NEW EXTENSIONS OF POPOVICIU’S INEQUALITY

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Dedicated to the memory of T. Popoviciu.

Abstract. Popoviciu’s inequality is extended to the framework of $h$-convexity and also to convexity with respect to a pair of quasi-arithmetic means. Several applications are included.

1. Introduction

Fifty years ago Tiberiu Popoviciu [23] published the following striking characterization of convex functions:

**Theorem 1.** A real-valued continuous function $f$ defined on an interval $I$ is convex if and only if it verifies the inequality

$$f(x) + f(y) + f(z) + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3} \left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right)$$

whenever $x, y, z \in I$.

He also noticed that inequality (Pop) has higher order analogues for each finite string of points (of length greater than or equal to 3). In [23], only the unweighted case was discussed, but Popoviciu’s argument covers the weighted case as well.

Popoviciu’s result has received a great deal of attention and many improvements and extensions have been obtained. The interested reader may consult the books of Mitrinović [11], Niculescu and Persson [16] and Pečarić, Proschan and Tong [21], as well as the recent papers by Niculescu and his collaborators [4], [10], [13], [16], [17], [18], [19] and [20].

Two easy extensions of Popoviciu’s inequality that escaped unnoticed refer to the case of convex functions with values in a Banach lattice and that of semiconvex functions (i.e., of the functions that become convex after the addition of a suitable smooth function). Using the phenomenon of semiconvexity one can state a Popoviciu type inequality for all functions of class $C^2$ :

**Proposition 1.** Suppose that $f \in C^2 ([a, b])$ and put

$$M = \sup \{f''(x) : x \in [a, b]\} \quad \text{and} \quad m = \inf \{f''(x) : x \in [a, b]\}.$$
Then
\[
\frac{M}{36}((x-y)^2 + (y-z)^2 + (z-x)^2) \geq \\
\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) - \frac{2}{3}\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right) \\
\geq \frac{m}{36}((x-y)^2 + (y-z)^2 + (z-x)^2)
\]
for all \(x, y, z \in [a, b]\).

Indeed, under the assumptions of Proposition 1, both functions \(\frac{M}{2}x^2 - f(x)\) and \(f(x) - \frac{M}{2}x^2\) are convex and Theorem 1 applies. The variant of Proposition 1 for strongly convex functions, that is for those functions \(f\) such that \(f - \frac{C}{2}x^2\) is convex for a suitable \(C > 0\) can be deduced in the same manner:
\[
\frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) - \frac{2}{3}\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right) \\
\geq \frac{C}{36}((x-y)^2 + (y-z)^2 + (z-x)^2).
\]

Since \(e^x \geq \frac{1}{2}x^2\) for \(x \geq 0\), this fact yields the inequality
\[
a + b + c + \sqrt{abc} - \frac{2}{3}\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right) \geq \frac{1}{36}\left(\log^2 \frac{a}{b} + \log^2 \frac{b}{c} + \log^2 \frac{c}{a}\right),
\]
for all \(a, b, c \geq 1\).

The aim of the present paper is to discuss Popoviciu’s inequality in the context of generalized convexity.

The next section deals with the case of convexity with respect to a pair of means. See Definition 1 below for details. Theorem 2 states the analogue of Popoviciu’s inequality in the context of quasi-arithmetic means, and its usefulness is illustrated by the case of the hypergeometric function and the volume function of the unit ball in \(L^p\) spaces of dimension \(n\). A counter-example shows that we cannot expect a full extension of Popoviciu’s inequality to the case of arbitrary convex functions with respect to a pair of means.

Section 3 deals with the case of \(h\)-convex functions in the sense of Varošanec [24]. We end our paper by noticing the availability of Popoviciu’s inequality in the general framework of \(h\)-Jensen pairs of functions.

2. THE CASE OF CONVEX FUNCTIONS RELATIVE TO A PAIR OF MEANS

Convexity relative to a pair of means was first considered by Aumann [3] in 1933, but its serious investigation started not until the 90s. By a mean on an interval \(I\) we understand any function \(M : I \times I \to \mathbb{R}\) such that
\[
\min \{x, y\} \leq M(x, y) \leq \max \{x, y\}
\]
for all \(x, y \in I\). The most used class of means is that of quasi-arithmetic means, which are associated to a continuous and strictly monotonic function \(\varphi : I \to \mathbb{R}\) by the formula
\[
\mathcal{M}_\varphi(x, y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right), \text{ for } x, y \in I.
\]
A particular case is that of power means of order \( p \in \mathbb{R} \),

\[
M_p(x, y) = \begin{cases} 
\min \{x, y\} & \text{if } p = -\infty \\
\left( \frac{x^p + y^p}{2} \right)^{1/p} & \text{if } p \neq 0 \\
\sqrt{xy} & \text{if } p = 0 \\
\max \{x, y\} & \text{if } p = \infty,
\end{cases}
\]

which corresponds to the function \( \varphi(x) = x^p \), if \( p \in \mathbb{R} \setminus \{0\} \) and \( \varphi(x) = \log x \), if \( p = 0 \). Notice that

\[
M_{-1} = H \quad (\text{the harmonic mean}) \\
M_0 = G \quad (\text{the geometric mean}) \\
M_1 = A \quad (\text{the arithmetic mean}).
\]

Remarkably, the quasi-arithmetic means \( M_\varphi \) admit natural extensions to the case of an arbitrary finite family of points \( x_1, \ldots, x_n \) endowed with weights \( \lambda_1, \ldots, \lambda_n \) of total mass \( 1 \),

\[
M_\varphi(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n) = \varphi^{-1} \left( \sum_{k=1}^{n} \lambda_k \varphi(x_k) \right)
\]

In order to simplify the notation, we put \( M_\varphi(x_1, \ldots, x_n; 1/n, \ldots, 1/n) = M_\varphi(x_1, \ldots, x_n) \).

**Definition 1.** Given a pair of intervals \( I \) and \( J \) endowed respectively with the means \( M \) and \( N \), a function \( f : I \to J \) is called \((M, N)\)-convex if it is continuous and

\[
((M, N)) \quad f(M(x, y)) \leq N(f(x), f(y)) \quad \text{for all } x, y \in I.
\]

The analogue of Jensen’s inequality works in the case of \((M_\varphi, M_\psi)\)-convex functions, so that for such functions we have

\[
f(M_\varphi(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n)) \leq M_\psi(f(x_1), \ldots, f(x_n); \lambda_1, \ldots, \lambda_n)
\]

for all \( x_1, \ldots, x_n \in I \) and \( \lambda_1, \ldots, \lambda_n \in [0, 1] \) with \( \sum \lambda_k = 1 \).

Clearly, the usual convex functions represent the case of \((A, A)\)-convex functions, while the log-convex functions are the same with \((A, G)\)-convex functions.

The importance and significance of other classes of generalized convex functions such as of \((G, A)\)-convex functions, \((G, G)\)-convex functions, \((H, A)\)-convex functions etc. is discussed in the book [15] and the paper of Anderson, M.K. Vamananurthy, M. Vuoirm [2].

Not all important means are quasi-arithmetic. Two examples are the **logarithmic mean**,

\[
L(a, b) = \begin{cases} 
\frac{a-b}{\ln a - \ln b} & \text{if } a \neq b \\
\frac{a}{a} & \text{if } a = b
\end{cases}
\]

and the **identric mean**,\n
\[
I(a, b) = \begin{cases} 
\frac{1}{e} \left( \frac{e^b}{a} \right)^{b-a} & \text{if } a \neq b \\
\frac{1}{a} & \text{if } a = b.
\end{cases}
\]

The theory of \((M_\varphi, M_\psi)\)-convex functions can be deduced from the theory of usual convex functions.
Lemma 1. (J. Aczel [1]). Let \( \varphi \) and \( \psi \) be two strictly monotonic functions defined respectively on the intervals \( I \) and \( J \), and let \( f : I \to J \) be an arbitrary function.

If \( \psi \) is strictly increasing, then \( f \) is \((\mathcal{M}_\varphi, \mathcal{M}_\psi)\)-convex/concave if and only if \( \psi \circ f \circ \varphi^{-1} \) is convex/concave on \( \varphi(I) \) in the usual sense.

If \( \psi \) is strictly decreasing, then \( f \) is \((\mathcal{M}_\varphi, \mathcal{M}_\psi)\)-convex/concave if and only if \( \psi \circ f \circ \varphi^{-1} \) is concave/convex on \( \varphi(I) \) in the usual sense.

Theorem 2. Suppose that \( f : I \to J \) is an \((\mathcal{M}_\varphi, \mathcal{M}_\psi)\)-convex function. If \( \psi \) is strictly increasing, then

\[
\mathcal{M}_\psi \left( f(x), f(y), f(z) \right) \leq \mathcal{M}_\psi \left( f(x), f(y), f(z) \right) f(\mathcal{M}_\varphi(x, y, z))
\]

for all \( x, y, z \in I \).

The inequality works in the reverse sense if the function \( \psi \) is strictly decreasing.

Proof. By Lemma 1, the function \( \psi \circ f \circ \varphi^{-1} \) is convex on the interval \( \varphi(I) \) so that one can apply Popoviciu’s inequality to it relative to the points \( a = \varphi(x), b = \varphi(y) \) and \( c = \varphi(z) \).

Then

\[
\frac{1}{3} \left( \psi \circ f \circ \varphi^{-1}(x) \right) + \frac{1}{3} \left( \psi \circ f \circ \varphi^{-1}(y) \right) + \frac{1}{3} \left( \psi \circ f \circ \varphi^{-1}(z) \right)
\]

\[
\geq \frac{2}{3} \left( \frac{\varphi(x) + \varphi(y) + \varphi(z)}{2} \right)
\]

that is,

\[
\frac{1}{2} \left( \psi(f(x)) + \psi(f(y)) + \psi(f(z)) \right) \geq \psi \left( \frac{\mathcal{M}_\varphi(x, y, z)}{3} \right).
\]

On the other hand,

\[
\frac{1}{3} \left( \psi(f(x)) + \psi(f(y)) + \psi(f(z)) \right) = \psi \left( \frac{1}{3} \left( \psi(f(x)) + \psi(f(y)) + \psi(f(z)) \right) \right)
\]

and the proof ends by applying \( \psi^{-1} \) to both sides. \( \square \)

This result can be extended to the case of an arbitrary finite family of points and weighted quasi-arithmetic means, but the details are tedious and will be omitted.

Example 1. The Gaussian hypergeometric function (of parameters \( a, b, c > 0 \)) is defined via the formula

\[
F(x) = {}_2 F_1 (a; b, c) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} x^n \quad \text{for} \ |x| < 1,
\]
where \((a, n) = a(a + 1) \cdots (a + n - 1)\) if \(n \geq 1\) and \((a, 0) = 1\). Anderson, Vamanamurthy and Vuorinen [2] proved that if \(a + b \geq c > 2ab\) and \(c \geq a + b - 1/2\), then the function \(1/F(x)\) is concave on \((0, 1)\). This implies

\[
F\left(\frac{x + y}{2}\right) \leq \frac{1}{\frac{1}{F(x)} + \frac{1}{F(y)}}\]

for all \(x, y \in (0, 1)\), whence it follows that the hypergeometric function is \((A, H)\)-convex. Taking into account that the harmonic mean is a quasi-arithmetic mean corresponding to the strictly decreasing function \(1/x\), we infer that \(F\) verifies the following analogue of Popoviciu’s inequality:

\[
\frac{1}{3} \left( \frac{1}{F(x)} + \frac{1}{F(y)} + \frac{1}{F(z)} \right) \leq \frac{1}{\frac{1}{F(\frac{x+y}{2})} + \frac{1}{F(\frac{x+z}{2})} + \frac{1}{F(\frac{y+z}{2})}}
\]

equivalently,

\[
\frac{1}{2} \left( \frac{1}{3} \left( \frac{1}{F(x)} + \frac{1}{F(y)} + \frac{1}{F(z)} \right) \right) \geq \frac{1}{3} \left( \frac{1}{F(\frac{x+y}{2})} + \frac{1}{F(\frac{x+z}{2})} + \frac{1}{F(\frac{y+z}{2})} \right).
\]

**Example 2.** D. Borwein, J. Borwein, G. Fee and R. Girgensohn [5] proved that the volume \(V_n(p)\) of the convex body \(E = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}\) is an \((H, G)\)-concave function on \([1, \infty)\). More precisely, given \(\alpha > 1\), the function

\[
V_\alpha(p) = 2^\alpha \frac{\Gamma(1 + 1/p)}{\Gamma(1 + \alpha/p)}
\]

verifies the inequality

\[
V_\alpha^{1-\lambda}(p)V_\alpha^\lambda(q) \leq V_\alpha \left( \frac{1}{\frac{1}{p} + \frac{\lambda}{q}} \right)
\]

for all \(p, q > 0\) and \(\lambda \in [0, 1]\). In this case, Popoviciu’s inequality becomes

\[
\sqrt[3]{V_\alpha(p)V_\alpha(q)V_\alpha(r)} \cdot V_\alpha \left( \frac{1}{\frac{1}{p} + \frac{\lambda}{q} + \frac{1}{r}} \right) \geq \sqrt[3]{V_\alpha \left( \frac{1}{\frac{1}{p} + \frac{\lambda}{q}} \right) \cdot V_\alpha \left( \frac{1}{\frac{1}{q} + \frac{\lambda}{r}} \right) \cdot V_\alpha \left( \frac{1}{\frac{1}{r} + \frac{\lambda}{p}} \right)}.
\]

A natural question is whether Popoviciu’s inequality works for an arbitrary \((M, N)\)-convex function.
We shall see that the answer is negative. Indeed, the log-convex functions are also \((A, L)\)-convex, because they verify the inequalities
\[
f\left(\frac{a + b}{2}\right) \leq \exp\left(\frac{1}{b-a} \int_a^b \log f(x) \, dx\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq L(f(a), f(b)) \leq \frac{f(a) + f(b)}{2}.
\]
See [14].

The logarithmic mean was extended to the case of an arbitrary finite family of points by Neuman in his paper [12]. An argument that Neuman’s extension is the “right” one can be found in [14]. For triplets, the logarithmic mean is given by the formula
\[
L(a, b, c) = \frac{2a}{\log \frac{a}{b} \log \frac{b}{c}} + \frac{2b}{\log \frac{b}{c} \log \frac{c}{a}} + \frac{2c}{\log \frac{c}{a} \log \frac{a}{b}}.
\]

The analogue of Popoviciu’s inequality in the case of \((A, L)\)-convex functions should be
\[
\frac{L(f(x), f(y), f(z)) - f\left(\frac{x + y + z}{3}\right)}{\log L(f(x), f(y), f(z)) - \log f\left(\frac{x + y + z}{3}\right)} \geq L\left(\frac{f\left(\frac{x + y}{2}\right)}{2}, \frac{f\left(\frac{y + z}{2}\right)}{2}, \frac{f\left(\frac{z + x}{2}\right)}{2}\right),
\]
for all \(x, y, z\) belonging to the domain of \(f\). However this does not work even in the case of the Gamma function,
\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \quad x > 0,
\]
that is known to be log-convex (see [15], Theorem 2.2.1, pp. 68-69). The Gamma function has a minimum at 1.461632..., so we will search around this point. Put
\[
E(x; y; z) = \frac{L\left(\Gamma(x), \Gamma(y), \Gamma(z)\right) - \Gamma\left(\frac{x + y + z}{3}\right)}{\log L\left(\Gamma(x), \Gamma(y), \Gamma(z)\right) - \log \Gamma\left(\frac{x + y + z}{3}\right)}
\]
\[
- L\left(\Gamma\left(\frac{x + y}{2}\right), \Gamma\left(\frac{y + z}{2}\right), \Gamma\left(\frac{z + x}{2}\right)\right),
\]
for \(x, y, z > 0\). A simple computation shows that
\[
E(1.40; 1.46; 1.47) = 65.92090117 - 108.64 < 0
\]
while
\[
E(0.30; 0.34; 0.35) = 2.711369453 - 2.709270 > 0.
\]
Therefore Popoviciu’s inequality does not always work for \((M, N)\)-convex functions.
3. The case of h-convex functions

In 2007, Varošanec [24] introduced a class of generalized convex functions that brings together several important classes of functions.

In order to enter into the details we have to fix a function $h : (0, 1) \to (0, \infty)$ such that

$$h(1 - \lambda) + h(\lambda) \geq 1 \text{ for all } \lambda \in (0, 1).$$

As above, $I$ will denote an interval.

**Definition 2.** A function $f : I \to \mathbb{R}$ is called $h$-convex if

$$f((1 - \lambda)x + \lambda y) \leq h(1 - \lambda)f(x) + h(\lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in (0, 1)$.

The role of the condition (3.1) is to assure that the function identically 1 is $h$-convex.

The usual convex functions represent the particular case of Definition 2 where $h$ is the identity function.

The $h$-convex functions corresponding to the case $h(\lambda) = \lambda^s$ (for a suitable $s \in (0, 1]$) are the $s$-convex functions in the sense of Breckner [6]. Their systematic study can be found in the papers of Hudzik and Maligranda [9] and Pinheiro [22].

An example of an $s$-convex function (for $0 < s < 1$) is given by the formula

$$f(t) = \begin{cases} a & \text{if } t = 0 \\ bt^s + c & \text{if } t > 0 \end{cases}$$

where $b \geq 0$ and $0 \leq c \leq a$. In particular, the function $t^s$ is $s$-convex on $[0, \infty)$ if $0 < s < 1$.

The nonnegative $h$-convex functions corresponding to the case $h(\lambda) = \lambda^s$ are the convex functions in the sense of Godunova-Levin [8]. They verify the inequality

$$f((1 - \lambda)x + \lambda y) \leq \frac{f(x)}{1 - \lambda} + \frac{f(y)}{\lambda}$$

for all $x, y \in I$ and $\lambda \in (0, 1)$. Every nonnegative monotonic function (as well as every nonnegative convex function) is convex in the sense of Godunova-Levin.

The $h$-convex functions corresponding to the case $h(\lambda) \equiv 1$ are the $P$-convex functions in the sense of Dragomir, Pečarić and Persson [7]. They verify inequalities of the form

$$f((1 - \lambda)x + \lambda y) \leq f(x) + f(y)$$

for all $x, y \in I$ and $\lambda \in (0, 1)$.

One can state the following analogue of Popoviciu’s inequality in the case of $h$-convex functions.

**Theorem 3.** If $h$ is concave, then every nonnegative $h$-convex function $f : I \to \mathbb{R}$ verifies the inequality

$$(hPop) \quad \max \{h(1/2), 2h(1/4)\} (f(x) + f(y) + f(z)) + 2h(3/4)f\left(\frac{x + y + z}{3}\right) \geq f\left(\frac{x + y}{2}\right) + f\left(\frac{y + z}{2}\right) + f\left(\frac{z + x}{2}\right)$$

for all $x, y, z \in I$. 
Proof. Without loss of generality we may assume that \( x \leq y \leq z \). If \( y \leq (x + y + z)/3 \), then
\[
(x + y + z)/3 \leq (x + z)/2 \leq z \quad \text{and} \quad (x + y + z)/3 \leq (y + z)/2 \leq z,
\]
which yields two numbers \( s, t \in [0, 1] \) such that
\[
\begin{align*}
\frac{x + z}{2} &= s \cdot \frac{x + y + z}{3} + (1 - s) \cdot z \\
\frac{y + z}{2} &= t \cdot \frac{x + y + z}{3} + (1 - t) \cdot z.
\end{align*}
\]
Summing up, we get \((x + y - 2z)(s + t - 3/2) = 0\). If \( x + y - 2z = 0 \), then necessarily \( x = y = z \), and the inequality \((h\text{Pop})\) is clear. If \( s + t = 3/2 \), by summing up the following three inequalities
\[
\begin{align*}
f \left( \frac{x + z}{2} \right) &\leq h(s) \cdot f \left( \frac{x + y + z}{3} \right) + h(1 - s) \cdot f(z) \\
f \left( \frac{y + z}{2} \right) &\leq h(t) \cdot f \left( \frac{x + y + z}{3} \right) + h(1 - t) \cdot f(z) \\
f \left( \frac{x + y}{2} \right) &\leq h(1/2) \cdot f(x) + h(1/2) \cdot f(y).
\end{align*}
\]
we get
\[
\begin{align*}
f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right) \\
&\leq (h(s) + h(t)) \cdot f \left( \frac{x + y + z}{3} \right) \\
&\quad + h(1/2) \cdot f(x) + h(1/2) \cdot f(y) + h(1 - s) \cdot f(z) + 2h(1/4) \cdot f(z) + 2h(3/4) f \left( \frac{x + y + z}{3} \right) \\
&\leq h(1/2) \cdot f(x) + h(1/2) \cdot f(y) + 2h(1/4) \cdot f(z) + 2h(3/4) f \left( \frac{x + y + z}{3} \right) \\
&\leq \max \{ h(1/2), 2h(1/4) \} \{ f(x) + f(y) + f(z) \} + 2h(3/4) f \left( \frac{x + y + z}{3} \right),
\end{align*}
\]
and the inequality \((h\text{Pop})\) is also clear.

The case where \((x + y + z)/3 < y\) can be treated in a similar way. \qed

As an application of Theorem 3 let us consider the case of the function \( t^{1/2} \) (which is \( s \)-convex for \( s = 1/2 \)). Then \( h(t) = t^{1/2} \), \( \max \{ h(1/2), 2h(1/4) \} = 1 \) and \( 2h(3/4) = \sqrt{3} \), which yields
\[
x^{1/2} + y^{1/2} + z^{1/2} + \sqrt{3} \left( \frac{x + y + z}{3} \right)^{1/2} \\
\geq \left( \frac{x + y}{2} \right)^{1/2} + \left( \frac{y + z}{2} \right)^{1/2} + \left( \frac{z + x}{2} \right)^{1/2}
\]
for all \( x, y, z \geq 0 \).

We end our paper with another Popoviciu type inequality for \( h \)-convex functions.

The basic ingredient is the Jensen-type inequality for the \( h \)-convex functions,
\[
f \left( \frac{x_1 + \cdots + x_n}{n} \right) \leq h \left( \frac{1}{n} \right) \left( f(x_1) + \cdots + f(x_n) \right)
\]
valid for arbitrary finite strings of points \( x_1, ..., x_n \) under the additional hypothesis that \( h \) is supermultiplicative in the sense that \( h(xy) \geq h(x)h(y) \) for all \( x, y \). See [24], Theorem 19. When \( f \) is \( h \)-concave and \( h \) is submultiplicative, the Jensen inequality takes the form

\[
 f \left( \frac{x_1 + \cdots + x_n}{n} \right) \geq h \left( \frac{1}{n} \right) (f(x_1) + \cdots + f(x_n)).
\]

**Theorem 4.** i) If \( h \) is supermultiplicative, with \( h(1/3) < 1 \), and \( f : I \to \mathbb{R} \) is an \( h \)-convex function, then

\[
 f(x) + f(y) + f(z) - f \left( \frac{x + y + z}{3} \right) \geq \frac{1 - h(1/3)}{2h(1/2)} \left( f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right) \right)
\]

for all \( x, y, z \in I \).

ii) If \( h \) is submultiplicative, with \( h(1/3) > 1 \), and \( f : I \to \mathbb{R} \) is an \( h \)-concave function, then

\[
 f \left( \frac{x + y + z}{3} \right) - (f(x) + f(y) + f(z)) \geq \frac{h(1/3) - 1}{2h(1/2)} \left( f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right) \right).
\]

**Proof.** i) In this case,

\[
 f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right) \leq 2h(1/2)(f(x) + f(y) + f(z))
\]

\[
 = \frac{2h(1/2)}{1 - h(1/3)} (f(x) + f(y) + f(z)) - \frac{2h(1/2)}{1 - h(1/3)} h(1/3) (f(x) + f(y) + f(z))
\]

\[
 \leq \frac{2h(1/2)}{1 - h(1/3)} (f(x) + f(y) + f(z)) - \frac{2h(1/2)}{1 - h(1/3)} f \left( \frac{x + y + z}{3} \right)
\]

\[
 = \frac{2h(1/2)}{1 - h(1/3)} \left( f(x) + f(y) + f(z) - f \left( \frac{x + y + z}{3} \right) \right).
\]

ii) Similarly,

\[
 f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right) \geq 2h(1/2)(f(x) + f(y) + f(z))
\]

\[
 = \frac{2h(1/2)h(1/3)}{h(1/3) - 1} (f(x) + f(y) + f(z)) - \frac{2h(1/2)h(1/3)}{h(1/3) - 1} (f(x) + f(y) + f(z))
\]

\[
 \geq \frac{2h(1/2)}{h(1/3) - 1} \left( f \left( \frac{x + y + z}{3} \right) - (f(x) + f(y) + f(z)) \right).
\]

As an application of Theorem 4, let us consider the case of the function \( t^{1/2} \) (which is \( s \)-convex for \( s = 1/2 \)). Then \( h(t) = t^{1/2} \) and \( h(1/3) = (1/3)^{1/2} = 0.577... < 1 \).
Therefore
\[
x^{1/2} + y^{1/2} + z^{1/2} - \left( \frac{x + y + z}{3} \right)^{1/2} \\
\geq \frac{1}{2} - \frac{(1/3)^{1/2}}{2} \left[ \left( \frac{x + y}{2} \right)^{1/2} + \left( \frac{y + z}{2} \right)^{1/2} + \left( \frac{z + x}{2} \right)^{1/2} \right]
\]
for all \( x, y, z \geq 0 \).

Last but not least it is worth noticing that Popoviciu’s inequality still works in the more general context of \( h \)-Jensen pairs \((f, g)\). These pairs are aimed to satisfy inequalities of the form
\[
f ((1 - \lambda)x + \lambda y) \leq h(1 - \lambda)g(x) + h(\lambda)g(y),
\]
for all \( x, y \in I \) and \( \lambda \in (0, 1) \); here \( I \) is a common domain of \( f \) and \( g \). An inspection of the argument of Theorem 3 easily yields the following result.

**Theorem 5.** Let \( h \) be concave and \((f, g)\) be an \( h \)-Jensen pair of positive functions \( f, g : I \to \mathbb{R} \). Then a Popoviciu type inequality holds:
\[
\max \left\{ h \left( \frac{1}{2} \right), 2h \left( \frac{1}{4} \right) \right\} (g(x) + g(y) + g(z)) + 2h \left( \frac{3}{4} \right)g \left( \frac{x + y + z}{3} \right) \\
\geq f \left( \frac{x + y}{2} \right) + f \left( \frac{y + z}{2} \right) + f \left( \frac{z + x}{2} \right).
\]

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