ON THE ONE-DIMENSIONAL CONTINUITY EQUATION WITH A NEARLY INCOMPRESSIBLE VECTOR FIELD

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Abstract. We consider the Cauchy problem for the continuity equation with a bounded nearly incompressible vector field \( b : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \ T > 0 \). This class of vector fields arises in the context of hyperbolic conservation laws (in particular, the Keyfitz–Kranzer system, which has applications in nonlinear elasticity theory).

It is well known that in the generic multi-dimensional case (\( d \geq 1 \)) near incompressibility is sufficient for existence of bounded weak solutions, but uniqueness may fail (even when the vector field is divergence-free), and hence further assumptions on the regularity of \( b \) (e.g. Sobolev regularity) are needed in order to obtain uniqueness.

We prove that in the one-dimensional case (\( d = 1 \)) near incompressibility is sufficient for existence and uniqueness of locally integrable weak solutions. We also study compactness properties of the associated Lagrangian flows.

1. Introduction. Let \( b \in L^\infty(I \times \mathbb{R}^d; \mathbb{R}^d) \) denote a time-dependent vector field on \( \mathbb{R}^d \), where \( I = (0, T), \ T > 0, \ d \in \mathbb{N} \). Consider the Cauchy problem for the continuity equation

\[
\begin{aligned}
\frac{\partial}{\partial t} u + \text{div}_x(ub) &= 0 \quad \text{in } I \times \mathbb{R}^d, \\
u|_{t=0} &= \bar{u} \quad \text{in } \mathbb{R}^d,
\end{aligned}
\]

where \( \bar{u} \in L^1_{\text{loc}}(\mathbb{R}^d) \) is the initial condition and \( u : I \times \mathbb{R}^d \rightarrow \mathbb{R} \) is the unknown. A function \( u \in L^1_{\text{loc}}(I \times \mathbb{R}^d) \) is called weak solution of (1) if it satisfies (1) in sense of distributions:

\[
\int \int u(t,x)(\partial_t \varphi(t,x) + b(t,x)\partial_x \varphi(t,x)) \, dx \, dt + \int \bar{u}(x)\varphi(0,x) \, dx = 0
\]

for any \( \varphi \in C^1_c([0,T) \times \mathbb{R}^d) \).

Existence and uniqueness of weak solution of (1) are well-known when the vector field \( b \) is Lipschitz continuous. However in connection with many problems in mathematical physics one has to study (1) when \( b \) is non-Lipschitz (in general).

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In particular, vector fields with Sobolev regularity arise in connection with fluid mechanics [15], and vector fields with bounded variation arise in connection with nonlinear hyperbolic conservation laws [3]. Therefore one would like to find the weakest assumptions on $b$ under which weak solution of (1) exists and is unique.

For a generic bounded vector field $b$ concentrations may occur and therefore the Cauchy problem (1) can have no bounded weak solutions. However under mild additional assumptions on $b$ existence of bounded weak solutions can be proved. Namely, the following class of vector fields was introduced in [10, 14]:

**Definition 1.1.** A vector field $b \in L^{\infty}(I \times \mathbb{R}^d; \mathbb{R}^d)$ is called nearly incompressible with density $\rho$: $I \times \mathbb{R}^d \to \mathbb{R}$ if there exists $C > 0$ such that $1/C \leq \rho(t,x) \leq C$ for a.e. $(t,x) \in I \times \mathbb{R}^d$ and $\rho$ solves $\partial_t \rho + \text{div}_x (\rho b) = 0$ (in sense of distributions).

This class of vector fields was considered [14] in connection with the model of propagation of forward (and backward) longitudinal and transverse waves in a stretched elastic string which moves in a plane [16]. For convenience of the reader let us formulate a 1-dimensional analog of this model, following [16].

Let $x$ be arc-length along the string measured from some origin when it is in a reference configuration of uniform tension $T_0 = 1$ and density $\rho_0 = 1$ (for simplicity). Let $w = w(t,x)$ denote the displacement at time $t$ of the point which was at distance $x$ from the origin in the reference state. Let $\varepsilon := |\partial_x w| - 1$ denote the strain and let $r := 1 + \varepsilon$. Assuming a stress-strain relation $T = T(\varepsilon)$, the equations of motion of the string can be written [16] in the form

$$\partial_t^2 w = \partial_x \left( \frac{T}{1 + \varepsilon} \partial_x w \right).$$

If we look for a forward wave solution of this equation, which has the form $w(t,x) = f(x - t)$, then $\partial_t w = -\partial_x w$. Hence $u := \partial_x w$ solves

$$\partial_t u + \partial_x (bu) = 0,$$

where $b := \frac{T}{1 + \varepsilon}$ is the “vector field”. Solutions with $r = 0$ are physically inadmissible [16], hence a reasonable assumption is that there exists a constant $c > 0$ such that $r \geq c$. On the other hand formally $r$ solves

$$\partial_t r + \partial_x (br) = 0$$

and it is realistic to assume that $r$ is bounded from above. Thus $b$ is nearly incompressible.

Apart from the model of forward and backward waves in elastic strings Problem (1) with a nearly incompressible vector field also arises as a model of transport of some passive tracer (e.g. salinity) embedded into compressible fluid under a physically natural assumption that the density of the fluid is bounded from below and above.

It is well-known that near incompressibility is sufficient for existence of bounded weak solutions of (1). However in the generic multidimensional case ($d \geq 2$) it is not sufficient for uniqueness. For example, there exists a bounded divergence-free autonomous vector field on the plane ($d = 2$), for which (1) has a nontrivial bounded weak solution with zero initial data [1].

Uniqueness of weak solutions has been established for some classes of weakly differentiable vector fields [15, 3]. Recently new uniqueness results were obtained for continuous vector fields [13, 8] (without explicit assumptions on weak differentiability). Note that in general a nearly incompressible vector field does not have
to be continuous (and vice versa). Uniqueness of locally integrable weak solutions has been studied in [11] for Sobolev vector fields under additional assumption of continuity (see also [18]).

Uniqueness of bounded weak solutions for nearly incompressible vector fields in the two-dimensional case \((d = 2)\) was also studied in [6]. In particular it was proved that uniqueness holds when \(b \neq 0\) a.e., or when \(b \in BV\).

Our main result is the following:

**Theorem 1.2.** Suppose that \(b \in L^\infty(I \times \mathbb{R}; \mathbb{R})\) is nearly incompressible. Then for any initial condition \(\bar{u} \in L^1_{loc}(\mathbb{R})\) the Cauchy problem (1) has a unique weak solution \(u \in L^1_{loc}(I \times \mathbb{R})\).

Existence of bounded weak solutions of (1) with bounded \(\bar{u}\) for nearly incompressible vector fields is well-known (see e.g. [12] for the case of vector fields with bounded divergence).

When \(b\) satisfies a one-sided Lipschitz condition, i.e. there exists \(\alpha \in L^1(0,T)\) such that \(\partial_x b(t,x) \leq \alpha(t)\) (in \(\mathcal{D}'\)), uniqueness was established in [9] in a much wider class of measure-valued solutions. However the one-sided Lipschitz condition is different from the near incompressibility assumption. For instance, denoting with \(1_A\) the indicator of a set \(A\), the vector field \(b = 1_{(-\infty,0)} + 1_{(0,1)} + 1_{(1,\infty)}\) is nearly incompressible with density \(\rho = 2 \cdot 1_{(-\infty,0)} + 1_{(0,1)} + 2 \cdot 1_{(1,\infty)}\), but does not satisfy the one-sided Lipschitz condition. And vice versa, \(b = 1_{(-\infty,0)}\) is not nearly incompressible, but satisfies the one-sided Lipschitz condition.

The case when \(b\) is piecewise continuous is also studied in [9], where a precise description of the solutions along the lines of discontinuity is given (in terms of the corresponding transport equation). However in general nearly incompressible vector fields are not piecewise continuous (for instance, \(b = 1 + 1_E\), where \(E\) is the Smith–Volterra–Cantor set, admits density \(\rho = 1/b\)).

Ultimately, when \(b\) has a non-negative density \(\rho \in L^\infty(I \times \mathbb{R})\), by Lemma 2.4.1 from [9] the problem (1) has at most one weak solution \(u\) satisfying \(|u| \leq \rho\). In particular, this implies uniqueness of bounded weak solutions when \(b\) is nearly incompressible. The novelty of Theorem 1.2 is that it applies to merely locally integrable weak solutions, and that the lower bound for \(\rho\) in the near incompressibility assumption can be relaxed to the a.e.-positivity assumption. Note that in the multidimensional setting uniqueness may fail for locally integrable weak solutions, even when \(b\) is divergence-free and belongs to some Sobolev space; we refer to [17] and [18] for the details.

2. Uniqueness of locally integrable weak solutions.

**Definition 2.1.** A non-negative function \(\rho \in L^1_{loc}(I \times \mathbb{R}^d; \mathbb{R})\) is called a density associated with a vector field \(b \in L^1_{loc}(I \times \mathbb{R}^d; \mathbb{R}^d)\) if \(\rho b \in L^1_{loc}(I \times \mathbb{R}^d; \mathbb{R}^d)\) and \(\partial_t \rho + \text{div}(\rho b) = 0\) in \(\mathcal{D}'(I \times \mathbb{R}^d)\).

**Remark 1.** If a vector field \(b \in L^\infty(I \times \mathbb{R}^d; \mathbb{R}^d)\) admits a density \(\rho\) and there exist strictly positive constants \(C_1,C_2\) such that \(C_1 \leq \rho(t,x) \leq C_2\) for a.e. \((t,x) \in I \times \mathbb{R}^d\) then \(b\) is nearly incompressible.

Suppose that a vector field \(b \in L^1_{loc}(I \times \mathbb{R}; \mathbb{R})\) admits a density \(\rho\). By Fubini’s theorem for a.e. \(x \in \mathbb{R}\) (in particular, for some \(\bar{x} \in \mathbb{R}\)) we have \(\rho(\cdot,\bar{x}),b(\cdot,\bar{x}) \in L^1_{loc}(I)\) and without loss of generality we may assume that \(\rho(t,\cdot) \in L^1_{loc}(\mathbb{R})\) for all
\( t \in I \) (otherwise one can redefine \( \rho \) e.g. with 1 on the corresponding negligible set). Then we define
\[
H(t, x) := \int_x^\infty \rho(t, s) \, ds - \int_0^t \rho(\tau, x) b(\tau, x) \, d\tau, \quad (t, x) \in I \times \mathbb{R}. \tag{3}
\]
It is easy to see that \( H \in W^{1,1}_{\text{loc}}((0, T) \times \mathbb{R}) \) and
\[
\partial_x H = \rho \quad \text{and} \quad \partial_t H = -\rho b. \tag{4}
\]
in \( \mathcal{C}'(I \times \mathbb{R}^d) \). Clearly the function \( H \) is uniquely defined a.e., up to an additive constant. Moreover, if \( \rho, b \in L^\infty(I \times \mathbb{R}) \) then \( H \in W^{1,\infty}(I \times \mathbb{R}) \), hence \( H \) can be redefined in such a way that it is Lipschitz continuous, i.e. \( H \in \text{Lip}([0, T] \times \mathbb{R}) \).

**Definition 2.2.** The function \( H \) defined above is called the **Hamiltonian associated with** \((\rho, b)\).

**Theorem 2.3.** Suppose that a vector field \( b \in L^\infty(I \times \mathbb{R}; \mathbb{R}) \) admits a density \( \rho \in L^\infty_{\text{loc}}(I \times \mathbb{R}; \mathbb{R}) \) such that \( \rho(t, x) > 0 \) for a.e. \((t, x) \in I \times \mathbb{R}\). If \( u \in L^1_{\text{loc}}(I \times \mathbb{R}; \mathbb{R}) \) is a weak solution of \((1)\) with \( \bar{u} \equiv 0 \) then \( u(t, x) = 0 \) for a.e. \((t, x) \in I \times \mathbb{R}\).

**Proof.** Step 1. Let \( H \) be the Hamiltonian associated with \((\rho, b)\). We would like to use test functions of the form \( \varphi(t, x) := f(H(t, x)) \) in the distributional formulation of \((1)\), where \( f \in C^\infty_\text{c}(\mathbb{R}) \). In general such functions could be not compactly supported, therefore we apply an approximation argument.

For any \((t, x) \in (-T, 0) \times \mathbb{R}\) let \( H(t, x) := H(t, x) \). Clearly \( \partial_x H = \tilde{\rho} \) and \( \partial_t H = \tilde{b} \rho \) \( \in \mathcal{C}'((-T, T) \times \mathbb{R}) \), where
\[
\tilde{\rho}(t, x) := \begin{cases} \rho(t, x), & t > 0, \\ \rho(-t, x), & t < 0; \end{cases} \quad \tilde{b}(t, x) := \begin{cases} b(t, x), & t > 0, \\ -b(-t, x), & t < 0. \end{cases}
\]

Let \( \varepsilon > 0 \) and let \( \omega_\varepsilon(z) := \varepsilon^{-2}\omega(\varepsilon^{-2}z) \), where \( \omega \in C^\infty_\text{c}(\mathbb{R}^2) \) is the standard mollification kernel. Let \( H_\varepsilon := H * \omega_\varepsilon \), where * denotes the convolution. Clearly
\[
\partial_x H_\varepsilon = \tilde{\rho}_\varepsilon \quad \text{and} \quad \partial_t H_\varepsilon = -\tilde{b}_\varepsilon \rho_\varepsilon. \tag{5}
\]

By Fubini’s theorem, since \( H_\varepsilon \rightarrow H \) a.e. as \( \varepsilon \to 0 \) and \( H(t, \cdot) \) is continuous (for any fixed \( t \in I \)), there exists a negligible set \( N_H \subset I \) such that for any \( t \in I \setminus N_H \) and any \( x \in \mathbb{R} \) we have \( H_\varepsilon(t, x) \rightarrow H(t, x) \) as \( \varepsilon \to 0 \).

Step 2. Let \( h \in \mathbb{R} \) be such that the level set \( L_{e, h} := \{(t, x) \in (-T+\varepsilon, T-\varepsilon) \times \mathbb{R} : H_\varepsilon(t, x) = h\} \) is not empty.

Suppose that \( \tau, \xi \in \mathbb{R} \) and \( \tau^2 + \xi^2 = 1 \). If \( |\xi| > ||b||_\infty |\tau| \) then the derivative of \( H_\varepsilon \) in the direction \( \nu := (\tau, \xi) \) satisfies \( \partial_\nu H_\varepsilon = \tau \partial_\tau H_\varepsilon + \xi \partial_\xi H_\varepsilon = -\nu \rho_\varepsilon \), hence
\[
\begin{cases}
\partial_\nu H_\varepsilon \geq (\xi - |\tau||b||_\infty)\rho_\varepsilon > 0, & \xi > ||b||_\infty |\tau| \\
\partial_\nu H_\varepsilon \leq (\xi + |\tau||b||_\infty)\rho_\varepsilon < 0, & \xi < -||b||_\infty |\tau|
\end{cases} \tag{6}
\]
therefore the level set \( L_{e, h} \) is contained in the cone
\[
L_{e, h} \subset \{(t', x') : |x' - x| \leq ||b||_\infty ||t' - t||\} \tag{7}
\]
for any \((t, x) \in L_{e, h}\).

Since \( \rho_\varepsilon > 0 \) for any \( t \in (-T+\varepsilon, T-\varepsilon) \) the function \( H_\varepsilon(t, \cdot) \) is strictly increasing. Hence \( H_\varepsilon(t, \mathbb{R}) \) is a nonempty open interval and for any \( h \in H_\varepsilon(t, \mathbb{R}) \) there exists a unique \( x = Y_\varepsilon(t, h) \) such that \( H_\varepsilon(t, x) = h \). Moreover, by continuity of \( H_\varepsilon(t, \cdot) \) and \((6)\) the image \( \mathcal{R}_\varepsilon := H(t, \mathbb{R}) \) is independent of \( t \in (-T+\varepsilon, T-\varepsilon) \).
Let us fix \( \varepsilon \in (0, T - \tau) \). By (4) the function \( x \mapsto H(t,x) \) is strictly increasing and continuous. Hence the image \( I_\tau := H(\tau, \mathbb{R}) \) is a nonempty open interval.

Consider \( f \in C^\infty_c(I_\tau) \) and let \( \varphi_\varepsilon(t, x) := f(H_\varepsilon(t, x)) \). We claim that there exists \( \varepsilon_1 > 0 \) and a compact \( K \subset [0, \tau] \times \mathbb{R} \) such that

\[ \text{supp } \varphi_\varepsilon \subset K \]

for any \( \varepsilon \in (0, \varepsilon_1) \).

Indeed, the support of \( f \) is contained in some finite interval \( (\alpha, \beta) \) such that \( [\alpha, \beta] \subset I_\tau \). Let us fix \( \alpha_1 \in I_\tau \setminus [\alpha, +\infty) \) and \( \beta_1 \in I_\tau \setminus (-\infty, \beta] \). By definition of \( I_\tau \) there exist \( x_1 \) and \( y_1 \) such that \( H(t, x_1) = \alpha_1 \) and \( H(t, y_1) = \beta_1 \). Since \( H_\varepsilon(t, x_1) \to H(t, x_1) \) and \( H_\varepsilon(t, y_1) \to H(t, y_1) \) as \( \varepsilon \to 0 \) we can find \( \varepsilon_0 > 0 \) such that \( R_\varepsilon \supset (\alpha, \beta) \) for any \( \varepsilon \in (0, \varepsilon_0) \).

Since \( x \mapsto H_\varepsilon(t, x) \) is strictly monotone and continuous, there exist unique \( x_0 \) and \( y_0 \) such that \( H(x_0, \tau) = \alpha \) and \( H(y_0, \tau) = \beta \). Since the support of \( f \) is a compact subset of \( (\alpha, \beta) \) and \( H_\varepsilon(x_0, \tau) \to \alpha \) and \( H_\varepsilon(y_0, \tau) \to \beta \) as \( \varepsilon \to 0 \), there exists \( \varepsilon_0 > 0 \) such that

\[ \text{supp } f \subset (H_\varepsilon(x_0, \tau), H_\varepsilon(y_0, \tau)) \]

whenever \( \varepsilon \in (0, \varepsilon_0) \). Hence the support of \( \varphi_\varepsilon \) (restricted to \( [0, \tau] \times \mathbb{R} \)) is confined by the level sets of \( H_\varepsilon \), passing through \( x_0 \) and \( y_0 \):

\[ \text{supp } \varphi_\varepsilon \subset \{(t, x) \mid t \in [0, \tau], \ x \in \left[ Y_\varepsilon(t, H_\varepsilon(\tau, x_0)), Y_\varepsilon(t, H_\varepsilon(\tau, y_0)) \right] \} \subset K, \]

where

\[ K := \{(t, x) \mid t \in [0, \tau], \ x \in [x_0 - ||b||_\infty(\tau - t), y_0 + ||b||_\infty(\tau - t)] \}. \]

Step 4. Now we are in a position to use \( \varphi_\varepsilon \) as a test function in (8). First we observe that

\[ \partial_t \varphi_\varepsilon + b \partial_x \varphi_\varepsilon = f'(H_\varepsilon(t, x)) (\partial_t H_\varepsilon + b \partial_x H_\varepsilon) = f'(H_\varepsilon(t, x)) (-(\tilde{p}(t) + \tilde{b}(t))) = 0 \]

a.e. on \( (0, \tau) \times \mathbb{R} \) as \( \varepsilon \to 0 \). Since \( \tilde{u} \equiv 0 \), by (8) and Lebesgue’s dominated convergence theorem

\[ \int_\mathbb{R} u(\tau, x) \varphi_\varepsilon(\tau, x) \, dx = \int_\mathbb{R} \int_K u \cdot (\partial_t \varphi_\varepsilon + b \partial_x \varphi_\varepsilon) \, dx \, dt \to 0 \]

as \( \varepsilon \to 0 \). (Indeed, \( |u \cdot (\partial_t \varphi_\varepsilon + b \partial_x \varphi_\varepsilon)| \leq \|f\|_{C^1} \|\rho\|_{L^\infty(K)} (1 + ||b||_\infty)|u| \in L^1(K) \).

Since \( H_\varepsilon(\tau, \cdot) \to H(\tau, \cdot) \) uniformly on \( [x_0, y_0] \), the left-hand side of the equality above converges to \( \int_\mathbb{R} u(\tau, x) f(H(t, x)) \, dx \). We have thus proved that

\[ \int u(\tau, x) f(H(\tau, x)) \, dx = 0 \]

for all \( f \in C^1_c(I_\tau) \). Approximating \( f \in C_c(I_\tau) \) with a sequence of functions from \( C^1_c(I_\tau) \) it is easy to see that (11) holds for any \( f \in C_c(I_\tau) \).
Fix $\psi \in C_c(\mathbb{R})$. Since $x \mapsto H(\tau, x)$ is strictly monotone and continuous, it has a continuous inverse, and therefore we can find $f \in C_c(I_T)$ such that $\psi(x) = f(H(\tau, x))$ for all $x \in \mathbb{R}$. Therefore by \eqref{eq:tilde_L}
\begin{equation}
\int u(\tau, x)\psi(x) \, dx = 0
\end{equation}
for any $\psi \in C_c(\mathbb{R})$. Hence $u(\tau, \cdot) \equiv 0$. Since this argument is valid for any $\tau \in (0, T) \setminus N$, we conclude that $u(\tau, \cdot) = 0$ a.e. for a.e. $\tau \in I$. \hfill \(\square\)

From the proof above one can also deduce the following result:

**Corollary 1.** Suppose that a vector field $b \in L^\infty(I \times \mathbb{R}; \mathbb{R})$ admits a density $\rho \in L^1_{\text{loc}}(I \times \mathbb{R}; \mathbb{R})$ such that $\rho(t, x) > 0$ for a.e. $(t, x) \in I \times \mathbb{R}$. If $u \in L^\infty_{\text{loc}}(I \times \mathbb{R}; \mathbb{R})$ is a weak solution of \eqref{eq:transport} with $\bar{u} \equiv 0$ then $u(t, x) = 0$ for a.e. $(t, x) \in I \times \mathbb{R}$.

The proof repeats the proof of Theorem 2.3. (Notice that in the proof of \eqref{eq:existence} we have not used the continuity of $H$ on $(-T+\varepsilon, T-\varepsilon) \times \mathbb{R}$. Only when passing to the limit in \eqref{eq:reg_time} we have to argue slightly differently. Namely, since $\tilde{\rho} \in L^1_{\text{loc}}([-T, T] \times \mathbb{R})$ it follows that
\begin{align*}
\|u \cdot (\partial_1 \varphi_x + v\partial_2 \varphi_x)\|_{L^1(K)} &\leq \|u\|_{L^\infty(K)} \cdot \|-(\tilde{\rho}\tilde{b})_x + \tilde{b}\tilde{\rho}_x\| \\
&\leq \|u\|_{L^\infty(K)} \cdot \|-(\tilde{\rho}\tilde{b})_x + \tilde{b}\tilde{\rho}_x\|_{L^1(K)} + \|-(\tilde{\rho}\tilde{b}) + b\tilde{\rho}_x\|_{L^1(K)} \to 0
\end{align*}
as $\varepsilon \to 0$. \hfill \(\square\)

3. Lagrangian flows and existence of weak solutions. In this section we discuss the corollaries of Theorem 2.3 at a Lagrangian level. Namely, we prove existence and uniqueness of the regular Lagrangian flow associated with a nearly incompressible vector field. Using this flow we establish existence of locally integrable solutions of the Cauchy problem for the continuity equation with locally integrable initial data.

Suppose that $b \in L^\infty(I \times \mathbb{R}; \mathbb{R})$ is a nearly incompressible vector field with density $\rho \in L^1_{\text{loc}}(I \times \mathbb{R}; \mathbb{R})$. Let $H \in \text{Lip}([0, T] \times \mathbb{R})$ be a Hamiltonian associated with $(\rho, b)$.

By \eqref{eq:transport} and Fubini’s theorem for a.e. $t \in I$ for all $x, y \in \mathbb{R}$ such that $x < y$ it holds that
\begin{equation}
C_1(y - x) \leq H(t, y) - H(t, x) \leq C_2(y - x),
\end{equation}
where $C_1, C_2$ are the constants from Definition 1.1. By continuity of $H$ \eqref{eq:transport} holds for all $t \in I$. Hence for any $t \in I$ the function $x \mapsto H(t, x)$ is strictly increasing and \textit{bilipschitz}. Consequently, for any $h \in \mathbb{R}$ there exists unique $Y(t, h) \in \mathbb{R}$ such that $H(t, Y(t, h)) = h$.

By \eqref{eq:transport} for any $t \in I$ there exists a function $\eta_t \in L^\infty(\mathbb{R})$ such that $C_1 \leq \eta_t \leq C_2$ a.e. and
\begin{equation}
\partial_x H(t, x) = \eta_t(x)
\end{equation}
in $D'(\mathbb{R})$. Note that by continuity of $H$ the function $I \ni t \mapsto \eta_t \in L^\infty(\mathbb{R})$ is $\ast$-weak continuous and therefore $\rho$ solves the Cauchy problem for the continuity equation \eqref{eq:transport} with the initial data $\rho_0$. In view of \eqref{eq:transport} for a.e. $t \in I$ we have $\rho(t, x) = \eta_t(x)$ for a.e. $x$. Since we can always redefine $\rho$ on a negligible set, for convenience we will assume that the last equality holds for \textit{all} $t \in [0, T]$.

**Lemma 3.1.** The function $Y$ is Lipschitz continuous on $[0, T] \times \mathbb{R}$. Moreover, there exists a negligible set $M \subset \mathbb{R}$ such that for all $h \in \mathbb{R} \setminus M$
\begin{equation}
\partial_t Y(t, h) = b(t, Y(t, h))
\end{equation}
in $\mathcal{D}'(I)$. Finally, for all $t \in [0,T]$ 
$$Y(t,\cdot)\# \mathcal{L} = \rho(t,\cdot)\mathcal{L}. \quad (15)$$

Here $f_\# \mu$ denotes the image of the measure $\mu$ under the map $f$ and $\mathcal{L}$ denotes the Lebesgue measure (we use the notation from [4]).

Proof. By (13) for any $h, h' \in \mathbb{R}$ it holds that 
$$C_1 |Y(t, h) - Y(t, h')| \leq |H(t, Y(t, h)) - H(t, Y(t, h'))| = |h - h'|$$

hence the function $h \mapsto Y(t, h)$ is Lipschitz continuous with Lipschitz constant $1/C_1$.

Fix $(t, x) \in I \times \mathbb{R}$. In view of (4) and Fubini’s theorem for a.e. $(t', x') \in I \times \mathbb{R}$ such that $|x' - x| > \|b\|_{\infty} |t' - t|$ it holds that

$$|H(t', x') - H(t, x)| \geq C_1 (|x' - x| - \|b\|_{\infty} |t' - t|). \quad (16)$$

By continuity of $H$, (16) holds for all $(t', x') \in I \times \mathbb{R}$. Hence for any $h \in \mathbb{R}$ and any $(t, x) \in H^{-1}(h)$ the level set $H^{-1}(h)$ is contained in a cone:

$$H^{-1}(h) \subset \{(t', x') \in I \times \mathbb{R} : |x' - x| \leq \|b\|_{\infty} |t' - t|\}, \quad (17)$$

therefore for any $h \in \mathbb{R}$ the function $t \mapsto Y(t, h)$ is Lipschitz continuous with Lipschitz constant $\|b\|_{\infty}$.

In view of Rademacher’s theorem the functions $H$ and $Y$ are differentiable a.e. on $I \times \mathbb{R}$. Hence by chain rule and taking into account (4) we obtain

$$0 = \partial_t h = \partial_t H(t, Y(t, h)) = \partial_t H(t, Y(t, h)) + \partial_x H(t, Y(t, h)) \partial_t Y(t, h)$$

$$= -\rho(t, Y(t, h)) b(t, Y(t, h)) \partial_t Y(t, h) + \rho(t, Y(t, h)) \partial_t Y(t, h).$$

and

$$1 = \partial_h h = \partial_h H(t, Y(t, h)) = \partial_x H(t, Y(t, h)) \partial_h Y(t, h)$$

$$= \rho(t, Y(t, h)) \partial_h Y(t, h)$$

for a.e. $(t, h) \in I \times \mathbb{R}$. Hence (14) holds and moreover for any $\varphi \in C_c(\mathbb{R})$

$$\int \varphi \, dY(t, \cdot)\# \mathcal{L} = \int \varphi(Y(t, h)) \, dh$$

$$= \int \varphi(Y(t, h)) \rho(t, Y(t, h)) \partial_h Y(t, h) \, dh = \int \varphi(y) \rho(t, y) \, dy$$

(by Area formula, see e.g. [4]). Thus (15) is proved.

We define the flow $X$ of $b$ as

$$X(t, x) := Y(t, H(0, x)). \quad (18)$$

Note that $X$ is independent of the additive constant in the definition of $H$. In order to show that $X$ is independent of the choice of $\rho$ we recall the definition of regular Lagrangian flow (see [12]) and the corresponding uniqueness result:

**Definition 3.2.** Let $b: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a bounded measurable vector field. We say that a map $X: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a regular Lagrangian flow relative to $b$ if

1. for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ the map $t \mapsto X(t, x)$ is an absolutely continuous integral solution of $\dot{\gamma}(t) = b(t, \gamma(t))$ for $t \in [0, T]$ with $\gamma(0) = x$;
2. there exists a constant $L > 0$ independent of $t$ such that $X(t, \cdot)\# \mathcal{L}^d \leq L\mathcal{L}^d$. 


Proposition 1 (see [12], Theorem 6.4.1). Let \( b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) be a bounded measurable vector field. Assume that the only weak solution \( u \in L^\infty(I \times \mathbb{R}^d) \) of (1) with \( \bar{u} = 0 \) is \( u = 0 \). Then the regular Lagrangian flow relative to \( b \), if it exists, is unique. Assume in addition that \( u \bar{u} = 1 \) has a positive solution \( u \in L^\infty(I \times \mathbb{R}^d) \). Then we have existence of a regular Lagrangian flow relative to \( b \).

By Lemma 3.1 the flow \( X \) defined in (18) is a regular Lagrangian flow of \( b \). Indeed, by 15
\[
X_\#(\rho(0, \cdot)\mathcal{L}) = Y(t, \cdot)_\#H(0, \cdot)_\#(\rho(0, \cdot)\mathcal{L}) = Y(t, \cdot)_\#\mathcal{L} = \rho(t, \cdot)\mathcal{L}.
\] (19)
Since Theorem 2.3 implies uniqueness of bounded weak solutions of (1), Proposition 1 immediately implies uniqueness of regular Lagrangian flow of \( b \). Hence \( X \) is independent of the choice of the density \( \rho \).

Theorem 3.3. Let \( b \in L^\infty(I \times \mathbb{R} ; \mathbb{R}) \) be nearly incompressible with the density \( \rho \). Let \( X \) be the flow of \( b \). Then for any \( \bar{u} \in L^1_{\text{loc}}(\mathbb{R}) \) there exists a function \( u \in L^1_{\text{loc}}(I \times \mathbb{R}) \) such that for a.e. \( t \in I \)

\[
u(t, \cdot)\mathcal{L} = X(t, \cdot)_\#(\bar{u}\mathcal{L})
\]
and the function \( u \) solves (1).

Proof. It is straightforward to check that for any \( t \in [0, T] \) the inverse \( X^{-1}(t, \cdot) \) of the function \( X(t, \cdot) \) is given by \( X^{-1}(t, x) = Y(0, H(t, x)) \). We define \( u(t, x) \) as follows:

\[
u(t, x) = \frac{\bar{u}(X^{-1}(t, x))}{\rho(0, X^{-1}(t, x))}\rho(t, x).
\]
Then

\[
u(t, \cdot)\mathcal{L} = \frac{\bar{u}(X^{-1}(t, \cdot))}{\rho(0, X^{-1}(t, \cdot))}X_\#(\rho(0, \cdot)\mathcal{L})
\]

\[
= X_\#\left(\frac{\bar{u}(\cdot)}{\rho(0, \cdot)}\rho(0, \cdot)\mathcal{L}\right) = X(t, \cdot)_\#(\bar{u}\mathcal{L})
\]
Therefore for any \( \varphi \in C^1_c([0, T] \times \mathbb{R}) \) by Definition 3.2

\[
\int_I \int_{\mathbb{R}} \left( \partial_t \varphi + b \partial_x \varphi \right) u(t, x) \, dx \, dt = \int_I \int_{\mathbb{R}} \left( \partial_t \varphi + b \partial_x \varphi \right) X(t, \cdot)_\#(\bar{u}\mathcal{L}) \, dt
\]

\[
= \int_I \int_{\mathbb{R}} \left[ (\partial_t \varphi)(t, X(t, x)) + b(t, X(t, x))(\partial_x \varphi)(t, X(t, x)) \right] \bar{u}(x) \, dx \, dt
\]

\[
= \int_I \int_{\mathbb{R}} \partial_t \varphi(t, X(t, x)) \bar{u}(x) \, dx \, dt
\]

\[
= -\int_{\mathbb{R}} \varphi(t, x) \bar{u}(x) \, dx \, dt.
\]

Proof of Theorem 1.2. Existence follows from Theorem 3.3 and uniqueness follows from Theorem 2.3.

In the context of the model of forward wave in an elastic string discussed above, Theorem 3.3 shows that (2) can be solved by the method of characteristics even under very weak assumptions on the regularity of \( b \). Theorem 3.3 can also be applied to the continuity equation for the density of coupled phase oscillators in the Kuramoto-Sakaguchi model (see e.g. [2] for the related well-posedness results) in the case when the density is bounded from below (e.g. for drifting oscillators in case of weak coupling).
4. Compactness of flows. In [10] Bressan has proposed the following conjecture:

**Conjecture 1 ([10]).** Consider a sequence of smooth vector fields $b_n : I \times \mathbb{R}^d \to \mathbb{R}^d$ which are uniformly bounded, i.e. $|b_n| \leq C$ for some $C > 0$ for all $n \in \mathbb{N}$. Let $X_n = X_n(t,x)$ denote the classical flow of $b_n$, i.e.

$$X_n(0,x) = x, \quad \partial_t X_n(t,x) = b_n(t,X_n(t,x)).$$

Suppose that there exist constants $C_1, C_2$

$$C_1 \leq \det(\nabla X(t,x)) \leq C_2, \quad (t,x) \in I \times \mathbb{R}^d,$$

$$\|\nabla b_n\|_{L^1} \leq C_3.$$

Then the sequence $X_n$ is strongly precompact in $L^1_{\text{loc}}(I \times \mathbb{R}^d; \mathbb{R}^d)$.

**Theorem 4.1.** Consider a sequence of one-dimensional vector fields $b_n \in L^\infty(I \times \mathbb{R}; \mathbb{R})$ which are uniformly bounded, i.e. $|b_n| \leq C$ for some $C > 0$ for all $n \in \mathbb{N}$. Let $X_n = X_n(t,x)$ denote the (regular Lagrangian) flow of $b_n$. Suppose that for each $n \in \mathbb{N}$ the vector field $b_n$ is nearly incompressible with density $\rho_n$ and there exist constants $C_1, C_2$ such that

$$C_1 \leq \rho_n \leq C_2$$

a.e. on $I \times \mathbb{R}$ for all $n \in \mathbb{N}$. Then the sequence $X_n$ is precompact in $C(K)$ for any compact $K \subset I \times \mathbb{R}$.

**Proof.** By (18) and the estimates from the proof of Lemma 3.1 one can easily deduce that for any $n \in \mathbb{N}$

$$|X_n(t,x) - X_n(t',x')| \leq \frac{C_2}{C_1} |x - x'| + \|b\|_{L^\infty} |t - t'|$$

for all $x, x' \in \mathbb{R}$ and $t, t' \in [0,T]$. Therefore it remains to apply Arzelà-Ascoli theorem.

**Remark 2.** Theorem 4.1 shows that in the one-dimensional case Conjecture 1 holds even without assuming BV bound (20). A quantitative version of Conjecture 1 assuming only the BV bound (20) (without near incompressibility) was established in [5]. The multi-dimensional version of Conjecture 1 is studied in [7].

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