CONCORDANCE INVARIANTS FROM HIGHER ORDER COVERS

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Abstract. We generalize the Manolescu-Owens smooth concordance invariant \(\delta(K)\) of knots \(K \subset S^3\) to invariants \(\delta_{p^n}(K)\) obtained by considering covers of order \(p^n\), with \(p\) a prime. Our main result shows that for any prime \(p \neq 2\), the thus obtained homomorphism \(\oplus_{n \in \mathbb{N}} \delta_{p^n}\) from the smooth concordance group to \(\mathbb{Z}^\infty\) has infinite rank. We also show that unlike \(\delta\), these new invariants typically are not multiples of the knot signature, even for alternating knots. A significant portion of the article is devoted to exploring examples.

1. Introduction

Many of the recent advances in our understanding of the smooth knot concordance group \(\mathcal{C}\) have been driven by the advents of two theories: Heegaard Floer homology and Khovanov homology. The former, discovered and developed by P. Ozsváth and Z. Szabó in a series of beautiful papers [19, 20, 21], has grown into a comprehensive package of invariants of low dimensional manifolds, including invariants of nullhomologous knots in arbitrary 3-manifolds. The latter has been discovered by M. Khovanov in [10] and further developed by Khovanov and L. Rosansky in [11], and is at present limited in scope to providing invariants for knots in \(S^3\).

Despite the rather different approaches taken in these two theories, they share a surprising amount of formal properties. One such similarity is a pair of spectral sequences associated to a knot \(K \subset S^3\). The \(E^2\) terms of these sequences are the knot Floer homology group \(\widehat{HF}(K)\) and the Khovanov homology group \(Kh(K)\) respectively, while their \(E^\infty\) terms are rather standard groups, namely \(\mathbb{Q}\) and \(\mathbb{Q}^2\) (when working with rational coefficients). These spectral sequences have been exploited in [24, 25] and [26] to define two epimorphisms \(\tau, s : \mathcal{C} \to \mathbb{Z}\) from the smooth knot concordance group \(\mathcal{C}\) to the integers. The two invariants exhibit a number of similar features and have at first been conjecture to be equal (indeed they agree for all quasi-alternating knots [16]) until the surprising article of M. Hedden and P. Ording [5] disproved this notion. Both invariants have been found to be powerful obstructions to smooth sliceness.

In [15] C. Manolescu and B. Owens defined yet another homomorphism \(\delta : \mathcal{C} \to \mathbb{Z}\) by exploiting a different feature of Heegaard Floer homology (thus far unparalleled in Khovanov homology). Namely, the Heegaard Floer homology package associates to a pair \((Y, s)\) consisting of a 3-manifold \(Y\) and a torsion spin\(^c\)-structure \(s\) on \(Y\) (by which we mean that \(c_1(s)\) is torsion in \(H^2(Y; \mathbb{Z})\)) a rational number \(d(Y, s)\) known as a correction term, cf. [21]. Manolescu and Owens define \(\delta(K) = 2d(Y_2(K), s_0)\) where \(Y_2(K)\) is the 2-fold cover of \(S^3\) with branching set \(K\) and \(s_0 \in Spin^c(Y_2(K))\) is the unique spin-structure on \(Y_2(K)\).

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Using properties of the correction terms, they show that $\delta$ descends to an epimorphism from $C$ to $\mathbb{Z}$.

The utility of the three homomorphisms $\tau, s, \delta : C \to \mathbb{Z}$ has been exploited by many authors and has led to substantial progress and new results about smooth knot concordance, see for example [6] for a survey. We would like to mention that $\tau, s, \delta$ have been found to be linearly independent epimorphisms, in fact they remain so even when restricted to the set of topologically slice knots as demonstrated by C. Livingston in [14]. Finally, we point out that for alternating knots, all three of $\tau, s, \delta$ agree with the knot signature, up to a multiplicative constant.

The compelling success of $\tau, s$ and $\delta$ in addressing knot concordance matters is the motivation for the present work. Specifically, the goal of this article is to introduce additional homomorphisms $\delta_{pn} : C \to \mathbb{Z}$ parametrized by a pair of positive integers $(p, n)$ of which $p$ is prime. Our construction of $\delta_{pn}$ exploits the Manolescu-Owens definition of $\delta$ (which in our notation corresponds to $\delta_2$) by using the $p^n$-fold cover of $S^3$ branched over $K$ rather than the 2-fold cover used in [15].

**Definition 1.1.** Let $K \subset S^3$ be a knot and $p$ a prime integer. Let $Y_{pn}(K)$ be the $p^n$-fold cover of $S^3$ with branching set $K$. We define the integer $\delta_{pn}(K)$ as

$\delta_{pn}(K) = 2d(Y_{pn}(K), s_0)\quad (\text{with } s_0 = s_0(K, p^n) \in \text{Spin}^c(Y_{pn}) \text{ being a spin-structure determined by } p^n \text{ and } K \text{ (see section 2 for a definition of } s_0).)$

We shall refer to $s_0$ as the canonical spin-structure associated to $(K, p^n)$.

It is not immediately clear that the thus defined $\delta_{pn}(K)$ is indeed an integer since correction terms typically take on rational, non-integral values. The normalization factor of 2 is introduced precisely for that purpose and indeed renders $\delta_{pn}(K)$ integral. This is demonstrated in section 4. With this definition in mind, the main result of this article is the next theorem.

**Theorem 1.2.** For each pair of positive integers $(p, n)$ with $p$ prime, $\delta_{pn}$ descends to a group homomorphism $\delta_{pn} : C \to \mathbb{Z}$ from the smooth knot concordance group $C$ to the integers. For a fixed prime $p \neq 2$, the homomorphism

$$\bigoplus_{n=1}^{\infty} \delta_{pn} : C \to \mathbb{Z}^\infty$$

is of infinite rank. Moreover $\delta_2 \oplus \delta_4 \oplus \delta_8 \oplus \delta_{16} : C \to \mathbb{Z}^4$ is of rank 4.

The much stronger statement of the above theorem in the case of $p \neq 2$ is based on an understanding of when the integers $p^{2n} + 1$ are prime powers. We give an answer to this question in section 4 for $p \neq 2$. When $p = 2$, then numbers $2^{2n} + 1$ are known as Fermat numbers and it is an open question in number theory which among them are prime powers. It is likely that $\bigoplus_{n=1}^{\infty} \delta_{2n} : C \to \mathbb{Z}^\infty$ is also of infinite rank but our proof doesn’t apply to this case. No effort was made to maximize the rank of $\bigoplus_{n=1}^{\infty} \delta_{2n}$; the rank of 4 from theorem 1.2 follows from examples 1.3, 1.7, 1.8 and 1.9 below. A slightly stronger statement of theorem 1.2 is given in theorem 4.5.
As already mentioned above, all three of $\tau(K), s(K), \delta(K)$ agree with the signature $\sigma(K)$ (up to a constant scalar factor) for the case of alternating knots $K$. To investigate the relation of the various $\delta_{pn}$ to the knot signature, we turn to examples.

1.1. Examples. A significant portion of the article is devoted to calculating the invariants $\delta_{pn}(K)$ for concrete knots $K$. We present 3 different computational techniques (discussed in sections 5, 6 and 7), for now however we contend ourselves with merely stating the results of these calculations. We let $T_{(a,b)}$ denote the $(a, b)$ torus knot. Throughout we assume that $a, b \in \mathbb{N}$ are relatively prime.

Example 1.3 (The right-handed trefoil $T_{(2,3)}$). For those integers $n > 0$ for which either of $6n \pm 1$ is a prime power, we get

$$\delta_{6n-1}(T_{(2,3)}) = 4 \quad \text{and} \quad \delta_{6n+1}(T_{(2,3)}) = 0$$

For example $n = 1, 2, 3, 4, 5$ give such prime powers. Additionally

$$\delta_{2n}(T_{(2,3)}) = \begin{cases} 3 & \text{if } n = 2k \text{ and } k \geq 1. \\ 1 & \text{if } n = 2k + 1 \text{ and } k \geq 0. \end{cases}$$

Example 1.4 (The torus knot $T_{(2,5)}$). For any integer $n > 0$ for which $10n \pm 1$ is a prime power (e.g. $n = 1, 3, 6, 8, 24$ etc.), we get

$$\delta_{10n-1}(T_{(2,5)}) = 4 \quad \text{and} \quad \delta_{10n+1}(T_{(2,5)}) = 0$$

In addition we also get

| $n$ | 3 | 7 | 13 | 17 |
|-----|---|---|----|----|
| $\delta_n(T_{(2,5)})$ | 4 | 0 | 4 | 0 |

Example 1.5 (The torus knot $T_{(2,9)}$). For any integer $n > 0$ making either of $18n \pm 1$ a prime power, one finds that

$$\delta_{18n-1}(T_{(2,9)}) = 8 \quad \text{and} \quad \delta_{18n+1}(T_{(2,9)}) = 0$$

Additionally we obtain

| $n$ | 5 | 7 | 11 | 13 |
|-----|---|---|----|----|
| $\delta_n(T_{(2,9)})$ | 4 | 4 | 0 | 0 |

Example 1.6 (The torus knot $T_{(3,4)}$). For any integer $n > 0$ with either of $12n \pm 1$ a prime power, we find that

$$\delta_{12n-1}(T_{(3,4)}) = 4 \quad \text{and} \quad \delta_{12n+1}(T_{(3,4)}) = 0$$

Likewise, we find

| $n$ | 5 | 7 | 17 | 19 |
|-----|---|---|----|----|
| $\delta_n(T_{(3,4)})$ | 0 | 4 | 0 | 4 |

Example 1.7 (The torus knot $T_{(3,5)}$). For any integer $n > 0$ rendering $15n \pm 1$ a prime power, one finds that

$$\delta_{15n-1}(T_{(3,5)}) = 8 \quad \text{and} \quad \delta_{15n+1}(T_{(3,5)}) = 0$$

Other examples are
Example 1.8 (The torus knot $T_{(3,7)}$). For any integer $n > 0$ rendering $21n \pm 1$ a prime power, one finds that
\[ \delta_{21n-1}(T_{(3,7)}) = 8 \quad \text{and} \quad \delta_{21n+1}(T_{(3,7)}) = 0 \]
Similarly we find that
\[
\begin{array}{c|cccc}
 n & 2 & 4 & 7 & 11 \\
\hline
\delta_n(T_{(3,5)}) & 4 & 0 & 4 & 4 \\
\end{array}
\]

Example 1.9 (The torus knot $T_{(5,7)}$). For any integer $n > 0$ for which $35n \pm 1$ is a prime power, one finds that
\[ \delta_{35n-1}(T_{(5,7)}) = 16 \quad \text{and} \quad \delta_{35n+1}(T_{(5,7)}) = 0 \]
In addition to these we also find
\[
\begin{array}{c|cccc}
 n & 2 & 4 & 5 & 8 & 16 \\
\hline
\delta_n(T_{(5,7)}) & 0 & 4 & 4 & 0 & 0 \\
\end{array}
\]

Example 1.10 (The torus knot $T_{(5,9)}$). For any integer $n > 0$ for which $45n \pm 1$ is a prime power, one finds that
\[ \delta_{35n-1}(T_{(5,9)}) = 0 \quad \text{and} \quad \delta_{35n+1}(T_{(5,9)}) = 0 \]
and for $n = 2, 4, 8, 16$ we obtain
\[
\begin{array}{c|cccc}
 n & 2 & 4 & 8 & 16 \\
\hline
\delta_n(T_{(5,9)}) & 4 & 4 & 0 & 0 \\
\end{array}
\]

Example 1.3 implies that

**Corollary 1.11.** For each integer $k \geq 0$, the homomorphism $\delta_{2k+1} : \mathcal{C} \to \mathbb{Z}$ is surjective.

**Remark 1.12.** As mentioned above, the homomorphisms $\tau, s, \delta : \mathcal{C} \to \mathbb{Z}$ agree with the signature of the knot (up to a multiplicative constant) for all alternating knots (see [16] for a stronger statement for quasi-alternating knots in the case of $\tau$ and $s$). The above examples illustrate that this is not the case for all $\delta_{p^n}$. For example
\[ \delta_7(T_{(2,3)}) = 0 \quad \delta_7(T_{(2,5)}) = 0 \quad \delta_7(T_{(2,9)}) = 4 \]
while the signatures of these knots are
\[ \sigma(T_{(2,3)}) = -2 \quad \sigma(T_{(2,5)}) = -4 \quad \sigma(T_{(2,9)}) = -8 \]

**1.2. Organization.** The remainder of the article is organized as follows. Sections 2 and 3 provide background material on spin$^c$-structures (with an emphasis on branched covers) and Heegaard Floer homology. Definition 2.3 in section 2 specifies the canonical spin-structure $s_0(K, p^n)$ alluded to in definition 1.1. Section 4 is devoted to the proof of theorem 1.2 and its slightly strengthened version, theorem 4.5. Sections 5 – 7 are devoted to exploring computational techniques. In particular, the results from examples 1.3 – 1.10 follow directly from the discussions in those sections.

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2. Spin\(^c\)-structures

2.1. Spin\(^c\)-structures on three and four manifolds. This section discusses spin\(^c\)-structures on 3 and 4 manifolds. Our exposition largely follows that from chapter 11 in Turaev’s book [31], see also [30]. To begin with, recall that the groups \(\text{Spin}(n)\) for \(n \geq 3\) are defined to be the universal covering spaces of \(\text{SO}(n)\) (it is well known that \(\pi_1(\text{SO}(n)) \cong \mathbb{Z}_2\) for \(n \geq 3\), see for example [4]) and \(\text{Spin}(n)\) is defined as

\[
\text{Spin}(n) = (\text{Spin}(n) \times U(1))/\mathbb{Z}_2
\]

where \(\mathbb{Z}_2 = \{\pm 1\}\) acts by diagonal multiplication. In the cases of \(n = 3, 4\) one obtains group isomorphisms \(\text{Spin}^c(3) \cong U(2)\) and \(\text{Spin}^c(4) \cong U(2) \times U(2)\). Similarly, there are diffeomorphisms \(\text{SO}(3) \cong \mathbb{R}P^3\) and \(\text{SO}(4) \cong S^3 \times \mathbb{R}P^3\), cf. [4]. For later use we point out that

\[
H^2(\text{SO}(n); \mathbb{Z}) \cong \mathbb{Z}_2 \quad \text{and} \quad H^1(\text{SO}(n); \mathbb{Z}_2) \cong \mathbb{Z}_2 \quad \text{for } n = 3, 4.
\]

If one thinks of \(\text{Spin}^c(n) \to \text{SO}(n)\) as a \(U(1)\)-bundle, then in both the cases of \(n = 3, 4\), the first Chern class of the bundle corresponds to the nontrivial element of \(H^2(\text{SO}(n); \mathbb{Z})\).

In the following we let \(Y\) be a 3-manifold and \(X\) a 4-manifold, both possibly with boundary. All manifolds are always assumed to be smooth, compact and oriented. For convenience we endow our manifolds with a Riemannian metric which we assume to be a product metric in a collar neighborhood of the boundary. By the frame bundle we shall mean the bundle of oriented orthonormal frames and we shall denote it by \(\text{Fr}_X\) or \(\text{Fr}_Y\). These are, of course, principal \(\text{SO}(3)\) and \(\text{SO}(4)\) bundles respectively.

A spin\(^c\)-structure \(s\) on \(Y\) is a principal \(\text{Spin}^c(3)\) bundle \(P_{\text{Spin}^c(3)} \to Y\) together with an bundle map \(\alpha : P_{\text{Spin}^c(3)} \to \text{Fr}_Y\) which fiberwise restricts to give the above map \(\text{Spin}^c(3) \to \text{SO}(3)\) (i.e. projection from \(\text{Spin}^c(3)\) to \(\text{Spin}(3)\), followed by the covering map to \(\text{SO}(3)\)). An alternative and equivalent point of view is to think of \(s\) as a \(U(1)\)-bundle over \(\text{Fr}_Y\) which restricts over each fiber \(\text{SO}(3) \subset \text{Fr}_Y\) to give the unique nontrivial \(U(1)\)-bundle over \(\text{SO}(3)\). Said differently, we can define a spin\(^c\)-structure \(s\) on \(Y\) as an element from \(H^2(\text{Fr}_Y; \mathbb{Z})\) which on each fiber \(\text{SO}(3) \subset \text{Fr}_Y\) restricts to the unique nontrivial element of \(H^2(\text{SO}(3); \mathbb{Z}) \cong \mathbb{Z}_2\). Fixing an orthogonal trivialization of \(TY\), we obtain the diffeomorphism \(\text{Fr}_Y \cong Y \times \text{SO}(3)\) and therefore

\[
H^2(\text{Fr}_Y; \mathbb{Z}) \cong H^2(Y; \mathbb{Z}) \oplus H^2(\text{SO}(3); \mathbb{Z}) \cong H^2(Y; \mathbb{Z}) \oplus \mathbb{Z}_2
\]

A spin\(^c\)-structure \(s\) on \(Y\) is thus an element of \(H^2(\text{Fr}_Y; \mathbb{Z})\) with nontrivial second coordinate in this decomposition. There is an obvious action of \(H^2(Y; \mathbb{Z})\) on \(\text{Spin}^c(3)\) given by the pullback map on second cohomology induced by the bundle map \(\text{Fr}_Y \to Y\) along with addition in \(H^2(\text{Fr}_Y; \mathbb{Z})\). This action is obviously free and transitive revealing that \(\text{Spin}^c(Y)\) is an \(H^2(Y; \mathbb{Z})\)-affine space.

\(\text{Spin}^c(Y)\) comes equipped with an involution sending and element in \(H^2(\text{Fr}_Y; \mathbb{Z})\) to its negative. We shall denote this map by \(s \mapsto \bar{s}\) and call \(\bar{s}\) the conjugate spin\(^c\)-structure of \(s\). Finally, note that if \(Y' \subset Y\) is a codimension zero submanifold, there is a natural restriction induced map \(\text{Spin}^c(Y) \to \text{Spin}^c(Y')\). In applications below, \(Y'\) will typically be the complement of a tubular neighborhood of a knot in \(Y\).

The same arguments apply verbatim to the 4-manifold \(X\) as well (though the case of \(X\) closed and \(TX\) nontrivial requires a slightly different argument to show that \(H^2(\text{Fr}_X; \mathbb{Z}) \cong \mathbb{Z}_2\) for \(n \geq 3\)).
$H^2(X;\mathbb{Z}) \oplus H^2(SO(4);\mathbb{Z})$, it involves a choice of a trivialization of $TX$ over $X$ minus a 4-ball).

A spin-structure on $Y$ is a principal $Spin(3)$-bundle $P_{Spin(3)} \to Y$ with a bundle map to $Fr_Y$ which fiberwise restricts to the double covering map $Spin(3) \to SO(3)$. We denote the set of spin-structures on $Y$ by $Spin(Y)$. Thinking of $P_{Spin(3)}$ as a (real) line bundle over $Fr_Y$, we can alternatively define a spin-structure on $Y$ as an element of $H^1(Fr_Y;\mathbb{Z}_2)$ which on each fiber $SO(3)$ restricts to the nontrivial element of $H^1(SO(3);\mathbb{Z}_2)$. Just as with the case of spin$^c$-structures, the obvious action of $H^1(Y;\mathbb{Z}_2)$ on $Spin(Y)$ gives the latter the structure of an affine $H^1(Y;\mathbb{Z}_2)$-space. Identical definitions and properties apply to $Spin(X)$, the space of spin-structures on $X$.

There is a natural homomorphism of affine spaces $Spin(Y) \to Spin^c(Y)$ given by the Bockstein map $H^1(Fr_Y;\mathbb{Z}_2) \to H^2(Fr_Y;\mathbb{Z})$ associated to the exact sequence $0 \to \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. Thus, under a compatible choice of origins in $Spin(Y)$ and $Spin^c(Y)$, a spin$^c$-structure $s \in H^2(Y;\mathbb{Z})$ is a spin-structure if and only if

$$s \in \text{Im} (H^1(Y;\mathbb{Z}_2) \to H^2(Y;\mathbb{Z})) = \text{Ker} (H^2(Y;\mathbb{Z}) \otimes \mathbb{Z} \to H^2(Y;\mathbb{Z}))$$

Note that distinct spin-structures may descend to give the same spin$^c$-structure. Similar considerations apply to $X$.

### 2.2. Spin$^c$-structures on connected sums.

Let $Y_0$ and $Y_1$ be two closed 3-manifolds and let $Y = Y_0 \# Y_1$ be their connected sum. Let $B_i \subset Y_i$ be the 3-balls used to perform the connected sum. An easy exercise in homological algebra reveals that the restriction maps $H^2(Y_i;\mathbb{Z}) \to H^2(Y_i - B_i;\mathbb{Z})$ are isomorphisms giving rise to the isomorphism (of affine spaces) $Spin^c(Y_1) \to Spin^c(Y_1 - B_1)$. We shall utilize these isomorphisms to identify spin$^c$-structures on $Y_1$ and $Y_1 - B_1$.

Yet another easy exercise shows that the restriction map

$$H^2(Y;\mathbb{Z}) \to H^2(Y_1 - B_1;\mathbb{Z}) \oplus H^2(Y_2 - B_2;\mathbb{Z})$$

is an isomorphism as well allowing us to identify $Spin^c(Y)$ with $Spin^c(Y_1 - B_1) \times Spin^c(Y_2 - B_2)$. Putting these two observations together yields the isomorphism

$$Spin^c(Y_1) \times Spin^c(Y_2) \cong Spin^c(Y_1 \# Y_2) \quad (s_1, s_2) \mapsto s_1 \# s_2 \quad (1)$$

### 2.3. Spin$^c$-structures on branched covers.

We now turn our attention to spin$^c$-structures on branched covers. We adopt the convention that whenever we consider a smooth map $f : Y_0 \to Y_1$ which is a local diffeomorphism, the Riemannian metric on $Y_0$ shall be the one obtained from the metric on $Y_1$ via pullback through $f$.

By way of notation, let $K \subset S^3$ be a knot and let $Y_{p^n}(K)$ be the $p^n$-fold branched cover of $S^3$ with branching set $K$. We always assume that $p$ is prime so that $Y_{p^n}(K)$ is a rational homology sphere (for a nice proof of this fact see [13]). We let $f : Y_{p^n}(K) \to S^3$ denote the branch covering map and we let $K' = f^{-1}(K)$. We shall write $N(K)$ and $N(K')$ to denote tubular neighborhoods of $K$ and $K'$ respectively.

Since $f : (Y_{p^n}(K) - N(K')) \to (S^3 - N(K))$ is a local diffeomorphism, it induces a push-forward map $f_* : Fr_{Y_{p^n}(K) - N(K')} \to Fr_{S^3 - N(K)}$, this map in turn induces a pull-back map

$$(f_*)^* : H^2(Fr_{S^3 - N(K)};\mathbb{Z}) \to H^2(Fr_{Y_{p^n}(K)};\mathbb{Z})$$
Picking a trivialization of the tangent bundle of \( S^3 - N(K) \) and then a compatible trivialization of the tangent bundle of \( Y - N(K') \) (the compatibility facilitated by \( f \) in the obvious way), it is easy to see that the restriction of \( (f_*)^\ast \) to \( H^2(SO(3); \mathbb{Z}) \) is injective. Therefore, \( (f_*)^\ast \) descends to a map

\[
(f_*): Spin^c(S^3 - N(K)) \to Spin^c(Y - N(K'))
\]

A similar discussion for spin-structure also yields a map, still denoted by \( f^* \), which with the previously defined one fits into the commutative diagram

\[
\begin{array}{ccc}
Spin(Y - N(K')) & \xrightarrow{f_*} & Spin^c(Y - N(K')) \\
\downarrow f^* & & \downarrow f^* \\
Spin(S^3 - N(K)) & \xrightarrow{f_*} & Spin^c(S^3 - N(K))
\end{array}
\]

We shall return to \( f^* \) after proving the next auxiliary lemma.

**Lemma 2.1.** Let \( f : Y_{p^n}(K) \to S^3 \) be the \( p^n \)-fold cyclic covering map with branching set the knot \( K \subset S^3 \) and set \( K' = f^{-1}(K) \). Then the restriction map \( H^2(Y_{p^n}(K); \mathbb{Z}) \to H^2(Y_{p^n}(K) - N(K'); \mathbb{Z}) \) is an isomorphism. Consequently, the restriction map

\[
Spin^c(Y_{p^n}(K)) \to Spin^c(Y_{p^n}(K) - N(K'))
\]

is likewise an isomorphism (of affine spaces).

**Proof.** Set \( Y = Y_{p^n}(K) \) and consider the Mayer-Vietoris sequence in cohomology for the decomposition \( Y = (Y - N(K')) \cup N(K') \):

\[
0 \to H^1(Y - N(K'); \mathbb{Z}) \oplus H^1(N(K'); \mathbb{Z}) \to H^1(\partial N(K'); \mathbb{Z}) \to H^2(Y; \mathbb{Z}) \to H^2(Y - N(K'); \mathbb{Z}) \to 0
\]

Since \( H^1(Y - N(K'); \mathbb{Z}) \) is generated by the Hom-dual of the meridian of \( K' \), we see that the map \( H^1(Y - N(K'); \mathbb{Z}) \oplus H^1(N(K'); \mathbb{Z}) \to H^1(\partial N(K'); \mathbb{Z}) \) is an isomorphism and therefore so is the map \( H^2(Y; \mathbb{Z}) \to H^2(Y - N(K'); \mathbb{Z}) \). \( \square \)

**Corollary 2.2.** Every spin\(^c\)-structure on \( Y_{p^n}(K) - N(K') \) extends, in a unique manner, to a spin\(^c\)-structure on \( Y_{p^n}(K) \).

**Definition 2.3.** Let \( K \) be a knot in \( S^3 \), \( p \) a prime integer and \( n \geq 1 \) a natural number. Let \( f : Y_{p^n}(K) \to S^3 \) be the \( p^n \)-fold branched covering map with branching set \( K \). We define \( s_0 = s_0(K, p^n) \in Spin^c(Y_{p^n}(K)) \) to be the unique spin\(^c\)-structure whose restriction to \( Y_{p^n}(K) - N(f^{-1}(K)) \) is the pull-back spin\(^c\)-structure \( f^*(s) \) (see [2]) of the unique spin\(^c\)-structure \( s \in Spin^c(S^3 - N(K)) \). We shall refer to \( s_0 \) as the canonical spin-structure of \( (K, p^n) \).

**Theorem 2.4.** The canonical spin-structure of a knot satisfies the following properties.

1. If \( K \subset S^3 \) is a smoothly slice knot with slice disk \( D^2 \hookrightarrow D^4 \) and if \( X \) is the \( p^n \)-fold branched cover of \( D^3 \) with branching set \( D^2 \) (so that \( \partial X = Y_{p^n}(K) \)), then \( s_0(K, p^n) \) lies in the image of the restriction map \( Spin^c(X) \to Spin^c(Y_{p^n}(K)) \).
2. If \( K_1, K_2 \subset S^3 \) are two knots, then (see [1])

\[
s_0(K_1 \# K_2, p^n) = s_0(K_1, p^n) \# s_0(K_2, p^n)
\]
3. $\mathfrak{s}_0(K, p^n)$ is a spin-structure on $Y$.

**Proof.** 1. Let us denote the slice disk $D^2 \hookrightarrow D^4$ by $\sigma$ and let $F : X \to D^4$ be the branched covering map and let $f = \partial F$. Set $\sigma' = F^{-1}(\sigma)$ and let $N(\sigma)$ and $N(\sigma') = F^{-1}(N(\sigma))$ denote tubular neighborhoods of $\sigma$ and $\sigma'$ respectively. Note that $N(\sigma) \cong D^2 \times D^2 \cong N(\sigma')$.

Since $F : X - N(\sigma') \to D^4 - N(\sigma)$ is a local diffeomorphism, it induces a pull-back map $F^* : \text{Spin}^c(D^4 - N(\sigma)) \to \text{Spin}^c(X - N(\sigma'))$. The proof of this analogous to that of $f^*$ discussed in equation [2], the details are omitted. Of course, $\text{Spin}^c(D^4)$ is a one-point set and we denote its sole member by $t$. By abuse of notation, we let $t$ also denote the only element of $\text{Spin}^c(D^4 - N(\sigma))$. Clearly $t\mid_{S^3 - N(K)} = s$, the unique spin$^c$-structure on $S^3 - N(K)$. Consider now the commutative diagram

$$
\begin{array}{ccc}
\text{Spin}^c(X) & \xrightarrow{r_1} & \text{Spin}^c(Y) \\
\downarrow r_2 & & \downarrow \cong \downarrow r_3 \\
\text{Spin}^c(X - N(\sigma')) & \xrightarrow{r_4} & \text{Spin}^c(Y - N(K')) \\
\uparrow F^* & & \uparrow f^* \\
\text{Spin}^c(D^4 - N(\sigma)) & \xrightarrow{r_5} & \text{Spin}^c(S^3 - N(K))
\end{array}
$$

where all $r_j$ stand for restriction maps. Then

$$
\mathfrak{s}_0(K, p^n) = r_5^{-1}(f^*(r_5(t))) = r_3^{-1}(r_4(F^*(t)))
$$

Thus, to prove point 1 of the theorem, we need to show that $F^*(t) \in Im(r_2)$. This in turn follows from the Mayer-Vietoris sequence for the decomposition $X = (X - N(\sigma')) \cup N(\sigma')$ by which the restriction map $H^2(X; \mathbb{Z}) \to H^2(X - N(\sigma'); \mathbb{Z})$ induces an isomorphism.

2. Consider two copies $S_1$ and $S_2$ of $S^3$ with $K_i \subset S_i$. Fix identifications of $N(K_i)$ with $S^1 \times D^2$ and pick small unknotted arcs $I_i \subset K_i$. We shall use the 3-balls $B_i = I_i \times D^2 \subset N(K_i)$ to perform the connected sum of $(S_1, K_1)$ and $(S_2, K_2)$, i.e.

$$(S^3, K_1 \# K_2) = ((S_1 - B_1) \cup_{\partial B_1 = \partial B_2} (S_2 - B_2), (K_1 - I_1) \cup_{\partial I_1 = \partial I_2} (K_2 - I_2))$$

Let $Y_i = Y_{p^n}(K_i)$ and $Y = Y_{p^n}(K_1 \# K_2)$ and let $f_i : Y_i \to S^3$ and $g : Y \to S^3$ be the $p^n$-fold branched covering maps. Note that $g = f_1 \# f_2$. As before, let $K_i' = f_i^{-1}(K_i)$ and define $I_i' \subset K_i'$ as $f_i^{-1}(I_i)$ and set $B_i' = I_i' \times D^2 = f_i^{-1}(B_i) \subset N(K_i')$. Observe that $Y_{p^n}(K_1 \# K_2) \cong Y_{p^n}(K_1) \# Y_{p^n}(K_2)$ where the connected sum is performed by removing $B_i'$ from $Y_{p^n}(K_i')$ and gluing them along their boundaries, exercising care so as to glue $K_i' - f_i^{-1}(I_i)$ to $K_i' - f_i^{-1}(I_i)$ in an orientation respecting manner.

The complement of the tubular neighborhood $N(K_1 \# K_2)$ in $S^3$ can be obtained from $S^3 - N(K_1)$ and $S^3 - N(K_2)$ by gluing them along $I_i \times \partial B_i$, a similar statement holds for $Y_{p^n}(K_1 \# K_2) - N(K_1' \# K_2')$:

$$
S^3 - N(K_1 \# K_2) = (S^3 - N(K_1)) \cup_{I_1 \times \partial B_1 = I_2 \times \partial B_2} (S^3 - N(K_2))
$$
$$
Y - N(K_1' \# K_2') = (Y_1 - N(K_1')) \cup_{I_1' \times \partial B_1' = I_2' \times \partial B_2'} (Y_2 - N(K_2'))
$$
A Mayer-Vietoris argument now shows that the restriction maps induce isomorphisms

\[ H^2(S^3 - N(K_1 \# K_2); \mathbb{Z}) \cong H^2(S^3 - N(K_1); \mathbb{Z}) \oplus H^2(S^3 - N(K_2); \mathbb{Z}) \]

\[ H^2(Y - N(K_1' \# K_2'); \mathbb{Z}) \cong H^2(Y_1 - N(K_1'); \mathbb{Z}) \oplus H^2(Y_2 - N(K_2'); \mathbb{Z}) \]

descending to affine isomorphisms between the corresponding spaces of spin\(^c\)-structures. This leads to the commutative diagram

\[
\begin{array}{ccc}
Spin^c(Y - N(K_1' \# K_2')) & \longrightarrow & Spin^c(Y_1 - N(K_1')) \times Spin^c(Y_2 - N(K_2')) \\
g^c & & f_1^* \times f_2^* \\
Spin^c(S^3 - N(K_1 \# K_2)) & \longrightarrow & Spin^c(S^3 - N(K_1)) \times Spin^c(S^3 - N(K_2))
\end{array}
\]

from which the proof of point 2 of the theorem follows. Namely, let \( s \in Spin^c(S^3 - N(K_1 \# K_2)) \) be the unique spin\(^c\)-structure on \( S^3 - N(K_1 \# K_2) \) so that \( s_0(K_1 \# K_2, p^n) \) is the unique extension to \( Y \) of \( g^*(s) \). But \( s_0(K_i, p^n) \) in turn is the unique extension to \( Y_i \) of \( f_i^*(s) \) and so the commutativity of the above diagram along with \( g = f_1 \# f_2 \), shows that \( s_0(K_1 \# K_2) = s_0(K_1, p^n) \# s_0(K_2, p^n) \), as claimed.

3. This is trivial, it follows for example from the commutative diagram (3). \( \square \)

**Remark 2.5.** Our definition of \( s_0(K, p^n) \) agrees, as a spin\(^c\)-structure, with that given by Grigsby, Ruberman and Strle in lemma 2.1 of [3]. The definition from [3] takes extra care to extend \( f^*(s) \) to \( Y_{p^n}(K) \) as a spin-structure and thus requires, in some cases, twisting by an element of \( H^1(Y; \mathbb{Z}_2) \). In those cases their definition differs from ours as a spin-structure, though not as a spin\(^c\)-structure.

### 3. Background material

This section gathers some supporting material to be used in section 4 in the proof of theorem 1.2

#### 3.1. Surgeries on torus knots

We start by recalling a beautiful theorem due to Louise Moser explaining which 3-manifolds are obtained by surgeries on torus knots.

**Theorem 3.1** (Moser [17]). Let \( a, b \in \mathbb{N} \) be two nonzero and relatively prime integers. Let \( T_{a,b} \subset S^3 \) denote the \((a, b)\) torus knot and let \( S^3_{p/q}(T_{a,b}) \) denote the 3-manifold obtained by \( p/q \)-framed Dehn surgery on \( T_{a,b} \). Then

\[
S^3_{p/q}(T_{a,b}) = \begin{cases} 
L(|p|, qb^2) & \text{if } |abq - p| = 1 \\
L(a, b) \# L(b, a) & \text{if } |abq - p| = 0 \\
S(a, b, |abq - p|) & \text{if } |abq - p| > 0
\end{cases}
\]

where \( S(x_1, x_2, x_3) \) with \( x_i \in \mathbb{Z} - \{0\} \) is a Seifert fibered space with 3 singular fibers of multiplicities \( x_1, x_2 \) and \( x_3 \).

In section 3 of [17], Moser outlines an algorithm to pin down the exact Seifert fibered space \( S(x_1, x_2, x_3) \) in case 3 of the above theorem. Rather than addressing how to do this in general, we focus on a special case of interest to us. Recall first that the Brieskorn sphere \( \Sigma(a, b, c) \) associated to a triple \( a, b, c \in \mathbb{N} \) of mutually prime integers, is the integral
homology 3-sphere obtained as the intersection $V(a, b, c) \cap S^5 \subset \mathbb{C}^3$ where $V(a, b, c)$ is the complex variety

$$V(a, b, c) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^a + z_2^b + z_3^c = 0\}$$

which is smooth away from the origin. Being the boundary of $V(a, b, c) \cap D^6$, $\Sigma(a, b, c)$ carries a natural orientation. The Brieskorn sphere $\Sigma(a, b, c)$ is realized as (see Kauffman [8]):

\begin{equation}
\Sigma(a, b, c) = \begin{cases}
\text{The } a\text{-fold branched cover of the torus knot } T_{(b,c)}. \\
\text{The } b\text{-fold branched cover of the torus knot } T_{(a,c)}. \\
\text{The } c\text{-fold branched cover of the torus knot } T_{(a,b)}. 
\end{cases}
\end{equation}

With the above orientation convention, we get

**Corollary 3.2** (Moser [17]). Let $n$ be a natural number and $(a, b)$ a pair of coprime positive integers. Then $\pm 1/n$ surgery on the torus knot $T_{(a,b)}$ yields the Brieskorn sphere $-\Sigma(a, b, abn \mp 1)$.

### 3.2. Heegaard Floer correction terms.

The material presented in this section can be found in [21].

Let $(Y, s)$ be a pair consisting of a rational homology 3-sphere $Y$ and a spin$^c$-structure $s \in Spin^c(Y)$. To such a pair, Ozsváth and Szabó [21] associate the rational number $d(Y, s) \in \mathbb{Q}$ called the correction term of $(Y, s)$. The correction terms satisfy a number of properties, some of which we point to in the next theorem.

**Theorem 3.3** (Ozsváth - Szabó, [21]). The correction terms satisfy the three properties:

1. If $X$ is a rational homology 4-ball with boundary $Y$ and if $s$ is a spin$^c$-structure on $Y$ lying in the image of the map $Spin^c(X) \to Spin^c(Y)$, then $d(Y, s) = 0$.
2. If $(Y_1, s_1)$ and $(Y_2, s_2)$ are two spin$^c$ rational homology 3-spheres, then

$$d(Y_1 \# Y_2, s_1 \# s_2) = d(Y_1, s_1) + d(Y_2, s_2)$$

3. If $-Y$ denotes $Y$ with reversed orientation, then $d(-Y, s) = -d(Y, s)$.

The correction terms are in general hard to compute. In select cases however, Ozsváth and Szabó provide easy to use formulae computing them. One such formula is

**Theorem 3.4** (Ozsváth - Szabó, [21]). Let $K \subset S^3$ be a knot and suppose that $p$-framed surgery on $K$, with $p > 0$, yields a lens space. For a rational number $r$ let $S^3_r(K)$ denote the result of $r$-framed Dehn surgery on $K$. Then, for the unique spin$^c$-structure $s_0$ on $S_{\pm 1/n}(K)$, we obtain

$$d(S^3_{1/n}(K), s_0) = -2t_0 \quad \text{and} \quad d(S^3_{-1/n}(K), s_0) = 0$$

where $t_0$ is the 0-th torsion coefficient computed from the symmetrized Alexander polynomial $\Delta_K(t) = a_0 + \sum_{j=1}^d a_j(t^i + t^{-1})$ of $K$ as $t_0 = \sum_{j=1}^d ja_j$.

**Corollary 3.5.** The correction terms of $\Sigma(a, b, nab \pm 1)$ are

\begin{equation}
\begin{aligned}
d(\Sigma(a, b, nab - 1)) &< 0 \quad \text{and} \quad d(\Sigma(a, b, abn + 1)) = 0
\end{aligned}
\end{equation}
Proof. This corollary is a direct consequence of theorem 3.1, corollary 3.2, and theorem 3.4 along with computing $t_0$ for torus knots. Since the Alexander polynomial $\Delta_{T(a,b)}(t)$ of the torus knot $T(a,b)$ equals
\[ \Delta_{T(a,b)}(t) = \frac{(t^{ab} - 1)(t - 1)}{(t^a - 1)(t^b - 1)} \cdot t^{-(a-1)(b-1)/2} \]
its coefficients are $\pm 1$ and alternate in sign. The leading coefficient of $\Delta_{T(a,b)}(t)$ is 1 rendering the 0-th torsion coefficient $t_0$ of $T(a,b)$ positive, and therefore making $-2t_0$ negative, as claimed. \(\square\)

4. Linear independence

Recall from the introduction that we defined $\delta_{p^n}(K)$ to be $2d(Y_{p^n}(K), s_0(K, p^n))$ as specified in definition 2.3. We first show that $\delta_{p^n}$ gives rise to a well defined homomorphism $\delta_{p^n} : C \rightarrow \mathbb{Z}$. The proofs of the proceeding two propositions heavily rely on Kauffman’s results from [8, 9].

**Proposition 4.1.** Let $K \subset S^3$ be a knot and $\tilde{\Sigma} \subset S^3$ be any Seifert surface of $K$. Let $\Sigma \subset D^4$ be obtained from $\tilde{\Sigma}$ by pushing the interior of the latter into $D^4$ (so that $\Sigma$ is properly embedded in $D^4$ and $\partial \Sigma = K$). Let $X_{\Sigma}$ be the $p^n$-fold branched cover of $D^4$ with branching set $\Sigma$. Then the following hold:

1. The signature $\sigma(X_{\Sigma})$ of $X_{\Sigma}$ is
\[ \sigma(X_{\Sigma}) = \sum_{i=0}^{p^n-1} \sigma_{\omega^i}(K) \]
where $\sigma_{\tau}(K)$ is the Tristram-Levine signature of $K$ associated to $\tau \in S^1$ and $\omega$ is a primitive $p^n$-th root of unity.
2. $X_{\Sigma}$ is a spin manifold and with a unique spin-structure.

**Proof.** This proposition is largely contained in the work of Kauffman [8] (see also [9]) where he extensively studies the algebraic topology of the manifold $X_{\Sigma}$. Specifically, the signature formula from the first claim of the proposition has been worked out by Kauffman (page 290 in [9]).

For the second claim we also rely on [8, 9]. In these works, Kauffman finds an explicit matrix representative for the intersection form on $H^2(X_{\Sigma}; \mathbb{Z})$ (in terms of the linking matrix of $K$ associated to $\Sigma$, see page 283 of [9]) from which one can readily pin down the second Stiefel-Whitney class $w_2(TX_{\Sigma})$ and finds the latter to be zero implying that $X_{\Sigma}$ is a spin manifold. Since $Spin(X)$ is an affine space on $H^1(X_{\Sigma}; \mathbb{Z}_2)$, the second claim of the proposition follows from $H^1(X_{\Sigma}; \mathbb{Z}_2) = 0$, another results from [9] (page 282). \(\square\)

**Proposition 4.2.** Let $p$ be any prime number. Then

1. For every knot $K \subset S^3$ the number $\delta_{p^n}(K)$ is an integer.
2. For any two knots $K_0, K_1 \subset S^3$ one obtains $\delta_{p^n}(K_0 \# K_1) = \delta_{p^n}(K_0) + \delta_{p^n}(K_1)$.
3. If $K$ is smoothly slice then $\delta_{p^n}(K) = 0$ for all choices of $p, n \in \mathbb{N}$ with $p$ prime.

In particular, $K \mapsto \delta_{p^n}(K)$ is a group homomorphism from $C \rightarrow \mathbb{Z}$. 

Proof. This proposition is largely contained in the work of Kauffman [8] (see also [9]) where he extensively studies the algebraic topology of the manifold $X_{\Sigma}$. Specifically, the signature formula from the first claim of the proposition has been worked out by Kauffman (page 290 in [9]).

For the second claim we also rely on [8, 9]. In these works, Kauffman finds an explicit matrix representative for the intersection form on $H^2(X_{\Sigma}; \mathbb{Z})$ (in terms of the linking matrix of $K$ associated to $\Sigma$, see page 283 of [9]) from which one can readily pin down the second Stiefel-Whitney class $w_2(TX_{\Sigma})$ and finds the latter to be zero implying that $X_{\Sigma}$ is a spin manifold. Since $Spin(X)$ is an affine space on $H^1(X_{\Sigma}; \mathbb{Z}_2)$, the second claim of the proposition follows from $H^1(X_{\Sigma}; \mathbb{Z}_2) = 0$, another results from [9] (page 282). \(\square\)
Proof. For the first statement of the proposition we recall a formula proved by Ozsváth and Szabó in [21]. To state the result, let $X$ be any smooth 4-manifold with $\partial X = Y_{p^n}(K)$ and let $t \in Spin^c(X)$ be any spin$^c$-structure with $t|_{Y_{p^n}(K)} = s_0(K, p^n)$. Then

$$d(Y_{p^n}(K), s_0(K, p^n)) \equiv \frac{c_1(t)^2 - \sigma}{4} \pmod{2}$$

Given a Seifert surface $\Sigma \subset S^3$ of $K$, let $X = X_\Sigma$ be the 4-manifold from proposition 4.1. Let $F : X_\Sigma \to D^4$ be the branched covering map and let $f : Y_{p^n}(K) \to S^3$ be $\partial F$. The restriction map $Spin^c(X) \to Spin^c(Y_{p^n}(K))$ is modeled on the map $H^2(X; \mathbb{Z}) \to H^2(Y_{p^n}(K); \mathbb{Z})$. The latter is surjective since $H^2(X, Y_{p^n}(K); \mathbb{Z}) \cong H_1(X; \mathbb{Z}) = 0$ by a result of Kauffman’s (page 282 in [9]). Thus every spin$^c$-structure on $Y_{p^n}(K)$ extends to a spin$^c$-structure on $X$.

To see that $s_0(K, p^n)$ extends to the unique spin-structure $t_0$ on $X$ (part 2 of proposition 4.1), consider the following isomorphisms

$$H^2(X, X - N(\Sigma'); \mathbb{Z}_2) \cong H^2(N(\Sigma'), \partial N(\Sigma); \mathbb{Z}_2) \cong H_2(N(\Sigma'); \mathbb{Z}_2) \cong 0$$

The first of these follows by excision and the second by Alexander-Poincaré duality. With this as input, consider the following portion of the exact sequence of the pair $(X, X - N(\Sigma'))$ with $\mathbb{Z}_2$-coefficients:

$$\ldots \to H^1(X; \mathbb{Z}_2) \to H^1(X - N(\Sigma'); \mathbb{Z}_2) \to H^2(X, X - N(\Sigma'); \mathbb{Z}_2) \to \ldots$$

As already pointed out in the proof of proposition 4.1, Kauffman’s results from [9] show that $H^1(X; \mathbb{Z}_2) = 0$ and so we conclude that $H^1(X - N(\Sigma'); \mathbb{Z}_2) = 0$ also. This shows that $Spin(X - N(\Sigma'))$ consists of a single spin-structure and that therefore the restriction map $Spin(X) \to Spin(X - N(\Sigma'))$ is an isomorphism. The fact that $s_0(K, p^n)$ lies in the image of $Spin(X) \to Spin(Y_{p^n}(K))$ now follows from the commutative diagram (with all $r_j$ being restriction maps):

$$
\begin{array}{ccc}
Spin^c(X) & \xrightarrow{r_1} & Spin^c(Y) \\
\downarrow r_2 & & \downarrow \cong r_3 \\
Spin^c(X - N(\Sigma')) & \xrightarrow{r_4} & Spin^c(Y - N(K')) \\
F^* & & F^* \\
Spin^c(D^4 - N(\Sigma)) & \xrightarrow{r_5} & Spin^c(S^3 - N(K))
\end{array}
$$

Namely, $s_0(K, p^n) = r_3^{-1}(F^*(r_2(t))) = r_3^{-1}(F^*(r_4(t)))$ where $t \in Spin^c(D^4 - N(\Sigma))$ is the unique spin-structure on $D^4 - N(\Sigma)$. Since $F^*(t)$ is also a spin-structure it must extend to the (unique) spin-structure $t_0$ on $X$ and so $s_0(K, p^n) = r_1(t_0)$ as claimed.

Finally, since $\delta_{p^n}(K) = 2d(Y_{p^n}(K), s_0(K, p^n))$ and the latter is congruent to $(c_1(t_0)^2 - \sigma)/2 = -\sigma/2$ modulo 2, we see that it must be an integer.

The second and third statement of the proposition follow directly from theorems 2.4 and 3.3.

With statements 1–3 proved, it is now automatic for $\delta_{p^n}$ to descend to a group homomorphism $\delta_{p^n} : C \to \mathbb{Z}$. First of, note that $\delta_{p^n}(-K) = -\delta_{p^n}(K)$ (where $-K$ is the reverse mirror of $K$) as follows from points 2 and 3 of the proposition along with the fact
that $K\#(-K)$ is smoothly slice. If $K_1$ is smoothly concordant to $K_2$ then $K_1\#(-K_2)$ is smoothly slice so that

$$0 = \delta_{p^n}(K_1\#(-K_2)) = \delta_{p^n}(K_1) + \delta_{p^n}(-K_2) = \delta_{p^n}(K_1) - \delta_{p^n}(K_2)$$

\[\square\]

**Definition 4.3.** A natural number $n$ is called a Fermat number if it is of the form $n = 2^{2^k} + 1$ for some $k \in \mathbb{N} \cup \{0\}$. Similarly, $n$ is called a Mersenne number if it looks like $n = 2^m - 1$ with $m \in \mathbb{N}$. A natural number $n$ is called a Fermat prime or a Fermat prime power (Mersenne prime or Mersenne prime power) if it is a Fermat number (Mersenne number) and prime or a prime power at the same time.

It is well known that if a Mersenne number $2^m - 1$ is a prime power, then $m$ must be a prime number itself [12]. It is unknown which Fermat numbers are either prime or prime powers. The only Fermat primes known to date are the first five, namely 3, 5, 17, 257 and 65537.

**Lemma 4.4.** Let $p$ be an odd prime. Then

1. $p^{2^n} - 1$ can only be a prime power if $p$ is a Fermat prime. If $p = 3$ this happens for $n = 0, 1$ and if $p > 3$ this can happen only for $n = 0$.
2. $p^{2^n} + 1$ can only be a prime power if $p^{2^n}$ is a Mersenne prime power. With $p$ fixed, this can happen for at most one $n \in \mathbb{N} \cup \{0\}$.

**Proof.** Pick and fix a prime $p \geq 3$ throughout the proof. Consider first when $p^{2^n} - 1$ is of the form $2^m$ for some integer $m$. When $n$ is at least 1, $p^{2^n} - 1$ factors as

$$p^{2^n} - 1 = (p^{2^{n-1}} - 1)(p^{2^{n-1}} + 1)$$

and so, in order for $p^{2^n} - 1$ to be a prime power, both $p^{2^{n-1}} \pm 1$ have to be powers of two. An easy analysis shows that this only happens if $p = 3$ and $n = 1$. If $n = 0$ and $p^{2^n} - 1 = p - 1$ is a prime power, say $2^m$, then $p = 2^m + 1$. Elementary number theory then shows that $m$ itself has to be a power of 2 (this observation was already made by Gauss) so that $p$ becomes a Fermat prime. Since $p = 3$ is itself a Fermat prime, the first statement of the lemma follows.

Turning to $p^{2^n} + 1$, suppose we found some $n$ for which $p^{2^n} + 1 = 2^m$. Then

$$p^{2^n+1} + 1 = (p^{2^n})^2 + 1 = (2^m - 1)^2 + 1 = 2(2^{2m-1} - 2^m + 1)$$

Since the second factor in the last term above is odd, we see that $p^{2^n+1} + 1$ cannot be a prime power. A similar argument shows the same to be true for $p^{2^{n+k}} + 1$ for any $k \geq 1$:

$$p^{2^{n+k}} + 1 = (p^{2^n})^{2^k} + 1 = (2^m - 1)^{2^k} + 1 = 1 + \sum_{j=0}^{2^k} \binom{2^k}{j}(-1)^j2^{mj}$$

$$= 2 \left(1 + \sum_{j=1}^{2^k} \binom{2^k}{j}(-1)^j2^{mj-1}\right) = 2(1 + \text{even number})$$

This shows that $p^{2^n} + 1$ can be a prime power for at most one $n \in \mathbb{N} \cup \{0\}$. Note that $p^{2^n} + 1 = 2^m$ can only happen if $p^{2^n} = 2^m - 1$, i.e. if $p^{2^n}$ is a Mersenne prime power. For
this to happen $m$ must itself be a prime number \[12\]. The choices of $p = 7$, $n = 0$ and $m = 3$ are an example.

The next theorem is a slightly strengthened version of theorem \[1.2\]

**Theorem 4.5.** Let $p \geq 3$ be a prime integer. If $p = 3$ set $\mathcal{F} = \{0, 1\}$, if $p \neq 3$ is a Fermat prime let $\mathcal{F} = \{0\}$ and otherwise let $\mathcal{F} = \emptyset$. If there exists an integer $n_0 \in \mathbb{N} \cup \{0\}$ such that $p^{2^{n_0}}$ is a Mersenne prime power (see definition \[4.3\]), let $\mathcal{M} = \{n_0\}$, otherwise let $\mathcal{M} = \emptyset$. Then the set

$$
\{\delta_{p^{2^n}} : \mathcal{C} \to \mathbb{Z} | n \in (\mathbb{N} \cup \{0\}) - (\mathcal{F} \cup \mathcal{M})\} \subset \text{Hom}(\mathcal{C}, \mathbb{Z})
$$

is linearly independent.

**Proof.** According to lemma \[4.4\], no element of the infinite set

$$
\{p^{2^n} \pm 1 | n \in (\mathbb{N} \cup \{0\}) - (\mathcal{F} \cup \mathcal{M})\}
$$

is a prime power. Let $n_i$, $i \in \mathbb{N}$ be the sequence enumerating (in increasing order) the elements of $(\mathbb{N} \cup \{0\}) - (\mathcal{F} \cup \mathcal{M})$. For simplicity of notation we shall write $m_i$ for $p^{2^n-i}$. Thus we need to prove that the set \(\{\delta_{m_i} : \mathcal{C} \to \mathbb{Z} | i \in \mathbb{N}\}\) is is linearly independent. To see this, suppose that some linear combination of $\delta_{m_i}$’s results in the zero homomorphism:

$$
\lambda_1 \delta_{m_1} + \lambda_2 \delta_{m_2} + \ldots \lambda_l \delta_{m_l} = 0
$$

Since $m_1 + 1$ is not a prime power (lemma \[4.4\]), we can find a pair of positive coprime integers $(a, b)$ such that

$$
m_1 = kab - 1
$$

According to corollary \[3.2\], $1/k$-framed Dehn surgery on the torus knot $T_{(a,b)}$ gives the Brieskorn sphere $-\Sigma(a,b,kab - 1)$. Since $kab - 1 = m_1 = p^{2n_1}$ we see that $\Sigma(a,b,m_1)$ is the $p^{2n_1}$-fold branched cover of $T_{(a,b)}$, cf. equation \[4\]. But according to corollary \[3.5\] (with the help of statement 3 from theorem \[3.3\] to address the change of orientation) we then know that

$$
\delta_{m_1}(T_{(a,b)}) < 0
$$

On the other hand, note that $m_1 - 1$ is a factor of $m_s - 1$ for all $s \geq 1$ since, if $n_s = n_1 + d$, then

$$
m_s - 1 = (m_1)^{2d} - 1 = \left((m_1)^{2d-1} - 1\right)\left((m_1)^{2d-1} + 1\right)
= \left((m_1)^{2d-2} - 1\right)\left((m_1)^{2d-2} + 1\right)\left((m_1)^{2d-1} + 1\right)
= \vdots
= (m_1 - 1) \cdot \prod_{j=0}^{d-1} \left((m_1)^{2j} + 1\right)
$$

Thus, for all $s \geq 2$ we get $m_s = \ell_s cd + 1$ for some integer $\ell_s$. With this, another use of corollary \[3.5\] show that

$$
\delta_{m_s}(T_{(a,b)}) = 0 \quad \forall s \geq 2
$$

Plugging $T_{(a,b)}$ into equation \[6\], with the help of equations \[7\] and \[8\], results in $\lambda_1 = 0$. 

\[\square\]
From here on one just repeats this argument \( \ell - 1 \) more times to obtain \( \lambda_i = 0 \) for all \( i = 1, \ldots, \ell \), we omit further details. \( \square \)

**Corollary 4.6.** Let \( p \) be a prime integer and \( n_i, i \in \mathbb{N} \) be the sequence enumerating the elements in \( \mathbb{N} \cup \{0\} \) – \( (\mathcal{F} \cup \mathcal{M}) \) from theorem 4.5. Then for every index \( i \) there is a pair of coprime positive integers \( (a_i, b_i) \) such that

\[
\delta_{p^{a_i}}(T(a_i, b_i)) = 0 \quad \forall j > i \quad \text{and} \quad \delta_{p^{a_i}}(T(a_i, b_i)) < 0
\]

5. Examples part 1 - Brieskorn spheres

In this section, the first of three, we start to address the question of how to evaluate the invariants \( \delta_p(K) \) for concrete knots \( K \) and concrete choices of \( p \) and \( n \). While this is a difficult task in general, we explore a number of techniques that are successful in such computations for a substantial set of examples of \( K, p \) and \( n \). The techniques stated in this section have already been exploited for the proof of theorem 4.5 and are only listed for emphasis.

Let \( a \) and \( b \) be two positive and relatively prime integers and consider the torus knot \( T(a, b) \). Let

\[
\Delta_{T(a, b)}(t) = \frac{(t^{ab} - 1)(t - 1)}{(t^a - 1)(t^b - 1)} \cdot t^{-(a-1)(b-1)/2}
\]

be its symmetrized Alexander polynomial. Let us write \( \Delta_{T(a, b)}(t) \) in the form \( \Delta_{T(a, b)}(t) = a_0 + f_{(a, b)}(t) + f_{(a, b)}(t^{-1}) \) with \( f_{(a, b)}(t) \in t \cdot \mathbb{Z}[t] \) so that the 0-th torsion coefficient \( t_0 \) of \( T(a, b) \) takes the form \( t_0(T(a, b)) = f'_{(a, b)}(1) \).

Since \( 1/n \) surgery on \( T(a, b) \) yields the Brieskorn sphere \( -\Sigma(a, b, abn - 1) \) and \( -1/n \) surgery on \( T(a, b) \) gives \( -\Sigma(a, b, abn + 1) \) (cf. corollary 3.2), theorems 3.1 and 3.4 imply that

\[
\delta_p(T(a, b)) = \begin{cases} 
2f'(1) & \text{; whenever } abm - 1 = p^n \text{ for some } m \in \mathbb{Z} \\
0 & \text{; whenever } abm + 1 = p^n \text{ for some } m \in \mathbb{Z}
\end{cases}
\]

In the above, \( p \) is always assumed to be a prime number. For example, taking \( a = 7 \) and \( b = 9 \) we obtain

\[
f_{(7, 9)}(t) = t - t^2 + t^3 - t^5 + t^6 - t^7 + t^8 - t^9 + t^{10} - t^{14} + t^{15} - t^{16} + t^{17} - t^{20} + t^{24}
\]

so that \( f'_{(7, 9)}(1) = 8 \). Therefore we obtain, for instance,

\[
\delta_{7^3}(T_{(7, 9)}) = 16, \ \delta_{251}(T_{(7, 9)}) = 16, \ \delta_{503}(T_{(7, 9)}) = 16, \ \delta_{881}(T_{(7, 9)}) = 16
\]

and

\[
\delta_{2^6}(T_{(7, 9)}) = 0, \ \delta_{127}(T_{(7, 9)}) = 0, \ \delta_{379}(T_{(7, 9)}) = 0, \ \delta_{631}(T_{(7, 9)}) = 0, \ \delta_{757}(T_{(7, 9)}) = 0.
\]

6. Examples part 2 - Negative definite plumbings

The computational methods from section 5 exclude many of the branched covers of torus knots. For instance, when \( K = T_{(2, 5)} \), formula 3.1 allows for a computation of \( \delta_9(K) \) and \( \delta_{11}(K) \) but not for example \( \delta_3(K) \) or \( \delta_7(K) \). Note that the 3-fold and 7-fold branched covers of \( T_{(2, 5)} \) are still Brieskorn spheres, namely \( \Sigma(2, 3, 5) \) and \( \Sigma(2, 5, 7) \) respectively.
Many Brieskorn spheres $\Sigma(a,b,c)$ bound negative definite plumbings, $\Sigma(2,3,5)$ and $\Sigma(2,5,7)$ are examples. When this happens, Ozsváth and Szabó [22] and A. Nemethi [18] provide formulae for computing the correction terms of $\Sigma(a,b,c)$, provided an additional restriction on the plumbing graph yielding $\Sigma(a,b,c)$ is met. Namely, given a weighted graph $G$, we shall call a vertex $x$ of $G$ a bad vertex if its valence $v(x)$ (the number of edges emanating from $x$) and weight $d(x)$ satisfy the inequality $d(x) < -v(x)$. The formulae from [22] provide a completely combinatorial algorithm for the computation of $d(\Sigma(a,b,c))$ for all those $\Sigma(a,b,c)$ which bound negative definite plumbing graphs with no more than two bad vertices. This algorithm, besides having been explained in [22], has been outlined in a number of articles, see for example [7] and [2]. We thus omit it here and instead focus on an example. We would like to point out that the results from examples 1.4–1.10 not covered by formula (9), have been computed in this manner. The results from example 1.3 are addressed in the next section.

Some of the negative definite plumbings with two or fewer bad vertices, utilized in examples 1.4–1.10 are described in figure 1. The remaining cases are left as an easy exercise. To obtain such plumbings we have followed the algorithm outlined in [28, 29] where we refer the interested reader for full details. By way of example, consider $\Sigma(2,5,7)$ which is the 7-fold branched cover of $T(2,5)$ (and of course the double branched cover of $T(5,7)$ and the 5-fold cover of $T(2,7)$). Set $a_1 = 2$, $s_2 = 5$ and $a_3 = 7$ and solve the equation

$$a_1a_2b_1 + a_1b_2a_3 + b_1a_2a_3 = 1$$

for $b_1$, $b_2$ and $b_3$. For example $b_1 = 1$, $b_2 = -1$ and $b_3 = -2$ will do. Find continued fraction expansions for $a_i/b_i$ next:

$$\frac{a_1}{b_1} = [-1, -1, -2, -2], \quad \frac{a_2}{b_2} = [-5], \quad \frac{a_3}{b_3} = [-4, -2]$$

where by $[x_1, x_2, ..., x_n]$ we mean

$$[x_1, x_2, ..., x_n] = x_1 - \frac{1}{x_2 - \frac{1}{x_3 - \frac{1}{\ddots - \frac{1}{x_n}}}}.$$

The plumbing diagram describing $\Sigma(2,5,7)$ is then obtained by drawing a single 3-valent vertex with framing zero from which 3 branches emerge whose vertices have framing coefficients given by the above 3 continued fraction expansion coefficients. This gives the plumbing diagram in figure 2. Repeatedly blowing down the $-1$ framed vertices from figure 2 leads to the plumbing description for $\Sigma(2,5,7)$ from figure 1. This weighted graph has only one bad vertex and its incidence matrix

$$\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 \\
0 & 0 & 1 & -2 & 0 \\
0 & 0 & 1 & 0 & -5
\end{bmatrix}$$
is easily verified to be negative definite (see for example [1]). The Ozsváth-Szabó algorithm from [22] now allows for a computation of $\delta_7(T_{(2,5)}) = \delta_2(T_{(5,7)}) = \delta_5(T_{(2,7)})$ and yields 0. We performed the needed computations by computer.
This section computes $\delta_{2n}(K)$ for all $n \in \mathbb{N}$ where $K$ is the right-handed trefoil. The results thus obtained are those from example 1.3. We compute these invariants by first finding an explicit Dehn surgery description of $Y_{2n}(K)$. By happenstance, each $Y_{2n}(K)$ turns out to be surgery on a single alternating knot, the knot itself depending on $n$. Such a description of $Y_{2n}(K)$ allows us to use the existing Heegaard Floer machinery \cite{23} to compute the correction terms $d(Y_{2n}(K), s_0(K, 2^n))$.

To arrive at a surgery description of $Y_{2n}(K)$, note that the right-handed trefoil $K$ can be described as the unknot in $S^3$ if $S^3$ is viewed as $+1$ surgery on another unknot as in figure 3. The $m$-fold cover $Y_m(K)$ of $S^3$ branched along $K$ can then be constructed by cutting $S^3$ open along the open disk $D$ bounded by $K$ (figure 3a), taking $m$ copies of $S^3 - D$ and gluing the $i$-th copy to the $(i + 1)$-st copy along their boundary components as prescribed by the orientation (and likewise gluing the $m$-th copy of $S^3 - D$ to the 1-st copy), see \cite{27} for full details.

The surgery description of $Y_m(K)$ is now easily obtained. Namely, in cutting $S^3$ open along $D$ (see again figure 3a) one also cuts the $+1$ framed unknot from figure 3 yielding the 2-component tangle from figure 4a. To obtain a surgery description of $Y_m(K)$, one concatenates $m$ copies of this tangle to obtain an $m$ component link $L = L_1 \sqcup ... \sqcup L_m$. The framing $\lambda_i$ of $L_i$ is determined from the equation $1 = \lambda_i + \sum_{j \neq i} (k(L_i, L_j)$ (see page 357 of \cite{9} for an explanation of this formula) and thus turns out to be $\lambda_i = -1$. Figure 4b illustrates the example of $m = 4$.

We see that $Y_m(K)$ is obtained by $-1$-framed surgery on an $m$ component “necklace”, each of whose components is an unknot. To simplify this picture, we perform a number of handle slides with the goal of reducing the number of components. The key idea is as depicted in figure 5. Figure 5 describes 3 handle slides by which one can replace 4 consecutive $-1$-framed components of the “necklace” by a single component, still with
framing $-1$, though this new component is given a single right-handed half-twist. The same holds if one starts with 4 consecutive $-1$-framed components that each contain $n$ half-twists (in what we may refer to as the $n$-twisted necklace): in this case one is left with a single component with framing $-1$ but now with $4n + 1$ right-handed half-twists. We shall refer to the sequence of handle slides from figure 5 reducing the number of components from 4 to 1, as the reduction procedure. The reduction procedure can be applied to any ($n$-twisted) necklace of at least 5 components.

We proceed by separately considering $\delta_{2k}^2(K)$ and $\delta_{2k+1}^2(K)$. To compute the former, one uses the reduction procedure described above to reduce the $-1$-framed, $4k$ component necklace describing $Y_{4k}(K)$, to the 4 component surgery diagram in figure 6a. Three further handle slides, akin to those from figure 5, reduce this description to $-3$-framed surgery on a single twist knot, see figure 6b.

On the other hand, to find a surgery description of $Y_{2k+1}(K)$, one repeatedly applies the reduction procedure to the $2^{2k+1}$ component necklace of $-1$-framed unknots to arrive at the 2-component Dehn surgery diagram from figure 7a. A single handle slide (followed by a slam-dunk) gives the description of $Y_{2k+1}(K)$ from figure 7b. In summary, we have:

**Proposition 7.1.** Let $K$ be the right-handed trefoil and $Y_{2^n}(K)$ be the $2^n$-fold branched cover of $S^3$ with branching set $K$. Let $T_m$ denote the twist knot with the clasp as in figure 8. Then

$$Y_{2^n}(K) = \begin{cases} 
-3\text{-framed surgery on } T_{(2^n-1)/3} & ; \ n = 2k, k \geq 1 \\
3\text{-framed surgery on } T_{(2^n-2)/3} & ; \ n = 2k + 1, k \geq 0 
\end{cases}$$

Using proposition 7.1 and the Heegaard Floer tools for computing correction terms for surgeries on a knot (as described by Ozsváth and Szabó in [23]), readily lead to a computation of all $\delta_{2^n}$ for the right-handed trefoil.
Figure 5. A sequence of handleslides and isotopies. Each box $\ell$ indicates $\ell$ half-twists; all components without indicated framing carry the framing $-1$. Applying two isotopies to the top picture yields the second one. From that one the third picture is obtained by sliding each of the two handles containing $2n$ half-twists over the handle to their immediate left (the handles without any twists in them). The fourth picture is then obtained by a further isotopy and picture 5 is gotten by sliding the handle containing the $4n$ half-twists over the handle to its left. The final picture is gotten by simple additional isotopy. This picture is drawn with $n$ odd in mind; the case of $n$ even works analogously. The components with $m$ and $k$ half-twist are allowed to be the same.
Figure 6. (a) The surgery description of $Y_{4^k}(K)$ obtained by applying the reduction procedure $k-1$ times to the $4^k$ component necklace of $-1$ framed unknots. The number of half-twists $\ell$ in each of the four boxes is given by $\ell = (4^k-1)/3$. (b) Three additional handle slides turn figure (a) into $-3$-framed surgery on a twist knot. The handle slides performed are almost identical to those from figure 5 and are left as an easy exercise.

Figure 7. (a) The surgery description of $Y_{2^{2k+1}}(K)$ obtained by applying the reduction procedure $k$ times to the $2^{2k+1}$ component necklace of $-1$ framed unknots. The number of half-twists $\ell$ in each of the two boxes on the left is: $\ell = (4^k - 1)/3$. (b) An additional handle slides turn figure (a) into $3$-framed surgery on a twist knot.
Theorem 7.2. Let $K$ be the right-handed trefoil knot. Then

$$\delta_{2^n}(K) = \begin{cases} 3 & \text{if } n = 2k \text{ and } k \geq 1. \\ 1 & \text{if } n = 2k + 1 \text{ and } k \geq 0. \end{cases}$$

This theorem is an application of Corollary 4.2 from [23]. The only inputs required by that corollary are the signatures $\sigma(T_m)$ and Alexander polynomials $\Delta_{T_m}(t)$ of the twists knots. These, in turn, are easily determined since all $T_m$ are alternating. For any integer $m \geq 1$ one finds:

$$\sigma(T_m) = \begin{cases} 0 & \text{if } m \text{ is even} \\ 2 & \text{if } m \text{ is odd} \end{cases}$$

and

$$\Delta_{T_m}(t) = \begin{cases} \frac{m}{2}t - (m + 1) + \frac{m}{2}t^{-1} & \text{if } m \text{ is even} \\ \frac{m+1}{2}t - m + \frac{m+1}{2}t^{-1} & \text{if } m \text{ is odd} \end{cases}$$

Finally, since $|H_1(Y_{2^n}(K); \mathbb{Z})| = 3$ for all $n \geq 1$, each $Y_{2^n}(K)$ possesses a unique spin-structure which by necessity has to equal $s_0(K, p^n)$. The corresponding correction term $d(Y_{2^n}(K), s_0(K, p^n))$ is distinguished by the fact that it is the only one, out of the 3 corrections terms of $Y_{2^n}(K)$, which yields an integer when multiplied by 2. This fortuitous coincidence makes the determination of $s_0(K, 2^n)$ easy.

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