MEAN-FIELD GAMES AND SWARMS DYNAMICS IN GAUSSIAN AND NON-GAUSSIAN ENVIRONMENTS

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ABSTRACT. The collective behaviour of stochastic multi-agents swarms driven by Gaussian and non-Gaussian environments is analytically discussed in a mean-field approach. We first exogenously implement long range mutual interactions rules with strengths that are weighted by the real-time distance separating each agent with the swarm barycentre. Depending on the form of this barycentric modulation, a transition between two drastically different collective behaviours can be unveiled. A behavioural bifurcation threshold due to the tradeoff between the desynchronisation effects of the stochastic environment and the synchronising interactions is analytically calculated. For strong enough interactions, the emergence of a soliton propagating wave is established. Alternatively, weaker interactions cannot overcome the environmental noise and evanescent diffusive waves result. In a second and complementary approach, we show that the emergent solitons can alternatively be interpreted as being the optimal equilibrium of mean-field games (MFG) models with ad-hoc running cost functions which are here exactly determined. These MFG’s soliton equilibria are therefore endogenously generated. Hence for the classes of models here proposed, an explicit correspondence between exogenous and endogenous interaction rules leading to similar collective effects is explicitly constructed. For non-Gaussian environments our results offer a new class of exactly solvable mean-field games dynamics.

1. Introduction. Yoshiki Kuramoto proposed for the first time in 1975 a fully analytic study describing the collective behaviour of a swarm of interacting Brownian phase oscillators [1]. In this now paradigmatic multi-agent model, each phase oscillator evolving on a one-dimensional compact state space (circle) interacts with all its neighbours (long-range type interactions). The dynamics is described by a set of coupled stochastic differential equations (SDE) driven by independent White Gaussian Noise (WGN). In the very large population limit, it is legitimate to use a mean-field (MF) approach enabling to summarise the swarm behaviour into a phase oscillator density measure. Thanks to the underlying WGN the density measure evolves according to a nonlinear deterministic Fokker-Planck equation (FPE)
which, in the stationary regime, can be analytically discussed. The trade-off between the desynchronising tendency due to the random environment and the synchronising effect due to the mutual interactions leads to a bifurcation threshold separating two drastically different swarm behaviours. Namely for low coupling to noise ratio, the collective motion is diffusive and disorganised. Conversely for large coupling to noise ratio, a collective synchronised swarm emerges. While for the basic Kuramoto model, the mutual interactions rule is exogenously given, a recent approach [27] shows how the Kuramoto’s corporative behaviour may alternatively be viewed as being an equilibrium of a mean-field games (MFG) framework as first introduced in [21, 22, 23, 17, 13]. In this MFG approach each agent (i.e. phase oscillator) minimises an individual cost function which itself depends on the whole swarm. This collective minimisation procedure leads to a global equilibrium which coincides with the Kuramoto’s synchronised phase. Hence when adopting the alternative MFG point of view, one may interpret the collective behaviour as being due to an endogenous rule.

The central goal in this paper is to construct new fully solvable classes of scalar multi-agents models evolving on a scalar state space \( \mathbb{R} \) instead of the compact Kuramoto’s circular state space. Basically, the dynamics of the swarm of agents will be stylised by a collection of \( N \) stochastic differential equations:

\[
\dot{X}_{k,t} = \mathcal{I}(X_t, X_{k,t}) + \zeta_{k,t}, \quad k = 1, 2, \ldots, N,
\]

where \( \mathcal{I}(X_t, X_{k,t}) \) with \( X_t = (X_{1,t}, X_{2,t}, \ldots, X_{N,t}) \) describes the interactions and \( \zeta_{k,t} \) are stochastic processes which model environmental noise sources. In the sequel, the interactions kernels will be defined either exogenously by implementing an “avoid-to-be-a-laggard (ABL) rule or endogenously as being derived from a mean-field game (MFG) dynamics where each individual minimises an objective function which itself depends on the whole swarm. The main goal of the paper is to construct appropriate exogenous versus exogenous algorithms which ultimately generate identical collective swarms behaviours. This program is repeated for two distinct random sources \( \zeta_{k,t} \) which will be either White Gaussian Noise (WGN) processes or alternatively two-states Markov chains in continuous time, (also known as the Telegraphic Noise (TN)). The organisation of the paper can be summarised as follows:

a) **WGN environment with exogenous ABL interactions**, (Proposition 1). We shall show how a soliton propagating wave emerges from a specific ABL rule. The ABL interaction kernel implements a barycentric weighting factor which measures the actual influence of agents on their fellows. This time-dependent weight depends on each agent’s position relative to the swarm’s barycentre. The collective evolution of the agents is described by their (empirical) probability distribution which evolves as a soliton wave, (it solves a Burgers’ type equation). We observe that the existence of the soliton is ensured only for a specific range of control parameters. This unveils the existence of a transition between two drastically different swarm’s evolutions and the bifurcation threshold is known exactly.

b) **WGN environment with MFG endogenous interactions**, (Proposition 2). We construct a MFG for which the resulting ergodic behaviour exactly coincides with the soliton evolution calculated in point a).

c) **TN environment with exogenous ABL interactions**, (Proposition 3). This is the counterpart to point a) with TN environmental noise sources. Here
instead of the Burgers’ type pde, the underlying dynamics is a fully solvable discrete velocity Boltzmann’s equation also solved by soliton waves. Again, the exact bifurcation threshold separating distinct collective behaviours can be calculated.

d) TN environment with MFG endogenous interactions, (Proposition 4). This is the counterpart to point b) with TN environmental noise sources. Hence, we construct a MFG which admits a stationary ergodic equilibrium similar to the soliton evolution calculated in point c). We obtain a new class of analytically solvable MFG’s (not belonging to the linear drift with quadratic costs) and therefore it completes the yet rather limited collection of known solvable illustrations as those to be found in [6, 4, 11, 19].

2. Nonlinear diffusive dynamics. Let us consider a set of $N$ scalar interacting diffusion processes $X_{k,t} \in \mathbb{R}$ with time $t \in \mathbb{R}^+$:

$$dX_{k,t} = I(X_t, X_{k,t})dt + \sigma dB_{k,t}, \quad k = 1, 2, \ldots, N,$$

with $X_t := (X_{1,t}, X_{2,t}, \ldots, X_{N,t})$, $\sigma \in \mathbb{R}^+$ and $dB_{k,t}$ are $N$ independent standard Brownian motions [10]. The drift $I(X_t, X_{k,t})$ defines a mutual-interaction kernel exogenously implementing the algorithm:

Avoid being a laggard algorithm (ABL).

i) For $k = 1, 2, \ldots, N$ and in real time, agent $A_k$ observes the positions $X_{j,t}$ of her fellows $A_j$ for $j \neq k$ and $j = 1, 2, \ldots, N$.

ii) For $k = 1, 2, \ldots, N$ agent $A_k$ accounts the number $n_k(t)$ of her leaders $A_j$ namely those for which $X_{j,t} \geq X_{k,t}$ and for $j \neq k$.

iii) For $k = 1, 2, \ldots, N$ agent $A_k$ implements her instantaneous drift according to the rule:

$$I(X_t, X_{k,t}) = \frac{n_k(t)}{N}.$$  \hspace{1cm} (2)

In view of Eq.(2), $A_k$ effectively avoids to remain a swarm’s laggard since the more leaders she finds, the higher is her incentive to increase the drift velocity.

In the sequel, we will systematically focus attention on large populations (i.e. $N \to \infty$) enabling us define a smooth empirical agents population density $\rho(x,t) \in [0,1]$ as

$$\rho(x,t) = \frac{1}{N} \sum_{j=1}^{N} \delta(X_{j,t} - x).$$  \hspace{1cm} (3)

Since the agents population is homogeneous, (i.e. $\mathcal{I}_k(\cdot) \equiv \mathcal{I}(\cdot)$), we may randomly select one representative (index independent) fellow $A$ located at $X_t \in \mathbb{R}$. For $A$, the ABL rule is formally implemented as:

$$dX_t = \left[ \int_{X_t}^{\infty} \rho(y,t)dy \right] dt + \sigma dB_t.$$  \hspace{1cm} (4)

**Remark 1.** (mean-field dynamics). Eq.(4) implements infinite range interactions since agent $A$ effectively takes into account the locations of the whole swarm population (except herself) to determine her own drift. This basically realises the mean-field approach of the swarm’s dynamics.
The probabilistic properties of the trajectories solving the (Markovian) stochastic differential equation (SDE) Eq.(4) can be found by solving the associated nonlinear Fokker-Planck equation (FPE)[10]:

\[
\frac{\partial \rho(x, t)}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \rho(x, t)}{\partial x^2} - \frac{\partial}{\partial x} \left( \rho(x, t) \left[ \int_x^\infty \rho(y, t) dy \right] \right). 
\]  

(5)

For later use, we now slightly generalise the ABL rule Eq.(4) by further introducing an (infinitely differentiable) barycentric weighting function \( G[X_t - \langle X(t) \rangle] : \mathbb{R} \rightarrow \mathbb{R}^+ \). Accordingly, Eq.(5) will be now generalised as:

\[
\begin{align*}
    &\frac{\partial \rho(x, t)}{\partial t} = \frac{\alpha^2}{2} \frac{\partial^2 \rho(x, t)}{\partial x^2} - \frac{\partial}{\partial x} \left\{ \rho(x, t) \int_x^\infty G[y - \langle X(t) \rangle] \rho(y, t) dy \right\}, \\
    &\int_\mathbb{R} \rho(x, t) dx = 1, \\
    &\langle X(t) \rangle = \int_\mathbb{R} x \rho(x, t) dx. 
\end{align*}
\]

(6)

The extra factor \( G(x) \) weights the relative influence of the leaders depending on their relative remoteness from the swarm barycentre. With \( x \)-increasing \( G(x) \), we effectively describe situations where leaders are more influential than fellows close to the swarm barycentre and conversely for decreasing \( G(x) \)'s. Observe nevertheless that in both situations the interaction mechanisms retain the long range character. For the nonlinear swarms dynamics given in Eq.(6), we now can establish:

**Proposition 1.** Assuming the class of kernel functions \( G_{\eta, \sigma}(x) := A(\eta, \sigma) \cosh(x)^\eta \) with parameters \( \sigma \in \mathbb{R}^+, \eta \in [-2, \infty] \) and the pre-factor \( A(\eta, \sigma) = \frac{\sigma^2(2+\eta)}{N(\eta)} \), Eq.(6) is solved by the normalised soliton stationary propagating wave:

\[
\begin{align*}
    &\rho(x, t) = N(\eta) \cosh^{-(2+\eta)}(x - \omega t), \quad (2 + \eta) > 0, \\
    &\omega = N'(\eta) A(\eta, \sigma), \\
    &N(\eta)^{-1} = B(1/2, 1 + \eta/2) = \frac{\sqrt{\pi} \Gamma(1 + \eta/2)}{\Gamma(1/2 + \eta/2)}. 
\end{align*}
\]

(7)

where \( B(x, y) := \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \) stands for the beta function.

**Proof.** (Proposition 1).

Introduce the change of variables \( x \mapsto \xi = (x - \omega t) \) and \( t \mapsto \tau \) and assume the \( \tau \)-independent stationary evolution \( \rho(\xi, \tau) = \rho(\xi) = N(\eta) \cosh^{-(2+\eta)}(\xi) \). Accordingly, we have:

\[
\begin{align*}
    &\langle X(t) \rangle = \int_\mathbb{R} x \rho(x, t) dx = \int_\mathbb{R} (\xi + \omega t) \rho(\xi) d\xi = \omega t, \\
    &\partial_t \mapsto -\omega \partial_\xi + \partial_\tau \quad \text{and} \quad \partial_x \mapsto \partial_\xi, \\
    &\int_\mathbb{R} [G_{\eta, \sigma}(y - \langle X(t) \rangle) \rho(y, t) dy] \mapsto \int_\xi^\infty G_{\eta, \sigma}(\xi) \rho(\xi) d\xi = \\
    &\int_\xi^\infty A(\sigma, \eta) N(\eta) \cosh^{-2}(\xi) d\xi. 
\end{align*}
\]

It follows that Eq.(6) can be rewritten as:

\[
\partial_\xi \left\{ \frac{\sigma^2}{2} \partial_\xi \rho(\xi) + \omega \rho(\xi) - \rho(\xi) \int_\xi^\infty A(\sigma, \eta) N(\eta) \cosh^{-2}(\xi) d\xi \right\} = 0. 
\]
Integrating once the last equation with respect to $\xi$ (with vanishing integration constant since to fulfill the normalisation constraint, one has to impose $\lim_{|\xi|\to\infty} \rho(\xi) = 0$). Then dividing by $\rho(\xi) > 0$, we straightforwardly obtain:

$$\frac{\sigma^2}{2} \frac{d}{d\xi} \log[\rho(\xi)] = -\omega + \int_{\xi}^{\infty} A(\sigma, \eta) N(\eta) \cosh^{-2}(\xi) d\xi.$$ 

Plugging $\rho(\xi) = N(\eta) \cosh^{-(2+\eta)}(\xi)$ into the last equation and using the relation

$$\int_{\xi}^{\infty} \cosh^{-2}(\xi) d\xi = 1 - \tanh(\xi),$$

yields

$$-\frac{\sigma^2}{2} (2 + \eta) \tanh(\xi) = -\omega + A(\sigma, \eta) N(\eta)[1 - \tanh(\xi)].$$

By direct identification, we see that we need to fulfil:

$$\begin{cases}
A(\sigma, \eta) = \frac{\sigma^2 (2+\eta)}{2N(\eta)}, \\
\omega = A(\sigma, \eta) N(\eta).
\end{cases} \tag{8}$$

The normalisation factor $N(\eta)$ imposes $(2 + \eta) > 0$ and, using [12], (see the entry 8.380.10), we have:

$$N(\eta)^{-1} = \int_{\mathbb{R}} \cosh^{-(2+\eta)}(x) dx = B \left( \frac{1}{2}, 1 + \frac{\eta}{2} \right). \tag{9}$$

**Remark 2.** Note that a normalised soliton cannot be generated for weights $G(z) \propto \cosh^{\eta}(z)$ for $\eta < -2$. This can be heuristically understood by the fact that decreasing $\eta$, reduces the cooperative influence of remote leaders which weakens the possibility to sustain a collective evolution. Hence $\eta = -2$ is a bifurcation threshold which separates two drastically different swarm’s evolution, namely: while for $\eta > -2$ one observes the emergence of a collective wave materialised by a soliton, this co-operative behaviour cannot be sustained for $\eta \leq -2$. In this last (diffusive like) regime, the noise source dominates thus leading to diffusive evanescent waves.

**Remark 3.** The transitions from a diffusive (evanescent) spatio-temporal pattern toward a stationary soliton driven by $\eta$ can be viewed as a phase transition similar to the one encountered in the Ising spin model of statistical physics$^1$. Basically, strong enough synchronising interactions (spin alignment) can beat the randomisation due to the environmental noise (stylised by WGN processes) and ultimately a coherent stationary pattern emerges (a permanent Ising ferro-magnet). For one-dimensional models, (like in our model where the agents evolve on $\mathbb{R}$), statistical physics teaches us that only infinite range interactions are strong enough to enable the emergence of stationary spatio-temporal patterns. In our present model, while keeping long range interactions (as those implemented in Eq.(6)) but progressively reducing the influence of far remote agents, we are able to sufficiently weaken the interactions to explore the onset of a phase transition reached at the threshold $\eta = -2$.

$^1$In statistical physics, the Ising model stylises how a ferromagnetic (permanent) macroscopic phase emerges from a background nominally fully disorganised individual microscopic spins.
2.1. Corresponding mean-field game dynamics. Consider the diffusive dynamics:
\[ dX_t = [f^*(X_t)] dt + \sigma dB_t, \]  
(10)
where the drift \( f^*(X_t) \) minimises the cost functional:
\[ J[X(\cdot), f(\cdot)] = \mathbb{E} \left\{ \int_0^T \mathcal{L}(X_s, \rho(X_s, s)) ds \right\} + C_T(X_T), \]  
(11)
on the time horizon \([0, T]\) and \( \mathbb{E} \{ \cdot \} \) stands for the expectation over the underlying WGN. In Eq.(11), we shall make the specific choice:
\[ \mathcal{L}(f(x), \rho(x, t)) := \frac{1}{2\mu} [f(x) - b]^2 - g [\rho(x, t)]^a. \]  
(12)
with \( g, \mu, a \in \mathbb{R}^+ \) and \( C_T(X_T) \) stands for a final cost. As shown in Appendix A, the cost structure defined in Eqs.(11) and (12) leads to a set of two nonlinear coupled pde’s which have to be simultaneously solved forward/backward in time \([21, 22, 23, 17, 13, 25]\):
\[
\begin{cases}
  f^*(x, t) = -\left[ \frac{1}{\mu} \partial_x u(x, t) - b \right], \\
  \partial_t \rho(x, t) = -\partial_x [f^*(x, t) \rho(x, t)] + \frac{\sigma^2}{2} \partial_{xx} \rho(x, t) \quad \text{(Fokker Planck)}, \\
  \partial_t u(x, t) + b \partial_x u(x, t) - \frac{1}{2\mu} [\partial_x u(x, t)]^2 + \frac{\sigma^2}{2} \partial_{xx} u(x, t) = g [\rho(x, t)]^a \quad \text{(HJB)},
\end{cases}
\]  
(13)
where \( u(x, t) \) stands for the value function of the Hamilton-Jacobi-Bellman (HJB) of the underlying optimal control problem. Assume now that we deal with sufficiently large time horizons \( T \) so that for the range of times \( 0 << t << T \), an ergodic regime \([25, 5]\) can be attained and for which we approximately have:
\[ u(x, t) \approx \epsilon t \]  
(14)
with \( \epsilon \) an a priori unknown constant. In this time range, the initial conditions and the final cost barely affect the solution of Eq.(13) and we can establish:

**Proposition 2.** Given the parameters \( a, b, \sigma \) and \( \mu \) in Eqs.(10) and (12) and for the amplitude \( g = \frac{\mu \sigma^4 (a+1)[\xi(a)]^{-2a}}{2a^2} \) in Eq.(12), the probability density \( \rho(x, t) \) and the value function \( u(x, t) \) associated with the stationary ergodic regime of the MFG dynamics defined by Eq.(13) reads:
\[ \begin{cases}
  \rho(x, t) = \frac{1}{B(1, \frac{1}{2}) \cosh(\frac{x - bt}{2\sigma})^{1/2}}, \\
  u(x, t) = -\mu \sigma^2 \ln \left[ \sqrt{\rho(x, t)} \right] - \epsilon t.
\end{cases} \]  
(15)

**Proof.** (Proposition 2).

We first introduce a couple of auxiliary scalar fields \( \Phi(x, t), \Psi(x, t) \) defined by:
\[ \begin{cases}
  \Phi(x, t) = e^{-\frac{u(x, t) - \epsilon t}{\mu \sigma^2}}, \\
  \Psi(x, t) = e^{\frac{u(x, t) - \epsilon t}{\mu \sigma^2}} \rho(x, t)
\end{cases} \]  
(16)
and hence \( \rho(x, t) = \Phi(x, t) \Psi(x, t) \). In terms of \( \Phi(x, t), \Psi(x, t) \), Eq.(13) can be rewritten as, (see Appendix B for the derivation of the second line of Eq.(17) below:
\[
\begin{align*}
\omega &= \text{ABL algorithm generates a collective behaviour which in the time range } 0 \ll (22) \text{ are identical. Therefore, one can directly assert that the exogenously given Exogenous versus endogenous interactions algorithm } \\
\text{Finally, since for this stationary ergodic regime } \Psi(\xi) \text{ the normalised solution of Eq.(20) reads as:} \\
\text{Using the identity } \cosh x - 1 = \sinh^2 x, \text{ it s straightforward to verify that provided we have:} \\
\text{the normalised solution of Eq.(20) reads as :} \\
\text{with the normalisation factor } \mathcal{N}(a) \text{ is given by } [12] : \\
\text{Finally, since for this stationary ergodic regime } \Psi(\xi) = \Phi(\xi), \text{ we have immediately:}
\end{align*}
\]

**Remark 4.** *Exogenous versus endogenous interactions algorithm.* By identifying \( \omega = b \) and \( 2a^{-1} = (2 + \eta) > 0 \), we see that both solitons arising in Eqs.(7) and (22) are identical. Therefore, one can directly assert that the exogenously given ABL algorithm generates a collective behaviour which in the time range } 0 \ll
$t << T$ coincides with the optimal ergodic equilibrium of the MFG with running costs function given by Eq. (12). Using this connection, we may further distinguish between two regimes, namely:

i) $a \in [0, 1]$ ($\Rightarrow \eta > 0$ in the $G_{\eta, \sigma}(x)$ factor of Proposition 1). When $\eta > 0$, the swarm’s top leaders strongly train the laggards (i.e. thus corresponding to a strong long range interactions). This ultimately gives rise to spread solitons. Viewed from the MFG picture, small values of parameter $a$ heuristically imply that the corresponding running costs are minimised for comparatively narrow shaped solitons.

ii) $a \in [1, \infty]$ ($\Rightarrow \eta \in [-2, 0]$ in the $G_{\eta, \sigma}(x)$ factor of Proposition 1). When $\eta < 0$, the swarm’s top leaders barely train the laggards (i.e. one effectively implements quasi short range interactions). This ultimately gives rise to wide waisted solitons. In the MFG picture for large $a$’s, one realises that the corresponding running costs are minimised for comparatively wide shaped solitons.

For potential calibration needs, let us mention that the variance $\Sigma^2$ of the resulting solitons can be computed as [9]:

\[ \Sigma^2 = \int_{\mathbb{R}} \frac{x^2 dx}{B \left( \frac{1}{2}, \frac{1}{2} \right) \left[ \cosh(x) \right]^2} = \sum_{n=0}^{\infty} \frac{1}{(a^{-1} + n)^2} \approx \int_{0}^{\infty} \frac{dx}{(a^{-1} + x)^2} \approx a. \]

**Remark 5.** Generic character of the modelling and soliton stability issue. The highly specific value of the $g$ pre-factor in the MFG dynamics Eq.(21) is here chosen with the goal to exactly reproduce the behaviour emerging from the exogenous ABL rule introduced in Eqs.(7) and (15).

Such highly specific $g$-factors, raise naturally questions regarding the generic character of these solutions. In other words, are such solutions stable under small perturbations of $g$. To answer this stability issue, let us reconsider Eq.(19) (first line) and introduce the following auxiliary definitions:

\[
\Phi(x) = \Phi_M S \left( \frac{x}{\Lambda(\epsilon)} \right) = \Phi_M S(y),
\]

\[
\Phi_M(\epsilon) := \left[ \frac{(a+1)\epsilon}{g} \right]^{1/2a},
\]

\[
\Lambda(\epsilon) = \sqrt{\frac{\mu \sigma^2}{2a^2 \epsilon}} \text{ characteristic length.}
\]

Observe that now $g$ is a free parameter whereas $\epsilon$ which is yet an unknown constant has to now be determined. Rewriting Eq.(19) in terms of Eq.(24), we obtain:

\[ a^2 [\partial_{yy} S(y)]^2 + (a + 1) [S(y)]^{2a+1} - S(y) = 0. \]  

(25)

which can be directly integrated to yield:

\[ S(y) = \left[ \cosh(y) \right]^{-\frac{1}{2}} \Rightarrow S(x) = \left[ \cosh \left( \frac{x}{\Lambda(\epsilon)} \right) \right]^{-\frac{1}{2}} \]

(26)

The normalisation constraint fixes finally the value of $\epsilon$, namely we have:

\[ \int_{\mathbb{R}} \Phi_M^2(\epsilon) \left[ \cosh \left( \frac{x}{\Lambda(\epsilon)} \right) \right]^{-\frac{2}{a}} dx = 1 \Rightarrow \]

\[ \epsilon = \frac{1}{2} \left[ B \left( \frac{1}{2}, \frac{1}{2} \right) \right]^{2a} \left[ \frac{g}{(a+1)} \right]^{\frac{2}{a}} \left[ \frac{2a^2}{\mu \sigma^2} \right]^{\frac{1}{a}}, \]  

(27)
where we used once again the definition in Eq.\((23)\). Observe now that Eq.\((27)\) exhibits a singularity at the value \(a = 2\). As discussed in [26], the \(a = 2\) threshold is the signature of an exchange of stability. More precisely, the authors in [26] show that while for \(a \in [0, 2]\) (corresponding to \(\eta \in [-1, +\infty)\)) in the ABL exogenous rule, the resulting solitons are stable, they become unstable for \(a \in [2, \infty)\) (i.e. for \(\eta \in [-2, -1]\)). This shows the central role played by the \(\eta\) parameter in the weighting factor \(G_{\eta, \sigma}(x)\) of Proposition 1. In view of this stability behaviour, one may now conclude that for \(a \in [0, 2]\), small perturbations of \(g\) value given in Proposition 2 will not alter the overall dynamic picture since the corresponding MFG soliton is stable. Conversely, for \(a \in [2, \infty]\), the resulting MFG soliton being not generically stable, small alterations of the \(g\)-value given in Proposition 2 are likely to strongly affect the overall MFG dynamics. As already notified in Remark 2, the instability can be traced back from the quasi short range nature of the interactions which result when \(\eta \in [-2, -1]\). For such one-dimensional multi-agents dynamics, quasi short range interactions offer a comparatively weak synchronisation capability.

3. Piecewise evolution dynamics and telegraphic Noise environments.

The exposition structure of section 2 will now be repeated for non-Gaussian environmental noise modelled by two states Markov chains in continuous time (i.e. telegraphic process, see in particular the section 9 in [16]). Instead of the diffusion dynamics given by Eq.\((1)\), we now consider a set of discrete two velocity Boltzman’s equation [24] with random Poisson switchings between the two velocities. Specifically, we consider a set of \(N\) agents evolving on \(\mathbb{R}\) driven by random velocities with values being either \(-1\) or \(+1\). The spontaneous velocity switches are triggered by a couple of Poisson processes with rates \(u_\pm \geq 0\). In addition to these random velocity switches, we implement a Boltzmann nonlinear collision mechanism which imposes that after a collision, a couple of particles initially with velocities \((-1, +1)\) emerge with velocities \((+1, +1)\). In the sequel, we assume that the \((-1, +1)\) spontaneous switching rate itself depends on the instantaneous configuration of the swarm of particles. For \(x \in \mathbb{R}\) and time \(t \in \mathbb{R}^+\), let \(P(x, t)dx\) (respectively \(Q(x, t)dx\) stand for the proportion of agents with velocities \(+1\) (respectively \(-1\)) located in \([x, x + dx]\). Instead of the Fokker-Planck Eq.\((6)\), the corresponding dynamics now reads as a generalised version of the velocity Boltzman’s equation which is also known as the Ruijgrok and Wu dynamics (RW) [24]:

\[
\begin{align*}
\dot{P}(x, t) + \partial_x P(x, t) &= -u_+ P(x, t) + u_- Q(x, t) + \Omega_T [P(x, t), Q(x, t)], \\
\dot{Q}(x, t) - \partial_x Q(x, t) &= +u_+ P(x, t) - u_- Q(x, t) - \Omega_T [P(x, t), Q(x, t)], \\
\Omega_T [P(x, t), Q(x, t)] &= a P(x, t) \int_x^\infty G [y - \langle X(t)\rangle] Q(y, t)dy + b Q(x, t) \int_x^\infty G [y - \langle X(t)\rangle] P(y, t)dy,
\end{align*}
\]

with \(u_-, u_+, a, b \in \mathbb{R}^+\) and initial conditions \(P_0(x)\) and \(Q_0(x)\). Associated with Eq.\((28)\), we have the normalisation constraint and the definition:

\[
\int_\mathbb{R} [P(x, t) + Q(x, t)] dx \equiv 1, \quad \text{(normalisation of probability mass)}
\]

\[
\langle X(t) \rangle := \int_\mathbb{R} x [P(x, t) + Q(x, t)] dx, \quad \text{(barycenter location of probability mass)}
\]
For the nonlinear dynamics Eq. (28), we will now establish:

**Proposition 3.** In the dynamics defined by Eq. (28), for the choices \( \omega \in [0, 1], \eta \in [-2, +\infty], (u_+, u_-) \) solving:

\[
\frac{u_+}{(1 - \omega)} - \frac{u_-}{(1 + \omega)} = 2 + \eta
\]

and the barycentre weight function:

\[
G(x) \equiv G_{a,b,\eta}(x) = 2B \left( \frac{1}{2}, 1 + \frac{\eta}{2} \right) \frac{(2 + \eta)}{(a + b)} \cosh^\eta(x),
\]

Eq. (28) is solved by the soliton wave:

\[
P(x - \omega t) = Q(x - \omega t) = \left[ 2B \left( \frac{1}{2}, 1 + \frac{\eta}{2} \right) \right]^{-1} \cosh(x - \omega t)^{-(2 + \eta)},
\]

with \( \eta \in [-2, +\infty] \).

**Proof. (Proposition 3).**

We introduce the change of variables \( t \mapsto \tau \) and \( x \mapsto \xi = (x - \omega t) \) and we focus on the stationary regime \( \partial_t P(\xi, \tau) = \partial_\xi Q(\xi, \tau) = 0 \). We assume the symmetry \( P(\xi) = P(-\xi) \) and \( Q(\xi) = Q(-\xi) \) and so Eq. (31) implies:

\[
\langle X(t) \rangle = \int_\mathbb{R} (\xi + \omega t) \left[ P(\xi) + Q(\xi) \right] d\xi = \omega t.
\]

In the stationary regime, Eq. (28) can be rewritten as:

\[
(1 - \omega) \partial_\xi P(\xi) = (1 + \omega) \partial_\xi Q(\xi) = a P(\xi) \int_\xi^\infty G_{a,b,\eta}(y) Q(y) dy
\]

\[
+ b Q(\xi) \int_\xi^\infty G_{a,b,\eta}(y) P(y) dy - u_+ P(\xi) + u_- Q(\xi).
\]

Introducing the rescaling factors:

\[
\begin{align*}
\{ & P(\xi) := (1 + \omega) \hat{P}(\xi) \quad \text{and} \quad Q(\xi) := (1 - \omega) \hat{Q}(\xi), \\
& u_+ := \hat{u}_+(1 - \omega) \quad \text{and} \quad u_- := \hat{u}_-(1 + \omega),
\end{align*}
\]

we can rewrite Eq. (32) as:

\[
\partial_\xi \hat{P}(\xi) = \partial_\xi \hat{Q}(\xi) = -\hat{u}_+ \hat{P}(\xi) + \hat{u}_- \hat{Q}(\xi) + a \hat{P}(\xi) \int_\xi^\infty G(a, b, \eta)(y) \hat{Q}(y) dy
\]

\[
+ b \hat{Q}(\xi) \int_\xi^\infty G(a, b, \eta)(y) \hat{P}(y) dy.
\]

In view of Eqs. (31) and (34), we now assume that \( G_{a,b,\eta}(\xi) = A(a, b, \eta) \cosh(\xi) \) and \( \hat{P}(\xi) = \hat{Q}(\xi) = \hat{N}(\eta) \cosh^{-(2 + \eta)}(\xi) \) and a direct substitution into Eq. (34) yields:

\[
\hat{N}(\eta)(2 + \eta) \cosh(\xi)^{-(\eta + 3)} \sinh(\xi) = [\hat{u}_+ - \hat{u}_-] \hat{N}(\eta) \cosh(\xi)^{-(2 + \eta)} + \hat{N}^2(\eta)(a + b) \cosh(\xi)^{-(2 + \eta)}  
\]

\[
\int_\xi^\infty \frac{A(a, b, \eta)}{\cosh(\xi)^2} d\xi, \quad (35)
\]

\[
A(a,b,\eta)(1 - \tanh(\xi))
\]
By direct identification, one concludes that one has to fulfil:

\[
\begin{aligned}
2 + \eta &= A(a, b, \eta)(a + b)\hat{N}(\eta) \Rightarrow A(a, b, \eta) = \frac{2 + \eta}{(a + b)\hat{N}(\eta)}, \\
[\hat{u}_- - \hat{u}_+] + A_{a, b, \eta}(a + b)\hat{N}(\eta) = 0 &\Rightarrow [\hat{u}_+ - \hat{u}_-] = 2 + \eta.
\end{aligned}
\]

(36)

Finally, the normalisation (using the first line of Eq.(29)), we use again Eq.(9) which leads to:

\[
\begin{aligned}
\left(2\hat{N}(\eta)\right)^{-1} &= \int_{\mathbb{R}} (1 + \omega) \frac{d\xi}{[\cosh(\xi)]^{2+\eta}} + \int_{\mathbb{R}} (1 - \omega) \frac{d\xi}{[\cosh(\xi)]^{2+\eta}} \\
&= B \left(\frac{1}{2}, 1 + \frac{\eta}{2}\right).
\end{aligned}
\]

\[\square\]

Remark 6. Here again one sees that a normalised soliton cannot be generated for modulation kernels \(G(x) \propto \cosh^\eta(x)\) for \(\eta < -2\) and therefore, similarly to section 2, one concludes that \(\eta = 2\) is the bifurcation threshold separating two different propagation modes.

3.1. Corresponding mean-field game dynamics for piecewise deterministic evolutions. In full similarity with the development adopted in section 2, we shall now construct a MFG dynamics in presence of a random environment modelled by a telegraphic noise process. Again, we shall focus on the ergodic regimes leading to solitons similar to those found in Proposition 3. Accordingly, let us consider the controllable piecewise deterministic evolution:

\[
\begin{aligned}
\partial_t P(x, t) + \partial_x P(x, t) &= -u_+(x, t)P(x, t) + u_-(x, t)Q(x, t), \\
\partial_t Q(x, t) - \partial_x Q(x, t) &= +u_+(x, t)P(x, t) - u_-(x, t)Q(x, t),
\end{aligned}
\]

(37)

which differs from Eq.(28) by the fact that the Poisson switching rates \(u_-(x, t)\) and \(u_+(x, t)\) are now explicitly \((x, t)\)-dependent. For a time horizon \(t \in [0, T]\), let us introduce a couple of cost functions \(J_{\pm}\) in the form discussed in [14]:

\[
\begin{aligned}
J_{\pm}[X(\cdot), u_{\pm}(\cdot)] &= \mathbb{E} \left\{ \int_0^T \left\{ \mathcal{L}(u_{\pm}(X(s), s) + W[P(\cdot), Q(\cdot), X(s), s])ds \right\} \\
&\quad + C_{\pm, T}(X(T)) \right\} \\
\mathcal{L}(u_{\pm}(x, t), t) &= u_{\pm}(x, t) \ln \left[u_{\pm}(x, t)\right] - u_{\pm}(x, t) + 1
\end{aligned}
\]

(38)

where \(C_{T}(X(T))\) stands for a final cost. In Eq.(38), observe that the running cost \(W[P(x, t), Q(x, t)]\) depends only on the probability densities \(P(x, t)\) and \(Q(x, t)\) which confers to the dynamics its MFG character. The objective is now to minimise the global costs \(J_{\pm}[X(\cdot), u_{\pm}(\cdot)]\) by optimally adjusting the switching rates \(u_{\pm}(x, t)\). Invoking the dynamic programming (DP) principle, we may now derive the associated Hamilton-Jacobi-Belman (HJB) equation and for the resulting couple of value functions \(V_{\pm}(x, t)\), we obtain (see the Proposition 1 in [14]):
Proposition 4. For given Eq. (39) are given by the soliton waves:

\[
\partial_t V_+ (x, t) + \partial_x V_+ (x, t) + \min_{u_+} \{ \mathcal{L}(u_+ (x, t), t) + u_+ [V_-(x, t) - V_+ (x, t)] \} + \mathcal{W}(P(x, t), Q(x, t)) = 0,
\]

\[
\partial_t V_- (x, t) - \partial_x V_- (x, t) + \min_{u_-} \{ \mathcal{L}(u_- (x, t), t) + u_- [V_+ (x, t) - V_- (x, t)] \} + \mathcal{W}(P(x, t), Q(x, t)) = 0.
\]

Performing the required minimisations, Eq. (39) becomes:

\[
\begin{align*}
\partial_t V_+ (x, t) + \partial_x V_+ (x, t) + & \left[ 1 - e^{+V_+ (x, t) - V_- (x, t)} \right] + \mathcal{W}(P(x, t), Q(x, t)) = 0, \\
\partial_t V_- (x, t) - \partial_x V_- (x, t) + & \left[ 1 - e^{-V_- (x, t) - V_+ (x, t)} \right] + \mathcal{W}(P(x, t), Q(x, t)) = 0
\end{align*}
\]

and the optimal switching rates \( u^*_+ (x, t) \) and \( u^*_- (x, t) \) are given by:

\[
\begin{align*}
u^*_+ (x, t) &= e^{+V_+ (x, t) - V_- (x, t)}, \\
u^*_- (x, t) &= e^{-V_- (x, t) - V_+ (x, t)}.
\end{align*}
\]

**Proposition 4.** For given \( \omega \in [0, 1] \) and for the class of running costs in Eq. (38) reading as:

\[
\mathcal{W}(P, Q) = g(q, \omega) [PQ]^q, \quad q > 0,
\]

\[
g(q, \omega) = \left[ \frac{q+1}{q(2-\omega)} \right]^{-2q} B \left( \frac{1}{2}, \frac{1}{2q} \right)^{2q},
\]

the probability densities \( P(x, t) \) and \( Q(x, t) \) solving the ergodic regime of the MFG Eq. (39) are given by the soliton waves:

\[
\begin{align*}
(1 - \omega)P(x - \omega t) = (1 + \omega)Q(x - \omega t) = \frac{\hat{N}(q)}{[\cosh(x - \omega t)]^{1/2}}, \\
\hat{N}(q) &= \frac{(\omega^2 - 1)}{2} B \left( \frac{1}{2}, \frac{1}{2q} \right).
\end{align*}
\]

**Proof.** (Proposition 4).

For \( \epsilon \in \mathbb{R}^+ \), let us introduce the following couple of alternative transformations:

\[
\begin{align*}
\varphi_B &= e^{-\epsilon t - V_+}, \quad \Gamma_B = P e^{\epsilon t + V_+} \Rightarrow P = \Gamma_B \varphi_B, \\
\varphi_A &= e^{-\epsilon t - V_-}, \quad \Gamma_A = Q e^{\epsilon t + V_-} \Rightarrow Q = \Gamma_A \varphi_A, \\
u^*_+ &= \frac{\varphi_A}{\varphi_B}, \quad \text{and} \quad u^*_- = \frac{\varphi_B}{\varphi_A}.
\end{align*}
\]

Using Eq. (44) and substituting the definitions of \( \varphi_A \) and \( \varphi_B \) into Eq. (40), we obtain:

\[
\begin{align*}
\partial_t [\varphi_A] - \partial_x [\varphi_A] - \varphi_A + \varphi_B + [\epsilon - \mathcal{W}] \varphi_A &= 0, \\
\partial_t [\varphi_B] + \partial_x [\varphi_B] + \varphi_A - \varphi_B + [\epsilon - \mathcal{W}] \varphi_B &= 0.
\end{align*}
\]

Introducing \( u^*_+ (x, t) \) and \( u^*_- (x, t) \) as given by Eqs. (41) and (44) into Eq. (37) and using once more Eq. (44), we obtain:

\[\text{In the sequel, for simplicity of the notation, we omit to repeat the ubiquitous } (x, t) \text{ argument.}\]
\[
\begin{align*}
\partial_t [\Gamma_A] - \partial_x [\Gamma_A] + \Gamma_A - \Gamma_B + [W - \epsilon] \Gamma_A &= 0, \quad (46a) \\
\partial_t [\Gamma_B] + \partial_x [\Gamma_B] - \Gamma_A + \Gamma_B + [W - \epsilon] \Gamma_B &= 0. \quad (46b)
\end{align*}
\]

In view of Eqs. (45) and (46) and the set of definitions introduced in Eq. (44), we can derive:

\[
\begin{align*}
\{ & [\text{Eq. (45a)}] \times \Gamma_A] + [\text{Eq. (46a)}] \times \varphi_A] \Rightarrow \partial_t Q - \partial_x Q + \{ \Gamma_A \varphi_B - \varphi_A \Gamma_B \} = 0. \\
& [\text{Eq. (45b)}] \times \Gamma_B] + [\text{Eq. (46b)}] \times \varphi_B] \Rightarrow \partial_t P + \partial_x P - \{ \Gamma_A \varphi_B - \varphi_A \Gamma_B \} = 0.
\end{align*}
\]

Performing the change of variables:

\[
\begin{align*}
t \mapsto \tau, \quad x \mapsto \xi = (x - \omega t), \\
\partial_t \mapsto \partial_\tau - \omega \partial_\xi, \quad \partial_x \mapsto \partial_\tau,
\end{align*}
\]

enables to rewrite Eq. (47) as:

\[
\begin{align*}
\frac{1}{(1 - \omega)} \partial_\tau \hat{P} + \partial_\xi \hat{P} - \{ \Gamma_A \varphi_B - \varphi_A \Gamma_B \} &= 0, \\
\frac{1}{(1 - \omega)} \partial_\tau \hat{Q} - \partial_\xi \hat{Q} + \{ \Gamma_A \varphi_B - \varphi_A \Gamma_B \} &= 0.
\end{align*}
\]

Focusing on the stationary regimes for which \(\partial_\tau \hat{P} = \partial_\tau \hat{Q} = 0\) ⇒ \(\hat{P}(\xi) = \hat{Q}(\xi) + c\), with \(c\) being a constant. Since the normalisation imposes that \(\lim_{|\xi| \to \infty} \hat{P}(\xi) = 0\) and \(\lim_{|\xi| \to \infty} \hat{Q}(\xi) = 0\), we have \(c = 0\) and therefore \(\hat{P}(\xi) = \hat{Q}(\xi)\) implying:

\[
\hat{P}(\xi) = (1 - \omega)P(\xi) = (1 - \omega)\varphi_B(\xi)\Gamma_B(\xi) = 0,
\]

\[
\hat{Q}(\xi) = (1 + \omega)Q(\xi) = (1 + \omega)\varphi_A(\xi)\Gamma_A(\xi),
\]

As \(\hat{P}(\xi) = \hat{Q}(\xi)\), we focus on \(\hat{P}(\xi)\) and differentiating the first lines in Eq. (49) yields:

\[
\partial_\xi \hat{P} - \partial_\xi \{ \Gamma_A \varphi_B - \varphi_A \Gamma_B \} = 0.
\]

In the stationary regime, Eqs. (45) and (46) enable to write straightforwardly the following expressions:

\[
\begin{align*}
\varphi_B \partial_\xi \Gamma_A &= + \varphi_B \Gamma_A \frac{1}{(1 + \omega)} - \varphi_B \Gamma_B \frac{1}{(1 + \omega)} - \varphi_B \Gamma_A \frac{W - \epsilon}{(1 + \omega)}, \\
\Gamma_A \partial_\xi \varphi_B &= - \varphi_A \Gamma_A \frac{1}{(1 - \omega)} + \varphi_A \Gamma_A \frac{1}{(1 - \omega)} + \varphi_B \Gamma_A \frac{W - \epsilon}{(1 + \omega)}, \\
\Gamma_B \partial_\xi \varphi_A &= - \varphi_A \Gamma_B \frac{1}{(1 + \omega)} + \varphi_A \Gamma_B \frac{1}{(1 + \omega)} - \varphi_A \Gamma_A \frac{W - \epsilon}{(1 + \omega)}, \\
\varphi_A \partial_\xi \Gamma_B &= - \varphi_A \Gamma_B \frac{1}{(1 - \omega)} + \varphi_A \Gamma_B \frac{1}{(1 - \omega)} + \varphi_A \Gamma_B \frac{W - \epsilon}{(1 - \omega)}.
\end{align*}
\]

This enables to write:

\[
\partial_\xi (\Gamma_A \varphi_B - \Gamma_B \varphi_A) = - \frac{2}{(\omega - 1)} \left\{ (\hat{P} + \hat{Q}) + (\epsilon - 1 - W) (\varphi_A \Gamma_B + \varphi_B \Gamma_A) \right\}
\]

(52)
Consistent with Eqs. (44) and (50), we now assume that:
\[ \varphi_B = \varphi_A \frac{1}{1 - \omega} \text{ and } \Gamma_B = \Gamma_A (1 + \omega) \]
\[ \Rightarrow (\varphi_A \Gamma_B + \varphi_B \Gamma_A) = \left( \frac{2 - \omega^2}{1 - \omega^2} \right) \dot{P}. \]
(53)

Since \( \dot{P}(\xi) = \ddot{Q}(\xi) \), Eqs. (51), (52) and (53) imply:
\[ \partial_{\xi} \ddot{P}(\xi) = \frac{2}{(1 - \omega^2)} \left[ 2 + (\epsilon - 1 - \mathcal{W}(\dot{P}, \dot{Q})) \left( \frac{2 - \omega^2}{1 - \omega^2} \right) \right] \ddot{P}(\xi) \]
(54)

We now focus on the set of running cost functions,:
\[ \mathcal{W}(P, Q) = \mathcal{W}(PQ) = \mathcal{W}(\frac{\dot{P} \dot{Q}}{1 - \omega^2}) := g \left( \ddot{P}(\xi) \right)^{2q}, \quad g, q \in \mathbb{R}^+. \]
(55)

For \( \omega \in [0, 1] \), it is immediate to realise that Eq. (54) exhibits the standard form of the nonlinear Schrödinger equation:
\[ \partial_{\xi} \ddot{P}(\xi) = 2\left[ \epsilon (2 - \omega^2) - \omega^2 \right] \dddot{P}(\xi) - 2g \frac{(2 - \omega^2)}{(1 - \omega^2)^2} \left[ \dot{P}(\xi) \right]^{2q+1}. \]
(56)

Provided appropriate constants \( g, q, \epsilon, \omega \) are chosen, Eq. (56) can be integrated to yield a soliton which coincides with the one found in Eq. (31). To see this, multiply both sides of Eq. (54) by \( \partial_{\xi} \dot{P} \) and integrate once with respect to \( \xi \) (with zero integration constant), we obtain:
\[ \left( \partial_{\xi} \ddot{P}(\xi) \right)^2 + 2\left[ \epsilon (2 - \omega^2) - \omega^2 \right] \dddot{P}(\xi) - 2g \frac{(2 - \omega^2)}{(1 - \omega^2)^2} \left[ \dot{P}(\xi) \right]^{2q+2} = 0. \]
(57)

Using once again the separation of variable technique, Eq. (57) leads to:
\[ \int_1^{\xi} \frac{d\ddot{P}(\xi)}{\sqrt{A_1(\epsilon, \omega)P^2(\xi) - A_2(g, \omega, q)\ddot{P}(\xi)^{2q+2}}} = \xi, \]
\[ A_1(\epsilon, \omega) := 2\left[ \epsilon (2 - \omega^2) - \omega^2 \right], \]
\[ A_2(g, \omega, q) := g \frac{(2 - \omega^2)}{(q + 1)(1 - \omega^2)^2}. \]
(58)

Now we can verify that the Ansatz \( \ddot{P}(\xi) = \frac{\hat{N}(q)}{\cosh(q)} \) with \( q > 0 \) solves Eq. (58) provided we impose:
\[ 1 = A_1(\epsilon, \omega) q^2 \Rightarrow \epsilon = \frac{1}{(2 - \omega^2)} \left[ (1 - \omega^2)^2 + \omega^2 \right] \]
\[ 1 = A_2(g, \omega, q) q^2 \hat{N}(q)^{2q}. \]
(59)

So given the couple constants \( g > 0 \) and \( \omega \in [0, 1] \), we choose \( g \) to satisfy the last line of Eq. (59) and then \( \epsilon \) follows. Since \( \ddot{P}(\xi) = \ddot{Q}(\xi) \), the normalisation factor \( \hat{N}(q) \) is given by:
\[ \int_{\mathbb{R}} \left[ \dot{P}(\xi) + \dot{Q}(\xi) \right] d\xi = 1 \Rightarrow \left[ \hat{N}(q) \right]^{-1} = \frac{2}{(1 - \omega^2)} \int_{\mathbb{R}} \frac{d\xi}{\cosh(q)} = \frac{2}{\left( \frac{1}{2}, \frac{1}{2q} \right).} \]
Remark 7. Fixing $\omega$ as given by Eq.(30) and choosing $1/q = (2 + \eta) > 0$, once more one observes that the solitons given in Eqs.(31) and (43) coincide. Hence under telegraphic noise environments, the soliton generated by exogenous interaction rule Eq.(6) coincides, in the ergodic regime, with the optimal solution of a MFG with an appropriate choice of the running cost function.

Remark 8. As shown in [16] (see Chapter 9), the WGN can be derived from the telegrapher’s process by an ad-hoc rescaling of the switching rates and the jumps sizes, (namely highly frequent very large jumps). Based on this observation, several contributions [15, 8] show how the parabolic Burgers’ equation for the diffusive dynamics results, via an ad-hoc limiting procedure, from the hyperbolic discrete velocity Boltzmann equation for the piecewise deterministic dynamics. In other words the RW dynamics [24] is merely a generalisation of the Burgers’s equation. Along the same lines, the assertions of Propositions 3 and 4 are mere generalisations of Propositions 1 and 2.

Remark 9. As expressed by Proposition 4, the dynamics is characterised by two stationary solitons travelling with identical speed $\omega$ ranging in $[0, 1]$. The telegraphic driving alternatives $\{-1\}, \{+1\}$ obviously imply a limited maximal velocity range of the resulting solitons. From Eqs.(44) and (53) we conclude that $u^*_+ = (1 - \omega) < u^*_- = (1 - \omega)^{-1}$ thus creating an optimal bias in the telegraphic noise effectively leading to an ABL dynamics. Finally, it is worth observing that in Eq.(12) the mean-field (MF) potential $-g \rho_a(x)$ is negatively defined so its minimum is located at the soliton maximum. Alternatively, Eq.(39) together with Eq.(55) imply a positive definite $+g \rho_a(x)$ MF coupling. This can be understood since remember that the global MF interaction is given by the product

$$W(PQ) = g \rho_a(x) = \frac{1-\omega^2}{2} \left[ -\frac{\sqrt{g}}{1-\omega} \rho^2_+ (x) \right] - \frac{\sqrt{g}}{1+\omega} \rho^2_- (x)$$

of two individual MF potentials acting respectively on $V_+$ and $V_-$ in Eq.(39). Hence the individual MF potentials simultaneously reach their minima at the solitons simultaneous maxima.

4. The “Avoid-to-Be-a-Laggard” (ABL) multi-agents dynamics and applications in finance and economy. The challenge to describe complex microscopic systems (involving huge number of variables) with macroscopic models (with a very few macroscopic variables) is illustrated by the link between statistical physics and thermodynamics where the Van der Walls (macroscopic) equation of gases can be derived on a purely microscopic ground. Far from being limited to physics, the micro-macro links connecting individual behaviours with self-organised collective evolutions is an ongoing research in many interdisciplinary domains like robotics, ethology, sociology, finance and economy to quote only but a few. Modelling the behaviour of gas particles is however a much simpler task than dealing with a “gas” of intelligent agents endowed with their own decision capability. The MFG dynamics in which each individuals optimises a private objective function offers a mathematical framework to address, in a highly stylised manner, the micro-macro challenge. In this general context and closely related to our present paper, we now point out a couple of recent contributions based ABL interactions algorithms and therefore for which a corresponding MFG can be associated.

- **Portfolio dynamics**

  The “Atlas model” introduced by E. Fernholz [7] describes the dynamics of portfolios, (basically a collection of assets) by using a collection of ranked-based drifted Brownian motion. For a collection of $N < \infty$ assets, the basic version of this dynamic reads:
\[
\begin{aligned}
\begin{cases}
  d \ln [Y_{k,t}] := dX_t = \gamma_k(t) dt + \sigma dW_{k,t}, & k = 1, 2 \ldots, N \\
  \gamma_k(t) := \begin{cases}
    nc & \text{if } X_{k,t} \text{ occupies the left most position on } \mathbb{R}, \\
    0 & \text{otherwise},
  \end{cases}
\end{cases}
\end{aligned}
\] 

where \( Y_{k,t} \in \mathbb{R}^+ \) stands for the \( k^{th} \) asset’s value at time \( t \), \( c \in \mathbb{R}^+ \) is a constant and \( \gamma_k(t) \) is a rank-dependent drift. As shown in [7], for this ranked-based dynamics the assets collection reaches a stationary regime with probability measure \( P_s(dy) d[\ln(y)] := \text{Prob} \{ y \in [y, y + dy] \} \sim e^{ry} d[\ln(y)] \) with negative slope \( r = -\frac{\sigma^2}{2g} \) thus exhibiting a fat tail log-log plot. In Eq.(60), the specific rank dependent drift \( \gamma_k(t) \) drives only the left-most laggard and thus effectively acts as a type of ABL algorithm. More complex ranked based drifts are studied in [3, 18] and the ABL rule used in our present paper belongs to this more general class of Atlas dynamics. Indeed, in our ABL dynamics, the agents drift amplitudes are proportional to the number of observed leaders, they are automatically of the ranked based type defined in [3, 18].

Observe once again that the stationary probability measures resulting from the ranked-based dynamics are due to long range interactions. Indeed, to determine their respective ranks, all agents need to permanently know the respective ranks of their fellows. Contrary to our present models, note that since the Atlas dynamics involves finite swarm’s population \( (N < \infty) \), the discussion does not rely on a mean-field approach.

- \textbf{Schumpeter’s dynamics and economic growth.}

As exposed in several recent contributions [9, 20], the interplay between innovation and imitation processes plays a determinant in economic growth. Starting with a Cobb-Douglas production framework [2], the production output \( Y_t \) of a firm is related to the capital \( K_t \) and the labor force \( L_t \) via the Cobb-Douglas relation:

\[
Y_t = A_t(K_t)^s(L_t)^{1-s} \quad \Rightarrow \quad y_t = \left[ \frac{Y_t}{L_t} \right] = \left[ \frac{A_t}{L_t} \right] \left[ \frac{K_t}{L_t} \right]^s := a_t k_t^s
\]

where \( s \in [0, 1] \) is an exogenous parameter and \( A_t \in \mathbb{R}^+ \) is the total factor of productivity and an economy is formed by a collection of \( N \) firms with individual \( a_{k,t} \) with \( k = 1, 2, \ldots, N \). The productivity of each firm grows as a result of technology improvements achieved either via in-house innovations or via the imitation of other firms. Adopting the the Schumpeter’s perspective, the firms innovation/imitation dynamics is stylised via a multi-agents diffusive like collective evolution:

\[
d [\ln(a_t)] := dX_t = \left[ bdt + \sigma dW_t \right] + \mathcal{I} \left[ X_{k,t}, \tilde{X}_t \right] dt, \quad \ln(a_{k,t}) \in \mathbb{R}, \quad (61)
\]

with \( k = 1, 2, \ldots, N \) and \( \tilde{X}_t := (X_{1,t}, X_{2,t}, \ldots, X_{N,t}) \) and the imitation kernel itself stylised by an ABL algorithm. Depending on the imitation kernel, the collective dynamics Eq.(61) exhibits either an evanescent diffusive behaviour or a stationary growth wave materialised here by a soliton dynamics similar to the one discussed in the present paper.

Note that in both illustrations above, the agents dynamics is expressed in terms of the logarithmic of the nominal variables. This leads to fat tails behaviours for
the resulting densities when expressed in log-log plots. Using similar logarithmic transformations, log-log plots of the hyperbolic secant densities as given Eqs. (7, 15, 31, 43) similarly exhibit fat tails asymptotic decays.

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Appendix A. To be self-contained, this appendix informally reproduces the basic steps leading to Eq.(13). To this aim, let us introduce a swarm of $N$ undistinguishable agents with dynamics:

$$dX_{k,t} = f(X_{k,t},t)dt + \sigma dW_{k,t}, \quad X_{k,t} \in \mathbb{R}, \quad k = 1, \cdots, N$$

(62)

where the $N$ stochastic processes $dW_{k,t}$ are chosen to be independent standard White Gaussian Noise (WGN). On the time interval $[0, T] \in \mathbb{R}^+$, an arbitrary chosen agent $X_t$ belonging to the swarm (hence the index is now deleted) determines the optimal drift $f^*(X_t,t) \in \Lambda$, ($\Lambda$ is a set of admissible drifts), which minimises the cost function $J(X_k, f(\cdot))$:

$$\begin{aligned}
& \mathcal{J}(X, f(X)) := \mathbb{E} \left\{ \int_0^T \mathcal{L}[X_s, \rho(\cdot, s), s; X_s] ds \right\} + c(T, X_T), \\
& \mathcal{L}[X_t, \rho(\cdot, t), t; X(t)] := \left[ c(X_t, f(X_t, t)) + V(\rho(\cdot); X_t) \right],
\end{aligned}$$

(63)

where $\mathbb{E} \{ \cdot \}$ stands for the expectation over the corresponding WGN process, the running cost $c[X_t, f(X_s, s)]$ is a convex function, $c_T(X_T)$ a terminal cost, the functional $V(\rho(\cdot); X_t) : [0, 1] \to \mathbb{R}^+$ is a mean-field type cost which depends on the agents’ density $\rho(x, t) \in [0, 1]$ \footnote{As in section 2, $\rho(x, t)$ is a smoothed version of the swarm’s empirical density.}. The notation $\mathcal{L}[X_t, \rho(\cdot, t), t; X_t]$ and $V(\rho(\cdot); X_t)$ means that both functionals of the density $\rho(\cdot, t)$ have to be evaluated at the value $x = X_t$. Subject to the drift $f^*(x, t)$, the corresponding swarm’s density $\rho(x, t)$ solves the Fokker-Planck equation:

$$\partial_t \rho(x, t) + \partial_x \left[ f^*(x, t) \rho(x, t) \right] = \frac{\sigma^2}{2} \partial_{xx} \rho(x, t)$$

(64)

and for $f^*(x, t)$, we now derive:

**Proposition 5.** The optimal drift $f^*(x, t)$ is given by:

$$f^*(x) = \max_{f(x) \in \Lambda} \left\{ -f(x, t) \partial_x \{ u(x, t) \} - c(x, f(x, t)) \right\},$$

(65)

where $u(x, t)$ is the value function solving the Hamilton-Jacobi-Bellman (HJB) equation:

$$-\partial_t u(x, t) - \frac{\sigma^2}{2} \partial_{xx} u(x, t) +$$

$$\max_{f(x, t) \in \Lambda} \left\{ -f(x, t) \partial_x u(x, t) - c(x, f(x, t)) \right\} = V(\rho(\cdot, t); x).$$

(66)

**Proof.** (Proposition 5).
Defining the value function $u(X_t, t)$ as:

$$ u(X_t, t) := \min_{f(\cdot) \in \Lambda} \int_t^T [c(X_s, f(X_s, s)) + V(\rho(X_s))] \, ds + C_T(X_T), $$

$$ u(x, T) = C_T(x), $$

the dynamic programming principle enables to write:

$$ u(X_t, t) = \min_{f(\cdot) \in \Lambda} \int_t^{t+dt} [c(X_s, f(X_s, s)) + V(\rho(X_s))] \, ds + u(X_{t+dt}, t + dt) $$

By Taylor expanding $u(X_{t+dt}, t + dt)$ to order $dt$ and using the properties:

$$ \mathbb{E}\{dW_t\} = 0 \quad \text{and} \quad \mathbb{E}\{(dW_t)^2\} = dt, $$

$$ \mathbb{E}\{\partial_X u(X_t, s) f(X_s, s) ds\} = f(x, s) \partial_x u(x, s) \quad \text{and} \quad \mathbb{E}\{(dX_t)^2\} = \sigma^2 dt, $$

we obtain:

$$ -\partial_t u(x, t) - \frac{\sigma^2}{2} \partial_{xx} u(x, t) + \max_{f(\cdot) \in \Lambda} \{-f(x, t) \partial_x u(x, t) - c[x, f(x, t)]\} = V[\rho(x, t)] $$

(67)

and hence $f^*(x, t)$ is given as in Eq. (65).

Coming back to Eqs. (12) and (11), the specific choices:

$$ \mathcal{L}(X_t, \rho(x, t), t) := c(X_t, t) - V[\rho(x, t)], $$

$$ c(X_t, t) = \frac{\mu}{2} \left[f(X_t, t) - b\right]^2 \quad \text{and} \quad V[\rho(x, t)] = g \rho(x, t)^a, $$

imply that $f^*(X_t, t) = -\left[\frac{1}{\nu} \partial_x u(x, t) - b\right]$. Finally, introducing this last relation into Eq. (67), we end with the HJB equation:

$$ -\partial_t u(x, t) - \frac{\sigma^2}{2} \partial_{xx} u(x, t) - b \partial_x u(x, t) + \frac{1}{2\nu} \left[\partial_x u(x, t)\right]^2 = -g \rho(x, t)^a $$

which finally coincides with Eq. (13).

**Appendix B.** The goal of this appendix is to derive the second line in Eq. (17). Using the fact that $\rho = \Phi \Psi$ and $u = -\mu \sigma^2 \ln \Phi - ct$, the FPE Eq. (13) reads:\footnote{All arguments of the scalar fields $\Phi(x, t)$ and $\Psi(x, t)$ are omitted on this Appendix.}

$$ (\partial_t \Phi) \Psi + \Phi (\partial_t \Psi) = \partial_x \left\{ \frac{1}{\nu} \left[\mu \sigma^2 \partial_x \Phi\right] \Phi \Psi - b \Phi \Psi \right\} + \frac{\sigma^2}{2} \Psi \partial_{xx} \Phi $$

$$ + \sigma^2 (\partial_x \Phi) (\partial_x \Psi) + \frac{\sigma^2}{2} \Phi \partial_{xx} \Psi $$

or equivalently:

$$ (\partial_t \Phi) \Psi + \Phi (\partial_t \Psi) = \partial_x \left\{ -b \Phi \Psi \right\} - \frac{\sigma^2}{2} \Psi \partial_{xx} \Phi. + \frac{\sigma^2}{2} \Phi \partial_{xx} \Psi. $$

This can be rewritten as:

$$ \Psi \left\{ \partial_t \Phi + b \partial_x \Phi + \frac{\sigma^2}{2} \partial_{xx} \Phi \right\} = -\Phi \left\{ \partial_t \Psi + b \partial_x \Psi - \frac{\sigma^2}{2} \partial_{xx} \Psi \right\}. $$

(68)
The first line in Eq. (17) implies:
\[
\left\{ \partial_t \Phi + b \partial_x \Phi + \frac{\sigma^2}{2} \partial_{xx} \Phi \right\} = \frac{1}{\mu \sigma^2} \left\{ \epsilon \Phi - g \left[ \Phi \Psi \right]^a \Phi \right\}.
\]
Hence Eq. (68) takes the form:
\[
\Psi \frac{1}{\mu \sigma^2} \left\{ \epsilon \Phi - g \left[ \Phi \Psi \right]^a \Phi \right\} = -\Phi \left\{ \partial_t \Psi + b \partial_x \Psi - \frac{\sigma^2}{2} \partial_{xx} \Psi \right\}
\]
or dividing by \(-\frac{\Phi}{\mu \sigma^2}\), we obtain:
\[
-\epsilon \Psi + g \left[ \Phi \Psi \right]^a \Psi = \mu \sigma^2 \left\{ \partial_t \Psi + b \partial_x \Psi - \frac{\sigma^2}{2} \partial_{xx} \Psi \right\}
\]
and hence the second line in Eq. (17) follows.

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