Non-local topological electromagnetic phases of matter

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In 2+1D, nonlocal topological electromagnetic phases are defined as atomic-scale media which host photonic monopoles in the bulk band structure and respect bosonic symmetries (e.g. time-reversal $T^2 = +1$). Additionally, they support topologically protected spin-1 edge states, which are fundamentally different than spin-$\frac{1}{2}$ and pseudo-spin-$\frac{1}{2}$ edge states arising in fermionic and pseudo-fermionic systems. The striking feature of the edge state is that all electric and magnetic field components vanish at the boundary - in stark contrast to analogs of Jackiw-Rebbi domain wall states. This surprising open boundary solution of Maxwell’s equations, dubbed the quantum gyroelectric effect [Phys. Rev. A 98, 023842 (2018)], is the supersymmetric partner of the topological Dirac edge state where the spinor wave function completely vanishes at the boundary. The defining feature of such phases is the presence of temporal and spatial dispersion in conductivity (the linear response function). In this paper, we generalize these topological electromagnetic phases beyond the continuum approximation to the exact lattice field theory of a periodic atomic crystal. To accomplish this, we put forth the concept of microscopic photonic band structure of solids - analogous to the traditional theory of electronic band structure. Our definition of topological invariants utilizes optical Bloch modes and can be applied to naturally occurring crystalline materials. For the photon propagating within a periodic atomic crystal, our theory shows that besides the Chern invariant $\chi \in \mathbb{Z}$, there are also symmetry-protected topological (SPT) invariants $\nu \in \mathbb{Z}_N$ which are related to the cyclic point group $C_N$ of the crystal $\nu \equiv \chi \mod N$. Due to the rotational symmetries of light $R(2\pi) = +1$, these SPT phases are manifestly bosonic and behave very differently from their fermionic counterparts $R(2\pi) = -1$ encountered in conventional condensed matter systems. Remarkably, the nontrivial bosonic phases $\nu \neq 0$ are determined entirely from rotational (spin-1) eigenvalues of the photon at high-symmetry points in the Brillouin zone. Our work accelerates progress towards the discovery of bosonic phases of matter where the electromagnetic field within an atomic crystal exhibits topological properties.

I. INTRODUCTION

From a material science standpoint, all known topological phases of matter to date have been characterized by electronic phenomena [1–2]. This is true for both time-reversal broken phases - often called Chern insulators [3–6] and time-reversal unbroken phases - known as topological insulators [7–9]. The signature of time-reversal broken phases is the quantum Hall conductivity $\sigma_{xy} = ne^2/h$, which is quantized in terms of the electronic Chern invariant $\chi = \sum g \sigma_{xy}(0, 0) = ne^2/h$. At high frequency $\omega \neq 0$ and short wavelength $k \neq 0$, the Hall conductivity $\sigma_{xy}(\omega, k)$ acquires new physical meaning. We have shown that the electromagnetic field itself becomes topological [24–25] and nonlocal Hall conductivity functions identically to a photonic mass [26–28] in the low-energy physics $\omega \approx 0$. These topological electromagnetic phases of matter depend on the global behavior of $\sigma_{xy}(\omega, k)$, over all frequencies and wave vectors.

As of yet, only the continuum topological theory of the aforementioned quantum gyroelectric effect (QGEE) has been solved [24–25]. Our goal is to extend this concept beyond the long wavelength approximation to the exact lattice field theory of optical Bloch waves. In this regime, we must consider not only the first spatial component $\sigma_{xy}(\omega, k) = \sigma_{xy}(\omega, k, 0)$ but all spatial harmonics of the crystal $g \neq 0$, to infinite order,

$$j^{\text{Hall}}_x(\omega, k) = \sum_g \sigma_{xy}(\omega, k, g)E_y(\omega, k + g).$$  (1)

$g \cdot R \in 2\pi\mathbb{Z}$ are the reciprocal lattice vectors and $R$ is the primitive vector of the crystal. In this case, $E_y$ is the microscopic electric field. The electromagnetic field must be described to the same scale as the electronic wave functions, i.e. for photon momenta on the order of the lattice constant $ka = \pi$, with $a \approx 5$ Å. Since topological invariants are fundamentally global properties, these astronomically deep subwavelength fields actually play a role in the topological physics.

The idea of lattice topologies in electromagnetism was first proposed by Haldane [29–30] in the context of photonic crystals [31–38]. These are artificial materials composed of two or more different constituents which form a macroscopic crystalline structure. A few important examples are gyrotropic photonic crystals [31–33]. Floquet topological insulators [39–41] and biaxialtopaxotropic metamaterials [42–56]. Instead, we focus on the microscopic domain and utilize the periodicity of the atomic lattice itself. Thus, the topological invariants in our theory are connected to the microscopic atomic lattice and not artificially engineered macroscopic structures. We stress that in the microscopic case, the electromagnetic theory is manifestly bosonic [57–60] (e.g. time-reversal $T^2 = +1$) and characterizes topological phases of matter fundamentally distinct from known fermionic and pseudo-fermionic phases.
With that in mind, this paper is dedicated to solving two longstanding problems, which is of interest to both photonics and condensed matter physics. The first, is developing the rigorous theory of optical Bloch modes in natural crystal solids. This problem gained significant interest in the 60’s and 70’s in the context of spatial dispersion (nonlocality) as it lead to qualitatively new phenomena - such as natural optical activity (gyrotropy) \[61\textendash}64\]. The current paper builds on our recent discovery of the quantum gyroelectric effect \[24\textendash}25\] where we have shown that nonlocality is also essential for topological phenomena and is a necessary ingredient in any long wavelength theory. However, since topological field theories are global constructs, a complete picture can only be achieved in the microscopic domain of Bloch waves. Most of the foundations have been summarized by Agronovich and Ginzburg in their seminal monograph on crystal optics \[65\]. Nevertheless, topological properties have never been tackled to date and a few fundamental quantities, such as the Bloch energy density, have not been defined.

This leads to the second problem - deriving the electromagnetic topological invariants of these systems given only the atomic lattice. We solve this problem and also provide a systematic bosonic classification of all 2+1D topological photonic matter. Utilizing the optical Bloch modes, we show that a Chern invariant \(c \in \mathbb{Z}\) can be found for any two-dimensional crystal and characterizes distinct topological phases. We then go one step further and classify these topological phases with respect to the symmetry group of the crystal - the cyclic point groups \(C_N\). These are known as symmetry-protected topological (SPT) phases \[66\textendash}77\] and the spin of the photon is critical to their definition. The rotational symmetries of light \(R(2\pi) = +1\) impart an intrinsically bosonic nature to these phases, which are fundamentally different than their fermionic counterparts \(R(2\pi) = 1\) encountered in conventional condensed matter systems. We illustrate this fact by directly comparing SPT bosonic and fermionic phases side-by-side. Our rigorous formalism of microscopic photonic band structure provides an immediate parallel with the traditional theory of electronic band structure in crystal solids.

This article is organized as follows. In Sec. \[II\] we develop the general formalism of 2+1D lattice electromagnetism. First we derive the generalized linear response function accounting for spatiotemporal dispersion to infinite order in the crystal’s spatial harmonics \(g\). Thereafter, we find the equivalent Hamiltonian that governs all light-matter Bloch excitations of the material. In Sec. \[III\] we study the discrete rotational symmetries (point groups) of the crystal and the implications on spin-1 quantization \[78\textendash}83\] of the photon. The following Sec. \[IV\] discusses the electromagnetic Chern number and its relationship to symmetry-protected topological (SPT) bosonic phases. The bosonic classification of each phase is related directly to integer quantization of the photon [Tab. \[I\]] and this is compared alongside their fermionic counterparts [Tab. \[II\]]. Sec. \[V\] presents our conclusions.

The focus of this paper is 2+1D topological electromagnetic (bosonic) phases of matter \(c \neq 0\) which requires breaking time-reversal symmetry. These bosonic Chern insulators are ultimately related to nonlocal gyrotrropic response (Hall conductivity) and show unidirectional, completely transverse electro-magnetic (TEM) edge states \[24\textendash}25\]. However, time-reversal symmetric topological phenomena can arise in higher dimensional systems in the context of nonlocal magnetoelectricity \[54\]. These time-reversal symmetric phases possess counter-propagating TEM edge states and are interpreted as two copies of a bosonic Chern insulator. Features of topological phenomena, such as spin-momentum locking \[85\textendash}89\], have also been reported in conventional surface state problems - surface plasmon-polaritons (SPPs), Dyakonov waves, etc. However, these traditional surface properties are not connected to any topologically protected edge states or nontrivial phases.

Note: Due to the frequency of integral formulas, all differential elements are assumed to be correctly normalized, such as Fourier transforms \(e^{i\textbf{\sigma} \cdot \textbf{r}}/\sqrt{V} \rightarrow e^{i\textbf{\sigma} \cdot \textbf{r}}\), where \(V\) is the unit cell area of a crystal.

II. LATTICE ELECTROMAGNETISM

A. 2+1D electrodynamics

In this paper we focus on two-dimensional materials and the topological electromagnetic phases associated with them. The preliminaries for 2+1D electromagnetism can be found in Appendix A of Ref. \[24\]. Conveniently, the restriction to 2D limits the degrees of freedom of both the electromagnetic field and the induced response of the material, such that strictly transverse-magnetic (TM) waves propagate. The corresponding wave equation reads,

\[
\mathcal{H}_0 \psi = i\partial_t g, \quad f = \begin{bmatrix} E_x \\ E_y \\ H_z \end{bmatrix}, \quad g = \begin{bmatrix} D_x \\ D_y \\ B_z \end{bmatrix}. \tag{2}
\]

\(f\) is the TM polarization state of the electromagnetic field and the material response is captured by the displacement field \(g\).

\[
\mathcal{H}_0(p) = p_x \hat{S}_x + p_y \hat{S}_y = \begin{bmatrix} 0 & 0 & -p_y \\ 0 & 0 & p_x \\ -p_y & p_x & 0 \end{bmatrix}. \tag{3}
\]

\(p = -i \nabla\) is the two-dimensional momentum operator. \(\hat{S}_x\) and \(\hat{S}_y\) are spin-1 operators that satisfy the angular momentum algebra \([\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k\).

\[
\hat{S}_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4}
\]

Here, \((\hat{S}_z)_{ij} = -i\epsilon_{ijk}\) is the generator of rotations in the \(xy\) plane and is represented by the antisymmetric matrix. In two dimensions, \(\hat{S}_z\) governs all rotational symmetries of the electromagnetic field.
B. 2+1D linear response theory

The effective electromagnetic properties of a material are very accurately described by a linear response theory - assuming nonlinear interactions are negligible. This is true for low intensity light $|f| \lesssim 10^8$ V/m that is sufficiently weak compared to the atomic fields governing the binding of the crystal itself. Our goal is to characterize the entire topological field theory in this regime. With this in mind, the most general linear response of a 2D material is nonlocal in both space and time coordinates,

$$g(t, r) = \int \! d^2r' \int_{-\infty}^t \! dt' \mathcal{M}(t, t', r, r') \, f(t', r').$$

Equation (7) implies energy conservation in Hermitian systems $\omega' = \omega$. However, a crystal is not translationally invariant in space - momentum is not conserved $k' \neq k$. Instead, the crystal is periodic and possesses discrete translational symmetry [65] [90].

$$\mathcal{M}(\omega, r, r') = \mathcal{M}(\omega, r + \mathbf{R}, r' + \mathbf{R}),$$

where $\mathbf{R}$ is the primitive lattice vector of the crystal. This admits a Fourier decomposition in the spatial harmonics of the crystal $g$,

$$\mathcal{M}(\omega, r, r') = \sum_{\mathbf{g}} \mathcal{M}_{\mathbf{g}}(\omega, r - \mathbf{R}) e^{-i\mathbf{r'} \cdot \mathbf{g}},$$

with $g \cdot R \in 2\pi \mathbb{Z}$ arbitrary integer combinations of the reciprocal lattice vectors.

Due to nonlocality, it is necessary to convert to the reciprocal space,

$$\mathcal{M}(\omega, k, k') = \int \! d^2r d^2r' \mathcal{M}(\omega, r, r') e^{-i\mathbf{r} \cdot \mathbf{k} + i\mathbf{r'} \cdot \mathbf{k'}}.$$
where \( \mathbf{k} + \mathbf{g} \rightarrow \mathbf{k} \). These are essentially the photonic structure factors of the two-dimensional crystal.

In this case, \( \mathbf{k} \) is the crystal momentum and is only uniquely defined within the Brillouin zone (BZ). Hence, the electromagnetic eigenstates of the medium are Bloch waves,

\[
\mathcal{H}_0(\mathbf{k}) f_k = \omega \int d^2 k' \mathcal{M}(\omega, \mathbf{k}, \mathbf{k}') f_{k'} = \omega \sum_{\mathbf{g}} \mathcal{M}_g(\omega, \mathbf{k}) f_{k+\mathbf{g}},
\]

where \( \mathcal{H}_0(\mathbf{k}) = \mathbf{k} \cdot \mathbf{S} \) are the vacuum Maxwell equations in momentum space. The Bloch photonic wave function \( f(\mathbf{k}, \mathbf{r}) = \langle \mathbf{r} | f_{\mathbf{k}} \rangle \) corresponds to the net propagation of all \( \mathbf{k} + \mathbf{g} \) scattered waves in the medium,

\[
f(\mathbf{k}, \mathbf{r}) = \sum_{\mathbf{g}} f_{\mathbf{k}+\mathbf{g}} e^{i \mathbf{g} \cdot \mathbf{r}},
\]

where \( f(\mathbf{k}, \mathbf{r} + \mathbf{R}) = f(\mathbf{k}, \mathbf{r}) \) is periodic in the crystal lattice. Note that Eq. (12) and (13) reduce to the continuum theory \cite{24, 25} when considering only the 0th order harmonic \( \mathbf{g} = 0 \).

### C. Generalized response function

Nevertheless, Eq. (12) poses a few serious problems; it does not represent a proper first-order in time Hamiltonian since all harmonics of the response function \( \mathcal{M}_g(\omega, \mathbf{k}) \) depend on the eigenvalue \( \omega \). Moreover, it is not evident that the Bloch waves in Eq. (13) are normalizable, as the system contains complex spatial and temporal dispersion. Due to these issues, it is advantageous to return to the more general form of \( \mathcal{M}(\omega, \mathbf{k}, \mathbf{k}') \) without assuming discrete translational symmetry. This will allow us to derive very robust properties of the response function that can also be applied to amorphous materials or quasicrystals.

First, we demand Hermiticity,

\[
\mathcal{M}(\omega, \mathbf{k}, \mathbf{k}') = \mathcal{M}^\dagger(\omega, \mathbf{k}', \mathbf{k}),
\]

such that the response is lossless. To account for normalizable electromagnetic waves, the energy density must be positive definite for all \( \omega \),

\[
U(\omega) = \int d^2 k' d^2 k'' f_k^\dagger \mathcal{M}(\omega, \mathbf{k}, \mathbf{k}') f_{k'} > 0,
\]

where \( \mathcal{M} \) describes the inner product space in a dispersive medium,

\[
\mathcal{M}(\omega, \mathbf{k}, \mathbf{k}') = \frac{\partial}{\partial \omega} \left[ \omega \mathcal{M}(\omega, \mathbf{k}, \mathbf{k}') \right].
\]

Notice that \( U(\omega) = U^*(\omega) \) is only real-valued when \( \mathcal{M} \) is Hermitian. For realistic materials, the energy density is also stable at static equilibrium \( \omega = 0 \),

\[
U(0) = \int d^2 k' d^2 k'' f_k^\dagger \mathcal{M}(0, \mathbf{k}, \mathbf{k}') f_{k'} > 0,
\]

with \( \mathcal{M}(0, \mathbf{k}, \mathbf{k}') = \hat{\mathcal{M}}(0, \mathbf{k}, \mathbf{k}') \) at zero frequency. To ensure the electromagnetic field is real-valued, i.e. represents a neutral particle, we always require the reality condition,

\[
\mathcal{M}(\omega, \mathbf{k}, \mathbf{k}') = \mathcal{M}^\dagger(-\omega, -\mathbf{k}, -\mathbf{k}').
\]

Furthermore, the response is transparent at high frequency \( \omega \rightarrow \infty \), as the material cannot respond to sufficiently fast temporal oscillations,

\[
\lim_{\omega \rightarrow \infty} \mathcal{M}(\omega, \mathbf{k}, \mathbf{k}') = \mathbb{I}_3 \delta_{\mathbf{k}-\mathbf{k}'}.
\]

\( \mathbb{I}_3 \) is the 3 \times 3 identity matrix and \( \delta_{\mathbf{k}-\mathbf{k}'} = \delta^2(\mathbf{k} - \mathbf{k}') \) is the momentum conserving delta function. Lastly, the response must be causal and satisfy the Kramers-Kronig relations.

Combining all the above criteria, we find that \( \mathcal{M} \) can always be decomposed as a discrete summation of oscillators \cite{29, 52, 91},

\[
\mathcal{M}(\omega, \mathbf{k}, \mathbf{k}') = \mathbb{I}_3 \delta_{\mathbf{k}-\mathbf{k}'} - \sum_{\alpha} \int d^2 k'' \frac{C_{\alpha k'' k' k} C_{\alpha k k' k'}}{\omega_{\alpha k''} - \omega_{\alpha k''}}.
\]

Any Hermitian (lossless) response function can be expressed in this form. Equation (20) is easily extended to 3D materials but our focus is on 2D topological field theories. In this case, \( \alpha \) labels an arbitrary bosonic excitation in the material, such as an exciton or phonon, which couples linearly to the electromagnetic fields via the 3 \times 3 tensor,

\[
C_{\alpha}(\mathbf{k}, \mathbf{k}') = \int d^2 r d^2 r' C_{\alpha}(\mathbf{r}, \mathbf{r}') e^{-i \mathbf{k} \cdot \mathbf{r}} e^{i \mathbf{k}' \cdot \mathbf{r}}.
\]

\( \omega_{\alpha k} \) is the resonant energy of the oscillator and corresponds to a first-order pole of the response function. Notice that \( \mathcal{M} \) itself contains an integral over \( \mathbf{k}' \). Microscopically, this constitutes the overlap with the electronic momentum to infinitesimally small scale \( k \rightarrow \infty \).

Substituting Eq. (20) into Eq. (15), we can exchange the order of integration \( \hat{U}(\omega) = \int d^2 k' \mathcal{M}(\omega, \mathbf{k}) \) and define,

\[
\hat{U}(\omega, \mathbf{k}) = |f_k|^2 + \sum_{\alpha} \left| \int d^2 k' \frac{C_{\alpha k k' k} f_{k'}}{\omega - \omega_{\alpha k}} \right|^2 > 0,
\]

which is positive definite for all \( \omega \) and \( \mathbf{k} \). Equation (22) is the generalized inner product for the electromagnetic field and represents the energy density at an arbitrary frequency and wave vector. We will now show that Eq. (20) is derived from a first-order in time Hamiltonian.

### D. Generalized Hamiltonian

To find the corresponding Hamiltonian, we expand the response function \( \mathcal{M} \) in terms of three-component matter oscillators \( \psi_{\alpha} \). These represent internal polarization and magnetization modes of the material,

\[
\omega_{\alpha k} \psi_{\alpha k} = \omega_{\alpha k} \psi_{\alpha k} + \int d^2 k' C_{\alpha k k'} f_{k'}. \]
Substituting Eq. (23) and (20) into Eq. (12) we obtain,
\[ \omega f_k = \mathcal{H}_0(k)f_k + \sum_{\alpha} \int \frac{d^2k'' d^2k'}{\omega_{ak''}} C^\dagger_{\alpha kk''} C_{\alpha kk'} f_{k'} \]
\[ + \sum_{\alpha} \int d^2k C^\dagger_{\alpha kk'} \psi_{\alpha k'}. \]  
(24)

which is manifestly Hermitian \( H(k, k') = H^\dagger(k', k) \).

We now define \( u_k \) as the generalized state vector of the electromagnetic problem; accounting for the photon \( f_k \) and all possible internal excitations \( \psi_{\alpha k} \),
\[ \int d^2k' H_{kk'} u_{k'} = \omega u_k, \quad u_k = \begin{bmatrix} f_k \\ \psi_{1k} \\ \psi_{2k} \\ \vdots \end{bmatrix}, \]  
(26)

which is a first-order wave equation. Notice that contraction of \( u_k \) naturally reproduces the energy density [Eq. (22)] upon summation over all degrees of freedom,
\[ u^\dagger_k u_k = |f_k|^2 + \sum_{\alpha} |\psi_{\alpha k}|^2 = U(\omega, k) \]
\[ = |f_k|^2 + \sum_{\alpha} \int d^2k' C^\dagger_{\alpha kk'} f_{k'} \left( \frac{1}{\omega - \omega_{\alpha k}} \right)^2. \]  
(27)

The complete set of eigenvectors and eigenvalues is represented by \( u_k \). We must define all relevant electromagnetic quantities in terms of this generalized state vector.

E. Crystal Hamiltonian

We are now ready to enforce crystal periodicity. Instead of expanding \( \mathcal{M} \) directly, we utilize the periodicity of the coupling tensors \( C_\alpha(r, r') = C_\alpha(r + \mathbf{R}, r' + \mathbf{R}) \), which is a discrete spectrum in \( g \).
\[ C_\alpha(k, k') = \sum_g C_{\alpha gg}(k) \delta^2(k + g - k'). \]  
(28)

\( C_{\alpha gg}(k) \) tells us the scattering amplitude of a photon \( f_{k + g} \) with momentum \( k + g \) into an internal mode of the material \( \psi_{\alpha k} \) at momentum \( k \), and vice versa. The crystal Hamiltonian accounts for all such scattering events,
\[ H(k, k') = \sum_g H_g(k) \delta^2(k + g - k'), \]  
(29)

with Hermiticity \( H_g(k) = H^\dagger_g(k + g) \) satisfied by definition. Note, the resonant energies \( \omega_{\alpha}(k + g) = \omega_{\alpha}(k) \) are generally periodic in \( k \), since they correspond to energy gaps in the electronic band structure. However, we do not need to assume this to define the optical Bloch excitations. A periodic coupling is sufficient.

The quasiparticle eigenstates of this Hamiltonian describe the complete spectrum of Bloch waves,
\[ \sum_g H_g(k) u_{nk + g} = \omega_{nk} u_{nk}, \quad \omega_n(k + g) = \omega_n(k), \]  
(30)

and the eigenenergies \( \omega_{nk} \) are periodic Bloch bands. \( n \) labels a particular energy band of the material with its associated Bloch eigenstate \( |u_{nk}\rangle \). The total wave function \( |u_{nk}\rangle \) contains the photon \( |f_{nk}\rangle \) and all internal degrees of freedom describing the linear response \( |\psi_{nk}\rangle \). This is expressed compactly in the Fourier basis \( u_n(k, r) = \langle r | u_{nk} \rangle \),
\[ u_n(k, r) = \sum_{g} u_{nk + g e^g r}, \quad u_{nk + g} = \begin{bmatrix} f_{nk + g} \\ \psi_{nk1 + g} \\ \psi_{nk2 + g} \\ \vdots \end{bmatrix}, \]  
(31)

where \( u_n(k, r + \mathbf{R}) = u_n(k, r) \) is periodic in the crystal lattice. In this basis, \( |u_{nk}\rangle \) is normalized to the energy density as,
\[ 1 = \langle u_{nk} | u_{nk} \rangle = \sum_g u^\dagger_{nk + g} u_{nk + g} \]
\[ = \sum_g \left( f_{nk + g}^\dagger f_{nk + g} + \sum_{\alpha} \psi_{nk + g}^\dagger \psi_{nk + g} \right) \]
\[ = \sum_g f_{nk + g}^\dagger \mathcal{M}_g - g \langle \omega_{nk}, k + g | f_{nk + g} \rangle. \]  
(32)

The bra-ket notation \( (| \rangle \) implies integration over the 2D unit cell and we have utilized the linear response theory to express \( \psi_{\alpha} \) in terms of the driving field \( f \),
\[ \psi_{nk + g} = \sum_g C_{\alpha gg}(k + g) f_{nk + g^* + g} \]
\[ \omega_{nk + g} = \frac{\sum_g C_{\alpha gg}(k + g) f_{nk + g^* + g}}{\omega_{nk} - \omega_{nk + g}}. \]  
(33)

\( \mathcal{M}_g(\omega, k) = \partial_\omega \left[ \omega \mathcal{M}_g(\omega, k) \right] \) is the contribution to the energy density arising from each spatial harmonic of the crystal.
Finally, the eigenenergies $\omega_{nk}$ are the $n$ nontrivial roots of the characteristic wave equation,

$$H_0(k)f_{nk} = \omega_{nk} \sum_g M_g(\omega_{nk}, k)f_{nk+g}, \quad (34)$$

which generates all possible photonic bands of the crystal. Note, the response function $M_g(\omega, k)$ is now expressed in terms of $C_{og}(k)$ and describes the net summation of all scattering and back-scattering events in the material,

$$M_g(\omega, k) = 1_3\delta_g - \sum_{\alpha g'} C^I_{\alpha-g}(k + g') C_{og-g'}(k + g').$$

This proves that the wave equation is derived from a first-order Hamiltonian, has real eigenvalues $\omega = \omega_{nk}$ for all momenta, and is normalizable in terms of $|n\rangle$.

III. DISCRETE ROTATIONAL SYMMETRY

A. Point groups in 2D

Point groups are the discrete analogs of continuous rotations and reflections. They represent the number of ways the atomic lattice can be transformed into itself [22][23]. Due to the crystallographic restriction theorem (CRT), there are ten such point groups in 2D. The first five are the cyclic groups $C_N$,

$$C_1, C_2, C_3, C_4, C_6. \quad (36)$$

For instance, $C_3$ implies threefold cyclic symmetry while $C_1$ is no symmetry. The last five are the dihedral groups $D_N$,

$$D_1, D_2, D_3, D_4, D_6. \quad (37)$$

The dihedral group $D_N$ contains $C_N$ plus reflections. However, it can be proven that the Chern number for all $D_N$ point groups vanish [69]. Therefore, we concern ourselves with only the cyclic groups $C_N$. The Brillouin zone of each point group is displayed in Fig. 1.

The defining characteristic of each cyclic group is the fermionic or bosonic representation. When we rotate the fields by $2\pi$, we take the particle into itself and acquire a phase,

$$\mathcal{R}(2\pi) = (-1)^F. \quad (38)$$

$F$ is twice the total spin of particle, or equivalently, the fermion number. Fermions with half-integer spin are antisymmetric under rotations $\mathcal{R}(2\pi) = -1$, while bosons with integer spin are symmetric $\mathcal{R}(2\pi) = +1$. Depending on the symmetries of the lattice, the topology fundamentally changes for fermions and bosons. We will understand the implications this has for spin-1 photons.

![FIG. 1. Brillouin zone of each cyclic point group $C_N$. (a), (b), (c), (d), and (e) correspond to $N = 2, 3, 4, 6, \text{and } \infty$ respectively. Due to rotational symmetry, the total Brillouin zone is equivalent to $N$ copies of the irreducible Brillouin zone (IBZ), which is represented by the blue quadrant. For continuous symmetry $N = \infty$, this is simply a line. The yellow circles label high-symmetry points $\mathcal{R}k_i = k$, where the crystal Hamiltonian is invariant under a certain rotation $\mathcal{R}$. At these specific momenta, a Bloch photonic wave function $\mathcal{R}_N|f(k_i)\rangle = \eta_N(k_i)|f(k_i)\rangle$ is an eigenstate of an an $N$-fold rotation $\eta_N(k_i) = (\pm \frac{2\pi}{N} m_N(k_i))$ such that the photon possesses quantized integer eigenvalues $m_N(k_i) \in \mathbb{Z}_N$. Since $m_N$ are discrete quantum numbers, their values cannot vary continuously if the crystal symmetry is preserved - they can only be changed at a topological phase transition.](image)

B. Spin-1 discrete symmetries

If the two-dimensional crystal belongs to a cyclic point group $C_N$, the Hamiltonian possesses discrete rotational symmetry about the $z$-axis,

$$\mathcal{R}^{-1} H_{\mathcal{R}g}(\mathcal{R}k) \mathcal{R} = H_g(k), \quad \omega_n(\mathcal{R}k) = \omega_n(k), \quad (39)$$

where $\mathcal{R}$ is any rotation in $C_N$. It is important to note that $\mathcal{R}$ is diagonal in $u$, meaning the photon and each oscillator is rotated individually, $f \rightarrow \mathcal{R} f$ and $\psi_\alpha \rightarrow \mathcal{R} \psi_\alpha$. This implies there is no mixing of fields. The symmetries of the Hamiltonian are endowed by the coupling tensors, which dictates the degrees of freedom of the material response,

$$C_{\alpha g}(\mathcal{R}k) = C_{\alpha g}(k), \quad \omega_\alpha(\mathcal{R}k) = \omega_\alpha(k). \quad (40)$$
After summation over all \( C_{\alpha g}(k) \), we can prove that the response function transforms identically under such a rotation,

\[
R^{-1} M R_{g}(\omega, Rk) R = M_{g}(\omega, k) \quad \text{(41)}
\]

Therefore, the photon inherits all symmetries of the crystal.

In this case, the \( R \) matrix represents a discrete rotation and can be expressed as the exponential of the spin-1 generator \((\hat{S}_{z})_{ij} = -i\epsilon_{ijz}\).

\[
\mathcal{R}_N = \exp \left( \frac{2\pi i}{N} \hat{S}_{z} \right) = \begin{bmatrix}
\cos \frac{2\pi i}{N} & \sin \frac{2\pi i}{N} & 0 \\
-\sin \frac{2\pi i}{N} & \cos \frac{2\pi i}{N} & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \text{(42)}
\]

where \( 2\pi/N \) is an \( N \)-fold rotation. We stress that every cyclic group for the photon is a vector representation, which is bosonic,

\[
\mathcal{R}(2\pi) = \mathcal{R}(N\theta_N) = (\mathcal{R}_N)^N = +1_3 \quad \text{(43)}
\]

The electromagnetic field returns in phase under cyclic revolution.

### C. High-symmetry points

The Bloch eigenstates \(|u_{nk}\rangle\) are essentially a collection of periodic vector fields. To rotate the fields, we must perform an operation on both the coordinates \( r \) and the polarization states \( f \) and \( \psi_{\alpha} \). In real space, the operation of a rotation \( \tilde{R} \) is preformed as,

\[
\langle r | \tilde{R} | u_{nk}\rangle = \mathcal{R} u_n(k, R^{-1}r) = n_{\eta}(k) u_n(\mathcal{R} k, r), \quad \text{(44)}
\]

where \( \mathcal{R} \) is a discrete rotation defined in Eq. (42). This implies the Fourier coefficients obey,

\[
\mathcal{R} u_n(k + R^{-1}g) = n_{\eta}(k) u_n(\mathcal{R} k + g). \quad \text{(45)}
\]

It follows from symmetry that the operation of \( \tilde{R} \) takes a wave function at \( k \) to \( \mathcal{R}k \) with the same energy \( \omega_n(k) = \omega_n(\mathcal{R}k) \) - but with a possibly different phase \( |n_{\eta}(k)|^2 = 1 \). Utilizing the linear response theory, we notice that the phase factor \( n_{\eta}(k) \) is governed entirely by the photon,

\[
\mathcal{R} \psi_{\eta n}(k + R^{-1}g) = \sum_g \frac{\mathcal{R} c_{\alpha g'}(k + R^{-1}g)f_{nk + g'} + R^{-1}g}{\omega_{nk} - \omega_{nk + R^{-1}g}} = \sum_g \frac{c_{\alpha g} \mathcal{R} c_{\alpha g'}(Rk + g)f_{nk + g'} + R^{-1}g}{\omega_{nk} - \omega_{nk + R^{-1}g}} \]

\[
= \sum_g \frac{c_{\alpha g} \mathcal{R} c_{\alpha g'}(Rk + g)n_{\eta}(k) f_{nk + g'} + g}{\omega_{nk} - \omega_{nk + R^{-1}g}} = n_{\eta}(k) \psi_{\eta n}(\mathcal{R}k + g). \quad \text{(46)}
\]

This is an incredibly convenient simplification and implies the precise coordinates of the matter oscillations \( \psi_{\alpha} \) are superfluous when discussing symmetries. The electromagnetic field \( f \) tells us everything.

Importantly, there are specific points in the Brillouin zone where \( k \) is invariant under a discrete rotation,

\[
\mathcal{R} k_i = k_i. \quad \text{(47)}
\]

This is because the crystal momentum only differs by a lattice translation at these points \( \mathcal{R} k_i = k_i + g \), which leaves a Bloch wave function unchanged,

\[
e^{i\mathcal{R} k_i\cdot r} u_n(\mathcal{R} k_i, r) = e^{i(k_i + g)\cdot r} u_n(k_i + g, r) = e^{iR k_i \cdot r} u_n(k_i, r). \quad \text{(48)}
\]

These are called high-symmetry points (HSPs); they occur at the center and certain vertices of the Brillouin zone. The crystal Hamiltonian is rotationally invariant at these momenta - i.e. it commutes with \( \mathcal{R} \). Therefore, the wave functions are simultaneous eigenstates of \( \mathcal{R} \) at HSPs,

\[
\mathcal{R} |u_n(k_i)\rangle = n_{\eta}(k_i)|u_n(k_i)\rangle, \quad \text{(49)}
\]

which immediately implies,

\[
\mathcal{R} |f_{n\eta}(k_i)\rangle = n_{\eta}(k_i)|f_{n\eta}(k_i)\rangle. \quad \text{(50)}
\]

Here, \( n_{\eta}(k_i) \) is the eigenvalue of \( \mathcal{R} \) at \( k_i \) for the \( n \)-th band.

### D. Spin-1 eigenvalues

Depending on the point group and the precise HSP, \( n_{\eta}(k_i) = \eta_{N,n}(k_i) \) can represent any \( N \)-th root of unity corresponding to the rotation operator \( \mathcal{R}_N \),

\[
\eta_{N,n}(k_i) = \exp \left[ \frac{2\pi i}{N} m_{N,n}(k_i) \right], \quad (\eta_{N,n})^N = +1. \quad \text{(51)}
\]

\( m_{N,n}(k_p) \in \mathbb{Z}_N \) is a modulo integer - it labels the \( N \) possible spin-1 eigenvalues at \( k_i \). In \( C_4 \) for example, the \( \Gamma \) and \( M \) points are invariant under \( \mathcal{R}_4 \) rotations, while the X and Y points are invariant under \( \mathcal{R}_2 \) rotations (inversion). This means there are 4 possible spin-1 charges located at \( m_{1,n}(\Gamma) \) & \( m_{1,n}(M) \in \mathbb{Z}_4 \) respectively and 2 possible charges located at \( m_{2,n}(X) = m_{2,n}(Y) \in \mathbb{Z}_2 \). A visualization of these topological charges is presented in Fig. [2] and is contrasted with their fermionic counterparts in Fig. [5]. In Sec. [IV] we will connect these rotational eigenvalues directly to the topological invariants.

### IV. TOPOLOGICAL ELECTROMAGNETIC (BOSONIC) PHASES OF MATTER

#### A. Electromagnetic Chern number

The Berry connection for a band \( n \) is found by varying the total Bloch wave function \(|u_{nk}\rangle\) with respect to the momentum,

\[
A_n(k) = -i\langle u_{nk}|\partial_k u_{nk}\rangle = -i \sum_g u^\dagger_{nk+g} \partial_k u_{nk+g}. \quad \text{(52)}
\]
***Symmetry-protected topological bosonic phases***

Nevertheless, even if we knew the specifics of the material, evaluating the Chern number by brute force would be a

As can be seen from Eq. (53), the Berry connection is only
defined within the Brillouin zone $A_{nk+g} = A_{nk} + \partial_k \chi_{nk}$,
up to a possible U(1) gauge. Hence, the gauge invariant Berry curvature is periodic $F_{nk+g} = F_{nk}$,
Let 

The Chern number is found by integrating the Berry curvature over the two-dimensional Brillouin zone,

which determines the winding number of the collective light-matter excitations over the torus $T^2 = S^1 \times S^1$. Equation (56)
is one of the central results of this paper. An electromagnetic Chern invariant can be found for any 2D crystal and characterizes
distinct topological phases of matter $\mathcal{C}_n \neq 0$.

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**B. Symmetry-protected topological bosonic phases**

Nevertheless, even if we knew the specifics of the material, evaluating the Chern number by brute force would be a
The interpretation of \( \nu \) is quite simple - it tells us the geometric phase around the irreducible Brillouin zone (IBZ) of the crystal,

\[
\exp \left( \frac{i}{N} \mathfrak{c}_n \right) = \exp \left( i \int_{\text{IBZ}} F_n(k) d^2k \right) = \exp \left( i \oint_{\partial \text{IBZ}} A_n(k) \cdot dk \right),
\]

where \( \partial \text{IBZ} \) is the path around IBZ. This follows from rotational symmetry of the Berry curvature \( F_n(k) = F_n(Rk) \). For instance, the path in \( C_4 \) is \( \partial \text{IBZ}_4 = \Gamma X MY \). Applying the logarithm, \( \nu_n \) is equivalent to,

\[
\nu_n = \frac{N}{2\pi} \oint_{\partial \text{IBZ}} A_n(k) \cdot dk \mod N.
\]

As we will see more explicitly, \( \nu_n \) is tied entirely to \( \eta_n \). The reason is subtle - any vortex within the interior of the IBZ contributes a Berry phase of \( 2\pi \), and by symmetry, there are \( N \) such vortices within the total Brillouin zone \( \mathfrak{c}_n \rightarrow \mathfrak{c}_n + N \). However, this has no effect on \( \nu_n \rightarrow \nu_n \). Only the vortices lying at HSPs contribute to \( \nu_n \) because these come in fractions of \( 2\pi \).

In the following sections we will discuss the bosonic classification of \( \nu_n \) for each cyclic point group and the SPT phases associated with them. We do not present the full derivations here since the rigorous proofs have been carried out by others (see Ref. [69]) - we simply state the salient results. For completeness, in Appendix A we also discuss the SPT fermionic phases associated with each point group. We do this to emphasize that fermionic and bosonic systems represent distinct topological field theories, with fundamentally different interpretations. These differences are highlighted with a few examples [Fig. 4 and 5].

### C. Twofold (inversion) symmetry: \( C_2 \)

For the \( C_2 \) point group, or simply inversion symmetry, the SPT phase is related to the Chern number by \( \nu_n = \mathfrak{c}_n \mod 2 \), which is a \( \mathbb{Z}_2 \) invariant. There is only one nontrivial SPT phase and it can be found modulo 2 from,

\[
\exp \left( \frac{i}{2} \mathfrak{c}_n \right) = \eta_{2,n}(\Gamma) \eta_{2,n}(X) \eta_{2,n}(Y) \eta_{2,n}(M).
\]
Applying the logarithm, this classification can be expressed equivalently in terms of $m_{2,n} \in \mathbb{Z}_2$ inversion eigenvalues,

$$\nu_n = m_{2,n}(\Gamma) + m_{2,n}(X) + m_{2,n}(Y) + m_{2,n}(M) \mod 2.$$  \hspace{1cm} (61)

If the summation of $m_{2,n}$ eigenvalues is odd, the SPT phase is nontrivial $\nu_n = 1$ and corresponds to an odd-valued Chern number. Likewise, $\nu_n = 0$ is an even-valued Chern number.

**D. Threefold symmetry: $C_3$**

$C_3$ is unique because it is the only point group with an odd rotational symmetry - i.e. it lacks inversion symmetry. This means the parity of Chern number (odd or even) is not restricted by the symmetries of the crystal. For $C_3$, the SPT phase is $\nu_n = \mathcal{C}_n \mod 3$ which is a $\mathbb{Z}_3$ invariant. There are two nontrivial SPT phases and they can be found modulo 3 from,

$$\exp\left(\frac{2\pi}{3} \mathcal{C}_n\right) = \eta_{3,n}(\Gamma) \eta_{3,n}(K) \eta_{3,n}(K').$$  \hspace{1cm} (62)

This classification is expressed equivalently in terms of quantized modulo 3 integers $m_{3,n} \in \mathbb{Z}_3$ at HSPs,

$$\nu_n = m_{3,n}(\Gamma) + m_{3,n}(K) + m_{3,n}(K') \mod 3.$$  \hspace{1cm} (63)

Note though, odd and even phases are not distinct $\nu = -2 = 1 \equiv 4 \mod 3$ under modulo 3.

**E. Fourfold symmetry: $C_4$**

For the $C_4$ point group, the SPT phase is related to the Chern number by $\nu_n = \mathcal{C}_n \mod 4$ which is a $\mathbb{Z}_4$ invariant. There are three nontrivial SPT phases and they can be found modulo 4 from,

$$\exp\left(\frac{2\pi}{4} \mathcal{C}_n\right) = \eta_{4,n}(\Gamma) \eta_{4,n}(M) \eta_{2,n}(Y).$$  \hspace{1cm} (64)

The classification is expressed equivalently in terms of spin-1 eigenvalues,

$$\nu_n = m_{4,n}(\Gamma) + m_{4,n}(M) + 2m_{2,n}(Y) \mod 4.$$  \hspace{1cm} (65)

where $m_{4,n}(\Gamma) \& m_{4,n}(M) \in \mathbb{Z}_4$ are modulo 4 integers and $m_{2,n}(Y) \in \mathbb{Z}_2$ is a modulo 2 integer. Examples of all SPT phases of the $C_4$ point group are displayed in Fig. 4 and these are compared with their fermionic counterparts in Fig. 5.

**F. Sixfold symmetry: $C_6$**

For the $C_6$ point group, the SPT phase is $\nu_n = \mathcal{C}_n \mod 6$ which is a $\mathbb{Z}_6$ invariant. There are five nontrivial SPT phases and they can be found modulo 6 from,

$$\exp\left(\frac{2\pi}{6} \mathcal{C}_n\right) = \eta_{6,n}(\Gamma) \eta_{3,n}(K) \eta_{2,n}(M).$$  \hspace{1cm} (66)

This is equivalent to the summation of spin-1 eigenvalues at the HSPs,

$$\nu_n = m_{6,n}(\Gamma) + 2m_{3,n}(K) + 3m_{2,n}(M) \mod 6.$$  \hspace{1cm} (67)

where $m_{6,n}(\Gamma) \in \mathbb{Z}_6$ is a modulo 4 integer, $m_{6,n}(K) \in \mathbb{Z}_3$ is a modulo 3 integer and $m_{2,n}(M) \in \mathbb{Z}_2$ is a modulo 2 integer. This completes the classification of all 2+1D topological electromagnetic (bosonic) phases of matter which is summarized in Tbl. I. These are compared alongside their fermionic counterparts in Tbl. II.

**G. Continuous symmetry: $C_\infty$**

To finish, we briefly discuss the continuum limit $g = 0$ and the topological phases that can be described by a long wavelength theory $k \approx 0$. The physics is significantly more tractable here and exactly solvable models are possible [23, 25]. In this limit, the rotational symmetry of the crystal is approximately continuous $C_\infty$. The SPT invariant $\nu_n$ and Chern number $\mathcal{C}_n$ are thus equivalent,

$$\nu_n = \mathcal{C}_n = m_n(0) - m_n(\infty).$$  \hspace{1cm} (68)

Note that $\nu_n \in \mathbb{Z}$ and $m_n \in \mathbb{Z}$ are not modulo integers in this limit and do not have the same interpretation as the lattice theory. This is because we have gained the full rotational symmetry in the continuum approximation. Clearly though, the eigenvalues must change at HSPs $m_n(0) \neq m_n(\infty)$ for a nontrivial phase to exist $\mathcal{C}_n \neq 0$. In the continuum regularization, $k_i = 0$ represents the $\Gamma$ point and $k_i = \infty$ is interpreted as mapping the vertices of the Brillouin zone into one another.

**V. CONCLUSIONS**

In summary, we have developed the complete 2+1D lattice field theory describing all symmetry-protected topological bosonic phases of the photon. To accomplish this, we analyzed the electromagnetic Bloch waves in microscopic crystals and derived the Chern invariant of these light-matter excitations. Thereafter, the rotational symmetries of the crystal were examined extensively and the implications these have on photonic spin. We have studied all two dimensional point groups $C_n$ with nonvanishing Chern number $\mathcal{C} \neq 0$ and linked the topological invariants directly to spin-1 quantized eigenvalues of the electromagnetic field - establishing the bosonic classification for each topological phase.

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where the eigenvalues at HSPs are related by,

$$\mathcal{R}_N \ket{\Psi(k_i)} = \zeta_N(k_i) \ket{\Psi(k_i)},$$  \hspace{1cm} (A1)

where the eigenvalues at HSPs are related by,

$$\zeta_N(k_i) = \exp \left( \frac{2\pi}{N} m_N(k_i) \right), \quad (\zeta_N)^N = -1. \hspace{1cm} (A2)$$

$$m_N(k_i) \in \mathbb{Z}_N + \frac{1}{2}$$ is a modulo half-integer and labels the \( N \) possible spin-$$\frac{1}{2}$$ eigenvalues. Notice that \( \zeta_N \) represents the \( N \)th roots of negative unity which is characteristic of a fermionic field.

The single-particle fermionic classification for \( C_2, C_3, C_4 \) and \( C_6 \) respectively is \([69, 76]\),

$$\exp \left( \frac{2\pi}{2} i \right) = \zeta_2(\Gamma) \zeta_2(X) \zeta_2(Y) \zeta_2(M),$$  \hspace{1cm} (A3a)

$$\exp \left( \frac{2\pi}{3} i \right) = -\zeta_3(\Gamma) \zeta_3(K) \zeta_3(K'),$$  \hspace{1cm} (A3b)

$$\exp \left( \frac{2\pi}{4} i \right) = -\zeta_4(\Gamma) \zeta_4(M) \zeta_2(Y),$$  \hspace{1cm} (A3c)

$$\exp \left( \frac{2\pi}{6} i \right) = -\zeta_6(\Gamma) \zeta_3(K) \zeta_2(M).$$  \hspace{1cm} (A3d)

Although the classification appears similar, the SPT fermionic phases constitute very different physics than their bosonic counterparts, which is alluded to by the antisymmetric phase factors \( \mathcal{R}(2\pi) = -1 \). We illustrate this with an example in \( C_4 \). Applying the logarithm - the classification for the SPT fermionic phase \( \nu = \mathcal{C} \mod 4 \) can be expressed as,

$$\nu = m_4(\Gamma) + m_4(M) + 2m_2(Y) + 2 \mod 4,$$  \hspace{1cm} (A4)

where \( m_4(\Gamma) \& m_4(M) \in \mathbb{Z}_4 + \frac{1}{2} \) are modulo 4 half-integers and \( m_2(Y) \in \mathbb{Z}_2 + \frac{1}{2} \) is a modulo 2 half-integer.

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