LQG Control and Sensing Co-Design

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Abstract—We investigate a linear-quadratic-Gaussian (LQG) control and sensing co-design problem, where one jointly designs sensing and control policies. We focus on the realistic case where the sensing design is selected among a finite set of available sensors, where each sensor is associated with a different cost (e.g., power consumption). We consider two dual problem instances: sensing-constrained LQG control, where one maximizes a control performance subject to a sensor cost budget, and minimum-sensing LQG control, where one minimizes a sensor cost subject to performance constraints. We prove that no polynomial time algorithm guarantees across all problem instances a constant approximation factor from the optimal. Nonetheless, we present the first polynomial time algorithms with per-instance suboptimality guarantees. To this end, we leverage a separation principle, which partially decouples the design of sensing and control. Then, we frame LQG codesign as the optimization of approximately supermodular set functions; we develop novel algorithms to solve the problems; and we prove original results on the performance of the algorithms and establish connections between their suboptimality and control-theoretic quantities. We conclude the article by discussing two applications, namely, sensing-constrained formation control and resource-constrained robot navigation.

Index Terms—Aerospace engineering, autonomous systems, algorithm design and analysis, computational complexity, multiagent systems, resource management.

I. INTRODUCTION

Tрадиционные подходы к системам контроля предполагают выбор сенсоров [1]. Сенсоры обычно выбираются на основе предварительного анализа требований и ограничений, например, по потреблению электрической энергии. Однако, в контексте приложений Internet of Things (IoT) и Battlefield Things (IoBT) [2] существуют серьезные ограничения на выполнение, связанные с выбором активации 

A. Related Work in Control

1) The authors in [1] and [8]–[14] assume all sensors are given and active (instead of choosing a few sensors to activate). They focus on the co-design of control and estimation over band-limited communication channels, and investigate tradeoffs between communication constraints (e.g., quantization), and control performance (e.g., stability). In more detail, they provide results on the impact of quantization [10], and for finite data rates [11, 12], as well as, on separation principles for LQG design with communication constraints [13]. Recent works also focus on privacy constraints [14]. For a comprehensive review on LQG control and estimation, we refer to [1], [8], and [9].

2) The authors in [15]–[21] extend the focus of the above-mentioned works, by focusing on the co-design of control, estimation, and sensing. Yet, the choice of each sensor can be arbitrary (instead, in our framework, a few sensors are activated from a given finite set of available ones). For example, the authors in [15]–[19] and [22] propose the optimization of steady-state LQG costs, subject to sparsity constraints on the sensor matrices and/or on the feedback control and estimation gains. Finally, the authors in [20] and [21] augment the LQG cost with an information-theoretic regularizer and design the sensors matrices using semidefinite programming.

3) The authors in [5]–[7], [22]–[41] focus on sensor selection, but they do not consider control aspects (with the exception of [7], [40], and [41], which we discuss...
below). Specifically, the authors in [22] studies sensor placement to optimize maximum likelihood estimation over static parameters, whereas [5], [6], [23]–[27] focus on optimizing Kalman filtering and batch estimation accuracy over nonstatic parameters. The authors in [28], [29], and [31]–[37] present sensor and actuator selection algorithms to optimize the average observability and controllability of systems; the authors in [30] focus on actuator placement for stability in uncertain systems. For additional relevant applications, we refer to [38]. The author in [42, Ch. 6.1.3] focuses on selecting a sensor for each edge of a consensus-type system for $\mathcal{H}_2$ optimization subject to sensor cost constraints and sensor noise considerations (instead, we consider general systems). The author in [39] select the location of a phasor measurement unit (PMU) on a single edge of an electrical network to minimize estimation error (each placement happens independently of the rest). The authors in [40] and [41] study sensor placement to optimize a steady-state LQG cost; although the latter case is similar to our framework (we optimize a finite horizon LQG cost, instead of a steady state), the authors focus only on a small-scale system with a few sensors, where a brute-force selection is viable, and no scalable algorithms are proposed (instead, our focus is on scalable approximation algorithms). Finally, the authors in [7] study an LQG control and scheduling codesign problem, where decoupled systems share a wireless sensor network, while power consumption constraints must be satisfied. Instead, we consider coupled systems, a framework that makes our codesign problem inapproximable in polynomial time, in contrast to [7]'s, which is optimally solved in polynomial time.

B. Related Work on Set Function Optimization

In this article, a few sensors must be activated among a set of available ones. This is a combinatorial problem, and we prove it inapproximable: across all problem instances, no polynomial time algorithm can guarantee a constant approximation factor from the optimal. Thus, to provide efficient algorithms with per-instance suboptimality bounds instead, we resort to tools from combinatorial optimization, which has been a successful paradigm on this front [38], [39], [43]–[46]. Specifically, the literature on combinatorial optimization includes investigation into (i) supermodular optimization subject to cardinality constraints (where only a prescribed number of sensors can be active) [47], [48]; (ii) supermodular optimization subject to cost constraints [49]–[51] (where only sensor combinations that meet a prescribed budget can be active—each sensor has a potentially different activation cost); and (iii) approximately supermodular optimization subject to cardinality constraints [43]–[46]. The literature does not cover approximately submodular optimization subject to cost constraints, which is the setup of interest in this article; hence, we herein develop algorithms and novel suboptimality bounds for this case.

C. Contributions to Control Theory

We address an LQG control and sensing codesign problem. The problem extends LQG control to the case where, besides designing an optimal controller and estimator, one has to decide which sensors to activate, due to sensor cost constraints and a limited budget. That is, the sensor choice is restricted to a finite selection from the available sensors, rather than being arbitrary (for arbitrary sensing design, see [15]–[18], [20]). In addition, each sensor has a cost that captures the penalty incurred for using the sensor. Since different sensors (e.g., lidars, radars, cameras, lasers) have different power consumption, bandwidth utilization, and/or monetary value, we allow each sensor to have a different cost. We formulate two dual instances of the LQG codesign problem. The first, sensing-constrained LQG control, involves the joint design of control and sensing to minimize the LQG cost subject to a sensor cost budget. The second, minimum-sensing LQG control, involves the joint design of control and sensing to minimize the cost of the activated sensors subject to a desired LQG cost.

To solve the proposed LQG problems, we first leverage a separation principle that partially decouples the control and sensor selection. As a negative result, we prove that the optimal sensor selection is inapproximable in polynomial time by a constant suboptimality bound across all problem instances. Therefore, we develop algorithms with per-instance suboptimality bounds instead. Particularly, we frame the sensor selection as the optimization of approximately supermodular set functions, using the notion of the supermodularity ratio introduced in [53] (see also [44]). Then, we provide the first polynomial time algorithms, which provably retrieve a close-to-optimal choice of sensors, and the corresponding optimal control policy. Specifically, the suboptimality gap of the algorithms depend on the supermodularity ratio $\gamma$ of the LQG cost, and we establish connections between $\gamma$ and control-theoretic quantities, providing computable lower bounds for $\gamma$.

D. Contributions to Set Function Optimization

To prove the aforementioned results, we extend the literature on supermodular optimization. Particularly, we provide the first efficient algorithm for approximately supermodular optimization (e.g., LQG cost optimization) subject to cost constraints for subset selection (e.g., sensor selection). To this end, we use the algorithm in [51], proposed for exactly supermodular optimization, and prove it maintains provable suboptimality bounds for even approximately supermodular optimization. Importantly, our bounds improve the previously known bounds for exactly supermodular optimization: our bounds become $1 - 1/e$ for supermodular optimization, tightening the known $1/2(1 - 1/e)$ [51]. Noticeably, $1 - 1/e$ is the best possible bound in polynomial time for the supermodular optimization subject to cardinality constraints [57]. That way, our analysis equates the approximation difficulty of cost and cardinality constrained optimization for the first time (among all algorithms with at most quadratic running time). That way, our results are relevant beyond sensing in control, such as in cost-effective outbreak detection in networks [62].

Similarly, we provide the first algorithm for a minimal cost subset selection subject to desired bounds on an approximately supermodular function. The algorithm relies on a simplification of the algorithm in [51]. Leveraging our novel bounds, we show the algorithm is the first with provable suboptimality bounds given approximately supermodular functions. Notably, for exactly supermodular functions, the bound recovers the well-known bound for cardinality minimization [48];
E. Application Examples

We demonstrate the effectiveness of the proposed algorithms in numerical experiments, by considering two application scenarios: sensing-constrained formation control and resource-constrained robot navigation. We present a Monte Carlo analysis for both, which demonstrates that (i) the proposed sensor selection strategy is near-optimal, and, particularly, the resulting LQG cost matches the optimal selection in all tested instances for which the optimal selection could be computed via a brute-force approach; (ii) a more naive selection that attempts to minimize the state estimation error [24] (rather than the LQG cost) has degraded LQG performance, often comparable to a random selection; and (iii) the selection of a small subset of sensors using the proposed algorithms ensures an LQG cost that is close to the one obtained by using all available sensors, hence, providing an effective alternative for control under sensing constraints.

F. Comparison With the Preliminary Results in [52] (Which Coincides With the Preprint [63])

This article (which coincides with the preprint [64]) extends the preliminary results in [52] and provides comprehensive presentation of the LQG codesign problem, by including both the sensing-constrained LQG control (introduced in [52]) and the minimum-sensing LQG control problem (not previously published). Moreover, we generalize the setup in [52] to account for any sensor costs (in [52] each sensor has a unit cost, whereas herein sensors have different costs). Also, we extend the numerical analysis accordingly. Moreover, we prove the inapproximability of the problem. Most of the technical results (Theorems 1–4, and Algorithms 2–4) are novel and have not been published.

Organization of the rest of this article. Section II formulates the LQG control and sensing codesign problems. Section III presents a separation principle, the inapproximability theorem, and introduces the algorithms for the codesign problems. Section IV characterizes the performance of the algorithms and establishes connections between their suboptimality bounds and control-theoretic quantities. Section V presents two examples of the codesign problems. Section VI concludes the article. All proofs are given in the Appendix.

Notation. Lowercase letters denote vectors and scalars (e.g., $v$); uppercase letters denote matrices (e.g., $A$). Calligraphic fonts denote sets (e.g., $S$). $I$ denotes the identity matrix.

II. PROBLEM FORMULATION: LQG CONTROL AND SENSING CODESIGN

Here, we formalize the LQG control and sensing codesign problem considered in this article. Specifically, we present two "dual" statements of the problem: the sensing-constrained LQG control and the minimum-sensing LQG control.

A. System, Sensors, and Control Policies

We start by introducing the article’s framework.

a) System. We consider a discrete-time time-varying linear system with additive Gaussian noise as

$$x_{t+1} = Ax_t + Bu_t + w_t, \quad t = 1, 2, \ldots, T$$

where $x_t \in \mathbb{R}^n$ is the system’s state at time $t$, $u_t \in \mathbb{R}^m$ is the control action, $w_t$ is the process noise, $A$ and $B$ are known matrices, and $T$ is a finite horizon. Also, $x_1$ is a Gaussian random variable with covariance $\Sigma_{x1}$, and $w_t$ is a Gaussian random variable with mean zero and covariance $\Sigma_w$, such that $w_t$ is independent of $x_1$ and $w_{t'}$ for all $t' \neq t$. $T$ is the measurement noise. We assume $v_i(t)$ to be a Gaussian random variable with mean zero and positive definite covariance $V_i$, such that $v_i$ is independent of $x_1$, and of $w_t$ for any $t' \neq t$, and independent of $v_i(t')$ for all $t' \neq t$, and any $i' \in V, t' \neq i$.

When only a subset $S \subseteq V$ of the sensors is active for all $t = 1, 2, \ldots, T$, then the measurement model becomes

$$y_t(S) = C_t(S)x_t + v_t(S)$$

b) Sensors. We consider the availability of a (potentially large) set $V$ of sensors, which can take noisy linear observations of the system’s state. Particularly,

$$y_{i,t} = C_{i,t}x_t + v_{i,t}, \quad i \in V$$

where $y_{i,t} \in \mathbb{R}^{h_i}$ is the measurement of sensor $i$ at time $t$, $C_{i,t}$ is a sensing matrix, and $v_{i,t}$ is the measurement noise. We assume $v_{i,t}$ to be a Gaussian random variable with mean zero and positive definite covariance $V_i$, such that $v_{i,t}$ is independent of $x_1$, and for any $t' \neq t$, and independent of $v_{i,t'}$ for all $t' \neq t$, and any $i' \in V, t' \neq i$.

Problem 1 (Sensing-constrained LQG control): Given a budget $b \geq 0$ on the sensor cost, find sensors $S$ and a policy $u_{1:T}(S)$ such that the sensor cost $c(S)$ is low and the finite horizon LQG cost $h(S, u_{1:T}(S))$ below is optimized:

$$h(S, u_{1:T}(S)) = \sum_{t=1}^{T} \mathbb{E} \left[ \|x_{t+1}(S)\|^2_{Q_t} + \|u_t(S)\|^2_{R_t} \right]$$

where $Q_1, \ldots, Q_T$ are known positive semidefinite matrices, $R_1, \ldots, R_T$ are known positive definite matrices, and the expectation is taken with respect to $x_1, u_{1:T}$, and $v_{1:T}(S)$. Particularly, the sensing-constrained LQG control minimizes the LQG cost subject to a sensor cost budget, and the dual minimum-sensing LQG control minimizes the sensor cost subject to a desired LQG cost.

Problem 2 (Minimum-sensing LQG control): Find a minimal cost sensor set $S$, and a policy $u_{1:T}(S)$, such that the finite horizon LQG cost $h(S, u_{1:T}(S))$ is at most $\kappa$, where $\kappa \geq 0$ is given as

$$\min_{S \subseteq V} h(S, u_{1:T}(S)), \quad \text{s.t.} \quad c(S) \leq \kappa.$$
Remark 1 (Case of uniform-cost sensors): When all sensors \( i \in \mathcal{V} \) have the same cost, say \( c(i) = \bar{c} > 0 \), the sensor budget constraint becomes a cardinality constraint

\[
c(S) \leq b \iff \sum_{i \in S} c(i) \leq b \iff |S| \leq \frac{b}{\bar{c}}.
\]

(8)

III. CODEPRINCIPLES, HARDNESS, AND ALGORITHMS

We leverage a separation principle to derive that the optimization of the sensor set \( S \) and of the control policy \( u_{1:T}(S) \) can happen in cascade. However, we show that optimizing for \( S \) is inapproximable in polynomial time. Nonetheless, we then present polynomial time algorithms for Problems 1 and 2 with provable per-instance suboptimality bounds. Particularly, the bounds are presented in Section IV.5

A. Separability of Optimal Sensing and Control Design

We characterize the jointly optimal control and sensing solutions for Problems 1 and 2, and prove they can be found in two separate steps, where first the sensor set is found, and then the control policy is computed.

Theorem 1 (Separability of optimal sensor set and control policy design): For any active sensor set \( S \), let \( \hat{x}_t(S) \) be the Kalman estimator of the state \( x_t \), and \( \Sigma_t(S) \) be \( \hat{x}_t(S) \)’s error covariance. Additionally, let the matrices \( \Theta_t \) and \( K_t \) be the solution of the following backward Riccati recursion:

\[
S_t = Q_t + N_t \quad N_t = A_t(S_{t-1} + B_t R_{t-1}^{-1} B_t^T)^{-1} A_t^T M_t = B_t^T S_t B_t + R_t K_t = -M_t^{-1} B_t^T S_t A_t \quad \Theta_t = K_t^T M_t K_t
\]

(9)

with boundary condition \( N_{T+1} = 0 \).

1) (Separability in Problem 1) Any optimal solution \((S^*, u_{1:T}^*)\) to Problem 1 can be computed in cascade:

\[
S^* \in \arg \min_{S \in \mathcal{C}} \sum_{t=1}^{T} \text{tr} \left[ \Theta_t \Sigma_t(S) \right], \text{ s.t. } c(S) \leq b
\]

(10)

\[
u_t^* = K_t \hat{x}_t(S^*), \quad t = 1, \ldots, T.
\]

(11)

2) (Separability in Problem 2) Define the constant \( \bar{c} \triangleq \kappa - \text{tr}(\Sigma_{11} N_1) - \sum_{t=1}^{T} \text{tr}(W_t S_t) \). Any optimal solution \((S^*, u_{1:T}^*)\) to Problem 2 can be computed in cascade:

\[
S^* \in \arg \min_{S \in \mathcal{C}} c(S), \text{ s.t. } \sum_{t=1}^{T} \text{tr} \left[ \Theta_t \Sigma_t(S) \right] \leq \bar{c}
\]

(12)

\[
u_t^* = K_t \hat{x}_t(S^*), \quad t = 1, \ldots, T.
\]

(13)

Remark 2 (Certainty equivalence principle): The control gain matrices \( K_1, K_2, \ldots, K_T \) are the same as the ones that make the controllers \( (K_1 \hat{x}_1, K_1 \hat{x}_2, \ldots, K_T \hat{x}_T) \) optimal for the perfect state-information version of Problem 1, where the state \( x_t \) is known to the controllers [1, Ch. 4].

Theorem 1 decouples the sensing design from the control policy design. Particularly, once an optimal sensor set \( S^* \) is found, then the optimal controllers are equal to \( K_t \hat{x}_t(S^*) \), which correspond to the standard LQG control policy.

An intuitive interpretation of the sensor design steps in (10) and (12) follows next.

Algorithm 1: Joint Sensing and Control Design for Problem 1.

Input: Horizon \( T \); system in (1); covariance \( \Sigma_{11} \); LQG cost matrices \( Q_t \) and \( R_t \) in (5); sensors in (2); sensor budget \( b \); sensor cost \( c(i) \), for all \( i \in \mathcal{V} \).

Output: Active sensors \( \hat{S} \), and controls \( \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_T \).

1: Compute \( \Theta_1, \Theta_2, \ldots, \Theta_T \) using (9).

2: Return \( \hat{S} \) returned by Algorithm 2, which finds a solution to the optimization problem in (10).

3: Compute \( K_1, K_2, \ldots, K_T \) using (9).

4: At each \( t = 1, \ldots, T \), compute the Kalman estimate of \( x_t \):

\[
\hat{x}_t \triangleq \mathbb{E}[x_t | y_1(\hat{S}), y_2(\hat{S}), \ldots, y_t(\hat{S})];
\]

5: At each \( t = 1, \ldots, T \), return \( \hat{u}_t = K_t \hat{x}_t \).

Remark 3 (Control-aware sensor design): To provide insight on the \( \sum_{t=1}^{T} \text{tr} \left[ \Theta_t \Sigma_t(S) \right] \) in (10) and (12), we rewrite it as

\[
\sum_{t=1}^{T} \text{tr} \left[ \Theta_t \Sigma_t(S) \right] = \sum_{t=1}^{T} \mathbb{E} \left[ (x_t - \hat{x}_t(S))^T \Theta_t (x_t - \hat{x}_t(S)) \right] = \sum_{t=1}^{T} \mathbb{E} \left[ \|K_t x_t - K_t \hat{x}_t(S)\|^2_{\Sigma_t} \right]
\]

(14)

since \( \Sigma_t(S) = \mathbb{E}[(x_t - \hat{x}_t(S))(x_t - \hat{x}_t(S))^T] \) and \( \Theta_t = K_t^T \Sigma_t K_t \). From (14), each \( \mathbb{E} \left[ \Theta_t \Sigma_t(S) \right] \) captures the mismatch between the imperfect state-information controller \( u_t(S) = K_t \hat{x}_t(S) \) (which is only aware of the measurements from the active sensors) and the perfect state-information controller \( K_t x_t \). That is, while standard sensor selection minimizes the estimation covariance, for instance by minimizing

\[
\sum_{t=1}^{T} \text{tr} \left[ \Sigma_t(S) \right] \triangleq \sum_{t=1}^{T} \mathbb{E} \left[ \|x_t - \hat{x}_t(S)\|^2 \right]
\]

(15)

the proposed LQG cost formulation selectively minimizes the estimation error focusing on the states that are most informative for control purposes. For example, the mismatch contribution in (14) of any \( x_t - \hat{x}_t(S) \) in the null space of \( K_t \) is zero; accordingly, the proposed sensor design approach has no incentive in activating sensors to observe states, which are irrelevant for control purposes.

B. Inapproximability of Optimal Sensing Design

Theorem 2 (Inapproximability): If \( NP \neq P \), then there is no polynomial time algorithm for Problems 1 and 2 that returns an approximate solution within a constant factor from the optimal. This remains true, even if all sensors have cost 1.

We prove the theorem by reducing the inapproximable problem in [65]—sensor selection with cost constraints for optimal steady-state Kalman filtering error—to (10).

Motivated by the inapproximability of Problems 1 and 2, we next present practical algorithms, which in Section IV, we prove to enjoy per-instance suboptimality bounds.

C. Codesign Algorithms for Problem 1

We present a practical algorithm for the sensing-constrained LQG control Problem 1 (see Algorithm 1). The algorithm follows Theorem 1: it first computes a sensing design, and then a control design, as described below.

1) Sensing Design for Problem 1: Theorem 1 implies an optimal sensor design for Problem 1 can be computed by solving (10). To this end, Algorithm 1 first computes \( \Theta_1, \Theta_2, \ldots, \Theta_T \) (see Algorithm 1’s line 1). Next, since (10) is inapproximable (see Theorem 2),
Algorithm 2: Sensing Design for Problem 1.

Input: Horizon $T$; system in (1); covariance $\Sigma_{1:1}$; LQG cost matrices $Q_t$ and $R_t$ in (5); sensors in (2); sensor budget $b$; sensor cost $c(i)$, for all $i \in V$.

Output: Set $\bar{S}$.

1. $\bar{S}_T \leftarrow \arg\min_{c(\bar{S}) \leq b} \sum_{t=1}^{T} \text{tr}(\Theta_t \Sigma_{t|t}^{-1}(\{\bar{S})})$;
2. $\bar{S}_0 \leftarrow \emptyset$; $\forall \bar{V} \not= \emptyset$;
3. while $\forall \bar{V} \not= \emptyset$ and $c(\bar{S}) \leq b$ do
4. for all $a \in \bar{V}$ do
5. $\bar{S}_{t+1} \leftarrow \bar{S}_t \cup \{a\}$; $\Sigma_{t|t}^{-1}(\bar{S}_{t+1}) \leftarrow \Sigma_{t|t}^{-1}(\bar{S}_t)$;
6. for all $t = 1, \ldots, T$ do
7. $\Sigma_{t|t}^{-1}(\bar{S}_{t+1}) \leftarrow A_t \Sigma_{t|t}(\bar{S}_{t+1}) A_t^T + W_t$;
8. $\Sigma_{t|t}(\bar{S}_{t+1}) \leftarrow [\Sigma_{t|t-1}(\bar{S}_{t+1})^{-1} + C_t(\bar{S}_{t+1})^T V_t(\bar{S}_{t+1})^{-1} C_t(\bar{S}_{t+1})]^{-1}$;
9. $\text{gain}_a \leftarrow \sum_{t=1}^{T} \text{tr}(\Theta_t \Sigma_{t|t}^{-1}(\bar{S}_{t+1}) - \Sigma_{t|t}^{-1}(\bar{S}_{t+1}))$;
10. end for
11. $s \leftarrow \arg\max_{i \in \bar{V}} \text{gain}_i / c(i)$;
12. $\bar{S}_t \leftarrow \bar{S}_t \cup \{s\}$;
13. $\forall \bar{V} \not= \emptyset$;
14. end while
15. if $c(\bar{S}) > b$ then
16. $\bar{S}_t \leftarrow \bar{S}_t \setminus \{s\}$;
17. end if
18. $\bar{S} \leftarrow \arg\min_{S \subseteq \bar{S}} \sum_{t=1}^{T} \text{tr}(\Theta_t \Sigma_{t|t}^{-1}(S))$.

Algorithm 3: Joint Sensing and Control Design for Problem 2.

Input: Horizon $T$; system in (1); covariance $\Sigma_{1:1}$; LQG cost matrices $Q_t$ and $R_t$ in (5); LQG cost bound $c$ sensors in (2); sensor cost $c(i)$, for all $i \in V$.

Output: Active sensors $\bar{S}$ and controls $u_1, u_2, \ldots, u_T$.

1. Compute $\Theta_1, \Theta_2, \ldots, \Theta_T$ using (9);
2. Return $\bar{S}$ returned by Algorithm 4, which finds a solution to the optimization problem in (7);
3. Compute $K_1, K_2, \ldots, K_T$ using (9);
4. At each $t = 1, \ldots, T$, compute the Kalman estimate of $x_t$:
$$\hat{x}_t \triangleq \mathbb{E}[x_t | y_1(\bar{S}), y_2(\bar{S}), \ldots, y_t(\bar{S})]$$
5. At each $t = 1, \ldots, T$, return $u_t = K_t \hat{x}_t$.

Particularly, as Algorithm 1 calls Algorithm 2 to solve (10), similarly, Algorithm 3 calls Algorithm 4 to solve (12). Algorithm 4 is similar to Algorithm 2, with the difference that Algorithm 4 selects sensors until the upper bound $\bar{s}$ in (12) is met (Algorithm 4’s line 3), whereas Algorithm 2 selects sensors up to the point the cost budget $b$ is violated (Algorithm 2’s line 3).

IV. PERFORMANCE GUARANTEES FOR LQG CODESIGN

We now quantify the suboptimality and running time of Algorithms 1 and 3. Particularly, we prove that both algorithms enjoy per-instance suboptimality bounds, and run in quadratic time. To this end, we present a notion of a supermodularity ratio (see Definition 3), which we use to prove the suboptimality bounds. We then establish connections between the ratio and control-theoretic quantities (see Theorem 5), and conclude that the algorithms’ suboptimality bounds are nonvanishing under control-theoretic conditions encountered in most real-world systems (see Theorem 6).

A. Supermodularity Ratio

To present the definition of the supermodularity ratio, we start by defining monotonicity and supermodularity.

Definition 1 (Monotonicity [47]): Consider any finite set $V$. The set function $f : 2^V \mapsto \mathbb{R}$ is nonincreasing if and only if for any sets $A \subseteq B \subseteq V$, it holds $f(A) \geq f(B)$.

Definition 2 (Supermodularity [47, Proposition 2.1]): Consider any finite set $V$. The set function $f : 2^V \mapsto \mathbb{R}$ is supermodular if and only if for any sets $A \subseteq B \subseteq V$, any element $v \in V$, it holds $f(A) - f(A \cup \{v\}) \geq f(B) - f(B \cup \{v\})$.

That is, $f$ is supermodular if and only if it satisfies a diminishing returns property: for any set $A \subseteq V$, the drop $f(A) - f(A \cup \{v\})$ diminishes as the set $A$ grows.

Definition 3 (Supermodularity ratio [53]): Consider any finite set $V$, and a nonincreasing set function $f : 2^V \mapsto \mathbb{R}$. We define the supermodularity ratio of $f$ as
$$\gamma_f \triangleq \min_{A \subseteq B \subseteq V, v \in V} \frac{f(A) - f(A \cup \{v\})}{f(B) - f(B \cup \{v\})}$$

The supermodularity ratio $\gamma_f$ measures how far $f$ is from being supermodular. Particularly, $\gamma_f$ takes values in $[0,1]$, and if $\gamma_f = 1$, then $f(A) - f(A \cup \{v\}) \geq f(B) - f(B \cup \{v\})$, i.e., $f$ is supermodular. Whereas, if $0 < \gamma_f < 1$, then $f(A) - f(A \cup \{v\}) \geq \gamma_f [f(B) - f(B \cup \{v\})]$, i.e., $\gamma_f$ captures how much it captures how much one needs to discount $f(B)$ from $f(B \cup \{v\})$, such that $f(A) - f(A \cup \{v\})$ is at least $f(B) - f(B \cup \{v\})$. In this case, $f$ is called approximately (or weakly) supermodular [66].

D. Codesign Algorithms for Problem 2

This section presents a practical algorithm for Problem 2 (see Algorithm 3). Since the algorithm shares steps with Algorithm 1, we focus on the different ones.

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Algorithm 4: Sensing Design for Problem 2.

Input: Horizon $T$; system in (1); covariance $Z_{11}$; LQG cost matrices $Q_t$ and $R_t$ in (5); LQG cost bound $\forall i$ sensors in (2); sensor cost $c(i)$, for all $i \in V$.

Output: Active sensors $\hat{S}$.
1. $\vec{k} \leftarrow \vec{k} - \text{tr}(Z_{11}N_1) - \sum_{t=1}^{T} \text{tr}(W_t S_t)$
2. $\bar{S} \leftarrow \emptyset$; $V' \leftarrow V$
3. while $V' \neq \emptyset$ and $\sum_{t=1}^{T} \text{tr}(\Theta_t \Sigma_{11}(\bar{S})) > \vec{k}$ do
4. for all $a \in V'$ do
5. $\bar{S}_a \leftarrow \bar{S} \cup \{a\}$; $\Sigma_{11}(\bar{S}_a) \leftarrow \Sigma_{11}$
6. for all $t = 1, \ldots, T$ do
7. $\Sigma_{t+1}(\bar{S}_a) \leftarrow A_t \Sigma_{t+1}(\bar{S}_a) A_t^T + W_t$
8. $\Sigma_{1t}(\bar{S}_a) \leftarrow 1
9. $\left[\Sigma_{tt-1}(\bar{S}_a) -1 + C_t(\bar{S}_a)^{-1}V_t(\bar{S}_a)^{-1}C_t(\bar{S}_a)^{-1}\right]^{-1}$
10. end for
11. $\text{gain}_a \leftarrow \sum_{t=1}^{T} \text{tr}(\Theta_t [\Sigma_{tt}(\bar{S}) - \Sigma_{tt}(\bar{S}_a)])$
12. end for
13. $s \leftarrow \arg \max_{a \in V'} [\text{gain}_a / c(a)]$
14. $\bar{S} \leftarrow \bar{S} \cup \{s\}$
15. $V' \leftarrow V' \setminus \{s\}$
16. end while

B. Performance Analysis for Algorithm 1

We quantify Algorithm 1’s running time and suboptimality, using the notion of the supermodularity ratio. We use the notation:

- $g(S)$ is the optimal value of $h[S, u_{1:T}(S)]$ across all $u_{1:T}(S)$, given any $S$:
  $$g(S) \triangleq \min_{u_{1:T}(S)} h[S, u_{1:T}(S)].$$ (16)

- $h^{*} \triangleq \min_{S, V, u_{1:T}(S)} h[S, u_{1:T}(S)]$, s.t. $c(S) \leq b$, i.e., the optimal value of Problem 1.

- $h^{\star} \triangleq \min_{S, V, u_{1:T}(S)} c(S)$, s.t. $h[S, u_{1:T}(S)] \leq \kappa$, i.e., the optimal value of Problem 2.

Theorem 3 (Performance of Algorithm 1): Algorithm 1 returns a sensor set $\hat{S}$ and control policies $u_{1:T}(\hat{S})$ such that

$$\frac{h[\emptyset, u_{1:T}(\emptyset)] - h[\hat{S}, u_{1:T}(\hat{S})]}{h[\emptyset, u_{1:T}(\emptyset)]} - h^{\star} \geq \max \left[ \frac{\gamma_g}{2} (1 - e^{-\gamma_g}), 1 - e^{-\gamma_g} c(S)/b \right]$$ (17)

where $\gamma_g$ is the supermodularity ratio of $g(S)$ in (16).

Moreover, Algorithm 1 runs in $O(\sqrt{2}Tn^{2.4})$ time.

In (17), $h[\emptyset, u_{1:T}(\emptyset)] - h[\hat{S}, u_{1:T}(\hat{S})]$ quantifies the gain from selecting $\hat{S}$, and in (17) the right-hand side guarantees that the gain is close to the optimal $h[\emptyset, u_{1:T}(\emptyset)] - h^{\star}$. Specifically, when either of the bounds in the right-hand side of (17) is 1, then the algorithm returns an optimal solution.

For comparison, in Fig. 1, we plot the bounds for $c(S)/b \in \{2/5, 1, 2\}$ and all $\gamma_g \in [0, 1]$. We observe that $1 - e^{-\gamma_g} c(S)/b$ dominates $\gamma_g/2(1 - e^{-\gamma_g})$ for $c(S)/b > 2/5$. Moreover, as $c(S)/b$ and $\gamma_g$ increase, then $1 - e^{-\gamma_g} c(S)/b$ tends to 1, in which case, Algorithm 1 returns an optimal solution.

Remark 4 (Novelty of algorithm and bounds): Algorithm 1 is the first scalable algorithm for Problem 1. Notably, although Algorithm 2 (used in Algorithm 1) is the same as [51, Algorithm 1], the latter was introduced for exactly supermodular optimization, instead of approximately supermodular optimization, which is the optimization framework in this article. Therefore, one of our contributions with Theorem 3 is to prove Algorithm 2 maintains suboptimality bounds even for approximately supermodular optimization. The novel bounds in Theorem 3 also improve upon the previously known [51], [62] for exactly supermodular optimization; particularly, our bounds can become $1 - 1/e$ for supermodular optimization (the closer $c(S)/b$ is to 1), tightening the known 1/2(1 - 1/e) [51], [62]. Noticeably, $1 - 1/e$ is the best possible bound in polynomial time for submodular optimization subject to cardinality constraints [47], instead of the general cost constraints in this article. That way, our analysis equates the approximation difficulty of cost and cardinality constrained optimization for the first time (among all algorithms with at most quadratic running time in the number of available elements in $V$, i.e., those in [47], [51], [62], and ours).

All in all, Theorem 3 guarantees that Algorithm 1 achieves a close-to-optimal solution for Problem 1, whenever $\gamma_g > 0$. In Section IV-D, we present conditions such that $\gamma_g > 0$.

Finally, Theorem 3 also quantifies the scalability of Algorithm 1. Particularly, Algorithm 1’s running time $O(\sqrt{2}Tn^{2.4})$ is in the worst case quadratic in the number of available sensors $V$ (when all must be chosen active), and linear in the Kalman filter’s running time: specifically, the multiplier $Tn^{2.4}$ is due to the complexity of computing all $\Sigma_{tt}$ for $t = 1, 2, \ldots, T$ [1, Appendix E].

C. Performance Analysis for Algorithm 3

Theorem 4 (Performance of Algorithm 3): Consider Algorithm 3 returns a sensor set $\hat{S}$ and control policies $u_{1:T}(\hat{S})$. Let $s_I$ be the last sensor added to $\hat{S}$. Then,

$$h[\hat{S}, u_{1:T}(\hat{S})] \leq \kappa$$ (18)

$$c(\hat{S}) \leq c(s_I) + \frac{1}{\gamma_g} \log \left( \frac{h[\emptyset, u_{1:T}(\emptyset)] - \kappa}{h[S_{t-1}, u_{1:T}(S_{t-1})] - \kappa} \right) b^{\gamma_g}$$ (19)

where $S_{t-1} \triangleq S \setminus \{s_I\}$.

Additionally, Algorithm 3 runs in $O(|V|^2 T n^{2.4})$ time.

Remark 5 (Novelty of algorithm and bound): Algorithm 3 is the first scalable algorithm for Problem 2. Importantly, Algorithm 4, used in Algorithm 3, is the first scalable algorithm with suboptimality guarantees for the problem of the minimal cost set selection where a bound to an approximately supermodular $g$ must be met. Particularly, Algorithm 4, generalizes previous algorithms [48] that focus instead on minimal cardinality set selection subject to bounds on an exactly

![Fig. 1. Plot of $f_i(\gamma_g)$, where $i = 1, 2, 3, 4$, for increasing values of $\gamma_g$ (each $f_i$ is defined in the figure’s legend). By Definition 3 of $\gamma_g$, $\gamma_g$ takes values between 0 and 1.](image-url)
supermodular function $g$ (in which case, $\gamma_g = 1$). Notably, for $\gamma_g = 1$, in (19)’s bound recovers the guarantee established in [48, Th. 1].

All in all, (18) implies Algorithm 3 returns a solution to Problem 2 with the prescribed LQG performance. And parallel to (17), (19) implies for $\gamma_g > 0$ that Algorithm 3 achieves a close-to-optimal sensor cost.

**D. Conditions for $\gamma_g > 0$**

We provide control-theoretic conditions such that $\gamma_g$ is nonzero, in which case both Algorithm 1 and Algorithm 3 guarantee a close-to-optimal performance. Particularly, we first prove that if $\sum_{t=1}^T \Theta_t > 0$, then $\gamma_g$ is nonzero. Afterwards, we show that the condition holds true in all problem instances one typically encounters in the real-world. Specifically, we prove $\sum_{t=1}^T \Theta_t > 0$ holds whenever zero control would result in a suboptimal behavior for the system; that is, we prove $\sum_{t=1}^T \Theta_t > 0$ holds in all systems where LQG control improves system performance.

**Theorem 5 (Nonzero computable bound for the supermodularity ratio $\gamma_g$):** For any sensor $i \in \mathcal{V}$, let $C_{i,t} \triangleq V_i^{-1/2} C_{i,t}$ be the whitened measurement matrix. If the strict inequality $\sum_{t'=1}^T \Theta_t > 0$ holds, then $\gamma_g \neq 0$. Additionally, if we assume $\text{tr}(C_i^T C_i) = 1$, and $\text{tr}[(\Theta_{i,t})^2] \leq \lambda_{\text{max}}[\Sigma_{i,t}(0)]$, then

$$
\gamma_g \geq \frac{\lambda_{\text{min}}(\sum_{t=1}^T \Theta_t)}{\lambda_{\text{max}}(\sum_{t=1}^T \Theta_t)} \min_{t \in \{1,2,\ldots,T\}} \frac{\lambda_{\text{min}}(\Sigma_{i,t}(0))^{1/2}}{\lambda_{\text{max}}(\Sigma_{i,t}(0))^{1/2}} \left(1 + \max_{t \in \{1,2,\ldots,T\}} \frac{\max_{i \in \mathcal{V}} \min_{t \in \{1,2,\ldots,T\}} C_{i,t}(\Sigma_{i,t}(0))^{1/2}}{\lambda_{\text{max}}(C_i^T C_i(0))^{1/2}} \right)^2 + \max_{t \in \{1,2,\ldots,T\}} \frac{\max_{i \in \mathcal{V}} \min_{t \in \{1,2,\ldots,T\}} C_{i,t}(\Sigma_{i,t}(0))^{1/2}}{\lambda_{\text{max}}(C_i^T C_i(0))^{1/2}} \right)^2 \right)
$$

Equation (20) suggests ways $\gamma_g$ can increase and, correspondingly, the bounds for Algorithm 1 and of Algorithm 3 can improve: when $\lambda_{\text{min}}(\sum_{t=1}^T \Theta_t)/\lambda_{\text{max}}(\sum_{t=1}^T \Theta_t)$ increases to 1, then the right-hand side in (20) increases. Therefore, since each $\Theta_t$ weight the states depending on their relevance for control purposes (see Remark 3), the right-hand side in (20) increases when all the directions in the state space become equally important for control purposes. Indeed, in the extreme case, where $\lambda_{\text{max}}(\Theta_t) = \lambda_{\text{min}}(\Theta_t) = \lambda$, the objective function in (10) becomes

$$
\frac{\sum_{t=1}^T \text{tr}[(\Theta_t \Sigma_{i,t}(0))] = \lambda \sum_{t=1}^T \text{tr}[(\Sigma_{i,t}(0))]}
$$

which matches the cost function in the classical sensor selection where all states are equally important [per (15)].

Theorem 5 states $\gamma_g$ is nonzero whenever $\sum_{t=1}^T \Theta_t > 0$. To provide insight on the type of control problems for which this result holds, next we translate the technical condition $\sum_{t=1}^T \Theta_t > 0$ into an equivalent control-theoretic condition.

**Theorem 6 (Control-theoretic condition for near-optimal codesign):** Consider the (noiseless, perfect state-information) LQG problem where at any $t = 1,2,\ldots,T$, the state $x_t$ is known to each controller $u_t$ and the process noise $w_t$ is zero, i.e., the optimal control problem

$$
\min_{u_{1:T}} \sum_{t=1}^T \left( ||x_{t+1}||_Q + ||u_t(x_t)||_R \right)_{x_{t+1}=W_t=0}. \tag{21}
$$

Let $A_t$ be invertible for all $t = 1,2,\ldots,T$; the strict inequality $\sum_{t=1}^T \Theta_t > 0$ holds if and only if for all nonzero initial conditions $x_1$, the all-zeros control policy $u_{1:T}\triangleq(0,0,\ldots,0)$ is not an optimal solution to (21)

$$
u_{1:T}^* \notin \arg \min_{u_{1:T}} \sum_{t=1}^T \left( ||x_{t+1}\Sigma_{t+1}^{1/2} + ||u_t||_R \right)_{x_{t+1}=W_t=0}. \tag{21}
$$

Theorem 6 suggests $\sum_{t=1}^T \Theta_t > 0$ holds if and only if for any zero initial condition $x_1$ the all-zeros control policy $u_{1:T}^* = (0,0,\ldots,0)$ is suboptimal for the noiseless, perfect state-information LQG problem. Intuitively, this encompasses most practical control design problems where a zero controller would result in a suboptimal behavior of the system (LQG control design itself would be unnecessary in the case where a zero controller, i.e., no control action, can already attain the desired system performance).

Overall, Algorithm 1 and Algorithm 3 are the first scalable algorithms for Problem 1 and Problem 2, respectively, and they achieve nonvanishing per-instance performance guarantees.

**V. NUMERICAL EVALUATIONS**

We consider two applications for the LQG control and sensing co-design framework: formation control and autonomous navigation. We present a Monte Carlo analysis for both, which demonstrates: (i) the proposed sensor selection strategy is near-optimal; particularly, the resulting LQG cost matches the optimal selection in all instances for which the optimal could be computed via a brute-force approach; (ii) a more naive selection, which attempts to minimize the state estimation covariance [24] (rather than the LQG cost) has degraded LQG performance, often comparable to a random selection; (iii) in the considered instances, a clever selection of a small subset of sensors can ensure an LQG cost that is close to the one obtained by using all available sensors, hence, providing an effective alternative for control under sensing constraints.

**A. Sensing-Constrained Formation Control**

**1) Simulation Setup:** The application scenario is illustrated in Fig. 2(a). A team of $n$ agents (blue triangles) moves in 2D. At $t = 1$, the agents are randomly deployed in a 10 m $\times$ 10 m square. Their objective is to reach a target formation shape (red stars); in Fig. 2(a) the desired formation has an equilateral polygon, while in general for a formation of $n$, the desired formation is an equilateral polygon with $n$ vertices. Each robot is modeled as a double-integrator, with state $x_i = [p_i, v_i]^T \in \mathbb{R}^4$ ($p_i$ is agent $i$’s position, and $v_i$ its velocity), and can control its acceleration $u_i \in \mathbb{R}^2$. The process noise is a diagonal matrix $W = \text{diag}(1e^{-2}, 1e^{-2}, 1e^{-4}, 1e^{-4})$. Each robot $i$ is equipped with a GPS, which measures the agent position $p_i$ with a covariance $V_{gps,i} = 2 \cdot I_2$. Moreover, the agents are equipped with lidars allowing each agent $i$ to measure the relative position of another agent $j$ with covariance $V_{lidar,ij} = 0.1 \cdot I_2$. The agents have limited on-board resources; hence, they want to activate only $k$ sensors.

For our tests, we consider two setups. In the first, named heterogeneous formation control, the LQG weight matrix $Q$ is a block diagonal matrix with $4 \times 4$ blocks, and each block $i$ chosen as $Q_i = 0.1 \cdot I_4$; since each block of $Q$ weights equally the tracking error of a robot, in the homogeneous case the tracking error of all agents is equally important. In the second setup, named heterogeneous formation control, $Q$ is chosen as above, except for one of the agents, say robot $1$, for which we choose $Q_1 = 10 \cdot I_4$; this setup models the case in which each agent has a different role or importance; hence, one weights differently the tracking error of the agents. In both cases, the matrix $R$ is chosen to be the identity matrix. The simulation is carried on over $T$ time steps, and $T$ is also chosen as LQG horizon. Results are averaged over 100 Monte
Carlo runs: at each run we randomize the initial estimation covariance $\Sigma_{1|1}$.

2) Compared Techniques: We compare five techniques. All techniques use an LQG-based estimator and controller, and they only differ by the selections of the active sensors. The first approach is the optimal sensor selection, denoted as optimal, which attains the minimum in (10), and which we compute by enumerating all possible subsets. The second approach is a pseudorandom sensor selection, denoted as random, which selects all the GPS measurements and a random subset of the lidar measurements. The third approach, denoted as logdet, selects sensors so to minimize the average logdet of the estimation covariance over the horizon; this approach resembles [24] and is agnostic to the control task. The fourth approach is the proposed sensor selection strategy (see Algorithm 2) and is denoted as $s$-LQG. Finally, we also report the LQG performance when all sensors are selected. This approach is denoted as allSensors.

3) Results: The results of the numerical analysis are reported in Fig. 3. When not specified otherwise, we consider a formation of $n = 4$ agents, which can only use a total of $k = 6$ sensors, and a control horizon $T = 20$. Fig. 3(a) shows the LQG cost for the homogeneous case and for increasing horizon. We note that, in all tested instances, the proposed approach $s$-LQG matches the optimal selection optimal, and both approaches are relatively close to allSensors, which selects all the available sensors. On the other hand, logdet leads to worse tracking performance, and is often close to random. These considerations are confirmed by the heterogeneous setup, in Fig. 3(b). In this case, the separation between our proposed approach and logdet becomes even larger; the intuition is that the heterogeneous case rewards differently the tracking errors at different agents; hence, while logdet attempts to equally reduce the estimation error across the formation, the proposed approach $s$-LQG selects sensors in a task-oriented fashion, since the matrices $\Theta_l$ for all $l = 1, 2, \ldots, T$ in the cost function in (10) incorporate the LQG weight matrices.

Fig. 3(c) shows the LQG cost attained for increasing number of selected sensors $k$ and for the homogeneous case. For increasing number of sensors all techniques converge to allSensors (since the entire ground set is selected). Fig. 3(d) shows the same statistics for the heterogeneous case. Now, $s$-LQG matches allSensors earlier, starting at $k = 7$; intuitively, in the heterogeneous case, adding more sensors may have marginal impact on the LQG cost (e.g., if the cost rewards a small tracking error for robot 1, it may be of a little value to take a lidar measurement between robots 3 and 4). This further stresses the importance of the proposed framework as a parsimonious way to control a system with minimal resources.

Fig. 3(e) and (f) shows the LQG cost attained by the compared techniques for increasing number of agents. optimal quickly becomes intractable to compute; hence, we omit values beyond $n = 4$. In both figures, the separation among the techniques increases with the number of agents, since the set of available sensors quickly increases with $n$. In the heterogeneous case, $s$-LQG remains relatively close to allSensors, implying that for the purpose of LQG control, using a cleverly selected small subset of sensors still ensures excellent tracking performance.

B. Resource-Constrained Robot Navigation

1) Simulation Setup: The second application scenario is illustrated in Fig. 2(b). An unmanned aerial robot (UAV) moves in a 3-D space, starting from a randomly selected location. The objective of the UAV is to land, and specifically, to reach $[0, 0, 0]$ with zero velocity. The UAV is modeled as a double-integrator, with state $x = [p \ v] \in \mathbb{R}^4$ ($p$ is the position, while $v$ its velocity), and can control its acceleration $u \in \mathbb{R}^3$. The process noise is $W = \mathbf{I}_6$. The UAV is equipped with multiple sensors. It has an on-board GPS, measuring the UAV position $p$ with a covariance $2 \cdot I_3$, and an altimeter, measuring only the last component of $p$ (altitude) with standard deviation $0.5$ m. Moreover, the UAV can use a stereo camera to measure the relative position of $\ell$ landmarks on the ground; we assume the location of each landmark to be known approximately and we associate to each landmark an uncertainty covariance [red ellipsoids in Fig. 2(b)], which is randomly generated at the beginning of each run. The UAV has limited on-board resources; hence, it wants to use only a few of sensing modalities. For instance, the resource-constraints may be due to the power consumption of the GPS and the altimeter, or may be due to computational constraints that prevent to run multiple object-detection algorithms to detect all landmarks on the ground. We consider two sensing-constrained scenarios: (i) all sensors to have the same cost (equal to 1), in which case, the UAV can activate at most $k$ sensors; (ii) the sensors to have heterogeneous costs: particularly, the GPS’s cost is set equal to 3; the altimeter’s cost is set equal to 2; and each landmark’s cost is set equal to 1.

We use $Q = \text{diag}(1e^{-3}, 1e^{-3}, 10, 1e^{-3}, 1e^{-3}, 10)$ and $R = I_3$. The structure of $Q$ reflects the fact that during landing we are particularly interested in controlling the vertical direction and the vertical velocity (entries with larger weight in $Q$), while we are less interested in controlling accurately the horizontal position and velocity (assuming a sufficiently large landing site). In the following, we present results averaged over 100 Monte Carlo runs: in each run, we randomize the covariances describing the landmark position uncertainty.

2) Compared Techniques: We consider the five techniques discussed in the previous section.

3) Results: The results of our numerical analysis are reported in Fig. 4 for the case where all sensors have the same sensor-cost, and in Fig. 5 for the case where sensors have different costs. When not
applications, ranging from large-scale networked systems to minia-
turized robotic networks. Motivated by the inapproximability of the
problem, we provided the first scalable algorithms with per-instance
suboptimality bounds. Importantly, the bounds are nonvanishing under
general control-theoretic conditions, encountered in most real-world
systems. To this end, we also extended the literature on supermod-
ular optimization: by providing scalable algorithms for optimizing
approximately supermodular functions subject to heterogeneous cost
constraints; and by providing novel suboptimality bounds that improve
the known bounds even for exactly supermodular optimization.

The article opens several avenues for future research. First, the
development of distributed implementations of the proposed algorithms
would offer computational speedups. Second, other codesign problems
are interesting to be explored, such as the codesign of control-sensing-
actuation. Third, while we provide bounds on an approximate sensor
design against optimal design, one could provide bounds against the
case where all sensors are used [67]. Finally, in adversarial or failure-
prone scenarios, one must account for sensor failures; to this end, one
could leverage recent results on robust combinatorial optimization [68].

**APPENDIX A**

**PRELIMINARY FACTS**

**Lemma 1** ([69, Proposition 8.5.5]): Consider two positive definite matrices $A_1$ and $A_2$. If $A_1 \preceq A_2$, then $A_1^{-1} \preceq A_2^{-1}$.

**Lemma 2** (Trace inequality [69, Proposition 8.4.13]): Consider a symmetric $A$, and a positive semidefinite matrix $B$. Then,

$$\lambda_{\min}(A)\tr(B) \leq \tr(AB) \leq \lambda_{\max}(A)\tr(B).$$

**Lemma 3** (Woodbury identity [69, Corollary 2.8.8]): Consider $A$, $C$, $U$, and $V$ such that $A$, $C$, and $A+UCV$ are invertible. Then,

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}. $$

**Lemma 4** ([69, Proposition 8.5.12]): Consider two symmetric matrices $A_1$ and $A_2$, and a positive semidefinite matrix $B$. If $A_1 \preceq A_2$, then $\tr(A_1B) \leq \tr(A_2B)$.

**Lemma 5** ([1, Appendix E]): For any sensors $S$, $\Sigma_{\iota\iota}(S)$ is the solution of the Kalman filtering recursion

$$\Sigma_{\iota+1|\iota}(S) = (\Sigma_{\iota\iota-1}(S)^{-1} + C_t(S)^T V_t(S)^{-1} C_t(S))^{-1}$$

with boundary condition $\Sigma_{\iota\iota}(S) = \Sigma_{11\iota}$.

**Lemma 6** ([63, Lemma 6]): Consider two sensor sets $S_1, S_2 \subseteq V$. If $S_1 \subseteq S_2$, then $\Sigma_{\iota\iota}(S_1) \succeq \Sigma_{\iota\iota}(S_2)$.

**Lemma 7** ([63, Corollary 1]): Let $\Sigma_{\iota\iota}$ be defined as in (22) with boundary condition $\Sigma_{\iota\iota}$; similarly, let $\Sigma_{\iota\iota}$ be defined as in (22) with boundary condition $\Sigma_{\iota\iota}$. If $\Sigma_{\iota\iota} \leq \Sigma_{\iota\iota}$, then $\Sigma_{\iota\iota+i+i} \preceq \Sigma_{\iota\iota+i+i}$ for any positive integer $i$.

**Lemma 8** ([63, Corollary 2]): Let $\Sigma_{\iota\iota}$ be defined as in (22) with boundary condition $\Sigma_{\iota\iota}$; similarly, let $\Sigma_{\iota\iota}$ be defined as in (22) with boundary condition $\Sigma_{\iota\iota}$. If $\Sigma_{\iota\iota} \leq \Sigma_{\iota\iota}$, then $\Sigma_{\iota\iota+i+i+1} \preceq \Sigma_{\iota\iota+i+i+1}$ for any positive integer $i$.

**Lemma 9**: Consider positive real numbers $a$, $b$, $\gamma$, $a_1, a_2, \ldots, a_n$ such that $\gamma_{a_1} = a$. Then,

$$f(a_1, a_2, \ldots, a_n) = 1 - \prod_{i=1}^{n} \left(1 - \gamma \frac{a_i}{a_i}ight)$$

**VI. CONCLUDING REMARKS**

We addressed an LQG control and sensing codesign problem, where
one jointly designs control and sensing policies under resource con-
straints. The problem is central in modern IoT and IoBT control

specified otherwise, we consider a total of $k = 3$ sensors to be selected,
and a control horizon $T = 20$. In Fig. 4(a), we plot the LQG cost
normalized by the horizon, which makes more visible the differences
among the techniques. Similarly to the formation control example,
s-LQG matches the optimal selection optimal, while logdet and
random have suboptimal performance. Fig. 4(b) shows the LQG cost
attained by the compared techniques for increasing number of selected
sensors $k$. All techniques converge to allSensors for increasing $k$,
but in the regime in which few sensors are used s-LQG still outperforms
alternative sensor selection schemes, and matches optimal.

Fig. 5 shows the LQG cost attained by the compared techniques for
increasing control horizon and various sensor cost budgets $b$. Similarly
to Fig. 4, s-LQG has the same performance as optimal, whereas
logdet and random have suboptimal performance. Notably, for $b = 15$ all sensors can be chosen; for this reason in Fig. 5(d) all compared
techniques (but the random) have the same performance.

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has its minimum at $a_1 = a_2 = \ldots = a_n = a/n$, and
\[
f(a/n, a/n, \ldots, a/n) = 1 - \left(1 - \frac{a}{b n}ight)^n \geq 1 - e^{-a/b}
\]

Proof of Lemma 9: The proof is obtained using the method of Lagrange multipliers, and is omitted (for a complete proof, see [64, Proof of Lemma 9]).

Lemma 10 (Monotonicity of cost function in (10) [63, Proposition 2]): Consider $\sum_{t=1}^T \text{tr}(\Theta_t \Sigma_{t|t}(S))$ in (10). If $S_1 \subseteq S_2$, then $\sum_{t=1}^T \text{tr}(\Theta_t \Sigma_{t|t}(S_1)) \geq \sum_{t=1}^T \text{tr}(\Theta_t \Sigma_{t|t}(S_2))$.

APPENDIX

PROOF OF THEOREM 1

A. Proof of Part (1) of Theorem 1

Lemma 11: Consider any $S$, and let $u_{1:T}^*(S)$ be the vector of control policies $(K_1 x_1(S), K_2 x_2(S), \ldots, K_T x_T(S))$. Then, $u_{1:T}^*(S)$ is an optimal control policy:
\[
u_{1:T}^*(S) \in \arg \min_{u_{1:T}(S)} h[S, u_{1:T}(S)] \tag{23}
\]
and, particularly, $u_{1:T}^*(S)$ attains an LQG cost equal to
\[
h[S, u_{1:T}^*(S)] = E(\|x_1\|N_1) + \sum_{t=1}^T \text{tr}(\Theta_t \Sigma_{t|t}(S)) + \text{tr}(W_t S_t).
\]

Proof of Lemma 11: The proof follows Lemma 1’s proof in [20], and can also be found in [64, Appendix B].

Proof of Part (1) of Theorem 1: Equation (10) is a direct consequence of (24), since the value of $S^*$ is an equal to $\min_{S \subseteq Y, \Sigma_{1:1} \subseteq \hat{b}} h[S, u_{1:T}^*(S)]$, and both $E(\|x_1\|N_1) = \text{tr}(\Sigma_{1:1} N_1)$ and $\sum_{t=1}^T \text{tr}(W_t S_t)$ are independent of $S$. Finally, (11) directly follows from (23).

B. Proof of Part (2) of Theorem 1

Lemma 12: $S^*$, and $u_{1:T}^*$ are a solution to Problem 2 if and only if they are a solution to
\[
\min_{S \subseteq Y, u_{1:T}(S)} c(S), \text{ s.t. } \min_{u_{1:T}(S)} h[S, u_{1:T}(S)] \leq \kappa.
\]

Proof of Lemma 12: We prove the lemma by contradiction. Particularly, let $S^*$ and $u_{1:T}^*$ be a solution to Problem 2, and assume by contradiction that they are not to (25), which instead has solution $S$ and $u_{1:T}$. By optimality of $S$ and $u_{1:T}^*$ (and suboptimality of $S^*$ and $u_{1:T}^*$) in (25), it follows $c(S) < c(S^*)$. In addition, $g(S) \leq \kappa$, since (25) must be feasible for (25). Therefore, $h(S, u_{1:T}) \leq \kappa$. Therefore, $(S, u_{1:T})$ is feasible for Problem 2 and has a better objective value with respect to the optimal solution $(S^*, u_{1:T}^*)$ (we already observed $c(S) < c(S^*)$), leading to contradiction.

For the other direction, now let $S^*$ and $u_{1:T}^*$ be a solution to (25), and assume that they are not to Problem 2, which instead has solution $(S, u_{1:T})$. By optimality of $(S, u_{1:T})$, and suboptimality of $S^*$ and $u_{1:T}^*$ in Problem 2, it follows $c(S) < c(S^*)$. In addition, $h(S, u_{1:T}) \leq \kappa$, since $(S, u_{1:T})$ must be feasible for Problem 2, and, as a result, $g(S) \leq \kappa$. Therefore, $(S, u_{1:T})$ is feasible for (25) and has a better objective value with respect to the optimal solution $(S^*, u_{1:T}^*)$ (we already observed $c(S) < c(S^*)$), leading to contradiction.

Proof of Part (2) of Theorem 1: The proof follows from Lemma 11 and Lemma 12.

APPENDIX C

PROOF OF THEOREM 2

Consider a problem instance for Problem 1 and Problem 2, where $T = 1$, and $A_1 = B_1 = C_1 = Q_1 = R_1 = I$. Then, $\Theta_1 = I/2$, and, as a result, the objective function in (10) becomes $1/2tr[\Sigma_{1:1} S]$. Now, choosing $\Sigma_{1:1}$ to be the steady-state Kalman filtering matrix defined in [65, Th. 2], as well as, $c(S)$, $b$ as as in [65, Th. 2], makes (10) and the optimization problem in [65] equivalent. But, the latter is inapproximable in polynomial time [65, Th. 2] (namely, unless NP=P, there is no polynomial time algorithm that guarantees a constant suboptimality bound). Therefore, (10) is too, and due to Theorem 1 both Problem 1 and Problem 2 as well.

APPENDIX D

PROOF OF THEOREM 3

For any $S$, let $f(S) = \sum_{t=1}^T \text{tr}(\Theta_t \Sigma_{t|t}(S))$ be the objective function in (10), $S^*$ be a solution in (10), and $b^* = c(S^*)$. Let $S_2$ be the set Algorithm 2 constructs by the end of line 19; let $\tilde{G} \subseteq \hat{S}_2$. Let $s_i$ be the ith element added to $\hat{G}$ during the ith iteration of Algorithm 2’s “while loop” (see lines 3–16). Let $\tilde{G}_i = \{s_1, s_2, \ldots, s_i\}$. Finally, consider Algorithm 2’s “while loop” terminates after $l + 1$ iterations.

Algorithm 2’s “while loop” terminates: (i) when $V = \emptyset$, that is, when all available sensors in $V$ can be chosen by Algorithm 2 as active while satisfying the budget constraint $b$; and (ii) when $c(G_i + 1) > b$, that is, when the addition of $s_i + 1$ in $G_i$ makes the cost of $G_i + 1$ to violate the budget $b$. Henceforth, we focus on the second scenario, which implies that $s_i + 1$ will be removed by the “if” statement in Algorithm 2’s lines 17–19, and, as a result, $\tilde{G}_i = \hat{S}_2$.

Lemma 13 (Generalization of [51, Lemma 2]): For
\[
i = 1, 2, \ldots, l + 1, \text{ it holds}
\]
\[
f(G_{i-1}) - f(G_i) \geq \gamma f(c(s_i))
\]

Proof of Lemma 13: Due to the monotonicity of the cost function $f$ in (10) (see Lemma 10),
\[
f(G_{i-1}) - f(S^*) \leq f(G_{i-1}) - f(S^* \cup \tilde{G}_{i-1})
\]
\[
f(G_{i-1}) - f(S^*) = f(G_{i-1}) - f([S^* \setminus \tilde{G}_{i-1}] \cup \tilde{G}_{i-1}].
\]

Let $\{z_1, z_2, \ldots, z_m\} \triangleq S^* \setminus \tilde{G}_{i-1}$, and also let
\[
d_j \triangleq f(G_{i-1} \cup \{z_1, z_2, \ldots, z_j\}) - f(G_{i-1} \cup \{z_1, z_2, \ldots, z_j\})
\]
for $j = 1, 2, \ldots, m$. Then, $f(G_{i-1}) - f(S^*) \leq \sum_{j=1}^m d_j$. Now,
\[
d_j \leq f(G_{i-1}) - f(G_{i-1} \cup \{z_j\}) = \frac{f(G_{i-1}) - f(G_i)}{\gamma f(c(z_j))}
\]
where the first inequality holds due to the Definition 3 of $\gamma_f$, and the second due to the greedy rule (see Algorithm 2 line 13) and the definitions of $G_i$ and $s_i$. Since $\sum_{j=1}^{m} c(z_j) \leq b^*$,

$$f(G_{i-1}) - f(S^*) \leq \sum_{j=1}^{m} d_j \leq b^* \frac{f(G_{i-1}) - f(G_i)}{\gamma_f c(s_i)}.$$  

Lemma 14 (Adaptation of [51, Lemma 3]): For $i = 1, 2, \ldots, l + 1$,

$$f(0) - f(G_i) \geq \left[ 1 - \prod_{j=1}^{i} \left( 1 - \frac{\gamma_f c(s_j)}{b^*} \right) \right] [f(0) - f(S^*)].$$  

Proof of Lemma 14: We complete the proof inductively. For $i = 1$, we need to prove $f(0) - f(G_1) \geq \gamma_f c(s_1)/b^* [f(0) - f(S^*)]$, which follows from Lemma 13 for $i = 1$. Then, for $i > 1$,

$$f(0) - f(G_i) = f(0) - f(G_{i-1}) + [f(G_{i-1}) - f(G_i)] \geq f(0) - f(G_{i-1}) +$$

$$\gamma_f c(s_i) \left( f(G_{i-1}) - f(S^*) \right)$$

$$= \left[ 1 - \frac{\gamma_f c(s_i)}{b^*} \right] [f(0) - f(G_{i-1})] +$$

$$\gamma_f c(s_i) \frac{f(0) - f(S^*)}{b^*} [f(0) - f(S^*)]$$

$$\geq \left[ 1 - \frac{\gamma_f c(s_i)}{b^*} \right] \left[ 1 - \prod_{j=1}^{i-1} \left( 1 - \frac{\gamma_f c(s_j)}{b^*} \right) \right] [f(0) - f(S^*)]$$

where we used Lemma 13 for the first inequality, and the induction hypothesis for the second. }

Proof of Part (1) of Theorem 3: To prove Algorithm 1’s approximation bound $\gamma_f/2(1 - e^{-\gamma_f})$, we let $b' \triangleq \sum_{j=1}^{l+1} c(s_j)$. Then,

$$f(0) - f(G_{i+1}) \geq \left[ 1 - \prod_{j=1}^{i+1} \left( 1 - \frac{\gamma_f c(s_j)}{b^*} \right) \right] [f(0) - f(S^*)]$$

$$\geq \left[ 1 - e^{-\gamma_f b'/b^*} \right] [f(0) - f(S^*)] \geq \left[ 1 - e^{-\gamma_f} \right] [f(0) - f(S^*)]$$  

Substituting (26) in (27), and rearranging gives

$$\max \left\{ f(0) - f(S_i), \gamma_f [f(0) - f(G_i)] \right\}$$

$$\geq \frac{\gamma_f}{2} (1 - e^{-\gamma_f}) [f(0) - f(S^*)]$$

which implies (since $\gamma_f$ takes values in $[0, 1]$)

$$\max \left\{ f(0) - f(S_i), f(0) - f(G_i) \right\} \geq \frac{\gamma_f}{2} (1 - e^{-\gamma_f}) [f(0) - f(S^*)].$$  

Finally, the bound $\gamma_g/2(1 - e^{-\gamma_g})$ follows from (28) as the combination of the following three observations: i) $S_i = S_2$, and, as a result, $f(0) - f(G_i) = f(0) - f(S_2)$, ii) Algorithm 2 returns $S$ such that $S \in \arg \max_{S \in (S_1, S_2)} [f(0) - f(S)]$ and, as a result, the previous observation, along with (28) gives

$$f(0) - f(S) \geq \frac{\gamma_f}{2} (1 - e^{-\gamma_f}) [f(0) - f(S^*)].$$  

iii) Finally, Lemma 11 implies that for any $S, S'$, $G(S) = f(S) + E(||x_1||_{N_1}) + \sum_{t=1}^{T} tr(W_t S_t)$, where $E(||x_1||_{N_1}) + \sum_{t=1}^{T} tr(W_t S_t)$ is independent of $S$. As a result, for any $S, S' \subseteq \mathcal{V}$, then $f(S) - f(S') = g(S) - g(S')$, which implies $\gamma_f = \gamma_g$ due to Definition 3. In addition, Lemma 11 implies for any $S \subseteq \mathcal{V}$ that $g(S) = h[S, u_{1:T}(S)]$ and $g^* = g(S^*)$. Therefore, for any $S$ that $f(0) - f(S) = g(0) - g(S) = h[0, u_{1:T}(0)] - h[S, u_{1:T}(S)]$ and $f(0) - f(S^*) = g(0) - g(S^*) = h[0, u_{1:T}(0)] - g^*$. Overall, (29) is written as

$$h[0, u_{1:T}(0)] - h[S, u_{1:T}(S)] \geq \frac{\gamma_f}{2} (1 - e^{-\gamma_f}) \{ h[0, u_{1:T}(0)] - g^* \}$$

which implies the bound $\gamma_f/2(1 - e^{-\gamma_f})$.

It remains to prove $1 - e^{-\gamma_f} c(S_l)/b^*$

$$f(0) - f(G_l) \geq \left[ 1 - \prod_{j=1}^{l+1} \left( 1 - \frac{\gamma_f c(s_j)}{b^*} \right) \right] [f(0) - f(G_l)]$$

$$\geq \left[ 1 - e^{-\gamma_f b'/b^*} \right] [f(0) - f(S^*)] \geq \left[ 1 - e^{-\gamma_f c(S_l)/b^*} \right] [f(0) - f(S^*)]$$

where the first inequality follows from Lemma 14, the second from Lemma 9, and in (26) from that $b'/b^* \geq 1$ and, as a result, $e^{-\gamma_f b'/b^*} \leq e^{-\gamma_f}$, that is, $1 - e^{-\gamma_f b'/b^*} \geq 1 - e^{-\gamma_f}$. Also, $f(0) - f(S_l) \geq \gamma_f [f(G_l) - f(G_{l+1})]$ due to the Definition 3 of $\gamma_g$ and, as a result,

$$\gamma_f [f(0) - f(G_l)]$$

where the first inequality holds due to the Definition 3 of $\gamma_f$, and the second due to the greedy rule (see Algorithm 2 line 13) and the definitions of $G_i$ and $s_i$. Since $\sum_{j=1}^{m} c(z_j) \leq b^*$,

$$f(G_{i-1}) - f(S^*) \leq \sum_{j=1}^{m} d_j \leq b^* \frac{f(G_{i-1}) - f(G_i)}{\gamma_f c(s_i)}.$$  

Proof of Part (2) of Theorem 3: The proof is parallel to that of [70, Th. 2].

APPENDIX E

PROOF OF THEOREM 4

We consider the notation in Appendix D. Also, let $S^*$ be a solution to Problem 2, and $b^* = c(S^*)$. Consider the computation
of the set $\tilde{S}$ in Algorithm 4, and let $G \triangleq \tilde{S}$ be the returned one. Let $s_i$ be the $i$th element added in $G$ during the $i$th iteration of Algorithm 4 “while loop.” Finally, let $G_i \triangleq \{s_1, s_2, \ldots, s_i\}$.

Lemma 15 (Adaptation of Lemma 13): For $i = 1, 2, \ldots, |G|$, 
\[ f(G_i) - f(G_i - 1) \geq \sum_{j=1}^{i} \left( 1 - \prod_{g=j}^{i} \left( 1 - \frac{f(c(s_j))}{b^*} \right) \right) [f(\emptyset) - f(S^*)]. \]

Proof: The proof is parallel to Lemma 13’s proof. \[ \blacksquare \]

Lemma 16 (Adaptation of Lemma 14): For $i = 1, 2, \ldots, |G|$, 
\[ f(\emptyset) - f(G_i) \geq \sum_{j=1}^{i} \left( 1 - \prod_{g=j}^{i} \left( 1 - \frac{f(c(s_j))}{b^*} \right) \right) [f(\emptyset) - f(S^*)]. \]

Proof: The proof is parallel to Lemma 14’s proof. \[ \blacksquare \]

Proof of Part (1) of Theorem 4: We first observe (18) holds since Algorithm 3 returns $S$ once $h(S, u_1, T(S)) \leq \kappa$ is satisfied.

It remains to prove (19). Let $l \triangleq |G|$; then, $G_l \triangleq G$, by the definition of $G_i$, and from Lemma 14 for $i = l - 1$,
\[ f(\emptyset) - f(G_l) \geq \sum_{j=1}^{i} \left( 1 - \prod_{g=j}^{i} \left( 1 - \frac{f(c(s_j))}{b^*} \right) \right) [f(\emptyset) - f(S^*)] \] (31)
where (31) follows from Lemma 9. Moreover, Lemma 11 implies that for any $S, S'$, it is $g(S) = f(S) + E(\|x_1\|_N) + \sum_{t=1}^{T} \text{tr}(W_tS_t)$, where $E(\|x_1\|_N) + \sum_{t=1}^{T} \text{tr}(W_tS_t)$ is independent of $\gamma$, and as a result, $f(S) - f(S') = g(S) - g(S')$, which implies $g_\gamma = g$. Moreover, Lemma 11 implies for any $S \subseteq S_1 \cup \{s_1\}$, $g(S) = h(S, u_1, T(S))$, and as a result, $f(\emptyset) - f(G_l) = h(\emptyset, u_1, T(\emptyset)) - h(G_l, u_1, T(G_l))$ and $f(\emptyset) - f(S^*) = h(\emptyset, u_1, T(\emptyset)) - h(S^*, u_1, T(S^*))$. In sum, (31) is the same as the inequality
\[ h(\emptyset, u_1, T(\emptyset)) - h(G_l, u_1, T(G_l)) \geq \sum_{j=1}^{i} \left( 1 - \prod_{g=j}^{i} \left( 1 - \frac{f(c(s_j))}{b^*} \right) \right) [f(\emptyset) - f(S^*)] \]
which, by letting $\beta \triangleq 1 - e^{-\gamma \sigma(G_l)/b^*}$ and rearranging, gives
\[ h(G_l, u_1, T(G_l)) \leq (1 - \beta)h(\emptyset, u_1, T(\emptyset)) + \beta h(S^*, u_1, T(S^*)) \]

where the second inequality holds because $S^*$ is a solution to Problem 2 and, as result, $h(S^*, u_1, T(S^*)) \leq \kappa$. Now, we recall Algorithm 4 returns $G = G_l$ when for $i = l$ it is the first time $h(G_l, u_1, T(G_l)) \leq \kappa$. Therefore, $h(G_l, u_1, T(G_l)) > \kappa$ and, as a result, there exists $\epsilon > 0$ such that $h(G_l, u_1, T(G_l)) = (1 + \epsilon)\kappa$, and (32) gives
\[ (1 + \epsilon)\kappa \leq (1 - \beta)h(\emptyset, u_1, T(\emptyset)) + \beta \kappa \Rightarrow \]

where the latter holds since $G = G_l \cup \{s_l\}$, due to the definitions of $G$, $G_l$, and $s_l$ and since $c(G) = c(G_l) + c(s_l)$. Finally, since the definition of $\epsilon$ implies $\epsilon \kappa = h(G_l, u_1, T(G_l)) - \kappa$, and the definition of $G$ is $G = \tilde{S}$, the proof of (18) is complete.

Proof of Part (2) of Theorem 4: The proof is similar to the proof of part (2) of Theorem 3.
where (34) holds due to Lemma 2. From (34), we have

\[ f_t(S) - f_t(S \cup \{v\}) \geq \]

\[ \lambda_{max}(I + \bar{C}_{v,t} \bar{\Theta}_t \bar{C}_{v,t}^T) \text{tr} \left( \bar{C}_{v,t} \bar{\Theta}_t \bar{C}_{v,t}^T \right) \]

\[ \geq \lambda_{max}(I + \bar{C}_{v,t} \Sigma_{t-1}(\bar{S} \cup \{v\})) \text{tr} \left( \bar{C}_{v,t} \Sigma_{t-1}(\bar{S} \cup \{v\}) \bar{C}_{v,t}^T \right) \]

\[ = \lambda_{max}(I + \bar{C}_{v,t} \Sigma_{t}(\bar{S} \cup \{v\})) \text{tr} \left( \bar{\Theta}_t \bar{C}_{v,t} \bar{C}_{v,t}^T \right) \]

\[ (35) \]

where we used \( \Sigma_t^{-1} \leq \Sigma_{t-1}(\bar{S} \cup \{v\}) \), which holds since \( \bar{\Theta}_t \) implies \( \bar{\Theta}_t \geq \Sigma_{t-1}(\bar{S} \cup \{v\}) \), and as a result, from Lemma 1 \( \bar{\Theta}_t \leq \Sigma_t^{-1} \leq \Sigma_{t-1}(\bar{S} \cup \{v\}) \). In addition, Corollary 8 and \( \Sigma_{t=1}(\bar{S} \cup \{v\}) \) holds due to Lemma 6, imply \( \Sigma_{t=1}(\bar{S} \cup \{v\}) \leq \Sigma_{t-1}(\bar{S} \cup \{v\}) \). Finally, from (22) in Lemma 5, \( \Sigma_{t=1}(\bar{S} \cup \{v\}) = \Sigma_t(\bar{S} \cup \{v\}) \). Overall, \( \bar{\Theta}_t \geq \Sigma_{t}(\bar{S} \cup \{v\}) \).

Consider \( t' \in \{1, 2, \ldots, T\} \) such that \( \bar{\Theta}_{t'} \bar{C}_{v,t'} \leq \bar{\Theta}_t \bar{C}_{v,t} \), for any \( t = 1, \ldots, T \). Also, let \( \Phi \triangleq \bar{\Theta}_{t'} \bar{C}_{v,t'} \bar{C}_{v,t} \), and let \( \min_{\bar{S}(\bar{S} \cup \{v\})} \lambda_{max}(I + \bar{C}_{v,t} \Sigma_{t}(\bar{S} \cup \{v\}) \bar{C}_{v,t}^T) \). Summing (35) across all \( t \in \{1, 2, \ldots, T\} \), and using Lemmata 2 and 4, we have

\[ g(S) - g(S \cup \{v\}) \geq \lambda_{min} \sum_{t=1}^{T} \text{tr} \left( \Theta_t \bar{C}_{v,t} \bar{C}_{v,t}^T \right) \]

\[ \geq \lambda_{min} \sum_{t=1}^{T} \text{tr} \left( \Phi \right) > 0 \]

which is nonzero because \( \sum_{t=1}^{T} \Theta_t > 0 \) and \( \Phi \) is a nonzero positive semidefinite matrix.

Finally, we lower bound \( \text{tr} \left( \Phi \right) \), using Lemma 2:

\[ \text{tr} \left( \Phi \right) \geq \lambda_{min} \lambda_{max}(I + \bar{C}_{v,t} \Sigma_{t}(\bar{S} \cup \{v\}) \bar{C}_{v,t}^T) \]

\[ \geq \lambda_{min} \lambda_{max}(I + \bar{C}_{v,t} \Sigma_{t}(\bar{S} \cup \{v\}) \bar{C}_{v,t}^T) \]

\[ (36) \]

where (36) holds because \( \lambda_{max}(I + \bar{C}_{v,t} \Sigma_{t}(\bar{S} \cup \{v\}) \bar{C}_{v,t}^T) \) is derived by applying Lemma 1 to \( \bar{C}_{v,t} \Sigma_{t}(\bar{S} \cup \{v\}) \bar{C}_{v,t}^T \). Particularly, \( \lambda_{max}(I + \bar{C}_{v,t} \Sigma_{t}(\bar{S} \cup \{v\}) \bar{C}_{v,t}^T) \) is derived by applying Lemma 1 to \( \bar{C}_{v,t} \Sigma_{t}(\bar{S} \cup \{v\}) \bar{C}_{v,t}^T \), where the equality holds by the definition of \( \bar{\Theta}_t \). In addition, due to Lemma 6, \( \Sigma_{t=1}(\bar{S} \cup \{v\}) \geq \Sigma_{t}(\bar{S} \cup \{v\}) \), and as a result, from Corollary 7, \( \Sigma_{t}(\bar{S} \cup \{v\}) \geq \Sigma_{t}(\bar{S} \cup \{v\}) \). Overall, \( \bar{\Theta}_t \geq \Sigma_{t}(\bar{S} \cup \{v\}) \).

b) Upper Bound for the Denominator of \( \gamma \). The proof follows similar ideas as mentioned above, and is omitted (for a complete proof, see [64, Proof of Th. 5]).

**APPENDIX G**

**PROOF OF THEOREM 6**

**Lemma 17 (System-condition for near-optimal cooperation):** Let \( N_1 \) be defined as in (9). The control policy \( u_{t-1} \) is suboptimal for the LQG problem in (21) for all nonzero initial conditions \( x_1 \) and only if

\[ \sum_{t=1}^{T} A_t^{\top} Q_t A_t \cdots A_1 \geq N_1. \]

(37)

**Proof of Lemma 17:** For any \( x_1 \), (24) in Lemma 11 implies for (21) as

\[ \min_{u_{t-1}} \sum_{t=1}^{T} \|x_{t+1} - u_t(x_t)\|_{R_t}^2 \|x_1\|_{R_1} = \frac{x_1^T N_1 x_1}{x_1^T N_1 x_1} \]

(38)
From (40), we have
\[
\sum_{t=1}^{T} \sum_{i=1}^{r} \mathbf{U}_t^T \Theta_i \mathbf{U}_i \geq 0
\]
where we let \( \xi_t \triangleq \mathbf{U}_t \mathbf{z} \). Consider time \( t' \) such that for any time \( t \in \{1, \ldots, T\} \), \( \xi_t \xi_t^T \geq \xi_{t'} \xi_{t'}^T \). From (40), we have
\[
\sum_{t=1}^{T} \sum_{i=1}^{r} \mathbf{U}_t^T \Theta_i \mathbf{U}_i \geq 0.
\]
\[
\mathbf{U}_t^T \Theta_i \mathbf{U}_i \geq 0, \quad t' \leq t.
\]

Proof of Theorem 6: Theorem 6 follows from the sequential application of Lemmata 17, 19, and 20.

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