Abstract
We write a 3D equation for three fermions by combining the three two-body potentials obtained by the reductions of the corresponding two-fermion Bethe-Salpeter equations to equivalent 3D equations, putting the spectator fermion on the mass shell. In this way, the cluster-separated limits are still exact, and the Lorentz invariance / cluster separability requirement is automatically satisfied, provided no supplementary approximation, like the Born approximation, is made. The use of positive free-energy projectors in the chosen reductions of the two-fermion Bethe-Salpeter equations prevents continuum dissolution in our 3D three-fermion equation. The potentials are hermitian and depend only slowly on the total three-fermion energy. The one high-mass limits are approximately exact.

In view of a possible perturbation calculation, correcting the remaining discrepancies with the three-fermion Bethe-Salpeter equation, we succeeded in deriving our 3D equation from an approximation of the three-fermion Bethe-Salpeter equation, in which the three-body kernel is neglected and the two-body kernels approached by positive-energy instantaneous expressions, with the spectator fermion on the mass shell. The neglected terms are transformed into corrections to the 3D equation. A comparison is made with Gross' spectator model.

PACS 11.10.Qr Relativistic wave equations.
PACS 11.10.St Bound and unstable states; Bethe-Salpeter equations.
PACS 12.20.Ds Specific calculations and limits of quantum electrodynamics.

Keywords: Bethe-Salpeter equations. Salpeter’s equation. Breit’s equation. Relativistic bound states. Relativistic wave equations.
1 Introduction

In the treatment of the three-body problem, there is a large gap between nonrelativistic quantum mechanics (Schrödinger equation) and relativistic quantum field theory (Bethe-Salpeter equation \([1, 2]\)). Starting from the Schrödinger equation one can of course replace the free part of the Hamiltonian by its relativistic form. Starting from the Bethe-Salpeter equation, one can try to eliminate the two relative time variables to get finally a Schrödinger equation with a compact potential plus a lot of correction terms of various origins. Instead of the Schrödinger equation, one can use the Faddeev equations. These equations can be derived from the Schrödinger equation, or, in a more general form, from the Bethe-Salpeter equation. Faddeev equations for the transition operator give the various scattering and reaction matrix elements and the poles (to be computed by basically nonperturbative methods) of this operator in the total energy give the spectrum of the three-body bound states.

Schrödinger’s equation is “relatively easy” to solve, but this zero-order approximation does not reflect several important properties and symmetries of the studied physical system. These could be recovered, in principle, by incorporating higher-order contributions (in general an infinity of them).

As an intermediate step between nonrelativistic quantum mechanics and quantum field theory we shall search for a 3D equation built with the sum of the relativistic free Hamiltonians plus two-body and perhaps also three-body interaction potentials. Such equations are closely related to the systems of coupled Dirac or Klein-Gordon equations of constraint theory \([3, 4, 5, 6, 7]\), which are still 4D equations but exhibit a simplified dependence in the relative times, which could be completely eliminated to get a single 3D equation. Besides containing relativistic free Hamiltonians, we shall try to make our 3D equation satisfy the following list of requirements:

— Correct nonrelativistic limit.
— Lorentz invariance and cluster separability. It is always possible to render an equation Lorentz invariant by working in the general rest frame (center of mass reference frame) and by building invariants with the total 4-momentum vector, although the result may be unelegant and artificial. The Lorentz invariance requirement becomes a tool when combined with the cluster separability requirement: when all mutual interactions are “switched off”, we must get a set of 3 free Dirac equations. This total separability can easily be obtained by using as Hamiltonian the sum of three free Dirac Hamiltonians and interaction terms. The real difficulty appears when only the interactions with fermion 3 (for example) are switched off. If we want a full cluster separability, the resulting equation for the (12) cluster can not
refer to the global center of mass frame anymore, as the momentum of fermion 3 enters in the definition of this frame.

Lorentz invariance and cluster separability can be explicit or implicit (via rearrangements). The best known example of implicit Lorentz invariance is a free Dirac equation solved with respect to the energy: it becomes explicitly covariant by multiplication with the \( \beta \) matrix. Other implicit Lorentz invariances are not that trivial. For example, the 3D reductions of a Bethe-Salpeter equation are implicitly covariant, provided the series generated by this reduction is not truncated.

— Hermiticity and total energy independence of the interaction terms. In the two-body problem, nonhermitian interaction terms can be hermitian with respect to a modified scalar product, or made hermitian via a rearrangement of the equation. In the three-body problem these rearrangements could be more complicated. The hermiticity and the independence in the total energy are linked features, as one of these is often achieved at the expense of the other one. Energy depending interaction terms destroy some of the advantages of the use of an hermitian hamiltonian, such as the mutual orthogonality of the solutions, and leads to modify the usual perturbation calculation methods. The 3D potentials deduced from field theory are generally energy dependent, at least in the higher-order terms. We shall require hermiticity and energy-independence (or slow energy dependence) in the lowest order terms at least.

— Correct heavy mass limits. When the mass of one of the fermions becomes infinite, its presence must be translated in the equations by a potential (Coulombian in QED) acting on the other fermions. For the two-body problem the "one-body limit" in QED is indeed a Dirac-Coulomb equation or a rearrangement of it (for example when a projection operator is introduced in order to avoid continuum dissolution - see below). In the Dirac-Coulomb equation, the Coulomb potential is already given by the limit of the Born term, as the higher-order crossed and ladder terms cancel mutually at the one-body limit. In contrast, the one-body limits of the rearrangements contain contributions from all terms.

— Perturbative approach. It must be possible to start with a manageable approximate equation and indefinitely improve the approximation of the measurable quantities (with respect to the uncalculated predictions of the here assumed exact Bethe-Salpeter equation) by adding higher-order contributions.

— Solution of the continuum dissolution problem. In the relativistic equations for several relativistic particles, the physical bound states are degenerate with a continuum of states combining asymptotically free particles with opposite energy signs. This often neglected fact forbids the building of normalizable solutions in the \( N > 2 \)-body problem (including the two-body plus potential problem. In the pure two-body problem the mixing is prevented by the conservation of the total momentum). The usual solution consists in including positive-energy projection operators into the zero-order propagator. The modified equations must of course continue to satisfy the other requirements, like the Lorentz invariance / cluster separability requirement.

Section 2 is devoted to the two-fermion problem, revisited in order to define the notations and present the building blocks and cluster-separated limits of our future three-fermion equations. In section 3 we write directly a cluster separable 3D equation, by combining the relativistic free hamiltonians and the 3D potentials obtained in the reduction of the two-fermion Bethe-Salpeter equations, putting in each 3D potential the spectator fermion on the mass shell. Switching off two of the three mutual interaction potentials gives then a free Dirac equation for the spectator fermion plus a two-body equation, equivalent to the two-fermion Bethe-Salpeter equation, for the two interacting fermions. This equivalence insures the Lorentz invariance / cluster separability property. Positive-energy projectors solve the continuum dissolution problem. Section 4 is devoted to the computation of the heavy-mass limits of the three-fermion equation, which are compared with the two-fermion in an external potential equations. In section 5, we show that our 3D equation can be obtained from the three-fermion Bethe-Salpeter equation, by neglecting the three-body kernel and replacing the two-body kernels by positive-energy instantaneous approximations, equivalent at the cluster-separated limits. The neglected terms are thus known at the Bethe-Salpeter equation level. A comparison is made with Gross’ spectator model. This section ends with an attempt of transforming the neglected terms in the Bethe-Salpeter equation into correction terms to the 3D equation, in view of a possible perturbation calculation. Section 6 is devoted to conclusions.

2 The two-fermion problem.
2.1 Notations.

We shall write the Bethe-Salpeter equation for the bound states of two fermions \[^{[1]}\] as

\[ \Phi = G_0 K \Phi, \] (1)

where \(\Phi\) is the Bethe-Salpeter amplitude, function of the positions \(x_1, x_2\) or of the momenta \(p_1, p_2\) of the fermions, according to the representation chosen. The operator \(K\) is the Bethe-Salpeter kernel, given in a non-local momentum representation by the sum of the irreducible two-fermion Feynman graphs.

The operator \(G_0\) is the free propagator, given by the product \(G_0 1 G_0 2\) of the two individual fermion propagators

\[ G_{0i} = \frac{1}{p_{0i} - h_i + i\epsilon h_i} \beta_i = \frac{p_{0i} + h_i}{p_i^2 - m_i^2 + i\epsilon \beta_i} \] (2)

where the \(h_i\) are the Dirac free hamiltonians

\[ h_i = \bar{\alpha}_i \cdot \vec{p}_i + \beta_i m_i \quad (i = 1, 2). \] (3)

The Bethe-Salpeter kernel \(K\) should contain charge renormalization and vacuum polarization graphs, while the propagators \(G_{0i}\) should contain self-energy terms (which can be transferred to \(K\) \[^{[19, 20]}\)).

In this work, we consider only the free fermion propagators in \(G_{0i}\) and the "skeleton" graphs in \(K\), and we hope that the inclusion of the various corrections would not change our conclusions.

We shall define the total (or external, CM, global) and relative (or internal) variables:

\[ X = \frac{1}{2} (x_1 + x_2), \quad P = p_1 + p_2, \] (4)

\[ x = x_1 - x_2, \quad p = \frac{1}{2} (p_1 - p_2). \] (5)

and give a name to the corresponding combinations of the free hamiltonians:

\[ S = h_1 + h_2, \quad s = \frac{1}{2} (h_1 - h_2). \] (6)

We know that, at the no-interaction limit, we shall have to get a pair of free Dirac equations:

\[ (p_{10} - h_1) \Psi = 0, \quad (p_{20} - h_2) \Psi = 0, \] (7)

where \(\Psi\) depends on \(x_1, x_2\). Let us also write their iterated version

\[ (p_{10}^2 - E_1) \Psi = 0, \quad (p_{20}^2 - E_2) \Psi = 0 \] (8)

with

\[ E_i = \sqrt{h_i^2} = (\vec{p}^2_i + m_i^2)^{\frac{1}{2}}. \] (9)

Interesting combinations can be obtained from the sum and differences of the equations (8) or of the iterated equations (5):

\[ (P_0 - S) \Psi = 0, \quad (p_0 - s) \Psi = 0, \] (10)

\[ H_0 \Psi = 0, \quad (p_0 - \mu) \Psi = 0 \] (11)

with

\[ H_0 = 2([p_1^2 - m_1^2] + [p_2^2 - m_2^2])_{p_0 = \mu} = P_0^2 - 2(E_1 + E_2) + 4\mu^2, \] (12)

\[ \mu = \frac{1}{2P_0} (E_1 - E_2) = \frac{1}{2P_0} (h_1^2 - h_2^2) = \frac{sS}{P_0}. \] (13)
2.2 3D reduction of the two-fermion Bethe-Salpeter equation.

The free propagator $G_0$ will be approached by a carefully chosen expression $G_{\delta}$, combining a constraint like $\delta(p_0 - \mu)$ fixing the relative energy, and a global 3D propagator like $-2i\pi(P_0 - S)^{-1}\beta_1\beta_2$. The argument of the $\delta$ and the inverse of the propagator should be combinations of the operators used in the free equations (at last approximately and for the positive-energy solutions). There exists an infinity of possible combinations.[8, 12, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. The best choice depends on the operator, the different choices of which result in different 3D operators applied on a common basic $\Psi$, and in different rearrangements of a common reduction series giving the 3D potential. The various 3D reductions of the literature can thus be classified according to the constraint they use. Once this constraint chosen, we can only write different equivalent forms (or sometimes projections) of the same 3D equation. It is the unavoidable truncation of the reduction series which makes the difference (numeric, if approximate) identical on the positive energy mass shell $\delta = 0$ and zero when they have opposite signs. It comes from the dependence of the sign ambiguity in the sum of the imaginary parts.

The first choice to be made in the 4D theory equations with QCD inspired potentials). In the pure two-fermion problem, this second choice has the advantage to fix the relative energy in the rest frame to a simple $\mu$ (or the $p_0 = \mu$) region. We can thus approach $G_0(p_0)$ with $G_{\delta}(p_0)$ defined by

$$G_{\delta}(p_0) = \delta(p_0 - s)G_S$$

(16) or by

$$G_{\delta}(p_0) = \delta(p_0 - \mu)G_S$$

(17) with

$$G_S = \int dp_0 G_0(p_0) = \frac{-2i\pi\tau}{P_0 - S} \beta_1\beta_2$$

(18)

$$\tau = \frac{1}{2}(\tau_1 + \tau_2), \quad \tau_i = \frac{h_i}{\sqrt{h_i^2}} = \frac{1}{E_i} = \text{sign}(h_i).$$

(19)

or

$$\tau = \Lambda^+ + \Lambda^-, \quad \Lambda^{ij} = \Lambda^i_1\Lambda^j_2, \quad \Lambda^\pm_i = \frac{E_i \pm h_i}{2E_i}.$$  

(20)

This operator $\tau$ has a clear meaning in the basis built with the free solutions: it is $+1$ for $h_1, h_2 > 0$, -1 for $h_1, h_2 < 0$ and zero when they have opposite signs. It comes from the dependence of the $p_0$ integral on the signs of the $i\epsilon h_i$. The denominator of $G_{\delta}$ must also contain an infinitesimal imaginary part, obtained by replacing $P_0$ by $P_0 + i\epsilon P_0$, as usual. The operator $\tau$ insures a common sign for $p_{01}, p_{02}$ and $P_0$, so that there is no sign ambiguity in the sum of the imaginary parts.

There exists of course an infinity of possible choices for $G_{\delta}$. They must however be (at least approximately) identical on the positive energy mass shell $P_0 = E_1 + E_2$. An obvious simplification can be made by replacing $\tau$ by 1 (its value for the physical free solutions) or by $\epsilon(P_0)$ (to get the correct value for the corresponding antiparticle states too). The merits of the $\tau$ or no-$\tau$ choice are the matter of an
old debate. The operator \( \tau \) brings apparently useless complications in the pure two-body case. In the two-body plus potential problem, however, the generalization of this operator prevents the "continuum dissolution" disease (see below). We shall refer to (18) as Salpeter’s propagator [21] and to this same expression without the operator \( \tau \) as the Breit propagator [35]. The Breit propagator is thus:

\[
G_B = -2i\pi \frac{1}{P_0 - S} \beta_1 \beta_2.
\]  

(21)

For a maximum ease of calculation with the zero-order approximation, one can choose the 3D propagator of Schrödinger. Many important physical properties are however lost at this approximation and can only be recovered by the inclusion of higher-order contributions. By contrast, the 3D propagator of Sazdjian, based on the second-order equations (11), is more complicated but leads to a covariant \( G_\delta \) when combined with the \( \delta(p_0 - \mu) \) constraint (11, 12, 19). In the remaining of this section we shall work with the approached propagator

\[
G_\delta = \delta(p_0 - s)AG_B
\]  

(22)

where \( A \) can be \( \tau \) (Salpeter), 1 (Breit) or \( \Lambda^+ \). We shall write the free propagator as the sum of the approached propagator, plus a rest:

\[
G_0 = G_\delta + G_R.
\]  

(23)

The Bethe-Salpeter equation becomes then the inhomogeneous equation

\[
\Phi = G_0 K\Phi = (G_\delta + G_R)K\Phi = \Psi + G_R K\Phi,
\]  

with

\[
\Psi = G_\delta K\Phi \quad (= G_\delta G_0^{-1}\Phi).
\]  

(25)

Solving (formally) the inhomogeneous equation (24) and putting the result into (23), we get

\[
\Psi = G_\delta K(1 - G_R K)^{-1}\Psi = G_\delta K_T\Psi
\]  

(26)

where

\[
K_T = K(1 - G_R K)^{-1} = K + KG_R K + ... = (1 - KG_R)^{-1}K
\]  

(27)

obeys

\[
K_T = K + KG_R K_T = K + K_T G_R K.
\]  

(28)

The reduction series (27) re-introduces in fact the reducible graphs into the Bethe-Salpeter kernel, but with \( G_0 \) replaced by \( G_R \). Equation (24) is a 3D equivalent of the Bethe-Salpeter equation. It depends on the choice of \( G_\delta \). The relative energy dependence of eq. (24) can be easily eliminated:

\[
\Psi = \delta(p_0 - s)\psi
\]  

(29)

and \( \psi \) obeys:

\[
\psi = \frac{A}{P_0 - S} V \psi
\]  

(30)

where \( V \) is proportional to \( K_T \) with the initial and final relative energies fixed to \( s \):

\[
V = -2i\pi \beta_1 \beta_2 K_T(s, s).
\]  

(31)

In less compact but more precise notations:

\[
\beta_1 \beta_2 K_T(s, s) \equiv \int dp'_0 dp_0 \delta(p'_0 - s)\beta_1 \beta_2 K_T(p'_0, p_0)\delta(p_0 - s).
\]  

(32)

Note that we write \((p'_0, p_0)\) but \((s, s)\), as we keep \( s \) in operator form. When \( A = 1 \) the operator \( AV \) is hermitian. When \( A = \Lambda^{++} \) the operator \( AV \) is hermitian in the \( \Lambda^{++} = 1 \) subspace, and we can write

\[
\psi = \frac{1}{P_0 - S} \Lambda^{++} V \Lambda^{++} \psi.
\]  

(33)
When $A = \tau$ the operator $AV$ is hermitian in the $\tau^2 = 1$ subspace, and we can write

$$\psi = \frac{1}{P_0 - S} \tau V \tau^2 \psi. \quad (34)$$

Let us recall that the potential $V$ depends on $G_\delta$ and is thus not the same for the different choices of $A$.

These 3D reductions can also be described in terms of transition operators. The 4D transition operator is

$$T = K + KG_0 K + \cdots \quad (35)$$

and the 3D transition operator is, when $A = \Lambda^{++}$, for example

$$T^{3D} = \Lambda^{++}V\Lambda^{++} + \Lambda^{++}V\Lambda^{++} - \frac{1}{P_0 - S + i\epsilon} \Lambda^{++}V\Lambda^{++} + \cdots \quad (36)$$

The relation between these two transition operators is simply

$$T^{3D} = -2i\pi \beta_1 \beta_2 <T>, \quad (37)$$

with

$$<T> \equiv \beta_1 \beta_2 \Lambda^{++} \beta_1 \beta_2 T(s,s) \Lambda^{++}. \quad (38)$$

We have indeed

$$\frac{-1}{2i\pi} \beta_1 \beta_2 T^{3D} = <K_T> + <K_T G_\delta K_T> + \cdots = <K_T (1 - G_\delta K_T)^{-1} >$$

$$= <K (1 - G R K)^{-1} (1 - G_\delta K (1 - G R K)^{-1})^{-1} >$$

$$= <K (1 - G R K - G_\delta K)^{-1} > = <K (1 - G_0 K)^{-1} > = <T>. \quad (39)$$

A Bethe-Salpeter equation leading directly to the same 3D reduction can be obtained by replacing the kernel $K$ by the instantaneous kernel $<K_T>$. The corresponding transition operator becomes in this case

$$T = <K_T> + <K_T G_0 K_T> + \cdots = <T>. \quad (40)$$

In contrast with the textbook cases, the potentials deduced from field theory are in general energy-dependent (although one starts often with an energy-independent approximation). The eigenvalue of the hamiltonian is then a function of the energy itself, and the energy spectrum is given by the solutions of the algebraic equations

$$E = E_\lambda(E) \quad (41)$$

but the corresponding wave functions are no more orthogonal with the the usual scalar product, so that the usual methods of perturbation calculation have to be revisited

Until now we did not specify a reference frame, and our equations are not manifestly covariant under the Lorentz group. The consequences of this must however be carefully discussed.

Through not manifestly covariant, equations like (26) can always be transformed back into the original Bethe-Salpeter equation, provided the reduction series (27) is not truncated. The inclusion of higher-order terms will thus, in principle, improve an approached covariance. Beyond this somewhat trivial consideration, it would of course be interesting to be able to simplify the equations in a way which preserves the covariance. If we use the covariant two-body Sazdjian propagator, based on the second-order equations (11), instead of Breit’s, it becomes possible to truncate the reduction series in a covariant way (one might for example keep only the first term of the reduction series at the ladder approximation, i.e. the Born term).

Even truncated, our equations could be made formally invariant by introducing a unit vector $n = (1, \vec{0})$ (we can call this vector the laboratory time unit) and by making all elements in our equations invariant by using scalar products with $n$. If we want more than a formal invariance, this unit vector can however not be external to the system: one must therefore define $n$ as $P/\sqrt{P^2}$ (the time unit of the center of mass reference frame), assuming $P^2 > 0$. 


– It must be noted that our non-covariant equations admit, among others, \( \vec{P} = 0 \) solutions which are identical to the \( P^2 > 0 \) solutions of the covariant equations in the center of mass reference frame. We could speak of “weak covariance”.

– An equation like (33) can be assumed to have been written in the center of mass reference frame and made explicitly covariant by using the vector \( P/\sqrt{P^2} \). This covariant form could be used to write the system of equations in other reference frames. There is no reason to do that in the pure two-body case, but it will become necessary when other objects (as an external potential or a third particle) are present. The \( G_B \delta(p_0 - s) \) combination written in the center of mass reference frame loses its simplicity when covariantized.

The cluster separability property is clearly satisfied, as the ”switching off” of the mutual interaction leads to a pair of free Dirac equations.

3 The three-fermion problem.

3.1 A cluster separable 3D equation.

A three-fermion 3D equation, inspired by the two-fermion equation (30) with \( A = 1 \) could be

\[
\psi = \frac{1}{P_0 - S} (V_{12} + V_{23} + V_{31}) \psi
\]

with \( S = h_1 + h_2 + h_3 \). The potentials are the two-body potentials defined by (31). Each two-body potential \( V_{ij} \) was a function of the partial energy \( P_{ij0} \), which we shall fix to its \((ij)+k\) cluster-separated limit \( P_0 - h_k \). At the \( V_{23} = V_{31} = 0 \) limit, for example, we shall thus get two completely independent equations:

\[
\begin{bmatrix}
P_{120} - h_1 - h_2
\end{bmatrix} \psi_{12} = V_{12}(P_{120}) \psi_{12}, \quad p_{30} \psi_3 = h_3 \psi_3
\]

with

\[
\psi = \psi_{12} \psi_3, \quad P_0 = P_{120} + p_{30}.
\]

Our 3D equation (42) satisfies thus clearly the cluster separability requirement. Furthermore, the three cluster-separated limits are exact equivalents of the corresponding two-fermion Bethe-Salpeter equations.

In the two-body problem all quantities can be defined in the center of mass reference frame. In the two-body plus potential problem we had to start in the laboratory frame but to consider also the center of mass reference frame at the no-external potential limit. In the three-body problem we must start in the three-body center of mass frame (unless we are satisfied with a “weak covariance”) and consider the center of mass reference frames of the three possible two-body subsystems obtained by cluster separation.

The fact that the three cluster-separated limits are exact insures in principle the Lorentz invariance / cluster separability requirement: at the cluster separated limits the two-body equation can indeed be transformed back into a covariant two-body Bethe-Salpeter equation. There is no necessity of introducing Lorentz boosts by hand.

Our two-body potentials are the sum of an infinity of contributions symbolized by Feynman graphs. Keeping only the first one (Born approximation) or a finite number of them renders the Lorentz covariance of the two-fermion clusters only approximate. A Born approximation preserving the Lorentz invariance / cluster separability property can be obtained by using another 3D reduction based on a covariant second-order two-body propagator of Sazdjian [12, 19] (combined with a covariant substitute of \( \Lambda^{++} \) to prevent continuum dissolution - see below). This leads to a 3D three-cluster equation which is covariantly Born approximable, but more complicated [37].

3.2 The continuum dissolution problem.

Unfortunately, our 3D equation (42) suffers of continuum dissolution. It is indeed possible to build a continuum of solutions with any a priori given total energy by combining asymptotically free fermions with opposite energy signs. Any physical bound state is thus degenerate with such a continuum and the building of normalisable bound state wave functions becomes impossible [13, 14, 15, 16, 17]. In the pure two-body case the energy of a system \((+, -)\) in the total rest frame is

\[
E_1 - E_2 = \sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2} = \frac{m_1^2 - m_2^2}{\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}}
\]
and lies thus between \( m_1 - m_2 \) (whichever the sign) and zero. We have thus no problem if we make the assumption that the energies of the bound states lie between \(|m_1 - m_2|\) and \(m_1 + m_2\). Below \(|m_1 - m_2|\) we meet the "strong field" problem. The strong field problem for the one-body plus potential and two-body systems, the continuum dissolution problem for the two-body plus potential and three-body systems are both consequences of the possibility of pair creation.

In the three-body case, the mixing with asymptotically separated (12)(3) subsystems of opposite energy signs is excluded by the total momentum conservation, as in the two-fermion case. For the mixing with three-fermion asymptotically free states, let us consider for example

\[
E_3 - E_1 - E_2 = \sqrt{p^2_3 + m_3^2} - \sqrt{p^2_1 + m_1^2} - \sqrt{p^2_2 + m_2^2}, \quad \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0. \tag{46}
\]

There is no lowest value (we can for example have \( \vec{p}_3 = 0 \) and \( \vec{p}_1 = -\vec{p}_2 \) arbitrarily large). The highest values are obtained when \( \vec{p}_1 \) and \( \vec{p}_2 \) have the same direction, and for \( \vec{p}_1/m_1 = \vec{p}_2/m_2 = -\vec{p}_3/(m_1 + m_2) \).

In this case we have

\[
E_3 - E_1 - E_2 = \sqrt{p^2_3 + m_3^2} - \sqrt{p^2_1 + m_1^2} = \sqrt{(m_1 + m_3)^2},
\]

so that \( E_3 - E_1 - E_2 \) lies finally between \(-\infty\) and \((m_3 - m_1 - m_2)\). Symmetrically, \( E_1 + E_2 - E_3 \) lies between \((m_1 + m_2 - m_3)\theta(m_3 - m_1 - m_2)\) and \(-\infty\). If we assume that the energies of the bound states lie between \((m_1 + m_2 - 2\ln(m_i))\) (i.e. above the highest negative-energy threshold) and \((m_1 + m_2 + m_3)\), there is no degenerescence with the "one plus-two minus" states. On the contrary, no weak field assumption could prevent the degenerescence with the "two plus-one minus" states.

This continuum dissolution problem can be cured by simply introducing the products of noncovariant projectors \( \Lambda_i^{++} = \Lambda_i^+ \Lambda_i^+ \) (on the eigenstates of the free \( h_i, h_j \) with positive eigenvalues) into the corresponding \( G_{hij} \), defining the two-body potentials, and around these potentials. This changes finally equation (42) into

\[
\psi = \frac{1}{P_0 - S} \left( \Lambda_{12}^{++} \psi_{12} + \Lambda_{23}^{++} \psi_{23} + \Lambda_{31}^{++} \psi_{31} \right) \psi \tag{48}
\]

built with the three two-fermion equations (43). There exist solutions of eq.(48) which are entirely contained in the \((+++)\) subspace. These solutions obey an equation with a spinless Salpeter free term:

\[
(P_0 - E_1 - E_2 - E_3) \psi = \Lambda^{++} \left( V_{12} + V_{23} + V_{31} \right) \Lambda^{++} \psi. \tag{49}
\]

It is also possible to build a three-fermion equation with three Salpeter equations:

\[
\psi = \frac{1}{P_0 - S} \left( \tau_{12} V_{12} \tau^2_{12} + \tau_{23} V_{23} \tau^2_{23} + \tau_{31} V_{31} \tau^2_{31} \right) \psi. \tag{50}
\]

This equation admits solutions lying entirely in the \((+++) \oplus (+-) \oplus (-+) \oplus (-++)\) subspace.

The remarks above about the cluster separability and Lorentz invariance of equation (42) remain true for equations (48) and (50).

In the remaining of this work we shall adopt the positive-energy equation (48) as our basic three-cluster 3D equation.

## 4 Heavy mass limits and external potentials.

### 4.1 Two fermions in an external potential.

The two-fermion plus potential problem can be approached in two ways which we shall call the two-body and the three-body approaches. In the two-body approach, the external potential is included in the definition of the creation and annihilation operators of field theory. Practically, we can keep the equations obtained in the treatment of the pure two-fermion problem, using simply a generalized definition of the \( h_i \) (14)

\[
h_i = \alpha_i \vec{p}_i + \beta_i m_i + V_i(\vec{x}_i) \tag{51}
\]

where \( V_i \) is the external potential acting on the fermion \( i \). All quantities can be expanded on the basis built with the eigenstates of \( h_1 \) and \( h_2 \). In the three-body approach, the free creation and annihilation operators are used and the external potential is treated as an heavy third body. This approach will be
presented as an heavy mass limit of the three fermion problem in subsection 4.3. Here we shall adopt the two-body approach.

The equations for two fermions in an external potential must be written in the laboratory reference frame (in which the external potential is defined) and are not Lorentz invariant. If we ”switch off” the mutual interaction we get a pair of uncoupled equations for two independent fermions in an external potential. If we switch off the external potential the equations remain written in the laboratory frame, which still refers to the vanished external potential. If the reduction series is not truncated, this no-external interaction limit of the equation is equivalent to a pure two-fermion covariant Bethe-Salpeter equation. If the reduction series is truncated, the pure two-body cluster equation is only ”weakly covariant”. If we want a truly cluster separable system, we must then use equations which are covariant at the vanishing external potential limit, such as Sazdjian’s equations with the generalization (51) of the $h_1$ $[8, 12, 19]$. In the two-body plus potential problem the total spatial momentum is no more conserved and the mixing with the continuum forbids the building of normalizable solutions. A easy way of seeing how this happens is to try to build these states by perturbations of the no-mutual interaction equations: the operators $h_1, h_2$ are then diagonal and the higher-order contributions contain denominators in $P_0 - h_1 - h_2$, which can vanish on a continuum. In Salpeter’s equation the operator $\tau$ kills the matrix elements of $\tau V$ between the $\tau^2 = 1$ states $(+, +), (-, -)$ and the $\tau^2 = 0$ states $(+, -), (-, +)$ in the basis built with the eigenstates of $(h_1, h_2)$, forbidding the mixing.

4.2 Heavy mass limits in the two-fermion problem.

When the mass of one of the fermions goes to infinity, we must find the equation of the other fermion in a potential (a Dirac equation with a Coulomb potential in QED). This limit can be obtained directly, or via a rearrangement of the equation. With Breit’s equation with the second-order constraint and with Sazdjian’s equation, the higher-order ladder and crossed terms cancel mutually at the one-body limit, so that the correct limit of the potential is already contained in the Born term. With Salpeter’s equation with a second-order constraint, the limit is also a Dirac-Coulomb equation, but solved with respect to the $\Lambda^+ \psi$ part of the wave function: the $\Lambda^- \psi$ part is transformed into higher-order contributions to the potential. The physical content remains identical to that of a Dirac-Coulomb equation, but the correct limit of the potential is no more entirely contained in the Born term, so that a truncation of the potential would spoil the one-body limit $[8, 12, 19]$.

4.3 Heavy mass limits in the three-fermion problem.

When $m_3$, for example, goes to infinity (two-body limit), we get, writing $P_0 = W_{12} + m_3$ in our basic three-cluster equation:

$$W_{12} \psi = (h_1 + h_2 + \Lambda_{12}^{++} V_{12} \Lambda_{12}^{++} + \Lambda_{2}^{+} V_{2}^{+} \Lambda_{2}^{+} + \Lambda_{1}^{+} V_{1}^{+} \Lambda_{1}^{+}) \psi$$

with $(P_0 - h_3)$ replaced by $W_{12}$ in $V_{12}$. The potential $V_{2}^{+}$ is given by the series

$$V_{2}^{+} = V_{2} + V_{2} \frac{\Lambda_{2}^{+}}{W_{12} - h_1 - h_2} V_{2} + ...$$

where $V_{2}$ is an external potential acting on fermion 2 (it is a Coulomb potential in QED, if we use the second-order constraint [17], but in general it could still depend on $(W_{12} - h_1)$). The potential $V_{1}^{+}$ is given by a similar formula.

The differences with the two-body approach of the two-body plus potential problem are:

- The projectors $\Lambda_{12}^{\pm}$ are the free ones.
- We have now projectors around the external potential terms $V_{2}^{+}, V_{2}^{+}$.
- These external potential themselves are now given by the series $[24]$.\n- $V_{2}$ could still depend on $(W_{12} - h_1)$ and vice-versa.

The cluster separability property in the three-body way survives the high-mass limit: switching off $V_{2}$ and $V_{1}$ leads to the equation for the $(12)$ two-fermion system, switching off $V_{12}$ and $V_{1}$ leads to a free Dirac equation for fermion 1 with the $\Lambda_{12}^{+}$ projection of the Dirac equation for fermion 2 in the external potential $V_{2}$. In the two-body approach of the two-fermion plus potential problem we had however more:
switching off only the mutual interaction led to a pair of independent equations for each fermion in the external potential. Here, in the three-body approach, the equation does not split perfectly into two parts: we can write \( W_{12} = W_1 + W_2 \), but we have \( W_{12} - h_1 \) instead of \( W_2 \) in the series defining \( V_2^+ \) and vice-versa. This is a consequence of the energy dependence introduced into (53) by the use of the anti-continuum dissolution projectors, combined with our choice of putting the spectator fermion on the mass shell in each two-body interaction, and neglecting the three-body terms which could balance this modification. The discrepancy is of order \( V^4 \). At the two-fermion plus potential level, a better but still not perfect equation would be obtained by replacing \( h_1 \) by \( h_1 + V_1 \) in the series defining \( V_2^+ \) and vice-versa.

5 Towards a perturbation calculation.

5.1 Bethe-Salpeter equivalent of our basic three-cluster equation.

The three-fermion Bethe-Salpeter equation can be written

\[
\Phi = \left[ G_{01} G_{02} K_{12} + G_{02} G_{03} K_{23} + G_{03} G_{01} K_{31} + G_{01} G_{02} G_{03} K_{123} \right] \Phi
\]

where \( K_{123} \) is given by the sum of the purely three-body irreducible contributions. We would like to get our basic three-cluster 3D equation by approximating the three-fermion Bethe-Salpeter equation, and to possibly recover the neglected contributions in a perturbation calculation afterwards. A three-fermion Bethe-Salpeter equation with instantaneous (i.e. independent of the relative energies) and positive-energy kernels can be transformed into a 3D Salpeter equation \([13]\). We shall thus neglect the three-body kernel and replace the two-body kernels by instantaneous positive-energy kernels (with the spectator fermions on their mass shell) which are equivalent at the cluster-separated limits:

\[
K_{12} \approx < K_{T12}(P_0 - h_3) > = -\frac{1}{2i\pi} \beta_1 \beta_2 \Lambda_{12}^{++} V_{12}(P_0 - h_3) \Lambda_{12}^{++}, \ldots.
\]

The Bethe-Salpeter equation becomes

\[
\Phi = -\frac{1}{2i\pi} G_{01} G_{02} G_{03} \beta_1 \beta_2 \beta_3 \left[ \Lambda_{12}^{++} V_{12} \Lambda_{12}^{++} \psi_{12} + \Lambda_{23}^{++} V_{23} \Lambda_{23}^{++} \psi_{23} + \Lambda_{31}^{++} V_{31} \Lambda_{31}^{++} \psi_{31} \right]
\]

where

\[
\psi_{ij}(p_{k0}) = \beta_k G_{0k}^{-1} \int dp_{j0} \Phi.
\]

This leads to a set of three coupled integral equations:

\[
\psi_{12}(p_{30}) = -\frac{1}{2i\pi} \int dp_{120} G_{01} G_{02} \beta_1 \beta_2 \left[ \Lambda_{12}^{++} V_{12} \Lambda_{12}^{++} \psi_{12}(p_{30}) + \Lambda_{23}^{++} V_{23} \Lambda_{23}^{++} \psi_{23}(p_{10}) + \Lambda_{31}^{++} V_{31} \Lambda_{31}^{++} \psi_{31}(p_{20}) \right]
\]

where \( p_{10}, p_{20} \) must be written in terms of \( P_0, p_{30}, p_{120} \):

\[
p_{10} = \frac{P_0 - p_{30}}{2} + p_{120}, \quad p_{20} = \frac{P_0 - p_{30}}{2} - p_{120}.
\]

and similarly for \( \psi_{23} \) and \( \psi_{31} \). We shall now search for solutions \( \psi_{ij}(p_{k0}) \) analytical in the lower \( \text{Im}(p_{k0}) < 0 \) half plane and perform the integration \([58]\) by closing the integration paths around these regions. The only singularities will then be the poles of the free propagators. The result is

\[
\psi_{12}(p_{30}) = \frac{\Lambda_{12}^{++}}{(P_0 - S) - (p_{30} - h_3) + i\epsilon} \left[ \Lambda_{12}^{++} V_{12} \Lambda_{12}^{++} \psi_{12}(p_{30}) + \Lambda_{23}^{++} V_{23} \Lambda_{23}^{++} \psi_{23}(h_1) + \Lambda_{31}^{++} V_{31} \Lambda_{31}^{++} \psi_{31}(h_2) \right], \ldots.
\]
Using these equations to compute the projections \(-2i\pi \Lambda_k^+ \psi_{ij}(h_k)\) we find that these three expressions are equal (let us call them \(\psi\)) and obey the \(\Lambda^{+++}\) projection of our three-cluster equation (48):

\[
\psi = \frac{\Lambda^{+++}}{P_0 - S} [V_{12} + V_{23} + V_{31}] \Lambda^{+++} \psi.
\]  

(61)

Furthermore, we have also

\[
\int dp_0 \Phi \equiv \int dp_0 dp_20 dp_{30} \delta(p_{10} + p_{20} + p_{30} - P_0) \Phi = \int dp_{k0} G_{0k} \beta_k \psi_{ij}(p_{k0})
\]

\[
= -2i\pi \Lambda_k^+ \psi_{ij}(h_k) = \psi.
\]  

(62)

Solving (60) with respect to \(\psi_{12}(p_{30})\) gives

\[
\psi_{12}(p_{30}) = \frac{1}{2i\pi} \frac{\Lambda^{+++}}{(P_0 - p_{30}) - (S_{12} + \Lambda_{12}^{++} V_{12} \Lambda_{12}^{++}) + i\epsilon} [V_{23} + V_{31}] \Lambda^{+++} \psi
\]  

(63)

which confirms the above assumption about the analyticity of the \(\psi_{ij}(p_{k0})\).

5.2 Relation with Faddeev formalism and Gross’ spectator model.

The nonrelativistic Faddeev equations can be obtained by transforming Schrödinger’s equation (in a first step, without three-body terms) into a set of three coupled equations for three parts \(T_{ij}\) of the transition operator (\(T_{ij}\) denotes the contribution of all graphs beginning by a (ij) interaction). The input is the set of the three two-body transition operators and the resulting series expansion contains only connected graphs (never twice the same two-body transition operator). This formalism is well adapted to the description of the various scattering processes, such as \((12) + 3 \rightarrow (12) + 3\) (elastic scattering), \((12) + 3 \rightarrow (12)^* + 3\) (excitation), \((12) + 3 \rightarrow 1 + 23\) (rearrangement), \((12) + 3 \rightarrow 1 + 2 + 3\) (breakup). The \((123)\) bound states correspond to the poles of this transition operator.

The structure of these equations can be generalized to relativistic equations which are not necessarily (exactly) reducible to a single 3D equation.

In Gross’ spectator model \([7, 8]\), Faddeev’s type equations are deduced from the Bethe-Salpeter equation and the transition operator is written in terms of the two-body transition operators. The relative time variables are then eliminated by putting, in each three-body propagator, all the “offmassshallness” on the only fermion which interacts before and after. The Lorentz invariance-cluster separability requirement is satisfied by applying suitable Lorentz boosts on the two-body transition operators.

The main difference between our approach and Gross’ approach comes from the fact that we are (presently) interested by the \((123)\) bound states only, to be computed principally by using a single 3D equation (such as the one from which Faddeev starts). Instead of working with the two-body transition operators, we work with the two-body potentials. We can however present our approach in terms of the two-body transition operators. Our approximation of each two-body transition operator is then unique and does not depend on the operators which come in front and behind it in the expansion of the Faddeev equations. This enables us to describe our model by a single 3D potential equation, by a set of three Faddeev equations, or by the expansion of the three-body transition operator in terms of the two-body ones. Another option of our model is to not introduce Lorentz boosts by hand: the Lorentz-invariance / cluster separability requirement is exactly satisfied if we do not truncate our potentials (practically, this means that the satisfaction of this requirement and the approximation of the potentials can be improved together). Let us go back to the three-fermion Bethe-Salpeter equation and neglect for simplicity the three-body irreducible kernel \(K_{123}\). Defining

\[
\Phi_{12} = G_{01} G_{02} K_{12} \Phi, \ldots \quad \Phi = \Phi_{12} + \Phi_{23} + \Phi_{31},
\]  

(64)

we get

\[
(1 - G_{01} G_{02} K_{12}) \Phi = \Phi_{23} + \Phi_{31}
\]  

(65)

\[
\Phi = (1 + G_{01} G_{02} T_{12}) (\Phi_{23} + \Phi_{31})
\]  

(66)

\[
\Phi_{12} = G_{01} G_{02} T_{12} (\Phi_{23} + \Phi_{31}).
\]  

(67)

Writing then

\[
\Phi_{12} = G_{01} G_{02} G_{03} \beta_1 \beta_2 \beta_3 \chi_{12}, \ldots
\]  

(68)
in order to factor out the propagators, we get
\[
\chi_{12} = \beta_1 \beta_2 T_{12} G_{01} G_{02} \beta_1 \beta_2 (\chi_{23} + \chi_{31}).
\] (69)

The (12) transition matrix element corresponding to our approximation (69) of \(K_{12}\) is
\[
T_{12}(p_{120}, p_{120}, p_0 - p_{30}) \approx T_{12}^0(p_{30})
\]
\[
= < K_{T12}(P_0 - h_3) > (1 - G_{01} G_{02} < K_{T12}(P_0 - h_3) > )^{-1}.
\] (70)

This \(T_{12}^0(p_{30})\) is analytical in the \(\text{Im}(p_{30}) < 0\) half plane and
\[
T_{12}^0(h_3) = < T_{12}(P_0 - h_3) > .
\] (71)

This approximation, combined with equation (69), implies that \(\chi_{12}(p_{120}, p_{30})\) is independent of \(p_{120}\) (let us write thus \(\chi_{12}(p_{30})\)). Equation (69) becomes then
\[
\chi_{12}(p_{30}) = \beta_1 \beta_2 T_{12}^0(p_{30}) \int dp_{120} G_{01} G_{02} \beta_1 \beta_2 [\chi_{23}(p_{10}) + \chi_{31}(p_{20})]
\] (72)

where \(p_{10}, p_{20}\) must be written in terms of \(P_0, p_{30}, p_{120}\). Equation (72) together with similar equations for \(\chi_{23}(p_{10})\) and \(\chi_{31}(p_{20})\), admits solutions which are analytical in the \(\text{Im}(p_{k0}) < 0\) half planes. At \(p_{k0} = h_k\), we get the 3D Faddeev equations
\[
\chi_{12}(h_3) = T_{12}^{3D}(P_0 - h_3) \frac{1}{P_0 - S} [\chi_{23}(h_1) + \chi_{31}(h_2)],
\] (73)

which can easily be transformed back into our basic three-cluster potential equation (by performing in the reverse order the transformations made above at the 4D level). Note that we would get the same result by approaching directly \(T_{12}\) by \(T_{12}^0(h_3)\) instead of \(T_{12}^0(p_{30})\). This means approaching \(K_{12}\) by a given function \(K_{12}^0(p_{30})\), analytical in the \(\text{Im}(p_{30}) < 0\) half plane and equal to \(< K_{T12}(P_0 - h_3) >\) at \(p_{30} = h_3\).

The key point of the manipulations above is the dominance of the positive-energy poles of \(G_{01} G_{02}\) in (38). This was obtained by approaching \(K_{T12}\) (or \(T_{12}\)) by a constant with \(\Lambda_{12}^{++}\) positive-energy projectors. We could try to insure the dominance of these two poles more economically. Let us come back to equation (38) without making any approximation on \(T_{12}\). The elements of (69) are then the operator and functions
\[
T_{12}(p_{120}, p_{120}, P_0 - p_{30})
\] (74)
\[
\chi_{12}(p_{120}, p_{30}), \chi_{23}(P_{230}, p_{10}), \chi_{31}(p_{310}, p_{20})
\] (75)

and (69) must be integrated with respect to \(p_{120}\), with \(p_{30}\) fixed. We must thus write \(p_{230}, p_{10}, p_{310}, p_{20}\) in terms of \(p_{120}, p_{30}\). Searching as above for solutions \(\chi_{ij}(p_{j0}, p_{k0})\) which are analytical in the \(\text{Im}(p_{k0}) < 0\) half planes, we shall close our integration path clockwise (counterclockwise) in front of \(\chi_{23}(\chi_{31})\) and keep thus the pole of \(G_{01}(G_{02})\), which puts fermion 1 (2) on its positive-energy mass shell. The elements of (69) are then replaced by
\[
T_{12}(p_{120}, P_0 - p_{30}) \to T_{12}(p_{120}, s_{12} = \frac{P_0 - S}{2} \pm \frac{p_{30} - h_3}{2}, P_0 - p_{30})
\] (76)
\[
\int dp_{120} G_{01} G_{02} \beta_1 \beta_2 \to \frac{-2i\pi}{(P_0 - S) - (p_{30} - h_3) + i\epsilon}
\] (77)
\[
\chi_{23}(p_{230}, p_{10}) \to \chi_{23}(s_{23} + \frac{P_0 - S}{2} - (p_{30} - h_3), h_1)
\] (78)
\[
\chi_{31}(p_{310}, p_{20}) \to \chi_{31}(s_{31} - \frac{P_0 - S}{2} + (p_{30} - h_3), h_2)
\] (79)

where we take the upper signs in (76) in front of \(\chi_{23}\), the lower signs in front of \(\chi_{31}\).

The manipulations above are submitted to some restrictions on the \(T_{ij}\). In the \(p_{k0}\) variable the \(T_{ij}\) must be analytical in the \(\text{Im}(p_{k0}) < 0\) half planes. In the \(p_{ij0}\) and \(p_{ij0}\) variables the \(T_{ij}\) must be analytical in the whole complex plane. Moreover, the \(T_{ij}\) must also be asymptotically bounded in the
three variables and contain $\Lambda_{ij}^{++}$ positive-energy projectors. These conditions are of course not satisfied by the exact transition matrix elements, but we shall assume that the singularities and the negative-energy parts of the $T_{ij}$ can be neglected in the computation of the integrals with respect to the relative times.

If we take then the equation for $\chi_{12}$ at $p_{30} = h_3$ and $p_{120} = s_{12}^+ = s_{12} \pm \frac{i}{2}(P_0 - S)$, etc..., we get a closed system of six 3D equations:

$$\chi_{12}^+ = -2i\pi \beta_1 \beta_2 \left( T_{12}^{++} - \frac{1}{P_0 - S} \chi_{23}^+ + T_{12}^{++} - \frac{1}{P_0 - S} \chi_{31}^+ \right)$$

and similarly for $\chi_{23}^+$ and $\chi_{31}^+$.

In each $T_{ij}$ two fermions are on the mass shell: the spectator fermion $k$ and the fermion which is not going to interact next left ($p'_{ij0}$) or next right ($p_{ij0}$). This is of course the philosophy of Gross’ spectator model [17, 18].

If we replace $s^\pm$ by $s$ in the $T_{ij}$ we get our basic three-cluster model. This supplementary approximation can be introduced by noticing that we already neglected the contributions of the singularities of $T_{ij}$ in the $p'_{ij0}$ and $p_{ij0}$ complex planes. The dependence on these variables can thus not be very strong, as we know that a function which is analytical and bounded on the whole complex sphere must necessarily be a constant. Using the same argument of the consistency of the approximations, we could also argue that $T_{ij}$ constant implies $K_{Tij}$ and $K_{ij}$ constant with $K_{Tij} = K_{ij}$. Our basic three-cluster equation is thus not a priori a worse approximation of the three-fermion Bethe-Salpeter equation than Gross’ equations, nor a better approximation than the simpler positive-energy instantaneous approximation (55).

We made several attempts to build a 3D perturbation calculation starting with our basic three-cluster model. Some of them led to interesting results, but not to the kind of 3D perturbation terms we wanted. In the two-fermion problem, a perturbation expansion can be built around an approximation of the propagator (as in section 2), of the kernel (as in [38] for example), or of both. Our approximation [38] for the three-fermion Bethe-Salpeter equation is an approximation of the kernels: the three-fermion kernel is neglected and the two-fermion kernels are replaced by series based on approximations of the two-fermion propagators. In order to transform the correction terms, we shall write, instead of (55),

$$\delta(p'_{30} - p_{30}) K_{12}(p'_{120}, p_{120}, P_0 - p_{30})$$

$$\approx \delta(p'_{30} - h_3) \Lambda_{3}^3 < K_{T12}(P_0 - h_3) > \equiv K_{12}^{0}(p'_{30}).$$

This gives directly the $(+++)_{++}$ projection of our basic three-cluster equation for the integral (32) of the Bethe-Salpeter amplitude with respect to the relative times. The unapproximated Bethe-Salpeter equation takes the form

$$\Phi = [G_{01}G_{02}K_{12}^{0} + G_{02}G_{03}K_{23}^{0} + G_{03}G_{01}K_{31}^{0} + R] \Phi$$

and leads to the 3D equation

$$\psi = \int dp_0 \Lambda_{++}^{+++} (1 - R)^{-1} \left[ G_{01}G_{02}K_{12}^{0} + G_{02}G_{03}K_{23}^{0} + G_{03}G_{01}K_{31}^{0} \right] \psi$$

5.3 Correction terms.

The contributions neglected above in the transformation of the Bethe-Salpeter equation into a three-cluster 3D equation are given at the Bethe-Salpeter level. We would like to transform them corrections to the 3D equation. However, the exact Bethe-Salpeter equation has no more the simple analyticity structure in the relative energies which allows the 3D reduction at the positive-energies instantaneous approximation [38].

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$$\approx \delta(p'_{30} - h_3) \Lambda_{3}^3 < K_{T12}(P_0 - h_3) > \equiv K_{12}^{0}(p'_{30}).$$

This gives directly the $(+++)_{++}$ projection of our basic three-cluster equation for the integral (32) of the Bethe-Salpeter amplitude with respect to the relative times. The unapproximated Bethe-Salpeter equation takes the form

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and leads to the 3D equation

$$\psi = \int dp_0 \Lambda_{++}^{+++} (1 - R)^{-1} \left[ G_{01}G_{02}K_{12}^{0} + G_{02}G_{03}K_{23}^{0} + G_{03}G_{01}K_{31}^{0} \right] \psi$$
This implies immediately that the terms ending like \( R \) calculation). For positive-energy instantaneous two-body kernels, we have although the "approximation" (82) led to our basic three-cluster equation, equation (86) could turn out to vanish. This is necessary if we want it to be of higher-order in the general case, but it is not automatic: approximation of positive-energy instantaneous two-body kernels, as (55), we expect \( \hat{\Lambda} \) in this last form the correction terms are more symmetric and less total energy dependent. At the\( \chi \) with \( R \): In terms of \( \chi = (1 - \hat{R})\psi \):

\[
(P_0 - S)\chi = \Lambda^{+++} (V_{12} + V_{23} + V_{31}) \Lambda^{+++} (1 - \hat{R})^{-1} \chi. \tag{86}
\]

In this last form the correction terms are more symmetric and less total energy dependent. At the approximation of positive-energy instantaneous two-body kernels, as (55), we expect \( \hat{\Lambda} \) to vanish. This is necessary if we want it to be of higher-order in the general case, but it is not automatic: although the "approximation" (82) led to our basic three-cluster equation, equation (86) could turn out to be a complicated equivalent of it (we met this situation in previous promising attempts of perturbation calculation). For positive-energy instantaneous two-body kernels, we have

\[
R = G_{01}G_{02}K_{12}^R + G_{02}G_{03}K_{23}^R + G_{03}G_{01}K_{31}^R = R_{12} + R_{23} + R_{31} \tag{87}
\]

\[
K_{12}(p_{30}, p_{30}) = K_{12} \left[ \delta(p_{30} - p_{30}) - \Lambda_3^+ \delta(p_{30} - h_3) \right]. \tag{88}
\]

This implies immediately that the terms ending like \( R_{12}G_{01}G_{02}K_{12}^0 \) are zero. For terms ending like \( R_{12}G_{02}G_{03}K_{23}^0 \) we have to examine the action of four possible operators at left:

\[
\int dp_0(...), \ R_{23}, \ R_{31}, \ R_{12}. \tag{89}
\]

The three first ones lead to contour integrals which give zero. With \( R_{12} \) we must perform the \( p_{120} \) integral and then apply again one of the four operators (82) at left, etc... This recurrence ends necessarily with one of the first three operators (82) which gives zero again.

In the general case our correction terms are thus really higher-order terms. Two aspects remain however to be investigated:

– It seems that our method brings corrections to the two-fermion potentials as well. At the cluster-separated limits, our two-fermions equations would then be replaced by equivalent equations, unless we could show that these two-fermion correction terms vanish.

– Our three-fermion correction terms are a priori not hermitian. Furthermore, they are total energy dependent, like the unperturbed potentials. As explained in the introduction, these two difficulties are not independent.

### 6 Conclusions

We have written a 3D equation for three fermions (our basic three-cluster equation) by combining the three two-body potentials obtained by an exact 3D reduction of the corresponding two-fermion Bethe-Salpeter equations, putting the spectator fermion on the mass shell. In this way, the cluster-separated limits are still exact, and the Lorentz invariance / cluster separability requirement is automatically satisfied, provided no supplementary approximation, like the Born approximation, is made. This equation can be written in terms of the two-body potentials, or in terms of two-body transition operators (Faddeev formalism). The use of positive free-energy projectors in the chosen reductions of the two-fermion Bethe-Salpeter equations prevents our 3D three-fermion equation from continuum dissolution. The potentials are hermitian and depend only slowly on the total three-fermion energy. The one high-mass limits of our "basic three-cluster equation" are approximately exact. The correction of the remaining discrepancy would demand the introduction of higher-order three-body terms.

Our combination of cluster separability and Lorentz invariance in the three fermion problem makes explicit use of the fact that the clusters can only be two-fermion and/or free fermion states. This is not directly adaptable to four and more fermion systems. In this respect, 3 is still not N.

The difference between our equation and Gross’ spectator model equations consists in higher-order three-body contributions. Supplementary investigations would be necessary to decide which approach provides the best approximation to the three-fermion Bethe-Salpeter equation.
A lot of work remains to be done in order to precise a possible perturbation calculation program, correcting the remaining discrepancies with the three-fermion Bethe-Salpeter equation. We succeeded in deriving our 3D equation from an approximation of the three-fermion Bethe-Salpeter equation, in which the three-body kernel is neglected while the two-body kernels are approached by positive-energy instantaneous expressions, with the spectator fermion on the mass shell. The correction terms are thus known at the Bethe-Salpeter level and can be transformed into corrections to the 3D equation, unfortunately not hermitian a priori.

There exists an infinity of ways of performing the 3D reduction of the two-fermion Bethe-Salpeter equation. The potentials generated by these reductions could all by used to build a three-fermion 3D equation, keeping however in mind the continuum dissolution problem. This leaves us a large freedom to suit phenomenological needs.

Our two-body potentials are the sum of an infinity of contributions symbolized by Feynman graphs. Keeping only the first one (Born approximation) or a finite number of them renders the Lorentz covariance of the two-fermion clusters only approximate. A Born approximation preserving the Lorentz invariance / cluster separability property can be obtained by using another 3D reduction based on a covariant second-order two-body propagator of Sazdjian, combined with a covariant substitute of $\Lambda^{++}$. This leads to a 3D three-cluster equation which is covariantly Born approximable, but more complicated \cite{17}.

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