Mesoscopic Effects in the Fractional Quantum Hall Regime: Chiral Luttinger Liquid versus Fermi Liquid

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We study tunneling through an edge state formed around an antidot in the fractional quantum Hall regime using Wen’s chiral Luttinger liquid theory extended to include mesoscopic effects. We identify a new regime where the Aharonov-Bohm oscillation amplitude exhibits a distinctive crossover from Luttinger liquid power-law behavior to Fermi-liquid-like behavior as the temperature is increased. Near the crossover temperature the amplitude has a pronounced maximum. This nonmonotonic behavior and novel high-temperature nonlinear phenomena that we also predict provide new ways to distinguish experimentally between Luttinger and Fermi liquids.

One of the most important outstanding questions in the study of the quantum Hall effect [1] concerns the nature of the transport in the fractional regime. It has been established that for integral filling factors, many aspects of the quantum Hall effect can be understood in terms of Halperin’s edge states of the two-dimensional noninteracting electron gas [2], and a useful description of this is provided by the Büttiker-Landauer formalism [3]. However, as was shown by Laughlin [4], the fractional quantum Hall effect (FQHE) occurs because interactions lead to the formation of highly correlated incompressible states at certain filling factors. In a large class of one-dimensional systems, interactions lead to a breakdown of Fermi liquid theory and to the formation of a Luttinger liquid with bosonic low-energy excitations [5,6]. Transport in a macroscopic Luttinger liquid was studied by Kane and Fisher [7], who have shown that the conductance in the presence of a weak impurity vanishes in the zero-temperature limit, in striking contrast to a Fermi liquid. The important connection between Luttinger liquids and the FQHE was made by Wen [8], who used Chern-Simons theory to show that the edge states there should be chiral Luttinger liquids (CLL). Wen’s proposal has stimulated a considerable effort to understand the properties of this exotic non-Fermi-liquid state [8–14].

The first observation of a CLL was reported by Milliken, Umbach, and Webb [15]. They measured the tunneling between FQHE edge states in a quantum-point-contact geometry. As the gate voltage was varied, resonance peaks in the conductance were observed that have the correct AB oscillation amplitude to vanish with temperature as $T^{2^q-2}$, in striking contrast with chiral Fermi liquid theory ($q = 1$). For $T$ near $T_0$, there is a pronounced maximum in the AB amplitude, also in contrast to a Fermi liquid. At high temperatures ($T \gg T_0$) we predict a crossover to a $T^{2q-1} e^{-q^2/2T}$ temperature dependence, which is qualitatively similar to chiral Fermi liquid behavior. Experiments in the strong-antidot-coupling regime should be able to distinguish between a chiral Fermi liquid and our predicted nearly Fermi liquid scaling. Finally, we predict a remarkable high-temperature nonlinear response regime, where the voltage $V$ satisfies $V > T > T_0$, which may also be used to distinguish between chiral Fermi liquid and CLL behavior.

To study mesoscopic effects associated with FQHE edge states, we extend CLL theory to include finite-size effects. Finite-size effects in nonchiral Luttinger liquids have been addressed in Refs. [6] and [19]. We
b Bosonize the electron field operators \( \psi \) according to the
convention \( \rho = \pm \partial_x \psi /2\pi \), where \( \rho \) is the
normal-ordered charge density and \( \phi \) is a chiral scalar field for
right (+) or left (−) movers. The dynamics is governed
by Wen’s Euclidean action [8]

\[
S = \frac{1}{4\pi g} \int_0^L dx \int_0^\beta d\tau \partial_x \phi (\pm i \partial_x \phi + v \partial_x \phi),
\]

where \( g = 1/q \) (with \( q \) odd) is the bulk filling factor. Here
\( L \) is the length of a given edge state. The field
theory (1) can be canonically quantized by imposing the
equal-time commutation relation

\[
[\phi(x), \phi(x')] = \pm i \pi g \text{ sgn}(x - x').
\]

We then decompose \( \phi \) into a nonzero-mode contribution \( \phi_0 \)
satisfying periodic boundary conditions that describes
the neutral excitations, and a zero-mode part \( \phi_0 \) that
contributes to the charged excitations, \( \phi = \phi_0 + \phi_0 \).
The nonzero-mode contribution may be expanded in a
basis of Bose annihilation and creation operators in the usual fashion, \( \phi_0^+ (x) = \sum_{k \neq 0} \theta (k) \times \sqrt{2 / \pi g / k} L (a_k \ee^{ikx} + a_k^\dagger \ee^{-ikx}) - e^{-|a|^2 / 4} \), with coefficients determined by the requirement that \( \phi_0^+ \) itself satisfies (2) as \( L \to \infty \). In a finite-size system, however,
\[
[\phi_0^+(x), \phi_0^+(x')] = \pm i \pi g \text{ sgn}(x - x') - \frac{2}{\pi g \text{ i}g}(x - x')/L.
\]
An occupation-number expansion for \( \phi_0 \) is constructed from (3)
and the requirement

\[
\phi_0^+(x + L) - \phi_0^+(x) = \pm 2\pi N_x,
\]

which follows from the bosonized expression for \( \rho \),
where \( N_x = \int_0^L dx \rho \) is the charge of an excited state
relative to the ground state. Conditions (3) and (4)
together determine \( \phi_0^+ \), up to an additive c-number constant,
as \( \phi_0^+(x) = \pm 2\pi N_x x/L - g \chi_x \), where \( \chi_x \) is an
Hermitian phase operator canonically conjugate to \( N_x \)
satisfying \( \{N_x, N_x\} = i \). The normal-ordered Hamiltonian may then be
written as

\[
H = \frac{v}{4\pi g} \int_{-L}^L (\partial_x \phi)^2
= \frac{\pi v}{gL} N_x^2 + \sum_k \theta (\pm k) v |k| a_k^\dagger a_k.
\]

What are the allowed eigenvalues of \( N_x \)? The answer may be determined by bosonization: To create an
electron, we need a \( \pm 2\pi \epsilon \) kink in \( \phi \). The electron
field operators can therefore be bosonized as \( \psi_\epsilon(x) = (2\pi a)^{-1/2} e^{i(\phi_\epsilon(x) + \pi x/L)}/g \). The c-number phase factor
is chosen for convenience. This implies \( \psi_\epsilon(x + L) = \psi_\epsilon(x)e^{2\pi i N_x}/g \). Periodic boundary conditions on \( \psi_\epsilon \) then lead to the important result that the allowed eigenvalues are given by \( N_x = ng \), which means that there exists
fractionally charged excitations, as expected in a FQHE
system.

Coupling to an additional AB flux \( \Phi \) is achieved by adding \( \int_0^L \partial_x \phi \) to the Lagrangian, where \( \phi = 0 \)
is the bosonized current density and \( A \) is a vector potential. The flux couples only to the zero modes, and results in the replacement \( N_x^2 \to (N_x \pm g \Phi/\Phi_0)^2 \) in
\( H \), where \( \Phi_0 = hc/e \). The grand-canonical partition
function of the mesoscopic edge state factorizes into a zero-mode contribution, \( Z^0 = \sum_0 e^{-\beta g N_x / |\Phi_0|^2} / |L_1|^2 \), and a flux-independent contribution from the nonzero modes
[20]. Note that if the \( N_x \) were restricted to be integral, then the partition function and the associated grand-canonical potential would be periodic functions of flux with period \( \Phi_0/g \). The fractionally charged excitations
are therefore responsible for restoring the AB period to the
proper value \( \Phi_0 \), as is known in other contexts [21].

We begin our study of transport by performing a perturbative
renormalization group (RG) analysis in the weak-
 antidot-coupling regime [see Fig. 1(a)]. In this case, \( S = S_0 + \delta S \), where \( S_0 = S_A + S_B + S_{\text{anti}} \) is the sum of actions of the form (1) for the left moving, right moving, and
antidot edge states, respectively, and \( \delta S = \sum_m \int x \left( V_+ + V_- + \text{c.c.} \right) \) is the weak coupling between them. Here \( V_{\pm}(x) =
\int_{-x}^{x} e^{i(|\Phi_0| \pm m \Phi)/2} / |Z|^2 \ee^{|x|} \) describes the tunneling
of \( m \) quasiparticles from an incident edge state into the
antidot edge state at point \( x \), with dimensionless amplitude
\( \Gamma(m) [7] \). We assume the leads, described by \( S_A \) and \( S_B \), to
be macroscopic, and we also assume for simplicity that
\( |\Gamma(m)| = |\Gamma(m)| \). We shall need the \( m \)-quasiparticle propagator
\( \langle e^{i|\Phi_0| \pm m \Phi}/|Z|^2 \ee^{|x|} \) taken with respect to \( S_0 \), which,
at \( T = 0 \) and for values of \( x \) such that \( x \ll L \), is given by
\( \left[ \pm ia/(x \pm i \nu \tau \pm ia \text{ sgn} \tau) \right]^{2\Delta} \), where \( \Delta = m^2 g/2 \)
is the scaling dimension of \( \ee^{|\Phi_0|} \).

FIG. 1. (a) Aharonov-Bohm effect geometry in the weak-
 antidot-coupling regime. The solid lines represent edge
states and the dashed lines denote weak tunneling points.
(b) Edge-state configuration in the strong-antidot-coupling
regime. Here the edge states are almost completely reflected.
(c) Temperature dependence of \( G_{AB} \) for the cases \( q = 1 \)
(dashed curve) and \( q = 3 \) (solid curve). Each curve is
normalized to have unit amplitude at its maximum.
Consider now the correlation function
\[
\langle V_+^1(\tau) V_\pm(0) \rangle = \left| \frac{\Gamma_{\phi}^{(m)}}{4\pi^2 a^2} \right|^2 \left( e^{-i\phi_{t}(x,\tau)} e^{i\phi_{t}(x,0)} \right) 
\times \langle e^{i\phi_{t}(x,\tau)} e^{-i\phi_{t}(x,0)} \rangle,
\]
which arises in a perturbative calculation of the total partition function \( Z = \int D\phi_t D\phi_R D\phi_A e^{-S} \). For \( Z \) to be invariant under a small increase in unit-cell size \( a \to a' = b a \), we need \( \Gamma' = b^{1-2\delta} \Gamma \), or \( \Gamma^{(m)}_{\phi}/\ln b = (1 - m^2 g) \Gamma^{(m)}_{\phi} \). These flow equations, which show that quasiparticle \((m = 1)\) backscattering processes are relevant and electron \((m = 1/2)\) backscattering is irrelevant when \( g = 1/3 \), were first derived by Kane and Fisher [7] using momentum-shell RG. Next consider the fourth order term \( \langle V_+^1(\tau) V_\pm(\tau') V_\pm(\tau'') V_\pm(0) \rangle \). A Wick expansion gives local terms as in (5), and, in addition, nonlocal antidot propagators like \( \langle e^{i\phi_{t}(x,\tau)} e^{-i\phi_{t}(x,0)} \rangle \) with \( x \neq 0 \). However, the nonlocal terms scale in the same way as the local terms. The Kane-Fisher flow equations are therefore valid in the antidot problem considered here.

This scaling analysis shows that off resonance [22] and at low enough temperatures the antidot will be in the strongly coupled regime shown in Fig. 1(b). Furthermore, if the antidot system starts in the strongly coupled regime, by an appropriate choice of gate voltages, it will stay in this regime because the \( m = 1 \) quasiparticle backscattering process (which would be relevant in the RG sense) is not allowed in this edge-state configuration and only electrons can tunnel. The strongly-coupled regime therefore admits a perturbative treatment [23], to which we now turn. Details shall be given elsewhere.

The current passing between edge states \( L' \) and \( R' \), driven by their potential difference \( V \), is defined by (restoring units) \( \Gamma = -e(N_{L'}(t)) \), where \( N_{L'} \) is the charge of edge state \( L' \) as defined above. The current is now evaluated for small tunneling amplitudes \( \Gamma_{i,j} \), for which simplicity are taken to be equal apart from AB phase factors. The result is \( \Gamma = 4\Gamma^2 T^3 \sinh^{-1}(T/T_0) \sinh^{-1}(V/2T_0) \). The corresponding conductances are \( G_{AB} = \frac{\Gamma^2}{\pi} \) and \( G_{AB}^{FL} = \frac{\Gamma^2}{\pi} T_0 \sin^{-1}(T/T_0) \).

The exact current-voltage relation for the \( q = 3 \) CLL is \( I_0 = (\Gamma^2/120\pi^2 V^3) (64\pi^4 T^4 V + 20\pi^2 T^2 V^3 + V^5) \), and

\[
I_{AB} = \frac{\Gamma^2 a^4 \pi^2}{\nu^6} \frac{T^3}{\sinh^2(T/T_0)} \left[ \frac{T}{T_0} \left( 1 - 3 \coth^2 \left( \frac{T}{T_0} \right) \right) \sin \left( \frac{V}{2\pi T_0} \right) + 6\pi V T \coth \left( \frac{T}{T_0} \right) \cos \left( \frac{V}{2\pi T_0} \right) \right].
\]

In the limit \( L \to 0 \), \( I_{AB} \) reduces to \( I_0 \). The AB conductance is

\[
G_{AB} = -\frac{2\pi^2}{\nu^6} \frac{T^4}{\sinh^2(T/T_0)} \left[ 3 \coth \left( \frac{T}{T_0} \right) + \left( \frac{T}{T_0} \right) \left[ 1 - 3 \coth^2 \left( \frac{T}{T_0} \right) \right] \right],
\]

which is shown in Fig. 1(c) along with the corresponding Fermi-liquid result.

The complete phase diagram is very rich and will be described in detail elsewhere. Here we shall summarize the transport properties for general \( q \) as a function of temperature for fixed voltage, first for \( V < T_0 \) and then for \( V > T_0 \).

\[
I_{AB} = \frac{-i\theta(t) \langle [B_i(t), B_j^d(0)] \rangle}{\nu^6} \text{ and } B_i = \psi_L(x_i) \psi_R^d(x_i) \text{ is an electron tunneling operator acting at point } x_i.
\]

This response function can be calculated using bosonization techniques, and the result for filling factor \( 1/q \) is \( X_{ij}(t) = -\theta(t) (a\pi)^{2q-2} \text{Im} [\sin^{-q}(y_+) \sin^{-q}(y_-)]/2L^2_T \), where \( L_T = \beta v \) is the thermal length and \( y_{\pm} = \pi(x_i - x_j \pm vt + ia)/L_T \). Each term \( X_{ij} \) in \( I \) corresponds to a process occurring with a probability \( \sim \Gamma_{i,j} \). The local terms \( X_{1i} = X_{2i} \) therefore describe independent tunneling at \( x_1 \) and \( x_2 \), respectively, whereas the nonlocal terms \( X_{12} = X_{21} \) describe coherent tunneling through both points. The AB phase naturally couples only to the latter. We shall see that the local contributions behave exactly like the tunneling in a quantum point contact. The AB effect, however, is a consequence of the nonlocal terms which lead to new non-Fermi-liquid mesoscopic effects.

We have Fourier transformed \( X_{ij}(t) \) exactly and find a crossover behavior in the nonlocal response functions when the thermal length \( L_T \) becomes less than \( |x_i - x_j| \).

The finite size of the antidot therefore provides an important new temperature scale as defined above. Note that \( T_0 \) is closely related to the energy level spacing \( \Delta \epsilon = 2\pi v/L \) for noninteracting electrons with linear dispersion in a ring of circumference \( L: T_0 = \Delta \epsilon/2\pi^2 \). The current in the strong-antidot-coupling regime can generally be written as \( I = I_0 + I_{AB} \cos(2\pi \Phi/\Phi_0) \), where \( I_0 \) is the direct contribution resulting from the local terms and \( I_{AB} \) is the AB contribution resulting from the nonlocal terms. Thus, the AB oscillations are always sinusoidal in this regime. The widths of the resonances are temperature independent so we shall focus entirely on their amplitude. For noninteracting electrons, the Büttiker-Landauer formula or our perturbation theory with \( q = 1 \) shows that \( I_{\text{FL}} = \Gamma^2 T^3 \pi \) and \( I_{\text{FL}}^{AB} = 2\Gamma^2 T^3 \sin^{-1}(T/T_0) \sinh^{-1}(V/2T_0) \). The corresponding conductances are \( G_{\text{FL}} = \frac{\Gamma^2}{\pi} \) and \( G_{\text{FL}}^{AB} = \frac{\Gamma^2}{\pi} T_0 \sin^{-1}(T/T_0) \).

The low-voltage \( (V < T_0) \) regime.---There are three temperature regimes here. When \( T < V < T_0 \), both \( I_0 \) and \( I_{AB} \) have nonlinear behavior, varying with voltage as \( V^{2q-1} \). When the temperature exceeds \( V \), the response becomes linear. When \( V < T < T_0 \), both \( G_0 \) and \( G_{AB} \) vary with temperature as \( G \propto (T/T_0)^{2q-2} \).
in striking contrast to a Fermi liquid \((q = 1)\). This is the same low-temperature power-law scaling predicted \([7,10,13]\) and observed \([15]\) in a quantum-point-contact tunneling geometry. Here \(T_F \equiv v/a\) is an effective Fermi temperature. Near \(T \approx 2T_0\) for the \(q = 3\) case, we find that \(G_{AB}\) displays a pronounced maximum, also in contrast to a Fermi liquid \([see \ Fig. 1(c)]\). Increasing the temperature further, however, we cross over into the regime that may also be used to distinguish between a Fermi liquid and our predicted nearly Fermi-liquid temperature dependence.

\[
G_{AB} \propto (T/T_0) \left( T/T_F \right)^{2q-2} e^{-qT/T_0}. \tag{9}
\]

Thus, the AB oscillation amplitude exhibits a crossover from the well-known \(T^{2q-2}\) CLL behavior to a new scaling behavior that is much closer to a chiral Fermi liquid. Careful measurements in this experimentally accessible regime should be able to distinguish between a Fermi liquid and our predicted nearly Fermi-liquid temperature dependence.

**High-voltage \((V \gg T_0)\) regime.**—Again there are three temperature regimes. At the lowest temperatures, \(T \ll T_0 \ll V\), the response is nonlinear. The direct contribution varies with voltage as \(I_0 \propto V^{q-1}\). The AB current is more complicated, involving Bessel functions of the ratio \(V/2\pi T_0\). As the temperature is increased further to \(T_0 \ll T \ll V\), we find a crossover to a remarkable high-temperature nonlinear regime. Here, \(I_0 \propto V^{2q-1}\) as before, but now \(I_{AB} \propto \left( T/T_0 \right)^q e^{-qT/T_0} V^{q-1} \sin(V/2\pi T_0)\). Note the additional \(V^{q-1}\) term that is not present in the corresponding Fermi-liquid result. Therefore, the nonlinear response can also be used to distinguish between Fermi liquid and CLL behavior, even at relatively high temperatures. When the temperature exceeds \(V\), the response finally becomes linear. When \(T_0 \ll V \ll T\), \(G_0\) scales as in \((8)\), whereas \(G_{AB}\) scales as in \((9)\). Thus, at high temperatures the low- and high-voltage regimes behave similarly.

In conclusion, we have studied the AB effect for filling factor \(1/q\) in the strong-antidot-coupling limit with CLL theory. The low-temperature linear response is similar to that in a quantum point contact. However, the AB oscillations are a mesoscopic effect and, as such, are diminished in amplitude above a crossover temperature \(T_0\) determined by the size of the antidot. Above \(T_0\), the temperature dependence of the AB oscillations is qualitatively similar to that in a chiral Fermi liquid \([see \ Fig. 1(c)]\). It is clear that a related crossover occurs in the weak-antidot-coupling regime as well. In addition, we have identified a new high-temperature nonlinear response regime that may also be used to distinguish between a Fermi and Luttinger liquid.

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*Note added.*—After this work was submitted for publication, we received a very interesting preprint by Chamon and co-workers \([24]\), where a double point-contact arrangement that allows one to measure the charge and statistics of FQHE quasiparticles is analyzed.

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[1] The Quantum Hall Effect, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1990).
[2] B. I. Halperin, Phys. Rev. B 25, 2185 (1982).
[3] M. Büttiker, Phys. Rev. Lett. 57, 1761 (1986); P. Streda, J. Kucera, and A. H. MacDonald, Phys. Rev. Lett. 59, 1973 (1987); J. K. Jain and S. A. Kivelson, Phys. Rev. Lett. 60, 1542 (1988); M. Büttiker, Phys. Rev. B 38, 9375 (1988).
[4] R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[5] For a review see J. Sólyom, Adv. Phys. 28, 201 (1979).
[6] F. D. M. Haldane, J. Phys. C 14, 2585 (1981).
[7] C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. 68, 1220 (1992); Phys. Rev. B 46, 15233 (1992).
[8] For reviews see X. G. Wen, Int. J. Mod. Phys. B 6, 1711 (1992); Adv. Phys. 44, 405 (1995).
[9] C. de C. Chamon and X. G. Wen, Phys. Rev. Lett. 70, 2605 (1993).
[10] K. Moon, H. Yi, C. L. Kane, S. M. Girvin, and M. P. A. Fisher, Phys. Rev. Lett. 71, 4381 (1993).
[11] V. L. Pokrovsky and L. P. Pryadko, Phys. Rev. Lett. 72, 124 (1994).
[12] C. L. Kane and M. P. A. Fisher, Phys. Rev. B 51, 13449 (1995).
[13] P. Fendley, A. W. W. Ludwig, and H. Saleur, Phys. Rev. Lett. 74, 3005 (1995).
[14] J. J. Palacios and A. H. MacDonald, Phys. Rev. Lett. 76, 118 (1996).
[15] F. P. Milliken, C. P. Umbach, and R. A. Webb, Solid State Commun. 97, 309 (1996).
[16] A. M. Chang, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. 77, 2538 (1996).
[17] J. D. F. Franklin et al., Surf. Sci. 361, 17 (1996).
[18] I. J. Maasilta and V. J. Goldman (unpublished).
[19] D. Loss, Phys. Rev. Lett. 69, 343 (1992).
[20] The chiral “persistent currents” associated with FQHE edge states have a universal non-Fermi-liquid temperature dependence and will be discussed elsewhere.
[21] P. A. Lee, Phys. Rev. Lett. 65, 2206 (1990); S. A. Kivelson, Phys. Rev. Lett. 65, 3369 (1990); D. J. Thouless and Y. Gefen, Phys. Rev. Lett. 66, 806 (1991).
[22] We define resonance here in the sense it is used in the Luttinger liquid literature: The system is on resonance when the AB flux inhibits transmission through the antidot. Thus, CLL theory at low temperature predicts peaks in the two-terminal conductance on resonance, and vanishing conductance elsewhere. This definition is opposite to what is usually used in the antidot literature, which defines resonance to occur when the flux allows tunneling through the antidot.
[23] X. G. Wen, Phys. Rev. B 44, 5708 (1991).
[24] C. de C. Chamon, D. E. Freed, S. A. Kivelson, S. L. Sondhi, and X. G. Wen (unpublished).