The pricing formulas of compound option based on the sub-fractional Brownian motion model

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Abstract. Based on the underlying asset driven by a sub-fractional Brownian motion, the formulas of pricing call option on a call option and other three kinds of compound options are derived by risk neutral valuation method. They are similar to the results based on the standard Brownian motion model and the fractional Brownian motion model.

1. Introduction
It is well known that the Black-Scholes model has become the most popular method of option pricing and its generalized version has provided mathematically beautiful and powerful results in option pricing. Nevertheless, classical mathematical models of financial assets are far from perfect. One apparent problem exists in the Black-Scholes formulation, namely that financial processes are not Markovian in distribution. In fact, behavioral finance as well as empirical studies shows that there exists long-range dependence in stock returns and verifies that long-range dependence is one of the genuine features of financial markets. Behavioral finance also suggests that the return distributions of stocks are leptokurtic and have longer and fatter tails than normal distribution and there exists long-range dependence in stock returns [1, 2]. These features have some differences with the standard Brownian motion, which are in accordance with the sub-fractional Brownian motion (sub-fBm for short) [3].

The sub-fBm is a continuous zero mean Gaussian process with non-stationary of increments. It has long-range dependency property which makes the sub-fBm a plausible model in mathematical finance. Tudor [4] has proved that there is no arbitrage in the market under sub-fBm with self-financing strategies, and deduced the European option pricing formula under sub-fBm by using the technology of quasi-martingale theory.

A compound option is an exotic contract whose underlying asset is an option, thus have two strike prices and two exercise dates. There are four main types of compound option, namely, a call on a call, a call on a put, a put on a call, a put on a put. This option has widely been applied in many financial practices. In this paper, we will derive the pricing formula of a call on a call under sub-fBm by the method of risk neutral pricing when the Hurst parameter H is greater than 0.5, and it can derive the pricing formula of other compound options in the same manner.

This paper is organized as followed. The next section is for basic assumptions. In section 3, we study the formulas of compound options. The final section is for concluding remarks.
2. Basic assumptions
Firstly, we give the following assumptions. Consider a sub-fractional Black-Scholes market that has two investment possibilities:

(H1) A money market account:
\[ dM(t) = rM(t)\,dt, \]  
where \( r \) represent the constant riskless interest rate.

(H2) A stock whose price satisfies the equation:
\[ dS(t) = \mu S(t) + \sigma S(t) dB_H^b(t), \]  
where \( \mu \) is the instantaneous expected return rate, \( \sigma \) is volatility, and \([B_H(t), 0 < H < \frac{1}{2}]\) is sub-fBm with respect to the market measure \( P \). More properties for sub-fractional Brownian motion can be founded in [5, 6, 7].

(H3) No transaction cost and tax.

(H4) The market is complete.

In Reference [4], it showed that this market does not have arbitrage and is complete with Hurst index \( H \in \left(\frac{1}{2}, 1\right) \). Under the risk neutral measure (denoted by \( Q \)) we have that:
\[ dS(t) = rS(t) + \sigma S(t) dB_H^b(t), \]  
The solution of this stochastic differential equation is:
\[ S(t) = S(0) \exp \left\{ r (1 - 2^{-2H+2}) \sigma^2 t^{2H} + \sigma B_H^b(t) \right\}. \]  

Lemma 1. (sub-fractional risk neutral evaluation) The price at every \( t \in [0, T] \) of a bounded \( F_T^H \) - measurable contingent claim \( F \in L^2(P) \) is given by
\[ F(t) = e^{-r(T-t)} \mathbb{E}_t[F], \]  
where \( \mathbb{E}_t[\cdot] \) is the quasi-conditional expectation with respect to the risk neutral measure, which is defined by [4].

Proof. According to sub-fractional Clark-Ocone representation formula [4] and the portfolio strategy, the conclusion can be derived. The course of proof is similar to [8].

Lemma 2. Consider a European call option with maturity \( T \) and the strike price \( K \) written on the stock whose price process evolves as in equation (2). The value of this call option is known from [4] and is given by
\[ C(t, S(t)) = S(t) \Phi(y_1) - K \, e^{-(T-t)} \Phi(y_2), \]  

where
\[ y_1 = \frac{\ln \frac{S}{K} + r(T-t) + \sigma^2 \left(1 - 2^{2H-2}\right) \left(T^{2H} - t^{2H}\right)}{\sigma \sqrt{\left(2 - 2^{2H-1}\right) \left(T^{2H} - t^{2H}\right)}} \]  
and
\[ y_2 = y_1 - \sigma \left(2 - 2^{2H-1}\right) \left(T^{2H} - t^{2H}\right)^{1/2}. \]  

\( \Phi(x) \) is the cumulative standard normal distribution function.

Remark 1. The conclusion of Lemma 2 can be found in [4].

3. The formulas of compound options with the sub-fBm model
Let \( CC_{T_1} \) denote the price of the compound option (a call on a call) at time \( t \in [0, T_1] \) under sub-fBm, whose terminal payoff with strike price \( K \) is
\[ CC_{T_i} = \max \left( C(T, S(T)) - K, 0 \right), \]

where \( S(t) \) denotes the stock price at time \( t \) \((0 < t < T)\); \( C(t, S(t)) \) is the price value of a European call option at time \( t \); \( T_i \) \((0 < T_i < T)\) is the maturity of a call on a call.

In the following text, our main results are given.

**Theorem 1.** The value of the compound option (a call on a call) at time \( t \) is
\[
CC(t, S(t)) = S(t) e^{-r(T-t)} \Phi_2(d_1 + m_1, d_2 + m_2, \rho) - Ke^{-r(T-t)} \Phi(d_1) - Ke^{-r(T-t)} \Phi_2(d_1, d_2; \rho).
\]

where

\[
m_1 = \sigma \left[ (2 - 2^{-2H-1})(T_i^{2H} - T^{2H}) \right]^{1/2},
\]
\[
m_2 = \sigma \left[ (2 - 2^{-2H-1})(T^{2H} - T_i^{2H}) \right]^{1/2},
\]
\[
d_1 = \frac{\ln S(t) + r(T_i - t) - \frac{1}{2} m_1^2}{m_1},
\]

\( X \) is the solution of \( S(T_i) \) in the following equation:
\[
C(T, S(T)) = K_*,
\]
and \( C(T, S(T)) \) is given by lemma 2.

\[
d_2 = \frac{\ln S(t) + r(T_i - t) - \frac{1}{2} m_2^2}{m_2},
\]

\( \Phi_2(x, y; \rho) \) is the two-dimensional cumulative standard normal distribution function with correlation coefficient \( \rho \).

**Proof.** \( C(t, S(t)) \) denote the value of a European call option with time \( t \), according to lemma 2, we can obtain that
\[
C(T, S(T)) = S(T) \Phi(\bar{y}_1) - Ke^{-r(T-t)} \Phi(\bar{y}_2),
\]
where
\[
m_3 = \sigma \left[ (2 - 2^{-2H-1})(T^{2H} - T_i^{2H}) \right]^{1/2}, \bar{y}_1 = \frac{\ln S(T) + r(T - T_i) + \frac{1}{2} m_3^2}{m_3}, \bar{y}_2 = y_1 - m_3.
\]

Let \( C(T, X) = K_*, \) then \( X \) satisfies the following equation:
\[
X \Phi(\bar{y}_1) - Ke^{-r(T-t)} \Phi(\bar{y}_2) = K_*,
\]
and the exercise condition this compound option obviously is \( S(T_i) > X \).

\[
\bar{y}_1 = \frac{\ln X + r(T - T_i) + \frac{1}{2} m_3^2}{m_3}, \bar{y}_2 = y_1 - m_3.
\]

According to the principle of risk neutral valuation, the value of this compound option is the expected present value as follows:
\[ CC(t, S(t)) = E_i \left[ e^{-r(T-t)} \left( C(T_t, S(T_t)) - K_t \right) \right] + e^{-r(T-t)}E_r \left[ \left( C(T_t, S(T_t)) - K_t \right) \right] 1_A \]  \hspace{1cm} (17) \\
\[ = e^{-r(T-t)}E_i \left[ C(T_t, S(T_t)) \right] + K_t - e^{-r(T-t)}E_r \left[ 1_A \right] \]  \hspace{1cm} (18)

where \[ A = \{ S(T_t), S(T_t) > X \} \] .

Noted that
\[ S(T) = S(t) \exp \left\{ r(T-t) - \frac{1}{2} m_t^2 + \sigma (2 - 2^{2H-1}) \right\}^{1/2} \left[ B_h(T_t) - B_h(t) \right] \}, \]  \hspace{1cm} (19)

then
\[ A = \{ S(t) \} \]  \hspace{1cm} (20)

so we have that
\[ CC(t, S(t)) = e^{-r(T-t)}E_i \left[ C(T_t, S(T_t)) \right] 1_A - K_t e^{-r(T-t)} \Phi (d_1), \]  \hspace{1cm} (21)

and
\[ C(T_t, S(T_t)) = e^{-r(T-t)}E_i \left[ \left( S(T_t) - K_t \right) \right] 1_{\{S(T_t) > K_t\}} , \]  \hspace{1cm} (22)

where
\[ S(T) = S(T_t) \exp \left\{ r(T-t) - \frac{1}{2} m_t^2 + \sigma (2 - 2^{2H-1}) \right\}^{1/2} \left[ B_h(T_t) - B_h(T_t) \right] \}, \]  \hspace{1cm} (23)

then the following equation can be derived, that is
\[ C(T_t, S(T_t)) = e^{-r(T-t)}E_i \left[ \left( S(T_t) - K_t \right) \right] 1_{\{S(T_t) > K_t\}} , \]  \hspace{1cm} (24)

where
\[ B = \{ (S(T_t), S(T_t)) \} \]  \hspace{1cm} (25)

and
\[ D(T_t) = -\frac{B_h(T_t) - B_h(t)}{T^{2H-1}} \], \hspace{1cm} (26)

we get that
\[ CC(t, S(t)) = e^{-r(T-t)}E_i \left[ \left( S(T_t) - K_t \right) \right] - K_t e^{-r(T-t)} \Phi (d_1) - K_t e^{-r(T-t)} \Phi_2 (d_1, d_2 ; \rho) . \]  \hspace{1cm} (27)

Then we can obtain the conclusion of Theorem 1.

By the similar argument, we may obtain the results on the other compound options ( a call on a put, a put on a call and a put on put, respectively ).

**Theorem 2.** The value of the compound option ( a put on a call ) at time \( t \) is
\[ PC(t, S(t)) = Ke^{-r(T-t)} \Phi_2 (-d_1, d_2 ; \rho) + Ke^{-r(T-t)} \Phi (-d_1) - S(t) e^{-r(T-t)} \Phi_2 (-d_1, m_1, d_2 + m_2 ; \rho). \]  \hspace{1cm} (28)

**Theorem 3.** The value of the compound option ( a call on a put ) at time \( t \) is
\[ CP(t, S(t)) = -S(t) e^{-r(T-t)} \Phi_2 (-d_1 - m_1, d_2 + m_2 ; \rho) + Ke^{-r(T-t)} \Phi (-d_1) + Ke^{-r(T-t)} \Phi_2 (-d_1, d_2 ; \rho). \]  \hspace{1cm} (29)

**Theorem 4.** The value of the compound option ( a put on a put ) at time \( t \) is
\[ PP(t, S(t)) = S(t) e^{-r(T-t)} \Phi_2 (d_1, d_2 + m_1 ; \rho) + Ke^{-r(T-t)} \Phi (d_1) - Ke^{-r(T-t)} \Phi_2 (d_1, d_2 ; \rho). \]  \hspace{1cm} (30)
where $d_1$, $d_2$, $m_1$, $m_2$, $\rho, \Phi$ and $\Phi_2$ are given by the above.

4. Summary
In this paper, we use the method of risk neutral valuation to obtain the formulas of four types of compound options in the sub-fractional Brownian motion model. They could be regarded by the generalization of the existing results under the standard fractional Brownian motion. The approach is also suitable for other types of options, such as European exchange options, better-of options et al.

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