An Explicit Model of Quark Mass Matrix

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Abstract

A physical model which describes the CKM matrix is analyzed. The elements of such a matrix are field-strength renormalization factors. Each column gives the probability amplitude for the field operators of the coupled Lagrangian to create a one-particle eigenfunction of definite energy. The total conserved charge is the sum of the flavor charges which are not conserved separately.

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I. INTRODUCTION

Although the Standard Model is a successful theory, its description of quark masses and their flavor mixing is uncertain. It is generally believed that physics beyond the Standard Model is necessary in order to acquire a complete understanding of the quark mass problem. In the absence of a satisfactory theory for describing the phenomenon of flavor mixing, a number of different parameterizations of the Cabibbo-Kobayashi-Maskawa matrix (CKM) [1,2], have been considered in the literature, see for example Refs. [3–9] and references there. For a detailed review of the different parameterizations see Ref. [10].

It is hoped that once the parameters of the CKM matrix will be known more precisely, they can give us some clue about the physics beyond the Standard Model. For example a given parameterization may be more useful than others in analyzing a given structure of a quark mass matrix. This because it may suggest different kinds of modifications when one tries to understand what is not yet understood.

In this paper an explicit model of the CKM rotation matrix is considered. This represents a somewhat different approach with respect to the previous ones and it should be considered as complementary to them. This model is obtained by the diagonalization of the Lagrangian

\[ \mathcal{L} = \bar{\psi}_{d'} (i\gamma \cdot \partial - m_{d'}) \psi_{d'} + \bar{\psi}_{s'} (i\gamma \cdot \partial - m_{s'}) \psi_{s'} + \bar{\psi}_{b'} (i\gamma \cdot \partial - m_{b'}) \psi_{b'} 
- (\delta \bar{\psi}_{d'} \psi_{s'} + \delta^* \bar{\psi}_{s'} \psi_{d'}) - (\varepsilon \bar{\psi}_{d'} \psi_{b'} + \varepsilon^* \bar{\psi}_{b'} \psi_{d'}) - (\eta \bar{\psi}_{s'} \psi_{b'} + \eta^* \bar{\psi}_{b'} \psi_{s'}), \]

where \( \psi_{d'}, \psi_{s'} \) and \( \psi_{b'} \) are quark flavor fields with masses respectively \( m_{d'}, m_{s'}, \) and \( m_{b'} \). The complex coupling constants \( \delta, \varepsilon \) and \( \eta \) describe flavor mixing.

The Lagrangian given by Eq. (1.1) describes quarks which are in a state of mixed flavors at any space time point. The rotation matrix is obtained by the diagonalization of the above Lagrangian. The elements of such a matrix are field strength renormalization factors and are functions of the quark masses \( m_{d'}, m_{s'}, \) and \( m_{b'} \) and their coupling constants. This point is certainly not obvious in the previous treatments and it gives some physical insight into the meaning of the CKM rotation matrix.
II. THE QUARK MASS MATRIX

1. Energy Eigenvalues

In order to diagonalize the Lagrangian given by Eq. (1.1) the procedure illustrated in ref. [11] is generalized to the three flavor case. For this case the following system of three coupled equations is diagonalized

\[ i \frac{\partial \psi_d'}{\partial t} = (\alpha \cdot \vec{p} + \beta m_d') \psi_d' + \beta \overline{\delta} \psi_{s'} + \beta \varepsilon \psi_{b'}, \]

\[ i \frac{\partial \psi_s'}{\partial t} = (\alpha \cdot \vec{p} + \beta m_s') \psi_s' + \beta \delta \overline{\psi_d'} + \beta \eta \overline{\psi_b'}, \]

\[ i \frac{\partial \psi_b'}{\partial t} = (\alpha \cdot \vec{p} + \beta m_b') \psi_b' + \beta \varepsilon \overline{\psi_d'} + \beta \eta \overline{\psi_s'}, \]

where \( \alpha \) and \( \beta \) are Dirac matrices. We notice here that the Hamiltonian is hermitian, therefore the eigenvalues are real. The energy eigenvalues are obtained by solving the cubic equation

\[ z^3 - z^2(3p^2 + h) + z(3p^4 + 2p^2h + l) - p^6 - p^4h - p^2l - f = 0, \quad (2.1) \]

where \( z = E^2 \), and the parameters \( h, f, l \) are given by

\[ h = m_d'^2 + m_s'^2 + m_b'^2 + 2(\overline{|\delta|^2} + \overline{|\varepsilon|^2} + \overline{|\eta|^2}), \]

\[ f = [r + m_d'm_{s'}m_{b'} - (m_d'|\eta|^2 + m_{s'}|\varepsilon|^2 + m_{b'}|\delta|^2)]^2, \]

\[ r = \varepsilon \delta^* \eta + \varepsilon^* \eta \delta, \quad (2.3) \]

\[ l = l_1 + l_2, \]

\[ l_1 = (\overline{|\delta|^2} - m_d'm_{s'})^2 + (\overline{|\varepsilon|^2} - m_d'm_{b'})^2 + (\overline{|\eta|^2} - m_{s'}m_{b'})^2, \]

\[ l_2 = 2[\overline{|\eta|^2}(\overline{|\varepsilon|^2} + m_d'^2) + \overline{|\varepsilon|^2}(\overline{|\delta|^2} + m_{s'}^2) + \overline{|\delta|^2}(\overline{|\eta|^2} + m_{b'}^2) - r(m_d' + m_{s'} + m_{b'})]. \]

Eq. (2.1) can be further simplified by the transformation

\[ z = E^2 = x + \frac{3p^2 + h}{3}, \quad (2.4) \]

which brings Eq. (2.1) to the simpler form

\[ x^3 + x(l - \frac{h^2}{3}) + \frac{hl}{3} - \frac{2h^3}{27} - f = 0. \quad (2.5) \]
If we assume

\[ a^3 + b^3 = \frac{hl}{3} - \frac{2h^3}{27} - f, \]
\[ ab = -\frac{1}{3}(l - \frac{h^2}{3}), \]  

the Eq. (2.5) becomes

\[ x^3 - 3abx + a^3 + b^3 = 0, \]

which has the roots

\[ -a - b, \quad -\omega a - \omega^2 b, \quad -\omega^2 a - \omega b, \]

where \( \omega \) is one of the complex cube roots of unity, i.e. \( \omega^3 = 1 \).

Then \( a^3 \) and \( b^3 \) are the roots of the quadratic equation

\[ y^2 - y \left( \frac{hl}{3} - \frac{2h^3}{27} - f \right) - \frac{1}{27} \left( l - \frac{h^2}{3} \right)^3 = 0, \]

with solutions

\[ y_{1,2} = \frac{1}{54} \left( -27f - 2h^3 + 9hl \pm 3\sqrt[3]{27f^2 + 4fh^3 - 18fhl - h^2l^2 + 4l^3} \right). \]  

We are looking for real eigenvalues, hence the discriminant in Eq. (2.10) must be negative. Therefore \( a^3 \) and \( b^3 \) are complex numbers, and in general they have to be found by using De Moivre's theorem. If we assume that \( a = m + in \) and \( b = m - in \) with

\[ m^3 - 3mn^2 = \frac{1}{54}(-27f - 2h^3 + 9hl), \]
\[ 3nm^2 - n^3 = \frac{1}{54} \left( 3\sqrt[3]{27f^2 + 4fh^3 - 18fhl - h^2l^2 + 4l^3} \right), \]

then the real roots of Eq. (2.5) are

\[ -2m, \quad m + n\sqrt{3}, \quad m - n\sqrt{3}. \]

Thus to find the three real roots of Eq. (2.5) we have to find the cube roots of two complex numbers. For practical purposes it is better to find one real root \( x_1 \) by a numerical method and then divide out the factor \( x - x_1 \) to obtain a quadratic equation.

The three energy eigenvalues can be formally written as

\[ E_{1,2,3}^2 = p^2 + m_{1,2,3}^2. \]
where the renormalized masses $m_{1,2,3}^2$ are given respectively by

$$
m_1^2 = \frac{h}{3} - 2m,
$$

$$
m_2^2 = \frac{h}{3} + m + n\sqrt{3},
$$

$$
m_3^2 = \frac{h}{3} + m - n\sqrt{3}.
$$

\[ (2.14) \]

2. Field-Strength Renormalization Constants

The energy eigenfunctions associated with the eigenvalues $E_{1,2,3}$ are of the type

$$
\psi_i = N_i \left( \begin{array}{c}
M_{i1} \\
M_{i2} \\
M_{i3}
\end{array} \right) \frac{1}{\sqrt{V}} \frac{u_i(s,p)}{\sqrt{2E_i}} e^{-iE_it} e^{ip\cdot x},
$$

\[ (2.15) \]

where $u_i$ is the Dirac spinor of mass $m_i$ and the coefficients $M_{i1,2,3}$ are given by

$$
M_{i1} = 1,
$$

$$
M_{i2} = \frac{C - m_i^2 D}{A - m_i^2 B},
$$

$$
M_{i3} = \frac{G + m_i^2 F - m_i^4}{A - m_i^2 B},
$$

\[ (2.16) \]

with

$$
A = \delta\eta(|\delta|^2 + |\eta|^2 - m_d m_s' - m_d m_v' - m_s m_v') + \varepsilon(m_s^2 (m_d' + m_v') + |\delta|^2 (m_d' - m_v') - |\eta|^2 (m_s' - m_d')) - \varepsilon^2 \delta^* \eta^* ,
$$

$$
B = \delta\eta + \varepsilon(m_d + m_s'),
$$

$$
C = \eta(m_d^2 (m_s' + m_v') + |\delta|^2 (m_d' - m_v') + |\varepsilon|^2 (m_s' - m_d'))
$$

$$
- \varepsilon^2 (m_d m_s' + m_d m_v' + m_s m_v' - |\delta|^2 - |\varepsilon|^2) - \delta^* \eta^* |\varepsilon|^2,
$$

$$
D = \varepsilon^* + \eta(m_s' + m_v'),
$$

$$
G = (m_d + m_s')(\delta\eta^* + \varepsilon\delta^* \eta^*) - (m_d m_s' - |\delta|^2)^2 - m_s^2 |\varepsilon|^2 - m_d^2 |\eta|^2 - |\delta|^2 |\varepsilon|^2 - |\delta|^2 |\eta|^2
$$

$$
F = m_d^2 + m_s^2 + 2|\delta|^2 + |\varepsilon|^2 + |\eta|^2.
$$

\[ (2.17) \]
\( N_i \) is the normalization constant given by

\[
N_i = \frac{|A - m_i^2 B|}{\sqrt{|A - m_i^2 B|^2 + |C - m_i^2 D|^2 + |G + m_i^2 F - m_i^4|^2}}. \quad (2.18)
\]

The eigenfunctions \( \psi_{1,2,3} \) represent states of given energy but of mixed flavors at any space time point. As discussed in ref. [11] in field theory the elements of the \( 3 \times 1 \) matrix in Eq. (2.15) are field-strength renormalization factors and give the probability amplitude for the field operators \( \psi_{d',s',b'} \) to create one-particle eigenfunctions of energy \( E_{1,2,3} \) and mixed flavors.

The rotation matrix between the flavor base and the mass base is given by these field-strength renormalization factors and it can be written as

\[
M = \begin{pmatrix}
N_1 & N_2 & N_3 \\
N_1 \frac{C-m_i^2 D}{A-m_i^2 B} & N_2 \frac{C-m_i^2 D}{A-m_i^2 B} & N_3 \frac{C-m_i^2 D}{A-m_i^2 B} \\
N_1 \frac{G+m_i^2 F-m_i^4}{A-m_i^2 B} & N_2 \frac{G+m_i^2 F-m_i^4}{A-m_i^2 B} & N_3 \frac{G+m_i^2 F-m_i^4}{A-m_i^2 B}
\end{pmatrix}, \quad (2.19)
\]

In the limit of only two flavor the rotation matrix reduces to

\[
U = \begin{pmatrix}
\frac{1}{\sqrt{1+M_1^2}} & \frac{1}{\sqrt{1+M_2^2}} \\
\frac{M_1}{\sqrt{1+M_1^2}} & \frac{M_2}{\sqrt{1+M_2^2}}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{1+M_1^2}} & \frac{M_1}{\sqrt{1+M_1^2}} \\
\frac{M_2}{\sqrt{1+M_2^2}} & -\frac{1}{\sqrt{1+M_2^2}}
\end{pmatrix}, \quad (2.20)
\]

where \( M_1 \) is given by

\[
M_1 = \frac{m_d - m_s'}{\delta} + \sqrt{(m_d - m_s')^2 + 4|\delta|^2} = - \frac{1}{M_2}. \quad (2.21)
\]

Therefore the matrix \( U \) is equivalent to the Cabibbo matrix.

Even if not manifestly, the matrix \( M \) is unitary, this because the wavefunctions given by Eq. (2.15) are orthonormal. The possibility of a quark rotation matrix without the assumption of unitarity has been discussed in [12,13]

The matrix \( M \) given by Eq. (2.17) also can describe neutrino flavor mixing, however only in the ultra-relativistic limit for the reasons discussed in [11].
III. SUMMARY

We have derived a physical representation of the CKM matrix. The conserved current is the sum of the three flavor currents and the elements of the rotation matrix are field-strength renormalization constants. Because of its simplicity it is hoped that this model can give some physical meaning to the CKM matrix especially in relation to its improvement.
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