Computation of Cohomology of Lie Algebra of Hamiltonian Vector Fields by Splitting Cochain Complex into Minimal Subcomplexes

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Computation of homology or cohomology is intrinsically a problem of high combinatorial complexity. Recently we proposed a new efficient algorithm for computing cohomologies of Lie algebras and superalgebras. This algorithm is based on partition of the full cochain complex into minimal subcomplexes. The algorithm was implemented as a C program LieCohomology. In this paper we present results of applying the program LieCohomology to the algebra of hamiltonian vector fields H(2|0). We demonstrate that the new approach is much more efficient comparing with the straightforward one. In particular, our computation reveals some new cohomological classes for the algebra H(2|0) (and also for the Poisson algebra Po(2|0)).

1 Introduction

Cohomology is defined by cochain complex

\[ 0 \to C^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{k-2}} C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \xrightarrow{d^{k+1}} \cdots. \] (1)

Here \( C^k \) are linear spaces (more generally, abelian groups), graded by the integer number \( k \), called dimension or degree (depending on the context). The elements of the spaces \( C^k \) are called cochains.

The linear mappings \( d^k \) are called differentials (or coboundary operators). The main property of these mappings is “their squares are equal to zero”: \( d^k \circ d^{k-1} = 0 \).

The elements of the space \( Z^k = \text{Ker} \ d^k \) are called cocycles. The elements of the space \( B^k = \text{Im} \ d^{k-1} \) are called coboundaries. Note that \( B^k \subseteq Z^k \).

The kth cohomology is the quotient space

\[ H^k = Z^k/B^k \equiv \text{Ker} \ d^k/\text{Im} \ d^{k-1}. \]

There are many cohomological theories designed for investigation of different mathematical structures and the space \( H^k \) carries important information about peculiarities in these structures. The only difference between cohomological theories lies in the constructions of the cochain spaces and coboundary operator. These constructions depend on the underlying mathematical structures.

The cohomology of the Lie (super)algebra \( A \) in the module \( X \) is defined via cochain complex (1) in which (see, e.g., [1]) the cochain spaces \( C^k = C^k(A; X) \) consist of super skew-symmetric \( k \)-linear mappings \( A \times \cdots \times A \to X \), \( C^0 = X \) by definition. Super
skew-symmetry means symmetry with respect to swapping of two adjacent odd cochain arguments and antisymmetry for any other combination of parities for adjacent pair.

The differential \( d^k \) takes the form\(^1\)

\[
(d^k c)(a_0, \ldots, a_k) = - \sum_{0 \leq i < j \leq k} (-1)^{s(a_i) + s(a_j) + p(a_i)p(a_j)} c([a_i, a_j], a_o, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_k) \\
- \sum_{0 \leq i \leq k} (-1)^{s(a_i)} a_i c(a_0, \ldots, \hat{a}_i, \ldots, a_k),
\]

where the functions \( c(\ldots) \) are elements of cochain spaces; \( a_i \in A; \) \( p(a_i) \) is the parity of \( a_i; \)
\( s(a_i) = i, \) if \( a_i \) is even element and \( s(a_i) \) is equal to the number of even elements in the sequence \( a_0, \ldots, a_{i-1}, \) if \( a_i \) is odd element. In the case of trivial module (i.e., if \( ax = 0 \) for all \( a \in A \) and \( x \in X \)) one uses as a rule the notation \( H^k(A). \)

In papers \([2, 3, 4, 5]\) we presented an algorithm for computation of Lie (super)algebra cohomologies. These papers contain also description of its C implementation and some results obtained with the help of codes designed. This algorithm computes cohomology of Lie (super)algebra \( A \) over module \( X \) in a straightforward way, i.e., for cochain complex \([\ref{1}]\) the algorithm constructs the full set of basis super skew-symmetric monomials forming the space \( C^k, \) generates subsequently all basis monomials in the space \( C^{k+1}, \) computes the differentials corresponding to these monomials to obtain the set of linear equations determining the space of cocycles

\[
Z^k = \text{Ker } d^k = \{ C^k | dC^k = 0 \},
\]

constructs the space of coboundaries

\[
B^k = \text{Im } d^{k-1} = \{ C^k | C^k = dC^{k-1} \}.
\]

Finally, the algorithm constructs the basis elements of quotient space

\[
H^k(A; X) = Z^k / B^k.
\]

This last step is based on the Gauss elimination procedure.

The main difficulty in computing cohomology results from the very high dimensions of the spaces \( C^k: \) for \( n \)-dimensional ordinary Lie algebra and \( p \)-dimensional module

\[
\dim C^k = p \binom{n}{k},
\]

and for \((n|m)\)-dimensional Lie superalgebra

\[
\dim C^k = p \sum_{i=0}^{k} \binom{n}{k-i} \binom{m+i-1}{i} \equiv p \binom{n}{k} + p \sum_{i=1}^{k} \binom{n}{k-i} \binom{m+i-1}{i}.
\]

In many cases it is possible to extract some easier to handle subcomplexes of the full cochain complex \([\ref{1}]\). The partition of cochain complex for a graded algebra and module into homogeneous components is a typical example. In many papers (see, e. g.,

\(^1\)This version of formula for differential corresponds to the algorithm used in the program \textit{LieCohomology}.}
more special subcomplexes were used successfully to obtain new results in the theory of cohomology of Lie (super)algebras.\footnote{10, 11, 12} The main idea of the new algorithm presented in [6, 7, 8] is to extract the minimal possible subcomplexes from complex (1) and to carry computations within these subcomplexes. There are two versions of the algorithm. One of them is applied when the cochain spaces under consideration are infinite-dimensional (or their dimensions are too large to fit the available memory), but the minimal subcomplexes contain finite-dimensional spaces of $k$-cochains. Another version of the algorithm is applied when it is possible to construct the full space $C^k$. Below we present this version in the pseudocode form.

### Algorithm: ComputeCohomology

| Input: $A$, Lie (super) algebra; $X$, module; $k$, cohomology degree; $g$, grade |
| Output: $BH^k_g$, set of basis cohomological classes |
| Local: $M^k_g$, full set of $k$-cochain monomials (basis of $C^k_g$); $s$, current subcomplex: $C^k_{g,s} \xrightarrow{d^k_g} C^k_{g,s} \xrightarrow{d^k_g} C^k_{g,s+1}$; $m^k_g \in M^k_g$, starting monomial for constructing subcomplex $s$; $M^k_g,s$, set of $k$-cochain monomials involved in subcomplex $s$; $BH^k_{g,s}$, set of basis cohomological classes in subcomplex $s$ |
| 1: $BH^k_g := \emptyset$ |
| 2: $M^k_g := \text{GenerateMonomials}(A, X, k, g)$ |
| 3: while $M^k_g \neq \emptyset$ do |
| 4: $m^k_g := \text{ChooseMonomial}(M^k_g)$ |
| 5: $\{s, M^k_g,s\} := \text{ConstructSubcomplex}(m^k_g)$ |
| 6: $BH^k_{g,s} := \text{ComputeCohomologyInSubcomplex}(s)$ |
| 7: if $BH^k_{g,s} \neq \emptyset$ then |
| 8: $BH^k_g := BH^k_g \cup BH^k_{g,s}$ |
| 9: fi |
| 10: $M^k_g := M^k_g \setminus M^k_g,s$ |
| 11: od |
| 12: return $BH^k_g$ |

Here the subalgorithm \texttt{GenerateMonomials} generates the full set $M^k_g$ of super skew-symmetric monomials

$$c(\alpha_{i_1}, \ldots, \alpha_{i_k}; \xi_\ell) \equiv c(\alpha_{i_1}) \wedge \cdots \wedge c(\alpha_{i_k}) \otimes \xi_\ell \equiv \alpha'_{i_1} \wedge \cdots \wedge \alpha'_{i_k} \otimes \xi_\ell$$

forming basis of the cochain space $C^k$ in the grade $g$; $\alpha_{i_j} \in A$ and $\xi_\ell \in X$ are basis elements of algebra and module; $\alpha'_{i}$ is the dual to $\alpha_{i}$ element. The subalgorithm \texttt{ChooseMonomial} takes some monomial $m^k_g \in M^k_g$. This monomial is starting monomial for

\footnote{The main trick consists in imposing some restrictions on the elements of $C^k$ and proving the invariance of these restrictions with respect to the differential.}
constructing the subcomplex $s$ by the subalgorithm \textbf{ConstructSubcomplex}. The subalgorithm \textbf{ComputeCohomologyInSubcomplex} computes basis cohomological classes $BH^k_{g,s}$ in the subcomplex $s$ by the straightforward algorithm described above.

2 Computation of $H^k_g(H(2|0))$

In this section we present the results of computation of cohomology in the trivial module for Lie algebra $H(2|0)$ of formal hamiltonian vector fields on the 2-dimensional symplectic manifold. We describe also cohomological classes up to grade 8 for the Poisson algebra $Po(2|0)$ which is a central extension of the algebra $H(2|0)$.

The hamiltonian algebra $H(2n|m)$ is an algebra of vector fields (see, e.g., [9]) acting on the $(2n|m)$ supermanifold and preserving the following 2-form

$$\sum_{i=1}^n dp_i \wedge dq_i + \sum_{j=1}^m d\theta_j \wedge d\theta_j,$$

where $p_1,\ldots,p_n; q_1,\ldots,q_n$ and $\theta_1,\ldots,\theta_m$ are even and odd local variables on the supermanifold, respectively. The elements of $H(2n|m)$ can be expressed in terms of generating function $f(p_1,\ldots,p_n; q_1,\ldots,q_n; \theta_1,\ldots,\theta_m)$ by the formula

$$\sum_{i=1}^n \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j=1}^m \frac{\partial f}{\partial \theta_j} \frac{\partial}{\partial \theta_j},$$

where $p(f)$ is parity of the function $f$ (this function is called usually hamiltonian). Thus one can consider the formal hamiltonian vector fields as linear combinations of monomials in the variables $p_i, q_i$ and $\theta_j$ (except of the monomial 1). Considering these monomials as basis elements of $H(2n|m)$ and using prescribed $\mathbb{Z}$-grading for the variables $p_i, q_i$ and $\theta_j$ one can impose $\mathbb{Z}$-grading $\text{gr}()$ on the algebra $H(2n|m)$. The standard grading is $\text{gr}(p_i) = \text{gr}(q_i) = \text{gr}(\theta_j) = 1$. For the standard grading the grade of algebra element corresponding to some monomial is equal to the grade of this monomial minus 2 (due to two differentiations in the terms of expression (6)).

Since the hamiltonian algebra is very important in both classical and quantum physics many efforts were applied to investigation of its cohomology. Most advanced results were obtained for the finite-dimensional algebras of the form $H(0|m)$ [13, 14]. Nevertheless the hamiltonian algebras on supermanifolds with nonzero even dimension are more important in applications but computation of their cohomologies is much more difficult task. Some results about cohomologies of such algebras were obtained in [10, 11, 12]. In the paper [11] some elements of $H^k_g(H(2|0))$ were calculated by considering special subcomplexes (and using computer partially). We present here all cohomological classes (without discussing their meaning and interpretation) from $H^k_g(H(2|0))$ up to grade 8.

The results of computation are summarized in Table 2. The boxes of this table corresponding to cochain degree $k$ and cochain grade $g$ contain the following information: $\dim C^k_g$, dimension of the full space of $k$-cochains in grade $g$; number of minimal subcomplexes $C^k_{g,s};$ maximum dimension of the subspace of $(k, g)$-cochains among all subcomplexes. The empty box means that $\dim C^k_g = 0$, i.e. the space of $(k, g)$-cochains is empty.
The boxes marked by the bullet • contain nontrivial 1-dimensional cohomological classes. For example, the box corresponding to the pair \((k, g) = (7, 8)\) tells that \(\dim C^7_{8,s} = 25488\), number of subcomplexes is 21, \(\max \dim C^7_{8,s} = 3148\) and \(\dim H^7_{8,s} = 1\).

More detailed information about computation in \((k, g) = (7, 8)\) is given in Table 1. In this table the columns \(\dim Z^k_{g,s}\), \(\dim B^k_{g,s}\) and \(\dim H^k_{g,s}\) contain dimensions of cocycle, coboundary and cohomology spaces in subcomplexes, respectively. On can see that there are 10 pairs of subcomplexes with repeated structure and the only single subcomplex containing nontrivial cohomological class.

The full set of nontrivial (1-dimensional) cohomological classes is: \(H^{2}_{-2}, H^{5}_{-2}, H^{7}_{0}\) (computed earlier) and \(H^{10}_{7,6}\) (computed by the new program). As to the Poisson algebra \(Po(2|0)\), the part of its cohomological classes up to grade 8 coincides with those for \(H(2|0)\) except of \(H^{2}_{-2}, H^{k}_{g\leq 8}(Po(2|0))\) contains also four additional classes: \(H^{6}_{-4}, H^{8}_{5,2}, H^{8}_{6}, H^{11}_{0}\). But all these classes are multiplicative consequences of the classes \(H^{5}_{-2}, H^{7}_{0}, H^{7}_{8}, H^{10}_{6}\). These classes can be expressed in the form \(H^k_g = H^{k-1}_{g+2} \land c(Z)\) due to the general property [4] of cohomology of algebras containing central element \(Z\).

### Table 1: Subcomplex structure for \((k, g) = (7, 8)\)

| \(\dim C^{k-1}_{g,s}\) | \(\dim C^k_{g,s}\) | \(\dim C^{k+1}_{g,s}\) | \(\dim Z^k_{g,s}\) | \(\dim B^k_{g,s}\) | \(\dim H^k_{g,s}\) | repeated |
|----------------|----------------|----------------|----------------|----------------|----------------|---------|
| 0 | 1 | 1 | 0 | 0 | 0 | 2 |
| 12 | 17 | 11 | 9 | 9 | 0 | 2 |
| 72 | 80 | 54 | 43 | 43 | 0 | 2 |
| 223 | 243 | 167 | 130 | 130 | 0 | 2 |
| 507 | 540 | 375 | 292 | 292 | 0 | 2 |
| 909 | 976 | 702 | 520 | 520 | 0 | 2 |
| 1406 | 1536 | 1120 | 813 | 813 | 0 | 2 |
| 1928 | 2117 | 1578 | 1114 | 1114 | 0 | 2 |
| 2382 | 2652 | 1992 | 1387 | 1387 | 0 | 2 |
| 2695 | 3008 | 2286 | 1568 | 1568 | 0 | 2 |
| 2806 | 3148 | 2391 | 1640 | 1639 | 1 | 1 |

3 Conclusion

Our new algorithm demonstrates substantially higher efficiency in comparison with the old one. For example, the program described in [2] computes the case \((k, g) = (6, 5)\) in 35 min 45 sec = 2145 sec whereas the new program takes 54 sec for this task. For both runs we used PC Pentium III, 667MHz, 256MB RAM. The superiority of the new program grows with increasing of the task complexity. Nevertheless, due to rapidly increasing computational complexity the presented results are not sufficient to derive any general idea about the structure of cohomology ring \(H^*_*(H(2|0))\). Our computation was carried over the field of rational numbers \(\mathbb{Q}\). As profiling shows, the most time consuming part

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3 The cocycle \(H^2_{-2}\) describes the central extension of the algebra \(H(2|0)\) to \(Po(2|0)\).
Table 2: Computation of $H^k_g$ for $(k, g) \in [1, \ldots, \infty) \otimes [-2, \ldots, 8]$

| $k\backslash g$ | -2 | -1 | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------------|----|----|----|----|----|----|----|----|----|----|----|
| 1               | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
| 2               | 1  | 6  | 11 | 22 | 33 | 52 | 71 | 100| 129| 170| 211|
| 3               | 1  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 |
| 4               | 1  | 3  | 8  | 13 | 22 | 34 | 52 | 72 | 100| 133| 177|
| 5               | 1  | 12 | 41 | 128| 315| 706|1432|2748|4949|8568|14240|
| 6               | 4  | 23 | 90 | 264| 688|1580|3382|6734|12766|23074|
| 7               | 6  | 32 | 135|412 |1128|2730|6132|12818|25488|
| 8               | 4  | 33 | 138|479 |1388|3606|8546|18963|
| 9               | 1  | 7  | 25 | 79 | 207| 510|1125|2391 |
| 10              | 1  | 4  | 17 | 62 | 188| 489|1188|
| 11              | 5  | 46 | 217|818 |2578|
| 12              | 10 | 7  | 54 |    |

Note: The table continues with similar entries for $k=10$ to $k=12$. The entries represent the computation of $H^k_g$ for various $(k, g)$ values within the specified range.
of computation by the program **LieCohomology** is multiprecision arithmetic. This is common difficulty for almost all problems in computer algebra. It seems that carrying computation over the finite fields, say $\mathbb{Z}_p$, we can go to the grade 40-50 for the problem considered here, but the results obtained in this way can be considered merely as hints.

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