Sequential constant rank constraint qualifications for nonlinear semidefinite and second-order cone programming with applications

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Abstract

We present new constraint qualification conditions for nonlinear semidefinite and second-order cone programming that extend some of the constant rank-type conditions from nonlinear programming. As an application of these conditions, we provide a unified global convergence proof of a class of algorithms to stationary points without assuming neither uniqueness of the Lagrange multiplier nor boundedness of the Lagrange multipliers set. This class of algorithm includes, for instance, general forms of augmented Lagrangian and sequential quadratic programming methods. We also compare these new conditions with some of the existing ones, including the nondegeneracy condition, Robinson’s constraint qualification, and the metric subregularity constraint qualification.

Keywords: Constraint qualifications, Semidefinite programming, Second-order cone programming.

1 Introduction

Constraint qualification (CQ) conditions play a crucial role in optimization. They permit to establish first- and second-order necessary optimality conditions for local minima and support the convergence theory of many practical algorithms (see, for instance, a unified convergence analysis for a whole class of algorithms by Andreani et al. [10, Thm. 6]). Some of the well-known CQs in nonlinear programming (NLP) are the constant-rank constraint qualification (CRCQ), introduced by Janin [25], and constant positive linear dependence (CPLD) condition. The latter was first conceptualized by Qi and Wei [29], and then proved to be a constraint qualification by Andreani et al. [14]. Moreover, it has been a source of inspiration for other authors to define even weaker constraint qualifications for NLP, such as the constant rank of the subspace component (CRSC) [11], and the relaxed versions of CRCQ [26] and CPLD [10]. Our interest in constant rank-type conditions is motivated, mainly, by their applications towards obtaining global convergence results of iterative algorithms to stationary points without relying on boundedness or uniqueness of Lagrange multipliers. However, several other applications that we do not pursue in this paper may be expected to be extended to the conic context, such as the computation of the derivative of the value function [25, 27] and the validity of strong second-order necessary optimality conditions that do not rely on the whole set of Lagrange multipliers [2]. Besides, their ability of dealing with redundant constraints, up to some extent, gives modellers some degree of freedom without losing regularity or convergence guarantees on algorithms. For instance, the standard NLP trick of replacing one nondegenerate equality constraint by two inequalities of opposite sign does not violate CRCQ, while violating the standard Mangasarian-Fromovitz CQ. As far as we know, no constant-rank type CQ has been proposed in conic programming until recent years. The first extension of CRCQ to nonlinear second-order cone programming (NSOCP) appeared in [35], but it was shown to be incorrect in [3]. A second proposal, presented in [9], consists of transforming some of the conic constraints into NLP constraints via a reduction function, whenever it was possible, and then demanding constant linear dependence of the reduced constraints, locally. This was considered by the authors a naive extension, since it basically avoids the main difficulties that are expected from a conic framework. What both these works have in common is that they somehow neglected the conic structure of the problem. In a recent article [8], we introduced weak notions of regularity for nonlinear semidefinite programming (NSDP) that were defined in terms of the eigenvectors of the constraints – therein called weak-nondegeneracy.
and weak-Robinson’s CQ. These conditions take into consideration only the diagonal entries of some particular transformation of the matrix constraint. Noteworthy, weak-nondegeneracy happens to be equivalent to the linear independence CQ (LICQ) when an NLP constraint is modeled as a structurally diagonal matrix constraint, unlike the standard nondegeneracy condition [33], which in turn is considered the usual extension of LICQ to NSDP. Moreover, the proof technique we employed in [8] induces a direct application in the convergence theory of an external penalty method. In this paper, we use these conditions to derive our extension proposals for CRCQ and CPLD to NSDP, which also recover their counterparts in NLP when it is modelled as a structurally diagonal matrix constraint. These CQs are called, in this paper, as weak-CRCQ and weak-CPLD, respectively.

However, to provide support for algorithms other than the external penalty method, we present stronger variants of these conditions, called seq-CRCQ and seq-CPLD respectively, by incorporating perturbations in their definitions. This makes them robust and easily connectible with algorithms that keep track of approximate Lagrange multipliers, but also more exigent. Nevertheless, seq-CRCQ is still strictly weaker than nondegeneracy, while seq-CPLD is strictly weaker than Robinson’s CQ. On the other hand, weak-CRCQ is strictly weaker than seq-CRCQ and independent of Robinson’s CQ, while weak-CPLD is strictly weaker than weak-CRCQ and seq-CPLD. Moreover, we show that seq-CPLD implies the metric subregularity CQ.

Due to the spectral structure of NSOCP problems, we propose analogues of weak-nondegeneracy and weak-Robinson’s CQ to this context as well, which are also used to extend CRCQ and CPLD, respectively, to NSOC. To achieve global convergence of a broader class of algorithms – namely all algorithms that generate sequences approximately satisfying the Karush-Kuhn-Tucker conditions (AKKT sequences, in the sense of [7]) – we propose stronger variants of the new CQs, that also imply metric subregularity CQ for NSOC. Generally speaking, all results obtained for NSDP were extended to NSOCP in an analogous, but nontrivial way.

The content of this paper is organized as follows: Section 2 introduces notation and some well-known theorems and definitions that will be useful in the sequel. In Section 3 we present our main results for NSDP in a constructive reasoning, starting from weak-nondegeneracy and weak-Robinson’s CQ, passing through weak-CRCQ and weak-CPLD, to finally arrive at seq-CRCQ and seq-CPLD, which are the main CQs of this paper. Section 4 is similarly structured, but concerns NSOCP problems. Lastly, some final remarks are given in Section 5. In the Appendices, we exhibit two algorithms that can be supported by the theory developed in this paper, together with some technical results that will be referenced throughout the paper.

2 A nonlinear conic programming review

There are many overlapping ideas in the classical theories of NSDP and NSOC, due to their conic nature. In fact, these problems can be seen as particular cases of a nonlinear conic programming (NCP) problem, usually stated as follows:

\[
\begin{align*}
\text{Minimize} & \quad f(x), \\
\text{subject to} & \quad G(x) \in K,
\end{align*}
\]

(NCP)

where \( f : \mathbb{R}^n \to \mathbb{R} \) and \( G : \mathbb{R}^n \to Y \) are continuously differentiable functions, \( Y \) is a finite dimensional linear space equipped with an inner product \( \langle \cdot, \cdot \rangle \), and \( K \subseteq Y \) is a closed convex cone. Although the general (NCP) is not the main object of our study, we will use it as a common language to present some common aspects of NSOC and NSDP; then, for simplicity, we will also assume that \( K \) is self-dual.

We omit equality constraints in (NCP) for simplicity of notation, but our definitions and results are flexible regarding inclusion of equality constraints, which should be done in the same way as in [9]. Moreover, throughout the whole paper, we will denote the feasible set of (NCP) by \( \mathcal{F} \).

For any given differentiable function \( F : \mathbb{R}^n \to Y \), we will denote its Jacobian at a given point \( x \in \mathbb{R}^n \) by \( DF(x) \), and the adjoint operator of \( DF(x) \) will be denoted by \( DF(x)^\ast \). In particular, if \( Y = \mathbb{R}^m \), then \( DF(x) \) can be represented as an \( m \times n \) matrix (using the canonical basis of \( \mathbb{R}^m \)), and its adjoint coincides with its transpose matrix, \( DF(x)^\top \). When \( Y = \mathbb{R} \), we will denote the gradient of \( F \) at \( x \) by \( \nabla F(x) \), for every \( x \in \mathbb{R}^n \). Moreover, the \( i \)-th partial derivative of \( F \) at \( x \) will be denoted by \( D_{x_i} F(x) \).

Given a sequence of sets \( \{S^k\}_{k \in \mathbb{N}} \), let us recall its outer limit (or upper limit) in the sense of Painlevé-Kuratowski (cf. [31, Def. 4.1] or [18, Def. 2.52]), defined as

\[
\limsup_{k \in \mathbb{N}} S^k = \left\{ y : \exists I \subseteq \mathbb{N}, \exists \{y^k\}_{k \in I} \to y, \forall k \in I, y^k \in S^k \right\},
\]

which is the collection of all cluster points of sequences \( \{y^k\}_{k \in \mathbb{N}} \) such that \( y^k \in S^k \) for every \( k \in \mathbb{N} \). The notation \( I \subseteq \mathbb{N} \) means that \( I \) is an infinite subset of the set of natural numbers \( \mathbb{N} \).

We also recall that the orthogonal projection (with respect to the induced norm \( \| \cdot \| \)) of an element \( Y \in Y \) onto \( K \), which is defined as

\[
\Pi_K(Y) = \arg\min_{Z \in K} \| Z - Y \|
\]

is a convex continuous function of \( Y \) since \( K \) is nonempty and self-dual. Furthermore, every \( Y \in Y \) has a Moreau decomposition [28, Prop. 1] in the form

\[
Y = \Pi_K(Y) - \Pi_K(-Y)
\]
with \((\Pi_K(Y), \Pi_K(-Y)) = 0\).

Additionally, for every \(Y \in \mathbb{Y}\) and every \(\tau > 0\), we denote by \(B(Y, \tau) = \{Z \in \mathbb{Y} : \|Y - Z\| < \tau\}\) the open ball centered at \(Y\) with radius \(\tau\), and its closure will be denoted by \(\text{cl}(B(Y, \tau))\).

### 2.1 Classical optimality conditions and constraint qualifications

As usual in continuous optimization, we drive our attention towards local solutions of (NCP) that satisfy the so-called Karush-Kuhn-Tucker (KKT) conditions, defined as follows:

**Definition 2.1.** We say that the Karush-Kuhn-Tucker conditions hold at \(\bar{x} \in F\) when there exists some \(\bar{Y} \in \mathbb{K}\) such that

\[
\nabla_y L(\bar{x}, \bar{Y}) = 0 \quad \text{and} \quad G(\bar{x}, \bar{Y}) = 0,
\]

where \(L(x, Y) = f(x) - (G(x), Y)\) is the Lagrangian function of (NCP). The latter condition is usually referred to as complementarity, and if \(G(\bar{x}) + \bar{Y} \in \text{int} \mathbb{K}\), we say that strict complementarity holds for \(\bar{x}\) and \(\bar{Y}\). The vector \(\bar{Y}\) is called a Lagrange multiplier associated with \(\bar{x}\), and the set of all Lagrange multipliers associated with \(\bar{x}\) will be denoted by \(\Lambda(\bar{x})\).

Of course, not every local minimizer satisfies KKT; this is the case however if the local minimizer satisfies a CQ. In order to recall some classical CQs, consider the (Bouligand) tangent cone to \(K\) at a point \(Y \in \mathbb{K}\), which is given by

\[
T_K(Y) = \{Z \in \mathbb{Y} : \text{dist}(Y + tZ, K) = o(t), \forall t > 0\}
\]

\[
= \{Z \in \mathbb{Y} : \exists(z^k)_{k \in \mathbb{N}} \rightarrow Z, \exists(t^k)_{k \in \mathbb{N}} \rightarrow 0, t^k > 0, Y + t^kZ^k \in \mathbb{K}, \forall k \in \mathbb{N}\}.
\]

Its lineality space, denoted by \(\text{lin}(T_K(Y))\), is defined as the largest subspace contained in \(T_K(Y)\). Since \(K\) is a convex cone, so is \(T_K(Y)\), and therefore we have \(\text{lin}(T_K(Y)) = T_K(Y) \cap (-T_K(Y))\). One of the most recognized constraint qualifications in conic programming is the nondegeneracy (or transversality) condition introduced by Shapiro and Fan [33], which can be characterized [18, Eq. 4.172] at a point \(\bar{x} \in F\) when the following relation is satisfied:

\[
\text{Im} DG(\bar{x}) + \text{lin}(T_K(G(\bar{x}))) = \mathbb{Y}.
\]

If \(\bar{x}\) is a local solution of (NCP) that satisfies nondegeneracy, then \(\Lambda(\bar{x})\) is a singleton, but the converse is not necessarily true unless strict complementarity holds [18, Prop. 4.75]. Another widespread constraint qualification is Robinson’s CQ [30], which can be characterized at \(\bar{x} \in F\) by the existence of some \(d \in \mathbb{R}^n\) such that

\[
G(\bar{x}) + DG(\bar{x})[d] \in \text{int} \mathbb{K}.
\]

where \(\text{int} \mathbb{K}\) stands for the topological interior of \(\mathbb{K}\), which is nonempty, since \(\mathbb{K}\) is assumed to be self-dual (cf. [19, Exerc. 2.31]). Alternatively, following [18, Prop. 2.97], Robinson’s CQ holds at \(\bar{x}\) if, and only if, the following relation holds:

\[
\text{Im} DG(\bar{x}) - G(\bar{x}) - \mathbb{K} = \mathbb{Y}.
\]

It is known that when \(\bar{x}\) is a local solution of (NCP), then \(\Lambda(\bar{x})\) is nonempty and compact if, and only if, Robinson’s CQ holds at \(\bar{x}\). Thus, the nondegeneracy condition can be considered an extension of the LICQ condition from NLP to NCP, while Robinson’s CQ is an extension of the well-known Mangasarian-Fromovitz CQ (MFCQ).

A weaker constraint qualification that is also of our interest in this paper is the so-called metric subregularity CQ (also known as the error bound CQ in NLP), defined as follows:

**Definition 2.2** (Def. 1.1 of [21]). We say that a feasible point \(\bar{x}\) of (NCP) satisfies the metric subregularity CQ when there exists some \(\gamma > 0\) and a neighborhood \(V\) of \(\bar{x}\) such that

\[
\text{dist}(x, F) \leq \gamma \|\Pi_K(-G(x))\|
\]

for every \(x \in V\). That is, when the set-valued mapping \(G : \mathbb{R}^n \rightrightarrows \mathbb{Y}\) that maps \(x \mapsto G(x) - K\) is metric subregular at \((\bar{x}, 0)\) in \(\text{graph}(G)\), where \(\text{dist}(x, F)\) denotes the distance between \(x\) and \(F\), and \(\text{graph}(G) \subset \mathbb{R} \times \mathbb{Y}\) is the graph of \(G\).

The metric subregularity CQ is implied by Robinson’s CQ, which in turn coincides with a similar condition called metric regularity CQ, and it has relevant implications on the stability analysis of optimization problems – for details, we refer to Ioffe’s survey [23, 24]. Besides, there are several works addressing the relationship between constant rank constraint qualifications and the metric subregularity CQ in NLP, such as Minchenko and Stakhovski [26], Andreani et al. [10], and others.

We will use the following sufficient condition for metric subregularity CQ to hold, which is a result that can be easily extended from Minchenko and Stakhovski [26, Thm. 2].

**Proposition 2.1.** Let \(\bar{x} \in F\) and for every given \(x \in \mathbb{R}^n\), let \(\Lambda_n(x)\) denote the set of Lagrange multipliers of the problem of minimizing \(||z - x||\) subject to \(G(z) \in K, z \in \mathbb{R}^n\). If there exist numbers \(\tau > 0\) and \(\delta > 0\) such that \(\Lambda_n(x) \cap \text{cl}(B(0, \tau)) \neq \emptyset\) for every \(x \in B(\bar{x}, \delta)\), then \(\bar{x}\) satisfies metric subregularity CQ.

The proof of Proposition 2.1 is included in Appendix B, for the sake of completeness.
2.2 Sequential optimality conditions and strict constraint qualifications

If we do not assume any CQ, every local minimizer of (NCP) can still be proved to satisfy at least a 
sequential type of optimality condition, which is deeply connected to the classical external penalty method. 
Namely:

**Theorem 2.1.** Let \( \overline{x} \) be a local minimizer of (NCP), and let \( \{\rho_k\}_{k \in \mathbb{N}} \to +\infty \). Then, there exists some 
\( \{x^k\}_{k \in \mathbb{N}} \to \overline{x} \), such that for each \( k \in \mathbb{N} \), \( x^k \) is a local minimizer of the regularized penalized function 
\[
F(x) = f(x) + \frac{1}{2}\|x - \overline{x}\|^2 + \frac{\rho_k}{2}\|\Pi_K(-G(x))\|^2.
\]

**Proof.** See the first part of the proof of [6, Thm. 2]. Alternatively, for a specialized proof for NSDP, see [12, 
Thm. 3.2]; and for a specialization to NSOCP, see [4, Thm. 3.1].

Note that Theorem 2.1 provides a sequence \( \{x^k\}_{k \in \mathbb{N}} \to \overline{x} \) such that each \( x^k \) satisfies, with an error 
\( x^k \to 0^+ \), the first-order optimality condition of the unconstrained minimization problem 
\[
\min_{x \in \mathbb{R}^n} f(x) + \frac{\rho_k}{2}\|\Pi_K(-G(x))\|^2,
\]
so \( \{x^k\}_{k \in \mathbb{N}} \) characterizes an output sequence of an external penalty method. Moreover, the sequence 
\( \{Y^k\}_{k \in \mathbb{N}} \subseteq K \), where 
\[
Y^k = \rho_k\Pi_K(-G(x^k))
\]
for every \( k \in \mathbb{N} \), consists of approximate Lagrange multipliers for \( \overline{x} \), in the sense that \( \nabla_s L(x^k, Y^k) \to 0 \) and complementarity and feasibility are approximately fulfilled, in view of Moreau’s decomposition – indeed, 

Note that Theorem 2.1 follows the same principle of Theorem 2.1, but it is designed to capture the output sequences of a 
big class of penalty-type algorithms – for instance, augmented Lagrangian methods and some of its 
variants (see Appendix A.1), SQP methods (see Appendix A.2), and interior-point methods [5, Sec. 3.2]. 
Let us recall one of its many characterizations\(^1\).

**Definition 2.3 (Def. 4 of [6]).** We say that a point \( \overline{x} \in \mathcal{F} \) satisfies the AKKT condition when there exist 
sequences \( \{x^k\}_{k \in \mathbb{N}} \to \overline{x} \) and \( \{Y^k\}_{k \in \mathbb{N}} \subseteq K \), and perturbation sequences \( \{\delta^k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}^n \) and \( \{\Delta^k\}_{k \in \mathbb{N}} \subseteq \mathcal{Y} \), such that:

1. \( \nabla_s L(x^k, Y^k) = \delta^k \), for every \( k \in \mathbb{N} \);
2. \( G(x^k) + \Delta^k \in K \) and \( G(x^k) + \Delta^k, Y^k = 0 \), for every \( k \in \mathbb{N} \);
3. \( \Delta^k \to 0 \) and \( \delta^k \to 0 \).

Again, note that \( \{Y^k\}_{k \in \mathbb{N}} \) is a sequence of approximate Lagrange multipliers of \( \overline{x} \), in the sense that \( Y^k \) 
is an exact Lagrange multiplier, at \( x = x^k \), for the perturbed problem 
\[
\min_{x \in \mathbb{R}^n} f(x) + \langle \overline{x} - x, \delta^k \rangle,
\]
subject to \( G(x) + \Delta^k \in K \).

It follows from Theorem 2.1 that the AKKT condition holds at every local minimizer of (NCP) regardless 
of the fulfilment of a CQ. Indeed, define \( Y^k = \rho_k\Pi_K(-G(x^k)) \), \( \delta^k = \overline{x} - x^k \), and \( \Delta^k \to \Pi_K(-G(x^k)) \) for 
every \( k \in \mathbb{N} \), and observe that 
\[
0 = \nabla F(x^k) = \nabla f(x^k) + DG(x^k)[\rho_k\Pi_K(-G(x^k))] + x^k - \overline{x} = \nabla_s L(x^k, Y^k) + x^k - \overline{x}.
\]

Then, use Moreau’s decomposition to arrive at \( G(x^k) + \Delta^k, Y^k = 0 \) and \( G(x^k) + \Delta^k \in K \) for each \( k \in \mathbb{N} \).

The main goal in enlarging the class of approximate Lagrange multipliers \( Y^k \) and perturbations \( \Delta^k \) as in 
Definition 2.3 instead of considering only the ones given by Theorem 2.1 is that while several algorithms will 
generate primal-dual sequences such as in Definition 2.3, a point that fulfills this definition can be proven to 
be a KKT point under constraint qualifications strictly weaker than nondegeneracy and Robinson’s CQ. 
In fact, this approach motivates the definitions of the new CQs we propose in this paper as, in view of 
Theorem 2.1, any condition that establishes that an AKKT point is also a KKT point is, in particular, a 
CQ; moreover, this CQ necessarily supports the global convergence of any aforementioned algorithm to a 
KKT point. Since these CQs must be generally stronger than usual CQs, such conditions are called in [13] 
as strict constraint qualifications, and we adopt the same terminology. As shown, for instance, in [4, 6], 
Robinson’s CQ and nondegeneracy are strict CQs.

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1 Definition 2.3 coincides with AKKT as presented in [12, Def. 3.1] and [4, Def. 3.3] when (NCP) is reduced to NSDP and 
NSOCP, respectively. See, for instance, [6, Prop. 4 and Prop. 5].
2.3 Reviewing constant rank-type constraint qualifications in NLP

This section is meant to be a brief review of the main results regarding the classical nonlinear programming problem:

\[
\begin{align*}
\text{Minimize} & \quad f(x), \\
\text{subject to} & \quad g_i(x) \geq 0, \ldots, g_m(x) \geq 0,
\end{align*}
\]

(NLP)

which is obtained from (NCP) when we consider \( \mathcal{Y} = \mathbb{R}^m, \quad \mathcal{K} = \mathbb{R}^m_+ \approx \{ Y \in \mathbb{R}^m : \forall i \in \{1, \ldots, m\}, Y_i \geq 0, \} \), and \( G(x) = (g_1(x), \ldots, g_m(x)) \) for every \( x \in \mathbb{R}^n \).

As far as we know, the first constant rank-type constraint qualification was introduced by Janin [25], to obtain directional derivatives for the optimal value function of a perturbed NLP problem. Janin’s condition is defined as follows:

**Definition 2.4.** Let \( \mathcal{F} \subset \mathcal{F} \). The constant rank constraint qualification for (NLP) (CRCQ) holds at \( \mathcal{F} \) if there exists a neighborhood \( \mathcal{V} \) of \( \mathcal{F} \) such that, for every subset \( J \subseteq \{ i \in \{1, \ldots, m\} : g_i(\mathcal{F}) = 0 \} \), the rank of the family \( \{ \nabla g_i(x) \}_{i \in J} \) remains constant for all \( x \in \mathcal{V} \).

As noticed by Qi and Wei [29] it is possible to rephrase Definition 2.4 in terms of the “constant linear dependence” of \( \{ \nabla g_i(x) \}_{i \in J} \) for every \( J \). That is, CRCQ holds at \( \mathcal{F} \) if, and only if, there exists a neighborhood \( \mathcal{V} \) of \( \mathcal{F} \) such that, for every \( J \subseteq \{ i \in \{1, \ldots, m\} : g_i(\mathcal{F}) = 0 \} \), if \( \{ \nabla g_i(x) \}_{i \in J} \) is linearly dependent, then \( \{ \nabla g_i(x) \}_{i \in J} \) remains linearly dependent for every \( x \in \mathcal{V} \). Based on this characterization, Qi and Wei proposed a relaxation of CRCQ, which they called constant positive linear dependence (CPLD) condition, but this was only proven to be a constraint qualification a few years later, in [14]. To properly define CPLD, recall that a family of vectors \( \{ z_i \}_{i \in J} \) of \( \mathbb{R}^n \) is said to be positively linearly independent when

\[
\sum_{i \in J} z_i \alpha_i = 0, \quad \alpha_i \geq 0, \quad \forall i \in J \Rightarrow \alpha_i = 0, \quad \forall i \in J.
\]

Next, we recall the CPLD constraint qualification:

**Definition 2.5.** Let \( \mathcal{F} \subset \mathcal{F} \). The constant positive linear dependence condition for (NLP) (CPLD) holds at \( \mathcal{F} \) if there exists a neighborhood \( \mathcal{V} \) of \( \mathcal{F} \) such that, for every \( J \subseteq \{ i \in \{1, \ldots, m\} : g_i(\mathcal{F}) = 0 \} \), if the family \( \{ \nabla g_i(x) \}_{i \in J} \) is positively linearly dependent, then \( \{ \nabla g_i(x) \}_{i \in J} \) remains linearly dependent for all \( x \in \mathcal{V} \).

Clearly, CPLD is implied by CRCQ, which is in turn implied by LICQ and independent of MFCQ. Moreover, CPLD is implied by MFCQ, and all those implications are strict [14, 25]. To show that our extensions of CRCQ and CPLD to NSDP and NSOCP are indeed constraint qualifications (Theorems 3.1 and 4.1), we shall take inspiration in [10], where the authors employ Theorem 2.1 together with the well-known Carathéodory’s Lemma:

**Lemma 2.1** (Exercise B.1.7 of [15]). Let \( z_1, \ldots, z_p \in \mathbb{R}^n \), and let \( \alpha_1, \ldots, \alpha_p \in \mathbb{R} \) be arbitrary. Then, there exists some \( J \subseteq \{1, \ldots, p\} \) and some scalars \( \tilde{\alpha}_i \) with \( i \in J \), such that \( \{ z_i \}_{i \in J} \) is linearly independent,

\[
\sum_{i=1}^p \alpha_i z_i = \sum_{i \in J} \tilde{\alpha}_i z_i,
\]

and \( \alpha_i \tilde{\alpha}_i > 0 \), for all \( i \in J \).

See also [22]. If one considers equality constraints in (NCP) separately, one should employ an adapted version of Carathéodory’s Lemma that fixes a particular subset of vectors, which can be found in [10, Lem. 2]. In our current setting, Lemma 2.1 will suffice as is.

Differently from NLP where CRCQ or CPLD can be used to arrive at a KKT point using either the sequences given by Theorem 2.1 or the more general ones given by Definition 2.3, these approaches will give rise to two different variants of CRCQ and CPLD in the conic context.

3 Nonlinear semidefinite programming

In this section, \( \mathbb{S}^m \) denotes the linear space of all \( m \times m \) real symmetric matrices equipped with the inner product defined as \( \langle M, N \rangle = \text{trace}(MN) = \sum_{i,j=1}^m M_{ij}N_{ij} \) for all \( M, N \in \mathbb{S}^m \). Consider the NSDP problem in standard form:

\[
\begin{align*}
\text{Minimize} & \quad f(x), \\
\text{subject to} & \quad G(x) \succeq 0,
\end{align*}
\]

(NSDP)

which can be seen as a particular case of (NCP) with \( \mathcal{K} = \mathbb{S}^m_+ \), where \( \mathbb{S}^m_+ \) is the cone of all symmetric positive semidefinite matrices in \( \mathbb{S}^m \) and \( \succeq \) is the partial order induced by \( \mathbb{S}^m_+ \); that is, \( M \succeq N \) if, and only if, \( M - N \in \mathbb{S}^m_+ \). Analogously, we say that \( M \succ N \) if, and only if, \( M - N \in \text{int} \mathbb{S}^m_+ \).

Recall that every \( M \in \mathbb{S}^m \) has a spectral decomposition in the form

\[
M = \lambda_1(M)u_1(M)u_1(M)^\top + \cdots + \lambda_m(M)u_m(M)u_m(M)^\top,
\]

where \( u_1(M), \ldots, u_m(M) \in \mathbb{R}^m \) are arbitrarily chosen orthonormal eigenvectors associated with the eigenvalues \( \lambda_1(M), \ldots, \lambda_m(M) \), respectively, which in turn are assumed to be arranged in non-increasing order.
Equivalently, we can write (2) as $M = UDU^T$, where $U$ is an orthogonal matrix whose $i$-th column is $u_i(M)$, and $D = \Diag(\lambda_1(M), \ldots, \lambda_m(M))$ is a matrix whose diagonal entries are $\lambda_1(M), \ldots, \lambda_m(M)$ and the remaining entries are zero.

A convenient property of the orthogonal projection onto $S^m_+$ is that, for every $M \in S^m$, we have $\Pi_{S^m_+}(M) = [\lambda_1(M)]_+u_1(M)u_1(M)^T + \ldots + [\lambda_m(M)]_+u_m(M)u_m(M)^T,$ where $[\cdot]_+ = \max\{\cdot, 0\}$.

The tangent cone to $S^m_+$ at a given $M \succeq 0$ with rank $r$ can be characterized in terms of any matrix $E \in \mathbb{R}^{m \times (m-r)}$ whose columns form an orthonormal basis of $\Ker M$ as follows

$$T_{S^m_+}(M) = \{ N \in S^m : E^TNE \succeq 0 \}.$$ This clearly yields $\lim(T_{S^m_+}(M)) = \{ N \in S^m : E^TNE = 0 \}.$

Regarding feasible points, say $\overline{x}$, of (NSDP), these matrices can be considered analogues of the “indices of active constraints” from NLP, in a certain sense. For instance, it is possible to characterize nondegeneracy at any feasible point $\overline{x}$ of (NSDP) by means of any given matrix $E$ with orthonormal columns that span $\Ker G(\overline{x})$. Indeed, following [18, Sec. 4.6.1], nondegeneracy holds at $\overline{x}$ if, and only if, either $\Ker G(\overline{x}) = \{0\}$ or the linear mapping $\psi_{\overline{x}} : \mathbb{R}^n \rightarrow \mathbb{R}^{m-r}$ given by

$$\psi_{\overline{x}}(\cdot) = E^T G(\overline{x})[\cdot]E$$ is surjective, which is in turn equivalent to saying that the vectors

$$v_{ij}(\overline{x}, E) = [e_i^T D x_1 G(\overline{x}) e_j, \ldots, e_i^T D x_m G(\overline{x}) e_j]^T, \quad 1 \leq i \leq j \leq m-r,$$

are all linearly independent [32, Prop. 6], where $e_i$ denotes the $i$-th column of $E$ and $r$ is the rank of $G(\overline{x})$.

### 3.1 Weak constant rank CQs for NSDP

Based on the relationship between LICQ and CRCQ, the most natural candidate for an extension of CRCQ to NSDP is to demand every subset of

$$\{v_{ij}(x, E) : 1 \leq i \leq j \leq m-r\}$$
to remain with constant rank (or constant linear dependence) in a neighborhood of $\overline{x}$. However, this candidate cannot be a CQ, as shown in the following counterexample, adapted from [3, Eq. 2]:

**Example 3.1.** Consider the problem to minimize $f(x) = -x$ subject to

$$G(x) = \begin{bmatrix} x & x^2 & x + x^2 \end{bmatrix} \succeq 0.$$

For this problem, $\overline{x} = 0$ is the only feasible point and, therefore, the unique global minimizer of the problem. Since $G(\overline{x}) = 0$, the columns of the matrix $E = I_2$ form an orthonormal basis of $\Ker G(\overline{x})$ (the whole space $\mathbb{R}^2$). For this choice of $E$, we have

$$v_{11}(x, E) = v_{22}(x, E) = 1 \quad \text{and} \quad v_{12}(x, E) = 1 + 2x.$$ Since they are all bounded away from zero, the rank of every subset of $\{v_{ij}(x, E) : 1 \leq i \leq j \leq 2\}$ remains constant for every $x$ around $\overline{x}$. However, Note that $\overline{x}$ does not satisfy the KKT conditions because any $\overline{Y} \in \Lambda(\overline{x})$ would necessarily be a solution of the system

$$\begin{align*}
\overline{y}_{11} & \geq 0, \\
\overline{y}_{22} & \geq 0, \\
\overline{y}_{11} \overline{y}_{22} - \overline{y}_{12} & \geq 0, \\
\overline{y}_{11} + 2\overline{y}_{12} + \overline{y}_{22} & = -1,
\end{align*}$$

which has no solution.

Besides, it is well-known that even if $G$ is affine, not all local minimizers of (NCP) satisfy KKT, but in this case every subfamily of $\{v_{ij}(x, E) : 1 \leq i \leq j \leq m-r\}$ remains with constant rank for every $x \in \mathbb{R}^n$.

What Example 3.1 tells us is that $E = I_2$ may be a bad choice of $E$. In fact, let us choose a different $E$, namely, denote the columns of $E$ by $e_1 = [a, b]^T$ and $e_2 = [c, d]^T$, and take $a = -1/\sqrt{2}$ and $b = c = d = 1/\sqrt{2}$.

This selection of $E$ happens to diagonalizes $G(x)$ for every $x$, but it follows that

$$\begin{align*}
v_{11}(x, E) &= 1 + 2ab(1+2x) = -2x; \\
v_{22}(x, E) &= 1 + 2cd(1+2x) = 2(1+x); \\
v_{12}(x, E) &= (ad + bc)(1+2x) = 0,
\end{align*}$$
and the rank of \( \{ v_i(x, E) : 1 \leq i \leq j \leq 2 \} \) does not remain constant in a neighborhood of \( \mathfrak{T} = 0 \).

In light of our previous work [8], the situation presented above is not surprising. Therein, we already noted that identifying the “good” matrices \( E \) allows us to obtain relaxed versions of nondegeneracy and Robinson’s CQ for NSDP. This identification is also crucial to extend constant-rank type conditions to NSDP and is the starting point for the results we will present in the current manuscript.

For the sake of completeness, let us quickly summarize a discussion raised in [8] before presenting the main contributions of this paper. Consider a feasible point \( \mathfrak{T} \in \mathcal{F} \) and denote by \( r \) the rank of \( G(\mathfrak{T}) \). Observe that \( \lambda_r(M) > \lambda_{r+1}(M) \) for every \( M \in S^n \) close enough to \( G(\mathfrak{T}) \). Thus, when \( r < m \), define the set

\[
E_r(M) = \left\{ E \in \mathbb{R}^{m \times m-r} : ME = E\text{Diag}(\lambda_{r+1}(M), \ldots, \lambda_m(M)) \right\},
\]

which consists of all matrices whose columns are orthonormal eigenvectors associated with the \( m-r \) smallest eigenvalues of \( M \), which is well defined whenever \( \lambda_r(M) > \lambda_{r+1}(M) \). By convention, \( E_r(M) = \emptyset \) when \( r = m \). By construction, \( E_r(M) \) is nonempty provided \( r < m \) and \( M \) is close enough to \( G(\mathfrak{T}) \). In particular, in this situation, \( E_r(G(\mathfrak{T})) \) is the set of all matrices with orthonormal columns that span \( \ker G(\mathfrak{T}) \).

We showed, in [8, Prop. 3.2], that nondegeneracy can be equivalently stated as the linear independence of the smaller family, \( \{ v_i(\mathfrak{T}, E) \}_{i \in \{1, \ldots, m-r\}} \), as long as this holds for all \( E \in E_r(G(\mathfrak{T})) \) instead of a fixed one. Similarly, Robinson’s CQ can be translated as the positive linear independence of the family \( \{ v_i(\mathfrak{T}, E) \}_{i \in \{1, \ldots, m-r\}} \) for every \( E \in E_r(G(\mathfrak{T})) \) [8, Prop. 5.1]. This characterization suggested a weak form of nondegeneracy (and Robinson’s CQ) that takes into account only a particular subset of \( E_r(G(\mathfrak{T})) \) instead of the whole set, which reads as follows:

**Definition 3.1** (Def. 3.2 and Def. 5.1 of [8]). Let \( \mathfrak{T} \in \mathcal{F} \) and let \( r \) be the rank of \( G(\mathfrak{T}) \). We say that \( \mathfrak{T} \) satisfies the:

- Weak-nondegeneracy condition for NSDP if either \( r = m \) or, for each sequence \( \{ x^k \}_{k \in \mathbb{N}} \to \mathfrak{T} \), there exists some \( E \in \limsup_{k \in \mathbb{N}} E_r(G(x^k)) \) such that the family \( \{ v_i(\mathfrak{T}, E) \}_{i \in \{1, \ldots, m-r\}} \) is linearly independent;
- Weak-Robinson’s CQ condition for NSDP if either \( r = m \) or, for each sequence \( \{ x^k \}_{k \in \mathbb{N}} \to \mathfrak{T} \), there exists some \( E \in \limsup_{k \in \mathbb{N}} E_r(G(x^k)) \) such that the family \( \{ v_i(\mathfrak{T}, E) \}_{i \in \{1, \ldots, m-r\}} \) is positively linearly independent.

Note that, in general, \( \limsup_{k \in \mathbb{N}} E_r(G(x^k)) \subseteq E_r(G(\mathfrak{T})) \), but the reverse inclusion is not always true, meaning \( E_r(G(x)) \) is not necessarily continuous at \( \mathfrak{T} \) as a set-valued mapping. It then follows that weak-nondegeneracy is indeed a strictly weaker CQ than nondegeneracy [8, Ex. 3.1]. Moreover, in contrast with nondegeneracy, weak-nondegeneracy happens to fully recover LICQ when \( G(x) \) is a structurally diagonal matrix in the form \( G(x) = \text{Diag}(g_1(x), \ldots, g_m(x)) \) [8, Prop. 3.3]. Similarly, weak-Robinson’s CQ is implied by Robinson’s CQ and coincides with MFCQ when \( G(x) \) is diagonal. We do not know whether Robinson’s CQ is equivalent to weak-Robinson’s CQ or not; however, we conjecture it is, and we give a partial answer to this open question in Appendix C.

A straightforward relaxation of weak-nondegeneracy and weak-Robinson’s CQ, likewise NLP, leads to our first extension proposal of CRCQ and CPLD to NSDP:

**Definition 3.2** (weak-CRCQ and weak-CPLD). Let \( \mathfrak{T} \in \mathcal{F} \) and let \( r \) be the rank of \( G(\mathfrak{T}) \). We say that \( \mathfrak{T} \) satisfies the:

- Weak constant rank constraint qualification for NSDP (weak-CRCQ) if either \( r = m \) or, for each sequence \( \{ x^k \}_{k \in \mathbb{N}} \to \mathfrak{T} \), there exists some \( E \in \limsup_{k \in \mathbb{N}} E_r(G(x^k)) \) such that, for every subset \( J \subseteq \{1, \ldots, m-r\} \); if the family \( \{ v_i(\mathfrak{T}, E) \}_{i \in J} \) is linearly dependent, then \( \{ v_i(x^k, E^k) \}_{i \in J} \) remains linearly dependent, for all \( k \in I \) large enough.
- Weak constant positive linear dependence constraint qualification for NSDP (weak-CPLD) if either \( r = m \) or, for each sequence \( \{ x^k \}_{k \in \mathbb{N}} \to \mathfrak{T} \), there exists some \( E \in \limsup_{k \in \mathbb{N}} E_r(G(x^k)) \) such that, for every subset \( J \subseteq \{1, \ldots, m-r\} \); if the family \( \{ v_i(\mathfrak{T}, E) \}_{i \in J} \) is positively linearly dependent, then \( \{ v_i(x^k, E^k) \}_{i \in J} \) remains linearly dependent, for all \( k \in I \) large enough.

For both definitions, \( I \subseteq \mathbb{N} \), and \( \{ E^k \}_{k \in \mathbb{N}} \) is the sequence converging to \( E \) and such that \( E^k \in E_r(G(x^k)) \) for every \( k \in I \), as required by the Painlevé-Kuratowski outer limit.

Clearly, weak-nondegeneracy implies weak-CRCQ, which in turn implies weak-CPLD. Similarly, it is also easy to see that weak-Robinson’s CQ implies weak-CPLD. However, Robinson’s CQ and its weak variant are both independent of weak-CRCQ. In fact, the next example shows that weak-CRCQ is not implied by neither (weak-)Robinson’s CQ nor weak-CPLD.

**Example 3.2.** Let us consider the constraint

\[
G(x) = \begin{bmatrix}
2x_1 + x_2^3 & -x_2^3 \\
-x_2^3 & 2x_1 + x_2^3
\end{bmatrix}
\]

and note that, for every orthogonal matrix \( E \) in the form

\[
E = \begin{bmatrix}
a & c \\
b & d
\end{bmatrix},
\]

and the rank of \( \{ v_i(x, E) : 1 \leq i \leq j \leq 2 \} \) does not remain constant in a neighborhood of \( \mathfrak{T} = 0 \).
we have
\[ v_{11}(x, E) = \begin{bmatrix} 2 \end{bmatrix} \quad \text{and} \quad v_{22}(x, E) = \begin{bmatrix} 2 \end{bmatrix}, \]

Then, at \( \mathbf{r} = 0 \), we have \( v_{11}(\mathbf{r}, E) = v_{22}(\mathbf{r}, E) = [2, 0]^T \), so they are linearly dependent, but positively linearly independent for all \( \mathbf{r} \in E(\mathbf{r}) \). However, choosing any sequence \( \{x^k\}_{k \in \mathbb{N}} \to 0 \) such that \( x^2_k \neq 0 \) for all \( k \), it follows that the eigenvalues of \( G(x_k) \):

\[ \lambda_1(G(x_k)) = 2(x_1 + x_2) \quad \text{and} \quad \lambda_2(G(x_k)) = 2x_1, \]

are simple, with associated orthonormal eigenvectors

\[ u_1(G(x_k)) = \left( -\frac{1}{\sqrt{2}}, 1, 1 \right) \quad \text{and} \quad u_2(G(x_k)) = \left( 1, 1, \frac{1}{\sqrt{2}} \right), \]

respectively, for every \( k \in \mathbb{N} \). Then, the only sequence \( \{E_k\}_{k \in \mathbb{N}} \) such that \( E_k \in E(\mathbf{r}(x_k)) \) for every \( k \), up to sign, is given by \( a = -1/\sqrt{2} \) and \( b = c = d = 1/\sqrt{2} \). However, keep in mind that \( v_{11}(x, E), i \in \{1, 2\} \), is invariant to the sign of the columns of \( E \), so \( v_{22}(x, E^k) = [2, 0]^T \) and \( v_{11}(x, E^k) = [2, 1, 2]^T \) are linearly independent for all large \( k \). Therefore, we conclude that (weak-)Robinson’s CQ holds at \( \mathbf{r} \), and consequently weak-CPLD also holds, but weak-CRCQ does not hold at \( \mathbf{r} \).

Conversely, we show with another counterexample, that weak-CRCQ does not imply (weak-)Robinson’s CQ, and neither does weak-CPLD.

**Example 3.3.** Let us consider the constraint

\[ G(x) = \begin{bmatrix} x & x^2 \\ x^2 & -x \end{bmatrix} \]

and the point \( \mathbf{r} = 0 \). Take any sequence \( \{x^k\}_{k \in \mathbb{N}} \to \mathbf{r} \) such that \( x^k \neq 0 \) for every \( k \), and consider two subsequences of it, indexed by \( I_+ \) and \( I_- \), such that \( x^k > 0 \) for every \( k \in I_+ \), and \( x^k < 0 \) for every \( k \in I_- \). Then, for every \( k \in I_+ \), we have that:

\[ \lambda_1(G(x_k)) = x \sqrt{(x^k)^2 + 1} \quad \text{and} \quad \lambda_2(G(x_k)) = -x \sqrt{(x^k)^2 + 1}, \]

are simple, with associated orthonormal eigenvectors uniquely determined (up to sign) by

\[ u_1(G(x_k)) = \frac{1}{\eta^k_1} \left( \frac{1 + \sqrt{(x^k)^2 + 1}}{x^k}, 1 \right) \quad \text{and} \quad u_2(G(x_k)) = \frac{1}{\eta^k_2} \left( \frac{-\sqrt{(x^k)^2 + 1}}{x^k}, 1 \right), \]

where

\[ \eta^k_1 = \left( \frac{1 + \sqrt{(x^k)^2 + 1}}{x^k} \right)^2 + 1 \quad \text{and} \quad \eta^k_2 = \left( \frac{-\sqrt{(x^k)^2 + 1}}{x^k} \right)^2 + 1. \]

Moreover, one can verify that whenever \( I_+ \) is an infinite set,

\[ \lim_{k \in I_+} u_1(G(x^k)) = (1, 0) \quad \text{and} \quad \lim_{k \in I_+} u_2(G(x^k)) = (0, 1). \]

Then, we have that for all \( E \in \text{Limsup}_{k \in I_+} E_k(G(x_k)) \), the vectors

\[ v_{11}(E, E) = 1 \quad \text{and} \quad v_{22}(E, E) = -1 \]

are positively linearly dependent. And, in addition, since \( \eta^k_1 \to \infty \) and \( \eta^k_2 \to 0 \), the vectors

\[ v_{11}(x^k, E^k) = \eta^k_1 + 4 \sqrt{(x^k)^2 + 1} - 2 \quad \text{and} \quad v_{22}(x^k, E^k) = \eta^k_2 - 4 \sqrt{(x^k)^2 + 1} - 2 \]

are nonzero and have opposite signs; and thus, remain positively linearly dependent, for all large \( k \in I_+ \).

For the indices \( k \in I_- \), the order of \( \lambda_1(G(x^k)) \) and \( \lambda_2(G(x^k)) \) is swapped, together with their respective eigenvectors, and we have \( \lim_{k \in I_-} u_1(G(x^k)) = (0, 1) \) and \( \lim_{k \in I_-} u_2(G(x^k)) = (-1, 0) \). Hence, for all \( E \in \text{Limsup}_{k \in I_-} E_k(G(x_k)) \), the vectors

\[ v_{11}(E, E) = -1 \quad \text{and} \quad v_{22}(E, E) = 1 \]

are also positively linearly dependent. The order of \( v_{11}(x^k, E^k) \) and \( v_{22}(x^k, E^k) \) is also swapped, so they remain positively linearly dependent for all large \( k \in I_- \).

By the above reasoning, observe that any sequence \( \{x^k\}_{k \in \mathbb{N}} \to \mathbf{r} \), such that \( x^k \neq 0 \) for every \( k \in \mathbb{N} \), shows that (weak-)Robinson’s CQ fails at \( \mathbf{r} \). Moreover, if \( x^2 = 0 \) for infinitely many indices, we may simply take \( E^k = [2, 0] \) for every \( k \), and then \( v_{11}(x^k, E^k) = v_{11}(E, E) = 1 \) and \( v_{22}(x^k, E^k) = v_{22}(E, E) = -1 \) are positively linearly dependent for every \( k \in \mathbb{N} \). This completes checking that weak-CPLD holds.

Thus, weak-CRCQ and weak-CPLD both hold at \( \mathbf{r} \), but (weak-)Robinson’s CQ does not.
Just as it happens in NLP, the weak-CPLD condition is strictly weaker than (weak-)Robinson’s CQ, and also weaker than weak-CRCQ, which are in turn, independent. Furthermore, let us establish a formal relationship between weak-CRCQ and weak-CPLD, and their NLP counterparts:

**Proposition 3.1.** Let \( G(x) \doteq \text{Diag}(g_1(x), \ldots, g_m(x)) \) be a structurally diagonal constraint and let \( \mathfrak{T} \) be such that \( g_1(\mathfrak{T}) \geq 0, \ldots, g_m(\mathfrak{T}) \geq 0 \). Then, the following statements are equivalent:

1. weak-CRCQ holds at \( \mathfrak{T} \);
2. For every \( J \subseteq A(\mathfrak{T}) \), if the set \( \{ \nabla g_i(\mathfrak{T}) : i \in J \} \) is linearly dependent, then \( \{ \nabla g_i(x) : i \in J \} \) is also linearly dependent, for every \( x \) close enough to \( \mathfrak{T} \);

where \( A(\mathfrak{T}) \doteq \{ i \in \{ 1, \ldots, m \} : g_i(\mathfrak{T}) = 0 \} \) is the set of active indices at \( \mathfrak{T} \).

**Proof.** Let \( r = \text{rank}(G(\mathfrak{T})) \), and note that the result follows trivially if \( m = r \). Hence, we will assume that \( r < m \). For simplicity, we will also assume that \( A(\mathfrak{T}) = \{ 1, \ldots, m-r \} \).

- \( 1 \Rightarrow 2 \): By contradiction, suppose that there is some \( J \subseteq A(\mathfrak{T}) \) and a sequence \( \{ x^k \}_{k \in \mathbb{N}} \to \mathfrak{T} \) such that \( \{ \nabla g_i(x^k) : i \in J \} \) is linearly independent for every \( k \), but \( \{ \nabla g_i(\mathfrak{T}) : i \in J \} \) is not. Let \( \{ E^k \}_{k \in \mathbb{N}} \) and \( \mathbf{E} \) be the sequence and its limit point described in Definition 3.2, for this particular \( \{ x^k \}_{k \in \mathbb{N}} \). Note that any other set \( J' \) that contains \( J \) such that \( \{ \nabla g_i(x^k) : i \in J' \} \) is linearly independent also fits this description, so let us assume that \( J \) is maximal.

Since \( G(x^k) \) is diagonal, every eigenvector \( v_i^k \) associated with an eigenvalue \( \lambda_i^k \) must satisfy \( G_i(x^k)v_i^k = \lambda_i^k v_i^k \) for every \( j \in \{ 1, \ldots, m \} \), which implies \( \lambda_i^k = G_j(x^k) \) or \( v_i^k = 0 \). Moreover, since \( G \) is continuous, the \( m-r \) smallest eigenvalues of \( G(x^k) \) converge to zero, and consequently, they are bounded from above by

\[
\alpha = \frac{1}{2} \min \{ G_i(\mathfrak{T}) : i \in \{ m-r+1, \ldots, m \} \}
\]

for \( k \) large enough. On the other hand, by continuity of \( G \) again, the \( r \) largest eigenvalues of \( G(x^k) \) are bounded from below by \( \alpha \) for all \( k \) large enough. Hence, it necessarily holds that \( v_i^k = 0 \) for all \( j \in \{ m-r+1, \ldots, m \} \) and for all \( k \) large enough. That is, \( E^k \) has the form

\[
E^k = \begin{bmatrix} Q^k \\ 0 \end{bmatrix}, \quad \text{where } Q^k \in \mathbb{R}^{m-r \times m-r} \text{ is orthogonal,}
\]

for every \( k \) large enough. A simple computation shows us that

\[
v_{ii}(x^k, E^k) = \sum_{j=1}^{m-r} \nabla g_j(x^k)(Q^k_{j,i})^2, \quad \text{and} \quad v_i(\mathfrak{T}, \mathbf{E}) = \sum_{j=1}^{m-r} \nabla g_j(\mathfrak{T})Q^2_{j,i},
\]

for every \( i \in \{ 1, \ldots, m-r \} \), where \( Q \) is the submatrix of \( \mathbf{E} \) correspondent to the indices of \( Q^k \). Observe that

\[
\text{span}(\{ \nabla g_i(x^k) : i \in J \}) = \text{span}(\{ \nabla g_i(x^k) : i \in \{ r+1, \ldots, m \} \})
\]

for all \( k \) large enough; otherwise, there would be a subsequence \( \{ x^{k_l} \}_{k \in \mathbb{N}} \subseteq \{ x^k \}_{k \in \mathbb{N}} \) and another index \( j' \notin J \) such that \( \{ \nabla g_i(x^{k_l}) : i \in J \cup \{ j' \} \} \) is linearly independent for every \( k \in I \), contradicting the maximality of \( J \). Hence, for every \( S \subseteq \{ 1, \ldots, m-r \} \) we have

\[
\text{span}(\{ v_{ii}(x^k, E^k) : i \in S \}) \subseteq \text{span}(\{ \nabla g_i(x^k) : i \in J \})
\]

for every large enough \( k \). In particular, there exists some \( S' \subseteq \{ 1, \ldots, m-r \} \) with the same cardinality as \( J \), such that (9) holds with equality for every large \( k \). On the other hand, it follows from (8) that

\[
\text{span}(\{ v_{ii}(\mathfrak{T}, \mathbf{E}) : i \in S' \}) \subseteq \text{span}(\{ \nabla g_i(\mathfrak{T}) : i \in J \})
\]

and this implies \( \text{span}(\{ v_{ii}(\mathfrak{T}, \mathbf{E}) : i \in S' \}) \) is a linearly dependent set. However, since \( \{ v_{ii}(x^k, E^k) : i \in S' \} \) is linearly independent for all \( k \), by weak-CRCQ, we obtain a contradiction.

- \( 2 \Rightarrow 1 \): Take \( Q^k = \mathbf{1}_{m-r} \) and \( E^k \) as in (7), so we have \( v_{ii}(x^k, E^k) = \nabla g_i(x^k) \) for every \( i \in \{ 1, \ldots, m-r \} \) and every \( k \in \mathbb{N} \), and the result follows immediately.

Using analogous arguments to the proposition above, we can also prove the following:

**Corollary 3.1.** Under the same hypotheses of the previous proposition, the following are equivalent:

1. weak-CPLD as in Definition 3.2 holds at \( \mathfrak{T} \);
2. For every \( J \subseteq A(\mathfrak{T}) \), if the set \( \{ \nabla g_i(\mathfrak{T}) : i \in J \} \) is positively linearly dependent, then \( \{ \nabla g_i(x) : i \in J \} \) is linearly dependent, for every \( x \) close enough to \( \mathfrak{T} \).

**Proof.** Note, in (8), that \( v_{ii}(x^k, E^k) \) is generated by a nonnegative linear combination of \( \nabla g_i(x^k) \), \( i \in \{ 1, \ldots, m-r \} \). Therefore, every argument in the proof of Proposition 3.1 can be adapted to prove Corollary 3.1. It suffices to consider positive linear independence, instead of linear independence; and the smallest cone generated by \( \{ v_{ii}(x^k, E^k) \}_{i \in S} \) instead of the smallest subspace.
Advancing to the main result of this section, which is to prove that weak-CPLD (and therefore, weak-CRCQ) guarantees the existence of Lagrange multipliers at all local solutions of (NSDP), we get inspiration in the proof of [14, Thm. 3.1] for NLP, and the proof of [8, Thm. 3.2]. That is, we analyse the sequence from Theorem 2.1 in terms of the spectral decomposition of its approximate Lagrange multiplier candidates, under weak-CPLD. Then, we use Carathéodory’s Lemma 2.1 to construct a bounded sequence from it, that converges to a Lagrange multiplier. As an intermediary step, we also obtain a convergence result of the external penalty method to KKT points under weak-CPLD, a fact that is emphasized in the statement of the next theorem.

**Theorem 3.1.** Let \( \{\rho_k\}_{k \in \mathbb{N}} \to \infty \) and \( \{x^k\}_{k \in \mathbb{N}} \to \overline{\mathbf{F}} \in \mathcal{F} \) be such that

\[
\nabla_x L \left( x^k, \rho_k \Pi_{\Sigma^0}(-G(x^k)) \right) \to 0.
\]

If \( \overline{\mathbf{F}} \) satisfies weak-CPLD, then \( \overline{\mathbf{F}} \) satisfies the KKT conditions. Moreover, every local minimizer of (NSDP) that satisfies weak-CPLD also satisfies KKT.

**Proof.** Let \( Y^k \doteq \rho_k \Pi_{\Sigma^0}(-G(x^k)) \), for every \( k \in \mathbb{N} \). Recall that we assume \( \lambda_1(-G(x^k)) \geq \ldots \geq \lambda_m(-G(x^k)) \), for every \( k \), and denote by \( r \) the rank of \( \text{Ker}(G(\overline{\mathbf{F}})) \). Note that when \( k \) is large enough, say greater than some \( k_0 \), we necessarily have \( \lambda_i(-G(x^k)) = -\lambda_{m+i-1}(G(x^k)) < 0 \) for all \( i \in \{m-r+1, \ldots, m\} \). Let \( I \subseteq \infty \mathbb{N} \), and \( \{E^k\}_{k \in I} \to \mathcal{E} \) be such that \( E^k \in \mathcal{E}_x(G(x^k)) \) for every \( k \in I \), as described in Definition 3.2. Then, for each \( k \in I \) greater than \( k_0 \), the spectral decomposition of \( Y^k \) is given by

\[
Y^k = \sum_{i=1}^{m-r} \alpha^k_i e^k_i (e^k_i)^\top,
\]

where \( \alpha^k_i \doteq |\rho_k \lambda(-G(x^k))|_i \geq 0 \) and \( e^k_i \) denotes the \( i \)-th column of \( E^k \), for every \( i \in \{1, \ldots, m-r\} \). Since \( \nabla_x L(x^k, Y^k) \to 0 \), we have

\[
\nabla f(x^k) - \sum_{i=1}^{m-r} \alpha^k_i DG(x^k)^* \left[ e^k_i (e^k_i)^\top \right] \to 0,
\]

but note that

\[
DG(x^k)^* \left[ e^k_i (e^k_i)^\top \right] = \begin{bmatrix} (D_1 G(x^k), e^k_i (e^k_i)^\top) \\ \vdots \\ (D_n G(x^k), e^k_i (e^k_i)^\top) \end{bmatrix} = \begin{bmatrix} (e^k_i)^\top D_1 G(x^k) e^k_i \\ \vdots \\ (e^k_i)^\top D_n G(x^k) e^k_i \end{bmatrix} = \nu_i(x^k, E^k),
\]

so we can rewrite (10) as

\[
\nabla f(x^k) - \sum_{i=1}^{m-r} \alpha^k_i \nu_i(x^k, E^k) \to 0.
\]

Using Carathéodory’s Lemma 2.1 for the family \( \{\nu_i(x^k, E^k)\}_{i \in \{1, \ldots, m-r\}} \), for each fixed \( k \in I \), we obtain some \( J^k \subseteq \{1, \ldots, m-r\} \) such that \( \{\nu_i(x^k, E^k)\}_{i \in J^k} \) is linearly independent and

\[
\nabla f(x^k) - \sum_{i=1}^{m-r} \alpha_i^k \nu_i(x^k, E^k) = \nabla f(x^k) - \sum_{i \in J^k} \alpha_i^k \nu_i(x^k, E^k),
\]

where \( \alpha_i^k \geq 0 \) for every \( k \in I \) and every \( i \in J^k \). By the infinite pigeonhole principle, we can assume \( J^k \) is the same, say equal to \( J \), for all \( k \in I \) large enough. We claim that the sequences \( \{\alpha_i^k\}_{k \in I} \) are all bounded. In order to prove this, suppose that

\[
m^k \doteq \max_{i \in J} \{\alpha_i^k\}
\]

is unbounded with \( k \in I \), divide (11) by \( m^k \) and note that \( m^k \to \infty \) implies that the vectors \( \nu_i(\overline{\mathbf{F}}, E^k) \), \( i \in J \), are positively linearly dependent. On the other hand, the vectors \( \nu_i(x^k, E^k) \), \( i \in J \), are linearly independent for all large \( k \), which contradicts weak-CPLD. Finally, note that every collection of limit points \( \{\alpha_i^k\}_{i \in J} \) of their respective sequences \( \{\alpha_i^k\}_{k \in \mathbb{N}}, i \in J \), generates a Lagrange multiplier associated with \( \overline{\mathbf{F}} \), which is \( \overline{\mathbf{F}} \doteq \sum_{i \in J} \alpha_i^k \nu_i(G(\overline{\mathbf{F}})) \). Thus, \( \overline{\mathbf{F}} \) is a KKT point.

The second part of the statement of the theorem follows directly from the analysis above and Theorem 2.1.

\[\square\]

Back to Example 3.1, observe that weak-CPLD does not hold at \( \overline{\mathbf{F}} = 0 \), as expected. Indeed, for any sequence \( \{x^k\}_{k \in \mathbb{N}} \to 0 \) such that \( x^k < 0 \) for all \( k \), the matrix \( G(x^k) \) has only simple eigenvalues, for all large \( k \), so \( E^k \in \mathcal{E}_x(G(x^k)) \) is unique up to sign. Without loss of generality, we can assume

\[
E^k = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},
\]

and then we have \( \nu_{11}(x^k, E^k) = -2x^k > 0 \) and \( \nu_{22}(x^k, E^k) = 2 + 2x^k > 0 \), which means they are positively linearly dependent, for all large enough \( k \). However, since \( \nu_{11}(\overline{\mathbf{F}}, E) = 0 \) and \( \nu_{22}(\overline{\mathbf{F}}, E) = 2 \) are positively linearly dependent, Definition 3.2 is not satisfied.
Remark 3.1. In [9], we presented a different extension proposal of CRCQ (and CPLD) to NSDP problems with multiple constraints, which is weaker than nondegeneracy (respectively, Robinson’s CQ) for a single constraint as in (NSDP) only when the zero eigenvalue of \(G(\overline{r})\) is simple. We called this definition the “naive extension of CRCQ (and CPLD)”. We remark that Definition 3.2 coincides with the naive extension of CRCQ (and CPLD) when zero is a simple eigenvalue of \(G(\overline{r})\), which makes Definition 3.2 an improvement of it, or a “non-naive variant” of it.

The phrasing of Theorem 3.1 was chosen to call the reader’s attention to the fact that it is, essentially, a convergence proof of the external penalty method to KKT points, under weak-CPLD. To obtain a more general convergence result, in the next section we introduce new constant rank-type CQs for NSDP that support every algorithm that generates AKKT sequences. Then, we provide several properties of these new conditions, and we compare them with weak-CPLD and weak-CRCQ.

3.2 Stronger sequential-type constant rank conditions for NSDP with applications

We begin this section by introducing a small perturbation in weak-CPLD and weak-CRCQ, which makes it stronger, but also brings some useful properties in return. At first, we present it in a form that most resembles Definition 3.2, for comparison purposes. Later, for convenience, we will provide a characterization of it without sequences.

Definition 3.3 (seq-CRCQ and seq-CPLD). Let \(\overline{r} \in \mathcal{F}\) and let \(r\) be the rank of \(G(\overline{r})\). We say that \(\overline{r}\) satisfies the

1. Sequential CRCQ condition for NSDP (seq-CRCQ) if \(r = m\) or, for all sequences \(\{x^k\}_{k \in \mathbb{N}} \rightarrow \overline{r}\) and \(\{\Delta^k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}^m\) with \(\Delta^k \rightarrow 0\), there exists \(\{E^k\}_{k \in \mathbb{N}} \rightarrow \overline{E}\), \(I \subseteq \mathbb{N}\), such that \(E^k \in \mathcal{E}_r(G(x^k) + \Delta^k)\) for every \(k \in I\) and, for every subset \(J \subseteq \{1, \ldots, m - r\}\), if the family \(\{v_i(\overline{r}, \overline{E})\}_{i \in J}\) is linearly dependent, then \(\{v_i(x^k, E^k)\}_{i \in J}\) remains linearly dependent, for all \(k \in I\) large enough.

2. Sequential CPLD condition for NSDP (seq-CPLD) if \(r = m\) or, for all sequences \(\{x^k\}_{k \in \mathbb{N}} \rightarrow \overline{r}\) and \(\{\Delta^k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}^m\) with \(\Delta^k \rightarrow 0\), there exists \(\{E^k\}_{k \in \mathbb{N}} \rightarrow \overline{E}\), \(I \subseteq \mathbb{N}\), such that \(E^k \in \mathcal{E}_r(G(x^k) + \Delta^k)\) for every \(k \in I\) and, for every subset \(J \subseteq \{1, \ldots, m - r\}\), if the family \(\{v_i(\overline{r}, \overline{E})\}_{i \in J}\) is positively linearly dependent, then \(\{v_i(x^k, E^k)\}_{i \in J}\) remains linearly dependent, for all \(k \in I\) large enough.

Note that the only difference between Definitions 3.2 and 3.3 is the perturbation matrix \(\Delta^k \rightarrow 0\). In particular, set \(\Delta^k \equiv 0\) for every \(k\) to see that seq-CRCQ and seq-CPLD imply weak-CRCQ and weak-CPLD, respectively. Moreover, both implications are strict, as we can see in the following example:

Example 3.4. Consider the constraint

\[
G(x) = \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix}
\]

at the point \(\overline{r} = 0\), so in this case \(r = 2\). For every non-constant sequence \(\{x^k\}_{k \in \mathbb{N}} \rightarrow \overline{r}\), we have

\[
\mathcal{E}_r(G(x^k)) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\};
\]

for every \(k \in \mathbb{N}\) large enough, and if \(x^k = \overline{r}\), then \(\mathcal{E}_r(G(x^k))\) is the set of all orthogonal \(2 \times 2\) matrices. Take \(E^k = \mathbb{I}_2\) for every \(k \in \mathbb{N}\) to see that both, weak-CRCQ and weak-CPLD, hold at \(\overline{r}\), since

\[
v_{11}(x^k, E^k) = 1 \quad \text{and} \quad v_{22}(x^k, E^k) = -1
\]

are nonzero and (positively) linearly dependent for every \(k \in \mathbb{N}\).

On the other hand, take

\[
\Delta^k = \frac{1}{1 + (x^k + 1)^2} \begin{bmatrix} -x^k(x^k - 1)^2 & x^k(x^k + 1) \\ x^k(x^k + 1) & x^k + 2x^k(x^k + 1)^2 \end{bmatrix},
\]

and note that the eigenvectors of \(G(x^k) + \Delta^k\) are uniquely determined up to sign. Then, since \(v_i(x, E), i \in \{1, 2\}\), is invariant to the sign of the columns of \(E\), we can assume without loss of generality that any \(E^k \in \mathcal{E}_r(G(x^k) + \Delta^k)\) has the form

\[
E^k = \frac{1}{\sqrt{1 + (x^k + 1)^2}} \begin{bmatrix} -1 & x^k + 1 \\ x^k + 1 & 1 \end{bmatrix}
\]

for every \(k \in \mathbb{N}\), if \(x^k \neq 0\). Then, for any sequence \(\{E^k\}_{k \in \mathbb{N}}\) such that \(E^k \in \mathcal{E}_r(G(x^k) + \Delta^k)\) for every \(k\), we have

\[
v_{11}(x^k, E^k) = 1 - (x^k + 1)^2 \quad \text{and} \quad v_{22}(x^k, E^k) = (x^k + 1)^2 - 1,
\]

which are both nonzero whenever \(x^k \neq 0\), but if \(E\) is a limit point of \(\{E^k\}_{k \in \mathbb{N}}\), then \(v_{11}(\overline{r}, \overline{E}) = v_{22}(\overline{r}, \overline{E}) = 0\). Thus, neither seq-CRCQ nor seq-CPLD hold at \(\overline{r}\).
Furthermore, since nondegeneracy can be characterized as the linear independence of \( v_{ii}(\overline{\tau}, \overline{E}) \), \( i \in \{1, \ldots, m-r\} \), for every \( \overline{E} \in \mathcal{E}_c(G(\overline{\tau})) \) [8, Prop. 3.2], we observe that it implies seq-CRCQ (see also Remark 3.3 at the end of this section), but Example 3.3 shows that this implication is also strict. In fact, recall that Example 3.3 analyses the constraint

\[
G(x) = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}
\]

at the point \( \overline{\tau} = 0 \). For any \( x \in \mathbb{R} \) and any arbitrary orthogonal matrix \( E \in \mathbb{R}^{2 \times 2} \), note that \( E \) has the form

\[
E = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \text{if} \quad \det(E) = 1 \quad \text{or} \quad E = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad \text{if} \quad \det(E) = -1
\]

where \( a^2 + b^2 = 1 \). In both cases, we have

\[
v_{11}(x, E) = v_{22}(x, E) = a^2 + b^2 = 1.
\]

That is, \( v_{11}(x, E) \) and \( v_{22}(x, E) \) are nonzero and linearly dependent, regardless of \( x \) and \( E \). Thus, seq-CRCQ holds at \( \overline{\tau} \), although nondegeneracy does not. Note that weak-nondegeneracy also fails at \( \overline{\tau} \), in this example.

By Example 3.2, we verify that Robinson’s CQ does not imply seq-CRCQ; because otherwise, it would also imply weak-CRCQ, contradicting the example. Moreover, it is worth mentioning that seq-CRCQ does not recover CRCQ via diagonal matrices (see Example 3.4), but it does when we consider a straightforward extension of it for (NSDP) with multiple constraints \( G_1(x) \geq 0, \ldots, G_q(x) \geq 0 \). Then, the most appropriate way of comparing it with other conditions is in the multifold case. The extension can be made similarly to [9]; that is, for all sequences \( \{x^k\}_{k \in \mathbb{N}} \to \overline{\tau} \), and \( \{\Delta^k\}_{k \in \mathbb{N}} \to 0 \), there must exist an infinite subset \( I \subseteq \mathbb{N} \) and convergence subsequences \( \{E^k\}_{k \in I} \to \overline{E} \), with \( E^k \in \mathcal{E}_c(G(x^k) + \Delta^k) \), such that for all subsets \( J_k \subseteq \{1, \ldots, m-r\} \), the (positive) linear dependence of

\[
\left\{ v_{ii}(\overline{\tau}, \overline{E^k}) \right\}_{i \in \{1, \ldots, q\}}
\]

implies the linear dependence of

\[
\left\{ v_{ii}(x^k, E^k) \right\}_{i \in \{1, \ldots, q\}}
\]

for every large \( k \in I \), where \( r_k = \text{rank}(G(x)) \), for each \( \ell \in \{1, \ldots, q\} \), and

\[
v_{ii}(x, E) = \left[ e_i^T D x_1 G_1(x) e_i, \ldots, e_i^T D x_q G_q(x) e_i \right]^T,
\]

where \( e_i \) denotes the \( i \)-th column of \( E \in \mathbb{R}^{m \times m-r} \), for each \( i \in \{1, \ldots, m-r\} \). Robinson’s CQ, on the other hand, holds for the multifold (NSDP) at \( \overline{\tau} \) if, and only if, there is some \( d \in \mathbb{R}^n \) such that \( G(\overline{\tau}) + DG(\overline{\tau})d \in \text{int}(K) \) for every \( \ell \in \{1, \ldots, q\} \). Considering these extensions, the constraint functions \( G_1(x) \leq x \) and \( G_2(x) \leq -x \) satisfy seq-CRCQ and seq-CPLD at \( \overline{\tau} = 0 \), while Robinson’s CQ fails at the same point. Therefore, Robinson’s CQ is strictly stronger than seq-CPLD and, consequently, it is independent of seq-CRCQ also.

We now show that seq-CPLD (and, consequently, seq-CRCQ) is a strict CQ with a small adaptation of the proof of Theorem 3.1.

**Theorem 3.2.** Let \( \overline{\tau} \in \mathcal{F} \) be an AKKT point that satisfies seq-CPLD. Then, \( \overline{\tau} \) satisfies KKT.

**Proof.** Let \( \{k\}_{k \in \mathbb{N}} \to \overline{\tau} \), \( \{Y^k\}_{k \in \mathbb{N}} \subseteq \mathbb{S}_+^m \), and \( \{\Delta^k\}_{k \in \mathbb{N}} \to 0 \) be the AKKT sequences from Definition 2.3. Since \( \lambda_i(G(x^k)) > 0 \) for every \( i \in \{1, \ldots, r\} \), where \( r \) is the rank of \( G(\overline{\tau}) \), then \( \lambda_i(G(x^k) + \Delta^k) > 0 \) and \( \lambda_{m-r+1}(Y^k) = 0 \) for every such \( i \) and all \( k \) large enough. Hence, the spectral decomposition of \( Y^k \) can be represented in the format

\[
Y^k = \sum_{i=1}^{m-r} \lambda_i(Y^k) u_i^k (u_i^k)^\top
\]

where \( u_1^k, \ldots, u_{m-r}^k \) are shared orthonormal eigenvectors between \( Y^k \) and \( G(x^k) + \Delta^k \), associated with \( m-r \) largest eigenvalues of \( Y^k \) and the \( m-r \) smallest eigenvalues of \( G(x^k) + \Delta^k \), respectively. Defining \( E^k = [u_1^k, \ldots, u_{m-r}^k] \) for every \( k \), we obtain

\[
\nabla_x L(x^k, Y^k) = \nabla f(x^k) - \sum_{i=1}^{m-r} \lambda_i(Y^k) v_{ii}(x^k, E^k) \to 0.
\]

For each \( k \in \mathbb{N} \), let \( P^k \in \mathbb{R}^{m \times r} \) be a matrix whose columns are orthonormal eigenvectors associated with the \( r \) largest eigenvalues of \( G(x^k) \), and construct

\[
M^k \doteq U^k \begin{bmatrix} \text{Diag}(\lambda_1(G(x^k)), \ldots, \lambda_r(G(x^k))) & 0 \\ 0 & \text{Diag}((r+1)||x^k - \overline{\tau}||, \ldots, m||x^k - \overline{\tau}||) \end{bmatrix}(U^k)^\top,
\]

(12)
where \( U^k \supseteq \{P^k, E^k \} \) for every \( k \in \mathbb{N} \). Note that \( M^k \to G(\mathcal{F}) \) and that the \( m - r \) smallest eigenvalues of \( M^k \) are simple, if \( x^k \neq \mathcal{T} \), meaning their associated eigenvectors are unique up to sign, when \( k \) is large enough. Consequently, \( v_i(x^k, E^k) \) is invariant to the choice of \( E^k \in \mathcal{E}_r(M^k) \), for all such \( k \), and every \( i \in \{1, \ldots, m - r \} \). The rest of this proof follows the exact same lines as the proof of Theorem 3.1. 

\[ \] 

**Remark 3.2.** An immediate consequence of Theorem 3.2 is the global convergence of all algorithms supported by AKKT, under seq-CPLD (and, therefore, under seq-CRCQ as well). For instance, the augmented Lagrangian method of Appendix A.1, the SQP method of Appendix A.2, or Yamashita, Yabe, and Harada’s primal-dual interior-point method for NSDP [34]. One can check the details in Theorem A.1, Proposition A.1, and [5, Sec. 3.2], respectively. Moreover, note that this convergence result neither assumes compactness of the Lagrange multiplier set nor that it is a singleton.

Besides convergence of algorithms, the CQs we present also have implications towards stability and error analysis. We make this link by means of establishing a relationship between seq-CPLD (and seq-CRCQ) and the metric subregularity CQ. To do this, we first need to show that they are robust, in the sense they are preserved in a neighborhood of the point of interest. This property may not be clear from Definition 3.3, but it becomes clear after we exhibit a characterization of it without sequences, as follows:

**Proposition 3.2.** Let \( \mathcal{T} \in \mathcal{F} \) and let \( r \) be the rank of \( G(\mathcal{T}) \).

- seq-CRCQ holds at \( \mathcal{T} \) if, and only if, \( r = m \) or, for every \( E \in \mathcal{E}_r(G(\mathcal{T})) \), there exists some neighborhood \( V \) of \( (\mathcal{T}, \mathcal{F}) \) such that for all \( J \subseteq \{1, \ldots, m - r \} \), we have that if the family \( \{v_i(\mathcal{T}, \mathcal{F})\}_{i \in J} \) is linearly dependent, then \( \{v_i(x, E)\}_{i \in J} \) remains linearly dependent for every \( (x, E) \in V \);

- seq-CPLD holds at \( \mathcal{T} \) if, and only if, \( r = m \) or, for every \( E \in \mathcal{E}_r(G(\mathcal{T})) \), there exists some neighborhood \( V \) of \( (\mathcal{T}, \mathcal{F}) \) such that for all \( J \subseteq \{1, \ldots, m - r \} \), we have that if the family \( \{v_i(\mathcal{T}, \mathcal{F})\}_{i \in J} \) is positively linearly dependent, then \( \{v_i(x, E)\}_{i \in J} \) remains linearly dependent for every \( (x, E) \in V \).

**Proof.** We will prove only item 1, since item 2 follows analogously. Let \( \mathcal{T} \) satisfy seq-CRCQ; by contradiction: suppose that there exists some \( E \in \mathcal{E}_r(G(\mathcal{T})) \), some \( J \subseteq \{1, \ldots, m - r \} \), and some sequence \( \{(x^k, E^k)\}_{k \in \mathbb{N}} \to (\mathcal{T}, \mathcal{F}) \) such that \( \{v_i(\mathcal{T}, \mathcal{F})\}_{i \in J} \) is linearly dependent, but \( \{v_i(x^k, E^k)\}_{i \in J} \) is linearly independent for every large \( k \in \mathbb{N} \). Let \( P^k \in \mathbb{R}^{n \times n} \) be a matrix whose columns are orthogonal eigenvectors associated with the \( r \) largest eigenvalues of \( G(x^k) \), define \( U^k \supseteq \{P^k, E^k\} \), and consider \( M^k \) as in (12). Set \( \Delta^k \supseteq M^k - G(x^k) \) and note that \( v_i(x^k, E^k) \) is invariant to \( E^k \in \mathcal{E}_r(\Delta^k + G(x^k)) \) when \( k \) is large, provided that \( x^k \neq \mathcal{T} \). Then, for each \( k \), consider \( \Delta^k \to \Delta^k \to M^k - G(x^k) \) to arrive at a contradiction with seq-CRCQ.

Conversely, let \( \{x^k\}_{k \in \mathbb{N}} \to \mathcal{T} \) and \( \Delta^k \to 0 \) be any sequences, and let \( J \subseteq \{1, \ldots, m - r \} \) be any subset. For each \( k \), pick any \( E^k \in \mathcal{E}_r(G(x^k) + \Delta^k) \) and consider the sequence \( \{E^k\}_{k \in \mathbb{N}} \), which is bounded. Let \( I \subseteq \infty \mathbb{N} \) and \( \mathcal{E} \) be arbitrary, as long as \( \{E^k\}_{k \in I} \to \mathcal{E} \). Then, by hypothesis, there exists a neighborhood \( V \) of \( \mathcal{T} \) such that \( \{v_i(\mathcal{T}, \mathcal{F})\}_{i \in J} \) is linearly dependent, then \( \{v_i(x^k, E^k)\}_{i \in J} \) is also linearly dependent for all large enough \( k \in I \), since \( (x^k, E^k) \in V \) for all such \( k \).

In light of the equivalence of Proposition 3.2, we obtain the robustness property:

**Proposition 3.3.** If seq-CPLD holds at \( \mathcal{T} \), then there exists a neighborhood \( V \) of \( \mathcal{T} \) such that seq-CPLD also holds for every \( x \in V \cap \mathcal{F} \). Moreover, the same property holds for seq-CRCQ.

**Proof.** Direct from Proposition 3.2.

Now, using Proposition 3.3, it is possible to prove that seq-CPLD (and seq-CRCQ) imply metric subregularity CQ. We shall do this in the same style as Andreani et al [10]:

**Theorem 3.3.** If seq-CPLD holds at \( \mathcal{T} \), then \( \mathcal{T} \) satisfies metric subregularity CQ.

**Proof.** Suppose that metric subregularity CQ does not hold at \( \mathcal{T} \). In view of Proposition 2.1, which was extended from NLP to NCP from Minchenko and Stakhovski’s article [26] with minor adaptations, there exist sequences \( \{x^k\}_{k \in \mathbb{N}} \to \infty \mathbb{N} \) and \( \{y^k\}_{k \in \mathbb{N}} \to \mathcal{T} \) such that \( \Lambda(y^k) \cap c(B(0, r^k)) = \emptyset \) for every \( k \in \mathbb{N} \).

Now let \( \{z^k\}_{k \in \mathbb{N}} \neq \emptyset \) be such that \( z^k = x^k + 1 \mathcal{T} \) for every \( k \) and note that \( z^k \to \mathcal{T} \). By the previous proposition, \( z^k \) satisfies metric subregularity for all \( k \) large enough. Consequently, there exists a sequence \( \{Y^k\}_{k \in \mathbb{N}} \subseteq S_+^n \) such that

\[
\frac{z^k - y^k}{\|z^k - y^k\|} - DG(z^k)[Y^k] = 0
\]

and \( (G(z^k), Y^k) = 0 \) for every \( k \), which implies that \( \lambda_i(Y^k) = 0 \) for every \( i \in \{m - r + 1, \ldots, m \} \) and every \( k \in \mathbb{N} \). Let \( U^k \) be an arbitrary matrix that diagonalizes \( Y^k \) and let \( E^k \) be the part of it that corresponds to the \( m - r \) smallest eigenvalues of \( G(z^k) \). So

\[
\frac{z^k - y^k}{\|z^k - y^k\|} = \sum_{i=1}^{m-r} \lambda_i(Y^k)v_i(x^k, E^k) = 0.
\]

Again, by Caratheodory’ lemma (cf. Lemma 2.1) and the infinite pigeonhole principle, we obtain a set \( J \subseteq \{1, \ldots, m - r \} \) such that \( \{v_i(x^k, E^k) : i \in J \} \) is linearly independent and \( \sum_{i=1}^{m-r} \lambda_i(Y^k)v_i(x^k, E^k) = \sum_{i \in J} a_i^k v_i(x^k, E^k) \) for every \( k \) where \( a_i^k \lambda_i(Y^k) > 0 \) for all \( i \in J \). Then, recall the definition that
$Y^k \in \Lambda(y^k)$, so $\|Y^k\| > \tau^k \to \infty$. Let $m^k = \max\{k^i : i \in J\}$ and divide (13) by $m^k$ to obtain that 
\{\nu_i(\mathcal{F}, \mathcal{E}) : i \in J\} is linearly dependent for every limit point $\mathcal{E}$ of \{\Delta^k\}_{k \in \mathbb{N}}$, which contradicts seq-CPLD at $\mathcal{F}$.

**Remark 3.3.** It is important to mention that the “perturbed versions” of weak-nondegeneracy and weak-Robinson’s CQ, in the sense of Definition 3.3, are nondegeneracy and Robinson’s CQ, respectively. In other words, nondegeneracy (respectively, Robinson’s CQ) holds at $\mathcal{F}$ if, and only if, for every sequence $\{x^k\}_{k \in \mathbb{N}} \to \mathcal{F}$ and every $\{\Delta^k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}^m$ such that $\Delta^k \to 0$, there is some $\mathcal{E} \in \operatorname{Lim sup}_{k \in \mathbb{N}} \mathcal{E}_r(G(x^k) + \Delta^k)$ such that $\{\nu_i(\mathcal{F}, \mathcal{E}) : i \in \{1, \ldots, m - r\}\}$ is (positively) linearly independent, where $r = \operatorname{rank}(G(\mathcal{F}))$. For more details, see [8, Rem. 3.1].

### 4 Nonlinear second-order cone programming

The material from the previous section leads to natural, but nontrivial analogies of weak-CRCQ and weak-CPLD for NSOCP problems. Although NSOCP problems admit a reformulation as NSDP problems via arrowhead matrices, we point out that several regularity properties may be lost in the reformulation process. For instance, nondegeneracy never holds for arrowhead matrix constraints, independently of its fulfillment for its equivalent NSOCP. It is elementary to verify that the same holds for weak-nondegeneracy, even with $m = 3$, which motivates a detailed analysis of the NSOCP case, presented in this section.

First, note that NSOCP problems can also be seen as particular cases of (NCP) with $Y \in \mathbb{R}^m$, and multiple constraints defined by $\mathcal{K} = \mathbb{L}_{m_1} \times \cdots \times \mathbb{L}_{m_q}$, and $G(x) = (G_1(x), \ldots, G_q(x))$ for every $x \in \mathbb{R}^n$, where

\[
\mathbb{L}_{m_\ell} = \{Z = (Z_0, \mathcal{Z}) \in \mathbb{R} \times \mathbb{R}^{m_\ell - 1} : Z_0 \geq \|\mathcal{Z}\| \in \mathbb{R}^{m_\ell}\},
\]

if $m_\ell > 1$, and $\mathbb{L}_1 = \mathbb{R}_+$, for all $\ell \in \{1, \ldots, q\}$, with $m_1 + \ldots + m_q = m$. Explicitly, we deal with the following problem:

\[
\begin{align*}
\text{Minimize} & \quad f(x), \\
\text{subject to} & \quad G_\ell(x) \in \mathbb{L}_{m_\ell}, \quad \forall \ell \in \{1, \ldots, q\}. \\
\end{align*}
\]

As usual in the study of NSOCP, for any $x \in \mathcal{F}$, we consider the following partition of $\{1, \ldots, q\}$:

\[
\begin{align*}
I_0(x) &= \{\ell : G_\ell(x) = 0\}, \\
I_B(x) &= \{\ell : G_\ell(x) \in \operatorname{bd} \mathbb{L}_{m_\ell}\}, \\
I_{\text{int}}(x) &= \{\ell : G_\ell(x) \in \operatorname{int} \mathbb{L}_{m_\ell}\},
\end{align*}
\]

(14)

which denote, respectively, the indices of the constraints that hit the vertex, the border excluding the origin, and the interior of their respective second-order cones.

The spectral decomposition of any element $Z$ of $\mathbb{R}^{m_\ell}$, $m_\ell > 1$, with respect to $\mathbb{L}_{m_\ell}$, has the form

\[
Z = \lambda_1(Z)u_1(Z) + \lambda_2(Z)u_2(Z),
\]

where

\[
\lambda_1(Z) = Z_0 + (-1)^i\|\mathcal{Z}\| \quad \text{and} \quad u_i(Z) = \begin{cases} 
\frac{1}{2} \left(1, (-1)^i\frac{\mathcal{Z}}{\|\mathcal{Z}\|}\right), & \text{if } \mathcal{Z} \neq 0, \\
\frac{1}{2} \left(1, (-1)^i\mathcal{w}\right), & \text{otherwise},
\end{cases}
\]

and $w \in \mathbb{R}^{m_\ell - 1}$ can be any unitary vector, with $i \in \{1, 2\}$. In this setting, just as in the NSDP case, $\lambda_i(Z)$ is said to be an eigenvalue of $Z$ associated with the eigenvector $u_i(Z)$, $i \in \{1, 2\}$. Moreover, it is known that the orthogonal projection of $Z$ onto $\mathbb{L}_{m_\ell}$ can be written in terms of the eigenvalues and eigenvectors of $Z$, as follows:

\[
\Pi_{\mathbb{L}_{m_\ell}}(Z) = [\lambda_1(Z)]_+ u_1(Z) + [\lambda_2(Z)]_+ u_2(Z).
\]

**Remark 4.1.** Due to notational issues regarding the spectral decomposition in NSOCP, from this point on, we will assume that $m_\ell > 1$ for every $\ell \in \{1, \ldots, q\}$. However, this is not a limitation of our results from the mathematical point of view. In fact, the spectral decomposition of $Z \in \mathbb{L}_1$ should be interpreted as $Z = \lambda_1(Z)u_1(Z)$, with $u_1(Z) = 1$ and $\lambda_1(Z) = Z$, since $\lambda_2(Z)$ and $u_2(Z)$ are not well defined in this case. To avoid cumbersome notation, we leave the reader in charge of the adjustments to the definitions and theorems to fit the case $m_\ell = 1$, which consist of simply disregarding all expressions involving $\lambda_2(Z)$ and $u_2(Z)$.

Before proceeding, we should recall a notion of conic linear independence, which will be used to characterize Robinson’s CQ later on:

**Definition 4.1.** Let $C_i \subseteq \mathbb{R}^{n_i}, i \in J$, be nonempty closed convex cones, and let $C = \prod_{i \in J} C_i$. A family of matrices $\{V_i\}_{i \in J}$, with $V_i \in \mathbb{R}^{n_i \times n}$ for all $i \in J$, is said to be $C$-linearly independent if

\[
\sum_{i \in J} V_i \alpha_i = 0, \quad \alpha_i \in C_i, \quad \forall i \in J \quad \Rightarrow \quad \alpha_i = 0, \quad \forall i \in \{1, \ldots, s\}.
\]
Observe that Definition 4.1 generalizes the idea of positive linear independence, which is recovered when \( a_i = 1 \) and \( C_i = \mathbb{R}_+ \) for every \( i \in J \). Now, in order to present a constructive reasoning for extending CRCQ and CPLD to the context of NSOCP, we recall the practical characterizations of nondegeneracy and Robinson’s CQ that were specialized to NSOCP by Bonnans and Ramírez [17].

**Proposition 4.1.** A point \( \mathbf{x} \in \mathcal{F} \) satisfies

- **Nondegeneracy** if, and only if, the family

\[
\left\{ DG_\ell(\mathbf{x})^\top G_\ell(\mathbf{x}) \right\}_{\ell \in I_0(\mathbf{x})} \cup \left\{ DG_\ell(\mathbf{x})^\top \right\}_{\ell \in I_0(\mathbf{x})},
\]

is \( \mathbb{R}^{\lceil |\ell| \rceil} \times \prod_{\ell \in I_0(\mathbf{x})} \mathbb{R}^{m_\ell} \)-linearly independent;

- **Robinson’s CQ** if, and only if, the family (16) is \( \mathbb{R}^{\lceil |\ell| \rceil} \times \prod_{\ell \in I_0(\mathbf{x})} \mathbb{L}_{m_\ell} \)-linearly independent;

where

\[
\Gamma_\ell \doteq \begin{bmatrix} 1 & 0 \\ 0 & -I_{m_\ell-1} \end{bmatrix}
\]

and \( I_{m_\ell-1} \) is the identity matrix of dimension \( m_\ell - 1 \).

**Proof.** For the first part, see the proof of [17, Prop. 19]. The second part follows from the general characterization of Robinson’s CQ in [18, Prop. 2.97 and Crlr. 2.98] using the fact \((Z_\ell, G_\ell(\mathbf{x})) = 0 \) with \( \ell \in I_B(\mathbf{x}) \) if, and only if, \( Z_\ell = \alpha \Gamma_\ell G_\ell(\mathbf{x}) \) for some \( \alpha \geq 0 \); and similarly, \((Z_\ell, G_\ell(\mathbf{x})) = 0 \) with \( \ell \in I_{m_\ell}(\mathbf{x}) \) if, and only if, \( Z_\ell = 0 \) [1, Lem. 15].

As in the NSDP case, the nondegeneracy condition as in Proposition 4.1 is fully reduced to LICQ from NLP when it is seen as an instance of (NSOCP) with \( m_1 = \ldots = m_\eta = 1 \). However, the existence of a neighborhood \( \mathcal{V} \subseteq \mathcal{F} \) such that, for all subsets \( J_B \subseteq I_B(\mathbf{x}) \) and \( J_0 \subseteq I_0(\mathbf{x}) \), the family

\[
\left\{ DG_\ell(\mathbf{x})^\top G_\ell(\mathbf{x}) \right\}_{\ell \in J_B} \cup \left\{ \nabla G_\ell,s(\mathbf{x}) \right\}_{\ell \in J_0, s \in \{1, \ldots, m_\ell\}}
\]

has constant rank for all \( \mathbf{x} \in \mathcal{V} \), where \( \nabla G_\ell,s(\mathbf{x}) \) denotes the \( s \)-th column of \( DG_\ell(\mathbf{x})^\top \), still does not characterize a CQ in NSOCP since it holds automatically for every (linear) SOCP with \( I_B(\mathbf{x}) = \emptyset \). In fact, this condition was once proposed as a CQ in [35, Def. 2.1] and then refined in [3].

In the previous section, we observed a similar issue, which was addressed by relaxing weak-nondegeneracy instead of nondegeneracy. Then, we will follow the same path of the previous section, meaning our next step is to present an analogue of weak-nondegeneracy (and weak-Robinson’s CQ) for NSOCP.

### 4.1 Weak-nondegeneracy and weak-Robinson’s CQ for NSOCP

Following [8], we begin by characterizing nondegeneracy and Robinson’s CQ at any \( \mathbf{x} \in \mathcal{F} \) in terms of a representation of \( G(\mathbf{x}) \) that considers the symmetric cone structure of \( \mathcal{K} \), instead of using the canonical basis of \( \mathbb{R}^n \). As in NSDP, this characterization is done in terms of all eigenvectors associated with the zero eigenvalues of each \( G_\ell(\mathbf{x}) \).

**Proposition 4.2.** Let \( \mathbf{x} \) be a feasible point of (NSOCP). Then:

1. **Nondegeneracy holds at** \( \mathbf{x} \) **if, and only if,** the family of vectors

\[
\left\{ DG_\ell(\mathbf{x})^\top u_1(G_\ell(\mathbf{x})) \right\}_{\ell \in I_B(\mathbf{x})} \cup \left\{ DG_\ell(\mathbf{x})^\top (1, -\overline{\ell}), \; DG_\ell(\mathbf{x})^\top (1, \overline{\ell}) \right\}_{\ell \in I_0(\mathbf{x})},
\]

is linearly independent for every \( \overline{\ell} \) such that \( ||\overline{\ell}|| = 1 \), \( \ell \in I_0(\mathbf{x}) \);

2. **Robinson’s CQ holds at** \( \mathbf{x} \) **if, and only if,** the family (17) is positively linearly independent for every \( \overline{\ell} \) such that \( ||\overline{\ell}|| = 1 \), \( \ell \in I_0(\mathbf{x}) \).

**Proof.** We shall prove only item 2, since the proof of item 1 is similar, but slightly simpler than the proof of item 2. Then, it suffices to prove that (16) is \( C \)-linearly independent with respect to

\[
C \doteq \mathbb{R}^{\lceil |\ell| \rceil} \times \prod_{\ell \in I_0(\mathbf{x})} \mathbb{L}_{m_\ell}
\]

if, and only if, (17) is positively linearly independent for all \( \overline{\ell} \), \( \ell \in I_0(\mathbf{x}) \), such that \( ||\overline{\ell}|| = 1 \).

Assume that (16) is a \( C \)-linearly independent family and pick any \( \overline{\ell} = (\overline{\ell}_2, \ldots, \overline{\ell}_m, \overline{m}) \in \mathbb{R}^{m_\ell-1}, \ell \in I_0(\mathbf{x}), \) such that \( ||\overline{\ell}|| = 1 \). Let \( \alpha_\ell, \beta_\ell \geq 0 \), \( \ell \in I_0(\mathbf{x}), \) and \( \gamma_\ell \geq 0 \), \( \ell \in I_B(\mathbf{x}), \) be such that

\[
0 = \sum_{\ell \in I_B(\mathbf{x})} \alpha_\ell DG_\ell(\mathbf{x})^\top (1, \overline{\ell}) + \beta_\ell DG_\ell(\mathbf{x})^\top (1, -\overline{\ell}) + \sum_{\ell \in I_0(\mathbf{x})} \gamma_\ell DG_\ell(\mathbf{x})^\top u_1(G_\ell(\mathbf{x}))
\]

\[
= \sum_{\ell \in I_B(\mathbf{x})} \left[ (\alpha_\ell + \beta_\ell) \nabla G_{\ell,s}(\mathbf{x}) + (\alpha_\ell - \beta_\ell) \sum_{s=2}^{m_\ell} \nabla G_{\ell,s}(\mathbf{x})\overline{\ell}_s \right] + \sum_{\ell \in I_0(\mathbf{x})} \frac{\gamma_\ell}{2||\overline{\ell}||} DG_\ell(\mathbf{x})^\top \ell G_\ell(\mathbf{x}),
\]

15
where $\omega_t,s$ denotes the $s$-th entry of $\omega_t$. Then, it follows from the $C$-linear independence of (16), since $a_\ell + \beta_\ell \geq |a_\ell - \beta_\ell||\omega_t||$ for every $\ell$ when $a_\ell, \beta_\ell \geq 0$, that
\[
\begin{align*}
\begin{cases}
a_\ell + \beta_\ell = 0, & \ell \in I_0(\omega) \\
\omega_t,s(a_\ell - \beta_\ell) = 0, & \ell \in I(t,\omega), \ s \in \{2, \ldots, m\}
\end{cases}
\end{align*}
\]
which implies $a_\ell = \beta_\ell = \gamma_\ell = 0$ for every $\ell$ since there is at least one non null entry $\omega_t,s$ for every $\ell \in I_0(\omega)$. Therefore, (17) is positively linearly independent.

Conversely, assume that (17) is positively linearly independent for all unitary vectors $\omega_t, \ell \in I_0(\omega)$, and suppose that there are some $a_\ell = (a_{\ell,1}, \ldots, a_{\ell,m_\ell}) = (a_{\ell,1}, \hat{a}_\ell) \in \mathbb{R} \times \mathbb{R}^{m_\ell-1}$, $\ell \in I(\omega)$, and $b_\ell \in \mathbb{R}, \ell \in I_B(\omega)$, not all zero, such that $a_\ell, 1 \geq \|\hat{a}_\ell\|$ and $b_\ell \geq 0$ for all $\ell$, and
\[
\sum_{\ell \in I(\omega)} DG(\omega)_{\ell} (\omega) + \sum_{\ell \in I_B(\omega)} b_\ell DG(\omega)_{\ell} (\omega) \Gamma_B(\omega) = 0. \tag{18}
\]
If $a_\ell = 0$ for all $\ell$, it would contradict the positive linear independence of (17). Hence, let us assume that there is at least one $a_\ell,s \neq 0$ and note that (18) is equivalent to
\[
\begin{align*}
\sum_{\ell \in I(\omega)} a_\ell DG(\omega)_{\ell} (\omega) + \beta_\ell DG(\omega)_{\ell} (\omega) + \|\hat{a}_\ell\| \sum_{\ell \in I_B(\omega)} 2b_\ell \|G(\omega)\|DG(\omega)_{\ell} (\omega) u_1(G(\omega)) = 0,
\end{align*}
\]
where $a_\ell = (a_{\ell,1} + \|\hat{a}_\ell\|)/2$, $\beta_\ell = (a_{\ell,1} - \|\hat{a}_\ell\|)/2$, and $\omega_t = \begin{cases} \hat{a}_\ell/\|\hat{a}_\ell\|, & \text{if } \hat{a}_\ell \neq 0 \\ \text{any unitary vector}, & \text{otherwise.} \end{cases}$

However, (19) implies $a_\ell = \beta_\ell = 0$, which in turn implies $a_\ell, 1 = \|\hat{a}_\ell\| = 0$ for every $\ell \in I_0(\omega)$, due to the positive linear independence of (17) (observe that $a_\ell \geq \beta_\ell \geq 0$, i.e., a contradiction. Thus, (16) is $C$-linearly independent.

**Remark 4.2.** In the same spirit of Remark 4.1, in view of the spectral decomposition, when $m_\ell = 1$ sets of the form $\{DG(\omega)_{\ell} (\omega)(1, -\omega) , DG(\omega)_{\ell} (\omega)(1, \omega)\}$ (for any $\omega_t, \ell \in \mathbb{R}^{m_\ell-1}$) correspond to the singleton $\{DG(\omega)_{\ell} (\omega)\}$ consisting of the gradient of $G_t$ at $\omega$.

In Proposition 4.2, not all $\omega_t$ are necessary for the linear independence of (17) to define a constraint qualification at $\omega$. Indeed, in light of [8], we can safely guess that only the limit points of sequences consisting of eigenvectors of $G(x)$, for some $\{x^k\}_{k \in \mathbb{N}} \to \omega$, are needed. This observation leads to two new constraint qualifications for NSOCP, based on nondegeneracy and Robinson's CQ:

**Definition 4.2** (Weak-nondegeneracy and weak-Robinson’s CQ for NSOCP). Let $\omega$ be a feasible point of (NSOCP). We say that $\omega$ satisfies:

- **Weak-nondegeneracy condition for NSOCP** if either $r = m$ or, for each sequence $\{x^k\}_{k \in \mathbb{N}} \to \omega$, there exists some $I \subseteq \mathbb{N}$ and convergent eigenvectors sequences $\{u_k(G(\omega)(x^k))\}_{k \in I} \to \frac{\omega}{\|\omega\|}(1, -\omega)$ and $\{u_2(G(\omega)(x^k))\}_{k \in I} \to \frac{\omega}{\|\omega\|}(1, \omega)$, for every $\ell \in I(\omega)$, such that (17) is linearly independent;

- **Weak-Robinson’s CQ condition for NSOCP** if either $r = m$ or, for each sequence $\{x^k\}_{k \in \mathbb{N}} \to \omega$, there exists some $I \subseteq \mathbb{N}$ and convergent eigenvectors sequences $\{u_k(G(\omega)(x^k))\}_{k \in I} \to \frac{\omega}{\|\omega\|}(1, -\omega)$ and $\{u_2(G(\omega)(x^k))\}_{k \in I} \to \frac{\omega}{\|\omega\|}(1, \omega)$, for every $\ell \in I(\omega)$, such that (17) is positively linearly independent;

Both conditions presented in Definition 4.2 will be proved to be CQs later on; let us first discuss their relationship with other CQs. Clearly, weak-nondegeneracy is implied by nondegeneracy, but the converses is not necessarily true, as the example below illustrates:

**Example 4.1.** Consider any NLP constraints
\[
g_0(x) - g_1(x) \geq 0, \quad g_0(x) + g_1(x) \geq 0
\]
such that LICQ holds at $\omega \in \mathbb{R}^n$ with $g_0(\omega) = g_1(\omega) = 0$. Now consider the equivalent NSOCP constraint
\[
G(x) \doteq (g_0(x), g_1(x), 0) \in \mathbb{L}_3.
\]
It is clear that nondegeneracy fails at $\omega$. However, weak-nondegeneracy holds at $\omega$. To see this, note that for every sequence $\{x^k\}_{k \in \mathbb{N}}$, such that $g_1(x^k) \neq 0$ for infinitely many $k$, the eigenvectors of $G(x)$ when $g_1(x^k) \neq 0$ are uniquely determined by
\[
u_1(G(x^k)) = \frac{1}{2} \left(1, -\frac{g_1(x^k)}{|g_1(x^k)|}, 0\right) \quad \text{and} \quad \nu_2(G(x^k)) = \frac{1}{2} \left(1, \frac{g_1(x^k)}{|g_1(x^k)|}, 0\right),
\]
which can be assumed to converge to $\nu_1 \doteq (1, -\omega)$ and $\nu_2 \doteq (1, \omega)$, respectively, where $\omega \doteq (1, 0)$. In this case,
\[
DG(\omega) \nu_1 = \frac{1}{2} (\nabla g_0(\omega) - \nabla g_1(\omega)) \quad \text{and} \quad DG(\omega) \nu_2 = \frac{1}{2} (\nabla g_0(\omega) + \nabla g_1(\omega))
\]
are linearly independent. If \( g(x^k) = 0 \) for every \( k \) large enough, we can choose \( w^k \doteq (1,0) \) to define eigenvectors

\[
  u_1(G(x^k)) = \frac{1}{2} \left( 1, -w^k \right) \quad \text{and} \quad u_2(G(x^k)) = \frac{1}{2} \left( 1, w^k \right),
\]

for every \( k \) large enough, which leads to the same \( DG(D) \) \( w_1 \) and \( DG(D) \) \( w_2 \) that are linearly independent.

Moreover, note that if the constraints \( g_0(x) - g_1(x) \geq 0 \) and \( g_0(x) + g_1(x) \geq 0 \) are modelled as \( G(x) \doteq (g_0(x), g_1(x), 0, \ldots, 0) \in \mathbb{R}_m \) for an arbitrary \( m \), then weak-nondegeneracy and weak-Robinson’s CQ still hold at \( \mathcal{F} \), while nondegeneracy does not, unless \( m = 2 \). In some sense, this effect resembles the way NLP constraints can be reformulated as structurally diagonal NSDP constraints without affecting weak-nondegeneracy, even though nondegeneracy can be lost.

Thus, weak-nondegeneracy for NSOCP is strictly weaker than its classical counterpart. We do not know whether weak-Robinson’s CQ is strictly weaker than Robinson’s CQ or not (see Appendix C for a partial answer in NSDP). Moreover, note that weak-nondegeneracy implies weak-Robinson’s CQ, but the converse is not true.

**Example 4.2.** Consider the constraint

\[
  G(x) \doteq (4x, 2x) \in \mathbb{R}_2
\]

and the point \( \mathcal{F} \doteq 0 \). Now, take any sequence \( \{x^k\}_{k \in \mathbb{N}} \rightarrow \mathcal{F} \), and note that the only possible eigenvectors for any \( G(x^k) \) are

\[
  u_1(G(x^k)) = \frac{1}{2} (1, -1) \quad \text{and} \quad u_2(G(x^k)) = \frac{1}{2} (1, 1)
\]

or vice-versa. Consequently, the only values of \( \mathcal{W} \) that are suitable for Definition 4.2 are \( \mathcal{W} = \pm 1 \). Let us assume that \( \mathcal{W} = 1 \), since the other case is analogous; then,

\[
  DG(D) \top (1, -\mathcal{W}) = 1 \quad \text{and} \quad DG(D) \top (1, \mathcal{W}) = 3
\]

are positively linearly independent, but not linearly independent. Thus, weak-Robinson’s CQ holds, while weak-nondegeneracy does not.

### 4.2 Weak constant rank-type conditions for NSOCP

With weak-nondegeneracy and weak-Robinson’s CQ for NSOCP at hand, we can present new extensions of CRCQ and CPLD for NSOCP by means of a simple relaxation of Definition 4.2, in the same lines as in NLP.

Basically, the idea is to demand every subfamily of (17) to remain locally (positively) linearly dependent, in some sense. So let us define, for any sets \( J_B, J_-, J_+ \subseteq \{1, \ldots, q\} \) the family of vectors

\[
  D(x, \nu) \doteq D(x, \nu, J_B, J_-, J_+) \doteq \left\{ DG(x) \top \nu \right\}_{\ell \in J_B \cup J_- \cup J_+}
\]

where \( \nu \doteq [\nu_\ell]_{\ell \in J_B \cup J_- \cup J_+} \). Above, the index set \( J_B \) refers to an arbitrary subset of \( I_B(\mathcal{F}) \), and the indices \( J_- \) and \( J_+ \) both refer to \( I_0(\mathcal{F}) \), but with distinct eigenvectors; see (17).

**Definition 4.3** (weak-CRCQ and weak-CPLD). We say that a feasible point \( \mathcal{F} \) of (NSOCP) satisfies the:

- **Weak constant rank constraint qualification for NSOCP** (weak-CRCQ) if \( r = m \) or the following holds: for every sequence \( \{x^k\}_{k \in \mathbb{N}} \rightarrow \mathcal{F} \), there exists some \( I \subseteq \mathbb{N} \), and convergent eigenvector sequences

\[
  \{ u_1(D(x^k)) \}_{k \in I} \rightarrow \frac{1}{2} (1, -\mathcal{W}) \quad \text{and} \quad \{ u_2(D(x^k)) \}_{k \in I} \rightarrow \frac{1}{2} (1, \mathcal{W}),
\]

for all \( \ell \in I_0(\mathcal{F}) \), such that for all subsets \( J_B \subseteq I_B(\mathcal{F}) \) and \( J_-, J_+ \subseteq I_0(\mathcal{F}) \), we have that: if the family of vectors \( D(\mathcal{F}, \mathcal{W}) \) is linearly dependent, then \( D(x^k, \nu^k) \) remains linearly dependent for all \( k \in I \) large enough, where

\[
  \mathcal{W} \doteq \begin{cases} 
    u_1(D(x^k)), & \text{if } \ell \in J_B, \\
    \frac{1}{2} (1, -\mathcal{W}), & \text{if } \ell \in J_-, \\
    \frac{1}{2} (1, \mathcal{W}), & \text{if } \ell \in J_+,
  \end{cases}
\]

and \( \nu^k \doteq \begin{cases} 
    u_1(D(x^k)), & \text{if } \ell \in J_B, \\
    u_1(D(x^k)), & \text{if } \ell \in J_+, \\
    u_2(D(x^k)), & \text{if } \ell \in J_-
  \end{cases} \)

- **Weak constant positive linear dependence condition for NSOCP** (weak-CPLD) if \( r = m \) or the following holds: for every sequence \( \{x^k\}_{k \in \mathbb{N}} \rightarrow \mathcal{F} \), there is some \( I \subseteq \mathbb{N} \), and convergent eigenvector sequences

\[
  \{ u_1(D(x^k)) \}_{k \in I} \rightarrow \frac{1}{2} (1, -\mathcal{W}) \quad \text{and} \quad \{ u_2(D(x^k)) \}_{k \in I} \rightarrow \frac{1}{2} (1, \mathcal{W}),
\]

for all \( \ell \in I_0(\mathcal{F}) \), such that for all subsets \( J_B \subseteq I_B(\mathcal{F}) \) and \( J_-, J_+ \subseteq I_0(\mathcal{F}) \), we have that: if \( D(\mathcal{F}, \mathcal{W}) \) is positively linearly dependent, then \( D(x^k, \nu^k) \) is linearly dependent for all \( k \in I \) large enough, where \( \mathcal{W} \) and \( \nu^k \) are as in (21).
There are some features about Definition 4.3 that should be highlighted for a better understanding of it. First, weak-CRCQ fully recovers CRCQ when we set \( m_\ell = 1 \) for every \( \ell \in \{1, \ldots, q\} \). See also Remarks 4.1 and 4.2 for clarifications about the notation in the case \( m_\ell = 1 \). Similarly, note that weak-CPLD recovers CPLD in the same setting. Second, in view of Proposition 4.2, we see that weak-CRCQ is implied by (weak-)nondegeneracy as in Definition 4.2, and weak-CPLD is implied by both (weak-)Robinson’s CQ and weak-CRCQ. However, due to the previous item, those implications are strict (see Example 4.4 and [14, Counterexample 4.2], respectively). Third, we point out that weak-CRCQ is not comparable with Robinson’s CQ [see [25, Ex. 2.1 and 2.2]].

**Remark 4.3.** To fix ideas, let us consider a single conic constraint \( G(x) \in \mathbb{L}_m \) at the point \( \varpi \in \mathcal{F} \). First, suppose that \( G(\varpi) = 0 \) and take any sequence \( \{x^k\}_{k \in \mathbb{N}} \rightarrow \varpi \). Let us analyse two partitions of it, indexed by the sets:

- \( \mathcal{N}_0 \doteq \{ k \in \mathbb{N} : \hat{G}(x^k) = 0 \} \). In this case, we have
  
  \[
  u_1(G(x^k)) = \frac{1}{2}(1, -w^k) \quad \text{and} \quad u_2(G(x^k)) = \frac{1}{2}(1, w^k),
  \]

  for any \( w^k \) such that \( \|w^k\| = 1 \), and every \( k \in \mathcal{N}_0 \). When \( \mathcal{N}_0 \) is infinite, weak-CRCQ simply demands the existence of some convergent sequence \( \{w^k\}_{k \in \mathcal{N}_0} \rightarrow \varpi \), such that \( DG(\varpi)\top (1, (-1)\top w^i) = 0 \) only if \( DG(x^k)\top (1, (-1)\top w^k) = 0 \) for all large \( k \in \mathcal{N}_0 \), \( i \in \{1, 2\} \); and if \( DG(\varpi)\top (1, -w^k) \) and \( DG(\varpi)\top (1, w^k) \) are linearly dependent, then \( DG(x^k)\top (1, -w^k) \) and \( DG(x^k)\top (1, w^k) \) must also be linearly dependent, for every sufficiently large \( k \in \mathcal{N}_0 \).

- \( \mathcal{N}_1 \doteq \{ k \in \mathbb{N} : \hat{G}(x^k) \neq 0 \} \). This case is similar to the previous one, except that \( w^k \) is uniquely determined by \( w^k = \hat{G}(x^k)/\|\hat{G}(x^k)\| \), for every \( k \in \mathcal{N}_1 \).

The reason why both eigenvectors are taken into consideration is that both eigenvalues of \( G(\varpi) \) are zero, in this case. Naturally, in case \( G(\varpi) \in \mathbb{B}_+ \mathbb{L}_m \), we have only one zero eigenvalue, which is \( \lambda_1(G(\varpi)) \), then weak-CRCQ simply demands the vector

\[
DG(x)\top u_1(G(x)) = \frac{1}{2}DG(x)\top (1, -\hat{G}(x)/\|\hat{G}(x)\|)
\]

to be either nonzero at \( \varpi \) or equal to zero in a whole neighborhood of \( \varpi \). Note that this coincides with the naive approach [9], obtained by reducing the problem to an NLP. This observation remains true for more than one conic constraint as long as \( I_0(\varpi) = \emptyset \). See also Remark 4.4.

Now, let us check how Definition 4.3 behaves when it is applied to example [3, Eq. 2], which was used to refute the CRCQ proposal of [35].

**Example 4.3 (Eq. 2 from [3]).** Consider the problem

\[
\begin{align*}
\text{Minimize} & \quad -x, \\
\text{subject to} & \quad G(x) \doteq (x, x + x^2) \in \mathbb{L}_2.
\end{align*}
\]

and its unique feasible point \( \varpi \doteq 0 \), which does not satisfy the KKT conditions. Our aim is to show that Definition 4.3 is not satisfied at \( \varpi \).

Firstly, note that \( J_\mathcal{F} = \emptyset \) for this case and, for all \( x \in (-1, 0) \cup (0, 1) \), the eigenvectors of \( G(x) \) are

\[
u_1(G(x)) = \frac{1}{2}(1, -\frac{x + x^2}{|x + x^3|}) \quad \text{and} \quad u_2(G(x)) = \frac{1}{2}(1, \frac{x + x^2}{|x + x^3|}).
\]

If \( x = 0 \), without loss of generality we can assume that \( u_1(G(x)) = \frac{1}{2}(1, -1) \) and \( u_2(G(x)) = \frac{1}{2}(1, 1) \), so

\[
u_1 \doteq \frac{1}{2}(1, -1) \quad \text{and} \quad \nu_2 \doteq \frac{1}{2}(1, 1)
\]

are the only possible limit points that suit Definition 4.3 for any sequence \( \{x^k\}_{k \in \mathbb{N}} \rightarrow \varpi \). Note that

\[
DG(\varpi)\top \nu_1 = 0.
\]

However,

\[
DG(x^k)\top u_1(G(x^k)) = -x^k,
\]

so for \( J_- \doteq \{1\} \) and \( J_+ \doteq \emptyset \) we have that \( \mathcal{D}(x^k, u_1(G(x^k))) = \{ -x^k \} \) is linearly independent for every \( x^k \neq 0 \), even though \( \mathcal{D}(\varpi, \varpi) = \{0\} \) is (positively) linearly dependent. Thus, neither weak-CRCQ nor weak-CPLD are satisfied at \( \varpi \).

As mentioned before, weak-nondegeneracy and weak-Robinson’s CQ are strictly stronger than weak-CRCQ and weak-CPLD, respectively. Let us prove this fact:
Example 4.4. Consider the constraint
\[ G(x) = (-x, x) \in \mathbb{L}_2, \]
and its unique feasible point \( \bar{x} = 0 \). The eigenvectors of \( G(x) \) when \( x \neq 0 \) are uniquely determined by
\[ u_1(G(x)) = \frac{1}{2} \left( 1, -\frac{x}{|x|} \right) \quad \text{and} \quad u_2(G(x)) = \frac{1}{2} \left( 1, \frac{x}{|x|} \right), \]
which implies the following:
- For every \( x > 0 \), we have \( DG(x)^\top u_1(G(x)) = 1 \) and \( DG(x)^\top u_2(G(x)) = 0 \);
- For every \( x < 0 \), we have \( DG(x)^\top u_1(G(x)) = 0 \) and \( DG(x)^\top u_2(G(x)) = 1 \).

Then, for every sequence \( \{x^k\}_{k \in \mathbb{N}} \to \bar{x} \) such that \( x^k \neq 0 \) for every \( k \), the family (17) will contain zero, making it (positively) linearly dependent. In particular, note that both, weak-nondegeneracy and weak-Robinson’s CQ, fail at \( \bar{x} \), while weak-CRCQ and weak-CPLD hold at \( \bar{x} \).

Example 4.4 can also be used to verify that weak-CRCQ does not imply Robinson’s CQ. In fact, Robinson’s CQ does not imply weak-CRCQ either, making them independent. Let us show this by another example.

Example 4.5. Consider the constraint
\[ G(x) = (2x_1, x_2^2) \in \mathbb{L}_2 \]
at \( \bar{x} = (0, 0) \). Without loss of generality, let us assume that
\[ u_1(G(x)) = \frac{1}{2} (1, -1) \quad \text{and} \quad u_2(G(x)) = \frac{1}{2} (1, 1), \]
since the other possibility consists of simply swapping the indices 1 and 2.
\[ DG(x)^\top u_1(G(x)) = (1, -x_2) \quad \text{and} \quad DG(x)^\top u_2(G(x)) = (1, x_2) \]
are linearly independent whenever \( x_2 \neq 0 \), but they are linearly dependent at \( \bar{x} \). Therefore, to see that weak-CRCQ fails at \( \bar{x} \), it suffices to consider any sequence \( \{x^k\}_{k \in \mathbb{N}} \to \bar{x} \) such that \( x^k_2 \neq 0 \) for every \( k \in \mathbb{N} \).

On the other hand, in view of Proposition 4.2, it is easy to check that Robinson’s CQ holds at \( \bar{x} \), since
\[ DG(x)^\top (1, -\bar{x}) = DG(x)^\top (1, \bar{x}) = (2, 0) \]
for every \( \bar{x} \in \{1, -1\} \), whence follows that they are positively linearly independent for every \( \bar{x} \in \{1, -1\} \).

Finally, we shall prove that weak-CPLD (and by consequence weak-CRCQ, weak-nondegeneracy, and weak-Robinson’s CQ) is a constraint qualification for (NSOCP), similarly to Theorem 3.1; that is, employing Theorem 2.1, taking spectral decompositions of its approximate Lagrange multiplier sequences, and then employing Carathéodory’s Lemma 2.1 to construct bounded sequences out of them.

Theorem 4.1. Let \( \{\rho_k\}_{k \in \mathbb{N}} \to \infty \) and \( \{x^k\}_{k \in \mathbb{N}} \to \bar{x} \in \mathcal{F} \) be such that
\[ \nabla \Phi L \left( x^k, \rho_k \Pi K(-G(x^k)) \right) \to 0, \]
and suppose that weak-CPLD holds at \( \bar{x} \). Then, \( \bar{x} \) satisfies the KKT conditions. Moreover, any local minimizer of (NSOCP) that satisfies weak-CPLD is a KKT point.

Proof. For each \( k \in \mathbb{N} \), define \( Y^k \equiv \rho_k \Pi K(-G(x^k)) \). Then, we have
\[ \nabla f(x^k) - \sum_{\ell=1}^q \left( DG_{\ell}(x^k)^\top Y^k_{\ell} \right) \to 0. \] (23)
Let us consider an arbitrary spectral decomposition of \( Y^k_{\ell} \):
\[ Y^k_{\ell} = \alpha^k_{\ell} u_1(G(x^k)) + \beta^k_{\ell} u_2(G(x^k)), \]
where \( \alpha^k_{\ell} = [-k\lambda_1(G(x^k))]_{+} \geq 0 \) and \( \beta^k_{\ell} = [-k\lambda_2(G(x^k))]_{+} \geq 0 \). Define
\[ \Psi^k = \sum_{\ell \in \mathcal{I}_{\ell}^h(\bar{x})} \alpha^k_{\ell} DG_{\ell}(x^k)^\top u_1(G(x^k)) + \sum_{\ell \in \mathcal{I}_{\ell}^h(\bar{x})} \alpha^k_{\ell} DG_{\ell}(x^k)^\top u_2(G(x^k)) \]
and note that (23) can be equivalently stated as \( \nabla f(x^k) - \Psi^k \to 0 \). By Carathéodory’s Lemma 2.1, for each \( k \in \mathbb{N} \), there exists some \( J^h_k \subseteq I_{B}(\bar{x}) \) and \( J^h_k \subseteq I_{\ell}(\bar{x}) \) such that
\[ J(x^k, \nu^k) = \left\{ DG_{\ell}(x^k)^\top u_1(G(x^k)) \right\}_{\ell \in \mathcal{I}_{\ell}^h(\bar{x})} \cup \left\{ DG_{\ell}(x^k)^\top u_2(G(x^k)) \right\}_{\ell \in J^h_k}. \]
is linearly independent, where \( \nu^k \) is as in (21), and
\[
\Psi^k \doteq \sum_{\ell \in J_B^\ell \cup J_\ell^k} \alpha^k_\ell D_G(x^k)^\top u_1(G_\ell(x^k)) + \sum_{\ell \in I_F^\ell} \beta^k_\ell D_G(x^k)^\top \! u_2(G_\ell(x^k)),
\]
for some new scalars \( \alpha^k_\ell \geq 0, \beta^k_\ell \geq 0, \ell \in J_B^\ell \cup J_\ell^k \cup J_F^\ell \). By the infinite pigeonhole principle, we can take a subsequence if necessary such that \( J_B^\ell \cup J_\ell^k \cup J_F^\ell \), do not depend on \( k \), which we denote by \( J_B \cup I_\ell \cup J_+ \doteq J_B^\ell \cup J_\ell^k \cup J_F^\ell \).

We claim that \( \{\alpha^k_\ell\}_{k \in \mathbb{N}} \) and \( \{\beta^k_\ell\}_{k \in \mathbb{N}} \) are bounded, for every \( \ell \in J_B \cup I_\ell \cup J_+ \). Indeed, by contradiction, suppose that the sequence \( \{m^k\}_{k \in \mathbb{N}}, \text{given by } m^k \doteq \max\{\alpha^k_\ell, \beta^k_\ell : \ell \in J_B \cup I_\ell \cup J_+\} \), diverges. Dividing (23) by \( m^k \), we obtain
\[
\sum_{\ell \in J_B \cup I_\ell} \frac{\alpha^k_\ell}{m} D_G(x^k)^\top u_1(G_\ell(x^k)) + \sum_{\ell \in J_+} \frac{\beta^k_\ell}{m} D_G(x^k)^\top u_2(G_\ell(x^k)) \to 0
\]
and since the sequences \( \{\alpha^k_\ell/m^k\}_{k \in \mathbb{N}} \) and \( \{\beta^k_\ell/m^k\}_{k \in \mathbb{N}} \) are bounded, we can assume without loss of generality, that they converge to, say, \( \overline{\alpha}_\ell \geq 0 \) and \( \overline{\beta}_\ell \geq 0 \), respectively, for all \( \ell \in J_B \cup I_\ell \cup J_+ \). Note that at least one element of \( \{\overline{\alpha}_\ell, \overline{\beta}_\ell : \ell \in J_B \cup I_\ell \cup J_+\} \) is nonzero, which makes the correspondent set \( D(\pi, \varpi) \) as in (21) linearly dependent for any limit point \( \varpi \) of any subsequence of \( \{\nu^k\}_{k \in \mathbb{N}} \), contradicting weak-CPLD.

Since \( \{\alpha^k_\ell\}_{k \in \mathbb{N}} \) and \( \{\beta^k_\ell\}_{k \in \mathbb{N}} \) are bounded, the sequence \( \{(Y^k_1, \ldots, Y^k_\ell)\}_{k \in \mathbb{N}} \subseteq \mathbb{K} \) defined by
\[
\hat{Y}^k_\ell \doteq \begin{cases} 
\alpha^k_\ell u_1(G_\ell(x^k)) + \beta^k_\ell u_2(G_\ell(x^k)) & \text{if } \ell \in J_- \cup J_+ \\
\beta^k_\ell u_1(G_\ell(x^k)) & \text{if } \ell \in J_B \cup (J_- \setminus J_+) \\
\alpha^k_\ell u_2(G_\ell(x^k)) & \text{if } \ell \in J_+ \setminus J_-
\end{cases}
\]
is also bounded. Finally, note that all limit points of \( \{(Y^k_1, \ldots, Y^k_\ell)\}_{k \in \mathbb{N}} \) are Lagrange multipliers associated with \( \varpi \), which completes the first part of the proof. The second part follows directly from Theorem 2.1.

Remark 4.4. It was proposed in [9, Sec. 5], some “naive extensions” of CRCQ (and CPLD) to NSOCP, which were obtained by replacing the conic constraints of (NSOCP) that satisfy \( G_\ell(\varpi) \in \text{bd}_1 \text{Lin}_1 \) with standard NLP constraints, via the reduction function
\[
\Phi_{\ell}(x) \doteq (G_\ell(x))_0^2 - ||\tilde{G}_\ell(x)||^2,
\]
and then applying CRCQ (respectively, CPLD) to those reduced constraints. In particular, when no constraint is reducible (that is, when \( I_B(\varpi) = \emptyset \)), the naive extension of CRCQ is equivalent to the nondegeneracy condition (similarly, weak-CPLD reduces to Robinson’s CQ in this case). When all constraints are reducible (that is, when \( I_B(\varpi) = \emptyset \)), then it is equivalent to Zhang and Zhang’s definition [35]. Moreover, when all constraints are reducible, the weak-CRCQ as in Definition 4.3 coincides with a naive approach using the reduction function
\[
\Phi_{\ell}(x) \doteq (G_\ell(x))_0 - ||\tilde{G}_\ell(x)||,
\]
instead of \( \Phi_{\ell}(x) \), since \( \nabla \Phi_{\ell}(x) = 2DG_\ell(x)^\top u_1(G_\ell(x)) \) for all \( x \) close enough to \( \varpi \) and \( \ell \in I_B(\varpi) \). As mentioned in [9, Rem. 5.1-c], using \( \Phi_{\ell} \) or \( \Phi_{\ell} \) characterize different approaches. Nevertheless, in the general case, the “naive extension” of CRCQ using \( \Phi_{\ell} \) strictly implies weak-CRCQ. An analogous relationship can be verified among the “naive extension” of CPLD using \( \Phi_{\ell} \), Robinson’s CQ, and weak-CPLD.

4.3 Stronger constant rank-type conditions for NSOCP with applications

In Section 3, we obtained convergence results of algorithms other than the external penalty method, by means of a stronger variant of weak-CPLD and weak-CRCQ. Here, we are motivated to follow the same path, since Theorem 4.1 only covers the external penalty method. We begin this section with an analogue of Definition 3.3 to NSOCP, which characterizes a perturbed version of weak-CRCQ and weak-CPLD.

Definition 4.4 (seq-CRCQ and seq-CPLD). We say that \( \pi \in \mathcal{F} \) satisfies the:

- Sequential CRCQ condition for NSOCP (seq-CRCQ) if either \( \ell = m \) or, for all sequences \( \{x^k\}_{k \in \mathbb{N}} \rightarrow \varpi \) and \( \{\Delta^k_\ell\}_{k \in \mathbb{N}} \subseteq \mathbb{R}^n, \ell \in I_B(\varpi) \cup I_\ell(\varpi) \), such that \( \Delta^k_\ell \rightarrow 0 \) for every \( \ell \), there exists some \( I \subseteq \mathbb{N} \), and convergent eigenvector sequences \( \{u_1(G_\ell(x^k) + \Delta^k_\ell)\}_{k \in \mathbb{N}} \rightarrow (1, -\overline{\omega}_\ell) \) and \( \{u_2(G_\ell(x^k) + \Delta^k_\ell)\}_{k \in \mathbb{N}} \rightarrow (1, \overline{\omega}_\ell) \), for all \( \ell \in I_B(\varpi) \), such that for all subsets \( J_B \subseteq I_B(\varpi) \) and \( J_- \cup J_+ \subseteq I_\ell(\varpi) \), we have that: if the family of vectors \( D(\varpi, \varpi) \) is linearly dependent, then \( D(x^k, \nu^k) \) remains linearly dependent for all \( k \in I \) large enough, where

\[
\varpi \doteq \begin{cases} 
u^k, & \ell \in J_B \\
1, & \ell \in J_-
\end{cases}, \quad \nu^k \doteq \begin{cases} u_1(G_\ell(x^k) + \Delta^k_\ell), & \ell \in J_B \\
u^k, & \ell \in J_-
\end{cases}
\]

and \( D(x, \nu) \) is as in (20).
Sequential CPLD condition for NSOCP (seq-CPLD) if either $r = m$ or, for all sequences \( \{x^k\}_{k \in \mathbb{N}} \rightarrow \mathfrak{T} \) and \( \{\Delta_k^j\}_{k \in \mathbb{N}} \subseteq \mathbb{R}^n \), \( \ell \in I_0(\mathfrak{T}) \cup I_H(\mathfrak{T}) \), such that \( \Delta_k^j \rightarrow 0 \) for every \( \ell \), there exists some \( I \subseteq \mathbb{N} \), and convergent eigenvector sequences \( \{u_1(G_k(x^k) + \Delta_k^j)\}_{k \in I} \rightarrow (1, -1) \) and \( \{u_2(G_k(x^k) + \Delta_k^j)\}_{k \in I} \rightarrow (1, 1, \mathfrak{T}) \), for all \( \ell \in I_0(\mathfrak{T}) \), such that for all subsets \( J_0 \subseteq I_0(\mathfrak{T}) \) and \( J_+ \subseteq I_+(\mathfrak{T}) \), we have that: if \( J(\mathfrak{T}, \mathfrak{T}) \) is positively linearly dependent, then \( D(x^k, \nu^k) \) remains linearly dependent for all \( k \in I \) large enough, where \( \mathfrak{T} \) and \( \nu^k \) are as in (25) and \( D(x, \nu) \) is defined in (20).

The nondegeneracy condition (as in Proposition 4.1) implies seq-CRCQ, whereas Robinson’s CQ implies seq-CPLD. Moreover, these implications are strict, as it is shown in the next counterexample:

**Example 4.6. Consider the constraint**

\[ G(x) = (\mathfrak{T}, x, 0) \in I_2 \]

at the point \( \mathfrak{T} \neq 0 \), which is the only feasible point of the problem. For any \( x \in \mathbb{R} \) and \( \Delta \in \mathbb{R}^2 \), we can assume that

\[ u_1(G(x) + \Delta) = \frac{1}{2}(1, -1) \quad \text{and} \quad u_2(G(x) + \Delta) = \frac{1}{2}(1, 1), \]

without loss of generality (the other possibility consists of swapping the indices 1 and 2), so

\[ DG(x)^\top u_1(G(x) + \Delta) = -1 \quad \text{and} \quad DG(x)^\top u_2(G(x) + \Delta) = 0 \]

are (positively) linearly dependent for every \( x \) and \( \Delta \). Therefore, seq-CPLD and seq-CRCQ both hold, while Robinson’s CQ and nondegeneracy do not.

Example 4.6 also shows that seq-CRCQ does not imply Robinson’s CQ, and the converse is also false; otherwise Robinson’s CQ would imply weak-CRCQ, contradicting Example 4.5. Further, note that Definition 4.4 is basically Definition 4.3 with the addition of some perturbation sequences \( \{\Delta_k^j\}_{k \in \mathbb{N}} \). Then, seq-CPLD implies weak-CPLD, and seq-CRCQ implies weak-CRCQ. The next example shows that these implications are both strict.

**Example 4.7. Consider the constraint**

\[ G(x) = (x^2, x, 0) \in I_3 \]

at \( \mathfrak{T} = 0 \). For any \( x \neq 0 \), we have \( \tilde{G}(x) = (x, 0) \neq 0, \) so

\[ u_1(G(x)) = \frac{1}{2} \left(1, -\frac{x}{|x|}, 0\right) \quad \text{and} \quad u_2(G(x)) = \frac{1}{2} \left(1, \frac{x}{|x|}, 0\right). \]

In particular:

1. If \( x > 0 \), we have \( Dg(x)^\top u_1(G(x)) = 2x - 1 \) and \( Dg(x)^\top u_2(G(x)) = 2x + 1 \);
2. If \( x < 0 \), we have \( Dg(x)^\top u_1(G(x)) = 2x + 1 \) and \( Dg(x)^\top u_2(G(x)) = 2x - 1 \);
3. If \( x = 0 \), we can choose \( u_1(G(x)) = (1, 1, 0) \) and \( u_2(G(x)) = (1, -1, 0) \) to get \( Dg(x)^\top u_1(G(x)) = 2x + 1 \) and \( Dg(x)^\top u_2(G(x)) = 2x - 1 \) as well.

Then, for every sequence \( \{x^k\}_{k \in \mathbb{N}} \rightarrow \mathfrak{T} \), let \( I_+, I_-, \) and \( I_0 \) be a partition of \( \mathbb{N} \), such that \( k \in I_+ \) if \( x^k > 0 \); \( k \in I_- \) if \( x^k < 0 \); and \( k \in I_0 \) if \( x^k = 0 \). Note that at least one among \( I_+, I_- \), and \( I_0 \) must be infinite. Take the subsequence of \( \{x^k\}_{k \in \mathbb{N}} \) indexed by such set, and in view of items 1, 2, and 3, we can conclude that weak-CRCQ and weak-CPLD both hold at \( \mathfrak{T} \), since \( 2x - 1 \) and \( 2x + 1 \) are nonzero and positively linearly dependent for every \( x \in \mathbb{R} \).

However, taking any sequence \( \{x^k\}_{k \in \mathbb{N}} \rightarrow \mathfrak{T} \) such that \( x^k > 0 \) for every \( k \in \mathbb{N} \), and the perturbation vector

\[ \Delta^k = (-x^k)^2, -x^k, x^k) \rightarrow 0, \]

we have that \( G(x^k) + \Delta^k \neq (0, 0, x^k) \), so its eigenvectors are uniquely determined by

\[ u_1(G(x^k)) = \frac{1}{2}(1, 0, -1) \quad \text{and} \quad u_2(G(x^k)) = \frac{1}{2}(1, 0, 1), \]

implying \( Dg(x^k)^\top u_1(G(x^k)) = 2x^k > 0 \) and \( Dg(x^k)^\top u_2(G(x^k)) = 2x^k > 0 \) for every \( k \in \mathbb{N} \). However, \( Dg(\mathfrak{T})^\top (1, 0, -1) = Dg(\mathfrak{T})^\top (1, 0, 1) = 0, \) so seq-CRCQ and seq-CPLD both fail at \( \mathfrak{T} \).

Note that the proof of Theorem 4.1 can be easily adapted, in the same style of the proof of Theorem 3.2, to achieve the following:

**Theorem 4.2.** If \( \mathfrak{T} \in \mathcal{F} \) satisfies AKKT and seq-CPLD, then \( \mathfrak{T} \) also satisfies KKT.

Theorem 4.2 tells us that every limit point of a sequence generated by an iterative algorithm that generates AKKT sequences is guaranteed to satisfy the KKT conditions, provided it satisfies seq-CPLD (or seq-CRCQ).

Now, to exhibit a more theoretical application of our constant rank conditions, let us characterize seq-CRCQ and seq-CPLD without sequences.

**Proposition 4.3.** Let \( \mathfrak{T} \in \mathcal{F} \).
1. seq-CRCQ holds at $\mathcal{T}$, and only if, for every $w \in \{w_\ell \mid \ell \in I_B(\mathcal{T}) \cup I_0(\mathcal{T})\}$, with $\|w_\ell\| = 1$, $\ell \in I_0(\mathcal{T})$, and $\overline{w_\ell} = \overline{G_\ell(\mathcal{T})}/\|G_\ell(\mathcal{T})\|$, $\ell \in I_B(\mathcal{T})$, there exists a neighborhood $V$ of $(\mathcal{T}, \overline{w})$, such that for every $J_B \subseteq I_B(\mathcal{T})$ and $J_- \subseteq I_0(\mathcal{T})$, if $\mathcal{D}(\mathcal{T}, \overline{w})$ is linearly dependent, then $\mathcal{D}(x, w)$ remains linearly dependent for every $(x, \nu) \in V$, where $\mathcal{D}(x, \nu)$ is as defined in (20), $\nu = \{w_\ell \mid \ell \in I_B(\mathcal{T}) \cup I_+\}$, 

$$\nu_\ell = \begin{cases} (1, -w_\ell), & \text{if } \ell \in J_B \cup J_- \cup I_+; \\ (1, w_\ell), & \text{if } \ell \in I_+, \end{cases}$$

and $\mathcal{T}$ is defined likewise.

2. seq-CPLD holds at $\mathcal{T}$, and only if, for every $w \in \{w_\ell \mid \ell \in I_B(\mathcal{T}) \cup I_0(\mathcal{T})\}$, with $\|w_\ell\| = 1$, $\ell \in I_0(\mathcal{T})$, and $\overline{w_\ell} = \overline{G_\ell(\mathcal{T})}/\|G_\ell(\mathcal{T})\|$, $\ell \in I_B(\mathcal{T})$, there exists a neighborhood $V$ of $(\mathcal{T}, \overline{w})$, such that for every $J_B \subseteq I_B(\mathcal{T})$ and $J_- \subseteq I_0(\mathcal{T})$, if $\mathcal{D}(\mathcal{T}, \overline{w})$ is positively linearly dependent, then $\mathcal{D}(x, w)$ remains linearly dependent for every $(x, \nu) \in V$, where $\mathcal{D}(x, \nu)$, $\nu$, and $\mathcal{T}$ are defined as above.

Proof. Similarly to Proposition 3.2, we shall prove only item 1, and item 2 follows analogously. Then, let $\mathcal{T} \in \mathcal{F}$ satisfy seq-CRCQ, and assume that there exists some $w = \{w_\ell \mid \ell \in I_B(\mathcal{T}) \cup I_0(\mathcal{T})\}$, with $\|w_\ell\| = 1$, $\ell \in I_0(\mathcal{T})$, and $\overline{w_\ell} = \overline{G_\ell(\mathcal{T})}/\|G_\ell(\mathcal{T})\|$, $\ell \in I_B(\mathcal{T})$, some subsets $J_B \subseteq I_B(\mathcal{T})$ and $J_- \subseteq I_0(\mathcal{T})$, and a sequence $\{x^k, w^k\}_{k \in \mathbb{N}} \rightarrow (\mathcal{T}, \overline{w})$, where $w^k = \{w_\ell^k \mid \ell \in I_B(\mathcal{T}) \cup I_0(\mathcal{T})\}$, such that $\mathcal{D}(\mathcal{T}, \overline{w})$ is linearly dependent, but $\mathcal{D}(x^k, w^k)$ is linearly independent for every $k \in \mathbb{N}$. Define, for each $k$ and $\ell$, the perturbation vector

$$\Delta_{\ell}^k = \begin{cases} \frac{1}{\|G_\ell \|} \{w_\ell, w_\ell^k\} - G_\ell(x^k), & \text{if } \ell \in I_0(\mathcal{T}) \\
\frac{1}{\|G_\ell \|} \{w_\ell^k, w_\ell\} - G_\ell(x^k), & \text{if } \ell \in I_B(\mathcal{T}), \end{cases}$$

which means $G_\ell(x^k) + \Delta_{\ell}^k \in \text{bd}_k L_{\infty}$ for every $\ell$ and its eigenvectors are uniquely determined by $(1, -w_\ell^k)$ and $(1, w_\ell)$, so this choice of $\Delta_{\ell}^k$ negates seq-CRCQ at $\mathcal{T}$.

Conversely, pick any sequences $\{x^k\}_{k \in \mathbb{N}} \rightarrow (\mathcal{T}, \overline{w})$ and $\{\Delta_{\ell}^k\}_{k \in \mathbb{N}} \rightarrow 0$, $\ell \in I_0(\mathcal{T}) \cup I_B(\mathcal{T})$. Then, for any $w_\ell$, such that $\lim_{k \in \mathbb{N}} u_{1k} G_\ell(x^k) + \Delta_{\ell}^k = (1, w_\ell)$ and $\lim_{k \in \mathbb{N}} u_{2k} G_\ell(x^k) + \Delta_{\ell}^k = (1, \overline{w_\ell})$, for some $I \subseteq \mathbb{N}$, we have that if $\mathcal{D}(\mathcal{T}, \overline{w})$ is linearly dependent, then $\mathcal{D}(x^k, w^k)$ is remains linearly dependent for every $k$ large enough, for all $J_B \subseteq I_B(\mathcal{T})$ and $J_- \subseteq I_0(\mathcal{T})$.

Finally, as a consequence, it follows that seq-CPLD and seq-CRCQ are robust, and they imply metric subregularity CQ.

**Theorem 4.3.** If $\mathcal{T} \in \mathcal{F}$ satisfies seq-CPLD, then:

1. There is a neighborhood $V$ of $\mathcal{T}$, such that every $x \in V \cap \mathcal{F}$ also satisfies seq-CPLD.

2. seq-CPLD implies metric subregularity CQ.

Moreover, the same holds for seq-CRCQ.

**Proof.** Item 1 follows directly from Proposition 4.3. Item 2 is analogous to Theorem 3.3, in view of Theorem 4.1.

### 5 Conclusion

There are few constraint qualifications available for conic programming, and as far as we know, the use of CQs in the global convergence of algorithms is somewhat limited to nondegeneracy and Robinson’s CQ. In contrast, several constraint qualifications have been defined for NLP in the past 20 years, mostly improving the global convergence of algorithms beyond the case when the set of Lagrange multipliers is bounded. We are in a path to extend these CQs to the conic context, that started in [9]. In fact, the results of this paper can be considered a significant improvement of [9] based on our previous developments in [8]. We introduced two weak constant rank CQs for NSOCP and NSDP, called weak-CRCQ and weak-CPLD, which are essentially “diagonal extensions” of their NLP counterparts, in the sense of Proposition 3.1 and Example 4.1. Namely, one can embed an NLP problem using a structurally diagonal (or sparse) conic constraint and both conditions are preserved. This is a fairly unusual property as this approach usually induces a degenerate conic problem; we however believe that this, in some sense, provides a sound mathematical consistency to our approach. These conditions were used to prove convergence of an external penalty method to stationary points, but any application beyond that, besides the mere existence of Lagrange multipliers, is still a subject for investigation. However, they were the starting points for introducing stronger constant rank CQs, called seq-CRCQ and seq-CPLD, with more interesting properties, such as the convergence theory of a larger class of algorithms such as augmented Lagrangians and sequential quadratic programming, and a property related with the ability to compute error bounds under these conditions. We believe that several other applications of constant rank CQs should appear in the literature, such as the computation of the derivative of the value function of a parameterized conic problem and the computation of second-order necessary optimality conditions. In NLP, constant rank CQs are used to define a strong second-order necessary optimality condition that depends on a single Lagrange multiplier, rather than on the full set of Lagrange multipliers, which we believe will be the case for conic problems as well. It is also the case that constant rank conditions provide the adequate
assumptions for guaranteeing global convergence of algorithms to second-order stationary points, which has not been considered yet in the conic programming literature.

This paper leaves several interesting open questions that can be addressed in a future work, such as the applications of weak-CRCQ and weak-CPLD towards algorithms other than external penalty, and stability. Besides, the relationships between seq-CRCQ and weak-nondegeneracy, and between seq-CPLD and weak-Robinson’s CQ is still open. It is also worth recalling that although our conditions were defined by means of sequences, which seems appropriate when talking about convergence of algorithms, we also provided characterizations of them without sequences, in a more classical way, which should foster new applications.

The relationship among the CQs we presented in this paper, and existing ones, is summarized in the following diagram, where (solid) arrows represent (strict) implications, existing CQs are in blue boxes, and new CQs are in green boxes.

![Diagram of constraint qualifications](https://example.com/diagram)

Figure 1: Relationship among the new constraint qualifications and some of the existing ones.

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A Two examples of general algorithms supported by AKKT

We present two general algorithms that are supported by the theory of this paper. These results can be regarded as conic extensions of the results obtained in [10, Sec. 5.2] for NLP. As a side note, we should mention that the conclusions in this section are still valid even if we do not assume that $K$ is self-dual, due to Moreau’s decomposition.

A.1 Example 1: A safeguarded augmented Lagrangian method

To illustrate the role of strict CQs and AKKT in the convergence of algorithms, we will briefly present a variant of the Powell-Hestenes-Rockafellar augmented Lagrangian algorithm that employs a safeguarding technique, which is the direct generalization of the one studied in [16]. The variant we use is also a generalization of [10, Pg. 13] and [12, Alg. 1], for instance.

For an arbitrary penalty parameter $\rho > 0$ and a safeguarded multiplier $\tilde{Y} \in K$, we define $L_{\rho, \tilde{Y}} : \mathbb{R}^n \rightarrow \mathbb{R}$ as the Augmented Lagrangian function of (NCP), which is given by

$$L_{\rho, \tilde{Y}}(x) = f(x) + \frac{\rho}{2} \left\| \Pi_K \left( -G(x) + \frac{\tilde{Y}}{\rho} \right) \right\|^2 - \frac{1}{2\rho} \left\| \tilde{Y} \right\|^2.$$  

Since it will be useful in the next proofs, we compute the gradient of $L_{\rho, \tilde{Y}}$ at $x$ below:

$$\nabla L_{\rho, \tilde{Y}}(x) = \nabla f(x) - DG(x)^\top \left[ \rho \Pi_K \left( -G(x) + \frac{\tilde{Y}}{\rho} \right) \right].$$  \hspace{1cm} (26)

Then, (26) leads to a natural choice of Lagrange multiplier update in the following algorithm:

**Algorithm 1** Safeguarded augmented Lagrangian method

**Input:** A sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ of positive scalars such that $\varepsilon_k \rightarrow 0$; a nonempty convex compact set $B \subset K$; real parameters $\tau > 1$, $\sigma \in (0, 1)$, and $\rho_1 > 0$; and initial points $(x^0, \tilde{Y}^1) \in \mathbb{R}^n \times B$. Also, define $\|V^0\| = \infty$.

Initialize $k \leftarrow 1$. Then:

**Step 1** *(Solving the subproblem):* Compute an approximate stationary point $x^k$ of $L_{\rho_k, \tilde{Y}^k}(x)$, that is, a point $x^k$ such that

$$\|\nabla L_{\rho_k, \tilde{Y}^k}(x^k)\| \leq \varepsilon_k;$$

**Step 2** *(Updating the penalty parameter):* Calculate

$$V^k := \Pi_K \left( -G(x^k) + \frac{\tilde{Y}^k}{\rho_k} \right) - \frac{\tilde{Y}^k}{\rho_k};$$  \hspace{1cm} (27)

Then,

- **a.** If $k = 1$ or $\|V^k\| \leq \tau \|V^{k-1}\|$, set $\rho_{k+1} := \rho_k$
- **b.** Otherwise, take $\rho_{k+1}$ such that $\rho_{k+1} \geq \gamma \rho_k$

**Step 3** *(Estimating a new safeguarded multiplier):* Choose any $\tilde{Y}^{k+1} \in B$, set $k \leftarrow k + 1$ and go to Step 1.

By the definition of projection we have that $Y^k := \Pi_K(\tilde{Y}^k - \rho_k G(x^k))$ if, and only if, $Y^k, G(x^k) \in K$ and $(\tilde{Y}^k, G(x^k)) = 0$, which means that $V^k = 0$ if, and only if, the pair $(x^k, \tilde{Y}^k)$ is primal-dual feasible and complementary. Moreover, note that Algorithm 1 does not necessarily keep a record of the approximate multiplier sequence associated with $\{x^k\}_{k \in \mathbb{N}}$. This sequence is, of course, $Y^k := \rho_k \Pi_K \left( -G(x^k) + \frac{\tilde{Y}^k}{\rho_k} \right)$,  \hspace{1cm} (28)

which also proves that any feasible limit point $\overline{x}$ of $\{x^k\}_{k \in \mathbb{N}}$ is an AKKT point. Since the proof is very simple, we shall state it below for completeness purposes. See [6] for a more detailed exposition:

**Theorem A.1.** Fix any choice of parameters in Algorithm 1 and let $\{x^k\}_{k \in \mathbb{N}}$ be the output sequence generated by it. If $\{x^k\}_{k \in \mathbb{N}}$ has a convergent subsequence $\{x^{k_l}\}_{l \in \mathbb{N}} \rightarrow \overline{x}$, then:
1. The point $\bar{x}$ is stationary of the function $\frac{1}{2}\|\Pi_K(-G(x))\|^2$.

2. If $\bar{x}$ is feasible, then $\bar{x}$ satisfies AKKT.

Proof. Let $\{x_k\}_{k \in \mathbb{N}} \to 0$, $\{Y^k\}_{k \in \mathbb{N}} \subseteq \mathcal{B} \subseteq \mathcal{K}$, $\tau > 1$, $\sigma \in (0, 1)$, and $\rho_1 > 0$ be the fixed input parameters of Algorithm 1. Moreover, let $\{\rho_k\}_{k \in \mathbb{N}}$ and $\{V^k\}_{k \in \mathbb{N}}$ computed as in Step 2. For simplicity, let us also assume that $I = \mathbb{N}$.

1. This part of the proof is standard; see, for instance, [6, Prop. 4.3];

2. Define $\{Y^k\}_{k \in \mathbb{N}}$ as in (28) and take $\Delta^k = V^k$ for all $k \in \mathbb{N}$, where $V^k$ is as given in (27). Then, it follows from Step 1 that $\nabla_x L(x^k, Y^k) = \nabla L_{\rho_k, \gamma_k}(x^k) \to 0$. We also have

$$G(x^k) + \Delta^k = \Pi_K \left( G(x^k) - \frac{V^k}{\rho_k} \right)$$

for every $k \in \mathbb{N}$, which yields $(Y^k, G(x^k) + \Delta^k) = 0$ for every $k$. If $\rho_k \to \infty$, then $V^k \to \Pi_K(-G(\bar{x}))$ by definition and $\Pi_K(-G(\bar{x})) = 0$ because $\bar{x}$ is assumed to be feasible; on the other hand, if $\rho_k$ remains bounded, then $V^k \to 0$ due to Step 2-a. Therefore, $\Delta^k \to 0$ and $\bar{x}$ is AKKT.

Theorem A.1 states that whenever Algorithm 1 does not fail completely, then all of its accumulation points are at least stationary for the feasibility measure $\frac{1}{2}\|\Pi_K(-G(x))\|^2$. And any global minimizer of $\frac{1}{2}\|\Pi_K(-G(x))\|^2$ will satisfy AKKT, provided that (NCP) is feasible.

Thus, when $\mathcal{K} = S_m^a$ (NSDP) or $\mathcal{K} = \mathbb{L}_m \times \cdots \times \mathbb{L}_m$, (NSOP), every feasible limit point generated by Algorithm 1 that satisfies seq-CPLD (or seq-CRCQ) will also satisfy the KKT conditions, which gives us a standard global convergence result for Algorithm 1 without assuming neither boundedness nor uniqueness of Lagrange multipliers. Note that the sequence of approximate Lagrange multipliers given by (28) may be unbounded while the primal limit point is still a KKT point.

A.2 Example 2: A sequential quadratic programming method

Next, we present a straightforward generalization of Correa and Ramírez’s [20] sequential quadratic programming (SQP) method:

**Algorithm 2** General SQP method

**Input:** A real parameter $\tau > 1$, a pair of initial points $(x^1, Y^1) \in \mathbb{R}^n \times \mathcal{K}$, and an approximation of $\nabla^2 L(x^1, Y^1)$ given by $H^1$.

Initialize $k \leftarrow 1$. Then:

**Step 1 (Solving the subproblem):** Compute a solution $d^k$, together its Lagrange multiplier $Y^{k+1}$, of the problem

$$\begin{align*}
\text{Minimize} & \quad d^T H^k d + \nabla f(x^k)^T d, \\
\text{subject to} & \quad G(x^k) + DG(x^k) d \in \mathcal{K},
\end{align*}$$

(Lin-QP)

and if $d^k = 0$, stop;

**Step 2 (Step corrections):** Perform line search to find a steplength $\alpha^k \in (0, 1)$ satisfying Armijo’s rule

$$f(x^k + \alpha^k d^k) - f(x^k) \leq \tau \alpha^k \nabla f(x^k)^T d^k$$

(29)

**Step 3 (Approximating the Hessian):** Set $x^{k+1} \leftarrow x^k + \alpha^k d^k$, compute an approximation $H^{k+1}$ of $\nabla^2 L(x^{k+1}, Y^{k+1})$, set $k \leftarrow k + 1$, and go to Step 1.

As expected, the SQP algorithm generates AKKT sequences as well.

**Proposition A.1.** If there is an infinite subset $I \subseteq \infty \mathbb{N}$ such that $\lim_{k \in I} d^k = 0$ and $\{\|H^k\|\}_{k \in I}$ is bounded, then any limit point $\bar{x}$ of $\{x^k\}_{k \in I}$ satisfies AKKT.

**Proof.** By the KKT conditions for (Lin-QP), there exists some $Y^k \in \mathcal{K}$ such that

$$\begin{align*}
\nabla f(x^k) + H^k d^k - DG(x^k)^* Y^k & = 0 \\
(G(x^k) + D G(x^k) d^k, Y^k) & = 0.
\end{align*}$$

(30)

(31)

Set $\Delta^k = DG(x^k) d^k$ for every $k \in I$ and since $d^k \to 0$, we obtain that $\lim_{k \in I} H^k d^k = 0$ and $\lim_{k \in I} \Delta^k = 0$. Moreover, since $d^k$ is feasible, $G(x^k) + \Delta^k \in \mathcal{K}$. Thus, $\bar{x}$ satisfies AKKT. ■
The hypothesis on the convergence of a subsequence of \( \{d^k\}_{k \in \mathbb{N}} \) to zero is somewhat standard regarding some types of SQP methods, as well as the boundedness of \( H^k \). Moreover, note that Proposition A.1 is very general and does not rely on the self-duality of \( \mathcal{K} \). Nevertheless, when \( \mathcal{K} \) is the semidefinite cone or the second-order cone, we obtain a convergence result of Algorithm 2 without assuming boundedness or uniqueness of Lagrange multipliers.

B On the metric subregularity CQ for self-dual cones

To compare metric subregularity CQ (Definition 2.2) with seq-CRCQ and seq-CPLD, we used Proposition 2.1, which is a sufficient condition for metric subregularity CQ to hold, originally proposed by Minchenko and Stakhovskii [26, Thm. 2]. Since it was originally proved for NLP problems, we made a simple extension of it to NCP. However, as far as we know, this extension has not been addressed before. It is worth mentioning, nevertheless, that the proof we present is essentially the same as the original one, with some minor adaptations to the general conic context via Moreau’s decomposition. Let us recall the statement of Proposition 2.1 and prove it.

**Proposition B.1.** Let \( \mathfrak{T} \in \mathcal{F} \) and for every given \( x \in \mathbb{R}^n \), let \( \Lambda(x) \) denote the set of Lagrange multipliers of the problem of minimizing \( \|z - x\| \) subject to \( G(z) \in \mathcal{K}, z \in \mathbb{R}^n \). If there exist numbers \( \tau > 0 \) and \( \delta > 0 \) such that \( \Lambda(x) \cap \text{cl}(B(0, \tau)) \neq \emptyset \) for every \( x \in B(\mathfrak{T}, \delta) \), then \( \mathfrak{T} \) satisfies metric subregularity CQ.

**Proof.** Let \( \tau \) and \( \delta \) be as described in the hypothesis. Following the proof of [26, Thm. 2], note if \( \mathfrak{T} \in \text{int}\mathcal{F} \), then it trivially satisfies metric subregularity CQ, so we will assume that \( \mathfrak{T} \in \text{bd}\mathcal{F} \). Let \( \delta_0 \in (0, \delta) \) be such that
\[
\frac{4}{\delta_0} 1_n - D^2 G(z)^* \geq 0
\]
for all \( z \in B(\mathfrak{T}, \delta) \) and all \( Y \in \text{cl}(B(0, 2\tau)) \). Let \( x \in B(\mathfrak{T}, \delta_0/2) \) such that \( x \notin \mathcal{F} \). Although \( \Pi_{\mathcal{F}}(x) \) may not be well-defined as a function of \( x \), we will use the notation \( \Pi_{\mathcal{F}}(x) \) to denote an arbitrary minimizer of \( \|z - x\| \) subject to \( G(z) \in \mathcal{K} \). Then, by definition, we have that \( \|\Pi_{\mathcal{F}}(x) - x\| \leq ||z - x|| < \delta_0/2 \), so \( \Pi_{\mathcal{F}}(x) \in B(x, \delta_0/2) \) and, therefore, \( \|\Pi_{\mathcal{F}}(x) - x\| \leq \|\Pi_{\mathcal{F}}(x) - x\| + \|x - \xi\| < \delta_0 \). Let \( h: \mathbb{R}^n \times \mathbb{B}^m \to \mathbb{R} \) be defined as
\[
h(z, Y) = \frac{(z - x, z - \Pi_{\mathcal{F}}(x))}{\|x - \Pi_{\mathcal{F}}(x)\|} - \langle G(z), Y \rangle
\]
and note that
\[
\nabla_h^2 h(z, Y) = \frac{2}{\|x - \Pi_{\mathcal{F}}(x)\|} [1_n - D^2 G(z)^* Y] \geq \frac{4}{\delta_0} [1_n - D^2 G(z)^* Y] \geq 0
\]
whenever \( z \in B(\mathfrak{T}, \delta) \) and \( Y \in \text{cl}(B(0, 2\tau)) \). Thus, \( h(z, Y) \) is convex with respect to its first variable \( z \in B(\mathfrak{T}, \delta) \), for every \( Y \in \text{cl}(B(0, 2\tau)) \). Now let us fix an arbitrary \( Y \in \Lambda(x) \cap \text{cl}(B(0, \tau)) \), which is nonempty by hypothesis. Recall that, by definition of the set \( \Lambda(x) \), we have that \( Y \) is a Lagrange multiplier of the projection problem associated with the point \( \Pi_{\mathcal{F}}(x) \). Hence, \( 2Y \) is a Lagrange multiplier of the problem:

Minimize \( f_\delta(z) = \|z - x\| + \frac{(z - x, z - \Pi_{\mathcal{F}}(x))}{\|x - \Pi_{\mathcal{F}}(x)\|}, \) subject to \( G(z) \in \mathcal{K} \) \hspace{1cm} (32)

associated with the point \( \Pi_{\mathcal{F}}(x) \), which is a local minimizer of \( f_\delta \) since it is elementary to check that \( f_\delta(\Pi_{\mathcal{F}}(x)) \geq \|z - x\| \) for every \( z \in \mathcal{F} \), by the definition of projection (for details, see [26, Lem. 3]), with equality at \( \Pi_{\mathcal{F}}(x) \). Writing the KKT conditions for the problem (32) at \( \Pi_{\mathcal{F}}(x) \) with the Lagrange multiplier \( 2Y \in \text{cl}(B(0, 2\tau)) \), we obtain
\[
\frac{2(\Pi_{\mathcal{F}}(x) - x)}{\|x - \Pi_{\mathcal{F}}(x)\|} - DG(\Pi_{\mathcal{F}}(x))^*[2Y] = 0
\]
with \( \langle G(\Pi_{\mathcal{F}}(x)), 2Y \rangle = 0 \), which yields
\[
\|x - \Pi_{\mathcal{F}}(x)\| = -\|x - \Pi_{\mathcal{F}}(x)\| - \langle DG(\Pi_{\mathcal{F}}(x))^*[2Y], x - \Pi_{\mathcal{F}}(x) \rangle \\
\leq \langle G(\Pi_{\mathcal{F}}(x)) - G(x), 2Y \rangle \\
= -\langle G(x), 2Y \rangle
\]
(34)
after taking inner products of both sides of (33) with \( x - \Pi_{\mathcal{F}}(x) \). The middle inequality follows from the definition of adjoint and the convexity of \( h(z, Y) \) in the first variable. Taking Moreau’s decomposition for \( G(x) \), we obtain from (34) that
\[
\|x - \Pi_{\mathcal{F}}(x)\| \leq \langle \Pi_K(G(x)), 2Y \rangle + \langle \Pi_K(\Pi_{\mathcal{F}}(x)), 2Y \rangle \leq \langle \Pi_K(\Pi_{\mathcal{F}}(x)), 2Y \rangle,
\]
because \( Y \in \mathcal{K} \), which is self-dual, so \( \langle \Pi_K(G(x)), 2Y \rangle \geq 0 \); then
\[
\text{dist}(x, F) = \|x - \Pi_{\mathcal{F}}(x)\| \leq \|2Y\|\|\Pi_K(\Pi_{\mathcal{F}}(x))\| \leq 2\|\Pi_K(\Pi_{\mathcal{F}}(x))\|.
\]
Since \( x \) was chosen arbitrarily, set \( \gamma = 2\tau \) and we are done.

Note that the proof of the proposition above does not necessarily remain true if one removes the assumption on the self-duality of \( \mathcal{K} \).
C Weak-Robinson’s CQ is equivalent to Robinson’s CQ in NSDP when $m = 2$

The relationship between Robinson’s CQ and weak-Robinson’s CQ was left as an open problem in [8]. In this section, we give a partial answer for it when $m = 2$. To do this, we start with a technical lemma, that says weak-Robinson’s CQ is invariant to linear conjugations.

**Lemma C.1.** If the constraint $G(x) \succeq 0$ satisfies weak-Robinson’s CQ at $\mathfrak{T} \in \mathcal{F}$, then the constraint $U^\top G(x)U \succeq 0$ also satisfies weak-Robinson’s CQ at $\mathfrak{T}$, for every orthogonal matrix $U \in \mathbb{R}^{m \times m}$.

**Proof.** Let $U \in \mathbb{R}^{m \times m}$ be an arbitrary orthogonal matrix, and let $\{x^k\}_{k \in \mathbb{N}} \to \mathfrak{T}$ be any sequence. If $\mathfrak{T}$ satisfies weak-Robinson’s CQ, then there exists some $\mathfrak{E} \in \operatorname{Lim sup}_{k \to \infty} E_i(G(x^k))$ such that the family of vectors $\{v_{ii}(\mathfrak{T}, \mathfrak{E})\}_{i \in \{1, \ldots, m - r\}}$ is positively linearly independent. Let $I \subseteq \mathbb{N}$ be such that $\{E^k\}_{k \in I} \to \mathfrak{T}$ and $E^k \in \mathcal{E}_i(G(x^k))$ for every $k \in \mathbb{N}$. Then, note that $W^k \doteq U^\top E^k \in \mathcal{E}_i(U^\top G(x^k)U)$, for every $k \in I$, and that $v_{ii}(\mathfrak{T}, \mathfrak{E}) = [\pi_i U^\top D_{x_i} G(\mathfrak{T}) U \pi_i]_{i \in \{1, \ldots, n\}}$ for all $i \in \{1, \ldots, m - r\}$, where $\pi_i$ denotes the $i$-th column of $W \doteq U^\top \mathfrak{T}$. Thus, weak-Robinson’s CQ holds at $\mathfrak{T}$ for the constraint $U^\top G(x)U \succeq 0$ as well.

As mentioned before, Robinson’s CQ always implies weak-Robinson’s CQ. Conversely,

**Theorem C.1.** If $m = 2$, then weak-Robinson’s CQ implies Robinson’s CQ.

**Proof.** Here, we assume that $G(\mathfrak{T}) = 0$, since this is the only nontrivial case. If Robinson’s CQ does not hold at $\mathfrak{T}$, then there exists some orthogonal matrix $U$ such that $\nabla G_{11}(\mathfrak{T})$ and $\nabla G_{22}(\mathfrak{T})$ are positively linearly dependent, where $G(x) \doteq U^\top G(x)U$, as a consequence of [8, Prop. 5.1]. Then, there exists some $\theta \geq 0$ such that $\nabla G_{22}(\mathfrak{T}) = -\theta \nabla G_{11}(\mathfrak{T})$. Now, we have three cases to consider:

1. If $\nabla G_{11}(\mathfrak{T}) = \nabla G_{12}(\mathfrak{T}) = 0$, then $\nabla G_{22}(\mathfrak{T}) = 0$ and, in this case, $v_{ii}(\mathfrak{T}, \mathfrak{E}) = \nabla G_{22}(\mathfrak{T}) = 0$ for every orthogonal matrix $E$.

2. If $\nabla G_{12}(\mathfrak{T}) = \xi \nabla G_{11}(\mathfrak{T})$ for some $\xi \in \mathbb{R}$ and $\nabla G_{11}(\mathfrak{T}) \neq 0$, let $\{x^k\}_{k \in \mathbb{N}} \to \mathfrak{T}$ be such that $d^k \to d \doteq \nabla G_{11}(\mathfrak{T})/\|\nabla G_{11}(\mathfrak{T})\|$. Then, we have

$$
DG(\mathfrak{T})[d] = \|\nabla G_{11}(\mathfrak{T})\| \begin{bmatrix}
\frac{1}{\xi} \\
\xi \\
\theta
\end{bmatrix},
$$

but since $\nabla G_{22}(\mathfrak{T})$ and $\nabla G_{12}(\mathfrak{T})$ are multiples of $\nabla G_{11}(\mathfrak{T})$, we also have

$$
D_{x_i} G(\mathfrak{T}) = D_{x_i} G_{11}(\mathfrak{T}) M,
$$

and since any limit point $\mathfrak{E}$ of any sequence $\{E^k\}_{k \in \mathbb{N}}$ of eigenvectors of $R^k$ must diagonalize $DG(\mathfrak{T})[d]$ (and, therefore, $M$), we have that

$$
v_{ii}(\mathfrak{T}, \mathfrak{E}) = \lambda_i(M) \nabla G_{11}(\mathfrak{T}),
$$

but note that $\lambda_2(M) < 0 < \lambda_1(M)$ to see that these vectors must be positively linearly dependent.

3. If $\nabla G_{11}(\mathfrak{T})$ and $\nabla G_{12}(\mathfrak{T})$ are linearly independent, there exists some $d$ such that $\nabla G_{11}(\mathfrak{T})^\top d = 0$ and $\nabla G_{12}(\mathfrak{T})^\top d = 0$. Pick any nonconstant sequence $\{x^k\}_{k \in \mathbb{N}} \to \mathfrak{T}$ such that $d^k \doteq \frac{1}{\|x^k - \mathfrak{T}\|} \to d$, consider the first-order expansion of $\tilde{G}(x^k)$ around $\mathfrak{T}$, given by

$$
\tilde{G}(x^k) = \|x^k - \mathfrak{T}\| \left( DG(x^k)[d^k] + o(\|x^k - \mathfrak{T}\|) \right)
$$

and note that $\tilde{G}(x^k)$ and $R^k$ have the same eigenvectors for every $k$, with eigenvalues scaled by a factor $\|x^k - \mathfrak{T}\|$. Since $d^k \to d$, we have $R^k \to DG(\mathfrak{T})[d]$. On the other hand, note that

$$
DG(\mathfrak{T})[d] = \operatorname{Diag}(\nabla G_{11}(\mathfrak{T})^\top d, -\theta \nabla G_{11}(\mathfrak{T})^\top d),
$$

implying it is simple, so its has a unique orthonormal basis of eigenvalues up to sign, which can be assumed, without loss of generality, to be the columns of $\mathfrak{T} = I_2$.

However, note that every limit point $E$, of any sequence of orthogonal matrices $\{E^k\}_{k \in \mathbb{N}}$ that diagonalize $R^k$, must also diagonalize $DG(\mathfrak{T})[d]$. Consequently, we have

$$
\tilde{v}_{ii}(\mathfrak{T}, E) = \left[e_i^\top D_{x_i} G(\mathfrak{T}) e_i, \ldots, e_i^\top D_{x_n} G(\mathfrak{T}) e_i\right] = \tilde{v}_{ii}(\mathfrak{T}, \mathfrak{E}) = \nabla G_{ii}(\mathfrak{T})
$$

where $e_i$ denotes the $i$-th column of $E$, for each $i \in \{1, 2\}$, and since these vectors are positively linearly dependent, weak-Robinson’s CQ does not hold at $\mathfrak{T}$. 28
Thus, weak-Robinson’s CQ does not hold for at \( \mathbf{x} \) for the constraint \( U^\top G(x)U \succeq 0 \) and the conclusion follows from Lemma C.1.

Note that extending the proof above for a general context may not be easy, and the case \( m > 2 \) remains as an open problem. However, we conjecture weak-Robinson’s CQ is equivalent to Robinson’s CQ even when \( m > 2 \).