal{M} \) is locally presentable; see also [15].

**Proposition 3.6.** Let \( \mathcal{M} \) be a combinatorial model category and \( \mathcal{I} \) a generating set of cofibrations between \( \kappa \)-presentable objects. Then the class of weak equivalences in \( \mathcal{M} \) is closed under \( \kappa \)-filtered colimits in \( \mathcal{M} \rightarrow \).

**Proof.** Let \( C \) be a small \( \kappa \)-filtered category and consider the category of \( C \)-diagrams \( \mathcal{M}^C \) with the projective model structure where weak equivalences and fibrations are defined pointwise. Then the colimit functor

\[
\text{colim}_C : \mathcal{M}^C \to \mathcal{M}
\]

is a left Quillen functor. It is required to show that the colimit functor preserves weak equivalences. Every weak equivalence in \( \mathcal{M}^C \) can be written as a composition of a projective trivial cofibration and a pointwise trivial fibration. The colimit of a projective trivial cofibrations is a trivial cofibration in \( \mathcal{M} \) because \( \text{colim}_C \) is left Quillen. Since both domains and codomains of the morphisms in \( \mathcal{I} \) are \( \kappa \)-presentable, it follows that the class of trivial fibrations \( \mathcal{I}^\square \) is closed under \( \kappa \)-filtered colimits. This means that \( \text{colim}_C \) preserves trivial fibrations too, and then the result follows. \( \square \)

4. **Proof of Theorem B**

By assumption the weak equivalences are closed under filtered colimits, so it suffices to show that every weak equivalence \( f : X \to Y \) is a \( \kappa \)-filtered colimit of weak equivalences between \( \kappa \)-presentable objects.

Since all objects in \( \mathcal{M} \) are cofibrant, \( X \) and \( Y \) are cofibrant. In fact, we can even assume that \( X \) is cellular. To see this note first that every cofibrant object is a retract of a cellular object \( X' \) where the retraction \( u : X' \to X \) is a trivial fibration. Thus \( f \) also is a retract of a weak equivalence \( fu : X' \to Y \) with cellular domain. If we can express \( fu \) as a \( \kappa \)-filtered colimit of a diagram with values weak equivalences between \( \kappa \)-presentable objects, then, following [11, 2.3.11], \( f \) is also a \( \kappa \)-filtered colimit of a diagram which has the same objects and takes the same values but has different morphisms. In particular, \( f \) is also a \( \kappa \)-filtered colimit of weak equivalences between \( \kappa \)-presentable objects.
Let $f : X \to Y$ be a weak equivalence in $\mathcal{M}$ where $X$ is cellular. Consider the mapping cylinder of $f$ defined by the pushout

$$
\begin{array}{c}
X \xrightarrow{j_0} \text{Cyl}(X) \\
\downarrow f \downarrow q \\
Y \xrightarrow{e} \sim M_f
\end{array}
$$

and the standard factorization of $f$ as follows

$$
X \xrightarrow{i} M_f \xrightarrow{p} Y
$$

where $i : X \xrightarrow{j} \text{Cyl}(X) \xrightarrow{q} M_f$ is a cofibration and $p$ is a weak equivalence such that $pe = \text{id}_Y$.

The “2-out-of-3” property implies that $i$ is a trivial cofibration. Let $\mathcal{J}$ denote a generating set of trivial cofibrations between $\kappa$-presentable objects. By Theorem 3.2 and Corollary 3.3, $i$ belongs to the cofibrant closure in $\mathcal{M}^\to$ of the set of morphisms $\chi_1 \cup \chi_2$ where

$$
\chi_1 = \{(id : A \to A) \xrightarrow{(x,x)} (id : B \to B) : x \in \mathcal{M}_\kappa^\to\}
$$

and

$$
\chi_2 = \{(id : A \to A) \xrightarrow{(id,j)} (y : A \to B) : j \in \mathcal{J}\}
$$

and thus it is a colimit of a $\kappa$-directed diagram

$$
F : P \to \mathcal{M}^\to
$$

whose values are weak equivalences between $\kappa$-presentable objects.

In the diagram

$$
\begin{array}{c}
\begin{array}{c}
\xymatrix{ X \ar[r]^{j_0} & \text{Cyl}(X) \ar[r]^s & X \\
Y \ar[r]_e & M_f \ar[r]^p & Y \\
\ar[d]_f \ar[d]_q \ar[d]_f \\
\end{array}
\end{array}
\end{array}
$$

where the horizontal compositions are identities, the outer rectangle and the left square are pushouts. Thus the right square is a pushout, which yields the following pushout in $\mathcal{M}^\to$

$$
\begin{array}{c}
\xymatrix{ (0 \to \text{Cyl}(X)) \ar[r]^{(0,q)} \ar[d]_{(id,s)} & (i : X \to M_f) \ar[d]^{(id,p)} \\
(0 \to X) \ar[r]^{(0,f)} & (f : X \to Y) \\
\end{array}
$$

Thus the morphism $f$ is obtained from $i$ by collapsing along the cylindrical direction.
Consider the set of morphisms in $\mathcal{M}^\to$
\[
\chi_3 = \{(\text{id}, s) : (0 \to Cyl(K)) \to (0 \to K) : K \in \mathcal{M}_\kappa\}.
\]

Since $X$ is cellular, it is a $\kappa$-directed colimit of $\kappa$-presentable subobjects $K_n$, $n \in I$ (see [14, 4.13 and 4.14(1)]). The corresponding monomorphisms will be denoted by $k_n : K_n \to X$. Thus there is an epimorphism
\[
k = \langle k_n \rangle : \bigsqcup_{n \in I} K_n \to X.
\]

Then it follows, by Lemma 2.5(a), that there is a pushout square
\[
\begin{array}{ccc}
\bigsqcup Cyl(K_n) & \xrightarrow{Cyl(k)} & Cyl(X) \\
\downarrow s_n & & \downarrow s \\
\bigsqcup K_n & \xrightarrow{k} & X
\end{array}
\]

The consequence is that
\[
(\text{id}, s) : (0 \to Cyl(X)) \to (0 \to X)
\]
is cellular with respect to $\chi_3$ and, consequently, that
\[
(\text{id}, p) : (i : X \to M_f) \to (f : X \to Y)
\]
is cellular with respect to $\chi_3$. Moreover, for purposes that will become clearer only later in the proof, we can give an explicit $\chi_3$-cellular presentation of this morphism. Let $(n_\alpha)_{\alpha < \zeta}$ be a well-ordering of $I$. Then $\bigsqcup_{\alpha < \zeta} K_{n_\alpha}$ is a transfinite composition of $\bigsqcup_{\beta < \alpha} K_{n_\beta}$ where the passage from $\alpha$ to $\alpha + 1$ is given by a pushout with $0 \to K_{n_\alpha}$. Since the functor $Cyl$ preserves colimits, $\bigsqcup_{\alpha < \zeta} s_{n_\alpha}$ is a transfinite composition of $\bigsqcup_{\beta < \alpha} s_{n_\beta}$ where the passage from $\alpha$ to $\alpha + 1$ is given by a pushout with $id_0 \to s_{n_\alpha}$. Thus $f$ is a composition of a smooth chain
\[
G : \zeta \to \mathcal{M}^\to
\]
where $G(0) = i$ and whose links $G(\alpha) \to G(\alpha + 1)$ are pushouts of
\[
(\text{id}_0, s_{n_\alpha}) : (0 \to Cyl(K_{n_\alpha})) \to (0 \to K_{n_\alpha}).
\]

Thus the arguments so far have shown that the morphism $f$ is cofibrant in $\mathcal{M}^\to$ with respect to the generating set $\chi_1 \cup \chi_2 \cup \chi_3$. The problem with finishing the argument by applying directly Theorem 3.2 is that cofibrant objects with respect to $\chi_3$ are not weak equivalences in general. Thus we need to look closer at the construction in the proof of Theorem 3.2 to make sure that this is indeed true in this case. This is essentially what follows in the proof.
The join of the diagrams $F$ and $G$ gives a new diagram:

$$H : P + \zeta \rightarrow \mathcal{M}^\rightarrow$$

whose colimit is the morphism $f$. Here, $P + \zeta$ is the ordinal sum of $P$ and $\zeta$ and $H(0) = i$. Then our goal will be achieved if we prove the following assertion.

**Assertion 4.1.** For every $\alpha \leq \zeta$ there exist a $\kappa$-directed poset $P_\alpha$ and a $\kappa$-directed diagram $D_\alpha : P_\alpha \rightarrow \mathcal{M}^\rightarrow$ such that:

(i) for every $\beta < \alpha$, $P_\beta \subseteq P_\alpha$ is an initial segment and $D_\beta$ is the restriction of $D_\alpha$,
(ii) the colimit of the restriction of $H$ to the subcategory $P + \alpha$ is a colimit of $D_\alpha$,
(iii) the values of $D_\alpha$ are morphisms between $\kappa$-presentable objects,
(iv) the values of $D_\alpha$ are weak equivalences.

We proceed to prove this by transfinite induction on $\alpha$. For $\alpha = 0$ we put $P_0 = P$ and $D_0 = F$. Let us assume that the assertion holds for all $\beta < \alpha$ where $0 < \alpha \leq \zeta$.

**Case 1:** $\alpha$ is a limit ordinal. Then we take the union $\tilde{P}_\alpha$ of all $P_\beta$, $\beta < \alpha$, and define $\tilde{D}_\alpha : \tilde{P}_\alpha \rightarrow \mathcal{M}^\rightarrow$ to be the obvious extension of the functors $D_\beta$. This does not yet satisfy the required assumptions because $\tilde{P}_\alpha$ will not be $\kappa$-directed in general when $\kappa$ is uncountable. Define $P_\alpha$ by adding, for each directed subset $S \subseteq \tilde{P}_\alpha$ of cardinality $< \kappa$ that does not have an upper bound, an element $p_S$ such that $s < p_s$ for each $s \in S$ and $p_S \leq p_{S'}$ when $S \subseteq S'$. Since a union of less than $\kappa$ subsets of cardinality less than $\kappa$ has cardinality less than $\kappa$, it can easily be seen that $P_\alpha$ is $\kappa$-directed. We define $D_\alpha : P_\alpha \rightarrow \mathcal{M}^\rightarrow$ by $D(p_S) = \colim_S \tilde{D}_\alpha$. Since we assume that weak equivalences are closed under directed colimits, the values of $D_\alpha$ are again weak equivalences. Since each $S$ has less than $\kappa$-elements, the values of $D_\alpha$ are morphisms between $\kappa$-presentable objects.

**Case 2:** $\alpha = \beta + 1$ is a successor ordinal. There is a pushout

$$
\begin{array}{ccc}
(0 \rightarrow \text{Cyl}(K)) & \longrightarrow & (H(\beta) : X \rightarrow N_\beta) \\
\downarrow & & \downarrow \\
(0 \rightarrow K) & \longrightarrow & (H(\alpha) : X \rightarrow N_\alpha)
\end{array}
$$
where \( k = k_{n\beta} : K = K_{n\beta} \to X \). Thus we have the pushout square

\[
\begin{array}{ccc}
\text{Cyl}(K) & \longrightarrow & N_{\beta} \\
\downarrow & & \downarrow \\
K & \longrightarrow & N_{\alpha}
\end{array}
\]

The object \( N_{\beta} \) is the quotient of \( M_f \) defined by the pushout of a diagram

\[
\bigsqcup_{\gamma + 1 < \beta} K_{n\gamma} \leftarrow \bigsqcup_{\gamma + 1 < \beta} \text{Cyl}(K_{n\gamma}) \longrightarrow M_f
\]

Let \( L \) be the image of \( l = \langle k_{n\gamma} \rangle : \bigsqcup_{\gamma + 1 < \beta} K_{n\gamma} \to X \). Consider the pushout

\[
\begin{array}{ccc}
\text{Cyl}(L) & \xrightarrow{\text{Cyl}(l)} & \text{Cyl}(X) \\
\downarrow s & & \downarrow r \\
L & \xrightarrow{\tau} & Z
\end{array}
\]

By Lemma 2.5(a), \( N_{\beta} \) is also the pushout of \( s : \text{Cyl}(L) \to L \) along \( q : \text{Cyl}(l) : \text{Cyl}(L) \to M_f \), hence we obtain a pushout

\[
\begin{array}{ccc}
\text{Cyl}(X) & \xrightarrow{q} & M_f \\
\downarrow r & & \downarrow \tau \\
Z & \xrightarrow{\tau} & N_{\beta}
\end{array}
\]

and consequently, also a composite pushout

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow r_{j_0} & & \downarrow \tau_e \\
Z & \xrightarrow{\tau} & N_{\beta}
\end{array}
\]

The morphism \( s : \text{Cyl}(X) \to X \) is the composition of \( r : \text{Cyl}(X) \to Z \) with an induced morphism \( Z \to X \). The composition \( r_{j_0} : X \to Z \) is a section of the latter morphism, and therefore a monomorphism.

By assumption (II), the subobject \( K \cap L \) of \( K \) is again \( \kappa \)-presentable. Let \( l' : K \cap L \to K \) denote the embedding and consider the following pushout

\[
\begin{array}{ccc}
\text{Cyl}(K \cap L) & \xrightarrow{\text{Cyl}(l')} & \text{Cyl}(K) \\
\downarrow s & & \downarrow \pi \\
K \cap L & \xrightarrow{\tau} & Z'
\end{array}
\]
The object \( Z' \) is also \( \kappa \)-presentable. Let \( k' : K \cap L \to L \) be the embedding. Since
\[
r \text{Cyl}(k) \text{Cyl}(l') = r \text{Cyl}(l) \text{Cyl}(k') = i s \text{Cyl}(k') = i k's,
\]
there is a unique morphism \( z : Z' \to Z \) such that \( z i' = i k' \) and \( z i = r \text{Cyl}(k) \).

**Claim 4.2.** The morphism \( z \) is a monomorphism.

**Proof of Claim 4.2.** Consider a pushout
\[
\begin{array}{ccc}
\text{Cyl}(K \cap L) & \xrightarrow{\text{Cyl}(l')} & \text{Cyl}(K) \\
\downarrow \text{Cyl}(k') & & \downarrow k' \\
\text{Cyl}(L) & \xrightarrow{\text{Cyl}(l)} & \text{Cyl}(K \cup L)
\end{array}
\]
where \( K \cup L \) denotes the union of \( K \) and \( L \) as subobjects of \( X \). We recall that the square
\[
\begin{array}{ccc}
\text{Cyl}(K \cap L) & \xrightarrow{\text{Cyl}(l')} & \text{Cyl}(K) \\
\downarrow \text{Cyl}(k') & & \downarrow \text{Cyl}(k) \\
\text{Cyl}(L) & \xrightarrow{\text{Cyl}(l)} & \text{Cyl}(X)
\end{array}
\]
is a pullback by Lemma 2.5(b). Consider the induced monomorphism \( z_1 : \text{Cyl}(K \cup L) \to \text{Cyl}(X) \) and the pushout
\[
\begin{array}{ccc}
\text{Cyl}(L) & \xrightarrow{i'} & \text{Cyl}(K \cup L) \\
\downarrow s & & \downarrow z' \\
L & \xrightarrow{i''} & Z''
\end{array}
\]
Then we obtain a composite pushout
\[
\begin{array}{ccc}
\text{Cyl}(K \cap L) & \xrightarrow{\text{Cyl}(l')} & \text{Cyl}(K) \\
\downarrow s \text{Cyl}(k') & & \downarrow z' k' \\
L & \xrightarrow{i''} & Z''
\end{array}
\]
Since
\[
r \text{Cyl}(k) \text{Cyl}(l') = r \text{Cyl}(l) \text{Cyl}(k') = i s \text{Cyl}(k'),
\]
there is a morphism \( z_2 : Z'' \to Z \) such that \( z_2 z' k' = r \text{Cyl}(k) \) and \( z_2 i'' = i \).
In the diagram

\[
\begin{array}{c}
\text{Cyl}(L) \xrightarrow{\tau'} \text{Cyl}(K \cup L) \xrightarrow{z_1} \text{Cyl}(X) \\
\downarrow s \quad \quad \downarrow z' \quad \quad \downarrow r \\
L \xrightarrow{l''} Z' \xrightarrow{z_2} Z
\end{array}
\]

the outer rectangle is a pushout because the top composition is \(\text{Cyl}(l)\) and the bottom one is \(\overline{t}l\). Since the square on the left side is a pushout, the square on the right side is also a pushout. Consequently, since \(z_1\) is a monomorphism, \(z_2\) is also a monomorphism. There is a morphism \(z_3 : Z' \to Z''\) such that \(z_3 \overline{s} = z'k'\) and the following diagram commutes

\[
\begin{array}{c}
\text{Cyl}(K \cap L) \xrightarrow{s} K \cap L \xrightarrow{k'} L \\
\downarrow \text{Cyl}(l') \quad \quad \downarrow \overline{t} \quad \quad \downarrow \tau'' \\
\text{Cyl}(K) \xrightarrow{\pi} Z' \xrightarrow{z_3} Z''
\end{array}
\]

The outer rectangle and the square on the left side are pushouts (because the top composition is \(s \text{Cyl}(k')\)). Thus the square on the right side is a pushout, which implies that \(z_3\) is a monomorphism because \(k'\) is. Since

\[
z_2z_3\overline{t} = z_2\overline{l''}k' = \overline{t}k'
\]

and

\[
z_2z_3\overline{s} = z_2z'\overline{k''} = r \text{Cyl}(k),
\]

it follows that \(z = z_2z_3\), i.e., more explicitly, the morphism \(z\) is the composition

\[
\text{Cyl}(K) \cup_{\text{Cyl}(K \cap L)} (K \cap L) \xrightarrow{z_3} \text{Cyl}(K \cup L) \cup_{\text{Cyl}(L)} L \xrightarrow{z_2} \text{Cyl}(X) \cup_{\text{Cyl}(L)} L.
\]

Therefore \(z\) is a monomorphism. \(\square\)

Consider the composition \(\overline{f}z : Z' \to N_\beta\). Since \(D_\beta : P_\beta \to \mathcal{M}^{\to}\) is \(\kappa\)-directed and \(Z'\) is \(\kappa\)-presentable, this composition factors through some earlier stage

\[
z_x : Z' \to \text{cod}(D_\beta(x)) = N^x_\beta.
\]

Then we apply the construction used in the proof of [13, 4.11] to define a new diagram \(D_\alpha : P_\alpha \to \mathcal{M}^{\to}\) with the required properties. The \(\kappa\)-directed poset \(P_\alpha\) is defined as follows:

- for every \(x \leq y\) in \(P_\beta\), we add a new object \(p_y\) with \(y < p_y\);
- for every \(x < y < y'\) in \(P_\beta\), we set \(p_y < p_{y'}\).
To define the value of $D_\alpha$ at an object $p_y \in P_\alpha$, we consider the pushout

$$
\begin{array}{ccc}
Z' & \xrightarrow{y} & N^y_eta \\
\downarrow{t} & & \downarrow{\tau} \\
K & \xrightarrow{\bar{y}} & \bar{N}_y
\end{array}
$$

where $t$ is a unique morphism satisfying $t\bar{l}' = l'$ and $t\bar{s} = s$ and the top morphism is the composition

$$
Z' \xrightarrow{z} N^x_\beta \rightarrow N^y_\beta
$$

where the second morphism is $D_\beta(x < y)$. Then we set

$$
D_\alpha(p_y) := (\text{dom}(D_\beta(y)) \rightarrow \bar{N}_y).
$$

The definition of $D_\alpha$ extends to morphisms in the obvious way, by taking the induced morphisms on pushouts. The new object $\bar{N}_y$ is a pushout of $\kappa$-presentable objects, so it is again $\kappa$-presentable.

It is easy to see that properties (i)-(iii) of Assertion 4.1 are satisfied. The rest of the proof deals with showing that property (iv) is also satisfied. By the “2-out-of-3” property of weak equivalences, this can now be stated as follows.

Claim 4.3. With the above notation, the morphism $t : N^y_\beta \rightarrow \bar{N}_y$ is a weak equivalence.

Consider the diagram

(3)

where:

- the top square on the right is the pushout square as defined previously. The top square on the left is its pullback along the morphism $N^y_\beta \rightarrow \bar{N}_\beta$. This is again a pushout square because $\mathcal{M}$ is locally cartesian closed (i.e., the base change functor along
$N^y_\beta \to N_\beta$ has a right adjoint) and the morphism $a_0 : A_0 \to A_1$ is (again) a monomorphism.

- the morphism $fz : Z' \to N_\beta$ factors through $Z$, which gives the dotted arrow $r'$ in the diagram. Since $z : Z' \to Z$ is a monomorphism, then so is $r' : Z' \to A_1$.

Consider the pushout

$$\begin{array}{ccc} Z' & \xrightarrow{r'} & A_1 \\ \downarrow t & & \downarrow a \\ K & \xrightarrow{r} & A_1' \end{array}$$

**Claim 4.4.** The morphism $a$ is a weak equivalence.

**Proof of Claim 4.4.** Since $\text{Cyl}(l')$ is a monomorphism and $s$ is a weak equivalence, left properness implies that $\overline{s}$ is a weak equivalence. Thus $t$ is a weak equivalence and, since $r'$ is a monomorphism, left properness implies that $a$ is a weak equivalence. \[\square\]

**Claim 4.5.** The composition $A_0 \xrightarrow{a_0} A_1 \xrightarrow{a} A_1'$ is a monomorphism.

**Proof of Claim 4.5.** Consider the diagram

$$\begin{array}{ccc} K & \xrightarrow{k_1} & A_0 \\ \overline{s}j_0 & \downarrow & \downarrow \overline{r}j_0 \\ Z' & \xrightarrow{r'} & A_1 \\ \downarrow t & & \downarrow a \\ X & \xrightarrow{r} & Z \end{array}$$

where the square on the right side is the pullback defining $a_0$. Since

$$r_j_0 k = r \text{Cyl}(k) j_0 = z \overline{s}j_0,$$

we get a morphism $k_1 : K \to A_0$ such that the square on the left side commutes. Moreover, the top composition is equal to $k : K \to X$ and the bottom composition is $z : Z' \to Z$. We will show that the outer rectangle is a pullback. We decompose this outer rectangle into two squares

$$\begin{array}{ccc} K & \xrightarrow{j_0} & \text{Cyl}(K) \\ \downarrow k & \text{Cyl}(k) & \downarrow \overline{s} \\ X & \xrightarrow{r} & Z \end{array}$$

$$\begin{array}{ccc} K & \xrightarrow{j_0} & \text{Cyl}(K) \\ \downarrow k & \text{Cyl}(k) & \downarrow \overline{s} \\ X & \xrightarrow{r} & Z \end{array}$$
where the square on the left side is a pullback by the properties of cylinder functors. The top morphism is a monomorphism because
\[ t\overline{s}j_0 = sj_0 = id_K. \]
We denote by \( t' : Z \to X \) the unique morphism satisfying \( t\overline{l} = l \) and \( t'r = s \). Since
\[ t'z\overline{s} = t'r \text{Cyl}(k) = s \text{Cyl}(k) = ks = kt\overline{s} \]
and
\[ t'z\overline{l} = t\overline{l}k' = lk' = kl' = kt\overline{l}, \]
we have
\[ t'z = kt. \]
Now, consider morphisms \( u : W \to X \) and \( v : W \to Z' \) such that \( rj_0u = zv \). We have
\[ \text{Cyl}(k)j_0tv = j_0ktv = j_0t'zv = j_0t'rj_0u = j_0sj_0u = j_0u. \]
Thus there is a unique morphism \( w : W \to K \) such that \( j_0w = j_0tv \) and \( kw = u \). Hence
\[ z\overline{s}j_0w = r \text{Cyl}(k)j_0w = rj_0kw = rj_0u = zv \]
and, because \( z \) is a monomorphism, \( \overline{s}j_0w = v \). The unicity of \( w \) follows from the fact that \( \overline{s}j_0 \) is a monomorphism. Thus the outer rectangle (5) is a pullback.

Therefore, the square on the left side in (4) is a pullback. Since it consists of monomorphisms, we can form the pushout
\[
\begin{array}{ccc}
K & \xrightarrow{k_1} & A_0 \\
\overline{s}j_0 & \downarrow \pi & \downarrow \pi \\
Z' & \xrightarrow{\overline{\kappa}_1} & A
\end{array}
\]
and obtain a monomorphism \( t'' : \overline{A} \to A_1 \) such that \( t''\overline{a} = a_0 \) and \( t''\overline{k}_1 = r' \). Moreover, since \( k_1\overline{s}j_0 = k_1 \), we have a morphism \( t''' : \overline{A} \to A_0 \) such that \( t'''\overline{k}_1 = k_1t \) and \( t'''\overline{a} = id_{A_0} \).

To finish the proof of the claim it suffices to prove that the following diagram is a pushout
\[
\begin{array}{ccc}
\overline{A} & \xrightarrow{t'''} & A_1 \\
t'' & \downarrow a & \downarrow a \\
A_0 & \xrightarrow{a_{00}} & A_1'
\end{array}
\]
This square commutes because
\[ at'' \overrightarrow{a} = aa_0 = aa_0 t'' \overrightarrow{a} \]
and
\[ at'' k_1 = ar' = r' t = \overrightarrow{r} t \overrightarrow{s} j_0 t = ar' \overrightarrow{s} j_0 t = aa_0 k_1 t = aa_0 t'' \overrightarrow{k}_1. \]

Consider the diagram

\[
\begin{array}{ccc}
Z' & \overset{k_1}{\longrightarrow} & A \\
\downarrow t & & \downarrow a \\
K & \overset{k_1}{\longrightarrow} & A_0 \\
\end{array}
\]

Since the top composition is \( r' \) and the bottom composition
\[ aa_0 k_1 = ar' \overrightarrow{s} j_0 = \overrightarrow{r} t \overrightarrow{s} j_0 = \overrightarrow{r}, \]
the outer rectangle is a pushout. Thus it suffices to prove that the diagram on the left side is a pushout. But this follows from the fact that in the following diagram

\[
\begin{array}{ccc}
K & \overset{\overrightarrow{s} j_0}{\longrightarrow} & Z' \\
\downarrow k_1 & & \downarrow t' \\
A_0 & \overset{\overrightarrow{s} j_0}{\longrightarrow} & A_0 \\
\end{array}
\]

both the top and the bottom compositions are identities and that the diagram of the left side is a pushout.

\[ \square \]

We can now complete the proof of Claim 4.3.

**Proof of Claim 4.3.** The morphism \( \overrightarrow{t} : N^y_\beta \rightarrow \tilde{N}_y \) is the canonical morphism induced on pushouts by the obvious natural transformation from the pushout

\[
\begin{array}{ccc}
A_0 & \overset{f_*}{\longrightarrow} & A_2 \\
\downarrow a_0 & & \downarrow e_* \\
A_1 & \overset{\overrightarrow{t}_*}{\longrightarrow} & N^y_\beta \\
\end{array}
\]
to the pushout

\[
\begin{array}{ccc}
A_0 & \xrightarrow{f} & A_2 \\
\downarrow{a} & & \downarrow{\bar{t}r_z} \\
A'_1 & \xrightarrow{\bar{t}} & N_y
\end{array}
\]

where \(\bar{t}\) is a unique morphism such that \(\bar{t}a = \bar{t}f\) and \(\bar{t}r' = \bar{z}_y\). The second square is a pushout because in the diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{r'} & A_1 \\
\downarrow{t} & & \downarrow{\bar{t}} \\
K & \xrightarrow{r} & A'_1 \\
\downarrow{\bar{t}} & & \downarrow{\bar{t}} \\
N & \xrightarrow{\bar{z}_y} & N_y
\end{array}
\]

the outer rectangle and the square on the left side are pushouts. By Claim 4.4 this natural transformation is a pointwise weak equivalence. By left properness and since the arrows on the left of the pushouts are cofibrations (Claim 4.5), it follows that the induced morphism \(\bar{t}\) between the pushouts is again a weak equivalence (see, e.g., [8, 13.5.4]).

The proof of Claim 4.3 completes the proof of Assertion 4.1 and so also of Theorem B.

5. Final Remarks

5.1. Strong generators. Recall that a set of objects \(\{S_\alpha\}\) in a category \(\mathcal{C}\) is a strong generator if the functors \(\mathcal{C}(S_\alpha, -) : \mathcal{C} \to \text{Set}\) are jointly faithful and jointly conservative (= isomorphism-reflecting). As observed in [4, 6.2], this definition is equivalent to the one in [2, 0.6] (see also [4, 6.3]). If a category \(\mathcal{K}\) is \(\kappa\)-accessible, then in particular it has a strong generator of \(\kappa\)-presentable objects. Conversely, every cocomplete category with a strong generator of \(\kappa\)-presentable objects is \(\kappa\)-accessible (see [2, 1.20]) and, as a consequence, it is \(\lambda\)-accessible for all \(\lambda \geq \kappa\).

However, having a strong generator of \(\kappa\)-presentable objects is weaker than \(\kappa\)-accessibility in general. Indeed since it is not true in general that a \(\kappa\)-accessible category is also \(\lambda\)-accessible (see [2, 2.11]), one cannot expect that any category with \(\kappa\)-filtered colimits and a strong generator consisting of \(\kappa\)-presentable objects is \(\kappa\)-accessible. This abstract argument does not work for \(\kappa = \aleph_0\). For an example in this case, consider the full subcategory \(\mathcal{K}\) of the category of sets with monomorphisms as...
morphisms, and consisting of the sets which are either one-element or infinite. This category has filtered colimits and a one-element set is finitely presentable and forms a strong generator. However, \( K \) is not finitely accessible.

In this subsection, we discuss the problem of finding sets of strong generators for \( W \) in cases which are not covered by Theorem B, but rather complement it. We begin with the following lemma.

**Lemma 5.1.** Let \( K \) be a \( \kappa \)-accessible category, \( \mathcal{L} \) a category with \( \kappa \)-filtered colimits and \( F : K \rightleftarrows \mathcal{L} : G \) an adjunction. Suppose that the right adjoint \( G \) is faithful, conservative and preserves \( \kappa \)-filtered colimits. Let \( S = \{ S_\alpha \} \) be a strong generator of \( \kappa \)-presentable objects in \( K \). Then its image under \( F \), \( F(S) = \{ F(S_\alpha) \} \), is also a strong generator of \( \kappa \)-presentable objects in \( \mathcal{L} \).

**Proof.** Since \( G \) preserves \( \kappa \)-filtered colimits, \( F \) preserves \( \kappa \)-presentable objects. Therefore \( F(S) \) consists of \( \kappa \)-presentable objects. It is easy to see that the functors \( \mathcal{L}(F(S_\alpha), -) : \mathcal{L} \to \text{Set} \) are jointly faithful (resp. conservative). Indeed, since \( G \) is faithful (resp. conservative), this follows from the fact that the functors \( \mathcal{K}(S_\alpha, -) : \mathcal{K} \to \text{Set} \) are jointly faithful (resp. conservative). \( \square \)

**Remark 5.2.** As remarked above, the existence of a strong generator of \( \kappa \)-presentable objects does not imply \( \kappa \)-accessibility in general. One would still need to demonstrate that for some such strong generator the canonical diagrams with respect to this set of objects are \( \kappa \)-filtered. More precisely, if \( K \) has \( \kappa \)-filtered colimits, \( S \) is a full subcategory spanned by a strong generator of \( \kappa \)-presentable objects such that in addition, for every \( X \in K \), the comma-category \( S/X \) is \( \kappa \)-filtered, then \( X \) is a \( \kappa \)-filtered colimit of the canonical diagram with respect to \( S \):

\[
\begin{align*}
S/X & \longrightarrow \mathcal{K} \\
(S_\alpha \to X) & \mapsto S_\alpha
\end{align*}
\]

and therefore \( \mathcal{K} \) is \( \kappa \)-accessible. The proof is similar to the proof of [2, 1.20].

**Proposition 5.3.** Let \( \mathcal{M} \) be a combinatorial model category, \( \mathcal{N} \) a model category and \( F : \mathcal{M} \rightleftarrows \mathcal{N} : G \) an adjuction such that both functors preserve weak equivalences. Suppose

1. the full subcategory of weak equivalences \( \mathcal{W}_\mathcal{M} \) in \( \mathcal{M}^{\rightarrow} \) is \( \kappa \)-accessible,
2. the full subcategory of weak equivalences \( \mathcal{W}_\mathcal{N} \) in \( \mathcal{N}^{\rightarrow} \) is closed under \( \kappa \)-filtered colimits,
3. \( G \) is faithful, conservative and preserves \( \kappa \)-filtered colimits.
Then $\mathcal{W}_N$ has a strong generator of $\kappa$-presentable objects.

**Proof.** Apply Lemma 5.1 to the adjunction $F : \mathcal{W}_M \rightleftarrows \mathcal{W}_N : G$. □

**Corollary 5.4.** Let $\mathcal{M}$ be a combinatorial model category such that every object is cofibrant and $C$ a small category. Suppose that the full subcategory of weak equivalences $\mathcal{W}$ in $\mathcal{M}^{\to}$ is $\kappa$-accessible. Then the full subcategory of pointwise weak equivalences $\mathcal{W}_C$ in $(\mathcal{M}^C)^{\to}$ has a strong generator of $\kappa$-presentable objects.

**Proof.** Let $C_0$ denote the set of objects of $C$, regarded as a discrete category. Then the subcategory of weak equivalences $\mathcal{W}_{C_0}$ in $(\mathcal{M}^{C_0})^{\to}$ is equivalent to $\prod_{C_0} \mathcal{W}$, and thus again $\kappa$-accessible. More explicitly, a strong generator of $\kappa$-presentable objects in $\prod_{C_0} \mathcal{W}$ is given by the collection of $C_0$-tuples $(X_i)_{i \in C_0}$ such that $X_i$ is the initial object except for $< \kappa$ entries where it is a $\kappa$-presentable object in $\mathcal{W}$.

There is adjunction

$$u_1 : \mathcal{M}^{C_0} \rightleftarrows \mathcal{M}^C : u^*$$

where $u : C_0 \to C$ is the inclusion of the objects. Since every object is cofibrant, it follows that the weak equivalences are closed under coproducts, hence the left adjoint $u_1$ preserves weak equivalences. Then the result follows from Proposition [5.3]. □

For a ring $R$, let $\text{Ch}^+(R)$ denote the category of non-negatively graded chain complexes of $R$-modules and $\text{Ch}(R)$ the category of all (unbounded) chain complexes of $R$-modules.

**Corollary 5.5.** The full subcategory of quasi-isomorphisms in $\text{Ch}^+(R)^{\to}$ has a strong generator of finitely presentable objects. Moreover, the same is also true for the full subcategory of quasi-isomorphisms in $\text{Ch}(R)^{\to}$.

**Proof.** Consider the adjunction $F : \text{SSet} \rightleftarrows \text{SMod}(R) : U$ where $F$ is the free $R$-module functor and $U$ the forgetful functor. By the Dold-Kan correspondence, the category $\text{SMod}(R)$ is equivalent to $\text{Ch}^+(R)$ and via this equivalence the quasi-isomorphisms correspond to the weak equivalences of the underlying simplicial sets. Then the first claim follows from Theorem A and Proposition [5.3]. Since every quasi-isomorphism $f : C_* \to D_*$ of unbounded chain complexes is a union of
quasi-isomorphisms between bounded below chain complexes:
\[
(\cdots \to C_n \xrightarrow{d} C_{n-1} \to \cdots \to C_k \to \im(d) \to 0 \to 0 \to \cdots)
\]
\[
\downarrow f_k
\]
\[
(\cdots \to D_n \xrightarrow{d} D_{n-1} \to \cdots \to D_k \to \im(d) \to 0 \to 0 \to \cdots)
\]
the property of being a strong generator can be checked on these bounded below pieces. Up to shifting the degree, these can be regarded as objects of \( \Ch^+(R) \), hence the second claim then follows from the first.

The category of quasi-isomorphisms in \( \Ch^+(R) \) has products and thus it has weak colimits by [2, 4.11]. Therefore this category is finitely accessible if and only if its full subcategory consisting of finitely presentable objects has weak finite colimits.

5.2. **Acyclic objects.** Under certain assumptions, it was shown in [14, 5.6] that every acyclic object in a \( \kappa \)-combinatorial model category \( \cM \) is a \( \kappa \)-directed colimit of \( \kappa \)-presentable acyclic objects, i.e., that the category of acyclic objects is \( \kappa \)-accessible. The assumptions were that the terminal object \( 1 \) is \( \kappa \)-presentable and the canonical morphism \( X \to 1 \) splits by a cofibration for every \( X \in \cM \). In \( SSet \), this recovers a theorem of Joyal and Wraith [9] – any acyclic simplicial set is a directed colimit of finite acyclic simplicial sets. It seems that the result about acyclic objects is much easier to prove than the result about weak equivalences, but at the same time, the former result does not follow from the latter as the following example demonstrates.

**Example 5.6.** Let \( \Set \) be the category of sets and consider the category \( \Set^{\aleph_0} \) with the discrete model structure where any morphism is a cofibration and weak equivalences are isomorphisms. This model category is finitely combinatorial and the terminal object \( 1 \) is the only acyclic object. Since \( 1 \) is not finitely presentable in \( \Set^{\aleph_0} \), it cannot be a directed colimit of finitely presentable acyclic objects. On the other hand, the identity on \( 1 \) is a directed colimit of weak equivalences between finitely presentable objects.

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