Enriched monoidal categories I: centers

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Abstract

This work is the first one in a series, in which we develop a mathematical theory of enriched (braided) monoidal categories and their representations. In this work, we introduce the notion of the $E_0$-center ($E_1$-center or $E_2$-center) of an enriched (monoidal or braided monoidal) category, and compute the centers explicitly when the enriched (braided monoidal or monoidal) categories are obtained from the canonical constructions. These centers have important applications in the mathematical theory of gapless boundaries of 2+1D topological orders and that of topological phase transitions in physics. They also play very important roles in the higher representation theory, which is the focus of the second work in the series.

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Introduction

Enriched categories were introduced long ago [EK66], and have been studied intensively (see a classical review [Kel82]). A category enriched in a symmetric monoidal category $\mathcal{A}$ is also called an $\mathcal{A}$-enriched category or simply an $\mathcal{A}$-category, and is denoted by $\mathcal{A}|_\mathcal{L}$. The category $\mathcal{A}$ is called the base category of the enriched category [Kel82]. In this work, we choose to call $\mathcal{A}$ the background category of $\mathcal{A}|_\mathcal{L}$. The $\mathcal{L}$ in $\mathcal{A}|_\mathcal{L}$ denotes the underlying category of $\mathcal{A}$.

The classical setup of the theory of enriched category is to consider a fixed symmetric monoidal category $\mathcal{A}$ and develop the theory within the 2-category of $\mathcal{A}$-categories, $\mathcal{A}$-functors and $\mathcal{A}$-natural transformations [Kel82]. Although it is quite obvious to consider the change of background category by a symmetric monoidal functor $\mathcal{A} \rightarrow \mathcal{A}'$, the classical setup is natural, convenient and sufficient for many purposes as Kelly wrote in his classical book [Kel82]:

Closely connected to this is our decision not to discuss the “change of base category” given by a symmetric monoidal functor $\mathcal{V} \rightarrow \mathcal{V}'$. . . The general change of base, important though it is, is yet logically secondary to the basic $\mathcal{V}$-category theory it acts on.

In recent years, however, there are strong motivations to go beyond the classical setting. First, mathematicians start to explore enriched monoidal categories with the background categories being only braided (instead of being symmetric) [JS93, For04, BM12, MP17, KZ18b, MPP18, JMPP19]. Secondly, in physics, recent progress in the study of gapless boundaries of 2+1D topological orders [KZ20, KZ21] and topological phase transitions [JKYZ20] demands us to consider enriched categories, enriched monoidal categories and enriched braided (or symmetric) monoidal categories such that the background categories vary from monoidal categories to braided monoidal categories and to symmetric monoidal categories, and to consider functors between enriched categories with different background categories, and to develop the representation theory of an $\mathcal{A}$-enriched (braided) monoidal category $\mathcal{A}|_\mathcal{L}$ in the 2-category of enriched (monoidal) categories with different background categories [Zhe17, KZ20, KZ21]. In other words, physics demands us to develop a generalized theory within the 2-category of enriched categories (with arbitrary background categories), generalized enriched functors that can change the background categories (see Definition 3.4) and generalized enriched natural transformations (see Definition 3.7).

This work is the first one in a series to develop this generalized theory. More precisely, in this work, we study enriched (braided) monoidal categories, and study the $E_1$-center (or the Drinfeld center or the monoidal center) of an enriched monoidal category and the $E_2$-center (or the Müger center) of an enriched braided monoidal category via their universal properties. We introduce the canonical constructions of enriched (braided/symmetric monoidal) categories, and compute their centers (see Theorem 4.37, 5.26, 6.3) and Corollary 4.39, 5.27, 6.10. These results generalize the main results in [KZ18b], where two of the authors introduced the notion of the Drinfeld center of an enriched monoidal category without studying its universal property. These centers play important roles in the (higher) representation theory of enriched (braided) monoidal categories. We will develop the representation theory in the second work in the series.
The layout of this paper is given as follows. In Section 2, we review some basic notions and set our notations, and for \( i = 0, 1, 2 \), we review the notion of the \( E_i \)-center of an \( E_i \)-algebra in a symmetric monoidal 2-category. In Section 3, we review the notion of an enriched category, and introduce the notions of an enriched functor and an enriched natural transformation. We also study the canonical construction of enriched categories as a locally isomorphic 2-functor. In Section 4, we study enriched monoidal categories and the canonical construction, and compute the \( E_0 \)-centers of enriched categories. In Section 5, we study enriched braided monoidal categories and the canonical construction, and compute the \( E_1 \)-centers of enriched monoidal categories. In Section 6, we study enriched symmetric monoidal categories and the canonical construction, and compute the \( E_2 \)-centers of enriched braided monoidal categories.

Throughout the paper, we choose a ground field \( k \) of characteristic zero when we mention a finite semisimple (or a multi-fusion) category. We use \( k \) to denote the symmetric monoidal category of finite-dimensional vector spaces over \( k \).

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2 2-categories

In this section, we recall some basic notions in 2-categories and examples, and set the notations along the way. We refer the reader to [JY20] for a detailed introduction to 2-categories and bicategories.

2.1 Examples of 2-categories

A bicategory is called a 2-category if the associators and the unitors of 1-morphisms are identities. We introduce simple notations for objects, 1-morphisms and 2-morphisms in a 2-category \( C \) as \( x, y \in C \), \((f, g : x \to y) \in C \) and \((\xi : f \Rightarrow g) \in C \), respectively.

Definition 2.1. A 2-functor \( F : C \to D \) is called locally equivalent (or fully faithful) if \( F_{x,y} : C(x, y) \to D(F(x), F(y)) \) is an equivalence for all \( x, y \in C \); it is called locally isomorphic if \( F_{x,y} \) is an isomorphism for all \( x, y \in C \). ■

Example 2.2. We give a few examples of 2-categories.

- **Cat**: the 2-category \( \text{Cat} \) of categories, functors and natural transformations. It is a symmetric monoidal 2-category with the tensor product given by the Cartesian product \( \times \) and the tensor unit given by \( * \), which consists of a single object and a single morphism.
- **\( \text{Alg}_{E_1}(\text{Cat}) \)**: the 2-category of monoidal categories, monoidal functors and monoidal natural transformations. We choose this notation because a monoidal category can be viewed as an \( E_1 \)-algebra in \( \text{Cat} \) (see for example [Lur]). It is a symmetric monoidal 2-category with the tensor product \( \times \) and the tensor unit \( * \).
- **\( \text{Alg}_{E_2}(\text{Cat}) \)**: the 2-category of braided monoidal categories, braided monoidal functors and monoidal natural transformations. It is a symmetric monoidal 2-category with the tensor product \( \times \) and the tensor unit \( * \).
- **\( \text{Alg}_{E_n}(\text{Cat}) \)**: the 2-category of symmetric monoidal categories, braided monoidal functors and monoidal natural transformations. It is a symmetric monoidal 2-category with the tensor product \( \times \) and the tensor unit \( * \).

In the rest of this subsection, we recall some basic notions and give a few more examples of 2-categories that are useful in this work. Let \( A \) and \( B \) be monoidal categories.
Definition 2.3. A lax-monoidal functor $F : \mathcal{A} \to \mathcal{B}$ is a functor equipped with a morphism $\mathbb{1}_\mathcal{B} \to F(1_\mathcal{A})$ and a natural transformation $F(x) \otimes F(y) \to F(x \otimes y)$ for $x, y \in \mathcal{A}$ satisfying the following conditions:

1. (lax associativity): for all $x, y, z \in \mathcal{A}$, the following diagram commutes:

$$
\begin{array}{ccc}
F(x) \otimes F(y) \otimes F(z) & \longrightarrow & F(x) \otimes F(y \otimes z) \\
\downarrow & & \downarrow \\
F(x \otimes y) \otimes F(z) & \longrightarrow & F(x \otimes y \otimes z)
\end{array}
$$

(2.1)

2. (lax unitality): for all $x \in \mathcal{A}$, the following diagrams commute:

$$
\begin{array}{ccc}
\mathbb{1}_\mathcal{B} \otimes F(x) & \longrightarrow & F(\mathbb{1}_\mathcal{A}) \otimes F(x), \\
\downarrow & & \downarrow \\
F(x) & \longleftarrow & F(\mathbb{1}_\mathcal{A} \otimes x)
\end{array}
\quad
\begin{array}{ccc}
F(x) & \longrightarrow & F(x) \otimes \mathbb{1}_\mathcal{A} \\
\downarrow & & \downarrow \\
F(x) & \longleftarrow & F(x \otimes \mathbb{1}_\mathcal{A})
\end{array}
$$

(2.2)

An oplax-monoidal functor $G : \mathcal{A} \to \mathcal{B}$ is a lax-monoidal functor $\mathcal{A}^{\text{op}} \to \mathcal{B}^{\text{op}}$.

Definition 2.4. A lax-monoidal natural transformation between two lax-monoidal functors $F, G : \mathcal{A} \to \mathcal{B}$ is a natural transformation $\eta_a : F(a) \to G(a)$ such that the following diagrams commute:

$$
\begin{array}{ccc}
\mathbb{1}_\mathcal{B} & \longrightarrow & F(\mathbb{1}_\mathcal{A}) \\
\downarrow & & \downarrow_{\eta_{\mathbb{1}_\mathcal{B}}} \\
G(\mathbb{1}_\mathcal{A}) & \longrightarrow & G(a \otimes \mathbb{1}_\mathcal{B})
\end{array}
\quad
\begin{array}{ccc}
F(a \otimes \mathbb{1}_\mathcal{B}) & \longrightarrow & F(a \otimes \mathbb{1}_\mathcal{A}) \\
\downarrow_{\eta_a \otimes \mathbb{1}_\mathcal{B}} & & \downarrow_{\eta_a} \\
F(a) \otimes G(b) & \longrightarrow & G(a \otimes b)
\end{array}
$$

(2.3)

An oplax-monoidal natural transformation between two oplax-monoidal functors is defined similarly (by flipping non-vertical arrows in (2.4)).

Using above two notions, we obtain some new symmetric monoidal 2-categories (all with the tensor product $\otimes$ and the tensor unit $\mathbb{1}$):

- $\text{Alg}_{\text{rel}}^{\text{lax}}(\text{Cat})$: the 2-category of monoidal categories, lax-monoidal functors and lax-monoidal natural transformations.
- $\text{Alg}_{\text{rel}}^{\text{oplax}}(\text{Cat})$: the 2-category of monoidal categories, oplax-monoidal functors and oplax-monoidal natural transformations.
- $\text{Alg}_{\text{rel}}^{\text{oplax}}(\text{Cat})$: the 2-category of braided monoidal categories, braided oplax-monoidal functors and oplax-monoidal natural transformations.

Definition 2.5. A left $A$-oplax-module is a category $\mathcal{L}$ equipped with an oplax-monoidal functor $\mathcal{A} \to \text{Fun}(\mathcal{L}, \mathcal{L})$, or equivalently, an action functor $\circ : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ equipped with

- an oplax-associator: a natural transformation $(- \otimes -) \circ - \to - \circ (- \otimes -)$ rendering the following diagram commutative.

$$
\begin{array}{ccc}
((a \otimes b) \otimes c) \circ x & \longrightarrow & (a \otimes b) \circ (c \circ x) \\
\downarrow & & \downarrow \\
(a \otimes (b \circ c)) \circ x & \longrightarrow & a \circ ((b \circ c) \circ x)
\end{array}
$$

(2.4)

- an oplax-unitor: a natural transformation $\mathbb{1} \circ - \to 1_\mathcal{M}$ rendering the following two diagrams commutative.

$$
\begin{array}{ccc}
(1 \otimes b) \circ x & \longrightarrow & 1 \circ (b \circ x) \\
\downarrow & & \downarrow \\
b \circ x & \longrightarrow & b \circ (1 \circ x)
\end{array}
\quad
\begin{array}{ccc}
(b \otimes 1) \circ x & \longrightarrow & b \circ (1 \circ x) \\
\downarrow & & \downarrow \\
b \circ x & \longrightarrow & b \circ (1 \circ x)
\end{array}
$$

(2.5)
If the oplax-associator (resp. the oplax-unitor) is a natural isomorphism, then the left $A$-oplax-module is called strongly associative (resp. strongly unital), and is called a left $A$-module if it is both strongly associative and strongly unital.

A left $A^{\text{lax}}$-module is defined by flipping the arrows of the oplax-associator and the oplax-unitor to give the lax-associator and the lax-unitor.

**Definition 2.6.** For two left $A^{\text{oplax}}$-modules $M$ and $N$, a lax $A^{\text{oplax}}$-module functor from $M$ to $N$ is a functor equipped with a natural transformation $\alpha_{b,m} : b \circ F(m) \to F(b \circ m)$ such that the following diagrams commute for all $a, b \in A$ and $m \in M$.

\[
\begin{array}{ccc}
  a \circ (b \circ F(m)) & \xrightarrow{\alpha_{b,m}} & a \circ F(b \circ m) \\
  \downarrow & & \downarrow \\
  (a \circ b) \circ F(m) & \xrightarrow{\alpha_{b,m}} & F(a \circ (b \circ m)) \\
  \downarrow & & \downarrow \\
  F((a \circ b) \circ m) & \xrightarrow{\alpha_{b,m}} & F(a \circ (b \circ m)) \\
\end{array}
\]

(2.6)

If $\alpha$ is a natural isomorphism, then $F$ is called an $A$-module functor.

An oplax $A^{\text{oplax}}$-module functor is defined by flipping the arrow $\alpha_{b,m}$. Similarly, a lax $A^{\text{lax}}$-module functor and an oplax $A^{\text{lax}}$-module functor can both be defined by flipping certain arrows.

**Definition 2.7.** For two lax $A^{\text{oplax}}$-module functors $F, G : M \to N$, a lax $A^{\text{oplax}}$-module natural transformation $\xi : F \Rightarrow G$ is a natural transformation rendering the following diagram commutative for all $a \in A$ and $m \in M$. An oplax $A^{\text{oplax}}$-module (or oplax $A^{\text{lax}}$-module, lax $A^{\text{lax}}$-module) natural transformation can be defined similarly.

**2.2 Centers in 2-categories**

A contractible groupoid is a non-empty category that has a unique morphism between every two objects. An object $x$ in a bicategory $C$ is called a terminal object if the hom category $C(y, x)$ is a contractible groupoid for every $y \in C$.

In the rest of this section, we use $C$ to denote a monoidal bicategory (i.e., a tricategory with one object) equipped with a tensor product $\times$ and a tensor unit $*$. By the coherence theorem of tricategories (see [GPS95, Gur13]), without loss of generality, we further assume that $C$ is a semistrict monoidal 2-category, which is also called a Gray monoid.

**Definition 2.8.** An $E_0$-algebra in $C$ is a pair $(A, u)$, where $A$ is an object in $C$ and $u : * \to A$ is a 1-morphism.

**Definition 2.9.** A left unital $(A, u)$-action on an object $x \in C$ is a 1-morphism $\cdot : A \times x \to x$, together with an invertible 2-morphism $\alpha$ as depicted in the following diagram:

Let $x \in C$. We define the 2-category of left unital actions on $x$ as follows:

- Objects are left unital actions on $x$. 

For $i = 1, 2$, let $((A_i, u_i), g_i, a_i)$ be a left unital $(A_i, u_i)$-action on $x$. A 1-morphism from $((A_1, u_1), g_1, a_1)$ to $((A_2, u_2), g_2, a_2)$ is a triple $(p, \sigma, \rho)$, where $p : A_1 \to A_2$ is a 1-morphism in $\mathcal{C}$ and $\sigma$ and $\rho$ are two invertible 2-morphisms in $\mathcal{C}$ (as depicted in the following diagrams) such that the following identity of 2-morphisms holds.

\[
\begin{array}{c}
\begin{array}{ccc}
A_2 \times x & \xrightarrow{p \times 1} & A_1 \times x \\
\downarrow{g_2} & & \downarrow{g_1} \\
\ast \times x & \xrightarrow{u_2 \times 1} & A_1 \times x
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
A_2 \times x & \xrightarrow{p \times 1} & A_1 \times x \\
\downarrow{g_2} & & \downarrow{g_1} \\
\ast \times x & \xrightarrow{u_2 \times 1} & A_1 \times x
\end{array}
\end{array}
\]

Remark 2.11. Let $(p, \sigma, \lambda, \rho)$ and $(q, \beta, \varphi) : ((A_1, u_1), g_1, a_1) \to ((A_2, u_2), g_2, a_2)$ be two 1-morphisms, a 2-morphism $(p, \sigma, \lambda, \rho) \Rightarrow (q, \beta, \varphi)$ is a 2-morphism $\xi : p \Rightarrow q$ such that

\[
\begin{array}{c}
\begin{array}{ccc}
A_2 & \xrightarrow{p} & A_1 \\
\downarrow{g_2} & & \downarrow{g_1} \\
\ast & \xrightarrow{u_1} & A_1
\end{array}
\end{array}
\]

Definition 2.10. Let $x$ be an object in $\mathcal{C}$. The $E_0$-center $\mathcal{Z}_0(x)$ of $x$ is the terminal object in the 2-category of left unital actions on $x$.

Remark 2.12. In general, the notion of an $E_0$-center can be defined for an $E_0$-algebra. Indeed, a left unital action of an $E_0$-algebra on another $E_0$-algebra $(x, \ast \to x)$ is a left unital action on the object $x$ “preserving” the $E_0$-algebra structures, and the $E_0$-center of $(x, \ast \to x)$ is terminal among all left unital actions. One can show that the $E_0$-center is independent of the $E_0$-algebra structure $\ast \to x$, thus in practice we only use the notion of the $E_0$-center of an object.

Definition 2.13. An $E_1$-algebra (or a pseudomonoid, see Section 3 in [DS97]) in $\mathcal{C}$ is a sextuple $(A, u, m, \alpha, \lambda, \rho)$, where $A \in \text{Ob}(\mathcal{C})$, $u : \ast \to A$ and $m : A \times A \to A$ are 1-morphisms, and $\alpha, \lambda, \rho$ are invertible 2-morphisms (i.e. the associator, the left unitor and the right unitor, respectively) as depicted in the following diagrams

\[
\begin{array}{c}
\begin{array}{@{}c@{}}
A \times A \times A \xrightarrow{m \times 1} A \times A \\
\downarrow{1 \times m} & & \downarrow{m} \\
A \times A & \xrightarrow{m} & A
\end{array}
\end{array}
\]

such that the following equations of pasting diagrams hold:

\[
\begin{array}{c}
\begin{array}{@{}c@{}}
A \times A \times A \xrightarrow{m \times 1} A \times A \\
\downarrow{m \times 1} & & \downarrow{m} \\
A \times A & \xrightarrow{m} & A
\end{array}
\end{array}
\]
where $A^4$ is an abbreviation for $A \times A \times A \times A$, and the unlabeled 2-morphism is induced by the monoidal structure of $C$. We often abbreviate an $E_1$-algebra to $(A, u, m)$ or $A$. ■

It is not hard to check the Gray monoid structure of $C$ induces a Gray monoid structure on the 2-category of left unital actions on $x \in C$. Then the following proposition is a simple corollary of the fact that a terminal object of a monoidal 2-category admits an $E_1$-algebra structure.

**Proposition 2.14.** The $E_0$-center $\mathcal{Z}_0(x)$ of $x \in C$ is an $E_1$-algebra.

**Remark 2.15.** According to the general theory of higher algebras by Lurie [Lur], if $C$ is a symmetric monoidal higher category, the $E_0$-center of an $E_n$-algebra $A$ in $C$ can be defined as the $E_0$-center of $A$ in the symmetric monoidal higher category $\mathcal{A}_{E_0}(C)$ of $E_n$-algebras in $C$. We expect that $\mathcal{A}_{E_0}(\mathcal{A}_{E_0}(C)) \simeq \mathcal{A}_{E_0}(C)$. Then an $E_n$-center is automatically an $E_{n+1}$-algebra. ⊤

**Example 2.16.** Consider the symmetric monoidal 2-category $(\text{Cat}, \times, \ast)$ of categories, functors and natural transformations. The $E_1$-algebras in Cat are monoidal categories.

1. The following diagram represents

\[
\begin{array}{ccc}
A \times \mathcal{L} & \xrightarrow{\alpha \otimes 1} & \mathcal{L} \\
\downarrow & & \downarrow \alpha \\
\ast \times \mathcal{L} & \xrightarrow{1 \otimes 1} & \mathcal{L}
\end{array}
\]

(2.7)

a left unital $(A, \mathbb{I}_A)$-action on $\mathcal{L}$ in Cat, where $\otimes : A \times \mathcal{L} \to \mathcal{L}$ is a functor and $\alpha$ is a natural isomorphism. The functor from $A$ to the category $\text{Fun}(\mathcal{L}, \mathcal{L})$ of endofunctors defined by $a \mapsto a \circ -$ is monoidal. We see that $\mathcal{Z}_0(\mathcal{L}) = \text{Fun}(\mathcal{L}, \mathcal{L})$.

2. For $(A, \mathbb{I}_A), (C, \mathbb{I}_C) \in \mathcal{A}_{E_0}(\text{Cat})$, let the diagram (2.7) depict a left unital $(A, \mathbb{I}_A)$-action on $\mathcal{L}$ in $\mathcal{A}_{E_0}(\text{Cat})$, i.e., $\otimes$ is a monoidal natural isomorphism. Then $\varphi := - \otimes \mathbb{I}_C : A \to \mathcal{Z}_1(\mathcal{L})$ is a well-defined monoidal functor, and the half-braiding of $\varphi(a)$ is defined by:

$$x \otimes \varphi(a) \xRightarrow{\alpha^{-1}\otimes 1} (\mathbb{I}_A \otimes x) \otimes \varphi(a) \xrightarrow{1 \otimes \mathbb{I}_C} (\mathbb{I}_A \otimes a) \otimes (x \otimes \mathbb{I}_C)$$

$$\xRightarrow{\alpha} (a \otimes (\mathbb{I}_\mathcal{L} \otimes x)) \xrightarrow{\varphi(a) \otimes (\mathbb{I}_A \otimes x)} \varphi(a) \otimes (\mathbb{I}_A \otimes x).$$

Then it is straightforward to check that the $E_0$-center of $\mathcal{L}$ in $\mathcal{A}_{E_0}(\text{Cat})$ is given by the Drinfeld center $\mathcal{Z}_1(\mathcal{L})$ of $\mathcal{L}$. Recall that the objects of the Drinfeld center $\mathcal{Z}_1(\mathcal{L})$ are pairs $(x, \gamma_x)$, where $\gamma_x : - \otimes x \to x \otimes -$ is a half braiding satisfying certain compatibility conditions (see for example section 7.13 in [EGNO15]).

3. For $A, \mathcal{L} \in \mathcal{A}_{E_0}(\text{Cat})$, the diagram (2.7) depicts a left unital $(A, \mathbb{I}_A)$-action on $\mathcal{L}$ in $\mathcal{A}_{E_0}(\text{Cat})$. Since $\mathbb{I}_A \otimes x = x$ and $\otimes$ is a monoidal natural isomorphism, $\varphi(-) := - \otimes \mathbb{I}_\mathcal{L}$ defines a braided monoidal functor from $A$ to the Müger center $\mathcal{Z}_2(\mathcal{L})$ of $\mathcal{L}$. Then it is straightforward to check that the $E_0$-center of $\mathcal{L}$ in $\mathcal{A}_{E_0}(\text{Cat})$ is given by the Müger center $\mathcal{Z}_2(\mathcal{L})$ of $\mathcal{L}$.

4. For $n \geq 3$, let $\mathcal{A}_{E_0}(\text{Cat}) \simeq \mathcal{A}_{E_0}(\mathcal{A}_{E_0}(\text{Cat}))$ be the symmetric monoidal 2-category of $E_n$-algebras in Cat. It is known that $\mathcal{A}_{E_0}(\text{Cat})$ is biequivalent to the symmetric monoidal 2-category of symmetric monoidal categories, symmetric monoidal functors and monoidal natural transformations. The $E_0$-center of a symmetric monoidal category $\mathcal{L}$ in $\mathcal{A}_{E_0}(\text{Cat})$ is given by $\mathcal{L}$ itself. ⊤

**Example 2.17.** Let $\mathcal{C}$ be a monoidal category and $x \in \mathcal{C}$. We can view $\mathcal{C}$ as a monoidal 2-category with only identity 2-morphisms and denote it by $C$. Then the $E_0$-center of $x \in \mathcal{C}$ can be defined as the $E_0$-center of $x$ in $C$. In this sense, the theory of centers in 2-categories can be viewed as a direct generalization of that of centers in 1-categories [Ost18, Dav18, KYZ21]. ⊤

**Remark 2.18.** Suppose $C$ is a monoidal 2-category and $M$ is a $C$-module. We can similarly define the $E_0$-center of an object $x \in M$ in $C$. More generally, if $C$ is an $E_{n+1}$-monoidal higher category and $M$ is an $E_n$-module over $C$, then $\mathcal{A}_{E_n}(M)$ is an $\mathcal{A}_{E_0}(C)$-module, and the $E_0$-center of an $E_n$-algebra $A \in \mathcal{A}_{E_n}(M)$ in $C$ can be defined as the $E_0$-center of $A$ in $\mathcal{A}_{E_0}(C)$. This should be a generalization of [KYZ21, Section 3.2]. ⊤
3 Enriched categories

3.1 Enriched categories, functors and natural transformations

Let \((\mathcal{A}, \otimes, 1_{\mathcal{A}})\) be a monoidal category.

**Definition 3.1 (Kelly).** An \(\mathcal{A}\)-enriched category \(\mathcal{A}\mathcal{L}\), or a category enriched in \(\mathcal{A}\), consists of the following data.

1. A set of objects \(\text{Ob}(\mathcal{A}\mathcal{L})\). If \(x \in \text{Ob}(\mathcal{A}\mathcal{L})\), we also write \(x \in \mathcal{A}\mathcal{L}\).
2. An object \(\mathcal{A}\mathcal{L}(x, y)\) in \(\mathcal{A}\) for every pair \(x, y \in \mathcal{A}\mathcal{L}\).
3. A distinguished morphism \(1_x : 1_{\mathcal{A}} \rightarrow \mathcal{A}\mathcal{L}(x, x)\) in \(\mathcal{A}\) for every \(x \in \mathcal{A}\mathcal{L}\).
4. A composition morphism \(\mathcal{A}\mathcal{L}(y, z) \otimes \mathcal{A}\mathcal{L}(x, y) \rightarrow \mathcal{A}\mathcal{L}(x, z)\) in \(\mathcal{A}\) for every triple \(x, y, z \in \mathcal{A}\mathcal{L}\).

They are required to make the following diagrams commutative for \(x, y, z, w \in \mathcal{A}\mathcal{L}\).

\[
\begin{array}{c}
\mathcal{A}\mathcal{L}(y, z) \otimes \mathcal{A}\mathcal{L}(x, y) \otimes \mathcal{A}\mathcal{L}(w, x) \\
\downarrow^{1_{\mathcal{A}}} \\
\mathcal{A}\mathcal{L}(x, y) \otimes \mathcal{A}\mathcal{L}(w, x)
\end{array}
\xrightarrow{1_{\mathcal{A}}} \begin{array}{c}
\mathcal{A}\mathcal{L}(y, z) \otimes \mathcal{A}\mathcal{L}(w, y) \\
\downarrow^{1_{\mathcal{A}}} \\
\mathcal{A}\mathcal{L}(w, z)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{A}\mathcal{L}(x, y) \otimes \mathcal{A}\mathcal{L}(y, z) \\
\downarrow^{1_{\mathcal{A}}} \\
\mathcal{A}\mathcal{L}(x, z)
\end{array}
\xrightarrow{1_{\mathcal{A}}} \begin{array}{c}
\mathcal{A}\mathcal{L}(x, y) \otimes 1_{\mathcal{A}} \\
\downarrow^{1_{\mathcal{A}}} \\
\mathcal{A}\mathcal{L}(x, y)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{A}\mathcal{L}(y, z) \otimes \mathcal{A}\mathcal{L}(x, y) \\
\downarrow^{1_{\mathcal{A}}} \\
\mathcal{A}\mathcal{L}(x, y)
\end{array}
\xrightarrow{1_{\mathcal{A}}} \begin{array}{c}
\mathcal{A}\mathcal{L}(y, z) \otimes \mathcal{A}\mathcal{L}(x, y) \\
\downarrow^{1_{\mathcal{A}}} \\
\mathcal{A}\mathcal{L}(x, y)
\end{array}
\]

In the rest of this work, the monoidal category \(\mathcal{A}\) is referred to as the **background category** of \(\mathcal{A}\mathcal{L}\). The **underlying category** of \(\mathcal{A}\mathcal{L}\), denoted by \(\mathcal{L}\), is defined by

\[
\text{Ob}(\mathcal{L}) := \text{Ob}(\mathcal{A}\mathcal{L}) \quad \text{and} \quad \mathcal{L}(x, y) := \mathcal{A}(1_{\mathcal{A}}, \mathcal{A}\mathcal{L}(x, y)), \quad \forall x, y \in \mathcal{A}\mathcal{L}.
\]

and the composition \(g \circ f\) of morphisms \(1_{\mathcal{A}} \xrightarrow{g} \mathcal{A}\mathcal{L}(y, z)\) and \(1_{\mathcal{A}} \xrightarrow{f} \mathcal{A}\mathcal{L}(x, y)\) is defined by

\[
1_{\mathcal{A}} \xrightarrow{1_{\mathcal{A}}} 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} \xrightarrow{g \otimes f} \mathcal{A}\mathcal{L}(y, z) \otimes \mathcal{A}\mathcal{L}(x, y) \xrightarrow{\circ} \mathcal{A}\mathcal{L}(x, z);
\]

the identity morphism of \(x\) in \(\mathcal{L}\) is precisely \(1_x : 1_{\mathcal{A}} \rightarrow \mathcal{A}\mathcal{L}(x, x)\).

**Remark 3.2.** Let \(\mathcal{A}\mathcal{L}\) be an enriched category. A morphism \(f \in \mathcal{L}(x, y)\) in the underlying category \(\mathcal{L}\) induces a morphism \(\mathcal{A}\mathcal{L}(w, f) : \mathcal{A}\mathcal{L}(w, x) \rightarrow \mathcal{A}\mathcal{L}(w, y)\) in \(\mathcal{A}\) for \(w \in \mathcal{A}\mathcal{L}\), defined by

\[
\mathcal{A}\mathcal{L}(w, x) \xrightarrow{1_{\mathcal{A}}} \mathcal{A}\mathcal{L}(w, x) \otimes \mathcal{A}\mathcal{L}(w, x) \xrightarrow{f \otimes 1_{\mathcal{A}}} \mathcal{A}\mathcal{L}(w, y).
\]

Similarly \(f\) also induces a morphism \(\mathcal{A}\mathcal{L}(f, w) : \mathcal{A}\mathcal{L}(y, w) \rightarrow \mathcal{A}\mathcal{L}(x, w)\) for \(w \in \mathcal{A}\mathcal{L}\). Moreover, if \(g \in \mathcal{L}(y, z)\), one can verify that \(\mathcal{A}\mathcal{L}(w, g) \circ \mathcal{A}\mathcal{L}(w, f) = \mathcal{A}\mathcal{L}(w, f \circ g)\). Thus \(\mathcal{A}\mathcal{L}(w, -)\) defines an ordinary functor \(\mathcal{L} \rightarrow \mathcal{A}\). Similarly, \(\mathcal{A}\mathcal{L}(-, w)\) is an ordinary functor \(\mathcal{L}^{\text{op}} \rightarrow \mathcal{A}\). Combining them together, we obtain a well-defined functor \(\mathcal{A}\mathcal{L}(-, -) : \mathcal{L}^{\text{op}} \times \mathcal{L} \rightarrow \mathcal{A}\).

**Remark 3.3.** For an \(\mathcal{A}\)-enriched category \(\mathcal{A}\mathcal{L}\), we define an \(\mathcal{A}^{\text{rev}}\)-enriched category \(\mathcal{A}^{\text{rev}}\mathcal{L}\) as follows:

- \(\text{Ob}(\mathcal{A}^{\text{rev}}\mathcal{L}) := \text{Ob}(\mathcal{A}\mathcal{L})\);
- For \(x, y \in \mathcal{A}^{\text{rev}}\mathcal{L}\), \(\mathcal{A}^{\text{rev}}\mathcal{L}(x, y) := \mathcal{A}\mathcal{L}(y, x)\);
- The identity is given by \(1_x : 1_{\mathcal{A}} \rightarrow \mathcal{A}^{\text{rev}}\mathcal{L}(x, x) = \mathcal{A}^{\text{rev}}\mathcal{L}(x, x)\);
the composition is defined by

\[ \mathcal{A}^{\text{rev}}(y, z) \otimes \mathcal{A}^{\text{rev}}(x, y) = \mathcal{A}(y, x) \otimes \mathcal{A}(z, y) \rightarrow \mathcal{A}(z, x) = \mathcal{A}^{\text{rev}}(x, z), \]

where \( \otimes \) is the reversed tensor product of \( \mathcal{A} \).

We call \( \mathcal{A}^{\text{rev}} \) the opposite category of \( \mathcal{A} \), i.e. \( \mathcal{A}^{\text{rev}} := \mathcal{A}^{\text{op}} \). We have

\[ \mathcal{A}^{\text{rev}}(1_A, \mathcal{A}^{\text{op}}(x, y)) = \mathcal{A}(1_A, \mathcal{A}(y, x)) = \mathcal{L}(y, x) = \mathcal{L}^{\text{op}}(x, y). \]

It means that the underlying category of \( \mathcal{A}^{\text{rev}} \) is \( \mathcal{L}^{\text{op}} \). This explains our notation.

\[ \Box \]

**Definition 3.4.** An enriched functor \( [F] : \mathcal{A} \to \mathcal{M} \) between two enriched categories consists of the following data:

1. A lax-monoidal functor \( \hat{F} : \mathcal{A} \to \mathcal{B} \), which is called the background changing functor;
2. a map \( F : \text{Ob}(\mathcal{L}) \to \text{Ob}(\mathcal{M}) \);
3. a family of morphisms \( [F_{x,y}] : \hat{F}([\mathcal{A}(x, y)]) \to [\mathcal{M}(F(x), F(y))] \) in \( \mathcal{B} \) for \( x, y \in \mathcal{L} \);

satisfying the following two axioms:

1. \( 1_{[F]} = (1_B \to \hat{F}(1_A) \xrightarrow{F(1)} \hat{F}([\mathcal{A}(x, x)]) \xrightarrow{[F_{x,x}]} [\mathcal{M}(F(x), F(x))] ; \)
2. the diagram

\[ \begin{array}{c}
\hat{F}([\mathcal{A}(y, z)] \otimes \hat{F}([\mathcal{A}(x, y)]) \\
\xrightarrow{[F_{x,y} \otimes F_{y,z}]} \\
\hat{F}([\mathcal{A}(z, y)] \otimes \hat{F}([\mathcal{A}(x, z)])) \\
\end{array} \]

(3.5)

\[ \xrightarrow{F(v)} \hat{F}([\mathcal{A}(x, z)])) \]

in \( \mathcal{B} \) is commutative.

\[ \blacksquare \]

**Remark 3.5.** We want to emphasize that if \( \mathcal{A} \) is the category \( \text{Set} \) of sets, the notion of an enriched functor defined in Definition 3.4 does not coincide with an ordinary functor unless we require \( \hat{F} = 1_A \) (see also Definition 3.10). In other words, even in the case \( \mathcal{A} = \text{Set} \), this work provides a non-trivial generalization of the usual category theory.

For every \( f : 1_A \to [\mathcal{A}(x, y)] \), we define

\[ F(f) := (1_B \to \hat{F}(1_A) \xrightarrow{F(f)} \hat{F}([\mathcal{A}(x, y)]) \xrightarrow{[F_{x,y}]} [\mathcal{M}(F(x), F(y))]. \]

(3.6)

The commutativity of the diagram (3.5) implies that \( F(gf) = F(g)F(f) \) for every \( f \in \mathcal{L}(x, y) \) and \( g \in \mathcal{L}(y, z) \). As a consequence, we obtain a functor \( F : \mathcal{L} \to \mathcal{M} \), which is called the underlying functor of \( [F] \).

**Remark 3.6.** Given an enriched functor \( [F] : \mathcal{A} \to \mathcal{M} \), for \( f \in \mathcal{L}(x, y) \) in the underlying category \( \mathcal{L} \), the commutativity of the diagram (3.5) implies that of the following diagram.

\[ \begin{array}{c}
\hat{F}([\mathcal{A}(w, x)]) \\
\xrightarrow{[F_{x,y}]} \\
\hat{F}([\mathcal{A}(w, y)]) \\
\end{array} \]

(3.3)

\[ \xrightarrow{F(v)} \hat{F}([\mathcal{A}(w, y)]) \]

In other words, \( [F_{w,-}] : \hat{F}([\mathcal{A}(w, -)]) \Rightarrow [\mathcal{M}(F(w), F(-))] \) is a natural transformation. Similarly, \( [F_{-,w}] : \hat{F}([\mathcal{A}(-, w)]) \Rightarrow [\mathcal{M}(F(-), F(w))] \) is also a natural transformation. Combining them together, we obtain a natural transformation \( [F_{-, -}] : \hat{F}([\mathcal{A}(-, -)]) \Rightarrow [\mathcal{M}(F(-), F(-))] \).

\[ \Box \]
Definition 3.7. An enriched natural transformation $\xi : \hat{F} \Rightarrow \hat{G}$ between two enriched functors $\hat{F}, \hat{G} : \mathcal{A} \rightarrow \mathcal{M}$ consists of a lax-monoidal natural transformation $\xi : \hat{F} \Rightarrow \hat{G}$, which is called the background changing natural transformation of $\xi$, and a family of morphisms $\xi_x : 1_B \rightarrow \mathcal{M}(F(x), G(x)), \forall x \in \mathcal{A}$, rendering the following diagram commutative.

$$
\begin{array}{ccc}
\hat{F}(\mathcal{A}(x, y)) & \xrightarrow{\xi_x \otimes \xi_y} & \mathcal{M}(F(x), F(y)) \\
\downarrow & & \downarrow \\
\hat{G}(\mathcal{A}(x, y)) & \xrightarrow{\xi_{y \otimes \xi_x}} & \mathcal{M}(G(x), G(y)) \\
\end{array}
$$

The family of morphisms $\{\xi_x\}$ automatically defines a natural transformation $\xi : \hat{F} \Rightarrow \hat{G}$, which is called the underlying natural transformation of $\xi$.

Remark 3.8. Two enriched natural transformations $\xi$ and $\eta$ are equal in $\text{lax}\mathcal{ECat}$ if and only if $\xi = \eta$ and $\xi = \eta$.

Remark 3.9. The diagram (3.7) can be rewritten as follows:

$$
\begin{array}{ccc}
\hat{G}(\mathcal{A}(x, y)) & \xrightarrow{\xi_x \otimes \xi_y} & \mathcal{M}(F(x), F(y)) \\
\downarrow & & \downarrow \\
\mathcal{M}(G(x), G(y)) & \xrightarrow{\mathcal{M}(\xi_x G(y))} & \mathcal{M}(F(x), G(y)) \\
\end{array}
$$

The commutativity of (3.8) guarantees that $\{\xi_x\}$ gives a well-defined natural transformation $\xi : \hat{F} \Rightarrow \hat{G}$.

Definition 3.10. An enriched functor $\hat{F} : \mathcal{A} \rightarrow \mathcal{M}$ is called an $\mathcal{A}$-functor if $\hat{F}$ is the identity functor $1_{\mathcal{A}}$ of $\mathcal{A}$. An enriched natural transformation $\xi : \hat{F} \Rightarrow \hat{G}$ between two $\mathcal{A}$-functors is called an $\mathcal{A}$-natural transformation if $\xi$ is the identity natural transformation of $1_{\mathcal{A}}$, i.e. $\xi = 1_{1_{\mathcal{A}}}$ [Kel82 Chapter 1].

Example 3.11. An $\mathcal{A}$-enriched category with only one object $*$ is determined by an algebra $(A, m_A : A \otimes A \rightarrow A, 1_A : 1 \rightarrow A)$ in $\mathcal{A}$ with the identity morphism $1 : 1 \rightarrow A$ defined by $1_A$ and the composition morphism $\circ : A \otimes A \rightarrow A$ defined by $m$. We use $\star^\mathcal{A}$ to denote this enriched category. If $\mathcal{A}$ is the monoidal category with only one object and one morphism, we simply denote $\star^\mathcal{A}$ by $\star$.

Let $\mathcal{B}$ be an algebra in a monoidal category $\mathcal{B}$. Then an enriched functor $\hat{F} : \star^\mathcal{A} \rightarrow \star^\mathcal{B}$ is defined by a lax monoidal functor $\hat{F} : \mathcal{A} \rightarrow \mathcal{B}$ and an algebra homomorphism $F_{\star^\mathcal{A}} : \hat{F}(A) \rightarrow B$.

Given another enriched functor $\hat{G} : \star^\mathcal{A} \rightarrow \star^\mathcal{B}$, an enriched natural transformations $\xi : \hat{F} \Rightarrow \hat{G}$ is defined by a lax-monoidal natural transformation $\xi : \hat{F} \Rightarrow \hat{G}$ and a morphism $\xi : 1_B \rightarrow B$ rendering the following diagram commutative.

$$
\begin{array}{ccc}
\hat{F}(A) & \xrightarrow{\xi \otimes \xi} & B \otimes B \\
\downarrow & & \downarrow \\
\hat{G}(A) & \xrightarrow{\xi \circ \xi} & B \otimes B \\
\end{array}
$$

For example, when $\xi = 1_B$ and $\xi$ satisfies the condition $G_{\star^\mathcal{A}} \circ \xi_{\star^\mathcal{A}} = F_{\star^\mathcal{A}}$, they define an enriched natural transformation.

Example 3.12. Let $\mathcal{A}$ be an $\mathcal{A}$-enriched category. For every $x \in \mathcal{A}$, we use $x$ to denote the enriched functor $\star \rightarrow \mathcal{A}$ defined by $x = 1_A, \star \mapsto x$ and $\star^x = 1_x$. The underlying functor of $x$, i.e. $\star \mapsto x$, is denoted by $x$. 

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\section{2-categories of enriched categories}

\begin{proposition}
Enriched categories (as objects), enriched functors (as 1-morphisms) and enriched natural transformations (as 2-morphisms) form a 2-category $\text{lax}\ECat$.
\end{proposition}

1. The composition \(\hat{G} \circ \hat{F}\) of 1-morphisms \(\hat{F} : \hat{A} \rightarrow \hat{B}\) and \(\hat{G} : \hat{B} \rightarrow \hat{C}\) is the enriched functor induced by the composition of lax-monoidal functors \(\hat{G} \circ \hat{F}\) and the composition of \(\hat{C}\)-natural isomorphisms if and only if both 1-morphisms are lax-monoidal functors \(\hat{F}\) and \(\hat{G}\).

\begin{equation}
\hat{G}(\hat{F}(x, y)) \stackrel{\hat{G}(\hat{F}(x, y))}{\longrightarrow} \hat{G}(\hat{F}(y, z)), \quad \forall x, y, z \in \hat{C}.
\end{equation}

2. The horizontal composition \(\eta \circ \xi = \hat{H} \circ \hat{F} \Rightarrow \hat{L} \circ \hat{G}\) is defined by the horizontal composition of natural transformations \(\eta \circ \xi = \hat{H} \circ \hat{F} \Rightarrow \hat{L} \circ \hat{G}\).

\begin{equation}
\begin{aligned}
\eta \circ \xi : L \circ F & \Rightarrow L \circ G, \\
\theta \circ \xi : \hat{L} \circ \hat{F} & \Rightarrow \hat{L} \circ \hat{G}.
\end{aligned}
\end{equation}

3. The vertical composition \(\beta \circ \gamma = \hat{H} \circ \hat{F} \Rightarrow \hat{K}\) is defined by the vertical composition of natural transformations \(\beta \circ \gamma = \hat{H} \circ \hat{F} \Rightarrow \hat{K}\).

\begin{equation}
\begin{aligned}
\beta \circ \gamma : & \hat{H} \circ \hat{F} \Rightarrow \hat{K}, \\
\beta \circ \gamma : & \hat{H} \circ \hat{F} \Rightarrow \hat{K}.
\end{aligned}
\end{equation}

We denote the sub-2-category of $\text{lax}\ECat$ consisting of enriched functors $\hat{F}$ such that $\hat{F}$ is monoidal by $\ECat$. Fix a monoidal category $\hat{A}$, and the tensor unit given by $\hat{A}$.

\begin{definition}
Let $\hat{\mathcal{M}}$ and $\hat{\mathcal{N}}$ be enriched categories. We define their Cartesian product $\hat{\mathcal{M}} \times \hat{\mathcal{N}}$ as a ($\mathcal{C} \times \mathcal{D}$)-enriched category as follows:

- The objects of $\hat{\mathcal{M}} \times \hat{\mathcal{N}}$ are the same as $\mathcal{M} \times \mathcal{N}$.
- For any $m_1, m_2 \in \mathcal{M}$ and $n_1, n_2 \in \mathcal{N}$, $$(\hat{\mathcal{M}} \times \hat{\mathcal{N}})((m_1, n_1), (m_2, n_2)) := (\mathcal{M}(m_1, m_2), \mathcal{N}(n_1, n_2)) \in \mathcal{C} \times \mathcal{D}.$$

- The composition is defined by

\begin{equation}
\begin{aligned}
(\hat{\mathcal{M}} \times \hat{\mathcal{N}})((m_1, m_2), (m_3, n_3)) & \otimes (\hat{\mathcal{M}} \times \hat{\mathcal{N}})((m_4, n_4), (m_5, n_5)) \\
= (\hat{\mathcal{M}}(m_2, m_3), \hat{\mathcal{N}}(n_2, n_3)) & \otimes (\hat{\mathcal{M}}(m_1, m_4), \hat{\mathcal{N}}(n_1, n_4)) \\
= (\hat{\mathcal{M}}(m_2, m_3) \otimes \hat{\mathcal{M}}(m_1, m_4), \hat{\mathcal{N}}(n_2, n_3) \otimes \hat{\mathcal{N}}(n_1, n_4)) \\
\overset{(\circ, \circ)}{\longrightarrow} (\hat{\mathcal{M}}(m_1, m_3), \hat{\mathcal{N}}(n_1, n_3)) = (\hat{\mathcal{M}} \times \hat{\mathcal{N}})((m_1, n_1), (m_3, n_3)).
\end{aligned}
\end{equation}

- The identity morphism $1_{(m, n)}$ defined by (1$_m$, 1$_n$):

$$1_{(m, n)} := (1_{(m, n)}, 1_{(m, n)}).$$

(Note that the underlying category of $\hat{\mathcal{M}} \times \hat{\mathcal{N}}$ is $\mathcal{M} \times \mathcal{N}$.)
\end{definition}

\begin{proposition}
We identify $\hat{\mathcal{M}} \times \hat{\mathcal{N}}$ with $\mathcal{M} \times \mathcal{N}$.
\end{proposition}

\begin{remark}
Note that $\hat{\mathcal{M}} \otimes \hat{\mathcal{N}}$ is not a monoidal category because the Cartesian product $\times$ is not defined in $\hat{\mathcal{M}} \otimes \hat{\mathcal{N}}$. We show in Example that if $\mathcal{A}$ is braided, then $\hat{\mathcal{M}} \otimes \hat{\mathcal{N}}$ has a natural monoidal structure with a non-trivial tensor product.
\end{remark}

\begin{remark}
There is a symmetric monoidal 2-functor from $\text{lax}\ECat$ to $\text{Cat}$ defined by $\hat{\mathcal{M}} \otimes \hat{\mathcal{N}}$.
\end{remark}

\begin{remark}
There is a symmetric monoidal 2-functor from $\text{lax}\ECat$ to $\text{Alg}_{\text{cat}}(\hat{\mathcal{M}})$.
\end{remark}

\begin{remark}
There is a symmetric monoidal 2-functor from $\text{lax}\ECat$ to $\text{Alg}_{\text{cat}}(\hat{\mathcal{M}})$ defined by $\hat{\mathcal{M}} \otimes \hat{\mathcal{N}}$.
\end{remark}
3.3 Pushforward 2-functors

Let \( A \) and \( B \) be monoidal categories and \( R : A \to B \) a lax-monoidal functor. Given an enriched category \( \mathcal{L} \), there is a \( B \)-enriched category \( R(\mathcal{L}) \) defined as follows:

1. \( \text{Ob}(R(\mathcal{L})) := \text{Ob}(\mathcal{L}) \);
2. \( R(\mathcal{L})(x, y) := R(\mathcal{L}(x, y)) \);
3. the identity morphism: \( 1_x := (1_B \to R(1_A) \circ R(1_x) = R(\mathcal{L}(x, x)) \);
4. the composition of morphisms is defined by the following composed morphisms in \( B \):
\[
R(\mathcal{L}(y, z)) \otimes R(\mathcal{L}(x, y)) \to R(\mathcal{L}(y, z) \otimes \mathcal{L}(x, y)) \to R(\mathcal{L}(x, z)).
\]

The two defining conditions (3.1) and (3.2) hold due to the lax-monoidalness of \( R \).

Given an \( A \)-functor \( F : \mathcal{L} \to \mathcal{M} \), i.e. a map \( F : \text{Ob}(\mathcal{L}) \to \text{Ob}(\mathcal{M}) \) together with \( F_{x,y} : \mathcal{L}(x, y) \to \mathcal{M}(F(x), F(y)) \) for \( x, y \in \mathcal{L} \). Then the same map \( F : \text{Ob}(\mathcal{L}) \to \text{Ob}(\mathcal{M}) \) together with
\[
R(F_{x,y}) : R(\mathcal{L}(x, y)) \to R(\mathcal{M}(F(x), F(y)))
\]
for \( x, y \in \mathcal{L} \) defines a \( B \)-functor \( R(F) : R(\mathcal{L}) \to R(\mathcal{M}) \).

Given an \( A \)-natural transformation \( \xi : F \Rightarrow G \) between two \( A \)-functors \( F, G : \mathcal{L} \to \mathcal{M} \).

Then the same map \( R(\mathcal{L}(x, y)) \) defines an \( A \)-natural transformation \( R(\xi) \).

Theorem 3.19. Given a lax-monoidal functor \( R : A \to B \), there is a well-defined 2-functor \( R_* : \mathcal{A} \mathcal{E} \mathcal{C} \mathcal{a} t \to \mathcal{B} \mathcal{E} \mathcal{C} \mathcal{a} t \) (called the pushforward of \( R \)) defined by \( \mathcal{A} \mathcal{L} \Rightarrow R(\mathcal{L}), F \mapsto R_*(F), \xi \mapsto R_*(\xi) \).

Example 3.20. We give a few examples of the pushforward 2-functor \( R_* \).

1. Given a monoidal category \( A \), the functor \( A(1_A, -) \) has a lax-monoidal structure defined by \( A(1_A, x) \times A(1_A, y) \to A(1_A \otimes 1_A, x \otimes y) \cong A(1_A, x \otimes y) \). Then the pushforward 2-functor \( A(1_A, -) \), maps an \( A \)-enriched category to its underlying category, an \( A \)-functor to its underlying functor, and an \( A \)-natural transformation to its underlying natural transformation.

2. Let \( A \) be a braided monoidal category. The tensor product \( \otimes : A \times A \to A \) is monoidal. Then \( \otimes_* : \mathcal{A} \times \mathcal{E} \mathcal{C} \mathcal{a} t \to \mathcal{E} \mathcal{C} \mathcal{a} t \) is a well-defined 2-functor. The Cartesian product \( \times \) defines a functor \( \times : \mathcal{E} \mathcal{C} \mathcal{a} t \times \mathcal{E} \mathcal{C} \mathcal{a} t \to \mathcal{E} \mathcal{C} \mathcal{a} t \). Then we obtain a composed functor
\[
\mathcal{A} \mathcal{E} \mathcal{C} \mathcal{a} t \times \mathcal{A} \mathcal{E} \mathcal{C} \mathcal{a} t \to \mathcal{E} \mathcal{C} \mathcal{a} t \to \mathcal{E} \mathcal{C} \mathcal{a} t.
\]
This functor \( \times \), together with the tensor unit \( \ast \), endows a monoidal structure on \( \mathcal{A} \mathcal{E} \mathcal{C} \mathcal{a} t \) [KZ18b].

3. Given a braided monoidal category \( A \), let \( B \) be a left monoidal \( A \)-module (i.e. \( B \) is equipped with a braided monoidal functor \( \phi : A \to \mathcal{J}_1(B) \)). Let \( \cap \) be the composed functor \( A \times B \to \mathcal{J}_1(B) \times B \). It defines a left unital \( A \)-action on \( B \). Moreover, \( \cap \) is monoidal, thus the pushforward \( \cap \), is well-defined. Therefore, we obtain a composed 2-functor
\[
\mathcal{A} \mathcal{E} \mathcal{C} \mathcal{a} t \times \mathcal{B} \mathcal{E} \mathcal{C} \mathcal{a} t \to \mathcal{A} \times \mathcal{B} \mathcal{E} \mathcal{C} \mathcal{a} t \to \mathcal{B} \mathcal{E} \mathcal{C} \mathcal{a} t,
\]
which endows \( \mathcal{B} \mathcal{E} \mathcal{C} \mathcal{a} t \) with a structure of a left \( \mathcal{A} \mathcal{E} \mathcal{C} \mathcal{a} t \)-module. This clarifies Remark 3.21 in [KZ21].

We obtain a new characterization of an enriched functor.
Proposition 3.21. An enriched functor $F : \mathcal{A} \to \mathcal{B}$ is precisely a pair $(\bar{F}, \bar{\gamma})$, where $\bar{F} : \mathcal{A} \to \mathcal{B}$ is a lax-monoidal functor and $\bar{\gamma} : \mathcal{F} \to \mathcal{B}$ is a $\mathcal{B}$-functor.

Let $R, R' : \mathcal{A} \to \mathcal{B}$ be two lax-monoidal functors and $\phi : R \Rightarrow R'$ be a lax-monoidal natural transformation. This $\phi$ induces an enriched functor $\phi_* : R_*(\mathcal{A}) \to R'_*(\mathcal{A})$, called the pushforward of $\phi$. More precisely, $\phi_*$ is a $\mathcal{B}$-functor defined as follows:

1. $\phi_*$ is the identity map on $\text{Ob}(\mathcal{L})$;
2. on morphisms: $(\phi_*)_{x,y} : R_*(\mathcal{A})(x, y) \to R'_*(\mathcal{A})(x, y)$ is defined by

$$(\phi_*)_{x,y} := \left( R_*(\mathcal{A})(x, y) \xrightarrow{\phi_*} R'_*(\mathcal{A})(x, y) \right).$$

The lax-monoidality and the naturalness of $\bar{\gamma}$ imply that such defined $\phi_*$ preserves the identities and the compositions. Therefore, $\phi_*$ is a well-defined $\mathcal{B}$-functor. We obtain a new characterization of an enriched natural transformation.

Proposition 3.22. Let $F, G : \mathcal{A} \to \mathcal{B}$ be two enriched functors. An enriched natural transformation $\xi : F \Rightarrow G$ is precisely a pair $(\hat{\xi}, \hat{\gamma})$, where $\hat{\xi} : \mathcal{F} \Rightarrow \mathcal{G}$ is a lax-monoidal functor and $\hat{\gamma} : F \Rightarrow G \circ \hat{\xi}$ is a $\mathcal{B}$-natural transformation.

3.4 Canonical construction

Throughout this subsection, $\mathcal{A}$ and $\mathcal{B}$ are monoidal categories, $\mathcal{L}$ is a left $\mathcal{A}$-oplax-module (recall Definition 2.5), and $\mathcal{M}$ is a left $\mathcal{B}$-oplax-module.

Recall that $\mathcal{L}$ is called enriched in $\mathcal{A}$ if the internal hom $[x, y]_\mathcal{A}$ exists in $\mathcal{A}$ for all $x, y \in \mathcal{L}$. We sometimes abbreviate $[x, y]_\mathcal{A}$ to $[x, y]$ for simplicity. We use $\text{ev}_x$ and $\text{coev}_x$ to denote the counit $\text{ev}_x : [x, y] \to y$ and unit $\text{coev}_x : a \to [x, a \circ x]$ of the adjunction

$$\mathcal{L}(a \circ x, y) \cong \mathcal{A}(a, [x, y]).$$

The pair $([x, y], \text{ev}_x)$ is terminal among all such pairs. For $f : x' \to x, g : y \to y'$, there exists a unique morphism $[f, g] : [x, y] \to [x', y']$ rendering the diagram commutative.

$$\begin{array}{ccc}
[x, y] \circ x' & \cong & [x', y'] \circ x' \\
\downarrow^{1 \circ f} & & \downarrow^{1 \circ f} \\
[x, y] \circ x & \xrightarrow{\text{ev}_x} & y \\
\downarrow^g & & \downarrow^{\text{ev}_x'} \\
[x, y] \circ x & \xrightarrow{\text{coev}_x} & y'
\end{array}$$

As a consequence, the internal homs define a bifunctor $[-, -] : \mathcal{L}^{op} \times \mathcal{L} \to \mathcal{A}$.

Remark 3.23. Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B}) : \eta : 1_\mathcal{B} \Rightarrow RL, \epsilon : LR \Rightarrow 1_\mathcal{A}$ be an adjunction such that $R$ is lax-monoidal. The functor $L$ is automatically oplax-monoidal. The left $\mathcal{A}$-oplax-module $\mathcal{L}$ pulls back to a left $\mathcal{B}$-oplax-module with the $\mathcal{B}$-action defined by $L(-) \circ - : \mathcal{B} \times \mathcal{L} \to \mathcal{L}$. If $\mathcal{L}$ is enriched in $\mathcal{A}$, then $\mathcal{L}$ is enriched in $\mathcal{B}$ with $[x, y]_\mathcal{B} = R([x, y]_\mathcal{A})$. The evaluation $[x, y]_\mathcal{B} \circ x \to y$ is defined by the composed morphism: $LR([x, y]_\mathcal{A}) \circ x \to [x, y]_\mathcal{A} \circ x \xrightarrow{\text{ev}_x} y$.

If $\mathcal{L}$ is enriched in $\mathcal{A}$, then $\mathcal{L}$ can be promoted to an $\mathcal{A}$-enriched category, denoted by $\hat{\mathcal{L}}$. More explicitly, the enriched category $\hat{\mathcal{L}}$ has the same objects as $\mathcal{L}$ and $\hat{\mathcal{L}}(x, y) = [x, y]$. The identity $1_x : 1 \to [x, x]$ and composition $\circ : [y, z] \circ [x, y] \to [x, z]$ are the unique morphisms rendering the following diagrams commutative:

$$\begin{array}{r}
1 \circ x & \xrightarrow{1} & [x, x] \circ x \\
\downarrow^{\text{ev}_x} & & \downarrow^{\text{ev}_x'} \\
[x, x] & \xrightarrow{[y, z] \circ [x, y]} & [y, z] \circ ([x, y] \circ x) \\
\downarrow^{\text{ev}_x} & & \downarrow^{\text{ev}_y} \\
[x, z] & \xrightarrow{1} & y
\end{array}$$

This construction of $\hat{\mathcal{L}}$ is called the canonical construction. If $\mathcal{L}$ is a strongly unital, i.e., $u_x : 1 \circ x \to x$ is invertible for every $x \in \mathcal{L}$, then $\mathcal{L}$ can be canonically identified with the underlying
category of \( \mathcal{A}\mathcal{L} \) via an isomorphism of categories defined by the identity map on objects and the map \( (f : y \to y') \mapsto (f : \mathbb{1}_\mathcal{A} \to [y, y']) \) on morphisms, where \( f \) is defined by the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{1}_\mathcal{A} \otimes y & \xrightarrow{\mathbb{1}_y} & [y, y'] \otimes y \\
\downarrow{\cong} & & \downarrow{\cong} \\
y & \xrightarrow{f} & y'
\end{array}
\] (3.12)

An enriched functor between two enriched categories from the canonical construction as explained in the following proposition.

Remark 3.26. We explain the duality between an \( \mathcal{A}\mathcal{L} \) on \( \mathcal{L} \).

When \( \mathcal{L} = \mathcal{B} \to \mathcal{A} \) be an \( \mathcal{B}\mathcal{L} \) on \( \mathcal{A} \) and a functor \( F : \mathcal{L} \to \mathcal{M} \). We introduce a new structure on \( F \) that is in some sense dual to the \( \mathcal{L}\mathcal{B} \mathcal{L} \) on \( \mathcal{F} \).

Definition 3.24. Given a lax-monoidal functor \( R : \mathcal{A} \to \mathcal{B} \) and a functor \( F : \mathcal{L} \to \mathcal{M} \). An \( \mathcal{L}\mathcal{B} \mathcal{L} \) on \( \mathcal{F} \) is defined by a natural transformation: \( \beta = \{\beta_{a,x} : R(a) \otimes F(x) \to F(a \otimes x)\}_{a \in \mathcal{A}, x \in \mathcal{L}} \) rendering the following diagrams commutative.

\[
\begin{array}{ccc}
R(a) \otimes (b \otimes F(x)) & \xrightarrow{\eta_{ab} \otimes \mathbb{1}_x} & F(a \otimes (b \otimes x)) \\
\downarrow{\beta_{a,b \otimes x}} & & \downarrow{\beta_{a,b \otimes x}} \\
R(a) \otimes (b \otimes F(x)) & \xrightarrow{\mathbb{1}_R \otimes \beta_{a,b \otimes x}} & F(a \otimes (b \otimes x)) \\
\downarrow{\mathbb{1}_R \otimes \beta_{a,b \otimes x}} & & \downarrow{\mathbb{1}_R \otimes \beta_{a,b \otimes x}} \\
R(a) \otimes F(x) & \xrightarrow{\mathbb{1}_R \otimes \beta_{a,b \otimes x}} & F(a \otimes x) \\
\downarrow{\mathbb{1}_R \otimes \beta_{a,b \otimes x}} & & \downarrow{\mathbb{1}_R \otimes \beta_{a,b \otimes x}} \\
R(a) \otimes F(x) & \xrightarrow{\mathbb{1}_R \otimes \beta_{a,b \otimes x}} & F(a \otimes x)
\end{array}
\] (3.13)

The functor \( F \) equipped with an \( \mathcal{L}\mathcal{B} \mathcal{L} \) is called an \( \mathcal{L}\mathcal{B} \mathcal{L} \) functor.

Remark 3.25. When \( \mathcal{A} = \mathcal{B}, R = 1_\mathcal{A} \), an \( \mathcal{L}\mathcal{B} \mathcal{L} \) functor becomes a lax \( \mathcal{B}\mathcal{L} \) module functor. In general, an \( \mathcal{L}\mathcal{B} \mathcal{L} \) functor \( F \) is not a "module functor" in any sense because \( \mathcal{A} \to \mathcal{B} \to \text{Fun}(\mathcal{M}, \mathcal{M}) \) (as the composition of a lax-monoidal functor and an \( \mathcal{L}\mathcal{B} \mathcal{L} \) module functor) is neither lax-monoidal nor \( \mathcal{L}\mathcal{B} \mathcal{L} \) module.

Remark 3.26. We explain the duality between an \( \mathcal{L}\mathcal{B} \mathcal{L} \) structure and an \( \mathcal{L}\mathcal{B} \mathcal{L} \) structure. Let \( (L : \mathcal{B} \to \mathcal{A}, \mathcal{A} : \mathcal{B} \to \mathcal{M}, \eta : 1_\mathcal{B} \Rightarrow RL \epsilon : LR \Rightarrow 1_\mathcal{A}) \) be an adjunction such that \( R \) is lax-monoidal. Then \( L \) is automatically \( \mathcal{L}\mathcal{B} \mathcal{L} \) with the \( \mathcal{L}\mathcal{B} \mathcal{L} \) maps defined by

\[
L(\mathbb{1}_\mathcal{B}) \to LR(1_\mathcal{A}) \xleftarrow{1} 1_\mathcal{A}, \quad L(a \otimes b) \xrightarrow{\eta_{ab} \otimes 1} LR(a \otimes b) \to LR(L(a) \otimes L(b)) \xrightarrow{} L(a) \otimes L(b).
\]

Given an \( \mathcal{L}\mathcal{B} \mathcal{L} \) structure on \( F \) \( \{\beta_{a,x} : R(a) \otimes F(x) \to F(a \otimes x)\}_{a \in \mathcal{A}, x \in \mathcal{L}} \), then

\[
\alpha_{b,x} := \{b \otimes F(x) \xrightarrow{\eta_{ab} \otimes 1} RL(b) \otimes F(x) \xrightarrow{\beta_{b,x} \otimes \mathbb{1}} F(L(b) \otimes x)\}, \quad \forall b \in \mathcal{B}, x \in \mathcal{L}
\]
define an \( \mathcal{L}\mathcal{B} \mathcal{L} \) structure on \( F \). Conversely, given an \( \mathcal{L}\mathcal{B} \mathcal{L} \) structure on \( F \) \( \{\beta_{a,x} : b \otimes F(x) \to F(L(b) \otimes x)\}_{b \in \mathcal{B}, x \in \mathcal{L}} \), then

\[
\beta_{a,x} := \{R(a) \otimes F(x) \xrightarrow{\alpha_{b,x} \otimes \mathbb{1}} F(LR(a) \otimes x) \xrightarrow{1 \otimes \beta_{a,b \otimes x}} F(a \otimes x)\}, \quad \forall a \in \mathcal{A}, x \in \mathcal{L}
\]
define an \( \mathcal{L}\mathcal{B} \mathcal{L} \) structure on \( F \). These two constructions are mutually inverse. In other words, there is a bijection between the set of \( \mathcal{L}\mathcal{B} \mathcal{L} \) structures on \( F \) and that of \( \mathcal{L}\mathcal{B} \mathcal{L} \) structures on \( F \).

The \( \mathcal{L}\mathcal{B} \mathcal{L} \) structure gives a characterization of an enriched functor between two enriched categories from the canonical construction as explained in the following proposition.
**Proposition 3.27.** Let $\hat{F} : A \rightarrow B$ be an enriched functor. The underlying functor $F : L \rightarrow M$ with the natural transformation

$$\beta_{a,x} := \left( \hat{F}(a) \circ F(x) \xrightarrow{\hat{F}(\text{coev}_y)} \hat{F}([x,a \otimes x]) \circ F(x) \xrightarrow{\hat{F}(\text{coev}_x)} [F(x), F(a \otimes x)] \circ F(x) \xrightarrow{\text{ev}_{(y)}} F(a \otimes x) \right) \quad (3.14)$$

is an $\hat{F}$-lax functor. Conversely, given a lax-monoidal functor $\hat{F} : A \rightarrow B$ and an $\hat{F}$-lax functor $F : L \rightarrow M$ with the $\hat{F}$-lax structure $\beta_{a,x} : \hat{F}(a) \circ F(x) \rightarrow F(a \otimes x)$, the morphisms

$$\|F_{x,y} := \left( \hat{F}([x,y]) \xrightarrow{\text{coev}_{(y)}} [F(x), \hat{F}([x,y]) \circ F(x)] \xrightarrow{\hat{F}(\text{coev}_x)} [F(x), F([x,y] \circ x)] \xrightarrow{\hat{F}([\text{ev}_y])} [F(x), F(y)] \right),$$

together with $\hat{F}$ and $F$, define an enriched functor $\|F : A \rightarrow B$.

**Proof.** Given an enriched functor $\|F : A \rightarrow B$, we need to show the natural transformation $\beta_{a,x}$ defined in (3.14) satisfies two diagrams in (3.13). The first diagram is the outer diagram of the following commutative diagram

![First diagram](image1)

where the pentagon $\boxtimes$ commutes because $\|F$ is an enriched functor. The second diagram is the outer diagram of the following commutative diagram

![Second diagram](image2)

where the subdiagram $\boxtimes$ commutes because $\|F$ is an enriched functor.

Conversely, suppose $\beta_{a,x}$ is an $\hat{F}$-lax structure of $F$, we need to show that the morphisms $\|F_{x,y}$, together with $\hat{F}$ and $F$, define an enriched functor $\|F : A \rightarrow B$. That $\|F$ preserves the identity morphisms follows from the commutativity of the outer diagram of the following commutative
defines an enriched natural transformation

\[ \text{Proposition 3.30.} \]

natural transformation. \( \xi \)

where \( \star \) commutes by the definition of \( \hat{F} \)-lax structure. Using the adjunction \( (- \circ x) \dashv [x, -] \), the condition \( (3.5) \) is equivalent to the commutativity of the outer square of the following diagram:

\[
\begin{array}{ccc}
\hat{F}([y, z]) \hat{F}([x, y]) F(x) & \xrightarrow{F(1)} & \hat{F}([y, z]) [x, y] F(x) \\
\xrightarrow{F(1)} & & \xrightarrow{F(1)} \\
\hat{F}([y, z]) \hat{F}([x, y]) y & \xrightarrow{F(1)} & \hat{F}([y, z]) [x, y] y \\
\xrightarrow{F(1)} & & \xrightarrow{F(1)} \\
\hat{F}([y, z]) y & \xrightarrow{F(1)} & \hat{F}([y, z]) y
\end{array}
\]

where the square \( \star \) commutes because \( \beta \) is an \( \hat{F} \)-lax structure of \( F \).

It is easy to verify that these two constructions are mutually inverse. \( \square \)

**Definition 3.28.** For \( i = 1, 2 \), let \( R_i : A \to B \) be a lax-monoidal functor and \( F_i : \mathcal{L} \to M \) an \( R_i \)-lax functor. Given a lax-monoidal natural transformation \( \xi : R_1 \Rightarrow R_2 \), a natural transformation \( \xi : F_1 \Rightarrow F_2 \) is \( \xi \)-lax or called a \( \hat{\xi} \)-lax natural transformation if the following diagram commutes

\[
\begin{array}{c}
R_1(a) \circ F_1(x) & \xrightarrow{\hat{\xi}(a, x)} & F_1(a \circ x) \\
\downarrow{\xi(a)} & & \downarrow{\xi(a)} \\
R_2(a) \circ F_2(x) & \xrightarrow{\hat{\xi}(a, x)} & F_2(a \circ x)
\end{array}
\]

where the unlabeled arrows are given by the \( R_i \)-lax structure on \( F_i \). \( \blacksquare \)

**Remark 3.29.** The definition \( 3.28 \) is motivated by the following fact. Let \( (L : B \to A, R : A \to B, \eta : 1_B \Rightarrow RL, \xi : LR \Rightarrow 1_A) \) be an adjunction such that \( R \) is lax-monoidal and \( F_1, F_2 : \mathcal{L} \to M \) be \( R \)-lax functors. By Remark \( 3.26 \), \( F_i \) is equipped with an \( L \)-oplax structure induced by the \( R \)-lax structure on \( F_i \). Then a natural transformation \( \xi : F_1 \Rightarrow F_2 \) is \( 1_R \)-lax if and only if \( \xi \) is a \( B \)-module natural transformation.

**Proposition 3.30.** For \( i = 1, 2 \), let \( \hat{F}_i : A \to B \) be a lax-monoidal functor and \( F_i : \mathcal{L} \to M \) an \( \hat{F}_i \)-lax functor. Then we have two enriched functors \( |F_1, F_2 : \mathcal{A} \Rightarrow M | : \mathcal{B} \Rightarrow M \). Given a lax-monoidal natural transformation \( \xi : \hat{F}_1 \Rightarrow \hat{F}_2 \), a natural transformation \( \hat{\xi} : F_1 \Rightarrow F_2 \) is \( \hat{\xi} \)-lax if and only if \( (\xi, \xi) \) defines an enriched natural transformation \( |\xi| : |F_1| \Rightarrow |F_2| \).

**Proof.** Given an enriched natural transformation \( |\xi| : |F_1| \Rightarrow |F_2| \), we need to show that \( (\hat{\xi}, \hat{\xi}) \) satisfies the diagram \( 3.15 \). This is the outer diagram of the following commutative diagram.

[Diagram omitted for brevity, but should include the commutative diagram as described in the text.]
where the pentagon $\star$ commutes because $\xi$ is an enriched natural transformation.

Conversely, suppose the natural transformation $\xi : F_1 \Rightarrow F_2$ is $\xi$-lax. Then the naturality of $\xi$ is the outer diagram of the following commutative diagram

\[
\begin{align*}
F_1([x,y]) & \xrightarrow{\text{coev}_{1(x)}} [F_1(x), F_1([x,y])F_1(x)] \xrightarrow{[1_{L}\xi_{x}, \xi_{y}]} [F_1(x), F_1([x,y])x] & \xrightarrow{[1_{L}\xi_{y}, \xi_{x}]} [F_1(x), F_1(x)] \\
\xi_{x,y} & \xrightarrow{[1_{L}\xi_{x}, \xi_{y}]} [F_1(x), F_1([x,y])F_2(x)] \xrightarrow{\text{coev}_{2(x)}} [F_2(x), F_2([x,y])F_2(x)] & \xrightarrow{[1_{L}\xi_{y}, \xi_{x}]} [F_1(x), F_1(x)]
\end{align*}
\]

where the square $\star$ commutes due to the diagram (3.15).

Let $\mathbf{LMod}$ be the 2-category defined as follows.

- The objects are pairs $(\mathcal{A}, \mathcal{L})$, where $\mathcal{A}$ is a monoidal category and $\mathcal{L}$ is a strongly unital left $\mathcal{A}$-oplax-module that is enriched in $\mathcal{A}$.
- A 1-morphism $(\mathcal{A}, \mathcal{L}) \to (\mathcal{B}, \mathcal{M})$ is a pair $(\hat{F}, F)$, where $\hat{F} : \mathcal{A} \to \mathcal{B}$ is a lax-monoidal functor and $F : \mathcal{L} \to \mathcal{M}$ is an $\hat{F}$-lax functor.
- A 2-morphism $(\hat{F}, F) \Rightarrow (\hat{G}, G)$ is a pair $(\hat{\xi}, \xi)$, where $\hat{\xi} : \hat{F} \Rightarrow \hat{G}$ is a lax-monoidal natural transformation and $\xi : F \Rightarrow G$ is a $\hat{\xi}$-lax natural transformation.

The horizontal/vertical composition is induced by the horizontal/vertical composition of functors and natural transformations. The 2-category $\mathbf{LMod}$ is symmetric monoidal with the tensor product defined by the Cartesian product and the tensor unit given by $(\star, \star)$.

By Proposition 5.27 and Proposition 5.30, we immediately obtain the following result.

**Theorem 3.31.** The canonical construction can be promoted to a symmetric monoidal locally isomorphic 2-functor from $\mathbf{LMod}$ to $\mathbf{Mod}_{\mathbf{ECat}}$ defined as follows.

- The image of $(\mathcal{A}, \mathcal{L})$ is the $\mathcal{A}$-enriched category $\mathcal{A}\mathcal{L}$ defined by the canonical construction.
- The image of a 1-morphism $(\hat{F}, F) : (\mathcal{A}, \mathcal{L}) \to (\mathcal{B}, \mathcal{M})$ is the enriched functor $\mathcal{A}\mathcal{L} \to \mathcal{B}\mathcal{M}$ defined by Proposition 5.27.
- The image of a 2-morphism $(\hat{\xi}, \xi) : (\mathcal{A}, \mathcal{L}) \to (\mathcal{B}, \mathcal{M})$ is the enriched natural transformation $\xi$ defined by the background changing natural transformation $\hat{\xi}$ and the underlying natural transformation $\xi$.

In physical applications, $\mathcal{A}$ and $\mathcal{L}$ are often finite semisimple. In this case, we obtain a stronger result of the canonical construction. A finite category over a ground field $k$ is a $k$-linear category $\mathcal{C}$ that is equivalent to the category of finite-dimensional modules over a finite-dimensional $k$-algebra $A$ (see [EGNO15, Definition 1.8.6] for an intrinsic definition). We say that $\mathcal{C}$ is semisimple if $A$ is semisimple. We use $\mathbf{fsCat}$ to denote the 2-category of finite semisimple categories, $k$-linear functors and natural transformations.

**Definition 3.32.** An enriched category $\mathcal{A}\mathcal{L}$ is called finite (semisimple) if $\mathcal{A}$ and $\mathcal{L}$ are both finite (semisimple) categories and the tensor product $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is $k$-bilinear.

**Remark 3.33.** Let $\mathcal{A}\mathcal{L}$ be an enriched category. If the background category $\mathcal{A}$ is a $k$-linear category such that the tensor product $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is $k$-bilinear, then underlying category $\mathcal{L}$ is also a $k$-linear category.

**Example 3.34.** Let $\mathcal{A}$ be a multi-fusion category and $\mathcal{L}$ a finite semisimple left $\mathcal{A}$-module. Then the canonical construction $\mathcal{A}\mathcal{L}$ is a finite semisimple enriched category.
Definition 3.35. Let $\mathcal{M}$ and $\mathcal{L}_i$ be finite enriched categories, $i = 1, \ldots, n$. An enriched functor $F : \mathcal{L}_1 \times \cdots \times \mathcal{L}_n \to \mathcal{M}$ is called multi-$k$-linear (or $k$-linear if $n = 1$) if the background changing functor $\hat{F} : \hat{\mathcal{L}}_1 \times \cdots \times \hat{\mathcal{L}}_n \to \hat{\mathcal{M}}$ is multi-$k$-linear.

Remark 3.36. Let $F : \mathcal{L}_1 \times \cdots \times \mathcal{L}_n \to \mathcal{M}$ be a multi-$k$-linear enriched functor. It is easy to see from (5.5) that the underlying functor $F : \mathcal{L}_1 \times \cdots \times \mathcal{L}_n \to \mathcal{M}$ is also multi-$k$-linear.

We denote the 2-category consisting of finite semisimple enriched categories, $k$-linear enriched functors and enriched natural transformations by $\text{lax}\mathcal{fsECat}$.

Let $\mathcal{fsLMod}$ be the 2-category defined as follows.

- An object is a pair $(\mathcal{A}, \mathcal{L})$, where $\mathcal{A}$ is a finite semisimple monoidal category and $\mathcal{L}$ is a strongly unital finite semisimple left $\mathcal{A}$-oplax-module that is enriched in $\mathcal{A}$ (i.e. the $\mathcal{A}$-action functor $\circ : \mathcal{A} \times \mathcal{L} \to \mathcal{L}$ is $k$-bilinear).
- A 1-morphism $(\mathcal{A}, \mathcal{L}) \to (\mathcal{B}, \mathcal{M})$ is a pair $(\hat{F}, F)$, where $\hat{F} : \mathcal{A} \to \mathcal{B}$ is a lax-monoidal $k$-linear functor and $F : \mathcal{L} \to \mathcal{M}$ is an $\hat{F}$-lax $k$-linear functor.
- A 2-morphism $(\hat{F}, F) \Rightarrow (\hat{G}, G)$ is a pair $(\xi, \zeta)$, where $\xi : \hat{F} \Rightarrow \hat{G}$ is a lax-monoidal natural transformation and $\zeta : F \Rightarrow G$ is a $\zeta$-lax natural transformation.

Theorem 3.37. The canonical construction defines a 2-equivalence from $\mathcal{fsLMod}$ to $\text{lax}\mathcal{fsECat}$. Moreover, this 2-equivalence is locally isomorphic.

Proof. Any $k$-linear functor between two finite semisimple categories is exact. In particular, $\mathcal{L}$ is enriched in $\mathcal{A}$ because the functor $a \mapsto \mathcal{L}(a \circ x, y) : \mathcal{A} \to 1$ is exact thus admits a right adjoint. Moreover, $\mathcal{A}^{\hat{\mathcal{L}}}(x, -) : \mathcal{L} \to \mathcal{A}$ is $k$-linear, thus admits a left adjoint. By [MPP18 Theorem 1.4] (see also [Lin81]), an enriched category $\mathcal{A}^{\hat{\mathcal{L}}}$ is equivalent to an enriched category obtained by the canonical construction of a strongly unital oplax module if and only if every functor $\mathcal{A}^{\hat{\mathcal{L}}}(x, -) : \mathcal{L} \to \mathcal{A}$ admits a left adjoint. Therefore, the 2-functor induced by canonical construction is essentially surjective on objects. By Proposition 3.22 and Proposition 3.30, it is also fully faithful on both 1-morphisms and 2-morphisms. Then the Whitehead theorem for 2-categories (see Theorem 7.5.8 in [JY20]) implies that this 2-functor is a 2-equivalence.

Remark 3.38. We use $\mathcal{fsECat}$ to denote the sub-2-category of $\text{lax}\mathcal{fsECat}$ consisting of those $k$-linear enriched functors $F$ such that $\hat{F}$ is monoidal. We will show in the second work in our series that $\mathcal{fsECat}$ has a monoidal structure with the tensor product given by an analogue of Deligne’s tensor product.

4 Enriched monoidal categories

An enriched monoidal category can be defined as an algebraic object in either $\mathcal{ECat}$ or $\text{lax}\mathcal{ECat}$. In this section, we restrict ourselves to the $\mathcal{ECat}$ case only. Namely, the background changing functor is always monoidal in this section. For a study of the $\text{lax}\mathcal{ECat}$ case, see [For04 BM12].

4.1 Definitions and examples

An enriched monoidal category should be an algebra (a pseudomonoid in [DS97]) in $\mathcal{ECat}$. An algebra in $\mathcal{ECat}$ is a collection $(\mathcal{L}, \otimes, 1, \alpha, \lambda, \rho)$ such that (recall Remark 3.18)

1. the background category $(\mathcal{A}, \otimes, I, \alpha, \lambda, \rho)$ is an algebra in the symmetric monoidal 2-category $\text{Alg}_{\mathcal{ECat}}(\mathcal{Cat})$ of monoidal categories;
2. the underlying category $(\mathcal{L}, \otimes, \mathcal{L}, \alpha, \lambda, \rho)$ is an algebra in the symmetric monoidal 2-category $\mathcal{Cat}$ of categories, i.e. a monoidal category.
However, in this work, we propose a slightly different definition of an enriched monoidal category for simplicity and convenience (see Remark 4.3). It is equivalent to [KZ18b] Definition 2.3 (see also [MP17] for a strict version). Note that the notion of an algebra in $\text{Alg}_{\mathcal{E}^2}(\text{Cat})$ is equivalent to that of a braided monoidal category.

**Definition 4.1.** Let $\mathcal{A}$ be a braided monoidal category. An enriched monoidal category consists of the following data:

- an enriched category $\mathcal{A}^L$;
- a tensor product enriched functor $\otimes : \mathcal{A}^L \times \mathcal{A}^L \to \mathcal{A}^L$ in $\text{ECat}$ such that $\hat{\otimes} = \otimes_{\mathcal{A}}$, where the monoidal structure on $\otimes_{\mathcal{A}}$ is given by Convention 4.2;
- a distinguished object $1^L_\mathcal{A} \in \mathcal{A}^L$ called the unit object, or equivalently, an enriched functor $1^L_{\mathcal{A}} : * \to \mathcal{A}^L$ in $\text{ECat}$ (see Example 3.12);
- an associator: an enriched natural isomorphism $\alpha : 1^L_\otimes (1^L \otimes 1) \Rightarrow 1^L_\otimes (1 \otimes 1)$ with $\hat{\alpha}$ given by the associator of the monoidal category $\mathcal{A}$ (i.e. $\hat{\alpha} = \alpha_{\mathcal{A}}$);
- two unitors: two enriched natural isomorphisms $\lambda : 1^L_\otimes (1 \otimes 1) \Rightarrow 1^L_{\mathcal{A}}$ and $\rho : 1^L_\otimes (1 \otimes 1) \Rightarrow 1^L_{\mathcal{A}}$ such that $\hat{\lambda}$ and $\hat{\rho}$ are given by the left and right unitors of the monoidal category $\mathcal{A}$, respectively (i.e. $\hat{\lambda} = \lambda_{\mathcal{A}}, \hat{\rho} = \rho_{\mathcal{A}}$);

such that $(\mathcal{A}, \otimes, 1^L_{\mathcal{A}}, \alpha, \lambda, \rho)$ is a monoidal category, which is called the underlying monoidal category of $\mathcal{A}^L = (\mathcal{A}^L, \otimes, 1^L_{\mathcal{A}}, \alpha, \lambda, \rho)$. The enriched monoidal category $\mathcal{A}^L$ is called strict if the underlying monoidal category $\mathcal{L}$ is strict.

**Convention 4.2.** Let $\mathcal{A}$ a braided monoidal category with the braiding $c_{a,b} : a \otimes b \to b \otimes a$. In the following, the monoidal structure on the tensor product functor $(a, b) \mapsto a \otimes b : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is always understood as the lax-monoidal structure induced by the braiding:

$$\otimes(a_1, b_1) \otimes (a_2, b_2) = a_1 \otimes b_1 \otimes a_2 \otimes b_2 \xrightarrow{1^L_{\otimes} \otimes 1^L_{\otimes}} a_1 \otimes a_2 \otimes b_1 \otimes b_2 = \otimes((a_1, b_1) \otimes (a_2, b_2)),$$

or equivalently, as the oplax-monoidal structure of $\otimes$ is defined by the anti-braiding.

**Remark 4.3.** Given an algebra $(\mathcal{A}^L, \otimes, 1^L_{\mathcal{A}}, \alpha, \lambda, \rho)$ in $\text{ECat}$, there is a canonical isomorphism $\phi : \otimes \xrightarrow{\hat{\phi}} \otimes_{\mathcal{A}},$ and $\hat{\alpha}, \hat{\lambda}, \hat{\rho}$ are compatible with $\alpha_{\mathcal{A}}, \lambda_{\mathcal{A}}, \rho_{\mathcal{A}}$ (through $\phi$), respectively. For convenience, we require $\hat{\otimes} = \otimes_{\mathcal{A}}$ and $\hat{\alpha} = \alpha_{\mathcal{A}}, \hat{\lambda} = \lambda_{\mathcal{A}}, \hat{\rho} = \rho_{\mathcal{A}}$ in Definition 4.1. It turns out that these additional requirements also endow Definition 4.1 with a new interpretation as an algebra in the monoidal category $\mathcal{A}^{\mathcal{E}^2}$ with the tensor product $\times$ (recall Example 3.20). This new interpretation is how the notion was defined in [KZ18b] Definition 2.3.

**Remark 4.4.** We use $\mathcal{A}$ to denote the same monoidal category $\mathcal{A}$ equipped with the anti-braiding. By Convention 4.2, the lax/oplax-monoidal structure of $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is induced by the anti-braiding/braiding of $\mathcal{A}$.

**Remark 4.5.** Let $\mathcal{A}^L$ be an enriched monoidal category. We use $\mathcal{L}^{\text{rev}}$ to denote the monoidal category obtained from $\mathcal{L}$ by reversing the tensor product. One can canonically construct an $\mathcal{A}$-enriched monoidal category $\mathcal{A}^{\mathcal{L}^{\text{rev}}}$, whose underlying monoidal category is $\mathcal{L}^{\text{rev}}$, as follows. As an enriched category $\mathcal{A}^{\mathcal{L}^{\text{rev}}} = \mathcal{A}^{\mathcal{L}}$. The tensor product enriched functor $\otimes^{\text{rev}} : \mathcal{A}^{\mathcal{L}^{\text{rev}}} \times \mathcal{A}^{\mathcal{L}^{\text{rev}}} \to \mathcal{A}^{\mathcal{L}^{\text{rev}}}$ is defined by the tensor product functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and the family of morphisms

$$\mathcal{A}^{\mathcal{L}}(x_1, x_2) \otimes^{\text{rev}} \mathcal{A}^{\mathcal{L}}(y_1, y_2) \xrightarrow{c_{x_1, y_1} \otimes^{\text{rev}} c_{x_2, y_2}} \mathcal{A}^{\mathcal{L}}(y_1, y_2) \otimes^{\text{rev}} \mathcal{A}^{\mathcal{L}}(x_1, x_2) \xrightarrow{\otimes^{\text{rev}} \otimes^{\text{rev}}} \mathcal{A}^{\mathcal{L}}(x_1 \otimes^{\text{rev}} y_1, x_2 \otimes^{\text{rev}} y_2).$$

The associator of $\mathcal{A}^{\mathcal{L}^{\text{rev}}}$ is defined by the associator of $\mathcal{A}$ and the inverse of the associator of $\mathcal{L}$. The left/right unitor of $\mathcal{A}^{\mathcal{L}^{\text{rev}}}$ is defined by the left/right unitor of $\mathcal{A}$ and the right/left unitor of $\mathcal{L}$. We refer to $\mathcal{A}^{\mathcal{L}^{\text{rev}}}$ as the reversed category of $\mathcal{A}^{\mathcal{L}}$, i.e. $(\mathcal{A}^{\mathcal{L}})^{\text{rev}} : = \mathcal{A}^{\mathcal{L}^{\text{rev}}}$. 
Remark 4.6. In the definition of the enriched functor $\Phi^{rev}$, we have used the braiding instead of the anti-braiding because $\Phi^{rev}$ is not well-defined if we use the anti-braiding. ☐

Definition 4.7. Let $\mathcal{L}$ and $\mathcal{M}$ be enriched monoidal categories. An enriched monoidal functor $F : \mathcal{L} \to \mathcal{M}$ consists of the following data:

- an enriched functor $F : \mathcal{L} \to \mathcal{M}$ in $\mathbf{ECat}$;
- an enriched natural isomorphism $\xi : F \times F \Rightarrow F \circ F$

with the background changing natural transformation $\xi$ given by the monoidal structure of $F$;

- an enriched natural isomorphism $\phi : F \Rightarrow F$ with the background changing natural transformation $\phi$ given by the monoidal structure of $F$;

satisfying the following conditions:

1. the background changing functor $\hat{F} = (\hat{F}, \hat{F}^2, \hat{F}^0) : \mathcal{A} \to \mathcal{B}$ is a braided monoidal functor;
2. the underlying functor $F : \mathcal{L} \to \mathcal{M}$, together with the underlying natural transformations $F^2$ and $F^0$, defines a monoidal functor. ■

Definition 4.8. Let $\mathcal{L}$ and $\mathcal{M}$ be enriched monoidal categories and $F, G : \mathcal{L} \to \mathcal{M}$ enriched monoidal functors. An enriched monoidal natural transformation $\xi : F \Rightarrow G$ such that the underlying natural transformation $F^2$ is defined by a monoidal natural transformation.

Remark 4.9. The symmetric monoidal 2-category of enriched monoidal categories, enriched monoidal functors and enriched monoidal natural transformations is a sub-2-category of the 2-category $\mathbf{Alg}_{\mathbf{Cat}}(\mathbf{ECat})$ of algebras in $\mathbf{ECat}$, and the inclusion is a biequivalence. ☐

Example 4.10. Let $(A, m_A, i_A)$ be a commutative algebra in a braided monoidal category $\mathcal{A}$. The $\mathcal{A}$-enriched category $\mathcal{A}$ is a strict enriched monoidal category with the tensor product functor induced by $m_A$. Conversely, every strict $\mathcal{A}$-enriched monoidal category with one object arises in this way (see also [KZ18b, Example 3.5]).

Let $(B, m_B, i_B)$ be a commutative algebra in a braided monoidal category $\mathcal{B}$ and $F : \mathcal{A} \to \mathcal{B}$ be an enriched monoidal functor. The enriched monoidal isomorphisms $\hat{F}^2, \hat{F}^0$ are determined by the underlying natural transformations $F^2, F^0$, and any morphism $(F^2)_*, (F^0)_*$ form an enriched natural transformation respectively since $B$ is commutative. The condition that $(F^2, F^0)$ is a monoidal functor is equivalent to say that $(F^2)$, is the inverse of $(F^0)$, in the underlying category of $\mathcal{A}$. Hence an enriched monoidal functor $F : \mathcal{A} \to \mathcal{B}$ is defined by a braided monoidal functor $\hat{F} : \mathcal{A} \to \mathcal{B}$, an algebra homomorphism $\xi : \hat{F}(A) \to B$ (see Example 5.11) and an isomorphism $(F^0)$, in the underlying category of $\mathcal{A}$.

Let $G : \mathcal{A} \to \mathcal{B}$ be another enriched monoidal functor defined by a braided monoidal functor $\hat{G}$, an algebra homomorphism $\xi : \hat{G}(A) \to B$ and an isomorphism $(G^0)$. Then an enriched natural transformation $\xi : F \Rightarrow G$ is an enriched monoidal natural transformation if and only if $(G^0)_* = \hat{G}(A) \Rightarrow B$, in the underlying category of $\mathcal{A}$, which further implies that $\xi$ is invertible. If $\xi$ is invertible, the commutativity of $B$ implies that the diagram $\xi$ is equivalent to

$$
(\hat{F}(A) \xrightarrow{\xi} B) = (\hat{F}(A) \xrightarrow{i_A} \hat{G}(A) \xrightarrow{\xi} B).
$$

As a conclusion, an enriched natural transformation $\xi : F \Rightarrow G$ is defined by a monoidal natural transformation $\xi$ such that $F^2 = \hat{G}_* \circ \hat{F}_*$ and the underlying natural transformation is given by $\xi_0 = (G^0)_* \circ (F^0)^{-1}$. ☐
4.2 The canonical construction

Let $\mathcal{A}$ be a braided monoidal category viewed as an algebra in $\text{Alg}_{E_1}^{\text{oplax}}(\text{Cat})$ by Convention \ref{convention}

**Definition 4.11.** A monoidal left $\mathcal{A}^{\text{oplax}}$-module is a left $\mathcal{A}^{\text{oplax}}$-module in $\text{Alg}_{E_1}^{\text{oplax}}(\text{Cat})$. A monoidal left $\mathcal{A}$-module is a left $\mathcal{A}$-module in $\text{Alg}_{E_1}^{\text{oplax}}(\text{Cat})$.

**Remark 4.12.** More explicitly, a monoidal left $\mathcal{A}^{\text{oplax}}$-module $\mathcal{L}$ consists of the following data:

- a monoidal category $\mathcal{L}$;
- a left $\mathcal{A}$-action given by an oplax-monoidal functor $\odot : \mathcal{A} \times \mathcal{L} \to \mathcal{L}$ (i.e. the functor $\odot$ is equipped with a natural transformation
  \[ (a \otimes b) \odot (x \otimes y) \to (a \otimes x) \otimes (b \otimes y) \quad \forall a, b \in \mathcal{A}, x, y \in \mathcal{L} \quad (4.1) \]
  and a morphism $1_{\mathcal{A}} \otimes 1_{\mathcal{L}} \to 1_{\mathcal{L}}$ rendering the following diagrams commutative;
  \[
  \begin{array}{ccc}
    ([a \otimes b] \odot [x \otimes y] \odot z) & \longrightarrow & [a \otimes (b \odot c)] \odot [x \otimes (y \odot z)] \\
    ↓ & & ↓ \\
    ([a \otimes b] \odot (x \otimes y) \odot (c \odot z)) & \longrightarrow & (a \otimes x) \odot ([b \odot c] \odot (y \odot z)) \\
    ↓ & & ↓ \\
    ([a \odot x] \odot (b \odot y) \odot (c \odot z)) & \longrightarrow & (a \odot x) \odot ([b \odot y] \odot (c \odot z))
  \end{array}
  \]

- two oplax-monoidal natural transformation:
  \[ (1_{\mathcal{A}} \odot a) \odot (1_{\mathcal{L}} \otimes x) \quad (a \odot x) \odot (1_{\mathcal{A}} \odot 1_{\mathcal{L}}) \quad (4.3) \]
  \[ (1_{\mathcal{A}} \odot 1_{\mathcal{L}}) \odot (a \otimes x) \quad (a \odot 1_{\mathcal{L}}) \odot (1_{\mathcal{A}} \otimes x) \quad (4.4) \]

- the associator $a \odot b) \odot x \to a \odot (b \odot x)$ and the unitor $1_{\mathcal{A}} \odot x \to x$ (i.e. the following diagrams are commutative);
  \[
  \begin{array}{ccc}
    ([a_1 \otimes a_2] \odot [b_1 \otimes b_2]) \odot (x \otimes y) & \longrightarrow & (a_1 \odot a_2) \odot [(b_1 \otimes b_2) \odot (x \otimes y)] \\
    \downarrow & & \downarrow \\
    (a_1 \odot (b_1 \odot b_2)) \odot (a_2 \odot (b_1 \odot b_2)) & \longrightarrow & (a_1 \odot b_2) \odot (a_2 \odot b_1) \\
    \downarrow & & \downarrow \\
    (1_{\mathcal{A}} \odot 1_{\mathcal{L}}) \odot 1_{\mathcal{L}} & \longrightarrow & 1_{\mathcal{A}} \odot (1_{\mathcal{L}} \odot 1_{\mathcal{L}}) \\
    \uparrow \text{oplax} & & \uparrow \text{oplax} \\
    1_{\mathcal{A}} \odot 1_{\mathcal{L}} & \longrightarrow & 1_{\mathcal{L}}
  \end{array}
  \]

where the arrow $[(a_1 \otimes a_2) \odot (b_1 \otimes b_2)] \odot (x \otimes y) \to [(a_1 \otimes b_1) \odot (a_2 \otimes b_2)] \odot (x \otimes y)$ is defined by the anti-braiding $c_{b_1,b_2}^{-1} : a_2 \otimes b_1 \to b_1 \otimes a_2$ (recall Convention \ref{convention});

such that $\mathcal{L}$, together with the left $\mathcal{A}$-action functor $\odot$, the associator and the unitor defined above, is a left $\mathcal{A}^{\text{oplax}}$-module.

If, in addition, the left $\mathcal{A}$-action functor $\odot$ is a monoidal functor, and the associator and the unitors are natural isomorphisms, then $\mathcal{L}$ is a monoidal left $\mathcal{A}$-module.

**Remark 4.13.** If, in addition, $\mathcal{L}$ is a strongly unital left $\mathcal{A}$-module, then the commutativity of the second diagram in \ref{diagram} follows from those of diagram \ref{diagram} and \ref{diagram}. Also, the commutativity of the first diagram in \ref{diagram} is a consequence of that of the second diagram in \ref{diagram} and the fact that $\mathcal{L}$ is an $\mathcal{A}^{\text{oplax}}$-module.

\footnote{The first diagrams in both \ref{diagram} and \ref{diagram} are the defining properties of the associator as an oplax-monoidal natural transformation. The other two are those of the unitors.}
Remark 4.14. By Definition 4.11, a monoidal structure on a left $A^{\oplus}$-module $L$ consists of a monoidal structure on $L$ and a natural transformation $\alpha, \alpha$ satisfying proper axioms. All the rest data can be included in the defining data of a left $A^{\oplus}$-module structure on $L$. 

In the following, we use $I$ to denote the forgetful functor $\mathcal{C}(L) \rightarrow L$ for any monoidal category $L$. If $\varphi : A \rightarrow \mathcal{C}(L)$ is a braided (oplax) monoidal functor, then $\varphi^\ast := I \circ \varphi$ is an (oplax) central functor [Bez04].

Example 4.15. Let $A$ be a braided monoidal category and $L$ a monoidal category. If $\varphi : A \rightarrow \mathcal{C}(L)$ is a braided oplax-monoidal functor, then $L$ is a monoidal left $A^{\oplus}$-module with the module action $\varphi^\ast(\cdot) \otimes - : A \times L \rightarrow L$. The oplax-monoidal structure $\varphi(a \otimes b) \otimes (x \otimes y) \rightarrow (\varphi(a) \otimes x) \otimes (\varphi(b) \otimes y)$ is given by

$$
\varphi^\ast(a \otimes b) \otimes (x \otimes y) \rightarrow (\varphi^\ast(a) \otimes \varphi^\ast(b)) \otimes (x \otimes y) \rightarrow (\varphi^\ast(a) \otimes (\varphi^\ast(b) \otimes x)) \otimes y
$$

where the third arrow is defined by the half-braiding of $\varphi(b)$.

Remark 4.16. When $L$ is a monoidal left $A$-module, it reduces to the usual definition (see for example [KZ18a, Definition 2.6.1]). Indeed, in this case, $\varphi := (\cdot \otimes 1_L)$ defines a braided monoidal functor $\varphi : A \rightarrow \mathcal{C}(L)$ (recall Example 2.16). Two monoidal left $A$-modules $(L, \varphi)$ and $(L, \varphi')$ are isomorphic. Hence, a monoidal left $A$-module $L$ can be equivalently defined by a monoidal category $A$ equipped with a braided monoidal functor $A \rightarrow \mathcal{C}(L)$. It is important to note that our convention is different from [KZ18a, Definition 2.6.1].

When $L$ is a monoidal left $A^{\oplus}$-module, the morphism $\alpha, \alpha$ is not necessarily invertible, thus does not induce a half-braiding.

Proposition 4.17. Given a pair $(A, L) \in \mathbf{LMod}$, let $A^L$ be the enriched category obtained from the pair $(A, L)$ via the canonical construction. There is a one-to-one correspondence between the monoidal structures on $A^L$ and the monoidal structures on the left $A^{\oplus}$-module $L$ as shown by the following two mutually inverse constructions:

$\Rightarrow$ Given a monoidal structure on $A^L$, i.e. a sextuple $(A^L, \otimes, 1_L, \alpha, \lambda, \rho)$, then the sextuple $(L, \otimes, 1_L, \alpha, \lambda, \rho)$ automatically defines a monoidal structure on $L$ by definition. A monoidal structure on the left $A^{\oplus}$-module $L$ is determined (recall Remark 4.14) if we further define the morphisms $\alpha, \alpha$ to be the one induced from the composed morphism:

$$(a \otimes b) \xrightarrow{\text{coev} \otimes \text{coev}} [x, a \otimes x] \otimes [y, b \otimes y] \xrightarrow{\varphi^\ast} [x \otimes y, (a \otimes x) \otimes (b \otimes y)].$$

$\Leftarrow$ Given a monoidal structure on the left $A^{\oplus}$-module $L$. A monoidal structure on $A^L$ is determined by defining the morphism $[x_1, x_2] \otimes [y_1, y_2] \rightarrow [x_1 \otimes y_1, x_2 \otimes y_2]$ to be the one induced by the following composed morphism

$$(x_1, x_2) \otimes (y_1, y_2) \xrightarrow{(\text{ev}_{x_1} \otimes \text{ev}_{y_1})} x_2 \otimes y_2,$$

where the first arrow is given by the oplax-monoidal structure of $\circ : A \times L \rightarrow L$.

Proof. It is clear that the diagram for a monoidal left $A^{\oplus}$-module $L$ commutes if and only if $\otimes_L$ is a $\otimes_A$-lax functor with the $\otimes_A$-lax structure given by the oplax-monoidal structure of $\circ$, and the diagram commutes if and only if $(a, a), (\hat{a}, \lambda), (b, \rho)$ is a 2-morphism in $\mathbf{LMod}$, respectively. Then the assertion follows from Theorem 5.3.

Example 4.18 (Construction 5.1 in [KZ18b]). Let $\varphi : A \rightarrow \mathcal{C}(L)$ be a strongly unital braided oplax-monoidal functor. Then $L$ equipped with the module action $\varphi^\ast(\cdot) \otimes - : A \times L \rightarrow L$ is a strongly unital monoidal left $A^{\oplus}$-module (see Example 4.15). If $L$ is also enriched in $A$, then $A^L$ is an $A$-enriched monoidal category.
Example 4.19. Let \( \mathcal{L} \) be a left \( A \)-module. We use \( \text{Fun}_A(\mathcal{L}, \mathcal{L}) \) to denote the category of left \( A \)-module functors and left \( A \)-module natural transformations. There is an obvious braided monoidal functor \( \hat{Z}_1(A) \to \mathcal{L}_1(\text{Fun}_A(\mathcal{L}, \mathcal{L})) \) defined by \( a \mapsto I(a) \circ - \). The left \( A \)-module functor \( I(a) \circ - \) is equipped with a half-braiding
\[
\gamma_{F,I(a)} : F(I(a) \circ -) \to I(a) \circ F(-), \quad \forall F \in \text{Fun}_A(\mathcal{L}, \mathcal{L}),
\]
defined by the left \( A \)-module functor structure on \( F \). It follows that \( \text{Fun}_A(\mathcal{L}, \mathcal{L}) \) is a monoidal left \( \hat{Z}_1(A) \)-module.

Definition 4.20. Let \( A, B \) be two braided monoidal categories. Let \( \mathcal{L} \) be a monoidal left \( A \)-oplax-module and \( \mathcal{M} \) a monoidal left \( B \)-oplax-module. Given a braided monoidal functor \( \hat{F} : A \to B \), a monoidal \( \hat{F} \)-lax functor \( F : \mathcal{L} \to \mathcal{M} \) is a monoidal functor and also an \( \hat{F} \)-lax functor such that the \( \hat{F} \)-lax structure \( \hat{F}(a) \circ F(x) \to F(a \circ x) \) is an oplax-monoidal natural transformation (i.e. the following diagrams commute).

\[
\begin{align*}
\hat{F}(a \circ b) \circ F(x \otimes y) & \to F((a \circ b) \circ (x \otimes y)) \\
\downarrow & \\
(F(a \circ F(b)) \circ (F(x) \otimes F(y)) & \to (F(a \circ x) \otimes (b \circ y)) \\
\downarrow & \\
(\hat{F}(a) \circ F(x)) \otimes (\hat{F}(b) \circ F(y)) & \to F(a \circ x) \otimes F(b \circ y) \\
\downarrow & \\
F(1 \circ 1) & \to F(1) \\
\end{align*}
\tag{4.7}
\]

\[
\begin{align*}
\hat{F}(1) \circ F(1) & \to F(1) \\
\downarrow & \\
\mathbb{1} \circ \mathbb{1} & \to \mathbb{1} \\
\downarrow & \\
F(1) & \to F(1) \\
\end{align*}
\tag{4.8}
\]

Remark 4.21. Let \( \varphi_i : A_i \to \hat{Z}_1(\mathcal{L}_i) \) be a braided oplax-monoidal functor, \( i = 1, 2 \). Then \( \mathcal{L}_i \) is a monoidal left \( A_i \)-oplax-module. Let \( \hat{F} : A_1 \to A_2 \) be a braided monoidal functor and \( F : \mathcal{L}_1 \to \mathcal{L}_2 \) be a monoidal functor. Suppose \( \theta_a : \varphi_2(\hat{F}(a)) \to F(\varphi_1(a)) \) is an oplax-monoidal natural transformation rendering the following diagram commutative,

\[
\begin{align*}
\varphi_2(\hat{F}(a)) \otimes F(x) & \xrightarrow{\theta_a \otimes 1} F(\varphi_1(a)) \otimes F(x) \xrightarrow{\sim} F(\varphi_1(a) \otimes x) \\
\downarrow & \\
F(x) \otimes \varphi_2(\hat{F}(a)) & \xrightarrow{1 \otimes \theta_a} F(x) \otimes F(\varphi_1(a)) \xrightarrow{\sim} F(x \otimes \varphi_1(a)) \\
\end{align*}
\tag{4.9}
\]

where two vertical arrows are the half-braidings of \( \varphi_2(\hat{F}(a)) \) and \( \varphi_1(a) \), respectively. Then \( F \) equipped with the natural transformation
\[
\beta_{a,x} := \left( \varphi_2(\hat{F}(a)) \otimes F(x) \xrightarrow{\theta_a \otimes 1} F(\varphi_1(a)) \otimes F(x) \xrightarrow{\sim} F(\varphi_1(a) \otimes x) \right)
\]
is a monoidal \( \hat{F} \)-lax functor.

We claim that every monoidal \( \hat{F} \)-lax structure on \( F \) arises in this way. Indeed, suppose \( \beta_{a,x} : \varphi_2(\hat{F}(a)) \otimes F(x) \to F(\varphi_1(a) \otimes x) \) is a monoidal \( \hat{F} \)-lax structure on \( F \). Then
\[
\theta_a := \left( \varphi_2(\hat{F}(a)) \xrightarrow{\beta_{a,x}} \varphi_2(\hat{F}(a)) \otimes F(1_{\mathcal{L}_1}) \xrightarrow{\beta_{1,a} \otimes 1} F(\varphi_1(a) \otimes \mathbb{1}) \xrightarrow{\sim} F(\varphi_1(a)) \right)
\]
is an oplax-monoidal natural transformation. Taking \( b = 1_{A_1} \) and \( x = 1_{\mathcal{L}_2} \) in the diagram (4.7), then we have
\[
\beta_{a,y} = \left( \varphi_2(\hat{F}(a)) \otimes F(y) \xrightarrow{\theta_a \otimes 1} F(\varphi_1(a) \otimes F(y)) \xrightarrow{\sim} F(\varphi_1(a) \otimes y) \right).
\]
By taking \( a = 1_{A_1} \) and \( y = 1_{\mathcal{L}_2} \) in the diagram (4.7), we can show that the oplax-monoidal natural transformation \( \theta_a \) renders the diagram (4.2) commutative. Hence, when \( A_1 = A_2 \) and \( \hat{F} = 1_{A_1} \), Definition 4.20 is precisely [KZ18a] Definition 2.6.6].
Proposition 4.22. Let \( \mathcal{A}, \mathcal{B} \) be braided monoidal categories. Suppose \( \mathcal{L} \) is a strongly unital monoidal left \( \mathcal{A}^{\text{oplax}} \)-module that is enriched in \( \mathcal{A} \), and \( \mathcal{M} \) is a strongly unital monoidal left \( \mathcal{B}^{\text{oplax}} \)-module that is enriched in \( \mathcal{B} \). Then the canonical construction gives enriched monoidal categories \( ^\mathcal{L}\mathcal{L} \) and \( ^\mathcal{B}\mathcal{M} \). Suppose \( \hat{F} : \mathcal{A} \to \mathcal{B} \) is a braided monoidal functor and \( F : \mathcal{L} \to \mathcal{M} \) is both a monoidal functor and an \( \hat{F} \)-lax functor. Then \( \hat{F} \) is a monoidal \( F \)-lax functor if and only if \( \hat{F} \) and \( F \) define an enriched monoidal functor \( \hat{F} : ^\mathcal{L}\mathcal{L} \to ^\mathcal{B}\mathcal{M} \).

Proof. By Theorem 3.31, the \( \hat{F} \)-lax functor \( F \) defines an enriched functor \( \hat{F} : ^\mathcal{A}\mathcal{L} \to ^\mathcal{B}\mathcal{M} \). Then we only need to show that \( \hat{F} \) is a monoidal \( F \)-lax functor if and only if \( (\hat{F}^2, F^2) \) and \( (\hat{F}^0, F^0) \) are 2-morphisms in \( \text{LMod} \). It is routine to check that \( F^2 \) is a 2-morphism in \( \text{LMod} \) if and only if the diagram (4.7) commutes, and \( (\hat{F}^0, F^0) \) is a 2-morphism in \( \text{LMod} \) if and only if the diagram (4.8) commutes. \( \blacksquare \)

Definition 4.23. Let \( \mathcal{A}, \mathcal{B} \) be two braided monoidal categories. Let \( \mathcal{L} \) be a monoidal left \( \mathcal{A}^{\text{oplax}} \)-module and \( \mathcal{M} \) be a monoidal left \( \mathcal{B}^{\text{oplax}} \)-module. Suppose \( \hat{F}_i : \mathcal{A} \to \mathcal{B} \) is a braided monoidal functor and \( F_i : \mathcal{L} \to \mathcal{M} \) is a monoidal \( \hat{F}_i \)-lax functor, \( i = 1, 2 \). Given a monoidal natural transformation \( \xi : \hat{F}_1 \Rightarrow \hat{F}_2 \), a monoidal \( \hat{F}_1 \)-lax natural transformation \( \hat{\xi} : F_1 \Rightarrow F_2 \) is a \( \hat{\xi} \)-lax natural transformation and also a monoidal natural transformation.

Remark 4.24. For \( i = 1, 2 \), let \( \mathcal{L}_i \) be a monoidal left \( \mathcal{A}_i^{\text{oplax}} \)-module induced by a braided oplax-monoidal functor \( \phi_i : \mathcal{A}_i \to ^{\mathcal{L}_i}\mathcal{L}_i \). Assume that \( \hat{F}_i : \mathcal{A}_i \to \mathcal{B}_i \) is a braided monoidal functor and \( F_i : \mathcal{L}_i \to \mathcal{M}_i \) is a monoidal \( \hat{F}_i \)-lax functor. Let \( \hat{\xi} : \hat{F}_1 \Rightarrow \hat{F}_2 \) and \( \xi : F_1 \Rightarrow F_2 \) be two monoidal natural transformations. Then \( \xi \) is a monoidal \( \hat{\xi} \)-lax natural transformation if and only if the following diagram commutes,

\[
\begin{array}{c}
\phi_2^{\mathcal{L}}(\hat{F}_1(a)) \downarrow \xi \\
\phi_2^{\mathcal{M}}(\hat{F}_2(a)) \downarrow \xi
\end{array}
\]

where \( \theta_1 : \phi_2^{\mathcal{L}} \circ \hat{F}_1 \Rightarrow F_1 \circ \phi_1^{\mathcal{L}} \) is the oplax-monoidal natural transformation that induces the monoidal \( \hat{F}_1 \)-lax module structure on \( F_1 \) (see Remark 4.21). \( \diamond \)

The proof of the following proposition is clear.

Proposition 4.25. Under the assumption of Proposition 4.22 let \( \hat{F}_i : \mathcal{A} \to \mathcal{B} \) be a braided monoidal functor and \( F_i : \mathcal{L} \to \mathcal{M} \) be a monoidal \( \hat{F}_i \)-lax functor, \( i = 1, 2 \). Then \( (\hat{F}_i, F_i) \) defines an enriched monoidal functor \( \hat{F}_i : ^\mathcal{L}\mathcal{L} \to ^\mathcal{B}\mathcal{M} \) for \( i = 1, 2 \). Suppose \( \hat{\xi} : \hat{F}_1 \Rightarrow \hat{F}_2 \) is a monoidal natural transformation and \( \xi : F_1 \Rightarrow F_2 \) is a \( \hat{\xi} \)-lax natural transformation. Then \( \xi \) is a monoidal \( \hat{\xi} \)-lax natural transformation if and only if \( (\hat{\xi}, \xi) \) defines an enriched monoidal natural transformation.

We use \( \text{ECat}^{\otimes} \) to denote the 2-category of enriched monoidal categories, enriched monoidal functors and enriched monoidal natural transformations. It is symmetric monoidal with the tensor product given by the Cartesian product \( \times \) and the tensor unit given by \( \ast \).

We define the 2-category \( \text{LMod}^{\otimes} \) as follows:

- The objects are pairs \( (\mathcal{A}, \mathcal{L}) \), where \( \mathcal{A} \) is a braided monoidal category and \( \mathcal{L} \) is a strongly unital monoidal left \( \mathcal{A}^{\text{oplax}} \)-module that is enriched in \( \mathcal{A} \).

- A 1-morphism \( (\mathcal{A}, \mathcal{L}) \to (\mathcal{B}, \mathcal{M}) \) is a pair \( (\hat{F}, F) \), where \( \hat{F} : \mathcal{A} \to \mathcal{B} \) is a braided monoidal functor and \( F : \mathcal{L} \to \mathcal{M} \) is a monoidal \( \hat{F} \)-lax functor.

- A 2-morphism \( (\hat{F}, F) \Rightarrow (\hat{G}, G) \) is a pair \( (\hat{\xi}, \xi) \), where \( \hat{\xi} : \hat{F} \Rightarrow \hat{G} \) is a monoidal natural transformation and \( \xi : F \Rightarrow G \) is a monoidal \( \hat{\xi} \)-lax natural transformation.

The horizontal/vertical composition is induced by the horizontal/vertical composition of functors and natural transformations. The 2-category \( \text{LMod}^{\otimes} \) is symmetric monoidal with the tensor product defined by the Cartesian product and the tensor unit given by \( (\ast, \ast) \).

By Proposition 4.17, Proposition 4.22 and Proposition 4.25 we obtain the following result.

Theorem 4.26. The canonical construction defines a symmetric monoidal 2-functor from \( \text{LMod}^{\otimes} \) to \( \text{ECat}^{\otimes} \). Moreover, this 2-functor is locally isomorphic.
4.3 The category of enriched endo-functors

Let $\mathcal{B}$ be a $\mathcal{B}$-enriched category. We use $\text{Fun}(\mathcal{B}, \mathcal{M})$ to denote the category of $\mathcal{B}$-functors from $\mathcal{B}$ to itself and $\mathcal{B}$-natural transformations between them (recall Definition 4.10). For every $[F,G] \in \text{Fun}(\mathcal{B}, \mathcal{M})$, we define a category $\mathcal{P}_{(F,G)}$ as follows:

- The objects are pairs $(a, [a] \in \mathcal{M})$, where $a \in \mathcal{Z}(\mathcal{B})$ and $a : I(a) \to \mathcal{M}(F(x), G(x))$ are morphisms in $\mathcal{B}$ such that the following diagram

$$
\begin{array}{c}
\mathcal{B}(x,y) \\
\downarrow a \otimes F_{r,s} \\
\mathcal{B}(F(y), G(y)) \\
\downarrow G_{x,y} \otimes a \\
\mathcal{B}(F(x), G(x))
\end{array}
\quad (4.10)
$$

commutes, where the unlabeled arrow is given by the half-braiding of $a$.

- A morphism $f : (a, [a]) \to (b, [b])$ is a morphism in $\mathcal{Z}(\mathcal{B})$ such that the following diagram

$$
\begin{array}{c}
\mathcal{B}(F(x), G(x)) \\
\downarrow \xi \\
\mathcal{B}(F(y), G(y))
\end{array}
\quad (4.11)
$$

commutes for every $x \in \mathcal{M}$.

**Definition 4.27.** We say $\mathcal{B}$ satisfies the condition (4) if:

for every $[F,G] \in \text{Fun}(\mathcal{B}, \mathcal{M})$, the category $\mathcal{P}_{(F,G)}$ has a terminal object. (*

We denote the terminal object of $\mathcal{P}_{(F,G)}$ by $([F,G], ([F,G])_{x \in \mathcal{M}})$.

**Remark 4.28.** A family of morphisms $(\xi_{x} : 1_{B} \to \mathcal{B}(F(x), G(x)))$ renders the diagram (4.10) commutative (with $I(a), a_{x}$ replaced by $1_{B}, \xi_{x}$, respectively) if and only if the family of morphisms $\{\xi_{x}\}$ defines a $\mathcal{B}$-natural transformation $\xi : F \Rightarrow G$. In particular, if $\mathcal{B}$ satisfies the condition (4), then the hom set $\mathcal{Z}(\mathcal{B})_{x \in \mathcal{M}}([F,G], [F,G])$ is isomorphic to the hom set $\text{Fun}(\mathcal{B}, \mathcal{M})([F,G], [F,G])$.

**Remark 4.29.** Let $(A, m, i)$ be an algebra in a monoidal category $\mathcal{A}$. By Example 3.11, $\text{Fun}(\mathcal{A}, \mathcal{A})$ is the category defined as follows:

- The objects are algebra homomorphisms $f : A \to A$.

- A morphism $f \to g$ is a morphism $\xi \in \mathcal{A}(1, A)$ such that the following diagram commutes.

$$
\begin{array}{c}
A \xrightarrow{i} A \otimes A \\
\downarrow \xi \otimes f \\
A \otimes 1 \xrightarrow{m} A
\end{array}
$$

It is also clear that the object $[1, \alpha, 1, \alpha]$ (if exists) is the full center of $A$ (see [Dav10] for the definition of the full center of an algebra).

**Lemma 4.30.** There are two well-defined functors.

1. $\mathcal{P}_{(G,H)} \times \mathcal{P}_{(F,H)} \to \mathcal{P}_{(F,G)}$ is defined by $((b, [b]), (a, [a])) \mapsto (b \otimes a, [b_{1} \circ [a_{1}])$, where $b_{1} \circ [a_{1}]$ is defined by the composed morphism:

$$
I(b \otimes a) = I(b) \otimes I(a) \xrightarrow{b \otimes \xi} \mathcal{B}(G(x), H(x)) \otimes \mathcal{B}(F(x), G(x)) \xrightarrow{\xi} \mathcal{B}(F(x), H(x)).
$$

Since $[F,H]$ is terminal in $\mathcal{P}_{(F,H)}$, we obtained a canonical morphism $b \otimes a \to [F,H]$. 25
(2) \( P_{(F',G)} \times P_{(F,G')} \to P_{(F',G')} \) defined by \((a, [a_1]) \otimes (a', [a'_1]) \mapsto (a \otimes a', [a_1] \bullet [a'_1])\), where \([a_1] \bullet [a'_1]\) is defined by the composed morphism:

\[
a \otimes a' \xrightarrow{a \otimes a' F_{G_0}(\text{id}_{a(a_1)})} 3\mathbb{M}(FG'(x), GG'(x)) \otimes 3\mathbb{M}(FF'(x), FG'(x)) \xrightarrow{\delta} 3\mathbb{M}(FF'(x), GG'(x)).
\]

Since \([FF', GG']\) is terminal in \( P_{(F,G')} \), we obtain a canonical morphism \( a \otimes a' \to [FF', GG']\).

**Proposition 4.31.** If \( 3\mathbb{M} \) satisfies the condition (3), then \( \text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M}) \) can be promoted to a strict \( 3(\mathbb{B}) \)-enriched monoidal category \( 3(\mathbb{B})\text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M}) \). More explicitly,

- the hom objects are \( 3(\mathbb{B})\text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M})(F,G) := [F, G] \);
- the identity morphism \( 1_{3\mathbb{B}} \to [F, F] \) is induced by the identity enriched natural transformation of \( F \) via the universal property of \( [F, F] \);
- the composition morphism \([G, H] \otimes [F, G] \to [F, H] \) is defined Lemma 4.30 (1);
- the tensor product morphism \([F, G] \otimes [F', G'] \to [FF', GG'] \) is defined by Lemma 4.30 (2).

**Proof.** It is routine to check that \( 3(\mathbb{B})\text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M}) \) is an \( 3(\mathbb{B}) \)-enriched category. Note that the underlying category of \( 3(\mathbb{B})\text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M}) \) is precisely \( \text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M}) \) by Remark 4.28. This explains the notation.

It remains to show that \( 3(\mathbb{B})\text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M}) \) is monoidal. We claim the following equation holds:

\[
\left( 3\mathbb{M}(F_2, G_3) \otimes [F_2, G_3] \otimes X \to [F_2, F_3] \otimes [F_1, F_2] \otimes [G_1, G_2] \right) \\
\xrightarrow{\delta} 3\mathbb{M}(F_1, F_2) \otimes [G_1, G_2] \to [F_1, F_3] \otimes [G_1, G_3] \\
\xrightarrow{\delta} 3\mathbb{M}(F_1, G_1, F_3, G_3) \\
= (3\mathbb{M}(F_2, G_3) \otimes [F_2, G_3] \otimes X \to [F_2, F_3] \otimes [F_1, F_2] \otimes [G_1, G_2] \to [F_2, G_1, F_3, G_3] \otimes [F_1, G_1, F_2, G_2] \to [F_1, G_1, F_3, G_3])
\]

where the first unlabeled arrow is induced by the half braiding of \([F_1, F_2]\). Indeed, by the definition of \([G, H] \otimes [F, G] \to [F, H] \) and \([F, G] \otimes [F', G'] \to [FF', GG'] \), it is not hard to check that both sides of the above equation are equal to

\[
\left( 3\mathbb{M}(G_2, G_3) \otimes [G_2, G_3, X] \to [G_2, F_3] \otimes [F_1, F_2] \otimes [G_1, G_2] \right) \\
\xrightarrow{\delta} 3\mathbb{M}(G_1, G_2) \otimes [G_1, G_2] \\
\xrightarrow{\delta} 3\mathbb{M}(G_2, G_3) \otimes [G_2, G_3, X] \to [G_2, F_3] \otimes [F_1, F_2] \otimes [G_1, G_2] \to [G_2, F_1, F_2, G_3] \otimes [F_1, G_1, F_2, G_2] \xrightarrow{\delta} [F_1, G_1, F_3, G_3] \\
\xrightarrow{\delta} 3\mathbb{M}(F_1, G_1, F_3, G_3).
\]

Similar, we can check that

\[
1_{FG(x)} = (1 \cdot 1 \xrightarrow{1 \otimes 1} [F, F] \otimes [G, G] \to [FG, FG] \xrightarrow{[FG, FG]} 3\mathbb{M}(FG(x), FG(x))).
\]

Therefore the morphisms \([F, G] \otimes [F', G'] \to [FF', GG'] \) define an enriched functor. Clearly the underlying monoidal category of \( 3(\mathbb{B})\text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M}) \) is \( \text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M}) \).

In the rest of this subsection, we discuss the construction of \( 3(\mathbb{B})\text{Fun}^{3\mathbb{M}}(3\mathbb{M}, 3\mathbb{M}) \) when \( 3\mathbb{M} = \mathbb{M} \) for \( (\mathbb{B}, \mathbb{M}) \in \text{LMod} \) and \( \mathbb{M} \) is a left \( \mathbb{B} \)-module. In particular, in this case, we give an easy-to-check condition that is equivalent to the condition (3) (see Lemma 4.32).

By Theorem 3.31 we can identify \( \text{Fun}^{3\mathbb{M}}(\mathbb{M}, \mathbb{M}) \) with \( \text{Fun}_{\mathbb{M}^L}(\mathbb{M}, \mathbb{M}) \), i.e., the category of lax \( \mathbb{B} \)-module functors and \( \mathbb{B} \)-module natural transformations. There is an obvious monoidal functor from \( 3(\mathbb{B}) \) into \( \text{Fun}_{\mathbb{M}^L}(\mathbb{M}, \mathbb{M}) \) which maps \( a \) to the \( \mathbb{B} \)-module functor \( I(a) \otimes - \). It is not hard to
check that $\text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M})$ is a strongly unital monoidal left $\mathcal{Z}_{1}(\mathcal{B})$-module with the module action induced by the monoidal functor $\mathcal{Z}_{1}(\mathcal{B}) \to \text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M})$ and the natural transformation

\[
I(a \otimes b) \circ FG(-) \sim I(a) \circ [I(b) \circ FG(-)] \to I(a) \circ F(I(b) \circ G(-)),
\]

where the first arrow is induced by the monoidal structure of $I$ and the left $\mathcal{B}$-module structure of $M$ and the second arrow is induced by the lax $\mathcal{B}$-module structure of $F$.

**Lemma 4.32.** When $(\mathcal{B}, \mathcal{M}) \in \text{LMod}$ and $\mathcal{M}$ is a left $\mathcal{B}$-module, $\mathcal{M}$ satisfies the condition $\mathcal{Z}_{2}$ if and only if $\text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M})$ is enriched in $\mathcal{Z}_{1}(\mathcal{B})$. In this case, the object $[F, G] \in \mathcal{Z}_{1}(\mathcal{B})$, and the morphism $[F, G]_{x} : [F, G] \to \mathcal{M}(F(x), G(x))$ is induced by $I([F, G]) \circ F(x) = ([F, G] \circ F)(x) \xrightarrow{(\text{ev}_{I})} G(x)$.

**Proof.** Since both $([F, G], [F, G]_{x})_{x \in M}$ and the internal hom $([F, G], \text{ev}_{F} = \{\text{ev}_{F}x\}_{x \in M})$ are terminal objects, it suffices to show that they satisfy the same universal property. Let $F, G \in \text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M})$ and $a \in \mathcal{Z}_{1}(\mathcal{B})$. Given a family of morphisms $\{\alpha_{x} : I(a) \to [F(x), G(x)]\}_{x \in M}$, define

\[
\tilde{\alpha}_{x} := \left(I(a) \circ F(x) \xrightarrow{\alpha_{x} \circ \mathcal{Z}_{1}} [F(x), G(x)] \circ F(x) \xrightarrow{\text{ev}_{I}} G(x)\right).
\]

Conversely,

\[
\alpha_{x} = \left(I(a) \xrightarrow{\text{coev}_{I}} [F(x), I(a) \circ F(x)] \xrightarrow{\mathcal{Z}_{1}1} [F(x), G(x)]\right).
\]

It is routine to check that $\{\alpha_{x}\}$ renders the diagram (4.10) commutative if and only if the following diagram commutes,

\[
\begin{array}{c}
(b \otimes I(a)) \circ F(x) \xrightarrow{\gamma_{a} \circ \mathcal{Z}_{1}} (I(a) \otimes b) \circ F(x) \xrightarrow{\alpha} I(a) \circ (b \circ F(x)) \\
\end{array}
\]

\[
\begin{array}{c}
\downarrow \gamma_{a} \circ \mathcal{Z}_{1} \\
\downarrow \mathcal{Z}_{1}1 \\
(b \otimes (I(a) \circ F(x)) \xrightarrow{1 \otimes \mathcal{Z}_{1}1} I(a) \circ (b \circ x) \xrightarrow{\alpha_{x}} G(b \circ x)
\end{array}
\]

(4.13)

i.e. $\tilde{\alpha} : a \circ F \Rightarrow G$ is a $\mathcal{B}$-module natural transformation. \hfill \Box

If $\mathcal{M}$ satisfies the condition $\mathcal{Z}_{2}$, by Proposition 4.31, $\text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M})$ can be promoted to a $\mathcal{Z}_{1}(\mathcal{B})$-enriched monoidal category $\mathcal{Z}_{1}(\mathcal{B})$-$\text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M})$. We obtain the following result.

**Proposition 4.33.** When $(\mathcal{B}, \mathcal{M}) \in \text{LMod}$ and $\mathcal{M}$ is a left $\mathcal{B}$-module, if $\text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M})$ is enriched in $\mathcal{Z}_{1}(\mathcal{B})$, then we obtain a monoidal equivalence:

\[
\mathcal{Z}_{1}(\mathcal{B})$-$\text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M}) \cong \mathcal{Z}_{1}(\mathcal{B})$-$\text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M}) \cong \mathcal{Z}_{1}(\mathcal{B})$-$\text{Fun}_{\mathcal{B}}^{\text{lax}}(\mathcal{M}, \mathcal{M}).
\]

**Proof.** By Lemma 4.32 we can set $[F, G] = [F, G]$. It suffices to verify (1) and (2).

(1) The composition $[G, \mathcal{I}] \otimes [F, \mathcal{I}] \to [F, \mathcal{I}]$ defined in Lemma 4.30 coincides with the canonical morphism $[G, \mathcal{I}] \otimes [F, \mathcal{I}] \to [F, \mathcal{I}]$. It is because both morphisms are induced from the same universal property according to Lemma 4.32.

(2) The tensor product morphism $[F, \mathcal{I}G] \otimes [F', \mathcal{I}G'] \to [F\mathcal{I}F', \mathcal{I}G'G]$ defined in Lemma 4.30 is induced by

\[
[F, G] \otimes [F', G'] \circ FF' \to [F, G] \circ F \otimes [F', G'] \circ F' \xrightarrow{\text{ev}_{I} \otimes \text{ev}_{I'}} GG',
\]

where the first arrow is induced by the lax $\mathcal{B}$-module functor structure of $F$. 

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In other words, we need to show the outer subdiagram of the following diagram

\[
\begin{array}{ccc}
[F,G] \otimes [F',G'] \otimes FF'(x) & \xrightarrow{1_{[F',G']}} & [F,G] \otimes [F'(x),G'(x)] \otimes FF'(x) \\
\downarrow{1_{(ev_{F'})}} & & \downarrow{1_{(ev_{F'}')}} \\
[F,G] \otimes F([F',G'] \otimes F'(x)) & \xrightarrow{1_{F[ev_{F'}]}} & [F,G] \otimes F([F'(x),G'(x)] \otimes F'(x)) \\
\downarrow{1_{[F,ev_{G'}]}} & & \downarrow{1_{F[ev_{F'}]}} \\
[F,G] \otimes FG'(x) & \xrightarrow{1_{[F,ev_{G'}]}} & GG'(x) \\
\downarrow{1_{(ev_{G'})}} & & \downarrow{ev_{G'}} \\
[F,G] \otimes [F'(x),G'(x)] \otimes FG'(x) & & \\
\end{array}
\]

commutes for each \(x \in M\). The upper left quadrangle commutes due to the naturality of the lax \(\mathcal{B}\)-module functor structure of \(F\), and the other subdiagrams commute by the adjunctions associated to internal homs. \(\square\)

When \(\mathcal{B}\) is rigid, we have \(\text{Fun}^{\text{lax}}_{\mathcal{B}}(\mathcal{M}, \mathcal{M}) = \text{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{M})\), i.e. the category of \(\mathcal{B}\)-module functors, because the morphism \(\eta : 1_{\mathcal{M}} \otimes \mathcal{M} \rightarrow \mathcal{M}\) now has an inverse given by \(F(a \otimes x) \rightarrow F(a \otimes x) \rightarrow a \otimes F(a \otimes x) \rightarrow a \otimes x\) for \(a \in \mathcal{B}, x \in \mathcal{M}\).

**Corollary 4.34.** When \((\mathcal{B}, \mathcal{M}) \in \text{LMod}\) and \(\mathcal{M}\) is a left \(\mathcal{B}\)-module, if \(\mathcal{B}\) is rigid and \(\text{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{M})\) is enriched in \(\mathcal{E}\), we have

\[
\text{Fun}^{\mathcal{B}}(\mathcal{M}, \mathcal{M}) \simeq \text{Fun}_{\mathcal{B}}(\mathcal{M}, \mathcal{M}).
\]

### 4.4 The \(E_0\)-centers of enriched categories in \(\text{ECat}\)

In this subsection, we prove that \(\text{Fun}^{\mathcal{B}}(\mathcal{M}, \mathcal{M})\) is the \(E_0\)-center of \(\mathcal{M}\) in \(\text{ECat}\) when \(\mathcal{M}\) satisfies the condition (3).

Recall that an \(E_0\)-algebra in \(\text{ECat}\) is a pair \((\mathcal{A}, U)\), where \(U : * \rightarrow \mathcal{A}\) is an enriched functor in \(\text{ECat}\). A left unital \((\mathcal{A}, U)\)-action on \(\mathcal{M}\) in \(\text{ECat}\) consists of an enriched functor \(\eta : \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M}\) in \(\text{ECat}\) and an enriched natural isomorphism \(\xi\) as depicted in the following diagram (recall Definition (2.9)).

\[
\begin{array}{ccc}
* \times \mathcal{M} & \xrightarrow{\eta} & \mathcal{M} \\
\downarrow{\xi} & & \downarrow{\xi} \\
\mathcal{M} \times \mathcal{M} & \xrightarrow{\eta \times 1_{\mathcal{M}}} & \mathcal{M} \\
\downarrow{1_{\mathcal{M}}} & & \downarrow{1_{\mathcal{M}}} \\
\mathcal{M} & \xrightarrow{1_{\mathcal{M}}} & \mathcal{M}
\end{array}
\]

We set \(1_{\mathcal{L}} := U(\ast) \in \mathcal{L}\). There exists a canonical enriched natural isomorphism \(\eta : 1_{\mathcal{L}} \Rightarrow U\) (recall Example (3.12) defined by \(1_{\mathcal{A}} \rightarrow U(\ast)\) (from the monoidal structure of \(U\)) and the identity underlying natural transformation. It defines a left unital \((\mathcal{A}, U)\)-action as depicted in the following diagram.

\[
\begin{array}{ccc}
* \times \mathcal{M} & \xrightarrow{\eta} & \mathcal{M} \\
\downarrow{\xi} & & \downarrow{\xi} \\
\mathcal{L} \times \mathcal{M} & \xrightarrow{\eta \times 1_{\mathcal{M}}} & \mathcal{M} \\
\downarrow{1_{\mathcal{L}}} & & \downarrow{1_{\mathcal{M}}} \\
\mathcal{M} & \xrightarrow{1_{\mathcal{M}}} & \mathcal{M}
\end{array}
\]

In particular, the left unital \((\mathcal{A}, 1_{\mathcal{L}})\)-action is isomorphic to that of \((\mathcal{A}, U)\)-action. Therefore, in order to show that \(\text{Fun}^{\mathcal{B}}(\mathcal{M}, \mathcal{M})\) is the \(E_0\)-center of \(\mathcal{M}\) in \(\text{ECat}\), it is enough to only consider the left unital \((\mathcal{A}, 1_{\mathcal{L}})\)-action on \(\mathcal{M}\) in \(\text{ECat}\) as depicted in the following diagram.

\[
\begin{array}{ccc}
* \times \mathcal{M} & \xrightarrow{\eta} & \mathcal{M} \\
\downarrow{\xi} & & \downarrow{\xi} \\
\mathcal{L} \times \mathcal{M} & \xrightarrow{\eta \times 1_{\mathcal{M}}} & \mathcal{M} \\
\downarrow{1_{\mathcal{L}}} & & \downarrow{1_{\mathcal{M}}} \\
\mathcal{M} & \xrightarrow{1_{\mathcal{M}}} & \mathcal{M}
\end{array}
\]
Example 4.35. If \( B^{1}M \) satisfies the condition (3), then there exists a canonical left unital action

\[
\begin{array}{c}
\text{Fun}^{B}(M, B^{1}M) \times B^{1}M \\
\downarrow \text{ev} \\
\times B^{1}M \\
\downarrow \alpha
\end{array}
\]

of the \( E_{0} \)-algebra \( (3, \text{Fun}^{B}(M, B^{1}M), 1_{\text{em}}) \) on \( B^{1}M \) in \( \text{ECat} \) defined as follows:

- The enriched functor \( \text{ev} \) is defined by the monoidal functor \( \text{ev} : \otimes : 3_{1}(B) \times B \rightarrow B \), the map of objects \( (F, x) \mapsto F(x) \) and the family of morphisms

\[
\text{ev}_{(F(x), G(y))} : (l(F, G)) \otimes B^{1}(M(x, y)) \xrightarrow{\otimes \text{Fun}(x, y)} B^{1}(M(F(x), G(y))) \otimes B^{1}(M(F(x), y));
\]

- The background changing natural transformation \( \hat{\alpha} \) of \( \alpha \) is given by the left unitor of \( B \) and the underlying natural transformation \( \alpha \) of \( \alpha \) is given by the identity natural transformation. ✤

It is routine to check the following fact. The proof is omitted.

Lemma 4.36. Let \( (\otimes, \mu) \) be a left unital \( (A^{1}L, 1_{L}) \)-action of the \( E_{0} \)-algebra on \( B^{1}M \) as depicted in the diagram (4.15). Then there exists a functor \( \Phi : L \rightarrow \text{Fun}^{B}(M, B^{1}M) \) defined as follows:

- For each \( a \in L \), \( \Phi(a) \) is the \( B \)-functor defined by the map \( x \mapsto a \otimes x \) and the family of morphisms

\[
B^{1}(M(x, y)) \xrightarrow{\ell_{x}} \mathbb{1}_{B} \hat{\otimes} B^{1}(M(x, y)) \xrightarrow{1_{B} \otimes \mu} A^{1}L(a, b) \hat{\otimes} B^{1}(M(x, y)) \xrightarrow{\|_{\text{Fun}(a, x)}} B^{1}(M(a \otimes x, a \otimes y)).
\]

- For each morphism \( f : \mathbb{1}_{A} \rightarrow A^{1}L(a, b) \) in \( L \), \( \Phi(f) \) is the \( B \)-natural transformation defined by the family of morphisms

\[
\mathbb{1}_{B} \xrightarrow{\ell_{1}} \mathbb{1}_{B} \hat{\otimes} \mathbb{1}_{B} \xrightarrow{f_{B}} A^{1}L(a, b) \hat{\otimes} B^{1}(M(x, x)) \xrightarrow{\|_{\text{Fun}(a, x)}} B^{1}(M(a \otimes x, b \otimes x)).
\]

Theorem 4.37. If \( B^{1}M \) satisfies the condition (3), then \( 3(\text{Fun}^{B}(M, B^{1}M)) \) is the \( E_{0} \)-center of \( B^{1}M \) in \( \text{ECat} \), i.e. \( 3_{0}(B^{1}M) = 3(\text{Fun}^{B}(M, B^{1}M)) \).

Proof. Let \( (\otimes, \mu) \) be a left unital action of an \( E_{0} \)-algebra \( (A^{1}L, 1_{L}) \) on \( B^{1}M \) as depicted in the diagram (4.15). We first show that there exist an enriched functor \( \Phi : A^{1}L \rightarrow 3(\text{Fun}^{B}(M, B^{1}M)) \) in \( \text{ECat} \) and two enriched natural isomorphisms \( \sigma \) and \( \rho \) such that the following pasting diagrams

\[
\begin{array}{c}
\xymatrix{ \text{Fun}^{B}(M, B^{1}M) \times B^{1}M \\
\downarrow \text{ev} \\
\times B^{1}M \\
\downarrow \alpha}
\end{array} = \begin{array}{c}
\xymatrix{ \text{Fun}^{B}(M, B^{1}M) \times B^{1}M \\
\downarrow \text{ev} \\
\times B^{1}M \\
\downarrow \alpha}
\end{array}.
\]

are equal in \( \text{ECat} \), where the right hand side of the equation is the canonical left unital action of \( 3(\text{Fun}^{B}(M, B^{1}M), 1_{\text{em}}) \) on \( B^{1}M \) defined in Example 4.35.

- Let \( \Phi : L \rightarrow \text{Fun}^{B}(M, B^{1}M) \) be the functor described in Lemma 4.36. Then \( \Phi \) can be promoted to an enriched functor \( \Phi : A^{1}L \rightarrow 3(\text{Fun}^{B}(M, B^{1}M)) \) in \( \text{ECat} \) with the background changing functor \( \hat{\Phi} : A \rightarrow 3_{1}(B) \) given by \( \hat{\otimes} \mathbb{1}_{B} \) (see Example 2.16) and the family of morphisms \( \|_{\Phi(a, x)} : A^{1}L(a, b) \hat{\otimes} B^{1}(M(x, x)) \rightarrow [\Phi(a), \Phi(b)] \) given by the unique morphisms rendering the following diagrams commutative.

\[
\begin{array}{c}
\xymatrix{ I(A^{1}L(a, b) \hat{\otimes} B^{1}(M(x, x))) \\
\downarrow I(1_{A^{1}L}) \\
I[\Phi(a), \Phi(b)]}
\end{array}
\]
The enriched natural isomorphism \( \delta \) is defined by the background changing natural transformation \( \delta = \tilde{e}^{-1}_\beta : \mathbb{I}_B \to \mathbb{I}_A \hat{\otimes} \mathbb{I}_B \) and the underlying natural transformation given by the unique morphism \( \sigma_\beta : \mathbb{I}_B \to [1_{\mathbb{E}_1 \Phi(\mathbb{I}_A \otimes \mathbb{I}_B) \mathbb{I}_M}] \) rendering the following diagram commutative.

\[
\begin{array}{c}
I(\mathbb{I}_B) \\
\downarrow \mathbb{E}_1^{-1} \\
I([1_{\mathbb{E}_1 \Phi(\mathbb{I}_A \otimes \mathbb{I}_B) \mathbb{I}_M}]) \\
\downarrow \beta \mathbb{I}_M(\mathbb{I}_\mathbb{L} \otimes \mathbb{I}_x) \\
\end{array}
\]

The enriched natural isomorphism \( \rho \) is defined by the background changing natural transformation

\[
I(\hat{\mathbb{G}}(-)) \otimes - = (\hat{\mathbb{G}} \mathbb{I}_B) \otimes - \xrightarrow{1_{\mathbb{E}_1} \mathbb{E}_1^{-1}} (\hat{\mathbb{G}} \mathbb{I}_B) \otimes (\mathbb{I}_A \hat{\otimes} \mathbb{I}_B \otimes -) \xrightarrow{\hat{\mathbb{G}}} (- \otimes \mathbb{I}_B \hat{\otimes} -) \xrightarrow{\hat{\mathbb{G}}} - \otimes -
\]

and the identity underlying natural transformation. Then it is not hard to check that the equation (4.17) holds.

Let \( (\Phi_i, \sigma_i, \rho_i), i = 1, 2 \), be two triples such that the similar equations as depicted by (4.17) hold. We only need to show that there exists a unique enriched natural isomorphism \( \hat{\beta} : \Phi_1 \Rightarrow \Phi_2 \) such that

\[
\begin{array}{ccc}
\begin{array}{c}
\text{Fun}(\mathbb{I}_M, \mathbb{I}_M) \\
\downarrow \mathbb{I}_L \end{array} & \xrightarrow{3(i)(\mathbb{I}_M, \mathbb{I}_M)} & \begin{array}{c}
\text{Fun}(\mathbb{I}_M, \mathbb{I}_M) \\
\downarrow \mathbb{I}_L \end{array} \\
\mathbb{I}_M & \xrightarrow{\Phi_1 \otimes \Phi_2} & \mathbb{I}_M \\
\end{array}
\end{array}
\]

Since \( 3_1(\mathbb{B}) \) is the \( E_0 \)-center of \( \mathbb{B} \) in \( \text{Alg}_{\mathbb{E}_1}(\mathbb{C}) \) (see Example 2.16), there exists a unique monoidal natural isomorphism \( \hat{\beta} : \Phi_1 \Rightarrow \Phi_2 \) such that

\[
\begin{array}{ccc}
\begin{array}{c}
\text{Fun}(\mathbb{I}_M, \mathbb{I}_M) \\
\downarrow \mathbb{I}_L \end{array} & \xrightarrow{3(i)(\mathbb{I}_M, \mathbb{I}_M)} & \begin{array}{c}
\text{Fun}(\mathbb{I}_M, \mathbb{I}_M) \\
\downarrow \mathbb{I}_L \end{array} \\
\mathbb{I}_M & \xrightarrow{\Phi_1 \otimes \Phi_2} & \mathbb{I}_M \\
\end{array}
\end{array}
\]

Define \( \beta_\sigma : \mathbb{I}_B \to [\Phi_1(\mathbb{I}_A), \Phi_2(\mathbb{I}_A)] \) to be the unique morphism rendering the following diagram commutative. Then the enriched natural isomorphism \( \hat{\beta} \) define by \( (\hat{\beta}, \{\beta_\sigma\}) \) is the unique enriched natural isomorphism such that the equations (4.18) and (4.19) hold.

\[
\begin{array}{c}
\begin{array}{ccc}
\begin{array}{c}
\mathbb{I}_M \\
\downarrow \mathbb{I}_L \end{array} & \xrightarrow{3(i)(\mathbb{I}_M, \mathbb{I}_M)} & \begin{array}{c}
\mathbb{I}_M \\
\downarrow \mathbb{I}_L \end{array} \\
\mathbb{I}_M & \xrightarrow{\Phi_1 \otimes \Phi_2} & \mathbb{I}_M \\
\end{array} \\
\end{array}
\end{array}
\]

Remark 4.38. It is straightforward to check that the monoidal structure of \( 3(i)(\mathbb{I}_M, \mathbb{I}_M) \) coincides with the \( E_1 \)-algebra structure of the \( E_0 \)-center of \( \mathbb{I}_M \) in \( \text{ECat} \) (see Proposition 2.14).

The following result follows immediately from Corollary 4.34 and Theorem 4.37.

Corollary 4.39. When \( (\mathbb{B}, \mathbb{M}) \in \text{LMod} \) and \( \mathbb{M} \) is a left \( \mathbb{B} \)-module, if \( \mathbb{B} \) is rigid and \( \text{Fun}_\mathbb{B}(\mathbb{M}, \mathbb{M}) \) is enriched in \( 3_1(\mathbb{B}) \), we have

\[
3_0(\mathbb{B}, \mathbb{M}) \simeq 3(i)(\mathbb{I}_M, \mathbb{I}_M).
\]

Remark 4.40. Corollary 4.39 has an important application in physics. In particular, it provides a mathematical foundation to Definition 3.18 and Remark 3.19 in [KZ21].
5 Enriched braided monoidal categories

5.1 Definitions and examples

For any two enriched categories \( \mathcal{A} \) and \( \mathcal{B} \), there is an obvious switching enriched functor \( \Sigma : \mathcal{A} \times \mathcal{B} \to \mathcal{B} \times \mathcal{A} \). It is clear that \( \Sigma \) has an obvious monoidal structure if \( \mathcal{A} \) and \( \mathcal{B} \) are enriched monoidal categories.

**Definition 5.1.** An enriched braided monoidal category consists of the following data:

- an enriched monoidal category \( \mathcal{A} \);
- an enriched natural transformation \( \hat{\epsilon} : \otimes \to \otimes \circ \Sigma \);

such that

1. The background category \( \mathcal{A} \) is symmetric.
2. The background changing natural transformation \( \hat{\epsilon} \) is equal to the braiding of \( \mathcal{A} \).
3. The underlying monoidal category \( \mathcal{L} \) equipped with the underlying natural transformation \( \epsilon \) of \( \hat{\epsilon} \) is a braided monoidal category (called the underlying braided monoidal category of \( \mathcal{A} \)).

**Definition 5.2.** An enriched monoidal functor \( F : \mathcal{A} \to \mathcal{B} \) between two enriched braided monoidal categories is an enriched braided monoidal functor if the underlying functor \( F : \mathcal{L} \to \mathcal{M} \) is a braided monoidal functor.

**Remark 5.3.** Let \( \mathcal{A} \) be an enriched braided monoidal category. There is an \( \mathcal{A} \)-enriched braided monoidal category \( \mathcal{A}_\otimes \) whose underlying braided monoidal category equals to \( \mathcal{T} \), defined as follows. As an enriched monoidal category \( \mathcal{A}_\otimes = \mathcal{A}_\otimes \mathcal{L} \), but the braiding of \( \mathcal{A}_\otimes \) is given by the anti-braiding \( \tilde{\epsilon}_{xy} = c^{-1}_{yx} \) of \( \mathcal{L} \) and the braiding of \( \mathcal{A} \). Since \( \mathcal{A} \) is symmetric, \( (\tilde{\epsilon}, \epsilon) \) is an enriched natural transformation if and only if \( (\tilde{\epsilon}, \epsilon) \) is.

**Example 5.4.** Let \( \mathcal{A} \) be a commutative algebra in a symmetric monoidal category \( \mathcal{A} \). Then the \( \mathcal{A} \)-enriched monoidal category \( \mathcal{A} \mathcal{A} \) is an enriched braided monoidal category with the braiding given by \( c_{*,*} = 1 \).

5.2 The canonical construction

Let \( \mathcal{A} \) be a symmetric monoidal category viewed as an algebra in \( \text{Alg}_{\mathcal{E}_2}^{\text{oplax}}(\text{Cat}) \).

**Definition 5.5.** A braided monoidal left \( \mathcal{A} \)-oplax-module is a left \( \mathcal{A} \)-oplax-module in \( \text{Alg}_{\mathcal{E}_2}^{\text{oplax}}(\text{Cat}) \).

**Remark 5.6.** More explicitly, a braided monoidal left \( \mathcal{A} \)-oplax-module \( \mathcal{L} \) is both a braided monoidal category and a monoidal left \( \mathcal{A} \)-oplax-module such that the action \( \circ : \mathcal{A} \times \mathcal{L} \to \mathcal{L} \) is a braided oplax-monoidal functor (i.e. the following diagram commutes,

\[
\begin{array}{ccc}
(a \otimes b) \circ (x \otimes y) & \longrightarrow & (a \circ x) \otimes (b \circ y) \\
\downarrow \epsilon_{xb,y} & & \downarrow \epsilon_{xb,y}
\end{array}
\]

\[(5.1)\]

where \( \epsilon \) is the braiding of \( \mathcal{L} \) and \( \hat{\epsilon} \) is the braiding of \( \mathcal{A} \).

If, in addition, \( \mathcal{L} \) is a monoidal left \( \mathcal{A} \)-module, then \( \mathcal{L} \) is called a braided monoidal left \( \mathcal{A} \)-module.

**Example 5.7.** Let \( \mathcal{A} \) be a symmetric monoidal category and \( \mathcal{L} \) be a braided monoidal category. If \( \varphi : \mathcal{A} \to \mathcal{Z}(\mathcal{L}) \) is a symmetric oplax-monoidal functor, then \( \mathcal{L} \) is a braided monoidal left \( \mathcal{A} \)-oplax-module with the module action \( \varphi(-) \otimes - : \mathcal{A} \times \mathcal{L} \to \mathcal{L} \). The monoidal left \( \mathcal{A} \)-oplax-module structure of \( \mathcal{L} \) is induced by the composite functor \( \mathcal{A} \to \mathcal{Z}(\mathcal{L}) \to \mathcal{Z}(\mathcal{L}) \).
Remark 5.8. If \((\mathcal{C}, \odot : A \times \mathcal{C} \to \mathcal{C})\) is a braided monoidal left\(A\)-module, the functor \(\varphi := (- \odot 1_{\mathcal{C}})\) is a symmetric monoidal functor \(\varphi : A \to \mathcal{Z}(\mathcal{C})\) (see Example 2.10). A braided monoidal left \(A\)-module \(\mathcal{L}\) can be equivalently defined by a braided monoidal category \(\mathcal{L}\) equipped with a symmetric monoidal functor \(\varphi : A \to \mathcal{Z}(\mathcal{L})\).

Proposition 5.9. Let \(A\) be a symmetric monoidal category and \(\mathcal{L}\) a braided monoidal category. Let \(\xi\) and \(\gamma\) be the braiding of \(A\) and \(\mathcal{L}\), respectively. If \(\mathcal{L}\) is also a strongly unital monoidal left \(A\)-\(\text{op-lax}\) module that is enriched in \(A\), then the pair \((\xi, \gamma)\) defines a braiding structure on \(\mathcal{L}\) if and only if \(\mathcal{L}\) is a braided monoidal left \(A\)-\(\text{op-lax}\) module.

Proof. The pair \((\xi, \gamma)\) is a 2-morphism in \(\mathcal{L}\text{-Mod}\) if and only if the diagram (5.1) commutes. \(\Box\)

Example 5.10. Let \(\varphi : A \to \mathcal{Z}(\mathcal{L})\) be a braided oplax-monoidal functor, where \(A\) is a symmetric monoidal category and \(\mathcal{L}\) is a braided monoidal category. Then \((\mathcal{C}, \varphi)\) is a monoidal \(A\)-\(\text{op-lax}\) module (see Example 5.7). If \((\mathcal{C}, \varphi)\) is strongly unital and \(\mathcal{L}\) is enriched in \(A\), then the enriched monoidal category \(A\) constructed in Example 4.18 (note that \(\mathcal{T} = A\)), together with the braiding in \(A\) and \(\mathcal{L}\), is an enriched braided monoidal category.

Definition 5.11. Let \(\mathcal{L}\) be a braided monoidal left \(A\)-\(\text{op-lax}\) module and \(M\) be a braided monoidal left \(B\)-\(\text{op-lax}\) module. Given a symmetric monoidal functor \(F : A \to B\), a monoidal \(F\)-lax functor \(F : \mathcal{L} \to M\) is called a braided monoidal \(F\)-lax functor if \(F\) is braided monoidal.

The proof of the following proposition is clear.

Proposition 5.12. Let \(A, B\) be symmetric monoidal categories. Suppose \(\mathcal{L}\) is a strongly unital braided monoidal left \(A\)-\(\text{op-lax}\) module that is enriched in \(A\), and \(M\) is a strongly unital oplax braided monoidal left \(B\)-module that is enriched in \(B\). Then the canonical construction gives enriched braided monoidal categories \(A\)-\(\mathcal{L}\) and \(B\)-\(\mathcal{M}\). Suppose \(\hat{F} : A \to B\) is a symmetric monoidal functor and \(F : \mathcal{L} \to M\) is a monoidal \(F\)-lax functor. Then \(F\) is a braided monoidal \(F\)-lax functor if and only if \(\hat{F}\), together with \(F\), defines an enriched braided monoidal functor \(F : \mathcal{L} \to \mathcal{M}\).

We use \(\text{ECat}^{br}\) to denote the 2-category of enriched braided monoidal categories, enriched braided monoidal functors and enriched monoidal natural transformations. The 2-category \(\text{ECat}^{br}\) is symmetric monoidal with the tensor product given by the Cartesian product and the tensor unit given by \(*\).

We define the 2-category \(\text{LMod}^{br}\) as follows:

- The objects are pairs \((A, \mathcal{L})\), where \(A\) is a symmetric monoidal category and \(\mathcal{L}\) is a strongly unital braided monoidal left \(A\)-\(\text{op-lax}\) module that is enriched in \(A\).
- A 1-morphism \((A, \mathcal{L}) \to (B, \mathcal{M})\) is a pair \((\hat{F}, F)\), where \(\hat{F} : A \to B\) is a symmetric monoidal functor and \(F : \mathcal{L} \to \mathcal{M}\) is a braided monoidal \(F\)-lax functor.
- A 2-morphism \((\hat{F}, F) \Rightarrow (\hat{G}, G)\) is a pair \((\xi, \gamma)\), where \(\xi : F \Rightarrow G\) is a monoidal natural transformation and \(\gamma : : F \Rightarrow G\) is a monoidal \(\xi\)-lax natural transformation.

The horizontal/vertical composition is induced by the horizontal/vertical composition of functors and natural transformations. The 2-category \(\text{LMod}^{br}\) is symmetric monoidal with the tensor product defined by the Cartesian product and the tensor unit given by \((*,*)\).

By Proposition 5.9, Proposition 5.12 and Proposition 4.25 we obtain the following result.

Theorem 5.13. The canonical construction induces a symmetric monoidal 2-functor \(\text{LMod}^{br} \to \text{ECat}^{br}\). Moreover, this 2-functor is locally isomorphic.

5.3 The Drinfeld center of an enriched monoidal category

Definition 5.14. Let \(A\)-\(\mathcal{L}\) be an enriched monoidal category. A half-braiding for an object \(x \in A\)-\(\mathcal{L}\) is an \(A\)-natural isomorphism \(\beta_{-x} : - \otimes x \to x \otimes -\) such that the underlying natural isomorphism \(\beta_{-x}\) is a usual half-braiding.
In other words, the half-braiding \( \beta_{-x} \) is a half-braiding \( \beta_{-y} \) on the underlying monoidal category such that the following diagram commutes.

\[
\begin{array}{c}
\text{A}^1\mathcal{L}(y,z) \xrightarrow{1_{\mathcal{L}}(A)} \text{A}^1\mathcal{L}(y,z) \otimes \text{A}^1\mathcal{L}(x,x) \\
\text{A}^1\mathcal{L}(x,x) \otimes \text{A}^1\mathcal{L}(y,z) \\
\text{A}^1\mathcal{L}(x \otimes y, x \otimes z) \xrightarrow{\text{A}^1\mathcal{L}(\beta_{-x},1)} \text{A}^1\mathcal{L}(y \otimes x, x \otimes z)
\end{array}
\]

Definition 5.15. Let \( \text{A}^1\mathcal{L} \) be an enriched monoidal category. Suppose \( x, y \in \text{A}^1\mathcal{L} \) and \( \beta_{-x}, \beta_{-y} \) are half-braiding. Define an object \( \{x, y, \beta_{-x}, \beta_{-y}\} \) \( \in \mathcal{Z}_2(A) \) to be the terminal one (if exists) among all pairs \( (a, \zeta) \), where \( a \in \mathcal{Z}_2(A) \) and \( \zeta : a \rightarrow \text{A}^1\mathcal{L}(x, y) \) is a morphism in \( A \) such that the diagram commutes,

\[
\begin{array}{c}
a \xrightarrow{1_{\mathcal{L}}(A)} \text{A}^1\mathcal{L}(z,z) \otimes \text{A}^1\mathcal{L}(x,x) \\
\zeta \cdot 1 \xrightarrow{\text{A}^1\mathcal{L}(\beta_{-z},1)} \text{A}^1\mathcal{L}(x \otimes z, y \otimes z) \xrightarrow{\text{A}^1\mathcal{L}(\beta_{-x},1)} \text{A}^1\mathcal{L}(z \otimes x, y \otimes z)
\end{array}
\] (5.2)

(i.e. \( a \rightarrow \text{A}^1\mathcal{L}(x, y) \) equalizes two morphisms \( \text{A}^1\mathcal{L}(x, y) \Rightarrow \text{A}^1\mathcal{L}(z \otimes x, y \otimes z) \), for every \( z \in \text{A}^1\mathcal{L} \).)

Remark 5.16. By the universal property, the morphism \( \{x, y, \beta_{-x}, \beta_{-y}\} \rightarrow \text{A}^1\mathcal{L}(x, y) \) is monic. Thus \( \{x, y, \beta_{-x}, \beta_{-y}\} \) is the maximal subobject \( a \) of \( \text{A}^1\mathcal{L}(x, y) \) in \( \mathcal{Z}_2(A) \) such that the diagram (5.2) commutes.

Definition 5.17. We say that an enriched monoidal category \( \text{A}^1\mathcal{L} \) satisfies the condition \( \mathfrak{A} \) if:

For all objects \( x, y \in \text{A}^1\mathcal{L} \) and half-braiding \( \beta_{-x}, \beta_{-y} \), the object \( \{x, y, \beta_{-x}, \beta_{-y}\} \) exists. \( \mathfrak{A} \)

Definition 5.18. Let \( \text{A}^1\mathcal{L} \) be an enriched monoidal category that satisfies the condition \( \mathfrak{A} \). Then we define an \( \mathcal{Z}_2(A) \)-enriched braided monoidal category \( ^{\mathcal{Z}_2(A)}\Gamma_1(\text{A}^1\mathcal{L}) \) as follows:

- The objects are pairs \( (x, \beta_{-x}) \), where \( x \in \text{A}^1\mathcal{L} \) and \( \beta_{-x} \) is a half-braiding.
- The hom objects are \( ^{\mathcal{Z}_2(A)}\Gamma_1(\text{A}^1\mathcal{L})((x, \beta_{-x}), (y, \beta_{-y})) := \{x, y, \beta_{-x}, \beta_{-y}\} \) \( \in \mathcal{Z}_2(A) \).
- The composition and identity morphisms are induced by those of \( \text{A}^1\mathcal{L} \).
- The monoidal structure is induced by that of \( \text{A}^1\mathcal{L} \).
- The braiding is given by \( (x, \beta_{-x}) \otimes (y, \beta_{-y}) \stackrel{\beta_{-x,y}}{\longrightarrow} (y, \beta_{-y}) \otimes (x, \beta_{-x}) \).

This enriched braided monoidal category \( ^{\mathcal{Z}_2(A)}\Gamma_1(\text{A}^1\mathcal{L}) \) is called the Drinfeld center of \( \text{A}^1\mathcal{L} \). Its underlying category is denoted by \( \Gamma_1(\text{A}^1\mathcal{L}) \).

Remark 5.19. This construction is different from the Drinfeld center defined in [KZ18b, Definition 4.2]. The hom objects defined in [KZ18b] are maximal subobjects in \( A \), not \( \mathcal{Z}_2(A) \). In some special cases these two definitions coincide (see [KZ18b, Theorem 4.7]).

Lemma 5.20. Under the condition \( \mathfrak{A} \), the Drinfeld center \( ^{\mathcal{Z}_2(A)}\Gamma_1(\text{A}^1\mathcal{L}) \) is a well-defined enriched braided monoidal category.
Proof. For any \((x, \beta_{-x}) \in 3\text{-}x\text{-}x\alpha_{x}^{\text{tr}}(\alpha_{x}^{\text{tr}})\) the diagram

\[
\begin{array}{ccccccc}
1 & \rightarrow & A_{x}L(z, z) \otimes A_{x}L(x, x) & \xrightarrow{\Delta_{z}} & A_{x}L(z \otimes z, x \otimes x) \\
1 & \otimes 1 & A_{x}L(x, x) \otimes A_{x}L(z, z) & \xrightarrow{\otimes} & A_{x}L(x \otimes z, x \otimes z) \\
\end{array}
\]

commutes because both two composite morphisms are equal to \(\beta_{-x}\). Thus the identity morphism \(1_{x} : 1 \rightarrow A_{x}L(x, x)\) factors through \([x, \beta_{-x}]\), \((x, \beta_{-x}) \in 3\text{-}x\text{-}x\alpha_{x}^{\text{tr}}(\alpha_{x}^{\text{tr}})\).

For any \((x, \beta_{-x}), (y, \beta_{-y}), (z, \beta_{-z}) \in 3\text{-}x\text{-}x\alpha_{x}^{\text{tr}}(\alpha_{x}^{\text{tr}})\), the diagram

\[
\begin{array}{ccccccc}
[(y, \beta_{-y}), (z, \beta_{-z})][(x, \beta_{-x})] & \rightarrow & A_{x}L(y, z) & \xrightarrow{\otimes} & A_{x}L(x, y) \\
A_{x}L(y, z) & \xrightarrow{\otimes} & A_{x}L(x, y) & \xrightarrow{\otimes} & A_{x}L(x, z) \\
A_{x}L(y, z) & \xrightarrow{\otimes} & A_{x}L(x, y) & \xrightarrow{\otimes} & A_{x}L(x, z) \\
\end{array}
\]

commutes: the upper left rectangle commutes by the definition of \([x, \beta_{-x}], (y, \beta_{-y})\), and the other subdiagrams commute due to the functoriality of \(\otimes\) and the associativity of the composition. Thus the morphism \([y, \beta_{-y}], (z, \beta_{-z})] \otimes [(x, \beta_{-x})] \rightarrow A_{x}L(y, z) \otimes A_{x}L(x, y) \xrightarrow{\otimes} A_{x}L(x, z)\) factors through \([x, \beta_{-x}], (z, \beta_{-z})\]. This shows that the identity and composition morphisms in \(A_{x}L\) induce those of \(3\text{-}x\text{-}x\alpha_{x}^{\text{tr}}(\alpha_{x}^{\text{tr}})\).

For the monoidal structure, consider the following diagram:

\[
\begin{array}{ccccccc}
[(x, \beta_{-x}), (z, \beta_{-z})][(y, \beta_{-y})] & \rightarrow & A_{x}L(x, z) & \xrightarrow{\otimes} & A_{x}L(y, w) \\
A_{x}L(x, z) & \xrightarrow{\otimes} & A_{x}L(y, w) & \xrightarrow{\otimes} & A_{x}L(xz, ayw) \\
A_{x}L(xz, ayw) & \xrightarrow{\otimes} & A_{x}L(xz, ayw) \\
\end{array}
\]

The upper quadrangle commutes by the definitions of \([x, \beta_{-x}], (z, \beta_{-z})\), the left quadrangle commutes by the definition of \([y, \beta_{-y}], (w, \beta_{-w})\], and the lower right rectangle is obviously commutative. It follows that the morphism

\[
[(x, \beta_{-x}), (z, \beta_{-z})] \otimes [(y, \beta_{-y}), (w, \beta_{-w})] \rightarrow A_{x}L(x, z) \otimes A_{x}L(y, w) \xrightarrow{\otimes} A_{x}L(x \otimes z, y \otimes w)
\]

factors through \([x \otimes z, \beta_{-x\otimes z}], (y \otimes w, \beta_{-y\otimes w})\]. It is not hard to see that this defines the monoidal structure of \(3\text{-}x\text{-}x\alpha_{x}^{\text{tr}}(\alpha_{x}^{\text{tr}})\).

For the braiding structure, note that the morphism \(1_{x} : A_{x}L(x \otimes y, y \otimes x) \rightarrow A_{x}L(a \otimes x \otimes y, y \otimes x \otimes a)\) because \(\beta_{-x\otimes y}\) is a half-braiding in the

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underlying category. Therefore, \( 1 \xrightarrow{\beta_{x,y}} A[I(x \otimes y, y \otimes x)] \) factors through \( [(x \otimes y, \beta_{x,y}), (y \otimes x, \beta_{y,x})] \) and thus gives the braiding morphism in \( 3[A[I_1(A[L]] \right) \), denoted by

\[
(x \otimes y, \beta_{x,y}) \xrightarrow{\beta_{x,y}} (y \otimes x, \beta_{y,x}).
\]

We need to verify that this braiding \( \beta_{x,y} \) is an enriched natural isomorphism, i.e. the diagram

\[
\begin{array}{ccc}
[(x, \beta_{x,y}), (z, \beta_{z})] \otimes [(y, \beta_{y,z}), (w, \beta_{w,z})] & \overset{c}{\longrightarrow} & [(y, \beta_{y,z}), (w, \beta_{w,z})] \otimes [(x, \beta_{x,y}), (z, \beta_{z})] \\
\| & \| & \| \\
[(x \otimes y, \beta_{x,y}), (z \otimes w, \beta_{z})] & \overset{[\beta_{x,y}]}{\longrightarrow} & [(x \otimes y, \beta_{x,y}), (z \otimes w, \beta_{z})]
\end{array}
\]

commutes, where \( c \) is the braiding of \( A \). Since the morphism \( [(x \otimes y, \beta_{x,y}), (w \otimes z, \beta_{w,z})] \rightarrow A[I(x \otimes y, w \otimes z)] \) is monic, it suffices to show that the diagram

\[
\begin{array}{ccc}
[(x, \beta_{x,y}), (z, \beta_{z})] & \mathbb{A}[L(x \otimes z, \beta_{x,y}, z)] & \mathbb{A}[L(x, z)] \\
\| & \| & \| \\
A[I_L(x, y, w)] & \mathbb{A}[L(x, w)] & \mathbb{A}[L(x, y, w)] \\
\| & \| & \| \\
A[I_L(x, y, w, z)] & \mathbb{A}[L(x, y, w, z)] & \mathbb{A}[L(x, y, w, z)] \\
\| & \| & \| \\
A[I_L(x, z, w)] & \mathbb{A}[L(x, z, w)] & \mathbb{A}[L(x, z, w)]
\end{array}
\]

commutes. The upper left rectangle commutes by the definition of \( [(y, \beta_{y,z}), (w, \beta_{w,z})] \), the lower right pentagon and the lower middle triangle commute due to the associativity of the composition \( \circ \), and the other two subdiagrams commute due to the functoriality of \( \otimes \). Hence, we conclude that \( 3[A[I_1(A[L]] \right) \) is a well-defined enriched braided monoidal category.

**Remark 5.21.** Let \( A[L] \) be an enriched monoidal category that satisfies the condition (33). The forgetful functor \( I : 3[A[I_1(A[L]] \rightarrow A[I_L] \) is an enriched monoidal functor whose background changing functor is the inclusion \( \mathbb{A}[L(A)] \rightarrow A \).

**Proposition 5.22.** The underlying category \( \Gamma_1(A[L]) \) is a full subcategory of \( \mathbb{A}[L(A)] \).

**Proof.** Suppose \((x, \beta_{x,y}), (y, \beta_{y,z}) \in 3[A[I_1(A[L]] \). By definition we have \( (x, \beta_{x,y}), (y, \beta_{y,z}) \in \mathbb{A}[L(A)] \). It suffices to show that

\[
\mathbb{A}[L(A)](1, [(x, \beta_{x,y}), (y, \beta_{y,z})]) = \mathbb{A}[L(A)](x, y) = \mathbb{A}[L(A)](z, y).
\]

By the universal property, the left hand side is isomorphic to the set of morphisms \( f : I \rightarrow A[I_L(x, y)] \) rendering the diagram

\[
\begin{array}{ccc}
I & \xrightarrow{1 \circ f} & A[I_L(x, z, y)] \\
\downarrow & & \downarrow \mathbb{A}[L(x, y)] \\
A[I_L(x, y)] & \xrightarrow{\otimes} & A[I_L(x, y) \otimes z, y] \\
\downarrow & & \downarrow \mathbb{A}[L(x, y)] \\
A[I_L(x, y) \otimes A[I_L(x, z, y)] & \xrightarrow{\otimes} & A[I_L(x, y) \otimes z, y \otimes z] \\
\downarrow & & \downarrow \mathbb{A}[L(x, y)] \\
A[I_L(x, y) \otimes A[I_L(x, z, y)] & \xrightarrow{\otimes} & A[I_L(x, y) \otimes z, y \otimes z]
\end{array}
\]

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commutative for every $z \in A^1_\cal{L}$. It means that the equation
\[
\beta_{z,y} \circ (1_z \otimes f) = (f \otimes 1_z) \circ \beta_{x,x}
\]
holds in the underlying category $\cal{L}$. In other words, $f$ is exactly a morphism $f : (x, \beta_{-x}) \to (y, \beta_{-y})$ in the Drinfeld center $\Gamma_1(\cal{L})$.

In the rest of this subsection, we compute the Drinfeld center of an enriched monoidal category $\cal{C}$ obtained via the canonical construction from the pair $\cal{C}, \cal{M} \in \bf{LMod}^{br}$ such that $\cal{M}$ is a monoidal left $\cal{C}$-module defined by a braided monoidal functor (recall Remark 3.16).

\[
\varphi : \cal{C} \to \Gamma_1(\cal{M}).
\]

Suppose $\cal{C} \cal{M}$ satisfies the condition $\Box$. Then the Drinfeld center $\Gamma_1(\cal{C} \cal{M})$ is a well-defined enriched braided monoidal category. By Proposition 5.22, $\Gamma_1(\cal{C} \cal{M})$ is a full subcategory of $\Gamma_1(\cal{M})$. It is straightforward to see that $\Gamma_1(\cal{C} \cal{M})$ in $\Gamma_1(\cal{M})$ factors through $\cal{Z}(\cal{C}) \to \cal{Z}(\cal{M}) \to \Gamma_1(\cal{M})$. Moreover, the image of $\varphi$ is contained in the Müger center $\cal{Z}(\cal{Z}(\cal{M}))$ of $\cal{Z}(\cal{M})$. Thus $\cal{Z}(\cal{M})$ is a braided monoidal left $\cal{Z}(\cal{C})$-module with the braided module structure defined by $\varphi : \cal{Z}(\cal{C}) \to \cal{Z}(\cal{Z}(\cal{M}))$.

**Lemma 5.23.** Let $\cal{C}$ be a braided monoidal category and $\cal{M}$ a strongly unital monoidal left $\cal{C}$-module that is enriched in $\cal{C}$. The object $[(x, \beta_{-x}), (y, \beta_{-y})]$ is precisely the internal hom in $\cal{Z}(\cal{C})$.

**Proof.** It is enough to show that $[(x, \beta_{-x}), (y, \beta_{-y})]$ satisfies the same universal property as the internal hom in $\cal{Z}(\cal{C})$. Given an object $a \in \cal{Z}(\cal{C})$ and a morphism $\zeta : a \to [x, y]_\cal{C}$, we can define a morphism $\tilde{\zeta} : q^f(a) \otimes x \to y$ in $\cal{M}$ by

\[
\tilde{\zeta} := \left( q^f(a) \otimes x \xrightarrow{q^f(1) \circ 1} q^f([x, y]_\cal{C}) \otimes x \xrightarrow{ev} y \right).
\]

Conversely, we have

\[
\zeta = \left( a \xrightarrow{coev} [x, q^f(a) \otimes x]_\cal{C} \xrightarrow{[1, 1]} [x, y] \right).
\]

It is routine to check that $\tilde{\zeta}$ renders the diagram (5.2) commutative if and only if the following diagram commutes,

\[
\begin{array}{ccc}
\cal{Z} \otimes q^f(a) \otimes x & \xrightarrow{\gamma_{z,\varphi} \otimes 1} & q^f(a) \otimes \cal{Z} \otimes x & \xrightarrow{\log \beta_{z,y}} & q^f(a) \otimes \cal{Z} \\
1_{\log \beta_{z,y}} & & \beta_{z,y} & & \\
\cal{Z} \otimes y & \xrightarrow{\beta_{z,y}} & y \otimes \cal{Z}
\end{array}
\]

(i.e. $\tilde{\zeta}$ is a morphism $\tilde{\zeta} : a \otimes (x, \beta_{-x}) \to (y, \beta_{-y})$ in $\cal{Z}(\cal{M})$).

**Corollary 5.24.** Let $\cal{C}$ be a braided monoidal category and $\cal{M}$ be a strongly unital monoidal left $\cal{C}$-module that is enriched in $\cal{C}$. If $\cal{C} \cal{M}$ satisfies the condition $\Box$, we have $\Gamma_1(\cal{C} \cal{M}) = \Gamma_1(\cal{M})$.

### 5.4 The $E_1$-centers of enriched monoidal categories in $\bf{ECat}$

In this subsection we prove that the Drinfeld center of an enriched monoidal category is the $E_1$-center in $\bf{ECat}$.

Let $\cal{C} \cal{M}$ be an enriched monoidal category that satisfies the condition $\Box$. Then the Drinfeld center $\Gamma_1(\cal{C} \cal{M})$ is a well-defined enriched braided monoidal category.
Suppose \( A^I \mathcal{L} \) is an enriched monoidal category. A left unital action of \( A^I \mathcal{L} \) on \( \mathcal{C}^I \mathcal{M} \) is an enriched monoidal functor \( \otimes \) and an enriched monoidal natural isomorphism \( u \) as depicted in the following diagram:

\[
\begin{array}{c}
\xymatrix{
A^I \mathcal{L} \times \mathcal{C}^I \mathcal{M} \ar[rd]_{\otimes} \ar[rr]^{\otimes} & & \mathcal{C}^I \mathcal{M} \\
\ast \times \mathcal{C}^I \mathcal{M} \ar[rr]_{\otimes} & & \mathcal{C}^I \mathcal{M}
}
\end{array}
\]

(5.3)

**Example 5.25.** There is an obvious left unital action of the Drinfeld center \( 3^{(E)} \Gamma_1(\mathcal{C}^I \mathcal{M}) \) on \( \mathcal{C}^I \mathcal{M} \):

\[
\begin{array}{c}
\xymatrix{
3^{(E)} \Gamma_1(\mathcal{C}^I \mathcal{M}) \times \mathcal{C}^I \mathcal{M} \ar[rd]_{\otimes} \ar[rr]^{\otimes} & & \mathcal{C}^I \mathcal{M} \\
\ast \times \mathcal{C}^I \mathcal{M} \ar[rr]_{\otimes} & & \mathcal{C}^I \mathcal{M}
}
\end{array}
\]

The enriched monoidal functor \( \otimes := \otimes \circ (I \times 1) \), where \( I : 3^{(E)} \Gamma_1(\mathcal{C}^I \mathcal{M}) \to \mathcal{C}^I \mathcal{M} \) is the forgetful functor (see Remark 5.21). The monoidal structure of \( I \) is induced by the braided monoidal structure of \( 3^{(E)} \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) and the monoidal structure of \( \mathcal{Z}_1(\mathcal{M}) \times \mathcal{M} \to \mathcal{M} \) (recall that \( \mathcal{Z}_1(\mathcal{M}) \) is a full subcategory of \( \mathcal{Z}_1(\mathcal{M}) \)). The background changing natural transformation of \( \beta \) is given by the left unitor of \( \mathcal{C} \) and the underlying natural transformation of \( \beta \) is given by the left unitor of \( \mathcal{M} \). ✷

**Theorem 5.26.** The Drinfeld center is the \( E_1 \)-center in \( ECat \), i.e. \( \mathcal{Z}_1(\mathcal{C}^I \mathcal{M}) \cong 3^{(E)} \Gamma_1(\mathcal{C}^I \mathcal{M}) \) as enriched braided monoidal categories.

**Proof.** Suppose \( (\otimes, u) \) is a left unital action of an enriched monoidal category \( A^I \mathcal{L} \) on \( \mathcal{C}^I \mathcal{M} \) as depicted in the diagram (5.3). First we show that there is an enriched monoidal functor \( \mathbf{P} : A^I \mathcal{L} \to \mathcal{E} \mathbf{C} \) and an enriched monoidal natural isomorphism \( \beta \) such that the following pasting diagrams

\[
\begin{array}{c}
\xymatrix{
3^{(E)} \Gamma_1(\mathcal{C}^I \mathcal{M}) \times \mathcal{C}^I \mathcal{M} \\
\ast \times \mathcal{C}^I \mathcal{M}
}
\end{array}
\]

are equal.

- Since the underlying functor \( \otimes \) of \( \otimes \) is a monoidal functor, for any \( x \in A^I \mathcal{L} \), the natural isomorphism

\[
\beta_{m,x} : m \otimes (x \otimes 1_M) \xrightarrow{\mu_c} (1_L \otimes m) \otimes (x \otimes 1_M) \cong x \otimes m \cong (x \otimes 1_M) \otimes (1_L \otimes m) \xrightarrow{\iota_M} (x \otimes 1_M) \otimes m
\]

is a half-braiding on \( x \otimes 1_M \in \mathcal{Z}_1(\mathcal{M}) \) (see Example 2.16). It is clear that \( \beta_{-,-} \) is a \( \mathcal{C} \)-natural isomorphism because it is the composition of 2-isomorphisms in \( ECat \) and the background changing natural transformation is identity. Hence \( x \otimes 1_M \) together with \( \beta_{-,-} \) is an object in \( 3^{(E)} \Gamma_1(\mathcal{E} \mathcal{M}) \). Then we define an enriched monoidal functor \( \mathbf{P} : A^I \mathcal{L} \to 3^{(E)} \Gamma_1(\mathcal{E} \mathcal{M}) \) as follows:

- The background changing functor \( \mathbf{P} \) is given by \( - \otimes 1_c : A \to \mathcal{Z}_2(\mathcal{C}) \), which is a braided monoidal functor (see Example 2.16).
- The object \( \mathbf{P}(x) \in 3^{(E)} \Gamma_1(\mathcal{E} \mathcal{M}) \) is defined by the object \( x \otimes 1_M \) together with the half-braiding \( \beta_{-,-} \).

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We only need to show that there exists a unique enriched monoidal natural isomorphism $\alpha$. It is not hard to check that the equation (5.4) holds.

Define $\alpha$: $\mathcal{E}$-structure of the natural isomorphism such that the equation (5.5) holds.

$\rho_{x,y}: (a \otimes \mathbb{1}_E) \otimes c \xrightarrow{1 \otimes \alpha} (a \otimes \mathbb{1}_E) \otimes (\mathbb{1}_M \otimes c) \xrightarrow{\alpha} a \otimes c$

and the universal property of $[P(x), P(y)]$. In other words, $\rho_{x,y}$ is the unique morphism rendering the following diagram

$$\xymatrix{ A(x, y) \otimes \mathbb{1}_E \ar[r]^{\rho_{x,y}} \ar[d]_{1 \otimes \alpha} & [P(x), P(y)] \ar[d]^\alpha \\
A(x, y) \otimes \mathbb{1}_M \otimes \mathbb{1}_E \ar[r] & \mathbb{1}_M(x \otimes \mathbb{1}_M, y \otimes \mathbb{1}_M) }$$

commutative.

The monoidal structure is induced by that of $\otimes$.

- The enriched monoidal natural isomorphism $\rho_{x,y}$ is induced from the universal property.

It is not hard to check that the equation (5.4) holds.

Let $(Q_i, \rho_i)$, $i = 1, 2$, be two pairs such that the similar equations as depicted by (5.4) hold. We only need to show that there exists a unique enriched monoidal natural isomorphism $\alpha: Q_1 \Rightarrow Q_2$ such that

$$\xymatrix{ 3d(C) \ar[r]^{\alpha} \ar[d]_{Q_1 \times 1} & 3d(C) \ar[d]_{Q_2 \times 1} \\
A \times \mathcal{E} \ar[r]_\phi & A \times \mathcal{E} }$$

and

$$\xymatrix{ 3d(C) \ar[r]^{\alpha} \ar[d]_{Q_1 \times 1} & 3d(C) \ar[d]_{Q_2 \times 1} \\
A \times \mathcal{E} \ar[r]_\phi & A \times \mathcal{E} }$$

Since $3d(C)$ is the $E_0$-center of $\mathcal{E}$ in $\text{Alg}_{\mathcal{E}}(\text{Cat})$ (see Example 2.16), there exists a unique monoidal natural isomorphism $\tilde{\alpha}: \tilde{Q}_1 \Rightarrow \tilde{Q}_2$ such that

$$\xymatrix{ 3d(C) \times \mathcal{E} \ar[r]^{\tilde{\alpha}} \ar[d]_{Q_1 \times 1} & 3d(C) \times \mathcal{E} \ar[d]_{Q_2 \times 1} \\
A \times \mathcal{E} \ar[r]_\phi & A \times \mathcal{E} }$$

Define $\alpha: \mathbb{1}_E \to [Q_1(a), Q_2(a)]$ to be the unique morphism rendering the following diagram

$$\xymatrix{ \mathbb{1}_E \ar[r]^{\alpha} \ar[d]_{(\rho_2)_{(1),3}(\rho)_{3,1}(\alpha)} & [Q_1(a), Q_2(a)] \\
\mathbb{1}_M(Q_1(a) \otimes \mathbb{1}_M, Q_2(a) \otimes \mathbb{1}_M) \ar[r]^\alpha & \mathbb{1}_M(Q_1(a), Q_2(a)) }$$

commutative. Then the enriched natural isomorphism $\alpha$ defined by $(\tilde{\alpha}, \{\alpha_a\})$ is the unique enriched natural isomorphism such that the equation (5.5) holds.

It is routine to check that the braiding structure of $3d(C) \Gamma_1(\mathcal{E})$ coincides with the $E_2$-algebra structure of the $E_1$-center of $\mathcal{M}$ in $\text{EAlgCat}$ induced from the universal property. \qed
Combining Corollary \ref{cor:5.24} and Theorem \ref{thm:5.26}, we obtain a corollary.

**Corollary 5.27.** Let \( \mathcal{C} \) be a braided monoidal category and \( \mathcal{M} \) a strongly unital monoidal left \( \mathcal{C} \)-module that is enriched in \( \mathcal{C} \). If \( \mathcal{C} \mathcal{M} \) satisfies the condition \( (\mathcal{C}) \), we have
\[
\mathcal{Z}(\mathcal{C} \mathcal{M}) = \mathcal{Z}(\mathcal{C})(\mathcal{Z}(\mathcal{C} \phi)).
\]

**Remark 5.28.** Corollary \ref{cor:5.27} has important applications in physics. When \( \mathcal{E} \) is a non-degenerate braided fusion category and \( \mathcal{M} \) is a multi-fusion left \( \mathcal{E} \)-module defined by a braided monoidal functor \( \phi : \mathcal{E} \rightarrow \mathcal{Z}(\mathcal{M}) \) \cite{KZ18a}, the condition \( (\mathcal{C}) \) is satisfied automatically. In this case, we obtain \( \mathcal{Z}(\mathcal{C} \mathcal{M}) = \mathcal{Z}(\phi) \). This recovers Corollary 5.4 in \cite{KZ18}. This result has important applications in the theory of gapless boundaries of 2+1D topological orders \cite{KZ20, KZ21}. Moreover, for an indecomposable multi-fusion category \( \mathcal{M} \) and a finite semisimple left \( \mathcal{A} \)-module \( \mathcal{L} \), then we have
\[
\mathcal{Z}(\mathcal{Z}(\mathcal{A} \mathcal{L})) \cong \mathcal{A},
\]
i.e. the center of a center is trivial. This is the mathematical manifestation of a physical fact: “the bulk of a bulk is trivial”.

## 6 Enriched symmetric monoidal categories

### 6.1 Definitions and canonical construction

**Definition 6.1.** An enriched braided monoidal category \( \mathcal{A} \mathcal{L} \) is called an enriched symmetric monoidal category if the underlying braided monoidal category \( \mathcal{L} \) is symmetric.

**Definition 6.2.** An enriched braided monoidal functor \( \mathcal{F} : \mathcal{A} \mathcal{L} \rightarrow \mathcal{M} \) between two enriched symmetric monoidal categories \( \mathcal{A} \mathcal{L} \) and \( \mathcal{M} \) is also called an enriched symmetric monoidal functor.

**Definition 6.3.** Let \( \mathcal{A} \) be a symmetric monoidal category and \( \mathcal{L} \) be a braided monoidal left \( \mathcal{A} \)-oplasx-module. If \( \mathcal{L} \) is symmetric, we say \( \mathcal{L} \) is a symmetric monoidal left \( \mathcal{A} \)-oplasx-module.

**Definition 6.4.** A braided monoidal \( \mathcal{F} \)-lax functor \( \mathcal{F} : \mathcal{L} \rightarrow \mathcal{M} \) between two symmetric monoidal categories \( \mathcal{L} \) and \( \mathcal{M} \) is also called a symmetric monoidal \( \mathcal{F} \)-lax functor.

Let \( \mathcal{ECat}^{\text{sym}} \) be a symmetric monoidal full sub-2-category of \( \mathcal{ECat}^{\mathcal{BR}} \) consisting of enriched symmetric monoidal categories. Let \( \mathcal{LMod}^{\text{sym}} \) be the symmetric monoidal full sub-2-category of \( \mathcal{LMod}^{\mathcal{BR}} \) consisting of pairs \( (\mathcal{A}, \mathcal{L}) \) such that \( \mathcal{L} \) is strongly unital symmetric monoidal left \( \mathcal{A} \)-oplasx-module that is enriched in \( \mathcal{A} \). We obtain a corollary of Theorem \ref{thm:5.13}.

**Corollary 6.5.** The canonical construction induces a symmetric monoidal 2-functor \( \mathcal{LMod}^{\text{sym}} \rightarrow \mathcal{ECat}^{\text{sym}} \). Moreover, this 2-functor is locally isomorphic.

**Definition 6.6.** Let \( \mathcal{A} \mathcal{L} \) be an enriched braided monoidal category. The Müger center \( \mathcal{A} \mathcal{L} \mathcal{M} \) of \( \mathcal{A} \mathcal{L} \) is defined by the subcategory of \( \mathcal{A} \mathcal{L} \) consisting of the transparent objects in \( \mathcal{L} \) and the same hom spaces as those in \( \mathcal{A} \mathcal{L} \) (i.e. \( \mathcal{A} \mathcal{L} \mathcal{M} = \mathcal{A} \mathcal{L} \mathcal{M} \)).

**Theorem 6.7.** Let \( \mathcal{A} \) be a symmetric monoidal category and \( \mathcal{L} \) be a strongly unital braided monoidal left \( \mathcal{A} \)-module that is enriched in \( \mathcal{A} \). Then we have \( \mathcal{A} \mathcal{L} \mathcal{M} = \mathcal{A} \mathcal{L} \mathcal{M} \).

### 6.2 The E2-centers of enriched braided monoidal categories in ECat

In this subsection we prove that the Müger center \( \mathcal{E} \mathcal{L} \mathcal{M} \) of an enriched braided monoidal category \( \mathcal{E} \mathcal{M} \) is its \( \mathcal{E} \)-2-center in \( \mathcal{ECat} \).

Suppose \( \mathcal{E} \mathcal{L} \) is an enriched braided monoidal category. A left unital action of \( \mathcal{A} \mathcal{L} \) on \( \mathcal{E} \mathcal{M} \) is an enriched braided monoidal functor \( \mathcal{U} \) and an enriched monoidal natural isomorphism \( \mathcal{U} \).
depicted in the following diagram:

\[
\begin{array}{c}
\text{Example 6.8. There is an obvious left unital action of the Müger center } \mathcal{E} \mathcal{T}_2(\mathcal{E} \mathcal{M}) \text{ on } \mathcal{E} \mathcal{M}:
\end{array}
\]

The enriched braided monoidal functor \( \otimes \) is given by the tensor product of \( \mathcal{E} \mathcal{M} \). The background changing natural isomorphism of \( \upsilon \) is given by the left unitor of \( \mathcal{E} \) and the underlying natural isomorphism of \( \upsilon \) is given by the left unitor of \( \mathcal{M} \).

\textbf{Theorem 6.9.} The Müger center \( \mathcal{E} \mathcal{T}_2(\mathcal{E} \mathcal{M}) \) is the \( E_2 \)-center of \( \mathcal{E} \mathcal{M} \) in \( \textbf{ECat} \).

\textbf{Proof.} Suppose \( (\otimes, \mu) \) is a left unital action of an enriched braided monoidal category \( \mathcal{E} \mathcal{L} \) on \( \mathcal{E} \mathcal{M} \) as depicted in the diagram \([6.1]\). First we show that there is an enriched braided monoidal functor \( \hat{P} : \mathcal{E} \mathcal{L} \to \mathcal{E} \mathcal{T}_2(\mathcal{E} \mathcal{M}) \) and an enriched monoidal natural isomorphism \( \hat{\rho} \) such that the following pasting diagrams

\[
\begin{array}{c}
\text{are equal.}
\end{array}
\]

- Since the underlying functor \( \otimes \) of \( \otimes \) is a braided monoidal functor, \( x \otimes 1 \mathcal{M} \in \mathcal{E} \mathcal{T}_2(\mathcal{E} \mathcal{M}) \) is transparent for any \( x \in \mathcal{E} \mathcal{L} \) (see Example \([2.16]\)). Then we define an enriched monoidal functor \( \hat{P} : \mathcal{E} \mathcal{L} \to \mathcal{E} \mathcal{T}_2(\mathcal{E} \mathcal{M}) \) by \( \hat{P} := \hat{\otimes} \mathcal{E} \). More explicitly:
  - The background changing functor \( \hat{P} \) is given by \( - \otimes \mathcal{E} \mathcal{L} : \mathcal{E} \mathcal{L} \to \mathcal{E} \mathcal{T}_2(\mathcal{E} \mathcal{M}) \).
  - The object \( P(x) \in \mathcal{E} \mathcal{T}_2(\mathcal{E} \mathcal{M}) \) is defined by the object \( x \otimes 1 \mathcal{M} \in \mathcal{E} \mathcal{T}_2(\mathcal{E} \mathcal{M}) \).
  - The morphism \( \hat{P}_{x,y} : \mathcal{E} \mathcal{L}(x, y) \otimes 1 \mathcal{E} \to \mathcal{E} \mathcal{M}(P(x), P(y)) \) is induced by the morphism

\[
\mathcal{E} \mathcal{L}(x, y) \otimes 1 \mathcal{E} \xrightarrow{1 \otimes \hat{P}_{1,1}} \mathcal{E} \mathcal{L}(x, y) \otimes \mathcal{E} \mathcal{M}(1 \mathcal{M}, 1 \mathcal{M}) \xrightarrow{\hat{\rho}} \mathcal{E} \mathcal{M}(x \otimes 1 \mathcal{M}, y \otimes 1 \mathcal{M}).
\]

- The braided monoidal structure is induced by that of \( \otimes \).

- The enriched monoidal natural isomorphism \( \hat{\rho} : \hat{\otimes} \circ (\hat{P} \times 1) \Rightarrow \otimes \) is defined by the background changing natural transformation

\[
\hat{\rho}_{x,c} : (a \otimes 1 \mathcal{E}) \hat{\otimes} c \xrightarrow{1 \otimes \hat{\rho}_{1,1}} (a \hat{\otimes} 1 \mathcal{E}) \hat{\otimes} (1 \mathcal{E} \otimes) \mathcal{C) \xrightarrow{\hat{\rho}} a \hat{\otimes} c}
\]

and the underlying natural transformation

\[
\rho_{x,m} : (x \otimes 1 \mathcal{M}) \otimes m \xrightarrow{1 \otimes \mu_{1,1}} (x \otimes 1 \mathcal{M}) \otimes (1 \mathcal{E} \otimes m) \xrightarrow{\hat{\rho}} x \otimes m.
\]
It is not hard to check that the equation (6.2) holds.

Let $(Q_i, p_i), \ i = 1, 2,$ be two pairs such that the similar equations as depicted by (6.2) hold. We only need to show that there exists a unique enriched monoidal natural isomorphism $\alpha : \hat{\mathcal{Q}}_1 \Rightarrow \hat{\mathcal{Q}}_2$ such that

\[
\begin{align*}
\hat{\mathcal{Q}}_1 & \xrightarrow{\hat{\mathcal{Q}}_1 \times \alpha_{Q_i}} \hat{\mathcal{Q}}_1 \times \hat{\mathcal{Q}}_2, \\
& \xrightarrow{Q_i \times 1} \hat{\mathcal{Q}}_2 \times \hat{\mathcal{Q}}_1, \\
\end{align*}
\]

(6.3)

Since $\mathcal{E}$ itself is the $E_{0}$-center of $\mathcal{E}$ in $\text{Alg}_{E_{0}}(\text{Cat})$ (see Example 2.16), there exists a unique monoidal natural isomorphism $\hat{\alpha} : \hat{Q}_1 \Rightarrow \hat{Q}_2$ such that

\[
\begin{align*}
\mathcal{E} \times \mathcal{E} & \xrightarrow{\hat{\alpha}_{\mathcal{E}}} \mathcal{E} \times \mathcal{E}, \\
& \xrightarrow{\hat{\mathcal{Q}}_1 \times 1} \mathcal{E} \times \mathcal{E}, \\
\end{align*}
\]

Define

\[
\alpha_a := \left( Q_1(a) \otimes Q_2(a) \otimes 1_M \xrightarrow{\hat{\rho}_2 \otimes 1} a \otimes 1 \xrightarrow{1 \otimes \hat{\rho}_1} Q_2(a) \otimes 1 \Rightarrow Q_2(a) \right).
\]

Then the enriched natural isomorphism $\alpha$ define by $(\hat{\alpha}, [\alpha_a])$ is the unique enriched natural isomorphism such that the equation (6.3) holds.

By Theorem 6.7, we obtain the following corollary.

**Corollary 6.10.** Let $\mathcal{A}$ be a symmetric monoidal category and $\mathcal{L}$ be a strongly unital braided monoidal left $\mathcal{A}$-module that is enriched in $\mathcal{A}$. Then we have $\mathcal{Z}_2(\mathcal{E}) = \mathcal{Z}_2(\mathcal{M})$.

**Example 6.11.** Let $\mathcal{E}$ be a symmetric fusion category, i.e. $\mathcal{E} = \text{Rep}(G)$ or $\text{Rep}(G, z)$ for a finite group $G$. Then the canonical construction $\mathcal{E}$ is an example of enriched monoidal category. We have $\mathcal{Z}_2(\mathcal{E}) \cong \mathcal{E}$.

**Remark 6.12.** When $\mathcal{A}$ is a non-degenerate braided fusion category and $\mathcal{L}$ is a multi-fusion left $\mathcal{A}$-module $[KZ18]$, we have $\mathcal{Z}_2(\mathcal{A}) \cong k$.

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