Multiple-scale analysis of open quantum systems

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Abstract. In this work, we present a multiple-scale perturbation technique suitable for the study of open quantum systems, which is easy to implement and in few iterative steps allows finding quite good approximate solutions. For any time-local master equation, whether markovian or non-markovian, in Lindblad form or not, we give a general procedure to construct analytical approximations to the corresponding dynamical map and, consequently, to the temporal evolution of the density matrix. As a simple illustrative example, we study an atom-cavity system described by a dissipative Jaynes-Cummings model, which does not have an exact analytical solution. Performing a multiple-scale analysis we obtain approximate analytical expressions for the strong and weak coupling regimes that allow us to identify characteristic time scales in the state of the physical system.

Keywords: perturbation technique, multiple-scale analysis, open quantum systems, time-local quantum master equations

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1. Introduction

The study of open quantum systems (OQS) has become mandatory when thinking about any technological application based on quantum mechanics [1]. However, its research is not limited to applications, but has allowed advancing in the study of fundamental physics [2]. In fact, any area of physics interested in the quantum nature of a system that interacts with its environment needs the theory of open quantum systems. Such is the case of quantum optics [3], quantum information [4], and quantum measurement [5], just to give some examples. Therefore, finding appropriate mathematical methods to determine the dynamics of OQS is as relevant as the study of physical systems themselves.

Traditionally, the approaches used to solve the dynamics and the steady state of an open quantum system are carried out by exact diagonalization of the Liouville
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superoperator (liouvillian) [6], or by simulations with quantum trajectories [7]. However, both approaches are very limited by the growth of the Hilbert space, and quickly become challenging numerical problems. The challenge increases when the liouvillian is time-dependent, where in general it is not possible to make an exact analytical diagonalization to obtain the exponential solution to the quantum master equation. In this case, it is common to use the Magnus series [8], which provides a series expansion to the solution of linear differential equations of linear operators. Although it provides approximate analytical solutions, it is difficult to use due to the complicated structure of the mathematical expressions involved. Therefore, it is necessary to look for other approaches to solve a wider spectrum of OQS that would otherwise be very difficult to study. For this purpose, there are a multitude of analytical and numerical methods, some of them are: Perturbative expansion of dissipative terms [9, 10], the numerical renormalization group [11, 12], the density matrix renormalization group [13], adiabatic elimination to obtain effective liouvillian superoperators [14, 15], Feshbach projection technique [16], variational method based on matrix product operators [17], Keldysh Green’s function approach [18, 19], and perturbation theory based on the Lindblad master equation (Lindblad perturbation theory) [20, 21]. These methods are either focused on obtaining solutions to the steady state or are numerical techniques to solve the dynamics of OQS. In any case, they do not provide approximate analytical expressions that describe the dynamics of OQS in both the transient and steady states. To this end, it is natural to think of a time-dependent perturbation theory, however, a usual perturbative expansion is often an insufficient approximation to solve classical and quantum dynamical problems. This, because it leads to secular terms in the approximate solutions, i.e., terms that grow as powers of the time variable, significantly limiting the range of validity of the approximate solution. These terms appear in differential equations with constant coefficients, when the non-homogeneous term is itself a solution to the associated homogeneous differential equation [22]. In classical mechanics, they appear for systems known to evolve harmonically, and in quantum mechanics they destroy the canonical commutation relations among the approximate operators [23].

Here, we propose an alternative perturbative approach based on the multiple-scale analysis of time-local master equations. However, multiple-scale methods have been around for a long time, in fact, the first of them was the Poincaré-Von Zeipel method, which was developed at the end of the 19th century for the study of celestial dynamics [24]. It approximates periodic solutions to ordinary differential equations without producing secular terms in the approximations, which inevitably appear in regular perturbation theory. The method provides asymptotic expansions free of secular terms, but lacks convergence. This, and other multiple-scale techniques were widely used, although the mathematical justification for its operation was not understood until the 1960s, when multiple-scale modeling was explained using coordinate transformations and invariant manifolds [25, 26]. Multiple-scale analysis (MSA) has been used successfully in the solution of linear and weakly nonlinear differential equations that describe the dynamics of classical systems [22, 27–29]. Moreover, there have been very
interesting proposals for their implementation in quantum systems, such as the quantum anharmonic oscillator [30–32], and a broad range of quantum optical systems [23, 33–35]. Besides, it has also been used in the study of time-dependent problems, such as the dynamical Casimir effect [36–39]. However, in the existing implementations the consistency of the solutions obtained is subject to restrictions that provide physical interpretation of each term in the perturbative expansion [40]. This does not allow the construction of a general scheme, but for each physical system a different implementation must be made. For example, in the solution of the Heisenberg equation using MSA [32], the integration constants in each iteration must guarantee the canonical commutation relations between creation and annihilation operators, which implies a step-by-step development that can hardly be generalized. Nevertheless, as we will show, it is possible to construct a multiple-scale perturbation technique (MSPT) that is free of such restrictions, and still preserves the physical properties of the system at each step of the perturbative iteration. Therefore, the aim of this work is to extend the MSA to the study of OQS described by time-local master equations (TLMEs). Our treatment is not limited to a specific physical system; on the contrary, it is applicable to a wide range of perturbations and open quantum systems. Besides, the presented method is not only a technique that avoids secular solutions, but a tool that allows us to obtain approximate solutions of fast convergence and stability with respect to initial conditions.

This paper is organized as follows: In section 2, we describe time-local quantum master equations for the density matrix operator, that are the kind of equations we attempt to solve using the method in this work. In section 3, we describe the multiple-scale perturbation technique. In section 4, as an illustration of the usefulness of the technique we give an example of its implementation. Finally, in section 5, we give some conclusions.

2. Time-local quantum master equations

The study of OQS appears as a more realistic approach to the quantum dynamics of systems that are interacting in a non-negligible way with an environment, or reservoir. Formally, the system-reservoir universe is itself a closed quantum system with Hilbert space $\mathcal{H}_T$, and evolving unitarily under the total hamiltonian $H_T(t)$. Then, the evolution of the total density matrix $\rho_T$ from an initial time $t_0$ to a posterior time $t$ will be described, in general, by the unitary operator $U(t,t_0)$ such that ($\hbar = 1$),

$$\rho_T(t) = U(t,t_0) \rho_T(t_0) U^\dagger(t,t_0), \quad (1)$$

$$U(t,t_0) = \mathcal{T} e^{i \int_{t_0}^t H_T(t') dt'}, \quad (2)$$

with $\mathcal{T}$ the Dyson time-ordering operator. However, this type of systems would be very difficult to study if all degrees of freedom are considered. Then, a partial trace is carried out over the reservoir, reducing the problem to an efficient set of parameters and
operators that describes the exact system dynamics,
\[ \rho(t) = \text{Tr}_R \{ U(t,t_0) \rho_{T}(t_0) U^\dagger(t,t_0) \}. \] (3)

If the reservoir is in a steady state with a given spectral decomposition \( \rho_R = \sum_j P_j |r_j\rangle \langle r_j| \), \( \sum_j P_j = 1 \), and the system and its environment are initially uncorrelated \( \rho_T(t_0) = \rho(t_0) \otimes \rho_R \), we can explicitly trace over the reservoir and separate the evolution of the system,
\[ \rho(t) = \sum_{i,j} P_j \langle r_i | U(t,t_0) | r_j \rangle \rho(t_0) \langle r_j | U^\dagger(t,t_0) | r_i \rangle, \] (4)

where it was defined \( K_{ij}(t,t_0) = \sqrt{P_j} \langle r_i | U(t,t_0) | r_j \rangle \). Equation (5) is the Kraus decomposition of the system density matrix evolution [41], with \( K_{ij}(t,t_0) \) the Kraus operators satisfying the time-independent completeness relation [42],
\[ \sum_{i,j} K_{ij}^\dagger(t,t_0) K_{ij}(t,t_0) = 1. \] (6)

It is clear, that the operator sum representation of the time-evolution in (5) is not unitary in general. However, it must describe a valid physical time-evolution, mapping a physical state \( \rho(t_0) \) into a physical state \( \rho(t) \) [43], where a valid density matrix representing a state, must be trace preserving and positive semidefinite [44]. Then, the operator sum represents a completely-positive trace-preserving (CPTP) dynamical map \( \Phi(t,t_0) \) [45, 46],
\[ \rho(t) = \Phi(t,t_0) \rho(t_0) = \sum_{i,j} K_{ij}(t,t_0) \rho(t_0) K_{ij}^\dagger(t,t_0). \] (7)

The trace-preserving property, \( \text{Tr}\{\rho(t)\} = \text{Tr}\{\rho(t_0)\} = 1 \), is a consequence of the completeness relation of Kraus operators (6), while the complete positivity is a consequence of the representation theorem for quantum operations, which establishes: If we have the operator sum representation (5), and the operators satisfy the completeness relation (6), then the dynamical map \( \Phi(t,t_0) \) is completely positive [47, 48]. In general, the time-evolution in (7) could be described by TLMEs of the form [49],
\[ \dot{\rho}(t) = \mathcal{L}(t) \rho(t), \] (8)

where the liouvillian \( \mathcal{L}(t) \) is a linear non-hermitian superoperator, and the infinitesimal generator of the dynamical map \( \Phi(t,t_0) \),
\[ \rho(t) = \Phi(t,t_0) \rho(t_0) = \left\{ T e^\int_{t_0}^{t} \mathcal{L}(t')dt' \right\} \rho(t_0). \] (9)

The dynamics described by TLMEs can be distinguished between markovian and non-markovian, where markovianity is defined by means of the divisibility of the dynamical map [44, 50, 51],
\[ \Phi(t,t_0) = \Phi(t,s) \Phi(s,t_0), \quad t \geq s \geq t_0. \] (10)
In many applications, it is justified the Born-Markov approximation, and therefore, to have master equations with time-independent liouvillians,

\[ \dot{\rho}(t) = \mathcal{L} \rho(t). \] (11)

Thus, \( \rho(t_1) = e^{\mathcal{L}(t_1-t_0)} \rho(t_0) \), and then, the evolution will be homogeneous in time and given by the single parameter, \( t = t_1 - t_0 \). Hence, the solution to (11) is given by,

\[ \rho(t) = \Phi(t) \rho(0) = e^{\mathcal{L} t} \rho(0), \] (12)

and therefore, the system dynamics is represented by a quantum dynamical semigroup \([52]\), i.e., it is represented by a single-parameter dynamical map \( \Phi(t) \), satisfying the semigroup property, \( \Phi(t+s) = \Phi(t) \Phi(s) \) (\( t, s \geq 0 \)). The most general type of markovian and time-homogeneous master equation describing non-unitary dynamics of the density matrix \( \rho \), is the celebrated Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equation \([47, 53]\), or simply, the master equation in Lindblad form,

\[ \dot{\rho}(t) = \mathcal{L}_{\text{GKLS}} \rho(t) = -i [H, \rho(t)] + \sum_n \gamma_n \mathcal{D}[O_n] \rho(t), \] (13)

\[ \mathcal{D}[O_n] \rho(t) = O_n \rho(t) O_n^\dagger - \frac{1}{2} (O_n^\dagger O_n \rho(t) + \rho(t) O_n^\dagger O_n), \] (14)

where \( H \) is the system’s hamiltonian, and \( \gamma_n \) are the parameters associated with the different dissipative and decoherent processes described by the operators \( O_n \). Besides, if \( \gamma_n > 0 \), then the dynamical map \( \Phi(t) = e^{\mathcal{L}_{\text{GKLS}} t} \) is completely-positive. A detailed derivation of the Lindblad master equation (13) can be found in references \([3, 54]\). It is worth mentioning that when the Born-Markov approximation can not be made, we are left with a quantum master equation that is, in general, non-local in time, i.e., the state of the system at a given time \( t \) depends on the physical state at all previous times. This represents a huge issue when looking for solutions, even approximate ones. However, through techniques such as the time convolution-less projection operator it is possible to bring a master equation describing a system initially uncorrelated with its reservoir, to a time-local form \([48, 49]\),

\[ \dot{\rho}(t) = \mathcal{L}(t) \rho(t). \] (15)

In fact, it has been shown that time-local quantum master equations can be brought into a canonical Lindblad form with time-dependent coefficients \([55, 56]\),

\[ \dot{\rho}(t) = -i [H(t), \rho(t)] + \sum_n \gamma_n(t) \left\{ O_n(t) \rho(t) O_n^\dagger(t) - \frac{1}{2} (O_n^\dagger(t) O_n(t) \rho(t) + \rho(t) O_n^\dagger(t) O_n(t)) \right\}, \] (16)

where all the parameters and operators involved, in general, depend on time. However, the rates \( \gamma_n \) can only be temporarily negative in order to preserve the complete positivity of the map, besides, \( \gamma_n(t) \geq 0 \) for all \( t \), is a necessary and sufficient condition for
the divisibility of the quantum operation [44]. Nevertheless, it is not possible, in general, to reduce the dynamical map \( \Phi(t, t_0) \) to a single-parameter representation, and consequently it is not a quantum dynamical semigroup. To simplify the notation, we will assume that the initial time is always zero \( (t_0 = 0) \), and then, \( \Phi(t, t_0 = 0) = \Phi(t) \); although, in general, we will not work with single-parameter dynamical maps.

Here, we will focus on systems whose dynamical map exists and is CPTP. In summary of the above, the description of the dynamics of OQS in terms of TLMEs and CPTP dynamical maps is based on the assumption of an initially uncorrelated state between system and reservoir. The markovianity of the system is defined by means of the divisibility of the dynamical map. Therefore, if the liouvillian \( \mathcal{L} \) is time-independent, then the dynamics is trivially markovian. On the other hand, if \( \mathcal{L} \) is time-dependent, markovianity depends on whether the map is divisible or not. However, the technique presented in this work, is suitable to obtain approximate analytical solutions to TLMEs describing both markovian and non-markovian dynamics.

3. Multiple-scale perturbation technique

As pointed out above, the key goal of this research is to illustrate a perturbation technique based on the multiple-scale analysis of open quantum systems whose evolution is represented by time-local quantum master equations, \( \dot{\rho}(t) = \mathcal{L}(t) \rho(t) \). Such multiple-scale perturbation technique is essentially a derivative expansion method [22, 32], which promotes the time variable \( t \) to an infinite collection \( \{ \tau_0, \tau_1, \tau_2, \ldots \} \) of independent variables \( \tau_n \), each one a function of the perturbative parameter \( \alpha \) such that \( \tau_n = \alpha^n t, \ n = 0, 1, 2, \ldots \), with \( 0 < \alpha < 1 \). In consequence, the temporal derivative transforms into an infinite sum provided by the chain rule,

\[
\frac{d}{dt} = \sum_n \alpha^n \mathcal{D}_n, \quad \mathcal{D}_n = \frac{\partial}{\partial \tau_n}. \tag{17}
\]

The allowed values of the perturbative parameter \( \alpha \) guarantee the positivity of the time scales \( \tau_n \) and their shortening with respect to the growth of the discrete variable \( n \), i.e., whenever \( n \) grows \( \tau_n \) will describe increasingly rapid changes in the physical quantity that is being resolved. In our implementation, we introduce an extended density matrix \( \rho(T) = \rho(\tau_0, \tau_1, \tau_2, \ldots) \) which is considered an extension of the true density matrix \( \rho(t) \), which is retrieved by restricting \( \rho(T) \): \( \rho(t) = \rho(T)|_{\tau_n = \alpha^n t} \). Expanding the density matrix asymptotically, we have,

\[
\rho(T) = \sum_n \alpha^n \varrho^{(n)}(T). \tag{18}
\]

It is worth mentioning that only \( \varrho^{(0)} \) represents a valid density matrix. In fact, it will be responsible of the unit-trace (all the other terms of the expansion are traceless), and the satisfaction of initial conditions, \( \varrho^{(0)}(0) = \rho(0) \) (while \( \varrho^{(n)}(0) = 0 \) for \( n > 0 \)). On the other hand, we can separate the liouvillian \( \mathcal{L}(t) \) in two parts: A solvable superoperator
\( \mathcal{L}_0 \) and a perturbation \( \mathcal{L}_1(t) \), such that \( \mathcal{L}(t) = \mathcal{L}_0 + \alpha \mathcal{L}_1(t) \). Thus, in the extended time we would have, \( \mathcal{L}(t) \to \mathcal{L}(\tau_0) \), \( \mathcal{L}(\tau_0) = \mathcal{L}_0 + \alpha \mathcal{L}_1(\tau_0) \). Putting the above expressions together in the TLME,

\[
\dot{\rho}(T) = \mathcal{L}(\tau_0) \rho(T),
\sum_{m,n} \alpha^{m+n} \mathcal{D}_m \rho^{(n)}(T) = \left( \mathcal{L}_0 + \alpha \mathcal{L}_1(\tau_0) \right) \sum_n \alpha^n \rho^{(n)}(T). \tag{19}
\]

Separating according to the powers of the perturbation parameter,

\[
\mathcal{O}(\alpha^0) : \mathcal{D}_0 \rho^{(0)} - \mathcal{L}_0 \rho^{(0)} = 0, \tag{20a}
\]

\[
\mathcal{O}(\alpha^1) : \mathcal{D}_0 \rho^{(1)} - \mathcal{L}_0 \rho^{(1)} = - \left( \mathcal{D}_1 \rho^{(0)} - \mathcal{L}_1(\tau_0) \rho^{(0)} \right), \tag{20b}
\]

\[
\mathcal{O}(\alpha^2) : \mathcal{D}_0 \rho^{(2)} - \mathcal{L}_0 \rho^{(2)} = - \left( \mathcal{D}_1 \rho^{(1)} - \mathcal{L}_1(\tau_0) \rho^{(1)} \right) - \mathcal{D}_2 \rho^{(0)}, \tag{20c}
\]

\[
\vdots
\]

\[
\mathcal{O}(\alpha^n) : \mathcal{D}_0 \rho^{(n)} - \mathcal{L}_0 \rho^{(n)} = - \left( \mathcal{D}_1 \rho^{(n-1)} - \mathcal{L}_1(\tau_0) \rho^{(n-1)} \right) - \sum_{m=0}^{n-2} \mathcal{D}_{n-m} \rho^{(m)}, \quad n \geq 2. \tag{20d}
\]

From the zero-order equation (20a), performing a Dyson integration as shown in Appendix A,

\[
\rho^{(0)}(T) = \{ e^{\mathcal{L}_0 \tau_0} C_{0,1}(T_1) \} \rho^{(0)}(0), \tag{21}
\]

where \( T_1 \) follows the notation, \( T_k = \{ \tau_k, \tau_{k+1}, \ldots \} \) (\( k \geq 1 \)), i.e., the new collection \( T_k \) includes all the time-scales from \( \tau_k \). Besides, \( C_{0,1}(T_1) \) is a constant superoperator with respect to the \( \tau_0 \) variable, whose dependence with respect to each time variable in \( T_k \) will be determined order-by-order in the perturbation theory. Equation (21) suggests that the description of the perturbative technique can also be performed by means of the dynamical map \( \Phi(T) \), then if,

\[
\rho^{(0)}(T) = \Phi^{(0)}(T) \rho(0), \tag{22}
\]

\[
\Phi^{(0)}(T) = \{ e^{\mathcal{L}_0 \tau_0} C_{0,1}(T_1) \},
\]

we can make a description independent of the initial conditions \( \rho(0) \). A description in terms of the extended dynamical map \( \Phi(T) \) will require to represent it as an asymptotic expansion as well,

\[
\Phi(T) = \sum_n \alpha^n \Phi^{(n)}(T), \tag{23}
\]

where only \( \Phi^{(0)}(T) \) represents a valid dynamical map satisfying the initial condition, \( \Phi^{(0)}(0) = \mathbb{1} \), while \( \Phi^{(n)}(0) = 0 \) for \( n > 0 \). The time-evolution of the system density matrix will be given by \( \rho(T) = \Phi(T) \rho(0) \), and therefore,

\[
\rho(T) = \sum_n \alpha^n \Phi^{(n)}(T) \rho(0), \tag{24}
\]

\[
\rho^{(n)}(T) = \Phi^{(n)}(T) \rho(0). \tag{25}
\]
Besides, we can separate the unperturbed solution from the expansion, which is analogous to moving to an interaction picture, \( \Phi^{(n)}(t) = e^{L_0 \tau_0} A_n(t) \), and then,

\[
\rho^{(n)}(t) = \{ e^{L_0 \tau_0} A_n(t) \} \rho(0),
\]

In terms of the \( A_n = A_n(t) \), equations (20) transform to,

\[
O(\alpha^0) : \mathcal{D}_0 A_0 = 0,
\]

\[
O(\alpha^1) : \mathcal{D}_0 A_1 = - (\mathcal{D}_1 A_0 - F_{1,0}(\tau_0) A_0), \tag{27b}
\]

\[
O(\alpha^2) : \mathcal{D}_0 A_2 = - (\mathcal{D}_1 A_1 - F_{1,0}(\tau_0) A_1) - \mathcal{D}_2 A_0, \tag{27c}
\]

\[
\vdots
\]

\[
O(\alpha^n) : \mathcal{D}_0 A_n = - (\mathcal{D}_1 A_{n-1} - F_{1,0}(\tau_0) A_{n-1}) - \sum_{m=0}^{n-2} \mathcal{D}_{n-m} A_m, \quad n \geq 2, \tag{27d}
\]

where, \( F_{1,0}(\tau_0) = e^{-L_0 \tau_0} L_1(\tau_0) e^{L_0 \tau_0} \). The solution procedure will be iterative order to order, satisfying solvability conditions before solving each order equation. The solvability conditions avoid the appearance of secular solutions, making zero the terms that resonate with the homogeneous equation. We can construct the solvability conditions by making time averages with respect to the \( \tau_0 \) variable, separating the terms that oscillate with zero frequency, which resonate with the homogeneous solution. In this way, the solution to (27a), which gives us the \( \tau_0 \) dependence of \( A_0 \), will be substituted into equation (27b), and here the solvability condition must be fulfilled before solving the first-order partial differential equation. Thus, the first-order equation (27b) could be written as, \( \mathcal{D}_0 A_1 = - \Psi_1 \), with \( \Psi_1 = (\mathcal{D}_1 A_0 - F_{1,0}(\tau_0) A_0) \). The solvability condition will be, \( \langle \Psi_1 \rangle_{\tau_0} = 0 \), which gives us the \( \tau_1 \) dependence of \( A_0 \). After this, the improved first-order equation must be solved. Therefore, the solvability conditions up to \( n \)th-order are,

\[
O(\alpha^1) : \langle \Psi_1 \rangle_{\tau_0} = (\mathcal{D}_1 A_0 - F_{1,0}(\tau_0) A_0)_{\tau_0} = 0, \tag{28a}
\]

\[
O(\alpha^2) : \langle \Psi_2 \rangle_{\tau_0} = (\mathcal{D}_1 A_1 - F_{1,0}(\tau_0) A_1 + \mathcal{D}_2 A_0)_{\tau_0} = 0, \tag{28b}
\]

\[
\vdots
\]

\[
O(\alpha^n) : \langle \Psi_n \rangle_{\tau_0} = \left( \mathcal{D}_1 A_{n-1} - F_{1,0}(\tau_0) A_{n-1} + \sum_{m=0}^{n-2} \mathcal{D}_{n-m} A_m \right)_{\tau_0} = 0, \quad n \geq 2, \tag{28c}
\]

where the time-averaging is given by,

\[
\langle \Psi_n \rangle_{\tau_0} = \frac{1}{T_n} \int_0^{T_n} \Psi_n \, d\tau_0, \tag{29}
\]

being \( T_n \) the period of the solvability condition, which can approach infinity if there is no well-defined period.
Figure 1. Flowchart schematizing the solution algorithm. We solve in each step first the solvability condition associated with the equation to solve. The solvability conditions will intercalate the equations for the different orders, and must be fulfilled if we require absence of secular terms.
In terms of the density matrix, it is possible to recognize the structure of the asymptotic sum up to a given level of precision. Therefore, $\rho^{(n)}$ will be the density matrix truncated to the $n$th-order, and $\varrho^{(m,n)}$ will be the $n$th correction to the $m$th term of the asymptotic expansion. For each order in the perturbative expansion, we will have,

**Zero-order:**

$$\rho^{(0)}(\tau_0) = \varrho^{(0,0)}(\tau_0).$$

**First-order:**

$$\rho^{(1)}(\tau_0, \tau_1) = \varrho^{(0,1)}(\tau_0, \tau_1) + \alpha \varrho^{(1,0)}(\tau_0, \tau_1).$$

**Second-order:**

$$\rho^{(2)}(\tau_0, \tau_1, \tau_2) = \varrho^{(0,2)}(\tau_0, \tau_1, \tau_2) + \alpha \varrho^{(1,1)}(\tau_0, \tau_1, \tau_2) + \alpha^2 \varrho^{(2,0)}(\tau_0, \tau_1, \tau_2).$$

**nth-order:**

$$\rho^{(n)}(\tau_0, \ldots, \tau_n) = \sum_{m=0}^{n} \alpha^m \varrho^{(m,n-m)}(\tau_0, \ldots, \tau_n), \quad n \geq 1.$$  

Then, the solution algorithm will go as follows (as schematized in figure 1): From the zero-order equation $(27a)$, $\varrho^{(0,0)}(\tau_0)$ is calculated. Then, the non-homogeneous term $\Psi_1$ in the first-order equation $(27b)$, defines the first order solvability condition $\langle \Psi_1 \rangle_{\tau_0} = 0$, which improves the previous solution giving the functional dependence with respect to the next time variable, $\varrho^{(0,1)}(\tau_0, \tau_1)$. Hence, the first order equation is solved, giving $\varrho^{(1,0)}(\tau_0, \tau_1)$. The approximate density matrix until this point is $\rho^{(1)}(\tau_0, \tau_1) = \varrho^{(0,1)}(\tau_0, \tau_1) + \alpha \varrho^{(1,0)}(\tau_0, \tau_1)$. In the same way, we can solve each order, improving the previous solution through the solvability condition, and then adding a new term to the asymptotic expansion. On the other hand, when we return to the usual time variable $t$, we will have,

**Zero-order:**

$$\rho^{(0)}(t) = \varrho^{(0,0)}(t)$$

**First-order:**

$$\rho^{(1)}(t) = \varrho^{(0,1)}(t) + \varrho^{(1,0)}(t)$$

**Second-order:**

$$\rho^{(2)}(t) = \varrho^{(0,2)}(t) + \varrho^{(1,1)}(t) + \varrho^{(2,0)}(t)$$

**nth-order:**

$$\rho^{(n)}(t) = \sum_{m=0}^{n} \alpha^m \varrho^{(m,n-m)}(t), \quad n \geq 1.$$  

Next, we will show the general expressions obtained from the solution to equations $(27)$ and $(28)$ for any open quantum system, with an arbitrary perturbation, described by a TLME.

**Zero-order:** From $(27a)$, as mentioned above, we have,

$$A_0(\tau) = C_{0,1}(\tau_1).$$

If we truncate to this order, the satisfaction of the initial conditions force that, $C^{(0)}_{0,1} = \mathbb{1}$. Then, the zero-order density matrix will be,

$$\rho^{(0)}(\tau_0) = \varrho^{(0,0)}(\tau_0),$$

$$\varrho^{(0,0)}(\tau_0) = e^{K_0 \tau_0} \rho(0), \quad K_0 = L_0.$$
First-order: The solvability conditions will allow us to determine the integration constants to the next level of precision, i.e., the functional dependence with respect to the next time variable. Then, from the solvability condition (28a), we can determine the \( \tau_1 \) dependence of \( C_{0,1}(T_1) \), i.e., the zero-order solution is improved to a next level of precision. Now, looking after the first solvability condition, we are going to separate the superoperator \( F_{1,0}(\tau_0) \) in a part that oscillates with zero frequency, \( \langle F_{1,0} \rangle_{\tau_0} \), which corresponds to the temporal average of the signal in a period \( T_{1,0} \), and another, \( \tilde{F}_{1,0}(\tau_0) \), that contains all the terms that oscillate with nonzero frequencies,

\[
F_{1,0}(\tau_0) = e^{-K_{0, \tau_0}} \mathcal{L}_1(\tau_0) e^{K_{0, \tau_0}} = \langle F_{1,0} \rangle_{\tau_0} + \tilde{F}_{1,0}(\tau_0); \tag{33}
\]

\[
\langle F_{1,0} \rangle_{\tau_0} = \frac{1}{T_{1,0}} \int_0^{T_{1,0}} F_{1,0}(\tau_0) \ d\tau_0. \tag{34}
\]

\[
C_{0,1}(T_1) = e^{K_{1, \tau_1}} C_{0,2}(T_2), \quad K_1 = \langle F_{1,0} \rangle_{\tau_0}. \tag{35}
\]

Furthermore, \( A_1(\tau) \) is calculated from (27b), and we have until first-order,

\[
A_0(\tau) = C_{0,1}(T_1) = e^{K_{1, \tau_1}} C_{0,2}(T_2), \tag{36}
\]

\[
A_1(\tau) = B_{1,0}(\tau_0) C_{0,1}(T_1) + C_{1,1}(T_1) = B_{1,0}(\tau_0) e^{K_{1, \tau_1}} C_{0,2}(T_2) + C_{1,1}(T_1); \tag{37}
\]

\[
B_{1,0}(\tau_0) = \int_0^{\tau_0} \tilde{F}_{1,0}(\tau'_0) \ d\tau'_0. \tag{38}
\]

If we truncate to this order, the determination of the integration constants will be subject to the initial conditions, i.e., \( \Phi(0) = \mathbb{1} \). Therefore, since \( B_{1,0}(0) = 0 \) (in general, \( B_{m,n}(0) = 0 \), \( C_{0,2}^{(1)} = \mathbb{1} \) and \( C_{1,1}^{(1)} = 0 \). Then, the first-order density matrix will be,

\[
\rho^{(1)}(\tau_0, \tau_1) = \varrho^{(0,1)}(\tau_0, \tau_1) + \varrho^{(1,0)}(\tau_0, \tau_1); \tag{39}
\]

\[
\varrho^{(0,1)}(\tau_0, \tau_1) = e^{K_{0, \tau_0}} e^{K_{1, \tau_1}} \rho(0), \tag{40}
\]

\[
\varrho^{(1,0)}(\tau_0, \tau_1) = e^{K_{0, \tau_0}} B_{1,0}(\tau_0) e^{K_{1, \tau_1}} \rho(0). \tag{41}
\]

Second-order: From the second-order solvability condition (28b), the \( \tau_2 \) dependence of \( C_{0,2}(T_2) \) and the \( \tau_1 \) dependence of \( C_{1,1}(T_1) \) are determined. Then, defining the new quantity \( F_{2,0}(\tau_0) \), we have,

\[
F_{2,0}(\tau_0) = F_{1,0}(\tau_0) B_{1,0}(\tau_0) - B_{1,0}(\tau_0) \langle F_{1,0} \rangle_{\tau_0} = \langle F_{2,0} \rangle_{\tau_0} + \tilde{F}_{2,0}(\tau_0); \tag{42}
\]

\[
\langle F_{2,0} \rangle_{\tau_0} = \frac{1}{T_{2,0}} \int_0^{T_{2,0}} F_{2,0}(\tau_0) \ d\tau_0. \tag{43}
\]

\[
F_{1,1}(\tau_1) = e^{-K_{1, \tau_1}} \langle F_{2,0} \rangle_{\tau_0} e^{K_{1, \tau_1}} = \langle F_{1,1} \rangle_{\tau_1} + \tilde{F}_{1,1}(\tau_1); \tag{44}
\]

\[
\langle F_{1,1} \rangle_{\tau_1} = \frac{1}{T_{1,1}} \int_0^{T_{1,1}} F_{1,1}(\tau_1) \ d\tau_1. \tag{45}
\]
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have until second-order,

In addition, the structure of the density matrix will be,

If we truncate to this order,

Consequently, we determined the general structure of

\begin{align}
C_{0,2}(T_2) &= e^{K_2 \tau_2} C_{0,3}(T_3), \\
K_2 &= \langle F_{1,1} \rangle_{\tau_1}, \\
C_{1,1}(T_1) &= e^{K_{1,1}} \{B_{1,1}(T_1) C_{0,2}(T_2) + C_{1,2}(T_2)\}, \\
&= e^{K_{1,1}} \{B_{1,1}(T_1) e^{K_2 \tau_2} C_{0,3}(T_3) + C_{1,2}(T_2)\}; \\
B_{1,1}(\tau_1) &= \int_0^{\tau_1} \tilde{F}_{1,1}(\tau'_1) d\tau'_1.
\end{align}

In addition, the structure of the \( A_2(\tau) \) superoperator is calculated from (27c), and we have until second-order,

\begin{align}
A_0(\tau) &= e^{K_{1,1}} e^{K_2 \tau_2} C_{0,3}(T_3), \\
A_1(\tau) &= B_{1,0}(\tau_0) \{e^{K_{1,1}} e^{K_2 \tau_2} C_{0,3}(T_3)\} \\
&\quad + e^{K_{1,1}} \{B_{1,1}(T_1) e^{K_2 \tau_2} C_{0,3}(T_3) + C_{1,2}(T_2)\}; \\
A_2(\tau) &= B_{2,0}(\tau_0) C_{0,1}(T_1) + B_{1,0}(\tau_0) C_{1,1}(T_1) + C_{2,1}(T_1), \\
&= B_{2,0}(\tau_0) e^{K_{1,1}} e^{K_2 \tau_2} C_{0,3}(T_3) \\
&\quad + B_{1,0}(\tau_0) e^{K_{1,1}} \{B_{1,1}(T_1) e^{K_2 \tau_2} C_{0,3}(T_3) + C_{1,2}(T_2)\} \\
&\quad + C_{2,1}(T_1); \\
B_{2,0}(\tau_0) &= \int_0^{\tau_0} \tilde{F}_{2,0}(\tau'_0) d\tau'_0.
\end{align}

If we truncate to this order, \( C_{0,3}^{(2)} = 1 \), \( C_{1,2}^{(2)} = 0 \), and \( C_{2,1}^{(2)} = 0 \). Then, the second-order density matrix will be,

\begin{align}
\rho^{(2)}(\tau_0, \tau_1, \tau_2) &= \rho^{(0,2)}(\tau_0, \tau_1, \tau_2) + \rho^{(1,1)}(\tau_0, \tau_1, \tau_2) + \rho^{(2,0)}(\tau_0, \tau_1, \tau_2); \\
\rho^{(0,2)}(\tau_0, \tau_1, \tau_2) &= e^{K_{0,\tau_0}} e^{K_{1,\tau_1}} e^{K_2 \tau_2} \rho(0), \\
\rho^{(1,1)}(\tau_0, \tau_1, \tau_2) &= e^{K_{0,\tau_0}} \{B_{1,0}(\tau_0) e^{K_{1,\tau_1}} e^{K_2 \tau_2} + e^{K_{1,\tau_1}} B_{1,1}(\tau_1) e^{K_2 \tau_2}\} \rho(0), \\
\rho^{(2,0)}(\tau_0, \tau_1, \tau_2) &= e^{K_{0,\tau_0}} \{B_{2,0}(\tau_0) e^{K_{1,\tau_1}} e^{K_2 \tau_2} + B_{1,0}(\tau_0) e^{K_{1,\tau_1}} B_{1,1}(\tau_1) e^{K_2 \tau_2}\} \rho(0).
\end{align}

\textit{nth-order:} Similar to the previous steps, in the \( nth \) iteration, the \( nth \) solvability condition (28c) provides the temporal dependence to a higher order of all the superoperators that appeared previously, and then the \( nth \)-order equation (27d) will allow us to find the structure of the \( A_n(\tau) \) superoperator. On the other hand, we can find more general expressions if we define,

\begin{align}
C_{m,0}(\tau) &= e^{K_{0,\tau_0}} A_m(\tau).
\end{align}

Consequently, we determined the general structure of \( A_m(\tau) \),

\begin{align}
A_m(\tau) &= \sum_{i=0}^m B_{(m-i),0}(\tau_0) C_{i,1}(T_1), \\
B_{0,0}(\tau_n) &= 1; \\
B_{(m-i),0}(\tau_0) &= \int_0^{\tau_0} \tilde{F}_{(m-i),0}(\tau'_0) d\tau'_0.
\end{align}
\[
C_{m,n}(T_n) = e^{\mathcal{K}_n \tau_n} \sum_{i=0}^{m} B_{(m-i),n}(\tau_n) C_{i,n+1}(T_{n+1}), \quad B_{0,n}(\tau_n) = 1. \tag{59}
\]

As a consequence of the above, the superoperators \( B_{m,n}(\tau_n) \) \((m > 0, n \geq 0)\) and \( \mathcal{K}_n \) \((n > 0)\) can be determined up to the desired order from the following expression, which summarizes equations (27) and (28),

\[
\sum_{i=0}^{n-1} D_0 B_{n-i,0}(\tau_0) C_{i,1}(T_1) = F_{1,0}(\tau_0) \sum_{i=0}^{n-1} B_{n-i-1,0}(\tau_0) C_{i,1}(T_1) - \sum_{m=0}^{n-1} \sum_{i=0}^{m} B_{n-i-1,0}(\tau_0) D_{n-m} C_{i,1}(T_1),
\]

\[n \geq 1. \tag{60}\]

It is worth mentioning, as is shown in Appendix B, that all superoperators \( \mathcal{K}_n \) \((n \geq 0)\) commute between them, \([\mathcal{K}_{n+1}, \mathcal{K}_n] = 0\). This means, that the zero term in the asymptotic expansion of the dynamical map,

\[
\Phi^{(0)}(T) = e^{\mathcal{K}_0 \tau_0} e^{\mathcal{K}_1 \tau_1} \ldots e^{\mathcal{K}_n \tau_n} \ldots
\]

is an exponential factorization between superoperators that commute with each other. Thus, if \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) commute, the MSA solution will correspond to the exact solution, \( \Phi(t) = e^{\mathcal{L}_0 t} e^{\mathcal{L}_1 t} \); where the exact solution is reached after solving the first-order solvability condition in the MSPT. Next, we will show a simple example of implementation of the MSPT, this with the intention of clarifying the ideas presented.

### 4. Illustrative example: The dissipative Jaynes-Cummings model

As a simple example of the implementation of the MSPT, let us consider the Jaynes-Cummings (JC) model with dissipation, which describes the quantum dynamics of a two-level system (TLS) interacting with a quantized single-mode of the electromagnetic field in a dissipative cavity. The system-reservoir interaction occurs by means of the leakage of photons through the cavity mirrors and the continuous and incoherent pumping of the TLS, a situation common in semiconductor cavity quantum electrodynamics [57]. This problem does not have an exact analytical solution, but by performing a MSA we obtained approximate analytical expressions for the strong and weak coupling regimes that allow us to study the competition between pumping and losses on the state of the physical system.

The JC Hamiltonian has two parts, the first one contains the information of the energy of the system,

\[
H_0 = \omega_c a^\dagger a + \frac{\omega_a}{2} \sigma_z,
\]

where, \( a \) and \( a^\dagger \) are the annihilation and creation photonic operators, \( \sigma_z \) is the third Pauli matrix, and \( \omega_c \) and \( \omega_a \) are respectively the energies of the photonic mode (cavity) and
the TLS (atom). The second part, has the information of the interaction between the involved subsystems, which is characterized by the radiation-matter interaction constant $g$. Therefore, near resonance ($\omega_c \approx \omega_a$), if $g$ is much smaller than the natural frequencies of the system ($\omega_c$, $\omega_a$), then, the rotating wave approximation (RWA) is justified and the interaction Hamiltonian is,

$$gV = g(a^\dagger \sigma_- + a \sigma_+),$$  

Thus, the complete JC Hamiltonian will be,

$$H = H_0 + gV,$$

$$H = \omega_c a^\dagger a + \frac{\omega_a}{2} \sigma_z + g (a^\dagger \sigma_- + a \sigma_+).$$  

On the other hand, in presence of leakage of photons through the cavity mirrors and continuous and incoherent pumping of the TLS, the master equation in Lindblad form is,

$$\dot{\rho}(t) = -i[H, \rho(t)] + \kappa \mathcal{D}[a] \rho(t) + P \mathcal{D}[\sigma_+] \rho(t),$$  

where, $\kappa$ is the rate of leakage of photons out of the cavity and $P$ is the amplitude of the continuous pump, and explicitly ($\rho = \rho(t)$),

$$\dot{\rho} = -i[H, \rho] + \frac{\kappa}{2} \left\{ 2a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a \right\} + \frac{P}{2} \left\{ 2\sigma_+ \rho \sigma_- - \sigma_- \sigma_+ \rho - \rho \sigma_- \sigma_+ \right\}. $$

If there is low pumping in the system ($P \ll g, \kappa$), according to the relationship between $g$ and $\kappa$ there are two clearly differentiated operating regimes [58–60]. The first one, known as the strong coupling (SC) regime, is characterized by the fact that the interaction constant is bigger than the system dissipation rate ($g > \kappa$). In the second one, known as the weak coupling (WC) regime, the opposite occurs, i.e., the interaction constant is smaller than the dissipation rate of the system ($g < \kappa$). For the description of any of these regimes, we assume that the atom can be in its ground $|g\rangle$ or excited $|e\rangle$ state, and the cavity photonic field can have zero $|0_c\rangle$ or one photon $|1_c\rangle$. Therefore, the atom-cavity system is described by the bare states, $|0\rangle = |g, 0_c\rangle$, $|1\rangle = |e, 0_c\rangle$, $|2\rangle = |g, 1_c\rangle$. The system of equations that describes the quantum dynamics in resonance ($\omega_c = \omega_a$) is the following,

$$\dot{\rho}_{0,0} = -P \rho_{0,0} + \kappa \rho_{2,2},$$  

$$\dot{\rho}_{1,1} = -ig(\rho_{2,1} - \rho_{1,2}) + P \rho_{0,0},$$  

$$\dot{\rho}_{2,2} = -ig(\rho_{1,2} - \rho_{2,1}) - \kappa \rho_{2,2},$$  

$$\dot{\rho}_{1,2} = -ig(\rho_{2,2} - \rho_{1,1}) - \frac{\kappa}{2} \rho_{1,2};$$  

where, $\rho_{j,i} = \rho_{i,j}^*$, and $\dot{\rho}_{0,0} = -(\dot{\rho}_{1,1} + \dot{\rho}_{2,2})$ is a consequence of the trace-preservation. We will consider that initially the atom is in its excited state and there are no photons in the cavity field, i.e., $\rho_{1,1}(0) = 1$ and all other entries of the density matrix will be zero when $t = 0$. 
4.1. Strong coupling

In the SC regime the perturbation will be associated with the Lindblad terms for incoherent pumping and leakage of photons through the cavity mirrors, then,

\[ \mathcal{L}_0 = -i[H, \rho], \]

\[ \mathcal{L}_1 = \kappa' \mathcal{D}[a] \rho + P' \mathcal{D}[\sigma_+] \rho, \]

where, \( \kappa = \alpha \kappa' \) and \( P = \alpha P' \), being \( \alpha \) the perturbation parameter. Therefore, we will solve the master equation,

\[ \dot{\rho}(t) = \{\mathcal{L}_0 + \alpha \mathcal{L}_1\} \rho(t). \]

Afterwards, we will briefly describe the solution procedure up to second-order in the MSPT, together with the approximate analytical expressions up to first-order in the MSPT. The shown analytical expressions were obtained using the general procedure developed in section 3, and they were used to obtain the curves in figure 2 and figure 3.

Zero-order (SC): As expected, the zero-order solution is the one to the unperturbed problem,

\[ \rho_{sc}^{(0)}(t) = \varrho_{sc}^{(0,0)}(t), \]

\[ \varrho_{sc}^{(0,0)}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos^2(gt) & \frac{i}{2} \sin(2gt) \\ 0 & \frac{-i}{2} \sin(2gt) & \sin^2(gt) \end{pmatrix}. \]

First-order (SC): To first-order in the MSPT, it is necessary to calculate \( F_{1,0}(\tau) \) and then its time-average \( \langle F_{1,0} \rangle_{\tau_0} \). For this purpose, the aforementioned superoperator was calculated and it was possible to recognize that its terms oscillate with a frequency \( \Omega_{1,0} = g \), and therefore the temporal average was calculated with respect to the period, \( T_{1,0} = 2\pi/g \). Using this result, we calculate \( \varrho_{sc}^{(0,1)}(t) \) and \( \varrho_{sc}^{(1,0)}(t) \), and therefore,

\[ \rho_{sc}^{(1)}(t) = \varrho_{sc}^{(0,1)}(t) + \varrho_{sc}^{(1,0)}(t), \]

\[ \varrho_{sc}^{(0,1)}(t) = \frac{\kappa}{2P + \kappa} \left( 1 - e^{-(2P+\kappa)t/2} \right), \]

\[ \varrho_{sc}^{(1,0)}(t) = \frac{1}{2(2P + \kappa)} \left( 2P + \kappa e^{-(2P+\kappa)t/2} + (2P + \kappa) e^{-\kappa t/2} \cos(2gt) \right), \]

\[ \varrho_{sc}^{(0,1)}(t) = \frac{i}{2} e^{-\kappa t/2} \sin(2gt); \]
\[ \rho_{SC0,0}^{(1,0)}(t) = -\frac{\kappa \sin(2gt)}{4g} e^{-\kappa t/2}, \]  
\[ \rho_{SC1,1}^{(1,0)}(t) = \frac{\kappa \sin(2gt)}{8g(2P + \kappa)} \left\{ (2P + \kappa)(1 + e^{-Pt})e^{-\kappa t/2} + 4P \left(1 - e^{-(2P + \kappa)t/2}\right) \right\}, \]  
\[ \rho_{SC1,2}^{(1,0)}(t) = i\frac{\kappa \sin^2(gt)}{4g(2P + \kappa)} \left\{ (2P + \kappa)e^{-(2P + \kappa)t/2} + 4P \left(1 - e^{-(2P + \kappa)t/2}\right) \right\}. \]

Up to first-order order in the perturbative expansion, the contribution to the frequency from \( \kappa \) and \( P \) appears simultaneously, providing the characteristic time scales, \( 2/\kappa \) and \( 2/(2P + \kappa) \). On the other hand, the approximate solution for the populations, up to this order, is in very good agreement with the numerical solution, as is shown in figure 2.

**Second-order (SC):** To this order the time-averages were made with respect to the periods, \( T_{2,0} = 2\pi/g \) and \( T_{1,1} = 2\pi i/P' \). Although we do not show the approximate analytic expressions, the imaginary part of coherence in the second-order solvability condition solution, was plotted in figure 3(b).

In figure 2 and figure 3, we compare approximate and numerical solutions for the SC regime. The parameters used in units of the coupling constant \( g \) were, \( \omega_c = \omega_a = 1000 g \), \( \kappa = 0.1 g \), \( P = 0.01 g \). The numerical solution was obtained by means of the numerical integration of the system of ordinary differential equations (67), using the aforementioned parameters. Figure 2(a) presents the population in the transitory regime for the three states considered, it shows that the first-order solution in the MSPT is in perfect agreement with the numerical solution obtained. Besides, figure 3(b) shows how the MSPT approaches the numerical solution for the population of the ground state. On the other hand, figure 3 shows the imaginary part of coherence between states \( |1\rangle \) and \( |2\rangle \). In particular, figure 3(a) shows the perfect agreement of the first-order solvability condition solution with the numerical one for very short times. In addition, figure 3(b) shows that for long times this agreement is not fulfilled until the second-order solvability condition.

### 4.2. Weak coupling

Although it is natural to think that the perturbative term will always be associated with Lindblad dissipators, in some cases it is useful to include hamiltonian terms in the perturbation. Such is the case of the WC regime, where the perturbation will be associated with the hamiltonian interaction term and with the Lindblad term for continuous and incoherent pumping,

\[ \mathcal{L}_0 = -i[H_0, \rho] + \kappa \mathcal{D}[a] \rho, \]  
\[ \mathcal{L}_1 = -i g'[V, \rho] + P' \mathcal{D}[\sigma_+] \rho, \]

where, \( g = \alpha g' \) and \( P = \alpha P' \) being \( \alpha \) the perturbation parameter. The plots of the approximate solutions for the WC regime are shown in figure 4. The parameters
used in terms of the dissipative rate $\kappa$ were, $\omega_c = \omega_a = 1000 \kappa$, $g = P = 0.01 \kappa$. Next, we will briefly describe the solution procedure together with the approximate analytical expressions, up to second-order in the MSPT. As before, the shown analytical expressions were obtained using the general procedure developed in section 3, and they were used to obtain the curves in figure 4.
Zero-order (WC): The solution to the unperturbed problem does not show time-evolution of the system,
\[ \rho_{WC}^{(0)}(t) = \varrho_{WC}^{(0,0)}(t), \]  
\[ \varrho_{WC}^{(0,0)}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]  

First-order (WC): In this case the time-average of \( F_{1,0}(\tau_0) \) is performed with respect to an imaginary period \( T_{1,0} = 4\pi i/\kappa \) in order to separate the zero-frequency terms. To first-order in the MSPT,
\[ \rho_{WC}^{(1)}(t) = \varrho_{WC}^{(0,1)}(t) + \varrho_{WC}^{(1,0)}(t), \]  
\[ \varrho_{WC}^{(0,1)}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]  
\[ \varrho_{WC}^{(1,0)}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2ig/\kappa (1 - e^{-\kappa t/2}) \\ 0 & -2ig/\kappa (1 - e^{-\kappa t/2}) & 0 \end{pmatrix}. \]  

Second-order (WC): To second-order the time-averages were calculated with respect to the periods, \( T_{2,0} = 4\pi i/\kappa \) and \( T_{1,1} = 2\pi i/P' \). The second-order approximate solution is,
\[ \rho_{WC}^{(2)}(t) = \varrho_{WC}^{(0,2)}(t) + \varrho_{WC}^{(1,1)}(t) + \varrho_{WC}^{(2,0)}(t), \]  
\[ \varrho_{WC}^{(0,2)}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \]  
\[ \varrho_{WC}^{(1,1)}(t) = \begin{pmatrix} \frac{3g^2}{P\kappa} (1 - e^{-Pt}) & 0 & 0 \\ 0 & -\frac{4g^2}{P\kappa} (1 - e^{-Pt}) & \frac{2ig}{\kappa} (1 - e^{-\kappa t/2}) \\ 0 & -\frac{2ig}{\kappa} (1 - e^{-\kappa t/2}) & 0 \end{pmatrix}; \]  
\[ \varrho_{WC}^{(2,0)}(0,0) = -\frac{4g^2}{\kappa^2} (3 - 4e^{-\kappa t/2} + e^{-\kappa t}), \]  
\[ \varrho_{WC}^{(2,0)}(0,1) = \frac{8g^2}{\kappa^2} (1 - e^{-\kappa t/2}), \]  
\[ \varrho_{WC}^{(2,0)}(0,2) = -\frac{8ig}{P\kappa^2} (1 - e^{-Pt} - e^{-\kappa t/2} + e^{-(2P+\kappa)t/2}). \]  

As can be concluded from the comparison between figures 2 and 3, and figure 4. The results for the SC regime are in better agreement with the numerical calculations than those for the WC regime. Then, a better approximation for WC would require going beyond in the perturbation technique.
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Figure 4. Results of the multiple-scale perturbation technique in the weak coupling regime. (a) Population of the ground state. (b) Imaginary part of coherence. In both plots, the circle markers show the numerical solution. The parameters used were, $\omega_c = \omega_a = 1000 \kappa$, $g = P = 0.01 \kappa$.

5. Conclusions

To conclude, we have presented a multiple-scale perturbation technique that allows us to find quite good approximate solutions to time-local master equations describing open quantum systems. Thus, it provides the time-evolution of the corresponding dynamical map and, consequently, the time-evolution of the system density matrix for arbitrary initial conditions, allowing to identify in each order, the characteristic time scales of the problem. The technique is easy to implement using the general expressions derived in this paper. Moreover, as is shown in the illustrative example, the method allows separating the problem according to the physical regimes that are being described, allowing the study the open quantum system from physical criteria. Besides, the presented method reaches the exact solution, in a single step in the perturbative theory, for systems where the solvable liouvillian and the perturbation commute. The description in terms of the dynamical map is suitable to obtain analytical expressions for physical observables, such as the energy density spectrum, and high-order correlation functions. As well, it allows calculating quantifiers and witnesses of entanglement, such as concurrence and negativity, for any initial condition, for a broad range of perturbations and different open quantum systems.

Finally, it is worth noting that the developed technique is a useful tool for the study of the dynamics of time-dependent quantum systems, both open and hamiltonian. This, as consequence of the time-averaging procedure in the multiple-scale analysis, which allows us to deal with time-dependent parameters in a surprisingly simple way. In fact, it allows finding general analytical expressions for time-dependent quantum systems, that otherwise it would be very difficult to calculate. In particular, the method is appropriate for systems from nonstationary cavity quantum electrodynamics, such as those related with the dynamical Casimir effect.
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Appendix A. Solution to the zero-order equation

It is worth noting, that given the nature of the mathematical objects involved in the zero-order equation (20a), the integration of the differential equation must be done in such a way that the temporal order is respected. Therefore, we are going to follow a procedure similar to that used for the construction of the Dyson series. From (20a), we have,

\[
\mathcal{D}_0 \varrho^{(0)}(T) - \mathcal{L}_0 \varrho^{(0)}(T) = 0 \quad (A.1)
\]

Being \(\varrho^{(0)}(T) = \Phi^{(0)}(T)\rho(0)\), we have then for the zero-order dynamical map,

\[
\mathcal{D}_0 \Phi^{(0)}(T) - \mathcal{L}_0 \Phi^{(0)}(T) = 0 \quad (A.2)
\]

\[
\int_0^{\tau_0} \frac{\partial \Phi^{(0)}(\tau'_0, \tau_1, \ldots)}{\partial \tau'_0} \, d\tau'_0 = \int_0^{\tau_0} \mathcal{L}_0 \Phi^{(0)}(\tau'_0, \tau_1, \ldots) \, d\tau'_0
\]

\[
\Phi^{(0)}(T) - \Phi^{(0)}(\tau'_0, \tau_1, \ldots) \bigg|_{\tau'_0=0} = \mathcal{L}_0 \int_0^{\tau_0} \Phi^{(0)}(\tau'_0, \tau_1, \ldots) \, d\tau'_0
\]

where we label \(\Phi^{(0)}(\tau'_0, \tau_1, \ldots) \bigg|_{\tau'_0=0} = C_{0,1}(T_1)\). The indexing of the \(C_{0,1}(T_1)\) superoperator is such that the first index is associated with the term of the expansion that is being determined, \(\Phi^{(0)}\) in this case; and the second index is associated with the minimum-labeled time-scale on which the superoperator depends, here \(\tau_1\). Hence, we have

\[
\Phi^{(0)}(T) = C_{0,1}(T_1) + \mathcal{L}_0 \int_0^{\tau_0} \Phi^{(0)}(\tau'_0, \tau_1, \ldots) \, d\tau'_0 \quad (A.3)
\]

then, we can solve this iteratively by replacing the integral form of \(\Phi^{(0)}(\tau'_0, \tau_1, \ldots)\) in (A.3),

\[
\Phi^{(0)}(T) = C_{0,1}(T_1) + \mathcal{L}_0 \tau_0 \, C_{0,1}(T_1)
\]

\[
+ \mathcal{L}_0 \int_0^{\tau_0} \int_0^{\tau''_0} \Phi^{(0)}(\tau''_0, \tau_1, \ldots) \, d\tau''_0 \, d\tau'_0 \quad (A.4)
\]
with \( \tau'_0 > \tau''_0 \). We continue in this way infinitely until the integration interval is so small that the integral is null and we get,

\[
\Phi^{(0)}(T) = \sum_n \frac{1}{n!}(L_0 \tau_0)^n C_{0,1}(T_1)
\]

(A.5)

\[
\Phi^{(0)}(T) = e^{L_0 \tau_0} C_{0,1}(T_1),
\]

(A.6)

where there is an implicit time ordering associated with the fact that \( \tau_0 > \tau_1 > \ldots > \tau_n \), which is guaranteed by the perturbative condition \( 0 < \alpha < 1 \). Finally,

\[
\varrho^{(0)}(T) = \{ e^{L_0 \tau_0} C_{0,1}(T_1) \} \rho(0)
\]

(A.7)

**Appendix B. Commutation relations between the \( \mathcal{K}_n \) superoperators**

Up to first-order in the MSPT, we have from (33) the definition of the superoperator \( F_{1,0}(\tau_0) \),

\[
F_{1,0}(\tau_0) = e^{-\mathcal{K}_0 \tau_0} \mathcal{L}_1(\tau_0) e^{\mathcal{K}_0 \tau_0}, \quad \mathcal{K}_1 = \langle F_{1,0} \rangle_{\tau_0}.
\]

(B.1)

Then, it is clear that the equation of motion for \( F_{1,0}(\tau_0) \) is given by,

\[
\mathcal{D}_0 F_{1,0}(\tau_0) = [F_{1,0}(\tau_0), \mathcal{K}_0], \quad F_{1,0}(0) = \mathcal{L}_1(0).
\]

(B.2)

Now, taking the time-average with respect to the \( \tau_0 \) variable, we have,

\[
\langle \mathcal{D}_0 F_{1,0} \rangle_{\tau_0} = \langle [F_{1,0}, \mathcal{K}_0] \rangle_{\tau_0} = \mathcal{K}_1, \quad \mathcal{K}_0
\]

(B.3)

\[
\mathcal{D}_0 F_{1,0}(\tau_0) d\tau_0 = \frac{F_{1,0}(T_{1,0}) - F_{1,0}(0)}{T_{1,0}} = 0,
\]

(B.5)

where \( T_{1,0} \) is the period of \( F_{1,0} \), and then,

\[
[\mathcal{K}_1, \mathcal{K}_0] = 0.
\]

(B.6)

On the other hand, up to the \( (n + 1) \)th-order in the MSPT \( (n \geq 1) \), we have,

\[
F_{1,n}(\tau_n) = e^{-\mathcal{K}_n \tau_n} \langle F_{2,n-1} \rangle_{\tau_{n-1}} e^{\mathcal{K}_n \tau_n}, \quad \mathcal{K}_{n+1} = \langle F_{1,n} \rangle_{\tau_n};
\]

(B.7)

\[
\mathcal{D}_n F_{1,n}(\tau_n) = [F_{1,n}(\tau_n), \mathcal{K}_n], \quad F_{1,n}(0) = \langle F_{2,n-1} \rangle_{\tau_{n-1}}.
\]

(B.8)

Analogously to the above, we take the time-average in a period \( T_{1,n} \) of the superoperator \( F_{1,n} \),

\[
\langle \mathcal{D}_n F_{1,n} \rangle_{\tau_n} = \langle [F_{1,n}, \mathcal{K}_n] \rangle_{\tau_n} = \langle \mathcal{D}_n F_{1,n} \rangle_{\tau_n} = 0;
\]

(B.9)

\[
[\mathcal{K}_{n+1}, \mathcal{K}_n] = 0.
\]

(B.10)

This shows that the zero term in the asymptotic expansion of the dynamical map, \( \Phi^{(0)}(T) = e^{\mathcal{K}_0 \tau_0} e^{\mathcal{K}_1 \tau_1} \ldots e^{\mathcal{K}_n \tau_n} \ldots \), is an exponential factorization of superoperators that commute between them.
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