LEAST ENERGY SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATION INVOLVING THE FRACTIONAL LAPLACIAN AND CRITICAL GROWTH

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Abstract. In this paper, we study a class of nonlinear Schrödinger equations involving the fractional Laplacian and the nonlinearity term with critical Sobolev exponent. We assume that the potential of the equations includes a parameter \( \lambda \). Moreover, the potential behaves like a potential well when the parameter \( \lambda \) is large. Using variational methods, combining Nehari methods, we prove that the equation has a least energy solution which, as the parameter \( \lambda \) large, localizes near the bottom of the potential well. Moreover, if the zero set \( \text{int} V^{-1}(0) \) of \( V(x) \) includes more than one isolated component, then \( u_\lambda(x) \) will be trapped around all the isolated components. However, in Laplacian case when \( s = 1 \), for \( \lambda \) large, the corresponding least energy solution will be trapped around only one isolated component and will become arbitrary small in other components of \( \text{int} V^{-1}(0) \). This is the essential difference with the Laplacian problems since the operator \((-\Delta)^s\) is nonlocal.

1. Introduction and main results. In this paper, we study the following nonlinear Schrödinger equation involving the fractional Laplacian and critical growth:

\[
\begin{aligned}
(-\Delta)^s u + (\lambda V(x) - \mu)u &= |u|^{2^*_s - 2}u, \quad x \in \mathbb{R}^N, \\
u(x) &\geq 0, \quad u(x) \in H^s(\mathbb{R}^N),
\end{aligned}
\]

where \( V(x) \) is the real-valued electric potential, \( 2^*_s = \frac{2N}{N-2s} \) for \( N > 5 + 2s \) and \( \mu > 0 \) will be specified later.

In recent years, much attention has been devoted to the study of the fractional Laplacian. The fractional powers of the Laplacian, which are called fractional Laplacians and correspond to Lévy stable processes, appear in anomalous diffusion phenomena in physics, biology as well as other areas. They occur in flame propagation, chemical reaction in liquids, population dynamics. Lévy diffusion processes have discontinuous sample paths and heavy tails, while Brownian motion has continuous sample paths and exponential decaying tails. These processes have been applied to American options in mathematical finance for modeling the jump processes of the financial derivatives such as futures, forwards, options, and swaps; see [2] and

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references therein. Moreover, they play important roles in the study of the quasigeostrophic equations in geophysical fluid dynamics.

There are many results which are concerned with the problems involving the fractional Laplacian. Firstly, we refer the readers to the work by Caffarelli and Silvestre \[13\], in which a new formulation of the fractional Laplacians through Dirichlet-Neumann maps was introduced. By this formulation, they transferred the nonlocal problem to a local problem defined in a higher half space. After their pioneering work, there have been many investigations of the fractional Laplacian problem by using variational methods. For example, using variational methods, Cabré and Tan \[11\] established the existence of positive solutions for fractional problems in a bounded domain with power-type nonlinearities in the subcritical case.

Recently, the nonlinear nonlocal elliptic equations, which are denoted by
\[
(-\Delta)^su = f(x,u) \text{ in } \Omega \subset \mathbb{R}^N, \ 0 < s < 1,
\] (1.2)
have been widely studied. We first introduce the fractional Brezis–Nirenberg problems on bounded domains
\[
\begin{cases}
(-\Delta)^su = |u|^{2^*_s-2}u + \mu u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\] (1.3)
Problem (1.3) was first studied by Tan \[26\] for \(s = 1/2\), where he obtained the existence of a positive solution for \(\mu > 0\) and he also considered the nonexistence of positive solutions to (1.3) for star-shaped domains when \(\mu = 0\). After that, Tan \[27\] also obtained the similar results for the general cases \(1/2 < s < 1\). For more general nonlinearity
\[
f(u) = u^{\frac{N+2s}{N-2s}} + \mu u^q
\] with \(s < \min\{N/2, 1\}\), \(\mu \in \mathbb{R}\) and \(q \in (0, \frac{N+2s}{N-2s})\), see also the work by Barrios et al. \[3\]. We also want to mention the paper by Choi, Kim and Lee \[15\], where they investigated the asymptotic behavior of solutions to (1.3). For the case \(\Omega = \mathbb{R}^N\) of (1.2), Felmer, Quaas and Tan \[17\] have obtained the existence of positive solutions.

For the following related fractional Schrödinger equations
\[
(-\Delta)^su + V(x)u = f(x,u), x \in \mathbb{R}^N,
\] (1.4)
with \(0 < s < 1\) and \(V: \mathbb{R}^N \to \mathbb{R}\) is an external potential function, there have been also many investigations; see also \[7, 8, 10, 12, 14, 16, 17, 18, 19, 21, 22, 24, 25, 28, 30, 32\] and references therein.

For the Laplacian, the following analog problems to (1.1)
\[
-\Delta u + (\lambda V(x) - \mu)u = f(x,u), x \in \mathbb{R}^N,
\] (1.5)
for different kinds of nonlinearities \(f\), have been the main subject of investigation in large amount of works in recent two decades. A lot of papers studied the existence of one-bump or multibump solutions of the problems related to (1.5), where \(f(x, u) = |u|^{p-2}u\); see \[11, 15, 6\] and references therein.

Now we are ready to present our main assumptions on \(V(x)\) and \(\mu\), we firstly assume that:
\begin{itemize}
\item[(V1)] \(V(x) \in C(\mathbb{R}^N, \mathbb{R})\), \(V(x) \geq 0\), and \(\Omega := \text{int } V^{-1}(0)\) is nonempty with smooth boundary and \(\overline{\Omega} = V^{-1}(0)\);
\item[(V2)] \(\liminf_{|x| \to \infty} V(x) > 0\).
\end{itemize}
where operator $(\cdot)$.

We give some remarks for the operator $(\cdot)$.

In section 2, we define $\mu$ to be the first eigenvalue of $A_s = (-\Delta)^s$ in $tr E_0$. We give the following further assumption on $\mu$:

$(V_3)$ $0 < \mu < \mu_1$. Namely, the operator $(-\Delta)^s - \mu$ is positively definite in $tr E_0$.

We give some remarks for the operator $(-\Delta)^s - \mu$ defined in $tr E_0$. Note that the operator $(-\Delta)^s : tr E_0 \to (tr E_0)^*$ is defined by

$$u \mapsto \partial_\nu^s U|_{\Omega \times \{0\}},$$

where $(tr E_0)^*$ denotes the dual space of $tr E_0$ and $U := h\text{-ext}(u) \in E_0$ is the s-harmonic extension of $u$ in $\mathbb{R}^{N+1}_+$ and $U(\cdot, 0)$ vanishes on $\Omega^c$. In other words, $U$ satisfies

$$\begin{cases}
\text{div}(t^{1-2s} \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
U = u & \text{on } \Omega \times \{0\}, \\
U = 0 & \text{on } \Omega^c \times \{0\}
\end{cases}$$

and for every $\xi \in tr E_0$,

$$\langle (-\Delta)^s u, \xi \rangle := \int_\Omega \partial_\nu^s U(\cdot, 0) \xi dx = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U \nabla \bar{\xi} dx dt,$$

where $\bar{\xi}$ is the s-harmonic extension of $\xi$. We take $\xi = u$, then

$$\langle (-\Delta)^s u, u \rangle := \int_\Omega \partial_\nu^s U(\cdot, 0) u dx = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U \nabla U dx dt \\
\geq \mu_1 \int_\Omega U(x, 0)^2 dx = \mu_1 \int_\Omega u(x)^2 dx.$$

Under assumption $(V_3)$, for any $u \in tr E_0$, $u \neq 0$, it holds that

$$\left(\langle (-\Delta)^s - \mu \rangle u, u \right) \geq (\mu_1 - \mu) \int_\Omega u(x)^2 dx \geq \delta_0 \|u\|_{L^2(\Omega)}^2$$

for some $\delta_0 > 0$, which implies that the operator $(-\Delta)^s - \mu$ is positively definite in $tr E_0$.

In this paper, we consider the fractional Schrödinger equation involving critical growth. We focus on the existence of least energy solutions, which localize near the potential well $V^{-1}(0)$ for $\lambda$ large. For similar investigations involving Laplacian and critical growth, we refer the reader to the second author’s paper.

Before stating our main results, we firstly give some notations and remarks.

To treat the nonlocal problem, we will study a corresponding extension problem in one more dimension, which allows us to investigate problem by studying a local problem via classical nonlinear variational methods.

The homogeneous fractional Sobolev space $D^s(\mathbb{R}^N)$ $(0 < s < 1)$ is given by

$$D^s(\mathbb{R}^N) := \left\{ u \in L^{3N/(2s)}(\mathbb{R}^N) : \|u\|_{D^s(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} < \infty \right\},$$

where $\hat{u}$ denotes the Fourier transform of $u$.

Note that $D^s(\mathbb{R}^N)$ is a Hilbert space equipped with an inner product

$$(u, v)_{D^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}(\xi) \hat{v}(\xi) d\xi.$$
We also define a fractional Laplacian operator on the whole space,
\((-\Delta)^s : D^s(\mathbb{R}^N) \to D^{-s}(\mathbb{R}^N)\)
by
\[
\langle (-\Delta)^s u, v \rangle_{D^{-s}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}(\xi) \hat{v}(\xi) d\xi,
\]
where \(D^{-s}(\mathbb{R}^N)\) is the dual space of \(D^s(\mathbb{R}^N)\).

Then, one can easily check that if \(u \in D^{2s}(\mathbb{R}^N)\), we have \((-\Delta)^s u \in L^2(\mathbb{R}^N)\) such that
\[
(-\Delta)^s u = \mathcal{F}^{-1} \left( |\xi|^{2s} \hat{u}(\xi) \right),
\]
where \(\mathcal{F}^{-1}\) denotes the inverse Fourier transform.

We see for \(u, v \in D^s(\mathbb{R}^N)\),
\[
\langle u, v \rangle_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} v dx,
\]
and assuming additionally \(u \in D^{2s}(\mathbb{R}^N)\) \(v \in L^2(\mathbb{R}^N)\), we can integrate by parts:
\[
\int_{\mathbb{R}^N} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} v dx = \int_{\mathbb{R}^N} (-\Delta)^s u \cdot v dx.
\]
Finally, the notation \(H^s(\mathbb{R}^N)\) denotes the standard fractional Sobolev space defined as
\[
H^s(\mathbb{R}^N) := D^s(\mathbb{R}^N) \cap L^2(\mathbb{R}^N),
\]
with norm
\[
\|u\|_{H^s(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |u(x)|^2 dx + \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2}.
\]

Similarly, it holds by taking trace that
\[
D^s(\mathbb{R}^N) = \{ u = \text{tr}|_{\mathbb{R}^N \times \{0\}} U : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt < \infty \},
\]
and
\[
\|U(\cdot, 0)\|_{D^s(\mathbb{R}^N)} \leq C \left( \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt \right)^{1/2}
\]
for some \(C > 0\) independent of \(U \in \{ U \in W^{1,1}_{\text{loc}}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt < \infty \}\).

Now we introduce the concept of \(s\)-harmonic extension of a function \(u \in D^s(\mathbb{R}^N)\), which provides a way to represent fractional Laplacian operators as a form of Dirichlet-to-Neumann map.

By works of Caffarelli-Silvestre \cite{CS} (for \(\mathbb{R}^N\)), it is known that there is one unique function \(U \in H(t^{1-2s}, \mathbb{R}^{N+1}_+) := \{ U : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt < \infty, \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |U|^2 dx dt < \infty \}\), which satisfies the equation
\[
\begin{cases}
\text{div}(t^{1-2s} \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
U(x, 0) = u & \text{for } x \in \mathbb{R}^N,
\end{cases}
\]
respectively in the distributional sense. Moreover, if \(u\) is compactly supported and smooth, then the following limits
\[
\partial_t^s U(x, 0) := -C_s^{-1} \left( \lim_{t \to 0^+} t^{1-2s} \frac{\partial U}{\partial t}(x, t) \right) \text{ with } C_s := \frac{2^{1-2s} \Gamma(1-s)}{\Gamma(s)}
\]
are well defined and one must have
\((-\Delta)^su = \partial^*_sU(x, 0).\)

We call \(U\) the \(s\)–harmonic extension of \(u\).

Let
\[
E := \left\{ U \in W^{1,1}_{\text{loc}}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2dxdt < \infty, U(\cdot, 0) \in L^2(\mathbb{R}^N) \right\},
\]
then \(E\) is the Hilbert space under the inner product
\[
(U, W)_E = \int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla U \cdot \nabla W dxdt + \int_{\mathbb{R}^N} U(x, 0)W(x, 0)dx,
\]
and the norm induced by the inner product \((\cdot, \cdot)\) is
\[
\|U\|_E = \left( \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2dxdt + \int_{\mathbb{R}^N} U(x, 0)^2dx \right)^{1/2}.
\]
Indeed for every \(U(x, t) \in E\), we denote by \(U(x, 0)\) the trace of \(U(x, t)\) on \(\mathbb{R}^N\) and we take
\[
tr_{\mathbb{R}^N}E := \{ U(x, 0) : U(x, t) \in E \}.
\]

Remark 1. For \(\lambda \geq \lambda_0 > 0\), \(V(x)\) satisfies assumptions \((V_1) - (V_3)\). Then \(E_\lambda\) is continuously embedded in \(E\).

We can study problem \((1.1)\) by variational methods for a local problem. More precisely, we will study the following boundary value problem in a half space:
\[
\begin{aligned}
\text{div}(t^{1-2s}\nabla U) &= 0 & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty), \\
\partial^*_sU(\cdot, 0) &= U^{2^*_s-1} - (\lambda V(x) - \mu)U & \text{on } \partial\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\},
\end{aligned}
\tag{1.8}
\]
with the norm
\[
\|U\|_\lambda := \left( \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2dxdt + \int_{\mathbb{R}^N} (\lambda V(x) - \mu)U(x, 0)^2dx \right)^{1/2}.
\]

The energy functional associated with \((1.8)\) is defined by
\[
J_\lambda(U) := \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2dxdt + \frac{1}{2} \int_{\mathbb{R}^N} (\lambda V(x) - \mu)U(x, 0)^2dx - \frac{1}{2s} \int_{\mathbb{R}^N} U^+(x, 0)^{2s}dx \quad \text{for } U \in E_\lambda.
\]
We define the Nehari manifold
\[ \mathcal{M}_\lambda := \left\{ U \in E_\lambda \setminus \{0\} : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt + \int_{\mathbb{R}^N} (\lambda V(x) - \mu) U(x,0)^2 dx = \int_{\mathbb{R}^N} U^+(x,0)^{2_s^*} dx \right\}, \]
and let
\[ c_\lambda := \inf \{ J_\lambda(U) : U \in \mathcal{M}_\lambda \} \quad (1.9) \]
be the infimum of \( J_\lambda \) on the Nehari manifold \( \mathcal{M}_\lambda \).

For \( \lambda \) large, the problem
\[ \begin{cases} (\lambda s) u(x) = |u(x)|^{2_s^*-2} u(x) + \mu u(x), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.10) \]
is some kind of limit problem of \( (1.1) \). We shall prove that there exists a least energy solution of \( (1.1) \) converging, for \( \lambda \to \infty \), to a least energy solution of \( (1.10) \).

Similarly, to consider the problem \( (1.10) \), we will study the following mixed boundary value problem in a half space:
\[ \begin{cases} \text{div}(t^{1-2s} \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty), \\ U = 0 & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}, \\ \partial_t U(.,0) = U^{2_s^*-1} + \mu U & \text{on } \Omega \times \{0\}, \\ U \geq 0 & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty), \end{cases} \quad (1.11) \]
where \( \partial_t^s U(x,0) := -C_s^{-1} \left( \lim_{t \to 0^+} t^{1-2s} \frac{\partial U}{\partial t}(x,t) \right) \) with \( C_s := \frac{2^{1-2s} \Gamma(1-s)}{\Gamma(s)} \). If \( U \) satisfies \( (1.11) \), then the trace \( u \) on \( \mathbb{R}^N \) of the function \( U \) is a solution of \( (1.10) \).

To consider problem \( (1.11) \), we define a subspace \( E_0 \) of \( E \) as follows:
\[ E_0 := \left\{ U(x,t) \in E : U(x,0) = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}. \quad (1.12) \]
The energy functional associated with \( (1.11) \) is defined by
\[ I(U) := \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt - \frac{\mu}{2} \int_{\Omega} U(x,0)^2 dx - \frac{1}{2s^*} \int_{\Omega} U^+(x,0)^{2_s^*} dx \quad \text{for } U \in E_0. \quad (1.13) \]

Comparing with the Nehari manifold \( \mathcal{M}_\lambda \), we define the Nehari manifold
\[ \mathcal{N}_0 := \left\{ U \in E_0 \setminus \{0\} : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt = \int_{\Omega} U^+(x,0)^{2_s^*} dx + \mu \int_{\Omega} U(x,0)^2 dx \right\} \]
and let
\[ c_0 := \inf \{ I(U) : U \in \mathcal{N}_0 \} \]
be the infimum of \( I \) on the Nehari manifold \( \mathcal{N}_0 \).

**Definition 1.1.** We say that a function \( u_\lambda(x) = U_\lambda(x,0) \) is a least energy solution of \( (1.1) \) if \( c_\lambda \) is achieved by some \( U_\lambda \in \mathcal{M}_\lambda \) which is a critical point of \( J_\lambda \). Similarly, we say that a function \( u(x) = U(x,0) \) is a least energy solution of \( (1.10) \) if \( c_0 \) is achieved by some \( U \in \mathcal{N}_0 \) which is a critical point of \( I \).

Our main results are the following:
**Theorem 1.2.** Suppose that assumptions \((V_1)-(V_3)\) hold. Then for \(\lambda\) large, problem \((1.1)\) has a least energy solution \(u_\lambda\). Furthermore, any sequence \(\lambda_n (\lambda_n \to \infty \text{ as } n \to \infty)\) has a subsequence such that \(u_{\lambda_n}\) converges in \(H^s(\mathbb{R}^N)\) along the subsequence to a least energy solution of \((1.10)\).

As in the case of the least energy solution of \((1.1)\), any solutions of \((1.1)\) converges, for \(\lambda \to \infty\), to a solution of \((1.10)\). More generally, we have the following result.

**Theorem 1.3.** Suppose that assumptions \((V_1)-(V_3)\) hold. Let \(u_n(x) = U_n(x, 0)\), \(n \in \mathbb{N}\) be a sequence of solutions of \((1.1)\) with \(\lambda_n \to \infty\) and such that
\[
\limsup_{n \to \infty} J_{\lambda_n}(U_n) < \infty.
\]
Then \(u_n(x) = U_n(x, 0)\) converges strongly along the subsequence in \(H^s(\mathbb{R}^N)\) to a solution of \((1.10)\).

Our paper is organized as follows: In Section 2, we present some results about the eigenvalues and eigenfunctions for the operators involving the fractional Laplacian. In Section 3, we give the Mountain Pass Geometry. Section 4 is devoted to the existence of the least energy solution to the limit problem. In Section 5, we prove the existence of the least energy solution. Section 6 contains the asymptotic behavior of the least energy solution and in Section 7 we give the proof of our main results.

We will use the same \(C\) to denote various generic positive constants and we will use \(o(1)\) to denote quantities that tend to 0 as \(\lambda\) (or \(n\)) tends to \(\infty\).

2. **Eigenvalues and eigenfunctions.** In this section, we present some results about the eigenvalues and eigenfunctions for the operators involving the fractional Laplacian. We consider the following boundary value problems:

\[
\begin{cases}
(-\Delta)^s u = \mu u & x \in \Omega, \\
u = 0 & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

(2.15)

To consider the above problem, we study the following boundary value problem in a half space:

\[
\begin{cases}
\text{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty), \\
U = 0 & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}, \\
\partial_{\nu}^s U(\cdot, 0) = \mu U & \text{on } \Omega \times \{0\},
\end{cases}
\]

(2.16)

where \(\partial_{\nu}^s U(x, 0) := -C_s^{-1} \left( \lim_{t \to 0^+} t^{1-2s} \frac{\partial U}{\partial t}(x, t) \right)\) with \(C_s := \frac{2^{1-2s} \Gamma(1-s)}{\Gamma(s)}\). If \(U\) satisfies \((2.16)\), then the trace \(u\) on \(\mathbb{R}^N\) of the function \(U\) is a solution of \((2.15)\). Firstly, we define

\[
\mu_1 := \inf_{M_1} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U(x, t)|^2 dxdt,
\]

where

\[
M_1 := \{U(x, t) \in E_0 : \int_{\Omega} U(x, 0)^2 dx = 1\}.
\]

Assume that there exists \(U_0(x, t) \in M_1\) which achieves \(\mu_1\). Then \(U_0(x, t)\) satisfies

\[
\begin{cases}
\text{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty), \\
U = 0 & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}, \\
\partial_{\nu}^s U(\cdot, 0) = \mu_1 U & \text{on } \Omega \times \{0\},
\end{cases}
\]

and \(\varphi_1(x) = U_0(x, 0)\) satisfies the following problem:
\[
\begin{cases}
(\Delta)^s \varphi_1 = \mu_1 \varphi_1 & x \in \Omega, \\
\varphi_1 = 0 & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Moreover, we call \(\mu_1 > 0\) the first eigenvalue of \((\Delta)^s\) in \(trE_0\) and \(\varphi_1(x) = U_0(x,0)\) the first eigenfunction corresponding to \(\mu_1\) in \(trE_0\).

Now we are going to show that \(\mu_1\) can be achieved by some \(U_0 \in M_1\). To show that, we firstly give an imbedding lemma which is standard.

**Lemma 2.1.** Let \(2 \leq p < 2^*_N := \frac{2N}{N-2} \) for \(N \geq 2\). Then \(tr\Omega E_0\) is compactly embedded in \(L^p(\Omega)\).

**Proof.** Note that \(tr\Omega E_0 \subset H^s(\Omega)\) and the fact that the embedding \(H^s(\Omega) \hookrightarrow L^p(\Omega)\) is compact for \(2 \leq p < \frac{2N}{N-2}\) for \(N \geq 2\) immediately implies the Lemma. \(\square\)

**Lemma 2.2.** \(\mu_1\) is achieved by some \(U_0(x,t)\).

**Proof.** We take a minimizing sequence \(\{U_k\}_{k=1}^\infty\) such that

\[
\int_{\Omega} U_k(x,0)^2 dx = 1, \quad \int_{\mathbb{R}^N_+} t^{1-2s}|\nabla U_k(x,t)|^2 dxdt \to \mu_1 \text{ as } k \to \infty.
\]

It is easy to see that \(U_k\) is bounded in \(E_0\). Since \(E_0\) is reflexive and \(tr\Omega E_0\) is compactly embedded in \(L^p(\Omega)\) by lemma 2.1, we conclude that there exists a subsequence (we still denote it by \(U_k\)) and a function \(U_0 \in E_0\) such that

\[
U_k \rightharpoonup U_0 \text{ weakly in } E_0, \\
U_k(\cdot,0) \to U_0(\cdot,0) \text{ in } L^2(\Omega), \\
U_k(\cdot,0) \to U_0(\cdot,0) \text{ a.e. in } \Omega.
\]

It follows that

\[
\int_{\mathbb{R}^N_+} t^{1-2s}|\nabla U_0(x,t)|^2 dxdt \leq \int_{\mathbb{R}^N_+} t^{1-2s}|\nabla U_k(x,t)|^2 dxdt
\]

and

\[
\int_{\Omega} U_0(x,0)^2 dx = 1.
\]

We have \(U_0 \in M_1\) and

\[
\int_{\mathbb{R}^N_+} t^{1-2s}|\nabla U_0(x,t)|^2 dxdt \leq \mu_1.
\]

Consequently, \(U_0\) is indeed a minimizer which achieves \(\mu_1\). \(\square\)

Now we consider the eigenvalue problems for the operator

\[
L_\lambda := (\Delta)^s + \lambda V(x).
\]

We define

\[
\mu_1^\lambda := \inf_{M_1^\lambda} \left\{ \int_{\mathbb{R}^N_+} t^{1-2s}|\nabla U(x,t)|^2 dxdt + \int_{\mathbb{R}^N} \lambda V(x)u(x)^2 dx \right\},
\]

where

\[
M_1^\lambda = \{ u(x) \in tr\mathbb{R}^N E_\lambda : \int_{\mathbb{R}^N} u(x)^2 dx = 1 \},
\]

and \(U(x,t)\) is the s-harmonic extension of \(u(x)\) with \(U(x,0) = u(x)\). We will show that \(\mu_1^\lambda\) is indeed an eigenvalue of the operator \(L_\lambda\). For this we only need to show \(\mu_1^\lambda\) is a discrete spectrum of \(L_\lambda\) for \(\lambda\) large. Indeed we have the following stronger result.
Lemma 2.3. Under the assumptions $(V_1)$ and $(V_2)$, the essential spectrum $\text{ess}(L_\lambda)$ of $L_\lambda$ is contained in $[\lambda V_\infty^2, +\infty)$, where

$$V_\infty := \liminf_{|x| \to \infty} V(x) > 0.$$ 

Remark 2. In fact, we can choose some $\varphi \in \text{tr}_{\mathbb{R}^N} E_\lambda$ which satisfies

$$\text{supp}\varphi \subseteq \Omega \text{ and } \int_{\mathbb{R}^N} \varphi(x)^2 dx = 1.$$ 

For such fixed $\varphi$,

$$\mu^1_\lambda \leq \int_{\mathbb{R}^{N+1}} t^{1-2s}|\nabla \hat{\varphi}(x, t)|^2 dx dt < \lambda V_\infty^2 \text{ for } \lambda \text{ large},$$

where $\hat{\varphi}(x, t)$ is the $s$-harmonic extension of $\varphi$. Thus the above lemma immediately indicates that $\mu^1_\lambda$ is a discrete spectrum of $L_\lambda$ for $\lambda$ large and thus is also an eigenvalue of $L_\lambda$, we denote the corresponding eigenfunction as $\psi^1_\lambda(x)$.

Now we give the proof of Lemma 2.3. A similar proof can be found in the paper by Bartsch, Pankov and Wang[4]. For the reader’s convenience, we give the details here.

Proof. We set

$$W_\lambda := \lambda \left(V(x) - \frac{V_\infty}{2}\right)$$

and write

$$W_\lambda^+ := \max\{W_\lambda, 0\}, W_\lambda^- := \min\{W_\lambda, 0\}.$$ 

For any $u(x) \in \text{tr}_{\mathbb{R}^N} E_\lambda \subseteq H^s(\mathbb{R}^N)$, we have

$$\left( -\Delta \right)^s u(x) + W_\lambda^+ u(x) + \frac{\lambda V_\infty}{2} u(x) = \int_{\mathbb{R}^{N+1}} t^{1-2s}|\nabla v(x, t)|^2 dx dt + \int_{\mathbb{R}^N} (W_\lambda^+ + \lambda \frac{V_\infty}{2}) u(x)^2 dx,$$

where $v(x, t)$ is the $s$-harmonic extension of $u(x)$ such that $v(x, 0) = u(x)$. Thus it is easy to see that

$$\sigma \left( -\Delta \right)^s + W_\lambda^+ + \lambda \frac{V_\infty}{2} \subset \left[ \lambda \frac{V_\infty}{2}, +\infty \right),$$

where $\sigma(L)$ denotes the spectrum of an operator $L$. Let us denote by

$$H_\lambda := \left(-\Delta\right)^s + W_\lambda^+ + \lambda \frac{V_\infty}{2}$$

the operator from $H^s(\mathbb{R}^N)$ to $H^{-s}(\mathbb{R}^N)$. Since $L_\lambda = H_\lambda + W_\lambda^-$, we only need to show that the operator of multiplication by $W_\lambda^-$ is a relatively compact perturbation of $H_\lambda$. Thus by the classical Weyl theorem (see [23] XIII.4, p. 117), we have

$$\sigma_{\text{ess}}(L_\lambda) = \sigma_{\text{ess}}(H_\lambda).$$

Here $\sigma_{\text{ess}}(L)$ denotes the essential spectrum of the operator $L$.

Thus to complete the proof of this lemma, we only need to show that

$$H^s(\mathbb{R}^N) \to H^{-s}(\mathbb{R}^N), \quad u \mapsto W_\lambda^- u$$

is compact. Indeed we will show that

$$H^s(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad u \mapsto W_\lambda^- u$$

(2.17)
is compact. Thus the boundedness of the following embedding
\[ H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \hookrightarrow H^{-s}(\mathbb{R}^N) \]
immediately implies that (2.17).

We set
\[ S := \{ W_\lambda^x u : u \in H^s(\mathbb{R}^N), \| u \|_{H^s(\mathbb{R}^N)} \leq 1 \}. \]
We only need to prove that \( S \) is precompact in \( L^2(\mathbb{R}^N) \). Set
\[ A_0 := \{ x \in \mathbb{R}^N : V(x) \leq \frac{V_{\infty}}{2} \}, \]
then by assumptions (V₁) and (V₂), we have \( A_0 \subset B_R(0) \) for some \( R > 0 \). Here \( B_R(0) \) denotes the ball center at origin with radius \( R \). Thus by the definition of \( W_\lambda^x \) we know that
\[ \text{supp} W_\lambda^x \subset A_0 \subset B_R(0). \]
Furthermore, there is a constant \( M > 0 \) which is independent of \( \lambda \) such that \( |W_\lambda^x| \leq M \) for all \( x \in \mathbb{R}^N \). Thus using the fact that \( H^s(\mathbb{R}^N) \hookrightarrow L^2(B_R(0)) \) is compact immediately implies that \( S \) is precompact in \( L^2(\mathbb{R}^N) \) and thus the proof of this Lemma is completed.

Now we give an asymptotic behavior result for \( \mu_1^\lambda \) and \( U_0^\lambda \), where \( U_0^\lambda \) is the s-harmonic extension of \( \psi_1^\lambda(x) \). The proof follows the similar arguments as that in [29].

**Lemma 2.4.** \( \mu_1^\lambda \to \mu_1, U_0^\lambda \to U_0 \) strongly in \( E \) as \( \lambda \to \infty \), where \( U_0 \in E_0 \) that achieves \( \mu_1 \).

**Proof.** We only need to show that for any sequence \( \{ \lambda_n \} (\lambda_n \to \infty \text{ as } n \to \infty) \),
\[ \mu_1^{\lambda_n} \to \mu_1 \text{ and } U_0^{\lambda_n} \to U_0 \text{ as } n \to \infty. \]
First of all, by the definition of \( \mu_1^{\lambda_n} \), it is easy to see that \( \mu_1^{\lambda_n} \leq \mu_1 \). Moreover \( U_0^{\lambda_n} \in E_{\lambda_n} \) such that
\[ \int_{\mathbb{R}^N} U_0^{\lambda_n}(x,0)^2dx = 1, \]
\[ \int_{\mathbb{R}_{+}^{N+1}} t^{1-2s}|\nabla U_0^{\lambda_n}(x,t)|^2dxdt + \int_{\mathbb{R}^N} \lambda_n V(x)U_0^{\lambda_n}(x,0)^2dx = \mu_1^{\lambda_n}. \]
It is easy to see that \( \{ U_0^{\lambda_n} \} \) is bounded in \( E \) and, without loss of generality, we assume that there exists a subsequence (we still denote \( U_0^{\lambda_n} \)) and a function \( U_0 \in E \) such that
\[ U_0^{\lambda_n} \rightharpoonup U_0 \text{ in } E, \]
\[ U_0^{\lambda_n}(\cdot,0) \to U_0(\cdot,0) \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}^N). \]

**Claim 1.** \( U_0(x,0) = 0 \) for \( x \in \mathbb{R}^N \setminus \bar{\Omega} \).

Indeed to show this claim, let us set \( C_m := \{ x \in \mathbb{R}^N : V(x) \geq \frac{1}{m} \} \), for \( n \) large we have
\[
\int_{C_m} U_0(x,0)^2dx \\
\leq \frac{m}{\lambda_n} \int_{\mathbb{R}^N} \lambda_n V(x)U_0^{\lambda_n}(x,0)^2dx \\
\leq \frac{m}{\lambda_n} \left( \int_{\mathbb{R}_{+}^{N+1}} t^{1-2s}|\nabla U_0^{\lambda_n}(x,t)|^2dxdt + \int_{\mathbb{R}^N} \lambda_n V(x)U_0^{\lambda_n}(x,0)^2dx \right)
\]
\[ \leq \frac{m}{\lambda_n} \mu_1 \to 0 \text{ as } n \to \infty. \]

Thus \( U_0(x, 0) = 0 \) on \( \bigcup_{m=1}^{\infty} C_m = \mathbb{R}^N \setminus \bar{\Omega} \) and Claim 1 is proved.

**Claim 2.** \( \int_{\bar{\Omega}} U_0(x, 0)^2 \, dx = 1. \)

To prove this, we only need to show that for any \( \varepsilon > 0, \)
\[ \int_{\Omega} U_0(x, 0)^2 \, dx \geq 1 - \varepsilon. \]

In fact, by the assumption \((V_2)\), we can take \( R > 0 \) large enough such that \( \bar{\Omega} \subseteq B_R(0) \) and \( \eta := \inf_{|x| \geq R} V(x) > 0, \)
where \( B_R(0) \) is the ball centered at the origin with radius \( R \). Thus
\[ \int_{\mathbb{R}^N \setminus B_R(0)} U_0^{\lambda_n}(x, 0)^2 \, dx \leq \lim_{n \to \infty} \eta \lambda_n \int_{\mathbb{R}^N \setminus B_R(0)} \lambda_n V(x) U_0^{\lambda_n}(x, 0)^2 \, dx \]
\[ \leq \lim_{n \to \infty} \frac{1}{\eta \lambda_n} \left( \int_{\mathbb{R}_{t+1}^{N+1}} t^{1-2s} |\nabla U_0^{\lambda_n}(x, t)|^2 \, dx \, dt + \int_{\mathbb{R}^N \setminus B_R(0)} \lambda_n V(x) U_0^{\lambda_n}(x, 0)^2 \, dx \right) \]
\[ \leq \lim_{n \to \infty} \frac{1}{\eta \lambda_n} \mu_1. \]

Thus for any \( \varepsilon > 0 \) small, there exists \( N_0 > 0 \) such that for all \( n \geq N_0 \) we have
\[ \int_{\mathbb{R}^N \setminus B_R(0)} U_0^{\lambda_n}(x, 0)^2 \, dx < \varepsilon, \]
which implies
\[ \int_{B_R(0)} U_0^{\lambda_n}(x, 0)^2 \, dx \geq 1 - \varepsilon, \text{ for } n \geq N_0. \]  \( (2.18) \)

Since \( U_0^{\lambda_n}(:, 0) \to U_0(:, 0) \) strongly in \( L^2(B_R(0)) \), by \( (2.18) \) and Claim 1, we have
\[ \int_{\Omega} U_0(x, 0)^2 \, dx = \int_{B_R(0)} U_0(x, 0)^2 \, dx \geq 1 - \varepsilon. \]

Thus we proved Claim 2.

Combining the weak convergence of \( U_0^{\lambda_n}(:, 0) \to U_0(:, 0) \) in \( L^2(\mathbb{R}^N) \) and Claim 2, we indeed have proved that
\[ U_0^{\lambda_n}(:, 0) \to U_0(:, 0) \text{ in } L^2(\mathbb{R}^N). \]

By assumption \((V_1)\) and Claim 1, we obtain that \( U_0 \in E_0 \) and \( \int_{\Omega} U_0(x, 0)^2 \, dx = 1. \)

Thus we have
\[ \mu_1 \leq \int_{\mathbb{R}_{t+1}^{N+1}} t^{1-2s} |\nabla U_0(x, t)|^2 \, dx \, dt \]
\[ \leq \lim_{n \to \infty} \left( \int_{\mathbb{R}_{t+1}^{N+1}} t^{1-2s} |\nabla U_0^{\lambda_n}(x, t)|^2 \, dx \, dt + \int_{\mathbb{R}^N} \lambda_n V(x) U_0^{\lambda_n}(x, 0)^2 \, dx \right) \]
\[ = \lim_{n \to \infty} \mu_1^{\lambda_n} \leq \mu_1, \]
which implies that \( \mu_1^{\lambda_n} \to \mu_1 \) and \( U_1^{\lambda_n} \to U_0 \) strongly in \( E \) as \( n \to \infty. \) \( \square \)
Remark 3. By our assumption \((V_3)\) and Lemma 2.4, we know that for \(\lambda\) large, the operator \(L_\lambda := (-\Delta)^s + \lambda V(x) - \mu\) is also positively definite in \(trE_\lambda\).

3. Mountain Pass Geometry. Let \(X\) be a Hilbert space and \(\varphi \in C^1(X, \mathbb{R})\). We call a sequence \(\{u_n\} \subset X\) a (Palais-Smale)\(_c\) sequence ((P.S.)\(_c\) sequence for shortness) of \(\varphi\) if it satisfies:

\[
\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0 \quad \text{in} \quad X^*,
\]

where \(X^*\) is the dual space of \(X\).

To obtain a (P.S.)\(_c\) sequence, we apply the well known Mountain Pass Lemma (see also [31]). More precisely, we show that the functional \(J_\lambda\) has a Mountain Pass Geometry for \(\lambda\) large. As a result, \(J_\lambda\) has a (P.S.)\(_c\) sequence for some \(c \in \mathbb{R}\).

**Lemma 3.1.** \(J_\lambda\) has a Mountain Pass Geometry for \(\lambda\) large.

**Proof.** We divide the proof into three steps.

**Step 1:** It is easy to verify that \(J_\lambda \in C^1(E_\lambda, \mathbb{R})\).

**Step 2:** \(J_\lambda(0) = 0\) and there exists \(r > 0, \rho_0 > 0\) such that \(J_\lambda(U) > \rho_0 > 0\) as \(\|U\|_\lambda = r\).

Indeed, applying the well known Sobolev trace inequality, which states for \(U \in H(t^{1-2s}, \mathbb{R}^{N+1})\),

\[
\left( \int_{\mathbb{R}^N} |U(x,0)|^{2^*_s} \, dx \right)^{\frac{1}{2^*_s}} \leq C_0 \left( \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla U|^2 \, dx \right)^{\frac{1}{2}},
\]

we get

\[
J_\lambda(U) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla U|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (\lambda V(x) - \mu)U(x,0)^2 \, dx \\
- \frac{1}{2^*_s} \int_{\mathbb{R}^N} U^+(x,0)^{2^*_s} \, dx \\
= \frac{1}{2} \|U\|_\lambda^2 - \frac{1}{2^*_s} \|U^+(x,0)\|_{2^*_s}^{2^*_s} \\
\geq \frac{1}{2} \|U\|_\lambda^2 - \rho_0 \|U\|_\lambda^2
\]

for large \(\lambda\). Then there exists \(r > 0, \rho_0 > 0\) such that \(J_\lambda(U) > \rho_0 > 0\) as \(\|U\|_\lambda = r\).

**Step 3:** There exists \(e \in E_\lambda\) such that \(J_\lambda(e) < 0\).

Fix \(U_0 \in E_\lambda, \ U_0 \geq 0\) and \(U_0 \neq 0\). Set

\[
g(l) := J_\lambda(lU_0) \\
= \frac{1}{2} l^2 \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla U_0|^2 \, dx + \frac{1}{2} l^2 \int_{\mathbb{R}^N} (\lambda V(x) - \mu)U_0(x,0)^2 \, dx \\
- \frac{1}{2^*_s} l^{2^*_s} \int_{\mathbb{R}^N} U_0^+(x,0)^{2^*_s} \, dx,
\]

it is easy to see that

\[
\lim_{l \to +\infty} g(l) = -\infty.
\]

Thus there exists \(L > 0\) large enough such that \(g(L) = J_\lambda(LU_0) < 0\) and \(\|LU_0\|_\lambda > r\). Then take \(e = LU_0\).

Combining Steps (1)–(3), we indeed have completed the proof of this lemma. □
Proposition 1. Suppose we have the following result which is the key part of this section.

\[
\text{admits a least energy solution } u \text{ to (4.20)}
\]

where

\[
c_\lambda = \inf_{\gamma} \max_{t \in [0, 1]} J_\lambda(\gamma(t))
\]

and

\[
\Gamma := \{ \gamma \in C([0, 1], E_\lambda) : \gamma(0) = 0, \gamma(1) = e \}.
\]

4. Existence of the least energy solution to the limit problem. In this section, we study the existence of the least energy solution of the limit problem (1.10). Namely, the following problem

\[
\begin{cases}
(\Delta)^s u(x) = |u(x)|^{2^*_s - 2}u(x) + \mu u(x), & x \in \Omega, \\
u(x) \geq 0, & x \in \Omega, \\
u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

We study the following mixed boundary value problem in a half space:

\[
\begin{cases}
\text{div}(t^{1-2s} \nabla U) = 0 & \text{in } \mathbb{R}_{+}^{N+1} = \mathbb{R}^N \times (0, \infty), \\
U = 0 & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}, \\
\partial^\sigma \nu U(\cdot, 0) = U^{2^*_s - 1} + \mu U & \text{on } \Omega \times \{0\}, \\
U \geq 0 & \text{in } \mathbb{R}_{+}^{N+1} = \mathbb{R}^N \times (0, \infty),
\end{cases}
\]

where \(\partial^\sigma \nu U(x, 0) := -C_s^{-1} \left( \lim_{t \to 0^+} t^{1-2s} \frac{\partial U}{\partial \nu}(x, t) \right)\) with \(C_s := \frac{2^{2s} - 2s}{4s}\). If \(U\) satisfies (4.21), then the trace \(u\) on \(\mathbb{R}^N\) of the function \(U\) is a solution of (4.20).

Again we define

\[
c_0 := \inf_{\mathcal{N}_0} I(U),
\]

where \(\mathcal{N}_0 = \{ U \in E_0 \setminus \{0\} : I(U) \cdot U = 0 \}\) and \(I(U)\) is defined as in (1.13). We have the following result which is the key part of this section.

**Proposition 1.** Suppose \(N > 5 + 2s, \mu\) satisfies assumption (V3). Then problem (4.20) admits a least energy solution \(u_0 = U_0(x, 0)\), where \(U_0\) achieves \(c_0\).

To begin with the proof of Proposition 1, we firstly present some notations and lemmas which are the main ingredients of the arguments. We denote

\[
S_\mu := \inf_{U \in E_0} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt - \mu \int_\Omega U(x, 0)^2 dx}{\left( \int_\Omega U^+(x, 0)^{2^*_s} dx \right)^{2/2^*_s}},
\]

\[
S := \inf_{U \in E_0} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt}{\left( \int_\Omega U^+(x, 0)^{2^*_s} dx \right)^{2/2^*_s}},
\]

\[
S_0 := \inf_{U \in H(t^{1-2s}, \mathbb{R}^{N+1})} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt}{\left( \int_{\mathbb{R}^N} U^+(x, 0)^{2^*_s} dx \right)^{2/2^*_s}}.
\]
Take
\[ Q_\mu(U) = \frac{\int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla U|^2 \, dx \, dt - \mu \int_\Omega U(x,0)^2 \, dx}{\left( \int_\Omega U^+(x,0)^{2t} \, dx \right)^{2/2t}}, \quad \mu \in \mathbb{R}, \quad \text{with } \mu \in [0, \mu_1). \] (4.26)

At first, we have the following Lemma 4.1, and the similar proof can be found in the paper by M. Gonzalez, J. Qing [20].

**Lemma 4.1.** Let \( \mu \in (0, \mu_1) \). Then we have
\[ S_\mu := \inf \{ Q_\mu(U) : U \in E_0 \} < S. \]

**Proof.** It is easy to see that \( S_0 \leq S \), then it suffices to show that \( S_\mu < S_0 \). It is known from [21] that \( S_0 \) is achieved by the extremal functions
\[ U_\varepsilon(x,t) := c_0 \beta(N,s) \int_{\mathbb{R}^N} \frac{t^{2s}}{|x-y|^2 + t^2} \left( \frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{N-2s}{2}} \, dy, \]
\[ U_\varepsilon(x,0) := c_0 \left( \frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{N-2s}{2}}, \]
where \( \varepsilon > 0 \) is arbitrary and \( c_0 = \left( \frac{c(N,s)2^{2s}}{S_0} \right)^{\frac{N-2s}{2}} \), which is (the) unique solution of the problem
\begin{align*}
\begin{cases}
\text{div}(t^{1-2s}\nabla U_\varepsilon) = 0 & \text{in } \mathbb{R}^{N+1} = \mathbb{R}^N \times (0, \infty), \\
\partial_s U_\varepsilon(x,0) = U_\varepsilon(x,0)^{2t-1} & \text{on } \mathbb{R}^N.
\end{cases} \tag{4.27}
\end{align*}

On the other hand, multiplying \( U_\varepsilon \) on both sides of equation (4.27) and integrating, we get
\[ \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla U_\varepsilon|^2 \, dx \, dt = \int_{\mathbb{R}^N} U_\varepsilon(x,0)^{2t} \, dx. \] (4.28)

Step 1. Computation of the energy in \( B_{2\rho}^+ \setminus B_{\rho}^+ \).

At first, we note that on the half-annulus,
\[ |\nabla V_\varepsilon|^2 \leq c |\nabla U_\varepsilon|^2 + \frac{c}{\rho^2} |U_\varepsilon|^2. \] (4.29)
One the other hand we have
\[
\int_{B_{2\rho}^+ \setminus B_{\rho}^+} t^{1-2s} U_\varepsilon^2 \, dx \, dt = \varepsilon^2 \int_{B_{2\rho/\varepsilon}^+ \setminus B_{\rho/\varepsilon}^+} t^{1-2s} U_1^2 \, dx \, dt \\
\leq \varepsilon^2 \left( \frac{\rho}{\varepsilon} \right)^{-3} \int_{B_{2\rho/\varepsilon}^+ \setminus B_{\rho/\varepsilon}^+} t^{1-2s} |(x, t)|^3 U_1^2 \, dx \, dt \\
\leq \varepsilon^5 \rho^{-3} o(1)
\]
and
\[
\int_{B_{2\rho}^+ \setminus B_{\rho}^+} t^{1-2s} |\nabla U_\varepsilon|^2 \, dx \, dt = \int_{B_{2\rho/\varepsilon}^+ \setminus B_{\rho/\varepsilon}^+} t^{1-2s} |\nabla U_1|^2 \, dx \, dt \\
\leq \varepsilon^5 \rho^{-5} \int_{B_{2\rho/\varepsilon}^+ \setminus B_{\rho/\varepsilon}^+} t^{1-2s} |(x, t)|^5 |\nabla U_1|^2 \, dx \, dt \\
\leq \varepsilon^5 \rho^{-5} o(1),
\]
where we have used the fact (see [20]) that
\[
\xi_5 := \int_{\mathbb{R}^{N+1}} t^{1-2s+5} |\nabla U_1|^2 \, dx \, dt < \infty
\]
and
\[
\int_{\mathbb{R}^{N+1}} t^{1-2s} |x|^5 |\nabla U_1|^2 \, dx \, dt < \infty,
\]
when \( N > 5 + 2s \). Thus from formula (4.29) we may estimate
\[
\int_{B_{2\rho}^+ \setminus B_{\rho}^+} t^{1-2s} |\nabla V_\varepsilon|^2 \, dx \, dt \leq c \varepsilon^5 \rho^{-5} o(1). \tag{4.30}
\]

**Step 2.** Conclusion.

By a direct computation, we obtain that
\[
\int_{\mathbb{R}^N} |U_\varepsilon(x, 0)|^{2^*_s} \, dx = c_0^{2^*_s} \int_{\mathbb{R}^N} \frac{\varepsilon^N}{(\varepsilon^2 + |x|^2)^N} \, dx \\
= c_0^{2^*_s} \omega_N \int_0^\infty \frac{r^{N-1}}{(1 + r^2)^N} \, dr =: K_1,
\]
where \( \omega_N \) is the surface area of unit sphere in \( \mathbb{R}^N \). We have
\[
\int_{\Omega \times \{0\}} |\eta U_\varepsilon|^{2^*_s} \, dx \, dt = c_0^{2^*_s} \int_{\mathbb{R}^N} \frac{\varepsilon^N \eta(x, 0)^{2^*_s}}{(\varepsilon^2 + |x|^2)^N} \, dx \\
= K_1 + c_0^{2^*_s} \int_{\mathbb{R}^N} \frac{\varepsilon^N (\eta(x, 0)^{2^*_s} - 1)}{(\varepsilon^2 + |x|^2)^N} \, dx \\
= K_1 + c_0^{2^*_s} \varepsilon \int_{\mathbb{R}^N \setminus B_\rho} \frac{\eta(x, 0)^{2^*_s} - 1}{(\varepsilon^2 + |x|^2)^N} \, dx \\
= K_1 + O(\varepsilon^N).
\]

Hence we have
\[
\left( \int_{\Omega \times \{0\}} |\eta U_\varepsilon|^{2^*_s} \, dx \, dt \right)^{2/2^*_s} = K_1^{2/2^*_s} + O(\varepsilon^N).
\]
Since $U_\varepsilon$ are minimizers for $S_0$ and \((4.28)\), we have that
\[ K_{12}^{\varepsilon} = S_0. \] (4.31)

On the other hand, we have for all \(\varepsilon \ll \rho\)
\[
\int_{\Omega \times \{0\}} |\eta U_\varepsilon|^2 dxdt = c_0^2 \int_{\Omega} \frac{\varepsilon^{N-2s}}{(\varepsilon^2 + |x|^2)^{-N-2s}} dx \\
\geq c_0^2 \int_{\{|x|<\rho\}} \frac{\varepsilon^{N-2s}}{(\varepsilon^2 + |x|^2)^{-N-2s}} dx \\
\geq c_0^2 \int_{\{|x|<\varepsilon\}} \frac{\varepsilon^{N-2s}}{(2\varepsilon^2)^{N-2s}} dx \\
+ c_0^2 \int_{\{|x|<\varepsilon\}} \frac{\varepsilon^{N-2s}}{(2|x|^2)^{N-2s}} dx \\
= c_1 \varepsilon^{2s} + c_2 \varepsilon^{N-2s} \int_{\varepsilon}^{\rho} r^{4s-1-N} dr \\
= c_3 \varepsilon^{2s} + c_4 \varepsilon^{N-2s}(\rho^{4s-N} - \varepsilon^{4s-N}) \\
= c_5 \varepsilon^{2s} + O(\varepsilon^{N-2s}),
\]
where \(c_1, c_2, c_3\) and \(c_5\) are positive constants.

Using the above estimates, we have
\[
Q_\mu(\eta U_\varepsilon) = \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla (\eta U_\varepsilon)|^2 dxdt - \mu \int_{\Omega \times \{0\}} |\eta U_\varepsilon|^2 dx}{(\int_{\Omega \times \{0\}} |\eta U_\varepsilon|^2 dx)^{2/2^*}}.
\]

In the case \(N > 5 + 2s\), it holds that
\[
Q_\mu(\eta U_\varepsilon) \leq \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_\varepsilon|^2 dxdt + c_5 \varepsilon^5 o(1) - \mu c_5 \varepsilon^{2s} + O(\varepsilon^{N-2s})}{K_1^{2/2^*} + O(\varepsilon^N)} \\
\leq \frac{K_1 + c_5 \varepsilon^5 o(1) - \mu c_5 \varepsilon^{2s} + O(\varepsilon^{N-2s})}{K_1^{2/2^*} + O(\varepsilon^N)} \\
\leq \frac{S_0 - \mu c_5 K_1^{-2/2^*} \varepsilon^{2s} + O(\varepsilon^5)}{1 + O(\varepsilon^N)}.
\]

Then
\[
Q_\mu(\eta U_\varepsilon) = S_0 - \mu \frac{c_5 \varepsilon^{2s}}{K_1^{2/2^*}} + O(\varepsilon^5) < S_0,
\]
if we take \(\varepsilon > 0\) small enough. Thus the proof of Lemma 4.1 is completed. \(\Box\)

**Lemma 4.2.** If \(0 < S_\mu < S\), then \(S_\mu\) can be achieved.

**Proof.** Let \(\{U_m\} \subset E_0\) be a minimizing sequence of \(S_\mu\), that is
\[ \|U_m^+\|_{L^{2^*}(\Omega \times \{0\})} = 1, \] (4.32)
\[ \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_m|^2 dxdt - \mu \int_{\Omega} U_m(x,0)^2 dx = S_\mu + o(1). \] (4.33)
Thus \(\{U_m\}\) is bounded in \(E_0\), we can find a subsequence of \(\{U_m\}\) (we still denote \(U_m\)) such that as \(m \to \infty\)
\[
U_m \rightharpoonup U \text{ weakly in } E_0, \\
U_m(\cdot,0) \to U(\cdot,0) \text{ in } L^2(\Omega), \\
U_m(\cdot,0) \to U(\cdot,0) \text{ a.e. in } \Omega,
\]
where \( \|U^+\|_{L^2_t(\Omega \times \{0\})} \leq 1 \). We consider a new sequence \( \{W_m\} \) with \( W_m = U_m - U \), then as \( m \to \infty \),

\[
W_m \to 0 \quad \text{weakly in } E_0, \\
W_m(\cdot,0) \to 0 \quad \text{in } L^2(\Omega), \\
W_m(\cdot,0) \to 0 \quad \text{a.e. in } \Omega.
\]

According to the definition of \( S \), we know that

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla W_m|^2 \, dx \, dt \geq S \| W_m^+ \|_{L^2_t(\Omega \times \{0\})}^2.
\]

From (4.33), we have,

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 \, dx \, dt + \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla W_m|^2 \, dx \, dt - \mu \| U \|_{L^2(\Omega \times \{0\})}^2 = S_\mu + o(1).
\]

Using Brezis-Lieb’s Lemma (See [9]), we have

\[
\| U^+ \|_{L^2_t(\Omega \times \{0\})}^2 + \| W_m^+ \|_{L^2_t(\Omega \times \{0\})}^2 + o(1) = 1,
\]

which implies

\[
\| U^+ \|_{L^2_t(\Omega \times \{0\})}^2 \geq \frac{1}{S} \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla W_m|^2 \, dx \, dt + o(1) \geq 1. \tag{4.35}
\]

We claim that

\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 \, dx \, dt - \mu \| U \|_{L^2(\Omega \times \{0\})}^2 \leq S_\mu \| U^+ \|_{L^2_t(\Omega \times \{0\})}^2. \tag{4.36}
\]

Indeed, (4.36) indicates that

\[
S_\mu \leq \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 \, dx \, dt - \mu \| U \|_{L^2(\Omega \times \{0\})}^2}{\| U^+ \|_{L^2_t(\Omega \times \{0\})}^2} \leq S_\mu.
\]

Namely,

\[
S_\mu = \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 \, dx \, dt - \mu \| U \|_{L^2(\Omega \times \{0\})}^2}{\| U^+ \|_{L^2_t(\Omega \times \{0\})}^2},
\]

that is, \( S_\mu \) is achieved by \( U \).

Thus, to finish the proof, we only need to show that (4.36) holds. In fact, by (4.35), we have

\[
S_\mu \leq S_\mu \| U^+ \|_{L^2_t(\Omega \times \{0\})}^2 + \frac{S_\mu}{S} \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla W_m|^2 \, dx \, dt + o(1). \tag{4.37}
\]

Thus combining (4.34) and (4.37) yields

\[
S_\mu \| U^+ \|_{L^2_t(\Omega \times \{0\})}^2 \geq \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 \, dx \, dt + \left( 1 - \frac{S_\mu}{S} \right) \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla W_m|^2 \, dx \, dt - \mu \| U \|_{L^2(\Omega \times \{0\})}^2 + o(1).
\]

This implies (4.36) if we take \( m \to \infty \). Hence the proof of Lemma 4.2 is completed. \( \square \)
Now we are ready to give the proof of Proposition 1.

**Proof of Proposition 1.** Let \( U \) be given by Lemma 4.2, that is \( U \in E_0 \) and
\[
\| U^+ \|_{L^2(\Omega \times \{0\})} = 1,
\]
\[
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 dx dt - \mu \| U \|^2_{L^2(\Omega \times \{0\})} = S_\mu.
\]

Assume \( U \geq 0 \) in \( \mathbb{R}^{N+1}_+ \), otherwise we can replace \( U \) by \( |U| \). Then using the Lagrange Multiplier Theorem, there exists a Lagrange multiplier \( \delta \in \mathbb{R} \) such that \( U \) satisfies
\[
\begin{cases}
\text{div}(t^{1-2s} \nabla U) = 0 & \text{in } \mathbb{R}^{N+1} = \mathbb{R}^N \times (0, \infty), \\
U = 0 & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}, \\
\partial_u U(\cdot, 0) = \delta U^{2^*_s - 1} + \mu U & \text{on } \Omega \times \{0\}, \\
U \geq 0 & \text{in } \mathbb{R}^{N+1} = \mathbb{R}^N \times (0, \infty),
\end{cases}
\]
(4.38)

A direct computation indicates that \( \delta = S_\mu \) in (4.38). Thus by choosing \( k = S_\mu^{\frac{N-2s}{\alpha^2}} \), \( kU \) satisfies (4.21) and \( U_0 := kU \) achieves \( c_0 \). Thus the proof of Proposition 1 is completed. \( \square \)

**Remark 4.** When the zero set \( \Omega = \text{int}V^{-1}(0) \) has more than one isolated components, for instance \( \Omega = \Omega_1 \cup \Omega_2 \) with \( \Omega_1 \cap \Omega_2 = \emptyset \). Suppose \( U \in N_0 \) is the least energy solution of (4.21) with \( U(x, 0) = 0 \) in \( \Omega_1 \) and \( U(x, 0) \geq 0 \) and \( U(x, 0) \neq 0 \) in \( \Omega_2 \).

Then we have
\[
(-\Delta)^s U(x, 0) = \int_{\mathbb{R}^N} \frac{U(x, 0) - U(y, 0)}{|x - y|^{N+2s}} dy < 0 \text{ in } \Omega_1.
\]

However, on the other hand
\[
(-\Delta)^s U(x, 0) = U^{2^*_s - 1}(x, 0) + \mu U(x, 0) = 0 \text{ for } x \in \Omega_1.
\]

This contradiction shows that the least energy solution \( U(x, y) \) of (4.21) satisfies \( U(x, 0) \geq 0 \) and \( U(x, 0) \neq 0 \) both in \( \Omega_1 \) and in \( \Omega_2 \). The phenomenon is totally different from the local operator Laplacian since in Laplacian case, \( u = 0 \) in \( \Omega \) immediately indicates that \( \Delta u = 0 \) in \( \Omega \) for any domain \( \Omega \). For fractional Laplacian case, it is not the case.

5. **Existence of the least energy solution.** After the above preliminaries, in this section, we are going to study the existence of the least energy solution of problem (1.1). Firstly, we show that any (P.S.) sequence of \( J_\lambda \) is bounded.

**Lemma 5.1.** There exists a positive constant \( \Lambda_0 > 0 \) such that if \( \lambda \geq \Lambda_0 \) and \( \{U_n\} \) is a (P.S.) sequence satisfying
\[
J_\lambda(U_n) \to c_\lambda, J'_\lambda(U_n) \to 0.
\]
(5.39)

Then there exists a constant \( C > 0 \) which is independent of \( \lambda \) and \( n \) such that
\[
\lim_{n \to \infty} \|U_n\|_\lambda \leq C.
\]
(5.40)
Proof. Assume \( \{U_n\} \) is a (P.S.) sequence of \( J_\lambda \), setting \( \varepsilon_n := \|\nabla J_\lambda(U_n)\| \) it follows from (5.40) that
\[
\left( \frac{1}{2} - \frac{1}{2^*} \right) \|U_n\|_{2^*}^2 = J_\lambda(U_n) - \frac{1}{2^*} J'_\lambda(U_n) \cdot U_n \\
\leq c_\lambda + o(1) + \varepsilon_n \|U_n\|_\lambda.
\]
Thus it is easy to see that there exists a constant \( C \) which is independent of \( \lambda \) and \( n \) such that \( \|U_n\|_\lambda \leq C \).

Now we show the compactness of the functional \( J_\lambda \) under certain level set. More precisely, we have the following lemma.

**Lemma 5.2.** Suppose \( \{U_n\} \) is a (P.S.) sequence of \( J_\lambda \) satisfying (5.39) with \( c_\lambda < s \text{NS}_{2^*} \) where \( \text{NS}_0 \) is the best Sobolev constant. Then there exists a subsequence of \( \{U_n\} \) which converges strongly in \( E_\lambda \) to a solution \( U_\lambda \) of (1.8) such that
\[
J_\lambda(U_\lambda) = c_\lambda.
\]

Proof. By Lemma 5.1, we know that \( \{U_n\} \) is bounded in \( E_\lambda \). Then there exists a function \( U_\lambda \in E_\lambda \) such that up to a subsequence,
\[
U_n \rightharpoonup U_\lambda \quad \text{weakly in} \quad E_\lambda,
U_n(\cdot, 0) \rightarrow U_\lambda(\cdot, 0) \quad \text{in} \quad L^2_t(\mathbb{R}^N),
U_n(\cdot, 0) \rightarrow U_\lambda(\cdot, 0) \quad \text{in} \quad L^2_{loc}(\mathbb{R}^N),
U_n(\cdot, 0) \rightarrow U_\lambda(\cdot, 0) \quad \text{a.e. in} \quad \mathbb{R}^N.
\]
It is easy to check \( J'_\lambda(U_\lambda) = 0, J\lambda(U_\lambda) \geq 0 \).

Let \( W_n = U_n - U_\lambda \), then
\[
W_n \rightarrow 0 \quad \text{weakly in} \quad E_\lambda,
W_n^+(\cdot, 0) \rightarrow 0 \quad \text{in} \quad L^2_t(\mathbb{R}^N),
W_n(\cdot, 0) \rightarrow 0 \quad \text{in} \quad L^2_{loc}(\mathbb{R}^N),
W_n(\cdot, 0) \rightarrow 0 \quad \text{a.e. in} \quad \mathbb{R}^N.
\]
Again using Brezis-Lieb’s Lemma, we can prove that \( \{W_n\} \) is also a (P.S.) sequence of \( J_\lambda \) satisfying
\[
J'_\lambda(W_n) \rightarrow 0,
\]
and
\[
\lim_{n \rightarrow \infty} J_\lambda(W_n) = c_\lambda - J_\lambda(U_\lambda) \leq c_\lambda < \frac{s}{N} S_{0}^\mathbb{R}^N.
\]
Now we prove that \( W_n \rightarrow 0 \) strongly in \( E_\lambda \). Indeed since \( J'_\lambda(W_n) \rightarrow 0 \), we only need to show that
\[
W_n^+(\cdot, 0) \rightarrow 0 \quad \text{in} \quad L^2_t(\mathbb{R}^N).
\]
We prove it by a contradiction argument and suppose on the contrary that
\[
W_n^+(\cdot, 0) \not\rightarrow 0 \quad \text{in} \quad L^2_t(\mathbb{R}^N).
\]
Without loss of generality, up to a subsequence, we assume that
\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} W_n^+(x, 0)^{2^*} \, dx = b > 0,
\]
it follows that
\[
\lim_{n \to \infty} \left( \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla W_n|^2 dx dt + \int_{\mathbb{R}^N} (\lambda V(x) - \mu)W_n(x,0)^2 dx \right)
= \lim_{n \to \infty} \int_{\mathbb{R}^N} W_n^+(x,0)^2 dx = b > 0.
\]
Since \( \{W_n\} \) is a \((P.S.)\) sequence of \( J_{\lambda} \), we have
\[
\lim_{n \to \infty} J_{\lambda}(W_n) = \lim_{n \to \infty} \left( \frac{1}{2} - \frac{1}{2t} \right) \int_{\mathbb{R}^N} W_n^+(x,0)^2 dx
= \frac{s}{N} b < \frac{s}{N} S_0^{N/2},
\]
which immediately implies
\[
b < S_0^{N/2s}. \tag{5.43}
\]
On the other hand, since
\[
\liminf_{|x| \to \infty} V(x) > 0,
\]
there exists \( R > 0 \) such that \( a_0 := \inf_{|x| \geq R} V(x) > 0 \). Thus there exists a constant \( \Lambda_0 > 0 \) such that when \( \lambda > \Lambda_0 \), \( \lambda V(x) - \mu \geq 0 \) for \( |x| \geq R \), so
\[
b = \lim_{n \to \infty} \left( \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla W_n|^2 dx dt + \int_{\mathbb{R}^N} (\lambda V(x) - \mu)W_n(x,0)^2 dx \right)
= \lim_{n \to \infty} \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla W_n|^2 dx dt + \lim_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R(0)} (\lambda V(x) - \mu)W_n(x,0)^2 dx
+ \lim_{n \to \infty} \int_{B_R(0)} (\lambda V(x) - \mu)W_n(x,0)^2 dx
\geq \lim_{n \to \infty} \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla W_n|^2 dx dt.
\]
Thus we obtain that
\[
S_0 \left( \int_{\mathbb{R}^N} W_n^+(x,0)^2 dx \right)^{2/2^*_s} \leq \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla W_n|^2 dx dt \leq b. \tag{5.44}
\]
Combining (5.42) and (5.44), we immediately have \( b \geq S_0^{N/2s} \). This contradicts with (5.43) and thus the proof of Lemma 5.2 is completed. \( \square \)

**Lemma 5.3.** For \( N > 5 + 2s \) and \( \lambda \geq \Lambda_0 \), we have \( c_{\lambda} < \frac{s}{N} S_0^{\frac{N}{s}} \), where \( c_{\lambda} \) is defined as in (1.9) and \( \Lambda_0 \) is taken as in Lemma 5.7.

**Proof.** By the definition of
\[
c_{\lambda} := \inf_{U \in E_{\lambda \setminus \{0\}}} \left( \frac{1}{2} - \frac{1}{2s} \right) \left( \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla U|^2 dx dt + \int_{\mathbb{R}^N} (\lambda V(x) - \mu)U(x,0)^2 dx \right),
\]
\[
c_0 := \inf_{U \in E_{0 \setminus \{0\}}} \left( \frac{1}{2} - \frac{1}{2s} \right) \left( \int_{\mathbb{R}^{N+1}} t^{1-2s} |\nabla U|^2 dx dt - \mu \int_{\mathbb{R}^N} U(x,0)^2 dx \right),
\]
and the fact that \( E_0 \subseteq E_{\lambda} \), we know that \( c_{\lambda} \leq c_0 \), thus to complete the proof, we only need to show that \( c_0 < \frac{s}{N} S_0^{\frac{N}{s}} \).
Let $U$ be the one in the proof of Proposition 1. Then the function $S^{N+2s}_\mu U \in \mathcal{N}_0$, and thus,  
\[ c_0 \leq I(S^{N+2s}_\mu U) = \frac{s}{N} S^{N}_\mu. \]
Since it has been proved in Lemma 4.1 that $S_\mu < S_0$, we have that $c_0 < \frac{s}{N} S^{N}_0$. \(\square\)

6. Asymptotic behavior of the least energy solutions. In this section, we study the asymptotic behavior of the least energy solutions of (1.1) as $\lambda \to \infty$. Firstly, we give the asymptotic behavior for $c_\lambda$ as $\lambda \to \infty$.

**Lemma 6.1.** $c_\lambda \to c_0$ as $\lambda \to \infty$.

**Proof.** By the definition of $c_\lambda$ and $c_0$, it is easy to see that $c_\lambda \leq c_0$ for all $\lambda \geq 0$. It is easy to show that $c_\lambda$ is monotone increasing of $\lambda$. We suppose on the contrary that there is a sequence $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$ such that

\[ \lim_{n \to \infty} c_{\lambda_n} = k < c_0. \]

Thus $k > 0$ and we take $U_n \in \mathcal{M}_{\lambda_n}$ as the least energy solution of problem (1.8) with $\lambda$ being replaced by $\lambda_n$ satisfying

\[ J_{\lambda_n}(U_n) = c_{\lambda_n}. \]

By a standard argument, it is easy to see that the norms $\|U_n\|_{\lambda_n}$ in $E_{\lambda_n}$ is bounded, which implies $\{U_n\}$ is bounded in $E$. As a consequence, up to a subsequence (we still denote $U_n$), there exists $U \in E$ such that

\[ U_n \rightharpoonup U \text{ weakly in } E, \]

\[ U_n(\cdot, 0) \to U(\cdot, 0) \text{ in } L^2(\mathbb{R}^N), \]

\[ U_n(\cdot, 0) \to U(\cdot, 0) \text{ in } L^2_{loc}(\mathbb{R}^N), \]

\[ U_n(\cdot, 0) \to U(\cdot, 0) \text{ a.e. in } \mathbb{R}^N. \]

We claim that $U(\cdot, 0)|_{\Omega^c} = 0$ and hence $U \in E_0$, where $\Omega^c := \mathbb{R}^N \setminus \Omega$. Indeed, if $U(\cdot, 0)|_{\Omega^c} \neq 0$, there exists a compact subset $F \subset \Omega^c$ with $\text{dist}(F, \Omega) > 0$ such that $U(\cdot, 0)|_F \neq 0$ and

\[ \int_F U_n(x, 0)^2 dx \to \int_F U(x, 0)^2 dx > 0. \]

From the fact that $\Omega = \text{int}V^{-1}(0)$, there exists $a_0 > 0$ such that $V(x) \geq a_0 > 0$ for all $x \in F$, which implies that

\[ J_{\lambda_n}(U_n) = \frac{1}{2} \int_{\mathbb{R}^N+1} t^{1-2s}|\nabla U_n|^2 dt + \frac{1}{2} \int_{\mathbb{R}^N} (\lambda_n V(x) - \mu)U_n(x, 0)^2 dx \]

\[ - \frac{1}{2s} \int_{\mathbb{R}^N} U_n^+(x, 0)^{2s} dx \]

\[ = \frac{s}{N} \int_{\mathbb{R}^N+1} t^{1-2s}|\nabla U_n|^2 dt + \frac{s}{N} \int_{\mathbb{R}^N} (\lambda_n V(x) - \mu)U_n(x, 0)^2 dx \]

\[ \geq \frac{s}{N} \int_F (\lambda_n V(x) - \mu)U_n(x, 0)^2 dx \]

\[ \geq \frac{s}{N} (\lambda_n a_0 - \mu) \int_F U_n(x, 0)^2 dx \]

\[ \to \infty \text{ as } n \to \infty. \]
This is a contradiction and thus $U(\cdot,0)|_{\Omega^c} = 0$. 
Next we show that $U_n^+(\cdot,0) \to U^+(\cdot,0)$ strongly in $L^{2s}(\mathbb{R}^N)$. Note that 
\[ c_{\lambda_n} \leq c_0 < \frac{s}{N} S_0^{\frac{2s}{N}}. \]
Now we take $W_n = U_n - U$ and assume on the contrary that 
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} W_n^+(x,0)^{2s} dx = b > 0. \]
As is done in the proof of Lemma 5.2, we can prove that $b \geq S_0^{N/2s}$, and it follows that 
\[ k = \lim_{n \to \infty} c_{\lambda_n} \geq \frac{s}{N} S_0^{\frac{2s}{N}}. \]
This is also a contradiction. Therefore the strong convergence of $U_n^+(\cdot,0) \to U^+(\cdot,0)$ in $L^{2s}(\mathbb{R}^N)$ is proved. Moreover 
\[ \liminf_{n \to \infty} \int_{\mathbb{R}^N} t^{1-2s}|\nabla U_n|^2 dx dt \geq \int_{\mathbb{R}^N} t^{1-2s}|\nabla U|^2 dx dt. \]
Since $U_n$ is a solution of problem (1.8), where $\lambda$ is instead of $\lambda_n$, hence 
\[ J_{\lambda_n}(U_n) \cdot U_n = \int_{\mathbb{R}^N} t^{1-2s}|\nabla U_n|^2 dx dt + \int_{\mathbb{R}^N} (\lambda_n V(x) - \mu) U_n(x,0)^2 dx \]
\[ - \int_{\mathbb{R}^N} U_n^+(x,0)^{2s} dx \to 0, \]
which implies that for $n$ large enough 
\[ \int_{\mathbb{R}^N} U^+(x,0)^{2s} dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} U_n^+(x,0)^{2s} dx \]
\[ = \lim_{n \to \infty} \int_{\mathbb{R}^N} t^{1-2s}|\nabla U_n|^2 dx dt + \lim_{n \to \infty} \int_{\mathbb{R}^N} (\lambda_n V(x) - \mu) U_n(x,0)^2 dx \]
\[ \geq \lim_{n \to \infty} \int_{\mathbb{R}^N} t^{1-2s}|\nabla U_n|^2 dx dt + \lim_{n \to \infty} \int_{\mathbb{R}^N} U_n(x,0)^2 dx \]
\[ \geq \int_{\mathbb{R}^N} t^{1-2s}|\nabla U|^2 dx dt + \int_{\mathbb{R}^N} U(x,0)^2 dx. \]
This implies 
\[ \int_{\mathbb{R}^N} U^+(x,0)^{2s} dx \geq \int_{\mathbb{R}^N} t^{1-2s}|\nabla U|^2 dx dt + \int_{\mathbb{R}^N} U(x,0)^2 dx. \]
As a consequence of $U(\cdot,0)|_{\Omega^c} = 0$, we have 
\[ \int_{\mathbb{R}^N} t^{1-2s}|\nabla U|^2 dx dt - \mu \int_{\Omega} U(x,0)^2 dx \]
\[ \leq \int_{\mathbb{R}^N} t^{1-2s}|\nabla U|^2 dx dt + \int_{\Omega} U(x,0)^2 dx \leq \int_{\Omega} U^+(x,0)^{2s} dx. \]
Then there exists $\alpha \in (0,1]$ such that $\alpha U \in \mathcal{N}_0$, that is 
\[ \int_{\mathbb{R}^N} t^{1-2s}|\nabla \alpha U|^2 dx dt - \mu \int_{\Omega} |\alpha U(x,0)|^2 dx = \int_{\Omega} |\alpha U^+(x,0)|^{2s} dx. \]
Furthermore,
\[ I(\alpha U) = \left( \frac{1}{2} - \frac{1}{2s} \right) \left( \int_{\mathbb{R}^{N+1}} t^{1-2s}|\nabla U|^2 dx dt - \int_{\Omega} \mu_0 |\alpha U(x,0)|^2 dx \right) \]
\[ \leq \frac{s}{N} \liminf_{n \to \infty} \left( \int_{\mathbb{R}^{N+1}} t^{1-2s}|\nabla U_n|^2 dx dt + \int_{\mathbb{R}^N} (\lambda_n V(x) - \mu) U_n(x,0)^2 dx \right) \]
\[ = k. \]
This implies \( k \geq c_0 \), this is a contradiction. Hence we proved that
\[ \lim_{\lambda \to \infty} c_\lambda = c_0 \]
and the proof of Lemma 6.1 is completed.

7. **Proof of the main results.** Now we give the proof of our main results.

**Proof of Theorem 1.2.** Combining Lemma 5.2 and Lemma 5.3 we have proved that for \( \lambda \) large, (1.8) has a least energy solution \( U_\lambda \) corresponding to \( c_\lambda \) and hence (1.1) has a least energy solution \( U_\lambda = U_\lambda (x,0) \). For any sequence \( \{\lambda_n\} (\lambda_n \to \infty \text{ as } n \to \infty) \), we denote by \( U_n \in E_{\lambda_n} \) the corresponding solution of (1.8) such that \( J_{\lambda_n} (U_n) = c_{\lambda_n} \). We will show that \( U_n \) converges (or along a subsequence when necessary) to a least energy solution \( U \) of (1.11) in \( E \).

Firstly, it is easy to see that \( \{U_n\} \) is bounded in \( E \). Therefore, we may assume that subject to a subsequence,
\[ U_n \to U \text{ in } E, \]
\[ U_n(\cdot ,0) \to U(\cdot ,0) \text{ in } L^2_t(\mathbb{R}^N), \]
\[ U_n(\cdot ,0) \to U(\cdot ,0) \text{ a.e. in } \mathbb{R}^N. \]
As is done in the proof of Lemma 6.1, we can obtain that \( U(\cdot ,0)|_{\partial \Omega} = 0 \), which implies that \( U \in E_0 \). On the other hand, with a similar argument as in the proof of Lemma 5.2, we also can prove that \( U_n(\cdot ,0) \to U(\cdot ,0) \) strongly in \( L^2_t(\mathbb{R}^N) \). Then it suffices to show that
\[ \lambda_n \int_{\mathbb{R}^N} V(x) U_n(x,0)^2 \to 0 \]
and
\[ \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_n|^2 dx dt \to \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 dx dt. \]
Suppose that either
\[ \limsup_{n \to \infty} \lambda_n \int_{\mathbb{R}^N} V(x) U_n(x,0)^2 dx > 0 \]
or
\[ \limsup_{n \to \infty} \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U_n|^2 dx dt > \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 dx dt. \]
In both cases, we can get that
\[ \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2 dx dt - \mu \int_{\Omega} U(x,0)^2 dx < \int_{\Omega} U^+(x,0)^{2^*_s} dx. \]
Then there exists \( \alpha \in (0,1) \) such that \( \alpha U \in \mathcal{N}_0 \), that is
\[ \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla(\alpha U)|^2 dx dt - \mu \int_{\Omega} U(x,0)^2 dx = \int_{\Omega} |\alpha U^+(x,0)|^{2^*_s} dx. \]
Furthermore,

$$I(\alpha U) = \left( \frac{1}{2} - \frac{1}{2s} \right) \left( \int_{\mathbb{R}^N} t^{1-2s}|\nabla\alpha U|^2 dx + \int_{\Omega} \mu |\alpha U(x,0)|^2 dx \right)$$

$$\leq \frac{s}{N} \left( \int_{\mathbb{R}^N} t^{1-2s}|\nabla U|^2 dx - \int_{\mathbb{R}^N} \mu U(x,0)^2 dx \right)$$

$$\leq \frac{s}{N} \liminf_{n \to \infty} \left( \int_{\mathbb{R}^{N+1}} t^{1-2s}|\nabla U_n|^2 dx + \int_{\mathbb{R}^N} (\lambda_n V(x) - \mu) U_n(x,0)^2 dx \right)$$

$$= \lim_{n \to \infty} J_{\lambda_n}(U_n) = c_0.$$ 

This is a contradiction. Then we have proved that the least energy solution $U_n$ of (1.8) converges (or along a subsequence when necessary) to a least energy solution $U$ of (1.11) in $E$ and hence the proof of Theorem 1.2 is completed.

**Proof of Theorem 1.3.** Suppose $u_n = U_n(x,0) \in H^s(\mathbb{R}^N)$ is a solution of (1.1), where $\lambda$ is replaced by $\lambda_n$. Similarly, such a sequence $U_n$ is also bounded in $E$, we may assume, going if necessary to a subsequence, that $u_n(x) \to U(x,0)$ in $L^2(\mathbb{R}^N)$. As in the proofs of Lemma 6.1 and Lemma 5.2, we can prove that $U \in E_0$ and $U_n^+(x,0) \to U^+(x,0)$ strongly in $L^2(\mathbb{R}^N)$. By Brezis-Lieb’s Lemma, we have

$$\int_{\mathbb{R}^N} t^{1-2s}|\nabla(U_n - U)|^2 dx + \int_{\mathbb{R}^N} (\lambda_n V(x) - \mu) U_n(x,0) - U(x,0)|^2 dx$$

$$= \int_{\mathbb{R}^N} t^{1-2s}|\nabla U_n|^2 dx + \int_{\mathbb{R}^N} (\lambda_n V(x) - \mu) U_n(x,0)^2 dx$$

$$- \int_{\mathbb{R}^N} t^{1-2s}|\nabla U|^2 dx - \int_{\mathbb{R}^N} (\lambda_n V(x) - \mu) U(x,0)^2 dx + o(1)$$

$$= \int_{\mathbb{R}^N} U_n^+(x,0)^2 dx - \int_{\Omega} U^+(x,0)^2 dx + o(1) = o(1).$$

Then the solution $U_n$ of (1.8) converges (or along a subsequence when necessary) to a solution $U$ of (1.11) in $E$. This implies that $u_n(x) = U_n(x,0)$ converges strongly along the subsequence in $H^s(\mathbb{R}^N)$ to a solution of (1.10). The proof of Theorem 1.3 is completed.

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