The formulation of vector potential evolution equation on rapidly rotating and accreting neutron star in the frame of Zero Angular Momentum Observers (ZAMO)

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Abstract. The equation for the dynamics of the magnetic field can explain the decrease in the magnetic field due to accretion. Maxwell’s relativistic equations formulate these equations. The formulation of the solution to the Maxwell equation is said to be correct if it is by the Schwarzschild space-time case by formulating the vector potential evolution equation. Maxwell's second relativistic equation, the covariance tensor of the electromagnetic field, and the vector potential are used to obtain the equation for the evolution of the vector potential. All equations are calculated in the Zero Angular Momentum Observers (ZAMO) framework. The purpose of this research is to formulate the vector potential evolution equation for each component in a rapidly rotating and accreting neutron star. The results obtained have covariant terms with the slow rotating neutron star. However, the equation for a rapidly rotating neutron star has a more complex description. There are terms that cannot be simplified, such as in slowly rotating neutron stars.

1. Introduction

Magnetic fields are one of the themes of study in neutron stars [1]. Initially, the neutron star was discovered as a pulsar with a very strong magnetic field. Magnetic field $B \sim 10^{10} - 10^{11} G$ belongs to the young neutron star [2]. Young neutron stars are at the center of the supernova remnant. Magnetic field strength is measured from spectral beams and X-rays. The surface of the neutron star has a strong magnetic field $B \sim 10^8 - 10^{13} G$ [3]. Neutron stars have a magnetic field of $\sim 10^{15} G$ in their interior [3]. Accreting X-ray pulsars, radio pulsars that emit infrared, and Low Mass X-Ray Binaries (LXMBs) are the sources by which magnetic fields can be detected. The magnetic field on the inside can also reach $B \sim 10^{16} - 10^{17} G$, yang berasal dari energi AXP (Anomalous X-ray pulsars) dan SGR (Strongly Magnetized but Slowly Rotating) [4]. The maximum magnetic field possessed by a neutron star is $B \sim 10^{18} G$. The results are obtained numerically based on the virial theorem [4].

The magnetic field changes in the neutron star. Significant changes are found in binary systems [3]. For example, in $5 \times 10^6$ years, the magnetic field changes from $\sim 10^{12} G$ to $\sim 10^9 G$ [5]. The neutron star accretes in binary system. Changes in the magnetic field created by the accretion process [2,6–10]. Ultimately the dynamics of the magnetic field became a topic of much research [2][11–13]. High Mass
X-ray Binaries (HMXBs) with $B \sim 10^{12}\text{G}$ and Low Mass X-ray Binaries (LMXBs) with $B \lesssim 10^{10}\text{G}$ are detectable sources of accreting neutron stars [7]. Changes in the magnetic field can be seen from LMXBs. The relationship between the accretion process and the decrease in the magnetic field, it can be seen from the LXMB in the binary system [7].

A mathematical relationship between magnetic field reduction and non-relativistic accretion has been obtained by comparing the Ohmic diffusion time and magnetic accretion time [7]. Equations for the dynamics of the magnetic field and the equations for the rate of accretion are needed to formulate the equation for the decrease in the magnetic field. The relativistic Maxwell equation describes the equations for the dynamics of the magnetic field. The solution of Maxwell's equations with a stationary electromagnetic field in Schwarzschild space-time has been obtained [14]. The curved spacetime produces a magnetic field that is generally larger than the Newtonian portion [14-15]. Curved space-time affects the time the magnetic field decreases [16]. Meanwhile, for rotating stars, relativistically different results are different from those that do not rotate [17]. These results indicate the importance of relativistically.

The equations for the dynamics of the magnetic field that can be applied in some cases of astrophysics are in the case of slowly rotating and non-accreting stars [18]. Slowly rotating and accreting neutron stars have an influence on the changing electric field [19]. The product is also willing for a rapidly rotating and accreting neutron star [20]. The solution of both equations has been obtained [21] and can be confirmed by a vector potential [18]. This form of confirmation is the equation for the evolution of vector potential. The purpose of this research is to formulate the vector potential evolution equation for each component in a rapidly rotating and accreting neutron star in the ZAMO framework.

2. Methods
This research method is a mathematical theory. This study using the metric of a rapidly rotating neutron star. Metrics for calculating tetrad, 1-form, and 4-speed. This third equation is for calculating the conductors for the velocity and tensor of the contravariant electromagnetic field. The current density four is obtained from the speed of the conductor four. The next step describes Maxwell's second relativistic equation. The vector potential evolution equation is obtained from the metric and tensor of the electromagnetic field. The relativistic solution of Maxwell's second equation is confirmed from the potential evolution equation. The research procedure is shown in Figure 1.

3. Results and discussion
Equation (1) is a rapidly rotating neutron star metrics [22]
\[
d s^2 = -(e^{2\phi} - \sin^2 \theta r^2 e^{2\lambda} \omega^2) dt^2 + e^{2\alpha} dr^2 + r^2 e^{2\lambda} d\theta^2 - 2 \sin^2 \theta r^2 e^{2\lambda} \omega dt d\phi + \sin^2 \theta r^2 e^{2\lambda} d\phi^2.
\] (1)
The function of $\lambda$, $\alpha$, and $\phi$ depends on $r$ and $\theta$ [22]. The component of the electromagnetic field tensor ($F_{\alpha\beta}$) is
\[
F_{\alpha\beta} \equiv \eta_{\alpha\beta\gamma\delta} u^\gamma B^\delta + 2u_{[\alpha} E_{\beta]}.
\] (2)
Whit $\eta_{\alpha\beta\gamma\delta}$ is pseudo tensor, $u^\gamma$ is four velocity vectors, $B^\delta$ is magnetic field component, and $E_{\beta}$ is electric field component. The four-velocity component in ZAMO frame is [20]
\[
u^\mu = \frac{e^{-\phi}}{\sqrt{1 - \omega^2}} (1,0,0,\omega).
\] (3)
Figure 1. Procedure for obtaining equations for the potential evolution of a rapidly rotating and accreting neutron star vector.

In the ZAMO framework, the tensors of the covariance electromagnetic field \( F_{\mu\nu} = -F_{\nu\mu} \) are [20]

\[
F_{\mu\nu} = 0, \mu = \nu
\]  
(4a)
\[
F_{01} = -\frac{e^a}{\sqrt{1-W^2}} \left[ e^\phi E^\theta + \omega r \sin \theta e^\lambda B^\phi \right], \quad (4b)
\]
\[
F_{02} = -\frac{e^a}{\sqrt{1-W^2}} \left[ e^\phi E^\phi + \omega r \sin \theta e^\lambda B^\theta \right], \quad (4c)
\]
\[
F_{03} = -r \sqrt{\frac{e^{2\phi} - e^{2\lambda r^2\omega^2 \sin^2 \theta}}{1-W^2}} \sin \theta e^\lambda E^\phi, \quad (4d)
\]
\[
F_{12} = r \left[ \sqrt{\frac{e^{2\phi} - e^{2\lambda r^2\omega^2 \sin^2 \theta}}{1-W^2}} e^{-\phi + 2\alpha B^\theta}, \right] \quad (4e)
\]
\[
F_{13} = \left[ \frac{e^{\alpha + \lambda}}{\sqrt{1-W^2}} r \sin \theta \right] B^\theta, \quad (4f)
\]
\[
F_{23} = r \left[ \frac{e^{\alpha + \lambda}}{\sqrt{1-W^2}} r \sin \theta \right] B^\theta, \quad (4g)
\]

with
\[
W^2 = e^{2\phi} \left[ \sin^2 \theta \ r^2 e^{2\lambda} (\Omega - \omega)^2 + e^{2\alpha} (v^r)^2 + r^2 e^{2\alpha} (v^\theta)^2 \right]. \quad (5)
\]

The formula for the solution of Maxwell's equations is said to be correct when using the Schwarzschild space-time case by formulating the vector potential evolution equation. This equation is obtained by the connection potential, the covariance tensor of the electromagnetic field, and Maxwell's second relativistic equation in the ZAMO framework. The vector potential equation associated with the tensor of the electromagnetic field is
\[
F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \quad (6)
\]
with \(A_i\) is vector potential. The relativistic Maxwell's second equation is
\[
F^{\mu\nu}_{\ ;\nu} = 4\pi J^\mu. \quad (7)
\]
Maxwell's second equation contains a covariant derivative. Equations involving vector potential and covariance derivatives produce ambiguous values because covariance derivatives are non-commutative [16]. Therefore, Maxwell's second equation can be changed to [18]
\[
\frac{1}{\sqrt{-g}} \left( \sqrt{-g} F^{\mu\nu} \right)_{\ ;\nu} = 4\pi J^\mu. \quad (8)
\]
Equation (8) does not contain covariance derivatives. The four conductor velocity vectors and Maxwell's first-second relativistic equations yield the equations [18]
\[
\frac{1}{4\pi \sigma u^0 \sqrt{-g}} \left( \sqrt{-g} F^{ij} \right)_{\ ;j} = \rho_e u^i + \sigma g^{ij} (F_{jk} u^k + F_{j0} u^0), \quad (9)
\]
with \(\sigma\) is electrical conductivity, and \(\rho_e\) is proper change density [18]. The equation (3.9) can be written as [18]
\[
F_{i0} = \frac{g_{ij}}{4\pi \sigma u^0 \sqrt{-g}} \left( \sqrt{-g} F^{jk} \right)_{\ ;k} - \frac{F_{ij} u^j}{u^0} - \frac{\rho_e g_{ij} u^j}{\sigma u^0}. \quad (10)
\]
The components of the electromagnetic field tensor \((F_{i0}\) when connected to the vector potential satisfying equation (6)
\[
F_{i0} = A_{0,i} - A_{i,0}. \quad (11)
\]
and \(A_{0,i} = 0\), then
\[
F_{i0} = -A_{i,0} = -\frac{g_{ij}}{4\pi \sigma u^0 \sqrt{-g}} \left( \sqrt{-g} F^{jk} \right)_{\ ;k} - \frac{F_{ij} u^i}{u^0} - \frac{\rho_e g_{ij} u^j}{\sigma u^0}. \quad (12)
\]
Equation (12) contains the component \(A_{i,0}\) which is the derivative of the vector potential with respect to time. Therefore, from equation (12) we get the dynamic vector potential equations for each component. According to equation (1), then the component \(g_{ij}\) it only has value for \(i = j\) with \(i, j = 1, 2, 3\). Meanwhile, according to equation (4a), the components are obtained \(F_{ij} = 0\) for \(i = j\). The equation for the partial vector evolution for each component is
\[
\frac{\partial A_\phi}{\partial t} = -\frac{g_{rr}}{4\pi \sigma u^0 \sqrt{-g}} \left( \sqrt{-g} F^{r\phi} \right)_\theta - \frac{g_{\theta r}}{4\pi \sigma u^0 \sqrt{-g}} \left( \sqrt{-g} F^{\theta \phi} \right)_r + F_{\phi \psi} u^\phi - \frac{\rho_e g_{\phi \psi}}{\sigma u^0}, \quad (13)
\]
\[
\frac{\partial A_\theta}{\partial t} = -\frac{g_{\theta \theta}}{4\pi \sigma u^0 \sqrt{-g}} \left( \sqrt{-g} F^{\theta \psi} \right)_r - \frac{g_{\phi \theta}}{4\pi \sigma u^0 \sqrt{-g}} \left( \sqrt{-g} F^{\phi \psi} \right)_r + F_{\theta \phi} u^\phi + \frac{\rho_e g_{\theta \phi}}{\sigma u^0}. \quad (14)
\]
\[
\frac{\partial A_\phi}{\partial t} = -\frac{g_{\phi\phi}}{4\pi\sigma u^0\sqrt{-g}}(\sqrt{-g} F^{\phi r})_r - \frac{g_{\phi\phi}}{4\pi\sigma u^0\sqrt{-g}}(\sqrt{-g} F^{\phi\theta})_\rho + \frac{\rho \partial g_{\phi\phi} u^\rho}{\sigma u^0}. \tag{15}
\]

Each contravariant electromagnetic field tensor in equations (13-15) is converted into a covariant electromagnetic field tensor. This change is used to make it easier to find a solution to the differential equation. This step is carried out to compare the results obtained by covariance with previous results [18]. But the shapes of the covariant and contravariant electromagnetic field tensors of a rapidly rotating neutron star are not as simple as those of a slow rotation. Slowly rotating neutron star and \(\Lambda\) function depends only on \(r\). While the rapidly rotating neutron star \(\phi\) and \(\Lambda\) function depends on \(r\) and \(\theta\). Therefore changing the tensor of the contravariant electromagnetic field in equations (13-15) to be covariant is not an effective way. On the other hand, the alternative solution given contains the component \(B^I\). Meanwhile, both covariant and contravariant electromagnetic field tensors contain \(B^I\). Therefore, equation (13-15) can still be written as

\[
\begin{align*}
\frac{\partial A_r}{\partial t} &= -\frac{e^{-\lambda_{\sqrt{-1}-W^2}}}{4\pi\sigma r^2 \sin \theta} \left\{ \left( (e^{\phi+2\alpha+\lambda r^2} \sin \theta) F^{r\theta} \right)_\rho + \left( (e^{\phi+2\alpha+\lambda r^2} \sin \theta) F^{r\phi} \right)_\rho \right\} + \omega F_{r\phi}, \tag{16}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial A_\theta}{\partial t} &= -\frac{e^{-\lambda_{\sqrt{-1}-W^2}}}{4\pi\sigma \sin \theta} \left\{ \left( (e^{\phi+2\alpha+\lambda r^2} \sin \theta) F^{r\theta} \right)_r + \left( (e^{\phi+2\alpha+\lambda r^2} \sin \theta) F^{r\phi} \right)_\rho \right\} + \omega F_{\phi r}, \tag{17}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial A_\phi}{\partial t} &= -\frac{e^{-\phi+2\alpha+\lambda_{\sqrt{-1}-W^2}}}{4\pi\sigma} \left\{ \left( (e^{\phi+2\alpha+\lambda r^2} \sin \theta) F^{\phi r} \right)_r + \left( (e^{\phi+2\alpha+\lambda r^2} \sin \theta) F^{\phi\theta} \right)_\rho \right\} + \omega \frac{\rho e^2}{\sigma} e^{2\lambda r^2} \sin^2 \theta. \tag{18}
\end{align*}
\]

The equations (16-18) are the vector potential evolution equation for each component of a neutron star that rotates rapidly and accreting in the ZAMO framework. There are terms in equation (16-18) that are covariant with a slowly rotating neutron star. Equations (16-18) contain a contravariant electromagnetic field tensor with solutions containing \(B^I\). The solution to this equation is comparable to a slowly rotating neutron star. Equation (16-18) has a more complex description than the slow-rotating neutron star equation, because the variables \(r\), \(\phi\), \(\lambda\), \(\omega\), and \(\alpha\) do not only depend on \(r\) but also depend on \(\theta\). In addition, the term \(e^{\phi+2\alpha+\lambda r^2} \sin \theta\) cannot be simplified by the component \(F^{ij}\). Therefore, going forward, solving this equation requires a numerical method.

4. Conclusion

The vector potential evolution equation is needed to confirm the equations for the dynamic solution of the magnetic field in the neutron star. This equation is obtained by relating the electromagnetic field tensor. The results obtained have covariant terms with the slow rotating neutron star. However, the equation for a rapidly rotating neutron star has a more complex description. There are terms that cannot be simplified, such as in slowly rotating neutron stars. The influencing factor is because the variables \(\phi\), \(\lambda\), \(\omega\), and \(\alpha\) depend on \(r\) and \(\theta\). In addition, the term \(e^{\phi+2\alpha+\lambda r^2} \sin \theta\) cannot be simplified by the component \(F^{ij}\).

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