Self – similar solutions of the Burgers hierarchy

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Abstract

Self — similar solutions of the equations for the Burgers hierarchy are presented.

1 Introduction

The Burgers hierarchy can be written in the form [1–4]

\[ u_t + \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} u + u \right)^n u = 0, \quad n = 0, 1, 2, \ldots \]  

Assuming \( n = 1 \) in Eq. (1) we have the Burgers equation

\[ u_t + 2 u u_x + u_{xx} = 0. \]  

Eq. (2) was firstly introduced in [5]. It is well known that this equation can be linearized by means of the Cole-Hopf transformation [6–8]. Exact solutions of Eq. (2) were considered in many papers (see, for example, [9–12]).

Assuming \( n = 2 \) in Eq. (1) we obtain the Sharma - Tasso - Olver equation

\[ u_t + u_{xxx} + 3 u_x^2 + 3 u u_{xx} + 3 u^2 u_x = 0. \]  

The Sharma - Tasso - Olver equation was derived in [1, 13]. Some exact solutions of this equation were presented in [14, 21].

At \( n = 3 \) and \( n = 4 \) we obtain the following fourth and fifth order partial differential equations

\[ u_t + u_{xxxx} + 10 u_x u_{xx} + 4 uu_{xxx} + 12 uu_x^2 + 6 u^2 u_{xx} + 4 u^3 u_x = 0, \]  

1
\begin{align*}
  u_t + u_{xxxx} + 10 u_{xx}^2 + 15 u_x u_{xxx} + 5 uu_{xxxx} + 15 u_x^3 + \\
  +50 uu_x u_{xx} + 10 u_x^2 u_{xxx} + 30 u_x^2 + 10 u_x^3 u_{xx} + 5 u_x^4 u_x = 0. \\
  \tag{5}
\end{align*}

Assuming
\begin{align*}
  x = L x', \quad u = C_0 u', \quad t = T t',
\end{align*}
we have that Eq. (1) is invariant under the dilation group in the case
\begin{align*}
  C_0 L = 1, \quad T = L^{n+1}.
\end{align*}

Assuming \( C_0 = e^{-a} \) in (7), we obtain the dilation group for the Burgers hierarchy (1) in the form
\begin{align*}
  u' = e^{-a} u, \quad x' = e^a x, \quad t' = e^{a(n+1)} t.
\end{align*}

From transformations (8) we have two invariants for Eq. (1)
\begin{align*}
  I_1 = u t^{n+1} = u' (t')^{n+1}, \quad I_2 = \frac{x}{t^{n+1}} = \frac{x'}{(t')^{n+1}}.
\end{align*}

Therefore we look for the solutions of the Burgers hierarchy taking into account the variables
\begin{align*}
  u(x,t) = \frac{A}{t^{n+1}} f(z), \quad z = \frac{B x}{t^{n+1}}.
\end{align*}

Substituting (10) into (1) we obtain the equation for \( f(z) \) at
\begin{align*}
  A = B = \frac{1}{(n + 1)^{n+1}},
\end{align*}
in the form
\begin{align*}
  \left( \frac{d}{dz} + f \right)^n f - z f + \beta = 0,
\end{align*}
where \( \beta \) is the constant of integration.

Solving Eq. (12) we obtain solutions of the Burgers hierarchy in the form
\begin{align*}
  u(x,t) = \frac{1}{(nt + t)^{n+1}} f(z), \quad z = \frac{x}{(nt + t)^{n+1}}.
\end{align*}

Let us study the solutions of nonlinear ordinary differential equation (12).
2 Exact solutions of equation (12)

First of all let us prove the following lemma.

**Lemma 1.** Equation (12) can be transformed to the linear equation of \((n + 1)\) -th order by means of transformation

\[
f = \frac{\psi_z}{\psi}
\]

**Proof.** The proof of this lemma can be given by means of the mathematical induction method.

Using the transformation (14) we have

\[
\left( \frac{d}{dz} + f \right) f = \frac{\psi_{zz}}{\psi}, \quad \left( \frac{d}{dz} + f \right)^2 f = \frac{\psi_{zzz}}{\psi}
\]

(15)

Assuming that there is equality

\[
\left( \frac{d}{dz} + f \right)^k f = \frac{\psi_{k+1,z}}{\psi}, \quad \psi_{k+1,z} = \frac{d^{k+1}\psi}{dz^{k+1}}.
\]

(16)

Differentiating Eq. (16) with respect to \(z\) we have

\[
\frac{d}{dz} \left( \frac{d}{dz} + f \right)^k f = \frac{\psi_{k+2,z}}{\psi} - \frac{\psi_z \psi_{k+1,z}}{\psi^2}.
\]

(17)

From Eq. (17) we obtain the equality

\[
\left( \frac{d}{dz} + f \right)^{k+1} f = \frac{\psi_{k+2,z}}{\psi}.
\]

(18)

Therefore we obtain the formula

\[
\left( \frac{d}{dz} + f \right)^{n} f = \frac{\psi_{n+1,z}}{\psi}.
\]

(19)

Taking this formula into account we have the equality

\[
\left( \frac{d}{dz} + f \right)^{n} f - zf + \beta = \frac{1}{\psi} (\psi_{n+1,z} - z \psi_z + \beta \psi).
\]

(20)

As result of this lemma we obtain that solutions of Eq. (12) can be found by the formula (14), where \(\psi(z)\) is the solution of the linear equation

\[
\psi_{n+1,z} - z \psi_z + \beta \psi = 0,
\]

(21)
Let us consider the partial cases. Assuming $\beta = 0$ in Eq. (21) we have

$$
\psi_{n+1,z} - z \psi_z = 0.
$$

(22)

Denoting $\psi_z = y$ we obtain

$$
y_{n,z} - z y = 0.
$$

(23)

In the case $n = 1$ we get solution of Eq. (23) in the form

$$
y(z) = C_2 e^{-\frac{z^2}{2}}.
$$

(24)

The general solution of Eq. (23) can be written as

$$
\psi(z) = C_3 + C_2 \int_0^z e^{-\frac{\xi^2}{2}} d\xi,
$$

(25)

where $C_2$ and $C_3$ are arbitrary constants. In the case $n = 2$ we obtain the general solution of Eq. (23) in the form

$$
y(z) = C_4 \sqrt{z} J_{\frac{3}{2}} \left( \frac{2}{3} z^{\frac{3}{2}} \right) + C_5 \sqrt{z} Y_{\frac{3}{2}} \left( \frac{2}{3} z^{\frac{3}{2}} \right),
$$

(26)

where $J_{\frac{3}{2}}$ and $Y_{\frac{3}{2}}$ are the Bessel functions.

In the case $n > 2$ solution of Eq. (23) has $n$ solutions

$$
y_j(z) = z^{j-1} E_{n,1+\frac{1}{n},1+\frac{1}{n}}(z^{n+1}), \quad j = 1, 2, \ldots, n,
$$

(27)

where $E_{n,m,l}$ is a Mittag-Leffler type special function defined by [22];

$$
E_{n,m,l}(z) = 1 + \sum_{k=1}^{\infty} b_k z^k, \quad b_k = \prod_{s=0}^{k-1} \frac{\Gamma(n(ms + l) + 1)}{\Gamma(n(ms + l + 1) + 1)}
$$

(28)

In the case $\beta \neq 0$ solutions of Eq. (23) can be referred to the type of the Laplace equations [23]. There are partial solutions $\psi(z) = -z^m$ of Eq. (21) at $\beta = m$, where $0 < m \leq n$ is integer. In the general case solutions of equations (21) can be found using the Laplace transformation or taking the expansions in the power series into account.

For example let us solve the Cauchy problem for linear ordinary differential equation (21) at $\beta = -1$. We have the following problem

$$
\psi_{n+1,z} - z \psi_z - \psi = 0,
$$

(29)

$$
\psi(z = 0) = b_0, \quad \psi_z(z = 0) = b_1, \ldots, \psi_{n-2,z} = b_{n-2} \quad \psi_{n-1,z} = b_{n-1}.
$$
Substituting
\[ \psi(z) = \sum_{m=0}^{\infty} a_m z^m \] (30)
into Eq.(29), we obtain the solution in the form
\[
\psi(z) = a_0 \sum_{k=0}^{\infty} \frac{z^{nk} \prod_{j=0}^{k} (n j + 1)}{(nk + 1)!} + a_1 \sum_{k=0}^{\infty} \frac{z^{nk+1} \prod_{j=0}^{k} (n j + 2)}{(nk + 2)!} + \\
+ 2a_2 \sum_{k=0}^{\infty} \frac{z^{nk+2} \prod_{j=0}^{k} (n j + 3)}{(nk + 3)!} + \ldots + \\
+(n - 2)! a_{n-2} \sum_{k=0}^{\infty} \frac{z^{nk+n-2} \prod_{j=0}^{k} (n j + n - 1)}{(nk + n - 1)!} + \\
+(n - 1)! a_{n-1} \sum_{k=0}^{\infty} \frac{z^{nk+n-1} \prod_{j=0}^{k} (n j + n)}{(nk + n)!}.
\] (31)

The value of coefficients \(a_0, a_1, a_2, \ldots, a_{n-2}\) and \(a_{n-1}\) are determined by the initial values \(b_0, b_1, b_2, \ldots, b_{n-2}\) and \(b_{n-1}\). We have
\[
a_0 = b_0, \quad a_1 = b_1, \quad a_2 = \frac{b_2}{(2!)^2}, \ldots, a_{n-1} = \frac{b_{n-1}}{((n - 1)!)^2}.
\] (32)

Let us present the partial cases of solution for equation (29). In the case \(n = 3\) we have solution in the form
\[
\psi(z) = a_0 \sum_{k=0}^{\infty} \frac{z^{3k} \prod_{j=0}^{k} (3 j + 1)}{(3k + 1)!} + a_1 \sum_{k=0}^{\infty} \frac{z^{3k+1} \prod_{j=0}^{k} (3 j + 2)}{(3k + 2)!} + \\
+ 2a_2 \sum_{k=0}^{\infty} \frac{z^{3k+2} \prod_{j=0}^{k} (3 j + 3)}{(3k + 3)!}.
\] (33)
Assuming $n = 4$ we obtain

$$\psi(z) = a_0 \sum_{k=0}^{\infty} z^{4k} \prod_{j=0}^{k} (4j + 1) \frac{1}{(4k+1)!} + a_1 \sum_{k=0}^{\infty} z^{4k+1} \prod_{j=0}^{k} (4j + 2) \frac{1}{(4k+2)!} +$$

$$\quad + 2a_2 \sum_{k=0}^{\infty} z^{4k+2} \prod_{j=0}^{k} (4j + 3) \frac{1}{(4k+3)!} + 6a_3 \sum_{k=0}^{\infty} z^{4k+3} \prod_{j=0}^{k} (4j + 4) \frac{1}{(4k+4)!}. \quad (34)$$

One can show that these power series are convered for any values $z$. Therefore self-similar solutions of equations for the Burgers hierarchy are found after substitution $(34)$ into formula $(14)$.

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