CONFORMAL KILLING VECTORS IN FIVE-DIMENSIONAL SPACE.

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Abstract

The solutions of generalized Killing equation have been obtained for line element with initial $t^2 \oplus so(3)$ symmetry. The coefficients of the metric $g$ corresponding to these vector fields are written down.

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1 Introduction

Conformal Killing vectors for line element (1) are being studies in the paper. In the differential geometry the invariance of some tensor field $T$ describes the conservation laws in the presence of the Killing vectors. The conservation laws for traceless tensors $T$ are connected with the conformal Killing vectors. Such fields are characterized by conserved quantities, whose number is equal to that of conformal Killing vectors [1]. In classical works [2] the conformal Killing vectors in the Riemannian space were investigated from the viewpoint of general principles. The role played by the vector fields in classifying the solutions of the gravitation equations [3], as well as in other applications [4] is well known.

In works [5] concrete series of isometric Killing vectors, derived from four-dimensional line element under its certain parameterization, were studied. A similar problem was solved in refs. [6,7], where isometry group generators in five-dimensional space were obtained for some line elements. In Kaluza-Klein type theory the five-dimensional space is distinguished due to the fact that out of all multi-dimensional spaces, just in this space the total manifold $M^5$ together with Kaluza-Klein type manifold $M^4 \times S^1$ satisfy classical Einstein equations [8]. Note, that the solution of the system of equations (3) for conformal Killing vectors brings us to more clear understanding of the nature of the restrictions on $g_{nn}$ for isometry groups obtained in refs. [6,7].

So, we study a five-dimensional space with the metric

$$
\text{ds}^2 = g_{11}(r)dt^2 - g_{22}(r)dr^2 - 4g_{33}(r)(\sigma_x^2 + \sigma_y^2) - 4g_{55}(r)\sigma_z^2,
$$

written down in the Cartan’s basis. Here $g_{11}$, $g_{22}$, $g_{33}$, and $g_{55}$ are the functions of the variable $r$; $d\sigma_x = 2\sigma_y \wedge \sigma_z$ (cyclic). Having introduced the variable

$$
z = \int \sqrt{g_{22}} dr
$$

we rewrite the line element in the form convenient for further studies:

$$
\text{ds}^2 = g_{11}(z)dt^2 - dz^2 - g_{33}(z)d\vartheta^2 - (g_{33}(z)\sin^2\vartheta + g_{55}(z)\cos^2\vartheta)d\varphi^2 - 2g_{55}(z)\cos\vartheta d\varphi d\psi - g_{55}(z) d\psi^2.
$$

In the Riemannian manifold, with the connection that is consistent with the metric $\tilde{g}$, the generalized Killing equation for the component $\xi^k$ of the vector $\zeta = \xi^k \partial_k$ looks like:

$$
\xi^k \partial_k \tilde{g}_{ij} + \tilde{g}_{ik} \partial_j \xi^k + \tilde{g}_{kj} \partial_i \xi^k = \Phi \tilde{g}_{ij}.
$$

Here $\partial_t = \frac{\partial}{\partial t}$, $\partial_r = \frac{\partial}{\partial z}$, $\partial_\vartheta = \frac{\partial}{\partial \vartheta}$, $\partial_\varphi = \frac{\partial}{\partial \varphi}$, $\partial_\psi = \frac{\partial}{\partial \psi}$.

Our goal is to obtain all series of vectors, that give the conformal motions in space (3). At the same time, the highest symmetries (as compared with the initial symmetry $L = t^2 \oplus so(3)$ of the line element) take place under certain restrictions imposed on $g_{nn}(z)$. In most cases these restrictions are explicitly solved w.r.t. $g_{nn}$, which gives us a possibility to write down suitable $g_{nn}$, that ensure the existence of the given series of vectors.
2 Solution of the System

Let us slightly transform the equations for \( \tilde{g}_{14} \) and \( \tilde{g}_{45} \) in system (3) so that in the R.H.S. of the equation, that corresponds to the component \( \tilde{g}_{14} \), we would have \( \frac{1}{2} \Phi \), and the R.H.S. of the equation for the component \( \tilde{g}_{45} \) would be equal to zero. Taking into account this modification, the equations in system (3) will be numerated by the indices \( \tilde{g}_{ij} \) : \((i, j) \). Besides we shall introduce notations for the additional equations obtained by subtracting from each other the equations, containing in the R.H.S. \( \Phi \) (i.e. the equations of the type \((n, n)\)) : 
\[
(i, k) = (i, i) - (k, k).
\]

One of the possible ways to solve system (3) is to apply the integrability conditions. Let us attempt to maximally simplify the procedure. By differentiating the equations with wave \( \partial_k \): \( \partial_k(i, n) \), where \( i \neq k, n \neq k \), \( i, k, n = 1, 2, 3, 5 \) and using the remaining equations \((j, l)\) of system (3), we shall obtain pairs of equations of the third order for the quantity \( \xi^k \), \( k = 1, 2, 3, 5 \):
\[
\begin{align*}
\partial_{ii}\xi^2 + g_{ii}(z)R_i(z)\partial_i\xi^2 &= 0, \\
\partial_{ii}\xi^2 + g_{ii}(z)\hat{R}_i(z)\partial_i\xi^2 &= 0, \quad i = 1, 3, 5;
\end{align*}
\]
and
\[
\begin{align*}
\partial_{222}\tilde{\xi}^i + 3F_i\partial_{22}\tilde{\xi}^i + \{(F_i' + 2F_i^2) + \hat{\gamma}R_i(z)\}\partial_2\tilde{\xi}^i &= 0, \\
\partial_{222}\tilde{\xi}^i + 3F_i\partial_{22}\tilde{\xi}^i + \{(F_i' + 2F_i^2) + \hat{\gamma}\hat{R}_i(z)\}\partial_2\tilde{\xi}^i &= 0, \quad i = 1, 3, 5; \\
\tilde{\xi}_k &= \xi^k, \quad k = 1, 3; \quad \xi^5 = \xi^5 + \cos \theta \xi^4.
\end{align*}
\]

Here \( \hat{\gamma} = 1 \) for \( i = 1 \) and \( \hat{\gamma} = -1 \) for \( i = 3, 5 \); \( F_i = \frac{g_i' g_{ii}}{2g_{ii}} \), \( i = 1, 3, 5 \). Here and in what follows the prime will denote the derivative over the variable \( z \). The quantities \( R_i \) and \( \hat{R}_i \) have the following form:
\[
\begin{align*}
R_i(z) &= (F_i - F_3)' + F_3(F_i - F_3), \\
\hat{R}_i(z) &= (F_i - F_5)' + F_5(F_i - F_5); \\
R_3(z) &= (F_1 - F_3)' + F_1(F_1 - F_3), \\
\hat{R}_3(z) &= \frac{1}{2}\{(F_5 - F_3)' + F_3(F_5 - F_3) + \frac{1}{g_{33}}(1 - \frac{g_{55}}{2g_{33}})\}; \\
R_5(z) &= (F_1 - F_5)' + F_1(F_1 - F_5), \\
\hat{R}_5(z) &= (F_3 - F_5)' + F_3(F_3 - F_5) + \frac{g_{55}}{4g_{33}}.
\end{align*}
\]

From formulae (4) and (5) we have:
\[
(R_i - \hat{R}_i)\partial_i\xi^2 = (R_i - \hat{R}_i)\partial_2\tilde{\xi}^i = 0, \quad i = 1, 3, 5.
\]

Besides from the combination of equations \((4)\) and \((5)\) there follows the restriction for \( \hat{R}_i \). The function \( \hat{R}_i \) should satisfy the condition
\[
\hat{R}_i = \frac{\text{const}}{g_{ii}},
\]
where \( \text{const} \) may also be zero.

The following remark should be made: from system (3) one can easily find out, that the equality to zero of the derivative \( \partial_2 \xi^i = 0 \) (or \( \partial_j \xi^2 = 0 \)), where \( i \) takes one of three values \( i = 3, 4, 5 \), entails the equality to zero of other similar quantities. Hence we have

\[
\partial_2 \xi^i = 0 \Rightarrow \{ \partial_2 \xi^i = 0 \} \quad \text{or} \quad \partial_j \xi^2 = 0, \quad \partial_j \xi^2 = 0, \quad \partial_k \xi^2 = 0, \quad \partial_2 \xi^k = 0; \quad i, j, k = 3, 4, 5. \tag{10}
\]

This observation greatly simplifies the problem, only demanding simultaneous dependence of \( \xi^2 \) on \( \theta, \varphi \) and \( \psi \). Formula (3) shows, that in the case of \( \xi^2 = \xi^2(\theta, \varphi, \psi) \) both equations \( R_3 = \dot{R}_3 \) and \( R_5 = \dot{R}_5 \) should only be fulfilled together.

First we shall look for the explicit dependence of the components \( \xi \), \( \xi^3 \) and \( \xi^5 \) on the variables \( z, \vartheta, \varphi \) and \( \psi \), i.e. we shall solve the equations (i, k), (i, k), where \( i, k \neq 1 \). Formulae (8), (9) and (10) make it clear that a nontrivial solution is possible only in the following cases:

either I. \( \partial_i \xi^2 = \partial_2 \xi^i = 0, \quad i = 3, 4, 5; \tag{11} \)

or II. \( R_3 = \dot{R}_3, \quad R_5 = \dot{R}_5, \) and

a). \( R_i = 0, \quad i = 3, 5; \) or

b). \( R_i = \rho_{ii}, \quad i = 3, 5, \) \( \rho_{ii} = \text{const} \neq 0. \tag{13} \)

In the simplest case I, when \( \xi^2 = \xi^2(t, z) \) and as a consequence equalities (12) are not fulfilled, conditions (11) bring us to three different series \( V_a, V_b \) and \( V_c \) for the components \( \xi^2 \) and \( \xi^3 \) for each of these series.

\[ V_a : \quad \{ F_3 - F_5 \neq 0, \quad R_3 \neq \dot{R}_3, \quad R_5 \neq \dot{R}_5 \} \]

\[ \xi^2 = 0, \quad \xi^3 = \xi^3_1(t) \sin \varphi + \xi^3_2(t) \cos \varphi. \]

\[ V_b : \quad \{ g_{33} = \tau_3 g_{55}, \quad \tau_3 = \text{const} \neq 1 \} \]

\[ \xi^2 = \sqrt{g_{33}} \xi^2_1(t), \quad \xi^3 = \xi^3_1; \quad i = 3, 4, 5. \]

\[ V_c : \quad \{ g_{33} = g_{55}, \quad R_3 \neq \dot{R}_3 \} \]

\[ \xi^2 = \sqrt{g_{33}} \xi^2_1(t), \quad \xi^3 = \xi^3_1 + \xi^3_2(t) \sin \psi - \xi^3_4(t) \cos \psi. \]

As for the second possibility, described with formula (12), we must consider four different cases: i) \( R_3 = R_5 = 0 \), ii) \( R_3 = 0, \quad R_5 = \rho_{33}, \) iii) \( R_3 = \rho_{33}, \quad R_5 = 0 \) and iv) \( R_3 = \rho_{33}, \quad R_5 = \rho_{55}. \)

The key component here is \( \xi^2 \). The dependence on \( \vartheta \) and \( \psi \) is fixed by equations (11). It is convenient to derive the dependence \( \xi^2 \) on the variable \( z \) from the analysis of equations (2, 3), (2, 5) and (3, 5). Using (2, 3) (2, 5), one can bring them to the form

\[
\partial_{22} \xi^2 - \partial_2 (F_3 \xi^2) + \frac{1}{g_{33}} \partial_{33} \xi^2 = 0, \tag{14}
\]

\[
\partial_{22} \xi^2 - \partial_2 (F_5 \xi^2) + \frac{1}{g_{55}} \partial_{55} \xi^2 = 0,
\]

\[
\partial_2 \{ (F_3 - F_5) \xi^2 \} - \frac{1}{g_{33}} \partial_{33} \xi^2 + \frac{1}{g_{55}} \partial_{55} \xi^2 = 0.
\]
However it can easily be seen that these equations for conditions i), ii) and iii) are consistent only in the trivial case \( \xi^2 \neq \xi^2(\vartheta, \varphi, \psi) \) and result in the series \( V_a, V_b, V_c \).

In case iv), \( R_3 = \frac{\rho_{33}}{g_{33}} \) and \( R_5 = \frac{\rho_{55}}{g_{55}} \), let \( \xi^2 \) be presented in the following way

\[
\xi^2(t, z, \vartheta, \varphi, \psi) = \sum_{i, k=0}^{2} \xi^{2(ik)}(t, z, \varphi)m_i(\vartheta)f_k(\psi). \tag{15}
\]

The explicit dependence on the variables \( \vartheta \) and \( \psi \) is given by the functions

\[
m_0 = 1, \quad m_1 = \begin{cases} \sin \sqrt{\rho_{33}} \vartheta, & \text{for } \rho_{33} > 0 \\ \frac{\cos \sqrt{\rho_{33}} \vartheta}{\sqrt{|\rho_{33}|}} \vartheta, & \text{for } \rho_{33} < 0 \end{cases}, \quad m_2 = \begin{cases} \cos \sqrt{\rho_{33}} \vartheta, & \text{for } \rho_{33} > 0 \\ \frac{\sin \sqrt{\rho_{33}} \vartheta}{\sqrt{|\rho_{33}|}} \vartheta, & \text{for } \rho_{33} < 0 \end{cases}
\]

and by similar \( f_i(\psi) \) but with the replacement \( \rho_{33} \rightarrow \rho_{55} \) and \( \vartheta \rightarrow \psi \).

Before starting to search for the dependence \( \xi^{2(ik)}(t, z, \varphi) \) on \( z \), we shall put down conditions \( (13) \) in their explicit form

\[
(F_5 - F_3)' + F_5(F_5 - F_3) + \frac{1 - 2\rho_{33}}{g_{33}} - \frac{g_{55}}{2g_{33}^2} = 0;
\]

\[
(F_3 - F_5)' + F_3(F_3 - F_5) - \frac{\rho_{55}}{g_{55}} + \frac{g_{55}}{4g_{33}^2} = 0;
\]

\[
(F_3 - F_5)^2 + \frac{1 - 2\rho_{33}}{g_{33}} - \frac{\rho_{55}}{g_{55}} - \frac{g_{55}}{4g_{33}^2} = 0.
\]

The third equation, that is a consequence of the first two equations, is given here for convenience. Equations \( (14) \) are analyzed separately for a) \( F_3 - F_5 \neq 0 \) b) \( F_3 - F_5 = 0 \).

At a) \( F_3 - F_5 \neq 0 \) equations \( (14) \) turn out to be inconsistent for the nonzero values of \( \xi^{2(ik)}(t, z, \varphi) \) and \( \xi^{2(0k)}(t, z, \varphi) \), \( k = 0, 1, 2 \), no matter what values \( \rho_{33} \) and \( \rho_{55} \) take. Only for \( \rho_{33} = \rho_{55} = 1/4 \), \( \xi^{2(\mu\nu)}(t, z, \varphi) \) \( (\mu, \nu = 1, 2) \) in \( (15) \) has a nontrivial solution. The function \( \sigma(z) \neq \text{const} \):

\[
g_{33} = \sigma(z)g_{55}, \quad \sigma'(z) \neq 0
\]

is introduced to make our further description more convenient. Note, that \( F_3 - F_5 = \frac{\sigma'(z)}{2\sigma(z)} \).

In the terms of the function \( \sigma(z) \) the solution of equation \( (14) \) for \( \xi^{2(\mu\nu)}(t, z, \varphi) \), alongside with the expressions for \( g_{nn} \) satisfying restrictions \( (12) \), has a simple form

\[
\xi^{2(\mu\nu)} = A^{\mu\nu}(t, \varphi)\frac{\sigma}{\sigma'}\sqrt{\frac{\sigma - 1}{\alpha}}, \quad \alpha = \text{sign}(\sigma - 1) \tag{18}
\]

and

\[
g_{11} = \text{const}\frac{\sigma^2(\sigma - 1)}{(\sigma')^2\alpha}, \quad g_{33} = \frac{\sigma(\sigma - 1)^2}{(\sigma')^2}, \quad g_{55} = \frac{(\sigma - 1)^2}{(\sigma')^2}. \tag{19}
\]

In the second case, b) \( F_3 - F_5 = 0 \), only \( \xi^{2(\mu\nu)} \) \( (\mu\nu = 1, 2) \) and \( \xi^{2(00)} \) are nontrivial solutions of the equations and only under equality of \( g_{33} = g_{55} \) and definite values of the constants \( \rho_{4i} \): \( \rho_{33} = \rho_{55} = 1/4 \). Elementary calculations yield

\[
\xi^2(t, z, \vartheta, \varphi, \psi) = \sqrt{g_{33}}\{\xi^{2(\mu\nu)}(t, \varphi)n_{\lambda}(\chi_{3})m_{\mu}(\vartheta)f_{\nu}(\psi) + B(t, \varphi) + G(t, \varphi)\chi_{3}\}, \quad \mu, \nu, \lambda = 1, 2. \tag{20}
\]
Here and in what follows $\chi_3 = \int \sqrt{g_{33}} dz$, $n_1(\chi_3) = \text{sh} \frac{1}{2} \chi_3$ $n_2(\chi_3) = \text{ch} \frac{1}{2} \chi_3$.

For both cases a) and b) we obtain the final dependence of the components $\xi^2$ $\xi^5$ on the variables $z, \vartheta, \varphi$ and $\psi$ applying equations (2, 2) $\div$ (5, 5) of system (2) to equations (18) and (20). Similar to the case $V_a $ $\div$ $V_c$, we shall write down only the second and third components: $V_f$ : \{ $F_3 - F_5 \neq 0$, $R_3 = \hat{R}_3$, $R_5 = \hat{R}_5$ \}

$$
\xi^2 = \frac{\sigma}{\sigma' \sqrt{(\sigma - 1)/\alpha N}(t) P_m(\vartheta, \varphi, \psi)},
\xi^3 = \frac{2}{\alpha} \sqrt{\sigma - 1} N_m \partial_3 P_m + \xi^3_{V_a}; \quad m = 8, 9, 10, 11.
$$

$V_g$ : \{ $g_{33} = g_{55}$, $R_3 = \hat{R}_3$ \}

$$
\xi^2 = \sqrt{g_{33}} \{ N^{m}(t) n_\mu(\chi_3) P_m(\vartheta, \varphi, \psi) + M(t) \},
\xi^3 = -2 \{ N^m n_2 + N^2 n_1 \} \partial_3 P_m + \xi^3_{V_a}; \quad \mu = 1, 2.
$$

Here $P_m(\vartheta, \varphi, \psi)$ signify the functions:

$$
P_8 = \cos \frac{\varphi - \psi}{2} \sin \vartheta/2, \quad P_9 = -\sin \frac{\varphi - \psi}{2} \sin \vartheta/2,
\quad P_{10} = \cos \frac{\varphi + \psi}{2} \cos \vartheta/2, \quad P_{11} = \sin \frac{\varphi + \psi}{2} \cos \vartheta/2.
$$

Note that $(\partial_{34} + \frac{1}{2} \sin \vartheta \partial_{55} + \cos \vartheta \partial_{35}) P_m = 0$.

For $V_a \div V_f$ the dependence on $t$ is given by equations (4). The final form of the series we get applying equations (1, 2) $\div$ (1, 5) to the sets $V_a \div V_f$. As in the case with formulae (11) $\div$ (13) equations (4), (5), (8) and (9) show, that for this one should take into consideration only two possibilities:

either I. $\partial_4 \xi^2 = \partial_5 \xi^1 = 0$, $i = 3, 4, 5$; (21)

or II. $R_1 = \hat{R}_1$, and (22)

a). $R_1 = 0$, or (23)

b). $R_1 = \frac{\rho_{11}}{g_{11}}$, $\rho_{11} = \text{const} \neq 0$. (24)

Let us successively use these restrictions for the vectors $V_a \div V_f$. Together with the final form of the series we shall write down corresponding $g_{nn}$, that are solutions of restrictions (13), (23), (24).

3 Results

First it should be noted that the series with minimal number of vectors, five and seven, are derived from $V_a$, $V_b$ and $V_c$ under condition (24) ($V_a$ leads to $V_5$ for all formulae (21) $\div$ (24)):
$V_5: \{ R_i \neq \hat{R}_i, i = 3, 5; (g_{33} = \tau_3 g_{55}, R_1 \neq 0, R_1 \neq \frac{\rho_{11}}{g_{11}}, or F_3 - F_5 \neq 0) \}$

$\xi^i_5 = D^7, \xi^2_5 = 0, \quad \xi^3_5 = D^1 \sin \varphi + D^2 \cos \varphi, \quad \xi^4_5 = \cot \vartheta (D^1 \cos \varphi - D^2 \sin \varphi) - D^5, \quad \xi^5_5 = -\frac{1}{\sin \vartheta} (D^1 \cos \varphi - D^2 \sin \varphi) + D^6.$

$V^0_7: \{ g_{33} = g_{55}, R_1 \neq 0, R_1 \neq \frac{\rho_{11}}{g_{11}}, R_3 \neq \frac{\rho_{33}}{g_{33}} \}$

$\xi^1_7 = D^7, \xi^2_7 = 0, \quad \xi^3_7 = -D^3 \cos \psi + D^4 \sin \psi + \xi^3_5, \quad \xi^4_7 = -\frac{1}{\sin \vartheta} (D^3 \sin \psi + D^4 \cos \psi) + \xi^5_5.$

In both cases $\Phi = 0$, i.e. $V_5$ and $V^0_7$ are generators of isometry groups. Here and in what follows $D^{mk} = \text{const}$, and basis vectors $\zeta_i$ in the series are built from the components $\xi^k$ in the following way

$$\zeta_i = \frac{\partial}{\partial D^i} \xi^k \sum_m D^m \partial_k.$$

Two sets $V^i_7, V^0_7, i = 1, 2, 3$, three series in each, consisting of seven and nine vectors, are obtained from $V_6, V_c$, respectively, under conditions (23) and (24). The first two components are equal at fixed $i$ for $V^i_7, V^0_7$, while three remaining components correspond to $V_5$ for $V^i_7$ and $V^0_7$ for $V^0_7$.

$V^1_7(V^0_7): \{ g_{11} = \tau_1 g_{33}, \tau_1 = \text{const} \}$

$$\xi^1 = \frac{\sqrt{3}}{\tau_1} D^{20} + D^7, \quad \xi^2 = \sqrt{g_{33}} (D^{20} t + D^{21});$$

$V^2_7(V^0_7): \{ g_{11} = g_{33} b \exp(2\hat{\delta} \chi_3), b, \hat{\delta} = \text{const}, (\hat{\delta}^2 \neq \frac{1}{4} \text{ for } V^2_7) \}$

$$\xi^1 = -\left( \frac{1}{2 b \hat{\delta}} \exp(-2\hat{\delta} \chi_3) + \frac{\hat{\delta}^2}{2} \right) D^{20} - \hat{\delta} t D^{21} + D^7, \quad \xi^2 = \sqrt{g_{33}} (D^{20} t + D^{21});$$

$V^3_7(V^0_7): \{ (F_1 - F_3)' + F_3 (F_1 - F_3) - \frac{\rho_{11}}{g_{11}} = 0 \}$

$$\xi^1 = \pm \sqrt{\frac{g_{33}}{|\rho_{11}|}} (F_1 - F_3) (D^{20} l_2(t) \mp D^{21} l_1(t)) + D^7, \quad \xi^2 = \sqrt{g_{33}} (D^{20} l_1(t) + D^{21} l_2(t)).$$

For other three components:

$$\xi^i = \xi^i_5, \text{ for } V^k_7 (g_{33} = \tau_3 g_{55}, \tau_3 \neq 1); \quad \xi^i = \xi^i_0, \text{ for } V^k_9 (g_{33} = g_{55}); \quad i = 3, 4, 5, \quad k = 1, 2, 3.$$

The functions $l_\mu$ are given by the formula similar to (20) but with the replacements $\vartheta \to t$ and $\rho_{33} \to \rho_{11}$. Two signs in $\xi^i$ for $V^0_7(V^0_9)$ are related to $\rho_{11} > 0$ and $\rho_{11} < 0$, respectively. In these series we have: $R_3 \neq \hat{R}_3, R_5 \neq \hat{R}_5$. The function $\Phi$ for the series $V^i_7(V^0_9), i = 1, 2, 3$, looks like $\Phi = 2 F_3 \xi^2$. 

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The other series are connected with the sets $V_f, V_g$, whose third components depend on the variable $z$. Since equation (22) for $V_f$ is not fulfilled, the only possible condition (21) applied to $V_f$ leads to the series, containing nine vectors:

$V_9^4: \{g_{mn} \text{ are given by formula (19)}\}$

\[
\xi^1 = D^7, \quad \xi^2 = \frac{\sigma}{\sqrt{\sigma - 1}} D^m P_m, \\
\xi^3 = \frac{2}{\sqrt{\sigma - 1}} D^m \partial_3 P_m + \xi_3^3, \\
\xi^4 = -\frac{4}{\alpha \sin \vartheta} \sqrt{\frac{\sigma}{\sigma - 1}} D^m \partial_3 P_m + \xi_4^4, \\
\xi^5 = \frac{2}{\sqrt{\sigma - 1}} D^m \{2 - \sigma\} \partial_3 P_m + 2 \cot \vartheta \partial_3 P_m \} + \xi_5^5; \\
\Phi = 2 F_1 \xi^2; \quad m = 8, 9, 10, 11.
\]

It is the set $V_g$, that gives the largest number of series. It yields a series consisting of eleven vectors and three series, containing twenty one conformal Killing vectors, i.e. the maximum possible number. The series consisting of eleven vectors $V_{11}$ are obtained from $V_g$ under condition (21):

$V_{11}: \{g_{33} = g_{55}, \quad R_1 \neq 0, R_1 \neq \frac{g_{11}}{g_{11}}, \quad (F_1 - F_3) + F_1 (F_1 - F_3) - \frac{1}{g_{33}} = 0\}$

\[
\xi^1 = D^7, \quad \xi^2 = \sqrt{g_{11}} D^m P_m, \quad \xi^3 = -4(F_1 - F_3) \sqrt{g_{11}} D^m \partial_3 P_m + \xi_3^3, \\
\xi^{4+i} = \frac{8}{\sin \vartheta} (F_1 - F_3) \sqrt{g_{11}} D^m \partial_{3(5-i)} P_m + \xi^{4+i}, \\
\Phi = 2 F_1 \xi^2; \quad i = 0, 1; \quad m = 8, 9, 10, 11.
\]

Before we start describing the remaining series, we shall make a small remark. As is seen from the condition $R_3 = R_3$, for $V_g$ we have $F_1 - F_3 \neq 0$. Similar to formula (17) this fact allows us to introduce a nontrivial function of the variable $z$: $g_{33} = \sigma_1(z)g_{33}, \quad \sigma_1(z) \neq 0$. Conditions (13), (23) and (24) are solved w.r.t. $\sigma_1$ and allow one to present the corresponding $g_{ii}$ in terms of this function.

As a result, applying condition (24) to $V_g$ we obtain two series with twenty one generalized Killing vectors

$V_1^1: \{g_{11} = \frac{(\sigma_1)^2}{\rho_{11}} \}, \quad g_{33} = g_{55} = \frac{\sigma_1(\sigma_1 - 4\rho_{11})}{(\rho_{11})^2}, \quad \sigma_1 \neq 0\}$

\[
\xi^1 = \sqrt{\frac{\rho_{11}}{\sigma_1}} \{2(D^m l_2(t) = D^m l_1(t)) P_m + \sqrt{\frac{\sigma_1 - 4\rho_{11}}{2\rho_{11}}} (D^2 l_2 \mp \\
D^2 l_1)\} + D^7, \quad \xi^2 = \sqrt{\frac{\sigma_1 - 4\rho_{11}}{(\sigma_1)^2}} \{(\sqrt{\sigma_1 - 4\rho_{11}} D^m l_2 + \\
\sqrt{\sigma_1} D^m P_m + D^2 l_1 + D^2 l_2)\}, \\
\xi^3 = -2(\sqrt{\sigma_1} D^m l_2 + \sqrt{\sigma_1 - 4\rho_{11}} D^m P_m + \xi_3^3, \\
\xi^{4+i} = \frac{4}{\sin \vartheta} \{(\sqrt{\sigma_1} D^m l_2 + \sqrt{\rho_{11}} D^m P_m + \xi^{4+i};
\]
\[\Phi = 2\left( (F_3 \sqrt{g_{33}(\sigma_1 - 4\rho_{11})}) + \frac{\sqrt{\sigma_1}}{2} \right)^2 D^\mu l_{\mu} P_m + \sqrt{g_{11}} F_1 \beta D^\alpha P_m + \sqrt{g_{33}} F_3 \beta (D^{\alpha} l_{\alpha} + D^{\beta} l_{\beta}); \quad \beta = \text{sign} \sigma_1; \quad i, \mu = 1, 2.\]

Two signs for \(\xi^i\) are related to \(\rho_{11} > 0\) and \(\rho_{11} < 0\), respectively.

And finally condition (23) leads to series \(V_{21}^2\):

\[V_{21}^2 : \quad \{g_{11} = \frac{(\sigma_1)^3}{(\sigma_1)^2}, \quad g_{33} = g_{55} = \frac{(\sigma_1)^2}{(\sigma_1)^3}, \quad a = \text{const} > 0\}\]

\[
\begin{align*}
\xi^1 &= -\frac{2}{a} \sqrt{\frac{a}{\sigma_1}} \left( D^m t + D^m \right) P_m - D^{21} \left( \frac{t^2}{4} + 1/\sigma_1 \right) - \frac{1}{2} D^{20} t + D^2, \\
\xi^2 &= \frac{\sigma_1}{\sigma_1} \left\{ \sqrt{\frac{a}{\sigma_1}} \left[ D^{mr} t^r / r! - (2/a - 2/\sigma_1) D^{m2} \right] P_m + D^{21} t + D^{20} \right\}, \\
\xi^3 &= -2 \sqrt{\frac{a}{\sigma_1}} \left\{ D^{mr} t^r / r! - (2/a + 2/\sigma_1) D^{m2} \right\} \partial_3 P_m + \xi^3_{3^i}, \\
\xi^{4+i} &= \frac{4}{\sin \theta} \sqrt{\frac{a}{\sigma_1}} \left\{ D^{mr} t^r / r! - (2/a + 2/\sigma_1) D^{m2} \right\} \partial_3 (5-i) P_m + \xi^{4+i}_{4^i}, \\
\Phi &= 2 \frac{\sigma_1}{\sigma_1} \left\{ \left( F_1 (t^2/2 - 2/a - 2/\sigma_1) + \frac{4}{\sigma_1} F_3 \right) D^m + F_1 (D^{m1} t + D^{m0}) \right\} P_m + F_3 (D^{21} t + D^{20}) \right\}; \quad r = 0, 1, 2.
\end{align*}\]

Thus, expressions \(V_5 \div V_{21}^2\) represent all series of generalized Killing vectors for line element (3). In accordance with theorem [9] there follows from the form of the function \(\Phi\) for the series \(V_7^i; \quad V_9^i; \quad i = 1, 2, 3; \quad V_9^4\) and \(V_{11}\) that these series are generators of trivial conformal groups, and, consequently, in a corresponding coordinate system they coincide with the generators of the isometry groups, described in [7].

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