On open and closed convex codes

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Abstract. Neural codes serve as a language for neurons in the brain. Convex codes, which arise from the pattern of intersections of convex sets in Euclidean space, are of particular relevance to neuroscience. Not every code is convex, however, and the combinatorial properties of a code that determine its convexity are still poorly understood. Here we find that a code that can be realized by a collection of open convex sets may or may not be realizable by closed convex sets, and vice versa, establishing that open convex and closed convex codes are distinct classes. We also prove that max intersection-complete codes (i.e. codes that contain all intersections of maximal codewords) are both open convex and closed convex, and provide an upper bound for their minimal embedding dimension. Finally, we show that the addition of non-maximal codewords to an open convex code preserves convexity.

1. Introduction.

The brain represents information via patterns of neural activity. Often, one can think of these patterns as strings of binary responses, where each neuron is “on” or “off” according to whether or not a given stimulus lies inside its receptive field. In this scenario, the

receptive field $U_i \subset X$ of a neuron $i$ is simply the subset of stimuli to which it responds, with $X$ being the entire stimulus space. A collection $U = \{U_1, \ldots, U_n\}$ of receptive fields for a population of neurons $[n] \overset{\text{def}}{=} \{1, \ldots, n\}$ gives rise to the combinatorial code

$$\text{code}(U, X) \overset{\text{def}}{=} \{\sigma \subseteq [n] \text{ such that } A^U_\sigma \neq \emptyset\} \subseteq 2^{[n]},$$

where $2^{[n]}$ is the set of all subsets of $[n]$, and the atoms $A^U_\sigma$ correspond to regions of the stimulus space carved out by $U$:

$$A^U_\sigma \overset{\text{def}}{=} \left(\bigcap_{i \in \sigma} U_i\right) \setminus \bigcup_{j \notin \sigma} U_j \subseteq X.$$ Here every stimulus $x \in A^U_\sigma$ gives rise to the same neural response pattern, or codeword, $\sigma \subseteq [n]$. By convention, $\cap_{i \in \emptyset} U_i = X$ and thus $A^U_\emptyset = X \setminus \bigcup_{i=1}^n U_i$, so that $\emptyset \in \text{code}(U, X)$ if and only if $X \neq \bigcup_{i=1}^n U_i$. Note that code$(U, X)$ may fail to be an abstract simplicial complex; see e.g. Figure 1.1

Definition 1.1. We say that a combinatorial code $C \subseteq 2^{[n]}$ is open convex if $C = \text{code}(U, X)$ for a collection $U = \{U_i\}_{i=1}^n$ of open convex subsets $U_i \subseteq X \subseteq \mathbb{R}^d$ for some $d \geq 1$. Similarly, we say that $C$ is closed convex if $C = \text{code}(U, X)$ for a collection of closed convex subsets $U_i \subseteq X \subseteq \mathbb{R}^d$. For an open convex code $C$, the embedding dimension $\text{odim} (C)$ is the minimal $d$ for which there exists an open convex realization of $C$ as code$(U, X)$. Similarly, for a closed convex code $C$, $\text{cdim} (C)$ is the minimal $d$ that admits a closed convex realization of $C$.

1A combinatorial code is any collection of subsets $C \subseteq 2^{[n]}$. Each $\sigma \in C$ is called a codeword.
Figure 1.1. An example of a cover $U = \{U_i\}$ and its code, $C = \text{code}(U, X) = \{\emptyset, 2, 3, 12, 23, 34, 123\}$, where $X = \mathbb{R}^2$. Here we denote a codeword $\{i_1, i_2, \ldots, i_k\} \in C$ by the string $i_1i_2\ldots i_k$; for example, $\{1, 2, 3\}$ is abbreviated to 123. Since $13 \notin C$ but $13 \subset 123$, $C$ is not a simplicial complex.

Convex codes have special relevance to neuroscience because neurons in a number of areas of mammalian brains possess convex receptive fields. A paradigmatic example is that of hippocampal place cells [11], a class of neurons in the hippocampus that act as position sensors. Here the relevant stimulus space $X \subset \mathbb{R}^d$ is the animal’s environment, with $d \in \{1, 2, 3\}$ [13]. Receptive fields can be easily computed when both the neuronal activity data and the relevant stimulus space are available. However, in many situations the relevant stimulus space for a given neural population may be unknown. This raises the natural question: how can one determine from the intrinsic properties of a combinatorial code whether or not it is an open (or closed) convex code? What is the embedding dimension of a code – that is, what is the dimension of the relevant stimulus space? How are open and closed convex codes related?

The code of a cover carries more information about the geometry/topology of the underlying space than the nerve of the cover. For example, it imposes more constraints on the embedding dimension than what is imposed by the nerve [3]. Arrangements of convex sets are ubiquitous in applied and computational topology, however all the standard constructions (e.g. the Čech complex) rely only on the nerve of the cover, and do not carry any information about the arrangement beyond the nerve. While the properties of nerves of convex covers were previously studied in [7, 8, 12], codes of convex covers are much less understood. Moreover, although any simplicial complex can be realized as the nerve of a convex cover (in high enough dimension), not all combinatorial codes can be realized from such convex set arrangements in Euclidean space.

There is currently little understanding of what makes a code convex beyond ‘local obstructions’ to convexity [5][2]. Furthermore, local obstructions can only be used to show that a code is not convex, and the absence of local obstructions does not guarantee convexity of the code [10]. To show that a code is convex, one must produce a convex realization, and there are few results that guarantee such an open (or closed) convex realization exists. Our first main result makes significant progress in this regard, as it provides a general condition for determining that a code is convex from combinatorial properties alone. Specifically, we show that max intersection-complete codes – i.e., codes that contain all intersections of their maximal codewords – are both open convex and closed convex.

**Theorem 1.2.** Suppose $C \subset 2^{[n]}$ is a max intersection-complete code. Then $C$ is both open convex and closed convex. Moreover, the embedding dimensions satisfy $\text{odim}(C) \leq \max\{2, (k - 1)\}$ and $\text{cdim}(C) \leq \max\{2, (k - 1)\}$, where $k$ is the number of maximal codewords of $C$.

\[^2\text{A codeword in } \sigma \in C \text{ is maximal if, as a subset } \sigma \subseteq [n], \text{ it is not contained in any other codeword of } C.\]
The fact that max intersection-complete codes are open convex was first hypothesized in [2], where it was shown that these codes have no local obstructions. In our proof we provide an explicit construction of the convex realizations and the upper bound for the corresponding embedding dimensions. Our next main result shows that open convex codes exhibit a certain type of monotonicity, in the sense that adding non-maximal codewords to an open convex code preserves convexity.

**Theorem 1.3.** Assume that a code $C \subset 2^{[n]}$ is open convex. If $D \supset C$ has the same maximal codewords as $C$, then $D$ is also open convex and has embedding dimension $\text{odim} D \leq \text{odim} C + 1$.

It is currently unknown if the monotonicity property holds for closed convex codes.

Finally, we establish that open convex codes and closed convex codes are distinct classes. This motivates us to define a non-degeneracy condition on the cover; we then show that this condition guarantees that the corresponding code is both open convex and closed convex (see Theorem 2.12, Section 2.3). This result suggests that combinatorial properties of convex codes are richer than originally believed. We propose that codes that are both open convex and closed convex are the most relevant to neuroscience, as the intrinsic noise in neural responses [9] makes it unclear whether receptive fields should be considered to be open or closed.

2. **Convex codes.**

We begin with observing that without sufficiently strong assumptions about the cover $U = \{U_i\}$, any code can be realized as $\text{code}(U, X)$.

**Lemma 2.1.** Every code $C \subset 2^{[n]}$ can be obtained as $C = \text{code}(U, X)$ for a collection of (not necessarily convex) $U_i \subset \mathbb{R}^1$.

**Proof.** It suffices to consider the case where each $i \in [n]$ appears in some codeword $\sigma \in C$. For each $\sigma \in C$, choose points $x_\sigma \in \mathbb{R}^1$ such that $x_\sigma \neq x_\tau$ if $\sigma \neq \tau$. Define $U_i = \{x_\sigma \mid i \in \sigma\}$ and $U = \{U_i\}_{i \in [n]}$. If $\emptyset \in C$, then $C = \text{code}(U, \mathbb{R}^1)$. Otherwise, $C = \text{code}(U, X)$, where $X = \cup_{\sigma \in C} \{x_\sigma\}$. □

The sets $U_i$ in the above proof are finite subsets of $\mathbb{R}^1$. However, even if one requires that the sets $U_i$ be open and connected, almost all codes can still arise as the code of such cover.

**Lemma 2.2.** Any code $C \subset 2^{[n]}$ that contains all singleton codewords, i.e. $\forall i \in [n], \{i\} \in C$, can be obtained as $C = \text{code}(U, X)$ for a collection of open connected subsets $U_i \subset \mathbb{R}^3$.

**Proof.** Similar to the proof of Lemma 2.1, one can place disjoint open balls $B_\sigma \subset \mathbb{R}^3$ for each $\sigma \in C$ and define $U_i = (\cup_{\sigma \in C} B_\sigma) \cup T_i$, where each $T_i \subset \mathbb{R}^3$ is a collection of open “narrow tubes” that connect all the balls $B_\sigma$ with $\sigma \ni i$. Because these sets are embedded in $\mathbb{R}^3$, the “tubes” $T_i$ can always be arranged so that for each $i \neq j$ the intersections $T_i \cap T_j$ are contained in the union of the balls $B_\sigma$. By construction, these $U_i$ are connected and open and $C = \text{code}(U, (\cup_{i=1}^n U_i) \cup B_\sigma)$. □

The condition of having all singleton words can not be relaxed without any further assumptions. For example, it can be easily shown that the code $C = \{\emptyset, 1, 2, 13, 23\}$, previously described in [3, 5] cannot be realized as a code of a cover by open connected sets.

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3Indeed, assuming the converse, it follows that $U_1 = (U_1 \cap U_3) \cup (U_2 \cap U_3)$ and, since this code does not contain a codeword $\sigma \ni 12$, we conclude that $U_1 \cap U_2 = \emptyset$ and $U_3$ is a union of two disjoint open sets, which yields a contradiction.
2.1. Local obstructions to convexity. Any combinatorial code \( C \subset 2^{[n]} \) can be completed to an abstract simplicial complex \( \Delta(C) \), the \textit{simplicial complex of the code}, which is the minimal simplicial complex containing \( C \). Note that \( \Delta(C) \) is determined solely by the maximal codewords of \( C \) (facets of \( \Delta(C) \)). A code can thus be thought of as a simplicial complex with some of its non-maximal faces ‘missing’. Moreover, given a collection of sets \( \mathcal{U} \) and \( X \), one can easily see that the simplicial complex of code(\( \mathcal{U}, X \)) is equal to the usual nerve of the cover \( \mathcal{U} \):

\[
\Delta(\text{code}(\mathcal{U}, X)) = \text{nerve}(\mathcal{U}) \overset{\text{def}}{=} \{ \sigma \subseteq [n] \text{ such that } \bigcap_{i \in \sigma} U_i \neq \emptyset \}.
\]

For example, Figure 11 depicts a code of the form \( C = \text{code}(\mathcal{U}, X) \) that differs from its simplicial complex \( \Delta(C) \) because the subset \( \{1, 3\} \) is missing. This results from the fact that \( U_1 \cap U_3 \subseteq U_2 \), a set containment that is not encoded in nerve(\( \mathcal{U} \)).

Not every code arises from a closed convex or open convex cover. For example, the code \( C = \{\emptyset, 1, 2, 13, 23\} \) above cannot be an open (or closed) convex code. The failure of this code to be convex is “local” in that it is missing the codeword 3, and adding new codewords which do not include \( i = 3 \) would not make this code convex.

**Definition 2.3.** For any \( \sigma \subset [n] \) the \textit{link} of \( C \) at \( \sigma \) is the code \( \text{link}_\sigma C \subset 2^{[n] \setminus \sigma} \subset 2^{[n]} \) on the same set of neurons, defined as

\[
\text{link}_\sigma C \overset{\text{def}}{=} \{ \tau \mid \tau \cup \sigma \subset C \text{ and } \tau \cap \sigma = \emptyset \}.
\]

Note that the link of a code is typically \textit{not} a simplicial complex, but the simplicial complex of a link is the usual \( \text{link}_\sigma \Delta = \{ \nu \in \Delta \mid \nu \cup \sigma \in \Delta, \text{ and } \nu \cap \sigma = \emptyset \} \) of the appropriate simplicial complex \( \text{link}_\sigma \Delta(C) = \text{link}_\sigma \Delta(\text{code}(\mathcal{U}, X)) \).

Moreover, it is easy to see that if \( C = \text{code}(\mathcal{U}, X) \), then for every non-empty \( \sigma \in \Delta(C) \)

\[
\text{link}_\sigma C = \text{code} \left( \{ U_j \cap U_\sigma \}_{j \in [n] \setminus \sigma}, U_\sigma \right), \quad \text{where } U_\sigma = \bigcap_{i \in \sigma} U_i.
\]

Since any intersection of convex sets is convex, we thus observe

**Lemma 2.4.** If \( C \) is an open (or closed) convex code, then for any \( \sigma \in \Delta(C) \), \( \text{link}_\sigma C \) is also an open (or closed) convex code.

Note that for \( \sigma \in \Delta(C) \),

\[
\sigma \in \Delta(C) \setminus C \iff \emptyset \notin \text{link}_\sigma C.
\]

We call the faces of \( \Delta(C) \), that are “missing” from the code, \textit{simplicial violators} of \( C \). If a code \( C \) is convex, and \( \sigma \) is a simplicial violator, then the convex code \( \text{link}_\sigma C = \text{code}(\{V_i\}, U_\sigma) \) is special in that the convex sets \( V_j = U_j \cap U_\sigma \) cover another convex set \( U_\sigma \) that is therefore contractible. The \textit{local obstructions} to convexity arise from a special case of the \textit{nerve lemma}.

**Lemma 2.5** (Nerve Lemma, [11][4]). For any finite cover \( \mathcal{V} = \{V_i\}_{i \in [n]} \) by convex sets \( V_i \subset \mathbb{R}^d \) that are either all open or all closed,[4] the abstract simplicial complex

\[
\text{nerve}(\mathcal{V}) \overset{\text{def}}{=} \{ \sigma \subseteq [n] \text{ such that } \bigcap_{i \in \sigma} V_i \neq \emptyset \} \subset 2^{[n]},
\]

\[\text{This is because both } \text{link}_\sigma C \text{ and } \text{link}_\sigma \Delta \text{ have the same set of maximal elements.}\]

\[\text{A formulation of the nerve lemma which applies to finite collections of closed, convex subsets of Euclidean space appears in [4], and follows from [11] Theorem 10.7.}\]
known as the \textit{nerve} of the cover is homotopy equivalent to the underlying space \( X = \bigcup_{i \in [n]} V_i \).

A simple corollary of Lemma 2.4 and the nerve lemma is the following observation (which first appeared in [5]) that provides a class of ‘local’ obstructions to being an open (or closed) convex code.

**Proposition 2.6.** Let \( \sigma \neq \emptyset \) be a simplicial violator of a code \( C \). If \( \text{link}_\sigma \Delta(C) \) is not a contractible simplicial complex, then \( C \) is not an open (or closed) convex code.

**Proof.** Assume the converse, i.e. \( C \) is open (or closed) convex and \( \sigma \) satisfies (\ref{eq:definition-local-obstruction}). Then the sets \( U_j \cap U_\sigma \) cover a convex and open (or closed) set \( U_\sigma \), and thus by the nerve lemma the simplicial complex

\[
\text{nerve}\left(\{U_j \cap U_\sigma\}_{j \in [n] \setminus \sigma}\right) = \Delta(\text{link}_\sigma C) = \text{link}_\sigma \Delta(C)
\]

is contractible. \( \square \)

As an example, consider \( C = \{\emptyset, 1, 2, 3, 4, 123, 124\} \). Then \( \sigma = 12 \) is a simplicial violator of \( C \) and \( \text{link}_\sigma C = \{3, 4\} \). Since \( \Delta(\text{link}_\sigma C) \) is not contractible, the code \( C \) is not the code of an open (or closed) convex cover. This is perhaps the minimal example of a non-convex code that can be still realized by an open cover by connected sets\(^6\).

Note that if the condition that all sets are open, or alternatively all sets are closed, is dropped, then (at the time of this writing) there are no known obstructions for a code to arise as a code of a convex cover. For instance, if one set is allowed to be of the “wrong kind”, the code \( \sigma \) (at the time of this writing) there are no known obstructions for a code to arise as a code of a convex cover. This is perhaps the minimal example of a non-convex code that can be still realized by an open cover by connected sets\(^6\).

### 2.2. Do truly “non-local” obstructions via nerve lemma exist?

The “local” obstructions to convexity in Proposition 2.6 equally apply to any open (or closed) good cover, i.e. a cover where each non-empty intersection \( U_\sigma = \bigcap_{i \in \sigma} U_i \) is contractible. Since this property stems from applying the nerve lemma to the cover of \( U_\sigma \) by the other contractible sets, it is natural to define a more general “non-local” obstruction to convexity that also stems from the nerve lemma.

**Definition 2.7.** We say that a non-empty subset \( \sigma \subseteq [n] \) \textit{covers} a code \( C \subseteq 2^{[n]} \) if for every \( \tau \in C \), \( \tau \cap \sigma \neq \emptyset \).

Note that any code covered by at least one non-empty set \( \sigma \) does not have the empty set. Moreover, \( \sigma \) covers \( C = \text{code}\left(\{U_i\}_{i \in [n]}; \bigcup_{i \in [n]} U_i\right) \) if and only if \( \bigcup_{i \in [n]} U_i = \bigcup_{j \in \sigma} U_j \).

**Lemma 2.8.** If there exist two non-empty subsets \( \sigma_1, \sigma_2 \subseteq [n] \), that both cover the code \( C \subseteq 2^{[n]} \), but the codes \( C \cap \sigma_a \defeq \{\tau \cap \sigma | \tau \in C \} \subseteq 2^{\sigma_a} \) for \( a \in \{1, 2\} \) have simplicial complexes \( \Delta(C \cap \sigma_a) \) that are \textit{not} homotopy equivalent, then \( C \) is not a code of a convex cover by open (or closed) sets in \( \mathbb{R}^d \).

**Proof.** If such convex cover existed, then the condition that each of the non-empty subsets \( \sigma_a \) covers the code \( C \) implies that that \( \bigcup_{i \in [n]} U_i = \bigcup_{j \in \sigma_a} U_j \) for each \( a \in \{1, 2\} \). Thus, by the nerve lemma, \( \Delta(C) \) has the same homotopy type as each of the complexes \( \Delta(C \cap \sigma_a) \). This yields a contradiction. \( \square \)

The above obstruction to convexity can be thought as “non-local” because it is conditioned on the homotopy type of a subset that covers the entire code. While it is straightforward to produce combinatorial codes with these “non-local” obstructions, we found that every such code that we

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\(^6\)In fact, all the non-convex codes on three neurons (these were classified in [3]) cannot be realized by open (or closed) connected sets. This is because the only obstruction to convexity is the “disconnection” of one set, similar to the case of the code \( C = \{\emptyset, 1, 2, 13, 23\} \).
have considered\textsuperscript{7} inevitably possesses a local obstruction for convexity. Perhaps the smallest such example is the code $C = \{23, 14, 123\}$ that meets the conditions of Lemma 2.8 with $\sigma_1 = \{12\}$, and $\sigma_2 = \{34\}$, but also has a local obstruction for the simplicial violator $\sigma = \{1\}$. The exact reason for the significant difficulty of finding a truly “non-local” obstruction is still unclear. Nevertheless, this provides some evidence for the conjecture that any code $C \subset 2^{[n]}$ that has a “non-local” obstruction (i.e. the conditions of Lemma 2.8 are met) must also have a “local” obstruction, i.e. a simplicial violator $\sigma \in \Delta(C) \setminus C$ such that $\Delta(\text{link}_\sigma C)$ is not a contractible simplicial complex.

2.3. The difference between open and closed convex codes. The homotopy type obstructions via the nerve lemma are obstructions to being a code of a good cover (as opposed to convex sets) and equally apply to both open and closed versions of the Definition 1.1. However, it turns out that the open and the closed convex codes are distinct classes of codes. Perhaps a minimal example of an open convex code that is not closed convex is the code

$$(2) \quad C = \{123, 126, 156, 456, 345, 234, 12, 16, 56, 45, 34, 23, \emptyset\} \subseteq 2^{[6]}.$$ 

This code is realizable by an open convex cover (Figure 2.1a) and also by an open or closed good cover (Figure 2.1b).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.1.png}
\caption{Two different realizations of the code $C$ in (2). In both realizations, each set $U_i$ is covered by the others, and is indicated by the colored arcs external to the sets; the colors of regions are combinations of the colors of the constituent sets. For example, in (a), $U_1$ is the open upper half-disk, while in (b) $U_1$ is the top right closed annular section.}
\end{figure}

\textbf{Lemma 2.9.} The code (2) is not closed convex.

The proof is given in the Appendix, Section 5.1. A different example,

$$(3) \quad C = \{2345, 124, 135, 145, 14, 15, 24, 35, 45, 4, 5\} \subseteq 2^{[5]},$$

was originally considered in [10], where it was proved that it is not open convex and possesses a realization by a good open cover (Figure 2.2b), thus does not have any “local obstructions” to

\textsuperscript{7}This included computer-assisted search among random codes.
convexity. However, it turns out that this code is closed convex (see a closed realization in Figure 2.2a).

![Figure 2.2. Two different realizations of the code in (3).](image)

The examples in (2) and (3) show that open convex and closed convex are distinct classes of codes. Moreover, they illustrate that one cannot generally “convert” an open convex realization into a closed convex realization or vice versa by simply taking closures or interiors of sets in a cover. Nevertheless, it is intuitive that open and closed versions of a “sufficiently non-degenerate” cover should yield the same code.

A natural candidate for such a condition would be that the sets in the cover \( U \) are in general position, i.e. there exists \( \varepsilon > 0 \) such that any cover \( V = \{ V_i \} \) whose sets \( V_i \) are no further than \( \varepsilon \) from \( U_i \) in the Hausdorff distance\(^8\), has the same code: \( \text{code}(U, \mathbb{R}^d) = \text{code}(V, \mathbb{R}^d) \). However, being in general position is too strong a condition. This is because there are covers of interest (such as those in Section 4) that are not in general position yet yield the same code after taking the closure or interior. We therefore consider the following weaker condition.

**Definition 2.10.** A cover \( U = \{ U_i \}_{i \in [n]} \), with \( U_i \subseteq \mathbb{R}^d \), is non-degenerate if the following two conditions hold:

(i) For all \( \sigma \in \text{code}(U, \mathbb{R}^d) \), the atoms \( A_{U_\sigma} \) are top-dimensional, i.e. any non-empty intersection with an open set \( B \subseteq \mathbb{R}^d \) has non-empty interior:

\[
B \text{ is open and } A_{U_\sigma} \cap B \neq \emptyset \implies \text{int}(A_{U_\sigma} \cap B) \neq \emptyset.
\]

(ii) For all non-empty \( \sigma \subseteq [n] \), \( \bigcap_{i \in \sigma} \partial U_i \subseteq \partial \left( \bigcap_{i \in \sigma} U_i \right) \).

Note that if a cover \( U \) is open, convex and in general position, then it is non-degenerate (see Lemma 5.3 in the Appendix), while an open convex and non-degenerate cover need not be in general position. We should also note that the two seemingly separate conditions (i) and (ii) in the above definition are motivated by the following observation.

**Lemma 2.11.** Assume that \( U = \{ U_i \} \) is a finite cover by convex sets. Then,

if all \( U_i \) are open and \( U \) satisfies the condition (ii), then it also satisfies the condition (i);

if all \( U_i \) are closed and \( U \) satisfies the condition (i), then it also satisfies the condition (ii).

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\(^8\)Recall that the Hausdorff distance between two subsets \( U \) and \( V \) of a Euclidean space is defined as

\[
d_H(U, V) = \max \{ \sup_{x \in U} \{ \inf_{y \in V} \| x - y \| \}, \sup_{y \in V} \{ \inf_{x \in U} \| x - y \| \} \}.
\]
The proof is given in the Appendix (Section 5.3, Lemmas 5.2 and 5.4). Note that if the sets $U_i$ are open and convex, then condition (i) does not imply condition (ii), similarly if the sets $U_i$ are closed and convex then condition (ii) does not imply condition (i).

For an open cover $U = \{U_i\}$, we denote by $\text{cl}(U)$ the cover by the closures $V_i = \text{cl}(U_i)$. Similarly, for a closed cover $U = \{U_i\}$ we denote by $\text{int}(U)$ the cover by the interiors $V_i = \text{int}(U_i)$. Recall that if a set is convex, then both its closure and its interior are convex.

**Theorem 2.12.** Assume that $U = \{U_i\}$ is a convex and non-degenerate cover, then
\[
\begin{align*}
U_i \text{ are open} & \implies \text{code}(U, \mathbb{R}^d) = \text{code} \left( \text{cl}(U), \mathbb{R}^d \right) ; \\
U_i \text{ are closed} & \implies \text{code}(U, \mathbb{R}^d) = \text{code} \left( \text{int}(U), \mathbb{R}^d \right).
\end{align*}
\]
The proof is given in the Appendix (Section 5.3). This theorem guarantees that if an open convex code is realizable by a non-degenerate cover, then it is also closed convex; similarly if a closed convex code is realizable by a non-degenerate cover, then it is also open convex. Non-degenerate covers are thus natural in the neuroscience context, where receptive fields (i.e. the sets $U_i$) should not change their code after taking closure or interior, since such changes in code would be undetectable in the presence of standard neuronal noise. This suggests that convex codes that arise from non-degenerate covers should serve as the standard model for convex codes in neuroscience-related contexts. Note that the existence of a non-degenerate convex cover realization is extrinsic in that it is not defined in terms of the combinatorics of the code alone. A combinatorial description of such codes is unknown at the time of this writing.

3. **MONOTONICITY OF OPEN CONVEX CODES.**

The set of all codes $C \subseteq 2^n$ with a prescribed simplicial complex $K = \Delta(C)$ forms a poset. It is easy to see that if $C$ is a convex code then its sub-codes can be non-convex. For example any non-convex code is a sub-code of its simplicial complex, and every simplicial complex is both an open and closed convex code (this follows from Theorem 1.12 in Section 4). It turns out that open convexity is a monotone increasing property.

**Theorem 1.13.** Assume that a code $C \subset 2^n$ is open convex. Then every code $D$ that satisfies $C \subseteq D \subseteq \Delta(C)$ is also open convex with open embedding dimension $\text{odim } D \leq \text{odim } C + 1$.

Note that the above bound on the embedding dimension is sharp. For example, the open convex code $C = \{123, 12, 1\}$ has embedding dimension $\text{odim } C = 1$, but its simplicial complex $D = \Delta(C)$ has embedding dimension $\text{odim } D = 2$. To prove this theorem we shall use the following lemma. Let $M(C)$ denote the facets of the simplicial complex $\Delta(C)$.

**Lemma 3.1.** Let $U = \{U_i\}$ be an open convex cover in $\mathbb{R}^d$, $d \geq 2$, with $C = \text{code}(U, X)$. Assume that there exists an open Euclidean ball $B \subset \mathbb{R}^d$ such that $\text{code}(\{B \cap U_i\}, B \cap X) = C$, and for every maximal set $\alpha \in M(C)$, its atom has non-empty intersection with the $(d - 1)$-sphere: $\partial B \cap A^\alpha \neq \emptyset$. Then for every $D$ such that $C \subseteq D \subseteq \Delta(C)$, there exists an open convex cover $V = \{V_i\}$ with $V_i \subseteq U_i$, such that $D = \text{code}(V, B \cap X)$. Moreover, if the cover $U$ is non-degenerate, then the cover $V$ can also be chosen to be non-degenerate.

For example, the cover by the open convex sets $U_1 = \{(x, y) \in \mathbb{R}^2 \mid y > x^2\}$ and $U_2 = \{(x, y) \in \mathbb{R}^2 \mid y < -x^2\}$ satisfies condition (i), but does not satisfy condition (ii). Similarly, the closed subsets of the real line, $U_1 = \{x \leq 0\}$, $U_2 = \{x \geq 0\}$ satisfy condition (ii), but do not satisfy condition (i).
The proof of this lemma is given in Section 5.4. Intuitively, the reason why this lemma holds is that one can “chip away” small pieces from the ball $B$ inside some atoms $A^U_\alpha$ to uncover only the atoms corresponding to the codewords in $D \setminus C$.

Proof of Theorem 1.3. Assume that $U$ is an open convex cover in $\mathbb{R}^d$ with $C = \text{code}(U, X)$. Since there are only finitely many codewords, there exists a radius $r > 0$ and an open Euclidean ball $B^d_r \subset \mathbb{R}^d$, of radius $r$, centered at the origin, that satisfy $\text{code}\{B^d_r \cap U_i\}, B^d_r \cap X = C$. Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be the standard projection. Let $\tilde{U}_i = \pi^{-1}(U_i)$. By construction, $\tilde{U} = \{\tilde{U}_i\}$ is an open, convex cover, such that each of its atoms has non-empty intersection with the sphere $\partial B$. Moreover, $\text{code}(\{B \cap \tilde{U}_i\}, B \cap \pi^{-1}(X)) = C$. Thus the conditions of Lemma 3.1 are satisfied for the cover $\tilde{U}$, and $D$ is an open convex code with $\text{odim} D \leq \text{odim} C + 1$. $\square$

Note that the proof of Lemma 3.1 (see Section 5.4) breaks down if one assumes that the convex sets $U_i$ are closed. Moreover, it is currently not known if the monotonicity property holds in the setting of the closed convex codes. The differences between the open convex and the closed convex codes (described in the previous section) leave enough room for either possibility.

4. MAX INTERSECTION-COMPLETE CODES ARE OPEN AND CLOSED CONVEX.

Here we introduce max intersection-complete codes and prove that they are open convex and closed convex. The open convexity of max intersection-complete codes was first hypothesized in [2].

Definition 4.1. The intersection completion of a code $C$ is the code that consists of all non-empty intersections of codewords in $C$:

$$\hat{C} = \{\sigma \mid \sigma = \bigcap_{\nu \in C'} \nu \text{ for some non-empty subcode } C' \subseteq C\}.$$ 

Note that the intersection completion satisfies $C \subseteq \hat{C} \subseteq \Delta(C)$.

Definition 4.2. Let $C \subset 2^{[n]}$ be a code, and denote by $M(C) \subset C$ the subcode consisting of all maximal codewords of $C$. A code $C$ is said to be

- intersection-complete if $\hat{C} = C$;
- max intersection-complete if $M(C) \subseteq C$.

Note that any simplicial complex (i.e. $C = \Delta(C)$) is intersection-complete and any intersection-complete code is max intersection-complete. Intersection-complete codes allow a simple construction of a closed convex realization that we describe in Section 5.5 (see Lemma 5.9). However, in order to prove that max intersection-complete codes are both open and closed convex, we need the following.

Proposition 4.3. Let $C \subset 2^{[n]}$ be a code with $k$ maximal elements. Then there exists an open convex and non-degenerate cover $U$ in $d = (k - 1)$-dimensional space whose code is the intersection completion of the maximal elements in $C$: $\text{code}(U, \mathbb{R}^d) = \hat{M(C)}$.

\[10\] Equivalently, the facets of $\Delta(C)$. 

\[11\] Equivalently, the facets of $\Delta(C)$. 

Proof. Denote the maximal codewords as \( M(C) = \{ \sigma_1, \sigma_2, \ldots, \sigma_k \} \). If \( k = 1 \) this statement is trivially true. Assume \( k \geq 2 \) and consider a regular geometric \((k - 1)\)-simplex \( \Delta^{k-1} \) in \( \mathbb{R}^{k-1} \) constructed by evenly spacing vertices \([k]\) on the unit sphere \( S^{k-2} \subseteq \mathbb{R}^{k-1} \). Construct a collection of hyperplanes \( \{ P_a \}_{a=1}^k \) in \( \mathbb{R}^{k-1} \) by taking \( P_a \) to be the plane through the facet of \( \partial \Delta^{k-1} \) which does not contain vertex \( a \). Denote by \( H^+_a \) the closed half-space containing the vertex \( a \) bounded by \( P_a \) and by \( H^-_a \) the complementary open half-space. Observe that this arrangement splits \( \mathbb{R}^{k-1} \) into \( 2^k - 1 \) disjoint, non-empty, convex chambers

\[ H_\rho = \bigcap_{a \in \rho} H^+_a \cap \bigcap_{b \notin \rho} H^-_b, \]

indexed by all non-empty\(^{11}\) subsets \( \rho \subseteq [k] \).

For every \( i \in [n] \) consider \( \rho(i) \) \( \overset{\text{def}}{=} \{ a \in [k] \mid \sigma_a \ni i \} \subset [k] \), i.e. the collection of indices of the maximal codewords \( \sigma_a \) that contain \( i \), and construct a collection of convex open sets \( \mathcal{U} = \{ U_i \} \)

\[ U_i \overset{\text{def}}{=} \bigcap_{\rho \subseteq \rho(i)} H_\rho. \]

To show that the sets \( U_i \) are convex and open, observe that the above construction implies that we have the disjoint unions

\[ \mathbb{R}^{k-1} = \bigcup_{\rho \neq \emptyset} H_\rho \quad \text{and} \quad H^+_b = \bigcup_{\rho \ni b} H_\rho, \]

thus

\[ \mathbb{R}^{k-1} \setminus U_i = \left( \bigcap_{\rho \neq \emptyset} H_\rho \right) \setminus \left( \bigcap_{\rho \subseteq \rho(i)} H_\rho \right) = \bigcup_{\rho \ni b(i)} H_\rho = \bigcup_{b \notin \rho(i)} \left( \bigcap_{\rho \ni b} H_\rho \right) = \bigcup_{b \notin \rho(i)} H^+_b. \]

\(^{11}\)The empty set is not included because under this definition, \( H_\emptyset = \emptyset \).
On open and closed convex codes

Therefore, by de Morgan’s Law,

\[ U_i = \mathbb{R}^{k-1} \setminus \left( \bigcup_{b \notin \rho(i)} H_b^+ \right) = \bigcap_{b \notin \rho(i)} H_b^-. \]

This is an intersection of open convex sets, and therefore open and convex. Note that if \( \rho(i) = [k] \), this is an intersection over an empty index, and we interpret this set as all of \( \mathbb{R}^{k-1} \).

To show that \( \text{code}(U, \mathbb{R}^{k-1}) = \hat{M}(C) \), observe that because the chambers of the hyperplane arrangement satisfy \( H_\rho \cap H_\nu \neq \emptyset \iff \rho = \nu \), the atoms of the cover \( \{U_i\} \) take the form

\[ A^U_\sigma = \bigcap_{i \in \sigma} U_i \setminus \left( \bigcup_{j \notin \sigma} U_j \right) = \left( \bigcup_{\rho \cap \cap \sigma} H_\rho \right) \setminus \left( \bigcup_{\nu \in \cup \sigma} H_\nu \right), \]

where each \( R_i \overset{\text{def}}{=} \{ \rho \subseteq \rho(i) \} \subseteq 2^{[k]} \setminus \emptyset \) is the collection of the non-empty subsets of \( \rho(i) \), and therefore

\[ \text{code}(\{U_i\}, \mathbb{R}^{k-1}) = \text{code}(\{R_i\}, 2^{[k]} \setminus \emptyset). \]

Now, observe that

\[ \rho \in \bigcap_{i \in \sigma} R_i \iff \forall i \in \sigma, \rho \subseteq \rho(i) \iff \forall i \in \sigma, \forall a \in \rho, i \in \sigma_a \iff \sigma \subseteq \bigcap_{a \in \rho} \sigma_a, \]

and also that,

\[ \rho \notin \bigcup_{j \notin \sigma} R_j \iff \forall j \notin \sigma, \rho \nsubseteq \rho(i) \iff \forall j \notin \sigma, \exists a \in \rho \text{ such that } j \notin \sigma_a \iff \sigma \supseteq \bigcap_{a \in \rho} \sigma_a \]

Therefore, \( \rho \in \bigcap_{i \in \sigma} R_i \setminus \left( \bigcup_{j \notin \sigma} R_j \right) \) if and only if \( \sigma = \bigcap_{a \in \rho} \sigma_a \) and thus

\[ \hat{M}(C) = \text{code}(\{R_i\}, 2^{[k]} \setminus \emptyset) = \text{code}(\{U_i\}, \mathbb{R}^{k-1}). \]

Lastly, we show that the cover \( \mathcal{U} \) is non-degenerate. By construction, the half-spaces \( H^-_a \) are open, convex and in general position. Thus Lemma 5.3 guarantees that the cover \( \mathcal{H} = \{H^-_a\} \) is non-degenerate and using Lemma 5.5 in the Appendix we conclude that for any non-empty \( \tau \subseteq [k] \), \( \bigcap_{a \in \tau} \text{cl}(H^-_a) = \text{cl}(\bigcap_{a \in \tau} H^-_a) \). For any non-empty subset \( \sigma \subseteq [n] \) we can combine this with the equality (11) to obtain

\[ \text{cl}(\bigcap_{i \in \sigma} U_i) = \text{cl}(\bigcap_{i \in \sigma} \bigcap_{a \notin \rho(i)} H^-_a) = \bigcap_{i \in \sigma} \text{cl}(H^-_a) = \bigcap_{i \in \sigma} \text{cl}(\bigcap_{a \notin \rho(i)} H^-_a) = \bigcap_{i \in \sigma} \text{cl}(U_i). \]

Since \( U_i \) are open we obtain

\[ \bigcap_{i \in \sigma} \partial U_i = \bigcap_{i \in \sigma} (\text{cl}(U_i) \setminus U_i) \leq \bigcap_{i \in \sigma} \left( \text{cl}(U_i) \setminus \bigcap_{i \in \sigma} U_i \right) = \left( \bigcap_{i \in \sigma} \text{cl}(U_i) \right) \setminus \bigcap_{i \in \sigma} U_i = \text{cl}(\bigcap_{i \in \sigma} U_i) \setminus \bigcap_{i \in \sigma} U_i. \]

Therefore by Lemma 2.11 the open and convex cover \( \mathcal{U} \) is also non-degenerate. \( \square \)
As a corollary we obtain the main result of this section:

**Theorem 1.2.** Suppose $C \subset 2^{[n]}$ is a max intersection-complete code. Then $C$ is both open convex and closed convex with the embedding dimension $d \leq \max\{2, (k-1)\}$, where $k$ is the number of facets of the complex $\Delta(C)$.

**Proof.** Note that the case of $k = 1$, i.e. $M(C) = \{[n]\}$, was proved in [2]. We first consider the case when the number of maximal codewords is $k \geq 3$ and begin by constructing convex regions $\{H_\rho\}_{\rho \in 2^{[k]} \setminus \emptyset}$ and the open convex cover $\{U_i\}_{i=1}^n$ as in the proof of Proposition 4.3 (see Figure 4.1). In this cover, every atom that corresponds to a maximal codeword is unbounded, therefore we can apply Lemma 3.2 using the open ball of radius 2 centered at the origin. This yields an open convex and non-degenerate cover, thus by Theorem 2.12 the code $C$ is both open convex and closed convex.

If $1 \leq k < 3$, we formally append $3-k$ empty maximal codewords $\{\gamma_j\}_{j=1}^{3-k}$ to $M(C)$ and apply the same construction. Because the $\gamma_i$ are empty, they serve only to “lift” the construction to $\mathbb{R}^2$. The sets $U_i$ are contained entirely in $\bigcap_{j=1}^{3-k} H_{\gamma_j}$, but the $\gamma_i$ have no other effect on their composition. This allows us to carry out the rest of the above proof in the same way as in the case of $k \geq 3$. \qed

### 5. Appendix: Supporting Proofs

#### 5.1. Proof of Lemma 2.9

**Proof.** Consider the code $C$ in (2) and assume that there exists a closed convex cover $U = \{U_i\}$ in $\mathbb{R}^d$, with code($U, \mathbb{R}^d$) = $C$. Without loss of generality, we can assume that the $U_i$ are compact.12 Because $U_i$ are compact and convex one can pick points $x_{123}$, $x_{345}$, and $x_{156}$ in the closed atoms $A_{123}^U$, $A_{345}^U$ and $A_{156}^U$ respectively so that for every $a \in A_{123}^U$, its distance to the closed line segment $M = x_{345}x_{156}$ satisfies $\text{dist}(a, M) \geq \text{dist}(x_{123}, M) \neq 0$, i.e. $x_{123}$ minimizes the distance to the line segment $M$. Moreover, the points $x_{123}, x_{156}, x_{345}$ cannot be collinear. For the rest of this proof we will consider only the convex hull of these three points (Figure 5.1).

![Figure 5.1](image)

---

12If $U_i$ are not compact, then one can intersect them with a closed ball of large enough radius to obtain the same code.

13Because $U_5$ is convex and contains the endpoints of $M$, $x_{123} \notin M$. Moreover, since both $M$ and $A_{123}^U$ are compact, the function $f(a) = \text{dist}(a, M)$ achieves its minimum on $A_{123}^U$. 
Consider the closed line segment \( L = x_{123}x_{156} \). Because \( U_1 \) is convex, \( L \subseteq U_1 \), therefore the code \( \mathcal{U} \) of the cover imposes that
\[
L \subseteq A_{123}^\mathcal{U} \sqcup A_{12}^\mathcal{U} \sqcup A_{156}^\mathcal{U} \sqcup A_{126}^\mathcal{U} \sqcup A_{16}^\mathcal{U} \sqcup A_{15}^\mathcal{U}.
\]
Because each of the atoms above is contained in either \( U_2 \) or \( U_6 \), \( L \subseteq U_2 \sqcup U_6 \). Since \( L \) is connected and the sets \( U_2 \cap L \) and \( U_6 \cap L \) are closed and non-empty, we conclude that \( U_2 \cap U_6 \cap L \subseteq A_{126}^\mathcal{U} \) must be non-empty, thus there exists a point \( x_{126} \in A_{126}^\mathcal{U} \cap L \) that lies in the interior of \( L \). By the same argument, there also exist points
\[
x_{234} \in A_{234}^\mathcal{U} \text{ in the interior of } x_{123}x_{345} \subset U_3, \text{ covered by } U_2 \text{ and } U_4.
\]
\[
y_{123} \in A_{123}^\mathcal{U} \text{ in the interior of } x_{234}x_{126} \subset U_2, \text{ covered by } U_1 \text{ and } U_3.
\]
and these points must lie on the interiors of their respective line segments (Figure 5.1).

Finally we observe that because the point \( y_{123} \in A_{123}^\mathcal{U} \) lies in the interior of a line segment \( x_{234}x_{126} \), it also lies in the interior of the closed triangle \( \Delta(x_{123}, x_{156}, x_{345}) \), and thus \( d(y_{123}, M) < d(x_{123}, M) \). This yields a contradiction, since we chose \( x_{123} \in A_{123}^\mathcal{U} \) to have the minimal distance to the line segment \( M \).

5.2. Proofs of lemmas, related to the non-degeneracy condition. We shall need the following several lemmas. The following lemma is well-known (see e.g. [6], exercises in Chapter 1), nevertheless we give its proof for the sake of completeness.

**Lemma 5.1.** For any finite cover \( \mathcal{U} = \{U_i\}_{i=1}^n \) and a subset \( \sigma \subseteq [n] \), the following hold:
\[
(5) \quad \text{cl}(\bigcup_{i \in \sigma} U_i) = \bigcup_{i \in \sigma} \text{cl}(U_i),
\]
\[
(6) \quad \text{cl}(\bigcap_{i \in \sigma} U_i) \subseteq \bigcap_{i \in \sigma} \text{cl}(U_i),
\]
\[
(7) \quad \text{int}(\bigcap_{i \in \sigma} U_i) = \bigcap_{i \in \sigma} \text{int}(U_i),
\]
\[
(8) \quad \text{int}(\bigcup_{i \in \sigma} U_i) \supseteq \bigcup_{i \in \sigma} \text{int}(U_i).
\]

**Proof.** Observe that since \( U_i \subseteq \text{cl}(U_i) \), we have \( \bigcup_{i \in \sigma} U_i \subseteq \bigcup_{i \in \sigma} \text{cl}(U_i) \) and thus
\[
(9) \quad \text{cl}(\bigcup_{i \in \sigma} U_i) \subseteq \text{cl} \left( \bigcup_{i \in \sigma} \text{cl}(U_i) \right) = \bigcup_{i \in \sigma} \text{cl}(U_i).
\]
Similarly, we find the inclusion \( (6) \). Using \( U_i \supseteq \text{int}(U_i) \), one also obtains the inclusion \( (8) \) and the inclusion
\[
(10) \quad \text{int}(\bigcap_{i \in \sigma} U_i) \supseteq \bigcap_{i \in \sigma} \text{int}(U_i).
\]
Observe that for any \( j \in \sigma \), \( \text{cl}(U_j) \subseteq \text{cl}(\bigcup_{i \in \sigma} U_i) \) and \( \text{int}(U_j) \supseteq \bigcap_{i \in \sigma} \text{int}(U_i) \), thus we obtain \( \bigcup_{i \in \sigma} \text{cl}(U_i) \subseteq \text{cl}(\bigcup_{i \in \sigma} U_i) \) and \( \bigcap_{i \in \sigma} \text{int}(U_i) \supseteq \text{int}(\bigcap_{i \in \sigma} U_i) \). These combined with \( (9) \) and \( (10) \) yield \( (5) \) and \( (7) \) respectively. \( \square \)
Lemma 5.2. Suppose $\mathcal{U} = \{U_i\}_{i \in [n]}$ is an open and convex cover such that for every non-empty subset $\tau \subseteq [n]$, $\bigcap_{i \in \tau} \partial U_i \subseteq \partial(\bigcap_{i \in \tau} U_i)$. Then every atom of $\mathcal{U}$ is top-dimensional.

Proof. Assume the converse, i.e. there exists non-empty subset $\sigma \subseteq [n]$ and an open set $B \subseteq \mathbb{R}^d$ such that that $A^\mathcal{U}_\sigma \cap B \neq \emptyset$ and $\text{int}(A^\mathcal{U}_\sigma \cap B) = \emptyset$. Let $x \in A^\mathcal{U}_\sigma \cap B$, and denote by $\tau \subseteq [n] \setminus \sigma$, the maximal subset such that $x \in \bigcap_{j \in \tau} \partial U_j$. Note that $\tau$ is non-empty,14 and therefore (using the assumption of the lemma) $x \in \partial(\bigcap_{j \in \tau} U_j)$. Denote by $\epsilon_0 > 0$ the maximal radius such that the open ball $B_{\epsilon_0}(x)$ satisfies (a) $B_{\epsilon_0}(x) \subset B \cap \bigcap_{i \in \sigma} U_i$ and (b) for every $l \not\in (\sigma \cup \tau)$, $B_{\epsilon_0}(x) \cap U_l = \emptyset$.

![Figure 5.2. Construction of points in $\text{int}(A^\mathcal{U}_\sigma \cap B)$ from the proof of Lemma 5.2](image)

Observe that for every point $y \in \bigcap_{j \in \tau} U_j$ and every $\epsilon \in (0, \epsilon_0)$, the point $z_\epsilon(y) = x + \epsilon \frac{x-y}{\|x-y\|}$ satisfies $z_\epsilon(y) \not\in U_j$ for every $j \not\in \sigma$. This is because for every $j \in \tau$, the open set $U_j$ is convex, thus if $x \in \partial U_j$, and $y \in U_j$, then $z_\epsilon(y) \not\in U_j$, as in Figure 5.2. We thus conclude that $z_\epsilon(y) \in A^\mathcal{U}_\sigma \cap B$. Since the intersection $\bigcap_{j \in \tau} U_j$ is open, the totality of all such points $z_\epsilon(y)$ form an open cone $C_x \subseteq A^\mathcal{U}_\sigma \cap B$. Therefore $\text{int}(A^\mathcal{U}_\sigma \cap B) \supseteq \text{int}(C_x) \neq \emptyset$, a contradiction.

Lemma 5.3. Suppose $\mathcal{U}$ is an open and convex cover in general position, then $\mathcal{U}$ is a non-degenerate cover.

Proof. Assume $\mathcal{U}$ is in general position, open, convex, yet not non-degenerate. Then, by Lemma 5.2 there exists a non-empty subset $\sigma \subseteq [n]$ so that $\bigcap_{i \in \sigma} \partial U_i \not\subseteq \partial(\bigcap_{i \in \sigma} U_i)$. Let’s choose $x \in (\bigcap_{i \in \sigma} \partial U_i) \setminus (\partial \bigcap_{i \in \sigma} U_i)$. Suppose there exists $z \in \bigcap_{i \in \sigma} U_i$, then the open line segment between $x$ and $z$ is contained in $\bigcap_{i \in \sigma} U_i$, and thus $x \in \partial(\bigcap_{i \in \sigma} U_i)$, a contradiction. Therefore, $\bigcap_{i \in \sigma} U_i = \emptyset$, and for every $\tau \supseteq \sigma$, $\tau \not\in \text{code}(\mathcal{U}, \mathbb{R}^d)$.

For any $\epsilon > 0$, define an open cover $\mathcal{V}(\epsilon) = \{V_i(\epsilon)\}$ by $V_i(\epsilon) = U_i \cup B_\epsilon(x)$ for $i \in \sigma$ and $V_j(\epsilon) = U_j$ otherwise. Notice that $\bigcap_{i \in \sigma} V_i(\epsilon) = B_\epsilon(x)$. Thus for any $\epsilon > 0$, there exists $\tau \supseteq \sigma$ with $\tau \not\in \text{code}(\mathcal{V}(\epsilon), \mathbb{R})$. Because $x$ lies in the boundary of $U_i$ for each $i \in \sigma$, each $V_i(\epsilon)$ is no more than $\epsilon$ away from $U_i$ w.r.t. the Hausdorff distance. Therefore $\mathcal{U}$ is not in general position, a contradiction.

Lemma 5.4. Assume that every atom of the cover $\mathcal{U} = \{U_i\}$ is top-dimensional, i.e. any non-empty intersection with an open set $B \subseteq \mathbb{R}^d$ has non-empty interior, and the subsets $U_i$ are closed and

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14If $x \not\in \partial U_j \forall j \not\in \sigma$, then (because $U_i$ are open) there exists a small open ball $B' \ni x$ such that $B' \subset A^\mathcal{U}_\sigma$, thus $\text{int}(A^\mathcal{U}_\sigma \cap B) \supseteq \text{int}(A^\mathcal{U}_\sigma \cap B \cap B') \neq \emptyset$, a contradiction.
convex, then for any non-empty \( \tau \subseteq [n] \),

\[
\bigcap_{i \in \tau} \partial U_i \subseteq \partial \left( \bigcup_{i \in \tau} U_i \right),
\]

(11) \[
\bigcap_{i \in \tau} \partial U_i \subseteq \partial \left( \bigcap_{i \in \tau} U_i \right).
\]

Proof. To show (11) assume the converse. Then there exist a point \( x \in (\bigcap_{i \in \tau} \partial U_i) \cap \operatorname{int} (\bigcup_{i \in \tau} U_i) \) at the interior, and an open ball \( B \ni x \), such that \( B \subseteq \bigcup_{i \in \tau} U_i \). First, let us show that these assumptions imply that

\[
B \cap \bigcap_{i \in \tau} \operatorname{int}(U_i) = \emptyset.
\]

Indeed, if there existed a point \( y \in B \cap \bigcap_{i \in \tau} \operatorname{int}(U_i) \), then for every \( \varepsilon > 0 \) such that \( z = x + \varepsilon(x - y) \in B \) and every \( i \in \tau, z \notin U_i \) by convexity of \( U_i \). This implies \( B \not\subseteq \bigcup_{i \in \tau} U_i \), a contradiction, thus (13) holds.

Denote by \( \rho \supseteq \tau \) the element of code \( (\{U_i\}, \mathbb{R}^d) \) such that \( x \in A_\rho^\mathcal{U} = \bigcap_{i \in \rho} U_i \setminus \bigcup_{j \notin \rho} U_j \). Because the sets \( U_j \) are closed, we can choose the open ball \( B \ni x \), that satisfies (13) so that it is disjoint from \( \bigcup_{j \notin \rho} U_j \). Therefore, using (7), we obtain

\[
\operatorname{int}(B \cap A_\rho^\mathcal{U}) = \operatorname{int}(B \cap \bigcap_{i \in \rho} U_i) \subseteq \operatorname{int}(B \cap \bigcap_{i \in \tau} U_i) = B \cap \bigcap_{i \in \tau} \operatorname{int}(U_i) = \emptyset.
\]

Since \( x \in B \cap A_\rho^\mathcal{U} \), this contradicts the non-degeneracy of \( \mathcal{U} \), and thus finishes the proof of (11).

To prove (12), consider \( x \in \bigcap_{i \in \tau} \partial U_i \subseteq \bigcap_{i \in \tau} U_i \). Because of \( (11) \), any open neighborhood \( O \ni x \) satisfies \( O \not\subseteq \bigcup_{i \in \tau} U_i \) and thus \( O \not\subseteq \bigcap_{i \in \tau} U_i \). Therefore \( x \in \partial (\bigcap_{i \in \tau} U_i) \). \( \square \)

Note that if the condition that the sets \( U_i \) are convex is violated, then the conclusions of the above lemma may not hold. For example, the sets \( U_1 = \{(x, y) \in \mathbb{R}^2 \mid y \leq x^2 \} \) and \( U_2 = \{(x, y) \in \mathbb{R}^2 \mid y \geq -x^2 \} \) do not satisfy the inclusion (11).

Lemma 5.5. If the cover \( \mathcal{U} = \{U_i\}_{i \in [n]} \) is non-degenerate, then for every non-empty subset \( \sigma \subseteq [n] \)

\[
U_i \text{ are closed and convex } \implies \operatorname{int} \left( \bigcup_{i \in \sigma} U_i \right) = \bigcup_{i \in \sigma} \operatorname{int}(U_i),
\]

(14) \[
U_i \text{ are open and convex } \implies \operatorname{cl} \left( \bigcap_{i \in \sigma} U_i \right) = \bigcap_{i \in \sigma} \operatorname{cl}(U_i).
\]

Proof. First, we show that if the cover \( \mathcal{U} \) is non-degenerate and closed convex, then

\[
\operatorname{int} \left( \bigcup_{i \in \sigma} U_i \right) \subseteq \bigcup_{i \in \sigma} \operatorname{int}(U_i).
\]

It suffices to show that if \( x \notin \bigcup_{i \in \sigma} \operatorname{int}(U_i) \), then \( x \in \partial \left( \bigcup_{i \in \sigma} U_i \right) \setminus \left( \mathbb{R}^d \setminus \bigcup_{i \in \sigma} U_i \right) \). If \( x \notin \bigcup_{i \in \sigma} U_i \), then this is true, thus we can assume that the set \( \tau \overset{\text{def}}{=} \{ i \in \sigma \mid x \in U_i \} \) is non-empty, and since \( x \notin \bigcup_{i \in \sigma} \operatorname{int}(U_i) \), we conclude that \( x \in \bigcap_{i \in \tau} \partial U_i \). Thus by Lemma 5.4 ((11)), \( x \in \partial(\bigcup_{i \in \tau} U_i) \). Now

\[\text{This is because } y \in \operatorname{int}(U_i), x \in \partial U_i \text{ and } U_i \text{ is convex, thus for every } \varepsilon > 0, z = x + \varepsilon(x - y) \notin U_i.\]

\[\text{Therefore, using (7), we obtain } \operatorname{int}(B \cap A_\rho^\mathcal{U}) = \operatorname{int}(B \cap \bigcap_{i \in \rho} U_i) \subseteq \operatorname{int}(B \cap \bigcap_{i \in \tau} U_i) = B \cap \bigcap_{i \in \tau} \operatorname{int}(U_i) = \emptyset.\]
observe that \( \bigcup_{i \in \sigma} U_i = A \cup B \) with \( A \overset{\text{def}}{=} \bigcup_{i \in \tau} U_i \) and \( B \overset{\text{def}}{=} \bigcup_{j \in \sigma \setminus \tau} U_j \). Since \( x \notin B \), and \( B \) is closed, there exists an open neighborhood \( O \ni x \) with \( O \cap B = \emptyset \). Therefore, using (7) we obtain that
\[
O \cap \text{int}(A) = \text{int}(O \cap A) = \text{int}(O \cap (A \cup B)) = O \cap \text{int}(A \cup B),
\]
and thus we conclude
\[
x \in \partial A \cap O = (A \setminus \text{int} A) \cap O = ((A \cup B) \setminus (\text{int}(A \cup B) \cap O)) \cap O = \partial (A \cup B) \cap O.
\]
Thus, \( x \in \partial (A \cup B) = \partial (\bigcup_{i \in \sigma} U_i) \), which proves (16). Combined with (8) in Lemma 5.1 this finishes the proof of (14).

To prove (15), taking into account (6), we need to show that \( \text{cl}(\bigcap_{i \in \sigma} U_i) \supseteq \bigcap_{i \in \sigma} \text{cl}(U_i) \). Assume the converse, then there exists \( x \in \bigcap_{i \in \sigma} \text{cl}(U_i) \) and \( r > 0 \) such that
\[
\forall \varepsilon \in (0, r) \text{ the open } \varepsilon\text{-ball } B_\varepsilon(x) \text{ satisfies } B_\varepsilon(x) \cap \bigcap_{i \in \sigma} U_i = \emptyset.
\]
Denote \( \tau \overset{\text{def}}{=} \{ i \in \sigma \mid x \notin \partial U_i \} \); we can assume that \( \tau \) is non-empty (otherwise, \( x \notin \text{cl}(\bigcap_{i \in \sigma} U_i) \)). Using the condition (ii) of Definition 2.11 we conclude \( x \in \bigcap_{i \in \tau} \partial U_i \subseteq \partial (\bigcap_{i \in \sigma} U_i) \), thus for every open \( \varepsilon\)-ball \( B_\varepsilon(x) \) centered at \( x \), \( B_\varepsilon(x) \cap \bigcap_{i \in \tau} U_i \neq \emptyset \). Because \( x \in \bigcap_{j \in \sigma \setminus \tau} U_j \), and \( U_j \) are open, for a sufficiently small \( \varepsilon \), \( B_\varepsilon(x) \subseteq \bigcap_{i \in \sigma \setminus \tau} U_j \). Thus \( B_\varepsilon(x) \cap \bigcap_{i \in \sigma} U_i \neq \emptyset \), which contradicts (17). This finishes the proof of (15). \( \square \)

5.3. Proof of Theorem 2.12

Proof. We need to show that if \( \mathcal{U} \) is convex and non-degenerate, then the cover of closures \( \text{cl}(\mathcal{U}) \overset{\text{def}}{=} \{ \text{cl}(U_i) \} \) and the cover of interiors \( \text{int}(\mathcal{U}) \overset{\text{def}}{=} \{ \text{int}(U_i) \} \) have the same code as \( \mathcal{U} \). First, we show that \( \text{code}(\mathcal{U}) = \text{code}(\text{cl}(\mathcal{U})) \). Let \( A^\mathcal{U}_\sigma \) denote an atom of \( \mathcal{U} \) and \( A^{\text{cl}(\mathcal{U})}_\sigma \) denote the corresponding atom of \( \text{cl}(\mathcal{U}) \). If \( A^\mathcal{U}_\sigma = \emptyset \), then using (15) and (5) we conclude that
\[
\bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \notin \sigma} U_j \implies \text{cl}(\bigcap_{i \in \sigma} U_i) \subseteq \text{cl}(\bigcup_{j \notin \sigma} U_j) \implies \bigcap_{i \in \sigma} \text{cl}(U_i) \subseteq \bigcup_{j \notin \sigma} \text{cl}(U_j),
\]
and thus \( A^{\text{cl}(\mathcal{U})}_\sigma = \emptyset \). Therefore, \( \text{code}(\text{cl}(\mathcal{U})) \subseteq \text{code}(\mathcal{U}) \). On the other hand, using (5) we obtain
\[
A^{\text{cl}(\mathcal{U})}_\sigma = \bigcap_{i \in \sigma} \text{cl}(U_i) \setminus \bigcup_{j \notin \sigma} \text{cl}(U_j) = \bigcap_{i \in \sigma} \text{cl}(U_i) \setminus \text{cl}(\bigcup_{j \notin \sigma} U_j)
\]
\[
= \left( \bigcap_{i \in \sigma} \text{cl}(U_i) \setminus \bigcup_{j \notin \sigma} U_j \right) \setminus \left( \text{cl}(\bigcup_{j \notin \sigma} U_j) \setminus \bigcup_{j \notin \sigma} U_j \right) \supseteq A^\mathcal{U}_\sigma \setminus \partial \left( \bigcup_{j \notin \sigma} U_j \right).
\]
Thus, if \( A^\mathcal{U}_\sigma \) is non-empty, since it is top-dimensional while \( \partial (\bigcup_{j \notin \sigma} U_j) \) is of codimension one, \( A^\mathcal{U}_\sigma \subseteq \partial \left( \bigcup_{j \notin \sigma} U_j \right) \), implying \( A^{\text{cl}(\mathcal{U})}_\sigma \neq \emptyset \), and thus, \( \text{code}(\mathcal{U}) = \text{code}(\text{cl}(\mathcal{U})) \).

Next, we show that \( \text{code}(\text{int}(\mathcal{U})) = \text{code}(\mathcal{U}) \). Let \( A^\mathcal{U}_\sigma \) be an atom of \( \mathcal{U} \) and \( A^{\text{int}(\mathcal{U})}_\sigma \) be the corresponding atom of \( \text{int}(\mathcal{U}) \). If \( A^\mathcal{U}_\sigma = \emptyset \), then using (7) and (14) we conclude that
\[
\bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \notin \sigma} U_j \implies \text{int}(\bigcap_{i \in \sigma} U_i) \subseteq \text{int}(\bigcup_{j \notin \sigma} U_j) \implies \bigcap_{i \in \sigma} \text{int}(U_i) \subseteq \bigcup_{j \notin \sigma} \text{int}(U_j),
\]
which implies \( A^\text{int}(\mathcal{U}) = \emptyset \). Therefore, \( \text{code}(\text{int}(\mathcal{U})) \subseteq \text{code}(\mathcal{U}) \). On the other hand, using (7), we obtain

\[
A^\text{int}(\mathcal{U}) = \bigcap_{i \in \sigma} \text{int}(U_i) \setminus \bigcup_{j \not\in \sigma} \text{int}(U_j) \supset \text{int}(\bigcap_{i \in \sigma} U_i) \setminus \bigcup_{j \not\in \sigma} U_j = \\
\left( \bigcap_{i \in \sigma} U_i \setminus \bigcup_{j \not\in \sigma} U_j \right) \setminus \partial \left( \bigcap_{i \in \sigma} U_i \right) = A^\text{int}(\mathcal{U}) \setminus \partial \left( \bigcap_{i \in \sigma} U_i \right).
\]

Thus, if \( A^\text{int}_\sigma \) is non-empty, since it is top-dimensional while \( \partial \left( \bigcap_{i \in \sigma} A_i \right) \) is of codimension one, \( A^\text{int}_\sigma \neq \emptyset \). Therefore, \( \text{code}(\mathcal{U}) = \text{code}(\text{int}(\mathcal{U})) \). \( \square \)

5.4. Proof of Lemma 3.1 In order to prove Lemma 3.1, we will need the following two lemmas.

**Lemma 5.6.** Let \( \mathcal{W} = \{W_i\} \) be a collection of sets, \( W_i \subseteq X \), and \( \mathcal{C} = \text{code}(\mathcal{W}, X) \). Assume that \( Q \) is a proper subset of some atom of \( \mathcal{W} \), i.e. \( \emptyset \neq Q \subseteq A^W_\alpha \), for a non-empty \( \alpha \in \mathcal{C} \). Then for any \( \sigma_0 \subseteq \alpha \), the cover \( \mathcal{V} = \{V_i\} \) by the sets

\[ (18) \quad V_i = \begin{cases} W_i, & \text{if } i \in \sigma_0, \\ W_i \setminus Q, & \text{if } i \not\in \sigma_0 \end{cases} \]

adds the codeword \( \sigma_0 \) to the original code, i.e. \( \text{code}(\mathcal{V}, X) = \text{code}(\mathcal{W}, X) \cup \{\sigma_0\} \).

**Proof.** Since \( Q \subseteq A^W_\alpha \), \( \text{code}(\{V_i \cap (X \setminus Q)\}, X \setminus Q) = \text{code}(\mathcal{W}, X) \). Moreover, because \( \sigma_0 \subseteq \alpha \), \( \text{code}(\{V_i \cap Q\}, Q) = \{\sigma_0\} \) by construction. Finally, observe that if \( X = Y \cup Z \), then \( \text{code}(\mathcal{V}, X) = \text{code}(\{V_i \cap Y\}, Y) \cup \text{code}(\{V_i \cap Z\}, Z) \), therefore we obtain

\[ \text{code}(\mathcal{V}, X) = \text{code}(\{V_i \cap (X \setminus Q)\}, X \setminus Q) \cup \text{code}(\{V_i \cap Q\}, Q) = \text{code}(\mathcal{W}, X) \cup \{\sigma_0\} \]. \( \square \)

Recall that \( M(\mathcal{C}) \subseteq \mathcal{C} \) denotes the set of maximal codewords of \( \mathcal{C} \). A subset \( A \cap B \) of a topological space is called *relatively open in B* if it is an open set in the induced topology of the subset \( B \).

**Lemma 5.7.** Let \( \mathcal{U} = \{U_i\} \) be an open convex cover in \( \mathbb{R}^d \), \( d \geq 2 \), with \( \mathcal{C} = \text{code}(\mathcal{U}, X) \). Assume that there exists an open Euclidean ball \( B \subseteq \mathbb{R}^d \) such that \( \text{code}(\{B \cap U_i\}, B \cap X) = \mathcal{C} \), and for every maximal set \( \alpha \in M(\mathcal{C}) \), the set \( \partial B \cap \text{cl}(\bigcap_{i \in \alpha} U_i) \) is non-empty and is relatively open in \( \partial B \). Then for any simplicial violator \( \sigma_0 \in \Delta(\mathcal{C}) \setminus \mathcal{C} \), there exists an open convex cover \( \mathcal{V} = \{V_i\} \) with \( V_i \subseteq U_i \), so that \( \text{code}(\mathcal{V}, B \cap X) = \mathcal{C} \cup \sigma_0 \), and the cover \( \mathcal{V} \) satisfies the same condition above with the same open ball \( B \). Moreover, if the cover \( \mathcal{U} = \{U_i\} \) is non-degenerate, then the cover \( \mathcal{V} \) can also be chosen to be non-degenerate.

**Proof.** Choose a facet \( \alpha \in M(\mathcal{C}) \) such that \( \alpha \supseteq \sigma_0 \). Because \( \alpha \) is a facet of \( \Delta(\mathcal{C}) \), the atom \( A^\text{int}_\alpha = \cap_{i \in \alpha} U_i \) is convex open and (by assumption) has a non-empty relatively open intersection with the Euclidean sphere \( \partial B \). This implies that we can always select an oriented and closed half-space \( \mathcal{P}^+ \subseteq \mathbb{R}^d \) such that \( \mathcal{P}^+ \cap B \subseteq A^\text{int}_\alpha \), and \( \{A^\text{int}_\alpha \cap B\} \setminus \mathcal{P}^+ \neq \emptyset \) has relatively open intersection with the sphere \( \partial B \) (see Figure 5.3).

We define two open covers, \( \mathcal{W} = \{W_i\} \), with \( W_i \buildrel \text{def} \over = U_i \cap B \) and \( \mathcal{V} = \{V_i\} \) via the equation (18), with \( Q = \mathcal{P}^+ \cap B \). We thus can use Lemma 5.6 and conclude that \( \text{code}(\mathcal{V}, X \cap B) = \ldots \).
and on the sphere \( \partial B \).}

\[ V \]

same condition should hold for the sets \( V_i \) are open and convex, moreover, the cover \( V \) automatically satisfies the same condition on the atoms of facets of \( \Delta(\mathcal{C}) \).

Finally, if \( U \) is non-degenerate, then \( V \) is also non-degenerate. Indeed, because \( A^U_\alpha \) is open, the only two atoms that were changed, \( A^V_\alpha = (A^U_\alpha \cap B) \setminus P^+ \) and \( A^V_{\sigma_0} = P^+ \cap B \) are also top-dimensional. Moreover, since the only new pieces of boundaries of \( V_i \subseteq U_i \) are introduced on the chord \( \partial P^+ \cap B \) and on the sphere \( \partial B \), if the condition that for all \( \sigma \subseteq [n] \), \( \bigcap_{i \in \sigma} \partial U_i \subseteq \partial (\bigcap_{i \in \sigma} U_i) \) holds then the same condition should hold for the sets \( V_i \).

A consecutive application of the above lemma to all the codewords in \( \mathcal{D} \setminus \mathcal{C} \) for any supra-code \( \mathcal{D} \) with the same simplicial complex yields Lemma 3.1.

**Proof of Lemma 3.1.** Let \( \mathcal{U} = \{U_i\} \) be an open convex cover in \( \mathbb{R}^d \), \( d \geq 2 \), with code \( \{U, X\} = \mathcal{C} \). Assume that there exists an open Euclidean ball \( B \subset \mathbb{R}^d \) such that \( \text{code}(\{B \cap U_i\}, B \cap X) = \mathcal{C} \), and for every maximal set \( \alpha \in M(\mathcal{C}) \), its atom has non-empty intersection with the \((d - 1)\)-sphere \( \partial B \). Let \( \mathcal{C} \subseteq \mathcal{D} \subseteq \Delta(\mathcal{C}) \) and denote \( \mathcal{D} \setminus \mathcal{C} = \{\sigma_1, \sigma_2, \ldots, \sigma_l\} \). Let \( \sigma_1 \subseteq \alpha \in M(\mathcal{C}) \). Because \( \alpha \in M(\mathcal{C}) \), \( A^U_\alpha = \bigcap_{i \in \alpha} U_i \) is open, and thus \( \partial B \cap \text{cl}(A^U_\alpha) \) is relatively open in \( \partial B \). We can now apply Lemma 5.7 to the “missing” codeword \( \sigma_1 \), and obtain a new cover \( \mathcal{V}^{(1)} \) that again satisfies the condition of Lemma 5.7. Consecutively applying Lemma 5.7 with \( \sigma_0 = \sigma_j \), \( j = 2, 3, \ldots l \), we obtain covers \( \mathcal{V}^{(j)} \), so that the last cover, \( \mathcal{V}^{(l)} \) is the desired cover of Lemma 3.1.

5.5. **A closed convex realization for an intersection-complete code.** Here we provide an explicit construction of a closed convex cover of an intersection-complete code. Intersection-complete codes are max intersection-complete, and thus Theorem 1.2 ensures that intersection-complete codes are both open convex and closed convex. Nevertheless, a different construction below may be useful for applications due to its simplicity.

**Definition 5.8.** The potential cover of the code \( \mathcal{C} \), is a collection \( \mathcal{V} = \{V_i\}_{i \in [n]} \) of closed convex sets \( V_i \subset \mathbb{R}^{[\mathcal{C}]} \), defined as follows. For each non-empty codeword \( \sigma \in \mathcal{C} \) let \( e_\sigma \) be a unit vector in \( \mathbb{R}^{[\mathcal{C}]} \) so that \( \{e_\sigma\} \) is a basis for \( \mathbb{R}^{[\mathcal{C}]} \). For each \( i \in [n] \), we define \( V_i \) as the convex hull

\[ V_i \triangleq \text{conv}\{e_\sigma \mid \sigma \in \mathcal{C}, \ \sigma \ni i\}. \]

Since this is a cover by convex closed sets, the code of the potential cover is closed convex. Note however, that this cover is not non-degenerate (Definition 2.10), and cannot be easily extended to an open convex cover.

**Lemma 5.9.** Let \( \mathcal{V} = \{V_i\} \) denote the potential cover of \( \mathcal{C} \), and \( X \triangleq \text{conv}\{e_\sigma \mid \sigma \in \mathcal{C}, \ \sigma \neq \emptyset\} \). Then the code of the potential cover of \( \mathcal{C} \) is the intersection completion of that code: \( \text{code}(\mathcal{V}, X) = \hat{\mathcal{C}} \).

\[ \text{Figure 5.3.} \] The oriented half space \( P^+ \) is chosen to intersect the ball \( B \) inside \( A^U_\alpha \).
Proof. Note that because the vectors $e_{\sigma}$ are linearly independent,
\[ \emptyset \notin \hat{C} \iff \exists i \in [n], V_i = X \iff X = \bigcup_{i \in [n]} V_i \iff \emptyset \notin \text{code}(\mathcal{V}, X). \]
Moreover,
\[ \bigcap_{i \in \sigma} V_i = \text{conv} \{ e_{\tau} \mid \tau \in \mathcal{C}, \tau \supseteq \sigma \}, \]
in particular, code$(\mathcal{V}, X) \subseteq \Delta(\mathcal{C})$. To show that code$(\mathcal{V}, X) \subseteq \hat{C}$, assume that a non-empty $\sigma \in \text{code}(\mathcal{V}, X)$, i.e. $A_{\sigma}^\mathcal{V} = \left( \bigcap_{i \in \sigma} V_i \right) \setminus \bigcup_{j \notin \sigma} V_j$ is non-empty. If there exists an index $j \in \left( \bigcap_{\tau \in \mathcal{C}} \tau \right) \setminus \sigma$, then by (19), $\bigcap_{i \in \sigma} V_i \subset V_j$, which contradicts $\sigma \in \text{code}(\mathcal{V}, X)$. Hence $\sigma = \bigcap_{\tau \in \mathcal{C}} \tau \supseteq \sigma$. Conversely, assume that a non-empty $\sigma \in \hat{C}$ and let $\sigma_1, \ldots, \sigma_k \in \mathcal{C}$ be code elements such that $\sigma = \bigcap_{\ell=1}^k \sigma_\ell$. Then the point $\frac{1}{k} \sum_{\ell=1}^k e_{\sigma_\ell} \in \left( \bigcap_{i \in \sigma} V_i \right) \setminus \bigcup_{j \notin \sigma} V_j$. Hence $\sigma \in \text{code}(\mathcal{V}, X)$. \qed

An immediate corollary is that any intersection-complete code is a closed convex code.

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