Quantum twist to complementarity: A duality relation

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Received November 7, 2012; Accepted February 19, 2013; Published April 1, 2013

Some recent works have introduced a quantum twist to the concept of complementarity, exemplified by a setup in which the which-way detector is in a superposition of being present and absent. It has been argued that such experiments allow measurement of particle-like and wave-like behavior at the same time. Here, we derive an inequality which puts a bound on the visibility of interference and the amount of which-way information that one can obtain, in the context of such modified experiments. As the wave aspect can only be revealed by an ensemble of detections, we argue that, in such experiments, a single detection can contribute only to one subensemble, corresponding to either wave aspect or particle aspect. This way, each detected particle behaves either as particle or as wave, never both, and Bohr’s complementarity is fully respected.

Subject Index

1. Introduction

The two-slit experiment carried out with particles is a testbed of various foundational ideas in quantum theory. It has been used to exemplify wave–particle duality and Bohr’s complementarity principle [1]. The two-slit experiment captures the essence of quantum theory in such a fundamental way that Feynman went to the extent of stating that it is a phenomenon “which has in it the heart of quantum mechanics; in reality it contains the only mystery” of the theory [2].

Niels Bohr had stressed that the wave nature of particles, characterized by interference, and their particle nature, characterized by the knowledge of which slit the particle passed through, are mutually exclusive. He argued that in a single experiment, one could see only one of these two complementary properties at a time. Bohr elevated this concept to the status of a separate principle, the principle of complementarity [1]. Two-slit interference experiments with a which-way detector and the Mach–Zehnder interferometer have been extensively used to study complementarity. It has been demonstrated in many different kinds of experiment that if one obtains the which-path information about a particle, the interference pattern cannot be obtained. Conversely, if the interference pattern is obtained, the which-path information is necessarily destroyed. In some clever experiments, the experimenter can choose to obtain the which-path information after the particle has been registered on the screen [3,4]. Such “delayed-choice” experiments present several conceptual difficulties. Nevertheless, it is generally accepted that, in an experiment, only one of the two aspects can be seen at a time, and that the two are mutually exclusive.
Fig. 1. Schematic diagram to illustrate a typical experiment to test complementarity with the introduction of a quantum device BS2. The beam splitter BS2 is in a superposition of being present in the path of the photon and being away from it.

2. Quantum twist to complementarity

Recently, a new kind of experiment to test complementarity has been proposed [5] and carried out [6–10], where the which-way detector is a quantum device which is prepared in a superposition of being present and absent. A typical such experiment is shown in Fig. 1. Here, if the beam splitter BS2 is absent, the two photon detectors will give which-way information about every photon detected. If the BS2 is present, the two paths are mixed and the which-way information is lost. However, the phases of the two paths can be tuned in such a way that they destructively interfere at (say) detector D1. Detector D1 not detecting any photons indicates interference. Now the setup is modified in such a way that BS2 is in a superposition of two locations, one of which is in the path of the photon, and the other is outside it.

If $|N\rangle$ represents the state of BS2 when it is in the path of the photon, and $|Y\rangle$ represents the state when it is outside it, the argument is that the combined state of BS2 and the photon can be written [5,6] as

$$|\psi\rangle = \sqrt{c}|Y\rangle|\text{particle}\rangle_S + \sqrt{1-c}|N\rangle|\text{wave}\rangle_S,$$

where $|\text{particle}\rangle_S$ represents the state of the photon where it behaves like a particle and $|\text{wave}\rangle_S$ represents its state when it behaves like a wave. The claim is that the wave and particle nature of the photon is present at the same time, in a superposition, which allows one to get some more information as compared to the conventional which-way experiments [5,6].

Here we carry out a detailed analysis of a gedanken setup which represents a typical such experiment, to explore what information such experiments can yield.

3. Path distinguishability and fringe visibility

Consider a conventional two-slit experiment with particles, with a single-bit which-way detector sitting in the path of slit A (see Fig. 2). The which-way detector is initially in the state $|d_2\rangle$. If the particle passes through slit A, the which-way detector comes to a state $|d_1\rangle$. Corresponding to the particle passing through slit B, the detector remains in the state $|d_2\rangle$. We can define the distinguishability of the two paths by $\mathcal{D} = (1 - |\langle d_1|d_2\rangle|)$, where $|d_1\rangle$ and $|d_2\rangle$ are assumed to be normalized, but not necessarily orthogonal to each other. Clearly, for completely orthogonal $|d_1\rangle$ and $|d_2\rangle$, $\mathcal{D} = 1$, and for identical $|d_1\rangle$ and $|d_2\rangle$, $\mathcal{D} = 0$. If $|d_1\rangle$ and $|d_2\rangle$ are orthogonal to each other, one can find an observable of the detector for which the two states can give two distinct eigenvalues. Measuring such an observable, one can find out which of the two slits the particle went through.

Next, we allow our which-way detector to be a quantum object in the sense that it can be in a superposition of two locations. The state $|Y\rangle_L$ corresponds to the detector being in front of slit A,
Fig. 2. A two-slit experiment with a one-bit path detector in front of slit A. The one-bit detector is in a superposition of being present in the path of the photon and being away from it.

and |\(N\rangle_L\) corresponds to it being away from slit A. Let the detector be prepared in an intial state

\[
|\phi_0\rangle_D = |d_2\rangle \left( \sqrt{c} |Y\rangle_L + \sqrt{1-c} |N\rangle_L \right),
\]

where \(c\) is a real constant between 0 and 1. The value of \(c\) being 1 means that the which-way detector is in front of slit A, and being 0 means it is away from the slit. The states |\(d_1\rangle, |d_2\rangle\) can give which-way information only when the which-way detector is in front of slit A. Keeping this in mind, we define the which-way distinguishability as

\[
\mathcal{D} = (1 - |\langle d_1 |d_2 \rangle|) D_\langle \phi_0 |(|Y\rangle \langle Y|)_L |\phi_0 \rangle_D.
\]

Here, we have assumed that when a particle passes through the double slit, the path through slit A gets correlated to |\(d_1\rangle\), and that through slit B gets correlated to |\(d_2\rangle\). For the state given by (2), distinguishability takes the form

\[
\mathcal{D} = c(1 - |\langle d_1 |d_2 \rangle|).
\]

As one can see, the two paths will be fully distinguishable when \(c = 1\) and \(\langle d_1 |d_2 \rangle = 0\).

Let us now assume that a particle traveling along the \(z\)-direction passes through the double slit, with a slit separation of \(d\), and also interacts with a which-path detector. Through a unitary process, the detector states get correlated with the states of the particle coming out of the two slits. The combined state of the particle and the which-path detector when the particle emerges from the double slit (time \(t = 0\)) is assumed to have the form

\[
\Psi(x, 0) = A \sqrt{c} \left( |d_1\rangle \exp \left[ -\frac{(x - d/2)^2}{4\epsilon^2} \right] + |d_2\rangle \exp \left[ -\frac{(x + d/2)^2}{4\epsilon^2} \right] \right) |Y\rangle_L
+ A \sqrt{1-c} \left( \exp \left[ -\frac{(x - d/2)^2}{4\epsilon^2} \right] + \exp \left[ -\frac{(x + d/2)^2}{4\epsilon^2} \right] \right) |d_2\rangle |N\rangle_L,
\]

where \(A = \frac{1}{\sqrt{2}} (2\pi\epsilon^2)^{-1/4}\). Here, we assume the states of the particle coming out of the slits to have a Gaussian form, with a width \(\epsilon\), centered at \(\pm d/2\). We do not explicitly consider the dynamics of the particle in the \(z\)-direction. We just assume that the wave packets are moving in the positive \(z\)-direction with an average momentum \(p_0 = \hbar/\lambda_d\), where \(\lambda_d\) is the de Broglie wavelength of the
particle. Thus, the distance \( L \) traveled by the particle in a time \( t_L \) is given by \( L = \frac{\hbar}{m\lambda_d} t_L \). This can be rewritten as \( \hbar L / m = \lambda_d L / 2\pi \).

After a time \( t \), the state of the particle and the detector evolves to

\[
\Psi(x, t) = A_t \sqrt{e} \left( |d_1\rangle \exp \left[ -\frac{(x - d/2)^2}{4\sigma_i^2} \right] + |d_2\rangle \exp \left[ -\frac{(x + d/2)^2}{4\sigma_i^2} \right] \right) |Y\rangle_L
+ A_t \sqrt{1 - c} \left[ \exp \left[ -\frac{(x - d/2)^2}{4\sigma_i^2} \right] + \exp \left[ -\frac{(x + d/2)^2}{4\sigma_i^2} \right] \right] |\langle Y|N\rangle_L, \tag{6}\]

where \( A_t = \frac{1}{\sqrt{2}} \left( \sqrt{2\pi} (\epsilon + i\hbar t/2me) \right)^{-1/2} \). The probability of finding the particle at position \( x \) on the screen is given by

\[
|\Psi(x, t)|^2 = |A_t|^2 \left( \exp \left[ -\frac{(x - d/2)^2}{2\sigma_i^2} \right] + \exp \left[ -\frac{(x + d/2)^2}{2\sigma_i^2} \right] \right)
+ (1 - c + c|\langle d_1|d_2\rangle|) \left[ \exp \left[ -\frac{x^2 + d^2/4}{2\sigma_i^2} \right] \exp \left[ -\frac{ixd\hbar t/2me^2}{\sigma_i^2} \right] \right]
+ (1 - c + c|\langle d_2|d_1\rangle|) \left[ \exp \left[ -\frac{x^2 + d^2/4}{2\sigma_i^2} \right] \exp \left[ -\frac{ixd\hbar t/2me^2}{\sigma_i^2} \right] \right]. \tag{7}\]

where \( \sigma_i^2 = \epsilon^2 + (\hbar t/2me)^2 \). Writing \( \langle d_2|d_1\rangle = |\langle d_2|d_1\rangle|e^{i\theta} \), and putting \( \hbar t/ m = \lambda_d L/2\pi \), the above can be simplified. Further, for simplicity we put \( \theta = 0 \), which reduces the above to

\[
|\Psi(x, t)|^2 = 2|A_t|^2 \exp \left[ -\frac{x^2 + d^2/4}{2\sigma_i^2} \right] \cosh(d\sigma_i^2 / 2\epsilon)
\times \left( 1 + (1 - c + c|\langle d_1|d_2\rangle|) \frac{\cos \left( \frac{xd\lambda_d L/2\pi}{4\epsilon + (\lambda_d \hbar L/2\pi)^2} \right)}{\cosh(x d/2\sigma_i^2)} \right). \tag{8}\]

Eq. (8) represents an interference pattern with a fringe width given by

\[
w = 2\pi \left( \frac{(\lambda_d L/2\pi)^2 + 4\epsilon^4}{\lambda_d d L/2\pi} \right) = \frac{\lambda_d L}{d} + \frac{16\pi^2 \epsilon^4}{\lambda_d L}. \tag{9}\]

For \( \epsilon^2 \ll \lambda_d L \) we get the familiar Young’s double-slit formula: \( w \approx \lambda_d L / d \).

Visibility of the interference pattern is conventionally defined as

\[
\mathcal{V} = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}, \tag{10}\]

where \( I_{\text{max}} \) and \( I_{\text{min}} \) represent the maximum and minimum intensity in neighboring fringes, respectively. In reality, fringe visibility will depend on many things, including the width of the slits. For example, if the width of the slits is very large, the fringes may not be visible at all. Maxima and minima of (8) will occur at points where the value of cosine is 1 and −1, respectively. The visibility can then be written down as

\[
\mathcal{V} = \frac{1 - c + c|\langle d_1|d_2\rangle|}{\cosh(x d/2\sigma_i^2)}. \tag{11}\]

Since \( \cosh(y) \geq 1 \), we get

\[
\mathcal{V} \leq 1 - c + c|\langle d_1|d_2\rangle|. \tag{12}\]

Using (4), the above equation gives a very important result:

\[
\mathcal{V} + \mathcal{D} \leq 1. \tag{13}\]

Eq. (13) can be considered as a quantitative statement of Bohr’s complementarity principle. It sets a bound on the which-path distinguishability and the visibility of interference that one can obtain
in a single experiment. It is similar in spirit to the well-known Englert–Greenberger duality relation [11,12], but clearly different from it.

In the context of the quantum twist to which-way experiments, if \(|d_1⟩\) and \(|d_2⟩\) are orthogonal, (12) tells us that the visibility \(V\) can be at most \(1 - c\). However, in that situation the path distinguishability \(D\), given by (4), is \(c\). So, even though the which-way detector is replaced by an equivalent quantum device, it doesn’t allow one to obtain simultaneously the which-path information and the interference more precisely than (13). In the special case \(c = 1\), the relation (13) will describe the bounds on distinguishability and visibility in the conventional complementarity experiments.

If one were to correlate every particle detected on the screen with a measurement result on the states \(|Y⟩\) and \(|N⟩\), every click will either give which-way information or it won’t. Only those clicks which do not yield which-way information will contribute to the interference pattern. So, which-way information and interference remain mutually exclusive. Of course, one can obtain fuzzy which-way information, but it will result in a fuzzy (not sharp) interference pattern.

4. A random quantum eraser

We now describe a gedanken experiment based on the so-called quantum eraser [13,14], which also achieves which-way information and interference in the same experimental setup, although with some differences. Let there be a setup of two-slit experiment with a one-bit which-way detector in front of one of the slits. The state that comes out of the slit can be written as

\[
\Psi(x) = \frac{1}{\sqrt{2}}(|d_1⟩\psi_A(x) + |d_2⟩\psi_B(x)),
\]

(14)

where \(|d_1⟩\), \(|d_2⟩\) are two orthonormal states of the which-way detector, and \(\psi_A(x)\), \(\psi_B(x)\) are the wave packets emerging from slits A and B, respectively. The probability of a particle falling at a position \(x\) on the screen is given by

\[
|\Psi(x)|^2 = \frac{1}{2}(|\psi_A(x)|^2 + |\psi_B(x)|^2),
\]

(15)

which gives no interference because of the mutual orthogonality of \(|d_1⟩\), \(|d_2⟩\). Measuring an observable, call it \(σ_z\), of the which-way detector whose eigenstates are \(|d_1⟩\), \(|d_2⟩\), in coincidence with detected particles, will give which-way information about each particle. The particle nature is brought out in such measurements. Let us now imagine that there is another observable, call it \(σ_x\), whose eigenstates are \(|d_+⟩\), \(|d_-⟩\) such that

\[
|d_+⟩ = (|d_1⟩ + |d_2⟩)/2, \quad |d_-⟩ = (|d_1⟩ - |d_2⟩)/2.
\]

(16)

In terms of these states, (14) can be written as

\[
\Psi(x) = \frac{|d_+⟩}{2}(\psi_A(x) + \psi_B(x)) + \frac{|d_-⟩}{2}(\psi_A(x) - \psi_B(x)).
\]

(17)

Now if one measures the observable \(σ_z\) in coincidence with the detected particles, the particles in coincidence with \(|d_+⟩\) will show an interference pattern, and those in coincidence with \(|d_-⟩\) will show a \(\pi\)-shifted interference pattern. The wave nature is brought out in such measurements. The measurement on the one-bit which-way detector can be done after the particle is detected, and one can choose to bring out either particle nature or wave nature. The wave nature and the particle nature are both present at the same time, until the measurement on the one-bit which-way detector is made. This is an example of the so-called quantum eraser [13,14].
Now, suppose one randomly measures $\sigma_z$ or $\sigma_x$ in coincidence with the particles falling on the screen. After detecting all the particles, one can separate out those which were in coincidence with $\sigma_z$ and $\sigma_x$, respectively. Thus, particle nature and wave nature can be explored in the same experiment. The result is quite similar to what has been proposed and shown in Refs. [5–10]. The major way in which the scheme [5–10] is different from our quantum eraser scheme is that in the former the choice between the particle nature and the wave nature is randomly made by the quantum nature of the position of the detector, whereas in the latter it is made by the experimenter. In both the schemes, every single particle detected has to clearly follow either wave nature or particle nature. Another difference is that in the quantum eraser scheme the which-way information is always carried by the which-way detector, but can be “erased” by the choice of observable of the which-way detector which is measured. In the scheme of [5–10], the which-way information is there only with a probability (say) $c$; there is a probability $1 - c$ that there is no which-way information.

5. Conclusions

In conclusion, we have theoretically analyzed the quantum twist to complementarity introduced recently in the context of some modified interference experiments. We have derived a duality relation in the context of such experiments. It puts a bound on the which-way information that one can extract, and the visibility of interference in the same experiment. We emphasize that Bohr’s complementarity continues to hold, and should be viewed in the context of individual outcomes. In each outcome, one can either get which-way information, in which case particle nature will emerge, or the detection will contribute to the interference pattern, which shows wave nature. Although interference builds up after registering many particles, only those detections will contribute to interference for which no which-way information was found. All those detections where which-way information is found will not contribute to the interference pattern. Thus, which-way information and interference remain mutually exclusive, although in the new clever experiments [5–10] both aspects can be seen in a single experimental setup. It is always possible to do imprecise which-way measurement and get a fuzzy interference for the same particle. But this was already known before, and has been the motivation for the Englert–Greenberger duality relation. Lastly, the new experiments where a quantum device is added to the interference experiments do help us understand Bohr’s complementarity principle better.

Acknowledgements

We are thankful to Lucas Chibebe Céleri for bringing this new aspect of complementarity to our notice. This work is supported by the University Grants Commission, India.

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