A Simple Volcano Potential with an Analytic, Zero-Energy, Ground State

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ABSTRACT

We describe a simple volcano potential, which is supersymmetric and has an analytic, zero-energy, ground state. (The KK modes are also analytic.) It is an interior harmonic oscillator potential properly matched to an exterior angular momentum-like tail. Special cases are given to elucidate the physics, which may be intuitively useful in studies of higher-dimensional gravity.

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1 Introduction

There has been recent excitement about the observation that extra dimensions in gravity could be of macroscopic size and yet not be in conflict with experiment [1]. This led to the proposal [2] of a scenario where one can have extra dimensions that are not compactified at all, but yet have negligible effect on normal-scale gravitational physics [2]. In one extra dimension, the problem effectively reduces to a Schrödinger-like equation for the graviton mass squared, $m^2$.

If such a model can yield a zero-energy ground state which is bound, then the massless graviton can be confined to a local region even though its quantum equation in the higher dimension is valid over the whole line:

$$\left[-\frac{d^2}{dx^2} + V(x)\right] \psi_m(x) = m^2,$$

(1)

One of the prime candidates for such a system is a “volcano” potential [2, 3].

Such a model should also be supersymmetric. Recall that to be supersymmetric [4], the zero-energy ground state and the potential are given by

$$\psi_0(x) = N \exp[-W(x)],$$

(2)

$$V(x) = [W'(x)]^2 - W''(x),$$

(3)

where $N$ is the normalization constant and $W$ is the superpotential.

A fair amount of effort has gone into studies of potentials with the above properties [2, 3, 4, 5]. These ‘volcano-like” potentials include i) a polynomial tail with a negative delta function at the origin [2]; ii) one from a superpotential $W = \ln[k^2x^2 + 1]$ [3]; and iii) one composed of a square-well box at the origin matched to two square steps on the sides [3]. Of course, none of these is a perfect toy model. The best understood analytically are the delta function and box potentials. (The latter, however, does not come from a superpotential).

Here, we will construct a supersymmetric potential which is simple and has a simple, exactly analytic solution for the zero-energy ground state. The excited states are complicated, but are known special functions of mathematical physics. Perhaps such a potential can be useful in studying the physics of unconfined extra dimensions.
2 The potential

The idea is to match, in the proper manner, a simple, exactly solvable, central, crater potential to a simple, exactly solvable, tail potential. The “appropriate match” and “simple” will yield a particular step function jump from the crater to the tail and a particular matching point.

Motivated by physical intuition and work on $E = 0$ bound state solutions for singular potentials [7, 8], we take the exterior potential to be an “angular-momentum-like” tail. This is the tail with the simplest analytic behavior. We take the central potential to be a harmonic oscillator. This is also the simplest analytically and somewhat intuitive in its universality.

Therefore, we consider

$$V(x) = V_i(x) = \omega^2 x^2 - \omega, \quad |x| < x_0,$$

$$V_e(x) = \frac{c(c+1)}{x^2}, \quad |x| > x_0.$$  

For the external region, which goes to infinity, we take the regular solution. From Eqs. (2), (3), and (5), one has

$$W_e(x) = c \ln x - b, \quad |x| > x_0,$$

$$\psi_{0,e}(x) = \frac{N e^b}{x^c}, \quad |x| > x_0.$$  

The constant $b$ is a relative normalization which will affect the location of the matching point and the relative couplings in the two regions. We only need one such constant, so none will be given for the internal region. For the internal region, we assume for now that one can similarly take the regular solution. Therefore,

$$W_i(x) = \frac{1}{2} \omega x^2, \quad |x| < x_0,$$

$$\psi_{0,i}(x) = N \exp \left[ -\frac{1}{2} \omega x^2 \right], \quad |x| < x_0.$$  

The superpotential, $W$, and its first derivative, $W'$, should also be continuous. From Eq. (2) and its derivative, this requirement is equivalent to the usual quantum-mechanical boundary conditions. Of course, one could consider other tail or other crater potentials, like a different inverse power of $x$ for the tail and/or the $|\cosh^{-2} x|$ potential for the crater.

We are ignoring the fact that one needs to handle the sign of $x$ in the negative-$x$ region carefully. However, since this problem is symmetric, there is no real concern. We just have to note that the phase in the wave function disappears if we let ($-x \to x$) in the original differential equation.
conditions that $\psi_0(x)$ and $\psi'_0(x)$ are continuous. Because we want $W$ and $W'$ to be continuous, we must carefully adjust the model at the meeting points of the regions, $\pm x_0$.

Performing the matching, one finds that $x_0$ is determined by two relationships among $c$, $b$, $\omega$, and $x_0$.

$$x_0 = \exp \left[ \frac{1}{2} + \frac{b}{\omega x_0^2} \right] = \exp \left[ \frac{1}{2} + \frac{b}{c} \right],$$

$$\omega = \frac{c}{x_0} = \exp \left[ 1 + \frac{2b}{c} \right].$$

Therefore, the general result is given by the wave functions of Eqs. (7) and (9), the matching conditions (10) and (11), and the normalization constant

$$N(b, c) = \frac{2}{x_0^{(2c-1)(2c-1)}} e^{2b} + x_0 e \sqrt{(\pi/c) \operatorname{erf}(\sqrt{c})}^{-1/2}. \quad (12)$$

As indicated, $V(x_0)$ is discontinuous. In particular, the potential starts at $V_i(0) = -\omega$ and rises in a parabola until $V_i(x_0) = c(c - 1)/x_0^2 = (c - 1)\omega$. It then discontinuously jumps to $V_e(x_0) = c(c + 1)/x_0^2 = (c + 1)\omega$, and finally falls off quadratically. $V$ being discontinuous means $W''$ is discontinuous which means the second derivative of $\psi_0(x_0)$ is discontinuous. However, this is all normal in quantum mechanics. Recall, for example, the particle in a box.

### 3 Examples

The natural meeting point of the two functional forms is given by

$$V_i(x_{0,nat}) = V_e(x_{0,nat}),$$

$$x_{0,nat} = \left[ \frac{2c(c + 1)}{\omega^2} \right]^{1/4}. \quad (13)$$

So why can this not be used, allowing our potential to be continuous?

The reason is our assumption that the interior solutions are regular (and therefore, simple, as we want). To make the potentials match at the $x_0$ of Eq. (14), one also would have had to use the irregular, interior solution of the Schrödinger equation. That is the parabolic cylinder function $D_{-1}(i\sqrt{2\omega} x) \quad [10]$. This would have made $W$ and $\psi_0$ more complicated, defeating
part of the purpose of this exercise. With our construction only the regular Gaussian solution, 
\( D_0(\sqrt{2\omega}x) \), is involved.

**The case \((b,c) = (0,1)\):** Having understood this, for a first example we look at the most elegant and simple special case. It is 
\[
(b,c) = (0,1), \quad \Rightarrow \quad x_0 = e^{1/2}, \quad \omega = e^{-1}, \quad N(0,1) = 0.582167. \tag{15}
\]
In Figure 1 we show the potential and the bound, \( E = 0 \), ground-state wave function for this \((b,c) = (0,1)\) case.

**The case \((b,c) = (3/2,3)\):** For comparison, in Figure 2 we show the case
\[
(b,c) = (3/2,3), \quad \Rightarrow \quad x_0 = e, \quad \omega = \frac{3}{e^2}, \quad N(0,1) = 0.598038. \tag{16}
\]
Observe that \( b \neq 0 \) means that \( x_0 \neq e^{1/2} \).

Given the height of the volcano peaks (this figure has a different vertical scale than the first), one can also ask about resonant states. They will be near the energies of the ordinary harmonic oscillator excited states, \( 2n\omega \). The resonant states will not be exactly at \( 2n\omega \) because the tail potential evolves wave functions differently than the full harmonic oscillator potential would. Further, as they first appear the widths of the resonances are broad. They only become narrow as the volcano wall becomes higher. One can approximately say that the number of resonant states is
\[
n \sim [(c + 1)/2], \quad c \geq 1. \tag{17}
\]

The case \((b,c) = (0,1)\) has the first resonant state about to appear and the case \((b,c) = (3/2,3)\) has the second resonant state about to appear. These resonant states and the rest of the continuum states are solvable, although complicated. In the interior one will obtain parabolic cylinder functions and in the exterior one will obtain Bessel functions. In this scenario the resonances do not occur at zero energy. (That would have been interesting as a model of higher-dimensional gravity whose effects reappear at large distances \([11]-[13]\).) To yield a zero-energy resonance would necessitate that the potential strictly go to zero at large distances.

By contrast, this type of system can yield unbound, zero-energy ground states.\(^4\) For \( c < 1/2,\)

\[^4\] Unbound \( E = 0 \) solutions were also discussed in Ref. \([\text{Ref.}]\).
Figure 1: For the case \((b, c) = (0, 1)\), we show the volcano potential (continuous line) and the wave function (dashed line) for the \(E = 0\) bound state.

Figure 2: For the case \((b, c) = (3/2, 3)\), we show the volcano potential (continuous line) and the wave function (dashed line) for the \(E = 0\) bound state.
the normalization of the wave function becomes power-wise divergent. In particular, note that the zero-energy bound/unbound regimes are bordered by the case \( c = 1/2 \).

![Graph](image)

Figure 3: For the case \((b, c) = (0, 1/2)\), we show the volcano potential (continuous line) and the wave function (dashed line) for the \( E = 0 \) state, which is logarithmically divergent. It is arbitrarily set so that \( \psi_{0,i}(0) = N(0, 1) = 0.582167 \).

**The case \((b, c) = (0, 1/2)\):** Here the normalization of the exterior wave function is logarithmically divergent. That \( c = 1/2 \) is the boundary between normalized and unnormalized ground states can also be seen by the factor \( 2/(2c-1) \) in the normalization constant of Eq. (12).

To visualize the broad nature of the wave function, in Fig. 3 we show the case \((b, c) = (0, 1/2)\). Since the wave function is not normalized, we set \( \psi_{0,i}(0) = N(0, 1) = 0.582167 \). Once again, \( x_0 = e^{1/2} \). This is an interesting contrast to the first case.

Perhaps this \( c = 1/2 \) case is of interest on its own for large scale gravity.

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