\textbf{Abstract.} We show that the $q$-Heun equation and its variants appear in the linear $q$-difference equations associated to some $q$-Painlevé equations by considering the blow-up associated to their initial-value spaces. We obtain the firstly degenerated Ruijsenaars-van Diejen operator from the linear $q$-difference equation associated to the $q$-Painlevé equation of type $E_8$. 

1. Introduction

The Painlevé transcendent is a transcendental solution of one of the six Painlevé equations and it is potentially a member of prospective special functions. The sixth Painlevé equation is a non-linear generalization of the hypergeometric equation, and the hypergeometric equation is a standard form of the second order Fuchsian differential equation with three singularities \{0, 1, \infty\}. Heun’s differential equation is a standard form of the second order Fuchsian differential equation with four singularities \{0, 1, t, \infty\}, and it is written as

\[
\frac{d^2 y}{dx^2} + \left( \gamma + \frac{\delta}{x-1} + \frac{\epsilon}{x-t} \right) \frac{dy}{dx} + \frac{\alpha \beta x - B}{x(x-1)(x-t)^2} y = 0,
\]

under the condition $\gamma + \delta + \epsilon = \alpha + \beta + 1$. The parameter $B$, which is independent from the local exponents, is called the accessory parameter. It is known that Heun’s differential equation is related with the sixth Painlevé equation (see [2], [12] and references therein).

A $q$-difference analogue of Heun’s differential equation was given by Hahn [3], and we may write it as

\[
(1.1) \quad (x - h_1 q^{1/2})(x - h_2 q^{1/2})g(x/q) + l_3 l_4(x - l_1 q^{-1/2})(x - l_2 q^{-1/2})g(qx) - \{(l_3 + l_4)x^2 + Ex + (l_1 l_2 l_3 l_4 h_1 h_2)^{1/2}(h_3^{1/2} + h_3^{-1/2})\}g(x) = 0.
\]

Note that the $q$-Heun equation was used to introduced the variants of the $q$-hypergeometric equation [4]. Several discrete Painlevé equations had been studied in the 1990’s. Sakai [10] proposed a list of the second order discrete Painlevé equations. In Sakai’s list, each member of the $q$-difference Painlevé equation was labelled by the affine root system from the symmetry. A relationship between the $q$-Heun equation and the $q$-Painlevé equation of type $D_5^{(1)}$ ($q$-$P(D_5^{(1)})$) was pointed out in [13], and it was called the $q$-Painlevé-Heun correspondence.
In this paper, we establish a further relationship between the \(q\)-Heun equation and the \(q\)-Painlevé equation by using the initial-value space of the \(q\)-Painlevé equation. Namely we derive the \(q\)-Heun equation by considering the Lax pair and the initial-value space.

Jimbo and Sakai [5] introduced a \(q\)-analogue of the sixth Painlevé equation by considering the connection preserving deformation of certain \(q\)-linear difference equations. It is essentially equivalent to a compatibility condition of a Lax pair, and the Lax pair, which was reformulated by Kajiwara, Noumi and Yamada [6], is written as

\[
L_1 = \left\{ \frac{z(q\nu_1 - 1)(q\nu_2 - 1)}{qg} - \frac{\nu_1\nu_2\nu_3\nu_4(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}{fg} \right\} + \frac{\nu_1\nu_2(z - q\nu_5)(z - q\nu_4)}{q(qf - z)}(g - T^{-1}_z) + \frac{(z - \nu_5/\nu_7)(z - \nu_1/\nu_8)}{q(f - z)}(\frac{1}{g} - T_z),
\]

\[
L_2 = \left( 1 - \frac{f}{z} \right) T + T_z - \frac{1}{g}.
\]

Here \(T_z\) represents the transformation \(z \to qz\) and \(T\) represents the time evolution. The parameters are constrained by the relation \(\kappa_1^2\kappa_2^2 = q\nu_1\nu_2 \ldots \nu_8\). We use the notation such as \(T(f) = \mathcal{T}\) and \(T^{-1}(g) = g\). The time evolution of the parameters is given by \(\mathcal{T}_1 = \kappa_1/q\), \(\mathcal{T}_2 = q\kappa_2\) and \(\mathcal{T}_i = \nu_i \ (i = 1, 2, \ldots, 8)\). It follows from the compatibility condition for the Lax operators \(L_1\) and \(L_2\) (see [6]) that the parameters \(f\) and \(g\) satisfies

\[
f\mathcal{T} = \nu_3\nu_4 \frac{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}{(g - 1/\nu_1)(g - 1/\nu_2)}, \quad gg = \frac{1}{\nu_1\nu_2} \frac{(f - \kappa_1/\nu_7)(f - \kappa_1/\nu_8)}{(f - \nu_3)(f - \nu_4)}.
\]

This is the \(q\)-Painlevé equation essentially introduced by Jimbo and Sakai [5], which we denote by \(q\)-\(P(D_5^{(1)})\). The time evolution of the variable \(f\) is given by

\[
(1.2) \quad \mathcal{T} = \nu_3\nu_4 \frac{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}{f(g - 1/\nu_1)(g - 1/\nu_2)}
\]

and the value \(\mathcal{T}\) is indefinite in the case \((f, g) = (0, \nu_5/\kappa_2)\) because of the form \(0/0\) in the right hand side of Eq. \((1.2)\). To resolve the indefiniteness, we consider the blow-up at \((f, g) = (0, \nu_5/\kappa_2)\). Namely we set \((f, g) = (f_1, f_1g_1 + \nu_5/\kappa_2)\). Then the point \((f, g) = (0, \nu_5/\kappa_2)\) corresponds to the line \(f_1 = 0\), and we have

\[
\mathcal{T} = \frac{\nu_3\nu_4g_1(f_1g_1 + (\nu_5 - \nu_6)/\kappa_2)}{(f_1g_1 + \nu_5/\kappa_2 - 1/\nu_1)(f_1g_1 + \nu_5/\kappa_2 - 1/\nu_2)}.
\]

Hence the indefiniteness is resolved. The indefiniteness may occur at the following eight points;

\[
P_i : (\infty, 1/\nu_i)_{i=1,2}, \quad (\nu_i, \infty)_{i=3,4}, \quad (0, \nu_i/\kappa_2)_{i=5,6}, \quad (\kappa_1/\nu_i, 0)_{i=7,8}.
\]

The space of initial values of \(q\)-\(P(D_5^{(1)})\) is defined by blowing up eight points \(P_1, \ldots, P_8\) of \(\mathbb{P}^1 \times \mathbb{P}^1\).

We now consider the blow-up at \(P_3 (0, \nu_5/\kappa_2)\) jointly with the \(q\)-difference equation \(L_1y(z) = 0\), where \(L_1\) is one of the Lax pair. Note that the equation \(L_1y(z) = 0\)
itself is different from the $q$-Heun equation in Eq. (1.1). We substitute $(f, g) = (f_1, f_1 g_1 + \nu_5/\kappa_2)$ into the equation $L_1 y(z) = 0$ and set $f_1 = 0$. Then we have

$$(z - q\nu_5)(z - q\nu_4)y(z/q) + \frac{1}{\nu_1 \nu_2} \left( z - \frac{\kappa_1}{\nu_7} \right) \left( z - \frac{\kappa_1}{\nu_8} \right) y(qz)$$

$- \left[ \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} \right) z^2 + \left\{ qg_1 \nu_3 \nu_4 \left( 1 - \frac{\nu_6}{\nu_5} \right) - \frac{q\nu_5(\nu_3 + \nu_4)}{\kappa_2} - \frac{\kappa_1 \kappa_2 (\nu_7 + \nu_8)}{\nu_1 \nu_2 \nu_3 \nu_7 \nu_8} \right\} z \right.$

$+ \left. \frac{q\kappa_1 \nu_3^{1/2} \nu_4^{1/2}}{\nu_1^{1/2} \nu_2^{1/2} \nu_7^{1/2} \nu_8^{1/2}} z \right\} y(z/q) = 0$

in $\mathbb{Q}[\mathcal{A}]$. Thus we obtain the $q$-Heun equation with the accessory parameter by considering the equation $L_1 y(z) = 0$ on the line $f_1 = 0$, which is an exceptional curve with respect to the blow-up at the point $P_5$ on the initial value space. We can also obtain the $q$-Heun equation by the blow-up at $P_i$ ($i = 1, \ldots, 8$).

We establish similar results on the $q$-Painlevé equation of type $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ in this paper. The corresponding linear $q$-difference equations for the cases $E_6^{(1)}$ and $E_7^{(1)}$ are the variants of $q$-Heun equation [14] (see Eqs. (2.23), (2.26)). Note that $q$-Heun equation and the variants are also obtained by degenerations of Ruijsenaars-van Diejen system [9] [15] [13]. The corresponding linear $q$-difference equations for the case $E_8^{(1)}$ is the firstly degenerated Ruijsenaars-van Diejen operator. Note that Noumi, Ruijsenaars and Yamada [8] obtained the non-degenerate Ruijsenaars-van Diejen system from the Lax formalism of the elliptic Painlevé equation $e-P(E_8^{(1)})$.

This paper is organized as follows. In section 2 we recall the $q$-Painlevé equations $q-P(D_5^{(1)})$, $q-P(E_6^{(1)})$ and $q-P(E_7^{(1)})$, their Lax pairs, and the corresponding initial-value spaces by following the review by Kajiwara, Noumi and Yamada [6]. We consider restriction of the variables of initial-value spaces and obtain the $q$-Heun equation and the variants. In section 3 we recall the $q$-Painlevé equation of type $E_8^{(1)}$ by following Yamada [10], and obtain the firstly degenerated Ruijsenaars-van Diejen operator by considering a suitable limit.

2. $q$-Painlevé Equation, Lax Pairs, Initial-Value Space of $q$-Painlevé Equation and $q$-Heun Equation

Lax pairs and the initial-value spaces for discrete Painlevé equations were reviewed by Kajiwara, Noumi and Yamada in [6]. We recall them for the $q$-Painlevé equations of type $D_5^{(1)}$, $E_6^{(1)}$ and $E_7^{(1)}$, and we apply the procedure of the blow-up associated with the the initial-value spaces to the linear $q$-difference equation related to the Lax pairs.

Throughout this section, we assume that the parameters $\kappa_1, \kappa_2, \nu_1, \nu_2, \ldots, \nu_8$ are non-zero and satisfy the relation $\kappa_1^2 \kappa_2^2 = q\nu_1 \nu_2 \ldots \nu_8$.

2.1. The case $D_5^{(1)}$.

The $q$-Painlevé equation of type $D_5^{(1)}$ ($q-P(D_5^{(1)})$) was given as

$$f \bar{f} = \nu_3 \nu_4 \frac{(g - \nu_5/\kappa_2)(g - \nu_6/\kappa_2)}{(g - 1/\nu_1)(g - 1/\nu_2)}, \quad gg = \frac{1}{\nu_1 \nu_2} \frac{f - \kappa_1/\nu_7}{f - \kappa_1/\nu_8}.$$
The space of initial condition for $q$-$P(D_{5}^{(1)})$ is realized by blowing up eight points $P_{1}, \ldots, P_{8}$ of $\mathbb{P}^{1} \times \mathbb{P}^{1}$, where

$$P_i : (\infty, 1/\nu_i)_{i=1,2}, \quad (\nu_i, \infty)_{i=3,4}, \quad (0, \nu_i/\nu_2)_{i=5,6}, \quad (\nu_1/\nu_4, 0)_{i=7,8}$$

On the other hand, $q$-$P(D_{5}^{(1)})$ is realized by the compatibility condition for the Lax pair $L_{1}$ and $L_{2}$, where

$$L_{1} = \left\{ \frac{z(g\nu_1 - 1)(g\nu_2 - 1)}{fg} - \frac{\nu_1\nu_2\nu_3\nu_4 (g - \nu_5/\nu_2)(g - \nu_6/\nu_2)}{fg} \right\}$$

$$+ \frac{\nu_1\nu_2(z - q\nu_3)(z - q\nu_4)}{q(qf - z)} g^{-1} + \frac{(z - \nu_1/q_1)(z - \nu_1/q_2)}{q(f - z)} \left( \frac{1}{g} - T_z \right)$$

$$L_{2} = \left( 1 - \frac{f}{g} \right) T + T_z - \frac{1}{g}$$

We track the linear differential equation $L_{1}y(z) = 0$ on the process of the blow-up of the point $P_{i}$ ($i = 1, 2, \ldots, 8$) respectively, and we investigate a relationship with the $q$-Heun equation

$$(2.1) \quad (x - h_{1}q^{1/2})(x - h_{2}q^{1/2})g(x/q) + l_{3}l_{4}(x - l_{1}q^{-1/2})(x - l_{2}q^{-1/2})g(qx)$$

$$- \{(l_{3} + l_{4})x^{2} + Ex + (l_{1}l_{3}l_{4}h_{1}h_{2})^{1/2}(h_{1}^{1/2} + h_{2}^{1/2})\}g(x) = 0.$$

The parameter $E$ is called the accessory parameter. Note that detailed calculation in the following subsections was performed in [11].

2.1.1. The points $P_{5} : (0, \nu_5/\nu_2)$ and $P_{6} : (0, \nu_6/\nu_2)$. We realize the blow-up at $(f, g) = (0, \nu_5/\nu_2)$ by setting $(f, g) = (f_{1}, f_{1}g_{1} + \nu_5/\nu_2)$. Then the point $P_{5} : (0, \nu_5/\nu_2)$ corresponds to the line $f_{1} = 0$. By the transformation $(f, g) = (f_{1}, f_{1}g_{1} + \nu_5/\nu_2)$, the operator $L_{1}$ is written as

$$L_{1} = - \frac{g_{1}\nu_{1}\nu_{2}\nu_{3}\nu_{4}(f_{1}g_{1}\nu_{2} + \nu_5 - \nu_6)}{f_{1}g_{1}\nu_{2} + \nu_5} + \frac{\nu_1\nu_2(q\nu_3 - z)(q\nu_4 - z)(f_{1}g_{1}\nu_{2} + \nu_5 - \nu_2T_z^{-1})}{f_{1}g_{1}\nu_{2} + \nu_5}$$

$$+ \frac{\nu_1\nu_2(z - q\nu_3)(z - q\nu_4)}{q(qf - z)} g^{-1} + \frac{\nu_1\nu_2(q\nu_3 - z)(q\nu_4 - z)(f_{1}g_{1}\nu_{2} + \nu_5 - \nu_2T_z^{-1})}{f_{1}g_{1}\nu_{2} + \nu_5}$$

$$- \frac{(\nu_1\nu_5 - \nu_2 + f_{1}g_{1}\nu_{2}nu_3)(\nu_2\nu_5 - \nu_2 + f_{1}g_{1}\nu_{2}nu_3)z}{\nu_2\nu_3(f_{1}g_{1}\nu_{2} + \nu_5)\nu_2\nu_3(f_{1} - z)}$$

We set $f_{1} = 0$ on the equation $L_{1}y(z) = 0$. Then we have

$$(2.2) \quad \nu_1\nu_2(z - q\nu_3)(z - q\nu_4)y(z/q) + \left( z - \frac{\nu_1}{\nu_7} \right) \left( z - \frac{\nu_1}{\nu_8} \right) g(qz)$$

$$- \left[ (\nu_1 + \nu_2)z^{2} + \left\{ qg_{1}\nu_{1}\nu_{2}\nu_{3}\nu_{4}(1 - \frac{\nu_6}{\nu_5}) - \frac{q\nu_1\nu_2\nu_3\nu_4\nu_5}{\nu_5\nu_7\nu_8}\right\} \right] y(z) = 0.$$
It follows from the relation $\kappa_1^2 \kappa_2^2 = q \nu_1 \nu_2 \ldots \nu_8$ that Eq. (2.2) is written as

\[
(z - q \nu_3)(z - q \nu_4)y(z/q) + \frac{1}{\nu_1 \nu_2} \left(z - \frac{\kappa_1}{\nu_2}\right) \left(z - \frac{\kappa_1}{\nu_5}\right)y(qz) \]

\[
- \left[\left(\frac{1}{\nu_1} + \frac{1}{\nu_2}\right)z^2 + \left(qg_1 \nu_3 \nu_4 \left(1 - \frac{\nu_6}{\nu_5}\right) - q \nu_5 (\nu_3 + \nu_4) - \frac{\kappa_1 \nu_2 (\nu_7 + \nu_8)}{\nu_1 \nu_2 \nu_5 \nu_7 \nu_8}\right)z\right] \]

\[
+ \frac{q \kappa_1 \nu_3^{1/2} \nu_4^{1/2}}{\nu_1^{1/2} \nu_2^{1/2} \nu_7^{1/2} \nu_8^{1/2}} \left\{ \frac{q \nu_6^{1/2}}{\nu_5} + \left(\frac{q \nu_6}{\nu_5}\right)^{-1/2}\right\} y(z) = 0.
\]

We compare it with the $q$-Heun equation given in Eq. (2.1). Then we obtain the following correspondence

\[
h_1 = q^{1/2} \nu_3, \quad h_2 = q^{1/2} \nu_4, \quad h_3 = \frac{q \nu_6}{\nu_5},
\]

\[
l_1 = \frac{q^{1/2} \kappa_1}{\nu_7}, \quad l_2 = \frac{q^{1/2} \kappa_1}{\nu_8}, \quad l_3 = \frac{1}{\nu_1}, \quad l_4 = \frac{1}{\nu_2}.
\]

We may regard the parameter $g_1$ as an accessory parameter.

On the blow-up at the point $P_6 : (0, \nu_6/\kappa_2)$, we obtain the $q$-Heun equation where the parameters $\nu_5$ and $\nu_6$ are exchanged.

2.1.2. The points $P_7 : (\kappa_1/\nu_7, 0)$ and $P_8 : (\kappa_1/\nu_8, 0)$.

We realize the blow-up at $(f, g) = (\kappa_1/\nu_7, 0)$ by setting $(f, g) = (f_1 g_1 + \kappa_1/\nu_7, g_1)$. Then the point $P_7 : (\kappa_1/\nu_7, 0)$ corresponds to the line $g_1 = 0$. By the transformation $(f, g) = (f_1 g_1 + \kappa_1/\nu_7, g_1)$, the operator $L_1$ is written as

\[
L_1 = - \frac{\nu_1 \nu_2 \nu_3 \nu_4 g_1 \kappa_1 \kappa_2 - \nu_5 (g_1 \kappa_2 - \nu_6) \nu_7}{g_1 \kappa_1^2 (\kappa_1 + f_1 g_1 \nu_7)} + \frac{(g_1 \nu_1 - 1)(g_1 \nu_2 - 1) z}{g_1 q} + \frac{\nu_1 \nu_2 \nu_3 (q \nu_7 - z)(q \nu_4 - z)(g_1 - T_z^{-1})}{q (\kappa_1 q + f_1 g_1 \nu_7 - \nu_7 z)} - \frac{(\kappa_1 - \nu_7 z)(\kappa_1 - \nu_8 z)(g_1 T_z - 1)}{g_1 q \nu_8 (\kappa_1 + f_1 g_1 \nu_7 - \nu_7 z)}.
\]

We consider the equation $L_1 y(z) = 0$. It follows from the relation $\kappa_1^2 \kappa_2^2 = q \nu_1 \nu_2 \ldots \nu_8$ that divergence as $g_1 \to 0$ in $L_1 y(z) = 0$ was avoided and we have

\[
- \frac{\kappa_1 - \nu_8 z}{q \nu_8} y(qz) - \frac{\nu_1 \nu_2 \nu_7 (z - q \nu_3)(z - q \nu_4)}{q (\kappa_1 q - \nu_7 z)} y(z/q) + \left\{ \frac{\nu_1 \nu_2 \nu_3 \nu_4 (\nu_5 + \nu_6)}{\kappa_2 (\kappa_1 - \nu_7 z)} - \frac{\kappa_1 (\nu_1 + \nu_2) z}{q (\kappa_1 - \nu_7 z)} - \frac{\nu_1 \nu_2 \nu_3 \nu_4 (\nu_5 + \nu_6) \nu_7 z}{\kappa_1 \kappa_2 (\kappa_1 - \nu_7 z)} \right\} y(z) = 0
\]
by setting $g_1 = 0$. To obtain the $q$-Heun equation, we transform the dependent variable $y(z)$ by setting $u(z) = y(z)/(\nu_7z - \kappa)$. Then it follows that

\[
(z - q\nu_3)(z - q\nu_4)u(z/q) + \frac{q^2}{\nu_1\nu_2}(z - \frac{\kappa_1}{\nu_7})(z - \frac{\kappa_1}{\nu_8})u(qz)
- \left[\left(\frac{q}{\nu_1} + \frac{q}{\nu_2}\right)z^2 + \left\{\frac{q}{\nu_1}\nu_2(1 - \frac{\nu_7}{\nu_8}) - \frac{\kappa_1q}{\nu_7}\left(\frac{1}{\nu_1} + \frac{1}{\nu_2}\right) - \frac{q^2\nu_3\nu_4\nu_7\nu_5}{\kappa_1\kappa_2}\right\}z
+ \frac{q^{3/2}\kappa_1\nu_3^{1/2}\nu_4^{1/2}}{\nu_1^{1/2}\nu_2^{1/2}\nu_7^{1/2}\nu_8^{1/2}}\left\{\left(\frac{\nu_6}{\nu_5}\right)^{1/2} + \left(\frac{\nu_6}{\nu_5}\right)^{-1/2}\right\}\right]u(z) = 0.
\]

We compare it with the $q$-Heun equation given in Eq. (2.1). Then we obtain the following correspondence

\[
h_1 = q^{1/2}\nu_3, \quad h_2 = q^{1/2}\nu_4, \quad h_3 = \frac{\nu_6}{\nu_5}
\]
\[
l_1 = \frac{\kappa_1}{q^{1/2}\nu_7}, \quad l_2 = \frac{q^{1/2}\kappa_1}{\nu_8}, \quad l_3 = \frac{q}{\nu_1}, \quad l_4 = \frac{q}{\nu_2}.
\]

We may regard $f_1$ as an accessory parameter.

On the blow-up at the point $P_3 : (\kappa_1/\nu_8, 0)$, we obtain the $q$-Heun equation where the parameters $\nu_7$ and $\nu_8$ are exchanged.

2.1.3. The points $P_1 : (\infty, 1/\nu_1)$ and $P_2 : (\infty, 1/\nu_2)$.

We consider the blow-up at the point $P_1 : (\infty, 1/\nu_1)$. Set $(f, g) = (1/f_0, g_0)$. Then the point $(f, g) = (\infty, 1/\nu_1)$ corresponds to the point $(f_0, g_0) = (0, 1/\nu_1)$. We realize the blow-up at $(f_0, g_0) = (0, 1/\nu_1)$ by setting $(f_0, g_0) = (f_1, f_1g_1 + 1/\nu_1)$. Then the point $P_1 : (\infty, \nu_1)$ corresponds to the line $f_1 = 0$. We substitute $(f, g) = (1/f_0, g_0)$ and $(f_0, g_0) = (f_1, f_1g_1 + 1/\nu_1)$ into the equation $L_1y(z) = 0$. We set $f_1 = 0$. By applying the relation $\kappa_1^2\kappa_2^2 = \nu_1\nu_2\ldots\nu_8$, we obtain

\[
(z - q\nu_3)(z - q\nu_4)y(z/q) + \frac{q}{\nu_1\nu_2}(z - \frac{\kappa_1}{\nu_7})(z - \frac{\kappa_1}{\nu_8})y(qz)
- \left[\left(\frac{1}{\nu_1} + \frac{q}{\nu_2}\right)z^2 + \left\{\frac{1}{\nu_1}\nu_2(1 - \frac{\nu_7}{\nu_8}) - \frac{\kappa_1q}{\nu_7}\left(\frac{1}{\nu_1} + \frac{1}{\nu_2}\right)\right\}z
+ \left(\frac{q^2\nu_3\nu_4\nu_7\nu_5}{\kappa_1\kappa_2}\right)^{1/2}\left\{\left(\frac{\nu_6}{\nu_5}\right)^{1/2} + \left(\frac{\nu_6}{\nu_5}\right)^{-1/2}\right\}\right]y(z) = 0.
\]

We compare it with the $q$-Heun equation given in Eq. (2.1). Then we obtain the following correspondence

\[
h_1 = q^{1/2}\nu_3, \quad h_2 = q^{1/2}\nu_4, \quad h_3 = \frac{\nu_6}{\nu_5}
\]
\[
l_1 = \frac{q^{1/2}\kappa_1}{\nu_7}, \quad l_2 = \frac{q^{1/2}\kappa_1}{\nu_8}, \quad l_3 = \frac{1}{\nu_1}, \quad l_4 = \frac{q}{\nu_2}.
\]

We may regard $g_1$ as an accessory parameter.

On the blow-up at the point $P_2 : (\infty, 1/\nu_2)$, we obtain the $q$-Heun equation where the parameters $\nu_1$ and $\nu_2$ are exchanged.
2.1.4. The points $P_3 : (\nu_3, \infty)$ and $P_4 : (\nu_4, \infty)$.

We consider the blow-up at the point $P_3 : (\nu_3, \infty)$. Set $(f, g) = (f_0, 1/g_0)$. Then the point $(f, g) = (\nu_3, \infty)$ corresponds to the point $(f_0, g_0) = (\nu_3, 0)$. We realize the blow-up at $(f_0, g_0) = (\nu_3, 0)$ by setting $(f_0, g_0) = (f_1, g_1)$. Then the point $P_3 : (\nu_3, \infty)$ corresponds to the line $g_1 = 0$. We substitute $(f, g) = (f_0, 1/g_0)$ and $(f_0, g_0) = (f_1, g_1 + \nu_3, g_1)$ into the equation $L_1 y(z) = 0$. Set $g_1 = 0$. Then we have

$$-rac{(\kappa_1 - \nu_7 z)(\kappa_1 - \nu_8 z)}{q \nu_7 \nu_8 (\nu_3 - z)} y(qz) - \frac{\nu_1 \nu_2 (q \nu_4 - z)}{q} y(z/q)$$

$$+ \frac{1}{\kappa_2 g(q \nu_3 - z)} \left[ q^2 \nu_1 \nu_2 \nu_3 \nu_4 (\nu_5 + \nu_6) + \{ q f_1 \nu_2 \nu_1 \nu_2 (1 - \nu_4/\nu_3) - q \nu_2 \nu_1 \nu_2 (\nu_5 + \nu_6) \} z + \kappa_2 (\nu_1 + \nu_2) z^2 \right] y(z) = 0.$$

To obtain the $q$-Heun equation, we set $u(z) = y(z)/(z - q \nu_3)$. Then

$$(z - q^2 \nu_3)(z - q \nu_4) u(z/q) + \frac{q^2}{\nu_1 \nu_2} \left( z - \frac{\nu_1}{\nu_7} \right) \left( z - \frac{\nu_1}{\nu_8} \right) u(qz)$$

$$- \left[ q \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} \right) z^2 + q^2 \left\{ f_1 \left( 1 - \frac{\nu_4}{\nu_3} \right) - \nu_3 \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} \right) - \frac{\nu_4}{\kappa_2} (\nu_5 + \nu_6) \right\} z$$

$$+ \frac{q^3 \nu_3 \nu_4 \nu_5^{1/2} \nu_6^{1/2}}{\kappa_2} \left\{ \left( \frac{\nu_6}{\nu_5} \right)^{1/2} + \left( \frac{\nu_6}{\nu_5} \right)^{-1/2} \right\} \right] u(z) = 0.$$

We compare it with the $q$-Heun equation given in Eq. (2.1). Then we obtain the following correspondence

$$h_1 = q^{3/2} \nu_3, \quad h_2 = q^{1/2} \nu_4, \quad h_3 = \frac{\nu_6}{\nu_5},$$

$$l_1 = \frac{q^{1/2} \nu_1}{\nu_7}, \quad l_2 = \frac{q^{1/2} \nu_1}{\nu_8}, \quad l_3 = \frac{q}{\nu_1}, \quad l_4 = \frac{q}{\nu_2}.$$

We may regard $f_1$ as an accessory parameter.

On the blow-up at the point $P_4 : (\nu_4, \infty)$, we obtain the $q$-Heun equation where the parameters $\nu_3$ and $\nu_4$ are exchanged.

2.1.5. The other cases related to the $q$-Heun equation.

We can obtain the $q$-Heun equation by other restrictions of the parameters. We substitute $f = \kappa_1/\nu_7$ to the equation $L_1 y(z) = 0$. Then we have

$$(z - q \nu_3)(z - q \nu_4) y(z/q) + \frac{1}{\nu_1 \nu_2} \left( z - \frac{q \kappa_1}{\nu_7} \right) \left( z - \frac{\kappa_1}{\nu_8} \right) y(qz) - \left[ \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} \right) z^2$$

$$+ \left\{ \frac{q (\nu_3 \nu_7 - \kappa_1) (\nu_4 \nu_7 - \kappa_1)}{\nu_7 \kappa_1} \right\} g - \kappa_1 q \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} \right) - \frac{q \nu_3 \nu_4 \nu_7 (\nu_5 + \nu_6)}{\kappa_1 \kappa_2} \right\} z$$

$$+ \frac{q^{3/2} \kappa_1 \nu_3 \nu_5^{1/2} \nu_6^{1/2}}{\nu_1^{1/2} \nu_2^{1/2} \nu_7^{1/2} \nu_8^{1/2}} \left\{ \left( \frac{\nu_6}{\nu_5} \right)^{1/2} + \left( \frac{\nu_6}{\nu_5} \right)^{-1/2} \right\} \right] y(z) = 0.$$

Hence we obtain the $q$-Heun equation, and we may regard $g$ as an accessory parameter. We have a similar result by substituting $f = \kappa_1/\nu_7$ to the equation $L_1 y(z) = 0$. 


We substitute $f = \nu_3$ to the equation $L_1y(z) = 0$. Then we have
\[
(z - \nu_3)(z - q\nu_4)y(z/q) + \frac{1}{\nu_1\nu_2}\left(z - \frac{\kappa_1}{\nu_7}\right)\left(z - \frac{\kappa_1}{\nu_8}\right)y(qz)
- \left[\left(\frac{1}{\nu_1} + \frac{1}{\nu_2}\right)z^2 - \frac{(\nu_3\nu_7 - \kappa_1)(\nu_4\nu_7 - \kappa_1)}{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7}\right]
\left(\frac{\nu_3}{\nu_4} + \frac{\nu_4}{\nu_3}\right)\left(\frac{\nu_3}{\nu_5} + \frac{\nu_5}{\nu_3}\right)\left(\frac{\nu_3}{\nu_6} + \frac{\nu_6}{\nu_3}\right)\left(\frac{\nu_3}{\nu_7} + \frac{\nu_7}{\nu_3}\right)\left(\frac{\nu_3}{\nu_8} + \frac{\nu_8}{\nu_3}\right)\right]y(z)
\]
\]
\[+ \frac{q^2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8}{\kappa_2}z^{-1/2}\nu_6^{1/2}z \left(\left(\frac{\nu_6}{\nu_5}\right)^{1/2} + \left(\frac{\nu_6}{\nu_5}\right)^{-1/2}\right) y(z) = 0.
\]

Hence we obtain the $q$-Heun equation, and we may regard $g$ as an accessory parameter. Note that this procedure was essentially considered in section 3.1 of [13]. We have a similar result by substituting $f = \nu_4$ to the equation $L_1y(z) = 0$.

### 2.2. The case $E_6^{(1)}$.

The $q$-Painlevé equation of type $E_6^{(1)}$ ($q$-$P(E_6^{(1)})$) in [6] was given as
\[
\frac{(fg-1)(fg-1)}{ff} = \frac{(g-1/\nu_1)(g-1/\nu_2)(g-1/\nu_3)(g-1/\nu_4)}{(g-\nu_5/\nu_2)(g-\nu_6/\nu_2)},
\]
\[
\frac{(fg-1)(fg-1)}{gg} = \frac{(f-\nu_1)(f-\nu_2)(f-\nu_3)(f-\nu_4)}{(f-\nu_1/\nu_7)(f-\nu_1/\nu_8)}.
\]

The space of initial condition for $q$-$P(E_6^{(1)})$ was realized by blowing up eight points $P_1, \ldots, P_8$ of $\mathbb{P}^1 \times \mathbb{P}^1$, where
\[
P_i : \left(\nu_i, \frac{1}{\nu_i}\right)_{i=1,2,3,4}, \quad \left(0, \frac{\nu_i}{\kappa_2}\right)_{i=5,6}, \quad \left(\frac{\kappa_1}{\nu_i}, 0\right)_{i=7,8}.
\]

On the other hand, $q$-$P(E_6^{(1)})$ is obtained by the compatibility condition for the Lax operators $L_1$ and $L_2$, where
\[
L_1 = \frac{z(g\nu_1 - 1)(g\nu_2 - 1)(g\nu_3 - 1)(g\nu_4 - 1)}{g(fg - 1)(gz - q)} - \frac{(g\nu_5 - 1)(g\nu_6 - 1)(g\nu_7 - 1)}{g\nu_8}(\nu_1 - z/q)(\nu_2 - z/q)(\nu_3 - z/q)(\nu_4 - z/q)
\]
\[\times \left(\frac{g}{1 - gz/q} - T^{-1}z\right)
\]
\[\times \frac{1}{q(f - z)} \left\{\left(\frac{1}{g} - z\right) - Tz\right\},\]

\[
L_2 = (1 - \frac{f}{z})T + Tz - \left(\frac{1}{g} - z\right).
\]

We track the linear differential equation $L_1y(z) = 0$ on the process of the blow-up of the point $P_i$ ($i = 1, 2, \ldots, 8$) respectively, and we investigate a relationship with the variant of the $q$-Heun equation of degree 3;
\[
\begin{align*}
(2.3) \quad (x - h_1q^{1/2})(x - h_2q^{1/2})(x - h_3q^{1/2})g(x/q) \\
+ (x - l_1q^{-1/2})(x - l_2q^{-1/2})(x - l_3q^{-1/2})g(qx) \\
+ \left\{-q^{1/2} + q^{-1/2}\right\}x^3 + (h_1 + h_2 + h_3 + l_1 + l_2 + l_3)x^2 \\
- Ex + (l_1l_2l_3h_1h_2h_3)^{1/2}(h_4^{-1/2} + h_4^{1/2})g(x) = 0.
\end{align*}
\]
Note that detailed calculation in the following subsections was performed in [11].

2.2.1. The points $P_5 : (0, \nu_5/\kappa_2)$ and $P_6 : (0, \nu_6/\kappa_2)$.

We realize the blow-up at $(f, g) = (0, \nu_5/\kappa_2)$ by setting $(f, g) = (f_1 g_1, g_1 + \nu_5/\kappa_2)$. Then the point $P_5 : (0, \nu_5/\kappa_2)$ corresponds to the line $g_1 = 0$. We substitute $(f, g) = (f_1 g_1, g_1 + \nu_5/\kappa_2)$ into the equation $L_1 y(z) = 0$, and set $g_1 = 0$. Then we have

\begin{equation}
(2.4) \quad \left( \frac{\kappa_1}{\nu_7} - z \right) \left( \frac{\kappa_1}{\nu_8} - z \right) y(qz) + \left( \frac{q \nu_1 - z}{q^2} \right) \left( \frac{q \nu_2 - z}{q^2} \right) \left( \frac{q \nu_3 - z}{q^2} \right) \left( \frac{q \nu_4 - z}{q^2} \right) \frac{y(z/q)}{q^2}
\end{equation}

To obtain a variant of the $q$-Heun equation, we apply gauge transformations. Set

\begin{equation}
(\alpha x ; q)_\infty = \prod_{i=0}^{\infty} (1 - q^i \alpha x) = (1 - \alpha x)(1 - q \alpha x)(1 - q^2 \alpha x) \cdots
\end{equation}

Proposition 2.1.

(i) If $y(x)$ satisfies $q^{-\lambda} a(x) g(x/q) + b(x) g(x) + q^\lambda c(x) g(q x) = 0$, then the function $h(x) = x^{-\lambda} y(x)$ satisfies $a(x) g(x/q) + b(x) g(x) + c(x) g(q x) = 0$.

(ii) If $y(x)$ satisfies $(1 - \alpha x) a(x) g(x/q) + b(x) g(x) + c(x) g(q x) = 0$, then the function $u(x) = (\alpha x q ; q)_\infty y(x)$ satisfies $a(x) g(x/q) + b(x) g(x) + (1 - \alpha q x) c(x) g(q x) = 0$.

Proof. If $h(x) = x^{-\lambda} y(x)$, then $y(q x) = q^{-\lambda} x^{-\lambda} h(q x)$ and $y(x/q) = q^\lambda x^{-\lambda} h(x/q)$. Hence the equation $q^{-\lambda} a(x) h(x/q) + b(x) g(x) + q^\lambda c(x) h(q x) = 0$ is rewritten as $a(x) y(x/q) + b(x) y(x) + c(x) y(q x) = 0$ and we obtain (i).

We can obtain (ii) similarly, as shown in [11].

We apply a gauge transformation in Proposition 2.1 (ii) to Eq. (2.4). Then we obtain

\begin{equation}
\frac{q^{1/2}}{\nu_1} \left( \nu_1 - z \right) \left( \frac{\kappa_1}{\nu_7} - z \right) \left( \frac{\kappa_1}{\nu_8} - z \right) \tilde{y}(qz) + \frac{\nu_1}{q^{1/2}} \left( q \nu_2 - z \right) \left( q \nu_3 - z \right) \left( q \nu_4 - z \right) \tilde{y}(z/q)
\end{equation}

\begin{equation}
+ q^{1/2} \left\{ - \frac{q^2 \nu_1 \nu_2 \nu_3 \nu_4 \nu_5}{\kappa_2} - \frac{\kappa_1^2 \kappa_2^2 \nu_5^2 \nu_7 \nu_8}{\kappa_2} + \left( \frac{q \nu_1 \nu_2 \nu_3 \nu_5}{\kappa_2} + \frac{q \nu_1 \nu_2 \nu_4 \nu_5}{\kappa_2} + \frac{q \nu_1 \nu_3 \nu_4 \nu_5}{\kappa_2} \right) \right. 
\end{equation}

\begin{equation}
+ \frac{q \nu_1 \nu_2 \nu_3 \nu_4 \nu_5}{\kappa_2} + \frac{q \nu_1 \nu_2 \nu_3 \nu_4 \nu_5}{\kappa_2} + \frac{\kappa_1 \kappa_2 \nu_5 \nu_7}{\nu_5 \nu_7 \nu_8} + \frac{\kappa_1 \kappa_2 \nu_5 \nu_8}{\nu_5 \nu_7 \nu_8} + \frac{\kappa_1 \kappa_2 \nu_7 \nu_8}{\nu_5 \nu_7 \nu_8} + \frac{\kappa_1 \kappa_2 \nu_7 \nu_8}{\nu_5 \nu_7 \nu_8}
\end{equation}

\begin{equation}
+ \left. \left( \nu_1 + \nu_2 + \nu_3 + \nu_4 + \frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8} \right) z^2 + \left( 1 + \frac{1}{q} \right) z^3 \right\} \tilde{y}(z) = 0.
\end{equation}
By applying the relation \( \kappa_1^2 \kappa_2^2 = q \nu_1 \nu_2 \ldots \nu_8 \) and a gauge transformation in Proposition 2.1 (i), we have

\[
(z - q \nu_2)(z - q \nu_3)(z - q \nu_4)u(z/q) + (z - \nu_1) \left( z - \frac{\kappa_1}{\nu_7} \right) \left( z - \frac{\kappa_1}{\nu_8} \right) u(qz)
\]

\[
+ \left[ -(q^{1/2} + q^{-1/2}) z^3 + q^{1/2} \left( \nu_1 + \nu_2 + \nu_3 + \nu_4 + \frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8} \right) z^2
\]

\[
- \frac{q^{3/2}}{\nu_5} \left( \frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8} \right) \right] \left( q^{3/2} \nu_2 \nu_3 \nu_4 \left( \frac{\nu_6}{\nu_5} - 1 \right) \left( \frac{1}{\nu_1} + \frac{\nu_5}{\kappa_2^2} \right) + \nu_1 \nu_2 \nu_3 \nu_4 \frac{\nu_5}{\kappa_2} \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} + \frac{1}{\nu_4} \right)
\]

\[
+ \frac{\kappa_2}{\nu_5} \left( \frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8} \right) \right] u(z) = 0.
\]

In fact, if \( y(z) \) is a solution of Eq. \( (\ref{2.1}) \) and the parameter \( \lambda \) satisfies \( q^\lambda = q^{1/2}/\nu_1 \), then the function \( u(z) = z^\lambda (z/\nu_1; q)_\infty y(z) \) satisfies Eq. \( (\ref{2.5}) \). We compare Eq. \( (\ref{2.5}) \) with the variant of the \( q \)-Heun equation of degree 3 given in Eq. \( (\ref{2.3}) \). Then we obtain the following correspondence

\[
\begin{align*}
& h_1 = q^{1/2} \nu_2, \quad h_2 = q^{1/2} \nu_3, \quad h_3 = q^{1/2} \nu_4, \quad h_4 = \frac{q \nu_5}{\nu_6}, \\
& l_1 = q^{1/2} \nu_1, \quad l_2 = \frac{q^{1/2} \kappa_1}{\nu_7}, \quad l_3 = \frac{q^{1/2} \kappa_1}{\nu_8}.
\end{align*}
\]

We may regard \( f_1 \) as an accessory parameter.

On the blow-up at the point \( P_6 \), we obtain the equation where the parameters \( \nu_5 \) and \( \nu_6 \) are exchanged.

2.2.2. The points \( P_7 : (\kappa_1/\nu_7, 0) \) and \( P_8 : (\kappa_1/\nu_8, 0) \).

We realize the blow-up at \( (f, g) = (\kappa_1/\nu_7, 0) \) by setting \( (f, g) = (f_1 g_1 + \kappa_1/\nu_7, g_1) \). Then the point \( P_7 : (\kappa_1/\nu_7, 0) \) corresponds to the line \( g_1 = 0 \). We substitute \( (f, g) = (f_1 g_1 + \kappa_1/\nu_7, g_1) \) into the equation \( L_1 y(z) = 0 \), and set \( g_1 = 0 \). Then we have

\[
\begin{align*}
& z - \kappa_1/\nu_8 \frac{y(qz)}{q} + \frac{(q \nu_1 - z)(q \nu_2 - z)(q \nu_3 - z)(q \nu_4 - z)}{q^2 (z - q \kappa_1/\nu_7)} y(z/q)
\]

\[
+ \left[ \frac{\kappa_1^2 q \left\{ - \kappa_2 (\nu_5 + \nu_6) + \nu_5 \nu_6 (1 - \nu_8/\nu_7) z \right\}}{q^2 \nu_5 \nu_6 \nu_7 \nu_8 (z - \kappa_1/\nu_7)} \right] + \frac{z \left\{ f_1 q (\nu_7 - \nu_8) + \nu_8 z ((q + 1) z - q (\nu_1 + \nu_2 + \nu_3 + \nu_4)) \right\}}{q^2 \nu_8 (z - \kappa_1/\nu_7)}
\]

\[
+ \frac{\kappa_1 z \left\{ \kappa_2 q (\nu_5 + \nu_6) \nu_7 + \nu_5 \nu_6 \left\{ -(q \nu_7 + \nu_8) z + q (\nu_1 + \nu_2 + \nu_3 + \nu_4) \nu_8 \right\} \right\}}{q^2 \nu_5 \nu_6 \nu_7 \nu_8 (z - \kappa_1/\nu_7)} \right] y(z) = 0.
\]
By setting \( \tilde{g}(z) = y(z)/(z - \kappa_1/\nu_7) \), we have
\[
\frac{(z - \kappa_1/\nu_8)(qz - \kappa_1/\nu_7)}{q} \tilde{g}(qz) + \frac{(q\nu_1 - z)(q\nu_2 - z)(q\nu_3 - z)(q\nu_4 - z)}{q^4} \tilde{g}(z/q) + \frac{\kappa_1^2 q \{-\kappa_2 (\nu_5 + \nu_6) + \nu_5 \nu_6 (1 - \nu_8/\nu_7) z\}}{q^2 \nu_5 \nu_6 \nu_7 \nu_8} + \frac{z \{f_1 q (\nu_7 - \nu_8) + \nu_8 z ((q + 1) z - q (\nu_1 + \nu_2 + \nu_3 + \nu_4))\}}{q^2 \nu_5 \nu_6 \nu_7 \nu_8} + \frac{\kappa_1 z \{\kappa_2 q (\nu_5 + \nu_6) \nu_7 + \nu_3 \nu_6 (- (q \nu_7 + \nu_8) z + q (\nu_1 + \nu_2 + \nu_3 + \nu_4) \nu_8)\}}{q^2 \nu_5 \nu_6 \nu_7 \nu_8} \right] \tilde{g}(z) = 0.
\]
Let \( \lambda \) be the parameter such that \( q^\lambda = q^{3/2}/\nu_1 \), then the function \( u(z) = z^\lambda(z/\nu_1; q) \tilde{y}(z) \) satisfies
\[
(z - q\nu_2)(z - q\nu_3)(z - q\nu_4)u(z/q) + (z - \nu_1) \left( z - \frac{\kappa_1}{\nu_7 q} \right) (z - \frac{\kappa_1}{\nu_8}) u(qz) + \left\{ -(q^{1/2} + q^{-1/2}) z^3 + q^{1/2} \left( \nu_1 + \nu_2 + \nu_3 + \nu_4 + \frac{\kappa_1}{\nu_7 q} + \frac{\kappa_1}{\nu_8} \right) z^2 \right. \\
- q^{1/2} \left\{ \left( f_1 + \frac{\kappa_1^2}{\nu_7^2} \right) \left( \nu_7 \right) - 1 \right\} + \frac{\kappa_1}{\nu_7} (\nu_1 + \nu_2 + \nu_3 + \nu_4) + \frac{\kappa_1}{\nu_8} \left( \frac{\kappa_2}{\nu_5} + \frac{\kappa_2}{\nu_6} \right) \right\} z \\
+ \left\{ \left( \frac{\nu_5}{\nu_6} \right)^{1/2} + \left( \frac{\nu_5}{\nu_6} \right)^{-1/2} \right\} \left\{ \left( \frac{\nu_5}{\nu_6} \right)^{1/2} + \left( \frac{\nu_5}{\nu_6} \right)^{-1/2} \right\} u(z) = 0.
\]
We compare it with the variant of the \( q \)-Heun equation of degree 3 given in Eq. (2.3). Then we obtain the following correspondence
\[
h_1 = q^{1/2} \nu_1, \quad h_2 = q^{1/2} \nu_3, \quad h_3 = q^{1/2} \nu_4, \quad h_4 = \frac{\nu_5}{\nu_6},
\]
\[
l_1 = q^{1/2} \nu_1, \quad l_2 = \frac{\kappa_1}{q^{1/2} \nu_7}, \quad l_3 = \frac{q^{1/2} \kappa_1}{\nu_8}.
\]
We may regard \( f_1 \) as an accessory parameter.

On the blow-up at the point \( P_8 \), we obtain the equation where the parameters \( \nu_7 \) and \( \nu_8 \) are exchanged.

2.2.3. The points \( P_i : (\nu_i, 1/\nu_i) \) \( (i = 1, 2, 3, 4) \).

We realize the blow-up at \( (f, g) = (\nu_1, 1/\nu_1) \) by setting \( (f, g) = (f_1 g_1 + \nu_1, g_1 + 1/\nu_1) \). Then the point \( P_1 : (\nu_1, 1/\nu_1) \) corresponds to the line \( g_1 = 0 \). We substitute \( (f, g) = (f_1 g_1 + \nu_1, g_1 + 1/\nu_1) \) into the equation \( L_1 y(z) = 0 \), and set \( g_1 = 0 \). Then we have
\[
\frac{(z - \kappa_1/\nu_7)(z - \kappa_1/\nu_8)}{q(z - \nu_1)} y(qz) + \frac{(z - q\nu_2)(z - q\nu_3)(z - q\nu_4)}{q^3} y(z/q) + \left\{ -\frac{\kappa_1^2 (1 - \kappa_2/(\nu_1 \nu_5))(1 - \kappa_2/(\nu_1 \nu_6))}{q\nu_7 \nu_8} + \frac{(z - \kappa_1/\nu_7)(z - \kappa_1/\nu_8)}{q} \right\} y(z) = 0.
\]
Let $\lambda$ be the parameter such that $q^\lambda = q^{1/2}/\nu_1$, then the function $u(z) = z^\lambda(z/\nu_1; q)_\infty y(z)$ satisfies

\[
(z - q\nu_2)(z - q\nu_3)(z - q\nu_4)u(z/q) + (z - q\nu_1)\left(z - \frac{\kappa_1}{\nu_7}\right)\left(z - \frac{\kappa_1}{\nu_8}\right)u(qz) \nonumber
\]

\[
+ \left[-(q^{-1/2} + q^{1/2})z^3 + q^{1/2}\left(q\nu_1 + \nu_2 + \nu_3 + \nu_4 + \frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8}\right)z^2 \right. 
\]

\[
- q^{3/2}\left\{ \frac{\nu_1(\nu_2 - \nu_1)(\nu_3 - \nu_1)(\nu_4 - \nu_1)}{f_1 + \nu_1^2} + \nu_1\left(\frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8}\right) \right. 
\]

\[
+ \nu_2\nu_3\nu_4\left( -\frac{1}{\nu_1} + \frac{1}{\nu_3} + \frac{1}{\nu_4} + \frac{\nu_5}{\kappa_2} + \frac{\nu_6}{\kappa_2}\right) \} z 
\]

\[
+ \left(q^{1/2}\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6 \right)^{1/2}\left\{ \left(\frac{\nu_5}{\nu_6}\right)^{1/2} + \left(\frac{\nu_5}{\nu_6}\right)^{-1/2} \right\} u(z) = 0. 
\]

We compare it with the variant of the $q$-Heun equation of degree 3 given in Eq. (2.3). Then we obtain the following correspondence

\[
h_1 = q^{1/2}\nu_2, \quad h_2 = q^{1/2}\nu_3, \quad h_3 = q^{1/2}\nu_4, \quad h_4 = \frac{\nu_5}{\nu_6}, 
\]

\[
l_1 = q^{3/2}\nu_1, \quad l_2 = \frac{q^{1/2}\kappa_1}{\nu_7}, \quad l_3 = \frac{q^{1/2}\kappa_1}{\nu_8}. 
\]

We may regard $f_1$ as an accessory parameter.

On the blow-up at the point $P_i$ ($i = 2, 3, 4$), we obtain the equation where the parameters $\nu_1$ and $\nu_i$ are exchanged.

2.2.4. The other cases related to the variant of the $q$-Heun equation of degree three.

We substitute $f = \kappa_1/\nu_7$ to the equation $L_1y(z) = 0$. Then we have

\[
(z - q\kappa_1/\nu_7)(z - \kappa_1/\nu_8)y(qz) + \frac{(q\nu_1 - z)(q\nu_2 - z)(q\nu_3 - z)(q\nu_4 - z)}{q^2}y(z/q) 
\]

\[
+ \left[(1 + 1/q)z^3 - \left(\nu_1 + \nu_2 + \nu_3 + \nu_4 + \frac{q\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8}\right)z^2 \right. 
\]

\[
+ q\left\{ (1 - \nu_1 \nu_7/\kappa_1)(1 - \nu_2 \nu_7/\kappa_1)(1 - \nu_3 \nu_7/\kappa_1)(1 - \nu_4 \nu_7/\kappa_1) + \frac{\kappa_1^2}{\nu_7\nu_8} \right. 
\]

\[
+ \frac{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6}{\kappa_1}\left( -\frac{1}{\nu_1} - \frac{1}{\nu_2} - \frac{1}{\nu_3} - \frac{1}{\nu_4} + \frac{\nu_5}{\kappa_2} + \frac{\nu_6}{\kappa_2} + \frac{\nu_7}{\kappa_1} + \sum_{1 \leq i < j \leq 4} \nu_i\nu_j \right) \} z 
\]

\[
- \frac{q\kappa_1^2\kappa_2(\nu_5 + \nu_6)}{\nu_5\nu_6\nu_7\nu_8} \} \right] y(z) = 0. 
\]

By applying a gauge transformation, we obtain the variant of the $q$-Heun equation of degree 3, and we may regard $g$ as an accessory parameter. We have a similar result by substituting $f = \kappa_1/\nu_8$ to the equation $L_1y(z) = 0$. 

We substitute \( f = \nu_i \) to the equation \( L_1 y(z) = 0 \). Then we have
\[
q(z - \kappa_1 / \nu_i)(z - \kappa_1 / \nu_8)y(qz) + \frac{(z - \nu_1)(z - q \nu_2)(z - q \nu_3)(z - q \nu_4)}{q} y(z/q)
\]
\[
+ \left[(q + 1)z^3 - \left(\nu_1 + q \nu_2 + q \nu_3 + q \nu_4 + \frac{q \kappa_1}{\nu_7} + \frac{q \kappa_1}{\nu_8}\right)z^2
\]
\[
+ q \left\{ \frac{-(\nu_1 - \kappa_1 / \nu_7)(\nu_1 - \kappa_1 / \nu_8)}{g \nu_1} + \nu_1 (\nu_2 + \nu_3 + \nu_4)
\]
\[
+ \frac{\kappa_1^2}{\nu_7 \nu_8} + \frac{\kappa_1^2}{\nu_1 \nu_7 \nu_8} \left( \frac{\kappa_2}{\nu_5} + \frac{\kappa_2}{\nu_6} \right) \right\} z - q \nu_1 \nu_2 \nu_3 \nu_4 (\nu_5 + \nu_6)
\]
\[
\frac{\kappa_2}{\nu_7 \nu_8} \} \right\} z - q^2 \nu_1 \nu_2 \nu_3 \nu_4 (\nu_5 + \nu_6)
\]
\[
\frac{\kappa_2}{\nu_7 \nu_8} \} y(z) = 0.
\]
By applying a gauge transformation, we obtain the variant of the \( q \)-Heun equation of degree 3, and we may regard \( g \) as an accessory parameter. We have similar results by substituting \( f = \nu_i \) \((i = 2, 3, 4)\) to the equation \( L_1 y(z) = 0 \).

2.3. The case \( E_7^{(1)} \).

The \( q \)-Painlevé equation of type \( E_7^{(1)} \) \((q-P(E_7^{(1)}))\) in [6] was given as
\[
\begin{align*}
\frac{(fg - \kappa_1 / \kappa_2)(g f - \kappa_1 / (g f))}{(fg - 1)(f g - 1)} &= \frac{(g - \nu_5 / \nu_2)(g - \nu_6 / \kappa_2)(g - \nu_7 / \kappa_2)(g - \nu_8 / \kappa_2)}{(g - 1 / \nu_1)(g - 1 / \nu_2)(g - 1 / \nu_3)(g - 1 / \nu_4)} ,
\end{align*}
\]
\[
\begin{align*}
\frac{(fg - \kappa_1 / \kappa_2)(g f - q \kappa_1 / \kappa_2)}{(fg - 1)(f g - 1)} &= \frac{(f - \kappa_1 / \nu_5)(f - \kappa_1 / \nu_6)(f - \kappa_1 / \nu_7)(f - \kappa_1 / \nu_8)}{(f - \nu_1)(f - \nu_2)(f - \nu_3)(f - \nu_4)} .
\end{align*}
\]
The space of initial condition for \( q-P(E_7^{(1)}) \) was realized by blowing up eight points \( P_1, \ldots, P_8 \) of \( \mathbb{P}^1 \times \mathbb{P}^1 \), where
\[
P_i : \left( \nu_i, \frac{1}{\nu_i} \right)_{i=1,2,3,4}, \left( \frac{\kappa_1}{\nu_i}, \frac{\nu_1}{\kappa_2} \right)_{i=5,6,7,8}.
\]
On the other hand, \( q-P(E_7^{(1)}) \) is obtained by the compatibility condition for the Lax operators \( L_1 \) and \( L_2 \), where
\[
L_1 = q(\kappa_1 - \kappa_2)(g \kappa_2 - \nu_5)(g \kappa_2 - \nu_6)(g \kappa_2 - \nu_7)(g \kappa_2 - \nu_8)
\]
\[
- \frac{g \kappa_1 \kappa_2}{g(f g-1)} \kappa_1 \nu_1 \nu_2 \nu_3 \nu_4 (g z - q)
\]
\[
+ \frac{(q \nu_1 - z)(q \nu_2 - z)(q \nu_3 - z)(q \nu_4 - z)}{q \kappa_1 \nu_1 \nu_2 \nu_3 \nu_4 (f q - z)(q - g z)} \{(g \kappa_2 z - \kappa_1 q) - \kappa_1 (g z - q) T_z^{-1}\}
\]
\[
- \frac{\kappa_1 z (g \kappa_2 - \nu_8 z)(\kappa_1 - \nu_6 z)(\kappa_1 - \nu_7 z)(\kappa_1 - \nu_8 z)}{\kappa_1 (g z - 1) - (g \kappa_2 z - \kappa_1) T_z},
\]
\[
L_2 = (1 - g \kappa_2 / \kappa_1) T_z - (1 - g z) + z (z - f) g T.
\]
We track the linear differential equation \( L_1 y(z) = 0 \) on the process of the blow-up of the point \( P_i \) \((i = 1, 2, \ldots, 8)\) respectively, and we investigate a relationship with the
variant of the $q$-Heun equation of degree 4;

\begin{equation}
(2.6) \quad (x - h_1 q^{1/2})(x - h_2 q^{1/2})(x - h_3 q^{1/2})(x - h_4 q^{1/2})g(x/q) \\
+ (x - l_1 q^{-1/2})(x - l_2 q^{-1/2})(x - l_3 q^{-1/2})(x - l_4 q^{-1/2})g(qx) \\
+ \left[ -(q^{1/2} + q^{-1/2})x^4 + (h_1 + h_2 + h_3 + h_4 + l_1 + l_2 + l_3 + l_4)x^3 + E x^2 \\
+ (h_1 h_2 h_3 h_4 l_1 l_2 l_3 l_4) \frac{1}{2} \{ (h_1^{-1} + h_2^{-1} + h_3^{-1} + h_4^{-1} + l_1^{-1} + l_2^{-1} + l_3^{-1} + l_4^{-1})x \\
- (q^{1/2} + q^{-1/2}) \} \right] g(x) = 0.
\end{equation}

Note that detailed calculation in the following subsections was performed in [III].

2.3.1. The points $P_i : (\nu, 1/\nu_i) \quad (i = 1, 2, 3, 4)$.

We realize the blow-up at $(f, g) = (\nu, 1/\nu)$ by setting $(f, g) = (f_1 g_1 + \nu, g_1 + 1/\nu)$. Then the point $P_1 : (\nu, 1/\nu_1)$ corresponds to the line $g_1 = 0$. We substitute $(f, g) = (f_1 g_1 + \nu, g_1 + 1/\nu)$ into the equation $L_1 g(z) = 0$, and set $g_1 = 0$. Then we have

\begin{equation}
\frac{q(\kappa_1 - \nu_5 z)(\kappa_1 - \nu_6 z)(\kappa_1 - \nu_7 z)(\kappa_1 - \nu_8 z)}{\kappa_1^4 (\nu - z)^2} y(z/q) \\
+ \frac{(q \nu_2 - z)(q \nu_3 - z)(q \nu_4 - z)}{q \nu_1 \nu_2 \nu_3 \nu_4} y(z/q) \\
+ \left\{ - \frac{q(\kappa_1 - \kappa_2)(\nu_1 - \nu_2)(\nu_1 - \nu_3)(\nu_1 - \nu_4)}{\kappa_1 (f_1 + \nu_1 \nu_2 \nu_3 \nu_4)} (q \nu_1 - z) \\
+ \frac{q(\kappa_2 - \nu_1 \nu_3)(\kappa_2 - \nu_1 \nu_5)(\kappa_2 - \nu_1 \nu_7)(\kappa_2 - \nu_1 \nu_8)}{\nu_1 \nu_2^2 \nu_3 \nu_4 (\nu_1 - z)^2} (\nu_1 \nu_2 \nu_3 \nu_4) \\
- \frac{q(\kappa_1 - \nu_5 z)(\kappa_1 - \nu_6 z)(\kappa_1 - \nu_7 z)(\kappa_1 - \nu_8 z)}{\kappa_1^3 \nu_1 \nu_2 \nu_3 \nu_4 (\nu_1 - z)^2} \right\} y(z) = 0.
\end{equation}

Let $\lambda$ be the parameter such that $q^\lambda = q^{3/2} \kappa_2 / \kappa_1$. Then the function $u(z) = z^\lambda y(z)/(q \nu_1 - z)$ satisfies

\begin{equation}
\begin{aligned}
(z - \frac{\kappa_1}{\nu_5})(z - \frac{\kappa_1}{\nu_6})(z - \frac{\kappa_1}{\nu_7})(z - \frac{\kappa_1}{\nu_8}) u(z/q) \\
+ (z - q^2 \nu_1)(z - q \nu_2)(z - q \nu_3)(z - q \nu_4) u(z/q) \\
+ \left[ -(q^{1/2} + q^{-1/2})z^4 + q^{1/2} (q \nu_1 + \nu_2 + \nu_3 + \nu_4 + \frac{\kappa_1}{\nu_5} + \frac{\kappa_1}{\nu_6} + \frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8}) z^3 \\
- q^{3/2} \frac{K_1}{\nu_2(f_1 + \nu_1^2)} + K_2 \right] z^2 + \frac{q^3 \nu_1 \nu_2 \nu_3 \nu_4}{\nu_2} \left\{ q^{-1/2} \left( \frac{1}{q \nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} + \frac{1}{\nu_4} \right) \\
+ \nu_5 \frac{\nu_6}{\kappa_1} + \nu_7 \frac{\nu_8}{\kappa_1} + \nu_8 \frac{\nu_8}{\kappa_1} \right\} z - (q^{1/2} + q^{-1/2}) \right\} u(z) = 0,
\end{aligned}
\end{equation}
Then we obtain the following correspondence

\[ \nu_f \]  

We may regard

\[
K_1 = \nu_1(\kappa_1 - \kappa_2)(\nu_1 - \nu_2)(\nu_1 - \nu_3)(\nu_4 - \nu_5), \\
K_2 = \nu_1(\frac{-\nu_1 + \nu_2 + \nu_3 + \nu_4}{\kappa_2} + \frac{1}{\nu_5} + \frac{1}{\nu_6} + \frac{1}{\nu_7} + \frac{1}{\nu_8}) \\
+ \nu_2\nu_3\nu_4\left( \frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} + \frac{\nu_5 + \nu_6 + \nu_7 + \nu_8}{\kappa_2} \right).
\]

Note that we used the relation \( \kappa_1^2 \kappa_2 = q\nu_2 \ldots \nu_8 \).

We compare it with the variant of the \( q \)-Heun equation of degree 4 given in Eq. (2.6). Then we obtain the following correspondence

\[
h_1 = q^{3/2} \nu_1, \quad h_2 = q^{1/2} \nu_2, \quad h_3 = q^{1/2} \nu_3, \quad h_4 = q^{1/2} \nu_4, \\
l_1 = \frac{q^{1/2} \kappa_1}{\nu_5}, \quad l_2 = \frac{q^{1/2} \kappa_1}{\nu_6}, \quad l_3 = \frac{q^{1/2} \kappa_1}{\nu_7}, \quad l_4 = \frac{q^{1/2} \kappa_1}{\nu_8}.
\]

We may regard \( f_1 \) as an accessory parameter.

On the blow-up at the point \( P_i \) (\( i = 2, 3, 4 \)), we obtain the equation where the parameters \( \nu_1 \) and \( \nu_i \) are exchanged.

2.3.2. The points \( P_i : (\kappa_1/\nu_i, \nu_i/\kappa_2) \) (\( i = 5, 6, 7, 8 \)).

We realize the blow-up at \((f, g) = (\kappa_1/\nu_5, \nu_5/\kappa_2)\) by setting \((f, g) = (f_1 g_1 + \kappa_1/\nu_5, g_1 + \nu_5/\kappa_2)\). Then the point \( P_5 : (\kappa_1/\nu_5, \nu_5/\kappa_2) \) corresponds to the line \( g_1 = 0 \).

We substitute \((f, g) = (f_1 g_1 + \kappa_1/\nu_5, g_1 + \nu_5/\kappa_2)\) into the equation \( L_1 y(z) = 0 \), and set \( g_1 = 0 \). Then we have

\[
\frac{q\nu_5(\kappa_1 - \nu_5 z)(\kappa_1 - \nu_7 z)(\kappa_1 - \nu_8 z)}{\kappa_1^3 z^2} y(z) \\
+ \frac{\nu_5(q\nu_1 - \nu_5 z)(q\nu_2 - z)(q\nu_3 - z)(q\nu_4 - z)}{\kappa_1 \nu_1 \nu_2 \nu_3 \nu_4 z^2(\kappa_1 q - \nu_5 z)} y(z/q) \\
+ \left\{ \frac{-q(\kappa_1 - \kappa_2)(\nu_5 - \nu_6)(\nu_5 - \nu_7)(\nu_5 - \nu_8)}{\kappa_1 (\kappa_1 \kappa_2 + f_1 \nu_5^2)(\kappa_1 - \nu_5 z)} \\
- \frac{q\kappa_1 \nu_1 \nu_2 \nu_3 \nu_4 \nu_5 z^2(\kappa_1 q - \nu_5 z)}{\kappa_1 \kappa_2 \nu_1 \nu_2 \nu_3 \nu_4 \nu_5 (\kappa_2 q - \nu_5 z)} \\
+ \frac{q(\kappa_2 - \nu_1 \nu_5)(\kappa_2 - \nu_2 \nu_5)(\kappa_2 - \nu_3 \nu_5)(\kappa_2 - \nu_4 \nu_5)}{\kappa_1 \kappa_2 \nu_1 \nu_2 \nu_3 \nu_4 \nu_5 (\kappa_2 q - \nu_5 z)} \\
- \frac{q\nu_5(\kappa_2 - \nu_5 z)(\kappa_1 - \nu_6 z)(\kappa_1 - \nu_7 z)(\kappa_1 - \nu_8 z)}{\kappa_1^3 \kappa_2 z^2(\kappa_1 - \nu_5 z)} \right\} y(z) = 0.
\]
Let $\lambda$ be the parameter such that $q^\lambda = q^{3/2}\kappa_2/\kappa_1$. Then the function $u(z) = z^\lambda y(z)/(\kappa_1 - \nu_5 z)$ satisfies
\[
(z - q\nu_3)(z - q\nu_4)(z - q\nu_5)(z - q\nu_4)u(z/q) + \left( z - \frac{\kappa_1}{q\nu_5} \right) \left( z - \frac{\kappa_1}{\nu_6} \right) \left( z - \frac{\kappa_1}{\nu_7} \right) \left( z - \frac{\kappa_1}{\nu_8} \right) u(qz)
\]
\[
+ \left[ -\left( q^{1/2} + q^{-1/2}\right) z^4 + q^{1/2} \left( \nu_1 + \nu_2 + \nu_3 + \nu_4 + \frac{\kappa_1}{q\nu_5} + \frac{\kappa_1}{\nu_6} + \frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8} \right) z^3
\]
\[
- q^{1/2} \left( \frac{\kappa_1}{\nu_1} \nu_2 \nu_3 \nu_4 \nu_5 \nu_6 \nu_7 \nu_8 \right) + K_2 \right] z^2 + q^2 \nu_1 \nu_2 \nu_3 \nu_4 \left\{ \frac{\kappa_1}{\nu_1} \nu_2 \nu_3 \nu_4 \right\} \left[ q^{-1/2} \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} + \frac{1}{\nu_4} + \frac{1}{\nu_5} + \frac{1}{\nu_6} + \frac{1}{\nu_7} + \frac{1}{\nu_8} \right) \right] u(z) + 0,
\]
where
\[
K_1 = q\nu_1 \nu_2 \nu_3 \nu_4 \kappa_1 \kappa_2 (\nu_5 - \nu_6) (\nu_5 - \nu_7) (\nu_5 - \nu_8),
\]
\[
K_2 = \frac{\kappa_1 \kappa_2}{\nu_5} \left( \frac{\nu_1}{\nu_5} + \frac{\nu_2}{\nu_5} + \frac{\nu_3}{\nu_5} + \frac{\nu_4}{\nu_5} - \frac{1}{\nu_5} + \frac{1}{\nu_6} + \frac{1}{\nu_7} + \frac{1}{\nu_8} \right)
\]
\[
+ q\nu_1 \nu_2 \nu_3 \nu_4 \left( \frac{\nu_5}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} + \frac{1}{\nu_4} + \frac{-\nu_5 + \nu_6 + \nu_7 + \nu_8}{\kappa_2} \right).
\]
Note that we used the relation $\kappa_1^2 \kappa_2^2 = q\nu_1 \nu_2 \ldots \nu_8$.

We compare it with the variant of the $q$-Heun equation of degree 4 given in Eq. (2.6). Then we obtain the following correspondence
\[
h_1 = q^{1/2} \nu_1, \quad h_2 = q^{1/2} \nu_2, \quad h_3 = q^{1/2} \nu_3, \quad h_4 = q^{1/2} \nu_4,
\]
\[
l_1 = \frac{\kappa_1}{q^{1/2} \nu_5}, \quad l_2 = \frac{q^{1/2} \nu_1}{\nu_6}, \quad l_3 = \frac{q^{1/2} \nu_1}{\nu_7}, \quad l_4 = \frac{q^{1/2} \nu_1}{\nu_8}.
\]
We may regard $f_1$ as an accessory parameter.

On the blow-up at the point $P_i$ ($i = 6, 7, 8$), we obtain the equation where the parameters $\nu_5$ and $\nu_i$ are exchanged.

2.3.3. The other cases related to the variant of the $q$-Heun equation of degree four.

We substitute $f = \nu_1$ to the equation $L_1 y(z) = 0$. Let $\lambda$ be the parameter such that $q^\lambda = q^{1/2}\kappa_2/\kappa_1$. Then the function $u(z) = z^\lambda y(z)$ satisfies
\[
(z - \frac{\kappa_1}{\nu_5}) \left( z - \frac{\kappa_1}{\nu_6} \right) \left( z - \frac{\kappa_1}{\nu_7} \right) \left( z - \frac{\kappa_1}{\nu_8} \right) u(qz)
\]
\[
+ (z - \nu_1)(z - q\nu_2)(z - q\nu_3)(z - q\nu_4)u(z/q)
\]
\[
+ \left[ -\left( q^{1/2} + q^{-1/2}\right) z^4 + q^{1/2} \left( \nu_1 + \nu_2 + \nu_3 + \nu_4 + \frac{\kappa_1}{q\nu_5} + \frac{\kappa_1}{\nu_6} + \frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8} \right) z^3
\]
\[
- q^{1/2} \left( \frac{\nu_1 - \kappa_1/\nu_5}{\nu_1 - \kappa_1/\nu_6}(\nu_1 - \kappa_1/\nu_7)(\nu_1 - \kappa_1/\nu_8)(\kappa_1 - \kappa_2) + K_2 \right) z^2
\]
\[
+ \frac{\nu_1^2 \nu_2 \nu_3 \nu_4}{\kappa_1} \left\{ q^{-1/2} \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} + \frac{1}{\nu_4} + \frac{\nu_5}{\kappa_1} + \frac{\nu_6}{\kappa_1} + \frac{\nu_7}{\kappa_1} + \frac{\nu_8}{\kappa_1} \right) z
\]
\[
- (q^{1/2} + q^{-1/2}) \right\} u(z) = 0,
\]
where $K_2$ is a value which does not depend on $z$ nor $g$. Hence we obtain the variant of the $q$-Heun equation of degree 4, and we may regard $g$ as an accessory parameter. We have similar results by substituting $f = \nu_i$ ($i = 2, 3, 4$) to the equation $L_1 y(z) = 0$.

We substitute $f = \kappa_1/\nu_i$ to the equation $L_1 y(z) = 0$. Let $\lambda$ be the parameter such that $q^\lambda = q^{1/2} \kappa_2/\kappa_1$. Then we have

$$
\begin{align*}
(z - q\nu_1)(z - q\nu_2)(z - q\nu_3)(z - q\nu_4)u(z/q) \\
+ \left( z - \frac{q\kappa_1}{\nu_5} \right) \left( z - \frac{\kappa_1}{\nu_6} \right) \left( z - \frac{\kappa_1}{\nu_7} \right) \left( z - \frac{\kappa_1}{\nu_8} \right) u(qz) \\
+ \left[ -(q^{1/2} + q^{-1/2}) z^4 + q^{1/2} \left( \nu_1 + \nu_2 + \nu_3 + \nu_4 + \frac{q\kappa_1}{\nu_5} + \frac{\kappa_1}{\nu_6} + \frac{\kappa_1}{\nu_7} + \frac{\kappa_1}{\nu_8} \right) z^3 \\
- q^{3/2} \left( \frac{(\kappa_1 - \kappa_2)(\kappa_1 - \nu_1\nu_5)(\kappa_1 - \nu_1\nu_6)(\kappa_1 - \nu_1\nu_7)(\kappa_1 - \nu_1\nu_8)}{\kappa_1^2 \kappa_2 \nu_5 (\nu_5 - \kappa_1 g)} + K_2 \right) z^2 \\
+ \frac{q^3 \kappa_1 \nu_1 \nu_2 \nu_3 \nu_4}{\kappa_2} \left( q^{-1/2} \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} + \frac{1}{\nu_4} + \frac{\nu_5}{q\kappa_1} + \frac{\nu_6}{\kappa_1} + \frac{\nu_7}{\kappa_1} + \frac{\nu_8}{\kappa_1} \right) z \\
- (q^{1/2} + q^{-1/2}) \right) \right] u(z) = 0.
\end{align*}
$$

where $K_2$ is a value which does not depend on $z$ nor $g$. Hence we obtain the variant of the $q$-Heun equation of degree 4, and we may regard $g$ as an accessory parameter. We have similar results by substituting $f = \kappa_1/\nu_i$ ($i = 6, 7, 8$) to the equation $L_1 y(z) = 0$.

3. **ON ASSOCIATED LINEAR $q$-DIFFERENCE EQUATION RELATED WITH $q$-$P(E_8^{(1)})$**

We review a Lax pair of the $q$-Painlevé equation of type $E_8^{(1)}$ ($q$-$P(E_8^{(1)})$) by following Yamada [16]. Let $u_1, u_2, \ldots, u_8, h_1$ and $h_2$ be the non-zero parameters which satisfy the condition $h_1^3 h_2^2 = qu_1 u_2 \ldots u_8$. Set

$$
U(z) = \prod_{i=1}^{8} (z - u_i),
$$

and define the quartic polynomials $P_a(x)$ and $P_d(x)$ by the condition

$$
\begin{align*}
(z - \frac{h_2}{z}) P_n \left( z + \frac{h_2}{z} \right) &= \frac{U(z)}{z^3} - \left( \frac{z}{h_2} \right)^3 U \left( \frac{h_2}{z} \right), \\
(z - \frac{h_2}{z}) P_d \left( z + \frac{h_2}{z} \right) &= z^5 U \left( \frac{h_2}{z} \right) - \left( \frac{h_2}{z} \right)^5 U(z).
\end{align*}
$$
Set
\[
\psi_n(f_1, f_2; g) = (f_1 - g)(f_2 - g) - \left( \frac{h_1}{q} - h_2 \right) (h_1 - h_2) \frac{1}{h_2},
\]
\[
\psi_d(f_1, f_2; g) = \left( \frac{f_1 q - g}{h_1} - \frac{f_2}{h_2} \right) - \left( \frac{g q - 1}{h_1} \right) \left( h_1 - h_2 \right) h_2,
\]
\[
V(f_1, f_2; g) = q \psi_n(f_1, f_2; g) P_d(g) - h_1^2 h_2^4 \psi_d(f_1, f_2; g) P_n(g),
\]
\[
f(u) = u + \frac{h_1}{u}, \quad g(u) = u + \frac{h_2}{u}, \quad \overline{f}(u) = u + \frac{h_1}{qu}, \quad \overline{g}(u) = u + \frac{qh_2}{u},
\]
\[
\varphi(f, g) = (f - g) \left( \frac{f}{h_1} - \frac{g}{h_2} \right) - (h_1 - h_2) \left( \frac{1}{h_1} - \frac{1}{h_2} \right).
\]

A Lax pair for the q-Painlevé equation of type \( E_8^{(1)} \) introduced in [16] is written as
\[
L_1 = \frac{q^5 U(z/q)}{(z^2 - h_1 q^2)(f - f(z/q))} \left[ T_{z^{-1}} - \frac{g - g(qh_1/z)}{g - g(z/q)} \right] + \frac{z^8 U(h_1/z)}{h_1^2 (z^2 - h_1)(f - f(z))} \left[ T_z - \frac{g - g(z)}{g - g(h_1/z)} \right] + \frac{(h_1 - h_2)z^2 (z^2 - qh_1) \overline{V}(f(z/q), f; g)}{q h_1^3 h_2^3 g \varphi(f, g) (g - g(h_1/z)) (g - g(z/q))} \cdot T_{z^{-1}}.
\]

Here \( T_z \) represents the transformation \( z \to qz \) and \( T \) represents the time evolution. The q-Painlevé equation of type \( E_8^{(1)} \) (q-P\( (E_8^{(1)}) \)) for the variables \( f \) and \( g \) was obtained by the compatibility condition for the operators \( L_1 \) and \( L_2 \) in [16], and the expression of q-P\( (E_8^{(1)}) \) was also described there. It seems that the definitions of the equation q-P\( (E_8^{(1)}) \) and the Lax pair in [6] is less explicit than those in [6], and we use the definition in [16].

We pick up the q-difference equation \( L_1 y(z) = 0 \), i.e.
\[
\frac{q^5 U(z/q)}{(z^2 - h_1 q^2)(f - f(z/q))} \left[ y(z/q) - \frac{g - g(qh_1/z)}{g - g(z/q)} y(z) \right] + \frac{z^8 U(h_1/z)}{h_1^2 (z^2 - h_1)(f - f(z))} \left[ y(z) - \frac{g - g(z)}{g - g(h_1/z)} y(z) \right] + \frac{(h_1 - h_2)z^2 (z^2 - qh_1) \overline{V}(f(z/q), f; g)}{q h_1^3 h_2^3 g \varphi(f, g) (g - g(h_1/z)) (g - g(z/q))} y(z) = 0,
\]
and consider a limit with respect to the parameters \( f \) and \( g \). Namely, set
\[
(3.3) \quad f = f(u_1 + \varepsilon c_1) = u_1 + \varepsilon c_1 + \frac{h_1}{u_1 + \varepsilon c_1}, \quad g = g(u_1 + \varepsilon c_2) = u_1 + \varepsilon c_2 + \frac{h_2}{u_1 + \varepsilon c_2}
\]
and consider the limit \( \varepsilon \to 0 \) in Eq. (3.2). Then
\[
f - f(z) = \frac{(z - u_1)(u_1 z - h_1)}{u_1 z} + O(\varepsilon), \quad g - g(z) = \frac{(z - u_1)(u_1 z - h_2)}{u_1 z} + O(\varepsilon)
\]
as \( \varepsilon \to 0 \). Write

\[
U_7(z) = \prod_{i=2}^{8}(z - u_i).
\]

Then \( U(z) = (z - u_1)U_7(z) \). On the coefficients of \( y(z/q) \) and \( y(qz) \) in Eq. (3.2), we have

\[
\frac{q^5U(z/q)}{(z^2 - h_1q^2)(f - f(z/q))} = \frac{-q^5u_1zU_7(z/q)}{(z^2 - h_1q^2)(u_1z - qh_1)} + O(\varepsilon),
\]

\[
\frac{z^8U(h_1/z)}{h_1^4(z^2 - h_1)(f - f(z))} = \frac{z^8u_1U_7(h_1/z)}{h_1^4(z^2 - h_1)(z - u_1)} + O(\varepsilon).
\]

The limit of the coefficient of \( y(z) \) is more complicated. By applying a gauge transformation for \( y(z) \) and calculating the limit of Eq. (3.2) as \( \varepsilon \to 0 \), we obtain the following theorem.

**Theorem 3.1.** Set

\[
y(z) = \frac{(z - qu_1)(u_1z - h_1)}{z} \tilde{y}(z).
\]

Write \( f \) and \( g \) as Eq. (3.3). Then Eq. (3.2) tends to

\[
B^{-}(z)\tilde{y}(z/q) + B^{+}(z)\tilde{y}(qz) + B^{0}(z)\tilde{y}(z) = \left(C_0\frac{c_2}{c_1 - c_2} + C'_0\right)\tilde{y}(z)
\]

as \( \varepsilon \to 0 \), where

\[
B^{-}(z) = \frac{(z - q^2u_1)}{q^2h_1z^2(z^2 - qh_1)(z^2 - q^2h_1)} \prod_{j=2}^{8}(z - qu_j),
\]

\[
B^{+}(z) = \frac{q(qu_1z - h_1)}{h_1^5z^2(z^2 - qh_1)(z^2 - h_1)} \prod_{j=2}^{8}(u_jz - h_1)
\]

\[
B^{0}(z) = \frac{-qz(h_1^{1/2} - qu_1)}{2h_1^{7/2}(z - h_1^{1/2})(z - qh_1^{1/2})} \prod_{j=2}^{8}(h_1^{1/2} - u_j)
\]

\[
+ \frac{qz(h_1^{1/2} + qu_1)}{2h_1^{1/2}(z + h_1^{1/2})(z + qh_1^{1/2})} \prod_{j=2}^{8}(h_1^{1/2} + u_j) - \frac{q^{3/2}}{h_1}(u_1u_2u_3u_4u_5u_6u_7u_8)^{1/2}
\]

\[
\cdot \left[(q + 1)\frac{z^2}{(qh_1)^2} + \frac{1}{z^2} - \left(\frac{z}{qh_1} + \frac{1}{z}\right)\left\{q\frac{u_1}{h_1} + \frac{1}{qu_1} + \sum_{j=2}^{8}(\frac{u_j}{h_1} + \frac{1}{u_j})\right\}\right],
\]

\[
C_0 = \frac{q(h_2 - h_1)}{h_1^2u_1(h_1 - u_1^2)(h_2 - u_1^2)} \prod_{j=2}^{8}(u_1 - u_j)
\]

and \( C'_0 \) is the constant which does not depend on \( z \), \( c_1 \) and \( c_2 \).
Proof. As $\varepsilon \to 0$, we have

$$\varphi(f, g) = -\frac{(h_1 - h_2)(h_1 - u_1^2)(h_2 - u_1^2)(c_1 - c_2)}{u_1^2 h_1 h_2} \varepsilon + O(\varepsilon^2),$$

$$\psi_n(f(z/q), f; g) = \frac{(h_1 - h_2)(u_1 z - qh_2)(h_2 z - h_1 u_1)}{q h_2 u_1^2 z} + O(\varepsilon),$$

$$\psi_d(f(z/q), f; g) = -\frac{(h_1 - h_2)(u_1 z - qh_2)(h_2 z - h_1 u_1)}{h_1^2 h_2^2 z} + O(\varepsilon)$$

and

$$(3.6) \quad \psi_n(f(z/q), f; g) = -\frac{h_1^2 h_2}{q u_1^2} \frac{h_2^2}{h_1^2} \left\{ \frac{h_2(h_1 - u_1^2)(h_2 + u_1^2)(u_1 z - h_1)(z - qu_1)}{u_1^2 h_1 - h_1)(u_1 z - h_1)(z - qu_1)} (c_1 - c_2) + 2c_2 \right\} \varepsilon + O(\varepsilon^2).$$

Set $u = u_1 + c_2 \varepsilon$. Then $g = u + h_2 / u$ and it follows from Eq. (3.1) that

$$V(f(z/q), f; g) = q \psi_n(f(z/q), f; g) P_d(u + h_2 / u) - h_1^2 h_2^2 \psi_d(f(z/q), f; g) P_n(h_2, u + h_2 / u)$$

$$= q \psi_n(f(z/q), f; g) \frac{u^3(u^2 - u_1^2) U(h_2 / u)}{u - h_2 / u}$$

$$+ q \psi_n(f(z/q), f; g) \frac{u^3 u_1^2 U(h_2 / u)}{u - h_2 / u} + h_1^2 h_2^2 \psi_d(f(z/q), f; g) \frac{u^3 U(h_2 / u)}{u - h_2 / u}$$

$$- q \psi_n(f(z/q), f; g) \frac{(h_2 / u)^2 U(u)}{u - h_2 / u} - h_1^2 h_2^2 \psi_d(f(z/q), f; g) \frac{U(u)}{u - h_2 / u}.$$ 

We define the equivalence by

$$A(\varepsilon) \simeq B(\varepsilon) \iff \lim_{\varepsilon \to 0} \frac{A(\varepsilon)}{B(\varepsilon)} = 1.$$

Then $U(u) = U(u_1 + \varepsilon c_2) \simeq \varepsilon c_2 U_7(u_1)$ and $U(h_2 / u) \simeq -(u_1 - h_2 / u_1) U_7(h_2 / u_1)$. Hence

$$q \psi_n(f(z/q), f; g) \frac{u^3(u^2 - u_1^2) U(h_2 / u)}{u - h_2 / u}$$

$$\simeq -2qu_1^4 U_7(h_2 / u_1)c_2 \psi_n(f(z/q), f; g) \varepsilon \simeq 2u_1^2 h_1^2 h_2 c_2 U_7(h_2 / u_1) \psi_d(f(z/q), f; g) \varepsilon.$$

It follows from Eq. (3.6) that

$$\psi_n(f(z/q), f; g) \frac{u^3 u_1^2 U(h_2 / u)}{u - h_2 / u} + h_1^2 h_2 \psi_d(f(z/q), f; g) \frac{u^3 U(h_2 / u)}{u - h_2 / u}$$

$$\simeq -\psi_d(f(z/q), f; g) U_7(h_2 / u_1)$$

$$\left\{ \frac{h_1^2 h_2^2 (h_1 - u_1^2)(h_2 + u_1^2)(u_1 z - h_1)(z - qu_1)}{(h_1 - h_2)(u_1 z - h_2 q)(h_2 z - h_1 u_1)} (c_1 - c_2) + 2h_1^2 h_2 u_1^2 c_2 \right\} \varepsilon.$$
Similarly, we have

\[-q\psi_n(f(z/q), f; g)(h_2/u)^3 U(u)\frac{(h_2/u)u_2^2\psi_d(f(z/q), f; g)U(u)/u^3}{u - h_2/u} - h_1^2 h_2^2 \psi_d(f(z/q), f; g)\frac{U(u)/u^3}{u - h_2/u}\]

\[\simeq qh_1 h_2 (h_2/u_1)^5 \varepsilon c_2 U_7(u_1) \frac{U_7(u_1)}{u_1 - h_2/u_1} \psi_d(f(z/q), f; g)\]

\[= -\frac{h_1^2 h_2^2 (u_1 + h_2/u_1)}{u_1^5} U_7(u_1) \psi_d(f(z/q), f; g)c_2\varepsilon.
\]

Thus

\[V(f(z/q), f; g) \simeq -h_1^2 h_2^2 (u_1^2 + h_2) U_7(u_1) \psi_d(f(z/q), f; g)c_2\varepsilon\]

\[-\psi_0(f(z/q), f; g) U_7(h_2/u_1) \frac{h_1^2 h_2^2 (-u_1^2 + h_1)(u_1 z - h_1)(z - qu_1)(u_2^2 + h_2)}{(h_1 - h_2)(u_1 z - h_2)(h_2 z - h_1 u_1)}(c_1 - c_2)\varepsilon\]

\[\simeq \frac{h_2^2 (h_1 - h_2)(u_1^2 + h_2)(u_1 z - h_1)(z - qu_1)}{u_1^6 z} U_7(u_1) c_2\varepsilon\]

\[+ \frac{(-u_1^2 + h_1)(u_1^2 + h_2)(u_1 z - h_1)(z - qu_1)}{z} U_7(h_2/u_1)(c_1 - c_2).\varepsilon.
\]

On the other hand, we have

\[\frac{(h_1 - h_2) z^2 (z^2 - q h_1)}{q h_1^3 h_2^3 h_2^4 \varphi(f, g)(g - g(h_1/z))(g - g(z/q))}\]

\[\simeq \frac{-z^4 (z^2 - q h_1) u_1^6}{h_1 h_2^2 (h_2 + u_1^2)(h_1 - u_1^2)(h_1 - u_1^2)(h_2 z - h_2 h_1)(z - qu_1)(u_1 z - h_1)(u_1 z - h_2)(c_1 - c_2)}.
\]

Therefore

\[\frac{(h_1 - h_2) z^2 (z^2 - q h_1) V(f(z/q), f; g)}{q h_1^3 h_2^3 h_2^4 \varphi(f, g)(g - g(h_1/z))(g - g(z/q))}\]

\[\simeq -\frac{(h_1 - h_2) U_7(u_1) z^3 (z^2 - q h_1) c_2}{u_1^6 U_7(h_2/u_1)(z^2 - q h_1) z^3 (z^2 - h_1 u_1)(z^2 - h_1 u_1)(z - qu_1)(u_1 z - h_1)(c_1 - c_2)}.
\]

By substituting it into Eq. (3.2), we have

\[\frac{-q^5 u_1 z U_7(z/q)}{(z^2 - h_1 q^2)(u_1 z - q h_1)} y(z/q) + \frac{q^5 u_1 z U_7(z/q)}{(z^2 - h_1 q^2)(u_1 z - q h_1)} y(qz)\]

\[+ \frac{z^8 u_1^7 U_7(h_1/z)}{h_1^4 (z^2 - h_1)(z - u_1^2)} y(qz) - \frac{z^8 u_1^7 U_7(h_1/z)}{h_1^4 (z^2 - h_1)(z - u_1^2)} y(z)\]

\[-\frac{(h_1 - h_2) U_7(u_1) z^3 (z^2 - qu_1)(u_1 z - h_1)(c_1 - c_2)}{u_1^6 U_7(h_2/u_1)(z^2 - q h_1) c_2}\]

\[-\frac{h_1 h_2 (h_2 - u_1^2)(h_2 z - h_1 u_1)(z^2 - qu_1)(u_1 z - h_1)(c_1 - c_2)}{u_1^6 U_7(h_2/u_1)(z^2 - q h_1) c_2} y(z) = 0.
\]
Recall that the function $\tilde{y}(z)$ was defined in Eq. (3.4) and it follows that $y(z/q) = \frac{1}{z}q(z/q - qu_1)(u_1 z - h_1)\tilde{y}(z/q)$ and $y(qz) = \frac{1}{z}q^{-1}(qz - qu_1)(qu_1 z - h_1)\tilde{y}(z/q)$. Then we obtain

\begin{equation}
(3.7) \quad \frac{-q^{-3}(z - q^2 u_1)}{z^2(z^2 - qh_1)(z^2 - h_1^2)}\tilde{y}(z/q) - \frac{(qu_1 z - h_1)}{h_1^2 z^2(z^2 - qh_1)(z^2 - h_1)}\tilde{y}(qz)
+ H(z)\tilde{y}(z) = \frac{1}{u_1 h_1(h_1 - u_1^2)(h_2 - u_1^2)(c_1 - c_2)}\tilde{y}(z) = 0,
\end{equation}

where

\begin{equation}
H(z) = \frac{q^5(h_2 z - qh_1 u_1)(u_1 z - h_1)U_7(z/q)}{h_1^2 z^2(z^2 - qh_1)(z^2 - q^2 h_1)(u_1 z - h_2 q)}
- \frac{z^5(u_1 z - h_2)(z - qu_1)U_7(h_1/z)}{h_1^2(z^2 - qh_1)(z^2 - h_1)(h_2 z - h_1 u_1)}
- \frac{u_1^5 U_7(h_2/u_1)(z - qu_1)(u_1 z - h_1)}{h_1^2 h_2^2(h_2 - u_1^2)(h_2 z - h_1 u_1)(u_1 z - qh_2)}.
\end{equation}

It is shown that the poles $z = \pm(qh_1)^{1/2}, h_1 u_1/h_2, qh_2/u_1$ of the function $H(z)$ are cancelled, and we have

\begin{equation}
(3.8) \quad H(z) = \frac{z(h_1^{1/2} - qh_1 u_1)(h_1^{1/2} - u_2)(h_1^{1/2} - u_3) \cdots (h_1^{1/2} - u_8)}{2h_1^{5/2}(z - h_1^{1/2})(z - qh_1^{1/2})}
- \frac{2h_1^{5/2}(z + h_1^{1/2})(z + qh_1^{1/2})}{h_1 h_2(q + 1)(z^2/(qh_1)^2 + z^{-2}) - (z/(qh_1) + z^{-1})h_2(qu_1 + u_2)
+ u_3 + \cdots + u_8 + h_1/(qu_1) + h_1/u_2 + h_1/u_3 + \cdots + h_1/u_8 + C},
\end{equation}

where $C$ is a constant which does not depend on the variable $z$ and the parameter $c_1$ and $c_2$. Recall that there is a relation $h_1^2 h_2^2 = qu_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8$. By substituting Eq. (3.8) and $h_2 = (qu_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8)^{1/2}/h_1$ into Eq. (3.7), we obtain the theorem. $\square$

Set $z = (qh_1)^{1/2}, \tilde{y}(z) = u(x), u_1 = q^{-1}h_1^{1/2} v_1, u_2 = h_1^{1/2} v_2, \ldots, u_8 = h_1^{1/2} v_8$ and $E = C_0 c_2/(c_1 - c_2) + C_0'$ in Eq. (3.3). Then we have

\begin{equation}
(3.9) \quad \frac{\prod_{j=1}^{8}(x - q^{1/2}v_j)}{qx^2(x^2 - 1)(x^2 - q^{-1})u(x/q)}u(x/q) + \frac{\prod_{j=1}^{8}(q^{1/2}v_jx - 1)}{qx^2(x^2 - 1)(qx^2 - 1)}u(qx)
+ \tilde{B}^0(x)u(x) = E u(x),
\end{equation}

where

\begin{equation}
\tilde{B}^0(x) = \frac{-q^{1/2}x}{2(x - q^{-1/2})(x - q^{1/2})} \prod_{j=1}^{8}(1 - v_j) + \frac{q^{1/2}x}{2(x + q^{-1/2})(x + q^{1/2})} \prod_{j=1}^{8}(1 + v_j)
- (v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8)^{1/2}\left[(q + 1)\left(x^2 + \frac{1}{x^2}\right) - q^{1/2}\left(x + \frac{1}{x}\right)\sum_{j=1}^{8}\left(v_j + \frac{1}{v_j}\right)\right].
\end{equation}

We compare it with the firstly degenerated Ruijsenaars-van Diejen operator. Let $\exp(\pm i a \partial_z)$ be the shift operator such that $\exp(\pm i a \partial_z)f(z) = f(z \pm ia)$. The
firstly degenerated Ruijsenaars-van Diejen operator $\tilde{A}^{(1)}(h; z)$ was discussed in [13], and it was expressed in [13] as

$$
\tilde{A}^{(1)}(h; z) = \tilde{V}^{(1)}(h; z) \exp(-ia_- \partial_z) + \tilde{W}^{(1)}(h; z) \exp(ia_- \partial_z) + U^{(1)}(h; z),
$$

where

$$
\tilde{V}^{(1)}(h; z) = \frac{\prod_{n=1}^{8} (1 - e^{-2\pi i z} e^{2\pi i h_n e^{-\pi a_-} - 4\pi i z})}{e^{-2\pi a_-} e^{-4\pi i z} (1 - e^{-4\pi i z}) (1 - e^{-2\pi i z} e^{-2\pi a_-})},
$$

$$
\tilde{W}^{(1)}(h; z) = \frac{\prod_{n=1}^{8} (1 - e^{2\pi i h_n e^{-\pi a_-}})}{e^{-2\pi a_-} e^{4\pi i z} (1 - e^{-4\pi i z}) (1 - e^{4\pi i z} e^{-2\pi a_-})},
$$

$$
U^{(1)}(h; z) = \frac{\prod_{n=1}^{8} (e^{2\pi i h_n} - 1)}{2(1 - e^{2\pi i z} e^{\pi a_-} - 2\pi i z) (1 - e^{-2\pi i z} e^{\pi a_-})} + \frac{\prod_{n=1}^{8} (e^{2\pi i h_n} + 1)}{2(1 + e^{2\pi i z} e^{\pi a_-}) (1 + e^{-2\pi i z} e^{\pi a_-})}
$$

$$
+ e^{-\pi a_-} \prod_{n=1}^{8} e^{\pi i h_n} \left[ (e^{2\pi i z} + e^{-2\pi i z}) \sum_{n=1}^{8} (e^{2\pi i h_n} - e^{-2\pi i h_n}) \right.
$$

$$
- (e^{\pi a_-} + e^{-\pi a_-}) (e^{4\pi i z} + e^{-4\pi i z}) \right].
$$

Set $x = e^{2\pi i z}$, $q = e^{-2\pi a_-}$ and $v_n = e^{2\pi i h_n}$. Then we have $\exp(\pm ia_- \partial_z) f(e^{2\pi i z}) = f(e^{2\pi i (z \pm ia_-)}) = f(q \pm 1)$. It follows from a straightforward calculation that the equation $\tilde{A}^{(1)}(h; z) f(z) = Ef(z)$ is equivalent to Eq. (3.9). Therefore we obtain the following theorem.

**Theorem 3.2.** The $q$-difference equation in Eq. (3.9) which was obtained from the associated linear equation of the $q$-Painlevé equation of type $E_8^{(1)}$ is equivalent to the equation $\tilde{A}^{(1)}(h; z) f(z) = Ef(z)$, where $\tilde{A}^{(1)}(h; z)$ is the firstly degenerated Ruijsenaars-van Diejen operator and $E$ is an arbitrary complex number.

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