ON THE WARRING RANK OF BINARY FORMS: THE BINOMIAL FORMULA AND A DIHEDRAL COVER OF RANK TWO FORMS

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ABSTRACT. Waring problem for forms is important and classical in mathematics. It has been widely investigated because of its wide applications in several areas. In this paper, we consider the Waring problem for binary forms with complex coefficients. Firstly, we give an explicit formula for the Waring rank of any binary binomial and several examples to illustrating it. Secondly, we prove that, up to scalar multiplication, there are exactly \( \binom{d+1}{2} \) binary forms of degree \( d \) with Waring rank two and multiple of three fixed distinct linear forms.

1. INTRODUCTION

This paper concerns symmetric tensor decomposition as a sum of rank one tensors which is also known as the Waring problem for forms. This topic has a rich history and recently received a huge interest mainly because of its wide applicability in areas as diverse as algebraic statistics, biology, quantum information theory, signal processing, data mining, machine learning, see [13, 24, 29].

Let \( \mathbb{k}[x, y] \) be the standard graded polynomial ring with coefficients in the field \( \mathbb{k} \subseteq \mathbb{C} \). For \( d \geq 0 \), we denote by \( \mathbb{k}[x, y]_d \) the \( \mathbb{k} \)-vector space of forms of degree \( d \) in \( \mathbb{k}[x, y] \). For every form \( F \in \mathbb{k}[x, y]_d \), by the classical Jordan Lemma (see [20, Appendix III]), there exist linear forms \( L_1, \ldots, L_r \in \mathbb{k}[x, y]_1 \) and scalars \( a_1, \ldots, a_r \in \mathbb{k} \) with \( r \leq d + 1 \) such that

\[
F = a_1 L_1^d + \cdots + a_r L_r^d.
\]

When \( \mathbb{k} = \mathbb{C} \) (resp. \( \mathbb{k} = \mathbb{R} \)), the least of such possible numbers \( r \) is called the Waring rank of \( F \) (resp. the real Waring rank of \( F \)) and we denote it by \( \text{rk}(F) \) (resp. \( \text{rk}_R(F) \)). Except for a few results comparing the Waring and
the real Waring rank (see Examples 3.3 and 3.4 and Theorem 4.13), we will consider $k = \mathbb{C}$ throughout this paper.

The longstanding problem of finding the Waring rank of a generic form in any number of variables has been solved recently by J. Alexander and A. Hirschowitz [1], after being remained open for more than one hundred years. However, solving the Waring problem for generic form of degree $d$ does not give any information about specific form of degree $d$.

There has been an intense research in finding the Waring rank of binary forms which goes back to the work of J. J. Sylvester. Sylvester [32, 33] gave an explicit algorithm for computing the Waring rank of a binary form. We refer to [31] for an excellent survey on the Waring problem for binary forms. The Waring problem is even more interesting and challenging when the coefficient field is different from the field of complex numbers, see [31]. Especially, because of the direct connection with the real world, there is a lot of interest in finding the decomposition of forms with real coefficients, see for instance [6].

As an immediate consequence of Sylvester’s algorithm one can give an explicit formula for the Waring rank of monomials in $\mathbb{C}[x, y]$. Moreover, an explicit formula for the Waring rank of monomials in any finite number of variables has been given recently in [8, Proposition 3.1]. This motives us to look beyond monomials for binary forms, so first we consider the binomial case and second forms with three distinct roots fixed.

The first main result of this paper gives an explicit formula for the Waring rank of binomials in $\mathbb{C}[x, y]$ (see Theorem 3.1). However, we remark that this is far from being an obvious generalization of the monomial case. More generally, it is difficult to describe the Waring rank of $F_1 + \cdots + F_k$ in terms of the Waring ranks of $F_1, \ldots, F_k$ as is evidenced by Strassen’s conjecture (see [7, 8, 9, 35]).

Our main tool in computing the Waring rank of a binomial form $F \in \mathbb{C}[x, y]$ is Sylvester’s algorithm (see Section 2.1). In order to apply this algorithm, we need to find a form of least degree in the apolar ideal $F^\perp$ (see Section 2.1). For this, we give a nonzero form $g_1$ in $F^\perp$ and, computing the Hilbert function (see Section 2.3) of $T/F^\perp$ in certain degrees, we are able to conclude that $g_1$ is of least degree in $F^\perp$. Hence we avoid to compute the entire apolar ideal $F^\perp$.

The second main goal is to study some geometric properties of the loci of binary forms with Waring rank two. For a fixed degree $d \geq 4$, we show that, up to scalar multiplication, there are $\binom{d-1}{2}$ forms with Waring rank two and multiple of three distinct linear forms $l_1, l_2, l_3$, or equivalently with three distinct roots fixed (see Corollary 4.8.1 and Theorem 4.12). For this, we build a dihedral cover of the set of forms with Waring rank two that are multiple of $l_1l_2l_3$ (see Theorem 4.7) and we show that there are $\binom{d-1}{2}$ different orbits in the cover space by the dihedral action (see Theorem 4.10).
Such dihedral covers depend on the linear forms $l_1, l_2, l_3$ and, ranging over all the three distinct linear forms, they cover the set of all binary forms with Waring rank two.

This cover also allows us to describe the set of all forms in $\mathbb{P} \mathbb{R}[x, y]_d$ with a maximal gap between their Waring and real Waring rank (see Theorem 4.13).

Finally we offer a second proof of Theorem 4.12 based on more classical results. Namely, the set of forms multiple of $l_1 l_2 l_3$ form a linear subspace of $\mathbb{P} S_d$ which has codimension three (see Corollary 4.5.1). We show that it intersects transversely the second secant variety $\Sigma$ of the rational normal curve in $\mathbb{P} S_d$, which has dimension three (see Corollary 4.3.1). Then, the claim follows by Bézout’s Theorem. To this intent, we use Teracini’s Lemma to parametrise the tangent space to a smooth point of $\Sigma$.

The paper is organized as follows: We fix some notation and gather some preliminary results needed for our purpose in Section 2. In Section 3 we give an explicit formula for the Waring rank of a binomial form. Section 4 is devoted to study the dihedral cover and the geometric properties of the loci of binary forms with Waring rank two.

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2. Preliminaries

Let $S, T$ denote respectively the standard graded polynomial rings $\mathbb{C}[x, y], \mathbb{C}[X, Y]$. For $i \geq 0$, denote respectively by $S_i, T_i$ the $i$-th graded component of $S, T$.

2.1 Apolarity theory and Sylvester’s algorithm. Consider the apolar action of $T$ on $S$, that is consider $S$ as a $T$-module by means of the differentiation action

$$
\circ : T \times S \longrightarrow S
$$

$$(g, F) \mapsto g \circ F = g(\partial_X, \partial_Y)(F).$$

Definition 2.1. Let $F$ be a form in $S_d$. A form $g \in T_{d'}$ is called apolar to $F$ when $g \circ F \in S_{d-d'}$ is the zero form. The apolar ideal to $F$, denoted as $F^\perp$,
is the homogeneous ideal in $T$ generated by all the forms apolar to $F$, or equivalently

$$F^\perp = \{ g \in T \mid g \circ F = 0 \} \subseteq T.$$  

The so called Sylvester’s algorithm below, an algorithm to compute Waring rank of binary forms, is a consequence of Sylvester’s Theorem developed in [32, 33]. Modern proofs of Sylvester’s Theorem may be found in [31, Theorem 2.1] (which is an elementary proof), [28, Section 5], and with further discussion in [25, 26, 27, 30]. Here we just state the final version of the algorithm, see [10, Remark 4.16], [2, Algorithm 2] or [12, Section 3].

**Theorem 2.2** (Sylvester’s algorithm). Let $F$ be a form in $S_d$. By the Structure Theorem (see [23, Theorem 1.44(iv)]), the apolar ideal $F^\perp$ to $F$ can be generated by two forms $g_1, g_2 \in T$ with $\deg(g_1) + \deg(g_2) = d + 2$. Suppose that such forms $g_1, g_2 \in T$ are given with $\deg(g_1) \leq \deg(g_2)$. Then,

$$\text{rk}(F) = \begin{cases} 
\deg(g_1) & \text{if } g_1 \text{ is square-free} \\
\deg(g_2) & \text{otherwise}.
\end{cases}$$

**2.2 Binary forms.** For the convenience of the reader, this section collects some well-known results on the Waring and the real Waring rank of binary forms that we will use.

Theorem 2.3 below is proved in [31, Theorems 4.9 and 4.10]. For $k = \mathbb{C}$, it is an immediate consequence of Sylvester’s Theorem. More generally, Theorem 2.3 is a consequence of [31, Corollary 2.2], an adaptation of Sylvester’s Theorem to forms with coefficients in a subfield $k$ of $\mathbb{C}$.

**Theorem 2.3.** Let $F$ be a form in $k[x, y]_d$ with $d \geq 0$ and $k$ a subfield of $\mathbb{C}$. Then, $\text{rk}_k(F) \leq d$.

For $d \geq 3$, Theorem 2.4 below gives all the forms $F \in S_d$ with maximal Waring rank, that is $\text{rk}(F) = d$. Though it is well-known that $\text{rk}(x^{d-1}y) = d$, to the best of our knowledge (cf. [37, Theorem 2.4]), the converse has been proven only later [22, Exercise 11.35] or [4, Corollary 3].

**Theorem 2.4.** Let $F \in S_d$ with $d \geq 3$. Then $\text{rk}(F) = d$ if and only if there are two distinct linear forms $l_1, l_2 \in S_1$ so that $F = l_1^d l_2$.

**Definition 2.5.** A form $F \in \mathbb{R}[x, y]_d$ is called *hyperbolic* if all its roots are real, that is it splits into linear factors over $\mathbb{R}$.

Sylvester in [34] already observed, by the dehomogenized case, that a hyperbolic binary form which is not a power of a linear form has real Waring rank $d$, which is maximal. See [31, Section 3] for more history of this problem. A partial converse of this statement for square free hyperbolic forms was established in [11, 14] and the full converse is proved in [5, Theorem 2.2].
**Theorem 2.6.** Let $F$ be a form in $\mathbb{R}[x,y]$ with $d \geq 3$. Suppose that $F$ is not a $d$-th power of a linear form. Then, the real Waring rank of $F$ is $d$ if and only if $F$ is hyperbolic.

### 2.3 Hilbert function.
Let $I$ be a homogeneous ideal in $T$. The Hilbert function of $T/I$ is an important numerical invariant associated to $T/I$ defined as

\[ \text{HF}_{T/I}(i) = \dim\mathbb{C}T_i - \dim\mathbb{C}I_i, \]

where $I_i$ denotes the $\mathbb{C}$-vector space of degree $i$ forms in $I$. In Section 3, our strategy is to compute the Hilbert function of $T/F^\perp$ and extract information about the Waring rank of $F$ from it using Sylvester’s algorithm.

For $F \in S_d$, let $\langle F \rangle$ denote the $\mathbb{C}$-submodule of $S_d$ generated by $F$, that is, the $\mathbb{C}$-vector space generated by $F$ and by the corresponding derivatives of all orders. It is well-known that $\langle F \rangle$ determines the Hilbert function of $T/F^\perp$ as follows (see [19]):

\[ (j) \quad \text{HF}_{T/F^\perp}(i) = \dim\mathbb{C}\langle F \rangle_i \]

where $\langle F \rangle_i$ denotes the $\mathbb{C}$-vector space generated by the forms of degree $i$ in $\langle F \rangle$.

### 2.4 Group actions.
We review some basic definitions and facts about a group acting on a set. Actually, we consider groups acting on a topological space but, since we will apply this definitions to finite topological spaces with the discrete topology, we may consider them as simply sets.

**Definition 2.7.** Let $X$ be a set. Let $\Omega$ and $\Omega'$ be two partitions of $X$. The partition $\Omega$ is finer than $\Omega'$ if for every $\omega \in \Omega$ there is $\omega' \in \Omega'$ such that $\omega \subseteq \omega'$.

Recall that to two partitions $\Omega$ and $\Omega'$ of a set $X$ are equal if and only if they are mutually finer.

Given a map of sets $f: X \rightarrow Y$, we denote by $X/f$ the partition of $X$ determined by the fibres of $f$.

**Definition 2.8.** Let $X$ be a set and $G$ a group acting on it. The orbit of $x$ by the action of $G$ is the set $\{g \cdot x\}_{g \in G}$ and it is denoted by $G \cdot x$. The set of all orbits is denoted by $X/G$, which is a partition of $X$.

Recall that a group $G$ acts faithfully on a set $X$ if there are no nontrivial $g \in G$ such that $g \cdot x = x$ for all $x \in X$.

**Definition 2.9.** Let $X$, $Y$ be finite sets and $G$ a group. A map $f: X \rightarrow Y$ is a $G$-cover of $Y$ if the group $G$ acts faithfully on $X$ and the partitions $X/f$ and $X/G$ are equal.

### 3. Waring rank of binomial forms

The main goal of this section is to give an explicit formula for the Waring rank of any binomial form (Theorem 3.1). In particular, we prove that the
Waring rank of a binomial form does not depend on its coefficients. We give several examples, Example 3.1 illustrates the result itself and Examples 3.2 and 3.4 that the Waring rank of a trinomial and the real Waring rank of a binomial may depend on its coefficients respectively.

Recall that, given a nonzero homogeneous ideal \( I \) of \( T \), the *initial degree of \( I \)* is the least integer \( i > 0 \) for which \( I \) is not zero.

**Theorem 3.1.** Let \( F \in S_d \) be a binomial form. That is, \( F = ax^r y^s + bx^{r+a} y^s \) for some \( a, b \in \mathbb{C} \setminus \{0\} \) and \( r, s, \alpha \in \mathbb{N} \). Assume (without loss of generality) that \( 0 \leq r \leq s \) and \( \alpha \geq 1 \). Let \( a, b \) be the unique nonnegative integers such that \( r = qa + j \) with \( 0 \leq j < \alpha \) and set \( \delta = r - \alpha + s \). Then the Waring rank of \( F \) can be computed using the following table.

| Conditions | \( \text{rk}(F) \) |
|------------|-----------------|
| (1) \( \delta \leq 0 \) | \( s + 1 \) |
| \( j = 0, r = s \) and \( \alpha > 1 \) | \( s + 2 \) |
| (2) \( \delta > 0 \) | \( s + 1 \) |
| \( j = \delta \) | \( r + \alpha + 1 \) |
| \( j > \delta \) | \( r + \alpha - j \) |
| otherwise | \( r + \alpha - j \) |

**Proof.** Since \( \text{rk}(F) \) is invariant by a linear change of coordinates, we may assume that \( a = \frac{(r+\alpha)s!}{r!(s+\alpha)!} \) and \( b = 1 \). For every pair of integers \( m, n \), we set \( \frac{m}{n} \) equal to \( \frac{m}{n} \) if \( m \geq n \geq 0 \) and equal to zero otherwise.

1. Suppose \( \delta \leq 0 \), that is \( s \geq r + \alpha \). Clearly, \( g_1 = X^{r+\alpha+1} \in F^\perp \). We claim that the initial degree of \( F^\perp \) is \( r + \alpha + 1 \). For \( 0 \leq i \leq s \), we have

\[
X^i Y^{s-i} F = a \cdot \left( \binom{s}{i} \cdot \binom{s+\alpha}{s-i} \cdot x^{r+s} + \binom{r+\alpha}{i} \cdot \binom{s}{s-i} \cdot x^{r-\alpha+i} + 1 \right).
\]

It is easy to see that the set \( \{ X^i Y^{s-i} F : 0 \leq i \leq r + \alpha \} \subseteq \langle F \rangle_{r+\alpha} \) is \( \mathbb{C} \)-linearly independent. Therefore by (1),

\[
\text{HF}_{T/F^\perp}(r + \alpha) = \dim_{\mathbb{C}} \langle F \rangle_{r+\alpha} = r + \alpha + 1.
\]

This implies that the nonzero homogeneous elements of \( F^\perp \) have degree at least \( r + \alpha + 1 \). Since \( g_1 \in F^\perp \), the initial degree of \( F^\perp \) is \( r + \alpha + 1 \).

Therefore \( g_1 \) is part of a minimal generating set of \( F^\perp \) and hence there exists \( g_2 \in T_{s+1}^\perp \) such that \( F^\perp = (g_1, g_2) \). As \( s \geq r + \alpha \) and \( g_1 \) is not square free, by Sylvester’s algorithm

\[
\text{rk}(F) = s + 1.
\]

2. Assume \( \delta > 0 \), that is \( s < r + \alpha \). In order find the Waring rank of \( F \) in this case, we compute the Hilbert function of \( T/F^\perp \) and use this information to compute the element of smallest degree in \( F^\perp \). Since this computation depends on \( j \), we divide the proof in the following four cases:

(i) \( 0 \leq j < \left\lfloor \frac{\delta - 1}{2} \right\rfloor \); (ii) \( \delta \) is odd and \( j = \left\lfloor \frac{\delta}{2} \right\rfloor \); (iii) \( \left\lfloor \frac{\delta - 1}{2} \right\rfloor < j < \delta - 1 \); (iv) \( \delta \) is even and \( j = \frac{\delta}{2} \); and (v) \( \delta - 1 \leq j \). Notice that the first row of (2) in the table is covered in Case (i), the second and the third rows of (2) of the table are
covered in Case (iv) whereas the last row of the table is covered in Cases (i),(ii),(iii) and (iv).

**Case i:** \(0 \leq j < \left\lceil \frac{r}{2} \right\rceil - 1\) First we show that the initial degree of \(F^\perp\) is \(s + j + 2\). For this it suffices to show that \(\text{HF}_{T/F^\perp}(s + j + 1) = s + j + 2\) and there exists a nonzero form of degree \(s + j + 2\) in \(F^\perp\). For \(0 \leq i \leq r + \alpha - j - 1\), we have

\[
X^{r+\alpha-j-1-i} Y^i \circ F
\]

By (1) for \(i = 0, 1, \ldots, \min\{s, \alpha - j - 2\}\), we get

\[
\{x^{i+1}y^s, x^{i+2}y^s-1, \ldots, x^{i+\min\{s, \alpha-j\}}y^s-\min\{s, \alpha-j-2\}\} \subseteq \langle F \rangle_{s+j+1}.
\]

By assumption on \(j\), we have \(2(j+1) \leq \delta\) and hence \((q+1)\alpha - (j+1) > s\). Therefore taking \(i = (q+1)\alpha - (j+1), (q+1)\alpha - j, \ldots, (q+1)\alpha - 1\) in (1), we get

\[
\{x^{\alpha}y^{s+j+1-q\alpha}, x^{qa+1}y^{s+j-q\alpha}, \ldots, x^{qa+j}y^{s-q\alpha+1}\} \subseteq \langle F \rangle_{s+j+1}.
\]

Now substituting \(i = \alpha - j - 1, \alpha - j, \ldots, (q+1)\alpha - (j+2)\) in (1), we conclude that all monomials of degree \(s + j + 1\) belong to \(\langle F \rangle\). Therefore by (i), \(\text{HF}_{T/F^\perp}(s + j + 1) = s + j + 2\). Hence the nonzero homogeneous elements of \(F^\perp\) have degree at least \(s + j + 2\).

We claim that

\[
g_1 = \sum_{i=0}^{q} (-1)^i X^{r+1-ia} Y^{q-s-r+j+1} \in (F^\perp)_{s+j+2}.
\]

If \(q = 0\), then \(r = j\). Therefore \(g_1 = X^{r+1} Y^{s+1}\) which clearly belongs to \(F^\perp\). Hence assume that \(q > 0\). Then

\[
g_1 \circ F = 0 + \left[ \begin{array}{c} r+a \\ r+1 \end{array} \right] \cdot \left[ \begin{array}{c} s \\ s-r-j+1 \end{array} \right] \cdot x^{a-1} Y^{r-j-1} + \sum_{i=1}^{q-1} (-1)^i \left( a \cdot \left[ \begin{array}{c} r \\ r+1-ia \end{array} \right] \cdot \left[ \begin{array}{c} s+a \\ i\alpha+s-r+j+1 \end{array} \right] \cdot x^{(i+1)\alpha-1} Y^{(i-1)\alpha+r-j-1} + \sum_{i=1}^{q-1} (-1)^i \left( a \cdot \left[ \begin{array}{c} r+a \\ r+1-ia \end{array} \right] \cdot \left[ \begin{array}{c} s \\ i\alpha+s-r+j+1 \end{array} \right] \cdot x^{(i+1)\alpha-1} Y^{(i-1)\alpha+r-j-1} \right) + (-1)^q a \cdot \left[ \begin{array}{c} r \\ r+1-q\alpha \end{array} \right] \cdot \left[ \begin{array}{c} s+a \\ q\alpha+s-r+j+1 \end{array} \right] \cdot x^{qa-1} Y^{(1-q)\alpha+r-j-1} + 0 = 0.
\]
Thus the initial degree of $F^\perp$ is $s + j + 2$. Therefore $F^\perp = (g_1, g_2)$ for some $0 \neq g_2 \in T_{r + a - j}$. Moreover,

$$g_1 = \begin{cases} \sum_{i=0}^{q} (-1)^{i} X^{r + 1 - i a} Y^{i a + s - r + j - 1} & \text{if } j \geq 1 \text{ OR } s - r > 0 \\ Y \sum_{i=0}^{q} (-1)^{i} X^{r + 1 - i a} & \text{if } j = 0 \text{ and } r = s. \end{cases}$$

Suppose that $j = 0$ and $r = s$. Then

$$\left( \sum_{i=0}^{q} (-1)^{i} X^{r + 1 - i a} Y^{i a} \right) (X^a + Y^a) = X^{r + a + 1} + (-1)^q XY^{r + a}.$$  

Since $X^{r + a + 1} + (-1)^q XY^{r + a}$ is square-free, $g_1$ is square-free if $j = 0$ and $r = s$. Clearly, if $j \geq 1$, OR $s - r > 0$, then $g_1$ is not square-free. Therefore by Sylvester’s algorithm

$$\text{rk}(F) = \begin{cases} r + a - j & \text{if } j \geq 1 \text{ OR } s - r > 0 \\ s + 2 & \text{if } j = 0 \text{ and } r = s. \end{cases}$$

**Case ii:** $\delta$ is odd and $j = \frac{\delta - 1}{2}$

First we prove that the initial degree of $F^\perp$ is $s + j + 1$. For $0 \leq i \leq r + a - j$, we have

$$X^{r + a - j - i Y^i} \circ F = a \cdot [r_{a-j-i}] \cdot [s^{a+1}] \cdot X^{i+a} Y^{i+a-i} + [r_{a-j-i}] \cdot [s] \cdot X^{i} Y^{s-i}.$$  

Substituting $i = 0, 1, \ldots, a - j - 1$ in (2), we get

\begin{align*}
\{x^i y^j, x^{i+1} y^{j-1}, \ldots, x^{a-1} y^{s-a+j+1}\} \subseteq \langle F \rangle_{s+j}.
\end{align*}

As $j + 1 = \delta - j = r + a - s - j$, we have $j + s + 1 - a = r - j = q a$. Therefore taking $i = s + 1, s + 2, \ldots, r + a - j - 1$ in (2) we get

\begin{align*}
\{x^a y^{q a + j}, x^{q a + 1} y^{q a + j - 1}, \ldots, x^{q a + j - 1} y^{s - q a + 1}\} \subseteq \langle F \rangle_{s+j}.
\end{align*}

Now substituting $i = a - j, a - j + 1, \ldots, s$ in (2) we conclude that all the monomials of degree $s + j$ are in $\langle F \rangle$. Hence by (1), $HF_{T/F}(s + j) = s + j + 1$. Therefore the nonzero homogeneous elements of $F^\perp$ have degree at least $s + j + 1$.

We claim that

$$g_1 = \sum_{i=0}^{q+1} (-1)^{i} X^{s+j+1-i a} Y^{i a} \in \langle F^\perp \rangle_{s+j+1}.$$
Indeed,
\[ g_1 \circ F = 0 + \left[ \begin{array}{c} r+a \\ s+j+1 \end{array} \right] \cdot \left[ \begin{array}{c} \alpha \\ 0 \end{array} \right] \cdot x^{r+a-s-j-1}y^s \\
+ \sum_{i=1}^{q} (-1)^i \left( a \cdot \left[ \begin{array}{c} r \\ s+j+1-ia \end{array} \right] \cdot x^{r+ia-s-j-1}y^{s+(1-i)a} + \\
+ \left[ \begin{array}{c} r+a \\ s+j+1-ia \end{array} \right] \cdot x^{r+(i+1)a-s-j-1}y^{s-ia} \right) \\
+ (-1)^{q+1} a \cdot \left[ \begin{array}{c} r \\ (q+1)a \end{array} \right] \cdot x^i y^{s-qa} + 0 \\
= 0. \]

Therefore there exists \( 0 \neq g_2 \in T_{r+a-j+1} \) such that \( F^\perp = (g_1, g_2) \). As
\[ g_1 (X^a + Y^a) = X^{s+j+1+a} + (-1)^q Y^{s+j+1+a}, \]
and \( X^{s+j+1+a} + (-1)^q Y^{s+j+1+a} \) is square-free, \( g_1 \) is also square-free. Hence by Sylvester’s algorithm \( \text{rk}(F) = s + j + 1 = r + a - j \).

**Case iii:** \( \left\lceil \frac{\delta - 1}{2} \right\rceil < j < \delta - 1 \) OR \( \delta \) is even and \( j = \frac{\delta}{2} \)

Let \( k = \delta - j \). Then \( 2 \leq k \leq \left\lfloor \frac{\delta - 1}{2} \right\rfloor \). We show that \( \text{rk}(F) = r + a - j = s + k \). For this it suffices to show that \( \text{HT}_{F^\perp}(s + k - 1) = s + k \) and that there exists a square-free polynomial of degree \( s + k \) in \( F^\perp \). For \( 0 \leq i \leq r + a - k + 1 \), we have
\[ X^{r+a-k+1-i}y^i \circ F \]
\[ = a \cdot \left[ \begin{array}{c} r+a-k+1-i \\ r \end{array} \right] \cdot \left[ \begin{array}{c} \alpha \\ i \end{array} \right] \cdot x^{k-1-i-a}y^{s+a-i} + \left[ \begin{array}{c} r+a \\ r+a-k+1-i \end{array} \right] \cdot \left[ \begin{array}{c} \alpha \\ i \end{array} \right] \cdot x^{k-1+i}y^{s-i}. \]

Substituting \( i = 0, 1, \ldots, a - k \) in (3) we get
\[ \{ x^{k-1}y^s, x^{k}y^{s-1}, \ldots, x^{a-1}y^{s-a+k} \} \subseteq \langle F \rangle_{s+k-1}. \]

Notice that \( r = qa + j = qa + (\delta - k) \).

Taking \( i = s + 1, s + 2, \ldots, (q + 1)a - 1 \) in (3) we get
\[ \{ x^a y^{s-qa+k-1}, x^{a+1} y^{s-qa+k-2}, \ldots, x^{a+k-2} y^{s-qa+1} \} \subseteq \langle F \rangle_{s+k-1}. \]

(Note that as \( k \leq \left\lfloor \frac{\delta - 1}{2} \right\rfloor \), \( k \leq \delta - k \) and hence \( (q + 1)a - 1 \leq r + a - k + 1 \).)

Now using (3) for \( \alpha - k < i \leq s \), we conclude that all the monomials of degree \( s + k - 1 \) are in \( \langle F \rangle \), and thus \( \langle F \rangle_{s+k-1} = S_{s+k-1} \). Hence by (i) \( \text{HT}_{F^\perp}(s + k - 1) = s + k \).

Next we claim that
\[ g_1 = \sum_{i=0}^{q+1} (-1)^i X^{s+k-ia} y^{ia} \in \langle F^\perp \rangle_{s+k}. \]
We have
\[ g_1 \circ F = 0 + \frac{r+a}{s+k} \cdot \frac{s}{0} \cdot x^{r+α-s-k}y^s \]
\[ + \sum_{i=1}^{q} (-1)^i \left( a \cdot \left[ \begin{array}{c} s \cr r \end{array} \right] \cdot \frac{s+a}{iα} \cdot x^{r-s-k+iα}y^{s+(1-i)α} \right) \]
\[ + \frac{r+a}{s+k-ia} \cdot \frac{s}{ia} \cdot x^{r-s-k+ia}y^{s-ia} \]
\[ + (-1)^q + 1 \cdot a \cdot \frac{r}{0} \cdot \frac{s+a}{(q+1)α} \cdot x^ry^{s-qa} + 0 \]
\[ = 0. \]

(Notice that \((q+1)α = qa + α = s - α + k + α = s + k > s\). Hence \(g_1 \in (F^\perp)_{s+k}\). Also
\[ g_1(X^α + Y^α) = X^{s+k+α} + (-1)^q Y^{s+k+α}. \]
As \(X^{s+k+α} + (-1)^q Y^{s+k+α}\) is square-free, \(g_1\) is also square-free. Therefore by Sylvester’s algorithm \(rk(F) = s + k = s + δ - j = r + α - j\).

**Case iv:** \(δ - 1 \leq j \leq α - 1\)

First we show that the initial degree of \(F^\perp\) is \(s + 1\). For this it suffices to show that \(HF_{\mathcal{T}/\mathcal{F}}(s) = s + 1\) and that there exists a nonzero form of degree \(s + 1\) in \(F^\perp\). For \(0 \leq i \leq r + α\), we have
\[ x^{r+α-i}y^i \circ F \]
\[ = a \cdot \left[ \begin{array}{c} r \cr r + α - i \end{array} \right] \cdot \frac{s+a}{i} \cdot x^{i-α}y^{s+α-i} + \left[ \begin{array}{c} r + α \cr r + α - i \end{array} \right] \cdot \left[ \begin{array}{c} i \cr \alpha \end{array} \right] \cdot x^iy^{s-i}. \]

Therefore
\[ \{x^iy^{s-i} : 0 \leq i < α\} \]
\[ \cup \{a \cdot \left[ \begin{array}{c} r \cr r + α - i \end{array} \right] \cdot \frac{s+a}{i} \cdot x^{i-α}y^{s+α-i} + \left[ \begin{array}{c} r + α \cr r + α - i \end{array} \right] \cdot \left[ \begin{array}{c} i \cr \alpha \end{array} \right] \cdot x^iy^{s-i} : α \leq i \leq s\} \subseteq \langle F \rangle_s. \]
This implies that \(\langle F \rangle_s = S_s\). Hence by (i),
\[ HF_{\mathcal{T}/\mathcal{F}}(s) = s + 1. \]

We claim that
\[ g_1 = \sum_{i=0}^{q+1} (-1)^i X^{s-(j-δ)-iα}Y^{jα+(j-δ)+1} \in (F^\perp)_{s+1}. \]
We have

\[
g_1 \circ F = 0 + \left[ \begin{array}{c} r+a \\ s-j+\delta \end{array} \right] \cdot \left[ \begin{array}{c} s \\ j-\delta+1 \end{array} \right] \cdot x^{r+s+j-\delta} y^{s-j+\delta-1} \\
+ \sum_{i=1}^g (-1)^i \left( a \cdot \left[ \begin{array}{c} r \\ s-j+\delta-i \end{array} \right] \cdot \left[ \begin{array}{c} s+a \\ ia+j-\delta+1 \end{array} \right] \cdot x^{r-s+i-\delta+ia} y^{s-j+\delta+(1-i)a-1} \\
+ \left[ \begin{array}{c} r+a \\ s-j+\delta-ia \end{array} \right] \cdot \left[ \begin{array}{c} s \\ ia+j-\delta+1 \end{array} \right] \cdot x^{r-s+j-\delta+(i+1)a} y^{s-j+\delta-ia-1} \right) \\
+ (-1)^{q+1} a \cdot \left[ \begin{array}{c} r \\ (q+1)a+(j-\delta+1) \end{array} \right] \cdot x^{r} y^{s-q\alpha-(j-\delta+1)} + 0
= 0.
\]

(Notice that \((q+1)a + j - \delta = r + \alpha - \delta = s\), hence \(s - (j - \delta) = (q+1)a = r + (\alpha - j) > r\). Hence \(g_1 \in (F^\perp)_{s+1}\). Therefore the initial degree of \(F^\perp\) is \(s+1\). Hence \(g_1\) is part of a minimal generating set of \(F^\perp\). Therefore there exists \(0 \neq g_2 \in T_{r+s+1}\) such that \(F^\perp = (g_1, g_2)\). Clearly, if \(j \geq \delta + 1\), then \(g_1\) is not square-free. If \(\delta - 1 \leq j \leq \delta\), then

\[
g_1(X^\alpha + Y^\alpha) = \begin{cases} 
X^{s+a+1} + (-1)^{q+1}Y^{s+a+1} & \text{if } j = \delta - 1 \\
Y(X^{s+a} + (-1)^{q+1}Y^{s+a}) & \text{if } j = \delta,
\end{cases}
\]

which implies that \(g_1\) is square-free. Hence by Sylvester’s algorithm

\[
\text{rk}(F) = \begin{cases} 
\frac{s+1}{r+\alpha+1} & \text{if } \delta - 1 \leq j \leq \delta \\
r+1 & \text{if } j \geq \delta + 1.
\end{cases}
\]

**Remark 3.1.1.** The generic rank of a form of degree \(r+s+\alpha\) is \([\frac{r+s+\alpha+1}{2}]\), see [1] or [10]. Theorem 3.1 illustrates that the Waring rank of a specific form behaves as weirdly as possible compared to the generic rank. In particular, the Waring rank of a specific form can be smaller or larger than the generic rank.

We illustrate Theorem 3.1 in the following example.

**Example 3.1.**

Let the notation be as in Theorem 3.1.

(0) \((r = s = 0 \text{ and } \alpha \geq 1)\) Let \(F = x^\alpha + y^\alpha\). Clearly,

\[
\text{rk}(F) = \begin{cases} 
1 & \text{if } \alpha = 1 \\
2 & \text{if } \alpha = 2.
\end{cases}
\]

On the other hand, since \(r = s = 0\), we have \(\delta = \alpha \geq 1\) and \(j = 0\). Therefore \(\text{rk}(F)\) coincides with the Waring rank stated in Theorem 3.1.

(1) \((r = s \text{ and } \alpha = 1)\) Let \(F = x^r y^{r+1} + x^{r+1} y^r\). In this case \(F^\perp = (g_1, g_2)\) where

\[
g_1 = X^r + X^r Y + \cdots + (-1)^i X^{r+1-i} Y^i + (-1)^{r+1} Y^{r+1}
\]

and \(g_2 = X^{r+2}\). Since \(g_1\) is square-free, by Sylvester’s algorithm \(\text{rk}(F) = r + 1\).
On the other hand, since \( r = s \), we have \( \delta = r + \alpha - s = \alpha = 1 \) and hence \( j = 0 \) for every nonnegative integer \( r \). Therefore \( \text{rk}(F) = r + 1 \) by Theorem 3.1 also.

(2) \( (r = 0, s > 0 \text{ and } \alpha \geq 1) \). Let \( F = ay^{s+\alpha} + x^\alpha y^s \) where \( a = \frac{\alpha!}{(s+\alpha)!} \). In this case \( \delta = r + \alpha - s = \alpha - s \) and \( j = 0 \) if \( \delta > 0 \). Hence, by Theorem 3.1,

\[
\text{rk}(F) = \begin{cases} 
    s + 1 & \text{if } \alpha \leq s \\
    \alpha & \text{if } \alpha > s.
\end{cases}
\]

This can be verified directly (without using Theorem 3.1) as follows: We have

\[
F^\perp = \begin{cases} 
    (X^{\alpha+1}, Y^{s+1} - X^\alpha Y^{s-\alpha+1}) & \text{if } \alpha \leq s \\
    (XY^{s+1}, X^\alpha - Y^s) & \text{if } \alpha > s
\end{cases}
\]

and hence by Sylvester’s algorithm \( \text{rk}(F) \) is as required.

The following example illustrates that the Waring rank of a trinomial may depend on its coefficients.

**Example 3.2.**

Consider the quadratic form \( F = x^2 + xy + y^2 \). Then

\[
F = [x \ y] A_F [x \ y]^T \text{ where } A_F = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}.
\]

It is a standard fact from linear algebra that \( \text{rk}(F) = \text{rank}(A_F) \). Hence \( \text{rk}(F) = 2 \). On the other hand, if \( G = x^2 + 2xy + y^2 = (x + y)^2 \), then \( \text{rk}(G) = 1 \). This shows that unlike the binomial case, even in the binary case the Waring rank of a trinomial may depend on its coefficients.

Now we compare the Waring and the real Waring rank of a binomial form.

**Example 3.3.**

Let \( F = x^r y^r (x + y) \) where \( r \geq 1 \). By Theorem 3.1 \( \text{rk}(F) = r + 1 \). Whereas, since \( F \) splits completely into linear factors in \( \mathbb{R} \), \( \text{rk}_R(F) = 2r + 1 \) by Theorem 2.6.

Moreover, unlike in the complex case, the real Waring rank of a binomial form may depend on its coefficients.

**Example 3.4.**

By Theorem 2.6, \( \text{rk}_R(x^3 - xy^2) = 3 \) whereas \( \text{rk}_R(x^3 + xy^2) = 2 \). But \( \text{rk}(x^3 \pm xy^2) = \text{rk}(y^3 \pm x^2 y) = 2 \) by Theorem 3.1 (take \( r = 0, s = 1 \) and \( \alpha = 2 \) in Theorem 3.1).

4. **On binary forms with Waring rank two**

This section studies some geometric properties of the loci of binary forms with Waring rank two. Fix an integer \( d \geq 4 \). We show that a form with
Waring rank two is square free (see Corollaries 4.1.1 and 4.1.2). Moreover, for every three distinct linear forms \( l_1, l_2, l_3 \in \mathbb{P}S_1 \) there are just a finite number of forms in \( \mathbb{P}S_d \) with Waring rank two and multiple of \( l_1 l_2 l_3 \) (see Corollary 4.8.1). This number does not depend on the linear forms \( l_1, l_2, l_3 \) and it is \((d-1)^2\) (see Theorem 4.12).

We build a dihedral cover of the set of forms with Waring rank two and multiples of \( l_1 l_2 l_3 \) (see Theorems 4.7 and 4.10, see also Theorem 2.9) and show that there are \((d-1)^2\) different orbits in the cover space by the dihedral action (see Theorem 4.12). This cover allows us also to describe the set of all forms in \( \mathbb{PR}[x, y]_d \) with a maximal gap between their Waring and real Waring rank (see Theorem 4.13). Finally we offer a second proof of Theorem 4.12 based on more classical results. Namely, the set of forms multiple of \( l_1 l_2 l_3 \) form a linear subspace of \( \mathbb{P}S_d \) (see Corollary 4.5.1). We show that it intersects transversely the second secant variety of the rational normal curve in \( \mathbb{P}S_d \) and that their dimensions are complementary, then result follows by Bezout’s theorem.

We denote by \( \mathbb{N}_d \) the set of integers \( l \) with \( 0 \leq l \leq d - 1 \). Let \( X \) be a set. We denote by \( |X| \) the cardinal of \( X \) and by \( \Delta(X) \) (or simply by \( \Delta \) when there is no risk of confusion) the big diagonal in \( X^3 \), that is \( X^3 \setminus \Delta \) is the set of triples of pairwise distinct elements of \( X \).

We consider the \( \mathbb{C} \)-vector space \( S_1 \) with basis \( \{ x, y \} \). This fixes an action of the general linear group \( \text{GL}_2(\mathbb{C}) \), or simply \( \text{GL}_2 \), on \( S_1 \). Namely, every \( \alpha \in \text{GL}_2 \) determines a unique invertible \( \mathbb{C} \)-linear map \( \alpha : S_1 \to S_1 \) and then \( \alpha \cdot l = \alpha(l) \) for all \( l \in S_1 \). Since \( S_d = \text{Sym}^d S_1 \), this action extends to an action of \( \text{GL}_2 \) on \( S_d \), that is given \( \alpha \in \text{GL}_2 \) and \( f \in S_d \),

\[
\alpha \cdot f = \text{Sym}^d(\alpha)(f).
\]

Observe that \( \text{Sym}^d(\alpha) \) is just the \( d \)-th graded component of the linear change of coordinates \( \text{Sym}(\alpha) : S \to S \).

This action determines an action of the projective general linear group \( \text{PGL}_2(\mathbb{C}) \), or simply \( \text{PGL}_2 \), on \( \mathbb{P}S_d \). Namely, given \( \varphi \in \text{PGL}_2 \), say the class of \( \alpha \in \text{GL}_2 \), we denote by \( \text{Sym}^d(\varphi) \) the class of \( \text{Sym}^d(\alpha) \in \text{GL}_{d+1} \) in \( \text{PGL}_{d+1} \) (which is clearly well defined). So, given \( F \in \mathbb{P}S_d \),

\[
\varphi \cdot F = \text{Sym}^d(\varphi)(F).
\]

**Proposition 4.1.** The set of forms \( f \in S_d \) with Waring rank two is equal to the orbit \( \text{GL}_2 \cdot (x^d - y^d) \).

**Proof.** Let \( f \in S_d \) be a form with Waring rank two. So, the form \( f \) decomposes as \( f = l_1^d - l_2^d \) and the linear forms \( l_1, l_2 \in \mathbb{P}S_1 \) are non-proportional, or equivalently, pairwise \( \mathbb{C} \)-linearly independent. Hence, via a linear change of coordinates, we may assume \( l_1 = x \) and \( l_2 = y \).

**Corollary 4.1.1.** Every form in \( S \) with Waring rank two splits as a product of distinct linear forms.
Remark 4.1.2. Given a form $F$ in a standard graded polynomial ring $k[x_1, \ldots, x_n]$ with coefficients in a field $k$, there is an analogue notion to Waring rank of $F$ (see [10, Definition 3.3]). Notice that in this section we have not used so far that $S$ is the polynomial ring over $C$ in two variables. Theorem 4.1 and Corollary 4.1.1 hold for every field $k$ and integer $d$ such that the Waring rank of $x_1^d - x_2^d \in k[x_1, \ldots, x_n]$ is two, for every element $a \in k$ there is $b \in k$ with $b^d = a$ and $k$ contains all the $d$-th roots of unity.

**Definition 4.2.** We denote by $C$ the rational normal curve in $\mathbb{P}S_d$ corresponding to the image of the map that sends $L \in \mathbb{P}S_1$ to $L^d \in \mathbb{P}S_d$.

**Definition 4.3.** We denote by $TC \subseteq \mathbb{P}S_d$ the union of all the lines tangent to $C$ and by $\Sigma \subseteq \mathbb{P}S_d$ the union of all the lines spanned by two different points of $C$.

**Remark 4.3.1.** The closure $\overline{\Sigma}$ is the second secant variety of $C$ and the set $\Sigma \setminus C$ is the set of all forms $F \in \mathbb{P}S_d$ with Waring rank two.

**Proposition 4.4.** We may classify the points of $\overline{\Sigma}$ into two different kinds.

- **The tangent points**, points lying on $TC$.
- **The secant points**, points $p$ lying on some line spanned by two different points of $C$ but $p \notin C$.

**Proof.** The first secant variety of $C$ is $C$ itself. Then, by [17, Proposition 10.15] a point in the subset $\overline{\Sigma} \setminus C$ of $\overline{\Sigma}$ lies on the span of a unique divisor of degree 2 of $C$. \hfill $\Box$

**Remark 4.4.1.** Let $p$ be a point of $\overline{\Sigma}$.

- If $p$ is a tangent point, there are $L, N \in \mathbb{P}S_1$ with $p = NL^{d-1}$ and then its Waring rank is either $d$ or 1. Furthermore, by Theorem 2.4, the converse implication also holds.
- If $p$ is a secant point, it corresponds to a form $F \in \mathbb{P}S_d$ with Waring rank 2 and then, by Corollary 4.1.1, $F$ is square free.

**Definition 4.5.** Given $\mathcal{L} = (L_1, L_2, L_3) \in (\mathbb{P}S_1)^3 \setminus \Delta$, we denote by $\Pi(\mathcal{L})$ the set of forms $F \in \mathbb{P}S_d$ that are multiple of $L_1L_2L_3$. We denote by $\Pi$ the particular case $\mathcal{L} = (x + y, x, y)$.

**Remark 4.5.1.** Given $\mathcal{L} = (L_1, L_2, L_3) \in (\mathbb{P}S_1)^3 \setminus \Delta$, consider $a_1, a_2, a_3 \in \mathbb{P}^1$ the respective roots of $L_1, L_2, L_3 \in \mathbb{P}S_1$. Observe that $\Pi(\mathcal{L}) \subseteq \mathbb{P}S_d$ is just the linear subspace of $\mathbb{P}S_d$ corresponding to the forms $F \in \mathbb{P}S_d$ with roots at $a_1, a_2, a_3 \in \mathbb{P}^1$. In particular, the codimension of $\Pi(\mathcal{L}) \subseteq \mathbb{P}S_d$ is three.

**Proposition 4.6.** Let $\mathcal{L} = (L_1, L_2, L_3) \in (\mathbb{P}S_1)^3 \setminus \Delta$. The Waring rank of every $F \in \Pi(\mathcal{L}) \cap \overline{\Sigma}$ is two.

**Proof.** Since a form $F \in \Pi(\mathcal{L}) \cap \overline{\Sigma}$ is a multiple of three distinct linear forms, by Corollary 4.4.1 the Waring rank of $F$ is two. \hfill $\Box$
We denote by $\Theta \subseteq \mathbb{C}$ the set of the $d$-th roots of the unity and we fix the usual anticlockwise order $\{\xi_l\}_{l \in \mathbb{N}_d}$ for $\Theta$ where $\xi_0 = 1$.

**Definition 4.7.** Fix $\mathcal{L} = (L_1, L_2, L_3) \in (\mathbb{P}S_1)^3 \setminus \Delta$. We define the map $\Gamma_{\mathcal{L}}: (\Theta^3 \setminus \Delta) \to \Pi(\mathcal{L}) \cap \Sigma$ as follows. Given $\xi = (\xi_i, \xi_j, \xi_k) \in \Theta^3 \setminus \Delta$, there is a unique $\varphi \in \text{PGL}_2$ with

$$
\varphi([x - \xi_i y]) = L_1, \quad \varphi([x - \xi_j y]) = L_2, \quad \varphi([x - \xi_k y]) = L_3.
$$

Then we set $\Gamma_{\mathcal{L}}(\xi) = \varphi \cdot [x^d - y^d]$. For the particular case $\mathcal{L} = (x, y, x, y)$, we denote $\Gamma_{\mathcal{L}}$ simply by $\Gamma$.

**Proposition 4.8.** For all $\mathcal{L} = (L_1, L_2, L_3) \in (\mathbb{P}S_1)^3 \setminus \Delta$, the map $\Gamma_{\mathcal{L}}$ is surjective.

**Proof.** Given $F \in \Pi(\mathcal{L}) \cap \Sigma$, first by definition of $\Pi(\mathcal{L})$, there is $G \in \mathbb{P}S_{d-3}$ with $F = L_1 L_2 L_3 G$ and, second by Theorem 4.6 and Theorem 4.1, there is $\varphi \in \text{PGL}_2$ with $\varphi \cdot [x^d - y^d] = F$.

So, for every $i = 1, 2, 3$ there is $\xi_i \in \Theta$ with $\varphi^{-1}(L_i) = [x - \xi_i y]$ and then $\Gamma_{\mathcal{L}}(\xi_i, \xi_j, \xi_k) = F$. \hfill $\square$

**Corollary 4.8.1.** The cardinal of $\Pi(\mathcal{L}) \cap \Sigma$ is at most $d(d-1)(d-2)$, the cardinal of $\Theta^3 \setminus \Delta$. In particular, it is finite.

Let us describe the map $\Gamma$ which has a particular neat description. Fix $\xi = (\xi_i, \xi_j, \xi_k) \in \Theta^3 \setminus \Delta$. Observe that the map $\varphi$ in the Theorem 4.7 is a Möbius transformation. Consider the affine chart $U \subseteq \mathbb{P}S_1$ corresponding to the coefficient of $x$ being different from zero. The map $\varphi|_U$ sends respectively $-\xi_i, -\xi_j, -\xi_k$ to $1, 0, \infty$. So, for all $l \in \mathbb{N}_d \setminus \{k\}$

$$
\varphi([x - \xi_l]y) = [x + (\xi_i, \xi_j; \xi_j, \xi_k)y],
$$

where $(\xi_i, \xi_j; \xi_j, \xi_k)$ denotes the cross ratio of four complex numbers, and then

$$
\Gamma(\xi) = [(x + y)xy \prod_{l \in \mathbb{N}_d \setminus \{i, k\}} (x + (\xi_i, \xi_j; \xi_j, \xi_k)y)] \in \Pi \cap \Sigma.
$$

In this way, by Theorem 4.8, we get an explicit description of $\Pi \cap \Sigma$,

$$
(4) \quad \Pi \cap \Sigma = \text{Im}(\Gamma) \subseteq \mathbb{P}S_d.
$$

For example when $d = 4$, the 4-th roots of the unity are the vertices of a square, hence their cross ratio is the harmonic ratio $-1$, or one of its conjugates 2 or $\frac{1}{2}$, and then

$$
\Pi \cap \Sigma = \{xy(x + y)(x - y), xy(x + y)(x + 2y), xy(x + y)(x + \frac{1}{2}y)\}.
$$

In this case we even know that the Waring rank of every $F \in \Pi \setminus (\Pi \cap \Sigma)$ is 3, since clearly it is not 1 nor 2 and it is not 4 by Theorem 2.4.
We denote by \( D \) the dihedral group of order \( 2d \), which is the group of symmetries of \( \Theta \) (see [16, Section 1.2]). We consider \( D \) acting component-wise on \( \Theta^3 \setminus \Delta \).

**Lemma 4.9.** The cardinal of \( (\Theta^3 \setminus \Delta) / D \) is \( \binom{d-1}{2} \).

**Proof.** It is a standard application of Burnside’s Lemma (see [16, Exercice 8]), but this case is particularly simple. Since every non-trivial \( \sigma \in D \) acting on \( \Theta \) fixes at most two points, for all \( \xi \in \Theta^3 \setminus \Delta \) the cardinals of the orbits \( D\xi \) are equal to the cardinal of \( D \). Hence,

\[
| (\Theta^3 \setminus \Delta) / D | = \frac{|\Theta^3 \setminus \Delta|}{|D|} = \left( \begin{array}{c} d-1 \\ 2 \end{array} \right).
\]

\( \Box \)

**Theorem 4.10.** For every \( L \in (PS_1)^3 \Delta \), the map \( \Gamma_L: \Theta^3 \setminus \Delta \longrightarrow \Pi(L) \cap \Sigma \) is a \( D \)-cover of \( \Pi(L) \cap \Sigma \). In particular, the cardinal of the partition \( (\Theta^3 \setminus \Delta) / \Gamma_L \) does not depend on \( L \in (PS_1)^3 \Delta \).

To prove Theorem 4.10 we introduce two maps \( n \) and \( m \) and we establish some of their elementary properties.

Let \( S^1 \subseteq \mathbb{C} \) denote the unit circle. For each \( \xi = (\xi_i, \xi_j, \xi_k) \in (\Theta^3 \setminus \Delta) \), first we denote by \( \gamma_\xi: S^1 \longrightarrow \mathbb{R} \cup \{\infty\} \) the map sending \( p \in S^1 \) to the cross ratio \( -(p, \xi_i; \xi_j, \xi_k) \in \mathbb{R} \cup \{\infty\} \). Second, we define \( S^1_{ij} \) as the unique connected component of the open subset \( S^1 \setminus \{\xi_i, \xi_j\} \subseteq S^1 \) such that \( \xi_k \notin S^1_{ij} \). Similarly, we define the open subsets \( S^1_{jk} \subseteq S^1 \) and \( S^1_{ki} \subseteq S^1 \) of \( S^1 \). Then, we define the maps \( n: \Theta^3 \setminus \Delta \longrightarrow \mathbb{N}^3 \) and \( m: \Theta^3 \setminus \Delta \longrightarrow \mathbb{N}^3 \) as sending \( \xi \in (\Theta^3 \setminus \Delta) \) to

\[
n(\xi) = \left( |\Theta \cap S^1_{ij}|, |\Theta \cap S^1_{jk}|, |\Theta \cap S^1_{ki}| \right)
\]

\[
m(\xi) = \left( |\gamma_\xi(\Theta \cap S^1_{ij})|, |\gamma_\xi(\Theta \cap S^1_{jk})|, |\gamma_\xi(\Theta \cap S^1_{ki})| \right).
\]

For example in Fig. 1, for \( \xi = (\xi_1, \xi_3, \xi_6) \) the value \( n(\xi) \) is \( (1,2,0) \) or for \( \xi = (\xi_2, \xi_5, \xi_1) \), it is \( n(\xi) = (2,1,0) \).

**Lemma 4.11.** Let \( n \) and \( m \) be the maps defined above.

1. The maps \( n \) and \( m \) are equal.
2. The partition \( (\Theta^3 \setminus \Delta) / \Gamma \) is finer than \( (\Theta^3 \setminus \Delta) / m \).
Theorem 4.11. The cardinal of the image of \( n \) is \( \binom{d-1}{2} \).

Proof. 1. For all \( \xi \in \Theta^3 \setminus \Delta \), the map \( \gamma_\xi \) is a Möbius transformation, so it is bijective.

2. For each \( \xi \in (\Theta^3 \setminus \Delta) \), the set \( \gamma_\xi(\Theta \setminus \{\xi\}) \subseteq \mathbb{R} \) is the set of roots of \( \Gamma(\xi)(x,1) \). Hence, given \( \xi, \xi' \in (\Theta^3 \setminus \Delta) \), if \( \Gamma(\xi) = \Gamma(\xi') \), then \( m(\xi) = m(\xi') \).

3. Observe that for all \( \xi \in (\Theta^3 \setminus \Delta) \), the sum of the integers \( n(\xi) \) is \( d-3 \) and, for all triplets \( (a, b, c) \in (\mathbb{N}_{d-3})^3 \) with \( a + b + c = d-3 \), trivially \( (a, b, c) = n(\xi_0, \xi_{a+1}, \xi_{(a+1)+(b+1)}) \). So,

\[
|\text{Im}(n)| = |\{(a, b, c) \in (\mathbb{N}_{d-3})^3 : a + b + c = d-3\}| = \binom{d-1}{2}. \quad \square
\]

Proof of Theorem 4.10. Given \( \mathcal{L}, \mathcal{L}' \in (\mathbb{P}S_1)^3 \setminus \Delta \) there is \( \varphi \in \text{PGL}_2 \) such that \( \Gamma_{\mathcal{L}} = \text{Sym}^d(\varphi) \circ \Gamma_{\mathcal{L}'} \). Since \( \text{Sym}^d(\varphi) \) is an isomorphism, the partition \( (\Theta^3 \setminus \Delta)/\Gamma_{\mathcal{L}} \) does not depend on \( \mathcal{L} \). So we restrict to the case \( \mathcal{L} = (x+y, x, y) \), that is \( \Gamma_{\mathcal{L}} = \Gamma \).

Given \( \sigma \in D \), the action of \( \sigma \) on \( \Theta \) is the restriction to \( \Theta \) of a symmetry of \( C \), which is a Möbius transformation. Then, for all pairwise distinct \( \xi_l, \xi_i, \xi_j, \xi_k \in \Theta \),

\[
(\xi_l, \xi_i; \xi_j, \xi_k) = (\sigma(\xi_l), \sigma(\xi_i); \sigma(\xi_j), \sigma(\xi_k)).
\]

Hence the map \( \Gamma \) is invariant by the action of \( D \) and then the partition \( (\Theta^3 \setminus \Delta)/D \) is finer than \( (\Theta^3 \setminus \Delta)/\Gamma \). Finally, by Theorem 4.11(2), the partition \( (\Theta^3 \setminus \Delta)/\Gamma \) is finer than \( (\Theta^3 \setminus \Delta)/m \) and, by Theorem 4.11(1,3) and Theorem 4.9, the cardinals of the partitions \( (\Theta^3 \setminus \Delta)/D \) and \( (\Theta^3 \setminus \Delta)/m \) are equal and finite, hence the partitions \( (\Theta^3 \setminus \Delta)/D \) and \( (\Theta^3 \setminus \Delta)/\Gamma \) are equal. \( \square \)

Proposition 4.12. The cardinal of \( \Pi(\mathcal{L}) \cap \Sigma \) is \( \binom{d-1}{2} \).

Proof. By Theorem 4.8 the map \( \Gamma_{\mathcal{L}} \) is surjective, so the cardinals of \( \Pi(\mathcal{L}) \cap \Sigma \) and \( (\Theta^3 \setminus \Delta)/\Gamma_{\mathcal{L}} \) are equal. Then the claim follows from Theorem 4.10 and Theorem 4.9. \( \square \)
**Proposition 4.13.** The set $A \subseteq \mathbb{PR}[x, y]_d$ of all forms with a maximal gap between their Waring and real Waring rank is equal to

$$B = \bigcup_{\mathcal{L} \in (\mathbb{PR}[x, y]_d)^3 \Delta} \text{Im}(\Gamma_{\mathcal{L}})$$

**Proof.** Notice that the real Waring rank of a form in $\mathbb{R}[x, y]_d$ is one if and only if its Waring rank is one as well. Hence, by Theorem 2.3, the maximal gap between the Waring and the real Waring rank of a form in $\mathbb{PR}[x, y]_d$ is at most $d - 2$. Clearly, by Theorem 2.6, this possible maximal gap is reached by the forms in $\Pi \cap \Sigma$, so

$$A = \{F \in \mathbb{PR}[x, y]_d : \text{rk}(F) = 2, \text{rk}_R(F) = d\}$$

and then, by Theorems 2.6 and 4.1, $A \subseteq B$.

Given $F \in B$, by Theorem 4.6, the Waring rank of $F$ is two. Hence, by Theorem 2.6, if all the roots of $F$ are real, then $F \in A$. But given $\mathcal{L} \in (\mathbb{PR}[x, y]_d)^3 \Delta$, via a linear change of coordinates of $\mathbb{PR}[x, y]_1$, we may assume $\mathcal{L} = (x + y, x, y)$ and then, for every $\xi \in \Theta^3 \Delta$, the roots of $\Gamma_{\mathcal{L}}(\xi)$ are real because they belong to the image of $\gamma_{\xi}$ (see discussion below Theorem 4.10). \qed

The following theorem summarises the results of this section.

**Theorem 4.14.** For all $\mathcal{L} = (L_1, L_2, L_3) \in (\mathbb{PS}_1)^3 \Delta$, there are exactly $\binom{d-1}{2}$ distinct forms in $\mathbb{PS}_d$ with Waring rank two and being multiple of $L_1L_2L_3$.

The union $\Pi(\mathcal{L}) \cap \Sigma$ for all $\mathcal{L} \in (\mathbb{PS}_1)^3 \Delta$ is equal to $\Sigma \setminus C$, the set of all forms $F \in \mathbb{PS}_d$ with Waring rank two.

**Proof.** Observe that each form $F \in \mathbb{PS}_d$ with Waring rank two and multiple of $L_1L_2L_3$ belongs to $\Pi(\mathcal{L}) \cap \Sigma$, so the first part follows from Theorem 4.12 and Theorem 4.6. The second part follows from the definition of $\Pi(\mathcal{L})$ and Corollary 4.1.1. \qed

The main result of this section, Theorem 4.14, follows mainly from Theorem 4.12. To finish we offer another proof for the latter using classical results instead of the maps $\Gamma_{\mathcal{L}}$.

First, we need Theorem 4.15, which is already known, but we prove it without the maps $\Gamma_{\mathcal{L}}$.

**Lemma 4.15.** Let $\mathcal{L} = (L_1, L_2, L_3) \in (\mathbb{PS}_1)^3 \Delta$. The cardinal of $\Pi(\mathcal{L}) \cap \Sigma$ is finite and it does not depend on the triplet $\mathcal{L}$.

**Proof.** By Theorem 4.1 and Theorem 4.6, every form $F \in \Pi(\mathcal{L}) \cap \Sigma$ belongs to the orbit $\text{PGL}_2 \cdot [x^d - y^d]$, but there are just finitely many $\varphi \in \text{PGL}_2$ with $\varphi \cdot [x^d - y^d] \in \Pi(\mathcal{L})$.

Given another triplet $\mathcal{L}' = (L'_1, L'_2, L'_3)$, there is $\varphi \in \text{PGL}_2$ such that $\varphi(L_i) = L'_i$ for all $i = 1, 2, 3$. Then, $\text{Sym}^d(\varphi)|_{\Pi(\mathcal{L})} : \Pi(\mathcal{L}) \to \Pi(\mathcal{L}')$ is an isomorphism which preserves the Waring rank. So, it determines a one-to-one correspondence between $\Pi(\mathcal{L}) \cap \Sigma$ and $\Pi(\mathcal{L}') \cap \Sigma$. \qed
Lemma 4.16. The variety $\Sigma$ is smooth away from $C$.

Proof. It is a particular case of some more general well-known results. See [15] or, for a more general setting, [39, Theorem 1.1 or Proposition 2.3] and [38, Lemma 3.1]. For another setting see [21, §8.15-8.16]. Recall [3], a seminal work of all of them.

Second proof of Theorem 4.12. By Theorem 4.15, we assume $L = (x + y, x, y)$. Since a form $F \in \Pi$ has at least three distinct roots, by Corollary 4.4.1, $F \not\in TC$ and $\Pi \cap \Sigma \subseteq (\Sigma \setminus TC)$. Now we show that $\Pi$ and $\Sigma$ intersect transversely. Let $P \in \Pi \cap \Sigma$. Since $P \in (\Sigma \setminus TC)$, first by Theorem 4.16, $P$ is a smooth point of $\Sigma$ and, second by Corollary 4.4.1, there are $l, t \in S_1$ with $[l] \neq [t] \in \mathbb{P}S_1$ such that $P = [l^d + t^d] \in \mathbb{P}S_d$. By Terracini’s Lemma (see [36] for the original statement, [10, Lemma 2.15] for a modern formulation and [18, Lemma 2.1] for a modern proof of a simplified version), the following morphism parametrises the projective tangent space of $\Sigma$ at $P$

$$\tau : \mathbb{P}(S_1 \times S_1) \to \mathbb{P}S_d,$$

$$[n : m] \to [nl^{d-1} + mt^{d-1}].$$

By Corollary 4.5.1, the projective tangent space of $\Pi$ at $P$ is $\Pi$ itself. So, the varieties $\Sigma$ and $\Pi$ intersect transversely at $P$ if the pull back by $\tau$ of the linear equations defining $\Pi$ are linearly independent. Recall from Corollary 4.5.1 that $\Pi$ is just the set of all forms with roots at $[1 : -1], [0 : 1], [1 : 0] \in \mathbb{P}^1$ (the roots of $x + y, x, y$). So, clearly such pull back just corresponds to the homogeneous linear equations on the coefficients of $[n : m] \in \mathbb{P}(S_1 \times S_1)$ determined by the following matrix

$$\begin{pmatrix}
  l^{d-1} (1, 0) & 0 & t^{d-1} (1, 0) & 0 \\
  0 & l^{d-1} (0, 1) & t^{d-1} (0, 1) & 0 \\
  t^{d-1} (1, 1) & -l^{d-1} (1, -1) & t^{d-1} (1, -1) & -t^{d-1} (1, -1)
\end{pmatrix}.$$

Using that $[l] \neq [t] \in \mathbb{P}S_1$ and $P = [l^d + t^d] \in \Pi$ we can see that the column rank is three.

Finally, since $\dim \Sigma + \dim \Pi = (3) + (d - 3) = d$ (cf. [22, Proposition 11.32] and Corollary 4.5.1), by [22, Theorem 18.3] the intersection of $\Sigma$ and $\Pi$ is a finite set with $\deg \Sigma = \binom{d-1}{2}$ (cf. [17, Theorem 10.16]) points. \hfill $\square$

5. To the future

5.1. To higher rank. Theorem 5.1 below shows that the family of linear subspaces $\Pi(L)$ contains all forms whose Waring rank is lower than the generic rank for degree $d$ forms.

Proposition 5.1. Let $F \in \mathbb{P}S_d \setminus C$ and denote $r = \text{rk}(F)$. If $r \leq \left\lfloor \frac{d+1}{2}\right\rfloor$, then $F$ has at least three distinct roots.
Proof. The form $F$ has not a single root since $F \notin C$. If $F$ has just two distinct roots, via a linear change of coordinates we may assume $F = x^ay^b$ with $a + b = d$ and then $\text{rk}(F) = \max\{a, b\} + 1 > r$. 

So, it could be interesting intersect higher secant varieties of $C$ with the linear subspaces $\Pi(L) \subseteq \mathbb{P}S_d$. One key point in order to describe such intersection for the rank two case is Theorem 4.4, which, by [12], looks generalizable to

$$
\sigma_r(C) \setminus \sigma_{r-1}(C) = (\mathcal{O}_{r-1} \setminus \mathcal{O}_{r-2}) \sqcup (\sigma_r(C) \setminus \mathcal{O}_{r-1})
$$

where $\mathcal{O}_r$ is the $r$-th osculating variety of $C$ (i.e. $\mathcal{O}_r = \bigcup_{P \in C} \langle rP \rangle$).

5.2. To higher dimensions. When we consider $S = \mathbb{C}[x_0, \ldots, x_n]$, the dimension of $C \subseteq \mathbb{P}S_d$ is $n$ and the dimension of $\Sigma \subseteq \mathbb{P}S_d$ is $2n + 1$ (assuming $d > 4$, for $d = 4$ the cases $n = 3, 4, 5$ are respectively 5, 9, 14-defectives). Observe that Theorem 4.4 and Corollary 4.4.1 are satisfied in any dimension. That is, the forms belonging to $\Sigma$ split as a product of linear forms. So, denoting $\mathbb{P}L_d$ the natural image of $S_1^{(d)}$ (the $n$-th symmetric power) into $\mathbb{P}S_d$, $\Sigma \subseteq \mathbb{P}L_d$. Now, $\dim \mathbb{P}L_d = nd$ and the subset of forms in $\mathbb{P}L_d$ multiples of two given linear forms is a subvariety of codimension $2n$. Hence, we could try to intersect $\Sigma \subseteq \mathbb{P}L_d$ with subspaces of forms with a fixed root $\alpha \in \mathbb{P}^n$ and multiples of two linear forms $l, t \in \mathbb{P}S_1$ where $\alpha$ is a root of neither $l$ nor $t$. Such varieties have the desired codimension in $\mathbb{P}L_d$.

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