An Exact Upper Bound on the $L^p$ Lebesgue Constant and The $\infty$-Rényi Entropy Power Inequality for Integer Valued Random Variables

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Abstract

In this paper, we proved an exact asymptotically sharp upper bound of the $L^p$ Lebesgue Constant (i.e. the $L^p$ norm of Dirichlet kernel) for $p \geq 2$. As an application, we also verified the implication of a new $\infty$-Rényi entropy power inequality for integer valued random variables.

1 Introduction.

The best constants of important operators of harmonic analysis is always an area of persistent investigation, for example, the norm of the Fourier transform (FT) on locally compact abelian (LCA) groups (some basic facts of Fourier analysis on LCA groups can be found in [9, 20]). In particular, for Euclidean space, the norm of FT from $L^p(\mathbb{R}^d)$ to $L^{p'}(\mathbb{R}^d)$ with $p'$ the Hölder dual index of $p$ and $p \in (1, 2]$ on Euclidean space is the content of Hausdorff-Young inequality, and the sharp constant is proven by Beckner in [5]. For some abstract LCA groups, the norm of FT is proven by Gilbert and Rzeszotnik in [8] for the case that the group is finite, and by Madiman and Xu in [15, 21] for the case that the group is infinite and discrete or compact.

Moreover, a lot of questions about estimating the $L^p$ norms of the FT of some special functions have been considered. For example, the upper bound of the $L^p$ norm of the FT of uniform probability distribution functions on intervals is proven by K. Ball in [3, 4] and by Nazarov and Podkorytov in [18] with the following sharp result:

Ball’s integral inequality: $\int_{\mathbb{R}} \left| \frac{\sin \pi x}{\pi x} \right|^p dx < \sqrt{\frac{2}{p}}$ for $p \geq 2$ (1)

As an application, Ball also derived the sharp constant for cube slicing inequality in [3, 4], which, together with Rogozin’s convolution inequality ([19]) and a rearrangement argument ([7, 24]), can also be used to derive the sharp constants for $\infty$-Rényi entropy power inequality (the proofs can be found in [6, 14, 22, 23], some basic facts about entropy power inequality can be found in a survey paper [13]) of the following form:

$$N_\infty(X_1 + \cdots + X_n) \geq \frac{1}{2} \sum_{i=1}^{n} N_\infty(X_i)$$ (2)

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for independent one-dimensional random variables $X_i$, with the notation of $\infty$-Rényi entropy power $N_{\infty}(X) := \|f\|_{\infty}^{-2}$, where $f$ is the density of $X$. As a further application, Ball’s integral inequality \[^{[1]}\] also plays a key role in deriving the sharp bounds for marginal densities of product one-dimensional measures (see \[^{[1]}\]).

On the other hand, the “discrete version” of Ball’s integral inequality, or equivalently the question about an exact upper bound of the $L^p$ norm of the FT of uniform probability mass function supported on the integer interval \{0, 1, \ldots, l − 1\} (this norm is also called $L^p$ Lebesgue constant) was still open. Note that this FT is precisely the normalized Dirichlet kernel of length \(l\) defined by

$$D_l(x) := \frac{\sin l \pi x}{l \sin \pi x} \cdot e^{i(l-1)\pi x}$$

supported on \([-1/2, 1/2]\). Before our work, some asymptotic estimates of the $L^p$ Lebesgue constant have been studied by, for example, \[^{[1]}\]^{[2]}^{[20]}$. Specifically, for $p = 1$, it is well known (see \[^{[20]}\])

$$\int_{-1/2}^{1/2} |D_l(x)| \, dx \simeq \frac{4 \log l}{\pi^2 l} \tag{3}$$

For the case that $p > 1$, Anderson et. al. in \[^{[1]}\], Lemma 2.1] proved an asymptotically sharp estimate:

$$\int_{-1/2}^{1/2} |D_l(x)|^p \, dx = \frac{2}{\pi} \int_0^\infty \frac{|\sin u|^p}{u} \, du + o_p \left( \frac{1}{l} \right) \leq \left( \sqrt{\frac{2}{p}} + o_p(1) \right) \cdot \frac{1}{l}. \tag{4}$$

This result also gives the connection between $L^p$ Legesgue constant and Ball’s integral inequality \[^{[1]}\]. However, no exact upper bound can be derived from these results.

In this paper, we provide the following new result: For $p \geq 2$ and \(l \geq 6\),

$$\int_{-1/2}^{1/2} |D_l(x)|^p \, dx < \sqrt{\frac{2}{p}} \cdot \sqrt{\frac{1}{l^2 - 1}}.$$  

it is easy to see that our upper bound coincides with the asymptotic estimation \[^{[4]}\].

We would like to mention something about the method in this paper. Motivated by the proof of \[^{[3]}\]^{[18]}$, the method is basically to compare the distribution functions of $D_l$ and a carefully truncated Gaussian function, and to use a similar argument as in \[^{[18]}\].

We will organize this paper as follows. In section 2, we will provide the proof of our main result (Theorem 2.1). In section 3, we will describe an application of the main result in deriving a new $\infty$-Rényi entropy power inequality for integer valued random variables (see Corollary 3.5).

2 An exact upper bound on the $L^p$ Lebesgue constant $p \geq 2$.

**Theorem 2.1.** Let \(l \geq 6\) be an integer. Then for $p \geq 2$, the normalized Dirichlet kernel defined by $D_l(x) := \frac{\sin l \pi x}{l \sin \pi x} \cdot e^{i(l-1)\pi x}$ supported on \([-1/2, 1/2]\) satisfies the following integral inequality:

$$\int_{-1/2}^{1/2} |D_l(x)|^p \, dx < \sqrt{\frac{2}{p(l^2 - 1)}} \tag{5}$$
Lemma 2.2. Let $l$ be a positive integer, for $x \in [0, 1/l]$,
\[
\frac{\sin l\pi x}{l \sin \pi x} < \exp \left(-\frac{\pi (l^2 - 1)x^2}{2}\right)
\] (6)

Proof. We have
\[
\frac{\sin l\pi x}{l \sin \pi x} = \frac{\sin \frac{\pi x}{l}}{\sin \frac{\pi x}{l}} \cdot \frac{\pi x}{l \sin \pi x} = \prod_{k=1}^{\infty} \left(1 - \frac{l^2 x^2}{k^2}\right) = \prod_{k=1}^{\infty} \frac{1 - \frac{x^2}{k^2}}{1 - \frac{l^2 x^2}{k^2}}
\]

Note that, for $x \in [0, 1/l^2]$,
\[
\exp \left(-\frac{\pi (l^2 - 1)x^2}{2}\right) \geq \exp \left(-\frac{\pi^2 (l^2 - 1)x^2}{6}\right) = \prod_{k=1}^{\infty} \exp \left(-\frac{(l^2 - 1)x^2}{k^2}\right) > \prod_{k=1}^{\infty} \left(1 - \frac{(l^2 - 1)x^2}{k^2}\right)
\]

Compare the right hand sides of these two expressions, it is sufficient to prove the inequality holds termwise, which is:
\[
\frac{1 - l^2 x}{1 - x} \leq 1 - (l^2 - 1)x, \text{ for } x \in [0, 1/l^2]
\] (7)

which is clearly true. \hfill \square

In order to prove Theorem 2.1, we will apply [18, Lemma on distribution functions], we state this lemma as follows.

Lemma 2.3. For a non-negative function $f : \mathbb{R} \to [0, \infty)$, its distribution function $F(y)$, $y > 0$ is defined by
\[
F(y) := \lambda\{x \in \mathbb{R} : f(x) > y\}
\]
where $\lambda$ is the Lebesgue measure. Let $f$ and $g$ be any two nonnegative measurable functions on $\mathbb{R}$. Let $F$ and $G$ be their distribution functions. Assume that both $F(y)$ and $G(y)$ are finite for every $y > 0$. Assume also that at some point $y_0$ the difference $F - G$ changes sign from $-$ to $+$. Let $S := \{x > 0 : f^p - g^p \in L^1(\mathbb{R})\}$, then the function
\[
\varphi(p) := \frac{1}{p y_0^p} \int_{\mathbb{R}} (f^p - g^p) d\lambda
\]
is increasing on $p$. In particular, if $\int_{\mathbb{R}} (f^{p_0} - g^{p_0}) d\lambda \geq 0$, then $\int_{\mathbb{R}} (f^p - g^p) d\lambda \geq 0$ for each $p > p_0$. The equality may hold only if the functions $F$ and $G$ coincide.

Proof of Theorem 2.1 In order to apply Lemma 2.3 we construct our functions $f$ as follows:

- If $l$ is even,
\[
f(x) := \begin{cases} 
\exp \left(-\frac{\pi (l^2 - 1)x^2}{2}\right) & \text{on } \left[0, \sqrt{\frac{2 \log \pi \left(\frac{1}{2} + \frac{1}{4}\right)}{\pi (l^2 - 1)}}\right] \\
0 & \text{otherwise}
\end{cases}
\]
• If $l$ is odd,

$$f(x) := \begin{cases} \exp \left(-\frac{\pi (l^2 - 1) x^2}{2}\right) & \text{on } \left[0, \sqrt{\frac{2 \log \pi (\lfloor \frac{l}{2} \rfloor + \frac{3}{2})}{\pi (l^2 - 1)}}\right] \\ 0 & \text{otherwise} \end{cases}$$

Note that $f$ is actually the truncated part of the Gaussian function $\exp \left(-\frac{\pi (l^2 - 1) x^2}{2}\right)$ with the values larger than $\frac{1}{\pi (l/2 + 1/2)}$ for even $l$ or larger than $\frac{1}{\pi (l/2 + 3/2)}$ for odd $l$. Now define $g(x)$ as follows:

$$g(x) := \begin{cases} \frac{\sin \pi x}{\pi x} & \text{on } [0, 1/2] \\ 0 & \text{otherwise} \end{cases}$$

Note that it suffices to prove that $\int_0^\infty g^p \leq \int_0^\infty f^p$ for $p \geq 2$, which implies that, for $p \geq 2$,

$$\int g^p \leq \int f^p \leq \int_{\mathbb{R}} \exp \left(-\frac{\pi p (l^2 - 1) x^2}{2}\right) dx = \frac{2}{p} \sqrt{\frac{1}{l^2 - 1}},$$

which provides the theorem. Note that for $p_0 = 2$, by Parseval’s identity (note that $g$ is actually the absolute value of the FT of the uniform probability distribution on $\{1, 2, \cdots, l\}$), we have $\int g^2 = 1/l$. On the other hand, we claim that $\int f^2 \geq 1/l$. In fact, for $l$ even,

$$\int f^2 = \frac{1}{\sqrt{l^2 - 1}} - 2 \int_{\mathbb{R}} \sqrt{\log \pi \frac{(\frac{l}{2} + \frac{1}{2})}{\pi (l^2 - 1)}} e^{-\pi (l^2 - 1) x^2} dx$$

So it suffices to prove that

$$2 \int_{\mathbb{R}} \sqrt{\log \pi \frac{(\frac{l}{2} + \frac{1}{2})}{\pi (l^2 - 1)}} e^{-\pi (l^2 - 1) x^2} dx \leq \frac{1}{\sqrt{l^2 - 1}} - \frac{1}{l} = \frac{1}{l (l + \sqrt{l^2 - 1}) \sqrt{l^2 - 1}}$$

In fact, the left hand side of (8) has the following estimation:

$$2 \int_{\mathbb{R}} \sqrt{\log \pi \frac{(\frac{l}{2} + \frac{1}{2})}{\pi (l^2 - 1)}} e^{-\pi (l^2 - 1) x^2} dx = \frac{2}{\sqrt{\pi (l^2 - 1)}} \int_{\mathbb{R}} e^{-x^2} dx \leq \frac{2}{\sqrt{\pi (l^2 - 1)}} \int_{\mathbb{R}} e^{-x^2} dx = \frac{1}{\sqrt{\pi (l^2 - 1)}} \int_{\mathbb{R}} e^{-x} dx = \frac{1}{\pi \frac{1}{2} (l + 1)^2 \sqrt{l^2 - 1}}$$

Comparing this with the right hand side of (8), it suffices to prove that

$$\frac{8}{\pi \frac{1}{2} (l + 1)^2} \leq \frac{1}{l \left(\frac{l + \sqrt{l^2 - 1}}{2}\right)},$$

In fact, the left hand side of (8) has the following estimation:
which is clearly true by the fact that \( l + 1 \geq l \geq \frac{1 + \sqrt{7}}{2} \) and that \( \frac{8}{\pi^2} < 1 \). For the case that \( l \) is odd, a similar argument will show that \( \int f^2 \geq 1/l \).

Now it is enough to show that the corresponding distribution functions \( F \) and \( G \) satisfy the conditions of Lemma 2.3 with \( s_0 = 2 \). Observe that both \( f \) and \( g \) are bounded above by 1 by the fact that \( |\sin(nx)| \leq n|\sin(x)| \). So we have

\[
F(y) = \begin{cases} 
F(0) = \sqrt{2 \log \frac{\pi (\frac{1}{2} + \frac{1}{l})}{\pi (l^2 - 1)}} & \text{for } l \text{ even} \\
\sqrt{2 \log \frac{\pi (\frac{1}{2} + \frac{1}{l})}{\pi (l^2 - 1)}} & \text{for } l \text{ odd}
\end{cases} \quad \text{for } y \geq 0
\]

\[
F(0) = \begin{cases} 
\sqrt{2 \log \frac{\pi (\frac{1}{2} + \frac{1}{l})}{\pi (l^2 - 1)}} & y \in \left[0, \frac{1}{\pi (l/2 + 1/2)}\right) \\
\sqrt{2 \log \frac{\pi (\frac{1}{2} + \frac{1}{l})}{\pi (l^2 - 1)}} & y \in \left[\frac{1}{\pi (l/2 + 1/2)} , 1\right]
\end{cases} \quad \text{for } l \text{ even}
\]

\[
F(y) = \begin{cases} 
\sqrt{2 \log \frac{\pi (\frac{1}{2} + \frac{1}{l})}{\pi (l^2 - 1)}} & y \in \left[0, \frac{1}{\pi ([l/2] + 3/2)}\right) \\
\sqrt{2 \log \frac{\pi (\frac{1}{2} + \frac{1}{l})}{\pi (l^2 - 1)}} & y \in \left[\frac{1}{\pi ([l/2] + 3/2)} , 1\right]
\end{cases} \quad \text{for } l \text{ odd}
\]

Now we will estimate \( G \). Note that \( g(x) \)'s graph is like a series of bumps with decreasing heights (see Figure 1 and 2). Consider \( y_m := \max_{[\frac{m}{l}, \frac{m+1}{l}] g} \) for \( m = \{1, \cdots , l/2 - 1\} \) for even \( l \) or \( m = \{1, \cdots , \lfloor l/2 \rfloor\} \) for odd \( l \) (Note that \( y_m \) is the peak of each bump). Clearly \( y_m \in \left[\frac{1}{\pi \sin \pi m / l}, \frac{1}{\pi \sin \pi m / l}\right] \). For \( x \in [0,1/l] \), by Lemma 2.2 \( g(x) < f(x) \), which means that for \( y \in (y_1, 1) \), \( G(y) < F(y) \). Combining this fact with \( F(0) < G(0) \), we claim that \( F - G \) must change sign at least once on \([0,1] \).

To prove that the change of sign of \( F - G \) occurs only once, it suffices to prove that \( F - G \) is increasing on \((0,y_1) \), which is enough to prove that \( |G'(y)| \geq |F'(y)| \) on \((0,y_1) \). Clearly, for each \( y \in (0,y_1) \) with \( y \neq y_j \),

\[
|G'(y)| = \sum_{x > 0: g(x) = y} \frac{1}{|g'(x)|}
\]

when \( y \in (y_{m+1}, y_m) \), the equation \( g(x) = y \) has one root in \((0,1/l) \) and two roots in \((k/l, (k + 1)/l) , k = 1, \cdots , \lfloor l/2 \rfloor - 1 \). In particular, if \( l \) is odd and \( m = \lfloor l/2 \rfloor \), then \( g(x) \) has possibly one root in \( \left(\frac{l/2}{1}, \frac{1}{2}\right) \) (see Figure 2 for the case that \( l \) is odd). We have,

1. If the root \( x \in (k/l, (k + 1)/l) \) for \( k \geq 1 \), we claim that

\[
|g'(x)| \leq \frac{\pi}{\sin \pi x} \cdot \frac{\pi x}{\sin k \pi / l} \leq \frac{l \pi^2}{4k}
\]
Figure 1: Graph of $g(x)$ and $f(x)$ with $l = 8$, the red curve is $g(x)$, the blue curve is $f(x)$, and the $y$-values of the dashed horizontal lines are $y_1, y_2, y_3, y_{last}$ from high to low, where $y_{last}$ is the lower bound of positive $f(x)$, see (14).

Figure 2: Graph of $g(x)$ and $f(x)$ with $l = 9$. The $y$-values of the dashed horizontal lines are $y_1, y_2, y_3, y_4$, and $y_{last}$ from high to low.

In fact,

$$|g'(x)| = \left| \frac{l^2 \pi \cos \pi x \sin \pi x - l \pi \sin \pi x \cos \pi x}{l^2 \sin^2 \pi x} \right|$$

$$= \left| \frac{\pi}{\sin \pi x} \left( \cos l\pi x - \frac{\cos \pi x \sin l\pi x}{l \sin \pi x} \right) \right|$$

$$\leq \frac{\pi}{\sin \pi x} \left( 1 + \frac{\left| \cos \pi x \sin(l\pi x - k\pi) \right|}{l \sin k\pi / l} \right)$$

$$\leq \frac{\pi}{\sin \pi x} \left( 1 + \frac{\pi x}{\sin k\pi / l} - \frac{k\pi}{l \sin k\pi / l} \right)$$

$$\leq \frac{\pi}{\sin \pi x} \cdot \frac{\pi x}{\sin k\pi / l} \leq \frac{l\pi^2}{4k}$$

where the last step is by the fact that $\sin \pi x \geq 2x$ for $x \in [0, 1/2]$.

2. If the root $x \in (0, 1/l)$, we have

$$|g'(x)| = \left| \frac{l^2 \pi \cos \pi x \sin \pi x - l \pi \sin \pi x \cos \pi x}{l^2 \sin^2 \pi x} \right|$$

$$= \frac{l \pi \cos \pi x \cos \pi x}{\sin^2 \pi x} |l \tan \pi x - \tan l\pi x|$$
We claim that, for \( x \in (0, 1/l) \), one always has

\[
|g'(x)| = \frac{l \pi \cos l \pi x \cos \pi x}{\sin^2 \pi x} (\tan l \pi x - l \tan \pi x)
\]

In fact, it is easy to prove that \( \tan l \pi x \geq l \tan \pi x \) for \( x \in (0, \frac{1}{2l}) \). On the other hand, if \( x \in (\frac{1}{2l}, \frac{1}{l}) \), \( \tan l \pi x < 0 \), \( l \tan \pi x > 0 \), but the common factor \( \cos l \pi x < 0 \). By this observation, we have

\[
|g'(x)| = \frac{l \pi \cos l \pi x \cos \pi x}{\sin^2 \pi x} (\tan l \pi x - l \tan \pi x)
\]

\[
= \frac{l \pi \cos \pi x}{\sin^2 \pi x} (\sin l \pi x - l \tan \pi x \cos l \pi x)
\]

By the fact that \( \tan \pi x \geq \pi x \) for \( x \in (0, 1/l) \), we claim that

\[
|g'(x)| \leq \frac{l \pi}{2} \left( \frac{\pi}{\sin \frac{\pi}{4}} \right)^2
\]

(13)

In fact,

\[
|g'(x)| \leq \frac{\pi \cos \pi x}{l \sin^2 \pi x} (\sin l \pi x - l \pi x \cos l \pi x)
\]

\[
= \frac{\pi \cos \pi x}{l \sin^2 \pi x} \int_0^{l \pi x} t \sin t dt
\]

\[
\leq \frac{\pi}{l \sin^2 \pi x} \int_0^{l \pi x} t dt
\]

\[
= \frac{l \pi (\pi x)}{2 (\sin \pi x)^2}
\]

\[
\leq \frac{l \pi (\pi/4)}{2 (\sin \pi/4)^2}
\]

\[
\leq \frac{l \pi (\pi/4)}{2 (\sin \pi/4)^2}
\]

\[
\leq 2l
\]

for \( l \geq 6 \).

Now in order to combine these two cases, we define an extra \( y_{last} \) (note that \( y_{last} \) is the lower bound of positive \( f(x) \), See Figure 1 and 2).

\[
y_{last} := \begin{cases} 
\frac{1}{\pi(l/2+1/2)} =: y_{l/2} & \text{for } l \text{ even} \\
\frac{1}{\pi(l/2)+3/2} =: y_{l/2} & \text{for } l \text{ odd}
\end{cases}
\]

(14)

we have two situations:

1. \( y \in [0, y_{last}] \). For this case, recall (9) and (10), \( F(y) \) is constant on \( [0, y_{last}] \), which means that \( F'(y) \) is 0 on this interval. So naturally we have \( |G'(y)| \geq |F'(y)| \) on this interval.
2. \( y \in (y_{\text{last}}, y_1) \). For this case, \( y \) must fall into some \((y_{m+1}, y_m)\), where \( y_{m+1} \) could be \( y_{\text{last}} \) as in our definition \((14)\). For every \( y \in (y_{m+1}, y_m) \), combine \((11), (12)\) and \((13)\),

\[
|G'(y)| \geq \frac{1}{2l} + 2 \sum_{k=1}^{m} \frac{4k}{l\pi^2} - \frac{4m}{l\pi^2}
\]

(15)

where the term with negative sign is to avoid the case that \( l \) is odd and \( m = \lfloor l/2 \rfloor \), where \( g(x) \) has only one root in \((\lfloor l/2 \rfloor, l/2)\) (see Figure 2). Thus,

\[
|G'(y)| \geq \frac{1}{2l} + \frac{4m^2}{l\pi^2}
\]

(16)

On the other hand, we have

\[
\frac{1}{|F'(y)|} = \frac{\sqrt{2\pi(l^2 - 1) \log \frac{1}{y}}}{y}
\]

One obtains

\[
\frac{|G'(y)|}{|F'(y)|} \geq \left( \frac{1}{2l} + \frac{4m^2}{l\pi^2} \right) y \sqrt{2\pi(l^2 - 1) \log \frac{1}{y}}
\]

Note that the function \( y \sqrt{\log \frac{1}{y}} \) is increasing on \((0, \frac{1}{\sqrt{e}})\) and decreasing on \((\frac{1}{\sqrt{e}}, 1)\). Now recall that \( y_1 \leq \frac{1}{\sin(\pi/l)} \leq \frac{1}{4\sin(\pi/4)} = \frac{1}{2\sqrt{2}} < \frac{1}{\sqrt{e}} \), hence \( y_1 \sqrt{\log \frac{1}{y}} \) increases on \((0, y_1)\).

Moreover, we claim that for \( y \in (y_{m+1}, y_m) \), one always has \( y \geq \frac{1}{\pi(m+\frac{3}{2})} \). In fact,

- For the case that \( y_{m+1} > y_{\text{last}} \), one has \( y \geq \frac{1}{\sin(\pi/l)} \geq \frac{1}{\pi(m+\frac{3}{2})} \).
- For the case that \( y_{m+1} = y_{\text{last}} \), which means that \( m = l/2 - 1 \) for \( l \) even or \( m = \lfloor l/2 \rfloor \) for \( l \) odd. Thus we surely have \( y \geq y_{\text{last}} = \frac{1}{\pi(m+\frac{3}{2})} \) by the definition \((14)\).

So we have, for \( l \geq 6 \),

\[
\frac{|G'(y)|}{|F'(y)|} \geq \left( \frac{2}{\pi} \right)^{5/2} \cdot \frac{\sqrt{l^2 - 1}}{l} \cdot \frac{(m^2 + \frac{\pi^2}{\pi}) \sqrt{\log \left( m + \frac{3}{2} \right)}}{m + \frac{3}{2}} \geq 0.3188 \cdot \frac{(m^2 + \frac{\pi^2}{\pi}) \sqrt{\log \left( m + \frac{3}{2} \right)}}{m + \frac{3}{2}}
\]

which is greater than 1 if \( m \geq 3 \).

Now we have only two cases left: \( y \in (y_3, y_2) \) or \( y \in (y_2, y_1) \). For the case that \( y \in (y_3, y_2) \) (i.e. \( m = 2 \)), note that if \( l \geq 6 \), then \( g(x) = y \) must have two roots on \((1/l, 2/l)\). So we can actually sharpen inequalities \((15)\) and \((16)\) by:

\[
|G'(y)| \geq \frac{1}{2l} + 2 \sum_{k=1}^{m} \frac{4k}{l\pi^2} = \frac{1}{2l} + \frac{4(m^2 + m)}{l\pi^2}
\]
Thus, by repeating the same steps, we obtain

\[ |G'(y)| \leq \frac{2 \pi}{l \pi} \left( \sin \frac{\pi}{l} \right)^2 + \frac{2 \sin \pi x \sin \frac{\pi}{l}}{\pi^2 x} \]  
\[ \geq \frac{2 \pi}{l \pi} \left( \sin \frac{\pi}{l} \right)^2 + \sin \frac{2\pi}{l} \sin \frac{\pi}{l} \]  
\[ = \frac{1}{l} \left( \sin \frac{\pi}{l} \right)^2 \left( \frac{2}{\pi} + 2 \cos \frac{\pi}{l} \right) \]

which is clearly greater than 1 for \( m = 2 \).

For the case that \( y \in (y_2, y_1) \), recall inequalities (12) and (13) and the fact that \( g(x) = y \) has one root in \((0, 1/l)\) and two roots in \((1/l, 2/l)\), we have

\[ |G'(y)| \leq \frac{1}{l} \left( \sin \frac{\pi}{l} \right)^2 \left( \frac{2}{\pi} + 2 \cos \frac{\pi}{l} \right) \sqrt{2\pi(l^2 - 1) \log \left( \frac{5\pi}{2} \right)} \]
\[ = \sqrt{\frac{l^2 - 1}{l^2}} \left( \sin \frac{\pi}{l} \right)^2 \left( \frac{4}{5\pi} + \frac{4}{5} \cos \frac{\pi}{l} \right) \sqrt{2\pi \log \left( \frac{5\pi}{2} \right)} \]

which is greater than 1 if \( l \geq 7 \). Now for \( l = 6 \), then by the series of inequalities after \( 12 \), we have, for \( x \in (1/6, 2/6) \),

\[ |g'(x)| \leq \frac{\pi}{\sin \pi x} \left( 1 + \cos \frac{\pi}{6} \cdot \left( \frac{\pi x}{\sin \pi x} - \frac{\pi}{6\sin \frac{\pi}{6}} \right) \right) \]

Note that the right hand side is increasing for \( x \in (1/6, 1/3) \) by computing the derivative. Thus we have

\[ |g'(x)| \leq \frac{\pi}{\sin \pi x} \left( 1 + \cos \frac{\pi}{6} \cdot \left( \frac{\pi x}{\sin \pi x} - \frac{\pi}{6\sin \frac{\pi}{6}} \right) \right) \leq 2\pi \left( \frac{1}{\sqrt{3}} + \frac{\pi}{6} \right) \]

Now by (13), we have

\[ |G'(y)| \geq \frac{1}{3\pi} \left( \sin \frac{\pi}{6} \right)^2 + \frac{1}{\left( \frac{1}{\sqrt{4}} + \frac{\pi}{6} \right)} \]

Thus,

\[ \frac{|G'(y)|}{F'(y)} \geq \frac{1}{3\pi} \left( \sin \frac{\pi}{6} \right)^2 + \frac{1}{\left( \frac{1}{\sqrt{4}} + \frac{\pi}{6} \right)} \sqrt{2\pi(l^2 - 1) \log \left( \frac{5\pi}{2} \right)} \approx 1.04598 > 1 \]

Now, by applying Lemma 2.3, we have that for any \( p \geq 2 \),

\[ \int_{-1/2}^{1/2} |g(x)|^p dx \leq \int_{-1/3}^{1/3} f(x)^p dx < \int_{\mathbb{R}} \exp \left( -p\pi(l^2 - 1)x^2 \right) dx = \sqrt{\frac{2}{p(l^2 - 1)}} \]

which provides the theorem. \( \square \)
3 An ∞-Rényi entropy power inequality (∞-EPI) for integer-valued random variables

Let us firstly introduce some notations. Let \( X \) be an integer valued random variable with probability mass function \( f \), denote by \( M(X) = M(f) := \|f\|_\infty \).

**Definition 3.1.** Let \( X \) be an integer valued random variable with probability mass function \( f \). Define the \( \infty \)-Rényi entropy \( H_\infty(X) \) by
\[
H_\infty(X) = H_\infty(f) := -\log \|f\|_\infty = -\log M(f).
\]
Define the \( \infty \)-Rényi entropy power by
\[
N_\infty(X) = N_\infty(f) := e^{2H_\infty(f)} = M(f)^{-2}.
\]

We would like to introduce our motivation for this section. In [14, 23], we derived a discrete version of Rogozin’s convolution inequality (the continuous Euclidean case can be found in [14, 19]). We provide the result as follows.

**Theorem 3.2 ([14, 19]).** Let \( X_1, \cdots, X_n \) be independent integer valued random variables with \( M(X_i) \in \left(\frac{1}{l+1}, \frac{1}{l}\right] \) for some positive integer \( l_i \), then
\[
M(X_1 + \cdots + X_n) \leq M(U_1 + \cdots + U_n),
\]
where \( U_i \)’s are independent integer valued random variables uniformly supported on \( \{1, 2, \cdots, l_i\} \).

This result enables us to reduce the estimation of \( M(X_1 + \cdots + X_n) \) to a discrete cube slicing problem. In particular, if \( l_i \)’s are the same (i.e. \( M(X_i) \)’s are not far from each other), then the corresponding \( U_i \)’s in (17) are i.i.d random variables uniformly distributed on \( \{1, 2, \cdots, l_i\} \). For this special case, a direct result by Mattner and Roos in [17, Theorem] (which proved a sharp upper bound of \( M(f^n) \) for \( f \) uniform probability mass function on a discrete interval) can be applied, which yields the following partial result.

**Theorem 3.3 ([14, 19]).** For independent integer valued random variables \( X_1, \cdots, X_n \) with \( M(X_i) \in \left(\frac{1}{l+1}, \frac{1}{l}\right] \) for some fixed integer \( l \geq 2 \),
\[
N_\infty \left( \sum_{i=1}^{n} X_i \right) \geq \frac{\pi}{6} \frac{l^2 - 1}{l + 1} \sum_{i=1}^{n} N_\infty(X_i).
\]
In particular, if all \( M(X_i) = l \),
\[
N_\infty \left( \sum_{i=1}^{n} X_i \right) \geq \frac{\pi}{6} \frac{l^2 - 1}{l^2} \sum_{i=1}^{n} N_\infty(X_i).
\]

**Remark 3.4.** The constants in Theorem 3.3 are asymptotically sharp as \( n \to \infty \) and \( l \to \infty \). In fact, as \( l \) large enough, then the constant \( \approx \frac{\pi}{6} \), which is the optimal constant by local central limit theorem.

However, for the case that \( M(X_i) \)’s are far from each other, the argument of Mattner and Roos fails to apply. We will have to use our main result [5]. We state this new ∞-EPI as follows.
Corollary 3.5. For independent integer valued random variables $X_1, \cdots, X_n$ with $M(X_i) \in \left(\frac{1}{l_i + 1}, \frac{1}{l_i + 1}\right]$ for some integers $l_i$, denote $l_{\min} := \min_i l_i$ and $l_{\max} := \max_i l_i$, and assume that $l_{\min} \geq 6$, then the following $\infty$-EPI holds

$$N_{\infty} \left( \sum_{i=1}^{n} X_i \right) \geq \frac{1}{2} \cdot \frac{l_{\min} - 1}{l_{\min} + 1} \sum_{i=1}^{n} N_{\infty}(X_i) \geq \frac{5}{14} \sum_{i=1}^{n} N_{\infty}(X_i) \quad (20)$$

In particular, if $M(X_i) = \frac{1}{l_i}$,

$$N_{\infty} \left( \sum_{i=1}^{n} X_i \right) \geq \frac{1}{2} \cdot \frac{l_{\min} - 1}{l_{\min} + 1} \sum_{i=1}^{n} N_{\infty}(X_i) \geq \frac{35}{72} \sum_{i=1}^{n} N_{\infty}(X_i) \quad (21)$$

Proof. By Theorem 3.2, we have $M(X_1 + \cdots + X_n) \leq M(U_1 + \cdots + U_n)$, where $U_i$’s are independent integer valued random variables uniformly supported on $\{1, 2, \cdots, l_i\}$. Let $g_i$ be the probability mass functions of $U_i$, thus by Hausdorff-Young inequality for discrete groups, we have

$$M(U_1 + \cdots + U_n) = M(g_1 * \cdots * g_n) \leq \left\| \prod_i D_{l_i} \right\|_1$$

Now we have two cases:

1. Case 1: $\frac{l_{\max}^2}{\sum_j l_j^2} \leq 1/2$

2. Case 2: $\frac{l_{\max}^2}{\sum_j l_j^2} > 1/2$

For Case 1, let $p_i := \frac{\sum_j l_j^2}{l_i^2}$, then clearly $p_i \geq 2$ and $\sum_i \frac{1}{p_i} = 1$, then by Hölder’s inequality and Theorem 2.1

$$\left\| \prod_i D_{l_i} \right\|_1^2 \leq \prod_{i=1}^{n} \left\| D_{l_i} \right\|_{p_i}^2 \leq \prod_{i=1}^{n} \left( \frac{2}{p_i(l_i^2 - 1)} \right)^{\frac{1}{p_i}} \leq \frac{2l_{\min}^2}{l_{\min}^2 - 1} \cdot \sum_{i=1}^{n} \frac{1}{l_i^2} \quad (21)$$

which is exactly (21) for this case. Furthermore, we have

$$N_{\infty} \left( \sum_{i=1}^{n} X_i \right) \geq \frac{1}{2} \cdot \frac{l_{\min}^2}{l_{\min}^2} \sum_{i=1}^{n} N_{\infty}(X_i) \geq \frac{1}{2} \cdot \frac{l_{\min}^2}{l_{\min}^2} \sum_{i=1}^{n} \frac{l_i^2}{(l_i + 1)^2} N_{\infty}(X_i) \geq \frac{1}{2} \cdot \frac{l_{\min} - 1}{l_{\min} + 1} \sum_{i=1}^{n} N_{\infty}(X_i)$$
which provides inequality (20) for this case.

For Case 2, by the fact that $N_\infty \left( \sum_{i=1}^{n} U_i \right) \geq N_\infty(U_j)$ for each $j$,

$$N_\infty \left( \sum_{i=1}^{n} U_i \right) \geq l_{\max}^2 > \frac{1}{2} \sum_{i=1}^{n} N_\infty(U_i)$$

which provides inequality (21) for this case. Furthermore,

$$N_\infty \left( \sum_{i=1}^{n} X_i \right) \geq N_\infty \left( \sum_{i=1}^{n} U_i \right) > \frac{1}{2} \sum_{i=1}^{n} N_\infty(U_i) \geq \frac{1}{2} \sum_{i=1}^{n} \frac{l_i^2}{(l_i+1)^2} N_\infty(X_i)$$

$$\geq \frac{1}{2} \cdot \frac{l_{\min}^2}{(l_{\min}+1)^2} \sum_{i=1}^{n} N_\infty(X_i) \geq \frac{1}{2} \cdot \frac{l_{\min}^2 - 1}{l_{\min} + 1} \sum_{i=1}^{n} N_\infty(X_i)$$

which provides inequality (20) for this case.

\[ \square \]

**Remark 3.6.** In Corollary 3.5, it is easy to see that as $l_{\min} \to \infty$, the constant is asymptotically $\frac{1}{2}$, which is asymptotically sharp in the sense that $N_\infty(X + X') = N(X)$ for the case that $X$ is uniformly distributed on a discrete interval and $X'$ is the independent copy of $X$.

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