Existence of Mild Solutions for a Class of Impulsive Hilfer Fractional Coupled Systems

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The aim of this paper is to give existence results for a class of coupled systems of fractional integrodifferential equations with Hilfer fractional derivative in Banach spaces. We first give some definitions, namely the Hilfer fractional derivative and the Hausdorff measure of noncompactness and the Sadovskii’s fixed point theorem.

1. Introduction

Fractional differential equations have been a good tool in many research areas in the last decade, such as engineering, mathematics, physics, and many other sciences [1, 2]. For some basic results on this theory, we refer the readers to the papers [3, 4] and the references therein.

There are many different definitions of fractional derivatives, each one with its importance and application, which helped justify the importance of fractional calculus. We mention here a few of the most notable definitions of fractional derivatives: Hadamard, Caputo–Hadamard, Hilfer, ψ-Hilfer, Caputo–Riesz, Grünwald–Letnikov, for more details we refer the readers to [5–9].

Recently, a lot of attention has been devoted to the existence of fractional differential problems with Hilfer fractional derivative, see [10–12]. The Hilfer fractional derivative, which is a generalization of the Riemann-Liouville fractional derivative, was introduced by nonother than Hilfer [1, 13].

The first results on the existence of general value problems involving Hilfer fractional derivative were investigated in [14] and after that in [12]. Following these results, Gu and Trujilo [15] gave the existence of solutions for fractional differential equations with Hilfer fractional derivative using the notion of measure of noncompactness. These equations are widely employed in the biomedical field.

On the other hand, the concept of noninstantaneous impulses was first introduced in [16] by Hernandez; these conditions appeared in the mathematical description of problems that experience abrupt changes during their evolution in time. In the established works, fractional differential equations (FDEs) involving Caputo’s fractional derivative are commonly considered with impulsive conditions for obtaining mild solutions [17–20]. However, in [21], Sousa obtained for the first time the mild solutions for Hilfer fractional differential equations with noninstantaneous impulses. To the best of our knowledge, there are few papers dealing with coupled systems, and on top of that, even fewer existence results for neutral Hilfer fractional differential equations. That is why, to make a little contribution to the already existing results, we consider in this paper a class of coupled systems of Hilfer fractional differential equations with not instantaneous impulses in a Banach space as follows:
where for \( j = 1, 2, l_0^{-1/\alpha_j} \) are the RL fractional integrals, \( H D^\beta_{\alpha_j} \) are the Hilfer fractional derivatives of order \( (\beta, \gamma_j) \) with

\[
0 \leq \beta \leq 1, 0 \leq \gamma_j \leq 1, \quad \text{and} \quad 0 \leq \alpha_j = \beta_j + \gamma_j - \beta_j \gamma_j \leq 1.
\]

The linear operators \( A_j : D(A_j) \subset X \rightarrow X \) are the infinitesimal generators of strongly continuous semigroups \( \{ T_j(t) \}_{t \geq 0} \) in a Banach space \( X \), \( 0 = t_0 < t_1 < t_2 < \cdots < t_i < t_{i+1} = T \) is a partition of \([0, T]\), \( T > 0, i = 1, 2, \ldots, m \).

The functions \( f_j : [0, T] \times X \times X \rightarrow X \) and \( g_j : [0, T] \times X \times X \times X \rightarrow X \) are satisfying some assumptions that will be given later, and the functions \( m_j : (t_i, s_i] \times X \rightarrow X \) and \( n_j : (t_i, s_i] \times X \rightarrow X \) characterize the impulsive conditions and \( x_{0_j} \in X \). The conditions on \( u_j : [0, T] \times [0, T] \rightarrow \mathbb{R} \) and \( \varphi_j : [0, T] \times [0, T] \times X \times X \rightarrow X \) are given in a later part.

This paper is organized as follows: we first give some preliminaries and notions that will be used throughout the work; after that, we will establish the existence results by means of the fixed point theory; last but not least, we will give an example that illustrates the ideas.

### 2. Preliminaries and Notations

Let \( C(J, X) \) be the complete normed linear space of all continuous functions \( x(t) \) defined on the interval \( J = [0, T] \) with

\[
\|x(t)\| = \sup_{t \in J} |x(t)|.
\]

We define the Banach space \( C_{1-a}(J, X) \)

\[
C_{1-a}(J, X) = \left\{ x(t) : J = [t_i, t_{i+1}] \subset \mathbb{R}, \lim_{t \to t_j} (t - t_j)^{1-a} x(t) \text{ exists} \right\}, \tag{2}
\]

for \( i = 1, 2, \ldots, m \) with the norm \( \|x\|_{C_{1-a}} = \sup_{i \in J} \|x(t)\| \).

The space \( X = PC_{1-a} \times PC_{1-a} \) equipped with the norm

\[
\| (x_1, x_2) \| = \max \{ \|x_1\|_{PC_{1-a}}, \|x_2\|_{PC_{1-a}} \}
\]

is also a Banach space.

By \( L(X) \), we denote the family of bounded linear operators defined on \( X \), and for \( j = 1, 2, \{ R_{\beta_j, \gamma_j}(t) \}_{t \geq 0} \) are the \( (\beta_j - \gamma_j) \)-resolvent operators generated by \( A_j \).

**Definition 1** (see [1]). The Hilfer fractional derivative of order \( n - 1 \leq \beta < n, n \in \mathbb{N} ; 0 \leq \gamma \leq 1 \), with lower limit \( a \) is defined as follows:

\[
D^\beta_{a} f(t) = I^\beta_{a} \frac{d}{dt} I^{(n-\beta)(\alpha-\gamma)}_{a} f(t) = I^\beta_{a} f(t) D^{\alpha-\beta \gamma}_{a} f(t), \tag{3}
\]

where \( I^\beta_{a} f(t) \) is the RL integral, and \( D^{\alpha-\beta \gamma}_{a} f(t) \) is the RL derivative.

**Lemma 2.** (see [14]). Let \( 0 < \beta < 1, 0 \leq \gamma \leq 1 \), and \( \alpha = \beta + \gamma - \beta \gamma \). If \( f \in C_{1-a}[a, b] \) is such that \( D^\beta_{a} f \in C_{1-a}[a, b] \), then

\[
I^\beta_{a} D^\beta_{a} f = I^\beta_{a} D^\beta_{a} f \text{ and } D^\beta_{a} I^\beta_{a} f = D^\beta_{a} f. \tag{4}
\]

**Lemma 3.** (see [3]). Let \( 0 < \beta < 1 \) and \( 0 \leq \alpha \leq 1 \). If \( f \in C_{1-a}[a, b] \), then

\[
I^\beta_{a} D^\alpha_{a} f(x) = f(x) - \frac{I^\beta_{a} f(a)}{I^\beta_{a} f}(x - a)^{\beta-1}, \forall x \in (a, b). \tag{5}
\]

**Definition 4.** (see [22]). The Hausdorff measure of noncompactness on a bounded subset \( \Omega_x \) of Banach space \( X \) is the mapping \( \mu : \mathcal{B} \subset \Omega_x \rightarrow [0, \infty) \) defined by

\[
\mu(\mathcal{B}) = \inf \{ \varepsilon > 0 : \mathcal{B} = \cup_{i=1}^{m} \mathcal{B}_i \text{ with radius of } \mathcal{B}_i \leq \varepsilon \text{ for } i = 1, 2, \ldots, m \}. \tag{6}
\]

We are going to look back on some properties of the measure of noncompactness.
Lemma 5 (see [22, 23]). The measure of noncompactness μ defined on bounded subsets \( A \) and \( B \) of a Banach space \( X \) has the properties:

1. \( \mu(A) = 0 \) if and only if \( A \) is a relatively compact set
2. \( A \subset B \) implies that \( \mu(A) \leq \mu(B) \)
3. \( \mu(A \cup B) = \max \{ \mu(A), \mu(B) \} \)
4. \( \mu(A) = |A| \mu(\mathcal{A}), \forall \mathcal{A} \in \mathcal{R} \)

Lemma 6 (see [22, 23]). For a bounded set \( D \subset X \), there is a countable set \( D_0 \subset D \) such that \( \mu(D_0) \leq \mu(D) \).

Lemma 7 (see [24]). For a bounded and equicontinuous function \( \mathcal{F} \in C(J, X) \), the Hausdorff measure of noncompactness \( \mu(\mathcal{F}(t)) \) is continuous on \( J \), and \( \mu(\mathcal{F}) = \max_{t \in J} \mu(\mathcal{F}(t)) \).

Lemma 8 (see [23]). Let \( D = \{ x_i \} \subset C(J, X) \) be a bounded countable subset of \( X \). Then, \( \mu(D(t)) \) is Lebesgue integrable on \( X \), and

\[
\mu \left( \int_{x_n(t)dt : n \in \mathbb{N}} \right) \leq \int \mu(x_n(t))dt.
\]

Lemma 9. (see [21]). We apply lemmas 2 and 3; then, we obtain an equivalent system of equations to the system 1 as follows:

\[
\begin{align*}
x_1(t) &= \left\{ \begin{array}{ll}
\frac{p-1}{\Gamma(\alpha_1)} \int_0^t (t-s)^\beta-1 \left( A_1 x_1(s) + g_1(s, x_1(s), x_2(s)) \right) ds, & t \in (0, t_1], \\
m_1(t, x_1(t)), & t \in (t_1, s_i), i = 1, 2, \ldots, m
\end{array} \right. \\
\quad + \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^\beta-1 \left( A_2 x_1(s) + g_2(s, x_1(s), x_2(s)) \right) ds, & t \in (s_i, t_1), i = 1, 2, \ldots, m, \\
m_1(t, x_1(t)), & t \in (t_1, s_i), i = 1, 2, \ldots, m
\end{align*}
\]

Remark 10. The Laplace transform of the Hilfer fractional derivative of a function \( f(t) \) of order \( 0 < \beta < 1 \) and \( 0 < \gamma < 1 \) is given in [13] by

\[
L\left\{ D^{\beta, \gamma}_{\alpha}f(t) \right\}(s) = s^{\beta-1} L\left\{ f(t) \right\}(s) - s^{(\beta-1)(1-\gamma)} f^{(1-\gamma)}(0^+),
\]

where \( f^{(1-\gamma)}(0^+) \) is the Riemann-Liouville derivative of order \( (1 - \beta)(1 - \gamma) \).

Now, we give a definition of a pair of mild solution to the problem 1, which is obtained by applying the Laplace transform of the Hilfer fractional derivative.

Definition 11 (see [15, 21]). A pair \( (x, y) \in PC_{1-a_1}(J, X) \times PC_{1-a_2}(J, X) \) is said to be a pair of mild solutions of the system (1) if the couple \( (x, y) \) satisfies the following coupled system:

\[
\begin{align*}
x_1(t) &= \left\{ \begin{array}{ll}
\frac{p-1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\beta-1} A_1 x_1(s) ds + \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta-1} A_2 x_1(s) ds + \int_0^t g_1(s, x_1(s), x_2(s)) ds, & t \in (0, t_1], \\
m_1(t, x_1(t)), & t \in (t_1, s_i), i = 1, 2, \ldots, m
\end{array} \right. \\
\quad + \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta-1} A_2 x_1(s) ds + \int_0^t g_2(s, x_1(s), x_2(s)) ds, & t \in (s_i, t_1), i = 1, 2, \ldots, m, \\
m_1(t, x_1(t)), & t \in (t_1, s_i), i = 1, 2, \ldots, m
\end{align*}
\]

\[
\begin{align*}
x_2(t) &= \left\{ \begin{array}{ll}
\frac{p-1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\beta-1} A_1 x_1(s) ds + \frac{1}{\Gamma(\beta_2)} \int_0^t (t-s)^{\beta-1} A_2 x_1(s) ds + \int_0^t g_3(s, x_1(s), x_2(s)) ds, & t \in (0, t_1], \\
m_2(t, x_1(t)), & t \in (t_1, s_i), i = 1, 2, \ldots, m
\end{array} \right. \\
\quad + \frac{1}{\Gamma(\beta_2)} \int_0^t (t-s)^{\beta-1} A_2 x_1(s) ds + \int_0^t g_4(s, x_1(s), x_2(s)) ds, & t \in (s_i, t_1), i = 1, 2, \ldots, m, \\
m_2(t, x_1(t)), & t \in (t_1, s_i), i = 1, 2, \ldots, m
\end{align*}
\]
for $j = 1, 2$, we have

$$R_{j, t}(v) = I_{
u}^{(1 - \beta_j)} S_{j, t}(v),$$

$$S_{j, t}(v) = t^{\alpha_j} Z_{j, t}(v),$$

$$Z_{j, t}(v) = \int_0^\infty \beta_j v^\nu W_{j, t}(v) T_j \left( t^{\beta_j} v \right) dv,$$

where $W_{j, t}(v)$ are the Wright functions defined as follows:

$$W_{j, t}(v) = \sum_{n=0}^\infty \left( -v \right)^{n-1}, 0 < \mu_j < 1, v_j \in C,$$

and satisfying

$$\int_0^\infty v^\rho W_{j, t}(v) dv = \frac{\Gamma(1 + \sigma_j)}{\Gamma(1 + \beta_j \sigma_j)}, v_j, \sigma_j \geq 0.$$  \hspace{1cm} (13)

From [15, 25], we can assume that for $j = 1, 2$

(i) The linear operators $\{ W_{j, t}(v) \}_{t > 0}$ and $\{ S_{j, t} \}_{t > 0}$ are strongly continuous and verify:

$$\| W_{j, t}(v) \| \leq \frac{M_j v^\sigma_j - 1}{\Gamma(\sigma_j)} \quad \text{and} \quad \| S_{j, t} \| \leq \frac{M_j v^{\beta_j - 1}}{\Gamma(\beta_j)} \quad \text{for} \ t > 0,$$

(ii) The norm continuity of the family $\{ T_j(t) \}$ for $t > 0$

3. Existence Results

In this section, we make some assumptions that are necessary to obtain our results:

$(H_1)$ For $j = 1, 2$, the functions $f_j : J \times X \times X \to X$ are bounded and Lipschitz continuous, that is, there exist $L_{f_j} > 0$ and $M_{f_j} \in (0, 1)$ such that

$$\| f_j(t, u, v) \| \leq L_{f_j},$$

$$\| f_j(t, u_1, v_1) - f_j(t, u_2, v_2) \| \leq M_{f_j} \| u_1 - u_2 \| + N_{f_j} \| v_1 - v_2 \|.$$  \hspace{1cm} (17)

$(H_2)$ For $j = 1, 2$, the functions $g_j : J \times X \times X \times D(\mathcal{B}) \times X$ are Caratheodory, that is, $g_j(t, x_1, x_2, x_3) : J \to D(\mathcal{B}) \times X$ is measurable for all $(x_1, x_2, x_3) \in X \times X \times X$, $g_j(t, x_1, x_2, x_3) : J \to D(\mathcal{B}) \times X$ is continuous a.e. for $t \in J$, and there exist $\psi_1, \psi_2 \in L_1(t, \mathbb{R}^+), 1/\delta > 1$, and a continuous function $\psi_3$ such that

$$\| g_j(t, x_1, x_2, x_3) \| \leq \psi_j(t) \| x_1 \| + \psi_{j, \beta}(t) \| x_2 \| + \psi_{j, \beta}(t) \| x_3 \|.$$  \hspace{1cm} (18)

for almost all $t \in J$.

$(H_3)$ For $j = 1, 2$, there exist functions $\tilde{\psi}_1, \tilde{\psi}_2 \in L_1(t, \mathbb{R}^+)$ and constants $M_{g_1}, M_{g_2} > 0$ such that

$$\| g_j(t, D_1, D_2) \| \leq M_{g_1} \tilde{\psi}_1 \mu(D_1) + M_{g_2} \tilde{\psi}_2 \mu(D_2), t \in J,$$

for any bounded, equicontinuous, and countable sets $D_i \subset X, i = 1, 2$.

$(H_4)$ The impulsive functions $m_j, n_j : [t_i, t_j] \times X \to X$ are Lipschitz continuous, that is, there exist $K_{m, j}, K_{n, j} > 0, i = 1, 2, \ldots, m$, such that for all $x, y \in X$, we have:

$$\| m_j(t, x) - m_j(t, y) \| \leq K_{m, j} \| x - y \|,$$

and

$$\| n_j(t, x) - n_j(t, y) \| \leq K_{n, j} \| x - y \|.$$  \hspace{1cm} (19)

$(H_5)$ For $j = 1, 2$, $\psi_j(t, s, \cdot, \cdot) : X \to X$ are caratheodory functions, and there exist $\xi_j : [0, T] \times [0, T] \to \mathbb{R}$ with

$$\xi_j^* = \sup_{t \in J} \left\{ \frac{\int_0^t \xi_j(t, s) ds + \xi_j(t, t)}{t - s} \right\} \leq \int_0^T \xi_j(t, s) \, ds < \infty$$

such that

$(H_6)$ For $j = 1, 2$, and for any bounded set $D_1 \subset X$ and $0 \leq s \leq t \leq T$, there exist functions $\tilde{\psi}_j : [0, T] \times [0, T] \to \mathbb{R}$ such that

$$\mu(\psi_j(t, s, D_1) \leq \tilde{\psi}_j(t, s) \mu(D_1) \quad \text{where} \quad \tilde{\psi}_j^* = \sup_{t \in J} \int_0^t \tilde{\psi}_j(t, s) ds < \infty.$$  \hspace{1cm} (20)

$(H_7)$ For $j = 1, 2, u_j^* = \sup \{ u_j(t, s) : 0 \leq s \leq t \}$ is bounded and measurable on $J$ along with the continuity of $u_j : J \to L_1(t, \mathbb{R}^+)$ defined by $u_j(t) = u_j(t, s).$

Theorem 12. The system (1) has a pair of solutions $(x_j(t), x_2(t))$ in the space $X = PC_{C_1, C_2}$ if the assumptions $(H_1)-(H_7)$ hold and the following conditions are verified, for $j = 1, 2$:

$$M_j \left[ K_j \frac{T^{1 - \gamma} \delta^\gamma}{\Gamma(\beta_j)} + T^{1 - \delta} \frac{1 - \delta}{\beta_j - \delta} \left( \| \psi_{j, \beta} \| + \| \psi_{j, \beta} \| \right) \right] + \left( \left| \beta_j \right| \frac{T^{1 - \delta} \delta^\delta}{\Gamma(\beta_j)} \right)^{1/q} < 1,$$
\[
\max_{j=1,2} \left\{ M_{fj} + M_{cj} + N_{fj} \right\} + \max_{j=1,2} \left( \frac{M_{j}^{T} \beta_{i}}{\beta_{i} - \delta} \right)^{1/3} \cdot \left( 1 - \delta \right)^{1/3} = \left( \frac{M_{j}^{T} \beta_{i}}{\beta_{i} - \delta} \right)^{1/3} \cdot \left( 1 - \delta \right)^{1/3} < 1
\]

(22)

Note that  \( K_{j} = K_{mi} \) and  \( K_{2} = K_{ni} \).

\[
P(x_{1},x_{2})(t) = \begin{cases}
R_{b_{j},a_{j}}(t)x_{01} - f_{1}(t,x_{1}(t),x_{2}(t)), t \in (0,t_{1}), \\
mg_{j}(t,x_{1}(t)), t \in (t_{1},s_{1}]), i = 1, 2, \ldots, m, \\
R_{b_{j},a_{j}}(t)m_{j}(s_{j},x_{1}(s_{j})) - f_{1}(t,x_{1}(t),x_{2}(t)), t \in (s_{j},t_{i+1}), i = 1, 2, \ldots, m,
\end{cases}
\]

(23)

\[
Q(x_{1},x_{2})(t) = \begin{cases}
R_{b_{j},a_{j}}(t)x_{02} - f_{2}(t,x_{1}(t),x_{2}(t)), t \in (0,t_{1}), \\
n_{j}(t,x_{2}(t)), t \in (t_{1},s_{1}]), i = 1, 2, \ldots, m, \\
R_{b_{j},a_{j}}(t)n_{j}(s_{j},x_{2}(s_{j})) - f_{2}(t,x_{1}(t),x_{2}(t)), t \in (s_{j},t_{i+1}), i = 1, 2, \ldots, m,
\end{cases}
\]

(24)

by splitting both (23) and (24), we have:

\[
P_{1}(x_{1},x_{2})(t) = \begin{cases}
\int_{s_{j}}^{t_{1}} S_{p_{j}}(t-s)g_{j}(s,x_{1}(s),x_{2}(s))ds + \int_{s_{j}}^{t} u_{j}(s-t)\varphi_{j}(s,t,x_{1}(t),x_{2}(t))d\tau, t \in (0,t_{1}), \\
R_{b_{j},a_{j}}(t)m_{j}(s_{j},x_{1}(s_{j})) - f_{1}(t,x_{1}(t),x_{2}(t)), t \in (s_{j},t_{i+1}), i = 1, 2, \ldots, m,
\end{cases}
\]

(25)

\[\text{Proof.} \text{ To prove the existence of solutions for system (1), we only have to prove the existence of solutions for the system (11) and (12) because they are equivalent.}
\]

Let us define  \( \Omega = \{ (x_{1},x_{2}) \in PC_{1-a_{j}}(J,X) \times PC_{1-a_{j}}(J,X) \} \), where  \( \| (x_{1},x_{2}) \| \leq r \) with fix radius  \( r, \Omega \) is a nonempty closed convex bounded subset of  \( X \).

Define the operator  \( S : X \to X \) such that  \( S(x_{1},x_{2})(t) = (P(x_{1},x_{2})(t), Q(x_{1},x_{2})(t)) \), where
The operator $S$ is continuous on the intervals $[t, s]$ and $[s, t]$, and similarly, we prove the continuity of $t^{1-a_i}Q(x_1, x_2)\left(t\right)$. To show that the operator $S$ is continuous on the intervals $[t, s]$, we use the continuity of noninstantaneous impulsive functions $m_i(x, x_i(t))$ and $n_i(t, x_2(t))$. Thus, we conclude that $S(x_1, x_2) \in \mathcal{X}$.}

\textbf{Step 2.} We show that $S : \Omega_r \to \Omega_r$, that is, $S(x_1, x_2) \in \Omega_r$, for $(x_1, x_2) \in \Omega_r.$

We first show that the operator $P$ is bounded, which means, $\|P(x_1, x_2)\| \leq M \|x_1\| + M \|x_2\|$, for $(x_1, x_2) \in \Omega_r$. Suppose the opposite, so there exist $(x_1, x_2) \in \Omega_r$ and $t \in I$ such that $\|P(x_1, x_2)\| \geq r$.

For $t \in [0, t_1]$, we have

\begin{align*}
\|P(x_1, x_2)\| &= \|t^{1-a_i}P(x_1, x_2)(t)\| \\
& \leq M \|x_1\| + M \|x_2\|.
\end{align*}

We make the substitution $t - s = s_1$ in the third and fifth terms, we get

\begin{align*}
\|P(x_1, x_2)(t)\| &\leq M \|x_1\| + M \|x_2\| \\
& + \int_{t_1}^{t} t^{1-a_i} \|P(x_1, x_2)(t)\| ds + \int_{t_1}^{t} t^{1-a_i} \|P(x_1, x_2)(t)\| ds,
\end{align*}

which implies that

\begin{align*}
\|P(x_1, x_2)(t)\| &\leq M \|x_1\| + M \|x_2\| \\
& + M \int_{t_1}^{t} t^{1-a_i} \|P(x_1, x_2)(t)\| ds + M \int_{t_1}^{t} t^{1-a_i} \|P(x_1, x_2)(t)\| ds,
\end{align*}

For $t \in [t_1, s]$ and $(s, t_1)$, we use the continuity of noninstantaneous impulsive functions $m_i(x, x_i(t))$ and $n_i(t, x_2(t))$. Thus, we conclude that $S(x_1, x_2) \in \mathcal{X}$. 

\begin{align*}
\|P(x_1, x_2)(t)\| &\leq M \|x_1\| + M \|x_2\| \\
& + M \int_{t_1}^{s} t^{1-a_i} \|P(x_1, x_2)(t)\| ds + M \int_{t_1}^{s} t^{1-a_i} \|P(x_1, x_2)(t)\| ds.
\end{align*}
which implies that

$$
\|t^{1-a_1}P(x_1, x_2)(t)\| \leq K_m r + V. \tag{32}
$$

For $t \in (s, t_{r+1})$,

$$
\|t^{1-a_1}P(x_1, x_2)(t)\| = \left| t^{1-a_1} \left[ \int_{s}^{t} S_{p}(t-s)A_1f_1(s, x_1(s), x_2(s))ds \right] \right| 
onumber
$$

$$
+ \left| \int_{s}^{t} S_{p}(t-s)A_1f_1(s, x_1(s), x_2(s))ds \right| \leq M_1 \left( \frac{K_m r + V}{\Gamma(\alpha_1)} \right) + L_{f_1} \nonumber
$$

$$
+ M_1 L_{f_1} \|A_1\| \|t-s\|^{\beta_1} \beta_1 

$$

\[ \frac{1}{\Gamma(\beta_1 + 1)} \right]\nonumber

\[ + \frac{r M_{1}^{\gamma_{1-a_1}}}{\Gamma(\beta_1)} \left[ \frac{1 - \delta}{\beta_1 - \delta} \right]^{1-\delta} T^{\beta_1-\delta} \|\psi_{11}\| \|L_{1_t}^{1-\delta} \left( \frac{\psi_{13}}{\beta_1} \right)^{k_{1}^{*}} \nonumber
$$

\[ + \frac{\psi_{13}}{\beta_1} \left( \frac{T^{\beta_1-\delta} u^*_1 \xi^*_1}{\beta_1} \right) \tag{33}\]

which implies that

$$
\|t^{1-a_1}P(x_1, x_2)(t)\| \leq L_{f_1} + \frac{M_1(K_m r + V)}{\Gamma(\alpha_1)} + \frac{M_1 L_{f_1} \|A_1\| \|t-s\|^{\beta_1}}{\Gamma(\beta_1 + 1)} \nonumber
$$

$$
+ \frac{r M_{1}^{\gamma_{1-a_1}}}{\Gamma(\beta_1)} \left[ \frac{1 - \delta}{\beta_1 - \delta} \right]^{1-\delta} T^{\beta_1-\delta} \|\psi_{11}\| \|L_{1_t}^{1-\delta} \left( \frac{\psi_{13}}{\beta_1} \right)^{k_{1}^{*}} \nonumber
$$

\[ + \frac{\psi_{13}}{\beta_1} \left( \frac{T^{\beta_1-\delta} u^*_1 \xi^*_1}{\beta_1} \right) \tag{34}\]

Combining the expressions (30), (32) and (34), we obtain

$$
\|P(x_1, x_2)(t)\|_{C_{1-a_1}} \leq L_{f_1} + \frac{M_1 \|x_{01}\|}{\Gamma(\alpha_1)} + \frac{M_1(K_m r + V)}{\Gamma(\alpha_1)} \nonumber
$$

$$
+ \frac{M_1 L_{f_1} \|A_1\| T_{1-t}^{1-a_1}}{\Gamma(\beta_1 + 1)} + \frac{r M_{1}^{\gamma_{1-a_1}}}{\Gamma(\beta_1)} \left[ \frac{1 - \delta}{\beta_1 - \delta} \right]^{1-\delta} T^{\beta_1-\delta} \|\psi_{11}\| \|L_{1_t}^{1-\delta} \left( \frac{\psi_{13}}{\beta_1} \right)^{k_{1}^{*}} \nonumber
$$

\[ + \frac{\psi_{13}}{\beta_1} \left( \frac{T^{\beta_1-\delta} u^*_1 \xi^*_1}{\beta_1} \right) \nonumber
$$

\[ + \frac{\psi_{13}}{\beta_1} \left( \frac{T^{\beta_1-\delta} u^*_1 \xi^*_1}{\beta_1} \right) \tag{35}\]

By our assumptions, we have

$$
\|P(x_1, x_2)(t)\|_{C_{1-a_1}} > r \quad \text{which implies that}
$$

$$
L_{f_1} + \frac{M_1 \|x_{01}\|}{\Gamma(\alpha_1)} + \frac{M_1(K_m r + V)}{\Gamma(\alpha_1)} + \frac{M_1 L_{f_1} \|A_1\| T_{1-t}^{1-a_1}}{\Gamma(\beta_1 + 1)} \nonumber
$$

$$
+ \frac{r M_{1}^{\gamma_{1-a_1}}}{\Gamma(\beta_1)} \left[ \frac{1 - \delta}{\beta_1 - \delta} \right]^{1-\delta} T^{\beta_1-\delta} \|\psi_{11}\| \|L_{1_t}^{1-\delta} \left( \frac{\psi_{13}}{\beta_1} \right)^{k_{1}^{*}} \nonumber
$$

\[ + \frac{\psi_{13}}{\beta_1} \left( \frac{T^{\beta_1-\delta} u^*_1 \xi^*_1}{\beta_1} \right) \geq \|P(x_1, x_2)(t)\|_{C_{1-a_1}} > r \tag{36}\]

Dividing both sides by $r$ and taking $\delta \to \infty$, we obtain

$$
M_1 \left[ \frac{K_m r + V}{\Gamma(\alpha_1)} + \frac{M_1 L_{f_1} \|A_1\|}{\Gamma(\beta_1 + 1)} \right] \nonumber
$$

\[ + \frac{r M_{1}^{\gamma_{1-a_1}}}{\Gamma(\beta_1)} \left[ \frac{1 - \delta}{\beta_1 - \delta} \right]^{1-\delta} T^{\beta_1-\delta} \|\psi_{11}\| \|L_{1_t}^{1-\delta} \left( \frac{\psi_{13}}{\beta_1} \right)^{k_{1}^{*}} \nonumber
$$

\[ + \frac{\psi_{13}}{\beta_1} \left( \frac{T^{\beta_1-\delta} u^*_1 \xi^*_1}{\beta_1} \right) \geq 1 \tag{37}\]

which is a contradiction. Hence, $\|P(x_1, x_2)(t)\|_{C_{1-a_1}} < r$.

Similarly, we show that $\|Q(x_1, x_2)(t)\|_{C_{1-a_1}} < r$.

Finally, $\|S(x_1, x_2)(t)\|_{C_{1-a_1}} = \max \{\|P(x_1, x_2)(t)\|_{C_{1-a_1}}, \|Q(x_1, x_2)(t)\|_{C_{1-a_2}}\} < r$.

That shows that the operator $S$ maps bounded sets to bounded sets.

Step 3. We prove that the operator $S_1$ is Lipschitz continuous. For $t \in [0, t_1)$, we have

$$
\|P_1(x_1, x_2) - P_1(y_1, y_2)\|_{C_{1-a_1}} \leq \left( M_{f_1} + N_{f_1} \right) \|(x_1 - y_1, x_2 - y_2)\|. \tag{38}\]

For $t \in (t, s]$, $i = 1, 2, \cdots, m$, we have

$$
\|P_1(x_1, x_2) - P_1(y_1, y_2)\|_{C_{1-a_1}} = \|m_i(t, x_1(t)) - m_i(t, y_1(t))\|_{C_{1-a_1}} \leq K_{mi} \|x_1 - y_1\|_{C_{1-a_1}}, \tag{39}\]

so we get

$$
\|P_1(x_1, x_2) - P_1(y_1, y_2)\|_{C_{1-a_1}} \leq K_{mi} \|(x_1 - y_1, x_2 - y_2)\|. \tag{40}\]

and for $t \in (t, s]$, we have

$$
\|P_1(x_1, x_2) - P_1(y_1, y_2)\|_{C_{1-a_1}} \leq \left( M_{K_{mi}} + M_{f_1} + N_{f_1} \right) \|(x_1 - y_1, x_2 - y_2)\|. \tag{41}\]

From (38), (40) and (41), we can say that the operator $P_1$ is Lipschitz continuous with constant $(M_{K_{mi}} + M_{f_1} + N_{f_1})$. 
and similarly, we show that the operator $Q_1$ is Lipschitz continuous with constant $(M_f K_{n1} + M_{f2} + N_{fj})$.

Finally, the operator $S_1$ is Lipschitz continuous with the constant max $\{ M_f K_{n1} + M_{f2} + N_{fj} ; M_f K_{n1} + M_{f2} + N_{fj} \}$.

**Step 4.** We show that $S_2$ is a continuous operator.

Let $\{x_{1n}, x_{2n}\}$ be a sequence in $\Omega_r$ such that $(x_{1n}, x_{2n}) \rightarrow (x_1, x_2)$ as $n \rightarrow \infty$.

The functions $f_1, g_1$ are continuous with respect to the second, third, and fourth variables, it follows that

$$\lim_{t \rightarrow \infty} f_1(t, x_{1n}(t), x_{2n}(t)) = f_1(t, x_1(t), x_2(t))$$

and

$$\lim_{t \rightarrow \infty} g_1 \left( t, x_{1n}(t), x_{2n}(t), \int_0^t u_1(s, \tau) \phi_1(t, \tau, x_{1n}(\tau), x_{2n}(\tau)) d\tau \right)$$

$$= g_1 \left( t, x_1(t), x_2(t), \int_0^t u_1(s, \tau) \phi_1(t, \tau, x_1(\tau), x_2(\tau)) d\tau \right)$$

by (H1) and (H2), we have

$$\| f_1(t, x_{1n}(t), x_{2n}(t)) - f_1(t, x_1(t), x_2(t)) \| \leq 2L_{f1} \| g_1 \left( t, x_{1n}(t), x_{2n}(t), \int_0^t u_1(s, \tau) \phi_1(t, \tau, x_{1n}(\tau), x_{2n}(\tau)) d\tau \right) \|$$

$$\leq 2\tau \| \psi_{11}(t) + \psi_{12}(t) + \psi_{13}(t) u_{11,2} \|.$$  (42)

Since $\psi_{11}, \psi_{12} \in L(1/\delta)[0, T]$ and $\psi_{13}(t)$ is continuous, the functions on the right hand side are integrable.

For all $t \in J$, $(x_{1n}, x_{2n}, x_1, x_2) \in \Omega_r$, we have

$$\| f_1(t, x_{1n}(t), x_{2n}(t)) - f_1(t, x_1(t), x_2(t)) \| \leq 2L_{f1} \| g_1 \left( t, x_{1n}(t), x_{2n}(t), \int_0^t u_1(s, \tau) \phi_1(t, \tau, x_{1n}(\tau), x_{2n}(\tau)) d\tau \right) \|$$

$$\leq 2\tau \| \psi_{11}(t) + \psi_{12}(t) + \psi_{13}(t) u_{11,2} \|.$$  (43)

$$\| f_1(t, x_{1n}(t), x_{2n}(t)) - f_1(t, x_1(t), x_2(t)) \|$$

$$\leq 2L_{f1} \| g_1 \left( t, x_{1n}(t), x_{2n}(t), \int_0^t u_1(s, \tau) \phi_1(t, \tau, x_{1n}(\tau), x_{2n}(\tau)) d\tau \right) \|$$

$$\leq 2\tau \| \psi_{11}(t) + \psi_{12}(t) + \psi_{13}(t) u_{11,2} \|.$$  (44)

By the Lebesgue dominated convergence theorem, we have

$$\| f_1(t, x_{1n}(t), x_{2n}(t)) - f_1(t, x_1(t), x_2(t)) \| \rightarrow 0 \text{ as } n \rightarrow \infty,$$  (45)

and by the same method, we show that

$$\| f_1(t, x_{1n}(t), x_{2n}(t)) - f_1(t, x_1(t), x_2(t)) \| \rightarrow 0 \text{ as } n \rightarrow \infty,$$  (46)

consequently, the operator $S_2$ is continuous.

**Step 5.** We show that the operator $S_2$ is equicontinuous.

For any $(x_1, x_2) \in \Omega_r$ and $s_i < t_1 < t_2 < t_{s,i}, i = 0, 1, \ldots, m$, we have

$$\| f_1(t, x_{1n}(t), x_{2n}(t)) - f_1(t, x_1(t), x_2(t)) \|$$

$$\leq 2L_{f1} \| g_1 \left( t, x_{1n}(t), x_{2n}(t), \int_0^t u_1(s, \tau) \phi_1(t, \tau, x_{1n}(\tau), x_{2n}(\tau)) d\tau \right) \|$$

$$\leq 2\tau \| \psi_{11}(t) + \psi_{12}(t) + \psi_{13}(t) u_{11,2} \|.$$  (47)

We substitute $(t_1 - s) = s_i$ in $I_1$ and $I_3$, we obtain

$$I_1 \leq \int_{t_1}^{t_2} \left( S_{\beta_1}(t_2 - s) - S_{\beta_1}(t_1 - s) \right) \| f_1(t, x_{1n}(t), x_{2n}(t)) - f_1(t, x_1(t), x_2(t)) \| ds_i,$$

$$I_3 \leq \int_{t_1}^{t_2} \left( S_{\beta_1}(t_2 - s) - S_{\beta_1}(t_1 - s) \right) \| f_1(t, x_{1n}(t), x_{2n}(t)) - f_1(t, x_1(t), x_2(t)) \| ds_i,$$

by the equicontinuity of $(\beta_1 - \gamma_1)$-resolvent operator nad Lebesgue dominated convergence theorem, the integrals $I_1, I_3 \rightarrow 0 \text{ as } t_2 \rightarrow t_1.$
And we have \( I_2, I_4 \to 0 \) as \( t_2 \to t_1 \); it follows that
\[
\left\| t^{-\alpha_0}(P_2(x_1, x_2)(t_2) - P_2(x_1, x_2)(t_1)) \right\| \to 0 \text{ as } t_2 \to t_1,
\]

we show with a similar method that
\[
\left\| t^{-\alpha_0}(Q_2(x_1, x_2)(t_2) - Q_2(x_1, x_2)(t_1)) \right\| \to 0 \text{ as } t_2 \to t_1.
\]

This proves the equicontinuity of the operator \( S_2 \).

**Step 6.** We prove that the operator \( S \) is condensing. Hence, we have to show that for any bounded subset \( D \subset \Omega_t, \mu(S(D)) < \mu(D) \).

Since \( S_1 \) is continuous, for any bounded set \( D \subset \Omega_t \), there exists a countable set \( D_0 = \{ (x_{in}, x_{2n}) \} \subset D \) such that \( \mu(S_1(D)) = \mu(S_2(D_0)) \).

We know that \( S_2 \) is bounded and equicontinuous; it follows that
\[
\mu(S_2(D_0)) = \max_{t \in [s, t+1]} \mu(S_2(D_0)(t)), i = 0, 1, \ldots, m
\]

we recall that \( S_2(x_1, x_2)(t) = (P_2(x_1, x_2), Q_2(x_1, x_2))(t) \) we have
\[
\mu(P_2(D_0)(t)) = \mu \left( \int_{s}^{t} S_{\beta_1}(t-s)g_1(s, x_{1n}(s), x_{2n}(s)) \right. \\
\left. \int_{s}^{t} u_1(s, \tau) \phi_1(s, \tau, x_{1n}(\tau), x_{2n}(\tau)) d\tau \right) \\
- \int_{s}^{t} S_{\beta_1}(t-s)A_{L_2}(s, x_{1n}(s), x_{2n}(s)) ds \\
\leq \frac{M_1}{T(\beta_1)} \int_{s}^{t} (t-s)^{\beta_1-1}g_1(s, x_{1n}(s), x_{2n}(s)) ds \\
- \int_{s}^{t} u_1(s, \tau) \phi_1(s, \tau, x_{1n}(\tau), x_{2n}(\tau)) d\tau \\
+ \frac{M_1}{T(\beta_1)} \int_{s}^{t} (t-s)^{\beta_1-1}A_{L_2}(s, x_{1n}(s), x_{2n}(s)) ds
\]
\[
\leq \frac{M_1}{T(\beta_1)} \mu(D_0(s)) \int_{s}^{t} (t-s)^{\beta_1-1} \left( M_{g_{11}} \right) \left( t \right) \\
+ \frac{M_1}{T(\beta_1)} \mu(D_0(s)) \int_{s}^{t} (t-s)^{\beta_1-1} \left( M_{g_{12}} \right) \left( t \right) \\
+ \frac{M_1}{T(\beta_1)} \mu(D_0(s)) \int_{s}^{t} (t-s)^{\beta_1-1} \left( M_{g_{13}} \right) \left( t \right) \\
+ \frac{M_1}{T(\beta_1)} \mu(D_0(s)) \int_{s}^{t} (t-s)^{\beta_1-1} \left( M_{g_{14}} \right) \left( t \right)
\]

which implies that
\[
\mu(P_2(D)) \leq \frac{M_1T^{e_1}\beta_1 - \delta}{\Gamma(\beta_1)} \left[ M_{g_{11}} || \psi_1 ||_{L^2[0,T]} + M_{g_{12}} u_2^* v_2^* || \psi_1 ||_{L^2[0,T]} \mu(D) \right]
\]

Similarly, we show that
\[
\mu(Q_2(D)) \leq \frac{M_1T^{e_1}\beta_1 - \delta}{\Gamma(\beta_1)} \left[ M_{g_{11}} || \psi_1 ||_{L^2[0,T]} + M_{g_{12}} u_2^* v_2^* || \psi_1 ||_{L^2[0,T]} \mu(D) \right]
\]

Hence, we get
\[
\mu(S_2(D)) \leq \max \left\{ \mu(P_2(D)), \mu(Q_2(D)) \right\}
\]

Since the operator \( S_1 \) is a Lipchitz operator with constant max \( \{ M_1K_{m_1} * M_{f_1} + N_{f_1} ; M_1K_{m_1} + M_{f_1} + N_{f_1} \} \) for any bounded set \( D \subset \Omega_t \), we have
\[
\mu(S_1(D)) \leq \max \left\{ M_1K_{m_1} + M_{f_1} + N_{f_1} ; M_1K_{m_1} + M_{f_1} + N_{f_1} \right\} \mu(D)
\]

As the operator \( S = S_1 + S_2 \), we obtain \( \mu(S(D)) \leq \mu(S_1(D)) + \mu(S_2(D)) \leq (\max \{ M_1K_{m_1} + M_{f_1} + N_{f_1} ; M_1K_{m_1} + M_{f_1} + N_{f_1} \} + \max \{ (M_1T^{e_1}\beta_1) / \Gamma(\beta_1)(1 - \delta/\beta_1 - \delta)^{-1} \} M_{g_{11}} || \psi_1 ||_{L^4[0,T]} + M_{g_{12}} u_2^* v_2^* || \psi_1 ||_{L^4[0,T]} + (M_1T^{e_1}\beta_1) / \Gamma(\beta_1 + 1) \)} \mu(D) \)

Thus, \( S : \Omega_t \to \Omega_t \) is a condensing operator. Hence, by Sadovski’s fixed point theorem [26], the operator \( S \) has at least a pair of solutions \( (x_1(t), x_2(t)) \). Therefore, the problem (1) has a pair of solutions \( (x_1(t), x_2(t)) \). This completes the proof.
4. Example

We consider in this example the following problem on $J = [0, 1]$

\[
\begin{align*}
H^{1}_{L_{2}}\left[ x_{1}(t, w) + \sin \left( x_{1}(t, w) \right) \right] &= \frac{\partial x_{1}(t, w)}{\partial t^2} + \frac{\partial x_{1}(t, w)}{\partial t^2} + \frac{x_{1}(t, w)}{10(1 + e^{t})^{1/4}} + \frac{x_{1}(t, w)}{10(1 + e^{t})^{1/4}} + \frac{e^{-t/2}}{5} \int_{0}^{t} (\sin (x_{1}(w, s)) + \sin (x_{2}(w, s))) ds, \\
H^{1}_{L_{2}}\left[ x_{2}(t, w) + \sin \left( x_{2}(t, w) \right) \right] &= \frac{\partial x_{2}(t, w)}{\partial t^2} + \frac{\partial x_{2}(t, w)}{\partial t^2} + \frac{x_{2}(t, w)}{10(1 + e^{t})^{1/4}} + \frac{x_{2}(t, w)}{10(1 + e^{t})^{1/4}} + \frac{e^{-t/2}}{5} \int_{0}^{t} (\sin (x_{1}(w, s)) + \sin (x_{2}(w, s))) ds, \\
x_{1}(t, 0) &= x_{1}(t, 1) = 0, t \in [0, 1], \\
x_{2}(t, 0) &= x_{2}(t, 1) = 0, t \in [0, 1], \\
I_{L_{2}}^{1, \alpha_{1}}(x_{1}(0, w) + f_{1}(0, x_{1}(0, w), x_{2}(0, w))) &= x_{1}, \\
I_{L_{2}}^{1, \alpha_{2}}(x_{2}(0, w) + f_{2}(0, x_{1}(0, w), x_{2}(0, w))) &= x_{2},
\end{align*}
\]

Let $X = L_{2}^{2}[0, 1]$ and for $j = 1, 2, A_{j}x_{j} = x'_{j}$ with $D(A_{j}) = \{ x \in X : x_{j}, x'_{j} \text{ are absolutely continuous}, \text{ and } x_{j}(0) = x_{j}(1) = 0 \}$.

The operators $A_{j}$ generates equicontinuous $C_{0}$-semi-groups $T_{j}(t), (t \geq 0)$ on $X$ with $\| T_{j}(t) \| \leq 1$ for $t \geq 0$.

We have the following for $j = 1, 2$

\[
\begin{align*}
\beta_{j} &= \frac{1}{2}, \gamma_{j} = \frac{1}{8}, \alpha_{j} = \frac{9}{16}, T = 1 \\
\varphi_{j}(t, x_{1}(s, w), x_{2}(s, w)) &= \frac{e^{-t}}{t^{2}} (\sin (x_{1}(s, w)) + \sin (x_{2}(s, w))), u_{j}(t, s) = 1 \\
f_{j}(t, x_{1}(t, w), x_{2}(t, w)) &= \frac{\sin (x_{1}(t, w)) + \sin (x_{2}(t, w))}{40} \\
m_{j}(t, x_{1}(t, w)) &= \frac{\cos \left\{ t \left[ x_{1}(t, w) \right] \right\}}{25 + \left\| x_{1}(t, w) \right\|^2} n_{j}(t, x_{2}(t, w)) \\
\end{align*}
\]

Taking $x_{j}(t, w) = x_{j}(t)$, we can see that

\[
\begin{align*}
\| \mathcal{G}_{j}(t, x_{1}(t), x_{2}(t), \int_{0}^{t} u_{j}(t, s) \varphi_{j}(t, s, x_{1}(s, w), x_{2}(s, w)) ds) \| &\leq \psi_{j1}(t) \| x_{1}(t) \| + \psi_{j2}(t) \| x_{2}(t) \| \\
&+ \psi_{j3}(t) \int_{0}^{t} \varphi_{j}(t, s, x_{1}(s), x_{2}(s)) ds \\
\psi_{j1}(t) &= \psi_{j2}(t) = \frac{1}{10(1 + e^{t})^{1/4}}, \psi_{j3}(t) = \frac{e^{-t/2}}{5}.
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\mu \left( \mathcal{G}_{j}(t, x_{1}(t), x_{2}(t), \int_{0}^{t} u_{j}(t, s) \varphi_{j}(t, s, x_{1}(s, w), x_{2}(s, w)) ds) \right) &\leq \tilde{\psi}_{j1}(t) \mu(x_{1}(t), x_{2}(t)) + \tilde{\psi}_{j2}(t) \mu \left( \int_{0}^{t} \varphi_{j}(t, s, x_{1}(s), x_{2}(s)) ds \right), \\
\tilde{\psi}_{j1}(t) &= \frac{1}{10(1 + e^{t})^{1/4}}, \tilde{\psi}_{j2}(t) = \frac{e^{-t/2}}{5}.
\end{align*}
\]

\[
\begin{align*}
\| \varphi_{j}(t, s, x_{1}(s, w), x_{2}(s, w)) \| &\leq \tilde{\xi}_{j}(t, s) \| (x_{1}(s, w), x_{2}(s, w)) \| \\
\tilde{\xi}_{j}(t, s) &= e^{-t/2} \text{ and } \tilde{\xi}_{j}^{*} = \sup_{t \in [0, 1]} \int_{0}^{t} \tilde{\xi}_{j}(t, s) ds = 0.63212,
\end{align*}
\]
We take $\delta = 1/4$

$$
\| \psi_j \|_{L^2[0,1]} = \left\| \psi_j \right\|_{L^2[0,1]} = \left( \int_0^1 \frac{1}{10(1 + e^t)^{1/4}} \, dt \right)^{1/2} = 0.063
$$

and

$$
\| \psi_j(t) \|_{L^2[0,1]} = \left( \int_0^1 \frac{e^{-t/2}}{5} \, dt \right)^{1/2} = 0.102.
$$

(61)

For $j = 1, 2, i = 1, 2, \ldots, m$, the functions $f_j(t, x_1(t), x_2(t))$, $m_i(t, x_1(t))$ and $n_i(t, x_2(t))$ are Lipschitz functions with constants:

$L_{f_j} = 1/20, M_{f_j} = N_{f_j} = 1/40$ and $K_{m_i} = K_{n_i} = 1/25$. Thus, we have

$$
\beta_j = \frac{1}{2}, \gamma_j = \frac{1}{8}, \alpha_j = \frac{9}{16}, \delta = \frac{1}{4}, M_j = 1, u_j^* = 1,
$$

$$
M_{g_{j,1}} = M_{g_{j,2}} = M_{g_{j,3}} = L_{f_j} = \frac{1}{40}, L_{m_i} = L_{n_i} = \frac{1}{25},
$$

$$
\| \psi_j \| = \| \psi_{j,2} \| = \| \psi_{j,3} \| = 0.063,
$$

$$
\| \psi_{j,1} \| = \| \psi_{j,3} \| = 0.102, \psi_{j,1}^* = \xi_{j,1}^* = 0.63212.
$$

(62)

For these values, the condition (1) of theorem 12 is satisfied: we have for $j = 1, 2:

$$
M_j \left[ \frac{K_j}{\Gamma(\alpha_j)} + \frac{T^{1-a_j+\beta_j-\delta}}{\Gamma(\beta_j)} \left( \frac{1 - \delta}{\beta_j - \delta} \right)^{1-\delta} \left( \| \psi_{j,1} \| + \| \psi_{j,2} \| \right) \right.
$$

$$
+ \frac{\| \psi_{j,1} \| T^{\beta_j-\delta} u_j^* \xi_{j,1}^*}{\beta_j \Gamma(\beta_j)} \right] = 0.260 < 1.
$$

(63)

The second condition is also verified:

$$
\max_{j=1,2} \left\{ M_j K_j + M_j + N_{f_j} \right\} + \max_{j=1,2} \left\{ M_j T^{\beta_j-\delta} \left( \frac{1 - \delta}{\beta_j - \delta} \right) \right\}
$$

$$
\cdot \left[ M_{g_{j,1}} \| \psi_{j,1} \|_{L^2[0,1]} + M_{g_{j,2}} u_j^* \right] \| \psi_{j,2} \|_{L^2[0,1]} \right]
$$

$$
+ \frac{M_j L_{f_j} T^{\beta_j-\delta}}{\Gamma(\beta_j + 1)} \right) \right\} \approx 0.3103 < 1
$$

(64)

Consequently, both conditions are satisfied, which means that the problem (1) has a couple of solutions $(x_1(t), x_2(t))$ in the space $X = PC_{1-a_1} \times PC_{1-a_2}$.

5. Conclusion

In this paper, we achieved the existence of solutions for a class of impulsive Hilfer fractional coupled systems by converting the problem to an integral form and then using the Sadovskiǐ's fixed point theorem. For future works, we can consider other fractional operators for example the $\psi$-Hilfer fractional operator for its new results and applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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