Book Embeddings of Graph Products

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Abstract

A $k$-stack layout (also called a $k$-page book embedding) of a graph consists of a total order of the vertices, and a partition of the edges into $k$ sets of non-crossing edges with respect to the vertex order. The stack number (book thickness, page number) of a graph is the minimum $k$ such that it admits a $k$-stack layout. A $k$-queue layout is defined similarly, except that no two edges in a single set may be nested.

It was recently proved that graphs of various non-minor-closed classes are subgraphs of the strong product of a path and a graph with bounded treewidth. Motivated by this decomposition result, we explore stack layouts of graph products. We show that the stack number is bounded for the strong product of a path and (i) a graph of bounded pathwidth or (ii) a bipartite graph of bounded treewidth and bounded degree. The results are obtained via a novel concept of simultaneous stack-queue layouts, which may be of independent interest.

1 Introduction

Embedding graphs in books is a fundamental problem in graph theory, which has been the subject of intense research since their introduction in 70s by Ollmann [25]. A book embedding (also known as a stack layout) of a graph $G = (V, E)$ consists of a total order, $\sigma$, of $V$ and an assignment of the edges to stacks (pages), such that no two edges in a single stack cross; that is, there are no edges $(u, v)$ and $(x, y)$ in a stack with $u <_{\sigma} x <_{\sigma} v <_{\sigma} y$.

The minimum number of pages needed for a book embedding of a graph $G$ is called its stack number (or book thickness or page number) and denoted by $\text{sn}(G)$.

Book embeddings have been extensively studied for various families of graphs. In particular, the graphs with stack number one are precisely the outerplanar graphs, while the graphs with stack number at most two are the subgraphs of planar Hamiltonian graphs [6]. The stack number of planar graphs is four [5], graphs of genus $g$ have stack number $O(\sqrt{g})$ [24], while for graphs of treewidth $tw$, it is at most $tw + 1$ [18]. More generally, all proper minor-closed graph families have a bounded stack number [7]. Non-minor-closed classes of graphs have also been investigated. Bekos et al. proved that 1-planar graphs have bounded stack number [3]. Recall that a graph is $k$-planar if it can be drawn in the plane with at most $k$ crossings per edge. Recently the result has been generalized to a wider family of $k$-framed graphs that admit a drawing with a planar skeleton, whose faces have degree at most $k \geq 3$ and whose crossing edges are in the interiors of the faces [4]. In general however, the best-known upper bound on the stack number of $k$-planar graphs is $O(\log n)$ [12].

We suggest to attack the problem of determining book thickness of non-planar graphs using graph products. Formally, let $A$ and $B$ be graphs. A product of $A$ and $B$ is a graph defined on a vertex set

$$V(A) \times V(B) = \{(v, x) : v \in V(A), x \in V(B)\}.$$
A potential edge, \((v,x),(u,y)\) \(\in V(A) \times V(B)\), can be classified as follows:

- **A-edge**: \(v = u\) and \((x,y) \in E(B)\), or
- **B-edge**: \(x = y\) and \((v,u) \in E(A)\), or
- **direct-edge**: \((v,u) \in E(A)\) and \((x,y) \in E(B)\).

The **cartesian product** of \(A\) and \(B\), denoted by \(A \Box B\), consists of A-edges and B-edges. The **direct product** of \(A\) and \(B\), denoted by \(A \times B\), consists of direct edges. The **strong product** of \(A\) and \(B\), denoted by \(A \boxtimes B\), consists of A-edges, B-edges, and direct-edges. Figure 1 illustrates examples of the defined graph products. Notice that all the products are symmetric. In this paper, we study stack layouts of strong products of a path and a bounded-treewidth graph (refer to Section 2 for a definition), focusing primarily on the following question:

**Open Problem 1.** Is stack number of \(P_n \boxtimes G\), where \(P_n\) is a path and \(G\) is a graph of treewidth \(tw \geq 1\), bounded by \(f(tw)\) for some function \(f\)?

Our motivation for studying stack layouts of graph products comes from a recent development of decomposition theorems for planar and beyond-planar graphs \([13, 15, 27]\). Dujmovič, Morin, and Wood \([15]\) recently show the following:

**Lemma 1** \(([15])\). Every \(k\)-planar graph is a subgraph of the strong product of a path and a graph of treewidth \(O(k^5)\).

Notice that Lemma 1 together with an affirmative answer to Open Problem 1 would provide a constant stack number for all \(k\)-planar graphs, thus resolving a long-standing open problem listed in a recent survey on graph drawing of beyond-planar graphs \([11]\). Furthermore, a similar decomposition exists for other classes of non-minor-closed families of graphs, such as map graphs, string graphs, graph powers, and nearest neighbor graphs, whose stack number is not known to be bounded by a constant; refer to \([15]\) for exact definitions. Interestingly, a negative answer to Open Problem 1 would resolve another question in the context of queue layouts that remains unsolved for more than thirty years.

A **queue layout** is a “dual” concept of a stack layout. For a graph \(G = (V,E)\), it consists of a total order, \(\sigma\), of \(V\) and an assignment of the edges to queues, such that no two edges in a single queue nest; that is, there are no edges \((u,v)\) and \((x,y)\) in a queue with \(u <_\sigma x <_\sigma y <_\sigma v\). The minimum number of queues needed in a queue layout of a graph is called its **queue number** and denoted by \(qn(G)\) \([20]\). As with stack
layouts, the queue number is known to be bounded for many classes of graphs, including planar graphs [13], graphs with bounded treewidth [14, 30], and all proper minor-closed classes of graphs [13, 15]. Queue layouts have been introduced by Heath, Leighton, and Rosenberg [19, 20], who tried to measure the power of stacks and queues to represent a given graph. Despite a wealth of research on the topic, a fundamental question of what is more “powerful” remains unanswered. That is, Heath et al. [19] ask whether the stack number of a graph is bounded by a function of its queue number, and whether the queue number of a graph is bounded by a function of its stack number. In a study of queue layouts of graph products, Wood [31] shows that for a path $P_n$ and all graphs $G$, $qn(P_n \boxtimes G) \leq 3 qu(G) + 1$. This result together with a negative answer to Open Problem 1 would provide an example of a graph (namely, the strong product of a path and a bounded-treewidth graph) that has a constant queue number but an unbounded stack number; thus, resolving one direction of the question posed by Heath et al. [19].

Results and Organization

In this paper we introduce and initiate an investigation of Open Problem 1. Our contribution is twofold. Firstly, we resolve the problem in affirmative for two subclasses of bounded-treewidth graphs. Secondly, we provide an evidence that the most “natural” approach cannot lead to a positive answer of the problem.

Positive Results. It is easy to verify that the stack number of $P_n \boxtimes G$ is bounded by a constant when $G$ is a “simple” graph such as a path, a star, or a cycle. Notice that the strong graph product consists of $n$ copies of $G$, which are connected by inter-copy edges. A natural approach is to layout each copy independently using a constant number of stacks and then join individual results into a final stack layout. In order to be able to embed inter-copy edges in a few stacks, one has to alternate direct and reverse vertex orders for the copies of $G$; refer to Figure 2 for the process of embedding $P_n \boxtimes P_m$ in four stacks.

The above technique can be generalized using the concept of simultaneous stack-queue layouts. Let $\sigma$ be a total order of $V$ for a graph $G = (V, E)$. A simultaneous $s$-stack $q$-queue layout consists of $\sigma$ together with (i) a partition of $E$ into $s$ stacks with respect to $\sigma$, and (ii) a partition of $E$ into $q$ queues with respect to $\sigma$. In such a layout every edge of $G$ is associated with a stack and with a queue. We stress the difference with so-called mixed layouts in which an edge belongs to a stack or to a queue [28].
In order to state the first main result of the paper, we use dispersable book embeddings in which the graphs induced by the edges of each page are 1-regular; see Figure 7a. The minimum number of pages needed in a dispersable book embedding of $G$ is called its dispersable stack number, denoted $dsn(G)$; it is also known as matching book thickness [1,6].

**Theorem 1.** Let $H$ be a bipartite graph and $G$ be a graph that admits a simultaneous $s$-stack $q$-queue layout. Then

(i) $sn(H \boxtimes G) \leq s + dsn(H)$,
(ii) $sn(H \times G) \leq 2q \cdot dsn(H)$,
(iii) $sn(H \boxdot G) \leq 2q \cdot dsn(H) + s + dsn(H)$.

What graphs admit simultaneous layouts for constant $s$ and $q$? We prove that graphs of bounded pathwidth (see Section 2 for a definition) have such a layout. Although it is known that both the stack number and the queue number of pathwidth-$p$ graphs is at most $p$ [14,29], Lemma 2 (in Section 3) shows that the bounds can be achieved using a common vertex order. As a direct corollary of the lemma, Theorem 1, and an observation that $dsn(P_n) = 2$, we get the following result.\(^1\)

**Corollary 1.** Let $G$ be a graph of pathwidth $p$. Then $sn(P_n \boxtimes G) \leq 5p + 2$.

Notice that Corollary 1 combined with Lemma 1 implies an alternative proof of the $O(\log n)$ upper bound for the stack number of $k$-planar graphs, since for every graph $G$, $pw(G) \in O(tw(G) \cdot \log n)$ [8].

Another corollary of Theorem 1 affirmatively resolves Open Problem 1 for the strong product of a path and a bounded-treewidth bipartite graph of bounded maximum vertex degree. For that case we bound the dispersable stack number of a bipartite graph by a function of its treewidth and the maximum vertex degree; see Lemma 3 in Section 3.

**Corollary 2.** Let $G$ be a bipartite graph of treewidth $tw$ with maximum vertex degree $\Delta$. Then $sn(P_n \boxtimes G) \leq 3(tw + 1)\Delta + 1$.

**Negative Results.** Next we investigate simultaneous stack-queue layouts. We prove that if a graph admits a simultaneous $s$-stack $q$-queue layout, then its pathwidth is bounded by a function of $s$ and $q$. In other words, the class of $O(1)$-pathwidth graphs coincides with the class of graphs admitting a simultaneous $O(1)$-stack $O(1)$-queue layout.

**Theorem 2.** Let $G$ be a graph admitting a simultaneous $s$-stack $q$-queue layout. Then $G$ has pathwidth at most $2s \cdot q$.

Corollaries 1 and 2 provide sufficient conditions for a graph $G$ to imply a bounded stack number of $P_n \boxtimes G$. Yet many relatively simple graphs of bounded treewidth (such as trees) have pathwidth $\Omega(\log n)$ and an unbounded vertex degree. A reasonable question is whether the conditions are necessary. Next we study the aforementioned natural approach to construct stack layouts of graph products, and prove that it cannot lead to a constant number of stacks for graphs with an unbounded pathwidth. Formally, call a stack layout of $P_n \boxtimes G$ separated if for at least two consecutive copies of $G$, $G_1$ and $G_2$, all vertices of $G_1$ precede all vertices of $G_2$ in the vertex order. The next result shows that a separated layout of $P_n \boxtimes G$ with a constant number of stacks implies a bounded pathwidth of $G$.

**Theorem 3.** Assume $P_n \boxtimes G$ has a separated layout on $s$ stacks. Then $G$ admits a simultaneous $s$-stack $s^2$-queue layout, and therefore, $pw(G) \leq 2s^3$.

\(^1\)Very recently, Dujmović, Morin, and Yelle [16] independently proved a result asymptotically equivalent to Corollary 1; see Section 4 for a discussion.
The remaining of the paper is organized as follows. After recalling basic definitions in Section 2, we prove the main results of the paper in Section 3. Section 4 is devoted to a discussion of related works on stack and queue layouts of graph products. Section 5 concludes the paper with possible future directions and interesting open problems.

2 Preliminaries

Throughout the paper, $G = (V(G), E(G))$ is a simple undirected graph. We denote a path with $n$ vertices by $P_n$. A vertex order, $\sigma$, of a graph $G$ is a total order of its vertex set $V(G)$, such that for any two vertices $u$ and $v$, $u <_\sigma v$ if and only if $u$ precedes $v$ in $\sigma$. Let $F$ be a set of $k \geq 2$ independent (that is, having no common endpoints) edges $(s_i, t_i), 1 \leq i \leq k$. If $s_1 <_\sigma \cdots <_\sigma s_k <_\sigma t_k <_\sigma \cdots <_\sigma t_1$, then $F$ is a $k$-rainbow, while if $s_1 <_\sigma \cdots <_\sigma s_k <_\sigma t_1 <_\sigma \cdots <_\sigma t_k$, then $F$ is a $k$-twist. Two independent edges forming a 2-twist (2-rainbow) are called crossing (nested).

A $k$-stack layout of a graph is a pair $(\sigma, \{S_1, \ldots, S_k\})$, where $\sigma$ is a vertex order and $\{S_1, \ldots, S_k\}$ is a partition of $E(G)$ into stacks, that is, sets of pairwise non-crossing edges. Similarly, a $k$-queue layout is $(\sigma, \{Q_1, \ldots, Q_k\})$, where $\{Q_1, \ldots, Q_k\}$ is a partition of $E(G)$ into sets of pairwise non-nested edges called queues. The minimum number of stacks (queues) in a stack (queue) layout of a graph is its stack number (queue number). It is easy to see that a $k$-stack layout ($k$-queue layout) cannot have a $k$-twist ($k$-rainbow).

In contrast, a vertex order without a $(k+1)$-rainbow corresponds to a $k$-queue layout [20]. Furthermore, a vertex order without a $(k + 1)$-twist may not produce a $k$-stack layout but corresponds to a $f(k)$-stack layout; the best-known function $f$ is quadratic [10].

A tree decomposition of a graph $G$ is given by a tree $T$ whose nodes index a collection $(B_x \subseteq V(G) : x \in V(T))$ of sets of vertices in $G$ called bags such that:

- For every edge $(u, v)$ of $G$, some bag $B_x$ contains both $u$ and $v$, and
- For every vertex $v$ of $G$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty connected subtree of $T$.

The width of a tree-decomposition is $\max_x |B_x| - 1$, and the treewidth of a graph $G$, denoted $tw(G)$, is the minimum width of any tree decomposition of $G$.

A path decomposition is a tree decomposition in which the underlying tree, $T$, is a path. Thus, it can be thought of as a sequence of subsets of vertices, called bags, such that each vertex belongs to a contiguous subsequence of bags and each two adjacent vertices have at least one bag in common. The pathwidth of a graph $G$, denoted $pw(G)$, is the minimum width of any path decomposition of $G$. We also use an equivalent definition of the pathwidth called the vertex separation number [8, 22]. Consider a vertex order $\sigma$ of a graph $G$. The vertex cut in $\sigma$ at a vertex $v \in V(G)$ is defined to be $C(v) = \{x \in V(G) : \exists (x, y) \in E(G), x <_\sigma v \leq_\sigma y\}$. The vertex separation number of $G$ is the minimum, taken over all vertex orders $\sigma$ of $G$, of a maximum cardinality of a vertex cut in $\sigma$.

3 Main Proofs

3.1 Positive Results

**Theorem 1.** Let $H$ be a bipartite graph and $G$ be a graph that admits a simultaneous $s$-stack $q$-queue layout. Then

(i) $\text{sn}(H \square G) \leq s + \text{dsn}(H),$
(ii) $\text{sn}(H \times G) \leq 2q \cdot \text{dsn}(H),$
(iii) $\text{sn}(H \boxtimes G) \leq 2q \cdot \text{dsn}(H) + s + \text{dsn}(H).$
Figure 3: An \((s + 4q + 2)\)-stack layout of the strong product of \(P_4\) and a graph \(G\) that admits a simultaneous \(s\)-stack \(q\)-queue layout using vertex order \(\sigma\). \(G_1, G_2, G_3,\) and \(G_4\) correspond to copies of \(G\) laid out by alternating \(\sigma\) and its reverse, \(\sigma^r\). Groups of stacks are colored differently.

Proof. For every pair of graphs, the set of edges of the strong product is the union of edges of the cartesian product and the direct product of the graphs. Therefore, claim (iii) of the theorem follows from claims (i) and (ii), which we prove next.

Let \(\pi\) be a vertex order of \(H\) in the dispersable stack layout, and let \(\sigma\) and \(\sigma^r\) be a vertex order of \(G\) and its reverse in the simultaneous stack-queue layout. We call the parts of the bipartition of \(H\) white and black, and denote by \(0 \leq \pi(v) < n\) the index of vertex \(v \in V(H)\) in \(\sigma\). To construct an order, \(\phi\), for the stack layout of a graph product, we start with \(\pi\) and replace each white vertex of \(H\) with \(\sigma\) and each black vertex of \(H\) with \(\sigma^r\). Formally, for two vertices \((v, x)\) and \((u, y)\) of a product, let \(\phi(v, x) < \phi(u, y)\) if and only if \(v \neq u\) and \(\pi(v) < \pi(u)\), or \(v = u\), \(v\) is white, and \(x < \sigma y\), or \(v = u\), \(v\) is black, and \(y < \sigma x\).

We emphasize that the same vertex order is utilized for all three graph products; see Figure 2 and Figure 3 for illustrations.

We first verify that \(su(H \square G) \leq s + dsn(H)\), thus proving claim (i) of the theorem. Since \(\sigma\) and \(\sigma^r\) are vertex orders of an \(s\)-stack layout of \(G\) and different copies of \(G\) are separated in \(\phi\), all \(G\)-edges are embedded in \(s\) stacks. Further, every edge of \(H\) is incident to a white and a black vertex of \(H\) that correspond to \(\sigma\) and \(\sigma^r\). Thus, \(H\)-edges between a pair of copies of \(G\) are non-crossing and can be assigned to the same stack. Since the edges of \(H\) require \(dsn(H)\) stacks and each stack consists of independent edges, all \(H\)-edges are embedded in \(dsn(H)\) stacks.

Next we show that direct-edges can be assigned to \(2q \cdot dsn(H)\) stacks, which we denote by \(S_i^j\) for \(1 \leq i \leq q\) and \(1 \leq j \leq 2dsn(H)\). To this end, partition all direct-edges into \(2dsn(H)\) groups and employ \(q\) stacks for each of the groups. A group of a direct-edge, \(e\), with endpoints \((v, x)\) and \((u, y)\) is determined by the stack of \((v, u) \in E(H)\) in the dispersable layout of \(H\) and by the relative order of \(x\) and \(y\) in \(\sigma\). Specifically,

\[ \begin{align*} 
&\text{if } x <_\sigma y, (v, u) \in S_j, \text{ and } (x, y) \in Q_i, \text{ then } e \in S_i^{2j}; \\
&\text{if } y <_\sigma x, (v, u) \in S_j, \text{ and } (x, y) \in Q_i, \text{ then } e \in S_i^{2j+1}. 
\end{align*} \]

Here \(\{S_1, \ldots, S_{dsn(H)}\}\) is the partition of \(E(H)\) in the dispersible stack layout of \(H\), and \(\{Q_1, \ldots, Q_q\}\) is the partition of \(E(G)\) in the \(q\)-queue layout of \(G\).

Let us verify that the direct-edges in a stack are non-crossing. For the sake of contradiction, assume two edges, \(e_1\) with endpoints \((v_1, x_1)\) and \((u_1, y_1)\), and \(e_2\) with endpoints \((v_2, x_2)\) and \((u_2, y_2)\), cross each other. We assume \(e_1\) and \(e_2\) belong to a group \(S_i^{2j}\) for some \(1 \leq i \leq q, 1 \leq j \leq dsn(H)\); the other case is symmetric. Since \(e_1\) and \(e_2\) cross,
Figure 4: An illustration for Lemma 2: creating (a) a 2-stack and (b) a 2-queue layout for a graph with pathwidth 2 using its vertex separation order.

\[ \pi(v_1) = \pi(v_2) \] and \( \phi(v_1, x_1) < \phi(v_2, x_2) < \phi(u_1, y_1) < \phi(u_2, y_2) \). By (C2), we have \( x_1 < x_2 \), and by (C3), we have \( y_2 < y_1 \). Hence, two edges of \( G \) from the same queue, \( (x_1, y_1) \) and \( (x_2, y_2) \), form a 2-rainbow in \( \sigma \); a contradiction.

Therefore, \( \exists G \) admits an \( (s + \text{dsn}(H)) \)-stack layout, \( H \times G \) admits a \( (2q \cdot \text{dsn}(H)) \)-stack layout, and \( H \boxtimes G \) admits a \( (2q \cdot \text{dsn}(H) + s + \text{dsn}(H)) \)-stack layout.

The bounds of Theorem 1 can be improved for certain families of graphs. For example, the stack number of the strong product of two paths is at most 4, while the theorem yields an upper bound of 7; see Figure 2. However, for a complete graph on 2 vertices, \( K_k \), it holds that \( \text{sn}(P_k \boxtimes K_k) \geq 3k - 1 \) (following from the density of the product [6]), while \( \text{sn}(K_k) = \text{qn}(K_k) = k \). Hence, the given bounds are asymptotically worst-case optimal.

Next we explore simultaneous linear layouts of bounded-pathwidth graphs. While it is known that the stack number and the queue number of pathwidth-\( p \) graphs is at most \( p \) \([14, 29]\), the existing proofs do not utilize the same vertex order for the stack and queue layouts. We show that the bounds can be achieved in a simultaneous stack-queue layout.

Lemma 2. A graph of pathwidth \( p \) has a simultaneous \( p \)-stack \( p \)-queue layout.

Proof. Consider a vertex order, \( \sigma \), of the given graph, \( G \), corresponding to its vertex separation number, which equals to the pathwidth, \( p \) \([8, 22]\). We prove that \( \sigma \) yields a \( p \)-stack layout of \( G \) and a \( p \)-queue layout of \( G \); see Figure 4.

Assume that edges of \( G \) form a rainbow of size greater than \( p \) with respect to \( \sigma \). That is, let \( \sigma \) be such that \( u_1 < \sigma \cdots < \sigma u_{p'} < \sigma \cdots < \sigma v_1 \) for some \( p' > p \) and \( (u_i, v_i) \in E(G) \) for all \( 1 \leq i \leq p' \). Then the vertex cut at \( v_{p'} \) has cardinality at least \( p' \), as \( u_1, \ldots, u_{p'} \in C(v_{p'}) \), which contradicts that the vertex separation is \( p \). Therefore, the queue number of \( G \) is at most \( p \).

To construct a \( p \)-stack layout, consider the vertices of \( G \) in the order \( v_1 < \sigma v_2 < \sigma \cdots < \sigma v_n \). Let \( E^i \) be the set of forward edges of \( v_i \), that is, \( E^i = \{(v_i, y) \in E(G) : v_i < \sigma y\} \). We process the vertices in the order and assign edges to \( p \) stacks while maintaining the following invariant for every \( 1 < i \leq n \):

- all edges \( E^1, \ldots, E^{i-1} \) are assigned to one of \( p \) stacks, and
- all edges from \( E^i \) for every \( 1 \leq j \leq i - 1 \) are in the same stack.

Clearly, the invariant is satisfied for \( i = 2 \) by assigning \( E^1 \) to a single stack. Suppose we obtained a stack assignment for all forward edges up to \( E^{i-1} \); let us process \( E^i \). Assume that \( E^i \neq \emptyset \), and observe that \( v_i \in C(v_{i+1}) \) and \( |C(v_{i+1}) \setminus \{v_i\}| \leq p - 1 \). Edges of \( E^i \) can cross only already processed edges incident to a vertex from \( C(v_{i+1}) \setminus \{v_i\} \). By the assumption of our invariant, such edges utilize at most \( p - 1 \) distinct stacks. Hence, we have an available stack, which we use for \( E^i \), and thus maintaining the invariant.

Corollary 1 follows from Theorem 1, Lemma 2, and an observation that \( \text{dsn}(P_n) = 2 \). In order to prove Corollary 2, we need the following auxiliary lemma.
Lemma 3. Let $G$ be a bipartite graph of maximum vertex degree $\Delta$ that admits an $s$-stack layout. Then $dsn(G) \leq s \cdot \Delta$.

Proof. Edges of every stack of the $s$-stack layout of $G$ form an outerplanar graph. Since $G$ is bipartite, the edges of each stack can be partitioned into at most $\Delta$ subgraphs, which are 1-regular. Thus the dispersable stack number of $G$ is at most $s \cdot \Delta$. \hfill \Box

Corollary 2 follows from Theorem 1, Lemma 3, and the fact that the stack number of a graph with treewidth $tw$ is at most $tw + 1$ [18]. Notice that in order to apply Theorem 1, we set $G$ to be a given path and $H$ to be a given bipartite bounded-treewidth graph. The bound of Corollary 2 can be reduced for low-treewidth graphs, whose dispersable stack number is lower than the one given by Lemma 3 [1].

3.2 Negative Results

In order to prove Theorem 2, we first consider the case when $s = q = 1$ and prove the existence of a path decomposition of width 2 with a certain property.

Lemma 4. Let $G$ be an $n$-vertex graph admitting a simultaneous 1-stack 1-queue layout with respect to a vertex order $\sigma = (v_1, v_2, \ldots, v_n)$. Then $G$ has pathwidth at most 2. Furthermore, the corresponding path decomposition consists of $n$ bags $B_1, \ldots, B_n$ such that $|B_x| \leq 3$ and $v_x \in B_x$ for all $1 \leq x \leq n$.

Proof. It is tempting to approach the lemma by arguing that for a vertex order, $\sigma$, the corresponding vertex separation number is bounded. However, a simultaneous 1-stack 1-queue layout of a star graph with its center at position $\lfloor n/2 \rfloor$ of $\sigma$ has an unbounded vertex cut. Therefore, we explicitly construct a path decomposition of $G$ to prove the claim. We use induction on the number of vertices in $G$; the base of the induction with $n = 1$ clearly holds.

Consider the last vertex in the vertex order, $v_n \in V(G)$, and let $d \geq 0$ be the degree of $v_n$. If $d = 0$ then we inductively construct a path decomposition for the first $n - 1$ vertices and append a bag containing the single vertex, $\{v_n\}$. Thus we may assume $d > 0$.

Let $v_i \in V(G)$ be the smallest (with respect to $\sigma$) neighbor of $v_n$ for some $1 \leq i < n$. Since $\sigma$ corresponds to a simultaneous 1-stack 1-queue layout, no edges of $G$ cross each other and no edges of $G$ nest each other. Thus, every vertex $x \in V(G)$ with $v_i <_\sigma x <_\sigma v_n$ is either (a) adjacent to $v_i$, or (b) adjacent to $v_n$, or (c) adjacent to both $v_i$ and $v_n$, or (d) an isolated vertex. Otherwise edge $(v_i, v_n)$ crosses or nests an edge $(x, y)$ for some $y \in V(G) \setminus \{v_i, v_n\}$; see Figure 5.

In order to build a desired path decomposition, we inductively apply the argument to a subgraph of $G$ induced by the vertices $v_1, v_2, \ldots, v_i$. Assume that the resulting path decomposition of the subgraph consists of bags $\{B_j : 1 \leq j \leq i\}$. We extend it to a path decomposition of $G$ by appending $n - i$ bags. Namely, if $d > 1$ then the extension is

\[
\{v_i, v_{i+1}\}, \ldots, \{v_i, v_{h-1}\}, \{v_i, v_h, v_n\}, \{v_{h+1}, v_n\}, \ldots, \{v_{n-1}, v_n\}, \{v_n\},
\]

where $v_h, i < h < n$ is the first neighbor of $v_n$ after $v_i$ in $\sigma$. Otherwise if $d = 1$ then we use bags

\[
\{v_i, v_{i+1}\}, \ldots, \{v_i, v_{n-1}\}, \{v_i, v_n\}.
\]

It is straightforward to verify that the constructed path decomposition of $G$ satisfies the requirements of the lemma. \hfill \Box
Figure 5: A graph admits a simultaneous 1-stack 1-queue layout if and only if its pathwidth is at most 2, since a non-neighbor of \( v_i \) and \( v_n, x \), between the two vertices creates either a crossing or a nested edge.

**Theorem 2.** Let \( G \) be a graph admitting a simultaneous \( s \)-stack \( q \)-queue layout. Then \( G \) has pathwidth at most \( 2 \cdot s \cdot q \).

**Proof.** Consider a vertex order, \( \sigma \), corresponding to the simultaneous \( s \)-stack \( q \)-queue layout. Every edge of \( G \) belongs to a stack and to a queue of the simultaneous layout. Thus all the edges can be partitioned into \( s \cdot q \) disjoint sets, denoted \( E^{i,j} \subseteq E(G) \) with \( 1 \leq i \leq s, 1 \leq j \leq q \), such that each set induces a simultaneous 1-stack 1-queue layout with vertex order \( \sigma \). By Lemma 4, for every (possibly disconnected) subgraph \( G^{i,j} = (V(G), E^{i,j}) \) of \( G \), there exists a path decomposition of width 2 whose bags are denoted by \( B_x^{i,j} \) for \( x \in V(G) \). Define a path decomposition of \( G \) to be \( \{ \cup_{i,j} B_x^{i,j} : x \in V(G) \} \), where the union is taken over all \( 1 \leq i \leq s, 1 \leq j \leq q \). Next we verify that the construction is indeed a path decomposition of \( G \):

- Every edge of \( G \) belongs to some set of the edge partition; thus, there is a bag in the corresponding path decomposition, which contains both endpoints of the edge.
- For every \( 1 \leq i \leq s, 1 \leq j \leq q \), a vertex \( x \in V(G) \) is in a continuous interval of bags of the path decomposition of \( G^{i,j} \). By Lemma 4, the interval contains bag \( B_x^{i,j} \); therefore, the union of such intervals taken over all path decompositions forms a continuous interval.

Finally we notice that for all \( 1 \leq i \leq s, 1 \leq j \leq q \), bag \( B_x^{i,j} \in \cup_i \cup_j B_x^{i,j} \subseteq V(G) \) consists of vertex \( x \) and possibly two more vertices of \( G \). Hence, \( |\cup_{i,j} B_x^{i,j}| \leq 2 \cdot s \cdot q + 1 \). That is, the width of the constructed path decomposition is at most \( 2 \cdot s \cdot q \). \( \square \)

Observe that the bound of the above theorem may not be tight when \( s > 1 \) or \( q > 1 \). It is even possible that for every graph \( G \), \( \text{pw}(G) \) is linear in \( (s + q) \).

Next our goal is to prove **Theorem 3**. To this end, we use an observation by Erdős and Szekeres [17] that for all \( a, b \in \mathbb{N} \), every sequence of distinct numbers of length \( a \cdot b + 1 \) contains a monotonically increasing subsequence of length \( a + 1 \) or a monotonically decreasing subsequence of length \( b + 1 \). We start with an auxiliary lemma.

**Lemma 5.** Let \( G \) be a graph with \( 2n \) vertices and \( n \) independent edges \( (u_i, v_i), 1 \leq i \leq n \). If \( G \) admits an \( s \)-stack layout in which \( u_1 < u_2 < \cdots < u_n < v_i \) for all \( 1 \leq i \leq n \), then there exists a subgraph of \( G \) with at least \( r = \lfloor n/s \rfloor \) edges such that \( u_{j_1} < u_{j_2} < \cdots < u_{j_r} < v_{j_r} < \cdots < v_{j_2} < v_{j_1} \) for some \( 1 \leq j_1 < \cdots < j_r \leq n \).

**Proof.** Assume that the \( s \)-stack layout of \( G \) is defined by the vertex order \( \sigma \):

\[
u_1 <_{\sigma} u_2 <_{\sigma} \cdots <_{\sigma} u_n <_{\sigma} v_{h_1} <_{\sigma} v_{h_2} <_{\sigma} \cdots <_{\sigma} v_{h_n},\]

where \( h_1, h_2, \ldots, h_n \) is a permutation of \( \{1, 2, \ldots, n\} \). An increasing subsequence of \( h_1, h_2, \ldots, h_n \) with length \( k > 0 \) corresponds to a \( k \)-twist in the stack layout of \( G \). Hence,
$k \leq s$. Now we can apply the result of Erdős and Szekeres [17] for $a = s$ and $b = \lceil n/s \rceil - 1$, as $s(\lceil n/s \rceil - 1) + 1 \leq n$ for all integers $n, s \geq 1$. Thus, the permutation contains a decreasing subsequence of length at least $b + 1 = \lceil n/s \rceil$, which completes the proof. \qed

Next we prove Theorem 3. Recall that a stack layout of $P_n \boxtimes G$ is separated if for two consecutive copies of $G$, denoted $G_1$ and $G_2$, all vertices of $G_1$ precede all vertices of $G_2$ in the vertex order.

**Theorem 3.** Assume $P_n \boxtimes G$ has a separated layout on $s$ stacks. Then $G$ admits a simultaneous $s$-stack $s^2$-queue layout, and therefore, $pw(G) \leq 2s^3$.

**Proof.** Suppose that $\sigma$ is a vertex order corresponding to the separated layout of $P_n \boxtimes G$, and $G_1 = (V(G_1), E(G_1)), G_2 = (V(G_2), E(G_2))$ are two copies of $G$ separated in $\sigma$. That is, $u \prec \sigma v$ for all $u \in V(G_1), v \in V(G_2)$.

Consider a suborder of $\sigma$ induced by the vertices of $V(G_1)$ and denote it by $\sigma_1$. If the largest rainbow formed by the edges of $E(G_1)$ with respect to $\sigma_1$ has the size at most $s^2$, then $\sigma_1$ corresponds to an $s^2$-queue layout [20]. In that case, $G$ admits a simultaneous $s$-stack $s^2$-queue layout using $\sigma_1$ as the vertex order, which proves the claim of the theorem. Therefore, we may assume that the largest rainbow in $G_1$ is of size $k > s^2$; see Figure 6.

Let $(u_i, v_i) \in E(G_1), 1 \leq i \leq k$ be such a $k$-rainbow with

$$u_1 \prec \sigma u_2 \prec \sigma \cdots \prec \sigma u_k \prec \sigma v_k \prec \sigma \cdots \prec \sigma v_2 \prec \sigma v_1.$$ 

Consider vertices $u'_1, \ldots, u'_k \in V(G_2)$ that are corresponding copies of $u_1, \ldots, u_k$ in graph $G_2$. Since the stack layout of $P_n \boxtimes G$ is separated, we have $u_1 \prec \sigma \cdots \prec \sigma u_k \prec \sigma u'_i$ for all $1 \leq i \leq k$. Hence, we may apply Lemma 5 for a graph induced by vertices $u_1, \ldots, u_k, u'_1, \ldots, u'_k$, which are connected by $k$ independent edges in the strong product. Therefore, we find a subset of $r = \lceil k/s \rceil > s$ edges in the graph such that

$$u_{j_1} \prec \sigma u_{j_2} \prec \sigma \cdots \prec \sigma u_{j_r} \prec \sigma u'_{j_1} \prec \sigma \cdots \prec \sigma u'_{j_2} \prec \sigma u'_{j_1},$$

where $1 \leq j_1, \ldots, j_r \leq k$. Finally, we observe that $(v_i, u'_i), 1 \leq i \leq k$ are direct-edges in the strong product, and vertices

$$v_{j_r} \prec \sigma v_{j_{r-1}} \prec \sigma \cdots \prec \sigma v_{j_1} \prec \sigma u'_{j_r} \prec \sigma \cdots \prec \sigma u'_{j_2} \prec \sigma u'_{j_1},$$

form an $r$-twist in the $s$-stack layout of $G$. This contradicts to our assumption that the largest rainbow in $G_1$ is of size greater than $s^2$.

The bound on the pathwidth of $G$ follows from Theorem 2. \qed

Figure 6: A large rainbow in a copy $G_1$ of a separated layout of $P_n \boxtimes G$ yields a large twist formed by inter-copy edges between $G_1$ and $G_2$. 

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\[10\]
4 Related Work

Although there exists numerous works on stack and queue layouts of graphs, layouts of graph products received much less attention. Wood [31] considers queue layouts of various graph products, and shows that the queue number of a product of graphs $H$ and $G$ is bounded by a function of the strict queue number of $H$ and the queue number of $G$. Here a queue layout with an order $\sigma$ is strict if for no pair of edges, $(u, v)$ and $(x, y)$, it holds that $u \leq_{\sigma} x <_{\sigma} y \leq_{\sigma} v$; see Figure 7b. Specifically, it is shown that for all $H$ and $G$, $qn(H \boxtimes G) \leq 2 sqn(H) \cdot qn(G) + sqn(H) + qn(G)$, where $sqn(H)$ is the strict queue number of $H$. Similar bounds are given for the cartesian and direct products of $H$ and $G$. It follows that $qn(P_n \boxtimes G) \leq 3 qn(G) + 1$. We stress that the result combined with a decomposition theorem for planar graphs [13,27] (such as one given by Lemma 1) and the fact that the queue number of planar 3-trees is bounded by a constant [2], yield a constant upper bound on the queue number of planar graphs.

Stack layouts of graph products have also been studied [6, 9, 21, 26], though most of the results are less complete as the problem is notoriously more difficult. Bernhart and Kainen [6] introduce the concept of dispersable (also known as matching) book embeddings in which the graphs induced by the edges of each page are 1-regular; see Figure 7a. The minimum number of pages needed in a dispersable book embedding of $G$ is called its dispersable stack number, denoted $dsn(G)$; it is also known as matching book thickness [1]. Clearly for every graph $G$ of maximum vertex degree $\Delta$, we have $dsn(G) \geq \Delta$. The authors of [6] observed that for every path, every tree, every cycle of an even length, every complete bipartite graph, and every binary hypercube, it holds that $dsn(G) = \Delta$. That made them conjecture that the equation holds for every regular bipartite graph. The conjecture was disproved in 2018 for every $\Delta \geq 3$ but it was shown that $dsn(G) = \Delta$ for every 3-connected 3-regular bipartite planar graph [1].

Bernhart and Kainen [6] show that for a bipartite graph $H$ and all graphs $G$, it holds that $sn(H \square G) \leq dsn(H) + sn(G)$; see [26] for an alternative proof. Our Theorem 3 generalizes the result. Several subsequent papers study book embeddings of cartesian products for special classes of graphs [9, 21, 23]; for example, when $H$ is a path and $G$ is a tree. However to the best of our knowledge, no results on stack layouts of direct and strong products of graphs have been published.

We remark that very recently, Dujmović, Morin, and Yelle [16] independently proved a result equivalent to Corollary 1. Specifically, they study stack layouts of graphs with bounded layered pathwidth. That is, a path decomposition with a layering of a graph (a mapping $\ell : V(G) \to \mathbb{Z}$ such that $|\ell(u) - \ell(v)| \leq 1$ for all $(u, v) \in E(G)$) in which the size of the intersection of a bag and a layer is bounded by a constant. It is shown that every graph of layered pathwidth $p$ has stack number at most $4p$. Since the strong product of a path and a pathwidth-$p$ graph has layered pathwidth $p + 1$, the result of [16] implies (asymptotically) Corollary 1. We emphasize that neither our work nor [16] provides a tight bound on the stack number of the class of graphs.

Figure 7: Examples of a dispersable stack layout and a strict queue layout.
5 Conclusion

In this paper we initiated the study of book embeddings of strong graph products. As explained in Section 1, resolving Open Problem 1 would either provide a constant upper bound on the stack number of several families of non-planar graphs, or it would answer a fundamental question of Heath et al. [19] on the relationship of stack and queue layouts.

Theorem 3 indicates that solving the open problem might be a challenging task. Thus we suggest to explore the problem for natural subclasses of bounded-treewidth graphs.

Open Problem 2. Is stack number of $P_n \boxtimes G$ bounded by a constant when $G$ is
(i) a tree (having an unbounded maximum degree)?
(ii) an outerplanar (1-stack) graph?
(iii) a planar graph with a constant treewidth, $\text{tw}(G) \geq 2$?
(iv) a bipartite graph with a constant treewidth, $\text{tw}(G) \geq 2$?

Notice that by the result of Wood [31], the queue number of $P_n \boxtimes G$ is a constant for all the aforementioned graph families.

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