Analytical description of thermal point-reactor parameters within particles birth and death model

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Abstract. Current work presents general principles for constructing a mathematical apparatus within the framework of a sub-Poisson distribution that describes the neutron generations emergence. On the basis of these principles, analytical expressions were obtained for describing the point reactor breeding medium main parameters. The corresponding dependencies for generation lifetime and appearance time were formed.

1. Introduction

Started in [1,2] the research direction in field of nuclear reactor physics aims to achieve common mathematical apparatus for simple analytic description of reactor’s breeding medium parameters. These studies are based primarily on the theory of Markov processes [3] having a well-developed mathematical apparatus. The description of the reactor operation within the framework of this theory makes it possible to consider it as an analog of an oscillatory system with events of nuclei-emitters emergence and decay occur in time.

The formalism of the Poisson distribution for describing the nuclei-emitters decay makes it possible to obtain simple analytic expressions for the breeding medium parameters and the processes occurring in it. However, from a physical point of view, it is incorrect due to discreteness and finite nature of the latter, for which it is appropriate to use a binomial distribution. The transition to the sub-Poisson distribution [4,5], which has the properties of both Poisson and binomial distributions, solved this problem and made it possible to treat the reactor as a harmonic oscillator.

For the purpose of creating a mathematical apparatus for physical processes inside nuclear reactor breeding medium a first step of describing a thermal point reactor has been made in current work.

2. General studies

The first implementation [2] of birth and model for describing processes in nuclear reactor as analog of ADS-system showed the probabilistic nature of this processes. Theory of probability as stated in [6,7] assumes that for majority physical processes instantaneous intensities can be built. In case of nuclear reactor breeding medium these intensities define processes of neutron appearance λ (birth) and neutron leakage μ (death). Defining system state change probability equation (1) represent the mentioned process.

$$\frac{dP_n}{dt} = \lambda_{n-1}(t) - [\lambda_n(t) + \mu_n(t)]P_n(t) + \mu_{n+1}(t)P_{n+1}(t)$$  (1)
Birth and death model describes neutron kinetics within reactor breeding medium in more generally way than it is done by solving kinetic equations described in [8] and similar subject literature. That gives possibility to consider some interesting questions concerning definition of a generation lifetime and average number of neutrons in the system.

In case of “breeding medium + neutron” the most valuable information should be the one describing total number of particles in the system – integral characteristic. Assuming instantaneous intensities this information can be gained from the integral function:

\[ M(t) = \exp\left(\int_{0}^{t} [\lambda(s) - \mu(s)] ds\right) \]  \hspace{1cm} (2)

The function under integral in equation (2) is the difference between the birth and death instantaneous intensities. For the case of nuclear reactor this under integral function must be related to some measurable parameters. These parameters were defined in [2] as prompt and delayed neutrons’ life time. Hence the equation (2) can be written as

\[ M(t) = \int_{0}^{t} \frac{e^{-s/c}}{a - b \cdot e^{-s/c}} ds \]  \hspace{1cm} (3)

Parameters \(a\), \(b\) and \(c\) are associated with \(\tau_p\) – prompt neutron life time, and \(\tau_d\) – delayed neutron life time. The association has the following form

\[ c = \tau_d \]
\[ b = c \cdot \beta \]
\[ a = b + \tau_p \]  \hspace{1cm} (4)

The integral can be derived from equations (3) and (4). This new derived formula gives us a total number \(n\) of already appeared generations depending on consideration time

\[ n(t) = \int_{0}^{t} \frac{e^{-s/c}}{(a - b \cdot e^{-s/c})} ds = \frac{c}{b} \cdot \ln[(a - b \cdot e^{-t/c}) / (a - b)] \]  \hspace{1cm} (5)

\[ n_{gen} \]

\[ (sec) \]

\[ t \]

Figure 1. Total generation number dependence on time.
The figure 1 represents the pattern of total neutron generation number dependence on time in equation (5).

It is noticeable that by a certain time the total generation number is fixed on a definite value. For example on a graph (figure 1) the total appeared number of generations $n$ equals $1.037E+03$ for such parameters of a point reactor breeding medium as $\tau_p = 0.0001$ (sec), $\tau_d = 13.0$ (sec) and $\beta = 0.0065$ (delayed neutron fraction). These numerical parameters can be calculated if the breeding medium composition is known [9].

That fact of reaching plateau on a graph (figure 1) describes one of the basic concepts in considering model: in a “critical” medium one neutron causes a chain process with events of birth and death, which leads to a neutron generation and respectively energy production. By the end of this process there will a single neutron in a medium as it was in the beginning. The presence of this neutron was achieved by $1.037E+03$ (in considered case) generations – events of birth and death.

2.1. Neutron generations properties

Before calculating point reactor parameters like $K_{eff}$ the basic properties of breeding medium must be determined. These properties can be derived from equation (5). First parameter of a breeding medium is generation appearance time:

$$t_k = -c \cdot \ln \left[ \frac{a - (a - b)e^{k/c}}{b} \right] \quad (6)$$

Generation appearance time as can be noticed depends on a generation number $k = 1,2,3...n$. This knowledge leads to another parameter – generation life time. If appearance time of two adjacent generations is known the generation life time of the first one can be found

$$\tau_k = t_{k+1} - t_k \quad (7)$$

Figure 2 (a) and (b) represents the pattern of generation appearance time (6) and generation life time (7). The data presented on a graphs is also derived with parameters $\tau_p = 0.0001$ (sec), $\tau_d = 13.0$ (sec) and $\beta = 0.0065$.

![Figure 2](image.png)

**Figure 2.** Generation appearance time (a) and lifetime (b).

Having these two parameters in equations (6) and (7) gives enough information for further analysis. For example, the probability of generation emerge can be found from equation
\[ p_k = \frac{\tau_p}{\tau_k} \]  

(8)

Emerge probability is an important characteristic of the process under investigation from the mathematical point of view: having a probability law one can predict a system state and its further behavior. The probability law from equation (8) is shown on the figure 3.

**Figure 3.** Generation emerge probability.

### 3. Point reactor parameters

To describe an operating reactor in terms of birth and death model it is necessary to find average number of neutrons on time dependence as well as neutron multiplication factor \( K \). The point-reactor suites the most for this purpose as it is an idealized model without any reactivity effects. For birth and death model equation (2) actually gives average number of neutrons in the medium in case of continuous timeline. To obtain the sampled time scale one should use equations (6) and (7). The assumption of subcritical state (on prompt neutrons and with \( \rho = -\beta \)) must be taken into account. The final form of equation for neutron average number for a point reactor was derived in details in [1] and can be used in current work for granted

\[ M(t_k) = M_0 \exp[(\rho + \beta(t_k)) \cdot t_k / \tau_k] \]  

(9)

The \( t_k \) and \( \tau_k \) in equation (9) are discrete generation appearance time and generation lifetime from equations (6–7). The \( \beta(t_k) \) can be found through the equation

\[ \beta(t_k) = \beta(1 - \exp[-t_k / \tau_c]) \]  

(10)

From the definition of neutron multiplication factor in which it equals the ratio of the neutrons average number in adjacent generations the next equation for \( K_{n,n+1} \) (neutron multiplication factor between two generations) can be obtained

\[ K_{k,k+1} = \frac{M(t_k)}{M(t_{k+1})} = \exp[(\rho - \beta(t_k)) \cdot t_k / \tau_k - (\rho - \beta(t_{k+1})) \cdot t_{k+1} / \tau_{k+1}] \]  

(11)

Taking into account that the last generation number is \( n \) the total multiplication factor is obviously can be calculated from equation

\[ K_{tot} = \frac{M(t_n)}{M_0} = \exp[(\rho + \beta(t_n)) \cdot t_n / \tau_n] \]  

(12)
Equations (9-12) give the connection between parameters of breeding medium and point reactor operating parameters.

If $K_{\text{tot}}$ is calculated for the last generation the case with $\tau_n \to \infty$ and $t_n \to \infty$ has place, as shown on figure 2 (a) and (b). Consequently, the asymptotic behavior of the coefficient is described by the equation

$$K_{\text{tot}} = \exp[\rho + \beta(t_n)]$$  \hspace{1cm} (13)

From this equation (13) with the corresponds to the equation (10) follows the condition of a self-sustaining reaction: $\rho = -\beta$. This result gives reason to believe that the thermal reactor is to be a safe stable system in general.

It is also interesting how the $K_{\text{tot}}$ behaves during the process of generating neutrons. The graphical representation of equation is shown on figure 4.

Figure 4. Effective neutron multiplication factor time dependence

Shown on figure 4 dependence explains the fact of a reactor “subcritical” state assumption. In general, that means that during the process of generating delayed neutrons the multiplication factor changes from generation to generation. At the point of $t_k = 141.065$ (sec) (for the same values of parameters $t_k$ and $\tau_k$) when the last delayed neutron is produced the $K_{\text{eff}}$ equals 1.00 and the process will start all over again.

4. Conclusions
To describe a thermal point reactor as a subcritical system with an inner source of neutrons the emitter-nuclei of prompt neutrons must be considered. The mathematical apparatus was obtained within particles birth and death model. The sub-Poisson formalism \[4,5\] with discretization of the time scale gives a reasonable physical description of the appearance process of delayed neutrons from emitter nuclei.

Dependences for lifetime of the neutron generation and generation appearance time were obtained within this mathematical apparatus. The equations obtained allows one to determine the parameters of the operation of the reactor under study.

The example of a point reactor shows the derivation of the equations for the multiplication factor (11–12) and the relation of the presented quantities to the reactivity (13). All the expressions obtained give an integral picture of the basic neutron interaction with a breeding medium. Further work aims to
scale current mathematical apparatus for different types critical and sub-critical systems in well-known experiments [10,11].

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