WZW-Toda Reduction using the Casimir Operator

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We construct a quantum Hamiltonian operator for the Wess-Zumino-Witten (WZW) model in terms of the Casimir operator. This facilitates the discussion of the reduction of the WZW model to Toda field theory at the quantum level and provides a very straightforward derivation of the quantum central charge for the Toda field theory.

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1. Introduction

Two-dimensional conformal field theories have been the object of much attention, in relation to string theory and also to statistical mechanics. The Wess-Zumino-Witten (WZW) models based on a Lie group \( G \) play a central role in the study and classification of conformal field theories, since a large variety of rational conformal field theories can be obtained by gauging the WZW model. On the one hand, the Goddard-Kent-Olive (GKO) construction (which, for instance, generates the minimal models) can be realised by gauging a diagonal subgroup of \( G \). On the other hand, Toda field theories can be obtained by gauging a nilpotent subgroup of a maximally non-compact group \( G \). In the Toda case, the function of the gauging is essentially to impose constraints on the Kac-Moody currents associated with the WZW model. The discussion of the GKO construction and the reduction of the WZW model to Toda theory can both be facilitated by the recent discovery of a free-field representation of the WZW model. This was found useful in the discussion of the BRST quantisation of the gauged WZW models corresponding to Toda field theory.

In particular, the simple form for the energy-momentum tensor of the WZW model in the free-field representation leads straightforwardly to the correct expression for the central charge of the Toda field theory at the quantum level. Our purpose here is to discuss the derivation of the Toda field theory from the WZW model in a way which parallels the free-field approach, but which is more rooted in group theory than in conformal field theory. The method is derived from techniques used recently to derive the exact metric and dilaton fields for string black holes (which are also obtained by gauging a WZW model). The idea is to construct the Hamiltonian operator for the WZW model, incorporating quantum effects using the Sugawara construction, and also including additional terms arising from the gauging. (This procedure is similar to results that have emerged in the discussion of the analogy between the Hamiltonians of quasi-exactly-soluble quantum mechanical systems and the energy-momentum tensor of conformal theories based on the generalised Sugawara construction.) The field equation derived from this Hamiltonian is then compared with the field equation deriving from the string effective action for a general non-linear \( \sigma \)-model, from which the metric and dilaton fields can be read off, and thence the action for the Toda field theory can be reconstructed. In the case of Witten’s string black hole, the method provided an exact solution whose validity has been checked up to fourth order in perturbation theory. The method has also been
used to construct more general exact string black hole solutions\cite{18}. Similar results have recently been obtained by considering the quantum effective action for the gauged WZW model\cite{19}. The Hamiltonian operator for the WZW model is (up to a factor) precisely the quadratic Casimir operator for the group $G$, the zero-modes of the Kac-Moody currents being the left- and right-invariant vector fields on the group manifold. A useful co-ordinate system on the group manifold for the present purpose is that corresponding to the Gauss decomposition of the group elements for a maximally non-compact group $G$. In this coordinate system, the quadratic Casimir operator assumes a particularly simple form, which can be given explicitly for a general maximally non-compact Lie group$^{20}$-$^{22}$. This simple form is analogous (though not exactly equivalent) to the free-field representation of the WZW Lagrangian. The correct quantum conformally-invariant action for the Toda field theory then follows almost immediately. We feel that this derivation of the Toda field theory from the WZW model is quicker and more explicit than the use of the free-field representation. Moreover, the method may be useful for discussing the so-called “non-abelian” Toda theories$^{23}$-$^{24}$ based on a non-canonical grading of the Lie algebra of $G$, which may be related to string black holes. Although the central charge of these theories has been calculated$^{25}$, the full quantum action does not appear to be known.

2. The Wess-Zumino-Witten model

The action for the Wess-Zumino-Witten model defined on a group manifold $G$ is given by$^{1}$

$$kS(g) = -\frac{k}{8\pi} \int d^2x \text{tr}(\partial_\mu g \partial^\mu g^{-1}) + \frac{k}{12\pi} \int \text{tr}(dg g^{-1})^3$$ \hspace{1cm} (2.1)

where $g \in G$. (We assume for convenience that the trace is normalised to agree with the trace in the adjoint representation.) This action is invariant under the transformations

$$g(z, \bar{z}) \to \Omega_L(z)g(z, \bar{z})\Omega_R(\bar{z})$$ \hspace{1cm} (2.2)

whose generators are the currents

$$J(z) = k\partial g g^{-1}, \quad \bar{J}(\bar{z}) = -kg^{-1}\partial \bar{g}$$ \hspace{1cm} (2.3)

which generate two commuting copies of the Kac-Moody algebra. We shall consider a maximally non-compact Lie group $G$ and make use of the Gauss decomposition of the group element $g \in G$, writing

$$g = g_0g_+$$ \hspace{1cm} (2.4)
where

\[ g_- = \exp\left( \sum_{\alpha \in \Phi_+} \varphi^\alpha E_{-\alpha} \right), \quad g_+ = \exp\left( \sum_{\alpha \in \Phi_+} \varphi^\alpha E_{\alpha} \right), \quad g_0 = \exp\left( \sum_{i=1}^{\text{rank} G} r^i H_i \right). \tag{2.5} \]

In Eq. (2.5), \( \Phi^+ \) is the set of positive roots \( \alpha \) and \( E_\alpha \) are the corresponding step operators, while \( \{ H_i, i = 1, 2, \ldots, \text{rank} G \} \) are the generators of the Cartan subalgebra. (See the Appendix for our Lie algebra conventions.) The parameters \( \{ \varphi^\alpha, \varphi^\alpha_+, r^i \} \) may be regarded as co-ordinates on the group manifold \( G \).

As mentioned in Section 1, the Toda field theory is obtained from the WZW model by gauging in order to impose certain constraints\[^5\][^6]. In the usual gauging of the WZW model\[^3\], one gauges the vector invariance \( g \to \gamma g \gamma^{-1}, \gamma(z, \bar{z}) \in H \) for some subgroup \( H \) of \( G \). However, owing to the nilpotency of the groups \( G_\pm \) generated by elements of the form \( g_\pm \) in Eq. (2.5), in this case we may gauge the invariance \( g \to \alpha g \beta^{-1}, \alpha(z, \bar{z}) \in G_-, \beta(z, \bar{z}) \in G_+ \). The gauged action is\[^3\]

\[ kS_G(g, A, \bar{A}) = kS(g) + \frac{k}{4\pi} \int d^2x \text{tr}(\bar{A} \partial g g^{-1} + Ag^{-1} \bar{\partial} g + \bar{A} g Ag^{-1} - \bar{A} \mu - A\nu). \tag{2.6} \]

The conventional vector gauging sets to zero the currents corresponding to the subgroup \( H \) of \( G \). Here, however, the presence of the matrices \( \mu, \nu \) defined by

\[ \mu = \sum_{\alpha \in \Delta} \mu^{-\alpha} E_{-\alpha}, \quad \nu = \sum_{\alpha \in \Delta} \nu^\alpha E_{\alpha} \tag{2.7} \]

(where \( \Delta \) denotes the set of simple roots) means that the effect of the gauging is to impose the constraints

\[ J_\alpha \equiv \text{tr}(E_\alpha J) = k\mu_\alpha, \quad \bar{J}_{-\alpha} \equiv \text{tr}(E_{-\alpha} \bar{J}) = -k\nu_{-\alpha} \quad (\alpha \in \Delta) \]

\[ J_\alpha = 0, \quad \bar{J}_{-\alpha} = 0 \quad (\alpha \in \Phi^+ \setminus \Delta). \tag{2.8} \]

The resultant Toda field theory action is given at the classical level by

\[ S = \frac{k}{8\pi} \int d^2x \left( \gamma_{ij} \partial_\mu r^i \partial_\nu r^j + \sum_{\alpha \in \Delta} M_\alpha e^{-r^i(\alpha(H_i))} \right), \tag{2.9} \]

where

\[ M_\alpha = \gamma^\alpha(-\alpha) \mu_\alpha \nu_{-\alpha} \tag{2.10} \]

and \( \gamma_{ij} \) is defined in Eq. (A.3). The energy-momentum tensor for the WZW model is constructed from the Kac-Moody currents according to the Sugawara construction\[^13\].
However with the imposition of the constraints Eq. (2.8), the energy-momentum tensor must be modified by an “improvement term” [5] to ensure that it commutes with the constraints. The result is

\[ T(z) = \frac{1}{k - h} (J, J) - 2 (\delta, \partial J) \]  

(2.11)

where \( h \) is the dual Coxeter number for the group \( G \) and where

\[ \delta = \sum_{\alpha \in \Phi^+} \frac{H_\alpha}{<\alpha, \alpha>} \]  

(2.12)

(with an analogous result for the antiholomorphic component of the energy-momentum tensor, \( \bar{T}(\bar{z}) \)). The important point to note is that this form of the energy-momentum tensor fully incorporates quantum effects. (The unconventional sign of \( h \) in Eq. (2.11) arises because we are using the conventions of Refs. [12], [11], appropriate for a non-compact group.) We now introduce vertex operators \( V(\varphi^a, \varphi^a+ r^i) \) for the WZW model. The zero modes \( J_0, \bar{J}_0 \) in Laurent expansions of the Kac-Moody currents can be expressed as operators \( \mathcal{J}^L, \mathcal{J}^R \) acting on the vertex operators [11]. The components of these operators, defined by

\[ \mathcal{J}^L_a = (T_a, \mathcal{J}^L), \quad \mathcal{J}^R_a = (T_a, \mathcal{J}^R), \]  

(2.13)

where \( T_a \) denotes a generic element of the Lie algebra, are in fact the generators of left and right multiplication by Lie algebra elements, \( \text{viz.} \)

\[ \mathcal{J}^L_a g = T_a g, \quad \mathcal{J}^R_a g = g T_a. \]  

(2.14)

The Virasoro operator \( L_0 \) can then also be expressed as a differential operator [11] by replacing the zero modes of the currents in \( T(z) \) in Eq. (2.11) by \( \mathcal{J}^a_L \), and similarly \( \bar{L}_0 \) may also be obtained by replacing the zero modes of the currents in \( \bar{T}(\bar{z}) \) by \( \mathcal{J}^a_R \). The results are

\[ L_0 = \frac{1}{k - h} (\mathcal{J}^L, \mathcal{J}^L) + 2 (\delta, \mathcal{J}^L), \]

\[ \bar{L}_0 = \frac{1}{k - h} (\mathcal{J}^R, \mathcal{J}^R) + 2 (\delta, \mathcal{J}^R). \]  

(2.15)

The expressions \( (\mathcal{J}^L, \mathcal{J}^L) \) and \( (\mathcal{J}^R, \mathcal{J}^R) \) are in fact both equal to the Casimir operator \( C \) for the group \( G \). Moreover, since \( \delta \) is in the Cartan subalgebra, \( (\delta, \mathcal{J}^L) = (\delta, \mathcal{J}^R) . \)
Hence $L_0$ and $\bar{L}_0$ coincide as operators. In string theory, the physical state conditions for tachyonic vertex operators read

\[(L_0 + \bar{L}_0 - 2)V(\varphi^\alpha_-, \varphi^\alpha_+, r^i) = 0, \quad (L_0 - \bar{L}_0)V(\varphi^\alpha_-, \varphi^\alpha_+, r^i) = 0.\] (2.16)

The second of these equations is automatically satisfied; the first becomes, using Eq. (2.14)

\[\left(\frac{c}{k-h} + 2(\delta, J_R) - 1\right)V(\varphi^\alpha_-, \varphi^\alpha_+, r^i) = 0.\] (2.17)

The operator in Eq. (2.17) may be regarded as the Hamiltonian operator for the Toda field theory, which already incorporates quantum effects via the Sugawara construction. We will see later how we can read off from the Hamiltonian the fully conformally invariant Toda field theory Lagrangian. Firstly, however, we devote the next Section to a discussion of the form of the Casimir operator.

### 3. The Casimir Operator

A thorough investigation of the form of the Casimir operator corresponding to various decompositions of a general Lie algebra $G$ has been undertaken by Leznov and Saveliev\cite{20}. The results are compiled in a recent book\cite{22}. For completeness, however, we will summarise the main points relating to the Gauss decomposition here, and we will also give more explicit results for some of the quantities involved.

As we mentioned in Section 2, the Casimir operator is given by

\[C = (J^L, J^L) = (J^R, J^R)\] (3.1)

where $J^L$ and $J^R$ are operators whose components, defined in Eq. (2.12), are the generators of left and right shifts respectively. Mathematically they may also be described as left- and right-invariant vector fields respectively, defined on the group manifold. Our first step, following Leznov and Saveliev, is to obtain expressions for these shift generators. It is sufficient to concentrate on, say, the right shift operators. We consider the component $J^R_a$ which generates multiplication on the right by $T_a$, where $T_a$ represents one of $E_\alpha$, $E_{-\alpha}$, or $H_i$. If we wish to be more specific, we shall denote the operators representing multiplication on the right by $E_\alpha$, $E_{-\alpha}$, and $H_i$ as $J^R_\alpha$, $J^R_{-\alpha}$ and $J^R_i$ respectively. We begin by introducing operators $X^L_{-\alpha}$, $X^R_{-\alpha}$, $X^L_{+\alpha}$, $X^R_{+\alpha}$ defined by

\[X^L_{-\alpha}g_- = E_{-\alpha}g_-, \quad X^R_{-\alpha}g_- = g_-E_{-\alpha},\]
\[X^L_{+\alpha}g_+ = E_\alpha g_+, \quad X^R_{+\alpha}g_+ = g_+E_\alpha.\] (3.2)
The operators $X_{-\alpha}^{L,R}$ only act on $g_-$ and contain only the variables $\varphi^\alpha$, and similarly the operators $X_{+\alpha}^{L,R}$ only act on $g_+$ and contain only the variables $\varphi^\alpha$. Hence we have

$$X_{-\alpha}^{L,R} g_+ = X_{+\alpha}^{L,R} g_+ = X_{-\alpha}^{L,R} g_- = X_{+\alpha}^{L,R} g_- = 0,$$

$$[X_{-\alpha}^{L,R}, X_{+\alpha}^{L,R}] = 0. \quad (3.3)$$

It is then clear that

$$\mathcal{J}_\alpha^R = X_+^{R\alpha}. \quad (3.4)$$

In general, though, we have

$$\mathcal{J}_\alpha^R g = g T_a = g g_0 (g^+ T_ag_+^{-1}) g_+. \quad (3.5)$$

It is convenient to define

$$E^\alpha = \gamma^{\alpha\beta} E_\beta, \quad H^i = \gamma^{ij} H_j, \quad (3.6)$$

where $\gamma^{ab}$ is the inverse of the Cartan-Killing form $\gamma_{ab}$, so that

$$(E^\alpha, E_\beta) = \delta^\alpha_\beta, \quad (E^\alpha, H_i) = 0,$$

$$(H^i, H_j) = \delta^i_j, \quad (H^i, E_{\pm\alpha}) = 0. \quad (3.7)$$

We can then write

$$g^+ T_ag_+^{-1} = \sum_{\beta \in \Phi^+} ((E^{-\beta}, g T_ag_+^{-1}) E^{-\beta}$$

$$+ (E^\beta, g T_ag_+^{-1}) E_\beta) + (H^i, g T_ag_+^{-1}) H_j. \quad (3.8)$$

Substituting Eq. (3.8) into Eq. (3.5), and using

$$g_0 E^{-\beta} = e^{-r^i, \beta(H_i)} E^{-\beta} g_0, \quad (3.9)$$

followed by Eq. (3.2), we obtain

$$\mathcal{J}_a^R = \sum_{\beta \in \Phi^+} ((E^{-\beta}, g T_ag_+^{-1}) e^{-r^i, \beta(H_i)} X_+^{R\beta} + (E^\beta, g T_ag_+^{-1}) X_-^{L\beta})$$

$$+ (H^j, g T_ag_+^{-1}) \frac{\partial}{\partial r^j}. \quad (3.10)$$
Note that in the case of $J_i^R$, the first term on the right-hand side of Eq. (3.11) is zero, and in the case of $J_\alpha^R$, the first and third terms on the right-hand side are zero. This expression for $J_\alpha^R$ agrees with that given earlier in Eq. (3.4) in virtue of the identity

$$X_{+\alpha}^R = \sum_{\beta \in \Phi^+} (E^\beta, g_+ E^\alpha g_+^{-1})X_{+\beta}^L. \quad (3.11)$$

Now

$$(gLg^{-1}, gMg^{-1}) = (L, M) \quad (3.12)$$

for any elements of the Lie algebra $L, M$ and any $g \in G$. So from Eq. (2.13) we have

$$J^R = \sum_{\beta \in \Phi^+} ((g_+^{-1}E^{-\beta}g_+)e^{-r^i\beta(H_i)}X_{-\beta}^R + (g_+^{-1}E^\beta g_+)X_{+\beta}^L)$$

$$+ (g_+^{-1}H^j g_+) \frac{\partial}{\partial r^j}.$$ \quad (3.13)

Using Eqs. (3.2), (3.3), we may also write Eq. (3.13) in the form

$$J^R = \sum_{\beta \in \Phi^+} (X_{-\beta}^R(g_+^{-1}E^{-\beta}g_+)e^{-r^i\beta(H_i)} + X_{+\beta}^L(g_+^{-1}E^\beta g_+))$$

$$+ g_+^{-1}H_\beta g_+) + \frac{\partial}{\partial r^j}(g_+^{-1}H^j g_+). \quad (3.14)$$

Substituting Eqs. (3.14) and (3.13) for the first and second occurrences respectively of $J^R$ on the right-hand side of Eq. (3.1), and using Eq. (3.12), we obtain finally

$$C = \sum_{\beta \in \Phi^+} (2\gamma^\beta(-\beta)e^{-r^i\beta(H_i)}X_{-\beta}^R X_{+\beta}^L + \gamma^{ij}\beta(H_i)\frac{\partial}{\partial r^j})$$

$$+ \gamma^{ij}\frac{\partial}{\partial r^i}\frac{\partial}{\partial r^j}. \quad (3.15)$$

In Eqs. (3.10) and (3.15) we have simple expressions for the shift operators $J_\alpha^R$ and the Casimir operator $C$ in terms of the operators $X_{-\alpha}^R$ and $X_{+\alpha}^L$. We can in fact obtain fairly explicit expressions for these operators. It is easy to show that

$$(\partial_{+\alpha}g_+)g_+^{-1} = \frac{e^{ad\varphi_+} - 1}{ad\varphi_+}E_\alpha \quad (3.16)$$

where

$$\varphi_+ = \sum_{\alpha \in \Phi^+} \varphi_\alpha^\alpha E_\alpha, \quad (ad\varphi_+)L = [\varphi_+, L] \quad (3.17)$$
for \( L \) in the Lie algebra of \( G \), and \( \partial_+ \equiv \frac{\partial}{\partial \varphi_+} \). Hence we have

\[
E_\alpha g_+ = \frac{\text{ad} \varphi_+}{e^{\text{ad} \varphi_+} - 1} \partial_+ \alpha g_+.
\]

(3.18)

Using

\[
\frac{x}{e^x - 1} = \sum b_n \frac{x^n}{n!},
\]

(3.19)

where \( b_n \) are the Bernoulli numbers, we find, comparing Eqs. (3.2) and (3.18),

\[
X^L_{+\alpha} = \sum b_n \frac{n!}{n!} (\text{ad} \varphi_+)^n \partial_+ \alpha.
\]

(3.20)

By considering the action of \( X^L_{+\alpha} \) on \( \varphi_+ \), we obtain the following more explicit expression:

\[
X^L_{+\alpha} = \sum \frac{b_n}{n!} N_{\alpha \rho_1 \ldots \rho_n} \varphi^\rho_1 \ldots \varphi^\rho_n \partial_{\alpha + \rho_1 + \ldots + \rho_n}
\]

(3.21)

where \( N_{\alpha \rho_1 \ldots \rho_n} \) is the coefficient of \( E_{\alpha + \rho_1 + \ldots + \rho_n} \) in \([E_{\rho_n}, \ldots [E_{\rho_2}, [E_{\rho_1}, E_{\alpha}]] \ldots]\). Similarly, using

\[
g_-^{-1} \partial_- g_- = -\frac{e^{-\text{ad} \varphi_-} - 1}{\text{ad} \varphi_-} E_- \alpha
\]

(3.22)

we can derive

\[
X^R_{-\alpha} = \sum \frac{b_n}{n!} (-\text{ad} \varphi_-)^n \partial_- \alpha
\]

(3.23)

where

\[
\varphi_- = \sum_{\alpha \in \Phi_+} \varphi^\alpha_+ E_\alpha, \quad \partial_- \alpha \equiv \frac{\partial}{\partial \varphi_-^\alpha}.
\]

(3.24)

Of course \( X^L_{-\alpha} \) and \( X^R_{+\alpha} \) may be obtained by interchanging the subscripts + and – in Eqs. (3.20), (3.23). In fact, it is easy to see that \( X^R_{-\alpha} \) may be obtained from \( X^L_{+\alpha} \) by replacing \( \varphi_+ \) by \((-\varphi_+)\), and \( X^L_{+\alpha} \) and \( X^R_{-\alpha} \) may be obtained from \( X^L_{+\alpha} \) and \( X^R_{+\alpha} \) respectively by replacing \( \varphi_+ \) by \((-\varphi_-)\), and hence explicit expressions may be obtained for all these operators starting from Eq. (3.21). Since the Lie algebras for \( G_- \) and \( G_+ \) are nilpotent, the summations in Eqs. (3.20), (3.21) and (3.23) terminate at the level of the highest root. Combined with the Baker-Campbell-Hausdorff formula

\[
g_+ E_{-\alpha} g_+^{-1} = \sum \frac{1}{n!} (\text{ad} \varphi_+)^n E_{-\alpha}
\]

(3.25)

(which of course also terminates) we may readily obtain explicit formulæ for the shift operators in Eq. (3.10) and the Casimir operator in Eq. (3.15) for any given Lie group. The
identity Eq. (3.12) relating $X^L_\alpha$ and $X^R_\alpha$ is a simple consequence of Eqs. (3.20), (3.23), (3.25) and the identity

$$b_n = \sum_{k=0}^{n} C(n,k) b_k \quad (n \neq 1). \quad (3.26)$$

Finally we note that we can obtain a simple completely explicit formula for $J^R_i$ as follows. Writing

$$gH_i = g - g_0 (g + H_i g +^{-1}) g_+,$$  

we have

$$g_+ H_i g_+^{-1} = H_i - \sum_{\alpha \in \Phi^+} \alpha(H_i) \sum_n \frac{1}{(n+1)!} (\text{ad} \varphi^+)^n E_\alpha$$

$$= H_i - \sum_{\alpha \in \Phi^+} \alpha(H_i) \frac{e^{\text{ad} \varphi^+} - 1}{\text{ad} \varphi^+} E_\alpha. \quad (3.28)$$

Hence, using Eq. (3.16), we have

$$J^h_\alpha = \frac{\partial}{\partial r^i} - \sum_{\alpha \in \Phi^+} \varphi_+^\alpha \alpha(H_i) \partial_+ \alpha.$$  

$$\quad (3.29)$$

4. Toda field theory

Toda field theory is obtained from the Wess-Zumino-Witten model by imposing the constraints Eq. (2.8). In the context of the vertex operators $V(\varphi^\alpha, \varphi_+^\alpha, r^i)$, Eqs. (2.8) are imposed as operator constraints derived again from the zero modes of the currents. The constraints Eq. (2.8) imply

$$J^R_\alpha V = \mu_\alpha V, \quad \tilde{J}^L_{-\alpha} V = -\nu_{-\alpha} V \quad (\alpha \in \Delta)$$

$$J^R_\alpha V = 0, \quad \tilde{J}^L_{-\alpha} V = 0 \quad (\alpha \in \Phi^+ \setminus \Delta). \quad (4.1)$$

Since $J^R_\alpha V = 0$ except for simple roots, we may use Eqs. (3.4), (3.26) (with $-\alpha \to \alpha$) to obtain

$$\partial_\alpha V = 0 \quad (\alpha \in \Phi^+ \setminus \Delta), \quad \partial_\alpha V = \mu_\alpha V \quad (\alpha \in \Delta) \quad (4.2)$$

by starting with the root(s) at the highest level and working downwards. Using analogous results for $\tilde{J}^L_{-\alpha}$, we find

$$\partial_{-\alpha} V = 0 \quad (\alpha \in \Phi^+ \setminus \Delta), \quad \partial_{-\alpha} V = -\nu_{-\alpha} V \quad (\alpha \in \Delta). \quad (4.3)$$
Hence we have, from Eqs. (3.20), (3.23), (and corresponding results for $\alpha \to -\alpha$)

$$X^R_{\alpha}V = X^L_{\alpha}V = 0, \quad (\alpha \in \Phi^+ \setminus \Delta) \quad X^R_{\alpha}V = X^L_{\alpha}V = \mu_{\alpha}V, \quad (\alpha \in \Delta)$$

$$X^R_{-\alpha}V = X^L_{-\alpha}V = 0, \quad (\alpha \in \Phi^+ \setminus \Delta) \quad X^R_{-\alpha}V = X^L_{-\alpha}V = \nu_{-\alpha}V, \quad (\alpha \in \Delta). \quad (4.4)$$

The physical state condition Eq. (2.17) becomes, using Eqs. (3.15) and (4.4)

$$\left(\frac{\partial}{\partial R}, \frac{\partial}{\partial R}\right) + 2\left(\frac{1}{\lambda} \delta + \lambda \rho, \frac{\partial}{\partial R}\right) - 2\left(\lambda^2 - \frac{1}{k}\right) \sum_{\alpha \in \Delta} M_{\alpha} e^{-\frac{1}{\lambda}(R,H_{\alpha}) - 1} V = 0 \quad (4.5)$$

where $M_{\alpha}$ is given by Eq. (2.10), $\delta$ is given in Eq. (2.12), and we define $\lambda$, $R$, $\frac{\partial}{\partial R}$ and $\rho$ by

$$\lambda = \frac{1}{\sqrt{k - h}}, \quad R = \frac{1}{\lambda} r^i H_i, \quad \frac{\partial}{\partial R} = \lambda H^i \frac{\partial}{\partial r^i}, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} H_{\alpha}, \quad (4.6)$$

with $H^i$ as given in Eq. (3.6). Following Ref. [11], we now wish to interpret the differential operator in Eq. (4.5) as the Hamiltonian operator derived from an action

$$S(r^i) = \frac{1}{8\pi} \int d^2 x \sqrt{\eta} \left(\eta^{\mu\nu} \partial_{\mu} r^i \partial_{\nu} r^j g_{ij} + V(r^i) + D(r^i) R^{(2)} \right) \quad (4.7)$$

where $R^{(2)}$ denotes the two-dimensional Ricci scalar. The conformal invariance conditions for the tachyon potential $V(r^i)$ in Eq. (4.7) are equivalent to the field equations derived from the string effective action, which is given to lowest order by

$$S(V) = \int dr e^{-2D(r)} \sqrt{g} \left(g^{ij} \partial_i V \partial_j V - 2V^2 + \ldots \right) \quad (4.8)$$

where the ellipsis represents a polynomial in $V$ which is determined by non-perturbative effects [26] [27]. However, these non-perturbative effects can be ignored for the purposes of reproducing results derived in conformal theory, which are predicated upon a renormalisation scheme in which non-perturbative effects are absent [28]. There are also higher-order perturbative corrections to $g_{ij}$ in Eq. (4.8), which can be deduced from the tachyon $\beta$-functions calculated in Ref. [29]. However, these corrections can be absorbed in a field redefinition of the metric and dilaton [17]. Identifying the physical state condition Eq. (4.5) with the $\sigma$-model conformal invariance conditions, we deduce that the derivative parts of the operator in Eq. (4.5) should be given by

$$\frac{-1}{e^{-2D} \sqrt{g}} \partial_i e^{-2D} \sqrt{g} g^{ij} \partial_j. \quad (4.9)$$
Comparing Eqs. (4.5), (4.9), we see that

\[
\partial_\mu r^i \partial_\nu r^j g_{ij} = (\partial_\mu R, \partial_\nu R)
\]

\[
D(r^i) = \left( \frac{1}{\lambda} \delta + \lambda \rho, R \right).
\]

(4.10)

However, it is not clear how to derive the correct quantum form of the potential term \(V(r^i)\) itself from a comparison of the field equations for Eq. (4.8) with Eq. (4.5). Previous calculations\[30\] indicate that the original exponential interactions in Eq. (2.9) are modified by multiplicative renormalisations of \(M_\alpha\) arising from one-loop perturbative effects. Other authors\[31\] have found non-perturbative renormalisations for exponential interactions with imaginary exponents, but these do not arise in the case of exponential interactions with real exponents such as we consider here\[27\]. We can now reconstruct the fully conformally invariant action for the Toda field theory as

\[
S(R) = \frac{1}{8\pi} \int d^2 x \sqrt{\eta} \left( \eta^{\mu\nu} (-\partial_\mu R, \partial_\nu R) + \left( \frac{1}{\lambda} \delta + \lambda \rho, R \right) R(2) + E(R) \right).
\]

(4.11)

where \(E(R)\) denotes the interaction potential. Using the well-known formula for the central charge for the Coulomb gas representation, we find the central charge for the Toda field theory to be given by\[5\]

\[
c = \text{rank} G + 12 \left( \frac{1}{\lambda} \delta + \lambda \rho, \frac{1}{\lambda} \delta + \lambda \rho \right).
\]

(4.12)

For the case of a simply-laced Lie algebra (with the roots normalised by \(\langle \alpha, \alpha \rangle = 2\)), for which \(\delta = \rho\), we have, upon using the Freudenthal-de Vries strange formula

\[
c = \text{rank} G + \left( \frac{1}{\lambda} + \lambda \right)^2 h \dim G,
\]

(4.13)

which is the correct central charge for the Toda field theory for a simply-laced Lie algebra at the quantum level\[10\].

Finally, we note that the canonical expression for the Toda action may be achieved by setting

\[
R = \sum_{\alpha \in \Delta} \tilde{r}^\alpha H_\alpha
\]

(4.14)

in Eq. (4.11), and using Eqns. (A.5), (A.7)-(A.9). We find

\[
S(\tilde{r}) = \frac{1}{8\pi} \int d^2 x \sqrt{\eta} \left( \sum_{\alpha, \beta \in \Delta} \eta^{\mu\nu} \frac{A_{\alpha\beta}}{\langle \alpha, \alpha \rangle} \partial_\mu \tilde{r}^\alpha \partial_\nu \tilde{r}^\beta 
\]

\[
+ \sum_{\alpha \in \Delta} \left( \frac{1}{\lambda} \delta + \frac{\lambda}{2} \right) \tilde{r}^\alpha R(2) + E(\tilde{r}) \right).
\]

(4.15)
5. Conclusions

We have obtained the correct central charge at the quantum level for the Toda field theory, for both the simply-laced\cite{10} and non-simply-laced\cite{5} cases, in what we believe is a succinct and elegant fashion. We note that the dilaton and central charge for the Toda theory were calculated in Ref.\cite{27} by assuming an action of the general Toda form and solving the conformal invariance condition for the tachyon (i.e., the field equation for Eq. (4.8)). Our approach is somewhat analogous since again the tachyon conformal invariance condition plays a central rôle. However the main feature of our method is that it shows how the correct quantum properties of the Toda theory can be derived from the WZW model. As we mentioned in the previous section, it is not clear if one can determine the quantum-corrected exponential interactions $E(R)$ from the non-derivative part of the operator in Eq. (4.5). This deserves further investigation.

In any case, Eq. (4.5) may have an interesting direct physical interpretation. The case of the Liouville model was discussed in Ref.\cite{11}. There it was pointed out that Eq. (4.5) was equivalent to a mini-superspace Wheeler-DeWitt equation for a loop amplitude, interpreted as a wave function for two-dimensional quantum gravity\cite{32}. In the same way as the Liouville model describes the effective induced action for two-dimensional quantum gravity in the conformal gauge, it appears that Toda field theory describes the effective induced action for $W$-gravity in the conformal gauge\cite{33}, and so one may speculate that Eq. (4.5) can be interpreted as the mini-superspace Wheeler-DeWitt equation for a wave-function in $W$-gravity.

The Gauss decomposition Eq. (2.3) corresponds to the canonical grading of the Lie algebra (in which elements of the form $g_0$ in Eq. (2.4) generate an abelian subgroup of $\mathcal{G}$). One may also consider non-canonical gradations of the Lie algebra which lead to a modified gauss decomposition for which $g_0$ generates a non-abelian subgroup. Upon going through the process of reducing the WZW model by imposing constraints \cite{5} \cite{25}, this yields the so-called non-abelian Toda field theory\cite{23}. At the classical level, the non-abelian toda theory based on the Lie algebra $B_2$ has a metric sector which reproduces Witten’s string black hole solution\cite{24} \cite{12}. This solution is only classically conformally invariant; the exact solution, conformally invariant to all orders, was constructed in Ref.\cite{11} using the methods we have used here. It would be interesting to apply these methods to the $B_2$ non-abelian Toda field theory to check whether the exact conformally invariant version of this theory agrees in the metric sector with the exact string black hole solution of Ref.\cite{11}. Moreover,
the central charge for a general non-abelian toda theory was calculated in Refs. [25], [3]. The methods we have used here should also provide an efficient means of reproducing these results.

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Appendix A. Lie Algebra conventions

We introduce a basis \( \{H_i, i = 1, \ldots, \text{rank} G \} \) for the Cartan subalgebra of \( G \), and also step operators \( \{E_\alpha, \alpha \in \Phi \} \) corresponding to the roots \( \alpha \). We then have

\[
[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha(H_i)E_\alpha
\] (A.1)

We define the Cartan-Killing form by

\[
(X, Y) = \text{tr}(\text{ad}X\text{ad}Y)
\] (A.2)

where \( X, Y \) are elements of the Lie algebra and \( \text{ad} \) is defined in Eq. (3.17). The Cartan-Killing form is then defined by

\[
\gamma_{ij} = (H_i, H_j), \quad \gamma_{\alpha\beta} = (E_\alpha, E_\beta)
\] (A.3)

We then define \( H_\alpha \) by

\[
(H_\alpha, H) = \alpha(H)
\] (A.4)

for all \( H \) in the Cartan sub-algebra. This enables one to define an inner product on the roots by

\[
<\alpha, \beta> = (H_\alpha, H_\beta)
\] (A.5)

and we also find

\[
[E_\alpha, E_{-\alpha}] = \gamma_{\alpha(-\alpha)}H_\alpha
\] (A.6)

The Cartan matrix \( A_{\alpha\beta} \) is then defined by

\[
A_{\alpha\beta} = 2\frac{<\alpha, \beta>}{<\beta, \beta>}
\] (A.7)
It can be shown that
\[ \rho = \sum_{\alpha, \beta \in \Delta} A^{\alpha \beta} H_\beta \] 
(A.8)

and
\[ \delta = 2 \sum_{\alpha, \beta \in \Delta} \frac{A^{\alpha \beta}}{<\alpha, \alpha>} H_\beta. \] 
(A.9)

We also make use of the Freudenthal-de Vries strange formula
\[ 24(\rho, \rho) = h <\psi, \psi> \dim G \] 
(A.10)

where \( \psi \) is the highest root and \( h \) is the dual Coxeter number.
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