SUPPORT-BASED LOWER BOUNDS FOR THE POSITIVE SEMIDEFINITE RANK OF A NONNEGATIVE MATRIX

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ABSTRACT. The positive semidefinite rank of a nonnegative \((m \times n)\)-matrix \(S\) is the minimum number \(q\) such that there exist positive semidefinite \((q \times q)\)-matrices \(A_1, \ldots, A_m, B_1, \ldots, B_n\) such that \(S(k, \ell) = \text{tr}(A_k^* B_\ell)\).

The most important lower bound technique on nonnegative rank only uses the zero/non-zero pattern of the matrix. We characterize the power of lower bounds on positive semidefinite rank based on the zero/non-zero pattern.

**Keywords:** Factorization rank; positive semidefinite rank; lower bounds on factorization ranks; poset embedding.

1. INTRODUCTION

In this paper, \(k\) is a subfield of the field \(\mathbb{C}\) of complex numbers. For a matrix \(A\) over \(k\), we denote its entries by \(A(k, \ell)\). As usual, \(A^*(k, \ell) = A(\ell, k)\) is the Hermitian transpose, and \(A\) is positive semidefinite if \(A\) is Hermitian and all eigenvalues are nonnegative. We let \(k_+ := k \cap \mathbb{R}_+\) denote the nonnegative numbers in \(k\). A matrix is nonnegative if all its entries are nonnegative.

Let \(S\) be an \(m \times n\) nonnegative matrix over \(k\). The **nonnegative rank** of \(S\), denoted by \(\text{rk}_+(S)\), is the smallest number \(q\) such that there exists a nonnegative factorization of \(S\) of size \(q\), i.e., vectors \(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n \in k_+^q\) such that \(S(k, \ell) = (\xi_k \mid \eta_\ell)\), where the latter is the standard inner product in \(k^q\). Similarly, the **positive semidefinite rank** of \(S\), denoted by \(\text{rk}_{\mathbb{P}}(S)\), is the smallest number \(q\) such that there exists a positive semidefinite factorization of \(S\) of size \(q\), i.e., positive semidefinite \((q \times q)\)-matrices \(A_1, \ldots, A_m, B_1, \ldots, B_n\) such that \(S(k, \ell) = \text{tr}(A_k^* B_\ell)\), the latter expression being the usual inner product of two square matrices. These two definitions are examples of the concept of **factorization rank**, where one wishes to write the entries of a matrix \(S\) as inner products of vectors in some Hilbert space, with diverse restrictions on the set of vectors which are allowed.

The nonnegative rank is a well-known concept in Matrix Theory, see e.g. [17, 12, 3]. Generalizations to other types of factorizations are of interest there, too, see e.g. [3, 2]. In [2], the factors \(\xi_\ell\) and \(\eta_k\) are required to be in \(R^q\), where \(R\) is some fixed semiring, e.g., a sub-semiring of \(\mathbb{R}_+\). To the best of our knowledge, replacing \(R^q\) by a cone (in some inner product space over an ordered field) which is not a product of 1-dimensional cones appears to be a new concept initiated by Gouveia, Parrilo, and Thomas [10].

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There is a beautiful connection between (1) factorization ranks, (2) linear mappings between convex cones, and (3) combinatorial optimization, which was first noted by Yannakakis [22] in 1991 for the nonnegative rank, and later extended by Gouveia, Parrilo, and Thomas [10]. Driven by these connections, the last several years have seen a surge of interest in factorization ranks, particularly the nonnegative rank, and recently also the positive semidefinite rank. As far as the link to combinatorial optimization is concerned, bounds—upper or lower—on the nonnegative or positive semidefinite rank provide corresponding bounds on the sizes of linear programming or semidefinite programming formulations of problems. Finding lower bounds on these factorization ranks is a difficult task, and draws on methods from combinatorial matrix theory and communication complexity.

For the nonnegative rank, the easiest, most successful, and more or less only method (for an exception see [8]) for obtaining lower bounds just considers the support of the matrix. The support of $S$ is the matrix obtained from $S$ by replacing every non-zero entry by 1. For an $m \times n$ matrix $S$ whose support is $M$, the best lower bound obtainable by considering only the support is

$$\min \{ \operatorname{rk}_+(T) \mid \text{supp}(T) = M, \ T \geq 0 \}.$$  

This turns out to be equal to the Boolean rank of $M$ [12], the smallest $r$ such that there are $r$ dimensional binary vectors $x_1, \ldots, x_m \in \{0,1\}^r$ and $y_1, \ldots, y_n \in \{0,1\}^r$ satisfying $M(k, \ell) = \bigvee_{j=1}^r x_k(j) y_\ell(j)$. The Boolean rank arises in many contexts, and is also known as rectangle covering number [6], biclique covering number [18] or, after taking $\log_2$, nondeterministic communication complexity [22]. Most lower bounds on nonnegative rank actually lower bound the Boolean rank, including for the recent result showing super-polynomial lower bounds on the size of linear programming formulations of the traveling salesman problem [7]. Notable exceptions to this rule include results of [22] and [14, 15].

This paper deals with the question of giving lower bounds for the positive semidefinite rank. Given the situation for nonnegative rank, it is natural to ask the following question.

**Question.** How good can support-based lower bounds for positive semidefinite rank be?

In the case of the nonnegative rank, there are plenty of examples where the Boolean rank is exponential in the rank. Moreover, it is not difficult to see that even the Boolean rank of the support of a rank-3 matrix can be unbounded [3]. In the case of the positive semidefinite rank, we will see that this is not the case: the best possible support-based lower bound for the positive semidefinite rank coincides with the minimum rank over all matrices with the same support.

**Theorem 1.1.** For all 0/1-matrices $M$, we have

$$\min \{ \operatorname{rk}_+(T) \mid \text{supp}(T) = M, \ T \geq 0 \} = \min \{ \operatorname{rk}(T) \mid \text{supp}(T) = M \}$$

The theorem answers completely the question what lower bound information can be gained about the positive semidefinite rank from the zero/non-zero pattern of a nonnegative matrix: the best possible bound is the minimum possible rank of a matrix with the given zero/non-zero pattern. De Wolf [21] calls this number the nondeterministic rank, and...
shows that the logarithm of the nondeterministic rank characterizes nondeterministic quantum communication complexity. We therefore have the pleasing parallel that the logarithm of the best support based lower bound for nonnegative rank is the nondeterministic communication complexity, while the logarithm of the best support based lower bound on positive semidefinite rank is the nondeterministic quantum communication complexity.

In the situation of the nonnegative rank, there is a connection between the Boolean rank and embeddings of posets: The Boolean rank of $M$ is the minimum number of co-atoms of a truncated Boolean lattice into which a certain poset defined by $M$ can be embedded. We prove a corresponding statement for the best-possible support-based lower bound for the positive semidefinite rank in Section 3.

2. Factorizations

There is a well-known connection between linear mappings between cones and factorizations of corresponding matrices. In this section, let $k$ be a subfield of the field $\mathbb{R}$ of real numbers. Let $S$ be a non-negative matrix, and suppose that $S = AX$ for an $(m \times d)$-matrix $A$ and an $(d \times n)$-matrix $X$, both of rank $d$. In other words, we are given a rank-$d$ factorization of $S$. Let $Q_0 \subseteq k^d$ be the polyhedral cone generated by the columns of $X$, and denote by $Q_1$ the polyhedral cone $\{x \in k^d \mid Ax \geq 0\}$. Clearly, since $S \geq 0$, we have $Q_0 \subseteq Q_1$. The rank condition on $A$ and $X$ is equivalent to $Q_0$, $Q_1$ having dimension $d$.

A linear extension of $Q_0 \subseteq Q_1$ of size $q$ is a polyhedral cone $\tilde{Q}$ in some $k^q$ with $q$ facets for which there exists a linear mapping $\pi : k^q \rightarrow k^d$ such that $Q_0 \subseteq \pi(\tilde{Q}) \subseteq Q_1$. The following is a well-known fact, going back to Yannakakis.

Theorem 2.1 ([22], c.f. [6]). The minimum size of a linear extension of $Q_0 \subseteq Q_1$ equals the nonnegative rank of $S$.

A positive semidefinite extension of $Q_0 \subseteq Q_1$ of size $q$ is the intersection $\tilde{Q}$ of a linear subspace of some $\mathbb{R}^q \times q$ with the set of all positive semidefinite $(q \times q)$-matrices, for which there exists a linear mapping $\pi : k^q \rightarrow k^d$ such that $Q_0 \subseteq \pi(\tilde{Q}) \subseteq Q_1$. The following fact is a straightforward generalization of a recent result by Gouveia, Parrilo, and Thomas.

Theorem 2.2 ([10], c.f. [20]). The minimum size of a positive semidefinite extension of $Q_0 \subseteq Q_1$ equals the positive semidefinite rank of $S$.

For the reader who wishes to know more about the combinatorial optimization point of view, we recommend [6].

3. Poset Embedding Ranks

In this section we give a more combinatorial interpretation of the number $\min\{\text{rk}_{\oplus}(S) \mid \text{supp} \, S = \text{supp} \, M\}$.

Definition 3.1. Let $S$ be an $(m \times n)$-matrix. We define the poset $\mathcal{P}(S)$ of $S$ as

$$
\mathcal{P}(S) := \left( \{0\} \times \{1, \ldots, m\} \cup \{1\} \times \{1, \ldots, n\}, \preceq \right),
$$
where
\[(i, k) \preceq (j, \ell) :\iff i = 0 \land j = 1 \land S(k, \ell) \neq 0.\]

In other words, \(\mathcal{P}(S)\) is the poset whose Hasse-diagram is the bipartite graph with lower level vertex set the row set of \(S\) and upper level vertex set the column set of \(S\), and a vertex \(k\) of the lower level adjacent to a vertex \(\ell\) of the upper level if and only if \(S(k, \ell) = 0\).

**Definition 3.2.** Let \(P, Q\) be posets. An embedding of \(P\) into \(Q\) is a mapping \(j: P \to Q\) such that \(x \leq y \iff j(x) \leq j(y)\) holds for all \(x, y \in P\).

**Definition 3.3.** Let \(S\) be a matrix, \(P\) a set of posets, and \(\mathcal{I}: P \to \mathbb{N}\). We define the \(P\)-embedding rank of \(S\) as the infimum over all \(\mathcal{I}(Q)\) such that there exists an embedding of \(\mathcal{P}(S)\) into \(Q\).

As mentioned in the introduction, the Boolean rank of a Boolean matrix \(S\) is equal to the \(P\)-embedding rank of \(\mathcal{P}(S)\), with \(P\) the set of truncated Boolean lattices \(\mathcal{I}(Q)\) the number of co-atoms of \(Q\) [6].

By a subspace lattice we mean the lattice of all linear subspaces of \(k^q\), for some \(q \in \mathbb{N}\). If \(Q\) is the lattice of all subspaces of \(k^q\), then we let \(\mathcal{I}(Q) := q\). With \(\mathcal{L}\) the set of all subspace lattices, it is clear that the \(\mathcal{L}\)-embedding rank, which we denote by \(\text{rk}_*(M)\), equals the minimum dimension of a vector space in which there exist subspaces \(U_1, \ldots, U_m\) and \(V_1, \ldots, V_n\) such that

\[U_k \subseteq V_\ell\text{ if, and only if, } S(k, \ell) = 0.\]  

(1)

In the proof of Theorem 1.1, we will prove en passant the following proposition.

**Proposition 3.4.** For all nonnegative \((m \times n)\)-matrices \(S\), we have
\[\text{rk}_*(S) = \min\{\text{rk}_*(T) \mid \text{supp}(T) = \text{supp}(S), T \geq 0\}.\]

More importantly,
(a) Every positive semidefinite factorization \(S(k, \ell) = \text{tr}(A_k^*B_\ell)\) gives rise to a subspace-lattice embedding of \(S\) of the same size by letting \(U_k := \ker A_k\) and \(V_\ell := \text{im} B_\ell\).
(b) If \(k = \mathbb{R}\), then in (a) we may assume that \(\dim U_k \leq (\sqrt{8m + 1})/2\) and \(\text{codim} V_\ell \leq (\sqrt{8n + 1})/2\).

It will become clear in the proof that, while the minimum in the subspace-lattice embedding rank is always attained by (co-)dimension 1 subspaces, this is not true for the subspace-lattice embedding arising from a positive semidefinite factorization.

The proposition also shows that the situation for positive semidefinite factorizations mirrors that for nonnegative factorizations. The subspace-lattice embedding rank is the minimum “size” \(\mathcal{I}(Q)\) of a poset \(Q\) of a certain type into which \(\mathcal{P}(S)\) can be embedded. The importance of such “poset embedding ranks” for factorization ranks has been noted before: it is implicit in [6] that the Boolean rank of a boolean matrix \(S\) is equal to the minimum number of co-atoms in a co-atomic poset\(^1\) into which \(\mathcal{P}(M)\) can be embedded.

\(^1\)Recall that a poset is co-atomic if every element is a meet of maximal elements. The maximal elements are then called co-atoms.
4. Proof of Theorem 1.1 and Proposition 3.4

In this section we prove Theorem 1.1 and Proposition 3.4. For this, we show the following four lemmas.

**Lemma 4.1.** For all nonnegative matrices \( S \) we have
\[
\text{rk}_\oplus(S) \geq \min\{ \text{rk}(T) \mid \supp(T) = \supp(M) \}.
\]

**Lemma 4.2.** For all matrices \( S \)
\[
\text{rk}(S) \geq \text{rk}_*(S).
\]

The subspaces \( U_k \) in the embedding can be chosen of dimension 1, and the subspaces \( V_\ell \) of co-dimension 1 (and vice-versa).

**Lemma 4.3.** For all 0/1 matrices \( M \), we have
\[
\text{rk}_*(M) \geq \min\{ \text{rk}_\oplus(T) \mid \supp(T) = M, \ T \geq 0 \}.
\]

**Lemma 4.4.** Let \( S \) be a nonnegative matrix. Every positive semidefinite factorization \( S(k, \ell) = \text{tr}(A_k^*B_\ell) \) gives rise to a subspace-lattice embedding of \( S \) of the same size by letting \( U_k := \ker A_k \) and \( V_\ell := \im B_\ell \).

**Lemma 4.5.** Suppose \( k = \mathbb{R} \), and \( S \) is a nonnegative \((m \times n)\)-matrix. If a factorization of \( S \) of size \( q \) exists, then there exists one \( S(k, \ell) = \text{tr}(A_k^*B_\ell) \) with \( \text{rk} A_k \leq (\sqrt{8m + 1})/2 \) and \( \text{rk} B_\ell \leq (\sqrt{8m + 1})/2 \) for all \( k, \ell \).

Theorem 1.1 and the equation in Proposition 3.4 now follow by sticking together the inequalities. Proposition 3.4(a) follows from Lemma 4.4, and Itemb follows with Lemma 4.5.

We start with Lemma 4.1. Before we prove it, we note the following easy fact.

**Lemma 4.6.** Suppose that \( S(k, \ell) = \text{tr} A_k^*B_\ell \), \( k = 1, \ldots, m, \ \ell = 1, \ldots, n \) is a positive semidefinite factorization of \( S \) with matrices of order \( q \). Then there exists a finite union \( H \) of proper sub-varieties of \([k^q]^{m+n} \) such that for any \( (\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) \in ([k^q]^{m+n} \setminus H \) we have:
\[
\left( A_k\xi_k \mid B_\ell\eta_\ell \right) = 0 \iff S(k, \ell) = 0
\]

In the case of \( k \in \{ \mathbb{R}, \mathbb{C} \} \) one can state more easily that \( H \) is a set of Lebesgue-measure zero.

**Proof of Lemma 4.6.** To have \( A_k\xi_k \mid B_\ell\eta_\ell \neq 0 \) for all \( (k, \ell) \) with \( S(k, \ell) \neq 0 \), we need to choose \( (\xi, \eta) \) which do not satisfy any of the following equations:
\[
(\xi_k \mid A_kB_\ell\eta_\ell) = 0; \quad (k, \ell) \text{ with } S(k, \ell) \neq 0.
\]

Each of these equations defines a proper sub-variety of \([k^q]^{m+n} \), since 0 \( \neq S(k, \ell) = \text{tr} A_k^*B_\ell \) implies \( A_kB_\ell \neq 0 \). (This is most easily seen by realizing that, for \( X := \sqrt{A}, \ Y := \sqrt{B} \), we have \( \text{tr} A^*B = ||XY||^2 \) where \( ||Z|| := \text{tr} Z^*Z \) refers to the Frobenius- (or Hilbert-Schmidt-) norm of the matrix \( Z \).)

We can now complete the proof of Lemma 4.1.
We have to show that for every nonnegative real matrix $S$ there exists a matrix $T$ with $\text{supp}(T) = \text{supp}(S)$ and $\text{rk}_\oplus S \leq \text{rk} T$.

Let $S$ be nonnegative and real with $\text{rk}_\oplus S = q$, and let $A_k, B_\ell, \xi_k$ and $\eta_\ell$, $k = 1, \ldots, m$, $\ell = 1, \ldots, n$ as in Lemma 4.6. The matrix $T$ defined by $T(k, \ell) := (A_k \xi_k \mid B_\ell \eta_\ell)$ has the same support as $S$ and rank at most $q = \text{rk}_\oplus S$. □

We have to show subspaces of a $q$-dimensional vector space $W$ satisfying (1).

For $k = 1, \ldots, m$, denote by $s_k \in \mathbb{k}^n$ the vector which constitutes the $k$-th row of $S$, i.e., $s_k = S(k, \ldots)\top$, and let $U_k := \langle s_k \rangle$, the linear subspace of $\mathbb{k}^n$ generated by $s_k$. The ambient space for our construction is $W := \sum_{k=1}^m U_k$, a vector space of dimension $q$. For $\ell \in \{1, \ldots, n\}$, let $K_\ell$ denote the set of columns indices $k$ with $S(k, \ell) = 0$, and define

$$V_\ell := \sum_{k \in K_\ell} U_k = \text{span}\{s_k \mid S(k, \ell) = 0\}. $$

Clearly, $U_1, \ldots, U_m, V_1, \ldots, V_n$ are linear subspaces of a real vector space of dimension $q$. Moreover, by construction, we have $U_k \subseteq V_\ell$ whenever $S(k, \ell) = 0$. But since

$$V_\ell \subseteq \{x \in \mathbb{k}^n \mid x(k) = 0 \ \forall \ k \in K_\ell\},$$

we have that $S(k, \ell) \neq 0$ implies $U_k \not\subseteq V_\ell$, and we conclude (1). □

We have to show $\text{rk}_\oplus (M) \geq \min\{\text{rk}_\oplus (T) \mid \text{supp}(T) = M, T \geq 0\}$ for all 0/1 matrices $M$. For this, from subspaces of $\mathbb{k}^q$ satisfying (1) with $S$ replaced by $M$, we construct a matrix $T$ and a positive semidefinite factorization with matrices of order $q$.

Let $U_1, \ldots, U_m, V_1, \ldots, V_n$ such a collection of subspaces. Fix any inner product of $\mathbb{k}^q$, and denote by $A_k$ the matrix of the orthogonal projection of $\mathbb{k}^q$ onto $U_k$ and by $B_\ell$ the matrix of the orthogonal projection of $\mathbb{k}^q$ onto $V_\ell$, by $\text{Id}$ the $q \times q$ identity matrix, and let $B_\ell \equiv \text{Id} - B_\ell^\top$. Clearly $A_k$ and $B_\ell$ are positive semidefinite, and we have $A_k B_\ell = 0$ if and only if $M(k, \ell) = 0$. Thus, $T$ defined by $T(k, \ell) := \text{tr} A_k^\top B_\ell$ is a matrix with $\text{supp}(T) = M$, and $A_k, B_\ell$ a positive semidefinite factorization. □

From a positive semidefinite factorization with matrices of order $q$, we will construct subspaces of $\mathbb{k}^q$ satisfying (1).

Let a positive semidefinite factorization of $S$ be given, i.e., let $A_1, \ldots, A_m, B_1, \ldots, B_n$ be $q \times q$ real positive semidefinite matrices with $S(k, \ell) = \text{tr} A_k^\top B_\ell$. Now, for positive semidefinite matrices $A, B$, the two statements $\text{tr} A^* B = 0$ and $\bar{A} \bar{B} = 0$ are equivalent. But $A_k B_\ell = 0$ is equivalent to $U_k := \text{im} A_k \subseteq \ker B_\ell =: V_\ell$. □

4.1. The case $\mathbb{k} = \mathbb{R}$. For positive semidefinite matrices with real entries, the following is well-known.

**Lemma 4.7** (E.g. [1]). Let $A_1, \ldots, A_m$ be square matrices, and $\alpha_1, \ldots, \alpha_m$ numbers. If there exists a real positive semidefinite matrix $X$ such that $\text{tr}(A_j^\top X) = \alpha_j$ for $j = 1, \ldots, m$, then there exists such a matrix $X$ with rank at most $(\sqrt{8m} + 1)/2$. 
Proof of Lemma 4.5. This lemma is an easy consequence of Lemma 4.7. We leave the easy
details to the reader. □

4.2. A corollary. We close this section by stating the following combinatorial corollary of
Theorem 1.1

Corollary 4.8. Let S be a nonnegative matrix. The triangular rank of S is a lower bound
to the positive semidefinite rank of S. □

5. Outlook

As we have shown, support-based lower bounds on the positive semidefinite rank of
a matrix will always be at most the rank. (In fact, one might wonder whether the rank
of a matrix is always an upper bound on its positive semidefinite rank, but for each r ≥
3, Corollary 4.16 in [10] gives families of matrices with rank r and unbounded positive
semidefinite rank.) We illustrate how lower bounds which move beyond considering the
support might be based on subspace-lattice embeddings via Proposition 3.4.

Example 5.1. With k := R, consider the (n × n)-matrix S_n where S_n(i, j) = (i − j −
1)(i − j − 2)/2. We have rk S_n = 3 for all n, which follows from the expansion

(i − j − 1)(i − j − 2) = (i^2 − 3i + 1) + (j^2 + 3j + 1) − (2ij),

as each term in parenthesis can be expressed as a rank one matrix.

We conjecture that the positive semidefinite rank of S_n grows unboundedly with n. (Note
that the bound in [10, Corollary 4.16] does not apply since S_n is not the slack-matrix of a
polytope.) We can prove the following.

Claim. If n ≥ 6, the positive semidefinite rank of S_n is at least 4.

Proof of the claim. By considering the upper-left 6 × 6 submatrix, it suffices to prove the
claim for n = 6:

\[ S_6 = \begin{bmatrix}
1 & 3 & 6 & 10 & 15 & 21 \\
0 & 1 & 3 & 6 & 10 & 15 \\
0 & 0 & 1 & 3 & 6 & 10 \\
1 & 0 & 0 & 1 & 3 & 6 \\
3 & 1 & 0 & 0 & 1 & 3 \\
6 & 3 & 1 & 0 & 0 & 1
\end{bmatrix} \]

By contradiction, assume that A_1, . . . , A_6, B_1, . . . , B_6 is a positive semidefinite factorization
of S_6 of order 3.

Let U_k, V_ℓ be subspaces of \mathbb{R}^3 as in Proposition 3.4. Since for k ≥ 3, the kth row
contains zeros and non-zeros, we have dim U_k ≥ 1 for these k. For the same reason, we
have dim V_ℓ ≤ 2 for ℓ ≤ 4. If we had dim U_k = 2 for any k ≥ 3, then, for ℓ, ℓ' with
S_6(k, ℓ) = S_6(k, ℓ') = 0, it would follow that V_ℓ = V_ℓ', which is impossible since the ℓth
column differs from the ℓ'th. Thus we conclude that dim U_k = 1 for k ≥ 3. Similarly, we
have dim V_ℓ = 2 for ℓ ≤ 4.
But this means that $A_k, k \geq 3$, and $B_\ell, \ell \leq 4$, are rank-1 matrices. Choose vectors $u_k, v_\ell \in \mathbb{R}^3, k = 3, \ldots, 6, \ell = 1, \ldots, 4$, such that $A_k = u_ku_k^\top$, and $B_\ell = v_\ell v_\ell^\top$. For these $k, \ell$, we have

$$S_6(k, \ell) = \text{tr}(u_k u_k^\top v_\ell v_\ell^\top) = (u_k^\top v_\ell)^2 = Y(k, \ell)^2,$$

where we define the rank-3 matrix $Y(k, \ell) := u_k^\top v_\ell$. Since $Y(k, \ell) = \pm \sqrt{S_6(k, \ell)}$, we may enumerate all the $2^9$ possible choices for $Y$. Doing this, we see that all possible choices for $Y$ have rank at least 4, so no such $Y$ can exist, a contradiction. (We note that, independently, the technique based on entry-wise square roots has been used and further developed in [11].)

This example shows how using additional structure of a positive semidefinite factorization—for example that if $S$ has a rank-one semidefinite factorization of dimension $k$ then there is a matrix $Y$ of rank $k$ whose entrywise square is $S$—can lead to improved lower bounds. The following concrete problems motivate finding more general methods that can show positive semidefinite rank lower bounds larger than the rank.

For a real matrix $S$, can the positive semidefinite rank over $\mathbb{k} := \mathbb{R}$ be larger than the positive semidefinite rank over $\mathbb{k} := \mathbb{C}$? This mirrors the corresponding problem posed by Cohen & Rothblum [3, Section 5] (cf. [2]) regarding the nonnegative rank over the reals of rational matrices.

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