Quantum Group Representations and Baxter Equation

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Dedicated to Memory of Sasha Belov

Abstract

In this article we propose algebraic universal procedure for deriving ”fusion rules” and Baxter equation for any integrable model with $U_q(sl_2)$ symmetry of Quantum Inverse Scattering Method. Universal Baxter Q-operator is got from the certain infinite dimensional representation called q-oscillator one of the Universal R-matrix for $U_q(sl_2)$ affine algebra (first proposed by V. Bazhanov, S.Lukyanov and A.Zamolodchikov\textsuperscript{3} for quantum KdV case). We also examine the algebraic properties of Q-operator.
0 Introduction

The problem of diagonalization of an infinite set of mutually commuting Integrals of Motion (IM) for some Integrable Quantum Theory is solved by the Quantum Inverse Scattering Method (QISM) [1]. The generating function for IM is trace of monodromy matrix \( T(\lambda) \) depending on some spectral parameter. The powerful method for finding eigenvalues of \( T(\lambda) \) was proposed by Baxter [2], where he used so called \( Q(\lambda) \)-operator satisfying Baxter eq.

\[
T(\lambda) Q(\lambda) = Q(q\lambda) + Q(q^{-1}\lambda)
\]  

(1)

The properties of \( Q \)-operator for quantum KdV system were examined in the paper [3].

Another approach for the objects of QISM such as \( Q \)-operator or monodromy matrix is based on so called universal \( R \)-matrix. An explicit formula for universal \( R \)-matrix for quantum affine algebras was found in [4]. Later we will consider only the case of \( U_q(\hat{sl}_2) \) algebra, though the proposed technique is applicable for general affine quantum algebras.

For \( U_q(\hat{sl}_2) \) algebra the universal \( R \)-matrix lies in the square \( R \in U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \) and satisfies the Yang-Baxter (YB) eq.

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},
\]

in the \( U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \). We write \( R = \sum_i A_i \otimes B_i \in U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \) and let

\[
R_{12} = \sum_i A_i \otimes B_i \otimes 1, \quad R_{13} = \sum_i A_i \otimes 1 \otimes B_i, \quad R_{23} = \sum_i 1 \otimes A_i \otimes B_i.
\]

Monodromy matrix can be got from the universal \( R \)-matrix \( R \in U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \) using matrix \( n \times n \) representation with a spectral parameter for the first \( U_q(\hat{sl}_2) \) algebra and some other representation (depending on a integrable model in consideration) for second \( U_q(\hat{sl}_2) \) algebra in the square \( U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \). First, it was noticed by L.Faddeev in [4]. In the same way, it is possible to get \( Q \)-operator from the universal \( R \)-matrix \( R \in U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \). For this we choose certain infinite dimensional representation for the first \( U_q(\hat{sl}_2) \) algebra.

Using \( Q \)-operator or monodromy matrix presentation from the universal \( R \)-matrix one clarifies the "fusion rules" and Baxter eq. The origin of these eqs. is in certain \( U_q(\hat{sl}_2) \) representation square. Namely, we have \( (M_1 \text{ and } M_2) \) two QISM objects (monodromy matrix or \( Q \)-operators) that got from universal \( R \)-matrix using two certain representations \( (W_1 \text{ and } W_2) \) for the first \( U_q(\hat{sl}_2) \) algebra in \( R \in U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \). Then, there is one-to-one correspondence between "fusion" like relation for \( \text{tr}_{W_1} M_1 \) and \( \text{tr}_{W_2} M_2 \) and the structure of the representation square \( W_1 \otimes W_2 \). We will explain how using the existence of a subrepresentation in square \( W_1 \otimes W_2 \) one can derive "fusion rules" and Baxter eq.

According to the methods of universal \( R \)-matrix for monodromy matrix or \( Q \)-operators we specify representation only of the first \( U_q(\hat{sl}_2) \) algebra for \( R \in U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \), and representation of the second one (depending on a concrete model in hand) is not fixed. So, we propose universal technique for deriving Baxter eq. as well as for algebraic understanding of Baxter \( Q \)-operator, which is very important in the Algebraic Bethe Ansatz.
The text is organized as follows. In Section 1 we remind some key moments of QISM and interpret monodromy matrix via universal $R$–matrix representations. We introduce algebraic method for constructing "fusion" like relations and give some examples for finite dimensional $U_q(\hat{sl}_2)$ representations in Section 2. We define infinite dimensional representation for Baxter $Q$- operator from universal $R$–matrix in Section 3. There we derive algebraically Baxter eq. and some other "fusion" like eqs. for $Q$- operator. The discussion and conclusion are presented in Section 4.

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1 Origin of QISM Ingredients from Universal R-matrix

In this section we use the notations from [5]. The quantum L-operator is the main object of the QISM. $L_{a,n}(\lambda)$ is a matrix in an auxiliary space $\nu_a$, the matrix elements of $L_{a,n}(\lambda)$ are operators in a Hilbert space $h_n$, $n = 1, \cdots, N$ associated with the site of the lattice, depending on the spectral parameter $\lambda$. The operators in different sites commute. In this way, the operator $L_{a,n}(\lambda)$ is the operator in the square $\nu_a \otimes h_n$.

The fundamental commutation relations for the matrix elements of $L_{a,n}(\lambda)$ is the YB eq.:

$$R_{a_1,a_2}(\lambda/\mu)L_{a_1,n}(\lambda)L_{a_2,n}(\mu) = L_{a_2,n}(\mu)L_{a_1,n}(\lambda)R_{a_1,a_2}(\lambda/\mu).$$

(2)

This is an equation in $\nu_{a_1} \otimes \nu_{a_2} \otimes h_n$. The indices $a_1$ and $a_2$ and the variables $\lambda$ and $\mu$ are associated with the auxiliary spaces $\nu_1$ and $\nu_2$, respectively. The matrix $R_{a_1,a_2}$ is one in the space $\nu_{a_1} \otimes \nu_{a_2}$.

Further, we consider only the special form of $R_{a_1,a_2}$– matrix, so called trigonometric R-matrix, $4 \times 4$. A lot of interesting models on a lattice (XXZ spin model, the lattice Sine- Gordon system, Volterra system etc.) and in continuum ( quantum mKdV system [8]) are described by the trigonometric R-matrix.

Another important object of the QISM is the monodromy matrix

$$M_a(\lambda) = L_{a,N}(\lambda)L_{a,N-1}(\lambda) \cdots L_{a,1}(\lambda).$$

(3)

The monodromy matrix satisfies commutation relations identical with the YB ones for L-operators [2]. It follows that the traces $T_a(\lambda) = \text{tr}_{\nu_a} M_a(\lambda)$ over the auxiliary spaces of the monodromy matrix with the different parameters commute $[T_{a_1}(\lambda), T_{a_2}(\mu)] = 0$. That is why the trace of the monodromy matrix is the generating function for the integrals of motion.

It is useful to consider so called fundamental L- operator [7], $L_{n_1,n_2}(\lambda)$, i.e. the operator in $h_{n_1} \otimes h_{n_2}$. In other words the auxiliary space coincides with the quantum one.
The YB eq. for the fundamental L-operator is following
\[ L_{a,n_1}(\lambda)L_{a,n_2}(\mu)L_{a,n_2}(\mu/\lambda) = L_{n_1,n_2}(\mu/\lambda)L_{a,n_2}(\mu)L_{a,n_1}(\lambda), \quad (4) \]
as the eq. in \( \nu_a \otimes h_{n_1} \otimes h_{n_2} \). The monodromy matrix for the fundamental L-operator gives the set of local integrals of motion.

It is possible to describe all the objects of QISM (trigonometric R-matrix, L-operator and monodromy matrix as well as fundamental L-operator) with the only algebraic object - universal \( R \)-matrix \([5]\).

We will briefly remind some facts about it \([4]\). Below, we consider the simple case of the affine quantum \( U_q(\hat{sl}_2) \) algebra with Cartan matrix \( A = (a_{ij}) \), \( i, j = 0, 1 \), \( A = \left( \begin{array}{cc} 2 & -2 \\ -2 & 2 \end{array} \right) \). This is an associative algebra with generators \( e_{\pm \alpha_i}, k_{\alpha_i}^\pm \), \( (i = 0, 1) \), and the defining relations
\[
[k_{\alpha_i}^\pm, k_{\alpha_j}^\pm] = 0, \quad k_{\alpha_i}^\pm e_{\pm \alpha_j} = q^{\pm (\alpha_i, \alpha_j)}e_{\pm \alpha_j}k_{\alpha_i},
\]
\[
[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q - q^{-1}},
\]
\[
(ad_q e_{\pm \alpha_i}) (1 - a_{ij}) e_{\pm \alpha_j} = 0 \quad \text{for} \quad i \neq j, \quad q' = q, q^{-1},
\]
where \( (ad_q e_{\alpha}) e_{\beta} \) is a q-commutator:
\[
(ad_q e_{\alpha}) e_{\beta} \equiv [e_{\alpha}, e_{\beta}]_q = e_{\alpha}e_{\beta} - q^{(\alpha, \beta)}e_{\beta}e_{\alpha}
\]
and \( (\alpha, \beta) \) is a scalar product of the roots \( \alpha \) and \( \beta \): \( (\alpha_i, \alpha_j) = a_{ij} \). The condition \( 1 = k_{\alpha_0}k_{\alpha_1} \) is also imposed.

We define a comultiplication in \( U_q(\hat{sl}_2) \) by the formulas
\[
\Delta(k_{\alpha_i}) = k_{\alpha_i} \otimes k_{\alpha_i},
\]
\[
\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes k_{\alpha_i} + 1 \otimes e_{\alpha_i},
\]
\[
\Delta(e_{-\alpha_i}) = e_{-\alpha_i} \otimes 1 + k_{\alpha_i}^{-1} \otimes e_{-\alpha_i},
\]
and the twisted comultiplication
\[
\Delta'(k_{\alpha_i}) = k_{\alpha_i} \otimes k_{\alpha_i},
\]
\[
\Delta'(e_{\alpha_i}) = k_{\alpha_i} \otimes e_{\alpha_i} + e_{\alpha_i} \otimes 1, \quad \Delta'(e_{-\alpha_i}) = 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_{\alpha_i}^{-1},
\]

By the definition, the universal \( R \)-matrix is an object \( R \) in \( U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \) such that \( \Delta'(g)R = R\Delta(g) \), for any \( g \in U_q(\hat{sl}_2) \) and
\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \quad (6)
\]
which is the universal form of the YB eq. in the \( U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \). We write
\[R = \sum_i A_i \otimes B_i \in U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2)\] and let
\[R_{12} = \sum_i A_i \otimes B_i \otimes 1, \quad R_{13} = \sum_i A_i \otimes 1 \otimes B_i, \quad R_{23} = \sum_i 1 \otimes A_i \otimes B_i.\]
In Appendix A we present the explicit form of the universal $R$-matrix for $U_q(\hat{sl}_2)$ following [4]. We know ([9]) that universal $R$-matrix belongs to the square of $U_q(b_+) \otimes U_q(b_-)$, where $b_{\pm}$ are positive (negative) borel subalgebras of $U_q(\hat{sl}_2)$, generated by $e_{\alpha_i}, k_{\alpha_i}^{\pm 1}$ and $e_{-\alpha_i}, k_{\alpha_i}^{\pm 1}$, $(i = 0, 1)$, respectively.

One can present QISM L-operators from the universal $R$-matrix using the certain representation of $U_q(\hat{sl}_2)$. For example, the trigonometric $4 \times 4$ $R$-matrix got from the universal $R$-matrix representing $U_q(\hat{sl}_2)$ in terms of matrix $2 \times 2$ with the spectral parameter.

It is possible to get the $L_{a,n}(\lambda)$ operator from the universal $R$-matrix $R \in U_q(b_+) \otimes U_q(b_-)$ representing the first algebra $U_q(b_+)$ in matrixes with the spectral parameter and the second one $U_q(b_-)$ in quantum (infinite dimensional) space. Substituting the co-multipled $\Delta^{(N-1)} U_q(\hat{sl}_2)$ quantum space generators in the $U_q(b_-)$ part of the universal $R$-matrix we get the monodromy matrix of N-sites ([3]).

Now we illustrate two different forms of the YB eq. (2), (4) from the point of view of the universal one (1). Eq. (3) is one for the algebraic elements in $U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2) \otimes U_q(\hat{sl}_2)$. To get the eq. (4) we represent the first and the second algebras $U_q(\hat{sl}_2)$ in finite dimensional (with respect to $U_q(sl_2)$) representation, say matrix with the spectral parameter and the third in quantum space $h_n$. The eq. (3) is the representation of the eq. (4) with the first algebra in matrixes with the spectral parameter and the second and the third algebras in the quantum spaces $h_{n_1}$ and $h_{n_2}$. Below we will consider the ingredients of QISM got from the universal $R$-matrix $R = \sum_i A_i \otimes B_i \in U_q(b_+) \otimes U_q(b_-)$ for certain representation of the first $U_q(\hat{sl}_2)$ algebra and for any representation of the second $U_q(\hat{sl}_2)$ one (for any integrable model with $U_q(\hat{sl}_2)$ symmetry).

2 "Fusion Rules" in Algebraic Presentation

The treatment of the origin of the QISM L-operators as the certain representation of the universal $R$-matrix gives possibility for an algebraic comprehension of the "fusion rules" and the Baxter eq. For these relations we use only the first algebraic elements $A_i$ of the universal $R$-matrix $R = \sum_i A_i \otimes B_i \in U_q(b_+) \otimes U_q(b_-)$, so nothing depends on a representation of the second $U_q(\hat{sl}_2)$ algebra (algebraic elements $B_i$) and, consequently, on the concrete model of the QISM.

Consider the evaluation representation for the $U_q(\hat{sl}_2)$ in terms of $U_q(sl_2) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$

$$ e_{\pm a_1} = e_{\pm a}, \quad e_{\pm a_0} = \lambda^{\pm} e_{\mp a}, \quad k_{a_1} = k, \quad (7) $$

where $U_q(sl_2)$ algebra has generators $e_{\pm a}$ and $k_{\pm 1}$ with the ordinary commutation relations

$$ ke_{\pm a} = q^{\pm 2} e_{\pm a} k, \quad [e_{a}, e_{-a}] = \frac{k - k^{-1}}{q - q^{-1}}, $$

Let $V_j$ be the irreducible $2j + 1$ -dimensional representation of $U_q(sl_2)$, $j = 0, \frac{1}{2}, 1, \cdots$. One can define the representation $\hat{V}_j(\lambda)$ of the affine $U_q(\hat{sl}_2)$ algebra from (7) and the representation $V_j$ of the simple $U_q(sl_2)$ algebra.

Let us denote by $M_j(\lambda)$ some L-operator or monodromy matrix got from universal $R$-matrix $R = \sum_i A_i \otimes B_i \in U_q(b_+) \otimes U_q(b_-)$ by following representation
• for the first algebra \( U_q(b_+) \) we take the evaluation representation \( V_j(\lambda) \) of \( U_q(sl_2) \) spin \( j \);

• the second algebra \( U_q(b_-) \) of the universal \( R \)-matrix is represented in some quantum space, depending on a physical model in consideration.

Let \( T_j(\lambda) = \text{tr}_{V_j(\lambda)} M_j(\lambda) \) be the trace of monodromy matrix. In fact,

\[
[T_j(\lambda), T_{j'}(\lambda')] = 0,
\]

that is why \( T_j(\lambda) \) for any spin \( j \) are generating functions for integrals of motion. There is an algebraic dependence between traces of monodromy matrixes with different spin \( j \), so called "fusion rules"

\[
T_j(q\lambda) T_j(q^{-1}\lambda) = 1 + T_{j-rac{1}{2}}(\lambda) T_{j+rac{1}{2}}(\lambda)
\]

Remark. The fusion procedure (8) differs from the ordinary notations by \( T(\lambda^2) = T'(\lambda) \). The origin of this is in different evaluation representations. For example, if we write the representation as

\[
e_{\pm\alpha_1} = \lambda^{\pm 1} e_{\pm\alpha}, \quad e_{\pm\alpha_0} = \lambda^{\pm 1} e_{\mp\alpha}, \quad k_{\alpha_1} = k,
\]

we get the ordinary form of fusion rules. Below for the reasons of algebraic simplicity we will consider the representation (7).

Let us illustrate the fusion rules (9) in algebraic way. We will claim that eq. (9) can be derived from a consideration of square of \( U_q(\hat{sl}_2) \) representations.

Indeed, there exists the trivial representation \( 1 \) in the square \( V = V_j(\lambda q^{-1}) \otimes V_j(\lambda q) \). The factor \( V/1 \) is isomorphic to

\[
V_{j-rac{1}{2}}(\lambda) \otimes V_{j+rac{1}{2}}(\lambda).
\]

Consider now two universal \( R \)-matrix \( R_{1,2} = \sum_i A_i^{(1)} \otimes B_i \) and \( R'_{1,2} = \sum_n A_n^{(1')} \otimes B_n \), acting in \( V_j(\lambda q^{-1}) \otimes h \) and \( V_j(\lambda q) \otimes h \), respectively. The Hilbert space \( h \) depending on a model in consideration is not fixed.

By the definition,

\[
T_j(q^{-1}\lambda) T_j(q\lambda) = \text{tr}_{V_j(\lambda q^{-1}) \otimes V_j(\lambda q)} \sum_{i,n} A_i^{(1)} \otimes A_n^{(1')} \otimes B_i B_n
\]

To take the trace over \( V = V_j(\lambda q^{-1}) \otimes V_j(\lambda q) \) one needs two steps:

• to calculate the trace over the trivial subrepresentation 1 of the representation \( V \) (it gives us 1 in fusion rules (9));

• take the trace over the factor space \( V/1 \cong V_{j-rac{1}{2}}(\lambda) \otimes V_{j+rac{1}{2}}(\lambda) \) (we get the term \( T_{j-rac{1}{2}}(\lambda) T_{j+rac{1}{2}}(\lambda) \) ).

6
In this way we derived the fusion procedure (9) extracting subrepresentation from square of certain $U_q(\widehat{sl}_2)$ representations. One can get other fusion rules from the following fact [8]:

if $b/a = q^{(2j+2k-4r+2)}$ for the square $V = V_j(a) \otimes V_k(b)$, \((j \geq k)\) of $U_q(\widehat{sl}_2)$ representations \((0 < r \leq \min(j,k))\), then $V$ has a unique proper subrepresentation $W$:

1. if $b/a = q^{(2j+2k-4r+2)}$, we have
   
   $$W \cong V_{j-r}(q^{2r}a) \otimes V_{k+r}(q^{2r}b),$$
   $$V/W \cong V_{r-j}(q^{2j-2r+1}a) \otimes V_{j+k-r+\frac{1}{2}}(q^{-(2j-2r+1)}b);$$

2. if $b/a = q^{-(2j+2k-4r+2)}$, we have
   
   $$W \cong V_{r-j}(q^{-(2j-2r+1)}a) \otimes V_{j+k-r+\frac{1}{2}}(q^{2j-2r+1}b),$$
   $$V/W \cong V_{j-r}(q^{2r}a) \otimes V_{k+r}(q^{2r}b).$$

For example, repeating procedure (10) for the first case of $b/a$ one gets the following fusion rules

$$T_j(a) T_k(b) = T_{j-r}(q^{2r}a) T_{k-r}(q^{2r}b) + T_{r-j}(q^{2j-2r+1}a) T_{j+k-r+\frac{1}{2}}(q^{-(2j-2r+1)}b). \quad (11)$$

This method is consistent with the commutativity (8). Namely, let us put $j = k$ in eq. (11). As we know the consideration of the square $V_j(a) \otimes V_j(b)$ leads us to eq. (11) for $j = k$. Another possibility to get rules (11) is to extract subrepresentation from twisted square $V_j(b) \otimes V_j(a)$. The difference with the ordinary square $V_j(a) \otimes V_j(b)$ is in interchanging of sub- and factor-representations. So, we get the same eq. (11) from the twisted square for commuting $T_j(a)$ and $T_j(b)$.

We see now that the procedure for deriving the fusion rules is identical to the extracting of some subrepresentation from the square of certain $U_q(\widehat{sl}_2)$ representations. Of course, it is not limited by the finite dimensional representations $V_j$. In the next section we will see how to construct Baxter operator $Q(\lambda)$ from the universal $R$–matrix using infinite dimensional $U_q(b_+)$ representations.

### 3 Infinite Dimensional Representations and Baxter Equation

An important object in QISM is so called Baxter operator $Q(\lambda)$ [4]. It can be defined in statistical and integrable models via Baxter eq.

$$Q(\lambda) T_{\frac{1}{2}}(\lambda) = Q(q^2 \lambda) + Q(q^{-2} \lambda) \quad (12)$$

(The difference with the standard Baxter eq. [4] is explained in Remark in Section 2).

The Baxter eq. plays an important role in Bethe Ansatz formalism, $Q(\lambda)$ has a simple eigenvalues on Bethe vectors. So, the algebraic interpretation of Baxter eq. is useful for
getting Bethe vectors in purely algebraic terms (i.e. in terms of $U_q(\mathfrak{sl}_2)$ representation space).

Using method of Section 2 and eq. (10) one can get $Q(\lambda)$ operator from $R = \sum_i A_i \otimes B_i \in U_q(b_+ \otimes U_q(b_-)$ by following representations of the first algebra $U_q(b_-)$

$$e_{a_0}e_{a_1} - q^2 e_{a_1}e_{a_0} = \frac{\lambda}{q^{-2} - 1} \quad \text{(the representation } V_-(\lambda))$$

and

$$e_{a_1}e_{a_0} - q^2 e_{a_0}e_{a_1} = \frac{\lambda}{q^{-2} - 1} \quad \text{(the representation } V_+(\lambda))$$

we have

$$Q_+(\lambda) = \text{tr}_{V_+(\lambda)} R \quad Q_-(\lambda) = \text{tr}_{V_-(\lambda)} R$$

for any representation of $U_q(b_-)$ (for any integrable model) \footnote{We choose the infinite dimensional representation space $V_{\pm}(\lambda)$ such that the trace $Q_{\pm}(\lambda) = \text{tr}_{V_{\pm}(\lambda)} R$ exists.} Such $Q$-operators satisfies Baxter eq. (12) and commute with each other and $T_j(\lambda)$. The representations (13-14) called q-oscillator algebras were first used in [3] for constructing of $Q$-operators in KdV system.

We illustrate now Baxter eq. (12) in terms of procedure (10) for $Q_+$- operator (analogously, for $Q_-$ operator.) Let two universal R- matrices

$$R_{1,2} = \sum_i A_i^{(1)} \otimes B_i \quad \text{and} \quad R_{1',2} = \sum_n A_n^{(1') \otimes B_n}.$$ 

act in spaces $V_+(\lambda) \otimes h$ and $V_{\frac{1}{2}}(\lambda) \otimes h$, respectively, (the Hilbert space $h$ is arbitrary). For traces

$$\text{tr}_{V_+(\lambda)} R_{1,2} \quad \text{and} \quad \text{tr}_{V_{\frac{1}{2}}(\lambda)} R_{1',2}$$

one has

$$Q_+(\lambda) T_{\frac{1}{2}}(\lambda) = \text{tr}_{V_+(\lambda) \otimes V_{\frac{1}{2}}(\lambda)} \sum_{i,n} A_i^{(1)} \otimes A_n^{(1') \otimes B_i B_n}$$

(15)

For calculating the trace over the square $V_+(\lambda) \otimes V_{\frac{1}{2}}(\lambda)$ we should examine it for an existence of some subrepresentation.

Indeed, it is possible to show (see Appendix B), that the square $V_+(\lambda) \otimes V_{\frac{1}{2}}(\lambda)$ has the subrepresentation $V_+(q^{-2}\lambda)$. The factor representation $V_+(\lambda) \otimes V_{\frac{1}{2}}(\lambda)/V_+(q^{-2}\lambda)$ is isomorphic to $V_+(q^2\lambda)$

$$V_+(\lambda) \otimes V_{\frac{1}{2}}(\lambda)/V_+(q^{-2}\lambda) \cong V_+(q^2\lambda).$$

(16)

To prove Baxter eq. from eqs. (13) and (16) one needs two usual steps

- to take trace over the subrepresentation $V_+(q^{-2}\lambda)$ (to get the term $Q_+(q^{-2}\lambda)$);
- to take trace over the factor-presentation $V_+(q^2\lambda)$ (to get the term $Q_+(q^2\lambda)$).
Baxter operator for the representations \([1,3]\) and \([4]\) was derived analytically by V.Bazhanov, S.Lukyanov and A.Zamolodchikov \([3]\) for Quantum KdV Model. To get this Q-operator from universal \(R\)-matrix \(R = \sum_i A_i \otimes B_i \in U_q(b_+) \otimes U_q(b_-)\) one needs to represent \(U_q(b_-)\) algebra via vertex operators

\[
e_{a_1} = \int du : e^{2\varphi(u)} : \quad e_{a_0} = \int du : e^{-2\varphi(u)} :,
\]

where \(\varphi(u)\) is the standard bosonic field \([3,4]\).

The result of A.Volokov \([3]\) (the coincidence of Q-operator and trace of fundamental L-operator \(L_{n_1,n_2}(\lambda)\) \([4]\) for Volterra model \([10]\)) can be interpreted in terms of representations of universal \(R\)-matrix. The quantum space (the Hilbert space \(h\)) generators for Volterra L-operator satisfies the relations \([13]\). So, Q-operator for Volterra model can be got from \(R = \sum_i A_i \otimes B_i \in U_q(b_+) \otimes U_q(b_-)\) by using the same representations for \(U_q(b_+)\) and \(U_q(b_-)\), and thus,

\[
Q(\lambda) = \text{tr}_{h_{n_1}} L_{n_1,n_2}(\lambda).
\]

Now the natural question arises: are there any subrepresentation in infinite dimensional squares \(V_+(a) \otimes V_+(b)\) or \(V_+(a) \otimes V_+(b)\) with certain \(a\) and \(b\)? In other words, are there any relations for squares of Baxter operators? The answer is positive.

Consider the square \(V_+(q^{-1}\lambda) \otimes V_-(q\lambda)\). It is possible to prove, that there exists the set of trivial subrepresentations \(1\). The factor over it gives

\[
V_+(q^{-1}\lambda) \otimes V_-(q\lambda)/1 \cong V_+(q\lambda) \otimes V_-(q^{-1}\lambda).
\]

Using our technique we derive the following eq.

\[
Q_+(q^{-1}\lambda) Q_-(q\lambda) = \text{const} + Q_+(q\lambda) Q_-(q^{-1}\lambda). \tag{17}
\]

Here the const depends on a normalization of representation spaces \(V_+(\lambda)\). Eq. \((17)\) was first obtained for KdV system in \([3]\).

From the eqs. \((12)\) and \((17)\) we can derive another dependence between \(T_2\) and Q-operators

\[
\text{const} \cdot T_2(\lambda) = Q_+(q^{-2}\lambda) Q_-(q^2\lambda) - Q_+(q^2\lambda) Q_-(q^{-2}\lambda). \tag{18}
\]

It is possible also to get this eq. considering representation square \(V_+(q^{-2}\lambda) \otimes V_-(q^2\lambda)\). Indeed, there exists subrepresentation, isomorphic to \(V_2(\lambda)\), and after factorizing we have

\[
V_+(q^{-2}\lambda) \otimes V_-(q^2\lambda)/V_2(\lambda) \cong V_+(q^2\lambda) \otimes V_-(q^{-2}\lambda).
\]

Eq. \((18)\) is the special case of the general formula for \(T_2(\lambda)\) \([3]\) for KdV system

\[
\text{const} \cdot T_2(\lambda) = Q_+(q^{-2j+1}\lambda) Q_-(q^{2j+1}\lambda) - Q_+(q^{2j+1}\lambda) Q_-(q^{-2j+1}\lambda),
\]

which can be proved factorizing the square \(V_+(q^{-2j+1}\lambda) \otimes V_-(q^{2j+1}\lambda)\) over the subrepresentation \(V_j(\lambda)\)

\[
V_+(q^{-2j+1}\lambda) \otimes V_-(q^{2j+1}\lambda)/V_j(\lambda) \cong V_+(q^{2j+1}\lambda) \otimes V_-(q^{-2j+1}\lambda).
\]

It is not clear now, are there other relations connecting \(Q_\pm\)-operators? It seems, that considering squares \(V_+(a) \otimes V_+(b)\) for proper \(a\) and \(b\) one can get sub- or factor representations differing from \(V_j(\lambda)\) and \(V_\pm(\lambda)\). That is why, one can treat \(V_\pm\) (or Baxter operators) as basic generating representations (operators).
4 Discussion and Concluding Remarks

In this article we proposed the universal procedure for constructing "fusion" like eqs. for quantum monodromy matrix and Baxter Q- operator for some Integrable system.

We got all the ingredients of QISM from universal $R$–matrix. The Q- operator was presented as certain infinite dimensional $U_q(\hat{sl}_2)$ representation (13-14) for the first algebra of $R = \sum_i A_i \otimes B_i \in U_q(b_+) \otimes U_q(b_-)$ (the second algebra representation in the square is arbitrary). This gave us opportunity to treat Baxter eq. in invariant terms, i.e. we derived it considering certain representation squares. The important point of the method is that we did not specify the model in hand, all the equations are valid for any Integrable Systems with the $U_q(\hat{sl}_2)$ symmetry.

The algebraic origin of QISM objects from universal $R$–matrix gives possibility to:

- treat $V_{\pm}(\lambda)$ representations ($Q_{\pm}$ operators) as basic representations (operators), and generate new infinite dimensional subrepresentation (corresponding to new objects of QISM) from the square $V_{\pm}(a) \otimes V_{\pm}(b)$;
- find Bethe eigenvectors in invariant terms (i.e. in terms of certain quantum representation space).

It seems that there are several directions for continuation:

- the examination of representation squares $V_{\pm}(a) \otimes V_{\pm}(b)$ for existing of subrepresentations;
- the examination of integrable models with $V_{\pm}(\lambda)$ symmetry (i.e. got from universal $R$–matrix by representation of second $U_q(b_-)$ algebra in $V_{\pm}$ spaces). In this case we have coincidence of fundamental R- matrix and Q- operator. The special case of these system is Volterra one (representation $V_+$ degenerates in commuting $e_{\alpha_1}$ and $e_{\alpha_0}$ generators, $e_{\alpha_1}e_{\alpha_0} = e_{\alpha_0}e_{\alpha_1} = \frac{\lambda}{(q-q^{-1})^2}$).
- the generalization of the results to other affine algebras.
Appendix

A Universal R- matrix for $U_q(\hat{sl}_2)$ algebra

In this Appendix we present the explicit form of the universal $R$–matrix for $U_q(\hat{sl}_2)$ following [4].

Along with the commutation relations for $U_q(\hat{sl}_2)$ we need an antiinvolution ($^*$), defined as $(k_{\alpha_i})^* = k_{\alpha_i}^{-1}$, $(e_{\pm \alpha_i})^* = e_{\mp \alpha_i}$, $(q)^* = q^{-1}$.

We use also the following standard notations

$$\exp_q(x) := 1 + x + \frac{x^2}{(2)q!} + \ldots + \frac{x^n}{(n)q!} + \ldots = \sum_{n\geq 0} \frac{x^n}{(n)q!},$$

$$(a)_q := \frac{q^a - 1}{q - 1}, \quad [a]_q := \frac{q^a - q^{-a}}{q - q^{-1}}, \quad q_\alpha := q^{-(\alpha,\alpha)}$$

We define the Cartan-Weyl generators of the $U_q(\hat{sl}_2)$. Let $\alpha \equiv \alpha_1$ and $\beta \equiv \alpha_0 = \delta - \alpha$ are simple roots for the affine algebra $\hat{sl}_2$ then $\delta = \alpha + \beta$ is a minimal imaginary root. We fix the following normal ordering in the system of the positive roots:

$$\alpha, \quad \alpha + \delta, \quad \alpha + 2\delta \ldots, \delta, \quad 2\delta, \ldots, \ldots, \beta + 2\delta, \beta + \delta, \beta \ .$$

We put

$$e'_\delta = e_\delta = [e_\alpha, e_\beta]_q ,$$

$$e_{\alpha + t\delta} = (-1)^t ([[\alpha, \alpha]]_q)^{-t} (ad e'_\delta)^t e_\alpha ,$$

$$e_{\beta + t\delta} = ([[\alpha, \alpha]]_q)^{-t} (ad e'_\delta)^t e_\beta ,$$

$$e'_{t\delta} = [e_{\alpha + (t-1)\delta}, e_\beta]_q$$

and, finally,

$$(q - q^{-1}) E(z) = \log \left( 1 + (q - q^{-1}) E'(z) \right)$$

where $E(z)$ and $E'(z)$ are generating functions for $e_{n\delta}$ and for $e'_{n\delta}$:

$$E(z) = \sum_{n\geq 1} e_{n\delta} z^{-n} ,$$

$$E'(z) = \sum_{n\geq 1} e'_{n\delta} z^{-n} .$$

The negative root vectors are given by the rule $e_{-\gamma} = e^{*}_{\gamma}$.

The universal $R$-matrix for $U_q(\hat{sl}_2)$ has the following form [4]:

$$R = \left( \prod_{n \geq 0} \exp_{q_\alpha} \left( (q - q^{-1}) e_{\alpha + n\delta} \otimes e_{-\alpha - n\delta} \right) \right) \cdot \exp \left( \sum_{n > 0} (q - q^{-1}) \frac{n(e_{n\delta} \otimes e_{-n\delta})}{n([\alpha, \alpha]]_q} \right) .$$

11
\[
\left( \prod_{n \geq 0} \exp_{q_n} \left( (q - q^{-1}) e_{\beta + n \delta} \otimes e_{-\beta - n \delta} \right) \right) \cdot \mathcal{K},
\]

where the order on \( n \) is direct in the first product and it is inverse in the second one. Factor \( \mathcal{K} \) is defined by the formula:

\[
\mathcal{K} = q^{\frac{h_{\alpha} \otimes h_{\alpha}}{(\alpha, \alpha)}}.
\]

**B The Structure of \( V_+^{(\lambda)} \otimes V_2^{(\lambda)} \) Square**

In this Appendix we prove Baxter eq. (12) for Q-operators (13-14) defined in Section 3. For simplicity we consider the special case of representations (13-14)

\[
e_{\alpha_1} = e, \quad e_{\alpha_0} = \frac{\lambda}{(q - q^{-1})^2} e^{-1}, \quad k_{\alpha_1} = K,
\]

where \( Ke = q^2 eK \). The representation (20) of \( U_q(b_-) \) algebra for \( R = \sum_i A_i \otimes B_i \in U_q(b_+) \otimes U_q(b_-) \) describes Volterra model.

Introduce the representation (20) space spanned by \( |j\rangle, \ j \in \mathbb{Z} \), with the action of \( e, K \) generators

\[
e |j\rangle = q^{-j} |j + 1\rangle \quad K |j\rangle = q^{2j} |j\rangle
\]

For describing the space \( V_+^{(\lambda)} \otimes V_2^{(\lambda)} \) one needs also the evaluation representation \( V_2^{(\lambda)} \) (11) with generators

\[
e_{\alpha_1} = e_{\alpha}, \quad e_{\alpha_0} = \lambda e_{-\alpha}, \quad k_{\alpha_1} = k.
\]

We have the action of \( U_q(sl_2) \) algebra on 2-dimensional \( U_q(sl_2) \) space (\(|+\rangle, \ |-\rangle \))

\[
e_\alpha |+\rangle = 0 \quad e_\alpha |-\rangle = |+\rangle,
\]

\[
e_{-\alpha} |+\rangle = |-\rangle \quad e_{-\alpha} |-\rangle = |+\rangle,
\]

\[
k |\pm\rangle = q^{\pm 1} |\pm\rangle.
\]

Algebra \( U_q(b_+) \) acts in square \( V_+^{(\lambda)} \otimes V_2^{(\lambda)} \) by comultiplication (5)

\[
\Delta(k_{\alpha_1}) = K \otimes k,
\]

\[
\Delta(e_{\alpha_1}) = e \otimes k + 1 \otimes e_{\alpha},
\]

\[
\Delta(e_{\alpha_0}) = \lambda \left( \frac{e^{-1}}{(q - q^{-1})^2} \otimes k^{-1} + 1 \otimes e_{-\alpha} \right)
\]

Now we should check that in the square \( V_+^{(\lambda)} \otimes V_2^{(\lambda)} \) spanned by vectors

\[
|j\rangle \otimes |\pm\rangle, \quad j \in \mathbb{Z}
\]

there exists subrepresentation isomorphic to \( V_+^{(\lambda q^{-2})} \).
For this we choose linear combinations
\[ U_j = \alpha_j \ket{j} \otimes \ket{+} + \beta_j \ket{j+1} \otimes \ket{-}, \quad j \in \mathbb{Z} \]
of vectors with the same grading from the space (23), and act on vectors \( U_j \) with generators \( \Delta(e_{\alpha_1}) \) and \( \Delta(e_{\alpha_0}) \) (22).

For \( \alpha_j = \frac{q^{-j+1}}{1-q^2} \) and \( \beta_j = q^{-2j}, \quad j \in \mathbb{Z} \),

one has
\[ \Delta(e_{\alpha_1}) U_j = q^{-j} U_{j+1}, \quad \Delta(e_{\alpha_0}) U_j = \frac{\lambda q^{-2}}{(q-q^{-1})^2} q^{j-1} U_{j-1}. \quad (24) \]

Comparing actions of generators (20-21) and (24) one concludes that there exists subrepresentation \( V_+^{\lambda q^{-2}} \) in the square \( V_+^{\lambda} \otimes V_+^{\frac{\lambda}{2}} \).

To take factor space \( V_+^{\lambda} \otimes V_+^{\frac{\lambda}{2}} / V_+^{\lambda q^{-2}} \) we act with generators \( \Delta(e_{\alpha_1}), \Delta(e_{\alpha_0}) \) on vectors
\[ W_j = q^j \ket{j} \otimes \ket{+}, \quad j \in \mathbb{Z} \quad \text{(modulo \( U_j \))} \]
and get
\[ \Delta(e_{\alpha_1}) W_j = q^{-j} W_{j+1}, \quad \Delta(e_{\alpha_0}) W_j = \frac{\lambda q^2}{(q-q^{-1})^2} q^{j-1} W_{j-1} \quad \text{(modulo \( U_j \)).} \]

So we prove that the factor \( V_+^{\lambda} \otimes V_+^{\frac{\lambda}{2}} / V_+^{\lambda q^{-2}} \) is isomorphic to \( V_+^{\lambda q^2} \).

The same procedure can be easily repeated for the general case of (13-14).

References

[1] L. Faddeev, E. Sklyanin and L. Takhtajan: Quantum inverse scattering method I. Theor. Math. Phys. 40, 194-219 (1979) (in Russian)
[2] R. Baxter: Exactly Solved Models in Statistical Mechanics. London: Academic Press 1982
[3] V. Bazhanov, S. Lukyanov and A. Zamolodchikov, II. Integrable Structure of Conformal Field Theory, Baxter Q-operator and Destry-De Vega Eq., to appear in hep/th
[4] S. Khoroshkin, A. Stolin and V. Tolstoy, Mod. Phys. Lett. A 10, # 19 (1995) 1375-1392,
[5] L. Faddeev, Int. J. Mod. Phys. A 10, # 13 (1995) 1845-1878
[6] V. Bazhanov, S. Lukyanov and A. Zamolodchikov, I. Integrable Structure of Conformal Field Theory, Quantum KdV theory and Thermodynamic Bethe Ansatz, hep/th 9412229;
[7] V. Tarasov, L. A. Takhtajan, L. D. Faddeev, Theor. Math. Phys. 57, (1983) 163.
[8] V. Chari and A. Pressley, Commun. Math. Phys. 142, 261-283 (1991)

[9] A. Volkov, Quantum lattice KdV equation, hep-th/9509024

[10] A. Volkov, Phys. Lett. A 167 (1992), 345-355