Possible Zero-Flux Transport induced by Density Waves in a Tube filled with Solid Helium

Kwang-Hua W. Chu [*]
Department of Physics, Xinjiang University, Urumqi 830046, PR China

Abstract

Macroscopic derivation of the entrainment in a supersolid cylinder induced by a surface elastic wave (of small amplitude) propagating along the flexible interface is conducted by considering the nonlinear coupling between the interface and the rarefaction effect. We obtain the critical bounds for zero-volume-flow-rate states corresponding to specific rarefaction measure and wave number which is relevant to the rather small critical velocity or disappearance of supersolid flows reported by Rittner and Reppy.

KEY WORDS : incommensurate quantum crystal, surface phonon, freezing

1 Introduction

In 1969, it was conjectured by Andreev and Lifshitz\(^1\) that at zero temperature, delocalized defects may exist in a quantum solid, as a result of which the number of sites of an ideal crystal lattice may not coincide with the total number of particles. Originally, this conjecture was proposed for three dimensional quantum solids made of atoms (\(^3\)He, \(^4\)He, \(\cdots\)) which do not interact via Coulomb repulsion. The proposed supersolid phase is believed to occur due to the quantum behavior of point defects, namely vacancies and interstitials, in this crystal of bosons\(^2-3\). Researchers have found that a small lattice model does not exhibit the mesoscopic signature of an intermediate phase separating the solid from the liquid, where the solid and the fluid would coexist\(^4\). Such a vacancy-solid phase was indeed suggested\(^1\) by Andreev and Lifshitz if the zero point motions of certain defects become sufficient to form waves propagating inside the solid. Castaing and Nozières have later considered\(^5\) such a possibility for spin polarized \(^3\)He. The statistics of the defects depend on their nature. For simple vacancies in the crystal, their statistics is given by the statistics of the particles out of which the solid is made. If the defects are bosons, they may form a condensate, giving rise to a superfluid coexisting with the solid. This supersolid phase is discussed in certain bosonic models\(^6\). If the defects are fermions, they may form a Fermi liquid\(^7\) coexisting with the solid, such that the system is neither a solid, nor a liquid. Two kinds of motion are possible in it; one possesses the properties of motion in an elastic solid, the second possesses the properties of motion in a liquid. This interesting issue motivates our present study.

Early theoretical work by Andreev and Lifshitz\(^1\) and Chester\(^2\) showed that solids may feature
a Bose-Einstein condensate of vacancies (or interstitial atoms) and thus possess superfluid (SF) properties. Quite recently one description of the quantum solid is as a density wave that has formed in the quantum fluid\textsuperscript{8–9}. The periodicity of this density wave need not match precisely to the particle density, so that the ground state may be incommensurate, with unequal densities of vacancies and interstitials. Whether or not the same is true for quantum fluctuations is not clear at this point. We noticed that previous theories imply a corresponding vacancy contribution to the specific heat that is as large as the phonon contribution near 1 Kelvin\textsuperscript{3,10}. Based on these considerations or phenomenological approaches, assuming the existence of small-amplitude density waves along the deformable boundaries, in this letter, we shall demonstrate that wavy flexible interface (between atoms and free vacancies or defects) or highly-pressured environments\textsuperscript{8} can produce elastically deformed interface or peristaltic motion will induce time-averaged transport in a Andreev-Lifshitz supersolid\textsuperscript{1,2}.

Theoretical studies of interphase nonlocal transport phenomena which appear as a result of a different type of nonequilibrium representing propagation of a surface elastic wave have been performed before\textsuperscript{11−12}. These are relevant to particles flowing along deformable elastic slabs with the dominated parameter being the Knudsen number (Kn = mean-free-path/$L_d$, mean-free-path (mfp) is the mean free path (of the particles) which is thus temperature dependent\textsuperscript{13,14}, $L_d$ is proportional to the characteristic distance between two boundaries)\textsuperscript{13−15}. The role of the Knudsen number is similar to that of the Navier slip\textsuperscript{15} parameter $N_s(= \mu S/L_d$; $S$ is a proportionality constant as $u_s = S \tau$, $\tau$ : the shear stress of the bulk velocity; $u_s$ : the dimensional slip velocity; for a no-slip case, $S = 0$, but for a no-stress condition. $S = \infty$, $\mu$ is the viscosity).

We shall choose a periodic domain to simplify our mathematical treatments. The deformable (elastic) interface is presumed. We adopt the macroscopic or hydrodynamical approach and simplify the original system of equations (related to the momentum and mass transport) to one single higher-order quasi-linear partial differential equation in terms of the unknown stream function. In this study, as the temperature is rather low and the phase is related to the supersolid (there might be weakly friction or shearing dissipation in-between) we shall assume that the governing equations are the incompressible Navier-Stokes equations which will be associated with the microscopically slip velocity boundary conditions along the interfaces\textsuperscript{13−15} (cf. Ref. 13 for the quantum slip case). To consider the originally quiescent environment for simplicity, due to the difficulty in solving a fourth-order quasi-linear complex ordinary differential equation (when the wavy boundary condition are imposed), we can finally get an analytically perturbed solution and calculate those physical quantities, like, time-averaged transport or entrainment, critical forcing corresponding to the freezed or zero-volume-flow-rate states. The latter might be relevant to those reported in Refs. [16-18] for very low flow rates\textsuperscript{16} or disappearance of supersolidity through an annealing of the solid helium sample\textsuperscript{17}. Note that, as also reported in Ref. 18, their results suggested that grain boundaries (GBs) are superfluid, so that $^4$He crystals
of medium quality are supersolid at the liquid/solid equilibrium pressure \( P_m \); that is, mass transport through them without dissipation is possible.

## 2 Formulations

To escape from the difficulties in treating many-body or many-particle problems together with the scattering with an elastic boundary, we adopt the hydrodynamical approach but use the microscopic quantum slip boundary condition which takes into account the mean free path\(^{13}\) of the dilute molecular gas corresponding to the nonzero slip velocity\(^{13}\) along the interface-wall.

We consider a circular cylindrical tube (that of the vortex core) of uniform radius filled with a homogeneous rarefied gas (Newtonian viscous fluid). The wall of the tube (or the interface between the inner vortex core and the outside part of it) is not absolutely rigid, on which is imposed axisymmetric travelling sinusoidal waves of moderate amplitude \( a \) (\( z \) is the axial coordinate in the wave propagation direction). The radial displacement from the mean position of the wall or the interface \( (r = r_w) \) is thus presumed to be \( \eta \), where \( \eta = a \cos[2\pi(z - ct)/\lambda] \), \( \lambda \) is the wave length, and \( c \) the wave speed (cf. Fig. 1). Axisymmetric motion is assumed with \( r \) measured in the direction normal to the mean position of the wall. \( u, v \) are the velocity components in the \( z \)- and \( r \)-directions, respectively.

Firstly, it is necessary to simplify these equations by introducing nondimensional variables. We have a characteristic velocity \( c \) and three characteristic lengths \( a, \lambda, \) and \( r_w \). Thus, the following variables based on \( c \) and \( r_w \) could be introduced:

\[
\begin{align*}
  r' &= \frac{r}{r_w}, & z' &= \frac{z}{r_w}, & u' &= \frac{u}{c}, & v' &= \frac{v}{c}, & \eta' &= \frac{\eta}{r_w}, & \psi' &= \frac{\psi}{c r_w^2}, & t' &= \frac{ct}{r_w}, & p' &= \frac{p}{\rho c^2},
\end{align*}
\]

where \( \psi \) is the stream function, \( \rho, p \) are the density and pressure of the fluid. The amplitude ratio \( \epsilon \) (presumed to be rather small), the wave number \( \alpha \), and the Reynolds number \( Re \) are defined by

\[
\begin{align*}
  \epsilon &= \frac{a}{r_w}, & \alpha &= \frac{2\pi r_w}{\lambda}, & Re &= \frac{c r_w}{\nu}.
\end{align*}
\]

Fig. 1 Schematic diagram of the wavy motion of the interface or tube wall. \( \epsilon \ll 1 \)
We shall seek a solution in the form of a series in the parameter $\epsilon$:

$$\psi = \psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots,$$

$$\frac{\partial p}{\partial z} = \left( \frac{\partial p}{\partial z} \right)_0 + \epsilon \left( \frac{\partial p}{\partial z} \right)_1 + \epsilon^2 \left( \frac{\partial p}{\partial z} \right)_2 + \cdots,$$

with $v = (\partial \psi/\partial z)/r$, $u = -(\partial \psi/\partial r)/r$.

The $r$- and $z$-momentum equations and the equation of continuity$^{19,20}$ could be in terms of the stream function $\psi$ if the pressure ($p$) term is eliminated. The final governing equation is

$$\frac{\partial}{\partial t} \hat{\nabla}^2 \psi + \frac{\psi_z}{r} \left[ \nabla^2 \psi - \frac{2}{r} \hat{\nabla}^2 \psi + \frac{\psi_z}{r^2} \right] - \psi_{rz} \hat{\nabla}^2 \psi_z = \frac{1}{Re} \hat{\nabla}^4 \psi, \quad \nabla^2 \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$$

and subscripts indicate the partial differentiation. Thus, we have

$$\frac{\partial}{\partial t} \hat{\nabla}^2 \psi_0 + \frac{\psi_{0z}}{r} \left[ \nabla^2 \psi_0 - \frac{2}{r} \hat{\nabla}^2 \psi_0 + \frac{\psi_{0z}}{r^2} \right] - \psi_{0rz} \hat{\nabla}^2 \psi_{0z} = \frac{1}{Re} \hat{\nabla}^4 \psi_0, \quad (2)$$

$$\frac{\partial}{\partial t} \hat{\nabla}^2 \psi_1 + \frac{\psi_{1z}}{r} \left[ \nabla^2 \psi_0 - \frac{2}{r} \hat{\nabla}^2 \psi_0 + \frac{\psi_{0z}}{r^2} \right] - \psi_{1rz} \hat{\nabla}^2 \psi_{1z} = \frac{1}{Re} \hat{\nabla}^4 \psi_1, \quad (3)$$

$$\frac{\partial}{\partial t} \hat{\nabla}^2 \psi_2 + \frac{\psi_{2z}}{r} \left[ \nabla^2 \psi_0 - \frac{2}{r} \hat{\nabla}^2 \psi_0 + \frac{\psi_{2z}}{r^2} \right] - \psi_{2rz} \hat{\nabla}^2 \psi_{2z} = \frac{1}{Re} \hat{\nabla}^4 \psi_2, \quad (4)$$

and other higher order forms. The fluid is subjected to boundary conditions imposed by the symmetric motion of the walls and the non-zero slip velocity$^{13,14}$ : $u = -Kn \, du/dr$, $v = \partial \eta/\partial t$ at $r = (1 + \eta)$. The boundary conditions may be expanded in powers of $\eta$ and then $\epsilon$:

$$[(\frac{-1}{r \, \partial r})(\psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots)]|_{r=1+\epsilon \cos \alpha(z-t)} = -Kn[(\frac{1}{r^2 \, \partial r} - \frac{\partial^2}{r \partial r^2})(\psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots)]|_{r=1+\epsilon \cos \alpha(z-t)}, \quad (5)$$

$$\psi_{0z}|_1 + \epsilon[\cos \alpha(z-t)\psi_{0z}|_1 + \psi_{1z}|_1] + \epsilon^2[\psi_{0z}+\frac{1}{2} \psi_{0z}|_1 \cos^2 \alpha(z-t) + \cos \alpha(z-t)\psi_{1z}|_1 + \psi_{2z}|_1] + \cdots = \epsilon \alpha \sin \alpha(z-t) + \epsilon^2 \alpha \cos \alpha(z-t) \sin \alpha(z-t). \quad (6)$$

Equations above, together with the condition of symmetry and a uniform pressure-gradient in the z-direction, $(\partial p/\partial z)_0=$constant, yield:

$$\psi_0 = K_0[r^2 - \frac{r^4}{2}], \quad K_0 = \frac{Re}{8} \frac{\partial p}{\partial z}, \quad (7)$$

$$\psi_1 = \phi(r)e^{i\alpha(z-t)} + \phi^*(r)e^{-i\alpha(z-t)} + \phi_0(r), \quad (8)$$

where the asterisk denotes the complex conjugate. A substitution of $\psi_1$ into equation (3) yields

$$\left\{ \frac{d^2}{dr^2} - \alpha^2 - \frac{d}{dr} + i\alpha Re[1 + 2K_0(1 - r^2)] \right\}\left( \frac{d^2}{dr^2} - \alpha^2 - \frac{d}{dr} \right) \phi = 0, \quad \hat{\nabla}^2 \hat{\nabla}^2 \phi_0 = 0. \quad (9)$$
The boundary conditions are
\[ \phi_r|_1 = 2K_0, \quad \phi_0|_1 = 4K_0 \text{Kn}, \quad \phi(1) = -\frac{1}{2}. \] (10)

The equations for \( \phi^* \) are conjugate to the foregoing and need not be written down. Similarly, with
\[ \psi_2 = D(r) + E(r)e^{i\alpha(z-t)} + E^*(r)e^{-i\alpha(z-t)} + G(r)e^{i\alpha(z-t)} + G^*(r)e^{-i\alpha(z-t)}, \] (11)
we have
\[ (\frac{d^2}{dr^2} - \frac{d}{dr})(\frac{d^2}{dr^2} - \frac{d}{dr})D = i\alpha \text{Re} \frac{d}{dr} \{ \frac{1}{r}[(\phi^*_{rr} - \phi^*_r) - 1/r(\phi^*_r - \phi_r \phi^*)] \} - \frac{i\alpha \text{Re}}{r^2} \]
\[ [\phi^*_{rr} - \phi^*_r] - \frac{1}{r}(r\phi^* - \phi_r \phi^*) \equiv i\alpha \text{Re} \{ \frac{d}{dr} \left( \frac{(S_1 - r^{-1}S_2)}{r} \right) - \frac{(S_1 - r^{-1}S_2)}{r^2} \}. \] (12)
\[ (\frac{d^2}{dr^2} - 4\alpha^2 - \frac{d}{dr})^2 E = -i2\alpha \text{Re}[1 + 2K_0(1 - r^2)](\frac{d^2}{dr^2} - 4\alpha^2 - \frac{d}{dr})E + \]
\[ \frac{i\alpha \text{Re}}{r} [\phi^*_r \phi_{rr} - \phi^*_r \phi_r - 3/2 \phi^*_r + 1/2(\phi^*_r)^2] - \frac{3/2 \phi^*_r}{r} + \frac{2/3 \phi^*_r}{r} + 2/3 \phi^*_r \phi^*_r]; \] (13)
\[ \frac{d^2}{dr^2} - \alpha^2 - \frac{d}{dr} + i\alpha \text{Re}[1 + 2K_0(1 - r^2)] \]
\[ (\frac{d^2}{dr^2} - \alpha^2 - \frac{d}{dr})G = \frac{i\alpha \text{Re}}{r} [\phi^*_0 \phi_{rr} - \]
\[ 3/2 \frac{\phi^*_0}{r} + 3/2 \frac{\phi^*_0}{r^2} + \alpha^2 \phi^*_r - \phi^*_0 \phi^*_r + \frac{\phi^*_0}{r}]. \] (14)
and the boundary conditions
\[ D_r(1) + \frac{1}{2}[\phi_{rr}(1) + \phi^*_r(1)] - 3K_0 = 0, \]
\[ E_r(1) + \frac{1}{2}[\phi_{rr}(1) - \frac{3K_0}{2} = 0, \quad E(1) + \frac{1}{4}\phi_r(1) = -\frac{1}{4}; \] (15)
\[ G_r(1) + \frac{\phi^*_0(1)}{2} = \text{Kn}[4K_0 - \phi_{rr}(1) + \frac{\phi_r(1)}{r}], \quad G(1) = \frac{\phi^*_0(1) - \phi_r(1)}{2} = 0. \] (16)

Here, \( S_1 \equiv S_1(r) = \phi^*_r - \phi^*_r \phi^*, \quad S_2 \equiv S_2(r) = \phi^*_r - \phi^*_r \phi^*. \) The equations for \( E^* \) are conjugate to those for \( E \). We can use these equations to determine the solution up to \( O(\epsilon^2) \). The boundary conditions are not linearized. Equations (14-16) account for the effect of satisfying the velocity-slip condition at the wavy interface, rather at the mean position of the interface.

To illustrate the nature of the solution, particularly with respect to the effects of nonlinearities in convective acceleration and boundary conditions, we shall consider the important case of pumping in the absence of the zeroth-order pressure-gradient; i.e., when \( (\partial p/\partial z)_0 = 0 \). However, a simple superposition of a pressure-driven flow corresponding to equation (7) with \( K_0 \sim O(\epsilon^2) \) is permissible.
2.1 Originally Quiescent Environment

To simplify the approach and obtain preliminary analytical solutions of above complicated equations and boundary conditions, we only consider the case in which \((\partial p/\partial z)_0\) vanishes or \(K_0 = \psi_0 = 0\). \(Kn\) is presumed to be comparable with the order of magnitude \(O(\epsilon)\) and will be rescaled firstly (like \(\hat{Kn}\)) and then adopted by the same representation in the following. Hence equations (9-10) become

\[
\frac{d^2}{dr^2} - \frac{d}{dr} - \alpha^2 \left( \frac{d^2}{dr^2} - \frac{d}{dr} \right) \phi = 0, \quad \tilde{\alpha}^2 = \alpha^2 - i\alpha \text{Re},
\]

\[
\phi_r(1) = 0, \quad \phi(1) = -\frac{1}{2};
\]

\[
\tilde{\nabla}^2 \tilde{\nabla}^2 \phi_0 = 0, \quad \phi_0(1) = 0;
\]

together with the condition that the velocity, i.e., \(\phi_r/r\)|\(r=0\) remains finite along the axis of the tube (at \(r = 0\)). After lengthy algebraic manipulations, we obtain

\[
\phi = Ar I_1(\tilde{\alpha}r) + Br I_1(\alpha r), \quad (20)
\]

where \(Q_0\) is an integration constant; \(A = A_0/det, B = B_0/det; A_0 = \alpha I_0(\alpha)/2, B_0 = -\tilde{\alpha} I_0(\tilde{\alpha})/2;\) and

\[
det = \tilde{\alpha} I_0(\tilde{\alpha}) I_1(\alpha) - \alpha I_0(\alpha) I_1(\tilde{\alpha}).
\]

Meanwhile, equation (12) becomes

\[
D_{rrrr} - \frac{2D_{rrr}}{r} + \frac{3D_{rr}}{r^2} - \frac{3D_r}{r^3} = -\frac{dL(r)}{dr} + \frac{L(r)}{r} = \tilde{S}(r), \quad L(r) \equiv -D_{rrrr} + \frac{D_{rr}}{r} - \frac{D_r}{r^2}, \quad (21)
\]

where \(\tilde{S}(r)\) denotes the right-hand-side (RHS) term of equation (12). Because of equations (12,20), equation (21) can be directly solved, if we denote \(\tilde{S}(r) = -\alpha^2 Re^2 S(r)\), so that we have

\[
D(r) = \int_0^r \tilde{S}(t) t^3 dt + \frac{r^2}{4} \int_0^r \tilde{S}(t) \ t \log(t) dt + \frac{r^4}{16} \int_0^r \tilde{S}(t) \ t^2 dt - \frac{\log(r)r^2}{4} \int_0^r \tilde{S}(t) dt + b_4 + b_3 r^4 + b_2 r^2 \log(r) + b_1 r^2, \quad (22)
\]

where \(b_i, i = 1, 2, 3, 4\) are integration constants. To obtain a simple solution which relates to the mean flow so long as only terms of \(O(\epsilon^2)\) are concerned, we see that if every term in the \(z\)-momentum equation is averaged over an interval of time equal to the period of oscillation [9], we obtain for our solution as given by above equations the mean pressure gradient

\[
\frac{\partial p}{\partial z} = \epsilon \left( \frac{\partial p}{\partial z_1} \right) + \epsilon^2 \left( \frac{\partial p}{\partial z_2} \right) + O(\epsilon^3) = \frac{\epsilon}{r \text{Re}} \left[ -\phi_{0rr} + \phi_{0r} \frac{1}{r} - \phi_{0} \frac{1}{r^2} \right] + \epsilon^2 \left\{ \frac{1}{r \text{Re}} \left( -D_{rrrr} + \frac{D_{rr}}{r} \right) \right\} \left( \phi \phi_r^* - \frac{1}{r} (\phi \phi_r^*) \right) + O(\epsilon^3).
\]

\[
(23)
\]
Thus, as far as the mean flow is concerned, $D$ is the only term which participates in the solution as long as only terms of $O(\epsilon)$ are retained. We have no need to consider $E$ when considering the mean free flow. Actually, with equations (12,21) and the rearrangement of both sides, we have
\[ \frac{\partial p}{\partial z} \bigg|_1 = \frac{4Q_0}{Re}, \quad \frac{\partial p}{\partial z} \bigg|_2 = \frac{P_0}{Re}, \]
where $P_0$ is an integration constant by considering the recombination of equations (12,21) and the integration hereafter (cf. equation (23)).

Now, from equation (14), we have
\[ D_r(1) = -\frac{1}{2}[\phi_{rr}(1) + \phi^*_{rr}(1)] \] (25)
here, we denote $\zeta = -[\phi_{rr}(1) + \phi^*_{rr}(1)]/2$.

From equation (22) and differentiating $D$ once, we obtain
\[ D_r(r) = (a_1 + a_2)r + 2a_2 r \ln r + a_0 r^3 + \bar{G}(r), \quad \bar{G}(r) \equiv -\alpha^2 Re^2 G(r), \] (26)
where $a_2$ must be zero as the axial velocity which is proportional to $D_r/r$ remains finite at $r = 0$.

Besides, from the expression of $D(r)$, we know that $\bar{G}(r)/r = 0$ at $r = 0$.

From equations (22-23,25), we know that $P_0 = -4a_0$. $a_0$ or $P_0$ is then proportional to the time-averaged pressure-gradient (on the axis) accompanying the peristaltic motion and now
\[ D_r(r) = [\zeta - \bar{G}(1)]r + a_0 [r^3 - r] + \bar{G}(r). \] (27)

With $a_0$ or $P_0$ specified, the solution for the mean axial velocity (averaged over time) is
\[ \bar{u} \equiv U(r) = -\epsilon \frac{\phi_{0r}}{r} - \epsilon^2 \frac{D_r}{r} = \epsilon[ -Q_0(1 - r^2) ] + \epsilon^2 \{ -[\zeta - \bar{G}(1)] - \frac{\bar{G}(r)}{r} + a_0(1 - r^2) \} = \epsilon[ -Q_0(1 - r^2) ] + \epsilon^2 \{ -R_0 + \bar{G}(1) - \frac{\bar{G}(r)}{r} + a_0(1 - r^2) \}. \] (28)

In practical applications we must determine $a_0$ from considerations of conditions at the ends of the channel. As usual, a critical reflux condition can be defined as the condition where there is zero velocity at the center of the tube. Using equations (22-23,27), we have
\[ \frac{\partial p}{\partial z} \bigg|_{cr} = \frac{4}{Re}[ -R_0 + \bar{G}(1)], \] (29)
where $R_0$ becomes $\zeta$ (cf. Ref. 19) as $Kn=0$ and is due to the nonlinear effect of the boundary condition.

### 3 Results and Discussion

Our numerical calculations confirm that the mean streamwise velocity distribution (averaged over time) due to the induced motion by the wavy elastic interface in the case of free (vacuum) pumping is dominated by $R_0$ (or $Kn$) and the parabolic distribution $-P_0(1 - r^2)$.
defines the boundary value of $D_r$ has its origin in the $y$-gradient of the first-order streamwise velocity distribution. Note that the Reynolds number here is based on the wave speed. The physical trend herein is also the same as those reported before$^{13−14,20}$ for the slip-flow effects. The slip produces decoupling with the inertia of the wavy interface.

Now, let us define a critical reflux condition as one for which the mean velocity $U(r)$ equals to zero at the center-line $r = 0$. With the equation of $U$, we have $P_{0,c,r} = Re(\partial p/\partial z)_2 = -4[\alpha^2 Re^2 G(1) + R_0]$, which means the critical reflux condition is reached when $P_0$ has above value. Pumping against a positive forcing greater than the critical value would result in a backward transport (reflux) in the central region of the stream. This critical value depends on $\alpha$, $Re$, and $Kn$. There will be no reflux if the pressure gradient is smaller than this $P_0$. Thus, for some $P_0$ values less than $P_{0,c,r}$, the superflow will keep moving forward. On the contrary, parts of the flow will move backward if $P_0 > P_{0,c,r}$.

As reported in Refs. 8 or 17, the rather small critical velocity ($\leq 20 \mu m/s$) observed shows an apparent dissipation or attenuation of the superflow. Thus, we present some of the values of $P_0(\alpha, Re; Kn = 0,0.15)$ corresponding to freezeed or zero-volume-flow-rate states (cf. Chu in Ref. 15) ($\int_0^1 U(r)dr = 0$) in Table 1 where the wave number ($\alpha$) has the range : $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and $0.9$; the Reynolds number ($Re$) $= 0.1, 1, 10, 25, 50$ and $100$. These freezeed states might be similar to that reported in Ref. 16 (the nonclassical rotational inertia signal (NCRI) is not a universal property of solid $^4$He but can be eliminated through an annealing of the solid helium sample).

We observe that as $Kn$ increases from zero to $0.15$, the critical $P_0$ decreases significantly (cf. Fig. 2). The quantum slip parameter$^{13}$ which is represented in terms of $Kn$ (via the mean free path$^{13,14}$) is temperature dependent in essence but is only relaxed in the boundaries (weakly compressible cases here). This kind of boundary-dominated flows resemble those reported in Ref. 18 by Sasaki et al. For the same $Kn$, once $Re$ is larger than $10$, critical reflux values $P_0$ drop rapidly and the wave-modulation effect (due to $\alpha$) appears. The latter observation might be interpreted as the strong coupling between the interface and the inertia of the streaming superflow. The illustration of the velocity fields for those zero-flux (zero-volume-flow-rate) or freezeed states are shown in Figure 3. There is one wave number : $\alpha = 0.8$ and the Reynolds number is $10$. Both no-slip and slip ($Kn=0.15$) cases are presented.

Some remarks could be made about these states : the transport being freezeed in the time-averaged sense for specific dissipations (in terms of Reynolds number which is the ratio of wave-inertia and viscous shearing effects) and wave numbers (due to the wavy interface or other fluctuations) for either no-slip and slip cases. These resemble those reported quite recently$^{16−18}$. Meanwhile, the time-averaged transport induced by the wavy interface is proportional to the square of the amplitude ratio (although the small amplitude waves being presumed), as can be seen in Eqn. (11), which is qualitatively the same as that presented in Ref. 11 for analogous
interfacial problems.
In brief summary, the entrained transport (either positive or negative and there is possibility: freezing) due to the wavy interface is mainly tuned by the $P_0$ for fixed Re. Meanwhile, $P_{0cr}$ depends strongly on the Knudsen number (Kn, a rarefaction measure) instead of Re or $\alpha$. These results (cf. Table 1 and Fig. 2) might explain why there are rather small critical velocities for superflows in the temperature range (similar to the effect represented by the quantum slip parameter: Kn here) where a supersolidity is observed\textsuperscript{[8,17]} or the disappearance of supersolidity through an annealing of the solid helium sample\textsuperscript{[16]}. We shall investigate much more complicated problems\textsuperscript{[21,22]} in the future.

References

\*[1] Correspondence after 2007-Aug-30 : 24, Lane 260, Section 1, Road Mucha, Taipei 11646, Taiwan, R. China and P.O. Box 39, Tou-Di-Ban, Road XiHong, Urumqi 830000, PR China.

[1] A.F. Andreev and I.M. Lifshitz, Zh. Eksp. Teor. Fiz. (Sov. Phys. JETP) 29, 1107 (1969).
[2] G.V. Chester, Phys. Rev. A 2, 256 (1970).
[3] P.W. Anderson, W.F. Brinkman, and D.A. Huse, Science 310, 1164 (2005).
[4] G. Katomeris, F. Selva, and J.-L. Pichard, Eur. Phys. J. B 31, 401 (2003). Z.Á. Németh and J.-L. Pichard, Eur. Phys. J. B 33, 87 (2003).
[5] B. Castaing and P. Nozières, J. Phys. France 40, 257 (1979)
[6] G.G. Batrouni, R.T. Scalettar, Phys. Rev. Lett. 84, 1599 (2000).
[7] I.E. Dzyaloshinskii, P.S. Kondratenko, V.S. Levchenkov, Sov. Phys. JETP 35, 823 (1972); ibid. 35, 1213 (1972).
[8] A.C. Clark and M.H.W. Chan, J. Low Temp. Phys. 138, 853 (2005). E. Kim and M.H.W. Chan, J. Low Temp. Phys. 138, 859 (2005).
[9] T. Leggett, Science 305, 1921 (2004). Z.Á. Németh and J.-L. Pichard, Eur. Phys. J B 45, 111 (2005). A.T. Dorsey, P.M. Goldbart and J. Toner, Phys. Rev. Lett. 96, 055301 (2006).
[10] C.A. Burns and J.M. Goodkind, J. Low Temp. Phys. 95, 695 (1994).
[11] V.D. Borman, S.Yu. Krylov, and A.M. Kharitonov, Sov. Phys. JETP 65, 935 (1987).
[12] M.S. Longuet-Higgins, Philos. Trans. R. Soc. London 345, 535 (1953). K.-H. W. Chu, J. Phys. A : Math. General. 36, 5817 (2003). K.-H. W. Chu, Eur. Phys. J Appl. Phys. 17 131 (2002).
[13] D. Einzel and J.M. Parpia, J. Low Temp. Phys. 109, 1 (1997).

[14] M.N. Kogan, Rarefied Gas Dynamics (Plenum Press, New York, 1969).

[15] P.G. de Gennes, Langmuir 18, 3013 (2002). A. K.-H. Chu, Biorheology 42(1/2), 116 (2005).

[16] A.S.C. Rittner and J.D. Reppy, Phys. Rev. Lett. 97, 165301 (2006).

[17] E. Kim and M.H.W. Chan, Phys. Rev. Lett. 97, 115302 (2006).

[18] S. Sasaki, R. Ishiguro, F. Caupin, H. J. Maris and S. Balibar, Science 313, 1098 (2006).

[19] L.D. Landau and E.M. Lifshitz, Fluid Mechanics (Pergamon Press, London, 1959). L.E. Malverm, Introduction to the Mechanics of a Continuous Medium (Prentice-Hall, Englewood Cliffs, NJ., 1969).

[20] K.-H. W. Chu, Phys. Scr. 65, 283 (2002). K.-H. W. Chu, Eur. Phys. J. AP 13, 147 (2001). K.-H. W. Chu, Preprint (2006).

[21] I.A. Todoshchenko, H. Alles, J. Bueno, H.J. Junes, A.Ya. Parshin and V. Tsepelin, Phys. Rev. Lett. 97, 165302 (2006). P. Noziéres, J. Low Temp. Phys. 142, 91 (2006).

[22] P.-G. de Gennes, C. R. Physique 7, 561 (2006).
Fig. 2 Demonstration of Kn, Re and \( \alpha \) effects on the \( \Pi_0 \) (zero-flux states). 

Re is the Reynolds number (the ratio of the wave-inertia and viscous shearing dissipation). 

\( \alpha \) is the wave number and Kn is the Knudsen number (a rarefaction measure).
Fig. 3 Demonstration of the zero-flux states: the mean velocity field $U(r)$ for wave numbers $\alpha = 0.8$. The Reynolds number is 10. $Kn$ is the rarefaction measure (the mean free path of the particles divided by the characteristic length). The integration of $U(r)$ w.r.t. $r$ for these velocity fields gives zero volume flow rate.
Table 1: Zero-flux or freezed states values \( (P_0) \) for a cylindrical wavy interface.

| Kn | \( \alpha \) | 0.1  | 0.2  | 0.4  | 0.5  | 0.6  | 0.8  | 0.9  |
|----|-------------|------|------|------|------|------|------|------|
| 0  |             | 1    | 10   | 25   | 50   | 100  |      |      |
| 0  | 0.1         | 24.2720 | 24.2545 | 24.1846 | 23.9435 | 23.0755 |      |      |
|    | 0.2         | 24.3293 | 24.3271 | 24.2753 | 24.0009 | 23.1347 | 20.8498 |      |
|    | 0.4         | 24.5712 | 24.5731 | 24.3596 | 23.3741 | 21.0765 | 17.8457 |      |
|    | 0.5         | 24.7615 | 24.7573 | 24.4307 | 23.0026 | 20.1985 | 17.1345 |      |
|    | 0.6         | 24.9884 | 24.9855 | 24.5116 | 22.6346 | 19.5060 | 16.7273 |      |
|    | 0.8         | 25.5830 | 25.5745 | 24.7686 | 22.0000 | 18.6721 | 16.4353 |      |
|    | 0.9         | 25.9523 | 25.9414 | 24.9380 | 21.8046 | 18.4807 | 16.4546 |      |
| 0.15| 0.1        | 11.9953 | 11.9945 | 11.9824 | 11.9298 | 11.7435 | 11.1254 |      |
|    | 0.2        | 12.0359 | 12.0359 | 11.9906 | 11.7865 | 11.1632 | 9.7550 |      |
|    | 0.4        | 12.2082 | 12.2066 | 12.0421 | 11.3259 | 9.8944 | 8.2432 |      |
|    | 0.5        | 12.3385 | 12.3366 | 12.0825 | 11.0743 | 9.4247 | 7.9058 |      |
|    | 0.6        | 12.4980 | 12.4952 | 12.1336 | 10.8404 | 9.0827 | 7.6943 |      |
|    | 0.8        | 12.9223 | 12.9152 | 12.2967 | 10.4917 | 8.7025 | 7.4861 |      |
|    | 0.9        | 13.1854 | 13.1763 | 12.4088 | 10.3877 | 8.6208 | 7.4413 |      |