Improved Rates for Derivative Free Gradient Play in Strongly Monotone Games∗

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Abstract— The influential work of Bravo et al. [1] shows that derivative free gradient play in strongly monotone games has complexity $O(d^2/\epsilon^2)$, where $\epsilon$ is the target accuracy on the expected squared distance to the solution. This paper shows that the efficiency estimate is actually $O(d^2/\epsilon^2)$, which reduces to the known efficiency guarantee for the method in unconstrained optimization. The argument we present simply interprets the method as stochastic gradient play on a slightly perturbed strongly monotone game to achieve the improved rate.

I. INTRODUCTION

Game theoretic abstractions are foundational in many application domains ranging from machine learning to reinforcement learning to control theory. For instance, in machine learning game theoretic abstractions are used to develop solutions to learning from adversarial or otherwise strategically generated data (see, e.g., [2–4]). Analogously, in reinforcement learning and control theory, game theoretic abstractions are used to develop robust algorithms and policies (see, e.g., [5–9]). Additionally, they are used to capture interactions between multiple decision making entities and to model asymmetric information and incentive problems (see, e.g., [7, 8, 10]).

In such game theoretic abstractions, each decision-maker or ‘player’ faces an optimization problem that is dependent on the decisions of other players in the game. Learning as a tool for finding equilibria, or explaining how players in a game arrive at an equilibrium through a process of tâtonnement, is a long studied phenomenon [11, 12]. Gradient-based learning algorithms form a very natural class of learning algorithms for games on continuous actions spaces with sufficiently smooth cost functions. Additionally, in both control theory and machine learning, typically gradient-based methods are used since they scale well.

There is an extensive body of literature—too vast to cite all relevant work—analyzing stochastic gradient play and its variants in different classes of games ranging from zero-sum to general-sum. The majority of the work on stochastic gradient play assumes access to a gradient oracle that provides an unbiased estimate of each player’s individual gradient—i.e., the partial gradient of a player’s cost with respect to their own choice variable.

Counter to this, we are motivated by settings in which non-cooperative players interact in extremely low-information environments: in this paper, we examine the long-run behavior of learning with so-called “bandit feedback” in strongly monotone games. Specifically, players have access only to a loss function oracle, and use responses to queries (of the loss function value) to construct a gradient estimate. The bandit feedback setting has been studied extensively in the single player case [13–16], and over the last few years it has been extended to the multi-agent setting [1, 17, 18].

In particular, the influential work of Bravo et al. [1] studies derivative free “gradient play” wherein players formulate a gradient estimate using a single-point query to their loss function. While in a general game such algorithms (even with perfect gradient information) may not converge, for strongly monotone games, which admit a unique Nash equilibrium, the authors show convergence to the Nash equilibrium. Moreover, they show that the iteration complexity is $O(d^2/\epsilon^3)$, where $\epsilon$ is the target accuracy on the expected squared distance to the solution, and $d$ is the problem dimension. It was conjectured in [1] that this rate should match that of single player optimization, which is known to be $O(1/\epsilon^2)$ in terms of target accuracy.

In this paper, we resolve this open question by showing that the iteration complexity is in fact $O(d^2/\epsilon^2)$. Our proof deviates significantly from the analysis in Bravo et al. [1]. In particular, we take the unique perspective that the update players are executing is simply stochastic gradient play on a slightly perturbed strongly monotone game, and this tighter analysis leads to the optimal rate result.

II. PROBLEM SETUP AND ALGORITHM

In this paper, we consider an $n$-player game defined by cost functions $f_i: \mathcal{X}_i \to \mathbb{R}$ and sets of strategies $\mathcal{X}_i \subset \mathbb{R}^{d_i}$. Thus each player $i \in \{1, \ldots, n\}$ seeks to solve the problem

$$\min_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}),$$

where $x_{-i}$ denotes the actions of all the players excluding player $i$. A vector of strategies $x^* = (x_1^*, \ldots, x_n^*)$ is a Nash equilibrium if each player $i$ has no incentive to unilaterally change their strategy, that is

$$x_i^* \in \arg \min_{x_i \in \mathcal{X}_i} f_i(x_i, x_{-i}).$$

The symbol $\nabla_i(\cdot)$ denotes the partial derivative of the argument $(\cdot)$ with respect to $x_i$. Set $d = \sum_{i=1}^n d_i$, and let $S_i$...
and $\mathbb{B}_i$ denote the unit sphere and unit ball in $\mathbb{R}^{d_i}$, respectively. Additionally, we impose the following convexity and smoothness assumptions.

**Assumption 1 (Standing):** There exist constants $\beta, L \geq 0$ and $\alpha > 0$ such that for each $i \in [n]$, the following hold:

(a) The set $X_i \subset \mathbb{R}^{d_i}$ is closed, convex, and bounded with non-empty interior, and there exist constants $r, R > 0$ satisfying $r \mathbb{B} \subset X \subset R \mathbb{B}$ where $X = X_1 \times \cdots \times X_n$.

(b) The function $f_i(x_i, x_{-i})$ is convex and $C^1$-smooth in $x_i$ and the gradient $\nabla_i f_i(x)$ is $\beta$-Lipschitz continuous in $x_i$.

(c) The Jacobian of the map $\nabla_i f_i(x)$ is $L$-Lipschitz continuous, meaning

$$\|\nabla(\nabla_i f_i(x)) - \nabla(\nabla_i f_i(x'))\|_{\text{op}} \leq L\|x - x'\|$$

for all $x, x' \in X$.

(d) The gradient map

$$g_i(x) := (\nabla_1 f_i(x), \nabla_2 f_i(x), \ldots, \nabla_n f_i(x))$$

is $\alpha$-strongly monotone on $X$, meaning

$$(g(x) - g(x'), x - x') \geq \alpha\|x - x'\|^2 \quad \forall x, x' \in X.$$  

(e) We set $F_s := \max_{i} \max_{x \in X} |f_i(x)|^2$ (which is finite, by assumption (a)).

Classical results, such as those in the seminal work by Rosen [19], guarantee that the game admits a unique Nash equilibrium under Assumption 1. Items (a)–(b) and (d)–(e) are identical to those in Bravo et al. [1]. In contrast, item (c) is not assumed in Bravo et al. [1], but will be important in what follows. We note that when applied to the single player setting $n = 1$, none of our results require item (c) and it may be dropped entirely.

### A. Examples of Strongly Monotone Games

Before moving on, we provide examples of classical strongly monotone games that satisfy Assumption 1(c).

**Example 1 (Cournot Competition):** Consider an $n$-player Cournot oligopoly model in which $X_i = [0, C_i]$ for some $C_i > 0$, and utility functions $u_i(x_i, x_{-i}) = x_i P(\bar{x}) - c_i x_i$ where $\bar{x} = \sum_{i=1}^n x_i$ and $P(z) = a - bq$ for positive constants $a, b > 0$. Then, the cost minimization game is defined by $(f_1, \ldots, f_n)$ where $f_i \equiv -u_i$. The individual gradient for player $i$ is given by

$$\nabla_i f_i(x) = 2b \cdot x_i - a + b \sum_{j \neq i} x_j + c_i,$$

so that $\nabla(\nabla_i f_i) = (b, \ldots, b, 2b, b, \ldots, b)$, where $2b$ is in the $i$-th entry of $\nabla(\nabla_i f_i)$ and all other $n - 1$ entries contain the element $b$. Thus, we deduce that

$$\|\nabla(\nabla_i f_i(x)) - \nabla(\nabla_i f_i(x'))\|_{\text{op}} \leq 0 \cdot \|x - x'\|$$

so that $L = 0$ in Assumption 1(c).

**Example 2 (Resource Allocation Auction):** Consider a resource allocation auction with $n$-bidders that place monetary bids on a set of resources $S = \{1, \ldots, S\}$. That is, bidder $i$ places bid $x_{is} \geq 0$ for the utilization of resources $s \in S$.

### Algorithm 1: Derivative Free Gradient Play [1]

**Input:** Horizon $T \in \mathbb{N}$, step-sizes $\eta_t > 0$, radius $\delta \in (0, r)$, initial strategies $x^0 \in (1 - \delta)X$.

```
for $t = 0, \ldots, T - 1$ do
    for $i = 1, \ldots, n$ do
        Sample $v^t_i \in S$ uniformly at random;
        Play $x^t_i + \delta v^t_i$;
        Compute $g^t_i = \frac{\partial}{\partial x_i} f_i(x^t_i + \delta v^t_i, x^{t-1}_i + \delta v^{t-1}_i)$;
        Update $x^{t+1}_i = \text{proj}_{(1-\delta)X_i}(x_i - \eta_t g^t_i)$;
    end
end
```

**Output:** $x^T = (x^T_1, \ldots, x^T_n)$

Each bidder has a budget $b_i$—that is, $\sum_s x_{is} \leq b_i$. Bidder $i$’s utility is given by

$$u_i(x_i, x_{-i}) = \sum_{s \in S} g_{is} x_{is} - x_{is},$$

where $g_s$ is the available units, $c_s$ is the cost for entry (minimum level bid), and $g_i$ is player $i$'s the marginal gain for acquiring a unit of a resource. For simplicity, let $n = 2$, and define the cost minimization game $(f_1, f_2)$ where $f_i \equiv -u_i$. Then,

$$\nabla_i f_i(x) = \left( g_{i} - \frac{g_s}{c_s + \sum_j x_{js}} - g_{i} \frac{q_{s} x_{is}}{(c_s + \sum_j x_{js})^2} \right)_{i \in S}.$$

A simple computation of derivatives yields bound

$$\|\nabla^2(\nabla_i f_i(x))\|_{\text{op}} \leq 4 \cdot S \cdot \frac{6g_i \max_{s \in S} q_s (b_i + 1)}{(\min_{s \in S} c_s)^p},$$

where $p = 4$ if $\min_{s} c_s > 1$ and otherwise $p = 3$. Taking the maximum over players $n = 1, 2$ yields a value of $L$ in Assumption 1(c).

### B. Algorithm and Convergence Guarantee

In this work, we study a derivative-free algorithm proposed by Bravo et al. [1] for finding the Nash equilibrium of the game. We note that the gradient estimator used by Bravo et al. [1] is motivated by the analogous construction introduced by Flaxman et al. [15] for the single player setting. The procedure is recorded as Algorithm 1.

Algorithm 1 in each iteration $t$ samples $v^t_i \in \mathbb{S}_1 \times \cdots \times \mathbb{S}_n$ uniformly at random and then declares

$$x^{t+1}_i = \text{proj}_{(1-\delta)X_i}(x^t_i - \eta_t g^t_i)$$

where

$$\hat{g}^t_i := (\hat{g}^t_1, \ldots, \hat{g}^t_n).$$

The reason for projecting onto the set $(1 - \delta)X$ is simply to ensure that in the next iteration $t + 1$, the action profile is valid in the sense that $x^{t+1}_i + \delta v^{t+1}_i$ lies in $X_i$ for each player $i \in \{1, \ldots, n\}$.
Bravo et al. [1] show that with appropriate parameter choices, Algorithm 1 will find a point \( x \) satisfying \( E\|x - x^*\|^2 \leq \varepsilon \) after \( O\left(\frac{d^2}{\varepsilon^2}\right) \) iterations, and leave it as an open question if this result is tight. We provide a different convergence argument that yields an improved efficiency estimate \( O\left(\frac{d^2}{\varepsilon^2}\right) \). The estimate matches the known rate of convergence of the method for unconstrained optimization problems (i.e., \( n = 1 \) and \( \mathcal{X} = \mathbb{R}^d \)) established by Agarwal et al. [13].

We note that compared to [1] our results do rely on a slightly stronger assumption on the second-order smoothness of the loss functions, summarized in item (c) of Assumption 1. This assumption is not needed in the single player setting \( n = 1 \).

**Theorem 1 (Informal):** For sufficiently small \( \varepsilon > 0 \), there exists a choice of \( \delta > 0 \) and \( \eta_i > 0 \) such that Algorithm 1 generates a sequence \( x^t \) satisfying

\[
E\|x^t - x^*\|^2 \leq \varepsilon
\]

for \( t \) on the order of \( \frac{d^2}{\varepsilon^2} \).

We provide the formal statement of the main result as well as the proof in the next section. A key idea is to interpret Algorithm 1 as stochastic gradient play applied to a slightly perturbed strongly monotone game.

**III. MAIN RESULT**

The starting point is the very motivation for the update (1), which is that the vector \( \hat{g} \) is an unbiased estimator of the gradient map for a different game. Namely, for each player \( i \), define the smoothed cost

\[
f_i^\delta(x_i, x_{-i}) := \mathbb{E}_{w \sim U_i} f_i(x_i + \delta w_i, x_{-i} + \delta w_{-i}),
\]

where \( U_i \) denotes the uniform distribution on

\[
\mathbb{E}_i \times \bigotimes_{j \neq i} S_j.
\]

The vector \( w \) is of size \( d \), where recall that \( d = \sum_{i=1}^n d_i \). The following is proved in [1, Lemma C.1] and follows closely the argument in [15].

**Lemma 1 (Unbiased gradient estimator):** For each index \( i \in \{1, \ldots, n\} \), let \( v_i \) be sampled uniformly from \( S_i \) and define the random vector

\[
\hat{g}_i = \frac{d_i}{\delta} f_i(x_i + \delta v_i, x_{-i} + \delta v_{-i}) v_i.
\]

The following equality holds:

\[
E[\hat{g}_i] = \nabla_i f_i^\delta(x).
\]

The path forward is now clear: we interpret Algorithm 1 as stochastic gradient play on the perturbed game defined by the losses \( f_i^\delta \) over the smaller set \( (1 - \delta)\mathcal{X} \). In particular, taking advantage of Assumption 1(c), we are able to bound the distance between the Nash equilibrium of game defined by the smoothed cost functions and the original game. Using this result, we can decompose the error between the iterates and the Nash equilibrium of the original game. To this end, define the perturbed gradient map

\[
g_i^\delta(x) = (\nabla_1 f_i^\delta(x), \ldots, \nabla_n f_i^\delta(x)).
\]

Lemmas 2 and 3 estimate the smoothness and strong monotonicity constants of the perturbed game, thereby allowing us to invoke classical convergence guarantees for stochastic gradient play on the perturbed game.

**Lemma 2 (Smoothness of the perturbed game):** For each index \( i \in \{1, \ldots, n\} \), the loss \( f_i^\delta \) is differentiable and the map \( x \mapsto \nabla_i f_i^\delta(x) \) is \( \beta \)-Lipschitz continuous. Moreover the following estimate holds,

\[
\|g(x) - g_i^\delta(x)\| \leq \beta \delta n \quad \forall x \in \mathcal{X}.
\]

**Proof:** For any points \( x, x' \in \mathcal{X} \), we successively estimate

\[
\|\nabla_i f_i^\delta(x) - \nabla_i f_i^\delta(x')\|
\leq \mathbb{E}_{w \sim U_i} \left[ \|\nabla_i f_i(x_i + \delta w_i, x_{-i} + \delta w_{-i}) - \nabla_i f_i(x_i' + \delta w_i, x_{-i}' + \delta w_{-i})\| \right]
\leq \beta\|x - x'\|.
\]

Thus, the individual gradient \( \nabla_i f_i^\delta \) is \( \beta \)-Lipschitz continuous. Next, we estimate

\[
\|\nabla_i f_i(x) - \nabla_i f_i^\delta(x)\| \leq \mathbb{E}_{w \sim U_i} \left[ \|\nabla_i f_i(x + \delta w) - \nabla_i f_i(x)\| \right]
\leq \beta \mathbb{E}_{w \sim U_i} \sqrt{\|w_i\|^2 + (n - 1)}
\leq \beta \sqrt{n},
\]

Therefore, we deduce that

\[
\|g(x) - g_i^\delta(x)\| = \left( \sum_{i=1}^n \|\nabla_i f_i(x) - \nabla_i f_i^\delta(x)\|^2 \right)^{1/2} \leq \beta \sqrt{n},
\]

as claimed.

Observe that in the single player case \( n = 1 \), the function \( f_i^\delta \) is trivially \( \alpha \)-strongly convex for any \( \delta > 0 \). For general \( n > 1 \), the following lemma shows that the perturbed map \( g_i^\delta \) is strongly monotone for all sufficiently small \( \delta \).

**Lemma 3 (Strong monotonicity of the smoothed game):** Choose \( \delta \leq \frac{c\alpha}{L \ln^{3/2} \bar{n}} \) for some constant \( c \in (0, 1) \). Then the gradient map \( g_i^\delta \) is strongly monotone over \( \mathcal{X} \) with parameter \( (1 - c)\alpha \).

**Proof:** Fix an index \( i \) and let us first estimate the Lipschitz constant of the difference map

\[
H_i(x) := \nabla_i f_i^\delta(x) - \nabla_i f_i(x).
\]

To this end, we compute

\[
\nabla H_i(x) = \mathbb{E}_{w \sim U_i} [\nabla(\nabla_i f_i)(x + \delta w) - \nabla(\nabla_i f_i)(x)].
\]
Taking into account that the map \( x \mapsto \nabla(\nabla f_i)(x) \) is \( L \)-Lipschitz continuous, we deduce that
\[
\|\nabla H_i(x)\|_{op} \leq \sum_{w \sim \delta} \frac{\|\nabla(\nabla f_i)(x + \delta w) - \nabla(\nabla f_i)(x)\|_{op}}{\delta L} \leq \delta L E_{w \sim \delta} \|w\| \leq \delta L \sqrt{n}.
\]
Thus, the map \( H_i \) is Lipschitz continuous with parameter \( \delta L \sqrt{n} \). In turn, we have that
\[
\langle g^\delta(x) - g^\delta(x'), x - x' \rangle = \sum_{i=1}^n \langle \nabla_i f_i^\delta(x) - \nabla_i f_i^\delta(x'), x_i - x_i' \rangle - \sum_{i=1}^n \langle H_i(x') - H_i(x), x_i - x_i' \rangle \geq (\alpha - L n^{3/2} \delta) \|x - x'\|^2.
\]
The proof is complete.

The last ingredient, summarized in Lemma 4, is to estimate the distance between the Nash equilibria of the original and the perturbed games. Henceforth, let \( x_\delta^* \) be the Nash equilibrium of the game with losses \( f_i^\delta \) over the set \((1 - \delta) \mathcal{X} \).

**Lemma 4 (Distance between equilibria):** Choose any \( \delta < \min \{ r, \frac{\alpha}{Ln^{3/2}} \} \). Then, the following estimate holds:
\[
\|x^* - x_\delta^*\| \leq \delta \left( 1 + \frac{\beta \sqrt{n}}{\alpha} \right) \|x^*\| + \frac{\beta n^{3/2}}{\alpha}.
\]
**Proof:** Lemma 3 and our choice of \( \delta \) ensures that \( g^\delta \) is strongly monotone and therefore that \( x_\delta^* \) is well-defined. There are two sources of perturbation: one replacing \( \mathcal{X} \) with \((1 - \delta) \mathcal{X} \) and the other in replacing \( f_i \) with \( f_i^\delta \). We deal with these in turn. To this end, set \( \gamma := 1 - \delta \) and let \( \tilde{x} \) be the Nash equilibrium of the original game defined by the losses \( f_i \) over the shrunk set \( \gamma \mathcal{X} \). Thus, the joint action \( \tilde{x} \) satisfies the inclusion
\[
0 \in g(\tilde{x}) + N_{\gamma \mathcal{X}}(\tilde{x}),
\]
where \( N_{\gamma \mathcal{X}}(\tilde{x}) \) denotes the normal cone to \( \gamma \mathcal{X} \) at \( \tilde{x} \), i.e., the set of all \( v \) such that \( \langle z - \tilde{x}, v \rangle \leq 0 \) for all \( z \in \gamma \mathcal{X} \) [20, Ch. 2]. The triangle inequality directly gives us that
\[
\|x^* - x_\delta^*\| \leq \|x^* - \tilde{x}\| + \|\tilde{x} - x_\delta^*\|.
\]
Let us bound the first term on the right side of (2). To this end, since the map \( x \mapsto g(x) + N_{\gamma \mathcal{X}}(x) \) is \( \alpha \)-strongly monotone, we deduce
\[
\alpha \|\tilde{x} - g^\delta(x^*)\| \leq \text{dist}(0, g(\gamma x^*) + N_{\gamma \mathcal{X}}(\gamma x^*)).
\]
Let us estimate the right-hand side of (3). Since \( x^* \) is a Nash equilibrium of the original game over \( \mathcal{X} \), the inclusion
\[
0 \in g(x^*) + N_{\mathcal{X}}(x^*)
\]
holds. Taking into account the identity \( N_{\gamma \mathcal{X}}(\gamma x^*) = N_{\mathcal{X}}(x^*) \), we deduce that
\[
d(0, g(x^*) + N_{\gamma \mathcal{X}}(x^*)) = d(0, g(x^*) + N_{\mathcal{X}}(x^*)) \leq \|g(x^*) - g(x^*)\| \leq \delta \beta \sqrt{n} \|x^*\|,
\]
where the last inequality follows from \( g \) being \( \beta \sqrt{n} \)-Lipschitz continuous. Appealing to (3) and using the triangle inequality, we therefore deduce that
\[
\|x^* - \tilde{x}\| \leq \|\tilde{x} - g^\delta(x^*)\| + \|g^\delta(x^*) - g(x^*)\| \leq \delta (1 + \beta \sqrt{n}) \|x^*\|.
\]
It remains to upper bound \( \|\tilde{x} - x_\delta^*\| \). The definition of \( \tilde{x} \) as a Nash equilibrium ensures
\[
\langle -g(\tilde{x}), x - \tilde{x} \rangle \leq 0, \ \forall x \in \gamma \mathcal{X}.
\]
Analogously, the definition of \( x_\delta^* \) as a Nash equilibrium ensures
\[
\langle -g^\delta(x_\delta^*), x - x_\delta^* \rangle \leq 0, \ \forall x \in \gamma \mathcal{X}.
\]
Then, by strong monotonicity of the game and estimates (5) and (6), we get that
\[
\alpha \|\tilde{x} - x_\delta^*\|^2 \leq \langle g(\tilde{x}) - g(x_\delta^*), \tilde{x} - x_\delta^* \rangle \leq \langle g^\delta(x_\delta^*) - g(x_\delta^*), \tilde{x} - x_\delta^* \rangle \leq \|g^\delta(x_\delta^*) - g(x_\delta^*)\| \cdot \|\tilde{x} - x_\delta^*\| \leq \beta \delta n \|\tilde{x} - x_\delta^*\|,
\]
where the last inequality follows from Lemma 2. Rearranging, we conclude
\[
\|\tilde{x} - x_\delta^*\| \leq \frac{\beta \delta n}{\alpha},
\]
which combined with (2) and (4) completes the proof.

With the observations that the game defined by smoothed functions \( f_i^\delta \) is strongly monotone and the individual gradients of the smoothed loss functions are Lipschitz, we arrive at the following efficiency guarantee.

**Theorem 2:** Suppose \( \delta \leq \min \{ r, \frac{\alpha}{Ln^{3/2}} \} \) and set \( \eta_t = \frac{2}{\alpha t} \). The following holds:
\[
E\left[\|x^t - x^*\|^2\right] \leq \frac{\max\{\delta^2 \alpha^2 \|x^1 - x^*_\delta\|^2, 8F_\delta d^2 n\}}{\delta^2 \alpha^2 t} + 2 \delta^2 \left( 1 + \frac{\beta \sqrt{n}}{\alpha} \right) \|x^*\|^2 + \frac{\beta n^{3/2}}{\alpha}.
\]
**Proof:** Using the inequality \((a + b)^2 \leq 2a^2 + 2b^2 \) and Lemma 4, we estimate the one step progress
\[
\frac{1}{2} \|x^{t+1} - x^*\|^2 \leq \|x^{t+1} - x^*_\delta\|^2 + \|x^*_\delta - x^*\|^2 + 2 \delta^2 \left( 1 + \frac{\beta \sqrt{n}}{\alpha} \right) \|x^*\|^2 + \frac{\beta n^{3/2}}{\alpha}.
\]
Next we continue with the standard estimate using non-expansiveness of the projection:
\[
E[\|x_{t+1} - x^\star\|^2] = E[\| \text{proj}_{A(1-\delta)X}(x^t - \bar{g}_\delta(x^t)) - x^\star \|^2]
\]
\[
\leq E[\|x^t - x^\star - \bar{g}_\delta(x^t)\|^2]
\]
\[
\leq E[\|x^t - x^\star\|^2] + \eta_t^2 E[\|\bar{g}_\delta(x^t)\|^2]
- 2\eta_t E[\bar{g}_\delta(x^t), x^t - x^\star]
\leq E[\|x^t - x^\star\|^2] - 2\eta_t (g^\delta(x^t), x^t - x^\star) + \eta_t^2 E[\|\bar{g}_\delta(x^t)\|^2].
\] (7)

Since \(g^\delta\) is strongly monotone with parameter \(\frac{\alpha}{2}\) (Lemma 3) and \(x^\star\) is the Nash equilibrium of the smoothed game, we deduce
\[
\langle g^\delta(x^t), x^t - x^\star \rangle \geq \langle g^\delta(x^t) - g^\delta(x^\star), x^t - x^\star \rangle \geq \frac{\alpha}{2} \|x^t - x^\star\|^2.
\]

Returning to (7), we conclude
\[
E[\|x_{t+1} - x^\star\|^2] \leq (1 - \alpha \eta_t) E[\|x^t - x^\star\|^2] + \eta_t^2 F \sigma^2 n^2 \delta^2.
\]

A standard inductive argument shows
\[
E[\|x^t - x^\star\|^2] \leq \sum_{i=1}^t \left( \max \left\{ \delta^2 \alpha^2 \|x^i - x^\star\|^2, 8F \sigma^2 n^2 \right\} \right)
\]
for all \(t \geq 1\). The proof is complete.

**Main Result.** The following is now the formal statement and proof of Theorem 1.

**Corollary 1 (Main Result):** Fix a target accuracy
\[
\varepsilon < ((\alpha + \beta \sqrt{n})R + \beta n)^2 \cdot \min \left\{ \frac{1}{L_{\text{n}}}, \frac{4 \pi^2}{\alpha^2} \right\},
\]
and set
\[
\delta = \frac{\alpha \sqrt{\varepsilon}}{4(\alpha + \beta \sqrt{n})R + \beta n}
\]
and \(\eta_t = \frac{2}{\sqrt{\alpha} t}\). Then, the estimate \(E[\|x^t - x^\star\|^2] \leq \varepsilon\) holds for all
\[
t \geq \max \left\{ \frac{32 \alpha^4 \varepsilon R^2}{C^2}, \frac{64((\alpha + \beta \sqrt{n})R + \beta n)^2 F \sigma^2 n^2}{\alpha^2 \varepsilon} \right\}.
\]

In the single player setting \(n = 1\), the conclusion holds for any \(\varepsilon < 4\pi^2((1 + \frac{\beta}{\alpha})R + \frac{\beta}{2} n)^2\) and Assumption 1 (item c) may be dropped.

**Proof:** The assumed upper bound on \(\varepsilon\) directly implies
\[
\delta \leq \frac{\alpha \sqrt{\varepsilon}}{2 L_{\text{n}} \beta R}, \quad \text{and} \quad \delta < r.
\]

An application of Theorem 2 yields the estimate
\[
E[\|x^t - x^\star\|^2] \leq \max \left\{ \delta^2 \alpha^2 \|x^t - x^\star\|^2, 8F \sigma^2 n^2 \right\} + \frac{\varepsilon}{2}.
\]

Setting the right side to \(\varepsilon\), solving for \(t\), and using the trivial upper bound \(\|x^t - x^\star\| \leq 2R\) completes the proof. The claims for the single player setting \(n = 1\) follows by noting that the conclusions of Lemmas 3 and 4 and Theorem 2 hold for any \(\delta \in (0, r)\), since \(F^\delta\) is strongly convex for any \(\delta > 0\).

In practice, it may be advantageous to not specify \(\varepsilon\) at the onset and instead allow the algorithm to run indefinitely. This can be easily achieved without sacrificing efficiency simply by restarting the algorithm periodically while shrinking \(\delta\) by a constant fraction. The resulting process and its convergence rate is summarized in the following corollary.

**Corollary 2 (Efficiency without target accuracy):** Define the following constants:
\[
A = \frac{8F \sigma^2 n}{\alpha^2}, \quad B = 2 \left( 1 + \frac{\beta \sqrt{n}}{\alpha} \right) R + \frac{\beta n}{\alpha},
\]
\[
\delta_t = \min \left\{ \frac{\alpha}{2}, \frac{\beta \sqrt{n}}{2Ln^3/2} \right\}, \quad T_1 = \max \left\{ \frac{A}{B \delta_t^4}, \frac{4R^2}{B \delta_t^2} \right\}.
\]

Fix a fraction \(q \in (0, 1)\), and set \(y^0 = x^0\) and \(\eta_t = \frac{2}{\sqrt{\alpha} t}\) for each index \(t \geq 1\). Consider the following process:
\[
\begin{align*}
y^k &= \text{DFO}\left(y^{k-1}, \{\eta_t\}_{t \geq 1}, \delta_t, T_k \right), \\
\delta_{k+1} &= q \cdot \delta_k, \\
T_{k+1} &= \max \left\{ \frac{A}{B \delta_{k+1}^4}, \frac{4R^2}{B \delta_{k+1}^2} \right\}.
\end{align*}
\]

For every \(k \geq 1\), the iterate \(y^k\) satisfies
\[
E[\|y^k - x^\star\|^2] \leq \varepsilon_k,
\]
where \(\varepsilon_k := 2B \delta_t^4 q^{2(k-1)}\), while the total number of steps of Algorithm 1 needed to generate \(y^k\) is at most
\[
1 + \frac{1}{2} \log(2B \delta_t^2 \varepsilon_k) + 4\sqrt{q} \varepsilon_k^{-2} \delta_t^{-2} + 4R^2 q^{-2} \varepsilon_k^{-2} - 1.
\]

In the single player setting (i.e., \(n = 1\)), we may instead set \(\delta_1 = r\).

**Proof:** Theorem 2 directly guarantees
\[
E[\|y^k - x^\star\|^2] \leq \frac{\max \{ A, 4R^2 \delta_t^2 \} }{\delta_t^4 T_k} + B \delta_t^2 \leq 2B \delta_t^2.
\]

The total number of steps of Algorithm 1 needed to generate \(y_k\) is bounded as follows:
\[
\sum_{i=1}^k T_i \leq \frac{A}{B} \delta_1^{-4} \cdot q^{-4(i-1)} + \frac{4R^2}{B} \delta_1^{-2} \cdot q^{-2(i-1)}
\]
\[
\leq k + \frac{A}{B} \delta_1^{-4} \sum_{j=0}^{k-1} q^{-4j} + k + \frac{4R^2}{B} \delta_1^{-2} \sum_{j=0}^{k-1} q^{-2j}
\]
\[
\leq k + \frac{A}{B} \delta_1^{-4} \cdot q^{-4k} + \frac{4R^2}{B} \delta_1^{-2} \cdot q^{-2k}.
\]

Rewriting the right-side in terms of \(\varepsilon_k\) completes the proof.

**IV. DISCUSSION AND FUTURE DIRECTIONS**

A promising future research direction is to examine the benefits of using a two-point (and multi-point) gradient estimator to improve the dimension dependence. In a two-point method, once a random perturbation direction is chosen, two function evaluations are performed along that direction. We note that asymptotic analysis of two-point updates has been performed for games with a very simplified structure (e.g., scalar action spaces, and symmetric externalities) [21]. An example of such an update is \(f(x + \delta u) - f(x - \delta u)\), as examined by Agarwal et al. [13] and by Nesterov and Spokoiny [16]. The use of this symmetric two-point estimate
can improve the constants in the convergence rate, however, the dependence of the rate on problem dimension $d$ still is not optimal. We remark that this symmetric expression yields an unbiased estimate of the gradient of $f$, and its extension to the game setting is straightforward; i.e., the proof approach in the current paper can be applied.

One point to note when considering the extension of two-point methods to the multi-player setting is to ensure each player can indeed evaluate their cost function $f_i(x_i, x_{-i})$ at two points—player $i$ can certainly vary its own action $x_i$, but can it do so while keeping the other players’ actions $x_{-i}$ fixed? Such a method requires more explicit coordination between players; however, there are practical settings where such coordination is possible. One example is multiplayer performative prediction—first introduced in [22]—wherein players have a loss function oracle and observe their competitors’ actions. This would enable players to form estimates using query responses of the form $f_i(x_i + \delta u_i, x_{-i})$.

The analysis in [23] improves the rate in [13] significantly by a factor of $\sqrt{d}$, by employing a one-sided two-point estimate of the gradient. This introduces additional bias in the gradient estimate that they cleverly handle. It is an interesting direction to adapt this algorithm to the game setting. However, due to the asymmetry and hence bias in the estimate of [23], a new proof approach is needed.

References

[1] M. Bravo, D. S. Leslie, and P. Mertikopoulos, “Bandit learning in concave n-person games,” Proceedings of the Conference on Neural Information Processing Systems, 2018.

[2] A. Madry, A. Makelov, L. Schmidt, D. Tsipras, and A. Vladu, “Towards deep learning models resistant to adversarial attacks,” Proceedings of the International Conference Learning Representation, 2018.

[3] T. Fiez, B. Chasnov, and L. Ratliff, “Implicit learning dynamics in Stackelberg games: Equilibria characterization, convergence analysis, and empirical study,” in International Conference on Machine Learning. PMLR, 2020, pp. 3133–3144.

[4] I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A. Courville, and Y. Bengio, “Generative adversarial nets,” Advances in neural information processing systems, vol. 27, 2014.

[5] L. J. Ratliff, S. A. Burden, and S. S. Sastry, “On the characterization of local Nash equilibria in continuous games,” IEEE Transactions on Automatic Control, vol. 61, no. 8, pp. 2301–2307, 2016.

[6] Z. Zhou, P. Mertikopoulos, A. L. Moustakas, N. Bambos, and P. Glynn, “Mirror descent learning in continuous games,” in Proceedings of the 56th IEEE Annual Conference on Decision and Control, 2017, pp. 5776–5783.

[7] L. J. Ratliff and T. Fiez, “Adaptive incentive design,” IEEE Transactions on Automatic Control, vol. 66, no. 8, pp. 3871–3878, 2020.

[8] N. Li and J. R. Marden, “Designing games for distributed optimization,” in Proceedings of the 50th IEEE Conference on Decision and Control, 2011.

[9] A. Yekkehkhany, H. Feng, and J. Lavaei, “Adversarial attacks on computation of the modified policy iteration method,” in Proceedings of the IEEE Conference on Decision and Control, 2021.

[10] Y. Savas, V. Gupta, M. Ornik, L. J. Ratliff, and U. Topcu, “Incentive design for temporal logic objectives,” in Proceedings of the 58th IEEE Conference on Decision and Control, 2019, pp. 2251–2258.

[11] D. Fudenberg, F. Drew, D. K. Levine, and D. K. Levine, The theory of learning in games. MIT press, 1998.

[12] N. Cesa-Bianchi and G. Lugosi, Prediction, learning, and games. Cambridge university press, 2006.

[13] A. Agarwal, O. Dekel, and L. Xiao, “Optimal algorithms for online convex optimization with multi-point bandit feedback,” in Proceedings of the Conference on Learning Theory, 2010, pp. 28–40.

[14] O. Shamir, “An optimal algorithm for bandit and zero-order convex optimization with two-point feedback,” The Journal of Machine Learning Research, vol. 18, no. 1, pp. 1703–1713, 2017.

[15] A. D. Flaxman, A. T. Kalai, and H. B. McMahan, “Online convex optimization in the bandit setting: gradient descent without a gradient,” Proceedings of the sixteenth annual ACM-SIAM Symposium on Discrete Algorithms, 2005.

[16] Y. Nesterov and V. Spokoiny, “Random gradient-free minimization of convex functions,” Foundations of Computational Mathematics, vol. 17, no. 2, pp. 527–566, 2017.

[17] T. Tatarenko and M. Kamgarpour, “Bandit online learning of Nash equilibria in monotone games,” arXiv preprint arXiv:2009.04258, 2020.

[18] ———, “Learning Nash equilibria in monotone games,” in Proceedings of the 58th IEEE Conference on Decision and Control, 2019, pp. 3104–3109.

[19] J. B. Rosen, “Existence and uniqueness of equilibrium points for concave n-person games,” Econometrica: Journal of the Econometric Society, pp. 520–534, 1965.

[20] R. T. Rockafellar, Convex analysis, 2nd ed. Princeton university press, 1972.

[21] S. Bervoets, M. Bravo, and M. Faure, “Learning with minimal information in continuous games,” Theoretical Economics, vol. 15, no. 4, pp. 1471–1508, 2020.

[22] A. Narang, E. Faulkner, D. Drusvyatskiy, M. Fazel, and L. J. Ratliff, “Learning in strongly monotone decision-dependent games,” Proceedings of the Artificial Intelligence and Statistics Conference, 2022.

[23] J. C. Duchi, M. I. Jordan, M. J. Wainwright, and A. Wibisono, “Optimal rates for zero-order convex optimization: The power of two function evaluations,” IEEE Transactions on Information Theory, vol. 61, no. 5, pp. 2788–2806, 2015.