On $\Gamma$-Interval Valued Fuzzification of Lagrange’s Theorem of $\Gamma$-Interval Valued Fuzzy Subgroups

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ABSTRACT In this paper, we present the idea of interval valued fuzzy subgroup defined over a certain t-conorm ($\Gamma$-IVFSG) and prove that every IVFSG is $\Gamma$-IVFSG. We use this ideology to define the concepts of $\Gamma$-IVF cosets, $\Gamma$-IVFNSG and formulate their various important algebraic characteristics. We also propose the study of the notion of level subgroups of $\Gamma$-IVFSG and investigate the condition under which a $\Gamma$-IVFS is $\Gamma$-IVFSG. Moreover, we extend the study of this phenomenon to introduce the concept of quotient group of a group $Z$ relative to the $\Gamma$-IVFNSG and acquire a correspondence between each $\Gamma$-IVF(N)SG of a group $Z$ and $\Gamma$-IVF(N)SG of its quotient group. Furthermore, we define the index of $\Gamma$-IVFSG and establish the $\Gamma$-interval valued fuzzification of Lagrange’s theorem of any $\Gamma$-IVFSG of a finite group $Z$.

INDEX TERMS $\Gamma$-interval valued fuzzy set ($\Gamma$-IVFS), $\Gamma$-interval valued fuzzy subgroup ($\Gamma$-IVFSG), $\Gamma$-interval valued fuzzy coset, $\Gamma$-interval valued fuzzy normal subgroup ($\Gamma$-IVFNSG), $\Gamma$-interval valued fuzzy quotient group.

I. INTRODUCTION

In the late eighteenth century, Lagrange’s Theorem appeared in the literature. It was basically discovered to resolve the problem of finding roots of the equation of degree greater than 4 and its association with symmetric volumes. In 1870, Lagrange’s expressed a modification of this theorem. Pietro Abbati gave the first complete proof of this theorem about thirty years after the Lagrange’s modification. This theorem has a significant role in the development of modern group theory. It is an incredible asset to investigate finite groups; as it gives an exact review about subgroups of a finite group. This theorem produces a very effective sign of Fermat’s Little Theorem, which is very valuable in cryptography and numerous different fields.

The theory of fuzzy logic offers a mathematical method to apprehend the uncertainty related to human cerebral process like thoughtful and intellectual. It also handles issues of uncertainty and lexical imprecision. In this theory the elements of a universe are permitted to be partially accommodated by the set. It is determined by indeterminate boundaries. Consequently, fuzzy set follows infinite-valued logic. Practically, the accomplishment of usage of the fuzzy set hypothesis depends upon a decision of membership function that we make. In spite of this, there are many physical problems in which scientist don’t have careful learning of the limit that should be taken. The limitation of this theory is the case when we do not have exact information of the membership function. In these cases, it is reasonable to declare each component of the fuzzy set of membership grades by methods of interval. These perceptions rise the development of fuzzy sets called the theory of IVFS. IVFSs are basically used in medical diagnosis, approximate reasoning and image processing.

In 1965, the idea of fuzzy sets was firstly presented by Zadeh [1]. In 1967, fuzzy sets were refined in terms of L-fuzzy set by Gougen [2]. The author [3] utilized this thought to innovate the hypothesis of fuzzy subgroups in 1971. Later on, Anthony and Sherwood [4] reviewed fuzzy subgroups on the basis of the study of t-norm. Das [5] derived level subgroups of a fuzzy group. The notions of fuzzy normal
subgroups and fuzzy cosets were presented by Mukherjee and Bhattacharya [6]. Bhattacharya [7] developed numerous fundamental characterizations of fuzzy subgroups. Liu [8] commenced the study of fuzzy invariant subgroups and fuzzy ideals. For more detail about the development of fuzzy subgroups, we refer to [9]–[11] and [12]. Gupta and Qi [13] reviewed the theory of t-norm and t-co-norm.

Zadeh [14] proposed the concept of IVFS in 1975. An important interpretation of IVFS was given Turksen [15] in 1986. Gorzalczyk [16] applied this particular concept to formulate an estimate technique for verbal choice into mathematical approximations. Atanassov and Gargov [17] initiated the study of IV intuitionistic fuzzy set in 1989. Roy and Biswas [18] described IVF relations and obtained their various important results in 1992. In 1995, the study of IVFSGs was initiated by Liu [8]. Mondal and Samanta [19] characterized the topology of IVFS and talked about its properties in 1999. Later on, Gehrke et al. [20] investigated the theory of IVFS defined over t-norms in 2001. Lee [21] gave a detailed comparison of IVFS with bipolar VFS and intuitionistic fuzzy set in 2001. Lee and Luo [31] gave useful methods to construct entropy of IVFS in 2015.

The rest of the paper is designed as: In the section 2 the ideas of IVFS and IVFSG are discussed. We present the concepts of $\Gamma$-IVFS and $\Gamma$-IVFSG and some of their basic algebraic properties in section 3. Section 4 deals with views of $\Gamma$-IVF coset, $\Gamma$-IVF quotient group and the index of $\Gamma$-IVFSG.

III. ALGEBRAIC CHARACTERISTICS OF $\Gamma$-INTERVAL VALUED FUZZY SUBGROUPS

This section is devoted to study the concept of $\Gamma$-IVFS and to investigate various fundamental algebraic characteristics of this phenomenon.

Definition 10: Suppose $M$ is an IVFS of a universe $Z$ and $\Gamma \in D(I)$. Then $M$ is said to be $\Gamma$-interval valued fuzzy set, denoted by $M_{\Gamma}$, of $Z$ w.r.t IVFS $M$ and is defined as:

$$M_{\Gamma}(x_1) = sp[M(x_1), \Gamma] = [M_L(x_1) spc, M^U(x_1) spd], \quad \forall x_1 \in Z,$$

where $M = [M_L, M^U]$ and $\Gamma = [c, d]$ with $0 \leq c < d \leq 1$.

Definition 11: Let $[v, \psi] \in [0, 1]$ with $v \leq \psi$. The $[v, \psi]$ cut set of an IVFS $M$ of a universe $Z$ is defined as:

$$M^{[v, \psi]} = \{x_1 \in Z : M^L(x_1) \geq v \text{ and } M^U(x_1) \geq \psi\}.$$

Definition 4 [13]: A function $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a $t$ co-norm if $t$ is commutative, associative, monotonic and satisfies the given boundary conditions:

$$t(x, 0) = x, t(x, 1) = 1, \quad x \in [0, 1].$$

Definition 5 [13]: A function $sp : [0, 1] \times [0, 1] \rightarrow [0, 1]$ defined by:

$$sp(x_1, x_2) = x_1 + x_2 - x_1 x_2, \quad \forall x_1, x_2 \in [0, 1]$$

satisfies axiomatic skeleton of a $t$–conorm and is called probabilistic sum.

Definition 6 [12]: A fuzzy set $M$ of group $Z$ is called a fuzzy subgroup, if $M$ admits the following conditions:

$$M(x_1 x_2) \geq \min[M(x_1), M(x_2)] \quad \text{and} \quad M(x_1^{-1}) \geq M(x_1), \quad \forall x_1, x_2 \in Z.$$
in shading pictures by choosing a suitable value of $\Gamma$. Our methodology depends on a solitary picture, where the amplified picture is gained by joining various developed rectangles. To make each rectangle we utilize the $\Gamma$-IVFS that we have recently connected with the picture, keeping up the power of the first pixel in the focal point of the rectangle and filling in the rest utilizing the connection between that pixel and its neighbors.

In the following theorem, we show that the intersection of any two $\Gamma$-IVFS is a $\Gamma$-IVFS.

**Theorem 13:** For any two $\Gamma$-IVFS $M_1$ and $N_1$ of $Z$, $(M \cap N)_1 = M_1 \cap N_1$.

**Proof:** In the view of definition (10), we have

$$(M \cap N)_1 = sp [\{sp [M (x_1), N (x_1)] : x \in I_1\} \cap [u_1, v_1]]$$

which implies that $M_1 \cap N_1 = \{M_1 (x_1), N_1 (x_1)\}$.

Consequently, $(M \cap N)_1 = M_1 \cap N_1$.

**Definition 14:** Let $M$ be a $\Gamma$-IVFS of a group $Z$ and $\Gamma \in D (I)$, then $M$ is called a $\Gamma$-interval valued fuzzy subgroup ($\Gamma$-IVFSG) w.r.t IVFS $M$, denoted by $M_1$, of $Z$ if $M_1$ admits following conditions:

(i) $M_1^L (x_1 x_2) \geq \min \{M_1^L (x_1), M_1^L (x_2)\}$ and $M_1^U (x_1 x_2) \geq \min \{M_1^U (x_1), M_1^U (x_2)\}$

(ii) $M_1^L (x_1) \geq M_1^L (x_1)$ and $M_1^U (x_1) \geq M_1^U (x_1)$,

Theorem 15: Let $M_1$ be a $\Gamma$-IVFSG of $Z$ and $x_1, x_2 \in Z$, then

(i) $M_1^L (x_1) = M_1^L (x_1)$

(ii) $M_1^L (e) \geq M_1^L (x_1)$ and $M_1^U (e) \geq M_1^U (x_1)$

(iii) $M_1^L (x_{1 x_2}) = M_1^U (e)$ implies $M_1^L (x_1) = M_1^L (x_2)$.

**Proof (i):** In the view of definition (14), we have

$$M_1^L (x_1^{-1}) \geq M_1^L (x_1)$$

which means that

$$M_1^L (x_1^{-1}) \geq M_1^L (x_1)$$

By the combination of (3.1) and (3.2), we obtain $M_1^L (x_1) = M_1^L (x_1)$. Similarly, the upper case can be proved.

**ii.** Note that, in the view of definition (14), we have

$$M_1^L (e) \geq M_1^L (x_1 x_1^{-1})$$

$$M_1^U (x_1) \geq \min \{M_1^L (x_1), M_1^L (x_1^{-1})\}$$

This shows that $M_1^L (e) \geq M_1^L (x_1), x_1 \in Z$. The upper case can be proved in the same way.

**Proof (iii):** For any $x_1, x_2 \in Z$. Consider

$$M_1^L (x_1) = M_1^L (x_1 x_2^{-1} x_2) \geq \min \{M_1^L (x_1 x_2^{-1}), M_1^L (x_2)\}.$$
Clearly, \( M \) is not IVFSG of \( Z \). Take \( \Gamma = [0.46, 0.49] \), we have

\[
M_\Gamma (x_1) = \begin{cases} 
[0.93, 0.99] & \text{if } x_1 = 1 \\
[0.81, 0.88] & \text{if } x_1 = -1 \\
[0.70, 0.75] & \text{if otherwise}
\end{cases}
\]

In view of definition (14), \( M_\Gamma \) is \( \Gamma \)-IVFSG of \( Z \). From the above discussion, we conclude that \( M \) is \( \Gamma \)-IVFSG of \( Z \). But \( M \) is not IVFSG of \( Z \).

In the following result, we show that any two \( \Gamma \)-IVFSG obey the intersection property.

**Theorem 18:** Intersection of two \( \Gamma \)-IVFSG is a \( \Gamma \)-IVFSG of \( Z \).

**Proof:** Let \( M_\Gamma \) and \( N_\Gamma \) be any two \( \Gamma \)-IVFSG of \( Z \). By definition (14), for each \( x_1, x_2 \in Z \), we have

\[
\left( M_\Gamma \cap N_\Gamma \right) (x_1, x_2) = \min \left\{ M_\Gamma (x_1, x_2), N_\Gamma (x_1, x_2) \right\}
\]

which implies that

\[
\left( M_\Gamma \cap N_\Gamma \right) (x_1, x_2) \geq \min \left\{ M_\Gamma (x_1), N_\Gamma (x_2) \right\},
\]

Moreover,

\[
\left( M_\Gamma \cap N_\Gamma \right) (x_1^{-1}) = \min \left\{ M_\Gamma (x_1^{-1}), N_\Gamma (x_1^{-1}) \right\}
\]

Consequently,

\[
\left( M_\Gamma \cap N_\Gamma \right) (x_1^{-1}) \geq \left( M_\Gamma \cap N_\Gamma \right) (x_1).
\]

Similarly, one can prove the above two inequalities for the upper case, that is,

\[
\left( M_\Gamma \cup N_\Gamma \right) (x_1, x_2) \geq \min \left\{ M_\Gamma (x_1), N_\Gamma (x_2) \right\},
\]

and \( \left( M_\Gamma \cup N_\Gamma \right) (x_1^{-1}) \geq \left( M_\Gamma \cup N_\Gamma \right) (x_1) \).

Thus, the intersection of any two \( \Gamma \)-IVFSG is \( \Gamma \)-IVFSG of \( Z \).

**Remark 19:** Union of two \( \Gamma \)-IVFSG may not be a \( \Gamma \)-IVFSG of a group \( Z \).

**Example 20:** Consider IVFS \( M \) and \( N \) of group of integers \( Z \) under addition as follows:

\[
M(x_1) = \begin{cases} 
[0.5, 0.7] & \text{if } x_1 \in 3Z \\
[0, 0.1] & \text{otherwise}
\end{cases}
\]

\[
N(x_1) = \begin{cases} 
[0.19, 0.2] & \text{if } x_1 \in 2Z \\
[0.07, 0.09] & \text{otherwise}.
\end{cases}
\]

The \( \Gamma \)-IVFSG \( M_\Gamma \) and \( N_\Gamma \) of \( Z \) corresponding to \( \Gamma = [0.3, 0.7] \) are given by:

\[
M_\Gamma (x_1) = \begin{cases} 
[0.65, 0.91] & \text{if } x_1 \in 3Z \\
[0.3, 0.7] & \text{otherwise}
\end{cases}
\]

\[
N_\Gamma (x_1) = \begin{cases} 
[0.43, 0.76] & \text{if } x_1 \in 2Z \\
[0.34, 0.72] & \text{otherwise}.
\end{cases}
\]

The values of \( M_\Gamma (9) \) and \( N_\Gamma (2) \) are given by [0.65, 0.91] and [0.43, 0.76].

Consider

\[
(M_\Gamma \cup N_\Gamma) (x_1) = \begin{cases} 
[0.65, 0.91] & \text{if } x_1 \in 3Z \\
[0.43, 0.76] & \text{if } x_1 \in 2Z - 3Z \\
[0.34, 0.72] & \text{otherwise}.
\end{cases}
\]

The values of \( M_\Gamma \cup N_\Gamma \) at \( x_1 = 9 \) and \( x_2 = 2 \) are given by

\[
(M_\Gamma \cup N_\Gamma) (9) = [0.65, 0.91] \quad \text{and} \quad (M_\Gamma \cup N_\Gamma) (2) = [0.43, 0.76].
\]

Moreover,

\[
\min \{(M_\Gamma \cup N_\Gamma) (9), (M_\Gamma \cup N_\Gamma) (2)\} = [0.43, 0.76] \quad \text{and} \quad (M_\Gamma \cup N_\Gamma) (9 - 2) = (M_\Gamma \cup N_\Gamma) (7) = [0.34, 0.72].
\]

It is quite evident from the above discussion that

\[
(M_\Gamma \cup N_\Gamma) (x_1, x_2) < \min \{(M_\Gamma \cup N_\Gamma) (x_1), (M_\Gamma \cup N_\Gamma) (x_2)\}.
\]

This shows that \( M_\Gamma \cup N_\Gamma \) is not a \( \Gamma \)-IVFSG of \( Z \).

In the following theorem, we show that a group cannot be written as a union of two proper \( \Gamma \)-IVFSG.

**Theorem 21:** A group \( Z \) cannot be a union of two proper \( \Gamma \)-IVFSG of a group \( Z \).

**Proof:** Let \( M_\Gamma \) and \( N_\Gamma \) be any two proper \( \Gamma \)-IVFSG of \( Z \), such that \( M_\Gamma \cup N_\Gamma = [1, 1] \), where \( M_\Gamma \neq [1, 1] \) and \( N_\Gamma \neq [1, 1] \). Since

\[
M_\Gamma \cup N_\Gamma = \left[ \max \left\{ M_\Gamma U, N_\Gamma U \right\}, \max \left\{ M_\Gamma U, N_\Gamma U \right\} \right] = [1, 1].
\]

Therefore, \( \max \left\{ M_\Gamma U, N_\Gamma U \right\} = 1 \) and \( \max \left\{ M_\Gamma U, N_\Gamma U \right\} = 1 \).

Then \( M_\Gamma (x_1) = 1 \) or \( N_\Gamma (x_1) = 1 \) and \( M_\Gamma U (x_1) = 1 \) or \( N_\Gamma U (x_1) = 1 \). But \( M_\Gamma \neq [1, 1] \) and \( N_\Gamma \neq [1, 1] \), which means that \( M_\Gamma U (x_1) \neq 1 \) or \( N_\Gamma U (x_1) \neq 1 \) for all \( x_1 \in Z \).

Thus, in either case, we find contradiction.
\textbf{Definition 22:} Let $M_G$ be a $\Gamma$-IVFSG of a group $G$ and for $[v, \psi] \in \text{Im} M_G$ such that $M_G^L (e) \geq v, M_G^U (e) \geq \psi$ and $v < \psi$, then $M_G^{[v, \psi]}$ is defined as:

$$M_G^{[v, \psi]} = \left\{ x_1 \in Z : M_G^L (x_1) \geq v, M_G^U (x_1) \geq \psi \right\}.$$ 

\textbf{Theorem 23:} Let $M_G$ be a $\Gamma$-IVFSG of a group $Z$, then $M_G \in \Gamma$-IVFSG$(Z)$ if and only if $M_G^{[v, \psi]}$ is a subgroup of $Z$ for each $[v, \psi] \in \text{Im} M_G$ with $v \leq M_G^L (e)$ and $\psi \geq M_G^U (e)$.

\textbf{Proof:} Let $x_1, x_2 \in M_G^{[v, \psi]}$, then in view of definition (22), we have $M_G^L (x_1) \geq v, M_G^U (x_1) \geq \psi$ and $M_G^L (x_2) \geq v, M_G^U (x_2) \geq \psi$. This implies that $M_G^L (x_1, x_2) = \min \{ M_G^L (x_1), M_G^U (x_2) \} = v$. The upper case $M_G^U (x_1, x_2)$ can be established in the same way, that is, $M_G^U (x_1, x_2) = \min \{ M_G^U (x_1), M_G^U (x_2) \} = \psi$.

Consequently, $x_1, x_2 \in M_G^{[v, \psi]}$.

Moreover, for any $x_1 \in M_G^{[v, \psi]}$, we have $M_G^L (x_1) \geq v$ and $M_G^U (x_1) \geq \psi$, which implies that $M_G^L (x_1^{-1}) = M_G^L (x_1) = v$ and $M_G^U (x_1^{-1}) = M_G^U (x_1) = \psi$. This means that $x_1^{-1} \in M_G^{[v, \psi]}$ and hence, $M_G^{[v, \psi]}$ is a subgroup of $Z$.

Conversely, for any $x_1, x_2 \in Z$, let $M_G (x_1) = [v, \psi]$, and $M_G (x_2) = [v_1, \psi_1]$. Then clearly, $x_1 \in M_G^{[v, \psi]}$ and $x_2 \in M_G^{[v_1, \psi_1]}$. Suppose $v < v_1$ and $\psi < \psi_1$. Then $M_G^{[v, \psi]} \subseteq M_G^{[v_1, \psi_1]}$. Moreover, for any $x_2 \in M_G^{[v_1, \psi_1]}$, since $M_G^{[v_1, \psi_1]}$ is a subgroup of $Z$, we have $x_1 x_2 \in M_G^{[v_1, \psi_1]}$.

Then $M_G^L (x_1 x_2) \geq v$ and $M_G^U (x_1 x_2) \geq \psi$ implying that $M_G^L (x_1 x_2) = \min \{ M_G^L (x_1), M_G^U (x_2) \}$ and $M_G^U (x_1 x_2) = \min \{ M_G^U (x_1), M_G^U (x_2) \}$. For any $x_1 \in Z$, let $M_G (x_1) = [v, \psi]$, we have $x_1 \in M_G^{[v, \psi]}$ and $x_1^{-1} \in M_G^{[v, \psi]}$. This implies that $M_G^L (x_1^{-1}) = M_G^U (x_1), M_G^U (x_1^{-1}) = M_G^U (x_1)$. Hence, $M_G$ is a $\Gamma$-IVFSG of $Z$.

\textbf{Theorem 24:} Let $x_1$ be some fixed element of $Z$ and $M_G$ be a $\Gamma$-IVFSG of $Z$, then for all $x_2 \in Z$, $M_G (x_1 x_2) = M_G (x_2)$ if and only if $M_G (x_1) = M_G (e)$.

\textbf{Proof:} Let $M_G (x_1 x_2) = M_G (x_2)$, $\forall x_2 \in Z$, then it is quite obvious that $M_G (x_1) = M_G (e)$.

Conversely, let $M_G (x_1) = M_G (e)$, in the light of theorem (15)(ii), we have $M_G (x_2) \geq M_G (x_1)$ for each $x_2 \in Z$. In view of definition (14), we get $M_G^U (x_1 x_2) \geq \min \{ M_G^U (x_1), M_G^U (x_2) \}$. The application of given condition in the above relation yields that

$$M_G^U (x_1 x_2) \geq M_G^U (x_2). \tag{3.5}$$

Moreover, $M_G^L (x_2) = M_G^L \left( x_1^{-1} x_1 x_2 \right)$.

\textbf{Definition 25:} Let $M_G$ be a $\Gamma$-IVFSG of $Z$ and $\Gamma \in \mathcal{D}(I)$. For any $x_1 \in Z$, the $\Gamma$-IVF right coset of $M_G$ in $Z$ is represented by $M_G x_1$ and is defined as:

$$M_G x_1 = \left\{ g \cdot x_1 : g \in G \right\}.$$
This means that
\[ M_{f}^{L}(g_{1}) \geq M_{f}^{L}(g_{2}). \tag{4.1} \]
Likewise
\[ M_{f}^{L}(g_{2}) \geq M_{f}^{L}(g_{1}). \tag{4.2} \]
In view of (4.1) and (4.2), we obtain
\[ M_{f}^{L}(g_{1}) = M_{f}^{L}(g_{2}). \]
Similarly, the above equality can be obtained for the upper case.

**Definition 28:** The \( \Gamma \)-interval valued fuzzy normal subgroup (\( \Gamma \)-IVFNSG) of a given \( \Gamma \)-IVFSG \( M_{f} \) of \( Z \) is defined as:
\[ M_{f}(x_{1}x_{2}) = M_{f}(x_{2}x_{1}) , \quad \text{for all } x_{1}, x_{2} \in Z. \]

**Theorem 29:** Let \( M_{f} \) be a \( \Gamma \)-IVFNSG of a group \( Z \), then for all elements in \( Z \), \( M_{f}:x_{1}(x_{1}g) = M_{f}:x_{1}(gx_{1}) = M_{f}(g) \).

**Proof:** Consider \( M_{f}:x_{1}(x_{1}g) = [M_{f}^{L}(x_{1})(x_{1}g), M_{f}^{U}(x_{1})(x_{1}g)] \), \( \forall x_{1} \in Z \). In the light of the definition (25), we have
\[ M_{f}:x_{1}(x_{1}g) = \left[ M_{f}^{L}(x_{1})(x_{1}g), M_{f}^{U}(x_{1})(x_{1}g) \right] , \]
which shows that \( M_{f}:x_{1}(x_{1}g) = M_{f}(g) \), for any \( g \in Z \). Similarly, we can prove \( M_{f}:x_{1}(gx_{1}) = M_{f}(g) \).

**Theorem 30:** Let \( M_{f} \) be a \( \Gamma \)-IVFSG of \( Z \), then \( M_{f} \in \Gamma \)-IVFNSG(Z) if and only if \( M_{f}^{L}(x_{1}, x_{2}) \geq M_{f}^{L}(x_{1}) \) and \( M_{f}^{U}(x_{1}, x_{2}) \geq M_{f}^{U}(x_{1}) \) for all \( x_{1}, x_{2} \in Z \).

**Proof:** Let \( M_{f} \in \Gamma \)-IVFNSG of \( Z \) and for any \( x_{1}, x_{2} \in Z \), we have
\[ M_{f}^{L}(x_{1}, x_{2}) = M_{f}^{L}(x_{1}^{-1}x_{2}^{-1}x_{1}x_{2}) = M_{f}^{L}(x_{2}^{-1}x_{1}x_{2}) \geq M_{f}^{L}(x_{1}^{-1}) , \]
\[ M_{f}^{U}(x_{1}, x_{2}) = M_{f}^{U}(x_{1}^{-1}x_{2}^{-1}x_{1}x_{2}) = M_{f}^{U}(x_{2}^{-1}x_{1}x_{2}) \geq M_{f}^{U}(x_{1}^{-1}) . \]
Therefore,
\[ M_{f}^{L}(x_{1}, x_{2}) \geq \min \left\{ M_{f}^{L}(x_{1}), M_{f}^{L}(x_{1}) \right\} . \]
Thus, \( M_{f}^{L}(x_{1}, x_{2}) \geq M_{f}^{L}(x_{1}), \forall x_{1} \in Z. \)

Similarly, the upper case can be established.

Conversely, let \( x_{1}, x_{2} \in Z \), \( M_{f}(x_{1}, x_{2}) \geq M_{f}(x_{1}) \) and \( M_{f}(x_{1}, x_{2}) \geq M_{f}(x_{1}) \). Consider \( z_{1} \in Z \), then
\[ M_{f}(x_{1}^{-1}z_{1}x_{1}) = M_{f}(z_{1}x_{1}^{-1}z_{1}x_{1}) = M_{f}(z_{1}z_{1}^{-1}x_{1}^{-1}z_{1}x_{1}) \geq \min \left\{ M_{f}(z_{1}), M_{f}(z_{1}) \right\} . \]
which implies that
\[ M_{f}(x_{1}^{-1}z_{1}x_{1}) \geq M_{f}(z_{1}) . \tag{4.3} \]

By using same arguments, we have \( M_{f}^{U}(x_{1}^{-1}z_{1}x_{1}) \geq M_{f}^{U}(z_{1}) \).

Moreover,
\[ M_{f}^{L}(z_{1}) = M_{f}(z_{1}x_{1}^{-1}z_{1}x_{1}^{-1}) = M_{f}^{L}\left( (x_{1}(z_{1}^{-1}x_{1}x_{1}^{-1}) \right) \geq \min \left\{ M_{f}(x_{1}), M_{f}(x_{1}^{-1}z_{1}x_{1}), M_{f}(x_{1}^{-1}) \right\} . \]

Therefore,
\[ M_{f}^{L}(z_{1}) \geq \min \left\{ M_{f}(x_{1}), M_{f}(x_{1}^{-1}z_{1}x_{1}) \right\} . \]

By using same argument, we have
\[ M_{f}^{U}(z_{1}) \geq \min \left\{ M_{f}(x_{1}), M_{f}(x_{1}^{-1}z_{1}x_{1}) \right\} . \]

**Case i):** Let
\[ \min \left\{ M_{f}^{L}(x_{1}), M_{f}^{L}(x_{1}^{-1}z_{1}x_{1}) \right\} = M_{f}^{L}(x_{1}) \quad \text{and} \quad \min \left\{ M_{f}^{U}(x_{1}), M_{f}^{U}(x_{1}^{-1}z_{1}x_{1}) \right\} = M_{f}^{U}(x_{1}) \]
Then \( M_{f}^{L}(z_{1}) \geq M_{f}^{L}(x_{1}) \) and \( M_{f}^{U}(z_{1}) \geq M_{f}^{U}(x_{1}) \), \( \forall x_{1}, z_{1} \in Z \). This means that \( M_{f} \) is a constant mapping, which implies that \( M_{f}(x_{1}z_{1}x_{2}) = M_{f}(x_{2}z_{1}x_{2}) \). Hence, \( M_{f} \in \Gamma \)-IVFNSG(Z).

**Case ii):** If
\[ \min \left\{ M_{f}^{L}(x_{1}), M_{f}^{L}(x_{1}^{-1}z_{1}x_{1}) \right\} = M_{f}^{L}(x_{1}^{-1}z_{1}x_{1}) \quad \text{and} \quad \min \left\{ M_{f}^{U}(x_{1}), M_{f}^{U}(x_{1}^{-1}z_{1}x_{1}) \right\} = M_{f}^{U}(x_{1}^{-1}z_{1}x_{1}) \]

Then,
\[ M_{f}^{L}(z_{1}) \geq M_{f}^{L}(x_{1}^{-1}z_{1}x_{1}) . \tag{4.4} \]
Using (4.3) and (4.4), we get \( M_{f}^{L}(x_{1}^{-1}z_{1}x_{1}) = M_{f}^{L}(z_{1}) \).

Similarly, according to same argument, we have \( M_{f}^{U}(x_{1}^{-1}z_{1}x_{1}) = M_{f}^{U}(z_{1}) \). Consequently, \( M_{f} \in \Gamma \)-IVFNSG(Z).

In the following theorem, we see that every IVFNSG is a \( \Gamma \)-IVFNSG of \( Z \).

**Theorem 31:** Every IVFNSG is a \( \Gamma \)-IVFNSG of \( Z \).

**Proof:** Let \( M \) be an IVFNSG of \( Z \) and \( x_{1} \) be a fixed element in \( Z \), then
\[ \left[ M_{f}^{L}(x_{1}^{-1}g), M_{f}^{U}(x_{1}^{-1}g) \right] = \left[ M_{f}^{L}(g_{x_{1}^{-1}}), M_{f}^{U}(g_{x_{1}^{-1}}) \right] . \]
The application of definition (25) yields that
\[ sp\left( [M_{f}^{L}(x_{1}^{-1}g), M_{f}^{U}(x_{1}^{-1}g)] \right) = \left[ M_{f}^{L}(g_{x_{1}^{-1}}), M_{f}^{U}(g_{x_{1}^{-1}}) \right] . \]
Therefore $x_1M_\Gamma = M_\Gamma x_1, \forall g \in Z$. Consequently, $M_\Gamma$ is a $\Gamma$-IVFNSG of $Z$.

**Theorem 32**: Suppose that $M_\Gamma \in \Gamma$-IVFNSG of $Z$ and $Z/M_\Gamma$ is the set of all $\Gamma$-IVF cosets of $M_\Gamma$ in $Z$. Define a binary operation $\ast$ on $Z/M_\Gamma$ in the following way: $M_\Gamma x_1 \ast M_\Gamma x_2 = M_\Gamma x_1 x_2$, where $x_1, x_2 \in Z$. Then $(Z/M_\Gamma, \ast)$ forms a group.

**Proof**: Let $x_0, y_0, x_1, x_2 \in Z$ and $M_\Gamma x_0, M_\Gamma y_0, M_\Gamma x_1$ and $M_\Gamma x_2 \in Z/M_\Gamma$ such that $M_\Gamma x_0 = M_\Gamma x_1$ and $M_\Gamma y_0 = M_\Gamma x_2$.

Then for each $g \in Z$, we have, $M_\Gamma x_1 x_2(g) = M_\Gamma(gx_1^{-1}x_2^{-1})$ and $M_\Gamma x_0 y_0(g) = M_\Gamma(gy_0^{-1}x_0^{-1})$.

Moreover,

$$M_\Gamma^L(gx_1^{-1}) = M_\Gamma^L(gy_0^{-1}x_0^{-1}x_1)$$

$$= M_\Gamma^L(gy_0^{-1}x_0 x_1)$$

$$\geq \min \left\{ M_\Gamma^L(gy_0^{-1}x_0), M_\Gamma^L(x_0x_1^{-1}) \right\}.$$ 

Since $M_\Gamma x_0 = M_\Gamma x_1$ and $M_\Gamma y_0 = M_\Gamma x_2$. Therefore $M_\Gamma(gx_1^{-1}) = M_\Gamma(gx_0^{-1})$ and $M_\Gamma(gx_2^{-1}) = M_\Gamma(gy_0^{-1})$.

Particularly, we have

$$M_\Gamma^L(x_0x_1^{-1}x_2^{-1}) = M_\Gamma(x_0x_1^{-1}x_2^{-1})$$

$$= M_\Gamma(x_0x_2^{-1})$$

$$= M_\Gamma(e).$$

This means that

$$M_\Gamma^L(x_0x_1^{-1}x_2^{-1}) = M_\Gamma^L(e).$$

Also, we know that $M_\Gamma^L(e) \geq M_\Gamma^L(gy_0^{-1}x_0^{-1})$ implying that

$$M_\Gamma^L(gx_2^{-1}) \geq M_\Gamma^L(gy_0^{-1}x_0^{-1}).$$

(4.5)

Similarly, we have

$$M_\Gamma^L(gy_0^{-1}x_0^{-1}) \geq M_\Gamma^L(gx_2^{-1}).$$

(4.6)

By comparing (4.5) and (4.6), we have

$$M_\Gamma^L(gy_0^{-1}x_0^{-1}) = M_\Gamma^L(gx_2^{-1}x_1^{-1}).$$

In the light of same arguments, we have, $M_\Gamma^U(gy_0^{-1}x_0^{-1}) = M_\Gamma^U(gx_2^{-1}x_1^{-1})$. Thus, $M_\Gamma x_0 y_0(g) = M_\Gamma x_1 x_2(g)$. Hence $\ast$ is well defined.

i) Furthermore $\ast$ is associative.

ii) $M_\Gamma x_1^{-1}$ is inverse of $M_\Gamma x_1$.

iii) $M_\Gamma e = M_\Gamma$ is identity of $Z/M_\Gamma$.

Hence, $(Z/M_\Gamma, \ast)$ is a group.

Note that $Z/M_\Gamma$ is known as the $\Gamma$- IVF quotient group induced by $M_\Gamma$.

**Theorem 33**: Let $M_\Gamma$ be a $\Gamma$-IVFNSG of $Z$, and $M_\Gamma: Z/M_\Gamma \rightarrow D(I)$ defined by $M_\Gamma(M_\Gamma x_1) = M_\Gamma x_1, x_1 \in Z$.

Then $M_\Gamma$ is a $\Gamma$-IVFSG of $Z/M_\Gamma$.

**Proof**: In view of given mapping and the application of definition (14), we have

$$M_\Gamma^L(M_\Gamma x_1 x_2) = M_\Gamma^L(M_\Gamma x_1 x_2)$$

$$= M_\Gamma^L(x_1 x_2) \geq \min \left\{ M_\Gamma^L(x_1), M_\Gamma^L(x_2) \right\}$$

$$= \min \left\{ M_\Gamma^L(x_1), M_\Gamma^L(x_2) \right\}.$$ 

Moreover,

$$M_\Gamma^L((M_\Gamma x_1)^{-1}) = M_\Gamma^L(M_\Gamma x_1^{-1})$$

$$= M_\Gamma^L(x_1^{-1}) \geq M_\Gamma^L(x_1)$$

$$= M_\Gamma^L(M_\Gamma x_1).$$

The upper case can be proved in the same way.

This completes the proof.

Note that $M_\Gamma$ is known as $\Gamma$-IVF sub-quotient group determined by $M_\Gamma$.

**Theorem 34**: Let $M_\Gamma$ be a $\Gamma$-IVFNSG of $Z$. The set $Z_{M_\Gamma} = \{ x_1 \in Z : M_\Gamma(x_1) = M_\Gamma(e) \}$ is normal subgroup of $Z$.

**Proof**: In view of definition (14), we have $M_\Gamma^L(x_1 x_2^{-1}) \geq \min \left\{ M_\Gamma^L(x_1), M_\Gamma^L(x_2^{-1}) \right\}, x_1, x_2 \in Z_{M_\Gamma}$, which implies that

$$M_\Gamma^L(x_1 x_2^{-1}) \geq M_\Gamma^L(e).$$

(4.7)

We also know that

$$M_\Gamma^L(e) \geq M_\Gamma^L(x_1 x_2^{-1}).$$

(4.8)

By the comparison of (4.7) and (4.8), we get

$$M_\Gamma^L(x_1 x_2^{-1}) = M_\Gamma^L(e).$$

Hence $Z_{M_\Gamma}$ is a subgroup of $Z$.

Moreover, by applying theorem (30) and using the normality of $M_\Gamma$ for any element $x_1 \in Z_{M_\Gamma}$ and $x_2 \in Z$, we have $M_\Gamma^L(x_1 x_2^{-1}) = M_\Gamma^L(x_1) = M_\Gamma^L(e)$. Similarly, one can established the upper case as well. Consequently, $x_2 x_1^{-1} \in Z_{M_\Gamma}$.

**Theorem 35**: Every $\Gamma$-IVFNSG $M_\Gamma$ of $Z$, admits the following properties:

i) $x_1 M_\Gamma = x_2 M_\Gamma$ iff $x_1^{-1} x_2 \in Z_{M_\Gamma}$

(ii) $M_\Gamma x_1 = M_\Gamma x_2$ iff $x_1 x_2^{-1} \in Z_{M_\Gamma}$, $\forall x_1, x_2 \in Z$.

**Proof (i)**: Suppose that $x_1 M_\Gamma = x_2 M_\Gamma$. In view of definition (25), we have

$$M_\Gamma^L(x_1^{-1} x_2) = (x_1 M_\Gamma)(x_2)$$

$$= (x_2 M_\Gamma)(x_2)$$

$$= M_\Gamma^L(x_1^{-1} x_2).$$

which shows that $x_1^{-1} x_2 \in Z_{M_\Gamma}$. 
Conversely, let $x_1^{-1}x_2 \in Z_{M\Gamma}$. By applying definition (25) on $x_1M_\Gamma^L$ for any element $z_1 \in Z$, we have
\[
\begin{align*}
( x_1M_\Gamma^L ) ( z_1 ) &= M_\Gamma^L ( x_1^{-1}z_1 ) \\
&= M_\Gamma^L \left( ( x_1^{-1}x_2 ) ( x_2^{-1}z_1 ) \right) \\
&\geq \min \left\{ M_\Gamma^L ( x_1^{-1}x_2 ) , M_\Gamma^L ( x_2^{-1}z_1 ) \right\} \\
&= \min \left\{ M_\Gamma^L ( e ) , M_\Gamma^L ( x_2^{-1}z_1 ) \right\} \\
( x_1M_\Gamma^L ) ( z_1 ) &\geq M_\Gamma^L ( x_2^{-1}z_1 ) ,
\end{align*}
\]
which implies that
\[
( x_1M_\Gamma^L ) ( z_1 ) \geq ( x_2M_\Gamma^L ) ( z_1 ) . \tag{4.9}
\]

The application of given condition in the above relation yields that
\[
( x_2M_\Gamma^L ) ( z_1 ) \geq ( x_1M_\Gamma^L ) ( z_1 ) . \tag{4.10}
\]

The comparison of (4.9) and (4.10) gives the required equality for $M_\Gamma^L$. Likewise, we can prove the upper case. Consequently, $x_1M_\Gamma = x_2M_\Gamma$.

**Proof (ii):** Suppose that $M\Gamma x_1 = M\Gamma x_2$. In view of definition (25), we have
\[
\begin{align*}
M_\Gamma^L ( x_1x_2^{-1} ) &= ( M_\Gamma^L ) ( x_1 ) ( x_2 ) \\
&= M_\Gamma^L ( x_1 ) \\
M_\Gamma^L ( x_1x_2^{-1} ) &= M_\Gamma^L ( e ) ,
\end{align*}
\]
which means that $x_1x_2^{-1} \in Z_{M\Gamma}$.

Conversely, let $x_1x_2^{-1} \in Z_{M\Gamma}$. By applying definition (25) on $M_\Gamma x_1$ for any element $z_1 \in Z$, we have
\[
\begin{align*}
( M_\Gamma x_2 ) ( z_1 ) &= M_\Gamma ( x_1^{-1}z_1 ) \\
&= M_\Gamma \left( ( x_1^{-1}x_2 ) ( x_2^{-1}z_1 ) \right) \\
&\geq \min \left\{ M_\Gamma ( x_1^{-1} ) , M_\Gamma ( x_2^{-1}z_1 ) \right\} \\
&= \min \left\{ M_\Gamma ( x_1^{-1} ) , M_\Gamma ( e ) \right\} \\
( M_\Gamma x_2 ) ( z_1 ) &\geq M_\Gamma ( x_2^{-1}z_1 ) ,
\end{align*}
\]
which implies that
\[
( M_\Gamma x_2 ) ( z_1 ) \geq ( M_\Gamma x_1 ) ( z_1 ) . \tag{4.11}
\]

The application of given condition in the above relation yields that
\[
( M_\Gamma x_1 ) ( z_1 ) \geq ( M_\Gamma x_2 ) ( z_1 ) . \tag{4.12}
\]

The comparison of relations (4.11) and (4.12) gives the required equality for $M_\Gamma^L$.

Likewise, the upper case can be proved in the similar way. Consequently, $M\Gamma x_1 = M\Gamma x_2$.

**Theorem 36:** The following postulate holds in each $\Gamma$-IVFNSG for any elements $x_1, x_2, m$ and $n \in Z$. If $( x_1M_\Gamma ) = ( mM_\Gamma )$ and $( x_2M_\Gamma ) = ( nM_\Gamma )$ then $( x_1x_2M_\Gamma ) = ( mnM_\Gamma )$.

**Proof:** The application of theorem (35) and using the given condition on $M_\Gamma$, we have $x_1^{-1}m$ and $x_2^{-1}n \in Z_{M_\Gamma}$. Consider
\[
( x_1x_2 )^{-1}mn = ( x_2^{-1}x_1^{-1} ) mn = x_2^{-1}( x_1^{-1}m ) n = x_2^{-1}( x_1^{-1}m ) ( x_2^{-1}n ) ,
\]
which implies that $( x_1x_2 )^{-1}mn \in Z_{M_\Gamma}$. Consequently, $x_1x_2M_\Gamma = mnM_\Gamma$.

In the following result, we establish a natural homomorphism between groups $Z$ and its quotient group by $\Gamma$-IVFNSG $M_\Gamma$.

**Theorem 37:** Let $M_\Gamma \in \Gamma$-IVFNSG and $x_1 \in Z$. The map $\emptyset : Z \to Z/M_\Gamma$ defined by $\emptyset ( x_1 ) = M_\Gamma x_1$ is a natural homomorphism with its kernel $Z_{M_\Gamma}$.

**Proof:** Consider
\[
\emptyset ( x_1x_2 ) = M_\Gamma x_1 \ast M_\Gamma x_2 \\
\emptyset ( x_1 ) \ast \emptyset ( x_2 ) ,
\]
which shows that $\emptyset$ is a natural homomorphism. Moreover,
\[
Ker. \emptyset = \{ x_1 \in Z : \emptyset ( x_1 ) = M_\Gamma e \} = \{ x_1 \in Z : M_\Gamma ( x_1 ) = M_\Gamma e \} = \{ x_1 \in Z : M_\Gamma ( x_1 ) = M_\Gamma e ( x_1 ) \} = \{ x_1 \in Z : M_\Gamma ( e ) = M_\Gamma ( x_1 ) \} = Z_{M_\Gamma}.
\]

We obtain a correspondence between each $\Gamma$-IVF(N)SG of $Z/M_\Gamma$ and $\Gamma$-IVF(N)SG of $Z$ in the following theorem.

**Theorem 38 (Correspondence Theorem):** Let $M_\Gamma \in \Gamma$-IVFNSG(Z), then each $\Gamma$-IVF(N)SG of $Z/M_\Gamma$ correlates in a usual way to a $\Gamma$-IVF(N)SG of $Z$.

**Proof:** Let $M_\Gamma$ be a $\Gamma$-IVFSG of $Z/M_\Gamma$. A mapping $S_\Gamma : Z \to D( I )$ defined by $S_\Gamma ( x_1 ) = M_\Gamma ( M_\Gamma x_1 )$, $x_1 \in Z$.

Consider
\[
S_\Gamma^L ( x_1x_2 ) = S_\Gamma^L ( M_\Gamma x_1x_2 ) = S_\Gamma^L ( M_\Gamma x_1 ) \ast S_\Gamma^L ( M_\Gamma x_2 ) \geq \min \left\{ S_\Gamma^L ( M_\Gamma x_1 ) , S_\Gamma^L ( M_\Gamma x_2 ) \right\} .
\]
Similarly, the above relation can be obtained for the upper case of $S_\Gamma$. 

96268
Consider
\[ S_\Gamma \left( x_1^{-1} \right) = \left[ S_\Gamma^I \left( x_1^{-1} \right), S_\Gamma^U \left( x_1^{-1} \right) \right] \]
\[ = \left[ M_\Gamma \left( M_\Gamma x_1^{-1} \right), M_\Gamma \left( M_\Gamma x_1^{-1} \right) \right] \]
\[ = \left[ M_\Gamma \left( M_\Gamma x_1 \right), M_\Gamma \left( M_\Gamma x_1 \right) \right] \]
\[ = \left[ S_\Gamma \left( x_1 \right), S_\Gamma^U \left( x_1 \right) \right] \]
which shows that \( S_\Gamma \in \Gamma-IVFS\Gamma \). Moreover, it is quite easy to show that if \( S_\Gamma \in \Gamma-IVFS\Gamma \), then \( S_\Gamma \) is \( \Gamma-IVFS\Gamma \) of \( Z \).

**Definition 39:** Let \( M_\Gamma \) be a \( \Gamma-IVFS\Gamma \) of finite group \( Z \). Then the cardinality \( |Z/M_\Gamma| \) of \( Z/M_\Gamma \) is known as the index of \( M_\Gamma \) in \( Z \).

**Theorem 40 (\( \Gamma \)-Interval Valued Fuzzification of Lagrange’s Theorem):** Let \( Z \) be a finite group, then the index of \( \Gamma-IVFS\Gamma \) of \( Z \) divides the order of \( Z \).

**Proof:** A natural homomorphism \( \pi : Z \to Z/M_\Gamma \) is obtained by using theorem (37). Consider a subgroup \( H_1 \) of \( Z \) from the form
\[ H_1 = \{ h_1 \in Z : M_\Gamma h_1 = M_\Gamma e \} . \]
By applying definition (25) on \( h_1 \in H_1 \) and \( g \in Z \), we have \( M_\Gamma h_1 \) \( g \), \( g \in Z \), \( M_\Gamma \) \( g \), \( g \) implies that \( M_\Gamma \left( gh_1^{-1} \right) = M_\Gamma \left( g \right) \).

Particularly, \( M_\Gamma \left( h_1^{-1} \right) = M_\Gamma \left( e \right) \).

Using the theorem (15)(i) in the above relation gives that \( M_\Gamma \left( h_1 \right) = M_\Gamma \left( e \right) \), which shows that \( h_1 \in Z/M_\Gamma \), and therefore
\[ H_1 \subseteq Z/M_\Gamma . \]

Now, for any element \( h_1 \in Z/M_\Gamma \) and using the fact that \( Z/M_\Gamma \) \( Z \), we have
\[ M_\Gamma \left( h_1^{-1} \right) = M_\Gamma \left( e \right) . \]
In the light of theorem (24) for the elements \( h_1^{-1} \) and \( g \) belong to \( Z/M_\Gamma \), we obtain the following relation \( M_\Gamma h_1 = M_\Gamma e \) which means that \( h_1 \in H_1 \) and hence
\[ Z/M_\Gamma \subseteq H_1 . \]
Therefore, in view of (4.13) and (4.14), we get
\[ H_1 = Z/M_\Gamma . \]
Now we partition the group \( Z \) into disjoint union of cosets. Consider
\[ Z = H_1 x_1 \cup H_1 x_2 \cup H_1 x_3 \ldots \cup H_1 x_n , \] (4.15)
where \( h_1 x_1 = H_1 \). Now, we prove that to each coset \( H_1 x_i \) in relation (4.15), there exists a \( \Gamma \)-IVF coset \( M_\Gamma x_i \) in \( Z/M_\Gamma \). For any element \( h_1 \in H_1 \) and coset \( H_1 x_i \), we have
\[ \pi \left( h_1 x_i \right) = M_\Gamma h_1 x_i \]
\[ = M_\Gamma h_1 \neq M_\Gamma x_i \]
\[ = M_\Gamma e \neq M_\Gamma x_i \]
\[ = M_\Gamma x_i , \]
Thus, \( \pi \) maps each element of \( H_1 x_i \) into \( \Gamma \)-IVF coset of \( M_\Gamma x_i \). Next, a natural correspondence \( \varpi \) between \( \{ H_1 x_i : 1 \leq i \leq k \} \) and \( Z/M_\Gamma \) can be developed by
\[ \varpi \left( H_1 x_i \right) = M_\Gamma x_i . \]
The correspondence \( \varpi \) is one-to-one.

For this, let \( M_\Gamma x_i = M_\Gamma y_j \), then \( M_\Gamma x_i x_j^{-1} = M_\Gamma e \). Thus, \( x_i x_j^{-1} \in H_1 \). So, \( H_1 x_i = H_1 x_j \). Hence, \( \varpi \) is one-to-one.

It is quite clear form the above discussion that \( |Z/M_\Gamma| = k \) which implies that \( |Z/M_\Gamma| \) divides the \( |Z| \) as \( k \) divides \( |Z| \).

The above algebraic fact is illustrated in the following example.

**Example 41:** Consider the finite group of order 6 as follows:
\[ S_3 = \{<a, b : a^3 = b^2 = e, ab = ba^2> \} . \]

The \( \Gamma \)-IVFS\Gamma \) \( M_\Gamma \) of \( S_3 \) corresponds to the value \( \Gamma \in [0.45, 0.6] \) is given by
\[ M_\Gamma \left( x_1 \right) = \begin{cases} 0.93, 0.98 & \text{if } x_1 = e \\ 0.78, 0.88 & \text{if } x_1 = a, a^2 \\ 0.67, 0.8 & \text{if otherwise} \end{cases} \]
The set of all distinct \( \Gamma \)-IVF right cosets of \( M_\Gamma \) in \( S_3 \) is given by \( \delta = [M_\Gamma e, M_\Gamma a, M_\Gamma b] \).

This shows that \( |S_3/M_\Gamma| = 3 \).

**V. CONCLUSION**

In this paper, we initiated the study of \( \Gamma \)-IVFS\Gamma \) and proved many algebraic characteristics of this notion. We defined \( \Gamma \)-IVFS\Gamma \), a quotient group of \( Z \) by \( \Gamma \)-IVFS\Gamma \) and established numerous fundamental algebraic aspects of these concepts. Moreover, we obtained a correspondence between each \( \Gamma \)-IVF(N)SG of a group \( Z \) and \( \Gamma \)-IVF(N)SG of its quotient group. Furthermore, we performed the \( \Gamma \)-interval valued fuzzification of Lagrange’s theorem of \( \Gamma \)-IVFS\Gamma \) of a finite group \( Z \).

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