FINITE MULTIPLE ZETA VALUES ASSOCIATED WITH 2-COLORED
ROOTED TREES

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Abstract. We define finite multiple zeta values (FMZVs) associated with some combinatorial
objects, which we call 2-colored rooted trees, and prove that FMZVs associated with 2-colored
rooted trees satisfying certain mild assumptions can be written explicitly as \(\mathbb{Z}\)-linear combina-
tions of the usual FMZVs. Our result can be regarded as a generalization of Kamano’s recent
work \cite{K} on finite Mordell-Tornheim multiple zeta values. As an application, we will give a new
proof of the shuffle relation of FMZVs, which was first proved by Kaneko and Zagier.

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1. Introduction

In the 1990’s, Hoffman \cite{H2} and Zhao \cite{Z} had started independently a theory of mod \(p\) multiple
harmonic sums. Recently, Kaneko and Zagier introduced a new “adelic” framework to describe
the work of Hoffman and Zhao. That is, they defined the \(\mathbb{Q}\)-algebra \(\mathcal{A} := \left( \prod_p F_p \right) / \left( \bigoplus_p F_p \right)\),
where \(p\) runs through all rational primes. Thus, an element of \(\mathcal{A}\) is represented by a family \((a_p)_p\)
of elements \(a_p \in F_p\), and two families \((a_p)_p\) and \((b_p)_p\) represent the same element of \(\mathcal{A}\) if and
only if \(a_p = b_p\) for all but finitely many rational primes \(p\).

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program 2013–2018 of the Faculty of Science and Technology at Keio University.
Definition 1.1 ([KZ]). Finite multiple zeta value is an element of $A$ defined by

$$\zeta_A(k_1, \ldots, k_r) := \left( \sum_{0 < n_1 < \cdots < n_r < p} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}} \right)_p.$$ 

We call the integers $k_1 + \cdots + k_r$ and $r$ the weight and the depth of $\zeta_A(k_1, \ldots, k_r)$, respectively.

In the following, we often denote an element $(a_p)_p$ of $A$ simply by $a_p$ omitting $(\ )_p$ if there is no fear of confusion. For example, the above definition is written as

$$\zeta_A(k_1, \ldots, k_r) = \sum_{0 < n_1 < \cdots < n_r < p} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}.$$ 

There exist many research of FMZVs, for example, [M], [O], [SS], [SW1], [SW2]. Our research in the present paper is motivated by Kamano’s work [K] on the finite Mordell Tornheim multiple zeta values (FMZVs of MT-type):

$$\zeta_A^{MT}(k_1, \ldots, k_r; k_{r+1}) := \sum_{m_1, \ldots, m_r \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^{k_{r+1}}} \in A,$$

where $k_1, \ldots, k_r$ are positive integers and $k_{r+1}$ is a non-negative integer. Kamano proved that FMZVs of MT-type can be written explicitly as $\mathbb{Z}$-linear combinations of the usual (i.e., Kaneko-Zagier’s) FMZVs [K Theorem 2.1]. Using this result, he obtained non-trivial $\mathbb{Z}$-linear relations, which we call Kamano’s relation, among FMZVs [K Theorem 3.2, Proposition 3.4].

In the present paper, we define FMZVs associated with 2-colored rooted trees and prove that, with mild assumptions, they can be written explicitly as a $\mathbb{Z}$-linear combination of the usual FMZVs. The following definition is inspired by Yamamoto’s work [Y].

Definition 1.2. A 2-colored rooted tree $(T, rt_T, V_\bullet)$ is a triple consisting of the following data.

(i) $T = (V, E)$ is a tree (in the graph theoretic sense) such that $\#V(= \#E + 1) < \infty$.

(ii) $rt_T \in V$ is a vertex, called the root.

(iii) $V_\bullet$ is a subset of $V$ containing all terminals of $T$. We set $V_0 := V \setminus V_\bullet$.

In the following, for a 2-colored rooted tree $(T, rt_T, V_\bullet)$ and an edge $e$ of $T$ and $(m_v) \in \mathbb{Z}_{\geq 1}^{V_\bullet}$, we set

$$L_e(rt_T, (m_v)) := \sum_{v \in V_\bullet \text{ s.t. } e \in P(rt_T, v)} m_v,$$

where $P(rt_T, v)$ is the path from $rt_T$ to $v$.

Definition 1.3. For a 2-colored rooted tree $X = (T, rt_T, V_\bullet)$ and a map $k : E \to \mathbb{Z}_{\geq 0}$, the FMZV $\zeta_A(X, k)$ associated with $X$ is defined as an element in $A$ by

$$\zeta_A(X, k) := \sum_{(m_v) \in \mathbb{Z}_{\geq 1}^{V_\bullet} \text{ s.t. } e \in E \atop \sum_{v \in V_\bullet} m_v = p} \prod_{e \in E} L_e(rt_T, (m_v))^{-k(e)}.$$
We call $k : E \to \mathbb{Z}_{\geq 0}$ an index on $X$.

Our main theorem is the following.

**Theorem 1.4.** Let $X$ be a 2-colored rooted tree and $k$ an index on $X$. Suppose that $\sum_{e \in P(v,v')} k(e)$ is positive for any $v, v' \in V_\bullet$. Then, the FMZV $\zeta_A(X,k)$ can be written explicitly as a $\mathbb{Z}$-linear combination of Kaneko-Zagier’s FMZVs.

Since a FMZV of MT-type coincides with $\zeta_A(X,k)$ for a specific $(X,k)$ (see Example 3.3 (i)), our main result is a generalization of Kamano’s result mentioned above.

As a corollary of our main theorem, we give another proof of the shuffle relation among FMZVs, which was first proved by Kaneko and Zagier [KZ] (see Corollary 4.1). Therefore our result can be regarded as a simultaneous generalization of both Kamano’s relation and the shuffle relation. We should note that this is very surprising since there were no obvious connections between these two classes of relations.

2. Basic properties of FMZV associated with 2-colored rooted trees

In this section, we will give two examples of 2-colored rooted trees and FMZVs associated with them. These examples show that the FMZV associated with a 2-colored rooted tree is a generalization of the usual FMZV and the FMZV of MT-type. Next, we prove the three basic properties of the FMZVs associated with 2-colored rooted trees. The first and second properties are about contracting certain edges of 2-colored rooted trees and the third is about changing the roots of the given 2-colored rooted trees. Using these properties, we define the notion “harvestable” for a pair consisting of a 2-colored rooted tree and an index on it. The proof of our main theorem will be reduced to the case when the pair is harvestable.

**Example 2.1.** We use diagrams to indicate 2-colored rooted trees $(T, rt_T, V_\circ, V_\bullet)$, with symbols $\circ$ and $\bullet$ corresponding to the vertices in $V_\circ$ and $V_\bullet$, respectively.

(i) Let $X = (T, rt_T, V_\bullet)$ be a 2-colored rooted tree and $k$ an index on $X$ as follows.

![Diagram of 2-colored rooted tree](image-url)
Here, \( k_i := k(e_i) \) and \( e_i \in E \) is an edge of \( T \) and \( rt_T = v_{r+1} \). If we set \( m_i := m_{v_i} \) (1 \leq i \leq r), since \( L_{e_i}(rt_T, (m_v)) = m_1 + \cdots + m_i \) (1 \leq i \leq r), we obtain

\[
\zeta_A(X, k) = \sum_{\substack{m_1, \ldots, m_{r+1} \geq 1 \\ m_1 + \cdots + m_{r+1} = p}} (m_1 + \cdots + m_r)^{-k_r} \cdots (m_1 + m_2)^{-k_2} m_1^{-k_1} = \sum_{\substack{m_1, \ldots, m_{r} \geq 1 \\ m_1 + \cdots + m_{r} = p-1}} \frac{1}{m_1^{k_1}(m_1 + m_2)^{k_2} \cdots (m_1 + \cdots + m_r)^{k_r}} = \zeta_A(k_1, \ldots, k_r).
\]

Thus, the usual FMZV coincides with the FMZV associated with the above 2-colored rooted tree.

(ii) Next, consider the following 2-colored rooted tree \( X = (T, rt_T, V_\bullet) \) and the index \( k \) on \( X \).

Assume that \( rt_T = v_{r+1} \) and \( k_i \geq 1 \) (1 \leq i \leq r). Since \( L_{e_i}(rt_T, (m_v)) = m_i \) (1 \leq i \leq r) and \( L_{e_{r+1}}(rt_T, (m_v)) = m_1 + \cdots + m_r \), we obtain

\[
\zeta_A(X, k) = \sum_{\substack{m_1, \ldots, m_{r+1} \geq 1 \\ m_1 + \cdots + m_{r+1} = p}} m_1^{-k_1} \cdots m_r^{-k_r} (m_1 + \cdots + m_r)^{-k_{r+1}} = \sum_{\substack{m_1, \ldots, m_{r} \geq 1 \\ m_1 + \cdots + m_{r} = p-1}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (m_1 + \cdots + m_r)^{k_{r+1}}} = \zeta_{MT}(k_1, \ldots, k_r; k_{r+1}).
\]

Thus we see that the FMZV of MT-type is a special case of the FMZV associated with the 2-colored rooted trees.

**Proposition 2.2.** Let \( X = (T, rt_T, V_\bullet) \) be a 2-colored rooted tree and \( k \) be an index on \( X \). Assume that there exists an edge \( e \in E \) satisfying that one of the end points of \( e \) is in \( V_\circ \) and \( k(e) = 0 \). Consider the quotient tree \( T := T/e = (V, E) \) obtained by contracting \( e \). Let \( V_\circ \) and \( \overline{rt_T} \in V_\circ \) be the images of \( V_\bullet \) and \( rt_T \) in \( T \), and take \( \overline{rt_T} := \overline{rt_T} \). These situations can be written as the following figures.
Here, $\times$ is $\circ$ or $\bullet$. Let $\overline{k}$ be an index on $\overline{X} := (\overline{T}, \overline{rT}, \overline{V_\star})$ defined by $\overline{k}(\overline{e}) := k(e')$ if $\overline{e} \in \overline{E}$ is the image of $e' \in E$. Then we have

$$\zeta_A(X, k) = \zeta_A(\overline{X}, \overline{k}).$$

**Proof.** Since $k(e) = 0$, we have

$$\zeta_A(X, k) = \sum_{(m_v) \in \mathbb{Z}_{\geq 1}^{V_\star} \text{ s.t. } \sum_{e \in V_\star} m_e = p} \prod_{e' \in E \setminus \{e\}} L_e(\overline{rT}, (m_e))^{-k(e')} \prod_{e' \in E} L_{e'}(\overline{rT}, (m_{e'}))^{-k(e')} = \zeta_A(\overline{X}, \overline{k}),$$

which completes the proof. \hfill \Box

**Proposition 2.3.** Let $X = (T, \mathbf{rt}_T, V_\star)$ be a 2-colored rooted tree and $k$ an index on $X$. Assume that there exist edges $e, e' \in E$ satisfying that $e$ and $e'$ are incident at a vertex $v \in V_\circ$ with $\text{deg}(v) = 2$. Consider the quotient tree $\overline{T} := T/e' = (\overline{V}, \overline{E})$ obtained by contracting $e$. Let $\overline{e} \in \overline{E}, \overline{V_\star}$ be the images of $e$ in $\overline{T}$ and $V_\star$ respectively, and $\overline{rT} \in \overline{V_\star}$ be the image of $\mathbf{rt}_T$ in $\overline{T}$. Set $\mathbf{rt}_T := \mathbf{rt}_{\overline{T}}$. Let $\overline{k} : \overline{E} \to \mathbb{Z}_{\geq 0}$ be the index on $\overline{X} := (\overline{T}, \mathbf{rt}_{\overline{T}}, \overline{V_\star})$ defined by, for $f \in \overline{E}$,

$$\overline{k}(\overline{f}) := \begin{cases} k(e) + k(e') & \text{if } \overline{f} = \overline{e}, \\ k(f) & \text{otherwise.} \end{cases}$$

These situations can be also written as the following figures.

Then we have

$$\zeta_A(X, k) = \zeta_A(\overline{X}, \overline{k}).$$

**Proof.** Since $v$ is a vertex in $V_\circ$, we have

$$\{v \in V_\star \mid e \in P(\mathbf{rt}_T, v)\} = \{v \in V_\star \mid e' \in P(\mathbf{rt}_T, v)\}.$$
Therefore, we obtain
\[ \zeta_A(X, k) = \sum_{(m_v) \in \mathbb{Z}_{21}^V \text{ s.t. } v \in V} \prod_{e \in E} L_e(\text{rt}_T, (m_v))^{-k(e)} \]
\[ = \sum_{(m_v) \in \mathbb{Z}_{21}^V \text{ s.t. } v \in V} L_e(\text{rt}_T, (m_v))^{-(k(e) + k(e'))} \prod_{e' \in E \setminus \{e, e'\}} L_{e'}(\text{rt}_T, (m_v))^{-(k(e')} \]
\[ = \sum_{(m_v) \in \mathbb{Z}_{21}^V \text{ s.t. } v \in V} L_{T_1}(\text{rt}_{T_1}, (m_v))^{-k(t_1)} \prod_{T \in E \setminus \{t_1\}} L_{T_1}(\text{rt}_{T_1}, (m_v))^{-k(T)} = \zeta_A(X, k), \]
which completes the proof. □

The following proposition, which is a generalization of \cite[Lemma 3.1]{K}, is a key in obtaining non-trivial relations among the usual FMZVs.

**Proposition 2.4.** For a tree \( T = (V, E) \), vertices \( v', v'' \in V \) and a subset \( V_\bullet \) of \( V \), let \( X' \) (resp. \( X'' \)) be the 2-colored rooted tree consisting of \( T \), \( rt_T = v' \) (resp. \( rt_T = v'' \)) and \( V_\bullet \). Then we have
\[ \zeta_A(X', k) = (-1)^{k(P(v', v''))} \zeta_A(X'', k) \]
for an index \( k \) on \( X' \). Here, \( k(P(v', v'')) := \sum_{e \in P(v', v'')} k(e) \).

**Proof.** Consider the path \( P(v', v'') \) from \( v' \) to \( v'' \). If \( e \in P(v', v'') \), \( V_\bullet \) is divided into two subsets as follows:
\[ V_\bullet = \{ v \in V_\bullet \mid e \in P(v', v) \} \sqcup \{ v \in V_\bullet \mid e \in P(v'', v) \}. \]
Therefore, we have \( L_e(v', (m_v)) = P - L_e(v'', (m_v)) \) for \( e \in P(v', v'') \). On the other hand, if \( e \not\in P(v', v'') \), we see that
\[ \{ v \in V_\bullet \mid e \in P(v', v) \} = \{ v \in V_\bullet \mid e \in P(v'', v) \}. \]
Thus, we have \( L_e(v', (m_v)) = L_e(v'', (m_v)) \) for \( e \not\in P(v', v'') \). Therefore, we obtain
\[ \zeta_A(X', k) \]
\[ = \sum_{(m_v) \in \mathbb{Z}_{21}^V \text{ s.t. } e \in P(v', v'')} \prod_{e \in P(v', v'')} \frac{1}{(P - L_e(v'', (m_v)))^{k(e)}} \prod_{e \not\in P(v', v'')} \frac{1}{L_e(v'', (m_v))^{k(e)}} \]
\[ = (-1)^{k(P(v', v''))} \zeta_A(X'', k), \]
which completes the proof. □
Example 2.5. Consider the 2-colored rooted tree $X'$ and the index $k$ in Example 2.1 (ii) with $r_T = v_{r+1}$. Then $\zeta_A(X', k)$ coincides with the FMZV $\zeta_A^{MT}(k_1, \ldots, k_r; k_{r+1})$ of MT-type. Let $X''$ be the 2-colored rooted tree whose root is $v_1$. Then, by Proposition 2.4, we have

$$\zeta_A^{MT}(k_1, \ldots, k_r; k_{r+1}) = \zeta_A(X', k) = (-1)^{k(P(v_{r+1}, v_1))} \zeta_A(X'', k)$$

which is Kamano's result [K, Lemma 3.1].

For the proof of our main theorem, we need the following definitions that the pair consisting of a 2-colored rooted tree and an index on it is harvestable and that an index on a 2-colored rooted tree is essentially positive.

Definition 2.6. Let $X = (T, r_T, V_\bullet)$ be a 2-colored rooted tree and $k$ an index on $X$. The pair $(X, k)$ is harvestable if the following conditions on $(X, k)$ holds.

(H1): The root $r_T$ is a terminal of $T$. In particular, $r_T$ is in $V_\bullet$.

(H2): $\deg(v) \leq 2$ for all $v$ in $V_\circ$ and $\deg(v) \geq 3$ for all $v$ in $V_\bullet$.

(H3): If an edge $e$ connects a branched point $v$ in $V_\circ$ and a child of $v$ in $V_\bullet$, $k(e)$ is positive.

Here, a branched point is a vertex whose degree is larger than or equal to 3.

Definition 2.7. For a 2-colored rooted tree $X = (T, r_T, V_\bullet)$, an index $k$ on $X$ is essentially positive if $k(P(v, v'))$ is positive for any $v, v' \in V_\bullet$.

Remark 2.8. If we cut the edges of 2-colored rooted trees which connect the branched point nearest to the root, the 2-colored rooted tree becomes decomposed into many parts. We take the upper part, and the parts under the branched point again become 2-colored rooted trees satisfying the conditions (H1), (H2) and (H3) after adding new roots to each part. We call this operation of taking the upper part as harvest, and this is the reason why we call 2-colored rooted trees satisfying (H1), (H2) and (H3) as being harvestable.

By using Proposition 2.2 and Proposition 2.3 we see that for a pair consisting of a 2-colored rooted tree and an essentially positive index on it, there exists a harvestable pair such that FMZVs associated with them coincides up to sign.

Proposition 2.9. Let $X = (T, r_T, V_\bullet)$ be a 2-colored rooted tree and $k$ an essentially positive index on $X$. Then, there exists a harvestable pair $(X_h, k_h)$ of the 2-colored rooted tree $X_h = (T_h = (V(X_h), E(X_h)), r_{T_h}, V_\bullet(X_h))$ and the index $k_h$ on $X_h$ satisfying

$$\zeta_A(X, k) = (-1)^{k(P(r_T, r_{T_h}))} \zeta_A(X_h, k_h).$$

Here $r_{T_h} \in V(X_h)$ is the image of $r_T$ in $X_h$.

Proof. By using Proposition 2.2 and 2.3 to contract edges that one of end points is in $V_\circ$ and $k(e) = 0$, and edges connecting $v' \in V_\circ$ with $\deg(v') = 2$, and again using Proposition 2.2 to
insert edges \( e' \) with \( k(e') = 0 \) and vertices in \( V_\circ \) into vertices in \( V_\bullet \) whose degree is greater than or equal to 3, we obtain a pair \( (X', k') \) of a 2-colored rooted tree \( X' \) and an index \( k' \) on \( X' \) satisfying the conditions (H2), (H3) and that FMZVs associated with them coincides. Further, by using Proposition 2.3 to move the root \( rt_T \) to a terminal, we obtain a desired harvestable pair \( (X_h, k_h) \) satisfying \( \zeta_A(X, k) = (-1)^{k_h(P(rt_T, rt_{T_h}))} \zeta_A(X_h, k_h) \), which completes the proof. □

**Definition 2.10.** For a 2-colored rooted tree \( X \) and an essentially positive index \( k \) on \( X \), we define a harvestable form of the pair \( (X, k) \) as the harvestable pair \( (X_h, k_h) \) satisfying Proposition 2.9.

**Remark 2.11.** For a 2-colored rooted tree \( X \) and an essentially positive index \( k \) on \( X \), a harvestable pair \( (X_h, k_h) \) of \( (X, k) \) is not unique. For example, consider the following 2-colored rooted tree \( X = (T, rt_T, V_\bullet) \) and an essentially positive index \( k \) on \( X \).

\[
\begin{array}{c}
\text{v}_1 \quad k_1 \quad \text{v}_2 \\
\text{rt}_T \\
k_2
\end{array}
\]

Then, we can take the following two 2-colored rooted trees and indices as harvestable forms of \( (X, k) \).

\[
\begin{array}{c}
\text{v}_1 \quad k_1 \quad \text{v}_2 \\
\text{rt}_T \\
k_2 \quad 0
\end{array}
\]

3. **Proof of our main theorem**

In this section, we will prove our main theorem (Theorem 1.4). If the pair \( (X, k) \) consisting of a 2-colored rooted tree \( X \) and an essentially positive index \( k \) on \( X \) is harvestable, our main theorem will be proved by induction on the sum of the index at the edges in paths from the branched point nearest to the root to all leaves (Proposition 3.2). The general case will be deduced to the harvestable case by using Proposition 2.9.

To explain that \( \zeta_A(X, k) \) associated with a harvestable pair \( (X, k) \) can be written explicitly as a \( \mathbb{Z} \)-linear combination of the usual FMZVs, we prepare some terminology on Hoffman’s algebra \( \mathcal{H} \). Let \( \mathcal{H} \) be the noncommutative polynomial ring \( \mathbb{Q}(x, y) \) in the variables \( x \) and \( y \) over \( \mathbb{Q} \), and we set \( \mathcal{H}^1 := \mathbb{Q} + y \mathcal{H} \). Note that \( \mathcal{H}^1 \) is generated by \( z_k := y u^{k-1} (k = 1, 2, \ldots) \) as a \( \mathbb{Q} \)-algebra. We define the map \( Z_A : \mathcal{H}^1 \rightarrow \mathcal{A} \) by sending \( z_{k_1} \cdots z_{k_r} \) to \( \zeta_A(k_1, \ldots, k_r) \) and extend it \( \mathbb{Q} \)-linearly.

**Definition 3.1.** We define the shuffle product \( \triangledown : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \) on \( \mathcal{H} \) by the following rule and \( \mathbb{Q} \)-bilinearity.

(i) \( w \triangledown 1 = 1 \triangledown w = w \) for all \( w \in \mathcal{H} \).

(ii) \( (w_1 u_1) \triangledown (w_2 u_2) = (w_1 \triangledown w_2 u_2) u_1 + (w_1 u_1 \triangledown w_2) u_2 \) for all \( w_1, w_2 \in \mathcal{H} \) and \( u_1, u_2 \in \{x, y\} \).
For instance, we have
\[ z_2 \tau z_2 = y x \tau y x = 2 y x y x + 4 y^2 x^2 = 2 z_2 z_2 + 4 z_1 z_3. \]

The next proposition is the harvestable case of our main theorem.

**Proposition 3.2.** Let \( X = (T, r_T, V_\tau) \) be a 2-colored rooted tree and \( k \) be an essentially positive index on \( X \). Assume that the pair \((X, k)\) is harvestable. Then \( \zeta(X, k) \) coincides with the image \( Z_{A}(w) \) of \( w \in \mathcal{H}^1 \) constructed by the following inductive method. Since \((X, k)\) is a harvestable pair, \((X, k)\) has the following shape.

Here, \( r \) and \( s \) are positive integers and \( k_i := k(e_i) \) \((1 \leq i \leq r)\), \( l_j := k(f_j) \) \((1 \leq j \leq s)\) and \( k' := k(e') \). \( T_1, \ldots, T_s \) are subtree of \( T \). Then we have
\[
  w = \left( \prod_{j=1}^{s} w_j \right) x^{k'} z_{k_1} \cdots z_{k_r}.
\]

Here, \( w_j \in \mathcal{H}^1 \) \((1 \leq j \leq s)\) are elements corresponding to the following harvestable pair \((X_j, k^{(j)})\).

For the proof of the above proposition, the following partial fraction decomposition is the key tool. For a positive integer \( s \) and indeterminates \( X_1, \ldots, X_s \), we have
\[
  \frac{1}{X_1 \cdots X_s} = \frac{1}{X_1 + \cdots + X_s} \sum_{i=1}^{s} \frac{1}{X_1 \cdots \hat{X}_i \cdots X_s}.
\]

**Proof of Proposition 3.2.** Let \( v' \in V \) be the branched point nearest to the root \( r_T \) in all the branched points. As \((X, k)\) is harvestable, \( v' \) is an element in \( V_\tau \). Set \( S = S(X, k) := \sum_e k(e) \), where \( e \) runs through edges in paths from \( v' \) to all leaves. If there exists no branched points, we set \( S = 0 \). We prove the statement by the induction on \( S \geq 0 \). If \( S = 0 \), as \((X, k)\) is harvestable,
we see that $\zeta_A(X, k)$ coincides with that associated with the following 2-colored rooted tree and the index on it with the root $v_{r+1}$:

Therefore, we have $\zeta_A(X, k) = \zeta_A(k_1, \ldots, k_r) = \mathcal{Z}_A(z_{k_1} \cdots z_{k_r})$, which completes the proof of the case $S = 0$. Next, assume that $S > 0$ and the statement holds for all the non-negative integers less than $S$. The assumption $S > 0$ means that there exists at least one branched point. Then the given 2-colored rooted tree $X = (T, r_T, V_*)$ and the index $k$ on $X$ can be written as follows:

Then we have

$$
\zeta_A(X, k) = \sum_{(m_v) \in \mathbb{Z}_{\geq 1}^{V_\bullet} \text{ s.t. } \sum_{v \in V_\bullet} m_v = p} \prod_{i=1}^{p} \frac{1}{L_{e_i}(r_T, (m_v))^{k_i}} \times \frac{1}{L_{e'}(r_T, (m_v))^{k'}}
\times \prod_{j=1}^{s} \prod_{e \in E(T_j)} \frac{1}{L_{e}(r_T, (m_v))^{k(e)}} \times \frac{1}{M_1^{l_1} \cdots M_s^{l_s}}.
$$

Here, for $1 \leq j \leq s$, we set

$$
M_j := \sum_{v \in V_\bullet \text{ s.t. } f_j \in \mathcal{P}(r_T, v)} m_v = L_{f_j}(r_T, (m_v)).
$$

By (2), we have

$$
\frac{1}{M_1^{l_1} \cdots M_s^{l_s}} = \frac{1}{M_1 + \cdots + M_s} \left( \frac{1}{M_1^{l_1-1} M_2^{l_2} \cdots M_s^{l_s}} + \cdots + \frac{1}{M_1 \cdots M_{s-1}^{l_{s-1}} M_s^{l_s-1}} \right).
$$

Therefore, since $L_{e'}(r_T, (m_v)) = M_1 + \cdots + M_s$, we obtain

$$
\zeta_A(X, k) = \sum_{j=1}^{s} \zeta_A(X, \alpha_j).
$$
Here, $\alpha_j$ is an index on $X$ defined by

$$\alpha_j(e) = \begin{cases} 
  l_j - 1 & e = f_j, \\
  k' + 1 & e = e', \\
  k(e) & \text{otherwise}, 
\end{cases}$$

and

$$\zeta_A(X, \alpha_j) = \sum_{(m_v) \in \mathbb{Z}^V_{\geq 1} \text{ s.t. } \sum_v m_v = p} \prod_{i=1}^r \frac{1}{L_{e_i}(rt_T, (m_v))^{k_i}} \times \frac{1}{L_{e'}(rt_T, (m_v))^{k'+1}} \times \prod_{j=1}^s \prod_{e \in E(T_j)} \frac{1}{L_{e}(rt_T, (m_v))^{k(e)}} \times \frac{1}{M_1^{l_1} \cdots M_j^{l_j - 1} \cdots M_s^{l_s}}.$$

To use the induction hypotheses, we need to consider two cases whether $(X, \alpha_j)$ is harvestable or not.

(i) First, consider the case that the pair $(X, \alpha_j)$ is harvestable. This is the case when $l_j > 1$ or the child of $v'$ incident to $f_j$ is in $V_0$. In this case, the pair $(X, \alpha_j)$ has the following shape.

\[ \begin{array}{c}
\text{rt}_T \\
\text{v}_r \\
\text{v}_2 \\
\text{v}_1 \\
\text{k}_1 \\
k' + 1 \\
l_j - 1 \\
l_j \\
l_s \\
\end{array} \]

Since $S(X, \alpha_j) = S(X, k) - 1 < S(X, k)$, by the induction hypotheses, we obtain

$$\zeta_A(X, \alpha_j) = Z_A \left( \prod_{a=1}^s \frac{w_a}{w_{a \neq j}} \right)^{x^{k'+1} + z_{k_1} \cdots z_{k_r}}.$$

Here $w'_j$ is the element of $\mathcal{H}^1$ corresponding to the following harvestable pair $(X_j, \beta_j)$.

\[ \begin{array}{c}
u_j \\
l_j - 1 \\
\end{array} \]

Note that $w_j = w'_j x$ in this case.
(ii) Next, consider the case that the pair \((X, \alpha_j)\) is not harvestable. This is the case when \(l_j = 1\) and the child of \(v'\) incident to \(f_j\) is in \(V_*\) because \(\alpha_j(f_j) = l_j - 1 = 0\). By using Proposition \(2.2\) to contract \(f_j\) and insert edges \(e'\) with \(\alpha_j(e') = 0\), we obtain a harvestable form \((X_h, \alpha_j, h)\) of \((X, \alpha_j)\) as follows.

Since \(S(X, \alpha_j) = S(X_h, \alpha_j, h) = S(X, k) - 1 < S(X, k)\), by the induction hypotheses, we obtain

\[
\zeta_A(X, \alpha_j) = \zeta_A(X_h, \alpha_j, h) = Z_A \left( \prod_{a=1}^{s} w_a \left[ u_j \right] w_j' x^0 z_k' z_{k_1} \cdots z_{k_r} \right)
\]

Here \(w_j'\) is the element of \(\mathcal{S}^1\) corresponding to the following harvestable pair \((X_j', \beta_j)\).

Note also that \(w_j = w_j'y\) in this case.

Therefore, by the definition of the shuffle product, we obtain

\[
\zeta_A(X, k) = \sum_{j=1}^{s} \zeta_A(X, \alpha_j) = Z_A \left( \prod_{j=1}^{s} w_j \right) x^k z_{k_1} \cdots z_{k_r}.
\]

Therefore, \(w := (\prod_{j=1}^{s} w_j) x^k z_{k_1} \cdots z_{k_r}\) is the desired element in \(\mathcal{S}^1\).

Proof of Theorem \(1.4\). By Proposition \(2.9\) for a given pair \((X, k)\) consisting of a 2-colored rooted tree \(X\) and an essentially positive index \(k\) on \(X\), there exists a harvestable pair \((X_h, k_h)\) satisfying \(\mathbf{II}\). Since the pair \((X_h, k_h)\) is harvestable, by Proposition \(3.2\) the right hand side of \(\mathbf{II}\) can
be written explicitly as a $\mathbb{Z}$-linear combination of the usual FMZVs, so can $\zeta_A(X, k)$. This completes the proof of our main theorem. $\square$

**Example 3.3.**

(i) For $1 \leq i \leq r$, consider the following 2-colored rooted tree $X$ and the essentially positive index $k$ on $X$.

\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) {$v_1$};
  \node (v2) at (1,1) {$v_2$};
  \node (va) at (2,2) {$v_a$};
  \node (va-1) at (2,-1) {$v_{a-1}$};
  \node (vb-1) at (3,-1) {$v_{b-1}$};
  \node (vb) at (4,1) {$v_b$};
  \node (vb') at (4,0) {$v_{b'}$};
  \node (v1b') at (5,0) {$v_{1b'}$};
  \node (va') at (3,1) {$v_{a'}$};
  \node (v1a') at (4,1) {$v_{1a'}$};
  \node (v1a) at (5,2) {$v_{1a}$};
  \node (v1a') at (5,1) {$v_{1a'}$};
  \node (v1b) at (5,0) {$v_{1b}$};
  \node (v1b') at (5,-1) {$v_{1b'}$};
  \node (v1c) at (6,0) {$v_{1c}$};
  \node (v_c+1) at (7,0) {$v_{c+1}$};
  \node (r) at (8,0) {$r$};
  \node (pr) at (8,1) {$p$};
  \node (q) at (8,-1) {$q$};
  \node (l) at (9,0) {$l$};
  \draw (v1) -- (v2);
  \draw (v2) -- (va);
  \draw (va) -- (va-1);
  \draw (va) -- (vb);
  \draw (vb) -- (vb');
  \draw (vb') -- (v1b');
  \draw (va') -- (va');
  \draw (va') -- (va);\node (g) at (10,0) {$g$};
  \draw (va) -- (g);
  \draw (vb) -- (vb');\node (h) at (10,1) {$h$};
  \draw (vb') -- (h);
  \draw (v1b) -- (v1b');\node (i) at (10,0) {$i$};
  \draw (v1b') -- (i);
  \draw (v1c) -- (v1c');;
ode (j) at (10,0) {$j$};
  \draw (v1c') -- (j);
  \draw (v1c) -- (r);
  \draw (r) -- (pr);
  \draw (pr) -- (q);
  \draw (q) -- (l);
\end{tikzpicture}
\end{center}

Set $rT = v_{c+1}$. Since $k$ is essentially positive, $k_j (1 \leq j \leq i)$ and $l_j (i + 1 \leq j \leq r)$ are positive. Then we have

\[
\zeta_A(X, k) = \sum_{m_1, \ldots, m_r \geq 1} \frac{1}{m_1 \cdots m_r \sum (m_1 + \cdots + m_i)^k \sum (m_1 + \cdots + m_r)^l} = Z_A((z_{k_1} \cdots z_{k_i}) x_1^{l_1} z_{l+1} \cdots z_{l_r}),
\]

which is Kamano’s result \[K\, Theorem 2.1\].

(ii) Consider the following 2-colored tree $X$ and the essentially positive index $k$ on $X$.

\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) {$v_1$};
  \node (v2) at (1,1) {$v_2$};
  \node (va) at (2,2) {$v_a$};
  \node (va-1) at (2,-1) {$v_{a-1}$};
  \node (vb-1) at (3,-1) {$v_{b-1}$};
  \node (vb) at (4,1) {$v_b$};
  \node (vb') at (4,0) {$v_{b'}$};
  \node (v1b') at (5,0) {$v_{1b'}$};
  \node (va') at (3,1) {$v_{a'}$};
  \node (v1a') at (4,1) {$v_{1a'}$};
  \node (v1a) at (5,2) {$v_{1a}$};
  \node (v1b) at (5,0) {$v_{1b}$};
  \node (v1b') at (5,-1) {$v_{1b'}$};
  \node (v1c) at (6,0) {$v_{1c}$};
  \node (v_c+1) at (7,0) {$v_{c+1}$};
  \node (r) at (8,0) {$r$};
  \node (pr) at (8,1) {$p$};
  \node (q) at (8,-1) {$q$};
  \node (l) at (9,0) {$l$};
  \draw (v1) -- (v2);
  \draw (v2) -- (va);
  \draw (va) -- (va-1);
  \draw (va) -- (vb);
  \draw (vb) -- (vb');
  \draw (vb') -- (v1b');
  \draw (va') -- (va');\node (g) at (10,0) {$g$};
  \draw (va) -- (g);
  \draw (vb) -- (vb');\node (h) at (10,1) {$h$};
  \draw (vb') -- (h);
  \draw (v1b) -- (v1b');\node (i) at (10,0) {$i$};
  \draw (v1b') -- (i);
  \draw (v1c) -- (v1c');;\node (j) at (10,0) {$j$};
  \draw (v1c') -- (j);
  \draw (v1c) -- (r);
  \draw (r) -- (pr);
  \draw (pr) -- (q);
  \draw (q) -- (l);
\end{tikzpicture}
\end{center}

Here, $a$, $b$ and $c$ are non-negative integers. Set $rT = v''_{c+1}$. Since $k$ is essentially positive, $p_x (1 \leq x \leq a)$, $q_y (1 \leq y \leq b)$ and $r_z (1 \leq z \leq c)$ are all positive. Then we have

\[
\zeta_A(X, k) = \sum_{0 < l_1 \cdots < l_b \atop 0 < m_1 \cdots < m_b \atop l_a + m_b < n_1 \cdots < n_c < p} \frac{1}{m_1 \cdots m_b \sum m_a \cdots m_b n_1 \cdots n_c} = Z_A((z_{p_1} \cdots z_{p_a} \cdots z_{q_1} \cdots z_{q_b}) z_{r_1} \cdots z_{r_c}),
\]

which is a finite analogue of a result of Komori, Matsumoto and Tsumura \[KMT\, Theorem 1\].
Remark 3.4. Since the only tool used to prove the main theorem (Theorem 1.4) is the partial fraction decomposition (2), the analogous statement of Theorem 1.4 for the MZVs holds. For example, we obtain a result

\[
\sum_{0 < l_1 < \cdots < l_a \atop 0 < m_1 < \cdots < m_b \atop l_a + m_b < n_1 < \cdots < n_c} \frac{1}{p_1^{n_1} \cdots p_a^{n_a} \cdot m_1^{n_1} \cdots m_b^{n_b} \cdot r_1^{n_1} \cdots r_c^{n_c}} = Z((z_{p_1} \cdots z_{p_a} \cdot z_{q_1} \cdots z_{q_b} \cdot z_{r_1} \cdots z_{r_c}))
\]

of Komori-Matsumoto-Tsumura [KMT, Theorem 1] by using our method. Here, \(Z: \mathcal{H}_0 := \mathbb{Q}(\{y\}) \rightarrow \mathbb{R}; \ z_{k_1} \cdots z_{k_r} \mapsto \zeta(k_1, \ldots, k_r)\) is a \(\mathbb{Q}\)-linear map. The left hand side of (3) can be regarded as a special value of the multiple zeta function \(\zeta(s; A_r)\) of the root system of type \(A_r\), which was first defined by Matsumoto and Tsumura in [MT]. Indeed, we have

\[
\sum_{0 < l_1 < \cdots < l_a \atop 0 < m_1 < \cdots < m_b \atop l_a + m_b < n_1 < \cdots < n_c} \frac{1}{p_1^{n_1} \cdots p_a^{n_a} \cdot m_1^{n_1} \cdots m_b^{n_b} \cdot r_1^{n_1} \cdots r_c^{n_c}} = Z((z_{p_1} \cdots z_{p_a} \cdot z_{q_1} \cdots z_{q_b} \cdot z_{r_1} \cdots z_{r_c})) = \zeta((k_{i,j}); A_r)
\]

for

\[
k_{ij} = \begin{cases} 
p_j-1 & i = 1, 2 \leq j \leq a + 1, 
q_j-(a+1) & i = a + 1, a + 2 \leq j \leq a + b + 1, 
r_j-(a+b+1) & i = 1, a + b + 2 \leq j \leq a + b + c + 1, 
0 & \text{otherwise.} \end{cases}
\]

4. Applications for non-trivial relations among FMZVs

In the final section, using the Theorem 1.4 and the Proposition 2.4, we give another proof of the shuffle relation among FMZVs, which was first proved by Kaneko and Zagier in [KZ].

Corollary 4.1. ([KZ]) For positive integers \(k_1, \ldots, k_r, l_1, \ldots, l_s\) and elements \(w := z_{k_1} \cdots z_{k_r}, w' := z_{l_1} \cdots z_{l_s} \in \mathcal{H}_1\), we have

\[
Z_A(w \shuffle w') = (-1)^{l_1+\cdots+l_s} Z_A(z_{k_1} \cdots z_{k_r}, z_{l_1} \cdots z_{l_s}).
\]
Proof. Consider the following two 2-colored rooted trees $X, X'$, whose root are $v$ and $v'_1$, and index $k$ on $X$ and $X'$.

Then, by Proposition 2.4 we have

\begin{equation}
\zeta_A(X, k) = (-1)^{k(P(v, v'_1))}\zeta_A(X', k).
\end{equation}

By Theorem 1.4 the left hand side of (4) coincides with

\[ Z_A(w \shuffle w'). \]

On the other hand, by Proposition 3.2 the right hand side of (4) coincides with

\[ (-1)^{l_1 + \cdots + l_s} Z_A(z_{k_1} \cdots z_{k_r} z_{l_1} \cdots z_{l_1}). \]

Therefore, we obtain the shuffle relation among FMZVs.

\begin{remark}

The case $r = s = 1, k_1 = 1$ and $l_1 = k - 1$ for $k > 1$, the Proposition 4.1 says that

\begin{equation}
Z_A(z_1 \shuffle z_{k-1}) = -Z_A(z_{k-1} z_1).
\end{equation}

The right hand side of (5) is $-\zeta_A(k - 1, 1)$, which is equal to $B_{p-k}$ by Hoffman’s result [H2 Theorem 6.1]. Here $B_{p-k} := (B_{p-k})_p \in \mathcal{A}$ and $B_n$ is the $n$-th Bernoulli number. On the other hand, since

\[ z_1 \shuffle z_{k-1} = z_1 z_{k-1} + \sum_{k_1, k_2 \geq 1, k_1 + k_2 = k} z_{k_1} z_{k_2}, \]

the left hand side of (5) is equal to

\[ \zeta_A(1, k - 1) + \sum_{k_1, k_2 \geq 1, k_1 + k_2 = k} \zeta_A(k_1, k_2). \]

Therefore, by [H2 Theorem 4.4], we have

\[ \sum_{k_1, k_2 \geq 1, k_1 + k_2 = k} \zeta_A(k_1, k_2) = -\zeta_A(k - 1, 1) + \zeta_A(1, k - 1) = 0. \]

\end{remark}
This equality is equivalent to the sum formula for double FMZVs [SW1, Theorem 1.4]. Indeed, by [SW1, Theorem 1.4] and [H2, Theorem 6.1], we have

$$
\sum_{k_1, k_2 \geq 1, k_i \geq 2, k_1 + k_2 = k} \zeta_A(k_1, k_2) = (-1)^{k+i}B_{p-k} = \begin{cases} 
-\zeta_A(1, k-1) & \text{if } i = 1, \\
-\zeta_A(k-1, 1) & \text{if } i = 2.
\end{cases}
$$

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