Stability of a fixed point in the replica action for the random field Ising model

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Abstract. We reconsider stability of the non-trivial fixed point in 6 − $\epsilon$ dimensional effective action for the random field Ising model derived by Brézin and De Dominicis. After expansion parameters of physical observables are clarified, we find that the non-trivial fixed point in 6 − $\epsilon$ dimensions is stable, contrary to the argument by Brézin and De Dominicis. We also computed the exponents $\nu$ and $\eta$ by the $\epsilon$ expansion. The results are consistent with the argument of the dimensional reduction at least in the leading order.

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1. Introduction

Recently, Brézin and De Dominicis derived effective scalar field theory for the Ising model in a Gaussian random field within the replica formalism [1]. They showed that the theory contains five $\phi^4$ coupling constants as well as the standard $\phi^4$ coupling constant. They also pointed out that we meet singular fluctuation on the critical surface below dimension eight when we take the zero-replica limit. In order to resolve this problem, they redefined the coupling constants such that the beta functions for the new coupling constants do not explicitly contain $n$, the number of the replica components.

The effective theory contains not only the five $\phi^4$ coupling constants $u_i$, ($i = 1, \ldots, 5$), but also a parameter $\Delta$ with the mass dimension 2, which is related to strength of the random field. Since the new coupling constants introduced in reference [1] are defined as linear combinations of $\Delta u_i$, they have the dimension $2 + (4 - d) = 6 - d$ in $d$-dimensional effective theory.

According to the beta functions corresponding to the new coupling constants, there is a non-trivial fixed point in $6 - \epsilon$ dimensions as in the case of the pure $\phi^4$ theory in $4 - \epsilon$ dimensions. However, aspects of flow is different from the pure theory because some flow is repelled by the fixed point. Therefore, they concluded that the non-trivial
fixed point is unstable and the $\epsilon$-expansion cannot be carried out near six dimensions. It indicates that the dimensional reduction \[4, 5\] breaks down near the upper critical dimension as well as the lower one \[6, 7\].

However, it depends on choice of coupling constants in this model whether a fixed point is attractive or repulsive. Hence it is important to study how physical observables are expanded by $u_i$ and $\Delta$, and to know what the effective coupling constants suitable for perturbation are. To make this point clear, it is instructive to consider the following simple example. Let $\lambda$ be a dimensionless, renormalized coupling constant. Suppose that the beta function is given as

$$\beta_\lambda(\lambda) = x\lambda - y\hat{\Delta}\lambda^2, \quad (1)$$

where $\hat{\Delta} \equiv \Delta m^{-2}$ and $m$ is the renormalized mass, the inverse of the correlation length. We denote by $x$ and $y$ dimensionless coefficients. The fixed point $\lambda^*$ is easily found as $x/(y\hat{\Delta})$. Then we have

$$\frac{d\beta_\lambda}{d\lambda} \bigg|_{\lambda=\lambda^*} = -x. \quad (2)$$

Next let us introduce a new coupling constant by

$$g \equiv \hat{\Delta}\lambda. \quad (3)$$

The beta function for $g$ is computed to be

$$\beta_g(g) = m \frac{dg}{dm} = \frac{dg}{d\lambda} \beta_\lambda - 2g. \quad (4)$$

Using (3) and (4), one finds

$$\frac{d\beta_g}{dg} \bigg|_{g=g^*} = -x + 2, \quad (5)$$

where $g^*$ is the zero of $\beta_g$. Suppose that $0 < x < 2$. Equation (2) shows that the fixed point is unstable. On the contrary, it seems stable from the result (5). This example shows that we have to clarify which coupling constants play a role of an expansion parameter for studying the stability. Note that the existence of the second term in the right-hand side of equation (4) shows breakdown of the covariance of the beta function\[3\]. It appears because the fixed dimensional parameter $\Delta$ enters the beta function and the definition (3).

Although Brézin and De Dominicis chose $\Delta u_i$ for all $i$ as the coupling constants in order to eliminate $n$ dependence, it has not yet confirmed whether the new coupling constants correspond to good expansion parameters. The main purpose of this paper is to obtain the good expansion parameters and to reconsider the instability studied in reference \[1\]. We conclude that their beta functions are not for the good expansion parameters.

This paper is organized as follows: in the next section we perform the naive perturbative expansion of the free-energy density in the disorder phase and determine the expansion parameters. Thanks to a finite mass in the disorder phase, there are
no singular behavior in the limit of \( n \to 0 \) discussed in reference [1]. Therefore we can take the zero-replica limit at the very beginning. We find that the good expansion parameters are obtained from the effective coupling constants \( \Delta^\alpha u_i \), where \( \alpha \) depends on \( i \) (see equation (18)). Our effective coupling constants have various mass dimensions at the tree level. One of them becomes marginal but the others are irrelevant in six dimensions. When the dimensions lower by \( \epsilon \), the marginal coupling constant becomes irrelevant at the non-trivial fixed point due to the one-loop correction, while the other coupling constants remain irrelevant.

In section 3, we turn to renormalized perturbation, where we define the renormalized effective coupling constants and compute their beta functions. Computing the scaling matrix (i.e., the derivative of beta functions at a fixed point), we find that the fixed point is stable.

The correlation-length exponent and the anomalous dimension of the correlation function in \( d = 6 - \epsilon \) are computed in section 4. The results are identical with the case of the \( d = 4 - \epsilon \) pure \( \phi^4 \) theory in the leading order of the \( \epsilon \)-expansion. It is consistent with \((d, d - 2)\) correspondence [4, 5].

The last section is devoted to summary and future problems.

2. Coupling dependence of the free energy

Here we observe the bare-parameter dependence of the free-energy density.

The \( d \)-dimensional effective action derived by Brézin and De Dominicis is given by

\[
S = S_0 + S_{\text{int}},
\]

\[
S_0 \equiv \int d^d x \sum_{\alpha, \beta = 1}^n \frac{1}{2} \phi_\alpha(x) \left\{ (-\partial^2 + t) \delta_{\alpha\beta} - \Delta \right\} \phi_\beta(x),
\]

\[
S_{\text{int}} \equiv \int d^d x \left( \frac{u_1}{4!} \sigma_4 + \frac{u_2}{3!} \sigma_3 \sigma_1 + \frac{u_3}{8} \sigma_2^2 + \frac{u_4}{4} \sigma_1^2 + \frac{u_5}{4!} \sigma_1^4 \right),
\]

(6)

where \( \alpha \) and \( \beta \), which run from 1 to \( n \), specify a replica component and

\[
\sigma_k \equiv \sum_{\alpha = 1}^n (\phi_\alpha(x))^k, \quad (k = 1, \ldots, 4).
\]

(7)

The interaction conjugate to \( u_1 \) contains one summation over the replica index while the other interactions have more than one, which apparently shows that the other coupling constants \( u_2, \ldots, u_5 \) are less relevant in the zero-replica limit.

The quadratic part \( S_0 \) defines the propagator:

\[
\hat{G}_{\alpha\beta}(p) = \frac{1}{p^2 + t} \delta_{\alpha\beta} + \frac{\Delta}{(p^2 + t)^2} + O(n),
\]

\[
\equiv A(p) \delta_{\alpha\beta} + B(p) + O(n).
\]

(8)

Note that \( B(p) \) dominates over \( A(p) \) in low-momentum region with small \( t \), while \( A(p) \) can be more relevant in the replica limit because \( \delta_{\alpha\beta} \) associated with it reduces powers of \( n \).
A simple dimensional analysis gives that the parameters of this theory have the following mass dimensions

\[ [u_i] = 4 - d \ (i = 1, \ldots, 5), \quad [t] = [\Delta] = 2. \]  

(9)

We note that equation (8) indicates that the correlation length \( \xi \) is proportional to \( t^{-1/2} \) at the tree level. We can see the critical phenomena by one-parameter tuning \( t = 0 \) despite that \( \Delta \) is a relevant coupling constant.

Let us look at the perturbative expansion of the free-energy density \( \bar{f} \), which is obtained as

\[ \bar{f} = \lim_{n \to 0} \frac{f_{\text{rep}}}{n}, \quad e^{-V f_{\text{rep}}} = \int \prod_{\alpha=1}^{n} \mathcal{D} \phi e^{-S}, \]  

(10)

where \( V \) is the volume of the system. The leading order is easily computed. The result is written as

\[ \bar{f} = \frac{1}{2} (1 - \Delta \frac{d}{dt}) \int_{|p| \leq \Lambda} \frac{d^d p}{(2\pi)^d} \log \left( p^2 + t \right). \]  

(11)

Here \( \Lambda \) is the ultraviolet cutoff. The integration is identical with the free-energy of the Gaussian model, whose singular part behaves as \( t^{d/2} \) when \( t \) is sufficiently small. Hence the singular part of \( \bar{f} \) is

\[ \bar{f}_{\text{sing}} \propto \Delta t^{d/2 - 1} (1 + O(\Delta^{-1} t)), \]  

(12)

where the \( O(\Delta^{-1} t) \) term is irrelevant near the criticality: \( t \sim 0 \) with fixed \( \Delta \).

Next, we compute the first-order correction

\[ \lim_{n \to 0} \frac{1}{nV} \langle S_{\text{int}} \rangle. \]  

(13)

For instance, the \( u_1 \) interaction gives

\[ \frac{u_1}{8} (\mathcal{A}_1 + \Delta \mathcal{A}_2)^2, \]  

(14)

where

\[ \mathcal{A}_k \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + t)^k}, \quad k = 1, 2. \]  

(15)

Extracting the singular part of \( \mathcal{A}_k \) from equation (14), we have

\[ \frac{u_1}{8} \left( a_1^{d/2 - 1} + \Delta a_2^{d/2 - 2} \right)^2 = \frac{u_1}{8} \Delta^2 a_2^{2d/2 - 3} \left( 1 + O(\Delta^{-1} t) \right), \]  

(16)

where \( a_1 \) and \( a_2 \) are some constants. Similarly, collecting the leading contribution from the each vertex that survives in the zero-replica limit, we get

\[ \bar{f}_{\text{sing}} = \Delta t^{d/2 - 1} \left\{ a_0 + \left( \frac{\Delta u_1}{8} a_2^{d/2 - 3} + \frac{2u_2 + u_3}{2} a_1 a_2 t^{d/2 - 2} + \frac{\Delta u_4}{2} a_1^2 t^{d/2 - 1} \right) \right\}, \]  

(17)

up to the first order of \( u_i \). Similar computation to the first few loops implies that the expansion parameters are \( \Delta^{\alpha_i} u_i t^{d/2 - 2 - \alpha_i} \). Here

\[ \alpha_1 = 1, \quad \alpha_2 = \alpha_3 = 0, \quad \alpha_4 = -1, \quad \alpha_5 = -2. \]  

(18)
We can see that the $\Delta u_1$ term survives while the other corrections in (17) vanish in $6 - \epsilon$ dimensions near the criticality.

Do our expansion parameters work for all diagrams in $\tilde{f}_{\text{sing}}$? Let us consider an arbitrary connected diagram in $\tilde{f}$. It contains two kinds of internal lines, $A(p)$ and $B(p)$. Let $\#A$ and $\#B$ be the number of $A(p)$ and $B(p)$ in the diagram respectively. Since $B(p)$ is proportional to $\Delta$, the diagram generates the following correction:

$$F(t, \Lambda) \Delta^{\#B} \prod_{i=1}^{5} u_i^{p_i},$$

where $F(t, \Lambda)$ represents momentum integrations and summations over the replica indices. We find from the dimensional analysis that (19) has the following form:

$$F(1, \Lambda/t) \Delta t^{d/2-1} \left( \Delta^{-1} t \right)^{p_0} \prod_{i=1}^{5} \left( \Delta^{\alpha_i} u_i t^{d/2-2-\alpha_i} \right)^{p_i}.$$  \hspace{1cm} (20)

Comparing (19) and (20) we find

$$p_0 = \left( \sum_{i=1}^{5} \alpha_i p_i \right) + 1 - \#B.$$ \hspace{1cm} (21)

We must show that

$$p_0 \geq 0.$$ \hspace{1cm} (22)

If $p_0 < 0$, the expansion may fail because the factor $\Delta t^{-1}$ grows near the criticality, even though the last factor gives small correction.

Here we prove (22). Because the number of the internal lines are $2 \sum p_i$,

$$\#A + \#B = 2 \sum_{i=1}^{5} p_i.$$ \hspace{1cm} (23)

From equation (3) we find that the interaction conjugate to $u_i$ has $2 - \alpha_i$ summations of the replica indices. Thus the number of the sums in $F$ is

$$\sum_{i=1}^{5} (2 - \alpha_i) p_i.$$ \hspace{1cm} (24)

It should be reduced up to 1 by Kronecker’s delta associated with $A(p)$: otherwise, $F$ becomes $O(n^2)$ and disappears in equation (17). Thus, we have the following restriction to $\#A$:

$$\#A \geq \sum_{i=1}^{5} (2 - \alpha_i) p_i - 1.$$ \hspace{1cm} (25)

Using (23) and (25), we obtain (22).
3. Renormalized perturbation

Now we are going to the renormalized perturbation. Using the renormalized 2-point vertex function \( \Gamma_{2\alpha\beta}(p) \), we introduce the renormalized parameters \( m \) and \( \Delta_R \) as

\[
\Gamma_{2\alpha\beta}(p) = \left(p^2 + m^2\right) \delta_{\alpha\beta} + \Delta_R m^2 + O(p^4). \tag{26}
\]

We have shown that the expansion parameters are \( \Delta_{\alpha i} u_i t^{d/2-2-\alpha_i} \), so that we put the following renormalization prescription by using the renormalization counterparts for them:

\[
\Delta_{R i} \Gamma_{4 i}(0)m^{d-4} = g_i, \quad (i = 1, \ldots, 5), \tag{27}
\]

where \( \Gamma_{4 i}(p) \) is the renormalized 4-point functions associated with \( u_i \). Then one can perform the renormalized perturbation with the small parameters \( g_i \).

The one-loop beta functions for \( g_i \) in \( 6 - \epsilon \) dimensions are easily computed by the usual manner. Following reference [1], we keep the terms proportional to

\[
\int \frac{d^d p}{(2\pi)^d (p^2 + m^2)^3}, \tag{28}
\]

which are expected to be important in \( 6 - \epsilon \) dimensions. See figure [1]. Here an internal line with a cross(x) represents \( B(p) \) and without crosses \( A(p) \).

Similar calculation performed in reference [1] gives

\[
\begin{align*}
\beta_1 &= -\epsilon g_1 + 3K g_1^2, \\
\beta_2 &= (2 - \epsilon) g_2 + 3K g_1 (g_2 + g_3), \\
\beta_3 &= (2 - \epsilon) g_3 + 5K g_1 (g_2 + g_3), \\
\beta_4 &= (4 - \epsilon) g_4 + 4K (g_2 + g_3)^2, \\
\beta_5 &= (6 - \epsilon) g_5 + 36K (g_2 + g_3)g_4,
\end{align*} \tag{29}
\]

where \( K = \Gamma(4 - d/2)/(4\pi)^{d/2} \). The non-trivial fixed point is, up to \( O(\epsilon) \),

\[
g_1^* = \frac{1}{3K} \epsilon, \quad g_2^* = \cdots g_5^* = 0. \tag{30}
\]

Now we compute eigenvalues of the scaling matrix

\[
\left. \frac{\partial \beta_i}{\partial g_j} \right|_{g=g^*}. \tag{31}
\]
The result is
\( \{ \epsilon, 2 - \epsilon, \frac{2}{3}(3 + \epsilon), 4 - \epsilon, 6 - \epsilon \}. \) (32)

The eigenvalues are all positive. Thus the non-trivial fixed point is stable, contrary to the conclusion in reference [1].

The discrepancy stems from the ways of redefinition of the coupling constants. Brézin and de De Dominicis have essentially chose \( \Delta u_i \) as the effective coupling constants. The linear terms of the beta functions correspond to \( \Delta u_i \) are \(-\epsilon\) for all \( i \), and negative eigenvalues emerge. However, the effective coupling constants that appear in perturbative expansion are \( \Delta^\alpha u_i \) as we have seen in the previous section. Since \( \Delta \) has the finite mass dimension, scaling behavior of \( \Delta u_i \) does not reflect the stability of the fixed point.

4. The \( \epsilon \)-expansion near six dimensions

Finally, we compute the leading order of the \( \epsilon \)-expansion for the critical exponents \( \nu \) and \( \eta \) at the non-trivial fixed point (30).

To compute the critical exponents one can take account of the leading infrared divergence only. Figure 2 represents the most singular Feynman diagrams contributing to the self-energy. One easily sees that only the \( u_1 \) term contributes to the most singular diagrams; vertices \( u_2, \ldots, u_5 \) combined with \( B(p) \) giving additional \( n \) factor are suppressed by the \( n \to 0 \) limit. Thus, from the renormalization conditions (26) and (27), the one-loop correction of the bare mass and the two-loop computation of the field renormalization factor \( Z_1 \) are written in terms of the renormalized parameters as follows:

\[
t = m^2 \left\{ 1 - \frac{1}{2} \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} g_1 + O(g_1^2) \right\},
\]

(33)

\[
Z_1 = \left\{ 1 + \frac{1}{12} \frac{\Gamma(6 - d)}{(4\pi)^d} g_1^2 \right\}^{-1}.
\]

(34)

From equations (29) and (33), the composite field renormalization factor \( Z_2 \) is written at the one-loop level as follows:

\[
Z_2 = \left. \frac{d m^2}{d t} \right|_{\Delta, u_i, \text{fixed}} = 1 - \frac{1}{2} \frac{\Gamma(3 - d/2)}{(4\pi)^{d/2}} g_1 + O(g_1^2).
\]

(35)

\[\begin{array}{c}
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\text{Figure 2. Feynman diagrams contributing to the self-energy.}
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Following the standard argument \[2, 3\], define
\[
\gamma_1(g_1, \ldots, g_5) \equiv m \frac{\partial}{\partial m} \ln Z_1 \bigg|_{\Delta, u_i: \text{fixed}},
\]
\[
\gamma_2(g_1, \ldots, g_5) \equiv m \frac{\partial}{\partial m} \ln Z_2 \bigg|_{\Delta, u_i: \text{fixed}}.
\] (36)

We can compute the critical exponents \(\nu\) and \(\eta\) from the following relationship:
\[
\nu = \frac{1}{2} - \gamma_2(g_1^*, \ldots, g_5^*),
\]
\[
\eta = \gamma_1(g_1^*, \ldots, g_5^*). \tag{37}
\]

Substituting the values of the fixed point (30) into the above equations, we obtain the leading order of the \(\epsilon\)-expansion for the critical exponents \(\nu\) and \(\eta\),
\[
\nu = \frac{1}{2} + \frac{1}{12} \epsilon + O(\epsilon^2),
\]
\[
\eta = \frac{1}{54} \epsilon^2 + O(\epsilon^3). \tag{38}
\]

These are exactly the same as the leading-order results in the \(d = 4 - \epsilon\) pure \(\phi^4\) theory \[1, 3\].

5. Summary and discussion

In this paper, we have clarified effective coupling constants for perturbative expansion of physical observables in effective theory for the random field Ising model and reconsidered the stability discussed in reference \[1\]. Our result ensures the \(\epsilon\) expansion near the upper critical dimension. We have also calculated the exponents \(\nu\) and \(\eta\). The result is consistent with the Parisi-Sourlas dimensional reduction in the leading-order computation. It is a nontrivial problem whether the consistency is preserved beyond the leading order, which will be reported elsewhere.

Our beta functions (29) tell us that the upper critical dimension is six, which is consistent with the rigorous result \[8\]. Further, they show that \(g_2\) and \(g_3\) change to relevant coupling constants when \(d\) becomes lower than four. Thus it does not surprise us that the dimensional reduction does not hold when \(d = 3\) \[7\].

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