We propose an experimentally feasible optomechanical scheme to realize a negative cavity photon spectral function (CPSF). The system under consideration is an optomechanical system (OMS) consisting of two mechanical (phononic) modes which are linearly coupled to a common cavity mode via the radiation pressure while parametrically driven through the coherent time-modulation of their spring coefficients. We find that, in the red-detuned and weak-coupling regimes, a frequency-dependent effective cavity damping rate (ECDR) is induced in the system. Furthermore, using the equations of motion for the cavity retarded Green’s function obtained in the framework of a generalized linear response theory (GLRT), we show that a negative ECDR corresponding to a negative CPSF can be realized by controlling the cooperativities and modulation parameters while the system still remains in the stable regime. Nevertheless, such a negativity never occurs in a standard cavity optomechanical system. Besides, we find that the presence of two modulated mechanical degrees of freedom provides more controllability on the negativity of CPSF with a smaller parametric drive in comparison to the setup with a single modulated mechanical oscillator (MO). Interestingly, the introduced negativity may open a new platform to realize an extraordinary (modified) optomechanically induced transparency (OMIT) leading to perfect tunable optomechanical filters, and a negative effective temperature (NET) corresponding, respectively, to the probe reflection above the unity and the coupled qubit-cavity population inversion.

Introduction

In the recent two decades, the field of quantum optomechanics, in which a mechanical oscillator (MO) is linearly or quadratically coupled to an optical/microwave mode via the radiation pressure, using state-of-the-art technologies has been significantly progressed both in theory and experiment. The optomechanical systems (OMSs) play a key role in quantum technologies and also can be employed as a controllable setup for the emergence of quantum effects on macroscopic scales or probing the fundamentals of physics.

OMSs have been applied in a variety range of applications including ultraprecision quantum sensing and measurement, quantum illumination or quantum radar, cooling of the MO, generation of entanglement, synchronization of MOs, generation of quadrature squeezing and amplification, realizing the dynamical Casimir effect-based nonclassical radiation source, optomechanically induced transparency (OMIT), which is analogous to the familiar phenomenon of electromagnetically induced transparency (EIT), and also the realization of microwave nonreciprocivity, unidirectional transport, isolator and circulator.

Very recently, by applying a generalized linear response theory (GLRT) to a standard driven-dissipative OMS as an open quantum system, the Green’s function equations of motion have been obtained from quantum Langevin equations (QLEs) in the Heisenberg picture to deeply explain a wealth of phenomena, including the anti-resonance, normal mode splitting, and OMIT. On the other hand, driven-dissipative quantum systems have recently attracted intensive research interests, particularly in the context of quantum control (for a very recent, comprehensive review, the reader is referred to Ref.64). Generally, the steady state of such systems can be far from equilibrium because of the unavoidable competition between driving and dissipation. The steady-state behaviors of such systems have been investigated theoretically by many authors (see, for example, 65–67). Besides the steady-state behavior, also the description of the system’s response to weak external perturbations is of great interest. It is well-known that the Green’s function method is a powerful technique for this purpose. To determine the Green’s functions of
Figure 1. (Color online) Schematic of an optomechanical system consisting of two similar MOs which are linearly coupled to the common electromagnetic (EM) mode via the radiation pressure. The spring coefficients of the MOs are parametrically driven at twice their natural frequencies so that \( k(t) = k + \delta k \cos(2\omega_{m(d)}t + \phi_{m(d)}) \).

In this paper, we will follow the method introduced in Ref.\(^{53}\) to obtain the cavity retarded Green’s function (CRGF) of an OMS composed of two parametrically-driven mechanical modes which are coupled to a common electromagnetic mode. We show that by controlling the system parameters a negative CPSF may be achieved in the stable regime of the system. In other words, we show that the coherent time-modulation of the spring coefficients of the mechanical modes leads to a mechanical parametric amplification which in turn induces a negative frequency-dependent effective cavity damping rate (ECDR) that corresponds to a negative cavity photon spectral function (CPSF). It should be pointed out that unlike other standard systems such as the degenerate parametric amplifier (DPA), the occurrence of a negative CPSF in the present system takes place in the stable regime. Furthermore, we briefly address how such a negative CPSF may result in the so-called extraordinary (modified) OMIT which corresponds to a probe reflectivity above the unity. Also, it may lead to the definition of a negative effective temperature (NET) for the intracavity mode which can emerge in qubit-cavity population inversion. These two important issues will be discussed in detail in our future paper.

The paper is organized as follows. In Sec. 1, we describe the basic model of the system under study and its effective Hamiltonian. In Sec. 2, by solving the QLEs in frequency space we find the self-energies and induced-parametric squeezing coefficients corresponding to the induced-effective damping rates. Using the solutions in the frequency space, we calculate the CPSF in Sec. 3, and show that by controlling the cooperativities and modulation parameters it can be negative when the ECDR becomes negative. In Sec. 4, we present a physical interpretation of the CPSF negativity in our system together with a comparison to an ordinary OPA. Finally, we summarize our concluding remarks and outlooks in Sec. 5. In particular, the manifestation of the negative CPSF in the modified OMIT and NET are briefly discussed in Sec. 5.

1 System Hamiltonian

As depicted in Fig. (1), we consider an OMS consisting of two MOs with the natural frequencies \( \omega_m \) and \( \omega_d \), and damping rates \( \gamma_m \) and \( \gamma_d \) interacting with the radiation pressure of the single optical mode of the cavity while the spring coefficients of the MOs are coherently modulated at twice their natural frequencies. Besides, the natural frequency of the optical mode inside the cavity and the cavity damping rate are, respectively, \( \omega_c \) and \( \kappa \). It is also assumed that the cavity is driven at the rate of \( E_L = \sqrt{\kappa}P_L/\hbar \omega_c \) by an external laser with frequency \( \omega_c \) and the input power \( P_L \). The total Hamiltonian of the system in the frame rotating at the driving laser frequency, \( \omega_c \), can be written as

\[
\hat{H} = \hbar \Delta \hat{\alpha} \hat{\alpha}^\dagger + \hbar \omega_m \hat{b} \hat{b}^\dagger + \hbar \omega_d \hat{d} \hat{d}^\dagger + i\hbar E_L (\hat{\alpha}^\dagger \hat{\alpha} - \hat{\alpha} \hat{\alpha}^\dagger) - \hbar g_0 \hat{a} \hat{a}^\dagger (\hat{b} + \hat{b}^\dagger) - \hbar G_0 \hat{a} \hat{a}^\dagger (\hat{d} + \hat{d}^\dagger) \\
+ \frac{i\hbar}{2} (\lambda_m \hat{b} \hat{b}^\dagger e^{-2i\omega_m t} - \lambda_m^* \hat{b}^\dagger \hat{b} e^{2i\omega_m t}) + \frac{i\hbar}{2} (\lambda_d \hat{d} \hat{d}^\dagger e^{-2i\omega_d t} - \lambda_d^* \hat{d}^\dagger \hat{d} e^{2i\omega_d t}).
\]

(1)

The first four terms in the Hamiltonian describe, respectively, the free energy of the optical mode, the free energy of the MOs and the coupling between the optical mode and the classical driving laser. Here, \( \hat{a} \) and \( \hat{b}(\hat{d}) \) are the annihilation operators of the optical and mechanical modes, and \( \Delta_c = \omega_c - \omega_L \) is the detuning of the optical mode from the driving laser frequency. The fifth...
and sixth terms denote, respectively, the optomechanical interactions between the optical mode and the two mechanical modes $\hat{b}$ and $\hat{d}$ with the single-photon optomechanical coupling strengths $g_0$ and $G_0$.

The seventh and eighth terms account for the parametric driving of the MOs spring coefficients at the twice of their natural frequencies giving rise to the time-dependent spring coefficient as $k_m(t) = k + \delta k \cos(2\omega_m t + \varphi_m(t))$, where $\varphi_m(t)$ is the phase of external modulation. These terms have been written in the rotating wave approximation (RWA) which is valid over the time scales longer than $\omega_m^{-1}$. Here, $\lambda_{m} = |\lambda_{m}|e^{i\varphi_{m}}$ is the mechanical modulation amplitude with $|\lambda_{m}| = \delta k x_{zp}^2/2h^{49,50}$ where $x_{zp} = \sqrt{\hbar/(2m\omega_{m})}$ is the zero-point position fluctuation. Note that by fixing the phase of modulation, it is always possible to take $\lambda_{m}$ as a real number. It is worth mentioning that the parametric mechanical driving term can be considered as the mechanical phonon analog of the degenerate parametric amplification which may lead to the dynamical Casimir effect of mechanical phonons$^{49}$. Furthermore, the parametric driving has been recently used for the preparation of an optomechanical system in a two-mode squeezed thermal state$^{33}$. It is pointed out that the parametric mechanical driving can be realized in a superconducting microwave optomechanical circuit by applying a combination of a static voltage and an oscillating voltage (ac-voltage) in order to simulate the modulation of the spring coefficient$^{34}$.

2 Dynamics of the system

By the assumption that the two MOs have equal natural frequencies, i.e., $\omega_m = \omega_d$, and in the red-detuned regime where $\Delta_0 = \omega_m = \omega_d$, in which $\Delta_0 = \Delta_0 - 2g_0\tilde{\alpha}_0 - 2G_0d$ is the effective cavity detuning and $\tilde{\alpha} \approx g_0\tilde{\alpha}_0/\omega_m$ and $\tilde{d} \approx -G_0\tilde{d}_0^2/\omega_d$ are the mean mechanical fields with $\tilde{\alpha} = E_1/\sqrt{k^2/4 + \Delta_0^2}$ being the steady-state mean value of the optical mode, the linearized QLEs for the quantum fluctuations of the system operators can be obtained in the RWA as the following set of equations$^{49,50}$

\begin{align}
\delta\dot{\hat{a}} &= -\frac{k}{2}\delta\hat{a} + ig\hat{b} + iG\delta\hat{d} + \sqrt{\kappa}\hat{a}_m, \quad (2a) \\
\delta\dot{\hat{b}} &= -\frac{\gamma_m}{2}\delta\hat{b} + ig\delta\hat{a} + \lambda_m\delta\hat{b}^\dagger + \sqrt{\gamma_m}\hat{b}_m, \quad (2b) \\
\delta\dot{\hat{d}} &= -\frac{\gamma_d}{2}\delta\hat{d} + iG\delta\hat{a} + \lambda_d\delta\hat{d}^\dagger + \sqrt{\gamma_d}\hat{d}_m, \quad (2c)
\end{align}

where $g(G) = g_0(G_0)\tilde{\alpha}$ is the enhanced-optomechanical coupling strength. Moreover, the optical and the Brownian mechanical input noises $\hat{a}_m (o = a, b, d)$ satisfy the Markovian correlation functions $\langle \hat{\alpha}_m(t)\hat{\alpha}_m^\dagger(t') \rangle = (1 + \tilde{n}_j^{\dagger})\delta(t - t')$ and $\langle \hat{\alpha}_m(t)\hat{\alpha}_m(t') \rangle = \tilde{n}_j^{\dagger}\delta(t - t')$ ($j = c, m, d$) where $\tilde{n}_j^{\dagger} = [\exp(\hbar\omega_j/k_B T) - 1]^{-1}$ is the mean number of thermal excitations of the cavity and mechanical modes at temperature $T$.

The linearized QLEs of motion, i.e., Eqs.(2a-2c) together with their Hermitian conjugates can be rewritten in the following compact form

$$
\frac{d}{dt}\hat{u}(t) = \chi_0\hat{u}(t) + \hat{u}_m(t),
$$

where $\hat{u}(t) = (\delta\hat{a}, \delta\hat{a}^\dagger, \delta\hat{b}, \delta\hat{b}^\dagger, \delta\hat{d}, \delta\hat{d}^\dagger)^T$, and $\hat{u}_m(t) = (\sqrt{\kappa}\hat{a}_m, \sqrt{\kappa}\hat{a}_m^\dagger, \sqrt{\gamma_m}\hat{b}_m, \sqrt{\gamma_m}\hat{b}_m^\dagger, \sqrt{\gamma_d}\hat{d}_m, \sqrt{\gamma_d}\hat{d}_m^\dagger)^T$ are the vector of the quantum field fluctuations and the vector of quantum noises, respectively. The drift matrix $\chi_0$ can be found easily from the equations of motion as follows

$$
\chi_0 = \begin{pmatrix}
-\frac{k}{2} & 0 & ig & 0 & iG & 0 \\
0 & -\frac{k}{2} & 0 & -ig & 0 & -iG \\
ig & 0 & -\frac{\gamma_m}{2} & \lambda_m & 0 & 0 \\
0 & ig & \frac{\gamma_m}{2} & -\frac{\gamma_m}{2} & 0 & 0 \\
0 & 0 & 0 & -iG & 0 & \lambda_d \\
0 & 0 & 0 & 0 & -iG & \frac{\gamma_d}{2}
\end{pmatrix}.
$$

It is obvious that one can solve Eq.(3) in the Fourier space as follows

$$
\hat{u}(\omega) = \chi(\omega)\hat{u}_m(\omega),
$$

where the susceptibility matrix $\chi(\omega)$ is generally defined as

$$
\chi(\omega) = \left(-i\omega I - \chi_0\right)^{-1},
$$

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in which $I$ is the $6 \times 6$ identity matrix. Now, based on Eq.
(5) the Fourier transform of the optical field can be obtained as follows

$$
\delta \hat{a}(\omega) = \sum_{\omega = \omega_{dr}} \sqrt{\kappa_0} \left[ \chi_{\omega\omega}(\omega) \delta \hat{a}_{\omega}(\omega) + \chi_{\omega\omega}^\dagger(\omega) \delta \hat{a}_{\omega}^\dagger(\omega) \right],
$$

with $\chi_{\omega\omega}(\omega)$ being the susceptibility matrix elements, the expressions of which will be given later [Eqs. (14a-14f)].

In the following, in order to compare the present modulated optomechanical system with the well-known DPA we write the solution to the Eq. (3) for the the optical field noise operator and its Hermitian conjugate as

$$
-i\omega \delta \hat{a}(\omega) = -\left( \kappa / 2 + i\Sigma_a(\omega) \right) \delta \hat{a}(\omega) + \delta \hat{a}(\omega) + \sqrt{\kappa} \delta \hat{a}_{\omega}(\omega),
$$

$$
-i\omega \delta \hat{a}^\dagger(\omega) = -\left( \kappa / 2 - i\Sigma_a^\dagger(-\omega) \right) \delta \hat{a}^\dagger(\omega) + \delta \hat{a}^\dagger(\omega) + \sqrt{\kappa} \delta \hat{a}_{\omega}^\dagger(\omega),
$$

where the cavity self-energy $\Sigma_a(\omega)$ and the induced frequency-dependent parametric amplification coefficient $\lambda_a(\omega)$, which is analogous to the gain factor in the conventional OPA, are, respectively, given by

$$
i\Sigma_a(\omega) = g^2 \left( \frac{\omega_m}{2} - i\omega \right) \left( \frac{\omega_m}{2} - \omega \right)^2 - |\lambda_m|^2 \right) + G^2 \left( \frac{\omega_m}{2} - i\omega \right)^2 - |\lambda_d|^2 \right),
$$

$$
\lambda_a(\omega) = \frac{g^2 \lambda_m}{(\gamma_m/2 - i\omega)^2 - |\lambda_m|^2} + \frac{G^2 \lambda_d}{(\gamma_d/2 - i\omega)^2 - |\lambda_d|^2}.
$$

Furthermore, the generalized cavity input-noise in the frequency space is given by (see Ref. 49)

$$
\delta \hat{a}_{in} = \hat{a}_{in} + ig \sqrt{\frac{\gamma_m}{\kappa} \left( \gamma_m/2 - i\omega \right)^2 - |\lambda_m|^2} \left[ \delta \hat{a}_{in} + \frac{\lambda_m}{\gamma_m/2 - i\omega} \delta \hat{a}_{in}^\dagger \right]
$$

$$
+ ig \sqrt{\frac{\gamma_d}{\kappa} \left( \gamma_d/2 - i\omega \right)^2 - |\lambda_d|^2} \left[ \delta \hat{a}_{in} + \frac{\lambda_d}{\gamma_d/2 - i\omega} \delta \hat{a}_{in}^\dagger \right].
$$

It should be noted that the susceptibility matrix of the system, $\chi$, can be found by rewriting the self-energies as $\Sigma_0(\omega) = i(\chi^{-1}_0(\omega) - \chi^{-1}(\omega))$ with $\chi^{-1}_0(\omega) = (\kappa/2 - i\omega)I$. As is evident, the cavity self-energy satisfies the relation $\Sigma_a(-\omega) = -\Sigma_a(\omega)$. As is evident, the cavity self-energy satisfies the relation $\Sigma_a(-\omega) = -\Sigma_a(\omega)$.

By solving the coupled Eqs. (8a) and (8b), one can find the intracavity fluctuation operator $\delta \hat{a}(\omega)$ in the following form

$$
\delta \hat{a}(\omega) = \sqrt{\kappa} \chi_{\omega\omega}(\omega) \left[ \hat{a}_{in}(\omega) + \mathcal{M}(\omega) \hat{a}_{in}^\dagger(\omega) \right],
$$

where

$$
\mathcal{M}(\omega) = \frac{\lambda_a(\omega)}{\kappa/2 - i(\omega + \Sigma_a(-\omega))}.
$$

and the matrix element $\chi_{\omega\omega}(\omega)$ is given by Eq. (14a) in which the modified cavity self-energy is as follows

$$
\tilde{\Sigma}_a(\omega) = \Sigma_a(\omega) - \frac{\lambda_a(\omega) \lambda_a^\dagger(-\omega)}{i\kappa/2 + (\omega + \Sigma_a(-\omega))}.
$$

It is straightforward to show that for $\lambda_a^\dagger(-\omega) = \lambda_a(-\omega)$, we have $\tilde{\Sigma}_a(-\omega) = -\tilde{\Sigma}_a(\omega)$ and $\mathcal{M}^\dagger(-\omega) = \mathcal{M}(\omega)$.
In the following, all the matrix elements of the susceptibility matrix \( \chi(\omega) \), which are derived from the solution to Eq. (3) in the frequency space, have been presented as:

\[
\chi_{aa}(\omega) = \frac{i}{\hbar^2 + (\omega - \Sigma_a(\omega))}, \quad \text{(14a)}
\]
\[
\chi_{ab}(\omega) = \chi_{ba}(\omega) = \chi_{ab}(\omega) = \chi_{ba}(\omega) = \frac{ig \left[ \frac{\hbar^2}{2m} - i\omega \right] - \lambda_m^* \mathcal{M}(\omega)}{(\frac{\hbar^2}{2m} - i\omega)^2 - |\lambda_m|^2}, \quad \text{(14b)}
\]
\[
\chi_{ad}(\omega) = \chi_{da}(\omega) = \frac{ig \left[ \lambda_a - \left( \frac{\hbar^2}{2m} - i\omega \right) \right]}{(\frac{\hbar^2}{2m} - i\omega)^2 - |\lambda_a|^2}, \quad \text{(14c)}
\]
\[
\chi_{ad}(\omega) = \frac{iG \left[ \frac{\hbar^2}{2m} - i\omega \right] \mathcal{M}(\omega)}{(\frac{\hbar^2}{2m} - i\omega)^2 - |\lambda_d|^2}, \quad \text{(14d)}
\]
\[
\chi_{ad}(\omega) = \frac{iG \left[ \frac{\hbar^2}{2m} - i\omega \right] \mathcal{M}(\omega)}{(\frac{\hbar^2}{2m} - i\omega)^2 - |\lambda_d|^2}. \quad \text{(14e)}
\]

It can be easily shown that for real parameters, \( \lambda_m(\omega) = \lambda_m^*(\omega) \), one has \( \chi_{aa}(-\omega) = \chi_{aa}(\omega) \).

### 3 Cavity retarded Green’s function: negative cavity photon spectral function

In this section we proceed to calculate the cavity retarded Green’s functions (CRGFs) which are defined as

\[
G_{aa}^R(t) = -i\theta(t) \langle [\hat{d}(t), \hat{d}(0)] \rangle, \quad \text{(15a)}
\]
\[
G_{aa}^R(t) = -i\theta(t) \langle [\hat{d}(t), \hat{d}^\dagger(0)] \rangle, \quad \text{(15b)}
\]

where \( \theta(t) \) is the Heaviside step function. It should be noted that the time evolution of the quantum field fluctuations in the Green’s function definitions of Eqs.(15a-15b) are obtained from the QLEs given by Eqs. (2a-2c) and the expectation values have been defined in the steady state of the system.

In the following we will calculate the CRGFs using their equations of motion which are obtained through the QLEs. To drive the equation of motion for \( G_{aa}^R(t) \) (\( G_{aa}^R(t) \)), one should multiply each of the equations in Eq. (3) by \( \hat{a}(t) \) (\( \hat{a}^\dagger(t) \)) on the left and on the right, subtract them from each other and then take their mean values. In this way, the Green’s functions equations of motion can be obtained as the following compact forms

\[
\frac{d}{dt} G_{aa}^R(t) = -i\delta(t) V_{aa} + \chi_{aa}^R(t), \quad \text{(16a)}
\]
\[
\frac{d}{dt} G_{aa}^R(t) = +i\delta(t) V_a + \chi_{aa}^R(t), \quad \text{(16b)}
\]

in which the six-dimensional Green’s functions vectors are defined as follows

\[
G_{aa}^R(t) = \begin{pmatrix} G_{aa}^R(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{aa}^R(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{aa}^R(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{aa}^R(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{aa}^R(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{aa}^R(t) \end{pmatrix};
\]
\[
G_{aa}^R(t) = \begin{pmatrix} G_{aa}^R(t) & G_{aa}^R(t) & G_{aa}^R(t) & G_{aa}^R(t) & G_{aa}^R(t) & G_{aa}^R(t) \end{pmatrix}^T.
\]

Besides, \( V_{aa} := (1, 0, 0, 0, 0, 0)^T \) and \( V_a := (0, 1, 0, 0, 0, 0)^T \) are fixed six-dimensional vectors in Eqs.(16a,16b).

Now, by taking the Fourier transforms of Eqs.(16a) and (16b) one can find the Green’s function vectors in the Fourier space as

\[
G_{aa}^R(\omega) = -i\chi(\omega)V_{aa}, \quad \text{(17a)}
\]
\[
G_{aa}^R(\omega) = +i\chi(\omega)V_a, \quad \text{(17b)}
\]

where \( \chi(\omega) \) is the susceptibility matrix defined by Eq.(6). As is seen from Eqs.(17a) and (17b) the CRGFs in the frequency space, i.e., the Fourier transform of Eqs.(15a) and (15b) can be obtained as

\[
G_{aa}^R(\omega) = -i\chi_{aa}(\omega) = \frac{1}{i\hbar^2 + (\omega - \Sigma_a(\omega))}, \quad \text{(18)}
\]
\[
G_{aa}^R(\omega) = -i\chi_{aa}^i(\omega) = G_{aa}^R(\omega) \mathcal{M}(\omega). \quad \text{(19)}
\]
Note that for $\lambda_{m(d)} = \lambda_{m(d)}$, it is easy to show that $G_{\omega\omega}^R(-\omega) = -G_{\omega\omega}^R(\omega)$ and $G_{\omega\omega}^R(-\omega) = -G_{\omega\omega}^R(\omega)$.

The CPSF plays the role of an effective density of photon states and is defined as\textsuperscript{35,63}

$$\mathcal{A}(\omega) := -2\text{Im}G_{\omega\omega}^R(\omega).$$

We will show in the following that in the present parametrically driven OMS the CPSF, $\mathcal{A}(\omega)$, can be negative by controlling the modulation parameters and cooperativities due to the parametric modulations of the phononic mechanical modes. The on-resonance CPSF can be easily calculated as follows

$$\mathcal{A}(0) = \frac{4}{\kappa} \frac{(\xi_m^2 - 1)(\xi_d^2 - 1) - \mathcal{C}_0(\xi_d^2 - 1) - \mathcal{C}_1(\xi_m^2 - 1)}{(\xi_m^2 - 1)(\xi_d^2 - 1) - 2\mathcal{C}_0\mathcal{C}_1(\xi_m\xi_d - 1) - \mathcal{C}_0'(\mathcal{C}_0 + 2)(\xi_d^2 - 1) - \mathcal{C}_1'(\mathcal{C}_1 + 2)(\xi_m^2 - 1)}$$

$$:= \frac{4}{\kappa} A_N = \frac{4}{\kappa} M' = \frac{1}{M},$$

where $\mathcal{C}_0 = 4\xi_0^2/\kappa \gamma_m$ and $\mathcal{C}_1 = 4\xi_0^2/\kappa \gamma_d$ are the optomechanical cooperativities associated with the two mechanical modes $b$ and $d$, and $\xi_{(m,d)} = 2\lambda_{(m,d)}/\gamma_{(m,d)}$ plays the role of an effective dimensionless amplitude of modulation. As is clearly seen, the on-resonance cavity photon spectral function ($\mathcal{A}(0)$) can be controlled by the effective amplitude of modulations $\xi_{m,d}$ as well as the optomechanical cooperativities $\mathcal{C}_0$ and $\mathcal{C}_1$. It can be easily shown that the denominator $A_D$ in Eq. (21) is always positive while the numerator $A_N$ can be negative in the stable regime. As has been shown in Ref.\textsuperscript{49}, based on the Routh-Hurwitz criterion for the on-resonance condition, the modulation parameters $\lambda_m$ and $\lambda_d$ should satisfy the condition

$$\lambda_{m(d)} \leq \gamma_{m(d)} + \Gamma_{m(d)} = \frac{\gamma_{m(d)}}{2} \left[ 1 + \mathcal{C}_{m(d)} \right],$$

where $\Gamma_{m(d)} = -2\text{Im}\Sigma_{m(d)}(0)$ is the maximum optomechanically induced damping rate in which the self-energies $\Sigma_b$ and $\Sigma_d$ are, respectively, given by Eqs. (26b) and (26c) of Ref.\textsuperscript{49}. Furthermore, $\mathcal{C}_{m(d)}(\mathcal{C}_d)$ is the collective optomechanical cooperativity which is given by

$$\mathcal{C}_{m(d)}(\mathcal{C}_d) = \mathcal{C}_0(1) \frac{1 + \mathcal{C}_1(0) - \xi_{d(m)}^2}{(1 + \mathcal{C}_1(0) - \xi_{d(m)}^2)^2 - \xi_{d(m)}^2} \xi_{d(m)}^2 \gamma_m^2.$$ \hspace{0.5cm} (23)

The maximum values of modulation amplitudes are obtained self-consistently from Eq. (22), as $\lambda_{m(d)}^{\text{max}} = \gamma_{m(d)}/2 \left[ 1 + \mathcal{C}_{m(d)} \right]$. It should be emphasized that so far as the modulation amplitudes are less than their maximum values ($\xi_{m(d)}^{\text{max}} = 2\lambda_{m(d)}/\gamma_{m(d)}$) the system is stable and the approximations we have made are valid because the system is in the weakly interacting regime.

To find the negativity condition or find the optimized parms $\xi_{m(d)}^{\text{opt}}$, for example in on-resonance frequency, one should simultaneously solve inequality $4A_N/A_D = M < 0$ together with the stability condition of Eq. (22); $\xi_{m(d)} \leq 1 + \mathcal{C}_{m(d)}$. In other words, to find the parms corresponding to the negative CPSF, one should require both inequalities

$$4A_N(\xi_{m(d)}^{\text{opt}} \xi_{d(m)}^{\text{opt}}) = M < 0, \quad \& \quad \xi_{m(d)}^{\text{opt}} \leq 1 + \mathcal{C}_{m(d)}.$$ \hspace{0.5cm} (24)

Note that $M$ ($M < M_{\text{max}}$, where $M_{\text{max}}$ is the maximum achievable negativity) is an arbitrary negative value which should satisfy the negativity conditions. Furthermore, the maximum negativity $M_{\text{max}}$ should also satisfy the negativity conditions (24).

Let us consider a special case where just one of the mechanical modes is parametrically driven while the other one remains unmodulated, i.e., $\xi_d = 0$. In this case, the on-resonance CPSF reads as

$$\mathcal{A}(0)|_{\xi_d=0} = \frac{4}{\kappa} \frac{1}{1 + \mathcal{C}_1} \frac{\xi_{m}^{\text{max}} - \xi_m^2}{(\xi_{m}^{\text{max}} - \xi_m^2)^2 - \xi_m^2},$$ \hspace{0.5cm} (25)

where $\xi_{m}^{\text{max}} = 1 + \mathcal{C}_0/(1 + \mathcal{C}_1)$. As is evident from Eq. (21) and (25), the denominators are always positive while the numerators can take negative values. For example, in Eq. (25) if the modulation amplitude lies in the interval $\sqrt{\xi_m^{\text{max}}} \leq \xi_m \leq \xi_{m}^{\text{max}}$ where the strong-coupling regime is avoided, the system is stable while the CPSF becomes negative near the resonance frequency. Note that in the absence of the parametric modulation, the cavity photon spectral function is given by

$$\mathcal{A}(0)|_{\xi_{m,d}=0} = \frac{4}{\kappa} \frac{1}{1 + \mathcal{C}_0},$$ \hspace{0.5cm} (26)

which never becomes negative.

Figure (2) shows the effect of modulation on the CPSF in the case of an OMS with a single MO, when $\mathcal{C}_0 = \lambda_d = \gamma_d = 0$. In Fig. 2(a), we have fixed the cooperativity and plotted the CPSF for different values of the relative amplitude $\xi_m^2/\xi_{m}^{\text{max}}$. As
is evident, in the absence of the modulation (green dot-dashed curve), $\xi_m = 0$, the CPSF remains positive at all frequencies. But, by turning on the parametric drive the CPSF can become negative at frequencies near the resonance frequency (blue dashed line). This negativity can be controlled by adjusting the amplitude of modulation such that for $\xi_m \geq \sqrt{\xi_{m\text{max}}}$ the CPSF becomes negative, $\kappa \omega (\omega) \leq 0$, and the amount of the negativity increases when the strength of the paramp increases until both conditions (24) are satisfied in the stable regime. To examine the effect of cooperativity on the behavior of the CPSF, we have plotted in Fig. 2(b) the CPSF for different values of cooperativity and a fixed value of the relative amplitude paramp $\xi_m/\xi_{m\text{max}} = 0.9$ and different values of cooperativity $\kappa_0 = 0.5, 2, 6.8$ corresponding, respectively, to the black-dotted, green-dot-dashed, blue-dashed and red-solid lines. Here, we have set $\kappa /\gamma_m = 10^4$. Note that in the case of a single driven MO we have $\xi_{m\text{max}} = 1 + \kappa_0$.

Figure 3 shows the effect of modulation when there are two parametrically driven mechanics in the system under consideration. To examine the effects of the amplitude of the modulation and the cooperativity on the CPSF, we have fixed the negativity at $M = -3$ in Fig. 3(a), and then, we have obtained the optimum paramps ($\xi_{m(d)}^{\text{opt}}$) to achieve this negativity through solving Eqs. (24). As is evident, when both paramps are turned on and fixed at their optimized values [see blue-solid and red-dashed curves], the same negativity with smaller paramps is achievable with respect to the case in which only one of the MOs is driven [see black double-dot-dashed curve]. This can be considered as an advantageous feature of two parametrically driven mechanics with respect to one parametrically driven MO, as far as the negativity of the CPSF is concerned. In other words, one can conclude that by increasing the number of parametrically driven modes, as the system degrees of freedom, the CPSF negativity can be amplified. This is in agreement with the results obtained in Refs.12,40, based on the theory of linear quantum amplifiers,76, for enhancing the functionality of a linear amplifier via introducing additional controllable degrees of freedom to the system. Moreover, at the fixed negativity, with increasing the cooperativity $\kappa_1$, smaller paramps are required to achieve that negativity [compare the blue-solid and red-dashed curves in Fig. 3(a)]. Also, when both modulated phononic modes are present, the system is very sensitive to the values of paramps needed to realize the desired negativity [see the green dot-dashed curve in Fig. 3(a) which has only 2% mismatch in its paramps with respect to the optimized one]. In addition, when both mechanical modes are parametrically driven the negative values of the spectral function occur over a very narrow range of frequencies around resonance. Figure 3(b) shows that for the fixed values of both cooperativities, one can achieve different values of negativity in the stable regime by slowly tuning and increasing both paramps. This implies the advantage of the present parametrically driven OMS with two phononic mode with respect to an OMS with a single driven MO.

Let us now explore the effect of the mechanical damping rates on the CPSF. For this purpose, we have illustrated in Fig. (4) the CPSF against the normalized frequency, $\omega / \kappa$, for different values of the ratio $\gamma_m/\gamma_d$. As is seen, for the fixed cooperativities and the optimized paramps, $\xi_{m(d)}^{\text{opt}}$, obtained through Eqs. (24) corresponding to an arbitrary fixed value of the negativity $M$, the negativity bandwidth is increased around the on-resonance frequency by decreasing the ratio $\gamma_m/\gamma_d$. In other words, one can see that in the off-resonance frequency region, larger negativity occurs for higher $\gamma_m/\gamma_d$. It means that in an OMS with two
Figure 3. (Color online) The effects of both modulation amplitudes on the CPSF for an OMS with two parametrically driven mechanics. Here, we have set $\mathcal{C}_0 = 2$. (a) CPSF for the fixed value of $M = -3$, and for different values of the modulation amplitudes ($\xi_m, \xi_d$) and the cooperativity $\mathcal{C}_1$. The black double-dot-dashed, blue-solid, and the red-dashed lines are corresponding, respectively, to ($\mathcal{C}_1 = 0$, $\xi_m = \xi_{m}^{op} = 0.15\xi_{m}^{max} = 1.390$, $\xi_d = \xi_{d}^{op} = 0.971\xi_{d}^{max} = 0.931$); and ($\mathcal{C}_1 = 0.25$, $\xi_m = \xi_{m}^{op} = 0.25\xi_{m}^{max} = 1.353$, $\xi_d = \xi_{d}^{op} = 0.975\xi_{d}^{max} = 0.879$). Also, the green dot-dashed curve shows the non-optimized case referred to ($\mathcal{C}_1 = 0.25$, $\xi_m = \xi_{m}^{op} = 0.98\xi_{m}^{op}$, $\xi_d = 0.98\xi_{d}^{op}$). (b) CPSF for different values of negativity $M = -3$ ($\xi_m = \xi_{m}^{op} = 1.390$, $\xi_d = \xi_{d}^{op} = 0.931$); $M = -10$ ($\xi_m = \xi_{m}^{op} = 1.393$, $\xi_d = \xi_{d}^{op} = 0.935$); $M = -15$ ($\xi_m = \xi_{m}^{op} = 1.394$, $\xi_d = \xi_{d}^{op} = 0.936$) and the fixed value of cooperativity $\mathcal{C}_1 = 0.25$, corresponding, respectively, to the black dot-dashed, blue-solid and the red-dashed lines. Here, in all curves we have set $\kappa/\gamma_m = 10^4$ and $\gamma_m/\gamma_d = 1$. Note that the optimized params are obtained by numerically solving Eqs. (24) which determine the optimized negativity conditions.

Figure 4. (Color online) The effect of damping rate ratio $\gamma_m/\gamma_d$ on the CPSF in the case of an OMS with two parametrically driven mechanics. The green-solid, red-dashed, and blue-dotted curves are, respectively, corresponding to $\gamma_m/\gamma_d = 0.2, 1.5$. All curves have plotted for the optimized paramps $\xi_m = 1.390$, $\xi_d = 0.931$ corresponding to the negativity $M = -3$ through solving Eq. (24) for the fixed cooperativities $\mathcal{C}_0 = 2$ and $\mathcal{C}_1 = 0.25$. Here, we have set $\kappa/\gamma_m = 10^4$. 

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parametrically driven mechanics, one can have more negativity for off-resonance frequencies by controlling the mechanical damping rates ratio.

Here, it is worth mentioning that in an experiment by measuring the output cavity spectra, one can directly reconstruct the scattering matrix \( s_{\text{cavity}} = 1 - \kappa \chi_{\text{cavity}} \) which consequently yields \( \mathscr{A}(\omega) = 2 \text{Re} \chi_{\text{det}}(\omega) \).

To sum up so far, it can be concluded that the OMS with two parametrically driven MOs provides more controllability to achieve the more negative CPSF in the stable regime with weaker paramps, as compared to the system with a single driven MO. It is worth mentioning that the occurrence of negative CPSF has also been predicted recently in a driven-dissipative nonlinear quantum van-der Pol oscillator, describing a single-mode cavity with Kerr nonlinearity subject to incoherent driving and two-photon losses. By using a generalized Lehmann representation of the Green’s function of Markovian open quantum systems governed by a Lindblad-type master equation, it has been shown that the interplay of interaction and non-equilibrium effects can give rise to a negative density of states, or equivalently a negative CPSF, which can be associated with an effective frequency-dependent negative temperature for cavity photons, even in the absence of steady-state population inversion in the system density matrix.

In the two following sections, firstly, we present the physical interpretation of the negative CPSF in our system by comparing to the OPA or DPA. After that, we shortly present some physical consequences of the negative CPSF, i.e., modified OMIT with probe reflectivity above the unity and negative effective temperature for the intracavity photons.

4 comparison to the OPA

4.1 negativity conditions for OPA

We are interested in comparison between the CPSF of the present system and that of an OPA since our system is effectively with \( F = \omega_0 \) and \( \omega_0 > 0 \) with \( \omega_0 = \omega_0 - \omega_\Delta \). Therefore, the CPSF for the detuned OPA can be found as follows

\[
\begin{align*}
\chi_{\text{det}}^{(\text{OPA})}(\omega) &= \frac{1}{(\kappa/2-i\omega_+)(\kappa/2-i\omega_-)-|\lambda|^2} \left( \frac{\kappa/2-i\omega_-}{\lambda^*} \frac{\lambda}{\kappa/2-i\omega_+} \right), \\
\mathscr{R}^{(\text{OPA})}_{\text{det}}(\omega) &= -2 \text{Im} \chi_{\text{det}}^{(\text{OPA})}(\omega) = -\frac{1}{\kappa \left( \kappa^2/4 - |\lambda|^2 - (\omega^2 - \Delta_p^2) \right)} \left[ \frac{\omega(\omega - \Delta_p)}{\kappa^2/4 - |\lambda|^2 - (\omega^2 - \Delta_p^2)} \right].
\end{align*}
\]

Therefore, the CPSF for the detuned OPA can be found as follows

\[
\mathscr{A}^{(\text{OPA})}_{\text{det}}(\omega) = -2 \text{Im} \mathscr{R}^{(\text{OPA})}_{\text{det}}(\omega) = -\kappa \frac{\left( \kappa^2/4 - |\lambda|^2 - (\omega^2 - \Delta_p^2) \right) - 2\omega(\omega - \Delta_p)}{\kappa^2/4 - |\lambda|^2 - (\omega^2 - \Delta_p^2) + \omega^2 \kappa^2}.
\]

The stability condition, \( \text{det} \chi_{\text{det}}^{(\text{OPA})} > 0 \), requires that the poles of the susceptibility lie in the lower half-plane. For the detuned-OPA the stability condition becomes \( |\lambda|^2 < \kappa^2/4 + \Delta_p^2 \). In order to examine the negativity of the CPSF for an OPA, let us rewrite Eq. (30) as follows

\[
\mathscr{A}^{(\text{OPA})}_{\text{det}}(\omega) = \frac{\kappa}{\kappa^2/4 - |\lambda|^2 - (\omega^2 - \Delta_p^2)} \left( \omega^2 - 2\Delta_p \omega + S \right) = \frac{\kappa}{\kappa^2/4 - |\lambda|^2 - (\omega^2 - \Delta_p^2)} \left( \omega^2 - 2\Delta_p \omega + S \right) > 0,
\]

with \( F(\omega) = \omega^2 - 2\Delta_p \omega + S \) and \( S = \kappa^2/4 + \Delta_p^2 - |\lambda|^2 > 0 \) according to the the stability condition. It follows from Eq. (31) that the stable negativity in CPSF occurs if \( F(\omega) < 0 \) for any positive frequency \( \omega > 0 \). Evidently, for the on-resonance condition we have \( F(0) = S \) which is always positive in the stability region and thus the spectral function never becomes negative in on-resonance frequency \( \mathscr{A}^{(\text{OPA})}_{\text{det}}(\omega = 0) > 0 \).
We first determine the ECDR for a detuned OPA whose self-energy matrix, \( \Delta \), the green thick-solid and purple loosely-dashed lines refer to zero detuning \( \Delta_p = 0 \), respectively, for \( \xi_k = 0 \) and \( \xi_k = 1 \). Red densely-dotted and blue double-dot dashed lines refer to the same paramp \( \xi_k = 1.2 \), respectively, for \( \Delta_k = 0.5 \) and \( \Delta_k = -0.5 \). Black densely-dashed and orange-solid lines refer to \((\xi_k = 1.25, \Delta_k = 0.5)\) and \((\xi_k = 1.2, \Delta_k = -0.6)\), respectively.

Let us now see what happens with the off-resonance frequencies for the OPA or detuned OPA. In this case, to find the negativity condition in off-resonance frequencies for the CPSF, one must solve inequality \( F(\omega > 0) < 0 \) which yields the constraint \( \kappa^2/4 < |\lambda|^2 < \kappa^2/4 + \Delta^2_p \). It is clear that the CPSF for nondetuned OPA, i.e., when \( \Delta_p = 0 \), never becomes negative in off-resonance frequencies. In other words, as the detuned OPA is stable the CPSF can be negative in off-resonance frequencies \( \Delta_p - \Delta \omega \leq \omega \leq \Delta_p + \Delta \omega \) where \( \Delta \omega = \sqrt{\Delta^2_p - \frac{5}{2}} = \sqrt{|\lambda|^2 - \kappa^2/4} \) when \( |\lambda|^2 < \kappa^2/4 + \Delta^2_p \).

In Fig. (5), we have plotted the CPSF for the nondetuned and detuned OPA (Eq.(30)). As is evident for the case of nondetuned OPA (see the green thick-solid and purple loosely-dashed lines), the CPSF never becomes negative and has a positive peak at the resonance frequency which decreases with increasing the effective paramp \((\xi_k = 2\lambda/\kappa)\) to its maximum value \( \xi_k = 1 \). Note that the case of zero detuning and absence of paramp corresponds to an empty cavity subjected to dissipation. However, for the detuned OPA, the CPSF can be negative in off-resonance frequencies as expected.

For a fixed value of the effective paramp \( \xi_k \) but opposite values of effective detuning \( \Delta_k \) (see red densely-dotted and blue double-dot dashed lines in Fig. (5)) the height of the peaks in CPSF is the same, while their locations are dependent on the sign of the detuning \( \Delta_k \). Moreover, the spectral function exhibits negativity over a range of positive (negative) frequencies for positive (negative) detuning \( \Delta_k \). On the other hand, for a fixed value of the effective detuning \( \Delta_k \) the spectral function negativity is enhanced with increasing the paramp \( \xi_k \) (see red densely-dotted and black densely-dashed lines in Fig. (5)). It is also seen from Fig. (5) (see orange-solid and blue double-dot dashed lines) that for a fixed value of the effective paramp \( \xi_k \) a larger value of detuning \( \Delta_k \) leads to smaller negativity of the spectral function.

Here, it is worth remarking that unlike the detuned OPA, the parametrically driven OMS considered in this paper can exhibit stable negative spectral function at on-resonance frequency \( \omega = 0 \). This is because by controlling the system parameters, in particular the effective induced paramp, the spectral function negativity can be achieved even at \( \omega = 0 \) due to the negative ECDR, which we will explain in the next subsection.

4.2 physical interpretation of the negative CPSF

In what follows, we are going to illustrate the physics behind the spectral function negativity in the stable regime in OPA and in our parametrically-driven OMS, and thus, compare them together.

As was shown in the previous subsection, the stable negativity in the cavity spectral function of an OPA occurs only in the detuned regime \( \Delta_p \neq 0 \) due to the extended stability region. However, we show that the negativity of the CPSF in our system originates from the negativity of the frequency dependent ECDR defined as

\[
\kappa_{\text{eff}}(\omega) := \kappa - 2 \text{Im} \Sigma_\omega(\omega) = \kappa + \kappa_{\text{opt}}(\omega). \tag{32}
\]

We first determine the ECDR for a detuned OPA whose self-energy matrix, \( \Sigma^{\text{(OPA)}}(\omega) = i(\chi^{-1}_0 - (\chi^{-1}_{\text{(det)}})^{-1}) \), on using Eq. (28),
is obtained as
\[ \Sigma^{(\text{OPA})}_{\text{det}}(\omega) = i \begin{pmatrix} 0 & \lambda \\ \lambda^* & 0 \end{pmatrix}, \] (33)
which is clearly frequency-independent and purely off-diagonal. Therefore, \( \kappa_{\text{eff}}^{(\text{OPA})} = \kappa > 0 \) indicating that the EDCR for a detuned OPA is always positive.

Despite of the OPA, we will show that in our system the ECDR can be negative, \( \kappa_{\text{eff}}(\omega) < 0 \), which reveals the physics behind this negativity in comparison to detuned OPA. In fact, the negative \( \kappa_{\text{eff}}(\omega) \) in our system is because of its frequency dependency through frequency dependency of the cavity self-energy \( \text{Im} \Sigma_\omega(\omega) \) due to \( \tilde{\lambda}_\omega(\omega) \) which originates from the coherent time-modulations of the mechanical spring coefficients of the MOs.

Now, we return to our system and calculate its ECDR. For this purpose, let us first make a comparison between the parameters for an ordinary OPA and the optomechanical system under consideration. In an ordinary OPA when its parametric drive strength satisfies the relation \( (\lambda)^* = \lambda^* \), i.e., the real paramp, the interaction is called a \textit{coherent interaction}.

Generally in our parametrically-driven system, a frequency-dependent effective parametric drive like \( \tilde{\lambda}_\omega(\omega) \) satisfies the relation \( (\tilde{\lambda}_\omega(\omega))^* = \tilde{\lambda}_\omega^*(-\omega) \). If \( \tilde{\lambda}_\omega(\omega) \) satisfies the relation \( (\tilde{\lambda}_\omega(\omega))^* = \tilde{\lambda}_\omega(\omega) \) the system behaves like an ordinary OPA in a coherent regime. In this situation, \( \tilde{\lambda}_\omega(\omega) \) can be considered as an effective coherent interaction strength whose real and imaginary parts might lead to the coherent and dissipative behaviors. In this way, two different regimes can be distinguished depending on which part is larger than the other. In the so-called \textit{coherent regime} where \( 35, 49 \),

\[ \left| \frac{\text{Im} \tilde{\lambda}_\omega(\omega)}{\text{Re} \tilde{\lambda}_\omega(\omega)} \right| \ll 1 \] (34)
is satisfied [for example, it can be satisfied in the largely different cooperativities regime with nonzero modulations such that \( \lambda_{\text{m(d)}}(\omega) > 1 \)]

Because of the frequency dependence of \( \tilde{\lambda}_\omega(\omega) \) and \( \Sigma_\omega(\omega) \), we can not directly map our system to an OPA. However, in the on-resonance case, the effective susceptibility matrix becomes exactly similar to an OPA but with the effective parametric strength \( \tilde{\lambda}_\omega(0) \) instead of \( \lambda \) and the effective cavity damping rate \( \kappa_{\text{eff}}(0) = \kappa - 2\text{Im} \Sigma_\omega(0) \) (with \( \kappa_{\text{opt}} = -2\text{Im} \Sigma_\omega(0) \)) instead of \( \kappa \).

The imaginary parts of the cavity self-energy and modified cavity self energy in on-resonance are, respectively, given by
\[ \text{Im} \Sigma_\omega(0) = -\left[ g^2 \frac{\nu_m}{4} |\lambda_m|^2 + g^2 \frac{\nu_d}{4} |\lambda_d|^2 \right] = -\frac{\kappa}{2} \left[ \frac{\gamma_0}{1 - \xi_m^2} + \frac{\gamma_1}{1 - \xi_d^2} \right] = -\frac{\kappa}{2} \gamma_a \] (35)
\[ \text{Im} \Sigma_\omega(0) = \text{Im} \Sigma_\omega(0) - \frac{[\tilde{\lambda}_\omega(0)]^2}{i \kappa / 2 - \Sigma_\omega(0)} = -\frac{\kappa}{2} \left[ \gamma_a - \frac{\gamma_a^2}{1 + \gamma_a} \right] = -\frac{\kappa}{2} \gamma_a'' \] (36)
where we have defined
\[ \tilde{\lambda}_\omega(0) = \frac{\kappa}{2} \left[ \frac{2\gamma_0 \xi_m}{1 - \xi_m^2} + \frac{2\gamma_1 \xi_d}{1 - \xi_d^2} \right] := \frac{\kappa}{2} \gamma_a' \] (37)
Note that the optomechanical cooperativities \( \gamma_a', \gamma_a'', \gamma_a \) are real parameters. In the stable regime and when the condition (34) is satisfied the parameters \( \gamma_a', \gamma_a'' \) become negative (for \( \lambda_{\text{m(d)}} > 1 \) we have \( -0.5 < \gamma_a < 0 \) and \( -2 < \gamma_a'' < -1 \)) which in turn lead to a negative optomechanically-induced cavity damping rate, \( \kappa_{\text{opt}} = -2\text{Im} \Sigma_\omega(0) < 0 \). Consequently, the effective cavity damping rate \( \kappa_{\text{eff}} = \kappa + \kappa_{\text{opt}}(\omega) = \kappa(1 + \gamma_a'') \) decreases, and can even become negative by controlling the system parameters. As Eq. (36) shows, the parameter \( \gamma_a'' \) plays the key role for the occurrence of this negativity. As a numerical example, for the parameters given in Fig. (3) corresponding to \( M = -3 \), we obtain \( \kappa_{\text{eff}} / \kappa \approx -1.3 \). Thus, in this analogy between our system and OPA in the on-resonance frequency the negative CPSF \( (\sigma < 0) \) is responsible for the negative ECDR \( (\kappa_{\text{eff}} < 0) \). Therefore, the on-resonance CPSF \( (\sigma < 0) \) for our system can be found by directly substituting \( \kappa \rightarrow \kappa_{\text{eff}}(0) \) and \( \lambda \rightarrow \tilde{\lambda}_\omega(0) \) into \( A_{\text{det}}^{(\text{OPA})}(\omega)_{\Delta p = 0} \). Note that this analogy cannot be seen as a coincidence, it only shows that the CPSF in our system looks exactly like that of a non-equilibrium OPA which can provide the link between the negative CPSF and associated negative photon temperature in our system.

5 concluding remarks, discussion and outlooks

5.1 summary and conclusion

In summary, as a novel feature of the OMSs we have theoretically proposed and investigated an experimentally feasible scheme to generate \textit{negative} CPSF in an OMS with two parametrically driven MOs in the red-detuned and weak coupling regimes through the coherent modulation of their spring coefficients.
Using the equations of motion for the CRGF, we have calculated the CPSF and have shown when the induced-frequency-dependent ECDR becomes negative, the CPSF becomes negative while the system still remains in the stable regime unlike the conventional OPA. This negativity can be controlled by the optomechanical cooperativities and amplitudes of modulation of MOs. In addition, the OMS with two parametrically driven MOs provides more controllability to achieve the more negative CPSFs in the stable regime with weaker parameters, as compared to the system with a single parametrically driven MO.

The comparison of our parametrically-driven OMS with the conventional OPA and detuned OPA shows that the CPSF negativity never occurs in the conventional OPA in the stable regime while occurs in the detuned OPA in the stable regime. However, the nature of the negativity in our system is totally different. This negativity originates from the negative ECDR because of its frequency dependence while the ECDR in the detuned OPA is zero and the CPSF negativity stems from the extension of the stability condition of the susceptibility matrix due to the presence of the cavity detuning.

5.2 Outlooks and Discussion

As an interesting outlook of the continuation of the present work, let us now briefly discuss how the negative CPSF may lead to some experimentally observable signatures such as modified OMIT and NET which are, respectively, corresponding to the reflectivity above the unity and qubit-cavity population inversion.

Optomechanically induced suppression of the effective density of photon states at the cavity resonance frequency, whose role is played by the CPSF $\alpha$, is known as the standard OMIT which can be interpreted as an effective density of single-particle states. This means that incident near-resonant photons ($\alpha_{\text{std}}(0) \to 0$) in the weakly coupled auxiliary mode are perfectly reflected ($R \approx 1$). Therefore, when probe photons go into the cavity there is no available states, and, thus, all photons perfectly reflect, i.e., a reflective narrow-band optomechanical filter is realized. In other words, in the standard-OMIT (when $\xi_{m,d} = 0$), the simultaneous injection of a strong control driving laser and a weak probe laser with coherence times longer than the effective mechanical damping rate into the red-detuned sideband of an OMS provides cavity resonance which leads to the transparency of the weak probe laser. On the other hand, the beat of the probe field and the driving laser induces a time-dependent radiation pressure force. If the beat frequency matches the frequency of the MO, then the MO is driven resonantly which in turn creates sidebands on the intracavity field where by considering the strong coupling laser in the resolved sideband regime, the lower sideband is far off cavity resonance and can be neglected. In contrast, the upper sideband of the driving laser, created by the mechanical motion, has precisely the same frequency as the probe field and is moreover phase-coherent with the probe field. This leads to an interference that yields a cancellation of the intracavity field on resonance, giving rise to the transparency window. The phenomenon thereby results from the destructive interference between reflection amplitudes for photons scattered from the driving laser and photons of the probe field. But, in the OMS presented here we assume that in addition to the red-detuned input/output driving laser port, the cavity is extremely weakly coupled to an auxiliary waveguide mode $\hat{c}$ at the small rate $\kappa' \ll \kappa$ with the output field $\hat{c}_{\text{out}}(\omega) = r_p(\omega)\hat{c}_{\text{in}}(\omega) - i\kappa'G^R(\omega)\hat{c}_{\text{out}}(\omega)$ with the amplitude of reflection as $r_p \equiv 1 - \kappa'G^R(\omega) = |r_p(\omega)|e^{i\delta_p(\omega)}$. If we consider a detuning between the frequency of the probe laser and that of the cavity, $\delta_p = \omega - \omega_0$ in the lab frame, one can ignore the second term proportional to $\chi_{\text{std}}^{(1)}$ because of $\langle \hat{c}_{\text{out}}^{\dagger}(\omega) \rangle \approx 0$ due to the RWA. Therefore, the reflection power of the probe mode from the cavity in the limit of $\kappa' \ll \kappa$ is $\mathcal{R}(\omega) \approx 1 - \kappa'\alpha(\omega)$ which also holds for $\omega = 0$ if one averages over the phase of the incident drive in the waveguide, that in this case, the contribution of the off-diagonal susceptibility in output field averages away. It is clear that if $\alpha(\omega) = 0$, the probe signal can be reflected with a coefficient larger than 1 ($\mathcal{R} > 1$) which never occurs in the standard-OMIT or even in the standard OPA which their CPSF is always positive. As we have shown, the OMS with two parametrically driven MOs offers more controllability to achieve the more negative CPSF compared to a system with a single driven MO. Therefore, we may expect that the parametrically driven OMS considered in this work provides the possibility of engineering a tunable optomechanical filter with the probe reflection above the unity. This issue will be addressed in detail in our future work. As is clear, the zero CPSF is responsible for the standard-OMIT or probe reflection near the unity at near resonance frequency, $\mathcal{R}(\omega \sim 0) \approx 1$. Interestingly, in the presented model by turning on two parametric drives through the time-modulation and by controlling the system parameters such as cooperativities, amplitudes of modulation and also the mechanical damping rates ratio, when CPSF becomes negative one can attain the probe reflection above the unity at near resonance frequency, i.e., extraordinary or modified OMIT. Let us interpret what happens in extraordinary OMIT: as the CPSF decreases, the available states for the incident probe mode to occupy decreases such that when $\alpha \to 0$, there are no available states for incident photons and thus perfectly reflection occurs, $\mathcal{R} \to 1$. While if the CPSF becomes negative, $\alpha < 0$, not only there are no available states for the incident probe photons but also there are not enough states for the intracavity photons. It means that in addition to the incident probe photons, some intracavity photons return and reflect yielding to reflection above the unity, $\mathcal{R} \geq 1$.

Let us now return to the CRGF in the lab frame where the negativity in CPSF and also OMIT phenomenon occur at near resonance $\omega \approx \omega_0$. Negative CPSF at the positive frequencies, i.e., $\alpha' (\omega > 0) < 0$, is responsible for a stationary population inversion between stationary eigenstates separated by $\hbar\omega$ for a time-independent Hamiltonian in a time-independent state $\rho_{\text{std}}$.

Although our system as an open quantum system with time-dependent Hamiltonian interacts with environment,
but the negative CPSF is still responsible for the population inversion. Therefore, it is worthwhile to address the effective temperature of the intracavity photons which can be obtained via the quantum noise properties of the system. As discussed in Refs. 35, 70, 76, the effective temperature can be defined using the quantum noise properties of photons field by comparing the size of the classical symmetrized photon correlation function to the size of the spectral function as $T_{\text{eff}}(\omega) = \frac{2\coth \frac{\omega}{2T}}{\omega G^k(\omega)}$ where $G^k(\omega) = i \int d\tau e^{i\omega \tau} \langle \{ \hat{a}(t + \tau), \hat{a}^\dagger(t) \} \rangle$ stands for the so-called Keldysh Green’s function 70, 77, and $\langle \{ \hat{a}(t + \tau), \hat{a}^\dagger(t) \} \rangle$ is a classical symmetrized photon correlation function. If the system is not in equilibrium, $T_{\text{eff}}$ depends explicitly on frequency, i.e., $T_{\text{eff}} = T_{\text{eff}}(\omega)$. In thermal equilibrium, effective temperature is exactly the temperature of the system at all frequencies, i.e., $T_{\text{eff}}(\omega) = T$ for all $\omega$. Out of equilibrium, for positive frequencies ($\omega > 0$) as $-iG^k(\omega)$ is always positive definite, a negative CPSF at positive frequency ($\omega > 0$) necessarily leads to a frequency-dependent NET for the intracavity photons, $T_{\text{eff}}(\omega > 0) < 0$ 35. As discussed in Refs. 35, 70, 76 the effect of NET can be directly probed by coupling the cavity field weakly to an auxiliary probe qubit with a splitting frequency equal to the cavity frequency. Then, the intracavity photons act as an effective engineered bath for the qubit. The steady-state population inversion of the qubit depends on the effective cavity temperature at cavity frequency as $\langle \hat{\sigma}_z \rangle = -\tanh (\hbar \beta_{\text{eff}}(\omega_c) / 2)$ with $\beta_{\text{eff}}(\omega_c) = k_B^{-1} T_{\text{eff}}^{-1}(\omega_c)$ 76. Thus, a NET for the intracavity photons can directly translate into a simple population inversion of the qubit. In light of this issue, a particular focus in our future work will be the investigation of realizing a controllable NET for the intracavity photons in the stable regime of a hybrid qubit-OMS with two parametrically driven mechanical modes.

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Author contributions statement

The authors’ contribution is as follows: AMF proposed and developed the primary idea, carried out all analytical calculations and numerical codes, and provided the first draft of the manuscript. AD and MHN checked the results, and participated in writing and editing the manuscript.

Competing Interests

The authors declare no competing interests.

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