Bifurcations in a convection problem with
temperature-dependent viscosity

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Abstract

A convection problem with temperature-dependent viscosity in an infinite layer is presented. As described, this problem has important applications in mantle convection. The existence of a stationary bifurcation is proved together with a condition to obtain the critical parameters at which the bifurcation takes place. For a general dependence of viscosity with temperature a numerical strategy for the calculation of the critical bifurcation curves and the most unstable modes has been developed. For an exponential dependence of viscosity on temperature the numerical calculations have been done. Comparisons with the classical Rayleigh-Bénard problem with constant viscosity indicate that the critical threshold decreases as the exponential rate parameter increases.

1 Introduction

A classical subject in fluid mechanics is the problem of thermoconvective instabilities in fluid layers driven by a temperature gradient \cite{2,10,11}. Gravity and/or capillary forces are responsible for the onset of motion when the temperature difference exceeds a certain threshold. The existence of a stationary bifurcation for the gravity driven case (the classical Rayleigh-Bénard problem) has been rigorously proved in \cite{4,11}. Above the threshold convective solutions consist of rolls \cite{10}. Most studies on the Rayleigh-Bénard problem consider a constant viscosity \cite{13,9,10}, although interest on convection problems with temperature dependent viscosity has increased \cite{19,3,17,6,7,14,20} due to the fact that this dependence is a fundamental feature of mantle convection. The formation of rocks indicates that viscosity in the interior of the Earth strongly depends on temperature, and this influences the thermal evolution of the mantle \cite{8,5}. Some theoretical or numerical studies with temperature dependent viscosity in finite boxes can be found for instance in Refs. \cite{19,17,16,12}. In \cite{18} several types of dependence have been used, but the most common is an exponential dependence \cite{14,7}.

This paper plans a theoretical and numerical study of the instabilities in a gravity driven convection problem with viscosity dependent on temperature. The domain, which is finite in the vertical coordinate and infinite in the horizontal plane, contains a viscous fluid which is heated uniformly from below. In this work, for a general dependence of viscosity on temperature, we prove the existence of a stationary bifurcation as the temperature gradient increases. And a condition to obtain the critical parameters at which the bifurcation takes place has been found. For a viscosity with an exponential dependence on temperature both the critical bifurcation
curves and the most unstable modes have been numerically computed. Comparisons with the classical Rayleigh-Bénard problem with constant viscosity indicate that the critical threshold decreases as the exponential rate parameter increases. We find the critical bifurcation curves for this case as a function of the exponential parameter.

The paper is organized as follows. In section 2 the formulation of the problem with the equations and boundary conditions (bc) and the conductive solution are presented. Section 3 develops the linear stability of the conductive solution and deduces the theoretical results on the existence and conditions of bifurcation. In section 4 numerical results on eigenmodes and critical curves are presented. Finally in Section 5 conclusions are detailed.

2 Formulation of the problem

The considered physical set-up is shown in figure 1. A fluid layer of depth \(d\) (\(z\) coordinate) is placed between two parallel boundless plates. On the bottom plate a temperature \(T_0\) is imposed and on the upper plate a temperature \(T_1 = T_0 - \Delta T = T_0 - \beta d\), where \(\beta\) is the vertical temperature gradient. The governing equations are the continuity equation

\[ \nabla \cdot \vec{v} = 0, \]  

the Navier-Stokes equations

\[ \partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\rho g \vec{e}_3}{\rho_0} - \frac{1}{\rho_0} \nabla P + \text{div}(\nu(T) \cdot (\nabla \vec{v} + (\nabla \vec{v})^T)), \]  

and the heat equation,

\[ \partial_t T + \vec{v} \cdot \nabla T = K \Delta T. \]  

Here \(\vec{v} = (v_x, v_y, v_z)\) \((x, y, z, t)\) is the velocity vector field, \(P\) is the pressure, \(T\) the temperature field, \(\rho\) is the density, which dependence on temperature is assumed to be \(\rho = \rho_1(1 - \alpha(T - T_1))\), being \(\rho_1\) the mean density at temperature \(T_1\), \(\alpha\) is the thermal expansion, \(K\) is the thermal diffusion, \(g\) is the acceleration of the gravity and \(\vec{e}_3\) is the unitary vector in the vertical direction. \(\nu(T)\) is the viscosity that is assumed to be a positive, convex and bounded function of temperature \(T\).

Now, the boundary conditions are introduced. The temperature on the bottom plate, \(z = -d/2\), and on the upper plate, \(z = d/2\), are respectively

\[ T \left( z = -\frac{d}{2} \right) = T_0, \quad T \left( z = \frac{d}{2} \right) = T_1. \]  

The boundary conditions for the velocity correspond to rigid, no-slip at the bottom plate and non-deformable and free-slip at the upper plate. They are expressed as follows,

\[ v_x = v_y = v_z = 0 \quad \text{on} \ z = -\frac{d}{2}, \quad \partial_z v_x = \partial_z v_y = v_z = 0 \quad \text{on} \ z = \frac{d}{2}. \]  

The simpler solution of the hydrodynamic equations is the conductive solution. For that solution the temperature only depends on the vertical component and the fluid is at rest,

\[ \partial_x T = \partial_y T = \partial_t T = 0, \quad \vec{v} = \vec{0}. \]  

Under these assumptions the hydrodynamic equations become,

\[ 0 = -\rho g \vec{e}_3 - \nabla P, \]  

\[ 0 = K \Delta T, \]  

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which solution is the conductive state

\[ T_b(z) = T_o - \beta \left( z + \frac{d}{2} \right), \quad (9) \]

\[ P_b(z) = P_o - \rho_o g \left[ z \left( 1 + \frac{\alpha \beta d}{2} \right) + \frac{\alpha \beta z^2}{2} \right], \quad (10) \]

\[ \vec{v}_b = \vec{0}, \quad (11) \]

where \( P_o \) is an arbitrary constant.

### 3 Linear stability of the conductive solution

Now, we are interested in the study of the linear stability analysis for the conductive solution. Therefore, we perturb it as follows,

\[ T(x, y, z, t) = T_b(z) + \theta(x, y, z, t), \quad (12) \]

\[ P(x, y, z, t) = P_b(z) + \delta P(x, y, z, t), \quad (13) \]

\[ \vec{v}(x, y, z, t) = \vec{v}_b + \vec{u}(x, y, z, t), \quad (14) \]

and introducing these expressions into the hydrodynamic equations the following equations for the perturbation fields are obtained,

\[ \nabla \cdot \vec{u} = 0, \]

\[ \partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = \alpha \theta g \vec{e}_3 - \frac{1}{\rho_o} \nabla (\delta P) + \text{div}(\nu(T) \cdot (\nabla \vec{u} + (\nabla \vec{u})^t)), \quad (15) \]

\[ \partial_t \theta + (\vec{u} \cdot \nabla) \theta = K \Delta \theta + \beta u_z. \]

Taking into account the following expressions,

\[ \nu(T) = \nu(T_b + \theta) = \nu(T_b) + \nu'(T_b) \theta + O(\theta^2), \]

\[ \nu(T_b) = \nu(T_0 - \beta(z + d/2)) = \nu(z), \]

\[ \text{div}(\nu(T) \cdot (\nabla \vec{u} + (\nabla \vec{u})^t)) = \tilde{L}\vec{u} + N(\vec{u}, \theta), \]

where \( \tilde{L}\vec{u} \) is the linear part and \( N(\vec{u}, \theta) \) includes the nonlinear terms as follows

\[ \tilde{L}\vec{u} = \text{div} \left( \nu(z) \cdot (\nabla \vec{u} + (\nabla \vec{u})^t) \right), \]

\[ N(\vec{u}, \theta) = \text{div} \left( (\nu'(T_b) \theta + O(\theta^2)) \cdot (\nabla \vec{u} + (\nabla \vec{u})^t) \right). \]

Linearizing eqs. (15) the following system of equations is obtained

\[ \nabla \cdot \vec{u} = 0, \]

\[ \partial_t \vec{u} = \alpha \theta g \vec{e}_3 - \frac{1}{\rho_o} \nabla (\delta P) + \tilde{L}\vec{u}, \quad (16) \]

\[ \partial_t \theta = K \Delta \theta + \beta u_z, \]

where displaying the operator \( \tilde{L}\vec{u} = \partial_z (\nu(z)\partial_z \vec{u}) + \nu(z)\Delta' \vec{u} + \nu'(z)\nabla u_z \) and \( \Delta' \) is the laplacian operator in the variables \( x \) and \( y \).

The system (16) together with its boundary conditions may be expressed in dimensionless form defining new variables \( \vec{r}' = \vec{r}/d, \vec{t}' = Kt/d^2, \vec{u}' = d\vec{u}/K, p' = d^2 p/(\rho_o K \nu_o), \theta' = (\theta - T_1)/\beta d. \)

After rescaling the variables and dropping the primes the system is left as follows,

\[ \nabla \cdot \vec{u} = 0, \]

\[ \frac{1}{Pr} \partial_t \vec{u} = -\nabla (\delta P) + \tilde{L}\vec{u} + R\theta \vec{e}_3, \quad (17) \]

\[ \partial_t \theta = \Delta \theta + u_z. \]
Here, the Rayleigh number is defined as \( R = d^4 \alpha g \beta / (\nu_0 K) \) and the Prandtl number is \( Pr = \nu_0 / K \), where \( \nu_0 = \nu(-1/2) \).

The dimensionless boundary conditions for fields in (17) are

\[
\theta = u_x = u_y = u_z = 0 \text{ on } z = -1/2, \quad \theta = \partial_z u_x = \partial_z u_y = u_z = 0 \text{ on } z = 1/2. \quad (18)
\]

### 3.1 Decomposition into normal modes

As the domain is infinite in the plane, the eigenvalue problem can be solved by using the normal modes method. For each mode a separate variable solution can be found. We look for solutions as follows,

\[
\begin{pmatrix}
  u_x \\
  u_y \\
  u_z \\
  \theta \\
  \delta P
\end{pmatrix} =
\begin{pmatrix}
  U_x(z) \\
  U_y(z) \\
  W(z) \\
  \Theta(z) \\
  \delta p(z)
\end{pmatrix} e^{i(\vec{k})t+i(k_x x+k_y y)},
\]

where \( \sigma(\vec{k}) \) are the eigenvalues, that initially can be complex numbers. It represents the growing rate of the normal mode at the wave number \((k_x, k_y) = \vec{k} \). If the solution belongs to \( L^2 \) in \( k_x, k_y \), the inverse Fourier transform of the normal modes can be considered the solution of the linear system.

Introducing the solution (19) into equations (17) we obtain the following equations:

- the continuity equation,
  \[
i (k_x U_x + k_y U_y) + DW = 0, \quad (20)
\]
- the Navier-Stokes equations,
  \[
  \frac{\sigma}{Pr} U_x = -ik_x \delta p + \mathcal{P} U_x + ik_x \nu'(z) W, \quad (21)
  \frac{\sigma}{Pr} U_y = -ik_y \delta p + \mathcal{P} U_y + ik_y \nu'(z) W, \quad (22)
  \frac{\sigma}{Pr} W = -D(\delta p) + \mathcal{P} W + \nu'(z) DW + R \Theta, \quad (23)
\]
- the heat equation,
  \[
  \sigma \Theta = (D^2 - k_x^2) \Theta + W, \quad (24)
\]

where the operators \( \mathcal{P} f = D(\nu D f) - \nu k^2 f \) and \( D f = df/dz \) and the wave number \( k^2 = k_x^2 + k_y^2 \) have been introduced.

The boundary conditions become

\[
\Theta = U_x = U_y = W = 0 \text{ on } z = -1/2; \quad \Theta = DU_x = DU_y = W = 0 \text{ on } z = 1/2. \quad (25)
\]

**Lemma 1** If \((U_x, U_y, W, \Theta, \delta P)\), such that \((U_x, U_y, W) \in H^2\), is a solution of problem (20) with b.c. (24), then it is a solution of the following problem

\[
\frac{\sigma}{Pr} Z = \nu(z) \left[ \left( D^2 - |k|^2 \right) Z \right] + \nu'(z) \cdot DZ, \quad (26)
\]
\[
\sigma \Theta = \left( D^2 - |k|^2 \right) \Theta + W, \quad (27)
\]
\[
\frac{\sigma}{Pr} (D^2 - |k|^2) W = -R |k|^2 \Theta - |k|^2 \mathcal{P} W + D(\mathcal{P}(DW)) + |k|^2 \nu''(z) W, \quad (28)
\]
\[
Z = W = DW = \Theta = 0, \text{ on } z = -1/2; \quad Z = W = D^2 W = \Theta = 0, \text{ on } z = 1/2. \quad (29)
\]

Where \( Z(z) = U_y(z) i k_x - U_x(z) i k_y \), and the velocity functions \( U_x \) and \( U_y \) can be calculated from \( W \) and \( Z \).
Proof. Pressure is perfectly determined by the velocity \((U_x, U_y, W)\) and the temperature \(\Theta\), so we proceed to eliminate the pressure applying \((ik_x, ik_y, D) \times (U_x, U_y, W)\) into the Navier-Stokes equations \((21-23)\) hence \(ik_x \frac{\sigma}{Pr} U_y - ik_y \frac{\sigma}{Pr} U_x\) leads to \((24)\).

On the other hand, from the equations \((21-22)\) and together the continuity equation \((20)\) we can obtain
\[- \frac{\sigma}{Pr} DW = |k|^2 \delta P - \mathcal{P}(DW).\] (30)

Hence one deduces the decoupled equation
\[\frac{\sigma}{Pr} |k|^2 W = \mathcal{P} |k|^2 W + R |k|^2 \Theta - D \left( \mathcal{P}(DW) - \frac{\sigma}{Pr} DW \right) - |k|^2 \nu''(z) W,\] (31)
which rewrites as \((28)\) and therefore the final system \((26-28)\) is obtained.

Finally, if we define the velocity functions \(U_x, U_y\) as,
\[U_x = ik_x \phi - ik_y \psi, \quad U_y = ik_y \phi + ik_x \psi,\] (32)
using the continuity equation \((20)\) and knowing that \(Z(z) = U_y(z) ik_x - U_x(z) ik_y\), we obtain that \(\phi(z)\) and \(\psi(z)\) are functions of \(W(z)\) and \(Z(z)\) respectively,
\[
\begin{align*}
U_x (z) &= i \left[ k_x \left( \frac{DW(z)}{k^2} \right) + k_y \left( \frac{Z(z)}{k^2} \right) \right], \\
U_y (z) &= i \left[ k_y \left( \frac{DW(z)}{k^2} \right) - k_x \left( \frac{Z(z)}{k^2} \right) \right]
\end{align*}
\] (33)
so, we can reduce the system \((20-24)\) in \((26-28)\) without the terms with \(U_x\) and \(U_y\). ■

3.2 Real eigenvalues

Theorem 2 The eigenvalues for problem \((26-29)\) are real.

Proof. Notice that, for any function \(f \in H^1_0 ([-\frac{1}{2}, \frac{1}{2}])\), we have the identity
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{P} f \overline{f} dz = -k^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \nu(z) \left[ |d_h f|^2 + |f|^2 \right] dz
\]
as well as the identity
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} (d_h^2 - 1) f \overline{f} dz = - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ |d_h f|^2 + |f|^2 \right] dz,
\]
where \(d_h = hD = D/k = d/dz/k\). We consider the equations \((27-28)\) and write them as follows,
\[
\begin{align*}
\sigma \Theta (z) &= k^2 (d_h^2 - 1) \Theta (z) + W(z), \\
-R \Theta (z) &= \frac{\sigma}{Pr} (d_h^2 - 1) W(z) - k^2 d_h^2 (\nu(z) d_h W(z)) + 2k^2 d_h (\nu(z) d_h W(z)) - k^2 (d_h (\nu(z) W(z)) - k^2 d_h (\nu(z) W(z)),
\end{align*}
\] (34)

where we have cleared the temperature function and the derivatives \(d_h^1 = h_D j = D_j/k_j = d_j/dz_j/k_j\).

\[
\int W \Theta dz = \sigma \int |\Theta|^2 dz + k^2 \int \left( |\Theta|^2 + |d_h \Theta|^2 \right) dz.
\] (36)
On the other hand, we consider the complex conjugate in the equation (35), we multiply it by \( W(z) \) and integrate, obtaining

\[
\int W \Theta dz = \frac{\pi}{PrR} \int \left( |W|^2 + |d_h W|^2 \right) dz + \frac{k^2}{R} \int \nu(z) \left( |W|^2 + 2|d_h W|^2 + |d_h^2 W|^2 + d_h^2 (|W|^2) \right) dz.
\]

(37)

We thus deduce

\[
\sigma \int |\Theta|^2 dz + k^2 \int (|\Theta|^2 + |d_h \Theta|^2) dz = \frac{\sigma}{PrR} \int \left( |W|^2 + |d_h W|^2 \right) dz + \frac{k^2}{R} \int \nu(z) \left( |W|^2 + 2|d_h W|^2 + |d_h^2 W|^2 + d_h^2 (|W|^2) \right) dz.
\]

(38)

Considering the imaginary part in relation (38) we obtain

\[
\text{Im} \sigma \left( \int |\Theta|^2 dz + \frac{1}{PrR} \int \left( |W|^2 + |d_h W|^2 \right) dz \right) = 0.
\]

(39)

So, if \( \sigma \) is not real, we deduce that \( \Theta \) and \( W \) are zero, hence is a trivial solution. The theorem is proved.

3.3 Existence of the eigenvalues

Before going into the proof of existence of eigenvalues, some previous lemmas are necessary. In what follows, we shall make use of the following inequalities:

**Lemma 3** For \( f \in H^1_0((-\frac{1}{2}, \frac{1}{2})) \), we have the Poincaré inequality as follows

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} |d_h f|^2 dz \geq \frac{\pi^2}{k^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f|^2 dz,
\]

(40)

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \nu(z) |d_h f|^2 dz \geq \frac{\delta_1}{k^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \nu(z) |f|^2 dz,
\]

(41)

where \( \delta_1 \) is a constant depending on \( \nu(z) \).

**Proof.** The inequality (40) is classical. It is obtained through the minimization of \( \int_{-\frac{1}{2}}^{\frac{1}{2}} |Df|^2 dz \) under the constraint \( \int_{-\frac{1}{2}}^{\frac{1}{2}} |f|^2 dz \) for \( f \) element of \( H^1_0((-\frac{1}{2}, \frac{1}{2})) \), the solution of this problem being given by the function \( \cos \pi z \), hence the inequality. On the other hand, we can deduce using (40) and \( \nu_{\min} \leq \nu(z) \leq \nu_{\max} \) the inequality as follows

\[
\int_{\Omega} \nu(z) |d_h f(z)|^2 dz \geq \nu_{\min} \int_{\Omega} |d_h f(z)|^2 dz \geq \frac{\pi^2 \nu_{\min}}{k^2} \int_{\Omega} |f(z)|^2 dz \geq \frac{\pi^2 \nu_{\min}}{k^2 \nu_{\max}} \int_{\Omega} \nu(z) |f(z)|^2 dz.
\]

(42)

So Lemma 3 is proved.

Now we introduce \( \hat{\sigma} = \sigma/k^2 \).

**Lemma 4** \( \forall \hat{\sigma} \in \mathbb{R} \) there exists \( k \in \mathbb{R} \) such that \( R(\hat{\sigma}, k) > 0 \).
Proof. Considering $\Omega = [-\frac{1}{2}, \frac{1}{2}]$ and using the Lemma 3 and the Hölder inequalities we obtain from the heat equation (42)

\[
\left(\tilde{\sigma} + 1 + \frac{\pi^2}{k^2}\right)\|\tilde{\Theta}\|_{L^2(\Omega)} \leq \|W\|_{L^2(\Omega)},
\]

where $\tilde{\Theta} = k^2\Theta$. From the decoupled equation (28),

\[
\left(\frac{\tilde{\sigma}}{Pr} + \min_{[-1/2,1/2]} \nu(z) \left(1 + \frac{\delta_1}{k^2}\right) + \min_{[-1/2,1/2]} d_h^2(\nu(z))\right)\|W\|_{H^1_0(\Omega)}^2 \leq \bar{R}\|\tilde{\Theta}\|_{L^2(\Omega)}\|W\|_{L^2(\Omega)},
\]

where $\bar{R} = R/k^4$. Hence we get the estimate

\[
\left(\frac{\tilde{\sigma}}{Pr} + \min_{[-1/2,1/2]} \nu(z) \left(1 + \frac{\delta_1}{k^2}\right) + \min_{[-1/2,1/2]} d_h^2(\nu(z))\right)\left(\tilde{\sigma} + 1 + \frac{\pi^2}{k^2}\right)\|W\|_{L^2(\Omega)}^2 \leq \bar{R}\|W\|_{L^2(\Omega)}^2.
\]

We must thus deduce from (45) that if $\tilde{\sigma} \geq 0$ then

\[
\left(\min_{[-1/2,1/2]} \nu(z) \left(1 + \frac{\delta_1}{k^2}\right) + \min_{[-1/2,1/2]} d_h^2(\nu(z))\right)\left(1 + \frac{\pi^2}{k^2}\right)\|W\|_{L^2(\Omega)}^2 \leq \bar{R}\|W\|_{L^2(\Omega)}^2.
\]

hence, using that $k > 0$

\[
R^*(k) = \left(\min_{[-1/2,1/2]} \nu(z)(k^2 + \delta_1) + k^2 \min_{[-1/2,1/2]} d_h^2(\nu(z))\right)(k^2 + \pi^2) \leq R(k),
\]

so $R(k) > 0$ because $d_h^2(\nu(z)) > 0 \forall z \in [-1/2,1/2]$. On the other hand, if $\sigma < 0$ we consider from (10) that

\[
|\tilde{\sigma}| < \min \left\{ \min_{[-1/2,1/2]} \nu(z) \left(1 + \frac{\delta_1}{k^2}\right), \min_{[-1/2,1/2]} d_h^2(\nu(z)), \left(1 + \frac{\pi^2}{k^2}\right) \right\}
\]

and we also obtain that $R(k) > 0$. 

Define the operator $\mathcal{M} = (1 - d_h^2 + \tilde{\sigma})$. We have

Lemma 5 Any solution of problem (20,26) satisfies the boundary conditions:

\[
\mathcal{M}^{1/2}\tilde{\Theta}(\pm \frac{1}{2}) = 0.
\]

Proof. First of all, we identify the eigenvalues and eigenvectors of the operator $\mathcal{M} = (1 - d_h^2 + \tilde{\sigma})$. For this purpose, we solve

\[
(1 + \tilde{\sigma} - d_h^2)T = \lambda T
\]

with boundary conditions $T(\pm \frac{1}{2}) = 0$. This rewrites

\[
(d_h^2 + \lambda - (1 + \tilde{\sigma}))\Theta = 0.
\]

As the equation $(d_h^2 - \omega^2)u = 0$ has not non-trivial solution in $H^1_0$ for $\omega$ real, we have necessarily

\[
\lambda > (1 + \tilde{\sigma}).
\]

We introduce $\omega$ such that $\omega^2 = \lambda - 1 - \tilde{\sigma}$. We obtain

\[
T(x) = A \cos(\omega k x) + B \sin(\omega k x),
\]

from which we deduce the condition to have a non zero non trivial solution $\sin \omega k = 0$, hence $\omega k = n \pi$. The associated eigenfunction is $\cos(n \pi x)$ for even $n$ and is $\sin(n \pi x)$ for odd $n$. We denote these functions by $T_n(x)$ and thanks to Fourier analysis, $L^2$ is described by a sum $\sum a_n T_n$ where $\sum |a_n|^2 < +\infty$ and $H^1_0$ is described by $\sum a_n T_n$ with $\sum n^2 |a_n|^2 < +\infty$. We have $\mathcal{M}(T_n) =$
Proof. We consider the equations (27-28) and we write them as follows:

\[ (n^2 + \vartheta)T_n = \lambda_n(\vartheta)T_n. \]

Hence the operator \( M^{1/2} \) is a symmetric operator with the eigenvalues \( \mu_n(\vartheta) = \sqrt{\lambda_n(\vartheta)} \) and for \( \hat{\Theta} \in H^1_0(-\frac{1}{2}, \frac{1}{2}) \) we have \( M\hat{\Theta} = \sum n_\vartheta \sqrt{\mu_n(\vartheta)}T_n \).

As \( M(\vartheta)\hat{\Theta} = W \) and \( W \in H^1_0, M^{1/2}\hat{\Theta} = M^{-1/2}W \) and \( M^{1/2}\hat{\Theta} \in H^1_0 \). Because \( W = \sum w_nT_n \) with \( \sum n^4|w_n|^2 < +\infty \) and \( \sum w_nT_n(\pm\frac{1}{2}) = 0 \). We deduce that \( M^{-1/2}W = \sum \mu_n^{-1/2}w_nT_n \).

This is an element of \( H^3 \), hence we can compute the trace on the boundary, and as the trace on the boundary of the eigenfunctions is zero, we have \( M^{-1/2}W(\pm\frac{1}{2}) = 0 \). Hence \( M^{1/2}\hat{\Theta}(\pm\frac{1}{2}) = 0 \).

\[ \text{Theorem 6} \] For all \( k \in \mathbb{R} \) and for all \( \hat{\vartheta} > 0 \), there exists a function \( R_\pm(k, \hat{\vartheta}) \) satisfying \( R_\pm(k, \hat{\vartheta}) \geq R^*(k) \) such that problem (27, 28) with the bc (29) has a unique solution \( (Z, \Theta, W) \in (H^1_0)^2 \times H^2_0 \); where \( H^2_0 \) are the functions of \( H^2 \) that fulfill the bc (29) for \( R = R_\pm(k, \hat{\vartheta}) \). This is in particular true for \( \hat{\vartheta} = 0 \).

**Proof.** We consider the equations (27, 28) and we write them as follows:

\[
\begin{align*}
\mathcal{M}(\hat{\vartheta})\tilde{\Theta}(z) &= W(z), \\
\mathcal{L}(\hat{\vartheta})W(z) &= \tilde{R}(k) \cdot \tilde{\Theta}(z),
\end{align*}
\]

where the operators \( \mathcal{M} \) and \( \mathcal{L} \) are defined as

\[
\begin{align*}
\mathcal{M} &= (1 - d_k^2 + \hat{\vartheta}), \\
\mathcal{L} &= \left[ d_h \left( Q - \frac{\hat{\vartheta}}{Pr} \right) d_h - \left( Q - \frac{\hat{\vartheta}}{Pr} \right) + d_h^2(\nu(z)) \right]
\end{align*}
\]

with \( Qf = Pf/k^2 = d_h(\nu(z) \cdot d_hf) - \nu(z) \cdot f \).

The equations (47, 48) lead to following eigenvalue problem

\[
\begin{align*}
\{ \mathcal{L}(\hat{\vartheta})\tilde{\Theta}(z) &= \tilde{R}(k) \cdot \tilde{\Theta}(z) \\
\Theta(-1/2) &= \Theta(1/2) = 0
\end{align*}
\]

where the operator \( \mathcal{L} \) is not a self-adjoint operator and is a composition expressed as follows

\[ L = \mathcal{L}\mathcal{M}. \]

Our operator \( L \) is not a self-adjoint operator, but \( \mathcal{L} \) and \( \mathcal{M} \) are self-adjoint operators. In fact \( \mathcal{M} \) is a positive operator,

\[
<\mathcal{M}\tilde{\Theta}, \tilde{\Theta}> = (1 + \hat{\vartheta}) \int |\tilde{\Theta}|^2 + \int |d_h\tilde{\Theta}|^2,
\]

so it is a symmetric operator. As \( \mathcal{M} \) is self-adjoint there is an orthogonal basis of \( H^1_0 \) with the eigenvalues \( \lambda_n \in \mathbb{R} \). Hence the operator \( M^{1/2} \) is a symmetric operator with eigenvalues \( \mu_n = \sqrt{\lambda_n} \in \mathbb{R} \). On the other hand \( M^{1/2} \) is a closed operator because Sobolev embedding, \( H^2 \hookrightarrow H^1_0 \rightarrow L^2 \). Therefore by Von Neumann theory \( M^{1/2} \) is a diagonal self-adjoint operator. The problem (51) is equivalent to the Sturm-Liouville problem as follows

\[
\begin{align*}
\{ H(\hat{\vartheta})\varphi(z) &= \tilde{R}(k) \cdot \varphi(z) \\
\varphi(-1/2) &= \varphi(1/2) = 0
\end{align*}
\]

where the Dirichlet boundary conditions can be used thanks to lemma [5] the unknown solution is \( \varphi = M^{1/2}\tilde{\Theta} \) and \( H = M^{1/2}\mathcal{L}M^{1/2} \) is a self-adjoint operator because \( M^{1/2} \) and \( \mathcal{L} \) are self-adjoint operators. \( \mathcal{L} \) is a self-adjoint operator from \( H^2 \) to \( H^{-2} \), hence \( H \) is a self-adjoint operator from \( H^2 \) to \( H^{-3} \). Therefore \( \tilde{R}(k) = 0 \) is not eigenvalue, so the Sturm-Liouville problem

\[
\begin{align*}
\{ H(\hat{\vartheta})\varphi(z) &= 0 \\
\varphi(-1/2) &= \varphi(1/2) = 0
\end{align*}
\]
has only the trivial solution as solution. Using the Fredholm Alternative theorem we can
deduce that there exists a function $R_1(k)$, which is the smallest eigenvalue of the operator $H$
with Dirichlet boundary conditions and the problem (51) has a unique and not trivial solution
$\Theta(z)$. Hence there exists a Green function $G(z, \xi)$ such that

$$\Theta(z) = \frac{R(k)}{k^4} \int_{-\frac{h}{2}}^{\frac{h}{2}} G(z, \xi) \Theta(\xi)d\xi,$$

and the theorem is proved. ■

### 3.4 Exchange of stability

**Theorem 7** There exists a $3 \times 3$ matrix $M(R, k)$ such that the values $R_o(k)$ for which $\sigma(R_o(k), k) = 0$ fulfill the condition $\det(M(R_o(k), k)) = 0$.

**Proof.** For any $k \in R$ we calculate the values of $R$ for which $\sigma(R(k), k) = 0$. From the equations (34-35), we consider the heat and decoupled equations as follows,

\[
(D^2 - k^2) \Theta + W = \sigma \cdot \Theta, \tag{55}
\]

\[
Rk^2\Theta = D \left( P - \frac{\sigma}{Pr} \right) DW - k^2 \left( P - \frac{\sigma}{Pr} - \nu''(z) \right) W, \tag{56}
\]

where $Pf = D(\nu(z) \cdot Df) - k^2(\nu(z) \cdot f)$. Homogenizing equations (55-56) we obtain

\[
(d_h^2 - 1) \tilde{\Theta} - \tilde{\sigma}\tilde{\Theta} + W = 0, \tag{57}
\]

\[
\tilde{R} \cdot \tilde{\Theta} = d_h \left( Q - \frac{\tilde{\sigma}}{Pr} \right) d_h W - \left( Q - \frac{\tilde{\sigma}}{Pr} - d_h^2(\nu(z)) \right) W, \tag{58}
\]

where $Qf = Pf/k^2$, $\tilde{R} = R/k^4$ and $\tilde{\Theta} = k^2 \Theta$.

Therefore, from the system (55-58), renaming the variables as follows

\[
W_o = W, \tag{59}
\]

\[
W_1 = d_h W = d_h W_o, \tag{59}
\]

\[
W_2 = \nu(z) \cdot d_h^2 W = \nu(z) \cdot d_h W_1, \tag{59}
\]

\[
W_3 = d_h W_2 - 2\nu(z) \cdot W_1, \tag{59}
\]

\[
\Theta_o = \Theta, \tag{59}
\]

\[
\Theta_1 = d_h \tilde{\Theta} = d_h \Theta_o, \tag{59}
\]

we obtain the system of ordinary differential equations for the velocity and the temperature functions

\[
d_h \Theta_1 = (1 + \tilde{\sigma}) \cdot \Theta_o - W_o, \tag{60}
\]

\[
\tilde{R} \cdot \Theta_o = d_h W_3 + \left( \frac{\tilde{\sigma}}{Pr} + \nu(z) + d_h^2(\nu(z)) \right) \cdot W_o - \left( \frac{\tilde{\sigma}}{\nu(z) Pr} \right) \cdot W_2. \tag{61}
\]

These equations admit the following matrix formulation

\[
d_h \begin{pmatrix}
W_o \\
W_1 \\
W_2 \\
W_3 \\
\Theta_o \\
\Theta_1
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\nu(z)} & 0 & 0 & 0 \\
0 & 0 & 2\nu(z) & 0 & 1 & 0 \\
0 & 0 & \tilde{\sigma} & 0 & \tilde{R} & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 + \tilde{\sigma} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \cdot 
\begin{pmatrix}
W_o \\
W_1 \\
W_2 \\
W_3 \\
\Theta_o \\
\Theta_1
\end{pmatrix} \tag{62}
\]
with boundary conditions

\[
\begin{align*}
W_o \left( z = -\frac{1}{2} \right) &= 0 \\
W_o \left( z = \frac{1}{2} \right) &= 0 \\
W_1 \left( z = -\frac{1}{2} \right) &= 0 \\
W_1 \left( z = \frac{1}{2} \right) &= 0 \\
\Theta_o \left( z = -\frac{1}{2} \right) &= 0 \\
\Theta_o \left( z = \frac{1}{2} \right) &= 0
\end{align*}
\]

In order to solve this problem we first consider the initial conditions on \( z = -1/2 \), where initial conditions for \( W_2, W_3 \) and \( \Theta_1 \) are deduced from the equations,

\[
\begin{align*}
W_2 \left( z = -\frac{1}{2} \right) &= \nu \left( -\frac{1}{2} \right) \cdot d_h W_1 \left( -\frac{1}{2} \right), \\
W_3 \left( z = -\frac{1}{2} \right) &= d_h W_2 \left( -\frac{1}{2} \right), \\
\Theta_1 \left( z = -\frac{1}{2} \right) &= d_h \Theta_o \left( -\frac{1}{2} \right).
\end{align*}
\]

As we do not know the values of \( d_h W_1(-1/2), d_h W_2(-1/2) \) and \( d_h \Theta_o(-1/2) \) such that the boundary conditions on \( 1/2 \) are fulfilled, we consider the following general initial conditions

\[
\begin{align*}
W_o \left( z = -\frac{1}{2} \right) &= 0 \\
W_1 \left( z = -\frac{1}{2} \right) &= 0 \\
W_2 \left( z = -\frac{1}{2} \right) &= \nu \left( -\frac{1}{2} \right) \frac{1}{k} A \\
W_3 \left( z = -\frac{1}{2} \right) &= \frac{1}{k} B \\
\Theta_o \left( z = -\frac{1}{2} \right) &= 0 \\
\Theta_1 \left( z = -\frac{1}{2} \right) &= \frac{1}{k} C
\end{align*}
\]

and we solve \( (62) \) with bc \( (67) \) for three different sets of values of \((A, B, C)\): \((1,0,0)\), \((0,1,0)\) and \((0,0,1)\). We call the respective solutions \( \tilde{W}^1, \tilde{W}^2 \) and \( \tilde{W}^3 \). The general solution can be written as a linear combination of the form

\[
\tilde{W} (z) = \lambda \cdot \tilde{W}^1 (z) + \mu \cdot \tilde{W}^2 (z) + \omega \cdot \tilde{W}^3 (z),
\]

where

\[
\tilde{W}^i \left( z \right) = (W_{i_o}, W_{i_1}, W_{i_2}, W_{i_3}, \Theta_{i_o}, \Theta_{i_1}) (z),
\]

We look for a solution of \( (62) \) with bc \( (63) \), therefore the solution \( \tilde{W} \) must verify the boundary conditions on \( z = 1/2 \)

\[
\begin{align*}
W_o \left( \frac{1}{2} \right) &= \lambda \cdot W_{1_o} \left( \frac{1}{2} \right) + \mu \cdot W_{2_o} \left( \frac{1}{2} \right) + \omega \cdot W_{3_o} \left( \frac{1}{2} \right) = 0 \\
W_2 \left( \frac{1}{2} \right) &= \lambda \cdot W_{1_2} \left( \frac{1}{2} \right) + \mu \cdot W_{2_2} \left( \frac{1}{2} \right) + \omega \cdot W_{3_2} \left( \frac{1}{2} \right) = 0 \\
\Theta_o \left( \frac{1}{2} \right) &= \lambda \cdot \Theta_{1_o} \left( \frac{1}{2} \right) + \mu \cdot \Theta_{2_o} \left( \frac{1}{2} \right) + \omega \cdot \Theta_{3_o} \left( \frac{1}{2} \right) = 0
\end{align*}
\]

This system can be expressed as

\[
M(R,k) \cdot \begin{pmatrix} \lambda \\ \mu \\ \omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\]

\( 10 \)
where the matrix $M(R,k)$ is defined by

$$
M(R,k) = \begin{pmatrix}
W_{1_o} \left( \frac{1}{2} \right) & W_{2_o} \left( \frac{1}{2} \right) & W_{3_o} \left( \frac{1}{2} \right) \\
W_{1_2} \left( \frac{1}{2} \right) & W_{2_2} \left( \frac{1}{2} \right) & W_{3_2} \left( \frac{1}{2} \right) \\
\Theta_{1_o} \left( \frac{1}{2} \right) & \Theta_{2_o} \left( \frac{1}{2} \right) & \Theta_{3_o} \left( \frac{1}{2} \right)
\end{pmatrix}
$$

(72)

Taking $\hat{\sigma} = 0$, problem (62-63) has solution, therefore the linear system (71) has a solution, i.e., there is a value of $\hat{R}$, $\hat{R}_o(k)$, such that the system (71) has a solution, i.e., there exist a value $\hat{R}_o(k)$ such that $\det(M(\hat{R}_o(k), k)) = 0$. Figure 2 shows this function $F(R) = \det(M(\hat{R}, k))$, in the case of exponential dependence of viscosity on temperature, for $\gamma = 3 \cdot 10^4$ and a fixed value of $k$, $k = 2.54$, crossing the axis at $\hat{R}_o(k)$.

**Corollary 8** For any $k \in \mathbb{R}$ there exists values of $\hat{R}$ for which $\hat{\sigma} > 0$ and values of $\hat{R}$ for which $\hat{\sigma} < 0$. Therefore, the exchange of stability holds.

**Proof.** If we consider $\hat{\sigma} > 0$, problem (62-63) has solution, therefore the linear system (71) has a solution, i.e., there is a value of $\hat{R}$, $\hat{R}_o(k)$, such that the system (71) has a solution, i.e., there exist a value $\hat{R}_o(k)$ such that $\det(M(\hat{R}_o(k), k)) = 0$. And the same applies to $\hat{\sigma} < 0$.

From here the marginal velocity and temperature fields are easily obtained. For $\hat{R}_o(k)$, the rank of the matrix is less than three and the system to solve is

$$
\begin{cases}
\lambda \cdot W_{1_o} \left( \frac{1}{2} \right) + \mu \cdot W_{2_o} \left( \frac{1}{2} \right) = -\omega \cdot W_{3_o} \left( \frac{1}{2} \right) \\
\lambda \cdot \Theta_{1_o} \left( \frac{1}{2} \right) + \mu \cdot \Theta_{2_o} \left( \frac{1}{2} \right) = -\omega \cdot \Theta_{3_o} \left( \frac{1}{2} \right)
\end{cases}
$$

(73)

with the free parameter $\omega$. The solution of this system is the growing perturbation after the bifurcation.

For all $k \in \mathbb{R}$ we calculate $\hat{R}_o(k)$, this function has a minimum, i.e., there is $k_c$ such that $\hat{R}_o(k_c) = \min \left\{ \hat{R}_o(k), k \in \mathbb{R} \right\}$. The real $k_c$ is the critical wave number and $\hat{R}_c = \hat{R}_o(k_c)$ is the critical Rayleigh number. As $\hat{R}$ increases this is the first value of $k$ for which an eigenvalue becomes positive. The corresponding mode grows and the conductive state becomes unstable. In fact, for any $\hat{R} < \hat{R}_c$, the conductive solution is stable, it loses stability at $\hat{R}_c$ and it is unstable for $\hat{R} > \hat{R}_c$. The curve $\hat{R}_o(k)$ is the marginal stability curve which is calculated in the next section for some values of the parameters.

### 4 Numerical Results

In this section the viscosity is assumed to have an exponential dependence on temperature $T$,

$$
\nu(T) = \nu_o \cdot \exp(-\gamma(T - T_1)),
$$

(74)

where $\nu_o$ is the viscosity at temperature $T_1$ and $\gamma$ the exponential rate. Figure 3 shows two different viscosity profiles for $\gamma = 10$ and $\gamma = 3 \cdot 10^4$, that correspond to almost constant viscosity and strong variable viscosity, respectively.

First we have calculated the critical values $R_o(k)$ for each $k$ looking for the first root of $\det(M(R,k))$. The marginal stability curves for $\gamma = 10$ and $\gamma = 3 \cdot 10^4$ can be seen in figure 4. The minima in the critical wave numbers $k_c = 2.68$ and $R_c = 1082.90$ for $\gamma = 10$ and $k_c = 2.16$ and $R_c = 73.51$ for $\gamma = 3 \cdot 10^4$ indicates the critical thresholds for the first bifurcation. In table I the critical Rayleigh and wave numbers for several values of $\gamma$ are displayed. The corresponding critical Rayleigh number, $R_c$, decreases when $\gamma$ increases. Therefore a higher variation of the
exponential rate in viscosity favors the instability. The critical wave number also depends on the viscosity factor $\gamma$. It decreases when $\gamma$ increases until $\gamma = 5 \cdot 10^4$ where it increases again. As $\gamma$ tends to zero, $\nu(z) \to \nu_0$ the results tend to the ones with constant viscosity, $k_c = 2.68$ and $R_c = 1100.65$. Figure 5 shows the corresponding velocity functions of the perturbation field for two cases: almost constant ($\gamma = 10$) and strongly variable viscosity ($\gamma = 3 \cdot 10^4$), for different Rayleigh numbers, below the critical one, at the critical threshold and above it. In the variable viscosity case the velocity is larger in the region where viscosity is smaller, while in the constant viscosity case velocity is more distributed along the cell. Results are similar to those in ref. [15] where a linear approximation to the exponential dependence is considered.

5 Conclusions

We have studied the thermoconvective instability problem in an infinite layer under the perspective of viscosity as an exponential function of the temperature. We have obtained the conductive solution and we have studied its linear stability analysis. We have demonstrated that the eigenvalues are real and the existence of eigenvalues and bifurcation.

With appropriate changes of variable we have considerably simplified the expressions to obtain a system of ordinary differential equations which is numerically manageable. We have obtained a practical condition which permits easily to calculate the critical bifurcation parameters.

Finally, we have obtained the stability curves for different values of the exponential rate, $\gamma$. As the exponential rates decrease the critical Rayleigh number increases. So a higher variation of the exponential rate in viscosity favors the instability. There is a non monotone variation of the critical wave number with the exponential rate. The vertical velocity for the growing mode exhibits more movement in the region where the viscosity is lower.

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Table captions

Table I
Critical wave and Rayleigh numbers \((k_c, R_c)\) for different values of \(\gamma\).

Figure captions

Figure 1
Problem set-up.

Figure 2
\(F(R) = \det(M(R, k = 2.54), k = 2.54)\) for \(\gamma = 3 \cdot 10^4\); this plot represents a transcritical bifurcation.

Figure 3
Viscosity profiles \(\nu(z)\) for \(\gamma = 3 \cdot 10^4\) (dashed line) and \(\gamma = 10\) (solid line).

Figure 4
Marginal stability curves for \(\gamma = 3 \cdot 10^4\) (dashed line) and \(\gamma = 10\) (solid line).

Figure 5
a) Velocity functions \(W\) of the growing mode for different values of the Rayleigh number for \(\gamma = 10\) and \(k = 2.68\); b) velocity functions \(W\) of the growing mode for different values of the Rayleigh number for \(\gamma = 3 \cdot 10^4\) and \(k = 2.16\).
Table I

| $\gamma$ | $(k_c, R_c)$  |
|---------|--------------|
| $10^{-4}$ | (2.68,1100.65) |
| 10      | (2.68,1082.90) |
| $10^3$  | (2.18,153.66)  |
| $3 \cdot 10^4$ | (2.16,73.51)  |
| $5 \cdot 10^4$ | (2.42,50.62)  |

Figure 1.

Figure 2.

Figure 3.
