INTRODUCTION

0.1. Let $k$ be an algebraically closed field of characteristic $p \geq 0$ and let $G$ be a possibly disconnected reductive algebraic group over $k$ with a fixed connected component $D$. The identity component $G^0$ of $G$ acts on $D$ by conjugation. In the case where $D = G^0$, Steinberg [St65] has defined the open set $D_{\text{reg}}$ of regular elements in $G^0$ (a Lie algebra analogue of this set was earlier defined by Kostant); in [L15] we have defined a partition of $G^0$ into finitely many Strata, one of which is $D_{\text{reg}}$. The goal of this paper is to define (without assuming $D = G^0$) a partition of $D$ into finitely many Strata, (each of which is a union of $G^0$-conjugacy classes of fixed dimension) generalizing the partition [L15] of $G^0$ into Strata. (We use the term “Stratum” of $D$ to distinguish it from “stratum” of $D$ in [L03, §3]. In fact, every Stratum of $D$ is a finite union of strata of $D$.) To define the Strata of $D$ we associate to any $g \in D$ an irreducible representation $E_g$ (a variant of Springer’s representation) of the subgroup of the Weyl group given by the fixed point set of the action of $D$ on the Weyl group. Then we say that $g, g'$ in $D$ are in the same Stratum whenever $E_g, E_{g'}$ are isomorphic. We show that the Strata of $D$ are indexed by a set defined purely in terms of the Weyl group and its automorphism defined by $D$ (thus extending a result of [L15]). A definition of Strata of $D$ (different from the one in this paper) was sketched without proof (under some additional assumptions on $G, D$) in [L15, 6.1].

0.2. Notation. We fix a prime number $l$ invertible in $k$. Let $\overline{\mathbb{Q}}_l$ be an algebraic closure of the field of $l$-adic numbers. For any finite group $\Gamma$ let $\text{Irr}(\Gamma)$ be the set of irreducible representations (over $\overline{\mathbb{Q}}_l$) of $\Gamma$ (up to isomorphism). For a Weyl group $W$ and for $E \in \text{Irr}(W)$ we denote by $b_E$ the smallest integer $\geq 0$ such that $E$ appears in the $b_E$-th symmetric power of the reflection representation of $W$.

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1. Definition of Strata

1.1. Let $\mathcal{B} = \mathcal{B}_{G^0}$ be the flag manifold of $G^0$. Let $\mathcal{T} = \mathcal{T}_{G^0}$ be the set of maximal tori of $G^0$. Let $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_{G^0}$ be the set of all pairs $(T, B)$ where $B, T \in \mathcal{T}, T \subset B$. Let $W = W_{G^0}$ be the Weyl group of $G^0$, viewed as the set of $G^0$-orbits on $B \times \tilde{\mathcal{B}}$; this is naturally a (finite) Coxeter group with length function $w \mapsto |w|$. For $(B_1, B_2) \in \mathcal{B} \times \mathcal{B}$ let $\text{pos}(B_1, B_2) \in W$ be the $G^0$-orbit of $(B_1, B_2)$. (The product of $\text{pos}(B_1, B_2)$ with $\text{pos}(B_2, B_3)$ is $\text{pos}(B_1, B_3)$ provided that some maximal torus of $G^0$ is contained in $B_1, B_2, B_3$.)

Now $G$ acts by inner automorphisms on $G^0$ and this induces an action of $G/G^0$ on $W$. In particular, $D$, viewed as an element of $G/G^0$, defines an automorphism $[D] : W \to W$ whose fixed point set is denoted by $W^D$. For each $[D]$-orbit $o$ on $\{s \in W; |s| = 1\}$ we denote by $s_o$ the element of maximal length in the subgroup of $W$ generated by $o$; then the elements $s_o \in W^D$ for various $o$ are the simple reflections for a Coxeter group structure on $W^D$. For $g \in G$ let $g_s$ be the semisimple part of $g$ and let $g_u$ be the unipotent part of $g$.

1.2. For $g \in G$ let $\tilde{B}_g = \{(T, B) \in \tilde{\mathcal{B}}; gTg^{-1} = T, gBg^{-1} = B\}$. Let $D_{un} := \{g \in D; g = g_u\}$. For any $e = (T, B) \in \tilde{\mathcal{B}}$ let $S = S_{e:D}$ be the set of all $g \in D$ such that $e \in \tilde{B}_g$. This is a single orbit of $T$ acting on $D$ by left multiplication (resp. by right multiplication). Now $T$ is uniquely determined by $S$ (it is the set of all $s's^{-1}$ where $(s, s') \in S \times S$). Let $\ast S_{e:D} = \{g \in S; Z_G(g_s)^0 \subset T\}$; this is an open dense subset of $S$, see [L04, 3.11]. Let $\ast D = \cup_{e \in \tilde{\mathcal{B}}} \ast S_{e:D}$. This is a stratum of $G$ in the sense of [L04, 3.1]. By [L04, 3.13, 3.16], $\ast D$ is an irreducible locally closed subset of $D$ of dimension $\dim(G^0/T) + \dim S = \dim(G^0) = \dim D$. Hence $\ast D$ is an open dense subset of $D$. In the case where $D = G^0$, $\ast D$ is the set of regular semisimple elements in $G^0$. (In the general case, $\ast D$ is the same as the subset of $D$ associated in [L87, (2.3.2)] to $(L^0, \Sigma)$ with $(L^0, \Sigma)$ as in [L87, (2.3.1)] with $L^0 \in \mathcal{T}$ and with the added requirement [L87, (2.3.5)] which, contrary to what is stated in loc. cit., is not an automatic consequence of the conditions [L87, (2.3.1)].) Now $\ast D$ has a finite unramified covering $\ast \pi : \ast \tilde{D} \to \ast D$, see [L04, 3.13] or [L87, 2.5]. In our case we have $\ast \tilde{D} = \{(g, B') \in \ast D \times \mathcal{B}; gB'g^{-1} = B'\}$, $\ast \pi(g, B') = g$. If $g \in S_{e:D}$ with $e = (T, B) \in \tilde{\mathcal{B}}$, then $\ast \pi^{-1}(g)$ is the finite set $\{(g, B'); B' \in \mathcal{B}, T \subset B', \text{pos}(B, B') \in W^D\}$. As stated in loc. cit., $\ast \pi$ is a principal covering whose group is in our case $W^D$, which acts freely on $\ast \tilde{D}$ by $w : (g, B') \mapsto (g, B'')$ where $B''$ is defined by $T \subset B''$, $\text{pos}(B', B'') = w$.

Let $\tilde{D} = \{(g, B') \in D \times \mathcal{B}; gB'g^{-1} = B'\}$. Define $\pi : \tilde{D} \to D$ by $\pi(g, B') = g$. This is a proper surjective morphism. According to [L87, 2.6], [L04] (see also [S04] in the case where $D_{un} \neq \emptyset$), this morphism is small. (When $D = G^0$, this is an observation of [L81].) It follows that $\pi_i(Q_\ell)$ is the intersection cohomology
complex of $D$ with coefficients in the local system $\pi_1(\overline{Q}_t)$ on $*D$, which has a natural action of $W^D$. Then $\pi_1(\overline{Q}_t)[\dim D] = \oplus_E E \otimes \pi_1(\overline{Q}_t)[\dim D]_E$ where $E$ runs through $\text{Irr}(W^D)$, $\pi_1(\overline{Q}_t)[\dim D]_E$ is a simple perverse sheaf on $D$ and the action of $W^D$ on $\pi_1(\overline{Q}_t)$ restricts for each $E$ to the obvious $W^D$ action on $E$ tensor the identity on $\pi_1(\overline{Q}_t)[\dim D]_E$.

1.3. In this subsection we assume that $D_{un} \neq \emptyset$. Let $g \in D_{un}$ and let $c \subset D_{un}$ be the $G^0$-conjugacy class of $g$.

(a) There is a unique $E = E_g \in \text{Irr}(W^D)$ such that $(\pi_1(\overline{Q}_t)[\dim D]_E)[D_{un}]$ is (up to shift) the intersection cohomology complex of the closure of $c$ with coefficients in $\overline{Q}_t$. 

In the case where $D = G^0$ this is proved in [L84a]. Similar arguments apply in the general case, see [LO4, 8.2(b)], [S04].

1.4. For any $g \in G$ let $\mathcal{B}_g = \{B \in \mathcal{B}; gBg^{-1} = B\}$; this is a closed nonempty subvariety of $\mathcal{B}$, see [St68]. Let $\overline{\mathcal{B}}_g$ be the set of irreducible components of $\mathcal{B}_g$.

Recall that an element $h \in G$ is said to be quasi-semisimple (qss) if $\overline{\mathcal{B}}_h \neq \emptyset$. In this subsection we fix a qss element $h$ of $G$; let $H = Z_G(h)$ (here $Z_*(-)$ denotes a centralizer).

(a) $H$ is reductive, see [Sp82, II, 1.17], and $(H \cap G^0)/H^0$ consists of semisimple elements, see [DM94, 1.8];

(b) if $B \in \mathcal{B}_h$, then $B \cap H^0 \subset B_{H^0}$; for any $Z \in \mathcal{B}_h$, $B \mapsto B \cap H^0$ is an isomorphism $\tau_Z : Z \rightarrow \mathcal{B}_{H^0}$, see [Sp82, II, 1.17].

(c) if $T' \in \mathcal{T}_{H^0}$ then $T := Z_{G^0}(T') \in \mathcal{T}$; moreover $T$ is the unique maximal torus of $G^0$ that contains $T'$, see [Sp82, II, 1.15], [DM94, 1.8];

(d) if $T \in \mathcal{T}$ is such that $hTh^{-1} = T, hBh^{-1} = B$ for some $B \in \mathcal{B}$ containing $T$ then $T' := T \cap H^0$ is in $\mathcal{T}_{H^0}$, see [Sp82, II, 1.15], [DM94, 1.8].

1.5. Let $h, T', T, H$ be as in 1.4(c). If $n \in H^0, nT'n^{-1} = T'$, then clearly $nTn^{-1} = T$. Thus we have a well defined inclusion $N_{H^0}(T') \subset N_{G^0}(T)$ (here $N_*(-)$ denotes a normalizer); this carries $T'$ into $T$ hence induces a homomorphism

(a) $N_{H^0}(T')/T' \rightarrow N_{G^0}(T)/T$.

This homomorphism is injective. It is enough to show that $N_{H^0}(T') \cap T \subset T'$; this follows from $H^0 \cap T = T'$ (see 1.4). Now $\text{Ad}(h)$ defines an automorphism of each side of (a) and the map (a) is compatible with these automorphisms; moreover the automorphism on the left hand side of (a) is the identity. It follows that (a) restricts to

(b) an imbedding $N_{H^0}(T')/T' \rightarrow (N_{G^0}(T)/T)^h$

where $^h$ denotes the fixed point set of $\text{Ad}(h)$.

1.6. Let $g \in D$. Note that $g_*$ is qss in $G$ (see [St68]). Hence the results in 1.4, 1.5 apply with $h = g_*$. Let $G' = Z_G(g_*)$. Let $D'$ be the connected component of $G'$ that contains $g$. Let $\mathcal{B}' = \mathcal{B}_{G^0}$, $W' = W_{G^0}$. We define $[D'] : W' \rightarrow W'$ and $W'_{D'}$ in terms of $\overline{W}', D'$ in the same way as $[D] : W \rightarrow W$ and $W_D$ were defined in terms of $W, D$. For any $Z \in \mathcal{B}_g$, the assignment
$G^0$ – orbit of $(B'_1, B'_2) \mapsto pos(t_Z^{-1}(B'_1), t_Z^{-1}(B'_2))$
is a map $\tau_Z : W' \to W$.

Let $B'_1 \in B'$ and let $T'$ be a maximal torus of $B'$. Let $B = t_Z^{-1}(B') \in Z$, and let $T$ be the unique maximal torus of $G^0$ that contains $T'$; we have $T \subset B$. We have a commutative diagram

$$
N_{G^0}(T')/T' \xrightarrow{j} N_{G^0}(T)/T
$$

$$
\xrightarrow{a'} \quad \xrightarrow{a} \quad \xrightarrow{\tau_Z}
$$

where $a'(nT') = G^0$ – orbit of $(B', nB'n^{-1})$, $a(nT) = G^0$ – orbit of $(B, nBn^{-1})$ and $j$ is as in 1.5(a) with $h = g_s$, $H = G'$. (We use that for $n \in N_{G^0}(T')$ we have $t_Z(nBn^{-1}) = nB'n^{-1}$.). Since $j$ is an injective homomorphism it follows that $\tau_Z$ is an injective homomorphism.

Now let $Z, Z'$ in $\mathfrak{B}_{g_s}$; let $w = w_{Z,Z'}$ be the unique $G^0$-orbit on $B \times B$ that contains $(t_Z^{-1}(B'), t_Z^{-1}(B'))$ for any $B' \in B'$ (we view $w$ as an element of $W$). We show that

(a) $\tau_Z(w') = w\tau_Z(w')$ for any $w' \in W'$.

It is enough to show that if $B'_1, B'_2$ are in $B'$ then $t_Z^{-1}(B'_1), t_Z^{-1}(B'_2), t_Z^{-1}(B'_2)$ contain a common maximal torus and that $t_Z^{-1}(B'_1), t_Z^{-1}(B'_2), t_Z^{-1}(B'_2)$ contain a common maximal torus. Now let $T'$ be a maximal torus of $G^0$ contained in $B'_1 \cap B'_2$. Let $T$ be the unique maximal torus of $G^0$ such that $T' \subset T$, see 1.4(c). We have $T \subset t_Z^{-1}(B'_1), T \subset t_Z^{-1}(B'_2), T \subset t_Z^{-1}(B'_1), T \subset t_Z^{-1}(B'_2)$. Our claim is proved: the orbit represented by either side of (a) is that of $(t_Z^{-1}(B'_1), t_Z^{-1}(B'_2))$.

Now $G'$ acts (by conjugation) on $\mathfrak{B}_{g_s}$; this induces an action $g' : Z \mapsto g'(Z)$ of $G'/G^0$ on $\mathfrak{B}_{g_s}$. From the definition, for any $Z \in \mathfrak{B}_{g_s}$ and any $g' \in G'/G^0$ we have $g'\tau_Z = \tau_{g'(Z)}g'$ as maps $W' \to W$. In particular, taking $g' = g$ we obtain $[D](\tau_Z(w')) = \tau_{g(Z)}([D](w'))$ for any $w' \in W'$. Let $(\mathfrak{B}_{g_s})_g = \{ Z \in \mathfrak{B}_{g_s} ; g(Z) = Z \}$. This set is nonempty; indeed, for some $B \in B$ we have $gBg^{-1} = B$ (see [St68, p.49]) so that we have also $g_s B g_s^{-1} = B$ and if $Z_0$ is the irreducible component of $\mathfrak{B}_{g_s}$ that contains $B$ then $Z_0 \in (\mathfrak{B}_{g_s})_g$. Note that for any $Z \in (\mathfrak{B}_{g_s})_g$ we have $[D]\tau_Z = \tau_Z[D]$ as maps $W' \to W$; hence $\tau_Z$ restricts to an (injective) homomorphism $\tau^D_Z : W'D' \to W^D$.

Assume now that $Z, Z'$ are in $(\mathfrak{B}_{g_s})_g$; let $w = w_{Z,Z'}$ be as above. We show that

(b) $w \in W^D$.

It is enough to show that if $B' \in B'$ then

$$(gt_Z^{-1}(B')g^{-1}, g t_Z^{-1}(B')g^{-1}) = (t_Z^{-1}(gB'g^{-1}), t_Z^{-1}(gB'g^{-1}))$$
is in the same $G^0$-orbit as $(t_Z^{-1}(B'), t_Z^{-1}(B'))$. This is clear since $gB'g^{-1}, B'$ are Borel subgroups of $G^0$ hence are in the same $G^0$-orbit.

From (a),(b) we see that

(c) the injective homomorphism $\tau^D_Z : W'D' \to W^D$ defined for $Z \in (\mathfrak{B}_{g_s})_g$ is
We show:

\[ B \text{ is in the same } G \text{-orbit as } (t_Z^{-1}(B'), t_Z^{-1}(B')) \]

independent of the choice of \( Z \), up to composition with an inner automorphism of \( W^D \).

1.7. Let \( h \in D \). Assume that \( h \) is qss in \( G \). Then the results in 1.4, 1.5 apply to \( h \) and \( H = Z_G(h) \). Let \( B' = B_{H^0}, W' = W_{H^0} \). For any \( Z \in B_h \), the assignment

\[ H^0 \text{-orbit of } (B'_1, B'_2) \mapsto \text{pos}(t_Z^{-1}(B'_1), t_Z^{-1}(B'_2)) \]

is a map \( \tau_Z : W' \to W \).

Let \( B' \in B' \) and let \( T' \) be a maximal torus of \( B' \). Let \( B = t_Z^{-1}(B') \in Z \), and let \( T \) be the unique maximal torus of \( G^0 \) that contains \( T' \); we have \( T \subseteq B \). We have a commutative diagram

\[
\begin{array}{ccc}
N_{H^0}(T')/T' & \xrightarrow{j} & N_{G^0}(T)/T \\
\downarrow{a'} & & \downarrow{a} \\
W' & \xrightarrow{\tau_Z} & W \\
\end{array}
\]

where \( a'(nT') = H^0 \text{-orbit of } (B', nB'n^{-1}) \), \( a(nT) = G^0 \text{-orbit of } (B, nBn^{-1}) \) and \( j \) is as in 1.5(a). Since \( j \) is an injective homomorphism it follows that \( \tau_Z \) is an injective homomorphism. Now \( j \) has image contained in \((N_{G^0}(T)/T)^h \), see 1.5(b).

It follows that \( \tau_Z \) has image contained in \( W^D \) so that \( t_Z \) restricts to an (injective) homomorphism \( \tau^D_Z : W' \to W^D \).

Now let \( Z, Z' \in B_h \); let \( w = w_{Z, Z'} \) be the unique \( G^0 \)-orbit on \( B \times B \) that contains \((t_Z^{-1}(B'), t_Z^{-1}(B'))\) for any \( B' \in B' \) (we view \( w \) as an element of \( W \)). As in 1.6 we see that

(a) \( \tau^Z_Z(w')w = w \tau_Z(w') \) for any \( w' \in W' \).

We show that

(b) \( w \in W^D \).

It is enough to show that if \( B' \in B' \) then

\[ (ht_Z^{-1}(B')h^{-1}, ht_Z^{-1}(B')h^{-1}) = (t_Z^{-1}(hB'h^{-1}), t_Z^{-1}(hB'h^{-1})) \]

is in the same \( G^0 \)-orbit as \((t_Z^{-1}(B'), t_Z^{-1}(B'))\). This is clear since \( hB'h^{-1}, B' \) are Borel subgroups of \( H^0 \) hence are in the same \( H^0 \)-orbit.

From (a),(b) we see that

(c) the injective homomorphism \( \tau^D_Z : W' \to W^D \) defined for \( Z \in B_h \) is independent of the choice of \( Z \), up to composition with an inner automorphism of \( W^D \).

1.8. Assume that \( h \in D \) is a unipotent qss element in \( G \). Then

(a) \( Z_{G^0}(h) \) is connected, see [DM94,1.33]; hence it is equal to \( H^0 \) where \( H = Z_G(h) \);

(b) if \( h' \in D \) is a unipotent qss element in \( G \) then \( h' \) is \( G^0 \)-conjugate to \( h \), see [Sp82, II, 2.21].

We show:

(c) there exists a pinning of \( G^0 \) preserved by \( \text{Ad}(h) \).

If \( p = 0 \) then \( h = 1 \) and the result is trivial. Assume now that \( p > 1 \). We fix
a pinning of $G^0$; let $\Gamma$ be the group of all $g \in G$ such that Ad$(g)$ preserves this pinning (it may permute non-trivially the simple root subgroups). Clearly $\Gamma$ is a closed subgroup of $G$ which meets any connected component of $G$; let $\gamma \in \Gamma \cap D$. The image of $\gamma_n$ in $G/G^0$ is semisimple and of order dividing $p$ hence $\gamma_n = 1$ and $\gamma = \gamma_u \in \Gamma \cap D$. Now Ad$(\gamma_u)$ preserves our pinning hence $\gamma_u$ is qss in $G$; it is also unipotent hence by (b) we have $x\gamma_u x^{-1} = h$ for some $x \in G^0$. Since Ad$(\gamma_u)$ preserves our pinning, it follows that Ad$(x\gamma_u x^{-1})$ preserves some pinning of $G^0$. Hence Ad$(h)$ preserves some pinning of $G^0$. This proves (c).

We show:

(d) $W_{H^0}$ is isomorphic to $W^D$.

We choose a pinning of $G^0$ preserved by Ad$(h)$. Let $T \in T_{G^0}$ be associated to this pinning. Let $T' = T \cap Z_G(h)^0$. Recall from 1.5(b) the imbedding $N_{H^0}(T')/T' \to (N_{G^0}(T)/T)^h$. We show that this is surjective. Let $I$ be the image of the Tits section $N_{G^0}(T)/T \to N_{C^0}(h)^0$; this is defined in terms of the pinning hence $hT^h = I$. Let $n \in N_{C^0}(T)$ be such that $hn^{-1} \in nT$. We have $n \in n_0T$ where $n_0 \in I$ and $hnh^{-1} \in n_0T \cap I$. But if $n_0 t \in I$, $t \in T$ for $t = 1$. Thus $hnh^{-1} = n_0$ so that $n_0 \in N_{Z_{G^0}(h)}$. Using (a) we deduce $n_0 \in N_{H^0}(T) = N_{H^0}(T')$. This shows that our map is surjective hence bijective. This proves (d).

In our case 1.7(c) implies (using (d)):

(e) For any $Z \in B_h$, $\tau^D_Z : W_{H^0} \to W^D$ is an isomorphism; moreover, it is independent of the choice of $Z$, up to composition with an inner automorphism of $W^D$.

1.9. To any $g \in D$ we associate $E_g \in \text{Irr}(W^D)$ as follows.

(i) Assume first that $g$ is unipotent. Then $E_g$ is defined as in 1.3(a).

(ii) Next we assume that $g_u$ is central in $G$. Let $D_1$ be the connected component of $G$ that contains $g_u$. We have $W^D = W^{D_1}$. Then $E_{g_u} \in \text{Irr}(W^{D_1}) = \text{Irr}(W^D)$ is defined as in (i) (with $G, D, g$ replaced by $G, D_1, g_u$). We set $E_g = E_{g_u}$. If $g$ is unipotent this definition agrees with the one in (i).

(iii) Consider the general case. Let $D'$ be the connected component of $G' := Z_G(g_s)$ which contains $g$ and let $W' = W_{G'}$. Then $W'^{D'}$ is defined. Now $g_s$ is central in $G'$. Let $E'_g \in \text{Irr}(W'^{D'})$ be defined as $E_g$ in (ii) by replacing $G, D, W, g$ by $G', D', W'$, $g$. Let $E_g = j_{W'^{D'}}(E'_g) \in \text{Irr}(W^D)$ where $j_-(-)$ is the $j$-induction [LS79] from $W'^{D'}$ to $W^D$. (We use any one of the imbeddings of $W'^{D'}$ into $W^D$ given by 1.6(c).) Note that $E'_g$ is a good representation of $W$, in the sense of [L15, 0.2] so the $j$-induction can be applied to it. (To see this we can assume that $G^0$ is simple; if moreover $D = G^0$, we use the explicit knowledge of $E_g$ in (ii) for connected groups, see [AL82], [L84a],[LS85],[Sh80], [Sp85]; if $D \neq G^0$, all relevant irreducible representations are automatically good.) If $g_s$ is central, this definition agrees with the one in (ii). (In this case, $G' = G, D' = D, W' = W$.)

1.10. In the case where $D_{un} \neq \emptyset$ we define $S_1(G, D)$ to be the subset of $\text{Irr}(W^D)$ consisting of irreducible representations $E$ such that $E = E_g$ for some $g \in D_{un}$. In the general case, let $S_2(G, D)$ be the subset of $\text{Irr}(W^D)$ consisting of irreducible
representations $E$ such that $E = E_g$ for some $g \in D$. Equivalently, $S_2(G, D)$ is the subset of $\text{Irr}(W^D)$ consisting of irreducible representations $E$ such that $E = E_g$ for some $g \in D$ with $g_s$ of finite order in $G$. (We use the fact that for any $g \in D$ we can find $g' \in D$ such that $Z_G(g_s) = Z_G(g'_s)$ and $g'_s$ has finite order in $G$.) The representations in $S_2(G, D)$ are said to be the 2-special representations of $W^D$. (This set is not attached to the Weyl group $W^D$ but rather to the pair $(G, D)$.) In the case where $D = G^0$, $S_2(G, D)$ coincides with the set $S_2(W)$ defined in [L15, 1.1]. This follows from the description of $S_2(W)$ given in [L15, 2.1].

1.11. From the definition, the representations of $W^D$ in $S_2(G, D)$ are exactly those obtained by applying $j_{W_s}^{W_D}$ to any representation of $W_s^{D_1}$ in $S_1(Z_G(s), D_1)$ where $s$ is any semisimple element in $G$ and $D_1$ is any connected component of $Z_G(s)$ such that $(D_1)_{\text{un}} \neq \emptyset$ and $sD_1 \subset D$; here $W_s = W_{Z_G(s)^0}$ and $W_s^{D_1}$ is viewed as a subgroup of $W^D$ as in 1.6(c).

1.12. We define a partition of $D$ into subsets (called Strata): we say that $g, g'$ in $D$ are in the same Stratum if $E_g = E_{g'}$. (In the case where $D = G^0$, this partition coincides with the partition defined in [L15, 2.3].) Let Str$(D)$ be the set of Strata of $D$. Now $g \mapsto E_g$ defines a bijection $\text{Str}(D) \sim \rightarrow S_2(G, D)$. One can show that any Stratum of $D$ is a constructible subset of $D$. It is likely that

(b) any Stratum of $D$ is locally closed in $D$.

When $D = G^0$ this is proved in [C20].

1.13. Let $Z_{G^0}$ be the centre of $G$. Let $G' = G/Z_{G^0}$ and let $D'$ be the image of $D$ under $G \rightarrow G'$ (a connected component of $G'$). From the definition, the Strata of $G$ are exactly the inverse images of the Strata of $D'$ under $G \rightarrow G'$. Thus $\text{Str}(D), \text{Str}(D')$ are naturally in bijection. The fixed point set of $[D']$ on $W_{G'^0} = W$ is equal to $W^D$. From the definition we have $S_2(G', D') = S_2(G, D)$.

1.14. Let $(W_1, \gamma)$ be a pair consisting of a Weyl group $W_1$ and a Coxeter group automorphism $\gamma : W_1 \rightarrow W_1$ which is ordinary in the sense that, whenever $s \neq s'$ are simple reflections in the same orbit of $\gamma$, the product $ss'$ has order 2 or 3. Let $W_1^\gamma$ be the fixed point set of $\gamma : W_1 \rightarrow W_1$. We state our main result.

**Theorem 1.15.** To any $(W_1, \gamma)$ as in 1.14 one can associate canonically a subset $S_2(W_1, \gamma)$ of $\text{Irr}(W_1^\gamma)$ such that, whenever $(W_1, \gamma) = (W, [D])$ with $G, D, W, [D]$ as above, we have $S_2(W_1, \gamma) = S_2(G, D)$.

The proof is given in §4.

1.16. Following [Sp82, I, 1.1] we set

$$\text{rk}_D(G) = \max_{g \in D} (\text{dimension of a maximal torus of } Z_G(g^0)).$$

Let $g \in D$. Let $D_1$ be the connected component of $G' := Z_G(g_s)$ which contains $g_s$. We have

(a) $\text{rk}_D(G') = \text{rk}_D(G)$.

See [Sp82, II, 1.14]. From [Sp82, II, 10.15] we have
We have clearly 

$$d \implies \lambda$$

From the theory of Springer correspondence it is known that 

$$\dim B'_{g_u} = b_{E_g} \quad \text{where} \quad E_g' \in \text{Irr}(W' D'), \quad W', D' \text{ are as in 1.9(iii)}. \quad \text{Since the} \quad b\text{-function is preserved by} \quad j\text{-induction}, \quad \text{we have also} \quad \dim B'_{g_u} = b_{E_g}. \quad \text{Thus (b) becomes (using also (a))}:$$

(c) $$\dim Z_{G'}(g_u) = 2b_{E_g} + \text{rk}_D(G).$$

We have clearly $$Z_{G'}(g_u) = Z_G(g) \text{ and dim } Z_G(g) = \dim Z_{G^0}(g). \quad \text{It follows that}$$

(d) $$\dim Z_{G^0}(g) = 2b_{E_g} + \text{rk}_D(G).$$

From this we see that $$\dim Z_{G^0}(g)$$ is constant when g varies in a fixed Stratum of D. We also see that

(e) any $$\Sigma \in \text{Str}(D)$$ is a union of $$G^0$$-orbits of fixed dimension, namely $$\dim G^0 - 2b_{E_g} - \text{rk}_D(G) \text{ where } g \in \Sigma.$$ In the case $$D = G^0$$ this is proved in [L15, 2.4].

2. Bipartitions

2.1. A bipartition is a sequence $$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$ in N such that $$\lambda_m = 0$$ for large m and $$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots, \lambda_2 \geq \lambda_4 \geq \lambda_6 \geq \ldots.$$ We write $$|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \ldots.$$ For $$n \in \mathbb{N}$$ let $$BP^n$$ be the set of bipartitions $$\lambda$$ such that $$|\lambda| = n.$$ Let $$(e, e') \in \mathbb{N} \times \mathbb{N}.$$ We say that a bipartition $$(\lambda_1, \lambda_2, \lambda_3, \ldots)$$ has excess $$(e, e')$$ if $$\lambda_i + e \geq \lambda_{i+1}$$ for $$i = 1, 3, 5, \ldots$$ and $$\lambda_i + e' \geq \lambda_{i+1}$$ for $$i = 2, 4, 6, \ldots.$$ Let $$BP_{e, e'}$$ be the set of bipartitions which have excess $$(e, e').$$ For $$n \in \mathbb{N}$$ let $$BP^n_{e, e'} = BP^n \cap BP_{e, e'}.$$ A partition is the same as a bipartition of excess $$(0, 0).$$ A bipartition is the same as an ordered pair of partitions $$(\lambda_1, \lambda_3, \lambda_5, \ldots), (\lambda_2, \lambda_4, \lambda_6, \ldots).$$

If $$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots), \quad \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$ are bipartitions, then $$\lambda + \lambda = (\lambda_1 + \lambda_1, \lambda_2 + \lambda_2, \lambda_3 + \lambda_3, \ldots)$$ is a bipartition; moreover if $$(e, e') \in \mathbb{N}_1, (e, e') \in \mathbb{N}$$ and $$\lambda \in BP_{e, e'}, \lambda \in BP_{e, e'}$$ then $$\lambda + \lambda \in BP_{e + e', e + e'}.$$

2.2. For $$(e, e') \in \mathbb{N} \times \mathbb{N}, \quad m \geq 1$$ let $$mBP^n_{e, e'}$$ be the set of all sequences $$\Lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{2m+1})$$ in N such that $$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{2m+1}, \quad \lambda_i - \lambda_{i+1} \geq e + e'$$ for $$i = 1, 2, \ldots, 2m - 1, \quad \lambda_{2m} = e', \quad \lambda_{2m+1} = 0;$$ let $$mBP^n_{e, e'}$$ be the set of all sequences $$\Lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{2m})$$ in N such that $$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{2m}, \quad \lambda_i - \lambda_{i+1} \geq e + e'$$ for $$i = 1, 2, \ldots, 2m - 2, \quad \lambda_{2m-1} = e, \quad \lambda_{2m} = 0.$$ Given $$n \in \mathbb{N}$$ we define $$m\kappa : BP^n_{e, e'} \to mBP^n_{e, e'}$$ by $$(\lambda_1, \lambda_2, \lambda_3, \ldots) \mapsto (\lambda_1, \lambda_2, \lambda_3, \ldots).$$

Then:

(a) a bijection of $$BP^n_{e, e'}$$ onto a subset $$mBP^n_{e, e'}$$ of $$mBP^n_{e, e'}.$$ Given $$n \in \mathbb{N}$$ we define $$m\kappa' : BP^n_{e, e'} \to mBP^n_{e, e'}$$ by $$(\lambda_1, \lambda_2, \lambda_3, \ldots) \mapsto (\lambda_1, \lambda_2, \lambda_3, \ldots)$$ where
\[\Lambda_1 = \lambda_1 + me + (m - 1)e', \Lambda_2 = \lambda_2 + (m - 1)(e + e'),\]
\[\Lambda_3 = \lambda_3 + (m - 1)e + (m - 2)e', \Lambda_4 = \lambda_4 + (m - 2)(e + e'), \ldots,\]
\[\Lambda_{2m-1} = \lambda_{2m-1} + e, \Lambda_{2m} = \lambda_{2m}.\]
(We choose \(m\) large enough so that \(\lambda_{2m-1} = \lambda_{2m} = 0\) for all \(\lambda \in BP^n_{e,e'}\).) Then \(m\kappa'\) defines \(\hat{m}\).

(b) a bijection of \(BP^n_{e,e'}\) onto a subset \(m\hat{BP}'^n_{e,e'}\) of \(m\hat{BP}_{e,e'}\).

**Proposition 2.3.** Let \(N \geq 2, u \geq 2\). Let \(c \in [0, N]\). We set \(\epsilon = 0\) if \(c = 0\), \(\epsilon = 1\) if \(c > 0\). Let \(A_1 \leq A_2 \leq \cdots \leq A_u\) be a sequence in \(\mathbb{N}\) such that \(A_1 = 0\), \(A_2 = c\), \(A_3 - A_1 \geq N, A_4 - A_2 \geq N, \ldots, A_u - A_{u-2} \geq N\). There exist two sequences \(B_1 \leq B_2 \leq \cdots \leq B_u, C_1 \leq C_2 \leq \cdots \leq C_u\) in \(\mathbb{N}\) such that \(A_i = B_i + C_i\) for all \(i\), \(B_1 = 0, B_2 = \epsilon, B_3 - B_1 \geq 1, B_4 - B_2 \geq 1, \ldots, B_u - B_{u-2} \geq 1, C_3 - C_1 \geq N - 1, C_4 - C_2 \geq N - 1, \ldots, C_u - C_{u-2} \geq N - 1\).

We say that \(i \in [1, u]\) is single if \(A_i\) appears exactly one in the sequence \(A_1, A_2, \ldots, A_u\). We define \(B_s \in \mathbb{N}\) by induction on \(s\). We set \(B_1 = 0, B_2 = \epsilon\).

Assume that \(s \in [3, u]\) and that \(B_i\) is already defined for \(i \in [1, s-1]\).

(i) If \(s\) is single and \(s-1\) is not single we set \(B_s = B_{s-1} + 1\).

(ii) If \(s\) and \(s-1\) are singles and \(s-2\) is not single we set \(B_s = B_{s-1}\).

(iii) If \(s, s-1, s-2\) are singles and \(B_{s-1} = B_{s-2}\) we set \(B_s = B_{s-1} + 1\).

(iv) If \(s, s-1, s-2\) are singles and \(B_{s-1} \neq B_{s-2}\) we set \(B_s = B_{s-1}\).

(v) If \(s\) is not single and \(s-1\) is single we set \(B_s = B_{s-1} + 1\).

(vi) If \(s\) is not single, \(s-1\) is not single and \(A_{s-1} < A_s\) we set \(B_s = B_{s-1} + 1\).

(vii) If \(s\) is not single, \(s-1\) is not single and \(A_{s-1} = A_s\) we set \(B_s = B_{s-1}\).

This completes the inductive definition of \(B_s\).

We show by induction on \(s \in [3, u]\) that

(a) \(A_s - A_{s-1} - B_s + B_{s-1} \geq 0\),

(b) \(A_s - A_{s-2} - B_s + B_{s-2} - (N - 1) \geq 0\).

Assume that \(s\) is as in (i) so that \(B_s = B_{s-1} + 1 = B_{s-2} + 1\). We have \(A_s \geq A_{s-1} + 1\), hence

\[A_s - A_{s-1} - B_s + B_{s-1} \geq 1 - B_s + B_{s-1} \geq 0,\]
\[A_s - A_{s-2} - B_s + B_{s-2} - (N - 1) = A_s - A_{s-2} - 1 - (N - 1) \geq N - N = 0.\]

Assume that \(s\) is as in (ii) so that \(B_s = B_{s-1} = B_{s-2} + 1\). We have

\[A_s - A_{s-1} - B_s + B_{s-1} = A_s - A_{s-1} \geq 0,\]
\[A_s - A_{s-2} - B_s + B_{s-2} - (N - 1) = A_s - A_{s-2} - N \geq 0.\]

Assume that \(s\) is as in (iii) so that \(B_s = B_{s-1} + 1 = B_{s-2} + 1\). We have

\[A_s \geq A_{s-1} + 1,\]
\[A_s - A_{s-1} - B_s + B_{s-1} \geq 1 - B_s + B_{s-1} \geq 0,\]
\[A_s - A_{s-2} - B_s + B_{s-2} - (N - 1) = A_s - A_{s-2} - N \geq 0.\]

Assume that \(s\) is as in (iv) so that \(B_s = B_{s-1} = B_{s-2} + 1\). We have

\[A_s - A_{s-1} - B_s + B_{s-1} = A_s - A_{s-1} \geq 0,\]
\[A_s - A_{s-2} - B_s + B_{s-2} - (N - 1) = A_s - A_{s-2} - N \geq 0.\]

Assume that \(s\) is as in (v) so that \(B_s = B_{s-1} + 1 = B_{s-2} + 1\). We have

\[A_s \geq A_{s-1} + 1,\]
\[A_s - A_{s-1} - B_s + B_{s-1} \geq 1 - B_s + B_{s-1} \geq 0,\]
\[A_s - A_{s-2} - B_s + B_{s-2} - (N - 1) = A_s - A_{s-2} - N \geq 0.\]
\[ A_s - A_{s-1} - B_s + B_{s-1} \geq 1 - B_s + B_{s-1} \geq 0, \]
\[ A_s - A_{s-1} - B_s + B_{s-1} = A_s - A_{s-1} - 1 \geq 0, \]
\[ A_s - A_{s-2} - B_s + B_{s-2} - (N - 1) = A_s - A_{s-2} - 1 - (N - 1) \geq 0. \]
Assume that \( s \) is as in (vi) so that \( B_s = B_{s-1} + 1, B_{s-1} = B_{s-2}, A_s > A_{s-1}. \) We have
\[ A_s - A_{s-1} - B_s + B_{s-1} = A_s - A_{s-1} = 1 \geq 0, \]
\[ A_s - A_{s-2} - B_s + B_{s-2} - (N - 1) = A_s - A_{s-2} - 1 - (N - 1) \geq 0. \]
This completes the inductive proof of (a),(b). We set \( C_i = A_i - B_i. \) Then \( B_i, C_i \)
satisfy the requirements of the proposition. (We have \( C_1 = 0, C_2 = c - \epsilon \geq C_1. \)) The proposition is proved.

**Corollary 2.4.** Let \((e, e') \in \mathbb{N} \times \mathbb{N}, (\tilde{e}, \tilde{e}') \in \mathbb{N} \times \mathbb{N}. \) Let \( \lambda \in BP_{e+\tilde{e}, e'+\tilde{e}'} \). There exist \( \lambda' \in BP_{e', \tilde{e}'} \) such that \( \lambda = \lambda' + \tilde{\lambda}. \)

Using 2.2(a) with \( m \gg 0, \) we see that it is enough to show that
(a) for any \( \Lambda \in mBP_{e+\tilde{e}, e'+\tilde{e}'} \) there exist \( \Lambda' \in \mathbb{mBP}_{e, \tilde{e}} \), \( \tilde{\lambda} \in \mathbb{mBP}_{\tilde{e}, \tilde{e}'} \) such that \( \Lambda = \lambda' + \tilde{\lambda} \) (addition coordinate by coordinate).

The following statement clearly implies (a):
(b) for any \((f, f') \in \mathbb{N} \times \mathbb{N} \) and any \( \Lambda = (A_1, A_2, \ldots, A_{2m+1}) \in mBP_{f,f'} \) there exist \( \Lambda^1, \Lambda^2, \ldots, \Lambda^f \) in \( mBP_{1,0} \) and \( \Lambda', \Lambda'' \) in \( mBP_{0,1} \) such that \( \Lambda = \Lambda^1 + \Lambda^2 + \cdots + \Lambda^f + \Lambda' + \Lambda'' + \Lambda^0 \). We prove (b). Using 2.3 with \( u = 2m + 1, N = f + f' \) and \((A_1, A_2, \ldots, A_{2m+1}) = (A_{2m+1}, A_{2m}, \ldots, A_1), c = f', \) we see that if \( f' \geq 1, \) then \( \Lambda = \Lambda^1 + \tilde{\Lambda} \) where \( \Lambda^1 \in mBP_{0,1}, \tilde{\Lambda} \in \mathbb{mBP}_{f,f'-1} \). Using this repeatedly, we see that \( \Lambda \) is of the form \( \Lambda^1 + \Lambda^2 + \cdots + \Lambda^f + \Lambda'' + \Lambda^0 \) where \( \Lambda^1, \Lambda^2, \ldots, \Lambda^f \) are in \( mBP_{0,1} \) and \( \tilde{\Lambda} \in \mathbb{mBP}_{0,0} \). Thus it is enough to prove (b) assuming in addition that \( f' = 0. \)

Using 2.3 with \( u = 2m + 1, N = f \) and \( (A_1, A_2, \ldots, A_{2m+1}) = (A_{2m+1}, A_{2m}, \ldots, A_1), c = f, \) we see that if \( f \geq 1, \) then \( \Lambda = \Lambda^1 + \tilde{\Lambda} \) where \( \Lambda^1 \in mBP_{1,0}, \tilde{\Lambda} \in \mathbb{mBP}_{f-1,0}. \)

2.5. Some special cases of 2.4 have been used (without proof) in three surjectivity statements in [L15] (see [L15, p.348, line 8],[L15, p.349, line 16 of 3.7], [L15, p.351, line 5]), namely that addition defines surjective maps \( BP_{1,1} \times BP_{1,1} \rightarrow BP_{2,2}, \)
\( BP_{2,0} \times BP_{0,2} \rightarrow BP_{2,2}, BP_{0,2} \times BP_{0,2} \rightarrow BP_{0,4} \). Other special cases of 2.4 are contained in [L09], namely that addition defines surjective maps \( BP_{1,0} \times BP_{0,1} \rightarrow BP_{1,1}, BP_{1,0} \times BP_{1,0} \rightarrow BP_{2,0}, BP_{0,1} \times BP_{0,1} \rightarrow BP_{0,2}. \)

3. Examples

3.1. Assume that \( p \neq 2, \) that \( G = O(V) \) where \( V \) is a \( k \)-vector space of even
dimension $N \geq 4$ with a fixed nondegenerate symmetric bilinear form and that $D = O(V) - SO(V)$. Let $g \in D$. For any $c \in \mathbb{k}^*$ let $V_c$ be the $c$-eigenspace of $g_s : V \to V$. Let $d_c = \dim V_c$. For any $c \in \mathbb{k}^*$ such that $c^2 \neq 1$ let $\lambda^c_1 \geq \lambda^c_2 \geq \ldots$ be the partition of $d_c$ whose nonzero terms are the sizes of the Jordan blocks of $g_u : V_c \to V_c$. For $c \in \mathbb{k}^*$ such that $c^2 = 1$ let $\nu^c_1 \geq \nu^c_2 \geq \ldots$ be the partition of (the odd number) $d_u$ whose nonzero terms are the sizes of the Jordan blocks of the unipotent element $g_u \in SO(V_c)$. Let $\lambda^c = (\lambda^c_1, \lambda^c_2, \lambda^c_3, \ldots)$ be the bipartition in $BP_{20}^{(d_u, -1)/2}$ associated to $\nu^c_1 \geq \nu^c_2 \geq \ldots$ in [L15, 3.6(a)]. Note that $\lambda^c$ is such that the Springer representation attached to the unipotent element $g_u \in SO(V_c)$ (an irreducible representation of a Weyl group of type $B$) is indexed by $\lambda^c$. (We use results in [L84a].) Define $g^c = (g^c_1, g^c_2, g^c_3, \ldots)$ by $g^c_j = \sum_c \lambda^c_j$ where $c$ runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbb{k}^*$. Note that $g^c \in BP_{40}^{(N-2)/2}$. Thus $g \mapsto g^c$ defines a map $D \to BP_{40}^{(N-2)/2}$. From the definitions we see that the fibres of this map are precisely the Strata of $D$. (We use the description of the $j$-induction in classical Weyl groups groups given in [L09].) This map is also surjective: indeed, by 2.4, for any $\mu \in BP_{40}^{(N-2)/2}$ we can find $\mu' \in BP_{20}, \mu'' \in BP_{20}$ such that $\mu = \mu' + \mu''$; we have $\mu' \in BP_{20}, \mu'' \in BP_{20}$ for some $k \geq 0, k' \geq 0$ such that $k + k' = (N - 2)/2$ and it remains to use that any bipartition in $BP_{20}^k$ (resp. $BP_{20}^{k'}$) represents the Springer representation associated to a unipotent element in $SO_{2k+1}$ (resp. $SO_{2k'+1}$); see [L15, 3.6(a)]. We see that

(a) $D \mapsto BP_{40}^{(N-2)/2}, g \mapsto g^c$ defines a bijection from the set of Strata of $D$ to $BP_{40}^{(N-2)/2}$. Thus, $S_2(G, D)$ can be identified with $BP_{40}^{(N-2)/2}$.

3.2. Assume that $V$ is a $\mathbb{k}$-vector space of dimension $N$. Let $V^*$ be the vector space dual to $V$. For $x \in V, \xi \in V^*$ we set $(x, \xi) = \xi(x)$. Let $D$ be the set of all linear isomorphisms $g : V \cong V^*$. For $g \in D$ we define an isomorphism $\hat{g} : V^* \to V$ by $(\hat{g}(\xi), g(x)) = (x, \xi)$ for any $x \in V, \xi \in V^*$. For $g \in GL(V)$ we define $\hat{g} \in GL(V^*)$ by $(\hat{g}(x), \hat{g}(\xi)) = (x, \xi)$ for any $x \in V, \xi \in V^*$. Let $G = GL(V) \sqcup D$. For $g, g' \in G$ we define $g*g' \in G$ to be $gg'$ if $g' \in GL(V)$ and $gg'$ if $g' \in D$. Then $(g, g') \mapsto g*g'$ defines a group structure on $G$. Note that $G$ has an obvious structure of algebraic group over $\mathbb{k}$ with $G^0 = GL(V)$ and with $D$ being another connected component. We now assume that this $G, D$ is the same as that in 1.1.

3.3. In the setup of 3.2 assume that $p \neq 2$. Let $s \in D$ be semisimple in $G$. Define a bilinear form $<,>_s : V \times V \to \mathbb{k}$ by $<x, x'>_s = (x, s(x'))$. For any $c \in \mathbb{k}^*$ let $V_c$ be the $c$-eigenspace of $s* : V \to V$; we have

$$V_c = \{x \in V; <x, y>_s = c <x, y>_s \text{ for any } y \in V\};$$

let $d_c = \dim V_c$. It follows that if $c \in \mathbb{k}^*, c' \in \mathbb{k}^*, cc' \neq 1$ and $x \in V_c, y \in V_{c'}$ then $<x, y>_s = cc'<x, y>_s$ so that $<x, y>_s = 0$. It follows that $<,>_s$ defines an isomorphism $V_c \to V^*_{c^{-1}}$. If $h \in Z_{G^0}(s)$ then $h$ restricts to an isomorphism $h_c : V_c \to V_c$ for any $c \in \mathbb{k}^*$. Now $h_{-1} : V_{-1} \to V_{-1}$ preserves the nondegenerate symplectic form $<,>_s$ restricted to $V_{-1}; h_{1} : V_1 \to V_1$ preserves the nondegenerate
symmetric bilinear form \(<,>_s\) restricted to \(V_1\); if \(c \in k^* - \{1,-1\}\) then \(h_c : V_c \rightarrow V_c\) is related to \(h_{c-1} : V_{c-1} \rightarrow V_{c-1}\) by \(<h_c(x), h_{c-1}(y)>_s = <x, y>_s\) for \(x \in V_c, y \in V_{c-1}\); moreover, \(h \mapsto (h_c)\) is an isomorphism of \(Z_{GO}(s)\) onto \(Sp(V_{-1}) \times O(V_1) \times \prod_{c \in C'} GL(V_c')\) where \(C'\) is a set of representatives for the orbits of \(c \mapsto c^{-1}\) on \(k^* - \{1,-1\}\).

Now assume that \(g \in D\) and \(s = g_s\). Then \(g_u \in Z_{GO}(s)\) gives rise to

- a unipotent element \((g_u)_{-1} \in Sp(V_{-1})\) with Jordan blocks of sizes \(\nu_1 \geq \nu_2 \geq \ldots\) (a partition of \(d_{-1}\));
- a unipotent element \((g_u)_1 \in O(V_1)\) with Jordan blocks of sizes \(\nu_1' \geq \nu_2' \geq \ldots\) (a partition of \(d_1\));
- a unipotent element \((g_u)_c \in GL(V_c)\) with Jordan blocks of sizes \(\lambda_1^c \geq \lambda_2^c \geq \ldots\) (a partition of \(d_c\)) for any \(c \in C'\).

Let \((\lambda_1, \lambda_2, \lambda_3, \ldots)\) be the bipartition in \(BP_{d_{-1}^{1/2}}^{d_{1}/2}\) associated to \(\nu_1 \geq \nu_2 \geq \ldots\) in [L15, 3.4(a)]. Note that \((\lambda_1, \lambda_2, \lambda_3, \ldots)\) is such that the Springer representation attached to the unipotent element \(g_u \in Sp(V_{-1})\) (an irreducible representation of a Weyl group of type \(C\)) is indexed by \((\lambda_1, \lambda_2, \lambda_3, \ldots)\). (We use results in [L84a].)

Let \((\lambda_1', \lambda_2', \lambda_3', \ldots)\) be the bipartition in \(BP_{d_{1}/2}^{d_{1}/2}\), if \(d_1\) is odd (resp. in \(BP_{0,2}^{d_{1}/2}\), if \(d_1\) is even) associated to \(\nu_1' \geq \nu_2' \geq \ldots\) in [L15, 3.6(a)] (resp. [L15, 3.6(b)]). Note that \((\lambda_1', \lambda_2', \lambda_3', \ldots)\) is such that the Springer representation attached to the unipotent element \(g_u \in SO(V_1)\) (an irreducible representation of a Weyl group of type \(B\) or \(D\)) is indexed by \((\lambda_1', \lambda_2', \lambda_3', \ldots)\). (We use results in [L84a].) Define \(g \lambda = (g \lambda_1, g \lambda_2, g \lambda_3, \ldots)\) by \(g \lambda_j = \lambda_j + \lambda_j' + \sum_{c \in C'} \lambda_j^c\). We have \(g \lambda \in BP_{3,1}^{(N-1)/2}\) (if \(N\) is odd), \(g \lambda \in BP_{1,3}^{N/2}\) (if \(N\) is even). Thus, \(g \mapsto g \lambda\) defines a map \(D \rightarrow BP_{3,1}^{(N-1)/2}\) (if \(N\) is odd) or \(D \rightarrow BP_{1,3}^{N/2}\) (if \(N\) is even). From the definitions we see that the fibres of these maps are precisely the Strata of \(D\). (We use the description of the \(j\)-induction in classical Weyl groups groups given in [L09].)

This map is also surjective: indeed, by 2.4, for any \(\mu \in BP_{3,1}^{(N-1)/2}\) if \(N\) is odd (resp. \(\mu \in BP_{1,3}^{N/2}\) if \(N\) is even) we can find \(\mu' \in BP_{1,1}\), \(\mu'' \in BP_{2,0}\) if \(N\) is odd (resp. \(\mu' \in BP_{1,1}\), \(\mu'' \in BP_{0,2}\) if \(N\) is even) such that \(\mu = \mu' + \mu''\); we have \(\mu' \in BP_{1,1}^{k}\) and \(\mu'' \in BP_{2,0}^{k'}\) if \(N\) is odd (resp. \(\mu'' \in BP_{0,2}^{k'}\) if \(N\) is even) for some \(k \geq 0, k' \geq 0\) such that \(k + k' = (N - 1)/2\) if \(N\) is odd (resp. \(k + k' = N/2\) if \(N\) is even) and it remains to use that any bipartition in \(BP_{2,0}^{k}\) (resp. \(BP_{0,2}^{k'}\) if \(N\) is odd or \(BP_{0,2}^{k'}\) if \(N\) is even) represents the Springer representation associated to a unipotent element in \(Sp_{2k}(k)\) (resp. \(SO_{2k'+1}(k)\) if \(N\) is odd or \(SO_{2k'}(k)\) if \(N\) is even); see [L15, 3.4(a)], [L15, 3.6(a)], [L15, 3.6(b)]. We see that

(a) the map \(g \mapsto g \lambda\) from \(D\) to \(BP_{3,1}^{(N-1)/2}\) (if \(N\) is odd) or to \(BP_{1,3}^{N/2}\) (if \(N\) is even) defines a bijection from the set of Strata of \(D\) to \(BP_{3,1}^{(N-1)/2}\) (if \(N\) is odd) or to \(BP_{1,3}^{N/2}\) (if \(N\) is even). Thus, \(S_2(G, D)\) can be identified with \(BP_{3,1}^{(N-1)/2}\) (if \(N\) is odd) or to \(BP_{1,3}^{N/2}\) (if \(N\) is even).
3.4. In the setup of 3.2, \( D \) is a union of two Strata, \( \Sigma \) and \( \Sigma' \). Now \( \Sigma \) is the union of all \( G^0 \)-conjugacy classes of dimension 8 in \( D \); \( \Sigma' \) is the union of all \( G^0 \)-conjugacy classes of dimension 6 in \( D \). More precisely, if \( p \neq 2 \), \( \Sigma' \) consists of two semisimple \( G^0 \)-orbits in \( D \); if \( p = 2 \), \( \Sigma' \) consists of a single \( G^0 \)-orbit in \( D \) (its elements are unipotent, qss). We have \( S_2(G, D) = \text{Irr}(W^D) \).

4. Proof of Theorem 1.15

4.1. In this section we assume (until the end of 4.11) that we are given \( \vartheta \in G \) of finite order \( \delta \) such that \( G = \sqcup_{i=1}^{\delta} G^0 \vartheta^i \), that \( D = G^0 \vartheta \), that \( G^0 \) is almost simple, simply connected, with a fixed pinning, that \( \text{Ad}(\vartheta) : G^0 \to G^0 \) has order \( \delta \) and it preserves the pinning of \( G^0 \). (We must have \( \delta \in \{1, 2, 3\} \).)

We now assume (until the end of 4.11) that \( \delta \geq 2 \). We are in one of the following cases (in each case we indicate the type of \( G \); the number \( \delta \) appears as an upper index): type \( 2A_{2n+1}, n \geq 1 \); type \( 2A_{2n}, n \geq 1 \); type \( 2D_n, n \geq 4 \); type \( 3D_4 \); type \( 2E_6 \).

Let \( (T, B) \in \widehat{B}_\vartheta \). Let \( Y = \text{Hom}(k^*, T), \mathcal{X} = \text{Hom}(T, k^*) = \text{Hom}(Y, Z) \). Let \( Y = Q \otimes Y, \mathcal{X} = Q \otimes \mathcal{X} = \text{Hom}_Q(Y, Q) \). Let \( \bar{\mathcal{R}} \subset \mathcal{Y} \subset \mathcal{Y} \) (resp. \( R \subset \mathcal{X} \subset \mathcal{X} \)) be the set of coroots (resp. roots) of \( G^0 \) with respect to \( T \). The natural bijection \( \bar{\mathcal{R}} \leftrightarrow R \) is denoted by \( \bar{\alpha} \leftrightarrow \alpha \). Let \( \{\alpha_i; i \in I\} \) be the set of simple roots in \( R \) determined by \( B \). (The corresponding root subgroups are contained in \( B \).) Now \( \text{Ad}(\vartheta) \) acts naturally on \( Y \) and \( \mathcal{X} \); this induces a permutation of \( R \), one of \( \bar{\mathcal{R}} \), one of \( I \) (these permutations are denoted by \( \vartheta \)). Let \( G^0 = Z_G(\vartheta) \), a reductive group with identity component \( G^0 \). Let \( T' = T \cap G^0 \); we have \( T' \in T_{G^0} \). We have \( T = Z_{G^0}(T') \). Let \( \mathcal{Y} = \text{Hom}(k^*, T') \subset \mathcal{Y}, \mathcal{X} = \text{Hom}(T', k^*) = \text{Hom}(\mathcal{Y}, Z) \). Let \( \mathcal{Y} = Q \otimes \mathcal{Y}, \mathcal{X} = Q \otimes \mathcal{X} \). The homomorphism \( \text{res} : \mathcal{X} \to \mathcal{X} \) (restriction to \( \mathcal{Y} \)) induces a linear surjective map \( \text{res} : \mathcal{X} \to \mathcal{X} \). Let \( \mathcal{R} = \text{res}(R) \subset \mathcal{X} \subset \mathcal{X} \). Now \( \text{res} : \mathcal{R} \to \mathcal{R} \) induces a bijection from the set of orbits of \( \vartheta : \mathcal{R} \to \mathcal{R} \) to \( \mathcal{R} \). For \( \beta \in \mathcal{R} \) let \( d^\beta_\vartheta \) be the cardinal of the corresponding \( \vartheta \)-orbit on \( R \); we set \( d^\beta_\vartheta = 2 \) if either \( 2 \beta \in R \) or \( (1/2) \beta \in R \) and \( d^\beta_\vartheta = 1 \); otherwise; let \( d^\beta_\vartheta = d^\beta_\vartheta d^\beta_\vartheta \). For \( \alpha \in R', \beta = \text{res}(\alpha) \), we set:

\[ \beta = \alpha \text{ if } d^\beta_\vartheta = 1 \text{ or if } d^{2\beta}_\vartheta = 2, \]
\[ \beta = \alpha + \vartheta(\alpha) \text{ if } d^\beta_\vartheta = 2 \text{ and } d^{2\beta}_\vartheta = 1, \]
\[ \beta = 2\alpha + \vartheta(\alpha) + \vartheta^2(\alpha) \text{ if } d^\beta_\vartheta = 3, \]
\[ \beta = 2\alpha + 2\vartheta(\alpha) \text{ if } d^\beta_\vartheta = 4. \]

Let \( \mathcal{R} = \{\beta; \beta \in \mathcal{R}\} \subset \mathcal{Y} \subset \mathcal{Y} \); this is in obvious bijection with \( \mathcal{R} \) and \( (\mathcal{R}', \mathcal{Y}, \mathcal{X}) \) is a (not necessarily reduced) root system. Let \( \mathcal{R}_0 \) be the set of elements in \( \mathcal{R} \) which are not in \( 2'R \); let \( \mathcal{R}_0 \) be the set of elements in \( \mathcal{R} \) which are not in \( (1/2)'\mathcal{R} \). Then \( (\mathcal{R}_0, \mathcal{R}_0, \mathcal{Y}, \mathcal{X}) \) is a (reduced) root system. Let \( \mathcal{I} \) be the set of orbits of the bijection \( \mathcal{I} \to \mathcal{I} \). For \( i \in \mathcal{I} \) let \( \beta_i = \text{res}(\alpha_i) \) where \( i' \) is any element of the orbit \( i \). Then \( \{\beta_i; i \in \mathcal{I}\} \) is a basis of the root system \( (\mathcal{R}_0, \mathcal{R}_0, \mathcal{Y}, \mathcal{X}) \). Let \( \mathcal{R} \) (resp. \( \bar{\mathcal{R}} \)) be the subset of \( \mathcal{X} \) (resp. \( \mathcal{Y} \)) consisting of the vectors \( d^\beta_\vartheta \beta \) (resp. \( d^{1/2}_\vartheta \beta \)) for various \( \beta \in \mathcal{R} \). Then \( (\mathcal{R}, \mathcal{R}, \mathcal{Y}, \mathcal{X}) \) is a
(reduced, irreducible) root system with basis \( \{ \gamma_i = d_{\beta_i} \beta_i; i \in \bar{I} \} \). There is a unique vector space isomorphism \( 'X \rightarrow 'Y \) which carries \( \gamma_i \) to \( \tilde{\beta}_i \) for any \( i \in \bar{I} \); from the definitions we see that this isomorphism carries \( \bar{R} \) onto \( \tilde{R}_0 \). Hence it carries \( \gamma_0 \in \bar{R} \), the negative of the highest root in \( \bar{R} \) relative to the basis \( \{ \gamma_i = d_{\beta_i} \beta_i; i \in \bar{I} \} \), to \( h_0 \), the negative of the highest coroot in \( \tilde{R}_0 \) relative to the basis \( \{ \tilde{\beta}_i; i \in \bar{I} \} \). We have \( h_0 = \tilde{\beta}_0 \) and \( \gamma_0 = d_{\beta_0} \beta_0 \) for a well defined \( \beta_0 \in \tilde{R}_0 \).

Setting \( \bar{I} = \bar{I} \cup \{ 0 \} \) there are unique integers \( n_i \in \mathbb{Z}_{>0} \), \( i \in \bar{I} \) such that \( n_0 = 1 \) and \( \sum_{i \in \bar{I}} n_i \gamma_i = 0 \). We define a subgroup \( Q_* \) of \( Q \) as follows: if \( p = 0 \) then \( Q_* = Q \); if \( p > 1 \), then \( Q_* \) consists of the rational numbers \( q \) such that \( \bar{N} q \in \mathbb{Z} \) for some integer \( N \) not divisible by \( p \). Let \( 'Y_* = Q_* \otimes 'Y \subset 'Y \). We define

\[
\mathcal{C} = \{ y \in 'Y_*; \sum_{i \in \bar{I}} n_i \gamma_i(y) = 0, \gamma_i(y) \in Q_{\geq 0} \text{ for } i \in \bar{I}, \gamma_0(y) + 1 \in Q_{\geq 0} \}.
\]

For \( y \in \mathcal{C} \) let \( \text{supp}(y) \subset \bar{I} \) be the set of all \( i \in \bar{I} \) such that either \( i \in \bar{I}, \gamma_i(y) > 0 \) or \( i = 0, \gamma_0(y) > -1 \). We have \( \mathcal{C} = \bigcap_{K \in \mathcal{P}(\bar{I})} \mathcal{C}_K \) where \( \mathcal{P}(\bar{I}) \) is the set of all subsets \( K \subset \bar{I} \) such that \( K \neq \emptyset \) and \( \mathcal{C}_K = \{ y \in \mathcal{C}; \text{supp}(y) = K \} \).

We define a subset \( \mathcal{P}'(\bar{I}) \) of \( \mathcal{P}(\bar{I}) \) as follows. If \( p = 0 \) we have \( \mathcal{P}'(\bar{I}) = \mathcal{P}(\bar{I}) \). If \( p > 1 \), \( \mathcal{P}'(\bar{I}) \) is the set of all \( K \in \mathcal{P}(\bar{I}) \) such that \( n_i/p \notin \mathbb{Z} \) for some \( i \in K \).

**Lemma 4.2.** Assume that \( p > 1 \) and that \( K \in \mathcal{P}(\bar{I}) - \mathcal{P}'(\bar{I}) \). Then \( \mathcal{C}_K = \emptyset \).

For any \( y \in 'Y_* \) and any \( j \in \bar{I} \) we have \( \gamma_j(y) \in Q_* \) since \( \gamma_j \) takes integer values on \( Y \). Assume now that \( y \in \mathcal{C}_K \). Since \( n_0 = 1 \), we have \( 0 \notin K \) so that \( \gamma_0(y) = -1 \).

For \( i \in K \) we have \( \gamma_i(y) = s_i/t_i, n_i = pn_i' \) where \( s_i, t_i \), \( n_i' \) are integers \( > 0 \) with \( p \) not dividing \( t_i \). We have \( \sum_{i \in K} pn_i' s_i/t_i - 1 = 0 \). It follows that \( p \) divides \( \prod_{i \in K} t_i \) hence for some \( i \in K \), \( p \) divides \( t_i \), contradiction.

**Lemma 4.3.** Assume that \( \delta \neq p \) and that \( K \in \mathcal{P}'(\bar{I}) \). Then \( \mathcal{C}_K \neq \emptyset \).

Assume first that \( 0 \notin K \). If \( p = 0 \) we choose any \( i \in K \cap \bar{I} \). If \( p > 1 \) we choose \( i \in K \cap \bar{I} \) such that \( n_i/p \notin \mathbb{Z} \). Let \( K' = K - \{ i \} \). We define \( c_j \in Q_* \) for \( j \in \bar{I} \) as follows: \( c_0 = -1 \); if \( j \in \bar{I} - K, j \neq 0 \) then \( c_j = 0 \); if \( j \in K' \), \( c_j \in Q_* \) is chosen so that \( 0 < c_j < 1/N \) where \( N \) is an integer such that \( \sum_{j' \in K', n_{j'} < N; c_i = (1 - \sum_{j' \in K'} n_{j'} c_{j'})/n_i. \) (Note that \( c_i \in Q_* \) and \( c_i > (1 - \sum_{j' \in K'} n_{j'} c_{j'})/n_i > 0. \))

Next we assume that \( 0 \in K \). We define \( c_j \in Q_* \) for \( j \in \bar{I} \) as follows: if \( j \in \bar{I} - K \) then \( c_j = 0 \); if \( j \in K \cap \bar{I} \) then \( c_j \in Q_* \) is chosen so that \( 0 < c_j < 1/N \) where \( N \) is an integer such that \( \sum_{j' \in K \cap \bar{I}} n_{j'} < N \); \( c_0 = -\sum_{j' \in K \cap \bar{I}} n_{j'} c_{j'} \). (Note that \( c_0 \in Q_* \) and \( c_0 > -\sum_{j' \in K \cap \bar{I}} n_{j'} c_{j'}/N > -1. \))

We now define \( y \in 'Y \) by \( \gamma_h(y) = c_h \) for all \( h \in \bar{I} \). Let \( 'X_1 \) be the subgroup of \( 'X \) generated by \( \{ \gamma_i; i \in \bar{I} \} \). Since \( c_h \in Q_* \), we see that for any \( \gamma \in 'X_1 \) we have \( \gamma(y) \in Q_* \). Using a case by case check we see that the index of \( 'X \) in \( 'X_1 \) is of the form \( \delta^m \) for some integer \( m \geq 1 \). It follows that for any \( \zeta \in 'X \) we have \( \delta^m \zeta \in 'X_1 \) hence \( \delta^m \zeta(y) \in Q_* \) hence \( \zeta(y) \in Q_* \). (Here we use that \( \delta \neq p \).) Since this holds for any \( \zeta \in 'X \) it follows that \( y \in 'Y_* \) so that \( y \in \mathcal{C}_K \). This completes the proof.
4.4. Let $\mu(k)$ be the group of roots of $1$ in $k$. We assume that an identification $Q_* / Z$ with $\mu(k)$ (as groups) is fixed. Then $\chi' / \mathcal{Y} = (Q_* / Z) \otimes \chi'$ is identified with $\mu(k) \otimes \chi'$ which can be identified (via $\lambda \otimes y \mapsto y(\lambda)$) with the group $T'_{fin}$ of elements of finite order in $T'$; thus we obtain a homomorphism $\iota : \chi' \rightarrow T'$ whose kernel is $\chi'$ and whose image is $T'_{fin}$. The following result is an adaptation of results in [L02, §6]. (In loc.cit the characteristic is assumed to be zero, but the same arguments apply assuming only that $\delta \neq p$, by replacing the Lie algebras of $T$ and $T'$ by $\chi', \chi'$.) Results of this kind go back to de Siebenthal’s paper [dS56] where conjugacy classes in disconnected compact Lie groups are discussed.

(a) Assume that $\delta \neq p$. Then $x \mapsto \iota(x)\vartheta$ defines a bijection of $C$ with a set of representatives for the $G^0$-conjugacy classes of semisimple elements in $D$ which have finite order in $G$.

Let $K \in \mathcal{P}(\tilde{I}), J = \tilde{I} - K$. Let $R_J$ be the set of all vectors in $'R$ which are $Z$-linear combinations of vectors in $\{\beta_i^i; i \in J\}$. Let $R_J$ ne the set of all $\beta \in 'R$ such that $'\beta \in R_J$. Then for any $x \in C_K$, $G_J := Z_{G^0}(\iota(x)\vartheta)$ is a connected reductive group (see [St68]) which depends only on $J$, not on $x$, and $(R_J, R_J, (\chi', \chi'))$ is the root system of $G_J$ relative to the maximal torus $T'$ of $G_J$. (Here $R_J$ is the set of roots and $R_J$ is the set of coroots.)

**Lemma 4.5.** Assume that $\delta = p$. Let $s \in G^0$ be semisimple. Then $W_{Z_G(s)\vartheta}$ is isomorphic to the fixed point set of $\text{Ad}(\vartheta)$ on $W_{Z_G(s)\vartheta}$.

Let $B, T$ in $G^0$ be preserved by $\text{Ad}(\vartheta)$. Then $(T \cap G^0, B \cap G^0) \in \tilde{B}_{G^0}$ and are preserved by $\text{Ad}(\vartheta)$. Hence $\vartheta$ is qss in $Z_G(s)$; it is also unipotent. We now use 1.8(a),(d) with $G$ replaced by $Z_G(s)$ and note that $Z_G(s)^0 \cap Z_G(\vartheta) = (Z_G(s)^0 \cap Z_G(\vartheta))^0 = Z_G(s\vartheta)^0$.

**Lemma 4.6.** Assume that $\delta = p$. Let $g \in D$. Then some $G^0$-conjugate of $g_s$ is in $Z_G(\vartheta)^0$.

The image of $g_s$ in $G / G^0$ is semisimple; since $G / G^0$ is a unipotent group this image must be $1$; thus, we have $g_s \in G^0$. Let $D' \subset G' = Z_G(g_s)$ be as in 1.6. Now $G'/G^0 \rightarrow G/G_0$ is injective (and carries $D'$ to $D$) since $Z_{G^0}(g_s)$ is connected. (Recall that $G^0$ is simply connected.) It follows that $D'$ has order $p$ in $G'/G^0$. Hence we can find $u \in D'$ unipotent and $(T', B') \in (\tilde{B}_{G^0})_u$. Let $Z$ be an irreducible component of $B_{g_s}$ such that $g(Z) = Z$. Let $B \in Z$ be the unique Borel subgroup such that $B \cap G^0 = B'$. Now $uB'u^{-1} \in Z$ (since $g(Z) = Z$) and $uB'u^{-1} \cap G^0 = uB'u^{-1} = B'$. By uniqueness we have $uB'u^{-1} = B$. Let $T$ be a maximal torus of $B$ containing $T'$. Now there is a unique maximal torus of $G^0$ containing $T'$. It must be equal to $T$ and $uT'u^{-1}$ is a maximal torus containing $T'$ hence is equal to $T$. Thus $uT'u^{-1} = T$. We have $u \in D$ and $u$ is unipotent and qss in $G$. Since $u \in D'$, we have $g_su = ug_s$. By 1.8(b), for some $h \in G^0$ we have $huh^{-1} = \vartheta$. Let $s' = hgh^{-1}$. Then $s'\vartheta = \vartheta s'$. Thus we have $s' \in G^0 \cap Z_G(\vartheta) = Z_{G^0}(\vartheta) = Z_G(\vartheta)^0$ (we use 1.8(a)).
4.7. Assume that $\delta \neq p$. Let $g \in D$. By replacing $g$ by a $G^0$-conjugate we can assume that for some $K \in \mathcal{P}'(\tilde{I})$ we have $g_s = \iota(x)\vartheta$ where $x \in \mathcal{C}_K$ (see 4.1, 4.4). Let $H = Z_G(g_s)$. With notation of 4.4 we have $Z_{G^0}(g_s) = G_{I-K} = G^0_{I-K} = H^0$. Let $W_{I-K} = W_{G_{I-K}}$. We have $g_s \in D$ hence $g_u \in G^0$. Thus $g_u \in G^0 \cap H$ so that $g_u \in H^0 = G_{I-K}$. From 1.11 we now see that

\[ (a) \quad S_2(G, D) = \bigcup_{K \in \mathcal{P}'(I)} j^{W_{I-K}}_{W_{I-K}}(S_1(G_{I-K}, G_{I-K})) \]

where $W_{I-K}$ is viewed as a subgroup of $W^D$ as in 1.6(c). Now let $K \in \mathcal{P}'(\tilde{I})$. We can find $i \in K$ such that $\{i\} \in \mathcal{P}'(\tilde{I})$. (If $p = 0$ any $i \in K$ satisfies our requirement; if $p > 0$ at least one $i \in K$ satisfies our requirement.) We have $G_{I-K} \subset G_{I-\{i\}}$ and in fact $G_{I-K}$ is a Levi subgroup of a parabolic subgroup of $G_{I-\{i\}}$ so that we can regard $W_{I-K}$ as a subgroup of $W_{\tilde{I} - \{i\}}$. If $E \in S_1(G_{I-K}, G_{I-K})$ then $j^{W_{I-K}}_{W_{I-\{i\}}}(E) = j^{W_{I-\{i\}}}_{W_{I-K}}(E')$ where $E' = j^{W_{I-\{i\}}}_{W_{I-K}}(E)$. It is known that $E' \in S_1(G_{I-K}, G_{I-K})$ (a property of induced unipotent classes). It follows that the union in (a) can be restricted to one elements subsets $K$ in $\mathcal{P}'(\tilde{I})$. Thus, we have

\[ (b) \quad S_2(G, D) = \bigcup_{i \in \tilde{I} - \{i\}} j^{W_{I-\{i\}}}_{W_{I-K}}(S_1(G_{I-\{i\}}, G_{I-\{i\}})) \]

4.8. Assume that $\delta = p$. Let $X$ be a set of representatives for the semisimple $G^{\vartheta_0}$-conjugacy classes in $G^{\vartheta_0}$. Let $g \in D$. By replacing $g$ by a $G^{\vartheta_0}$-conjugate we can assume that $g_s \in X$ (see Lemma 4.6). We set $s = g_s$, $H = Z_G(s)$. Since $s \in G^0$ we have $g_u \in D$. Let $D_s$ be the connected component of $H$ which contains $\vartheta$; we have $g_u \in D_s$ (indeed, $g_u \vartheta^{-1} \in G^0$ and $g_u \vartheta^{-1} \in H$ hence $g_u \vartheta^{-1} \in G^0 \cap H$ which equals $H^0$ by [St68]). Hence $W^D_{H^0}$ is equal to the fixed point of $\text{Ad}(\vartheta)$ on $W^D_{H^0}$ which by Lemma 4.5 is the same as $W_{Z_G(s)^0(\vartheta)}$ that is, $W_{Z_G(s)^0}$. (Indeed, $Z_G(s)^0 = (Z_G(s) \cap G^0)^0 = Z_{G^{\vartheta_0}}(s)^0$.) From Lemma 4.5 we see also that $W^D$ is the same as $W_{Z_G(s)^0} = W_{Z_{G^{\vartheta_0}}}$. (We use 1.8(a)). From 1.11 we now see that

\[ (a) \quad S_2(G, D) = \bigcup_{s \in X} j^{W_{Z_{G^{\vartheta_0}(s)^0}}}_{W_{Z_{G^{\vartheta_0}}}(s)^0}(S_1(Z_G(s), D_s)) \]

where $W_{Z_{G^{\vartheta_0}(s)^0}}$ is viewed as a subgroup of $W_{Z_{G^{\vartheta_0}}} = W^D$ as in 1.6(c). As in 4.7(b), the union in (a) can be restricted to a subset $X'$ of $X$, namely:

\[ (b) \quad S_2(G, D) = \bigcup_{s \in X'} j^{W_{Z_{G^{\vartheta_0}(s)^0}}}_{W_{Z_{G^{\vartheta_0}}}(s)^0}(S_1(Z_G(s), D_s)) \]

where $X'$ consists of those $s \in X$ such that $Z_{G^{\vartheta_0}(s)^0}$ has the same semisimple rank as $G^{\vartheta_0}$.

4.9. In this subsection we assume that we are in type $2E_6$ (see 4.1) so that $G^{\vartheta_0}$ is of type $F_4$. We write the set $\tilde{I}$ in 4.1 as $\{1, 2, 3, 4\}$ in such a way that $\beta_1 + \beta_2$, $\beta_2 + \beta_3$, $\beta_2 + 2\beta_3$, $\beta_3 + \beta_4$ are roots of $G^{\vartheta_0}$ with respect to $T'$; the corresponding simple reflections in $W^D = W_{G^{\vartheta_0}}$ are denoted by $s_1, s_2, s_3, s_4$ respectively. Let $\omega : W^D \to W^D$ be the Coxeter group automorphism given by $s_1 \mapsto s_4, s_2 \mapsto s_3, s_3 \mapsto s_2, s_4 \mapsto s_1$. The reflection with respect to the highest root of $G^{\vartheta_0}$ is the element of $W_D$ given by $s'_0 := s_1s_2s_3s_4s_2s_3s_4s_2s_3s_4$. Let $s_0 = \omega(s'_0) \in W^D$.

If $p \neq 2$, the one element subsets $K$ in $\mathcal{P}'(\tilde{I})$ (see 4.1), the corresponding $G_{I-K}$
(see 4.4) and \( W_{\tilde{I} - K} \subset W^D \) (see 4.7) are as follows:

(i) \( K = \{0\} \) with \( G_{I-K} \) of type \( F_4 \) with \( W_{I-K} \) generated by \( s_1, s_2, s_3, s_4 \);

(ii) \( K = \{4\} \) with \( G_{I-K} \) of type \( A_1 \times B_3 \) with \( W_{I-K} \) generated by \( s_1, s_2, s_3, s_0 \);

(iii) \( K = \{3\} \) with \( G_{I-K} \) of type \( A_2 \times A_2 \) with \( W_{I-K} \) generated by \( s_1, s_2, s_4, s_0 \) (if \( p \neq 3 \));

(iv) \( K = \{2\} \) with \( G_{I-K} \) of type \( A_3 \times A_1 \) with \( W_{I-K} \) generated by \( s_1, s_3, s_4, s_0 \);

(v) \( K = \{1\} \) with \( G_{I-K} \) of type \( C_4 \) with \( W_{I-K} \) generated by \( s_2, s_3, s_4, s_0 \).

In each case \( S_1(G_{I-K}, G_{I-K}) \subset \text{Irr}(W_{I-K}) \) is explicitly known: in case (i), see [Sh80]; in case (ii),(v) see [L84a]; in case (iii),(iv) it is equal to \( \text{Irr}(W_{I-K}) \).

From this and 4.7(b) we see that:

(a) if \( p \neq 2 \) then \( S_2(G, D) \) consists of \( 10, 4_1, 9_2, 12_4, 16_5, 9_{10}, 4_{13}, 1_{24} \); both \( 8_3, 8_3 \); both \( 8_9, 8_9 \); one of the two \( 6_0, 6_6 \), namely the one of the form \( j_{W_{I-(3)}}^{W^D} (16) \); one of the two \( 9_6, 9_6 \), namely the one of the form \( j_{W_{I-(2)}}^{W^D} (16) \); one of the two \( 4_7, 4_7 \), namely the one of the form \( j_{W_{I-(2)}}^{W^D} (17) \); one of the two \( 1_{12}, 1_{12} \), namely the one of the form \( j_{W_{I-(1)}}^{W^D} (112) \); one of the two \( 2_{16}, 2_{16} \), namely the one of the form \( j_{W_{I-(1)}}^{W^D} (116) \).

Here \( d_n \) denotes an irreducible representation \( E \) of a Weyl group of degree \( d \) with \( b_E = n \). (For a Weyl group of type \( F_4 \) there are at most two irreducible representations with a given \( d, n \).)

If \( p = 2 \), the set \( X' \) in 4.8 has two elements \( s = 1 \) and \( s = g \) where \( g \in G^{g_0} \) has order 3 with \( Z_{G^{g_0}}(g)^0 \) of type \( A_2 \times A_2 \); \( Z_G(g) \) is of type \( A_2 \times A_2 \times A_2 \) with \( \text{Ad}(\vartheta) \) interchanging two of these \( A_2 \)-factors; \( W_{Z_{G^{g_0}}(g)^0} \) can be identified with the subgroup of \( W^D \) generated by \( s_0, s_1, s_3, s_4 \).

In each case \( S_1(Z_G(s), D_s) \subset \text{Irr}(W_{Z_{G^{g_0}}(s)^0}) \) is explicitly known: for \( s = 1 \) see [M05]; for \( s = g \) we have \( S_1(Z_G(s), D_s) = \text{Irr}(W_{Z_{G^{g_0}}(s)^0}) \). From this and 4.8(b) we see that:

(b) if \( p = 2 \), then \( S_2(G, D) \) consists of the same irreducible representations of \( W^D \) as those in (a).

Note that

(c) in the union 4.8(b) the term corresponding to \( s = g \) is contained in the term corresponding to \( s = 1 \).

From (a),(b) we see that

(d) \( S_2(G, D) \) is independent of \( k \).

**4.10.** In this subsection we assume that we are in type \( ^3D_4 \) (see 4.1) so that \( G^{g_0} \) is of type \( G_2 \). We write the set \( \tilde{I} \) in 4.1 as \( \{1, 2\} \) in such a way that \( \beta_1 + \beta_2, \beta_1 + 2\beta_2, \beta_1 + 3\beta_3 \) are roots of \( G^{g_0} \) with respect to \( T' \); the corresponding simple reflections in \( W^D = W_{G^{g_0}} \) are denoted by \( s_1, s_2 \) respectively. Let \( \omega : W^D \to W^D \) be the Coxeter group automorphism given by \( s_1 \mapsto s_2, s_2 \mapsto s_1 \). The reflection with respect to the highest root of \( G^{g_0} \) is the element of \( W^D \) given by \( s_0 : = s_1 s_2 s_1 s_2 s_1 \).

Let \( s_0 = \omega(s_0') \in W^D \).

If \( p \neq 3 \), the one element subsets \( K \) in \( \mathcal{P}'(\tilde{I}) \) (see 4.1), the corresponding \( G_{I-K} \) (see 4.4) and \( W_{I-K} \subset W^D \) (see 4.7) are as follows:
(i) $K = \{0\}$ with $G_{I-K}$ of type $G_2$ with $W_{I-K}$ generated by $s_1, s_2$;
(ii) $K = \{2\}$ with $G_{I-K}$ of type $A_1 \times A_1$ with $W_{I-K}$ generated by $s_1, s_0$ (if $p \neq 2$)
(iii) $K = \{1\}$ with $G_{I-K}$ of type $A_2$ with $W_{I-K}$ generated by $s_2, s_0$.
In each case, $S_1(G_{I-K}, G_{I-K}) \subset \text{Irr}(W_{I-K})$ is explicitly known: in case (i), see [Sp85]; in case (ii),(iii) it is equal to $\text{Irr}(W_{I-K})$. From this and 4.7(b) we see that:

(a) if $p \neq 3$ then $S_2(G, D)$ consists of $1_0, 2_1, 2_2, 1_6$; one of the two $1_3, 1_3$, namely the one of the form $j^{W_D}_{W_{I-(1)}}(1_3)$.
(Notation as in 4.9(a).)
If $p = 3$, the set $X'$ in 4.8 has two elements $s = 1$ and $s = g$ where $g \in G_{G_0}$ has order 2 with $Z_{G_{G_0}}(g)^0$ of type $A_1 \times A_1$; $Z_G(g)$ is of type $A_1 \times A_1 \times A_1 \tau A_1$ with $\text{Ad}(\theta)$ permuting cyclically three of these $A_1$-factors; $W_{Z_{G_{G_0}}(g)^0}$ can be identified with the subgroup of $W^D$ generated by $s_0, s_2$.
In each case $S_1(Z_G(s), D_s) \subset \text{Irr}(W_{Z_{G_{G_0}}(s)^0})$ is explicitly known: for $s = 1$ see [M05]; for $s = g$ we have $S_1(Z_G(s), D_s) = \text{Irr}(W_{Z_{G_{G_0}}(s)^0})$. From this and 4.8(b) we see that:

(b) if $p = 3$, then $S_2(G, D)$ consists of the same irreducible representations of $W^D$ as those in (a).
Note that
(c) in the union 4.8(b), the term corresponding to $s = g$ is contained in the term corresponding to $s = 1$.
From (a),(b) we see that
(d) $S_2(G, D)$ is independent of $k$.

4.11. In this subsection we assume that we are in type $2A_{2n+1}, n \geq 1$, or type $2A_{2n}, n \geq 1$, or type $2D_{n}, n \geq 4$. If $p \neq 2$, the Strata of $D$ and the set $S_2(G, D)$ are determined by 3.1(a), 3.3(a) (we use also 1.13); we see that $S_2(G, D)$ can be identified with $BP_{m,1}^n, BP_{1,3}^n, BP_{4,0}^{n-1}$ respectively. If $p = 2$, from 4.8(b) we have

(a) $S_2(G, D) = S_1(G, D)$
and this is identified in [L04,§13], [MS04] with $mBP_{3,1}^n, mBP_{1,3}^n, mBP_{4,0}^{n-1}$ respectively (with $m \gg 0$). (The proof of this identification in loc.cit. is very similar to the analogous statement for connected classical groups in characteristic 2 given in [LS85].) Using the identifications in 2.2, we see that $S_2(G, D)$ is identified with $BP_{3,1}^n, BP_{1,3}^n, BP_{4,0}^{n-1}$ respectively. We now see that

(b) $S_2(G, D)$ is independent of $k$.

4.12. We prove Theorem 1.15. By a sequence of reductions as in [L04, 12.1-12.7] we see that we can assume that $G, D, \delta$ are as in the first sentence of 4.1. If $\delta = 1$, we have $G = G^0 = D$ and the desired result follows from [L15]. Assume now that $\delta \geq 2$. In this case the desired result follows from 4.9(d), 4.10(d), 4.11(b). The theorem is proved.
5. Further results

5.1. In this subsection we assume that $D = G^0 = G$ and $k = C$. Let $\text{sgn}_W$ be the sign representation of $W$. The following is well known:

(a) If $g \in G$ is unipotent then $E_g = \text{sgn}_W$ if and only if $g = 1$.

Let $^0S_2(W)$ be the set of all $E \in \text{Irr}(W)$ such that $E = j_{W'}^W(\text{sgn}_{W'})$ where $W'$ is the Weyl group of $Z_G(s)^0$ (for some semisimple $s \in G$) viewed as a subgroup of $W$. We have $^0S_2(W) \subset S_2(W)$ (we use (a)). Note that $^0S_2(W)$ parametrizes a subset of the set of unipotent classes of the group of type dual to that of $G$.

Let $\Sigma \in \text{Str}(G)$ be corresponding to $E \in S_2(W)$. From (a) and the definitions we deduce:

(b) $\Sigma$ contains some semisimple element if and only if $E \in ^0S_2(W)$.

5.2. We preserve the setup of 5.1. Let $G_0$ be a maximal compact subgroup of $G$. Then $G_0$ is partitioned into subsets defined by the intersections of the various Strata of $G$ with $G_0$. The Strata of $G$ which have nonempty intersection with $G_0$ are precisely those in 5.1(b).

5.3. We preserve the setup of 5.1. For $a \in \mathbb{N}$ we set

$Y_a = (0, 1, 0, 1, 0, 1, \ldots, 0, 1, 0, 0, 0, \ldots) \in BP^a_{1,0}$,

$Y'_a = (1, 0, 1, 0, 1, \ldots, 0, 1, 0, 0, 0, \ldots) \in BP^a_{0,1}$.

Assume that $G = Sp_{2n}(k)$, $n \geq 2$. We identify $S_2(W)$ with $BP^a_{2,2}$ as in [L15, 3.5(b)]. Then $^0S_2(W)$ becomes the set of bipartitions of the form $Y_a + Y'_b + C$ where $C \in BP^{a-b}_{0,0}$, $a + b \leq n$.

Assume that $G = SO_{2n+1}(k)$, $n \geq 2$. We identify $S_2(W)$ with $BP^a_{2,2}$ as in [L15, 3.5(b)]. Then $^0S_2(W)$ becomes the set of bipartitions of the form $Y_a + Y'_b + C$ where $C \in BP^{a-b}_{0,0}$, $a + b \leq n$.

Assume that $G = SO_{2n}(k)$, $n \geq 4$. We identify $S_2(W)$ (modulo the action of $O_{2n}(k)$) with $BP^a_{2,4}$ as in [L15, 3.10(b)]. Then $^0S_2(W)$ (modulo the action of $O_{2n}(k)$) becomes the set of bipartitions of the form $Y_a' + Y'_b + C$ where $C \in BP^{a-b}_{0,0}$, $a + b \leq n$.

5.4. We no longer assume that $D = G^0 = G$ but we assume that $p = 0$. Let $s_0 \in D$ be a semisimple element and let $T_0$ be a maximal torus of $Z_G(s_0)^0$. It is known that $T_0s_0$ consists of semisimple elements and any semisimple element in $D$ is $G^0$-conjugate to an element of $T_0s_0$ (see [L03, 1.14]). In this case the description of $S_2(G, D)$ in 1.11 simplifies as follows: the representations of $W^D$ in $S_2(G, D)$ are exactly those obtained by applying $j_{W_s}^{W^D}$ to any representation of $W_s$ in $S_1(Z_G(s)^0, Z_G(s)^0)$ where $s$ is any element in $T_0s_0$; here $W_s = W_{Z_G(s)^0}$ is viewed as a subgroup of $W^D$ as in 1.6(c). (We use that $D_1$ in 1.11 is now $Z_G(s)^0$.) For $s, W_s$ above we consider a point $\xi$ on the torus $T_0^*$ dual to $T_0$ and we denote by $W_s, \xi$ the subgroup of $W_s$ generated by the reflections which keep $\xi$ fixed (in the natural action of $W_s$ on $T_0^*$). From [L09] it is known that the representations of $W_s$ in $S_1(Z_G(s)^0, Z_G(s)^0)$ are exactly those obtained by applying $j_{W_s, \xi}^{W_s}$ to any special
representation of \( W_{s, \xi} \). Let \( \mathcal{P}(G, D) \) be the collection of reflection subgroups of \( W^D \) which are conjugate to a subgroup of the form \( W_{s, \xi} \) for some \( s \in T_0 s_0 \) and some \( \xi \in T_0^* \); the subgroups in \( \mathcal{P}(G, D) \) are said to be the 2-parabolic subgroups of \( W^D \). It follows that the representations of \( W^D \) in \( S_2(G, D) \) are exactly those obtained by applying \( j_{W_1}^{W^D} \) (with \( W_1 \in \mathcal{P}(G, D) \)) to any special representation of \( W_1 \).

The set \( \mathcal{P}(G, D) \) consists of the reflection subgroups of \( W^D \) of type

(i) \( B_a \times B_b \times B_c \times B_d \times \mathcal{A} \) if \( (G, D) \) is of type \( 2D_{n+1}, n \geq 3 \) (see 4.1);

(ii) \( B_a \times B_b \times B_c \times D_d \times \mathcal{A} \) if \( (G, D) \) is of type \( 2A_{2n}, n \geq 1 \), (see 4.1);

(iii) \( B_a \times B_b \times D_c \times D_d \times \mathcal{A} \) if \( G = G^0 = D \) is \( Sp_{2n}(k) \) or \( SO_{2n+1}(k), n \geq 2 \);

(iv) \( B_a \times D_b \times D_c \times D_d \times \mathcal{A} \) if \( (G, D) \) is of type \( 2A_{2n-1}, n \geq 2 \), (see 4.1);

(v) \( D_a \times D_b \times D_c \times D_d \times \mathcal{A} \) if \( G = G^0 = D \) is \( SO_{2n}(k), n \geq 4 \).

Note that in (i)-(iv), \( W^D \) is of type \( B_n \) while in (v), \( W^D = W \) is of type \( D_n \); in each case we have \( (a, b, c, d) \in \mathbb{N}^4, a + b + c + d \leq n \); \( \mathcal{A} \) stands for a product of reflection subgroups of type \( A \).

5.5. In this subsection we assume that \( G, D, \delta \) are as in 4.1 with \( p = \delta \in \{2, 3\} \). We show:

(a) Each Stratum of \( D \) contains a unique unipotent \( G^0 \)-orbit.

It is enough to show that \( S_2(G, D) = S_1(G, D) \). If \( G, D, \delta \) is as in 4.11 this follows from 4.11(a). If \( G, D, \delta \) is as in 4.9 (resp. 4.10) this follows from 4.9(c) (resp. 4.10(c)).

5.6. Let \( S(W) \) be the subset of \( \text{Irr}(W) \) consisting of special representations. We define a map \( \tilde{\zeta} : \text{Str}(D) \to S(W) \) as the composition \( \text{Str}(D) = S_2(W, [D]) \subset \text{Irr}(W) \) of \( \tilde{\zeta} \) is defined as follows. If \( D = G^0 \), \( \tilde{\zeta} : \text{Irr}(W) \to S(W) \) associates to any \( E \in \text{Irr}(W) \) the special representation in the same family as \( E \). In the general case, we can find a connected reductive group \( \mathcal{G} \) defined over \( F_q \) with connected centre, with Weyl group \( W \) and with Frobenius map acting on \( W \) as \([D]\). Then \( \text{Irr}(W^D) \) is in natural bijection with the set of irreducible representations of \( \mathcal{G}(F_q) \) which have nonzero invariants under a Borel subgroup defined over \( F_q \); this is a subset of the set of unipotent representations of \( \mathcal{G}(F_q) \) which by its known parametrization [L84b] is decomposed into pieces indexed by the families of \( W \) which are stable under \([D]\); in this way we can associate to each representation in \( \text{Irr}(W^D) \) a family of \( W \) and hence the unique special representation of \( W \) in that family. This defines \( \tilde{\zeta} \) and hence \( \zeta \).

5.7. We define a partition of \( D \) into “special pieces” indexed by \( S(W) \). For \( E \in S(W) \), the special piece corresponding to \( E \) is \( \bigcup_{\Sigma \in \text{Str}(D) : \zeta(\Sigma) = E} \Sigma \). We expect that this special piece is locally closed in \( D \). As supporting evidence, we note that, when \( G = G^0 = D, p = 0 \), the intersection of this special piece with the unipotent variety of \( G \) coincides with a special piece of that unipotent variety considered in [L97]; in particular it is locally closed in the unipotent variety.
5.8. In this subsection we assume that that \( k \) is an algebraic closure of a finite field \( F_q \) and that \( G = G^0 = D \) has connected centre; we also assume that \( G \) has a fixed \( F_q \)-rational structure. Let \( G^* \) be a connected reductive group over \( k \) of type dual to that of \( G \) with the induced \( F_q \)-structure. Let \( sc(G^*) \) be the set of all special conjugacy classes in \( G^* \) which are defined over \( F_q \). It is known [L84b, 13.2] that there is a natural map \( \tau : \text{Irr}(G(F_q)) \rightarrow cs(G^*) \). For any \( \Sigma \in \text{Str}(G^*) \) let \( \text{Irr}(G(F_q))_\Sigma \) be the set of all \( E \in \text{Irr}(G(F_q)) \) such that \( \tau(E) \subset \Sigma \). Thus we have a partition

(a) \( \text{Irr}(G(F_q)) = \bigsqcup_{\Sigma \in \text{Str}(G^*)} \text{Irr}(G(F_q))_\Sigma \).

We have natural bijections \( a : \text{Str}(G^*) \xrightarrow{\sim} \text{Str}(G) \), \( b : \text{Str}(G) \xrightarrow{\sim} \text{Str}(G_C) \) where \( G_C \) denotes a connected reductive group over \( C \) of the same type as that of \( G \) (see [L15]: each of the sets \( \text{Str}(G^*), \text{Str}(G), \text{Str}(G_C) \) is indexed by \( S_2(W) \)). From the results in [L84b, 13.3], we see that for \( \Sigma \in \text{Str}(G^*) \), \( \text{Irr}(G(F_q))_\Sigma \) is empty unless \( ba(\Sigma) \) contains some unipotent class in \( G_C \). Thus (a) can be regarded as a partition of \( \text{Irr}(G(F_q)) \) into pieces indexed by the Strata \( \Sigma' \) of \( G \) such that \( b(\Sigma') \) contains a unipotent class in \( G_C \). (In loc.cit. this partition is defined only under the assumption that \( p \) is a good prime for \( G \) but the same definition applies without this assumption; if \( p \) is a bad prime for \( G \), not all \( \Sigma' \) as above contribute to the partition.)

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