AFFINE VECTOR FIELDS ON FINSLER MANIFOLDS

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Abstract. We give characterizations of affine transformations and affine vector fields in terms of the spray. By utilizing the Jacobi type equation that characterizes affine vector fields, we prove some rigidity theorems of affine vector fields on compact or forward complete non-compact Finsler manifolds with non-positive total Ricci curvature.

1. Introduction

It is well known that every affine vector field on a compact orientable Riemannian manifold is a Killing vector field [5]. In the noncompact case, Junichi Hano proved that if the length of an affine vector field in a complete Riemannian manifold is bounded, then its affine vector field is a Killing vector field [4]. This result was generalized later by Shinuke Yorozu, who proved that every affine vector field with finite norm on a complete noncompact Riemannian manifold is a Killing vector field. Moreover, if the Riemannian manifold has non-positive Ricci curvature, then every affine vector field with finite global norm on it is a parallel vector field [9]. We generalize this result to Finsler manifolds. We investigate affine vector fields on Finsler manifolds and prove some rigidity theorems of affine vector fields on compact and forward complete non-compact Finsler manifolds with non-positive total Ricci curvature.

Theorem 1.1. Let $(M, F)$ be an $n$-dimensional compact Finsler manifold with non-positive total Ricci curvature. Then every affine vector field $V$ on $M$ is a linearly parallel vector field.

Theorem 1.2. Let $(M, F)$ be an $n$-dimensional forward complete non-compact Finsler manifold with non-positive total Ricci curvature and bounded reversibility. Then every affine vector field $V$ on $M$ with finite global norm is parallel.

The proofs of the above theorems follow a typical Finslerian style. We extensively use knowledge of the tangent bundle and sphere bundle, which are by no means necessary in Riemannian geometry. Moreover, since the sphere bundle is always orientable, we can drop the orientable condition of the manifold that is used in Riemannian case.

In Riemannian geometry, affine vector field $V$ is characterized by the fact that the Lie derivative of the Riemannian metric $g$ is parallel, namely,

$$\nabla(L_V g) = 0,$$

where $\nabla$ is the Levi-Civita connection [10]. In Finsler geometry, there are many choices of connections but the characterization of an affine vector field is much more concise and without any connection. We show that a vector field $V$ is affine if and
only if its flow commutes with the geodesic flow. Equivalently, $\hat{V}$, the complete lift of $V$, commutes with the Finsler spray $G$, i.e.,

$$\mathcal{L}_{\hat{V}} G = 0.$$  

By expanding the above Lie bracket, we obtain a Jacobi type equation, thus Bochner type techniques could be applied. The crucial difference between the Riemannian case and Finsler case is that, in Riemannian geometry one needs to compute the Laplacian of the energy of $V$, but in Finsler geometry we don’t need further computation.

The organization of the paper is as follows. Firstly, we review some basic facts of Finsler geometry and give a definition of the total Ricci curvature in Section 2. Then we define affine transformations and affine vector fields on Finsler manifolds respectively in Section 3. Several characterizations of affine vector fields are provided, which are useful to the theorems. Finally, the proofs of the above theorems are presented in Section 4.

2. Preliminaries

In this section, we give a brief description of some basic materials that are needed to prove Theorem 1.1.

2.1. Spray and Riemann curvature tensor. Let $M$ be an $n$-dimensional smooth manifold. A smooth function $F$ on the punctured tangent bundle $TM_0 := TM \setminus \{0\}$ is called a Finsler metric, if the restriction $F|_{T_xM \setminus \{0\}}$ is a Minkowski norm for every $x$ in $M$.

The natural projection $\pi : TM_0 \to M$ induces a pull back bundle $\pi^*TM$ over $TM_0$, whose fiber at each point $y \in TM_0$ is just a copy of $T_xM$, where $x = \pi(y)$. For each fixed $y \in TM_0$, one can define an inner product $g_y$ on the fibre $T_xM$ as follows

$$g_y = g_{ij} \, dx^i \otimes dx^j,$$

where we have used the natural local coordinate system $(x^i, y^j)$ on $TM_0$. When $y$ varies, the inner products $g_y$ become a globally defined tensor field on $\pi^*TM$, called the fundamental tensor [7].

The following set of functions are defined locally and are called the (nonlinear) connection coefficients.

$$N^i_{jk} = [G^i]_{x^k} + [g_{kj}]_{x^i} - [g_{jk}]_{x^i} y^j y^k.$$  

The local functions $N^i_{jk}$ are called the (nonlinear) connection coefficients.

Let

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_{jk} \frac{\partial}{\partial y^j},$$
then $HTM = \text{span}\{\delta_{\delta x^i}\}$ and $VTM = \text{span}\{\partial_{\delta y^j}\}$ are well-defined subbundles of $TTM_0$ and $TTM_0 = HTM \oplus VTM$.

Direct computation yields

$$
\left[ G, \frac{\partial}{\partial y^j} \right] = -\frac{\delta_{\delta x^i}}{\delta x^i} + N^j_i \frac{\partial}{\partial y^j},
$$

(2.2)

and

$$
\left[ G, \frac{\delta}{\delta x^i} \right] = R^j_i \frac{\partial}{\partial y^j} + N^j_i \frac{\delta}{\delta x^i},
$$

(2.3)

where the functions $R^j_i$ are given by

$$
R^j_i = 2\left[ G_j \right]_{\delta x^i} - G(N^j_i) - N^j_i N^k_i.
$$

For each fixed $y \in T M_0$, the $(1,1)$ tensor $R_y = R^j_i \frac{\partial}{\partial x^j} \otimes d x^i$ is called the Riemann curvature tensor.

2.2. Sphere bundle and volume form. The set $SM = \{ y \in T M_0 | F(y) = 1 \}$ is called the unit sphere bundle or indicatrix bundle. Let $\omega = F_y d x^i = g_{ij} y^j / F d x^i$, then $\omega$ is a globally defined contact form on $SM$ and it is called the Hilbert form. By straightforward computation, one can show that the vector field $\xi = G / F$ satisfies

$$
\omega(\xi) = 1, \quad d \omega(\xi, \cdot ) = 0.
$$

(2.4)

Henceforth, $\xi$ is called the Reeb field according to the contact terminology.

Although the manifold $M$ could be non-oriented, the sphere bundle $SM$ is always oriented because it carries the following volume form

$$
d \nu = c_n \omega \wedge (d \omega)^{n-1},
$$

where the constant $c_n = (-1)^{-1+n(n+1)/2}/(n-1)!$.

The mean Ricci curvature $\tilde{Ricci}$ is defined in [6] with the help of an auxiliary Riemannian metric. For our purposes, we define the total Ricci curvature $T(V)$ as follows

$$
T(V) = \int_{SM} \frac{1}{F^2} g_y(R_y(V), V) \, d \nu.
$$

where $V$ is any vector field on $M$. When $M$ is oriented, $T(V)$ is the integration of $\tilde{Ricci}(V)$ over $M$, where $M$ has been assigned the volume form of the auxiliary Riemannian metric.

2.3. Dynamical derivative. Berwald connection is a linear connection on the vector bundle $\pi^*TM$ over $T M_0$, whose connection coefficients are given by $\Gamma^i_{jk} = [G^i]_{\delta x_j \delta x_k}$. Using Berwald connection, one can take covariant derivatives of any tensor fields on $T M_0$. For example, if $T = T^i_{\delta x^i} \otimes d x^j$, then the horizontal covariant derivative is given by

$$
T^i_{j|k} = \frac{\delta T^i_j}{\delta x^k} + T^i_j \Gamma^i_{jk} - T^i_j \Gamma^i_{jk}.
$$

The dynamical derivative is just the horizontal covariant derivative along the direction $G = y^k \frac{\partial}{\partial x^k}$. For example, the dynamical derivative of the above tensor field $T$ is given by $T^i_{j|0} = T^i_{j|0} \frac{\partial}{\partial x^j} \otimes d x^j$, where

$$
T^i_{j|0} = G(T^i_j) + T^i_j N^i_j - T^i_j N^i_j.
$$
It should be remarked that, although we introduce the dynamical derivative using Berwald connection, this special derivative is actually independent of any named connection. One may consult [3] for another treatment of this concept.

For every smooth function \( f \) on \( T^0M \), the dynamical derivative of \( f \) is given by

\[
f_{0}(x, y) = \frac{d}{dt} f(\gamma(t), \dot{\gamma}(t)) \bigg|_{t=0}, \quad \forall (x, y) \in T^0M,
\]

where \( \gamma \) is the unique geodesic with \( \gamma(0) = x, \dot{\gamma}(0) = y \).

As a concrete example, the dynamical derivative of the Finsler function \( F \) is zero. Another example is the fundamental tensor; its dynamical derivative is also zero, because

\[
g_{ij|0} = G(g_{ij}) - g_{kj} N^k_i - g_{ik} N^k_j = 0. \tag{2.5}
\]

Sometimes, the dynamical derivative is also defined to be horizontal covariant derivative along the direction \( \xi = G/F \). When using this convention, we shall denote the dynamical derivative of a tensor field \( T \) by \( \dot{T} \). For example, if \( V = V^i \partial \partial x^i \), then

\[
\dot{V} = \frac{V^i}{F} \frac{\partial}{\partial x^i}.
\]

Since every vector field \( V \) on \( M \) can be thought of as a smooth section of \( \pi^*TM \), the symbol \( V_{00} \) or \( \dot{V} \) makes sense. In this case \( \dot{V} = 0 \) if and only if \( V_{00} = 0 \), if and only if \( V_i = 0 \), i.e., \( V \) is a linearly parallel vector field [2].

3. Affine transformation and affine vector field

3.1. Affine transformation. In affine differential geometry, affine transformation is a special kind of projective transformation, which preserves the (parametrized) geodesics on a manifold with an affine connection. In Euclidean space, affine transformation consists of translation, scaling, homothety, similarity transformation, reflection, rotation, shear mapping, and compositions of the above in any combination or sequence. This concept can also be studied in the realm of spray geometry.

Definition 1. Let \( M \) be an \( n \) dimensional manifold with a spray \( G \). A diffeomorphism \( \phi : M \to M \) is called affine transformation if for any geodesic \( \gamma : (a, b) \to M \), the curve \( \phi \circ \gamma \) is also a geodesic.

Since the geodesics are defined by the spray, we can prove affine transformation preserves the spray.

Lemma 3.1. A diffeomorphism \( \phi \) is an affine transformation iff

\[
\tilde{\phi}_* G = G, \tag{3.1}
\]

where \( \tilde{\phi} : TM \to TM \) is the lift of \( \phi \) defined by

\[
\tilde{\phi}(x, y) = (\phi(x), \phi_*(y)), \quad \forall x \in M, \quad y \in T_x M.
\]

Proof. For any \( (x, y) \in TM_0 \), let \( \gamma \) be the unique geodesic with \( \gamma(0) = x, \dot{\gamma}(0) = y \). Let \( \sigma = \dot{\gamma} \), then \( \sigma \) is an integral curve of \( G \), namely, \( G_\sigma = \dot{\sigma} \).

If \( \phi \) is an affine transformation, then \( \phi \circ \gamma \) is also a geodesic. In this case, \( (\phi \circ \gamma)' = \phi \circ \sigma \) is also an integral curve of \( G \), i.e.,

\[
G_{\tilde{\phi} \circ \sigma} = (\tilde{\phi} \circ \sigma)' = \tilde{\phi}_* \sigma = \tilde{\phi}_* G_\sigma.
\]
Taking $t = 0$ in the above identity, then we have $\hat{\phi}_x G_{(x,y)} = G_{\hat{\phi}(x,y)}$.

Conversely, if $\hat{\phi}_x G = G$, then $\hat{\phi}$ maps every integral curve of $G$ to an integral curve. Consequently it will map every geodesic to a geodesic. As a result, $\hat{\phi}$ is affine.

\[\square\]

### 3.2. Affine vector field

Let $V$ be a vector field on $M$ which generates a local one-parameter group $\phi_t$, $t \in (-\varepsilon, \varepsilon)$. Let $\hat{\phi}_t$ be the lift of $\phi_t$, then $\hat{\phi}_t$ is also a local one-parameter group on $TM$. So there is a vector field $\hat{V}$ on $TM$ induced by $\hat{\phi}_t$; it is called the \textit{complete lift} of $V$. Locally, if $V = V^i \frac{\partial}{\partial x^i}$, then we have $\hat{V} = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i}$.

**Definition 2.** On a spray manifold $(M, G)$, a vector field $V$ is called affine vector field, if the local one-parameter group $\phi_t$ generated by $V$ consists of affine transformations.

Based on the property of affine transformation, we can deduce the following characterizations of affine vector fields.

**Proposition 3.2.** Let $V$ be a vector field on a spray manifold $(M, G)$. Then the following assertions are mutually equivalent.

1. $V$ is an affine vector field;
2. $\mathcal{L}_V G = 0$;
3. $V^i |_{0} + V^k R^i_k = 0$.

**Proof.** By definition, $V$ is an affine vector field, if and only if the corresponding one-parameter group $\phi_t$ consists of affine transformations. This is also equivalent to $\hat{\phi}_t G = G$ by Lemma 3.1. Taking derivative with respect to $t$ at $t = 0$, we have $\mathcal{L}_V G = 0$. So the implication (1)$\Rightarrow$(2) is proved. The converse is clear by the definition of Lie derivative (or one may consult [5, Prop. 1.7]).

Now we prove the equivalence of (2) and (3) by some local computation. First, note that

$$\hat{V} = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i} = V^i \left( \frac{\delta}{\delta x^i} + N^j_i \frac{\partial}{\partial y^j} \right) + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

$$= V^i \frac{\delta}{\delta x^i} + y^j \left( \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i} + V^k R^i_k \right) \frac{\partial}{\partial y^i} = V^i \frac{\delta}{\delta x^i} + V^i |_0 \frac{\partial}{\partial y^i}.$$

Using the above equation, we deduce

$$[G, \hat{V}] = [G, V^i \frac{\delta}{\delta x^i} + V^i |_0 \frac{\partial}{\partial y^i}]$$

$$= G(V^i) \frac{\delta}{\delta x^i} + G(V^i |_0) \frac{\partial}{\partial y^i} + V^i \cdot [G, \frac{\delta}{\delta x^i}] + V^i |_0 \cdot [G, \frac{\partial}{\partial y^i}]$$

$$= G(V^i) \frac{\delta}{\delta x^i} + G(V^i |_0) \frac{\partial}{\partial y^i} + V^i \left( R^i_k \frac{\partial}{\partial y^j} + N^j_i \frac{\delta}{\delta x^j} \right)$$

$$+ V^i |_0 \left( - \frac{\delta}{\delta x^i} + N^j_i \frac{\partial}{\partial y^j} \right)$$

$$= \left( G(V^i |_0) + V^i |_0 N^j_i + V^k R^i_k \right) \frac{\partial}{\partial y^i}$$

$$= (V^i |_0 + V^k R^i_k) \frac{\partial}{\partial y^i}.$$
where we have used (2.3) and (2.2). It is clear that $\mathcal{L}_V G = 0$ if and only if $V^i_{00} + V^k R^i_k = 0$. Thus the proposition is proved. □

**Remark.** The third assertion can be written as

$$V_{00} + R_g(V) = 0, \quad \text{or} \quad \dot{V} + R_g(V)/F^2 = 0.$$  

This equation is the same as Jacobi equation when restricted to a geodesic. So one can conclude that $V$ is an affine vector field, if and only if the restriction of $V$ to any geodesic is a Jacobi field.

4. **Affine vector fields on Finsler manifolds**

The main purpose of this section is to prove Theorems 1.1 and 1.2. To that end, we need a technical lemma.

**Lemma 4.1.** Let $(M, F)$ be a Finsler manifold without boundary. Let $d\nu = c_n \omega \wedge (d\omega)^{n-1}$ be the volume form of $SM$. Then for any compactly supported smooth function $f$ on $SM$, we have

$$\int_{SM} f \, d\nu = 0.$$  

**Proof.** It is well-known [1] that we can choose local coframe field

$$\omega^1, \omega^2, \ldots, \omega^{n-1}, \omega^n = \omega_1, \omega_2, \ldots, \omega_2^{n-1},$$

on $SM$, such that $d\omega = \sum_{\alpha=1}^{n-1} \omega^\alpha \wedge \omega^{n+\alpha}$. Thus we have

$$(d\omega)^{n-1} = (n-1)! \omega^1 \wedge \omega^{n+2} \wedge \cdots \wedge \omega^{n-2} \wedge \omega^{2n-1}.$$  

Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n-1}$ be the dual frame field, then $e_n = \xi$ (compare (2.3)). If we write

$$df = f_1 \omega^1 + \cdots + f_n \omega^n + \cdots + f_{2n-1} \omega^{2n-1},$$

then it is clear that $f_n = \xi(f) = \dot{f}$. So we have

$$\dot{f} \omega \wedge (d\omega)^{n-1} = df \wedge (d\omega)^{n-1} = df \wedge (d\omega)^{n-1}.$$  

In other words, the form to be integrated is exact. By Stokes theorem, the integration is zero since $M$ is boundaryless. □

4.1. **Compact case.** We restate Theorem 1.1 as follows.

**Theorem 4.2.** Let $(M, F)$ be an $n$-dimensional compact Finsler manifold and let $V$ be an affine vector field. If $\mathcal{T}(V) \leq 0$, then $V$ is a linearly parallel field.

**Proof.** First do some computation on $TM_0$. Let $f = \frac{1}{F} g_{ij} V^i V^j$, then we have

$$f_{00} = \frac{1}{F} (g_{ij} V^i_{00} V^j_0 + g_{ij} V^i V^j_{00}) = \frac{1}{F} (g_{ij} V^i_0 V^j_0 - g_{ij} V^i V^j R^i_k),$$

where we have used the characterization (3) of affine vector field in Prop. 3.2. Since $f$ is $y$-homogeneous of order 0, it can be thought of as a function on $SM$. The above equation can be interpreted as

$$\dot{f} = g_y(\dot{V}, \dot{V}) - \frac{1}{F^2} g_y(R_g(V), V).$$

Taking integral of both sides on $SM$ and using Lemma 4.1 yield

$$0 = \int_{SM} g_y(\dot{V}, \dot{V}) \, d\nu - \int_{SM} \frac{1}{F^2} g_y(R_g(V), V) \, d\nu \geq 0.$$
Therefore, $\dot{V} = 0$, which means $V$ is a linearly parallel field.

\[\square\]

4.2. Forward complete non-compact case. In this section, we consider affine vector fields on forward complete non-compact Finsler manifolds. Again, we state a precise version of Theorem 1.2 as follows.

**Theorem 4.3.** Let $(M, F)$ be an $n$-dimensional forward complete non-compact Finsler manifold. Assume that

(1) $V$ is an affine vector field with finite global norm, i.e.,
$$\int_{SM} g_y(V, V) \, d\nu < +\infty;$$

(2) The total Ricci curvature is non-positive, i.e., $T(V) \leq 0$;

(3) The reversibility $\lambda(F) := \sup_{y \in SM} \frac{F(x, -y)}{F(x, y)} < +\infty$.

Then $V$ is a linearly parallel vector field.

**Proof.** Let $p$ be a fixed point of $M$. For each point $x \in M$, we denote by $d(p, x)$ the forward geodesic distance from $p$ to $x$. Let $\sigma : [0, +\infty) \to [0, 1]$ be a smooth cut-off function such that
$$\sigma(t) = \begin{cases} 1, & t \in [0, 1], \\ 0, & t \in [2, +\infty). \end{cases}$$

Fix a positive real number $\alpha$ and let $\mu(x) = \sigma(\frac{d(p, x)}{\alpha})$. Then we have the following

**Claim.** There is a positive constant $A$ such that $\dot{\mu}^2 \leq \frac{A \lambda(F)^2}{\alpha^2}$.

**Proof of the claim.** Set $\rho(x) = d(p, x)$. We first show that $\dot{\rho}$ is bounded. Recall that the value of $\dot{\rho}$ at $(x, y) \in SM$ is given by
$$\dot{\rho}(x, y) = \frac{d}{dt} \rho(\gamma(t)) \bigg|_{t=0} = \frac{d}{dt} d(p, \gamma(t)) \bigg|_{t=0}$$
$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} (d(p, \gamma(\epsilon)) - d(p, \gamma(0))) \leq \lim_{\epsilon \to 0} \frac{1}{\epsilon} d(\gamma(0), \gamma(\epsilon)) = F(x, y) = 1.$$

In a similar manner, we have
$$\dot{\rho}(x, y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (d(p, \gamma(\epsilon)) - d(p, \gamma(0)))$$
$$\geq -\lim_{\epsilon \to 0} \frac{1}{\epsilon} d(\gamma(\epsilon), \gamma(0)) = -F(x, -y) \geq -\lambda(F).$$

So $\dot{\mu} = \sigma'(\rho/\alpha) \cdot \dot{\rho}/\alpha$ satisfies $\dot{\mu}^2 \leq A \cdot \lambda(F)^2/\alpha^2$, where $A$ is the upper bound of $|\sigma'(t)|^2$. Thus the claim is proved.

Now, consider the function $f = \mu^2 g_y(V, V)$ on $SM$. We have
$$\dot{f} = 2\mu \dot{\mu} g_y(V, V) + \mu^2 g_y(\dot{V}, V) + \mu^2 g_y(V, \dot{V}).$$

Expanding the inequality $\frac{1}{2} g_y(\mu \dot{V} + 2\mu V, \mu \dot{V} + 2\mu V) \geq 0$ yields
$$2\mu \dot{\mu} g_y(V, V) \geq -2\mu^2 g_y(V, V) - \frac{1}{2} \mu^2 g_y(V, V).$$
Substituting this inequality into (4.1), we get

\[
\dot{f} \geq -2\mu^2 g_y(V, V) + \frac{1}{2} \mu^2 g_y(\dot{V}, \dot{V}) + \mu^2 g_y(V, \ddot{V}) \\
\geq -\frac{2A \cdot \lambda(F)^2}{\alpha^2} g_y(V, V) + \frac{1}{2} \mu^2 g_y(\dot{V}, \dot{V}) - \frac{\mu^2}{F^2} g_y(V, R_y(V)).
\]

Taking integral of both sides on \( SM \) and using Lemma 4.1 yield

\[
0 \geq -\frac{2A \cdot \lambda(F)^2}{\alpha^2} \int_{SM} g_y(V, V) \, d\nu + \frac{1}{2} \int_{SM} \mu^2 g_y(\dot{V}, \dot{V}) \, d\nu \\
- \int_{SM} \frac{\mu^2}{F^2} g_y(V, R_y(V)) \, d\nu.
\]

It follows that

\[
\frac{2A \cdot \lambda(F)^2}{\alpha^2} \int_{SM} g_y(V, V) \, d\nu \\
\geq \frac{1}{2} \int_{SM} \mu^2 g_y(\dot{V}, \dot{V}) \, d\nu - \int_{SM} \frac{\mu^2}{F^2} g_y(V, R_y(V)) \, d\nu.
\]

Letting \( \alpha \to \infty \), then the left hand side approaches 0 by the conditions (1) and (3), while the right hand side approaches \( \frac{1}{2} \int_{SM} g_y(\dot{V}, \dot{V}) \, d\nu - T(V) \geq 0 \). Consequently, we must have \( \dot{V} = 0 \). So \( V \) is linearly parallel. \( \square \)

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