A TWO-PARAMETER FAMILY OF INFINITE-DIMENSIONAL DIFFUSIONS IN THE KINGMAN SIMPLEX

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INTRODUCTION

The main result of the present paper is to construct a two-parameter family of Markov processes $X_{\alpha,\theta}(t)$ in the infinite-dimensional Kingman simplex

$$\nabla_\infty = \left\{ x = (x_1, x_2, \ldots) \in [0, 1]^\infty : x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i=1}^\infty x_i \leq 1 \right\}.$$  

In the topology of coordinatewise convergence $\nabla_\infty$ is a compact, metrizable and separable space. Denote by $C(\nabla_\infty)$ the algebra of real continuous functions on $\nabla_\infty$ with pointwise operations and the supremum norm.

In $C(\nabla_\infty)$ there is a distinguished dense subspace $\mathcal{F} := \mathbb{R}[q_1, q_2, \ldots]$ generated (as a commutative unital algebra) by algebraically independent continuous functions $q_k(x) := \sum_{i=1}^\infty x_i^{k+1}, k = 1, 2, \ldots, x \in \nabla_\infty$.

For each $0 \leq \alpha < 1$ and $\theta > -\alpha$ we define an operator $A: \mathcal{F} \to \mathcal{F}$ which can be written as a formal differential operator of second order with respect to the generators of the algebra $\mathcal{F}^1$

$$A = \sum_{i,j} (i+1)(j+1)(q_{i+j} - q_i q_j) \frac{\partial^2}{\partial q_i \partial q_j} + \sum_{i=1}^\infty \left[ -(i+1)(i+\theta)q_i + (i+1)(i-\alpha)q_{i-1} \right] \frac{\partial}{\partial q_i},$$

or (subject to certain restrictions, see Remarks 5.3 and 5.4 below) as a differential operator in natural coordinates:

$$A = \sum_{i=1}^\infty x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j=1}^\infty x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^\infty (\theta x_i + \alpha) \frac{\partial}{\partial x_i}.$$  

It is worth noting that the operator $A$ is defined only on $\mathcal{F}$, and a direct application of the right-hand sides of (1) and (2) to other functions requires caution.

Main theorem. (1) The operator $A$ is closable in $C(\nabla_\infty)$;

(2) Its closure $\overline{A}$ generates a diffusion process $\{X_{\alpha,\theta}(t)\}_{t \geq 0}$ in $\nabla_\infty$, that is, a strong Markov process with continuous sample paths;

(3) The two-parameter Poisson-Dirichlet measure $\text{PD}(\alpha, \theta)$ on $\nabla_\infty$ is a unique invariant probability measure for $X_{\alpha,\theta}(t)$. The process is reversible with respect to that measure;

1Here, by agreement, $q_0 = 1$.  

(4) The spectrum of the generator $\Lambda$ is described explicitly in §4.2 below. Due to existence of a spectral gap, the process $X_{\alpha,\theta}(t)$ is ergodic with respect to the measure $\text{PD}(\alpha, \theta)$.

Fix arbitrary $0 \leq \alpha < 1$ and $\theta > -\alpha$. We construct the process in the simplex $\nabla_\infty$ as a limit for a sequence of finite Markov chains with growing number of states. The state space for $n$th chain is the set of all partitions of $n$. Denote this set by $\mathcal{K}_n$. Each $\mathcal{K}_n$ is equipped with a certain probability measure $\mathcal{M}_n$ (also depending on $\alpha$ and $\theta$) which is explicitly written out. The system $\{\mathcal{M}_n\}$ is a partition structure. A bijection between partition structures and probability measures on $\nabla_\infty$ was established in [Ki3]. The Poisson-Dirichlet measure $\text{PD}(\alpha, \theta)$ on $\nabla_\infty$ corresponds to the two-parameter Ewens-Pitman’s partition structure $\{\mathcal{M}_n\}$. In the case $\alpha = 0$ this partition structure was introduced in [Ew], the two-parameter generalization is due to Pitman [Pi1]. A special case $\alpha = 0, \theta = 1$ of $\{\mathcal{M}_n\}$ was considered in [VS] in connection with limit behaviour of certain functionals on the symmetric group $\mathfrak{S}_n$ as $n \to \infty$. Also in [VS] the measure $\text{PD}(0, 1)$ was studied. Two-parameter Ewens-Pitman’s partition structures were studied in [Pi2, Pi3, Pi4, GP, PY, DGP] and many other works. See also [Pi4] for more bibliography.

It should be noted that the first example of a Markov process in $\nabla_\infty$ having the two-parameter Poisson-Dirichlet measure as a unique invariant symmetrizing probability measure was constructed in [Be]. Diffusions in the infinite-dimensional unordered simplex preserving the GEM-distribution (Griffiths-Engen-McCloskey, see [Pi4]) are considered in a recent paper [FW]. The GEM-distribution maps into the Poisson-Dirichlet measure if we reorder the coordinates in a descending order.

The case $\alpha = 0$ is relatively well understood and has applications to population genetics. The measure $\text{PD}(0, \theta)$ appeared in [Ki1]. The process $X_{0,\theta}(t)$ was constructed in [EK1] as a limit for diffusion processes in finite-dimensional simplexes as their dimension grows. In §5.3 we show that these finite-dimensional diffusions arise as a particular case of the process $X_{\alpha,\theta}(t)$ when $\alpha = -\beta < 0$ and $\theta = N\beta$, $N = 2, 3, \ldots$. Note also that when $\alpha = 0$ the operator $A$ is equal to the operator $[EK1, (2.10)]$ multiplied by two.

In [EK1] it was also proved that $X_{0,\theta}(t)$ is a limit process for a certain Moran-type population model (about this model see also [Wa2] and [KMG, Model II]). This Moran-type model is a sequence of finite Markov chains on partitions. They differ from the ones considered in the present paper but have the same limit.

The process $X_{0,\theta}(t)$ (when $\alpha = 0$) is called the infinitely many neutral alleles diffusion model. It was also studied in [E1, Sch] and other works. The results of the present paper extend some of the results of [EK1] and [E1] to the case $\alpha \neq 0$. However, it seems that in the general case there is no connection between the model $X_{\alpha,\theta}(t)$ and population genetics. Namely, both finite Markov chains and the process in $\nabla_\infty$ have no known interpretation in terms of population genetics.

In the present paper the construction of the process $X_{\alpha,\theta}(t)$ in $\nabla_\infty$ involves only partition structures and up/down Markov chains. The up/down chains first appeared in [Fu1]. They were also studied in the papers [BO, Fu2]. The setting of the problem studied in the present paper and the general approach to it were inspired by [BO]. However, the concrete computations here are performed in a different way. Monomial symmetric functions in the present paper play the same role as in the papers [BO, Fu2].
role as Schur functions in \([BO]\). The former are simpler than the latter, and the final result is achieved by simpler means. In addition to the results similar to \([BO]\) one can compute the operator \(A\) in natural coordinates \([2]\) and describe the process \(X_{\alpha,\theta}(t)\) when \(\alpha = \beta < 0\) and \(\theta = N\beta\). Let us describe the organization of the paper.

In \([1.1]\) we recall some notation concerning partitions. In \([1.2]\) the definitions of partition structures and up/down Markov chains related to them are recalled. In \([1.3]\) we give the definition of Ewens-Pitman’s partition structures. We also recall some of their properties.

In \([2]\) we deal with some properties of symmetric functions in the coordinates \((x_1, x_2, \ldots)\) of a point \(x \in \nabla_\infty\). These symmetric functions form the algebra \(\mathcal{F}\) defined above. In these terms we formulate the Kingman theorem about the one-to-one correspondence between partition structures and probability measures on \(\nabla_\infty\).

In \([3]\) we obtain an explicit expression for the action of operators \(T_n\) (each \(T_n\) is a transition operator of the \(n\)th up/down Markov chain) on symmetric functions in the coordinates \(\lambda_1, \lambda_2, \ldots, \lambda_\ell\) of a partition \(\lambda \in \mathbb{K}_n\).

In \([4.1]\) we perform a limit transition from the up/down Markov chains to the process \(X_{\alpha,\theta}(t)\) on \(\nabla_\infty\). First, we use the connection of symmetric functions in the coordinates of a partition with symmetric functions on \(\nabla_\infty\) to explicitly compute the limit of the operators \(n^2(T_n - 1)\). This limit is an operator \(A: \mathcal{F} \to \mathcal{F}\) described above. After that we use some general results from the book \([EK2]\) to establish the convergence of discrete semigroups \(\{1, T_n, T_n^2, \ldots\}\) to the continuous semigroup \(\{T(t)\}_{t \geq 0}\). This semigroup is generated by the closure \(\overline{A}\) of the operator \(A: \mathcal{F} \to \mathcal{F}\). In \([4.2]\) we formulate the remaining results of the Main Theorem except the result about the continuity of the process \(X_{\alpha,\theta}(t)\)'s sample paths.

In \([5.1]\) we derive \([2]\). We also give explanations why the RHS of this formula should not be understood literally. In \([5.2]\) we derive the formula \([1]\) for the operator \(A\). Using this formula it is possible to prove the continuity of the sample paths of the process in \(\nabla_\infty\). In \([5.3]\) we deal with the case of degenerate values of the parameters, namely, \(\alpha = \beta, \theta = N\beta\). In this case as a limit of up/down Markov chains instead of the diffusion in \(\nabla_\infty\) we obtain a diffusion in the \((N - 1)\)-dimensional simplex. This diffusion coincides with the one studied in \([EK1]\).

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1. Ewens-Pitman’s partition structures

1.1. Notation. In this subsection we give some combinatorial notation which is used throughout the paper.

A partition is a sequence of the form

\[\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots), \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0,\]

where \(\lambda_i \in \mathbb{Z}_{>0}\) and only a finite number of elements differs from zero. Partitions are identified with Young diagrams as in \([Ma]\). We denote them by same letters. The number of boxes in \(\lambda\) is denoted by \(|\lambda| = \lambda_1 + \cdots + \lambda_\ell\). The number of rows in \(\lambda\) is called the length of a diagram and is denoted by \(\ell(\lambda)\). Also let \(\emptyset\) denote the empty Young diagram.
If a diagram $\lambda$ is obtained from a diagram $\mu$ by adding one box, then we write $\mu \not\rightarrow \lambda$ or, equivalently, $\lambda \not\subseteq \mu$. Denote this box (that distinguishes $\lambda$ and $\mu$) by $\lambda/\mu$.

Let $\square$ be an arbitrary box. By $r(\square)$ denote its column number counting from left to right. This number does not depend on a partition containing the box.

Fix an arbitrary nonempty diagram $\lambda$. The following important properties hold. First, for each $i = 1, \ldots, \ell(\lambda)$ there exists a unique box with $r(\square) = \lambda_i$ (denoted by $\square(\lambda_i)$) such that it can be removed from $\lambda$ and the result is again a Young diagram. The resulting diagram is denoted by $\lambda - \square(\lambda_i)$. Every diagram $\mu$ such that $\mu \not\rightarrow \lambda$ has this form. Second, for each $i = 1, \ldots, \ell(\lambda)$ there exists a unique box with $r(\square) = \lambda_i + 1$ (denoted by $\square(\lambda_i + 1)$) such that it can be added to $\lambda$ and the result is again a Young diagram. Denote the resulting diagram by $\lambda + \square(\lambda_i + 1)$. Every diagram $\nu$ such that $\nu \not\subseteq \lambda$ has either the form $\lambda + \square(\lambda_i + 1)$ for some $i = 1, \ldots, \ell(\lambda)$ or the form $\lambda + \square(1)$, where $\lambda + \square(1)$ is the diagram obtained from $\lambda$ by adding a one-box row.

For a natural $k$ by $[\lambda : k]$ denote the number of rows in $\lambda$ of length $k$. It is a nonnegative integer. For two diagrams $\mu, \lambda$ such that $|\lambda| = |\mu| + 1$ we set

$$\kappa(\mu, \lambda) := \left\{ \begin{array}{ll} [\lambda : r(\lambda/\mu)], & \text{if } \mu \not\rightarrow \lambda; \\ 0, & \text{otherwise.} \end{array} \right.$$ 

All Young diagrams are organized in a graded set $\mathbb{K} := \bigsqcup_{n=0}^\infty \mathbb{K}_n$, where $\mathbb{K}_n = \{ \lambda : |\lambda| = n \}, \ n \in \mathbb{Z}_{\geq 0}$, $\mathbb{K}_0 = \{ \emptyset \}$. We introduce the structure of a graded graph on this set. This graph has edges only between consecutive “floors” $\mathbb{K}_n$ and $\mathbb{K}_{n+1}$. If $\mu \in \mathbb{K}_n$ and $\lambda \in \mathbb{K}_{n+1}$ for some $n \geq 0$, then we draw $\kappa(\mu, \lambda)$ edges between $\mu$ and $\lambda$. Let edges be oriented in the direction from $\mathbb{K}_n$ to $\mathbb{K}_{n+1}$. This graph differs from the Young graph. Namely, the latter has the same edges without multiplicities.

By $g(\mu, \lambda)$ denote the total number of oriented paths from $\mu$ to $\lambda$ in the graph $\mathbb{K}$. Clearly, $g(\mu, \lambda)$ vanishes unless $\mu \subseteq \lambda$ as diagrams (the sets of boxes in the plane). Set $g(\lambda) := g(\emptyset, \lambda)$, it appears that $g(\lambda) = |\lambda|!/((\lambda_1! \ldots \lambda_{\ell(\lambda)}!))$.

We will also need Pochhammer symbols

$$(a)_k := a(a + 1) \ldots (a + k - 1), \quad k = 1, 2, \ldots, \quad (a)_0 := 1$$

and factorial powers

$$a^{-1}_k := a(a - 1) \ldots (a - k + 1), \quad k = 1, 2, \ldots, \quad a^{-1}_0 := 1.$$ 

1.2. The up/down Markov chains. The graph structure on $\mathbb{K}$ is important, but in this paper it is needed only to use the recurrent relations (5) for $g(\mu, \lambda)$ below.

One of the most important objects under consideration are the up/down Markov chains. They were studied in [BO, Fu1, Fu2]. We use the formalism of [BO §1] to define them.

The down transition function for $\mu, \lambda \in \mathbb{K}$ such that $|\lambda| = |\mu| + 1$ is defined as

$$p^\downarrow(\lambda, \mu) := \frac{g(\mu)}{g(\lambda)} \kappa(\mu, \lambda).$$

It can be easily checked that

- $p^\downarrow(\lambda, \mu) > 0$ for all $\mu, \lambda \in \mathbb{K}$ such that $|\lambda| = |\mu| + 1$;
- $p^\downarrow(\lambda, \mu)$ vanishes unless $\mu \not\rightarrow \lambda$;

The papers [Fu1, Fu2] introduced and studied the down/up chains, but their difference from the up/down chains is minor.
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- If $|\lambda| = n \geq 1$, then $\sum_{|\mu| = n-1} p^i(\lambda, \mu) = 1$.

The object $(K, p^i)$ gives rise to partition structures $\{M_n\}_{n \geq 0}$, where $M_n$ is a probability measure on $K_n$ for every $n \geq 0$ and

$$M_n(\mu) = \sum_{\lambda: |\lambda|, \mu} M_{n+1}(\lambda)p^i(\lambda, \mu) \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \text{ and } \mu \in K_n.$$  

Here by $M_n(\mu)$ we denote the measure of a singleton $\{\mu\}$.

Fix a partition structure $\{M_n\}$. The up transition function for $\lambda, \nu \in K$ such that $|\nu| = |\lambda| + 1$ and $M_n(\lambda) \neq 0$ is defined as

$$p^i(\lambda, \nu) := \frac{M_{n+1}(\nu)}{M_n(\lambda)}p^i(\nu, \lambda).$$

This function depends on the choice of a partition structure. Moreover, $\{M_n\}$ and $p^i$ are consistent in a sense similar to (3):

$$M_{n+1}(\nu) = \sum_{\lambda: \lambda \neq \nu, \lambda \in \text{supp}(M_n)} M_n(\lambda)p^i(\lambda, \nu) \quad \text{for all } n \in \mathbb{Z}_{\geq 0} \text{ and } \nu \in K_{n+1}.$$

Let $\{M_n\}$ be a partition structure and $M_n(\lambda) > 0$ for all $n \geq 0$ and $\lambda \in K_n$. Let us define a Markov chain $T_n$ on each $K_n$, $n \geq 1$ with the following transition matrix:

$$T_n(\lambda, \tilde{\lambda}) := \sum_{\nu: |\nu| = n+1} p^i(\lambda, \nu)p^i(\nu, \tilde{\lambda}), \quad \lambda, \tilde{\lambda} \in K_n.$$

This is the composition of the up and down transition functions, from $K_n$ to $K_{n+1}$ and then back to $K_n$. From the definitions above it follows that $M_n$ is a stationary distribution for $T_n$. It can be readily shown that the matrix $M_n(\lambda)T_n(\lambda, \tilde{\lambda})$ is symmetric with respect to the substitution $\lambda \leftrightarrow \tilde{\lambda}$. This means that the chain $T_n$ is reversible with respect to $M_n$.

1.3. Ewens-Pitman’s partition structures. In the present paper we deal with a special two-parameter family of Ewens-Pitman’s partition structures. It is defined as follows.

Let $\alpha$ and $\theta$ be arbitrary parameters. We set

$$M_n(\lambda) := \frac{n!}{(\theta)_n} \cdot \frac{\theta(\theta + \alpha) \ldots (\theta + (\ell(\lambda) - 1)\alpha)}{\prod_{k=1}^{\infty} [\lambda: k!] \cdot \prod_{i=1}^{\ell(\lambda)} \lambda_i!} \prod_{\square \in \lambda, r(\square) \geq 2} (r(\square) - 1 - \alpha).$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in K_n$.

It can be proved that for all $\alpha$ and $\theta$ such that $M_n$ is well-defined $M_n$ satisfies (3) and that for such parameters we have

$$\sum_{\lambda: |\lambda| = n} M_n(\lambda) = 1 \quad \text{for all } n \in \mathbb{Z}_{\geq 0}.$$

It also can be checked that $M_n(\lambda)$ is nonnegative for all $n \geq 0$ and $\lambda \in K_n$ if and only if

- (principal series) $0 \leq \alpha < 1$ and $\theta > -\alpha$;

- (degenerate series) $\alpha = -\beta < 0$ and $\theta = N\beta$ for some $N = 2, 3, \ldots$. 


In the principal series the support of each $M_n$ is the whole $\mathbb{K}_n$. In degenerate series $M_n(\lambda) > 0$ only for $\lambda$ of length $\leq N$. It follows that when the parameters are of either principal or degenerate series, the system $\{M_n\}_{n \geq 0}$ is a partition structure. It is called the Ewens-Pitman’s partition structure. It can be easily shown that the up transition function for this partition structure equals

$$p^\uparrow(\lambda, \nu) = \begin{cases} \frac{\lambda_i - \alpha}{n + \theta} \lambda_i, & \text{if } \nu = \lambda + \Box(\lambda_i + 1) \text{ for } 1 \leq i \leq \ell(\lambda); \\ \frac{\theta + \ell(\lambda) \alpha}{n + \theta}, & \text{if } \nu = \lambda + \Box(1); \\ 0, & \text{otherwise,} \end{cases}$$

where $|\lambda| = n \in \mathbb{Z}_{\geq 0}$.

Everywhere below except \[\text{we assume that the parameters } \alpha \text{ and } \theta \text{ are of principal series. It was explained in } \text{[L2]} \text{ that in this case due to the positivity of } M_n \text{ on the whole } \mathbb{K}_n \text{ we can consider the up/down Markov chains on } \mathbb{K}_n, n \geq 1.\]

2. Symmetric functions and the Kingman simplex

2.1. Symmetric functions. Let $\Lambda$ be the (real) algebra of symmetric functions in the formal variables $y_1, y_2, \ldots$. We will need the following functions:

- **Newton power sums** $p_k = \sum_{i=1}^{\infty} y_i^k$, $k = 1, 2, \ldots$. These elements are algebraically independent and generate $\Lambda$ as a commutative unital algebra: $\Lambda = \mathbb{R} [p_1, p_2, p_3, \ldots]$.

- **Monomial functions** $m_\lambda, \lambda \in \mathbb{K}$ are defined as $\sum y_{i_1}^{\lambda_1} \cdots y_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}$, where the sum is taken over all distinct monomials, the indexes $i_1, \ldots, i_{\ell(\lambda)}$ are pairwise distinct and run from one to infinity. We also need multiples of $m_\lambda$ of the form $m_\lambda := (\prod_{k \geq 1} [\lambda : k!] m_\lambda$. They can be viewed as similar sums $\sum y_{i_1}^{\lambda_1} \cdots y_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}$, where the sum is taken over all collections of pairwise distinct indexes $i_1, \ldots, i_{\ell(\lambda)}$ from one to infinity. It is sometimes simpler to deal with $m_\lambda$ instead of $m_\lambda$. Each of the systems $\{m_\lambda\}_{\lambda \in \mathbb{K}}$ and $\{m_\lambda\}_{\lambda \in \mathbb{K}}$ is a basis for $\Lambda$ as a vector space over $\mathbb{R}$.

- **Factorial functions** $m_\lambda^!$ and $m_\lambda^{!*}$, $\lambda \in \mathbb{K}$ are obtained from $m_\lambda$ and $m_\lambda^!$, respectively, by substituting each power of a variable $y_i^k$ by the factorial power $y_i^k!$. Hence the homogeneous component of $m_\lambda^!$ and $m_\lambda^{!*}$ of maximal degree $|\lambda|$ is equal to $m_\lambda$ and $m_\lambda^!$, respectively. It follows that each of the systems $\{m_\lambda^!\}_{\lambda \in \mathbb{K}}$ and $\{m_\lambda^{!*}\}_{\lambda \in \mathbb{K}}$ is also a basis for $\Lambda$ as a vector space over $\mathbb{R}$.

Let $I := (p_1 - 1)\Lambda$ be the principal ideal in $\Lambda$ generated by $p_1 - 1$. Set $\Lambda^o := \Lambda/I$. To every element $f \in \Lambda$ corresponds an image in $\Lambda^o$ denoted by $f^o$. In particular, $p_1^o = 1$ and $\Lambda^o$ is freely generated (as a commutative unital algebra) by the elements $p_k^o, k = 2, 3, \ldots$. Moreover,

$$\Lambda = \mathbb{R} [p_1, p_2, p_3, \ldots] = I \oplus \mathbb{R} [p_2, p_3, \ldots].$$

It follows that $\Lambda^o \cong \mathbb{R} [p_2, p_3, \ldots]$. It can be easily checked that the basis for the latter algebra over $\mathbb{R}$ is $\{m_\lambda\}_{\lambda \in \mathbb{K}, |\lambda| = 0}$. Therefore, the basis for $\Lambda^o$ over $\mathbb{R}$ is $\{m_\lambda^!\}_{\lambda \in \mathbb{K}, |\lambda| = 0}$.

2.2. The Kingman simplex and moment coordinates. In the Introduction we described the Kingman simplex

$$\nabla_\infty = \left\{ x = (x_1, x_2, \ldots) \in [0, 1]^\infty : x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1 \right\}.$$
It is a compact, metrizable and separable space in the topology of coordinatewise
convergence (we use this topology throughout the paper). The simplex $\nabla_\infty$ contains
a distinguished dense subspace $\nabla_\infty := \{ x = (x_1, x_2, \ldots) \in \nabla_\infty : \sum_{i=1}^{\infty} x_i = 1 \}$. The use of the symbols $\nabla_\infty$ and $\nabla_\infty$ follows the work [EK1]. By $C(\nabla_\infty)$ denote
the algebra of real continuous functions on $\nabla_\infty$ with pointwise operations and the
supremum norm.
To every point $x \in \nabla_\infty$ we assign a probability measure

$$\nu_x := \sum_{i=1}^{\infty} x_i \delta_{x_i} + \gamma(x) \delta_0$$

on the segment $[0,1]$, where $\delta_s$ is the Dirac measure at a point $s$ and $\gamma(x) := 1 - \sum_{i=1}^{\infty} x_i$. By $q_k(x)$ denote the $k$th moment of $\nu_x$:

$$q_k(x) := \int_0^1 u^k \nu_x(du) = \sum_{i=1}^{\infty} x_i^{k+1}, \quad k = 1, 2, \ldots.$$ 

These functions are continuous on $\nabla_\infty$ because $x_i \leq i^{-1}$ for every $i = 1, 2, \ldots$. It
is worth noting that the function $\gamma(x)$ is not continuous on $\nabla_\infty$. The functions
$q_k(x), k \geq 1$ also separate points of $\nabla_\infty$ (because the measure on $[0,1]$ in uniquely
determined by its moments) and are algebraically independent. Following [BO] we

Let $\mathcal{F} = \mathbb{R}[q_1, q_2, \ldots]$ be the commutative unital algebra generated by the mo-
ment coordinates. By the Stone-Weierstrass theorem, $\mathcal{F}$ is a dense subalgebra of
$C(\nabla_\infty)$. 

2.3. Symmetric functions on the simplex. The Kingman theorem. The correspondence $p_2^\circ \rightarrow g_1(x), p_3^\circ \rightarrow g_2(x), \ldots$ establishes an isomorphism of the
algebra $\Lambda^\circ$ described in 2.2 and the algebra $\mathcal{F}$ from 2.2. Hence to each element
$f^\circ \in \Lambda^\circ$ corresponds a continuous function on $\nabla_\infty$. Denote this function by $f^\circ(x)$. In particular, $p_k^\circ(x) = \sum_{i=1}^{\infty} x_i^k, k = 2, 3, \ldots$ and $p_1^\circ(x) \equiv 1$.

Now we recall the Kingman theorem about partition structures. Consider for
$n = 1, 2, \ldots$ the following embeddings

$$\iota_n : \mathbb{K} \hookrightarrow \nabla_\infty, \quad \iota_n : (\lambda_1, \ldots, \lambda_\ell) \mapsto \left( \frac{\lambda_1}{n}, \ldots, \frac{\lambda_\ell}{n}, 0, 0, \ldots \right) \in \nabla_\infty,$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{K}$. The next remark will be useful in 4.1

Remark 2.1. The sets $\iota_n(\mathbb{K})$ approximate the space $\nabla_\infty$ in the sense that any
open subset of $\nabla_\infty$ has a nonempty intersection with $\iota_n(\mathbb{K})$ for all $n$ large enough.

Theorem 2.2 ([K3, KO]). For every partition structure $\{M_n\}$ on $\mathbb{K}$ there exists
a Borel probability measure $P$ on $\nabla_\infty$ (called the boundary measure of a partition
structure) such that

$$P = \lim_{n \to \infty} \iota_n(M_n).$$

Conversely, any partition structure can be reconstructed from its boundary measure
as follows:

$$M_n(\lambda) = g(\lambda) \int_{\nabla_\infty} m^\circ_\lambda(x) P(dx) \quad \text{for all } \lambda \in \mathbb{K}.$$ 

Hence the partition structures on $\mathbb{K}$ and Borel probability measures on $\nabla_\infty$ are in
one-to-one correspondence.
To the Ewens-Pitman’s partition structure (§1.3) with parameters α and θ of principal or degenerate series corresponds the well-known Poisson-Dirichlet measure PD(α, θ). We have

\[ \int_{\nabla_{\infty}} m_\lambda^*(x) \text{PD}(\alpha, \theta)(dx) = \frac{(-\theta/\alpha)^{\ell(\lambda)}}{(\theta)_{|\lambda|}} \prod_{i=1}^{\ell(\lambda)} (-\alpha)_{\lambda_i} \quad \text{for all} \ \lambda \in \mathbb{K}. \]

When α = 0, the value of the RHS is obtained by continuity.

It is known (see, e.g., [PPY, Pi3]) that the measure PD(α, θ) is concentrated on the dense subset \( \nabla_{\infty} \subset \nabla_\infty \) (the definition of \( \nabla_{\infty} \) see in (2.2)).

3. The up/down Markov chains transition operators’ action on symmetric functions

Here we assume that the parameters α and θ are of principal series. Let us introduce some extra notation first.

For every set \( \mathcal{X} \) denote by \( \text{Fun}(\mathcal{X}) \) the algebra of real functions on \( \mathcal{X} \) with pointwise operations.

Consider an embedding of the algebra of symmetric functions \( \Lambda \) into the algebra \( \text{Fun}(\mathbb{K}) \) defined on the generators \( p_k, k = 1, 2, \ldots \) as follows: \( p_k \to p_k(\lambda) := \sum_{i=1}^{\ell(\lambda)} \lambda_i^k \). Thus, to every element \( f \in \Lambda \) corresponds a function from \( \text{Fun}(\mathbb{K}) \). Denote this function by \( f(\lambda) \).

Let \( f \in \Lambda \). By \( f_n \) denote the restriction of the function \( f(\cdot) \in \text{Fun}(\mathbb{K}) \) to \( \mathbb{K}_n \subset \mathbb{K} \). It can be easily checked that the subalgebra \( \Lambda \subset \text{Fun}(\mathbb{K}) \) separates points. Therefore, the functions of the form \( f_n \), with \( f \in \Lambda \), exhaust the (finite-dimensional) space \( \text{Fun}(\mathbb{K}_n) \).

The aim of this section is to prove the following proposition which is needed for Lemma 3.1 below.

**Proposition 3.1.** Consider the transition operator of the nth up/down Markov chain \( T_n : \text{Fun}(\mathbb{K}_n) \to \text{Fun}(\mathbb{K}_n) \) which corresponds to the two-parameter Ewens-Pitman’s partition structure. Its action on the functions \( m^*_\mu, \mu \in \mathbb{K} \) looks as follows:

\[
(T_n - 1)(m^*_\mu)_n = -\frac{k(k-1+\theta)}{(n+1)(n+\theta)}(m^*_\mu)_n \\
+ \frac{n+1-k}{(n+1)(n+\theta)} \sum_{i=1}^{\ell(\mu)} \mu_i (\mu_i - 1 - \alpha) (m^*_{\mu-\square(\mu_i)})_n \\
+ \frac{n+1-k}{(n+1)(n+\theta)} [\mu : 1] (\theta + \alpha(\ell(\mu) - 1)) (m^*_{\mu-\square(1)})_n,
\]

where \( 1 \) denotes the identity operator and \( k = |\mu| \).

**Remark 3.2.** The Proposition states that \((T_n - 1)(m^*_\mu)_n\) is a linear combination of the functions \((m^*_\mu)_n\) (the first summand in the RHS) and the functions of the form \((m^*_\nu)_n\) for all \( \nu : \nu \neq \mu \). The sum over \( i \) in the RHS deals with \( \nu \) of the same length as \( \mu \). Note that this sum contains similar terms if \( \mu \) has equal rows of length \( \geq 2 \). The last summand is nonzero if \( [\mu : 1] > 0 \) and in this case corresponds to the diagram \( \nu = \mu - \square(1) \) obtained from \( \mu \) by deleting a one-box row.

---

4The functions \( m_\lambda \) are used instead of \( m_\lambda \) to simplify the notation.
Let us give two formulas concerning factorial functions and the numbers of paths $g(\mu, \lambda)$ defined in (1.1). We use these formulas below to proof the Proposition 3.1.

Let $|\lambda| = n \geq m = |\mu|$, $\lambda, \mu \in \Lambda$. Then

$$g(\mu, \lambda) = \left(\frac{m!}{(n-1)! \cdots (n-m+1)}\right)^{\mu}_\lambda,$$

and

$$g(\mu, \lambda) = \sum_{\kappa(\lambda) \neq \mu} \kappa(\lambda)g(\mu, \lambda).$$

The formula (4) can be checked directly, and (5) follows from the definitions of $g(\mu, \lambda)$ and $\kappa(\lambda, \mu)$, see (1.1).

In order to prove Proposition 3.1 let us write the operator $T_n$ as a composition of “down” $D_{n+1,n}: \text{Fun}(\mathbb{K}_n) \to \text{Fun}(\mathbb{K}_{n+1})$ and “up” $U_{n,n+1}: \text{Fun}(\mathbb{K}_{n+1}) \to \text{Fun}(\mathbb{K}_n)$ operators acting on functions.

The operator $D_{n+1,n}$ is constructed using the down transition probabilities and does not depend on the parameters $\alpha$ and $\theta$. The operator $U_{n,n+1}$ is constructed using the up transition probabilities and depends on the parameters. Namely,

$$(D_{n+1,n}f_n)(\lambda) := \sum_{\mu: \mu \neq \lambda} p^1(\lambda, \mu)f_n(\mu), \quad \lambda \in \mathbb{K}_{n+1}$$

and

$$(U_{n,n+1}f_{n+1})(\mu) := \sum_{\lambda: \lambda \neq \mu} p^1(\mu, \lambda)f_{n+1}(\lambda), \quad \mu \in \mathbb{K}_n.$$  

These operators are adjoint to the corresponding operators acting on measures. The latter act in agreement with their names, e.g., $D^*_{n+1,n}: \mathcal{M}(\mathbb{K}_{n+1}) \to \mathcal{M}(\mathbb{K}_n)$, where $\mathcal{M}(\mathcal{X})$ is the space of measures on $\mathcal{X}$.

It clearly follows from the definition of the $n$th up/down Markov chain (1.2) that $T_n = U_{n,n+1} \circ D_{n+1,n}: \text{Fun}(\mathbb{K}_n) \to \text{Fun}(\mathbb{K}_n)$, $n \in \mathbb{Z}_{>0}$. Proposition 3.1 directly follows from

**Lemma 3.3.** (1) There exists a unique operator $\bar{D}: \Lambda \to \Lambda$ such that

$$D_{n+1,n}f_n = \frac{1}{n+1}(\bar{D}f)n+1$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $f \in \Lambda$. In the basis $\{\mathbf{m}_\mu\}_{\mu \in \mathbb{K}}$ for the algebra $\Lambda$ this operator has the form

$$\bar{D}\mathbf{m}_\mu = (p_1 - |\mu|)\mathbf{m}_\mu.$$  

(2) There exists a unique operator $\bar{U}: \Lambda \to \Lambda$ such that

$$U_{n,n+1}g_{n+1} = \frac{1}{n+\theta}(\bar{U}g)n$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $g \in \Lambda$. In the basis $\{\mathbf{m}_\mu\}_{\mu \in \mathbb{K}}$ for the algebra $\Lambda$ this operator has the form

$$\bar{U}\mathbf{m}_\mu = (p_1 + \theta + |\mu|)\mathbf{m}_\mu + \sum_{i=1}^{\ell(\mu)} \mu_i(\mu_i - 1 - \alpha)\mathbf{m}_\mu - 1(\mu_i)$$

$$+ \left[\mu : 1\right](\theta + \alpha(\ell(\mu) - 1))\mathbf{m}_\mu - 1(1).$$
Note that in the RHS of (5) there is a sum similar to the one explained in Remark 3.2.

Proof. (1) Let us show that the operator $\tilde{D}$ defined by (7) is the desired one. We use the connection (4) of factorial functions with the numbers of paths $g(\mu, \lambda)$ and the recurrent relations (5) for the latter.

Let $\mu \in K$, $\nu \in K_{n+1}$. We suppose that $n \geq |\mu|$, because otherwise $(m^*_\mu)_n = 0$. We have

$$D_{n+1,n}(m^*_\mu)(\nu) = \sum_{\lambda: \lambda \succ \nu} p^1(\nu, \lambda)(m^*_\lambda)_n(\lambda)$$

$$= \sum_{\lambda: \lambda \succ \nu} \frac{g(\lambda)}{g(\nu)} \kappa(\lambda, \nu)(m^*_\lambda)_n(\lambda) = \sum_{\lambda: \lambda \succ \nu} \frac{g(\lambda)}{g(\nu)} \kappa(\lambda, \nu) n^{\;|\mu|} \frac{g(\mu, \lambda)}{g(\lambda)}$$

$$= \frac{1}{g(\nu)} \sum_{\lambda: \lambda \succ \nu} \kappa(\lambda, \nu) g(\mu, \lambda) = n^{\;|\mu|} \frac{g(\mu, \nu)}{g(\nu)} = \frac{n+1 - |\mu|}{n+1} (m^*_\mu)_{n+1}(\nu).$$

Thus,

$$\frac{1}{n+1} \tilde{D}m^*_\mu(\nu) = \frac{1}{n+1} ((p_1 - |\mu|)m^*_\mu)(\nu),$$

because $p_1(\nu) = |\nu| = n+1$. It follows that in the basis $\{m^*_\mu\}_{\mu \in K}$ the operator $\tilde{D}$ has the form $\tilde{D}m^*_\mu = (p_1 - |\mu|)m^*_\mu$. If we multiply both sides by $\prod_{k=1}^{|\mu|} [\mu : k]!$, then we get the desired expression in the basis $\{m^*_\mu\}_{\mu \in K}$. The uniqueness of the operator $\tilde{D}$ follows from the fact that $\Lambda$ is embedded into $\text{Fun}(K)$.

(2) Fix $\mu \in K$ and denote $l := \ell(\mu)$. Let $\nu \in K_n$. We have an explicit expression

$$(m^*_\mu)_n(\nu) = \sum_{j_1 \;\cdots\; j_l} \nu_{j_1} \cdots \nu_{j_l},$$

where the sum is taken over all pairwise distinct $j_1, \ldots, j_l$ from 1 to $\ell(\nu)$. Thus, we can verify (5) directly.

From the definition of the operator $U_{n,n+1}$ (6) and the formula for the up transition probabilities of the Ewens-Pitman’s partition structure (3.3) it follows that

$$U_{n,n+1}(m^*_\mu)(\nu) = \frac{1}{n+\theta} \left[ \sum_{i=1}^{\ell(\nu)} (\nu_i - \alpha) (m^*_\mu)_{n+1}(\nu + \Box(\nu_i + 1)) + (\theta + \ell(\nu)\alpha) (m^*_\mu)_{n+1}(\nu + \Box(1)) \right].$$

Let us transform the RHS and obtain (5).

1. First, note two simple properties:

   1a. $(a+1) b^a - a^b = b \cdot a^{(b-1)}$,  
   1b. $a \cdot a^b = a^{(b+1)} + b \cdot a^b$.

2. Let us deal with the standalone summand in the RHS of (5). We show that

$$\frac{n}{n+1} (m^*_\mu)(\nu + \Box(1)) = (m^*_\mu)(\nu) + |\mu| \left(\sum_{i=1}^{\ell(\nu)} (\nu_i - \alpha) (m^*_\mu)_{n+1}(\nu + \Box(\nu_i + 1)) + (\theta + \ell(\nu)\alpha) (m^*_\mu)_{n+1}(\nu + \Box(1)) \right).$$

It is clear that $(m^*_\mu)(\nu + \Box(1)) = \sum (\nu + \Box(1))^{j_{\mu_1}} \cdots (\nu + \Box(1))^{j_{\mu_l}}$, where the sum is taken over all pairwise distinct $j_1, \ldots, j_l$ from 1 to $\ell(\nu) + 1$. To every combination of indexes $j_1, \ldots, j_l$ we assign a number $c \in \{1, \ldots, l\}$ defined by the condition $j_c = \ell(\nu) + 1$. It follows that $(\nu + \Box(1))^{j_{\mu_c}} = 1^{\mu_c}$, and for other $j_k$
we have \((\nu + \square(1))_{j_k} = \nu_{j_k}\). Let us combine the summands of the form \((\nu + \square(1))_{j_1}^{\mu_1} \ldots (\nu + \square(1))_{j_l}^{\mu_l}\) with equal \(c\), i.e., let us write

\[
(m^*_\mu)_{n+1}(\nu + \square(1)) = (m^*_\mu)_n(\nu) + \sum_{c=1}^l \nu_i^{\mu_c} \left( \sum_{w/o j_c} \nu_{j_1}^{\mu_1} \ldots \nu_{j_{c-1}}^{\mu_{c-1}} \nu_{j_{c+1}}^{\mu_{c+1}} \ldots \nu_{j_l}^{\mu_l} \right).
\]

The last sum \(\sum_{w/o j_c}\) means the sum taken over all pairwise distinct indexes \(j_1, \ldots, j_{c-1}, j_{c+1}, \ldots, j_l\) from 1 to \(\ell(\nu)\). We denote this sum by \(S_c\) to simplify the notation. Note that \(1^{1^b} = 0\) if \(b \geq 2\), and \(1^{1^1} = 1\). Therefore,

\[
\sum_{c=1}^l \nu_i^{\mu_c} S_c = [\mu : 1](m^*_\mu - \square(1))_n(\nu),
\]

and this implies (10).

3. Now we deal with the sum over \(i\) in the RHS of (9).

First, fix an arbitrary \(i\) from 1 to \(\ell(\nu)\). We have

\[
(m^*_\mu)_{n}(\nu) = \sum_{c=1}^l \nu_i^{\mu_c} S_c + \sum_{j_1, \ldots, j_l \neq i} \nu_i^{\mu_1} \ldots \nu_{j_l}^{\mu_l};
\]

\[
(m^*_\mu)_{n+1}(\nu + \square(\nu_i + 1)) = \sum_{c=1}^l (\nu_i + 1)^{\mu_c} S_c + \sum_{j_1, \ldots, j_l \neq i} \nu_i^{\mu_1} \ldots \nu_{j_l}^{\mu_l}.
\]

Using 1a, we get

\[
(m^*_\mu)_{n+1}(\nu + \square(\nu_i + 1)) - (m^*_\mu)_n(\nu) = \sum_{c=1}^l \mu_c \nu_i^{\mu_c - 1} S_c.
\]

Thus, the sum over \(i\) in the RHS of (9) becomes

\[
\sum_{i=1}^{\ell(\nu)} (\nu_i - \alpha)(m^*_\mu)_{n+1}(\nu + \square(\nu_i + 1)) = (|\nu| - \ell(\nu)\alpha)(m^*_\mu)_n(\nu) + \sum_{i=1}^{\ell(\nu)} \sum_{c=1}^l (\nu_i - \alpha) \mu_c \nu_i^{\mu_c - 1} S_c.
\]

Now fix an arbitrary \(c\) from 1 to \(l\). Using 1b, we get

\[
\sum_{i=1}^{\ell(\nu)} (\nu_i - \alpha) \mu_c \nu_i^{\mu_c - 1} S_c
\]

\[
= S_c \left[ \sum_{i=1}^{\ell(\nu)} \mu_i \nu_i^{\mu_i - 1} + \mu_c (\mu_c - 1) \sum_{i=1}^{\ell(\nu)} \nu_i^{\mu_c - 1} - \alpha \mu_c \sum_{i=1}^{\ell(\nu)} \nu_i^{\mu_c - 1} \right]
\]

\[
= \left\{ \begin{array}{ll}
\mu_c (m^*_\mu)_n(\nu) + \mu_c (\mu_c - 1 - \alpha)(m^*_\mu - \square(\mu_c))_n(\nu), & \text{if } \mu_c \geq 2; \\
(m^*_\mu)_n(\nu) - \alpha (\ell(\nu) - (l - 1))(m^*_\mu - \square(1))_n(\nu), & \text{if } \mu_c = 1.
\end{array} \right.
\]
4. If we put together 2 and 3 and recall that $|\nu| = n$ and $\ell(\mu) = l$, we get
\[
(U_{n}^{\pm}(m^*_{\mu})_{n+1})_{n}(\nu) = (m^*_{\mu})_{n}(\nu) + \frac{1}{p_{1}(\nu) + \theta} [\mu |(m^*_{\mu})_{n}(\nu)]
\]
\[
+ \sum_{\mu \geq 2} \mu_{c} (\mu_{c} - 1 - \alpha) (m^*_{\mu-\square(\mu_{c})})_{n}(\nu)
\]
\[
+ [\mu : 1] (\theta + \alpha (\ell(\mu) - 1)) (m^*_{\mu-\square(1)})_{n}(\nu).\]

This coincides with the desired expression (3). The uniqueness of the operator $\tilde{U}$ again follows from the fact that $\Lambda \hookrightarrow \text{Fun}(\mathbb{K})$ is embedding.

4. Convergence of the up/down Markov chains

In this section we consider the limit, as $n \to \infty$, of the Markov chains $T_{n}$ on $\mathbb{K}_{n}$. We assume that the parameters $\alpha$ and $\theta$ are of principal series. The limit is a continuous time Markov process on $\nabla_{\infty}$ denoted by $X_{\alpha, \theta}(t)$. The argument in this section uses the results of the book [EK2]. The application of these results is based on the algebraic calculations of §3 and is similar to [BO, §1]. Due to this similarity the proofs repeating those from [BO] are not given.

4.1. The construction of the process in the simplex. First, let us introduce essential notation.

Each operator $T_{n}$ acts in a finite-dimensional space of functions $\text{Fun}(\mathbb{K}_{n}), n = 1, 2, \ldots$. These spaces can be viewed as Banach spaces with the supremum norm $\| \cdot \|_{n}$. Recall the embeddings $\iota_{n}: \mathbb{K}_{n} \hookrightarrow \nabla_{\infty}$ introduced in §2.3. Let $\pi_{n}$ denote the corresponding projections of function spaces:
\[
(\pi_{n}(f))(\lambda) := f(\iota_{n}(\lambda)), \quad \lambda \in \mathbb{K}_{n}, \quad f \in C(\nabla_{\infty}).
\]

In §2 we introduced a dense subalgebra $\mathcal{F} \subset C(\nabla_{\infty})$. This algebra $\mathcal{F}$ admits an ascending filtration by finite-dimensional subspaces
\[
\mathcal{F}^{0} \subset \mathcal{F}^{1} \subset \mathcal{F}^{2} \cdots \subset \mathcal{F}, \quad \bigcup_{m=0}^{\infty} \mathcal{F}^{m} = \mathcal{F}.
\]

Recall that $\mathcal{F} \cong \Lambda^{\circ} = \Lambda / (p_{1} - 1)\Lambda$. We define the filtration $\{\mathcal{F}^{m}\}$ of $\Lambda^{\circ}$ as the image of the filtration of the algebra $\Lambda$ by degrees of polynomials (in the formal variables $y_{i}$, see §2.2).

From Remark 2.1 and the fact $\dim \mathcal{F}^{m} < \infty$ it follows that each projection $\pi_{n}$ is one-to-one on $\mathcal{F}^{m}$ (for fixed $m$) for all $n$ large enough.

Now we prove that the generators of the up/down Markov chains converge to some operator $A$ in the space $C(\nabla_{\infty})$.

**Lemma 4.1.** Each $\pi_{n}(\mathcal{F}^{m})$ is invariant under the operator $T_{n}$ (for fixed $m$) for all $n$ large enough. Under the identification $\mathcal{F}^{m} = \pi_{n}(\mathcal{F}^{m})$ there exists a limit
\[
\lim_{n \to \infty} n^{2} (T_{n} - 1) f = Af \quad \text{for all } f \in \mathcal{F}
\]

in any finite-dimensional space $\mathcal{F}^{m}$. In this way we obtain an operator $A: \mathcal{F} \to \mathcal{F}$ with the property $A\mathcal{F}^{m} \subset \mathcal{F}^{m}, m \geq 0$. The action of this operator on the functions

5Here by 1 we denote the identity operator.
\( m^\circ_\mu \in \Lambda^\circ \cong \mathcal{F}, \mu \in \mathbb{K} \) (they were defined in [2.7]) has the following explicit form:

\[
A m^\circ_\mu = -|\mu|(|\mu| - 1 + \theta) m^\circ_\mu + \sum_{c=1}^{\ell(\mu)} \mu_c (\mu_c - 1 - \alpha) m^\circ_{\mu-\square(\mu_c)} + |\mu : 1| (\theta + \alpha (\ell(\mu) - 1)) m^\circ_{\mu-\square(1)}.
\]

(12)

Note that \( A \) sends the function \( m^\circ_\mu \equiv 1 \) to 0, as it should be.

**Proof.** It is clear that for all \( n \) large enough the space \( \pi_n(\mathcal{F}^m) \) is the linear span of the functions \( (m^\circ_{\mu})_n \) with \( |\mu| \leq m \). Therefore, the invariance of \( \pi_n(\mathcal{F}^m) \) under \( T_n \) follows from Proposition 3.1 because it is clear that the operator \( T_n \) does not increase the degree of each function \( (m^\circ_{\mu})_n \in \text{Fun}(\mathbb{K}_n) \) (this degree is equal to \( |\mu| \)).

Thus, identifying \( \pi_n(\mathcal{F}^m) \) with \( \mathcal{F}^m \) (which makes sense for fixed \( n \) and \( n \) large enough), we may say that each \( \mathcal{F}^m \) is invariant under \( T_n \). Now we can prove (11) and (12) together.

Consider the map \( \Lambda \rightarrow \text{Fun}(\mathbb{K}_n) \) defined as

\[
f_{[n]}(\lambda) := \pi_n(f^\circ(\lambda)) = f^\circ \left( \frac{\lambda_1}{n}, \frac{\lambda_2}{n}, \ldots, \frac{\lambda_{\ell(\lambda)}}{n}, 0, 0, \ldots \right), \quad \lambda \in \mathbb{K}_n, \ f \in \Lambda.
\]

This is the restriction of an element \( f^\circ \in C(\nabla_\infty) \) to \( \iota_n(\mathbb{K}_n) \subset \nabla_\infty \). Let \( G_s : \Lambda \rightarrow \Lambda \) for all \( s > 0 \) be an automorphism of the algebra defined on its basis as

\[
G_s m_\mu = s^{|\mu|} m_\mu, \quad \mu \in \mathbb{K}.
\]

On the homogeneous component of degree \( k, \ k = 0, 1, 2, \ldots \) this automorphism reduces to multiplication by the number \( s^k \).

Therefore, we have the following expression for \( f_n \), where \( f \in \Lambda \):

\[
f_n(\lambda) = (G_n f)_{[n]}(\lambda), \quad \lambda \in \mathbb{K}_n.
\]

(13)

Indeed, since \( \iota_n(\mathbb{K}_n) \subset \nabla_\infty \), it follows that for all \( g \in \Lambda \) and \( \lambda \in \mathbb{K}_n \) the value \( g^\circ(\iota_n(\lambda)) = g_{[n]}(\lambda) \) is equal to the formal evaluation of the element \( g \in \Lambda \) on the coordinates of the point \( \iota_n(\lambda) = (\lambda_1/n, \ldots, \lambda_{\ell(\lambda)}/n, 0, 0, \ldots) \in \nabla_\infty \) (see Remark 5.2 below). In [2] we pointed out that the homogeneous component of maximal degree of \( m^\circ_\mu \) is \( m_\mu \). Thus, for all \( \mu \in \mathbb{K} \) we have

\[
\lim_{n \to \infty} n^{-|\mu|} G_n m^\circ_\mu = m_\mu.
\]

Now, to conclude the proof of the Lemma, express each function of the form \( f_n(\lambda) \) in the formula for \( (T_n - 1)(m^\circ_\mu)_n \) from Proposition 3.1 in terms of (13). It remains to multiply the result by \( n^{-|\mu|} \) and take the limit as \( n \to \infty \).

\[\square\]

Note that the functions \( m^\circ_\mu, \ \mu \in \mathbb{K} \) are not linearly independent. However, from the proof of the Lemma it follows that the operator \( A \) is well-defined by (12).

Now proceed to the convergence of discrete semigroups \( \{1, T_n, T_n^2, \ldots \} \) to a conservative Markov semigroup \( \{T(t)\}_{t \geq 0} \) in the Banach space \( C(\nabla_\infty) \) constructed using the operator \( A \). Recall that a conservative Markov semigroup is a strongly continuous semigroup of contraction operators in \( C(\nabla_\infty) \) preserving positive functions and the constant 1.

---

6The homogeneous component of degree \( k \) in the algebra \( \Lambda \) consists of all homogeneous symmetric functions of degree \( k \). This set is the linear span of the elements \( m_\mu, \ |\mu| = k \).
We use the definition from [EK2, Chapter 1, Section 6] to introduce strict sense to the concept of convergence of discrete semigroups to a continuous one:

**Definition 4.2.** We say that a sequence of functions \( \{f_n \in \text{Fun}(K_n)\} \) converges to a function \( f \in C(\nabla_\infty) \) if \( \|f_n - \pi_n(f)\|_n \to 0 \) as \( n \to \infty \). Here \( \|\cdot\|_n \) is the supremum norm of the space \( \text{Fun}(K_n) \). In this case we write \( f_n \to f \).

To establish a convergence of the semigroups \( \{1, T_n, T_n^2, \ldots\} \) to \( \{T(t)\}_{t \geq 0} \) one must perform a natural scaling of time: one step of the \( n \)th Markov chain corresponds to a small time interval of order \( n^{-2} \). This convergence is established in

**Proposition 4.3.** (1) The operator \( A: \mathcal{F} \to \mathcal{F} \) defined in Lemma 4.1 is closable in the space \( C(\nabla_\infty) \):

(2) The closure \( \overline{A} \) of the operator \( A \) generates a conservative Markov semigroup \( \{T(t)\}_{t \geq 0} \) in \( C(\nabla_\infty) \):

(3) Discrete semigroups \( \{1, T_n, T_n^2, \ldots\} \) converge, as \( n \to \infty \), to the semigroup \( \{T(t)\}_{t \geq 0} \) in the following sense:

\[
T_n^{[n^2]} \pi_n(f) \to T(t)f \quad \text{for all } f \in C(\nabla_\infty)
\]

(the limit in understood according to Definition 1.2) for all \( t \geq 0 \) uniformly on bounded intervals.

This Proposition can be proved similarly to [BO, Proposition 1.4] and follows from the convergence of the generators (Lemma 4.1). The proof uses general statements [EK2, Chapter 1, Theorem 6.5] and [EK2, Chapter 1, Lemma 2.11].

Next, it directly follows from [EK2, Chapter 4, Theorem 2.7] that \( \{T(t)\}_{t \geq 0} \) is a semigroup corresponding to a strong Markov process with càdlàg sample paths which can start from any point and any probability distribution.

We will call the operator \( A \) the pre-generator of both the semigroup \( \{T(t)\}_{t \geq 0} \) and the process \( X_{t, \theta}(t) \). Thus, for every pair of parameters \( \alpha, \theta \) of principal series we have constructed a Markov process \( X_{\alpha, \theta}(t) \) in \( \nabla_\infty \) which can start from any point and any probability distribution.

### 4.2. Some properties of the process in the simplex.

In this subsection we formulate and comment the properties of the process \( X_{t, \theta}(t) \) which are similar to the ones stated in [BO].

First, we formulate the properties of the process that directly follow from its construction as a limit of finite Markov chains \( T_n \) preserving measures \( M_n \) on \( K_n \). To prove them we use the convergence of measures \( M_n \) (the Kingman theorem, §2.3) and the properties of the up/down chains (§1.2).

**Invariant measure** (Cf. [BO, Proposition 1.6]). *The Poisson-Dirichlet distribution \( \text{PD}(\alpha, \theta) \) is an invariant measure for the process \( X_{\alpha, \theta}(t) \).*

This follows from the fact that each chain \( T_n \) preserves the measure \( M_n \).

**Reversibility of the process** (Cf. [BO, Proposition 1.7 and Theorem 7.3 (2)]). *The process \( X_{\alpha, \theta}(t) \) is reversible with respect to the measure \( \text{PD}(\alpha, \theta) \).*

This follows from the fact that each chain \( T_n \) is reversible with respect to \( M_n \).

**Convergence of finite-dimensional distributions** (Cf. [BO, Proposition 1.8]). *Let \( X_{t, \theta}(t) \) and all the chains \( T_n \) are viewed in equilibrium (that is, starting from the invariant distribution). Then the finite-dimensional distributions for the \( n \)th chain converge, as \( n \to \infty \), to the corresponding finite-dimensional distributions.*
of the process $X_{\alpha, \theta}(t)$. Here we assume a natural scaling of time described before Proposition 4.3.

We proceed to the properties that follow from the expression (12) for the pre-generator $A$ of the process $X_{\alpha, \theta}(t)$.

The spectrum of the Markov generator in $L^2(\nabla_\infty, PD(\alpha, \theta))$. The pre-generator $A$ acts in the space $\mathcal{F} \cong \Lambda^\circ$. Define an inner product in it:

$$(f, g)_{PD(\alpha, \theta)} := \int_{\nabla_\infty} f(x)g(x)PD(\alpha, \theta)(dx).$$

Then the space $\Lambda^\circ$ can be decomposed into the orthogonal direct sum of eigenspaces of the operator $A$. The spectrum of the operator $A$ looks as follows:

$$\{0\} \cup \{-\sigma_m : m = 2, 3, \ldots\}, \quad \sigma_m = m(m - 1 + \theta).$$

The eigenvalue 0 is simple and the multiplicity of each $-\sigma_m$ is the number of partitions of $m$ without parts equal to 1.

The existence of such a decomposition of $\Lambda^\circ$ follows from the fact that the pre-generator $A$ is symmetric (see the reversibility property above) and preserves the filtration $\{F_m\}$ of the space $\mathcal{F} \cong \Lambda^\circ$. The fact that the operator $A$ is triangle in the basis $\{m^\circ_{\mu}\}_{\mu \in K, [\mu : 1] = 0}$ (compatible with the filtration $\{F_m\}$) implies the facts about the eigenstructure.

The uniqueness of the invariant measure. The measure $PD(\alpha, \theta)$ is a unique invariant measure for the process $X_{\alpha, \theta}(t)$. This can be proved similar to [BO, Theorem 7.3 (1)].

Ergodicity. The process $X_{\alpha, \theta}(t)$ is ergodic with respect to the measure $PD(\alpha, \theta)$, that is,

$$\lim_{t \to +\infty} \|T(t)f - \int_{\nabla_\infty} f(x)PD(\alpha, \theta)(dx)\| = 0 \quad \text{for all } f \in C(\nabla_\infty),$$

where $\| \cdot \|$ is the supremum norm of the space $C(\nabla_\infty)$. This follows from the existence of a spectral gap of the process’ generator, see the eigenstructure above. A detailed proof is given in [BO, Theorem 7.3 (3)].

5. THE PRE-GENERATOR OF THE PROCESS IN THE SIMPLEX AS A DIFFERENTIAL OPERATOR

This section performs a more detailed study of the properties of the pre-generator $A: \mathcal{F} \to \mathcal{F}$ defined in Lemma 4.1. In §5.1 and §5.2 we assume the parameters $\alpha$ and $\theta$ to be of principal series, and in §5.3 we study degenerate values of them.

5.1. Pre-generator in natural coordinates. Recall that the process $X_{\alpha, \theta}(t)$ in $\nabla_\infty$ has the generator $\overline{A}$ (the closure of the pre-generator $A: \mathcal{F} \to \mathcal{F}$). The algebra $\mathcal{F} \cong \Lambda^\circ$ is defined in [2], and the operator $A$ is given by (12) in the basis $\{m^\circ_{\mu}\}_{\mu \in K, [\mu : 1] = 0}$. It turns out that in this basis the operator $A$ can be written as a second order differential operator in natural coordinates $x_1, x_2, \ldots$ on $\nabla_\infty$. First, let us express the functions $m^\circ_{\mu}(x)$ in these coordinates.

**Proposition 5.1.** For all $\mu \in K$ such that $[\mu : 1] = 0$ the function $m^\circ_{\mu}(x)$ on $\nabla_\infty$ can be written as

$$m^\circ_{\mu}(x) = \sum_{i_1, \ldots, i_{[\mu : 1]} \in K} x_1^{i_1} \cdots x_{{[\mu : 1]}}^{i_{[\mu : 1]}}, \quad x = (x_1, x_2, \ldots) \in \nabla_\infty.$$
where the sum is taken over all pairwise distinct indexes $i_1, \ldots, i_{\ell(\mu)}$ from one to infinity.

Note that if $[\mu : 1] > 0$ and $\sum_{i=1}^{\infty} x_i < 1$, then this statement is false.

**Proof.** The Proposition follows from a more general result by S. Kerov [Ke]. This result allows to evaluate any function of the form $m_\lambda^\circ$, $\lambda \in \mathbb{K}$ at any point $x \in \nabla_\infty$. By $f(x_1, x_2, \ldots)$ denote the formal evaluation of the element $f \in \Lambda$ at the point $x \in \nabla_\infty$. Let $\lambda \in \mathbb{K}$, $[\lambda : 1] = r \geq 0$. Then

$$m_\lambda^\circ(x) = \sum_{k=0}^{r} C^k_r (\gamma(x))^k m_{\lambda-k \cdot \square(1)}(x_1, x_2, \ldots), \quad x \in \nabla_\infty.$$ 

Here $m_\lambda^\circ(x)$ is the value of the function $m_\lambda^\circ$ at the point $x$ (this value was constructed in [2] using the factorization of the algebra $\Lambda$); $\gamma(x) = 1 - \sum_{i=1}^{\infty} x_i$; and $\lambda - k \cdot \square(1)$ is the diagram obtained from $\lambda$ by deleting $k$ one-box rows.

It is clear that the Proposition is a special case of this formula. \hfill \Box

**Remark 5.2.** It follows from the proof that the function $g^\circ(x)$ on $\nabla_\infty$ for every element $g$ from the (non-factorized) algebra $\Lambda$ can be constructed not only using the factorization of $\Lambda$ (as explained in [2]). This can also be done directly. Namely, for every point $x \in \nabla_\infty$ (for which $\sum_{i=1}^{\infty} x_i = 1$) we set $g^\circ(x)$ to be the formal evaluation of the symmetric function $g \in \Lambda$ at the coordinates of the point $x = (x_1, x_2, \ldots)$. We then extend the function $g^\circ(x)$ to the whole simplex $\nabla_\infty$ by continuity. This method is suggested by the comment after the formula [EKI] (2.10).

Now it is not hard to compute the pre-generator $A$ in natural coordinates. Let $D$ denote the following formal expression:

$$D = \sum_{i=1}^{\infty} x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j=1}^{\infty} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{\infty} (\theta x_i + \alpha) \frac{\partial}{\partial x_i}.$$ 

It can be easily checked (using Proposition 5.1) that for all $\mu \in \mathbb{K}$ for which $[\mu : 1] = 0$ we have

$$Dm_\mu^\circ = Am_\mu^\circ = -[\mu](|\mu| - 1 + \theta)m_\mu^\circ + \sum_{c=1}^{\ell(\mu)} \mu_c (\mu_c - 1 - \alpha)m_{\mu - \square(\mu_c)}^\circ.$$ 

It follows that $Df = Af$ for all $f \in \mathcal{F} \subset C(\nabla_\infty)$. Therefore, the formula (2) from Introduction is justified.

**Remark 5.3.** The RHS of (14) can be understood in two equivalent ways as follows. Let $f \in \mathcal{F}$ and we want to compute $Df = Af$. Then we can either

1. express $f$ as a linear combination of vectors of the basis $\{m_\mu^\circ\}_{\mu \in \mathbb{K}, [\mu : 1] = 0}$ for the algebra $\mathcal{F}$, and then apply the operator $D = A$ to each $m_\mu^\circ$ separately; or

2. compute $Af(x)$ first for $x \in \nabla_\infty$ directly applying to $f$ the RHS of (14), and then extend $Af(x)$ to the whole $\nabla_\infty$ by continuity (cf. this method with Remark 5.2).

**Remark 5.4.** One might want to apply $D$ (the RHS of (14)) to functions $f$ not in $\mathcal{F}$. However, if for a function $f \in C(\nabla_\infty)$ the expression $Df$ has a meaning, it is
unevident that \( f \) enters the domain of the generator \( \overline{A} \). It is even less evident that the expression \( Df \) coincides with \( \overline{A}f \).

Consider an example \( f = x_1 + \frac{\theta}{\theta} \), \( \theta \neq 0 \). It is clear that \( Df = -\theta f \). Had the equality \( Df = \overline{A}f \) held, then \( f \) would be an eigenfunction of the generator \( \overline{A} \) corresponding to the eigenvalue \( -\theta \). Since \( f \in L^2(\nabla_\infty, \mathbb{P}D(\alpha, \theta)) \) and \( -\theta \) is not an eigenvalue of \( \overline{A} \) in this space (see (12)), we get a contraction.

As shown in the paper [Schm], in the case \( \alpha = 0 \) the function \( f = x_1 \) enters the domain of the Dirichlet form corresponding to the operator \( \overline{A} \) (and, therefore, the domain of the operator \( \overline{A}^{1/2} \)). Putting aside the question whether the function \( f = x_1 + \frac{\theta}{\theta} \) (and, therefore, the coordinate function \( x_1 \)) enters the domain of \( \overline{A} \), we conclude that the equality \( Df = \overline{A}f \) is impossible.

This argument explains the following seeming paradox. If we treat the RHS of (14) as a generator of a diffusion process (similarly to the finite-dimensional situation), then the drift vector at the point \( x = (0, 0, \ldots) \) is equal to \( (-\alpha, -\alpha, \ldots) \) and therefore is directed “away” from the simplex \( \nabla_\infty \). But the action of the generator \( \overline{A} \) on the coordinate functions \( x_1, x_2, \ldots \) is known not to be given by the RHS of (14). It follows that in our situation the coefficients of the first-degree derivatives in (14) cannot be interpreted as components of the drift vector.

5.2. The pre-generator in moment coordinates. First, recall the following definition.

Suppose we have a commutative algebra. We say that every operator of multiplication by an element of the algebra has zero order. We say that an operator \( Q \) has order \( n \) if the commutator of \( Q \) with any operator of multiplication by an element of the algebra has order \( n - 1 \).

**Lemma 5.5.** The pre-generator \( A \) is a second order operator in the algebra \( \mathcal{F} \).

**Proof.** We must show that

\[
[[[A, H_1], H_2], H_3] f(x) = 0
\]

for all \( h_1, h_2, h_3, f \in \mathcal{F} \) and \( x \in \nabla_\infty \), where by \( H_j \) we denote the operator of multiplication by \( h_j \), \( j = 1, 2, 3 \).

From Remark 5.3 (2) it follows that (15) holds for every \( x \in \nabla_\infty \). Indeed, in this case the operator \( A \) acts according to the RHS of (14), and due to the linearity it remains to refer to the fact that \( [[[\partial^2/\partial x_i \partial x_j], H_1], H_2], H_3] f(x) = 0 \) and \( [[[\partial/\partial x_i], H_1], H_2], H_3] f(x) = 0 \) for all \( 1 \leq i, j < \infty \).

For a general \( x \in \nabla_\infty \) the claim (15) holds by continuity. \( \square \)

Now, the moment coordinates \( q_1, q_2, \ldots \) are algebraically independent generators of the algebra \( \mathcal{F} \) and the operator \( A \) has second order in \( \mathcal{F} \). Therefore, the action of \( A \) on \( \mathcal{F} \) is completely determined by the elements \( A_1, Aq_i, A(q_iq_j), i, j = 1, 2, \ldots \). It is clear that \( q_i q_j = q_{i+j+1} + m^0_{(i+1,j+1)}, i \geq j \). Using (12), we find

\[
\begin{align*}
A_1 &= 0, \\
Aq_i &= -(i+1)(i+\theta)q_i + (i+1)(i-\alpha)q_{i-1}; \\
A(q_iq_j) &= q_iAq_j + q_jAq_i + 2(i+1)(j+1)(q_{i+j} - q_iq_j),
\end{align*}
\]

where, by agreement, \( q_0 = 1 \).
Using the “basis” formal differential operators \( \frac{\partial}{\partial q_i} \) and \( \frac{\partial^2}{\partial q_i \partial q_j} \) in the algebra \( \mathcal{F} = \mathbb{R} \{q_1, q_2, \ldots \} \), we can write

\[
A = \sum_{i,j=1}^{\infty} (i + 1)(j + 1)(q_{i+j} - q_iq_j) \frac{\partial^2}{\partial q_i \partial q_j} + \sum_{i=1}^{\infty} [-(i + 1)(i + \theta)q_i + (i + 1)(i - \alpha)q_{i-1}] \frac{\partial}{\partial q_i}.
\]

This form of the operator \( A \) allows to show that almost all sample paths of the process \( X_{\alpha, \theta}(t) \) are continuous. This can be proved similarly to [BO Corollary 6.4 and Theorem 7.1].

5.3. Degenerate values of parameters. Let us make mention what happens when the parameters \( \alpha \) and \( \theta \) are of degenerate series (see [13] for the definition). Consider the finite-dimensional ordered simplex

\[
\nabla_N := \{(x_1, \ldots, x_N) \in \mathbb{R}^N: x_1 \geq \ldots \geq x_N \geq 0, \sum_{i=1}^{N} x_i = 1\}, \quad N = 2, 3, \ldots
\]

We can view \( \nabla_N \) as a subset in \( \nabla_\infty \subset \nabla_\infty \). The measure \( \text{PD}(\alpha, \theta) \) is concentrated on \( \nabla_N \subset \nabla_\infty \) for all values of \( \alpha \) and \( \theta \) of principal or degenerate series.

Proposition 5.6 ([14] p. 62]). When \( \alpha = -\beta < 0 \) and \( \theta = N\beta \), the measure \( \text{PD}(-\beta, N\beta) \) is concentrated on \( \nabla_N \subset \nabla_\infty \). It coincides with the measure on \( \nabla_N \) that has the following density with respect to the Lebesgue measure:

\[
\text{PD}(-\beta, N\beta)(dx) = \frac{N! \Gamma(N\beta)}{(\Gamma(\beta))^N} x_1^{\beta-1} \ldots x_N^{\beta-1} dx_1 \ldots dx_{N-1}.
\]

Moreover, as \( \beta \to +0 \) and \( N \to +\infty \) such that \( N\beta \to \eta > 0 \), the following weak convergence of measures on \( \nabla_\infty \) holds: \( \text{PD}(-\beta, N\beta) \to \text{PD}(0, \eta) \).

It turns out that this result expands to the process \( X_{\alpha, \theta}(t) \) in \( \nabla_\infty \). First, we recall the diffusion processes considered in the finite-dimensional simplex \( \nabla_N \).

We are interested in processes that are called approximating diffusions for the Wright-Fisher genetic model with symmetric mutation in [1]. They were also studied in [W2, Gr] and other papers. A detailed construction of more general finite-dimensional diffusions related to population genetics is explained in [EK2 Chapter 10].

Let \( \mathcal{F}_N = \mathbb{R} \{q_1, \ldots, q_{N-1}\} \) be a commutative unital algebra freely generated by the moment coordinates on \( \nabla_N \): \( q_k(x) = \sum_{i=1}^{N} x_i^{k+1}, k = 1, \ldots, N-1 \). It is clear that \( \mathcal{F}_N \subset C(\nabla_N) \) is a dense subalgebra. For each parameter \( \eta > 0 \) we define an operator in \( \mathcal{F}_N \):

\[
A_{\eta} := \sum_{i=1}^{N} x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j=1}^{N} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\eta}{N-1} \sum_{i=1}^{N} (1 - N x_i) \frac{\partial}{\partial x_i}.
\]

Note that this operator (contrary to its infinite-dimensional analogue from [15,1]) can be defined by this formula on a wider subspace of \( C(\nabla_N) \). This subspace consists of twice continuously differentiable under certain boundary conditions, see [EK1 (2.8)]. Denote this extension by \( \tilde{A}_{\eta} \).
Proposition 5.7 (EK1). (1) The operator $\tilde{A}_{N,\eta}$ (and, therefore, $A_{N,\eta}$) is closable in $C(\nabla_N)$. Denote by $\overline{A}_{N,\eta}$ its closure (and, therefore, the closure of $A_{N,\eta}$);
(2) The closure $\overline{A}_{N,\eta}$ generates a diffusion process (that is, a strong Markov process with continuous sample paths which can start from any point and any probability distribution) in $\nabla_N$;
(3) This process preserves the measure $PD\left(-\frac{\eta}{N-1}, \frac{\eta N}{N-1}\right)$ defined above, and is reversible with respect to this measure.

Denote this process by $Y_{N,\eta}(t)$.

Proposition 5.8 (EK1 Theorem 2.5]). As $N \to \infty$, the processes $Y_{N,\eta}(t)$ converge$^4$ to the process $X_{0,\eta}(t)$ constructed in \[2.3\].

It appears that the process $X_{\alpha,\theta}(t)$ (which is constructed and studied throughout the present paper) can be considered also for degenerate parameters. First, we describe how the up/down Markov chains should be modified in this case.

Let $K(N)$ consist of all diagrams $\lambda \in K$ such that $|\lambda| \leq N$. It is again a graded graph. Let $\alpha = -\beta$, $\theta = N\beta$. In this case each of the measures $M_\theta$ defined in $\{1,3\}$ is positive everywhere on $K_n(N) := K(N) \cap K_n$. It follows that we can consider the up/down Markov chains on $K_n(N)$ similarly to $\{1.2\}$. The place of the algebra $\Lambda$ is taken by the algebra $\Lambda_N$ of symmetric functions in $N$ variables. It is clear that $\Lambda_N/I \cong F_N$, where $I = \langle f_1 - 1 \rangle$ is the ideal of the algebra $\{2.1\}$. Each embedding $t_n$, $n = 1, 2, \ldots$ (they were defined in $\{2.3\}$ now maps $K_n(N)$ into $\nabla_N$.

Proposition 5.9. (1) Let $T_n(N)$ be the transition operator of the $n$th up/down Markov chain. In the same sense as in Lemma $\{2.1\}$ there holds a convergence
\[
\lim_{n \to \infty} n^2(T_n(N) - 1)f = A_{N,\beta(N-1)}f \quad \text{for all } f \in F_N.
\]

(2) The discrete semigroups $\{1, T_n(N), T_n^2(N), \ldots\}$ converge, as $n \to \infty$ (in the same sense as in Proposition $\{2.3\}$), to a continuous semigroup in the space $C(\nabla_N)$ generated by the operator $\overline{A}_{N,\beta(N-1)}$.

Thus, it is natural to say that the process $X_{-\beta,N\beta}(t)$ coincides with the process $Y_{N,\beta(N-1)}(t)$, that is, $X_{-\beta,N\beta}(t)$ is a finite-dimensional diffusion process in $\nabla_N$ with the pre-generator $A_{N,\beta(N-1)}: F_N \to F_N$. It follows that the diffusions in the finite-dimensional simplexes studied in EK1 and many other works arise as a special case of the two-parameter family of diffusions $X_{\alpha,\theta}(t)$ in $\nabla_\infty$.

Proposition $\{5.8\}$ can be interpreted as a convergence (in the sense of EK1 Theorem 2.5]) of processes $X_{-\beta,N\beta}(t)$ in $\nabla_N$ to the process $X_{0,\eta}(t)$ in $\nabla_\infty$, as $\beta \to +0$ and $N \to +\infty$ such that $N\beta \to \eta > 0$. This convergence strengthens the second claim of Proposition $\{5.6\}$ concerning the convergence of invariant measures.

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