INCOHERENT COXETER GROUPS

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(Communicated by Kevin Whyte)

Abstract. We use probabilistic methods to prove that many Coxeter groups are incoherent. In particular, this holds for Coxeter groups of uniform exponent > 2 with sufficiently many generators.

1. Introduction

A Coxeter group $G$ is given by the presentation

$$(a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^{m_{ij}} : 1 \leq i < j \leq r)$$

where $m_{ij} \in \{2, 3, \ldots, \infty\}$ and where $m_{ij} = \infty$ means no relator of the form $(a_i a_j)^{m_{ij}}$. Throughout this paper all presentations of Coxeter groups are of the above form. It is traditional to encode the above data for $G$ in terms of an associated labelled graph $\Gamma_G$, whose vertices correspond to the generators and where an edge labelled by $m_{ij}$ joins vertices $a_i, a_j$ when $m_{ij} < \infty$. We omit an edge for $m_{ij} = \infty$.

Definition 1.1. A group $G$ is coherent if every finitely generated subgroup of $G$ is finitely presented. Otherwise, $G$ is incoherent.

Our main result which is stated and proven as Theorem 3.3 is the following:

Theorem 1.2. For each $M$ there exists $R = R(M)$ such that if $K$ is a Coxeter group with $3 \leq m_{ij} \leq M$ and rank $r \geq R$, then $K$ is incoherent.

Our result joins a similar result for groups acting properly and cocompactly on Bourdon buildings [Wis11] and we expect that there is more to come in this direction.

2. Preliminaries on Coxeter groups, walls, and Morse theory

2.1. Euler characteristic and compression. Let $G$ be a Coxeter group given by

$$(a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^{m_{ij}} : 1 \leq i < j \leq r)$$

and let $X$ be the standard 2-complex associated to this presentation. Consider an index $d$ torsion-free subgroup $G'$ of $G$. Let $\tilde{X} \to X$ be a cover of $X$ corresponding to $G'$. All edges embed in $\tilde{X}$, since all generators are torsion elements, and all 2-cells embed since each proper subword of $(a_i a_j)^{m_{ij}}$ is a torsion element. Consider the complex $\tilde{X}$ obtained from $\tilde{X}$ by first collapsing 2-cells corresponding to $a_i^2$ relators.
to 1-cells and second collapsing $2m_{ij}$ copies of $2m_{ij}$-gons with the same boundary when $m_{ij} \neq \infty$. The complex $\mathcal{X}$ is the compression of $\hat{\mathcal{X}}$. See Figure 1 for the compression arising from $\langle a, b \mid a^2, b^2, (ab)^3 \rangle$.

We say $G$ has dimension $\leq 2$ when $\mathcal{X}$ is aspherical. This holds precisely when $1/m_{ij} + 1/m_{jk} + 1/m_{ki} \leq 1$ for each $i, j, k$. Indeed, there is then a natural metric of nonpositive curvature on $\mathcal{X}$ induced by metrizing each 2-cell as a regular Euclidean polygon. However if some 3-generator subgroup is finite, then $\mathcal{X}$ contains a copy of $S^2$. We focus on Coxeter groups of dimension $\leq 2$, in which case the following discussion of $\chi(G)$ is sensible.

The complex $\mathcal{X}$ has one 0-cell, $r$ 1-cells and one 2-cell for each pair of generators $\{i, j\}$ with $m_{ij} < \infty$. As degree $(\hat{\mathcal{X}} \rightarrow \mathcal{X}) = d$, the complex $\hat{\mathcal{X}}$ has $d$ 0-cells, $dr$ 1-cells and $d$ 2-cells for each pair $\{i, j\}$ with $m_{ij} < \infty$. The Euler characteristic of $G$ is:

$$\chi(G) = \chi(\mathcal{X})/|G:G'| = 1/d \left(d - \frac{dr}{2} + \sum_{\{i,j\}} \frac{d}{2m_{ij}}\right) = 1 - \frac{r}{2} + \sum_{\{i,j\}} \frac{1}{2m_{ij}}.$$

This is independent of the choice of a finite index torsion-free subgroup. We thus have:

$$\chi(G_{(r,m)}) = 1 - \frac{r}{2} + \frac{(r-1)r}{4m}.$$

Thus if $m$ is fixed, then $\chi(G_{(r,m)}) > 0$ for all sufficiently large $r$.

2.2. Walls. Let $K$ be a combinatorial 2-complex with the property that each 2-cell has an even number of sides (we have in mind $K = \overline{\mathcal{X}}$ as defined in the previous section). Two 1-cells in the attaching map $\partial_p C \rightarrow K^1$ of a 2-cell $C$ are parallel if they are images of opposite edges in $\partial_p C$. An abstract wall is an equivalence class of 1-cells in the equivalence relation generated by parallelism. A wall $W$ associated to an abstract wall $\bar{W}$ is a graph with a locally injective map $\phi : W \rightarrow K$ defined as follows:

- for each 1-cell $a$ in $\bar{W}$ there is a vertex $v_a$ in $W$,
- $\phi(v_a)$ is the center of $a$,
- for each pair of 1-cells $a, a'$ in $\bar{W}$ and each 2-cell $C$ in which $a, a'$ are parallel, there is an edge $(v_a, v_{a'})$ in $W$,
- the edge $(v_a, v_{a'})$ is sent by $\phi$ to an arc in $C$ joining $\phi(v_a)$ and $\phi(v_{a'})$.

The wall $W$ is dual to each 1-cell in $\bar{W}$. The wall $W$ is adjacent to $x$ at a vertex $v$ of $\text{link}(x)$, if it is dual to the 1-cell corresponding to $v$. The wall $W$ is adjacent to $x$ at an edge $e$ of $\text{link}(x)$, if $W$ is not adjacent at either endpoint of $e$ but is dual...
to a pair of 1-cells in $\partial_p C$ where $C$ is a 2-cell corresponding to $e$. We say that the wall $W$

- *embeds* if $W \to K$ is injective,
- *is two-sided* if $W \to K$ extends to an embedding $W \times (-1, 1) \to K$,
- *self-osculates at $x$* if it is adjacent to $x$ at more than one vertex and/or edge of $\text{link}(x)$. See Figure 2.

### 2.3. Orientation of walls

An embedded wall $W \to K$ is two-sided if and only if there is a globally consistent orientation of its dual 1-cells such that parallel 1-cells in any 2-cell $C$ have opposite orientations in $\partial_p C$. An orientation of a two-sided wall $W$ is one of two globally consistent orientations of its dual 1-cells. Let $W$ be the set of all walls in $K$. An orientation on $W$ is a choice of orientation on each $W \in W$.

### 2.4. Bestvina-Brady Morse theory

An affine complex $K$ has cells that are convex Euclidean polyhedra, which metrically agree on their faces. A map $f : K \to \mathbb{R}$ is a Morse function if it is linear on each cell $C$, constant on $C$ if and only if $\dim C = 0$, and the image $f(K^0)$ of the 0-skeleton is a closed discrete subset of $\mathbb{R}$. It follows that the restriction of $f$ to a cell has a unique minimum and maximum.

Let $x \in K^0$. A vertex $v \in \text{link}(x)$ is ascending (resp. descending) if the corresponding 1-cell is oriented away from $x$ (resp. toward $x$). An edge $e \in \text{link}(x)$ is ascending (resp. descending) if each wall passing through the corresponding 2-cell is oriented away from $x$ (resp. toward $x$). The ascending link $\text{link}_a(x)$ (resp. descending link $\text{link}_d(x)$) is the subgraph of $\text{link}(x)$ consisting of all ascending (resp. descending) vertices and edges.

We will employ the following result of Bestvina-Brady proven in [BB97, Thm 4.1]:

**Theorem 2.1.** Let $K$ be a finite (aspherical) affine cell complex. Consider a map $K \to S^1$ that lifts to a Morse function $\tilde{K} \to \mathbb{R}$. If $\text{link}_a(x)$ and $\text{link}_d(x)$ are nonempty and connected for each $x \in \tilde{K}^0$, then $\ker(\pi_1 K \to \mathbb{Z})$ is finitely generated.

### 2.5. An orientation induces a combinatorial map $K^1 \to S^1$

Let $S^1$ have a cell structure with one 0-cell and one (oriented) 1-cell. Each orientation on $W$ determines an orientation preserving combinatorial map $K^1 \to S^1$. The map $\partial_p C \to S^1$ is null-homotopic for each 2-cell $C$, since pairs of opposite 1-cells in $\partial_p C$ travel in opposite directions around $S^1$. Thus the map $K^1 \to S^1$ extends to $K \to S^1$. The map $K \to S^1$ lifts to $\tilde{K} \to \tilde{S}^1 \simeq \mathbb{R}$, but the restriction of this map to a 2-cell does not necessarily have a unique minimum or maximum.

The lawful subcomplex $Y \subset K$ is the subcomplex of $K$ obtained by discarding 2-cells whose attaching maps cannot be expressed as the concatenation $\alpha \beta^{-1}$ where $\alpha \to K^1$, $\beta \to K^1$ are positively directed paths. The restriction $Y \to S^1$ of the map $K \to S^1$ lifts to $\tilde{Y} \to \mathbb{R}$ which is a Morse function in the sense of Bestvina-Brady.
3. Main theorem

The Coxeter group of uniform exponent $m$ and rank $r$ is the Coxeter group $G_{(r,m)}$ with the following presentation:

$$\langle a_1, \ldots, a_r \mid a_i^2, (a_i a_j)^m : 1 \leq i < j \leq r \rangle.$$  

The standard 2-complex of the above presentation for $G_{(r,m)}$ is denoted by $X_{(r,m)}$.

**Theorem 3.1.** For each $m \geq 3$ there exists $R_m$ such that for all $r \geq R_m$ the group $G_{(r,m)}$ has a finite index torsion-free subgroup $G'$ that admits an epimorphism $G' \to \mathbb{Z}$ whose kernel $N$ is finitely generated.

**Corollary 3.2.** For $m \geq 3$, the group $G_{(r,m)}$ is incoherent for all sufficiently large $r$.

**Proof.** A result of Bieri in [Bie81] states that a nontrivial finitely presented normal subgroup of a group of cohomological dimension $\leq 2$ is either free or of finite index. Since $[G' : N] = \infty$ it remains to exclude the case where $N$ is free, whence:

$$\chi(G') = \chi(N) \cdot \chi(\mathbb{Z}) = (1 - \text{rank}(N)) \cdot 0 = 0.$$  

This is impossible for all sufficiently large $r$, since then $\chi(G') > 0$ (see Section 2.1).

A Coxeter subgroup is generated by a subset of the generators of $G$. It is presented by those generators together with all relators in those generators appearing in the presentation of $G$ [Dav08]. We now prove the main result stated in the introduction:

**Theorem 3.3.** For each $M$ there exists $R = R(M)$ such that if $K$ is a Coxeter group with $3 \leq m_{ij} \leq M$ and rank $r \geq R$, then $K$ is incoherent.

**Proof.** The multi-color version of Ramsey’s theorem [GRS80] states that given a number of colors $c$ and natural numbers $n_1, \ldots, n_c$ there exists a number $R = R(n_1, \ldots, n_c)$ such that if the edges of a complete graph $\Gamma$ of order at least $R$ are colored with $c$ colors, then for some $i$ there exists a complete subgraph of $\Gamma$ of order $n_i$ with edges of color $i$. Let $c = M$ and $n_i = R_i$ of Theorem 3.1. Consequently there exists a uniform exponent Coxeter subgroup $G_{(r,m)}$ of $K$ for some $m \leq M$ and $r = R_m$. By Corollary 3.2 the subgroup $G_{(r,m)}$ is incoherent and hence so is $K$.

The above results lend credence to the following:

**Conjecture 3.4.** Let $G$ be a finitely generated infinite Coxeter group of dimension $\leq 2$. If $\chi(G) > 0$, then $G$ is incoherent.

3.1. A polynomial degree finite cover of $X_{(r,m)}$ with good walls. The goal of this subsection is to prove the following:

**Proposition 3.5.** There is a homomorphism $\beta : G_{(r,m)} \to Q^{k(r)}$ such that the compression $\overline{X}_{(r,m)}$ of the induced cover $\tilde{X}_{(r,m)} \to X_{(r,m)}$ has the following property: each wall is 2-sided, embedded and has no self-osculation.

Moreover $|\overline{X}_{(r,m)}|/|Q|^{k(r)} \leq |Q|^r C$ for some constant $C$.

The proof of Proposition 3.5 appears at the end of this subsection.

A partition of a set $S$ is a map $p : S \to \{1, 2, 3, 4\}$. The partition $p$ separates $a, b, c, d$ if $p(a), p(b), p(c), p(d)$ are distinct.
Lemma 3.6. Let $S$ have cardinality $r \geq 4$. There is a collection of
\[ k = k(r) = \left\lfloor \frac{\log (r^4)}{\log \frac{32}{29}} \right\rfloor \]
partitions such that each quadruple of distinct elements of $S$ is separated by this collection.

Proof. Let $M$ denote the set of all partitions of $S$, and note that $|M| = 4^r$. Let $\mathcal{M}_k$ denote the collection of cardinality $k$ subsets of $M$ and note that $|\mathcal{M}_k| = \binom{4^r}{k}$. Let $\mathcal{N}_k \subset \mathcal{M}_k$ be the subcollection consisting of sets of $k$ partitions that do not separate some quadruple. We want to show that $|\mathcal{N}_k| < |\mathcal{M}_k|$. Let $\mathcal{N}_k(\{a, b, c, d\}) \subset \mathcal{M}_k$ be the subcollection of sets that fail to separate $a, b, c, d \in S$. We have
\[ |\mathcal{N}_k(\{a, b, c, d\})| = \left( \frac{29}{32} \cdot 4^r \right)_k \]

since there are $\binom{r^4}{4}$ quadruples $\{a, b, c, d\}$ of distinct elements of $S$. There are $4^1 \cdot 4^{r-4} = 6 \cdot 4^{r-3}$ partitions that separate $a, b, c, d$. Thus there are $4^r - 6 \cdot 4^{r-3} = \frac{29}{32} \cdot 4^r$ partitions that do not separate $a, b, c, d$. We thus have
\[ |\mathcal{N}_k(\{a, b, c, d\})| = \left( \frac{29}{32} \cdot 4^r \right)_k. \]

Observe that we have the following:
\[ \left( \frac{29}{32} \cdot 4^r \right)_k < \left( \frac{29}{32} \cdot 4^r \right)_k. \]

Since $k \geq \log (r^4)/\log \frac{32}{29}$ we have
\[ \binom{r}{4} \left( \frac{29}{32} \right)_k \leq 1. \]

Altogether we have
\[ |\mathcal{N}_k| \leq \binom{r}{4} |\mathcal{N}_k(\{a, b, c, d\})| < \binom{r}{4} \left( \frac{29}{32} \cdot 4^r \right)_k = |\mathcal{M}_k(P)|. \]

Proof of Proposition 3.5. There is a finite quotient $\psi : G_{(4,m)} \to Q$ such that ker $\psi$ is torsion-free, and the compression $\overline{X}_{(4,m)}$ of the induced cover $\hat{X}_{(4,m)} \to X_{(4,m)}$ has the following property: each wall in $\overline{X}_{(4,m)}$ is two-sided, embedded and has no self-osculation. This follows from the separability of wall stabilizers [HW10].

Let $S = \{1, \ldots, r\}$. Each partition $p : S \to \{1, 2, 3, 4\}$ defines a homomorphism $\phi_p : G_{(r,m)} \to G_{(4,m)}$ induced by $\phi_p(a_i) = a_{p(i)}$. Let $\beta = (\psi \circ \phi_{p_1}, \ldots, \psi \circ \phi_{p_k}) : G_{(r,m)} \to Q^{k(r)}$ where $(p_1, \ldots, p_k)$ is a collection of partitions from Lemma 3.6. For each partition $p$ there is a map $\overline{\phi}_p : X_{(r,m)} \to X_{(4,m)}$ induced by $\phi_p$. We will show that a “wall pathology” in $X_{(r,m)}$ would project to a wall pathology in $X_{(4,m)}$ for a suitable $p$ and hence there are no such wall pathologies. Suppose there is a wall $W$ in $X_{(r,m)}$ that self-intersects within a 2-cell $C$. Let $a_i, a_j$ be the generators of $G_{(r,m)}$ labelling $C$. Let $p \in \{p_1, \ldots, p_k\}$ separate $i$ and $j$. The image $\overline{\phi}_p(W)$ is a wall in $X_{(4,m)}$ that self-intersects, which is a contradiction. Thus walls in $X_{(r,m)}$ embed. We now show that no wall in $X_{(r,m)}$ has a self-osculation. Suppose $W$ in $X_{(r,m)}$ has a self-osculation at some 0-cell $x$, and let $C, C'$ be 2-cells adjacent to $x$ such that
\( W \) is dual to edges in both \( C, C' \). Let \( a_i, a_j, a'_i, a'_j \) be generators that label the boundaries of \( C, C' \). If \( i, j, i', j' \) are distinct consider a partition \( p \) that separates them. The image \( \phi_p(W) \) is a wall in \( X_{(4,m)} \) that has a self-osculation, which is a contradiction. Otherwise \( C \) and \( C' \) share one label and we let \( p \) be a partition that separates the three distinct generators, and the argument is similar. Hence walls do not have self-osculations in \( X_{(r,m)} \). Finally, the fact that all walls of \( X_{(r,m)} \) are two-sided follows by considering a single \( \phi_p : X_{(r,m)} \to X_{(4,m)} \).

Finally, to see that the degree is bounded by a polynomial we observe that:

\[
|Q|^k \leq |Q|^{\log \left( \frac{r}{4} \right) \log |Q| \log \log |Q|} = \left| Q \right|^{r \log |Q| / \log 32} \leq |Q|^{r \log |Q| / \log 32}. 
\]

\[\square\]

3.2. Probability of an empty or disconnected link is exponentially small.

Let \( \Gamma \) be a complete graph on \( r \) vertices. Consider assigning a vertex to be ascending [respectively descending] with probability \( \frac{1}{2} \). Furthermore, for an edge whose vertices are ascending [descending] assign it to be ascending [descending] with probability \( \frac{1}{2} - \frac{1}{2} m \). Let \( \Gamma^\uparrow \) [\( \Gamma^\downarrow \)] be the subgraph of \( \Gamma \) consisting of all ascending [descending] vertices and edges.

Observe that \( \Gamma^\uparrow \) is assured to be nonempty and connected if

1. there exists an ascending vertex in \( \Gamma \), and
2. for each pair of distinct vertices \( v_1, v_2 \in \Gamma^\uparrow \) there is a third vertex \( v_3 \in \Gamma^\uparrow \) such that \((v_1, v_3)\) and \((v_2, v_3)\) are edges in \( \Gamma^\uparrow \).

Let \( P_1 \) denote the probability that condition (1) fails to be satisfied. If \( \Gamma \) fails to be nonempty and connected, then at least one of (1), (2) is not satisfied. Consequently

\[ \mathbb{P}(\Gamma^\uparrow \text{ fails}) \leq P_1 + P_2. \]

Similarly,

\[ \mathbb{P}(\Gamma^\downarrow \text{ fails}) \leq P_1 + P_2. \]

Lemma 3.7. \( P_1 = \frac{1}{2^r} \).

Proof. Since no wall in \( \mathcal{W} \) has a self-osculation, each wall is adjacent to \( x \) at at most one vertex of \( \Gamma \). Each of the \( r \) vertices in \( \Gamma \) is descending with probability \( \frac{1}{2} \) and these probabilities are independent. Hence \( P_1 = \frac{1}{2^r}. \) \[\square\]

Lemma 3.8. \( P_2 \leq \left( \frac{r}{2} \right) \frac{1}{4} (1 - \frac{1}{2^{m-3}})^{r-2} \).

Proof. For distinct vertices \( v_1, v_2 \in \Gamma^\uparrow \) the edge \((v_1, v_2)\) is ascending with probability \( \frac{1}{2} - \frac{1}{2} m \). For a triple \( v_1, v_2, v_3 \) of distinct vertices in \( \Gamma \), where \( v_1, v_2 \) are ascending the probability that \( v_3 \) is also ascending and both edges \((v_1, v_3), (v_2, v_3)\) are ascending is

\[ \frac{1}{2^{2m-3}}. \]

For \( v_1, v_2 \in \Gamma^\uparrow \) the probability that there is no connecting \( v_3 \) as above equals

\[ (1 - \frac{1}{2^{2m-3}})^{r-2}. \]

Thus

\[ P_2 \leq \sum_{v_1, v_2 \in \Gamma} \frac{1}{4} (1 - \frac{1}{2^{2m-3}})^{r-2} = \left( \frac{r}{2} \right) \frac{1}{4} (1 - \frac{1}{2^{2m-3}})^{r-2}. \]

\[\square\]
Consider orientations on the set of all walls $W$ of $X_{(r,m)}$. We orient each wall randomly, assigning probability $\frac{1}{2}$ to each of two orientations for each wall $W \in W$. For each 0-cell $x \in X_{(r,m)}$ the graph $\text{link}(x)$ is complete on $r$ vertices. No self-osculations in $X_{(r,m)}$ provide that walls adjacent to two distinct edges and/or vertices of $\text{link}(x)$ are distinct. Thus every vertex of $\text{link}(x)$ is ascending [descending] with probability $\frac{1}{2}$ and each edge of $\text{link}(x)$ whose edges are ascending [descending] is ascending [descending] with probability $\frac{1}{2^m-2}$. We thus have the following:

**Corollary 3.9.** $\mathbb{P}(\text{link}↑(x) \text{ or link}↓(x) \text{ fails})$ is exponentially decreasing. Specifically

$$
\mathbb{P}(\text{link}↑(x) \text{ or link}↓(x) \text{ fails}) \leq \mathbb{P}(\text{link}↑(x) \text{ fails}) + \mathbb{P}(\text{link}↓(x) \text{ fails}) \leq 2(P_1 + P_2) \leq \frac{1}{2^{r-1}} + \left(\frac{r}{2}\right)\left(\frac{1}{2} - \frac{1}{2^m-3}\right)^{r-2}.
$$

3.3. **Proof of Theorem 3.1**

**Proof.** Proposition 3.5 provides a finite cover $\hat{X}_{(r,m)}$ whose degree is bounded by a polynomial in $r$, and such that the compression $\bar{X} = X_{(r,m)}$ has the property that its walls are two-sided and have no self-osculations.

To apply Theorem 2.1 we need to find an orientation on $W$ such that $\text{link}↑(x)$ and $\text{link}↓(x)$ are nonempty and connected for each $x \in X^0$. We orient each $W \in W$ randomly assigning probability $\frac{1}{2}$ to each of two orientations of $W$. We need to prove that

$$
\mathbb{P}(\text{link}↑(x) \text{ or link}↓(x) \text{ fails for some } x \in X^0) < 1.
$$

Since the left hand side is bounded above by

$$
\sum_{x \in X^0} \mathbb{P}(\text{link}↑(x) \text{ or link}↓(x) \text{ fails})
$$

it suffices to prove that for each $x \in \overline{X}^0$

$$
(\ast) \quad \mathbb{P}(\text{link}↑(x) \text{ or link}↓(x) \text{ fails}) < \frac{1}{|X^0|}.
$$

$|X^0|$ is bounded by a polynomial in $r$, but by Corollary 3.9 the probability on the left decreases exponentially in $r$, hence the inequality $(\ast)$ holds for all $r$ greater than some $R(m)$.

After finding an orientation on $W$ such that $\text{link}↑(x)$ and $\text{link}↓(x)$ are nonempty and connected, we consider the lawful subcomplex $Y \subset \bar{X}$ and the map $\bar{X} \xrightarrow{\phi} S^1$ induced by the orientation whose restriction to $Y$ lifts to a Morse function $\bar{Y} \rightarrow \mathbb{R}$. By Theorem 2.1, the group $\ker(\pi_1 Y \rightarrow \mathbb{Z})$ is finitely generated. Consequently, its quotient $N = \ker(\pi_1 \bar{X} \rightarrow \mathbb{Z})$ is also finitely generated. To see that $\pi_1 \bar{X} \rightarrow \mathbb{Z}$ is nontrivial, observe that $X^1$ has a positively directed closed path since $\bar{X}$ is compact and each $\text{link}↑(x)$ is nonempty. \qed
4. Local quasiconvexity and Coxeter groups with nonpositive sectional curvature

4.1. Negative sectional curvature and local quasiconvexity.

**Definition 4.1** (Sectional curvature). An angled 2-complex is a 2-complex $Y$ with an angle $\angle(e) \in \mathbb{R}$ assigned to each edge $e$ of link$(y)$ for each $y \in Y^0$. As edges in link$(y)$ correspond to corners of 2-cells at $y$, we regard the angles as assigned to corners of 2-cells at $y$. The curvature at a 2-cell $f$ of $Y$ is given by

$$\kappa(f) = 2\pi - \sum_{e \in \text{Corners}(f)} \text{def}(e)$$

where $\text{def}(e) = \pi - \angle(e)$. The curvature of $Y$ at $y$ is given by

$$\kappa(y) = 2\pi - \pi \chi(\text{link}(y)) + \sum_{e \in \text{Corners}(y)} \angle(e) = (2 - v)\pi + \sum \text{def}(e).$$

A section of a combinatorial 2-complex $Y$ at the 0-cell $y$ is a combinatorial immersion $(S, s) \to (Y, y)$. A section is regular if link$(s)$ is finite, connected, nonempty, with no valence $\leq 1$ vertex. Pulling back the angles at a corner at $y$ to corners at $s$, the curvature of a section $(S, s) \to (Y, y)$ is defined to be $\kappa(s)$. We say that $Y$ has sectional curvature $\leq \alpha$ at $y$ if all regular sections of $Y$ at $y$ have curvature $\leq \alpha$. Finally, $Y$ has sectional curvature $\leq \alpha$ if each 2-cell has curvature $\leq \alpha$ and $Y$ has sectional curvature $\leq \alpha$ at each 0-cell.

**Definition 4.2** (Quasiconvexity). Let $G$ be a group with a finite generating set $S$ and the Cayley graph $\Gamma(G, S)$. A subgroup $H$ of $G$ is quasiconvex if there is a constant $L \geq 0$ such that every geodesic in $\Gamma(G, S)$ between two elements of $H$ lies in the $L$-neighborhood of $H$. When $G$ is hyperbolic, the quasiconvexity of $H$ is independent of the generating set of $G$ [Sho91]. A group $G$ is locally quasiconvex if every finitely generated subgroup of $G$ is quasiconvex. Every quasiconvex subgroup of a hyperbolic group is finitely presented [Sho91]. Thus a locally quasiconvex hyperbolic group is coherent.

The main result about negative sectional curvature is as follows [Wis04, MPW13]:

**Theorem 4.3.** If $Y$ is a compact, piecewise Euclidean nonpositively curved 2-complex whose associated angles have negative sectional curvature, then $\pi_1 Y$ is locally quasiconvex.

The following is known about locally quasiconvex Coxeter groups:

**Proposition 4.4.** For each $r \geq 3$ there exists $N(r)$ such that for all $m > N(r)$ the group $G_{(r,m)}$ is locally quasiconvex.

We briefly review two ways of proving Proposition 4.4: One method to prove Proposition 4.4 is from [MW05] or [Sch03, Thm IV] and shows that a Coxeter group $G_{(r,m)}$ is locally quasiconvex whenever $m \geq \frac{3}{2}r$. We shall focus on reviewing conditions ensuring negative sectional curvature so that Theorem 4.3 provides Proposition 4.4.

As in Section 2.1, let $X$ be the standard 2-complex of the presentation of $G = G_{(r,m)}$ and let $\bar{X}$ be the compression of a finite cover of $X$ corresponding to a finite index torsion-free subgroup of $G$. If each 3-generator Coxeter subgroup of $G$ is infinite (i.e. $\frac{1}{m_{ij}} + \frac{1}{m_{jk}} + \frac{1}{m_{ki}} \leq 1$), then there is a natural metric of nonpositive curvature...
curvature on $\overline{X}$ induced by metrizing each 2-cell as a regular Euclidean polygon. The previous condition is equivalent to the nonpositive curvature of all sections $(S, s) \to (\overline{X}, x)$ where $S$ is a disc. Thus we say that $G$ has nonpositive planar sectional curvature, when all 3-generator Coxeter subgroups are infinite. Finally, if all exponents satisfy $m_{ij} > \frac{r(r-1)}{2(r-2)}$, then $\overline{X}$ has negative sectional curvature [Wis04, Thm 13.3].

4.2. Nonpositive sectional curvature. Let $X$ denote the standard 2-complex of the presentation of Coxeter group $G$ and let $\overline{X}$ denote the compression of a cover of $X$ corresponding to a finite index torsion-free subgroup. There is a surprisingly elegant characterization of nonpositive sectional curvature of $\overline{X}$ in terms of the Euler characteristic of Coxeter subgroups of $G$.

Theorem 4.5. The following are equivalent:

1. $\overline{X}$ has nonpositive sectional curvature,
2. $\chi(H) \leq 0$ for each nontrivial Coxeter subgroup $H \subset G$ whose associated graph $\Upsilon_H$ is connected but not a tree.

Proof. $(1) \Rightarrow (2)$: Suppose $\chi(H) > 0$ and $\Upsilon_H$ is connected and not a tree. We can assume that $\Upsilon_H$ has no valence 1 vertex, since the Coxeter subgroup $H'$ associated to the subgraph $\Upsilon_{H'}$ of $\Upsilon_H$ obtained by removing a valence 1 vertex satisfies $\chi(H') \geq \chi(H)$ by equation (1). A section at a 0-cell of $\overline{X}$ whose vertices correspond to the generators of $H$ has curvature $2\pi\chi(H)$ by comparing equations (1) and (2).

$(2) \Rightarrow (1)$: Let $x$ be a 0-cell of $\overline{X}$. It suffices to consider sections corresponding to the full subgraphs of $\text{link}(x)$. Indeed $\text{def}(e) > 0$ for each edge $e$ since each angle is $< \pi$ and thus adding edges increases $\kappa$ by the second part of equation (2). Any regular section corresponding to a full subgraph is isomorphic to the associated graph $\Upsilon_H$ of a Coxeter subgroup $H$ and the curvature of the section equals $2\pi\chi(H)$. Thus if the section has positive curvature, then $\chi(H) > 0$.

Problem 4.6. Let $G$ have a nonpositive planar sectional curvature with $\chi(G) > 0$ and $\Upsilon_G$ connected and not a tree. Is it true that $\pi_1G$ is incoherent?

We hope that the methods used here can be applied to an appropriate finite index subgroup. An affirmative answer to Problem 4.6 would be a step in proving the following:

Conjecture 4.7. If $G$ has nonpositive planar sectional curvature, then the following are equivalent:

1. $G$ is coherent,
2. $\overline{X}$ has nonpositive sectional curvature.

References

[BB97] Mladen Bestvina and Noel Brady, Morse theory and finiteness properties of groups, Invent. Math. 129 (1997), no. 3, 445–470, DOI 10.1007/s002220050168. MR1465330

[Bie81] Robert Bieri, Homological dimension of discrete groups, 2nd ed., Queen Mary College Mathematical Notes, Queen Mary College, Department of Pure Mathematics, London, 1981. MR6715779 (84h:20047)

[Dav08] Michael W. Davis, The geometry and topology of Coxeter groups, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR2360474 (2008k:20091)
[GRS80] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer, *Ramsey theory*, John Wiley & Sons, Inc., New York, 1980. Wiley-Interscience Series in Discrete Mathematics; A Wiley-Interscience Publication. MR591457 (82b:05001)

[HW10] Frédéric Haglund and Daniel T. Wise, *Coxeter groups are virtually special*, Adv. Math. 224 (2010), no. 5, 1890–1903, DOI 10.1016/j.aim.2010.01.011. MR2646113 (2011g:20060)

[MPW13] Eduardo Martínez-Pedroza and Daniel T. Wise, *Coherence and negative sectional curvature in complexes of groups*, Michigan Math. J. 62 (2013), no. 3, 507–536, DOI 10.1307/mmj/1378757886. MR3102528

[MPW13] Eduardo Martínez-Pedroza and Daniel T. Wise, *Coherence and negative sectional curvature in complexes of groups*, Michigan Math. J. 62 (2013), no. 3, 507–536, DOI 10.1307/mmj/1378757886. MR3102528

[MW05] J. P. McCammond and D. T. Wise, *Coherence, local quasiconvexity, and the perimeter of 2-complexes*, Geom. Funct. Anal. 15 (2005), no. 4, 859–927, DOI 10.1007/s00039-005-0525-8. MR2221153 (2007k:20087)

[Sch03] Paul E. Schupp, *Coxeter groups, 2-completion, perimeter reduction and subgroup separability*, Geom. Dedicata 96 (2003), 179–198, DOI 10.1023/A:1022155823425. MR1956839 (2003m:20052)

[Sho91] Hamish Short, *Quasiconvexity and a theorem of Howson’s*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 168–176. MR1170365 (93d:20071)

[Wis04] D. T. Wise, *Sectional curvature, compact cores, and local quasiconvexity*, Geom. Funct. Anal. 14 (2004), no. 2, 433–468, DOI 10.1007/s00039-004-0463-x. MR2062762 (2005j:53043)

[Wis11] Daniel T. Wise, *Morse theory, random subgraphs, and incoherent groups*, Bull. Lond. Math. Soc. 43 (2011), no. 5, 840–848, DOI 10.1112/blms/bdr023. MR2854555

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