Inversion formulas and their finite-dimensional analogs for multidimensional Volterra equations of the first kind

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Abstract. The paper focuses on solving one class of Volterra equations of the first kind, which is characterized by the variability of all integration limits. These equations were introduced in connection with the problem of identifying nonsymmetric kernels for constructing integral models of nonlinear dynamical systems of the "input-output" type in the form of Volterra polynomials. The case when the input perturbation of the system is a vector function of time is considered. To solve the identification problem, previously introduced test signals of duration $h$ (mesh step) are used in the form of linear combinations of Heaviside functions with deviating arguments. The paper demonstrates a method for obtaining the desired solution, developing a method of steps for a one-dimensional case. The matching conditions providing the desired smoothness of the solution are established. The mesh analogs of the studied integral equations based on the formulas of middle rectangles are considered.

1. Introduction

Volterra integral equations are used in various applied problems. A detailed review of such applications is presented in the monograph [1]. The class of integral equations considered in this paper arises when modeling the response of a nonlinear dynamical system $y(t)$ to an input signal $x(t)$ in the form of a Volterra polynomial (a segment of an integro-power series) [2]. Integral models based on Volterra polynomials attract the attention of many researchers and have an extensive field of applications (a review of the current state of research is given in [3]). Of greatest interest in terms of applications is the case where the input signal $x(t) = (x_1(t),...,x_p(t))^T$ is a vector function of time:

$$y(t) = \sum_{m=1}^{n} \sum_{1 \leq i_1 \leq ... \leq i_m \leq p} f_{i_1...i_m}(t), \quad t \in [0,T],$$ (1)

$$f_{i_1...i_m}(t) = \int_0^t \cdots \int_0^t K_{i_1...i_m}(s_1,...,s_m) \prod_{j=1}^m x_j(t-s_j)ds_j. \quad (2)$$

In (1) $y(t)$ is a scalar function of time, $y(0) = 0$, $y'(t) \in C_{[0,T]}$. The functions $K_{i_1...i_m}$ in (2) are called Volterra kernels and they are symmetric only in those variables that correspond to the
coinciding indices \( i_1 \ldots i_m \). The key problem in constructing a model of nonlinear dynamical system of input-output type in form (1), (2) lies in the identification of Volterra kernels. The absence of the symmetry property of the functions \( K_{i_1 \ldots i_m} \) complicates the problem of constructing (1). This paper develops the approach [4] which is based on setting \((m - 1)-\)parametric families of test signals in the form of combinations of Heaviside functions with deviating arguments.

Confine ourselves to \( x(t) = (x_1(t), x_2(t))^T \) and consider the case which is most widely used in practice, when in (1) \( n = 2 \). Suppose further that the problem of decomposing the response (1) into components (2) is somehow solved and consider the problem of identifying an nonsymmetric kernel \( K_{12} \).

2. The problem statement

To solve the problem of identification of nonsymmetrical kernel \( K_{12} \) in

\[
f_{12}(t) = \int_0^t \int_0^t K_{12}(s_1, s_2)x_1(t - s_1)x_2(t - s_2)ds_1ds_2, \quad t \in [0, T], \tag{3}
\]

introduce two series of test signals [6]:

\[
\begin{align*}
\begin{cases}
  x_1(t) = e(t) - e(t - h), \\
x_2_v(t) = e(t - v) - e(t - v - h),
\end{cases} & t \in [h, T], \quad v \leq t - h, \tag{4} \\
\begin{cases}
  x_{1_v}(t) = e(t - v) - e(t - v - h), \\
x_2(t) = e(t) - e(t - h),
\end{cases} & t \in [h, T], \quad v \leq t - h, \tag{5}
\end{align*}
\]

where \( h > 0 \) is a sampling interval of the output signal, \( T = Nh, N = \text{const} \). Figures 1 and 2 illustrate test signals (4), (5), respectively.

![Figure 1. Test signals \( x_1(t) \) and \( x_{2_v}(t) \).](image)

![Figure 2. Test signals \( x_{1_v}(t) \) and \( x_2(t) \).](image)
Substituting (4), (5) into (3) we obtain a paired Volterra equation of the first kind

\[
\int_{t-h}^{t} ds_1 \int_{t-v}^{t-h} ds_2 K_{12}(s_1, s_2) = f(t, v),
\]

where \(t \in [h, T], f(t, v)\) denote responses to (4) and (5), respectively. It is seen from (6), (7) that for \(v = 0, t \in [h, T]\), \(f(t, 0) = f(t, 0) = f(t, 0) = f(t, 0)\) holds true.

In this case, the definition of a solution of (6), (7) needs to be refined. By virtue of \(t - h \geq 0\) within the low limits of integration, the domain of the sought function \(K_{12}\) is segment \([0, T]\), including segment \([0, h]\). Therefore, paired equation (6), (7) makes sense only in the case where \(K_{12}\) on \([0, h]\) is known. By analogy with [4], for one-dimensional integral equations with "prehistory", require that

\[
K_{12}(s_1, s) = K_{12}^{(0)}(s_1, s), K_{12}(s, s_2) = K_{12}^{(0)}(s, s_2), s_1, s_2 \in [0, h], s \in [0, T].
\]

Consider further the procedure of obtaining the sought solution to (6), (7). The procedure develops the method of steps for the one-dimensional case [4] (pp. 58–62). This method is a classical method in the theory of differential equations with a deviating argument [5]. It has proven to be efficient for constructing a solution to the one-dimensional Volterra equation of the first kind with two variables of integration [4].

3. Development of the steps method

We introduce

\[
\Delta_k = \{t, v : v + h \leq t, kh < t < (k + 1)h\}, \ \Delta_{N+1} = \{t, v : v + h \leq t, Nh \leq t \leq T\},
\]

\[
\Delta_0 = \{t, v : D_1 \cup D_2, v \geq 0\}, \ D_1 = \{t, v : v \leq t, 0 \leq t < h, h > 0\},
\]

\[
D_2 = \{t, v : t - h < v \leq t, h \leq t \leq T, h > 0\}, \ N = \frac{T}{h}, k = \frac{1}{N},
\]

such that \(\Delta_0\) coincides with the "prehistory" and

\[
\bigcup_{k=1}^{N+1} \Delta_k = \{t, v : v + h \leq t, h \leq t \leq T, v \geq 0, h > 0\}.
\]

Let \(N(t, v)\) be a point of a plain with Cartesian coordinates. We will show that the condition for solvability of (6), (7) at the initial point \(N(h, 0) \in \Delta_1\) is met:

\[
\int_{t-h}^{t} ds_1 \int_{t-h}^{t} ds_2 K_{12}(s_1, s_2) = \int_{t-h}^{h} ds_1 \int_{t-h}^{t} ds_2 K_{12}(s_1, s_2) + \int_{h}^{t} ds_1 \int_{h}^{t} ds_2 K_{12}(s_1, s_2) +
\]

\[
+ \int_{h}^{t} ds_1 \int_{t-h}^{h} ds_2 + \int_{h}^{t} ds_1 \int_{t-h}^{h} ds_2 = \int_{t-h}^{h} ds_1 \int_{t-h}^{h} ds_2 K_{12}^{(0)}(s_1, s_2).
\]
\[ + \int_{t-h}^{t} ds_1 \int_{h}^{t} K_{12}^{(0)}(s_1, s_2)ds_2 + \int_{h}^{t} ds_1 \int_{t-h}^{t} K_{12}^{(0)}(s_1, s_2)ds_2 + \int_{h}^{t} ds_1 \int_{h}^{t} K_{12}(s_1, s_2)ds_2 \]

and by (8)

\[ \int_{h}^{t} ds_1 \int_{h}^{t} K_{12}(s_1, s_2)ds_2 = f(t, 0) - \int_{h}^{t} ds_1 \int_{t-h}^{t} K_{12}^{(0)}(s_1, s_2)ds_2 - \int_{t-h}^{t} ds_1 \int_{h}^{t} K_{12}^{(0)}(s_1, s_2)ds_2 - \int_{h}^{t} ds_1 \int_{h}^{t} K_{12}(s_1, s_2)ds_2 \equiv f_1(t, 0). \]

Note, that

\[ f_1(h, 0) = f(h, 0) - \int_{0}^{h} ds_1 \int_{0}^{h} K_{12}^{(0)}(s_1, s_2)ds_2 = 0. \tag{9} \]

Let \( M(p, q) \in \Omega_k(N(t, v)) \), \( k = \Gamma, N \), is a point of the plain with Cartesian coordinates \((p, q)\), \( t < v \leq q \leq p \leq t \), \( \bar{M} \) is a point from \( \Omega_k(N(t, v)) \), symmetrical to \( M \) with respect to diagonal \( p = q \), so that \( t - v \leq p \leq q \leq t \). Assuming that \( f(t, v), f(t, v) \in C_{\Delta_1}^{(2)} \), we solve (6), (7) by differentiation with respect to \( t \) and \( v \), so that for \( N(t, v) \in \Delta_1 \)

\[ K_{12}(M) = D_2 \int (t, v) + K_{12}^{(0)}(t, t - v - h) + K_{12}^{(0)}(t - h, t - v) - K_{12}^{(0)}(t - h, t - v - h), \tag{10} \]

\[ K_{12}(M) = D_2 \int (t, v) + K_{12}^{(0)}(t - v - h, t) + K_{12}^{(0)}(t - v, t - h) - K_{12}^{(0)}(t - v, t - h - h), \tag{11} \]

\[ D_2 \int (t, v) = - (f_{h,t,v}^{(1)} + f_{h,t,v}^{(1)}), \quad D_2 \int (t, v) = - (f_{h,t,v}^{(2)} + f_{h,t,v}^{(2)}), \]

where \( M(p, q) \in \Omega_1(N(t, v)) \).

We denote this solution by \( K_{12}^{(i)}(M) \) and rewrite (10), (11) in the following form:

\[ K_{12}^{(i)}(M) = D_2 \int (N(t, v)) + K_{12}^{(i-1)}(t, t - v - h) + K_{12}^{(i-1)}(t - h, t - v) - K_{12}^{(i-1)}(t - h, t - v - h), \tag{12} \]

\[ K_{12}^{(i)}(M) = D_2 \int (N(t, v)) + K_{12}^{(i-1)}(t - v - h, t) + K_{12}^{(i-1)}(t - v, t - h) - K_{12}^{(i-1)}(t - v, t - h - h), \tag{13} \]

where \( i = 1 \).

The condition of simultaneous continuity of the initial function \( K_{12}^{(0)} \) and the desired solution \( K_{12}^{(1)} \) at points \( M, \bar{M} \in \Omega_1(N(t, v)) \) for \( h \leq t < 2h, v = t - h \) follows from (12), (13):

\[ D_2 \int (N(t, v)) = K_{12}^{(i)}(t, t - v) - K_{12}^{(i-1)}(t, t - v - h) - K_{12}^{(i-1)}(t - h, t - v) - K_{12}^{(i-1)}(t - h, t - v - h) = \]

\[ = K_{12}^{(i-1)}(t, t - v) - K_{12}^{(i-1)}(t, t - v - h) - K_{12}^{(i-1)}(t - h, t - v) + K_{12}^{(i-1)}(t - h, t - v - h), \tag{14} \]

\[ D_2 \int (N(t, v)) = K_{12}^{(i)}(t - v, t) - K_{12}^{(i-1)}(t - v - h, t) - K_{12}^{(i-1)}(t - v, t - h) + K_{12}^{(i-1)}(t - v - h, t - h) = \]

\[ = K_{12}^{(i-1)}(t - v, t) - K_{12}^{(i-1)}(t - v - h, t) - K_{12}^{(i-1)}(t - v, t - h) + K_{12}^{(i-1)}(t - v - h, t - h). \tag{15} \]
In particular, from (14), (15) for $\mathbf{M}(h, h) = \mathbf{M}(h, h)$ ($t = h, v = 0$), we have

$$D_2 f |_{\mathbf{N}(h, 0)} = K_{12}^{(0)}(h, h) - K_{12}^{(0)}(h, 0) - K_{12}^{(0)}(0, h) + K_{12}^{(0)}(0, 0),$$

$$D_2 f |_{\mathbf{N}(h, 0)} = D_2 f |_{\mathbf{N}(h, 0)} = D_2 f |_{\mathbf{N}(h, 0)}.$$

If $K_{12}^{(0)}$ is continuous on $\Omega_0$, then (12), (13) implies that $K_{12}^{(1)}$ is continuous on $\Omega_1$.

Now let $\mathbf{N}(t, 0) \in \Delta_2$. Then, using the same procedure, we have

$$\int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}(s_1, s_2) ds_2 = \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}(s_1, s_2) ds_2 + \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}(s_1, s_2) ds_2 +$$

$$+ \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}(s_1, s_2) ds_2 + \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}(s_1, s_2) ds_2 = \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}^{(1)}(s_1, s_2) ds_2 +$$

$$+ \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}^{(1)}(s_1, s_2) ds_2 + \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}^{(1)}(s_1, s_2) ds_2 + \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}(s_1, s_2) ds_2,$$

so that (6), (7) implies

$$\int_{2h}^{t} ds_1 \int_{2h}^{t} K_{12}(s_1, s_2) ds_2 = f(t, 0) - \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}^{(1)}(s_1, s_2) ds_2 -$$

$$- \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}^{(1)}(s_1, s_2) ds_2 - \int_{t-h}^{t} ds_1 \int_{t-h}^{t} K_{12}^{(1)}(s_1, s_2) ds_2 \equiv f_2(t, 0).$$

The solvability condition for (17) $f_2(2h, 0) = 0$ is obviously satisfied, since

$$f_2(2h, 0) = f(2h, 0) - \int_{h}^{2h} ds_1 \int_{h}^{2h} K_{12}^{(1)}(s_1, s_2) ds_2 =$$

$$= \int_{h}^{2h} ds_1 \int_{h}^{2h} K_{12}^{(1)}(s_1, s_2) ds_2 - \int_{h}^{2h} ds_1 \int_{h}^{2h} K_{12}^{(1)}(s_1, s_2) ds_2 \equiv 0.$$

Therefore, under the assumption that $f_2(t, v) \in C_{\Delta_2}^{(2)}$ we have (12), (13), where $\mathbf{N}(t, v) \in \Delta_2, \ i = 2$.

Make sure that conditions (14), (15) for $i = 2$ provide continuity, firstly, of functions $K_{12}^{(2)}$ and $K_{12}^{(0)}$ at $2h \leq t < 3h, v = t - h$, and, secondly, of functions $K_{12}^{(1)}$, $K_{12}^{(2)}$ at $t = 2h, 0 \leq v < h$. Indeed, in the first case it follows from (14) that

$$D_2 f |_{\mathbf{N}(t, v)} = K_{12}^{(2)}(t, h) - K_{12}^{(1)}(t, 0) - K_{12}^{(1)}(t - h, h) + K_{12}^{(1)}(t - h, 0) =$$

$$= K_{12}^{(2)}(t, h) - K_{12}^{(0)}(t, 0) - K_{12}^{(0)}(t - h, h) + K_{12}^{(0)}(t - h, 0) =$$
where

\[ \epsilon, t \in \mathbb{R} \]

in (14), (15) gives

\[ f \mid_{N(t, v)} = K^{(0)}_{12}(h, t) - K^{(0)}_{12}(0, t) - K^{(0)}_{12}(h, t - h) + K^{(0)}_{12}(0, t - h), \]

so that

\[ \lim_{\epsilon \to 0} K^{(0)}_{12}(t, h - \epsilon) = K^{(2)}_{12}(t, h), \]

\[ \lim_{\epsilon \to 0} K^{(0)}_{12}(h, t - \epsilon) = K^{(2)}_{12}(h, t). \]

In the second case, the substitution of \( i = 2, t = 2h \) in (14), (15) gives

\[ \lim_{\epsilon \to 0} K^{(0)}_{12}(2h - \epsilon, 2h - \epsilon) = K^{(2)}_{12}(2h - \epsilon, 2h - \epsilon), \]

\[ \lim_{\epsilon \to 0} K^{(0)}_{12}(2h - \epsilon, 2h - \epsilon) = K^{(2)}_{12}(2h - \epsilon, 2h - \epsilon). \]

Extending by [4] this process to \( \Delta_k, k = 1, N + 1 \), we find a solution to (6), (7):

\[ K_{12}(t, t - h) = K^{(0)}_{12}(t, h) - K^{(0)}_{12}(0, t) - K^{(0)}_{12}(t, h - h) + K^{(0)}_{12}(0, t - h), \]

so that

\[ \lim_{\epsilon \to 0} K^{(0)}_{12}(2h - \epsilon, 2h - \epsilon) = K^{(2)}_{12}(2h - \epsilon, 2h - \epsilon), \]

\[ \lim_{\epsilon \to 0} K^{(0)}_{12}(2h - \epsilon, 2h - \epsilon) = K^{(2)}_{12}(2h - \epsilon, 2h - \epsilon). \]

Consider an illustrative example.

**Example 1.** Let

\[ \int_{t-1}^{t} ds_1 \int_{t-v}^{t-v} ds_2 K_{12}(s_1, s_2) ds_2 = -t^2 - v^2 + 2tv + 2t - v - \frac{5}{6}, \quad (19) \]

\[ \int_{t-v}^{t} ds_1 \int_{t-1}^{t} ds_2 K_{12}(s_1, s_2) ds_2 = -t^2 + 2t - v - \frac{5}{6}, \quad (20) \]

where \( t \in [1, 3], v \leq t - 1, \) and

\[ K^{(0)}_{12}(s_1, s_2) = s_1 - \frac{2}{3}s_2, \quad K^{(0)}_{12}(s_2, s_1) = s_2 - \frac{2}{3}s_1, \quad s_1 \in [0, 3], s_2 \in [0, 1], s_1 \geq s_2. \quad (21) \]
Now check the feasibility of the conditions (9), (16) and (18). When calculating $f_1(1,0)$ from (9), let us use the equality
\[
\int_{\lambda_1}^{\lambda_2} ds_1 \int_{\lambda_1}^{\lambda_2} K_{12}(s_1, s_2) ds_2 = \frac{\lambda_2}{\lambda_1} \int_{\lambda_1}^{\lambda_2} K_{12}(s_2, s_1) ds_2 = \frac{1}{2} \int_{\lambda_1}^{\lambda_2} K_{12}(s_1, s_2) + K_{12}(s_2, s_1)) ds_2,
\]
which, taking into account (21), has the form
\[
\int_{0}^{1} ds_1 \int_{0}^{1} \left(s_1 - \frac{2}{3} s_2\right) ds_2 = \int_{0}^{1} ds_1 \int_{0}^{1} \left(s_2 - \frac{2}{3} s_1\right) ds_2 = \frac{1}{6} \int_{0}^{1} ds_1 \int_{0}^{1} (s_1 + s_2) ds_2.
\]
As a result, we get that for
\[f(t,0) = -t^2 + 2t - \frac{5}{6}, \ t \in [1,2),\]
the matching condition with the initial function is satisfied:
\[f_1(1,0) = \frac{1}{6} - \frac{1}{6} \int_{0}^{1} ds_1 \int_{0}^{1} (s_1 + s_2) ds_2 = \frac{1}{6} - \frac{1}{6} \equiv 0.\]
Therefore, according to (10), (11), for $\mathbf{N}(t,v) \in \Delta_1 = \{t,v : v + 1 \leq t, 1 \leq t < 2\}$ we have
\[K_{12}^{(1)}(\mathbf{M}) = \frac{1}{3} t + \frac{2}{3} v = s_1 - \frac{2}{3} s_2, \quad K_{12}^{(1)}(\mathbf{M}) = \frac{1}{3} t - v = s_2 - \frac{2}{3} s_1,
\]
where $\mathbf{M}(p,q) \in \Omega_1(\mathbf{N}(t,v))$, $s_1 = t$, $s_2 = t - v$. Since the matching condition (16) of the right-hand side is also satisfied, then $K_{12}^{(1)}$ is continuous on $\Omega_1$.

Further, let $\mathbf{N}(t,v) \in \Delta_2 = \{t,v : v + 1 \leq t, 2 \leq t < 3\}$. Since
\[f(2,0) = -\frac{5}{6} \neq \frac{1}{6} \int_{0}^{1} ds_1 \int_{0}^{1} (s_1 + s_2) ds_2 = \frac{1}{2},\]
the problem (19)–(21) is not solvable in the class of functions bounded on $\Omega_2(\mathbf{N}(t,v)$, due to the fact that the condition (18) is not met.

4. Numerically solving (6), (7)

Let us solve numerically (6), (7). For simplicity, denote $K_{12}(t_{i-\frac{1}{2}}, t_{i-j-\frac{1}{2}})$ by $K_{i-\frac{1}{2},i-j-\frac{1}{2}}$. Introduce for $h \leq t \leq T$ a uniform mesh by coordinating integer nodes with the points of discontinuities on (4), (5)
\[t_i = ih, \ t_{i-\frac{1}{2}} = \left(i - \frac{1}{2}\right) h, \ v = jh, \ j = 0,i-2, \ i = 2N, \ T = Nh.
\]
Applying the quadrature formula of the middle rectangles for the approximation of the integrals in the left-hand side of (6), (7) we find that the discrete analog of (6), (7) has the form
\[
h^2 \sum_{l=i}^{i-j} \sum_{m=i-j}^{i-j} K_{l-\frac{1}{2},m-\frac{1}{2}}^{(1)} = f_{i,j}, \ i > j, \ i = 2N, \ j = 0,i-2, \ (22)
\]
\[ h^2 \sum_{l=i-j}^{i} \sum_{m=i}^{i-j} K_{l-m}^{h} = f_{i,j}^{h}, \quad i > j, \quad i = 3, N, \quad j = 1, i-2, \]

where \( f_{i,j}^{h} = f^{h}(ih, jh) \) ((23) factors in the equality \( f_{i,0}^{h} = f_{i,0}^{h} = f_{i,0}^{h} \) for \( i = 1, N \)).

Inverse formulas of a paired system of linear algebraic equations (22), (23) have the form

\[
K_{i-\frac{1}{2},j-\frac{1}{2}}^{h} = \frac{f^{h}(ih, 0)}{h^2}, \quad K_{i-\frac{1}{2},j-\frac{1}{2}}^{h} = \frac{f^{h}(ih, jh)}{h^2}, \quad K_{i-j-\frac{1}{2},i-\frac{1}{2}}^{h} = \frac{f^{h}(ih, jh)}{h^2}, \quad j = 1, i-2. \] (24)

We will demonstrate numerical calculations using (22), (23) on an example.

**Example 2.** Let an exact solution to (6), (7) be the function

\[ K(s_1, s_2) = as_2^2 - bs_2, \quad a, b = \text{const}. \]

Then the right-hand sides of (6), (7) have the form

\[
(1) \quad f(t, \upsilon) = \frac{ah^2}{3} \left(3(t-h) + h^2\right) + \frac{bh^2}{2} (h - 2(t - \upsilon)),
\]

\[
(2) \quad f(t, \upsilon) = \frac{ah^2}{3} \left(3(t - \upsilon)(t - \upsilon - h) + h^2\right) + \frac{bh^2}{2} (h - 2t).
\]

Determine the difference approximation \( K^{h} \) using (22), (23) and find errors

\[ \varepsilon_1 = \max_{i,j} |K_{i-\frac{1}{2},j-\frac{1}{2}}^{h} - K(t_{i-\frac{1}{2}}, t_{j-\frac{1}{2}})|, \quad \varepsilon_2 = \max_{i,j} |K_{i-j-\frac{1}{2},i-\frac{1}{2}}^{h} - K(t_{i-j-\frac{1}{2}}, t_{j-\frac{1}{2}})|. \]

Table 1 presents the results of the calculations of \( \varepsilon = \max\{\varepsilon_1, \varepsilon_2\} \) for \( a = 4, \quad b = -1 \). Source of Table 1: author’s calculations performed using MAPLE.

**Table 1.** The quadrature method of middle rectangles.

| \( h \)  | \( \varepsilon \)  |
|--------|--------------------|
| 0.2500 | 0.00520            |
| 0.1250 | 0.00130            |
| 0.0625 | 0.00033            |

As can be seen from the Table 1, the numerical method has the second order of convergence, i.e. with a decrease in the mesh step by half, \( \varepsilon \) decreases by a factor of 4.

**Remark 1.** Note that to construct a model in the form (1), (2), we can use an approach based on solving the problem of identifying integrals of Volterra kernels employing the product integration method [7]. When applied to the mesh analogue of (3), we have

\[
f_{12}(t_i) = \sum_{j=1}^{i} \sum_{k=1}^{i} g_{12,j,k} x_1(t_{i-j+\frac{1}{2}}) x_2(t_{i-k+\frac{1}{2}}), \quad i = 1, N, \] (25)

where

\[ g_{12,j,k} = \int_{(j-1)h}^{jh} \int_{(k-1)h}^{kh} K_{12}(s_1, s_2) ds_1 ds_2. \]
The problem of finding $g_{12,j,k}$ was solved in [6] (pp. 113–116) using test signals of the form (4), (5). Matching the discontinuity points in the test signals with the nodes of the uniform mesh ($t_i = ih$, $v = jh$, $Nh = T$), we have instead of (25)

$$g_{12,i,j-1}^{(1)} = f^h(ih,jh), \quad g_{12,i-1,j}^{(2)} = f^h(ih,jh),$$

where $f^h(ih,jh)$ and $f^h(ih,jh)$ denote responses to discrete analogue of inputs (4) and (5), respectively. Approximating the integrals in (26) for $h \leq t \leq T$ by the midpoint rule, we finally obtain (24).

**Remark 2.** Further development of the work is associated with the study of the numerical solution of (6), (7) using a Runge-Kutta type method. Application of this method will take into account the presence of "prehistory" and will require particular consideration. The case of a one-dimensional integral equation with variable limits of integration was considered in [4]. The development of this approach as applied to (6), (7) was started in [8].

### 5. Conclusion

The paper considers solving the paired two-dimensional Volterra integral equation of the first kind arising in the problem of identification of nonsymmetric Volterra kernels. The method of obtaining the desired solution develops the method of steps for the one-dimensional case. The coordination conditions that ensure the continuity of the solution are indicated. A mesh analog of the solution obtained on the basis of cubes of middle rectangles is given.

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