Dynamics of various population groups in a two-dimensional spatial economy model.

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Abstract. The system of parabolic partial differential equations, which describes the dynamics of various population groups in an urban environment, is considered. The model takes into account both the interaction between different groups in the process of their cohabitation and the diffusion processes of population migration. Various examples of world megacities are considered and some general principles of urban development are formulated.

1. Introduction

Modern urbanism requires taking into account the dynamics of various population groups, taking into account the spatial distribution. In the process of development of the urban environment, natural processes of population diffusion arise, for example, the desire of high-income groups to move to more environmentally friendly zones of the urban environment. Of interest is the study of models of the interaction of various population groups in an urban environment, the geometry of which can strongly depend on the geographical location of megacities. Some cities, such as Moscow, Berlin, Paris, London are located on the plains, some are located on the banks of large rivers, for example, all major cities of the Volga region and Siberia. There are cities that are "sandwiched" between the sea and the mountains, for example, Barcelona and other coastal cities. The model of interaction between different population groups in a megacity requires consideration of both their interaction and diffusion processes. At present, this has become particularly relevant given the large migration flow from Africa and Asia to prosperous countries in western Europe. In this paper, we propose a two-dimensional spatial model describing such an interaction for cities with a "square" geometry, as well as having an optimal geometric configuration according to Christaller, that is, in the form of a regular hexagon. For simplicity, two population groups are considered. The evolution of each population is described by a nonlinear differential equation of a parabolic type.

The resulting system of two-dimensional stationary equations with boundary conditions of the second kind, reflecting the absence of population overflow across the boundary of the region, is solved numerically by the Peaceman-Rachford method. A computational experiment is carried out for different values of the interaction coefficients of different groups of the population and different types of power dependence for diffusion coefficients. Modes of absorption of one population of another are found, as well as modes of coexistence of different groups of the population. Relationships between groups of the population can be friendly, unfriendly and "neutral".
2. Mathematical model

Either the square $\{(x,y): 0 \leq x \leq 1; 0 \leq y \leq 1\}$ or the regular hexagon with the radius equal to 1 is considered as the domain $D$ (Figures 1-2).

Denote by $u = u(x,y,t) = u(r, \varphi, t), \ v = v(x,y,t) = v(r, \varphi, t)$ – density of the first and second populations. $(x,y)$ – cartesian coordinates, $(r, \varphi)$ – polar coordinates:

\[
\begin{aligned}
&x = r \cos(\varphi) \\
y = r \sin(\varphi)
\end{aligned}
\]

The evolution of both populations can be described as follows [1].

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha u(a - bu - cv) - d_1 uv + \text{div}(k_1(u)\text{grad}(u)), \\
\frac{\partial v}{\partial t} &= \beta v(a - bu - cv) - d_2 uv + \text{div}(k_2(v)\text{grad}(v)).
\end{aligned}
\]  

(1)

(2)

In cartesian coordinates:

\[
\begin{aligned}
\text{div}(k_1(u)\text{grad}(u)) &= \frac{\partial}{\partial x}(k_1(u)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(k_1(u)\frac{\partial u}{\partial y}), \\
\text{div}(k_2(v)\text{grad}(v)) &= \frac{\partial}{\partial x}(k_2(v)\frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(k_2(v)\frac{\partial v}{\partial y}).
\end{aligned}
\]  

(3)

In polar coordinates, in which it is convenient to consider the deviation from central symmetry:

\[
\begin{aligned}
\text{div}(k_1(u)\text{grad}(u)) &= \frac{1}{r\partial r}(rk_1(u)\frac{\partial u}{\partial r}) + \frac{1}{r^2\partial \varphi}(k_1(u)\frac{\partial u}{\partial \varphi}), \\
\text{div}(k_2(v)\text{grad}(v)) &= \frac{1}{r\partial r}(rk_2(v)\frac{\partial v}{\partial r}) + \frac{1}{r^2\partial \varphi}(k_2(v)\frac{\partial v}{\partial \varphi}).
\end{aligned}
\]  

(4)

$r$ – distance from the central district (central business district) to the place of residence, $\varphi$ – azimuth angle.

Note that in the work of Zhang these operators are written incorrectly.

Terms $\text{div}(k_1(u)\text{grad}(u))$ and $\text{div}(k_2(v)\text{grad}(v))$ describe the effect of geographical population diffusion. Geographic diffusion terms measure people’s tendency to live in less populated areas. The diffusion coefficients $k_1$ and $k_2$ may in fact depend on the unknown functions $u, v$ and the independent $(x,y,t)$. We have considered the simplest and, perhaps, the most interesting power dependence on the desired functions.

\[
k_1(u) = \kappa_1 u^{\sigma_1}, \ k_2(v) = \kappa_2 v^{\sigma_2}
\]  

(5)
Note that for $\sigma = 1$, we obtain the Darcy equation, which describes the leakage of a liquid in a porous medium.

$$\frac{\partial \omega}{\partial t} = \text{div}(\omega \ \text{grad}(\omega)) + f(\omega)$$

The term $\alpha u(a - bu - cv)$ describes the population’s response to economic conditions. Zhang interprets $a$ as the "physical" capacity of urban space at point $r$. When the parameter $\alpha$ is constant, the physical capacity is uniform in space. If we assume that $(bu + cv)$ is a quantitative measure of the space occupied by both groups, then the value $(a - bu - cv)$ can be considered as an excess of the supply of physical capacity. When this value at some point becomes greater than zero, then this place of residence is more attractive to the population. Obviously, when it is zero, and the $d_{1uv}$ terms and diffusion effects are negligible, population migration stops. The factor $au$ in equation (1) is the rate at which the equilibrium distribution of the population in group 1 is established: if the population density is high, then the establishment of equilibrium is slow because the system is less informed. The $d_{1uv}$ term is used to measure the interaction of groups. This term is not related to economic factors, reflecting social interaction. The $d_{1}$ coefficient can be both positive and negative, and zero. If it is positive, group 1 does not like living with group 2. If $d_{1} = 0$, there are no "social" biases. If $d_{1}$ is negative, the high density of group 2 attracts the population of group 1. For example, we can classify the population according to educational level, and less educated people may tend to live in an area dominated by more educated.

Similarly, we can interpret equation (2).

We chose Neumann conditions as the boundary conditions (we assume that there are no population flows on the border of the region):

$$\frac{\partial u}{\partial n}|_{\partial D} = 0, \frac{\partial v}{\partial n}|_{\partial D} = 0.$$  \hspace{1cm} (6)

At the initial moment of time

$$u|_{t=0} = u_{0}(x,y), \ v|_{t=0} = v_{0}(x,y)$$  \hspace{1cm} (7)

Or

$$u|_{t=0} = u_{0}(r,\phi), \ v|_{t=0} = v_{0}(r,\phi)$$  \hspace{1cm} (8)

3. Results of computer modeling

For numerical solution of the system of equations (1), (2), (6), (7), a homogeneous difference scheme with linearization at the previous step is used [2].

Consider a square and a hexagon as a domain $D$.

Let us assign initial distributions in the form of two regions in which the population density is specified by a paraboloid of rotation (Figures 3-4):

$$u_{0}(x,y) = \begin{cases} U_{0} - h((x-x_{0})^{2} + (y-y_{0})^{2}), \quad (x-x_{0})^{2} + (y-y_{0})^{2} \leq \frac{u_{0}}{h} \\ 0, \quad (x-x_{0})^{2} + (y-y_{0})^{2} > \frac{u_{0}}{h} \end{cases}$$

$$v_{0}(x,y) = \begin{cases} V_{0} - h((x-x_{0})^{2} + (y-y_{0})^{2}), \quad (x-x_{0})^{2} + (y-y_{0})^{2} \leq \frac{v_{0}}{h} \\ 0, \quad (x-x_{0})^{2} + (y-y_{0})^{2} > \frac{v_{0}}{h} \end{cases}$$  \hspace{1cm} (9)
By modeling equations (1), (2), (6), (7) with different sets of parameters, we get different solutions.

\[
\alpha = 1, \beta = 1, a = 1, b = 1, c = 1, d_1 = 0, d_2 = 1, \kappa_1 = 1, \kappa_2 = 1, \sigma_1 = 1, \sigma_2 = 5 \text{ (Figures 5-6)}
\]

\[
\alpha = 1, \beta = 1, a = 1, b = 1, c = 1, d_1 = -1, d_2 = 1, \kappa_1 = 1, \kappa_2 = 1, \sigma_1 = 3, \sigma_2 = 1 \text{ (Figures 7-8)}
\]
\( \alpha = 0, \beta = 1, a = 1, b = 1, c = 1, d_1 = 1, d_2 = 1, \kappa_1 = 1, \kappa_2 = 1, \sigma_1 = 1, \sigma_2 = 1 \) (Figures 9-10)

\[ \begin{align*}
\text{Figure 9.} \quad \text{Equation (1-2,6-7) at time 250.} \\
\text{Figure 10.} \quad \text{Equation (1-2,6-7) at time 600.}
\end{align*} \]

\( \alpha = 1, \beta = 1, a = 1, b = 1, c = 1, d_1 = 1, d_2 = -1, \kappa_1 = 1, \kappa_2 = 0, \sigma_1 = 1, \sigma_2 = 1 \) (Figures 11-12)

\[ \begin{align*}
\text{Figure 11.} \quad \text{Equation (1-2,6-7) at time 250.} \\
\text{Figure 12.} \quad \text{Equation (1-2,6-7) at time 600.}
\end{align*} \]

4. Conclusion

The paper considers the evolution of diverse urban models in the framework of approximation of dynamic spatial interaction. A simulation of one of the most important aspects of the development of the city, namely, the interaction of various population groups. A large number of computer experiments were carried out at various coefficients characterizing the mathematical model, and some features of the coexistence of certain population groups were noted.

References

[1] Zhang W-B 1991 Synergetic Economics: Time and Change in Nonlinear Economics (Berlin: Springer)

[2] Samarskii A A 2001 The Theory of Difference Schemes (New York: Marcel Dekker)