Indirect Controllability of Quantum Systems; A Study of Two Interacting Quantum Bits

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May 5, 2014

Abstract

A quantum mechanical system $S$ is indirectly controlled when the control affects an ancillary system $A$ and the evolution of $S$ is modified through the interaction with $A$ only. A study of indirect controllability gives a description of the set of states that can be obtained for $S$ with this scheme. In this paper, we study the indirect controllability of quantum systems in the finite dimensional case. After discussing the relevant definitions, we give a general necessary condition for controllability in Lie algebraic terms. We present a detailed treatment of the case where both systems, $S$ and $A$, are two-dimensional (qubits). In particular, we characterize the dynamical Lie algebra associated with $S+A$, extending previous results, and prove that complete controllability of $S+A$ and an appropriate notion of indirect controllability are equivalent properties for this system. We also prove several further indirect controllability properties for the system of two qubits, and illustrate the role of the Lie algebraic analysis in the study of reachable states.

1 INTRODUCTION

Indirect controllability of a quantum system $S$ refers to the situation where two quantum systems $S$ and $A$ interact, and the externally applied control is allowed to influence the evolution of $A$ only, while the state of $S$ is of
interest. The systems $S$ and $A$ are referred to as the target system and accessor system, or ancillary system, respectively. Therefore, the state of $S$ is indirectly controlled through the interaction with $A$. The problem of indirect controllability is to analyze to what extent it is possible to modify the state of $S$ with this scheme in various situations.

The question of indirect controllability is of theoretical interest for the further development of quantum control and of quantum information, as indirect controllability is a signature of the capability of the dynamics to generate entanglement [20]. In fact, if there is no entanglement between the two systems, $S$ and $A$, they evolve separately and indirect controllability is impossible. This question is also of practical relevance since, in many experimental setups, it is much easier to control the accessor system $A$ rather than the relevant part $S$. There might be several reasons for that. For instance, we might be able to use only control fields tuned at the resonance frequency of some of the particles which make up a multi-particle systems or we might be able to access only systems in certain locations. As a physical example of indirect control, protocols of short-distance quantum communication have been described using spin chains [1,2]. The state to be transmitted is put in interaction with one end of the chain, the controlled part, and it is reproduced at the other side of the chain, without physical transmission. So, control at one end of the chain realizes the task of state transfer. It has been proved theoretically [7] that control at the end of the chain is sufficient to realize a reliable transfer. This can be seen as indirect control. The target system $S$ is the last spin at one end of the chain whose state we want to manipulate (and make equal to the one at the other end). The rest of the spin system can be seen as the accessor system. Spin chain schemes can be implemented through super-conducting devices based on Josephson junctions [27] and are potential tools for the implementation of quantum technologies.

There is a large literature studying the control of quantum systems, starting with establishing the connection with geometric control theory [3,16,21] until more recent studies for closed (see, e.g., [3,4,9,23,25]) and open (see e.g., [5,13,26]) quantum systems. Nevertheless, an algebraic characterization of indirect controllability is still lacking. In fact, most investigations on indirect controllability have given conditions so that the full system $S+A$ is completely controllable, that is, every unitary transformation can be performed on it. This implies in particular that every unitary operation can be performed on $S$. In these cases, controllability can be proved by verifying the standard Lie algebraic conditions for closed systems, that is, calculating the so-called dynamical Lie algebra (see definition in Section 3), which fully characterizes the dynamics of the total system [10]. However, as we shall prove, there are cases where indirect controllability on $S$, in an appropriate sense, can be achieved even without complete controllability of the total system, and this motivates a more detailed study of this property.
The goal of this paper is to provide a general framework for the treatment of indirect controllability of quantum systems, with special emphasis on the case of two coupled qubits $S$ and $A$. Because of the introduction of the partial trace (see definition in the next section) to describe the dynamics of the system $S$, much of the machinery of Lie transformation groups cannot be used in this case. Nevertheless Lie algebraic techniques play a crucial role in determining many indirect controllability properties.

We briefly summarize this work. In Section 2 we define the mathematical set up and give the basic notations and mathematical tools. In Section 3 we define the main notions for indirect controllability and give a general necessary criterion for indirect controllability at the Lie algebra level (Theorem 1). In Sections 4 and 5 we present a detailed treatment of the case of two interacting quantum bits, $S$ and $A$. In particular, in Section 4 we give a description of the dynamical Lie algebra associated with this system. This extends a result of [15] which only gives a sufficient condition for complete controllability. Based on the characterization of the dynamical Lie algebra, in Section 5 we prove various indirect controllability properties for the two-qubit system presented in Theorems 4, 5, in Proposition 5.2 and in Example 5.1. We draw some conclusions and give directions for further research in Section 6.

2 BASIC FACTS CONCERNING THE DYNAMICS OF INTERACTING SYSTEMS

The states of the target and ancillary systems, $S$ and $A$, are described by density operators $\rho_S$ and $\rho_A$, respectively, that is, positive semi-definite, trace class operators\(^1\) with unit trace on Hilbert spaces $S$ and $A$, associated with $S$ and $A$, respectively \([6, 20]\). The total system $S + A$ is described by the density operator $\rho_T$ on the Hilbert space $S \otimes A$. The sets of density operators on $S$, $A$, and $S \otimes A$ are convex, the boundary consisting of pure states, that is, projectors of rank one of the form $\rho = |\psi\rangle\langle\psi|$, with $|\psi\rangle$ in $S$, $A$, or $S \otimes A$, respectively\(^2\). The state of a subsystem ($\rho_S$ or $\rho_A$) can be extracted from the state of the total system $\rho_T$ through the partial trace operation, $\rho_S = \text{Tr}_A(\rho_T)$ and $\rho_A = \text{Tr}_S(\rho_T)$. We recall that, if we express an Hermitian operator $\rho \in S \otimes A$ as

$$\rho = \sum_j \alpha_j \rho_j^S \otimes \rho_j^A,$$  \hspace{1cm} (1)

\(^1\)A trace class operator is a bounded operator $A$ such that $\text{Tr}(A^\dagger A)^{\frac{1}{2}}$ is finite. We shall deal with the finite dimensional case where this requirement is automatically satisfied.

\(^2\)We shall make use of the Dirac notation of quantum mechanics denoting a general vector in a Hilbert space ($ket$) by $|\cdot\rangle$ and the associated dual linear operator ($bra$) by $\langle\cdot|$.\
with Hermitian operators $\rho^S_j \in S$ and $\rho^A_j \in A$, and real coefficients $\alpha_j$, the partial trace operation is a linear map from the space of Hermitian operators on $S \otimes A$ to the space of Hermitian operators on $S$ (A) defined as

$$\text{Tr}_A(\rho) := \sum_j \alpha_j \rho^S_j \text{Tr}(\rho^A_j), \quad \text{Tr}_S(\rho) := \sum_j \alpha_j \rho^A_j \text{Tr}(\rho^S_j).$$  

(2)

The main property of the partial trace which will be used in the following is that, for unitary operators $F$ and $G$ acting on $S$ and $A$ respectively, it holds

$$\text{Tr}_A \left( F \otimes G \rho F^\dagger \otimes G^\dagger \right) = F \text{Tr}_A(\rho) F^\dagger.$$  

(3)

In our discussion we are going to assume finite dimensional systems $S$ and $A$, that is, both $S$ and $A$ are finite dimensional vector spaces. Some of the definitions and notions we shall introduce hold for the infinite dimensional case as well or can be naturally extended. We shall denote by $n_S$ and $n_A$ the dimensions of $S$ and $A$, respectively, and the total system $S + A$ has dimension $n_S n_A$, so $\rho_S$, $\rho_A$ and $\rho_T$ are represented by Hermitian, positive semi-definite, unit trace matrices of dimensions $n_S \times n_S$, $n_A \times n_A$, $n_S n_A \times n_S n_A$, respectively.

The evolution of the total system is determined by a Hermitian operator $H_T$, called the Hamiltonian, which can be conveniently separated in four parts as follows

$$H_T = H_S + H_A + H_I + H_C.$$  

(4)

The Hamiltonian $H_S$ determines the evolution of the system $S$ alone. It is of the form $H_S = B \otimes 1_{n_A}$, for some Hermitian operator $B$ on $S$. Analogously $H_A = 1_{n_S} \otimes C$ for some Hermitian operator $C$ on $A$. $H_I$ models the two-body interaction between $S$ and $A$, and it has the general form

$$H_I = \sum_{j=1}^m B_j \otimes C_j,$$  

(5)

for Hermitian operators $B_j$ and $C_j$ on $S$ and $A$ respectively. Finally the Hamiltonian $H_C$ models the interaction with the external control. Since in the indirect controllability scheme we assume only control on $A$, $H_C$ will be written as

$$H_C = \sum_{k=1}^n u_k(t) 1_{n_S} \otimes L_k.$$  

(6)

\[\text{Without loss of generality we shall assume in the following that all Hamiltonians have zero trace, since we can always decompose a Hamiltonian $H$ as $H = \frac{1}{n} \text{Tr}(H) 1_n + \tilde{H}$ where $\tilde{H}$ has zero trace and $n$ is the dimension of the underlying system. The term $\frac{1}{n} \text{Tr}(H) 1_n$ affects the dynamics only for a global phase factor which is irrelevant in quantum mechanics, and thus can be neglected. Here and in the following $1$ denotes the identity operator. We shall write $1_n$ if the identity operator acts on a space of dimension $n$, when we want to highlight such dimension.}\]
for some control functions \(u_k(t)\) and Hermitian operators \(L_k\). Assuming units such that \(\hbar = 1\), the total system \(S + A\) evolves according to the Schrödinger operator equation

\[
\dot{X} = -iH_T(u(t))X, \quad X(0) = 1_{n_S n_A},
\]

and the density operator \(\rho_T\) varies as

\[
\rho_T(t) = X(t)\rho_T(0)X^\dagger(t).
\]

The initial state is assumed to be uncorrelated, that is \(\rho_T(0) = \rho_S(0) \otimes \rho_A(0)\). By combining this evolution with the partial trace (2), we obtain the dynamics of the relevant system \(S\),

\[
\rho_S(t) = \text{Tr}_A \left( X(t)\rho_S(0) \otimes \rho_A(0)X^\dagger(t) \right).
\]

In the following, we shall study the map, for given \(\rho_A\),

\[
\rho_S \rightarrow \text{Tr}_A(X\rho_S \otimes \rho_A X^\dagger),
\]

as \(X\) varies in the set of all unitary operators available as solutions of (7) with suitable control actions.

### 3 Indirect Controllability for General Systems

We now give several definitions of controllability, that might be appropriate in different contexts. The following definition is standard for the controllability of the composite system \(S + A\).

**Definition 3.1** The system \(S + A\) is called completely controllable (CC) if, for any special unitary evolution \(P \in SU(n_S n_A)\) there exists a time \(T \geq 0\) and a set of control functions, \(u_1(t), \ldots, u_m(t)\), defined in \([0, T]\), such that the solution of (7) satisfies \(X(T) = P\).

In dealing with indirect controllability of \(S\), it is important to take into account two processes: (i) the preparation of \(\rho_A\); (ii) the dynamical control of the full system \(S + A\). The process of preparation of \(\rho_A\) is typically an irreversible process completely separated from the unitary evolution of the
total system. This preparation is performed before dynamically controlling the system. Therefore, in the definition of indirect controllability we need to include the information on the set of possible initial states for the system $A$. This set is denoted by $\Lambda_A$ in the following definitions.

**Definition 3.2** A state $\rho'_S$ is said to be reachable from $\rho_S$ given $\Lambda_A$, if there exists $\rho_A \in \Lambda_A$, and a set of controls $u_1(t), \ldots, u_m(t)$, defined in $[0, T]$, such that the solution $X = X(T)$ of (11) satisfies

$$\rho'_S = \text{Tr}_A \left( X(T) \rho_S \otimes \rho_A X^\dagger(T) \right). \tag{11}$$

If $\mathcal{R}$ is the set of unitary evolutions which can be obtained for the total system, the set of reachable states from $\rho_S$ given $\Lambda_A$ is

$$\Lambda_S(\rho_S) := \{ X \rho_S \otimes \rho_A X^\dagger | \rho_A \in \Lambda_A, X \in \mathcal{R} \}. \tag{12}$$

In the previous definition, $\Lambda_S(\rho_S)$ depends on $\Lambda_A$, and the choice of $\rho_A$ in $\Lambda_A$ is part of our control strategy. Special cases are when $\Lambda_A$ consists of a single state, that is the accessor admits only a specific initial configuration, or when it consists of states that have special features. The case of pure states is particularly important, both from an application viewpoint, and because of some results we shall prove in this paper (cf. proposition 5.2). Nonetheless, we will not a priori put any restriction on $\Lambda_A$.

**Definition 3.3** The system $S$ is called fully indirectly controllable (FIC) given $\Lambda_A$ if, for every initial state $\rho_S$, the reachable set $\Lambda_S(\rho_S)$ is the set of all density matrices of $S$. The system $S$ is called unitary indirectly controllable (UIC) given $\Lambda_A$ if, for any given initial state $\rho_S$, the reachable set $\Lambda_S(\rho_S)$ is the set of all density matrices unitarily equivalent to $\rho_S$.

By definition, FIC implies UIC given the same set $\Lambda_A$. Moreover, if FIC (or UIC) holds given $\Lambda_A$, it also holds given any $\Lambda'_A$ such that $\Lambda_A \subseteq \Lambda'_A$. It is a known result in quantum information theory that, if the system is CC and $n_A \geq n_S^2$, the system $S$ is FIC given the set $\Lambda_A$ consisting of one pure state for $A$. We shall prove in Theorem 5 that if $n_S = 2$, $n_A = 2$ is sufficient to have this property.

If UIC or FIC hold independently of the set $\Lambda_A$, we shall refer to these properties as strong. For instance, strong UIC means that, independently of the initial state $\rho_A$, for all $X_S \in SU(n_S)$ we are able to find a control

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<ref>6</ref>This is equivalent to $\Lambda_A$ being the set of all pure states since under the CC condition we can transfer the state of $A$ between two arbitrary pure states. So this statement is equivalent to pure FIC. It is proved that an arbitrary completely positive map can be performed on $S$ (dilation theorem for completely positive maps<sup>20</sup>) and, according<sup>24</sup> to, every possible state transformation can be realized through a completely positive map.
strategy such that the solution of (9) is \( X_\mathcal{S} \rho_\mathcal{S} X_\mathcal{S}^\dagger \). Complete controllability of the total system implies the strong UIC notion since every transformation of the form \( X_\mathcal{S} \otimes 1 \) is, in particular, a unitary transformation, and gives the desired state transfer \( \rho_\mathcal{S} \rightarrow X_\mathcal{S} \rho_\mathcal{S} X_\mathcal{S}^\dagger \), independently of the initial state \( \rho_A \). For this reason, most of the investigations on indirect controllability have focused on giving conditions for complete controllability (see, e.g., [15]). When UIC or FIC holds for the set \( \Lambda_A \) of pure states for \( A \) we shall refer to these properties as \textit{pure}. Clearly strong implies pure for both properties.

\textbf{Remark 3.4} We remark that all the notions of indirect controllability which have been introduced, and the criteria which will be presented in the following, are invariant with respect to unitary changes of coordinates which affect the system \( S \) and \( A \) separately (local transformations). They are not invariant under general unitary changes of coordinates that affect both systems \( S \) and \( A \).

According to classical results of geometric control theory [16] applied to quantum mechanics [8], [18], [21], the set of all available unitary transformations on the total system \( S + A \) is dense in the connected Lie group \( e^\mathcal{L} \) corresponding to the Lie algebra \( \mathcal{L} \) generated by the set

\[ \mathcal{F} := \{ iH_S + iH_I + iH_A, i1 \otimes L_1, \ldots, i1 \otimes L_m \}, \] (13)

where \( H_S, H_I \) and \( H_A \) were defined in (4) - (6), and it is equal to \( e^\mathcal{L} \) if \( e^\mathcal{L} \) is compact.\(^7\) Therefore the Lie algebra \( \mathcal{L} \) is of crucial importance in determining the structure of the reachable set for the total system and the system \( S \). In particular the set \( \Lambda_S(\rho_S) \) defined in (12) can be written as

\[ \Lambda_S(\rho_S) = \{ \rho'_S | \rho_A \in \Lambda_A, X \in e^\mathcal{L} \}. \] (14)

The Lie algebra \( \mathcal{L} \) is referred to as the \textit{dynamical Lie algebra} associated with the system. We shall also consider the \textit{control Lie algebra}, \( \mathcal{B} \) which is generated by \( \{ i1 \otimes L_1, \ldots, i1 \otimes L_m \} \); it turns out that \( \mathcal{L} \) is generated by \( \mathcal{B} \) and \( iH_S + iH_I + iH_A \).

For a subspace \( \mathcal{V} \) of \( u(n_{SN_A}) \), let \( \text{Tr}_A(\mathcal{V}) \) denote the image of \( \mathcal{V} \) under \( \text{Tr}_A \) in \( u(n_S) \). Consider an initial state for the total system, \( \rho_S \otimes \rho_A \), and the subspace of \( u(n_{SN_A}) \),

\[ \mathcal{V} := \bigoplus_{k=0}^{\infty} \text{ad}^k_{\mathcal{L}}(\text{span}\{ i\rho_S \otimes \rho_A \}), \] (15)

\(^7\)We shall not insist in the following on this distinction and in fact all the Lie groups we shall encounter are compact.
where \( \text{ad}_L^0 P = P \) for any space \( P \) and, recursively, \( \text{ad}_L^k P := [L, \text{ad}_L^{k-1} P] \).

Theorem 1: Consider the nontrivial case \( \rho_S \neq \frac{1}{n_S} 1_{n_S} \). Assume that for all \( X \in SU(n_S) \) there exists \( U \in eL \) such that
\[
\text{Tr}_A(U \rho_S \otimes \rho_A U^\dagger) = X \rho_S X^\dagger.
\]
Then \( \text{Tr}_A(V) = u(n_S) \).

Proof. Since \( V \) is invariant under \( L \), it is also invariant under the action of \( eL \) by conjugation, i.e., \( Q \in V \implies U Q U^\dagger \in V \) for any \( U \in eL \). Now, from (16) we have that, for every \( X \in SU(n_S) \), \( X i \rho_S X^\dagger \in \text{Tr}_A(V) \). This implies that
\[
\text{span}\{X i \rho_S X^\dagger | X \in SU(n_S)\} \subseteq \text{Tr}_A(V).
\]
Since \( \rho_S \neq \frac{1}{n_S} 1_{n_S} \), span\(\{X i \rho_S X^\dagger | X \in SU(n_S)\}\) = \( u(n_S) \), it follows \( u(n_S) = \text{Tr}_A(V) \). \( \square \)

Theorem 1 can be applied without calculating \( V \) when we recognize that \( i \rho_S \otimes \rho_A \) belongs to a subspace \( \tilde{V} \) invariant under commutation with \( L \), which might in general include properly \( V \). If \( \text{Tr}_A(\tilde{V}) \neq u(n_S) \), we can exclude the UIC property. One case is when \( i \rho_S \otimes \rho_A \in L \otimes \text{span}\{i 1_{n_S} \otimes 1_{n_A}\} \) and \( \text{Tr}_A(L \otimes \text{span}\{i 1_{n_S} \otimes 1_{n_A}\}) \neq u(n_S) \). An application of this will be used in Theorem 1. Another possible case is when \( \tilde{V} = L \perp i 1_{n_S} \otimes 1_{n_A} \), which is also invariant under \( L \). We have found an example showing that the converse of Theorem 1 is not true in general and it is an interesting open problem to understand how the condition of the theorem can be modified to have a necessary and sufficient condition.

4 THE DYNAMICAL LIE ALGEBRA FOR THE INDIRECT CONTROL OF TWO QUBITS

In this and the following section, we explore the case in which target \( S \) and accessor \( A \) are both two-level systems. We start with a study of the dynamical Lie algebra in all cases. We shall treat separately the two cases in which the control Lie algebra \( B \) is the full \( 1 \otimes \mathfrak{su}(2) \), and in which it is one-dimensional. These are the only possible cases since \( B \) is always a Lie subalgebra of (a Lie algebra isomorphic to) \( \mathfrak{su}(2) \) and there are no two-dimensional subalgebras in \( \mathfrak{su}(2) \). We shall also assume throughout that the interaction Hamiltonian \( H_I \) is non-zero since the case of trivial interaction is clearly non controllable and consists of the two systems evolving separately.

\[ \text{The space } V \text{ can be calculated recursively from a basis of } L, \text{ and because of } u(n_S n_A) \text{ being finite dimensional, there exists a } k \text{ such that } V = \bigoplus_{k=0}^L \text{ad}_L^k \text{ (span}\{i \rho_S \otimes \rho_A\}). \text{The main (in fact, defining) property of } V \text{ is that it is the smallest subspace of } u(n_S n_A) \text{ which is invariant under } L \text{ and contains } i \rho_S \otimes \rho_A. \]
4.1 Algebra with Pauli matrices

We recall for future reference the definition of the Pauli matrices

\[
\tilde{\sigma}_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\sigma}_y := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \tilde{\sigma}_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (18)

The related matrices

\[
\sigma_x := \frac{i}{2} \tilde{\sigma}_x, \quad \sigma_y := \frac{i}{2} \tilde{\sigma}_y, \quad \sigma_z := \frac{i}{2} \tilde{\sigma}_z,
\] (19)

span the standard two-dimensional representation of \(su(2)\) and satisfy the commutation relations

\[
[\sigma_x, \sigma_y] = \sigma_z, \quad [\sigma_y, \sigma_z] = \sigma_x, \quad [\sigma_z, \sigma_x] = \sigma_y,
\] (20)

and the anti-commutation relations

\[
\{\sigma_j, \sigma_k\} = -\frac{1}{2} \delta_{jk} \mathbf{1}, \quad \{\sigma_j, \mathbf{1}\} = 2\sigma_j \quad j, k = x, y, z.
\] (21)

Anti-commutation relations are useful when using the formula

\[
[A \otimes B, C \otimes D] = \frac{1}{2} [A, C] \otimes [B, D] + \frac{1}{2} \{A, C\} \otimes [B, D]
\] (22)
in calculations.

4.2 Model for two interacting qubits

We specialize to the two qubits case the general model \([4, 5, 6]\), and write \(1 = 1_2\). The Hamiltonian of the system \(S\) has the form \(H_S = B \otimes \mathbf{1}\), with \(iB \in su(2)\), and we can make a change of coordinates (cf. Remark 3.4) to diagonalize \(B\). This does not modify the form of \(H_I\) nor the dimension of \(B\). The Hamiltonian of the accessor system alone has the form \(H_A := \mathbf{1} \otimes C\) with \(iC \in su(2)\). Therefore we have

\[
iH_S := \omega_S \sigma_z \otimes \mathbf{1},
\] (23)

\[
iH_I := i\sigma_a \otimes \sigma_x + i\sigma_b \otimes \sigma_y + i\sigma_c \otimes \sigma_z,
\] (24)

\[
iH_A := i\mathbf{1} \otimes C, \quad iC \in su(2).
\] (25)

Here and in the following, \(\sigma_a\), with \(a \neq x, y, z\) indicates a matrix determined by a vector \(\vec{a} := [a_x, a_y, a_z]^T \in \mathbb{R}^3\) and given by

\[
\sigma_a := a_x \sigma_x + a_y \sigma_y + a_z \sigma_z,
\] (26)

\(\text{The commutator } [B, C] \text{ is defined as } [B, C] := BC - CB \text{ and the anti-commutator } \{B, C\} \text{ is defined as } \{B, C\} := BC + CB.\)
that is a general matrix in $\mathfrak{su}(2)$. Define the matrix

$$K := \begin{pmatrix} \vec{a}^T \\ \vec{b}^T \\ \vec{c}^T \end{pmatrix},$$

and write it as

$$K := (D F),$$

with $D$ of dimension $3 \times 2$ and $F$ of dimension $3 \times 1$.

In this notation the model is determined by the matrix $K$, the parameter $\omega_S$, the skew-Hermitian matrix $iC$ and the control Lie algebra $B$. We are interested in the dynamical Lie algebra generated by the set

$$S := \{\omega_S \sigma_z \otimes 1 + i\sigma_a \otimes \sigma_x + i\sigma_b \otimes \sigma_y + i\sigma_c \otimes \sigma_z + i1 \otimes C, B\}. \quad (29)$$

In the following subsection we characterize $L$ in all cases when $B$ is the full Lie algebra $1 \otimes \mathfrak{su}(2)$. Then, we shall give a necessary and sufficient controllability condition for the case where $B$ is 1-dimensional and $\omega_S = 0$, that is, the system $S$ does not have dynamics by itself.

### 4.3 The dynamical Lie algebra in the case $B = \mathfrak{su}(2)$

**Theorem 2** Assume $\dim B = 3$. Let $K$, $D$, $F$ and $\omega_S$ defined as above. Then we have the following cases for the dynamical Lie algebra $L$ generated by $S$ in (29). If $\omega_S \neq 0$, we have:

1a) If $D \neq 0$ and $F \neq 0$, the system is completely controllable, that is, $L = \mathfrak{su}(4)$.

1b) If $D \neq 0$ and $F = 0$, then

$$L = \text{span}\{\sigma_z \otimes 1, 1 \otimes \mathfrak{su}(2), i(\sigma_x, \sigma_y) \otimes \mathfrak{su}(2)\}, \quad (30)$$

which is 10-dimensional.

1c) If $D = 0$ and $F \neq 0$, then

$$L = \text{span}\{i\sigma_z \otimes \mathfrak{su}(2), \sigma_z \otimes 1, 1 \otimes \mathfrak{su}(2)\}, \quad (31)$$

which is 7-dimensional.

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10In this correspondence between vectors in $\mathbb{R}^3$ and elements in $\mathfrak{su}(2)$, we have $\sigma_x \leftrightarrow \vec{i}$, $\sigma_y \leftrightarrow \vec{j}$, $\sigma_z \leftrightarrow \vec{k}$. This correspondence between $\mathfrak{su}(2)$ and $\mathbb{R}^3$ is, in fact, a Lie algebra isomorphism, where in $\mathbb{R}^3$ the Lie bracket is replaced by the cross product. There is a correspondence also between inner products in $\mathfrak{su}(2)$ and $\mathbb{R}^3$ and orthogonal vectors in $\mathbb{R}^3$ correspond to orthogonal matrices in $\mathfrak{su}(2)$.

11Notice that since we have assumed nontrivial interaction at least one between $F$ and $D$ must be nonzero.
If $\omega_S = 0$, we have:

2a) If $\text{rank}(K) = 1$, there exists $\sigma \in \text{su}(2)$ such that

\[
\mathcal{L} := \text{span}\{i\sigma \otimes \text{su}(2), \mathbf{1} \otimes \text{su}(2)\}. \tag{32}
\]

2b) If $\text{rank}(K) = 2$, $\mathcal{L}$ is conjugate (and therefore isomorphic) to the Lie algebra $\mathcal{L}$ of point 1b) above.

2c) If $\text{rank}(K) = 3$, the system $S + A$ is completely controllable.

The condition $\text{rank}(K) = 3$, or equivalently $\det(K) \neq 0$, is the complete controllability sufficient condition of [15] for both the cases $\omega_S = 0$ and $\omega_S \neq 0$. It is clear that there are cases included in 1a) where this condition is not verified and the system is nevertheless completely controllable.

It is interesting to analyze the structure of the Lie algebra $\mathcal{L}$ in the cases which are not completely controllable, in particular, in cases 1b) and 2b). This Lie algebra is isomorphic (and therefore conjugate, see Lemma 4.2 in [3]) to $\mathfrak{sp}(2)$ as it can be shown for example by using the test of Theorem 8 in [3]. Therefore the total system $S + A$ is pure state controllable, which means that every transfer is possible between pure states in $S \otimes A$. This implies that, if $\rho_S \otimes \rho_A$ is a pure state, and the final desired state for $\rho_S$ is also a pure state, this transfer will be possible in the indirect control scheme. We shall see in Proposition 5.2 that a stronger property actually holds in this case, that is, Pure UIC. We shall consider this case also in the Example 5.1, in a setting of interest for applications. We now give the proof of Theorem 2.

Proof. We notice that since $B = \mathbf{1} \otimes \text{su}(2)$ we can cancel $\mathbf{1} \otimes C$ in (25) with an element of $B$. We also notice that, in any case, the dynamical Lie algebra $\mathcal{L}$ is the same as the Lie algebra $\mathcal{L}'$ generated by the set

\[
\mathcal{S}' := \{\omega_S \sigma_z \otimes \mathbf{1}, i\sigma_a \otimes \text{su}(2), i\sigma_b \otimes \text{su}(2), i\sigma_c \otimes \text{su}(2), \mathbf{1} \otimes \text{su}(2)\}. \tag{33}
\]

The fact that $\mathcal{L} \subseteq \mathcal{L}'$ is obvious since the generators of $\mathcal{L}$ can be obtained as linear combinations of the generators of $\mathcal{L}'$. To show the converse inclusion we need to show that every generator of $\mathcal{L}'$ listed in (33) is in $\mathcal{L}$. To do this take the Lie bracket of $\omega_S \sigma_z \otimes \mathbf{1} + i\sigma_a \otimes \sigma_x + i\sigma_b \otimes \sigma_y + i\sigma_c \otimes \sigma_z \in \mathcal{L}$ with $\mathbf{1} \otimes \sigma_z$. This gives $-i\sigma_a \otimes \sigma_y + i\sigma_b \otimes \sigma_x$. Taking the Lie bracket of this with $\mathbf{1} \otimes \sigma_x$ gives $i\sigma_a \otimes \sigma_z$ and with Lie brackets with $\mathbf{1} \otimes \sigma_y$ and linear combinations, we obtain all elements of the form $i\sigma_a \otimes \sigma$, with arbitrary $\sigma \in \text{su}(2)$. Analogously we obtain elements in $i\sigma_b \otimes \text{su}(2)$ and $i\sigma_c \otimes \text{su}(2)$. We therefore analyze the Lie algebra generated by the set in (33) in all cases.

Assume first $\omega_S \neq 0$. Assume both $D$ and $F$ are different from zero. Since $D \neq 0$, there exists one among $\sigma_a$, $\sigma_b$ and $\sigma_c$ with a nontrivial component perpendicular to $\sigma_z$. Assume, without loss of generality, that it is $\sigma_a$. 
We can perform a rotation (i.e., a unitary similarity transformation) on the first qubit so that this component is proportional to $\sigma_y$ without modifying $\sigma_z$, that is, we can assume without loss of generality that $\sigma_a = \alpha \sigma_y + \beta \sigma_z$, with $\alpha \neq 0$ which we can therefore assume equal to 1. Taking the Lie bracket of $i\sigma_a \otimes \sigma$ for any $\sigma \in \mathfrak{su}(2)$ with $\sigma_z \otimes 1$, we obtain therefore $i\sigma_x \otimes \sigma$. Taking the Lie bracket with $\sigma_z \otimes 1$ again, we obtain $i\sigma_y \otimes \sigma$. This means that the span of $i\sigma_x \otimes \sigma$ and $i\sigma_y \otimes \sigma$ is in $L$. Now since $F \neq 0$ there is one of the $\sigma_{a,b,c}$ which has nonzero component along $\sigma_z$ (recall that our change of coordinates did not modify the components along $\sigma_z$ since it amounts to a rotation in the $x-y$ plane). Therefore we have every element of the form $i\sigma_1 \otimes \sigma_2$, $\sigma_1 \in \mathfrak{su}(2)$ and $\sigma_2 \in \mathfrak{su}(2)$. The remaining terms of the standard basis of $\mathfrak{su}(4)$ can be obtained by Lie brackets of the type $[i\sigma_1 \otimes \sigma_z, i\sigma_2 \otimes \sigma_z]$, for appropriate $\sigma_1$ and $\sigma_2$ in $\mathfrak{su}(2)$. In the case $D \neq 0$ but $F = 0$, the argument is the same but it stops at the point where we have generated all elements in the Lie algebra described in (30). If $D = 0$ but $F \neq 0$, there is at least one element among $\sigma_a, \sigma_b$ and $\sigma_c$ which has nonzero component along $\sigma_z$ and this is the only possible nonzero component. Therefore we have already all the elements listed in (31) which span a proper Lie subalgebra of $\mathfrak{su}(4)$.

Consider now the case where $\omega_S = 0$, i.e., there is no natural dynamics of the target system $S$. If rank($[D \ F]$) = 1 then there exists a $\sigma \in \mathfrak{su}(2)$ such that $\sigma_{a,b,c} = k_{a,b,c} \sigma$, for coefficients $k_{a,b,c}$. Therefore we obtain the Lie algebra described in (32). If the rank is 2 there are two among $\sigma_a, \sigma_b$ and $\sigma_c$ which are linearly independent while the third one is a linear combination of these two and therefore the corresponding matrices $M \otimes \sigma$, with $M$ free, do not contribute to the Lie algebra. Assume, without loss of generality that the linearly independent matrices are $\sigma_a$ and $\sigma_b$. By a rotation again, up to a nonzero unimportant proportionality factor, we can assume that $\sigma_a = \sigma_x$ and $\sigma_b = \sigma_y + \alpha \sigma_x$. Therefore $i(\sigma_x, \sigma_y) \otimes \mathfrak{su}(2)$ is included in the Lie algebra $L$. By taking the Lie bracket $[i\sigma_x \otimes \sigma_z, i\sigma_y \otimes \sigma_z]$ one obtains the remaining basis element $\sigma_z \otimes 1$ in (30). In the case 2c) (rank 3) one adds to the previously generated elements another set of elements $i\sigma_c \otimes \sigma$ with arbitrary $\sigma \in \mathfrak{su}(2)$ and notice that $\sigma_c$ must have, in the chosen coordinates, non-zero component along $\sigma_z$. Therefore one obtains as above the remaining elements in the standard basis of $\mathfrak{su}(4)$. Alternatively, and more quickly, one can use the fact pointed out after the statement of the theorem, that the Lie algebra (30) is conjugate to $\mathfrak{sp}(2)$ and every such Lie algebra is known to be a maximal subalgebra [14] in $\mathfrak{su}(4)$, that is, the addition of every nonzero element outside the Lie algebra but inside $\mathfrak{su}(4)$, causes the generation of the whole Lie algebra $\mathfrak{su}(4)$. □
4.4 Complete controllability when $\mathcal{B}$ is 1-dimensional and $\omega_S = 0$

In the case where the control Lie algebra $\mathcal{B}$ is one-dimensional, that is, there is only one independent control, the analysis becomes more complicated. Among the other issues, one has to consider that the independent dynamics of the accessor system (i.e., the Hamiltonian $H_A$ in (4)) cannot be canceled in general by the action of the control and play a significant role in determining the dynamics of the total system. Nevertheless, we can use the result of Theorem 2 jointly with the fact that, with the same parameters for the system, the dynamical Lie algebra in the case of $\mathcal{B}$ 1-dimensional is always a subalgebra (not necessarily proper) of the corresponding dynamical Lie algebra for the case of $\mathcal{B}$ 3-dimensional. We restrict ourselves to the case where $\omega_S = 0$.

**Theorem 3** Assume $\omega_S = 0$ and $\dim \mathcal{B} = 1$. Then the system $S + A$ is completely controllable if and only if the following two conditions are verified:

C1) $\det(K) \neq 0$;

C2) The components of $\text{Tr}_A(iH_IH_C)$ and $\text{Tr}_S(H_A)$ perpendicular to $\text{Tr}_S(H_C)$ are not both zero.

Notice that condition C1 is the same as the condition 2c in the $\dim \mathcal{B} = 3$ case. To that, we have to add the generically satisfied condition C2 to have a necessary and sufficient condition in the case of $\mathcal{B}$ 1-dimensional.

**Proof.** It is useful to make a change of coordinates on the system $A$, so as to make $\mathcal{B} = \text{span}\{1 \otimes \sigma_z\}$. In these coordinates, the part of $iC$ in (29) which is parallel to $\sigma_z$ can be neglected because it is already contained in $\mathcal{B}$. Therefore, we can consider only the part perpendicular to $\sigma_z$ which can be taken equal to $\omega_A \sigma_y$, for a certain real parameter $\omega_A$. In these coordinates, we need to redefine $\sigma_a, \sigma_b$ and $\sigma_c$ and therefore $K$. Moreover we can make a change of coordinates on the system $S$ to make $\sigma_b = \beta \sigma_y, \sigma_a = (\alpha \sigma_x + \gamma \sigma_y)$ and $\sigma_c = x \sigma_x + y \sigma_y + z \sigma_z$, for real coefficients $\alpha, \beta, \gamma$ and $x, y, z$.

Conditions C1 and C2, in these coordinates, become

C1')
$$
\begin{vmatrix}
\alpha & \gamma & 0 \\
0 & \beta & 0 \\
x & y & z
\end{vmatrix} \neq 0.
$$

(34)

C2')
$$
\omega_A^2 + x^2 + y^2 \neq 0.
$$

(35)

The condition (34) is necessary because it is needed in the $\mathcal{B}$ 3-dimensional case. If condition (35) is not satisfied, the set $S$ in (29) can be taken, in these coordinates,

$$
S := \{i\alpha \sigma_x \otimes \sigma_x + i\gamma \sigma_y \otimes \sigma_x + i\beta \sigma_y \otimes \sigma_y + iz \sigma_z \otimes \sigma_z, 1 \otimes \sigma_z\},
$$

(36)
and both elements in this set are in the Lie subalgebra

\[ L' := \text{span}\{1 \otimes \sigma_z, \sigma_z \otimes 1, i\sigma_y \otimes \sigma_x, i\sigma_x \otimes \sigma_y, i\sigma_y \otimes \sigma_y, i\sigma_x \otimes \sigma_x, i\sigma_z \otimes \sigma_z\}. \]  

(37)

We now prove the sufficiency of the conditions C1 and C2 for complete controllability. We examine the Lie algebra generated by the set \( S \) which can be written in the given coordinates

\[ S := \{i\sigma_a \otimes \sigma_x + i\sigma_b \otimes \sigma_y + i\sigma_c \otimes \sigma_z + \omega_A 1 \otimes \sigma_y, 1 \otimes \sigma_z\}. \]  

(38)

We notice that the Lie algebra generated by \( S \) in (38) is the same as the Lie algebra generated by the set with three elements \( S' \) defined as

\[ S' := \{i\sigma_c \otimes \sigma_z, i\sigma_a \otimes \sigma_x + i\sigma_b \otimes \sigma_y + \omega_A 1 \otimes \sigma_y, 1 \otimes \sigma_z\}. \]  

(39)

This can be easily seen because, if we take the Lie bracket of the first element of \( S \) in (38) with the second element \((1 \otimes \sigma_z)\) two times, we obtain the second element in \( S' \) in (39), and by subtracting this from the first element of \( S \), we obtain the first element in \( S' \). We first characterize the Lie algebra \( L'' \) generated by the second and third element in \( S' \) in (39), i.e., the Lie algebra generated by

\[ S'' := \{i\sigma_a \otimes \sigma_x + i\sigma_b \otimes \sigma_y + \omega_A 1 \otimes \sigma_y, 1 \otimes \sigma_z\}. \]  

(40)

Then we examine what happens when we include the first element of \( S' \) and see that, under the given assumptions, we generate the whole Lie algebra \( \text{su}(4) \).

In the given coordinates, we have

\[ S' := \{i(x\sigma_x + y\sigma_y + z\sigma_z) \otimes \sigma_z, 1 \otimes \sigma_z, i(\alpha\sigma_x + \gamma\sigma_y) \otimes \sigma_x + i\beta\sigma_y \otimes \sigma_y + \omega_A 1 \otimes \sigma_y\}, \]  

(41)

and \( S'' \) is given by the last two terms listed in (41).

Define

\[ k := \alpha^2 + 4\omega_A^2, \]  

(42)

which is always non-vanishing since \( \alpha \neq 0 \) from (34).

Define the following matrices:

\[ \Gamma_x^\pm := \left(i\sigma_y \otimes \sigma_x \pm \frac{1}{\sqrt{k}}(\alpha i\sigma_x \otimes \sigma_y - \omega_A 1 \otimes \sigma_x)\right), \]  

(43)

\[ \Gamma_y^\pm := -\frac{1}{2} \left(1 \otimes \sigma_z \pm \frac{1}{\sqrt{k}}(\alpha \sigma_z \otimes 1 - 4\omega_A i\sigma_y \otimes \sigma_z)\right), \]  

(44)

\[ \Gamma_z^\pm := \left(i\sigma_y \otimes \sigma_y \pm \frac{1}{\sqrt{k}}(\alpha i\sigma_x \otimes \sigma_x + \omega_A 1 \otimes \sigma_y)\right). \]  

(45)
It is straightforward to verify that
\[ [\Gamma^+_x, \Gamma^+_y] = \Gamma^+_z, \quad [\Gamma^+_y, \Gamma^+_z] = \Gamma^+_x, \quad [\Gamma^+_z, \Gamma^+_x] = \Gamma^+_y, \] (46)
and
\[ [\Gamma^-_x, \Gamma^-_y] = \Gamma^-_z, \quad [\Gamma^-_y, \Gamma^-_z] = \Gamma^-_x, \quad [\Gamma^-_z, \Gamma^-_x] = \Gamma^-_y. \] (47)
Therefore, the Lie algebras \( L^+ := \text{span}\{\Gamma^+_x, \Gamma^+_y, \Gamma^+_z\} \) and \( L^- := \text{span}\{\Gamma^-_x, \Gamma^-_y, \Gamma^-_z\} \), are both isomorphic to \( \mathfrak{su}(2) \) (cf. (20)), and one can verify that
\[ [L^+, L^-] = 0, \] (48)
so that the sum \( L^+ \oplus L^- \) is, in fact, a direct sum of Lie algebras. Consider the generators of \( L'' \). We have
\[ L_1 = 1 \otimes \sigma_z = - (\Gamma^+_y + \Gamma^-_y), \] (49)
and
\[
L_2 = i \alpha \sigma_x \otimes \sigma_x + i \gamma \sigma_y \otimes \sigma_x + i \beta \sigma_y \otimes \sigma_y + \omega A 1 \otimes \sigma_y \\
= \frac{1}{2} \left[ \gamma (\Gamma^+_x + \Gamma^-_x) + \sqrt{k} (\Gamma^-_z - \Gamma^+_z) + \beta (\Gamma^+_z + \Gamma^-_z) \right].
\] (50)
From these expressions it follows that \( L'' \) is always a subalgebra of \( L^+ \oplus L^- \). Using (46), (47), (48), we calculate
\[
[[L_1, L_2], L_2] = \frac{1}{4} \left[ \left( \gamma^2 + (\beta - \sqrt{k})^2 \right) \Gamma^+_y + \left( \gamma^2 + (\beta + \sqrt{k})^2 \right) \Gamma^-_y \right],
\] (51)
and this, along with the fact that \( \beta \) and \( k \) are both different from zero, and the expression of \( L_1 \) in (49), shows that both \( \Gamma^+_y \) and \( \Gamma^-_y \) belong to \( L'' \). Using this fact, it is easy to verify, using (46) and (47), that there are only three possibilities:

1. \( \gamma = 0 \) and \( \beta = \sqrt{k} \) and \( L'' \) is given by
\[
L'' = \text{span}\{\Gamma^+_x, \Gamma^-_y, \Gamma^-_z, \Gamma^+_y\}
\] (52)

2. \( \gamma = 0 \) and \( \beta = - \sqrt{k} \) and \( L'' \) is given by
\[
L'' = \text{span}\{\Gamma^+_x, \Gamma^+_y, \Gamma^+_z, \Gamma^-_y\}
\] (53)

3. At least one between \( \gamma \) and \( |\beta| - \sqrt{k} \) is different from zero and \( L'' \) is equal to \( L^+ \oplus L^- \).
The task is to examine, under the given assumptions, what happens when
we add to these Lie algebras the matrix \( i x \sigma_x \otimes \sigma_z + i y \sigma_y \otimes \sigma_z + i z \sigma_z \otimes \sigma_z := \) \( i \sigma_c \otimes \sigma_z \). Let us assume without loss of generality that \( \{ \sigma_c, \sigma_c \} = -\frac{1}{2} \mathbf{1} \) (cf. (21)). The case (52) is considered in the Appendix. The case (53) is similar.

In the case where \( L' = L^+ \oplus L^- \), we calculate, defining \( T := i x \sigma_x \otimes \sigma_z + i y \sigma_y \otimes \sigma_z + i z \sigma_z \otimes \sigma_z \),

\[
\left[ (\Gamma_z^+ + \Gamma_z^-), T \right] = \frac{y}{4} \mathbf{1} \otimes \sigma_x.
\] (54)

If \( y \neq 0 \), the result follows as before by comparison with the case of \( B \) 3-dimensional. If \( y = 0 \) at least one between \( x \) and \( \omega_A \) is different from zero.

We calculate

\[
\frac{1}{\sqrt{k}} \left[ 8 \omega_A [\Gamma_z^+, T] + \alpha x (\Gamma_z^+ - \Gamma_z^-), T \right] = \left( \frac{x^2}{4} + \frac{\omega_A^2}{k} z^2 \right) \mathbf{1} \otimes \sigma_y,
\] (55)

and the result follows as before by comparison with the case of \( B \) 3-dimensional. \( \square \)

5 INDIRECT CONTROLLABILITY FOR THE TWO QUBIT SYSTEM

The following theorem says that strong unitary indirect controllability is equivalent to complete controllability for the system of two qubits. Nonetheless, we shall see that weaker notions of indirect controllability are still possible without complete controllability.

**Theorem 4** A system of two qubits with \( B = \mathbf{1} \otimes \mathfrak{su}(2) \) is strong UIC if and only if it is CC.

**Proof.** We have already noticed in Section 3 how complete controllability implies strong UIC. Assume now that complete controllability is not verified. The dynamical Lie algebra is (modulo similarity transformations acting separately on the two qubits) one of the Lie algebras (30), (31) or (32). Therefore, it is sufficient to consider these cases. In the case (31), consider for \( S \) an initial density matrix of the form \( \rho_S = \mathbf{1} + \kappa i \sigma_z \). Every element in \( L \) has the form

\[
A = \mathbf{1} \otimes \sigma_1 + a \sigma_2 \otimes \mathbf{1} + i \sigma_3 \otimes \sigma_2,
\] (56)
with $a$ a real number, and $\sigma_1$ and $\sigma_2$ elements of $\mathfrak{su}(2)$. Since $e^L$ is a compact Lie group, every element $X$ can be written as the exponential of a matrix $A \in L$. Therefore we have

$$X \rho_S \otimes \rho_A X^\dagger = e^A \rho_S \otimes \rho_A e^{-A} = \sum_{j=0}^{\infty} \frac{1}{j!} \text{ad}^j_A \rho_S \otimes \rho_A.$$  \hspace{1cm} (57)

Now, by induction on $j$, given the form of $\rho_S$ as a diagonal matrix, it is easily seen that $\text{ad}^j_A \rho_S \otimes \rho_A$ is always the sum of a finite number of tensor products where the first factor is a diagonal element. Taking the partial trace with respect to $A$ we still obtain a diagonal matrix. Therefore, from diagonal density matrices we can never reach matrices which are not diagonal.

In the case of $L$ as in (32), the proof is the same. In fact, the Lie algebra (32) becomes a subalgebra of (31) after a change of coordinates on $S$ which diagonalizes $\sigma$.

In the case of $L$ as in (30), consider $\rho_S = \frac{1}{2}(1 + ki\sigma_z)$ for some real $k$. With this choice, $i\rho_S \otimes \rho_A$ belongs to $\tilde{L} = (\text{span}(i1 \otimes 1)) \oplus L$ which is invariant under $L$. The result follows by applying Theorem 1, as $\text{Tr}_A(\tilde{L}) \neq u(n_S)$ (see the discussion after the theorem). \hfill $\square$

The proof of the Theorem in the cases (31), (32) works to show that UIC given a set $\Lambda_A$ is not verified, even even if we fix the set $\Lambda_A$ a priori. It is interesting to investigate if this is the case also for (30). In the next proposition we will show that if $\Lambda_A$ is chosen as the set of all pure states, then the UIC property given $\Lambda_A$ is verified in the case (30). We first present an example in order to physically motivate the study of the Lie algebra (30). This example also illustrates the use of Lie group decomposition techniques in a direct analysis of the reachable set (14).

**Example 5.1** Consider the system of two qubits $S$ and $A$ interacting via an Ising interaction and with full control on the system $A$. It is described by

$$iH_S := \sigma_z \otimes 1,$$  \hspace{1cm} (58)

$$iH_C := u_x(t)1 \otimes \sigma_x + u_y(t)1 \otimes \sigma_y,$$  \hspace{1cm} (59)

$$H_I := \sigma_y \otimes \sigma_y.$$  \hspace{1cm} (60)

\hspace{1cm}

\textsuperscript{12}Notice that this argument also holds if we consider specific sets $\Lambda_A$ for $\rho_A$. Therefore UIC given $\Lambda_A$ is never verified, no matter what $\Lambda_A$ is.

\textsuperscript{13}In the case (30), $L$ is conjugated to the symplectic Lie algebra $\mathfrak{sp}(2)$ and therefore any transfer is possible between pure states. If we want to find an example to show that strong UIC is not verified we have to use $\rho_S \otimes \rho_A$ non-pure.
An application of Theorem 2 (or a direct calculation) shows that the dynamical Lie algebra $L$ is given by (30). To parametrize the elements of the corresponding Lie group $e^L$, we use the Cartan decomposition \[17\] of $L$

\[L = K \oplus P,\] (61)

with

\[K := \text{span}\{\sigma_x \otimes 1, 1 \otimes \sigma_z, i\sigma_y \otimes \sigma_z\},\]

\[P := \text{span}\{1 \otimes \sigma_x, 1 \otimes \sigma_y, i\sigma_x \otimes \sigma_y, i\sigma_y \otimes \sigma_x\},\] (62)

satisfying

\[[K, K] \subseteq K, \quad [P, P] \subseteq K, \quad [P, K] \subseteq P.\] (63)

Every element $X$ in $e^L$ can be written as $X = K_1 \tilde{A} K_2$ where $K_1$ and $K_2$ are in the Lie group $e^K$ and $\tilde{A}$ is in the Lie group $e^\tilde{A}$ where $\tilde{A}$ is a maximal Abelian subalgebra of $L$ included in $P$. Since $K$ is isomorphic to $u(2)$, we can use an Euler decomposition for $K_1$ and $K_2$ and write

\[K_1 = e^{t_1 \sigma_x \otimes 1} e^{t_2 \sigma_z \otimes \sigma_z} e^{t_3 \sigma_y \otimes \sigma_z} e^{t_4 \sigma_z \otimes 1},\] (64)

\[K_2 = e^{s_1 \sigma_x \otimes 1} e^{i s_2 \sigma_z \otimes \sigma_x} e^{i s_3 \sigma_x \otimes 1} e^{i s_4 \sigma_z \otimes 1},\] (65)

for real coefficients $t_1, \ldots, t_4, s_1, \ldots, s_4$. For the Abelian subalgebra $\tilde{A}$, we can choose

\[\tilde{A} := \text{span}\{1 \otimes \sigma_x, i\sigma_y \otimes \sigma_x\},\] (66)

so that

\[\tilde{A} = e^{a_1 \sigma_x \otimes 1} e^{i a_2 \sigma_z \otimes \sigma_x},\] (67)

for real coefficients $a_1$ and $a_2$. Notice that we have used the freedom in ordering the factors in a different way in (64) and (65), so as to minimize the number of parameters which are relevant for the indirect controllability problem. The first factor in (64) can be pulled out of the partial trace (cf. (14) and (3)) while the second factor is a unitary local operation on $A$ which does not affect the partial trace and can be neglected. The last two terms in (65) can be included in the initial state $\rho_S \otimes \rho_A$. In conclusion, we have to consider a transformation $\rho_S \otimes \rho_A \rightarrow Y \rho_S \otimes \rho_A Y^\dagger$ with the matrix $Y$ of the form

\[Y = e^{t_1 \sigma_x \otimes 1} e^{t_2 \sigma_z \otimes \sigma_z} e^{t_3 \sigma_y \otimes \sigma_z} e^{t_4 \sigma_z \otimes 1} e^{s_1 \sigma_x \otimes \sigma_x} e^{i s_2 \sigma_x \otimes \sigma_x} e^{i s_3 \sigma_z \otimes 1} e^{i s_4 \sigma_z \otimes \sigma_z}.\] (68)

\footnote{When analyzing a quantum mechanical system whose dynamical Lie algebra $L$ is not the full $su(n)$ one approach is to first decompose it according to the Levi decomposition \[12\] into simple and Abelian Lie subalgebras \[11\]. In our case $L$ is isomorphic to $sp(2)$ and therefore it is simple. Levi decomposition only gives a simple component and no Abelian subalgebras.}
The reachable set starting from \( \rho_S \otimes \rho_A \) will be

\[
\Lambda_S(\rho_S) = \left\{ e^{t_1 \sigma_z} (\text{Tr}_A(\Omega)) e^{t_1 \sigma_z} \mid t_1, s_3, s_4 \in \mathbb{R} \right\}, \tag{69}
\]

\[
\Omega := \left( Ye^{s_3 \sigma_3} \rho_S e^{-s_3 \sigma_3} \otimes e^{s_4 \sigma_4} \rho_A e^{-s_4 \sigma_4} Y^\dagger \right),
\]

with \( Y \) varying as in (68). Each term of \( Y \) in (68) can be expressed in terms of a basis of \( iu(4) \). We have for example

\[
e^{it_3 \sigma_x} \otimes \sigma_z = \cos \alpha_1 \mathbf{1} \otimes \mathbf{1} + i \sin \alpha_1 \tilde{\sigma}_x \otimes \tilde{\sigma}_z, \quad \alpha_1 := -\frac{t_3}{4}, \tag{70}
\]

and the other expressions are collected in the Appendix, see (80). Combining all the expressions for the factors in (68), we obtain for \( Y \) a formula of the type

\[
Y := C_0 \otimes \mathbf{1} + C_x \otimes \tilde{\sigma}_x + C_y \otimes \tilde{\sigma}_y + C_z \otimes \tilde{\sigma}_z, \tag{71}
\]

for matrices \( C_0, C_x, C_y, \) and \( C_z \), functions of 6 real arbitrary parameters \( \alpha_1, \ldots, \alpha_6 \). We have calculated the expressions of these matrices and the results are reported in the Appendix.

Formulas such as (69), (71) allow in principle to calculate an expression of the reachable set for every initial state but in practice are typically very complicated and are useful only with the help of numerical simulations. We have plotted the reachable set starting from a state

\[
\rho_S \otimes \rho_A = \left( \frac{1}{2} (1 + s_x \tilde{\sigma}_x + s_z \tilde{\sigma}_z) \right) \otimes \left( \frac{1}{2} (1 + a_z \tilde{\sigma}_z) \right) \tag{72}
\]

in several situations, see Fig. 1-4. This illustrates the variety of cases that can arise depending on the underlying algebraic conditions.

**Proposition 5.2** The system of two qubits with Lie algebra \( L \) in (30) is pure UIC.

**Proof.** Inspection of the basis of \( L \) in (30) tells us that the Lie group \( e^L \) contains elements of the form

\[
Z \otimes \mathbf{1}, \quad \mathbf{1} \otimes P, \quad e^{i \sigma_x \otimes \sigma_z t}, \tag{73}
\]

where \( Z \) is an arbitrary element of the form \( e^{\sigma_z \alpha} \) with \( \alpha \) real, \( P \) is any arbitrary element in \( SU(2) \) and \( t \) an arbitrary real number. We start with an initial state for \( \rho_S \otimes \rho_A \) with \( \rho_A \) pure. By applying a transformation \( \mathbf{1} \otimes P \), we can always assume that

\[
\rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} := E_1 \tag{74}
\]
Consider now a general matrix $\rho'_S \otimes E_1$ and calculate
\[
e^{it\sigma_x \otimes \sigma_z} \rho'_S \otimes E_1 e^{-it\sigma_x \otimes \sigma_z} = \left( e^{-\frac{it}{2} \sigma_x} \rho'_S e^{\frac{it}{2} \sigma_x} \right) \otimes E_1 = X \rho'_S X^\dagger \otimes E_1,
\] (75)
where $X$ denotes a generic transformation of the form $e^{\sigma_x \alpha}$ for $\alpha$ real. Using (75) with $\rho'_S = Z_1 \rho_S Z_1^\dagger$, we have
\[
T \rho_S \otimes E_1 T^\dagger = \left( Z_2 X Z_1 \rho_S Z_1^\dagger X^\dagger Z_2^\dagger \right) \otimes E_1,
\]
and taking the partial trace, we obtain $Z_2 X Z_1 \rho_S Z_1^\dagger X^\dagger Z_2^\dagger$. The claim follows because, from Euler decomposition of $SU(2)$, every element of $SU(2)$ can be written in the form $Z_2 X Z_1$.

We remark that, given the structure of the Lie algebra $\mathcal{L}$, we could have chosen a specific (arbitrary) pure state, and the proof of the previous proposition would have gone through.
5.1 Full Indirect Controllability

We conclude our investigation of the indirect controllability of a system of two qubits studying Full Indirect Controllability. In general, complete controllability does not imply strong FIC since the state $\rho_S \otimes \rho_A = \frac{1}{4}(1 \otimes 1)$ is a fixed state for the total unitary dynamics. If we consider the set $\Lambda_A$ to be the set of pure states for $A$, we have however the following result.

**Theorem 5** Assume the system $S + A$ is CC. Then $S$ is pure FIC.

**Proof.** We write the initial state of $S$ as $\rho_S := \sum_{j=1,2} r_j |j\rangle \langle j|$, with $\{|j\rangle\}$ an orthonormal basis in $S$, $\sum_{j=1,2} r_j = 1$, and the initial state of $A$ as $\rho_A := |\psi\rangle \langle \psi|$. Therefore,

$$\rho_S \otimes \rho_A = \sum_j r_j |\psi_j\rangle \langle \psi_j|,$$

where $|\psi_j\rangle = |j\rangle \otimes |\psi\rangle$, and $\langle \psi_j|\psi_k\rangle = \delta_{jk}$. An arbitrary unitary transformation of the composite system, denoted by $U$, will act as

$$U \rho_S \otimes \rho_A U^\dagger = \sum_{j=1,2} r_j |\phi_j\rangle \langle \phi_j|,$$

where $|\phi_j\rangle = |\psi_j\rangle \otimes |\phi\rangle$ and $\langle \phi_j|\phi_k\rangle = \delta_{jk}$.
with $\langle \phi_j | \phi_k \rangle = \delta_{jk}$, and $|\phi_j\rangle$ arbitrary orthonormal vectors in $S \otimes A$. Now, consider the following particular choices of $U$:

1. $U$ is a unitary operator such that every vector $|\phi_j\rangle$ is maximally entangled. This is always possible, because there are 4 maximally orthonormal entangled vectors in $S \otimes A$ (cf. [20]), and we need only 2 of them. In this case,

$$\rho'_S = \sum_{j=1,2} r_j \text{Tr}_A (|\phi_j\rangle \langle \phi_j|) = \frac{1}{2} 1,$$

that is, the initial $\rho_S$ is mapped to the maximally mixed state;

2. $U$ is the SWAP operator, that is $U \rho_S \otimes \rho_A U = \rho_A \otimes \rho_S$. In this case, the initial $\rho_S$ is mapped to the pure state $|\psi\rangle \langle \psi|$. Since the unitary group is path-connected, there is certainly a continuous path in the space of states of the system $S$, connecting the maximally mixed state to $|\psi\rangle \langle \psi|$, and representing reachable states. By further using local unitary operations acting on $S$, every final state can be obtained. \qed

Figure 3: Simulation of the reachable set as described in the main text, with $s_z = 1/2$, $s_x = 0$ and $a_z = 1$. The simulation consists of 729 points randomly distributed; the big spot marks the initial state.
Figure 4: Simulation of the reachable set as described in the main text, with $s_z = 1/2$, $s_x = 0$ and $a_z = 0$. The simulation consists of 729 points randomly distributed; the big spot marks the initial state. The situation is similar to the one in Figure 2 since now the initial state is, modulo the identity, in $i \mathcal{C}$ which is also invariant under $\mathcal{L}$. This explains why all reachable points are in the $x-y$ plane.

6 CONCLUSIONS

In this paper, we have formally introduced the problem of indirect controllability for quantum systems and we have given tools and tests to describe the reachable set. In particular, we have introduced physically motivated notions and proved the general Theorem 1 to conclude lack of indirect controllability. This theorem is a test at the Lie algebra level and therefore its application implies only operations from linear algebra. The case where both the target system and the accessor system are two level systems is the simplest non trivial case which is also of great importance in applications. To treat this system we have made full use of techniques in the Lie algebraic approach to quantum control. Our first step has been to describe the possible dynamical Lie algebras that can arise (Theorems 2 and 3). These results are of interest on their own, and generalize previous results in the literature. With this information at hand, we have proven in Theorem 4 that we need complete controllability of the whole system in order to have unitary indirect controllability given any state of the accessor system. However, if we are allowed to consider only pure states for the probe, we can have unitary indirect controllability without complete controllability (Proposition 5.2).
When the state of the accessor system is pure and we have complete controllability, we can have full controllability on the target system, i.e., move the state between arbitrary density matrices (Theorem 5).

This paper is a first step in the study of indirect controllability and several questions are of interest for future research. Among these, the extension of our results to systems with general dimensions and an analysis of how indirect controllability of the system $S$ through $A$ depends on the relative dimensions of $S$ and $A$. In this work we have given tools but also indicated the difficulties in pursuing this study including the fact that much of the criteria and results have to be necessarily dependent on the coordinates. We believe that a closer connection with the theory of entanglement in quantum information would be fruitful in further developing this important topic.

7 Expression of $C_0$, $C_x$, $C_y$ and $C_z$ in example 5.1

Recall that $\alpha_1$ is defined in (70). We have

$$e^{\sigma_x \otimes 1} = c_2 \mathbf{1} \otimes 1 + is_2 \tilde{\sigma}_z \otimes 1, \quad \alpha_2 := \frac{t_1}{2},$$

$$e^{1 \otimes \sigma_x \alpha_1} = c_3 \mathbf{1} \otimes 1 + is_3 \tilde{\sigma}_x \otimes \tilde{\sigma}_x, \quad \alpha_3 := \frac{a_1}{2},$$

$$e^{i\sigma_x \otimes \sigma_x \alpha_2} = c_4 \mathbf{1} \otimes 1 + is_4 \tilde{\sigma}_x \otimes \tilde{\sigma}_x, \quad \alpha_4 := \frac{-a_2}{4},$$

$$e^{\sigma_x \otimes 1 \sigma_1} = c_5 \mathbf{1} \otimes 1 + is_5 \tilde{\sigma}_z \otimes \tilde{\sigma}_z, \quad \alpha_5 := \frac{s_1}{2},$$

$$e^{i\sigma_x \otimes \sigma_z \sigma_2} = c_6 \mathbf{1} \otimes 1 + is_6 \tilde{\sigma}_x \otimes \tilde{\sigma}_z, \quad \alpha_6 := \frac{-s_2}{4},$$

having defined $c_j := \cos(\alpha_j)$, $s_j := \sin(\alpha_j)$ for $j = 3, 4$. We have for the matrices $C_0$, $C_x$, $C_y$ and $C_z$ in example 5.1,

$$C_0 = c_3 c_4 c_2 + 5 c_1 + 6 \mathbf{1} - s_3 s_4 c_2 - 5 c_1 + 6 \tilde{\sigma}_x - s_3 s_4 s_2 - 5 c_1 - 6 \tilde{\sigma}_y + ic_3 s_4 c_2 + 5 c_1 - 6 \tilde{\sigma}_x,$$

$$C_x = is_3 c_4 c_2 + 5 c_1 - 6 \mathbf{1} + ic_3 s_4 c_2 - 5 c_1 - 6 \tilde{\sigma}_x + ic_3 s_4 s_2 - 5 c_1 + 6 \tilde{\sigma}_y - s_3 s_4 s_2 + 5 \tilde{\sigma}_z,$$

$$C_y = ic_3 s_4 c_2 - 5 c_1 - 6 \mathbf{1} + ic_3 s_4 c_2 + 5 c_1 - 6 \tilde{\sigma}_x - is_3 c_4 s_2 + 5 s_1 + 6 \tilde{\sigma}_y + c_3 s_4 s_2 - 5 s_1 + 6 \tilde{\sigma}_z,$$

$$C_z = -is_3 s_4 c_2 - 5 s_1 + 6 \mathbf{1} + ic_3 c_4 c_2 + 5 s_1 + 6 \tilde{\sigma}_x - ic_3 s_4 s_2 + 5 s_1 - 6 \tilde{\sigma}_y - s_3 s_4 s_2 - 5 s_1 - 6 \tilde{\sigma}_z,$$

where we have defined $c_{i \pm j} := \cos(\alpha_i \pm \alpha_j)$, and $s_{i \pm j} := \sin(\alpha_i \pm \alpha_j)$. 

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8 Proof of controllability in the case (52)

Using (42), we define \( \cos(\theta) := \frac{c}{\sqrt{k}} \) and \( \sin(\theta) := \frac{s}{\sqrt{k}} \). For brevity we denote by \( c \) and \( s \), \( \cos(\theta) \) and \( \sin(\theta) \) respectively. The goal is to show that the Lie algebra generated by the matrices \( \{ \Gamma^-_{x,y,z}, \Gamma^+_y \} \) and the matrix \( P := i\sigma_c \otimes \sigma_z \) is the full Lie algebra \( su(4) \). This is the Lie algebra generated by the matrices

\[
\Gamma^-_x := i\sigma_y \otimes \sigma_x - c\sigma_x \otimes \sigma_y + \frac{s}{2} 1 \otimes \sigma_x, \quad (82)
\]

\[
\Gamma^-_z := i\sigma_y \otimes \sigma_y + c\sigma_x \otimes \sigma_x + \frac{s}{2} 1 \otimes \sigma_y, \quad (83)
\]

\[
Z := 1 \otimes \sigma_z, \quad (84)
\]

\[
A := c\sigma_z \otimes 1 - 2si\sigma_y \otimes \sigma_z, \quad (85)
\]

\[
P := i\sigma_c \otimes \sigma_z, \quad (86)
\]

with the first four matrices forming a subalgebra. Applying the standard algorithm to calculate the Lie algebra generated by a set of matrices (see, e.g., [10]) we first calculate all the Lie brackets of depth 1. We however do not report Lie brackets which give zero and Lie brackets among the first terms \((82)-(85)\) since they do not give any new directions. Moreover we also skip the Lie bracket with \( \Gamma^+_x \). Since \( \Gamma^-_x \) is the Lie bracket of a linear combination of \( A \) and \( Z \), say \( L \), from the Jacobi identity, for any matrix \( M \) we have \( [M, \Gamma^-_x] = -[L, [\Gamma^-_x, M]] - [\Gamma^-_x, [M, L]] \). So it is enough to consider Lie brackets with \( Z \), \( A \) and \( \Gamma^-_z \) only. We calculate

\[
Q_1 := [P, A] = ic(y\sigma_x \otimes \sigma_z - x\sigma_y \otimes \sigma_z) + \frac{s}{2}(z\sigma_x \otimes 1 - x\sigma_z \otimes 1), \quad (87)
\]

\[
Q_2 := 4[P, \Gamma^-_z] = -y1 \otimes \sigma_x + cx1 \otimes \sigma_y - 2is\sigma_c \otimes \sigma_x. \quad (88)
\]

At step 2 we take the Lie brackets obtained at step 1 \((Q_1 \text{ and } Q_2)\) with the generating matrices \((83)-(86)\). We scale and eliminate directions that are already achieved. From \([Q_1, P]\), since \( z \neq 0 \) from C1’, we obtain

\[
R_1 = c\sigma_c \otimes 1. \quad (89)
\]

From \([Q_1, A]\), we obtain

\[
R_2 := ic^2z\sigma_z \otimes \sigma_z + is^2y\sigma_y \otimes \sigma_z - \frac{sc}{2}(y\sigma_z \otimes 1 + z\sigma_y \otimes 1). \quad (90)
\]

From \([Q_1, \Gamma^-_z]\), we obtain

\[
R_3 := \frac{c^2y}{4} 1 \otimes \sigma_y - \frac{is}{2}y\sigma_x \otimes \sigma_x + \frac{cx}{4} 1 \otimes \sigma_x +
+ \frac{is}{2}(z\sigma_z \otimes \sigma_y + x\sigma_x \otimes \sigma_y). \quad (91)
\]
From \([Q_2, P]\) we obtain
\[
R_4 := iy\sigma_c \otimes \sigma_y + icx\sigma_c \otimes \sigma_x + \frac{s}{2} 1 \otimes \sigma_y. \tag{92}
\]

From \([Q_2, Z]\), we obtain
\[
R_5 := y1 \otimes \sigma_y + cx1 \otimes \sigma_x + 2is\sigma_c \otimes \sigma_y. \tag{93}
\]

From \([Q_2, \Gamma_z^-]\), we obtain
\[
R_6 := -ic^2x\sigma_x \otimes \sigma_z - iy\sigma_y \otimes \sigma_z + sc^2 (y\sigma_z \otimes 1 - z\sigma_y \otimes 1). \tag{94}
\]

Now calculate
\[
S_1 := [R_4, Z] = iy\sigma_c \otimes \sigma_x - icx\sigma_c \otimes \sigma_y + \frac{s}{2} 1 \otimes \sigma_x. \tag{95}
\]

We have
\[
2scxyS_1 - 2sy^2R_4 + (c^2x^2y + y^3)R_5 = y^2(y^2 + c^2x^2 - s^2)1 \otimes \sigma_y + cxy(c^2x^2 + y^2 + s^2)1 \otimes \sigma_x. \tag{96}
\]

We notice that \(c \neq 0\), otherwise C1’ is violated. Moreover, if \(x = y = 0\), we have \(z = 1\), and \(s \neq 0\) to fulfill C2’; in this case it is sufficient to consider \([P, Q_2] = -\frac{t}{2}1 \otimes \sigma_y\), which is the extra desired local term. It remains to consider the cases when \(x = 0\) and \(y \neq 0\), or viceversa. If \(x = 0\), \([Q_2, R_4]\) gives the needed extra local term unless \(y = \pm s\). In this case, the local term \(1 \otimes \sigma_z\) is obtained from \([Q_2, R_4]\) by discarding the component along \(P\). Finally, when \(y = 0\) and \(x \neq 0\), it is always possible to obtain the local term \(1 \otimes \sigma_y\) by linearly combining \(Q_2\) and \(R_4\), since \(c, s\) and \(x\) are all real numbers.

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