Optimal Point Movement for Covering Circular Regions

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Abstract

Given \(n\) points in a circular region \(C\) in the plane, we study the problems of moving the \(n\) points to its boundary to form a regular \(n\)-gon such that the maximum (min-max) or the sum (min-sum) of the Euclidean distances traveled by the points is minimized. The problems have applications, e.g., in mobile sensor barrier coverage of wireless sensor networks. The min-max problem further has two versions: the decision version and optimization version. For the min-max problem, we present an \(O(n \log^2 n)\) time algorithm for the decision version and an \(O(n \log^3 n)\) time algorithm for the optimization version. The previously best algorithms for the two problem versions take \(O(n^{3.5})\) time and \(O(n^{3.5} \log n)\) time, respectively. For the min-sum problem, we show that a special case with all points initially lying on the boundary of the circular region can be solved in \(O(n^2)\) time, improving a previous \(O(n^4)\) time solution. For the general min-sum problem, we present a 3-approximation \(O(n^2)\) time algorithm, improving the previous \((1 + \pi)\)-approximation \(O(n^2)\) time algorithm. A by-product of our techniques is an algorithm for dynamically maintaining the maximum matching of a circular convex bipartite graph; our algorithm can handle each vertex insertion or deletion on the graph in \(O(\log^2 n)\) time. This result is interesting in its own right.

1 Introduction

Given \(n\) points in a circular region \(C\) in the plane, we study the problems of moving the \(n\) points to its boundary to form a regular \(n\)-gon such that the maximum (min-max) or the sum (min-sum) of the Euclidean distances traveled by the points is minimized. The problems have applications, e.g., in mobile sensor barrier coverage of wireless sensor networks. The problems have been studied before. In this paper we present new algorithms that significantly improve the previous solutions for the problems.

1.1 Problem Definitions

Let \(|ab|\) denote the Euclidean length of the line segment with two endpoints \(a\) and \(b\) in the plane. Let \(C\) be a circular region in the plane. Given a set of \(n\) points \(S = \{A_0, A_1, \ldots, A_{n-1}\}\) in \(C\) (i.e., in its interior or on its boundary), we wish to move all sensors to \(n\) points \(A'_0, A'_1, \ldots, A'_{n-1}\) on the boundary of \(C\) that form a regular \(n\)-gon. The min-max problem aims to minimize the maximum Euclidean distance traveled by all points, i.e., \(\max_{0 \leq i \leq n-1} |A_iA'_i|\). The min-sum problem aims to minimize the sum of the Euclidean distances traveled by all points, i.e., \(\sum_{i=0}^{n-1} |A_iA'_i|\).

Further, given a value \(\lambda \geq 0\), the decision version of the min-max problem is to determine whether it is possible to move all points in \(S\) to the boundary of \(C\) to form a regular \(n\)-gon such that the distance traveled by each point is no more than \(\lambda\). Indeed, let \(\lambda_C\) be the maximum distance traveled by the points in an optimal solution for the min-max problem. Then, the answer to the

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feasibility problem is “yes” if and only if \( \lambda_C \leq \lambda \). For discrimination, we refer to the original min-max problem as the \textit{optimization version} of the min-max problem.

For the min-sum problem, if the points in \( S \) are given initially all lying on the boundary of \( C \), then this case is referred to as the \textit{boundary case} of the min-sum problem.

1.2 Applications in Wireless Sensor Networks

A Wireless Sensor Network (WSN) is composed of a large number of sensors which monitor some surrounding environmental phenomenon. Usually, the sensors are densely deployed either inside the target phenomenon or are very close to it [1]. Each sensor is equipped with a sensing device with limited battery-supplied energy. The sensors process data obtained and forward the data to a base station. A typical type of WSN applications is concerned with security and safety systems, such as detecting intruders (or movement thereof) around infrastructure facilities and regions. Particularly, it is often used to monitor a protected area so as to detect intruders as they penetrate the area or as they cross the area border. For example, research efforts have been under way to extend the scalability of wireless sensor networks to the monitoring of international borders [10, 13].

The study of \textit{barrier coverage} using mobile sensors was originated in [5, 13] and later in [2]. Different from the traditional concept of \textit{full coverage}, it seeks to cover the deployment region by guaranteeing that there is no path through the region that can be traversed undetectedly by an intruder, i.e., all possible crossing paths through the region are covered by the sensors [2, 5, 13]. Hence, an interesting problem is to reposition the sensors quickly so as to repair the existing security hole and thereby detect intruders [2]. Since barrier coverage requires fewer sensors for detecting intruders, it gives a good approximation of full area coverage. The planar region on which the sensors move is sometimes represented by a circle. Since sensors have limited battery-supplied energy, we wish to minimize their movement. Thus, if each sensor is represented as a point, the problem is exactly our optimal point movement min-max (the optimization version) or min-sum problem. Further, if each sensor has energy \( \lambda \) and we want to determine whether this level of energy is sufficient to form a barrier coverage, then the problem becomes the decision version of the min-max problem.

1.3 Previous Work and Our Results

For the min-max problem, Bhattacharya \textit{et al.} [2] proposed an \( O(n^{3.5}) \) time algorithm for the decision version and an \( O(n^{3.5} \log n) \) time algorithm for the optimization version, where the decision algorithm is based on some observations and brute force and the optimization algorithm is based on parametric search approach [7, 15]. Recently, it was claimed in [17] that these two problem versions were solvable in \( O(n^{2.5}) \) time and \( O(n^{2.5} \log n) \) time, respectively. However, it seems that the announced algorithms in [17] contain errors (which might be fixed, say, by using the methods given in this paper). In this paper, we solve the decision version in \( O(n \log^2 n) \) time and the optimization version in \( O(n \log^3 n) \) time, which significantly improve the previous results. The improvements of our algorithms are based on new observations and interesting techniques.

A by-product of our techniques that is interesting in its own right is an algorithm for dynamically maintaining the maximum matchings of \textit{circular convex bipartite graphs}. Our algorithm handles each (online) vertex insertion or deletion on an \( n \)-vertex circular convex bipartite graph in \( O(\log^2 n) \) time. This matches the performance of the best known dynamic matching algorithm for \textit{convex bipartite graphs} [3]. Note that convex bipartite graphs are a subclass of circular convex bipartite graphs [13]. To our best knowledge, no dynamic matching algorithm for circular convex bipartite graphs [14].
graphs was known before. Since dynamically maintaining the maximum matching of a graph is a basic problem, our result may find other applications.

For the min-sum problem, an $O(n^2)$ time approximation algorithm with approximation ratio $1+\pi$ was given in [2]. A PTAS approximation algorithm, which has a substantially larger polynomial time bound, was also given in [2]. In this paper, we present an $O(n^2)$ time approximation algorithm with approximation ratio 3, which improves the $(1+\pi)$-approximation result in [2]. However, whether the general min-sum problem is NP-hard is still left open.

For the boundary case of the min-sum problem, an $O(n^4)$ time (exact) algorithm was given in [17]. We show that the time bound of that algorithm can be reduced to $O(n^2)$.

The rest of this paper is organized as follows. Our algorithm for the decision version of the min-max problem is given in Section 2 and our algorithm for the optimization version is presented in Section 3. The min-sum problem is discussed in Section 4.

To distinguish from a normal point in the plane, in the following paper we refer to each point $A_i \in S$ as a sensor.

## 2 The Decision Version of the Min-max Problem

For simplicity, we assume the radius of the circle $C$ is 1. Denote by $\partial C$ the boundary of $C$. Let $\lambda_C$ be the maximum distance traveled by the sensors in $S$ in an optimal solution for the min-max problem, i.e., $\lambda_C = \min\{\max_{0 \leq i \leq n-1}\{|A_iA'_i|\}\}$. Since the sensors are all in $C$, $\lambda_C \leq 2$. In this section, we consider the decision version of the min-max problem on $C$: Given a value $\lambda$, determine whether $\lambda_C \leq \lambda$. We present an $O(n \log^2 n)$ time algorithm for this problem.

### 2.1 An Algorithm Overview

We first discuss some concepts. A bipartite graph $G = (V_1, V_2, E)$ with $|V_1| = O(n)$ and $|V_2| = O(n)$ is convex on the vertex set $V_2$ if there is a linear ordering on $V_2$, say, $V_2 = \{v_0, v_1, \ldots, v_{n-1}\}$, such that if any two edges $(v, v_j) \in E$ and $(v, v_k) \in E$ with $v_j, v_k \in V_2$, $v \in V_1$, and $j < k$, then $(v, v_l) \in E$ for all $j \leq l \leq k$. In other words, for any vertex $v \in V_1$, the subset of vertices in $V_2$ connected to $v$ forms an interval on the linear ordering of $V_2$. For any $v \in V_1$, suppose the subset of vertices in $V_2$ connected to $v$ is $\{v_j, v_{j+1}, \ldots, v_k\}$; then we denote $\text{begin}(v, G) = j$ and $\text{end}(v, G) = k$. Although $E$ may have $O(n^2)$ edges, it can be represented implicitly by specifying $\text{begin}(v, G)$ and $\text{end}(v, G)$ for each $v \in V_1$. A vertex insertion on $G$ is to insert a vertex $v$ into $V_1$ with an edge interval $[\text{begin}(v, G), \text{end}(v, G)]$ and implicitly connect $v$ to every $v_i \in V_2$ with $\text{begin}(v, G) \leq i \leq \text{end}(v, G)$. Similarly, a vertex deletion on $G$ is to delete a vertex $v$ from $V_1$ as well as all its adjacent edges.

A bipartite graph $G = (V_1, V_2, E)$ is circular convex on the vertex set $V_2$ if there is a circular ordering on $V_2$ such that for each vertex $v \in V_1$, the subset of vertices in $V_2$ connected to $v$ forms a circular-arc interval on that ordering. Precisely, suppose such a clockwise circular ordering of $V_2$ is $v_0, v_1, \ldots, v_{n-1}$. For any two edges $(v, v_j) \in E$ and $(v, v_k) \in E$ with $v_j, v_k \in V_2$, $v \in V_1$, and $j < k$, either $(v, v_j) \in E$ for all $j \leq l \leq k$, or $(v, v_l) \in E$ for all $k \leq l \leq n - 1$ and $(v, v_l) \in E$ for all $0 \leq l \leq j$. For each $v \in V_1$, suppose the vertices of $V_2$ connected to $v$ are from $v_j$ to $v_k$ clockwise on the ordering, then $\text{begin}(v, G)$ and $\text{end}(v, G)$ are defined to be $j$ and $k$, respectively. Vertex insertions and deletions on $G$ are defined similarly.

A maximum matching in a convex bipartite graph can be found in $O(n)$ time [9, 11, 16]. The same time bound holds for a circular convex bipartite graph [14]. Brotal et al. [3] designed a data structure for dynamically maintaining the maximum matchings of a convex bipartite graph that can
support each vertex insertion or deletion in \(O(\log^2 n)\) amortized time. For circular convex bipartite graphs, however, to our best knowledge, we are not aware of any previous work on dynamically maintaining their maximum matchings.

The main idea of our algorithm for the decision version of the min-max problem is as follows. First, we model the problem as finding the maximum matchings in a sequence of \(O(n)\) circular convex bipartite graphs, which is further modeled as dynamically maintaining the maximum matching of a circular convex bipartite graph under a sequence of \(O(n)\) vertex insertion and deletion operations. Second, we develop an approach for solving the latter problem. Specifically, we show that the maximum matching of a circular convex bipartite graph of \(O(n)\) vertices can be dynamically maintained in \(O(\log^2 n)\) time (in the worst case) for each vertex insertion or deletion. Note that this result is of independent interest.

In the following, we first present the problem modeling and then give our algorithm for dynamically maintaining the maximum matching of a circular convex bipartite graph.

### 2.2 The Problem Modeling

Recall that in the decision version of the min-max problem, our goal is to determine whether \(\lambda_C \leq \lambda\). Let \(P\) be an arbitrary regular \(n\)-gon with its vertices \(P_0, P_1, \ldots, P_{n-1}\) ordered clockwise on \(\partial C\). We first consider the following sub-problem: Determine whether we can move all sensors to the vertices of \(P\) such that the maximum distance traveled by the sensors is at most \(\lambda\). Let \(G_P\) be the bipartite graph between the sensors \(A_0, \ldots, A_{n-1}\) and the vertices of \(P\), such that a sensor \(A_i\) is connected to a vertex \(P_j\) in \(G_P\) if and only if \(|A_i P_j| \leq \lambda\). The next lemma is immediate.

**Lemma 1** The bipartite graph \(G_P\) is circular convex.

**Proof:** This simply follows from the fact that the boundary of any circle of radius \(\lambda\) can intersect \(\partial C\) at most twice. \(\square\)

To solve the above sub-problem, it suffices to compute a maximum matching \(M\) in the circular convex bipartite graph \(G_P\) (by using the algorithm in [14]). If \(M\) is a perfect matching, then the answer to the sub-problem is “yes”; otherwise, the answer is “no”. Thus, the sub-problem can be solved in \(O(n)\) time (note that the graph \(G_P\) can be constructed implicitly in \(O(n)\) time, after \(O(n \log n)\) time preprocessing). If the answer to the sub-problem is “yes”, then we say that \(P\) is feasible with respect to the value \(\lambda\).

If \(P\) is feasible, then clearly \(\lambda_C \leq \lambda\). If \(P\) is not feasible, however, \(\lambda_C > \lambda\) does not necessarily hold, because \(P\) may not be positioned “right” (i.e., \(P\) may not be the regular \(n\)-gon in an optimal solution of the optimization version of the min-max problem). To further decide whether \(\lambda_C \leq \lambda\), our strategy is to rotate \(P\) clockwise on \(\partial C\) by an arc distance at most \(2\pi/n\). Since the perimeter of \(C\) is \(2\pi\), the arc distance between any two neighboring vertices of \(P\) is \(2\pi/n\). A simple yet critical observation is that \(\lambda_C \leq \lambda\) if and only if during the rotation of \(P\), there is a moment (called a feasible moment) at which \(P\) becomes feasible with respect to \(\lambda\). Thus, our task is to determine whether a feasible moment exists during the rotation of \(P\).

Consider the graph \(G_P\). For each sensor \(A_i\), denote by \(E(A_i) = \{P_j, P_{j+1}, \ldots, P_k\}\) the subset of vertices of \(P\) connected to \(A_i\) in \(G_P\), where the indices of the vertices of \(P\) are taken as module by \(n\). We assume that \(E(A_i)\) does not contain all vertices of \(P\) (otherwise, it is trivial). Since the arc distance from \(P_{j-1}\) to \(P_j\) is \(2\pi/n\), during the (clockwise) rotation of \(P\), there must be a moment after which \(P_{j-1}\) becomes connected to \(A_i\), and we say that \(P_{j-1}\) is added to \(E(A_i)\); similarly, there must be a moment after which \(P_k\) becomes disconnected to \(A_i\), and we say that \(P_k\) is removed from \(E(A_i)\). Note that these are the moments when the edges of \(A_i\) (and thus the graph \(G_P\)) are
changed due to the rotation of $P$. Also, note that during the rotation, all vertices in $E(A_i) \setminus \{P_k\}$ remain connected to $A_i$ and all vertices in $P \setminus \{E(A_i) \cup \{P_{j-1}\}\}$ remain disconnected to $A_i$. Hence throughout this rotation, there are totally $n$ additions and $n$ removals on the graph $G_P$. If we sort all these additions and removals based on the time moments when they occur, then we obtain a sequence of $2n$ circular convex bipartite graphs, and determining whether there exists a feasible moment is equivalent to determining whether there is a graph in this sequence that has a perfect matching. With the $O(n)$ time maximum matching algorithm for circular convex bipartite graphs of $n$ vertices in $[14]$, a straightforward solution for determining whether there is a feasible moment would take $O(n^2)$ time.

To obtain a faster algorithm, we further model the problem as follows. Consider the addition of $P_{j-1}$ to $E(A_i)$. This can be done by first deleting the vertex of $G_P$ corresponding to $A_i$ and then inserting a new vertex corresponding to $A_i$ with its edges connecting to the vertices in $\{P_{j-1}\} \cup E(A_i)$. The removal of $P_k$ from $E(A_i)$ can be handled similarly. Thus, each addition or removal on $E(A_i)$ can be transformed to one vertex deletion and one vertex insertion on $G_P$. If we sort all vertex updates (i.e., insertions and deletions) by the time moments when they occur, then the problem of determining whether there is a feasible moment is transformed to determining whether there exists a perfect matching in a sequence of vertex updates on the graph $G_P$. In other words, we need to dynamically maintain the maximum matching in a circular convex bipartite graph to support a sequence of $2n$ vertex insertions and $2n$ vertex deletions. This problem is handled in the next subsection.

### 2.3 Dynamic Maximum Matching in a Circular Convex Bipartite Graph

In this subsection, we consider the problem of dynamically maintaining the maximum matching in a circular convex bipartite graph to support vertex insertions and deletions. We treat all vertex updates in an online fashion.

Let $G = (V_1, V_2, E)$ with $|V_1| = O(n)$ and $|V_2| = O(n)$ be a circular convex bipartite graph on the vertex set $V_2$, i.e., the vertices of $V_2$ connected to each vertex in $V_1$ form a circular-arc interval on the sequence of the vertex indices of $V_2$. Suppose $V_2 = \{v_0, v_1, \ldots, v_{n-1}\}$ is ordered clockwise. Recall that a vertex insertion on $G$ is to insert a vertex $v$ into $V_1$ with an edge interval $[begin(v, G), end(v, G)]$ such that $v$ is (implicitly) connected to all vertices of $V_2$ from $begin(v, G)$ clockwise to $end(v, G)$. A vertex deletion is to delete a vertex $v$ from $V_1$ and all its adjacent edges (implicitly). Our task is to design an algorithm for maintaining the maximum matching of $G$ to support such update operations (i.e., vertex insertions and deletions) efficiently. Below, we present an algorithm with an $O(\log^2 n)$ time per update operation.

Our approach can be viewed as a combination of the data structure in $[3]$ for dynamically maintaining the maximum matching in a convex bipartite graph and the linear time algorithm in $[14]$ for computing a maximum matching in a circular convex bipartite graph. We refer to them as the BGHK data structure $[3]$ and the LB algorithm $[14]$, respectively. We first briefly describe the BGHK data structure and the LB algorithm.

The BGHK data structure $[3]$ is a binary tree $T$, and each node of $T$ maintains a balanced binary tree. This data structure can be constructed in $O(n \log^2 n)$ time and can support each vertex insertion or deletion in $O(\log^2 n)$ amortized time. Consider a vertex insertion, i.e., inserting a vertex $v$ into $V_1$. Let $M'$ (resp., $M$) be the maximum matching in the graph before (resp., after) the insertion. Let $|M|$ denote the number of matched pairs in $M$. After the data structure is updated (in $O(\log^2 n)$ amortized time), the value $|M|$ can be reported in $O(1)$ time and $M$ can be reported in $O(|M|)$ time. We can also determine in $O(1)$ time whether $v$ is matched in $M$. 


Further, if another vertex \( v' \in V_1 \) was matched in \( M' \) but is not matched in \( M \), then it is easy to see that \( v \) must be matched in \( M \). When this case occurs, we say that \( v \) replaces \( v' \) and \( v' \) is called the replacement, and the data structure is able to report the replacement in \( O(1) \) time. Note that as shown in [3], although an update on the graph can cause dramatic changes on the maximum matching, the sets of the matched vertices in \( V_1 \) (and \( V_2 \)) can change by at most one vertex. Thus, there is at most one such replacement \( v' \). Similarly, consider deleting a vertex \( v \) from \( V_1 \). After the data structure is updated, the value \( |M| \) can be reported in \( O(1) \) time and \( M \) can be reported in \( O(|M|) \) time. We can also find out whether \( v \) was matched in \( M' \) in \( O(1) \) time. If a vertex \( v' \in V_1 \) was not matched in \( M' \) but is matched in \( M \), then it is easy to see that \( v \) must be matched in \( M' \).

When this case occurs, we say \( v' \) is the supplement, which can be determined in \( O(1) \) time.

The LB algorithm [14] finds a maximum matching in a circular convex bipartite graph \( G = (V_1, V_2, E) \) by reducing the problem to two sub-problems of computing the maximum matchings in two convex bipartite graphs \( G_1 \) and \( G_2 \). Some details are summarized below. For any vertex \( v \in V_1 \), if \( \text{begin}(v, G) \leq \text{end}(v, G) \), then \( v \) is called a non-boundary vertex. Otherwise, \( v \) is a boundary vertex; the edges connecting \( v \) to \( v_{\text{begin}(v, G)}, v_{\text{begin}(v, G)+1}, \ldots, v_{n-1} \) in \( V_2 \) are called lower edges, and the other edges connecting \( v \) are upper edges. Based on the graph \( G \), a convex bipartite graph \( G_1 = (V_1, V_2, E_1) \) is defined as follows. Both its vertex sets are the same as those in \( G \). For each vertex \( v \in V_1 \) in \( G \), \( \text{begin}(v, G_1) = \text{begin}(v, G) \); if \( v \) is a non-boundary vertex, then \( \text{end}(v, G_1) = \text{end}(v, G) \), and otherwise \( \text{end}(v, G_1) = n - 1 + \text{end}(v, G) \) (note that this value of \( \text{end}(v, G_1) \) is used only for comparison in the algorithm although there are not so many vertices in \( V_2 \)). The LB algorithm has two main steps. The first step is to compute a maximum matching in \( G_1 \), which can be done in \( O(n) \) time [9, 11, 16]. Let \( M(G_1) \) be the maximum matching of \( G_1 \). Next, another convex bipartite graph \( G_2 = (V_1, V_2, E_2) \) is defined based on \( M(G_1) \) and \( G_2 \), as follows. Both its vertex sets are the same as those in \( G \). For each non-boundary vertex \( v \in V_1 \) in \( G \), \( \text{begin}(v, G_2) = \text{begin}(v, G) \) and \( \text{end}(v, G_2) = \text{end}(v, G) \). For each boundary vertex \( v \in V_1 \) in \( G \), there are two cases: If \( v \) is matched in \( M(G_1) \), then \( \text{begin}(v, G_2) = \text{begin}(v, G) \) and \( \text{end}(v, G_2) = n - 1 \); otherwise, \( \text{begin}(v, G_2) = 0 \) and \( \text{end}(v, G_2) = \text{end}(v, G) \). The second step of the LB algorithm is to compute a maximum matching in \( G_2 \) (in \( O(n) \) time), denoted by \( M(G_2) \). It was shown in [14] that \( M(G_2) \) is also a maximum matching of the original graph \( G \).

We now discuss our algorithm for dynamically maintaining a maximum matching in the circular convex bipartite graph \( G \). As preprocessing, we first run the LB algorithm on \( G \), after which both the convex bipartite graphs \( G_1 \) and \( G_2 \) of \( G \) are available. We then build two BGHK data structures for \( G_1 \) and \( G_2 \), denoted by \( T(G_1) \) and \( T(G_2) \), respectively, for maintaining their maximum matchings. This completes the preprocessing, which takes \( O(n \log^2 n) \) time. In the following, we discuss how to perform vertex insertions and deletions.

Consider a vertex insertion, i.e., inserting a vertex \( v \) into \( V_1 \) with the edge interval \([\text{begin}(v, G), \text{end}(v, G)]\). To perform this insertion, intuitively, we need to update the two BGHK data structures \( T(G_1) \) and \( T(G_2) \) in a way that mimics some behavior of the LB algorithm. Specifically, we first insert \( v \) into the graph \( G_1 \) by updating \( T(G_1) \). Based on the results on \( G_1 \) (e.g., whether there is a replacement) and the behavior of the LB algorithm, we modify \( G_2 \) by updating \( T(G_2) \) accordingly. In this way, the maximum matching maintained by \( T(G_2) \) is the maximum matching of \( G \) after the insertion. The details are given below.

Let \( G'_1 \) and \( G'_2 \) be the two graphs that would be produced by running the LB algorithm on \( G \) with the new vertex \( v \) (and its adjacent edges). Let \( M(G_1), M(G_2), M(G'_1), \) and \( M(G'_2) \) be the maximum matchings of \( G_1, G_2, G'_1, \) and \( G'_2 \), respectively. Depending on whether \( v \) is a boundary vertex, there are two main cases.
• If \( v \) is a non-boundary vertex (i.e., \( \text{begin}(v, G) \leq \text{end}(v, G) \)), then \( G'_1 \) can be obtained by inserting \( v \) into \( G_1 \). Hence we insert \( v \) into \( T(G_1) \). Depending on whether there is a replacement, there are two cases.

  - If no replacement, then \( G'_2 \) can be obtained by inserting \( v \) into \( G_2 \). Thus, we simply insert \( v \) into \( T(G_2) \) and we are done.
  - Otherwise, let \( v' \) be the replacement. So \( v' \) was matched in \( M(G_1) \) but is not matched in \( M(G'_1) \). Depending on whether \( v' \) is a boundary vertex, there are two subcases.
    * If \( v' \) is a non-boundary vertex, then again, \( G'_2 \) can be obtained by inserting \( v \) into \( G_2 \). We thus insert \( v \) into \( T(G_2) \) and we are done.
    * If \( v' \) is a boundary vertex, then since \( v' \) was matched in \( M(G_1) \), according to the LB algorithm, \( v' \) with the edge interval \( [\text{begin}(v', G), n - 1] \) is in \( G_2 \). After the insertion of \( v \) into \( G_1 \), \( v' \) is not matched in \( M(G'_1) \). Thus, according to the LB algorithm, \( G'_2 \) can be obtained by deleting \( v' \) (with the edge interval \( [\text{begin}(v', G), n - 1] \)) from \( G_2 \), inserting \( v' \) with the edge interval \( [0, \text{end}(v', G)] \) into \( G_2 \), and finally inserting \( v \) into \( G_2 \).

    In summary, for this subcase, we delete \( v' \) (with the edge interval \( [\text{begin}(v', G), n - 1] \)) from \( T(G_2) \) and insert \( v' \) with the edge interval \( [0, \text{end}(v', G)] \) into \( T(G_2) \). Finally, we insert \( v \) into \( T(G_2) \), and we are done.

• If \( v \) is a boundary vertex (i.e., \( \text{begin}(v, G) > \text{end}(v, G) \)), then according to the LB algorithm, \( G'_1 \) can be obtained by inserting \( v \) with the edge interval \( [\text{begin}(v, G), n - 1 + \text{end}(v, G)] \) into \( G_1 \). Thus we insert \( v \) with the edge interval \( [\text{begin}(v, G), n - 1 + \text{end}(v, G)] \) into \( T(G_1) \). Depending on whether there is a replacement, there are two cases.

  - If no replacement, then depending on whether \( v \) is matched in \( M(G'_1) \), there are two subcases.
    * If \( v \) is matched, then according to the LB algorithm, \( G'_2 \) can be obtained by inserting \( v \) with the edge interval \( [\text{begin}(v, G), n - 1] \) into \( G_2 \). Thus, we insert \( v \) with the edge interval \( [\text{begin}(v, G), n - 1] \) into \( T(G_2) \), and we are done.
    * If \( v \) is not matched, then according to the LB algorithm, \( G'_2 \) can be obtained by inserting \( v \) with the edge interval \( [0, \text{end}(v, G)] \) into \( G_2 \). We thus insert \( v \) with the edge interval \( [0, \text{end}(v, G)] \) into \( T(G_2) \), and we are done.
  - Otherwise, there is a replacement \( v' \). So \( v' \) was matched in \( M(G_1) \) but is not matched in \( M(G'_1) \), and \( v \) is matched in \( M(G'_1) \). Depending on whether \( v' \) is a boundary vertex, there are two subcases.
    * If \( v' \) is a non-boundary vertex, then since \( v \) is matched in \( M(G'_1) \), \( G'_2 \) can be obtained by inserting \( v \) with the edge interval \( [\text{begin}(v, G), n - 1] \) into \( G_2 \). We thus insert \( v \) with the edge interval \( [\text{begin}(v, G), n - 1] \) into \( T(G_2) \).
    * If \( v' \) is a boundary vertex, then according to the LB algorithm, \( G'_2 \) is the graph obtained by deleting \( v' \) (with the edge interval \( [\text{begin}(v', G), n - 1] \)) from \( G_2 \), inserting \( v' \) with the edge interval \( [0, \text{end}(v', G)] \) into \( G_2 \), and finally inserting \( v \) with the edge interval \( [\text{begin}(v, G), n - 1] \) into \( G_2 \).

    Thus, we delete \( v' \) (with the edge interval \( [\text{begin}(v', G), n - 1] \)) from \( T(G_2) \), and insert \( v' \) with the edge interval \( [0, \text{end}(v', G)] \) into \( T(G_2) \). Finally, we insert \( v \) with the edge interval \( [\text{begin}(v, G), n - 1] \) into \( T(G_2) \).
This completes the description of our procedure for handling a vertex insertion.

Next, consider a vertex deletion, i.e., deleting a vertex \( v \) from \( V_1 \) of \( G \). Our procedure for this operation proceeds in a manner symmetric to the insertion procedure, and we briefly discuss it below. Define the two graphs \( G'_1 \) and \( G'_2 \) similarly as above.

- If \( v \) is a non-boundary vertex, then we delete \( v \) from \( T(G_1) \). If no supplement, then we delete \( v \) from \( T(G_2) \) and we are done. Otherwise, let \( v' \) be the supplement. So \( v' \) was not matched in \( M(G_1) \) but is matched in \( M(G'_1) \). Depending on whether \( v' \) is a boundary vertex, there are two cases.
  - If \( v' \) is a non-boundary vertex, then we delete \( v \) from \( T(G_2) \) and we are done.
  - If \( v' \) is a boundary vertex, then we delete \( v' \) with the edge interval \([0, end(v', G)]\) from \( T(G_2) \) and insert \( v' \) with the edge interval \([\text{begin}(v', G), n - 1]\) into \( T(G_2) \). Finally, delete \( v \) from \( T(G_2) \), and we are done.

- If \( v \) is a boundary vertex, then we delete \( v \) (with the edge interval \([\text{begin}(v, G), n - 1 + end(v, G)]\)) from \( T(G_1) \). Depending on whether there is a supplement, there are two cases.
  - If no supplement, then depending on whether \( v \) was matched in \( M(G_1) \), there are two subcases. If \( v \) was matched, then we delete \( v \) (with the edge interval \([\text{begin}(v, G), n - 1]\)) from \( T(G_2) \); otherwise, we delete \( v \) (with the edge interval \([0, end(v, G)]\)) from \( T(G_2) \).
  - Otherwise, let \( v' \) be the supplement. So \( v' \) was not matched in \( M(G_1) \) but is matched in \( M(G'_1) \), and \( v \) was matched in \( M(G_1) \). Since \( v \) was matched in \( M(G_1) \), according to the LB algorithm, \( G_2 \) contains \( v \) with the edge interval \([\text{begin}(v, G), n - 1]\). If \( v' \) is a non-boundary vertex, then we delete \( v \) (with the edge interval \([\text{begin}(v, G), n - 1]\)) from \( T(G_2) \) and we are done. Otherwise, since \( v' \) was not matched in \( M(G_1) \), according to the LB algorithm, \( G'_2 \) contains \( v' \) with the edge interval \([0, end(v', G)]\); since \( v' \) is matched in \( M(G'_1) \), according to the LB algorithm, \( G'_2 \) should contain \( v' \) with the edge interval \([\text{begin}(v', G), n - 1]\). Therefore, we delete \( v' \) (with the edge interval \([0, end(v', G)]\)) from \( T(G_2) \), insert \( v' \) with the edge interval \([\text{begin}(v', G), n - 1]\) into \( T(G_2) \), and finally delete \( v \) (with the edge interval \([\text{begin}(v, G), n - 1]\)) from \( T(G_2) \).

This completes the description of our vertex deletion procedure.

As shown in Subsection 2.2, the decision version of the min-max problem can be transformed to the problem of dynamically maintaining the maximum matching in a circular convex bipartite graph subject to a sequence of vertex insertions and deletions. Hence, the correctness of our algorithm for the decision version hinges on the correctness of our dynamic maximum matching algorithm for circular convex bipartite graphs. Yet, the correctness of our (online) dynamic maximum matching algorithm for circular convex bipartite graphs can be seen quite easily. This is because our procedures for performing vertex insertions and deletions are both based on the fact that they simply mimic the behavior of the LB algorithm (while implementing their processing by the means of the BGHK data structures).

For the running time of our algorithm, each update operation involves at most two vertex insertions and two vertex deletions on \( T(G_1) \) and \( T(G_2) \), each of which takes \( O(\log^2 n) \) amortized time \( [8] \); thus, it takes \( O(\log^2 n) \) amortized time in total. Actually, the BGHK data structure in \( [8] \) supports vertex insertions and deletions not only on \( V_1 \) but also on \( V_2 \). Inserting vertices on \( V_2 \) may make the tree unbalanced, and that is why its running time is amortized. However, if vertices
are inserted only on $V_1$, then the tree will never become unbalanced and thus each update takes $O(\log^2 n)$ time in the worst case. In our problem formulation, the vertex updates indeed are only on $V_1$. Denote by $M(G)$ the maximum matching in $G$. We then have the following result.

**Theorem 1** A data structure on a circular convex bipartite graph $G = (V_1, V_2, E)$ can be built in $O(n \log^2 n)$ time for maintaining its maximum matching $M(G)$ so that each online vertex insertion or deletion on $V_1$ can be done in $O(\log^2 n)$ time in the worst case. After each update operation, $|M(G)|$ can be reported in $O(1)$ time and $M(G)$ can be reported in $O(|M(G)|)$ time.

Since the decision version of the min-max problem has been reduced to dynamically maintaining the maximum matching in a circular convex bipartite graph under a sequence of $2n$ vertex insertions and $2n$ vertex decisions, we solve the dynamic maximum problem as follows. After each update operation, we check whether $|M(G)| = n$, and if this is true, then we report $\lambda_C \leq \lambda$ and halt the algorithm. If all $4n$ updates have been processed but it is always $|M(G)| < n$, then we report $\lambda_C > \lambda$. Based on Theorem 1, we have the result below.

**Theorem 2** Given a value $\lambda$, we can determine whether $\lambda_C \leq \lambda$ in $O(n \log^2 n)$ time for the decision version of the min-max problem.

### 3 The Optimization Version of the Min-max Problem

In this section, we consider the optimization version of the min-max problem, and present an $O(n \log^3 n)$ time algorithm for it. The main task is to compute the value $\lambda_C$.

Let $o$ be the center of $C$. For simplicity of discussion, we assume that no sensor lies at $o$. Denote by $X_i$ and $Y_i$ the two points on $\partial C$ which are closest and farthest to each sensor $A_i$, respectively. Clearly, $X_i$ and $Y_i$ are the two intersection points of $\partial C$ with the line passing through $A_i$ and the center $o$ of $C$ (see Figure 1(a)). The lemma below has been proved in [17], and for self-containment of this paper, we include that proof in Appendix A.

**Lemma 2** [17] Suppose an optimal solution for the min-max optimization problem is achieved with $\lambda_C = |A_iA'_i|$ for some $i \in \{0, \ldots, n-1\}$. Then either $A'_i$ is the point $X_i$, or there is another sensor $A_j$ ($j \neq i$) such that $\lambda_C = |A_jA'_j|$ also holds. In the latter case, any slight rotation of the regular $n$-gon that achieves $\lambda_C$ in either direction causes the value of $\lambda_C$ to increase (i.e., it makes one of the two distances $|A_iA'_i|$ and $|A_jA'_j|$ increase and the other one decrease).

The points on $\partial C$ satisfying the conditions specified in Lemma 2 may be considered as those defining candidate values for $\lambda_C$, i.e., they can be considered as some vertices of possible regular
Lemma 3 The two distances referred to as $d_i$ in at most one point at which the two curves cross each other. (Such a set of curves is sometimes without having to compute these $k$-th highest vertex of $A_F$.)

Proof: We apply a technique, called binary search in sorted arrays \cite{6}, as follows. Given $M$ arrays $A_i, 1 \leq i \leq M$, each containing $O(N)$ elements in sorted order, the task is to find a certain element $\delta \in A = \bigcup_{i=1}^{M} A_i$. Further, assume that there is a “black-box” decision procedure $\Pi$ available, such that given any value $a$, $\Pi$ reports $a \leq \delta$ or $a > \delta$ in $O(T)$ time. An algorithm is given in \cite{6} to find the sought element $\delta$ in $A = \bigcup_{i=1}^{M} A_i$ in $O((M + T) \log(NM))$ time. We use this technique to find $d_k$ and $d_{k+1}$, as follows.

Consider a sensor $A_i$. Let $S(A_i)$ be the set of distances between $A_i$ and all vertices of $P$. In $O(\log n)$ time, we can implicitly partition $S(A_i)$ into two sorted arrays in the following way. By binary search, we can determine an index $j$ such that $X_i$ lies on the arc of $\partial C$ from $P_j$ to $P_{j+1}$ clockwise (the indices are taken as module by $n$). Recall that $X_i$ is the point on $\partial C$ closest to $A_i$. If a vertex of $P$ is on $X_i$, then define $j$ to be the index of that vertex. Similarly, we can determine an index $h$ such that $Y_i$ (i.e., the farthest point on $\partial C$ to $A_i$) lies on the arc from $P_h$ to $P_{h+1}$ clockwise. If a vertex of $P$ is on $Y_i$, then define $h$ to be the index of that vertex. Both $j$ and
h can be determined in \(O(\log n)\) time, after which we implicitly partition \(S(A_i)\) into two sorted arrays: One array consists of all distances from \(A_i\) to \(P_j, P_{j-1}, \ldots, P_{h+1}\), and the other consists of all distances from \(A_i\) to \(P_{j+1}, P_{j+2}, \ldots, P_h\) (again, all indices are taken as module by \(n\)). Note that both these arrays are sorted increasingly and each element in them can be obtained in \(O(1)\) time by using its index in the corresponding array.

Thus, we obtain \(2n\) sorted arrays (represented implicitly) for all \(n\) sensors in \(O(n \log n)\) time, and each array has no more than \(n\) elements. Therefore, by using the technique of binary search in sorted arrays, with our algorithm in Theorem 2 as the black-box decision procedure, both \(d_k\) and \(d_{k+1}\) can be found in \(O(n \log^3 n)\) time. The lemma thus follows.

By applying Lemma 3 we have \(\lambda_C \in (d_k, d_{k+1})\). Below, for simplicity of discussion, we assume \(\lambda_C \neq d_{k+1}\). Thus \(\lambda_C \in (d_k, d_{k+1})\). Since \(\max_{0 \leq h \leq n-1} |A_hX_h| < \lambda_C\), we redefine \(d_k := \max\{d_k, \max_{0 \leq h \leq n-1} |A_hX_h|\}\). We still have \(\lambda_C \in (d_k, d_{k+1})\). Let \(D_2'\) be the set of all critical equal distances in the range \((d_k, d_{k+1})\). Then \(\lambda_C \in D_2'\). We show below that \(|D_2'| = O(n^2)\) and \(\lambda_C\) can be found in \(O(n \log^3 n)\) time without computing \(D_2'\) explicitly.

Suppose we rotate the regular \(n\)-gon \(P = (P_0, P_1, \ldots, P_{n-1})\) on \(\partial C\) clockwise by an arc distance \(2\pi/n\) (this is the arc distance between any two adjacent vertices of \(P\)). Let \(A_i(P_h(t))\) denote the distance function from a sensor \(A_i\) to a vertex \(P_h\) of \(P\) with the time parameter \(t\) during the rotation. Clearly, the function \(A_i(P_h(t))\) increases or decreases monotonically, unless the interval of \(\partial C\) in which \(P_h\) moves contains the point \(X_i\) or \(Y_i\); if that interval contains \(X_i\) or \(Y_i\), then we can further divide the interval into two sub-intervals at \(X_i\) or \(Y_i\), such that \(A_i(P_h(t))\) is monotone in each sub-interval. The functions \(A_i(P_h(t))\), for all \(P_h\)'s of \(P\), can thus be put into two sets \(S_{11}\) and \(S_{12}\) such that all functions in \(S_{11}\) monotonically increase and all functions in \(S_{12}\) monotonically decrease. Let \(m = |S_{11}|\). Then \(m \leq n\). Denote by \(d_1 \leq d_2 \leq \cdots \leq d_m\) the sorted sequence of the initial values of the functions in \(S_{11}\). Also, let \(d_0 = 0\) and \(d_{m+1} = 2\) (recall that the radius of \(C\) is 1). It is easy to see that the range \((d_k, d_{k+1})\) obtained in Lemma 3 is contained in \([d_j, d_{j+1}]\) for some \(0 \leq j \leq m\). The same discussion can be made for the distance functions in the set \(S_{12}\) as well.

Since we rotate \(P\) by only an arc distance \(2\pi/n\), during the rotation of \(P\), each sensor \(A_i\) can have at most two distance functions (i.e., one decreasing and one increasing) whose values may vary in the range \((d_k, d_{k+1})\). We can easily identify these at most \(2n\) distance functions for the \(n\) sensors in \(O(n \log n)\) time. Denote by \(F'\) the set of all such distance functions. Clearly, all critical equal distances in the range \((d_k, d_{k+1})\) can be generated by the functions in \(F'\) during the rotation of \(P\). Because every such distance function either increases or decreases monotonically during the rotation of \(P\), each pair of one increasing function and one decreasing function can generate at most one critical equal distance during the rotation. (Note that by Lemma 2 a critical equal distance cannot be generated by two increasing functions or two decreasing functions.) Since \(|F'| \leq 2n\), the total number of critical equal distances in \((d_k, d_{k+1})\) is bounded by \(O(n^2)\), i.e., \(|D_2'| = O(n^2)\). For convenience of discussion, since we are concerned only with the critical equal distances in \((d_k, d_{k+1})\), for each function in \(F'\), we restrict it to the range \((d_k, d_{k+1})\) only.

Let the time \(t\) be the \(x\)-coordinate and the function values be the \(y\)-coordinates of the plane. Then each function in \(F'\) defines a curve segment that lies in the strip of the plane between the two horizontal lines \(y = d_k\) and \(y = d_{k+1}\). We refer to a function in \(F'\) and its curve segment interchangeably, i.e., \(F'\) is also a set of curve segments. Clearly, a critical equal distance generated by an increasing function and a decreasing function is the \(y\)-coordinate of the intersection point of the two corresponding curve segments. Note that every function in \(F'\) has a simple mathematical description. Below, we simply assume that each function in \(F'\) is of \(O(1)\) complexity. Thus, many operations on them can each be performed in \(O(1)\) time, e.g., computing the intersection of a
decreasing function and an increasing function.

The set $D'_2$ can be computed explicitly in $O(n^2)$ time, after which $\lambda_C$ can be easily found by binary search. Below, we develop a faster solution without computing $D'_2$ explicitly, by utilizing the property that each element of $D'_2$ is the $y$-coordinate of the intersection point of a decreasing function and an increasing function in $F'$ and generalizing the techniques in [8].

A slope selection algorithm for a set of points in the plane was given in [S]. We will extend this approach to solve our problem. The following lemma is needed.

**Lemma 4** For any two increasing (resp., decreasing) functions in $F'$, if the curve segments defined by them are not identical to each other, then the two curve segments intersect in at most one point and they cross each other at their intersection point (if any).

**Proof:** We only prove the decreasing case. The increasing case can be proved similarly. Let $A_i(P_a(t))$ and $A_j(P_b(t))$ be two decreasing curves in $F'$, where $A_i(P_a(t))$ (resp., $A_j(P_b(t))$) is the distance function between the sensor $A_i$ (resp., $A_j$) and the vertex $P_a$ (resp., $P_b$) of the regular $n$-gon $P$, and the two curve segments defined by $A_i(P_a(t))$ and $A_j(P_b(t))$ are not the same. Since each sensor has at most one decreasing function in $F'$, we have $A_i \neq A_j$. We assume that during the (clockwise) rotation of $P$, $A_i(P_a(t)) = A_j(P_b(t))$ at the moment $t = t_1$ and $t_1$ is the first such moment. Below, we prove that $A_i(P_a(t)) = A_j(P_b(t))$ cannot happen again for any $t > t_1$ in the rotation. There are two cases: $P_a = P_b$ and $P_a \neq P_b$.

For any two points $p$ and $q$, let $l(p, q)$ denote the line passing through the two points and $\overline{pq}$ denote the line segment with endpoints $p$ and $q$ whose length is $|pq|$. Recall that $o$ is the center of the circle $C$. Let $P_a(t_1)$ and $P_b(t_1)$ be the positions of $P_a$ and $P_b$ at the moment $t_1$, respectively.

- **$P_a = P_b$.** Clearly, $P_a(t_1) = P_b(t_1)$. Let $l$ be the perpendicular bisector of the line segment $A_iA_j$. At the moment $t_1$, since $|A_iP_a(t_1)| = |A_jP_a(t_1)|$, $P_a(t_1)$ is at one of the two intersection points of $l$ and $\partial C$. Further, since $A_i(P_a(t))$ is a decreasing function, $P_a(t_1)$ must be on the right side of the line $l(A_i, o)$ if we walk from $A_i$ to $o$ (see Fig. 2(a)). Similarly, $P_a(t_1)$ must be on the right side of the line $l(A_j, o)$ (going $A_j$ to $o$). Let $z$ be the other intersection point of $l$ and $\partial C$. It is easy to see that $z$ is on the left side of either the line $l(A_i, o)$ or the line $l(A_j, o)$. Note that $d_k \geq \max_{0 \leq k \leq n-1} |A_kX_k|$. Thus, $d_k \geq |A_kX_k|$ and $d_k \geq |A_jX_j|$. During the rotation of $P$, since both $A_i(P_a(t))$ and $A_j(P_b(t))$ are always larger than $d_k$, $P_a(t)$ cannot pass any of $X_i$ and $X_j$, and thus $P_a(t)$ cannot arrive to the position $z$ during the rotation. Hence, $A_i(P_a(t)) = A_j(P_b(t))$ cannot happen again after $t_1$.

Further, recall that $t_1$ is the first moment from the beginning of the rotation with $A_i(P_a(t)) = A_j(P_b(t))$. Without loss of generality, we assume $A_i(P_a(t)) < A_j(P_b(t))$ for any time $t < t_1$ (as the example shown in Fig. 2(a)). It is easy to see that $A_i(P_a(t)) > A_j(P_b(t))$ for any time $t > t_1$, which implies that the two functions cross each other at their intersection point.

- **$P_a \neq P_b$.** At the moment $t_1$, we have $|A_iP_a(t_1)| = |A_jP_b(t_1)|$. Assume to the contrary that at some moment $t_2 > t_1$, we also have $A_i(P_a(t_2)) = A_j(P_b(t_2))$. Suppose at the moment $t_2$, $P_a(t_2)$ is at the position $P'_a$ and $P_b(t_2)$ is at the position $P'_b$ (see Fig. 2(b)). Then $|A_iP'_a| = |A_jP'_b|$. Since $P_a$ and $P_b$ are rotated simultaneously, the arc distance from $P_a(t_1)$ to $P'_a$ is equal to the arc distance from $P_b(t_1)$ to $P'_b$, and thus $|P_a(t_1)P'_a| = |P_b(t_1)P'_b|$. Consider the two triangles $\triangle P_b(t_1)A_jP'_a$ and $\triangle P_a(t_1)A_jP'_b$ (shown with red solid segments in Fig. 2(b)). Since $|A_iP_a(t_1)| = |A_jP_b(t_1)|$, $|A_iP'_a| = |A_jP'_b|$, and $|P_a(t_1)P'_a| = |P_b(t_1)P'_b|$, $\triangle P_a(t_1)A_jP'_a$ is congruent to $\triangle P_b(t_1)A_jP'_b$. Thus, the two angles $\angle A_iP_a(t_1)P'_a = \angle A_jP_b(t_1)P'_b$. Further, it is easy to see $\angle oP_b(t_1)P'_a = \angle oP_a(t_1)P'_b$. Consequently, we have $\angle oP_a(t_1)A_i = \angle oP_b(t_1)A_j$. 

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But, if $\angle oP_a(t_1)A_i = \angle oP_b(t_1)A_j$, then we can show that the two functions $A_i(P_a(t))$ and $A_j(P_b(t))$ define exactly the same curve segment. The proof is nothing but the inverse of the above argument. Specifically, consider any time moment $t_3 > t_1$ before the end of the rotation. Suppose at the moment $t_3$, $P_a$ is at the position $P''_a$ and $P_b$ is at the position $P''_b$. Since $\angle oP_a(t_1)A_i = \angle oP_b(t_1)A_j$ and $\angle oP_a(t_1)P''_a = \angle oP_b(t_1)P''_b$, we have $\angle A_iP_a(t_1)P''_a = \angle A_jP_b(t_1)P''_b$. Further, since $|A_iP_a(t_1)| = |A_jP_b(t_1)|$ and $|P_a(t_1)P''_a| = |P_b(t_1)P''_b|$, $\triangle P_b(t_1)A_jP''_b$ is congruent to $\triangle P_a(t_1)A_iP''_a$. Thus, $|A_iP''_a| = |A_jP''_b|$, i.e., $A_i(P_a(t)) = A_j(P_b(t))$ at any time $t = t_3 > t_1$. Similarly, we can also show that at any time moment $t_3 < t_1$, $A_i(P_a(t_3)) = A_j(P_b(t_3))$. Hence, $A_i(P_a(t))$ and $A_j(P_b(t))$ define exactly the same curve segment. But this contradicts with the fact that the curve segments defined by these two functions are not the same. This implies that $A_i(P_a(t)) = A_j(P_b(t))$ cannot happen again at any moment $t > t_1$.

Further, without loss of generality, we assume $A_i(P_a(t)) < A_j(P_b(t))$ for any time $t < t_1$ (as the example shown in Fig. 2(b)). We then show that $A_i(P_a(t_3)) > A_j(P_b(t_3))$ for any time $t_3 > t_1$, which means that the two functions cross each other at their intersection point. We briefly discuss this. Again, suppose at the moment $t_3$, $P_a$ is at the position $P''_a$ and $P_b$ is at the position $P''_b$. First, since $A_i(P_a(t)) < A_j(P_b(t))$ for any time $t < t_1$, it must be $|oA_j| > |oA_i|$ (this can be proved by similar techniques as above and we omit the details). Consider the two triangles $\triangle oA_iP_a(t_1)$ and $\triangle oA_jP_b(t_1)$ (at the moment $t_1$). Since $|A_iP_a(t_1)| = |A_jP_b(t_1)|$, $|oP_a(t_1)| = |oP_b(t_1)|$, and $|oA_j| > |oA_i|$, we have $\angle oP_b(t_1)A_j > \angle oP_a(t_1)A_i$, which further implies $\angle A_jP_b(t_1)P''_b > \angle A_iP_a(t_1)P''_a$. Consider the triangles $\triangle P_b(t_1)A_jP''_b$ and $\triangle P_a(t_1)A_iP''_a$. Due to $|A_iP_a(t_1)| = |A_jP_b(t_1)|$, $|P_a(t_1)P''_a| = |P_b(t_1)P''_b|$, and $\angle A_jP_b(t_1)P''_b > \angle A_iP_a(t_1)P''_a$, it must be $|A_jP''_b| < |A_iP''_a|$. In other words, $A_i(P_a(t)) > A_j(P_b(t))$ at any time $t = t_3 > t_1$.

The lemma thus follows.
the vertices of $A_F$ explicitly. By generalizing some techniques in [8], we have the following lemma.

**Lemma 5** The value $|A_F|$ can be computed in $O(n \log n)$ time. Given an integer $k$ with $1 \leq k \leq |A_F|$, the $k$-th highest vertex of $A_F$ can be found in $O(n \log^2 n)$ time.

**Proof:** First of all, because every function in $F'$ is of $O(1)$ complexity, we can determine in $O(1)$ time whether a curve segment in $F'$ intersects a given horizontal line, and if “yes”, then compute the intersection. Thus, for every curve in $F$, we can also compute its intersection with any horizontal line in $O(1)$ time. Let $N = |F| \leq 2n$.

Recall that the curve segments in $F'$ are all in the horizontal strip between $y = d_k$ and $y = d_{k+1}$. Thus, all vertices of $A_F$ above the horizontal line $y = d_k$ are intersections of the newly attached half-lines. We can easily determine the highest vertex of $A_F$ in $O(n \log n)$ time, e.g., by using the approach in [8]. Let $l$ be a horizontal line higher than the highest vertex. Denote by $f_1, f_2, \ldots, f_N$ the sequence of the curves of $F$ sorted in increasing order of the $x$-coordinates of their intersections with $l$. Similarly, we can determine the lowest vertex of $A_F$ in $O(n \log n)$ time. Let $f_{\pi(1)}, f_{\pi(2)}, \ldots, f_{\pi(N)}$ be the sequence of the curves of $F$ sorted in increasing order of the $x$-coordinates of their intersections with a horizontal line below the lowest vertex of $A_F$. Since the curves in $F$ can be viewed as a set of pseudolines, as in [8], the number of inversions in the permutation $\pi$, which can be computed in $O(n \log n)$ time, is equal to $|A_F|$. In summary, we can compute $|A_F|$ in $O(n \log n)$ time.

To compute the $k$-th highest vertex of $A_F$, we choose to generalize the $O(n \log^2 n)$ time algorithm in [8]. Let $L$ be a set of $n$ lines in the plane and $A_L$ be the arrangement of $L$. An $O(n \log^2 n)$ time algorithm was given in [8] for computing the $k$-th highest vertex of $A_L$ ($1 \leq k \leq |A_L|$) in $O(n \log^2 n)$ time based on parametric search [7, 15]. The main property used in the algorithm [8] is the following one. Denote by $l_1, l_2, \ldots, l_n$ the sequence of lines in $L$ sorted in increasing order of their intersections with a horizontal line above the highest vertex of $A_L$. Given any horizontal line $l'$, let $l_{\pi(1)}, l_{\pi(2)}, \ldots, l_{\pi(n)}$ be the sequence of lines of $L$ sorted in increasing order of their intersections with $l'$. Then, the number of vertices of $A_L$ above $l'$ is equal to the number of inversions in the permutation $\pi$.

In our problem, since any two curves in $F$ can intersect each other in at most one point and they cross each other at their intersection point, the above property still holds for $A_F$. Thus, the $O(n \log^2 n)$ time algorithm in [8] is applicable to our problem. Therefore, we can find the $k$-th highest vertex of $A_F$ in $O(n \log^2 n)$ time, and the lemma follows. □

A remark: An optimal $O(n \log n)$ time algorithm was also given in [8] (and in [12]) for finding the $k$-th highest vertex of $A_L$. However, these algorithms are overly complicated. Although we think that the $O(n \log n)$ time approach in [8] may be made work for our problem, it does not benefit our overall solution for the optimization version of the min-sum problem because its total time is dominated by other parts of the algorithm. Hence, the much simpler $O(n \log^2 n)$ time solution (for finding the $k$-th highest vertex of $A_F$) suffices for our purpose.

Recall that $\lambda_C$ is the $y$-coordinate of a vertex of $A_F$. Our algorithm for computing $\lambda_C$ then works as follows. First, compute $|A_F|$. Next, find the $(|A_F|/2)$-th highest vertex of $A_F$, and denote its $y$-coordinate by $\lambda_m$. Determine whether $\lambda_C \leq \lambda_m$ by the algorithm in Theorem 2, after which one half of the vertices of $A_F$ can be pruned away. We apply the above procedure recursively on the remaining vertices of $A_F$, until $\lambda_C$ is found. Since there are $O(\log n)$ recursive calls to this procedure, each of which takes $O(n \log^2 n)$, the total time for computing $\lambda_C$ is $O(n \log^3 n)$.

**Theorem 3** The min-max optimization problem is solvable in $O(n \log^3 n)$ time.
4 The Min-sum Problem

In this section, we present our new algorithms for the min-sum problem. We show that the boundary case of this problem is solvable in $O(n^2)$ time, which improves the $O(n^4)$ time result in [17]. We also give an $O(n^2)$ time approximation algorithm with approximation ratio 3, which improves the $(1 + \pi)$-approximation $O(n^2)$ time algorithm in [2].

For the boundary case, the $O(n^4)$ time algorithm in [17] uses the $O(n^3)$ time Hungarian algorithm to compute a minimum weight perfect matching in a complete bipartite graph. However, the graph for this case is very special in the sense that all its vertices lie on the boundary of a circle. By using the result in [4], we can actually find a minimum weight perfect matching in such a graph in $O(n)$ time. Therefore, if we follow the algorithmic scheme in [17] but replace the Hungarian algorithm by the algorithm in [4], the boundary case can be solved in $O(n^2)$ time. For completeness, more details on the proof of the following theorem are given in Appendix [3].

**Theorem 4** The boundary case of the min-sum problem can be solved in $O(n^2)$ time.

Next, we discuss our approximation algorithm for the general min-sum problem.

Let $A_0, \ldots, A_{n-1}$ be the sensors in $C$. Our approximation algorithm works as follows. (1) For each sensor $A_i$, $i = 0, \ldots, n-1$, compute the point $X_i$ on $\partial C$ that is closest to $A_i$. (2) By using the algorithm in Theorem 4 solve the following min-sum boundary case problem: Viewing the $n$ points $X_0, X_1, \ldots, X_{n-1}$ as pseudo-sensors (which all lie on $\partial C$), find $n$ points on $\partial C$ as the goal positions for the pseudo-sensors such that the sum of the distances traveled by all $n$ pseudo-sensors is minimized. Let $X'_i$ be the goal position for each $X_i$ ($0 \leq i \leq n-1$) in the optimal solution thus obtained. We then let $X'_i$ be the goal position for each sensor $A_i$, $0 \leq i \leq n-1$, for our original min-sum problem. This completes the description of our approximation algorithm.

Clearly, with Theorem 4 the time complexity of the above approximation algorithm is $O(n^2)$. The lemma below shows that the approximation ratio of this algorithm is 3.

**Lemma 6** The approximation ratio of our approximation algorithm is 3.

**Proof:** Let $\Delta = \sum_{i=0}^{n-1} |A_iX'_i|$. Let $A'_0, A'_1, \ldots, A'_{n-1}$ be the goal positions of all sensors (i.e., $A'_i$ is the goal position for each sensor $A_i$, $0 \leq i \leq n-1$) in an optimal solution for the min-sum problem. Let $\Delta_C = \sum_{i=0}^{n-1} |A_iA'_i|$. Our task is to prove $\Delta \leq 3 \cdot \Delta_C$.

First, $\sum_{i=0}^{n-1} |X_iX'_i| \leq \sum_{i=0}^{n-1} |X_iA'_i|$, and $|A_iX_i| \leq |A_iA'_i|$ holds for each $0 \leq i \leq n-1$. Then,

\[
\Delta = \sum_{i=0}^{n-1} |A_iX'_i| \leq \sum_{i=0}^{n-1} (|A_iX_i| + |X_iX'_i|) \quad \text{(triangle inequality)}
\]

\[
= \sum_{i=0}^{n-1} |A_iX_i| + \sum_{i=0}^{n-1} |X_iX'_i| \leq \sum_{i=0}^{n-1} |A_iX_i| + \sum_{i=0}^{n-1} |X_iA'_i|
\]

\[
\leq 2 \cdot \sum_{i=0}^{n-1} |A_iX_i| + \sum_{i=0}^{n-1} |A_iA'_i| \quad \text{(triangle inequality)}
\]

\[
\leq 3 \cdot \sum_{i=0}^{n-1} |A_iA'_i| = 3 \cdot \Delta_C.
\]

The lemma thus follows. Hence, we conclude with the following result.
Theorem 5 There exists an $O(n^2)$ time approximation algorithm for the min-sum problem with approximation ratio $3$.

References

[1] I. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci. Wireless sensor networks: A survey. *Computer Networks*, 38(4):393–422, 2002.

[2] B. Bhattacharya, B. Burmester, Y. Hu, E. Kranakis, Q. Shi, and A. Wiese. Optimal movement of mobile sensors for barrier coverage of a planar region. *Theoretical Computer Science*, 410(52):5515–5528, 2009.

[3] G. Brodal, L. Georgiadis, K.A. Hansen, and I. Katriel. Dynamic matchings in convex bipartite graphs. In *Proc. of the 32nd International Symposium on Mathematical Foundations of Computer Science*, volume 4708 of *Lecture Notes in Computer Science*, pages 406–417. Springer-Verlag, 2007.

[4] S. Buss and P. Yianilos. Linear and $O(n \log n)$ time minimum-cost matching algorithms for quasi-convex tours. *SIAM Journal on Computing*, 27(1):170–201, 1998.

[5] A. Chen, S. Kumar, and T. Lai. Designing localized algorithms for barrier coverage. In *Proc. of the 13th Annual ACM International Conference on Mobile Computing and Networking*, pages 63–73, 2007.

[6] D.Z. Chen, C. Wang, and H. Wang. Representing a functional curve by curves with fewer peaks. *Discrete and Computational Geometry*, 46(2):334–360, 2011.

[7] R. Cole. Slowing down sorting networks to obtain faster sorting algorithms. *Journal of the ACM*, 34(1):200–208, 1987.

[8] R. Cole, J. Salowe, W. Steiger, and E. Szemerédi. An optimal-time algorithm for slope selection. *SIAM Journal on Computing*, 18(4):792–810, 1989.

[9] H. Gabow and R. Tarjan. A linear-time algorithm for a special case of disjoint set union. *Journal of Computer and System Sciences*, 30:209–221, 1985.

[10] S. Hu. ‘Virtual Fence’ along border to be delayed. *Washington Post*, February 28, 2008.

[11] W. Lipski Jr. and F.P. Preparata. Efficient algorithms for finding maximum matchings in convex bipartite graphs and related problems. *Acta Informatica*, 15(4):329–346, 1981.

[12] M. Katz and M. Sharir. Optimal slope selection via expanders. *Information Processing Letters*, 47(3):115–122, 1993.

[13] S. Kumar, T. Lai, and A. Arora. Barrier coverage with wireless sensors. *Wireless Networks*, 13(6):817–834, 2007.

[14] Y. Liang and N. Blum. Circular convex bipartite graphs: Maximum matching and Hamiltonian circuits. *Information Processing Letters*, 56:215–219, 1995.

[15] N. Megiddo. Applying parallel computation algorithms in the design of serial algorithms. *Journal of the ACM*, 30(4):852–865, 1983.

[16] G. Steiner and J. Yeomans. A linear time algorithm for maximum matchings in convex, bipartite graphs. *Computers and Mathematics with Applications*, 31(2):91–96, 1996.

[17] X. Tan and G. Wu. New algorithms for barrier coverage with mobile sensors. In *Proc. of the 4th International Workshop on Frontiers in Algorithmics*, volume 6213 of *Lecture Notes in Computer Science*, pages 327–338. Springer-Verlag, 2010.
Appendix

A The Proof of Lemma 2

Lemma 2 [17] Suppose an optimal solution for the min-max optimization problem is achieved with \( \lambda_C = |A_iA'_i| \) for some \( i \in \{0, \ldots, n-1\} \). Then either \( A'_i \) is the point \( X_i \), or there is another sensor \( A_j \) (\( j \neq i \)) such that \( \lambda_C = |A_jA'_j| \) also holds. In the latter case, any slight rotation of the regular \( n \)-gon that achieves \( \lambda_C \) in either direction causes the value of \( \lambda_C \) to increase (i.e., it makes one of the two distances \( |A_iA'_i| \) and \( |A_jA'_j| \) increase and the other one decrease).

Proof: First assume that in an optimal solution, the sensor \( A_i \) is the only one satisfying \( \lambda_C = |A_iA'_i| \), but \( A'_i \) is not the point \( X_i \). Thus \( |A_iX_i| < |A_iA'_i| \) holds (see Figure 1(a)). Then, we rotate the regular \( n \)-gon that achieves \( \lambda_C \) by moving the vertex \( A'_i \) towards \( X_i \), with a very small distance \( \epsilon \). Clearly, the distance function between \( A_i \) and \( A'_i \) decreases monotonically during this rotation of that \( n \)-gon. Denote by \( A''_0', A''_1', \ldots, A''_{n-1} \) the new positions of the sensors after the rotation stops. Since \( \epsilon \) is arbitrarily small and \( A_i \) is the only sensor satisfying \( \lambda_C = |A_iA'_i| \), we have \( |A_iA''_k'| \geq |A_kA''_k| \) for all \( k \neq i \); moreover, \( |A_iA''_i| < |A_iA'_i| \) holds. But, this contradicts with the assumption that \( \lambda_C = |A_iA'_i| \) gives an optimal solution to the min-max optimization problem.

Suppose now there exists another sensor \( A_j \) such that the optimal value \( \lambda_C = |A_jA'_j| \) (\( j \neq i \)) also holds (see Figure 1(b)). A slight rotation of the regular \( n \)-gon that achieves \( \lambda_C \) in either direction cannot make \( |A_iA''_k'| < |A_iA'_i| \) and \( |A_jA''_j| < |A_jA'_j| \) both occur, where \( A''_0', A''_1', \ldots, A''_{n-1} \) are the new positions of \( A'_i \) and \( A'_j \) after the rotation stops (otherwise, it would contradict with the assumption that \( \lambda_C = |A_iA'_i| = |A_jA'_j| \) gives an optimal solution). Hence, the rotation of the regular \( n \)-gon that achieves \( \lambda_C \) increases one of the two distances \( |A_iA'_i| \) and \( |A_jA'_j| \), while decreasing the other. The proof is thus complete.

B The Proof of Theorem 4

Recall that in the boundary case of the min-sum problem, all sensors are on the boundary \( \partial C \) of \( C \). Let \( A_0, A_1, \ldots, A_{n-1} \) denote the initial positions of the \( n \) sensors on \( \partial C \), and \( A'_0, A'_1, \ldots, A'_{n-1} \) denote their goal positions on \( \partial C \) that form a regular \( n \)-gon. Denote by \( \Delta_C \) the sum of the distances traveled by all \( n \) sensors in an optimal solution of the min-sum problem, i.e., \( \Delta_C = \min \sum_{i=0}^{n-1} |A_iA'_i| \). The following lemma has been proved in [17].

Lemma 7 [17] There exists an optimal solution for the boundary case of the min-sum problem with the following property: There exists a sensor \( A_i \) which does not move, i.e., \( A_i = A'_i \).

Based on Lemma 7 the boundary case can be solved as follows. For each sensor \( A_i \), \( 0 \leq i \leq n-1 \), let \( P(A_i) \) be the regular \( n \)-gon on \( \partial C \) such that \( A_i \) is one of its vertices. Denote by \( H_i \) the complete bipartite graph between the set of all sensors and the set of all vertices of \( P(A_i) \) such that the weight of an edge connecting a sensor and a vertex of \( P(A_i) \) is defined as their Euclidean distance. We compute a minimum weight perfect matching \( M_i \) in \( H_i \), for each \( 0 \leq i \leq n-1 \), and the one that gives the minimum weight defines an optimal solution for our original problem. Here, the weight of a perfect matching is the sum of all edge weights of the matching.

The running time of the above algorithm is dominated by the step of computing the minimum weight perfect matchings in the graphs \( H_i \). The algorithm in [17] uses the \( O(n^3) \) time Hungarian algorithm for computing such matchings in the graphs \( H_i \).
Let $H$ be a complete bipartite graph with two vertex sets of cardinalities $n_1$ and $n_2$, respectively, such that all its vertices lie on the boundary of a circle and each edge weight is the Euclidean distance between two such vertices (the edges are represented implicitly). A maximum cardinality matching of $H$ consists of $\min\{n_1, n_2\}$ edges. An algorithm was given in [4] for computing a minimum weight maximum cardinality matching in $H$ in $O(n_1 + n_2)$ time (i.e., the total sum of edge weights in the output maximum cardinality matching is as small as possible).

Since in our algorithm, all vertices of every complete bipartite graph $H_i$ lie on $\partial C$, the linear time algorithm in [4] can be applied to compute a minimum weight maximum cardinality matching of $H_i$ in $O(n)$ time, for $0 \leq i \leq n - 1$. (Note that a maximum cardinality matching in the graph $H_i$ is a perfect matching, and vice versa.) Consequently, the total running time of our algorithm is $O(n^2)$. Theorem 4 thus follows.