Spontaneous emission of light from atoms: the model

P. Marecki\(^1\) and N. Szpak\(^2\)

\(^1\) Wyzsza Szkola Informatyki i Zarzadzania, Bielsko-Biala, Poland
\(^2\) Institute for Theoretical Physics, J.W. Goethe University, Frankfurt/Main, Germany

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We investigate (non-relativistic) atomic systems interacting with quantum electromagnetic field (QEF). The resulting model describes spontaneous emission of light from a two-level atom surrounded by various initial states of the QEF. We assume that the quantum field interacts with the atom via the standard, minimal-coupling Hamiltonian, with the \(A^2\) term neglected. We also assume that there will appear at most single excitations (photons). By conducting the analysis on a general level we allow for an arbitrary initial state of the QEF (which can be for instance: the vacuum, the ground state in a cavity, or the squeezed state). We derive a Volterra-type equation which governs the time evolution of the amplitude of the excited state. The two-point function of the initial state of the QEF, integrated with a combination of atomic wavefunctions, forms the kernel of this equation.

1 Introduction

The phenomenon of emission of photons during the transition between atomic energy-levels is one of the most important, and well understood aspect of contemporary theoretical physics. The explanations of this process which have already been presented in the early days of quantum theory [1], vary in their level of sophistication. The desirable formulation, in the spirit of quantum theory, describes the process as a unitary evolution of the full system which is composed of the atom and the quantum electromagnetic field. It is disappointing that modern literature in most cases reports models which either do not fit to this basic requirement of quantum theory, or at some point involve unjustified operations with (unnecessarily)

* Corresponding author E-mail: piotrm@wsi.edu.pl
singular expressions\(^1\). One has an unpleasant feeling that the outcome (the exponential decay of the excited-state’s amplitude) is simply assumed at some point of the investigations. While we do not claim that no clean model exists, we believe a simple field-theoretical model which is additionally more general with respect to the quantum electromagnetic field (QEF), is worth presenting.

It is generally recognized that the atomic emission of radiation is influenced by both: the atomic structure (in particular: the energies and wavefunctions of the levels under consideration) and the initial state of the QEF (which can be a strong, coherent state or a vacuum, for instance).

With respect to the latter we note that there exists a great variety of states of QEF. On the level of local quantum field theory \cite{3}, it is known that even disjoint representations of the algebra of fields (the Canonical Commutation Relations (CCR)-algebra) can be constructed\(^2\). Each such representation carries its own notion of excitations (“photons”). A general method of Gelfand, Naimark and Segal (GNS) allows to construct a representation once a single “reference state” \(\Omega\) is specified. If we deal with fields propagating on the Minkowski spacetime, the Fock representation, based on the usual vacuum, is physically distinguished. However, if we were interested in some generic spacetimes (or time-dependent environments) there would be no privileged “reference state”. On the mathematical level, one is content with a picture of many permissible representations, each based on a different “reference state” \(\Omega\), although such a state of affairs is certainly conceptually worrying (there is no unique answer to the question what a photon is).

On the other hand, experimental advances in the area of quantum optics have produced a variety of states of light which exhibit non-classical features. The so-called squeezed states or the ground states of the QEF in small cavities provide environments which are much different from the usual Minkowski-spacetime vacuum. In particular, the experiments with atoms surrounded by squeezed states, \cite{4}, revealed a modification of their spectroscopic characteristics.

One of the simplest ways to test the properties of various states of QEF is to let the quantum field interact with a quantum system of relatively simple structure. As an example, we can consider a “two-level atom”, assume a simple minimal-coupling interaction with the QEF, and investigate the time-evolution of various initial states of the full system. Such an approach is also experimentally viable, as this evolution is nowadays fully accessible to measurements (see e.g. \cite{5}). In order to illustrate the potential of such an approach we remark on a possibility to measure the Unruh effect with the help of an electron, namely, a simple perturbative calculation shows that the state of a two-level quantum system coupled to the QEF initially in the KMS (thermal) state thermalizes to the temperature of the KMS state. As originally suggested by Bell and Leinaas \cite{6}, a spin of a linearly-accelerated electron might serve as a detector of the Unruh effect.

\(^1\) See, for instance, \cite{2} chapter 6.3.
\(^2\) It is possible, for instance, to prescribe two states of the QEF which cannot be expressed as density-operator states in a common Fock space.
temperature\(^3\). It is one of the aims of this paper to pursue further this type of models in which test atomic systems are employed as detectors of properties of various states of the QEF.

The paper contains a field-theoretical model of an interaction of a two-level atom with the QEF. We assume the atom to be initially in an excited state, and let it interact with the QEF; the dynamics of such a system is usually referred to as the \textit{spontaneous emission of radiation}. It has been investigated in the past by many authors with various emphasis on mathematical rigor, on the one hand, and on physical concreteness on the other. We shall briefly summarize these attempts here, in order to put our model into a context.

The simplest way to estimate the dynamics of the full system is to use the time-dependent perturbation theory w.r.t. the interaction of the QEF and the atom (Eq. (3)). By doing so, one can approximate the initial dynamics of the emission of a photon, however, the long-term behavior of this process cannot be estimated in this way. Another method, due to Weisskopf and Wigner \cite{1}, establishes an ordinary integro-differential equation for the time-dependence of the amplitude of the excited state. The model we shall present here generalizes the Weisskopf-Wigner model with respect to the allowed initial states of the quantum radiation field.

With an alternative approach, Bach and collaborators \cite{7} have investigated systems of non-relativistic particles coupled to the QEF from the functional-analytic point of view. Without any assumption on the number of atomic energy levels, they worked with the full minimal coupling Hamiltonian (together with the quadratic term, and an ultraviolet cutoff), and have been able to prove the existence of a ground state of the coupled system. Moreover, an estimate has been derived for the life-times of the states of the atom which were stationary when the interaction was absent.

A common feature of all these developments is, that the choice of the initial state of the QEF is made at a very early stage of investigations. Consequently, given the final equation for the excited state’s amplitude, it is difficult to discern which features of this equation are related to the initial state of the QEF, and which are influenced by the atomic wavefunctions\(^5\).

In this paper we attempt to generalize the Weisskopf-Wigner model; in particular we proceed in a general way, so that the initial state of the QEF can be left unspecified to the very end. By doing so, we prove that the spontaneous emission process is governed by a Volterra integro-differential equation (Eq. 3).

\(^3\) The usual vacuum state, when observed by an accelerated observer, appears to be a thermal state, with an acceleration-dependent temperature.

\(^4\) See, for instance, the textbook of Scully and Zubairy \cite{2}.

\(^5\) In an interesting paper Bondarev and Lambin \cite{8}, investigate the spontaneous emission of light from atoms placed inside of small cavities (carbon nanotubes). Their explicit numerical results complement our analysis and exemplify the possible complexity of the emission process.
the kernel of which is composed of the two-point function, \((\Omega, A_i(x)A_j(y)\Omega)\), of the initial state of the radiation field, smeared with test functions which are derived from the wavefunctions of the ground- and excited-state of the atom. As the two-point functions are the foundation of the GNS construction, and on the other hand are known for physically interesting states, this opens up a possibility to investigate various initial states of radiation on an equal basis.

2 Formulation of the model

2.1 General setting

A natural Hilbert space for a system of an atom (non-relativistic) coupled to the quantum radiation field is

\[
\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F},
\]

where \(\mathcal{F}\) denotes the physical (transversal) Fock space of the radiation field. This space will be later restricted to a subspace, consisting of a two-dimensional Hilbert space of atomic states, tensor-multiplied with the vacuum- and one-excitation-subspace of \(\mathcal{F}\). The single excitations (over the reference state \(\Omega\)) of the quantum radiation field, denoted by the symbol \(|f\rangle\), will be described by

\[
|f\rangle = A(f)\Omega = \int d^3x f_i(x)A^i(0, x)\Omega,
\]

where \(A_i(0, x)\) is the (unsmearred) vector-potential operator at \(t = 0\). The complex-valued test function \(f_i(x)\) will carry the information about the "wave packet" of the excitation\(^6\).

The interaction between the atom and the QEF will be generated by

\[
V(t, x) = -\sqrt{\alpha} \ A^i(t, x) \cdot p_i,
\]

where\(^7\) \(p_i = -i\partial_i\), \(\alpha\) is the fine-structure constant, and \(A^i(t, x)\) denotes the vector-potential operator, in the radiation gauge, with the free time-evolution already implemented\(^8\). This is a standard interaction Hamiltonian with the \(A^2\) term neglected.

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\(^6\) If \(\Omega\) is the vacuum on the Minkowski spacetime, the excitations are photons. In this case \(A(f)\Omega = \int \frac{d^3p}{(2\pi)^3} f^\alpha(p)a_\alpha^\dagger(p)\Omega\), where \(a_\alpha^\dagger(p)\) is the creation operator for \(\alpha\)-th polarization and \(f^\alpha(p) = \int d^3x f^i(x)\epsilon_i^\alpha(p) e^{-ipx}\). The symbols \(\epsilon_i^\alpha(p)\) denote the transversal polarization vectors.

\(^7\) In the paper we use dimensionless units (see appendix A). By the letter \(p\) we denote the frequency: \(p = |p|\).

\(^8\) Which means that the free time-evolution of \(A^i(t, x)\) is governed by the Maxwell equations.
2.2 Dynamics of the restricted system

In this section we will consider the evolution of the above system in the situation where the radiation field is initially in the state $\Omega$, about which we only assume that the expectation value of an odd number of field operators in this state vanishes\(^9\), and the atom is initially excited.

The Hilbert space of the atomic motion will be restricted to a two-dimensional subspace (two energy levels), and the Fock space will be restricted to the zero- and one-excitation-subspace\(^10\).

We start by restricting the interaction (3) to the subspace of only two energy-levels. To do so, we substitute the interaction operator $V$ by $V_2$:

$$V_2 = \sum_{ij=0,1} (\psi_i, V \psi_j)_{L^2} |\psi_i\rangle \langle \psi_j|,$$

$$(\psi_i, V \psi_j)_{L^2} = \int d^3x \overline{\psi_i(x)} V \psi_j(x),$$

where we have employed the Dirac’s notation. We obtain\(^11\):

$$V_2 = i\sqrt{\alpha} [\psi_1 \langle \psi_0 | \otimes A_t(\chi) + |\psi_0\rangle \langle \psi_1| \otimes A_t(\varphi)],$$

(4)

where

$$\chi_1(x) = \overline{\psi_1(x)} \partial_i \psi_0(x),$$

$$\varphi_1(x) = \overline{\psi_0(x)} \partial_i \psi_1(x)$$

$$A_t(\chi) = \int d^3x A_i(t, x) \chi_i(x).$$

Because of $A_t(\chi) = -A_t(\varphi)$, the interaction can be written as

$$V_2 = \sqrt{\alpha} \sigma_2 \otimes A_t(\chi),$$

(5)

which is a selfadjoint operator on $C^2 \otimes F$. At this point we remark that the process of “spontaneous emission of radiation” will be generated by a selfadjoint Hamiltonian, and thus will be governed by a unitary evolution. In this way the basic requirement of the quantum theory will be fulfilled.

We shall now additionally restrict the investigations to the subspace of at most one excitation of the QED\(^12\). Later, we will employ the interaction picture and derive a system of evolution equations. At the

\(^9\) The vacuum, the ground state, the squeezed state and a general class of quasi-free states fulfill this requirement.

\(^10\) Every vector $v$ of such a space can be written as: $v = c\Omega + A(f)\Omega$ with a complex number $c$ and complex, vector-valued functions $f^i(x)$.

\(^11\) The terms $|\psi_1\rangle \langle \psi_1|$ and $|\psi_0\rangle \langle \psi_0|$ drop out, if the wavefunctions are real (which we hereby assume), because $\int \overline{\psi_0(\partial_i \psi_0)} A^i = 0$, as a consequence of the chosen gauge condition $\partial_i A^i = 0$.

\(^12\) This restriction allows us to derive a single equation for the evolution of the excited-state’s amplitude (Eq. (15)). We expect such a restriction to be justified in some cases, e.g. where one-photon transitions are known (from experiments) to be dominant.
time \( t \), the state of the full system, in the restricted space, can always be parameterized by two complex, vector-valued functions \( f_t(x), g_t(x) \) (excitation wavepackets), and two complex numbers \( c(t), d(t) \):

\[
S = c(t) \psi_1 \otimes \Omega + d(t) \psi_0 \otimes \Omega + \psi_0 \otimes A_0(f_t)\Omega + \psi_1 \otimes A_0(g_t)\Omega. \tag{6}
\]

We note that we speak of excitations because the state \( \Omega \) is still unspecified. We use the following initial conditions

\[
c(0) = 1,
\]

\[
f_0(x), g_0(x), d(0) = 0.
\]

For such conditions the functions \( g_t(x) \) and \( d(t) \) never acquire non-zero values, and will be omitted from now on. In the interaction picture, we reparameterize the state of the full system as follows:

\[
S_I(t) = c(t) \psi_1 \otimes \Omega + \psi_0 \otimes A(f_t)\Omega. \tag{7}
\]

The time evolution of \( c(t) \) and \( f_t \) needs to be determined; it will be generated by the operator \( V_2^I \), which is the operator \( V_2 \) in the interaction picture:

\[
V_2^I = i \sqrt{\alpha} \left( e^{i\omega t}|\psi_1\rangle\langle\psi_0| - e^{-i\omega t}|\psi_0\rangle\langle\psi_1| \right) \otimes A_t(\chi), \tag{8}
\]

where \( \omega = E_1 - E_0 \) is the energy difference of both states of the atom. We shall introduce a decomposition of \( f_t \) in terms of an orthonormal basis \( \{f_m\} \) of the one-excitation space

\[
A(f_t)\Omega = \sum_{m=1}^{\infty} c_m(t) A(f_m)\Omega, \tag{9}
\]

where \( f_m \) are complex functions, orthonormal w.r.t. the scalar product

\[
(f, g) = \int d^3x \omega_{ij}(0, x, 0, y) f^i(x)g^j(y). \tag{10}
\]

The evolution equation,

\[
\frac{dS_I}{dt}(t) = V_2^I(t) S_I(t), \tag{11}
\]

after taking scalar products with \( \psi_1 \otimes \Omega \) and \( \psi_0 \otimes A(f_n)\Omega \), leads to an infinite system of ordinary differential equations:

\[
i \dot{c}(t) = i \sqrt{\alpha} e^{i\omega t} \cdot \sum_{m=1}^{\infty} c_m(t) \langle \Omega, A_t(\chi)A(f_m)\Omega \rangle, \tag{12a}
\]

\[
i \dot{c}_n(t) = -i \sqrt{\alpha} e^{-i\omega t} \cdot c(t) \langle \Omega, A(f_n)A_t(\chi)\Omega \rangle. \tag{12b}
\]

\[\text{13} \]: Here, the two-point function of the initial state of the QEF, \( \omega_{ij}(t, x, s, y) = \langle \Omega, A_i(t, x)A_j(s, y)\Omega \rangle \), enters the investigations. The one-excitation space is separable for reasonable initial states \( \Omega \).
Integrating the second equation(s), (12b), from 0 to \( t \), with the initial condition \( c_n(0) = 0 \) (for all \( n \)), yields
\[
c_n(t) = -\sqrt{\alpha} \int_0^t ds \ e^{-i\omega s} c(s) \ (\Omega, A(f_n)A_\omega(\chi)\Omega).
\] (13)

After substituting this into the equation for \( c(t) \), (12a), we obtain
\[
\dot{c}(t) = -\alpha \int_0^t ds \ c(s) \ e^{i\omega(t-s)} \sum_{m=1}^\infty (\Omega, A(f_m)A_\omega(\chi)\Omega) (\Omega, A_t(\chi)A(f_m)\Omega),
\] (14)

and finally, as a consequence of the completeness of the \( \{ f_m \} \) in the one-excitation subspace of \( \mathcal{F} \), we get
\[
\dot{c}(t) = -\alpha \int_0^t ds \ c(s) \ e^{i\omega(t-s)} (\Omega, A_t(\chi)A_\omega(\chi)\Omega) = -\alpha \int_0^t ds \ c(s) e^{i\omega(t-s)} S(t,s).
\] (15)

with
\[
S(t,s) = \int d^3x \ d^3y \ \omega_{ij}(t,x,s,y) \chi^i(x)\chi^j(y).
\] (16)

The equation (15), valid for arbitrary initial states of the QEF, is a Volterra integro-differential equation of the second type. In the important case of a stationary initial state \( \Omega \), the smeared two-point function \( \omega(t,\chi,s,\chi) \) depends only on the difference \(^{14} t - s \), and therefore in this case we obtain a convolution Volterra integro-differential equation. Such an equation can be brought into the form of an integral equation. To show this, we integrate it w.r.t. the variable \( t \) from zero to \( T \) and (with an appropriate change of variables) obtain
\[
c(T) = 1 - \int_0^T Z(T - s)c(s) \ ds,
\] (17)

where
\[
Z(\tau) = \alpha \int_0^\tau dt \ S(t,0)e^{i\omega t}.
\] (18)

An alternative way to approach the equation (15), in the case of stationary initial states, is to perform a Laplace transform of it. For the transformed amplitude, \( \hat{c}(s) \), we obtain a relation
\[
\hat{c}(s) = \frac{1}{s + \alpha \hat{S}(s - i\omega)},
\] (19)

where \( \hat{S}(s) = \int_0^\infty \exp(-st) S(t,0) \ dt \). The inverse Laplace transform gives the original amplitude which we intend to compute:
\[
c(t) = \frac{1}{2\pi i} \int_C \frac{e^{st}}{s + \alpha \hat{S}(s - i\omega)} \ ds,
\] (20)

\(^{14} \) As a consequence, \( S(t,s) = S(t-s,0) \).
where the contour of integration, $C$, needs to be chosen parallel to the imaginary axis, to the right of all poles. In our case, the dynamics of the total system is unitary, and consequently there will be no poles of the integrand for $\text{Re}(s) > 0$. The standard procedure is to attempt to close the contour $C$ from the left ($\text{Re}(s) < 0$). The dominant long-term asymptotic behavior of $c(t)$ comes from the singularities of $\hat{c}(s)$ with the greatest $\text{Re}(s)$. In general, the integrand, analytically continued to $\text{Re}(s) < 0$, may exhibit branch points (which often lead to power-law decays) and simple poles (which lead to exponential decays). It is not possible to be any more specific here, without specifying $\Omega$, as the analytic properties $\hat{S}(s - i\omega)$ obviously depend crucially on it.

In any case, even for non-stationary initial states, the kernel of the equation (15) is a smooth function of its arguments (because the test functions $\chi$ are smooth), and therefore our problem always reduces to solving a Volterra-type integro-differential equation of the second kind with a smooth kernel. The numerical experience with this equation, for various initial states, shows that the solutions exhibit a non-trivial initial dynamics which is followed by a relatively uncomplicated, monotonic-decay phase\textsuperscript{15}.

We wish to defer a systematic study of the solutions of (15) to a future publication\textsuperscript{11}.

2.3 Special case: Minkowski vacuum as the initial state $\Omega$

The simplest case of great physical importance arises if one takes the usual vacuum state $\Omega$ as the initial state of QEF. Then, the electromagnetic field operator can be expressed in terms of the creation and annihilation operators,

$$A_i(t, x) = \frac{1}{\sqrt{2\pi^3}} \int \frac{d^3k}{\sqrt{2k^0}} \epsilon_\alpha^\ast(k) \left\{ a^\ast_\alpha(k) e^{ikx} + a_\alpha(k) e^{-ikx} \right\}, \quad (21)$$

where $\epsilon_\alpha^\ast(k)$ denote the two ($\alpha = 1, 2$) $k$-dependent, transversal polarization vectors, and the vacuum is defined via

$$a_\alpha(k) \Omega = 0. \quad (22)$$

The respective two-point function is well-known:

$$\omega_{ij}(t, s, x, y) = (\Omega, A_i(t, x) A_j(s, y) \Omega) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p} e^{-ip(t-s)} \left( \delta_{ij} - \frac{p_i p_j}{p^2} \right) e^{ip(x-y)}. \quad (23)$$

Denoting by $\chi_i(p)$ the Fourier transform of $\chi_i(x)$,

$$\chi_i(p) = \int d^3x e^{-ipx} \overline{\psi_1(x)} \partial_i \psi_0(x), \quad (24)$$

\textsuperscript{15} If the atom is placed inside of a reflecting cavity, however, $c(t)$ exhibits a completely different dynamics, due to the interaction with the emitted photon. In an extreme case this can even lead to oscillations of $|c(t)|^2$ which are called the vacuum Rabi oscillations [8, 9].
we get
\[ S(t - s) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p} e^{-ip(t-s)} \left( \delta^{ij} - \frac{p^i p^j}{p^2} \right) \chi_i(p)\chi_j(p) \] (25)

The function \( S(t - s) \) is evidently a smooth function of its arguments\(^{16}\), because \( \chi(p) \) is a smooth function of rapid decay\(^{17}\).

\[ \begin{align*}
\text{2.4 Another special case: squeezed state as the initial state} Ω \\
\text{The squeezed states are examples of states of the QEF which can be produced experimentally [2], and exhibit exotic, inherently quantum properties}^{18}. \text{These states appear as a coherent superposition of pairs of photons. The photons constituting a pair, in the case of a degenerate squeezed state, possess the same wavepacket, } f_α(p). (\text{We assume this function to be smooth and rapidly decaying.)} \text{The squeezed states are important in the context of this paper, as their presence is known (experimentally) to influence the atomic dynamics [4].}
\end{align*} \]

A squeezed state, \(|S⟩\), is a vector state created in the Fock space constructed upon the (distinguished, Minkowski-spacetime) vacuum with the help of a unitary operator \( S(f, r) \):
\[ |S⟩ = S(f, r)Ω = e^{r[a(f)a(f) - a^⋆(f)a^⋆(f)]}Ω \] (26)

where \( r \) is a real number which we interpret as the amplitude of squeezed light\(^{19}\). The operator \( a(f) \) denotes here the smeared annihilation operator\(^{20}\):
\[ a(f) = \int d^3p \overline{f_α(p)}a^α(p). \] (27)

The following (commutation) relations hold for an arbitrary test function \( g(p) \), and facilitate the calculation of various expectation values:
\[ \begin{align*}
S^*a(g)S &= a(g) - (g, f) a(f) + (g, f) [a(f)cosh r - a^*(f) sinh r] \quad \text{(28)} \\
S^*a^*(g)S &= a^*(g) - (f, g) a(f) + (f, g) [a^*(f)cosh r - a(f) sinh r], \quad \text{(29)}
\end{align*} \]

\(^{16}\) In particular, the limit \( s \to t \) is finite.

\(^{17}\) The smoothness of \( \chi(p) \) assures, that the point \( p = 0 \) is an integrable singularity. The decay property for large \(|p|\) makes the integral convergent for large momenta, even when the time derivatives of \( S(t - s) \) are taken.

\(^{18}\) Such as, for instance, a sub-vacuum level of electric field fluctuations.

\(^{19}\) As the arguments of \( S(f, r) \) will not be varied in the following discussion, we will briefly write \( S \) instead of \( S(f, r) \).

\(^{20}\) The index \( α \) is summed over, \( α = 1, 2 \), whenever it appears twice.
where \((f, g) = \int d^3p \overline{f_\alpha(p)}g^\alpha(p)\) stands for the scalar product of \(f^\alpha\) and \(g^\alpha\). With the help of these relations we find

\[
(S\Omega, a^\alpha_\alpha(p) a^\beta(k) S\Omega) = \overline{f_\alpha(p)} f_\beta(k) \sinh r \cosh r
\]

The two-point function of the radiation field, in the squeezed state, can now be computed (here \(x \equiv (x_0, x)\))

\[
\omega^S_{ij}(x, y) = (\Omega, A_i(x) A_j(y) \Omega) = \omega^0_{ij}(x, y) + \int \frac{d^3p d^3k}{2\sqrt{pk}} e^{-ipx - iky + c.c.} \sinh r \cosh r + \left[\overline{f_\alpha(p)} f_\beta(k) e^{ipx - iky + c.c.}\right] \sinh^2 r.
\]

Evidently, it differs from the vacuum-two-point function, \(\omega^0_{ij}(x, y)\), by a function which is smooth in both arguments.

If the wavepacket \(f_\alpha(p)\) is concentrated around a specific momentum \(q\), then the two-point function of the squeezed state simplifies to

\[
\omega^S_{ij}(x, y) = \omega^0_{ij}(x, y) - \left\{\left[\overline{d_i d_j e^{-iq(x+y)}} + c.c.\right] \sinh r \cosh r + \left[\overline{d_i d_j e^{iq(x-y)}} + c.c.\right] \sinh^2 r\right\},
\]

where \(d\) is a polarization vector (orthogonal to \(q\)). From the above expression one realizes that the difference between \(\omega^S\) and \(\omega^0\) becomes negative for some periods of time. This fact corresponds to the sub-vacuum fluctuations of the vector potential\(^{21}\) \(A_i\); the expectation values of the squared electric-field operator, and the energy-density operator behave similarly. We note that the squeezed state is not a stationary state, and therefore the Volterra equation (15) is will not be a convolution-type equation in this case.

\section{Example}

\subsection{Hydrogen atom, 2P \(\rightarrow\) 1S transition}

The wavepackets of the initial and final state of the atom are

\[
\psi_0 = \frac{\alpha^{3/2}}{\sqrt{\pi}} e^{-\alpha r}, \quad \psi_1 = \frac{\alpha^{5/2}}{4\sqrt{2\pi}} r \cos \theta e^{-\alpha r/2}.
\]

We will compute the smearing functions,

\[
\chi_i(p) = \int d^3x e^{-ip\cdot x} \overline{\psi_1(x)} \partial_i \psi_0(x).
\]

\(\kappa\) The square of \(A^i(x)\) is defined via normal ordering w.r.t. the vacuum \(\Omega\).
The differentiation w.r.t. $x^i$ leads to a factor $x^i$ (multiplied with functions of $r$ only) which can be expressed as a derivative $i\partial/\partial p_i$. Similarly, the term $z = r \cos(\theta)$, can be expressed as a derivative w.r.t. the $p_z$. In this way we obtain

$$\chi_i(p) = \frac{\alpha^5}{4\sqrt{2\pi}} \frac{\partial^2}{\partial p_z \partial p_i} \int d^3x \, e^{-3/2\alpha r} r,$$

and consequently

$$\chi_i(p) = \alpha^5 \sqrt{2} \left[ \frac{4 \, p_i \, p_z}{(p^2 + 9/4 \alpha^2)^2} - \frac{\delta_{i2}}{(p^2 + 9/4 \alpha^2)^2} \right].$$

The first term in the square bracket is longitudinal (proportional to $p_i$) and will be canceled by the projection on the transversal directions in the two-point function.

### 3.2 Spontaneous emission in the presence of vacuum

We now compute the kernel of the equation (15) for the $2P \rightarrow 1S$ transition ($\omega = \alpha^2 \left( \frac{1}{2} - \frac{1}{8} \right)$ in dimensionless units) in the presence of the vacuum state. The smeared two-point function, $S(t - s)$, of equation (25) will be employed. After integrating out the angle variables, we get

$$S(t - s) = \frac{\alpha^2}{3\pi^2} \int_0^\infty \frac{p \, dp}{[(p/\alpha)^2 + 9/4]^2} e^{-ip(t-s)}.$$

From this point, the investigation of the spontaneous emission process, in particular of the type of decay-curve, and of the life-time of the $2P$ state, can be addressed with the help of mathematical methods for Volterra equations [10].

### 4 Conclusions and outlook

We have presented a simple model of the spontaneous emission of light. While the restriction to the one-excitation (photon) transitions is the most obvious limitation of this model, it offers, on the other hand, a generalization with respect to the allowed initial states of the quantum electromagnetic field. It asserts that the characteristics of the emission process are not intrinsic or unique properties of atoms, but rather they depend on the initial state of the quantum electromagnetic field. Two future lines of research seem interesting: firstly one might ask, whether the emission process in the distant (cosmological) past was similar to the one we observe in laboratories nowadays. The two-point function of the initial state of the quantum electromagnetic field, at that epoch, was certainly different from the (Minkowski-) vacuum two-point function. The second direction, which we regard as promising, is to endeavor to use simple quantum...
systems (e.g. some probe-atoms) as detectors of subtle non-classical properties of quantum electromagnetic fields. The temporal dependence of the excited-state’s amplitude, if measured with sufficient resolution, might reveal fingerprints of the, for instance, negative energy-densities of the field.

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A Dimensionless units

In this paper we use dimensionless units. They are defined as follows:

\[ E = mc^2 E_D, \]
\[ x = \frac{h}{mc} x_D, \]
\[ t = \frac{h}{mc^2} t_D, \]
\[ p = mc p_D, \]
\[ \psi = \psi_D \left( \frac{h}{mc} \right)^{-3/2}, \]
\[ A = A_D \sqrt{\frac{m^2 c^3}{\hbar}}, \]

where the subscript $D$ denotes the dimensionless quantities. The only constant which can appear in the equations is the fine-structure constant $\alpha$. If we take the interaction term as in the equation (3), then the quantities corresponding to the free electromagnetic field, in particular: the two-point function, should not contain $\alpha$. As the unit of time is particularly important in the context of this paper, we note that $t_D = 1$ corresponds to $\hbar/m_e c^2 \approx 1.25 \cdot 10^{-21}$s.

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\[ As in the paper we use only the dimensionless expressions, this subscript is dropped for brevity.\]
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