CRITICAL SYSTEM INVOLVING FRACTIONAL LAPLACIAN

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ABSTRACT. In this paper, we study the following critical system with fractional
Laplacian:

\[
\begin{aligned}
(-\Delta)^s u &= \mu_1 |u|^{2^*-2} u + \frac{\alpha \gamma}{2^*} |u|^\alpha |v|^\beta u
&\quad \text{in } \mathbb{R}^n, \\
(-\Delta)^s v &= \mu_2 |v|^{2^*-2} v + \frac{\beta \gamma}{2^*} |u|^\alpha |v|^\beta v
&\quad \text{in } \mathbb{R}^n, \\
u, v &\in D_s(\mathbb{R}^n).
\end{aligned}
\]

By using the Nehari manifold, under proper conditions, we establish the
existence and nonexistence of positive least energy solution of the system.

1. Introduction. Recently, a great attention has been focused on the study of
equations or systems involving the fractional Laplacian with nonlinear terms, both
for their interesting theoretical structure and their concrete applications(see [3,
13, 21, 5, 6, 30, 26, 4, 31, 27, 17] and references therein). This type of operator
arises in a quite natural way in many different contexts, such as, the thin obstacle
problem, finance, phase transitions, anomalous diffusion, flame propagation and
many others(see [1, 16, 22, 28] and references therein).

Compared to the Laplacian problem, the fractional Laplacian problem is nonlocal
and more challenging. In 2007, L. Caffarelli and L. Silvestre [5] studied an extension
problem related to the fractional Laplacian in \(\mathbb{R}^n\), which can transform the nonlocal
problem into a local problem in \(\mathbb{R}^{n+1}\). This method can be extended to bounded
regions and is extensively used in recent articles. For example, B. Barrios, E.
Colorado, A. de Pablo and U. Sánchez [3] studied the following nonhomogeneous
equation involving fractional Laplacian,

\[
\begin{aligned}
(-\Delta)^s u &= \lambda u^q + u^{\frac{n+s}{n-2s}} &\quad \text{in } \Omega, \\
u &= 0 &\quad \text{on } \partial \Omega,
\end{aligned}
\]

and proved the existence and multiplicity of solutions under suitable conditions of \(s\)
and \(q\). In the above, the fractional Laplacian operator \((-\Delta)^s\) is defined through the
spectral decomposition using the powers of the eigenvalues of the positive Laplace
operator \((-\Delta)\) with zero Dirichlet boundary data.

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Furthermore, E. Colorado, A. de Pablo and U. Sánchez [13] studied the following fractional equation with critical Sobolev exponent

\[
\begin{aligned}
(-Δ)^s u &= |u|^{2^* - 2}u + f(x) \quad \text{in } Ω, \\
\quad \text{on } ∂Ω,
\end{aligned}
\]

where the existence and the multiplicity of solutions were proved under appropriate conditions on the size of \( f \). For more recent advances on this topic, see [15, 18, 23] and references therein.

It is also natural to study the coupled system of equations. X. He, M. Squassina and W. Zou [19] considered the following fractional Laplacian system with critical nonlinearities on a bounded domain in \( \mathbb{R}^n \)

\[
\begin{aligned}
(-Δ)^s u &= \lambda |u|^{q-2}u + \frac{2α}{α+β}|u|^{α-2}u|v|^β \quad \text{in } Ω, \\
(-Δ)^s v &= μ|v|^{q-2}v + \frac{2α}{α+β}|u|^α|v|^β-2v \quad \text{in } Ω, \\
\quad \text{on } ∂Ω, \\
u = 0, \quad v = 0
\end{aligned}
\]

using variational methods and a Nehari manifold decomposition, they prove that the system admits at least two positive solutions when the pair of parameters \((λ, μ)\) belongs to certain subset of \( \mathbb{R}^2 \). For the above system, when the boundary conditions are replaced by \( u = 0, v = 0 \) in \( \mathbb{R}^n \setminus Ω \), by using fiber maps and the Nehari manifold, when the pair of parameters \((λ, μ)\) belongs to certain subset of \( \mathbb{R}^2 \), W. Chen, S. Deng [8] also showed that the system admits at least two non-semi-trivial solutions.

When \( Ω = \mathbb{R}^n \), the Dirichlet-Neumann map used in [3, 13, 19] provides a formula for the fractional Laplacian in the whole space, which is equivalent to that obtained from Fourier Transform [5], where the operator has explicit expression,

\[
(-Δ)^s u(x) = C(n, s) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,
\]

with

\[
C(n, s) = \left( \int_{\mathbb{R}^n} \frac{1 - \cos(k_1)}{|k|^{n+2s}} \, dk \right)^{-1} = 2^{2s}π^{−\frac{n}{2s}} \frac{Γ\left(\frac{n+2s}{2}\right)}{Γ(2-s)} s(1-s), \quad 0 < s < 1.
\]

In [17], Z. Guo, S. Luo and W. Zou studied the following critical system involving fractional Laplacian

\[
\begin{aligned}
(-Δ)^s u &= μ_1 |u|^{2^*-2}u + \frac{αγ}{2} |u|^{α-2}u|v|^γ \quad \text{in } \mathbb{R}^n, \\
(-Δ)^s v &= μ_2 |v|^{2^*-2}v + \frac{βγ}{2} |u|^α|v|^β-2v \quad \text{in } \mathbb{R}^n, \\
\quad \text{in } D_\ast(\mathbb{R}^n), \\
u, v \in D_\ast(\mathbb{R}^n),
\end{aligned}
\]

under the condition of

\[
(H) = \begin{cases} 
1 < α, \ β < 2, & \text{if } 4s < N < 6s, \\
α, \ β > 1, & \text{if } N ≥ 6s,
\end{cases}
\]

they showed the existence of positive least energy solution, which is radially symmetric with respect to some point in \( \mathbb{R}^n \) and decays at infinity with certain rate. Q. Wang [30] studied a special case where \( α = β = \frac{2}{7} \) and \( \frac{2α}{α+β} = \frac{αγ}{2} = β \), the author also showed the existence of positive least energy solution under suitable conditions.

In this paper, we study the existence of the least energy solutions for the system (1) with critical exponent. We assume

\[
μ_1, \ μ_2 > 0, \ 2^* = \frac{2n}{n-2s}, \ n > 2s, \ 0 < s < 1, \ α, \ β > 1, \ \text{and } α + β = 2^*.
\]
Let $D_s(\mathbb{R}^n)$ be Hilbert space as the completion of $C_c^\infty(\mathbb{R}^n)$ equipped with the norm
\[
\|u\|_{D_s(\mathbb{R}^n)}^2 = \frac{C(n, s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^2 |y - x|^{n+2s} \, dx \, dy.
\]

Let
\[
S_s = \inf_{u \in D_s(\mathbb{R}^n) \setminus \{0\}} \frac{\|v\|_{D_s(\mathbb{R}^n)}^2}{\left( \int_{\mathbb{R}^n} |u|^{2^*} \, dx \right)^{\frac{n}{2s}}}
\]
be the sharp imbedding constant of $D_s(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ and $S_s$ is attained (see [14]) in $\mathbb{R}^n$ by $\tilde{u}_{\epsilon,y} = \kappa(\epsilon^2 + |x-y|)^{-\frac{n-2s}{2s}}$, where $\kappa \neq 0 \in \mathbb{R}$, $\epsilon > 0$ and $y \in \mathbb{R}^n$. That is
\[
S_s = \frac{\|\tilde{u}_{\epsilon,y}\|_{D_s(\mathbb{R}^n)}^2}{\left( \int_{\mathbb{R}^n} |\tilde{u}_{\epsilon,y}|^{2^*} \, dx \right)^{\frac{n}{2s}}}.
\]

We normalize $\tilde{u}_{\epsilon,y}$ as follow, let
\[
\pi_{\epsilon,y}(x) = \frac{\tilde{u}_{\epsilon,y}(x)}{\|\tilde{u}_{\epsilon,y}\|^{2^*}}.
\]

By Lemma 2.12 in [17], $U_{\epsilon,y}(x) = (S_s)^{\frac{n-2s}{2s}} \pi_{\epsilon,y}(x)$ is a positive ground state solution of
\[
(-\Delta)^s u = |u|^{2^*-2} u \quad \text{in} \quad \mathbb{R}^n
\]
and
\[
\|U_{\epsilon,y}\|_{D_s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |U_{\epsilon,y}|^{2^*} \, dx = (S_s)^\frac{n}{2s}.
\]

Note that, the energy functional associated with (1) is given by
\[
E(u, v) = \frac{1}{2}(\|u\|_{D_s(\mathbb{R}^n)}^2 + \|v\|_{D_s(\mathbb{R}^n)}^2) - \frac{1}{2^*} \int_{\mathbb{R}^n} (\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^{\beta})dx.
\]

Define the Nehari manifold
\[
\mathcal{N} = \{(u, v) \in D_s(\mathbb{R}^n) \times D_s(\mathbb{R}^n) : u \neq 0, v \neq 0, \}
\]
\[
\|u\|_{D_s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (\mu_1|u|^{2^*} + \frac{\alpha\gamma}{2^*}|u|^\alpha|v|^{\beta})dx,
\]
\[
\|v\|_{D_s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (\mu_2|v|^{2^*} + \frac{\beta\gamma}{2^*}|u|^\alpha|v|^{\beta})dx.
\]

\[
A := \inf_{(u, v) \in \mathcal{N}} E(u, v) = \inf_{(u, v) \in \mathcal{N}} S \left( \|u\|_{D_s(\mathbb{R}^n)}^2 + \|v\|_{D_s(\mathbb{R}^n)}^2 \right)
\]
\[
= \inf_{(u, v) \in \mathcal{N}} \frac{S}{n} \int_{\mathbb{R}^n} (\mu_1|u|^{2^*} + \mu_2|v|^{2^*} + \gamma|u|^\alpha|v|^{\beta})dx.
\]

Finally, we say that $(u, v)$ is a nontrivial solution of (1) if $u \neq 0, v \neq 0$ and $(u, v)$ solves (1). Any nontrivial solution of (1) is in $\mathcal{N}$. It is easy to see that when the following algebra system (5) has a solution $(k, l)$, then $(\sqrt{k}U_{\epsilon,y}, \sqrt{l}U_{\epsilon,y})$ is a nontrivial solution of (1).

In this paper, we get the existence and nonexistence of least energy solutions of (1) under certain conditions of $\gamma$, $n$ and $s$. Our existence results strongly depend
on the following algebra system
\[
\begin{align*}
\mu_1 k^{2-\frac{s}{n}} + \frac{\alpha}{2} k^{2-\frac{s}{n}} l^2 &= 1, \\
\mu_2 l^{2-\frac{s}{n}} + \frac{\beta}{2} k^{2-\frac{s}{n}} l^{\frac{s}{n}} &= 1, \\
k, l > 0.
\end{align*}
\tag{5}
\]

Our main results are:

**Theorem 1.1.** If $\gamma < 0$, then $A = \frac{n}{\gamma} (\mu_1 - \frac{n-2s}{2} + \mu_2 - \frac{n-2s}{2}^2) S_\gamma^{\frac{n}{2}}$ and $A$ is not attained.

**Theorem 1.2.** If $2s < n \leq 4s$, $\alpha > 2$, $\beta > 2$ and
\[
0 < \gamma \leq \frac{4ns}{(n-2s)^2} \min\left\{ \frac{\mu_1}{\alpha} (\frac{\alpha-2}{\alpha-2})^{\frac{n}{2s}}, \frac{\mu_2}{\beta} (\frac{\alpha-2}{\beta-2})^{\frac{n}{2s}} \right\},
\tag{6}
\]

or $n > 4s$, $1 < \alpha, \beta < 2$ and
\[
\gamma \geq \frac{4ns}{(n-2s)^2} \max\left\{ \frac{\mu_1}{\alpha} (\frac{2-\beta}{2-\alpha})^{\frac{n}{2s}}, \frac{\mu_2}{\beta} (\frac{2-\alpha}{2-\beta})^{\frac{n}{2s}} \right\},
\tag{7}
\]

then $A = \frac{n}{\gamma} (k_0 + l_0) S_\gamma^{\frac{n}{2}}$ and $A$ is attained by $(\sqrt{k_0 U_{\varepsilon,y}}, \sqrt{l_0 U_{\varepsilon,y}})$, where $(k_0, l_0)$ satisfies $(5)$ and $k_0 = \min\{k : (k, l) satisfies (5)\}$.

**Theorem 1.3.** Assume $n > 4s$ and $1 < \alpha, \beta < 2$ hold, there exists a
\[
\gamma_1 \in (0, \frac{4ns}{(n-2s)^2} \max\left\{ \frac{\mu_1}{\alpha} (\frac{2-\beta}{2-\alpha})^{\frac{n}{2s}}, \frac{\mu_2}{\beta} (\frac{2-\alpha}{2-\beta})^{\frac{n}{2s}} \right\},
\]

such that for any $\gamma \in (0, \gamma_1)$, there exists a solution $(k(\gamma), l(\gamma))$ of $(5)$, satisfying $E(\sqrt{k(\gamma) U_{\varepsilon,y}}, \sqrt{l(\gamma) U_{\varepsilon,y}}) > A$ and $(\sqrt{k(\gamma) U_{\varepsilon,y}}, \sqrt{l(\gamma) U_{\varepsilon,y}})$ is a positive solution of $(1)$.

**Remark 1.4.** Z. Guo, S. Luo and W. Zou [17] already showed the existence of positive least energy solutions for the system $(1)$ with $n > 4s$ and $1 < \alpha, \beta < 2$ for all $\gamma > 0$. Theorem 1.2 and Theorem 1.3 tell that if
\[
\gamma \geq \gamma_0 := \frac{4ns}{(n-2s)^2} \max\left\{ \frac{\mu_1}{\alpha} (\frac{2-\beta}{2-\alpha})^{\frac{n}{2s}}, \frac{\mu_2}{\beta} (\frac{2-\alpha}{2-\beta})^{\frac{n}{2s}} \right\},
\]

the positive least energy solution of $(1)$ has to have the form $(\sqrt{k(\gamma) U_{\varepsilon,y}}, \sqrt{l(\gamma) U_{\varepsilon,y}})$; whereas, if $0 < \gamma < \gamma_1$, $(\sqrt{k(\gamma) U_{\varepsilon,y}}, \sqrt{l(\gamma) U_{\varepsilon,y}})$ is not the least energy solution. However, it should be interesting to know whether $\gamma_1 = \gamma_0$ or not. If $\gamma_1 \neq \gamma_0$, what happens when $\gamma \in [\gamma_1, \gamma_0]$.

The paper is organized as follows. In section 2, we introduce some preliminaries that will be used to prove theorems. In section 3, we prove Theorem 1.1. In section 4, we prove Theorem 1.2. The proof of Theorem 1.3 is given in section 5.

2. Some preliminaries. For the case of $\gamma < 0$, the following Lemma 2.1 shows that if the energy functional attains its minimum at some point $(u, v) \in \mathbb{N}$, then $(u, v)$ is a nontrivial solution of $(1)$.

**Lemma 2.1.** Assume $\gamma < 0$, if $A$ is attained by a couple $(u, v) \in \mathbb{N}$, then $(u, v)$ is a nontrivial solution of $(1)$. 

Proof. Define
\[ N_1 = \{(u, v) \in D_\alpha(\mathbb{R}^n) \times D_\alpha(\mathbb{R}^n) : u \neq 0, v \neq 0 \} \]
\[ G_1(u, v) = \|u\|^2_{D_\alpha(\mathbb{R}^n)} - \int_{\mathbb{R}^n} (\mu_1|u|^{2^*-2}u\varphi_1 + \mu_2|v|^{2^*-2}v\varphi_2)dx = 0 \],
\[ N_2 = \{(u, v) \in D_\alpha(\mathbb{R}^n) \times D_\alpha(\mathbb{R}^n) : u \neq 0, v \neq 0 \} \]
\[ G_2(u, v) = \|v\|^2_{D_\alpha(\mathbb{R}^n)} - \int_{\mathbb{R}^n} (\mu_2|v|^{2^*-2}v\varphi_2 + \beta|u|^{2^*-2}u\varphi_2)dx = 0 \}.
Then \( \mathbb{N} = N_1 \cap N_2 \). By the Fréchet derivative, for any \( \varphi_1, \varphi_2 \in D_\alpha(\mathbb{R}^n) \), we have
\[ E'(u, v)(\varphi_1, \varphi_2) = (u, \varphi_1) + (v, \varphi_2) \]
\[ - \int_{\mathbb{R}^n} (\mu_1|u|^{2^*-2}u\varphi_1 + \mu_2|v|^{2^*-2}v\varphi_2)dx \]
\[ - \frac{\gamma}{2^*} \int_{\mathbb{R}^n} (\alpha|u|^{2^*-2}u\varphi_1 + \beta|u|^{2^*-2}v\varphi_2)dx, \]
\[ G_1'(u, v)(\varphi_1, \varphi_2) = 2(u, \varphi_1) - 2^* \int_{\mathbb{R}^n} \mu_1|u|^{2^*-2}u\varphi_1 dx \]
\[ - \frac{\alpha\gamma}{2^*} \int_{\mathbb{R}^n} (\alpha|u|^{2^*-2}u\varphi_1 + \beta|u|^{2^*-2}v\varphi_2)dx, \]
\[ G_2'(u, v)(\varphi_1, \varphi_2) = 2(v, \varphi_2) - 2^* \int_{\mathbb{R}^n} \mu_2|v|^{2^*-2}v\varphi_2 dx \]
\[ - \frac{\beta\gamma}{2^*} \int_{\mathbb{R}^n} (\alpha|u|^{2^*-2}u\varphi_1 + \beta|u|^{2^*-2}v\varphi_2)dx, \]
where
\[ (u, \varphi_1) = \frac{C(n, s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi_1(x) - \varphi_1(y))}{|y - x|^{n+2s}} dxdy. \]
Then
\[ E'(u, v)(u, 0) = G_1(u, v) = 0, \]
\[ E'(u, v)(0, 0) = G_2(u, v) = 0, \]
\[ G_1'(u, v)(u, 0) = 2\|u\|^2_{D_\alpha(\mathbb{R}^n)} - 2^* \int_{\mathbb{R}^n} \mu_1|u|^{2^*} dx - \frac{\alpha\gamma}{2^*} \int_{\mathbb{R}^n} \alpha|u|^\alpha v^\beta dx \]
\[ = -(2^* - 2) \int_{\mathbb{R}^n} \mu_1|u|^{2^*} dx + (2 - \alpha) \int_{\mathbb{R}^n} \frac{\alpha\gamma}{2^*} |u|^\alpha v^\beta dx, \]
\[ G_1'(u, v)(0, v) = -\frac{\alpha\gamma}{2^*} \int_{\mathbb{R}^n} \beta|u|^\alpha v^\beta dx > 0, \]
\[ G_2'(u, v)(u, 0) = -\frac{\beta\gamma}{2^*} \int_{\mathbb{R}^n} \alpha|u|^\alpha v^\beta dx > 0, \]
\[ G_2'(u, v)(0, v) = 2\|v\|^2_{D_\alpha(\mathbb{R}^n)} - 2^* \int_{\mathbb{R}^n} \mu_2|v|^{2^*} dx - \frac{\beta\gamma}{2^*} \int_{\mathbb{R}^n} \beta|u|^\alpha v^\beta dx \]
\[ = -(2^* - 2) \int_{\mathbb{R}^n} \mu_2|v|^{2^*} dx + (2 - \beta) \int_{\mathbb{R}^n} \frac{\beta\gamma}{2^*} |u|^\alpha v^\beta dx. \]
Suppose that \((u, v) \in \mathbb{N}\) is a minimizer for \(E\) restricted to \(\mathbb{N}\), then by the standard minimization theory, there exist two Lagrange multipliers \(L_1, L_2 \in \mathbb{R}\) such that,

\[
E'(u, v) + L_1 G'_1(u, v) + L_2 G'_2(u, v) = 0.
\]

Then we have

\[
L_1 G'_1(u, v)(u, 0) + L_2 G'_2(u, v)(u, 0) = 0,
\]

and

\[
G'_1(u, v)(u, 0) + G'_1(u, v)(0, v) = -(2^* - 2)||u||^2_{D_2(\mathbb{R}^n)} \leq 0,
\]

\[
G'_2(u, v)(0, 0) + G'_2(u, v)(u, v) = -(2^* - 2)||v||^2_{D_2(\mathbb{R}^n)} \leq 0.
\]

Since

\[
||u||^2_{D_2(\mathbb{R}^n)} > 0, \quad ||v||^2_{D_2(\mathbb{R}^n)} > 0,
\]

hence

\[
G'_1(u, v)(u, 0) = -G'_1(u, v)(u, 0) > G'_1(u, v)(0, v),
\]

\[
G'_2(u, v)(0, v) = -G'_2(u, v)(0, v) > G'_2(u, v)(u, 0).
\]

Define the matrix

\[
M = \begin{pmatrix} G'_1(u, v)(u, 0), & G'_2(u, v)(u, 0) \\ G'_1(u, v)(0, v), & G'_2(u, v)(0, v) \end{pmatrix},
\]

then

\[
det(M) = |G'_1(u, v)(u, 0)||G'_2(u, v)(0, v)| - G'_1(u, v)(0, v)G'_2(u, v)(u, 0) > 0,
\]

which means \(L_1 = L_2 = 0\), that is \(E'(u, v) = 0\). \(\square\)

Define functions

\[
F_1(k, l) = \mu_1 k^{\frac{2^{-*}}{2}} + \sigma_1 k^{\frac{2^{-*} - 2}{2}} l^\frac{\alpha}{2} - 1, \quad k > 0, l \geq 0,
\]

\[
F_2(k, l) = \mu_2 l^{\frac{2^{-*}}{2}} + \sigma_2 k^{\frac{2^{-*} - 2}{2}} l^\frac{\beta}{2} - 1, \quad l \geq 0, k > 0,
\]

\[
l(k) = \left(\frac{2^{-*}}{2}\right) \frac{\alpha}{2} k^{\frac{2^{-*} - 2}{2}} (1 - \mu_1 k^{\frac{2^{-*}}{2}}) \frac{\alpha}{2}, \quad 0 < k \leq \mu_1^{- \frac{2^{-*}}{2}},
\]

\[
k(l) = \left(\frac{2^{-*}}{2}\right) \frac{\beta}{2} l^{\frac{2^{-*} - 2}{2}} (1 - \mu_2 l^{\frac{2^{-*}}{2}}) \frac{\beta}{2}, \quad 0 < l \leq \mu_2^{- \frac{2^{-*}}{2}},
\]

then

\[
F_1(k, l(k)) \equiv 0, \quad F_2(k(l), l) \equiv 0.
\]

**Remark 2.2.** If \((k, l)\) satisfies (5), then \((\sqrt{k}U_{e,y}, \sqrt{l}U_{e,y})\) is a nontrivial solution of (1), where \(U_{e,y}\) satisfy (3) and (4). Hence the main work is to establish the existence of solutions to (5).

In order to prove the existence results for (5), we have the following Lemma 2.3.

**Lemma 2.3.** Assume that \(n > 4s, 1 < \alpha, \beta < 2, \gamma > 0\), then

\[
F_1(k, l) = 0, \quad F_2(k, l) = 0, \quad k, l > 0
\]

has a solution \((k_0, l_0)\) such that

\[
F_2(k, l(k)) < 0, \quad \forall \, k \in (0, k_0),
\]

where \((k_0, l_0)\) satisfies \(k_0 = \min\{k : (k, l)\ \text{satisfies (5)}\}\). Similarly, (9) has a solution \((k_1, l_1)\), such that

\[
F_1(k(l), l) < 0, \quad \forall \, l \in (0, l_1).
\]
Proof. Solving $F_1(k, l) = 0$ for $k, l > 0$, we have $l = l(k)$ for all $k \in (0, \mu_1^{-\frac{2s}{\alpha+2}})$. Then, substituting this into $F_2(k, l) = 0$, we have

\[
\mu_2 \left( \frac{2^*}{\alpha \gamma} \right)^\frac{\alpha}{\gamma} \left( 1 - \mu_1 k^{\frac{2s}{2s-2}} \right) - \frac{2^*}{\alpha \gamma} \left( \frac{2s}{2s-2} \right) k^{\frac{2s}{2s-2}} \left( 1 - \mu_1 k^{\frac{2s}{2s-2}} \right) = 0.
\]

(12)

Let

\[
f(k) = \mu_2 \left( \frac{2^*}{\alpha \gamma} \right)^\frac{\alpha}{\gamma} \left( 1 - \mu_1 k^{\frac{2s}{2s-2}} \right) - \frac{2^*}{\alpha \gamma} \left( \frac{2s}{2s-2} \right) k^{\frac{2s}{2s-2}} \left( 1 - \mu_1 k^{\frac{2s}{2s-2}} \right),
\]

then (12) has a solution is equivalent to $f(k) = 0$ has a solution in $(0, \mu_1^{-\frac{2s}{\alpha+2}})$. Since $1 < \alpha, \beta < 2$, we obtain

\[
\lim_{k \to 0^+} f(k) = -\infty, \quad f(\mu_1^{-\frac{2s}{\alpha+2}}) = \frac{\beta \gamma}{\alpha \gamma} > 0,
\]

then by the intermediate value theorem, there exists

\[
k_0 \in (0, \mu_1^{-\frac{2s}{\alpha+2}}) \text{ s. t., } f(k_0) = 0
\]

and

\[
f(k) < 0, \; \forall \; k \in (0, k_0).
\]

Let $l_0 = l(k_0)$, then $(k_0, l_0)$ is the solution of (9) and satisfies (10). Similarly, we can show (9) has a solution $(k_1, l_1)$, such that

\[
F_1(k_1, l_1) < 0, \; \forall \; l \in (0, l_1).
\]

Remark 2.4. From the proof of Lemma 2.9 in the next a few pages, it is easy to see that system (5) has only one solution $(k, l) = (k_0, l_0)$ under the assumption that $2s < n \leq 4s$, $\alpha > 2$, $\beta > 2$ and (6).

Remark 2.5. Obviously, if $(\sqrt{k_0} U_{x,y}, \sqrt{l_0} U_{x,y}) \in \mathbb{N}$, then

\[
A \leq E(\sqrt{k_0} U_{x,y}, \sqrt{l_0} U_{x,y}) = \frac{s}{n} (k_0 + l_0) S_{k_0}^{\frac{n}{2}}.
\]

Next, in order to show $A \geq \frac{s}{n} (k_0 + l_0) S_{k_0}^{\frac{n}{2}}$, we require the following lemmas.

Case 1. $n > 4s$, $1 < \alpha, \beta < 2$ and (7) hold.

Lemma 2.6. Assume $n > 4s$, $1 < \alpha, \beta < 2$ and (7) hold, then

\[
l(k) + k \text{ is strictly increasing in } [0, \mu_1^{-\frac{2s}{\alpha+2}}],
\]

\[
k(l) + l \text{ is strictly increasing in } [0, \mu_2^{-\frac{2s}{\alpha+2}}].
\]

Proof. Since

\[
l'(k) = \left( \frac{2^*}{\alpha \gamma} \right)^\frac{\alpha}{\gamma} \left( \frac{2s}{2s-2} - \mu_1 k^{\frac{2s}{2s-2}} - \mu_1 \beta \right) - \frac{2s}{2s-2} \left( \frac{2 - \alpha}{2} k^{\frac{2s}{2s-2}} - \frac{\mu_1 \beta}{2} k^{\frac{2s}{2s-2}} \right)
\]

\[
= \left( \frac{2^* \mu_1}{\alpha \gamma} \right)^\frac{\alpha}{\gamma} \left( \frac{2s}{2s-2} - \mu_1 k^{\frac{2s}{2s-2}} - \mu_1 \beta \right) - \frac{2s}{2s-2} \left( \frac{2 - \alpha}{2} k^{\frac{2s}{2s-2}} \right)
\]

\[
k'(l) = \left( \frac{2^*}{\beta \gamma} \right)^\frac{\alpha}{\gamma} \left( \frac{2s}{2s-2} - \mu_2 l^{\frac{2s}{2s-2}} - \mu_2 \alpha \right) - \frac{2s}{2s-2} \left( \frac{2 - \beta}{2} l^{\frac{2s}{2s-2}} - \mu_2 \alpha \right)
\]
By (7), we obtain
\[ l'(\frac{2 - \alpha}{\mu_1 \beta})\frac{2 - 2\beta}{\alpha} = l'(\mu_1^{-\frac{2 - 2\beta}{\alpha}}) = 0, \]
\[ k'(\frac{2 - \beta}{\mu_2 \alpha})\frac{2 - 2\beta}{\alpha} = k'(\mu_2^{-\frac{2 - 2\beta}{\alpha}}) = 0, \]
and
\[ l'(k) > 0 \Leftrightarrow k \in (0, \frac{2 - \alpha}{\mu_1 \beta}^{-\frac{2 - 2\beta}{\alpha}}), \]
\[ l'(k) < 0 \Leftrightarrow k \in (\frac{2 - \alpha}{\mu_1 \beta}^{-\frac{2 - 2\beta}{\alpha}}, \mu_1^{-\frac{2 - 2\beta}{\alpha}}), \]
\[ k'(l) > 0 \Leftrightarrow l \in (0, \frac{2 - \beta}{\mu_2 \alpha}^{-\frac{2 - 2\beta}{\alpha}}), \]
\[ k'(l) < 0 \Leftrightarrow l \in (\frac{2 - \beta}{\mu_2 \alpha}^{-\frac{2 - 2\beta}{\alpha}}, \mu_2^{-\frac{2 - 2\beta}{\alpha}}). \]

Next, we compute the second derivatives,
\[ l''(k) = \frac{2 - \beta}{\beta} \left( \frac{2\pi \mu_1}{\alpha \gamma} \right)^2 k^{2\alpha-\alpha} \left( \frac{\alpha (2 - \alpha)}{\mu_1 \beta} - k \frac{\alpha (2 - \alpha)}{\mu_1 \beta} \right), \]
\[ k''(l) = \frac{2 - \alpha}{\alpha} \left( \frac{2\pi \mu_2}{\beta \gamma} \right)^2 l^{2\alpha-\alpha} \left( \frac{\beta (2 - \beta)}{\mu_2 \alpha} - l \frac{\beta (2 - \beta)}{\mu_2 \alpha} \right), \]
we have
\[ l''(k) = 0 \Leftrightarrow k = \left( \frac{2 (2 - \alpha)}{\mu_1 \beta (4 - 2\alpha)} \right)^{\frac{2 - 2\beta}{\alpha}} \Leftrightarrow k \in (\frac{2 - \alpha}{\mu_1 \beta}^{-\frac{2 - 2\beta}{\alpha}}, \mu_1^{-\frac{2 - 2\beta}{\alpha}}), \]
\[ k''(l) = 0 \Leftrightarrow l = \left( \frac{2 (2 - \beta)}{\mu_2 \alpha (4 - 2\alpha)} \right)^{\frac{2 - 2\beta}{\alpha}} \Leftrightarrow l \in (\frac{2 - \beta}{\mu_2 \alpha}^{-\frac{2 - 2\beta}{\alpha}}, \mu_2^{-\frac{2 - 2\beta}{\alpha}}). \]

By (7), we obtain
\[ l'(k)_{\text{min}} = l'(k_1) = -\left( \frac{2\pi (2^* - 2)}{2 \alpha \gamma} \right)^{2 - \alpha} \frac{2 - \beta}{2 - \alpha} \geq -1, \]
\[ k'(l)_{\text{min}} = k'(l_1) = -\left( \frac{2\pi (2^* - 2)}{2 \beta \gamma} \right)^{2 - \alpha} \frac{2 - \alpha}{2 - \beta} \geq -1. \]
Hence
\[ l'(k) > -1, \quad \forall \ k \in (0, \mu_1^{-\frac{2 - 2\beta}{\alpha}}), \]
\[ k'(l) > -1, \quad \forall \ l \in (0, \mu_2^{-\frac{2 - 2\beta}{\alpha}}), \]
which means \( l(k) + k \) is strictly increasing in \([0, \mu_1^{-\frac{2 - 2\beta}{\alpha}}]\) and \( k(l) + l \) is strictly increasing in \([0, \mu_2^{-\frac{2 - 2\beta}{\alpha}}]\).
Lemma 2.7. Assume $n > 4s$, $1 < \alpha, \beta < 2$ and (7) hold, $(k_0, l_0)$ is obtained in Lemma 2.3. Then
\[
(k_0 + l_0) \frac{2s - 2}{2} \max \{\mu_1, \mu_2\} < 1, \tag{13}
\]
and
\[
F_2(k, l(k)) < 0, \quad \forall \, k \in (0, k_0); \quad F_1(k(l), l) < 0, \quad \forall \, l \in (0, l_0). \tag{14}
\]

Proof. Since $k_0 < \mu_1 \frac{2s}{\mu_2}$. By Lemma 2.6, we have
\[
\mu_1 \frac{2s}{\mu_2} = l(\mu_1 \frac{2s}{\mu_2}) + \mu_1 \frac{2s}{\mu_2} \geq l(k_0) + k_0 = l_0 + k_0.
\]
That is
\[
\mu_1(k_0 + l_0) \frac{2s - 2}{2} \leq 1.
\]
Similarly
\[
\mu_2(k_0 + l_0) \frac{2s - 2}{2} \leq 1.
\]
Hence
\[
(k_0 + l_0) \frac{2s - 2}{2} \max \{\mu_1, \mu_2\} < 1.
\]
To prove (14), by Lemma 2.3, we only need to show that $(k, l_0) = (k_1, l_1)$. By (10), (11), we have $l_0 \geq l_1$, $k_1 \geq k_0$. Suppose by contradiction that $k_1 > k_0$, then $l(k_1) + k_1 > l(k_0) + k_0$, hence
\[
l_1 + k(l_1) = l(k_1) + k_1 > l(k_0) + k_0 = l_0 + k(l_0).
\]
Since $k(l) + l$ is strictly increasing for $l \in [0, \mu_2 \frac{2s - 2}{\mu_1}]$, therefore $l_1 > l_0$, which contradicts to $l_0 \geq l$, then we have $k_1 = k_0$. Similarly, $l_1 = l_0$. \hfill \Box

Lemma 2.8. Assume $n > 4s, 1 < \alpha, \beta < 2$ and (7) hold, then
\[
\begin{cases}
 k + l \leq k_0 + l_0, \\
 F_1(k, l) \geq 0, \quad F_2(k, l) \geq 0, \\
 k, l > 0, \quad (k, l) \neq (0, 0),
\end{cases} \tag{15}
\]
has an unique solution $(k, l) = (k_0, l_0)$.

Proof. Obviously $(k_0, l_0)$ satisfies (15). Suppose $(\tilde{k}, \tilde{l})$ is another solution of (15). Without loss of generality, we may assume that $\tilde{k} > 0$, then $\tilde{l} > 0$. In fact, if $l = 0$, then $\tilde{k} \leq k_0 + l_0$ and
\[
F_1(\tilde{k}, 0) = \mu_1 \tilde{k} \frac{2s - 2}{2} - 1 \geq 0.
\]
Therefore
\[
1 \leq \mu_1 \tilde{k} \frac{2s - 2}{2} \leq \mu_1(k_0 + l_0) \frac{2s - 2}{2},
\]
which contradicts to Lemma 2.7. In the following, we prove that $\tilde{k} = k_0$. Suppose by contradiction that $\tilde{k} < k_0$, by the proof of Lemma 2.6, we have $k(l)$ is strictly increasing on $(0, \frac{2s-\beta}{\mu_2 \alpha} \frac{2s-2}{2})$, and strictly decreasing on $((\frac{2s-\beta}{\mu_2 \alpha} \frac{2s-2}{2}, \mu_2 \frac{2s-2}{\mu_1})$.

On the one hand, since $k(0) = k(\mu_2 \frac{2s-2}{\mu_1}) = 0$ and $0 < \tilde{k} < k_0$, therefore, there exist $l_1, l_2$ satisfying $0 < l_1 < l_2 < \mu_2 \frac{2s-2}{\mu_1}$, such that $k(l_1) = k(l_2) = \tilde{k}$ and
\[
F_2(\tilde{k}, l) < 0 \iff \tilde{k} < k(l) \iff l_1 < l < l_2. \tag{16}
\]
Furthermore, $F_1(\tilde{k}, \tilde{l}) \geq 0$, $F_2(\tilde{k}, \tilde{l}) \geq 0$, we have $\tilde{l} > l(\tilde{k})$, $\tilde{l} \leq l_1$ or $\tilde{l} \geq l_2$.

By (14), we see $F_2(\tilde{k}, l(\tilde{k})) < 0$, by (16), we obtain $l_1 < l < l_2$, therefore $\tilde{l} \leq l_2$. 

On the other hand, let \( l_3 = k_0 + l_0 - \tilde{k} \), then \( l_3 > l_0 \) and
\[
k(l_3) + k_0 + l_0 - \tilde{k} = k(l_3) + l_3 > k(l_0) + l_0 = k_0 + l_0,
\]
that is \( k(l_3) > \tilde{k} \). By (16), we have \( l_1 < l_3 < l_2 \). Since \( \tilde{k} + \tilde{l} \leq k_0 + l_0 \), we obtain that \( \tilde{l} \leq k_0 + l_0 - \tilde{k} = l_3 < l_2 \). This contradicts to \( \tilde{l} \geq l_2 \), the proof completes. \( \square \)

**Case 2.** \( 2s < n \leq 4s, \alpha > 2, \beta > 2 \) and (6) hold.

**Lemma 2.9.** Assume \( c, d \in \mathbb{R} \) satisfy
\[
\begin{align*}
\mu_1 k^{\frac{2s-2}{2}} + \frac{\alpha^2 k}{2} l_2^\gamma & \geq 1, \\
\mu_2 l^{\frac{2s-2}{2}} + \frac{\beta^2 l}{2} k^\gamma l_2^\gamma & \geq 1, \\
k, l & > 0.
\end{align*}
\] (17)

If \( 2s < n \leq 4s, \alpha, \beta > 2 \) and (6) hold, then \( c + d \geq k + l \), where \((k, l) \in \mathbb{R}^2 \) is the unique solution of (5).

**Proof.** Let \( y = c + d, x = \frac{\alpha}{\beta}, y_0 = k + l, x_0 = \frac{k}{l} \), by (17) and (5) we have
\[
y^{\frac{2s-2}{2}} \geq \frac{(x + 1)^{\frac{2s-2}{2}}}{\mu_1 x^{\frac{2s-2}{2}} + \frac{\alpha^2 x}{2} l_2^\gamma} \equiv f_1(x), \quad \frac{2s-2}{\mu_2} \leq f_2(x, y_0^{\frac{2s-2}{2}} = f_2(x_0).
\]

Then
\[
\begin{align*}
f_1'(x) &= \frac{\alpha \gamma x + 1)^{\frac{2s-2}{2}} x^{\frac{\alpha^4}{2}}}{2^{2s} \left( \mu_1 x^{\frac{2s-2}{2}} + \frac{\alpha^2 x}{2} l_2^\gamma \right)^2} \left[ -\frac{2s(2s - 2) \mu_1}{\alpha \gamma} x^\gamma + \beta x - (\alpha - 2) \right], \\
f_2'(x) &= \frac{\beta \gamma x + 1)^{\frac{2s-2}{2}} x^{\frac{\alpha^4}{2}}}{2^{2s} \left( \mu_2 + \frac{\beta^2 x}{2} l_2^\gamma \right)^2} \left[ (\beta - 2) x^\gamma - \alpha x^{\frac{\alpha^4}{2}} + \frac{2s(2s - 2) \mu_2}{\beta \gamma} \right].
\end{align*}
\]

Let
\[
\begin{align*}
g_1(x) &= -\frac{2s(2s - 2) \mu_1}{\alpha \gamma} x^\gamma + \beta x - (\alpha - 2), \\
g_2(x) &= (\beta - 2) x^\gamma - \alpha x^{\frac{\alpha^4}{2}} + \frac{2s(2s - 2) \mu_2}{\beta \gamma},
\end{align*}
\]
then
\[
\begin{align*}
g_1'(x) &= 0 \Leftrightarrow x = \left( \frac{2\alpha \gamma}{2s(2s - 2) \mu_1} \right)^{\frac{1}{2s-2}} := x_1, \\
g_1'(x) > 0 \Leftrightarrow x < \left( \frac{2\alpha \gamma}{2s(2s - 2) \mu_1} \right)^{\frac{1}{2s-2}}, \\
g_1'(x) < 0 \Leftrightarrow x > \left( \frac{2\alpha \gamma}{2s(2s - 2) \mu_1} \right)^{\frac{1}{2s-2}}, \\
g_2'(x) &= 0 \Leftrightarrow x = \frac{\alpha - 2}{\beta - 2} := x_2, \\
g_2'(x) > 0 \Leftrightarrow x > \frac{\alpha - 2}{\beta - 2}, \quad g_2'(x) < 0 \Leftrightarrow x < \frac{\alpha - 2}{\beta - 2}.
\end{align*}
\]
Therefore, $g_1(x)$ increases in the interval $(0, (\frac{2\alpha \gamma}{2\gamma'(2-2\mu_1)})^{\frac{1}{\gamma'}})$ and decreases in the interval $((\frac{2\alpha \gamma}{2\gamma'(2-2\mu_1)})^{\frac{1}{\gamma'}}, \infty)$. $g_2(x)$ decreases in the interval $(0, \frac{\alpha - 2}{\beta - 2})$ and increases in the interval $(\frac{\alpha - 2}{\beta - 2}, \infty)$.

Hence

$$g_1(x)_{\max} = g_1(x_1) = (\beta - 2)(\frac{2\alpha \gamma}{2\gamma'(2-2\mu_1)})^{\frac{1}{\gamma'}} - (\alpha - 2),$$

$$g_2(x)_{\min} = g_2(x_2) = -2(\frac{\alpha - 2}{\beta - 2})^{\frac{1}{\gamma'}} + (\frac{2\alpha \gamma}{\beta - 2})^{\frac{1}{\gamma'}}.$$

Then, by (6), we have

$$g_1(x)_{\max} \leq 0, \quad g_2(x)_{\min} \geq 0.$$

Therefore, $f_1(x)$ is strictly decreasing in $(0, +\infty)$ and $f_2(x)$ is strictly increasing in $(0, +\infty)$. Due to the fact that

$$\lim_{x \to 0} f_1(x) = +\infty, \quad \lim_{x \to +\infty} f_2(x) = +\infty,$$

there exists a unique $x_0 > 0$, such that $f_1(x_0) = f_2(x_0)$, which gives the uniqueness of $(k, l)$.

Since $f_1(x) \geq f_1(x_0)$ for $x \leq x_0$ and $f_2(x) \geq f_2(x_0)$ for $x \geq x_0$, we get

$$y^\frac{2^* - 2}{2} \geq \max\{f_1(x), f_2(x)\} = f_1(x_0) = y_0^\frac{2^* - 2}{2},$$

which means $c + d \geq k + l$. \qed

3. Proof of Theorem 1.1.

Proof. By Lemma 2.1, we obtain that when $-\infty < \gamma < 0$ and if $A$ is attained by a couple $(u, v) \in \mathbb{N}$, then $(u, v)$ is a solution of (1). For any $(u, v) \in \mathbb{N}$,

$$||u||^2_{D_\gamma(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \mu_1 |u|^2 + \frac{\alpha \gamma}{2\gamma} |u|^\gamma |v|^\delta \, dx$$

$$\leq \int_{\mathbb{R}^n} \mu_1 |u|^{2^*} \, dx \leq \mu_1 s_{2^*}^{\frac{\alpha \gamma}{2\gamma}} (||u||^2_{D_\gamma(\mathbb{R}^n)})^{\frac{\delta}{2\gamma}}.$$

Therefore

$$||u||^2_{D_\gamma(\mathbb{R}^n)} \geq \mu_1 \frac{\alpha \gamma}{2\gamma} - S_\gamma^{\frac{\delta}{2\gamma}}.$$

Similarly

$$||v||^2_{D_\gamma(\mathbb{R}^n)} \geq \mu_2 \frac{\alpha \gamma}{2\gamma} - S_\gamma^{\frac{\delta}{2\gamma}}.$$

Hence

$$A \geq \frac{s}{n} (||u||^2_{D_\gamma(\mathbb{R}^n)} + ||v||^2_{D_\gamma(\mathbb{R}^n)}) \geq \frac{s}{n} (\mu_1 \frac{\alpha \gamma}{2\gamma} + \mu_2 \frac{\alpha \gamma}{2\gamma}) - S_\gamma^{\frac{\delta}{2\gamma}}.$$

By lemma 2.12 in [17], we know $w_{\mu_i} = (\frac{2^*}{\mu_i})^{\frac{\delta}{2\gamma}} - \mu_i = (\frac{1}{\mu_i})^{\frac{\delta}{2\gamma}} U_{e,y}$ is the solution of the equation

$$(-\Delta)^s u = \mu_i |u|^{2^*-2} u, \quad \text{in } \mathbb{R}^n \text{ for } i = 1, 2.$$

Let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$, $(U(x), V_R(x)) = (w_{\mu_1}(x), w_{\mu_2}(x + Re_1))$, where $R$ is a positive constant. Since $V_R(x) \in D_\gamma(\mathbb{R}^n)$ is a solution of

$$(-\Delta)^s u = \mu_2 |u|^{2^*-2} u \quad \text{in } \mathbb{R}^n,$$
we have, $V_R(x) \to 0$ in $L^{2^*}(\mathbb{R}^n)$ as $R \to +\infty$, hence
\[
\lim_{R \to +\infty} \int_{\mathbb{R}^n} U^\alpha V_R^\beta dx = \lim_{R \to +\infty} \int_{\mathbb{R}^n} U^\alpha V_R^{\frac{2^*_\alpha}{2^*}} V_R^{\frac{2^*_\beta}{2^*}} dx,
\]
\[
\leq \lim_{R \to +\infty} \left( \int_{\mathbb{R}^n} U^{2^*-1} V_R dx \right)^{\frac{\alpha}{2^*}} \left( \int_{\mathbb{R}^n} V_R^{2^*} dx \right)^{\frac{\beta}{2^*}} \to 0.
\]

We will show that for $R > 0$ sufficiently large, the system
\[
\begin{cases}
||U||_{D_s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \mu_1 |U|^{2^*} dx \\
(\mu_1 + \mu_2 V_R^{2^*} + (t_R)^{\frac{\alpha}{2^*}} (s_R)^{\frac{\beta}{2^*}}) \int_{\mathbb{R}^n} \alpha \frac{U^{\alpha}}{2^*} V_R^\beta dx,
\end{cases}
\]
\[
||V||_{D_s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \mu_2 |V|^{2^*} dx
\]
\[
(\mu_1 + \mu_2 V_R^{2^*} + (t_R)^{\frac{\alpha}{2^*}} (s_R)^{\frac{\beta}{2^*}}) \int_{\mathbb{R}^n} \alpha \frac{V^{\alpha}}{2^*} U^\beta dx,
\]
has a solution $(t_R, s_R)$ with
\[
\lim_{R \to +\infty} ((t_R - 1) + |s_R - 1|) = 0 \quad \text{(\star)}
\]
which implies that
\[
(\sqrt{t_R} U, \sqrt{s_R} V_R) \in \mathbb{N}.
\]

Let us assume $(\star)$ first, then
\[
A = \inf_{(u,v) \in \mathbb{N}} E(u,v) \leq E(\sqrt{t_R} U, \sqrt{s_R} V_R),
\]
\[
= \frac{s}{n} (t_R ||U||_{D_s(\mathbb{R}^n)}^2) + s_R ||V||_{D_s(\mathbb{R}^n)}^2,
\]
\[
\leq \frac{s}{n} (t_R \mu_1 \frac{n-2^*}{2^*} ||U_{\epsilon,\gamma}||_{D_s(\mathbb{R}^n)}^2 + s_R \mu_2 \frac{n-2^*}{2^*} ||U_{\epsilon,\gamma}||_{D_s(\mathbb{R}^n)}^2).
\]

Let $R \to +\infty$, we get
\[
A \leq \frac{s}{n} (\mu_1 \frac{n-2^*}{2^*} + \mu_2 \frac{n-2^*}{2^*}) S^{\frac{n}{2^*}}.
\]

Therefore
\[
A = \frac{s}{n} (\mu_1 \frac{n-2^*}{2^*} + \mu_2 \frac{n-2^*}{2^*}) S^{\frac{n}{2^*}}. \quad (19)
\]

Suppose that $A$ is attained by some $(u,v) \in \mathbb{N}$, then $E(u,v) = A$. By Lemma 2.1, we know $(u,v)$ is a nontrivial solution of (1). By Strong maximum principle for fractional Laplacian (see, Proposition 2.17 in [28], Lemma 6 in [26]) and comparison principle in [24], we may assume that $u > 0$, $v > 0$ and
\[
\int_{\mathbb{R}^n} |u|^\alpha |v|^\beta dx > 0,
\]
then
\[
||u||_{D_s(\mathbb{R}^n)}^2 < \int_{\mathbb{R}^n} \mu_1 |u|^{2^*} dx \leq \mu_1 s^{\frac{2^*}{n}} (||u||_{D_s(\mathbb{R}^n)}^2)^{\frac{2^*}{n^*}}.
\]

Hence
\[
||u||_{D_s(\mathbb{R}^n)}^2 > \mu_1 \frac{n-2^*}{2^*} S^{\frac{n}{2^*}}.
\]

Similarly, we have
\[
||v||_{D_s(\mathbb{R}^n)}^2 > \mu_2 \frac{n-2^*}{2^*} S^{\frac{n}{2^*}}.
\]

Then
\[
A = E(u,v) = \frac{s}{n} (||u||_{D_s(\mathbb{R}^n)}^2 + ||v||_{D_s(\mathbb{R}^n)}^2) > \frac{s}{n} (\mu_1 \frac{n-2^*}{2^*} + \mu_2 \frac{n-2^*}{2^*}) S^{\frac{n}{2^*}},
\]
which contradicts to (19). Therefore, $A$ is not be obtained.
Now, we claim (*).

**Proof of (*)&.** Let

$$\theta \triangleq \frac{\int_{\mathbb{R}^n} U^\alpha V^\beta \, dx}{\int_{\mathbb{R}^n} \mu_1 |U|^{2^*} \, dx},$$

we have \(\theta \to 0\) as \(R\) sufficiently large. Then (18) has a solution equivalent to that the following system has a solution at the neighbourhood of point \((1,1)\).

$$\begin{cases}
    k \frac{x_1^2}{2} + \frac{\alpha \gamma}{2} k \frac{x_2^2}{l^2} \theta = 1, \\
    l \frac{x_2^2}{2} + \frac{\beta \gamma}{2} k \frac{x_1^2}{l^2} \theta = 1.
\end{cases}$$

By Taylor expansion at \((1,1)\), we have

$$\begin{align*}
    &\left(1 + \frac{2}{2^*} (k - 1) + O((k - 1)^2)\right) \\
    &\left(1 + \frac{2}{2^*} (l - 1) + O((l - 1)^2)\right) = 1,
\end{align*}$$

Therefore

$$\begin{align*}
    &\left\{ \begin{array}{l}
        k - 1 = \frac{\alpha \gamma}{2} \left(\frac{2}{2^*} - 2\right) (k - 1) \theta - \frac{\beta \gamma}{2} \left(\frac{2}{2^*} - 2\right) (l - 1) \theta - \frac{\alpha \gamma}{2} \left(\frac{2}{2^*} - 2\right) \theta \\
        l - 1 = -\frac{\beta \gamma}{2} \left(\frac{2}{2^*} - 2\right) (k - 1) \theta - \frac{\alpha \gamma}{2} \left(\frac{2}{2^*} - 2\right) (l - 1) \theta - \frac{\beta \gamma}{2} \left(\frac{2}{2^*} - 2\right) \theta
    \end{array} \right.
\end{align*}$$

Let

$$\begin{align*}
    &a_1 = -\frac{\alpha \gamma}{2^*} \left(\frac{2}{2^*} - 2\right), b_1 = -\frac{\alpha \gamma}{2^*} \left(\frac{2}{2^*} - 2\right), c_1 = -\frac{\alpha \gamma}{2^*} \left(\frac{2}{2^*} - 2\right), \\
    &a_2 = -\frac{\beta \gamma}{2^*} \left(\frac{2}{2^*} - 2\right), b_2 = -\frac{\beta \gamma}{2^*} \left(\frac{2}{2^*} - 2\right), c_2 = -\frac{\beta \gamma}{2^*} \left(\frac{2}{2^*} - 2\right),
\end{align*}$$

we have that

$$\begin{align*}
    &\left\{ \begin{array}{l}
        k - 1 = (a_1 (k - 1) + b_1 (l - 1) + c_1) \theta + O((k - 1)^2 + (l - 1)^2), \\
        l - 1 = (a_2 (k - 1) + b_2 (l - 1) + c_2) \theta + O((k - 1)^2 + (l - 1)^2).
    \end{array} \right.
\end{align*}$$

(20)

Let

$$x = \begin{pmatrix} k - 1 \\ l - 1 \end{pmatrix}, \quad B = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

then

$$x = \theta B x + c \theta + O(x^2).$$

Let

$$T(x) \triangleq \theta B x + c \theta + O(x^2),$$

then

$$\|T(x)\| \leq \theta \|B\| \|x\| + \|c\| \theta + O(\|x\|^2).$$

If we choose \(x \in \mathbb{R}^2\) with \(\|x\| \leq 2\|c\| \theta\), we get

$$\|T(x)\| \leq (2\|B\| \|c\| + \|c\| + O(\theta)) \theta \leq 2\|c\| \theta,$$

when \(\theta\) is small enough. Since \(T\) is continuous and \(\theta \to 0\), as \(R \to \infty\), by Brouwer’s fixed point theorem, we get that the system (18) has a solution \((t_R, s_R)\) for all large \(R\) with

$$\lim_{R \to +\infty} (|t_R - 1| + |s_R - 1|) = 0.$$

\(\square\)
4. Proof of Theorem 1.2.

**Proof.** By Remark 2.5, we have
\[ A \leq E(\sqrt{k_0}U_{\varepsilon,y}, \sqrt{t_0}U_{\varepsilon,y}) = \frac{s}{n} (k_0 + l_0) S_n^{\frac{2}{2s}}. \]  
(21)

Let \((u_i, v_i) \in \mathbb{N}\) be a minimizing sequence for \(A\), that is \(E(u_i, v_i) \to A\) as \(n \to \infty\). Define
\[ c_i = (\int_{\mathbb{R}^n} |u_i|^{2s} dx)^{\frac{1}{2s}}, \quad d_i = (\int_{\mathbb{R}^n} |v_i|^{2s} dx)^{\frac{1}{2s}}, \]
then
\[ S_n c_i \leq ||u_i||_{D_\varepsilon(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \mu_1 |u_i|^{2s} + \frac{\alpha \gamma}{2s} |u_i|^\alpha |v_i|^{\beta} dx \]
\[ \leq \mu_1 (c_i)^{\frac{2s}{\alpha \gamma}} + \frac{\alpha \gamma}{2s} \left( \int_{\mathbb{R}^n} |u_i|^{2s} \right)^{\frac{\alpha \gamma}{2s}} \left( \int_{\mathbb{R}^n} |v_i|^{2s} \right)^{\frac{\beta}{2s}} \]
\[ \leq \mu_1 (c_i)^{\frac{2s}{\alpha \gamma}} + \frac{\beta}{2s} (d_i)^{\frac{\beta}{2s}}, \]
\[ S_n d_i \leq ||v_i||_{D_\varepsilon(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \mu_1 |v_i|^{2s} + \frac{\alpha \gamma}{2s} |u_i|^\alpha |v_i|^{\beta} dx \]
\[ \leq \mu_2 (d_i)^{\frac{2s}{\alpha \gamma}} + \frac{\beta}{2s} (d_i)^{\frac{\beta}{2s}}. \]

Dividing both side of inequality by \(S_n c_i\) and \(S_n d_i\). Let
\[ \bar{c}_i = \frac{c_i}{S_n^{\frac{2s}{2s}}}, \quad \bar{d}_i = \frac{d_i}{S_n^{\frac{2s}{2s}}}, \]
we get
\[ \begin{cases} 
\mu_1 \bar{c}_i \bar{c}_i^{\frac{2s-\alpha}{\alpha \gamma}} + \frac{\alpha \gamma}{2s} \bar{c}_i \bar{d}_i^{\frac{\beta}{2s}} \geq 1, \\
\mu_2 \bar{c}_i \bar{c}_i^{\frac{2s-\beta}{\beta \gamma}} + \frac{\beta}{2s} \bar{c}_i \bar{d}_i^{\frac{\beta}{2s}} \geq 1, 
\end{cases} \]
that is
\[ F_1(\bar{c}_i, \bar{d}_i) \geq 0, \quad F_2(\bar{c}_i, \bar{d}_i) \geq 0. \]

Consequently, for the case \(2s < n \leq 4s, \alpha > 2, \beta > 2\) and (6) hold, Lemma 2.9 ensures that
\[ \bar{c}_i + \bar{d}_i \geq k + l = k_0 + l_0. \]
For the case \(n > 4s, 1 < \alpha, \beta \leq 2\) and (7) hold, Lemma 15 ensures
\[ \bar{c}_i + \bar{d}_i \geq k + l = k_0 + l_0. \]
Therefore
\[ c_i + d_i \geq (k_0 + l_0) S_n^{\frac{2s}{2s}} = (k_0 + l_0) S_n^{\frac{n-2s}{2s}}. \]

Since \((u_i, v_i) \in \mathbb{N}\) is a minimizing sequence for \(A\), we have
\[ E(u_i, v_i) = \frac{s}{n} (||u_i||_{D_\varepsilon(\mathbb{R}^n)}^2 + ||v_i||_{D_\varepsilon(\mathbb{R}^n)}^2). \]  
(22)

By the definition of \(S_n\) and (22), we have,
\[ S_n (c_i + d_i) \leq (||u_i||_{D_\varepsilon(\mathbb{R}^n)}^2 + ||v_i||_{D_\varepsilon(\mathbb{R}^n)}^2) = \frac{n}{s} E(u_i, v_i) = \frac{n}{s} A + o(1). \]

Since
\[ S_n (c_i + d_i) \geq (k_0 + l_0) S_n^{\frac{n}{2s}} \quad \text{and} \quad A \leq \frac{s}{n} (k_0 + l_0) S_n^{\frac{n}{2s}}, \]
we have
\[ (k_0 + l_0) S_n^{\frac{n}{2s}} \leq S_n (c_i + d_i) \leq (k_0 + l_0) S_n^{\frac{n}{2s}}, \]
that is
\[ c_i + d_i \to (k_0 + l_0)S_s^{\frac{2s}{n}} \text{ as } i \to +\infty, \]
thus
\[ A = \lim_{i \to +\infty} E(u_i, v_i) \geq \lim_{i \to +\infty} \frac{s}{n}S_s(c_i + d_i) \geq \frac{s}{n}(k_0 + l_0)S_s^{\frac{2s}{n}}. \]

Hence
\[ A = \frac{s}{n}(k_0 + l_0)S_s^{\frac{2s}{n}} = E(\sqrt{k_0}U_{\varepsilon}, \sqrt{l_0}V_{\varepsilon}). \]

5. Proof of Theorem 1.3. In order to proof Theorem 1.3, we use a result of Z. Guo, S. Luo and W. Zou in [17].

**Theorem 5.1** (Theorem 1.1 of [17]). Assume \((H)\) holds, where
\[
(H) = \begin{cases} 
1 < \alpha, \beta < 2, & \text{if } 4s < N < 6s; \\
\alpha, \beta > 1, & \text{if } N \geq 6s.
\end{cases}
\]

Then \((1)\) has a positive ground state solution \((U, V)\) for all \(\gamma > 0\), which is radially symmetric decreasing with the following decay condition
\[ U(x), V(x) \leq C(1 + |x|)^{2s-n}. \]

That is \(E(U, V) = A\), where
\[ A = \inf_{(u,v)\in\mathcal{N}} E(u,v), \]
\[ \mathcal{N} = \left\{ (u, v) \in D_s(\mathbb{R}^n) \times D_s(\mathbb{R}^n) \setminus \{(0, 0)\} : \|u\|_{D_s(\mathbb{R}^n)}^2 + \|v\|_{D_s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (\mu_1|u|^{2s} + \mu_2|v|^{2s} + \gamma|u|^{\alpha}|v|^{\beta}) \right\}. \]

**Lemma 5.2** (A result after Lemma 3.1 in [17]). Let \(u_{\mu} = (\frac{S_s}{n})^{\frac{2s}{n}}S_s^{\frac{2s}{n}}\) is a positive ground state solution of
\[ (-\Delta)^s u = \mu|u|^{2s-2}u, \text{ in } \mathbb{R}^n \]
then
\[ A < \min\left\{ \inf_{(u,v)\in\mathcal{N}} E(u,0), \inf_{(u,v)\in\mathcal{N}} E(0,v) \right\} \]
\[ = \min\{E(u_{\mu_1},0), E(0,u_{\mu_2})\} \]
\[ = \min\left\{ \frac{s}{n}\mu_1 \frac{n-2s}{2s} S_s^{\frac{2s}{n}}, \frac{s}{n}\mu_2 \frac{n-2s}{2s} S_s^{\frac{2s}{n}} \right\} \]

**Remark 5.3.** The Nahri manifold used in [17] is different from the Nahri manifold used in our paper. However, the positive solutions of \((1)\) are certainly in both of the Nahri manifolds, therefore, Z. Guo, S. Luo and W. Zou’s result: Theorem 5.1 ensures that
\[ A = \inf_{(u,v)\in\mathcal{N}} E(u,v) = \inf_{(u,v)\in\mathcal{N}} E(u,v). \]

**Proof of Theorem 1.3.** To obtain the existence of \((k(\gamma), l(\gamma))\) for \(\gamma > 0\), we define functions
\[ F_1(k,l,\gamma) = \mu_1 k^{\frac{2s-2}{n}} + \frac{\alpha\gamma}{2s}k^{\frac{n-2s}{n}}l^2 - 1 \quad k, l > 0, \]
\[ F_2(k,l,\gamma) = \mu_2 l^{\frac{2s-2}{n}} + \frac{\beta\gamma}{2s}k^{\frac{n-2s}{n}}l^2 - 1 \quad k, l > 0, \]
let
\[ k(0) = \mu_1^{-\frac{2s}{n-2s}}, \quad l(0) = \mu_2^{-\frac{2s}{n-2s}}, \]
then

\[ F_1(k(0), l(0), 0) = F_2(k(0), l(0), 0) = 0, \]
\[ \partial_k F_1(k(0), l(0), 0) = \frac{2^s - 2}{2} \mu_1 k^{\frac{s}{s-4}} > 0, \]
\[ \partial_k F_1(k(0), l(0), 0) = \partial_k F_2(k(0), l(0), 0) = 0, \]
\[ \partial F_2(k(0), l(0), 0) = \frac{2^s - 2}{2} \mu_2 l^{\frac{s}{s-4}} > 0. \]

Denote

\[ D = \left( \begin{array}{ccc}
\partial_k F_1(k(0), l(0), 0), & \partial_k F_1(k(0), l(0), 0), & \partial_k F_2(k(0), l(0), 0) \\
\partial_k F_2(k(0), l(0), 0), & \partial_k F_2(k(0), l(0), 0), & \partial F_2(k(0), l(0), 0) 
\end{array} \right), \]

since \( \det(D) > 0 \), by implicit function theorem, we see that \( k(\gamma), l(\gamma) \) are well defined and of class \( C^1 \) in \( (-\gamma_2, \gamma_2) \), for some \( \gamma_2 > 0 \) and

\[ F_1(k(\gamma), l(\gamma), \gamma) = F_2(k(\gamma), l(\gamma), \gamma) = 0. \]

Thus \( (\sqrt{\gamma} U_{x,y}, \sqrt{\gamma} U_{x,y}) \) is a solution of (1).

Since

\[ \lim_{\gamma \to 0} (k(\gamma) + l(\gamma)) = k(0) + l(0) = \mu_1 \frac{\alpha-2s}{n} + \mu_2 \frac{\alpha-2s}{n}, \]

there exists a \( \gamma_1 \in (0, \gamma_2) \), such that

\[ k(\gamma) + l(\gamma) > \min\{\mu_1 \frac{\alpha-2s}{n}, \mu_2 \frac{\alpha-2s}{n}\}, \forall \gamma \in (0, \gamma_1). \]

By Lemma 5.2 and Remark 5.3, we get

\[ E(\sqrt{\gamma} U_{x,y}, \sqrt{\gamma} U_{x,y}) = \frac{s}{n} (k(\gamma) + l(\gamma)) S_n^{\frac{\alpha}{s}} \]

\[ > \min\left\{ \frac{s}{n} \mu_1 \frac{\alpha-2s}{n} S_n^{\frac{\alpha}{s}}, \frac{s}{n} \mu_2 \frac{\alpha-2s}{n} S_n^{\frac{\alpha}{s}} \right\} \]

\[ > A = \inf_{(u,v) \in \mathbb{N}} E(u, v), \]

that is \( (\sqrt{\gamma} U_{x,y}, \sqrt{\gamma} U_{x,y}) \) is another positive solution of (1). \( \square \)

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