RESOLUTIONS OVER KOSZUL ALGEBRAS

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Abstract. In this paper we show that if \( \Lambda = \prod_{i \geq 0} \Lambda_i \) is a Koszul algebra with \( \Lambda_0 \) isomorphic to a product of copies of a field, then the minimal projective resolution of \( \Lambda_0 \) as a right \( \Lambda \)-module provides all the information necessary to construct both a minimal projective resolution of \( \Lambda_0 \) as a left \( \Lambda \)-module and a minimal projective resolution of \( \Lambda \) as a right module over the enveloping algebra of \( \Lambda \). The main tool for this is showing that there is a comultiplicative structure on a minimal projective resolution of \( \Lambda_0 \) as a right \( \Lambda \)-module.

Introduction and preliminaries

Let \( \Lambda = \prod_{i \geq 0} \Lambda_i \) be a Koszul algebra over a field \( k \) with \( \Lambda_0 \) a product of copies of \( k \), where we recall the definition of Koszul later in this section. Denote by \((L, e)\) a minimal (graded) projective resolution of \( \Lambda_0 \) as a right \( \Lambda \)-module. We show that \((L, e)\) contains all the information needed to construct a minimal projective resolution of \( \Lambda \) as a right \( \Lambda^e \)-module, where \( \Lambda^e = \Lambda^\text{op} \otimes_k \Lambda \). The resolution \((L, e)\) is shown to have a “comultiplicative structure”. This structure is used to prove that one can obtain a minimal projective resolution of \( \Lambda_0 \) over \( \Lambda \) as a left \( \Lambda \)-module from the knowledge of \((L, e)\). We apply these results to prove an unpublished result of E. L. Green and D. Zacharia that \( \Lambda \) is a Koszul algebra if and only if \( \Lambda \) is a linear module as a right module over \( \Lambda^e \). In [2], the comultiplicative structure is applied to give the multiplicative structure of the Hochschild cohomology ring of a Koszul algebra and also the structure constants for a basis for the Koszul dual.

The rest of the section is devoted to recalling definitions, results, and terminology relevant to this paper. Let \( \Lambda = \prod_{i \geq 0} \Lambda_i \) be a graded algebra over a field \( k \). Assume that (i) \( \Lambda_0 \) is a product of copies of \( k \), that (ii) each \( \Lambda_i \) is finite dimensional over \( k \), and that (iii) \( \Lambda \) as an algebra is generated in degrees 0 and 1. Such an algebra \( \Lambda \) is isomorphic to a quotient of the path algebra \( kQ/I \), where \( kQ \) is isomorphic to the tensor algebra \( T_{\Lambda_0}(\Lambda_1) = \prod_{i \geq 0} \Lambda_1 \otimes_{\Lambda_0} \cdots \otimes_{\Lambda_0} \Lambda_1 \). Conversely, if \( Q \) is a quiver and \( I \) is an ideal generated by length homogeneous elements in \( kQ \), then \( \Lambda = kQ/I \) is a graded algebra over \( k \) satisfying the conditions above. Throughout this paper \( \Lambda \) denotes a graded algebra having properties (i)–(iii).

Let \( r = \prod_{i \geq 1} \Lambda_i \), which is the graded Jacobson radical of \( \Lambda \). If \((P, d)\):

\[
\cdots \rightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \rightarrow 0
\]

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is a graded projective resolution of a graded \( \Lambda \)-module \( M \), then it is \textit{minimal} if \( \text{Im} d^n \subseteq t^{P^n-1} \) for \( n \geq 1 \). It is well known that graded modules over graded algebras have minimal graded projective resolutions. We say that a graded projective resolution

\[ \cdots \rightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} M \rightarrow 0 \]

is \textit{linear}, and \( M \) is a \textit{linear module} if, for \( n \geq 0 \), the graded module \( P^n \) is generated in degree \( n \). Note that a linear resolution is a minimal projective resolution. A graded algebra \( \Lambda \) is a \textit{Koszul algebra} if \( \Lambda_0 \) is a linear module; that is, \( \Lambda_0 \) has a linear (graded) projective resolution \((\mathbb{L}, e)\):

\[ \cdots \rightarrow L^2 \xrightarrow{e^2} L^1 \xrightarrow{e^1} L^0 \xrightarrow{e^0} \Lambda_0 \rightarrow 0 \]

as a right \( \Lambda \)-module.

Before giving the precise results, we introduce notation and recall results from [5] which are used throughout the paper. For ease of notation, let \( R = kQ \), let \( \mathcal{B} \) be the set of all paths in the quiver \( Q \), and denote by \( B_t \) all the paths of length \( t \).

There exist integers \( \{t_n\}_{n \geq 0} \) and elements \( \{f^n_i\}_{i=0}^{t_n} \) in \( R \) such that a minimal right projective resolution \((\mathbb{L}, e)\) of \( \Lambda_0 \) can be given in terms of a filtration of right ideals

\[ \cdots \subseteq \Pi_{i=0}^{t_n} f^n_i R \subseteq \Pi_{i=0}^{t_n-1} f^{n-1}_i R \subseteq \cdots \subseteq \Pi_{i=0}^{t_1} f^1_i R \subseteq \Pi_{i=0}^{t_0} f^0_i R = R \]

in \( R \). Then \( L^n = \Pi_{i=0}^{t_n} f^n_i R / \Pi_{i=0}^{t_n} f^n_i R \) and the differential \( e \) is induced by the inclusion \( \Pi_{i=0}^{t_n} f^n_i R \subseteq \Pi_{i=0}^{t_n-1} f^{n-1}_i R \). This inclusion gives elements \( h_{j_i}^{n,1,n} \) in \( R \) such that

\[ f^n_i = \sum_{j=0}^{t_n-1} f^{n-1}_j h_{j_i}^{n,1,n} \]

for all \( i = 0, 1, \ldots, t_n \) and all \( n \geq 1 \), so that

\[ e^n f^n_i (\underline{t}_i^n) = (h_{0_i}^{n-1,n}, h_{1_i}^{n-1,n}, \ldots, h_{t_n-1,i}^{n-1,n}) \]

for all \( n \geq 1 \), where \( \underline{t}_i^n \) denotes the natural residue class of \( \ast \) modulo \( I \). It is shown in [5] that the \( f^n_i \)'s can be chosen so that \((\mathbb{L}, e)\) is a minimal resolution of \( \Lambda_0 \) over \( \Lambda \). We point out that an algorithmic construction of the elements \( f^n_i \)'s can be found in [4].

An important property of the elements \( \{f^n_i\}_{i=0}^{t_n} \) is that there exist elements \( f^{n+1}_j \) in \( \Pi_{i=0}^{t_n-1} f^n_i R \) such that

\[ (\Pi_{i=0}^{t_n} f^n_i R) \cap (\Pi_{i=0}^{t_n-1} f^{n-1}_i R) = (\Pi_{i=0}^{t_n+1} f^{n+1}_i R) \cap (\Pi_{j=0}^{t_n} f^{n+1}_j R). \]

Recall that an element \( x \) in \( R \) is called \textit{uniform} if \( x \) is non-zero and there exist vertices \( u \) and \( v \) in \( Q \) such that \( x = uv \). If \( x \) is a uniform element with \( x = uv \), then we write \( \mathfrak{a}(x) = u \) and \( \mathfrak{t}(x) = v \). The elements \( f^n_i \) can all be chosen uniform for \( i = 0, 1, \ldots, t_n \) and all \( n \geq 0 \), and we assume that they are.

Note that \( t_0 + 1 \) is the number of non-isomorphic graded simple right \( \Lambda \)-modules, and that \( \{f^0_i\}_{i=0}^{t_0} \) is the set of vertices of \( Q \). Moreover, \( t_1 + 1 \) is the number of arrows of \( Q \) and \( \{f^1_i\}_{i=0}^{t_1} \) is chosen to be the set of arrows of \( Q \). The set \( \{f^2_i\}_{i=0}^{t_2} \) is a set of uniform length homogeneous minimal generators for \( I \).

In case \( \Lambda \) is a Koszul algebra, we have the following additional property of the elements \( f^n_i \) in \( R \); namely each \( f^n_i \) is a linear combination of paths in \( B_n \) for
i = 0, 1, \ldots, t_n and the length of each path occurring in $f_{ni}^n$ is at least $n + 1$. By length considerations, $h_{ji}^{n-1,n}$ are all linear combinations of elements in $B_i$.

In section 1 we prove that the elements \{f_{ni}^n\}_{i=0, n \geq 0} have the following “comultiplicative structure”, which is used in [2] to give the multiplicative structure of the Hochschild cohomology ring of a Koszul algebra and the structure constants for the basis associated to the elements \{f_{ni}^n\} for the Koszul dual.

**Theorem.** Let $\Lambda = kQ/I$ be a Koszul algebra. Then for each $r$, with $0 \leq r \leq n$, and $i$, with $0 \leq i \leq t_n$, there exist elements $c_{pq}(n, i, r)$ in $k$ such that

$$f_{ni}^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n, i, r)f_p^nf_{n-r}^q$$

for all $n \geq 1$, all $i$ in \{0, 1, \ldots, t_n\} and all $r$ in \{0, 1, \ldots, n\}.

Viewing $\Lambda_0$ as a left module over $\Lambda$, it also has a minimal graded projective resolution given by \{g_{ni}^n\}_{i=0}^{s_n}$, where $g_{ni}^n$s are the left analogue of the right $f_{ni}^n$s in $R$. The above result is used to prove that one can choose the elements $g_{ni}^n$s to be the same as the elements $f_{ni}^n$s and then the formula $f_{ni}^n = \sum_{p=0}^{t_i} c_{pq}(n, i, 1)f_p^nf_{n-1}^q$ gives the differential in the projective resolution of $\Lambda_0$ as a left $\Lambda$-module. Thus the knowledge of the minimal projective resolution $(L, e)$ via the elements $f_{ni}^n$ contains all the information needed to construct a minimal projective resolution of $\Lambda_0$ as a left $\Lambda$-module.

In the final section of the paper, the elements $f_{ni}^n$s are shown to provide all the information needed to construct a minimal projective resolution of $\Lambda$ as a right $\Lambda^e$-module. In particular, we prove the following.

**Theorem.** Let $\Lambda = kQ/I$ be a Koszul algebra, and let $\{f_{ni}^n\}_{i=0}^{t_n}$ be defined as above for $\Lambda_0$ as a right $\Lambda$-module. A minimal projective resolution $(P, \delta)$ of $\Lambda$ over $\Lambda^e$ is given by

$$P^n = \bigoplus_{i=0}^{t_n} \Lambda \circ \circ_{k} t(f_{ni}^n)$$

for $n \geq 0$, where $j$-th component of the differential $\delta^n: P^n \to P^{n-1}$ applied to the $i$-th generator $\circ \circ_{k} t(f_{ni}^n)$ is given by

$$\sum_{p=0}^{t_i} c_{pq}(n, i, 1)f_p^nf_{n-1}^q \circ \circ_{k} t(f_{ni}^n) + (-1)^{n-1}\sum_{q=0}^{t_{n-1}} c_{pq}(n, i, n-1)f_q^nf_{n-1}^q$$

for $j = 0, 1, \ldots, t_{n-1}$ and $n \geq 1$, and $\delta^n: \bigoplus_{i=0}^{t_n} \Lambda e_i \circ \circ_{k} e_i \Lambda \to \Lambda$ is the multiplication map.

As mentioned earlier, the final result of the paper is that $\Lambda$ is a Koszul algebra if and only if $\Lambda$ is a linear module as a right $\Lambda^e$-module.

1. A resolution with comultiplicative structure

In this section Theorem 1.1 provides a comultiplicative structure to a minimal projective resolution of $\Lambda_0$ as a right $\Lambda$-module. This result is then applied to show that the knowledge of a minimal projective resolution of $\Lambda_0$ as a right $\Lambda$-module is sufficient to construct a minimal projective resolution of $\Lambda_0$ as a left $\Lambda$-module.

Let $\Lambda = kQ/I$ be a graded algebra over a field $k$. Let $\{t_n\}_{n \geq 0}$ and $\{f_{ni}^n\}_{i=0, n \geq 0}$ be as in the introduction. We say that $\{f_{ni}^n\}_{i=0, n \geq 0}$ defines a minimal resolution if the resolution described in the introduction is minimal.
The next result shows that the elements \( \{f^n_i\} \) have a comultiplicative structure for a Koszul algebra.

**Theorem 1.1.** Let \( \Lambda = kQ/I \) be a Koszul algebra, and assume that \( \{f^n_i\}_{i=0}^t \) defines a minimal resolution of \( \Lambda_n \) as a right \( \Lambda \)-module. Then for each \( r \), with \( 0 \leq r \leq n \), and \( i \), with \( 0 \leq i \leq t_n \), there exist elements \( c_{pq}(n,i,r) \) in \( k \) such that

\[
f^n_i = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n,i,r) f^n_p f^{n-r}_q.
\]

**Proof.** For any \( n \), and \( r \) equal to 0 or \( n \), the result follows from \( f^n_i = f^n_i \frac{t_i f^n_i}{t_i} = f^n_i f^n_i f^n_i \) for \( i = 0, 1, \ldots, t_n \). Also, this proves the result in the case \( n \) is equal to 1.

Next we discuss the case \( n = 2 \). As we have remarked, each \( f^2_1 = \sum_{j=0}^{t_1} f^1_j h_{j_1}^{1,2} \). Since \( \Lambda \) is Koszul, each \( f^2_2 \) is a linear combination of paths in \( B_2 \), and hence \( h_1^{n,n} \) is a linear combination of elements in \( B_1 \). This gives the result for \( n = 2 \).

Now we proceed by induction on \( n \) and assume that the result is true for \( l < n \) and \( n \geq 3 \). We have that \( f^n_i = \sum_{j=0}^{t_{n-1}} f^n_j h_{j_1}^{n-1,n} \). As in our discussion for \( n = 2 \), we see that \( h_{j_1}^{n-1,n} \) is a linear combination of elements in \( B_1 \). There exist elements \( c_{ijs} \) in \( k \) such that

\[
f^n_i = \sum_{j=0}^{t_{n-1}} \sum_{s=0}^{t_1} c_{ijs} h^n_i f^n_j f^n_s.
\]

By induction, there exist elements \( c'_{juv} \) in \( k \) such that

\[
f^n_{j} = \sum_{s=0}^{t_{n-r-1}} \sum_{v=0}^{t_{n-r-1}} c'_{juv} f^n_j f^n_{s-1} f^n_{t-1}
\]

for any \( r \), with \( 0 \leq r \leq n - 1 \). Hence

\[
f^n_i = \sum_{j=0}^{t_{n-1}} \sum_{s=0}^{t_{n-1}} \sum_{u=0}^{t_{n-1}} \sum_{v=0}^{t_{n-1}} c_{ijs} c'_{juv} f^n_j f^n_{s-1} f^n_{t-1} f^n_i
\]

for any \( r \), with \( 0 \leq r \leq n - 1 \). The term after \( f^n_i \) is

\[
A = \sum_{j=0}^{t_{n-1}} \sum_{s=0}^{t_{n-1}} \sum_{u=0}^{t_{n-1}} \sum_{v=0}^{t_{n-1}} c_{ijs} c'_{juv} f^n_j f^n_{s-1} f^n_{t-1} f^n_i.
\]

Theory tells us that

\[
f^n_i = \sum_{w=0}^{t_{n-2}} f^n_{w-2} z_w,
\]

where \( z_w \) is in \( I \). Again by length considerations each \( z_w \) is a linear combination of \( f^2_i \). Hence, there exist elements \( c''_{iwx} \) in \( k \) such that

\[
f^n_i = \sum_{w=0}^{t_{n-2}} \sum_{x=0}^{t_{n-2}} c''_{iwx} f^n_{w-2} f^n_{x-2} f^n_i.
\]

By induction each \( f^n_{w-2} \) is a linear combination of \( f^n_{w} f^n_{x-2} f^n_{w} \). We obtain

\[
f^n_i = \sum_{w=0}^{t_{n-2}} \sum_{x=0}^{t_{n-2}} \sum_{y=0}^{t_{n-2}} c'''_{iwx} c''_{wxy} f^n_{w-2} f^n_{x-2} f^n_{w}.
\]
for some \( c''_{wuy} \) in \( k \). So the term after \( f^n_u \) in this expression is

\[
B = \sum_w \sum_x \sum_u c''_{wux} f^{n-r-2}_x.
\]

Since \( \sum u f^n_u R \) is a direct sum, we see that formulas (1) and (2) are equal. The equation (1) implies that \( A \) is in \( \Pi_{i=0}^{n-r-1} f^n_{r-1} R \), and the equation (2) implies that \( A \) is in \( \Pi_{i=0}^{n-r-2} f^n_{r-2} I \). It follows that \( A \) is contained in \( (\Pi_{i=0}^{n-r} f^n_{r-T} R) \Pi (\Pi_{i=0}^{n-r} f^n_{r-T} R)^{\prime} \). By length arguments we infer that \( A \) is in \( \Pi_{i=0}^{n-r} f^n_{r-T} R \) and that \( A \) is a \( k \)-linear combination of the \( f^n_{r-T} s \). Hence we conclude that \( f^n_i \) is a \( k \)-linear combination of \( f^n_x f^n_{r-T} \), and this completes the proof of the result. \( \square \)

Since the maps in the minimal projective resolution of \( \Lambda_0 \) as a right \( \Lambda \)-module are given by the \( h^{n-1}_{ji} \), we explicitly point out the following relationship.

**Corollary 1.2.** Keeping the notation of Theorem 1.1, we have

\[
h^{n-1}_{ji} = \sum_{i=0}^{t_1} f^1_{ij} c_{ji}(n, i, n-1)
\]

for \( n \geq 1 \), and \( i \) and \( j \), with \( 0 \leq i \leq t_n \), and \( 0 \leq j \leq t_{n-1} \).

Before applying Theorem 1.1 we need the following lemma, where \( J \) denotes the ideal generated by the arrows in \( Q \).

**Lemma 1.3.** Let \( \{x_i\}_{i \in I} \) be a set of elements in the linear span of \( B_s \). Suppose that \( \{x_i\}_{i \in I} \) is linearly independent viewed as vectors over \( k \). Then \( \sum_{i \in I} R x_i \) and \( \sum_{i \in I} x_i R \) are direct sums.

**Proof.** Suppose that \( \sum_i \sum_{j \in I} c_{ij} q_{ij} x_j = 0 \) in \( R \) for some elements \( c_{ij} \) in \( k \) and some paths \( q_{ij} \) in \( R \). Since all the paths occurring in any \( x_i \) have the same length, we can assume without loss of generality that the paths \( q_{ij} \) all have the same length, say \( t \). Since \( J^t = \Pi_{i \in I} q R_i \), fixing \( q \) in \( B_t \), it follows that \( \sum_i q_{ij} = 0 \), which implies that \( \sum_i q_{ij} = c_{ij} x_j = 0 \). By assumption, we have that \( c_{ij} = 0 \) for all \( q_{ij} = q \). Hence we infer that \( \sum_{i \in I} R x_i \) is a direct sum. Similarly, \( \sum_{i \in I} x_i R \) is a direct sum. \( \square \)

We now show that the \( \{f^n_i\} \) obtained from a right minimal projective resolution of \( \Lambda_0 \) and the \( \{g^n_i\} \) obtained from a left minimal projective resolution of \( \Lambda_0 \) can be chosen to be the same.

**Proposition 1.4.** Let \( \Lambda = k Q / I \) be a Koszul algebra. Let \( \{f^n_i\}_{i=0}^{t_n} \) and \( \{g^n_i\}_{i=0}^{s_n} \) define a minimal resolution of \( \Lambda_0 \) as a right \( \Lambda \)-module and as a left \( \Lambda \)-module, respectively. Then \( s_n = t_n \) for all \( n \geq 0 \) and the set \( \{g^n_i\}_{i=0}^{t_n} \) can be chosen to be equal to the set \( \{f^n_i\}_{i=0}^{s_n} \) for all \( n \geq 0 \).

**Proof.** For \( n \) equal to 0, 1, or 2 the result is clear. Let \( n \geq 3 \). We proceed by induction on \( n \) and assume that the result is true for all \( i < n \). By Theorem 1.1, for each \( i \) with \( 0 \leq i \leq t_n \) the equalities

\[
f^n_i = \sum_{p,q} c_{pq} f^1_p f^{n-1}_q = \sum_{p',q'} c_{p',q'} f^2_{p'} f^{n-2}_q
\]

hold for some \( c_{pq} \) and \( c_{p',q'} \) in \( k \). Hence \( f^n_i \) is in \( (\Pi_{q} R f^{n-1}_q) \cap (\Pi_{r} R f^{n-2}_r) \), which, by induction, is equal to \( (\Pi_{q=0}^s R g^n_q) \Pi (\Pi_{r=0}^s R g^n_q) \). Since \( \sum f^n_i R \) is direct, the set
\{f_i^n\} is linearly independent as vectors over \(k\), and therefore the sum \(\sum Rf_i^n\) is direct by Lemma 1.3. By length considerations, \(\{f_i^n\}_{i=0}^{t_n}\) is contained in the \(k\)-linear span of \(\{g_p^n\}_{p=0}^{s_n}\). Therefore \(t_n \leq s_n\). By switching the roles of \(\{f_i^n\}\) and \(\{g_p^n\}\) and using the argument above, we conclude that \(\{g_p^n\}_{p=0}^{s_n}\) is linearly independent and each \(g_p^n\) is in \(k\)-linear span of \(\{f_i^n\}_{i=0}^{t_n}\). Hence \(s_n = t_n\). By Lemma 1.3 it follows that \(\Pi_{i=0}^{t_n} Rf_i^n = \Pi_{i=0}^{s_n} Rg_i^n\). This shows that we can choose the set \(\{g_p^n\}_{p=0}^{s_n}\) equal to \(\{f_i^n\}_{i=0}^{t_n}\).

Proposition 1.4 implies that, given a minimal projective resolution of \(\Lambda_0\) as a right \(\Lambda\)-module in the form of \(\{f_i^n\}\), we have all the information to construct a minimal projective resolution of \(\Lambda_0\) as a left \(\Lambda\)-module. More precisely, take the \(\{f_i^n\}\) as the \(\{g_p^n\}\), and the maps in the left resolution are given by \(g_p^n \mapsto \sum_{t=0}^{s_n-1} c_{pq}(n, i, 1)g_q^{i}g_q^{-1}\).

2. A Minimal Projective Bimodule Resolution of \(\Lambda\)

In this section we turn our attention to the construction of a minimal projective \(\Lambda^e\)-resolution of \(\Lambda\). This construction uses the comultiplicative structure of the minimal projective resolution of \(\Lambda_0\) as a right \(\Lambda\)-module found in Theorem 1.1. This is applied to show an unpublished result of E. L. Green and D. Zacharia that \(\Lambda\) is a Koszul algebra if and only if \(\Lambda\) is a (right) linear module over \(\Lambda^e\).

The following result also shows that the knowledge of the \(\{f_i^n\}\) from a minimal projective resolution of \(\Lambda_0\) as a right \(\Lambda\)-module is sufficient to explicitly give the projective modules and the differentials in a minimal projective resolution of \(\Lambda\) as a right \(\Lambda^e\)-module. The structure of the projective modules in a minimal projective resolution of \(\Lambda\) as right \(\Lambda^e\)-module was first given in [6]. Recall that the notation \(\overline{\cdot}\) denotes the natural residue class of \(\ast \mod \ I\). Let \(\{c_{pq}(n, i, r)\}\) be as in Theorem 1.1.

**Theorem 2.1.** Let \(\Lambda = kQ/I\) be a Koszul algebra, and let \(\{f_i^n\}_{i=0}^{t_n}\) define a minimal resolution of \(\Lambda_0\) as a right \(\Lambda\)-module. A minimal projective resolution \((P, \delta)\) of \(\Lambda\) over \(\Lambda^e\) is given by

\[
P^n = \Pi_{i=0}^{t_n} \Lambda_0 (f_i^n) \otimes_k \mathfrak{l}(f_i^n) \Lambda
\]

for \(n \geq 0\), where the \(j\)-th component of the differential \(\delta^n: P^n \to P^{n-1}\) applied to the \(i\)-th generator \(\mathfrak{o}(f_i^n) \otimes (f_i^n)\) is given by

\[
\sum_{p=0}^{t_i} c_{pj}(n, i, 1)f_j^{p-1}\mathfrak{o}(f_j^{p-1}) \otimes (f_j^{p-1}) + (-1)^n \sum_{q=0}^{t_k} c_{pq}(n, i, n-1)\mathfrak{o}(f_q^{n-1}) \otimes (f_q^{n-1}) f_q^n
\]

for \(j = 0, 1, \ldots, t_n-1\) and \(n \geq 1\), and \(\delta^0: \Pi_{i=0}^{t_0} \Lambda e_i \otimes_k e_i \Lambda \to \Lambda\) is the multiplication map.

In particular, \(\Lambda\) is a linear module over \(\Lambda^e\).

**Proof.** Direct computations show that \((\delta)^2 = 0\), so that \((P, \delta)\) is a linear complex. In addition, note that \((\Lambda_0 \otimes \Lambda \mathfrak{F}, 1_{\Lambda_0} \otimes \delta)\) is a minimal resolution of \(\Lambda_0\) as a right \(\Lambda\)-module.

In our setting, we have that \(\Lambda^e/\text{rad}\Lambda^e \simeq \text{Hom}_k(\Lambda_0, \Lambda_0)\). Let \((\mathfrak{F}, d)\) be a minimal projective resolution of \(\Lambda\) as a right \(\Lambda^e\)-module. Then by [3, Chap. IX, Proposition
4.3] we have that 
\[ \text{Hom}_{\Lambda^c/\text{rad } \Lambda^c} (F^n / F^n \text{ rad } \Lambda^c, \Lambda^c / \text{ rad } \Lambda^c) \cong \text{Ext}_{\Lambda^c}^n (\Lambda, \Lambda^c / \text{ rad } \Lambda^c) \]
\[ \cong \text{Ext}_{\Lambda}^n (\Lambda_0, \Lambda_0) \]
\[ \cong \text{Hom}_{\Lambda_0} (\Pi_{i=0}^n f_i^n R / \Pi_{i=0}^n f_i^n J, \Lambda_0) \]
for all \( n \geq 0 \). In particular, \( P^n \cong F^n \) as \( \Lambda^c \)-modules for all \( n \geq 0 \), and hence \( P^n / P^n \text{ rad } \Lambda^c \cong F^n / F^n \text{ rad } \Lambda^c \) as \( \Lambda^c / \text{ rad } \Lambda^c \)-modules for all \( n \geq 0 \). Note that these need not be isomorphic as graded modules, but, we in fact show that this is the case.

Since \((P, \delta)\) is a complex, we obtain the following commutative diagram

\[
\begin{array}{ccccccccc}
\cdots & \rightarrow & P^2 & \delta^2 & P^1 & \delta^1 & P^0 & \delta^0 & \Lambda & \rightarrow & 0 \\
\downarrow \alpha^2 & & \downarrow \alpha^1 & & \downarrow \alpha^0 & & \downarrow \alpha^0 & & \downarrow \Lambda & & 0 \\
\cdots & \rightarrow & F^2 & \delta^2 & F^1 & \delta^1 & F^0 & \delta^0 & \Lambda & \rightarrow & 0 \\
\end{array}
\]

Clearly \( \alpha^0 : P^0 \rightarrow F^0 \) is an isomorphism, and we get an isomorphism \( \alpha^0 |_{\text{Ker } \delta^0} : \text{Ker } \delta^0 \rightarrow \text{Ker } \delta^0 \). Hence \( \text{Ker } \delta^0 / \text{Ker } \delta^0 \text{ rad } \Lambda^c \cong F^1 / F^1 \text{ rad } \Lambda^c \). Since \( \text{Im } \delta^1 \) is contained in \( \text{Ker } \delta^0 \), this induces a map \( \beta^1 : P^1 \rightarrow \text{Ker } \delta^0 / \text{Ker } \delta^0 \text{ rad } \Lambda^c \). If \( \beta^1 \) is an isomorphism, then \( \alpha^1 \) is an isomorphism and we have exactness at \( P^0 \). Suppose that \( \beta^1 \) is not an isomorphism. Since \( P^1 \) is generated in degree 1, there is some projective summand of \( P^1 \) which is mapped to zero by \( \delta^1 \). Using the observation that \((\Lambda_0 \otimes_\Lambda F, 1_{\Lambda_0} \otimes \delta)\) is a minimal resolution of \( \Lambda_0 \) as a right \( \Lambda \)-module, we obtain a contradiction. Hence \( \beta^1 \) is an isomorphism.

Since \( \alpha^1 \) is an isomorphism, we can use the above argument replacing \( \alpha^0 \) by \( \alpha^1 \) to show that \( \alpha^2 \) is an isomorphism and exactness at \( P^1 \). By induction we infer that \((P, \delta)\) is exact. Since the terms \( \overline{f}_q \otimes (f_{j}^{n-1}) \otimes t(f_{j}^{n-1}) \) and \( \alpha(f_{j}^{n-1}) \otimes t(f_{j}^{n-1}) \overline{f}_q \) are elements of degree one in \( \Lambda^c \), we conclude that \((P, \delta)\) is a minimal linear projective resolution of \( \Lambda \) over \( \Lambda^c \). This also implies that \( \Lambda \) is a linear module over \( \Lambda^c \). The proof is now complete.

As a consequence we obtain the next corollary which was first proved by E. L. Green and D. Zacharia.

**Corollary 2.2.** Let \( \Lambda = kQ/I \) be a graded algebra. Then \( \Lambda \) is a Koszul algebra if and only if \( \Lambda \) as a (right) \( \Lambda^c \)-module is a linear module.

**Proof.** Suppose that \( \Lambda \) is a Koszul algebra. Then Theorem 2.1 implies that \( \Lambda \) has a linear projective \( \Lambda^c \)-resolution, and hence \( \Lambda \) is a linear module over \( \Lambda^c \).

Suppose that \( \Lambda \) is a linear module over \( \Lambda^c \) as a right module. Let \((P, \delta)\) be a linear projective resolution of \( \Lambda \) as a right \( \Lambda^c \)-module. Tensoring \( P \) with \( \Lambda_0 \), we obtain \( \Lambda_0 \otimes_\Lambda P \). But \( \Lambda_0 \otimes_\Lambda P \) is a linear projective resolution of \( \Lambda_0 \) as a right \( \Lambda \)-module. Hence \( \Lambda \) is a Koszul algebra, and we are done. □

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