THE CONICAL KÄHLER-RICCI FLOW WITH WEAK INITIAL DATA ON FANO MANIFOLD

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Abstract. In this paper, we prove the long-time existence and uniqueness of the conical Kähler-Ricci flow with weak initial data which admits $L^p$ density for some $p > 1$ on Fano manifold. Furthermore, we study the convergence behavior of this flow.

1. Introduction

Conical Kähler-Einstein metric plays an important role in solving the Yau-Tian-Donaldson’s conjecture (see [6,7,8,45]). There has been renewed interest in conical Kähler-Einstein metric in recent years, see references [1,3,4,18,24,26,30,43] etc. On the other hand, the conical Kähler-Ricci flow was introduced to attack the existence problem of conical Kähler-Einstein metric. The long-time existence and limit behaviour of the conical Kähler-Ricci flow has been widely studied. Chen-Wang [11] studied the strong conical Kähler-Ricci flow and obtained the short-time existence, Y.Q. Wang [48] and the authors [34] got the long-time existence of the conical Kähler-Ricci flow respectively. In [34], the authors also considered the convergence of this flow on Fano manifold with positive twisted first Chern class, they proved that, for any cone angle $0 < 2\pi \beta < 2\pi$, the conical Kähler-Ricci flow converges to a conical Kähler-Einstein metric if there exists one. Chen-Wang [12] obtained the convergence result of this flow when the twisted first Chern class is negative or zero. Later, by adopting the smooth approximation used by the authors in [34], L.M. Shen [38][39] generalized Song-Tian’s result [41] and Tian-Zhang’s result [40] to the unnormalized conical Kähler-Ricci flow, and G. Edwards [17] obtained the uniform scalar curvature bound when the twisted first Chern class is negative on the foundation of Song-Tian’s result [42].

In [34], the authors studied the conical Kähler-Ricci flow which starts with a model conical Kähler metric

$$\omega_\beta = \omega_0 + \sqrt{-1k} \partial \bar{\partial} |s|^\frac{2\beta}{h}$$

on Fano manifold, where $\omega_0 \in c_1(M)$ is a smooth Kähler metric, $s$ is the defining section of a smooth divisor $D \in |-\lambda K_M|$ and $h$ is a smooth Hermitian metric on $-\lambda K_M$ with curvature $\lambda \omega_0$. In [11][12][48], Chen-Wang studied the existence of the conical Kähler-Ricci flow from initial $(\alpha, \beta)$ metric or weak $(\alpha, \beta)$ metric with other assumptions.

In this paper, we mainly study the long-time existence, uniqueness and convergence of the conical Kähler-Ricci flow with some weak initial data which admits $L^p$
density with $p > 1$ on Fano manifold. We consider the conical Kähler-Ricci flow by using smooth approximation of the twisted Kähler-Ricci flows as that in [34].

Let $M$ be a Fano manifold with complex dimension $n$, $\omega_0 \in c_1(M)$ be a smooth Kähler metric. For any $p \in (0, \infty]$, we define the class

\[
\mathcal{E}_p(M, \omega_0) = \{ \phi \in \mathcal{E}(M, \omega_0) \mid (\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi)^n \omega_0^n \in L^p(M, \omega_0) \},
\]

where the class

\[
\mathcal{E}(M, \omega_0) = \{ \phi \in PSH(M, \omega_0) \mid \int_M (\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi)^n = \int_M \omega_0^n \}
\]

defined by Guedj-Zeriahi in [21] is the largest subclass of $PSH(M, \omega_0)$ on which the operator $(\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi)^n$ is well defined and the comparison principle is valid. When $p > 1$, by S. Kolodziej’s $L^p$-estimate [29] and S. Dinew’s uniqueness theorem [14] (see also Theorem B in [21]), we know that the functions in $\mathcal{E}_p(M, \omega_0)$ are Hölder continuous with respect to $\omega_0$ on $M$.

Let $D$ be a divisor on $M$. By saying a closed positive $(1,1)$-current $\omega \in 2\pi c_1(M)$ with locally bounded potential is a conical Kähler metric with cone angle $2\pi \beta$ ($0 < \beta < 1$) along $D$, we mean that $\omega$ is a smooth Kähler metric on $M \setminus D$, and near each point $p \in D$, there exist local holomorphic coordinate $(z^1, \cdots, z^n)$ in a neighborhood $U$ of $p$ such that locally $D = \{ z^n = 0 \}$, and $\omega$ is asymptotically equivalent to the model conical metric

\[
\sqrt{-1}|z^n|^{2\beta - 2} dz^n \wedge d\bar{z}^n + \sqrt{-1} \sum_{j=1}^{n-1} dz^j \wedge d\bar{z}^j \quad \text{on} \quad U.
\]

Assume that $D \in | - \lambda K_M |$ ($\lambda \in \mathbb{Q}$), $\mu = 1 - (1 - \beta) \lambda$, $\tilde{\omega} \in c_1(M)$ is a Kähler current which admits $L^p$ density with respect to $\omega_0^n$ for some $p > 1$ and $\int_M \tilde{\omega}^n = \int_M \omega^n$. We study the long-time existence, uniqueness and convergence of the following conical Kähler-Ricci flow with weak initial data $\tilde{\omega}$

\[
\begin{cases}
\quad \frac{\partial \omega(t)}{\partial t} = -\text{Ric}(\omega(t)) + \mu \omega(t) + (1 - \beta)[D], \\
\quad \omega(t) \big|_{t=0} = \tilde{\omega}.
\end{cases}
\]

From now on, we denote the Kähler current $\omega = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi := \omega_{\varphi_0}$ with $\varphi_0 \in \mathcal{E}_p(M, \omega_0)$ for some $p > 1$.

**Definition 1.1.** We call $\omega(t)$ a long-time solution to the conical Kähler-Ricci flow (1.5) if it satisfies the following conditions.

- For any $[\delta, T] \ (\delta, T > 0)$, there exist constant $C$ such that

  \[
  C^{-1} \omega_\beta \leq \omega(t) \leq C \omega_\beta \quad \text{on} \quad [\delta, T] \times (M \setminus D);
  \]

- On $(0, \infty) \times (M \setminus D)$, $\omega(t)$ satisfies the smooth Kähler-Ricci flow;

- On $(0, \infty) \times M$, $\omega(t)$ satisfies equation (1.5) in the sense of currents;

- There exists metric potential $\varphi(t) \in C^0((0, \infty) \times M) \cap C^\infty((0, \infty) \times (M \setminus D))$ such that $\omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t)$ and $\lim \| \varphi(t) - \varphi_0 \|_{L^\infty(M)} = 0$;

- On $[\delta, T]$, there exist constant $\alpha \in (0, 1)$ and $C^*$ such that the above metric potential $\varphi(t)$ is $C^\alpha$ on $M$ with respect to $\omega_0$ and $\| \frac{\partial \varphi(t)}{\partial t} \|_{L^\infty(M \setminus D)} \leq C^*$. 

In Definition 1.1 by saying $\omega(t)$ satisfies equation (1.5) in the sense of currents on $(0, \infty) \times M := M_{\infty}$, we mean that for any smooth $(n-1, n-1)$-form $\eta(t)$ with compact support in $(0, \infty) \times M$, we have
\[
\int_{M_{\infty}} \frac{\partial \omega(t)}{\partial t} \wedge \eta(t, x) dt = \int_{M_{\infty}} (-Ric(\omega(t)) + \mu \omega(t) + (1 - \beta)[D]) \wedge \eta(t, x) dt,
\]
where the integral on the left side can be written as
\[
\int_{M_{\infty}} \frac{\partial \omega(t)}{\partial t} \wedge \eta(t, x) dt = -\int_{M_{\infty}} \omega(t) \wedge \frac{\partial \eta(t, x)}{\partial t} dt
\]
it the sense of currents.

We study the conical Kähler-Ricci flow (1.5) by using the following twisted Kähler-Ricci flow with weak initial data $\omega_{\varphi_0}$.
\[
\begin{cases}
\frac{\partial \omega(t)}{\partial t} = -Ric(\omega(t)) + \mu \omega(t) + \theta_{\epsilon}, \\
\omega_{\epsilon}(t)|_{t=0} = \omega_{\varphi_0},
\end{cases}
\]
(1.6)
where $\theta_{\epsilon} = (1 - \beta)(\lambda \omega_0 + \sqrt{-1} \partial \bar{\partial} \log(e^2 + |s_h^2|^2))$ is a smooth closed positive (1,1)-form, $s$ is the definition section of $D$ and $h$ is a smooth Hermitian metric on $-\lambda K_M$ with curvature $\lambda \omega_0$. The smooth case of the twisted Kähler-Ricci flow was studied in [13, 17, 19, 20, 32, 33, 34, 38, 48], etc.

There are some important results on the Kähler-Ricci flow (as well as its twisted versions with smooth twisting form) from weak initial data, such as Chen-Ding [5], Chen-Tian [9], Chen-Tian-Zhang [10], Guedj-Zeriahi [23], Lott-Zhang [35], Nezza-Lu [39], Song-Tian [41], Székelyhidi-Tosatti [44], Z. Zhang [49]. Here, we study the Kähler-Ricci flow which is twisted by non-smooth twisting form, i.e. the flow (1.5). We first obtain the long-time existence, uniqueness and regularity of the flow (1.6) by following Song-Tian’s arguments in [41]. Then we study the long-time existence of the conical Kähler-Ricci flow (1.5) by approximating method. In this process, in addition to getting the locally uniform regularity of the twisted Kähler-Ricci flow (1.6), the most important step is to prove that $\varphi(t)$ converges to $\varphi_0$ in $L^\infty$-norm as $t \to 0^+$ (i.e the 4th property in Definition 1.1), where $\varphi(t)$ is a metric potential of $\omega(t)$ with respect to metric $\omega_0$. Here we need a new idea because of Song-Tian’s method in [41] is invalid. At the same time, we prove the uniqueness of the conical Kähler-Ricci flow by Jeffress’ trick [25] and an improvement of the arguments in [48]. In fact, we obtain the following theorem.

**Theorem 1.2.** Let $M$ be a Fano manifold with complex dimension $n$, $\omega_0 \in c_1(M)$ be a smooth Kähler metric on $M$, divisor $D \in \mathbb{Z} \lambda K_M$ $(\lambda \in \mathbb{Q})$ and $\hat{\omega} \in c_1(M)$ be a Kähler current which admits $L^p$ density with respect to $\omega_0^n$ for some $p > 1$ and $\int_M \hat{\omega}^n = \int_M \omega_0^n$. Then for any $\beta \in (0, 1)$, there exists a unique solution $\omega(t, \cdot)$ to the conical Kähler-Ricci flow (1.5) with weak initial data $\hat{\omega}$.

Then we consider the convergence of the conical Kähler-Ricci flow (1.5). When $\lambda > 0$ and there is no nontrivial holomorphic field which is tangent to $D$ along $D$, Tian-Zhu [47] proved a Moser-Trudinger type inequality for conical Kähler-Einstein manifold and gave a new proof of Donaldson’s openness theorem [16]. Using the Moser-Trudinger type inequality in [47] and following the arguments in [34], we obtain the following convergence result of the conical Kähler-Ricci flow (1.5).
Theorem 1.3. Assume that $\lambda > 0$ and there is no nontrivial holomorphic field on $M$ tangent to $D$, if there exists a conical Kähler-Einstein metric with cone angle $2\pi \beta$ ($0 < \beta < 1$) along $D$, then the conical Kähler-Ricci flow (1.5) must converge to this conical Kähler-Einstein metric in $C^\infty_{\text{loc}}$ topology outside divisor $D$ and globally in the sense of currents on $M$.

Remark 1.4. In this paper, we only study the convergence with positive twisted first Chern class, i.e. $\mu = 1 - (1 - \beta)\lambda > 0$. When $\mu \leq 0$, one can also get the convergence of the conical Kähler-Ricci flow by following Chen-Wang’s argument in [12].

The paper is organized as follows. In section 2, we prove the long-time existence and uniqueness of the twisted Kähler-Ricci flow (1.6) by adopting Song-Tian’s methods in [41]. In section 3, we obtain the existence of a long-time solution to the conical Kähler-Ricci flow (1.5) by limiting the twisted Kähler-Ricci flows, and prove that $\varphi(t)$ converges to $\varphi_0$ in $L^\infty$-norm as $t \to 0^+$, where $\varphi(t)$ is a metric potential of $\omega(t)$ with respect to the metric $\omega_0$. We also prove the uniqueness of the conical Kähler-Ricci flow with weak initial data $\omega_{\varphi_0}$. In section 4, by using the uniform Perelman’s estimates along the twisted Kähler-Ricci flows obtained in [34], we prove the convergence theorem under the assumptions in Theorem 1.3.

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2. THE LONG-TIME EXISTENCE OF THE TWISTED KÄHLER-RICCI FLOW WITH WEAK INITIAL DATA

In this section, we prove the long-time existence and uniqueness of the twisted Kähler-Ricci flow (1.6) by following Song-Tian’s arguments in [41]. For further consideration in the next section, we shall pay attention to the estimates which are independent of $\varepsilon$. In the following arguments, for the sake of brevity, we only consider the flow (1.6) in the case of $\lambda = 1$ (i.e. $\mu = \beta$), where $\beta \in (0, 1)$. Our arguments are also valid for any $\lambda$, only if the coefficient $\beta$ before $\omega(t)$ in the case of $\lambda = 1$ is replaced by $\mu = 1 - (1 - \beta)\lambda$. We denote

$$F = \frac{(\omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi_0)^n}{\omega_0^n} \in L^p(M, \omega_0^n) \text{ for } p > 1.$$  

Recall that $C^\infty(M)$ is dense in $L^p(M, \omega_0^n)$. Therefore there exists a sequence of positive functions $F_j \in C^\infty(M)$ such that $\int_M F_j \omega_0^n = \int_M \omega_0^n$ and

$$\lim_{j \to \infty} \|F_j - F\|_{L^p(M)} = 0.$$  

By considering the complex Monge-Ampère equation

$$\begin{equation}
(\omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi_{0,j})^n = F_j \omega_0^n
\end{equation}$$

and using the stability theorem in [28] (see also [15] or [22]), we have

$$\begin{equation}
\lim_{j \to \infty} \|\varphi_{0,j} - \varphi_0\|_{L^\infty(M)} = 0,
\end{equation}$$

where $\varphi_{0,j} \in PSH(M, \omega_0) \cap C^\infty(M)$ satisfy $\sup_M (\varphi_{0,j} - \varphi_0) = \sup_M (\varphi_{0,j} - \varphi_0)$.  

Let $\omega_{\varphi_{0,j}} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{0,j}$. We prove the long-time existence of the twisted Kähler-Ricci flow by using a sequence of smooth twisted Kähler-Ricci flows
\begin{equation}
\left\{ \begin{array}{l}
\frac{\partial \omega_{\varphi_{t,j}}(t)}{\partial t} = -\text{Ric}(\omega_{\varphi_{t,j}}(t)) + \beta \omega_{\varphi_{t,j}}(t) + \theta_{\varepsilon}, \\
\omega_{\varphi_{t,j}}(t)|_{t=0} = \omega_{\varphi_{0,j}}.
\end{array} \right.
\end{equation}
(2.3)

Since the twisted Kähler-Ricci flow preserves the Kähler class, we can write the flow as the parabolic Monge-Ampère equation on potentials,
\begin{equation}
\left\{ \begin{array}{l}
\frac{\partial \varphi_{t,j}(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{t,j}(t))^n}{\omega_0^n} + F_0 + \beta \varphi_{t,j}(t) + \log(\varepsilon^2 + |s|^2) \right. \\
\varphi_{t,j}(0) = \varphi_{0,j}
\end{array} \right.
\end{equation}
(2.4)
where $F_0$ satisfies $-\text{Ric}(\omega_0) + \omega_0 = \sqrt{-1} \partial \bar{\partial} F_0$, $\frac{1}{M} \int_M e^{-F_0} dV_0 = 1$ and $dV_0 = \frac{\omega_0^n}{m}$. By using the function
\begin{equation}
\chi(\varepsilon^2 + |s|^2) = \frac{1}{\beta} \int_0^{\beta \chi(\varepsilon^2 + |s|^2)} \frac{(\varepsilon^2 + r)^{1-\beta}}{r} dr
\end{equation}
(2.5)
which was given by Campana-Guenancia-Păun in \cite{4}, we can rewrite the flow as
\begin{equation}
\frac{\partial \varphi_{t,j}(t)}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{t,j}(t))^n}{\omega_0^n} + F_0 + \beta (\varphi_{t,j}(t) + k \chi(\varepsilon^2 + |s|^2)),
\end{equation}
(2.6)
where $\varphi_{t,j}(t) = \varphi_{t,j}(t) - k \chi(\varepsilon^2 + |s|^2)$, $\omega_0 = \omega_0 + \sqrt{-1} \partial \bar{\partial} \chi(\varepsilon^2 + |s|^2)$, $F_0 = F_0 + \log \frac{(\omega_0^n}{\omega_0^n} (\varepsilon^2 + |s|^2)^{1-\beta})$. We know that $\chi(\varepsilon^2 + |s|^2)$ and $F_0$ are uniformly bounded (see (15) and (25) in \cite{4}).

**Proposition 2.1.** For any $T > 0$, there exists constant $C$ depending only on $\|\varphi_0\|_{L^\infty(M)}$, $\beta$, $n$, $\omega_0$ and $T$ such that for any $t \in [0, T]$, $\varepsilon > 0$ and $j \in \mathbb{N}^+$,
\begin{equation}
\|\varphi_{t,j}(t)\|_{L^\infty(M)} \leq C.
\end{equation}
(2.7)
Furthermore, for any $j, l$, we have
\begin{equation}
\|\varphi_{t,j}(t) - \varphi_{t,l}(t)\|_{L^\infty([0,T] \times M)} \leq e^{\beta T} \|\varphi_{0,j} - \varphi_{0,l}\|_{L^\infty(M)}.
\end{equation}
(2.8)
In particular, $\{\varphi_{t,j}(t)\}$ satisfies
\begin{equation}
\lim_{j,l \to \infty} \|\varphi_{t,j}(t) - \varphi_{t,l}(t)\|_{L^\infty([0,T] \times M)} = 0.
\end{equation}
(2.9)

**Proof:** From equation (2.6), we have
\[
\frac{\partial e^{-\beta t} \varphi_{t,j}(t)}{\partial t} = e^{-\beta t} \log \frac{(e^{-\beta t} \omega_0 + \sqrt{-1} \partial \bar{\partial} e^{-\beta t} \varphi_{t,j}(t))^n}{(e^{-\beta t} \omega_0)^n} + e^{-\beta t} (F_0 + k \beta \chi(\varepsilon^2 + |s|^2))
\leq e^{-\beta t} \log \frac{(e^{-\beta t} \omega_0 + \sqrt{-1} \partial \bar{\partial} e^{-\beta t} \varphi_{t,j}(t))^n}{(e^{-\beta t} \omega_0)^n} + C e^{-\beta t},
\]
which is equivalent to
\[
\frac{\partial}{\partial t} (e^{-\beta t} (\varphi_{t,j}(t) + \frac{C}{\beta})) \leq e^{-\beta t} \log \frac{(e^{-\beta t} \omega_0 + \sqrt{-1} \partial \bar{\partial} e^{-\beta t} (\varphi_{t,j}(t) + \frac{C}{\beta}))^n}{(e^{-\beta t} \omega_0)^n}.
\]
where constant $C$ depends only on $\|\varphi\|_{L^\infty(M)}$, $\beta$, $n$ and $\omega_0$.

For any $\delta > 0$, we denote $\hat{\phi}_{\varepsilon,j}(t) = e^{-\beta t}(\phi_{\varepsilon,j}(t) + C) - \delta t$. Let $(t_0, x_0)$ be the maximum point of $\hat{\phi}_{\varepsilon,j}(t)$ on $[0, T] \times M$. If $t_0 > 0$, by maximum principle, we have

$$0 \leq \frac{\partial}{\partial t}(e^{-\beta t}(\phi_{\varepsilon,j}(t) + c) - \delta)(t_0, x_0) - \delta \leq \frac{\partial}{\partial t} \left( e^{-\beta t} \log \left( \frac{e^{-\beta t}(\phi_{\varepsilon,j}(t) + c) - \delta}{e^{-\beta t} \omega_\varepsilon} \right) \right)(t_0, x_0) - \delta \leq -\delta,$$

which is impossible. Hence $t_0 = 0$, then

$$\phi_{\varepsilon,j}(t) \leq e^{\beta t} \sup_M \phi_{\varepsilon,j}(0) + \delta T e^{\beta T} + C(e^{\beta T} - 1).$$

Let $\delta \to 0$, we obtain

$$\phi_{\varepsilon,j}(t) \leq e^{\beta t} \sup_M \phi_{\varepsilon,j}(0) + \frac{C}{\beta}(e^{\beta T} - 1). \tag{2.10}$$

By the same arguments, we can get the lower bound of $\phi_{\varepsilon,j}(t)$

$$\phi_{\varepsilon,j}(t) \geq e^{\beta t} \inf_M \phi_{\varepsilon,j}(0) - \frac{C}{\beta}(e^{\beta T} - 1). \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$\|\phi_{\varepsilon,j}(t)\|_{L^\infty(M)} \leq e^{\beta T} \|\phi_{\varepsilon,j}(0)\|_{L^\infty(M)} + \frac{C}{\beta}(e^{\beta T} - 1) \leq C,$$

where constant $C$ depends only on $\|\varphi_0\|_{L^\infty(M)}$, $\beta$, $n$, $\omega_0$ and $T$.

Let $\psi_{\varepsilon,j}(t) = \phi_{\varepsilon,j}(t) - \phi_{\varepsilon,j}(t)$, then $\psi_{\varepsilon,j}$ satisfies the following equation

$$\begin{cases}
\frac{\partial \psi_{\varepsilon,j}}{\partial t} = \log \left( \frac{\omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t) + \beta \bar{\partial} \Phi_{\varepsilon,j}(t)(0)}{\omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t)} \right) + \beta \psi_{\varepsilon,j}(t), \\
\psi_{\varepsilon,j}(0) = \varphi_{0,j} - \varphi_{0,l}.
\end{cases} \tag{2.12}$$

By the same arguments as that in the first part, we have

$$\|\psi_{\varepsilon,j}(t)\|_{L^\infty([0,T] \times M)} \leq e^{\beta T} \|\varphi_{0,j} - \varphi_{0,l}\|_{L^\infty(M)}.$$

Since $\{\varphi_{0,j}\}$ is a Cauchy sequence in $L^\infty$-norm, we conclude the limit (2.9).

We now prove the uniform equivalence of the volume forms along the complex Monge-Ampère flow (2.6).

**Lemma 2.2.** For any $T > 0$, there exists constant $C$ depending only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\omega_0$ and $T$ such that for any $t \in (0, T]$, $\varepsilon > 0$ and $j \in \mathbb{N}^+$,

$$\frac{t^n}{C} \leq \frac{(\omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \phi_{\varepsilon,j}(t))^n}{\omega_\varepsilon^n} \leq e^{\bar{\Phi}}. \tag{2.13}$$

**Proof:** Let $\Delta_{\varepsilon,j}$ be the Laplacian operator associated to the Kähler form $\omega_{\varepsilon,j}(t) = \omega_\varepsilon + \sqrt{-1} \partial \bar{\partial} \phi_{\varepsilon,j}(t)$. Straightforward calculations show that

$$\left( \frac{\partial}{\partial t} - \Delta_{\varepsilon,j} \right) \hat{\phi}_{\varepsilon,j}(t) = \beta \hat{\phi}_{\varepsilon,j}(t). \tag{2.14}$$
Let $H_{\varepsilon,j}^+(t) = t\dot{\phi}_{\varepsilon,j}(t) - A\phi_{\varepsilon,j}(t)$, where $A$ is a sufficiently large number (for example $A = \beta T + 2$). Then $H_{\varepsilon,j}^+(0) = -A\phi_{\varepsilon,j}(0)$ is uniformly bounded by a constant $C$ depending only on $\|\varphi_0\|_{L^\infty(M)}$, $\beta$, $n$, $\omega_0$ and $T$.

$$
\left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)H_{\varepsilon,j}^+(t) = (1 + \beta t - A)\dot{\phi}_{\varepsilon,j}(t) + A\Delta_{\varepsilon,j}\phi_{\varepsilon,j}(t)
$$

(2.15) \leq (1 + \beta t - A)\dot{\phi}_{\varepsilon,j}(t) + A.

By the maximum principle, $H_{\varepsilon,j}^+(t)$ is uniformly bounded from above by a constant $C$ depending only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\omega_0$ and $T$.

Let $H_{\varepsilon,j}^-(t) = \dot{\phi}_{\varepsilon,j}(t) + \phi_{\varepsilon,j}(t) - n \log t$. Then $H^-(t)$ tends to $+\infty$ as $t \to 0^+$ and

$$
\left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)H_{\varepsilon,j}^-(t) = (\beta + 1)\dot{\phi}_{\varepsilon,j}(t) + tr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon,j} - n - \frac{n}{t}.
$$

(2.16)

Assume that $(t_0, x_0)$ is the minimum point of $H_{\varepsilon,j}^-(t)$ on $[0, T] \times M$. We conclude that $t_0 > 0$ and there exists constant $C_1$, $C_2$ and $C_3$ such that

$$
\left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)H_{\varepsilon,j}^-(t)\big|_{(t_0, x_0)} \geq \left( C_1 \frac{\omega_{\varepsilon,j}^n(t)}{\omega_{\varepsilon,j}^n(t)} \right)^t + C_2 \log \frac{\omega_{\varepsilon,j}^{n}(t)}{\omega_{\varepsilon,j}^{n}} - \frac{C_3}{t})\big|_{(t_0, x_0)},
$$

(2.17)

where constant $C_1$ depends only on $n$, $C_2$ depends only on $\beta$ and $C_3$ depends only on $n$, $\omega_0$, $\|\varphi_0\|_{L^\infty(M)}$, $\beta$ and $T$. In inequality (2.17), without loss of generality, we assume that $\frac{\omega_{\varepsilon,j}^n(t)}{\omega_{\varepsilon,j}^n(t)} > 1$ and $C_2 \left(\frac{\omega_{\varepsilon,j}^{n}(t)}{\omega_{\varepsilon,j}^{n}}\right)^t + C_2 \log \frac{\omega_{\varepsilon,j}^{n}(t)}{\omega_{\varepsilon,j}^{n}} \geq 0$ at $(t_0, x_0)$. By the maximum principle, we have

$$
\omega_{\varepsilon,j}^{n}(t_0, x_0) \geq C_4 t^{n_{\varepsilon,j}}(x_0),
$$

(2.18)

where $C_4$ independent of $\varepsilon$ and $j$. Then it easily follows that $H_{\varepsilon,j}^-(t)$ is bounded from below by a constant $C$ depending only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\omega_0$ and $T$. \hfill $\Box$

In the following lemma, we prove the uniform equivalence of the metrics along the twisted Kähler-Ricci flow (2.3).

**Lemma 2.3.** For any $T > 0$, there exists constant $C$ depending only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\omega_0$ and $T$ such that for any $t \in (0, T]$, $\varepsilon > 0$ and $j \in \mathbb{N}^+$,

$$
e^{-\Psi_{\varepsilon,\rho}}\omega_{\varepsilon,j}(t) \leq \omega_{\varepsilon,j}(t) \leq e^{\Psi_{\varepsilon,\rho}}\omega_{\varepsilon,j}.
$$

(2.19)

**Proof:** Let

$$
\Psi_{\varepsilon,\rho} = B \frac{1}{\rho} \int_0^{[\varepsilon^2 + r]^\rho} \frac{\rho r - \varepsilon^2}{r} dr
$$

(2.20)

be the uniform bound function introduced by Guenancia-Păun in [24]. By choosing suitable $B$ and $\rho$, and following the arguments in section 2 of [34], we have

$$
\left(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j}\right)(t \log tr_{\omega_{\varepsilon,j}}(t) + t\Psi_{\varepsilon,\rho})
$$

(2.21)

$$\leq \log tr_{\omega_{\varepsilon,j}}(t) +Ctr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon,j} + C,
$$

where constant $C$ depends only on $n$, $\beta$, $\omega_0$ and $T$. 


Let $H_{\varepsilon,j}(t) = t \log tr_{\omega_{\varepsilon,j}}(t) + t\Psi_{\varepsilon,p} - A\phi_{\varepsilon,j}(t)$, $A$ be a sufficiently large constant and $(t_0, x_0)$ be the maximum point of $H_{\varepsilon,j}(t)$ on $[0, T] \times M$. We need only consider $t_0 > 0$. By the inequality
\begin{equation}
(2.22) \quad tr_{\omega_{\varepsilon,j}}(t) \leq \frac{1}{(n-1)!} (tr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon})^{n-1}\frac{\omega_{\varepsilon,j}^n(t)}{\omega_{\varepsilon}^n},
\end{equation}
we conclude that
\begin{equation}
(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j})H_{\varepsilon,j}(t) \leq \log tr_{\omega_{\varepsilon,j}}(t) + Ctr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon} - A\phi_{\varepsilon,j}(t) - Atr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon} + C
\end{equation}
\begin{equation}
\leq (n-1)\log tr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon} - \frac{A}{2}tr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon} - (A-1)\log \frac{\omega_{\varepsilon,j}^n(t)}{\omega_{\varepsilon}^n} + C,
\end{equation}
where constant $C$ depends only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\omega_0$ and $T$.

Without loss of generality, we assume that $-\frac{A}{4}tr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon} + (n-1)\log tr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon} \leq 0$ at $(t_0, x_0)$. Then at $(t_0, x_0)$, by Lemma 2.2, we have
\begin{equation}
(2.23) \quad (\frac{\partial}{\partial t} - \Delta_{\varepsilon,j})H_{\varepsilon,j}(t) \leq -\frac{A}{4}tr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon} - C\log t + C.
\end{equation}
By the maximum principle, at $(t_0, x_0)$,
\begin{equation}
(2.24) \quad tr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon} \leq C\log \frac{1}{t} + C.
\end{equation}
By using inequality (2.22), at $(t_0, x_0)$,
\begin{equation}
(2.25) \quad tr_{\omega_{\varepsilon,j}}(t) \leq C(\log \frac{1}{t} + 1)^{n-1}e^\frac{C}{t} \leq e^{\frac{2C}{t}},
\end{equation}
where constant $C$ depends only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\omega_0$ and $T$. Hence we have
\begin{equation}
(2.26) \quad tr_{\omega_{\varepsilon,j}}(t) \leq e^{\frac{C}{t}}
\end{equation}
for some constant $C$ depending only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\omega_0$ and $T$.

Furthermore, by inequality (2.22) again, we know
\begin{equation}
(2.27) \quad tr_{\omega_{\varepsilon,j}}(t)\omega_{\varepsilon} \leq e^{\frac{C}{t}},
\end{equation}
where constant $C$ depends only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\omega_0$ and $T$. From (2.26) and (2.27), we prove the lemma.

By Lemma 2.3 and the fact that $\omega_{\varepsilon} > \gamma\omega_0$ for some uniform constant $\gamma$ (see inequality (24) in [4]), we have
\begin{equation}
(2.28) \quad e^{-\frac{C}{t}}\omega_0 \leq \omega_{\varepsilon,j}(t) \leq C\omega_0 e^{\frac{C}{t}},
\end{equation}
on $(0, T) \times M$, where $C$ is a uniform constant and $C_\varepsilon$ depends on $\varepsilon$. We next prove the Calabi’s $C^3$-estimates. Denote
\begin{equation}
(2.29) \quad S_{\varepsilon,j} = |\nabla_{\omega_0}\omega_{\varepsilon,j}(t)|_{\omega_{\varepsilon,j}(t)}^2 = g_{\varepsilon,j}^m g_{\varepsilon,j}^n g_{\varepsilon,j}^p \nabla_0(g_{\varepsilon,j})_{kl} \nabla_0(g_{\varepsilon,j})_{pq}.
\end{equation}

Lemma 2.4. For any $T > 0$ and $\varepsilon > 0$, there exist constants $C_\varepsilon$ and $C$ such that for any $t \in (0, T]$ and $j \in \mathbb{N}^+$,
\begin{equation}
(2.30) \quad S_{\varepsilon,j} \leq C\omega_0 e^{\frac{C}{t}},
\end{equation}
where constant $C$ depends only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\omega_0$ and $T$, and constant $C_\varepsilon$ depends in addition on $\varepsilon$. 
By the maximum principle, we have
\begin{equation}
\frac{\partial}{\partial t} \Delta_{\varepsilon,j} (e^{-\frac{2\varepsilon}{\gamma} S_{\varepsilon,j}}) \leq C_{\varepsilon} e^{-\frac{2\varepsilon}{\gamma} S_{\varepsilon,j}} + C_{\varepsilon},
\end{equation}
By choosing \( A_{\varepsilon} = C_{\varepsilon}(C_{\varepsilon} + 1) \) and \( \alpha = 3\gamma \),
\begin{equation}
\frac{\partial}{\partial t} \Delta_{\varepsilon,j} (e^{-\frac{2\varepsilon}{\gamma} \text{tr}_{\omega_0} \omega_{\varepsilon,j}(t)}) \leq C_{\varepsilon} - C_{\varepsilon} e^{-\frac{2\varepsilon}{\gamma} S_{\varepsilon,j}}.
\end{equation}
By the maximum principle, we have
\begin{equation}
S_{\varepsilon,j} \leq C_{\varepsilon} e^{\frac{\varepsilon}{\gamma}} \quad \text{on} \quad (0, T) \times M
\end{equation}
for some constant \( C \) depending only on \( \|\varphi_0\|_{L^\infty(M)} \), \( n \), \( \beta \), \( \omega_0 \) and \( T \), and constant \( C_{\varepsilon} \) depending in addition on \( \varepsilon \).

By using the Schauder regularity theory and equation (2.34), we get the high order estimates of \( \varphi_{\varepsilon,j}(t) \).

\textbf{Proposition 2.5.} For any \( 0 < \delta < T < \infty \), \( \varepsilon > 0 \) and \( k \geq 0 \), there exists constant \( C_{\varepsilon,\delta,T,k} \) depending only on \( \delta \), \( T \), \( \varepsilon \), \( k \), \( n \), \( \beta \), \( \omega_0 \) and \( \|\varphi_0\|_{L^\infty(M)} \), such that for any \( j \in \mathbb{N}^+ \),
\begin{equation}
\|\varphi_{\varepsilon,j}(t)\|_{C^k([\delta,T] \times M)} \leq C_{\varepsilon,\delta,T,k}.
\end{equation}

By (2.34), for any \( T > 0 \), \( \varphi_{\varepsilon,j}(t) \) converges to \( \varphi_{\varepsilon}(t) \in L^\infty([0,T] \times M) \) uniformly in \( L^\infty([0,T] \times M) \). For any \( 0 < \delta < T < \infty \) and \( \varepsilon > 0 \), \( \varphi_{\varepsilon,j}(t) \) is uniformly bounded (depends on \( \varepsilon \)) in \( C^\infty([\delta,T] \times M) \). Therefore \( \varphi_{\varepsilon,j}(t) \) converges to \( \varphi_{\varepsilon}(t) \) in \( C^\infty([\delta,T] \times M) \). Hence for any \( \varepsilon > 0 \), \( \varphi_{\varepsilon}(t) \in C^\infty((0,\infty) \times M) \).

\textbf{Proposition 2.6.} For any \( \varepsilon > 0 \), \( \varphi_{\varepsilon}(t) \in C^0((0,\infty) \times M) \) and
\begin{equation}
\lim_{t \to 0^+} \|\varphi_{\varepsilon}(t) - \varphi_0\|_{L^\infty(M)} = 0.
\end{equation}

By (2.22), we have
\begin{equation}
\lim_{j \to \infty} \|\varphi_{0,j}(t,z) - \varphi_0(z)\|_{L^\infty(M)} = 0.
\end{equation}
For any \( \varepsilon > 0 \), there exists \( N \) such that for any \( j > N \),
\begin{equation}
\sup_{[0,T] \times M} |\varphi_{\varepsilon}(t,z) - \varphi_{\varepsilon,j}(t,z)| < \frac{\varepsilon}{3},
\end{equation}
\begin{equation}
\sup_{M} |\varphi_{0,j}(z) - \varphi_0(z)| < \frac{\varepsilon}{3}.
\end{equation}
On the other hand, fix such $j$, there exists $0 < \delta < T$ such that

\[(2.40) \quad \sup_{[0, \delta] \times M} |\varphi_{\epsilon, j}(t, z) - \varphi_{0, j}| < \frac{\epsilon}{3}.\]

Combining the above estimates together, for any $t \in [0, \delta]$ and $z \in M$,

\[(2.41) \quad |\varphi_{\epsilon}(t, z) - \varphi_{0}(z)| < \epsilon.

This completes the proof of the lemma. \hfill \Box

**Proposition 2.7.** $\varphi_{\epsilon}(t)$ is the unique solution to the parabolic Monge-Ampère equation

\[
\begin{cases}
\frac{\partial \varphi_{\epsilon}(t)}{\partial t} = \log \left( \frac{\omega_{0} + \sqrt{-1} \partial \bar{\partial} \psi_{\epsilon}(t)}{\omega_{0}^{n}} \right)^{n} + F_{0} \\
+ \beta \varphi_{\epsilon}(t) + \log(\epsilon^{2} + |s_{\theta}^{2}|^{1-\beta}) , & (0, \infty) \times M
\end{cases}
\]

in the space of $C^{0}(\{0, \infty\} \times M) \cap C^{0}(\{0, \infty\} \times M)$.

**Proof:** By proposition 2.6, we only need to prove the uniqueness. Suppose there exists another solution $\tilde{\varphi}_{\epsilon}(t) \in C^{0}(\{0, \infty\} \times M) \cap C^{0}(\{0, \infty\} \times M)$ to the Monge-Ampère equation (2.42). Let $\psi_{\epsilon}(t) = \tilde{\varphi}_{\epsilon}(t) - \varphi_{\epsilon}(t)$. Then

\[
\begin{cases}
\frac{\partial \psi_{\epsilon}(t)}{\partial t} = \log \left( \frac{\omega_{0} + \sqrt{-1} \partial \bar{\partial} \psi_{\epsilon}(t)}{\omega_{0}^{n}} \right)^{n} + \beta \psi_{\epsilon}(t) \\
\psi_{\epsilon}(0) = 0
\end{cases}
\]

For any $T > 0$, by the same arguments as that in the proof of Proposition 2.1, we have

\[\|\psi_{\epsilon}(t)\|_{L^{\infty}(\{0, T\} \times M)} \leq e^{\beta T} \|\psi_{\epsilon}(0)\|_{L^{\infty}(M)} = 0.\]

Hence $\psi_{\epsilon}(t) = 0$, that is $\tilde{\varphi}_{\epsilon}(t) = \varphi_{\epsilon}(t)$. \hfill \Box

By the similar arguments as that in [41], we prove the uniqueness theorems of the twisted Kähler-Ricci flow.

**Theorem 2.8.** Let $M$ be a Fano manifold with complex dimension $n$, $\omega_{0} \in c_{1}(M)$ be a smooth Kähler metric on $M$ and $\hat{\omega} \in c_{1}(M)$ be a Kähler current which admits $L^{p}$ density with respect to $\omega_{0}^{n}$ for some $p > 1$ and $\int_{M} \hat{\omega}^{n} = \int_{M} \omega_{0}^{n}$. Then there exists a unique solution $\omega_{\epsilon}(t) \in C^{0}(\{0, \infty\} \times M) \cap C^{0}(\{0, \infty\} \times M)$ such that $\omega_{\epsilon}(t) = \omega_{0} + \sqrt{-1} \partial \bar{\partial} \varphi_{\epsilon}(t)$ and

\[(2.44) \quad \lim_{t \to 0^{+}} \|\varphi_{\epsilon}(t) - \varphi_{0}\|_{L^{\infty}(M)} = 0,

where $\varphi_{0} \in E_{p}(M, \omega_{0})$ is a metric potential of $\hat{\omega}$ with respect to $\omega_{0}$. In particular, $\omega_{\epsilon}(t)$ converges in the sense of distribution to $\hat{\omega}$ as $t \to 0$.\]
Therefore, we obtain
\[
\lim_{t \to 0^+} \| \varphi(t) - \varphi_0 \|_{L^\infty(M)} = 0
\]
for some metric potential \( \varphi_0 \in \mathcal{E}_p(M, \omega_0) \) of \( \hat{\omega} \) with respect to \( \omega_0 \). Suppose that there is another solution \( \hat{\omega}_\varepsilon(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon(t) \) to the twisted Kähler-Ricci flow (1.6) with initial data \( \hat{\omega}_\varepsilon \). Then \( \varphi_\varepsilon(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times M) \) satisfies

\[
\frac{\partial \varphi_\varepsilon(t)}{\partial t} = \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon(t)}{\omega_0} \right) + F_0 + \beta \varphi_\varepsilon(t) + \log(\varepsilon^2 + |s|_{\delta}^{2\beta-1}) + f_\varepsilon(t)
\]

on \((0, \infty) \times M\) for a smooth function \( f_\varepsilon(t) \) on \((0, \infty)\) and

\[
\lim_{t \to 0^+} \| \varphi_\varepsilon(t) - \varphi_0 \|_{L^\infty(M)} = 0,
\]
where \( \varphi_0 \in \mathcal{E}_p(M, \omega_0) \) is also a metric potential of \( \hat{\omega} \) with respect to \( \omega_0 \). At the same time, we have \( \varphi_0 = \varphi_0 + \hat{C} \).

Let \( \hat{\varphi}(t) = \hat{\varphi}(t) + \hat{C} e^{\beta t} \). It is obvious that \( \hat{\varphi}_\varepsilon(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times M) \) is a solution to equation (2.46) and satisfies

\[
\lim_{t \to 0^+} \| \hat{\varphi}_\varepsilon(t) - \varphi_\varepsilon(t) \|_{L^\infty(M)} = 0.
\]

Now we consider the function \( \psi_\varepsilon(t) = \hat{\varphi}_\varepsilon(t) - \varphi_\varepsilon(t) \).

\[
\frac{\partial \psi_\varepsilon(t)}{\partial t} = \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon(t)}{\omega_0} \right)^n + \beta \psi_\varepsilon(t) + f_\varepsilon(t),
\]

(2.47)

For any \( 0 < t_1 < t_2 < \infty \), by the same arguments as that in the proof of Proposition 2.1, we have

\[
\sup_M \psi_\varepsilon(t_2) \leq e^{\beta(t_2-t_1)} \sup_M \psi_\varepsilon(t_1) + \int_{t_1}^{t_2} e^{\beta(t_2-t)} f_\varepsilon(t) dt,
\]

\[
\inf_M \psi_\varepsilon(t_2) \geq e^{\beta(t_2-t_1)} \inf_M \psi_\varepsilon(t_1) + \int_{t_1}^{t_2} e^{\beta(t_2-t)} f_\varepsilon(t) dt.
\]

Therefore, we obtain

\[
\inf_M \psi_\varepsilon(t_2) \geq \sup_M \psi_\varepsilon(t_2) \geq e^{\beta(t_2-t_1)}(\sup_M \psi_\varepsilon(t_1) - \inf_M \psi_\varepsilon(t_1)).
\]

Let \( t_1 \to 0^+ \), we have

\[
\inf_M \psi_\varepsilon(t_2) \geq \sup_M \psi_\varepsilon(t_2).
\]

By equation (2.47), \( \psi_\varepsilon(t) = \int_t^t e^{\beta(t-s)} f_\varepsilon(s) ds \). Hence \( \hat{\omega}_\varepsilon(t) = \omega_\varepsilon(t) \). \( \square \)
3. The long-time existence of the conical Kähler-Ricci flow with weak initial data

In this section, we study the long-time existence of the conical Kähler-Ricci flow (1.1) by the smooth approximation of the twisted Kähler-Ricci flows. We also prove the uniqueness of the conical Kähler-Ricci flow (1.1).

By Proposition 2.1 Lemma 2.3 and Proposition 2.5 we conclude that for any \( T > 0 \), there exists constants \( C_1 \) and \( C_2 \) depending only on \( \| \varphi_0 \|_{L^\infty(M)} \), \( \beta \), \( n \), \( \omega_0 \) and \( T \), such that for any \( \varepsilon > 0 \),

\[
\| \phi_\varepsilon(t) \|_{L^\infty([0,T] \times M)} \leq C_1, \\
e^{-\varepsilon^2} \omega_\varepsilon \leq \omega_\varepsilon(t) \leq e^{\varepsilon^2 M} \omega_\varepsilon \quad \text{on} \quad (0,T) \times M.
\]

We first prove the local uniform Calabi’s \( C^3 \)-estimate and curvature estimate along the flow (2.3). Our proofs are similar as that in [34] (see section 2 in [34] or section 3 in [40]), but we need some arguments to handle the weak initial data case.

**Lemma 3.1.** For any \( T > 0 \) and \( B_r(p) \subset M \setminus D \), there exist constants \( C, C' \) and \( C'' \) such that for any \( \varepsilon > 0 \) and \( j \in \mathbb{N}^+ \),

\[
S_{\varepsilon,j} \leq \frac{C'}{r^2} e^{\frac{C}{r^2}}, \\
|Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)}^2 \leq \frac{C''}{r^4} e^{\frac{C}{r^2}}
\]

on \( (0,T) \times B_2^\varepsilon(p) \), where constants \( C, C' \) and \( C'' \) depend only on \( \| \varphi_0 \|_{L^\infty(M)} \), \( n \), \( \beta \), \( T \), \( \omega_0 \) and \( \text{dist}_{\omega_0}(B_r(p),D) \).

**Proof:** By Lemma 2.3 there exists uniform constat \( C \) depending only on \( \| \varphi_0 \|_{L^\infty(M)} \), \( n \), \( \beta \), \( T \), \( \omega_0 \) and \( \text{dist}_{\omega_0}(B_r(p),D) \), such that

\[
e^{-\frac{C}{r^2}} \omega_\varepsilon \leq \omega_\varepsilon(t) \leq e^{\frac{C}{r^2}} \omega_\varepsilon, \quad \text{on} \quad B_r(p) \times (0,T).
\]

Let \( r = r_0 > r_1 > \frac{\varepsilon}{2} \) and \( \psi \) be a nonnegative \( C^\infty \) cut-off function that is identically equal to 1 on \( B_{r_1}(p) \) and vanishes outside \( B_r(p) \). We may assume that

\[
|\partial \psi|^2_{\omega_0} \leq \frac{C}{r^2} \quad \text{and} \quad |\sqrt{-1} \partial \bar{\partial} \psi|_{\omega_0} \leq \frac{C}{r^2}.
\]

Straightforward calculations show that

\[
(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j})(\psi^2 S_{\varepsilon,j}) \leq \frac{C'}{r^2} e^{\frac{C}{r^2}} S_{\varepsilon,j} + C e^{\frac{C}{r^2}}.
\]

By choosing sufficiently large \( \alpha \), \( \gamma \) and \( A \), we get

\[
(\frac{\partial}{\partial t} - \Delta_{\varepsilon,j})(e^{-\frac{\alpha}{r^2}} \psi^2 S_{\varepsilon,j}) + Ae^{-\frac{\gamma}{r^2} \text{tr}_{\omega_0}(\omega_{\varepsilon,j}(t))} \leq - \frac{1}{r^2} e^{-\frac{\alpha}{r^2}} S_{\varepsilon,j} + \frac{C}{r^2},
\]

where \( \alpha = 3\gamma \), \( A = \frac{C}{r^2} \), constat \( C \) depends only on \( \| \varphi_0 \|_{L^\infty(M)} \), \( n \), \( \beta \), \( T \), \( \omega_0 \) and \( \text{dist}_{\omega_0}(B_r(p),D) \). By the maximum principle, we conclude that

\[
S_{\varepsilon,j} \leq \frac{C'}{r^2} e^{\frac{C}{r^2}} \quad \text{on} \quad (0,T) \times B_2^\varepsilon(p).
\]
Now we prove that $|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}$ is uniformly bounded. Through computation, there exist uniform constants $C$ such that

$$\frac{d}{dt} - \Delta_{\varepsilon,j}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)} \leq C|Rm_{\varepsilon,j}|^3_{\omega_{\varepsilon,j}(t)} + Ce^{\hat{\tau}}|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)} + Ce^{\hat{\tau}} \hat{S}_{\varepsilon,j} |Rm_{\varepsilon,j}|_{\omega_{\varepsilon,j}(t)} + Ce^{\hat{\tau}} S_{\varepsilon,j} Rm_{\varepsilon,j}.$$

Next, we show that $|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}$ is uniformly bounded. We fix a smaller radius $r_2$ satisfying $r_1 > r_2 > \frac{\delta}{2}$. Let $\rho$ be a cut-off function identically equal to 1 on $\overline{B_{r_2}}(p)$ and identically equal to 0 outside $B_{r_1}$. We also let $\rho$ satisfy

$$|\partial \rho|^2_{\omega_0}, |\sqrt{-1} \partial \bar{\rho}|_{\omega_0} \leq \frac{C}{r_2}$$

for some uniform constant $C$. From the former part we know that $S_{\varepsilon,j}$ is bounded by $\frac{\delta}{2} e^{\hat{\tau}}$ on $B_{r_2}(p)$. Let $K_t = \frac{\delta^2}{2} e^{\hat{\tau}}$, $k$ and $C$ be constants which are large enough such that $\frac{\delta}{2} e^{\hat{\tau}} \leq K_t - S_{\varepsilon,j} \leq K_t$. We consider

$$F_{\varepsilon,j} = \rho^2 e^{-\frac{4 \varepsilon}{2}} \frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t - S_{\varepsilon,j}} + Ae^{-\frac{4 \varepsilon}{2}} S_{\varepsilon,j}.$$

By computing, we have

$$\frac{d}{dt} - \Delta_{\varepsilon,j} F_{\varepsilon,j} = e^{-\frac{4 \varepsilon}{2}} \left( (-\Delta_{\varepsilon,j}) \rho^2 \frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t - S_{\varepsilon,j}} + \rho^2 \frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{(K_t - S_{\varepsilon,j})^2} \left( \frac{d}{dt} - \Delta_{\varepsilon,j} \right) S_{\varepsilon,j} + \rho^2 \frac{\hat{C}}{r_2 (K_t - S_{\varepsilon,j})^2} \frac{K_t - S_{\varepsilon,j}}{t^2} e^{\hat{\tau}} \right)$$

We only consider an inner point $(t_0, x_0)$ which is a maximum point of $F_{\varepsilon,j}$ achieved on $[0, T] \times B_{r_1}(p)$. We use the fact that $\nabla_{\varepsilon,j} F_{\varepsilon,j} = 0$ at this point,

$$e^{-\frac{4 \varepsilon}{2}} \frac{2 \rho \nabla_{\varepsilon,j} \rho}{K_t - S_{\varepsilon,j}} + \rho^2 \frac{\nabla_{\varepsilon,j} |Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)}}{K_t - S_{\varepsilon,j}} + \rho^2 \frac{|Rm_{\varepsilon,j}|^2_{\omega_{\varepsilon,j}(t)} \nabla_{\varepsilon,j} S_{\varepsilon,j}}{(K_t - S_{\varepsilon,j})^2}$$

Combining the above two equalities, we have
\[
\left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) F_{\epsilon,j} = \left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) F_{\epsilon,j}
\]

\[
= e^{-2\frac{\epsilon}{\delta}} \left( (-\Delta_{\epsilon,j} + \frac{\rho^2}{K_t} - S_{\epsilon,j}) + \frac{\rho^2}{K_t} \left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) S_{\epsilon,j} + \frac{\rho^2}{K_t} \left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) \right) + 2 \frac{\rho}{K_t} \delta \left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) S_{\epsilon,j} + \frac{A \rho^2}{K_t \delta} e^{-2\frac{\epsilon}{\delta}} S_{\epsilon,j}
\]

\[
+ \frac{\rho^2}{K_t} \left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) \left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) S_{\epsilon,j}
\]

Our goal is to show that at \((t_0, x_0)\) we have \(e^{-2\frac{\epsilon}{\delta}} |RM_{\epsilon,j}|^2_{\omega_{\epsilon,j}}(t) \leq \frac{\epsilon}{\delta} \) for some uniform constant \(C\) and \(\delta\). Without loss of generality, we assume that \(|RM_{\epsilon,j}|^3_{\omega_{\epsilon,j}}(t) \geq e^{\frac{\epsilon}{\delta}} + \frac{1}{C} e^{\frac{\epsilon}{\delta}} |RM_{\epsilon,j}|^2_{\omega_{\epsilon,j}}(t)\) at \((t_0, x_0)\).

\[
\left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) |RM_{\epsilon,j}|^2_{\omega_{\epsilon,j}}(t) \leq C \left( \left| RM_{\epsilon,j} \right|^3_{\omega_{\epsilon,j}}(t) - \left| \nabla_{\epsilon,j} RM_{\epsilon,j} \right|^2_{\omega_{\epsilon,j}}(t) - |\nabla \varphi RM_{\epsilon,j}|^2_{\omega_{\epsilon,j}}(t) \right),
\]

\[
\left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) S_{\epsilon,j} \leq \frac{C}{\epsilon} e^{\frac{\epsilon}{\delta}} - \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t) - \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t)
\]
on \(B_{\epsilon,1}(p)\). We also note that

\[
\left| \nabla_{\epsilon,j} \right|^2_{\omega_{\epsilon,j}}(t) \leq \left| RM_{\epsilon,j} \right|^2_{\omega_{\epsilon,j}}(t) \left( \left| \nabla_{\epsilon,j} \right|^2_{\omega_{\epsilon,j}}(t) + \left| \nabla_{\epsilon,j} \right|^2_{\omega_{\epsilon,j}}(t), \right.
\]

\[
\left. \left| \nabla_{\epsilon,j} S_{\epsilon,j} \right|^2_{\omega_{\epsilon,j}}(t) \leq 2S_{\epsilon,j} \left( \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t) + \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t) \right) \right).
\]

By using the above inequalities, at \((t_0, x_0)\), we have

\[
\left( \frac{d}{dt} - \Delta_{\epsilon,j} \right) F_{\epsilon,j} \leq -A e^{-2\frac{\epsilon}{\delta}} \left( \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t) + \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t), \right.
\]

\[
\left. \left| \nabla_{\epsilon,j} S_{\epsilon,j} \right|^2_{\omega_{\epsilon,j}}(t) \leq 2S_{\epsilon,j} \left( \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t) + \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t) \right) \right).
\]

Let \(\hat{C}\) be sufficiently large so that \(\frac{8AS_{\epsilon,j}}{K_t} \leq \frac{\epsilon}{2}\), where we denote \(Q = \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t) + \left| \nabla_{\epsilon,j} X \right|^2_{\omega_{\epsilon,j}}(t)\). Then
\[
\frac{C^2|\Omega_{\tau,\varepsilon}|^3_{\omega_{\tau,\varepsilon}}(t)}{K_t} \leq \frac{\rho^2|\Omega_{\tau,\varepsilon}|^4_{\omega_{\tau,\varepsilon}}(t)}{2K_t^2} + C\rho^2|\Omega_{\tau,\varepsilon}|^2_{\omega_{\tau,\varepsilon}}(t) \\
= \frac{\rho^2|\Omega_{\tau,\varepsilon}|^2_{\omega_{\tau,\varepsilon}}(t)Q}{K_t^2} + Ce^{-\delta \frac{\rho^2}{2}}|\Omega_{\tau,\varepsilon}|^2_{\omega_{\tau,\varepsilon}}(t)
\]
(3.7)

Let \( k = 1, \delta = 2\sigma \) and \( \tau - 2\sigma < 0 \), where \( \sigma \) is sufficiently large. We conclude that the evolution equation of \( F_{\varepsilon,j} \) can be controlled as follows,
\[
\left( \frac{d}{dt} - \Delta_{\varepsilon,j} \right) F_{\varepsilon,j} \leq -\frac{Ac}{2} + \frac{AC}{2} e^{-\frac{\rho^2}{2}} \frac{\tau}{\delta} + \frac{AC}{2} e^{-\frac{\rho^2}{2}} e^{\frac{\rho^2}{2}} + Ce^{-\frac{\rho^2}{2}}|\Omega_{\tau,\varepsilon}|^2_{\omega_{\tau,\varepsilon}}(t)
\]
\[
\leq -\frac{Ac}{2} + \frac{AC}{2} e^{-\frac{\rho^2}{2}} \frac{\tau}{\delta} + \frac{AC}{2} e^{-\frac{\rho^2}{2}} e^{\frac{\rho^2}{2}} + Ce^{-\frac{\rho^2}{2}} + Ce^{-\frac{\rho^2}{2}}.
\]

Now we choose a sufficiently large \( A \) such that \( A = 2(\sigma + 1) \) and obtain
\[
e^{-\frac{\rho^2}{2}} \leq \frac{C}{\sigma^2}
\]
at \((t_0, x_0)\). This implies that \( e^{-\frac{\rho^2}{2}}|\Omega_{\tau,\varepsilon}|^2_{\omega_{\tau,\varepsilon}}(t) \leq \frac{C}{\sigma^2} \) at this point, where \( C \) depends only on \( \|\varphi_0\|_{L^\infty(M), n, \beta, T, dist_w(B_r(p), D), \|\theta\|_{C^2(B_r(p))}} \) and \( \omega_0 \). Following that we conclude that \( F_{\varepsilon,j} \) is bounded by \( \frac{C}{\sigma^2} \) at \((t_0, x_0)\). Hence on \([0, T] \times \overline{B_{\varepsilon}(p)}\), we obtain
\[
|\Omega_{\tau,\varepsilon}|^2_{\omega_{\tau,\varepsilon}}(t) \leq \frac{C}{\sigma^4} e^{\frac{\rho^2}{2}}
\]
(3.8)

where \( C, \delta \) and \( \tau \) depend only on \( \|\varphi_0\|_{L^\infty(M), n, \beta, T, \omega_0} \). By using the standard parabolic Schauder regularity theory [31], we obtain the following proposition.

**Proposition 3.2.** For any \( 0 < \delta < T < \infty, k \in \mathbb{N}^+ \) and \( B_r(p) \subset \subset M \setminus D \), there exists constant \( C_{k,T,k,p} \) depends only on \( \|\varphi_0\|_{L^\infty(M), n, \beta, T, dist_w(B_r(p), D)} \) and \( \omega_0 \), such that for any \( \varepsilon > 0 \) and \( j \in \mathbb{N}^+ \),
\[
\|\varphi_{\varepsilon,j}(t)\|_{C^2([\delta,T]\times B_r(p))} \leq C_{k,T,k,p,r}.
\]

Through a further observation to equation (2.42), we prove the monotonicity of \( \varphi_{\varepsilon}(t) \) with respect to \( \varepsilon \).

**Proposition 3.3.** For any \((t, x) \in [0, T] \times M \), \( \varphi_{\varepsilon}(t, x) \) is monotone decreasing as \( \varepsilon \searrow 0 \).

**Proof:** For any \( \varepsilon_1 < \varepsilon_2 \), let \( \psi_{1,2}(t) = \varphi_{\varepsilon_1}(t) - \varphi_{\varepsilon_2}(t) \). Then we have
\[
\frac{\partial}{\partial t}(e^{-\beta t}\psi_{1,2}(t)) \leq e^{-\beta t} \log \left( e^{-\beta t}\omega_0 + \sqrt{-1} e^{-\beta t} \varphi_{\varepsilon_2}(t) + \sqrt{-1} e^{-\beta t} e^{-\beta t}\psi_{1,2}(t) \right)^n.
\]
(3.10)

Let \( \tilde{\psi}_{1,2}(t) = e^{-\beta t}\psi_{1,2}(t) - \delta t \) with \( \delta > 0 \) and \((t_0, x_0)\) be the maximum point of \( \tilde{\psi}_{1,2}(t) \) on \([0, T] \times M \). If \( t_0 > 0 \), by maximum principle, at this point, we have
\[
0 \leq \frac{\partial}{\partial t}\tilde{\psi}_{1,2}(t) = \frac{\partial}{\partial t}(e^{-\beta t}\psi_{1,2}(t)) - \delta \leq -\delta
\]
(3.11)
which is impossible, hence \( t_0 = 0 \). So for any \((t, x) \in [0, T] \times M\),
\[
(3.12) \quad \psi_{1,2}(t, x) \leq e^{\beta t} \sup_M \psi_{1,2}(0, x) + Te^{\beta T} \delta = Te^{\beta T} \delta.
\]
Let \( \delta \to 0 \), we conclude that \( \varphi_{\varepsilon_1}(t, x) \leq \varphi_{\varepsilon_2}(t, x) \). \(\square\)

For any \([\delta, T] \times K \subset (0, \infty) \times M \setminus D\) and \( k \geq 0 \), \( \|\varphi_{\varepsilon,j}(t)\|_{C^k([\delta, T] \times K)} \) is uniformly bounded by Proposition 3.2. Let \( j \) approximate to \( \infty \), we obtain that \( \|\varphi_{\varepsilon}(t)\|_{C^k([\delta, T] \times K)} \) is uniformly bounded. Then let \( \delta \) approximate to \( 0 \), \( T \) approximate to \( \infty \) and \( K \) approximate to \( M \setminus D \), by diagonal rule, we get a sequence \( \{\varepsilon_i\} \), such that \( \varphi_{\varepsilon_i}(t) \) converges in \( C^\infty_{\text{loc}} \) topology on \((0, \infty) \times (M \setminus D)\) to a function \( \varphi(t) \) that is smooth on \( C^\infty ((0, \infty) \times (M \setminus D)) \) and satisfies equation
\[
(3.13) \quad \frac{\partial \varphi(t)}{\partial t} = \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t)}{\omega_0^n} \right) + F_0 + \beta \varphi(t) + \log |s|^{2(1-\beta)}
\]
on \((0, \infty) \times (M \setminus D)\). Since \( \varphi_{\varepsilon}(t) \) is monotone decreasing as \( \varepsilon \to 0 \), we conclude that \( \varphi_{\varepsilon}(t) \) converges in \( C^\infty_{\text{loc}} \) topology on \((0, \infty) \times (M \setminus D)\) to \( \varphi(t) \). Combining the above arguments with (3.11) and (3.2), for any \( T > 0 \), we have
\[
(3.14) \quad \|\varphi(t)\|_{L^\infty([0, T] \times (M \setminus D))} \leq C_1,
\]
\[
(3.15) \quad e^{-C_2} \omega_\beta \leq \omega(t) \leq e^{C_2} \omega_\beta \quad \text{on} \quad (0, T] \times (M \setminus D),
\]
where \( \omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t) \), constants \( C_1 \) and \( C_2 \) depend only on \( \|\varphi_0\|_{L^\infty(M)}, \beta, n, \omega_0 \) and \( T \).

**Proposition 3.4.** For any \( t > 0 \), \( \varphi(t) \) is Hölder continuous on \( M \) with respect to the metric \( \omega_0 \).

**Proof:** We assume that \( t \in [\delta, T] \) for some \( \delta \) and \( T \) satisfying \( 0 < \delta < T < \infty \). By (3.13) we have
\[
(3.16) \quad C^{-1} \omega_\beta \leq \omega(t) \leq C \omega_\beta \quad \text{on} \quad [\delta, T] \times (M \setminus D),
\]
where constant \( C \) depends only on \( \|\varphi_0\|_{L^\infty(M)}, \beta, T, n, \omega_0 \) and \( \delta \). Combining this estimate and the fact that \( \log \frac{\omega_0 |s|^{2(1-\beta)}}{\omega_0^n} \) is bounded uniformly on \( M \setminus D \), we obtain
\[
(3.17) \quad \|\log \frac{\omega_0^n(t) |s|^{2(1-\beta)}}{\omega_0^n} \|_{L^\infty([\delta, T] \times (M \setminus D))} \leq C
\]
for some uniform constant \( C \) independent of \( t \). Therefore, \( \|\frac{\partial \varphi(t)}{\partial t}\|_{L^\infty([\delta, T] \times (M \setminus D))} \) is uniformly bounded by equation (3.13) and estimate (3.14). We rewrite equation (3.13) as
\[
(3.18) \quad (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t))^n = e^{\frac{\partial \varphi(t)}{\partial t} - F_0} - \beta \varphi(t) \frac{\omega_0^n}{|s|^{2(1-\beta)}}.
\]
The function on the right side of equation (3.18) is \( L^p \) integrable with respect to \( \omega_0^n \) for some \( p > 1 \). By S. Kolodziej’s \( L^p \)-estimates [23], we know that \( \varphi(t) \) is Hölder continuous on \( M \) with respect to \( \omega_0 \) for any \( t > 0 \). \(\square\)

Next, by using the monotonicity of \( \varphi_{\varepsilon}(t) \) with respect to \( \varepsilon \) and constructing auxiliary function, we prove the continuity of \( \varphi(t) \) as \( t \to 0^+ \).
Proposition 3.5. \( \varphi(t) \in C^0([0, \infty) \times M) \) and

\[
\lim_{t \to 0^+} \| \varphi(t) - \varphi_0 \|_{L^\infty(M)} = 0.
\]

**Proof:** Through the above arguments, we only need prove (3.19). By the monotonicity of \( \varphi(\varepsilon) \) with respect to \( \varepsilon \), for any \( (t, z) \in (0, T] \times M \), we have

\[
\varphi(t, z) - \varphi_0(z) \leq |\varphi_{\varepsilon_1}(t, z) - \varphi_0(z)|
\]

\[
\leq |\varphi_{\varepsilon_1}(t, z) - \varphi_{\varepsilon_1,j}(t, z)| + |\varphi_{\varepsilon_1,j}(t, z) - \varphi_{0,j}(z)|
\]

\[
+ |\varphi_{0,j}(z) - \varphi_0(z)|.
\]

Since \( \varphi_{\varepsilon_1,j}(t) \) is a Cauchy sequence in \( L^\infty([0, T] \times M) \),

\[
\lim_{j \to \infty} \| \varphi_{\varepsilon_1,j}(t, z) - \varphi_{\varepsilon_1,j}(t, z) \|_{L^\infty([0, T] \times M)} = 0.
\]

From (2.22), we have

\[
\lim_{j \to \infty} \| \varphi_{0,j}(z) - \varphi_0(z) \|_{L^\infty(M)} = 0.
\]

For any \( \varepsilon > 0 \), there exists \( N \) such that for any \( j > N \),

\[
\sup_{[0, T] \times M} |\varphi_{\varepsilon_1}(t, z) - \varphi_{\varepsilon_1,j}(t, z)| < \frac{\varepsilon}{3},
\]

\[
\sup_M |\varphi_{0,j}(z) - \varphi_0(z)| < \frac{\varepsilon}{3}.
\]

Fix such \( \varepsilon_1 \) and \( j \), there exists \( 0 < \delta_1 < T \) such that

\[
\sup_{[0, \delta_1] \times M} |\varphi_{\varepsilon_1,j}(t, z) - \varphi_{0,j}| < \frac{\varepsilon}{3}.
\]

Combining the above estimates together, for any \( t \in (0, \delta_1] \) and \( z \in M \),

\[
\varphi(t, z) - \varphi_0(z) < \varepsilon.
\]

On the other hand, by S. Kolodziej’s results [27], there exists a smooth solution \( u_{\varepsilon,j} \) to the equation

\[
(\omega_0 + \sqrt{-1} \partial \bar{\partial} u_{\varepsilon,j})^n = e^{-F_0 - \beta \varphi_0,j} + \hat{C} \frac{\omega_0^n}{(\varepsilon^2 + |s|^2_h(1 - \beta))},
\]

and \( u_{\varepsilon,j} \) satisfies

\[
\| u_{\varepsilon,j} \|_{L^\infty(M)} \leq C,
\]

where \( \hat{C} \) is a uniform normalization constant independent of \( \varepsilon \) and \( j \), constant \( C \) depends only on \( \| \varphi_0 \|_{L^\infty(M)} \), \( \beta \) and \( F_0 \).

We define function

\[
\psi_{\varepsilon,j}(t) = (1 - te^{\beta t}) \varphi_{0,j} + te^{\beta t} u_{\varepsilon,j} + h(t) e^{\beta t},
\]

where

\[
h(t) = -t \| \varphi_{0,j} \|_{L^\infty(M)} - t \| u_{\varepsilon,j} \|_{L^\infty(M)} + \beta t \log t - t e^{-\beta t}
\]

\[
+ \beta n \int_0^t e^{-\beta s} \log s ds - \frac{\hat{C}}{\beta} e^{-\beta t} + \frac{\hat{C}}{\beta}
\]

and \( h(0) = 0 \).
Straightforward calculations show that
\[
\frac{\partial}{\partial t} \psi_{\varepsilon,j}(t) - \beta \psi_{\varepsilon,j}(t) = -\beta \varphi_{0,j} - e^{\beta t} \varphi_{0,j} + e^{\beta t} \varphi_{z,j} + e^{\beta t} \frac{\partial}{\partial t} \left[ h(t) + n \log t - \beta n(t \log t - t) + \beta n t \log t + \tilde{C} \right].
\]

Therefore, we have
\[
e^{\frac{\partial}{\partial t} \psi_{\varepsilon,j}(t) - \beta \psi_{\varepsilon,j}(t)} \omega_0^n \leq t^n e^{\beta t} e^{-\beta \varphi_{0,j} + \tilde{C} \omega_0^n}.
\]

When \( t \) is sufficiently small,
\[
\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t) = (1 - te^{\beta t}) (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{0,j}) + te^{\beta t} (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{z,j})
\geq te^{\beta t} (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{z,j}).
\]
Combining the above inequalities,
\[
(\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t))^n \geq t^n e^{\beta t} (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{z,j})^n
= t^n e^{\beta t} e^{-\beta \varphi_{0,j} + \tilde{C}} \left( \frac{\omega_0^n}{(\epsilon^2 + |s|^2)^{1-\beta}} \right)
\geq e^{-F_0} e^{\frac{\partial}{\partial t} \psi_{\varepsilon,j}(t) - \beta \psi_{\varepsilon,j}(t)} \frac{\omega_0^n}{(\epsilon^2 + |s|^2)^{1-\beta}}.
\]
This equation is equivalent to
\[
\begin{align*}
\frac{\partial}{\partial t} \psi_{\varepsilon,j}(t) &\leq \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t)}{\omega_0} \right)^n + \beta \psi_{\varepsilon,j}(t) \\
\psi_{\varepsilon,j}(0) &= \varphi_{0,j}
\end{align*}
\] (3.29)

Let \( \tilde{\psi}_{\varepsilon,j}(t) = \varphi_{\varepsilon,j}(t) - \psi_{\varepsilon,j}(t) \), then
\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{\psi}_{\varepsilon,j}(t) &\geq \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \tilde{\psi}_{\varepsilon,j}(t)}{\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_{\varepsilon,j}(t)} \right) + \beta \tilde{\psi}_{\varepsilon,j}(t) \\
\tilde{\psi}_{\varepsilon,j}(0) &= 0
\end{align*}
\] (3.30)

By the similar arguments as that in the proof of Proposition 3.3 for any \((t, z) \in [0, T] \times M\),
\[
\tilde{\psi}_{\varepsilon,j}(t, z) \geq 0,
\] (3.31)

That is, for any \((t, z) \in [0, T] \times M\)
\[
\varphi_{\varepsilon,j}(t, z) - \varphi_{0,j}(z) \geq -te^{\beta t} \varphi_{0,j} + te^{\beta t} \varphi_{z,j} + h(t)e^{\beta t}
\] (3.32)

where \( h_1(t) = -n(t \log t - t)e^{-\beta t} + \beta n \int_0^t e^{-\beta s} s \log s ds \), constant \( C \) depends only on \( \| \varphi_0 \|_{L^\infty(M)} \), \( \beta \) and \( F_0 \). Let \( j \rightarrow \infty \) and then \( \varepsilon \rightarrow 0 \), we have
\[
\varphi(t, z) - \varphi_0(z) \geq -C(e^{\beta t} - 1) + h_1(t)e^{\beta t}.
\] (3.33)

There exists \( \delta_2 \) such that for any \( t \in [0, \delta_2] \),
\[
-C(e^{\beta t} - 1) + h_1(t)e^{\beta t} > -\epsilon.
\] (3.34)
Let $\delta = \min(\delta_1, \delta_2)$, then for any $t \in (0, \delta]$ and $z \in M$,
\begin{equation}
- \epsilon < \varphi(t, z) - \varphi_0(z) < \epsilon.
\end{equation}
This completes the proof of the proposition. \hfill \Box

**Theorem 3.6.** $\omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t)$ is a long-time solution to the conical Kähler-Ricci flow \((1.5)\).

**Proof:** We should only prove that $\omega(t)$ satisfies equation \((1.5)\) in the sense of currents on \([0, \infty) \times M\).

Let $\eta = \eta(t, x)$ be a smooth \((n-1, n-1)\)-form with compact support in \((0, \infty) \times M\). Without loss of generality, we assume that its compact support included in \((\delta, T) (0 < \delta < T < \infty)\). On \([\delta, T] \times M\), by \((3.11), (3.12), (3.14)\) and \((3.15)\), we know that $\log \frac{\omega^n_t(x^2 + |s|^2_h)}{\omega^n_0}$, $\varphi_\varepsilon(t)$ and $\varphi(t)$ are uniformly bounded by constants depending only on $\|\varphi_0\|_{L^\infty(M)}$, $n$, $\beta$, $\delta$ and $T$. On \([\delta, T]\), we have
\begin{align*}
\int_M \frac{\partial \omega_t}{\partial t} \wedge \eta &= \int_M \sqrt{-1} \partial \bar{\partial} \frac{\partial \varphi_t}{\partial t} \wedge \eta \\
&= \int_M \sqrt{-1} \partial \bar{\partial} \left( \log \frac{\omega^n_t(x^2 + |s|^2_h)}{\omega^n_0} + F_0 + \beta \varphi_\varepsilon(t) \right) \wedge \eta \\
&= \int_M \left( \log \frac{\omega^n_t}{\omega^n_0} + F_0 + \beta \varphi(t) + \log |s|^{2(1-\beta)} \right) \sqrt{-1} \partial \bar{\partial} \eta \\
&\overset{\varepsilon \to 0}{\longrightarrow} \int_M \left( \log \frac{\omega^n_t}{\omega^n_0} + F_0 + \beta \varphi(t) + \log |s|^{2(1-\beta)} \right) \sqrt{-1} \partial \bar{\partial} \eta \\
&= \int_M (-\text{Ric}(\omega(t)) + \beta \omega(t) + 2\pi (1 - \beta) [D]) \wedge \eta. \hfill (3.36)
\end{align*}

At the same time, there also holds
\begin{align*}
\int_M \omega_t \varphi_t \wedge \frac{\partial \eta}{\partial t} &= \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \sqrt{-1} \partial \bar{\partial} \varphi_t \wedge \frac{\partial \eta}{\partial t} \\
&= \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \varphi_t \sqrt{-1} \partial \bar{\partial} \frac{\partial \eta}{\partial t} \\
&\overset{\varepsilon \to 0}{\longrightarrow} \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \varphi(t) \sqrt{-1} \partial \bar{\partial} \frac{\partial \eta}{\partial t} \\
&= \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \sqrt{-1} \partial \bar{\partial} \varphi(t) \frac{\partial \eta}{\partial t} \\
&= \int_M \omega(t) \wedge \frac{\partial \eta}{\partial t}. \hfill (3.37)
\end{align*}

On the other hand, $\varphi_\varepsilon(t)$ and $\frac{\partial \varphi_\varepsilon(t)}{\partial t}$ are uniformly bounded on $[\delta, T] \times M$, $\varphi(t)$ and $\frac{\partial \varphi(t)}{\partial t}$ are uniformly bounded on $[\delta, T] \times (M \setminus D)$, therefore
\begin{align*}
\frac{\partial}{\partial t} \int_M \omega \varphi \wedge \eta &= \int_M \varphi \sqrt{-1} \partial \bar{\partial} \frac{\partial \eta}{\partial t} \\
&+ \int_M \frac{\partial \varphi(t)}{\partial t} \sqrt{-1} \partial \bar{\partial} \eta + \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t}
\end{align*}
\[ \lim_{t \to 0} \int_M \varphi(t) \sqrt{-1} \partial \bar{\partial} \eta(t) = \int_M \frac{\partial \varphi}{\partial t} \sqrt{-1} \partial \bar{\partial} \eta(t) + \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} \]

Combining equality (3.38)

\[ = \frac{\partial}{\partial t} \int_M \omega(t) \wedge \eta. \]

Integrating form 0 to \( \infty \) on both sides,

\[ \int_{M \times (0, \infty)} \frac{\partial \omega(t)}{\partial t} \wedge \eta \, dt = - \int_{M \times (0, \infty)} \omega(t) \wedge \frac{\partial \eta}{\partial t} \, dt = - \int_{0}^{\infty} \int_{M} \omega(t) \wedge \frac{\partial \eta}{\partial t} \, dt \]

\[ = \int_{0}^{\infty} \int_{M} (-Ric(t) + \beta \omega(t) + 2\pi(1-\beta) [D]) \wedge \eta \, dt \]

\[ = \int_{M \times (0, \infty)} (-Ric(t) + \beta \omega(t) + 2\pi(1-\beta) [D]) \wedge \eta \, dt. \]

By the arbitrariness of \( \eta \), we prove that \( \omega(t) \) satisfies the conical Kähler-Ricci flow (1.3) in the sense of currents on \((0, \infty) \times M\). \( \square \)

Now we are ready to prove the uniqueness of the parabolic Monge-Ampère equation (3.13) starting with \( \varphi_0 \in \mathcal{E}_p(M, \omega_0) \) for some \( p > 1 \).

**Theorem 3.7.** Let \( \varphi_i(t) \in C^0((0, \infty) \times M) \cap C^\infty((0, \infty) \times (M \setminus D)) \) \( (i = 1, 2) \) be two long-time solutions to the parabolic Monge-Ampère equation

\[ \frac{\partial \varphi_i}{\partial t} = \log \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_i(t)^n}{\omega_0} + F_0 + \beta \varphi_i(t) + \log |s|_h^{2(1-\beta)} \]

on \((0, \infty) \times (M \setminus D)\). If \( \varphi_i \) \( (i = 1, 2) \) satisfy

- For any \( 0 < \delta < T < \infty \), there exists uniform constant \( C \) such that
  \[ C^{-1} \omega_\beta \leq \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_i(t) \leq C \omega_\beta \]
  on \([\delta, T] \times (M \setminus D)\);
- On \([\delta, T] \), there exist constant \( \alpha > 0 \) and \( C^* \) such that \( \varphi_i(t) \) is \( C^* \) on \( M \) with respect to \( \omega_0 \) and \( \| \frac{\partial \varphi_i}{\partial t} \|_{L^\infty(M \setminus D)} \leq C^* \);
- \( \lim_{t \to 0^+} \| \varphi_i(t) - \varphi_0 \|_{L^\infty(M)} = 0 \).

Then \( \varphi_1 = \varphi_2 \).

**Proof:** We apply Jeffres’ trick [25] in the parabolic case. For any \( 0 < t_1 < T < \infty \) and \( a > 0 \). Let \( \phi_1(t) = \varphi_1(t) + a|s|_h^{2q} \), where \( 0 < q < 1 \) is determined later. The
evolution of \( \phi_1 \) is
\[
\frac{\partial \phi_1(t)}{\partial t} = \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_1(t)}{\omega_0^n} \right)^n + F_0 + \beta \phi_1(t) - a \beta |s|_{h}^{2q} + \log |s|_{h}^{2(1 - \beta)}.
\]

Denote \( \psi(t) = \phi_1(t) - \phi_2(t) \) and \( \hat{\Delta} = \int_{0}^{1} g_{s\phi_1 + (1-s)\phi_2} \frac{\partial^2}{\partial s^2} ds, \) \( \psi(t) \) evolves along the following equation
\[
\frac{\partial \psi(t)}{\partial t} = \hat{\Delta} \psi(t) - a \hat{\Delta} |s|_{h}^{2q} + \beta \psi(t) - a \beta |s|_{h}^{2q}.
\]

By the equivalence of the metrics and the equation
\[
\sqrt{-1} \partial \bar{\partial} |s|_{h}^{2q} = q^2 |s|_{h}^{2q} \sqrt{-1} \partial \bar{\partial} \log |s|_{h}^{2q} + q|s|_{h}^{2q} \sqrt{-1} \partial \bar{\partial} \log |s|_{h}^{2q},
\]
we obtain the estimate
\[
\hat{\Delta} |s|_{h}^{2q} \geq q|s|_{h}^{2q} g_{\phi_1 + (1-s)\phi_2} \left( \frac{\partial^2}{\partial z \partial \bar{z}} \log |s|_{h}^{2q} \right)
\]
\[
= -q|s|_{h}^{2q} g_{\phi_1 + (1-s)\phi_2} g_{0 ij}
\]
\[
\geq -Cq|s|_{h}^{2q} g_{\phi_1 + (1-s)\phi_2} g_{0 ij}
\]
\[
\geq -C
\]
on \( M \setminus D \), where constant \( C \) independent of \( a \), and we apply the fact that \( \omega_\beta \geq \gamma \omega_0 \) on \( M \setminus D \) for some constant \( \gamma \). Then we obtain
\[
\frac{\partial \psi(t)}{\partial t} \leq \hat{\Delta} \psi(t) + \beta \psi(t) + aC.
\]

Let \( \tilde{\psi} = e^{-\beta(t-t)} \psi + 4C e^{-\beta(t-t)} - \epsilon(t-t) \). By choosing suitable \( 0 < q < 1 \), we can assume that the space maximum of \( \tilde{\psi} \) on \([t_1, T] \times M \) is attained away from \( D \). Let \((t_0, x_0)\) be the maximum point. If \( t_0 > t_1 \), by the maximum principle, at \((t_0, x_0)\), we have
\[
0 \leq \left( \frac{\partial}{\partial t} - \hat{\Delta} \right) \tilde{\psi}(t) \leq -\epsilon,
\]
which is impossible, hence \( t_0 = t_1 \). Then for \((t, x) \in [t_1, T] \times M \), we obtain
\[
\psi(t, x) \leq e^{\beta T} \| \varphi_1(t_1, x) - \varphi_2(t_1, x) \|_{L^\infty(M)} + aCe^{\beta T} + \epsilon Te^{\beta T}
\]
Let \( a \to 0 \) and then \( t_1 \to 0^+ \), we get
\[
\varphi_1(t) - \varphi_2(t) \leq \epsilon Te^{\beta T}.
\]
It shows that \( \varphi_1(t) \leq \varphi_2(t) \) after we let \( \epsilon \to 0 \). By the same reason we have \( \varphi_2(t) \leq \varphi_1(t) \), then we prove that \( \varphi_1(t) = \varphi_2(t) \). \( \square \)

**Lemma 3.8.** Assume that on \((0, \infty) \times (M \setminus D)\), \( f(t, x) \) is a smooth function and satisfies \( \sqrt{-1} \partial \bar{\partial} f(t, x) = 0 \). If
\[
(3.41) \quad \| f(t, x) \|_{L^\infty([3, T] \times (M \setminus D))} \leq C_3 T,
\]
where \( 0 < \delta < T < \infty \). Then \( f(t, x) = f(t) \), i.e. function \( f(t, x) \) depends only on \( t \) on \((0, \infty) \times (M \setminus D)\).
\textbf{Proof:} We prove this lemma by the logarithmic cutoff trick. Fix a cutoff function \( \eta : [0, \infty) \to [0, 1] \) such that \( \eta(s) = 1 \) for \( s < 1 \) and \( \eta(s) = 0 \) for \( s > 2 \). We define
\begin{equation}
(3.42) 
\gamma(x) = \eta\left(\frac{\log r}{\log \varepsilon}\right),
\end{equation}
where \( r \) is the the distance function from the divisor. Then \( \gamma = 0 \) for \( r < \varepsilon^2 \) and \( \gamma = 1 \) for \( r > \varepsilon \). Straightforward calculations shows that
\[
\int_M |\partial \gamma|^2_{g_0} dV_0 = \int_M \left(\eta'\right)^2 \frac{1}{r^2(\log \varepsilon)^2} |\partial r|^2_{g_0} dV_0 
\leq C \int_{\varepsilon^2}^{\infty} \frac{1}{r^2(\log \varepsilon)^2} rdr = -C \frac{1}{\log \varepsilon},
\]
so we have \( \int_M |\partial \gamma|^2_{g_0} dV_0 \to 0 \) as \( \varepsilon \to 0 \). On the other hand, when \( t \in [\delta, T] \),
\[
0 = \int_M \text{div}_{g_0} (\gamma^2 f \nabla_{g_0} f) dV_0 
= 2 \int_M \gamma f < \partial \gamma, \nabla f > dV_0 + \int_M \gamma^2 |\partial f|^2_{g_0} dV_0 
\geq \frac{1}{2} \int_M \gamma^2 |\partial f|^2_{g_0} dV_0 - 2 \int_M f^2 |\partial \gamma|^2_{g_0} dV_0,
\]
which implies
\begin{equation}
(3.43) 
\int_M \gamma^2 |\partial f|^2_{g_0} dV_0 \leq C \int_M |\partial \gamma|^2_{g_0} dV_0,
\end{equation}
where constant \( C \) depends only on \( \|f\|_{L^\infty([\delta, T] \times (M \setminus D))} \). Passing to the limit and then combining with the arbitrary choice of \( \delta \) and \( T \), we obtain
\begin{equation}
(3.44) 
\int_M |\partial f|^2_{g_0} dV_0 = 0 \quad \text{on} \quad (0, \infty).
\end{equation}
Hence function \( f(t, x) \) depends only on \( t \). \( \square \)

\textbf{Theorem 3.9.} \( \omega_{\phi(t)} = \omega_0 + \sqrt{-1 \partial \bar{\partial} \phi(t)} \) is the unique long-time solution to the conical \( \text{Kähler-Ricci flow} \) \((3.43)\).

\textbf{Proof:} Suppose there is another solution \( \omega_{\phi(t)} = \omega_0 + \sqrt{-1 \partial \bar{\partial} \phi(t)} \) to the conical \( \text{Kähler-Ricci flow} \) \((1.3)\). It follows from Lemma 3.8 that
\begin{equation}
(3.45) \frac{\partial \phi(t)}{\partial t} = \log \left(\frac{\omega_0 + \sqrt{-1 \partial \bar{\partial} \phi(t)}}{\omega_0^n} \right) + F_0 + \beta \phi(t) + \log |s|^2(1-\beta) + f(t)
\end{equation}
on \((0, \infty) \times (M \setminus D)\) for a smooth function \( f(t) \) defined on \((0, \infty)\), and \( \phi(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times (M \setminus D)) \) satisfies
\begin{itemize}
  \item For any \( 0 < \delta < T < \infty \), there exists uniform constant \( C \) such that \( C^{-1} \omega_\beta \leq \omega_0 + \sqrt{-1 \partial \bar{\partial} \phi(t)} \leq C \omega_\beta \) on \([\delta, T] \times (M \setminus D)\);
  \item On \([\delta, T]\), there exist constant \( \alpha > 0 \) and \( C \) such that \( \phi(t) \) is \( C^\alpha \) on \( M \) with respect to \( \omega_0 \) and \( \|\frac{\partial \phi(t)}{\partial t}\|_{L^\infty(M \setminus D)} \leq C; \)
  \item \( \lim_{t \to 0^+} \|\phi(t) - \varphi_0\|_{L^\infty(M)} = 0. \)
\end{itemize}
For any $0 < t_1 < T < \infty$ and $a > 0$. Let $\psi(t) = \phi(t) + a|s|^{2q}_{h} - \varphi(t)$, where $0 < q < 1$ is determined later. Then

$$\frac{\partial \psi(t)}{\partial t} = \Delta \psi(t) - a\Delta|s|^{2q}_{h} + \beta \psi(t) - a\beta|s|^{2q}_{h} + f(t).$$

By the same arguments as that in the proof of Proposition 3.7, for any $(t, x) \in [t_1, T] \times M$, we have

$$\psi(t, x) \leq e^{\beta(t-t_1)}\|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} + aCe^{\beta(t-t_1)} + e(t - t_1)e^{\beta(t-t_1)} + e^{\beta(t-t_1)}\int_{t_1}^{t} e^{-\beta(s-t_1)}f(s)ds.$$

Let $a \to 0$, we obtain

$$\phi(t) - \varphi(t) \leq e^{\beta(t-t_1)}\|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} + e(t - t_1)e^{\beta(t-t_1)} + e^{\beta(t-t_1)}\int_{t_1}^{t} e^{-\beta(s-t_1)}f(s)ds.$$

By the similar arguments, we can obtain

$$\varphi(t) - \phi(t) \leq e^{\beta(t-t_1)}\|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} + e(t - t_1)e^{\beta(t-t_1)} - e^{\beta(t-t_1)}\int_{t_1}^{t} e^{-\beta(s-t_1)}f(s)ds.$$

Therefore, for any $t > t_1 > 0$, we have

$$\inf_{M}(\phi(t) - \varphi(t)) \geq \sup_{M}(\phi(t) - \varphi(t)) - 2e^{\beta(t-t_1)}\|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} - 2e(t - t_1)e^{\beta(t-t_1)} \geq \sup_{M}(\phi(t) - \varphi(t)) - 2e\beta T\|\phi(t_1, x) - \varphi(t_1, x)\|_{L^\infty(M)} - 2\epsilon Te^{\beta T}.$$

Let $t_1 \to 0^+$ and then $\epsilon \to 0$, we conclude that $\phi(t) = \varphi(t) + e^{\beta t}\int_{0}^{t} e^{-\beta s}f(s)ds$. Then $\omega_{\phi(t)} = \omega_{\varphi(t)}$ on $(0, \infty) \times (M \setminus D)$.

4. The convergence of the conical Kähler-Ricci flow with weak initial data

In this section, we study the convergence of the conical Kähler-Ricci flow (1.5) on Fano manifold with positive twisted first Chern class. Our discussion is very similar as that in [34], but we need new arguments on estimates of the twisted Ricci potential $u_{\varepsilon}(t)$ and the term $|\hat{\varphi}_{\varepsilon}|$ when we handle the weak initial data case.

Without loss of generality, we assume $\lambda = 1$ (i.e. $\mu = \beta$). We first prove the uniform Perelman's estimates along the twisted Kähler-Ricci flow

$$\left\{ \begin{array}{l}
\frac{\partial \omega_{\varepsilon}(t)}{\partial t} = -Ric(\omega_{\varepsilon}(t)) + \beta \omega_{\varepsilon}(t) + \theta_{\varepsilon} \\
\omega_{\varepsilon}(t)|_{t=0} = \omega_{\varphi_0}
\end{array} \right.$$

(4.1)

By the same argument as Proposition 4.1 in [34], we have
Proposition 4.1. \( t^2(R(g_{ε,j}(t)) - tr_{g_{ε,j}(t)}θ_ε) \) is uniformly bounded from below along the twisted Kähler-Ricci flow \( (2.3) \), i.e. there exists a uniform constant \( C \), such that

\[ t^2(R(g_{ε,j}(t)) - tr_{g_{ε,j}(t)}θ_ε) \geq -C \]

for any \( t \geq 0 \), \( j \in \mathbb{N}^+ \) and \( ε > 0 \), while the constant \( C \) only depends on \( β \) and \( n \). In particular,

\[ R(g_{ε,j}(t)) - tr_{g_{ε,j}(t)}θ_ε \geq -C \]

when \( t \geq \frac{1}{2} \).

Remark 4.2. By Proposition \( 2.5 \), we know that there exists constant \( C \) only depending on \( β \) and \( n \), such that

\[ R(g_{ε}(t)) - tr_{g_{ε}(t)}θ_ε \geq -C \]

along the twisted Kähler-Ricci flow \( (4.1) \) for any \( ε > 0 \) when \( t \geq \frac{1}{2} \).

Straightforward calculation shows that the twisted Ricci potential \( u_ε(t) \) with respect to \( Ω_ε(t) \) at \( t = \frac{1}{2} \) can be written as

\[ u_ε(\frac{1}{2}) = \log \frac{\omega_0(\frac{1}{2})(ε^2 + |s|^2)^{1-β}}{ω_0} + F_0 + βφ_ε(\frac{1}{2}) + C_ε. \]

where \( C_ε \) is a normalization constant such that \( \frac{1}{N} \int_M e^{-u_ε(t)}dV_{ε,t} = 1 \). By \( (3.1) \) and \( (3.2) \), we conclude that \( C_ε, \frac{1}{2} \) and \( u_ε(\frac{1}{2}) \) are uniformly bounded. Let \( a_ε(t) = \frac{1}{N} \int_M u_ε(t)e^{-u_ε(t)}dV_{ε,t} \), then by Lemma 4.4 in \( [34] \), we have

**Lemma 4.3.** There exists a uniform constant \( C \), such that

\[ |a_ε(t)| \leq C \]

for any \( t \geq \frac{1}{2} \) and \( ε > 0 \).

Now we consider the twisted Kähler-Ricci flows \( (4.1) \) starting at \( t = \frac{1}{2} \). Using the estimates \( (1.4) \), \( (4.0) \) and following the arguments in section 4 of \( [34] \), we have the following uniform Perelman’s estimates.

**Theorem 4.4.** Let \( g_ε(t) \) be a solution of the twisted Kähler Ricci flow, i.e. the corresponding form \( Ω_ε(t) \) satisfies the equation \( (4.1) \) with initial metric \( Ω_ε_0 \), \( u_ε(t) \in C^∞((0,∞) \times M) \) is the twisted Ricci potential satisfying

\[ -Ric(Ω_ε(t)) + βΩ_ε(t) + θ_ε = \sqrt{-1}∂\bar{∂}u_ε(t) \]

and \( \frac{1}{N} \int_M e^{-u_ε(t)}dV_{ε,t} = 1 \), where \( θ_ε = (1-β)(ω_0 + \sqrt{-1}∂\bar{∂}log(ε^2 + |s|^2)) \). Then for any \( β \in (0,1) \), there exists a uniform constant \( C \), such that

\[ |R(g_ε(t)) - tr_{g_ε(t)}θ_ε| \leq C, \]

\[ ||u_ε(t)||_{C^1(g_ε(t))} \leq C, \]

\[ diam(M,g_ε(t)) \leq C \]

hold for any \( t \geq 1 \) and \( ε > 0 \), where \( R(g_ε(t)) - tr_{g_ε(t)}θ_ε \) and \( diam(M,g_ε(t)) \) are the twisted scalar curvature and diameter of the manifold respectively with respect to the metric \( g_ε(t) \).
If $\varphi_\varepsilon(t)$ is a solution to the Monge-Ampère equation
\begin{equation}
\frac{\partial \varphi_\varepsilon(t)}{\partial t} = \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\varepsilon(t)}{\omega_0^n} \right) + F_0 + \beta \varphi_\varepsilon(t) + \log(\varepsilon^2 + |s_h|^2)^{1-\beta}
\end{equation}
on (0, \infty) \times M with initial value $\varphi_\varepsilon(0) = \varphi_0$, it is obvious that $\phi_\varepsilon(t) = \varphi_\varepsilon(t) + C e^{\beta t}$ is a solution to equation (4.11) with initial value $\phi_\varepsilon(0) = \varphi_0 + C$. At the same time, $\omega_{\phi_\varepsilon(t)} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi_\varepsilon(t)$ is also a solution to the twisted Kähler-Ricci flow (4.11) with initial value $\omega_{\phi_0}$.

From (4.11), we know that $\varphi_\varepsilon(t)$ is uniformly bounded on $[0, T] \times M$ by a constant $C$ which depends only on $\|\varphi_0\|_{L^\infty(M)}$, $\beta$ and $T$. Now, we consider the solution $\psi_\varepsilon(t) = \varphi_\varepsilon(t) + \tilde{C}_{\varepsilon,1} e^{\beta t}$ to the equation
\begin{equation}
\begin{cases}
\frac{\partial \psi_\varepsilon(t)}{\partial t} = \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_\varepsilon(t)}{\omega_0^n} \right) + F_0 + \beta \psi_\varepsilon(t) + \log(\varepsilon^2 + |s_h|^2)^{1-\beta} \\
\psi_\varepsilon(0) = \varphi_0 + \tilde{C}_{\varepsilon,1}
\end{cases}
on \text{on } (0, \infty) \times M,
\end{equation}
where
\[ \tilde{C}_{\varepsilon,1} = e^{-\beta} \frac{1}{\beta} \left( \int_1^{+\infty} e^{-\beta t} \|\nabla u_\varepsilon(t)\|^2_{L^2} dt - \frac{1}{V} \int_M (F_{\varepsilon,1} + \beta \varphi_\varepsilon(1)) dV_{\varepsilon,1} \right), \]
\[ F_{\varepsilon,1} = F_0 + \log(\frac{\omega_0^n}{\omega_0^m} \cdot \varepsilon^2 + |s_h|^2)^{1-\beta} \] and $dV_{\varepsilon,1} = \frac{\omega_0^n(1)}{m}$. By (4.11), the above uniform Perelman’s estimates (4.9), we know that the constant $\tilde{C}_{\varepsilon,1}$ is well-defined and uniformly bounded. Straightforward calculation shows that the twisted Ricci potential $u_\varepsilon(1)$ with respect to $\omega_\varepsilon(1)$ can be written as
\begin{equation}
u_\varepsilon(1) = \log \left( \frac{\omega_\varepsilon^n(1)(\varepsilon^2 + |s_h|^2)^{1-\beta}}{\omega_0^n} \right) + F_0 + \beta \varphi_\varepsilon(1) + C_{\varepsilon,1},
\end{equation}
where $C_{\varepsilon,1}$ is a normalization constant such that $\frac{1}{V} \int_M e^{-u_\varepsilon(1)} dV_{\varepsilon,1} = 1$. Then
\begin{equation}
C_{\varepsilon,1} = \log \left( \frac{1}{V} \int_M e^{-F_0 - \beta \varphi_\varepsilon(1)} \frac{dV_0}{(\varepsilon^2 + |s_h|^2)^{1-\beta}} \right).
\end{equation}
By (4.11) and (4.12), we conclude that $C_{\varepsilon,1}$ and $u_\varepsilon(1)$ are uniformly bounded.

Let $u_\varepsilon(t) = \psi_\varepsilon(t) + c_\varepsilon(t)$. By equation (4.12) and equality (4.13), we have
\begin{equation}
c_\varepsilon(1) = C_{\varepsilon,1} - \beta e^{\beta} \tilde{C}_{\varepsilon,1}.
\end{equation}

**Proposition 4.5.** There exists a uniform constant $C$ such that
\[ \|\psi_\varepsilon(t)\|_{C^0} \leq C \]
for any $\varepsilon > 0$ and $t \geq 1$.

**Proof:** As in [37], when $t \geq 1$, we let
\begin{equation}
\alpha_\varepsilon(t) = \frac{1}{V} \int_M \dot{\psi}_\varepsilon(t) dV_{\varepsilon,t} = \frac{1}{V} \int_M u_\varepsilon(t) dV_{\varepsilon,t} - c_\varepsilon(t).
\end{equation}
Through computing, we have
\begin{equation}
\frac{d}{dt} \alpha_\varepsilon(t) = \beta \alpha_\varepsilon(t) - \|\nabla \dot{\psi}_\varepsilon\|^2_{L^2},
\end{equation}
\[ e^{-\beta(t-1)}\alpha_\varepsilon(t) = \alpha_\varepsilon(1) - \int_1^t e^{-\beta(s-1)}\|\nabla\psi_\varepsilon\|^2_{L^2} ds \]

(4.18)

\[ = \frac{1}{V} \int_M u_\varepsilon(1)dV_{\varepsilon,1} - c_\varepsilon(1) - \int_1^t e^{-\beta(s-1)}\|\nabla\psi_\varepsilon\|^2_{L^2} ds. \]

Putting (4.13) and (4.14) into (4.18), we have

\[ e^{-\beta(t-1)}\alpha_\varepsilon(t) = \int_t^{+\infty} e^{-\beta(s-1)}\|\nabla\dot{\psi}_\varepsilon\|^2_{L^2} ds. \]

By Theorem 4.4 we conclude that

\[ 0 \leq \alpha_\varepsilon(t) = \int_t^{+\infty} e^{\beta(t-s)}\|\nabla\dot{\psi}_\varepsilon\|^2_{L^2} ds \leq C. \]

Then we conclude that \( \dot{\psi}_\varepsilon(t) \) is uniformly bounded by the uniform Perelman’s estimates when \( t \geq 1 \).

We recall Aubin’s functionals, Ding’s functional and the twisted Mabuchi K-energy functional.

\[ I_{\omega_0}(\phi) = \frac{1}{V} \int_M \phi(dV_0 - d\phi), \]

\[ J_{\omega_0}(\phi) = \frac{1}{V} \int_0^1 \int_M \phi_t(dV_0 - d\phi_t) dt, \]

where \( \phi_t \) is a path with \( \phi_0 = c, \phi_1 = \phi \).

\[ F_{\omega_0}^0(\phi) = J_{\omega_0}(\phi) - \frac{1}{V} \int_M \phi dV_0, \]

\[ F_{\omega_0,\theta}(\phi) = J_{\omega_0}(\phi) - \frac{1}{V} \int_M \phi dV_0 - \frac{1}{\beta} \log \left( \frac{1}{V} \int_M e^{-u_{\omega_0} - \beta \theta} dV_0 \right), \]

\[ M_{\omega_0, \theta}(\phi) = -\beta(J_{\omega_0}(\phi) - J_{\omega_0}(\phi)) - \frac{1}{V} \int_M u_{\omega_0}(dV_0 - d\phi) \]

\[ + \frac{1}{V} \int_M \log \frac{\omega_0^g}{\omega_0^\theta} dV_\theta, \]

where \( u_{\omega_0} \) is the twisted Ricci potential of \( \omega_0 \), i.e. \( -\text{Ric}(\omega_0) + \beta \omega_0 + \theta = \sqrt{-1} \partial \bar{\partial} u_{\omega_0} \) and \( \frac{1}{V} \int_M e^{-u_{\omega_0}} dV_{\omega_0} = 1 \).

**Proposition 4.6.** For any \( t \geq 1 \), the solution \( \psi_\varepsilon(t) \) to equation (4.12) satisfies:

(i) \( M_{\omega_0, \theta}(\psi_\varepsilon(t)) - \beta F_{\omega_0}^0(\psi_\varepsilon(t)) - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(t)dV_{\varepsilon,t} = C_\varepsilon, \)

(ii) \( M_{\omega_0, \theta}(\psi_\varepsilon(1)) \) is uniformly bounded,

where \( C_\varepsilon \) in (i) can be bounded by a uniform constant \( C \).

**Proof:** Following the argument in [34], since

\[ \frac{d}{dt}(M_{\omega_0, \theta}(\psi_\varepsilon(t)) - \beta F_{\omega_0}^0(\psi_\varepsilon(t)) - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(t)dV_{\varepsilon,t}) = 0, \]

(4.25)
we obtain that

\[
\mathcal{M}_{\omega_0, \theta_0}(\psi_\varepsilon(t)) - \beta F_{\omega_0}^0(\psi_\varepsilon(t)) - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(t) dV_{\varepsilon, t} = \mathcal{M}_{\omega_1, \theta_0}(\psi_\varepsilon(1)) - \beta F_{\omega_0}^0(\psi_\varepsilon(1)) - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(1) dV_{\varepsilon, 1}
\]

\[
= \frac{1}{V} \int_M \log \frac{\omega_{\varepsilon}^n(1)(|s|_h^n + \varepsilon^2)^{1-\beta}}{e^{-F_0 \omega_0^n}} dV_{\varepsilon, 1} + \frac{\beta}{V} \int_M \psi_\varepsilon(1) dV_{\varepsilon, 1}
\]

\[
- \frac{1}{V} \int_M F_0 + \log(|s|_h^n + \varepsilon^2)^{1-\beta} dV_0 - \frac{1}{V} \int_M \dot{\psi}_\varepsilon(1) dV_{\varepsilon, 1}.
\]

where the last equality can be bounded by a uniform constant. This gives a proof of (i). Furthermore, by the definition of \( \mathcal{M}_{\omega_0, \theta_0} \), we have

\[
\mathcal{M}_{\omega_0, \theta_0}(\psi_\varepsilon(1)) = \frac{1}{V} \int_M \log \frac{\omega_{\varepsilon}^n(1)(|s|_h^n + \varepsilon^2)^{1-\beta}}{e^{-F_0 \omega_0^n}} dV_{\varepsilon, 1} - \beta I_{\omega_0}(\psi_\varepsilon(1)) + \beta J_{\omega_0}(\psi_\varepsilon(1))
\]

\[
- \frac{1}{V} \int_M F_0 + \log(|s|_h^n + \varepsilon^2)^{1-\beta} dV_0.
\]

Since \( I_{\omega_0}(\psi_\varepsilon(1)) \) is uniformly bounded and \( \frac{1}{n} I_{\omega_0} \leq \frac{1}{n+1} J_{\omega_0} \leq J_{\omega_0} \), we prove (ii). \( \square \)

Using Proposition 4.5 and 4.6, by following the arguments in section 5 of [34], we obtain the following uniform \( C^\infty \) estimate of \( \psi_\varepsilon(t) \) along the equation (4.12) under the assumption that the twisted Mabuchi \( K \)-energy functional \( \mathcal{M}_{\omega_0, \theta_0} \) is uniformly proper on the space

\[
(4.26) \quad \mathcal{H}(\omega_0) = \{ \phi \in C^\infty(M) \mid \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi > 0 \}.
\]

**Theorem 4.7.** Let \( \psi_\varepsilon(t) \) be a solution of the flow (4.12). If the twisted Mabuchi \( K \)-energy functional \( \mathcal{M}_{\omega_0, \theta_0} \) is uniformly proper on \( \mathcal{H}(\omega_0) \), i.e. there exists a uniform function \( f \) such that

\[
(4.27) \quad \mathcal{M}_{\omega_0, \theta_0}(\phi) \geq f(J_{\omega_0}(\phi))
\]

for any \( \varepsilon \) and \( \phi \in \mathcal{H}(\omega_0) \), where \( f(t) : \mathbb{R}^+ \rightarrow \mathbb{R} \) is some monotone increasing function satisfying \( \lim_{t \to +\infty} f(t) = +\infty \), then there exists a uniform constant \( C \) such that for any \( \varepsilon > 0 \) and \( t \geq 0 \)

\[
(4.28) \quad \|\psi_\varepsilon(t)\|_{C^0} \leq C.
\]

We study the flow (4.11) start at \( t = \frac{1}{2} \). We obtain \( C^{-1} \omega_\varepsilon \leq \omega_\varepsilon(\frac{1}{2}) \leq C \omega_\varepsilon \) on \( M \) and \( \|\psi_\varepsilon(\frac{1}{2})\|_{C^4(K)} \leq C_{k,K} \) on \( K \subset M \setminus D \) for some uniform constants \( C \) and \( C_{k,K} \) in section 3, after getting the uniform bound of \( \dot{\psi}_\varepsilon(t) \) and \( \psi_\varepsilon(t) \), we can prove the uniform Laplacian \( C^2 \) estimates and local uniform estimates for any \( t \geq 1 \) and \( \varepsilon > 0 \) by the arguments in [34] (see Proposition 2.1 and 2.3 in [34]). In fact, we prove the following theorem.

**Theorem 4.8.** Under the assumption in Theorem 4.7, for any \( k \in \mathbb{N}^+ \) and \( K \subset M \setminus D \), there exists constant \( C_{k,K} \) depending only on \( \|\varphi_0\|_{L^\infty(M)}, n, \beta, k, \omega_0 \) and \( \text{dist}_{\omega_0}(K, D) \), such that for any \( \varepsilon > 0 \) and \( t \geq 1 \), we have

\[
(4.29) \quad \|\psi_\varepsilon(t)\|_{C^k(K)} \leq C_{k,K}.
\]
Now we assume that there exists a conical Kähler-Einstein metric with cone angle $2\pi\beta$ along $D$. When $\lambda > 0$ and there is no nontrivial holomorphic field which is tangent to $D$ along $D$, Tian-Zhu [47] obtained the following Moser-Trudinger type inequality

$$(4.30) \quad F_{\omega_0, (1-\beta)D}(\phi) \geq \delta J_{\omega_0}(\phi) - C, \quad \forall \phi \in \mathcal{H}(\omega_0)$$

for some constants $\delta$ and $C$, where

$$F_{\omega_0, (1-\beta)D}(\phi) = J_{\omega_0}(\phi) - \frac{1}{V} \int_M \phi dV_0 - \frac{1}{\beta} \log \left( \frac{1}{V} \int_M \frac{1}{|s|^2 e^{(1-\beta)\varepsilon}} dV_0 \right)$$

is defined in [47] (see also [30]).

**Remark 4.9.** When $\lambda \geq 1$, R. Berman [1], Li-Sun [30] proved that there is no nontrivial holomorphic vector field on $M$ tangent to divisor $D$, and Li-Sun also proved that the existence of conical Kähler-Einstein metric can deduce the properness of the Log Mabuchi $K$-energy functional (see also Song-Wang’s results in [43]).

By the definition of $F_{\omega_0, \theta_\varepsilon}$ and $F_{\omega_0, (1-\beta)D}$, we have

$$(4.31) \quad F_{\omega_0, \theta_\varepsilon}(\phi) - F_{\omega_0, (1-\beta)D}(\phi) = \frac{1}{\beta} \log \left( \frac{1}{V} \int_M e^{-F_0 - C_0 - \beta \phi} \frac{dV_0}{|s|^{2(1-\beta)}} \right)
- \frac{1}{\beta} \log \left( \frac{1}{V} \int_M e^{-F_0 - C_\varepsilon - \beta \phi} \frac{dV_0}{(\varepsilon^2 + |s|^2)^{(1-\beta)}} \right)
\geq -C,$$

where $C_0$ and $C_\varepsilon$ are two normalized constants, and $C$ is a constant independent of $\varepsilon$. So the Ding’s functional $F_{\omega_0, \theta_\varepsilon}$ is uniform proper. By the normalization and Jensen’s inequality, we have

$$(4.32) \quad \frac{1}{V} \int_M -u_{\omega_\varepsilon} dV_\varepsilon \leq \log \left( \frac{1}{V} \int_M e^{-u_{\omega_\varepsilon}} dV_\varepsilon \right) = 0.$$

Then we have the following inequalities by (4.23), (4.24) and (4.32).

$$(4.33) \quad M_{\omega_0, \theta_\varepsilon}(\phi) = \beta F_{\omega_0, \theta_\varepsilon}(\phi) + \frac{1}{V} \int_M u_{\omega_\varepsilon} dV_\varepsilon - \frac{1}{V} \int_M u_{\omega_0} dV_0
\geq \beta F_{\omega_0, \theta_\varepsilon}(\phi) - \frac{1}{V} \int_M F_0 + C_\varepsilon + (1 - \beta) \log(\varepsilon^2 + |s|^2) dV_0
\geq \beta F_{\omega_0, \theta_\varepsilon}(\phi) - C,$$

where constant $C$ independent of $\varepsilon$. Hence we deduce the uniform properness of the twisted Mabuchi $K$-energy functional by (4.30), (4.31) and (4.33), i.e.

$$(4.34) \quad M_{\omega_0, \theta_\varepsilon}(\phi) \geq C_1 J_{\omega_0}(\phi) - C_2, \quad \forall \phi \in \mathcal{H}(\omega_0)$$

for some uniform constants $C_1$ and $C_2$. At the same time, we have the uniqueness theorem of conical Kähler-Einstein metric (proved by B. Berndtsson in [2]) under the assumption that there is no nontrivial holomorphic field which is tangent to $D$. Using the above $C^0$ estimate and the uniqueness theorem, we can apply the arguments in section 6 of [54] to obtain the convergence result of the conical Kähler-Ricci flow (1.3), i.e. Theorem 1.3.
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