A TOPOLOGICAL INTERPRETATION OF VIRO’S
\textit{gl}(1|1)-ALEXANDER POLYNOMIAL OF A GRAPH

YUANYUAN BAO

Abstract. This is a sequel to [2]. For an oriented trivalent graph \( G \) without source or sink embedded in \( S^3 \), we prove that the \textit{gl}(1|1)-Alexander polynomial \( \Delta(G, c) \) defined by Viro satisfies a series of relations, which we call MOY-type relations in [2]. As a corollary we show that the Alexander polynomial \( \Delta_{(G,c)}(t) \) studied in [2] coincides with \( \Delta(G, c) \) for a positive coloring \( c \) of \( G \), where \( \Delta_{(G,c)}(t) \) is constructed from certain regular covering space of the complement of \( G \) in \( S^3 \) and it is the Euler characteristic of the Heegaard Floer homology of \( G \) that we studied before. When \( G \) is a planar graph, we provide a topological interpretation to the vertex state sum of \( \Delta_{(G,c)}(t) \) by considering a special Heegaard diagram of \( G \) and the Fox calculus on the Heegaard surface.

1. Introduction

The Alexander polynomial of a knot was first studied by J. W. Alexander in 1920s. It is a well-known fact that it can be interpreted in several different ways. For example, it can be defined via the universal abelian covering space of the knot complement. There is a definition for it from the classical Burau representation of the braid group. By applying the representation theory of quantum groups (see J. Murakami [6, 5], Kauffman and Saleur [3], Rozansky and Saleur [9], and Reshetikhin [8]), people found that it is also a \textit{gl}(1|1)- and \textit{sl}(2)-quantum invariant. In recent years this invariant has come to draw a lot of attention for the reason that it is the Euler characteristic of the knot Floer homology introduced by Ozsváth and Szabó, and Rasmussen independently.

Unlike the Alexander polynomial of a knot, there is no standard definition for the Alexander polynomial of a spatial graph. In [2], we studied a version of it. For an oriented trivalent graph \( G \) without source or sink embedded in \( S^3 \) and a positive coloring defined on it, we constructed a polynomial \( \Delta_{(G,c)}(t) \). It was originally studied in [1], where we studied the Heegaard Floer homology for a balanced bipartite graph with a proper orientation and showed that a multi-variable version of \( \Delta_{(G,c)}(t) \) is the Euler characteristic of the homology.

In this paper, we show that Viro’s \textit{gl}(1|1)-Alexander polynomial of a graph \( \Delta(G, c) \) defined in [10] coincides with \( \Delta_{(G,c)}(t) \). We use two approaches to analyze the coincidence. The first approach is based on MOY-type relations. In [2] we showed that \( \Delta_{(G,c)}(t) \) satisfies a series of relations, which we call MOY-type relations since they are analogous to MOY’s relations in [4], and proved that these MOY-type relations characterize \( \Delta_{(G,c)}(t) \). In Section 2 we prove that \( \Delta(G, c) \) satisfies an adapted version of MOY-type relations. By comparing these relations with those for \( \Delta_{(G,c)}(t) \), in Section...
3 we provide a precise relation between \( \Delta(G, c) \) and \( \Delta(G, c)(t) \). The other approach is based on a special Heegaard diagram of \( G \) when \( G \) is a planar graph. In this case we show that the morphism around a trivalent vertex, which determines \( \Delta(G, c) \), can be obtained from the Fox calculus on the Heegaard diagram. The morphism was originally obtained by scaling the Clebsch-Gordan morphisms for irreducible \( U_q(gl(1|1)) \)-modules of dimension \((1|1)\).

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2. The MOY-type relations for Viro’s \( gl(1|1) \)-Alexander polynomial

2.1. Viro’s \( gl(1|1) \)-Alexander polynomial. Viro [10] defined a functor from the category of colored framed trivalent graph to the category of finite dimensional modules over the \( q \)-deformed universal enveloping algebra \( U_q(gl(1|1)) \). The \( gl(1|1) \)-Alexander polynomial is constructed from the functor. Instead of recalling a full definition of the functor, we recall the definition and calculation of the polynomial.

Viro defined the Alexander polynomial for any oriented trivalent graph. Here we restrict our discussion to those graphs without source or sink. Let \( G \) be an oriented trivalent graph without source or sink, and let \( E \) be the set of edges and \( V \) be the set of vertices of \( G \). Consider a map which we call a coloring

\[
c : E \to \mathbb{Z} \setminus \{0\} \oplus \mathbb{Z} \quad e \mapsto (j, J).
\]

The first number \( j \) is called the *multiplicity* and the second number \( J \) is called the *weight*. Around a vertex, suppose the three edges adjacent to it are colored by \((j_1, J_1)\), \((j_2, J_2)\) and \((j_3, J_3)\). Let \( \epsilon_i = -1 \) if the \( i \)-th edge points toward the vertex and \( \epsilon_i = 1 \) otherwise. The coloring \( c \) needs to satisfy the following conditions, which are called admissible conditions in [10].

**Admissible conditions**

\[
\sum_{i=1}^{3} \epsilon_i j_i = 0,
\]

\[
\sum_{i=1}^{3} \epsilon_i J_i = -\prod_{i=1}^{3} \epsilon_i.
\]

The pair \((j, J)\) corresponds to two irreducible \( U_q(gl(1|1)) \)-modules of dimension \((1|1)\), which are denoted by \((j, J)_+\) and \((j, J)_-\). These two modules are dual to each other. The module \((j, J)_+\) (resp. \((j, J)_-\)) is generated by two elements \( e_0 \) (boson) and \( e_1 \) (fermion). For details of their definitions please see Appendix 1 of [10].

A **framing** of \( G \) is an embedded compact surface \( F \subset S^3 \) in which \( G \) is sitting as a deformation retract. More precisely, in \( F \) each vertex of \( G \) is replaced by a disk where the vertex is the center, and each edge of \( G \) is replaced by a strip \([0, 1] \times [0, 1]\) where \([0, 1] \times \{0, 1\}\) is attached to the boundaries of its adjacent vertex disks and \( \{\frac{1}{2}\} \times [0, 1] \) is the given edge of \( G \). See Fig. 1 for an example.

A **framed graph** is a graph with a framing. By an **isotopy** of a framed graph we mean an isotopy of the graph in \( S^3 \) which extends to an isotopy of the framing. A graph diagram of \( G \) in \( \mathbb{R}^2 \) can be equipped with a framing such that the tubular neighborhood
of the graph diagram in $\mathbb{R}^2$ is an immersion of the framing. It is called the blackboard framing.

The difference between a generic framing and the blackboard framing can be represented on the diagram by introducing the half-twist symbols $\leftarrow$ and $\rightarrow$, which mean a positive half twist and a negative half twist respectively at the fragment. Therefore we can use a graph diagram with $\leftarrow$ and $\rightarrow$ to represent a framed graph, as shown in Fig. 1.

Now we review the definition of $\Delta(G,c)$ for a framed graph $G$ with a coloring $c$. Choose a graph diagram of $G$ in $\mathbb{R}^2$. The diagram divides $\mathbb{R}^2$ into several regions, one of which is unbounded. Choose an edge of $G$ on the boundary of the unbounded region and cut the edge at a generic point. Suppose the color of the edge is $(j,J)$. Deform the graph diagram under isotopies of $\mathbb{R}^2$ to make it in a Morse position under a given orthogonal coordinate system of $\mathbb{R}^2$ so that the two endpoints created by cutting have heights zero and one and the critical points, the crossings, the half-twist symbols and the vertices of the diagram have different heights between zero and one. Namely after deformation the diagram can be divided into several pieces by horizontal lines so that each piece is a disjoint union of trivial vertical segments with one of the eight elements in Fig. 2. Each piece connects a sequence of endpoints on its bottom to a sequence of endpoints on its top.

Under Viro’s functor, each piece is mapped to a morphism between tensor products of irreducible $U_q(gl(1|1))$-modules of dimension $(1|1)$. Suppose the sequence of endpoints for a given piece on the bottom (resp. top) is $(p_1,\cdots,p_k)$ for $k \geq 1$ (resp. $(q_1,\cdots,q_l)$ for $l \geq 1$) where the subindices respect the $z$-coordinates of the endpoints. Then $(p_1,\cdots,p_k)$ (resp. $(q_1,\cdots,q_l)$) corresponds to the tensor product $(j_1,J_1)_{e_1} \otimes \cdots \otimes (j_k,J_k)_{e_k}$ (resp. $(i_1,I_1)_{e_1} \otimes \cdots \otimes (i_l,I_l)_{e_l}$), where $(j_r,J_r)$ (resp. $(i_s,I_s)$) is the color of the edge containing $p_r$ (resp. $q_s$) and $e_r = +$ (resp. $e_s = -$) when the edge points downward and $e_r = -$ (resp. $e_s = +$) otherwise for $1 \leq r \leq k$ (resp. $1 \leq s \leq l$).

The morphism is defined in the language of Boltzmann weights. Simply speaking, each module $(j_r,J_r)$ (resp. $(i_s,I_s)$) has two generators $e_0$ (boson) and $e_1$ (fermion), and therefore $(j_1,J_1)_{e_1} \otimes \cdots \otimes (j_k,J_k)_{e_k}$ (resp. $(i_1,I_1)_{e_1} \otimes \cdots \otimes (i_l,I_l)_{e_l}$) is generated by $\{ \otimes_{r=1}^k e_{\delta_r} \}_{\delta_r=0,1}$ (resp. $\{ \otimes_{s=1}^l e_{\delta_s} \}_{\delta_s=0,1}$). The morphism is represented by a matrix under the above choice of generators, and the Boltzmann weights are the entries of the matrix.

The composition of two pieces (attaching them by identifying the top of one piece with the bottom of the other piece) corresponds to the composition of their morphisms.
for $U_q(gl(1|1))$-modules. As a consequence, the graph diagram in Morse position with two endpoints of heights zero and one is mapped to a morphism from $(j,J)_+$ to $(j,J)_+$ (or $(j,J)_-$ to $(j,J)_-$ depending the orientation of $G$ at the endpoints), which acts as the multiplication of a rational function on $q$ ([10, 5.1.A]). Recall that $(j,J)$ is the color of the edge which was cut. Then dividing the rational function by $q^{2j} - q^{-2j}$ we get $\Delta(G,c)$. To calculate $\Delta(G,c)$, we can take the following steps provided by Viro in [10, Section 6.4].

**Proposition 2.1** (vertex state sum representation in [10]). $\Delta(G,c)$ can be calculated as follows.
Choose an orthogonal coordinate system for $\mathbb{R}^2$ and consider a graph diagram of $G$ in a Morse position in $\mathbb{R}^2$. Choose a generic point $\delta$ on the leftmost edge of $G$, which we call initial point. Suppose the color of the edge is $(j, J)$.

(ii) Assign the generator $e_0$ to the edge with initial point, and assign $e_0$ or $e_1$ to each of the other edges. Such an assignment is called a state. In a state, if an edge is assigned with $e_0$ (resp. $e_1$), we represent it by a dotted (resp. solid) line, as in Fig. 3.

(iii) For each state, take the product of the Boltzmann weights at the critical points of the diagram, the crossings, the half-twist symbols, and the vertices of the graph. The Boltzmann weights for all possible assignments are defined in Tables 1 and 2 of [10]. See Tables 1 and 2 for part of the data which we will use later.

(iv) Take the sum of the products for all the states. Multiplying the sum by $\frac{q^{(-1)^{\theta_2}}}{q^{\theta_1} - q^{-\theta_1}}$ we get $\Delta(G, c)$, where $\theta = 0$ (resp. 1) if the edge segment around $\delta$ points downward (resp. upward).

Remark 2.2. Viro defined a multi-variable version of the Alexander polynomial, where each edge is colored by an irreducible module over a subalgebra $U^1$ of $U_q(gl(1|1))$. Here we consider a single variable version. In this case it suffices to consider $U_q(gl(1|1))$-modules, as we described above, and the Boltzmann weights for the single variable version are given in Tables 1 and 2 of [10].

2.2. MOY-type relations for $\Delta(G, c)$. In this section, we discuss a series of relations that $\Delta(G, c)$ satisfies, which are adapted versions of the MOY-type relations in [2]. These relations were inspired by Murakami-Ohtsuki-Yamada’s work in [4], where they provided a graphical definition for the $U_q(sl_n)$-polynomial invariants of a link for any $n \geq 2$.

Before stating the relations, we make the following agreement.

- If several diagrams are involved in one relation, they are identical outside the local diagrams drawn in the relation.
- If the orientations of some edges are omitted in a relation, they may be recovered in any consistent way that does not create source or sink.
- If the multiplicities or weights are omitted in a relation on some edges, they may be recovered in any consistent way that respects the admissible conditions.
- In the definition of the coloring $c$, the multiplicity of an edge is not allowed to be zero. In the following relations, if zero appears as the multiplicity of an edge which is entirely included in a local diagram, we formally define the Alexander polynomial as below, where $\{k\}_q = q^k - q^{-k}$ for $k \in \mathbb{Z}\setminus\{0\}$. 
Table 2. Boltzmann weights at a vertex from Viro’s Table 2.
A TOPOLOGICAL INTERPRETATION OF VIRO’S $gl(1|1)$-ALEXANDER POLYNOMIAL

\[
\begin{pmatrix}
  i & j \\
  j & 0 \\
  i & j
\end{pmatrix} := \{2j\}_q \cdot \begin{pmatrix}
  i & j \\
  i & j
\end{pmatrix}, \quad \begin{pmatrix}
  i & j \\
  j & 0 \\
  i & j
\end{pmatrix} := \{2i\}_q \cdot \begin{pmatrix}
  i & j \\
  i & j
\end{pmatrix}.
\]

**Theorem 2.3.** Viro’s $gl(1|1)$-Alexander polynomial $\Delta(G, c)$ satisfies the following relations, where $(D)$ represents $\Delta(D, c)$.

(i) \( \begin{pmatrix}
  i \\
  i
\end{pmatrix} = \frac{1}{\{2i\}_q}. \)

(ii) \( (D) = 0 \) if $D$ is a disconnected diagram.

(iii) \( \begin{pmatrix}
  i & I \\
  i & I
\end{pmatrix} = q^{-iI} \cdot \begin{pmatrix}
  i & i \\
  i & i
\end{pmatrix}, \quad \begin{pmatrix}
  i & I \\
  j & I
\end{pmatrix} = q^{iI} \cdot \begin{pmatrix}
  i & i \\
  j & j
\end{pmatrix}. \)

(iv) \( \begin{pmatrix}
  i \\
  i
\end{pmatrix} = \{2i\}_q \cdot \begin{pmatrix}
  i \\
  i
\end{pmatrix}. \)

(v) \( \begin{pmatrix}
  j & i + j \\
  i & j + i \\
  j & i + j
\end{pmatrix} = \{2i + 2j\}_q \cdot \begin{pmatrix}
  j & i \\
  j & i \\
  j & j
\end{pmatrix} + \{2j\}_q^2 \cdot \begin{pmatrix}
  i - j \\
  j \\
  i
\end{pmatrix}. \)

(vi) \( \begin{pmatrix}
  i & j & k \\
  i + j & k \\
  i + j + k
\end{pmatrix} = \frac{\{2i + 2j\}_q}{\{2j + 2k\}_q} \cdot \begin{pmatrix}
  i & j & k \\
  j + k \\
  i + j + k
\end{pmatrix}. \)
\[
\begin{align*}
(i + j + k) & (i + j) = (i + j + k) \\
\begin{pmatrix}
 i + k - l & j + l - k \\
 i + k & j - k \\
 i & j \\
\end{pmatrix} & = \frac{\{2j\}_q\{2l\}_q}{\{2j + 2l - 2k\}_q} \cdot \begin{pmatrix}
 i + k - l & j + l - k \\
 i & j \\
\end{pmatrix} \\
\begin{pmatrix}
 j & (i, I) \\
\end{pmatrix} & = q^{j - i - j + j - l} \frac{\{2i\}_q}{\{2j\}_q} \cdot \begin{pmatrix}
 (j, J) & (i, I) \\
 (i, I) & (j, J) \\
\end{pmatrix} + q^{-i - j - i - j - l} \frac{\{2i\}_q}{\{2j\}_q} \cdot \begin{pmatrix}
 (j, J) & (i, I) \\
 (i, I) & (j, J) \\
\end{pmatrix} \\
\begin{pmatrix}
 j & (i, I) \\
\end{pmatrix} & = q^{j - i + j + j - l} \frac{\{2i\}_q}{\{2j\}_q} \cdot \begin{pmatrix}
 (j, J) & (i, I) \\
 (i, I) & (j, J) \\
\end{pmatrix} + q^{i + j + i + j + l} \frac{\{2i\}_q}{\{2j\}_q} \cdot \begin{pmatrix}
 (j, J) & (i, I) \\
 (i, I) & (j, J) \\
\end{pmatrix} \\
\begin{pmatrix}
 j & (i, I) \\
\end{pmatrix} & = \begin{pmatrix}
 j & (i, I) \\
\end{pmatrix} \\
\begin{pmatrix}
 j & (i, I) \\
\end{pmatrix} & = \begin{pmatrix}
 j & (i, I) \\
\end{pmatrix}
\end{align*}
\]
that the morphisms between relations can be proved by a straightforward application of the Boltzmann weights in weights do not vanish. the same relations. Proofs for the remaining relations are left to the interested readers.

Relations (i), (ii), (iii) and (iv) were proved in [10, Sections 5, 9]. The other proofs.

Proof. Relations (i), (ii), (iii) and (iv) were proved in [10]. The other relations can be proved by a straightforward application of the Boltzmann weights in Tables [1] and [2]. We prove (v), (vii) and the first relation of (viii). It is enough to show that the morphisms between $U_q(gl(1|1))$-modules defined by the local diagrams satisfy the same relations. Proofs for the remaining relations are left to the interested readers.

(v) The left-hand diagram has the following local states on which the Boltzmann weights do not vanish.

Therefore the morphism defined by the left-hand diagram is as below. For simplicity we omit the subindex $q$ from the expression $\{k\}_q$.

$$
\begin{array}{l}
e_0 \otimes e_0 \mapsto q^{-2i}e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_0 \\
\quad \mapsto q^{-2i}([2i + 2j]q^{2i})e_0 \otimes e_0 + [2j]e_1 \otimes e_1 \\
\quad \mapsto q^{-2i}([2i + 2j]q^{2i})e_0 \otimes e_0 + (-q^{2(i+j)})e_0 \otimes e_1 \otimes e_1 + [2j]e_1 \otimes e_1 \otimes e_0 \\
\quad \mapsto \{2i + 2j\}[\{2i + 2j\}e_0 \otimes e_0 - q^{2(i+j)}\{2j\}e_1 \otimes e_1] - \{2j\}\{2i\}q^{-2(i+j)}e_0 \otimes e_0 \\
\quad = \{2i + 2j\}^2 - \{2i\}\{2j\}q^{-2(i+j)}e_0 \otimes e_0 - \{2i + 2j\}\{2j\}q^{2(i+j)}e_1 \otimes e_1; \\
\end{array}
$$

$$
\begin{array}{l}
e_0 \otimes e_1 \mapsto e_1 \otimes e_1 \otimes e_1 \\
\quad \mapsto \{2i\}e_1 \otimes e_0 \\
\quad \mapsto \{2i\}(-q^{2(i+j)})e_1 \otimes e_1 \otimes e_1 \\
\quad \mapsto \{2i\}(-q^{2(i+j)})(-\{2i\}q^{-2(i+j)})e_0 \otimes e_1 = \{2i\}^2e_0 \otimes e_1; \\
\end{array}
$$

$$
\begin{array}{l}
e_1 \otimes e_0 \mapsto e_0 \otimes e_1 \otimes e_0 \\
\quad \mapsto \{2j\}e_0 \otimes e_1 \\
\quad \mapsto \{2j\}e_0 \otimes e_1 \otimes e_0 \\
\quad \mapsto \{2j\}e_0 \otimes e_1 \otimes e_0; \\
\end{array}
$$

$$
\begin{array}{l}
e_1 \otimes e_1 \mapsto e_0 \otimes e_1 \otimes e_1 \\
\quad \mapsto \{2i\}e_0 \otimes e_0 \\
\quad \mapsto \{2i\}([e_0 \otimes e_0 \otimes e_0 + (-q^{2(i+j)})e_0 \otimes e_1 \otimes e_1] \\
\quad \mapsto \{2i\}\{2i + 2j\}e_0 \otimes e_0 + (-q^{2(i+j)})\{2j\}e_1 \otimes e_1 \\
\quad = \{2i\}\{2i + 2j\}e_0 \otimes e_0 - \{2i\}\{2j\}q^{2(i+j)}e_1 \otimes e_1. \\
\end{array}
$$
The matrix for the morphism under the basis \( \{e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1\} \) is

\[
A = \begin{pmatrix}
\{2i + 2j\}^2 - \{2i\}\{2j\}q^{-2(i+j)} & 0 & 0 & -\{2i + 2j\}\{2j\}q^{2(i+j)} \\
0 & \{2i\}^2 & 0 & 0 \\
0 & 0 & \{2j\}^2 & 0 \\
\{2i\}\{2i + 2j\} & 0 & 0 & -\{2i\}\{2j\}q^{2(i+j)}
\end{pmatrix}.
\]

The two local diagrams on the right-hand side have the following local states on which the Boltzmann weights do not vanish:

\[
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics{local_diagram1}} \\
\text{\includegraphics{local_diagram2}}
\end{array}
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics{local_diagram3}} \\
\text{\includegraphics{local_diagram4}}
\end{array}
\end{array}
\]

The matrices for the morphisms are

\[
B = \begin{pmatrix}
\{2i\}q^{2j} & 0 & 0 & -\{2j\}q^{2(i+j)} \\
0 & \{2i - 2j\} & 0 & 0 \\
0 & 0 & 0 & 0 \\
\{2i\} & 0 & 0 & -\{2j\}q^{2i}
\end{pmatrix}
\]

and the identity matrix \( I \) respectively. It is easy to check that \( A = \{2i + 2j\} \cdot B + \{2j\}^2 \cdot I \).

(vii) The left-hand local diagram has the following local states with non-vanishing Boltzmann weights.

\[
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics{local_diagram5}} \\
\text{\includegraphics{local_diagram6}} \\
\text{\includegraphics{local_diagram7}} \\
\text{\includegraphics{local_diagram8}} \\
\text{\includegraphics{local_diagram9}}
\end{array}
\end{array}
\]

Therefore the morphism defined by the left-hand diagram is as below.

\[
\begin{array}{c}
\begin{array}{c}
\text{\includegraphics{local_diagram10}} \\
\text{\includegraphics{local_diagram11}} \\
\text{\includegraphics{local_diagram12}} \\
\text{\includegraphics{local_diagram13}} \\
\text{\includegraphics{local_diagram14}}
\end{array}
\end{array}
\]

\[
e_0 \otimes e_0 \mapsto \{2j\}e_0 \otimes e_0 \otimes e_0 \\
\mapsto \{2j\}e_0 \otimes e_0 \\
\mapsto \{2j\}\{2i + 2k\}e_0 \otimes e_0 \otimes e_0 \\
\mapsto \{2j\}\{2i + 2k\}e_0 \otimes e_0; \\
e_0 \otimes e_1 \mapsto \{2j - 2k\}e_0 \otimes e_0 \otimes e_1 + \{2k\}q^{-2(j-k)}e_0 \otimes e_1 \otimes e_0 \\
\mapsto \{2j - 2k\}e_0 \otimes e_1 + \{2k\}q^{-2(j-k)}q^{2i}e_1 \otimes e_0 \\
\mapsto \{2j - 2k\}\{2i + 2k\}e_0 \otimes e_0 \otimes e_1 \\
+ \{2k\}q^{-2(j-k)}q^{2i}\{2l\}e_0 \otimes e_1 \otimes e_0 + \{2i + 2k - 2l\}q^{-2l}e_1 \otimes e_0 \otimes e_0 \\
\mapsto \{2j - 2k\}\{2i + 2k\}q^{2l}e_0 \otimes e_1.
\]
The matrices for the morphisms are
\[
\begin{align*}
\{2j\} \{2i + 2k\} &
\begin{pmatrix}
0 & 0 & 0 \\
\{2j\} \{2l\} & \{2j\} \{2i + 2k - 2l\} q^{-2l} & 0 \\
0 & 0 & 0 \\
0 & 0 & \{2j\} \{2i + 2k - 2l\}
\end{pmatrix},
\end{align*}
\]
where \(a_{22} = \{2j - 2k\} \{2i + 2k\} q^{-2l} + \{2k\} \{2l\} q^{2(i-j+k)}\) and \(a_{23} = \{2i+2k-2l\} \{2k\} q^{2(i-j+k-l)}\).

The two diagrams on the right-hand side have the following local states with non-vanishing Boltzmann weights:

The matrices for the morphisms are
\[
\begin{align*}
\{2i + 2j\} &
\begin{pmatrix}
0 & 0 & 0 \\
\{2j + 2l - 2k\} q^{2i} & \{2i + 2k - 2l\} q^{-2(j+l-k-i)} & 0 \\
0 & \{2j + 2l - 2k\} & \{2i + 2k - 2l\} q^{-2(j+l-k)} \\
0 & 0 & 0
\end{pmatrix},
\end{align*}
\]
and
\[
\begin{align*}
\{2j\} &
\begin{pmatrix}
0 & 0 & 0 \\
\{2j + 2l - 2k\} & \{2k - 2l\} q^{-2(j+l-k-i)} & 0 \\
0 & \{2j\} & 0 \\
0 & 0 & \{2j + 2l - 2k\}
\end{pmatrix}.
\end{align*}
\]
We see that \( A = \frac{2i}{2i+2j} \cdot B + \frac{2i+2k}{2i+2j-2k} \cdot C. \)

(viii) We prove the first equality of (viii). From Table 1, it is easy to see that the matrix for the diagram on the left-hand side is

\[
A = \begin{pmatrix}
q^{i+j-i-J-jI} & 0 & 0 & 0 \\
0 & (q^i - 1)q^{j-i-i-j-jI} & 0 & 0 \\
0 & q^{j-i-j-i-jI} & 0 & 0 \\
0 & 0 & 0 & -q^{i-j-j-j}I
\end{pmatrix}.
\]

For the diagrams on the right-hand side, the morphisms for them have been calculated in the proof of (vii). Let \( k - l = j - i \). The matrices of the morphisms are respectively

\[
B = \begin{pmatrix}
\{2i+2j\} & 0 & 0 & 0 \\
0 & \{2i\}q^{2j} & \{2j\} & 0 \\
0 & \{2i\} & \{2j\}q^{-2j} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

and

\[
C = \begin{pmatrix}
\{2j\} & 0 & 0 & 0 \\
0 & \{2i\} & \{2j-2i\} & 0 \\
0 & 0 & \{2j\} & 0 \\
0 & 0 & 0 & \{2i\}
\end{pmatrix}.
\]

It is easy to see that \( A = \frac{q^{i-j-j-j-I}}{2i}B + \frac{-q^{j-i-j-j-I}}{2i}C. \)

3. Relations between \( \Delta_{(G,c)}(t) \) and \( \tilde{\Delta}(G,c) \)

For an oriented trivalent graph \( G \) without source or sink embedded in \( S^3 \), consider a coloring \( c \) of \( G \) for which the weight on each edge is one. It is easy to see that the admissible condition for the weights is satisfied. In this case, we simply suppress the weights and the coloring becomes a map \( c : E \to \mathbb{Z} \) which sends each edge to its multiplicity. The admissible conditions simply become one condition \( \sum_{i=1}^3 \epsilon_i j_i = 0 \), which coincides with the definition of balanced coloring in \([2]\). For a balanced positive coloring \( c \), in \([2]\) we defined a single-variable polynomial \( \Delta_{(G,c)}(t) \) and studied its MOY-type relations.

We briefly review the definition of \( \Delta_{(G,c)}(t) \). For more details and properties of it, please refer to \([2]\). The balanced coloring \( c \) of \( G \) naturally defines a homomorphism

\[
\phi_c : \pi_1(S^3 \setminus G, x_0) \to H_1(S^3 \setminus G; \mathbb{Z}) \to \mathbb{Z}(t)
\]

oriented meridian of \( e \mapsto t^{c(e)}, \)

where \( \mathbb{Z}(t) \) is the abelian group generated by \( t \) and \( e \) is any edge of \( G \). Let \( X = S^3 \setminus G \). Then \( \ker(\phi_c) \) corresponds to a regular covering space of \( X \), which we call \( p : \tilde{X} \to X \). Consider the \( \mathbb{Z}[t^{-1}, t] \)-module \( H_1(\tilde{X}, p^{-1}(\partial_0(X))) \), where \( \partial_0(X) := \bigcup_{v \in V} \partial_0(v) \subset \partial(X) \) and \( \partial_0(v) \) is a subsurface around vertex \( v \) bounded by meridians of the edges pointing toward \( v \) and the “meridian” around \( v \). The polynomial \( \Delta_{(G,c)}(t) \) is the 0-th
characteristic polynomial of a presentation matrix of the module. It is a topological invariant of $G$.

A multi-variable version of $\Delta_{(G,c)}(t)$ was first studied in [1], where we showed that it is the Euler characteristic of the Heegaard Floer homology of $G$ studied there. This polynomial was further studied intensively in [2], where we provided a normalized state sum formula for it, proved its topological invariance in combinatorial way and studied its MOY-type relations.

To state the relation between $\Delta_{(G,c)}(t)$ and $\Delta(G,c)$, we divide the vertices of $G$ into odd and even types.

\[\text{odd type} \qquad \text{even type}\]

Define the color of a vertex $v$, which we call $c(v)$, to be the sum of the colors of the edges pointing out of $v$, which by admissible condition equals the sum of colors of the edges pointing toward $v$.

**Theorem 3.1.** For a coloring $c$ whose multiplicities are positive and weights are one, we have

\[\Delta_{(G,c)}(q^{-4}) = \prod_{v: \text{even type}} \{2c(v)\} q^{\mathcal{A}(G,c)},\]

where $|V|$ is the number of vertices in $G$.

**Proof.** It was proved in [2, Theorem 4.3] that the MOY-type relations in [2, Theorem 4.1] uniquely characterize $\Delta_{(G,c)}(t)$. Using the same argument we can show that the relations in Theorem 2.3 also characterize $\Delta(G,c)$. Comparing the relations in [2, Theorem 4.3] with those in Theorem 2.3 it is easy to see that both sides of the equation above satisfy exactly the same MOY-type relations. \(\square\)

### 4. A concrete correspondence when $G$ is a planar graph

In this section we show that when $G$ is a planar graph and $c$ is a positive coloring the vertex state sum formula for $\Delta(G,c)$ has a topological interpretation, where a state is identified with a tuple of intersection points of the $\alpha$-curves and $\beta$-curves on a Heegaard diagram associated with $G$ and the Boltzmann weights come from the Fox calculus on the Heegaard surface.

In Section 4.1 we give a simplified version of the vertex state sum for $\Delta(G,c)$ in Prop. 2.1. In Section 4.2 we construct a Heegaard diagram associated with $G$ and show a one-one correspondence between the states for $\Delta(G,c)$ and the intersection points of $\alpha$-curves and $\beta$-curves on the Heegaard diagram. In Section 4.3 we study a vertex state sum formula for $\Delta_{(G,c)}(t)$ and compare it with that for $\Delta(G,c)$.

#### 4.1. Vertex state sum formula for $\Delta(G,c)$ when $G$ is a planar graph.

For a planar graph $G$, when all the multiplicities of the coloring $c$ are positive and the framing is the blackboard framing, we can simplify the vertex state sum formula for $\Delta(G,c)$ in Prop. 2.1 as follows.

Choose an initial point $\delta$ on an outermost edge of $G$. A state is a map $s : E \rightarrow \{0, 1\}$ which sends the edge containing $\delta$ to zero and satisfies the condition that at each vertex,
the sum of $s(e)$ for all the edges $e$ pointing toward the vertex equals that for the edges pointing out of the vertex. Let $\mathcal{S}$ be the set of states. By definition, around each vertex we have the following six possibilities under a state, where the dotted edges are those which are sent to zero and the solid edges are those which are sent to one.

It is easy to see that the solid edges constitute a collection of simple closed curves. We call them solid curves. Define the sign of a state $s$ to be

$$\text{sign}(s) = (-1)^{\text{the number of solid curves for } s}.$$ 

From $G$ we can obtain a collection of oriented simple closed curves by replacing each edge $e$ with $c(e)$ copies of parallel edges. Define $\text{Rot}(G, c, \delta)$ to be the sum of the rotation numbers of those simple closed curves which are disjoint with $\delta$. Here the rotation number of a simple closed curve is defined to be 1 (resp. $-1$) if it is clockwise (resp. counter-clockwise).

We define the weight of a state at a vertex $v$ as in Table 3. Then we get the following state sum formula for $\Delta(G, c)$.

**Proposition 4.1.**

\[
\Delta(G, c) = \frac{q^{2\text{Rot}(G, c, \delta)}}{q^{2j} - q^{-2j}} \sum_{s \in \mathcal{S}} \text{sign}(s) \prod_{v \in V} Wt(v; s),
\]

where $j$ is the color of the edge where $\delta$ lies.

**Proof.** By an isotopy of $\mathbb{R}^2$ we can assume that the edges around each vertex of $G$ point upward and the edge containing $\delta$ is a leftmost edge of $G$. Namely $G$ is in the following position around a vertex.

It is easy to see that the right-hand side of (1) does not change under the isotopy. We show that for such a graph diagram the vertex state sum of $\Delta(G, c)$ that we stated in Prop. 2.1 coincides with the right-hand side of (1).

Recall that a state in Prop. 2.1 is defined to be an assignment of $e_0$ to the edge containing $\delta$ and $e_0$ or $e_1$ to any of the other edges. Therefore $\mathcal{S}$ is a subset of the set

| State | $\text{Wt}(v; s)$ |
|-------|------------------|
| $i + j$ | $\{2i + 2j\}q$ |
| $i + j$ | $\{2j\}q$ |
| $i + j$ | $\{2i\}q^{2j}$ |
| $i + j$ | $q^{2i}$ |
| $i + j$ | $1$ |

**Table 3.** The weight of a state at a vertex for $\Delta(G, c)$. 

or
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of states in Prop. 2.1. From the Boltzmann weights in Table 2 it is easy to see that any state which is not in $S$ has no contribution to $\Delta(G,c)$. Therefore we only need to consider the states in $S$.

For a planar graph with blackboard framing, there is no crossings and half-twist symbols on the diagram. We only need to consider the Boltzmann weights at vertices and at critical points. For $s \in S$, the product of Boltzmann weights at vertices is exactly $\prod_{v \in V} Wt(v,s)$. To finish the proof we need to show that the product of the Boltzmann weights at critical points is

$$\text{sign}(s)q^{2[\text{Rot}(G,c,\delta) - (-1)^{\theta_j}]},$$

where $\theta$ is defined in Prop. 2.1. The proof follows from the fact that for an oriented simple closed curve $U$ in $\mathbb{R}^2$, colored by $j \in \mathbb{Z}\{0\}$, if we assign $e_i$ to it, the product of the Boltzmann weights at critical points is $(-1)^i q^{\text{Rot}(U)2j}$ for $i = 0, 1$, where $\text{Rot}(U)$ is the rotation number of $U$.

4.2. A Heegaard diagram for $G$. For a planar oriented graph $G$ without source or sink, we introduce a Heegaard diagram, which is inspired by the Heegaard diagram in [7, Section 5]. The construction is as follows.

(i) The Heegaard surface is $S^2$ where $G$ is embedded as a planar graph.

(ii) Choose an initial point $\delta$ on an edge of $G$. We regard this initial point as a new vertex of $G$.

(iii) At each vertex, introduce a base point $w$, and on each edge of $G$ introduce a base point $z$. Let $w$ and $z$ be the set of $w$’s and $z$’s respectively.

(iv) Around each vertex $v$, introduce a curve $\alpha_v$ which encloses the base point $w$ at $v$ and the base point(s) $z$’s on the edge(s) pointing to $v$. Introduce a curve $\beta_v$ which encloses the base point $w$ at $v$ and the base point(s) $z$’s on the edge(s) pointing out of $v$.

(v) Remove the $\alpha$-curve and $\beta$-curve around $\delta$.

As a result, we get the following data $H = (S^2, \{\alpha_v\}_{v \in V}, \{\beta_v\}_{v \in V}, w, z)$. See Fig. 4 for an example.

The data $H$ constructed above is a Heegaard diagram for $(G,\delta)$. To be precise, from $(G,\delta)$ we construct a new graph $\tilde{G}$ as below by inserting a thick edge at each vertex and at $\delta$ and splitting the vertex into two vertices.

\[
\text{\begin{tikzpicture}
    \node[circle, draw] (v1) at (0,0) {$v$};
    \node[circle, draw] (v2) at (-1,0) {$v$};
    \node[circle, draw] (v3) at (1,0) {$v$};
    \draw[thick] (v1) -- (v2);
    \draw[thick] (v1) -- (v3);
    \draw[thick] (v2) -- (v3);
    \end{tikzpicture}} \Rightarrow
\text{\begin{tikzpicture}
    \node[circle, draw] (v1) at (0,0) {$v$};
    \node[circle, draw] (v2) at (-1,0) {$v$};
    \node[circle, draw] (v3) at (0,-1) {$v$};
    \node[circle, draw] (v4) at (1,0) {$v$};
    \draw[thick] (v1) -- (v2);
    \draw[thick] (v1) -- (v3);
    \draw[thick] (v1) -- (v4);
    \draw[thick] (v2) -- (v3);
    \end{tikzpicture}} \Rightarrow
\text{\begin{tikzpicture}
    \node[circle, draw] (v1) at (0,0) {$v$};
    \node[circle, draw] (v2) at (-1,0) {$v$};
    \node[circle, draw] (v3) at (-1,1) {$\delta$};
    \node[circle, draw] (v4) at (1,0) {$v$};
    \draw[thick] (v1) -- (v2);
    \draw[thick] (v1) -- (v3);
    \draw[thick] (v1) -- (v4);
    \draw[thick] (v2) -- (v3);
    \end{tikzpicture}} \Rightarrow
\text{\begin{tikzpicture}
    \node[circle, draw] (v1) at (0,0) {$v$};
    \node[circle, draw] (v2) at (-1,0) {$v$};
    \node[circle, draw] (v3) at (1,0) {$v$};
    \node[circle, draw] (v4) at (0,-1) {$v$};
    \draw[thick] (v1) -- (v2);
    \draw[thick] (v1) -- (v3);
    \draw[thick] (v1) -- (v4);
    \draw[thick] (v2) -- (v3);
    \end{tikzpicture}}
\]

Then $\tilde{G}$ becomes a bipartite graph and these thick edges define a balanced orientation (see [1, Definition 2.5]) for $\tilde{G}$. Then it is easy to check that $H$ is a Heegaard diagram for $\tilde{G}$ (see [1, Definition 3.3]). For simplicity we suppress the thick edges and say that $H$ is a Heegaard diagram for $(G,\delta)$.

Consider the intersection points of $\alpha$-curves and $\beta$-curves. Let

$$\mathcal{T} := \{\{x_v\}_{v \in V} | x_v \in \alpha_{\sigma(v)} \cap \beta_v \text{ for a bijection } \sigma \text{ of } V\}.$$

Let $\sigma_x$ be the bijection that defines $x \in \mathcal{T}$. It is easy to see that the intersection points always appear in pairs around a base point. Two elements $x = \{x_v\}_{v \in V}$ and $y = \{y_v\}_{v \in V}$
are said to be equivalent \((x \sim y)\) if \(x_v\) and \(y_v\) are around the same base point for any \(v \in V\). Let \([x]\) be the equivalence class of \(x\). It is obvious that \(\sigma_x\) keeps invariant within the equivalence class of \(x\).

**Proposition 4.2.** There is an identification between \(T/\sim\) and \(S\).

**Proof.** The identification \(\psi: T/\sim \to S\) is constructed as follows. For \(x = \{x_v\}_{v \in V}\), \(\psi([x])\) sends an edge to one if there is \(x_v \in x\) appearing around the base point \(z\) on that edge. The other edges are sent to zero.

To prove that \(\psi([x])\) is an element of \(S\), it is enough to show that the following situations can not occur under \(\psi([x])\). Around an even vertex \(v\), the following cases can not appear under \(\psi([x])\):

![Diagram](image)

The first and second cases can not appear since \(\alpha_v \cap x\) can not be around two base points of \(z\). The third and forth cases do not appear since \(\beta_v \cap x\) is not empty. The fifth case can not appear since \(\alpha_v \cap x\) is not empty.

Around an odd vertex \(v\), the following cases can not appear under \(\psi([x])\) for similar reasons:

![Diagram](image)

It is easy to see that \(\psi\) is an injective map. For \(s \in S\), from the construction of \(\psi\) it is easy to find the pre-image of \(s\). Therefore it is also a surjective map.

**Lemma 4.3.** Under the identification \(\psi\) we have

\[
\text{sign}(\psi([x])) = \text{sign}(\sigma_x) \prod_{v \in V} (-1)^{\delta_{v,\delta_x(v)} + 1},
\]

for any \(x \in T\), where \(\delta_{v,\delta_x(v)}\) is the Kronecker delta.

**Proof.** Each solid curve consisting of \(k\) edges corresponds to a \(k\)-cycle of \(\sigma_x\). Its contribution to \(\text{sign}(\sigma_x)\) is \((-1)^{k-1}\). On the other hand, each solid edge contributes minus one
to \(\prod_{v \in V} (-1)^{d_{v, \sigma z(v)}+1}\). Therefore
\[
\text{sign}(\sigma_x) \prod_{v \in V} (-1)^{d_{v, \sigma z(v)}+1} = \prod_{\text{solid curve of length } k} (-1)^{k-1}(-1)^k
\]
\[
= \prod_{\text{solid curve of length } k} (-1)
\]
\[
= (-1)^{\text{the number of solid curves for } \psi([x])} = \text{sign}(\psi([x])).
\]

4.3. **A vertex state sum formula for** \(\Delta_{(G,c)}(t)\). From the Heegaard diagram \(H\) we can calculate \(\Delta_{(G,c)}(t)\) by applying Fox calculus as we discussed in [1, Section 6.3]. We follow the method in [1, Section 6.3] to calculate \(\det(\phi_c(\frac{\partial \beta_v}{\partial \alpha_u})))_{u,v \in V}\) and then state its relation with \(\Delta_{(G,c)}(t)\).

Around an even vertex \(v\) with the following coloring we have \(\beta_v = \alpha_u c_3 \alpha_u^{-1} \alpha_v c_1^{-1} c_2^{-1} \alpha_v^{-1}\),

where \(c_i\) is an oriented simple arc on \(S^2\) connecting the base point \(z\) on an edge, say \(e\), to the base point \(w\) at the vertex that \(e\) points to. The geometric meaning of \(c_i\) was explained in [1, Section 6.3], from where we see \(\phi_c(c_i) = t^{c(e)}\), where \(\phi_c\) is defined in Section 3. Therefore
\[
\phi_c(\frac{\partial \beta_v}{\partial \alpha_u}) = \phi_c(1 - \alpha_u c_3 \alpha_u^{-1}) = 1 - \phi_c(c_3) = 1 - t^{i+j},
\]
\[
\phi_c(\frac{\partial \beta_v}{\partial \alpha_v}) = \phi_c(\alpha_u c_3 \alpha_u^{-1} (1 - \alpha_v c_1^{-1} c_2^{-1} \alpha_v^{-1}))
\]
\[
= \phi_c(c_3)(1 - \phi_c(c_1^{-1} c_2^{-1})) = t^{i+j} (1 - t^{-i} t^{-j}) = t^{i+j} - 1.
\]

For each of the other \(\alpha\)-curves that are disjoint with \(\beta_v\), the Fox calculus is zero. Multiplying each of the elements in the row of \((\phi_c(\frac{\partial \beta_v}{\partial \alpha_u})))_{u,v \in V}\) that corresponds to \(\beta_v\) by \(t^{-(i+j)/2}\), the determinant does not change modulo \(\mathbb{Z}(t^{1/2})\). Summarizing the discussion above we have
\[
\phi_c(\frac{\partial \beta_v}{\partial \alpha_u}) = \begin{cases} t^{(i+j)/2} - t^{-(i+j)/2} & u = v \\ -(t^{(i+j)/2} - t^{-(i+j)/2}) & u \text{ and } v \text{ are joined by the edge with color } i+j \\ 0 & \text{otherwise.} \end{cases}
\]

Around an odd vertex \(v\) of the following coloring we have \(\beta_v = \alpha_v c_3^{-1} \alpha_v^{-1} \alpha_v c_2^{-1} \alpha_u c_1 \alpha_u^{-1}\). Therefore
\[
\phi_c(\frac{\partial \beta_v}{\partial \alpha_u}) = \phi_c(1 - \alpha_v c_3^{-1} \alpha_v^{-1}) = 1 - \phi_c(c_3^{-1}) = 1 - t^{-(i+j)},
\]
State $\psi([x])$

| $i+j$ | $i+j$ | $i+j$ | $i+j$ | $i+j$ | $i+j$ |
|-------|-------|-------|-------|-------|-------|
| $\{i+j\}$ | $\{\frac{i+j}{2}\}t^{-i/2}$ | $\{\frac{i+j}{2}\}t^{j/2}$ | $\{\frac{i+j}{2}\}t$ | $\{\frac{i+j}{2}\}t$ | $\{\frac{i+j}{2}\}t$ |

Table 4. The weight $\tilde{W}_t(v; [x])$ where $\{k\}_t := t^k - t^{-k}$ for $k \in \mathbb{Z}$.

For each of the other $\alpha$-curves that are disjoint with $\beta_v$, the Fox calculus is zero. Multiplying each of the elements in the row of $\phi_c(\frac{\partial \beta_v}{\partial \alpha_u})_{u,v \in V}$ that corresponds to $\beta_v$ by $t^{(i+j)/2}$, the determinant does not change modulo $\mathbb{Z}(t^{1/2})$. Summarizing the discussion above we have

$$
\phi_c(\frac{\partial \beta_v}{\partial \alpha_u}) = \begin{cases} 
\frac{t^{(i+j)/2} - t^{-i-j/2}}{t^{i/2} - t^{-i-j/2}} & u = v \\
\frac{-t^{i/2} - t^{j/2}}{t^{i/2} - t^{-j/2}} & u \ and \ v \ are \ joined \ by \ the \ edge \ with \ color \ i \\
0 & u \ and \ v \ are \ joined \ by \ the edge \ with \ color \ j \\
& \text{otherwise}.
\end{cases}
$$

Note that $\frac{\partial \beta_v}{\partial \alpha_u}$ is calculated by tracing the intersection points of $\beta_v$ with $\alpha$-curves.

For a bijection $\sigma$ of $V$ we have

$$
\prod_{v \in V} \phi_c(\frac{\partial \beta_v}{\partial \alpha_{\sigma(v)}}) = \sum_{[x] \in T/\sim, \sigma_v = \sigma} \prod_{v \in V} (-1)^{\delta_{v,\sigma(v)}+1} \tilde{W}_t(v; [x]),
$$

where $\tilde{W}_t(v; [x])$ is defined as in Table 4 and the term $(-1)^{\delta_{v,\sigma(v)}+1}$ comes from the fact that if $v \neq \sigma(v)$ there is a minus sign in $\phi_c(\frac{\partial \beta_v}{\partial \alpha_{\sigma(v)}})$. 

\[ \phi_c(\frac{\partial \beta_v}{\partial \alpha_u}) = \phi_c(\alpha_v c_3^{-1} \alpha_v^{-1}(1 - \alpha_w c_2 \alpha_w^{-1})) = \phi_c(c_3^{-1})(1 - \phi_c(c_2)) = t^{-(i+j)}(1 - t^j), \]

\[ \phi_c(\frac{\partial \beta_v}{\partial \alpha_u}) = \phi_c(\alpha_v c_3^{-1} \alpha_v^{-1} \alpha_w c_2 \alpha_w^{-1}(1 - \alpha_u c_1 \alpha_u^{-1})) = \phi_c(c_3^{-1} c_2)(1 - \phi_c(c_1)) = t^{-(i+j)} t^j (1 - t^i) = t^{-i} (1 - t^i). \]
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State \( \psi([x]) \)

| \( i \) | \( j \) | \( i+j \) | \( i=j \) | \( i=j \) | \( i=j \) | \( i=j \) |
|---|---|---|---|---|---|---|
| \( \hat{W}_t(v; [x]) \) | \( \frac{i+j}{2} \) | \( \frac{j}{2} \) | \( \frac{i}{2} t^{j/2} \) | \( \frac{i+j}{2} t^{i/2} \) | \( \frac{i+j}{2} t^{-i/2} \) | \( \frac{i+j}{2} t \) |

Table 5. The weight \( \hat{W}_t(v; [x]) \).

**Lemma 4.4.** For any \([x] \in T/\sim\) we have

\[
\prod_{v \in V} \hat{W}_t(v; [x]) = \prod_{v \in V} \hat{W}_t(v; [x]),
\]

where \( \hat{W}_t(v; [x]) \) is defined in Table 5.

**Proof.** The difference of \( \hat{W}_t(v; [x]) \) and \( \hat{W}_t(v; [x]) \) appears around solid curves of the state \( \psi([x]) \). Precisely

\[
\frac{\hat{W}_t(v; [x])}{\hat{W}_t(v; [x])} = \begin{cases} 
\frac{t^{-i/2}}{1} & \text{if } \psi([x]) \text{ has the form around } v; \\
\frac{t^{i/2}}{1} & \text{if } \psi([x]) \text{ has the form around } v; \\
1 & \text{otherwise.}
\end{cases}
\]

Therefore

\[
\prod_{v \in V} \frac{\hat{W}_t(v; [x])}{\hat{W}_t(v; [x])} = \prod_{c: \text{solid curve of } \psi([x])} \left( \frac{\hat{W}_t(v; [x])}{\hat{W}_t(v; [x])} \right) = \prod_{c: \text{solid curve of } \psi([x])} 1 = 1,
\]

where the second equality follows from the fact that for each solid curve \( c \), the product of the meridians for dotted edges pointing toward it equals the product of the meridians for dotted edges pointing out of it, as illustrated in the following figure.

null-homologous in \( H_1(S^3 \setminus G) \)
Summarizing the discussion above, we have

\[
\det(\phi_c(\frac{\partial \beta_v}{\partial \alpha_u}))_{u,v \in V} = \sum_{\sigma: \text{ bijection of } V} \text{sign}(\sigma) \prod_{v \in V} \phi_c\left(\frac{\partial \beta_v}{\partial \alpha_{\sigma(v)}}\right)
= \sum_{[x] \in T/\sim} \text{sign}(\sigma_x) \prod_{v \in V} (-1)^{\delta_v,\sigma(v)+1} \hat{W}t(v; [x])
= \sum_{[x] \in T/\sim} \text{sign}(\sigma_x) \prod_{v \in V} (-1)^{\delta_v,\sigma(v)+1} \hat{W}t(v; [x])
= \sum_{s \in S} \text{sign}(s) \prod_{v \in V} \hat{W}t(v; s),
\]

where the third equality follows from Lemma 4.3 and the forth one follows from Lemma 4.3. As a conclusion, we have the following relation.

**Proposition 4.5.**

\[
(t^{1/2} - t^{-1/2}) |V|^{-1} \Delta_{(G,c)}(t) = \frac{1}{t^{j/2} - t^{-j/2}} \det(\phi_c(\frac{\partial \beta_v}{\partial \alpha_u}))_{u,v \in V}
\]

\[
= \frac{1}{t^{j/2} - t^{-j/2}} \sum_{s \in S} \text{sign}(s) \prod_{v \in V} \hat{W}t(v; s),
\]

(2)

where \(\equiv\) indicates that the values connected are equal modulo \(\pm \mathbb{Z}(t^{1/2})\) and \(j\) is the color of the edge containing \(\delta\).

**Proof.** The determinant \(\det(\phi_c(\frac{\partial \beta_v}{\partial \alpha_u}))_{u,v \in V}\) calculates the Alexander polynomial of \(\tilde{G}\). On the other hand, the state sum \(\langle G|\delta\rangle\) in [2] also calculates the Alexander polynomial of \(\tilde{G}\) using a different Heegaard diagram. Therefore we have \(\langle G|\delta\rangle = \det(\phi_c(\frac{\partial \beta_v}{\partial \alpha_u}))_{u,v \in V}\). The relation follows from [2] (2), the definition of \(\Delta_{(G,c)}(t)\). \(\square\)

**Corollary 4.6 (A weaker form of Theorem 3.1).** For each state \(s \in S\), we have

\[
\prod_{v \in V} \hat{W}t(v; s)|_{t=q^{-4}} = (\prod_{v: \text{ even type}} \{2c(v)\}_q) \prod_{v \in V} Wt(v; s).
\]

As a corollary

\[
\Delta_{(G,c)}(t)|_{t=q^{-4}} = \prod_{v: \text{ even type}} \{2c(v)\}_q (q^2 - q^{-2})^{|V|} \Delta(G, c).
\]

**Proof.** Compare the values of \(Wt(v; s)\) and \(\hat{W}t(v; s)\) in Tables 3 and 5. The relation between them follows from that \(t^{i/2} - t^{-i/2}|_{t=q^{-4}} = -(q^{2i} - q^{-2i})\) and that there are even number of vertices in \(G\) (since the vertices of even type equals that of odd type). \(\square\)

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Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, Japan

E-mail address: bao@ms.u-tokyo.ac.jp