Anomalous Current Due to Weyl Anomaly for Conformal Field Theory

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Abstract

Recently it is found that Weyl anomaly leads to new anomalous currents in an external electromagnetic field in curved spacetime. For simplicity, the initial works mainly focus on weak gravitational fields and the anomalous current is obtained for conformally flat space with small scale factors. In this paper, we generalize the results to the case with arbitrary scale factors. Firstly, we give a holographic derivation of the transformation law of current under Weyl transformation, from which one can read off the anomalous current in general conformally flat space. Secondly, by using the effective anomalous action, we prove the transformation law of current for general CFTs. As an application of our result, we provide a new derivation of the Weyl-anomaly-induced current near the boundary. We also show that there are close relations between the two kinds of anomalous currents in the literature. Finally, we extend the discussions to n-form fields and find similar anomalous currents.

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1 Introduction

The anomaly-induced current has drawn much attention in the past few years [1]. The famous ones are chiral magnetic effect (CME) [2, 3, 4, 5, 6] and chiral vortical effect (CVE) [7, 8, 9, 10, 11, 12, 13], which refer to the generation of currents due to an external magnetic field and the rotational motion in the charged fluid, respectively. Recently, it is found that Weyl anomaly can also produce anomalous current in an external electromagnetic field [14, 15, 16, 17]. See also [18, 19, 20, 21] for following works. It is remarkable that a direct measurement of the beta function is proposed in [18].

Weyl anomaly measures the breaking of scaling symmetry of conformal field theory (CFT) due to quantum effects [22],

$$A = \partial_\sigma I_{\text{eff}}[\epsilon^{2\sigma} g_{ij}]|_{\sigma=0}, \quad (1)$$

where $I_{\text{eff}}$ is the effective action of CFT and $g_{ij}$ are the metrics. For 4d CFTs, Weyl anomaly takes the form [22]

$$A = \int_M d^4x \sqrt{\bar{g}} [c \, C^{ijkl} C_{ijkl} - a \, E_4 + b_1 F_{ij} F^{ij}] \quad (2)$$

where $C_{ijkl}$ are the Weyl tensors, $E_4 = R_{ijkl} R^{ijkl} - 4 R_{ij} R^{ij} + R^2$ is the Euler density, $F = dA$ is the field strength and $c, a, b_1$ are central charges.
There are two kinds of anomalous current induced by Weyl anomaly in an external electromagnetic field [14, 15, 16, 17]. The first kind occurs in a curved spacetime without boundaries [14, 15], while the second kind occurs in the general spacetime (including flat space) with boundaries [16, 17]. Below we call them Type I anomalous current and Type II anomalous current, respectively.

For a CFT defined in a conformally flat space
\[ ds^2 = e^{2\sigma(x)} \delta_{ij} dx^i dx^j, \]  
(3)

Type I anomalous current takes the form [14, 15]
\[ J^i = 4b_1 F_{ij} \partial_j \sigma + O(\sigma^2). \]  
(4)

Note that [14, 15] have assumed that the current vanish in flat space (at least in some region), and they derive only the anomalous current in a weak gravitational background with small scale factor \( \sigma \). It should be mentioned that a formal formula of the Type I anomalous current is obtained for general background fields in [15]. However, since the Green function is unknown generally, it is not clear how to derive the exact result of current from the formal formula of [15].

For a CFT defined in a space with a boundary (BCFT) [16, 17], Type II anomalous current takes the universal form [16, 17]
\[ J^i = \frac{4b_1 F^{ij} n_j x}{x} + \ldots, \quad x \sim 0, \]  
(5)

where \( x \) is the proper distance to the boundary, \( n_i \) are the normal vectors and \( \ldots \) denote higher order terms in \( O(x) \). Note that there are boundary contributions to the current density, which can exactly cancel the apparent “divergence” in the bulk current (5) at \( x = 0 \) and define a finite total current [16]. Note also that (5) actually applies to the general boundary quantum field theory instead of only BCFT [16]. For simplicity, we focus on CFT/BCFT in this paper.

In this paper, we aim to obtain the Type I anomalous current for general conformally flat space and try to reveal relations between these two kinds of anomalous currents. By applying AdS/CFT [23], we find that the current of CFTs transform as
\[ J'_i = e^{-2\sigma} J_i + 4b_1 \nabla'_j (F'_{i j} \sigma), \]  
(6)

under the Weyl transformations
\[ g'_{ij} = e^{2\sigma} g_{ij}. \]  
(7)

1In fact every spacetime has a conformal boundary. In this sense, all CFTs are also BCFTs. However, the conformal boundary is a boundary located at infinity. While in this paper, by BCFT we mean the CFT defined in a manifold with the boundary which is located at a finite place. Note that BCFTs have fewer symmetries than CFTs due to the obstruction of boundaries.
Here $\nabla'$ and $F'$ are the covariant derivative and the field strength defined by the metric $g'_{ij}$, respectively. Furthermore, by using the effective anomalous action, we prove the transformation law \([8]\) for general CFTs. Note that the first term of \([8]\) depends on the states and the temperature of the theory, while the second term of \([8]\) is universal, which depends on only the central charge (beta function). From \([8]\) and the assumption that $J_i = 0$ in some region of flat space \([14, 15]\), we obtain Type I anomalous current

$$J_{\text{anomaly}}^i = 4b_1 \nabla'_j (F^{ij} \sigma) \quad (8)$$

in the same region \([3]\) of conformally flat space with arbitrary scale factor. It is remarkable that BCFT in a half space is conformally equivalent to CFT in the Poincare patch of AdS. As a result, we can derive the Type II anomalous current \([5]\) in a half space from the Type I anomalous current \([5]\) with $\sigma = \ln x$. Please see section 4 for details. This shows that there are close relations between these two kinds of anomalous currents. Finally, we generalize our results to $n$-form fields in $2(n+1)$ dimensions and find similar anomalous currents

$$J'_{i_1...i_n} = e^{-2\sigma} J_{i_1...i_n} - 2(n+1)b_n \nabla'_j (H^{ij}_{i_1...i_n} \sigma), \quad (9)$$

where $H = dB$ is the field strength of $n$-form field and $b_n$ are the central charges of Weyl anomaly \([45]\).

The paper is organized as follows. In section 2, we derive the transformation law of current under Weyl transformation in AdS/CFT. From this transformation law, we obtain the anomalous current in arbitrary conformally flat space. In section 3, we give a field-theoretical proof of the transformation law of current. In section 4, by using the conformal equivalence between AdS and flat space, we give a new derivation of Weyl anomaly induced current for BCFT. In section 5, we extend our results to $n$-form fields. Finally, we conclude with some open questions in section 6. For simplicity, we focus on Euclidean signature in this paper, the generalization to Lorentzian signature is straightforward.

## 2 A holographic derivation of the anomalous current

In this section, we take Penrose-Brown-Henneaux (PBH) transformation \([24, 25]\) to study how the current transform under Weyl transformations in AdS/CFT \([23]\). From this transformation law, we get the anomalous current in general conformally flat space. According to \([24, 25]\), the Weyl transformations of the boundary metric can be understood as a certain subgroup of bulk diffeomorphisms in AdS/CFT. This is the so-called PBH transformation. For simplicity, the PBH transformation is worked out up to the linear order of scale factor in \([24, 23]\). For our purpose, we need to generalize the results of \([24, 25]\) to non-perturbative scale factor.

\(^2\)Suppose that $J_i = 0$ in the region $f(x') \leq 1$ in a flat space. By “the same region” in conformally flat space, we means the same coordinate region $f(x') \leq 1$. 

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Let us start with the general higher derivative gravity and Maxwell theory in the bulk, whose action is given by

\[ I = \int_M dX^5 \sqrt{G} \left[ f(R, \nabla R, ...) + b_1 \mathcal{F}_{\mu
u} \mathcal{F}^{\mu\nu} \right], \]  

(10)

where \( X^\mu = (\rho, x^i) \) are the bulk coordinates, \( b_1 \) is the central charge of Weyl anomaly, \( \mathcal{F} = dA \) is the field strength, \( R_{\mu\nu\rho\sigma} \) are the curvatures and ‘...’ denote higher derivatives of curvatures. For simplicity, we have ignored the indexes of curvatures. We assume that the asymptotically AdS are solutions to the above theory. In Feferman-Graham gauge, the metric of asymptotically AdS takes the form

\[ ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{\hat{g}_{ij}(x, \rho) dx^i dx^j}{\rho}, \]  

(11)

where \( \hat{g}_{ij} = g_{ij} + \rho g_{(1)ij} + ... \). Similarly, we can choose a gauge for the bulk gauge field

\[ A_\rho = 0, \]

(12)

\[ A_i = A_i + \rho [A_{(1)i} + \tilde{A}_{(1)i} \ln \rho] + ... \]  

(13)

Here \( g_{ij} \) and \( A_i \) are boundary metrics and vectors, respectively. Note that \( \tilde{A}_{(1)i} \) can be derived from either the bulk Maxwell equations or the Weyl anomaly

\[ \tilde{A}_{(1)i} = -\frac{1}{4} \nabla_j F^i_j. \]  

(14)

Please see the appendix for the details. According to [26], the holographic current of CFT is given by

\[ J^i = \lim_{\rho \to 0} -\frac{4b_1}{\sqrt{g}} \sqrt{G} F^{\mu i}. \]  

(15)

From (11,12,13,15), we get

\[ J_i = -8b_1 A_{(1)i}, \]  

(16)

where we have subtracted the finite and log divergent term proportional to \( \tilde{A}_{(1)i} \) following the standard approach of holographic renormalization. Please refer to the appendix for the details of calculations.

We aim to derive the transformation law of current under the Weyl transformations. According to [24], the Weyl transformations can be realized by suitable bulk diffeomorphisms. Inspired by [24], we take the ansatz

\[ \rho = \rho' e^{-2\sigma(x')} \left( 1 + \sum_{n=1}^{\infty} \rho^n b_{(n)}(x') \right) \]

(17)

\[ x^i = x'^i + \sum_{n=1}^{\infty} \rho^n a_{(n)}(x') \]  

(18)
We require that the above diffeomorphisms leave the form of bulk metric (11) invariant, i.e.,
\[
G'_{\rho\rho} = \frac{\partial X^\mu}{\partial \rho'} \frac{\partial X^\nu}{\partial \rho'} G_{\mu\nu} = \frac{1}{4 \rho'^2},
\]
(19)
\[
G'_{\rho i} = \frac{\partial X^\mu}{\partial \rho'} \frac{\partial x^n}{\partial x'^i} G_{\mu\nu} = 0.
\]
(20)
Substituting \([17, 18]\) into \([20]\), we derive
\[
G'_{\rho i} = \frac{1}{\rho'} \left( -\frac{1}{2} \partial_i \sigma + a_{(1)}^i g'_{ij} \right) + O(\rho'^0) = 0,
\]
(21)
from which we get
\[
a_{(1)}^i = \frac{1}{2} g_{ij} \partial_j \sigma.
\]
(22)
Note that \( g'_{ij} = e^{-2\sigma} g_{ij} \) is nonperturbative in the scale factor. Similarly, we can calculate \( a_{(n)}^i \) and \( b_{(n)} \) order by order from \([19, 20]\). Since they are irrelevant to the calculations of holographic current, we do not discuss them here.

Now we are ready to derive the transformation law of current under Weyl transformation. Under the diffeomorphisms \([17, 18]\), the bulk gauge fields become
\[
A'_\rho(\rho', x') = \frac{\partial X^\mu}{\partial \rho'} A_\mu(\rho, x) = a_{(1)}^i A_i + O(\rho'),
\]
(23)
and
\[
A'_i(\rho', x') = \frac{\partial X^\mu}{\partial x'^i} A_\mu(\rho, x)
\]
\[
= (\delta_i^j + \rho' \partial_\rho a_{(1)}^j) \left( A_j(x) + \rho [A_{(1)j}(x) + \bar{A}_{(1)j}(x') \ln(\rho')] \right) + O(\rho'^2)
\]
\[
= (\delta_i^j + \rho' \partial_\rho a_{(1)}^j) \left( A_j(x') + \rho' e^{-2\sigma} [A_{(1)j}(x') + \bar{A}_{(1)j}(x') \ln(\rho' e^{-2\sigma})] \right) + O(\rho'^2)
\]
\[
= A_i(x') + \rho' \left( e^{-2\sigma} A_{(1)i}(x') + \partial_\rho a_{(1)}^j A_j(x') + a_{(1)}^j \partial_j A_i(x') - 2e^{-2\sigma} \bar{A}_{(1)i}(x') \right) + O(\rho' \ln(\rho'), \rho'^2).
\]
(24)
Note that we have used the gauge \( A_\rho = 0 \) in the above derivations. However \( A_\rho' \) becomes non-zero after the coordinate transformations. As a result, we cannot use the formula like \([16]\) to calculate the current \( J'_i \). Instead, we make use of the general formula like \([15]\)
\[
J'^i = \lim_{\rho' \to 0} \frac{-4b_1}{\sqrt{g'}} \sqrt{G'} F'^{\rho'i},
\]
(25)
from which we get
\[
J'_i = -8b_1 \left( A'_{(1)i} - \partial_i A'_{(0)\rho} \right),
\]
(26)
where $A'_{(0)\rho} = \lim_{\rho' \to 0} A'_{\rho}$. Another method to derive the holographic current (26) is to perform a gauge transformation to make $A'_{\rho} = 0$, i.e.,

$$A'_\mu \to A'_\mu - \partial_\mu \left( A'_{(0)\rho} \rho' + O(\rho'^2) \right).$$

(27)

And then we can use the formula like (16) to get the holographic current (26). Substituting (14, 16, 22, 23, 24) into (26), we finally obtain

$$J'_i = e^{-2\sigma} J_i + 4b_1 \nabla'_j (F'_{i j} \sigma),$$

(28)

where $F'_{ij} = F_{ij} = \partial_i A_j - \partial_j A_i$ and $F'_{i j} = F_{il} g^{lj} = e^{-2\sigma} F_{i j}$. Now we finish the derivations of the Weyl transformation law of current for holographic CFTs.

Comments of the transformation law of current (28) are in order. First, (28) works for the general gravitational background and scale factor. In particular, the scale factor $\sigma$ need not be small. Second, the first term of LHS of (28) is the ordinary transformation law, while the second term is due to Weyl anomaly and we call it the anomalous current

$$J'_{\text{anomaly}} i = 4b_1 \nabla'_j (F'_{i j} \sigma).$$

(29)

It is remarkable that the anomalous current is universal, in the sense that it depends on only the central charge of CFT instead of the states and temperature of the theory. Third, consider the case where $J_i = 0$ in some region in flat space, then the anomalous current contribute to all of the currents in the same region in general conformally flat space. This is a generalization of the works of [14, 15] to the general scale factor. Fourthly, (28) implies that the currents obey the Wess-Zumino-like condition

$$[\delta_{\sigma_1}, \delta_{\sigma_2}] J_i = 0,$$

(30)

which is consistent with the fact that Weyl transformation is abelian. This can be regarded as a test of our result. Finally, although (28) is derived for only the strongly-coupled CFTs dual to the general higher derivative gravity, it actually applies to the most general CFTs. We will give a field-theoretical proof of (28) in the next section.

3 A field-theoretical proof of the anomalous current

In this section, by using the effective anomalous action, we prove the transformation law of current (28) and the anomalous current (29) for general CFTs.

According to [27, 28, 29, 30], the effective anomalous action generated by one-loop quantum corrections takes the form

$$S_{\text{anomaly}}[g, A] = \frac{1}{8} \int d^4x \sqrt{g(x)} \int d^4y \sqrt{g(y)}$$

(31)
\[ H(x)G_4(x, y) \left[ 2cC^2(y) - aH(y) + 2b_1 F_{ij}(y) F^{ij}(y) \right], \]

where \( C^2 = C_{ijkl}C^{ijkl} \) is the squared Weyl tensor, \( H = E_4 - \frac{2}{3} \Box R, E_4 = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2 \) is the Euler density and \( G_4(x, y) \) is the Green function of the differential operator \( \Delta_4 = \nabla_i \left( \nabla^i \nabla^j + 2R^{ij} - \frac{2}{3} Rg^{ij} \right) \nabla_j. \)

By definition, we have

\[ \Delta_4(x)G_4(x, y) = \frac{\delta^{(4)}(x - y)}{\sqrt{g}}. \tag{32} \]

From the anomalous action (31), one can derive a formal expression of the anomalous current as \[ J^i(x) = \frac{1}{\sqrt{g(x)}} \frac{\delta S_{\text{anomaly}}}{\delta A_i(x)} = \frac{b_1}{\sqrt{g(x)}} \frac{\partial}{\partial x^i} \left[ \sqrt{g(x)} F^{ij}(x) \right] \cdot \int d^4y \sqrt{g(y)}G_4(x, y) \left( E(y) - \frac{2}{3} R(y) \right), \tag{33} \]

which applies to general gravitational background. However, since the exact expression of the Green function \( G_4 \) is unknown, one cannot use (33) directly to derive the anomalous current. Fortunately, we do not need the exact expression of Green function \( G_4 \) to prove the transformation law of current (28). Note that \( \Delta_4 \) is a conformal differential operator, which transforms

\[ \Delta_4 \to \Delta'_4 = e^{-4\sigma} \Delta_4 \tag{34} \]

under the Weyl transformation \( g_{ij} \to g'_{ij} = e^{2\sigma} g_{ij} \). From (32) and (34), it is not difficult to observe that the Green function is invariant under Weyl transformation, i.e.,

\[ G_4 \to G'_4 = G_4. \tag{35} \]

After some calculations, we obtain the Weyl transformation of \( E - \frac{2}{3} \Box R \) as

\[ (E - \frac{2}{3} \Box R) \to (E' - \frac{2}{3} \Box' R') = e^{-4\sigma}(E - \frac{2}{3} \Box R + 4\Delta_4 \sigma). \tag{36} \]

The above two equations yield

\[ \int d^4y \sqrt{g(y)}G'_4(x, y) \left( E'(y) - \frac{2}{3} \Box' R'(y) \right) = 4\sigma(x) + \int d^4y \sqrt{g(y)}G_4(x, y) \left( E(y) - \frac{2}{3} \Box R(y) \right). \tag{37} \]
Now we are ready to prove the transformation law of current. By using (33, 37) together with $F'_{ij} = e^{-4\sigma} F_{ij}$ and $\sqrt{g'} = e^{4\sigma} \sqrt{g}$, we finally obtain

$$J'^{i} = e^{-4\sigma} J^{i} + 4b_{1} \nabla_{j}' (F'^{ij} \sigma),$$

(38)

which exactly agrees with (28). Note that in the above proof, we focus on the Weyl transformation of anomalous current (33). In general, the total currents include the normal current $J^{i}_{n}$ (such as the classical current $J^{i}_{n} = \nabla_{j} F_{ji}$) which transforms trivially as $J'^{i}_{n} = e^{-4\sigma} J^{i}_{n}$, and the anomalous current (33). To prove that the total currents obey the transformation law (28, 38), it is sufficient to prove the anomalous current (33) do. This is just what we have done above. Now we finish the proof of the transformation law of current (28, 38) under Weyl transformations for general CFTs.

4 Relations between the two kinds of anomalous current

As an application of the transformation law of current (28), we give a new derivation of the Type II anomalous current (5) for BCFT in this section. For simplicity, we focus on the bulk region of BCFT, where the transformation law of current (6) derived for CFT still applies. Besides, since there is no boundary contribution to Weyl anomaly from the background field strength, it is expected that the boundary current transforms trivially under Weyl transformations. Below we focus on the bulk current for BCFT. A key observation is that BCFT in a flat space with a plane boundary

$$ds^2 = dx^2 + dy^2, \ x \geq 0$$  

(39)

is conformally equivalent to CFT in the Poincare patch of AdS

$$ds^2 = \frac{dx^2 + dy^2}{x^2}, \ x \geq 0.$$  

(40)

where $x = 0$ are the boundary of half space for BCFT and the conformal boundary of AdS for CFT, respectively. Note that these two kinds of boundaries are quite different, since the proper distance from an inside point to the boundary is finite in a half space, while is infinite in AdS.

Let us first discuss the current in an external electromagnetic field in AdS (40). By definition of current, we have

$$\delta I_{\text{ren}} = \int_{M} dx^4 \sqrt{g} J^i \delta A_i = \int_{M} dx^4 \frac{J^i}{x^4} \delta A_i,$$  

(41)

where $I_{\text{ren}}$ is the renormalized effective action, which is finite by renormalization. Since both the effective action and the vectors are finite, so do their variations. As a result, $\frac{J^i}{x^4}$
is finite from (41). Performing the Weyl transformation (7) with $e^{2\sigma} = x^2$ and using the transformation law (38), we get the current in the half space (39),

$$J_{BCFT}^i = \frac{J^i}{x^4} + \frac{4b_1}{x} F'^{ix} x + \frac{4b_1}{x} \nabla'_{ij} F'^{ij} \ln x = \frac{4b_1}{x} F'^{ix} x + \ldots$$  \hspace{1cm} (42)$$

which exactly agrees with the anomalous current (5) near the boundary. In the above derivation we have used the fact that $\frac{J^i}{x^4}$ is finite as implied by (41). Now we finish the derivation of type II anomalous current (5). This can be regarded as a support of the transformation law of current (38).

The above discussions can be easily generalized to curved space. Following the above approach and taking into account the conformal equivalence between the curved half space in Gauss normal coordinates

$$ds^2 = dx^2 + h_{ab}(x,y) dy^a dy^b, \quad x \geq 0,$$

and asymptotically AdS metrics

$$ds^2 = dx^2 + h_{ab}(x,y) dy^a dy^b, \quad x \geq 0,$$

we can derive (42) for the general curved case. Note that the anomalous current $\frac{4b_1}{x} F'^{ix} x$ is universal, and the finite part $\frac{J^i}{x^4}$ of (42) depends on the states and temperature of the theory.

5 Anomalous current for n-form fields

In this section, we discuss the anomalous current for n-form fields in 2($n + 1$)-dimensions. The Weyl anomaly due to a background of n-form field is given by

$$A = \int_M dx^d \sqrt{g} b_n H^{i_1 i_2 \ldots i_{n+1}} H_{i_1 i_2 \ldots i_{n+1}},$$

(45)

where $d = 2(n + 1)$, $b_n$ is the central charge and $H = dB$ is the field strength of n-form $B_{i_1 i_2 \ldots i_n}$.

Following sect. 2, we first give a holographic derivation of the Type I anomalous current for n-form fields. Similar to (11), we take the following bulk action

$$I = \int_M dX^{d+1} \sqrt{G} \left[ f(R, \nabla R, \ldots) + b_n H_{\mu_1 \ldots \mu_{n+1}} \mathcal{H}^{\mu_1 \ldots \mu_{n+1}} \right],$$

(46)

which yields the expected Weyl anomaly (45). Please refer to the appendix of [20] for the calculations of holographic Weyl anomaly for 2-form fields. The generalizations to n-form fields are straightforward. From action (46), we can derive the holographic current as

$$J^{i_1 \ldots i_n} = \frac{1}{\sqrt{g}} \frac{\delta I}{\delta B_{i_1 \ldots i_n}} = \lim_{\rho \to 0} -\frac{2(n + 1)b_n}{\sqrt{g}} \frac{\sqrt{G} \mathcal{H}^{\mu_1 \ldots \mu_{n+1}}}{\rho},$$

(47)
Similar to the case of one form, we have

$$B_{\rho i_1...i_{n-1}} = 0, \quad B_{1...i_n} = B_{1...i_n} + \rho[B_{(1)i_1...i_n} + \bar{B}_{(1)i_1...i_n} \ln \rho] + ...$$  \quad (48)

$$B_{(1)i_1...i_n} = -\frac{1}{4} \nabla^j H_{ji_1...i_n} = -\frac{1}{4} e^{2\sigma} \nabla^j H'_{ji_1...i_n}.$$  \quad (50)

After suitable holographic renormalization, the holographic current \((47)\) becomes

$$J_{1...i_n} = -4(n+1)b_n B_{(1)i_1...i_n}.$$  \quad (51)

Performing the diffeomorphisms \((17,18)\), we have

$$\mathcal{B}'_{\rho i_1...i_{n-1}}(\rho', x')(\rho', x') = \frac{\partial X^\mu}{\partial \rho'} \frac{\partial X^{\mu_1}}{\partial x'_{i_1}} ... \frac{\partial X^{\mu_{n-1}}}{\partial x'_{i_{n-1}}} \mathcal{B}_{\mu_1...\mu_{n-1}}(\rho, x)$$

$$= \alpha_{(1)} x_{i_{i_1...i_{n-1}}}(x') + O(\rho'),$$  \quad (52)

and

$$\mathcal{B}'_{1...i_n}(\rho', x') = \frac{\partial X^\mu}{\partial x'_{i_1}} ... \frac{\partial X^{\mu_n}}{\partial x'_{i_n}} \mathcal{B}_{\mu_1...\mu_n}(\rho, x)$$

$$= (\delta^i_{i_1} + \rho \partial_{i_1} a^i_{(1)}) ...(\delta^i_{i_n} + \rho \partial_{i_n} a^i_{(1)}) (B_{1...i_n}(x) + \rho[B_{(1)i_1...i_n}(x) + \bar{B}_{(1)i_1...i_n}(x) \ln \rho]) + O(\rho^2)$$

$$= (\delta^i_{i_1} + \rho \partial_{i_1} a^i_{(1)}) ...(\delta^i_{i_n} + \rho \partial_{i_n} a^i_{(1)}) (B_{1...i_n}(x') + \rho a^k_{(1)} \partial_k B_{1...i_n}(x')) + O(\rho^2)$$

$$+ (\delta^i_{i_1} + \rho \partial_{i_1} a^i_{(1)}) ...(\delta^i_{i_n} + \rho \partial_{i_n} a^i_{(1)}) (\rho e^{2\sigma} B_{1...i_n}(x') + \rho e^{2\sigma} \bar{B}_{1...i_n}(x')) \ln(\rho e^{-2\sigma})$$

$$= B_{1...i_n}(x') + \rho \left( \sum_{p=1}^{n} \partial_{i_p} a_{(1)}^{i_p} B_{1...i_p...i_{n-1}}(x') + \alpha_{(1)}^i \partial_i B_{1...i_{n-1}}(x') \right)$$

$$+ \rho (e^{-2\sigma} B_{1...i_1...i_n}(x') - 2\sigma e^{-2\sigma} \bar{B}_{1...i_1...i_n}(x')) + O(\rho \ln \rho, \rho^2).$$  \quad (53)

Note that \(\mathcal{B}'_{\rho i_1...i_{n-1}}\) become non-zero after the diffeomorphisms. To preserve the gauge \(\mathcal{B}'_{\rho i_1...i_{n-1}} = 0\) \((48)\), let us perform a gauge transformation

$$\mathcal{B}'_{\mu_1...\mu_{n+1}} \rightarrow \mathcal{B}'_{\mu_1...\mu_{n+1}} - n \partial_{\mu_1} \left( a_{(1)}^{i_1} B_{i_1...i_{n+1}} \rho' + O(\rho^2) \right),$$  \quad (54)

which yields \(\mathcal{B}'_{\rho i_1...i_{n-1}} = 0\) and

$$\mathcal{B}'_{1...i_n}(\rho', x') = B_{1...i_n}(x') + \rho \left( e^{-2\sigma} [B_{1...i_1...i_n} - 2\sigma \bar{B}_{1...i_1...i_n}](x') + a_{(1)}^{i_1} H_{1...i_{n+1}}(x') \right)$$

$$+ O(\rho \ln \rho, \rho^2).$$  \quad (55)
Substituting (22, 50, 55) into the formula (51), we finally get the transformation law of current under Weyl transformations for n-form fields

$$J_{i_1...i_n}^\prime = e^{-2\sigma}J_{i_1...i_n} - 2(n + 1)b_n\nabla_j^\prime (H'^{j}_{i_1...i_n})$$

(56)

where the first term comes from the usual Weyl transformation of current and the second term originates in Weyl anomaly. We name the second term of (56) as the anomalous current

$$J_{\text{anomaly } i_1...i_n} = -2(n + 1)b_n\nabla_j^\prime (H'^{j}_{i_1...i_n})$$

(57)

Suppose $J_{i_1...i_n} = 0$ in some region of a flat space, then all of the currents are given by the anomalous currents (57) in the same region of a conformally flat space (3).

Recall that BCFT in the flat half space (39) is conformally equivalent to CFT in the Poincare patch of AdS (40). Following approaches of section 4, from (56) we can derive Type II anomalous current near the boundary as

$$J_{i_1...i_n} = -2(n + 1)b_n\frac{H'^{i_1...i_n}}{x} + ... , \ x \sim 0,$$

(58)

where $x$ is the distance to the boundary and ... denotes higher orders of $O(x)$. Note that (58) agrees with the result of [20, 21] for two-form fields. This can be regarded as a check of the transformation law (56).

To end this section, let us briefly discuss the field-theoretical derivations of anomalous current. Let us first derive the anomalous action for general n-form fields. For simplicity, we focus on the contributions relevant to the background field strength. From the definition of Weyl anomaly (1) and (45), we have

$$\delta_\sigma I_{\text{eff}} = IA_\sigma = b_n \int_M dx^d \sqrt{g}H'^{i_1...i_{n+1}}H_{i_1...i_{n+1}}\delta\sigma(x).$$

(59)

Note that $\sqrt{g}H'^{i_1...i_{n+1}}H_{i_1...i_{n+1}}$ is independent of the scale factor. Performing the integral in (59), we get

$$I'_{\text{eff}} = I_{\text{eff}} + b_n \int_M dx^d \sqrt{g}H'^{i_1...i_{n+1}}H_{i_1...i_{n+1}}\sigma(x).$$

(60)

The above effective action can also be obtained by the method of dimensional regularization. Following [31], we have

$$I'_{\text{eff}} - I_{\text{eff}} = \lim_{\epsilon \to 0} \frac{b_n}{\epsilon} \int_M dx^{d+\epsilon} \left( \sqrt{g}H'^{i_1...i_{n+1}}H_{i_1...i_{n+1}} - \sqrt{g}H^{i_1...i_{n+1}}H_{i_1...i_{n+1}} \right)$$

(61)

which reproduces (60). From (35, 36), it is not difficult to observe that the effective action (31) for the one-form field agrees with our general results (60). This can be regarded as a support of our above discussions. From the anomalous action (60), we finally get

$$J'^{i_1...i_n} = \frac{1}{\sqrt{g'}} \frac{\delta I'_{\text{eff}}}{\delta B_{i_1...i_n}} = e^{-2(n+1)\sigma}J^{i_1...i_n} - 2(n + 1)b_n\nabla_j^\prime (H'^{j}_{i_1...i_n})$$

(62)

which agrees with the holographic result (56).
6 Conclusions and Discussions

In this paper, we investigate the anomalous current due to Weyl anomaly for CFTs. In particular, we obtain the transformation law of current under Weyl transformations. We first give a holographic derivation of this transformation law, and then find a solid proof of it by using the effective anomalous action. From the transformation law of current, we can read off the anomalous current in general conformally flat space. This is a generalization of the works of [14, 15], which mainly discuss the case of weak gravity with small scale factor. Furthermore, by applying the transformation law of current, we give a new derivation of the anomalous current near the boundary [16, 17], and reveal that there are close relations between the Type I anomalous current [13, 15] and Type II anomalous current [16, 17]. Finally, we extend our discussions to n-form fields and find similar anomalous currents. We notice that our results agree with Type II anomalous current for two-form field [20, 21]. For simplicity, in this paper we discuss only the current induced by an external electromagnetic field in four dimensions. It is interesting to study the anomalous current in higher dimensions. Besides, it is also interesting to search for the applications of our results to cosmology and condensed matter. Since our results work for arbitrary conformally flat space, now we can study the current for general Robertson-Walker metrics in cosmology. We hope we could address these problems in the future.

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A Holographic renormalization of current

In this appendix, we derive eqs. (14-16) for the holographic current in section 2. The current of CFT is defined by

\[ j^i = \frac{1}{\sqrt{g}} \frac{\delta I_{\text{eff}}}{\delta A_i} \]  

(63)

where \( I_{\text{eff}} \) is the non-renormalized effective action of CFTs, which takes the form [16]

\[ I_{\text{eff}} = \ldots + A \ln \frac{1}{\epsilon} + I_{\text{ren}}, \]  

(64)
where ... denote divergent terms, \( A \) is the Weyl anomaly, \( \epsilon \) is the cutoff and \( I_{\text{ren}} \) is the renormalized finite effective action. From (63), we get the non-renormalized current

\[
J^i = ... + \frac{1}{\sqrt{g}} \frac{\delta A}{\delta A_i} \ln \frac{1}{\epsilon} + J^i_{\text{ren}}. \tag{65}
\]

One need performing suitable renormalization in order to obtain the finite current \( J^i_{\text{ren}} \). Below we focus on the holographic renormalization for 4d CFTs.

In AdS/CFT, the effective action of CFTs is given by the on-shell gravitational action (10). From (63), we get the holographic current (15)

\[
J^i = \lim_{\rho \to 0} \frac{1}{\sqrt{g}} \frac{\delta I}{\delta A_i} = \lim_{\rho \to 0} -4b_1 \sqrt{G} F^{oi} \tag{66}
\]

Substituting the bulk vectors (12,13) into (66), we get

\[
J^i = -8b_1 [A^{(1)i} + \bar{A}^{(1)i} + \bar{A}^{(1)i}] \ln \epsilon^2 \tag{67}
\]

where we have chosen the cut-off \( \rho = \epsilon^2 \). Comparing the log divergent terms in (65) and (67), we obtain

\[
16b_1 \bar{A}^{(1)i} = \frac{1}{\sqrt{g}} \frac{\delta A}{\delta A_i} = -4b_1 \nabla_j F^{ij}, \tag{68}
\]

where we have used (2) in the last equation of (68). From (68), we obtain (14) of section 2,

\[
\bar{A}^{(1)i} = -\frac{1}{4} \nabla_j F^{ij} = -\frac{1}{4} e^{2\sigma} \nabla_j F^{ij}. \tag{69}
\]

Now we finish the derivation of \( \bar{A}^{(1)i} \) from Weyl anomaly. Note that \( \bar{A}^{(1)i} = e^{2\sigma} \bar{A}'^{(1)i} \) transforms trivially under Weyl transformations.

Following the standard approach of holographic renormalization, we can add suitable boundary counter terms to cancel the divergence of gravitational action and holographic current. Let us focus on the counter terms related to gauge fields. The renormalized gravitational action is given by [32]

\[
I_{\text{reg}} = \int_M dX^5 \sqrt{G} [f(R, \nabla R, ...) + b_1 F_{\mu\nu} F^{\mu\nu}] - \mathcal{A} \ln \frac{1}{\epsilon} + \int_{\rho=\epsilon^2} dx^4 \sqrt{\hat{g}} (\xi F_{ij} F_{kl} \hat{g}^{ik} \hat{g}^{jk}), \tag{70}
\]

where the first counter term is designed to cancel the log divergent term of the holographic current (67), and the second counter term is finite, which depends on the choices of renormalization schemes labeled by the parameter \( \xi \). From the renormalized gravitational action (70), we derive the renormalized holographic current

\[
J^i = -8b_1 A^{(1)i} + (16\xi - 8b_1) \bar{A}^{(1)i}, \tag{71}
\]
which reduces to (16) of section 2 for $\xi = b_1/2$. One can easily check that the choices of $\xi$ do not affect the derivations of the transformation law of current (28) in section 2. That is because $\bar{A} = e^{2\sigma} \bar{A}'$ transforms trivially under Weyl transformations. Finally, it is straightforward to generalize the above discussions to n-form fields and derive eqs. (50, 51) of section 5.

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