Research Article

The Existence of Normalized Solutions for a Nonlocal Problem in $\mathbb{R}^3$

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In this paper, we study the following fractional Schrödinger equation in $\mathbb{R}^3$:

$$(-\Delta)^{\alpha} u - \lambda u = |u|^{p-2} u, \text{ in } \mathbb{R}^3,$$

where $\sigma \in (0, 1)$, $p \in (2 + \sigma, 2 + (4/3)\sigma)$, and $\lambda \in \mathbb{R}$. Here, the fractional Laplacian $(-\Delta)^{\alpha}$ in $\mathbb{R}^3$ is defined by

$$(-\Delta)^{\alpha} u = C_{n, \sigma} PV \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x-y|^{n+2\sigma}} \, dy,$$

where $PV$ stands for the Cauchy principal value and $C_{n, \sigma}$ is a normalization constant.

In the present paper, the motivation for studying such equations comes from mathematical physics: searching for the form of standing wave $\psi = e^{-iht} u$ of the evolution equation

$$i \frac{\partial \psi}{\partial t} + (-\Delta)^{\alpha} \psi + (\lambda - h) \psi = |\psi|^{p-2} \psi \text{ in } \mathbb{R}^+ \times \mathbb{R}^3$$

leads to looking for solutions of (1). Here, $i$ is the imaginary unit and $h \in \mathbb{R}$. This class of Schrödinger-type equations is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy processes. A path integral over the Lévy flight paths and a fractional Schrödinger equation of fractional quantum mechanics are formulated by Laskin [1, 2] from the idea of Feynman and Hibbs’s path integrals.

On the other hand, problem (1) has attracted considerable attention in the recent period. Part of the motivation is to consider $h \in \mathbb{R}$ as a fixed parameter and then to search for a solution $u \in H^u(\mathbb{R}^3)$ solving (1). In this direction, mainly by variational methods, many researches have been devoted to the study of the existence, multiplicity, uniqueness, regularity, and asymptotic decay properties of the solutions to fractional Schrödinger equation (1). For this information, we can refer to [3–11] and the references therein. Besides, some more complicated fractional equations and systems were also studied, and indeed, some interesting results were obtained. Nearly, Mingqi et al. [12] investigated a critical Schrödinger–Kirchhoff type systems driven by non-local integro-differential operators and by applying the mountain pass theorem and Ekeland’s variational principle; the authors obtained the existence and asymptotic behavior of solutions for this system under some suitable assumptions. Later, in [13], the same authors as in [12] studied a diffusion model of Kirchhoff-type. Under some appropriate conditions, by employing the Galerkin method, the local existence of nonnegative solutions was obtained, and then by virtue of a differential inequality technique, they proved that the local nonnegative solutions blow up in finite time with arbitrary negative initial energy and suitable initial values. Moreover, in [14], Mingqi et al. concerned with a class of fractional...
Kirchhoff-type problems with the Trudinger-Moser nonlinearity. By applying minimax techniques combined with the fractional the Trudinger-Moser inequality, they found the existence of a ground state solution with positive energy and the existence of nonnegative solutions with negative energy by using Ekeland’s variational principle. In [15], the three authors considered a fractional Choquard-Kirchhoffer-type problem involving an external magnetic potential and a critical nonlinearity and established a fractional version of the concentration-compactness principle with magnetic field, and then together with the mountain pass theorem, they verified the existence of nontrivial solutions in nondegenerate and degenerate cases. Furthermore, Mingqi et al. [16] concerned the Schrödinger-Kirchhoff-type problems involving the fractional \( p \)-Laplacian and critical exponent. By using the concentration-compactness principle in fractional Sobolev spaces, they showed the existence of \( m \) pairs of solutions for any \( m \in \mathbb{N} \), and by applying Krasnoselskii’s genus theory, they also got the existence of infinitely many solutions under some suitable conditions for the parameter. For more information on this direction, one can refer to [17–24] and the references therein.

In the present paper, inspired by the fact that physicists are often interested in normalized solutions, we look for solutions in \( H^p(\mathbb{R}^3) \) having a prescribed \( L^2 \) norm to equation (1). Such types of problems were studied extensively in recent years for the classical Schrödinger equations with the standard Laplacian operator. We refer the interested reader to [25–31] and to the references therein. But up to our knowledge, not much is obtained for the existence of normalized solutions of equation (1) in \( H^p(\mathbb{R}^3) \) with a fractional Laplacian operator. So, in this paper, the aim is to get the normalized solutions of equation (1). Here, we give the definition of prescribed \( p \)-L^2 norm solutions. For fixed \( p > 0 \), if \( u_p \in H^p(\mathbb{R}^3) \) is a solution of problem (1) such that

\[
\|u_p\|_2 := \left( \int_{\mathbb{R}^3} |u_p|^2 \right)^{1/2} = p, \tag{4}
\]

we call it a prescribed \( p \)-L^2 norm solution. Naturally, a prescribed \( p \)-L^2 norm solution \( u_p \in H^p\mathbb{R}^3 \) of (1) can be a constrained critical point of the functional

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left( |(-\Delta)^{\sigma/2} u|^2 - 1 \right) \left| \int_{\mathbb{R}^3} |u|^p \right|, \tag{5}
\]

on the \( L^2 \) sphere \( B_p \) in \( H^p(\mathbb{R}^3) \), where

\[
B_p = \{ u \in H^p(\mathbb{R}^3) : \|u\|_2 = p, p > 0 \}. \tag{6}
\]

Note that for any \( p \in (2, 6/(3 - 2\sigma)) \), \( I(u) \) is a well-defined and \( C^1 \) functional. Set

\[
m_p = \inf_{B_p} I(u). \tag{7}
\]

It is standard that if \( u_p \) is a minimizer of (7), then \( u_p \) is a solution of (1) with prescribed \( p \)-L^2 norm with the constraint \( \lambda_p \in \mathbb{R} \) being the Lagrange multiplier. However, it is worth mentioning that dealing with this kind of problem, one has to face the main difficulty concerning with the lack of compactness of the minimizing sequence \( \{u_n\} \subset B_p \). In fact, we will encounter two possible bad scenarios that \( u_n \rightarrow 0 \) and \( u_n \rightarrow \hat{u} \neq 0 \) with \( \|u\|_2 < p \). In order to avoid the possible cases and to get that the infimum is obtained, we prove an important lemma (Lemma 6) that guarantees the compactness of minimizing sequence. As a consequence of this lemma, setting

\[
\rho^*_1 = \inf \{ \rho > 0 : m_p < 0 \},
\]

\[
\rho^*_2 = \inf \{ \rho > 0 : \exists u \in B_p \text{ such that } I(u) \leq 0 \}, \tag{8}
\]

we can get our main result as follows:

**Theorem 1.** If \( p \in (2 + \sigma, 2 + (4/3)\sigma) \), then \( m_p \) has a minimizer if and only if \( p \in [\rho^*_2, +\infty) \).

In particular, there is a prescribed \( p \)-L^2 norm solution \( u_p \in H^p(\mathbb{R}^3) \) of (1) with the constraint \( \lambda_p \in \mathbb{R} \). But, when \( p = 2 + (4/3)\sigma \), \( m_p \) has no minimizer for any \( p \in (\rho^*_2, +\infty) \).

**Remark 2.** In fact, for any \( p > 0 \), we can infer that \( m_p \leq 0 \). To see this, letting \( u \in B_{p^0} \) be arbitrary and considering \( u^\theta(y) = \theta^{5/2} u(\theta y) \) for any \( \theta > 0 \), we find \( u^\theta \in B_p \) and

\[
I\left(u^\theta\right) = \frac{1}{2} \theta^{2 + 2\sigma} \int_{\mathbb{R}^3} \left| (-\Delta)^{\sigma/2} u^\theta \right|^2 - \frac{1}{p} \theta^{\sigma(5/2)p - 3} \int_{\mathbb{R}^3} |u|^p. \tag{9}
\]

Hence, \( I(u^\theta) \rightarrow 0 \) as \( \theta \rightarrow 0 \) and the conclusion is as follows:

Finally, we give the following notations which can be used in this paper:

(i) \( H^\sigma(\mathbb{R}^3) \) is the usual Sobolev space endowed with the standard norm

\[
\|u\|_{H^\sigma}^2 = \int_{\mathbb{R}^3} \left( |(-\Delta)^{\sigma/2} u|^2 + u^2 \right). \tag{10}
\]

(ii) Denote \( \|u\|_{L^p}^2 := \int_{\mathbb{R}^3} |(-\Delta)^{\sigma/2} u|^2 \)

(iii) \( \|u\|_{q} \) is the norm of the Lebesgue space \( L^q(\mathbb{R}^3) \) for \( 1 < q < \infty \)

(iv) Denote \( C > 0 \) by various positive constants which may vary from one line to another and which are not important for the analysis of the problem

This paper is organized as follows: In Section 2, we will give some preliminary results which are crucial to prove...
2. Preliminaries

In this part, we give some important results. First, similar to the classical Gagliardo-Nirenberg inequality to the Laplacian operator, we introduce the fractional version of Gagliardo-Nirenberg inequality as follows:

**Lemma 3** (see [32]). Let $1 \leq q, p_2 < \infty, 0 < \alpha < q < \infty, 0 < \sigma < n$, and $1 < p_1 < (n/\sigma)$. We have

$$
\|u\|_{L^q(\mathbb{R}^n)} \leq C_0^{\alpha/q} \|(-\Delta)^{\alpha/2} u\|_{L^{p_1}(\mathbb{R}^n)}^{\alpha/q} \|u\|_{L^{p_2}(\mathbb{R}^n)}^{(q-\alpha)/q},
$$

where $a((1/p_1) - (\sigma/n)) + ((q-\alpha)/p_2) = 1$.

Furthermore, if we now know that

$$
\eta(u^*) = (3A'(-u) - D'(-u))|_{u^*} = 0.
$$

Using Lemma 3, the estimate (13) leads to the following fact:

**Lemma 5.** If $p \in (2 + \sigma, 2 + (4/3)\sigma)$, then for any $\rho > 0$, functional $I$ is bounded from below and coercive on $B_\rho$.

**Proof.** If $\|u\|_2 = \rho$, from Lemma 3, we have

$$
\|u\|_p \leq C\|u\|^{(3(p-2))/2\sigma}.
$$

So,

$$
I(u) = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\sigma/2} u|^2 - \frac{1}{p} \int_{\mathbb{R}^n} |u|^p \geq \frac{1}{2} \|u\|^{2\sigma} C\|u\|^{(3(p-2))/2\sigma}.
$$

Since $p < 2 + (4/3)\sigma$ and ($(3(p-2))/2\sigma) < 2$, our result follows.

From the above Lemma, we can prove that

**Lemma 6.** If $p \in (2 + \sigma, 2 + (4/3)\sigma)$, then

(i) there exists $\rho_i > 0$ such that for all $\rho \in (\rho_i, \infty)$, $m_\rho < 0$.

(ii) for any $\rho > 0$ such that $m_\rho < 0$, $m_\rho$ admits a minimizer.

(iii) the functional $\rho \mapsto m_\rho$ is continuous about each $\rho > 0$.

**Proof.**

(i) Letting $u^\theta(y) = \beta^{1/(3\theta)} u(\beta^{-\theta} y)$ such that $\|u^\theta\|_2 = \beta$ with $\beta, \theta \in \mathbb{R}$, we see

$$
\int_{\mathbb{R}^n} |(-\Delta)^{\sigma/2} u^\theta|^2 = \beta^{-2\sigma} \int_{\mathbb{R}^n} |(-\Delta)^{\sigma/2} u|^2,
$$

$$
\int_{\mathbb{R}^n} |u^\theta|^p = \beta^{p(1/(3\theta)+3\theta)} \int_{\mathbb{R}^n} |u|^p.
$$

If we take $\theta = -2$, one has

$$
I(u^\theta) = \frac{1}{2} \beta^{2(1+2\sigma)} \int_{\mathbb{R}^n} |(-\Delta)^{\sigma/2} u|^2 - \frac{1}{p} \beta^{p(4-p)} \int_{\mathbb{R}^n} |u|^p.
$$
Since $p > 2 + \sigma$ and $4p - 6 > 2 + 4\sigma$, for $\beta$ is large enough, we get $I(u^\beta) < 0$ and then (i) has been proved.

(ii) We will divide it into three steps to show (ii).

Step 1. We claim that for any $0 < \nu < \rho$, $m_\nu < m_{\nu + \sqrt{\rho^2 - \nu^2}}$. To see this, let $\{u_n\}$ be a minimizing sequence on $B_\rho$ for $m_\rho$. Since $m_\rho < 0$, from (22), we can get

$$0 < C_1 < \|u_n\|_\sigma^2 < C_2,$$

$$0 < C_3 < \|u_n\|_p^p < C_4.$$  

On the other hand, applying (24), we obtain that for $\theta = -2$,

$$I(u_n^\theta) = \frac{1}{2} \beta^{2(1 + 2\varepsilon)} \int_{\mathbb{R}^3} \left| (-\Delta)^{\sigma/2} u_n \right|^2 - \frac{1}{p} \beta^{2(p - 8)} \int_{\mathbb{R}^3} |u_n|^p,$$

where

$$f(\beta, u_n) = \frac{1}{2} (\beta^{\sigma} - 1) \int_{\mathbb{R}^3} \left| (-\Delta)^{\sigma/2} u_n \right|^2 - \frac{1}{p} (4p - 8) < 0,$$

with $4p - 8 > 4\sigma$. Moreover, using $I(u_n) < 0$, we can deduce that

$$\frac{df(\beta, u_n)}{d\beta} \bigg|_{\beta = 1} = 2\sigma \int_{\mathbb{R}^3} \left| (-\Delta)^{\sigma/2} u_n \right|^2 - \frac{1}{p} (4p - 8) < 0,$$

and for all $\beta > 1$,

$$\frac{d^2f(\beta, u_n)}{d\beta^2} = 2\sigma(4\sigma - 1) \beta^{4\sigma - 2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\sigma/2} u_n \right|^2 - \frac{1}{p} (4p - 8)(4p - 9) \beta^{4p - 10} \int_{\mathbb{R}^3} |u_n|^p < 0.$$  

Thus, combining (28) and (29), for all $\beta > 1$, we find $f(\beta, u_n) < 0$ and from (26),

$$m_{\rho\nu} < \beta^2 m_\rho.$$  

As a result, if $\sqrt{\rho^2 - \nu^2} < \nu < \rho$,

$$m_\rho = m_{(\rho, \nu)} = \frac{\rho^2 - \nu^2 + \nu^2}{\nu^2} m_{\nu} = \frac{\rho^2 - \nu^2}{\nu^2} m_{\nu} < m_{\sqrt{\rho^2 - \nu^2} + \nu}.$$  

and if $\nu > \sqrt{\rho^2 - \nu^2} < \rho$,

$$m_\rho = m_{(\nu, \rho)} = \frac{\rho^2 - \nu^2 + \nu^2}{\nu^2} m_{\nu} = \frac{\rho^2 - \nu^2 + \nu^2}{\nu^2} m_{\nu} < m_{\sqrt{\rho^2 - \nu^2} + \nu}.$$  

It follows from (31) and (32) that the claim holds.

Step 2. We will show that all the minimizing sequences $\{u_n\}$ for $m_\rho$ have a weak limit, up to translations, different from zero. Let $\{u_n\}$ be a minimizing sequence on $B_\rho$ for $m_\rho$. Note that for any sequence $\{y_n\} \subset \mathbb{R}^3$, $u_n(\cdot + y_n)$ is still a minimizing sequence for $m_\rho$. So the proof of this step can be finished if we can prove the existence of a sequence $\{y_n\} \subset \mathbb{R}^3$ such that the weak limit of $u_n(\cdot + y_n)$ is different from zero.

Applying Lion’s lemma, we know that if

$$\limsup_{n \to \infty} \int_{B(0, 1)} |u_n|^q \Gamma dy = 0,$$

then $u_n \rightharpoonup 0$ in $L^q(\mathbb{R}^3)$ for any $q \in (2, 6/(3 - 2\sigma))$, where $B(a, r) = \{x \in \mathbb{R}^3 : |x - a| \leq r\}$. Since $m_\rho < 0$, we see that

$$\int_{B(0, 1)} |u_n(\cdot + y_n)|^\gamma \Gamma dy \geq \delta > 0.$$  

Therefore, it follows the compactness of the embedding $H^\sigma(B(0, 1)) \hookrightarrow L^2(B(0, 1))$ that the weak limit of the sequence $u_n(\cdot + y_n)$ is not the trivial function.

Step 3. Finally, we verify that $m_\rho$ has a minimizer for $m_\rho < 0$. Suppose $\{u_n\}$ is a minimizing sequence on $B_\rho$ for $m_\rho$ with $m_\rho < 0$. Then by Lemma 5, $\{u_n\}$ is bounded in $H_\sigma(\mathbb{R}^3)$ and $L^q(\mathbb{R}^3)$ for any $q \in [2, 6/(3 - 2\sigma)]$. So there exists $\bar{u} \in H^\sigma(\mathbb{R}^3)$ such that $u_n \rightharpoonup \bar{u}$ in $H^\sigma(\mathbb{R}^3)$ and then we can get

$$\frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\sigma/2} u_n \right|^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u_n|^p = m_\rho + o(1),$$

$$\frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\sigma/2} (u_n - \bar{u}) \right|^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u_n - \bar{u}|^p + \frac{1}{2} \int_{\mathbb{R}^3} \left| (-\Delta)^{\sigma/2} \bar{u} \right|^2 - \frac{1}{p} \int_{\mathbb{R}^3} |\bar{u}|^p = m_\rho + o(1).$$  

If we set $\nu = \|\bar{u}\|_2$ and $\epsilon_n = (\sqrt{\rho^2 - \nu^2})/\|u_n - \bar{u}\|_2^2$, then by the Step 2, $0 < \nu < \rho$. Now, we want to prove that
v = ρ. To see this, we assume that 0 < ν < ρ. From (36), we find
\[
\frac{1}{2} \| e_n(u_n - \bar{u}) \|_2^2 - \frac{1}{p} \| e_n(u_n - \bar{u}) \|_p^p + I(\bar{u}) = m_p + o(1),
\]
(37)

since \( e_n \to 1 \). Noting that \( \| e_n(u_n - \bar{u}) \|_2 = \| \rho^2 - \nu^2 \|_2 \), (37) tells that
\[
m \sqrt{\rho^2 - \nu^2} + m \leq m_p + o(1),
\]
(38)

which contradicts to (31) and (32) and then \( v = \rho \).

Since \( \bar{u} \in B_\rho \), we have \( \| u_n - \bar{u} \|_2 = o(1) \). Hence, if we would verify that \( u_n \to \bar{u} \) in \( H^\sigma (\mathbb{R}^3) \), it remains to show that \( \| u_n - \bar{u} \|_2 = o(1) \) up to a subsequence. First, by assumption, there is \( \{ \lambda_n \} \subset \mathbb{R} \) such that
\[
\langle I'(u_n) - \lambda_n u_n, \psi \rangle = o(1), \quad \forall \psi \in H^\sigma (\mathbb{R}^3).
\]
(39)

So,
\[
\langle I'(u_n) - \lambda_n u_n, u_n \rangle = o(1),
\]
(40)

which implies \( \{ \lambda_n \} \) is bounded and up to a subsequence; there exists \( \lambda \in \mathbb{R} \) with \( \lambda_n \to \lambda \).

On the other hand, we find \( n, m \to \infty \),
\[
\langle I'(u_n) - I'(u_m) - \lambda_n u_n + \lambda_m u_m, u_n - u_m \rangle = o(1),
\]
(41)

Hence,
\[
\| u_n - u_m \|_{\sigma}^2 - \int_{\mathbb{R}^3} |u_n|^{p-2} u_n |u_m|^{p-2} u_m (u_n - u_m) - \lambda_n \| u_n - u_m \|_2^2 = o(1).
\]
(42)

Notice that by the interpolation inequality, we get
\[
\| u_n - u_m \|_p \leq C \| u_n - u_m \|_2^{1/2} \| u_n - u_m \|_{\alpha}^{1/2},
\]
(43)

with \( (\gamma/2) + (1 - \gamma)/(2\sigma) = 1/p \) and \( 2\sigma = 6/(3 - 2\sigma) \), and then
\[
\| u_n - u_m \|_p = o(1).
\]
(44)

As a result,
\[
\int_{\mathbb{R}^3} |u_n|^{p-2} u_n |u_m|^{p-2} u_m (u_n - u_m) \leq \int_{\mathbb{R}^3} |u_n|^{p-2} u_n |u_n|^{p-2} u_n \| u_n - u_m \|_p^p + \int_{\mathbb{R}^3} |u_m|^{p-2} u_m |u_n|^{p-2} u_n \| u_n - u_m \|_p^p \]
\[
\leq \left[ \left( \int_{\mathbb{R}^3} |u_n|^{p-2} u_n |u_n|^{p-2} u_n \right)^{(p-1)/p} \right]^p \| u_n - u_m \|_p^p + \left( \int_{\mathbb{R}^3} |u_m|^{p-2} u_m |u_n|^{p-2} u_n \right)^{(p-1)/p} \| u_n - u_m \|_p^p
\]
\[
\leq \left( \int_{\mathbb{R}^3} |u_n|^{p-2} u_n |u_n|^{p-2} u_n \right)^{(p-1)/p} + \left( \int_{\mathbb{R}^3} |u_m|^{p-2} u_m |u_n|^{p-2} u_n \right)^{(p-1)/p} \| u_n - u_m \|_p^p
\]
(45)

Using \( \| u_n - u_m \|_2 = o(1) \), \( \lambda_n \to \lambda \), and (45), one can get that \( \{ u_n \} \) is a Cauchy sequence in \( H^\sigma (\mathbb{R}^3) \) and hence \( \| u_n - \bar{u} \|_{H^\sigma (\mathbb{R}^3)} \to 0 \) as \( n \to \infty \).

(iii) Now, we come to prove that if \( \rho_n \to \rho \), then \( \lim_{n \to \infty} m_{p_n} = m_\rho \). For any \( n \in \mathbb{N}^* \), let \( w_n \in B_{\rho_n} \) such that \( I(w_n) < m_{p_n} + (1/n) \). Using Lemma 5, we deduce that \( \{ u_n \} \) is bounded in \( H^\sigma (\mathbb{R}^3) \) and then \( \| u_n \|_{\sigma} \) and \( \| u_n \|_p \) are bounded. So it is easy to find that
\[
m_p \leq I \left( \frac{\rho_n}{\rho} w_n \right) = \frac{1}{2} \left( \frac{\rho_n}{\rho} \right)^2 \| w_n \|_p^2 - \frac{1}{p} \left( \frac{\rho_n}{\rho} \right)^p \| u_n \|_p^p
\]
\[
= I(w_n) + o(1) < m_{p_n} + o(1).
\]
(46)

On the other hand, letting \( \{ v_n \} \subset B_\rho \) be a minimizing sequence for \( m_\rho \), we have
\[
m_{p_n} = I(\rho_n v_n) \leq I(v_n) + o(1) = m_\rho + o(1).
\]
(47)

Hence, from (46) and (47), \( \lim_{n \to \infty} I_{p_n} = m_\rho \) follows.

3. Proof of the Main Result

To prove our main theorem, we first give the following important results:

**Lemma 7.** When \( p \in (2 + \sigma, 2 + (4/3)\sigma) \), there exists \( \rho_1 > 0 \) such that \( m_\rho \) has no minimizer for all \( \rho \in (0, \rho_1) \).

**Proof.** We prove it by contradiction and suppose that there exist \( \{ \rho_n \} \subset \mathbb{R}^+ \) with \( \rho_n \to \rho^* \) as \( n \to \infty \) and \( \{ u_n \} \subset B_{\rho_n} \) such that \( I(u_n) = m_{p_n} \).

Since for any \( \rho > 0, m_\rho \leq 0 \), we have \( m_{p_n} \leq 0 \) and then by Lemma 3,
\[
\frac{1}{2} \| u_n \|_{\sigma}^2 \leq \frac{1}{p} \| u_n \|_p^p \leq C \| u_n \|_\sigma^{(3(p-2)/2)p/(p-\theta)} \| u_n \|_{\sigma}^{(3(p-2)/2)p/(p-\theta)},
\]
(48)

with \( \theta = (3(p-2)/2p) \). Due to \( p < 2 + (4/3)\sigma \) and \( (3(p-2))/\sigma < 2 \), (48) tells us that
\[
\| u_n \|_{\sigma} \to 0 \quad \text{as} \quad n \to \infty.
\]
(49)
So, from (48) and Lemma 4, we infer that
\[ \eta(u_n) = 2\|u_n\|_\sigma^2 - \frac{3(p-2)}{p} \|u_n\|_p^p \geq 2\|u_n\|_\sigma^2 - C\|u_n\|_\sigma^{(3(p-2)/2a)} p^{(1-8)} > 0, \]  
which is impossible since \( \eta(u_n) = 0 \) from Lemma 4. Recall that \( \rho_1^* = \inf \{ \rho > 0 : m_\rho < 0 \} \), \( \rho_2^* = \inf \{ \rho > 0 : \exists u \in B_p \text{ such that } I(u) \leq 0 \} \).  

We have

**Lemma 8.** If \( p = 2 + (4/2)\sigma \), then \( \rho_2^* \in (0, +\infty) \).

**Proof.** First, it follows from Lemma 3 that for any \( u \in B_p \), if \( p = 2 + (4/3)\sigma \)
\[ \|u\|_p^p \leq C\|u\|_\sigma^2 \rho^2. \]  
So,
\[ I(u) \geq \frac{1}{2} \|u\|_\sigma^2 - \frac{C}{p} \|u\|_\sigma^2 \rho^2. \]  
Thus, \( I(u) > 0 \) for any \( u \in B_p \) with \( \rho > 0 \) small enough and then \( \rho_2^* > 0 \) follows.

On the other hand, taking \( u_1 \in B_1 \), then \( u_1^\theta(y) = \theta^{(5/2)} u_1(\theta y) \in B_\theta \) for all \( \theta > 0 \) and
\[ I(u_1^\theta) = \frac{1}{2} \theta^{2+2a} \|u_1\|_\sigma^2 - \frac{1}{p} \theta^{(5/2)p-3} \|u_1\|_p^p \]  
This implies that if \( \theta > 0 \) is large enough, \( I(u_1^\theta) < 0 \) and our result has been proved.

**Lemma 9.** When \( p \in (2 + \sigma, 2 + (4/3)\sigma) \), we have the following:

(i) \( \rho_1^* \in (0, \infty) \)

(ii) \( m_\rho = 0 \) if \( \rho \in (0, \rho_1^*) \)

(iii) \( m_\rho < 0 \) and is strictly decreasing about \( \rho \) if \( \rho \in (\rho_1^*, +\infty) \)

(iv) Moreover, when \( p = 2 + (4/3)\sigma \), then \( \rho_2^* \in (0, \infty) \) and \( m_\rho = 0 \) as \( \rho \in (0, \rho_2^*) \) and \( m_\rho = -\infty \) if \( \rho \in (\rho_2^*, +\infty) \)

**Proof.**

(i) We prove it by contradiction. Suppose that \( \rho_1^* = 0 \), and then from the definition of \( \rho_1^* \), for any \( \rho > 0 \), we can get \( m_\rho < 0 \). Hence, it follows from Lemma 6 (ii) that \( m_\rho \) has a minimizer for any \( \rho > 0 \). But this gives a contradiction with Lemma 7. On the other hand, using Lemma 6 (i), we can infer that \( \rho_1^* < \infty \) and so (i) follows.

(ii) First, from the definition of \( \rho_1^* \) and \( m_\rho \leq 0 \) for any \( \rho > 0 \), we know that \( m_\rho = 0 \) for \( \rho \in (0, \rho_1^*) \). Furthermore, using the continuity of \( \rho \rightarrow m_\rho \) (see Lemma 6 (iii)), we have \( m_{\rho_1^*} = 0 \) and (ii) is proved.

(iii) By the definition of \( \rho_1^* \), we have \( m_\rho < 0 \) for \( \rho \in (\rho_1^*, +\infty) \). So, by Lemma 6 (ii), \( m_\rho \) admits a minimizer \( u_\rho \in B_p \). Setting with \( u_\rho^\theta = \theta^{(5/2)} u_\theta(\theta y) \in m_\theta \), with \( \theta > 1 \), we can find that
\[ m_{\theta_\rho} \leq I(u_\rho^\theta) = \frac{1}{2} \theta^{2+2a} \|u_\rho\|_\sigma^2 - \frac{1}{p} \theta^{(5/2)p-3} \|u_\rho\|_p^p \]
\[ = \theta^{2+2a} \left( \frac{1}{2} \|u_\rho\|_\sigma^2 - \frac{1}{p} \theta^{(5/2)p-5-2a} \|u_\rho\|_p^p \right) \]
\[ < \theta^{2+2a} \left( \frac{1}{2} \|u_\rho\|_\sigma^2 - \frac{1}{p} \|u_\rho\|_p^p \right) = \theta^{2+2a} I(u_\rho), \]
which implies that \( m_{\theta_\rho} < \theta m_\rho < m_\rho \) since \( m_\rho < 0 \) and \( \theta > 1 \). This tells that (iii) holds.

(iv) It follows from Lemma 8 that \( \rho_2^* \in (0, +\infty) \). On the other hand, from the definition of \( \rho_2^* \), it is direct to see that \( m_\rho = 0 \) for \( \rho \in (0, \rho_2^*) \). Now, it remains to show that \( m_\rho = -\infty \) if \( \rho \in (\rho_2^*, +\infty) \). Here, we claim that for any \( \rho \in (\rho_2^*, +\infty) \), there exists \( u^* \in B_p \) such that \( I(u^*) \leq 0 \). In fact, suppose that \( I(u) > 0 \) for all \( \rho \in (\rho_2^*, +\infty) \). Then for an arbitrary \( \rho \in (\rho_2^*, \rho) \), taking \( \tilde{v} \in B_p \) and \( \tilde{v}_0 = \theta^{(5/2)} \tilde{v}(\theta y) \) for \( \theta = \rho/\rho^* \), we find \( \tilde{v}_0 \in B_p \) and from (55),
\[ 0 < I(\tilde{v}_0) < \theta^{2+2a} I(\tilde{v}), \]
which tells that \( I(\tilde{v}) > 0 \) for \( \tilde{v} \in B_p \). This contradicts the definition of \( \rho_2^* \) since \( \rho \in (\rho_2^*, \rho) \) and so the claim holds.

Now we consider another scaling \( u_\rho^* = \theta^{(3/2)} u^*(\theta y) \) for all \( \theta > 0 \). Then \( u_\rho^* \in B_p \) and
\[ I(u_\rho^*) = \frac{1}{2} \theta^{2a} \|u^*\|_\sigma^2 - \frac{1}{p} \theta^{(3/2)p-3} \|u^*\|_p^p. \]

This yields \( I(u_\rho^*) \rightarrow -\infty \) as \( \theta \rightarrow +\infty \), and we get our result.

**Proof of Theorem 1.** We will divide it into three steps to prove Theorem 1.

Step 1. First, we prove that if \( p \in (2 + \sigma, 2 + (4/3)\sigma) \), \( m_\rho^* \) has a minimizer. Set \( \ell_n = \rho_1^* + (1/n) \) for all \( n \in \mathbb{N}^* \). Then
\( \ell_n \rightarrow \rho_1^* \) and using Lemmas 6 (iii) and 9 (ii), we find
\( m_{\ell n} \rightarrow m_{\rho_1^*} = 0. \)

So applying Lemmas 9 (iii) and 6 (ii), \( m_{\ell n} < 0 \) and \( m_{\ell n} \) have a minimizer, denoted by \( v_n \). By Lemma 5, \( \{v_n\} \) is bounded in \( H^p(\mathbb{R}^3) \).

Now, we claim that \( \|v_n\|_p \to 0 \) as \( n \to \infty \). Suppose that \( \|v_n\|_p \to 0 \) as \( n \to \infty \). Since \( I(v_n) < 0 \), similar to (49) and (48), we find
\[
\|v_n\|_2 \to 0, \quad I(v_n) \geq \frac{1}{2} \|v_n\|^2 - \frac{C}{p} \|v_n\|^{(3p-2)/2} \|\nabla v_n\|^{(1-\theta)},
\]
with \( \theta = (3(p-2))/2p \sigma \), which, combining (58), implies that \( I(v_n) \geq 0 \) for \( n \) large enough. This contradicts \( I(v_n) < 0 \) and then the claim holds.

With the same argument as the proof of (34), there exists \( \delta > 0 \) and a sequence \( \{y_n\} \subset \mathbb{R}^3 \) such that
\[
\int_{\mathbb{R}^3} |v_n(y_1 + y_n)|^2 dy \geq \delta > 0.
\]

Letting \( v_n = v_n(y_1 + y_n) \), then \( v_n \) is bounded in \( H^p(\mathbb{R}^3) \) and there exists \( v_0 \in H^p(\mathbb{R}^3) \) such that \( v_n \to v_0 \) in \( H^p(\mathbb{R}^3) \), and \( v_n \to v_0 \) in \( L^1(\mathbb{R}^3) \).

Thus, by (60), we can check that \( v_0 \neq 0 \). Finally, we come to show \( v_0 \) is a minimizer of \( m_{\rho_1^*} \). First, we have
\[
\lim_{n \to \infty} \|v_n\|_2 = \|v_0\|_2 + \lim_{n \to \infty} \|v_n - v_0\|_2 = \rho_1^*,
\]
and then by Lemma 6 (iii) and 9 (ii),
\[
\lim_{n \to \infty} I(v_n - v_0) \geq \lim_{n \to \infty} m_{\|v_n - v_0\|_2} = m_{\rho_1^*} \|v_0\|_2 = 0.
\]

On the other hand, from Lemma 9 (ii), we have
\[
\|v_0\|_2 \leq \rho_1^* \text{ and } m_{\|v_0\|_2} = 0 \text{ and then } I(v_0) < 0 \text{ is impossible.}
\]

So by (61) and (62), we can get \( I(v_0) = 0 = m_{\rho_1^*} \) and \( v_0 \) is a minimizer of \( m_{\|v_0\|_2} = 0 \). Now if we assume that \( \|v_0\|_2 < \rho_1^* \), using (55), one has
\[
m_{\rho_1^*} = m_{\rho_1^*} / \|v_0\|_2 \leq \frac{\rho_1^*}{\|v_0\|_2} \|v_0\|_2 = 0.
\]

This yields a contradiction with \( m_{\rho_1^*} = 0 \), and thus
\[
\|v_0\|_2 = \rho_1^*.
\]

Step 2. We will prove that \( m_{\rho} \) has a minimizer if and only if \( \rho \in (\rho_1^*, +\infty) \) if \( \rho \in (2 + \sigma, 2 + (4/3) \sigma) \).

Suppose that there is \( \rho_0 \in (0, \rho_1^*) \) such that \( m_{\rho_0} \) has a minimizer. Then from the definition of \( \rho_1^* \) and (55), we get that \( m_{\rho_0} = 0 \) and \( m_{\rho} < 0 \) for any \( \rho > \rho_0 \). This contradicts the definition of \( \rho_1^* \).

On the other hand, if \( \rho \in (\rho_1^*, +\infty) \), using Lemmas 9 (iii) and 6 (ii), \( m_{\rho} \) admits a minimizer.

Step 3. Finally, it follows from the definition of \( \eta(u) \) that for any \( u \in B_\rho \) and if \( \rho = 2 + (4/3) \sigma \),
\[
I(u) - \frac{1}{3(\rho - 2)} \eta(u) = \left( \frac{1}{2} - \frac{2\sigma}{3(\rho - 2)} \right) \|(-\Delta)^{3/2} u\|^2. \tag{64}
\]

Thus, if we assume that \( m_{\rho} \) has a minimizer \( u_\rho \in B_\rho \) for some \( \rho \in (\rho_1^*, +\infty) \), then from (64) and Lemma 9 (iv), we have
\[
-\infty = m_{\rho} = I(u_\rho) = 0, \tag{65}
\]
which completes the Proof of Theorem 1.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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References

[1] N. Laskin, “Fractional quantum mechanics and Lévy path integrals,” Physics Letters A, vol. 268, pp. 29–305, 2000.
[2] N. Laskin, “Fractional Schrödinger equation,” Physical Review E, vol. 66, no. 5, pp. 31–35, 2002.
[3] C. O. Alves and O. H. Miyagaki, “Existence and concentration of solution for a class of fractional elliptic equation in \( \mathbb{R}^N \) via penalization method,” Calculus of Variations and Partial Differential Equations, vol. 55, no. 3, 2016.
[4] V. Ambrosio, “Ground states for superlinear fractional Schrödinger equations in \( \mathbb{R}^N \),” Annales Academiae Scientiarum Fennicae Mathematica, vol. 41, pp. 745–756, 2016.
[5] G. M. Bisci and V. D. Radulescu, “Ground state solutions of scalar field fractional Schrödinger equations,” Calculus of Variations and Partial Differential Equations, vol. 54, no. 3, pp. 2985–3008, 2015.
[6] L. Caffarelli and L. Silvestre, “An extension problem related to the fractional Laplacian,” Communications in Partial Differential Equations, vol. 32, no. 8, pp. 1245–1260, 2007.
[7] G. M. Figueiredo and G. Siciliano, “A multiplicity result via Ljusternick-Schnirelmann category and Morse theory for a fractional Schrödinger equation in \( \mathbb{R}^N \),” Nonlinear Differential Equations and Applications, vol. 23, pp. 1–22, 2016.
[8] X. He and W. Zou, “Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities,” Calculus of Variations and Partial Differential Equations, vol. 55, no. 4, 2016.
[9] S. Secchi, “Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$,” *Journal of Mathematical Physics*, vol. 54, no. 3, p. 031501, 2013.

[10] X. Shang and J. Zhang, “Concentrating solutions of nonlinear fractional Schrödinger equation with potentials,” *Journal of Differential Equations*, vol. 258, no. 4, pp. 1106–1128, 2015.

[11] X. Zhang, B. Zhang, and D. Repovš, “Existence and symmetry of solutions for critical fractional Schrödinger equations with bounded potentials,” *Nonlinear Analysis*, vol. 142, pp. 48–68, 2016.

[12] X. Mingqi, V. D. Rădulescu, and B. Zhang, “Combined effects for fractional Schrödinger-Kirchhoff systems with critical nonlinearities,” *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 24, no. 3, pp. 1249–1273, 2018.

[13] X. Mingqi, V. D. Rădulescu, and B. Zhang, “Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions,” *Nonlinearity*, vol. 31, no. 7, pp. 3228–3250, 2018.

[14] X. Mingqi, V. D. Rădulescu, and B. Zhang, “Fractional Kirchhoff problems with critical Trudinger–Moser nonlinearity,” *Calculus of Variations and Partial Differential Equations*, vol. 58, no. 2, 2019.

[15] X. Mingqi, V. D. Rădulescu, and B. Zhang, “A critical fractional Choquard–Kirchhoff problem with magnetic field,” *Communications in Contemporary Mathematics*, vol. 21, no. 4, p. 1850004, 2019.

[16] M. Xiang, B. Zhang, and V. D. Rădulescu, “Superlinear Schrödinger–Kirchhoff type problems involving the fractional $p$–Laplacian and critical exponent,” *Advances in Nonlinear Analysis*, vol. 9, no. 1, pp. 690–709, 2019.

[17] J. Giacomoni, T. Mukherjee, and K. Sreenadh, “Positive solutions of fractional elliptic equation with critical and singular nonlinearity,” *Advances in Nonlinear Analysis*, vol. 6, no. 3, pp. 327–354, 2017.

[18] C. E. Torres Ledesma, “Multiplicity result for non-homogeneous fractional Schrodinger–Kirchhoff-type equations in $\mathbb{R}^n$,” *Advances in Nonlinear Analysis*, vol. 7, no. 3, pp. 247–257, 2018.

[19] D. B. Wang, Y. M. Ma, and W. Guan, “Least-energy sign-changing solutions for Kirchhoff–Schrödinger–Poisson systems in $\mathbb{R}^3$,” *Boundary Value Problems*, vol. 2019, 2019.

[20] D. B. Wang, T. J. Li, and X. Hao, “Least-energy sign-changing solutions for Kirchhoff–Schrödinger–Poisson systems in $\mathbb{R}^3$,” *Boundary Value Problems*, vol. 2019, 2019.

[21] D. B. Wang, H. B. Zhang, and W. Guan, “Existence of least-energy sign-changing solutions for Schrodinger-Poisson system with critical growth,” *Journal of Mathematical Analysis and Applications*, vol. 479, no. 2, pp. 2284–2301, 2019.

[22] M. Xiang and F. Wang, “Fractional Schrodinger-Poisson-Kirchhoff type systems involving critical nonlinearities,” *Nonlinear Analysis*, vol. 164, pp. 1–26, 2017.

[23] M. Xiang, B. Zhang, and V. D. Rădulescu, “Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian,” *Nonlinearity*, vol. 29, no. 10, pp. 3186–3205, 2016.

[24] M. Xiang, B. Zhang, and V. D. Rădulescu, “Existence of solutions for perturbed fractional $p$–Laplacian equations,” *Journal of Differential Equations*, vol. 260, no. 2, pp. 1392–1413, 2016.

[25] T. Bartsch and S. De Valeriola, “Normalized solutions of nonlinear Schrödinger equations,” *Archiv der Mathematik*, vol. 100, no. 1, pp. 75–83, 2013.

[26] J. Bellazzini, L. Jeanjean, and T. Luo, "Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations," *Proceedings of the London Mathematical Society*, vol. 107, no. 2, pp. 303–339, 2013.

[27] M. Colin, L. Jeanjean, and M. Squassina, "Stability and instability results for standing waves of quasilinear Schrödinger equations," *Nonlinearity*, vol. 23, no. 6, pp. 1353–1385, 2010.

[28] L. Jeanjean, “Existence of solutions with prescribed norm for semilinear elliptic equations,” *Nonlinear Analysis*, vol. 28, no. 10, pp. 1633–1659, 1997.

[29] L. Jeanjean and T. Luo, "Sharp nonexistence results of prescribed L 2-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations," *Zeitschrift für angewandte Mathematik und Physik*, vol. 64, no. 4, pp. 937–954, 2013.

[30] L. Jeanjean, T. Luo, and Z.-Q. Wang, "Multiple normalized solutions for quasilinear Schrödinger equations,” *Journal of Differential Equations*, vol. 259, no. 8, pp. 3894–3928, 2015.

[31] T. Luo, "Multiplicity of normalized solutions for a class of nonlocal Schrödinger–Poisson–Slater equations,” *Journal of Mathematical Analysis and Applications*, vol. 416, no. 1, pp. 195–204, 2014.

[32] H. Hajaiej, X. Yu, and Z. Zhai, "Fractional Gagliardo-Nirenberg and Hardy inequalities under Lorentz norms," *Journal of Mathematical Analysis and Applications*, vol. 396, no. 2, pp. 569–577, 2012.

[33] X. Ros-Oton and J. Serra, "The Pohozaev identity for the fractional Laplacian,” *Archive for Rational Mechanics and Analysis*, vol. 213, no. 2, pp. 587–628, 2014.