Dynamic Event-Triggered Consensus of Multi-agent System on Matrix-weighted Networks

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Abstract

This paper examines the event-triggered consensus of the multi-agent system on matrix-weighted networks, where the interdependencies among higher-dimensional states of neighboring agents are characterized by matrix-weighted edges in the network. Specifically, a novel distributed dynamic event-triggered coordination strategy is proposed for this category of generalized networks, in which an auxiliary system is employed for each agent to dynamically adjust the triggering threshold, which plays an essential role in guaranteeing that the triggering time sequence does not exhibit Zeno behavior. Distributed event-triggered control protocols are proposed to guarantee leaderless and leader-follower consensus for multi-agent systems on matrix-weighted networks, respectively. Remarkably, the spectrum of matrix-valued weights is crucial in event-triggered mechanism design for matrix-weighted networks, generalizing those results only applicable for scalar-weighted networks. The proposed approach allows each agent to broadcast and receive information only at its triggering instants. Finally, simulation examples are provided to demonstrate the theoretical results.

Keywords: Matrix-weighted networks, dynamic event-triggered mechanism, consensus, multi-agent systems.

1. Introduction

Reaching a consensus is a paramount routine in distributed coordination of multi-agent systems Mesbahi and Egerstedt [14], Olfati-Saber et al. [17], DeGroot [5]. Although the consensus problem has been extensively investigated, the ties among agents are assumed to be characterized by scalar-weighted networks, which fail in characterizing interdependencies among higher-dimensional states of neighboring agents.

Recently, a broader category of networks termed matrix-weighted networks has been introduced which is an immediate generalization of scalar-weighted networks Sun and Yu [27], Pan et al. [23, 20], Trinh et al. [28], Pan et al. [21, 22], Wang et al. [30], Pan et al. [19]. In fact, matrix-weighted networks naturally become relevant in scenarios such as graph effective resistance based distributed control and estimation Barooah and Hespanha [2], logical inter-dependency of multiple topics in opinion evolution Friedkin et al. [8], bearing-based formation control Zhao and Zelazo [37], array of coupled LC oscillators Tuna [29] as well as consensus and synchronization on matrix-weighted networks Trinh et al. [28], Pan et al. [20].

As opposed to scalar-weighted networks, connectivity alone does not translate to achieving consensus for matrix-weighted networks. To this end, properties of weight matrices play an important role in characterizing consensus. For instance, positive definiteness and positive semi-definiteness of weight matrices have been employed to provide consensus conditions in Trinh et al. [28]; negative definiteness and negative semi-definiteness of weight matrices

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are further introduced in Pan et al. [20], Su et al. [26]. In the meanwhile, the notion of network connectivity

are further introduced in Pan et al. [20], Su et al. [26]. In the meanwhile, the notion of network connectivity
can be further extended for matrix-weighted networks. For instance, one can identify edges with positive/negative
definite matrices as “strong” connections; whereas an edge weighted by positive/negative semi-definite matrices can
be considered a “weak” connection Trinh et al. [28], Pan et al. [20].

During the multi-agent coordination process, simultaneous information exchange and transmission between
neighboring agents can be expensive from the perspective of both communication and computation. The event-
trigger mechanism turns out to be efficient in handling this issue, where the control actuation or the information
transmission was determined by the designed event Ding et al. [7], Nowzari et al. [16]. Decentralized event-triggered
control for the first-order multi-agent system was initially proposed in Dimarogonas et al. [6] which can efficiently
reduce the control updates of agents. However, the event-triggered function for the agent depends on the continuous
information monitoring of its neighbors. In order to overcome this limitation, the distributed event-triggered
functions were proposed in Nowzari and Cortés [15] where the state of neighboring agents at the last event triggering
time was employed to avoid the continuous information exchange between neighboring agents. The periodically
checked event-triggered coordination strategies were addressed in Meng and Chen [13], Nowzari and Cortés [15],
in which the triggering functions were only evaluated at the sampling instants avoiding the Zeno phenomenon
automatically.

Note that the thresholds in the aforementioned results were state-dependent. The events are triggered when the
measurement error equals or exceeds the threshold, which can be regarded as the static triggering conditions. At
the beginning of system evolution, the static triggering conditions can effectively reduce the communication cost,
as they are not easy to be satisfied. However, as time goes by, it can be triggered frequently since the threshold
becomes smaller and smaller, leading to unnecessarily triggered instants. In Yi et al. [33], distributed event-triggered
consensus of the first-order multi-agent system was examined, where a dynamic parameter associated with agents’
states was introduced in the triggering conditions. The consensus of the second-order multi-agent system was
investigated in Sheng et al. [25] with a centralized dynamic triggering condition. The distributed adaptive dynamic
event-triggered strategies for general linear multi-agent systems are proposed in He et al. [10]. It was shown that
dynamic parameters ensure fewer triggering instants and played essential roles in avoiding Zeno behaviors. Recently,
the event-trigger mechanism is ubiquitously employed in distributed control of network systems. For instance, secure
synchronization of network systems using double event-triggering mechanisms subject to actuator fault is examined in
Xu et al. [31]. Event-triggered security controller design of cyber-physical systems subject to cyber-attacks are
investigated in Zha et al. [34, 35], Liu et al. [12]. The event-triggered dynamic output quantization controller is
designed for switched T-S fuzzy systems with unstable modes in Yang et al. [32]. Stabilization of markovian jump
boolean networks via event-triggered control is provided in Chen et al. [4]. The event-triggered bipartite consensus
problem for coupled general linear systems is investigated in Pan et al. [18], Zhang et al. [36], Cai et al. [3]. For
more details about the event-triggered problem of multi-agent systems, one can refer to the recent survey papers
and references therein Ding et al. [7], Nowzari et al. [16].

Although the event-triggered consensus problem for scalar-weighted networks has been extensively investigated,
since matrix-weighted networks introduce more complexity in the design and analysis of the event-trigger mecha-
nism, the related results are unfortunately still in their infancy. In general, a crucial challenge in dealing with the
convergence analysis using the Lyapunov function method for matrix-weighted networks is the loss of commuta-
tive property of edge weight compared with scalar-weighted networks. The extension to matrix-weighted networks
is not trivial since the specific properties of weight matrices associated with edges have to be extracted for the
convenience/stability analysis of the matrix-weighted networks under the event-triggered interaction protocol. Re-
cently, the event-triggered consensus problem on matrix-weighted networks is examined in Pan et al. [24], where
the triggering condition is periodically checked which, however, can be computationally inefficient.

This paper proposes event-triggered bipartite consensus strategy for multi-agent system on matrix-weighted
networks whose edge weight allows both positive semi-definite/definite and negative semi-definite/definite matrices. First, a distributed event-triggered scheme with dynamic parameters is introduced for both leaderless and leader-following cases, the updating law of each dynamic parameter depends on the measurement error and the relative errors between neighbors’ states and its own state at triggering instants and the eigenvalue of matrix weights between neighbors. Some sufficient conditions are derived to guarantee leaderless and leader-following bipartite networks respectively, which is followed by the theoretical results. Finally, simulation examples are given to verify triggered consensus algorithms for scalar-weighted networks cannot be applied to matrix-weighted networks but can be viewed as special cases of the algorithm proposed in this paper. The remainder of this paper is organized as follows. The preliminaries of matrix analysis and graph theory are introduced in §2 as well as functionality facts on consensus problem on the multi-agent system on matrix-weighted networks. Then, the main results on the design of event-triggered consensus protocol for leaderless matrix-weighted networks and leader-follower matrix-weighted networks are provided in §3 and §4, respectively, which is followed by the numerical simulation in §5. The concluding remarks are finally given in §6.

2. Preliminaries

2.1. Notations

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}_+$ be the set of real numbers, natural numbers and positive integers, respectively. Denote $\mathbb{Z} = \{1, 2, \ldots, n\}$ for an $n \in \mathbb{Z}_+$. For a symmetric matrix $M$, if $M$ is positive definite (resp., negative definite), we write $M > 0$ (resp., $M < 0$); if $M$ is positive (resp., negative) semi-definite, we write $M \succeq 0$ (resp., $M \preceq 0$). Define the absolute value of a symmetric matrix $M \in \mathbb{R}^{n \times n}$, denoted by $|M|$, such that $|M| = M$ if $M > 0$ or $M \succeq 0$ and $|M| = -M$ if $M < 0$ or $M \preceq 0$. The null space of a matrix $M \in \mathbb{R}^{n \times n}$ is $\text{null}(M) = \{z \in \mathbb{R}^n | Mz = 0\}$. Let $\mu(M)$ denote the largest eigenvalue of a symmetric matrix $M \in \mathbb{R}^{n \times n}$.

2.2. Matrix-weighted Networks

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a matrix-weighted network where the node set and the edge set of $\mathcal{G}$ are denoted by $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, respectively. The matrix-valued weight for an edge $(i, j) \in \mathcal{E}$ in $\mathcal{G}$ is a symmetric matrix $A_{ij} \in \mathbb{R}^{d \times d}$ such that $|A_{ij}| \geq 0$ or $|A_{ij}| > 0$, and $A_{ij} = 0_{d \times d}$ for all $(i, j) \notin \mathcal{E}$. Note that a matrix-weighted network becomes a scalar-weighted network when $d = 1$. Thereby, the matrix-valued adjacency matrix $A = [A_{ij}] \in \mathbb{R}^{dn \times dn}$ is a block matrix such that the block located in the $i$-th row and the $j$-th column is $A_{ij}$. We shall assume that $A_{ij} = A_{ji}$ for all $i \neq j \in \mathcal{V}$ and $A_{ii} = 0$ for all $i \in \mathcal{V}$, which are analogous to the assumptions of undirected and simple graph in a normal sense. The neighbor set of an agent $i \in \mathcal{V}$ is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$. Denote $D = \text{diag}\{D_1, D_1, \ldots, D_n\} \in \mathbb{R}^{dn}$ as the matrix-valued degree matrix of a graph where $D_i = \sum_{j \in \mathcal{N}_i} |A_{ij}| \in \mathbb{R}^{d \times d}$. The matrix-valued Laplacian matrix of a matrix-weighted graph is defined as $L(\mathcal{G}) = D - A$, which is symmetric. A path from a node $i \in \mathcal{V}$ to a node $j \in \mathcal{V}$ in a graph $\mathcal{G}$ is a concatenation of edges $\mathcal{P}_{i,j} = \{(i, i_1), (i_1, i_2), \cdots, (i_{p-1}, j)\} \subset \mathcal{E}$, where all nodes $i_1, i_2, \ldots, i_p \in \mathcal{V}$ are distinct; a node $i \in \mathcal{V}$ is reachable from a node $j \in \mathcal{V}$ if there exists a path $\mathcal{P}_{i,j}$ in $\mathcal{G}$. A graph is connected if each pair of nodes in $\mathcal{G}$ are reachable from each other.

Consider a multi-agent system on a matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ with $n \in \mathbb{Z}_+$ nodes. The state of each agent $i \in \mathcal{V}$ is denoted by $x_i(t) = [x_{i1}, x_{i2}, \ldots, x_{id}]^T \in \mathbb{R}^d$ where $d \in \mathbb{N}$ is the size of the corresponding weight matrix. The control protocol for each agent admits,

$$\dot{x}_i(t) = u_i(t), i \in \mathcal{V},$$

(1)
where
\[ u_i(t) = - \sum_{j \in N_i} |A_{ij}|(x_i(t) - sgn(A_{ij})x_j(t)), i \in \mathcal{V}, \]  
the sign function \( sgn(\cdot) : \mathbb{R}^{n \times n} \mapsto \{0, -1, 1\} \) satisfies \( sgn(A_{ij}) = 1 \) if \( A_{ij} \geq 0 \) or \( A_{ij} > 0 \), \( sgn(A_{ij}) = -1 \) if \( A_{ij} \leq 0 \) or \( A_{ij} < 0 \), and \( sgn(A_{ij}) = 0 \) if \( A_{ij} = 0_{d \times d} \).

The overall dynamics of the multi-agent system \((1)\) can be characterized by the associated matrix-valued Laplacian,
\[ \dot{x}(t) = -Lx(t), \]  
where \( x(t) = [x_1^T(t), x_2^T(t), \ldots, x_n^T(t)]^T \in \mathbb{R}^{dn} \).

**Definition 1.** A bipartition of node set \( \mathcal{V} \) of matrix-weighted network \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) is two subsets of nodes \( \mathcal{V}_1 \subset \mathcal{V} \) where \( i \in 2 \) such that \( \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \) and \( \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset \).

In signed networks, the concept of structural balance (can be tracked back to the seminal work Harary et al. [9]) turns out to be an important graph-theoretic object playing a critical role in bipartite consensus problems Altafini [1]. This concept has been extended to the matrix-weighted networks in Pan et al. [20].

**Definition 2.** Pan et al. [20] A matrix-weighted network \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) is structurally balanced if there exists a bipartition of the node set \( \mathcal{V} \), say \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \), such that the matrix weights on the edges within each subset is positive definite or positive semi-definite, but negative definite or negative semi-definite for the edges between the two subsets. A matrix-weighted network is structurally imbalanced if it is not structurally balanced.

Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) be a matrix-weighted network with a node bipartition \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) and \( d \in \mathbb{N} \) represent the dimension of edge weight. The Gauge transformation for this node bipartition \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) is performed by the diagonal matrix \( D^* = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \) where \( \sigma_i = I_d \) if \( i \in \mathcal{V}_1 \) and \( \sigma_i = -I_d \) if \( i \in \mathcal{V}_2 \). If the matrix-weighted network \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) is structurally balanced, then it satisfies that \( D^*AD^* = ||A_{ij}|| \in \mathbb{R}^{dn \times dn} \).

The following result characterizes the structure of the null space of matrix-valued Laplacian for matrix-weighted networks, that in turn, determine the steady-state of the multi-agent system \((3)\).

**Lemma 1.** Pan et al. [20] Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) be a structurally balanced matrix-weighted network. Then the Laplacian matrix \( L \) of \( \mathcal{G} \) is positive semi-definite and its null space can be characterized by
\[ \text{null}(L) = \text{span}\{\mathcal{R}, \mathcal{H}\}, \]  
where
\[ \mathcal{R} = \text{range}\{D^*(1 \otimes I_d)\} \]  
and
\[ \mathcal{H} = \{v = [v_1^T, v_2^T, \ldots, v_n^T]^T \in \mathbb{R}^{dn} | (v_i - sgn(A_{ij})v_j) \in \text{null}(|A_{ij}|), (i, j) \in \mathcal{E}\}. \]  

In the following discussion, we proceed to design event-triggered mechanisms for multi-agent system \((3)\) on matrix-weighted networks such that bipartite consensus can be guaranteed.

3. Leaderless Matrix-weighted Networks

In this section, a distributed dynamic event-triggered coordination strategy in the leaderless multi-agent system on the matrix-weighted networks is discussed. First, the definition of bipartite consensus is given.
Definition 3. For $i \in \mathcal{V}$ and an arbitrary $x_i(0) \in \mathbb{R}^d$, the multi-agent system (2) is said to admit bipartite consensus if
\[
\lim_{t \to \infty} |x(t)| = \alpha > 0.
\]

Denote by $\hat{x}_i(t)$ as the last broadcast state of agent $i \in \mathcal{V}$ at any given time $t \geq 0$, we consider the following event-triggered control protocol,
\[
u_i(t) = \frac{\dot{q}_i(t)}{\sqrt{|A_{i\ell}|(\hat{x}_i(t) - \text{sgn}(A_{i\ell})\hat{x}_\ell(t))}},
\]
let $\hat{\mathbf{x}}(t) = [\hat{x}_1^T(t), \hat{x}_2^T(t), \ldots, \hat{x}_n^T(t)]^T \in \mathbb{R}^{dn}$, then the system (11) can be written in a compact form as
\[
\dot{\mathbf{x}}(t) = -L\hat{\mathbf{x}}(t).
\]

Define the state-based measurement error between the last broadcast state of agent $i \in \mathcal{V}$ and its current state at time $t \geq 0$ as
\[
e_i(t) = \hat{x}_i(t) - x_i(t),
\]
and the system-wise measurement error is denoted by $e(t) = [e_1^T(t), e_2^T(t), \ldots, e_n^T(t)]^T$.

For agent $i \in \mathcal{V}$, the triggering time sequence is initiated from $t_1^i = 0$ and subsequently determined by
\[
t_{k+1}^i = \max_{r > t_k^i} \left\{ r : \theta_i(\bar{\mu}_i | N_i \| e_i(t) \|^2 - \sum_{j \in N_i} \frac{\sigma_i}{4} \sqrt{|A_{ij}|p_{ij} \|^2} \leq \chi_i(t), \forall t \in (t_k^i, r) \right\}, k \in \mathbb{Z}_+,
\]
where
\[
p_{ij} = \hat{x}_i(t) - \text{sgn}(A_{ij})\hat{x}_j(t),
\]
$\bar{\mu}_i = \max_{j \in N_i} \{\mu(|A_{ij}|)\}$, and $\sigma_i \in [0, 1)$, $\theta_i$ is the parameter to be designed and $\chi_i(t)$ is an auxiliary system for each agent $i \in \mathcal{V}$ such that
\[
\dot{\chi}_i(t) = -\beta_i\chi_i(t) + \delta_i \left( \frac{\sigma_i}{4} \sum_{j \in N_i} |A_{ij}|p_{ij} \|^2 - \bar{\mu}_i | N_i \| e_i(t) \|^2 \right),
\]
with $\chi_i(0) > 0$, $\beta_i > 0$ and $\delta_i \in [0, 1]$.

Remark 1. Note that the triggering condition (10) is totally distributed, it just relates to the maximal eigenvalues of the absolute value of matrix weights between the agent and its neighbors, it does not depend on any overall information of the system. Also, it can be seen that only neighbors’ states at triggering instants are required which reduces the communication burden effectively.

Before we proceed to present the main results in this part, the following assumption will be employed.

Assumption 1. The matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ is connected and structurally balanced and there exists a Gauge transformation $D^*$ such that $\text{null}(D^*LD^*) = \mathcal{R}$.

Remark 2. The Assumption 1 eventually guarantees that the multi-agent network (3) admits a bipartite consensus solution Pan et al. [20] and the bipartite solution of (3) is
\[
\tilde{x} = D^*(1 \otimes \left(1_T \otimes I_d\right)D^*x(0))
\]
A notable distinction of matrix-weighted networks is that the network connectivity cannot translate into achieving
consensus in a multi-agent system. In this case, even if the matrix-weighted networks is connected, the null space of a matrix-valued Laplacian may be not equal to $\mathbb{R}^n$ (under a proper Gauge transformation), i.e., only the connectivity from a graph-theoretic perspective cannot guarantee the consensus of multi-agent system in the matrix-weighted networks, the property of the weight matrices has to be involved in the general graph theoretic condition. Under this assumption, we shall give the main results of this part.

**Theorem 1.** Consider the multi-agent system (8) under the matrix-weighted network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ satisfying **Assumption 1.** The triggering time sequence for agent $i$ is determined by (10). Let $\theta_i$ be such that $\theta_i > \frac{1-\delta_i}{\beta_i}$ for all $i \in \mathcal{V}$, then the multi-agent system (8) admits a bipartite consensus solution. Moreover, the Zeno behavior can be avoided.

**Proof.** For arbitrary $t \geq 0$, according to the inequalities in (10) and (12), one has

$$\dot{\chi}_i(t) \geq -\beta_i \chi_i(t) - \frac{\delta_i}{\theta_i} \chi_i(t),$$

thus,

$$\chi_i(t) \geq \chi_i(0) e^{-\left(\beta_i + \frac{\delta_i}{\theta_i}\right)t} > 0.$$ (15)

Consider the following Lyapunov function candidate $V(t) = V_1(t) + V_2(t)$, where

$$V_1(t) = \frac{1}{2} (x(t) - \tilde{x})^T (x(t) - \tilde{x}),$$

and $V_2(t) = \sum_{i=1}^n \chi_i(t)$. Note that

$$x(t) = -e(t) + \tilde{x}(t)$$ (17)

and

$$p_{ij} = \tilde{x}_i(t) - \text{sgn}(A_{ij}) \tilde{x}_j(t).$$ (18)

Then, computing the time derivative of $V_1(t)$ along with (8) yields,

$$\dot{V}_1(t) = \frac{1}{2} \dot{x}^T(t) x(t) + \frac{1}{2} x^T(t) \dot{x}(t)$$

$$= -\frac{1}{2} \dot{x}^T(t) L x(t) - \frac{1}{2} x^T(t) \dot{L} \tilde{x}(t)$$

$$= -\tilde{x}^T(t) \dot{L} \tilde{x}(t) + e^T(t) \dot{L} \tilde{x}(t)$$

$$= -\sum_{i=1}^N \sum_{j \in N_i} \left( \frac{1}{2} p_{ij}^T |A_{ij}| p_{ij} - e_i^T(t) |A_{ij}| p_{ij} \right)$$

$$\leq -\sum_{i=1}^N \left( -\bar{\mu}_i |N_i| \| e_i(t) \|^2 + \sum_{j \in N_i} \frac{1}{4} \| |A_{ij}| p_{ij} \| ^2 \right),$$ (19)

where $\bar{\mu}_i = \max_{j \in N_i} \{ \mu(|A_{ij}|) \}$.

Now, we are in position to consider the Lyapunov function candidate $V(t)$. According to the definition of $V(t)$, one has,
\[
\dot{V}(t) = \dot{V}_i(t) + \sum_{i=1}^{n} \dot{\chi}_i(t)
\]

\[
\leq -\sum_{i=1}^{N} \left( -\mu_i \parallel N_i \parallel e_i(t) \parallel^2 + \sum_{j \in N_i} \frac{1}{4} \parallel [A_{ij}] p_{ij} \parallel^2 \right) + \sum_{i=1}^{n} \delta_i \left( -\mu_i \parallel N_i \parallel e_i(t) \parallel^2 + \frac{\sigma_i}{4} \sum_{j \in N_i} \parallel [A_{ij}] p_{ij} \parallel^2 \right) - \sum_{i=1}^{n} \beta_i \chi_i(t)
\]

\[
\leq \sum_{i=1}^{n} \frac{1-\delta_i}{\theta_i} \chi_i(t) - \sum_{i=1}^{n} \beta_i \chi_i(t) - \sum_{i=1}^{n} \frac{1-\sigma_i}{4} \sum_{j \in N_i} \parallel [A_{ij}] p_{ij} \parallel^2 - \sum_{i=1}^{n} \left( \beta_i - \frac{1-\delta_i}{\theta_i} \right) \chi_i(t)
\]

\[
\leq -\frac{1-\sigma_{\text{max}}}{2} \hat{x}(t)^T L \hat{x}(t) - k_1 \sum_{i=1}^{n} \chi_i(t),
\]

where \( \sigma_{\text{max}} = \max_i \{ \sigma_i \} \) and \( k_1 = \min_i \left\{ \beta_i - \frac{1-\delta_i}{\theta_i} \right\} \).

Note that

\[
x^T(t)Lx(t) = (\hat{x}(t) - e(t))^T L(\hat{x}(t) - e(t))
\]

\[
\leq 2\hat{x}(t)^T L\hat{x}(t) + 2 \parallel L \parallel \parallel e(t) \parallel^2
\]

\[
\leq 2\hat{x}(t)^T L\hat{x}(t) + \sum_{i=1}^{n} \frac{2 \parallel L \parallel \chi_i(t)}{\theta_i \mu_i | N_i |} + \sum_{i=1}^{n} \frac{2 \parallel L \parallel \sigma_i}{4 \mu_i | N_i |} \sum_{j \in N_i} \parallel [A_{ij}] p_{ij} \parallel^2
\]

\[
\leq \left( 2 + \frac{\sigma_{\text{max}} \parallel L \parallel}{\min_i \{ \mu_i | N_i | \}} \right) \hat{x}(t)^T L\hat{x}(t) + \frac{2 \parallel L \parallel}{\min_i \{ \mu_i | N_i | \theta_i \}} \sum_{i=1}^{n} \chi_i(t)
\]

\[
\leq k_2 \hat{x}(t)^T L\hat{x}(t) + \frac{2 \parallel L \parallel}{\min_i \{ \mu_i | N_i | \theta_i \}} \sum_{i=1}^{n} \chi_i(t),
\]

where

\[
k_2 = \max \left\{ 2 + \frac{\sigma_{\text{max}} \parallel L \parallel}{\min_i \{ \mu_i | N_i | \}} \frac{2(1-\sigma_{\text{max}}) \parallel L \parallel}{k_1 \min_i \{ \mu_i | N_i | \theta_i \}} \right\}.
\]

Therefore,

\[
\hat{x}(t)^T L\hat{x}(t) \geq \frac{1}{k_2} x^T(t) Lx(t) - \frac{2 \parallel L \parallel}{k_2 \min_i \{ \mu_i | N_i | \theta_i \}} \sum_{i=1}^{n} \chi_i(t),
\]

\[
\geq \frac{1}{k_2} x^T(t) Lx(t) - \frac{k_1}{(1-\sigma_{\text{max}})} \sum_{i=1}^{n} \chi_i(t).
\]

Then,

\[
-\frac{(1-\sigma_{\text{max}})}{2} \hat{x}(t)^T L\hat{x}(t) \leq -\frac{(1-\sigma_{\text{max}})}{2k_2} x^T(t) Lx(t) + \frac{k_1}{2} \sum_{i=1}^{n} \chi_i(t).
\]

Thus, one further has
\[
\dot{V}(t) \leq -\frac{(1 - \sigma_{\text{max}})}{2k_2} x^T(t) L x(t) - \frac{k_1}{2} \sum_{i=1}^{n} \chi_i(t)
\]
\[
\leq -\frac{\rho_2(L)(1 - \sigma_{\text{max}})}{k_2} V_1(t) - \frac{k_1}{2} \sum_{i=1}^{n} \chi_i(t),
\]
\[
\leq -k_3 V(t),
\]
(25)

where
\[
k_3 = \min \left\{ \frac{\rho_2(L)(1 - \sigma_{\text{max}})}{k_2}, \frac{k_1}{2} \right\}. \tag{26}
\]

Hence,
\[
V_1(t) < V(t) \leq V(0) e^{-k_3 t}, \tag{27}
\]
for any \( t \geq 0 \). Therefore, the multi-agent system (8) admits a bipartite consensus solution.

In the following, we shall prove that there is no Zeno behavior. By contradiction, suppose that there exists Zeno behavior. Then, there at least exists one agent \( i \) such that \( \lim_{k \to \infty} t^i_k = T_0 \), where \( T_0 > 0 \). From the above analysis, we know that there exists a positive constant \( M_0 > 0 \) satisfying \( \| x_i(t) \| \leq M_0 \) for all \( t \geq 0 \) and \( i \in \mathcal{W} \). Then, for any \( t \geq 0 \), one has
\[
\| u_i(t) \| \leq 2M_0 \sum_{j \in \mathcal{N}_i} \| A_{ij} \|. \tag{28}
\]
Choose
\[
\varepsilon_0 = \left( 2M_0 \sum_{j \in \mathcal{N}_i} \| A_{ij} \| \right)^{-1} \sqrt{\frac{\chi_i(0)}{\theta_i \bar{\mu}_i} | \mathcal{N}_i |} e^{-\frac{1}{2} (\beta_i + \delta_i) T_0}. \tag{29}
\]
Then, from the definition of limits, there exists a positive integer \( N(\varepsilon_0) \) such that for any \( k \geq N(\varepsilon_0) \),
\[
t^i_k \in [T_0 - \varepsilon_0, T_0]. \tag{30}
\]
Noting that \( \sum_{j \in \mathcal{N}_i} \frac{\alpha}{4} \| \sqrt{A_{ij}} p_{ij} \|^2 \geq 0 \), we can conclude that one sufficient condition to guarantee that the inequality in (11) holds is
\[
\| e_i(t) \| \leq \sqrt{\frac{\chi_i(0)}{\theta_i \bar{\mu}_i} | \mathcal{N}_i |} e^{-\frac{1}{2} (\beta_i + \delta_i) t}. \tag{31}
\]
In addition,
\[
\| e_i(t) \| = \| \hat{x}_i(t^i_k) - x_i(t) \|
= \| x_i(t^i_k) - x_i(t) \|
= \left\| \int_{t^i_k}^{t} \dot{x}_i(t)d(t) \right\|
\leq \int_{t^i_k}^{t} \| \dot{x}_i(t) \| d(t)
\leq 2(t - t^i_k) M_0 \sum_{j \in \mathcal{N}_i} \| A_{ij} \|, \tag{32}
\]
for any \( t \geq 0 \).
then another sufficient condition to guarantee that the inequality in (10) holds is

\[ 2(t - t^i_k)M_o \sum_{j \in \mathcal{N}_i} \| A_{ij} \| \leq \sqrt{\frac{\chi_i(0)}{\theta_i \bar{\mu}_i | \bar{N}_i |}} e^{-\frac{1}{\rho} (\beta_i + \frac{\tilde{\beta}_i}{\rho}) t}. \] (33)

Let \( t^i_{N(\varepsilon_0) + 1} \) and \( \tilde{t}^i_{N(\varepsilon_0) + 1} \) denote the next triggering time determined by the inequalities in (10) and (33), respectively. Then,

\[
t^i_{N(\varepsilon_0) + 1} - t^i_{N(\varepsilon_0)} \geq \tilde{t}^i_{N(\varepsilon_0) + 1} - t^i_{N(\varepsilon_0)} = 2M_o \sum_{j \in \mathcal{N}_i} \| A_{ij} \| \leq \sqrt{\frac{\chi_i(0)}{\theta_i \bar{\mu}_i | \bar{N}_i |}} e^{-\frac{1}{\rho} (\beta_i + \frac{\tilde{\beta}_i}{\rho}) T_0} \geq 2 \varepsilon_0, \quad (34)
\]

which contradicts with \( t^i_k \in [T_0 - \varepsilon_0, T_0] \) for any \( k \geq N(\varepsilon_0) \). Therefore, Zero behavior is excluded. \( \square \)

**Remark 3.** The proposed event-triggered algorithm here is not only applicable to the matrix-weighted networks but also to the scalar-weighted networks. Note that (10) degenerates into the scalar-weighted case when \( A_{ij} = a_{ij} I \) where \( a_{ij} \in \mathbb{R} \) and \( I \) denotes the \( d \times d \) identity matrix and in this case, one can choose

\[ \bar{\mu}_i = \max_{j \in \mathcal{N}_i} \{ |a_{ij}| \}. \quad (35) \]

If the scalar-valued weights associated to all edges are equal, the triggering function (10) is the same as the case in the scalar-weighted networks Yi et al. \[33\]; otherwise, the triggering function (10) is easier to be triggered than those triggering functions that are only applicable to the scalar-weighted networks.

### 4. Leader-follower Matrix-weighted Networks

Besides the leaderless network, there also exists another popular paradigm where a subset of agents are selected as leaders or informed agents to steer the network state to a desired one which is referred to as leader-follower network. In a leader-follower network, a subset of agents are referred to as leaders (or informed agents), denoted by \( \mathcal{V}_{\text{leader}} \subset \mathcal{V} \), who can be directly influenced by the external input signal, the remaining agents are referred to as followers, denoted by \( \mathcal{V}_{\text{follower}} = \mathcal{V} \setminus \mathcal{V}_{\text{leader}} \). The set of external input signal is denoted by \( \mathcal{U} = \{ \mathbf{u}_1, \ldots, \mathbf{u}_m \} \) where \( \mathbf{u}_i \in \mathbb{R}^d \), \( l \in \mathcal{U} \) and \( m \in \mathbb{Z}_+ \). In the following discussion, we shall assume that the input signal is homogeneous, i.e., \( \mathbf{u}_{l_1} = \mathbf{u}_{l_2} = 0 \) for all \( l_1, l_2 \in \mathcal{U} \). Denote by the edge set between external input signals and the leaders as \( \tilde{\mathcal{E}} \), and a corresponding set of matrix weights as \( B = [B_{il}] \in \mathbb{R}^{nd \times nd} \) where \( |B_{il}| \geq 0 \) or \( |B_{il}| > 0 \) if agent \( i \) is influenced by the input \( \mathbf{u}_l \) and \( B_{il} = 0 \) otherwise. The graph \( \tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{A}) \) is directed with \( \tilde{\mathcal{V}} = \mathcal{V} \cup \mathcal{U} \), \( \tilde{\mathcal{E}} = \mathcal{E} \cup \tilde{\mathcal{E}} \), \( \tilde{A} = A \cup B \). Consider the following leader-follower control protocol,

\[ \dot{x}_i(t) = u_i(t), \quad i \in \mathcal{V}, \quad (36) \]
where
\[
\mathbf{u}_i(t) = -\sum_{j \in \mathcal{N}_i} |A_{ij}|(\mathbf{x}_i(t) - \text{sgn}(A_{ij})\mathbf{x}_j(t)) - \sum_{l=1}^m |B_{il}|(\mathbf{x}_i(t) - \text{sgn}(B_{il})\mathbf{u}_l), \quad i \in \mathcal{V}.
\] (37)

The collective dynamics of (36) can subsequently be characterized by
\[
\dot{x} = -L_B(\mathcal{G})x + Bu,
\] (38)
where \(x = (x_1^T(t), \ldots, x_n^T(t))^T \in \mathbb{R}^{nd}, u = (u_1^T, \ldots, u_m^T)^T \in \mathbb{R}^{md}\) and
\[
L_B(\mathcal{G}) = L(\mathcal{G}) + \text{blkdiag}(\sum_{l=1}^m |B_{il}|).
\] (39)

**Definition 4.** For \(i \in \mathcal{V}\) and an arbitrary \(x_i(0) \in \mathbb{R}^d\), the multi-agent system (38) is said to admit bipartite leader-follower consensus if \(\lim_{t \to \infty} |x(t)| = |u_0|\).

Similar to the leaderless case, consider the following event-triggered control protocol,
\[
\mathbf{u}_i(t) = \hat{q}_i(t)
= -\sum_{j \in \mathcal{N}_i} |A_{ij}|(\hat{x}_i(t) - \text{sgn}(A_{ij})\hat{x}_j(t))
- \sum_{l=1}^m |b_{il}|(|\hat{x}_i(t) - \text{sgn}(b_{il})\mathbf{u}_l|), \quad i \in \mathcal{V},
\] (40)
and the system (36) can be written in a compact form as
\[
\dot{x}(t) = -L_B\hat{x}(t) + Bu,
\] (41)
where \(\hat{x}(t) = (\hat{x}_1^T(t), \hat{x}_2^T(t), \ldots, \hat{x}_n^T(t))^T \in \mathbb{R}^{dn}\). Define the state-based measurement error between the last broadcast state of agent \(i \in \mathcal{V}\) and its current state at time \(t \geq 0\) as
\[
e_i(t) = \hat{x}_i(t) - x_i(t),
\] (42)
and the system-wise measurement error is denoted by \(e(t) = [e_1^T(t), e_2^T(t), \ldots, e_n^T(t)]^T\).

For agent \(i \in \mathcal{V}\), the triggering time sequence is initiated from \(t_1^i = 0\) and subsequently determined by
\[
t_{k+1}^i = \max_{r \geq t_k^i} \{ r : \theta_i(\gamma_i \| e_i(t) \|^2 - \sigma_i \| \hat{q}_i(t) \|^2) \leq \chi_i(t), \forall t \in [t_k^i, r]\}, \quad k \in \mathbb{Z}_+,
\] (43)
where \(\sigma_i \in [0, 1], \theta_i\) and \(\gamma_i\) are the design parameters and \(\chi_i(t)\) is an auxiliary system for each agent \(i \in \mathcal{V}\) such that
\[
\dot{\chi}_i(t) = -\beta_i\chi_i(t) + \delta_i(\sigma_i \| \hat{q}_i(t) \|^2 - \gamma_i \| e_i(t) \|^2),
\] (44)
with \(\chi_i(0) > 0, \beta_i > 0\) and \(\delta_i \in [0, 1]\). We shall also denote \(\sigma_{\text{max}} = \max_i \{\sigma_i\}\) in the subsequent discussions.
Assumption 2. The matrix-weighted network \( \bar{G} = (\bar{V}, \bar{E}, \bar{A}) \) is structurally balanced and \( \sum_{i=1}^{m} \sum_{l=1}^{n} |B_{il}| \) is positive definite.

The Assumption 1 and Assumption 2 together guarantee that the leader-follower multi-agent network (38) admits a bipartite leader-follower consensus. In the following, we shall give the main results of this part.

Theorem 2. Consider the multi-agent system (41) under the matrix-weighted network \( G = (V, E, A) \) satisfying Assumptions 1 and 2. Let \( \theta_i \) and \( \gamma_i \) be such that \( \theta_i > 1 - \frac{\delta_i}{\beta_i} \) and

\[
\gamma_i = n \left( \sum_{j \in N_i} \mu(|A_{ij}|) + \sum_{l=1}^{m} \mu(|B_{il}|) \right)^2 + n \sum_{j \in N_i} \mu^2(|A_{ij}|). \tag{45}
\]

for all \( i \in V \), the triggering time sequence is determined by (43) for agent \( i \) with \( \chi_i(t) \) defined in (44). Then the multi-agent system (41) admits a bipartite leader-follower consensus. Moreover, there is no Zeno behavior.

Proof. For \( \forall t \geq 0 \), from the inequalities in (43) and (44), one has

\[
\dot{\chi}_i(t) \geq -\beta_i \chi_i(t) - \frac{1}{\theta_i} \chi_i(t), \tag{46}
\]

thus,

\[
\chi_i(t) \geq \chi_i(0) e^{-(\beta_i + \frac{1}{\theta_i})t} > 0. \tag{47}
\]

Let \( \xi(t) = x(t) - D^* (1_n \otimes u_0) \), where \( D^* \) is the Gauge transformation corresponding to the matrix-weighted network \( \bar{G} = (\bar{V}, \bar{E}, \bar{A}) \). Then one has

\[
\dot{\xi}(t) = -L_B \xi(t). \tag{48}
\]

Consider the Lyapunov function candidate as follows

\[
V(t) = V_1(t) + V_2(t), \tag{49}
\]

where

\[
V_1(t) = \xi^T(t) L_B \xi(t), \tag{50}
\]

and

\[
V_2(t) = \sum_{i=1}^{n} \chi_i(t). \tag{51}
\]

Computing the time derivative of \( V_1(t) \) along with (48) yields,

\[
\dot{V}_1(t) = \dot{\xi}^T(t) L_B \xi(t) + \xi(t)^T L_B \dot{\xi}(t) = 2 \xi(t)^T L_B \dot{q}(t) = -2q(t)^T \dot{q}(t). \tag{52}
\]
Define $e_{q(t)}(t) = \hat{q}(t) - q(t)$, then one has

$$
\dot{V}_1(t) = -2q^T(t)\hat{q}(t) + 2e_{q(t)}(t)^T\hat{q}(t)
$$

$$
= - \sum_{i=1}^{N} 2q_i^T(t)\hat{q}_i(t) + \sum_{i=1}^{N} 2e_{q_i(t)}(t)^T\hat{q}_i(t)
$$

$$
\leq - \sum_{i=1}^{N} q_i^T(t)\hat{q}_i(t) + \sum_{i=1}^{N} e_{q_i(t)}(t)^T e_{q_i(t)}(t). \quad (53)
$$

Recall that

$$
e_{q_i(t)}(t) = \sum_{j \in N_i} |A_{ij}| (\text{sgn}(A_{ij})e_j(t) - e_i(t)) - \sum_{l=1}^{m} |B_{il}|e_l(t), \quad (54)
$$

thus,

$$
\| e_{q_i(t)}(t) \| \leq \left( \sum_{j \in N_i} \| A_{ij} \| + \sum_{l=1}^{m} \| B_{il} \| \right) \| e_i(t) \| + \sum_{j \in N_i} \| A_{ij} \| \| e_j(t) \|. \quad (55)
$$

Note that

$$
\left( \sum_{i=1}^{N} x_i \right)^2 \leq N \sum_{i=1}^{N} x_i^2, \quad (56)
$$

therefore,

$$
\| e_{q_i(t)}(t) \|^2 \leq n \left( \sum_{j \in N_i} \mu(|A_{ij}|) + \sum_{l=1}^{m} \mu(|B_{il}|) \right)^2 \| e_i(t) \|^2
$$

$$
+ n \sum_{j \in N_i} \mu(A_{ij})^2 \| e_j(t) \|^2, \quad (57)
$$

hence,

$$
\sum_{i=1}^{n} e_{q_i(t)}(t)^T e_{q_i(t)}(t) \leq \sum_{i=1}^{n} \left( \sum_{j \in N_i} \mu(|A_{ij}|) + \sum_{l=1}^{m} \mu(|B_{il}|) \right)^2
$$

$$
\| e_i(t) \|^2 + \sum_{i=1}^{n} n \sum_{j \in N_i} \mu^2(|A_{ij}|) \| e_j(t) \|^2
$$

$$
= \sum_{i=1}^{n} \gamma_i \| e_i(t) \|^2. \quad (58)
$$

Then,

$$
\dot{V}_1(t) \leq \sum_{i=1}^{n} \gamma_i \| e_i(t) \|^2 - \sum_{i=1}^{n} q_i^T(t)\hat{q}_i(t). \quad (59)
$$
Now, we are in position to consider the Lyapunov function candidate $V(t)$, one has

$$
\dot{V}(t) = \dot{V}_i(t) + \sum_{i=1}^{n} \dot{\chi}_i(t)
$$

$$
\leq \sum_{i=1}^{n} \gamma_i \| e_i(t) \|^2 \sum_{i=1}^{n} \hat{q}_i^T(t) \hat{q}_i(t)
$$

$$
+ \sum_{i=1}^{n} \left( -\beta_i \chi_i(t) + \delta_i(\sigma_i \| \hat{q}_i(t) \|^2 - \gamma_i \| e_i(t) \|^2) \right)
$$

$$
= -\sum_{i=1}^{n} (1 - \delta_i) \sigma_i \| \hat{q}_i(t) \|^2 + \sum_{i=1}^{n} (1 - \delta_i) \gamma_i \| e_i(t) \|^2 - \sum_{i=1}^{n} \beta_i \chi_i(t)
$$

$$
\leq -\sum_{i=1}^{n} \left( \beta_i - \frac{1 - \delta_i}{\theta_i} \right) \chi_i(t) - (1 - \sigma_{max}) \sum_{i=1}^{n} \| \hat{q}_i(t) \|^2
$$

Due to $V(t) \geq 0$ and $\dot{V}(t) \leq 0$, which implies that $\lim_{t \to \infty} \dot{V}(t) = 0$, then one has,

$$
0 = \lim_{t \to \infty} \dot{V}(t)
$$

$$
\leq -\sum_{i=1}^{n} \left( \beta_i - \frac{1 - \delta_i}{\theta_i} \right) \chi_i(t) - (1 - \sigma_{max}) \sum_{i=1}^{n} \| \hat{q}_i(t) \|^2
$$

$$
\leq 0,
$$

(60)

thus, $\lim_{t \to \infty} \chi_i(t) = 0$ and $\lim_{t \to \infty} \hat{q}_i(t) = 0$. Due to

$$
0 \leq \| e_i(t) \|^2 \leq \sigma_i \| \hat{q}_i(t) \|^2 + \chi_i(t),
$$

(62)

therefore, $\lim_{t \to \infty} e_i(t) = 0$. Then,

$$
\dot{V}_i(t) = \dot{\xi}_i^T(t) L_B \xi(t) + \xi_i(t)^T L_B \dot{\xi}(t)
$$

$$
= -2\xi_i(t)^T L_B L_B \xi(t) + e(t)
$$

$$
= -\xi_i(t)^T L_B L_B \xi(t) - \xi_i(t)^T L_B L_B e(t),
$$

(63)

thus, $\lim_{t \to \infty} L_B \xi(t) = 0$, when the interaction graph is connected, the matrix $L_B$ is positive definite, therefore,

$$
\lim_{t \to \infty} \xi(t) = 0,
$$

(64)

then the multi-agent system admits a bipartite leader-follower consensus.

In the following discussion, we shall prove that there is no Zeno behavior. By contradiction, suppose that there
exists Zeno behavior when applying the proposed event-trigger control protocol \(^{(40)}\) to the multi-agent system. Then, there at least exists one agent \(i\) such that \(\lim_{k \to \infty} t^i_k = T_0\) where \(T_0 > 0\). From the above analysis, we know that there exists a positive constant \(M_0 > 0\) satisfying \(\| x_i(t) \| \leq M_0 \) for all \(t \geq 0\) and \(i \in n\). Then one has \(\| p_i(t) \| \leq 2M_o \sum_{j \in N_i} \| A_{ij} \| + (M_0 + \| u_0 \|) \sum_{l=1}^m \| B_{il} \|, \tag{65}\) for any \(t \geq 0\). Choose \(\varepsilon_0 = \left(2M_o \sum_{j \in N_i} \| A_{ij} \| (M_0 + \| u_0 \|) \sum_{l=1}^m \| B_{il} \| \right)^{-1} \sqrt{\chi_i(0) e^{-\frac{1}{2} (\beta_i + \theta_i) T_0}}. \tag{66}\) Then, according to the definition of limits, there exists a positive integer \(N(\varepsilon_0)\) such that for any \(k \geq N(\varepsilon_0)\), \(t^i_k \in [T_0 - \varepsilon_0, T_0]. \tag{67}\) Note that \(\sigma_i \| p_i(t) \|^2 \geq 0\), then one sufficient condition to guarantee the inequality in \((43)\) is \(\| e_i(t) \| \leq \sqrt{\chi_i(0) e^{-\frac{1}{2} (\beta_i + \theta_i) t}}. \tag{68}\) In addition, \[\| e_i(t) \| = \| \hat{x}_i(t^i_k) - x_i(t) \| = \| x_i(t^i_k) - x_i(t) \| \leq \int_{t^i_k}^t \| \dot{x}_i(t) \| \, dt \leq \int_{t^i_k}^t \| \dot{x}_i(t) \| \, dt \leq (t - t^i_k)(2M_o \sum_{j \in N_i} \| A_{ij} \| + (M_0 + \| u_0 \|) \sum_{l=1}^m \| b_{il} \|), \tag{69}\] then another sufficient condition to guarantee that the inequality in \((43)\) holds if \[(t - t^i_k)(2M_o \sum_{j \in N_i} \| A_{ij} \| + (M_0 + \| u_0 \|) \sum_{l=1}^m \| b_{il} \|) \leq \sqrt{\chi_i(0) e^{-\frac{1}{2} (\beta_i + \theta_i) t}}. \tag{70}\] Let \(t^i_{N(\varepsilon_0) + 1}\) and \(\hat{t}^i_{N(\varepsilon_0) + 1}\) denote the next triggering time determined by the inequalities in \((43)\) and \((70)\), respectively. Then,
Figure 1: A six-node structurally balanced matrix-weighted network $\mathcal{G}_1$. The solid lines represent the edges weighted by (positive or negative) definite matrices, the dashed lines represent the edges weighted by (positive or negative) semi-definite matrices. The blue lines represent edges weighted by positive (semi-)definite matrices, and red lines represent edges weighted by negative (semi-)definite matrices.

\[
t_{\mathbf{N}(\varepsilon_0)} + 1 - t_{\mathbf{N}(\varepsilon_0)}^i \geq t_{\mathbf{N}(\varepsilon_0)}^i + 1 - t_{\mathbf{N}(\varepsilon_0)}^i \\
= \left(2M_0 \sum_{j \in \mathcal{N}_i} \| A_{ij} \| (M_0 + \| u_0 \|) \sum_{l=1}^{m} \| B_{il} \| \right)^{-1} \sqrt{\chi_i(0) e^{-\frac{1}{2} \left(\beta_i + \frac{\varepsilon_i}{\delta_i} \right) T_0}} \\
\geq \left(2M_0 \sum_{j \in \mathcal{N}_i} \| A_{ij} \| + (M_0 + \| u_0 \|) \sum_{l=1}^{m} \| B_{il} \| \right)^{-1} \sqrt{\chi_i(0) e^{-\frac{1}{2} \left(\beta_i + \frac{\varepsilon_i}{\delta_i} \right) T_0}} \\
= 2\varepsilon_0,
\]

which contradicts with the equation in (67). Therefore, Zeno behavior is excluded.

Remark 4. Similar to the leaderless case, the multi-agent system (38) degenerates into the scalar-weighted case when $A_{ij} = a_{ij}I$, in this case,

\[
\gamma_i = n \left( \sum_{j \in \mathcal{N}_i} | a_{ij} | + \sum_{l=1}^{m} | b_{il} | \right)^2 + n \sum_{j \in \mathcal{N}_i} a_{ij}^2.
\]

Therefore, the event-triggered strategy proposed for the matrix-weighted leader-follower system can be applied for the scalar-weighted leader-follower case directly.

5. Simulations

In this section, we proceed to provide two simulation examples to demonstrate the theoretical results in this paper.

5.1. Leaderless Matrix-weighted Networks

First, consider the leaderless multi-agent system (38) on the structurally balanced matrix-weighted network $\mathcal{G}_1$ with the node bipartition $\mathcal{V}_1 = \{1, 2, 6\}$ and $\mathcal{V}_2 = \{3, 4, 5\}$ as shown in Figure 1.

In this case, the state dimension of each agent is $d = 4$, and all agents adopt event-triggered control protocol (7). The weight matrices on edges in $\mathcal{G}_1$ are
Figure 2: Entry-wise trajectory of each agent for the multi-agent system $\mathcal{G}$ under the structurally balanced matrix-weighted network $\mathcal{G}_1$ in Figure 1.

Figure 3: The event-based control protocol $\hat{q}_i(t)$ of each agent $i \in \mathcal{V}$ for the multi-agent system $\mathcal{G}$ under the structurally balanced matrix-weighted network $\mathcal{G}_1$.

$$A_{12} = \begin{bmatrix} 0.0975 & 0.9649 & 0.4854 & 0.9157 \\ 0.2785 & 0.1576 & 0.8003 & 0.7922 \\ 0.5469 & 0.9706 & 0.1419 & 0.9595 \\ 0.9575 & 0.9572 & 0.4218 & 0.6557 \end{bmatrix} > 0,$$
Each agent is randomly chosen from $\mathcal{G}_1$. The protocol for each agent are illustrated in Figure 3. Sequences of triggering time for each agent are illustrated in Figure 4. Bipartite consensus can be achieved in an element-wise manner, as shown in Figure 2. The dimensions of control \[ A_{16} = \begin{bmatrix} 8.1684 & 1 & -0.1160 & 0.3328 \\ 1 & 6.7945 & 1.2264 & 0.4473 \\ -0.1160 & 1.2264 & 7.4303 & 0.2236 \\ 0.3328 & 0.4473 & 0.2236 & 8.0775 \end{bmatrix} > 0, \]

\[ A_{26} = \begin{bmatrix} 4.6211 & 0.8971 & 0.8392 & 2.7045 \\ 0.8971 & 1.1161 & 2.1934 & 0.0274 \\ 0.8392 & 2.1934 & 4.5295 & -0.5815 \\ 2.7045 & 0.0274 & -0.5815 & 1.8457 \end{bmatrix} \geq 0, \]

\[ A_{23} = \begin{bmatrix} -6.6469 & 0.4166 & 0.044 & 0.2922 \\ 0.4166 & -8.2131 & 0.1152 & -0.3055 \\ 0.044 & 0.1152 & -6.2339 & -0.1434 \\ 0.2922 & -0.3055 & -0.1434 & -6.6147 \end{bmatrix} \leq 0, \]

\[ A_{56} = \begin{bmatrix} -4.7176 & -1.6485 & 1.5246 & -3.1114 \\ -1.6485 & -6.7837 & -1.3214 & 0.9421 \\ 1.5246 & -1.3214 & -6.4716 & -2.6201 \\ -3.1114 & 0.9421 & -2.6201 & -6.0166 \end{bmatrix} \leq 0, \]

\[ A_{35} = \begin{bmatrix} 4.8630 & -0.9583 & -1.0002 & 0.6242 \\ -0.9583 & 4.9516 & 1.1961 & -0.8268 \\ -1.0002 & 1.1961 & 6.5071 & -2.4257 \\ 0.6242 & -0.8268 & -2.4257 & 6.4197 \end{bmatrix} > 0, \]

\[ A_{34} = \begin{bmatrix} 4.6843 & -0.5024 & 1.2292 & 0.5247 \\ -0.5024 & 6.2876 & 0.5766 & 0.0968 \\ 1.2292 & 0.5766 & 5.2446 & 0.0118 \\ 0.5247 & 0.0968 & 0.0118 & 6.2167 \end{bmatrix} > 0, \]

and

\[ A_{45} = \begin{bmatrix} 0.7899 & 1.5860 & -0.3137 & -0.498 \\ 1.5860 & 3.2857 & -1.0541 & -1.5607 \\ -0.3137 & -1.0541 & 1.9019 & 2.5477 \\ -0.498 & -1.5607 & 2.5477 & 3.4211 \end{bmatrix} \geq 0. \]

Moreover, $A_{ij} = A_{ji}$ for all $(i,j) \in \mathcal{E}(\mathcal{G}_1)$. Choose $\sigma_i = 0.9$, $\delta_i = 1$, $\beta_i = 1$, and $\chi_i(0) = 0.5$. According to Theorem 2 choose $\theta_i = 0.5$ which satisfies $\theta_i > \frac{1}{\beta_i}$. Each dimension of initial value corresponding to each agent is randomly chosen from $[-1, 1]$. By computing the eigenvalues of the weight matrices, one can get $\mu_1 = 9.2047$, $\mu_2 = 8.396$, $\mu_3 = 9.7599$, $\mu_4 = 6.7454$, $\mu_5 = 9.7599$, $\mu_6 = 9.3996$. Using the above parameters, the bipartite consensus can be achieved in an element-wise manner, as shown in Figure 2. The dimensions of control protocol for each agent are illustrated in Figure 3. Sequences of triggering time for each agent are illustrated in Figure 4.
5.2. Leader-follower Matrix-weighted Networks

Consider the leader-follower multi-agent system (41) on the leader-follower network $G_1^\prime$ in Figure 5, where agents 1 and 6 are the leaders influenced by the inputs $u_1$ and $u_2$, respectively. The edge weights in the matrix-weighted network $G_1^\prime$ are the same as the leaderless case above, the influence weights by the inputs $u_1$ and $u_2$ are $B_{11} = A_{45} \geq 0$ and $B_{62} = A_{16} > 0$, respectively. In this case, choose $\sigma_i = 0.9$, $\delta_i = 1$, $\beta_i = 1$, $\chi_i(0) = 0.5$, and $u_1 = u_2 = [0.2, 0.4, 0.6, 0.8]^T$. By computing the eigenvalues of the weight matrices, one has $\gamma_1 = 9.2047$, $\gamma_2 = 8.396$, $\gamma_3 = 9.7599$, $\gamma_4 = 6.7454$, $\gamma_5 = 9.7599$, $\gamma_6 = 9.3996$. According to Theorem 2, choose $\theta_i = 1$ satisfying $\theta_i > \frac{1-\delta_i}{\beta_i}$. Under these parameters, the bipartite leader-follower consensus can be achieved as shown in Figure 6.
Sequences of triggering time for each agent are demonstrated in Figure 7. The dimensions of control protocol for each agent are illustrated in Figure 8.
Figure 8: The event-based control protocol $\hat{q}_i(t)$ of each agent $i \in \mathcal{V}$ for the multi-agent system (41) under the leader-follower network $G_i^l$ in Figure 5.

6. Conclusion

The event-triggered bipartite consensus strategies for both leaderless and leader-follower multi-agent system on matrix-weighted networks are discussed in this paper. By introducing an additional variable, which is generated by an auxiliary system, for each agent to adjust its threshold dynamically, the proposed distributed dynamic event-triggered strategies are proposed for the matrix-weighted networks and the Zeno behavior for the triggering time sequence can be avoided. In the proposed event-triggered strategies, each agent only needs to broadcast at its own triggering instants, and listen to incoming information from its neighbors at their triggering instants. Thus, continuous measurement of neighbors’ states can be avoided. Simulation examples are provided to demonstrate the theoretical results. The future work includes the extension of the proposed algorithm to directed networks and event-triggered consensus problem of high-order matrix-weighted networks.

7. Appendix

Lemma 2. Let $x, y \in \mathbb{R}^d$ and $\alpha > 0$. Then

$$x^Ty \leq \frac{x^Tx}{2\alpha} + \frac{\alpha y^Ty}{2}.$$ 

Lemma 3. Horn and Johnson [11] Let $M \in \mathbb{R}^{n \times n}$ be Hermitian with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Let $x_{i_1}, \cdots, x_{i_k}$ be mutually orthonormal vectors such that $Mx_{i_p} = \lambda_{i_p}x_{i_p}$, where $i_p \in \mathbb{N}$, $p \in \mathcal{V}$ and $1 \leq i_1 < \cdots < i_k \leq n$. Then

$$\lambda_{i_1} = \min_{\{x \neq 0, x \in S\}} \frac{x^TMx}{x^Tx},$$

and

$$\lambda_{i_k} = \max_{\{x \neq 0, x \in S\}} \frac{x^TMx}{x^Tx},$$
where $S = \text{span}\{x_{i_1}, \ldots, x_{i_k}\}$.

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