ON THE ESSENTIAL \((p)\)-DIMENSION OF PARABOLIC BUNDLES ON CURVES

AJNEET DHILLON AND DINESH VALLURI

Abstract. We study the essential dimension and essential \(p\)-dimension of the moduli stack of vector bundles over a smooth orbifold curve containing a rational point. We improve the known bounds on this essential dimension and obtain an equality modulo the famous conjecture of Colliot-Thelene, Karpenko and Merkurjev. In the case of essential \(p\)-dimension we obtain an equality.

1. Introduction

Roughly the essential dimension of a family of algebraic objects is the number of parameters needed to parameterise a generic family of such objects. This heuristic definition points to its central role it plays in moduli problems. A precise definition can be obtained by observing that there are two ways of defining the dimension of an algebraic variety. Firstly, there is the Krull dimension and secondly one can define dimension as transcendence degree of the function field over the base field. By, lifting the second definition to the category of algebraic stacks one arrives at the precise notion of essential dimension. This intriguing invariant is difficult to compute. There is a variation known as the essential \(p\)-dimension, which is roughly the essential dimension ignoring prime to \(p\) information, that is easier to compute. We will recall both of these definitions in section 4 below, see also [17] and [15].

The purpose of this paper is to study these invariants for the moduli of stack vector bundles on an orbifold curve, with coarse moduli of genus at least two. We extend the results of [5] in two ways. Firstly, we extend them to essential \(p\)-dimension. This has the virtue of being able to state and prove an equality that is not conjectural, see [6,3]. Secondly we consider smooth projective curves with an orbifold structure, i.e. certain kinds of root stacks. The problem is divided into two pieces as in [5]. Given a vector bundle \(\mathcal{E}\) on an orbifold curve, it corresponds to a point of a moduli stack of vector bundles. Hence there is a corresponding residual gerbe \(\mathcal{G}(\mathcal{E})\) with coarse moduli space \(k(\mathcal{E})\), the field of moduli of \(\mathcal{E}\). The essential dimension of \(\mathcal{E}\) breaks down into two component pieces. That of understanding the essential dimension of the residual gerbe over the field of moduli and then understanding the transcendence degree of the field of moduli over the base field.

The second of these two steps is carried out in section 5. Roughly amounts to understanding the tangent space to the automorphism group of a parabolic bundle. We present a new approach to this, different to that in [5], using deformation theory and filtered derived categories. This approach is useful in that it has potential to generalise to principal bundles over groups other than the general linear group.

2000 Mathematics Subject Classification. 14D23, 14D20.

Keywords and phrases. Essential dimension, parabolic vector bundle, curve.
The results of this paper are confined to dimension one, due to the fact that the dimensions of the stacks that we consider can be computed via Euler characteristics. This is no longer the case in higher dimension. As we are considering orbifold curves Toen’s extension of Riemann-Roch to Deligne-Mumford, see [19], stacks plays a pivotal role. We recall this theorem in section 3.

The essential dimension of vector bundle on an orbifold curve was first considered in [4]. The results of this paper give a vast improvement over the results in [4]. For example, let’s consider a smooth projective curve $X$ with a single orbifold point $x \in X$ point with orbifold structure $\mathbb{Z}/n\mathbb{Z}$. Vector bundles on this curve acquire an action of the group $\mathbb{Z}/n\mathbb{Z}$ over the orbifold point. Hence by ordering the eigenvalues of the action we obtain a filtration of the orbifold point, that is a parabolic bundle. Let $n_i$ be the dimensions of these vector spaces in this filtration so that $n_0 = \text{rank}$ of the vector bundle being considered. Suppose that the vector bundle has degree $d$ and set $h = \gcd(n_1, r, d)$. Let $\mathcal{Bun}_{n,d}^r$ be the moduli stack of vector bundles with prescribed data where $n = (n_0 \geq n_1 \geq \ldots \geq n_e)$. In this paper we show that

$$\text{ed}(\mathcal{Bun}_{n,d}^r) \leq r^2(g - 1) + 1 + \text{Flag}_n + \sum_{p|h}(p^{\nu_p(h)} - 1).$$

The last sum is over primes dividing $h$. The bound in [4] is more difficult to describe, but roughly it is of the form

$$\text{ed}(\mathcal{Bun}_{n,d}^r) \leq r^2(g - 1) + 1 + \text{Flag}_n + F(r),$$

where the function $F(r)$ is quadratic in the rank $r$, see [4, 12.1] for details.

In section 2 of the paper we start by recalling the parabolic-orbifold correspondence. This is an equivalence of categories between vector bundles on a root stack and vector bundles with filtration on its coarse moduli space. The third section is an overview of Riemann-Roch for orbifolds. We perform some calculations that will be useful later. In section 4 we recall essential dimension and its variant essential $p$-dimension. We recall the conjecture in [10] and state its $p$-analogue, see 4.1. Some results from [5] are recalled and extended to essential $p$-dimension and orbifolds. The fifth section studies the field of moduli of a parabolic bundle. We use deformation theory methods to understand and bound the transcendence degree of the field of moduli. This is in contrast to the global methods in [5]. As stated earlier, this may prove to be useful as these local calculations are more apt to generalisation to other groups. The final section, section six, states and proves our main result 6.3.

Acknowledgements

The first named author would like to thank Kirill Zainoulline for suggesting that we consider essential $p$-dimension.

2. The parabolic-orbifold correspondence

Let $X$ be a scheme and $\mathcal{L}$ a line bundle on $X$ with section $s \in H^0(X, \mathcal{L})$. If $e$ is a positive integer, we may form the root stack

$$q : X_{\mathcal{L},s,e} \to X,$$

see [3]. A lift of an $S$-point, $f : S \to X$ to the root stack amounts to a line bundle with section $(M, t)$ on $S$ and an isomorphism $\alpha : M^e \to f^*\mathcal{L}$ sending $t^e$ to $s$. The automorphisms are the obvious ones. It follows that there is a universal root line
bundle $\mathcal{N}$ on $\mathcal{X}_{\mathcal{L},s,e}$ whose $e$th power is the pullback of $\mathcal{L}$. We will refer to $e$ as the
*ramification index* of the construction.

On the other hand the data $(\mathcal{L},t)$ and $e$ determine a notion of *parabolic vector bundle* on $X$. This is a vector bundle $\mathcal{E}$ together with a filtration

$$\mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \ldots \supseteq \mathcal{E}_e$$

and an isomorphism $\mathcal{E} \otimes \mathcal{L}^{-1} \cong \mathcal{E}_e$ such that the composition

$$\mathcal{E} \otimes \mathcal{L}^{-1} \cong \mathcal{E}_e \hookrightarrow \mathcal{E}_0 \cong \mathcal{E} \otimes \mathcal{O}$$

arises from the section. There is a corresponding category $\text{Par}(\mathcal{L},s,e)$ of parabolic vector bundles. We refer the reader to [6] for details.

**Theorem 2.1.** Let $X$ be a noetherian scheme. There is an equivalence of categories

$$\text{Par}(\mathcal{L},s,e) \cong \text{Vect}(X,\mathcal{L},s,e)$$

**Proof.** See [6, 3.13]. Let us remark here that the parabolic bundle associated to a vector bundle $\mathcal{E}$ on $X$ is obtained by considering the sheaves $q_*(\mathcal{E} \otimes \mathcal{N}^i)$. □

**Remark 2.2.** The root stack admits a nice local description which explains the above correspondence quickly. Suppose that $X = \text{Spec}(R)$ is affine and $\mathcal{L}$ is trivial. Then $s \in R$. The scheme

$$R[X]/\langle X^e - s \rangle$$

has an action of the group scheme $\mu_e$. The quotient stack is the root stack. The correspondence comes from the fact that $\mu_e$-equivariant objects are just graded objects. For details see [8].

The root stack construction is easily seen to be functorial in the following sense, given $f : Y \to X$ then the root stack $Y_{f^*\mathcal{L},e}$ is the 2-pullback of $X_{\mathcal{L},e}$ along the morphism $f : Y \to X$. In other words, there is a 2-cartesian diagram

$$
\begin{array}{ccc}
Y_{f^*\mathcal{L},e} & \xrightarrow{g} & X_{\mathcal{L},e} \\
\downarrow g' & & \downarrow g \\
Y & \xrightarrow{f} & X
\end{array}
$$

**Proposition 2.3.** In the above situation, suppose that $Y \to X$ is flat. Suppose that $\mathcal{F}$ is a vector bundle on $X_{\mathcal{L},e}$ with corresponding parabolic vector bundle $\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \ldots \supseteq \mathcal{F}_e$. Then the parabolic vector bundle corresponding to $g^*\mathcal{F}$ is $f^*\mathcal{F}_0 \supseteq f^*\mathcal{F}_1 \supseteq \ldots \supseteq f^*\mathcal{F}_e$.

**Proof.** Recall that vector bundle $\mathcal{F}_i = q_*(\mathcal{F} \otimes \mathcal{N}^i)$ so that this result amounts to essentially flat base change. To make this precise, the root stack is locally on $X$ a $\mu_e$ quotient stack, [6, 3.4]. The result now follows from flat base change and the fact that the push forward $q_*$ amounts to taking $\mu_e$-invariants of an equivariant sheaf. □

Now assume $X$ is a projective variety over our ground field $k$. We fix Cartier divisors $D_1, D_2, \ldots, D_l$ and positive integers $e_1, e_2, \ldots, e_l$ coprime to $\text{char}(k)$. We let

$$\mathcal{X} = X_{(D_1,e_1),\ldots,(D_l,e_l)}$$
be the corresponding root stack construction. Corresponding to this there are root line bundles (see \cite{8}) written $N_i$ on the root stack $\mathcal{X}$. We write $q : \mathcal{X} \rightarrow \text{Spec}(k)$ for the structure map.

**Lemma 2.4.** The morphism $q_*$ is exact where $q$ is the coarse moduli map $q : \mathcal{X} \rightarrow X$.

**Proof.** This follows from the local description of the root stack see \cite{22}. Note that the stack is tame. □

**Corollary 2.5.** The derived functor $Rq_*$ preserves the amplitude of a bounded complex.

**Theorem 2.6.** The stack of coherent sheaves on $X$, written $\mathcal{Coh}_X$ is algebraic.

**Proof.** The standard proof in \cite{13} can be made to work when combined with the following observations. Let $\mathcal{F}$ be a coherent sheaf on $X$. We can find integers $n_{ij}$ and vector spaces $H_{ij}$ so that we have epimorphisms

$$\mathcal{O}_X(-n_{ij}) \otimes H_{ij} \rightarrow \pi_*(\mathcal{F} \otimes N_{ij}^{-1})$$

where $0 \leq j \leq e_i$. By adjointness we obtain a morphism

$$N_{ij}^{-1}\mathcal{O}_X(-n_{ij}) \otimes H_{ij} \rightarrow \mathcal{F}.$$ 

Taking a direct sum of these maps we obtain an epimorphism, this follows from the local description, see \cite{22} or \cite{7}. Further we can arrange for the appropriate higher cohomology to vanish using Serre vanishing. The needed presentation comes from considering open subsets of Quot schemes.

For the existence of quot scheme in the current setting, see \cite{18}. □

We will be interested in the case where $X$ is a smooth projective curve over a field $k$. In this case a closed point $p \in X$ determines a Cartier divisor. The corresponding notion of parabolic vector bundle amounts to vector bundle $E$ on $X$ and a $k(p)$-point of a flag variety $\text{Flag}((E|_{k(p)}, n_1, n_2, \ldots, n_{e-1}))$ parameterising subspaces

$$E_{k(p)} \supseteq V_1 \supseteq \ldots \supseteq V_{e-1}$$

with $\dim_{k(p)} V_i = n_i$. If $E$, thought of as a vector bundle on the root stack via the previous theorem, is allowed to vary in a flat family the numbers $n_i$ along with $r = \text{rk}(E)$ and $d = \text{deg}(E)$ do not change. We will refer to the collection

$$(r, d, (p, n_1, n_2, \ldots, n_{e-1}))$$

as a parabolic datum. It will be convenient to set $n_0 = \text{rk}(E)$ and $n_e = 0$.

We will have occasion to consider many such points $p_1, \ldots, p_l$ with ramification indices $e_i$ and integers

$$(n_{0i} = \text{rk}(E), n_{1i} \geq n_{2i} \geq \ldots \geq n_{e_i} = 0) = n_i.$$

A moduli stack of parabolic bundles, denoted

$$\mathcal{Bun}_{n, X}^{r,d}$$

is obtained, here $n = (n_1, \ldots, n_l)$. As the forgetful morphism

$$\mathcal{Bun}_{n, X}^{r,d} \rightarrow \mathcal{Bun}^{r,d}$$

is represented by Weil restrictions of flag varieties, we obtain an alternate proof that the stack is algebraic.
Remark 2.7. There is an internal hom object in the category of parabolic vector bundles. Indeed there is one in the category of vector bundles on the root stack, hence the assertion follows from the correspondence 2.1. We would like to describe the parabolic datum associated to the endomorphism bundle as this will be used later. Let \( \mathcal{F} \) be a parabolic bundle with datum \((n_0, n_1, \ldots, n_{e-1})\) at the Cartier divisor \( D \). Then \( \mathcal{H}om(\mathcal{F}, \mathcal{F}) \) has datum \((m_0, m_1, \ldots, m_{e-1})\) where

\[
m_d = \sum_{d \leq \lambda < e, \lambda = i-j \mod e} (n_i - n_{i+1})(n_j - n_{j+1}).
\]

This can be seen by looking at the \( \mu_e \)-action on a module of the form \( M \otimes M^\vee \) in the local description, 2.2 and observing that \( n_i - n_{i+1} \) is the dimension of the space where the action has weight \( \zeta^i \) for some primitive \( e \)th root of unity \( \zeta \).

3. Riemann-Roch on Deligne-Mumford stacks

We assume that \( \mathcal{X} \) is a tame Deligne-Mumford stack and \( k \) contains the roots of unity throughout this section.

Let’s start by recalling Toen’s Riemann-Roch theorem for smooth, proper, tame Deligne-Mumford stacks.

**Theorem 3.1.** Let \( \mathcal{X} \) be a Deligne-Mumford stack, tame and proper over \( \text{Spec}(k) \) with a projective coarse moduli space. If \( \mathcal{F} \) is a vector bundle on \( \mathcal{X} \) then

\[
\chi(\mathcal{X}, \mathcal{F}) = \int_{\mathcal{X}} \text{ch}^\text{rep}(\mathcal{F}) \text{td}^\text{rep}(\mathcal{X}).
\]

**Proof.** See [20, §3] and [19] for a proof and the definition of the relevant terms. A summary can also be found in [6]. □

If \( X \) is a smooth projective curve and \( p_i \) are closed points on \( X \) then the root stack

\[
\mathcal{X} = X((p_1, e_1), \ldots, (p_m, e_m))
\]
satisfies the hypothesis of the theorem when \( e_i \) are coprime to the characteristic of the ground field. We would like to give a more explicit description of the right hand side of the theorem in this instance. So let \( \mathcal{F} \) be a vector bundle on \( \mathcal{X} \) with datum

\[
(n_{i0}, n_{i1}, \ldots, n_{ie_i}).
\]

We are interested in a formula for the Euler characteristic, \( \chi(\mathcal{F}) \). By base change we may assume that the ground field is algebraically closed, \( k = \bar{k} \). See 3.3 below.

The characteristic classes present in Toen’s theorem live inside some bigraded cohomology theory \( H^*(\mathcal{X}, *) \) see [20, §3]. Toen defines the **cohomology in representations** of \( \mathcal{X} \) to be

\[
H^*_\text{rep}(\mathcal{X}, *) := H^*(I_{\mathcal{X}}, *)
\]

where \( I_{\mathcal{X}} \) is the inertia stack of \( \mathcal{X} \).

In our particular case, the inertia stack can be written as

\[
I_{\mathcal{X}} = \bigcup_{i=1}^m \mathbb{BZ}/\mathbb{Z}e_i.
\]
Consequently, we have
\[ H^0_{\text{rep}}(X, 0) \cong H^0(X, 0) \bigoplus_{i=1}^m K(\mathbb{Z}/\mathbb{Z}e_i) \]
and
\[ H^1_{\text{rep}}(X, 1) \cong H^1(X, 1). \]

The Chern character with coefficients in representations is given by
\[ \text{ch}^{\text{rep}} : K_0(\mathcal{X}) \xrightarrow{\phi_{\mathcal{X}}} K_0^{\text{rep}}(\mathcal{X}) \xrightarrow{\text{ch}} H^*_\text{rep}(\mathcal{X}, *), \]
the reader is referred to [19] or [6] for a description of \( \phi_{\mathcal{X}} \). Recall \( K_0^{\text{rep}}(\mathcal{X}) := K_0(I_\mathcal{X}) \). Since \( I_\mathcal{X} = \coprod_{i=1}^m B(\mathbb{Z}/e_i) \), we have
\[ K_0^{\text{rep}}(\mathcal{X}) = K_0(\mathcal{X}) \bigoplus_{i=1}^m K_0(B(\mathbb{Z}/e_i)) \]

The Chern character of \( \mathcal{F} \) in the cohomology in representations is given by
\[ \text{ch}^{\text{rep}}(\mathcal{F}) = (\text{ch}(\mathcal{F}), V_1, \ldots, V_l) \]
where \( \text{ch}(\mathcal{F}) \) is the Chern character in the ordinary cohomology \( H^*(X, \mathcal{F}) \) and \( V_i \) are the \( \mathbb{Z}/e_i \)-representation over \( k \) induced by \( \mathcal{F} \) at \( x_i \). Since \( \mathbb{Z}/e_i \) is diagonalizable, each \( V_i \) splits into characters. So we may write
\[ V_i = \bigoplus_{d=0}^{e_i-1} (\chi_i^d)^{n_{id}-n_{id+1}} \]

Where \( \chi_i^d : \mathbb{Z}/e_i \to k \) is given by \( 1 \mapsto \zeta_i^d \) and \( (n_{0i}, n_{1i}, \ldots, n_{e_i-1}) \) is the parabolic datum of \( \mathcal{F} \) at the point \( p_i \) and we define \( n_{e_i} = 0 \).

The Todd class in the cohomology in representations is given by
\[ \text{td}^{\text{rep}}(\mathcal{X}) = (\text{td}(T_X), (\text{td}_1, \ldots, \text{td}_l)) \]
Where \( \text{td}_i : \mathbb{Z}/e_i \to k \) are the characters given by \( 1 \mapsto \frac{1}{1-\zeta_i^d} \), where \( \zeta_i \) are the \( e_i \)-th roots of unity.

Combining these calculations we obtain :
\[ \text{ch}^{\text{rep}}(\mathcal{F}) \text{td}^{\text{rep}}(\mathcal{X}) = (\text{ch}(\mathcal{F}) \text{td}(T_X), [V_1] \text{td}_1, \ldots, [V_l] \text{td}_l) \]

Hence we get
\[ \int_X \text{ch}^{\text{rep}}(\mathcal{F}) \text{td}^{\text{rep}}(\mathcal{X}) = \int_X \text{ch}(\mathcal{F}) \text{td}(T_X) + \sum_{i=1}^m \sum_{d=0}^{e_i-1} \int_{B(\mathbb{Z}/e_i)} (|\chi_i^d|^{n_{id}-n_{id+1}})(|\text{td}_i|). \]

The push-forward \( p_{\mathcal{F}*} : H^*(B(\mathbb{Z}/e_i)) \to H^*(\text{Spec}(k)) \) of the class \( |\chi_i^d|T_{d} \) is the average of their image as characters \( \frac{1}{e_i} \sum_{a=0}^{e_i-1} \frac{\zeta_i^{ad}}{1-\zeta_i^a} \).

On the other hand, we have the usual formulas for the Chern character and Todd class \( \text{ch}(\mathcal{F}) = \text{rk}(\mathcal{F}) + c_1(\mathcal{F}) \) and \( \text{td}(T_X) = 1 + \frac{1}{2}c_1(T_X) \). Hence we get
\[ \int_X \text{ch}(\mathcal{F}) \text{td}(T_X) = \frac{\text{rk}(\mathcal{F})}{2} \int_X c_1(T_X) + \int_X c_1(\mathcal{F}). \]
and
\[
\int_X \text{ch}^{r_{\mathcal{E}}}(\mathcal{F}) \text{td}^{r_{\mathcal{E}}}(X) = \frac{\text{rk}(\mathcal{F})}{2} \int_X c_1(T_X) + \int_X c_1(\mathcal{F}) + \sum_{i=1}^{l} \frac{1}{e_i} \sum_{d=0}^{e_i-1} \sum_{a=0}^{e_i-1} \frac{(n_{id} - n_{i,d+1})\zeta^a_i}{1 - \zeta^a_i}
\]
Furthermore, we have the following formulae:
\[
\frac{1}{2} \int_X c_1(T_X) = (1 - g) + \sum_i \frac{r_i - 1}{2}\eta_i
\]
\[
\sum_{a=0}^{r_i-1} \frac{\zeta^{ad}_i}{1 - \zeta^a_i} = -\frac{r_i - 1}{2} - d
\]
\[
\sum_{a=0}^{r_i-1} \frac{\zeta^{ad}_i + 1}{1 - \zeta^a_i} = -d
\]

**Theorem 3.2.** In the above situation we have
\[
\chi(\mathcal{F}) = \deg(\mathcal{F}) + (1 - g) \text{rk}(\mathcal{F}) - \sum_i \sum_{d=0}^{e_i-1} \frac{d(n_{id} - n_{i,d+1})}{e_i}.
\]
In this formula \(\deg(\mathcal{F})\) is the degree of the ordinary vector bundle underlying the parabolic vector bundle corresponding to \(\mathcal{F}\).

*Proof.* The is a combination of the calculations above. \(\square\)

**Corollary 3.3.** Suppose that \(k\) is not algebraically closed and the \(p_i\) are closed points of \(X\). Then
\[
\chi(\mathcal{F}) = \deg(\mathcal{F}) + (1 - g) \text{rk}(\mathcal{F}) - \sum_i \sum_{d=0}^{e_i-1} \frac{d(n_{id} - n_{i,d+1})}{e_i}.
\]

*Proof.* One can base change to an algebraically closed field. Note that in the diagram
\[
\begin{array}{c}
\text{Flag}_{n_i}(V) \\
\downarrow \pi \\
X_k \xrightarrow{f} X
\end{array}
\]
the functors \(f^*, g^*, \pi^*, \pi_*\) are all exact so that flat base change applies, see 2.3. \(\square\)

Let \(\mathcal{F}\) be a parabolic vector bundle on \(X\) with parabolic datum as specified earlier given by
\[
n_i = (n_{i0} = \text{rk}(\mathcal{F}) \geq n_{i1} \geq \ldots \geq n_{ie_i} = 0).
\]
at the points \(p_i\) as previously specified.

Recall that in (2.7) we described the parabolic datum on \(\text{Hom}(\mathcal{F}, \mathcal{F})\). Let’s write down the Euler characteristic of this bundle under the hypothesis \(\bar{k} = k\). To simplify the statement it will be helpful to introduce the notation Flag_{\mathfrak{n}_i}(V) for the flag variety parameterising sequences of subspaces
\[
V_0 \supseteq V_1 \supseteq \ldots \supseteq V_{d_i} = 0
\]
of a fixed vector space $V$ with $\dim V = \dim V_0 = \text{rk}(\mathcal{F})$ such that $\dim V_i = n_i$.

**Proposition 3.4.** In the above setting, in particular $k = \bar{k}$, we have

$$\chi(\text{Hom}(\mathcal{F}, \mathcal{F})) = (1 - g) \text{rk}(\mathcal{F})^2 - \sum_{i=1}^l \dim \text{Flag}_{n_i}(\mathcal{F}|_{p_i}).$$

**Proof.** We have $\deg(\text{Hom}(\mathcal{F}, \mathcal{F})) = 0$ by the usual properties of chern classes and $\text{rk}(\text{Hom}(\mathcal{F}, \mathcal{F})) = \text{rk}(\mathcal{F})^2$. To simplify notation we may assume that $\mathcal{F}$ is ramified only at a single point $p$ with parabolic data $n_i$, the general result follows by an easy induction. By 2.7 and 3.3 it is enough to prove that

$$\frac{e - 1}{2} \sum_{d=0}^{e-1} \frac{d(m_d - m_{d+1})}{e} = \dim \text{Flag}_{n}(\text{rk}(\mathcal{F}|_p)),$$

where $m_d$ is defined in remark 2.7. To simplify notation, we write $\delta_i = n_i - n_{i+1}$. We get

$$\frac{e - 1}{2} \sum_{d=0}^{e-1} \frac{d(m_d - m_{d+1})}{e} = \frac{1}{2} \sum_{d=1}^{e-1} \left( \frac{d}{e} \sum_{i-j=d \mod e} \delta_i \delta_j + (e-d) \sum_{i-j=-d \mod e} \delta_i \delta_j \right)$$

$$= \frac{1}{2} \left( \sum_{d=1}^{e-1} \sum_{i-j=d \mod e} \delta_i \delta_j \right)$$

$$= \frac{1}{2} \sum_{i \neq j} \delta_i \delta_j$$

$$= \sum_{0 \leq i < j \leq e-1} (n_i - n_{i+1})(n_j - n_{j+1})$$

The dimension of the flag variety is

$$\dim \text{Flag}_{n}(\text{rk}(\mathcal{F}|_p)) = \sum_{i=1}^{e-1} n_i(n_{i-1} - n_i).$$

One checks that these two formulas agree, recall $n_e = 0$. \hfill $\square$

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are coherent sheaves over $X_K$, for a field $K \ni k$, we denote

$$\chi(\mathcal{E}_2, \mathcal{E}_1) := \dim_K \text{Hom}(\mathcal{E}_2, \mathcal{E}_1) - \dim_K \text{Ext}(\mathcal{E}_2, \mathcal{E}_1).$$

When $\mathcal{E}_1$ and $\mathcal{E}_2$ are vector bundles, $\chi(\mathcal{E}_2, \mathcal{E}_1)$ coincides with $\chi(\text{Hom}(\mathcal{E}_2, \mathcal{E}_1)).$

**Remark 3.5.** Recall, 2.2, that locally that the root stack is a quotient of $R[X]/X^e - s$ by $\mu_e$. The ring $R[X]/X^e - s$ is $\mathbb{Z}/e$-graded. An equivariant module over this module amounts to a graded module.

In our present situation of a root stack over a smooth curve, we see that a coherent sheaf $\mathcal{E}$ over $X_K$ can be written as a direct sum $\mathcal{F} \oplus \mathcal{F}$, where $\mathcal{F}$ is a vector bundle and $\mathcal{F}$ a torsion sheaf supported at finitely many points. We say
that \( n \) is the parabolic datum of \( E \) if \( n \) is the parabolic datum of \( F \), in particular \( \operatorname{rk}(E|_p) := \operatorname{rk}(F|_p) \) for a point \( p \) in \( X_K \).

**Lemma 3.6.** With the notation in 3.5 we have \( \chi(F, T) = -\chi(T, F) \) and \( \chi(T, T) = 0 \).

**Proof.** The question is local on the base so we may assume that our curve is \( \text{Spec}(R) \) where \( R \) is a DVR. Let \( t \) be a uniformizing parameter of \( R \). For modules over a DVR the result is true by elementary calculations. The result for the root stack follows from the previous remark by passage to graded modules. \( \square \)

**Lemma 3.7.** For a coherent sheaf \( E \) over \( X_K \),

\[
\chi(E, E) = (1 - g) r^2 - \sum_{i=1}^l \dim \operatorname{Flag}_{n_i}(E|_p_i)
\]

**Proof.** From 3.5 it follows that \( \chi(E, E) = \chi(F, F) + \chi(F, T) + \chi(T, F) + \chi(T, T) \). Now the lemma follows from 3.4 and 3.6. \( \square \)

### 4. Essential Dimension

Let \( \text{Fields}/k \) be the category of field extensions of \( k \). Consider a functor

\[
F : \text{Fields} \to \text{Sets}
\]

Given a field extension \( L/k \) and \( x \in F(L) \) we say that a subextension \( K \subseteq L \) is a field of definition of \( x \), and write \( x \sim K \) if there is an \( x' \in F(K) \) with \( F(i)(x') = x \) where \( i : K \hookrightarrow L \). We define the essential dimension of \( x \) by

\[
ed x = \inf_{x \sim K} \deg_k K,
\]

where the infimum is over all possible fields of definition. The essential dimension of \( F \) is defined to be

\[
ed F = \sup_{x \in F(L)} \inf_{L \in \text{Fields}_k} \deg_k K,
\]

Let \( p \) be a prime number. We will consider the following variation obtained by throwing away prime to \( p \) data, see [15]. In the above situation, we say that \( K \) is a \( p \)-field of definition of \( x \) and write \( x \sim_p K \) if there are inclusions in \( \text{Fields}/k \)

\[
K \longleftarrow K' \longleftarrow L
\]

with \( K'/L \) a finite extension of degree prime to \( p \) so that \( x' \) and \( x \) have the same image in \( F(K') \). The essential \( p \)-dimension of \( x \) is then defined by

\[
ed_p x = \inf_{x \sim_p K} \deg_k K,
\]

where the infimum is over all possible \( p \)-fields of definition. Finally, the essential \( p \)-dimension of \( F \) is defined to be

\[
ed_p F = \sup_{x \in F(L)} \inf_{L \in \text{Fields}_k} \deg_k K,
\]
An algebraic stack produces such a functor $F$ by considering isomorphism classes of objects, the essential dimension of which we refer to as the essential dimension of the stack.

**Example 4.1.** Consider a $\mathbb{G}_m$-gerbe $\mathcal{G}$ over a field $k$ of index $n = p_1^{a_1} \cdots p_\alpha^{a_\alpha}$ with $p_i$ prime. Then we have

$$\text{ed } \mathcal{G} \leq \sum_{i=1}^\alpha (p_i^{a_i} - 1),$$

and this is conjecturally an equality. It is known to be an equality when $\alpha = 1$ or $n = 6$. See [10], [15] and [12]. The situation for the essential $p$-dimension is simpler,

$$\text{ed } _p \mathcal{G} = v_p(\text{ind } \mathcal{G}) - 1.$$  

This is easily reduced to the previous case by remarking that prime to $p$-torsion in the Brauer group can be removed by passing to a prime to $p$ extension.

In this article we will be concerned with the essential ($p$-) dimension of $\text{Bun}_{r,d,n,X}$ where $X$ is a smooth projective curve and the parabolic points are closed points. We will recall some theorems from [5] that will be useful in our context.

For now, let's work in a slightly general context. Let $X$ be a projective scheme and choose a collection $D_1, D_2, \ldots, D_l$ of effective Cartier divisors and some positive integers $e_1, \ldots, e_l$ and form the corresponding root stack $q : X \to X$.

Consider a vector bundle $F$ on the root stack defined over some field $l$ containing $k$. Let $G(F)$ be the residual gerbe in $\text{Bun}_{r,d,n,X}$ of a parabolic bundles $F$. The coarse moduli space of this gerbe is called the field of moduli of $F$ and is denoted $k(F)$. There is a finite extension $L/k(F)$ so that $G(L) \neq \emptyset$, so we may find a parabolic vector bundle $F'$ that is a form of $F$ that is defined over $L$. Following [5], we consider $A := \text{End}(p_*F')$ where $p : X_L \to X_{k(F)}$ is the projection. This is just the algebra of the ordinary vector bundle underlying $F$ which preserves the parabolic structure.

One of the main results (stated for projective schemes) of [5] is:

**Theorem 4.2.** In the above situation, consider a field extension $K \supseteq k(F)$. Set $d = [L : k(F)]$. There is an equivalence of categories between the category of projective $A \otimes_{k(F)} K$-modules of rank $1/d$ and the groupoid $G(F)_K$.

**Proof.** As stated above, this is [5, 5.3], and we assert that the proof goes through in our more general context of root stacks. We describe here quickly the functors in each direction. To produce a module from a point of $G(F)_K$, say $\mathcal{E}$ consider the module

$$M = \text{Hom}(p_*(F) \otimes K, \mathcal{E}).$$

To see that $M$ is projective of the correct rank, consider a field $L$ containing $l$ and $K$, so that after base change to $L$, $\mathcal{E}$ and $F$ are isomorphic. Observe that

$$M \otimes_K L \cong \text{Hom}(p_*(F) \otimes K, \mathcal{E}) \otimes_K L \cong p_*\text{Hom}(p_*(F) \otimes L, \mathcal{E} \otimes L) \cong \text{Hom}(p_*(F) \otimes L, F \otimes L),$$

using [2.3]. It follows that $M$ is projective of the correct rank.
In the opposite direction, given a module $M$ over $A_L$, one considers the sheaf
\[ \pi_* \mathcal{F}_L \otimes_{A_L} M. \]
As per the argument in [5, 5.3] these functors give the required equivalence. \qed

Armed with this result, we are reduced to studying the essential dimension of
the functor of projective modules over a finite dimensional algebra. We recall here
some pertinent definitions and results from [5]. All proofs and further details can
be found there. We fix for now a finite dimensional (noncommutative) $k$-algebra
$A$. Let $j(A)$ be its Jacobson radical. Given a nonnegative rational number $r$, we
denote by $\text{Mod}_{A,r}$ the category of projective modules over $A$ of rank $r$. Recall that
$r$ is defined by $dr = m$ where $P$ is a projective module with $P^d = A^m$. The functor
\[ \text{Mod}_{A,r} : \text{Fields}_k \to \text{Sets} \]
that sends a field to isomorphism classes of projective $A \otimes_k K$-modules of rank $r$, is a determination functor, that is
\[ \text{Mod}_{A,r} = \begin{cases} \{\ast\} & \text{a singleton} \\ \emptyset. & \end{cases} \]

**Proposition 4.3.**
1. If $n$ is a nilpotent two-sided ideal of $A$. Then $\text{Mod}_{A,r} = \text{Mod}_{A/n,r}$.
2. If $A \cong B_1 \times B_2$ then
   \[ \text{Mod}_{B_1,r} \times \text{Mod}_{B_2,r} \cong \text{Mod}_{A,r}. \]
3. For coprime integers $n$ and $d$ we have
   \[ \text{Mod}_{A,1/d} \cong \text{Mod}_{A,n/d}. \]

**Proof.** See [5]. \qed

**Proposition 4.4.** Let $l/k$ be a prime to $p$ extension. Consider the functors
\[ \text{Mod}_{A,r} : \text{Fields}/k \to \text{Sets} \]
\[ \text{Mod}_{A_l,r} : \text{Fields}/l \to \text{Sets}. \]

The $\text{ed}_p(\text{Mod}_{A,r}) = \text{ed}_p(\text{Mod}_{A_l,r})$.

**Proof.** The inequality $\text{ed}_p(\text{Mod}_{A,r}) \geq \text{ed}_p(\text{Mod}_{A_l,r})$ is clear. Take $M \in \text{Mod}_{A,r}(L)$. If
\[ K \longrightarrow K' \]
\[ \downarrow \]
\[ L \]
is a $p$-field of definition for $M$ then
\[ lK \longrightarrow lK' \]
\[ \downarrow \]
\[ llL \]
is a $p$-field of definition for $M \otimes L lL$ where $lL$ is the compositum. Further,
\[ \text{trdeg}_l lK = \text{trdeg}_K K. \]

**Proposition 4.5.** Fix a prime $p$. Let $E$ be an indecomposable vector bundle on $X_K$ and suppose $X$ has a $k$-point. Then
\[ \dim_k \text{End}(E)/j(E) \leq \text{rk}(E) \quad \text{and} \quad v_p(\dim_k \text{End}(E)/j(E)) \leq v_p(\text{rk}(E)), \]
where $j(E)$ is the Jacobson radical of $\text{End}(E)$.

**Proof.** This is [5, Lemma 4.2] in the first case. In the second case, it suffice to observe that $\text{End}(E)/j(\text{End}(E))$ is a division ring and the fiber over the rational point is a module over it.

**Proposition 4.6.** Suppose that $E$ is vector bundle on $X$ of rank $r$. Then
\[ \text{ed}_{k(E)}(E) \leq r - 1 \quad \text{ed}_{k(E), p}(E) \leq v_p(r) - 1 \]

**Proof.** The first assertion is proved in [5, 5.5] so we concentrate on the second. By the arguments in loc. cit. we can assume that $K/k(E)$ is a finite extension of degree $d$. We write $\pi : X_K \to X_{k(E)}$ for the projection. As in loc. cit. we can decompose
\[ \pi_* \mathcal{E} \cong \bigoplus_i \mathcal{E}_i^{n_i} \]
where $\mathcal{E}_i$ is indecomposable and $\text{End}(\mathcal{E}_i)/j(\mathcal{E}_i) \cong D_i$ for some division ring. Further we have a decomposition
\[ \text{End}(\pi_* \mathcal{E})/j(\mathcal{E}) \cong \prod_i \text{Mat}_{n_i \times n_i}(D_i). \]

Each of the division rings have a primary decomposition in the Brauer group. For $q$ a prime, we write $D_{i, q}$ for the $q$-primary component. So the division ring decomposes as
\[ D_i \cong \bigotimes_i D_{i, q}. \]

Note that we have by 4.5 $\dim_k(E) D_{i, q} = v_q(\dim_k(E) D_i) \leq v_q(\text{rk}(E_i))$, the last inequality is by the prior proposition. We have
\[ \text{ed}_p(\text{Mod}_{\text{End}(\pi_* \mathcal{E}), 1/d}) \leq \sum \text{ed}_p(\text{Mat}_{n_i \times n_i}(D_i), 1/d) \quad \text{by 4.5} \]
\[ = \sum \text{ed}_p(\text{Mat}_{D_{i, n_i/d}}) \quad \text{by 4.3} \]
\[ = \sum \text{ed}_p(\text{Mat}_{D_{i, p, n_i/d}}) \quad \text{by 4.3} \]
\[ < \sum \frac{n_i}{d} v_p(\dim_k(E) D_i) \quad \text{by the above and 3.7 of loc. cit.} \]
\[ \leq \sum \frac{n_i}{d} v_p(\text{rk}(E_i)) \quad \text{by 4.5} \]

\[ \square \]
5. The field of moduli of a parabolic bundle

In this section we will consider sheaves \( E \) with a finite decreasing filtration
\[
E = F^0 E \subseteq F^1 E \subseteq \ldots \subseteq F^n E = 0.
\]
There is a shifted filtration given by
\[
F^n E(m) = F^{n+m} E.
\]
Given a sheaf \( E \) with such a filtration we can form the associated graded sheaf that is denoted by \( \text{gr}(E) \).

We can form a category of such filtered sheaves by considering morphisms that preserve the filtration. This is an additive category and we may form the associated derived category. We will be interested in a full subcategory of this derived category whose objects are bounded complexes with a finite filtration. Details of its construction and its properties can be found in \([11]\). This is known as the filtered derived category and will be denoted by \( D_{\text{filt}}^b(X) \). We will make use of the filtered extension groups \( \text{Ext}^n_{\text{filt}}(E, F) \) and their variants.

Let \( E \) be a filtered sheaf. We will say that \( E \) is \emph{filtered locally free} if each piece of the filtration is a coherent sheaf that is locally free. We record a few results that are natural generalisations of standard results in the non-filtered setting to the filtered setting pertaining to filtered locally free sheaves. We will make use of these results later.

The following proposition records an extension of a known result to the filtered case.

**Proposition 5.1.** Let \( E \) and \( F \) be filtered locally free sheaves. Then
\[
\text{Hom}_{\text{filt}}(E, F) \cong E^\vee \otimes F
\]
and
\[
\text{Ext}^1_{\text{filt}}(E, F) \cong \text{Ext}^1_{\text{filt}}(\text{Hom}(F, E), \mathcal{O})
\]

**Proof.** This first result is a standard extension of the known result to the filtered case. It follows immediately from definitions.

The second assertion is a standard extension of the known result to the filtered case. It follows immediately from definitions.

The result is now a corollary of the adjunction \([11] \text{ Ch. V, 2.3.1.5}\) which implies
\[
R\text{Hom}(E \otimes F^\vee, \mathcal{O}) \cong R\text{Hom}(E, F)
\]

\[\square\]

**Proposition 5.2.** Let \( \theta : F \to E \) be a filtered morphism of filtered locally free sheaves. The following diagram commutes:
\[
\begin{array}{ccc}
\text{Ext}^1_{\text{filt}}(\text{Hom}(F, E), \mathcal{O}) & \xrightarrow{\theta^*} & \text{Ext}^1_{\text{filt}}(\text{Hom}(E, E), \mathcal{O}) \\
\downarrow & & \downarrow \\
\text{Ext}^1_{\text{filt}}(E, F) & \xrightarrow{\theta^*} & \text{Ext}^1_{\text{filt}}(E, E)
\end{array}
\]

**Proof.** This is a standard diagram chase. \[\square\]
Remark 5.3. One way of obtaining a filtered sheaf is by starting with a sheaf $E$ equipped with a nilpotent morphism $\theta$. We define $F^iE = \text{Im} \theta^i$. With this filtration we obtain a filtered morphism

$$\theta : E \to E(1).$$

Proposition 5.4. Let $X$ be a smooth projective curve. Let $E$ be a vector bundle on $X$ and let $\theta : E \to E(1)$ be as in the previous remark. Then the morphism

$$\text{Ext}^1(\text{gr} E_{\geq 0}, \text{gr} E_{\geq 0}) \to \text{Ext}^1(\text{gr} E_{1\geq 0}, \text{gr} E(1)_{\geq 0})$$

given by $\theta_* - \theta^*$ is surjective.

The symbol $\text{gr} E_{\geq 0}$ refers to the graded sub-object of $\text{gr} E$ consisting of graded pieces in nonnegative degrees.

Proof. We induct on the integer $n$ that is minimal so that $\theta^n = 0$.

When $n = 0$ the statement is trivial as $\text{gr} E(1)_{\geq 0}$ vanishes.

Now we inducted. The induction hypothesis applies to $\theta|_{F^1E}$ so we can assume the results for it. Now the terms of $\text{Ext}^1(\text{gr} E_{\geq 0}, \text{gr} E_{\geq 0})$ that are not terms of $\text{Ext}^1(\text{gr} F^1E_{\geq 0}, \text{gr} F^1E_{\geq 0})$ are of the form

$$\text{Ext}^1(F^0(E)/F^1(E), F^i(E)/F^{i+1}(E)) \quad i \geq 0$$

and those of $\text{Ext}^1(\text{gr} E_{\geq 0}, \text{gr} E(1)_{\geq 0})$ that are new are of the form

$$\text{Ext}^1(F^0(E)/F^1(E), F^i(E)/F^{i+1}(E)) \quad i \geq 1.$$

We have a diagram

$$\begin{array}{ccc}
\text{Ext}^1(F^0(E)/F^1(E), F^i(E)/F^{i+1}(E)) & \to & \text{Ext}^1(F^0(E)/F^1(E), F^{i+1}(E)/F^{i+2}(E)) \\
& & \uparrow \\
& & \text{Ext}^1(F^1(E)/F^2(E), F^{i+1}(E)/F^{i+2}(E))
\end{array}$$

By induction we have chosen classes in $\text{Ext}^1(F^1(E)/F^2(E), F^{i+1}(E)/F^{i+2}(E))$. To finish the proof it suffices to observe that we have a short exact sequence

$$0 \to \text{kernel} \to F^i(E)/F^{i+1}(E) \xrightarrow{\delta} F^{i+1}(E)/F^{i+2}(E) \to 0.$$

The result now follows from the associated long exact sequence and by observing that we are on a curve.

□

In this section we preserve our notation from the start of the previous section. In particular,

$$\mathcal{X} = X_{(p_1, e_1), \ldots, (p_t, e_t)}$$

is a root stack construction. Corresponding to this there are root line bundles (see [8]) written $N_i$ on the root stack $\mathcal{X}$.

Let $\mathcal{N}_{\mathcal{X}}$ denote the stack of nilpotent coherent sheaves on $\mathcal{X}$ defined as follows. For a $k$-scheme $S$,
• Objects of \( \mathcal{N}I_{n,X}(S) \) are pairs \((\mathcal{E}, \phi)\) where \(\mathcal{E}\) is a coherent sheaf on \(X\) and \(\phi\) is a nilpotent endomorphism of \(\mathcal{E}\). Each of the sheaves \(\text{coker } \phi^i\) are assumed to be flat over \(S\).

• A morphism between \((\mathcal{E}, \phi)\) and \((\mathcal{F}, \psi)\) is an isomorphism of sheaves \(\alpha : \mathcal{E} \to \mathcal{F}\) with \(\alpha\phi = \psi\alpha\).

Our goal in this section is to prove that \(\mathcal{N}I_{n,X}\) is a smooth stack and find its dimension at a given \(K\)-point \((E, \phi)\) for a field \(K \supset k\). We give a proof that is different to the one in [5]. The proof that we give is based upon deformation theory arguments rather than the global decomposition in loc. cit.

Consider a filtered vector bundle \(\mathcal{E}\) on \(X\). The obstruction to deforming \(\mathcal{E}\) so that \(\mathcal{E}/F^1\mathcal{E}\) are flat over the base ring is controlled by \(F^0R\text{Hom}_{\text{filt}}(E, \mathcal{E})\), see [13]. This fact is stated there without proof but the proof is a routine generalisation of the fact that \(R\text{Hom}(E, \mathcal{E})\) controls the deformation theory of the ordinary vector bundle \(E\), see for example [11].

**Theorem 5.5.** In the above situation the stack \(\mathcal{N}I_{n,X}\) is smooth.

**Proof.** Consider a square zero extension of local Artin \(k\)-algebras

\[0 \to I \to B \to A \to 0.\]

Suppose that \((\mathcal{E}, \theta)\) is an \(A\)-point of \(\mathcal{N}I_{X,n}\). Recall that the pair \((\mathcal{E}, \theta)\) produces a filtered object \(\mathcal{E}\) that we also denote by \(\mathcal{E}\). The obstruction to deforming the filtered object lies inside

\[H^2(F^0R\text{Hom}_{\text{filt}}(E, \mathcal{E})) \otimes I = \text{Ext}^2_{\text{filt}}(\mathcal{E}, \mathcal{E}) \otimes I.\]

To see that it vanishes it suffices to show that \(\text{gr Ext}^2_{\text{filt}}(\mathcal{E}, \mathcal{E})\) vanishes but

\[\text{gr Ext}^2_{\text{filt}}(\mathcal{E}, \mathcal{E}) = \text{Ext}^2_{\text{filt}}(\text{gr } E, \text{gr } E),\]

by [11]. This last group vanishes as \(\dim X = 1\). The obstruction to lifting the nilpotent and morphism \(\theta\) amounts to the vanishing of the class of the extension under

\[\theta_- - \theta^* : \text{Ext}^2_{\text{filt}}(\mathcal{E}, \mathcal{E}) \otimes I \to \text{Ext}^2_{\text{filt}}(\mathcal{E}, \mathcal{E}(1)) \otimes I.\]

But this map is surjective by the previous proposition.

Consider the two term filtered complex

\[\theta_- - \theta^* : \text{Hom}(\mathcal{E}(1), \mathcal{E}) \to \text{Hom}(\mathcal{E}, \mathcal{E})\]

concentrated in degrees -1 and 0. We will denote this complex by \(P(E, \theta)\). We refer the reader to [9] for a description of the groupoid \(\text{Ext}(P(E, \theta), \mathcal{O}_X)\).

We may also view \((E, \theta)\) as a \(k\)-point of the stack \(\mathcal{N}I_{n,X}\). We denote the associated tangent groupoid by \(T_{\mathcal{N}I_{n,X}}(E, \theta)\). It is a sub-groupoid of \(\mathcal{N}I_{n,X}(k[\epsilon])\).

**Proposition 5.6.** There is an equivalence \(\text{Ext}(P(E, \theta), \mathcal{O}_X) \cong T_{\mathcal{N}I_{n,X}}(E, \theta)\).

**Proof.** We modify the proof in [9, §10] for our needs. Observe that given an object of the tangent groupoid we obtain a filtered extension, ie an element of \(\text{Ext}^1_{\text{filt}}(E, E)\). The class vanishes under \(\theta^* - \theta_-\) and hence produces an object of the category \(\text{Ext}(P(E, \theta), \mathcal{O}_X)\). This extends to a fully faithful functor.

To see that it is essentially surjective we need to verify that a flat extension amounts to an extension class over \(\mathcal{O}_X\) rather than \(\mathcal{O}_X[\epsilon]\). This is standard, see [11, IV], particularly page 248. Note that in the notation of this work, in order for
the lift to be flat we must have $J = E$ and $u = 1$, this is the local criterion for flatness.

\[ \text{Theorem 5.7. The stack } \mathcal{N}_n,X \text{ is smooth over } k. \text{ Its dimension at the } K\text{-valued point given by a coherent sheaf } E \text{ on } X_K \text{ and } \phi \in \text{End}(E) \text{ with } \phi^n = 0 \text{ is} \]

\[ \dim_{(E,\phi)}(\mathcal{N}_n,X) = (g - 1) \sum_{i=1}^n r_i^2 + \sum_{i=1}^n \sum_j \dim_K \text{Flag}_{n_j} \left( \frac{\text{Im} \phi^{i-1}}{\text{Im} \phi^i} |_{p_j} \right), \]

where $r_i$ denotes the rank of the coherent sheaf $\text{Im}(\phi^{i-1})/\text{Im}(\phi^i)$ and $n_j^{(i)}$ the parabolic datum of $\text{Im}(\phi^{i-1})/\text{Im}(\phi^i)$ at $p_j$.

\[ \text{Proof. By the above, we have that the dimension is given by } \chi(E,E) - \chi(E,E(1)). \text{ We may pass to associated graded objects as we are only interested in dimensions. Further the cross terms cancel and we are reduced to 3.3} \]

\[ \text{Lemma 5.8. Let } \mathcal{C} \text{ be the closure of a point by } E \text{ in } \mathcal{B}_{n,\mathbf{r},d}^{\mathcal{E}}, \text{ then} \]

\[ \dim_k \mathcal{C} = \text{trdeg}_k(k(E)) - \dim_K \text{End}(E) \]

\[ \text{Proof. The stack } \mathcal{B}_{n,\mathbf{r},d}^{\mathcal{E}} \text{ is locally a quotient stack as it is for ordinary vector bundles. One way to prove this is to observe that the map forgetting the parabolic structure is representable in flag varieties. Hence, we may assume } \mathcal{C} \text{ to be a quotient stack } [U/H] \text{ for some scheme } U \text{ and some algebraic group } H. \text{ Let } \mathcal{G} \hookrightarrow \mathcal{C} \text{ be the residual gerbe. We have the following Cartesian square} \]

\[ \begin{array}{ccc} R & \longrightarrow & U \\ \downarrow H & & \downarrow H \\ \mathcal{G} & \longrightarrow & \mathcal{C} \end{array} \]

We have $\dim_k(U) - \dim_k(\mathcal{C}) = \dim_k H$ and $\dim_k(\mathcal{G}) R - \dim_k(\mathcal{G}) \mathcal{G} = \dim_k H$. Combining them with the equations $\dim_k R = \dim_k U$ (since $R$ is an open dense subscheme of $U$) and $\dim_k U = \text{trdeg}_k(k(U)) = \text{trdeg}_k(k(\mathcal{G})) + \dim_k(\mathcal{G}) U$ we get the required formula.

\[ \text{Remark 5.9. The category of parabolic vector bundles satisfies the bichain conditions in [1]. In particular by Lemma 6 in loc. cit., every endomorphism of an indecomposable bundle is either nilpotent or an automorphism.} \]

\[ \text{Corollary 5.10. Assume that } K \text{ is algebraically closed and that } \mathcal{E} \text{ be an indecomposable vector bundle with parabolic datum } \mathbf{n} \text{ on } X_K. \text{ Take } \phi \text{ to be a general element } \phi \text{ of the Jacobson radical } j(\mathcal{E}). \text{ Let } r_i \text{ be the rank of } \text{Im}(\phi^{i-1})/\text{Im}(\phi^i). \text{ Then} \]

\[ \text{trdeg}_k(k(\mathcal{E})) \leq 1 + (g - 1) \sum_i r_i^2 + \sum_j \dim_K \text{Flag}_n(\mathcal{E}|_{p_j} \right) \]
Proof. By the above remark we have \( \text{End}_X(E)/j(E) = K \), notice that \( K \) is algebraically closed.

Let \( \mathcal{C} \) be the closure of a point given by \( E \) in \( \mathcal{C}h_{X_K} \). By the previous lemma,

\[
\dim_k \mathcal{C} = \text{trdeg}_k \kappa(E) - \dim_K \text{End}_X(E)
\]

We can find a natural number so that \((j(E))^n = 0\). Let \( \mathcal{N} \subset \mathcal{N}_{j,0} \) be the closure of the points \((E, \phi)\) with \( \phi \in j(E) \) such that each of the sheaves \( \text{Im}(\phi^i)/\text{Im}(\phi^j) \) has rank \( r_i \).

There is a forgetful morphism \( \mathcal{N} \to \mathcal{C} \) whose generic fiber an open dense subscheme of \( j(E) \). So we have,

\[
\dim_k \mathcal{N} \geq \dim_k \mathcal{C} + \dim_K j(E) = \dim_k \mathcal{C} + \dim_K \text{End}_X(E) - 1 = \text{trdeg}_k \kappa(E) - 1
\]

From the previous theorem, 5.5, and [4, 11.1] we have,

\[
\text{trdeg}_k \kappa(E) \leq 1 + (g - 1) \sum_i r_i^2 + \sum_j \dim_K \text{Flag}_{n_j}(E|_{p_j})
\]

\( \square \)

Remark 5.11. We have that \( \kappa(E) = \kappa(E \otimes_K L) \) for any field \( L \supset K \).

Lemma 5.12. We assume that \( g(X) \geq 2 \). Let \( E \) be a vector bundle of rank \( r \), degree \( d \) and parabolic datum \( n \) over \( X_K \). If \( E \) is not simple, in other words \( E \) has an endomorphism that is not a scalar, then

\[
\text{trdeg}_k \kappa(E) \leq (g - 1)(r^2 - r) + 2 + \sum_i \dim_K \text{Flag}_{n_i}(E|_{p_i})
\]

Proof. By the above remark we may assume \( K \) is algebraically closed. Hence by Krull-Schmidt \( E \) can be written as a direct sum of indecomposable vector bundles \( E_\alpha \) over \( X_K \) of rank \( r_\alpha \geq 1 \) and parabolic data \( n^{(\alpha)} \) such that \( n^{(\alpha)} = (n^{(\alpha)}_1 \cdots n^{(\alpha)}_i) \), where \( n^{(\alpha)}_j \) is the parabolic datum at the point \( p_j \). The above corollary says that

\[
\text{trdeg}_k \kappa(E_\alpha) \leq 1 + (g - 1) \sum_i r_{i,\alpha}^2 + \sum_j \dim_K \text{Flag}_{n^{(\alpha)}_j}(E_\alpha|_{p_j})
\]

for some integers \( r_{i,\alpha} \geq 1 \) such that \( \sum_i r_{i,\alpha} = r_\alpha \). We also have that

\[
\sum_\alpha \sum_j \dim_K \text{Flag}_{n^{(\alpha)}_j}(E_\alpha|_{p_j}) \leq \sum_j \dim_K \text{Flag}_{n_j}(E|_{p_j}),
\]

see [4, 11.1]. Using

\[
\text{trdeg}_k \kappa(E) \leq \sum_j \text{trdeg}_k \kappa(E_j)
\]

we have

\[
\text{trdeg}_k \kappa(E) \leq \sum_j 1 + (g - 1) \sum_{i,\alpha} r_{i,\alpha}^2 + \sum_j \dim_K \text{Flag}_{n_j}(E|_{p_j})
\]

Note that the sum \( \sum_{i,\alpha} r_{i,\alpha} = r \) has at least two terms, (cf [4, 6.5]). Hence

\[
\text{trdeg}_k \kappa(E) \leq (g - 1)(r^2 - r) + \sum_j \dim_K \text{Flag}_{n_j}(E|_{p_j}) + 2 - (g - 2)(r - 2)
\]
\[ r \geq 2 \text{ and } g \geq 2 \implies (g - 2)(r - 2) \geq 0. \] So,
\[
\deg_k k(E) \leq (g - 1)(r^2 - r) + \sum_j \dim K \Flag_{n_j}(E|_{p_j}) + 2
\]

6. Essential dimension on curves

In this section \( X \) is a smooth projective, geometrically connected curve over \( k \) with \( X(k) \neq \emptyset \). We fix some closed points \( p_1, \ldots, p_l \) with \( \deg p_i = f_i \). We will consider vector bundles of rank \( r \) and degree \( d \) and ramification indices \( e_i \) at each \( p_i \). We fix parabolic data \( r = n_{i0} \geq n_{i1} \geq \ldots \geq n_{ie_i} = 0 \) at each \( p_i \). Let \( n = (n_1, \ldots, n_l) \) be the corresponding parabolic datum. The arguments in this section are modeled by those in [5].

A parabolic vector bundle \( \mathcal{F} \) on \( X \) is said to be \textit{simple} if \( \End(\mathcal{F}) = K \). If \( \mathcal{F} \) is simple then the morphism from the residual gerbe to its moduli space
\[
\mathcal{G}(\mathcal{F}) \to \text{Spec}(k(\mathcal{F}))
\]
is banded by \( G_m \). As a generic parabolic bundle is simple this is in fact the generic gerbe of the moduli stack. We wish to understand the index of this gerbe.

**Proposition 6.1.** In the above situation, set \( h = \gcd(r, d, n_{ij}) \). Then \( h = \ind(\mathcal{G}(\mathcal{F})) \) where \( \mathcal{G}(\mathcal{F}) \) is the generic gerbe.

**Proof.** The index of this gerbe divides \( h = \gcd(r, d, n_{ij}) \). To see this, we can construct twisted sheaves of ranks \( r \) and \( n_{ij} \) on \( \mathbb{B}un^{r, d}_n \times X \) and then pull them back to the gerbe using the fact that \( X(k) \neq \emptyset \). By [14] we see that the index divides \( r \) and the \( n_{ij} \). We have another twisted sheaf obtained by taking \( \pi_\ast \) of a sufficiently ample twist of the universal bundle. Its rank is computed by Riemann-Roch, see [3.2], and hence the index divides \( h \).

Now choose a particular simple parabolic bundle \( \mathcal{F}_0 \). We can consider the moduli stack of \( \mathbb{B}un^{r, d}_n, \det(\mathcal{F}_0) \) of parabolic bundles where the underlying bundle has determinant \( \det(\mathcal{F}) \). The stack \( \mathbb{B}an^{r, d}_n \) is a Grassman bundle over the moduli stack of ordinary vector bundles and hence by [16, Theorem 6.1] and [2] the generic gerbe of \( \mathbb{B}an^{r, d}_n, \det(\mathcal{F}_0) \) has index \( h \). As the index can only drop by base change, it follows that our original gerbe had index \( h \). In the case of a ground field of characteristic \( 0 \), one could apply the main theorem of [3]. \( \square \)

**Proposition 6.2.** Let \( \mathcal{F} \) be a simple parabolic vector bundle with rank \( r \), degree \( d \) and specified parabolic data. Then
\[
ed_k \mathcal{F} \leq r^2(g - 1) + 1 + \sum_{i=1}^l f_i \dim k(p_i) \Flag_{n_i} + \sum_{p/h} p^n p(h) - 1
\]
and
\[
ed_k, p \mathcal{F} = r^2(g - 1) + 1 + \sum_{i=1}^l f_i \dim k(p_i) \Flag_{n_i} + p^n p(h) - 1.
\]

**Proof.** One combines the above proposition with [5.10] and [4.1]. \( \square \)
**Theorem 6.3.** Set $h = \gcd(r,d,n_{ij})$. We have

$$\text{ed } \text{Bun}^{r,d}_{\mathbf{n}} \leq r^2(g-1) + 1 + \sum_{i=1}^{l} f_i \dim_{k(p_i)} \text{Flag}_{\mathbf{n}_i} + \sum_{p|h} p^{\nu_p(h)} - 1$$

and Further,

$$\text{ed}_p \text{Bun}^{r,d}_{\mathbf{n}} = r^2(g-1) + 1 + \sum_{i=1}^{l} f_i \dim_{k(p_i)} \text{Flag}_{\mathbf{n}_i} + p^{\nu_p(h)} - 1.$$ 

When the main conjecture of [10] holds for $r$ then the first inequality is an equality.

**Proof.** Using [4.6] and its proof that $\text{ed}_{k(F)} F \leq r - 1$, for every parabolic bundle.

The case where $F$ is not simple we can combine this remark with [5.12] to obtain the result inequalities in the assertions of the theorem.

The case of a simple bundle is the prior proposition.

The conjecture of [10] relates the essential dimension of a gerbe to its index so that the equality is a consequence of [6.1]. \qed

**References**

[1] Atiyah, M. On the Krull-Schmidt theorem with application to sheaves. *Bull. Soc. Math. France* 84 (1956), 307–317.

[2] Balaji, Vikraman and Biswas, Indranil and Gabber, Ofer and Nagaraj, Donihakkalu S. Brauer obstruction for a universal vector bundle. *C. R. Math. Acad. Sci. Paris* 345 (2007), no. 5, 265–268.

[3] Biswas, Indranil; Dey, Arijit. Brauer group of a moduli space of parabolic vector bundles over a curve. *J. K-Theory* 8 (2011), no. 3, 437–449.

[4] Biswas, Indranil and Dhillon, Ajneet and Lemire, Nicole. The essential dimension of stacks of parabolic vector bundles over curves. *J. K-Theory* 10 (2012), no. 3, 455–488.

[5] Biswas, Indranil and Dhillon, Ajneet and Hoffmann, Norbert, On the essential dimension of coherent sheaves, *J. Reine Angew. Math.* 735 (2018).

[6] Borne, Niels. Fibrés paraboliques et champ des racines. *Int. Math. Res. Not.* (2007).

[7] Borne, Niels and Vistoli, Angelo. Parabolic sheaves on logarithmic schemes. *Adv. Math.* 231 (2012), no. 3-4.

[8] Cadman, Charles. Using stacks to impose tangency conditions on curves *Amer. J. Math.* 129 (2007).

[9] Casalaina-Martin, Sebastian and Wise, Jonathan. An introduction to moduli stacks, with a view towards Higgs bundles on algebraic curves [arXiv:1708.08124]

[10] Colliot-Thélène, J.L and Karpenko, N. and Merkurjev, A. Rational surfaces and the canonical dimension of the group PGL_n. *Algebra i Analiz*, 19 (2007).

[11] Illusie, L. Complexe cotangent et déformations. I. Lecture Notes in Mathematics, 239, (1971).

[12] Karpenko, Nikita. On anisotropy of orthogonal involutions, *J. Ramanujan Math. Soc.*, 15, (2000).

[13] Laumon, Gérard and Moret-Bailly, Laurent. Champs algébriques. (French) Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.

[14] Lieblich, Max. Twisted sheaves and the period-index problem. *Compos. Math.* 144 (2008), no. 1, 1–31.

[15] Merkurjev, Alexander. Essential dimension: a survey. *Transform. Groups*, 18, 2013.

[16] Hoffmann, Norbert. Rationality and Poincaré families for vector bundles with extra structure on a curve. *Int. Math. Res. Not.* 2007, no. 3.

[17] Reichstein, Zinovy. Essential dimension, *Proceedings of the International Congress of Mathematicians, Volume II*, 2010.
[18] Olsson, Martin and Starr, Jason, Quot functors for Deligne-Mumford stacks *Comm. Algebra*, 31 (2003)

[19] Toen, Bertrand, K-theory and cohomology of algebraic stacks: Riemann-Roch theorems, D-modules and GAGA theorems [arXiv:math/9908097]

[20] Toen, Bertrand. Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford. *K-Theory* 18 (1999), no. 1.

Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada
E-mail address: adhill3@uwo.ca

Department of Computer Science, University of Western Ontario, London, Ontario N6A 5B7, Canada
E-mail address: dvalluri@uwo.ca