Robustness of Interval-Valued Intuitionistic Fuzzy Reasoning Quintuple Implication Method

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ABSTRACT The interval-valued intuitionistic fuzzy quintuple implication algorithm, as an extension of the fuzzy reasoning algorithm, may better characterize and deal with uncertainty in the reasoning, but how to select distance measure and analyze the algorithm’s robustness is an important and unsolved topic. In this paper, a novel distance measure of interval-valued intuitionistic fuzzy sets is constructed based on interval-valued intuitionistic fuzzy biresiduum similarity. The unified form of the conclusion about the robustness of interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm for interval-valued intuitionistic fuzzy modus ponens (IVIFMP) and interval-valued intuitionistic fuzzy modus tollens (IVIFMT) is obtained. Especially, the robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm based on Gödel, Lukasiewicz, and Goguen implication operators is presented. An application example and experiment are offered to demonstrate the validity of the obtained conclusion. Furthermore, the new distance metric is compared to traditional distances, and its benefits and limits are discussed. The results show that our approach to research the robustness is simpler and more representative, and the robustness of the algorithm based on other implication operators can be obtained by simple substitution.

INDEX TERMS Interval-valued intuitionistic fuzzy reasoning, quintuple implication algorithm, biresiduum similarity, robustness.

I. INTRODUCTION
Fuzzy reasoning is the theoretical foundation of fuzzy control technology [1]. Fuzzy Modus Ponens (FMP) and Fuzzy Modus Tollens (FMT) are the two most fundamental forms of fuzzy reasoning [2]. Professor Zadeh, the father of fuzzy theory, presented the compositional rule of inference (CRI) algorithm to handle the problem of FMP and FMT [3], which has been widely utilized in time series forecasting, image processing, and target detection [4]–[6]. However, the logic of the CRI algorithm is flawed. The full implication Triple I algorithm and the reverse Triple I algorithm [7], [8] substituted the conjunction operation in the CRI method with the implication operator, which improved the logic in the process of reasoning. On the other hand, the Triple I and CRI methods have several restrictions that may result in deceptive computation results in some rare instances [9]. The fuzzy reasoning quintuple implication algorithm, proposed in [9], successfully solved the above limitations. The robustness and approximation of the fuzzy reasoning quintuple implication method were explored in [10]. The interval-valued fuzzy reasoning approach has been the subject of several research investigations [11]–[13].

Intuitionistic fuzzy sets (IFSs) and interval-valued intuitionistic fuzzy sets (IVIFSs), both of which are extensions of fuzzy sets, can retain fuzzy information better than fuzzy sets and have been successfully applied in the fields of cluster analysis, multi-attribute decision-making, and pattern recognition [14]–[18]. To measure uncertainty in multi-attribute decision-making, Liu and Jiang [19] converted IVIFSs to three interval vectors and proposed a new distance measure of IVIFSs based on the distance of interval number. Verma and Merigo [20] proposed a cosine similarity measure and used it to solve real-world decision problems with interval-valued intuitionistic fuzzy information. Because of the diversity of intuitionistic fuzzy implication operators and triangular
modules [21, 22], fuzzy reasoning is further extended to intuitionistic fuzzy reasoning [23]–[25]. Quintuple implication principle (QIP) and triple implication principle (TIP) on IVIFSs have been presented by Jin et al. [26], which is the extension of fuzzy reasoning quintuple implication algorithm on IVIFSs. Interval-valued Intuitionistic Fuzzy Modus Ponens (IVIFMP) and Interval-valued Intuitionistic Fuzzy Modus Ponens (IVIFMT) are two reasoning forms of interval-valued intuitionistic fuzzy reasoning.

In the fuzzy reasoning system, because the selection of membership function types and parameters is subjective, the rule and new input will deviate from ideal values to some extent, resulting in a deviation of reasoning output. If the reasoning system is sensitive to deviation, the reasoning output will most likely be erroneous. Therefore, the sensitivity of the fuzzy reasoning system to the perturbation in rules and input, referred to as the system’s robustness, is an important indicator for assessing the applicability of the fuzzy reasoning system. It is meaningful to quantify the robustness mathematically [27]. The distance measure is an important tool for investigating the robustness of fuzzy reasoning algorithms. Hui et al. [28] employed the natural distance of the IFSs to define the sensitivity and analyzed the robustness of the intuitionistic fuzzy reasoning (1,2,2)-α type universal triple I algorithm. Zhou and Luo [29] used the normalized Minkowski distance and Minkowski inequality to assess the robustness of interval-valued fuzzy reasoning quintuple implication algorithm. In order to further stress the relevance between fuzzy reasoning and implication operator, especially the robustness of the interval-valued intuitionistic fuzzy reasoning, it can better represent vague information. Nevertheless, the robustness of interval-valued intuitionistic fuzzy reasoning has not been studied because of its structural complexity and the difficulty in distance measure selection. It is feasible to use the traditional distance to analyze the robustness of fuzzy reasoning algorithm based on implication operator, but the process is complicated, especially the robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm. In addition, the unified form of the conclusion about robustness under distinct implication operators is not available. A novel distance measure of IVIFSs, which is an interval-valued intuitionistic fuzzy number, is constructed to address these issues, and the unified form of the conclusion concerning the robustness of interval-valued intuitionistic fuzzy reasoning quintuple implication is obtained in this paper. The main work and contributions of this thesis we have done are as follows:

i) The fuzzy biresiduum is extended to interval-valued intuitionistic fuzzy biresiduum, which is utilized to define the similarity of IVIFSs, and its property is analyzed.

ii) A novel distance measure of IVIFSs is constructed by the interval-valued intuitionistic fuzzy biresiduum, and the robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm is investigated. Especially, the robustness of the algorithm based on Gödel, Lukasiewicz, and Goguen implication operators is given.

iii) An example and experiment are provided to validate the correctness of our conclusion. Furthermore, the proposed distance measure is compared to the traditional distance measure, and its advantages and drawbacks are discussed.

This paper is organized as follows: Section 2 reviews some conclusions and concepts that can be used in this paper; Section 3 defines interval-valued intuitionistic fuzzy biresiduum and constructs a new distance; Section 4 discusses the robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm; Section 5, a computational experiment is provided to demonstrate the correctness of our conclusions, and the advantages and limitations of our method are analyzed. The final section concludes the conclusion and the future research.

II. PRELIMINARIES

Proposition 1 [23], [26]: Let triangular norm(t-norm) be left continuous, then there exists a binary operation on [0, 1] such that (⊙, ⊕) is co-adjoint pair; that is, c ≤ a ⊕ b if and only if c ⊗ b ≤ a, where ⊗ is given by b ⊗ a = ∧ {x ∈ [0, 1]|b ≤ x ⊕ a}.

Lemma 1 [23], [26]: Let (⊙, →) be a pair of adjoint, (⊙, ⊗) be a pair of co-adjoint, and ⊗, ⊕ be dual of each other; That is →, ⊙ and ⊕ are associated operator of ⊗. ∀a, b ∈ [0, 1], we have

\[ b ⊕ a = 1 − (1 − a) → (1 − b) . \]

Definition 1 [26]: Let X be a non-empty universe, and an interval-valued intuitionistic fuzzy set A on X can be expressed in the following form:

\[ A(x) = \{(x, (A_r(x), A_f(x))) | x \in X\} , \]

where

\[ A_r(x) = [A^r_r(x), A^r_f(x)] \subseteq [0, 1] , \]

\[ A_f(x) = [A^f_r(x), A^f_f(x)] \subseteq [0, 1] , \]

\[ A^r_r(x) ≤ A^f_r(x), A^f_r(x) ≤ A^r_f(x) , \]

\[ 0 ≤ A^r_f(x) + A^f_f(x) ≤ 1 . \]

\[ A_r(x) \text{ and } A_f(x) \text{ denote the range of membership and the range of non-membership, respectively. If } A^r_r(x) = A^f_f(x) = A_r(x) \text{ and } A^r_f(x) = A^f_r(x) = A_f(x) , \text{ the interval-valued intuitionistic fuzzy set } A(x) \text{ degrades into the intuitionistic fuzzy set. Meanwhile, if } A_r(x) = 1 − A_f(x) \text{ is tenable, the interval-valued intuitionistic fuzzy set } A(x) \text{ is further degraded into the ordinary fuzzy set. In this paper, the interval-valued intuitionistic fuzzy set on } X \text{ is denoted as IVIFS}(X) . \]

The interval-valued intuitionistic fuzzy set is the extension of the fuzzy sets and intuitionistic fuzzy set, which extends the value of the characteristic function.
from the unit interval $[0, 1]$ to the domain $SL = \{(a, b), (c, d)\} | 0 \leq a, b, c, d \leq 1, b + d \leq 1\).

Let $X$ and $Y$ both be the non-empty universe, $A(x), A^*(x) \in IVIFS(X), B(y), B^*(y) \in IVIFS(Y)$, the problem of IVIFMP and IVIFMT can be expressed as (1) and (2):

| Rule | IF $A(x)$ THEN $B(y)$ | Premise $A^*(x)$ |
|------|------------------------|------------------|
| Calculate output | $B^*(y)$ |                     |
| Rule | IF $A(x)$ THEN $B(y)$ | Premise $A^*(x)$ |
| Calculate output | $B^*(y)$ |                     |

**Definition 2 [26]:** Let $α = ((a_1, b_1), [c_1, d_1]) \in SL, β = ([a_2, b_2], [c_2, d_2]) \in SL$, a partial order on $SL$ can be given as follows: $α \leq β$ iff $a_1 \leq a_2, b_1 \leq b_2$ and $c_1 \geq c_2, d_1 \geq d_2$. Obviously,

$$\begin{align*}
α \wedge β &= ([a_1 \wedge a_2, b_1 \wedge b_2], [c_1 \vee c_2, d_1 \wedge d_2]), \\
α \vee β &= ([a_1 \vee a_2, b_1 \vee b_2], [c_1 \wedge c_2, d_1 \wedge d_2]).
\end{align*}$$

Here, $0^{SL} = ([0, 0], (1, 1))$ and $1^{SL} = ([1, 1], [0, 0])$ represent the smallest and largest elements to $SL$, respectively. We call $(SL, \leq_{SL})$ a complete lattice.

**Definition 3 [26]:** Suppose $⊗$ is a t-norm on $[0, 1]$, $⊕$ is the dual t-conorm of t-norm $⊗$, for $α, β \in SL, α = ([a_1, b_1], [c_1, d_1]), β = ([a_2, b_2], [c_2, d_2])$, two operations $⊗_{SL}$ and $⊕_{SL}$ can be defined on $SL$ as follows:

$$\begin{align*}
α⊗_{SL}β &= ([a_1 \otimes a_2, b_1 \otimes b_2], [c_1 \oplus c_2, d_1 \oplus d_2]), \\
α⊕_{SL}β &= ([a_1 \oplus a_2, b_1 \oplus b_2], [c_1 \otimes c_2, d_1 \otimes d_2]).
\end{align*}$$

**Example 1:** The three classical left-continuous t-norm $⊗$ and t-conorm $⊕$ and their corresponding interval-valued intuitionistic t-norm $⊗_{SL}$ and t-conorm $⊕_{SL}$ are as follows:

1. Let $a, b \in [0, 1]$, the G"{o}del t-norm $⊗_G$ and t-conorm $⊕_G$ have the following form:

$$\begin{align*}
α⊗_Gβ &= a \wedge b, \\
α⊕_Gβ &= a \vee b.
\end{align*}$$

Let $a = ([a_1, b_1], [c_1, d_1]), b = ([a_2, b_2], [c_2, d_2]) \in SL$, the G"{o}del interval-valued intuitionistic t-norm $⊗_{G, SL}$ and t-conorm $⊕_{G, SL}$ can be obtained from Definition 3:

$$\begin{align*}
α⊗_{G, SL}β &= ([a_1 \otimes a_2, b_1 \otimes b_2], [c_1 \oplus c_2, d_1 \oplus d_2]), \\
α⊕_{G, SL}β &= ([a_1 \oplus a_2, b_1 \oplus b_2], [c_1 \otimes c_2, d_1 \otimes d_2]).
\end{align*}$$

2. Let $a, b \in [0, 1]$, the Lukasiewicz t-norm $⊗_L$ and t-conorm $⊕_L$ have the following form:

$$\begin{align*}
α⊗_Lβ &= (a + b - 1) \wedge 0, \\
α⊕_Lβ &= (a + b) \vee 1.
\end{align*}$$

Let $a = ([a_1, b_1], [c_1, d_1]), b = ([a_2, b_2], [c_2, d_2]) \in SL$, the Lukasiewicz interval-valued intuitionistic t-norm $⊗_{L, SL}$ and t-conorm $⊕_{L, SL}$ can be obtained from Definition 3:

$$\begin{align*}
α⊗_{L, SL}β &= ([a_1 + a_2 - 1] \wedge 0, (b_1 + b_2 - 1) \wedge 0), \\
α⊕_{L, SL}β &= ([a_1 + a_2 - 1] \vee 0, (b_1 + b_2 - 1) \vee 0).
\end{align*}$$

(3) Let $a, b \in [0, 1]$, the Goguen t-norm $⊗_G$ and t-conorm $⊕_G$ have the following form:

$$\begin{align*}
α⊗_Gβ &= ab, \\
α⊕_Gβ &= a + b - ab.
\end{align*}$$

Let $a = ([a_1, b_1], [c_1, d_1]), b = ([a_2, b_2], [c_2, d_2]) \in SL$, the Goguen interval-valued intuitionistic t-norm $⊗_{G, SL}$ and t-conorm $⊕_{G, SL}$ can be obtained from Definition 3:

$$\begin{align*}
α⊗_{G, SL}β &= ([a_1 a_2, b_1 b_2], [c_1 + c_2 - c_1 c_2, d_1 + d_2 - d_1 d_2]), \\
α⊕_{G, SL}β &= ([a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2], [c_1 c_2, d_1 d_2]).
\end{align*}$$

**Theorem 1 [26]:** Let $⊗_{SL}$ be an interval-valued intuitionistic t-norm derived from a left-continuous t-norm $⊗$, and $⊗_{SL}→ SL$ be an interval-valued intuitionistic adjoint pair. Then there exists an operation $→ SL$ on $SL$ such that

$$\gamma⊗_{SL}α ≤ β, \text{ iff } γ ≤ α→ SL β,$$

and $→ SL$ can be expressed as follows:

$$α→ SL β = \vee \{η ∈ SL | α⊗_{SL}η ≤ β\}.$$

**Proposition 2 [26]:** Let $⊗_{SL}$ be an interval-valued intuitionistic t-norm derived from a left-continuous t-norm $⊗$ and $⊗_{SL}→ SL$ be an interval-valued intuitionistic adjoint pair. Then $η, γ, β \in SL$, we have:

1. $γ→ SL η = 1_{SL}$, iff $γ ≤ η$.
2. $η ≤ γ→ SL β$, iff $γ ≤ η→ SL β$.
3. $γ→ SL (η→ SL β) = η→ SL (γ→ SL β)$.
4. $1_{SL}→ SL η = η$.
5. $γ→ SL (γ→ SL β) = (γ→ SL β)$.
6. $(γ→ SL β) ∈ SL β = (γ→ SL β)$.
7. $γ→ SL β$ is antitone in the first variable $γ$ and isotone in the second variable $β$.

**Theorem 2 [26]:** Let $a = ([a_1, b_1], [c_1, d_1]), b = ([a_2, b_2], [c_2, d_2]) \in SL$ and $→ SL$ be an interval-valued intuitionistic implication derived from a left-continuous t-norm $⊗$. Besides, $→$, $⊗$ and $⊕$ are associated operator of $⊗$, we have that:

$$\begin{align*}
α→ SL β &= ([((a_1 → a_2) ∧ (b_1 → b_2) ∧ (1 - c_2 ⊕ c_1)) ∧ (1 - d_2 ⊕ d_1), (a_1 → a_2) ∧ (1 - c_2 ⊕ c_1)) ∧ (1 - d_2 ⊕ d_1), (c_2 ⊕ c_1, (c_2 ⊕ c_1) ∧ (d_2 ⊕ d_1))).
\end{align*}$$

**Theorem 3 [26]:** Suppose $→ SL$ is the residual interval-valued intuitionistic implication on $SL$ derived from a left-continuous t-norm $⊗$, then the quintuple implication solution $B^*(y)$ for IVIFMP is the smallest element on $IVIFS(Y)$ satisfying

$$\begin{align*}
(A(x) → SL B(y)) → SL (A^*(x) → SL A(x)) → SL (A^*(x) → SL B^*(y)) = 1_{SL}.
\end{align*}$$
And the solution $B^*(y)$ has the following form:

$$B^*(y) = \bigvee_{x \in X} \left[ A^*(x) \otimes_{SL} \left( A^*(x) \rightarrow_{SL} A(x) \right) \right] \otimes_{SL} \left( A(x) \rightarrow_{SL} B(y) \right), \quad \forall y \in Y.$$

**Theorem 4 [26]:** Suppose $\rightarrow_{SL}$ is the residual interval-valued intuitionistic fuzzy implication on $SL$ derived from a left-continuous t-norm $\otimes$, then the quintuple implication solution $A^*(x)$ for IVIFMT is the smallest element on IVIFS $(X)$ satisfying

$$(A(x) \rightarrow_{SL} B(y)) \rightarrow_{SL} (B(x) \rightarrow_{SL} B^*(y))$$

$$\rightarrow_{SL} (A(x) \rightarrow_{SL} A^*(x)) = 1_{SL}.$$

And the solution $A^*(x)$ has the following form:

$$A^*(x) = \bigvee_{y \in Y} \left[ A(x) \otimes_{SL} \left( A(x) \rightarrow_{SL} B(y) \right) \right] \otimes_{SL} \left( B(y) \rightarrow_{SL} B^*(y) \right), \quad \forall x \in X.$$

**Definition 4 [30]:** Let $\rightarrow$ be a regular implication on $[0,1]$, $\forall a, b \in [0,1]$, $a \leftrightarrow b$ is called as the biresiduum of $a$ and $b$, where $a \leftrightarrow b = (a \rightarrow b) \land (b \rightarrow a)$.

**Proposition 3 [30]:** Let $\rightarrow$ be the residual implication on $[0,1]$ derived from a left-continuous t-norm $\otimes$, $\forall a, b, c, d \in [0,1]$, the following properties hold:

1. $(a \leftrightarrow b) \land (c \leftrightarrow d) \leq (a \otimes c) \leftrightarrow (b \otimes d)$.
2. $(a \leftrightarrow b) \lor (c \leftrightarrow d) \leq (a \rightarrow c) \leftrightarrow (b \rightarrow d)$.

**Proposition 4 [30]:** Suppose $X$ be a non-empty universe, $f$ and $g$ are both functions on $[0,1]$, then some relations are satisfied:

1. $(\land_{x \in X} f(x)) \leftrightarrow (\land_{x \in X} g(x)) \leq (\land_{x \in X} f(x) \leftrightarrow g(x)).$
2. $(\lor_{x \in X} f(x)) \leftrightarrow (\lor_{x \in X} g(x)) \leq (\lor_{x \in X} f(x) \leftrightarrow g(x)).$

### III. THE SIMILARITY BETWEEN INTERVAL-VALUED INTUITIONISTIC FUZZY SETS

In this section, the similarity of interval-valued intuitionistic fuzzy sets is constructed using biresiduum, and the operation relation of the interval-valued intuitionistic fuzzy biresiduum is analyzed.

**Definition 5:** Let $\rightarrow_{SL}$ be the residual interval-valued intuitionistic implication on $SL$ derived from a left-continuous interval-valued intuitionistic t-norm $\otimes_{SL}$, $\forall a, b \in SL$, we have $a \rightarrow_{SL} b = (a \rightarrow_{SL} b) \land (b \rightarrow_{SL} a)$ and call it as interval-valued intuitionistic fuzzy biresiduum.

**Lemma 2:** Suppose $\rightarrow_{SL}$ is the interval-valued intuitionistic fuzzy biresiduum related to the interval-valued intuitionistic fuzzy implication $\rightarrow_{SL}$. Let $A, B, C, D \in SL$, $A = ([a_1, b_1], [c_1, d_1]), B = ([a_2, b_2], [c_2, d_2]), C = ([a_3, b_3], [c_3, d_3]), D = ([a_4, b_4], [c_4, d_4])$, the following properties are satisfied:

1. $(A \rightarrow_{SL} B) \otimes_{SL} (C \rightarrow_{SL} D) \leq (A \otimes_{SL} C) \rightarrow_{SL} (B \otimes_{SL} D)$.
2. $(A \rightarrow_{SL} B) \otimes_{SL} (C \rightarrow_{SL} D) \leq (A \rightarrow_{SL} C) \rightarrow_{SL} (B \rightarrow_{SL} D)$.

**Proof:**

1. For convenience, let $c_1 = 1 - c_1$, $c_2 = 1 - c_2$, $c_3 = 1 - c_3$, $c_4 = 1 - c_4$, $d_1 = 1 - d_1$, $d_2 = 1 - d_2$, $d_3 = 1 - d_3$, $d_4 = 1 - d_4$.

   $E = ([e_1, e_2], [e_3, e_4]) = (A \rightarrow_{SL} B) \otimes_{SL} (C \rightarrow_{SL} D)$,

   $F = ([f_1, f_2], [f_3, f_4]) = (A \otimes_{SL} C) \rightarrow_{SL} (B \otimes_{SL} D)$.

   According to Proposition 3 and Theorem 2, we have that:

   $$E \leq (A \rightarrow_{SL} B) \otimes_{SL} (C \rightarrow_{SL} D)$$

   $$= [(A \rightarrow_{SL} B) \land (B \rightarrow_{SL} A)] \otimes_{SL} [(C \rightarrow_{SL} D) \land (D \rightarrow_{SL} C)]$$

   $$= \left[ \left( ((a_1 \leftrightarrow a_2) \land (b_1 \leftrightarrow b_2) \land (c_1 \leftrightarrow c_2) \land (d_1 \leftrightarrow d_2)) \otimes ((a_3 \leftrightarrow a_4) \land (b_3 \leftrightarrow b_4) \land (c_3 \leftrightarrow c_4) \land (d_3 \leftrightarrow d_4)) \right) \land \left( ((b_1 \leftrightarrow b_2) \land (c_1 \leftrightarrow c_2) \land (d_1 \leftrightarrow d_2)) \otimes ((b_3 \leftrightarrow b_4) \land (c_3 \leftrightarrow c_4) \land (d_3 \leftrightarrow d_4)) \right) \right] \land \left( ((b_2 \leftrightarrow b_3) \land (c_2 \leftrightarrow c_3) \land (d_2 \leftrightarrow d_3)) \right).$$

   $$= ([e_1, e_2], [e_3, e_4])$$

   $$F = (A \otimes_{SL} C) \rightarrow_{SL} (B \otimes_{SL} D)$$

   $$= ([a_2 \otimes a_4, b_2 \otimes b_4], [c_2 \otimes c_4, d_2 \otimes d_4])$$

   $$= \left( \left( ((a_2 \otimes a_4) \leftrightarrow (a_2 \otimes a_4)) \land ((b_2 \otimes b_4) \leftrightarrow (b_2 \otimes b_4)) \land ((1 - c_2 \otimes c_4) \leftrightarrow (1 - c_1 \otimes c_3)) \land (1 - d_2 \otimes d_4) \leftrightarrow (1 - d_1 \leftrightarrow d_3)) \right) \land \left( ((b_2 \otimes b_4) \leftrightarrow (b_2 \otimes b_4)) \land ((1 - c_2 \otimes c_4) \leftrightarrow (1 - c_1 \otimes c_3)) \land (1 - d_2 \leftrightarrow d_3) \land (1 - d_1 \leftrightarrow d_3)) \right) \land \left( ((b_2 \otimes b_4) \leftrightarrow (b_2 \otimes b_4)) \land ((1 - c_2 \otimes c_4) \leftrightarrow (1 - c_1 \otimes c_3)) \land (1 - d_2 \leftrightarrow d_3) \land (1 - d_1 \leftrightarrow d_3)) \right).$$

   $$= ([f_1, f_2], [f_3, f_4])$$

   To facilitate the analysis of the relationship between $E$ and $F$, and the following four parts of $E$ are deduced and analyzed respectively:

   (i) The part $e_1$ in $E$.

   $e_1 = (a_1 \leftrightarrow a_2) \land (b_1 \leftrightarrow b_2) \land (c_1 \leftrightarrow c_2)$

   $$\land (d_1 \leftrightarrow d_2) \otimes ((a_3 \leftrightarrow a_4) \land (b_3 \leftrightarrow b_4) \land (c_3 \leftrightarrow c_4) \land (d_3 \leftrightarrow d_4))$$

   $$\leq ((a_1 \leftrightarrow a_2) \otimes (a_3 \leftrightarrow a_4)) \land (b_1 \leftrightarrow b_2) \otimes (b_3 \leftrightarrow b_4) \otimes ((c_1 \leftrightarrow c_2) \otimes (c_3 \leftrightarrow c_4)) \land (d_1 \leftrightarrow d_2) \otimes (d_3 \leftrightarrow d_4).$$

   $$\leq ((a_1 \otimes a_3) \leftrightarrow (a_2 \otimes a_4)) \land (b_1 \otimes b_3) \leftrightarrow (b_2 \otimes b_4) \land (c_1 \otimes c_3) \leftrightarrow (c_2 \otimes c_4) \land (d_1 \otimes d_3) \leftrightarrow (d_2 \otimes d_4).$$

   $$= ([e_1, e_2], [e_3, e_4])$$

   (ii) The relationship between $e_2$ in $E$ and $f_2$ in $F$ is $e_2 \leq f_2$, and the proof is similar to (i).
The part $e_3$ in $E$.

$$e_3 = ((c_2 \oplus c_1) \lor (c_1 \oplus c_2))$$

$$= [(1-c_3 \rightarrow c_7) \lor (1-c_4 \rightarrow c_3)]$$

$$= (1 - (c_7 \rightarrow c_2) \lor (c_2 \rightarrow c_7))$$

$$= (1 - (c_7 \rightarrow c_2) \lor (c_3 \rightarrow c_7))$$

$$= 1 - (c_7 \rightarrow c_2) \lor (c_3 \rightarrow c_7)$$

$$= 1 - (c_7 \rightarrow c_2) \lor (c_8 \rightarrow c_7)$$

$$= 1 - (1 - c_1 \oplus c_3) \lor (1 - c_2 \oplus c_4)$$

$$= (1 - (1 - c_1 \oplus c_3) \lor (1 - c_2 \oplus c_4))$$

$$= [1 - (c_2 \oplus c_4) \lor (c_1 \oplus c_3)] \lor [1 - (c_2 \oplus c_4) \lor (c_1 \oplus c_3)]$$

$$= f_3$$

(iv) The relationship between $e_4$ in $E$ and $f_4$ in $F$ is $e_4 \geq f_4$, and the proof is similar to (iii).

According to the results of (i)-(iv), we have that $e_1 \leq f_1, e_2 \leq f_2, e_3 \geq f_3, e_4 \geq f_4 \iff E \leq F$. The proof of (2) is similar to (1).

**Lemma 3**: Let $X$ be a non-empty universe, $\forall A(x), A'(x) \in IVIFS (X)$, they have the following properties:

1. $\left( \land_{x \in X} A(x) \right) \leftrightarrow_{SL} \left( \land_{x \in X} A'(x) \right)$
2. $\left( \lor_{x \in X} A(x) \right) \leftrightarrow_{SL} \left( \lor_{x \in X} A'(x) \right)$

**Proof**: The proof of Lemma 3 is similar to Lemma 2.

**IV. THE ROBUSTNESS OF INTERVAL-VALUED INTUITIONISTIC FUZZY REASONING QUINTUPLE IMPLICATION METHOD**

The distance measure is an important tool to analyze the robustness of fuzzy reasoning. This part constructs a novel distance measure $d_{SL}^A$ and analyzes the robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication method. Fig.1 depicts the analytic process about the robustness of the algorithm for IVIFMP. When the new input and output become $B^*(y)$ ($B^*(y)$) and $A^*(x)$ ($A^*(x)$) respectively, the procedure in Fig.1 becomes the analysis process of the algorithm for IVIFMT.

**Definition 6**: Suppose $A, B, C \in SL$, a mapping $d_{SL}^A : SL \times SL \rightarrow SL$ is called the distance between IVIFSs, if $d_{SL}^A$ has the following properties:

1. $([0, 0]) \leq d_{SL}^A (A, B) \leq ([1, 1][0, 0])$;
2. $d_{SL}^A (A, B) = ([0, 0][1, 1]),$ if $A = B$;
3. $d_{SL}^A (A, B) = d_{SL}^A (B, A)$;
4. If $A \subseteq B \subseteq C$, then $d_{SL}^A (A, B) \leq S (A, C)$, and $d_{SL}^A (A, C) \geq d_{SL}^A (B, C)$.

**Definition 7**: Suppose $X$ is a non-empty universe, $A(x), A'(x) \in IVIFS (X), \sigma \in SL, \forall x \in X, S_{SL}^A (A, A')$ is the similarity between $A(x)$ and $A'(x), \sigma$ is the largest IVIFS satisfying $S_{SL}^A (A, A') = \land_{x \in X} (A(x) \leftrightarrow_{SL} A'(x)) \geq \sigma$. Then we call $A$ and $A'$ are $\sigma$-equalities, denoted as $A \equiv_{SL}^\sigma (A')$.

Due to the similarity of fuzzy sets having the opposite meaning to distance measure, and they are a pair of dual concepts [31], we can further define a similar dual distance between interval-valued intuitionistic fuzzy sets.

**Definition 8**: Suppose $X$ is a non-empty universe, $A(x), A'(x) \in IVIFS (X), \forall x \in X$, the novel distance between $A(x)$ and $A'(x)$ is that:

$$d_{SL}^A (A, A') = N(S_{SL}^A (A, A'))$$

where $N$ is the complement of IVIFSs. Let $f = ([f_1, f_2], [f_3, f_4])$, and we take $N(f) = ([f_3, f_4], [f_1, f_2])$ in this paper.

**Proof**: Suppose $A(x), B(x), C(x) \in IVIFS (X)$, we have that:

1. It is obvious that $d_{SL}^A (A, B) \leq ([0, 0][1, 1]) \leq ([1, 1][0, 0])$.
2. If $d_{SL}^A (A, B) = ([0, 0][1, 1])$, we have that:

$$A(x) \rightarrow SL B(x) = ([1, 1][0, 0])$$

$$B(x) \rightarrow SL A(x) = ([1, 1][0, 0])$$

According to property (1) in Proposition 2, it is easy to have $A(x) \leq B(x), B(x) \leq A(x) \iff A(x) = B(x)$.

3. If $d_{SL}^A (A, B) = d_{SL}^B (B, A)$ is true.
4. If $A(x) \leq B(x) \subseteq C(x)$, according to (7) in Proposition 2, it is satisfied that $A(x) \rightarrow SL B(x) \geq B(x) \rightarrow SL A(x)$, $A(x) \rightarrow SL C(x) \geq C(x) \rightarrow SL A(x)$ and $B(x) \rightarrow SL C(x) \geq C(x) \rightarrow SL B(x)$. So, we have that:

$$S_{SL}^A (A, B) = \land_{x \in X} (A(x) \rightarrow SL B(x) \land (B(x) \rightarrow SL A(x)))$$

$$S_{SL}^A (A, C) = \land_{x \in X} (A(x) \rightarrow SL C(x) \land (C(x) \rightarrow SL A(x)))$$

![FIGURE 1. The analysis process about robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm for IVIFMP.](image-url)
Further, the $S_{\text{SL}}(A, B)$ and $S_{\text{SL}}(B, C)$ can be defined as:

$$
S_{\text{SL}}(A, B) = \wedge_{x \in X} (B(x) \rightarrow SL A(x)) \wedge (C(x) \rightarrow SL B(x))
$$

and

$$
S_{\text{SL}}(B, C) = \wedge_{x \in X} (B(x) \rightarrow SL C(x)) \wedge (C(x) \rightarrow SL B(x))
$$

and $S_{\text{SL}}(C, A)$ can be obtained. In addition, $d_{\text{SL}}(A, B) \leq d_{\text{SL}}(A, C)$ and $d_{\text{SL}}(B, C) \leq d_{\text{SL}}(A, C)$ are satisfied.

If $S_{\text{SL}}(A, A') \geq \sigma$, then $S_{\text{SL}}(A, A') = N(S_{\text{SL}}(A, A')) \leq N(\sigma) = \delta$. The distance $d_{\text{SL}}(A, A')$ represents the degree belonging to "large" of the distance between $A$ and $A'$. The higher the similarity between $A$ and $A'$, the smaller the distance, that is, the smaller the deviation.

Theorem 5: Suppose $A(x), A'(x), A^* (x) \in IVIFS(X), B(y), B'(y), B^* (y) \in IVIFS(Y)$, and $A \equiv_{\text{SL}} (\sigma_1) A', B \equiv_{\text{SL}} (\sigma_2) B', A^* \equiv_{\text{SL}} (\sigma_3) A^*, B^* (y)$ and $B^* (y)$ represent the solutions of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm for IVIFMP($A, B, A^*$) and IVIFMT($A', B', B^*$) derived from Theorem 3 respectively, then we have that

$$
B^* \equiv_{\text{SL}} (\sigma_1 \otimes_{\text{SL}} \sigma_2 \otimes_{\text{SL}} \sigma_3 \otimes_{\text{SL}} \sigma_3) A^*.
$$

Proof: According to Lemma 2 and Lemma 3, we have

$$
S_{\text{SL}}(B^*, B^*) = \wedge_{y \in Y} (B^*(y) \rightarrow SL B^*(y))
$$

and

$$
S_{\text{SL}}(B, B^*) = \wedge_{y \in Y} \left( \wedge_{x \in X} (B(x) \rightarrow SL (B^*(y) \rightarrow SL (A(x) \rightarrow SL B(y)))) \wedge (A(x) \rightarrow SL B(y)) \rightarrow SL (A^*(x) \rightarrow SL B(y)) \right)
$$

and

$$
\equiv_{\text{SL}} (\sigma_1 \otimes_{\text{SL}} \sigma_2 \otimes_{\text{SL}} \sigma_3 \otimes_{\text{SL}} \sigma_3) A^*.
$$

In Corollary 1, we can get that when $\delta_1, \delta_2$, and $\delta_3$ are all close to 0, the $d_{\text{IVIFMP}}$ is also close to 0.

Theorem 6: Suppose $A(x), A'(x), A^* (x) \in IVIFS(X), B(y), B'(y), B^* (y) \in IVIFS(Y)$, and $A \equiv_{\text{SL}} (\sigma_1) A', B \equiv_{\text{SL}} (\sigma_2) B', B^* \equiv_{\text{SL}} (\sigma_3) B^*$, $A^*(y)$ and $A^*(y)$ represent the solutions of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm for IVIFMT($A, B, B^*$) and IVIFMT($A', B', B^*$) derived from Theorem 4 respectively, then we have that

$$
A^* \equiv_{\text{SL}} (\sigma_1 \otimes_{\text{SL}} \sigma_1 \otimes_{\text{SL}} \sigma_2 \otimes_{\text{SL}} \sigma_2 \otimes_{\text{SL}} \sigma_3) A^*.
$$

Proof: according to Lemma 2 and Lemma 3, we have that

$$
S_{\text{SL}}(A^*, A^*) = \wedge_{x \in X} (A^*(x) \rightarrow SL A^*(x))
$$

and

$$
\equiv_{\text{SL}} (\sigma_1 \otimes_{\text{SL}} \sigma_1 \otimes_{\text{SL}} \sigma_2 \otimes_{\text{SL}} \sigma_2 \otimes_{\text{SL}} \sigma_3) A^*.
$$

Corollary 2: Suppose $d_{\text{SL}}(A, A') \leq \delta_1$, $d_{\text{SL}}(B, B') \leq \delta_2$, $d_{\text{SL}}(B^*, B^*) \leq \delta_3$, $A^* \equiv_{\text{SL}} (\sigma_1) A^*$ and $A^* \equiv_{\text{SL}} (\sigma_2) A^*$ represent the solutions of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm for problems IVIFMP($A, B, A^*$) and IVIFMT($A', B', B^*$) derived from Theorem 4 respectively, then we have that $d_{\text{SL}}(A^*, A^*) \leq d_{\text{IVIFMP}}(A, B, A^*)$
robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm for IVIFMT ($A, B, A^*$) based on Gödel, Lukasiewicz, and Goguen implication operators by using corollary 1.

**Corollary 3:** Suppose $\otimes_{SL} = \otimes_{G-SL}, d^{G-SL}(A, A') \leq \delta_1, d^{G-SL}(B, B') \leq \delta_2, d^{G-SL}(A^*, A^*) \leq \delta_3$, we have that:

\[
d^{G-SL}(B^*, B'^*) \leq \left(\{[\delta_{11} + \delta_{21} + \delta_{31}, \delta_{12} + \delta_{22} + \delta_{32}], [\delta_{13} + \delta_{23} + \delta_{33}, \delta_{14} + \delta_{24} + \delta_{34}]\}\right)\]

**Corollary 4:** Suppose $\otimes_{SL} = \otimes_{G-SL}, d^{L-SL}(A, A') \leq \delta_1, d^{L-SL}(B, B') \leq \delta_2, d^{L-SL}(A^*, A^*) \leq \delta_3$, we have that:

\[
d^{L-SL}(B^*, B'^*) \leq \left(\{(2\delta_{11} + \delta_{21} + 2\delta_{31}) \land 1, (2\delta_{12} + \delta_{22} + 2\delta_{32}) \land 1, \right.
\] 
\[
(2\delta_{13} + 2\delta_{23} + 2\delta_{33} - 4) \lor 0, (2\delta_{14} + 2\delta_{24} + 2\delta_{34} - 4) \lor 0\}\right)\]

**Corollary 5:** Suppose $\otimes_{SL} = \otimes_{G-SL}, d^{GO-SL}(A, A') \leq \delta_1, d^{GO-SL}(B, B') \leq \delta_2, d^{GO-SL}(A^*, A^*) \leq \delta_3$, we have that:

\[
d^{GO-SL}(B^*, B'^*) \leq \left(\{(3\delta_{11} + \delta_{21} + 3\delta_{31}), 2\delta_{12} + \delta_{22} + 2\delta_{32} \land 1, \right.
\] 
\[
(2\delta_{13} + 2\delta_{23} + 2\delta_{33} - 4) \lor 0, (2\delta_{14} + 2\delta_{24} + 2\delta_{34} - 4) \lor 0\}\right)\]

Analogously, we can get the conclusions about the robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm for IVIFMT ($A, B, A^*$) based on Gödel, Lukasiewicz, and Goguen implication operators.

**Corollary 6:** Suppose $\otimes_{SL} = \otimes_{G-SL}, d^{G-SL}(A, A') \leq \delta_1, d^{G-SL}(B, B') \leq \delta_2, d^{G-SL}(B^*, B'^*) \leq \delta_3$, we have that:

\[
d^{G-SL}(A^*, A'^*) \leq \left(\{[\delta_{11} + \delta_{21} + \delta_{31}, \delta_{12} + \delta_{22} + \delta_{32}], [\delta_{13} + \delta_{23} + \delta_{33}, \delta_{14} + \delta_{24} + \delta_{34}]\}\right)\]

**Corollary 7:** Suppose $\otimes_{SL} = \otimes_{G-SL}, d^{L-SL}(A, A') \leq \delta_1, d^{L-SL}(B, B') \leq \delta_2, d^{L-SL}(B^*, B'^*) \leq \delta_3$, we have that:

\[
d^{L-SL}(A^*, A'^*) \leq \left(\{(2\delta_{11} + \delta_{21} + 2\delta_{31}) \land 1, (2\delta_{12} + \delta_{22} + 2\delta_{32}) \land 1, \right.
\] 
\[
(2\delta_{13} + 2\delta_{23} + 2\delta_{33} - 4) \lor 0, (2\delta_{14} + 2\delta_{24} + 2\delta_{34} - 4) \lor 0\}\right)\]

**Corollary 8:** Suppose $\otimes_{SL} = \otimes_{G-SL}, d^{GO-SL}(A, A') \leq \delta_1, d^{GO-SL}(B, B') \leq \delta_2, d^{GO-SL}(B^*, B'^*) \leq \delta_3$, we have that:

\[
d^{GO-SL}(A^*, A'^*) \leq \left(\{3\delta_{11} + 2\delta_{21} + 3\delta_{31} - 2\delta_{12} + 2\delta_{22} + 3\delta_{32}, \right.
\] 
\[
- \delta_{13} - 2\delta_{23} + 3\delta_{33} - 2\delta_{14} - 2\delta_{24} + 2\delta_{34} \}\right)\]

Next, we provide an example of interval-valued intuitionistic fuzzy reasoning with a single rule to prove the robustness in Corollary 4.

**Example 2:** Suppose there is a patient in the hospital who needs to be diagnosed based on four symptoms: body temperature (represented as $x_1$), headache (represented as $x_2$), stomachache (represented as $x_3$), cough (represented as $x_4$), and chest tightness (represented as $x_5$). Let $X$ be symptoms set and $Y$ be disease set. $X = \{x_1, x_2, x_3, x_4, x_5\}$, $Y = \{y_1, y_2, y_3, y_4, y_5\}$. The symptoms of the patient and the type of disease can be represented by quintuple interval-valued intuitionistic fuzzy numbers.

(1) Suppose there is a rule “If $A(x)$, then $B(y)$” based on expert knowledge as follows:

$A(x) = \{(x_1, (0.80, 0.85), (0.10, 0.13)), (x_2, (0.50, 0.60), (0.14, 0.20)), (x_3, (0.20, 0.23), (0.70, 0.74)), (x_4, (0.56, 0.60), (0.20, 0.30)), (x_5, (0.10, 0.14), (0.70, 0.80))\}$.

$B(y) = \{(y_1, (0.23, 0.25), (0.50, 0.55)), (y_2, (0.45, 0.50), (0.23, 0.30)), (y_3, (0.20, 0.30), (0.30, 0.35)), (y_4, (0.10, 0.20), (0.40, 0.50)), (y_5, (0.12, 0.15), (0.50, 0.60))\}$.

The symptoms of the patient can be expressed as:

$A_s(x) = \{(x_1, (0.60, 0.65), (0.10, 0.20)), (x_2, (0.50, 0.55), (0.40, 0.43)), (x_3, (0.30, 0.33), (0.40, 0.40)), (x_4, (0.70, 0.80), (0.10, 0.12)), (x_5, (0.45, 0.50), (0.40, 0.40))\}$.

Let $(\otimes_{SL}, \rightarrow_{SL}) = (\otimes_{G-SL}, \rightarrow_{L-SL})$, the next step is to compute $B^*(y)$ by Theorem 3, where

\[
a \otimes_{L-SL} b = \{(a_1 + a_2 - 1) \lor 0, (b_1 + b_2 - 1) \lor 0\}, \quad (c_1 + c_2) \land 1, (d_1 + d_2) \land 1\}
\]

\[
a \rightarrow_{L-SL} b = \{(1 - a_1 + a_2) \land (1 - b_1 + b_2) \land (1 - c_2 + c_1) \land (1 - d_2 + d_1) \land 1, (c_2 - c_1) \lor 0, (c_2 - c_1) \lor 0\}
\]

And we can get that:

$B^*(y) = \{(y_1, (0.15, 0.25), (0.50, 0.60)), (y_2, (0.40, 0.50), (0.23, 0.33)), (y_3, (0.20, 0.30), (0.30, 0.40)), (y_4, (0.10, 0.20), (0.40, 0.50)), (y_5, (0.05, 0.15), (0.50, 0.60))\}$.
(2) Suppose there is a deviation in the reasoning system, then the new reasoning rule is If $A'(x)$, then $B'(y)$, where

$$A'(x) = \{(x_1, [0.78, 0.85], [0.10, 0.13]), (x_2, [0.55, 0.60], [0.15, 0.20]), (x_3, [0.20, 0.30], [0.68, 0.72]), (x_4, [0.55, 0.60], [0.25, 0.30]), (x_5, [0.10, 0.15], [0.70, 0.80])\}\).  

$$B'(y) = \{(y_1, [0.25, 0.28], [0.50, 0.60]), (y_2, [0.50, 0.50], [0.25, 0.30]), (y_3, [0.30, 0.33], [0.35, 0.40]), (y_4, [0.06, 0.20], [0.40, 0.60]), (y_5, [0.15, 0.15], [0.55, 0.60])\}\).  

The new input is that:

$$A^*(x) = \{(x_1, [0.54, 0.63], [0.15, 0.23]), (x_2, [0.50, 0.55], [0.40, 0.43]), (x_3, [0.25, 0.33], [0.40, 0.40]), (x_4, [0.62, 0.75], [0.10, 0.15]), (x_5, [0.34, 0.54], [0.40, 0.43])\}\).  

Analogously, let $(\otimes_{SL} \rightarrow SL) = (\otimes_{L-SL} \rightarrow L-SL)$, the next step is to compute $B^*(y)$ by Theorem 3 and we have

$$B^*(y) = \{(y_1, [0.13, 0.26], [0.50, 0.60]), (y_2, [0.40, 0.50], [0.25, 0.30]), (y_3, [0.23, 0.33], [0.35, 0.40]), (y_4, [0.10, 0.20], [0.40, 0.60]), (y_5, [0.05, 0.15], [0.55, 0.60])\}\).  

(1) Next, we analyze the deviation of the input and rule between cases (1) and (2). By definition 7, the similarity and distance of $B^*(y)$ and $B^*(y)$ can be obtained as follows:

$$S_{L-SL}(B^*, B^*) = [(0.90, 0.90), (0.05, 0.10)] \)  

$$d_{L-SL}(B^*, B^*) = [(0.05, 0.10), (0.90, 0.90)] \)  

Analogously, we can get the dual similar distance of input and rule:

$$d_{L-SL}(A, A') = [(0.05, 0.05), (0.95, 0.95)] \)  

$$d_{L-SL}(B, B') = [(0.05, 0.10), (0.90, 0.90)] \)  

$$d_{L-SL}(A, A') = [(0.05, 0.05), (0.89, 0.95)] \)  

Then let us verify Corollary 4. We take

$$\delta_1 = [(0.05, 0.05), (0.93, 0.93)] \)  

$$\delta_2 = [(0.05, 0.10), (0.90, 0.90)] \)  

$$\delta_3 = [(0.05, 0.05), (0.89, 0.95)] \)  

And we have

$$\delta_{IVIFMP} = [(2\delta_{11} + \delta_{21} + 2\delta_{31}) \land 1, (2\delta_{12} + \delta_{22} + 2\delta_{32}) \land 1] 

[(2\delta_{13} + \delta_{23} + 2\delta_{33} - 4) \lor 0, (2\delta_{14} + \delta_{24} + 2\delta_{34} - 4) \lor 0] 

= [(0.25, 0.30), (0.54, 0.66)].$$  

Obviously, according to definition 2, the following relationship can be obtained:

$$d^L_{SL}(B^*, B^*) \leq N(N(\delta_1) \otimes L-SL N(\delta_2) \otimes L-SL N(\delta_3)).$$  

V. EXPERIMENT AND DISCUSSION

To further verify the correctness of the robustness conclusions we obtained, this section calculates the solution of the interval-valued intuitionistic quintuple implication algorithm for IVIFMP with different degrees of perturbation and analyzes the relationship between $d_{SL}(B^*, B^*)$ and $\delta_{IVIFMP}$, where $B^*$ is the ideal solution and $B^*$ is the perturbed solution. In addition, the distance measure we proposed is compared with the traditional distance, and the advantages and limits of our analysis approach are discussed.

A. THE REASONING IN THE IDEAL CASE

In the fusion of MRI and CT medical images, the brighter areas of the two images will be extended into the fusion image. The image pixels can be divided into three gray levels based on the gray value of the image: Dark, Normal, and Bright, abbreviated as $D$, $N$, and $B$. The degree of image pixels belonging to the gray level $(D, N, B)$ can be represented by the IVIFS $(\{u_n, v_n\}, \{u^+_n, v^+_n\})$, which can be obtained using the Gaussian membership function as follows:

$$u_n = 0.745e^{-\frac{(x-c_n)^2}{2a^2}}, \quad u^+_n = 0.855e^{-\frac{(x-c_n)^2}{2a^2}}, \quad v_n = 1 - 0.945e^{-\frac{(x-c_n)^2}{2a^2}}, \quad v^+_n = 1 - 0.9e^{-\frac{(x-c_n)^2}{2a^2}}.$$  

$n$ represents one of the gray levels $D$, $N$, and $B$, and $c_n$ is the mean term of the gray level $n$. In the ideal scenario, set the variance $a$ to 0.5 and the mean parameter $c_n$ of the IVIFS corresponding to the three gray levels $(D, N, B)$ to 0, 0.5, and 1, respectively, for input and output.

Brighter pixels have higher fusion priority in MRI and CT image fusion [32], so the rule selected for this experiment is “If MRI $A(x_1)$ is $D$, and CT $A(x_2)$ is $N$, then the output $B(y)$ is $N$”.

The rule is concreted using the upper and lower bounds of the membership and non-membership functions to simplify calculations. If an expert gives a judgment based on his own experience: “if the normalized gray value in MRI is 0.2, and the normalized gray value in CT is 0.6, then the normalized gray value of the output is 0.55”; we substitute the expert experience into the membership and non-membership functions and obtain that

$$If A(x_1) = ([0.6877, 0.7893], [0.1277, 0.1692]), \quad A(x_2) = ([0.7302, 0.8381], [0.0737, 0.1178]), \quad then B(y) = ([0.7413, 0.8507], [0.0597, 0.1045]).$$  

If the normalized gray value $x_1$ of MRI is 0.4, and the normalized gray value $x_2$ of CT is 0.3 in the new input, we can
obtain the interval-valued intuitionistic fuzzy set of the input:

\[ A^*(x_1) = ([0.5410, 0.6209], [0.3138, 0.3465]) , \]
\[ A^*(x_2) = ([0.6877, 0.7893], [0.1277, 0.1692]) \]

Let \( \otimes_{SL} = \otimes_{L-SL} \rightarrow_{SL} = \rightarrow{L-SL} \), according to Theorem 4, the solution of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm under ideal conditions can be calculated as follows:

\[ B^*(y) = ([0.6887, 0.7893], [0.1277, 0.1692]) \].

**B. THE REASONING WITH PERTURBATION**

The different parameter selections of membership and non-membership functions will result in rules and input values that deviate from the ideal values to different degrees. Hence, our distance measure is compared to the traditional distance measure (the continuous intuitionistic fuzzy-number ordered weighted distance(C-IFOWD)) \( d^{C-IFOWD} \), Hamming distance \( d^H \), and Euclidean distance \( d^E \), which are all introduced in [19]), and the relationship of the robustness they obtained is analyzed.

We know that our distance is an interval-valued intuitionistic fuzzy number, and the traditional distance is an ordinary fuzzy number from the definition of distance. From another perspective, ordinary fuzzy numbers can also be expressed as interval-valued intuitionistic fuzzy numbers. For example, if \( d = 0.3 \), \( d \) can also be expressed as \([0.3, 0.3],[0.7, 0.7]\), so the traditional distance measure can also be regarded as a special interval-valued intuitionistic fuzzy distance. Based on this, we compare our distance with the traditional distance in four dimensions \([\mu^-, u^+],[v^-, v^+]\), which is the lower bound of membership degree, the upper bound of membership degree, the lower bound of non-membership degree and the upper bound of non-membership degree respectively.

The results of the comparison are shown in Fig.3. From Fig.3, the distance measure we proposed has the same tendency as the traditional distance measure, and it is more sensitive to perturbation. Hence, our distance measure is more suitable for measuring the difference between IVIFSs. Respectively, Figure 3(a) and Figure 3(b) are the lower bound and upper bound of membership under different variance parameters, and it’s clear that \( d^H \leq d^E \leq d^{C-IFOWD} \leq d^{L-SL} \leq d^H \). Similarly, \( d^H \geq d^E \geq d^{C-IFOWD} \geq d^{L-SL} \geq d^H \) can be obtained from Figure 3(c) and Figure 3(d). According to the partial order in IVIFSs, we can get \( d^H \leq d^E \leq d^{C-IFOWD} \leq d^{L-SL} \leq d^H \), which illustrates that our conclusion in Corollary 4 is correct, and the C-IFOWD, Hamming distance, and Euclidean distance also satisfies our conclusion. The robustness we obtained can represent the robustness based on C-IFOWD, Hamming distance, and Euclidean distance.

In summary, our approach to research the robustness has the following advantages:

(i) The distance measure we presented is composed of the interval-valued intuitionistic fuzzy implication operator (IVIFIO), consistent with the interval-valued intuitionistic fuzzy reasoning algorithm. The property of IVIFIO makes...
FIGURE 3. The comparison between \(d^L\), \(d^C\), \(d^H\), \(d^E\), and \(d^{IVIFMP}\) under different variance parameters \(a\).

Our analytical method is easier than the method based on the traditional distance measure.

(ii) The conclusion we obtained is a unified form, and the robustness of the reasoning algorithms based on the other residual implication operators can be obtained by simple substitution (the robustness based on Lukasiewicz, Gödel, and Goguen operators are obtained in our work).

(iii) The proposed distance measure is more sensitive to perturbation than other traditional distance measures. It is more suitable for measuring the difference between IVIFSs. Furthermore, the robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm we obtained is more representative than the traditional distance measure.

However, our method also has a limitation: the method is too dependent on implication operators, and the algorithm’s robustness based on complicated implication operators is difficult to obtain, and its structure is also complex, as in Corollary 5 and Corollary 8.

VI. CONCLUSION

This paper proposes a new distance measure constructed by interval-valued intuitionistic fuzzy biresiduum. Then, based on the properties of interval-valued intuitionistic fuzzy biresiduum, the robustness of the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm is analyzed using the new distance measure. The analysis process is uncomplicated and easy to understand. The corollary demonstrates that the interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm for IVIFMP and the algorithm for IVIFMT has different robustness. Especially, the robustness of interval-valued intuitionistic fuzzy reasoning quintuple implication algorithm based on Lukasiewicz, Gödel and Goguen implication operators are obtained. The interval-valued intuitionistic fuzzy reasoning quintuple implication algorithms based on other implication operators can derive their robustness through our unified conclusion. One numerical example and experiment are given to demonstrate the correctness of our conclusion. The proposed distance measure is compared with other traditional distance measures in the end and the distance measure is more sensitive to disturbance than traditional distance. The algorithm’s robustness based on the distance metric we represent is representative.

In future research, we shall try to extend the fuzzy reasoning quintuple implication algorithm to Pythagorean fuzzy sets and interval-valued Pythagorean fuzzy sets [33]–[35]. The distance measure proposed in this paper could also be extended, and the extended distance measure could be used to analyze the robustness of the Pythagorean fuzzy reasoning quintuple implication algorithm or the interval-valued Pythagorean fuzzy reasoning quintuple implication algorithm. In addition, our another future research direction is to apply fuzzy reasoning algorithm based on different fuzzy sets and implication operators to group decision making.

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