Membranes with a Boundary

David S. Berman\textsuperscript{1}, and Daniel C. Thompson\textsuperscript{2}
Queen Mary, University of London,
Department of Physics,
Mile End Road,
London, E1 4NS, England

Abstract

We investigate the recently developed theory of multiple membranes. In particular, we consider open membranes, i.e. the theory defined on a membrane world volume with a boundary.

We first restrict our attention to the gauge sector of the theory. We obtain a boundary action from the Chern-Simons terms.

Secondly, we consider the addition of certain boundary terms to various Chern-Simons theories coupled to matter. These terms ensure the full bulk plus boundary action has the correct amount of supersymmetry. For the ABJM model, this construction motivates the inclusion of a boundary quartic scalar potential. The boundary dynamics obtained from our modified theory produce Basu-Harvey type equations describing membranes ending on a fivebrane.

The ultimate goal of this work is to throw light on the theory of fivebranes using the theory of open membranes.

\textsuperscript{1}email: D.S.Berman@qmul.ac.uk
\textsuperscript{2}email: D.C.Thompson@qmul.ac.uk
## Contents

1 Introduction ................................................................. 2

2 Boundary Theory of The Gauge Sector ................................. 5

3 $\mathcal{N} = 1$ Supersymmetry with Boundary - General Theory 8

4 $\mathcal{N} = 1$ Super Chern-Simons with Boundary ................ 10
   4.1 EL boundary conditions and WZW model revisited ............. 14

5 $\mathcal{N} = 1$ Super Chern-Simons Matter Theory .................. 16

6 $\mathcal{N} = 2$ Supersymmetry with Boundary - General Theory .... 18

7 $\mathcal{N} = 2$ Chern-Simons with Boundary ......................... 20

8 $\mathcal{N} = 2$ Matter with Boundary .................................. 23

9 ABJM with Boundary ....................................................... 28
   9.1 ABJM review .......................................................... 28
   9.2 $U(1) \times U(1)$ ABJM with boundary .......................... 29
   9.3 Towards $U(N) \times U(N)$ ABJM with boundary .............. 31
   9.4 $\mathcal{N} = (2,0)$ supersymmetry ................................ 32
   9.5 $\mathcal{N} = (1,1)$ supersymmetry ................................ 34

10 Discussion ................................................................. 35

11 Acknowledgements ....................................................... 36

12 Appendix 1: $\mathcal{N} = 1$ Supersymmetry Conventions .......... 36

13 Appendix 2: $\mathcal{N} = 2$ Supersymmetry Conventions .......... 38

14 Appendix 3: Multiplet Decomposition .............................. 39
1 Introduction

For over a decade since the discovery of M-theory, the search for an adequate description of multiple and coincident membranes, was fruitless. For a review of branes in M-theory see [1]. Recently, a significant breakthrough in this direction was made by Bagger and Lambert (BL) [2–4], and separately by Gustavsson [5], with the remarkable discovery of a 3-dimensional maximally supersymmetric (\( \mathcal{N} = 8 \)) theory endowed with \( SO(8) \) R-symmetry. The BL theory is based on a novel algebraic structure called a three-algebra. However, the conditions placed on the structure constants of this algebra, namely complete antisymmetry and a generalization of the Jacobi identity, prove to be very restrictive [6]. Essentially, there is a unique finite-dimensional three-algebra with a positive definite metric. With this choice of algebra, the BL can also be described in a more conventional way as supersymmetric Chern-Simons gauge theory with a \( SU(2) \times SU(2) \) product gauge group and bi-fundamental matter [7]. This theory is thought to describe two membranes in a non-trivial orbifold-like background [8,9].

Much work has gone into generalising the BL theory to account for an arbitrary number of membranes. One approach is to relax the positivity condition on the three-algebra. This leads to the so-called Lorentzian BL models [10–12]. Another approach, which has received much attention, is the ‘ABJM’ model [13]. This shares many features with BL theory and, indeed, can be built from a three-algebra by relaxing the condition of complete antisymmetry on the structure constants [14]. Like the BL theory this is a super Chern-Simons theory with bi-fundamental matter but with a more general \( G = U(N) \times U(N) \) product gauge group. Unlike the BL theory, the only R-symmetry manifestly is \( SU(4) \) and therefore the theory only possesses \( \mathcal{N} = 6 \) supersymmetry. The theory is characterized by an integer parameter, the Chern-Simons level \( k \). At a general level \( k \) this theory is thought to describe membranes
whose transverse space is an orbifold $\mathbb{C}^4/\mathbb{Z}_k$ and is conjectured to be the CFT dual to $AdS_4 \times S_7/\mathbb{Z}_k$ geometry. In the case of $k = 1$ it is proposed that the supersymmetry is enhanced to $\mathcal{N} = 8$ and that the theory describes $N$ membranes in flat space with a geometric dual of $AdS_4 \times S_7$. Interestingly, there is also a ’t Hooft limit given by taking both $N$ and $k$ large with $\lambda = N/k$ fixed. In this limit the theory admits a dual geometric description of $AdS_4 \times \mathbb{C}P_3$.

Alongside the M2 stands another extended object of M-theory; the M5-brane. The world volume theory of even a single M5 brane is a complicated affair. A key reason for this complexity is the self dual two-form contained in the (2,0) tensor multiplet describing the world volume fields. Generalizing this to any non-abelian version is harder still and it seems highly unlikely that there is a straightforward way to do so. A full theory of multiple M5-branes currently remains a distant prospect.

An important tool in the study of M5 branes is the membrane. Open membranes end on M5 branes much as open strings end on D-branes [15, 16]. By studying open membranes one could hope to learn about the M5 brane. A striking example of this idea is the derivation of the M5 brane equations of motion from demanding $\kappa$-symmetry in the open membrane action [17]. In the context of BL theory there has already been some interesting work in attempting to extract information about the M5 brane [18].

The system of a membrane ending on a fivebrane has two alternative view points. From the membrane perspective it is conjectured, in analogy to the D1-D3 system in string theory, that the ending can be described as a fuzzy $S^3$ funnel solution of the Basu-Harvey equation [19, 20] and as a solution of the mass deformed ABJM model [21]. From the perspective of the M5, the M2 brane appears as a string like soliton of the non-linear world volume equations of motion. This soliton is known as the self dual string (SDS) [22].

This paper will work towards a further description of the open membrane. We will attempt to generalize the recently developed theory of interacting membranes to the describe multiple membranes whose world volume possess a boundary. This boundary should correspond to self dual strings.
In section 2, we shall restrict our attention to the gauge sector of ABJM theory with a boundary. We find that this sector can be expressed as a non-chiral WZW-type model with a central charge that scales like $N$. This result is consistent with the expected scaling of the degrees of freedom of the self dual string.

In the remainder of the paper we consider the full ABJM theory with a boundary. The presence of a boundary should break half the supersymmetry; the boundary obviously breaks translational symmetry and, since supersymmetry closes on translations, it is inevitable that the presence of boundary will also break supersymmetry. A few moments of contemplation will reveal that only a half the supersymmetry can be preserved. We employ and extend the method of [23] to supplement the ABJM theory with an appropriate boundary action so that the correct amount of symmetry is preserved. We will find the BPS Basu-Harvey type equations are rediscovered as boundary conditions for the open membranes. (This is a fairly involved procedure and before tackling ABJM we address some less supersymmetric Chern-Simons matter theories.)

In section 3, we outline the procedure of [23] which allows for the construction of $1/2$ supersymmetric actions for manifolds with a boundary. In section 4, we recap how this procedure applies to the $\mathcal{N} = 1$ abelian Chern-Simons theory and pay special attention to boundary conditions. In section 5, we consider the $\mathcal{N} = 1$ Chern-Simons matter theory. In sections 6,7,8, we generalize this construction to $\mathcal{N} = 2$ superspace and apply this to abelian Chern-Simons matter theory. In section 9, we apply these considerations to the ABJM model formulated in $\mathcal{N} = 2$ superspace [30]. In this $\mathcal{N} = 2$ description we are readily able to lift the construction of $\mathcal{N} = 1$ case and draw some interesting conclusions. A downside of working with $\mathcal{N} = 2$ is that we will only ever have partial knowledge of the consequence of a boundary since not all the supersymmetry is manifest. There have been some formulations of both Bagger-Lambert and ABJM using more extended superspaces [31–34] and it would be interesting to address the issue of a boundary in these formalisms. Although we do not tackle the full non-abelian gauge sector in this paper we are able to motivate the inclusion of a certain boundary potential for the matter fields. This paper should be
viewed as the first step in a complete rigorous study of membrane boundary theories whose eventual goal is to learn about the fivebrane.

2 Boundary Theory of The Gauge Sector

As a warm up to the ideas involved we examine the pure gauge sector. The Chern-Simons action is not gauge invariant in the presence of a boundary unless specific boundary conditions are added that ultimately induce physical degrees of freedom on the boundary. This is an example of the generic idea we wish to explore. We will produce a boundary action that allows the preservation of the right amount of symmetries (supersymmetry) but at first we will explore the idea with the well known case of Chern-Simons theory.

The Chern-Simons action for a Lie algebra valued gauge field, $\mathbf{A}$, is given by

$$ S[\mathbf{A}] = \frac{k}{4\pi} \int_M \text{Tr} \left( \mathbf{A} \wedge d\mathbf{A} + \frac{2}{3} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A} \right), $$

(1)

where $M$ is a 3-dimensional manifold. Demanding invariance of the path under large gauge transformations requires the Chern-Simons level $k$ be quantized. When the theory has no boundary it is purely topological; the metric does not enter into the definition of the theory\(^3\). If $M = \Sigma \times \mathbb{R}$ and thus $\partial M = \partial \Sigma \times \mathbb{R}$, one can show that $S[\mathbf{A}]$ depends only on the values of the gauge field on the boundary and can be expressed as a WZW model [25–27]. Consider the simplest case where $\Sigma$ is a disc with coordinates $(r, \theta)$ and the $\mathbb{R}$ direction is identified with time. In this case the boundary of $M$ is a cylinder. After choosing $A_0 = 0$ as an Euler-Lagrange boundary condition one finds that $A_0$ is a Lagrange multiplier enforcing a Gauss’ law constraint on the remaining components of the gauge field. Solving the constraint and a little algebra yields the WZW model,

$$ S_{\text{wzw}}[U] = \frac{k}{4\pi} \int_{\partial M} d\theta \tau \text{Tr} (U^{-1} \partial_\theta U U^{-1} \partial_\tau U) + \frac{k}{12\pi} \int_M \text{Tr} (U^{-1} dU)^3, $$

(2)

where $U$ is group valued. The second term at first seems to suggest that the theory depends on the value of $U$ across the whole of $M$. This is illusory; under a change

\(^3\text{though at a quantum level it is more subtle to show the topological nature}\)}
of extension the action differs only by an integer multiple of $2\pi$ and leaves a path integral unaltered [24]. An important observation is that this is a chiral WZW model. The kinetic term is non standard since it is first order in time derivatives. Careful consideration of the symmetries of this theory shows that only a left action of the group given by $U(\theta, t) \rightarrow V(\theta)U(\theta, t)$ is a true global symmetry and this gives rise to a chiral current algebra [26]. We note however, that although the kinetic term is non-standard, we do not obviously have the equation of motion for a chiral boson [44]. We will return to this issue presently.

The gauge kinetic term of the ABJM (and BL) theory is similar to the above Chern-Simons action. However, the gauge group is now a product $G = U(N) \times U(N)$, and there are two gauge fields, $A$ and $\hat{A}$; one for each factor. The kinetic term is now

$$S = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) - \frac{k}{4\pi} \int_M \text{Tr} \left( \hat{A} \wedge d\hat{A} + \frac{2}{3} \hat{A} \wedge \hat{A} \wedge \hat{A} \right),$$

with the essential difference in sign between the two factors. The case of BL $SU(2) \times SU(2)$ theory was discussed in [36] where it was proposed that the associated WZW model has a six-dimensional target space. This target space could perhaps be related to the M5 world volume geometry back reacted by multiple self-dual strings. That analysis was, of course, predicated on the conjecture that the $SU(2) \times SU(2)$ theory of the BL was unique and could describe multiple M2s. The subsequent emergence of the more general ABJM model, and the understanding that BL describes only two membranes, invites us to revise such an interpretation.

A key observation to make is that the Chern-Simons action (1) is not parity invariant. We will denote our coordinates in three dimensions as $x^\mu = \{x^0, x^1, x^3\}$. The action of parity is a reflection in either spatial coordinate and is given by [27]

$$\mathcal{P} : x = \{x^0, x^1, x^3\} \mapsto x' = \{x^0, x^1, -x^3\},
$$

$$\mathcal{P} : A = \{A_0(x), A_1(x), A_3(x)\} \mapsto A' = \{A_0(x'), A_1(x'), -A_3(x')\}. $$

On spinors this acts with the multiplication by $\gamma^3$. Under this action the Chern-Simons term picks up a minus sign. The ABJM theory is then parity invariant with the additional identification that parity also swaps the two gauge fields [7].
When we have a boundary, which we will henceforth assume to be in the $x^3$ direction, the Lorentz symmetry is broken and it makes a difference which spatial coordinate one chooses to reflect in. If, instead of the above parity operation (which we shall denote as $P^{(3)}$) we choose to reflect the $x^1$ direction with a corresponding operator $P^{(1)}$ then we can develop a notion of chirality. We may define chirality projectors on spinors as $P_\pm = \frac{1}{2}(1 + \gamma^3)$. Then $P^{(1)}$ acts essentially by swapping plus with minus i.e. it switches chirality and swaps over light cone coordinates $x_\pm = x_0 \pm x_1$.

For example on a spinor, $P^{(1)}: \psi_+ = P_+ \psi \mapsto P_+ \gamma^1 \psi = \gamma^1 P_- \psi = \gamma^1 \psi_-$. Again for the ABJM model these actions are to be combined with a switch of fields.

With this in mind we understand that the $G = U(N) \times U(N)$ gauge kinetic terms of the ABJM model produces two chiral WZW models. The three dimensional parity transformation descends to a two dimensional chirality transformation and has the effect of switching the two WZW factors. In short, the two gauge fields and relative sign of the levels are exactly what is needed to construct a full non-chiral string. A useful observation at this stage is that in deriving the WZW model we choose a boundary condition $A_0 = \dot{A}_0 = 0$. This boundary condition is compatible with parity invariance and as a result the final boundary theory is non-chiral.

It is interesting to note that one would arrive at a similar non-chiral string by considering a single gauge group but when $\Sigma$ has the topology of an annulus rather than a disc [26].

The fact that the left and right moving sectors are decoupled means that the central charge is given by the usual result (see e.g. [37])

$$c = \frac{k \text{dim} SU(N)}{k + \tilde{h}_{SU(N)}} = \frac{k(N^2 - 1)}{k + N}.$$  

For large $N$ and fixed $k$ this scales linearly in $N$. It is worth viewing this in terms of the 't Hooft coupling in the ABJM model. It was demonstrated in [13] that the effective 't Hooft coupling of the theory is $\lambda = \frac{N}{k}$.

There are two natural limits to examine, large and small 't Hooft coupling. In the large 't Hooft coupling limit,

$$\lambda \gg 1 \quad c \rightarrow kN = k^2 \lambda.$$  

7
and in the small ’t Hooft coupling limit where

\[ \lambda \ll 1 \quad c \to N^2 = k^2 \lambda^2. \quad (8) \]

As explained in the introduction, open membranes can end on five-branes. The M5 world volume description of this is the self dual string soliton solution to the non-linear five-brane equations of motion. One can calculate the absorption cross section of scalar fluctuations of the five-brane equations of motion in the SDS background. This indicates that the number of degrees of freedom of the SDS scales linearly with the SDS charge [38] or, equivalently, linearly with the number of M2 branes. More subtle anomaly considerations confirm this result [39] and can also indicate how the degree of freedom count depends on the number of five-branes (for a review of all of this see [1]).

The linear scaling with \( N \) of the central charge derived above suggests we are on the right track in trying to interpret the boundary theory of open membranes as describing \( N \) coincident self dual strings in the large ’t Hooft coupling limit.

It remains a (realistic) calculational challenge to reproduce the \( N^2 \) scaling of the boundary in the weak ’t Hooft coupling theory.

Of course, the derivation of the WZW model is innately tied to the topological nature of the Chern-Simons theory. Considering the full ABJM model makes such an interpretation much harder due to the decidedly non-topological matter sector. Nevertheless, it seems that degrees of freedom associated to the self dual string may arise as the remnants of the would-be non-propagating pure gauge degrees of freedom in the membrane.

\section{\( \mathcal{N} = 1 \) Supersymmetry with Boundary - General Theory}

We now move on to discuss the supersymmetric theory. We will use supersymmetry to motivate the inclusion of certain boundary terms for the case of open membranes. In particular, we shall try to build an action that is automatically supersymmetric in the presence of a boundary. This construction requires no boundary condition and thus holds off-shell i.e. without the imposition of Euler-Lagrange boundary conditions.
We shall ultimately use the $\mathcal{N} = 2$ superspace form of the ABJM model introduced in [30], but, before exploring that, we review the approach for some simpler models. We first introduce a formalism employed by Belyaev and van Nieuwenhuizen, [23], for $\mathcal{N} = 1$ supersymmetry and apply this Chern-Simons matter theories. We then develop this idea to tackle $\mathcal{N} = 2$ supersymmetry and the ABJM theory.

In [23] it is shown how to build 3-dimensional supersymmetric actions when space-time has a boundary and we now review this procedure. A $\mathcal{N} = 1$ scalar superfield is given by

$$\Phi = a + \theta \psi - \theta^2 f,$$

and can be integrated over superspace to form an action

$$S_0 = \int d^3 x \int d^2 \theta \Phi = \int d^3 x f.$$  \hfill (9)

The supersymmetry transformations,

$$\delta \Phi = \epsilon Q \Phi \Rightarrow \begin{cases} 
\delta a = \epsilon \psi \\
\delta \psi_\alpha = -\epsilon_\alpha f + (\gamma^\mu \epsilon)_\alpha \partial_\mu a \\
\delta f = -\epsilon \gamma^\mu \partial_\mu \psi ,
\end{cases} \hfill (11)$$

ensure that the action varies to a total derivative under rigid supersymmetry. When space-time has no boundary such terms can be safely ignored using generic arguments and the action is thus supersymmetric. In the case that space-time has a boundary, which we will assume throughout to be spatial and lie at $x_3 = 0$, we must pay attention to this surface term. Without any other considerations we have broken supersymmetry as $\delta S_0 = -\partial_\mu (\epsilon \gamma^\mu \psi)$.

One might stop here and say that supersymmetry is recovered by imposing some boundary conditions. However, we should be clear in distinguishing Euler-Lagrange boundary conditions which are associated with equations of motion and the kind of boundary condition that are required to enforce supersymmetry off-shell. Instead the approach advocated by [23] is to build actions that are supersymmetric without the need for any boundary conditions. Only then, having built such a bulk + boundary

---

4See appendix for details of supersymmetry conventions.
supersymmetric action, should one go ahead and calculate the EL field equations and boundary conditions if so desired.

Consider the following boundary action

$$S_1 = - \int d^3x \partial_3 \Phi|_{\theta=0} = - \int d^3x \partial_3 a,$$

with supersymmetry variation

$$\delta S_1 = - \int d^3x \partial_3 (\epsilon \psi).$$

Then the combination $S_0 \pm S_1$ has variation

$$\delta[S_0 \pm S_1] = \mp \int d^3x \partial_3 [\epsilon (1 \pm \gamma^3) \psi] = \mp \int d^3x \partial_3 [2\epsilon_\mp \psi_\pm]$$

where we have defined projected spinors $\psi_\pm \equiv P_\pm \psi \equiv \frac{1}{2}(1 \pm \gamma^3)\psi$. Then

$$\delta[S_0 \pm S_1] = 0 \iff \epsilon_\mp = 0.$$ 

Hence the modified action preserves half ($\mathcal{N} = (1, 0)$ or $(0, 1)$) of the supersymmetry generated by $\epsilon_\mp Q_\pm$.

We may augment this minimal process by including an extra $\mathcal{N} = (1, 0)$ 2-dimensional theory defined solely on the boundary. To this end it is helpful to relate 3-dimensional $\mathcal{N} = 1$ multiplets to 2-dimensional $\mathcal{N} = (1, 0)$ multiplets. This is detailed in the appendix.

## 4 $\mathcal{N} = 1$ Super Chern-Simons with Boundary

Let us apply this formalism to $\mathcal{N} = 1$ abelian Chern-Simons theory. This is addressed in [23] and we recapitulate this here for convenience and to clarify a few subtleties which will become important in the more involved scenarios we consider later on.

We begin with a spinor superfield,

$$\Gamma_\alpha = \chi_\alpha - \theta_\alpha M + (\gamma^\mu \theta)_{\alpha \nu} \mu - \theta^2[\lambda + \gamma^\mu \partial_\mu \chi]_\alpha,$$
which contains the 3-dimensional vector field as one of its components. The notion of
gauge transformation is extended to superspace
\[
\delta_G \Gamma_\alpha = \Gamma_\alpha \Rightarrow \begin{cases} 
\delta_G \chi = \psi \\
\delta_G M = f \\
\delta_G v_\mu = \partial_\mu a \\
\delta_G \lambda = 0 .
\end{cases}
\]

When an action is invariant under this gauge transformation the arbitrary shifts in
\( M \) and \( \chi \) allow the WZ gauge choice \( M = \chi = 0 \). The gauge invariant field strength
is given as
\[
W_\beta = D_\alpha D_\beta \Gamma_\alpha = \lambda_\beta + 2\epsilon^{\mu\nu\rho}(\theta\gamma_\rho)_\beta \partial_\mu v_\nu + \theta^2(\gamma^\mu \partial_\mu \lambda)_\alpha .
\]

The Chern-Simons action is given by
\[
S^{CS}_0 = \int d^3 x \int d^2 \theta \Gamma^\alpha W_\alpha \\
= \int d^3 x \lambda \lambda - 4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho - \partial_\mu (\chi \gamma^\mu \lambda) .
\]
Notice that the auxiliary field \( M \) is entirely absent and that \( \chi \) enters only as a total
derivative. This action is gauge invariant only up to a total derivative
\[
\delta_G S^{CS}_0 = \int d^3 x \partial_\mu [\lambda \gamma^\mu \psi + 4\epsilon^{\mu\nu\lambda} \partial_\nu a v_\lambda] .
\]
When we have a boundary we must be careful about such terms. In keeping with our
overall philosophy we do not impose a boundary condition just to restore a symmetry.
For the moment we take the view point that we have destroyed the gauge symmetries
\[
\delta_G \chi = \psi , \quad \delta_G v_m = \partial_m a ,
\]
where \( m = \{0, 1\} \). The symmetry associated to the absent field \( M \) remains (trivially)
as does that corresponding to \( v_3 \). The supersymmetry is also destroyed.

We now follow the procedure of the preceding section and add a supersymmetric
restorative term
\[
S^{CS}_1 = - \int d^3 x \partial_3 (\Gamma^\alpha W_\alpha)|_{\theta=0} = - \int d^3 x \partial_3 (\chi^\alpha \lambda_\alpha)
\]
Then

\[ S_0^{CS} + S_1^{CS} = \int d^3 x \lambda \lambda - 4 \epsilon^{\mu \rho \nu} v_\mu \partial_\nu v_\rho - \partial_3 (2 \lambda + \chi) \] (24)

is, by construction, invariant under \( \epsilon + Q_+ \) supersymmetry transformations. Notice that by the elimination of \( \chi_+ \) we have also restored some gauge symmetry namely

\[ \delta G \chi_+ = \psi_+ . \] (25)

Unfortunately this modified action is a little awkward since the non-propagating gaugino has a coupling on the boundary. This prevents us from straightforwardly integrating out the gaugino which is something we may wish to do when we couple to matter. We would like to remove this term without resorting to an ad-hoc off-shell boundary condition on \( \lambda \) and thereby voiding the construction of supersymmetry without boundary conditions.

We are at liberty to supplement this construction with any two-dimension \( \mathcal{N} = (1,0) \) theory defined on the boundary. Suppose this system is built from the same fields we already have. Then we can find appropriate actions by decomposing \( \Gamma_\alpha \) into co-dimension 1 superfields as is detailed in the appendix. One finds that

\[ \hat{\Gamma}_\alpha^- = \chi_\alpha + (\gamma^m \theta_+)_\alpha v_m \] (26)

\[ \hat{\Sigma}_m^+ = v_m + \theta_+ [\frac{1}{2} \gamma^m \lambda_+ + \partial_m \chi_-] \] (27)

are two such 2D \( \mathcal{N}=(1,0) \) superfields. From these we can form the boundary action

\[ S_2^{CS} = -2 \int d^3 x \partial_3 \{ \int d\theta_+ \gamma^m_\alpha \hat{\Gamma}_\beta^- \hat{\Sigma}_m^+ \} \] (28)

\[ = 2 \int d^3 x \partial_3 [\chi_- \lambda_+ + \chi_- \gamma^m \partial_m \chi_- + v_m v^m] . \] (29)

We combine this with our previous expressions to find a total action for Chern-Simons with a boundary

\[ S_{tot}^{CS} = S_0^{CS} + S_1^{CS} + S_2^{CS} = \int d^3 x \lambda \lambda - 4 \epsilon^{\mu \rho \nu} v_\mu \partial_\nu v_\rho + 2 \partial_3 [\chi_- \gamma^m \partial_m \chi_- + v_m v^m] \] (30)
which preserves $\epsilon_+$ supersymmetry and which contains a non-propagating gaugino which can be easily integrated out. It is interesting to note the appearance of a dynamical fermion on the boundary. This field, which would have been pure gauge in the WZ sense, has been promoted to become dynamical. This sort of behavior is analogous to the purely bosonic case we looked at before. It should be noted that the kinetic term for this field is really chiral; one simply makes use of the identity $\gamma^0\gamma^1 = \gamma^3$ to show

$$\chi^-\gamma^m\partial_m\chi^- = \chi^-\gamma^1\partial_+\chi^- .$$

(31)
The new combined action has gauge transformation

$$\delta_G S_{CS}^{CS} = \int d^3x 4\partial_3[\psi^-\gamma^m\partial_m\chi^- + (\eta^{mn} + \epsilon^{mn})\partial_mav_n]$$

(32)

$$= \int d^3x 4\partial_3[\psi^-\gamma^m\partial_m\chi^- - \partial_+av_-] ,$$

(33)

where we have defined light cone combinations $v_\pm = v_0 \pm v_1$. So for the final action the full set of gauge symmetries are

$$\delta v_- = \partial_- a , \quad \delta v_3 = \partial_3 a ,$$

(34)

$$\delta M = f , \quad \delta \chi_+ = \psi_+ ,$$

(35)

(the last two are somewhat trivial since those fields are absent).

In summary we started with $S_0$, (19), an action that was neither gauge invariant nor supersymmetric in the presence of a boundary. We added a suitable term $S_1$, (23) to restore $\epsilon_+$ susy. We then added a separate boundary action $S_2$, (28) which in itself is $\epsilon_+$ supersymmetric. The final result is a combined action, $S_{CS}^{CS} = S_0 + S_1 + S_2$, (30), which preserves half the supersymmetry and ‘half’ the gauge symmetry. Since $\chi^-$ appears in the final theory as a propagating boundary field it seems that we might have to be cautious about adopting Wess-Zumino gauge. We shall return to this point when we discuss couplings to matter.
4.1 EL boundary conditions and WZW model revisited

We are now in a position to perform an Euler Lagrange variation of the total action. We find the bulk variation yields the usual equations of motion

$$\lambda = \epsilon^\mu_{\nu\rho} \partial_\mu v_\rho = 0$$

and the boundary variation requires

$$(v_+ \delta v_- + \delta \chi_- \gamma^m \partial_m \chi_-)_{\partial M} = 0.$$  

With respect to the transverse direction we view this equation as having the form $p \delta q$. Neumann conditions are of the form $p = 0$ and Dirichlet are $q = \text{const}$. The correct supersymmetric boundary conditions can be easily read off from the boundary $\mathcal{N} = (1, 0)$ multiplets we have used earlier (27). The Neumann condition is, given in terms of those multiplets, $\hat{\Sigma}^+_{m=+} = 0$ and the Dirichlet is $\hat{\Gamma}^-_\alpha = \text{const}$. For the gauge field these are just $v_+ = 0$ for Neumann and $v_- = 0$ for Dirichlet\(^5\).

Even in the bosonic sector the $v_m v^m$ boundary term in (30) is a departure from the standard CS theory. This has an implication for the boundary model obtained. If we compare to the discussion in section 2, we see that we have a different boundary condition. Instead of $v_0 = 0$, we may choose $v_+ = v_0 + v_1 = 0$. In the bosonic action

$$S_{\text{bos}} = \int_M d^3x - 4\epsilon^\mu_{\nu\rho} v_\mu \partial_\nu v_\rho + 2\partial_3 [v_m v^m]$$

we may make use of the boundary condition and carry out some integration by parts to obtain

$$S_{\text{bos}} = -4\int_M d^3x \epsilon^{0ij} (v_0 (\partial_i v_j - \partial_j v_i) - v_i v_j + \partial_j (v_i v_0))$$

with $x^i = \{x^1, x^3\}$. In the standard derivation of the boundary WZW action one would now use the b.c. to eliminate the total derivative term in the above, and having done so, $v_0$ becomes a simple Lagrange multiplier. Although we can not do\(^5\)The Dirichlet condition on a superfield allows the lowest component to be a non-zero constant but other components must be zero.
exactly this we can still use the boundary condition to modify the total derivative term by replacing $v_0$ with $-v_1$ on the boundary. The action is equivalent to

$$S_{bos} = -4 \int_M d^3x \epsilon^{0ij} (2v_0 \partial_i v_j - v_i \dot{v}_j - \partial_j (v_i v_1)) . \quad (40)$$

We can now use the field equation for $v_0$ to invoke the pure gauge constraint $v_i = \partial_i U$. Plugging into the action we are left with

$$S_{bos} = -4 \int_M d^3x \epsilon^{0ij} (-\partial_j (\partial_i U \partial_0 U) - \partial_j (\partial_i U \partial_1 U)) \quad (41)$$

$$= 4 \int_{\partial M} dx^0 dx^1 (\partial_1 U \partial_0 U + \partial_1 U \partial_1 U) . \quad (42)$$

The result appears surprising since manifest Lorentz symmetry appears to be lost and the kinetic term looks somewhat unconventional. However this sort of two-dimensional action is not unknown; it is the Floreanini Jackiw (FJ) action [43] for a chiral field. The field equations show that $\partial_0 \partial_1 U = -\partial_1 \partial_1 U$, and so after integrating\(^6\) one sees that $U$ is indeed a chiral boson. The boundary Majorana-Weyl fermion term is the natural superpartner for this chiral boson.

Although we have only addressed the abelian CS theory it seems very plausible that similar considerations in the non-abelian context would result in the supersymmetric chiral WZW of Sonnenschein [44]; the kinetic term becomes FJ like, the Wess-Zumino term is unaltered and the super-partner is a free adjoint Majorana-Weyl fermion.

We now comment on how this might generalize to the ABJM model. First let us think about the bosonic sector. We saw in section 2 that we should anticipate a non-chiral WZW model for the ABJM theory. However we have changed the boundary condition used in the derivation of the WZW model. To construct a non-chiral boundary string theory we must choose boundary conditions that do not break the parity invariance. Therefore the appropriate parity preserving boundary conditions are $v_+ = 0$ for one gauge field and $\dot{v}_- = 0$ for the other. In the bosonic sector this choice would indeed result in chiral and anti chiral FJ action for a boundary boson.

\(^6\)The arbitrary function that arises upon integration is set to zero by invoking suitable boundary conditions.
However, such a choice is not compatible with preserving $\mathcal{N} = (1, 0)$ supersymmetry. Our bulk + boundary construction applied to the case of two gauge group factors would yield two propagating boundary fermions with the same chirality. This is quite clearly not the correct supersymmetric completion of two chiral bosons of opposite chirality. Of course, this is really a triviality. Breaking half of $\mathcal{N} = 1$ supersymmetry in necessarily chiral.

5 $\mathcal{N} = 1$ Super Chern-Simons Matter Theory

We begin with a pure matter theory based on a real scalar superfield $\Phi = (\phi, \psi, f)$. The action is given as

$$S^M_0 = \int d^3x \int d^2\theta - \frac{1}{2} D^\alpha \Phi D_\alpha \Phi$$

$$= \int d^3x \partial \phi \partial^\mu \phi - \psi \gamma^\mu \partial_\mu \psi$$

and the extra contribution

$$S^M_1 = \int d^3x \partial \phi \left[ \frac{1}{2} (D^\alpha + i \Gamma^\alpha) \Phi^* (D_\alpha - i \Gamma_\alpha) \Phi \right]$$

is such that $S_0 + S_1$ has $\epsilon_+ \, \text{SUSY}$. Since no auxiliary field occurs on the boundary there is no particular motivation to add any additional boundary action to this. We now consider the $U(1)$ gauged version of this action for complex fields with a Chern-Simons kinetic term for the gauge field. The two matter terms to consider are

$$S^M_0 = \int d^3x \int d^2\theta - \frac{1}{2} (D^\alpha + i \Gamma^\alpha) \Phi^* (D_\alpha - i \Gamma_\alpha) \Phi ,$$

$$S^M_1 = \int d^3x \partial \phi \left[ \frac{1}{2} (D^\alpha + i \Gamma^\alpha) \Phi^* (D_\alpha - i \Gamma_\alpha) \Phi \right]_{\theta = 0} .$$

Both of these terms are completely invariant under the superspace generalization of gauge symmetry:

$$\Gamma_\alpha \rightarrow \Gamma_\alpha + D_\alpha \Lambda ,$$

$$\Phi \rightarrow \exp(i\Lambda) \Phi .$$
By construction $S_0 + S_1$ is now $\epsilon_+$ supersymmetric. At this stage we must think a little about how we expand these actions into components. One would normally adopt Wess-Zumino gauge ($\chi = M = 0$) however, we have established that $\chi_-$ occurs as a propagating boundary field in the Chern-Simons kinetic term and the WZ symmetry associated to this field is broken. We ask to what extent can we work in WZ gauge when evaluating the above actions?

Let us answer this question in the abstract. Suppose we have some fields donated by $\chi, \phi$ and some symmetry $U$ which acts as a shift on $\chi$ i.e.

$$U : \begin{cases} \chi \mapsto \chi^U = \chi + U \\ \phi \mapsto \phi^U \end{cases}$$

For an action $S[\chi, \phi]$ that is invariant under these transformations we can gauge fix in the standard way. We introduce a fiducial choice $\hat{\chi} = 0$ and insert this into the path integral; the Fadeev Popov determinant is trivial, and gauge invariance allows us to perform the integration over the gauge group$^7$.

Now let us suppose that there is an addition action $S'[,\phi]$ that is not invariant under the symmetry $U$. This is exactly the situation we have found ourselves in. Let us go on regardless with the gauge fixing procedure and see where we end up. This time we are not able to perform the integration over the gauge group. It may seems as though we have not achieved anything but, the key point is that for the portion of the action $S$ that does not break the symmetry we can adopt the gauge fixing choice. Applied to our Chern-Simons matter theory this shows that we are allowed to consider the matter sector in the WZ gauge$^8$.

Expanding in components we find that the total $N = (1, 0)$ Chern-Simons matter

---

$^7$One must also check if there are any constraints that we must enforce on the Hilbert space due to ‘missing’ equations of motion (for example, in string theory fixing conformal gauge requires that the stress tensor must act as zero on the Hilbert space). In the cases that we are interested in however the remaining equations of motion after gauge fixing automatically imply the ‘missing’ equation from WZ fields.

$^8$Of course, there are the regular issues of the WZ gauge breaking supersymmetry but this is not pertinent to our discussion.
theory with a boundary is given by

\begin{equation}
S_{tot}^{CS-M} = S_{tot}^{CS} + S_0^M + S_1^M
\end{equation}

\begin{equation}
= \int d^3x \kappa \lambda \lambda - 4\kappa \epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho + f f^* - \nabla^\mu \phi \nabla_\mu \phi^* - \psi^* \gamma^\mu \partial_\mu \psi
\end{equation}

\begin{equation}
+ \frac{i}{2}(\psi \lambda \phi^* - \psi^* \lambda \phi)
\end{equation}

\begin{equation}
+ \partial_3 [2\kappa \chi - \gamma^m \partial_m \chi + 2\kappa v_m v^m + \psi^* \psi]
\end{equation}

Now the importance of removing the boundary gaugino couplings becomes clear; we can integrate out the gaugino to generate the potential. The result (generalized to include capital Roman flavor indices) is given by

\begin{equation}
S_{tot}^{CS-M} = \int d^3x \kappa \lambda \lambda - 4\kappa \epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho - \nabla^\mu \phi I \nabla_\mu \phi^*_I - \psi^*_I \gamma^\mu \partial_\mu \psi_I
\end{equation}

\begin{equation}
+ \frac{1}{16\kappa} (\psi^*_I \psi^*_J \phi_I \phi_J + \psi_I \psi_J \phi^*_I \phi^*_J - 2\psi^*_I \psi_J \phi_I \phi^*_J)
\end{equation}

\begin{equation}
+ \partial_3 [2\kappa \chi - \gamma^m \partial_m \chi + 2\kappa v_m v^m + \psi^*_I \psi_I]
\end{equation}

In this case the boundary terms are not especially interesting; they are just the combination of the ones obtained for Chern-Simons theory and ungauged matter separately.

6 \textbf{\( \mathcal{N} = 2 \) Supersymmetry with Boundary - General Theory}

We now move up in complexity by considering extended supersymmetry. In three-dimensions \( \mathcal{N} = 2 \) superspace is realized by taking the Grassman coordinates to be complex:

\begin{equation}
\theta_\alpha = \frac{1}{\sqrt{2}} (\theta_{1\alpha} + i\theta_{2\alpha})
\end{equation}

We wish to generalize the procedure used in the \( \mathcal{N} = 1 \) case to restore supersymmetry with boundaries. It is easiest to do this by working in a basis where we decompose the \( \mathcal{N} = 2 \) supersymmetry into two copies of the real \( \mathcal{N} = 1 \) symmetry [47] with the

\footnote{We have included a normalisation factor \( \kappa \) for the Chern-Simons term. In the non-abelian theory, invariance under large gauge transformations requires \( \kappa \) to obey a quantisation condition. With traces normalised so that \( Tr(T^a T^b) = \delta_{ab} \) we have that \( \kappa \in \frac{Z}{32\pi} \).}
algebra$^{10}$

$$\{Q_i, Q_j\} = 2\delta_{ij} \gamma^\mu \partial_\mu \quad i = \{1, 2\}.$$ (59)

Under this decomposition the supersymmetry parameters are two 2-component Majorana spinors $\epsilon_1^\alpha$ and $\epsilon_2^\alpha$. Thus we may repeat the construction for $\mathcal{N} = 1$ SUSY with a boundary twice. We start with the most general $\mathcal{N} = 2$ action

$$S_0 = -\int d^3x d^2\theta d^2\bar{\theta} \, L[\theta, \bar{\theta}] = \int d^3x d^2\theta_1 d^2\theta_2 \, L[\theta_1, \theta_2].$$ (60)

We augment this action with another term

$$S_1 = -\int d^3x d^2\theta_1 \partial_3 L|_{\theta_2=0},$$ (61)

so that $S_0 + S_1$ has the $\epsilon_2^+, Q_2^-$ supersymmetry. There is now an apparent choice for the remaining supercharge as to whether we preserve the same chirality supersymmetry $\epsilon_1^+, Q_1^-$ or the opposite $\epsilon_1^-, Q_1^+$. We consider two further terms

$$S_2 = -\int d^3x \int d^2\theta_2 \, \partial_3 L|_{\theta_1=0},$$ (62)

$$S_3 = \int d^3x \, \partial_3 \partial_3 L|_{\theta_1=\theta_2=0}$$ (63)

Then $S_0 + S_1 + S_2 + S_3$ preserves $(\epsilon_1^+, \epsilon_2^+)$ supersymmetry and we shall describe this choice as $\mathcal{N} = (2, 0)$. This can be expressed as a projection condition on the complex spinor

$$P_+ \epsilon = P_+ \frac{1}{\sqrt{2}} (\epsilon_1 + i\epsilon_2) = \epsilon.$$ (64)

On the other hand $S_0 + S_1 - S_2 - S_3$ preserves $(\epsilon_1^-, \epsilon_2^+)$ supersymmetry and we shall call this choice $\mathcal{N} = (1, 1)$.

In $\mathcal{N} = 2$ theories we can also have superpotentials of the form

$$S^W = \int d^3x \int d^2\theta W(\Phi) + \text{c.c.}$$ (65)

$^{10}$See appendix for details.
where \( \Phi \) is an \( \mathcal{N} = 2 \) chiral superfield. We apply the above procedure and find

\[
S_W^0 = \int d^3x d^2\theta d^2\bar{\theta} W(\Phi) = \int d^3x d^2\theta d^2\bar{\theta}(-\bar{\theta}^2)W(\Phi)
\]

(66)

\[
= \int d^3x d^2\theta_1 d^2\theta_2 \frac{1}{2}(\theta_1^2 - \theta_2^2 - i\theta_1\theta_2)W(\Phi)
\]

(67)

and

\[
S_W^1 = \int d^3x d^2\theta_1 d^2\theta_3 [\frac{1}{2}(\theta_1^2 - \theta_2^2 - i\theta_1\theta_2)W(\Phi)]_{\theta_2=0} = \int d^3x \frac{1}{2}\partial_3 W(a),
\]

(68)

\[
S_W^2 = \int d^3x d^2\theta_2 d^2\theta_3 [\frac{1}{2}(\theta_1^2 - \theta_2^2 - i\theta_1\theta_2)W(\Phi)]_{\theta_1=0} = -\int d^3x \frac{1}{2}\partial_3 W(a),
\]

(69)

\[
S_W^3 = -\int d^3x \partial_3 \partial_3 [\frac{1}{2}(\theta_1^2 - \theta_2^2 - i\theta_1\theta_2)W(\Phi)]_{\theta_1=\theta_2=0} = 0.
\]

(70)

In which \( a \) is the lowest component of \( \Phi \). The implication is that if we are to preserve \( \mathcal{N} = (2,0) \) supersymmetry a superpotential term requires no boundary contribution however to preserve \( \mathcal{N} = (1,1) \) supersymmetry we must add to the lagrangian a boundary term of \( \partial_3 W(a) \).

7 \( \mathcal{N} = 2 \) Chern-Simons with Boundary

The abelian \( \mathcal{N} = 2 \) Chern-Simons theory is described by an action \([47,49]\)

\[
S_0 = \int d^3x d^2\theta d^2\bar{\theta} V D^a \bar{D}_a V.
\]

(71)

The vector superfield \( V \) can be expanded in to \( \mathcal{N} = 1 \) component superfields as

\[
V(\theta_1, \theta_2) = A(\theta_1) + \theta_2 \Gamma(\theta_1) - \theta_2^2 (B(\theta_1) - D^2_1 A)
\]

(72)

with components summarized by

\[
A = (a, \psi, f), \quad B = (b, \eta, g), \quad \Gamma = (\chi, M, v, \lambda).
\]

(73)

The extended supersymmetric gauge transformation allow the ‘Ivanov’ gauge choice whereby we set \( A = 0 \) and invoke standard WZ gauge for the spinor multiplet namely
\( \Gamma = (0, 0, v, \lambda) \) \[47\]. Note that this choice differs from the gauge choice adopted in some of the rest of the literature e.g. \[30, 45\].

We expand \( S_0 \) into \( N = 1 \) superfields

\[
S_0^{CS} = \int d^3x d^2\theta_1 (BB + \Gamma^a W_\alpha + \frac{1}{2} D_1^a (D_1 A - BD_1 A)). \tag{74}
\]

In this action the components of \( A \) occur only inside a total derivatives which can easily been seen from the identity \( D^2 D^\alpha = \gamma^\mu D^\alpha \partial_\mu \). Thus, without a boundary, the only difference between this and \( N = 1 \) Chern-Simons theory is the appearance of an auxiliary multiplet. In components

\[
S_0^{CS} = \int d^3 x 2gb + \eta\eta + \lambda\lambda - 4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho \tag{75}
+ \partial_3 (\lambda \gamma^3 \chi + \eta \gamma^3 \psi + \partial^2 ba - b \partial^3 a). \tag{76}
\]

Following the rules of section 6, we build the extra terms

\[
S_1^{CS} = \int d^3 x \partial_3 (-ag - bf - \eta\psi), \tag{77}
\]
\[
S_2^{CS} = \int d^3 x \partial_3 (-bb - \lambda\chi - ag + bf), \tag{78}
\]
\[
S_3^{CS} = \int d^3 x \partial_3 (\partial_3 ab + a \partial_3 b). \tag{79}
\]

Hence we form a \( N = (2, 0) \) action

\[
S^{CS}_{(2,0)} = S_0^{CS} + S_1^{CS} + S_2^{CS} + S_3^{CS} \tag{80}
= \int d^3 x 2gb + \eta\eta + \lambda\lambda - 4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho \tag{81}
+ \partial_3 (-2\chi_-\lambda_+ - 2\psi_-\eta_+ - bb - 2a(g - \partial_3 b)), \tag{82}
\]

and a \( N = (1, 1) \) action

\[
S^{CS}_{(1,1)} = S_0^{CS} + S_1^{CS} - S_2^{CS} - S_3^{CS} \tag{83}
= \int d^3 x 2gb + \eta\eta + \lambda\lambda - 4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho \tag{84}
+ \partial_3 (2\chi_+\lambda_- - 2\psi_-\eta_+ + bb - 2(f + \partial_3 a)b). \tag{85}
\]
In both of these cases we find that we have a gaugino coupling on the boundary. Motivated by the \( \mathcal{N} = 1 \) example we now construct some 2d boundary terms that can be used to remove this term.

For the \( \mathcal{N} = (2, 0) \) case we consider the addition of

\[
S_{(2,0)}^b = \int d^2 x \int d\theta_1 + \gamma^m d\theta_2 + 2\tilde{V}\tilde{V}_m,
\]

where the half supersymmetric superfields are defined and constructed in the appendix. Expanding out into components one finds that

\[
S^{CS}_{(2,0)} - S_{(2,0)}^b = \int d^3 x 2gb + \eta\lambda - 4\epsilon^{\mu\nu\rho}v_\mu \partial_v v_\rho
\]

\[
+ 2\partial_3 (v_\mu v_\mu + \chi_+\gamma^m \partial_m \chi_- + \psi_-\gamma^m \partial_m \psi_- + \partial_m a \partial^m a - \frac{1}{2}bb) \tag{88}
\]

In this case we have removed both the gaugino couplings from the boundary at the expense of introducing some propagating boundary fields. We have two propagating fermions of the same chirality; this indicates the non-chiral \((2, 0)\) nature of the symmetry.

Also notice that we have not eliminated the auxiliary field \( b \) from the boundary. This will actually prove to be a source of interest to us. Notice that the other auxiliary scalar \( g \) serves as a Lagrange multiplier enforcing \( b \) to take a particular value. We then have no need to eliminate \( b \) by its equations of motion and so it is not a problem that it enters in our boundary term. Moreover, when we couple to matter this will provide an interesting boundary interaction term for the matter fields.

For the \( \mathcal{N} = (1, 1) \) case we consider

\[
S_{(1,1)}^b = \int d^2 x \int d\theta_1 - d\theta_2 + 2\tilde{U}^\alpha\tilde{V}_\alpha \tag{89}
\]

where again we refer the reader to the appendix. We find that

\[
S_{(1,1)}^b = -\int d^3 x 2\partial_3 [\chi_+\lambda_+ + \psi_-\eta_+ + \chi_+\gamma^m \partial_m \chi_+ + \psi_-\gamma^m \partial_m \psi_- + b(f + \partial_3 a) - (f - \partial_3 a)^2]. \tag{90}
\]

Unlike the \( \mathcal{N} = (2, 0) \) case we can no longer eliminate both the gaugino boundary
couplings. The best we can do is

$$S_{(1,1)}^{CS} - S_{(1,1)}^b = \int d^3x \left[ S_{(1,1)}^{CS} - S_{(1,1)}^b \right] = \int d^3x 2gb + \eta \eta + \lambda \lambda - 4\epsilon^{\mu\nu\rho} v_\mu \partial_\nu v_\rho$$

$$+ 2\partial_3 (v_n v^n + \chi^m \partial_\mu \chi^\mu + 2\chi^\lambda - \psi^- \gamma^m \partial_\mu \psi^- - (f + \partial_3 a)^2 + \frac{1}{2} bb) \right].$$

On the boundary the two fermions of opposite chirality seems to indicate the (1,1) nature of the theory.

We see that the $\chi^+ \lambda^-$ interaction remains. This is not a necessarily a problem; it simply means that the auxiliary field is needed for off-shell supersymmetry without boundary conditions. If we are interested in on-shell effects we may choose a boundary condition to eliminate this term providing it is compatible with the preserved supersymmetry. Such a boundary condition is be encoded in the (1,1) multiplets. For instance we may choose as a boundary condition for the gauge sector

$$0 = \tilde{U}_\alpha = \chi^+ - (\theta_1 \gamma^m) v_m - \theta_2 \chi^- (b - (f + \partial_3 a)) + \theta_2 \theta_1 (\eta^+ + \gamma^m \partial_\mu \psi^-) \right].$$

This choice of boundary condition suggests that $\chi^+ = 0$. This boundary condition would certainly eliminate any concerns about the $\chi^+ \lambda^-$ interaction; on-shell we could freely integrate out $\lambda$ from the bulk. It also suggests that $\chi^+$ may not actually be a propagating degree of freedom at all.

We also have a non-propagating scalar squared term on the boundary given by $(f + \partial_3 a)^2$. A naive Euler Lagrange variation would suggest setting this term to zero, at least on-shell. However we can see form the boundary multiplet (92) that compatibility with supersymmetry also requires that $b = 0$. Similarly we see that $\psi^-$ obeys a simple 2d fermion equation of motion provided that $\eta^+ = 0$ on the boundary.

8 $\mathcal{N} = 2$ Matter with Boundary

$\mathcal{N} = 2$ matter is describe by chiral and anti chiral superfields with $\mathcal{N} = 1$ expansion

$$Z(\theta_1, \theta_2) = \frac{1}{2} Z(\theta_1) + \frac{1}{2} i \theta_2 D_{1\alpha} Z + \frac{1}{2} \theta_2^2 D_1^2 Z,$$ (93)

$$\bar{Z}(\theta_1, \theta_2) = \frac{1}{2} Z^*(\theta_1) - \frac{1}{2} i \theta_2 D_{1\alpha} Z^* + \frac{1}{2} \theta_2^2 D_1^2 Z^*,$$ (94)
where the factors of half are for later convenience and the components of the $\mathcal{N} = 1$ superfields are summarized by $Z(\theta_1) = (Z, \xi, F)$. With our conventions the kinetic term for gauged matter is given by

$$S_0 = \int d^3x d^4\theta \bar{Z} e^{2V} Z.$$  \hfill (95)

According to our earlier consideration of gauge fixing we evaluate this action using ‘Ivanov’ gauge

$$V(\theta_1, \theta_2) = \theta_2 \Gamma(\theta_1) - \theta_2^2 B(\theta_1),$$  \hfill (96)

with $\Gamma = (0, 0, v_\mu, \lambda)$ and $F = (b, \eta, g)$.

First we consider the $\mathcal{N} = (2, 0)$ case. The construction of section 6, yields the following bulk + boundary action

$$S^M_{(2,0)} = \int d^3x L_{\text{bulk}} + \partial_3 [\xi^* \xi_+ + \frac{1}{2} Z^* b Z]$$  \hfill (97)

where the bulk lagrangian is given by

$$L_{\text{bulk}} = F^* F - \nabla_\mu Z^* \nabla^\mu Z - \xi^* \gamma^\mu \nabla_\mu \xi$$

$$- \frac{1}{2} (Z^* b F + Z^* (\eta + i\lambda) \xi) + c.c. $$

$$- \frac{1}{2} Z^* g Z - \frac{1}{2} \xi^* b \xi.$$  \hfill (98)

We observe that in this case we have a new gauge-matter coupling on the boundary of the form $Z^* b Z$. This will lead to new boundary interactions once the auxiliary field $b$ is eliminated. If we now couple this matter sector to the $\mathcal{N} = (2, 0)$ Chern-Simons term, generalize to include flavor indices and integrate out auxiliary fields, we find

$$S_{(2,0)}^{CSM} = \kappa S_{(2,0)\text{tot}}^{CS} + S^M_{(2,0)}$$

$$= \int L_{\text{kin}} + L_{\text{int}} + L_{\text{pot}} + \partial_3 L_{(2,0)\text{bound}}$$  \hfill (101)
where

\begin{align}
\mathcal{L}_{\text{kin}} &= -4\kappa\epsilon^{\mu\nu\rho}v_\mu\partial_\nu v_\rho - \nabla_\mu Z_A^*\nabla^\mu Z^A - \xi^*_A\gamma^\mu\nabla_\mu\xi^A \\
\mathcal{L}_{\text{int}} &= -\frac{1}{8\kappa} \left( 2\xi^*_A Z^A Z^*_B\xi^B + \xi^*_A\xi^A Z^*_B Z^B \right) \\
\mathcal{L}_{\text{pot}} &= -\frac{1}{64\kappa^2} (Z^*_A Z^A)^3 \\
\mathcal{L}_{(2,0)\text{ bound}} &= 2\kappa [v_\mu v^\mu + \chi_-\gamma^m \partial_m \chi_- + \psi_-\gamma^m \partial_m \psi_- + a\partial^m a] \\
&\quad + \xi^*_A\xi^A + \frac{1}{16\kappa} Z^*_A Z^A Z^*_B Z^B
\end{align}

The most striking new feature is the emergence of a scalar potential on the boundary. One possible interpretation for such a term can be found in the classical literature [50]. Consider a classical membrane whose boundary is attached to the equilibrium displacement by means of zero natural length springs as displayed in figure 1.

Figure 1: A membrane attached via springs at its boundary giving rise to boundary potential

It is clear that such a system requires the inclusion of a boundary potential due to the potential energy stored in the elastic displacement of these springs. Applying this
thinking to our above action suggests that the boundary can be thought of as being attached to some sort of elastic material which displays a non-linear restorative force to displacement. We interpret this as being due to the fivebrane. Further understanding of this term from the fivebrane perspective would be very desirable.

If one now performs an Euler Lagrange variation we find a boundary condition for the scalar

\[ \partial_3 Z^A - \frac{1}{8\kappa} Z^A Z_B Z^B = 0. \] (108)

How should we interpret this?

Let us consider searching for 1/2 supersymmetric bosonic vacuum solutions of the closed membrane theory and forget for a moment about the boundary terms. This is most easily done by looking at the Hamiltonian and employing the Bogomolny trick. We demand that the scalar fields are in a static configuration and only vary in the \( x^3 \) direction and the gauge fields are unexcited. Then the Hamiltonian is given by

\[ H = \partial_3 Z^A \partial_3 Z_A^* + \frac{1}{64\kappa^2} (Z_A^* Z_A)^3 \] (109)

\[ = |\partial_3 Z^A - \frac{1}{8\kappa} Z_B Z_B Z^A|^2 + \frac{1}{16\kappa} \partial_3 (Z_A^* Z_A Z_B^* Z_B). \] (110)

Then the minimum energy configuration satisfies the BPS bound

\[ \partial_3 Z^A - \frac{1}{8\kappa} Z_A^* Z_B Z^B = 0. \] (111)

So we see that our ‘natural’ boundary condition obtained from the generalized theory corresponds exactly to the BPS equation.

We now turn to the \( \mathcal{N} = (1, 1) \) case. Here things are slightly different. In the gauged matter sector we find

\[ S_{(1,1)}^M = \int d^3 x \mathcal{L}_{\text{bulk}} + \partial_3 \left[ \frac{1}{2} \xi^* \gamma^3 \xi + \frac{1}{2} F^* Z + \frac{1}{2} Z^* F + \frac{1}{2} \partial_3 (Z^* Z) \right] \] (112)

where the bulk Lagrangian is unchanged. We observe that the non-propagating scalar \( F \) appears on the boundary. Unlike the similar situations encountered before it seems to be impossible to eliminate all of these terms from the boundary by the addition of an
extra boundary action. This established by a detailed examination of the $\mathcal{N} = (1, 1)$ boundary multiplets described in the appendix. This means that the auxiliary field is required for the action to have supersymmetry without boundary conditions.

The full result for the $\mathcal{N} = (1, 1)$ Chern-Simons Matter theory is

$$S_{(1,1)}^{CSM} = \kappa S_{(1,1)}^{CS tot} + S_{(1,1)}^M$$

$$= \int \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} + \mathcal{L}_F + \partial_3 \mathcal{L}_{(1,1) \text{bound}}$$

where the kinetic and bose-fermi interaction terms are as before. The occurrence of $F$ on the boundary prevents simply integrating out $F$ and so we have

$$\mathcal{L}_F = F^*F - \frac{1}{8\kappa} (F^*(Z^*Z)Z + Z^*(Z^*Z)F).$$

The boundary terms are given by

$$\mathcal{L}_{(1,1) \text{bound}} = 2\kappa v_n v^n + 2\kappa \chi^m \gamma^m \partial_m \chi_+ + 4\kappa \lambda_- \chi_+ + 2\kappa \psi_- \gamma^m \partial_m \psi_-$$

$$-2\kappa (f + \partial_3 a)^2 + \kappa b b$$

$$\frac{1}{2} \xi^* \gamma^3 \xi + \frac{1}{2} F^* Z + \frac{1}{2} Z^* F + \frac{1}{2} \partial_3 (Z^* Z)$$

If we go on-shell, we could perform a naive EL variation in $F$ we come to the conclusion that $Z$ must be zero on the boundary. Alternatively we might choose $F = 0$ as a boundary condition upfront, but after looking at the bulk equation for $F$ we come to the same conclusion that $Z = 0$ as a boundary condition. More sophisticated would be to look at the boundary $\mathcal{N} = (1, 1)$ multiplets and see that $\partial_3 Z + F$ is an appropriate boundary condition. Since this choice eliminates $F$ from the boundary action, we may simply integrate out the $F$ term and generate the bosonic potential for on-shell fields. Also following the discussion in the section 7 (i.e. pure chern-simons) it is natural to choose gauge sector boundary conditions in which $b = f + \partial_3 a = 0$. In this case, the lagrange multiplier which enforces $b = \frac{1}{4\kappa} Z^* Z$ means that the $b = 0$ boundary condition is equivalent to fixing $Z = F = 0$ on the boundary. This is a little disheartening since the restrictions appear so strong. However, we shall see that when we have a superpotential that this result changes.

27


9 ABJM with Boundary

9.1 ABJM review

With a single gauge group factor the maximal extension of Chern-Simons matter theories seems to be \( \mathcal{N} = 3 \) [46]. From an \( \mathcal{N} = 2 \) superspace perspective this theory is built from pairs of chiral matter fields transforming in conjugate representations of the gauge group and a particular superpotential [45]. In component form this can be recast in a way with manifest \( SU(2)_R \) symmetry.

The ABJM model generalizes the \( \mathcal{N} = 3 \) theory by having two gauge group factors and and two conjugate sets of bi-fundamental matter. We summarize this action in \( \mathcal{N} = 2 \) superspace as formulated in [30].

The gauge fields are contained in two \( U(N) \) adjoint superfields (\( V \) and \( \hat{V} \)). We will suppress all gauge indices. The kinetic terms for these gauge fields are Chern-Simons but at opposite levels \( k \) and \(-k\). In superspace the non-abelian Chern-Simons action is a little complicated and is given by

\[
S^{CS}[V] = \kappa \int d^3x \int d^4\theta \int_0^1 dt \text{Tr} V \hat{D}^{\dot{\alpha}} (e^t V \hat{D}_{\alpha} e^{-4V}).
\] (119)

The matter is described by bi-fundamental chiral superfields \( Z^A \) and \( \mathcal{W}_A \) transforming respectively in the \((N, \bar{N})\) and \((\bar{N}, N)\) of the group. The global flavor index takes values \( A = \{1, 2\} \). The components of \( Z^A \) (\( \mathcal{W}_A \)) are a complex scalar, \( Z^A \) (\( \mathcal{W}_A \)), a fermion, \( \xi^A_\alpha (\omega_{\alpha A}) \), and an auxiliary complex scalar \( F^A \) (\( G_A \)). The four complex scalars \( Z^A \) and \( \mathcal{W}_A \) encode the transverse position of the membrane:

\[
Z_1 = X^1 + iX^5 \quad W^1 = X^{3\dagger} + iX^{7\dagger}
\]

\[
Z_2 = X^2 + iX^6 \quad W^1 = X^{4\dagger} + iX^{8\dagger}.
\] (120) (121)

Making this split of the transverse scalars means giving up on having manifest \( SO(8) \) R-symmetry.

The kinetic terms for the matter fields are, in our conventions,

\[
S^{Mat} = \int d^3x \int d^4\theta \text{Tr} \left( \bar{Z}_A e^{2V} Z^A e^{-2V} + \bar{\mathcal{W}}^A e^{2V} \mathcal{W}_A e^{-2V} \right)
\] (122)
The superpotential is given by

\[ S_{\text{pot}} = \int d^3x \int d^2\theta W(Z, W) + \int d^3x \int d^2\bar{\theta} W(\bar{Z}, \bar{W}), \]  

(123)

where

\[ W = \frac{1}{\kappa} \epsilon_{AC} \epsilon_{BD} \text{Tr} \left( Z^A W_B Z^C W_D \right), \quad \bar{W} = \frac{1}{\kappa} \epsilon_{AC} \epsilon_{BD} \text{Tr} \left( \bar{Z}_A \bar{W}^B \bar{Z}_C \bar{W}^D \right). \]  

(124)

This superpotential has a manifest \( SU(2) \times SU(2) \) global symmetry and since we are working in \( \mathcal{N} = 2 \) superspace there is also a \( U(1)_R \). In fact, with the correct normalisation for the superpotential, which depends on the Chern-Simons level, the theory enjoys an enhanced \( SU(4)_R \) symmetry [30]. The ABJM model is therefore \( \mathcal{N} = 6 \) supersymmetric. The geometric reason for this \( \frac{3}{4} \) maximal supersymmetry is that the transverse scalars actually describe a \( \mathbb{Z}_k \) orbifold of \( \mathbb{C}_4 \). For \( k = 1, 2 \) this quotient should preserve all the supersymmetry [13], however the details of this enhancement are subtle.

In summary the full ABJM model is then given by

\[ k S^{CS}[V] - k S^{CS}[\hat{V}] + S^{Mat} + S^{Pot}. \]  

(125)

The full bulk action can be found in [30] and is characterized by a quartic bose-fermi interaction and sextic bosonic potential.

### 9.2 \( U(1) \times U(1) \) ABJM with boundary

Given the complexity of the non-abelian Chern-Simons term it is natural to start with the most basic \( U(1) \times U(1) \) theory. In this case the superpotential obviously vanishes. Also, because the fields commute, it turns out that all of the matter interactions disappear once auxiliary fields are integrated out. Without boundary the theory is simple and free;

\[ S_{\text{bulk}}^{U(1)\times U(1)} = \int d^3x 4 \kappa \epsilon^{\mu \nu \rho} \left( v_\mu \partial_\nu v_\rho - \hat{v}_\mu \partial_\nu \hat{v}_\rho \right) \]  

(126)

\[ + F^A F^*_A - D^\mu Z^*_A D_\mu Z^A - \xi^*_A \gamma^\mu D_\mu \xi^A \]  

(127)

\[ + G_A G^*_A - D^\mu W^*_A D_\mu W_A - \omega^*_A \gamma^\mu D_\mu \omega_A, \]  

(128)
where the covariant derivative acts as $D_\mu Z = \partial_\mu Z + iv_\mu Z - iZv_\mu$ and with opposite charges on $W$. In the above expression we have eliminate all auxiliary fields except $F$ and $G$ for reasons that will become clear shortly.

With a boundary we can essentially read off the additional terms me must add to restore half the supersymmetry from section 7.

In the $\mathcal{N} = (2, 0)$ procedure we find we must include the following boundary terms:

$$
\mathcal{L}_{U(1) \times U(1)}^{(2,0)\text{bound}} = 2k \left( v_n v^n + \chi_- \gamma^m \partial_m \chi_- + \psi_- \gamma^m \partial_m \psi_- + \partial_m a \partial^m a \right) 
$$

$$
-2k \left( \dot{v}_n \dot{v}^n + \dot{\chi}_- \gamma^m \partial_m \dot{\chi}_- + \dot{\psi}_- \gamma^m \partial_m \dot{\psi}_- + \partial_m \dot{a} \partial^m \dot{a} \right) 
$$

$$
+ \xi_A^* \gamma^A \xi_- + \omega_A^* \omega^{-A} 
$$

In this expression, we have used the the Lagrange multiplier equation for $g$ and $\hat{g}$ to give values to the auxiliary fields $b$ and $\hat{b}$. These provide canceling contributions in the abelian case. Since we have chiral $\mathcal{N} = (2, 0)$ supersymmetry, it comes as no surprise that we have a chiral action with propagating fermions of the same chirality.

In the $\mathcal{N} = (1, 1)$ case

$$
\mathcal{L}_{U(1) \times U(1)}^{(1,1)\text{bound}} = 2k \left( v_n v^n + 2\chi_+ \lambda_- + \chi_+ \gamma^m \partial_m \chi_+ + \psi_- \gamma^m \partial_m \psi_- - (f + \partial_3 a)^2 + \frac{1}{2} bb \right) 
$$

$$
-2k \left( \dot{v}_n \dot{v}^n + \dot{\chi}_+ \dot{\lambda}_- + \dot{\chi}_+ \gamma^m \partial_m \dot{\chi}_+ + \dot{\psi}_- \gamma^m \partial_m \dot{\psi}_- - (\dot{f} + \partial_3 \dot{a})^2 + \frac{1}{2} \dot{bb} \right) 
$$

$$
+ \frac{1}{2} \xi_A^* \gamma^A \xi_- + \frac{1}{2} F_A^* Z_A + \frac{1}{2} Z_A^* F^A + \frac{1}{2} \partial_3 (Z_A^* Z^A) 
$$

$$
+ \frac{1}{2} \omega_A^* \gamma^2 \omega_A + \frac{1}{2} G_A^* W_A + \frac{1}{2} W_A^* G_A + \frac{1}{2} \partial_3 (W_A^* W^A) 
$$

(132)

Here we see that $G_A$ and $F_A^*$ occur as boundary couplings.

We have preserved $(1,1)$ supersymmetry in the parity invariant ABJM model. This strongly suggests that boundary theory should have equal number of left and right movers. It is not chirality invariant because these are different fields.

To gain an immediate physical understanding we go on-shell. We pick boundary conditions that are consistent with the parity invariance of the ABJM model and the supersymmetry. The boundary $(1,1)$ superfields detailed in the appendix readily tell
us how to choose supersymmetric boundary conditions\textsuperscript{11}.

For the vector multiplet $V$ we choose the following boundary condition:

$$0 = V_\alpha = \psi_\alpha + \theta_1 (f + \partial_3 a) - \theta_2 \gamma_1 v_\alpha + \theta_2 \theta_1 (\lambda - \gamma_1 \partial_\alpha \chi) .$$

(133)

For the other vector multiplet $\hat{V}$ we choose:

$$0 = \hat{U}_\alpha = \hat{\chi}_\alpha - \theta_1 \gamma_1 \hat{v}_\alpha + \theta_2 \hat{v}_\alpha = (\hat{f} + \partial_3 \hat{a}) + \theta_2 \theta_1 (\hat{\eta} + \gamma_1 \partial_+ \hat{\psi}) .$$

(134)

Notice that we have $v_\alpha = 0$ and $\hat{v}_\alpha = 0$, which is compatible with parity. These conditions are exactly what we have seen is required to produce the combination of a chiral and anti-chiral FJ action in the pure gauge (no matter) theory.

The lowest component of these two boundary superfields show that $\psi_\alpha$ and $\hat{\chi}_\alpha$ are constrained to zero. We may also set the gauginos appearing in the boundary conditions to zero, i.e. $\lambda = \hat{\eta} = 0$.

For the auxiliary scalars demanding $f + \partial_3 a = \hat{b} - (\hat{f} + \partial_3 \hat{a}) = 0$ can only be compatible with parity if both $\hat{b}$ and $b$ are also set to zero on the boundary.

As we saw in the earlier example in section 8, the appropriate boundary conditions on the matter seem to be $F + \partial_3 Z = G + \partial_3 W = 0$. In the Abelian scenario this, as before, forces $Z = W = 0$ on the boundary. This will not be true in the non-abelian case because there is a superpotential.

If we plug in the trivial i.e. algebraic and non-derivative boundary conditions in to the (1,1) boundary terms (132) we are simply left with

$$\mathcal{L}^{(1,1) \text{bound}}_{U(1) \times U(1)} = 2k \chi_+ \gamma^m \partial_m \chi_+ - 2k \hat{\psi}_- \gamma^m \partial_m \hat{\psi}_-$$

(135)

This boundary theory quite clearly has two propagating fermions of opposite chirality and is essentially non-chiral.

9.3 Towards $U(N) \times U(N)$ ABJM with boundary

As we have seen the non-abelian kinetic term is a very complicated affair. In principle we could, by following the procedure of section 6, construct the boundary action to

\textsuperscript{11}we drop the tilde-hat notation of the appendix and understand that hatted quantities correspond to the hatted vector multiplet
preserve half the supersymmetry. (For a treatment of $\mathcal{N} = 1$ non-abelian super Chern-Simons using similar techniques to us see [52]). However, if one wanted to look at this in component form it would take significant effort. There are also additional complications concerning field redefinitions and gauge fixing. Furthermore, we would have to establish the correct additional terms required to remove the gaugino boundary interactions.

In this paper, we don’t intend to complete all of the above. Instead, we will make a couple of sensible assumptions that will allow us to learn about the matter sector. We assume that if, and only if, we were able to eliminate an auxiliary field boundary interaction in the abelian case through the addition of separate boundary actions we will be able to do so for the non-abelian case. The only difference will be the obvious inclusion of a trace.

Although we will only have partial knowledge of the gauge sector boundary terms (e.g. we do not know any commutator terms) we will have full knowledge of the matter sector. This will be enough to inform us about a boundary potential for the bosonic matter fields.

9.4 $\mathcal{N} = (2, 0)$ supersymmetry

We first consider the case where we preserve manifest $\mathcal{N} = (2, 0)$ supersymmetry.

In what follows we shall turn our attention to the just the boundary contributions for the bosonic matter fields. We find the following boundary terms

$$L_{\text{bound}}^{(2, 0)} = -\kappa b^a b^a + \frac{1}{2} b^a \text{Tr} \left( T^a (Z Z^\dagger - W W^\dagger) \right) + \frac{1}{2} \hat{b}^a \hat{b}^a - \frac{1}{2} \hat{b}^a \text{Tr} \left( T^a (Z Z^\dagger - W W^\dagger) \right) + \ldots (136)$$

where the dots indicate boundary contributions from fermions in the matter multiplet and terms generated by the gauge multiplet which don’t interact with the matter. The abelian contribution to these omitted terms can be read off from the constructions in the preceding section. Because we are preserving $\mathcal{N} = (2, 0)$ supersymmetry there is no boundary contribution from the superpotential.

In the bulk, we find that $g$ and $\hat{g}$ are Lagrange multipliers enforcing $b$ and $\hat{b}$ to
take a particular value given by
\[ b^a = \frac{1}{4\kappa} \text{Tr}(T^a(ZZ^\dagger - WW^\dagger)) \], \quad \hat{b}^a = \frac{1}{4\kappa} \text{Tr}(T^a(Z^\dagger Z - WW^\dagger)) \, . \tag{137} \]

We may make the above replacement into the boundary terms and find
\[ \mathcal{L}^{(2,0)}_{\text{bound}} = -\frac{1}{16\kappa} \text{Tr}[(Z^\dagger Z - WW^\dagger)^2 - (ZZ^\dagger - W^\dagger W)^2] + \ldots . \tag{138} \]

A key observation is that we now have a quartic boundary scalar potential. We interpret this as being the effect of a five-brane. The consequence of this boundary potential can be seen in the field equations and Euler-Lagrange boundary conditions. The matter field \( Z \) obeys a natural boundary condition of the form
\[ \partial_3 Z^A + \frac{1}{8\kappa} [Z^A(Z^\dagger Z - WW^\dagger) - (ZZ^\dagger - W^\dagger W)Z^A] = 0 \, . \tag{139} \]

It is helpful to introduce a three-bracket given by
\[ [A, B; C] = AC^\dagger B - BC^\dagger A \tag{140} \]
in order to make contact with the Bagger-Lambert formulation of the ABJM model [14]. One can re-write the boundary condition using this bracket as
\[ \partial_3 Z^A - \frac{1}{8\kappa} ([Z^B, Z^A; Z^\dagger_B] + [Z^A, W^\dagger_B; W_B]) = 0 \, . \tag{141} \]

This equation (together with the similar contribution for \( W \)) can be seen in the ‘D-term’ BPS equation found in [42] by looking at the Hamiltonian of the ABJM model. However, the full corresponding BPS equations also include a constraint \( \epsilon_{AC} \epsilon^{BD} W_B Z^C W_D = 0 \) which we have not observed. When only half the scalars, e.g. the \( Z^A \), are excited this constraint is solved and the remaining BPS equation simplifies to
\[ \partial_3 Z^A - \frac{1}{8\kappa} [Z^B, Z^A; Z^\dagger_B] = 0 \, . \tag{142} \]

This equation should yield fuzzy funnel solutions describing the membrane ending on the five-brane. However, the symmetry of this equation is only \( SU(2) \times U(1) \) whereas
the Basu-Harvey equation describing fuzzy three-spheres has $SO(4)$ symmetry. Solutions of (142) have been found and are thought to represent fuzzy $S^3/\mathbb{Z}_k$ [42]. In [55] fluctuations of the fuzzy funnel were analyzed in a large $k$ limit where a perturbation theory can be used. This indicated an underlying fuzzy $S^2$ structure rather than the perhaps expected fuzzy $S^3$.

9.5 $\mathcal{N} = (1, 1)$ supersymmetry

We turn to $\mathcal{N} = (1, 1)$ case. In this case we find the following contributions to the bosonic boundary term

$$
\mathcal{L}_{\text{bound}}^{(1,1)} = \frac{1}{2} Z^\dagger (F + \partial_3 Z) + \frac{1}{2} W (W^\dagger + \partial_3 G^\dagger + \text{h.c.})
$$

(143)

$$
- \frac{1}{2} b (Z Z^\dagger - W W^\dagger) + \frac{1}{2} \hat{b} (Z^\dagger Z - WW^\dagger) - \kappa b b + \kappa \hat{b} \hat{b}
$$

(144)

$$
- \frac{1}{8 \kappa} \epsilon_{AC} \epsilon^{BD} Z^A W^A Z^C W^D + \text{h.c.} + \ldots ,
$$

(145)

where again the dots indicate the fermions and the decoupled gauge sector.

As with the abelian scenario the presence of auxiliary fields in this action makes it hard to understand the on-shell nature of the theory. If we choose the same boundary conditions as the abelian case i.e.

$$
0 = b = \hat{b} = F^A + \partial_3 Z^A = G_A + \partial_3 W_A
$$

(146)

then the boundary action reduces to

$$
\mathcal{L}_{\text{bound}}^{(1,1)} = - \frac{1}{8 \kappa} \epsilon_{AC} \epsilon^{BD} Z^A W^A Z^C W^D + \text{h.c.} + \ldots ,
$$

(147)

Unlike the abelian case however, the $b$ boundary condition does not require that $Z = W = 0$. It does constrain the matter fields to obey

$$
b^a = \frac{1}{4 \kappa} \text{Tr} (T^a (Z Z^\dagger - W W^\dagger)) = 0, \quad \hat{b}^a = \frac{1}{4 \kappa} \text{Tr} (T^a (Z^\dagger Z - WW^\dagger)) = 0
$$

(148)

on the boundary. We may also make use of the bulk equation for $F$ and $G$ together with the $b$ boundary condition to write the matter boundary condition as

$$
\partial_3 Z^A - \frac{1}{4 \kappa} \epsilon^{AC} \epsilon_{BD} W^{\dagger B} Z^C W^{\dagger D} = 0, \quad \partial_3 W_A + \frac{1}{4 \kappa} \epsilon_{AC} \epsilon^{BD} Z^B W^{\dagger C} Z^D = 0
$$

(149)

34
These equations can be recognised as the ‘F-term’ BPS equations found in [42] by the Bogomolny completion of the Hamiltonian. The constraints (148) also imply the constraints found by the Bogomolny trick, although here they are a little stronger.

Note that after invoking these boundary conditions there remains a quartic boundary potential, which is not set to zero, and is given by (147). Upon performing an EL variation of the bulk+boundary action the total derivative picked up from varying the scalar kinetic terms combines with the variation of the boundary potential to reproduce the boundary conditions.

10 Discussion

This paper is the first step towards the study of open interacting membranes. The ultimate aim is to gain insight into the fivebrane as a theory of open membranes though as yet we are still far from that goal. In spite of this, the reproduction of the BPS equations as supersymmetric boundary equations encourages us that we are on the right path to understanding more about the interacting self-dual string.

(As an aside, the gauge sector of the theory is interesting in its own right as its boundary theory produces an interesting WZW model. The role of parity and the resulting chirality in the WZW model is particularly interesting).

There is still a great deal to understand. In particular one would like to see the role of the quartic potential on the boundary from the fivebrane perspective. A more direct question is to understand the boundary equation arising from the (1,1) supersymmetry. As stated this has been observed before as a BPS equation but so far the solutions are not known and the brane interpretation is open. Solving this equation and interpreting the solutions would hopefully provide some insight. The relation between the two choices of supersymmetry are also interesting and perhaps there is a map between solutions.

The most important limitation to this work was that using the superspace method described above meant that it was easiest to deal with only a manifest $N = 2$ supersymmetry. Obviously extending these results to higher supersymmetry would be of
great interest though somewhat technically demanding. Essentially, one would like a complete classification of the boundaries preserving different amounts of supersymmetry, beginning with membranes that also preserve differing amounts of supersymmetry. We also did not carry out a rigorous derivation using the full non-abelian superspace action; although we expect no surprises it would be good to have this further developed.

There is a very interesting complimentary approach to this work that we have as yet not explored. To study open interacting membranes, one could use an ABJM style brane set up in IIB with the addition of a suitable boundary and thus the inclusion of an additional NS5 brane. One could then be able to utilise the work of Gaiotto and Witten [53] in this ABJM configuration to learn about the open membrane. It would be interesting to see from this perspective whether the different choices of supersymmetry preserved by the boundary are related to symmetries in the branes set up. (The changing of the type of supersymmetry seems reminiscent of T-duality).

11 Acknowledgements

We wish to acknowledge helpful discussions with James Bedford, Oren Bergman, Sergey Cherkis, Neil Lambert, Andrew Low, George Papadopoulos, Constantinos Papageorgakis and Sanjaye Ramgoolam. We also wish to thank Will Black for help with the tex file and illustrations. DCT acknowledges the support of an STFC doctoral grant. DSB acknowledges the support of an STFC rolling grant and thanks DAMTP for hospitality during completion of this work.

12 Appendix 1: $\mathcal{N} = 1$ Supersymmetry Conventions

We broadly follow Superspace. Index contraction and manipulation is given by

$$\theta^\alpha = C^{\alpha\beta} \theta_\beta, \quad \theta_\beta = \theta^\alpha C_{\alpha\beta}, \quad \theta_\alpha \theta_\beta = -C_{\alpha\beta} \theta^2 = -\frac{1}{2} C^\gamma_{\alpha\beta} \theta_\gamma \theta_\gamma, \quad (150)$$

where

$$C_{\alpha\beta} = -C_{\beta\alpha} = -C^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C^\gamma_{\alpha\beta} C^\gamma_{\delta\epsilon} = \delta^\gamma_{\alpha} \delta^\delta_{\beta} \delta^\epsilon_{\gamma}. \quad (151)$$
When spinor indices are suppressed index contraction is always top right to bottom left \((\nwarrow)\). With this convention it is unnecessary to show conjugation with an overbar. Gamma matrices obey the Clifford algebra

\[
\{\gamma^\mu, \gamma^\nu\}_{\alpha\beta} = 2g^{\mu\nu}C_{\alpha\beta}
\]

where

\[
\gamma^\mu_{\alpha\beta} \equiv \gamma^\nu\gamma C_{\gamma\beta} = \gamma^\mu_{\alpha}, \quad \gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu\rho}\gamma_\rho.
\]

Differentiation and integration is summarized by

\[
\partial_\alpha \theta_\beta = C_{\alpha\beta}, \quad \int d^2 \theta \theta^2 = -1.
\]

The SUSY charge and covariant derivative are

\[
Q_\alpha = \partial_\alpha - (\gamma^\mu \theta)_\alpha \partial_\mu, \quad D_\alpha = \partial_\alpha + (\gamma^\mu \theta)_\alpha \partial_\mu,
\]

and the algebra is

\[
\{Q_\alpha, Q_\beta\} = -\{D_\alpha D_\beta\} = 2\gamma^\mu_{\alpha\beta} \partial_\mu.
\]

The covariant derivatives satisfy the following identities

\[
D_\alpha D_\beta = -\gamma^\mu_{\alpha\beta} \partial_\mu - C_{\alpha\beta} D^2, \quad D^2 D_\alpha = -D_\alpha D^2 = (\gamma^\mu D)_\alpha \partial_\mu, \quad (D^2)^2 = \Box, \quad D^\alpha D_\beta D_\alpha = 0.
\]

A scalar superfield is given by

\[
\Phi = a + \theta \psi - \theta^2 f = (a, \psi, f),
\]

and a spinor superfield by

\[
\Gamma_\alpha = \chi_\alpha - \theta_\alpha M + (\gamma^\mu \theta)_\alpha v_\mu - \theta^2 [\lambda_\alpha + (\gamma^\mu \partial_\mu \chi)_\alpha] = (\chi, M, v_\mu, \lambda).
\]

We use early Greek letters to denote spinor indices, late Greek for 3-dimensional space-time indices (with \(x_\mu = (x_0, x_1, x_3)\)) and Latin indices for two-dimensional space-time (\(x_m = (x_0, x_1)\)). We assume Lorentzian \((+++)\) signature and \(\epsilon^{013} = +1\).
### 13 Appendix 2: $\mathcal{N} = 2$ Supersymmetry Conventions

In three-dimensions $\mathcal{N} = 2$ superspace is realized by taking the Grassman coordinates to be complex. For our purposes it is convenient to express these in terms of an $\mathcal{N} = 1$ decomposition by writing

$$\theta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2), \quad \bar{\theta} = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2)$$

(164)

so that the $\mathcal{N} = 2$ superspace covariant derivatives

$$D_\alpha = \partial_\alpha + (\gamma^\mu \bar{\theta})_\alpha \partial_\mu, \quad \bar{D}_\alpha = \bar{\partial}_\alpha + (\gamma^\mu \theta)_\alpha \partial_\mu.$$  

(165)

are decomposed as

$$D_\alpha = \frac{1}{\sqrt{2}}(D_1 - iD_2), \quad \bar{D}_\alpha = \frac{1}{\sqrt{2}}(D_1 + iD_2)$$

(166)

with

$$D_i = \frac{\partial}{\partial \theta_i} + (\gamma^\mu \theta_i)_\alpha \partial_\mu \quad i = \{1, 2\},$$

(167)

satisfying the algebra

$$\{D_i \alpha, D_j \beta\} = -2\delta_{ij} \gamma_\alpha^\mu \partial_\mu.$$  

(168)

The $\mathcal{N} = 2$ chiral superfield obeys

$$\bar{D}\Phi = 0 \Leftrightarrow D_2 \Phi = iD_1 \Phi$$

(169)

and can be expressed as

$$\Phi = X(\theta_1) + i\theta_2 D_1 X + D_1^2 X \theta_2^2$$

(170)

where $X(\theta_1) = a + \theta_1 \psi - f\theta_1^2$ is a complex $\mathcal{N} = 1$ superfield. The $\mathcal{N} = 2$ vector field is a real superfield obeying

$$V = A(\theta_1) + \theta_2 \Gamma - \theta_2^2 (B - D_1^2 A)$$

(171)
with \( A \) and \( B \) real \( \mathcal{N} = 1 \) scalar superfields and \( \Gamma_\alpha \) a real \( \mathcal{N} = 1 \) spinor superfield. The \( \mathcal{N} = 2 \) gauge transformations are

\[
\delta_{\text{gauge}} V = i[\Phi - \bar{\Phi}] \Rightarrow \begin{cases} 
\delta_{\text{gauge}} A = i[X - \bar{X}] \\
\delta_{\text{gauge}} \Gamma_\alpha = -D_{1\alpha}(X + \bar{X}) \\
\delta_{\text{gauge}} B = 0
\end{cases}.
\]

(172)

The arbitrary shift in \( A \) is usually used to gauge fix this field to zero.

14 Appendix 3: Multiplet Decomposition

14.1 \( \mathcal{N} = 1 \to \mathcal{N} = (1,0) \)

There is a simple procedure to obtain half supersymmetric multiplets from a fully supersymmetric one. We introduce projectors

\[
(P_\pm) = \frac{1}{2}(C_{\alpha\beta} \pm \gamma^3_{\alpha\beta})
\]

(173)

and write the scalar multiplet as

\[
A = a + \theta \psi - \theta^2 f
\]

(174)

\[
= \exp(-\theta_+ \theta_- \partial_3) \left( \hat{A}(\theta_+) + \theta^\alpha \hat{A}_\alpha(\theta_+) \right)
\]

(175)

\[
= \exp(+\theta_+ \theta_- \partial_3) \left( \bar{A}(\theta_-) + \theta^\alpha \bar{A}_\alpha(\theta_-) \right)
\]

(176)

where hatted objects are now 1+1 dimension superfields whose supersymmetry is generated by \( \epsilon_+ Q'_+ \equiv \epsilon_+ [\partial_3 - \gamma^m \theta_+ \partial_m] \) and tilded objects are 1+1 dimension superfields whose supersymmetry is generated by \( \epsilon_- Q'_- \equiv \epsilon_- [\partial_3 - \gamma^m \theta_- \partial_m] \). A useful relation is \( \frac{\partial}{\partial \theta_+^\beta} \theta_+^\beta = P_{-\alpha}^\beta \).

One can take the spinor multiplet project with \( P_\pm \) and perform a similar decomposition

\[
\Gamma_\pm^\alpha = \exp(-\theta_+ \theta_- \partial_3) \left( \hat{\Gamma}_\pm^\alpha(\theta_+) + \theta^\beta \hat{\Gamma}_{\beta\alpha}^\pm (\theta_+) \right)
\]

(177)

\[
= \exp(+\theta_+ \theta_- \partial_3) \left( \bar{\Gamma}_\pm^\alpha(\theta_-) + \theta^\beta \bar{\Gamma}_{\beta\alpha}^\pm (\theta_-) \right).
\]

(178)

In principle the \( \hat{\Gamma}_{\beta\alpha}^\pm \) and \( \bar{\Gamma}_{\beta\alpha}^\pm \) are Lorentz reducible multiplets so we must take symmetric and anti-symmetric parts to discover the correct irreducible superfields. However
one find that these fields are either scalar or vector like and not a combination of both. To be explicit we can write
\[
\theta^\beta \hat{\Gamma}_{\beta\alpha}^+ = -\theta^\beta \gamma^m \hat{\Sigma}^+_m, \quad \theta^\beta \hat{\Gamma}_{\beta\alpha}^- = -\theta_{-\alpha} \hat{\Sigma}^-,
\]
\[
\theta^\beta \tilde{\Gamma}_{\beta\alpha}^+ = -\theta_{+\alpha} \tilde{\Sigma}^+, \quad \theta^\beta \tilde{\Gamma}_{\beta\alpha}^- = -\theta^\beta \gamma^m \tilde{\Sigma}^-_m.
\]

In components we have
\[
\hat{A}_\alpha = a + \theta_+ \psi_+
\]
\[
\hat{A}_+ = \psi_+ + \theta_+ (-f + \partial_3 a)
\]
\[
\hat{\Gamma}_\alpha^+ = \chi_+ + \theta_+ (v_3 - M)
\]
\[
\hat{\Sigma}_m^+ = v_m + \theta_+ \left( \frac{1}{2} \gamma_m \lambda_+ + \partial_m \chi_+ \right)
\]
\[
\hat{\Gamma}_\alpha^- = \chi_- - \theta_+ \gamma^m v_m
\]
\[
\hat{\Sigma}^- = M + v_3 - \theta_+ (\lambda_- - 2 \partial_3 \chi_- + \gamma^m \partial_m \chi_+)
\]
\[
\tilde{A}_\alpha = a + \theta_- \psi_+
\]
\[
\tilde{A}_+ = \psi_- - \theta_- (f + \partial_3 a)
\]
\[
\tilde{\Gamma}_\alpha^+ = \chi_- - \theta_- \gamma^m v_m
\]
\[
\tilde{\Sigma}_m^+ = M - v_3 - \theta_- (\lambda_- + 2 \partial_3 \chi_- + \gamma^m \partial_m \chi_-)
\]
\[
\tilde{\Gamma}_\alpha^- = \chi_- - \theta_- (M + v_3)
\]
\[
\tilde{\Sigma}^-_m = v_m + \theta_- \left( \frac{1}{2} \gamma_m \lambda_- + \partial_m \chi_+ \right)
\]

To get to these forms of the superfields one actually must do a little work and use identities like
\[
(\theta_- \gamma^m)_{\alpha} \theta_+ \gamma_m \gamma^n \partial^n \chi_- = 2(\theta_- \gamma^m)_{\alpha} \theta_+ \partial_m \chi_-.
\]

Also when calculating the correct boundary conditions one should bear in mind identities like, for example,
\[
(\theta_- \gamma^m)_{\alpha} \hat{\Sigma}^+_m = (\theta_- \gamma^1)_{\alpha} \left( \hat{\Sigma}^+_0 + \hat{\Sigma}^+_1 \right) = (\theta_- \gamma^1)_{\alpha} \tilde{\Sigma}^+_m = 0.
\]
\[ \mathcal{N} = 2 \rightarrow \mathcal{N} = (2,0) \text{ or } \mathcal{N} = (1,1) \]

Starting with the most general superfield

\[ V = A(\theta_1) + \theta_2 \Gamma(\theta_1) - \theta_2^2 C(\theta_1) \quad (195) \]

we decompose this into 2d \( \mathcal{N} = (1,1) \) or \( \mathcal{N} = (2,0) \) multiplets given by

\[ V = \exp(-\theta_2 + \theta_2 - \partial_3) \exp(+\theta_1 + \theta_1 - \partial_3) \left( \hat{V}(\theta_1, \theta_2) + \theta_1^a \hat{\Gamma}_a + \theta_2^m \hat{C}^m + \partial_3 \hat{A}_a \right) \quad (196) \]

We can, in turn, express the components of these expansions in terms of the \( \mathcal{N} = 1 \) decompositions defined previously. We have for \( \mathcal{N} = (1, 1) \)

\[ \hat{V} = \hat{A}(\theta_1) + \theta_2^a \hat{\Gamma}_a \quad (198) \]
\[ \hat{V}_a = \hat{A}_a - (\gamma^m \theta_{2+}) \hat{\Sigma}^m \quad (199) \]
\[ \hat{U}_a = \hat{\Gamma}_a^+ + \theta_2^\alpha (-\hat{C}_3 + \partial_3 \hat{A}_a) \quad (200) \]
\[ \hat{U} = -\hat{\Sigma}^+ + \theta_2^\alpha (\hat{C}_\beta - \partial_3 \hat{A}_\beta) \quad (201) \]

and for \( \mathcal{N} = (2,0) \):

\[ \hat{V} = \hat{A}(\theta_1) + \theta_2^a \hat{\Gamma}_a \quad (202) \]
\[ \hat{V}_a = \hat{A}_a - \theta_2^\alpha \hat{\Sigma}^\alpha \quad (203) \]
\[ \hat{U}_a = \hat{\Gamma}_a^+ + \theta_2^\alpha (-\hat{C}_3 + \partial_3 \hat{A}_a) \quad (204) \]
\[ \hat{V}_m = \Sigma_m^+ \frac{1}{2} (\theta_2 + \gamma_m)^\alpha (\hat{C}_\alpha - \partial_3 \hat{A}_\alpha) \quad (205) \]

For the case of the vector field

\[ V(\theta_1, \theta_2) = A(\theta_1) + \theta_2 \Gamma(\theta_1) - \theta_2^2 (B(\theta_1) - D_1^2 A) \quad (206) \]

with components

\[ A = (a, \psi, f), \quad B = (b, \eta, g), \quad \Gamma = (\chi, M, v, \lambda). \quad (207) \]
the half supersymmetric multiplets are

\[ \hat{V} = a + \theta_1^+\psi_+ + \theta_2^+\chi_- + \theta_2^+\gamma^m\theta_1^+v_m \] (208)
\[ \hat{V}_\alpha = \psi_+^\alpha + \theta_1^+\alpha(-f + \partial_3a) - \theta_2^+\alpha(M + v_3) + \theta_2^+a\theta_1^+\alpha_1^+(\lambda_- - 2\partial_3\chi_- + \gamma^m\partial_m\chi_+)_\beta \] (209)
\[ \hat{U}_\alpha = \chi_+^\alpha + \theta_1^+\alpha(v_3 - M) + \theta_2^+\alpha(f + \partial_3a - b) - \theta_2^+\gamma_m\theta_1^+\gamma^m_\alpha(\eta_- + \gamma_m\partial_m\psi_+ - 2\partial_3\eta_-) \] (210)
\[ \hat{V}_m = v_m + \theta_1^+(\frac{1}{2}\gamma_m\lambda_+ + \partial_m\chi_-) - \theta_2^+(\frac{1}{2}\gamma_m\eta_+ + \partial_m\psi_-) \]
\[ -\theta_2^+\gamma_m\theta_1^+(\frac{1}{2}(-g + \partial_3b + \partial_m\partial_ma)) \] (211)

\[ \tilde{\hat{V}} = a + \theta_1^-\psi_+ + \theta_2^+\chi_- - \theta_2^-\theta_1^-(M + v_3) \] (212)
\[ \tilde{\hat{V}}_\alpha = \psi_-^\alpha - \theta_1^-\alpha(f + \partial_3a) + (\theta_2^+\gamma^m_\alpha)v_m + \theta_2^+\theta_1^-\alpha_1^+(\lambda_- + \gamma^m\partial_m\chi_+) \] (213)
\[ \tilde{\hat{U}}_\alpha = \chi_-^\alpha - (\theta_1^-\gamma^m_\alpha)v_m - \theta_2^+\alpha(b - (f + \partial_3a)) + \theta_2^+\theta_1^-\alpha_1^+(\eta_+ + \gamma^m\partial_m\psi_-) \] (214)
\[ \tilde{\hat{U}} = v_3 - M + \theta_1^-(-\lambda_+ + 2\partial_3\chi_+ + \gamma^m\partial_m\chi_-) + \theta_2^+\eta_+ + \gamma^m\partial_m\psi_+ - 2\partial_3\eta_- \]
\[ -\theta_2^-\theta_1^-(g - \partial_3\partial_m\partial_ma - 2\partial_3\partial_3a + 2\partial_3f - \partial_3b) \] (215)

References

[1] D. S. Berman, “M-theory branes and their interactions,” Phys. Rept. 456 (2008) 89 [arXiv:0710.1707 [hep-th]].

[2] J. Bagger and N. Lambert, “Modeling multiple M2’s,” Phys. Rev. D 75 (2007) 045020 [arXiv:hep-th/0611108].

[3] J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” arXiv:0711.0955 [hep-th].

[4] J. Bagger and N. Lambert, “Comments On Multiple M2-branes,” JHEP 0802 (2008) 105 [arXiv:0712.3738 [hep-th]].

[5] A. Gustavsson, “Algebraic structures on parallel M2-branes,” arXiv:0709.1260 [hep-th].
[6] G. Papadopoulos, “M2-branes, 3-Lie Algebras and Plucker relations,” JHEP 0805 (2008) 054 [arXiv:0804.2662 [hep-th]]. J. P. Gauntlett and J. B. Gutowski, “Constraining Maximally Supersymmetric Membrane Actions,” arXiv:0804.3078 [hep-th].

[7] M. Van Raamsdonk, “Comments on the Bagger-Lambert theory and multiple M2-branes,” JHEP 0805 (2008) 105 [arXiv:0803.3803 [hep-th]].

[8] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, “M2-branes on M-folds,” JHEP 0805 (2008) 038 [arXiv:0804.1256 [hep-th]].

[9] N. Lambert and D. Tong, “Membranes on an Orbifold,” Phys. Rev. Lett. 101 (2008) 041602 [arXiv:0804.1114 [hep-th]].

[10] S. Benvenuti, D. Rodriguez-Gomez, E. Tonni and H. Verlinde, “N=8 superconformal gauge theories and M2 branes,” arXiv:0805.1087 [hep-th].

[11] J. Gomis, G. Milanesi and J. G. Russo, “Bagger-Lambert Theory for General Lie Algebras,” JHEP 0806 (2008) 075 [arXiv:0805.1012 [hep-th]].

[12] M. A. Bandres, A. E. Lipstein and J. H. Schwarz, “Ghost-Free Superconformal Action for Multiple M2-Branes,” JHEP 0807 (2008) 117 [arXiv:0806.0054 [hep-th]].

[13] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP 0810 (2008) 091 [arXiv:0806.1218 [hep-th]].

[14] J. Bagger and N. Lambert, “Three-Algebras and N=6 Chern-Simons Gauge Theories,” Phys. Rev. D 79 (2009) 025002 [arXiv:0807.0163 [hep-th]].

[15] P. K. Townsend, “D-branes from M-branes,” Phys. Lett. B 373 (1996) 68 [arXiv:hep-th/9512062].
[16] A. Strominger, “Open p-branes,” Phys. Lett. B 383, 44 (1996) [arXiv:hep-th/9512059].

[17] C. S. Chu and E. Sezgin, “M-fivebrane from the open supermembrane,” JHEP 9712, 001 (1997) [arXiv:hep-th/9710223].

[18] P. M. Ho and Y. Matsuo, “M5 from M2,” JHEP 0806 (2008) 105 [arXiv:0804.3629 [hep-th]].

[19] A. Basu and J. A. Harvey, “The M2-M5 brane system and a generalized Nahm’s equation,” Nucl. Phys. B 713, 136 (2005) [arXiv:hep-th/0412310].

[20] D. S. Berman and N. B. Copland, “Five-brane calibrations and fuzzy funnels,” Nucl. Phys. B 723, 117 (2005) [arXiv:hep-th/0504044]. D. S. Berman and N. B. Copland, “A note on the M2-M5 brane system and fuzzy spheres,” Phys. Lett. B 639 (2006) 553 [arXiv:hep-th/0605086].

[21] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, JHEP 0809 (2008) 113 [arXiv:0807.1074 [hep-th]].

[22] P. S. Howe, N. D. Lambert and P. C. West, “The self-dual string soliton,” Nucl. Phys. B 515, 203 (1998) [arXiv:hep-th/9709014].

[23] D. V. Belyaev and P. van Nieuwenhuizen, “Rigid supersymmetry with boundaries,” JHEP 0804, 008 (2008) [arXiv:0801.2377 [hep-th]].

[24] E. Witten, “Nonabelian bosonization in two dimensions,” Commun. Math. Phys. 92, 455 (1984).

[25] E. Witten, “Quantum field theory and the Jones polynomial,” Commun. Math. Phys. 121, 351 (1989).

[26] S. Elitzur, G. W. Moore, A. Schwimmer and N. Seiberg, “Remarks On The Canonical Quantization Of The Chern-Simons-Witten Theory,” Nucl. Phys. B 326, 108 (1989).
[27] G. V. Dunne, “Aspects of Chern-Simons theory,” arXiv:hep-th/9902115.

[28] G. W. Moore and N. Seiberg, “Taming the Conformal Zoo,” Phys. Lett. B 220, 422 (1989).

[29] B. Ezhuthachan, S. Mukhi and C. Papageorgakis, “D2 to D2,” JHEP 0807 (2008) 041 [arXiv:0806.1639 [hep-th]].

[30] M. Benna, I. Klebanov, T. Klose and M. Smedback, “Superconformal Chern-Simons Theories and AdS$_4$/CFT$_3$ Correspondence,” JHEP 0809 (2008) 072 [arXiv:0806.1519 [hep-th]].

[31] S. Cherkis, V. Dotsenko and C. Saemann, “On Superspace Actions for Multiple M2-Branes, Metric 3-Algebras and their Classification,” arXiv:0812.3127 [hep-th].

[32] I. L. Buchbinder, E. A. Ivanov, O. Lechtenfeld, N. G. Pletnev, I. B. Samsonov and B. M. Zupnik, “ABJM models in N=3 harmonic superspace,” JHEP 0903, 096 (2009) [arXiv:0811.4774 [hep-th]].

[33] M. Cederwall, “N=8 superfield formulation of the Bagger-Lambert-Gustavsson model,” JHEP 0809, 116 (2008) [arXiv:0808.3242 [hep-th]].

[34] M. Cederwall, “Superfield actions for N=8 and N=6 conformal theories in three dimensions,” JHEP 0810, 070 (2008) [arXiv:0809.0318 [hep-th]].

[35] J. Bedford and D. Berman, “A note on Quantum Aspects of Multiple Membranes,” Phys. Lett. B 668 (2008) 67 [arXiv:0806.4900 [hep-th]].

[36] D. S. Berman, L. C. Tadrowski and D. C. Thompson, “Aspects of Multiple Membranes,” Nucl. Phys. B 802 (2008) 106 [arXiv:0803.3611 [hep-th]].

[37] P. Di Francesco, P. Mathieu and D. Senechal, “Conformal Field Theory,” Springer (1997).
[38] D. S. Berman and P. Sundell, “AdS(3) OM theory and the self-dual string or membranes ending on the five-brane,” Phys. Lett. B 529, 171 (2002) [arXiv:hep-th/0105288]. D. S. Berman, “Aspects of M-5 brane world volume dynamics,” Phys. Lett. B 572 (2003) 101 [arXiv:hep-th/0307040].

[39] D. S. Berman and J. A. Harvey, “The self-dual string and anomalies in the M5-brane,” JHEP 0411, 015 (2004) [arXiv:hep-th/0408198].

[40] S. Mukhi and C. Papageorgakis, “M2 to D2,” JHEP 0805 (2008) 085 [arXiv:0803.3218 [hep-th]].

[41] A. Gustavsson, “Selfdual strings and loop space Nahm equations,” JHEP 0804 (2008) 083 [arXiv:0802.3456 [hep-th]].

[42] S. Terashima, “On M5-branes in N=6 Membrane Action,” JHEP 0808 (2008) 080 [arXiv:0807.0197 [hep-th]].

K. Hanaki and H. Lin, “M2-M5 Systems in N=6 Chern-Simons Theory,” JHEP 0809, 067 (2008) [arXiv:0807.2074 [hep-th]].

[43] R. Floreanini and R. Jackiw, “Selfdual Fields As Charge Density Solitons,” Phys. Rev. Lett. 59, 1873 (1987).

[44] J. Sonnenschein, “Chiral Bosons,” Nucl. Phys. B 309 (1988) 752.

[45] D. Gaiotto and X. Yin, “Notes on superconformal Chern-Simons-matter theories,” JHEP 0708, 056 (2007) [arXiv:0704.3740 [hep-th]].

[46] H. C. Kao, “Selfdual Yang-Mills Chern-Simons Higgs systems with an N=3 extended supersymmetry,” Phys. Rev. D 50, 2881 (1994).

[47] E. A. Ivanov, “Chern-Simons matter systems with manifest N=2 supersymmetry,” Phys. Lett. B 268, 203 (1991).
[48] L. V. Avdeev, D. I. Kazakov and I. N. Kondrashuk, “Renormalizations in supersymmetric and nonsupersymmetric nonAbelian Chern-Simons field theories with matter,” Nucl. Phys. B 391, 333 (1993).

[49] L. V. Avdeev, G. V. Grigorev and D. I. Kazakov, “Renormalizations in Abelian Chern-Simons field theories with matter,” Nucl. Phys. B 382, 561 (1992).

[50] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol I., Interscience Press, 1953.

[51] E. Bergshoeff, D. S. Berman, J. P. van der Schaar and P. Sundell, “A noncommutative M-theory five-brane,” Nucl. Phys. B 590 (2000) 173 [arXiv:hep-th/0005026]. D. S. Berman and B. Pioline, “Open membranes, ribbons and deformed Schild strings,” Phys. Rev. D 70 (2004) 045007 [arXiv:hep-th/0404049].

C. S. Chu and D. J. Smith, “Towards the Quantum Geometry of the M5-brane in a Constant C-Field from Multiple Membranes,” arXiv:0901.1847 [hep-th].

[52] N. Sakai and Y. Tanii, “SuperWess-Zumino-Witten models from superChern-Simons theories,” Prog. Theor. Phys. 83, 968 (1990).

[53] D. Gaiotto and E. Witten, “Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory,” arXiv:0804.2902 [hep-th]. D. Gaiotto and E. Witten, “Janus Configurations, Chern-Simons Couplings, And The Theta-Angle in N=4 arXiv:0804.2907 [hep-th]. D. Gaiotto and E. Witten, “S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory,” arXiv:0807.3720 [hep-th].

[54] A. M. Low, “N=6 Membrane Worldvolume Superalgebra,” arXiv:0903.0988 [hep-th].

[55] H. Nastase, C. Papageorgakis and S. Ramgoolam, “The fuzzy $S^2$ structure of M2-M5 systems in ABJM membrane theories,” arXiv:0903.3966 [hep-th].