ON THE SECOND HANKEL DETERMINANT OF CONCAVE
FUNCTIONS

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In dedication to Professor Karl-Joachim Wirths on his 70th birthday.

Abstract. In the present paper, we will discuss the Hankel determinants
\( H(f) = a_2a_4 - a_3^2 \) of order 2 for normalized concave functions
\( f(z) = z + a_2z^2 + a_3z^3 + \ldots \) with a pole at \( p \in (0,1) \). Here, a meromorphic function
is called concave if it maps the unit disk conformally onto a domain whose
complement is convex. To this end, we will characterize the coefficient body
of order 2 for the class of analytic functions \( \varphi(z) \) on \( |z| < 1 \) with
\( |\varphi| < 1 \) and \( \varphi(p) = p \). We believe that this is helpful for other extremal problems
concerning \( a_2, a_3, a_4 \) for normalized concave functions with a pole at \( p \).

1. Introduction

A meromorphic function \( f \) on the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \) of the complex
plane \( \mathbb{C} \) is called concave if \( f \) is univalent and if \( \mathbb{C} \setminus f(\mathbb{D}) \) is convex. Such
functions are intensively studied by Avkhadiev, Bhowmik, Pommerenke, Wirths
and others in recent years, see \([1, 2, 3, 4, 6]\). For \( p \in \mathbb{D} \setminus \{0\} \), we denote by
\( \mathcal{C}_p \) the set of concave functions \( f \) with a pole at \( p \) normalized by
\( f(0) = 0 \) and \( f'(0) = 1 \). By a suitable rotation, we will assume without loss of generality that
\( 0 < p < 1 \) in what follows. Each function \( f \) in \( \mathcal{C}_p \) can be expanded in the form
\( f(z) = z + a_2z^2 + a_3z^3 + \ldots \) for \( |z| < p \). We sometimes write \( a_n = a_n(f) \) to
indicate that the coefficients belong to the function \( f \).

By \( \text{End}(\mathbb{D}) \) we denote the set of analytic endomorphisms (self-maps) of
the unit disk \( \mathbb{D} \). Let \( \mathcal{B}_p \) stand for the class of \( \varphi \in \text{End}(\mathbb{D}) \) fixing the point \( p \). The
first author gave the following characterization of the functions in \( \mathcal{C}_p \) in \([10]\).

Theorem A. Let \( 0 < p < 1 \). For \( f \in \mathcal{C}_p \), there exists a \( \varphi \in \mathcal{B}_p \) such that
\begin{equation}
(1.1) \quad f'(z) = \frac{p^2}{(z-p)^2(1-pz)^2} \exp \int_0^z \frac{-2\varphi(t)}{1-t\varphi(t)} \, dt, \quad z \in \mathbb{D}.
\end{equation}

Conversely, for a given \( \varphi \in \mathcal{B}_p \), there exists a function \( f \in \mathcal{C}_p \) satisfying \((1.1)\).

We remark that the condition \( \varphi(p) = p \) comes from the demand that \( f'(z) \)
should have no residue at \( z = p \).

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For $a \in \mathbb{D}$, the Möbius transformation

$$T_a(z) = \frac{a - z}{1 - \bar{a}z} = -[z, a]$$

is an analytic involution of $\mathbb{D}$ interchanging $0$ and $a$. Here $[z, w] = (z - w)/(1 - \bar{w}z)$ denotes the complex pseudo-hyperbolic distance introduced by Beardon and Minda [3]. Let $\zeta \in \partial \mathbb{D}$. Then, the conjugation $\rho_\zeta$ of the rotation $z \mapsto \zeta z$ by $T_\rho$ is an analytic automorphism of $\mathbb{D}$ contained in $\mathcal{B}_p$. More explicitly, $\rho_\zeta$ is expressed by

$$\rho_\zeta(z) = T_\rho(\zeta T_\rho(z)) = \frac{(\zeta - p^2)z + (1 - \zeta)p}{-(1 - \zeta)pz + 1 - p^2\zeta} = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots,$$

where

$$(1.2) \quad \alpha_0 = \frac{(1 - \zeta)p}{1 - p^2\zeta}, \quad \text{and} \quad \alpha_k = \frac{\zeta(1 - p^2)^2(1 - \zeta)^{k-1}p^{k-1}}{(1 - p^2\zeta)^{k+1}}, \quad k = 1, 2, 3, \ldots.$$

Obviously, $\rho_\zeta$ can be defined for $\zeta \in \partial \mathbb{D}$ as an analytic endomorphism of $\mathbb{D}$. Noting the fact that

$$\frac{-2\rho_\zeta(z)}{1 - z\rho_\zeta(z)} = \frac{(\zeta(z) - p)^2 - (1 - p^2)\zeta}{\zeta(z) - p)^2 - (1 - p^2)},$$

we see that the function determined by (1.1) with the choice $\varphi = \rho_\zeta$ is given by

$$(1.3) \quad F_\zeta(z) = \frac{z - T_\rho(p\zeta)z^2}{(1 - z/p)(1 - p)^2} = \frac{z}{1 - p^2\zeta} \left[ \frac{1}{1 - z/p} - \frac{p^2}{1 - p^2} \right] = \sum_{n=1}^\infty \frac{1 - p^{2n}}{p^{n-1}(1 - p^2)} z^n =: \sum_{n=1}^\infty A_n(\zeta)z^n.$$

Thus we see that the coefficient region $\{a_n(f) : f \in \mathcal{C}_p\}$ contains the set $A_n(\mathbb{D}) = \{A_n(\zeta) : \zeta \in \mathbb{D}\}$. We note that $A_n(\mathbb{D})$ is the closed disk $|w - (1 - p^{2n+2})/p^{n-1}(1 - p^4)| \leq (p^2 - p^{2n})/p^{n-1}(1 - p^4)$. Indeed, Avkhadiev and Wirths [4] proved the following.

**Theorem B.** Let $0 < p < 1$ and $n \geq 2$. Then

$$\{a_n(f) : f \in \mathcal{C}_p\} = A_n(\mathbb{D}) = \left\{ w : \left| w - \frac{1 - p^{2n+2}}{p^{n-1}(1 - p^4)} \right| \leq \frac{p^2 - p^{2n}}{p^{n-1}(1 - p^4)} \right\}.$$ 

Moreover, for $f \in \mathcal{C}_p$, $a_n(f) \in \partial A_n(\mathbb{D})$ if and only if $f = F_\zeta$ for some $\zeta \in \partial \mathbb{D}$.

Note that for each $\zeta \in \partial \mathbb{D}$, $T_\rho(p\zeta) = (1 + e^{i\theta})p/(1 + p^2)$ for some $\theta \in \mathbb{R}$ and vice versa. In the present paper, we consider the second Hankel determinant of order 2 for $f(z) = z + a_2z^2 + \ldots$, which is defined by

$$H(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$
Especially, we will take a closer look at the variability region $H(C_0) = \{ H(f) : f \in C_0 \}$ for $0 < p < 1$. The second Hankel determinant of general order was studied by Pommerenke [12] and Hayman [9] and many others in recent years. A straightforward computation yields

$$H(F_\zeta) = A_2(\zeta) A_4(\zeta) - A_3(\zeta)^2 = -\frac{(1 - p^2)^2 \zeta^2}{(1 - p^2)^2} - \frac{(1 - p^2)^2}{p^2} K(p^2 \zeta),$$

where

$$K(z) = \frac{z}{(1 - z)^2}$$

is the Koebe function. Set

$$(1.4) \quad \Omega_p = \{ H(F_\zeta) : |\zeta| \leq 1 \} = -\frac{(1 - p^2)^2}{p^2} K(p^2 \mathbb{D}).$$

This set has the following property.

**Proposition 1.1.** $\Omega_p \subset \Omega_q$ for $0 < q < p < 1$ and

$$\bigcup_{0 < p < 1} \Omega_p = \mathbb{D} \cup \{-1\} \quad \text{and} \quad \bigcap_{0 < p < 1} \Omega_p = \{- (1 + z)^2 / 4 : |z| \leq 1\}.$$

Note that the set $\{- (1 + z)^2 / 4 : |z| \leq 1\}$ is a closed Jordan domain, bounded by a cardioid with an inward-pointing cusp at the origin.

By the above observations, we have $\Omega_p \subset H(C_0)$. In view of the coefficient regions of $a_n$ for $C_0$, one might suspect that $H(C_0) = \Omega_p$ for $0 < p < 1$ and, in particular, $H(C_0) \subset \mathbb{D}$. This is, however, not the case. To state our result, we set

$$M(p) = \sup \{|H(f)| : f \in C_0\}.$$

**Theorem 1.2.** Let $0 < p < 1$. Then $M(p) > 1$. Moreover,

$$\frac{1}{3p} < M(p) < \frac{1}{3p} + \frac{2}{3}.$$

In Section 3, we will prove the above proposition and the theorem. Indeed, we give a description of the variability region of $H(f)$ for $f \in C_0$ in Proposition 3.1 below. As a preliminary, we give an explicit form of the coefficient body of order 2 for the class $B_p$ in Section 2. Our basic idea is to employ an higher-order analogue of Dieuonne’s lemma.

2. Higher-order analogue of Dieudonné’s lemma and its application

We expand a function $\varphi \in B_p$ in the form

$$(2.1) \quad \varphi(z) = c_0 + c_1 z + c_2 z^2 + \ldots, \quad |z| < 1.$$
Then the early coefficients of the function \( f(z) = z + a_2z^2 + a_3z^3 + \ldots \) determined by (1.1) are given by
\[
\begin{align*}
a_2 &= P - c_0, \\
a_3 &= P^2 + \frac{-c_1 + c_0^2 - 4Pc_0 - 2}{3}, \\
a_4 &= P^3 + \frac{-c_2 + c_0c_1 + 6c_0 - 9P - 9P^2c_0 + 3Pc_0^2 - 3Pc_1}{6},
\end{align*}
\]
where we put
\[
P = p + \frac{1}{p} = \frac{1 + p^2}{p}.
\]

By making use of this type of relations, a coefficient problem for \( C_0p \) reduces in principle to that of \( B_p \). Based on this idea, in [11], the authors solved the extremal problem on \(|a_3 - \mu a_2^2|\) for a function \( f(z) = z + a_2z^2 + a_3z^3 + \ldots \) in \( C_0p \) and a real constant \( \mu \). The key ingredient in [11] was the determination of the coefficient body \( X_1(B_p) \) of order 1 for \( B_p \). Here, for \( n \geq 0 \), the coefficient body \( X_n(F) \) of order \( n \) for a class \( F \) of analytic functions at the origin is defined by
\[
X_n(F) = \{(c_0, c_1, \ldots, c_n) \in \mathbb{C}^{n+1} : \quad \varphi(z) = c_0 + c_1z + \cdots + c_nz^n + O(z^{n+1}) \text{ for some } \varphi \in F\}.
\]

The goal of the present section is to show the following description of the coefficient body \( X_2(B_p) \) of order 2.

**Theorem 2.1.** Let \( 0 < p < 1 \). A triple \((c_0, c_1, c_2)\) of complex numbers is contained in the coefficient body \( X_2(B_p) \) if and only if
\[
c_0 = P^{-1}(1 - \sigma_0), \quad \text{and} \quad c_1 = P^{-2}\left[1 + (P^2 - 2)\sigma_0 + \sigma_0^2\right] + P^{-1}(1 - |\sigma_0|^2)\sigma_1
\]
and
\[
c_2 = P^{-3}(1 - \sigma_0)\left[1 + (P^2 - 2)\sigma_0 + \sigma_0^2\right] - P^{-2}(P^2 - 2 + 2\sigma_0)(1 - |\sigma_0|^2)\sigma_1 + \varepsilon P^{-1}(1 - |\sigma_0|^2)|\sigma_0\sigma_1|^2 + P^{-1}(1 - |\sigma_0|^2)(1 - |\sigma_1|^2)|\sigma_2|
\]
for some \( \sigma_0, \sigma_1, \sigma_2 \in \overline{D} \), where \( P = (1 + p^2)/p > 2 \) and \( \varepsilon = |1 + p^2\sigma_0|/(1 + p^2\sigma_0) \in \partial D \).

Let us now recall Dieudonné’s lemma (see, for instance, [8, p. 198]).

**Lemma 2.2** (Dieudonné’s lemma). Let \( z_0, \tau_0 \in \mathbb{D} \) with \(|\tau_0| \leq |z_0| \neq 0\). Then the variability region of \( \tau_1 = \psi'(z_0) \) for \( \psi \in \text{End}(\mathbb{D}) \) with \( \psi(0) = 0, \psi(z_0) = \tau_0 \) is the closed disk given by
\[
|\tau_1 - \tau_0| \leq \frac{|z_0|^2 - |\tau_0|^2}{|z_0|(1 - |z_0|^2)}.
\]

We remark that equality holds in Dieudonné’s lemma if and only if \( \psi \) is a finite Blaschke product of degree at most 2 (cf. [7, Theorem 3.6]). The following result can be regarded as Dieudonné’s lemma of the second order (see [7, Theorem 3.7]).
Lemma 2.3. Let $z_0, \tau_0 \in \mathbb{D}$ with $|\tau_0| < |z_0| \neq 0$ and suppose that $\tau_1 \in \mathbb{C}$ satisfies (2.3). Then the variability region of $\tau_2 = \psi''(z_0)/2!$ for $\psi \in \text{End}(\mathbb{D})$ with $\psi(0) = 0, \psi(z_0) = \tau_0$ and $\psi'(z_0) = \tau_1$ is the closed disk described by
\[
|\tau_2 - \frac{\tau_1 - \tau_0}{z_0(1 - |z_0|^2)} + \frac{\tau_0(\tau_1 - \tau_0/\overline{z_0})^2}{|z_0|^2 - |\tau_0|^2} + \frac{|z_0|(|\tau_1 - \tau_0/\overline{z_0}|^2)}{|z_0|^2 - |\tau_0|^2}| \leq \frac{|z_0|(1 - |\tau_0/\overline{z_0}|^2)}{(1 - |z_0|^2)^2}.
\]

We remark that equality holds precisely when $\psi$ is a finite Blaschke product of degree at most 3. In [7, Theorem 3.7], the above inequality is stated as a necessary condition. For sufficiency, a construction is given in the proof below.

In order to prove Theorem 2.3, we show a preliminary form of the characterization of $X_2(B_p)$.

Lemma 2.4. Let $0 < p < 1$. A triple $(c_0, c_1, c_2)$ of complex numbers is contained in the coefficient body $X_2(B_p)$ if and only if
\[
c_0 = \frac{p - pw_0}{1 - p^2w_0} \quad \text{and} \quad c_1 = \frac{(1 - p^2)^2w_0 + p(1 - p^2)(1 - |w_0|^2)w_1}{(1 - p^2w_0)^2}
\]
and
\[
c_2 = \frac{(1 - p^2)}{1 - p^2w_0} \left[ p(1 - p^2)(1 - w_0)w_0 - (1 - p^2)(1 + p^2w_0)(1 - |w_0|^2)w_1 + p(\overline{w_0} - p^2)(1 - |w_0|^2)w_1^2 + p(1 - p^2w_0)(1 - |w_0|^2)(1 - |w_1|^2)w_2 \right]
\]
for some $w_0, w_1, w_2 \in \mathbb{T}$.

Proof. When $\varphi = \rho_\zeta$ for some $\zeta \in \partial\mathbb{D}$, the coefficients $a_0, a_1, a_2$ are given as $c_0, c_1, c_2$ with $(w_0, w_1, w_2) = (\zeta, 0, 0)$ (see (1.2)).

We next suppose that a function $\varphi(z) = c_0 + c_1z + c_2z^2 + \ldots$ in $B_p$ is not of the form $\rho_\zeta$, $\zeta \in \partial\mathbb{D}$. Then $\psi = T_p \circ \varphi \circ T_p \in \text{End}(\mathbb{D})$ satisfies $\psi(0) = 0$ but is not a rotation about 0. It is straightforward to check the formulæ:
\[
\tau_0 := \psi(p) = T_p(c_0) = \frac{p - c_0}{1 - pc_0};
\]
\[
\tau_1 := \psi'(p) = \frac{c_1}{(1 - pc_0)^2};
\]
\[
\tau_2 := \frac{\psi''(p)}{2} = \frac{-(1 - pc_0) c_2 + pc_1(1 - pc_0 - c_1)}{(1 - p^2)(1 - pc_0)^3}.
\]
Hence,
\[
c_0 = T_p(\tau_0) = \frac{p - \tau_0}{1 - p\tau_0}; \quad (1 - pc_0)(1 - p\tau_0) = 1 - p^2,
\]
\[
c_1 = (1 - pc_0)^2\tau_1 = \frac{(1 - p^2)^2\tau_1}{(1 - p\tau_0)^2},
\]
\[
c_2 = \frac{-(1 - p^2)^3(1 - p\tau_0)\tau_2 + p(1 - p^2)^2\tau_1(1 - p\tau_0 - \tau_1 + p^2\tau_1)}{(1 - p\tau_0)^3}.
\]
By Schwarz’s lemma and Dieudonné’s lemma (with \( z_0 = p \)), we have
\[
(2.5) \quad w_0 := \frac{\tau_0}{p} \in \mathbb{D} \quad \text{and} \quad w_1 := \frac{\tau_1 - \tau_0/p}{(p^2 - |\tau_0|^2)/p(1 - p^2)} = \frac{(1 - p^2)(\tau_1 - w_0)}{p(1 - |w_0|^2)} \in \mathbb{D}.
\]

When \(|w_1| = 1\), by the remark after Lemma 2.3, \( \psi(z) \) is of the form \( z \omega([z, p]) \), where \( \omega(z) = [w_1z, -w_0] \). Then the first three Taylor coefficients of \( \psi(z) \) about \( z = 0 \) are given by \( c_0, c_1, c_2 \) in (1.2) with \( w_2 = 0 \).

Finally, suppose that \(|\omega_1| < 1\). Then, by Lemma 2.3, we see that
\[
(2.6) \quad w_2 := \left( \frac{\tau_2 - \tau_0/p}{p(1 - p^2)} + \frac{\tau_0(\tau_1 - \tau_0/p)^2}{p^2 - |\tau_0|^2} \right) \div \left( \frac{p(1 - |\tau_0|^2)}{(1 - p^2)^2} - \frac{p|\tau_1 - \tau_0/p|^2}{p^2 - |\tau_0|^2} \right)
= \frac{(1 - p^2)^2\tau_2 - (1 - |w_0|^2)(1 - pw_0w_1)w_1}{p(1 - |w_0|^2)(1 - |w_1|^2)} \in \mathbb{D}.
\]

Here, note that the denominator does not vanish because of \(|w_1| < 1\). Conversely, for \( w_0, w_1 \in \mathbb{D} \) and \( w_1 \in \overline{\mathbb{D}} \), the function \( \psi(z) = z\omega([z, p]) \) fulfills the relations in (2.5) and (2.6), where \( \omega(z) = [z|w_2z, -w_1], -w_0] \). (Note that this construction shows the sufficiency part of Lemma 2.3.) We now obtain
\[
\tau_0 = p\omega_0,
\tau_1 = w_0 + \frac{p(1 - |w_0|^2)w_1}{1 - p^2},
\tau_2 = \frac{1 - |w_0|^2}{(1 - p^2)^2} [(1 - pw_0w_1)w_1 + p(1 - |w_1|^2)w_2].
\]

Substitution of these expressions into (2.4) proves the lemma.

**Proof of Theorem 2.1.** For \( w_0, w_1, w_2 \in \overline{\mathbb{D}} \), we put
\[
\sigma_0 = [w_0, p^2] = \frac{w_0 - p^2}{1 - p^2w_0}, \quad \sigma_1 = \frac{|1 - p^2w_0|^2}{(1 - p^2w_0)^2}w_1, \quad \sigma_2 = \frac{|1 - p^2w_0|^2}{(1 - p^2w_0)^2}w_2.
\]

Then \( \sigma_j \in \overline{\mathbb{D}} \) for \( j = 0, 1, 2 \) and vice versa. Noting the elementary relations
\[
w_0 = [\sigma_0, -p^2] = \frac{\sigma_0 + p^2}{1 + p^2\sigma_0}, \quad (1 + p^2\sigma_0)(1 - p^2w_0) = 1 - p^4
\]
and
\[
(1 - |\sigma_0|^2)(1 - p^2w_0)^2 = (1 - p^4)(1 - |w_0|^2),
\]
the formulae of \( c_j \) in Lemma 2.4 can be expressed in terms of \( \sigma_0, \sigma_1, \sigma_2 \) through tedious but straightforward computations. We finally replace \( p + 1/p \) by \( P \) to prove Theorem 2.1. \( \square \)
3. Proof of main results

By the relations (2.2), we can express $H(f)$ for $f \in C_{o_p}$ in terms of $c_j$'s as follows:

\begin{equation}
18H(f) = 3(c_0 - P)c_2 - 2c_1^2 + (c_0^2 - 4Pc_0 + 3P^2 - 8)c_1
- (c_0^2 - Pc_0 + 1)(2c_0^2 - 5Pc_0 + 3P^2 + 8).
\end{equation}

We further substitute the formula in Theorem 2.1 into (3.1) to obtain

\begin{equation}
18P^3H(f) = -18P[1 + (P^2 - 2)\sigma_0 + \sigma_0^2]
+ 3[1 - 7P^2 + 2P^4 + (3P^2 - 2)\sigma_0 + \sigma_0^2](1 - |\sigma_0|^2)\sigma_1
+ P[2(1 - |\sigma_0|^2) + 3e^{2i\theta_0}(P^2 - 1 + \sigma_0)](1 - |\sigma_0|^2)\sigma_1^2
- 3P(P^2 - 1 + \sigma_0)(1 - |\sigma_0|^2)(1 - |\sigma_1|^2)\sigma_2
=: \Phi_p(\sigma_0, \sigma_1, \sigma_2),
\end{equation}

where $\varepsilon = |1 + p^2\sigma_0|/(1 + p^2)\sigma_0$. At this stage, we have obtained the following description of the set $H(C_{o_p})$.

**Proposition 3.1.** Let $0 < p < 1$. Then the variability region of the second Hankel determinant $H(f)$ of order 2 for $f \in C_{o_p}$ is given by

$$H(C_{o_p}) = \{\Phi_p(\sigma_0, \sigma_1, \sigma_2)/18P^3 : \sigma_0, \sigma_1, \sigma_2 \in \overline{D}\}.$$ 

We note that the function $F_\zeta$ given in (1.3) corresponds to the parameters $(\sigma_0, \sigma_1, \sigma_2) = ([\zeta, p^2], 0, 0)$. Since $\Phi_p(\sigma, 0, 0) = -18P(1 + (P^2 - 2)\sigma + \sigma_2)$, as a by-product, we have the following description of the set $\Omega_p$ defined by (1.4).

**Lemma 3.2.**

$$\Omega_p = \{-P^{-2}[1 + (P^2 - 2)\sigma + \sigma_2] : \sigma \in \overline{D}\},$$

where $P = (1 + p^2)/p$, $0 < p < 1$.

The description of the Lemma can now be used to show Proposition 3.1.

**Proof of Proposition 3.1.** Put $t = P^2 > 4$ and write $f_t(z) = -t^{-1}[1 + (t - 2)z + z^2]$.

Then $\Omega_p = f_t(\overline{D})$ by Lemma 3.2. To show the monotonicity of $\Omega_p$, it is enough to prove that $f_t(\mathbb{D}) \subset f_{t'}(\mathbb{D})$ for $4 < t < t'$. We note that $f_t(z)$ is univalent for each $t > 4$. This is implied by the elementary fact that $f(z) = z + az^2$ is univalent (indeed, starlike) if and only if $|a| \leq 1/2$. Hence, $\gamma_t(\theta) = f_t(e^{i\theta})$, $0 \leq \theta \leq 2\pi$, gives a smooth parametrization of the boundary curve of the Jordan domain $\Omega_p$.

We first observe that $f_t(1) = -1$ is stationary with respect to $t$. In order to show that $f_t(\mathbb{D})$ is an increasing family of domains, it is enough to see that the flow $t \mapsto \gamma_t(\theta)$ is outgoing from $f_{t_0}(\mathbb{D})$ at the time $t = t_0$ for each $\theta \in (0, 2\pi)$. Since an outer normal vector of the boundary curve $\partial \Omega_p$ at $\gamma_t(\theta)$ is given by $\gamma_t'(\theta)/i$,
we should show that \(|\arg [i\gamma'(\theta)/\gamma'_{\Omega}(\theta)]| < \pi/2\), where \(\gamma_{\Omega}(\theta) = \partial \gamma(\theta)/\partial t\). By a straightforward computation, we obtain
\[
\text{Re} \left( \frac{\gamma'(\theta)}{\gamma(\theta)} \right) = \text{Re} \left( \frac{-t^{-1}(t - 2 + 2e^{i\theta})e^{i\theta}}{t^{-2}(1 - e^{i\theta})^2} \right) = \frac{t(t - 2 + 2 \cos \theta)}{4 \sin^2(\theta/2)} > 0
\]
for \(0 < \theta < 2\pi\). Thus we have shown the monotonicity of \(\Omega_{\rho}\) in \(p\). The other assertion easily follows from the facts that \(\lim_{t \to 4} f_t(z) = -(1 + z)^2/4\) and that \(\lim_{t \to \infty} f_t(z) = -z\).

Finally, we prove our main result.

**Proof of Theorem 1.2.** Considering \(\sigma_0 = t, \sigma_1 = -1\) and \(\sigma_2 = 0\) with \(t \in [0, 1]\) in Proposition 3.1 above, we obtain
\[
\Phi_{\rho}(t, -1, 0) = -18P \left[ 1 + (P^2 - 2)t + t^2 \right]
- 3 \left[ 1 - 7P^2 + 2P + (3P^2 - 2)t + t^2 \right] (1 - t^2)
+ P \left[ 2(1 - t^2) + 3t(P^2 - 1 + t) \right] (1 - t^2).
\]
Setting \(h_p(t) := -\Phi_{\rho}(t, -1, 0)/18P^3\) gives
\[
18P^3 h_p(t) = -(P + 3)t^4 - (3P^3 + 9P^2 - 3P - 6)t^3 - (6P^4 - 21P^2 - 17P)t^2
+ 3(7P^3 + 3P^2 - 13P - 2)t + (6P^4 - 21P^2 + 20P + 3)
\]
and
\[
18P^3 h'_p(t) = -4(P + 3)t^3 - 3(3P^3 + 9P^2 - 3P - 6)t^2
+ 2 \left( 6P^4 - 21P^2 - 17P \right) t + 3(7P^3 + 3P^2 - 13P - 2),
\]
which leads to \(h_p(1) = 1\) and \(h'_p(1) = -2(P - 2)(P + 1)/3P < 0\). Thus the function \(h_p(t)\) is strictly decreasing at \(t = 1\) and therefore \(h_p(t_0) > 1\) for \(t_0 < 1\) sufficiently close to 1. Hence, \(|H(f)| = h_p(t_0) > 1\) for the function \(f \in \mathcal{C}_p\) corresponding to the parameter triple \((t_0, -1, 0)\). Thus \(M(p) > 1\) follows.

To show the inequality \(M(p) > 1/3p\), we use the lower estimate
\[
M(p) \geq h_p \left( \frac{1}{3p} \right) = \frac{P}{3} + g \left( \frac{1}{3p} \right) = \frac{1}{3p} + g \left( \frac{1}{3p} \right),
\]
where
\[
g(x) = -\frac{7x}{48} + \frac{143x^2}{72} - \frac{121x^3}{128} - \frac{427x^4}{1152} + \frac{343x^5}{384} + \frac{5831x^6}{4608} - \frac{2401x^7}{1536}.
\]
We note that \(p/3 > 1/3P\). It is not difficult to see that \(1/3P + g(1/P) = x/3 + g(x) > 0\) for \(x = 1/P \in (0, 1/2)\). Therefore, \(M(p) > 1/3p\).

Finally, we show \(M(p) < (1 + 2p)/3p\). In view of (3.2), one can estimate \(\Phi_{\rho}\) as in
\[
|\Phi_{\rho}(\sigma_0, \sigma_1, \sigma_2)| \leq B_0 + B_1 x + B_2 x^2 + B_3 (1 - x^2)
= (B_2 - B_3)x^2 + B_1 x + B_0 + B_3
\]
with $x = |\sigma_1|$, where
\[
B_0 = 18P[1 + (P^2 - 2)y + y^2],
B_1 = 3[1 - 7P^2 + 2P^4 + (3P^2 - 2)y + y^2](1 - y^2),
B_2 = P[2(1 - y^2) + 3y(P - 1 + y)](1 - y^2),
B_3 = 3P(P^2 - 1 + y)(1 - y^2)
\]
with $y = |\sigma_0|$. Since
\[
[2(1 - y^2) + 3y(P^2 - 1 + y)] - 3P(P^2 - 1 + y) = (1 - y)(1 - y - 3P^2) \leq 0
\]
for $P > 2$, $0 \leq y \leq 1$, we have $B_2 - B_3 \leq 0$. Thus, we have
\[
|\Phi_p(\sigma_0, \sigma_1, \sigma_2)| \leq B_0 + B_1 + B_3 = 18P^3 + 6P^2 (P^2 - P - 2)2t - 3P(2P^3 + P^2 + 2P - 4)t^2 + 3(3P^2 + P + 2)t^3 - 3t^4,
\]
where $t = 1 - y \in [0, 1]$.

Using the inequalities $2t \leq 1 + t^2$ and $6t^3 - 3t^4 \leq 3t^2$ for $0 \leq t \leq 1$, we obtain
\[
B_0 + B_1 + B_3 \leq 6(P^4 + 2P^3 - 2P^2) + 3(-3P^3 - 6P^2 + 4P + 1)t^2 + 3P(3P + 1)t^3 =: G_p(t).
\]

The function $G_p(t)$ has a maximum at $t = 0$ (and a minimum at $t = 2(3P^3 + 6P^2 - 4P - 1)/(3P(3P + 1)) > 1$ for all $P > 2$).
Therefore we have
\[
\sup_{\sigma_0, \sigma_1, \sigma_2 \in \mathbb{D}} |\Phi_p(\sigma_0, \sigma_1, \sigma_2)| \leq \max_{0 \leq t \leq 1} G_p(t) = G_p(0) = 6P^2 (P^2 + 2P - 2),
\]
which implies according to Proposition 3.1
\[
M(p) \leq \frac{P^2 + 2P - 2}{3P} = \frac{1 + 2p}{3p} - \frac{p(1 - p^2)}{1 + p^2} < \frac{1}{3p} + \frac{2}{3p}
\]
This completes the proof. □

References

[1] F.G. Avkhadiev, Ch. Pommerenke and K.-J. Wirths, Sharp inequalities for the coefficients of concave schlicht functions, Comment. Math. Helv. 81 (2006), 801–807.
[2] F.G. Avkhadiev and K.-J. Wirths, Convex holes produce lower bounds for coefficients, Complex Variables 47 (2002), 553–563.
[3] F.G. Avkhadiev and K.-J. Wirths, Concave schlicht functions with bounded opening angle at infinity, Lobachevskii J. of Math. 17 (2005), 3–10.
[4] F.G. Avkhadiev and K.-J. Wirths, A proof of the Livingston conjecture, Forum Math. 19 (2007), 149–157.
[5] A. F. Beardon and D. Minda, A multi-point Schwarz-Pick Lemma, J. Anal. Math. 92 (2004), 81–104.
[6] B. Bhowmik, S. Ponnusamy and K.-J. Wirths, Domains of variability of Laurent coefficients and the convex hull for the family of concave univalent functions, Kodai Math. Journal 30 (2007), 385–393.
[7] K. H. Cho, S.-A Kim, and T. Sugawa, On a multi-point Schwarz-Pick lemma, Comput. Methods Funct. Theory 12 (2012), 483–499.
[8] P. L. Duren, Univalent Functions, Springer-Verlag, 1983.
[9] W. K. Hayman, *On the second Hankel determinant of mean univalent functions*, Proc. London Math. Soc. (3) **18** (1968), 77–94.

[10] R. Ohno, *Characterizations for concave functions and integral representations*, Topics in Finite or Infinite Dimensional Complex Analysis, Tohoku University Press, Sendai (2013), 203–216.

[11] R. Ohno and T. Sugawa, *Coefficient estimates of analytic endomorphisms of the unit disk fixing a point with applications to concave functions*, Preprint.

[12] Ch. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc. **41** (1966), 111–122.

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