Functional Mellin Transforms

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Abstract

Functional integrals are defined in terms of locally compact topological groups and their associated Banach-valued Haar integrals. This approach generalizes the functional integral scheme of Cartier and DeWitt-Morette. The definition allows a construction of functional Mellin transforms. In turn, the functional Mellin transforms can be used to define functional traces, logarithms, and determinants. The associated functional integrals are useful tools for probing function spaces in general and $C^*$-algebras in particular. Several interesting aspects are explored.

1 Introduction

Rigorous functional integration had its beginnings in the study of stochastic processes — particularly the Wiener process — and was therefore deeply rooted in probability theory. Subsequently, functional integrals were found to be useful in the study of partial differential equations in general and quantum theory in particular, and so they have been extensively developed in the applied mathematics literature.

One consequence of this heritage is that functional integration methods in mathematical physics borrow heavily from probability constructs and are mostly confined to transformations or expansions/perturbations of quadratic-type functional integrals. These archetypical integrals can be characterized by the statement “the Fourier transform of a gaussian is a gaussian”. More precisely, the characteristic function of a gaussian probability distribution is again a gaussian, and the probability analogy is used to carry this notion over to the context of quantum physics. This allows Feynman ‘path integrals’ in physics to be interpreted as functional Fourier transforms between dual Banach spaces.

But the probability analogy can be both inspirational and restrictive. On one hand, probability theory is a useful complement to intuition, and it is easy to imagine that functional integrals based on probability distributions other than gaussian are useful in mathematical physics. On the other hand, expressing probability distributions through their characteristic function can lead to an overemphasis on Fourier transform as a guiding principle.

Without abandoning the probability analogy, we wish to make the simple observation that a probability distribution can also be characterized by its cumulative distribution function. In the functional context, this observation leads us to propose a rather broad definition of functional integral that goes beyond the idea of Fourier transform in the sense that it does not depend on a duality structure between Banach spaces. Instead, the definition (which resembles the functional analog of a cumulative distribution function) is based on the structure of locally compact topological groups and their associated Haar measures. Significantly, useful dualities are still present; but this time in the richer context of Banach algebras.
We start then with a brief exposition of relevant results concerning locally compact topological groups and their associated integral operators on Banach algebras. Then we propose the idea that a functional integral, whose domain is some topological group $G$, represents a family of well-defined integrals over locally compact topological groups that are all homomorphic to $G$. This allows construction of linear integral operators on a Banach algebra of functionals — and eventually representation of $C^*$-algebras.

After laying the necessary groundwork and presenting the new topological group-based scheme for functional integration, it is shown that the Cartier/De Witt-Morette scheme for functional integration [1] is a special case of our construction. The remainder of the paper concentrates on an important subclass of functional integrals that are functional analogs of the Mellin transform. In the functional context, Mellin is not just a simple transformation of Fourier. So developing and investigating the infinite dimensional analog of Mellin transforms is worthwhile. Like functional Fourier, it will turn out that functional Mellin also encodes a duality — but an algebraic rather than group duality. We use the functional Mellin transform to define functional analogues of trace, log and determinant. Contained in a key theorem based on these definitions is the functional generalization of $\exp tr M = \det \exp M$ that, roughly stated, says the Mellin transform and exponential map commute under appropriate conditions.

There are good reasons — beyond representing $C^*$-algebras — to expect that functional Mellin transforms will be useful in applied mathematics. To give just a few: Functional integrals based on the gamma probability distribution show up in the study of constrained function spaces, and these are particular functional Mellin transforms [2]. The functional properties of the Mellin transform allow efficient analytic treatment of harmonic integrals, asymptotic analysis of harmonic sums, and Fuchsian type partial differential equations [3],[4]. Finally, it is well-known that spectral properties of operators associated with a $C^*$-algebra are closely related to Mellin transforms through functional traces and determinants.

2 Topological groups

The proposed scheme for functional integration is based on topological groups. So this entire section comprises a selection of particularly pertinent definitions and theorems (which we state without proof) all of which can be found in [5],[6],[7].

**Definition 2.1** A Hausdorff topological group $G$ is a group endowed with a topology such that; (i) multiplication $G \times G \to G$ by $(g,h) \mapsto gh$ and inversion $G \to G$ by $g \mapsto g^{-1}$ are continuous maps, and (ii) $\{e\}$ is closed.

$G$ is locally compact if every $g \in G$ has a neighborhood basis comprised of compact sets.

$G$ is a Lie group if there exists a neighborhood $U$ of $\{e\}$ such that, for every subgroup $H$, if $H \subseteq U$ then $H = \{e\}$.

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1In this context, given a topological group and target Banach $*$-algebra, the framework furnishes a functional integral realization of algebraic quantization; where integrable functionals comprise the quantum $C^*$-algebra and traces of functional integrals represented as operators on a Hilbert space correspond to quantum states. We will report on this elsewhere.

2Unfortunately, we must necessarily omit a host of interesting and deep results from these references.

3A neighborhood basis at $g \in G$ is a family $\mathcal{N}$ of neighborhoods such that given any neighborhood $U$ of $g$ there exists an $N \in \mathcal{N}$ such that $N \subseteq U$. 
Remark that the closure hypothesis on \( \{ e \} \) together with the topology and group structure allow the closure property to be ‘transported’ to every element in \( G \). That is, \( G \) is Hausdorff iff \( \{ e \} \) is closed. Moreover, since \( G \) is Hausdorff, it is locally compact iff every \( g \in G \) possesses a compact neighborhood.

The motivation for the following definition comes from analogy with the exponential map for finite Lie groups.

**Definition 2.2** ([5] ch. 5) A one-parameter subgroup \( \phi : \mathbb{R} \to G \) of a topological group is the unique extension of a continuous homomorphism \( f \in \text{Hom}_C(I \subseteq \mathbb{R}, G) \) such that \( f(t+s) = f(t)f(s) \) and \( f(0) = e \in G \). Let \( \mathfrak{L}(G) \) denote the set of one-parameter subgroups \( \text{Hom}_C(\mathbb{R}, G) \) endowed with the uniform convergence topology on compact sets in \( \mathbb{R} \). The exponential function is defined by

\[
\exp_G : \mathfrak{L}(G) \to G \quad \text{with} \quad \exp_G(\phi) := \phi_1(1).
\]

where \( \mathfrak{L}(G) \ni \phi : \mathbb{R} \to G \). Similarly, \( \log_G := \exp^{-1}_G \) is a well-defined function.

In particular, if \( G \) is an abelian topological group, then \( \mathfrak{L}(G) \) is a topological vector space with the uniform convergence topology on compact sets.

Let \( B \) be the group of units of some (unital) Banach algebra \( \mathfrak{B} \). Then \( B \) is a topological Lie group and there exists a homeomorphism \( \eta : \mathfrak{B} \to \mathfrak{L}(B) \) such that \( \phi_{\eta(b)}(1) = \exp_G(\eta(b)) =: \exp_B(b) \). Moreover, for any Lie subgroup \( G^B \subseteq B \), there is an induced homeomorphism \( \eta_{G^B} : \mathfrak{B} \to \mathfrak{L}(G^B) \). Consequently, the exponential function extends to the algebra level \( \exp_\mathfrak{B} : \mathfrak{B} \to B \) by \( b \mapsto \exp_\mathfrak{B}(tb) \), and it enjoys the standard properties if \( \mathfrak{B} \) is endowed with a Lie bracket.

**Definition 2.3** ([5] def. 5.1) Let \( B_1(1) \) be the unit ball about the identity element \( 1 \in \mathfrak{B} \). The exponential, \( \exp_\mathfrak{B} : \mathfrak{B} \to B \), and logarithm, \( \log_\mathfrak{B} : B_1(1) \to \mathfrak{B} \), are defined by

\[
\exp_\mathfrak{B}(b) := \sum_{n=0}^{\infty} \frac{1}{n!} b^n \quad \forall b \in \mathfrak{B},
\]

\[
\log_\mathfrak{B}(1 + b) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} b^n \quad \text{for} \| b \| < 1.
\]

The two functions are absolutely convergent for the indicated \( b \in \mathfrak{B} \), and they are analytic.

**Proposition 2.1** ([5] prop. 5.3)

Let \( N_0 \) be the connected component of the 0-neighborhood of \( \exp^{-1}_\mathfrak{B} B_1(1) \). Then

i) \( \log_\mathfrak{B}(\exp_\mathfrak{B} b) = b \quad \forall b \in N_0 \).

ii) \( \exp_\mathfrak{B}(\log_\mathfrak{B} b) = b \quad \forall b \in B_1(1) \).

iii) \( \exp_\mathfrak{B}|_{N_0} : N_0 \to B_1(1) \) is an analytic homeomorphism with analytic inverse \( \log_\mathfrak{B} : B_1(1) \to N_0 \). (3)

\[4\]Denote *continuous* group homomorphisms \( G \to H \) by \( \text{Hom}_C(G, H) \).
Definition 2.4  Let $\mathfrak{B}_L \subseteq (\mathfrak{B}, [\cdot, \cdot])$ be a closed Lie subalgebra of some Banach algebra $(\mathfrak{B}, [\cdot, \cdot])$ equipped with a Lie bracket. Let $B_1$ be a subgroup of the multiplicative group of units of $\mathfrak{B}_L$ such that $\exp_{\mathfrak{B}_L}$ is a homeomorphism mapping a neighborhood of $\{0\} \in \mathfrak{B}_L$ into a neighborhood of $\{e\} \in B_1$. A topological group is a linear Lie group if it is isomorphic to $B_1$.

Proposition 2.2  If $G$ is a linear Lie group, then the set $\mathcal{L}(G)$ is a completely normable topological real Lie algebra and $\exp^G$ is a homeomorphism from a $0$-neighborhood of $\mathcal{L}(G)$ to an $e$-neighborhood in $G$.

There are of course many other interesting and useful structures regarding topological group structures (e.g. for subgroups, quotients, product groups, etc.) that will only be recorded as needed. The most important for our application is the following well-known result:

Theorem 2.1  If $G$ is locally compact, then there exists a unique (up to positive scalar multiplication) Haar measure. If $G$ is compact, then it is unimodular. If $G$ is a locally compact linear Lie group then it is a dim-$\mathcal{L}(G)$ manifold.

Evidently, locally compact topological groups can be used as a footing on which to ground functional integration: They supply measure spaces on which to model functional integral domains and their associated integrators.

Accordingly, we require Banach-valued integration on locally compact topological groups:

Proposition 2.3  Let $G$ be a locally compact topological group, $\mu$ its associated Haar measure, and $\mathfrak{B}$ a Banach space possibly with an algebraic structure. Then the set of integrable functions $L^1(G, \mathfrak{B})$ of functions equal almost everywhere with norm $\|f\|_1 := \int_G \|f(g)\|d\mu(g) \leq \|f\|_\infty \mu(\text{supp } f) < \infty$, is a Banach space. Moreover, $f \mapsto \int_G f(g)d\mu(g)$ is a linear map such that

$$\|\int_G f(g)d\mu(g)\| \leq \|f\|_\infty \mu(\text{supp } f)$$

for all $f \in L^1(G, \mathfrak{B})$,

$$\varphi \left( \int_G f(g)d\mu(g) \right) = \int_G \varphi(f(g))d\mu(g)$$

for all $\varphi \in \mathfrak{B}'$, and

$$L_B \left( \int_G f(g)d\mu(g) \right) = \int_G L_B(f(g))d\mu(g)$$

for bounded linear maps $L_B : \mathfrak{B}_1 \to \mathfrak{B}_2$. Moreover, Fubini’s theorem holds for all equivalence classes $f \in L^1(G_1 \times G_2, \mathfrak{B})$.

Corollary 2.1  Let $\mathfrak{B}^*$ be a $C^*$-algebra, $\rho : \mathfrak{B}^* \to L_B(\mathcal{H})$ a representation with $L_B(\mathcal{H})$ the algebra of bounded linear operators on Hilbert space $\mathcal{H}$. Then

$$\left< \rho \left( \int_G f(g)d\mu(g) \right) v | w \right> = \int_G \left< \rho \left( f(g) \right) v | w \right> d\mu(g),$$

(7)
\[
\left( \int_G f(g)d\mu(g) \right)^* = \int_G f(g)^*d\mu(g),
\]

(8)

and

\[
a \int_G f(g)d\mu(g)b = \int_G af(g)bd\mu(g)
\]

(9)

where \( v, w \in \mathcal{H} \) and \( a, b \in M(\mathfrak{B}^*) \) with \( M(\mathfrak{B}^*) \) the multiplier algebra\(^5\) of \( \mathfrak{B}^* \).

It can be shown (\[6\], Appendix B) that \( L^1(G, \mathfrak{B}^*) \) is a Banach \(*\)-algebra when equipped with the \( \| \cdot \|_1 \) norm, the convolution

\[
f_1 * f_2(g) := \int_G f_1(h)f_2(h^{-1}g)d\mu(h),
\]

(10)

and the involution

\[
f^*(g) := f(g^{-1})^*\Delta(g^{-1})
\]

(11)

where \( \Delta \) is the modular function on \( G \).

3 Functional integration scheme

Experience with constrained functional integrals coupled with the properties of locally compact topological groups motivates our proposed scheme for functional integration.

3.1 The definition

Start with the data \((G, \mathfrak{B}, G_\Lambda)\) where \( G \) is a Hausdorff topological group, \( \mathfrak{B} \) is a Banach space that may have additional algebraic structure\(^6\), and \( G_\Lambda \) := \( \{G_\lambda, \lambda \in \Lambda\} \) is a family of locally compact topological groups indexed by continuous homomorphisms \( \lambda : G \to G_\lambda \).

The idea is to use the rigorous \( \mathfrak{B} \)-valued integration theory associated with \( \{G_\lambda, \lambda \in \Lambda\} \) to define and characterize functional integration on \( G \)\(^7\).

In physics applications (where the notion of dynamical evolution seems to require local compactness), the set \( \Lambda \) typically represents some kind of boundary\(^8\) conditions/constraints, invariance under symmetry, or homological/cohomological constructs. Such restrictions are conjectured here to lead to a localization in the topological sense of the previous section; i.e. restrictions characterize locally compact subgroups \( G_\lambda \hookrightarrow G \), usually through homomorphisms like \( G|_{\text{constraints}} \to G_\lambda \) or \( G/G_{\text{symm}} \to G_\lambda \)\(^9\).

\(^5\)Recall the multiplier algebra can be characterized as the set of adjointable linear operators \( L_*(\mathfrak{B}_*^*) \) on \( \mathfrak{B}_*^* \) viewed as a right Hilbert module over itself. If \( \mathfrak{B}_*^* \) is unital then \( M(\mathfrak{B}_*^*) = \mathfrak{B}_*^* \).

\(^6\)In this case is is convenient to view \( \mathfrak{B} \) as a Banach module over itself.

\(^7\)It is probably fruitful to consider \( G \) as a topological groupoid and \( \mathfrak{B} \) an algebroid, but this would add a layer of complexity that is better left as a separate investigation.

\(^8\)Boundary here includes initial and final ‘time’.

\(^9\)Since observation/measurment can be viewed as a kind of constraint, we can postulate that observation/measurement helps determine \( \Lambda \).
Definition 3.1 Let $\nu$ be a left Haar measure\footnote{The Haar measure $\nu$ does not necessarily have unit normalization. Recall that if $\nu$ and $\mu$ are left and right Haar measures respectively, then $\nu(G_\lambda) = \mu(G_\lambda^{-1})$ and $d\nu(g_\lambda) = \Delta(g_\lambda^{-1}) d\mu(g_\lambda^{-1})$.} on $G_\lambda$, and $L^1(G_\lambda, \mathcal{B})$ be the Banach space of $\mathcal{B}$-valued integrable functions with respect to $\nu$. Let $F : G \to \mathcal{B}$ and denote the set of integrable functionals by $\mathcal{F}(G) \ni F$. A family (indexed by $\lambda$) of integral operators $\int_\lambda : \mathcal{F}(G) \to \mathcal{B}$ is defined by\footnote{Compare to the idea expressed by equation (2.20) and related ideas in \cite{8}.}

$$\int_\lambda(F) = \int_G F(g) D_\lambda g := \int_{G_\lambda} f(g_\lambda) \, d\nu(g_\lambda)$$

(12)

where $F = f \circ \lambda$ with $f \in L^1(G_\lambda, \mathcal{B})$, and $\lambda$ runs over the set $\Lambda$. We say that $D_\lambda g$ represents an integrator family.

The functional $\ast$-convolution and $\ast$-convolution are defined by

$$(F_1 \ast F_2)_\lambda(g) := \int_G F_1(\tilde{g}) F_2(\tilde{g}^{-1} g) D_\lambda \tilde{g}$$

(13)

and

$$(F_1 \ast F_2)_\lambda(g) := \int_G F_1(\tilde{g} g) F_2(\tilde{g} \tilde{g}) D_\lambda \tilde{g}$$

(14)

for each $\lambda \in \Lambda$.

For any given $\lambda$, the integral operator is linear and bounded according to

$$\|\int_\lambda(F)\| = \int_{G_\lambda} \|f(g_\lambda)\| \, d\nu(g_\lambda) = \|f\|_1 < \infty .$$

(15)

This suggests to define the norm $\|F\| := \sup_\lambda \|F\|_\lambda$ where

$$\|F\|_\lambda := \int_G \|F(g)\| \, D_\lambda g := \int_{G_\lambda} \|f(g_\lambda)\| \, d\nu(g_\lambda) = \|f\|_1 < \infty .$$

(16)

The definition implies

$$\|F_1 \ast F_2\|_\lambda = \int_{G_\lambda} \int_{G_\lambda} \|f_1(\tilde{g}_\lambda) f_2(\tilde{g}_\lambda^{-1} g_\lambda)\| \, d\nu(\tilde{g}_\lambda, g_\lambda)$$

$$\leq \int_{G_\lambda} \int_{G_\lambda} \|f_1(\tilde{g}_\lambda) f_2(\tilde{g}_\lambda)\| \, d\nu(\tilde{g}_\lambda) d\nu(g_\lambda)$$

$$\leq \|F_1\|_\lambda \|F_2\|_\lambda$$

(17)

where the second line follows from left-invariance of the Haar measure and the last line follows from functional Fubini. Moreover, a similar computation (using left-invariance and Fubini) establishes $(F_1 \ast F_2) \ast F_3 = F_1 \ast (F_2 \ast F_3)$. Consequently, $\mathcal{F}(G)$ equipped with the $\ast$-convolution is a Banach algebra when completed w.r.t. the norm $\|F\| := \sup_\lambda \|F\|_\lambda$\footnote{There are delicate mathematical issues stemming from the fact that $f(g_\lambda)$ fails to be norm continuous in general. We refer to § 1.5.1, \cite{6} for a discussion.}

When $\mathcal{B}$ is a Banach $\ast$-algebra, $\mathcal{F}(G)$ and $\int_\lambda$ have important structures.
Proposition 3.1 If $\mathfrak{B} \equiv \mathfrak{B}^*$ a Banach $*$-algebra, then $\mathcal{F}(G)$ endowed with the involution $F^*(g) := F(g^{-1})^* \Delta(g^{-1})$ and complete w.r.t. the norm $\|F^*\| = \|F\|$ is a Banach $*$-algebra.

Proof: Linearity of the $*$-operation is obvious. Next,

$$
(F^*)^*(g) := F^*(g^{-1})^* \Delta(g^{-1}) = (F(g)^*)^* \Delta(g) \Delta(g^{-1}) = F(g)
$$

and

$$
(F_1^* * F_2^*)_\lambda (g) := \int_{G_\lambda} f_1^*(\tilde{g}_\lambda)f_2^*(\tilde{g}_\lambda^{-1}g_\lambda) \, d\nu(\tilde{g}_\lambda)
= \int_{G_\lambda} (f_2(g_\lambda^{-1}\tilde{g}_\lambda)\Delta(g_\lambda^{-1}\tilde{g}_\lambda)f_1(\tilde{g}_\lambda^{-1})\Delta(\tilde{g}_\lambda^{-1}))^* \, d\nu(\tilde{g}_\lambda)
= \left(\int_{G_\lambda} f_2(g_\lambda^{-1}\tilde{g}_\lambda)f_1(\tilde{g}_\lambda^{-1})\Delta(\tilde{g}_\lambda^{-1}) \, d\nu(\tilde{g}_\lambda)\right)^*
= ((F_2 * F_1)_\lambda(g^{-1}))^* \Delta(g^{-1})
= (F_2 * F_1)_\lambda^*(g)
$$

where we used the definition of involution, left-invariance of the Haar measure, and the fact that $\Delta$ is a homomorphism.

For the norm,

$$
\|F^*\|_\lambda := \int_G \|F^*(g)\| \, D_\lambda g
= \int_G \|F(g^{-1})^* \Delta(g^{-1})\| \, D_\lambda g
= \int_G \|F(g^{-1})^*\| \, \Delta(g^{-1}) \, D_\lambda g
= \int_G \|F(g)^*\| \, D_\lambda g
= \|F\|_\lambda
$$

where the fourth line follows by virtue of the Haar measure. Delicate issues regarding integrability do not arise because $f_1 * f_2$ is well-defined ([6] Appendix B).

Proposition 3.2 If $\mathfrak{B} \equiv \mathfrak{B}^*$ a Banach $*$-algebra, then $\text{int}_\lambda$ is a $*$-representation.

Proof:

$$
\text{int}_\lambda(F^*) = \int_G F^*(g) \, D_\lambda g := \int_{G_\lambda} f^*(g_\lambda) \, d\nu(g_\lambda)
= \int_{G_\lambda} f(g_\lambda^{-1})^* \Delta(g_\lambda^{-1}) \, d\nu(g_\lambda)
= \int_{G_\lambda} f(g_\lambda)^* \, d\nu(g_\lambda)
= \left(\int_{G_\lambda} f(g_\lambda) \, d\nu(g_\lambda)\right)^*
= \text{int}_\lambda(F)^*
$$
\[
\int_{\lambda} (F_1 \ast F_2)_{\lambda} := \int_{G_{\lambda}} \int_{G_{\lambda}} f_1(\tilde{g}_{\lambda}) f_2(\tilde{g}_{\lambda}^{-1} g_{\lambda}) \, d\nu(\tilde{g}_{\lambda}, g_{\lambda}) \\
= \int_{G_{\lambda}} \int_{G_{\lambda}} f_1(\tilde{g}_{\lambda}) f_2(g_{\lambda}) \, d\nu\tilde{g}_{\lambda}) d\nu(g_{\lambda}) \\
= \int_{\lambda} (F_1) \int_{\lambda} (F_2) \tag{22}
\]

where we used left-invariance of the Haar measure and Fubini. \(\square\)

**Corollary 3.1** If \(\mathfrak{B}^*\) is a C*-algebra, then \(\mathcal{F}(G)\) completed w.r.t. the norm \(\sup_\lambda \| F \|_\lambda\) is a C*-algebra.

It should be emphasized that a functional integral does not determine a unique element in \(\mathfrak{B}\) unless we have specified \(\lambda\). In a loose sense, specifying \(\lambda\) amounts to evaluating a functional integral through ‘localization’ in \(G\). Without specifying \(\lambda\), a functional integral represents a subset of \(\mathfrak{B}\), and so formal manipulations of \(\int_{\lambda} (F)\) amount to transformations on \(\mathfrak{B}\). This situation is reminiscent of solutions of differential equations: there exists a family of solutions, but a particular solution requires some boundary conditions or other restrictions. However, unlike particular solutions of differential equations, not all items of interest invoke a specific \(\lambda\). There can be objects (e.g. propagators in quantum mechanics) associated with a subset or even summation/integration over \(\Lambda\).

**Example 3.1** Cartier/DeWitt-Morette functional integration scheme \([1]\):
The CDM scheme for functional integration corresponds to the particular case of \(\mathfrak{B} \cong \mathbb{C}\) and \(G\) an abelian group. More precisely, \(G =: X_0\) is the abelian (additive) topological group of pointed maps \(x : [t_a, t_b] \subseteq \mathbb{R} \to \mathbb{C}^n\) such that \(x(t_a) = 0\). In this case, \(\mathcal{L}(X_0)\) is a topological vector space. Since \(X_0\) is abelian, the dual group \(\hat{X}_0 =: X'\) is compact abelian and unimodular with respect to its Haar measure \(\mu\). Hence, \(\mathcal{L}(X_0)'\) is a Banach space and \(L^1(X')\) is a Banach algebra under convolution.

The space \(\mathcal{F}(X_0)\) is the set of functionals defined by\(^1\)

\[
F_\mu(x) := \int_{X'} \Theta(x, x') \, d\mu(x') \tag{23}
\]

where \(\Theta(x, x') : X_0 \times X' \to \mathbb{C}\) is continuous, bounded and integrable. Then \(\mathcal{F}(X_0)\) is a Banach space with an induced norm defined as the total variation of \(\mu\). Bounded linear integral operators \(\int_X D_\lambda x\) with \(\int_X F_\mu D_\lambda x \leq \| F_\mu \|\) on \(\mathcal{F}(X_0)\) are defined by

\[
\int_{X_0} F_\mu(x) D_\lambda x := \int_{X'} \widehat{F}_\lambda(x') \, d\mu(x') \tag{24}
\]

where

\[
\int_{X_0} \Theta(x, x') D_\lambda x := \widehat{F}_\lambda(x') \tag{25}
\]

\(^1\)Since Haar measures can only differ by a scalar multiple, the \(\mu\) designation of \(F_\mu\) can be dropped if we agree to use the normalized Haar measure.
defines the integrator family $D_x$. Typically, $\lambda$ represents mean and covariances (relative to a Gaussian integrator family) associated with the elements in $X$. It is evident that a choice of $\lambda$ corresponds to a choice of Haar measure relative to $\Theta(x,x')$. Also, a simple affine transformation $x \mapsto x + x_a$ along with the invariance $D_\lambda(x + x_a) = D_\lambda x$ yields integration on $X$.

To handle spaces (which are not topological groups) of pointed maps $M_a$ where now $m : [t_a, t_b] \to \mathbb{U} \subseteq \mathbb{M}$ with $m(t_a) = m_a$ and $\mathbb{U} \subseteq \mathbb{M}$ an open neighborhood of an arbitrary Riemannian manifold, CDM uses the left-invariant vector field Lie algebra $\mathfrak{g}_a$ at $m_a$ to identify the non-abelian linear Lie group $\tilde{G}$ underlying $M_a$. In this case, the Lie algebra morphism $\Phi : \mathfrak{L}(X_0) \to \mathfrak{L}(\tilde{G})$ defines a morphism $\text{Exp} : \mathfrak{L}(X_0) \to M_a$ by $\text{Exp}(f) = (\exp_{\tilde{G}} \circ \Phi)(f)$. Given $\text{Exp}$ and the fact that $M_a$ is contractible since it is a pointed space, the parametrization $P : X_0 \to M_a$ by $x \mapsto \text{Exp}(\log_{X_0}(x))$ allows the integral on $M_a$ to be defined by

$$\int_{M_a} F(m)D_\lambda m := \int_{X_0} F_\mu(P(x))D_\lambda(P(x)) := \int_{X_0} F_\mu(P(x)) |\text{Det}_\lambda P'(x)| D_\lambda x \; .$$

The left-hand side furnishes the path integral route to quantum mechanics. Note that it has limited applicability if $\mathbb{M}$ is not geodesically complete.

On the other hand, if $\mathbb{M} = G$ happens to be a Lie group manifold, then (26) can be readily used since the Lie algebra is already available:

$$\int_{M_a} F(m)D_\lambda m := \int_{X_0} F_\mu(P(x))D_\lambda(P(x)) = \int_G f_\lambda(g) \, d\nu(g) \; .$$

In particular, this means that the free point-to-point propagator on a group manifold is ‘exact’ in the sense that it can be expressed as a sum over relevant $\lambda$ of finite dimensional integrals.

Alternatively, CDM uses the soldering form $\theta$ on the frame bundle $F(\mathbb{M})$ equipped with a connection to construct a parametrization $\text{Dev} : \tilde{M}_a \to M_a$ where $\tilde{M}_a$ is the abelian topological group of pointed maps $\tilde{M}_a$ such that $\tilde{m} : [t_a, t_b] \to \mathbb{U} \subseteq T_{m_a}(\mathbb{M})$ with $\tilde{m}(t_a) = 0$. The explicit construction of the development map uses the identification $\theta(\text{hor}(v_p)) = \tilde{z}$ where $\tilde{z} \in \mathbb{C}^n$ and $\text{hor}(v_p) \in T_p(F(\mathbb{M}))$ is tangent to the horizontal lift $\tilde{m}(T)$ of $m(T)$. Then $\text{Dev}(\tilde{m}) = (\pi \circ \tilde{m})$ where $\pi$ is the projection on the frame bundle and

$$\int_{M_a} F(m)D_\lambda m := \int_{M_a} F_\mu(\text{Dev}(\tilde{m}))D_\lambda(\text{Dev}(\tilde{m})) \; .$$

When $\mathbb{M} = G$ and the connection is Riemannian, $\text{Dev}$ and $\text{Exp}$ amount to the same thing.

It should be noted that [1] already suggested generalizing the space of paths in their scheme to include locally compact abelian groups. Also, the CDM scheme has been extended to the more general case of maps between arbitrary Riemannian manifolds [II]. Such maps can be included under the new scheme by suitable parametrizations, but they also suffer limited applicability.

### 3.2 Duality

But if locally compact groups are to supply the foundation for functional integration, it is their duality structure that renders functional integrals powerful computational tools by allowing them to be manipulated as functional transforms. Already at the abelian level there is a rich, intricate structure encompassing generalized Fourier transforms.
Theorem 3.1 ([6] ch. 7) Let \( \hat{G} = \text{Hom}_C(G, \mathbb{T}) \) denote the topological dual group of an abelian topological group \( G \) where \( \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \). If \( \hat{G} \) is endowed with the uniform convergence topology and \( G \) is locally compact; then \( \hat{G} \) is also locally compact, the map \( \eta : G \rightarrow \hat{G} \) is an isomorphism, and \( \eta \) and the evaluation map \( \varepsilon : \hat{G} \times G \rightarrow \mathbb{T} \) by \( (g', g) \mapsto \langle g', g \rangle \) are continuous. Moreover, \( G = E \oplus H \) where \( E \cong \mathbb{R}^n \) and \( H \) is a locally compact abelian subgroup that possesses a compact open subgroup, and \( \mathcal{L}(G) = \mathbb{R}^n \oplus \mathcal{L}(H) \).

This theorem illustrates the important role played by abelian topological groups. There are corresponding structures for non-abelian topological groups. But to access these dualities — and thereby enable generalized Fourier transforms — requires one more step.

Let \( G \) be locally compact and \( U : G \rightarrow \mathcal{U}(\mathcal{H}) \) be a unitary representation furnished by a Hilbert space \( \mathcal{H} \). For generalized Fourier, we need integrals of the type \( \int_G f(g)U(g)d\nu(g) \). Unfortunately, \( f(g)U(g) \) is not a continuous function in general. To fix the problem, it is enough to equip the multiplier algebra with the strict topology since then \( f(g)U(g) \) is continuous for \( f(g) \in M_s(\mathcal{B}^\ast) \) where \( M_s(\mathcal{B}^\ast) \) denotes \( M(\mathcal{B}^\ast) \) endowed with the strict topology ([6], §1.5).

Proposition 3.3 ([6] lemma 1.101) For \( f \in C_C(G, M_s(\mathcal{B}^\ast)) \) and \( \overline{\theta} : M(\mathcal{B}^\ast) \rightarrow L_B(\mathcal{H}) \), there exists a linear map \( f \mapsto \int_G f(g)d\nu(g) \) from \( C_C(G, M_s(\mathcal{B}^\ast)) \) to \( M(\mathcal{B}^\ast) \) such that

\[
\left\langle \overline{\theta} \left( \int_G f(g)d\nu(g) \right) v \right| w \right\rangle = \int_G \left\langle \overline{\theta} (f(g)) v \right| w \right\rangle d\nu(g),
\]

and

\[
\overline{\theta} \left( \int_G f(g)d\nu(g) \right) = \int_G \overline{\theta}(f(g)) d\nu(g)
\]

for \( \overline{\theta} : M(\mathcal{B}^\ast_1) \rightarrow M(\mathcal{B}^\ast_2) \) a non-degenerate homomorphism.

With the understanding that the target Banach algebra gets replaced by its multiplier algebra endowed with the strict topology, the integral of interest \( \int_G f(g)U(g)d\nu(g) \) has meaning and

Proposition 3.4 ([6] ch. 2.3)

\[
\overline{\theta} \rtimes U(f) := \int_G \overline{\theta}(f(g)) U(g)d\nu(g)
\]

defines a *-representation of \( C_C(G, \mathcal{B}^\ast) \) on \( \mathcal{H} \).

This proposition embodies the crossed-product approach to algebraic quantization [7].

The duality structure allows one to view a functional integral as an isomorphism between ‘dual’ topological groups (as opposed to defining a family of integrators). For example, consider the abelian case, and let \( g' \in \hat{G}_\lambda \) such that \( \langle g', g \rangle \in \mathbb{T} \). Then \( \int_{G_\lambda} \langle g', g \rangle d\nu(g) \) represents the Fourier transform between \( L^1(G_\lambda) \) and \( L^\infty(\hat{G}_\lambda) \), and the family indexed by \( \lambda \)

\[
\hat{F}_\lambda(g') := \int_G F(g)\langle g', g \rangle \mathcal{D}_\lambda g
\]

\(^{14}\)Proposition 2.3 and consequently Definition 3.1 were stated for \( f \in L^1(G, \mathcal{B}) \). In order to follow [6] precisely and thereby avoid introducing technical difficulties, we restrict here to \( f \in C_C(G, \mathcal{B}^\ast) \) with \( \mathcal{B}^\ast \) a \( C^\ast \)-algebra noting that \( C_C(G, \mathcal{B}^\ast) \) is dense in \( L^1(G, \mathcal{B}^\ast) \) when \( G \) is locally compact.
is the functional dual; i.e. the functional Fourier transform. The inverse functional Fourier transform would be given by the set
\[ F(g) = \int_{\mathcal{G}_\lambda} \hat{F}_\lambda(g') \langle g', g \rangle \, d\nu(g') \tag{33} \]
with a proper choice of \( \nu \) to maintain normalization. Note that this picture is essentially CDM slightly generalized.

A more symmetrical treatment is to define the inverse functional Fourier transform from \( L^1(\mathcal{G}_\lambda) \) to \( L^\infty(\mathcal{G}_\lambda) \);
\[ F_{\lambda', \lambda}(g) := \int_{\mathcal{G}} \hat{F}_\lambda(g') \langle g', g \rangle \mathcal{D}_{\lambda', \lambda} g' \tag{34} \]
where \( \lambda, \lambda' \) must be compatible in the sense that \( F_{\lambda', \lambda}(g) = F(g) \forall g \in \mathcal{G}_\lambda \).

The next level of abstraction is for \( \mathcal{G}_\lambda \) non-abelian but compact. Then Fourier provides the isomorphism \( C^*(\mathcal{G}_\lambda) \cong \bigoplus_{g' \in \mathcal{G}} M_{d_g} \), where \( M_{d_g} \) is the \( C^* \)-algebra of \( d_g \times d_g \) matrices with the isomorphism given by
\[ \hat{F}_\lambda(g') := \int_{\mathcal{G}} F(g) \pi_{g'}(g) \mathcal{D}_\lambda g \tag{35} \]
where \( \pi \) is a sub-representation of the left-regular representation \( \rho : \mathcal{G}_\lambda \to \mathcal{U}(L^2(\mathcal{G}_\lambda)) \) where \( \mathcal{U}(L^2(\mathcal{G}_\lambda)) := \{ U \in L_B(L^2(\mathcal{G}_\lambda)) \mid UU^* = \text{Id} \} \) with \( L_B(L^2(\mathcal{G}_\lambda)) \) the set of bounded linear operators on \( L^2(\mathcal{G}_\lambda) \). Finally, for \( \mathcal{G}_\lambda \) non-abelian and locally compact, the functional integral represents an isomorphism between \( C^*(\mathcal{G}_\lambda) \) and its primitive ideal space.

A point worth emphasizing is that the rich duality structure of locally compact groups provides powerful methods to manipulate functional integrals — as evidenced in the relevant (vast) literature. No doubt; a lot can be learned by studying the duality structure of the various integrator families that have been introduced, but we will move on to define functional Mellin transforms. We will learn that Mellin transforms offer a duality of their own — duality between certain Banach algebras.

## 4 Functional Mellin transform

The duality picture together with the close relationship between Fourier and Mellin transforms suggests introducing the functional analog of the Mellin transform. However, contrary to the finite dimensional case, the functional analog of the Mellin transform does not seem to be related to the functional Fourier transform: it is not clear how the change of variable in the finite dimensional case carries over into the duality structure. Instead, we propose to interpret the functional Mellin transform in terms of one-parameter subgroups of a topological group.

To define the functional Mellin transform, we restrict the functional integral data to the case of \( \mathfrak{B} = \mathfrak{C}^* \) a \( C^* \)-algebra and \( G = G_B \) a topological linear Lie group isomorphic to the group (or subgroup) of units \( B_l \) of some Banach *-algebra \( \mathfrak{B}^* \). (The subscript \( B \) reminds that \( G \) is a (sub)group of units and hence a topological linear Lie (sub)group. In order to utilize results from reference [5], we will analytically continue one-parameter subgroups of a topological linear Lie group that were outlined in [2].
4.1 The definition

Recall the definition of the exponential function on $G$: it associates an element $g^1 \in G$ with some $\phi_0 \in \mathcal{L}(G)$ by $\phi_0(1) = \exp_G(g) =: g^1$. Then, by the definition of one-parameter subgroups, $\phi_0(t) = \exp_G(tg) =: g^t$ with $t \in \mathbb{R}$, and the short-hand notation $g^t$ can formally be interpreted as the $t$-th power of $g$ in the sense that $g^t = \exp_G(t\log_G g^1)$.

Now let $\gamma_0 : \mathbb{R} \to \mathbb{C}$ by $t \mapsto z \in \mathbb{C}$ be a continuous injective homomorphism such that $\gamma_0(0) = 0$. Denote the set of one-parameter complex subgroups $\tilde{\phi}_g := \phi_g \circ \gamma_0^{-1}$ by $\mathcal{L}(G)^c := \text{Hom}_C(\mathbb{C}, G)$ such that $g^1 = \exp_G(g) = \tilde{\phi}_g(\gamma_0(1))$. Then $\tilde{\phi}_g(z) = \exp_G(zg) =: g^z$ is the complex analytic extension of a one-parameter subgroup of $G$ parametrized by the complex parameter $z$ and subject to the condition $\tilde{\phi}_g(\gamma_0(1)) = \exp_G(g)$.

We thus have a complex analytic exponential map of the complex group $G^c_B$. Formally interpret $g^z$ as a complex power. This is what is needed for functional Mellin: so the functional integral data is now $(G^c_B, \mathcal{C}^*, G^c_B, \lambda)$ in this entire section.

**Definition 4.1** Let $\mathcal{C}^*$ be a $C^*$-algebra, the map $\rho : G^c_B \to M(\mathcal{C}^*)$ be a continuous, injective homomorphism, and $\pi : \mathcal{C}^* \to L_B(\mathcal{H})$ be a non-degenerate $\ast$-homomorphism. Define continuous functionals $\mathcal{F}(G^c_B) \ni F : G^c_B \to \mathcal{C}^*$ equivariant under right-translations according to $F(gh) = F(g)\rho(h)$. Then the functional Mellin transform $\mathcal{M}_\lambda : \mathcal{F}(G^c_B) \to M(\mathcal{C}^*)$ is defined by

$$
\mathcal{M}_\lambda[F; \alpha] := \int_{G^c_B} F(gg^\alpha) \, d\lambda g = \int_{G^c_B} F(g)\rho(g^\alpha) \, d\lambda g 
$$

(36)

where $\alpha \in \mathcal{S} \subseteq \mathbb{C}$, $g^\alpha := \exp_G(\alpha \log_G g)$ and $\pi(F(g)\rho(g^\alpha)) \in L_B(\mathcal{H})$ where the space of bounded linear operators $L_B(\mathcal{H})$ is given the $\ast$-strong operator topology. Denote the space of Mellin integrable functionals by $\mathcal{F}_S(G^c_B)$.

Since we don’t have a definition of Mellin transform for generic locally compact topological groups, we have first defined the functional Mellin transform. Then, according to Definition 3.1 necessary conditions for the functional Mellin transform to exist are $f \in L^1(G^c_{B,\lambda}, \mathcal{C}^*)$ and $\|f(g^{1+\alpha})\|$ integrable precisely when $\alpha \in \mathcal{S}$. This will supply us with a Mellin transform that extends the usual definition to the case of Banach-valued integrals over locally compact topological groups:

**Definition 4.2** Let $f \in L^1(G^c_{B,\lambda}, \mathcal{C}^*)$ such that $\|f(g^{1+\alpha})\|$ is integrable for all $\alpha \in \mathcal{S} \subseteq \mathbb{C}$, and $\rho : G^c_{B,\lambda} \to M(\mathcal{C}^*)$ be a continuous injective homomorphism. Then $f$ is Mellin integrable

$$
\|\mathcal{M}[f(g\lambda); \alpha]\| := \int_{G^c_{B,\lambda}} \|f(g\lambda)\rho(g^\alpha_{\lambda})\| \, d\nu(g_{\lambda}) < \infty \quad \alpha \in \mathcal{S}.
$$

(37)

We say the Mellin transform $\tilde{f}(\alpha) := \mathcal{M}[f(g\lambda); \alpha]$ exists in the fundamental region $\mathcal{S}$.

---

15 This prescription is for left-invariant Haar measures. For right-invariant Haar measures impose equivariance under left-translations.

16 The class of functional Mellin transforms defined here includes the crossed products of [6] as a special case. To relate crossed products to functional Mellin transforms, require $\pi \circ \rho$ to be a strongly continuous unitary representation $U : G^c_{B,\lambda} \to L_B(\mathcal{H})$. Then $\pi(f(g\lambda h\lambda)) = \pi(f(g\lambda)\rho(h\lambda)) = \pi(f(g\lambda))U(h\lambda)$ and the crossed product $\pi \times U(f)$ is equivalent to $\pi(\mathcal{M}_\lambda[F; 1])$. 
Identifying the Lie algebra $\mathfrak{s}_B^C$ of $G_B^C$ with $\mathfrak{g}(G_B^C)$, the Mellin integral can be explicitly formulated as

$$
\int_{G_{B,\lambda}^C} f(g_\lambda) \rho(g_\lambda) \, d\nu(g_\lambda) = \int_{\mathfrak{g}(G_{B,\lambda}^C)} f(\exp_{G_{B,\lambda}^C}(g)) \rho(\exp_{G_{B,\lambda}^C}(\alpha g)) \left| \det d_0 \exp_{G_{B,\lambda}^C}(g) \right| \, dg .
$$

(38)

Roughly speaking, the functional Mellin transform is a family of integrals represented by the right-hand side of (38) which can be interpreted as a generalized Laplace transform.

Stating the definitions is easy: the hard work involves determining $S$ given $f$, $\rho$ and $\lambda$. To emphasize that the fundamental region depends on $\lambda$, we will sometimes write $S_\lambda$. For example, let $\mathfrak{e}^* = \mathbb{C}$ and $\lambda : G_B^C \to \mathbb{R}_+$ the strictly positive reals. Choose the standard normalization for the Haar measure $\nu(g_\lambda) = \log(g_\lambda).$ Then $M_\lambda[F;\alpha]$ reduces to the standard finite-dimensional Mellin transform for suitable $f$ (see appendix A):

$$
M_H[F;\alpha] = \int_0^\infty f(x) x^\alpha \, dx = \int_0^\infty f(x) x^{\alpha-1} \, dx \quad \alpha \in \langle a, b \rangle_H
$$

(39)

where the subscript $H$ indicates the normalized Haar measure.

**Example 4.1** For a less trivial example, consider the Mellin transform of the heat kernel of a free particle on $\mathbb{R}^n$. In this context, $\mathfrak{e}^* = \mathbb{R}$ and $f$ is the heat kernel $f(g) \equiv e^{-t(\bar{x}(g))}$ with the ‘effective action’ functional given by

$$
\Gamma(\bar{x}(g)) = \frac{\pi |x_{a'} - x_a|^2}{g} + n/2 \log g .
$$

(40)

Physically, $x$ represents a path $x : R_+ \to \mathbb{R}^n$ which dictates $\lambda : G_B^C \to \mathbb{R}_+$. With the choice of the usual Haar normalization, $M_H [E^{-\Gamma(\bar{x})}; 1]$ is the elementary kernel of the Laplacian $\Delta$ on $\mathbb{R}^n$ where $\bar{x}$ is an appropriate average path. Following standard arguments, we have $\bar{x}(t) = x_a + t(x_{a'} - x_a)/t_{a'}$ for some fixed $t_{a'} \in \mathbb{R}_+$.

Consequently, the elementary kernel of the Laplacian on $\mathbb{R}^n$ is given by

$$
K(x_a, x_{a'}) := M_H [E^{-\Gamma(\bar{x})}; 1] = \int_{\mathbb{R}_+} \exp \left\{ -\frac{\pi |x_{a'} - x_a|^2}{g} \right\} g^{-n/2} \, dg
$$

$$
= \left\{ \begin{array}{ll}
-2 \log |x_{a'} - x_a| & n = 2 \\
\pi^{1-n/2} \Gamma(n/2 - 1)|x_{a'} - x_a|^2 & n \neq 2
\end{array} \right..
$$

(41)

Define a norm on $F_S(G_B^C)$ by $\|F\| := sup_{\alpha}\|F\|_\alpha$ where

$$
\|F\|_\alpha := \sum_{\lambda} \|M[f(g_\lambda);\alpha]\| < \infty , \quad \alpha \in S .
$$

(42)

Assume that $F_S(G_B^C)$ can be completed w.r.t. this (or some other suitably defined) norm. Not surprisingly, $F_S(G_B^C)$ inherits an algebraic structure from $F(G)$.

**Proposition 4.1** If $F \in F_S(G_B^C)$, then

$$
M_H^\alpha [F;\alpha] := \pi(M_\lambda [F;\alpha]) = M[\pi(f(g_\lambda));\alpha]
$$

(43)
Proof: By definition, \( F \in \mathcal{F}_S(G_B^\infty) \) implies \( f \) is Mellin integrable for some \( S_\lambda \). So

\[
\pi(\mathcal{M}_\lambda [F; \alpha]) = \pi \left( \int_{G_B^\infty} F(gg^\alpha) D\lambda g \right)
= \pi \left( \int_{G_B^\infty_\lambda} f(g_\lambda g^\alpha) \, d\nu(g_\lambda) \right)
= \int_{G_B^\infty_\lambda} \pi \left( f(g_\lambda g^\alpha) \right) \, d\nu(g_\lambda)
\tag{44}
\]

and the third line follows from Proposition 2.3. □

Proposition 4.2 \( \mathcal{F}_S(G_B^\infty) \) is a Banach \( \ast \)-algebra with \( \|F^\ast\|_\alpha = \|F\|_\alpha \) when endowed with an involution defined by \( F^\ast(g^{1+\alpha}) := F(g^{-1-\alpha})^\ast \Delta(g^{-1}) \).

Proof: Linearity and \( (F^\ast)^\ast = F \) are obvious. Next,

\[
(F_1^\ast \ast F_2^\ast)_\lambda (g^{1+\alpha}) := \int_{G_\lambda} f_1^\lambda (\tilde{g}_\lambda) f_2^\lambda (\tilde{g}_\lambda^{-1} g_\lambda^{1+\alpha}) \, d\nu(\tilde{g}_\lambda)
= \int_{G_\lambda} (f_2(g_\lambda^{-1-\alpha} \tilde{g}_\lambda) \Delta(g_\lambda^{-1} \tilde{g}_\lambda^{-1}) f_1(\tilde{g}_\lambda^{-1}))^\ast \, d\nu(\tilde{g}_\lambda)
= \left( \int_{G_\lambda} f_2(g_\lambda^{-1-\alpha} \tilde{g}_\lambda) f_1(\tilde{g}_\lambda^{-1}) \, d\nu(\tilde{g}_\lambda) \right)^\ast \Delta(g_\lambda^{-1})
= ((F_2 \ast F_1)_\lambda (g^{-1-\alpha}))^\ast \Delta(g^{-1})
= (F_2 \ast F_1)_\lambda^\ast (g^{1+\alpha})
\tag{45}
\]

where we used left-invariance of the Haar measure to put \( \tilde{g}_\lambda \rightarrow g_\lambda^{1+\alpha} \tilde{g}_\lambda \) in the fourth line.

Finally,

\[
\|F^\ast\|_\alpha := \sum_\lambda \int_G \|F^\ast(g^{1+\alpha})\| \, D\lambda g
= \sum_\lambda \int_G \|\rho(g^{-1})^\ast F(g^{-1})^\ast \Delta(g^{-1})\| \, D\lambda g
= \sum_\lambda \int_G \|\rho(g^\alpha) F(g^1)\| \, D\lambda g
= \sum_\lambda \int_G \|F(g^{1+\alpha})\| \, D\lambda g
= \|F\|_\alpha .
\tag{46}
\]

□

More importantly, the functional Mellin transform inherits some useful properties from \( \text{int}_\lambda \) that follow from Theorem 2.3, Definition 3.1, and equivariance. First note that if \( \alpha = 0 \in \mathbb{S} \) then \( \mathcal{M}_\lambda \rightarrow \text{int}_\lambda \), so we will not consider this case any longer.
Proposition 4.3 If \( F \in \mathcal{F}_S(G_B^C) \), then
\[
\mathcal{M}_\lambda^*[F; \alpha] := (\mathcal{M}_\lambda[F; \alpha])^* = \mathcal{M}_\lambda[F^*; \alpha]
\]
(47)

Proof:
\[
\left( \int_{G_B^C} (F(gg^\alpha)) \mathcal{D}_\lambda g \right)^* = \int_{G_B^C} (f(g\lambda g^\alpha))^* \, d\nu(g_\lambda)
\]
\[
= \int_{G_B^C} \rho(g_\alpha^\alpha)^* (f(g_\lambda))^* \, d\nu(g_\lambda)
\]
\[
= \int_{G_B^C} \rho(g_\lambda^{-\alpha})^* f(g_\lambda^{-1})^* \Delta(g_\lambda^{-1}) \, d\nu(g_\lambda)
\]
\[
= \int_{G_B^C} F^*(gg^\alpha) \mathcal{D}_\lambda g
\]
(48)
where we used \( f^*(g\lambda g^\alpha) = \rho(g_\lambda^{-\alpha})^* f(g_\lambda^{-1})^* \Delta(g_\lambda^{-1}) = \rho(g_\lambda^{-\alpha})^* f^*(g_\lambda) \). \( \square \)

Corollary 4.1 \( \mathcal{M}_\lambda^*(\mathcal{F}_S(G_B^C)) \) is a \(*\)-algebra when complete w.r.t. the norm \( \|F\| := \sup_\alpha \|F\|_\alpha \).

Proof: This is a consequence of \( \mathcal{M}_\lambda \) a \(*\)-homomorphism and \( \mathcal{F}_S(G_B^C) \) a Banach \(*\)-algebra. \( \square \)

Under certain conditions, \( \mathcal{M}_\lambda \) is a \( \ast \)-representation:

Proposition 4.4 If \( \mathcal{C}^* \) is commutative, then
\[
\mathcal{M}_\lambda [(F_1 \ast F_2)_\lambda; \alpha] = \mathcal{M}_\lambda [F_1; \alpha] \mathcal{M}_\lambda [F_2; \alpha]
\]
(49)

and
\[
\mathcal{M}_\lambda [(F_1 \ast F_2)_\lambda; \alpha] = \mathcal{M}_\lambda [F_1; \alpha] \mathcal{M}_\lambda [F_2; 1 - \alpha] .
\]
(50)

Proof: For \( \mathcal{C}^* \) commutative,
\[
\rho((gh)^\alpha) = e^{\alpha \rho'(\log g + \log h)} = e^{\alpha \rho'(g) + \alpha \rho'(h)} = e^\alpha \rho'(g) e^{\alpha \rho'(h)} = \rho(g)^\alpha \rho(h^\alpha) .
\]
(51)

Now take the functional Mellin transform of the \(*\)-convolution and put \( g \to \tilde{g}g \). The functional integral is invariant under this transformation by virtue of the left-invariant Haar measure,
\[
\mathcal{M}_\lambda [(F_1 \ast F_2); \alpha] = \int_{G_B^C \times G_B^C} F_1(\tilde{g}) F_2(g\tilde{g}^\alpha) \mathcal{D}_\lambda \tilde{g} \mathcal{D}_\lambda g
\]
\[
= \int_{G_B^C \times G_B^C} f_1(\tilde{g}) \rho(\tilde{g}_\lambda^\alpha) f_2(g_\lambda) \rho(g_\lambda^\alpha) \, d\nu_\lambda(\tilde{g}_\lambda) \, d\nu_\lambda(g_\lambda)
\]
\[
= \int_{G_B^C \times G_B^C} F_1(\tilde{g}g^\alpha) F_2(gg^\alpha) \mathcal{D}_\lambda \tilde{g} \mathcal{D}_\lambda g
\]
\[
= \mathcal{M}_\lambda [F_1; \alpha] \mathcal{M}_\lambda [F_2; \alpha]
\]
(52)
where the last equality follows from functional Fubini which follows from Theorem 2.8 and Definition 3.1. The proof for the *-convolution follows similarly. □

**Corollary 4.2** If \( \mathcal{C}^* \) is commutative, then

\[
\mathcal{M}_\lambda \left[ |F|^2 ; \alpha \right] = |\mathcal{M}_\lambda [F; \alpha]|^2
\]

where \( |F|^2 := (F \ast F^*)_\lambda \) and \( |\mathcal{M}_\lambda [F; \alpha]|^2 = \mathcal{M}_\lambda [F; \alpha] \mathcal{M}_\lambda^* [F; \alpha] \).

Likewise, if \( \Re(\alpha) = 1/2 \) and \( \mathcal{C}^* \) is commutative, then

\[
\mathcal{M}_\lambda \left[ |F|^2 ; \alpha \right] = |\mathcal{M}_\lambda [F; \alpha]|^2
\]

where \( |F|^2 := (F \ast F^*)_\lambda \) and \( |\mathcal{M}_\lambda [F; \alpha]|^2 := \mathcal{M}_\lambda [F; \alpha] \mathcal{M}_\lambda^* [F; \alpha^*] \).

Evidently these characterize functional Plancherel-type relationships.

**Proposition 4.5** If \( \mathcal{C}^* \) is noncommutative, but \( G^C_B \) is abelian and \( \rho \) is real, then

\[
\mathcal{M}_\lambda \left[ (F_1 \ast F_2)_\lambda ; \alpha_3 \right] = \mathcal{M}_\lambda [F_1; \alpha_3] \mathcal{M}_\lambda [F_2; \alpha_3]
\]

and

\[
\mathcal{M}_\lambda \left[ (F_1 \ast F_2)_\lambda ; \alpha_3 \right] = \mathcal{M}_\lambda [F_1; \alpha_3] \mathcal{M}_\lambda [F_2; 1 - \alpha_3]
\]

where \( \alpha_3 = -\alpha_3^* \), i.e. \( \alpha_3 \in i\mathbb{R} \).

**Proof:** If \( G^C_{B,\lambda} \) is abelian,

\[
\rho((gh)\alpha) = \rho(e^{\alpha(\log g + \log h)}) = \rho(e^{\alpha \log g} e^{\alpha \log h}) = \rho(g^\alpha)\rho(h^\alpha) = \rho(h^\alpha)\rho(g^\alpha).
\]

Since \( \rho \) is assumed real, \( \rho'(g) = \rho'(\log g) \) is Hermitian and \( \rho(g^{-\alpha_3})^* = e^{-\alpha_3^*}\rho'(g)^* = \rho(g^{\alpha_3}) \).

Clearly, if instead \( \rho \) is assumed unitary then \( \alpha_3 \) gets replaced by \( \alpha_3 \in \mathbb{R} \).

Hence,

\[
\int_{G^C_B \times G^C_B} F_1(\tilde{g})(F_2^*)^*(g(\tilde{g}g)\alpha) D_\lambda \tilde{g} D_\lambda g
\]

\[
= \int_{G^C_{B,\lambda} \times G^C_{B,\lambda}} f_1(\tilde{g}_\lambda)\rho(\tilde{g}_\lambda^\alpha)\rho(g_\lambda^{-\alpha})^*(f_2^*)^*(g_\lambda) d\nu_\lambda(\tilde{g}_\lambda) d\nu_\lambda(g_\lambda)
\]

\[
= \int_{G^C_B \times G^C_B} F_1(\tilde{g}\bar{g}) \alpha) F_2(gg^\alpha) D_\lambda \tilde{g} D_\lambda g
\]

where the second line follows from \( \rho(g^{-\alpha_3})^* = \rho(g^{\alpha_3}) \). Again, the *-convolution result follows similarly. □

An easy observation; if \( \rho(g) \) is in the center of \( \mathcal{C}^* \) then there is no restriction on \( \alpha \in S \).

**Proposition 4.6** If \( \mathcal{C}^* \) is noncommutative and \( G^C_B \) is non-abelian, but \( \rho \) is unitary, then

\[
\mathcal{M}_\lambda \left[ (F_1 \ast F_2)_\lambda ; 1 \right] = \mathcal{M}_\lambda [F_1; 1] \mathcal{M}_\lambda [F_2; 1]
\]

and

\[
\mathcal{M}_\lambda \left[ (F_1 \ast F_2)_\lambda ; 1 \right] = \mathcal{M}_\lambda [F_1; 1] \mathcal{M}_\lambda [F_2; 0]
\]
Proof: Obviously $\rho(gh) = \rho(g)\rho(h)$; then follow the previous argument. \hfill \Box

It is convenient to denote by $R_\lambda^{(\alpha)}$ functional Mellin under the various conditions that render it a $*$-representation, and $R_\lambda^{(\alpha)}(G^C_B)$ the corresponding space of integrable functionals. For example, if $C^*$ is non-commutative, $R_\lambda^{(\alpha\beta)}$ denotes functional Mellin for an abelian group and unitary $\rho$, and $R_\lambda^{(\alpha)}(G^C_B)$ is a $C^*$-algebra when completed w.r.t. $\|F\| := \sup_\alpha \|F\|_\alpha$. On the other hand, if $C^*$ is commutative, $R_\lambda^{(\alpha)}(G^C_B)$ and $\mathcal{F}_\mathbb{R}(G^C_B)$ coincide.

These inherited properties of the functional Mellin transform (at least partially) explain the utility of the resulting integrals in probing the local structure of linear Lie groups $G^C_B$ and $*$-algebras $\mathcal{F}_\mathbb{R}(G^C_B)$. More importantly, (and this is the main theme of this paper) since it is not merely a restatement of Fourier transform it provides an independent tool to investigate these spaces. The following subsection is exemplary.

### 4.2 Mellin functional tools

Before extracting useful tools from the functional Mellin transform, it is a good idea to gain some experience and insight by analyzing its reduction to finite integrals under various conditions. Appendix B contains several examples. They suggest how to define Mellin functional counterparts of resolvents, traces, logarithms, and determinants.

From now on to clean up notation, no distinction will be made between $g$, $g_\lambda$, and $\rho(g_\lambda)$ when it will not cause confusion.

#### 4.2.1 Functional resolvent

Analysis of the functional exponential in Appendix B suggests a resolvent operator can be represented by a functional Mellin transform:

**Definition 4.3** Let $\beta' \in \text{Mor}_C(G^C_B, C^*)$ such that $E^{-\beta'} \in \mathcal{F}_\mathbb{R}(G^C_B)$. The functional resolvent of $\beta'$ is defined by

$$R_{\alpha,\beta'}(z) := M_\lambda \left[ E^{-\beta'-Id_\alpha} ; \alpha \right]. \quad (61)$$

where $z \in \mathbb{C}$.

For a quick example, let $\mathfrak{B}^* = C^* \equiv L_B(L^2(\mathbb{C}^n))$. Suppose $\beta'(g) = \beta g$ with $\beta \in C^*$ invertible positive-definite and $\lambda : G^C_B \to \phi_{\log\beta}(\mathbb{R})$ the real one-parameter subgroup generated by $\log\beta$. The functional resolvent reduces to

$$R_{\alpha,\beta'}(z) = \int_{\phi_{\log\beta}(\mathbb{R})} e^{-(\beta-zId)\alpha} g^\alpha \, d\nu(g_\alpha) = (\beta-zId)^{-\alpha}, \quad \alpha \in (0, \infty)_\mathbb{R}. \quad (62)$$

To see this, note that $\beta$ invertible implies $\beta \in G^C_{B,\lambda}$ and $\log\beta \in \mathfrak{G}_{B,\lambda}$. So $\beta-zId$ commutes with all $g \in \phi_{\log\beta}(\mathbb{R})$ and can be extracted from the integral. The remaining integral is normalized (see Appendix B).
This example is a significant simplification because $\beta'$ is linear on $G^C_B$ and $G^C_{B,\lambda}$ is a rather drastic reduction to a one-dimensional abelian subgroup. In the generic case, it is much harder to exhibit a closed form for $R_{\alpha,\beta'}(z)$ (if not impossible).

Specializing the functional resolvent affords a definition of the complex power of elements in $M(\mathcal{C}^*)$:

**Definition 4.4** The functional complex power of $\beta'$ is defined by

$$\beta'^{-\alpha}_\lambda := (\beta'^{-\alpha})_\lambda := R_{\alpha,\beta'}(0) = \mathcal{M}_\lambda \left[ E^{-\beta'}; \alpha \right].$$

(64)

Note that the $\lambda$ dependence of $\beta'^{-\alpha}_\lambda$ refers to both the normalization and the averaging group $G_\lambda$. Functional complex powers will play a central role in the next subsection.

**Definition 4.3** and the characterization of delta functionals in [2], suggest that, for $\alpha \in \mathbb{N}$, the functional complex power implicitly includes derivatives of Dirac delta functionals if $z = 0 \in \sigma(\beta'_\lambda)$. If on the other hand $\beta'_\lambda$ has no non-trivial kernel, then we can define a functional zeta:

**Definition 4.5** Suppose $\beta'^{-\alpha}_\lambda$ is trace class with $0 \notin \sigma(\beta'_\lambda)$. The functional zeta is defined by

$$\zeta_{\beta'^{-\alpha}}(\alpha) := \text{tr} \left( \beta'^{-\alpha}_\lambda \right) = \text{tr} \mathcal{M}_\lambda \left[ E^{-\beta'}; \alpha \right].$$

(65)

Of course this trace is not always a well-defined object for all $\alpha$ in the fundamental region of $\beta'^{-\alpha}_\lambda$. Rather, the fundamental region containing $\alpha$ is dictated by $\lambda$ and is often restricted. An effective strategy to quantify the restriction is to move the trace inside the integral. Presumably the trace of the integrand will be well-defined for certain choices of $\lambda$ and the properties of the functional Mellin transform will allow the domain of $\alpha$ to be determined. This strategy leads us to the next subsection.

### 4.2.2 Functional trace, logarithm, and determinant

**Definition 4.6** Let $\beta' \in \text{Mor}_C(G^C_B, \mathfrak{B}^*)$ with $E^{-\beta'} \in F_S(G^C_B)$ for some $\lambda$-dependent fundamental region $\alpha \in S_\lambda$. Require $\rho = \text{ch}$ to be a multiplicative character. Define the functional trace of $\beta'$ by

$$\text{Tr} \beta'^{-\alpha}_\lambda := (\text{Tr} \beta'^{-\alpha})_\lambda := \mathcal{M}_\lambda \left[ \text{tr} E^{-\beta'}; \alpha \right] := \int_{G^C_B} \text{tr} \left( e^{-\beta'(g)} \right) \text{ch}(g^\alpha) \mathcal{D}_\lambda g.$$

(66)

$^{17}$When $\beta'$ is linear on $G^C_B$ it is useful to generalize for any $\mathcal{C}^*$ the delta functional defined in [2] and to formally write (under appropriate conditions on $\beta'$)

$$R_{\alpha,\beta'}(z) = \text{p.v.} \frac{1}{(\beta' - z\text{Id})^\alpha_\lambda} - \pi i \delta^{(\alpha)}(\beta' - z\text{Id})_\lambda.$$

(63)

Accordingly, p.v.$(\beta' - z\text{Id})^\alpha_\lambda$ and $\delta(\beta' - z\text{Id})_\lambda$ correspond to the functional resolvent set and functional spectrum respectively. The appearance of distributions here motivates extending the theory of Mellin transforms of distributions (briefly outlined in the appendix) to the functional context, but this will be left to future work.

$^{18}$We stress that the functional form of $\beta'^{-\alpha}_\lambda$ will not resemble the functional form of $\beta'$ in general.
Remark that, as a consequence of Theorem 2.3, the interchange of the ordinary trace and functional integral is valid only for $E^{-\beta'} \in F_S(G_B^C)$ and appropriate $\lambda$. Then according to the definition,

$$
\text{tr} (\beta'^{-\alpha}) = \text{tr} \mathcal{M}_\lambda [E^{-\beta'}; \alpha] = \text{tr} \left( \int_{G_B^C} e^{-\beta'(g)} \rho(g^\alpha) \mathcal{D} \lambda g \right), \quad \forall \alpha \in S_{\lambda}
$$

$$
= \int_{G_B^C} \text{tr} (e^{-\beta'(g)}) \text{ch}(g^\alpha) \mathcal{D} \lambda g, \quad \forall \alpha \in \tilde{S}_{\lambda}
$$

$$
= \mathcal{M}_\lambda \left[ \text{Tr} E^{-\beta'}; \alpha \right], \quad \forall \alpha \in \tilde{S}_{\lambda}.
$$

(67)

Evidently, when $\rho$ is a character the functional trace and ordinary trace possess the same functional form. But the fundamental region of the functional trace depends on the chosen normalization, and taking the ordinary trace inside the integral often requires a restriction on the fundamental region of $\mathcal{M}_\lambda [\text{Tr} E^{-\beta'}; \alpha]$. The point is we can turn the calculation around and (with appropriate normalization/regularization) give meaning to the object $\mathcal{M}_\lambda [\text{Tr} E^{-\beta'}; \alpha]$ through the ordinary trace and an adjustment to $S_{\lambda}$.

In particular, the functional zeta can be represented as

$$
\zeta_{\beta'^{-\lambda}}(\alpha) \equiv \mathcal{M}_\lambda \left[ \text{Tr} E^{-\beta'}; \alpha \right] \ (68)
$$

but only for appropriate $\lambda$ and $\alpha$. For example,

**Example 4.2** Suppose $\mathcal{B}^*$ is unital with a separable representation Hilbert space $\mathcal{H}$. Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint with $\sigma(A) = N_+$, and let $\{|i\rangle, \varepsilon_i\}$ with $i \in \{1, \ldots, \infty\}$ denote the set of orthonormal eigenvectors and associated eigenvalues of $A$. Choose $\lambda : G_B^C \to \mathbb{R}_+$ and $\rho$ such that $g \mapsto g \cdot \text{Id}$. The Riemann zeta function associated with $A$ can be defined by

$$
\zeta_{A_{\Gamma}}(\alpha) = \text{tr} \int_{\mathbb{R}_+} e^{-Ag} \rho(g^\alpha) \, d\nu(g_{\Gamma})
$$

$$
= \sum_{i=1}^{\infty} \int_{\mathbb{R}_+} e^{-\varepsilon_i g + \alpha (\log g)} \langle i | i \rangle \, d\nu(g_{\Gamma})
$$

$$
= \int_{\mathbb{R}_+} \sum_{i=1}^{\infty} e^{-\varepsilon_i g + \alpha (\log g)} \, d\nu(g_{\Gamma}), \quad \alpha \in \langle 1, \infty \rangle_{\Gamma}
$$

$$
= \int_{0}^{\infty} \frac{1}{e^g - 1} g^\alpha \, d\nu(g_{\Gamma}), \quad \alpha \in \langle 1, \infty \rangle_{\Gamma}
$$

$$
= \mathcal{M}_{\Gamma} \left[ \text{Tr} E^{-A'}; \alpha \right], \quad \alpha \in \langle 1, \infty \rangle_{\Gamma}
$$

(69)

where $\nu(g_{\Gamma}) := \nu(g) / \Gamma(\alpha)$. Note that the integral in the first line is valid for $\alpha \in \langle 0, \infty \rangle$, so exchanging summation and integration here comes with the price of restricting the fundamental strip.

If instead $G_B^C \to \mathbb{C}$ a smooth contour in $\mathbb{C} \setminus \{0\}$, this has the well-known representation

$$
\zeta_{A_{\Gamma_c}}(\alpha) = \int_{\mathbb{C}} \frac{1}{e^g - 1} g^\alpha \, d\nu(g_{\Gamma_c}), \quad \alpha \in \langle 0, \infty \rangle \setminus \{1\}_{\Gamma_c}
$$

$$
= \mathcal{M}_{\Gamma_c} \left[ \text{Tr} E^{-A'}; \alpha \right], \quad \alpha \in \langle 0, \infty \rangle \setminus \{1\}_{\Gamma_c}
$$

(70)
where \( \nu(g_{\Gamma C}) := \frac{\pi \csc(\pi \alpha)}{2\pi i} \nu(g)/\Gamma(\alpha) \) and the contour starts at \(+\infty\) just above the real axis, passes around the origin counter-clockwise, and then continues back to \(+\infty\) just below the real axis.\(^{19}\) This is an explicit illustration of the fact that \(S\) depends on \(\lambda\) through both the normalization and the averaging group.

**Example 4.3** Again for \(A \in \mathcal{L}(\mathcal{H})\) with \(A = A^*\) and \(\sigma(A) = \mathbb{N}_+\), the Dirichlet eta function associated with \(A\) is given by

\[
\eta_A(\alpha) = \text{tr} \int_{\mathbb{R}^+} \left( -e^{-A(g+i\pi)} \rho(g^\alpha) \right) \, d\nu(g) \\
= -\int_{\mathbb{R}^+} \sum_{i=1}^{\infty} e^{-\varepsilon_i(g+i\pi)+\alpha(\log g)} \, d\nu(g), \quad \alpha \in (0, \infty)_{\Gamma} \\
= \int_0^\infty \frac{1}{e^g + 1} g^\alpha \, d\nu(g), \quad \alpha \in (0, \infty)_{\Gamma} \\
= \mathcal{M}_{\Gamma} \left[ \text{Tr} E^{-A'_n}; \alpha \right], \quad \alpha \in (0, \infty)_{\Gamma} \\
= \mathcal{M}_{\Gamma_n} \left[ \text{Tr} E^{-A'_n}; \alpha \right], \quad \alpha \in (0, \infty)_{\Gamma_n} \quad (71)
\]

where \(A'_n(g) = A \rho(g+i\pi)\) and \(\nu(g_{\Gamma_n}) := (1-2^{1-\alpha})\nu(g)/\Gamma(\alpha)\). Going from the first to second line uses the fact that \(\sum \int e^{-\varepsilon_i g^\alpha} \, d\nu(g)\) converges for \(\alpha \in (0, \infty)\).

Notice that the same functional \(\text{Tr} E^{-A'}\) corresponds to different objects in \(\mathcal{E}^*\) depending on the choice of \(\lambda\).

Continuing with this strategy, define the functional logarithm:

**Definition 4.7** Let \(\beta' \in \text{Mor}(G^C_{\mathcal{B}}, \mathcal{E}^*)\). Suppose that \(E^{-\beta'} := (E^{-\beta'} - E^{-1\beta'}) \in \mathcal{F}_\Sigma(G^C_{\mathcal{B}})\) for some fundamental region that contains the point \(\{0\}\). The functional logarithm of \(\beta'\) is defined by

\[
\log \beta'^{-1}_\lambda := (\log \beta'^{-1})_\lambda := \mathcal{M}_\lambda \left[ E^{-\log \beta'}; 0 \right] := \int_{G^C_{\mathcal{B}}} e^{-\beta'(g)} \log(g) \, \mathcal{D}_\lambda g \\
= \frac{d}{d\alpha} \mathcal{M}_\lambda \left[ E^{-\beta'}; \alpha \right] \bigg|_{\alpha = 0} \\
= \frac{d}{d\alpha} \beta'^{-\alpha} \bigg|_{\alpha = 0} \\
= \int_{G^C_{\mathcal{B}}} \phi^{-\beta'}(g) \, \mathcal{D}_\lambda g \\
=: \mathcal{M}_\lambda \left[ E^{-\beta'}; 0 \right]. \quad (72)
\]

Notice that \((\log (E^{-\beta'})^{-1})_\lambda = (\log (E^{\beta'}))_\lambda = \beta'_\lambda\) for appropriate \(\lambda\).

\(^{19}\)Of course \(\pi \csc(\pi \alpha)/\Gamma(\alpha)\) can be analytically continued to the left-half plane thus obtaining a meromorphic representation of Riemann zeta.
Example 4.4 In particular let $\beta'(g) = \beta g$ with $\beta$ positive-definite invertible, and suppose $G_B^C \to \mathbb{R}_+$ and $\mathfrak{c}^* = \mathbb{C}$. Choose the Haar normalization to get

$$
\log \beta'^{-1}_H = \int_{\mathbb{R}_+} e^{-\beta g} \, d\nu(g_H) = - \log \beta = \log \mathcal{M}_H \left[ E^{-\beta'}; 1 \right].
$$

(73)

Functional Log can be extended to other $\alpha \in \mathbb{C}_+$ by choosing $\nu(g_{\Gamma_\alpha}) := \Gamma(\alpha+1)\nu(g_H)/\Gamma(\alpha)$ to get

$$
\log \beta'^{-\alpha}_H := \log \beta'^{-1}_{\Gamma_\alpha} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} \int_{\mathbb{R}_+} e^{-\beta'(g)} \, d\nu(g_H), \quad \alpha \in (0, \infty).$

(74)

This yields the expected behavior for exponents when $\beta'(g) = \beta g$. However, functional Log does not seem as useful as the functional trace and determinant, because its complex extension is simply determined by a normalization.

The final component of our triad is the functional determinant of a complex power which will be defined in terms of the functional trace:

Definition 4.8 Let $\beta' \in \text{Mor}(G_B^C, \mathfrak{b}^*)$ with $\beta'(g)$ trace class and $0 \notin \sigma(\beta'_\lambda)$. Suppose $E^{-\text{Tr} \beta'} \in \mathcal{F}_S(G_B^C)$ for $\alpha \in \mathcal{S}_\lambda$. Define the functional determinant of $\beta'$ by

$$
\det(\beta'^{-\alpha})_{\lambda} := (\det(\beta'^{-\alpha}))_{\lambda} := \mathcal{M}_\lambda \left[ \det E^{-\beta'}; \alpha \right] := \int_{G_B^C} \det \left( e^{-\beta'(g)} g^\alpha \right) \mathcal{D}_\lambda g
$$

$$
= \int_{G_B^C} e^{-\text{tr}(\beta'(g))} \det(g^\alpha) \mathcal{D}_\lambda g
$$

$$
=: \mathcal{M}_\lambda \left[ E^{-\text{Tr} \beta'}; \alpha \right]
$$

(75)

where $\rho : G_B^C \to \mathbb{C}$ by $g \mapsto \det(g^\alpha) := e^{\alpha \text{tr}(g)}$ and the trace is with respect to $\mathfrak{b}^*$.

As with the functional trace and logarithm, the determinant essentially can be ‘taken outside the integral’ only for special choices of $\lambda$ for which the trace in the integrand is well-defined given a particular representation. It is important to realize that in general the definition implies:

Proposition 4.7 If $\beta'_1(g) \neq \beta'_2(g)$ are trace class, then

$$
\det(\beta'_1 \ast \beta'_2)^{-\alpha}_{\lambda} \neq \det(\beta'_1)^{-\alpha}_{\lambda} \det(\beta'_2)^{-\alpha}_{\lambda}.
$$

(76)

Proof: Let $\beta'(g) := (\beta'_1 \ast \beta'_2)(g)$ be trace class. Then

$$
\det(\beta'_1 \ast \beta'_2)^{-\alpha}_{\lambda} = \mathcal{M}_\lambda \left[ \left( \det E^{-\beta'_1} \ast \det E^{-\beta'_2} \right); \alpha \right]
$$

$$
= \mathcal{M}_\lambda \left[ \left( E^{-\text{Tr} \beta'_1} \ast E^{-\text{Tr} \beta'_2} \right); \alpha \right]
$$

$$
\neq \mathcal{M}_\lambda \left[ E^{-\text{Tr} \beta'}; \alpha \right] = \det(\beta'_1 \ast \beta'_2)^{-\alpha}_{\lambda}.
$$

(77)
in general.

On the other hand, since $\mathcal{C}^*$ is commutative, putting $\beta_2' = \beta_1'$ implies $\text{Det}(\beta' * \beta')_\lambda = \text{Det}\beta'_{\lambda} \text{Det}\beta^{-\alpha}_\lambda$ using Proposition 4.4, still don’t have $\text{Det}\beta'_{\lambda} = (\text{Det}\beta^{-1}_\lambda)^{\alpha}$ in general. □

However, there are some special cases for which the functional determinant will have the multiplicative property with respect to $\alpha$:

**Proposition 4.8** Suppose $\beta'(g) = \beta g$ with $\beta \in G_{B,\lambda}^C$ and $\beta g$ trace class. Then,

$$\text{Det}\beta^{-\alpha}_\lambda = \mathcal{N}_\lambda(\alpha) (\text{det}^{-1})^{\alpha} e^{i\varphi(\alpha)}, \quad \alpha \in \mathbb{S}_\lambda$$  \quad (78)

where $\mathcal{N}_\lambda(\alpha)$ is a $\lambda$-dependent normalization.

**Proof**: Using the invariance of the Haar measure with $g \rightarrow \beta^{-1}g$ yields

$$\text{Det}\beta^{-\alpha}_\lambda = \mathcal{M}_\lambda \left[ \text{Det}E^{-\beta'}; \alpha \right] = \int_{G_{B,\lambda}^C} e^{-\text{tr}(g)} \text{det}((\beta^{-1}g)\alpha) \, d\nu(g\lambda)$$

$$= \int_{G_{B,\lambda}^C} e^{-\text{tr}g} (\text{det}(\beta^{-1}g))^{\alpha} e^{i\varphi(\alpha)} \, d\nu(g\lambda)$$

$$= (\text{det}^{-1})^{\alpha} e^{i\varphi(\alpha)} \int_{G_{B,\lambda}^C} e^{-\text{tr}g} \, d\nu(g\lambda)$$

$$=: (\text{det}^{-1})^{\alpha} e^{i\varphi(\alpha)} \mathcal{N}_\lambda(\alpha), \quad \alpha \in \mathbb{S}_\lambda$$  \quad (79)

where $\varphi(\alpha) = \alpha(\arg\beta^{-1} + 2n\pi)$. □

The normalization $\mathcal{N}_\lambda(\alpha) := \text{Det}Id^{-\alpha}_\lambda$ requires scrutiny. The definition of functional determinant assumes $\beta'(g)$ is trace class. However, $Id'(g) = g$ will not be trace class for generic $G_{B,\lambda}^C$. If it is not, then we must regulate $\mathcal{N}_\lambda(\alpha)$ with a positive-definite invertible fixed element $R \in G_{B,\lambda}^C$ such that $Rg$ is trace class and $e^{-\text{tr}(Rg)} \in \mathcal{F}_{\gamma}(G_{B}^C)$. Let $\mathcal{H}_B$ furnish a representation $\pi$ of $\mathcal{B}^*$. Pick a basis in $\mathcal{H}_B$ for which $R$ is diagonal. Then

$$\mathcal{N}_\lambda(R; \alpha) := \int_{G_{B,\lambda}^C} e^{-\text{tr}(Rg)} \, d\pi g \, d\nu(g\lambda) = \prod_{i=1}^{d} \mathcal{R}_i^{\alpha}, \quad \alpha \in \mathbb{S}_\lambda. \quad (80)$$

where $\dim(\mathcal{H}_B) = d$. Even if $d = \infty$, this product can be rendered finite and well-defined if a suitable regulator $R$ exists.

On the other hand, we can simply choose the normalization/regularization associated with the choice of $\lambda$ to set $\mathcal{N}_\lambda(R; \alpha) = 1$ — which amounts to formally dividing out this factor from the functional determinant (similarly to what is done with $\Gamma(\alpha)$). The corresponding
regularized functional determinant of an operator $\pi(O)$ can then be defined by

$$\overline{\text{Det}}_\pi O^\prime_{\lambda}^{-\alpha} := \frac{\text{Det}_\pi O_{\lambda,R}^{-\alpha}}{N_{\lambda}(R; \alpha)} = \frac{1}{N_{\lambda}(R; \alpha)} \int_{G_{B,\lambda}^C} e^{-\text{tr}_{S}(ROR^{-1})g} \det_{\pi} g^{\alpha} d\nu(g_{\lambda})$$

$$= \frac{1}{N_{\lambda}(R; \alpha)} \left( \frac{\det_{\pi} O^{-\alpha}}{\det_{\pi} R^{-\alpha}} \right) \int_{G_{B,\lambda}^C} e^{-\text{tr}_{S}(Rg)} \det_{\pi} g^{\alpha} d\nu(g_{\lambda})$$

$$= e^{i\alpha \nu_{O,R}(\alpha)} \left( \frac{\det_{\pi} O^{-1}}{\det_{\pi} R^{-1}} \right)^{\alpha}$$

$$=: \overline{\det}_\pi(O^{-\alpha}) \quad (81)$$

In essence, the functional determinant is formally regularized by dividing out the possibly infinite factor $N_{\lambda}(\alpha)$. This is common practice: One knows how $\pi(R)$ acts on $\mathcal{H}_B$ and then defines the regularized determinant of $\pi(\beta)$ by $\overline{\text{Det}}_{\pi} \beta^{-1}_\Gamma := \text{Det}_{\pi} \beta^{-1}_\Gamma / \text{Det}_{\pi} R^{-1}_\Gamma = \det_{\pi}(\beta) / \det_{\pi}(R)$. So in particular, if $\det_{\pi}(R) = 1$ and $\arg \beta^{-1} = 0 \mod 2\pi$, then

$$\overline{\text{Det}}_{\pi} \beta^{-1}_\lambda \overline{\text{Det}}_{\pi} \beta^{-1}_\lambda = \overline{\text{Det}}_{\pi} \beta^{-2}_\lambda = (\det_{\pi}\beta)^{-2} \quad (82)$$

and the regularized determinant enjoys the usual multiplicative property — in this case at least.

### 4.3 Commuting Mellin and the exponential

The fundamental relationship between exponentials, determinants, and traces in finite dimensions, i.e. $\exp \tr M = \det \exp M$, also characterizes the functional analogs. In order to relate functional trace and determinant, consider $\beta' = \log F'$. Then formally, for appropriate $F'$ and choice of $\lambda$,

$$e^{\tr \log F'^{-1}} = e^{\int_{G_{B}^C} \tr e^{-F'(g)} \mathcal{D}g} \sim \int_{G_{B}^C} e^{-\tr F'(g)} \det g \mathcal{D}g = \text{Det}_{\lambda} (F'^{-1}) \quad (83)$$

This important relation signifies a deep connection between Poisson processes and functional Mellin transforms/gamma integrators (as indicated for example in [2]). It is a particular case of the following theorem:

**Theorem 4.1** Let $\beta' \in \text{Mor}(G_{B}^C, \mathfrak{B}^*)$ by $g \mapsto \beta g$ with $0 \notin \sigma(\beta)$ and $F' \in \text{Mor}(G_{B}^C, \mathfrak{B}^*)$ by $g \mapsto e^{-\beta} g$ with $e^{-\beta} \in G_{B}^C$ 20. Assume that $\tr E^{-\beta'} \in F_{S}(G_{B}^C)$ and $E^{-\tr F'} \in F_{S}(G_{B}^C)$ for a common domain $\alpha \in S_{\lambda}$. Then

$$e^{-\tr \beta'^{-1}} = \det(E^{-\beta'})^{-\alpha}, \quad \alpha \in S_{\lambda} \quad (84)$$

20 Notice that if $F' = E^{-\beta'}$ then $(\log F')(g) = (\log E^{-\beta'})(g) = -\beta'(g) = -\beta g = -\log(e^\beta)g$. 


Proof: First, recall that an immediate consequence of the definitions and the relationship between \((\exp \tr)\) and \((\det \exp)\) is

\[
\mathcal{M}_\lambda \left[ \Det E^{-F'}; \alpha \right] = \int_{G_B^c} \det \left( e^{-F'(g)} g^\alpha \right) \mathcal{D}_\lambda g
\]

\[
= \int_{G_B^c} e^{-\tr F'(g) \det g^\alpha} \mathcal{D}_\lambda g
\]

\[
= \mathcal{M}_\lambda \left[ E^{-\tr F'}; \alpha \right]
\]

where the second line follows as soon as \(F'(g)\) is trace class.

Lemma 4.1 Let \(\tr F' \in \text{Mor}(G_B^c, \mathbb{C})\). Suppose the functional Mellin transforms of \(\tr F'\) and \(E^{-\tr F'}\) exist for common \(\alpha \in S_\lambda\) for a given \(\lambda\). Then

\[
\mathcal{M}_\lambda \left[ E^{-\tr F'}; \alpha \right] = e^{-\mathcal{M}_\lambda[\tr F'; \alpha]}, \quad \alpha \in S_\lambda.
\]

Proof: Since the Mellin transform of \(\tr F'\) exists by assumption, then \(e^{-\mathcal{M}_\lambda[\tr F'; \alpha]}\) represents an absolutely convergent series for \(\alpha \in S_\lambda\). Hence,

\[
e^{-\mathcal{M}_\lambda[\tr F'; \alpha]} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{M}_\lambda[\tr F'; \alpha]^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{M}_\lambda[(\tr F')^n; \alpha]
\]

\[
= \mathcal{M}_\lambda \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\tr F')^n; \alpha \right]
\]

\[
= \mathcal{M}_\lambda \left[ E^{-\tr F'}; \alpha \right]
\]

where moving the power of \(n\) into the functional Mellin transform in the first line follows from induction on Proposition 4.4 because the multiplication represented by \((\tr F')^n\) is the *-convolution and \(\tr F'(g) \in \mathbb{C}\), i.e. \(\mathcal{C}^*\) is commutative. Equality between the first and second lines follows from the absolute convergence of \(e^{-\mathcal{M}_\lambda[\tr F'; \alpha]}\), the existence of the integral \(\mathcal{M}_\lambda \left[ E^{-\tr F'}; \alpha \right]\) for the common domain \(\alpha \in S_\lambda\), and the fact that \(e^{-\tr F'(g)}\) is analytic.

It should be stressed that the equality in the lemma holds only for \(\alpha \in S_\lambda\) properly restricted. \(\square\)

Corollary 4.3 Under the conditions of Lemma 4.1, replace \(\tr F' \in \text{Mor}(G_B^c, \mathbb{C})\) with \(V' \in \text{Mor}(G_B^c, \mathcal{C}^*)\) where now \(G_B^c\) is abelian and \(V'(g)\) is self-adjoint, then the lemma together with Proposition 4.3 imply

\[
\mathcal{M}_\lambda \left[ E^{-V'}; \alpha \right] = e^{-\mathcal{M}_\lambda[V'; \alpha]}, \quad \alpha \in i\mathbb{R}.
\]

To finish the proof, put \(F' \equiv E^{-\beta'}\) in the lemma and recall that \(F'(g) = E^{-\beta'}(g) = e^{-\beta}g\) and \(-\beta(g) = (\text{Log}(E^{-\beta'}))(g) = -\log(e^{\beta})g = -\beta g\) so

\[
\mathcal{M}_\lambda \left[ E^{-\tr E^{-\beta'}}; \alpha \right] = \int_{G_B^c} e^{-\tr(e^{-\beta}g)} \det g^\alpha \mathcal{D}_\lambda g = \Det(E^{-\beta'})^{-\alpha}_\lambda
\]
and
\[
\mathcal{M}_\lambda \left[ \text{Tr} \, E^{-\beta} ; \alpha \right] = \int_{G_B^C} \text{tr} \ (e^{-\beta g}) \, \text{ch} \, g^\alpha \, D \chi g = \text{Tr} \, E^{-\beta} \chi^{-\alpha} .
\]  
(90)

Hence
\[
\text{Det}(E^{-\beta})^{-\alpha}_\lambda = e^{-\text{Tr} \, \beta \chi^{-\alpha}} .
\]  
(91)

\[\square\]

If \( E^{-\beta} \) is self-adjoint and \( G_B^C \) abelian, then the corollary implies
\[
(E^{-\beta})^{-\alpha}_\lambda = \mathcal{M}_\lambda \left[ E^{-E^{-\beta}} ; \alpha \right] = e^{-\mathcal{M}_\lambda [E^{-\beta} ; \alpha]} = e^{-\beta \chi^{-\alpha}}
\]  
(92)

which leads to the remarkable property
\[
e^{-\text{Tr} \, \beta \chi^{-\alpha}} = \text{Det}(E^{-\beta})^{-\alpha}_\lambda = \text{Det} \left[ e^{-\beta \chi^{-\alpha}} \right] , \quad \alpha \in i\mathbb{R} .
\]  
(93)

It should perhaps be emphasized that, for appropriate choice of \( \lambda \), Theorem 4.1 and the above property are derived from well-defined Haar integrals, and they should be interpreted in the spirit of Definition 3.1: That is, they are a family of statements at the functional level that \emph{may} be explicitly realized for appropriate choices of \( \lambda \).

5 Conclusion

A scheme for functional integration has been proposed based on locally compact topological groups and the Haar measures associated with them. The key concept is the identification of a functional integral with a \emph{family} of rigorously defined Haar integrals: A particular realization within the family comprises a restriction of a topological group to a locally compact topological group which often includes a choice of regularization. The proposed definition supplies a conceptual and mathematical framework on which to construct and understand functional Mellin transforms — in much the same way as functional Fourier transforms have long been understood.

Functional Mellin transforms can be used to define functional traces, determinants, and logarithms. The resulting functional integrals yield powerful tools and techniques for analyzing \( C^\ast \)-algebras. More generally and usually under restrictive conditions, Mellin can transfer well-defined functions between Banach algebras and even maintain their algebraic structures. Several important aspects of ‘Mellin transfer’ were considered: The most notable being the functional analog of \( \exp \text{ tr} \, A = \text{det} \exp A \) where the functional setting allows the analog to be extended to a functional relation defined on an appropriate domain in the complex plane.

From a more general perspective, the functional Mellin transform is complementary to the functional Fourier transform. Whereas the Fourier transform relates dual topological groups, Mellin relates ‘dual algebras’. This is particularly useful when one of the algebras is comprised of functionals \( \mathcal{F}(G) = C_C(G, \mathcal{C}^\ast) \). In a loose sense, functional Mellin encodes a correspondence between the algebra of Mellin functionals \( \mathcal{F}_B(G_B^C) \) and a family of multiplier algebras \( M(\mathcal{C}^\ast) \) (under suitable restrictive conditions). The correspondence is a \( \ast \)-representation for generic \( G \) and \( \mathcal{C}^\ast \) only when \( \alpha = 1 \); in which case functional Mellin boils down to crossed products. However, there are mathematically and physically interesting
objects associated with abelian $G$ and/or commutative $\mathcal{E}^*$ where the restriction on $\alpha$ can be loosened or even removed altogether. This suggests a functional Mellin approach to algebraic quantization would be fruitful.

There are several directions open for extending this formalism both from a physics and a mathematics perspective. A few specific ideas have already been indicated in passing, and the reader will likely be able to see many more.

## A Mellin transforms

### A.1 Basics

Analysis of the Mellin transform can be found in many references. Most of the following, which includes some non-standard aspects, can be found in [3] and [4].

**Definition A.1** Let $f : (0, \infty) \to \mathbb{C}$ be a function such that $f \in L^1(\mathbb{R}_+)$ with limits

\[ \lim_{x \to 0^+} f(x) = \mathcal{O}(x^{-a}) \quad \text{and} \quad \lim_{x \to \infty} f(x) = \mathcal{O}(x^{-b}) \]

for $a, b \in \mathbb{R}$. Then the Mellin transform $\tilde{f}(\alpha)$ with $\alpha \in (a, b) := (a, b) \times i\mathbb{R} \subset \mathbb{C}$ is defined by

\[
\tilde{f}(\alpha) := \mathcal{M}[f(x); \alpha] := \int_0^{\infty} f(x)x^{\alpha-1}dx.
\]  

(94)

The fundamental strip $(a, b) \subset \mathbb{C}$ indicates the domain of convergence. Since, by definition,

\[ f(x)|_{x \to 0^+} = \mathcal{O}(x^{-a}) \quad \text{and} \quad f(x)|_{x \to \infty} = \mathcal{O}(x^{-b}) \]

(95)

then $\tilde{f}(\alpha)$ exists in $(a, b)$ where it is holomorphic and absolutely convergent. More precisely,

\[
|\tilde{f}(\alpha)| \leq \int_0^1 |f(x)|x^{\Re \alpha-1} \, dx + \int_1^{\infty} |f(x)|x^{\Re \alpha-1} \, dx \\
\leq M_1 \int_0^1 x^{\Re \alpha-1-a} \, dx + M_2 \int_1^{\infty} x^{\Re \alpha-1-b} \, dx
\]

(96)

for some finite constants $M_1, M_2$.

From the definition follows some important properties (for suitable fundamental strips);

\[
c^{-\alpha} \tilde{f}(\alpha) = \mathcal{M}[f(cx); \alpha] \quad c > 0
\]

\[
\tilde{f}(\alpha + d) = \mathcal{M}[x^df(x); \alpha] \quad d > 0
\]

\[
\frac{1}{|r|} \tilde{f}\left(\frac{\alpha}{r}\right) = \mathcal{M}[f(x^r); \alpha] \quad r \in \mathbb{R} - \{0\}, \alpha \in (ra, rb)
\]

\[
\frac{d^n}{d\alpha^n} \tilde{f}(\alpha) = \mathcal{M}[(\log x)^nf(x); \alpha] \quad n \in \mathbb{N}
\]

\[
-\alpha \tilde{f}(\alpha) = \mathcal{M}\left[\left(\frac{d}{dx}\right)f(x); \alpha\right]
\]

\[
-\tilde{f}(\alpha - 1) = \mathcal{M}\left[\frac{d}{dx}f(x); \alpha\right]
\]

\[
-\frac{1}{\alpha} \tilde{f}(\alpha + 1) = \mathcal{M}\left[\int_0^xf(x') \, dx'; \alpha\right]
\]

(97)
The last three can be extended by iteration:

\[ (-1)^n \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \tilde{f}(\alpha) = \mathcal{M} \left[ \left( x \frac{d}{dx} \right)^n f(x); \alpha \right] \]  

for \( n \in \mathbb{N} \) and \( x^{\alpha+n-m} f^{(n-m)}(x) |_{0}^{\infty} = 0 \ \forall m \in \{1, \ldots, n\} \),

\[ (-1)^n \frac{\Gamma(\alpha)}{\Gamma(\alpha - n)} \tilde{f}(\alpha - n) = \mathcal{M} \left[ f^{(\alpha)}(x); \alpha \right] \]  

for \( n \in \mathbb{N} \) and \( x^{\alpha-n-1+m} f^{(n-m)}(x) |_{0}^{\infty} = 0 \ \forall m \in \{1, \ldots, n\} \), and

\[ (-1)^n \frac{\Gamma(\alpha)}{\Gamma(\alpha + n)} \tilde{f}(\alpha + n) = \mathcal{M} \left[ \left( \int_{0}^{x} f(x') \ dx' \right)^n; \alpha \right] \]  

where \( \left( \int_{0}^{x} f(x') \ dx' \right)^n \) defines an iterated integral

\[ \left( \int_{0}^{x} f(x') \ dx' \right)^n := \int_{0}^{x} \cdots \int_{0}^{x} f(x_n) \cdots f(x_1) \ dx_1 \cdots dx_n . \]  

The last two relations show that (for functions with appropriate asymptotic conditions) the Mellin transforms of derivatives and integrals are symmetrical under \( n \rightarrow -n \). Indeed, this is the basis of the definition of fractional derivatives. This suggests an application to pseudo-differential symbols of the type \( A(x, d/dx) = \sum_{i=0}^{n} a_i(x) d^i/dx^i \).

The Mellin transform is directly related to the Fourier and (two-sided) Laplace transforms by

\[ \mathcal{M}[f(x); \alpha] = \mathcal{F}[f(e^x); -i\alpha] = \mathcal{L}[f(e^{-x}); \alpha] . \]  

From these relationships, the inverse Mellin transform can be deduced;

\[ f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-\alpha} \tilde{f}(\alpha) \ d\alpha \]  

where \( c \in (a, b) \) (provided \( \tilde{f}(\alpha) \) is integrable along the path). The almost everywhere (a.e.) designation can be dropped if \( f(x) \) is continuous. Moreover, if \( f(x) \) is of bounded variation about \( x_0 \), then

\[ \frac{f(x_0^+) + f(x_0^-)}{2} = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} x^{-\alpha} \tilde{f}(\alpha) \ d\alpha . \]

Using the inversion formula, the Parseval relation for the Mellin transform follows from

\[ \int_{0}^{\infty} g(x) h(x) x^{\alpha-1} \ dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{g}(\alpha') \tilde{h}(\alpha - \alpha') \ d\alpha' , \]  

assuming the necessary conditions on \( g(x) \) and \( h(x) \) to allow for the interchange of integration order. In particular,

\[ \mathcal{M}[g(x) h(x); 1] = \int_{0}^{\infty} g(x) h(x) \ dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{g}(\alpha') \tilde{h}(1 - \alpha') \ d\alpha' . \]

Similarly,

\[ \mathcal{M}[g(x) \star h(x); \alpha] := \int_{0}^{\infty} \int_{0}^{\infty} g(x') h \left( \frac{x}{x'} \right) x^{\alpha-1} \ dx' \ dx = \tilde{g}(\alpha) \tilde{h}(\alpha) , \]  

and

\[ \mathcal{M}[g(x) \star h(x); \alpha] := \int_{0}^{\infty} \int_{0}^{\infty} g(x x') h(x') x^{\alpha-1} \ dx' \ dx = \tilde{g}(\alpha) \tilde{h}(1 - \alpha) . \]
A.2 Expansions

Definition A.2 The singular expansion of a meromorphic function \( f(b) \) with a finite set \( \mathcal{P} \) of poles is a sum of its Laurent expansions to order \( O(b^0) \) about each pole, i.e.

\[
f(b) \asymp \sum_{\varepsilon \in \mathcal{P}} \text{Laur}[f(b), \varepsilon; O(b^0)].
\]  

(109)

Theorem A.1 (ii) Let \( f(x) \) have Mellin transform \( \tilde{f}(\alpha) \) in \( \langle a, b \rangle \). Assume 

\[
f(x)|_{x \to 0^+} = \sum_{\varepsilon, k} c_{\varepsilon, k} x^\varepsilon (\log x)^k + O(x^M) \]  

(110)

where \(-M < -\varepsilon \leq a \) and \( k \in \mathbb{N} \). Then \( \tilde{f}(\alpha) \) can be continued to a meromorphic function in \( \langle -M, b \rangle \) where it has the singular expansion

\[
\tilde{f}(\alpha) \asymp \sum_{\varepsilon, k} c_{\varepsilon, k} \frac{(-1)^k k!}{(\alpha + \varepsilon)^{k+1}}.
\]  

(111)

Likewise, if 

\[
f(x)|_{x \to \infty} = \sum_{\varepsilon, k} c_{\varepsilon, k} x^{-\varepsilon} (\log x)^k + O(x^{-M})
\]  

(112)

where \( b \leq \varepsilon < M \) and \( k \in \mathbb{N} \). Then \( \tilde{f}(\alpha) \) can be continued to a meromorphic function in \( \langle a, M \rangle \) where it has the singular expansion

\[
\tilde{f}(\alpha) \asymp \sum_{\varepsilon, k} c_{\varepsilon, k} \frac{(-1)^{k+1} k!}{(\alpha - \varepsilon)^{k+1}}.
\]  

(113)

More generally, it can be shown that for \( \tilde{f}(\alpha) \) meromorphic in \( \langle -M, b \rangle \) (respectively \( \langle a, M \rangle \)) whose poles lie to the right (respectively left) of \( M \), then

\[
f(x)|_{x \to 0^+} = \sum_{\varepsilon_k \in \mathcal{P}} \text{Res} \left[ \tilde{f}(\alpha)x^\alpha, \alpha = \varepsilon_k \right] + O(x^M)
\]

\[
f(x)|_{x \to \infty} = -\sum_{\varepsilon_k \in \mathcal{P}} \text{Res} \left[ \tilde{f}(\alpha)x^{-\alpha}, \alpha = \varepsilon_k \right] + O(x^{-M})
\]  

(114)

if \( f(x) \) is at least twice differentiable.

Conversely,

Theorem A.2 (iii) Let \( f(x) \) have Mellin transform \( \tilde{f}(\alpha) \) in \( \langle a, b \rangle \). Assume \( \tilde{f}(\alpha) \) is meromorphic in \( \langle -M, b \rangle \) such that 

\[
\tilde{f}(\alpha)|_{|\alpha| \to \infty} = O(|\alpha|^{-r}), \quad r > 1
\]  

(115)

and

\[
\tilde{f}(\alpha) \asymp \sum_{k, \varepsilon} \frac{c_{k, \varepsilon}}{(\alpha - \varepsilon)^{k+1}}.
\]  

(116)
Then
\[ f(x)\big|_{x\to 0^+} = \sum_{k,\varepsilon} \frac{(-1)^k}{k!} c_{k,\varepsilon} x^{-\varepsilon} (\log x)^k + O(x^M). \quad (117) \]

Likewise, if \( \tilde{f}(\alpha) \) is meromorphic in \( \langle a, M \rangle \), then
\[ f(x)\big|_{x\to \infty} = \sum_{k,\varepsilon} \frac{(-1)^{k+1}}{k!} c_{k,\varepsilon} x^{-\varepsilon} (\log x)^k + O(x^{-M}). \quad (118) \]

### A.3 Holomorphic functions

The definition of Mellin transform can be extended to include certain holomorphic functions. The following theorems and definitions are taken from [4] with slight modification.

**Theorem A.3** Let \( F(b) \) be analytic in a sector \( S_{\theta_1,\theta_2} := \{ b : \theta_1 < \arg(b) < \theta_2 \} \) based at the point \( a' \in \mathbb{R} \) with \( \theta_1 < 0 < \theta_2 \) and \( (\theta_1, \theta_2) \in \mathbb{R} \). Assume that, for all \( b \in S_{\theta_1,\theta_2} \),
\[ F(b)\big|_{b\to 0,\infty} = O(b^{-r}) \quad (119) \]
where \( r \in (a, b) \). Then:

(i) The complex Mellin transform
\[ \mathcal{M}[F(b); \alpha] = \tilde{F}(\alpha) := \int_{c_{a'}} F(b) b^{\alpha-1} \, db \quad (120) \]
exists and does not depend on the pointed contour \( C_{a'} \) (based at \( b = a' \)) inside the sector \( S_{\theta_1,\theta_2} \).

(ii) The complex Mellin transform equals the real Mellin transform in the fundamental strip.

(iii) \( \mathcal{M}[F(wb); \alpha] = w^{-\alpha} \tilde{f}(\alpha) \) in the sector \( S_{\theta_1-\arg(w),\theta_2-\arg(w)} \) for \( w \in S_{\theta_1,\theta_2} \). (iv) The inverse transform exists and is given by
\[ F(b) = \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} b^{-\alpha} \tilde{F}(\alpha) \, d\alpha = \int_{t-i\infty}^{t+i\infty} b^{-\alpha} \tilde{f}(\alpha) \, d\alpha. \quad (121) \]

**Theorem A.4** If \( \tilde{f}(\alpha) \) is the Mellin transform of \( f(x) \) in \( \langle a, b \rangle \) such that \( \tilde{f}(\rho + i\sigma)e^{\theta_1 \sigma} \) and \( \tilde{f}(\rho + i\sigma)e^{\theta_2 \sigma} \) are absolutely integrable, then \( f(x) \) can be analytically continued to \( F(b) \) in the sector \( S_{\theta_1,\theta_2} \) such that \( F(b)\big|_{b\to 0,\infty} = O(b^{-r}) \) for all \( r \in (a, b) \). Moreover, if \( \tilde{f}(\alpha) \) possesses regular isolated poles in \( \langle a', b' \rangle \) where \( a' < a < b < b' \), then the asymptotic expansion of \( F(b) \) agrees with the expansion of \( f(x) \) up to order \( O(b^{-a'}) \) and \( O(b^{-b'}) \) at \( b = 0 \) and \( b = \infty \) respectively.

### A.4 Mellin distributions

The relationship between Mellin and Fourier transforms allows the development of Mellin distributions. And the extension to holomorphic functions allows development of Mellin distributions on paracompact \( C^\infty \) complex manifolds. Following [4]:
**Definition A.3** Let \( f_I : \mathbb{R}^n_+ := \{ y \in \mathbb{R}^n : 0 < y < \infty \} \to \mathbb{C} \) be a function with support \( I := \{ x \in \mathbb{R}^n_+ : 0 < x \leq x_0 \text{ for some } x_0 \in \mathbb{R}^n_+ \} \). Take \( f_I \in L^1(\mathbb{R}^n_+) \) with limits \( \lim_{x \to 0^+} f_I(x) \to O(x^{-a}) \) and \( \lim_{x \to \infty} f_I(x) \to O(x^{-b}) \) for \( a, b \in \mathbb{R}^n \). Then the Mellin transform \( \tilde{f}(\alpha) \) with \( \alpha \in \langle a, b \rangle := (a, b) \times i\mathbb{R}^n \subset \mathbb{C}^n \) is defined by (the analytic extension of)

\[
\tilde{f}(\alpha) := \mathcal{M}[f_I(x); \alpha] := \int_{\mathbb{R}^n_+} f_I(x)x^{-\alpha - 1}dx .
\]

The notation \( (a, b) \) denotes a poly-interval \( \{ y \in \mathbb{R}^n : a < y < b \} \) and \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

Now, the tight relationship with the Fourier transform motivates the definition

**Definition A.4** Let \( v \in \mathbb{R}^n \) and define

\[
M_v(I) := \{ \psi \in C^\infty(I) : \sup_{x \in I} |x^{v+1}(x\partial_x)^\lambda \psi(x)| < \infty \}
\]

where \( \lambda \in \mathbb{N}_0^n \) and \( \mathbb{N}_0^n \) is the set of non-negative multi-indices. Endow \( M_v(I) \) with the topology defined by the sequence of seminorms

\[
\rho_{v, \lambda}(\psi) = \sup_{x \in I} |x^{v+1}(x\partial_x)^\lambda \psi(x)| .
\]

Then \( M_{\omega}(I) \) for \( \omega \in \mathbb{R}^n_\infty := (\mathbb{R} \cup \{ \infty \})^n \) is defined to be the inductive limit of \( M_v(I) \), i.e.

\[
M_{\omega}(I) = \lim_{v \nearrow \omega} \ M_v(I) .
\]

The dual space \( M'_{\omega}(I) \) is comprised of Mellin distributions and the total space of Mellin distributions is

\[
M'(I) = \bigcup_{w \in \mathbb{R}^n_\infty} M'_{\omega}(I) .
\]

Finally, the Mellin transform of a distribution \( T \in M'_{\omega}(I) \) is defined by

\[
\tilde{T}(\alpha) := \mathcal{M}[T; \alpha] := \langle T, x^{-\alpha - 1} \rangle , \quad \Re \alpha < \omega .
\]

Note the topological inclusions

\[
D(I) \subset M_v(I) \subset M_{\omega}(I) \subset D'(I) ,
\]

and \( \tilde{T}(\alpha) \) is well-defined on the set

\[
\Omega_T := \bigcup_{\{ v : T \in M'_v(I) \}} [\Re \alpha < v] .
\]

Some of the important properties of the 1-dimensional Mellin transform have their analogues for distributions:

**Theorem A.5** The Mellin transform \( \tilde{T}(\alpha) \) is holomorphic on \( \Omega_T \) and

\[
\frac{\partial}{\partial \alpha_i} \tilde{T}(\alpha) = \langle T, x^{-\alpha - 1}(-\log x_i) \rangle
\]

\[
\tilde{T}(\alpha - \beta) = \mathcal{M}[x^\beta T; \alpha] \quad \Re \alpha - \Re \beta < \omega
\]

\[
\alpha^\gamma \tilde{T}(\alpha) = \mathcal{M}[x^{\partial_x \gamma} T; \alpha] \quad \gamma \in \mathbb{N}_0^n , \Re \alpha < \omega
\]

\[
(\alpha^\gamma + 1) \tilde{T}(\alpha + \gamma) = \mathcal{M}[(\partial_x)^\gamma T; \alpha] \quad |\gamma| = 1 , \Re \alpha < \omega - \gamma .
\]

\[\text{The substitution } \alpha \to -\alpha \text{ in the exponent of } x \text{ conforms with reference [4].}\]
B Exponential exercises

The exponential function plays a prominent role in ordinary Mellin transforms, so we want to develop and characterize the functional counterpart by looking at some specific cases of reduction to finite dimensional groups.

Let $E := \exp F_{S \lambda}$ stand for the exponential on $\mathcal{F}(G_B^C)$ defined (via Definition 2.3) with the product given by the $\ast$–convolution. Suppose $\mathcal{C}^* = \mathbb{C}$, $G_B^C \to \mathbb{R}_+$, and $F = E^{-\beta}$ such that $F(g) = e^{-\beta g}$ where $\beta \in \mathbb{C}_+ := \mathbb{R}_+ \times i\mathbb{R}$. For the choice of $\lambda$ that corresponds to the standard Haar measure, this is just the usual exponential Mellin transform

$$M_H \left[ E^{-\beta}; \alpha \right] := \int_{\mathbb{R}_+} e^{-\beta g} g^{\alpha - 1} dg = \frac{\Gamma(\alpha)}{\beta^\alpha}, \quad \alpha \in (0, \infty)_H .$$  \tag{130}

In particular,

$$M_H \left[ E^{-Id}; \alpha \right] = \Gamma(\alpha), \quad \alpha \in (0, \infty)_H .$$  \tag{131}

As a quick exercise, use Proposition 4.4 with $F_1(\tilde{g}g) = e^{-\tilde{g}g}$ and $F_2(\tilde{g}) \rho(\tilde{g}) = e^{-\tilde{g}g}$ to deduce (for $\beta' = Id'$ and a choice of $\lambda$ corresponding to standard normalization)

$$\Gamma(\alpha) \Gamma(1 - \alpha) = M_H \left[ \int_{\mathbb{R}_+} e^{-\tilde{g}g} e^{-\tilde{g}g} d(\log \tilde{g}); \alpha \right]$$

$$= \int_0^\infty \frac{t^{\alpha - 1}}{1 + t} dt$$

$$= \pi \csc(\pi \alpha), \quad \alpha \in (0, 1)_H .$$  \tag{132}

Notice the reduction in $S_\lambda$. Simple manipulations yield the standard results $\pi \alpha \csc(\pi \alpha) = \Gamma(1 + \alpha) \Gamma(1 - \alpha)$ and $\Gamma(1 + \alpha)/\Gamma(\alpha - 1) = \alpha(\alpha - 1)$.

However, the functional Mellin transform provides a mechanism to regularize; and with a suitable choice of $\lambda$,

$$M_\Gamma \left[ E^{-\beta'}; \alpha \right] := \int_{\mathbb{R}_+} e^{-\beta g} g^{\alpha} d\nu(g) = \frac{1}{\beta^\alpha}, \quad \alpha \in (0, \infty)_\Gamma$$  \tag{133}

for $\nu(g) := \log(g)/\Gamma(\alpha) = \nu(g)/\Gamma(\alpha)$ where $\nu(g)$ is the normalized Haar measure on $\mathbb{R}_+$.

To extend the fundamental strip to the left of the imaginary axis, one can use

$$M_{\Gamma^p} \left[ E^{-\beta'}; \alpha \right] := \frac{1}{\Gamma(\alpha + p)} \int_{\mathbb{R}_+} (\beta g)^p e^{-\beta g} g^{\alpha} d\nu(g)$$

$$= \frac{(-1)^p}{\Gamma(\alpha + p)} M_\Gamma \left[ \left( g \frac{d}{dg} \right)^p E^{-\beta'}; \alpha \right], \quad \alpha \in (-p, \infty)_{\Gamma^p}, \quad p \geq 0 .$$  \tag{134}

There are other ways to extend the fundamental strip to the left of the imaginary axis. For example, defining $\varphi^{-\beta g} := e^{-\beta g} - e^{-g}$ yields

$$M_H \left[ \varphi^{-\beta}; \alpha \right] := \int_{\mathbb{R}_+} \varphi^{-\beta g} g^{\alpha} d\nu(g) = \frac{\Gamma(\alpha)}{\beta^\alpha} - \Gamma(\alpha), \quad \alpha \in (-1, \infty)_H .$$  \tag{135}
At $\alpha = 0$ this gives
\[
\int_{\mathbb{R}} e^{g} \beta^{-g} d\nu(g) = -\log(\beta) ,
\]
and therefore
\[
\mathcal{M}_H \left[ E^{-\beta}; 0 \right] = \frac{d}{d\alpha} \mathcal{M}_H \left[ E^{-\beta}; \alpha \right] |_{\alpha = 0}
\]
which suggests the definition
\[
\mathcal{M}_\lambda' [F; 0] := \frac{d}{d\alpha} \mathcal{M}_\lambda [F; \alpha] |_{\alpha = 0} = \int_{G_B^C} F(g) \log g \, d\lambda g .
\]

Generalizing to the non-abelian case, take $G_B^C \rightarrow GL(n, \mathbb{C})$ and $\mathfrak{c}^* = \mathbb{C}$. Define the functional $E^{-\text{Tr} \beta g} : G_B^C \rightarrow \mathbb{C}$ by $g \mapsto e^{-\text{Tr} \beta g}$ with $\beta \in GL(n, \mathbb{C})$, and take $\rho : G_B^C \rightarrow \mathbb{C}$ by $g \mapsto \det g$. Then we get
\[
\mathcal{M}_{\Gamma_n} \left[ E^{-\text{Tr} \beta g}; \alpha \right] := \int_{GL(n, \mathbb{C})} e^{-\text{Tr} \beta g} \det (g^\alpha) \, d\nu(g_{\Gamma_n}), \quad \alpha \in \mathcal{S}_{\Gamma_n}
\]
\[
= \int_{GL(n, \mathbb{C})} e^{-\text{Tr} \beta g} (\det g)^\alpha \, d\nu(g_{\Gamma_n}), \quad \alpha \in \mathcal{S}_{\Gamma_n}
\]
\[
= \int_{GL(n, \mathbb{C})} e^{-\text{Tr} \beta^{-1} g} (\det g)^\alpha \, d\nu(g_{\Gamma_n}), \quad \alpha \in \mathcal{S}_{\Gamma_n}
\]
\[
= \det(\beta^{-\alpha}), \quad \alpha \in \mathcal{S}_{\Gamma_n}
\]
where $\varphi(\alpha)$ is a complex phase, $\nu(g_{\Gamma_n}) := \nu(g)/\Gamma_n(\alpha)$ with $\nu(g)$ the Haar measure on $GL(n, \mathbb{C})$, and $\Gamma_n(\alpha)$ a complex multi-variate gamma function defined by
\[
\Gamma_n(\alpha) := \mathcal{M}_H \left[ E^{-\text{Tr} \beta Id}; \alpha \right] := \int_{GL(n, \mathbb{C})} e^{-\text{Tr} g} \det (g^\alpha) \, d\nu(g), \quad \alpha \in \mathcal{S}_H .
\]
In particular, if $\alpha = 1 \in \mathcal{S}_{\Gamma_n}$, then
\[
\det(\beta^{-1}) = \int_{GL(n, \mathbb{C})} e^{-\text{Tr} \beta g} (\det g) \, d\nu(g_{\Gamma_n}) = (\det \beta)^{-1}
\]
Remark that $\Gamma_n(\alpha)$ is not a well-defined object unless one restricts to a compact subgroup of $GL(n, \mathbb{C})$. Otherwise, the price of extracting $\det(\beta^{-\alpha})$ from the integral comes with the price of regularizing this possibly singular normalization.

Generalizing further, suppose $G_B^C \rightarrow GL(n, \mathbb{C})$ but now $\mathfrak{c}^* = L_B(\mathbb{C}^n)$ the space of bounded linear maps on $\mathbb{C}^n$ and $E^{-\beta g} : G_B^C \rightarrow L_B(\mathbb{C}^n)$ by $g \mapsto e^{-\beta g}$ with $\beta \in GL(n, \mathbb{C})$. The Haar normalized functional Mellin transform of the exponential yields
\[
\mathcal{M}_H \left[ E^{-\beta}; \alpha \right] := \int_{GL(n, \mathbb{C})} e^{-\beta g} g^\alpha \, d\nu(g), \quad \alpha \in \mathcal{S}_H
\]
\[
= \int_{GL(n, \mathbb{C})} e^{-g} (\beta^{-1} g)^\alpha \, d\nu(g), \quad \alpha \in \mathcal{S}_H
\]
\[
=: \beta_H^{-\alpha}, \quad \alpha \in \mathcal{S}_H
\]
which defines the element $\beta_H^\alpha \in M(\mathcal{C}^*)$ for $\alpha \in S_H$.

Unless $\beta$ is in the center of $GL(n, \mathbb{C})$ or we restrict to a subgroup of $GL(n, \mathbb{C})$, this can’t be reduced further, i.e. $\beta_H^\alpha \neq (\beta)^{-\alpha}$ in general. However, various restrictions allow for various degrees of simplification. For example, if $\beta$ is positive-definite Hermitian and $G^\mathbb{C}_B$ is restricted to a real one-parameter subgroup generated by $\log \beta \in \mathfrak{gl}(n, \mathbb{C})$, then more can be done. So let us take $G^\mathbb{C}_B \to \phi_{\log, \beta}(\mathbb{R})$. Then, since $\beta \in \phi_{\log, \beta}(\mathbb{R})$,

$$M_\Gamma \left[ E^{-\beta'}; \alpha \right] = \int_{\phi_{\log, \beta}(\mathbb{R})} e^{-\beta g} g^\alpha \, d\nu(g_T), \quad \alpha \in \mathbb{S}_\Gamma$$

$$= \int_{\phi_{\log, \beta}(\mathbb{R})} e^{-g(\beta^{-1} g)} \, d\nu(g_T), \quad \alpha \in \mathbb{S}_\Gamma$$

$$= \beta^{-\alpha} \int_{\phi_{\log, \beta}(\mathbb{R})} e^{-g} g^\alpha \, d\nu(g_T), \quad \alpha \in \mathbb{S}_\Gamma$$

$$= \beta^{-\alpha} \text{Id} \int_{\mathbb{R}_+} e^{-t} t^\alpha \, d\nu(t_T), \quad \alpha \in (0, \infty)_\Gamma$$

$$= \beta^{-\alpha}, \quad \alpha \in (0, \infty)_\Gamma.$$

The second line follows from the left-invariance of the Haar measure, the third line from the fact that $\beta$ and $g$ commute, and the fourth from $\phi_{\log, \beta}(\mathbb{R}) \cong \mathbb{R}_+ \times \mathbb{C}^{a^2}$.

References

[1] Cartier, P., DeWitt-Morette, C., Functional Integration: Action and Symmetries, Cambridge University Press, Cambridge (2006).

[2] LaChapelle, J., Functional Integration on Constrained Function Spaces, arXiv:math-ph/1212.0502 (2012).

[3] Flajolet, P., Gourdon, X., Phillipe, D., Mellin Transforms and Asymptotics: Harmonic Sums, Theo. Comp. Sci. 144, 3–58 (1995).

[4] Szmydt, Z., Ziemian, B., The Mellin transformation and Fuchsian type partial differential equations, Kluwer Academic Publishers, Boston (1992).

[5] Hofmann, K.H., Morris, S.A., The Structure of Compact Groups, Walter de Gruyter, Berlin (1998).

[6] Williams, D.P., Crossed Products of $C^*$-algebras, American Mathematical Society, Providence, Rhode Island (2007).

[7] Landsman, N.P., Rieffel induction as generalized quantum Marsden-Weinstein reduction, J. Geo. and Phys. 15, 285-319, (1995).

[8] Blau, M., Thompson, G., Localization and Diagonalization, J. Math. Phys. 36, 2192 (1995).

[9] LaChapelle, J., Functional Integration for Quantum Field Theory, Integration: Mathematical Theory and Applications, 1(4), 1–21 (2008).