Localized solutions of extended discrete nonlinear Schrödinger equation

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Abstract. We consider the extended discrete nonlinear Schrödinger (EDNLSE) equation which includes the nearest neighbor nonlinear interaction in addition to the on-site cubic and quintic nonlinearities. This equation describes nonlinear excitations in dipolar Bose-Einstein condensate in a periodic optical lattice. We are particularly interested with the existence and stability conditions of localized nonlinear excitations of different types. The problem is tackled numerically, by application of Newton methods and by solving the eigenvalue problem for linearized system near the exact solution. Also the modulational instability of plane wave solution is discussed.

1. Introduction and main equations

Study of nonlinear evolution equations and their solutions is a continuously expanding domain of modern applied mathematics with numerous applications in physics and engineering [1, 2]. One of these equations, the discrete nonlinear Schrödinger equation (DNLSE) [3] attracts much interest because of its wide applicability as a discrete lattice model in different fields of physics [3, 4, 5, 6]. Mathematically DNLSE is the system of \( n \) coupled ordinary differential equations for \( n \) complex variables. The DNLSE even in its simplest, standard form, which include nearest neighbor linear interaction and cubic (Kerr) on site nonlinearity is not integrable and no analytical solutions have been found for it, but because of its generality and importance for applications, considerable efforts were maid to investigate its properties and to find numerical and semianalitical solutions. In particular, the possibility of modulational instability of nonlinear discrete plane wave solutions were demonstrated [7]. Also different types of discrete localized states were found and their stability were investigated numerically and analytically [8, 3, 9]. The natural generalization of DNLSE were to include the higher order terms, which take into account nonlocality of nonlinearity, non-Kerr on site nonlinearities, etc. In this paper we consider the particular extended DNLSE (EDNLSE), where the nearest neighbor nonlinear interaction in addition to the on-site cubic and quintic nonlinearities are taken into account. It has the following form in normalized, dimensionless units,

\[
iu_{n,t} + \kappa (u_{n+1} + u_{n-1}) + g |u_n|^2 u_n + q (|u_{n+1}|^2 + |u_{n-1}|^2) u_n + \alpha |u_n|^4 u_n = 0, \tag{1}
\]
where $i$ is the imaginary unit, $u_n(t)$ is the complex dependent variable at site $n$, $n$ is discrete variable taking integer values, $t$ is evolution parameter (time), and $t$ in subscript means derivative with respect $t$, parameters $\kappa$ is for tunneling, $g$ for cubic nonlinearity, $q$ for nonlocal interaction and $\alpha$ for quintic nonlinearity. When $q = \alpha = 0$, the equation reduces to the standard DNLSE. When only $q = 0$, we have cubic-quintic DNLSE, and its solution were discussed in the papers [10, 11, 12]. In case of $\alpha = 0$ the reduced equation describes the dynamics of dipolar Bose-Einstein condensate in deep optical lattice, and the properties and solutions of this equation also were discussed in the literature [14, 15, 16]. The EDNLS we consider in this paper includes combined effect of quintic and nonlocal nonlinearity, in addition to the cubic nonlinearity. It can be applied to describe evolution of dipolar Bose-Einstein condensate in deep optical lattice [17, 18, 19], when both two body and three body interactions are important. Also, it can be useful to study light beam propagation in the system of nonlinearly coupled optical waveguides [20] when both cubic and quintic nonlinearities have to be taken into account. The EDNLS (1) has two conserved quantities, the norm

$$N = \sum_n |u_n|^2$$

and Hamiltonian

$$H = \sum_n \left[ \kappa(u_n^* u_{n+1} + u_n u_{n+1}^*) + \frac{g}{2} |u_n|^4 + \frac{\alpha}{3} |u_n|^6 + q|u_{n+1}|^2 |u_{n-1}|^2 \right].$$

These quantities can be applied to control a precision of numerical calculations. In the next sections we present the results on the modulational instability of nonlinear plane wave solution of EDNLS, also the existence and stability of nonlinear localized modes of Eq.(1) will be examined.

2. Modulational instability of plane wave solution

Let us start with the derivation of plane wave solution of Eq.(1). We first, insert the plane wave ansatz $u_n = u_0 e^{i(k n + \omega t)}$ into Equation (1) to obtain the nonlinear dispersion relation

$$\omega = 2\kappa \cos(k) + (g + 2q) u_0^2 + \alpha u_0^4. \quad (2)$$

Next, to study the stability of nonlinear plane wave, we set a perturbed solution in the form $u_n(t) = [u_0 + \delta u_n(t)] e^{i(k n + \omega t)}$ and once again substitute into Equation (1). Collecting linear terms in $\delta u_n$ only, gives the evolution equation for the perturbation

$$i \delta u_n + \left[2 (g + q) u_0^2 - \omega + 3 \alpha u_0^4\right] \delta u_n + \kappa \left( \delta u_{n+1} e^{i k} + \delta u_{n-1} e^{-i k} \right) + \left( q u_0^2 + 2\alpha u_0^3 u_0^1 \right) \delta u_n^* + g u_0^2 \left( \delta u_{n+1} + \delta u_{n-1} + \delta u_{n+1}^* + \delta u_{n-1}^* \right) = 0. \quad (3)$$

Furthermore, setting

$$\delta u_n(t) = u_1 e^{i(Q n + \Omega t)} + u_2 e^{-i(Q n + \Omega t)} \quad (4)$$

leads to the linear system

$$[-\Omega - \omega + a^+] u_1 + b u_2 = 0$$
$$b u_1 + [\Omega - \omega + a^+] u_2 = 0 \quad (5)$$

where $a^\pm = 2 (g + q) u_0^2 + 2\kappa \cos(Q \pm k) + 2 g u_0^2 \cos(Q) + 3 \alpha u_0^4$ and $b = q u_0^2 + 2 g u_0^2 \cos(Q) + 2 \alpha u_0^4$. This homogeneous system has a nontrivial solution if its determinant equal to zero; thus from the determinant yields

$$\Omega = \frac{1}{2} \left[ d \pm \sqrt{d^2 - 4b^2 + 4 (\omega - a^+) (\omega - a^-)} \right] \quad (6)$$

where $d = a^+ - a^-$. The instability gain is defined as $G = \text{Im} (\Omega)$ and the nonlinear plane wave experiences modulational instability when $G \neq 0$. 


3. Numerical simulations of nonlinear localized modes

The objective of this section is to investigate numerically the properties of localized solutions of EDNLSE (1). We assume that these solutions are placed at the center of long discrete lattice, so the effects of boundaries can be neglected. As we have mentioned in the previous section, the particular cases of the EDNLSE(1), when: a) $\alpha = q = 0$, b) only $q = 0$ and c) only $\alpha = 0$ have been studied previously. For all these cases it was shown that very diverse and reach families of localized solutions exist, for small values of $\kappa$, and we expect that the equation under consideration (1) also has many different types of localized solutions. In this work we consider only some of the possible solutions, namely those that can be continued from anti-continuum limit ($\kappa = 0$), when each site effectively is decoupled from others. We concentrate on three particular types of solutions: symmetric on site, symmetric inter site, and asymmetric on site strongly localized modes. Following the paper [11], values of coefficients $g$ and alpha are fixed as $g = 2$, $\alpha = -1$, so competing focusing cubic and defocusing quintic nonlinearities are considered, while the the coefficient $q$ is considered as a positive real parameter. The coefficient $\kappa$ will be continued from 0 up to 0.15. In our numerical calculations the range of $\mu$ we keep as $-50 \leq \mu \leq 50$. We are looking for localized solutions of EDNLSE (1) in a time periodic form

$$u_n(t) = a_n \exp(i\mu t),$$

where $a_n$ and $\mu$ are real numbers. Substitution of this anzatz to the Eq.(1) will reduce it to the following system of nonlinear algebraic equations.

$$\mu a_n = \kappa(a_{n+1} + a_{n-1}) + ga_n^3 + q(a_{n+1}^2 + a_{n-1}^2)a_n + \alpha a_n^5.$$  \hspace{1cm} (7)

Solutions of the Eq.(7) can be found numerically by Newton method if appropriate initial guess is given. The stability of the localized modes are checked by perturbing the solutions of Eq.(1) as

$$u_n = e^{i\mu t}(a_n^s + b_n e^{\lambda t} + c_n e^{\lambda t}),$$

where $a_n^s$ is a solution of Eq.(7) and the coefficients of perturbation $b_n, c_n$ are assumed to be small, and solving resulting linearized eigenvalue problem for $(b_n, c_n, \lambda)$

$$\lambda b_n = [\mu b_n + \kappa(b_{n+1} + b_{n-1}) + g(a_n^s)^2 c_n^s + 2|a_n^s|^2 b_n + q(|a_{n+1}^s|^2 b_n + |a_{n-1}^s|^2 b_n + a_n^s a_{n+1}^s c_{n-1}^s + a_n^s (a_{n+1}^s)^* b_{n+1} + a_n^s (a_{n-1}^s)^* b_{n-1}) + \alpha(2|a_n^s|^2 (a_n^s)^2 c_n^s + 3|a_n^s|^4 b_n)]$$ \hspace{1cm} (8)

$$\lambda c_n = [\mu c_n + \kappa(C_{n+1}^s + b_{n-1}^s) + g(a_n^s)^2 b_n + 2|a_n^s|^2 c_n^s + q(|a_{n+1}^s|^2 c_n^s + |a_{n-1}^s|^2 c_n^s + a_n^s a_{n+1}^s b_{n+1} + a_n^s (a_{n+1}^s)^* c_{n+1}^s + a_n^s (a_{n-1}^s)^* c_{n-1})^* + \alpha(2|a_n^s|^2 (a_n^s)^2 b_n + 3|a_n^s|^4 c_n^s)].$$ \hspace{1cm} (9)
The solution $a_n^*$ is unstable if the imaginary part of an eigenvalue $\lambda$ takes the negative values. To initiate the iteration in the Newton method, we substitute nonzero amplitudes at the central sites $n = 0, +1, -1$, and keep other amplitudes equal to zero. We begin with $\kappa = 0$ and then continue solution by increasing $\kappa$ with the step size $\Delta\kappa = 0.001$, and starting Newton iteration, with increased $\kappa$, with the solution that has been found in the previous cycle. By this way the procedure of continuation proceeds up to $\kappa = 0.15$. Simultaneously for each solution found the eigenvalue problem is solved and the linear stability is determined. The frequency is chosen as $\mu = 1$.

**Figure 2.** Left panel: Odd tall symmetric solution of Eq.(7) for $\mu = 1, \kappa = 0.15, g = 2, \alpha = -1, q = 0.15$. Right panel: Odd asymmetric solution of Eq.(7) for the values of parameters the same as in left panel.

**Figure 3.** Left panel: Even tall and even short symmetric solutions of Eq.(7) for $\mu = 1, \kappa = 0.15, g = 2, \alpha = -1, q = 0.12$. Right panel: Evolution of even short symmetric solution from the left panel perturbed by small noise. The values of parameters are the same as in left panel.
Let us describe now the results of numerical calculations. First we start with the odd (site centered) symmetric solutions. The iteration procedure begins by substitution of the initial guess in the following form \(a_n = (0, \ldots, 0, 1.2, 0, 0, \ldots, 0)\). The continuation and subsequent solution of linearized eigenvalue problem shows the the solutions exist and stable for values of \(q\) from \(q = 0\) up to \(q = 0.17\). We label that solutions as a tall, since another family (short) of site centered symmetric solutions exists, with maximal amplitude smaller than the first one, and can be found if initial guess is chosen as \(a_n = (0, \ldots, 0, 0.8, 0, \ldots, 0)\). These solutions are stable and exist even for large values of \(q\). We check up to \(q = 1\). The substitution \(a_n = (0, \ldots, 0, 0.6, 1.2, 0, 0, \ldots, 0)\) allows to get site-centered asymmetric solutions. The simulations show that these solutions also exist and linearly stable in the interval of \(0 \leq q \leq 0.17\). Examples of stable site centered tall symmetric and site centered asymmetric solutions shown in the Fig.2.

Now we proceed with even (inter site centered) symmetric solutions. These solutions can be found by substituting initial guess as \(a_n = (0, \ldots, 0, A, A, \ldots, 0, 0, \ldots, 0)\). Again tall and short solutions exist, for tall \(A = 1.2\), and for short \(A = 0.8\). Tall solutions are stable and exist for \(0 \leq q \leq 0.12\), while short one linearly unstable but exist for large values of \(q\), again we check up to \(q = 1\). The examples of tall and short even solutions are presented in the Fig.3, left panel. The evolution of unstable short even solution we check by perturbing it by small noise (the amplitude of noise \(\epsilon = 0.001\)) and solving with this initial conditions the Eq.(1) for some particular values of parameters. The result shown in the Fig.3, right panel. As one can notice, the instability develops quite fast, at \(t \simeq 20\), since in this case the imaginary part of \(\lambda\) is negative and its absolute value is relatively large \(\text{Im}(\lambda) = -0.4\). After instability starts the evolution becomes nonstationary, and the pulse oscillates between tall odd symmetric and asymmetric modes. Also one can observe that the small amplitude radiation propagates away from the main pulse.

4. Conclusions
The objective of this paper was to introduce the extended discrete nonlinear Schrödinger equation, which includes the cubic and quintic, as well as nonlocal nonlinearities. We have derived the analytical expression which describes the modulational instability of plane waves in this system. Also we have considered the case when the linear nearest neighbor interaction strength is small and analyzed numerically the existence and stability of strongly localized solutions with periodic time dependence. We have found two types, tall and short, of odd symmetric and two types, also tall and short even symmetric solutions. In addition the odd asymmetric solutions have been found. It was found that odd symmetric modes of both types, as well as odd asymmetric and tall even symmetric modes are linearly stable, and the existence range for these solutions were identified. It was shown than short even symmetric solutions are unstable, and the time evolution of perturbed unstable solution was investigated. In our opinion, the future work may include more detailed numerical investigation of Eq.(1), covering wider range of parameters and searching other types of localized solutions, as well as considering long time evolution of both stable and unstable solutions and their interaction with each other and with impurities. Also represents interest to develop semianalitical approach, which enables to explain, at least partially the obtained localized solutions.

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