A Closed Expression for the UV-Divergent Parts of One-Loop Tensor Integrals in Dimensional Regularization

G. Sulyok
Institute of Atomic and Subatomic Physics, Vienna University of Technology, Vienna, 1020 Austria
e-mail: gsulyok@ati.ac.at
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Abstract—Starting from the general definition of a one-loop tensor N-point function, we use its Feynman parametrization to calculate the ultraviolet (UV-) divergent part of an arbitrary tensor coefficient in the framework of dimensional regularization. In contrast to existing recursion schemes, we are able to present a general analytic result in closed form that enables direct determination of the UV-divergent part of any one-loop tensor N-point coefficient independent from UV-divergent parts of other one-loop tensor N-point coefficients. Simplified formulas and explicit expressions are presented for A-, B-, C-, D-, E-, and F-functions.

Keywords: quantum field theory, high energy physics phenomenology, dimensional regularization, tensor integrals, ultraviolet divergences

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1. INTRODUCTION
Quantum field theory provides a perturbation series expansion which allows the calculation of physical observables in particles processes up to an arbitrary accuracy [1, 2]. Due to high precision measurements of decay rates and cross sections at collider experiments, present calculations include at least quantum corrections of first order. The evaluation of radiative corrections contains integrals over the inner momentum. These loop integrals have been investigated and classified by t’Hooft, Veltman and Passarino [3, 4], partly elaborating an idea of Brown and Feynman [6]. Their works focus on loop integrals with up to four internal propagators. Nevertheless, their methods can be extended leading to a compact notation for all one loop tensor N-point functions and coefficients, presented in [5] and exhaustively used in [7].

The divergent behaviour of these loop integrals necessitates the renormalization procedure rendering all physical observables finite. To quantify the divergences of the various loop integrals a regularization scheme is needed. Dimensional regularization has become the standard method to deal with ultraviolet (UV-) divergences [8–10], i.e., divergences that arise due to large inner momenta. In current literature, one can find extensive tables of UV-divergent parts of specific tensor coefficients [7, 13] and generally valid recursive evaluation schemes [7, 14], but no closed analytic formula is given yet. Using Feynman parametrization, we are able to provide such an expression for the UV-divergent part of an arbitrary one-loop tensor N-point coefficient.

Although the number of physically relevant tensor functions is limited by dimensional considerations or renormalizability, tensor coefficients of arbitrary high ranks can appear as mathematical by-product as, for example, in the reduction schemes for one loop tensor integrals in case of vanishing Gram determinants [7]. These schemes provide recursive formulas that converge for infinite iteration steps. To get high order approximations for a certain tensor coefficient, one needs lower order expressions for tensor coefficients of a higher rank. In particular, the divergent parts of these high rank tensor coefficients are necessary as well. Since it is not possible to decide a priori which approximation order suffices for a specific process, a generally valid formula for divergent parts represents a useful tool.

We have organized this work as follows: First, all occurring quantities are defined and a compact notation is introduced. Then, we outline the key steps of the calculation and present the general result. In addition, we provide simplified formulas for A-, B-, C-, and D-functions and in the appendix, we give explicit expressions for some B-, C-, D-, E-, and F- tensor coefficients.
2. DEFINITIONS AND NOTATION

The general form of a one-loop tensor N-point integral, graphically illustrated in Fig. 1, reads

$$T^{N, \mu_1 \ldots \mu_P}(p_1, \ldots, p_{N-1}, m_0, \ldots, m_{N-1}) := \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int \frac{q^{\mu_1} \ldots q^{\mu_P}}{N_0 N_1 \ldots N_{N-1}}$$

with denominators

$$N_k = (q + p_k)^2 - m_k^2 + i\eta \quad (p_0 \equiv 0),$$

where \(i\eta \ (\eta > 0)\) denotes an infinitesimally small imaginary part, \(\mu\) is a mass parameter and \(D\) the non-integer space-time dimension defined as \(D = 4 - \varepsilon\). Following common convention, we abbreviate \(T^1 = A, T^2 = B, T^3 = C, T^4 = D, \ldots\). For the decomposition of the tensor integral in its Lorentz-covariant structures we use the same notation as in [7]

$$N_{00} = \sum_{i_1, \ldots, i_P} P_{i_1}^{\mu_1} \ldots P_{i_P}^{\mu_P} T_{i_1 \ldots i_P}^N + \sum_{i_1, \ldots, i_P} \{gp\}_{i_1 \ldots i_P}^{\mu_1 \mu_P} T_{i_1 \ldots i_P}^N + \cdots + \sum_{i_1, \ldots, i_P} \{g\ldots g\}_{i_1 \ldots i_P}^{\mu_1 \ldots \mu_P} T_{i_1 \ldots i_P}^N,$$

for \(P \) odd,

$$\sum_{i_1, \ldots, i_P} \{g\ldots g\}_{i_1 \ldots i_P}^{\mu_1 \ldots \mu_P} T_{i_1 \ldots i_P}^N,$$

for \(P \) even.

The curly brackets stand for symmetrization with respect to Lorentz indices in such a way that all non-equivalent permutations of the Lorentz indices on metric tensors \(g\) and a generic momentum \(p\) contribute with weight one. In covariants with \(n\) momenta \(p_i^{\mu_i} \) (\(i = 1, \ldots, n\)) only one representative out of the \(n!\) permutations of the indices \(i_j\) is kept, e.g.

$$\{gg\}_{i_j \ldots i_P}^{\mu_1 \mu_P} = g^{\nu_1 \rho_1} \ldots g^{\nu_P \rho_P} + \sum_{\sigma_1 \ldots \sigma_P} g^{\nu_1 \mu_1} g^{\nu_2 \mu_2} g^{\sigma_1 \nu_2} g^{\sigma_2 \nu_3} \ldots g^{\sigma_P \mu_P},$$

$$\{g\ldots g\}_{i_1 \ldots i_P}^{\mu_1 \ldots \mu_P} = g^{\nu_1 \mu_1} g^{\nu_2 \mu_2} \ldots g^{\nu_P \mu_P} + \sum_{\sigma_1 \ldots \sigma_P} g^{\nu_1 \mu_1} g^{\nu_2 \mu_2} \ldots g^{\nu_P \mu_P} g^{\sigma_1 \nu_2} g^{\sigma_2 \nu_3} \ldots g^{\sigma_P \nu_P}.$$

3. FEYNMAN PARAMETRIZATION

The investigation of the divergent behaviour is most easily done by means of Feynman parametrization. From its most general form

$$\frac{1}{A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n}} = \int dx_1 dx_2 \ldots dx_n \delta \left(1 - \sum_{i=1}^n x_i\right) \prod_{i=1}^n \frac{\Gamma(m_i + x_i)}{\Gamma(m_i)} \Gamma(m_n)$$

we obtain for the denominators of Eq. (1)

$$\frac{1}{N_0 \ldots N_{N-1}} = \int dx_0 dx_1 \ldots dx_{N-1} \delta \left(1 - \sum_{i=0}^{N-1} x_i\right) \frac{\Gamma(N)}{\left(\sum_{i=0}^{N-1} N_i x_i\right)^N}$$

$$= (N-1)! \int dx_1 \int dx_2 \ldots \int dx_{N-1} \delta \left(1 - \sum_{i=1}^{N-1} x_i\right) \prod_{i=1}^{N-1} N_i \left(1 - x_i \right) \prod_{j=1}^{N-1} x_j^{N_j}.$$

By denoting the integration over the \(N\)-dimensional simplex as \(\int dS_N\), explicitly inserting the \(N_i\)'s, completing the square and performing a shift in the integration variable \(q\) we can write the general one loop tensor \(N\)-point integral Eq. (1) as
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with

\[ A^{(N)}(x_i, (p_i, p_j), m_i^2) = \sum_{i=1}^{N-1} (p_i x_i)^2 \]

\[ + \sum_{i=1}^{N-1} \sum_{j \neq i}^{N-1} 2p_ip_j x_ix_j - \sum_{i=1}^{N-1} x_if_i + m_0^2 \]

and

\[ f_i = p_i^2 - m_i^2 + m_0^2 \]

for the further evaluation of the divergent parts.

4. CALCULATION OF THE UV-DIVERGENT PARTS

Regularization of loop integrals can be done with the aid of various techniques. Dimensional regularization has become the standard scheme to deal with UV divergences \([8–10]\). One could in principle also handle IR divergences in the framework of dimensional regularization, but then both are parameterized by \(\varepsilon\)-terms and become indistinguishable. To separate them we assume the introduction of small regulator-masses \(\lambda_i\) which guarantee IR finiteness. Thus, we need not to worry about the infra-red behaviour in the following calculation.

At first, we investigate the UV-divergent part of a tensor coefficient proportional to metric tensors only. Later, we reduce the calculation for coefficients belonging to momenta and metric tensors to the upper case. In order to extract the coefficient proportional to metric tensors only from the Feynman parametrization of a general one-loop tensor integral (with an even number of indices) one just drops all terms \(\sum x_i p_i\) in Eq. (8), that is

\[ T_{\mu_1, ..., \mu_D}^{\nu_1, ..., \nu_D} = \frac{(2\pi\mu)^{D-4}}{i\pi^2} (N-1)! \int d^Dq \left[ q - \sum_{i=1}^{N-1} x_i p_i \right]^{\mu_1} \cdots \left[ q - \sum_{i=1}^{N-1} x_i p_i \right]^{\mu_D} \]

\[ \int dS_{N-1} \left[ q^2 - A^{(N)}(x_i, (p_i, p_j), m_i^2) + i\eta \right]^{-N}. \]

Contracting both sides of Eq. (11) with \(n\) metric tensors yields

\[ g_{\mu_1 \nu_1} \cdots g_{\mu_n \nu_n} \{ g \cdot g \}^{\mu_1 \cdots \mu_n} = D(D+2) \cdots (D+2(n-1)) \]

on the left hand side and generates \(n\) times \(q^2\) on the right hand side. Using

\[ T_{0, 0}^{N} = \frac{(2\pi\mu)^{D-4}}{i\pi^2} (N-1)! \int d^Dq \left[ q^2 - A^{(N)} + i\eta \right]^{-N}. \]

The next step consists of carrying out the \(q\) integration. Therefore, we transform the integrand with the help of the binomial theorem (for better readability we leave out the upper index in \(A^{(N)}\) until Eq. (22))

\[ \left( q^2 \right)^n \left[ q^2 - A + i\eta \right]^{-N} = \left( q^2 - A + i\eta + A - i\eta \right)^n \left[ q^2 - A + i\eta \right]^{-N} = \]

\[ \sum_{k=0}^{n} \binom{n}{k} \left( A - i\eta \right)^k \left[ q^2 - A + i\eta \right]^{n-k} \left[ q^2 - A + i\eta \right]^{-N}, \]

\[ \Rightarrow \int d^Dq \left[ q^2 - A^{(N)} + i\eta \right]^{-N}. \]
which yields for the tensor coefficient

\[
T_{0,0}^{N} = 2^{-\alpha} \frac{(1 + O(\varepsilon))}{(n + 1)!} \left( \frac{1}{\pi^2} \right) (N - 1)! \int dS_{N-1} \sum_{k=0}^{n} \binom{n}{k} (A - i\eta)^k \int d^D q \{ q^2 - A + i\eta \}^{-N-k+n}.
\]

To perform the \( q \)-integration we distinguish between two cases

1. Case: \(-N - k + n = \beta \geq 0\)

The pole prescription becomes unnecessary in this case and we use the rules for integration over a \( D \)-dimensional space, that can be found for example in [11].

Explicitly, we exploited linearity and applied the result \( \int d^D q (q^2)^\alpha = 0, \forall \alpha \).

\[
\int d^D q \{ q^2 - A \}^\beta = \int d^D q \sum_{\alpha=0}^{\beta} c_\alpha(A) (q^2)^\alpha = \sum_{\alpha=0}^{\beta} c_\alpha(A) \int d^D q (q^2)^\alpha = 0.
\]

In dimensions close to 4, UV divergences only occur for \( I_1 \) and \( I_2 \), i.e. for \( k = n - N + 1 \) and \( k = n - N + 2 \), the other \( I_j \)'s are UV-finite. Thus, we are left with

\[
T_{0,0}^{N} = 2^{-\alpha} \frac{1}{(n + 1)! \pi^2} (N - 1)! \int dS_{N-1} \left[ \binom{n}{N - 1} + \binom{n}{n - N + 1} \binom{n}{n - N + 2} \right] + \text{UV-finite terms}.
\]

After inserting the explicit forms of \( I_1 \) and \( I_2 \) and using \( \pi^2 = 1 + O(\varepsilon) \) and \( \Gamma(z + 1) = z \Gamma(z) \) we arrive at

\[
T_{0,0}^{N} = 2^{-\alpha} \frac{1}{(n + 1)! \pi^2} (N - 1)! \int dS_{N-1} \left[ \binom{n}{N - 1} + \binom{n}{n - N + 1} \frac{1}{(n - N + 1)!} \frac{n!(N - 1)!}{2} \right] \times (A - i\eta)^{n-N+1} I_1(A) \times \binom{n}{n - N + 2} (A - i\eta)^{n-N+2} I_2(A) + \text{UV-finite terms}.
\]

By ignoring UV-finite terms, the prefactor can be further simplified yielding

\[
T_{0,0}^{N} = 2^{-\alpha} \frac{1}{(n - N + 1)! \pi^2} (N - 1)! \int dS_{N-1} (A - i\eta)^{n-N+1} \frac{1}{2} \Gamma \left( \frac{\varepsilon}{2} \right) + \text{UV-finite terms}.
\]

By taking into account that

\[
(A - i\eta)^{\frac{\varepsilon}{2}} = 1 - \frac{\varepsilon}{2} \ln(A - i\eta) + ...\]

and using

\[
\Gamma \left( \frac{\varepsilon}{2} \right) = \frac{2}{\varepsilon} - \gamma_E + O \left( \frac{\varepsilon}{2} \right), \quad \text{for } \varepsilon \to 0,
\]

\( \gamma_E = 0.57721 \) (Euler’s constant)

We get

\[
T_{0,0}^{N} = 2^{-\alpha} \frac{1}{(n - N + 1)! \pi^2} (N - 1)! \int dS_{N-1} (A - i\eta)^{n-N+2} + \text{UV-finite terms}
\]

and finally for the UV-divergent part (we can omit the \( i\eta \) now, because for all one loop N-point tensor functions that are UV-divergent the relation \( n - N + 2 \geq 0 \) holds)

\[
(D - 4)\pi^2 \frac{2^{-\alpha}}{(n - N + 2)!} \int dS_{N-1} (A - i\eta)^{n-N+2} + \text{UV-finite terms}.
\]

With the help of this special result we can now investigate the divergence of a general tensor coefficient that we extract from Eq. (8).
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\[ \left\{ g_i^a \right\}_{n}^{1-N} p_1^{2a_{n+1}} \cdots p_{N-1}^{2a_{N-n}} p_{N-N-1}^{2a_{N+1-k}} \cdots p_{N-1}^{2a_{N-k-1}} \int d^{2D} q \int dS_{N-1} q^{\mu_1} \cdots q^{\mu_{N-1}} (-x_1 p_1)^{\mu_{N-n+1}} \cdots (-x_1 p_1)^{\mu_{N-k-1}} \cdots \left\{ q^2 - A(N) + i\eta \right\}^{-N}. \]

The external momenta only determine the form of the polynomial in the \( x_i \)'s. By comparing both sides one gets

\[ \left\{ g_i^a \right\}_{n}^{1-N} p_1^{2a_{n+1}} \cdots p_{N-1}^{2a_{N-n}} p_{N-N-1}^{2a_{N+1-k}} \cdots p_{N-1}^{2a_{N-k-1}} \int d^{2D} q \int dS_{N-1} q^{\mu_1} \cdots q^{\mu_{N-1}} \left\{ q^2 - A(N) + i\eta \right\}^{-N}. \]

where the multinomial coefficient is given by

\[ \binom{m}{s} = \frac{m!}{s_1! \cdot s_2! \cdots s_{N(N+1)}!} \]

and the summation runs over all combinations of \( s_i \) that sum up to \( m \) denoted as

\[ |s| := s_1 + s_2 + \cdots + s_{N(N+1)} = m. \]

By defining a vector \( \vec{b} \) consisting of the summands \( a_i \) without the \( x_i \), i.e.

\[ \vec{b} = (p_1^2, \ldots, p_{N-1}^2, 2p_1p_2, \ldots, 2p_1p_{N-1}, 2p_2p_3, \ldots, 2p_{N-2}p_{N-1}, \ldots, 2p_{N-1}p_{N-2}, 0)^T, \]

we rewrite

\[ (D-4)T_{0,1,\ldots,N-1}^{N} = \sum_{i=1}^{N-1} b_i \int dS_{N-1} \prod_{i=1}^{N-1} x_i^{k_i}, \]

where the collected exponents belonging to a certain \( x_i \) read in general

\[ t_i = k_i + 2s_i + \sum_{j=1}^{i-1} s_{\frac{1}{2}(N-1-j)+i} \]

\[ + \sum_{j=1}^{N-1-i} s_{\frac{1}{2}(N-1-j)+i} + s_{\frac{1}{2}(N-1-i)+i}. \]
We denote the sum over the $t_i$ as $u$

$u := \sum_{i=1}^{N-1} t_i = \sum_{i=1}^{N-1} k_i + 2 \sum_{i=1}^{N(N+1)-1} s_j + \sum_{i=1}^{N(N+1)-1} s_j$ \hspace{1cm} (33)

and make use of

$$\int dS_k \prod_{i=1}^{N} x_i^N = \left(\frac{N + \sum r_i}{N!}\right)! \hspace{1cm} (34)$$

to finally obtain for the UV-divergent part of a general one loop tensor coefficient

$$(D-4)T_{B,C,\ldots}^N = \frac{(-1)^{k_j+1}}{2^{(n-1)n} m!} \sum_{n=1}^{N-1} \left(\prod_{i=1}^{N} b_i^N\right) \left(\prod_{i=1}^{N-1} t_i!\right)$$

$$\left(\frac{N + \sum r_i}{N!}\right)! \hspace{1cm} (35)$$

where $m = n - N + 2$ and all necessary abbreviations and definitions are given by Eqs. (26), (28)—(30), (32), and (33).

5. EXPLICIT FORMULAE FOR $N = 1, 2, \ldots, 6$

The result Eq. (24) is very compact and can be applied to all one loop N-point tensor integrals, but for practical purpose the explicit formulae for the $A_-, B_-, C_-, \ldots$ functions can be useful as well.

5.1. $A$-functions

For 1-point tensor integrals $T^1 = A_-$ there are no external momenta and by recognizing

$k_i = 0 \forall i, \hspace{0.5cm} m = n + 1, \hspace{0.5cm} s = s_1, \hspace{0.5cm} \bar{b} = m_0^2,$

$t_i = 0 \forall i, \hspace{0.5cm} u = 0$

we get from Eq. (35)

$$(D-4)A_{0,0} = \frac{(-1)^{k_j+1}}{2^{(n-1)n} (n+1)!} \sum_{s_1=m+1}^{n+1} \left(\prod_{i=1}^{n} s_i\right) \left(\prod_{i=1}^{n+1} m_0^2\right) \left(\prod_{i=1}^{n+1} t_i!\right)$$

$$(D-4)A_{0,0} = -\frac{m_0^{2n+2}}{2^{(n-1)n} (n+1)!} \hspace{1cm} (36)$$

Note that the UV-divergent part of the scalar integral $A_0$ is obtained by setting $n = 0$.  

5.2. $B$-functions

For 2-point tensor coefficients $T^2 = B_-$ we have

$m = n, \hspace{0.5cm} \bar{s} = \{s_1, s_2, s_3\}, \hspace{0.5cm} \bar{b} = \{p_1^2, -f_1, m_0^2\},$

$t_i = k_1 + 2s_1 + s_2, \hspace{0.5cm} u = t_i$

yielding

$$(D-4)B_{0,0,1,1} = \frac{(-1)^{k_j+1}}{2^{(n-1)n} s_{1,2,3}} \sum_{s_1, s_2, s_3} \left(\prod_{i=1}^{s_1} p_1^2\right)\left(\prod_{i=1}^{s_2} -f_1\right)\left(\prod_{i=1}^{s_3} m_0^2\right)$$

$$\left(\prod_{i=1}^{s_1} v_1, \prod_{i=1}^{s_2} v_2, \prod_{i=1}^{s_3} v_3\right)^{(1)} v_4 u_4 \hspace{1cm} (37)$$

This expression can be also found in [15]. Since 2-point tensor functions are usually denoted with arguments $B_-(p_1^2, m_1^2, m_2^2)$ we reinsert $f_1 = p_1^2 - m_1^2 + m_0^2$ and get

$$(D-4)B_{0,0,1,1} = \frac{(-1)^{k_j+1}}{2^{(n-1)n} s_{1,2,3}} \sum_{s_1, s_2, s_3} \left(\prod_{i=1}^{s_1} p_1^2\right)\left(\prod_{i=1}^{s_2} -f_1\right)\left(\prod_{i=1}^{s_3} m_0^2\right)$$

$$\left(\prod_{i=1}^{s_1} v_1, \prod_{i=1}^{s_2} v_2, \prod_{i=1}^{s_3} v_3\right)^{(1)} v_4 u_4 \hspace{1cm} (38)$$

To avoid the combinatorics required for the summation in the multinomial theorem, it is possible to alternatively write Eq. (37) with binomial coefficients

$$(D-4)B_{0,0,1,1} = \frac{(-1)^{k_j+1}}{2^{(n-1)n} s_{1,2,3}} \sum_{s_1, s_2, s_3} \left(\prod_{i=1}^{s_1} p_1^2\right)\left(\prod_{i=1}^{s_2} -f_1\right)\left(\prod_{i=1}^{s_3} m_0^2\right)$$

$$\left(\prod_{i=1}^{s_1} v_1, \prod_{i=1}^{s_2} v_2, \prod_{i=1}^{s_3} v_3\right)^{(1)} v_4 u_4 \hspace{1cm} (39)$$

In the appendix, UV-divergent parts of $B$-functions up to rank 8 are listed explicitly.

5.3. $C$-functions

For 3-point tensor coefficients $T^3 = C_-$ we have

$N = 6$ and the quantities used in Eq. (35) become

$$m = n - 1, \hspace{0.5cm} \bar{s} = \{s_1, s_2, s_3, s_4, s_5, s_6\}, \hspace{0.5cm} \bar{b} = \{p_1^2, p_2^2, 2p_1p_2, -f_1, f_2, m_0^2\},$$

$$t_1 = k_1 + 2s_1 + s_2, \hspace{0.5cm} t_2 = k_2 + 2s_2 + s_3, \hspace{0.5cm} u = t_1 + t_2 \hspace{1cm} (40)$$

Thus, the UV-divergent part reads

$$(D-4)C_{0,0,1,1,2,2} = \frac{(-1)^{k_j+k_i+1}}{2^{(n-1)n} (n+1)!} \sum_{s_1, s_2, s_3, s_4, s_5, s_6} \left(\prod_{i=1}^{s_1} p_1^2\right)\left(\prod_{i=1}^{s_2} -f_1\right)\left(\prod_{i=1}^{s_3} m_0^2\right)$$

$$\left(\prod_{i=1}^{s_1} v_1, \prod_{i=1}^{s_2} v_2, \prod_{i=1}^{s_3} v_3\right)^{(1)} v_4 u_4 \hspace{1cm} (41)$$

$$\left(\prod_{i=1}^{s_4} v_4, \prod_{i=1}^{s_5} v_5, \prod_{i=1}^{s_6} v_6\right)^{(1)} v_7 u_7 \hspace{1cm} (42)$$
which can be also found in [15]. If the $f_i = p_i^2 - m_i^2 + m_0^2$ are reinserted and $2p_1p_2$ is written as $p_1^2 + p_2^2 - p_{12}$, where $p_{12} = (p_1 - p_2)^2$, we obtain with $\bar{v} = \{v_1, v_2, \ldots, v_{12}\}$ and $|\bar{v}| := \sum v_i$$

(D - 4) C_{0, 01, -12, -3} \frac{(-1)^{k_1+k_2+1}}{2^{(n-1)(n-1)!}} \sum_{|\bar{v}|=n-1} \binom{n-1}{\bar{v}} (-1)^{v_1+v_2+v_3+v_4} \times (p_1^2)^{v_1}(p_2^2)^{v_2}(p_1 p_2)^{v_3}(m_1^2)^{v_4}(m_2^2)^{v_5}(m_0^2)^{v_6+v_7+v_8} (k_2 + 2v_1 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8) l_{k_2 + 2v_1 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8} (k_2 + 2v_1 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8) ! \times (k_1 + 2v_1 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8) l_{k_1 + 2v_1 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8} (k_1 + 2v_1 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8) !$$

An alternative expression for Eq. (42) using binomial coefficients is given by

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\begin{equation}
(D - 4) C_{0, 01, -12, -3} = \frac{(-1)^{k_1+k_2+1}}{2^{(n-1)(n-1)!}} \sum_{n=0}^{l_1=0} \sum_{l_2=0}^{l_3=0} \sum_{l_4=0}^{l_5=0} \binom{n-1}{l_1 l_2 l_3 l_4} (p_1^2)^{l_1} (p_2^2)^{l_2} (p_1 p_2)^{l_3} (-f_1)^{l_4} (-f_2)^{l_5} (m_0^2)^{l_6-l_1-l_2} \times (k_1 + 2l_3 - l_4 + l_5) l_{k_2 - 2l_3 + l_4 + l_5 - l_2 + l_1} ! \times (k_1 + 2l_3 - l_4 + l_5) l_{k_2 - 2l_3 + l_4 + l_5 - l_2 + l_1} ! (k_1 + 2l_3 - l_4 + l_5) l_{k_2 - 2l_3 + l_4 + l_5 - l_2 + l_1} !$$
\end{equation}

In the appendix, UV-divergent parts of C-functions up to rank $\bar{s}$ are shown explicitly.

$$m = n - 2, \quad \bar{s} = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\},$$

$$\bar{b} = \{p_1^2, p_2^2, p_3^2, p_4^2, p_5^2, p_6^2, p_7^2, p_8^2, p_9^2, p_{10}^2\},$$

$$t_1 = k_1 + 2s_1 + s_4 + s_5 + s_7,$$

$$t_2 = k_2 + 2s_2 + s_4 + s_5 + s_8,$$

$$t_3 = k_3 + 2s_3 + s_5 + s_6 + s_9,$$

$$u = t_1 + t_2 + t_3$$

yielding

$$\begin{equation}
(D - 4) D_{0, 01, -12, -3} = \frac{(-1)^{k_1+k_2+1}}{2^{(n-1)(n-1)!}} \sum_{s_1, \ldots, s_{10}} (p_1^2)^{s_1} (p_2^2)^{s_2} (p_1 p_2)^{s_3} (2p_1 p_3)^{s_4} (2p_2 p_3)^{s_5} (-f_1)^{s_6} (-f_2)^{s_7} (m_0^2)^{s_8} (k_2 + 2s_1 + s_4 + s_5 + s_7) l_{k_2 + 2s_1 + s_4 + s_5 + s_7 + 2s_3 + s_5 + s_6 + s_9} ! (k_1 + k_2 + k_3 + 2s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8 + s_9 + 3) !$$
\end{equation}

This expression can be rewritten using binomial coefficients as

$$\begin{equation}
(D - 4) D_{0, 01, -12, -3} = \frac{(-1)^{k_1+k_2+1}}{2^{(n-1)(n-1)!}} \sum_{l_1, \ldots, l_{10}} \binom{n-2}{l_1 l_2 l_3 l_4 l_5 l_6 l_7 l_8 l_9} (p_1^2)^{l_1} (p_2^2)^{l_2} (p_1 p_2)^{l_3} (2p_1 p_3)^{l_4} (2p_2 p_3)^{l_5} (-f_1)^{l_6} (-f_2)^{l_7} (m_0^2)^{l_8} (k_2 + 2l_3 - l_4 + l_5) l_{k_2 - 2l_3 + l_4 + l_5 - l_2 + l_1} ! (k_1 + 2l_3 - l_4 + l_5) l_{k_2 - 2l_3 + l_4 + l_5 - l_2 + l_1} ! (k_1 + 2l_3 - l_4 + l_5) l_{k_2 - 2l_3 + l_4 + l_5 - l_2 + l_1} ! (k_1 + 2l_3 - l_4 + l_5) l_{k_2 - 2l_3 + l_4 + l_5 - l_2 + l_1} !$$
\end{equation}

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In the appendix, UV-divergent parts of D-functions up to rank 8 are listed explicitly.

5.5. E- and F- Functions

In a four dimensional space-time, 6-point integrals $T_6 = F_6$ can be reduced to 5-point integrals $T_5 = E_5$ that can be further reduced to 4-point functions [7, 13, 16]. Nevertheless, we want to provide the explicit form of the quantities required in the generic expression Eq. (35) in order to demonstrate the general applicability of our result. Additionally, UV-divergent parts of high ranked coefficients can occur in reduction procedures for $E$- and $F$-functions [7].

For $E$-functions, the length of the index vector $\tilde{s}$ and of $\tilde{b}$ amounts to $\frac{N}{2}(N + 1) = 15$ and we get for the abbreviation used in Eq. (35)

$$m = n - 3,$$

$$\tilde{s} = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}\},$$

$$\tilde{b} = \left\{p_1^2, p_2^2, p_3^2, p_4^2, 2p_1p_2, 2p_1p_3, 2p_1p_4, 2p_2p_3, 2p_2p_4, 2p_3p_4, -f_1, -f_2, -f_3, -f_4, m_0^2\right\},$$

$$t_1 = k_1 + 2s_1 + s_5 + s_6 + s_7 + s_{11}, \quad t_2 = k_2 + 2s_2 + s_5 + s_8 + s_9 + s_{12},$$

$$t_3 = k_3 + 2s_3 + s_5 + s_8 + s_{10} + s_{13}, \quad t_4 = k_4 + 2s_4 + s_7 + s_9 + s_{10} + s_{14},$$

$$u = t_1 + t_2 + t_3 + t_4.$$  

For $F$-functions, the length of the index vector $\tilde{s}$ and of $\tilde{b}$ amounts to $\frac{N}{2}(N + 1) = 21$ and we obtain

$$\tilde{s} = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}\},$$

$$\tilde{b} = \left\{p_1^2, p_2^2, p_3^2, p_4^2, 2p_1p_2, 2p_1p_3, 2p_1p_4, 2p_2p_3, 2p_2p_4, 2p_3p_4, 2p_3p_5, 2p_4p_5, -f_1, -f_2, -f_3, -f_4, -f_5, m_0^2\right\},$$

$$m = n - 4,$$

$$t_1 = k_1 + 2s_1 + s_6 + s_7 + s_8 + s_{16}, \quad t_2 = k_2 + 2s_2 + s_6 + s_{10} + s_{11} + s_{12} + s_{17},$$

$$t_3 = k_3 + 2s_3 + s_7 + s_{10} + s_{11} + s_{14} + s_{18}, \quad t_4 = k_4 + 2s_4 + s_8 + s_{11} + s_{13} + s_{15} + s_{19},$$

$$t_5 = k_5 + 2s_5 + s_9 + s_{12} + s_{14} + s_{15} + s_{20}, \quad u = t_1 + t_2 + t_3 + t_4 + t_5.$$  

6. CONCLUSIONS

In this work, the UV-divergent term of an arbitrary one-loop tensor coefficient in dimensional regularization is evaluated by using Feynman parametrization. The resulting formula is very compact and can be easily implemented\(^2\). The computational effort is mainly determined by finding all possible index combinations over which the summation in the multinomial theorem runs.

Compared to recursive schemes existing for the calculation of these UV-divergent parts, our result can be implemented with less programming effort and benefits from the obvious advantages of a closed expression over a recursion formula, i.e. no start values needed or direct access to coefficient functions of high rank.

Such a generally valid result contributes to the completion of investigations on UV-divergent parts of one-loop tensor coefficients. In addition, for practical purposes, the formula proves its usefulness in reduction procedures or for approximate formulas of tensor coefficients whose required accuracy is not known a priori.

In the appendix, we provide the most extensive tables currently available in literature allowing to straightforwardly look up UV-divergent parts of tensor coefficients of $B$, $C$, $D$, $E$-functions up to rank 8 and of $F$-functions up to rank 10.

APPENDIX

Here, we want to give the explicit expressions for the UV-divergent parts of $B$, $C$, $D$, and $E$-functions up rank 8 and for $F$-functions up the rank 10 thus representing the most extensive collection of UV-divergent parts currently available in literature.

The abbreviation $p_i^2$ always stands for $(p_i - p_j)^2$.

A. UV-Divergent Parts of B-functions up to Rank 8

The explicit formulae for 2-point tensor coefficients up to rank 8 read ($B_0$ is obtained from Eq. (37) for $n = 0$)
A CLOSED EXPRESSION FOR THE UV-DIVERGENT PARTS

\[(D - 4)B_0 = -2, \quad (D - 4)B_1 = 1,\]

\[(D - 4)B_{00} = \frac{1}{6} \left(-3m_0^2 - 3m_1^2 + p_1^2\right), \quad (D - 4)B_{11} = -\frac{2}{3},\]

\[(D - 4)B_{001} = \frac{2m_0^2 + 4m_1^2 - p_1^2}{12}, \quad (D - 4)B_{111} = \frac{1}{2},\]

\[(D - 4)B_{0000} = \frac{-10m_1^2 - 10m_1^2 + 5m_1^2 p_1^2 - p_1^4 + 5m_0^2 \left(-2m_1^2 + p_1^2\right)}{120},\]

\[(D - 4)B_{0011} = \frac{-5m_0^2 + 3\left(-5m_1^2 + p_1^2\right)}{60}, \quad (D - 4)B_{1111} = -\frac{2}{5},\]

\[(D - 4)B_{0001} = \frac{5m_0^4 + 15m_1^4 - 6m_1^2 p_1^2 + p_1^4 + 2m_0^2 \left(5m_1^2 - 2p_1^2\right)}{240},\]

\[(D - 4)B_{00111} = \frac{3m_0^2 + 12m_1^2 - 2p_1^2}{60}, \quad (D - 4)B_{11111} = \frac{1}{3},\]

\[(D - 4)B_{00000} = \frac{-35m_0^6 - 35m_1^6 + 21m_1^4 p_1^2 - 7m_1^2 p_1^4 + p_1^6}{3360} + \frac{-7m_0^4 \left(5m_1^2 - 3p_1^2\right) - 7m_1^2 \left(5m_1^4 - 4m_1^2 p_1^2 + p_1^4\right)}{3360},\]

\[(D - 4)B_{00011} = \frac{-7m_0^4 + 7m_0^2 \left(-3m_1^2 + p_1^2\right) - 2\left(21m_1^4 - 7m_1^2 p_1^2 + p_1^4\right)}{840},\]

\[(D - 4)B_{00111} = \frac{-7m_0^4 + 5\left(-7m_1^2 + p_1^2\right)}{210}, \quad (D - 4)B_{111111} = -\frac{2}{7},\]

\[(D - 4)B_{000000} = \frac{14m_0^6 + 56m_1^6 - 28m_1^4 p_1^2 + 8m_1^2 p_1^4 - p_1^6}{6720} + \frac{14m_0^4 \left(2m_1^2 - p_1^2\right) + m_1^2 \left(42m_1^4 - 28m_1^2 p_1^2 + 6p_1^4\right)}{6720},\]

\[(D - 4)B_{000011} = \frac{14m_0^4 + 8m_1^2 \left(7m_1^2 - 2p_1^2\right) + 5\left(28m_1^4 - 8m_1^2 p_1^2 + p_1^4\right)}{3360},\]

\[(D - 4)B_{001111} = \frac{4m_0^4 + 24m_1^2 - 3p_1^2}{168}, \quad (D - 4)B_{1111111} = \frac{1}{4},\]

\[(D - 4)B_{0000000} = \frac{-126m_0^8 - 126m_1^8 + 84m_1^6 p_1^2 - 36m_1^4 p_1^4 + 9m_1^2 p_1^6 - p_1^8}{120960} + \frac{-42m_0^6 \left(3m_1^2 - 2p_1^2\right) - 18m_0^4 \left(7m_1^4 - 7m_1^2 p_1^2 + 2p_1^4\right)}{120960} + \frac{-9m_0^2 \left(14m_1^6 - 14m_1^4 p_1^2 + 6m_1^2 p_1^4 - p_1^6\right)}{120960},\]

\[(D - 4)B_{0000001} = \frac{120960 \left(-42m_0^6 - 18m_0^4 \left(7m_1^2 - 3p_1^2\right) - 9m_0^2 \left(28m_1^4 - 16m_1^2 p_1^2 + 3p_1^4\right)\right)}{60480} + \frac{5 \left(-84m_1^6 + 36m_1^4 p_1^2 - 9m_1^2 p_1^4 + p_1^6\right)}{60480},\]

\[(D - 4)B_{0000111} = \frac{-12m_0^4 + 15m_0^2 \left(-4m_1^2 + p_1^2\right) - 5 \left(36m_1^4 - 9m_1^2 p_1^2 + p_1^4\right)}{5040}, \quad (D - 4)B_{11111111} = -\frac{2}{9},\]
B. UV-Divergent Parts of $C$-Functions up to Rank 8

Tensor coefficients with $n = 0$ are UV-finite, $p_{12}$ denotes $(p_1 - p_2)^2$.

\[(D - 4)C_{00} = -\frac{1}{2}, \quad (D - 4)C_{0ii} = \frac{1}{6}, \quad (D - 4)C_{000} = -\frac{4m_0^2 - 4m_1^2 - 4m_2^2 + p_1^2 + p_{12} + p_2^2}{48}, \]

\[(D - 4)C_{00ii} = -\frac{1}{12}, \quad (D - 4)C_{00ij} = -\frac{1}{24}. \]

\[(D - 4)C_{0000} = \frac{1}{240}\left[5m_0^2 - 2p_{12} + \sum_{n=1}^{2}\left(5m_n^2 - p_n^2\right)\left[1 + \delta_{in}\right]\right], \]

\[(D - 4)C_{000i} = \frac{1}{20}, \quad (D - 4)C_{00ij} = \frac{1}{60}, \]

\[(D - 4)C_{00000} = -\frac{15m_0^4 - 15m_1^4 - 15m_2^4 + 3m_3^2 p_1^2 - p_1^4 + 6m_2^2 p_{12} - p_1^2 p_{12}}{1440} + \frac{p_1^2 + 6m_2^2 p_2^2 - p_1^2 p_2^2 - p_{12}^2 p_2^2 + 3m_1^2}\left(-5m_2^2 + 2p_1^2 + 2p_{12}^2 + 2p_2^2\right)}{1440} + \frac{3m_0^2\left(-5m_1^2 - 2m_2^2 + 2p_1^2 + p_{12} + 2p_2^2\right)}{1440}, \]

\[(D - 4)C_{0000i} = -\frac{1}{720}\left[6m_0^2 - 3p_{12} + \sum_{n=1}^{2}\left(6m_n^2 - p_n^2\right)\left[1 + 2\delta_{in}\right]\right], \]

\[(D - 4)C_{0000j} = -\frac{1}{720}\left[3m_0^2 + 2p_{12} + \sum_{n=1}^{2}\left(6m_n^2 - p_n^2\right)\right], \]

\[(D - 4)C_{000ii} = -\frac{1}{30}, \quad (D - 4)C_{000ij} = -\frac{1}{120}. \]

\[(D - 4)C_{000000} = -\frac{1}{10080}\left[3p_{12}^2 + 21m_0^4 - 7p_{12}m_0^2 + p_{12}\sum_{n=1}^{2}\left(p_n^2 - 7m_n^2\right)\left[2 + \delta_{in}\right]\right] + 7m_0^2\sum_{n=1}^{2}\left(3m_n^2 - p_n^2\right)\left[1 + \delta_{in}\right] + \sum_{n,m=1}^{2}\left(p_n^2 p_m^2 - 7p_n^2 m_m^2 + 21m_n^2 m_m^2\right)\left[1 + 2\delta_{in}\right], \]

\[(D - 4)C_{00000i} = -\frac{1}{1680}\left[7m_0^2 - 4p_{12} + \sum_{n=1}^{2}\left(7m_n^2 - p_n^2\right)\left[1 + 3\delta_{in}\right]\right], \]

\[(D - 4)C_{0000ij} = \frac{1}{5040}\left[7m_0^2 - 6p_{12} + \sum_{n=1}^{2}\left(7m_n^2 - p_n^2\right)\left[2 + \delta_{in}\right]\right], \]

\[(D - 4)C_{0000ii} = \frac{1}{42}, \quad (D - 4)C_{0000ij} = \frac{1}{210}. \]
\[(D - 4)C_{00000000}^{\text{ii}} = \frac{-168m_0^6 - 168m_1^6 - 168m_2^6 + 28m_1^4p_1^2 - 8m_2^4p_1^4 + 3p_1^6}{161280} \\
+ \frac{84m_0^4p_{12} - 16m_1^2p_1^2p_{12} + 3p_1^6p_{12} - 24m_2^2p_1^2p_{12} + 3p_1^2p_{12}^2}{161280} \\
+ \frac{3p_1^2 + 84m_1^4p_1^2 - 16m_2^2p_1^2 + 3p_1^4p_1^2 - 24m_2^2p_1p_{12}^2}{161280} \\
+ \frac{4p_1^2 + 3p_1^2p_{12}^2 + 3p_1^2p_1^2 - 24m_2^2p_1^4 + 3p_1^2p_1^2 + 3p_1^2p_{12}^2 + 3p_1^6}{161280} \\
- \frac{28m_0^4(6m_1^2 + 6m_2^2 - p_1^2 - p_{12}^2 - 3p_2^2) - 28m_1^4(6m_2^2 - 3p_1^2)}{161280} \\
- \frac{3p_{12}^2 - p_2^2}{8} - \frac{8m_0^2\{21m_1^4 + 21m_2^4 + 3p_1^4 + 2p_1^2p_1^2 + p_{12}^2\}}{161280} \\
+ \frac{+3p_1p_{12}^2 + 2p_{12}p_2^2 + 3p_2^2 + 7m_1^2\{3m_2^2 - 2p_1^2 - p_{12}^2 - p_2^2\}}{161280} \\
+ \frac{-7m_2^2\{p_1^2 + p_{12} + 2p_2\} - 8m_1^2\{21m_2^4 + 3p_1^4 + 3p_{12}^2\}}{161280} \\
+ \frac{2p_{12}p_2^2 + p_2^4 - 7m_2^2\{p_1^2 + 2p_{12} + p_2^2\} + p_1^2\{3p_{12} + 2p_2^2\}}{161280}, \]

\[(D - 4)C_{00000000}^{\text{ii}} = \frac{1}{40320} \left[ -6p_{12}^2 + 12p_{12}m_0^2 - p_{12}^2 \sum_{n=1}^{2} \{3p_n^2 - 24m_n^2\}(1 + \delta_{in}) \right] - \frac{28m_0^4 - 4m_0^2\sum_{n=1}^{2}\{7m_n^2 - 2p_n^2\}(1 + 2\delta_{in})}{161280} \]

\[-\frac{1}{2} \sum_{n,m=1}^{2} \left\{ p_n^2p_m^2 - 8p_n^2m_m^2 + 28m_n^2m_m^2 \right\}(2 + \delta_{in} + \delta_{im} + 8\delta_{in}\delta_{im}) \]

\[(D - 4)C_{00000000}^{\text{ii}} = \frac{-28m_0^4 - 84m_1^4 - 84m_2^4 + 16m_1^2p_1^2 - 3p_1^4 + 48m_2^2p_1^2}{80640} \]

\[+ \frac{-6p_1^2p_{12}^2 - 9p_1^2p_{12}^2 + 24m_2^2p_2^2 - 4p_2^2p_1^2 - 6p_2p_1^2 - 3p_2^4}{80640} \]

\[+ \frac{-8m_0^2\{7m_1^4 + 7m_2^4 - 2(p_1^2 + p_{12} + p_2^2)\}}{80640} \]

\[+ \frac{-8m_1^2\{14m_2^2 - 3p_1^2 - 2(3p_{12} + 2p_2^2)\}}{80640} \]

\[(D - 4)C_{00000000}^{\text{ii}} = \frac{-1}{3360} \left[ 8m_0^2 - 5p_{12}^2 + \sum_{n=1}^{2} \{8m_n^2 - p_n^2\}(1 + 4\delta_{in}) \right] \]

\[(D - 4)C_{00000000}^{\text{ii}} = \frac{-1}{6720} \left[ 4m_0^2 - 4p_{12}^2 + \sum_{n=1}^{2} \{8m_n^2 - p_n^2\}(1 + \delta_{in}) \right] \]

\[(D - 4)C_{00000000}^{\text{ii}} = \frac{-1}{20160} \left[ 8m_0^2 - 9p_{12}^2 + 3\sum_{n=1}^{2} \{8m_n^2 - p_n^2\} \right] \]

\[(D - 4)C_{00000000}^{\text{ii}} = \frac{-1}{56} \quad (D - 4)C_{00000000}^{\text{ii}} = \frac{-1}{336} \quad (D - 4)C_{00000000}^{\text{ii}} = \frac{-1}{840} \quad (D - 4)C_{00000000}^{\text{ii}} = \frac{-1}{1120} \]
C. UV-Divergent Parts of D-Functions up to Rank 8

Coefficients with \( n < 2 \) are UV-finite, \( p_{ij} \) stands for \((p_i - p_j)^2\).

\[
(D - 4)D_{0000} = -\frac{1}{12}, \quad (D - 4)D_{0000i} = \frac{1}{48},
\]

\[
(D - 4)D_{000000} = \frac{-5m_0^2 - 5m_1^2 - 5m_2^2 - 5m_3^2 + p_1^2 + p_{12} + p_{13} + p_2^2 + p_{23} + p_3^2}{480},
\]

\[
(D - 4)D_{00000i} = -\frac{1}{120}, \quad (D - 4)D_{00000j} = -\frac{1}{240},
\]

\[
(D - 4)D_{0000000} = \frac{1}{2880} \left[ 6 \sum_{n=0}^{3} m_n^2 (1 + \delta_{in}) - \sum_{n=1}^{3} p_n^2 (1 + \delta_{in}) - \sum_{m>n,n=1}^{3} p_{nm} (1 + \delta_{in} + \delta_{im}) \right]
\]

\[
(D - 4)D_{0000000i} = \frac{1}{240}, \quad (D - 4)D_{0000000j} = \frac{1}{720}, \quad (D - 4)D_{0000000k} = \frac{1}{1440},
\]

\[
(D - 4)D_{00000000} = \frac{1}{40320} \left[ -42 \sum_{m,n=0}^{3} m_m^2 m_m^2 + 7 \sum_{k=0}^{3} \sum_{m>n,n=1}^{3} m_k^2 p_{nm} (1 + \delta_{kn} + \delta_{km}) - 14 m_0^3 \sum_{n=1}^{3} p_n^2 ight.
\]

\[
\left. - 2 \sum_{m,n=0}^{3} p_m^2 p_n^2 + \sum_{l>k, k=1}^{3} \sum_{m>n,n=1}^{3} p_{kl} p_{mn} (1 + \delta_{kn} + \delta_{km}) + 14 m_0^3 \sum_{n=1}^{3} p_n^2 \right]
\]

\[
(D - 4)D_{00000000} = \frac{1}{10080} \left[ -7 \sum_{n=0}^{3} m_n^2 (1 + 2 \delta_{in}) + \sum_{n=1}^{3} p_n^2 (1 + 2 \delta_{in}) + \sum_{m>n,n=1}^{3} p_{nm} (1 + 2 \delta_{in} + 2 \delta_{im}) \right],
\]

\[
(D - 4)D_{0000000ij} = \frac{1}{20160} \left[ -7 \sum_{n=0}^{3} m_n^2 (1 + \delta_{in} + \delta_{jn}) + \sum_{n=1}^{3} p_n^2 (1 + \delta_{in} + \delta_{jn}) + 2 \sum_{m>n,n=1}^{3} p_{nm} (1 + \delta_{in} \delta_{jm}) \right],
\]

\[
(D - 4)D_{00000000i} = -\frac{1}{420}, \quad (D - 4)D_{00000000j} = -\frac{1}{1680}, \quad (D - 4)D_{00000000k} = -\frac{1}{2520}, \quad (D - 4)D_{00000000ij} = -\frac{1}{5040}.
\]

D. UV-Divergent Parts of E-Functions and F-Functions

UV-divergent expressions occur for \( n \geq 3 \) (E-functions) and \( n \geq 4 \) (F-functions) respectively, \( p_{ij} \) denotes \((p_i - p_j)^2\).

\[
(D - 4)E_{0000} = -\frac{1}{96}, \quad (D - 4)E_{000000} = \frac{1}{480},
\]

\[
(D - 4)E_{00000000} = \frac{1}{5760} \left[ -6m_0^2 + \sum_{n=1}^{4} (p_n^2 - 6m_n^2) + \sum_{m>n,n=1}^{4} p_{nm} \right],
\]

\[
(D - 4)F_{000000} = -\frac{1}{1440}, \quad (D - 4)F_{00000000} = -\frac{1}{2880},
\]

\[
(D - 4)F_{00000000} = -\frac{1}{960}, \quad (D - 4)F_{0000000000} = \frac{1}{5760},
\]

\[
(D - 4)F_{00000000000} = \frac{1}{80640} \left[ -7m_0^2 + \sum_{n=1}^{5} (p_n^2 - 7m_n^2) + \sum_{m>n,n=1}^{5} p_{nm} \right],
\]

\[
(D - 4)F_{000000000000} = -\frac{1}{20160}, \quad (D - 4)F_{00000000000000} = -\frac{1}{40320}.
\]
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