THE MONOIDAL CENTRE FOR GROUP-GRADED CATEGORIES

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Abstract. Let $G$ be a group and $k$ be a commutative ring. Our aim is to ameliorate the $G$-graded categorical structures considered by Turaev and Virelizier by fitting them into the monoidal bicategory context. We explain how these structures are monoidales in the monoidal centre of the monoidal bicategory of $k$-linear categories on which $G$ acts. This provides a useful example of a higher version of Davydov’s full centre of an algebra.

Introduction

As well as being a mathematical discipline with its own structures, calculations and significant theorems, category theory provides a framework for understanding analogies between concepts in different disciplines as examples of the same concept interpreted in different categories. Bicategories do the same for categorical concepts, and so on, iterating the microcosm principle\(^1\). An example of this unification process is provided in Section 4.8 of Street [31] where the full centre of an algebra, as defined and constructed by Davydov [9], is shown to be a monoidal centre in the sense of Street [29] in the monoidal bicategory of pointed categories.

We show here how structures and constructions defined by Turaev and Virelizier [33] (also see [32, 20, 2]) for $G$-graded categories, where $G$ is a group, are precisely the usual monoidal structures taken in an appropriate monoidal bicategory. To some extent, we replace the monoidal category of $k$-modules by the monoidal bicategory $\text{Mod}^{op}$ whose morphisms are two-sided modules on which categories act.

There are several benefits gained by working at the bicategorical level. While $k$-modules only have monoidal duals if they are finitely generated and projective, all objects of $\text{Mod}^{op}$ have duals. While $k$-modules only have finite direct sums, all small direct sums exist in $\text{Mod}^{op}$. While the monoidal centre of the category of all $k$-linear representations of a group $G$ is not tortile (also called “ribbon”), the monoidal centre of the bicategory of $\text{Mod}^{op}$-representations of $G$ is tortile. Moreover, a higher version of the Davydov full centre is involved in the key example mentioned in [33].

At this stage we are not saying anything about the main theorem of [33] which is a $G$-graded version of a theorem of Müger [24]. However, note that the invertibility of the $S$-matrix in the $G$-graded case uses Müger’s result.

\(^1\)so named by Baez and Dolan; see footnote 2 on page 100 of [11]
1. Basics on bicategories

Bicategories as defined by Bénabou [3] are “monoidal categories with several objects” in the sense that additive categories are “rings with several objects” [23]. For bicategories \( \mathcal{A} \) and \( \mathcal{B} \), we write \( \text{Ps}_*(\mathcal{A}, \mathcal{B}) \) for the bicategory of pseudofunctors, pseudo-natural transformations, and modifications (in the terminology of Kelly-Street [19]). We will use monoidal bicategory terminology from Day-Street [11] except that we now use “monoidale” in preference to “pseudomonoid”.

A good example of an autonomous symmetric monoidal bicategory is \( \text{Mod} \). The objects are categories. The homcategories are defined by \( \text{Mod} \coprod \mathcal{A}, \mathcal{B} \rightarrow \text{Mod}(\mathcal{A}, \mathcal{C}) \)

objects of these homs are called modules. Composition

is defined by

The identity module \( \mathcal{A} \rightarrow \mathcal{A} \) is the hom functor of \( \mathcal{A} \). Tensor product is finite cartesian product of categories; it is not the product in \( \text{Mod} \). Coproduct in \( \text{Mod} \) is coproduct of categories; it is also product in \( \text{Mod} \). We identify each functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) with the module \( F : \mathcal{A} \rightarrow \mathcal{B} \) defined by \( F(B, A) = \mathcal{B}(B, FA) \); this gives a faithful locally-full pseudofunctor \( (\cdot)_* : \text{Cat} \rightarrow \text{Mod} \) which is the identity on objects, strong monoidal, and coproduct preserving. The dual of \( \mathcal{A} \) in \( \text{Mod} \) is the opposite category \( \mathcal{A}^{\text{op}} \).

Recall (for example from [11]) that monoidales in \( \text{Mod}^{\text{op}} \) are promonoidal categories \( \mathcal{A} \) in the sense of Day [10]. The tensor product is a module \( P : \mathcal{A} \rightarrow \mathcal{A} \oplus \mathcal{A} \) and the unit is a module \( J : \mathcal{A} \rightarrow 1 \). The Day convolution monoidal structure on the functor category \( [\mathcal{A}, \text{Set}] \) has unit \( J : \mathcal{A} \rightarrow \text{Set} \) and tensor product \( F \ast G \) defined by

Suppose \( \mathcal{V} \) is a complete cocomplete closed symmetric monoidal category. We have a symmetric (weak) monoidal pseudofunctor

taking each category \( \mathcal{A} \) to the \( \mathcal{V} \)-enriched functor category \( [\mathcal{A}, \mathcal{V}] \). Therefore, each promonoidal category is taken to a convolution monoidal \( \mathcal{V} \)-category. In the case \( \mathcal{V} = \text{Set} \), (1.1) is the contravariant representable pseudofunctor \( \text{Mod}(\cdot, 1) \).

Let \( G \) be a group regarded as a one-object category. The object will be denoted by \( o_G \). The morphisms of this category will also be called elements of \( G \); so \( a \in G \) will mean, as usual, that \( a \) is an element. We also use the conjugation notation \( cac^{-1} \) and \( ac^e = c^{-1}ac \).

Like every category, \( G \) has a promonoidal structure for which the convolution structure on \( [G, \text{Set}] \) is pointwise cartesian product. In this case, the unit \( J \) is constant at the one-point set and the module \( P : G \rightarrow G \times G \) is defined by

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As we expound the theory, we will carry along an example based on Example 3.5 of [33] which, in turn, was based on an example in [22]. At this point it is just to show some naturally occurring lax functors and pseudofunctors.

**Example 1.1.** A group morphism \( \pi : H \to G \) is a functor. According to Bénabou [4] (see [28]), any functor over a category \( G \) corresponds to a normal lax functor \( H : G \to \text{Mod}^{op} \). The functor \( \pi \) is a fibration if and only if it is an opfibration if and only if it is Giraud-Conduché [16, 6] if and only if it is a surjective group morphism. In this case, the normal lax functor is actually a pseudofunctor. Let us now describe it. The fibre of the functor \( \pi \) over \( o_G \) is the kernel \( K \) of the group morphism \( \pi \); so \( H_{o_G} = K \) as a one-object category. For \( a : o_G \to o_G \) in \( G \), the module \( Ha : K \to K \) is defined by

\[
(Ha)(o_K, o_K) = \{ x \in H : \pi(x) = a \}.
\]

In particular, \( H1 = K \) is the identity module. For a composable pair \( o_G \xrightarrow{b} o_G \xrightarrow{a} o_G \) in \( G \), the composite module \( Hb \circ Ha \) is seen to be the set

\[
(Hb \circ Ha)(o_K, o_K) = \left( \{ x \in H : \pi(x) = a \} \times \{ y \in H : \pi(y) = b \} \right)/K
\]

of orbits under the action of \( K \) on the right in the first factor of the product and on the left in the second. The composition constraint is the bijection

\[
(Hb \circ Ha)(o_K, o_K) \to H(ab)(o_K, o_K), \quad [x, y] \mapsto xy.
\]

A cleavage for the fibration \( H \to G \) amounts to a splitting \( \sigma \) of \( \pi \) as a function on elements; that is, \( \pi\sigma a = a \) for all \( a \in G \). The chosen cartesian morphism over \( a : o_G \to o_G \) is \( \sigma(a) : o_H \to o_H \). The pseudofunctor \( H : G \to \text{Cat}^{op} \) corresponding to this cloven fibration is defined by \( H_{o_G} = K \), \( (Ha)k = k^{\sigma(a)} \) for \( a \in G \) and \( k \in K \), while the component at \( o_K \) of the invertible composition constraint \( (Kb)(Ka) \Rightarrow K(ab) \) is equal to \( \sigma(a)\sigma(b)\sigma(ab)^{-1} \in K \). We use the same symbol for the pseudofunctors \( H \) since the first is equivalent to the composite of the second with the canonical pseudofunctor \((-)_*\).

The groupoid of automorphisms in \( G \) will be denoted by \( G^\text{aut} \). The objects are elements \( a \in G \). A morphism \( f : a \to b \) is an element \( f \in G \) such that \( fa = bf \). If \( G_{\text{conj}} \) denotes the \( G \)-set whose elements are those of \( G \) and whose \( G \)-action is by conjugation then there is an equivalence of categories

\[
[G, \text{Set}]/G_{\text{conj}} \cong [G^\text{aut}, \text{Set}]. \tag{1.3}
\]

Since \( G_{\text{conj}} \) is a monoid in \([G, \text{Set}]\), there is a monoidal structure on the left hand side of (1.3) whose tensor product takes cartesian product of the morphisms over \( G_{\text{conj}} \) followed by the monoid multiplication. This monoidal structure is closed (on both sides) and so transports to a promonoidal structure on \( G^\text{aut} \). Recall from the end of Section 4 of [13] that this promonoidal structure is defined by

\[
P(a, b; c) = \{(u, v) \in G \times G : u^a v^b = c\} \quad \text{and} \quad Jc = \begin{cases} 1 & \text{if } c = 1 \\ \emptyset & \text{if } c \neq 1 \end{cases}
\]

and that there is a braiding

\[
\gamma_{a,b,c} : P(a, b; c) \xrightarrow{\sim} P(b, a; c), \quad (u, v) \mapsto (u^av, u).
\]
It also has a twist
\[ \tau_a = a : a \to a ; \]
compare Section 2 of [27]. The reader is invited to check the commutativity of (1.4) which is the main twist condition.

Furthermore, \( G^{\text{aut}} \) is a *-autonomous promonoidal category in the sense of [12]: we have the natural isomorphisms

\[ P(a, b; c^{-1}) \cong P(b, c; a^{-1}) , \quad (u, v) \leftrightarrow (u^{-1}v, u^{-1}) ; \]

\[ P(a, b; c^{-1}) \cong P(b^{-1}, a^{-1}; c)^{\text{op}} , \quad (u, v) \leftrightarrow (v, u) . \]

**Proposition 1.2.** [29, 13, 14] The braided monoidal \( G^{\text{aut}} \) is the monoidal centre of the monoidal \( G \) in \( \text{Mod}^{\text{op}} \). The convolution braided monoidal category \( [G^{\text{aut}}, \text{Set}] \) is braided monoidal equivalent to the monoidal centre \( \mathfrak{3}[G, \text{Set}] \) of the cartesian monoidal category \( [G, \text{Set}] \) of \( G \)-sets.

We note that the equivalence becomes balanced on transport of the convolution twist to the monoidal centre.

**Example 1.3.** We return to our surjective group morphism \( \pi : H \to G \). Since fibrations in \( \text{Cat} \) are preserved by 2-functors of the form \([\mathcal{D}, -]\), we have the fibration \( \pi^{\text{aut}} : H^{\text{aut}} \to G^{\text{aut}} \). The corresponding pseudofunctor \( H^{\text{aut}} : G^{\text{aut}} \to \text{Mod}^{\text{op}} \) is defined as follows. For each \( a \in G \), the category \( H^{\text{aut}}a \) has objects those \( x \in H \) with \( \pi(x) = a \); morphisms \( k : x \to x_1 \) are those \( k \in K \) with \( kx = x_1k \); composition is that of \( K \). For \( f : b \to a \) in \( G^{\text{aut}} \), the module \( H^{\text{aut}}f : H^{\text{aut}}a \to H^{\text{aut}}b \) is defined by

\[ (H^{\text{aut}}f)(y, x) = \{ h \in H^{\text{aut}}(y, x) : \pi(h) = f \} . \]

The splitting \( \sigma \) for \( \pi \) also gives a cleavage for \( \pi^{\text{aut}} \). The corresponding pseudofunctor \( H^{\text{aut}} : G^{\text{aut}} \to \text{Cat}^{\text{op}} \) has the same value on objects as in the last paragraph. For \( f : b \to a \) in \( G^{\text{aut}} \), the functor \( H^{\text{aut}}f : H^{\text{aut}}a \to H^{\text{aut}}b \) takes \( k : x \to x_1 \) in \( H^{\text{aut}}a \) to \( k\sigma(f) : x\sigma(f) \to x_1\sigma(f) \). The invertible composition constraint \( (H^{\text{aut}}g)(H^{\text{aut}}f) \to H^{\text{aut}}(fg) \) has component at \( x \in H^{\text{aut}}a \) equal to \( \sigma(f)\sigma(g)\sigma(fg)^{-1} \).

2. **Monoidales in convolution bicategories**

One virtue of the promonoidal groupoid \( G^{\text{aut}} \) over the monoid \( G_{\text{conj}} \) is that we can obtain convolution balanced monoidal structures on functors from \( G^{\text{aut}} \), not only into \( \text{Set} \) but, into any nice enough monoidal category; or even on pseudofunctors from \( G^{\text{aut}} \) into any nice enough monoidal bicategory.

Let \( \mathcal{K} \) be a monoidal bicategory with coproducts preserved by horizontal composition in each variable. The tensor product will be denoted by \(- \otimes - : \mathcal{K} \times \mathcal{K} \to \mathcal{K} \).

\(^2\)We are using \( \mathfrak{3} \) for the centre of a monoidal category since we like the fact that it is the first letter of the German words for both centre and braid.
with unit object $\mathcal{I}$. Think of $G^{\text{aut}}$ as a bicategory with only identity 2-cells. We will make explicit the convolution monoidal structure on the bicategory $\text{Ps}(G^{\text{aut}}, \mathcal{K})$.

Take $S, T \in \text{Ps}(G^{\text{aut}}, \mathcal{K})$. Put
\[
(S \ast T)c = \sum_{ab=c} S a \otimes T b \quad (\simeq \int_{\text{Ps}}^{a,b} P(a, b; c) \cdot S a \otimes T b)
\]
and, for $f : c \to c_1$ in $G^{\text{aut}}$, define $(S \ast T)f$ by commutativity in
\[
\begin{align*}
S a \otimes T b & \xrightarrow{\text{in}_{a,b}} \sum_{ab=c} S a \otimes T b \\
Sf \otimes Tf & \xrightarrow{\sum_{a_1b_1=c_1}} (S \ast T)f \\
Sf a \otimes T f b & \xrightarrow{\text{in}_{fa, fb}} \sum_{a_1b_1=c_1} S a_1 \otimes T b_1.
\end{align*}
\]
This defines the tensor product $S \ast T$ for a monoidal structure on $\text{Ps}(G^{\text{aut}}, \mathcal{K})$ with unit $J : G^{\text{aut}} \to \mathcal{K}$ defined by
\[
J_c = \begin{cases} 
\mathcal{I} & \text{if } c = 1 \\
0 & \text{if } c \neq 1
\end{cases}
\]
which becomes functorial on noting that, for $f : c \to c'$, if $c = 1$ then $c' = 1$.

We can now contemplate monoidales $M$ in $\text{Ps}(G^{\text{aut}}, \mathcal{K})$. Such a monoidale consists of a pseudofunctor $M : G^{\text{aut}} \to \mathcal{K}$ equipped with morphisms
\[
I : \mathcal{I} \to M 1 \quad \text{and} \quad \circ_{a,b} : M a \otimes M b \to M(ab)
\]
in $\mathcal{K}$ and invertible 2-cells
\[
\begin{align*}
M a \otimes M b \otimes M c & \xrightarrow{1 \otimes \circ_{b,c}} M a \otimes M(bc) \\
\circ_{a,b} \otimes 1 & \xrightarrow{\circ_{a,b,c}} M(ab) \otimes M c \\
M(ab) \otimes M c & \xrightarrow{\circ_{ab,c}} M(abc)
\end{align*}
\]
\[
\begin{align*}
M 1 \otimes M a & \xrightarrow{\lambda_a} M a \otimes M 1 \\
1 \otimes M a & \xrightarrow{1 \otimes \mathcal{I}} M a \otimes M a \\
\circ_{1,a} & \xrightarrow{\circ_{a,1}} M a \\
\circ_{a,1} & \xrightarrow{\circ_{1,a}} M a
\end{align*}
\]
all subject to pseudonaturality
\[
\begin{align*}
M a \otimes M b & \xrightarrow{\circ_{a,b}} M c \\
M f \otimes M f & \xrightarrow{\circ_{f,a,f}} M f \\
M f a \otimes M f b & \xrightarrow{\circ_{fa,fb}} M f
\end{align*}
\]
modificationality and coherence conditions.

\[\text{The coend here is in the pseudo-sense appropriate to bicategories.}\]
Example 2.1. The $H$ of Example 1.1 is a monoidale in $Ps(G, \text{Mod}^{op})$. The monoidal structure is provided by the modules

$$\Box : H_0G \to H_0G \times H_0G \quad \text{and} \quad I : H_0G \to 1$$

defined by

$$\Box(o_K, o_K; o_K) = \{(u, v) : u, v \in K \}, \quad \Box(k, \ell; m)(u, v) = (m u k, m v \ell), \quad I = 1.$$ 

3. The Turaev-Virelizier Structures

Definition. [33] A $G$-graded category over $k$ is a $k$-linear monoidal category $C$, with finite direct sums, endowed with a system of pairwise disjoint full $k$-linear subcategories $C_a$, $a \in G$, with finite direct sums, such that

(a) each object $X \in C$ splits as a direct sum $\bigoplus a X_a$ where $X_a \in C_a$ and $a$ runs over a finite subset of $G$;

(b) if $X \in C_a$ and $Y \in C_b$ then $X \otimes Y \in C_{ab}$;

(c) if $X \in C_a$ and $Y \in C_b$ with $a \neq b$ then $C(X, Y) = 0$;

(d) the tensor unit $I$ of $C$ is in $C_1$.

Turaev-Virelizier call an object $X$ of a $G$-graded category $C$ homogeneous when there exists a (necessarily unique) $a \in G$ such that $X \in C_a$; this $a$ is denoted by $|X|$.

They write $\hat{G}$ for the discrete monoidal category of elements of $G$ with the multiplication as tensor product. They write $\text{Aut}(\hat{C})$ for the monoidal category of monoidal endo-equivalences of the monoidal $k$-linear category $C$ and monoidal natural isomorphisms; the tensor product is composition of functors.

Definition. [33] A $G$-crossed category $C$ is a $G$-graded category over $k$ equipped with a strong monoidal functor $\varphi : \hat{G} \to \text{Aut}(\hat{C})$ such that $\varphi_a(C_b) \subseteq C_{a^{-1}ba}$ for all $a, b \in G$.

Proposition 3.1. Let $\mathcal{V}$ be the monoidal category of modules over a fixed commutative ring. Then monoidales in $Ps(G^{\text{aut}}, \mathcal{V}\text{-Cat})$ are equivalent to the $G$-crossed categories of [33].

Proof. Take a monoidale $M$ in $Ps(G^{\text{aut}}, \mathcal{V}\text{-Cat})$. Let $C$ be the $\mathcal{V}$-category obtained by taking the completion, with respect to finite direct sums, of the coproduct $C_{\text{hom}} = \sum_a Ma$ in $\mathcal{V}\text{-Cat}$. Then $C$ is a $G$-graded category with $\varphi_f = Mf^{-1}$.

Conversely, take a $G$-graded category $\mathcal{C}$ and define $Ma$ to be the full sub-$\mathcal{V}$-category $\mathcal{C}_a$ of $C$ consisting of the objects homogeneous over $a \in G$. Then $M$ is a monoidale in $Ps(G^{\text{aut}}, \mathcal{V}\text{-Cat}).$ \qed

Corollary 3.2. With $\mathcal{V}$ as in Proposition 3.1, any monoidale in $Ps(G^{\text{aut}}, \text{Mod}^{op})$ delivers a $G$-crossed category on application of the pseudofunctor (1.1).

Definition. [33] A $G$-braided category $\mathcal{C}$ is a $G$-crossed category $(\mathcal{C}, \varphi)$ equipped with a natural family of isomorphisms

$$\gamma_{X,Y} : X \otimes Y \to Y \otimes \varphi_{Y}(X),$$

for $X, Y \in \mathcal{C}$ and $Y$ homogeneous, subject to three axioms.

In the next section, we will see how this fits into our theory of braided monoidales.
4. Internal homs, biduals and braidings

If $\mathcal{K}$ is left closed, it is straightforward to see that so too is $\text{Ps}^p(G^{\text{aut}}, \mathcal{K})$:

$$[T, U]a = \prod_b [Tb, U(ab)] .$$

(4.11)

**Proposition 4.1.** Suppose in $\mathcal{K}$ that direct sums indexed by the elements of $G$ exist and that each $Tb$ has a left bidual $(Tb)^\vee$. Then $T$ has a left bidual

$$T^\vee a = [T, \mathbb{1}]a = (Ta^{-1})^\vee$$

(4.12)

in $\text{Ps}^p(G^{\text{aut}}, \mathcal{K})$.

**Proof.** Taking (4.12) as the definition of $T^\vee$, we need to prove that the canonical morphism $S \cdot T^\vee \to [T, S]$ is an equivalence for all $S$. The component of this canonical pseudonatural transformation at $c$ is

$$\sum_{ab=c} Sa \otimes T(b^{-1})^\vee \cong \sum_d S(cd) \otimes T(d)^\vee \cong \sum_d [Td, S(cd)] \xrightarrow{\text{canom}} \prod_d [Td, S(cd)]$$

in which the arrow is an equivalence because of our assumption about direct sums. □

**Corollary 4.2.** All biduals exist in $\text{Ps}^p(G^{\text{aut}}, \text{Mod}^{\text{op}})$; that is, the monoidal bicategory is autonomous (also called “compact” or “rigid”).

If $\mathcal{K}$ is equipped with a braiding $\gamma_{X,Y} : X \otimes Y \to Y \otimes X$ then we obtain a braiding $\gamma_{S,T} : S \cdot T \to T \cdot S$ on $\text{Ps}^p(G^{\text{aut}}, \mathcal{K})$ as defined by the commutative pentagon (4.13).

$$\begin{array}{ccc}
Sa \otimes Tb & \xrightarrow{\gamma_{Sa,Tb}} & Tb \otimes Sa \\
\text{in}_{a,b} & & \text{in}_{b,a} \\
\sum_{ab=c} Sa \otimes Tb & \xrightarrow{\gamma_{S,Tc}} & \sum_{ab'=c} Tb' \otimes Sa' \\
\end{array}$$

(4.13)

If $\mathcal{K}$ is balanced then so too is $\text{Ps}^p(G^{\text{aut}}, \mathcal{K})$ with twist

$$\theta_{S,a} = (Sa \xrightarrow{\theta_{Sa}} Sa \xrightarrow{\theta_{a}} Sa) .$$

Recall that, if $\mathcal{K}$ is symmetric, we choose its twist to be the identity.

We already have the example $G^{\text{aut}}$ of a $*$-autonomous balanced monoidal bicategory in $\text{Mod}^{\text{op}}$.

**Proposition 4.3.** The monoidal bicategory $\text{Ps}^p(G^{\text{aut}}, \text{Mod}^{\text{op}})$ is tortile.

Moreover, with $\text{Ps}^p(G^{\text{aut}}, \mathcal{K})$ a braided monoidal bicategory, according to [29], we can contemplate monoidal centres $\mathcal{Z}M$ for monoidales $M$ therein. Since the centre is a limit, it is formed pointwise in $\mathcal{K}$. From [29], we know that $\mathcal{Z}M$ is a braided monoidal bicategory in $\text{Ps}^p(G^{\text{aut}}, \mathcal{K})$.

**Example 4.4.** The $H^{\text{aut}}$ of Example 1.3 is a balanced monoidal in $\text{Ps}^p(G^{\text{aut}}, \text{Mod}^{\text{op}})$. The monoidal structure is provided by the modules

$$e_{a,b} : H^{\text{aut}}(ab) \to H^{\text{aut}}a \times H^{\text{aut}}b \quad \text{and} \quad I : H^{\text{aut}}1 \to 1$$
defined by
\[ \diamond_{a,b}(x, y; z) = \{(u, v) : u, v \in K, \ u^* x^* y = z\} \text{ and } I_z = \begin{cases} 1 & \text{if } z = 1 \\ \varnothing & \text{if } z \neq 1 \end{cases}. \]

The braiding is
\[ \gamma_{x,y,z} : \diamond_{a,b}(x, y; z) \xrightarrow{\cong} \diamond_{b,a}(y, x; z), \quad (u, v) \mapsto (u^* x^* y, u). \]

The twist \( \tau : H^{\text{aut}} \to H^{\text{aut}} \) is given by \( \tau_a = H^{\text{aut}} a : H^{\text{aut}} a \to H^{\text{aut}} a \). We suspect this example is also tortile [25] (also called “ribbon”).

Let us consider the case of \( \mathcal{K} = \mathcal{V}\text{-Cat} \) where \( \mathcal{V} \) is a complete cocomplete closed symmetric monoidal category. For \( S \in \text{Ps}(G^{\text{aut}}, \mathcal{V}\text{-Cat}), f : a \to b \) in \( G^{\text{aut}} \) and \( A \in S \), we put \( fA = (Sf)A \in Sb \).

Let \( M \) be a monoidale in \( \text{Ps}(G^{\text{aut}}, \mathcal{V}\text{-Cat}) \). The tensor product consists of \( \mathcal{V}\text{-functors} \diamond_{a,b} : Ma \otimes Mb \to M(ab) \). The unit is an object \( I \) of \( M1 \). The associativity constraint consists of a \( \mathcal{V}\text{-natural family} \)
\[ \alpha_{a,b,c} : (A \diamond_{a,b} B) \diamond_{ab,c} C \to A \diamond_{a,b,c} (B \diamond_{b,c} C). \quad (4.14) \]

A braiding for \( M \) consists of a \( \mathcal{V}\text{-natural family} \)
\[ \gamma_{a,b} : A \diamond_{a,b} B \to aB \diamond_{ab,a} A. \quad (4.15) \]

**Proposition 4.5.** In the setting of Proposition 3.1, the braided monoidales in \( \text{Ps}(G^{\text{aut}}, \mathcal{V}\text{-Cat}) \) are equivalent to the \( G\text{-braided categories of [33].} \)

According to Section 3 of [29], since pseudolimits are formed pointwise, the monoidal centre \( 3M \) of a monoidale \( M \) in \( \text{Ps}(G^{\text{aut}}, \mathcal{V}\text{-Cat}) \) is constructed as follows. The \( \mathcal{V}\text{-category} \ (3M)a \) has objects pairs \((A, v)\) where \( A \) is an object of \( Ma \) and \( v \) is a half \( G\text{-braiding} \) for \( A \) consisting of a \( \mathcal{V}\text{-natural family of isomorphisms} \)
\[ v_b : A \diamond_{a,b} B \to aB \diamond_{ab,a} A. \quad (4.16) \]

such that \( v_1 : A \diamond_{a,1} I \to I \diamond_{1,a} A \) transports the right unit constraint into the left unit constraint and the following hexagon commutes.

For \( f : a \to f' a, \) we have \( 3M(A, v) = (fA, v') \) where \( v'_b \) for \( B \in Mb \) is the composite
\[ fA \diamond_{f,a} B = f(A \diamond_{a, f^{-1}b} f^{-1}B) \xrightarrow{f_{v_b}^{-1}f} f(a f^{-1}B \diamond_{a(f^{-1}b), a} A) \cong f B \diamond_{f^{-1}b, f} fA. \]
5. Full centres

Davydov [9] defined the full centre of a monoid $M$ in a (not necessarily braided) monoidal category $\mathcal{E}$ to be a commutative monoid $\mathfrak{Z}M$ in the (braided) monoidal centre $\mathfrak{Z}\mathcal{E}$ of $\mathcal{E}$ satisfying an appropriate universal property. Street [31] pointed out that the pair $(\mathfrak{Z}\mathcal{E}, \mathfrak{Z}M)$ is the monoidal centre of the monoidal $\mathcal{E}$, $M$ in the monoidal bicategory of pointed categories.

We can lift this concept of full centre from the monoidal category level to the monoidal bicategory level. The full centre of a monoidale $\mathbf{M}$ in a monoidal bicategory $\mathcal{K}$ is a braided monoidale $\mathfrak{Z}\mathbf{M}$ in the monoidal centre $\mathfrak{Z}\mathcal{K}$ (see [1, 8, 21]) of $\mathcal{K}$ satisfying an appropriate universal property.

Proposition 5.1. (i) The braided monoidal bicategory $\operatorname{Ps}(G^\text{aut}, \text{Mod}^{\text{op}})$ is the monoidal centre of the monoidal bicategory $\operatorname{Ps}(G, \text{Mod}^{\text{op}})$.

(ii) The full centre of the monoidale $\mathbf{H}$ in $\operatorname{Ps}(G, \text{Mod}^{\text{op}})$ (see Example 2.1) is the braided monoidale $\mathbf{H}^\text{aut}$ (see Example 4.4) in $\operatorname{Ps}(G^\text{aut}, \text{Mod}^{\text{op}})$.

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