Spectrum of the heat equation with memory

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Abstract

We consider the system

\[
\partial_t \theta(x, t) = \int_0^t k(t-s) \partial_{xx} \theta(x, s) \, ds \quad x \in (0, \pi), \ t > 0 \quad \theta(0, \cdot) = \xi(\cdot),
\]

with homogeneous Dirichlet boundary condition. Here

\[
k(t) = \sum_{1}^{\infty} a_k e^{-b_k t}
\]

with positive \(a_k\), \(0 \leq b_1 < b_2 \ldots\) and

\[
\sum_{1}^{\infty} a_k < \infty, \ b_k \uparrow +\infty.
\]

Assuming an additional condition to \(b_k\), e.g., \(b_{k+1} - b_k \geq \delta > 0\), we obtain the structure of the spectrum of the system.

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1 Introduction. Notations. Main results

Gurtin and Pipkin in [4] introduce a model of heat transfer with finite propagation speed. A linearized model with the zero memory at \( t = 0 \) can be written [2] as:

\[
\dot{\theta}(x, t) = \int_0^t k(t-s)\theta''(x, s) \, ds, \quad x \in (0, \pi), \quad t > 0, \quad \theta(0, \cdot) = \xi(\cdot) \in L^2(0, \pi). \tag{1}
\]

with homogeneous DBC. Here \( \dot{\theta} = \frac{\partial}{\partial t} \theta, \theta' = \frac{\partial}{\partial x} \theta \). Regularity of this equation in more general setting is studied in [2], but in one-dimensional case it is possible to obtain regularity results directly using Fourier method. Two kind of controllability of this system are studied in [2], [1].

We will study this equation with the kernel in the form

\[
k(t) = \sum_{k=1}^{\infty} a_k e^{-b_k t}
\]

with positive \( a_k, 0 \leq b_1 < b_2 \ldots \) and

\[
\sum_{k=1}^{\infty} a_k = \alpha^2 < \infty, \quad b_k \uparrow +\infty. \tag{2}
\]

Note that for \( k(t) = c^2 \) we have in fact the wave equation \( \ddot{\theta} = c^2 \theta'' \).

First, apply the Fourier method: we set \( \varphi_n = \sqrt{\frac{2}{\pi}} \sin nx \) and expand the solution and the initial data in series in \( \varphi_n \)

\[
\theta(x, t) = \sum_{k=1}^{\infty} \theta_n(t) \varphi_n(x), \quad \xi(x) = \sum_{k=1}^{\infty} \xi_n \varphi_n(x).
\]

For the components we obtain

\[
\dot{\theta}_n(t) = -n^2 \int_0^t k(t-s)\theta_n(s)ds, \quad t > 0, \quad \theta_n(0) = \xi_n. \tag{3}
\]

We will denote the Laplace image by the capital characters. Applying the Laplace Transform to System (3) we find

\[
z \Theta_n(z) - \xi_n = -n^2 K(z) \Theta_n(z)
\]

or

\[
\Theta_n(z) = \frac{1}{z + n^2 K(z)} \xi_n.
\]

We will need to study zeros of the function \( G_n(z) = z + n^2 K(z) \). Let \( \Lambda_n, n = 1, 2, \ldots \), be the set of zeros of \( G_n \), \( \Lambda := \{ \Lambda_n \} \).
**Definition 1** The set $\Lambda$ is called the spectrum of System (1).

The Laplace image of $k(t)$ is

$$K(z) = \sum_{1}^{\infty} \frac{a_k}{z + b_k}$$

This is a meromorphic function real valued on the real axis and mapping the upper half plane to the lower one and visa versa.

**Lemma 1** $K(z)$ has only real zeros $-\mu_j$, $j = 1, 2, \ldots$, such that

$$b_1 < \mu_1 < b_2 < \mu_2 < b_3 < \ldots$$

**Proof of the lemma** Since

$$\Im K(x + iy) = -y \sum_{1}^{\infty} \frac{a_k}{(x + b_k)^2 + y^2},$$

all zeros are real. The lemma follows now from the fact that $K(x)$ runs the real axis on every interval $(-b_j, -b_{j-1})$, see the figure.

![Graph of K(x) with b_1=1, b_2=2, b_3=3](image-url)
Theorem 1 \ Let 
\[
\sup_k \{b_k(b_{k+1} - b_k)\} = \infty.
\] (4)

Then the sets \( \Lambda_n \) can be represented as
\[
\Lambda_n = \{\lambda_{nj}\}_{j=1}^{\infty} \cup \{\lambda^+\} \cup \{\lambda^-\},
\]
with
\[
\begin{align*}
(i) \quad & \lambda_{nj} = -\mu_j + o(1), \; j = 1, 2, \ldots; \\
(ii) \quad & \{\lambda_{nj}\} \text{ are real and for fixed } j \text{ we have } \lambda_{nj} \uparrow -\mu_j \text{ and the sequence } \\
& \{\lambda_{nj}\}_{n=1}^{\infty} \text{ is in } (-b_{j+1}, -\mu_j).
\end{align*}
\]

and the spectrum is in the left half plane.

Remark 1 In [3] regularity of the Gurtin-Pipkin equation of the second order in time is studied in more general situation. In [5] the authors consider this equation with model kernel and find the asymptotic of complex zeros of the function
\[
z^2/n^2 + 1 - K(z), \; n \in \mathbb{N},
\]
where
\[
a_k = 1/k^\alpha, \; b_k = k^\beta, \; 0 < \alpha \leq 1, \; \alpha + \beta > 1.
\]

In this case
\[
\sum_{1}^{\infty} c_k = \infty, \quad \sum_{1}^{\infty} \frac{c_k}{a_k} < \infty.
\]

Remark 2 The authors are grateful to Prof. A.E. Eremenko for very helpful consultations.

2 Proof of the main theorem

First show that the spectrum is in the left half plane. If \( z + n^2 K(z) = 0 \), \( z = x + iy \), then
\[
x + iy + n^2 \sum a_k (x + b_k) (x + b_k)^2 + y^2 - iyn^2 \sum \frac{a_k}{(x + b_k)^2 + y^2} = 0
\]

For the real part we have
\[
x + n^2 \sum \frac{a_k (x + b_k)}{(x + b_k)^2 + y^2} = 0.
\]
If \( x \geq 0 \), then this expression is positive.

We are going to apply the Argument Principle. Fix \( n \) and take the rectangle contour \( \Gamma \) with the vertices \((\pm X, \pm Y), X, Y > 0\). Let

\[
 f(z) = K(z), \quad g(z) = z/n^2.
\]

We show that we can take \( X \), and \( Y \) in such a way that

\[
 |f|_{\Gamma} < |g|_{\Gamma}.
\]

Consider the side \( \Gamma_1 = \{ \Re z = -X, |\Im z| \leq Y \} \). For \( z \in \Gamma_1 \) we have

\[
 |K(z)| \leq \sum a_k | -X + iy + b_k | \leq \sum a_k | -X + b_k | =: q(X). \tag{6}
\]

Take \( X = X_N = (b_{N+1} + b_N)/2 \) where we choose \( N \) later.

**Lemma 2**

\[
 \frac{1}{X_N} q(X_N) \leq \frac{2}{b_N \delta_N} \alpha^2. \tag{7}
\]

**Proof of the lemma.** From (6) we conclude

\[
 q(X_N) = \sum_{1}^{N} \frac{a_k}{X_N - b_k} + \sum_{N+1}^{\infty} \frac{a_k}{b_k - X_N}.
\]

Since

\[
 X_n - b_k = \frac{1}{2}(b_{N+1} - b_k + b_N - b_k) \geq \frac{1}{2}(b_{N+1} - b_k), \quad k = 1, 2, \ldots, N,
\]

and

\[
 b_k - X_N \geq \frac{1}{2}(b_k - b_N), \quad k = N + 1, N + 2, \ldots,
\]

we have

\[
 q(X_N) \leq 2 \left( \sum_{1}^{N} \frac{a_k}{b_{N+1} - b_k} + \sum_{N+1}^{\infty} \frac{a_k}{b_k - b_N} \right). \tag{8}
\]

Set

\[
 \delta_n = b_{n+1} - b_n.
\]

Evidently

\[
 b_{N+1} - b_k \geq b_{N+1} - b_N = \delta_N, \quad k = 1, 2, \ldots, N, \quad b_k - b_N \geq \delta_N, \quad k = N + 1, N + 2, \ldots, N.
\]
Now (8) gives
\[ q(X_N) \leq 2 \frac{1}{\delta_N} \sum_{k=1}^{\infty} a_k = 2 \frac{1}{\delta_N} \alpha \]
and then
\[ \frac{1}{X_N} q(X_N) \leq \frac{2}{b_N \delta_N} \alpha^2. \]
This proves the lemma.

Let us obtain (5) on \( \Gamma_1 \). Choose \( N \) such that
\[ \frac{1}{n^2} > \frac{2}{b_N \delta_N} \alpha^2, \]
what is possible by the assumption (4). Then
\[ |g|_{\Gamma_1} \geq \frac{X_N}{n^2} > X_N \frac{2}{b_N \delta_N} \alpha^2 \geq X_N \frac{1}{X_N} q(X_N) \geq |K|_{\Gamma_1}. \]
Consider the side \( \Gamma_2 = \{|\Re z| < X, \Im z = Y\} \). Here \( \inf |g| = \inf |z/n^2| = Y/n^2 \) and
\[ |K(z)| \leq \sum \frac{a_k}{|x + iY + b_k|} \leq \sum \frac{a_k}{Y} = \frac{\alpha^2}{Y}. \] (10)
Let \( Y > n\alpha \). Then
\[ \frac{Y}{n^2} > \frac{\alpha^2}{Y} \]
and (10) gives (5).

The estimate on \( \Gamma_2 \) is the same as for \( \Gamma_1 \) because
\[ \tilde{K}(z) = K(\bar{z}). \] (11)
On the rest side \( \Re z = X, |\Im z| \leq Y \) we have
\[ |K| \leq \sum \frac{a_k}{X} = \frac{\alpha^2}{X}. \]
Since \( |z/n^2| \geq X/n^2 \) we have (5) if \( X > n\alpha \). Therefore, for \( X, Y > n\alpha \) and with (4) we have (5) and we can conclude that
\[ N(g) - P(g) = N(f + g) - P(f + g) \]
where \( N \) and \( P \) denote respectively the number of zeros and poles of the function inside the contour \( \Gamma \), with each zero and pole counted as many times as its multiplicity and order respectively. Evidently, \( N(g) - P(g) = 1 \).
Inside $\Gamma$ the function $f + g$ has $N$ simple poles at $-b_N, -b_{N-1}, \ldots, -b_1$. Therefore $N(f + g) = N + 1$.

Show that $G_n$ has $N - 1$ real zeros $\lambda_{nj}$ inside $\Gamma_n$ satisfying the theorem. For fixed $n$ the graph of one branch of $K(x)$ and of $-x/n^2$ is

![Graph of K(x) and -x/n^2](image)

We see that the straight line $-x/n^2$ intersects the graph of $K(x)$ in $(-b_{j+1}, -\mu_j)$ and monotonically approaches $-\mu_j$ as $n \to \infty$, $j = 1, 2, \ldots$. Thus, there exists $N$ or $N - 1$ real zeros of $G_n$ inside $\Gamma_n$ depending whether $-(b_{N+2} + b_N)/2$ is more or less $\lambda_{N,n}$. In the first case we have exactly one complex zero of $G_n$ what is impossible by (11).

Thus, there exists two (complex conjugated) zeros. Denote these zeros by $\lambda^+_n$ and $\lambda^-_n = \overline{\lambda^+_n}$.

Let us localize these roots. Consider the case $b_1 = 0$. Then for $z = iy$

$$G_n(iy) = iy + n^2 \sum_{k=2}^{\infty} \frac{a_k b_k}{b_k^2 + y^2} - in^2 y \sum_{k=1}^{\infty} \frac{a_k}{b_k^2 + y^2}.$$  

We find that $G_n(iy) = 0$ only if $b_2 = b_3 = \cdots = 0$, i.e., $K(z) = \alpha^2/z$ and $\lambda^+_n = \pm i \sqrt{\alpha n}$. Consider the case $K(z) \neq \alpha^2/z$.

Take $\varepsilon > 0$ and the rectangle contour $G$ with the vertices $(-\varepsilon\alpha, i\alpha(1 - \varepsilon)), (-\varepsilon\alpha, i\alpha(1+\varepsilon)), (\varepsilon\alpha, i\alpha(1+\varepsilon)), (\varepsilon\alpha, i\alpha(1-\varepsilon))$. Write $z/n^2 + K(z)$
as
\[
\left[ \frac{z}{n^2} + \frac{\alpha^2}{z} \right] + \left[ K(z) - \frac{\alpha^2}{z} \right] = g + f.
\]

**Lemma 3** Let
\[ |\arg z| < \pi - \delta, \tag{12} \]
Then
\[ K(z) = \frac{\alpha^2}{z} (1 + o(z)). \tag{13} \]

Proof of the lemma. Show first that for
\[ |z + x|^2 \asymp |z|^2 + x^{2b}, \quad x \geq 0, \ b > 0. \tag{14} \]
Set \( z = re^{i\varphi}, \ \rho = |z| \)
\[ |z + x|^2 = (\rho \cos \varphi + x)^2 + \rho^2 \sin^2 \varphi = \rho^2 + 2x \rho \cos \varphi + x^{2b}. \]
Evidently
\[ \rho^2 + 2x \rho \cos \varphi + x^{2b} \leq (\rho + x)^2 \leq 2(\rho^2 + x^{2b}) \]
and
\[ \rho^2 + 2x \rho \cos \varphi + x^{2b} \geq \rho^2 - 2x \rho \cos \delta + x^{2b} \geq (\rho^2 + x^{2b}) - (\rho^2 + x^{2b}) \cos \delta \]
\[ = (1 - \cos \delta)(\rho^2 + x^{2b}). \]
This gives (14)

Now we can apply the Weierstrass theorem to the series
\[ zK(z) = \sum \frac{a_kz}{z + b_k}. \]
Indeed, by (14)
\[ \frac{|a_kz|}{|z + b_k|} \ll \frac{|a_k| |z|}{|z| + b_k} \leq a_k \]
and
\[ \frac{a_kz}{z + b_k} \to a_k, \quad z \to \infty. \]
Therefore
\[ zK(z) = \lim_{z \to \infty} \sum \frac{a_kz}{z + b_k} = \sum_{1}^{\infty} a_k = \alpha^2. \]
The lemma is proved.
Find the estimate of $g|_G$ from below. For $\varepsilon < 1$

$$|g|_G = \left|\frac{z + i\alpha n}{z} \right| \geq \frac{n\alpha(2 - \varepsilon)\alpha n}{z^2|n^2|} > \frac{\alpha^2\varepsilon}{|z|}.$$  

From Lemma 3 we have in the sector $\{12\}$

$$f(z) = h(z)\frac{1}{z}, \ h(z) = o(1).$$

Take $n$ large enough in order that

$$|h(z)|_G < \varepsilon.$$

Then

$$|g|_G > |f|_G.$$

Now $N(f + g) = N(g) = 1$. We have proved that $\lambda^+_n$ is in the box centered at $i\alpha n$ and with the diameter $O(\varepsilon n)$.

The theorem is proved.

**Remark 3** If we know asymptotic or estimate of the parameters $\{a_k, b_k\}$, we can find a more precise expression of points of $\Lambda$, especially of $\lambda^+_n$, see [5].

**Remark 4** The condition (4) is evidently fulfilled if $b_{k+1} - b_k \geq \delta > 0$. For a model case $b_k = ck^\alpha$

$$b_k(b_{k+1} - b_k) \asymp k^{2\alpha - 1}$$

and (4) is true if and only if $\alpha > 1/2$. Using the proposed approach we have to show that

$$\frac{1}{b_N} \sum \frac{a_k}{|b_k - (b_{N+1} - b_N)/2|} \to 0, \text{ as } N \to \infty.$$  

This is not true for slowly increasing $b_k$, say, $b_k = \log \log k$. Nevertheless we have the conjecture that main theorem is true for every kernel satisfying (2).
References

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