Supersymmetric brane actions from interpolating dualisations

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Abstract

We explore the possibility of constructing $p$-brane world-volume actions with the requirements of $\kappa$-symmetry and gauge invariance as the only input. In the process, we develop a general framework which leads to actions interpolating between Poincaré-dual descriptions of the world-volume theories. The method does not require any restrictions on the on-shell background configurations or on the dimensions of the branes. After some preliminary studies of low-dimensional cases we apply the method to the type IIB five-branes and, in particular, construct a $\kappa$-symmetric action for the type IIB NS5-brane with a world-volume field content reflecting the fact that the D1-, D3- and D5-branes can end on it.

1 Introduction

It is by now well established that extended objects play a fundamental rôle in the non-perturbative completion of the superstring theories. A few years ago this fact led to renewed interest in the study of effective world-volume actions describing such objects. In particular, while actions for the fundamental strings and for the M-theory membrane have been known for a long time, actions for the D-branes were constructed more recently in refs. A further development was the observation that these branes can equivalently be described by actions where the tension is generated dynamically by a gauge-invariant world-volume $p+1$-form, $F_{p+1}$. These actions have the generic form

$$ S = \int d^{p+1}\xi \sqrt{-g} \lambda [1 + \Phi(\{F_i\}) - (\ast F_{p+1})^2], $$

where $\{F_i\}$ collectively denotes the world-volume field strengths (excluding $F_{p+1}$), and $\lambda$ is a Lagrange multiplier for the constraint $1 + \Phi(\{F_i\}) - (\ast F_{p+1})^2 \approx 0$. The world-volume field strengths (including $F_{p+1}$) have the schematic form $F = dA - C - "C \wedge F"$. Here "$C \wedge F$" denotes corrections to the canonical, minimally coupled form determined by the requirement of gauge invariance. The equation of motion for the tension gauge potential $A_p$ leads to $\lambda \ast F_{p+1} = \text{const}$; this constant is identified with the tension.

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An appealing feature of the formulation (1.1) is that, in all known cases, $\Phi$ is a polynomial function of its arguments. Notice also that there is no Wess–Zumino term in the above action; this term is instead incorporated in the $p+1$-form field strength $F_{p+1}$. Whenever convenient, one can return to the formulation without the Lagrange multiplier $\lambda$ and the tension gauge potential $A_p$ by solving their equations of motion, thereby regaining the Wess–Zumino term in its traditional form.

Actions of the form (1.1) were constructed for the M2-brane and the fundamental strings in refs [7, 8] and more recently for the D-branes in ref. [9]. For the type IIB branes it is known [10] that the fundamental string (charge (1,0)) and the Dirichlet string (charge (0,1)) belong to a doublet of $(p,q)$ strings under the non-perturbative SL(2,$\mathbb{Z}$) symmetry [11] of the type IIB theory, and an action for these $(p,q)$ strings which is manifestly covariant under the SL(2,$\mathbb{Z}$) symmetry [12, 13] has been constructed. The D3-brane on the other hand is a singlet under SL(2,$\mathbb{Z}$). In the action which makes this property manifest, two world-volume two-form field strengths are required [14]. In order to get the correct counting for the degrees of freedom, an auxiliary duality relation between these two fields has to be imposed at the level of the equations of motion. In ref. [15] an action for the M5-brane was constructed which circumvents the problems [16] associated with the self-dual world-volume three-form by implementing the self-duality relation of this three-form as an auxiliary condition at the level of the equations of motion. Thus, in the last two cases duality relations had to be imposed to compensate for the fact that too many fields appear in the actions. For the three-brane the introduction of extra fields was necessary to make a symmetry manifest, and for the M5-brane to circumvent a topological restriction. In this paper auxiliary duality relations will be a recurrent theme.

A crucial requirement for supersymmetric boson-fermion matching of the above brane actions is that of $\kappa$-symmetry, a local world-volume symmetry for which the variation parameter $\kappa$ is a target-space spinor which can be written as $\kappa = P_+ \zeta = \frac{1}{2}(1 + \Gamma)\zeta$, where $P_\pm$ are mutually orthogonal projection operators, each reducing the number of independent components of a spinor by half. These properties translate into the requirements $\text{tr}(\Gamma) = 0$, and $\Gamma^2 = 1$. A long-standing problem has been the construction of a $\kappa$-symmetric action for the type IIB five-brane. While it is known [17] that the theory on the world-volume is described (on-shell) by a six-dimensional vector multiplet, there has been a debate in the literature about whether this multiplet should be realised in the action as a two-form or as a four-form field strength (these are Poincaré dual, and hence have the same number of degrees of freedom). Recently, a proposal based on T-duality considerations for the bosonic part of the action for the type IIB NS5-brane action was put forward [18]. One of the main results of the present paper is the construction of a $\kappa$-symmetric action for the type IIB NS5-brane in a general curved supergravity background, and with a world-volume field content which reflects the fact that the D1-, D3- and D5-branes can end on it. The bosonic part of our action differs from the one given in ref. [18]; we will comment on this fact in later sections.

In ref. [19] it was observed that in order to relate the action for the directly dimensionally reduced M2-brane to the action for the D2-brane, which describes the same physical
object, one has to perform a world-volume Poincaré dualisation of the one-form on the
world-volume of the dimensionally reduced membrane to a two-form, which can be inter-
preted as the two-form field-strength on the D2-brane world-volume, and vice versa.
Such dualisations were shown to be required also in other interconnections between p-
brane actions \[20, 21\]. There are however limitations on the applicability of this method.
The extension for \(p > 2\) to general backgrounds, where all form fields are non-constant,
was recently accomplished in ref. \[22\]. In the present paper we also work with general
backgrounds. The method of refs \[20, 21\] runs into more serious problems when \(p > 4\),
since one then encounters fifth-order algebraic equations.

In this paper we propose a general method for constructing world-volume actions with
the requirements of \(\kappa\)-symmetry and gauge invariance as the only input. To be able
to apply the method one does not need to know even the bosonic part of the action
beforehand. We work within the framework where the brane actions take the form (1.1).
The method furthermore leads to a new way of handling Poincaré dualisation, which in
a certain sense circumvents the above mentioned problem. The dualisation is also more
general in that it is not a \(\mathbb{Z}_2\) transformation; rather a whole set of interpolating theories
is constructed. The interpolating actions depend on certain parameters which control
the dualisation. For intermediate values of the parameters the world-volume fields are
“doubled,” i.e., they come in pairs with their Poincaré duals. In these formulations it
can be arranged so that there is a world-volume field corresponding to every possible
brane-ending-on-another-brane case \[23, 24\], thus realising an equality among the various
possible gauge invariant world-volume field strengths on the branes. In order to obtain the
correct number of degrees of freedom for intermediate values of the parameters auxiliary
duality relations are imposed. As a byproduct of our results we show that the form of the
usual D-brane actions is determined by super- and \(\kappa\)-symmetry alone (supplemented by
gauge invariance).

The fundamental requirements we impose are that the world-volume fields should be
gauge invariant and satisfy Bianchi identities on the world-volume. For the type IIB
branes we also require our actions to be invariant under the perturbative Peccei–Quinn
symmetry (this requirement is closely related to the condition that the world-volume field
strengths should satisfy Bianchi identities). All hitherto known supersymmetric brane
actions are invariant under this symmetry.

The paper is organised as follows. In the next section we illustrate the method for the
D2-brane in the type IIA theory. In section \[3\] we discuss the strings and the D3-brane
in the type IIB theory. The construction of the action for the type IIB NS5-brane is
presented in section \[4\]; this section also contains a discussion about the D5-brane.

Finally, in the appendices we describe our notation and conventions, list some proper-
ties of the ten-dimensional type II supergravities in superspace relevant for our discussion,
and give some details about the method used to verify \(\kappa\)-symmetry of our actions. In the
latter context, we list a number of essential identities involving the world-volume field
strengths. These identities should also be useful in applications of our results.
2 An introductory example: the D2/M2-brane

In this section we will describe the method in a simple, but non-trivial, setting—the D2-brane in type IIA. This brane can also be described as the dimensionally reduced M2-brane, the two actions being related via a Poincaré-duality transformation of the world-volume fields [19]. This is, in fact, the way the $\kappa$-symmetric action for the D2-brane was first constructed [19]. The action for the dimensionally reduced M2-brane contains the world-volume field $F_{\text{M2}}^3 = dA_2 - C_3 + B_2 F_1$ which generates the tension (here $F_1 = dA_0 - C_1$). Similarly, an action for the D2-brane where the tension is replaced by the world-volume field $F_{\text{D2}}^3 = dA_2 - C_3 - C_1 F_2$ has been constructed [9] (here as usual $F_2 = dA_1 - B_2$).

Before we continue we will briefly discuss some facts about the background type IIA supergravity theory (more details can be found in appendix B). The gauge-invariant field strengths in the type IIA theory which are relevant for the discussion in this section are $R_2$, $H_3$ and $R_4$. These fields satisfy the Bianchi identities

\begin{align}
\text{d}R_2 &= 0, \quad \text{d}H_3 = 0, \quad \text{d}R_4 = H_3 R_2. \tag{2.1}
\end{align}

The first two relations are “solved” by $R_2 = dC_1$ and $H_3 = dB_2$, whereas there is an ambiguity in the definition of $R_4$; if one requires $R_4$ to satisfy the above Bianchi identity one arrives at the natural “solution” $R_4 = dC_3 + x B_2 R_2 - (1-x) C_1 H_3 = d(C_3 + x C_1 B_2) - C_1 H_3$. From the last equality we see that the parameter $x$ arises from an ambiguity in the definition of $C_3$, the most general natural field redefinition being $C_3 \rightarrow C_3 + x C_1 B_2$. This innocent field redefinition will play a central role in the sequel. Different values of the parameter $x$ describe the same physics, but, as we will see later, the corresponding descriptions of the world-volume theories can be quite different. The background field strengths are invariant under the gauge transformations

\begin{align}
\delta C_1 &= dL_0, \quad \delta B_2 = dL_1, \\
\delta C_3 &= dL_2 - x R_2 L_1 - (1-x) H_3 L_0. \tag{2.2}
\end{align}

We now turn to the construction of gauge-invariant world-volume field strengths satisfying Bianchi identities. The general form of these field strengths is $F = dA - C - "C \wedge F"$. Here $C$ collectively denotes the background potentials (both the RR and the NS-NS ones) and “$C \wedge F$” denotes possible corrections to the minimally coupled form, required by gauge invariance. It is easy to see that $F_1 = dA_0 - C_1$, and $F_2 = dA_1 - B_2$ are invariant under the gauge transformations (2.2) together with $\delta A_0 = c + L_0$ and $\delta A_1 = dl_0 + L_1$ (here $c$ is a constant), and that they satisfy the Bianchi identities

\begin{align}
\text{d}F_1 &= -R_2, \quad \text{d}F_2 = -H_3. \tag{2.3}
\end{align}

\[\text{In order to obtain the M2-brane action from the D2-brane action one has to perform a world-volume Poincaré-duality transformation followed by an S-duality transformation, which, given the relation between the eleventh dimension and the dilaton [24, 27], corresponds to lifting the action to eleven dimensions.}\]
Furthermore, using the above form of $R_4$, we see that the world-volume three-form field strength
\[ F_3 = dA_2 - C_3 + x B_2 F_1 - (1-x) C_1 F_2 \] (2.4)
obey the Bianchi identity
\[ dF_3 = -R_4 + x F_1 H_3 - (1-x) F_2 R_2, \] (2.5)
and is invariant under the above gauge transformations combined with $\delta A_2 = dl_1 + L_2 + x L_1 F_1 + (1-x) L_0 F_2$. Thus, on the world-volume the field redefinition in the background has a more profound implication: it determines which world-volume field strengths appear. We see that for $x = 1$ we obtain the correct tension field for the description of the dimensionally reduced M2-brane, whereas for $x = 0$ we instead find the D2-brane description. The limits $x = 0$ and $x = 1$ thus lead to sensible results, but what happens for other values of $x$? The answer to this question is given below, where we construct an action which interpolates between the two limiting cases discussed at the beginning of the section. The interpolation is controlled by the real parameter $x$ introduced above. In order for the kinetic term to be positive semi-definite, it appears desirable to impose the physical restriction $x \in [0, 1]$. We make a general Ansatz for the action of the form
\[ S = \int d^3\xi \sqrt{-g} \lambda \left[ 1 + \Phi(e^{\frac{3}{4} \phi} F_1, e^{-\frac{1}{2} \phi} F_2) - e^{\frac{3}{4} \phi} (* F_3)^2 \right]. \] (2.6)
The dilaton dependence in this expression can be motivated by considering the dilaton-scaling of the supergravity constraints, which, via the basic $\kappa$-variations, directly determine the dilaton-factor in front of each occurrence of the world-volume field strengths. We will from now on suppress the dilaton factors; they can be reinstated at any time by using the rules $F_1 \rightarrow e^{\frac{3}{4} \phi} F_1$, $F_2 \rightarrow e^{-\frac{1}{2} \phi} F_2$ and $F_3 \rightarrow e^{\frac{3}{4} \phi} F_3$. The equation of motion for $A_2$ is $d[\lambda * F_3] = 0$, whereas the ones for $A_0$ and $A_1$ are
\[ d\left[ \lambda \left\{ \frac{\delta \Phi}{\delta F_1} + 2x B_2 * F_3 \right\} \right] = 0, \]
\[ d\left[ \lambda \left\{ \frac{\delta \Phi}{\delta F_2} - 2(1-x) C_1 * F_3 \right\} \right] = 0. \] (2.7)
By using the Bianchi identities for $F_1$ and $F_2$ together with the result $d[\lambda * F_3] = 0$ one can eliminate the explicit appearance of the background field strengths:
\[ d\left[ \lambda \left\{ \frac{\delta \Phi}{\delta F_1} - 2x F_2 * F_3 \right\} \right] = 0, \]
\[ d\left[ \lambda \left\{ \frac{\delta \Phi}{\delta F_2} + 2(1-x) F_1 * F_3 \right\} \right] = 0. \] (2.8)
It is thus consistent with the equations of motion and the Bianchi identities to impose the “duality” relations
\[ -2x * F_3 * F_2 = K_1 := \frac{\delta \Phi}{\delta F_1}, \]
\[ 2(1-x) * F_3 * F_1 = K_2 := \frac{\delta \Phi}{\delta F_2}, \] (2.9)
where $\Phi$ is yet to be determined. These relations reduce the number of degrees of freedom by half, and thus remedy the doubling of fields in the action. Auxiliary duality relations of this kind have been used before in the literature [14,15,28]. For the two limiting values $x = 0$ and $x = 1$ the duality relations (2.3) become degenerate, so that we have the same number of degrees of freedom for all values of $x$, namely that of a one-form $F_1$ (or, equivalently, of its Poincaré dual $F_2$).

Turning next to the $\kappa$-symmetry properties of the action (2.6), it can be shown that the $\kappa$-transformations of the world-volume fields are

$$
\delta_\kappa g_{ij} = 2E_i^a E_j^b \kappa^a T_{ab}, \quad \delta_\kappa \phi = \kappa^a \partial_\phi \kappa^a, \\
\delta_\kappa F_1 = -i_\kappa R_1, \quad \delta_\kappa F_2 = -i_\kappa H_3, \\
\delta_\kappa F_3 = -i_\kappa R_4 + x F_1 i_\kappa H_3 - (1-x) F_2 i_\kappa R_2.
$$

(2.10)

Notice the close correspondence between the variations of the world-volume field strengths and their respective Bianchi identities given in eqs (2.3) and (2.3). This correspondence holds for all cases considered in this paper. We would now like to check whether the action (2.6) is invariant under these transformations. To this end, it is necessary and sufficient to show that the variation of the constraint $\Upsilon = 1 + \Phi - (*F_3)^2 \approx 0$ is zero. However, we do not know the form of the function $\Phi$. The way out of this impasse [14,15] is to note that if we assume that the scalar functional $\Phi$ is formed out of contractions between the world-volume field strengths and the metric only,\footnote{We thus exclude WZ-type terms linear in $\epsilon^{ijk}$, which is reasonable since in the formulation we are using such terms are contained in $F_3$.} a simple scaling argument shows that the variation of $\Upsilon$ can be written\footnote{The part of this expression that is proportional to the dilaton variation was derived by temporarily reinstating the suppressed dilaton-dependence of the action (2.6) by means of the previously given substitution rules $F_1 \rightarrow e^{\frac{1}{2}\phi} F_1$, $F_2 \rightarrow e^{-\frac{1}{2}\phi} F_2$ and $F_3 \rightarrow e^{\frac{3}{2}\phi} F_3$.}

$$
\delta_\kappa \Upsilon = K^i \delta_\kappa F_i + \frac{1}{2} K_{ijk} \delta_\kappa F_{ij} + \frac{2}{3} F^{ijk} \delta_\kappa F_{ijk} - \frac{1}{2} K^{(i} F^{j)} - \frac{1}{2} K^{l(i} F^{j)l} + \frac{3}{2} \frac{1}{2} F^{l(m} F^{j)n} \delta_\kappa g_{ij} + \frac{3}{2} K_1 F_1 - \frac{1}{2} K_2 F_2 + \frac{2}{3} F_3 F_3 \delta_\kappa \phi.
$$

(2.11)

This improves matters considerably since from (2.3) we have explicit expressions for $K_1$ and $K_2$, which are valid when the duality relations are imposed. Hence, we are in the fortunate situation that although we do not know the action we know its variation under a $\kappa$-transformation. By demanding that $\Upsilon$ be $\kappa$-invariant, we can exploit this knowledge to derive the action, as we shall demonstrate next.

Inserting the expressions for the variations of the world-volume fields, together with the background constraints leads to

$$
(\delta_\kappa \Upsilon)^{(1/2)} = -2 * F_3 \tilde{A} \left[ \Xi + 2 * F_1 \gamma_2 \gamma_{11} + 3 * F_2 \gamma_1 \gamma_{11} + (3-x) *(F_1 \wedge F_2) + * F_3 \right] \kappa, \\
(\delta_\kappa \Upsilon)^{(0)} = 4 i * F_3 \tilde{E}_{ij} \left[ \frac{\epsilon^{ijk}}{2 \sqrt{-g}} \gamma_{jk} - * F_1 \gamma_2 \gamma_{11} + * F_2 \gamma_1 \gamma_{11} + x g^{ij} F_1 * F_2 \gamma_j + F_1 * F_2 \gamma_j \right]
$$

+ $g^{ij} * F_3 \gamma_j \kappa.

(2.12)
These expressions should vanish when \( \kappa \) is of the form \( P_+ \zeta = \frac{1}{2} (\mathbb{1} + \Gamma) \zeta \), which leads us to another question: how do we determine \( P_+ \)? This is a less serious problem since there are not that many “natural” terms that can appear. In all known cases one can write \( \ast P_+ \) as a \( p+1 \)-form expression involving the world-volume fields and the totally antisymmetric products of \( \gamma \)-matrices, \( \gamma_{i_1 \ldots i_r} \), considered as forms. By making a natural Ansatz for \( P_+ \), inserting it into the above variations, and then using the formula (A.9) to expand the products of \( \gamma_{i_1 \ldots i_r} \)’s one obtains two expressions which are linear combinations of \( \gamma_{i_1 \ldots i_r} \)’s, for various values of \( r \). Requiring these expressions to vanish forces each tensor component to vanish separately, since the \( \gamma_{i_1 \ldots i_r} \)’s are linearly independent (for generic embeddings). These conditions determine expressions for the duality relations (2.9), which can be integrated to give \( \Phi \); for further details, see appendix C. The projection operator which makes the above variation vanish is

\[
2 \ast F_3 P_\pm = \ast F_3 \mathbb{1} \mp [\Xi - x \ast F_1 \cdot \gamma_2 \gamma_{11} + (1-x) \ast F_2 \cdot \gamma_1 \gamma_{11}], \tag{2.13}
\]

and the result for the action is

\[
S = \int d^3 \xi \sqrt{-g} \lambda \left[ 1 + x F_1 \cdot F_1 + (1-x) F_2 \cdot F_2 + x (1-x) F_1 \cdot F_1 F_2 \cdot F_2 - (\ast F_3)^2 \right], \tag{2.14}
\]

supplemented by the duality relations

\[
\ast F_3 \ast F_1 = F_2 + x (F_1 \cdot F_1) F_2, \\
-\ast F_3 \ast F_2 = F_1 + (1-x) (F_2 \cdot F_2) F_1. \tag{2.15}
\]

At this point we would like to make a few comments. Notice the similarity between the Bianchi identity for \( F_3 \) and the form of \( \ast P_+ \), which becomes even closer if we use the formal rules \( R_{2n} \rightarrow -(-)^n \gamma_{2n-1} (\gamma_{11})^n \), \( H_3 \rightarrow \gamma_2 \gamma_{11} \). A similar correspondence holds for all hitherto known brane actions. If assumed to hold in all cases, it reduces the amount of guesswork involved (basically we only had to guess the form for \( P_+ \)) and makes the method more algorithmic. We will comment further on this issue in later sections. Let us also remark that in the simple case considered above one could alternatively have made progress by making an Ansatz for \( \Phi \); for higher-dimensional cases this approach is less feasible, however.

To summarise, in the two limits \( x = 0 \) and \( x = 1 \) we recover known results with correct expressions for the projection operator \( P_+ \). For other values of \( x \) we obtain new \( \kappa \)-symmetric formulations of the D2-brane action. In particular, for the choice \( x = \frac{1}{2} \) one obtains a formulation which treats the two world-volume fields in a symmetric fashion. We would like to stress that these actions are all equivalent (i.e., they describe the same physical object) as is obvious from the way the parameter \( x \) was introduced.
3 Some further examples

The type IIB D-string

The D-string in the type IIB theory couples minimally to the two-form potential $C_2$. Possible world-volume fields satisfying Bianchi identities are

$$
F_0 = \mu - C_0, \quad dF_0 = R_1, \\
F_2 = dA_1 - B_2, \quad dF_2 = H_3,
$$

(3.1)

where $\mu$ is a constant. In order for $F_0$ to be invariant under the Peccei-Quinn symmetry, $\mu$ has to change to compensate for the constant shift of $C_0$. We will comment on this fact later on. Given the above two fields, one can construct the following expressions for the tension form $\tilde{F}_2$:

$$
\tilde{F}_2 = d\tilde{A}_1 - C_2 + x B_2 F_0 - (1-x) C_0 F_2, \\
d\tilde{F}_2 = -R_3 + H_3 F_0 - (1-x) R_1 F_2.
$$

(3.2)

Here the background field strength $R_3$ is defined as

$$
R_3 = dC_2 + x B_2 dC_0 - (1-x) C_0 dB_2,
$$

(3.3)

and satisfies the usual Bianchi identity, $dR_3 = R_1 H_3$. The action is taken to be of the form

$$
S = \int d^2\xi \sqrt{-g} \lambda [1 + \Phi(F_0, F_2) - (\ast \tilde{F}_2)^2].
$$

(3.4)

In order to be able to apply the method of the previous section it is convenient to formally regard $\mu$ as the exterior derivative of a “$-1$”-form, $A_{-1}$. This procedure will be justified later in this section. Using the just mentioned formal trick, the duality relations can be shown to be determined by

$$
2(1-x) \ast \tilde{F}_2 \ast F_0 = K_2 := \frac{\delta \Phi}{\delta F_2}, \\
-2x \ast \tilde{F}_2 \ast F_2 = K_0 := \frac{\delta \Phi}{\delta F_0}.
$$

(3.5)

As in the case of the D2-brane, the $\kappa$-variation of the constraint $\Upsilon = 1 + \Phi - (\ast \tilde{F}_2)^2 \approx 0$ can be written in terms of the $K$’s:

$$
\delta_\kappa \Upsilon = K_0 \delta_\kappa F_0 + \frac{1}{2} K^{ij} \delta_\kappa F_{ij} + \frac{2}{27} \tilde{F}^{ij} \delta_\kappa \tilde{F}_{ij} - \left( \frac{1}{2} K^{(ij} F^{j)l} + \tilde{F}^{(ij} \tilde{F}^{j)l} \right) \delta_\kappa g_{ij} \\
+ \frac{1}{27} (\frac{1}{2} K^{ij} \tilde{F}_{ij} - \frac{1}{2} K^{ij} F_{ij}) \delta_\kappa \phi.
$$

(3.6)

The $\kappa$-variations of the fields are $\delta_\kappa F_0 = -i_\kappa R_1$, $\delta_\kappa F_2 = -i_\kappa H_3$ and $\delta_\kappa \tilde{F}_2 = -i_\kappa R_3 + F_0 i_\kappa H_3 - (1-x) F_2 i_\kappa R_1$ (the variations of the metric and the dilaton are the same as in

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6As for the D2-brane we suppress the dilaton dependence; it can be reinstated by using the rules: $F_0 \rightarrow e^{-\phi} F_0$, $F_2 \rightarrow e^{-\frac{1}{2}\phi} F_2$ and $\tilde{F}_2 \rightarrow e^{\frac{1}{2}\phi} \tilde{F}_2$. 

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the D2-brane case). Using these expressions together with the supergravity constraints (see appendix [3]) one then obtains the expressions

\[
(\delta \kappa \Upsilon)^{(1/2)} = 2 * \tilde{F}_2 \Lambda \left[ - \Xi J + F_0 \Xi K - * F_2 I - (1+x) F_0 * F_2 - * \tilde{F}_2 \right] \kappa ,
\]

\[
(\delta \kappa \Upsilon)^{(0)} = 4i * \tilde{F}_2 E_i \left[ \frac{\epsilon^{ij}}{\sqrt{-g}} \gamma_j J + F_0 \frac{\epsilon^{ij}}{\sqrt{-g}} \gamma_j K - (1-x) F_0 * F_2 g^{ij} \gamma_j + g^{ij} * \tilde{F}_2 \gamma_j \right] \kappa .
\]

(3.7)

The projection \( \kappa = P_{\pm} \) which makes the above variations vanish turns out to be

\[
P_{\pm} \propto - * (\tilde{F}_2 - \alpha F_0 F_2) \mathbf{1} \pm [\Xi J + (x+\alpha) F_0 \Xi K + (1-(x+\alpha)) * F_2 I] ,
\]

(3.8)

where the parameter \( \alpha \) is undetermined by the requirement of \( \kappa \)-symmetry. Following the same approach as for the D2-brane one obtains the action

\[
S = \int d^2 \xi \sqrt{-g} \lambda \left[ 1 + x F_0^2 + (1-x) F_2 \cdot F_2 + x (1-x) F_0^2 F_2 \cdot F_2 \right] .
\]

(3.9)

The duality relations which supplement this action are

\[
* \tilde{F}_2 * F_0 = F_2 + x F_0^2 F_2 ,
\]

\[
- * \tilde{F}_2 * F_2 = F_0 + (1-x) F_2 \cdot F_2 F_0 .
\]

(3.10)

It is fairly straightforward to show that the second relation follows from the first one and the constraint \( 1 + \Phi = (* \tilde{F}_2)^2 \), thus showing that it can be consistently imposed and justifying the earlier formal derivation.

Let us close this subsection by discussing some simple limiting cases of the above action. When \( x = 0 \), we recover the conventional \( \kappa \)-symmetric D-string action. In the opposite limit, \( x = 1 \), \( \Phi \) equals \( 1 + F_0^2 = 1 + (\mu - C_0)^2 \). This result agrees with the well-known result of refs [20,21,29], where it was shown that \( \mu \) has the correct transformation property under the Peccei-Quinn symmetry (which in particular transforms \( C_0 \) to \( C_0 + 1 \)) to make \( F_0 \) invariant. It was furthermore shown that the action could be interpreted as the action for the fundamental string in an \( \text{SL}(2,\mathbb{Z}) \)-transformed background, thus establishing the S-duality connection between the D- and F-strings at the level of their world-volume actions.

**The type IIB D3-brane**

The complete \( \kappa \)-symmetric action for the super-D3-brane in the type IIB theory was constructed in ref. [3]. It is known that the D3-brane is a singlet under the \( \text{SL}(2,\mathbb{Z}) \) symmetry

\[\text{Note the similarity with the result for the D2-brane in the previous section, a fact which should be derivable from T-duality considerations.}\]
of the type IIB theory. This fact was demonstrated at the level of the world-volume action in refs \[20,22,30\], and an action which makes the \(SL(2,\mathbb{Z})\) symmetry manifest was constructed in ref. \[14\].

In this section we will apply our method to the type IIB D3-brane, and compare our findings with previously known results. The parameterisations for the background field strengths which we will use are

\[
R_1 = dC_0, \quad R_3 = dC_2 - C_0 H_3, \\
H_3 = dB_2, \quad R_5 = dC_4 + x B_2 dC_2 - (1-x) C_2 H_3. \tag{3.11}
\]

Notice that we have not introduced a parameter in the definition for \(R_3\), in contrast to the case of the D-string. We only introduce parameters which lead to (Poincaré) dual pairs of world-volume field strengths. Although this is \textit{a priori} a restriction, it is unclear whether it excludes any physically interesting cases. The introduction of a parameter in the expression for \(R_3\) would have lead to the appearance of \(F_0\) in various places, but there is no corresponding four-form to which it can be dual (recall that for the D-string there where two two-forms). Given the above parameterisation the possible gauge invariant world-volume fields and their Bianchi identities become

\[
F_2 = dA_1 - B_2, \quad dF_2 = -H_3, \\
\tilde{F}_2 = d\tilde{A}_1 - C_2 - C_0 F_2, \quad d\tilde{F}_2 = -R_3 - R_1 F_2, \\
F_4 = dA_3 - C_4 + x B_2 \tilde{F}_2 - (1-x) C_2 F_2 + x C_0 B_2 F_2 + (x-\frac{1}{2}) C_0 F_2 F_2, \\
dF_4 = -R_5 + x H_3 \tilde{F}_2 - (1-x) F_2 R_3 + (x-\frac{1}{2}) R_1 F_2 F_2. \tag{3.12}
\]

There is an alternative way of looking at the parameter \(x\) in \(F_4\) above. Consider the expression \(F_4^{D3} = x F_2 \tilde{F}_2\), where \(F_4^{D3}\) is the canonical D-brane expression, \(F_4^{D3} = dA_3 - C_4 - C_2 F_2\). By making the field redefinitions \(C_4 \rightarrow C_4 + x B_2 C_2\) together with \(A_3 \rightarrow A_3 - x A_1 d\tilde{A}_1\), we recover the expression for \(F_4\) given above. Thus, at the world-volume level the dualisation is performed by adding the term \(F_2 \tilde{F}_2\) to \(F_4\). This is analogous to the way one usually proceeds. Looking at the dualisation this way leads to an easy way of determining which field parameterisations are needed for the background fields.

The variations of the world-volume form fields under a \(\kappa\)-symmetry transformation are

\[
\delta_\kappa F_2 = -i_\kappa H_3, \\
\delta_\kappa \tilde{F}_2 = -i_\kappa R_3 - F_2 i_\kappa R_1, \\
\delta_\kappa F_4 = -i_\kappa R_5 + x \tilde{F}_2 i_\kappa H_3 - (1-x) F_2 i_\kappa R_3 - (\frac{1}{2}-x) F_2 F_2 i_\kappa R_1. \tag{3.13}
\]

We make a general Ansatz for the action of the form\[8\]

\[
S = \int d^4\xi \sqrt{-g} \lambda \left[ 1 + \Phi(F_2, \tilde{F}_2) - (*F_4)^2 \right]. \tag{3.14}
\]

\[\text{The dilaton dependence is as usual suppressed, but can be reinstated using the rules: } F_2 \rightarrow e^{-\frac{2}{3}\phi} F_2, \text{ } \tilde{F}_2 \rightarrow e^{\frac{2}{3}\phi} \tilde{F}_2 \text{ and } F_4 \rightarrow F_4.\]
It is often convenient to rewrite this action in “form language” as

\[ S = - \int \lambda \left[ *1 + \Phi(F_2, \tilde{F}_2) + F_4 \ast F_4 \right]. \] (3.15)

This form of the action is better suited for the derivation of the duality relations consistent with the Bianchi identities and the equations of motion. In the present case these duality relations must be of the form

\[
2(1-x) F_4 \ast \tilde{F}_2 = K_2 := \frac{\delta \Phi}{\delta F_2},
\]

\[-2x F_4 \ast F_2 = \tilde{K}_2 := \frac{\delta \Phi}{\delta F_2}.
\] (3.16)

In order for the action (3.14) to be \( \kappa \)-invariant it is necessary and sufficient that the constraint \( \Upsilon = 1 + \Phi(F_2, \tilde{F}_2) - (\ast F_4)^2 \approx 0 \) is invariant. By using the same argument as before we can rewrite the \( \kappa \)-variation of the constraint in terms of the \( K \)’s as

\[
\delta_\kappa \Upsilon = \frac{1}{4!} \tilde{K}^{ij} \delta_\kappa F_{ij} + \frac{1}{2!} \tilde{K}^{i} \delta_\kappa \tilde{F}_{i} + \frac{2}{2!} F^{ijkl} \delta_\kappa F_{ijkl} - \left( \frac{1}{2} \tilde{K}^{ij} F_{j} \right) + \frac{1}{2} \tilde{K}^{i} F_{i} - \frac{1}{2} \tilde{K}^{ij} F_{ij} \delta_\kappa \phi.
\] (3.17)

Inserting the expressions (3.16) for the \( K \)’s and using the variations (3.13) of the world-volume fields together with the on-shell constraints for the background fields, we then obtain

\[
(\delta_\kappa \Upsilon)^{(1/2)} = 2 \ast F_4 \tilde{A} \left[ \frac{1}{2} e^{\phi/2} \ast \tilde{F}^{ij} \gamma_{ij} K - \frac{1}{2} e^{-\phi/2} \ast F^{ij} \gamma_{ij} J - e^{-\phi} \ast (\tilde{F} \wedge \tilde{F}) I - \ast (F \wedge \tilde{F}) \right] \kappa,
\]

\[
(\delta_\kappa \Upsilon)^{(0)} = 4i \ast F_4 \tilde{A} \left[ \frac{\epsilon^{ijkl}}{1! \sqrt{-g}} \gamma_{ijkl} I + e^{\phi/2} \ast \tilde{F}^{ij} \gamma_{ij} K + e^{-\phi/2} \ast F^{ij} \gamma_{ij} J - (1-x) g^{ij} \ast (\tilde{F}_2 \wedge F_2) \gamma_{ij} + F^{(i} \tilde{F}^{j)l} \gamma_{ij} + g^{ij} \ast F_4 \gamma_{ij} \right] \kappa.
\] (3.18)

The correct projection operator can be shown to be (more details can be found in appendix C)

\[
P_\pm \propto \left[ F_4 - \alpha (F_2 \wedge \tilde{F}_2) \right] \mathbb{1} \mp \left[ \Xi I + \frac{1}{2} (1 - (x + \alpha)) \ast F_2^{ij} \gamma_{ij} J, + \frac{1}{2} (x + \alpha) \ast \tilde{F}_2^{ij} \gamma_{ij} K + \frac{1}{2} (1 - 2(x + \alpha)) \ast (F_2 \wedge F_2) I \right].
\] (3.19)

Again we note the similarity with the Bianchi identity; from this perspective the free parameter \( \alpha \) can be understood from the fact that the Bianchi identity for \( F_4 \) can be rewritten as \( d(F_4 - \alpha F_2 \tilde{F}_2) = -R_5 + (x + \alpha) H_3 \tilde{F}_2 - (1 - (x + \alpha)) F_2 R_3 + [(x + \alpha) - \frac{3}{2}] R_1 F_2 F_2 \). Inserting the above expression for \( \kappa = P_+ \zeta \) into the variations (3.13) and using the expansion formula (3.14b) for the product of two \( \gamma \)-matrices, we obtain a set of component expressions which must each vanish in order for \( \kappa \)-symmetry to hold. The final results of
the analysis of these expressions are the duality relations (for further details, see appendix C)

\[ -\ast F_4 \ast F_2 = \tilde{F}_2 - (1-x) \ast (F_2 \wedge \tilde{F}_2) \ast F_2 + \frac{1}{2} (1-x) \ast (F_2 \wedge F_2) \ast \tilde{F}_2 - \frac{1}{2} x \ast (\tilde{F}_2 \wedge \tilde{F}_2) \ast \tilde{F}_2 , \]

\[ \ast F_4 \ast \tilde{F}_2 = F_2 - x \ast (F_2 \wedge \tilde{F}_2) \ast \tilde{F}_2 + \frac{1}{2} x \ast (\tilde{F}_2 \wedge \tilde{F}_2) \ast F_2 - \frac{1}{2} (1-x) \ast (F_2 \wedge F_2) \ast F_2 , \]

(3.20)

which can be derived from the \(\kappa\)-invariant action

\[ S = \int \sqrt{-g} \frac{d^4 \xi}{4} \left[ 1 + (1-x) F_2 \cdot F_2 + x \tilde{F}_2 \cdot \tilde{F}_2 - x (1-x) \ast (F_2 \wedge \tilde{F}_2) \ast (F_2 \wedge \tilde{F}_2) \right. \]

\[ + \frac{1}{2} x (1-x) \ast (F_2 \wedge F_2) \ast (\tilde{F}_2 \wedge \tilde{F}_2) - \frac{1}{4} (1-x)^2 \ast (F_2 \wedge F_2) \ast (F_2 \wedge F_2) \]

\[ - \frac{1}{4} x^2 \ast (\tilde{F}_2 \wedge \tilde{F}_2) \ast (\tilde{F}_2 \wedge \tilde{F}_2) \right] . \]

(3.21)

The value \(x = 0\) corresponds to the usual D3-brane action with the standard projection operator, a result we have arrived at using only the requirements of \(\kappa\) and gauge invariance. In this limit there is no need for the duality relations since they just define \(\tilde{F}_2\) in terms of \(F_2\), but \(\tilde{F}_2\) appears neither in the action nor in the projection operator. As a consequence, the duality relations can simply be dropped in this case. Another interesting limit is \(x = 1\), in which the action is also of Born-Infeld form, but with \(\tilde{F}_2\) as the world-volume field. However, unlike \(\tilde{F}_2\) in the limit \(x = 0\), \(F_2\) does not decouple completely, since the action depends implicitly on \(F_2\) through \(\tilde{F}_2\) and \(F_4\). The explicit dependence of \(F_2\) in \(P_+\) can be removed by using the identity (C.3), but the implicit dependence remains. It is nevertheless true that the equation of motion for \(A_1\) becomes non-dynamical, as can be demonstrated in the following way. When \(x = 1\), the equation of motion for \(A_1\) is

\[ d \left[ \lambda \left\{ 2 \ast F_4 \ast F_2 - \frac{\delta \Phi}{\delta \tilde{F}_2} \right\} C_0 \right] = 0 . \]

(3.22)

Here the expression in curly brackets vanishes trivially as a consequence of the duality relations. The action in the limit \(x = 1\) differs from the dual action derived in refs \[20, 21\]. The reason for this discrepancy is that the dual field of refs \[20, 21\] does not satisfy a Bianchi identity, nor is it invariant under the Peccei-Quinn symmetry. It thus violates two of our basic assumptions. We can reproduce the results of refs \[20, 21\] by solving the duality relations to eliminate \(F_2\) at the level of the equations of motion and then integrate the result to an action; this action agrees with the one derived earlier using conventional Poincaré dualisation.

Yet another interesting case is \(x = \frac{1}{2}\). This is the most symmetrical choice for the parameter \(x\), and the corresponding action is, as one might suspect, related to the manifestly SL(2,\(Z\))-covariant action constructed in ref. \[14\]. More precisely, the action (3.21) with \(x = \frac{1}{2}\) is a gauge-fixed version of the manifestly SL(2,\(Z\))-covariant action constructed in ref. \[14\] and can easily be lifted to the latter. In the manifestly covariant formulation the two world-volume field strengths are combined into a single complex field, \(\mathcal{F} = U^r F_{2;r}\), which by construction is invariant under SL(2,\(Z\)) but transforms under the (local) U(1)
symmetry of the background theory.

Here \( F_{2,r} = dA_{1,r} - C_{2,r} \) transforms as a doublet under \( SL(2,\mathbb{Z}) \), where \( C_{2,1} = B_2 \), \( C_{2,2} = C_2 \) and similarly for \( A_{1,r} \). By choosing the \( U(1) \) gauge \( \mathcal{U}^1 = -e^{1/2\phi}C_0 + ie^{-1/2\phi} \) and \( \mathcal{U}^2 = e^{1/2\phi} \), one gets \( \mathcal{F} = e^{1/2\phi}F_2 + ie^{-1/2\phi}F_2 \). Implementing this relation in the action leads to complete agreement with the results of ref. [14] (after taking into account the differences in conventions for \( F_4 \) and \( R_5 \)). We have thus explicitly shown in a simple way that the usual D3-brane action and the manifestly covariant action of ref. [14] describe the same physical object—the D3-brane—which is a singlet under \( SL(2,\mathbb{Z}) \).

4 The type IIB 5-branes

In type IIB supergravity in its doubled formulation there are two six-form gauge potentials, \( C_6 \) and \( B_6 \), which are dual to the two-form potentials \( C_2 \) and \( B_2 \), respectively. The six-form potentials couple minimally to two different branes: the D5-brane and the NS5-brane. Below we discuss these two objects using the method outlined in previous sections. We will be somewhat more elaborate in our treatment of the NS5-brane as this is the most interesting and novel case.

The D5-brane

The D5-brane couples minimally to the six-form potential \( C_6 \). The standard form of the associated world-volume six-form is

\[
F_6 = dA_5 - C_6 - C_4 F_2 - \frac{1}{2} C_2 F_2 F_2 - \frac{1}{6} C_0 F_2 F_2 F_2 .
\] (4.1)

In addition to this tension form only \( F_2 \) appears in the action. As in previous sections, this action can be generalised by introducing parameters into the definitions of the background field strengths. In analogy with the case of the D3-brane, we only introduce dual pairs of world-volume fields, i.e., fields that are related by auxiliary duality relations. In other words, we are dualising the two-form field strength \( F_2 \) into the four-form field strength \( F_4 \). In world-volume language this means that we are redefining \( F_6 \) as \( F_6 \to F_6 - yF_2 F_4 \). We let the background fields \( R_1 \), \( R_3 \) and \( R_5 \) have the canonical form \( R = dC - CH \), whereas \( R_7 \) is given by

\[
R_7 = dC_6 + y dC_4 B_2 - (1-y) C_4 H_3 .
\] (4.2)

These field strengths all obey Bianchi identities of the form \( dR - RH = 0 \). The corresponding world-volume fields are

\[
F_2 = dA_1 - B_2 ,
F_4 = dA_3 - C_4 - C_2 F_2 - \frac{1}{2} C_0 F_2 F_2 ,
F_6 = dA_5 - C_6 + y B_2 F_4 - (1-y) C_4 F_2 - \frac{1}{2} - y C_2 F_2 F_2 ,
+y B_2 C_2 F_2 - \frac{1}{2} (\frac{1}{2} - y) C_0 F_2 F_2 F_2 + \frac{1}{2} y C_0 B_2 F_2 F_2 .
\] (4.3)

\[9\text{See appendix B for a brief account of some relevant properties of type IIB supergravity.}\]
satisfying the Bianchi identities

\[ dF_2 = -H_3, \]
\[ dF_4 = -R_5 - R_3 F_2 - \frac{1}{2} R_1 F_2 F_2, \]
\[ dF_6 = -R_7 + y H_3 F_4 - (1-y) R_5 F_2 - \frac{1}{2} (\frac{1}{3} - y) R_3 F_2 F_2 - \frac{1}{2} (\frac{1}{3} - y) R_1 F_2 F_2. \] (4.4)

To verify that \( F_4 \) is an appropriate dual field in the conventional sense one can apply the methods of refs \[20, 21\] to the usual DBI D-brane action. One then obtains exactly the first duality relation given in (4.6) below, after the identification \( \Phi(F_2) = \frac{\det(g + F)}{\det(g)} - 1. \)

Our Ansatz for the action is

\[ S = \int \lambda \left[ *1 + *\Phi(F_2, F_4) + F_6 *F_6 \right]. \] (4.5)

The duality relations compatible with the Bianchi identities and the equations of motion can then be shown to be

\[ -2y *F_6 *F_2 = K_4 := \frac{\delta \Phi}{\delta F_4}, \]
\[ 2(1-y) *F_4 *F_6 = K_2 := \frac{\delta \Phi}{\delta F_2}. \] (4.6)

Furthermore, the variation of the constraint \( \Upsilon = 1 + \Phi(F_2, F_4) - (*F_6)^2 \approx 0 \) is

\[ \delta_\kappa \Upsilon = \frac{1}{2!} K^{ij} \delta_\kappa F_{ij} + \frac{1}{3!} K^{ijkl} \delta_\kappa F_{ijkl} + \frac{2}{6!} F^{ijklmn} \delta_\kappa F_{ijklmn} - \frac{1}{2} K^{(i} F^{j)l} \delta_\kappa g_{ij} \]
\[ -\frac{1}{12} K^{(i} \delta_\kappa g_{ij} - \frac{1}{2!} F_{(i} \delta_\kappa g_{j)mn} F_{j)lmnpq} \delta_\kappa g_{ij} - \frac{1}{4} K^{ij} \delta_\kappa F_{ij} \phi \]
\[ -\frac{1}{6!} F^{ijklmn} \delta_\kappa \phi. \] (4.7)

By inserting the explicit expressions (16) for the \( K \)'s, as well as the supergravity on-shell constraints (see appendix [3]), we obtain

\[ (\delta_\kappa \Upsilon)^{(1/2)} = 2 *F_6 \Lambda \left[ \Xi J + *F_4 \cdot \gamma_2 K - \frac{1}{2} *(F_2 \wedge F_2) \cdot \gamma_2 J - \frac{1}{3} *(F_2 \wedge F_2 \wedge F_2) I \right. \]
\[ - (1-y) *(F_2 \wedge F_4) + *F_6 \right] \kappa, \]
\[ (\delta_\kappa \Upsilon)^{(0)} = 4i *F_6 E_i \left[ (*\gamma_3)^i I + \frac{1}{3!} F^{ijkl} \gamma_{jkl} I + *F_4 \gamma_j K + \frac{1}{2} *(F_2 \wedge F_2)^{ij} \gamma_j J \right. \]
\[ + y g^{ij} *(F_2 \wedge F_4) \gamma_j + *F_4 (F_2) \gamma_j + *F_6 \gamma_j \right] \kappa. \] (4.8)

In analogy with the previously considered cases, the calculation proceeds by inserting the projected spinor-parameter \( \kappa = F_4 \zeta \) using an appropriate Ansatz for the projection operator and examining the irreducible components of the expression obtained by application

\textsuperscript{10}As usual we make the assumption that \( \Phi \) can be constructed from only the form fields and the metric. Moreover, the dilaton variation is obtained using the rules \( F_2 \to e^{-\Phi} F_2, F_4 \to e^{0} \Phi F_4 \) and \( F_6 \to e^{-\Phi} F_6. \)
of the $\gamma$-matrix product identity (A.9); for details we refer to appendix C quoting here only the results.

The projection operator is found to be

$$ P_\pm \propto -*(F_5 - \alpha F_2 \wedge F_4) \pm \left[ \Xi J + (1-(y+\alpha)) *F_2 \cdot \gamma_1 I + (y+\alpha) *F_4 \cdot \gamma_2 K \\
+ (\frac{1}{2}-(y+\alpha)) *(F_2 \wedge F_4) \cdot \gamma_2 J + \frac{1}{2} (\frac{1}{2}-(y+\alpha)) *(F_2 \wedge F_2 \wedge F_2) I \right]. \quad \text{(4.9)} $$

Here we note, in particular, that the correspondence with the tension field strength holds also in this case. Moreover, the explanation for the free parameter $\alpha$ is the same as in the D3-brane case.

The final expression for the action is

$$ S = \int d^6 \xi \sqrt{-g} \lambda \left[ 1 + (1-y) F_2 \cdot F_2 + y F_4 \cdot F_4 - y (1-y) *(F_2 \wedge F_4) *(F_2 \wedge F_4) \\
+ \frac{1}{2} y (F_2 \wedge F_2) \cdot *(F_4 \wedge *F_4) + \frac{1}{2} (\frac{1}{2}-y) (F_2 \wedge F_2) \cdot (F_2 \wedge F_2) \\
- \frac{1}{12} (\frac{1}{3} - y) *(F_2 \wedge F_2 \wedge F_2) *(F_2 \wedge F_2 \wedge F_2) - *(F_6)^2 \right], \quad \text{(4.10)} $$

which is to be supplemented with the duality relations

$$ - *F_6 *F_2 = F_4 - (1-y) *(F_2 \wedge F_4) *F_2 + \frac{1}{2} *(F_2 \wedge F_2) \wedge *F_4, \\
*F_6 *F_4 = F_2 - y *(F_2 \wedge F_4) *F_4 - \frac{1}{4} *[*(F_2 \wedge F_2 + *F_4 \wedge *F_4) \wedge F_2] \\
- \frac{1}{4} \frac{1-3y}{1-y} \left\{ *[F_2 \wedge F_2 - *F_4 \wedge *F_4 - \frac{1}{3} *(F_2 \wedge F_2 \wedge F_2) \wedge F_2] \wedge F_2 \right\}. \quad \text{(4.11)} $$

In the last line of the second relation we have extracted a linear combination of terms which can be shown to vanish identically when both duality relations are satisfied (see appendix C, in particular eq. (C.12)). Hence, in applications of the D5-brane formulation under discussion one may simply drop these terms and use the simplified expression.

As expected, we recover the usual D5-brane action for $y = 0$. In the opposite limit, $y = 1$, we get a dual (but equivalent) description. However, $F_2$ does not decouple completely from the action in this limit; unlike the situation for the D3-brane, in addition to the implicit dependence, $F_2$ also appears explicitly in the action. In order to fully eliminate $F_2$ one needs to solve an algebraic equation similar to the one encountered in ref. [21]. The difference compared to earlier approaches is that we have the additional information of knowing the equations of motion and the action of the dual theory, so we can in principle determine the dynamics of the three-form potential. Another interesting limit appears to be $y = \frac{1}{3}$, where the action becomes quartic. One would like to relate the action for the D5-brane given above to an action for the NS5-brane; we will return to this issue once we have constructed an action for the NS5-brane.

**The NS5-brane**

It has been known for several years [17] that the world-volume field theory of the type IIB NS5-brane is described (on-shell) by a six-dimensional vector multiplet. However,
there has been a debate in the literature about whether the action should be formulated in terms of a two-form field strength (as for the D5-brane) or in terms of a four-form field strength (in keeping with the prescription of accompanying a target-space S-duality transformation with a world-volume Poincaré duality transformation); for discussions see e.g. refs [31,32]. It is furthermore known that the D5-, D3- and D1-branes can end on the NS5-brane (see e.g. ref. [24]). From this perspective one expects a six-form, a four-form and a two-form to be present in the world-volume theory of the NS5-brane. The six-form in question is different from the one describing the tension and the need for it has been discussed in ref. [33]. Below we will construct an action whose world-volume field content reflects the above mentioned facts.

First we need to choose a proper parameterisation of the background field strengths. Since it is known that the fundamental string cannot end on the NS5-brane, we use as a guiding principle the condition that \( F_2 \) should not appear (not even implicitly) in the world-volume action. This restriction leads to the following expressions for the RR background field strengths:

\[
\begin{align*}
R_1 &= dC_0 , \\
R_3 &= dC_2 + B_2 dC_0 , \\
R_5 &= dC_4 + B_2 dC_2 + \frac{1}{2} B_2 B_2 dC_0 , \\
R_7 &= dC_6 + B_2 dC_4 + \frac{1}{2} B_2 B_2 dC_2 + \frac{1}{6} B_2 B_2 B_2 dC_0 .
\end{align*}
\]  

The corresponding world-volume field strengths become\(^\text{11}\)

\[
\begin{align*}
F_0 &= dA_{-1} - C_0 , \\
\tilde{F}_2 &= dA_1 - C_2 + F_0 B_2 , \\
F_4 &= dA_3 - C_4 + B_2 \tilde{F}_2 - \frac{1}{2} B_2 B_2 F_0 , \\
F_6 &= dA_5 - C_6 + B_2 F_4 - \frac{1}{2} B_2 B_2 \tilde{F}_2 + \frac{1}{6} B_2 B_2 B_2 F_0 .
\end{align*}
\]

These relations can be succinctly written as \( R = e^B dC \) and \( e^{-B} F = dA - C \), where the last definition is iterative and \( F = F_0 + \tilde{F}_2 + F_4 + F_6 \). In the same compact notation the Bianchi identities are

\[
dF = -R + H_3 F .
\]

The gauge-invariant field strength of the NS-NS six-form gauge potential is chosen as

\[
\begin{align*}
H_7 &= dB_6 - (1-y) C_6 dC_0 + y C_0 dC_6 - (1-x) C_2 F_4 + x C_4 dC_2 , \\
dH_7 &= R_7 R_1 - R_5 R_3 .
\end{align*}
\]

The associated tension form is then found to be

\[
\begin{align*}
\tilde{F}_6 &= d\tilde{A}_5 - B_6 - (1-y) C_6 F_0 + y C_0 F_6 - (1-x) C_2 F_4 + x C_4 \tilde{F}_2 + (1-x-y) B_2 F_0 F_4 \\
&\quad + (\frac{1}{2} - x) B_2 \tilde{F}_2 F_2 + (1-x) B_2 C_2 \tilde{F}_2 - x B_2 C_4 F_0 - y B_2 C_0 F_4 \\
&\quad - (1-\frac{3}{2} x - \frac{3}{2} y) B_2 B_2 F_0 \tilde{F}_2 - \frac{1}{2} (1-x) B_2 B_2 B_2 F_0 + \frac{1}{2} y B_2 B_2 C_0 \tilde{F}_2 \\
&\quad + (\frac{1}{3} - \frac{1}{2} x - \frac{1}{2} y) B_2 B_2 B_2 F_0^2 - \frac{1}{6} y B_2 B_2 B_2 C_0 F_0 ,
\end{align*}
\]

\(^{11}\)Here, as in the case of the D-string, \( dA_{-1} \) formally denotes an exact zero-form, i.e. a constant.
with Bianchi identity

\[
\mathrm{d}\tilde{F}_6 = -H_7 - (1-y) R_7 F_0 + y R_1 F_0 - (1-x) R_3 F_4 + x R_5 \tilde{F}_2 \\
+ (1-x-y) H_3 F_0 F_4 + \left(\frac{1}{2} - x\right) H_3 \tilde{F}_2 \tilde{F}_2.
\]

(4.17)

The fact that one has a non-dynamical six-form in addition to the tension form makes the situation analogous to the D-string discussion, where two field strengths of maximal degree are present together with a non-dynamical scalar \( F_0 \).

Given the above field content, the Ansatz for the action takes the by now familiar form:

\[
S = \int \lambda \left[ \ast 1 + \ast \Phi(F_0, \tilde{F}_2, F_4, F_6) + \tilde{F}_6 \ast \Phi \right].
\]

(4.18)

Compatibility between the equations of motion and the Bianchi identities leads to the following expressions for the duality relations:

\[
2y \ast \tilde{F}_6 F_0 = \ast K_6 := \frac{\delta \ast \Phi}{\delta F_6}, \\
-2(1-x) \ast \tilde{F}_6 F_2 = \ast K_4 := \frac{\delta \ast \Phi}{\delta F_4}, \\
2x \ast \tilde{F}_6 F_4 = \ast K_2 := \frac{\delta \ast \Phi}{\delta F_2}, \\
-2(1-y) \ast \tilde{F}_6 F_6 = \ast K_0 := \frac{\delta \ast \Phi}{\delta F_0}.
\]

(4.19)

The derivation of the last relation is at this point formal, but will be justified by the final result. The total \( \kappa \)-variation of the constraint \( \Upsilon = 1 + \Phi(F_0, \tilde{F}_2, F_4, F_6) - (\ast \tilde{F}_6)^2 \approx 0 \) is

\[
\delta_\kappa \Upsilon = K_0 \delta_\kappa F_0 + \frac{1}{21} \tilde{K}^{ij} \delta_\kappa \tilde{F}_{ij} + \frac{1}{4!} K^{ijkl} \delta_\kappa F_{ijkl} + \frac{1}{6!} K^{ijklmn} \delta_\kappa F_{ijklmn} + \frac{2}{6!} \tilde{F}^{ijklmn} \delta_\kappa \tilde{F}_{ijklmn} \\
- \frac{1}{21} \tilde{K}^{(i} \tilde{F}^{j)l} + \frac{2}{4!} K^{(lmn} F^{j)lmn} + \frac{2}{6!} K^{(lmnopq} F^{j)lmnopq} \delta_\kappa g_{ij} + \frac{1}{21} [K_0 F_0 + \frac{1}{2} \tilde{K}^{ij} \tilde{F}_{ij} \\
- \frac{1}{2} K^{ij} F_{ij} - \frac{1}{2} \tilde{K}^{ijklmn} F_{ijklmn} + \frac{2}{6!} \tilde{F}^{ijklmn} \tilde{F}_{ijklmn}] \delta_\kappa \phi.
\]

(4.20)

Inserting the duality relations (4.19) and the expressions for the variations of the worldvolume fields with the background constraints imposed leads to

\[
(\delta_\kappa \Upsilon)^{(1/2)} = 2 \ast \tilde{F}_6 \Lambda \left[ \Xi K + F_0 \Xi J - \ast F_4 \gamma_2 J + [F_0 \ast F_4 - \frac{1}{2} \ast (F_2 \wedge F_2)] \gamma_2 K \\
+ 2 \ast F_6 J + (2-y) F_0 \ast F_6 - \ast (\tilde{F}_2 \wedge F_4) - \ast \tilde{F}_6 \right] \kappa, \\
(\delta_\kappa \Upsilon)^{(0)} = 4i \ast \tilde{F}_6 \tilde{E}_i \left[ - (\ast \gamma_5)^i K + F_0 (\ast \gamma_5)^i J - \frac{1}{3} \ast \tilde{F}^{ijkl} \gamma_{jkl} I + \ast F_4^{ij} \gamma_{j} J \\
+ [F_0 \ast F_4^{ij} - \frac{1}{2} \ast (\tilde{F}_2 \wedge \tilde{F}_2)^{ij}] \gamma_{j} K + [- (1-x) g^{ij} \ast (\tilde{F}_2 \wedge F_4) \\
+ \ast F_4^{i} \tilde{F}_2^{j}] + y \ast \gamma_{j} F_0 \ast F_6 + \ast g^{ij} \ast \tilde{F}_6 \gamma_{j} \right].
\]

(4.21)

\footnote{Again, we suppress the dilaton dependence. When deriving the \( \kappa \)-variation it is temporarily reinstated by means of the rules \( F_{2k} \to e^{\left(\frac{1}{2} - \frac{1}{2k}\right) \phi} F_{2k} \) (here \( F_2 = \tilde{F}_2 \)) and \( \tilde{F}_6 \to e^{\frac{1}{2} \phi} \tilde{F}_6 \).}
After a rather long and in parts somewhat intricate calculation (outlined in appendix C) it is possible to show that the above variations vanish when \( \kappa = P_+ \zeta \) and the projection operator is

\[
P_+ \propto \pm \left[ \Xi K + \left( \frac{1}{2} + 3\alpha \right) F_0 \Xi J + \left( \frac{1}{2} + \alpha \right) \ast \tilde{F}_2 \cdot \gamma_4 I - \left( \frac{1}{2} - \alpha \right) \ast F_4 \cdot \gamma_2 J \right. \\
- 2 \alpha \left[ F_0 \ast F_4 - \frac{1}{2} \ast (\tilde{F}_2 \wedge \tilde{F}_2) \right] \cdot \gamma_2 K + \left( \frac{1}{2} - 3\alpha \right) \ast F_6 I \right] \\
+ \ast \left[ \tilde{F}_6 - \left( \frac{1}{2} - 3\alpha - y \right) F_0 F_6 - \left( \frac{1}{2} + \alpha - x \right) \tilde{F}_2 \wedge F_4 \right] 1, \\
(4.22)
\]

where \( \alpha \) is arbitrary. Moreover, the action is given by (4.18) with

\[
\ast \Phi = (1-y) F_0 \ast F_0 + x \tilde{F}_2 \wedge \ast \tilde{F}_2 + (1-x) F_4 \wedge \ast F_4 + y F_6 \ast F_6 \\
- y (1-y) (\ast F_6)^2 F_0 \ast F_0 + x (1-x) (\tilde{F}_2 \wedge F_4) \ast (\tilde{F}_2 \wedge F_4) \\
+ \frac{1}{2} (1-x) (\ast F_4 \wedge \ast F_4) \wedge \tilde{F}_2 \wedge \tilde{F}_2 + \frac{1}{2} (x - \frac{1}{2}) (\tilde{F}_2 \wedge \tilde{F}_2) \wedge \tilde{F}_2 \wedge \tilde{F}_2 \\
+ \frac{1}{3} y F_6 \tilde{F}_2 \wedge \tilde{F}_2 \wedge \tilde{F}_2 - \frac{1}{2} (x - y) F_0 \ast F_4 \wedge \tilde{F}_2 \wedge \tilde{F}_2 + \left( 1 - x - y \right) \left( F_0 \right)^2 \ast F_4 \wedge \ast F_4 \\
- \frac{1}{6} \left( 2 - 3x + y \right) F_0 \ast (\ast F_4 \wedge \ast F_4) \wedge F_4 - 2 x y F_0 \ast F_6 \tilde{F}_2 \wedge F_4 \\
- \frac{1}{6} \left( 2 - 3x - y \right) \left[ (F_0 F_4 - \frac{1}{2} \tilde{F}_2 \wedge \tilde{F}_2) \right]^3.
(4.23)
\]

Although this expression looks rather intimidating, it simplifies somewhat in certain limits of parameter space. The duality relations supplementing the action are readily obtained by inserting the appropriate functional derivatives of this expression in the equations (4.19); equivalent expressions are listed in appendix C. It is also worth noting that one has the freedom of adding to the action expressions whose variations vanish as a consequence of the duality constraints, e.g. quadratic combinations of the constraints given in eq. (C.18).

Finally, let us comment on the relation of our formulation of the NS5-brane to the one obtained in ref. [18] and to the formulation of the D5-brane above. By inspection of (4.23) one notices that there is no way to consistently eliminate the four-form field strength at the level of the action. In ref. [18], T-duality considerations led to an action for the NS5-brane which is related to the D5-brane by a simple S-duality transformation of the supergravity background, without the need for any Poincaré-duality transformations in the world-volume. However, as for the dual D3-brane action of refs [20, 21], the world-volume field strength of ref. [18] does not satisfy a Bianchi identity, nor is it invariant under \( C_0 \rightarrow C_0 + 1 \); this is the primary source for the discrepancy. However, if both actions are to describe the same object, it must be possible to obtain the results of ref. [18] from ours at the level of the equations of motion. In order to accomplish this one must eliminate \( F_4 \) and \( F_6 \) from the equation of motion for \( A_1 \). As a first step in this direction one has to choose \( x = 1 \) and \( y = 0 \). It turns out to be rather involved (if at all possible) to algebraically eliminate \( F_4 \), and we therefore temporarily limit our considerations to backgrounds for which \( C_0 = 0 \) and set \( \mu = 0 \) so that \( F_0 = 0 \). Under these simplifying assumptions we find agreement with the results of ref. [18]. It would be of interest to investigate whether this is still the case when \( C_0 \neq 0 \). Furthermore, implementing the restriction \( F_0 = 0 \) into the expression for \( \Phi \) leads to complete agreement with the action for the D5-brane
in eq. (4.10), provided we let \( x \to 1 - y \), \( \tilde{F}_2 \to -F_2 \), \( F_{4}^{\text{NS}} \to F_{4}^{\text{D5}} \) and \( \tilde{F}_6 \to F_6 \). These transformations follow from making an SL(2,\( \mathbb{Z} \)) transformation of the background; under SL(2,\( \mathbb{Z} \)) the combinations \((B_2, C_2), (A_1, \tilde{A}_1), (B_0, C_0)\) and \((\tilde{A}_5, A_5)\) transform as doublets whereas \( C_4 \) and \( A_3 \) are singlets. In particular, for the SL(2,\( \mathbb{Z} \)) transformation \[ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (4.24) \]

the field strengths have the required transformation properties. Reinstating the dilaton factors one finds that these also behave correctly under the transformation \( e^\phi \to e^{-\phi} \) associated with (4.24).

5 Discussion

We would like to comment briefly on some of the cases we have not treated in this paper. For the D4-brane in the type IIA theory, the relevant world-volume field strengths are \( F_2 \) and \( F_3 \); since there is no possible dual to \( F_1 \) we choose the parameters so that it does not appear. The tension form is given by

\[
F_5 = dA_4 - C_5 + x B_2 F_3 - (1-x) C_1 F_2 + x C_1 B_2 F_2 + (x-\frac{1}{2}) C_1 F_2 F_2, \quad (5.1)
\]

and satisfies the Bianchi identity \( dF_5 = R_6 - x H_3 F_3 - (1-x) R_4 F_2 + (x-\frac{1}{2}) R_2 F_2 F_2 = 0 \). The duality relations take the same canonical form as for the other D-branes. For \( x = 0 \) we recover the usual D4-brane description in terms of \( F_2 = dA_1 - B_2 \), whereas for \( x = 1 \) we obtain a dual description in terms of \( F_3 = dA_2 - C_3 - C_1 F_2 \). In the dual case the action can be related to the one given in ref. [21] by completely eliminating \( F_2 \) at the expense of losing manifest gauge invariance. For the symmetric choice \( x = \frac{1}{2} \), we obtain an action which can also be obtained from the M5-brane action given in ref. [13] by double dimensional reduction. For the type IIA NS5-brane the relevant world-volume fields are \( F_1, F_3 \) and \( F_5 \), while \( F_2 \) cannot have a dual and is thus not introduced; this meshes nicely with the fact that the fundamental string cannot end on the NS5-brane. The field strength \( F_3 \) is self-dual, whereas there is a parameter controlling the dualisation \( F_1 \leftrightarrow F_3 \). In particular, for a certain choice of the parameter the action can be related to the direct dimensional reduction of the action in ref. [15]. By applying the methods developed in this paper to the D6-brane, it may be possible—for certain values of the parameters—to lift the solution to eleven dimensions. Since it is known that the D6-brane is obtained from the \( D=11 \) Kaluza–Klein monopole, one would in this way obtain an action for the latter object. The known action [33] for the KK-monopoles gives the standard D6-brane action upon direct dimensional reduction. If it is possible to lift the D6-brane action for other choices of the parameters one would presumably obtain other formulations of the action for the KK-monopole in \( D = 11 \). A similar situation holds for the IIA D8-brane, where again it may be possible to lift the solution to eleven dimensions for certain values of the parameters and thus make contact with work on the M9-brane [36]. In order to treat the last two
cases it may be necessary to introduce more general parameters than those considered in this paper. It would also be of interest to extend the formalism to include the $D = 10$ KK monopoles. Another issue involves the extension to massive branes \[37\]. We have seen that there is a correspondence between the possible brane-ending-on-another-brane configurations and the restriction to only introducing world-volume fields which have duals. Invoking this restriction there seems to be a world-volume field for every possible brane-ending-on-another-brane case. Conversely, every world-volume field (except for the tension form) can “arise” from the end of a brane. In addition, one also has the configurations which arise from dualisations of the background. For the type IIB branes an interesting outstanding problem concerns the construction of a manifestly $\text{SL}(2,\mathbb{Z})$-covariant action for the type IIB $(p,q)$ five-branes. However, the problem of constructing such an action turns out to be significantly more involved than the cases treated in this section; nevertheless some progress can be made \[34\].

There exist formulations of the ten-dimensional $N = 2$ supergravity theories where all the usual bosonic fields (except for the metric) are “doubled,” i.e., supplemented with their Poincaré duals; see e.g. refs \[38, 39\]. In particular, the dilaton is supplemented by a nine-form field strength. For each field allowed in the doubled formulations there appears to be an associated brane. The question therefore arises if there exist branes which couple to the dual of the dilaton (“NS 7-branes”). In Hull’s general brane scan \[32\], which is based on considerations of the supersymmetry algebra of the background superspace, there appears to be no place for such objects. However, in the type IIB theory the D7-brane must transform under $\text{SL}(2,\mathbb{Z})$, since the potential to which it couples does. In particular, it is possible to transform it into a brane which couples to the dual of the dilaton. It is furthermore known \[10\] that in a manifestly $\text{SL}(2,\mathbb{Z})$-covariant formulation there must appear a triplet of seven-branes coupling to the dual eight-form potentials of the three scalars which belong to the $\text{SL}(2,\mathbb{R})/\text{U}(1)$ coset. After gauge fixing there remain two scalars and two dual eight-form potentials. The branes coupling to these eight-form potentials are the usual D7-brane and an NS7-brane. In the type IIA theory, on the other hand, there is only one eight-form potential, which in this case should couple to an NS7-brane. This object appears to be different from the KK-type seven-brane of the IIA theory discussed in ref. \[11\]. It would be interesting to see how (or indeed whether) it fits into the U-duality structure of M-theory (see e.g. ref. \[12\]).

It would also be desirable to have a more uniform description for the various branes within the framework of this paper, along the lines of those available for the D-branes both in their original formulation \[3–6\] and in the one with dynamically generated tension \[9\]. It should also be possible to construct T-duality rules which relate actions of the general form considered in this paper. Presumably it is possible to choose the parameters so that they are left inert under T-dualisation, as suggested by the similarity between the D1- and D2-brane actions presented earlier. A more thorough investigation of the correspondence between the Bianchi identity for $F_{p+1}$ and $P_+$ which seems to hold in all cases, might lead to a deeper understanding of $\kappa$-symmetry. Our actions should also find applications in the study of world-volume solitons \[13–15\].
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A Notation, conventions, and useful formulæ

Throughout the paper we use metrics of signature \((-, +, \cdots, +)\). Moreover, all our brane actions are written in Einstein frame, i.e., the frame where there is no dilaton factor in front of the Einstein–Hilbert term in the action for the background supergravity theory.

For superspace forms we use the conventions

\[
\Omega_{n} = \frac{1}{n!}dZ^{M_{n}} \cdots dZ^{M_{1}} \Omega_{M_{1} \cdots M_{n}} = \frac{1}{n!}E^{A_{n}} \cdots E^{A_{1}} \Omega_{A_{1} \cdots A_{n}} ,
\]

(A.1)

with the exterior derivative \(d\) as well as the interior product \(i_{V}\), acting from the right, so that

\[
d(\Omega_{m} \wedge \tilde{\Omega}_{n}) = \Omega_{m} \wedge d\tilde{\Omega}_{n} + (-1)^{n}d\Omega_{m} \wedge \tilde{\Omega}_{n} ,
\]

\[
i_{V}(\Omega_{m} \wedge \tilde{\Omega}_{n}) = \Omega_{m} \wedge i_{V}\tilde{\Omega}_{n} + (-1)^{n}i_{V}d\Omega_{m} \wedge \tilde{\Omega}_{n} ,
\]

(A.2)

where \(V\) is an arbitrary super-vector field. (We usually suppress the symbol \(\wedge\) when no confusion should arise.) World-volume forms are defined analogously and hence obey the same rules. We do not distinguish notationally between a target-space form \(\Omega_{n}\) and its pull-back to the world-volume, the components of which are given by

\[
\Omega_{i_{1} \cdots i_{n}} = E_{i_{1}}^{A_{n}} \cdots E_{i_{n}}^{A_{1}} \Omega_{A_{1} \cdots A_{n}} := \partial_{i_{n}} Z^{M_{n}} E_{M_{n}}^{A_{n}} \cdots \partial_{i_{1}} Z^{M_{1}} E_{M_{1}}^{A_{1}} \Omega_{A_{1} \cdots A_{n}} .
\]

(A.3)

The Hodge dual of a world-volume \(n\)-form is defined by

\[
(* \Omega_{n})^{i_{1} \cdots i_{d-n}} = \frac{1}{n! \sqrt{-g}} \epsilon^{i_{1} \cdots i_{d}} \Omega^{i_{d-n+1} \cdots i_{d}} ,
\]

(A.4)

where \(g\) is the determinant of the induced metric \(g_{ij} = E^{a}_{i} E^{b}_{j} \eta_{ab}\) and \(\epsilon^{i_{1} \cdots i_{d}}\) is the totally antisymmetric tensor density satisfying \(\epsilon^{01 \cdots (d-1)} = +1\) and \(\epsilon^{i_{1} \cdots i_{d}} \epsilon^{j_{1} \cdots j_{d}} = d! g^{i_{1} \cdots i_{d}} g_{j_{1} \cdots j_{d}}\), \(d = p + 1\) being the dimension of the world-volume. Here we have defined the tensor

\[
g^{i_{1} \cdots i_{m}} \ partial_{m} = g^{[i_{1} | j_{1} | i_{2} | j_{2} | \cdots | i_{m} ] j_{m}} ,
\]

(A.5)

satisfying, in particular,

\[
g^{i_{1} \cdots i_{m}} g^{j_{1} \cdots j_{m}} = \frac{n!(m-n)!}{m!} g^{i_{1} \cdots i_{m-n} j_{1} \cdots j_{m-n}} (m \geq n) ,
\]

(A.6)

as well as \(g^{i_{1} \cdots i_{m}} F_{j_{1} \cdots j_{m}} = F^{i_{1} \cdots i_{m}}\), for any \(m\)-form \(F_{m}\).
World-volume $\gamma$-matrices are defined as the pull-backs $\gamma_i = E_i^a \Gamma_a$, thus inheriting the Clifford algebra $\{\gamma_i, \gamma_j\} = 2g_{ij} \mathbb{1}$ from the target-space. They can be combined into the forms
\[ \gamma_n = \frac{1}{n!} d\xi^{i_1} \wedge \ldots \wedge d\xi^{i_n} \gamma_{i_1\ldots i_n}, \quad (A.7) \]
where $\gamma_{i_1\ldots i_n} = \gamma_{[i_1 \ldots \gamma_{i_n]}$ and the antisymmetrisation is of weight one. In particular, we have the world-volume scalar matrix
\[ \Xi := \star \gamma_d = \frac{1}{d!\sqrt{-g}} e^{i_1\ldots i_d} \gamma_{i_1\ldots i_d}, \quad (A.8) \]
which satisfies $\Xi^2 = (-1)^{\frac{1}{2}d(d-1)+1} \mathbb{1}$. The latter identity is a special case of the following identity, crucial for the $\kappa$-symmetry calculations:
\[ \gamma_{i_1\ldots i_m} \gamma_{j_1\ldots j_n} = c_{m,n}^{m,n} g_{[i_1\ldots i_q} \gamma_{j_{q+1}\ldots i_m]} g_{j_{q+1}\ldots j_n]} \quad (A.9) \]
with the expansion coefficients given by
\[ c_{m,n}^{m,n} = \frac{m!}{q!(m-q)!} \frac{n!}{q!(n-q)!} g!(-1)^{(m-q)q+\frac{q(q-1)}{2}}. \quad (A.10) \]
Finally, the scalar product of two world-volume $n$-forms is defined as
\[ A_n \cdot B_n = \frac{1}{n!} A_{i_1\ldots i_n} B_{i_1\ldots i_n}. \quad (A.11) \]

B  The $D = 10$ type II supergravities in superspace

The maximally supersymmetric supergravity theories in ten dimensions were originally formulated in superspace language in refs [46] and [47] for type IIB and type IIA, respectively. However, when dealing with higher-dimensional $p$-branes ($p \geq 4$) it is necessary to use formulations in which the Poincaré duals to all field strengths contained in the respective bosonic sectors are included on an equal footing. Since the constraints imposed on the superfields force the theories on-shell, this is indeed possible to do. In fact, one can go even further and consider “doubled” formulations, in which every bosonic super-field except for the vielbein (but including the dilaton) is accompanied by its Poincaré-dual field (see e.g., refs [38, 39]).

In its doubled formulation the type IIA theory comprises in its bosonic sector a vielbein $e_{\mu}^a$, a dilaton $\phi$ and the gauge potentials $B_2$, $B_6$ and $B_8$, as well as $C_{2k+1}$ ($k = 0, \ldots, 4$). (In the massive theory one also has zero- and ten-form field strengths.) In the superspace formulation each of the above fields (as well as the corresponding field strengths) becomes the leading component of a superfield. In the type IIB theory we again have vielbein and dilaton superfields and the potentials are $B_2$, $B_6$, $B_8$ and $C_{2k}$ ($k = 0, \ldots, 4$). The type IIB theory is chiral and has an SO(2) R-symmetry group (U(1) in a complex formulation) under which the two Majorana–Weyl spinorial superspace coordinates transform as a doublet.
We work with real Majorana spinors. The γ-matrices acting on the spinor space are \(\gamma^a \otimes \{1, \gamma_{11}\}\) (type IIA) and \(\gamma^a \otimes \{1, I, J, K\}\) (type IIB), where \(\gamma^a = (\gamma^a)^{\alpha\beta}\) are real; \(\gamma_{11} = \gamma_0\gamma_1\cdots\gamma_9\) and squares to \(1\); the \(2\times2\) matrices \(I, J\) and \(K\) anticommute pairwise and satisfy \(I^2 = -1, J^2 = 1, K^2 = 1, IJ = K, IK = J\) and \(JK = -I\). (Notice in particular the minus-sign in the last relation.) The matrices \((\gamma_{a_1\cdots a_n})_{\alpha\beta}\) are antisymmetric for \(n = 0, 3, 4, 7, 8\) and symmetric for \(n = 1, 2, 5, 6, 9\), while \((\gamma_{11})_{\alpha\beta}\) is antisymmetric. Furthermore, the \(2\times2\) matrix \(I\) is antisymmetric, whereas \(J\) and \(K\) are symmetric.

Spinor indices are raised and lowered with the antisymmetric charge conjugation matrix \(C_{\alpha\beta}\) and its inverse \(C^{\alpha\beta}\), according to the rules \(\psi_\alpha = C_{\alpha\beta}\tilde{\psi}_\beta\) and \(M_\alpha^{\beta*} = C_{\alpha\lambda}M^\lambda_{\rho}C^{\rho\beta}\). Additional useful information regarding the conventions we use can be found in ref. [4].

In the type IIB theory, the two physical scalars, \(\phi\) and \(C_0\), belong to the coset space \(SL(2,R)/U(1)\). However, the scalar fields can be made to transform linearly under \(SL(2,R)\) by combining \(\phi\) and \(C_0\) into a \(2\times2\) matrix \((U^r, \tilde{U}^r)\) on which \(SL(2,R)\) acts from the left and \(U(1)\) acts locally from the right. The \(U^r\)'s satisfy \(\frac{i}{2}\epsilon_{rs}U^r\tilde{U}^r = 1\). By a suitable fixing of the \(U(1)\) gauge symmetry one can remove the additional scalar field that has been introduced in the process and regain the physical scalars. For further details on this construction, see ref. [16].

The on-shell supergravity constraints needed in this paper are\(^\text{14}\)

\[
\text{IIA&B: } T^{a\beta} = 2i(\gamma^a)_{\alpha\beta}, \quad T_a^c = 0, \quad \Lambda_\alpha = \frac{i}{2}\partial_\alpha \phi
\]

\[
\text{IIA: } H_{a\beta\gamma} = -2i e^{\frac{i}{4}\phi}(\gamma_{11}\gamma_a)_{\beta\gamma}
\]

\[
H_{ab\gamma} = e^{\frac{i}{4}\phi}(\gamma_{ab}\gamma_{11})_{\gamma}
\]

\[
\text{IIB: } H_{a\beta\gamma} = -2i e^{\frac{i}{4}\phi}(K\gamma_a)_{\beta\gamma}
\]

\[
H_{ab\gamma} = e^{\frac{i}{4}\phi}(\gamma_{ab}K\Lambda)_{\gamma}
\]

\[
\text{IIA: } R_{a_1\ldots a_{n-2}a\alpha} = 2i e^{\frac{i}{4}\phi}(\gamma_{a_1\ldots a_{n-2}}(\gamma_{11})^{\frac{3}{2})}_{a\beta}
\]

\[
R_{a_1\ldots a_{n-2}\alpha} = -\frac{n-5}{2}e^{\frac{n-5}{4}\phi}(\gamma_{a_1\ldots a_{n-1}}(-\gamma_{11})^{\frac{n-3}{2})}\Lambda)_{a}
\]

\[
\text{IIB: } R_{a_1\ldots a_{n-2}a\alpha} = 2i e^{\frac{i}{4}\phi}(\gamma_{a_1\ldots a_{n-2}}(K)^{\frac{n+1}{2})}_{a\beta}
\]

\[
R_{a_1\ldots a_{n-2}\alpha} = -\frac{n-5}{2}e^{\frac{n-5}{4}\phi}(\gamma_{a_1\ldots a_{n-1}}(K)^{\frac{n+1}{2})}\Lambda)_{a}
\]

\[
\text{IIA: } H_{a_1\ldots a_5a\beta} = -2i e^{-\frac{i}{4}\phi}(\gamma_{a_1\ldots a_5}K)_{a\beta}
\]

\[
H_{a_1\ldots a_6\alpha} = -e^{-\frac{i}{4}\phi}(\gamma_{a_1\ldots a_6}K\Lambda)_{\alpha}
\]

\[\text{B.1}\]

\[\]
each case, we also list some identities which follow from the duality relations. These identities are crucial for proving \( \kappa \)-invariance and should also be useful in applications of our results. Before delving into the specific cases considered, however, we first describe some general features of the calculations.

After inserting a suitably chosen Ansatz for \( \kappa = P_+ \zeta \) into the \( \kappa \)-variation of the constraint \( \Upsilon = 1 + \Phi(\{ F_i \}) - (\ast F_{p+1})^2 \approx 0 \), and utilising the formula (A.9), the variation takes the generic form

\[
\delta_{P_+ \kappa} \Upsilon = \sum_{Q,k} \left[ \tilde{\Lambda}_M^{i_1 \ldots i_k} \gamma_{i_1 \ldots i_k} + \tilde{\bar{E}}_i N_{Q_i}^{i_2 \ldots i_k} \gamma_{i_2 \ldots i_k} \right] e^Q .
\]  
(C.1)

Here \( \{ e^Q \} = \{ 1, \gamma_{11} \} \) (type IIA) or \( \{ e^Q \} = \{ 1, I, J, K \} \) (type IIB). Since the \( \gamma_{i_1 \ldots i_m} \)'s are all linearly independent (for generic embeddings), each component has to vanish separately, i.e.

\[
M_Q^{i_1 \ldots i_k} = 0 \quad (\text{dim } 1/2) , \quad N_Q^{i_2 \ldots i_k} = 0 \quad (\text{dim } 0) .
\]  
(C.2)

The \( \text{SO}(1,p) \) tensors \( N_{Q_i}^{i_2 \ldots i_k} \) at dimension 0 can be further decomposed into irreducible parts, each of which has to vanish separately. We will now give some additional details case by case.

**D2**

The final result of the requirement that all the relevant \( M_Q^{i_1 \ldots i_k} \) and \( N_Q^{i_2 \ldots i_k} \) tensors that occur should vanish is contained in the duality relations

\[
\ast F_3 \ast F_1 = F_2 + x (F_1 \cdot F_1) F_2 , \quad -\ast F_3 \ast F_2 = F_1 + (1-x) (F_2 \cdot F_2) F_1 .
\]  
(C.3)

We will now outline some of the main steps of how to arrive at this result. The projection operator which makes the corresponding \( \kappa \)-variations vanish is\(^{15}\)

\[
2 \ast F_3 P_\pm = \ast F_3 \mathbb{1} \mp \left[ \Xi - (1-x) F_1 \cdot \gamma_2 \gamma_{11} + x F_2 \cdot \gamma_1 \gamma_{11} \right] \quad (C.4)
\]

Let us first consider the condition \( M_4^i = 0 \), which reads \(- (3-x) F_{i}^{i_k} F_k = (3-x) \ast F_{i_k} \ast F_k = 0 \). There are two ways this expression can vanish: either \( x = 3 \) or \( F_{i}^{i_k} F_k = 0 \). It turns out that the latter is the case. Next we consider the requirement \( M_{i_1 i_1}^i = 0 \). This condition becomes

\[
\frac{1}{2} (3-x) \left[ \ast F_3 \ast F_1^{i_j} - F_2^{i_j} + x \ast (F_1 \wedge F_2) \ast F_1^{i_j} \right] = 0 .
\]  
(C.5)

By using the identity \( \epsilon^{ijk} \epsilon^{lmn} = 3! g_{ijklm} \) (see appendix [A]), the last term can be rewritten as \(- (F_1 \cdot F_1) F_2^{i_j} - 2 F_k \ast F_{i_k} \ast F_1^{i_j} \). Using the result of the condition \( M_4^i = 0 \), we see that (C.3)

---

\(^{15}\)In general it is advantageous to make an Ansatz for \( P_+ \) with arbitrary coefficients which are then determined by the calculation. To make the presentation more readable we will from the outset use the correct coefficients.
reduces to the first of the duality relations given in (C.3). By similarly analysing the condition \( M_{ij}^{kl} = 0 \), one obtains the second duality relation. For consistency one then has to show that \( F^{jk} F_k = 0 \) follows from the duality relations. This is indeed the case, as can be seen by taking the wedge product of the second duality relation in (C.3) with \( F_1 \). To complete the programme one must analyse the remaining components at dimensions 1/2 and 0 and show that they vanish using the information obtained so far; one also has to show that \( P_+^2 = P_+ \). We will, however, not give the details here.

**D3**

For the D3-brane case the \( \kappa \)-variation of the constraint \( \Upsilon = 1 + \Phi - (F_3)^2 \approx 0 \) can be written as

\[
\delta_{F_{+\kappa}} \Upsilon = \sum_{Q=0}^{3} \left[ \Lambda M_Q + \frac{1}{2} \tilde{\Lambda} M_Q^{ij} \gamma_{ij} + \tilde{\Lambda} M_Q^{ijkl} \gamma_{ijkl} + \tilde{E}_i N_Q^{ij} \gamma_j + \tilde{E}_i N_Q^{ijkl} \gamma_{ijkl} \right] e^Q ,
\]

where \( \{ e^Q \} = \{ 1, I, J, K \} \). By systematically analysing the components of this expression one can deduce the duality relations

\[
- *F_4 * F_2 = \tilde{F}_2 - (1 - x) *(F_2 \wedge \tilde{F}_2) * F_2 + \frac{1}{2} (1 - x) *(F_2 \wedge F_2) * \tilde{F}_2 - \frac{1}{2} x *(\tilde{F}_2 \wedge \tilde{F}_2) * F_2 ,
\]

\[
*F_4 * \tilde{F}_2 = F_2 - x *(F_2 \wedge \tilde{F}_2) * \tilde{F}_2 + \frac{1}{2} x *(\tilde{F}_2 \wedge \tilde{F}_2) * F_2 - \frac{1}{2} (1 - x) *(F_2 \wedge F_2) * F_2 .
\]

The associated constraint \( \Upsilon \approx 0 \) becomes

\[
1 + (1 - x) F_2 \cdot F_2 + x \tilde{F}_2 \cdot \tilde{F}_2 - x (1 - x) *(F_2 \wedge \tilde{F}_2) * (F_2 \wedge \tilde{F}_2)
\]

\[
+ \frac{1}{2} x (1 - x) *(F_2 \wedge F_2) * (\tilde{F}_2 \wedge \tilde{F}_2) - \frac{1}{4} (1 - x)^2 *(F_2 \wedge F_2) * (F_2 \wedge F_2)
\]

\[
- \frac{1}{4} x^2 *(\tilde{F}_2 \wedge \tilde{F}_2) * (\tilde{F}_2 \wedge \tilde{F}_2) - (*F_4)^2 \approx 0 .
\]

We would like to stress that the above expressions for the duality relations and the constraint are the result of requiring the components of (C.6) to vanish. The vanishing of \( M_1^{ijkl} \) and the \( N_1^{ijkl} \) part of \( N_1^{ijkl} \) lead to the ubiquitous identity

\[
F_2 \wedge F_2 + \tilde{F}_2 \wedge \tilde{F}_2 = 0 .
\]

Furthermore, modulo this identity, one obtains the first duality relation from the requirement \( M_1^{ij} = 0 \) and also from \( N_1^{ij} = 0 \). Similarly from the \( M_1^{ij} = 0 \) and \( N_1^{ij} = 0 \) conditions one gets the second duality relation. The identity (C.9) can easily be shown to follow from the duality relations. As a further example we consider the implications of the condition \( N_1^{ij} = 0 \). By multiplying the expression for the variation with the projection operator one obtains the expression

\[
N_1^{ij} = [1 - *F_4 * F_4 - \alpha (1 - x) *(F_2 \wedge \tilde{F}_2) * (F_2 \wedge \tilde{F}_2) + (1 - x + \alpha) * F_4 *(F_2 \wedge \tilde{F}_2)] g^{ij}
\]

\[
- (1 - (x + \alpha)) * F_1 F^j l - (x + \alpha) * \tilde{F}_i j l + \alpha *(F_2 \wedge \tilde{F}_2) * F^i l \tilde{F}^j l
\]

\[
+ \alpha *(F_2 \wedge \tilde{F}_2) * F^{(i} \tilde{F}^{j)l} - *F_4 * F^{(i} \tilde{F}^{j)l} .
\]
This expression consists of three irreducible parts which can be investigated separately: \( N_{1}^{[ij]} \), \( g_{ij} g^{kl} N_{1}^{kl} \) and \( N_{1}^{(ij)} = N_{1}^{[ij]} - \frac{1}{4} g_{ij} N_{1}^{kl} g^{kl} \). The antisymmetric part vanishes using the identity \( F^{[i} \tilde{F}^{j]l} = 0 \) (see below). The trace part is proportional to

\[
1 - *F_{4} *F_{4} + \frac{1}{2} (1-(x+\alpha)) F_{2} \cdot F_{2} + \frac{1}{2} (x+\alpha) \tilde{F}_{2} \cdot \tilde{F}_{2} + \alpha (x-\frac{1}{2}) (F_{2} \wedge \tilde{F}_{2}) (F_{2} \wedge \tilde{F}_{2})
+ (\frac{1}{2}-x+\alpha) \left[ F_{2} \cdot F_{2} - x (F_{2} \wedge \tilde{F}_{2}) (F_{2} \wedge \tilde{F}_{2}) - \frac{1}{2} (F_{2} \wedge \tilde{F}_{2}) (F_{2} \wedge \tilde{F}_{2}) \right].
\]

This fact, which readily follows from the duality relations, means that they can be simultaneously diagonalised. Finally we note that one has the freedom to add the identity \( (C.3) \) exactly reproduces \( (C.8) \).

Next, we note that it follows from the duality relations that \( x K_{2} \cdot F_{2} + (1-x) \tilde{K}_{2} \cdot \tilde{F}_{2} = 0 \), which implies \( F_{2} \cdot F_{2} + \tilde{F}_{2} \cdot \tilde{F}_{2} - (F_{2} \wedge \tilde{F}_{2})^{2} = 0 \). Adding \(-\frac{1}{2} (\alpha-x)\) times this identity to the expression \( (C.11) \) and using \( (C.9) \) exactly reproduces \( (C.8) \).

Finally, to show that the symmetric traceless part is zero it is convenient to write \( *F_{4} *F^{(i} \tilde{F}^{j]l} \) as \( (x+\alpha) (\ast F_{4} \ast F^{(i} \tilde{F}^{j]l} + (1-(x+\alpha)) F^{(i} \ast F_{4} \ast \tilde{F}^{j]l}) \). After inserting the duality relations and using the identity \( *A^{i} B^{j} + *B^{i} A^{j} = g^{ij} A \ast B \), valid for arbitrary \( A_{ij} \) and \( B_{ij} \), the resulting expression for the symmetric traceless part vanishes.

Some further identities which follow from the duality relations, and which are instrumental in verifying \( \kappa \)-symmetry, include \( F^{[i} \tilde{F}^{j]l} = 0 \), \( *F^{[i} \tilde{F}^{j]l} = 0 \), \( F^{[i} \ast \tilde{F}^{j]l} = 0 \) and \( *F^{[i} \ast \tilde{F}^{j]l} = 0 \), i.e., \( F_{2} \), \( \tilde{F}_{2} \ast F_{2} \) and \( \tilde{F}_{2} \) all commute when considered as matrices. This fact, which readily follows from the duality relations, means that they can be simultaneously diagonalised. Finally we note that one has the freedom to add the identity \( (C.9) \) squared to the action without changing the equations of motion; we have fixed this freedom in a way that yields simple expressions in the limiting cases \( x = 0 \), \( \frac{1}{2} \) and 1.

D5

The principal difference between the three-brane and the five-brane as far as establishing \( \kappa \)-symmetry is concerned, is the increased level of complexity in the latter case. Otherwise, the basic steps of the computations are essentially the same and we will therefore try to focus on the main difficulties.

First, the condition that \( P_{+} \) (given in \( (4.3) \)) have the properties required of a half-maximal rank projection operator gives three independent constraints which must be fulfils once the duality relations have been applied; in addition to an implicit expression for \( \Phi \) (similar in structure to the trace-part of the expression given in eq. \( (C.10) \)), these are the condition \( F^{[i[k]} (F_{4})_{k]}^{j} = 0 \) (i.e., \( F_{2} \) and \( *F_{4} \) must commute when considered as matrices) and the identity

\[
F_{2} \wedge F_{2} - *F_{4} \wedge *F_{4} - \frac{1}{3} (*F_{2} \wedge F_{2} \wedge F_{2}) *F_{2} = 0.
\]

When turning to the analysis of the conditions for \( \kappa \)-invariance, it is very useful to assume the last two properties from the outset and at the end show that they follow from the action and the duality relations that are derived in the process.

Indeed, all the information needed to determine the action and the duality relations is contained in the requirement that \( \delta_{P_{+}} \Upsilon \approx 0 \). At dimension 1/2 this gives 16 constraint
equations—8 scalar equations from the \( \Xi \)- and \( I \)-components plus 8 two-form equations from the \( \gamma_2 \) and \( \gamma_4 \) ones. At dimension 0, each \( \gamma_1 \), \( \gamma_3 \), and \( \gamma_5 \)-component can be decomposed into three \( \text{SO}(1,5) \)-irreducible tensors which must vanish independently. For \( \gamma_1 \) and \( \gamma_5 \) these can be written as the antisymmetric, the traceless symmetric and the trace part of a second-rank tensor. Similarly, the \( \gamma_3 \)-components give rank-four tensors with the symmetries \( N^{[ijkl]} \), \( N^{(ij)[kl]} \) and \( g^{ij} N_{m^k l^m} \), where \(-\) denotes the traceless part. Hence, demanding that \( (\delta_{P_4 \kappa} \Upsilon)^{(0)} \approx 0 \) gives in total \( 3 \times 4 \times 3 = 36 \) separate constraint equations, to add to the 16 coming from the corresponding requirement at dimension 1/2. While some of these 52 constraint equations are more or less trivially satisfied, most require some effort to be shown to hold; they can roughly be divided into six separate categories: trivial ones; those that vanish for purely algebraic reasons; those proportional to \( [F_2, *F_4] \) or the identity \((C.12)\); those that give expressions for the duality relations for \( F_2 \) or \( F_4 \); and those that give expressions for \( \Phi \). (Not all of the constraints are, however, quite as clear-cut, but rather combinations of the above listed categories.)

To illustrate the kind of reasoning involved in the analysis, let us consider the dimension-0 component\footnote{Here \( N^{(ij)[kl]} \) can be further decomposed into self-dual and anti-self-dual parts. If we write \( N^{ijkl} := N^{(ij)[kl]} \), these are \( N^{(ij)[kl]} = \frac{1}{2}(N^{ijkl} \pm \epsilon^{ijklmnp} N_{mnp}) \). However, only either \( N^{ijkl}_+ + N^{ijkl}_- \) or \( N^{ijkl}_+ - N^{ijkl}_- \) appear in any given component constraint.}

\[
\delta_{P_4 \zeta} \Upsilon |_{\gamma(3)R} = -2i(y + \alpha) F_6 \tilde{E}_i [g^{ij} F^{kl} + \frac{1}{6} (F_2 \wedge F_2)_m * F_2^{mijkl}] \\
- \frac{1}{2} \{(F_4 F_2)^{(ij)} + g^{ij} (F_6 + y F_4 \wedge F_2) \} * F_4^{kl} \gamma_{jkl} K \zeta.
\]

(C.13)

The fully antisymmetric part can be rewritten as \( N_{\gamma_3 K}^{[ij][kl]} \propto \epsilon^{ijklmn} \left([F_2, *(F_2 \wedge F_2)]\right)_{mn} \) and thus vanishes identically. Using the algebraic identity \( A^{(i}_m * B^{j]klm} = -B^{(i}_m * A^{j]klm} \) as well as the identity \((\text{C.12})\), the symmetric, traceless part can be seen to vanish:

\[
N_{\gamma_3 K}^{(ij)[kl]} \propto F_2 F_4^{i}_m * F_4^{m[ji]} F_4^{kl} + \frac{1}{6} (F_2 \wedge F_2) F_2^i_m * F_2^j_{klm} \\
= \frac{1}{6} F_2^i_m (F_2 \wedge F_2 - *F_4 \wedge *F_4) F_2^j_{klm} \\
= \frac{1}{18} (F_2 \wedge F_2 F_2) F_2^i_m * F_2^j_{klm} = 0.
\]

(C.14)

The remaining component is reminiscent of a duality relation:

\[
(N_{\gamma_3 K})_j^{[kl]} \propto F_6 F_4 F_4^{[kl} - F_2 F_4^{kl} + y *(F_2 \wedge F_4) F_4^{kl} + \frac{3}{4} (F_4 F_2)_j^{[kl} F_4^{kl]} \\
+ \frac{1}{4}(F_2 \wedge*(F_2 \wedge F_2))^{kl}.
\]

(C.15)

Indeed, employing the algebraic identity \( (*F_4 F_2)_j^{[kl} F_4^{kl]} = \frac{1}{3} *[F_2 \wedge *(F_4 \wedge F_4)]^{kl} \) (valid when \( [F_2, *F_4] = 0 \)) the requirement that \( (N_{\gamma_3 K})_j^{[kl]} = 0 \) becomes a simplified version of the duality relation for \( F_4 \) given in eq. \((4.11)\). (Actually, the duality relations are most readily obtained at dimension \( \frac{1}{2} \); more precisely, the components \( \gamma_4 I \) and \( \gamma_2 K \) give expressions for \( *F_6 *F_2 \) and \( *F_6 *F_4 \), respectively.)
Equipped with expressions for the duality relations one can make use of the fact that these relations involve the derivatives of the scalar functional $\Phi$ in order to determine the action. When doing this, one has to take into account the further fact that the expressions derived from the component analysis are valid only modulo the identity (C.12). The condition that the duality relations both should give the same $\Phi$ upon integration fixes the expressions and the end result is the one displayed in (4.10).

Once the action and the duality relations have been determined, it remains to show that the latter imply the property that $F_2$ and $*F_4$ commute as well as the identity (C.12).

The first property is readily proven by taking the commutator of the Hodge-dual of the first duality relation in eq. (4.11) with $*F_4$ and subtracting $\frac{1-2y}{1-y}$ times the commutator of the second one with $F_2$. Then, by using the last of the identities in (C.17) below, one is left with an expression proportional to $[F_2, *F_4]$, which consequently vanishes. Note that one must use the proper duality relations for this calculation (i.e., the full expressions given in eq. (4.11)) in order to be allowed to use the result that $[F_2, *F_4] = 0$ when proving the identity (C.12). We will now turn to this problem. We start by subtracting the exterior product between $*F_4$ and the second duality relation in eq. (4.11) from the exterior product between $F_2$ and the Hodge-dual of the first one. One can then show that one is left with an expression which implies the identity (C.12). In establishing this result the following algebraic identity (valid when $[F_2, *F_4] = 0$) is useful:

$$
\frac{1}{3} [\star (A \wedge A \wedge A) \wedge A - \frac{1}{3} (A \wedge A) \wedge (A \wedge A)] = 0,
$$

$$
*[\star (A \wedge A) \wedge A] + (A \cdot A) A \wedge A - \frac{1}{3} (A \wedge A \wedge A) \wedge A = 0,
$$

$$
A^i_m (\star B)^{jklm} + B^i_m (\star A)^{jklm} - 3 g^{ij} \star (A \wedge B)^{kl} = 0,
$$

$$
*[\star (A \wedge B)^{ijkl} + (B \wedge A)^{ijkl}] = 0.
$$

When deriving identities of this kind it is convenient to use the freedom of choice of basis to minimise the number of non-vanishing tensor components. Also, for the more involved identities, the assistance of a computer algebra package can be helpful.

**NS5**

The $\kappa$-symmetry analysis for the NS5-brane is in many respects similar to the D5-brane analysis sketched above. Of course, the fact that there are now four instead of two world-volume field strengths does make the analysis rather more involved and lengthy.
One quickly finds that the matrices $\tilde{F}_2$ and $*F_4$ must commute when the duality relations are satisfied. Moreover, combining the condition from the dimension-$\frac{1}{2}$ component $\gamma_4$ with that from the totally antisymmetric part of $\gamma_4$ at dimension 0, leads to the two identities

\[
0 = F_0 F_4 - *F_6 \cdot \tilde{F}_2 + \frac{1}{2} *F_4 \wedge *F_4 - \frac{1}{2} \tilde{F}_2 \wedge \tilde{F}_2, \\
0 = F_0 F_4 - *F_6 \cdot \tilde{F}_2 - \frac{1}{2} (F_0 F_4 - \frac{1}{2} \tilde{F}_2 \wedge \tilde{F}_2) \wedge (*F_0 F_4 - \frac{1}{2} \tilde{F}_2 \wedge \tilde{F}_2). \quad (C.18)
\]

Expressions for the duality relations—all valid modulo these identities—are obtained from various component constraints. In particular, the components $\gamma_6 J$, $\gamma_4 I$, $\gamma_2 J$ and $I$ give

\[
*\tilde{F}_6 F_0 = *F_6 + \frac{1}{6} (*F_2 \wedge \tilde{F}_2 \wedge \tilde{F}_2) + (1-y) (F_0)^2 *F_6 - x F_0 (*F_2 \wedge F_4), \\
-*\tilde{F}_6 \tilde{F}_2 = *F_4 - (F_0)^2 *F_4 + \frac{1}{2} F_0 (*F_2 \wedge \tilde{F}_2) + (y-2) *F_6 F_0 \tilde{F}_2 \\
\quad + x (*F_2 \wedge F_4) \tilde{F}_2 + \frac{1}{6} *[F_4 \wedge (*F_2 \wedge \tilde{F}_2)] - F_0 (*F_4 \wedge *F_4), \\
*\tilde{F}_6 F_4 = *F_2 - y *F_6 F_0 F_4 + (1-x)(*F_2 \wedge F_4) F_4 - F_0 \tilde{F}_2 \wedge *F_4 \\
\quad + \frac{1}{6} \tilde{F}_2 \wedge (*F_2 \wedge \tilde{F}_2) - \frac{1}{2} *F_6 \tilde{F}_2 \wedge \tilde{F}_2, \\
-*\tilde{F}_6 F_6 = *F_0 + \frac{1}{3} F_0 F_4 \wedge *F_4 - \frac{1}{6} \tilde{F}_2 \wedge \tilde{F}_2 \wedge *F_4 - y (*F_6)^2 *F_6 \\
\quad + (x-\frac{2}{3}) *F_6 \tilde{F}_2 \wedge F_4. \quad (C.19)
\]

Similar expressions for each of the above duality relations are obtained twice at dimension 0. In addition, for other components one obtains a mixture of the duality relations as well as various identities following from them, which are somewhat intricate to disentangle. By inserting the above expressions in trivial identities of the kind $0 \equiv (*\tilde{F}_6 F_0) F_4 - F_0 (*\tilde{F}_6 F_4)$, one finds the following identities which also enter in the $\kappa$-symmetry computations:

\[
0 = *F_6 F_4 - F_0 *\tilde{F}_2 - F_0 (*\tilde{F}_2 \wedge F_4) F_4 + F_0 *F_6 [F_0 F_4 - \frac{1}{2} \tilde{F}_2 \wedge \tilde{F}_2] \\
- \frac{1}{2} F_0 \tilde{F}_2 \wedge (*F_4 \wedge *F_4) + \frac{1}{6} (*F_2 \wedge \tilde{F}_2 \wedge \tilde{F}_2) F_4, \\
0 = \tilde{F}_2 \wedge *\tilde{F}_2 + F_4 \wedge *F_4 + (*\tilde{F}_2 \wedge F_4) \tilde{F}_2 \wedge F_4 + \frac{3}{4} \tilde{F}_2 \wedge \tilde{F}_2 \wedge *F_4 \wedge *F_4 \\
\quad + (F_0)^2 F_4 \wedge *F_4 - F_0 *F_4 \wedge \tilde{F}_2 \wedge \tilde{F}_2 + \frac{1}{4} \tilde{F}_2 \wedge \tilde{F}_2 \wedge *F_2 \wedge \tilde{F}_2, \\
0 = F_0 \tilde{F}_0 + F_6 *F_6 - \frac{1}{2} \tilde{F}_6 F_0 \tilde{F}_2 \wedge F_4 - (*F_6)^2 F_0 *F_0 \\
\quad + \frac{1}{6} F_6 \tilde{F}_2 \wedge \tilde{F}_2 \wedge \tilde{F}_2 + \frac{1}{3} (F_0)^2 F_4 \wedge *F_4 - \frac{1}{6} F_0 *F_4 \wedge \tilde{F}_2 \wedge \tilde{F}_2. \quad (C.20)
\]

The correct expression for $\Phi$ is derived by integrating the expressions for its functional derivatives obtained from the four duality relations (C.19) via the equations (4.19). Again, before integrating one must take care to incorporate the freedom allowed for in the expressions (C.19) as a consequence of the identities (C.18). Most of this freedom is fixed by mutual consistency requirements and one is left with the expression for $\Phi$ given in eq. (4.23). Having obtained an expression for $\Phi$ in this way, one must show that the identities used to arrive at the result follow from the duality relations. In particular, one
must derive the condition that $\tilde{F}_2$ and $*F_4$ commute, and in addition the identities (C.18). In proving the first statement it is convenient to consider $[*K_4, *F_4] + [\tilde{F}_2, \tilde{K}_2]$ and use the algebraic identity

$$\tilde{F}^{[i|m]}(\tilde{F}_2 \wedge *F_4)_m j - \frac{1}{2}(\tilde{F}_2 \wedge \tilde{F}_2)^{[i|m]} *F_m j = 0 .$$

(C.21)

Finally, the derivation of (C.18) is performed by considering the trivially satisfied relations

$$0 = y F_0 *K_4 + (1 - x) *K_6 \tilde{F}_2 ,$$
$$0 = \frac{1}{(1-y)} K_0 *F_4 + \frac{1}{(1-x)} *F_6 \tilde{K}_2 + \frac{1}{2} (1-x) *K_4 \wedge *F_4 - \frac{1}{2} \tilde{K}_2 \wedge \tilde{F}_2 ,$$

(C.22)

for which the expressions on the right-hand-side can be shown to be linear in the identities (C.18).

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