SIMPSON TYPE INEQUALITIES FOR FIRST ORDER DIFFERENTIABLE PREINVEX AND PREQUASIINVEX FUNCTIONS

M. EMİN ÖZDEMİR • AND MERVE AVCI ARDIC •

Abstract. In this paper, we obtain some inequalities for functions whose first derivatives in absolute value are preinvex and prequasiinvex.

1. Introduction and preliminaries

Suppose \( f : [a, b] \to \mathbb{R} \) is a four times continuously differentiable mapping on \((a, b)\) and \( \|f^{(4)}\|_{\infty} = \sup |f^{(4)}(x)| < \infty \). The following inequality

\[
\left\lfloor \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] - \frac{1}{b - a} \int_a^b f(x)dx \right\rfloor \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b - a)^4
\]

is well known in the literature as Simpson’s inequality.

For some results about Simpson inequality see [1]-[5].

Let \( K \) be a nonempty closed set in \( \mathbb{R}^n \). We denote by \( \langle ., . \rangle \) and \( \| . \| \) the inner product and norm respectively. Let \( f : K \to \mathbb{R} \) and \( \eta : K \times K \to \mathbb{R} \) be continuous functions.

Definition 1. (See [7]) Let \( u \in K \). Then the set \( K \) is said to be invex at \( u \) with respect to \( \eta (. , . ) \), if

\[
u + t\eta(v, u) \in K, \quad \forall u, v \in K, \quad t \in [0, 1].
\]

\( K \) is said to be invex set with respect to \( \eta \), if \( K \) is invex at each \( u \in K \). The invex set \( K \) is also called a \( \eta \)-connected set.

Remark 1. (See [6]) We would like to mention that the Definition 1 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point \( u \) which is contained in \( K \). We don’t require that the point \( v \) should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that \( v \) should be an end point of the path for every pair of points, \( u, v \in K \), then \( \eta(v, u) = v - u \) and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to \( \eta(v, u) = v - u \), but the converse is not necessarily true.

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Corresponding Author.
Definition 2. (See [7]) The function $f$ on the invex set $K$ is said to be preinvex with respect to $\eta$, if

$$f(u + t\eta(v, u)) \leq (1 - t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

The function $f$ is said to be preconcave if and only if $-f$ is preinvex. Note that every convex function is a preinvex function, but the converse is not true. For example, the function $f(u) = -|u|$ is not a convex function, but it is a preinvex function with respect to $\eta$, where

$$\eta(v, u) = \begin{cases} v - u, & \text{if } v \leq 0, u \leq 0 \text{ and } v \geq 0, u \geq 0 \\ u - v, & \text{otherwise.} \end{cases}$$

Definition 3. (See [8]) The function $f$ on the invex set $K$ is said to be prequasiinvex with respect to $\eta$, if

$$f(u + t\eta(v, u)) \leq \max \{f(u), f(v)\}, \quad \forall u, v \in K, \quad t \in [0, 1].$$

In this paper, we establish some new Simpson type inequalities for preinvex and prequasiinvex functions.

2. SIMPSON TYPE INEQUALITIES FOR PREINVEX FUNCTIONS

We used the following Lemma to obtain our main results.

Lemma 1. Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : I \times I \rightarrow \mathbb{R}_+$ and $f : I \rightarrow \mathbb{R}$ be an absolutely continuous mapping on $I$ with $\eta(b, a) \neq 0$. If $f'$ is integrable on $\eta$-path $P_{ac}$, $c = a + \eta(b, a)$, following equality holds:

$$\int_0^1 \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)dx \right| dt$$

$$= \eta(b, a) \int_0^1 m(t)f'(a + t\eta(b, a))dt,$$

where

$$m(t) = \begin{cases} t - \frac{1}{6}, & t \in [0, \frac{1}{6}) \\ t - \frac{5}{6}, & t \in [\frac{1}{6}, 1]. \end{cases}$$

Proof. Since $a, b \in I$ and $I$ is an invex set with respect to $\eta$, it is obvious that $a + t\eta(b, a) \in I$ for $t \in [0, 1]$. Integrating by parts implies that

$$\int_0^{\frac{1}{6}} (t - \frac{1}{6}) f'(a + t\eta(b, a))dt + \int_{\frac{1}{6}}^1 (t - \frac{5}{6}) f'(a + t\eta(b, a))dt$$

$$= \left. \left( t - \frac{1}{6} \frac{f(a + t\eta(b, a))}{\eta(b, a)} \right|_0^{\frac{1}{6}} - \int_0^{\frac{1}{6}} \frac{f(a + t\eta(b, a))}{\eta(b, a)}dt \right)$$

$$+ \left. \left( t - \frac{5}{6} \frac{f(a + t\eta(b, a))}{\eta(b, a)} \right|_\frac{1}{6}^1 - \int_{\frac{1}{6}}^1 \frac{f(a + t\eta(b, a))}{\eta(b, a)}dt \right)$$

$$= \frac{1}{6\eta(b, a)} \left[ f(a) + 4f\left(\frac{2a + \eta(b, a)}{2}\right) + f(a + \eta(b, a)) \right]$$

$$- \frac{1}{\eta(b, a)} \left[ \int_0^{\frac{1}{6}} f(a + t\eta(b, a))dt + \int_{\frac{1}{6}}^1 f(a + t\eta(b, a))dt \right].$$
If we change the variable \( x = a + t\eta(b, a) \) and multiply the resulting equality with \( \eta(b, a) \) we get the desired result. \( \square \)

**Theorem 1.** Let \( I \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : I \times I \to \mathbb{R}_{+} \) and \( f : I \to \mathbb{R} \) be an absolutely continuous mapping on \( I, a, b \in I \) with \( \eta(b, a) \neq 0 \). If \(|f'|\) is preinvex then the following inequality holds:

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(x)dx \right| 
\leq \frac{5}{72} \eta(b, a) \|f'(a)| + |f'(b)|\,.
\]

**Proof.** From Lemma [1] and using the preinvexity of \(|f'|\) we have

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(x)dx \right| 
\leq \eta(b, a) \left\{ \int_{0}^{\frac{1}{6}} \left( 1 - t \right) \left| f'(a) \right| + t \left| f'(b) \right| dt 
+ \int_{\frac{1}{6}}^{\frac{5}{6}} \left( 1 - t \right) \left| f'(a) \right| + t \left| f'(b) \right| dt 
+ \int_{\frac{5}{6}}^{1} \left( 1 - t \right) \left| f'(a) \right| + t \left| f'(b) \right| dt \right\}
\]

If we compute the above integrals, we get the desired result. \( \square \)

**Theorem 2.** Let \( I \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : I \times I \to \mathbb{R}_{+} \) and \( f : I \to \mathbb{R} \) be an absolutely continuous mapping on \( I, a, b \in I \) with \( \eta(b, a) \neq 0 \). If \(|f'|\) is preinvex for some fixed \( q > 1 \) then the following inequality holds

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(x)dx \right| 
\leq \eta(b, a) \left( \frac{1 + 2^{p+1}}{6^{p+1}(p + 1)} \right)^{\frac{1}{q}}
\times \left\{ \left( \frac{3}{8} |f'(a)|^{q} + \frac{1}{8} |f'(b)|^{q} \right)^{\frac{1}{q}} + \left( \frac{1}{8} |f'(a)|^{q} + \frac{3}{8} |f'(b)|^{q} \right)^{\frac{1}{q}} \right\}
\]

where \( p = \frac{q}{q+1} \).
Proof. From Lemma 1 and using the Hölder inequality, we have

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)dx \right|
\]

\[
\leq \eta(b, a) \left\{ \left( \int_0^\frac{1}{2} t \left( 1 - \frac{1}{6} \right) dt \right)^\frac{1}{p} \left( \int_0^\frac{1}{2} |f'(a + t\eta(b, a))|^q dt \right)^\frac{1}{q} + \left( \int_\frac{1}{2}^1 t \left( \frac{5}{6} - t \right) dt \right)^\frac{1}{p} \left( \int_\frac{1}{2}^1 |f'(a + t\eta(b, a))|^q dt \right)^\frac{1}{q} \right\}.
\]

Since \( |f'|^q \) is preinvex, we obtain

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)dx \right|
\]

\[
\leq \eta(b, a) \left\{ \left( \int_0^\frac{1}{2} \frac{1}{6} - t \right)^p dt + \int_\frac{1}{2}^1 \left( 1 - \frac{1}{6} \right) dt \right)^\frac{1}{p} \left( \int_0^\frac{1}{2} (1 - t)|f'(a)|^q + t|f'(b)|^q dt \right)^\frac{1}{q} + \left( \int_\frac{1}{2}^1 \frac{5}{6} - t \right)^p dt + \int_\frac{1}{2}^1 \left( 1 - \frac{5}{6} \right) dt \right)^\frac{1}{p} \left( \int_\frac{1}{2}^1 (1 - t)|f'(a)|^q + t|f'(b)|^q dt \right)^\frac{1}{q} \right\}
\]

\[
= \eta(b, a) \left( \frac{1 + 2p+1}{6^{p+1}(p + 1)} \right)^\frac{1}{p} \left\{ \left( \frac{3}{8} |f'(a)|^q + \frac{1}{8} |f'(b)|^q \right)^\frac{1}{q} + \left( \frac{1}{8} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right)^\frac{1}{q} \right\}.
\]

The proof is completed. \( \Box \)

**Theorem 3.** Under the assumptions of Theorem 2 we have the following inequality

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)dx \right|
\]

\[
\leq \eta(b, a) \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p + 1)} \right)^\frac{1}{p} \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^\frac{1}{q}.
\]
Proof. From Lemma \[\text{I}\] preinvexity of \(|f'|^q\) and using the Hölder inequality, we have

\[
\frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)dx \geq 0.
\]

\[
\leq \eta(b, a) \left[ \int_0^1 |m(t)||f'(a + t\eta(b, a))| dt \right]
\]

\[
\leq \eta(b, a) \left( \int_0^1 |m(t)|^p dt \right)^\frac{1}{p} \left( \int_0^1 |f'(a + t\eta(b, a))|^q dt \right)^\frac{1}{q}
\]

\[
\leq \eta(b, a) \left( \int_0^1 t - \frac{1}{6} |t|^p dt + \int_{\frac{1}{2}}^1 t - \frac{5}{6} |t|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 [(1 - t)|f'(a)|^q + t|f'(b)|^q] dt \right)^{\frac{1}{q}}
\]

\[
= \eta(b, a) \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}
\]

where we used the fact that

\[
\int_0^\frac{1}{2} t - \frac{1}{6} |t|^p dt = \int_{\frac{1}{2}}^1 t - \frac{5}{6} |t|^p dt = \frac{(1 + 2^{p+1})}{6^{p+1}(p+1)}.
\]

The proof is completed. \qed

Theorem 4. Let \(I \subseteq \mathbb{R}\) be an open invex subset with respect to \(\eta : I \times I \to \mathbb{R}_+\) and \(f : I \to \mathbb{R}\) be an absolutely continuous mapping on \(I, a, b \in I\) with \(\eta(b, a) \neq 0\). If \(|f'|\) is preinvex for some fixed \(q \geq 1\) then the following inequality holds

\[
\frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)dx \geq 0.
\]

\[
\leq \eta(b, a) \left( \frac{5}{72} \right)^{1-\frac{1}{q}}
\]

\[
\times \left\{ \frac{61|f'(a)|^q + 29|f'(b)|^q}{1296} \right\}^{\frac{1}{q}} + \left( \frac{29|f'(a)|^q + 61|f'(b)|^q}{1296} \right)^{\frac{1}{q}}
\]

Proof. From Lemma \[\text{I}\] and using the power-mean inequality, we have

\[
\frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)dx \geq 0.
\]

\[
\leq \eta(b, a)
\]

\[
\times \left\{ \left( \int_0^\frac{1}{2} t - \frac{1}{6} |t|^p dt \right)^{1-\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 t - \frac{5}{6} |t|^p dt \right)^{\frac{1}{q}} \right\}^{\frac{1}{p}}
\]

\[
+ \left( \int_{\frac{1}{2}}^1 t - \frac{1}{2} |t|^p dt \right)^{1-\frac{1}{q}} \left( \int_{\frac{1}{2}}^1 t - \frac{5}{6} |t|^p dt \right)^{\frac{1}{q}}
\]

\[
\}
\]
Since $|f'|^q$ is preinvex function we have
\[
\int_{\frac{5}{6}}^{\frac{1}{6}} \left| t - \frac{1}{6} \right|^q |f'(a + t\eta(b, a))|^q \, dt
\]
\[
\leq \int_{\frac{5}{6}}^{\frac{1}{6}} \left( \frac{1}{6} - t \right) \left[ (1 - t) |f'(a)|^q + t |f'(b)|^q \right] \, dt
\]
\[+ \int_{\frac{5}{6}}^{\frac{1}{6}} \left( t - \frac{1}{6} \right) \left[ (1 - t) |f'(a)|^q + t |f'(b)|^q \right] \, dt
\]
\[= \frac{61 |f'(a)|^q + 29 |f'(b)|^q}{1296}
\]
and
\[
\int_{\frac{5}{6}}^{\frac{1}{6}} \left| t - \frac{5}{6} \right|^q |f'(a + t\eta(b, a))|^q \, dt
\]
\[
\leq \int_{\frac{5}{6}}^{\frac{1}{6}} \left( \frac{5}{6} - t \right) \left[ (1 - t) |f'(a)|^q + t |f'(b)|^q \right] \, dt
\]
\[+ \int_{\frac{5}{6}}^{\frac{1}{6}} \left( t - \frac{5}{6} \right) \left[ (1 - t) |f'(a)|^q + t |f'(b)|^q \right] \, dt
\]
\[= \frac{29 |f'(a)|^q + 61 |f'(b)|^q}{1296}
\]
Combining all the above inequalities gives us the desired result. □

3. Simpson type inequalities for prequasiinvex functions

In this section, we obtained Simpson type inequalities for prequasiinvex functions.

**Theorem 5.** Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : I \times I \rightarrow \mathbb{R}_+$ and $f : I \rightarrow \mathbb{R}$ be an absolutely continuous mapping on $I, a, b \in I$ with $\eta(b, a) \neq 0$. If $|f'|$ is prequasiinvex for some fixed $q \geq 1$ then the following inequality holds
\[
\left| \frac{1}{6} \left( f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \right|
\]
\[
\leq \frac{5}{36} \eta(b, a) \left[ \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right]^{\frac{1}{q}}.
\]

**Proof.** From Lemma 1 and using the power-mean inequality, we have
\[
\left| \frac{1}{6} \left( f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \right|
\]
\[
\leq \eta(b, a)
\]
\[
\times \left\{ \left( \int_{\frac{5}{6}}^{\frac{1}{6}} \left| t - \frac{1}{6} \right|^q \, dt \right)^{\frac{1}{q}} \left( \int_{\frac{5}{6}}^{\frac{1}{6}} \left| t - \frac{1}{6} \right|^q |f'(a + t\eta(b, a))|^q \, dt \right)^{\frac{1}{q}}
\right.
\]
\[
+ \left( \int_{\frac{5}{6}}^{\frac{1}{6}} \left| t - \frac{5}{6} \right|^q \, dt \right)^{\frac{1}{q}} \left( \int_{\frac{5}{6}}^{\frac{1}{6}} \left| t - \frac{5}{6} \right|^q |f'(a + t\eta(b, a))|^q \, dt \right)^{\frac{1}{q}} \right\}.
\]
Since \( |f'|^q \) is prequasiinvex function we have

\[
\int_0^{1/2} \left| t - \frac{1}{6}\right|^q |f'(a + t\eta(b,a))|^q \, dt \\
\leq \int_0^{1/2} \left( \frac{1}{6} - t \right) \left[ \max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right] \, dt \\
+ \int_0^{1/2} \left( t - \frac{1}{6} \right) \left[ \max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right] \, dt \\
= \frac{5}{72} \left[ \max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right]
\]

and

\[
\int_0^{1/2} \left| t - \frac{5}{6}\right|^q |f'(a + t\eta(b,a))|^q \, dt \\
\leq \int_0^{1/2} \left( \frac{5}{6} - t \right) \left[ \max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right] \, dt \\
+ \int_0^{1/2} \left( t - \frac{5}{6} \right) \left[ \max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right] \, dt \\
= \frac{5}{72} \left[ \max \left\{ \left| f'(a) \right|^q, \left| f'(b) \right|^q \right\} \right].
\]

From the above inequalities we get the desired result. \(\square\)

**Corollary 1.** In Theorem 5 if we choose \( q = 1 \) we obtain

\[
\left| \frac{1}{6} \left[ f(a) + 4f\left( \frac{2a + \eta(b,a)}{2} \right) + f(a + \eta(b,a)) \right] - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) \, dx \right| \\
\leq \frac{5}{36} \eta(b,a) \max \left\{ |f'(a)|, |f'(b)| \right\}.
\]

**Corollary 2.** In Theorem 5 if we choose \( f(a) = f\left( \frac{2a + \eta(b,a)}{2} \right) = f(a + \eta(b,a)) \) we obtain the midpoint type inequality as follows:

\[
\left| f\left( \frac{2a + \eta(b,a)}{2} \right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) \, dx \right| \\
\leq \frac{5}{36} \eta(b,a) \max \left\{ |f'(a)|, |f'(b)| \right\}.
\]

**Theorem 6.** Let \( I \subseteq \mathbb{R} \) be an open invex subset with respect to \( \eta : I \times I \to \mathbb{R}_+ \) and \( f : I \to \mathbb{R} \) be an absolutely continuous mapping on \( I, a, b \in I \) with \( \eta(b,a) \neq 0 \).
If $|f'|$ is prequasiinvex for some fixed $q > 1$ then the following inequality holds
\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\
\leq 2\eta(b, a) \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{q}} \left( \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}
\]
where $p = \frac{q}{q-1}$.

Proof. From Lemma 1 and using the Hölder inequality, we have
\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\
\leq \eta(b, a) \left\{ \left( \int_0^1 \left| \frac{1}{6} - t \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} + \left( \int_0^1 \left| \frac{5}{6} - t \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(a + t\eta(b, a))|^q dt \right)^{\frac{1}{q}} \right\}.
\]
Since $|f'|^q$ is prequasiinvex, we obtain
\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(x) dx \right| \\
\leq \eta(b, a) \left\{ \left( \int_0^\frac{1}{6} \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}} \times \left( \int_0^\frac{1}{6} \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}} + \left( \int_0^\frac{5}{6} \left( \frac{5}{6} - t \right)^p dt + \int_0^\frac{1}{6} \left( \frac{5}{6} - t \right)^p dt \right)^{\frac{1}{p}} \right\} \\
= 2\eta(b, a) \left( \frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{q}} \left( \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}}.
\]
We used
\[
\int_0^\frac{1}{6} \left( \frac{1}{6} - t \right)^p dt + \int_\frac{5}{6}^1 \left( \frac{5}{6} - t \right)^p dt = \int_0^\frac{1}{6} \left( \frac{5}{6} - t \right)^p dt + \int_\frac{5}{6}^1 \left( \frac{5}{6} - t \right)^p dt = \frac{1 + 2^{p+1}}{6^{p+1}(p+1)}
\]
in the above inequality to complete the proof. \qed
Theorem 7. Under the assumptions of Theorem 4 we have the following inequality
\[
\frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx
\]
\[
\leq \eta(b, a) \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p + 1)} \right) \left[ \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right]^{\frac{1}{q}}.
\]

Proof. From Lemma 1, prequasiinvexity of \(|f'|^q\) and using the Hölder inequality, we have
\[
\frac{1}{6} \left[ f(a) + 4f \left( \frac{2a + \eta(b, a)}{2} \right) + f(a + \eta(b, a)) \right] - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x) dx
\]
\[
\leq \eta(b, a) \left( \int_0^1 |m(t)| |f'(a + t\eta(b, a))| dt \right)
\]
\[
\leq \eta(b, a) \left( \int_0^1 |m(t)|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}}
\]
\[
\leq \eta(b, a) \left( \int_0^1 \left| t - \frac{1}{6} \right|^p dt + \int_0^1 \left| t - \frac{5}{6} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}}
\]
\[
= \eta(b, a) \left( \frac{2(1 + 2^{p+1})}{6^{p+1}(p + 1)} \right) \left[ \max \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right]^{\frac{1}{q}}
\]
where we used the fact that
\[
\int_0^1 \left| t - \frac{1}{6} \right|^p dt = \int_{\frac{1}{2}}^1 \left| t - \frac{5}{6} \right|^p dt = \frac{(1 + 2^{p+1})}{6^{p+1}(p + 1)}.
\]
The proof is completed. \(\square\)

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