THE GLOBAL DIMENSION OF THE FULL
TRANSFORMATION MONOID

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Abstract. The representation theory of the symmetric group has been
intensively studied for over 100 years and is one of the gems of modern
mathematics. The full transformation monoid $T_n$ (the monoid of all
self-maps of an $n$-element set) is the monoid analogue of the symmetric
group. The investigation of its representation theory was begun by
Hewitt and Zuckerman in 1957. Its character table was computed by
Putcha in 1996 and its representation type was determined in a series of
papers by Ponizovskii, Putcha and Ringel between 1987 and 2000. From
their work, one can deduce that the global dimension of $C T_n$ is $n - 1$
for $n = 1, 2, 3, 4$. We prove in this paper that the global dimension is
$n - 1$ for all $n \geq 1$ and, moreover, provide an explicit minimal projective
resolution of the trivial module of length $n - 1$.

1. Introduction

The character theory of the symmetric group [14, 19, 28, 29, 37, 39] is an
elegant piece of mathematics, featuring a beautiful blend of algebra and
combinatorics, with applications to such diverse areas as probability [24, 25]
and mathematical physics.

The analogue in monoid theory of the symmetric group is the full
transformation monoid $\Sigma_n$. This is the monoid of all self-maps of an $n$-element
set. In 1957, Hewitt and Zuckerman initiated the study of the representation
theory of $\Sigma_n$, showing that the simple $C\Sigma_n$-modules are parameterized by
partitions of $r$ where $1 \leq r \leq n$ [34]. However, very few of their results were
specific to $\Sigma_n$, as witnessed by the fact that Munn immediately generalized
their main theorem to arbitrary finite monoids [47].

In a tour de force work [51], Putcha computed the character table of $\Sigma_n$
and gave an explicit description of all the simple $C\Sigma_n$-modules except for
one family. It was the discovery of an explicit description of this second
family via exterior powers that led to this paper. Putcha, in fact, knows

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Grant.
Recall that a finite dimensional algebra \( A \) has finite representation type if there are only finitely many isomorphism classes of finite dimensional indecomposable \( A \)-modules. Ponizovskii proved that \( \mathbb{C} \Sigma_n \) has finite representation type for \( n \leq 3 \) and conjectured that this was true for all \( n \) \[50\]. Putcha disproved Ponizovskii’s conjecture by showing that \( \mathbb{C} \Sigma_n \) does not have finite representation type for \( n \geq 5 \) \[52\]. He did this by computing enough of the quiver of \( \mathbb{C} \Sigma_n \) to see that \( \mathbb{C} \Sigma_n / \text{rad}^2(\mathbb{C} \Sigma_n) \) already does not have finite representation type. Putcha also computed the quiver of \( \mathbb{C} \Sigma_4 \) and observed that \( \mathbb{C} \Sigma_4 / \text{rad}^2(\mathbb{C} \Sigma_4) \) does have finite representation type. Ringel \[54\] computed a quiver presentation for \( \mathbb{C} \Sigma_4 \) and proved that it is of finite representation type. It is an open question to compute the quiver of \( \mathbb{C} \Sigma_n \) in full generality.

Ringel also proved that \( \mathbb{C} \Sigma_4 \) has global dimension 3 by exhibiting the minimal projective resolutions of the simple modules. It is easy to check that the global dimension of \( \mathbb{C} \Sigma_n \) is also \( n - 1 \) for \( n = 1, 2, 3 \) using the results of \[51\]. The main result of this paper is that the global dimension of \( \mathbb{C} \Sigma_n \) is \( n - 1 \) for all \( n \geq 1 \). As a byproduct of the proof, we also establish that the quiver of \( \mathbb{C} \Sigma_n \) is acyclic. In fact, these results hold mutatis mutandis over any ground field of characteristic 0 because \( \mathbb{Q} \) is a splitting field for all symmetric groups and hence for all full transformation monoids.

We prove the main theorem using homological techniques for working with the algebra of a von Neumann regular monoid developed by the author and Margolis in \[43\]. One could alternatively use the theory of quasi-hereditary algebras \[22,26\]. From that point of view, the key point is that outside of a single family the standard modules are simple.

The paper begins with a review of monoid representation theory and the character theory of the full transformation monoid. The following section proves the main result.

The reader is referred to \[21,27,35,38,53\] for basic semigroup theory. We use \[2,4,9\] as our primary references for the theory of finite dimensional algebras. In this paper, all modules are assumed to be left modules and all monoids are assumed to act on the left of sets unless otherwise indicated.

In recent years, there has been a revival of interest in the representation theory of finite monoids (eg. \[1,10,23,36,43,46,59,60\]), due in a large part to applications to Markov chains \[3,5,8,11,13,15,17,20,55\] and data analysis \[40,42\]. This paper is a further contribution to our understanding of monoid representation theory.

2. The Representation Theory of Finite Monoids

This section reviews the necessary background results for the paper.

2.1. Finite monoids. Fix a finite monoid \( M \). An ideal \( I \) of \( M \) is a non-empty subset \( I \) such that \( MIM \subseteq I \). Left and right ideals are defined analogously. If \( m \in M \), then \( Mm, \ mM \) and \( MmM \) are the principal left,
right and two-sided ideals generated by \( m \), respectively. We put
\[
I(m) = \{ n \in M \mid m \notin MnM \}.
\]
If \( I(m) \neq \emptyset \), then it is an ideal.

The following equivalence relations are three of Green’s relations \([32]\). Set, for \( m_1, m_2 \in M \),
1. \( m_1 \not\mathcal{J} m_2 \) if and only if \( Mm_1M = Mm_2M \);
2. \( m_1 \not\mathcal{L} m_2 \) if and only if \( Mm_1 = Mm_2 \);
3. \( m_1 \not\mathcal{R} m_2 \) if and only if \( m_1M = m_2M \).

The \( \mathcal{J} \)-class of an element \( m \) is denoted by \( J_m \), and similarly the \( \mathcal{L} \)-class and \( \mathcal{R} \)-class of \( m \) are denoted \( L_m \) and \( R_m \), respectively.

Let us describe Green’s relations on \( T_n \); this is a standard exercise in semigroup theory (cf. \([21, 30]\)). The rank of a mapping \( f \in T_n \) is the cardinality of its image. Let \( f, g \in T_n \). Then \( f \not\mathcal{J} g \) if and only if they have the same rank, \( f \not\mathcal{R} g \) if and only if they have the same image and \( f \not\mathcal{L} g \) if and only if they induce the same partition of the domain into fibers.

The set of idempotents of a monoid \( M \) is denoted \( E(M) \). If \( e \in E(M) \), then \( eMe \) is a monoid with identity \( e \). Let \( G_e \) be the group of units of \( eMe \). It is called the maximal subgroup of \( M \) at \( e \). If \( e, f \in E(M) \) with \( MeM = MfM \), then \( eMe \cong fMf \) and \( G_e \cong G_f \) (cf. \([61]\)).

The group of units of \( T_n \) is the symmetric group \( S_n \). If \( e \in T_n \) has rank \( r \), then it is well known and easy to show that \( eT_n \cong T_r \) and hence \( G_e \cong S_r \).

A standard fact in the theory of finite monoids is the following (cf. \([61]\)).

**Proposition 2.1.** Let \( e \in E(M) \). Then \( J_e \cap eMe = G_e \).

The maximal subgroup \( G_e \) at an idempotent \( e \) acts freely on the right of \( L_e \) on and on the left of \( R_e \), respectively, by multiplication and two elements of one of these sets are in the same \( G_e \)-orbit if and only if they are \( \mathcal{R} \)-equivalent, respectively \( \mathcal{L} \)-equivalent; see \([21]\) or \([53]\) Appendix A).

A monoid \( M \) is (von Neumann) regular if, for all \( m \in M \), there exists \( m' \in M \) with \( mm'm = m \). For example, it is well known that the full transformation monoid \( T_n \) is regular \([21, 30]\), as is the monoid \( M_n(\mathbb{F}) \) of all \( n \times n \) matrices over a field \( \mathbb{F} \) \([21]\). A finite monoid is regular if and only if each \( \mathcal{J} \)-class contains an idempotent, cf. \([38]\) or \([53]\) Appendix A).

### 2.2. Simple modules.

Let \( k \) be a field. We continue to hold fixed a finite monoid \( M \). If \( X \subseteq M \) we let \( kX \) denote the \( k \)-linear span of \( X \) in the monoid algebra \( kM \). If \( I \subseteq M \) is a left (respectively, right or two-sided) ideal of \( M \), then \( kI \) is a left (respectively, right or two-sided) ideal of \( kM \).

Let \( S \) be a simple \( kM \)-module. We say that an idempotent \( e \in E(M) \) is an apex for \( S \) if \( eS \neq 0 \) and \( kI(e)S = 0 \). One has that \( mS = 0 \) if and only if \( m \in I(e) \). It follows that if \( e, f \) are apexes of \( S \), then \( MeM = MfM \).

Fix an idempotent \( e \in E(M) \) and put \( A_e = kM/kI(e) \). Observe that \( eA_e \cong k[eMe]/k[eI(e)e] \cong kG_e \) by Proposition 2.1. A simple \( kM \)-module \( S \) with apex \( e \) is the same thing as a simple \( A_e \)-module \( S \) with \( eS \neq 0 \). One
can then apply the theory [33, Chapter 6] to classify these modules. This was done in [31].

Notice that as a $k$-vector space, $A_e \cong kL_e$ and $eA_e \cong kR_e$ [31]. The corresponding left $kM$-module structure on $kL_e$ is defined by

$$m \circ \ell = \begin{cases} m \ell, & \text{if } m \ell \in L_e \\ 0, & \text{else} \end{cases}$$

for $m \in M$ and $\ell \in L_e$. From now on we will omit the symbol “$\circ$.” The right $kM$-module structure on $kR_e$ is defined dually. In the semigroup theory literature, $kL_e$ and $kR_e$ are known as left and right Schützenberger representations. Note that $kL_e$ is a free right $kG_e$-module and $kR_e$ is a free left $kG_e$-module because $G_e$ acts freely on the right of $L_e$ and on the left of $R_e$. In fact, $kL_e$ is a $kM$-$kG_e$-bimodule and dually $kR_e$ is a $kG_e$-$kM$-bimodule. Thus we can define functors

$$\text{Ind}_{G_e} : kG_e\text{-mod} \to kM\text{-mod}$$
$$\text{Coind}_{G_e} : kG_e\text{-mod} \to kM\text{-mod}$$
$$\text{Res}_{G_e} : kM\text{-mod} \to kG_e\text{-mod}$$
$$K_e : kM\text{-mod} \to kM\text{-mod}$$
$$N_e : kM\text{-mod} \to kM\text{-mod}$$

by putting

$$\text{Ind}_{G_e}(V) = A_e \otimes_{eA_e} V = kL_e \otimes_{kG_e} V$$
$$\text{Coind}_{G_e}(V) = \text{Hom}_{A_e}(eA_e, V) = \text{Hom}_{kG_e}(kR_e, V)$$
$$\text{Res}_{G_e}(V) = eV = \text{Hom}_{A_e}(A_e, V) = eA_e \otimes_{A_e} V$$
$$K_e(V) = kM e V$$
$$N_e(V) = \{v \in V \mid e kM v = 0\}.$$

One has that $\text{Res}_{G_e}(\text{Ind}_{G_e}(V)) \cong V \cong \text{Res}_{G_e}(\text{Coind}_{G_e}(V))$ for any $kG_e$-module, cf. [31]. Note that the functors $\text{Ind}_{G_e}$ and $\text{Coind}_{G_e}$ are exact [31] because $kL_e$ and $kR_e$ are free $kG_e$-modules (on the appropriate sides).

If $V$ is a module over a finite dimensional algebra $A$, then $\text{rad}(V)$ denotes the radical of $V$ and $\text{soc}(V)$ the socle of $V$ (see [2] for the definitions).

We now state the fundamental theorem of Clifford-Munn-Ponizovskii theory [21, Chapter 5], as formulated in [31].

**Theorem 2.2.** Let $M$ be a finite monoid and $k$ a field.

(i) There is a bijection between isomorphism classes of simple $kM$-modules with apex $e \in E(M)$ and isomorphism classes of simple $kG_e$-modules given by

$$S \mapsto \text{Res}_{G_e}(S) = eS$$
$$V \mapsto V^e = \text{Ind}_{G_e}(V)/N_e(\text{Ind}_{G_e}(V)) = \text{Ind}_{G_e}(V)/\text{rad}(\text{Ind}_{G_e}(V))$$

$$\cong \text{soc}(\text{Coind}_{G_e}(V)) \equiv K_e(\text{Coind}_{G_e}(V)) \equiv \text{Res}_{G_e}(V).$$
for $S$ a simple $\mathbb{k}M$-module with apex $e$ and $V$ a simple $\mathbb{k}G_e$-module.

(ii) Every simple $\mathbb{k}M$-module has an apex (unique up to $\mathcal{J}$-equivalence).

(iii) If $V$ is a simple $\mathbb{k}G_e$-module, then every composition factor of $\text{Ind}_{G_e}(V)$ and $\text{Coind}_{G_e}(V)$ has apex $f$ with $MeM \subseteq MfM$. Moreover, $V^\sharp$ is the unique composition factor of these two modules with apex $e$.

If we denote the set of isomorphism classes of simple $\mathbb{k}M$-modules by $\text{Irr}_\mathbb{k}(M)$, then there is the following parametrization of the irreducible representations of $M$.

**Corollary 2.3.** Let $e_1, \ldots, e_s$ be a complete set of representatives of the $\mathcal{J}$-classes of idempotents of $M$. Then there is a bijection between $\text{Irr}_\mathbb{k}(M)$ and the disjoint union $\bigcup_{i=1}^s \text{Irr}_\mathbb{k}(G_{e_i})$.

If $M$ is regular and $\mathbb{k}$ has characteristic 0, then the modules $\text{Ind}_{G_e}(V)$ and $\text{Coind}_{G_e}(V)$ with $e \in E(M)$ and $V \in \text{Irr}_\mathbb{k}(G_e)$ form the standard and costandard modules, respectively, for the structure of a quasi-hereditary algebra on $\mathbb{k}M$ [52]. See [22, 26] for quasi-hereditary algebras.

2.3. **Homological aspects.** Let $A$ be a finite dimensional $\mathbb{k}$-algebra. The *projective dimension* $\text{pd} V$ of a module $V$ is the minimum length (possibly infinite) of a projective resolution of $V$. Each finite dimensional $A$-module $V$ has a unique minimal projective resolution (minimal in both length and in a certain categorical sense); see [2, 4, 9]. Formally, a projective resolution $P_\bullet \to V$ is *minimal* if each boundary map $d_n: P_n \to d_n(P_n)$ is a projective cover. The following well-known proposition is stated in the context of group algebras in [18, Proposition 3.2.3], but the proof there is valid for any finite dimensional algebra.

**Proposition 2.4.** Let $A$ be a finite dimensional $\mathbb{k}$-algebra, $M$ a finite dimensional $A$-module and let $P_\bullet \to M$ be a projective resolution. Then the following are equivalent.

1. $P_\bullet \to M$ is the minimal projective resolution of $M$.
2. $\text{Hom}_A(P_q, S) \cong \text{Ext}^q_A(M, S)$ for any $q \geq 0$ and simple $A$-module $S$.

The *global dimension* of $A$ is

$$\text{gl. dim } A = \sup \{ \text{pd } V \mid V \text{ is an } A\text{-module} \}.$$ 

An equivalent definition, which is the one that we shall mostly use in this paper, is

$$\text{gl. dim } A = \sup \{ n \in \mathbb{N} \mid \text{Ext}^n_A(S, S') \neq 0, \text{ for some simple modules } S, S' \}$$

where the supremum could be infinite [2, 4, 9]. This reformulation relies on the following well-known lemma, which is proved by induction on the number of composition factors using the long exact sequence associated to the Ext-functors.

**Lemma 2.5.** Let $V, W$ be finite dimensional $A$-modules. Then one has that $\text{Ext}^n_A(V, W) = 0$ if either of the following two conditions hold.
(i) \( \text{Ext}^n_A(V, S) = 0 \) for each composition factor \( S \) of \( W \).
(ii) \( \text{Ext}^n_A(S', W) = 0 \) for each composition factor \( S' \) of \( V \).

The following result is \([43, \text{Lemma 3.3}]\).

**Lemma 2.6.** Let \( M \) be a finite regular monoid and \( k \) a field. Let \( I \) be an ideal of \( M \). Then the isomorphism

\[
\text{Ext}^n_{kM}(V, W) \cong \text{Ext}^n_{kM/I}(V, W)
\]

holds for any \( kM/kI \)-modules \( V, W \) and all \( n \geq 0 \).

The author and Margolis proved in \([43, \text{Lemma 3.5}]\) the following lemma in the same vein as the Eckmann-Shapiro lemma from group cohomology.

**Lemma 2.7.** Let \( M \) be a finite regular monoid and \( k \) a field. Let \( e \in E(M) \) and \( I = MeM \setminus J_e \). Then, for any \( kG_e \)-module \( V \) and \( kM/kI \)-module \( W \), one has natural isomorphisms

\[
\text{Ext}^n_{kM}(\text{Ind}_e(V), W) \cong \text{Ext}^n_{kG_e}(V, \text{Res}_e(W))
\]

\[
\text{Ext}^n_{kM}(W, \text{Coind}_e(V)) \cong \text{Ext}^n_{kG_e}(\text{Res}_e(W), V)
\]

for all \( n \geq 0 \).

Since \( kG_e \) is semisimple whenever \( k \) is of characteristic zero, we obtain the following corollary.

**Corollary 2.8.** Let \( M \) be a finite regular monoid and \( k \) a field of characteristic 0. Let \( e \in E(M) \) and \( I = MeM \setminus J_e \). Then, for any \( kG_e \)-module \( V \) and \( kM/kI \)-module \( W \), one has

\[
\text{Ext}^n_{kM}(\text{Ind}_e(V), W) = 0
\]

\[
\text{Ext}^n_{kM}(W, \text{Coind}_e(V)) = 0
\]

for all \( n \geq 1 \).

Nico \([48, 49]\) proved that the global dimension of a regular monoid over a field of characteristic zero is always finite.

**Theorem 2.9** (Nico). Let \( M \) be a finite regular monoid and let \( k \) be a field of characteristic 0. Then \( \text{gl. dim} \, kM \) is bounded by \( 2(m - 1) \) where \( m \) is the length of the longest chain of non-zero principal ideals of \( M \).

### 2.4. The character theory of the full transformation monoid

The character theory of \( \Sigma_n \) has a very long history, beginning with the work of Hewitt and Zuckerman \([34]\). A complete computation of the character table of \( \Sigma_n \) was finally achieved by Putcha nearly 40 years later \([51, \text{Theorem 2.1}]\). To formulate his result, first let \( e_r \in \Sigma_n \) be the idempotent given by

\[
e_r(i) = \begin{cases} i, & \text{if } i \leq r \\ 1, & \text{if } i > r \end{cases}
\]
and note that $e_1, \ldots, e_n$ form a complete set of idempotent representatives of the $\mathcal{S}$-classes of $\mathcal{T}_n$ and $e_r \mathcal{T}_n e_r \cong \mathcal{T}_r$, whence $G_{e_r} \cong \mathcal{S}_r$. The isomorphism takes $f \in e_r \mathcal{T}_n e_r$ to $f|_{[r]}$ where $[r] = \{1, \ldots, r\}$.

The reader is referred to [37] for the representation theory of the symmetric group. If $\lambda$ is a partition of $n$, then $S_{\lambda}$ will denote the corresponding simple module (Specht module). Let us use $1^k$ as short hand for a sequence of $k$ ones occurring in a partition. With this notation, $S_{(1^r)}$ is the sign representation of $\mathcal{S}_r$. On the other hand, $S_{(r)}$ is the trivial representation of $\mathcal{S}_r$. Thus, we consider $S_{(1)}$ to be both the trivial representation and the sign representation of $\mathcal{S}_1$.

In the next theorem, we retain the notation of Theorem 2.10.

**Theorem 2.10** (Putcha). Fix $n \geq 1$ and let $S_{\lambda} \in \text{Irr}_C(\mathcal{S}_r)$ for $1 \leq r \leq n$.

(i) If $S_{\lambda}$ is not the sign representation $S_{(1^r)}$, then $\text{Ind}_{\mathcal{S}_r}(S_{\lambda})$ is simple (and hence equal to $S_{\lambda}^\sharp$). Moreover, its restriction to $\mathcal{S}_n$ is isomorphic to the induced module $\mathcal{S}_n \otimes_C [\mathcal{S}_r \times S_{n-r}] (S_{\lambda} \otimes S_{(n-r)})$.

(ii) If $S_{(1^r)}$ is the sign representation, then the restriction of $S_{(1^r)}^\sharp$ to $\mathcal{S}_n$ is the simple module $S_{(n-r+1,1^{r-1})}$ of dimension $\binom{n-1}{r-1}$.

Let us remark that Putcha uses very different notation than ours. If $\theta$ is an irreducible representation of $\mathcal{S}_r$ afforded by a simple module $V$, then Putcha uses $\theta^+$ to denote the representation afforded by $\text{Ind}_{\mathcal{S}_r}(V)$ and $\theta^-$ to denote the representation afforded by $\text{Coind}_{\mathcal{S}_r}(V)$. He uses $\bar{\theta}$ for the representation afforded by $V^\sharp$.

We proceed to give a new proof of Theorem 2.10(ii), which is simpler than Putcha’s, by exhibiting the simple module. Putcha has informed the author that he is aware of this construction. The key observation is that $S_{(n-r+1,1^{r-1})}$ is an exterior power of the standard representation of $\mathcal{S}_n$.

Note that $\mathbb{C}^n$ is a $\mathcal{S}_n$-module by defining $f v_i = v_{f(i)}$, for $f \in \mathcal{T}_n$, where $v_1, \ldots, v_n$ denotes the standard basis for $\mathbb{C}^n$. Moreover, the augmentation

$$\text{Aug}(\mathbb{C}^n) = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$$

is a $\mathcal{S}_n$-submodule and $\mathbb{C}^n / \text{Aug}(\mathbb{C}^n)$ is the trivial module.

**Theorem 2.11.** Let $n \geq 1$ and let $V = \text{Aug}(\mathbb{C}^n)$. Then the exterior power $\Lambda^{r-1}(V)$, for $1 \leq r \leq n$, is a simple $\mathcal{S}_n$-module with apex $e_r$ and with $e_r \Lambda^{r-1}(V) \cong S_{(1^r)}$ the sign representation of $G_{e_r} \cong \mathcal{S}_r$.

**Proof.** By [29] Proposition 3.12, the exterior power $W = \Lambda^{r-1}(V)$ is a simple $\mathcal{S}_n$-module and hence a simple $\mathcal{S}_n$-module. If $r = 1$, then $W$ is the trivial module, which has apex $e_1$ and $e_1 W$ is the trivial $\mathcal{S}_1$-module, which is also the sign representation in this case. So assume that $r > 1$. Notice that $\mathbb{C}^n / V$ is the trivial $\mathcal{S}_n$-module. It follows easily that the rank of $f \in \mathcal{T}_n$ as an operator on $V$ is $1$ less than the rank of $f$ as a mapping. Thus $W$ is annihilated by all mappings $f$ of rank less than $r$ (since each exterior power drops the rank by $1$). Therefore, $e_r$, which is a rank $1$ operator on $W$,
is an apex. Recall that \( W \) has basis \( v_{i_1} \wedge \cdots \wedge v_{i_r} \) where \( i_1 < \cdots < i_r \). Then \( e_rW \) is the one-dimensional subspace spanned by the vector \( v = v_{i_1} \wedge \cdots \wedge v_{i_r} \). Moreover, if \( g \in G_{e_r} \cong \mathfrak{S}_r \), then \( gv = v_{g(1)} \wedge \cdots \wedge v_{g(r)} = \text{sgn}(g|_r)v \) and so \( e_rW \) is the sign representation of \( G_{e_r} \cong \mathfrak{S}_r \).

3. The global dimension of \( \mathbb{C} \mathfrak{S}_n \)

In this section, we prove that the global dimension of \( \mathbb{C} \mathfrak{S}_n \) is \( n - 1 \). Our first goal is to provide the minimal projective resolutions of the exterior powers of \( \mathbb{C}^n \). We retain the notation of the previous sections.

3.1. Minimal projective resolutions of the exterior powers. Recall that \( \mathbb{C}^n \) is a \( \mathbb{C} \mathfrak{S}_n \)-module. It turns out that each exterior power of \( \mathbb{C}^n \) is a projective indecomposable module.

**Proposition 3.1.** For each \( r \) with \( 1 \leq r \leq n \), the \( \mathbb{C} \mathfrak{S}_n \)-module \( \Lambda^r(\mathbb{C}^n) \) is a projective module.

**Proof.** Let \( e_r \) be the idempotent from (2.1) and \( W = \mathbb{C}^n \). Again denote by \( v_1, \ldots, v_n \) the standard basis for \( W \). Observe that \( \mathfrak{S}_n e_r \) consists of all the mappings \( f \in \mathfrak{S}_n \) with \( f(x) = f(1) \) for \( x > r \). We can thus define an isomorphism of \( \mathbb{C} \mathfrak{S}_n \)-modules \( \rho : \mathbb{C} \mathfrak{S}_n e_r \rightarrow W^{\otimes r} \) by

\[
\rho(f) = v_{f(1)} \otimes \cdots \otimes v_{f(r)}
\]

for \( f \in \mathfrak{S}_n e_r \). We can identify \( G_{e_r} \) with the symmetric group \( \mathfrak{S}_r \) via the isomorphism \( g \mapsto g|_r \). Under this identification, we see that if \( g \in G_{e_r} \) and \( f \in \mathfrak{S}_n e_r \), then

\[
\rho(fg) = v_{f(g(1))} \otimes \cdots \otimes v_{f(g(r))} = \rho(f)g|_r
\]

where \( \mathfrak{S}_r \) acts on the right of \( W^{\otimes r} \) by permuting the tensor factors.

Let \( \eta = \frac{1}{r!} \sum_{g \in G_{e_r}} \text{sgn}(g|_r)g \). Then \( \eta \) is a primitive idempotent of \( \mathbb{C} G_{e_r} \) and \( \mathbb{C} G_{e_r} \eta \) affords the sign representation of \( G_{e_r} \cong \mathfrak{S}_r \). By definition, \( \Lambda^r(W) = W^{\otimes r} \otimes_{\mathbb{C} G_{e_r}} S_{(1^r)} \) and hence

\[
\Lambda^r(\mathbb{C}^n) \cong \mathbb{C} \mathfrak{S}_n e_r \otimes_{\mathbb{C} G_{e_r}} S_{(1^r)} = \mathbb{C} \mathfrak{S}_n e_r \otimes_{\mathbb{C} G_{e_r}} \mathbb{C} G_{e_r} \eta \cong \mathbb{C} \mathfrak{S}_n \eta.
\]

Since \( \eta \in \mathbb{C} \mathfrak{S}_n \) is an idempotent, we deduce that \( \Lambda^r(\mathbb{C}^n) \) is a projective \( \mathbb{C} \mathfrak{S}_n \)-module.

We shall need the following lemma about exterior powers to see that \( \Lambda^r(\mathbb{C}^n) \) is indecomposable and to compute its radical.

**Lemma 3.2.** Let \( 1 \leq r \leq n \) and let \( V = \text{Aug}(\mathbb{C}^n) \). Then there is a short exact sequence of \( \mathbb{C} \mathfrak{S}_n \)-modules

\[
0 \longrightarrow \Lambda^r(V) \longrightarrow \Lambda^r(\mathbb{C}^n) \longrightarrow \Lambda^{r-1}(V) \longrightarrow 0.
\]
Proof. Clearly, $\Lambda^r(V)$ is a submodule of $\Lambda^r(C^n)$. Let $v_1, \dots, v_n$ be the standard basis for $C^n$ and put $w_i = v_i - v_n$ for $i = 1, \dots, n-1$. Then $w_1, \dots, w_{n-1}$ is a basis for $V$. We claim that $\Lambda^r(C^n)/\Lambda^r(V) \cong \Lambda^{r-1}(V)$.

Put $w_n = v_1 + \cdots + v_n$. Then $w_1, \dots, w_n$ is a basis for $C^n$. Define $\rho: \Lambda^r(C^n) \to \Lambda^{r-1}(V)$ on the basis of $r$-fold wedge products of $w_1, \dots, w_n$ by

$$\rho(w_{i_1} \wedge \cdots \wedge w_{i_r}) = \begin{cases} w_{i_1} \wedge \cdots \wedge w_{i_{r-1}}, & \text{if } i_r = n \\ 0, & \text{else.} \end{cases}$$

for $1 \leq i_1 < \cdots < i_r \leq n$. This is clearly a surjective linear map with kernel $\Lambda^r(V)$. Let us check that it is a $C\Sigma_n$-module homomorphism. If $g \in \Sigma_n$, then since $C^n/V$ is the trivial module, it follows that $g w_n + V = w_n + V$ and so $g w_n = w_n + v$ with $v \in V$. Therefore, if $1 \leq i_1 < \cdots < i_{r-1} \leq n - 1$, then

$$g(w_{i_1} \wedge \cdots \wedge w_{i_{r-1}} \wedge w_n) = gw_{i_1} \wedge \cdots \wedge gw_{i_{r-1}} \wedge w_n + gw_{i_1} \wedge \cdots \wedge gw_{i_{r-1}} \wedge v$$

$$\in gw_{i_1} \wedge \cdots \wedge gw_{i_{r-1}} \wedge w_n + \Lambda^r(V).$$

We conclude that

$$g\rho(w_{i_1} \wedge \cdots \wedge w_{i_{r-1}} \wedge w_n) = gw_{i_1} \wedge \cdots \wedge gw_{i_{r-1}} = \rho(g(w_{i_1} \wedge \cdots \wedge w_{i_{r-1}} \wedge w_n)).$$

This completes the proof. \hfill $\square$

Let us deduce the following important corollary.

**Corollary 3.3.** For $1 \leq r \leq n$, the exterior power $\Lambda^r(C^n)$ is a projective indecomposable module with $\text{rad} (\Lambda^r(C^n)) = \Lambda^r(Aug(C^n))$ and simple top $\Lambda^{r-1}(Aug(C^n))$.

**Proof.** Proposition 3.1 yields that $P = \Lambda^r(C^n)$ is projective. As the modules $\Lambda^r(Aug(C^n))$ and $\Lambda^{r-1}(Aug(C^n))$ are simple by Theorem 2.11 Lemma 3.2 yields that $\text{rad} (P) = \Lambda^r(Aug(C^n))$ and $P/\text{rad} (P) \cong \Lambda^{r-1}(Aug(C^n))$. Since a projective module is indecomposable if and only if it has simple top \cite{2}, we deduce that $P$ is a projective indecomposable module, as required. \hfill $\square$

Lemma 3.2 allows us to construct the minimal projective resolution of $\Lambda^{r-1}(Aug(C^n))$ for $1 \leq r \leq n$.

**Corollary 3.4.** Let $v_1, \dots, v_n$ be the standard basis for $C^n$ and let $w_i = v_i - v_n$ for $1 \leq i \leq n - 1$ and let $w_n = v_1 + \cdots + v_n$. Let $V = Aug(C^n)$. Then, for $1 \leq r \leq n$, the minimal projective resolution of the simple module $\Lambda^{r-1}(V)$ is

$$0 \to P_{n-r} \xrightarrow{d_{n-r}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} \Lambda^{r-1}(V) \to 0$$

where $P_q = \Lambda^{q+r}(C^n)$ and

$$d_q(w_{i_1} \wedge \cdots \wedge w_{i_{q+r}}) = \begin{cases} w_{i_1} \wedge \cdots \wedge w_{i_{q+r-1}}, & \text{if } i_{q+r} = n \\ 0, & \text{else.} \end{cases}$$
Therefore, $pd \Lambda^{r-1}(V) = n - r$ for $1 \leq r \leq n$. In particular, the projective dimension of the trivial $\mathbb{C}S_n$-module is $n - 1$.

Proof. The exactness of the resolution follows from repeated application of Lemma 3.2 (and its proof). Corollary 3.3 implies that each mapping $d_q : \Lambda^n(C^n) \rightarrow d_q(\Lambda^n(C^n)) = \Lambda^{n-1}(V)$ is a projective cover and so the resolution is a minimal projective resolution. The final statement holds because $\Lambda^0(V)$ is the trivial $\mathbb{C}S_n$-module. □

It follows from Corollary 3.3 that the global dimension of $\mathbb{C}S_n$ is at least $n - 1$. The next subsection will show that this lower bound is tight. In fact, Corollary 3.4 yields that the cohomological dimension of $\Lambda^n(C^n)$ is $n - 1$ (the cohomological dimension of a monoid $M$ over a base ring $R$ is the projective dimension of the trivial $RM$-module).

3.2. A computation of the global dimension. We first compute $\text{Ext}^n$ from an exterior power of the augmentation submodule of $\mathbb{C}^n$.

Proposition 3.5. For $1 \leq k, r \leq n$ and $\lambda$ a partition of $k$, one has that

$$\text{Ext}^n_{\mathbb{C}S_n}(S^\lambda_{(1^r)}, S^k_\lambda) \cong \begin{cases} \mathbb{C}, & \text{if } \lambda = (1^{r+m}) \\ 0, & \text{else.} \end{cases}$$

Proof. Let $V = \text{Aug}(\mathbb{C}^n)$. Recall that $S^k_{(1^r)} \cong \Lambda^{r-1}(V)$ by Theorem 2.11. Using the minimal projective resolution for $\Lambda^{r-1}(V)$ from Corollary 3.3 and Proposition 2.4, we deduce that

$$\text{Ext}^n_{\mathbb{C}S_n}(\Lambda^{r-1}(V), S^k_{\lambda}) \cong \text{Hom}_{\mathbb{C}S_n}(\Lambda^{r+m}(\mathbb{C}^n), S^k_{\lambda})$$

$$\cong \text{Hom}_{\mathbb{C}S_n}(\Lambda^{r+m-1}(V), S^k_{\lambda})$$

where the last isomorphism uses that $\Lambda^{r+m}(\mathbb{C}^n)$ has simple top $\Lambda^{r+m-1}(V)$ by Corollary 3.3. Recalling that $\Lambda^{r+m-1}(V) \cong S^k_{(1^{r+m})}$ by Theorem 2.11 the result follows. □

We next prove a vanishing result when the first variable is not one of the exterior powers of the augmentation submodule of $\mathbb{C}^n$.

Proposition 3.6. Let $n \geq 1$ and $1 \leq k, r \leq n$. Let $S_\lambda$ be a simple $\mathbb{C}S_r$-module with $\lambda \neq (1^r)$ and let $S_\mu$ be a simple $\mathbb{C}S_k$-module. If $k > r$, then we have $\text{Ext}^m_{\mathbb{C}S_n}(S^\lambda_{(1^r)}, S^k_\mu) = 0$ for all $m \geq 0$. If $k \leq r$, then $\text{Ext}^m_{\mathbb{C}S_n}(S^\lambda_{(1^r)}, S^k_\mu) = 0$ for $m > r - k$ and, in particular, for $m > n - 1$.

Proof. Suppose first that $k > r$. Let $I_{r-1} \subseteq \mathbb{T}_n$ be the ideal of mappings of rank at most $r - 1$, where we take $I_0 = \emptyset$, and put $A = \mathbb{C}S_n/I_{r-1}$. Since $S^k_\lambda = \text{Ind}_e(\lambda)$ by Theorem 2.10 and $S^k_\mu$ is an $A$-module, we conclude that $\text{Ext}^m_{\mathbb{C}S_n}(S^\lambda_{(1^r)}, S^k_\mu) = 0$ for all $m \geq 1$ by Corollary 2.3. Since there are no homomorphisms from $S^\lambda_{(1^r)}$ to $S^k_\mu$ if $k > r$, this completes the first case.
Next suppose that \( k \leq r \). We proceed by induction on \( r - k \). If \( r = k \), then the argument of the previous paragraph shows that \( \text{Ext}^m_{\mathbb{C}T_n}(S^2_\lambda, S^2_\mu) = 0 \) for all \( m > 0 = r - k \). Assume that the result is true for \( k' \) with \( k \leq k' \leq n \) and that \( k < r \). Let \( A = \mathbb{C}T_n/CI_{k-1} \) and consider the exact sequence of \( A \)-modules

\[
0 \longrightarrow S^2_\mu \longrightarrow \text{Coind}_{e_k}(S_\mu) \longrightarrow \text{Coind}_{e_k}(S_\mu)/S^2_\mu \longrightarrow 0.
\]

Since \( S^2_\lambda \) is an \( A \)-module, Corollary 2.8 implies that, for \( m > r - k \geq 1 \),

\[
\text{Ext}^m_{\mathbb{C}T_n}(S^2_\lambda, S^2_\mu) \cong \text{Ext}^{m-1}_{\mathbb{C}T_n}(S^2_\lambda, \text{Coind}_{e_k}(S_\mu)/S^2_\mu) = 0
\]

where the last equality uses that each composition factor of \( \text{Coind}_{e_k}(S_\mu)/S^2_\mu \) has apex \( e_\mu \) with \( \mu \geq k \) by Theorem 2.2, induction and Lemma 2.5. This completes the proof.

We are now prepared to prove the main result of the paper.

**Theorem 3.7.** The global dimension of \( \mathbb{C}T_n \) is \( n - 1 \) for all \( n \geq 1 \).

**Proof.** By Proposition 3.5 we have that \( \text{Ext}^{n-1}_{\mathbb{C}T_n}(S^2_{(1)}, S^2_{(1^n)}) \cong \mathbb{C} \). Proposition 3.5 and Proposition 3.6 yield that \( \text{Ext}^m_{\mathbb{C}T_n}(S^2_\lambda, S^2_\mu) = 0 \) for all simple modules \( S^2_\lambda, S^2_\mu \) and \( m \geq n \). This establishes that \( \text{gl. dim} \mathbb{C}T_n = n - 1 \).

In fact, since \( \mathbb{Q} \) is a splitting field for all symmetric groups, and hence for all full transformation monoids, the above argument works mutatis mutandis to prove \( k\Sigma_n \) has global dimension \( n - 1 \) for all fields \( k \) of characteristic 0.

Recall that if \( A \) is a finite dimensional algebra over an algebraically closed field, then the quiver of \( A \) is the directed graph with vertices the isomorphism classes of simple \( A \)-modules and edges as follows. If \( S_1 \) and \( S_2 \) are simple \( A \)-modules, the number of directed edges from the isomorphism class of \( S_1 \) to the isomorphism class of \( S_2 \) is \( \dim \text{Ext}^1_A(S_1, S_2) \). See [2, 4, 9] for details.

**Corollary 3.8.** The quiver of \( \mathbb{C}T_n \) is acyclic for all \( n \geq 1 \).

**Proof.** Proposition 3.5 implies that the only arrow exiting \( S^2_{(1^r)} \) is the arrow \( S^2_{(1^r)} \rightarrow S^2_{(1^{r+1})} \) for \( 1 \leq r \leq n - 1 \) and that \( S^2_{(1^n)} \) is a sink. Proposition 3.6 implies that all other arrows go from a module with apex \( e_k \) to a module with apex \( e_j \) with \( j < k \). It follows immediately that the quiver of \( \mathbb{C}T_n \) is acyclic.

It is an open question to compute the quiver of \( \mathbb{C}T_n \) for \( n \geq 5 \). The quiver for \( 1 \leq n \leq 4 \) can be found in [52].

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