Fermion Mapping for Orthogonal and Symplectic Ensembles

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March 23, 2022

Abstract

The circular orthogonal and circular symplectic ensembles are mapped onto free, non-hermitian fermion systems. As an illustration, the two-level form factors are calculated.

March 23, 2022

1 Introduction

Dyson introduced the orthogonal, unitary, and symplectic ensembles of random matrices; different ensembles correspond to different physical problems according to the presence or absence of time reversal symmetry. The joint probability distribution function for the eigenvalues of matrices in these ensembles is equivalent to the statistical weight of a configuration of charges which repel each other with a logarithmic interaction at an appropriate temperature. By integrating over all levels except for some finite number, level-level correlation functions can be found[1].

Although the integrals can be carried out directly, it is of interest to find a physical system which reproduces the desired probability distribution function. This has previously been done using the Calogero-Sutherland model[2], a model of interacting fermions. In this paper, physical systems will be presented which use non-interacting, but non-hermitian, fermion systems to produce the desired results.
2 Circular Ensembles

The circular orthogonal and symplectic ensembles are defined by considering a system of $N$ points on the unit circle in the complex plane. Each configuration of points is weighted by a factor $\prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^\beta$, where $\beta = 1$ for the orthogonal ensemble and $\beta = 4$ for the symplectic ensemble. The points are located at the positions $e^{i\theta_i}$, for $i = 1$ to $N$. Both ensembles will be mapped onto fermion systems.

The partition function will be given for arbitrary chemical potentials, which will permit the calculation of correlation functions, reproducing the Dyson formula. Although this formula has been obtained previously by other means, and none of the final results are new, the fermion mapping for this problem provides a simple trick for calculating correlation functions.

3 Orthogonal Ensemble

First, we will consider the orthogonal ensemble. We wish to calculate the following partition function

$$\prod_{i=1}^{N} \int d\theta_i \prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|$$

Then, we will introduce a chemical potential and take functional derivatives to calculate correlation functions.

Up to a sign, for given $\theta_i$, we may write

$$\prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}| = \pm \prod_{i<j} e^{i(\theta_i - \theta_j)} \frac{1}{\sqrt{e^{i\theta_i} e^{i\theta_j}}}$$

We may write eq. (2), up to a constant multiplier in front, as a correlation function in a two-dimensional free Euclidean fermionic field theory. We take the $x$ coordinate to be the same as the $\theta$ coordinate, and thus periodic with period $2\pi$, and take correlation functions to be analytic in $\tau + ix$. We introduce the destruction operator $\psi(x, \tau)$ which destroys a particle at $(x, \tau)$, and the creation operator $\psi^\dagger(x, \tau)$. The field $\psi$ has the action $S = \int dx \, d\tau \psi^\dagger(x, \tau) \partial_\tau + i \partial_x \psi(x, \tau)$.

Then we write eq. (2) as

$$\langle \prod_{i=1}^{N} \psi(x_i, 0) \{ \psi^\dagger(0, -\infty) \}^{N/2} \{ \psi^\dagger(0, \infty) \}^{N/2} \rangle$$

(3)
where the operator \( \{ \psi^\dagger(0,-\infty) \}^{N/2} \) represents \( N/2 \) creation operators, slightly separated from each other in space, and located very far away in time from the line \( \tau = 0 \). The constant multiplier relating eqs. (2) and (3) will depend on the precise separation between the creation points, and the precise distance between those points and \( \tau = 0 \).

We may then forget about the two-dimensional field theory and write eq. (3) as the expectation value

\[
\langle V^- | \prod_{i=1}^{N} \psi(x_i) | V^+ \rangle
\]  

(4)

where the state \( |V^+ \rangle \) is the state obtained by taking the vacuum state and adding particles to the \( N/2 \) lowest unoccupied energy states, and the state \( |V^- \rangle \) is the state obtained by taking the vacuum state and removing particles from the \( N/2 \) highest occupied energy states. This expectation value is the correct one since these two states are the ones projected out by the long time separation used in eq. (3). It should be noted that the constant factor relating eq. (4) to eq. (2) is \( N \)-dependent, and, unless we take care to determine the constant, we will only be able to calculate correlation functions, and not the total free energy.

The sign needed to relate eq. (2) to the product in eq. (1) may be obtained by introducing an extra field \( \eta \). This field is \( x \)-dependent but confined to the line \( \tau = 0 \). For the \( \eta \)-field, \( x \) plays the role of time. This field has the action \( S = \int dx \eta \eta \). Then the sign is simply given by

\[
\langle \prod_{i=1}^{N} \{ \eta(x_i) + \bar{\eta}(x_i) \} \rangle
\]  

(5)

Then we write introduce a chemical potential \( f(x) \), weighting each configuration of points by \( \prod_{i=1}^{N} f(x_i) \), and write the partition function as

\[
\int [d\eta] \langle V^- | e^{\int dx f(x) \psi(x) (\eta(x) + \bar{\eta}(x)) - \eta \eta} | V^+ \rangle
\]  

(6)

and integrate out \( \eta \) to obtain

\[
\langle V^- | e^{\int dx dx' f(x) \psi(x) f(x') \psi(x') \text{sign}(x-x')} | V^+ \rangle
\]  

(7)

where \( \text{sign}(x-x') \) is equal to

\[
\sum_{k} e^{ik(x-x')} \frac{1}{k}
\]  

with \( k \) equal to half an odd integer.
It is convenient to rewrite eq. (7) for \( f(x) = 1 \) as

\[
\langle V^- | e^{\sum_k a(k) a(-k) \frac{i}{k}} | V^+ \rangle \tag{9}
\]

This is clearly equal to

\[
\frac{(N-1)/2}{(N/2)!} \prod_{k=1/2}^{2/k}
\]

We may calculate the two-point correlation function of levels by taking two derivatives of eq. (7) with respect to \( f(x) \). The result splits into two pieces, eqs. (11) and (13).

Calculating the correlation function of levels at points \( x \) and \( x' \), one piece of the correlation function is obtained by inserting the operator

\[
\frac{1}{2} \psi(x) \psi(x') \text{sign}(x - x') \tag{11}
\]

into the expectation value of eq. (7). Expanding the \( \psi \) operators in Fourier components \( a(k), a(k') \), and noting that the result is non-vanishing only if \( k = -k' \), the correlation function is

\[
\frac{1}{2} \frac{(N-1)/2}{\sum_{k=-(N-1)/2}^{(N-1)/2}} \sum_{l}\frac{1}{l} e^{i(l)(x-x')}
\]

where the sum is carried out over all half odd values of \( l \). The factor of \( k \) arises due to a missing factor of \( \frac{1}{k} \) in the product of eq. (10).

The second piece of the correlation function is obtained by inserting the operator

\[
\frac{1}{2} \int dy dy' \psi(x) \psi(y) \text{sign}(x - y) \psi(x') \psi(y') \text{sign}(x' - y') \tag{13}
\]

We again expand each \( \psi \) operator in \( a(k) \) operators, and then pair off operators \( a(k) \) with each other, so that two operators in a pair have opposite momenta. If we pair off the first with the second and the third with the fourth, the result is just the product of one-level correlators.

If we pair off the first operator with the fourth operator, and the second with the third, we obtain

\[
\frac{1}{2} \sum_{k=-(N-1)/2}^{(N-1)/2} \sum_{l=-(N-1)/2}^{(N-1)/2} e^{i(l)(x-x')} \tag{14}
\]

If we pair off the first operator with the third operator, and the second with the fourth, we obtain

\[
\frac{1}{2} - \sum_{k=-(N-1)/2}^{(N-1)/2} k e^{i(k)(x-x')} \sum_{l=-(N-1)/2}^{(N-1)/2} \frac{1}{l} e^{i(l)(x-x')} \tag{15}
\]
The average spacing between particles is $2\pi/N$. We rescale the $x$-coordinate to $Nx$ and rescale the momenta $k$ and $l$ to $k/N$ and $l/N$, to calculate the two-level correlator in units of average particle spacing. As $N$ becomes large, we may replace the sums by integrals, reproducing the Dyson formula. We calculate the $m$-th Fourier component of the form factor of the two-level correlator, where $m = k + l$.

Then, from equation (12) we obtain

$$-1 + |m| \ln\left(\frac{2|m| + 1}{2|m| - 1}\right)$$

From eqs. (14,15) we obtain a non-vanishing result only if $|m| < 1$, in which case eq. (14) gives

$$1 - |m|$$

and eq. (15) gives

$$1 - |m| + |m| \ln(2|m| - 1)$$

So the form factor is given by eq. (16) for $|m| > 1$ and, for $|m| < 1$, is equal to

$$1 - 2|m| + |m| \ln(2|m| + 1)$$

It is also possible to obtain the correlation functions of levels which belong to the same alternate series, by introducing separate chemical potentials for operators $\psi\eta$ and $\psi\eta$. This may help relate this technique to the more standard method of integration over alternate variables [4].

4 Symplectic Ensemble

For the symplectic ensemble, the desired partition function is

$$\prod_{i=1}^{N} \int d\theta_i \prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^4$$

For given $\theta_i$, we may again write the weight of the configuration as a correlation function in a two-dimensional field theory. Here, the desired correlation function is

$$\langle (\prod_{i=1}^{N} \psi(x_i, 0)^2)\{\psi^\dagger(0, -\infty)\}^N \{\psi^\dagger(0, \infty)\}^N \rangle$$
where now $N$ particles must be created at plus and minus infinite time instead of $N/2$ particles. The operator $\psi(x_i, 0)^2$ represents the operator

$$\lim_{\epsilon \to 0} \frac{1}{2i\epsilon} \psi(x_i + \epsilon, 0) \psi(x_i - \epsilon, 0)$$  \hspace{1cm} (22)$$

This is equal to

$$\lim_{\epsilon \to 0} \frac{1}{2i\epsilon} \sum_{k,k'} e^{i(k+k')x} e^{i(k-k')x} a(k) a(k')$$  \hspace{1cm} (23)$$

which is

$$\sum_{k,k'} e^{i(k+k')x} \frac{k - k'}{2} a(k) a(k')$$  \hspace{1cm} (24)$$

Taking $|V^+\rangle$ to be the state with $N$ particles added to the vacuum, and $|V^-\rangle$ to be the state with $N$ particles removed from the vacuum, eq. (20) becomes, up $N$-dependent factors,

$$\langle V^- | e^{\int dx f(x) \psi(x)^2} | V^+ \rangle$$  \hspace{1cm} (25)$$

where a chemical potential $f(x)$ has been introduced. For $f(x) = 1$, this is equal to

$$\frac{2^{N-1/2}}{N!} \prod_{k=1/2}^{2k} 2k$$  \hspace{1cm} (26)$$

The calculation of the two-level form factor at this point is almost identical to that in the orthogonal ensemble. We must introduce the operator $\frac{1}{2} \psi(x)^2 \psi(x')^2$ into the expectation value, expand in Fourier components, and pair off the annihilation operators in all possible ways.

The non-trivial pairings of annihilation operators involve pairing the first with the third or the first with the fourth. These two possibilities give the same contribution. This yields

$$-\frac{1}{4} e^{i(k+l)(x-x')} \frac{(k-l)^2}{kl}$$  \hspace{1cm} (27)$$

which is zero for $|m| > 2$ and equal to, in the large $N$ limit,

$$2 - |m| + \frac{|m|}{2} \ln(|m| - 1)$$  \hspace{1cm} (28)$$

for $|m| < 2$. The factor of $1/4$ in eq. (27) arises from the the factor of $1/2$ in the operator inserted into the expectation value, times the factor of 2 from the two different pairings, times the factor of $1/4$ from the two terms missing in eq. (26).

A fermion mapping has been given for the orthogonal and symplectic ensembles. Although closely related to the technique of integrating over alternate variables, this technique provides a physical system, the fermionic field theory, which handles the various integrals and determinants.
References

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[4] M.L. Mehta, Random Matrices, 2nd edition, (Academic Press, New York, 1991).