Huge Casimir effect at finite temperature in electromagnetic Rindler space

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Abstract

We investigate the Casimir effect at finite temperature in electromagnetic Rindler space, and find the Casimir energy is proportional to \(\frac{T^4}{d^2}\) in the high temperature limit, where \(T \approx 27^\circ C\) is the temperature and \(d \approx 100\) nm is a small cutoff. We propose to make metamaterials to mimic Rindler space and measure the predicted Casimir effect. Since the parameters of metamaterials we proposed are quite simple, this experiment would be easily implemented in laboratory.

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I. INTRODUCTION

In recent years, fields of metamaterials and transformation optics have highly developed. Metamaterials can be used to design many interesting devices such as invisibility devices [1–4], perfect lens [5], illusion devices [6] and perfect confinement devices [7]. Besides, metamaterials can also be used to mimic black holes [8, 9], cosmos [10–15], string theory [16] and even to model time [17]. So far, to our best knowledge, most works in these fields are done on the classical aspects except [11, 12, 18–21]. In [11, 12], one author of this paper and his collaborators compute the quantum Casimir energy of electromagnetic de Sitter space and find it is proportional to the size of horizon, the same order as dark energy. They also suggest to make metamaterials to mimic de Sitter space and measure the predicted Casimir energy. This experiment is of great importance in two aspects. First, it could detect an unusual large Casimir effect. Second, it may provide an alternative of the origin of dark energy for our accelerating universe.

However, the experiment they proposed is difficult to be practiced for several reasons. First, the permittivity and permeability of metamaterials they designed are very complicated. Second, due to the spherical symmetry of de Sitter space, one has to make the corresponding metamaterials one spherical shell by one spherical shell and finally assemble them together, which is, of course, a hard task. Third, they do not consider the temperature effect which is an important factor in an actual experiment. In this letter, we try to overcome those problems. We consider the Rindler space instead, which is much simpler than de Sitter space but shares almost all the important features such as Hawking radiation, area law of entropy, infinite red-shift on horizon and, in particular, the huge Casimir effect. We use the natural unit system with $\hbar = c = k_B = 1$ below.

II. BRIEF REVIEW OF RINDLER SPACE

We first give a brief review of Rindler space. Rindler space with the metric

$$ds^2 = -a^2 x^2 dt^2 + dx^2 + dy^2 + dz^2, \; 0 < x,$$

(1)

is a flat spacetime experienced by observers with constant acceleration $a$. Rindler space is related with the Minkowski space by suitable coordinate transformations, however physics in those two spaces are completely different. From the viewpoint of observers in Rindler space,
the vacuum state of Minkowski space is a thermal state with temperature \( \frac{\hbar a}{2\pi} \) instead. This means that one would feel a small temperature \( \frac{\hbar a}{2\pi c k_B} \) when running with an acceleration. In addition, there is an event horizon at \( x = 0 \) in Rindler space and the entropy on this horizon is \( \frac{4}{a} \), which is the same as that of a black hole or de Sitter space.

According to [21–23], the behavior of light in gravity field \( g_{\mu\nu} \) is exactly the same as that in metamaterials with the following permeability and permittivity

\[
\varepsilon^{ij} = \mu^{ij} = \sqrt{-g} \frac{g^{ij}}{-g_{00}}, \quad i, j = 1, 2, 3.
\]

(2)

We note that the above permeability and permittivity remain invariant under a conformal transformation of the metric \( g_{\mu\nu} \rightarrow \omega(x)^2 g_{\mu\nu} \), which is consistent with the conformal invariance of Maxwell theory in 4 dimensions. Thus, metamaterials can mimic the gravitational metrics up to an arbitrary conformal factor. From eqs. [21–23], we find that metamaterials with

\[
\varepsilon = \mu = \frac{1}{ax}, \quad x \in [d, \frac{1}{a})
\]

(3)

can be used to mimic the behavior of light in Rindler space, where we have set a physical cutoff \( d \) for metamaterials to avoid the singularity. Since the characteristic length of Rindler space is \( \frac{1}{a} \), for simplicity, we choose the length of metamaterials to be \( \frac{1}{a} \). Now it is clear that this kind of metamaterial is much easier to be made than that of [11]. Firstly, its parameters (3) are very simple. Secondly, one can make it layer by layer rather than spherical shell by spherical shell. However, these parameters are still hard to be satisfied by common material. Thanks to the developments of metamaterials, now we can make such singular materials artificially. It should be mentioned that the Unruh radiation in material with parameters (3) is studied in [24]. While in this letter, we shall focus on the finite-temperature Casimir effect.

III. FINITE TEMPERATURE CASIMIR EFFECT IN ELECTROMAGNETIC RINDLER SPACE

Let us start with the Maxwell equations in a curved space

\[
\partial_{[\mu} F_{\nu\lambda]} = 0, \quad \partial_{\mu} H^{\mu\nu} = 0,
\]

(4)
where $F_{\mu\nu}$ is the field strength, $H^{\mu\nu} = \sqrt{-g}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}$. The electric field $E_i$ and magnetic field $H_i$ are related with $F_{\mu\nu}$ by

$$E_i = F_{0i}, \quad H_i = -\frac{\epsilon_{ijk}H_{jk}}{2},$$

where the Levi-Civita symbol $\epsilon_{ijk}$ is +1 for all even permutations of $\epsilon_{123}$. For static space, with the help of optical metric

$$\gamma^{ij} = -g_{00}g^{ij},$$

we can rewrite eq. (5) in a spatial covariant form

$$\nabla_i E^i = 0, \quad \partial_t E^i - \frac{\epsilon^{ijk}}{\sqrt{\gamma}} \partial_j H_k = 0,$$

$$\nabla_i H^i = 0, \quad \partial_t H^i + \frac{\epsilon^{ijk}}{\sqrt{\gamma}} \partial_j E_k = 0,$$

where $\nabla_i$ is the covariant derivative defined by $\gamma^{ij}$, and all the indices are lowered and raised by $\gamma^{ij}$. By spatial covariance, we mean eq.(7) is covariant under the spatial coordinate transformations $t' = t, x' = x' (x, y, z), y' = y' (x, y, z), z' = z' (x, y, z)$. Though eqs.(3-6) are not covariant in 4 dimensions, they are spatial covariant in 3 dimensions in the sense that eq.(2,3) are spatial tensor densities, eq.(4) can be rewritten as spatial covariant eq.(7), eq.(5) are spatial vectors and eq.(6) is the spatial metric. From eq. (7), one can derive the field equations of $E_i$ and $H_i$ as

$$-\partial_t^2 V_i = D_i^j V_j = (-\nabla^2 + R^{ij})V_j, \quad \nabla^i V_i = 0,$$

where $V_i$ denotes $E_i$ and $H_i$, $R^{ij}$ is the Ricci curvature tensor. These are the key equations we shall use to calculate the partition function of electromagnetic field in Rindler space.

According to the standard thermal field theory, the partition function $Z(\beta)$ satisfies the identity

$$\ln Z(\beta) = -\frac{1}{2} Tr \ln(-\Delta)$$

$$= \frac{1}{2} \int_0^\infty ds \int d^4 x \sqrt{|g|} K(s; x, x'),$$

where $\beta = 1/T$, $\Delta$ is the Laplace-Beltrami operator for corresponding field, and $K(s; x, x') = < x | e^{-s\Delta} | x' >$ is the heat kernel (we omit indices for non-scalar field here). Form eq. (8), we get

$$\Delta = -(\partial_t^2 + \nabla^2)\gamma^{ij} + R^{ij},$$
for electromagnetic field, where we have perform a Wick rotation of time \( t = -i\tau \) which is a general method to derive the partition function in the thermal field theory. From the definition of heat kernel above, one can easily find that the heat kernel of electromagnetic field satisfies

\[
\partial_s K^i_j(s; x, x') - \left[ (\partial^2_x + \nabla^2)\delta^i_l - R^i_l \right] K^l_j(s; x, x') = 0. \tag{11}
\]

According to the thermal field theory, the heat kernel of electromagnetic field should satisfy the periodic boundary condition

\[
K^i_j(s; x, x') \big|_{\tau=\tau+\beta} = K^i_j(s; x, x') \big|_{\tau=\tau} . \tag{12}
\]

Set \( K^i_j(s; x, x') = K_1(s; \tau, \tau')K_3^i_j(s; x_i, x'_i) \), we can separate the above equation into two independent equations

\[
\partial_s K_1(s; \tau, \tau') - \partial^2_\tau K_1(s; \tau, \tau') = 0, \tag{13}
\]

\[
\partial_s K_3^i_j(s; x_m, x'^i_m) - (\nabla^2\delta^i_l - R^i_l) K^l_j(s; x_m, x'_m) = 0. \tag{14}
\]

The solution of eq. (13) is

\[
K_1(s; \tau, \tau') = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \exp\left[ -\frac{4\pi^2 n^2}{\beta^2} s + \frac{i2\pi n}{\beta}(\tau - \tau') \right], \tag{15}
\]

which satisfies the periodic boundary condition eq. (12) and has the correct zero-temperature limit

\[
\lim_{\beta \to \infty} K_1(s; \tau, \tau') = \frac{1}{\sqrt{4\pi s}} e^{-\frac{(\tau - \tau')^2}{4s}}. \tag{16}
\]

From eq. (15), we can derive

\[
K_1(s; \tau, \tau) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \exp\left( -\frac{4\pi^2 n^2}{\beta^2} s \right) = \frac{1}{\beta} \theta_3(0) \left( \frac{i4\pi s}{\beta^2} \right) = \frac{1}{\beta} \frac{\theta_3(0) i\beta^2}{\sqrt{4\pi s}} = \frac{1}{\sqrt{4\pi s}} \sum_{n=-\infty}^{\infty} \exp\left( -\frac{\beta^2 n^2}{4s} \right), \tag{17}
\]

where \( \theta_3(z; \sigma) \) is the third Jacobi \( \theta \) function satisfying the identity \(( -i\sigma)^{1/2} \theta_3(0; \sigma) = \theta_3(0; -\frac{1}{\sigma})\). Notice that a regularization procedure is hidden in the above transformation.
Without this transformation, we can also derive eq. (20) below if we regularize the result at the end.

Note that the optical metric of Rindler space
\[ ds^2 = \frac{dx^2 + dy^2 + dz^2}{a^2 x^2} \]  
(18)
is just the Euclidean anti-de Sitter space whose heat kernel has been studied in details in [25]. From eqs. (3.4,3.5) of [25], we have
\[ \gamma^{ij} K_{3ij}(s; x_m, x_m) = \frac{2 + 4a^2 s}{(4\pi s)^{3/2}}, \]  
(19)
where we have subtracted a non-physical term of ghost field. Substituting eqs. (17,19) into eq. (9), we can derive
\[ \ln Z(\beta) = \frac{\hat{V} \pi^2}{45 \beta^3} [1 + \frac{15(a\beta)^2}{2\pi^2}], \]  
(20)
where \( \hat{V} = \int \sqrt{\gamma} dx^3 = \frac{A}{2\pi^2} (\frac{1}{d^2} - \frac{1}{L^2}) \), \( d \) is a small cutoff in metamaterials and \( L \) is the length of metamaterials along x-axis. In the above derivation, we have dropped an infinite constant coming from the \( n = 0 \) term in the sum. Note that, in the flat-space limit (\( \gamma = 1, a = 0 \)), eq. (20) exactly reduces to the partition function of photon gas in flat space. Applying the partition function, we can obtain various thermodynamic quantities. We only list the free energy \( F \) and the internal energy \( U \) (finite-temperature Casimir energy) below:
\[ F(\beta) = -\frac{A\pi^2 a}{90(a\beta)^4} \left( \frac{1}{d^2} - \frac{1}{L^2} \right) [1 + \frac{15(a\beta)^2}{2\pi^2}], \]  
(21)
\[ U(\beta) = \frac{A\pi^2 a}{30(a\beta)^4} \left( \frac{1}{d^2} - \frac{1}{L^2} \right) [1 + \frac{5(a\beta)^2}{2\pi^2}], \]  
(22)
Those quantities are very huge. Let us make an estimation of \( U(\beta) \) below. Choose \( L = \frac{1}{a} \approx 0.1 m \), the characteristic temperature in metamaterials becomes \( T_e = \frac{a}{2\pi} \approx 0.0036 K \). Suppose the temperature \( T \approx 300 K \) in laboratory, it is easy to observe that the usual temperature of laboratory corresponds to the high-temperature limit \( T/T_e \approx 8 \times 10^4 \). Set \( A = L^2, d \approx 100 nm \), in this high-temperature limit, we get
\[ U(\beta) \approx \frac{A\pi^2 a}{30d^2(a\beta)^4} \approx 3 \times 10^3 J, \]  
(23)
which is much larger than that of photon gas in flat space (\( U_f = \frac{\pi^2 V}{16\pi^3 \hbar^3 \beta} \))
\[ \frac{U(\beta)}{U_f} = \frac{L^2}{2d^2} = 5 \times 10^{11}. \]  
(24)
Following the standard thermal field theory, one can also derive the radiation spectrum of photon gas in the electromagnetic Rindler space as

$$U(\omega, \beta) d\omega = \frac{\hat{V}}{\pi^2 c^3} \frac{\hbar \omega (\omega^2 + \frac{\omega^2}{\gamma})}{e^{\hbar \omega / \beta} - 1} d\omega,$$

which is nearly a blackbody spectrum except a small correction and a large effective volume $$\hat{V} = \frac{A}{2a^3} (\frac{1}{\beta^2} - \frac{1}{L^2}).$$ Applying eq. (25), one can again obtain eqs. (20-22).

It should be stressed that it is the large effective volume $$\hat{V}$$ of the optical metric from the viewpoint of gravity, or equivalently, the large permittivity and permeability of metamaterials ($$\varepsilon^{ij} = \mu^{ij} = \sqrt{\gamma} \gamma^{ij} = \frac{1}{ax} \delta^{ij}$$) from the viewpoint of material that leads to the above huge Casimir effects.

**IV. PROPOSALS OF EXPERIMENT**

We suggest to detect the large energy flux density $$J(\beta) = \frac{c U(\beta)}{4V}$$ (V=AL) and radiation spectrum eq. (25). Notice that we mean the radiation from the thermal material rather than the Unruh radiation studied in [24]. Since metamaterials have frequency dispersion, the designed permittivity and permeability are effective only to frequencies in certain brand. Thus, the above ideal discussions on finite-temperature Casimir effects seem invalid. However, according to [12], there is a typical frequency whose contribution to Casimir effect is dominating. Thus, to perform this experiment, the parameters of metamaterials eq. (3) only need to be valid around the typical frequency $$\omega_t \approx \frac{2a}{\beta},$$ which can be derived from eq. (25). Metamaterials are also dissipative by virtue of the principle of causality. Fortunately, one can compensate the dissipation near the typical frequency by gain to make it very small [26, 27]. We hope that one day the metamaterials we proposed can be made with suitable size and effective for the typical frequency. Then the predicted huge Casimir effects can be measured. This experiment is important for studying gravity in laboratory and may open a window for the use of vacuum energy.

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