A Model for Crack Propagation and Contact Line Depinning

Peter B. Thomas and Maya Paczuski

Department of Physics, Brookhaven National Laboratory, Upton, New York, 11973

email: thomas@cmth.phy.bnl.gov maya@cmth1.phy.bnl.gov

submitted to Phys. Rev. Lett. Jan 31, 1996

A model for crack propagation and contact line depinning is studied. Although the model contains nonlocal interactions, it obeys general scaling relations for depinning via localized bursts or avalanches. Our numerical result for the roughness exponent in one dimension, \( \chi = 0.49 \pm 0.05 \), agrees with recent experiments on cracks measuring the in-plane roughness \( \chi \approx 0.5 - 0.6 \), as well as mean field arguments giving \( \chi = 1/2 \), but is significantly higher than the functional renormalization group prediction \( \chi = 1/3 \). 

PACS number(s): 68.45.-v, 05.40.+j, 64.60.Ht

Despite its enormous practical importance and intense efforts to model the phenomenon, the physics of crack propagation is poorly understood \cite{1}. Recently, Bouchaud \textit{et al.} \cite{2} proposed that crack propagation may be related to the dynamics of interface depinning by quenched random impurities \cite{3}. Subsequently, Schmittbuhl \textit{et al.} \cite{4} argued that the propagation of a crack in a certain idealized geometry has the same in-plane dynamics as contact line (CL) depinning on a dirty substrate. This mapping would apply specifically to a slowly advancing crack confined within an easy plane between two elastic solids that are being slowly pulled apart. They numerically measured an in-plane roughness index \( \chi \approx 0.35 \), in seeming agreement with the functional renormalization group (RG) \cite{5,6} prediction by Ertaç and Kardar \cite{7} that \( \chi = (2 - d)/3 \) exactly \cite{9}. Daguier, Bouchaud, and Lapasset \cite{8} have actually measured the in-plane roughness index of slowly moving cracks in metallic alloys and for stopped cracks found \( \chi \approx 0.5 - 0.6 \), in apparent contrast to the theoretical predictions. Since the model \cite{4} clearly represents a vastly simplified description of crack propagation, the discrepancy might be due to an inherent deficiency. Here we present results which lead to a different conclusion.

Our main numerical result from simulations of the model is \( \chi = 0.49 \pm 0.05 \) with scaling over almost three decades in length scale. Our result is inconsistent with the functional RG prediction, but compares reasonably well with the experimental finding. We also show that despite the nonlocal interactions in the model motion near the depinning transition takes place in terms of localized bursts, or avalanches. As a result the critical dynamics obeys general scaling relations for avalanche phenomena in systems out of equilibrium \cite{11,12}. Finally, based on numerical evidence, we conjecture that the critical exponents for CL depinning and the related model for crack propagation are given by a simple mean field theory giving \( \chi = 1/2, D = 3/2, \tau = 4/3, \nu = 2, \gamma = 2, z = 1, d_f = 1/2, \beta = 1, \) and \( \pi = 2 \) in one dimension.

Applying a sufficient force to an interface causes it to move through a random medium with a finite velocity which vanishes continuously at a depinning transition where quenched impurities pin the interface into a static configuration. Rather than exhibiting a smooth continuous motion, the dynamics near the transition point takes place in terms of intermittent, localized bursts. A scaling theory has been developed that relates the critical dynamics of interface motion near the depinning transition to the spatiotemporal structure of avalanches \cite{10}. This theory encompasses systems with local interactions \cite{11,12} such as the motion of magnetic domain walls in the presence of quenched disorder, fluid invasion in a porous media, extinguishing flame fronts \cite{13}, invasion percolation, and flux creep \cite{14}. Here we propose that it also includes contact line depinning and the related phenomena in crack propagation.

Unlike the examples mentioned above, CL motion is governed by a nonlocal integral equation. This comes about because the capillary energy associated with small deformations of the CL has the long wavelength limit

\[
U_{cap} = \frac{\gamma \Theta^2}{2} \int \frac{dq}{2 \pi} d_q [q] h(q) \overset{2}{\frac{d_q}{(d_q^2)^{z/2}}},
\]

associated with increasing the entire area of the liquid-vapor interface, rather than the usual line tension energy \( q^2 [h(q)]^2 \) \cite{15}. Here \( h(x) \), the height of the CL profile at position \( x \), is assumed to be single valued. \( \Theta \to 0 \) is the macroscopic contact angle that the liquid-vapor interface makes with the substrate at the CL, \( \gamma \) is the liquid-vapor interfacial tension, \( a \) is a small scale cutoff, and \( L \) is the linear extent of the one dimensional CL. The unusual capillary energy has a significant effect on the observed height fluctuations both in equilibrium and out of equilibrium \cite{14}.

If the CL is driven slowly and most of the dissipation occurs in its vicinity, the capillary energy enters into the
equation of motion for the height profile as an applied force \( F_{\text{int}}(x, t) = -\frac{\delta U(x, t)}{\delta h(x, t)} \), where \( t \) is time. In addition, there is a random contribution to the local force density from the quenched impurities \( \eta(x, h(x, t)) \). The CL is driven with an external force \( F \).

In our numerical simulations the variables \((h, x, t)\) are discretized to integers. We impose a geometry where the CL is periodic in \( x \), i.e. \( h(x + mL) = h(x) \), where \( m \) is integer and \( L \) is the system size studied. This requires a summation of the nonlocal kernel over all periods and modifies the \( 1/x^2 \) interaction to be \( (\pi x/L)^2 / \sin^2 (\pi x/L) \). In the thermodynamic \( L \rightarrow \infty \) limit the summed interaction simplifies to \( 1/x^2 \). The total force density \( f(x, t) \)

\[
-\nu \left( \frac{\pi x}{L} \right)^2 \sum_{x' \neq x} \frac{(h(x, t) - h(x', t))}{\sin^2 \left( \frac{\pi (x-x')}{L} \right)} - \eta(x, h) + F \quad ,
\]
where random forces \( \eta \) are chosen independently from a flat distribution between zero and one, and the sum is over all lattice sites. Essentially the same equation was proposed \[1\] to describe the in-plane dynamics of a crack propagating within an easy plane of an elastic block that is being slowly wedged open. Of course, this represents a very idealized picture for crack propagation since, for instance, it ignores the out-of-plane meandering of the propagating crack and the out-of-plane roughness of the resulting crack surface \[4\].

A constant force depinning transition is implemented as follows: At each time step, the force at each site is evaluated. If \( f(x, t) > 0 \), then the site is unstable or “active” and the height at that site is advanced by one unit \( h \rightarrow h + 1 \); otherwise the site is pinned and the height is unchanged. After the heights at all active sites have been advanced by one unit, time is advanced by one unit \((t \rightarrow t + 1)\), the local forces are re-evaluated, and the process is repeated. For \( F > F_c \), the interface moves with a finite average velocity \( v = n_{\text{act}} / L \) where \( n_{\text{act}} \) is the average number of unstable, moving sites.

The critical value \( F_c \) can be found by locating the site with the largest value of \( f(x, t) \) in a pinned configuration and increasing \( F \) until that site becomes unstable. This generates a burst of activity, or an avalanche, during which the interface moves. Eventually the burst dies out and the interface becomes stuck, with all sites frozen in a new configuration. The closer \( F \) gets to \( F_c \), the larger is the average spurt of growth, or avalanche size, \( s \). The average avalanche size diverges at the depinning transition \( F = F_c \). In order to obtain statistics for the steady state properties, one can set a value \( F = F_c - \Delta F \), where \( \Delta F \ll 1 \). At a moment in time when there are no active sites with \( f(x, t) > 0 \), the current \( F \) avalanche stops. Then a new \( F \) avalanche is initiated by advancing the height of the site with the largest value \( f(x, t) \) by one step. Starting from a flat configuration, \( h(x, t = 0) = 0 \) for all \( x \) and repeating this process many times, the CL will reach a steady state in a time \( t_{\text{ss}} \sim L^z \). All of the steady state properties of the depinning transition studied in this work were obtained using this method.

In Fig. 1, we show an actual realization of the height profile in the steady state near the depinning transition. The method of estimating the roughness exponent \( \chi \) which works best for this model, is to calculate the power spectrum \( P(k) \) for the height profile in the steady state. Fig. 2 shows power law behavior \( P(k) \sim k^{-1-2\chi} \). The asymptotic slope is \(-1.98\), which fits the data over 2.5 decades and gives the result \( \chi = 0.49 \).

One can compare Fig. 2 to the equivalent Fig. 3. in Ref. \[4\] where a slope for \( P(k) \) of \(-1.7\) was measured with much fewer samples on a somewhat smaller system size \( L = 2048 \) than we study. On inspection of their figure, it is apparent that there is a systematic deviation from their fitted slope at small values of \( k \) (large values of \( x \)) where the asymptotic behavior dominates. Their data points at small \( k \) do not fall near the “best fit” line and actually give an apparent slope which is closer to what we measure.

The equal time, height-height correlation function is

\[
C(r) = \langle (h(x + r, t) - \langle h(t) \rangle)(h(x, t) - \langle h(t) \rangle) \rangle ,
\]
where the angular brackets denote an average over time \( t \) and over \( x \), and \( \langle h(t) \rangle \) is the average height of the interface at time \( t \). In Fig. 3, we plot \( C(r)/L^x \) versus \( r/L \), where \( \chi \) assumes its mean field value \( 1/2 \), so that \( C(0) \sim L \). The large deviation for small system size \( L = 128 \) for \( r \) near zero indicates that such a small system size is not in the asymptotic regime. However we find reasonable good data collapse for \( L = 1024 - 8192 \). Our main numerical result is that \( \chi \) is bounded by \( \chi = 0.49 \pm 0.05 \).

We now discuss the scaling behavior of avalanches in the model. The size \( s \) of an \( F \) avalanche is the integrated motion, or the area between the initial configuration and the final configuration of the avalanche. In an experimental situation, perhaps, \( F \) would be increased slightly for each subsequent avalanche, but for a sufficiently large system, many avalanches would occur within a narrow interval centered on \( F \). The statistics of those avalanches are described by the probability distribution \( P(s, \Delta F) \).

In analogy with other depinning problems \[10\], it is plausible that the probability distribution of avalanche sizes is given by

\[
P(s, \Delta F) \sim s^{-\tau} g(s \Delta F^{\omega_s})
\]

near the critical point \( \Delta F = 0 \). Fig. 4 shows the measured distribution for small \( \Delta F \); it decays as a power law over four decades with a characteristic exponent \( \tau = 1.31 \pm 0.06 \) up to a cutoff determined by the system size.

In spite of the non-local nature of the interactions, we find that similar to other interface models which have been studied \[1\] the dynamics during an
avalanche is local. Fig. 5 shows the location of the active, moving sites vs. time, for one avalanche of moderate size. It is clear from the figure that the avalanche spreads out in time. In fact, the distance spread \( r(t) \sim t^{1/2} \). If the projection of the completed avalanche onto the sites that it covered is compact, and the avalanche size \( s \sim r^D \), where \( r \) is the spatial extent of the avalanche, then \( D = d + \chi \). In general \( D \neq z \).

For the same avalanches used in Fig. 4, we measured \( D = 1.5 \pm 0.15 \) which is consistent with the scaling relation and our result for \( \chi \).

We now discuss a simple mean field theory to explain these results. The characteristic dimension of the nonlocal interaction term in Eq. (2) at length scale \( l \) is \( h/l \) where \( h \sim l^x \). The sum of random forces at this scale makes a contribution \( l^{1/2}/l \) per unit length. Balancing these two terms gives \( \chi = 1/2 \). Since the velocity \( v(l) \sim h/t \sim f(l) \), the time \( t \sim l \) so that the dynamical exponent \( z = 1 \). Defining the exponent \( \beta \) by \( v \sim \Delta F^\beta \) and the cutoff \( l_\infty \sim \Delta F^{-\nu} \), one obtains the scaling relation \( \beta = \nu(z-\chi) \). The exponent \( \beta \) can be obtained by integrating the equation of motion over the entire system, so that the nonlocal part vanishes. This gives

\[
v = F + \frac{1}{L} \int dx \eta(x, h) = F + \eta(F) ,
\]

where the average \( \eta(F) \) over the CL is evaluated at a particular value of \( F \) in the steady state. At \( F = F_c \), \( v = 0 \) so that \( \eta(F_c) = -F_c \). In general for \( F = F_c + \Delta F \) we have \( \eta(F) = -F_c + g(\Delta F) \). Substituting this last result into Eq. (2), we get

\[
v = F - F_c + g(F - F_c) .
\]

Assuming \( g \) has a power series expansion gives \( \beta = 1 \). The scaling relation for \( \beta \) then gives \( \nu = 2 \).

In general, for systems with local avalanche dynamics the exponent \( \tau \) for the distribution of avalanche sizes obeys the relation \( \tau = 1 + \frac{d-1}{d-\chi} \), and substituting the mean field values for one dimension \( (d=1) \) gives \( \tau = 4/3 \) in agreement with our numerics. The exponent \( \gamma \) for the divergence of the average avalanche size is \( \gamma = \nu D(2-\tau) \), and \( \gamma = 2 \) in mean field. In the constant velocity depinning transition the critical exponent \( b \) in Ref. [3] characterizing the power law distribution between subsequent activity obeys \( b = \pi - 1 = D(2 - \tau) \) according to the scaling theory. The exponent \( b \) was measured to be \( b = 0.9 \pm 0.05 \) which is consistent with the scaling relation and our numerical values for \( D \) and \( \tau \). The mean field prediction is that \( b = \pi - 1 \). Also, in the constant velocity case, there are scaling relations for the fractal dimension of active sites \( d_f = D(\tau - 1) \) and the dynamical exponent \( z = D(2 - \tau) \) which in mean field take the values \( d_f = 1/2 \) and \( z = 1 \). This last result suggests that all of the critical exponents are the same in the constant velocity and constant force case for this problem.

In addition to CL depinning and crack propagation, exact results have been put forward using the functional RG for interface depinning with purely local interactions, where Narayan and Fisher predicted \( \chi = (4 - d)/3 \). Numerical simulations of related lattice models in both one and two dimensions give \( \chi = 1.23 \) in \( d = 1 \) and \( \chi = 0.72 \) in \( d = 2 \). In all cases thus far where a comparison can be made, including our result, the numerical simulations give a roughness exponent higher than the functional RG prediction.

Our theoretical picture for depinning phenomena is based on avalanche dynamics. To our knowledge, avalanches are not explicitly contained in the functional RG theory. We suspect that the apparent failure of the functional RG to reproduce numerical results for interface depinning may express an inadequacy of these methods to describe avalanche dynamics. The agreement between our numerical results and experiments measuring the in-plane roughness index of cracks supports the model for crack propagation proposed by Schmittbuhl, Roux, Villette, and Målay.

We thank V. Emery, P. Bak, and D. Dhar for useful discussions. This work was supported by the U.S. Department of Energy Division of Materials Science, under contract DE-AC02-76CH00016. MP thanks the U.S. Department of Energy Distinguished Postdoctoral Research Program for partial financial support.

[1] B.B. Mandelbrot, D.E. Passoja, and A.J. Paullay, Nature (London) 308, 721 (1984); Statistical Models for the Fracture of Disordered Media, edited by H.J. Herrmann and S. Roux (Elsevier, Amsterdam, 1990); P. Meakin, Phys. Rep. 235, 191 (1993).
[2] J.P. Bouchaud, E. Bouchaud, G. Lapasset, and J. Planèes, Phys. Rev. Lett. 71, 2240 (1993).
[3] T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. 254, 215 (1995).
[4] J. Schmittbuhl, S. Roux, J.P. Villette, and K.J. Målay, Phys. Rev. Lett. 74, 1877 (1995).
[5] T. Nattermann, S. Stepnow, L.-H. Tang, and H. Leschhorn, J. Phys. (France) II 2, 1483 (1993); O. Narayan and D.S. Fisher, Phys. Rev. B 46, 11520 (1992).
[6] O. Narayan and D.S. Fisher, Phys. Rev. B 48, 7030 (1993).
[7] D. Ertaq and M. Kardar, Phys. Rev. E 49, 2532 (1994).
[8] For a \( d = 1 \) dimensional CL or crack, \( \chi = 1/3 \).
[9] P. Daguiu, E. Bouchaud, and G. Lapasset, Europhys. Lett. 31, 367 (1995).
[10] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E 53, 414 (1996).
[11] K. Sneppen, Phys. Rev. Lett. 69, 3539 (1992); H. Leschhorn and L.-H. Tang, Phys. Rev. E 49, 1238 (1994).
FIG. 1. An actual realization of the height profile $h(x, t)$ for $L = 8192$ [20].

FIG. 2. A log-log plot of the power spectrum of the height profile versus $k$ for $L = 8192$. The height profile was sampled every $t = 273$ time steps and over $10^6$ samples were used to compute $P(k)$. The straight line is a best fit to all the points $k < 0.1$. It has a slope of $-1.98$.

FIG. 3. Rescaled plot of the equal time height-height correlation function $(C(r)/L)$ vs. $r/L$ for $L = 128, F = 0.83$ (open circles), $L = 1024, F = 0.828$ (triangles), and $L = 8192$ (filled circles). This collapse of the data for $L = 1024 - 8192$ is consistent with $\chi = 1/2$.

FIG. 4. The avalanche size distribution near the depinning transition for $L = 8192$. Over $5 \times 10^4$ avalanches were measured. The slope gives an estimate of $\tau = 1.31 \pm 0.06$.

FIG. 5. The location (marked as dots) of the active, moving sites as a function of time, for $L = 1024$, during an avalanche which begins at $t = 0$ and ends at $t = 545$. The entire avalanche has a size $s = 9814$. The avalanches are local and spread out in time as $r \sim t^{1/z}$.
log (Distribution of Avalanches of size $s$)
