Worldline Influence Functional: Abraham-Lorentz-Dirac-Langevin Equation from QED

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Abstract

We present a stochastic theory of charges moving in an electromagnetic field using nonequilibrium quantum field theory. We give a first principles’ derivation of the Abraham-Lorentz-Dirac-Langevin equation which depicts the quantum expectation value for a particle’s trajectory and its stochastic fluctuations by combining the worldline path integral quantization with the Feynman-Vernon influence functional or closed-time-path effective action methods [1, 2]. At lowest order, the equations of motion are approximated by a stochastic Lorentz-Dirac equation.

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1 Introduction

1.1 Particle versus field formulation, fixed versus dynamic background

The contrasting paradigms of particles versus fields give very different representations for physical processes. The success of quantum field theory for the past century has propelled the field concept to the forefront, with the particle interpreted as excitations of field degrees of freedom. In recent years, the use of the quantum-mechanical path integral in string theory has inspired a renewed effort towards particle-centric quantum formalisms. The so-called worldline quantization method – where the particle’s spacetime coordinate $x^\mu (\tau)$ is effectively quantized – has been used for calculating high-loop processes in non-dynamical classical background fields.

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The background field approximation is useful and well-defined in conditions where quantum fluctuations of the fields are small. However, many problems require the inclusion of field dynamics beyond the fixed-background approximation. A simple example is radiation reaction. More examples are found in semiclassical gravity, where spacetime playing the role of ‘particle’ is treated classically in its interaction with quantum fields.

The proper treatment of radiation-reaction requires, at the least, consistency between the particle’s (quantum) average trajectory and the average radiation. When one treats the particle as quantum mechanical, but coupled only to the mean-field background (and not the quantum field fluctuations) one is working within the semiclassical regime. (When quantum fluctuations of the particle motion are sufficiently small, one recovers the classical regime). In our program, we go beyond the semiclassical approximation to include the influence of quantum field fluctuations on the moving particle, thus entering the stochastic regime. To do this properly, we must begin with the full quantum theory for both particle and field, since a particle can never fully decouple from the field. Also since most field degrees of freedom remain unobserved, we treat the particle as a quantum open system with the field acting as an environment. This is how nonequilibrium and stochastic mechanics ideas enter, and why the self-consistent quantum particle evolution is generally described by stochastic quantum dynamics.

The open system concept and the accompanying method of coarse-grained effective action and influence functional have been applied to interacting quantum fields and semiclassical gravity. The new task we undertake is the adoption of the worldline quantization method for relativistic charged particles treated as an open system. This is an alternative formulation of QED in terms of quantized worldlines and quantum fields. The worldline structure is highly efficient for describing the particle-like degrees of freedom (such as spacetime position), while the field structure most easily describes processes like radiation.

1.2 Radiation reaction, causality, and the Abraham-Lorentz-Dirac equation

The conceptual and technical scheme we construct is advantageous for carrying out calculations which conventional methods find difficult, and for clarifying confusions in existing theories. Consider the classical problem of radiation reaction. A relativistic solution to classical radiation reaction is known as the Abraham-Lorentz-Dirac (ALD) equation, whose solutions give rise to famous paradoxes including: runaway solutions, pre-acceleration, and the need for higher-derivative initial data. These difficulties with the classical theory may be avoided by replacing point particles with extended objects; but a relativistic treatment is cumbersome. Moniz and Sharp have shown that non-relativistic QED for extended objects can be causal and runaway free. Low has shown that runaways apparently don’t occur in spin 1/2 QED. Others have shown how the lowest order-perturbation expansion of the ALD equation can be inferred within conventional relativistic-QED field theory. But most of these earlier works lack a comprehensive theoretical foundation for addressing all these issues of quantum field theoretic and nonequilibrium (in fact non-Markovian) nature, and for dealing with their interrelations consistently and completely. We show, for instance, that the equations of motion...
for the particle’s mean trajectory are causal and runaway-free regardless of fine structures like spin. We highlight the role of the field-environment in decohering the particle trajectory. Going beyond the result in [14] which is to order $e^4$ in the particle charge, we find the ALD equation to all orders in $e$ for the semiclassical limit. This shows the significant advantage of the worldline formulation together with the loop expansion in the influence action. The inclusion of spin and color are important to make full use of this approach in QED and QCD.

Beyond demonstrating the conceptual and technical consistency of the particle equations of motion, we describe the influence of quantum field fluctuations on the particle trajectory. Quantum field fluctuations play dual roles: they are responsible for the decoherence in the quantum particle (system), leading to the emergent semiclassical and classical behaviors, and they provide an effective classical stochastic noise in the particle motion [6]. We show that the one loop equations of motion for the particle trajectory are approximated by stochastic differential equations featuring colored noise that encodes the influence of the quantum statistics of the field. When there is sufficient decoherence, these equations give excellent approximations to the particle motion. Even in the case of weak decoherence they provide a one-loop approximation to the particle correlation functions, and may be extended self-consistently to higher-loop approximations via the N-particle-irreducible (NPI) effective action [15]. In the second part of this paper, we show the main steps in our derivation of the ALD-Langevin equations describing the stochastic dynamics of particle motion.

2 Coarse-grained effective action and equations of motion

2.1 The in-in generating functional

We begin with the construction of a generating functional for the particle worldline correlation functions. We assume an initially factorized density matrix for the particles plus field at initial time $t_i$ with the form

$$\hat{\rho}(t_i) = \hat{\rho}_{\text{particles}}(t_i) \otimes \hat{\rho}_{\text{field}}(t_i) \equiv \hat{\rho}_z(t_i) \otimes \hat{\rho}_\psi(t_i).$$

For simplicity, consider only one particle worldline and assume that the initial particle state is a positive frequency configuration space state $\hat{\rho}_z(t_i) = |z_i^{(+)}\rangle \langle z_i^{(+)}|$, where $|z_i^{(+)}\rangle \equiv |z_i, t_i; +\rangle = \hat{\psi}^{(+)}(z_i) |\text{vac}\rangle_{\text{particle}}$. These are quasi-localized, physical, relativistic one-particle states; we treat more general initial states in [7, 10]. A basis of field states is $|A\rangle$. A basis of field states is $|A(x)\rangle$; we define a direct product basis for the particles plus field by $|zA\rangle = |z\rangle \otimes |A\rangle$.

To find the generating functional for the correlation functions defined as

$$\langle \hat{z}^\mu(\tau_1) ... \hat{z}^\nu(\tau_n) \rangle = \text{Tr}_{A_z} (\hat{z}^\mu(\tau_1) ... \hat{z}^\nu(\tau_n) \hat{\rho}_{in}),$$

where $\hat{z}^\mu(\tau)$ are Heisenberg operators and the trace is over all final particle and field states, we need the time evolution of the density matrix:

$$\hat{\rho}(t_f) = \hat{U}(t_f, t_i) \hat{\rho}(t_i) \hat{U}(t_f, t_i)^\dagger.$$
In field theory, one could introduce a functional representation for the unitary evolution operators $\hat{U}$ in terms of the sum over all field-histories between $t_i$ and $t_f$, consistent with the initial and final state boundary conditions.

Instead, in a hybrid particle-field theory we start with the action

$$S[z,A] = S_z[z] + S_A[A] + S_{int}[z,A]$$

$$= \int d\tau \left( \dot{z}^\mu \dot{z}_\mu / N - N \dot{\tau}^2 + h_\mu(\tau) \dot{z}_\mu(\tau) \right) - \int d^4x F^2 / 4$$

$$- \int d^4x j^\mu(x;z) \left( A_\mu(x) + \bar{A}_\mu(x) \right).$$

The quantum degrees of freedom are the field $A_\mu(x)$, the particle coordinates $z^\mu(\tau)$, and the 'lapse function' $N(\tau)$. The action $S[z,A]$ is invariant under reparametrizations $\tau \rightarrow \tau' = \tau + \varepsilon(\tau)$; this is a crucial symmetry of the relativistic worldline formulation. These issue are discussed in detail in [7]. The $F^2 / 4$ term is the usual free electromagnetic field action. We allow for the possibility of an additional, non-dynamical background field $\bar{A}$. The particle couples to the field through the current $j^\mu(x;z)$, which is given by

$$j^\mu(x;z) = e \int d\tau \dot{z}^\mu(\tau) \delta(x - z(\tau)).$$

In [3] we add a source term $h_\mu(\tau)$ whose role will be made clear below.

The in-in or closed-time-path (CTP) generating functional is given by [10]

$$Z_{\text{in-in}}[h,h'] = \text{Tr}_{A \bar{A}} \left( \hat{U}_h(t_f,t_i) \hat{\rho}(t_f,t_i) \hat{U}_{h'}^\dagger(t_f,t_i) \right)$$

$$= \int dz_f dz'_f dA_f dA'_f d\rho \delta(z_f - z'_f) \left( \partial'_{t_f} - \partial_{t_f} \right)$$

$$\times \langle A_f z_f | A_i z_i^{(+)} \rangle_h \langle A_f z'_f | A'_i z'_i^{(+)} \rangle_{h'} \rho(A_i,A'_i,t_i) \big|_{t_f = t_{f'}}.$$ (6)

We have implicitly assumed in writing (5) that the sole particle worldline begins at $z_i$ on the initial time hypersurface $\Sigma(t_i)$ and ends somewhere on the final time hypersurface $\Sigma(t_f)$. We then integrate over all possible $z_f$ on $\Sigma(t_f)$. A more general form, with more general boundary conditions, is given in [3, 10]. For example, a worldline that begins and ends on $\Sigma(t_i)$ ($\Sigma(t_f)$) describes pair-annihilation (creation).

We may now use the worldline formalism to give a path integral representation for the propagators $\langle A_f z_f | A_i z_i^{(+)} \rangle_h$ and $\langle A'_i z'_i^{(+)} | A_f z'_f \rangle_{h'} = \langle A_f z'_f | A'_i z'_i^{(+)} \rangle_{h'}$. They are given by

$$\langle A_f z_f | A_i z_i^{(+)} \rangle_h = \int_{z_i A_i}^{z_f A_f} Dz DA \exp \left\{ \frac{i}{\hbar} S[z,A,h] \right\},$$

and similarly for $\langle A'_i z'_i^{(+)} | A_f z'_f \rangle_{h'}$. The measure is defined by

$$\int_{z_i A_i}^{z_f A_f} Dz DA = \prod_i \prod_{\mu,j} \int_{t_i}^{t_f} dz_i^0 \int_{-\infty}^{\infty} dz_i \int dA_\mu(x_j).$$ (8)
The generating functional then takes the form

\[
Z_{\text{in-in}}[h, h'] = \int dz_f \int_{z_i}^{z_f} Dz Dz' \left( \partial_{tf} - \partial_{t_f} \right) e^{i(S_z[z,h]-S_z[z',h'])} \bigg|_{t_f=t_f'}^{}
\]

\[
\times \int_{A_i, A_i'}^{} dA_i dA_i' dA_i DA DA' \rho(A_i, A_i'; t_i)
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \left(S_A[A] + S_{\text{int}}[z, A] - S_A[A'] + S_{\text{int}}[z', A'] \right) \right\}. \tag{9}
\]

The sum over histories in (7) include all worldlines that begin at \(z_i\), end at \(z_f\), and are bounded between the initial and final times, \(t_i\) and \(t_f\), respectively, including those that are spacelike and that change direction in time \(7\).

2.2 Feynman-Vernon influence functional

Evaluating the generating functional for our action (4) is virtually impossible. The interaction term \(S_{\text{int}}\), while linear in \(A\), is highly non-linear in the worldline coordinate \(z\). However, if the initial state of the field \(\hat{\rho}_z(A_i, A_i'; t_i)\) is Gaussian, the entire integrand in the second line of (9) is Gaussian in the variables \(A_i\). Thus, we can then do the \(A_i, A_i'\) path integrals exactly. Initial Gaussian field states include thermal, squeezed, and coherent states, and therefore provides a fairly rich set of interesting and physical examples. The result of the \(A_i, A_i'\) functional integrals is the Feynman-Vernon influence functional \(1\):

\[
F[z, z'] = \int_{A_i, A_i'}^{} dA_i dA_i' DA DA' \rho(A_i, A_i'; t_i)
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \left(S_A[A] + S_{\text{int}}[z, A] - S_A[A'] + S_{\text{int}}[z', A'] \right) \right\}. \tag{10}
\]

Notice that initial factorizability (see (1)) makes this result independent of the initial particle state \(\hat{\rho}_z(t_i)\). The generating functional is now

\[
Z_{\text{in-in}}[h, h'] = \int dz_f \int_{z_i}^{z_f} Dz Dz' \left( \partial_{tf} - \partial_{t_f} \right) \left. e^{i(S_z[z,h]-S_z[z',h'])} \right|_{t_f=t_f'}^{}
\]

\[
\times \int_{A_i, A_i'}^{} dA_i dA_i' DA DA' \rho(A_i, A_i'; t_i)
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \left(S_z[z, h] - S_z[z', h'] + S_{IF}[z, z'] \right) \right\}, \tag{11}
\]

where the influence action \(S_{IF}\) is defined via \(F[z, z'] = \exp \left( \frac{i}{\hbar} S_{IF}[z, z'] \right)\). For the Gaussian case, the influence action is known; it is

\[
S_{IF} = \frac{1}{\hbar} \int \int dxdx' g^{\mu\nu} j^\pm_\mu (x) G_R(x, x') j^\pm_\nu (x') \tag{12}
\]

\[
+ \frac{i}{\hbar} \int \int dxdx' g^{\mu\nu} j^\pm_\mu (x) G_H(x, x') j^\pm_\nu (x').
\]

5
The $j^\pm$ are defined by

$$j^\pm_\mu (x; z, z') = \int d\tau \left( \dot{z}_\mu \delta (x - z) \pm \dot{z'}_\mu \delta (x - z') \right), \quad (13)$$

$G_R (x, x')$ is the retarded Green’s function

$$G_R (x, x') = \text{Tr}_A \left( \left[ \hat{A}_\mu (x), \hat{A}_\nu (x') \right] \hat{\rho}_A (t_i) \right) \theta (t, t'), \quad (14)$$

and $G_H (x, x')$ is the Hadamard function

$$G_H (x, x') = \text{Tr}_A \left( \left\{ \hat{A}_\mu (x), \hat{A}_\nu (x') \right\} \hat{\rho}_A (t_i) \right). \quad (15)$$

The coarse-grained effective action is defined by [2]

$$S_{CGEA} [z^\pm, h^\pm] = S_z [z, h] - S_z [z', h'] + S_{IF} [z^\pm], \quad (16)$$

where $h^\pm_\mu (\tau) = (h_\mu (\tau) \pm h'_\mu (\tau))$.

### 2.3 Decoherence and the (modified) loop expansion

To evaluate the non-linear $z, z'$ path integrals, we need to use an approximation method. In perturbative scattering theory, one expands $e^{iS_{IF}/\hbar} = 1 + iS_{IF}/\hbar + \mathcal{O} (\epsilon^4)$. (Lowest order interactions are $\mathcal{O} (\epsilon^2)$ because we are working in a in-in formalism.) But, to evaluate quantum corrections to the semiclassical result, the loop expansion (or even better, the NPI effective action [13]) is the most suitable. Observe that

$$|F [z^\pm]| = |e^{iS_{CGEA}/\hbar}| = e^{-\text{Im} S_{IF} [z^\pm]} \quad (17)$$

It is not difficult to show that $\text{Im} (S_{IF} [z^\pm]) \geq 0$ for all $z^\pm$. For large $z^- = z - z'$, it is often the case that $|F| \ll 1$. This suppression of the generating functional for large $z^-$ is a manifestation of decoherence; it occurs due to the large number of degrees of freedom $A (x)$ that are integrated out (traced over) in finding the influence action. We now assume that decoherence effectively suppresses large $z^-$ (to what extent this is true depends on the details of the system, environment, and their mutual coupling) and expand $S_{CGEA}$ in powers of $z^-$. We define $S^R_{CGEA} \equiv \text{Re} (S_{CGEA})$ and $S^I_{CGEA} \equiv \text{Im} (S_{CGEA})$. Dropping the $\mathcal{O} (z^-)^3$ terms gives a Gaussian approximation to the $z^-$ path integrals. Then the $Dz^-$ integration may be performed giving

$$Z_{\text{in-in}} [h, h'] = \int d\mathbf{z}_f \int_{{z^+_f = z_i}}^{z^+_f = z_f} Dz^+ \int_{{z^-_f = 0}}^{z^-_f = 0} Dz^- \left( \partial_{t_f} - \partial_{t_f} \right) e^{i \int d\tau h^-_\mu z^+ (\tau)}$$

$$\times e^{i \int \left( \frac{S^R_{CGEA}}{\hbar} \right) z^-_\mu d\tau} e^{- \int d\tau d\tau' g^{\mu \nu} z^-_\mu (\tau) G_H (z^+ (\tau), z^+ (\tau')) z^-_\nu (\tau')}$$

$$= \int d\mathbf{z}_f \left( \partial_{t_f} - \partial_{t_f} \right) \int_{{z^+_f = z_i}}^{z^+_f = z_f} Dz^+ \times \exp \left\{ \frac{i}{\hbar} \int d\tau h^-_\mu z^+ (\tau) - \int d\tau d\tau' \eta_\mu (\tau) G_H (z^+, z^+') \eta^\nu (\tau') \right\},$$
where we have defined the variable $\eta_\mu (\tau)$ by
\[
\eta_\mu (\tau) = \frac{\delta S_{CGEA}^R [z^+, z^+]}{\delta Z^-_\mu} \bigg|_{z^- = 0}.
\] (19)

We now have only the $z^+$ integration left. Note that (19) is an equation for $z^+$ as a functional of $\eta (\tau)$. We use this fact to change integration variables from $z^+ \to \eta$, yielding $Dz^+ = \text{(Jacobian)} \times D\eta = D\eta$, since the Jacobian equals 1. This gives us our final result, the generating function
\[
Z_{\text{in-in}} [h^-_\mu] = \int D\eta P[\eta] \exp \left\{ \frac{i}{\hbar} \int d\tau h^-_\mu (\tau) \quad z^{\mu+} (\tau) \right\}
\] (20)

with
\[
P[\eta] = \exp \left\{ - \int d\tau d\tau' \eta_\mu (\tau) \quad G_H (z^+, z^{\prime+}) \quad \eta^\mu (\tau') \right\}. \quad (21)
\]

$P[\eta]$ may now be treated as a probability distribution for $\eta (\tau)$, thus rendering $\eta_\mu (\tau)$ a stochastic noise variable. (19) is therefore a Langevin equation for $z^{\mu+} (\tau)$ with noise $\eta_\mu (\tau)$. Derivatives $\delta^n Z_{\text{in-in}} [h^-_\mu] / \delta h^-_\mu (\tau_1) \ldots \delta h^-_\mu (\tau_n)$ give the correlation functions (2). Because $z^+$ is a nonlinear function of $\eta$, as determined by (19), and $G_H (z^+, z^{\prime+})$ is a function of $z^+$, the distribution $P[\eta]$ is not Gaussian. In (10), we show how to perform a cumulant expansion on the Langevin equation. The order $e^2$ term then generates Gaussian noise, but more generally, one recognizes that even after decoherence has suppressed large quantum fluctuations in $z^-$, the effective noise will be non-Gaussian for nonlinear Langevin equations. Only when the Langevin equation is linear is the noise strictly Gaussian; at high temperatures it is approximately white.

3 The ALD-Langevin equation

To apply this general result to charged particles in the quantum electromagnetic field requires regularizing $S_{CGEA}$ by introducing a high-frequency cutoff $\Lambda$ in the Green’s functions $G_{R,H}^\Lambda$. We perform this analysis in (10). The resulting Langevin equations of motion found from (19) are
\[
m(r) \ddot{z}_\mu (\tau) = e \dot{z}_\mu F_{\mu\nu}^{\text{ext}} + e^2 \sum_{n=2}^{\infty} g^{(n)}(r) u^{(n)}_\mu + \eta_\mu (\tau).
\] (22)

$F_{\mu\nu}^{\text{ext}}$ is included as a possible classical, external background field. The variable $r = \tau - \tau_i$ is the elapsed proper-time from the initial time $\tau_i$.

Quantum field fluctuations dress the particle state giving it an effective mass $m(r)$. At $r = 0$ (corresponding to the initial time when the field and particle are uncorrelated) $m(0) = m_0$, the bare mass of the particle. The mass is then renormalized on the cutoff timescale $1/\Lambda$, quickly reaching its late-time value $m(\infty) = m_0 + a_0 \Lambda$. The constant $a_0$ depends on the details of the
cutoff since the high-energy theory ultimately determines the precise relationship between bare and effective mass.

Radiation reaction is given by the second term on the RHS of (22). The
\[ u^{(n)}_{\mu} = u^{(n)}_{\mu}(\dot{z}, \ddot{z}, ..., \frac{d^{n+1}z}{d\tau^n}) \]
(23)
are combinations of higher time-derivatives of \( z(\tau) \). The time dependent coefficients \( g^{(n)}(r) \) determined by the regulated retarded Greens function \( G^R_\Lambda \) scale as \( \Lambda^{2-n} \) with the cutoff. Therefore, the \( n > 2 \) terms are high-energy corrections to the radiation reaction force and as such they are strongly suppressed at low-energies. The late-time behavior of the \( n = 2 \) term gives
\[ f^{R,R, (2)}_{\mu} = \frac{e^2}{2\pi} (\dot{z}_\mu \ddot{z}^2 + \dot{z}_\mu \ddot{z}) , \]
(24)
which is just the usual ALD results. In \[10\], the expansion (22) is shown to be convergent for all \( r \), including \( r \to \infty \) and \( r \to 0 \). The short-time behavior is particularly significant: \( \lim_{r \to 0} g^{(n)}(r) = 0 \) for all \( n \); therefore, the radiation reaction force identically vanishes at \( t_i \), and the usual Newtonian initial data (i.e. position and velocity) uniquely determine the equations of motion. In \[10\], it is further shown that the equations of motion are strictly causal and without runaways. The non-Markovian time-dependence of the radiation reaction force reflects the fact that in any nonequilibrium quantum setting it takes time for the particle’s self-field to adjust to changes in its motion; it can not respond faster than the effective high-energy cutoff \( \Lambda \). This lag is always enough to preserve causality. To the extent that QED is a good effective field theory, these results demonstrate the consistency of the low-energy physics, including the ALD equation as the proper low-energy limit for radiation reaction.

Finally, the noise is given by
\[ \langle \{\eta_{\mu}(\tau), \eta_{\nu}(\tau')\} \rangle_s = g_{\mu\nu} G^A_H(\tau, \tau') , \]
(25)
where \( \langle \rangle_s \) is the noise average with respect to the distribution \( P[\eta] \). Consistency with the fluctuation-dissipation relation derived in \[10\] requires mutually consistent regularization of both \( G_R \) and \( G_H \). Therefore, the noise will also depend on the cutoff \( \Lambda \). In fact, the noise kernel is modified at early times by the restriction that the path integrals over \( z^\pm \) are bounded between the surfaces \( \Sigma(t_i) \) and \( \Sigma(t_f) \). This modification of the noise is balanced by the early time radiation reaction behavior. At late times, the restriction on the paths has little effect on the integrals and the noise quickly approaches the form (25) just as the radiation reaction force approaches its late-time limit.

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