Invariant Differential Operators for Non-Compact Lie Groups: 
the Reduced $SU(3,3)$ Multiplets\textsuperscript{1,2}

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Abstract—In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact algebras $su(n, n)$. Earlier were given the main multiplets of indecomposable elementary representations for $n \leq 4$, and the reduced ones for $n = 2$. Here we give all reduced multiplets containing physically relevant representations including the minimal ones for the algebra $su(3, 3)$. Due to the recently established parabolic relations the results are valid also for the algebra $sl(6, \mathbb{R})$ with suitably chosen maximal parabolic subalgebra.

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1. INTRODUCTION

Invariant differential operators play very important role in the description of physical symmetries. In a recent paper \cite{1} we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

In the present paper we focus on the algebra $su(3, 3)$. The algebras $su(n, n)$ belong to a narrow class of algebras, which we call “conformal Lie algebras,” which have very similar properties to the canonical conformal algebras of Minkowski space-time. This class was identified from our point of view in \cite{2}. The same class was identified independently from different considerations and under different names in \cite{3, 4}.

This paper is a sequel of \cite{5}, and due to the lack of space we refer to it and to \cite{6} for motivations and extensive list of literature on the subject.

2. PRELIMINARIES

Let $G$ be a semisimple non-compact Lie group, and $K$ a maximal compact subgroup of $G$. Then we have an Iwasawa decomposition $G = K A_0 N_0$, where $A_0$ is abelian simply connected vector subgroup of $G$, $N_0$ is a nilpotent simply connected subgroup of $G$ preserved by the action of $A_0$. Further, let $M_0$ be the centralizer of $A_0$ in $K$. Then the subgroup $P_0 = M_0 A_0 N_0$ is a minimal parabolic subgroup of $G$. A parabolic subgroup $P = MAN$ is any subgroup of $G$ which contains a minimal parabolic subgroup.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of $G$ \cite{7–9}.

Let $\nu$ be a (non-unitary) character of $A$, $\nu \in \mathcal{A}^\times$, let $\mu$ fix an irreducible representation $D^\mu$ of $M$ on a vector space $V_\mu$.

We call the induced representation $\chi = \mathrm{Ind}^G_P (\mu \otimes \nu \otimes 1)$ an elementary representation of $G$ \cite{10}. Their spaces of functions are:

\begin{equation}
\mathcal{E}_\chi = \{ \mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(g \operatorname{man}) = e^{-\chi(H)} \cdot D^\mu(m^{-1}) \mathcal{F}(g) \},
\end{equation}

where $a = \exp(H) \in A$, $H \in \mathcal{A}$, $m \in M$, $n \in N$. The representation action is the left regular action:

\begin{equation}
(\mathcal{F} \chi(g))(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G.
\end{equation}

For our purposes we need to restrict to maximal parabolic subgroups $P$, so that rank $A = 1$. Thus, for our representations the character $\nu$ is parameterized by a real number $d$, called the conformal weight or energy.

An important ingredient in our considerations are the highest/lowest weight representations of $\mathcal{G}$. These can be realized as (factor-modules of) Verma modules $V^\Lambda$ over $\mathcal{G}^\mathbb{C}$, where $\Lambda \in (\mathcal{H}^\mathbb{C})^\times$, $\mathcal{H}^\mathbb{C}$ is a Cartan subalgebra of $\mathcal{G}^\mathbb{C}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from $\chi$ \cite{11, 12}.

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Actually, since our ERs will be induced from finite-dimensional representations of $\mathcal{M}$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use generalized Verma modules $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector $v_0$ is taken by the space $V^\Lambda_{-n_0}$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight $d$. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called multiplets [12, 13]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair $(\beta, m)$, where $\beta$ is a (non-compact) positive root of $G^C$, $m \in \mathbb{N}$, such that the BGG [14] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$ \Lambda + \rho, \beta^\vee = m, \quad \beta^\vee = 2\beta/(\beta, \beta). \quad (2.3) $$

When (2.3) holds then the Verma module with shifted weight $V^\Lambda - m\beta$ (or $\tilde{V}^\Lambda - m\beta$ for GVM and $\beta$ non-compact) is embedded in the Verma module $V^\Lambda$ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector $v_\beta$ determined by a polynomial $P_m(\beta)$ in the universal enveloping algebra $(U(G))v_0$, $G$ is the subalgebra of $G^C$ generated by the negative root generators [15].

More explicitly, [12], $v_\beta = P_m(\beta)v_0$ (or $v_\beta = P_m(\beta)v_0$ for GVMs). Then there exists [12] an intertwining differential operator

$$ D_\beta : \mathcal{E}_\chi(\Lambda) \longrightarrow \mathcal{E}_\chi(\Lambda - m\beta) \quad (2.4) $$

given explicitly by:

$$ D_\beta = P_m(\beta) \quad (2.5) $$

where $\mathcal{E}$ denotes the right action on the functions $F$, cf. (2.1).

3. THE NON-COMPACT LIE ALGEBRA $su(3, 3)$

Let $G = su(3, 3)$. This algebra has discrete series representations and highest/lowest weight representations since the maximal compact subalgebra is $\mathbb{H} \equiv u(1) \oplus sl(3, \mathbb{C}) \oplus sl(3, \mathbb{C})$. Thus, the factor $\mathbb{H}$ has the same finite-dimensional (non-unitary) representations as the finite-dimensional (unitary) representations of the semi-simple subalgebra of $\mathbb{H}$.

We label the signature of the ERs of $G$ as follows:

$$ \chi = \{ n_1, n_2, n_4, n_5; c \}, \quad n_j \in \mathbb{Z}_+, \quad c = d - 3, \quad (3.6) $$

where the last entry of $\chi$ labels the characters of $G$, and the first 4 entries are labels of the finite-dimensional nonunitary irreps of $\mathbb{H}$ when all $n_j > 0$ or limits of the latter when some $n_j = 0$.

Below we shall use the following conjugation on the finite-dimensional entries of the signature:

$$ (n_1, n_2, n_4, n_5)^* = (n_4, n_5, n_1, n_2). \quad (3.7) $$

The ERs in the multiplet are related also by intertwining integral operators introduced in [16]. These operators are defined for any ER, the general action being:

$$ G_{KS} : \mathcal{E}_\chi \longrightarrow \mathcal{E}_{\chi'}, \quad (3.8) $$

$$ \chi = \{ n_1, n_2, n_4, n_5; c \}, \quad \chi' = \{ (n_1, n_2, n_4, n_5)^*; -c \}. $$

For the classification of the multiplets we shall need one more conjugation for the entries of the $\mathcal{M}$ representations:

$$ (n_1, n_2, n_4, n_5)^* = (n_5, n_4, n_2, n_1). \quad (3.9) $$

Further, we need the root system of the complexification $G^C = (6, \mathbb{C})$. The positive roots in terms of the simple roots are given standardly as:

$$ \alpha_{ij} = \alpha_i + \ldots + \alpha_j, \quad 1 \leq i < j \leq 5, \quad (3.10) $$

$$ \alpha_{ij} = \alpha_i, \quad 1 \leq j \leq 5. $$

From these the compact roots are those that form (by restriction) the root system of the semisimple part of $G^C$, the rest are noncompact, i.e.,

$$ \text{noncompact} : a_{ij}, \quad 1 \leq i \leq 3, \quad 3 \leq j \leq 5. \quad (3.11) $$

Further, we give the correspondence between the signatures $\chi$ and the highest weight $\Lambda$. The connection is through the Dynkin labels:

$$ m_i = (\Lambda + \rho, \alpha_i') = (\Lambda + \rho, \alpha_i), \quad i = 1, \ldots, 5, \quad (3.12) $$

where $\Lambda = \Lambda(\chi)$, $\rho$ is half the sum of the positive roots of $G^C$. The explicit connection is:

$$ m_i = m_i, \quad c = -\frac{1}{2}(m_4 + m_3) $$

$$ = -\frac{1}{2}(m_1 + m_2 + 2m_3 + m_4 + m_5), \quad (3.13) $$

where $\bar{a} = \alpha_1 + \ldots + \alpha_5$ is the highest root.

We shall use also the so-called Harish–Chandra parameters:

$$ m_{ik} = (\Lambda + \rho, \alpha_{jk}) = m_j + \ldots + m_k, \quad j < k, \quad m_{jj} = m_j. \quad (3.14) $$
Finally, we remind that according to [6] the above results for $su(3, 3)$ are valid also for the algebra $sl(6, \mathbb{R})$ with parabolic $M$-factor $sl(6, \mathbb{R}) \ominus sl(3, \mathbb{R})$.

4. MULTIPLETS OF $su(3, 3)$

4.1. Main Multiplets

There are two types of multiplets: main and reduced. The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $su(3, 3)$, i.e., they are labelled by the five positive Dynkin labels $m_i \in \mathbb{N}$. In [5] we have given explicitly the main multiplets for $n = 2, 3, 4$, and the reduced for $n = 2$.

A main multiplet contains 20 ERs/GVMs whose signatures can be given in the following pair-wise manner [5]:

$$\begin{align*}
\chi_0^+ &= \{ (m_1, m_2, m_4, m_5)^\pm \pm 1/2(m_6 + m_3) \}, \\
\chi_0^- &= \{ (m_1, m_3, m_5, m_6)^\pm \pm 1/2(m_4 - m_2) \}, \\
\chi_a^+ &= \{ (m_{12}, m_3, m_4, m_5)^\pm \pm 1/2(m_{14} m_{45}) \}, \\
\chi_b^+ &= \{ (m_1, m_{24}, m_3, m_5)^\pm \pm 1/2(m_{14} m_{45}) \}, \\
\chi_b^- &= \{ (m_{12}, m_3, m_4, m_5)^\pm \pm 1/2(m_{14} m_{45}) \}, \\
\chi_c^+ &= \{ (m_1, m_3, m_{14}, m_5)^\pm \pm 1/2(m_{45} - m_1) \}, \\
\chi_c^- &= \{ (m_{12}, m_{34}, m_{23}, m_{45})^\pm \pm 1/2(m_{12} - m_3) \}, \\
\chi_d^+ &= \{ (m_1, m_{23}, m_3, m_4)^\pm \pm 1/2(m_{12} + m_3) \}, \\
\chi_d^- &= \{ (m_1, m_{23}, m_3, m_4)^\pm \pm 1/2(m_{12} + m_3) \}, \\
\chi_e^+ &= \{ (m_1, m_{35}, m_{15}, m_4)^\pm \pm 1/2(m_1 + m_3) \}, \\
\chi_e^- &= \{ (m_1, m_{35}, m_{15}, m_4)^\pm \pm 1/2(m_1 + m_3) \},
\end{align*}$$

where $(k_1, k_2, k_3, k_4)^\pm = (k_1, k_2, k_3, k_4), (k_1, k_2, k_3, k_4)^\pm = (k_1, k_2, k_3, k_4)^\pm$. They are given explicitly in Fig. 1 (first in [5]). The pairs $\Lambda^\pm$ are symmetric w.r.t. to the bullet in the middle of the figure—this represents the Weyl symmetry realized by the Knapp–Stein operators (3.8): $G_{KS}^\pm (\chi_k^\pm \rightarrow \chi_k^\pm)$.

Matters are arranged so that in every multiplet only the ER with signature $\chi_0^+$ contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $E$. The latter corresponds to the finite-dimensional irrep of $su(3, 3)$ with signature $\{m_1, ..., m_3\}$. The subspace $E$ is annihilated by the operator $G^+$, and is the image of the operator $G^-$. The subspace $E$ is annihilated also by the intertwining differential operator acting from $\chi_0^+$ to $\chi_0^-$. When all $m_i = 1$ then $\dim E = 1$, and in that case $E$ is also the trivial one-dimensional UIR of the whole algebra $\mathfrak{g}$.

Furthermore in that case the conformal weight is zero: $d = 3 + c = 3 + 1/2(m_1 + m_2 + 2m_3 + m_4 + m_5) |_{m_i = 1} = 0$.

Analogously, in every multiplet only the ER with signature $\chi_0^+$ contains holomorphic discrete series representation. This is guaranteed by the criterion [11] that for such an ER all Harish–Chandra parameters for non-compact roots must be negative, i.e., in our situation, $m_i < 0$. [That this holds for our $\chi^+$ can be easily checked using the signatures (4.15).]

Note that the ER $\chi_0^+$ contains also the conjugate anti-holomorphic discrete series. The direct sum of the holomorphic and the anti-holomorphic representations are realized in an invariant subspace $\mathcal{D}$ of the ER $\chi_0^+$. That subspace is annihilated by the operator $G^-$, and is the image of the operator $G^+$. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holographic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series. The conformal weight of the ER $\chi_0^+$ has the restriction $d = 3 + c = 3 + 1/2(m_1 + m_2 + 2m_3 + m_4 + m_5) \geq 6$.

In Fig. 1 and below we use the notation: $\Lambda^\pm = \Lambda(\chi^\pm)$. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{jk}^\pm$ encoding the root $a_{jk}$ and the number $m_{a_{jk}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data $\beta, m_\beta$, which is involved in the embedding $V^\Lambda \rightarrow V^{\Lambda - m_\beta}$ turns out to involve only the $m_i$ corresponding to simple roots, i.e., for each $\beta, m_\beta$ there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, ..., 5\}$,
such that $m_0 = m_e$. Hence the data $\alpha_{jk}$, $m_{\alpha_{jk}}$ is represented by $i_{jk}$ on the arrows.

4.2. Reduced Multiplets

There are five types of reduced multiplets, $R_a$, $a = 1, \ldots, 5$, which may be obtained from the main multiplet by setting formally $m_0 = 0$. Multiplets of type $R_4$, $R_5$, are conjugate to the multiplets of type $R_2$, $R_3$, resp., as follows. First we make the conjugation on the roots and exchange all indices: $1 \leftrightarrow 5, 2 \leftrightarrow 4$. With this operation we obtain the diagrams of the conjugated cases from one another. For the entries of the $M$ representation we have further to employ the conjugation (3.9). Then we obtain the signatures of the conjugated cases from one another. Thus, we give explicitly only first three types.

The reduced multiplets of type $R_3$ contain 14 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\chi_0^\pm = \{(m_1, m_3, m_4, m_5)^\pm; \pm 1/2(m_{12} + m_{34}),$$

$$\chi_b^\pm = \{(m_{12}, 0, m_{24}, m_5)^\pm; \pm 1/2m_{1,45},$$

$$\chi_{b'}^\pm = \{(m_1, m_{24}, 0, m_{35})^\pm; \pm 1/2m_{1,2,3},$$

$$\chi_c^\pm = \{(m_2, 0, m_{14}, m_5)^\pm; \pm 1/2(m_{45} - m_1),$$

$$\chi_{c'}^\pm = \{(m_1, m_{25}, 0, m_3)^\pm; \pm 1/2(m_{12} - m_3),$$

$$\chi_d^\pm = \{(m_2, m_4, m_{12}, m_{45})^\pm; \pm 1/2(m_5 - m_1),$$

$$\chi_e^\pm = \{(m_2, m_{45}, m_{12}, m_4)^\pm; \pm 1/2(m_1 + m_5).$$

These multiplets are given in Fig. 2. They may be called the main type of reduced multiplets since here in $\chi_0^\pm$ are contained the limits of the (anti)holomorphic discrete series.

The reduced multiplets of type $R_2$ contain 14 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\chi_0^\pm = \{(m_1, 0, m_4, m_5)^\pm; \pm 1/2(m_{12} + m_{34}),$$

$$\chi_b^\pm = \{(m_1, m_3, m_{34}, m_5)^\pm; \pm 1/2m_{1,45},$$

$$\chi_c^\pm = \{(0, m_3, m_{14}, m_5)^\pm; \pm 1/2(m_{45} - m_1),$$

$$\chi_d^\pm = \{(m_1, m_{35}, m_3, m_4)^\pm; \pm 1/2(m_1 + m_3),$$

$$\chi_e^\pm = \{(0, m_{35}, m_{13}, m_4)^\pm; \pm 1/2(m_1 + m_3).$$

These multiplets are given in Fig. 3.
There are further reductions of the multiplets denoted by $\alpha, \beta = 1, \ldots, 5$, $\alpha < \beta$, which may be obtained from the main multiplet by setting formally $m_\alpha = m_\beta = 0$. From these ten reductions four (for $(\alpha, \beta) = (1, 2), (2, 3), (3, 4), (4, 5)$) do not contain representations of physical interest, i.e., induced from finite-dimensional irreps of the $\mathfrak{su}(\lambda_1, \lambda_2)$ subalgebra. From the others and $\chi_c$ are conjugated to $\chi_b$ and $\chi_d$ resp., as explained above. Thus, we present explicitly only four types of multiplets.

The reduced multiplets of type $\chi_c$ contain 10 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\chi_c^\pm = \left\{ \begin{array}{l}
(0, m_{2,45}, 0, m_4)^\pm; \pm \frac{1}{2}(m_2 - m_5) \\
(0, m_{2,45}, 0, m_4)^\pm; \pm \frac{1}{2}(m_2 - m_5)
\end{array} \right\},$$

(4.19)

The multiplets are given in Fig. 5.

Note that the differential operator from $\chi_c^-$ to $\chi_c^+$ is a reduction of an integral Knapp-Stein operator.

The reduced multiplets of type $\chi_d$ contain 10 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\chi_d^\pm = \left\{ \begin{array}{l}
(0, m_{2,45}, 0, m_4)^\pm; \pm \frac{1}{2}(m_2 - m_5) \\
(0, m_{2,45}, 0, m_4)^\pm; \pm \frac{1}{2}(m_2 - m_5)
\end{array} \right\},$$

(4.20)

The multiplets are given in Fig. 6.
The reduced multiplets of type $R_{14}^3$ contain 10 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\chi^\pm_0 = \{0, m_2, m_4, 0\}^\pm; \frac{\pm}{2}(m_{24} + m_3),$$

$$\chi^\pm_a = \{0, m_{23}, m_{34}, 0\}^\pm; \frac{\pm}{2}m_{24},$$

$$\chi^\pm_b = \{0, m_{24}, m_3, 0\}^\pm; \frac{\pm}{2}m_2,$$

$$\chi^\pm_c = \{m_2, m_3, 0\}^\pm; \frac{\pm}{2}m_4,$$

$$\chi^\pm_d = \{m_2, m_{34}, m_{23}, m_4\}^\pm; 0.$$ (4.21)

The multiplets are given in Fig. 7.

We note a peculiarity on the last case, namely, the operator between $\chi^+_d$ is not a differential operator. It is a reduction of the Knapp-Stein operator which does not change the conformal weight, but only conjugates the signature of $\mathcal{M}$.

The reduced multiplets of type $R_{15}^3$ contain 10 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\chi^\pm_0 = \{m_1, 0, 0, m_5\}^\pm; \frac{\pm}{2}(m_{13} + m_3),$$

$$\chi^\pm_b = \{m_1, m_3, m_3, m_5\}^\pm; \frac{\pm}{2}m_{15},$$

$$\chi^\pm_c = \{0, m_3, m_{35}, m_3\}^\pm; \frac{\pm}{2}(m_3 - m_1),$$

$$\chi^\pm_d = \{0, m_3, 0, m_{35}\}^\pm; \frac{\pm}{2}(m_1 - m_3),$$

$$\chi^\pm_e = \{0, 0, m_{35}, m_{35}\}^\pm; \frac{\pm}{2}m_{15}.$$ (4.22)

The multiplets are given in Fig. 8.

### 4.4. Last Reduction of Multiplets

There are further reductions of the multiplets—triple and quadruple, but only one triple reduction contains representations of physical interest. Namely, this is the
The multiplets are given in Fig. 9. The representation $\chi^d$ is a singlet, not in a pair, since it has zero weight $c$, and the $M$ entries are self-conjugate under (3.7). It is placed in the middle of the figure as the bullet. That ER contains the minimal irreps of $SU(3, 3)$ characterized by two positive integers which are denoted in this context as $m_2, m_4$. Each such irrep is the kernel of the two invariant differential operators $\mathcal{D}^{m_2}_{14}$ and $\mathcal{D}^{m_4}_{25}$, which are of order $m_2, m_4$, resp., and correspond to the noncompact roots $\alpha_{14}, \alpha_{25}$, resp., cf. (2.5).

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