A Recursion Operator for the Universal Hierarchy Equation via Cartan’s Method of Equivalence

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Abstract. We apply Cartan’s method of equivalence to find a Bäcklund autotransformation for the tangent covering of the universal hierarchy equation. The transformation provides a recursion operator for symmetries of this equation.

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1. Introduction

Recursion operators provide an important tool for the study of nonlinear partial differential equations PDEs. They are linear operators acting on a symmetries of a PDE and generating infinite hierarchies of new (nonlocal) symmetries. The presence of infinite series of symmetries allows one to apply different techniques to study a given PDE, see [53, 17, 21, 38, 2] and references therein. Typically, recursion operators are considered as integro-differential operators, [37, 12, 56, 10, 9, 11, 47, 1, 38, 16, 48, 46, 55], although this interpretation is accompanied by a number of difficulties, e.g., discussed in [14, 23]. Another definition is proposed in [40, 14, 19, 20] (see also [44] and references therein) and developed in [27, 48, 32, 28, 29, 31, 30]. It treats recursion operators as the Bäcklund autotransformations of the tangent (or linearized) coverings of the PDEs. In [33], M. Marvan and A. Sergyeyev proposed the method for constructing recursion operators of PDEs of any dimension from their linear coverings of a special form. By this method they found recursion operators for a number of PDEs of physical and geometrical significance.

In the present paper we adapt the technique of [36] for the problem of finding recursion operators of PDEs. This technique is based on Élie Cartan’s structure theory of Lie pseudo-groups, [4, 7, 52, 51], and allows one to find coverings and Bäcklund transformations for nonlinear PDEs by means of contact integrable extensions of their symmetry pseudo-groups. We consider the universal hierarchy equation [45, 25, 26]

\[ u_{yy} = u_y u_{tx} - u_x u_{ty}. \]  

In accordance with [22, §5] we identify the tangent covering for (1) with the system of PDEs constituted by (1) and the defining equation for its symmetries

\[ v_{yy} = u_y v_{tx} + v_y u_{tx} - u_x v_{ty} - v_x u_{ty}. \]  

The standard procedures of Élie Cartan’s method of equivalence give Maurer–Cartan forms and the structure equations of the pseudo-group of contact symmetries for system (1), (2). Then we find a contact integrable extension for these structure equations. The corresponding contact form provides the Bäcklund auto-transformation for Eq. (2). This transformation defines a recursion operator for symmetries of Eq. (1) and its inverse operator.

2. Preliminaries

2.1. Coverings of PDEs

Let \( \pi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), \( \pi: (x^1, \ldots, x^n, u^1, \ldots, u^m) \mapsto (x^1, \ldots, x^n) \), be a trivial bundle, and \( J^\infty(\pi) \) be the bundle of its jets of the infinite order. The local coordinates on \( J^\infty(\pi) \) are \( (x^i, u^a, u^a_I) \), where \( I = (i_1, \ldots, i_n) \) is a multi-index, and for every local section \( f: \mathbb{R}^n \to \mathbb{R}^n \) the corresponding infinite jet \( j_\infty(f) \) is a section \( j_\infty(f): \mathbb{R}^n \to J^\infty(\pi) \) such that \( u^a_I(j_\infty(f)) = \frac{\partial^{\#I} f^\alpha}{\partial x^I} = \frac{\partial^{i_1 + \ldots + i_n} f^\alpha}{(\partial x^1)^{i_1} \ldots (\partial x^n)^{i_n}} \). We put...
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where \( u^\alpha = u^\alpha_{(0,\ldots,0)} \). Also, in the case of \( n = 3, m = 1 \) we denote \( x^1 = t, \ x^2 = x, \ x^3 = y \), and \( u^1_{(i,j,k)} = u_{t,x,x,y-y} \) with \( i \) times \( t, \ j \) times \( x, \) and \( k \) times \( y \).

The vector fields

\[
D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I \geq 0} \sum_{\alpha=1}^m u^\alpha_I \frac{\partial}{\partial u^\alpha_I}, \quad k \in \{1, \ldots, n\},
\]

\((i_1, \ldots, i_k, \ldots, i_n) + 1_k = (i_1, \ldots, i_k + 1, \ldots, i_n)\), are called total derivatives. They commute everywhere on \( J^{\infty}(\pi) \): \([D_{x^i}, D_{x^j}] = 0\).

The evolutionary differentiation associated to an arbitrary vector-valued smooth function \( \varphi : J^{\infty}(\pi) \to \mathbb{R}^m \) is the vector field

\[
E_{\varphi} = \sum_{\#I \geq 0} \sum_{\alpha=1}^m D_I(\varphi^\alpha) \frac{\partial}{\partial u^\alpha_I}, \quad (3)
\]

with \( D_I = D_{(i_1, \ldots, i_n)} = D_{i_1} \circ \ldots \circ D_{i_n} \).

A system of pdes \( F_r(x^i, u^\alpha_I) = 0, \ #I \leq s, \ r \in \{1, \ldots, R\} \), of the order \( s \geq 1 \) with \( R \geq 1 \) defines the submanifold \( \mathcal{E} = \{(x^i, u^\alpha_I) \in J^{\infty}(\pi) \mid D_K(F_r(x^i, u^\alpha_I)) = 0, \ #K \geq 0\} \) in \( J^{\infty}(\pi) \).

A function \( \varphi : J^{\infty}(\pi) \to \mathbb{R}^m \) is called a \((\text{generator of an infinitesimal})\) symmetry of \( \mathcal{E} \) when \( E_{\varphi}(F) = 0 \) on \( \mathcal{E} \). The symmetry \( \varphi \) is a solution to the defining system

\[
\ell_{\mathcal{E}}(\varphi) = 0, \quad (4)
\]

where \( \ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}} \) with the matrix differential operator

\[
\ell_F = \left( \sum_{\#I \geq 0} \frac{\partial F_r}{\partial u^\alpha_I} D_I \right).
\]

Denote \( \mathcal{W} = \mathbb{R}^\infty \) with coordinates \( w^s, \ s \in \mathbb{N} \cup \{0\} \). Locally, an \((\text{infinite-dimen-}\) sional) differential covering of \( \mathcal{E} \) is a trivial bundle \( \tau : J^{\infty}(\pi) \times \mathcal{W} \to J^{\infty}(\pi) \) equipped with the extended total derivatives

\[
\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^{\infty} T^s_k(x^i, u^\alpha_I, w^j) \frac{\partial}{\partial w^s}, \quad (5)
\]

such that \([\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0\) for all \( i \neq j \) whenever \((x^i, u^\alpha_I) \in \mathcal{E}\). We define the partial derivatives of \( w^s \) by \( w^s_{x^k} = \tilde{D}_{x^k}(w^s) \). This yields the system of covering equations

\[
w^s_{x^k} = T^s_k(x^i, u^\alpha_I, w^j). \quad (6)
\]

This over-determined system of pdes is compatible whenever \((x^i, u^\alpha_I) \in \mathcal{E}\).

Denote by \( \tilde{E}_{\varphi} \) the result of substitution \( \tilde{D}_{x^k} \) for \( D_{x^k} \) in \((3)\). A \text{shadow of nonlocal symmetry} of \( \mathcal{E} \) corresponding to the covering \( \tau \) with the extended total derivatives \((5)\), or \( \tau\text{-shadow}, \) is a function \( \varphi \in C^\infty(\mathcal{E} \times \mathcal{W}) \) such that

\[
\tilde{E}_{\varphi}(F) = 0 \quad (7)
\]
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is a consequence of equations $D_K(F) = 0$ and (6). A *nonlocal symmetry* of $E$ corresponding to the covering $\tau$ (or $\tau$-symmetry) is the vector field

$$
\tilde{E}_{\varphi,A} = \tilde{E}_{\varphi} + \sum_{s=0}^{\infty} A^s \frac{\partial}{\partial w_s},
$$

with $A^s \in C^\infty(E \times W)$ such that $\varphi$ satisfies to (7) and

$$
\tilde{D}_k(A^s) = \tilde{E}_{\varphi,A}(T_k^s)
$$

for $T_k^s$ from (5), see [2, Ch. 6, §3.2].

**Remark 1.** In general, not every $\tau$-shadow corresponds to a $\tau$-symmetry, since Eqns. (2) provide an obstruction for existence of (5). But for any $\tau$-shadow $\varphi$ there exists a covering $\tau_{\varphi}$ and a nonlocal $\tau_{\varphi}$-symmetry whose $\tau_{\varphi}$-shadow coincides with $\varphi$, see [2, Ch. 6, §5.8].

A *recursion operator* $R$ for $E$ is a $\mathbb{R}$-linear map such that for each (local or nonlocal) symmetry $\varphi$ of $E$ the function $R(\varphi)$ is a (local or nonlocal) symmetry of $\varphi$ of $E$.

The tangent covering for pde $E$ is defined as follows, [22]. Consider the trivial bundle $\sigma: J^\infty(\pi) \times V \to J^\infty(\pi)$ with coordinates $v_I^\alpha$, $\#I \geq 0$, on the fiber $V$ equipped with the extended total derivatives

$$
\hat{D}_x^k = D_x^k + \sum_{\#I \geq 0} \sum_{\alpha=1}^{m} v_I^\alpha \frac{\partial}{\partial v_I^\alpha}.
$$

Then for $\hat{D}_I = \hat{D}_{x_1}^1 \circ \ldots \circ \hat{D}_{x_n}^n$ define

$$
\hat{\ell}_F = \left( \sum_{\#I \geq 0} \frac{\partial F_r}{\partial v_I^\alpha} \hat{D}_I \right),
$$

and put

$$
\mathcal{T}(E) = \{(x^i, u_i^\alpha, v_I^\alpha) \in J^\infty(\pi) \times V \mid D_K(F(x^i, u_i^\alpha)) = 0, \hat{D}_K(\hat{\ell}_F(v^\alpha)) = 0, \#K \geq 0\}.
$$

The *tangent covering* is the restriction of $\sigma$ to $\mathcal{T}(E)$. A section $\varphi: E \to \mathcal{T}(E)$ of the tangent covering is a symmetry of $E$. The extended total derivatives of this covering are

$$
\tilde{D}_x^k = \hat{D}_x^k |_{\mathcal{T}(E)}.
$$

**Example 1.** We write Eq. (1) in the form $u_{yy} - u_y u_{tx} + u_x u_{ty} = 0$. Then we have

$$
\ell_F(\varphi) = D_y^2(\varphi) - u_y D_t D_x(\varphi) - u_{tx} D_y(\varphi) + u_x D_t D_y(\varphi) + u_{ty} D_x(\varphi)
$$

and

$$
\hat{\ell}_F(v) = v(0,0,2) - u_y v(1,1,0) - u_{tx} v(0,0,1) + u_x v(1,0,1) + u_{ty} v(0,1,0).
$$
The fiber of the tangent covering has local coordinates \( v_{(i,j,0)} \) and \( v_{(i,j,1)} \). The extended total derivatives of the tangent covering are

\[
\begin{align*}
\tilde{D}_t &= D_t + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( v_{(i+1,j,0)} \frac{\partial}{\partial v_{(i,j,0)}} + v_{(i+1,j,1)} \frac{\partial}{\partial v_{(i,j,1)}} \right), \\
\tilde{D}_x &= D_x + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( v_{(i,j+1,0)} \frac{\partial}{\partial v_{(i,j,0)}} + v_{(i,j+1,1)} \frac{\partial}{\partial v_{(i,j,1)}} \right), \\
\tilde{D}_y &= D_y + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v_{(i,j,1)} \frac{\partial}{\partial v_{(i,j,0)}} \\
&\quad + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{D}_x^i \tilde{D}_x^j \left( u_y v_{(1,1,0)} + u_{tx} v_{(0,0,1)} - u_x v_{(1,0,1)} - u_{ty} v_{(0,1,0)} \right) \frac{\partial}{\partial v_{(i,j,1)}}.
\end{align*}
\]

**Remark 2.** Abusing the notation, we write \( v_{t...tx...xy...y} \) with \( i \) times \( t \), \( j \) times \( x \), \( k \) times \( y \) instead of \( v_{(i,j,k)} \) in what follows. Also, we identify the tangent covering with the coverings equations \( \hat{\ell}_F(v) = 0 \), i.e., with Eqs. (1), (2).

### 2.2. Cartan's structure theory of Lie pseudo-groups

Let \( M \) be a manifold of dimension \( n \). A local diffeomorphism on \( M \) is a diffeomorphism \( \Phi: \mathcal{U} \to \hat{\mathcal{U}} \) of two open subsets of \( M \). A pseudo-group \( \mathfrak{G} \) on \( M \) is a collection of local diffeomorphisms of \( M \), which is closed under composition when defined, contains an identity and is closed under inverse. A Lie pseudo-group is a pseudo-group whose diffeomorphisms are local analytic solutions of an involutive system of partial differential equations called defining system.

Elie Cartan’s approach to Lie pseudo-groups is based on a possibility to characterize transformations from a pseudo-group in terms of a set of invariant differential 1-forms called Maurer–Cartan (MC) forms. In a general case, MC forms \( \omega^1, ... , \omega^m \) of an infinite-dimensional Lie pseudo-group \( \mathfrak{G} \) are defined on a direct product \( M \times \hat{M} \times G \), where \( \hat{M} \) is the coordinate space of parameters of prolongation. [39, Ch. 12], \( G \) is a finite-dimensional Lie group, and \( m = \dim M + \dim \hat{M} \). The forms \( \omega^i \) are independent and include differentials of coordinates on \( M \times \hat{M} \) only, while their coefficients depend also on coordinates of \( G \). These forms characterize the pseudo-group \( \mathfrak{G} \) in the following sense: a local diffeomorphism \( \Phi: \mathcal{U} \to \hat{\mathcal{U}} \) on \( M \) belongs to \( \mathfrak{G} \) whenever there exists a local diffeomorphism \( \Psi: \mathcal{W} \to \hat{\mathcal{W}} \) on \( M \times \hat{M} \times G \) such that \( \rho \circ \Psi = \Phi \circ \rho \) for the projection \( \rho: M \times \hat{M} \times G \to M \) and the forms \( \omega^j \) are invariant w.r.t. \( \Psi \), that is,

\[
\Psi^*(\omega^j|_{\mathcal{W}}) = \omega^j|_{\mathcal{W}}.
\]  

(10)

Expressions for the exterior differentials of the forms \( \omega^i \) in terms of themselves give Cartan’s structure equations of \( \mathfrak{G} \):

\[
d\omega^i = A^i_{\gamma j} \pi^\gamma \wedge \omega^j + B^i_{jk} \omega^j \wedge \omega^k, \quad B^i_{jk} = -B^i_{kj}.
\]  

(11)

The forms \( \pi^\gamma, \gamma \in \{1, ... , \dim G\} \), are linear combinations of MC forms of the Lie group \( G \) and the forms \( \omega^i \). The coefficients \( A^i_{\gamma j} \) and \( B^i_{jk} \) are either constants or functions.
of a set of invariants $U^\kappa: M \to \mathbb{R}$, $\kappa \in \{1, \ldots, l\}$, $l < \dim M$, of the pseudo-group $\mathcal{G}$, so $\Phi^*(U^\kappa|_U) = U^\kappa|_U$ for every $\Phi \in \mathcal{G}$. In the latter case, the differentials of $U^\kappa$ are invariant 1-forms, so they are linear combinations of the forms $\omega^j$,

$$dU^\kappa = C^\kappa_j \omega^j,$$

where the coefficients $C^\kappa_j$ depend on the invariants $U^1, \ldots, U^l$ only.

Eqs. (11) must be compatible in the following sense: we have

$$d(d\omega^i) = 0 = d(A^i_{\gamma j} \pi^\gamma \wedge \omega^j + B^i_{jk} \omega^j \wedge \omega^k),$$

therefore there must exist expressions

$$d\pi^\gamma = W^\gamma_j \lambda^\lambda \wedge \omega^j + X^\gamma_{\beta \epsilon} \pi^\beta \wedge \pi^\epsilon + Y^\gamma_j \pi^\beta \wedge \omega^j + Z^\gamma_{jk} \omega^j \wedge \omega^k,$$

with some additional 1-forms $\chi^\lambda$ such that the right-hand side of (13) is identically equal to zero after substituting for (11), (12), and (14). Also, from (12) it follows that the right-hand side of the equation

$$d(dU^\kappa) = 0 = d(C^\kappa_j \omega^j)$$

must be identically equal to zero after substituting for (11) and (12).

The forms $\pi^\gamma$ are not invariant w.r.t. the pseudo-group $\mathcal{G}$. Respectively, the structure equations (11) are not changing when replacing $\pi^\gamma \mapsto \pi^\gamma + z^\gamma_j \omega^j$ for certain parametric coefficients $z^\gamma_j$. The dimension $r^{(1)}$ of the linear space of these coefficients satisfies the following inequality

$$r^{(1)} \leq n \dim G - \sum_{k=1}^{n-1} (n - k) s_k,$$

where the reduced characters $s_k$ are defined by the formulas

$$s_1 = \max_{u_1 \in \mathbb{R}^n} \text{rank } A_1(u_1),$$

$$s_k = \max_{u_1, \ldots, u_k \in \mathbb{R}^n} \text{rank } A_k(u_1, \ldots, u_k) - \sum_{j=1}^{k-1} s_j, \quad k \in \{1, \ldots, n - 1\},$$

$$s_n = \dim G - \sum_{j=1}^{n-1} s_j,$$

with the matrices $A_k$ inductively defined by

$$A_1(u_1) = (A^i_{\gamma j} u^j), \quad A_l(u_1, \ldots, u_l) = \left( \begin{array}{c} A_{l-1}(u_1, \ldots, u_{l-1}) \\ A^i_{\gamma j} u^j \end{array} \right), \quad l \in \{2, \ldots n - 1\},$$

see [4, §5], [39, Def. 11.4] for the full discussion. The system of forms $\omega^k$ is involutive when both sides of (16) are equal, [4, §6], [39, Def. 11.7].

Cartan’s fundamental theorems, [4, §§16, 22–24], [7], [52, §§16, 19, 20, 25, 26], [51, §§14.1–14.3], state that for a Lie pseudo-group there exists a set of $mc$ forms whose structure equations satisfy the compatibility and involutivity conditions; conversely, if Eqs. (11), (12) meet the compatibility conditions (13), (15) and the involutivity condition, then there exists a collection of 1-forms $\omega^1, \ldots, \omega^m$ and functions $U^1, \ldots, U^l$
which satisfy (11) and (12). Eqs. (10) then define local diffeomorphisms from a Lie pseudo-group.

3. Symmetry pseudo-group of the tangent covering of the universal hierarchy equation

Using the procedures of Élie Cartan’s method of equivalence, we find the Maurer–Cartan forms and their structure equations for the symmetry pseudo-group of system (11), (12). The full set of involutive structure equations for this pseudo-group consists of two parts:

\[
\begin{align*}
\text{d} \theta_0 &= \theta_0 \wedge (\theta_3 - \xi^1 - U_2 \xi^2 - U_1 \xi^3 - \theta_{12}) + \xi_1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3, \\
\text{d} \theta_1 &= \eta_1 \wedge \theta_1 + \theta_0 \wedge (\theta_{13} + (U_1 + 1) \xi^2 + \xi^3) + \xi^1 \wedge \theta_{11} + \xi^2 \wedge \theta_{12} + \xi^3 \wedge \theta_{13}, \\
\text{d} \theta_2 &= \theta_2 \wedge \eta_1 + \xi^1 \wedge (\theta_{12} - \theta_{13}) + \xi^2 \wedge \theta_{22} + \xi^3 \wedge (\theta_{23} - \theta_2), \\
\text{d} \theta_3 &= \xi^1 \wedge \theta_{13} + \xi^2 \wedge \theta_{25} + \xi^3 \wedge \theta_{12}, \\
\text{d} \xi_1 &= (\theta_{12} - \eta_1 - \theta_{3} + U_2 \xi^2 + U_1 \xi^3) \wedge \xi^1, \\
\text{d} \xi_2 &= (\eta_1 + \theta_1 + \xi^1 + (U_1 + 1) \xi^3) \wedge \xi^2, \\
\text{d} \xi_3 &= (\theta_2 + \xi^1) \wedge \xi^2 + (\theta_{12} + \xi^1 + U_2 \xi^2) \wedge \xi^3, \\
\text{d} \theta_{11} &= \eta_1 \wedge (\xi^2 + 2 \theta_{11}) + \eta_2 \wedge (\theta_0 + \xi^3) + \eta_3 \wedge \xi^1 + \theta_0 \wedge (\theta_{13} + 2 (U_1 + 1) \xi^2 + 2 \xi^3) + \theta_{13} \wedge (U_1 + 1) \xi^3, \\
\text{d} \theta_{12} &= \eta_1 \wedge \xi^1 + (\eta_3 + \theta_2 - U_2 \theta_{13}) \wedge \xi^3 + \eta_4 \wedge \xi^2 + (\theta_3 - \theta_{12}) \wedge (\xi^1 - (U_1 + 1) \xi^3), \\
\text{d} \theta_{13} &= (\eta_1 + \theta_3) \wedge (\xi^3 + \theta_{13}) + \eta_2 \wedge \xi^1 + \eta_3 \wedge \xi^2 + \xi^3 \wedge (2 \theta_{12} - U_1 \theta_{13}) - \theta_{12} \wedge \theta_{13}, \\
\text{d} \theta_{22} &= (\eta_3 + U_2 \theta_{2} - U_2 \theta_{12} + \theta_{22} + 2 \theta_{23}) \wedge \xi^1 + \eta_7 \wedge \xi^2 - (2 \eta_1 + \theta_{12}) \wedge \theta_{22} + (\eta_6 + (U_1 + 2) \theta_{22} - U_2 \theta_{23}) \wedge \xi^3 + \theta_2 \wedge (U_2 \xi^3 - \xi^1 - \theta_{23}), \\
\text{d} \xi_{23} &= (\eta_3 + \theta_1 - U_2 \theta_{13} + \theta_{23}) \wedge \xi^1 + \eta_6 \wedge \xi^2 + (\eta_4 - U_2 \theta_{12} + (U_1 + 1) \theta_{23}) \wedge \xi^3 - (\eta_1 + \theta_{12}) \wedge \theta_{23} - \theta_2 \wedge \theta_{12}, \\
\text{d} \eta_1 &= 0, \\
\text{d} \eta_2 &= \eta_2 \wedge (\theta_{12} - \eta_1 - \theta_3 + U_2 \xi^2 + U_1 \xi^3) \wedge \xi^3 \wedge (U_1 \theta_{13} - 4 \theta_{12}) + \eta_3 \wedge (2 (U_1 + 1) \xi^2 + 2 \xi^3 + \theta_{13}) + \theta_{13} \wedge \theta_{23} \\
& \quad + (\eta_3 - 2 (U_1 + U_1^2 - U_2) \xi^3 - 2 (U_1 + 1) \theta_{12}) \wedge \xi^2 - (\eta_1 + \theta_3) \wedge (\theta_{13} + 2 (U_1 + 1) \xi^2 + 3 \xi^3), \\
\text{d} \eta_3 &= \eta_1 \wedge (\theta_2 - (2 U_1 + 1) \xi^1 - U_2 \xi^3) - \eta_4 \wedge (2 (U_1 + 1) \xi^2 + 2 \xi^3 + \theta_{13}) + \theta_{13} \wedge \theta_{23} + \eta_3 \wedge (2 \theta_{12} - \theta_3 + U_2 \xi^2 + (2 U_1 + 1) \xi^3) - \theta_3 \wedge ((2 U_1 + 1) \xi^1 + U_2 \xi^3) + U_2 \eta_2 \wedge \xi^1 + \theta_2 \wedge (2 \theta_{12} - \theta_3 - (3 U_1 + 2 U_1^2 - 2 U_2 + 1) \xi^2 - U_1 \theta_{13}) \\
& \quad - \xi^1 \wedge ((3 U_1 + 2 U_1^2 - 3 U_2 + 1) \xi^2 + 2 (U_1 + U_1^2 - U_2) \xi^3 + 2 U_1 \theta_{12} + U_1 \theta_{13}) + \xi^2 \wedge ((U_1 + 1) \theta_{23} - U_2 (3 U_1 + 2 U_1^2 - 2 U_2 + 1) \xi^3 - \theta_{22}) + \xi^2 \wedge (2 - U_2 \theta_{23} - U_2 U_1 \theta_{13}), \\
\text{d} \eta_4 &= \eta_3 \wedge (\theta_2 + \xi^1 - U_2 \xi^3) + (\eta_4 + U_2 \xi^1) \wedge \eta_1 + \eta_4 \wedge (\theta_{12} - U_2 \xi^2) - \eta_6 \wedge \xi^2.
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\[ d\eta = \eta_1 + (3 \eta_5 - 3 \xi^2 - \eta_1 - 2 \eta_2 \wedge (\theta_0 + \xi_1 + \xi^2 + \xi^3) + 2 \eta_5 \wedge (\theta_12 - \theta_5 + U_2 \xi^2) + U_1 \eta_5 \wedge \xi^3 + \eta_8 \wedge (\theta_0 + \xi^3) + \eta_9 \wedge \xi^1 - \theta_0 \wedge (\theta_13 + 4 (U_1 + 1) \xi^2 + 4 \xi^3) + \xi^2 \wedge (U_1 \theta_11 - \theta_12 - \theta_1 \wedge (\theta_13 + 3 (U_1 + 1) \xi^2 + 3 \xi^3) - \theta_3 \wedge (\theta_11 + 3 \xi^2) + \xi^2 \wedge (U_2 \theta_11 - 4 \theta_12 - \theta_13) - \theta_11 \wedge \theta_12, \]

\[ d\eta_0 = \eta_6 \wedge (2 \eta_1 + 2 \xi^1 + (2 U_1 + 1) \xi^3 + 2 \theta_{12}) + \eta_4 \wedge (2 \theta_2 - U_1 \xi^1 - 3 U_2 \xi^3 - \theta_23) + \theta_2 \wedge (U_2 \theta_{12} - (3 U_1 + 2 U^2 - 2 U_2 + 1) \xi^1 - U_2 \xi^3 - (U_1 + 1) \theta_{23}) + \eta_{10} \wedge \xi^2 + \xi^2 \wedge (U_2 \theta_{12} - U_2 (U_1 + 2 U^2 - 2 U_2 + 1) \xi^3 - U_2^2 \theta_{13} - \theta_{22} + U_2 \theta_{23}) + \xi^2 \wedge (U_2 (U_1 + 1) \theta_{23} - U_2^2 \theta_{12} - U_1 \theta_{22}) - \theta_{12} \wedge \theta_{22} - U_2 \eta_3 \wedge \xi^1, \]

\[ d\eta_7 = \eta_7 \wedge (3 \eta_1 + 2 \xi^1 + (3 + 2 U_1) \xi^3 + 2 \theta_{12}) - \eta_4 \wedge (\theta_{22} + 3 U_2 \xi^1) + \eta_{10} \wedge \xi^3 + \eta_6 \wedge (2 \theta_2 + 2 \xi^1 - 2 U_2 \xi^3) + \theta_2 \wedge (U_2 \xi^1 - U_2^2 \xi^3 - (U_1 + 2) \theta_{22} + U_2 \theta_{23}) + \eta_{11} \wedge \xi^2 + \xi^1 \wedge (U_2^2 (\theta_3 - \xi^3 - \theta_{12}) - (2 U_1 - U_2 + 3) \theta_{22} + 4 U_2 \theta_{23}) + \theta_{22} \wedge \theta_{23} + U_2 \xi^3 \wedge ((U_1 + 3) \theta_{22} - U_2 \theta_{23}), \]

and

\[ d\omega_0 = \omega_0 \wedge (\omega_3 - \theta_{12} + V_1 \xi^1 + V_2 \xi^2 + V_3 \xi^3) + \xi^1 \wedge \omega_1 + \xi^2 \wedge \omega_2 + \xi^3 \wedge \omega_3, \]

\[ d\omega_1 = \omega_0 \wedge \theta_{13} + \omega_1 \wedge (\omega_3 - \eta_1 - \theta_3) + \eta_{12} \wedge \xi^1 + \eta_{13} \wedge \xi^2 + \eta_{14} \wedge \xi^3, \]

\[ d\omega_2 = \omega_2 \wedge (\eta_1 + (V_1 + 1) \xi^1 + \omega_3) + \omega_3 \wedge (\theta_2 + \xi^1) + \eta_{15} \wedge \xi^2 + \eta_{16} \wedge \xi^3 + (\eta_{13} - (U_1 + 1) \omega_0 - (U_2 + V_2) \omega_1) \wedge \xi^1, \]

\[ d\omega_3 = \omega_3 \wedge ((V_1 + 1) \xi^1 + (U_2 + V_2) \xi^2 + (U_1 + V_3) \xi^3) - \omega_0 \wedge (\xi^1 + (U_1 + 1) \xi^3) - \omega_1 \wedge ((U_1 + V_3) \xi^1 + (U_2 + V_2) \xi^3) + \eta_{14} \wedge \xi^1 + (\eta_{16} - (U_1 + V_3 + 1) \omega_2) \wedge \xi^2 + (\eta_{13} + (V_1 + 1) \theta_2 - (U_1 + V_3) \theta_3 - \theta_{12} - V_4 \theta_{13}) \wedge \xi^3, \]

\[ d\eta_{12} = \eta_{17} \wedge \xi^1 + (V_1 + 1) (\omega_1 \wedge \omega_3 + \eta_{13} \wedge \xi^2 + \eta_{14} \wedge \xi^3) + \xi^3 \wedge (\eta_2 - (V_1 + 2) \theta_{13}) + \omega_1 \wedge (\eta_{14} + \xi^2 + V_1 \xi^3) + \omega_0 \wedge (\omega_1 - 2 (U_1 + 1) \xi^2 - 2 \xi^3 + V_1 \theta_{13} + \eta_2) + \eta_{12} \wedge (\omega_3 - 2 \eta_1 - 2 \theta_3 + (2 U_2 + V_2) \xi^2 + (V_3 + 2 U_1) \xi^3 + \theta_{12}) + \xi^2 \wedge ((2 V_4 + U_1 (V_1 + 2) + V_1 - V_3 + 3) \xi^3 - V_4 \eta_2 - (U_1 + V_3 - V_4) \theta_{13}), \]

\[ d\eta_{13} = \omega_0 \wedge (\eta_3 - 2 (U_1 + 1) (\xi^1 + 2 U_2 \xi^2) - 2 U_2 \xi^3 + V_2 \theta_{13}) + \omega_1 \wedge (\eta_{16} + \theta_{23} + \xi^1) + \omega_1 \wedge ((U_2 + V_2) \omega_3 - (U_1 + V_3 + 1) \omega_2 - (V_4 (U_1 + 1) - 2 (U_2 + V_2)) \xi^3) + 2 \eta_{16} \wedge \xi^2 - \omega_2 \wedge (\theta_{13} + 2 (2 U_1 + V_3 + 2) \xi^2 + \xi^3) - \omega_3 \wedge (\eta_{13} - (U_1 + 2) \xi^3) + \eta_{13} \wedge (\theta_{12} - \theta_3 + (V_1 + 2) \xi^1 + (3 U_2 + 2 V_2) \xi^2 + (2 U_1 + V_3 + 1) \xi^3) + \eta_{14} \wedge (\theta_2 + \xi^1 + (U_2 + V_2) \xi^3) + \theta_2 \wedge ((U_1 + V_3 - 2) \xi^2 - V_1 \xi^3) + \theta_3 \wedge (2 (U_2 + V_2) \xi^2 + \xi^3) \wedge (2 \theta_{12} - (U_2 + V_2) \theta_{13} - (V_1 + 1) \theta_{23}) + (V_4 \eta_2 + (U_1 + V_3 - V_4) \theta_{13}) \wedge \xi^1 + \xi^2 \wedge ((V_1 + 1) \theta_{22} + 2 (U_2 + V_2) \theta_{12})}
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\[ + \xi^1 \wedge ((U_2 + V_2) \eta_{12} + (2 V_4 (U_1 + 1) + 3 U_1 + U_2 (V_1 - 1) - 2 V_2 + V_3 + 2) \xi^2) + (2 V_4 + V_1 (U_1 - 1) - V_3 - 2) \xi^1 \wedge \xi^3 + \xi^2 \wedge (V_5 \theta_{13} - (U_1 + V_3 + 2) \theta_{23}) + (V_5 + (U_1 (2 U_2 + 3 V_2 + 2 V_4) - U_2 (V_3 + 4) - 2 V_2 + 2 V_4) \xi^2 \wedge \xi^3, \]

\[ d\eta_{14} = \omega_0 \wedge (\eta_1 + \theta_3 - 2 \xi^1 - 2 U_2 \xi^2 - (2 U_1 + 1) \xi^3 - 2 \theta_{12} + (U_1 + V_3) \theta_{13}) + \omega_1 \wedge ((V_1 + 1) \theta_2 - (U_1 + V_3) \theta_3 - (V_4 (U_1 + 1) - 2 (U_2 + V_2)) \xi^2 - (V_4 - 1) \xi^3) + \omega_1 \wedge (\eta_{13} - (U_1 + 1) \omega_0 + (U_1 + V_3) \omega_3 - V_4 \theta_{13} + V_1 \xi^1) - (V_1 + 1) \xi^2 \wedge \theta_{23} - (\eta_2 - (U_1 + V_3) \eta_{12}) \wedge \xi^1 - (\omega_2 - (U_1 + V_3) \eta_{13} + V_1 \theta_2) \wedge \xi^2 + \eta_1 \wedge (\eta_{14} - \xi^3) \]

\[ + \eta_{14} \wedge (\theta_{12} - \theta_3 + (V_1 + 2) \xi^1 + (2 U_2 + V_2) \xi^2 + (3 U_1 + 2 V_3 + 1) \xi^3) - \xi^1 \wedge ((V_1 + 2) \theta_{13} + (2 (U_1 + V_1) + 5) \xi^2) - \xi^3 \wedge (2 \theta_{12} + (U_1 + V_3 + 1) \theta_{13}) + \xi^2 \wedge (2 \theta_{12} + (U_1 (U_1 + V_3 + 5) - U_2 (V_1 - 1) + 2 V_3 + 3) \xi^3 - (U_2 + V_2) \theta_{13}) - \omega_3 \wedge (\eta_{14} + (2 + U_1) \xi^2 + \xi^3 + \theta_{13}) + \theta_3 \wedge (\xi^2 - \xi^3), \]

\[ d\eta_{15} = (U_2 + V_2) (\eta_{13} + 2 \theta_3 - (U_1 + 1) \omega_0 - (U_2 + V_2) + \omega_1) \wedge \xi^1 + \eta_6 \wedge (V_4 \xi^2 - \xi^3) + \omega_2 \wedge (\eta_{16} + (V_2 + U_2) \omega_3 - (U_1 + V_3 + 1) \theta_2 - (V_4 (U_1 + 1) - V_2) \xi^3) + (V_1 + 1) (U_2 + V_2) - 3 (U_1 + V_3) - 2) \omega_2 \wedge \xi^1 - \eta_7 \wedge \xi^2 - \omega_3 \wedge (\eta_{15} + \theta_{22} - (U_2 + V_2) \theta_2 - V_2 \xi^1) + \eta_{16} \wedge (2 \theta_2 + 4 \xi^1 + (U_2 + V_2) \xi^3) + \eta_{15} \wedge (2 \eta_1 + \theta_{12} + (V_1 + 2) \xi^1 + (3 U_2 + 2 V_2) \xi^2 + (2 U_1 + V_3 + 2) \xi^3) + \theta_2 \wedge ((U_1 + V_3 - 2) \xi^1 + (2 U_2 V_4 + 3 V_5) \xi^2 + (2 V_4 (U_1 + 1) - 5 (U_2 + V_2)) \xi^3) + \xi^1 \wedge (2 (U_2 + V_2) \theta_{12} + V_5 \theta_{13} - (U_1 + 1) \theta_{22} - (U_1 + V_3 + 2) \theta_{23}) - (2 U_1 (U_2 V_4 + V_5) - U_2 (7 U_2 - 5 V_2 - V_4) - V_3) \xi^1 \wedge \xi^2 + (U_1 (7 U_2 + 5 V_2 + 4 V_4) - 2 U_2 - 4 (V_2 - V_4) + V_4) \xi^1 \wedge \xi^3 + \xi^2 \wedge ((3 V_5 + 2 U_2 V_4) (\theta_{23} + U_2 \xi^3) - (4 U_2 + 3 V_2 + V_4) \theta_{22}) - \xi^3 \wedge ((U_1 + V_3) \theta_{22} + (4 U_2 + 3 V_2) \theta_{23}), \]

\[ d\eta_{16} = ((U_1 + 1) \omega_0 + (U_2 + V_2) \omega_1) \wedge (\omega_2 - \theta_2 - (U_1 + V_3 + 2) \xi^1) - \eta_1 \wedge \eta_{16} + \theta_2 \wedge \theta_{12} + \omega_2 \wedge ((V_1 + 1) \theta_2 - (U_1 + V_3) \theta_3 + ((U_1 + V_3) (V_1 + 1) + V_1) \xi^1 - \theta_{12} - V_4 \theta_{13}) + \omega_2 \wedge (\eta_{13} + (U_1 + V_3) \omega_3 - (V_4 (U_1 + 1) - V_2) \xi^2 - (5 U_1 + 3 V_3 + 4 + 3) \xi^3) - \omega_3 \wedge (\eta_{16} - (U_1 + V_3 + 1) \theta_2 - (V_3 - 1) \xi^1 + \theta_{23}) + \eta_{13} \wedge (\theta_2 + (U_1 + V_3 + 2) \xi^1) - \xi^2 \wedge ((U_1 + V_3 + 1) \eta_{15} - (V_5 (U_1 + 1) + U_2 (2 V_4 (U_1 + 1) - 3 (U_2 + V_2)))) \xi^3) + \xi^2 \wedge (\eta_6 - (U_1 + V_3) \theta_{22} - (4 U_2 + 3 V_2) \theta_{23}) + \xi^1 \wedge (3 \theta_2 + (U_1 + V_3 - 1) \theta_3) + \xi^2 \wedge (\eta_4 + V_4 \eta_5 - V_4 (U_1 + 1) \theta_3 + V_2 \theta_{12} - U_2 V_4 \theta_{13} - 2 (U_1 + V_3 + 1) \theta_{23}) + \eta_{16} \wedge ((V_1 + 2) \xi^1 + (2 U_2 + V_2) \xi^2 + (3 U_1 + 2 V_3 + 4) \xi^3 + \theta_{12}) + \theta_2 \wedge ((U_1 + V_3) \theta_3 + \xi^1 + (2 V_4 (U_1 + 1) - 5 (U_2 + V_2)) \xi^2 + (2 U_1 + V_3) \xi^3) + \xi^1 \wedge ((U_1 (2 (V_2 + V_4) + 5 U_2) + U_2 (V_3 + 2) - 2 (V_2 + V_4)) \xi^2 - (V_1 + 1) \theta_{23}) + \xi^1 \wedge (U_1 (U_1 + V_3 + 2 V_4 + 8) - 2 (V_2 + V_4) + 3 V_3 + 5) \xi^3 + (V_4 \theta_2 - (U_2 + V_2 - V_4) \xi^1) \wedge \theta_{13}. \]
Eqns. (17) are the involutive structure equations for the symmetry pseudo-group of Eq. (14). The invariants $U_1, U_2, V_1, V_2, V_4$ in (17), (18) have the following expressions

$$U_1 = \frac{u_x (u_y^2 u_{txy} + S_2^2 + u_x S_1)}{u_x S_2^2},$$

$$U_2 = S_1 \left( \frac{u_y}{u_x S_2^3} (u_y u_{tx} - u_x u_{xy}) + \frac{u_y^3}{u_x S_2^2} (u_x u_{txx} - u_{tx} u_{xx}) + \frac{S_1 u_x^2}{S_2^4} + \frac{u_x}{S_2^2} (2 U_1 + 1) \right),$$

$$V_1 = \frac{u_x u_y (v_{ty} S_2 - v_y S_2)}{v_y (u_x u_y S_2 t - u_y u_{tx} S_2 - S_2^2)},$$

$$V_2 = \frac{u_2^2 (v_x - v_y) v_{ty}}{v_2^2 S_2} + \frac{(V_1 + 1) (u_x v_y - u_y v_x) S_0}{S_2} - U_1,$$

$$V_4 = (u_y v_x - u_x v_y) S_0 S_2^{-1},$$

where

$$S_0 = (u_y u_x S_2 t - u_y u_{tx} S_2 - S_2^2) (u_x v_y S_2)^{-1},$$

$$S_1 = u_x u_y u_{ty} - u_x^2 u_{ttx} + u_y u_{tx} u_{ty} - u_x u_{ty}^2,$$

$$S_2 = u_x u_{ty} - u_y u_{tx},$$

while the formulas for $V_3$ and $V_5$ are too big to write them in full here. The differentials of the invariants satisfy the equations

$$dU_1 = U_2 \theta_{13} - \eta_3 - \theta_2 + (U_1 + 1) \theta_1 - \xi - \frac{1}{2} U_1^2 + 3 U_1 - 2 U_2 + 1) \xi^3 - 2 (U_1 + 1) \theta_{12},$$

$$dU_2 = - U_1 \theta_2 - (2 U_1 + 1) \xi^1 - \theta_{23} - \eta_4 - U_2 (\eta_1 + \theta_{12} + \xi^3),$$

$$dV_1 = \omega_0 + (U_1 + V_3) \omega_1 + (V_1 + 1) (\eta_1 - \omega_3 + \theta_3 - \theta_1) - \eta_{14} - \theta_{13} - (V_1^2 + 3 V_1 + 2) \xi^1 - (U_1 + 2 U_2 + V_3) (V_1 + 1) + 2) \xi^2 - (V_1 + 1)(2 U_1 + V_2 + 1) \xi^3,$$

$$dV_2 = (U_1 + V_3 + 1) \omega_2 - V_2 \eta_1 - (U_2 + V_2) \omega_3 - \eta_{16} + \eta_4 - V_3 \theta_2 - V_2 \theta_{12} - (U_1 (2 U_2 + V_3 - 2 V_4) + 2 U_2 (V_2 + 1) + V_3 (V_2 + 1) - V_4) \xi^2 - ((U_1 + V_2) (V_1 + 2) + V_1) \xi^1 - (V_2 (V_2 + 3 U_1 + 1) + 2 (U_2 - U_1) - V_4) \xi^3,$$

$$dV_3 = (U_1 + 1) \omega_0 + \eta_3 - \eta_{13} + (U_2 + V_2) \omega_1 - (U_1 + V_3) \omega_3 - V_1 \theta_2 + (V_3 - 1) \theta_3 + (U_1 - V_3 + 2) \theta_{12} + (V_4 - U_2) \theta_{13} - ((U_2 + V_3) (V_1 + 2) + V_2 + 1) \xi^1 - (U_1 (2 U_2 + 2 V_3 - V_4) + U_2 (V_2 + 2) + V_3 (V_2 + 3) - V_4) \xi^3 - (U_2 + V_3) (2 U_2 + V_3) \xi^2,$$

$$dV_4 = \theta_2 - \omega_2 - V_4 (\eta_1 + \omega_3) - (V_4 (V_4 + 1) + U_1 + V_3 - 1) \xi^1 + (V_5 - V_4 (U_2 + V_2)) \xi^2 - (V_4 (U_1 + V_3 + 1) + U_2 + V_2) \xi^3,$$

$$dV_5 = \eta_{15} + \theta_{22} - (U_2 + V_2) \omega_2 - V_4 \theta_{23} - V_5 (\omega_3 + 2 \eta_1 + \theta_{12}) + (V_4 + 2 (U_2 + V_2)) \theta_2 + (5 U_2 + 4 V_2 - U_1 V_4 - V_5 (V_1 + 2)) \xi^1 - (U_2 V_4 + V_5 (2 U_1 + V_3 + 2)) \xi^3 - V_5 (2 U_2 + V_2) \xi^2.$$

In what follows we need explicit formulas only for the mc forms

$$\theta_0 = \frac{u_y u_{tx} - u_x u_{ty}}{u_x^2} (du - u_t dt - u_x dx - u_y dy),$$
\[ \theta_1 = -\frac{S_2^2}{u_y S_1} (du_t - u_{tt} dt - u_{tx} dx - u_{ty} dy) + \frac{u_{tx} S_2}{S_1} \theta_0, \]

\[ \theta_2 = \frac{S_1}{u_y S_2^2} (u_x du_y - u_y du_x - S_2 dt - (u_y u_{xx} - u_x u_{xy}) dx + (u_y u_{xy} + u_x S_2) dy), \]

\[ \theta_{12} = \frac{1}{S_2} \left( dS_2 + u_{tx} du_y + u_{ty} du_x - \frac{u_x S_1 + u_y u_{tx} S_2 + S_2^2}{u_x u_y} dt - (u_x u_{txy} - u_y u_{txx}) dx \right. 
\[ \left. + \frac{u_y^2 u_{txy} + u_x S_1 + u_y u_{tx} S_2 + S_2^2}{u_y} dy \right) - \frac{S_2}{S_1 u_{ty}} \theta_2, \]

\[ \xi^1 = \frac{S_1}{u_y S_2} dt, \quad \xi^2 = \frac{S_3}{u_y^2 S_1} dx, \quad \xi^3 = -\frac{S_2}{u_y^2} (u_x dx + u_y dy), \]

\[ \omega_0 = -\frac{S_2}{u_y v_y} (dv - v_t dt - v_x dx - v_y dy), \]

\[ \omega_1 = -\frac{S_2}{v_y S_1} (dv_t - vt dt - v_{tx} dx - v_{ty} dy) - \frac{S_2 u_{ty}}{S_1} \omega_0, \]

\[ \omega_3 = \frac{1}{v_y} (dv_y - v_{ty} dt - v_{xy} dx - (u_y u_{tx} - u_x u_{ty} + v_y u_{tx} - v_x u_{ty}) dy). \] (19)

4. Contact integrable extensions

For applying Élie Cartan’s structure theory of Lie pseudo-groups to the problem of finding coverings of PDEs we use the notion of integrable extension. It was introduced in \([2]\) for the case of PDEs with two independent variables and finite-dimensional coverings. The generalization of the definition to the case of infinite-dimensional coverings of PDEs with more than two independent variables was proposed in \([36]\). In contrast to \([54, 3]\), the starting point of our definition is the set of Maurer–Cartan forms of the symmetry pseudo-group of a given PDE, and all the constructions are carried out in terms of invariants of the pseudo-group. Therefore, the effectiveness of our method increases when it is applied to equations with large symmetry pseudo-groups.

Let \( \mathfrak{G} \) be a Lie pseudo-group on a manifold \( M \). Let \( \omega^1, ..., \omega^m, m = \dim M, \) be its Maurer–Cartan forms with the compatible and involutive structure equations \([11], \[12]\). Consider the system of exterior differential equations

\[ d\zeta^q = D^q_{\rho \sigma} \mu^\rho \wedge \zeta^\sigma + E^q_{\rho \sigma} \zeta^\rho \wedge \zeta^\sigma + F^q_{\rho \beta} \zeta^\rho \wedge \pi^\beta + G^q_{\beta \gamma} \zeta^\beta \wedge \omega^\gamma + H^q_{\beta \gamma} \pi^\beta \wedge \omega^\gamma \]

\[ + I^q_{jk} \omega^j \wedge \omega^k, \] (20)

\[ dV^\epsilon = J^\epsilon_j \omega^j + K^\epsilon_q \zeta^q, \] (21)

for unknown 1-forms \( \zeta^q, q \in \{1, ..., Q\}, \nu^\rho, \rho \in \{1, ..., R\}, \) and unknown functions \( V^\epsilon, \epsilon \in \{1, ..., S\} \) with some \( Q, R, S \in \mathbb{N} \). The coefficients \( D^q_{\rho \sigma}, ..., K^\epsilon_q \) in equations \([20], [21]\) are supposed to be functions of \( U^\lambda \) and \( V^\kappa \).

**Definition 1.** The system \([20], [21]\) is called an integrable extension of the system \([11], [12]\), if equations \([20], [21]\), \([11], [12]\) together meet the involutivity conditions.
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and the compatibility conditions
\[ d(d\zeta^q) \equiv 0, \quad d(dV^\epsilon) \equiv 0. \] (22)

Equations \( (22) \) give an over-determined system of PDEs for the coefficients \( D_{\rho\tau}^\epsilon, \ldots, K^\epsilon_q \) in equations \( (20), (21) \). Suppose this system is satisfied. Then we apply the third and the second inverse fundamental Lie’s theorems in Cartan’s form. \[ 1, \{4, 52\}, \{16–24\}, \{7, 51\}, \{14.1–14.3\} \] The third inverse fundamental theorem ensures the existence of the forms \( \zeta^q \) and the functions \( V^\epsilon \), the solutions to equations \( (20), (21) \). In accordance with the second inverse fundamental theorem, the forms \( \zeta^q, \omega^i \) are Maurer–Cartan forms for a Lie pseudo-group \( \mathcal{G} \) acting on \( M \times \mathbb{R}^Q \).

**Definition 2.** The integrable extension \( (20), (21) \) is called trivial, if there exists a change of variables on the manifold of action of the pseudo-group \( \mathcal{G} \) such that in the new coordinates the coefficients \( F^q_{\rho\beta}, G^q_{\rho\tau}, H^q_{\beta\tau}, I^q_{jk} \) and \( J^\epsilon_j \) are identically equal to zero, while the coefficients \( D_{\rho\tau}^q, E^q_{\rho\tau} \) and \( K^\epsilon_q \) are independent of \( U^\lambda \). Otherwise, the integrable extension is called nontrivial.

Let \( \theta^\alpha_i \) and \( \xi^j \) be a set of Maurer–Cartan forms of a symmetry pseudo-group \( \mathfrak{Lie}(\mathcal{E}) \) of a PDE \( \mathcal{E} \) such that \( \xi^j \) are horizontal forms, that is, \( \xi^1 \wedge \ldots \wedge \xi^n \neq 0 \) on each solution of \( \mathcal{E} \), while \( \theta^\alpha_i \) are contact forms, that is, they are equal to 0 on each solution.

**Definition 3.** Nontrivial integrable extension of the structure equations for the pseudo-group \( \mathfrak{Lie}(\mathcal{E}) \) of the form
\[ dw^\alpha = \Pi^\alpha_r \wedge \omega^r + \xi^j \wedge \Omega^q_j, \] (23)
\( q, r \in \{1, \ldots, N\} \), \( N \geq 1 \), is called a contact integrable extension, if the following conditions are satisfied:

(i) \( \Omega^q_j \in \langle \theta^\alpha_i, \omega^r \rangle_{11n} \) for some additional 1-forms \( \omega^r \);

(ii) \( \Omega^q_j \notin \langle \omega^r \rangle_{11n} \) for some \( q \) and \( j \);

(iii) \( \Omega^q_j \notin \langle \theta^\alpha_i \rangle_{11n} \) for some \( q \) and \( j \);

(iv) \( \Pi^r_i \in \langle \theta^\alpha_i, \xi^j, \omega^r, \omega^r \rangle_{11n} \).

(v) The coefficients of expansions of the forms \( \Omega^q_j \) with respect to \( \{\theta^\alpha_i, \omega^r\} \) and the forms \( \Pi^r_i \) with respect to \( \{\theta^\alpha_i, \xi^j, \omega^r, \omega^r\} \) depend either on the invariants of the pseudo-group \( \mathfrak{Lie}(\mathcal{E}) \) alone, or they depend also on a set of some additional functions \( W_\rho \), \( \rho \in \{1, \ldots, \Lambda\} \), \( \Lambda \geq 1 \). In the latter case, there exist functions \( P^I_\rho, Q^\rho_q, R^i_\rho q \) and \( S^\rho_j \) such that
\[ dW_\rho = P^I_\rho \theta^\alpha_I + Q^\rho_{pq} \omega^q + R^i_\rho q \omega^q_j + S^\rho_j \xi^j, \] (24)
and the set of equations \( (24) \) satisfies the compatibility conditions
\[ d(dW_\rho) = d \left( P^I_\rho \theta^\alpha_I + Q^\rho_{pq} \omega^q + R^i_\rho q \omega^q_j + S^\rho_j \xi^j \right) \equiv 0. \] (25)

**Example 2.** Eqns. \( (18) \) are a cie for Eqns. \( (17) \) with the additional forms \( \eta_{12}, \ldots, \eta_{17} \) and the additional invariants \( V_1, \ldots, V_5 \).
5. Bäcklund auto-transformation for the tangent covering of the universal hierarchy equation

We apply Definition 3 to the structure equations (17), (18). We restrict our analysis to cies of the form

\[
d\omega_4 = \sum_{k=0}^{4} \left( \sum_{i=0}^{3} A_{ik} \theta_i + \sum_{s=1}^{7} C_{sk} \eta_s + \sum_{i=0}^{16} C_{sk} \eta_s + \sum_{j=1}^{3} D_{jk} \xi^j \right) \wedge \omega_k
\]

\[
+ \sum_{k=0}^{4} E_k \omega_k \wedge \omega_5 + \sum_{k=1}^{3} \left( \sum_{i=0}^{3} F_{ik} \theta_i + \sum_{j=1}^{3} G_{ijk} \theta_{ij} + H_k \omega_5 \right) \wedge \xi^k
\]

\[
+ \sum_{k=0}^{3} M_k \omega_k \wedge \omega_4
\]

(26)

with one additional form \( \omega_5 \) mentioned in the part (i) of Definition 3. In (26), \( \sum^* \) means summation for all \( i, j \in \mathbb{N} \) such that \( 1 \leq i \leq j \leq 3 \), \( (i, j) \neq (3, 3) \). These equations together with equations (17), (18) satisfy the requirement of involutivity. We assume that the coefficients of (26) depend on the invariants \( U_1, U_2, V_1, \ldots, V_5 \).

REMARK 3. We consider (18), (26) together as a single cie for (17). This defines the form of the r.h.s. of (26).

Definition 3 gives an over-determined system of pdes for the coefficients of (26). The analysis of this system gives a cie. Then, since the mc forms included in Eqns. (17), (18) are known explicitly, we use the third inverse fundamental Lie’s theorem and find form \( \omega_4 \) by means of integration. We obtain

THEOREM. The structure equations (17), (18) of the contact symmetry pseudo-group of the tangent covering of Eq. (1) have the cie

\[
d\omega_4 = (\eta_1 - \omega_3 + \theta_{12} - V_2 \xi^2 - V_3 \xi^3) \wedge \omega_4 + \omega_5 \wedge \xi^1 + \omega_1 \wedge \theta_0 - \omega_0 \wedge \theta_1
\]

\[
+ (\omega_3 - \omega_0 - (U_1 + V_3) \theta_0 - V_4 \theta_1) \wedge \xi^2 + (\omega_1 + \theta_1 - (V_1 + 1) \theta_0) \wedge \xi^3.
\]

Each solution of this equation is contact-equivalent to the form

\[
\omega_4 = \frac{(u_y u_{tx} - u_x u_{ty})^3}{u_y^2 v_y S_1} (dw - w_t dt - (v_y + u_x v_t - u_{tx} v) dx - (u_y v_t - u_{ty} v) dy)
\]

\[- \frac{v (u_y u_{tx} - u_x u_{ty})}{u_y v_y} \theta_1 - \frac{(u_y v_t - u_{ty} v) (u_y u_{tx} - u_x u_{ty})^2}{u_y v_y S_1} \theta_0.
\]

The form \( \omega_4 \) is equal to zero on solutions of (1) whenever \( w \) solves the system

\[
\begin{align*}
    w_x &= v_y + u_x v_t - u_{tx} v, \\
    w_y &= u_y v_t - u_{ty} v.
\end{align*}
\]

(27)
This system is compatible on solutions to (2). Thus it defines a Bäcklund transformation from (2) to a certain pde. To get this pde, we solve (27) for \( v_t \) and \( v_y \):
\[
\begin{align*}
  v_t &= (w_y + u_{ty}v)u_y^{-1}, \\
  v_y &= ((u_yu_{tx} - u_xu_{ty})v + u_yw_x - u_xw_y)u_y^{-1}.
\end{align*}
\] (28)

The compatibility condition of this system turns out to be a copy of (2) with \( w \) substituted for \( v \). Therefore, (27) and (28) define a Bäcklund auto-transformation for (2). Since solutions to (2) are identified with local symmetries or shadows of nonlocal symmetries for Eq. (1), we have

**Corollary.** Equations
\[
\begin{align*}
  D_x(\psi) &= u_xD_t(\varphi) - u_{tx}\varphi + D_y(\varphi), \\
  D_y(\psi) &= u_yD_t(\varphi) - u_{ty}\varphi.
\end{align*}
\]

define a recursion operator \( \psi = R(\varphi) \) for symmetries of the universal hierarchy equation (7). The inverse recursion operator \( \varphi = R^{-1}(\psi) \) is defined by equations
\[
\begin{align*}
  D_t(\varphi) &= (D_y(\psi) + u_{ty}\varphi)u_y^{-1}, \\
  D_y(\varphi) &= ((u_yu_{tx} - u_xu_{ty})\varphi + u_yD_x(\psi) - u_xD_y(\psi))u_y^{-1}.
\end{align*}
\]

**6. Conclusion**

We have showed the possibility to find recursion operators for symmetries of nonlinear PDEs by means of Cartan’s method of equivalence. While this approach is computationally involved, it does not require any preliminary information about linear coverings of the PDE under study. Its applicability to other PDEs is an interesting problem for the further research.

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