AN ACTIVE SET SOLVER FOR CONSTRAINED $H_\infty$ OPTIMAL CONTROL PROBLEMS WITH STATE AND INPUT CONSTRAINTS

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Abstract. This paper proposes an active set solver for $H_\infty$ min-max optimal control problems involving linear discrete-time systems with linearly constrained states, controls and additive disturbances. The proposed solver combines Riccati recursion with dynamic programming. To deal with possible degeneracy (i.e. violations of the linear independence constraint qualification), constraint transformations are introduced that allow the surplus equality constraints on the state at each stage to be moved to the previous stage together with their Lagrange multipliers. In this way, degeneracy for a feasible active set can be determined by checking whether there exists an equality constraint on the initial state over the prediction horizon. For situations when the active set is degenerate and all active constraints indexed by it are non-redundant, a vertex exploration strategy is developed to seek a non-degenerate active set. If the sampled state resides in a robust control invariant set and certain second-order sufficient conditions are satisfied at each stage, then a bounded $l_2$ gain from the disturbance to controlled output can be guaranteed for the closed-loop system under some standard assumptions. Theoretical analysis and numerical simulations show that the computational complexity per iteration of the proposed solver depends linearly on the prediction horizon.

1. Introduction. Although a rich theory has been developed for the robust control of unconstrained linear systems, few results are available for linear systems with constraints. In [5], an $H_\infty$ min-max optimal control problem was investigated for linear discrete-time systems with linearly constrained control inputs and additive

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disturbances. An efficient optimization method was proposed to compute a model predictive control (MPC) law for the problem, which guarantees stability and a bounded $l_2$ gain from the disturbance to the output. This method, originated in [7], formulates the optimization problem in the framework of multi-parametric quadratic programming (mp-QP), and finds the optimal active set for the current measured state by a line search over the state space. Riccati recursion is integrated into the multi-stage optimization procedure. As a result, the computational complexity at each iteration depends linearly on the length of the prediction horizon. Moreover, a warm-start strategy can be easily integrated into this method to reduce the number of iterations. Thus, the computational expense of this method is comparable to that of the fast MPC law in [18], which exploits the special structure of the weight matrices, and much less than that of the scenario search method (e.g. [16]), where the decision variables grow exponentially with the length of the prediction horizon. Explicit MPC (e.g., [1, 2, 17, 13]) is applicable only to small problems and short horizons as it requires exploring on-line a large number of polyhedral regions to find the one that contains the current state. There are also a variety of methods (e.g., [12, 9, 15]) that generate computationally tractable robust control strategies, which are, however, suboptimal.

The active set method in [5] was extended in [4, 6] to problems with linear state constraints. When state constraints are involved, the optimization problem may exhibit degeneracy [2], specifically, violations in the linear independence constraint qualification (LICQ). A characterization of the solution to this constrained $H_\infty$ optimal control problem involving state and control constraints was presented in [13], where mp-QP was incorporated in each stage of the dynamic programming procedure. The main obstacle to this method is that, if state constraints are involved, then the continuous differentiability of the value function is not always invariant in the dynamic programming procedure. It was proved in [2, 13] that this differentiability property could be ensured under some non-degeneracy conditions. In [4], Riccati recursion was integrated into dynamic programming to reduce the computational complexity. A set of equality constraints on the state, called compatibility conditions, was added into the optimization problem, and a scalar free parameter was determined to find a non-degenerate active set in the presence of degeneracy. However, a scalar parameter is often insufficient in reality. In fact when the LICQ is violated, the Lagrange multipliers may depend on multiple free parameters.

In this paper, we revisit the $H_\infty$ MPC problem addressed in [4, 6], and propose a new active set solver by investigating the continuity properties of the polyhedral partitions of the state space and their corresponding solutions. Compared to [4, 6], our contributions are as follows. First, we introduce transformations for the active constraints and their associated Lagrange multipliers. If the LICQ is violated at a stage, then surplus state constraints together with their multipliers can be moved to the outer problem or to the previous stage. If certain second-order optimality conditions are satisfied, then the solution to the optimal control problem at each iteration is in the form of a piecewise-affine function of the initial state for the prediction horizon and the Lagrange multiplier associated with the equality constraint on the initial state. With these transformations, incorporating the compatibility conditions in [4] is unnecessary. We also introduce a new concept, called degeneracy degree, and prove that the optimization problem at each iteration exhibits degeneracy (LICQ violations) if the degeneracy degree is greater than zero. The situation discussed in [4, 6] corresponds to the special case of 1-degree degeneracy.
Second, we prove that, in the presence of degeneracy (LICQ violations), if the active constraints are non-redundant (see Definition 3.4), then the feasible initial states are confined to an affine subspace of reduced dimension in the state space. For each initial state in this subspace, the feasible multiplier resides on the boundary of a convex polyhedron. In the presence of this type of degeneracy, if we select a vertex of this polyhedron and remove all the active constraints whose multipliers are zero at this vertex, then the derived active set is non-degenerate under certain conditions.

Finally, we present a new on-line active set solver for the $H_\infty$ MPC problem, which combines the strategy in [5, 4] with a vertex exploration strategy to deal with degeneracy. In the presence of degeneracy (LICQ violations), the vertex exploration strategy is guaranteed to find an active set that is non-degenerate. In the special case of 1-degree degeneracy, this strategy reduces to the scalar parameter search strategy proposed in [4, 6].

The remainder of this paper is organized as follows. Section 2 presents the problem formulation, and Section 3 presents the solution algorithm, where our emphasis is on the constraint transformations, Riccati recursion and the continuity properties of polyhedral partitions of the state space. After that, we investigate the computational complexity of the proposed solver in Section 4. Then, numerical simulations are presented in Section 5. Finally, we conclude the paper in Section 6.

2. Problem formulation. Let $\mathbb{N} \triangleq \{0, 1, 2, \ldots \}$ denote the set of natural numbers. $\mathbb{N}_{[a, b]} \triangleq \{a, a+1, \ldots, b\}$ and $\mathbb{N}_b \triangleq \mathbb{N}_{[0, b]}$ for $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $a \leq b$. Consider the following linear discrete-time system:

$$x_{k+1} = Ax_k + B_u u_k + B_w w_k, \quad k \in \mathbb{N}, \quad (1)$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ and $w_k \in \mathbb{R}^{n_w}$ are, respectively, the state, control and disturbance; and $A \in \mathbb{R}^{n_x \times n_x}$, $B_u \in \mathbb{R}^{n_x \times n_u}$ and $B_w \in \mathbb{R}^{n_x \times n_w}$ are known matrices. Furthermore, $x_k$, $u_k$ and $w_k$ are subject to the constraints $x_k \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$, $u_k \in \mathcal{U} \subset \mathbb{R}^{n_u}$ and $w_k \in \mathcal{W} \subset \mathbb{R}^{n_w}$, respectively, where $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{W}$ are convex polytopes containing the origin in their interiors. In this paper, a convex polyhedron is a closed set described by a finite set of linear inequalities; a convex polytope is a bounded polyhedron.

Let $0 < N \in \mathbb{N}$ be the length of a receding horizon. For $t \in \mathbb{N}_N$, $x_{k+t}$ represents the state measured at sampling time $k + t$. For simplicity, denote $x_{t|k} \triangleq x_{k+t}$, $u_{t|k} \triangleq u_{k+t}$, $w_{t|k} \triangleq w_{k+t}$ for $t \in \mathbb{N}_{N-1}$, and $x_{N|k} \triangleq x_{k+N}$. Let $Q$, $R$ and $P$ be real matrices satisfying $Q = Q^T \succeq 0$, $R = R^T \succ 0$ and $P = P^T \succeq 0$, i.e. $Q$ and $P$ are symmetric positive semi-definite while $R$ is symmetric positive definite, and let $0 < \gamma \in \mathbb{R}$ be a constant. Let $\mathcal{X}_I \subseteq \mathcal{X}$ denote a terminal constraint set, which is a convex polytope with the origin in its interior. We investigate the following $N$-stage closed-loop min-max optimal control problem as considered in [13, 9, 4]:

$$J^*_t(x_{t|k}) \triangleq \min_{u_{t|k} \in \mathcal{U}_t} J_t(x_{t|k}, u_{t|k}), \quad (2a)$$

subject to

$$Ax_{t|k} + B_u u_{t|k} \in \mathcal{X}_{t+1} \ominus B_w \mathcal{W} \quad (2b)$$
and

\[
J_t(x_{t|k}, u_{t|k}) \triangleq \max_{u_{t|k} \in \mathcal{W}} \left\{ \frac{1}{2} \| x_{t|k} \|^2_Q + \| u_{t|k} \|^2_R - \gamma^2 \| w_{t|k} \|^2 \right\} + J_{t+1}^* (Ax_{t|k} + B^u u_{t|k} + B^w w_{t|k}) \right \}
\]

for \( t \in \mathbb{N}_{N-1} \) with boundary conditions

\[
J_N^* (x_{N|k}) \triangleq \frac{1}{2} \| x_{N|k} \|^2_P
\]

and

\[ \mathcal{X}_t \triangleq \mathcal{X}_f, \]

where \( \mathcal{X}_t \) denotes the set of \( x_{t|k} \) for which (2a)-(2c) is feasible,

\[
\mathcal{X}_t \triangleq \{ x_{t|k} \in \mathcal{X} : \exists u_{t|k} \in \mathcal{U}, \ Ax_{t|k} + B^u u_{t|k} \in \mathcal{X}_{t+1} \ominus B^w \mathcal{W} \}. \]

Here, ‘\( \ominus \)’ denotes the Pontryagin difference [11]; \( \| z \| \triangleq \sqrt{z^T z} \) and \( \| z \|_M \triangleq \sqrt{z^T M z} \) for real vector \( z \) and real matrix \( M \equiv M^T \geq 0 \).

We want to find a state-feedback solution to Problem (2). Specifically, we require a function \( u_{t|k}^* : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u} \) mapping the state \( x_{t|k} \) to the corresponding optimal control \( u_{t|k}^* \) for \( t \in \mathbb{N}_{N-1} \), and apply

\[ u_k \triangleq u_{0|k}(x_k) \]

to system (1) as the MPC law at sampling time step \( k \).

For Problem (2) to be well-posed, we impose the following assumption [4].

**Assumption 1.** The constant \( 0 < \gamma \in \mathbb{R} \) is selected such that the min-max problem (2) is strictly convex in \( u_{t|k} \in \mathcal{U} \) and strictly concave in \( u_{t|k} \in \mathcal{W} \) for \( t \in \mathbb{N}_{N-1} \) and \( k \in \mathbb{N} \).

Denote

\[
y_k \triangleq \left[ \begin{array}{c} Q^\frac{1}{2} \\ 0_{n_u \times n_x} \\ R^\frac{1}{2} \end{array} \right] x_k + \left[ \begin{array}{c} 0_{n_x \times n_u} \\ R^\frac{1}{2} \end{array} \right] u_k
\]

as the controlled output, where \( 0_{n_u \times n_x} \in \mathbb{R}^{n_u \times n_x} \) and \( 0_{n_x \times n_u} \in \mathbb{R}^{n_x \times n_u} \) are both zero matrices. As in [13, 9, 4], the following assumptions are needed.

**Assumption 2.** Let \( (A, B^u) \) and \( (Q^\frac{1}{2}, A) \) be stabilizable and detectable respectively, and suppose that \( (Q^\frac{1}{2}, A, B^u) \) has no zeros on the unit circle. Matrix \( P = P^T \) in (2d) satisfies

\[
\| x_0 \|^2_P = \sum_{k=0}^{\infty} (\| y_k \|^2 - \gamma^2 \| w_k \|^2)
\]

with \( u_k = u_{0|k}(x_k) \) and \( w_k = w_{0|k}(x_k, u_k) \), where \( u_{0|k}(\cdot) \) and \( w_{0|k}(\cdot, \cdot) \) constitute the optimal solution to (2) in the limit \( N \rightarrow \infty \) without considering constraints [10].

**Assumption 3.** The terminal constraint set \( \mathcal{X}_f \subseteq \mathcal{X} \) in (2c) is robust positive invariant for \( x_{k+1} = Ax_k + B^u u_{0|k}(x_k) + B^w w_k \), where \( w_k \in \mathcal{W} \) and \( k \in \mathbb{N} \). Specifically, \( Ax_k + B^u u_{0|k}(x_k) + B^w w \subseteq \mathcal{X}_f \) for every \( x_k \in \mathcal{X}_f \), where \( k \in \mathbb{N} \) [11]. In addition, \( u_{0|k}(\mathcal{X}_f) \subseteq \mathcal{U} \) and \( w_{0|k}(\mathcal{X}_f, u_{0|k}(\mathcal{X}_f)) \subseteq \mathcal{W} \).
Under Assumption 3, for \( t \in \mathbb{N}_{N-1} \), \( \mathcal{X}_t \) defined in (2f) is robust control invariant and nested (i.e. \( \mathcal{X}_t \supseteq \mathcal{X}_{t+1} \) for \( t \in \mathbb{N}_{N-1} \)) [13].

Similar to [5, 4], we use the following equivalent form of system (1):

\[
\begin{align*}
x_{t+1|k} &= \hat{x}_{t+1|k} + B^w w_{t|k}, \quad (4a) \\
\hat{x}_{t+1|k} &= A x_{t|k} + B^u u_{t|k}. \quad (4b)
\end{align*}
\]

Let \( \lambda_{t|k} \in \mathbb{R}^{n_x} \) and \( \hat{\lambda}_{t|k} \in \mathbb{R}^{n_x} \) be, respectively, the Lagrange multipliers associated with (4a) and (4b).

In addition, we define, for \( t \in \mathbb{N}_{N-1} \),

\[
\begin{align*}
\mathcal{U} &\triangleq \{ u_{t|k} \in \mathbb{R}^{n_u} : F u_{t|k} \leq f \}, \quad (5a) \\
\mathcal{W} &\triangleq \{ w_{t|k} \in \mathbb{R}^{n_w} : G w_{t|k} \leq g \}, \quad (5b)
\end{align*}
\]

where \( F \in \mathbb{R}^{n_F \times n_u} \), \( G \in \mathbb{R}^{n_G \times n_w} \), \( f \in \mathbb{R}^{n_F} \) and \( g \in \mathbb{R}^{n_G} \) are known matrices or vectors. Let \( \mu_{t|k} \in \mathbb{R}^{n_F} \) and \( \eta_{t|k} \in \mathbb{R}^{n_G} \) be, respectively, the Lagrange multipliers associated with the constraints \( F u_{t|k} \leq f \) and \( G w_{t|k} \leq g \).

From the definitions of convex sets \( \mathcal{X}_N \), \( \mathcal{X}_t \) for \( t \in \mathbb{N}_{N-1} \), and \( \mathcal{W} \), we can derive

\[
\mathcal{X}_{t+1} \cap B^w \mathcal{W} = \{ \hat{x}_{t+1|k} \in \mathbb{R}^{n_x} : E_t \hat{x}_{t+1|k} \leq e_t \}, \quad (5c)
\]

where \( E_t \in \mathbb{R}^{n_x \times n_x} \) and \( e_t \in \mathbb{R}^{n_x} \). Let \( v_{t|k} \in \mathbb{R}^{n_x} \) be the Lagrange multiplier associated with \( E_t \hat{x}_{t+1|k} \leq e_t \).

3. Solution procedure. For Problem (2), define the following Hamiltonians:

\[
\begin{align*}
\hat{H}_t(\hat{x}_{t+1|k}, w_{t|k}, \eta_{t|k}, \lambda_{t|k}, x_{t+1|k}, \ldots, x_{N|k}) &\triangleq \frac{1}{2} \gamma^2 \| u_{t|k} \|^2 + \eta_{t|k}^T (g - G w_{t|k}) - \lambda_{t|k}^T (x_{t+1|k} - (\hat{x}_{t+1|k} + B^w w_{t|k})) \\
&+ H_{t+1} \left( x_{t+1|k}, u_{t+1|k}, v_{t+1|k}, \lambda_{t+1|k}, \hat{x}_{t+2|k}, \ldots, x_{N|k} \right) \tag{6a}
\end{align*}
\]

and

\[
\begin{align*}
H_t(x_{t|k}, u_{t|k}, \mu_{t|k}, v_{t|k}, \hat{\lambda}_{t|k}, \hat{x}_{t+1|k}, \ldots, x_{N|k}) &\triangleq \frac{1}{2} \left( \| x_{t|k} \|^2_Q + \| u_{t|k} \|^2_R \right) - \mu_{t|k}^T (f - F u_{t|k}) - v_{t|k}^T (e_t - E_t \hat{x}_{t+1|k}) \\
&- \lambda_{t|k}^T (\hat{x}_{t+1|k} - (A x_{t|k} + B^u u_{t|k})) \\
&+ \hat{H}_t(\hat{x}_{t+1|k}, w_{t|k}, \eta_{t|k}, \lambda_{t|k}, x_{t+1|k}, \ldots, x_{N|k}), \tag{6b}
\end{align*}
\]

for \( t \in \mathbb{N}_{N-1} \) with boundary condition

\[
H_N(x_{N|k}) \triangleq J_N^* (x_{N|k}) = \frac{1}{2} \| x_{N|k} \|^2_P. \tag{6c}
\]

Then, from dynamic programming (see [3]), we obtain the following Karush-Kuhn-Tucker (KKT) conditions similar to those in [4].

**Theorem 3.1.** The KKT conditions, i.e. the first-order necessary conditions, for the optimal solution of Problem (2) over an \( N \)-stage horizon with initial state \( x_k \) are:

\[
\begin{align*}
\gamma^2 w_{t|k} + G^T \eta_{t|k} - (B^w)^T \lambda_{t|k} &= 0, \tag{7a} \\
G w_{t|k} \leq g, \quad 0 \leq \eta_{t|k}, \quad \eta_{t|k}^T (g - G w_{t|k}) &= 0, \tag{7b} \\
R u_{t|k} + F T \mu_{t|k} + (B^u)^T \hat{\lambda}_{t|k} &= 0, \tag{7c} \\
E_t \hat{x}_{t+1|k} \leq e_t, \quad 0 \leq v_{t|k}, \quad v_{t|k}^T (e_t - E_t \hat{x}_{t+1|k}) &= 0, \tag{7d}
\end{align*}
\]
be, respectively, the corresponding Lagrange multipliers. Let $\lambda_{t|k}$ be the $t$th constraint in (7) that is active at time $t$. Let $h_{t+1|k} = x_{t+1|k}$ and $v_{t+1|k} = B^w u_{t|k}$, for $t \in \mathbb{N}_{N-1}$ with boundary conditions  
\begin{equation}
 x_{0|k} = x_k
 \end{equation}
and  
\begin{equation}
 \lambda_{N-1|k} = px_N|k,
 \end{equation}
where $0 \triangleq [0, \ldots, 0]^T$.

**Remark 1.** Note that in [4], additional equality constraints on the state $x_{t|k}$ or $\hat{x}_{t|k}$, called compatibility conditions, were introduced into Problem (2). Thus, in contrast with (7), the KKT conditions in [4] involve additional terms relating to the compatibility conditions and their multipliers. We will show later that these constraints for compatibility are not necessary. With these constraints, the Lagrange multipliers in [4] are duplicated and some of them need to be set to zero artificially.

### 3.1. Constraint transformation.

For a given initial state $x_{0|k}$, assume that the active constraints in (7b), (7d) and (7c) corresponding to the optimal solution are represented by  
\begin{equation}
 G\dot{t}|kw_{t|k} = \tilde{g}_t|k,
 \end{equation}
\begin{equation}
 E\dot{t}|k \hat{x}_{t+1|k} = \tilde{v}_t|k
 \end{equation}
and  
\begin{equation}
 F\dot{t}|ku_{t|k} = \tilde{f}_t|k,
 \end{equation}
for $t \in \mathbb{N}_{N-1}$, where $G\dot{t}|k$, $\tilde{g}_t|k$, $E\dot{t}|k$, $\tilde{v}_t|k$, $F\dot{t}|k$ and $\tilde{f}_t|k$ are, respectively, the reduced parts of the matrices or vectors $G$, $g$, $E$, $e$, $F$ and $f$ corresponding to the inequality constraints in (7) that are active at time $k + t$. Let $\tilde{g}_t|k \geq 0$, $\tilde{v}_t|k \geq 0$, $\mu_t|k \geq 0$ be, respectively, the corresponding Lagrange multipliers. Let $A^*(x_{0|k})$ denote the index set of these active constraints. Correspondingly, let  
\begin{equation}
 \tilde{G}\dot{t}|kw_{t|k} < \tilde{g}_t|k, \quad \tilde{E}\dot{t}|k \hat{x}_{t+1|k} < \tilde{v}_t|k, \quad \tilde{F}\dot{t}|ku_{t|k} < \tilde{f}_t|k
 \end{equation}
represent all the other constraints that are inactive.

In the presence of state inequality constraints, Problem (2) may be degenerate [1, 2, 8, 4]. For the KKT conditions (7) with active set $A^*(x_{0|k})$, we transform the active constraints (8) at stage $t \in \mathbb{N}_{N-1}$, and move the surplus constraints from stage $t$ to stage $t - 1$ such that the degeneracy can be determined at stage $t = 0$.

First, inspired by [8], we introduce the following two transformations for the active constraints, where $t \in \mathbb{N}_{N-1}$:

1) Let  
\begin{equation}
 Z_{t|k} \triangleq \begin{bmatrix} \hat{Z}_{11|k} & \hat{Z}_{12|k} \\ \hat{Z}_{21|k} & \hat{Z}_{22|k} \end{bmatrix}
 \end{equation}
be an orthogonal matrix that satisfies  
\begin{equation}
 \begin{bmatrix} (\hat{Z}_{11|k})^T & (\hat{Z}_{12|k})^T \\ (\hat{Z}_{21|k})^T & (\hat{Z}_{22|k})^T \end{bmatrix} \begin{bmatrix} -\tilde{C}_{t|k}B^w \\ \tilde{G}_{t|k} \end{bmatrix} = \begin{bmatrix} D_{t|k}^w \\ 0 \end{bmatrix},
 \end{equation}
where $D_{t|k}^w$ has full row rank and $\tilde{C}_{t|k}$, which satisfies $\tilde{v}_{t|k} + \tilde{C}_{t|k}x_{t+1|k} = 0$ for $t \in \mathbb{N}_{N-2}$, is defined later in the next transformation (refer to (11c)). At stage $t = N - 1$, there is no such constraint. Then, $\tilde{C}_{N-1|k}$ and $\tilde{v}_{N-1|k}$ are set
as empty matrices, and hence the corresponding matrices $\hat{Z}_{t+1}^{11}$ and $\hat{Z}_{t+1}^{12}$ are also empty. Since $\hat{G}_{N-1|k}$ is of full row rank, $\hat{Z}_{t+1}^{22}$ is empty and $\hat{Z}_{t+1}^{23}$ is an identity matrix of consistent dimension. With $\hat{Z}_{t|k}$ constructed, let

\[
\begin{align*}
D_{t|k}^p & \triangleq - (\hat{Z}_{t|k}^{11})^T \hat{C}_{t|k}, & d_{t|k} & \triangleq (\hat{Z}_{t|k}^{11})^T \hat{c}_{t|k} + (\hat{Z}_{t|k}^{21})^T \tilde{g}_{t|k}, \\
C_{t|k} & \triangleq - (\hat{Z}_{t|k}^{12})^T \hat{C}_{t|k}, & c_{t|k} & \triangleq (\hat{Z}_{t|k}^{12})^T \hat{c}_{t|k} + (\hat{Z}_{t|k}^{22})^T \tilde{g}_{t|k}.
\end{align*}
\]

(10c)

We then have

\[
\hat{Z}_{t|k}^T \left[ \begin{array}{c} \hat{c}_{t|k} + \hat{C}_{t|k} x_{t+1|k} \\ \tilde{g}_{t|k} - \hat{G}_{t|k} w_{t|k} \end{array} \right] = \left[ \begin{array}{c} d_{t|k} - D_{t|k}^p \hat{x}_{t+1|k} - D_{t|k}^w w_{t|k} \\ c_{t|k} - C_{t|k} \hat{x}_{t+1|k} \end{array} \right].
\]

(10d)

Constraints $\hat{c}_{t|k} + \hat{C}_{t|k} x_{t+1|k} = 0$ and $\tilde{g}_{t|k} - \hat{G}_{t|k} w_{t|k} = 0$ are, therefore, transformed into

\[
d_{t|k} - D_{t|k}^p \hat{x}_{t+1|k} - D_{t|k}^w w_{t|k} = 0 \quad \text{and} \quad c_{t|k} - C_{t|k} \hat{x}_{t+1|k} = 0.
\]

Let $\hat{\xi}_{t|k}$ and $-\hat{\zeta}_{t|k}$ be, respectively, Lagrange multipliers associated with these two new constraints. We assume that

\[
\left[ \begin{array}{c} \xi_{t|k} \\ \eta_{t|k} \end{array} \right] = \hat{Z}_{t|k} \left[ \begin{array}{c} \hat{\xi}_{t|k} \\ -\hat{\zeta}_{t|k} \end{array} \right],
\]

(10c)

where $\xi_{t|k}$ is the Lagrange multiplier associated with $\hat{c}_{t|k} + \hat{C}_{t|k} x_{t+1|k} = 0$ (refer to (11e)). Note that this transformation possibly generates a state constraint, $c_{t|k} - C_{t|k} \hat{x}_{t+1|k} = 0$ for $t \in \mathbb{N}_{N-2}$, which will be combined with constraints (8b) and (8c) for the minimization problem at the current stage $t$. We will show this operation next. When $t = N - 1$, from the relation previously established, we have

\[
d_{N-1|k} = \tilde{g}_{N-1|k}, \quad D_{N-1|k}^p = 0, \quad D_{N-1|k}^w = \hat{G}_{N-1|k}, \quad \eta_{N-1|k} = \hat{\xi}_{N-1|k}
\]

(10f)

2) Let

\[
\hat{Z}_{t|k} = \left[ \begin{array}{c} Z_{t|k}^{11} \\ Z_{t|k}^{12} \\ Z_{t|k}^{21} \\ Z_{t|k}^{22} \\ Z_{t|k}^{31} \\ Z_{t|k}^{32} \end{array} \right]
\]

(11a)

be another orthogonal matrix which ensures that

\[
\left[ \begin{array}{ccc} (Z_{t|k}^{11})^T & (Z_{t|k}^{12})^T & (Z_{t|k}^{31})^T \\ (Z_{t|k}^{21})^T & (Z_{t|k}^{22})^T & (Z_{t|k}^{32})^T \end{array} \right] \left[ \begin{array}{c} C_{t|k} B^w \\ \hat{E}_{t|k} B^w \\ \hat{F}_{t|k} \end{array} \right] = \left[ \begin{array}{c} \hat{D}_{t|k}^w \\ 0 \end{array} \right],
\]

(11b)

where $\hat{D}_{t|k}^w$ has full row rank and $C_{t|k}$, which satisfies $c_{t|k} - C_{t|k} \hat{x}_{t+1|k} = 0$ for $t \in \mathbb{N}_{N-2}$, is derived from (10c). Since transformation (10) does not produce this constraint at stage $t = N - 1$, $C_{N-1|k}$ and $c_{N-1|k}$ together with its multiplier $\hat{\zeta}_{N-1|k}$ and the transforming matrices $Z_{t|k}^{11}$ and $Z_{t|k}^{12}$ are set as empty matrices. Let

\[
\left\{ \begin{array}{l}
\hat{D}_{t|k}^p \triangleq (Z_{t|k}^{11})^T C_{t|k} A + (Z_{t|k}^{21})^T \hat{E}_{t|k} A, \\
\hat{d}_{t|k} \triangleq (Z_{t|k}^{11})^T \hat{c}_{t|k} + (Z_{t|k}^{21})^T \tilde{f}_{t|k}, \\
\hat{C}_{t-1|k} \triangleq - (Z_{t|k}^{12})^T C_{t|k} A - (Z_{t|k}^{22})^T \hat{E}_{t|k} A, \\
\hat{c}_{t-1|k} \triangleq (Z_{t|k}^{12})^T \hat{c}_{t|k} + (Z_{t|k}^{22})^T \tilde{f}_{t|k} + (Z_{t|k}^{32})^T \tilde{f}_{t|k}.
\end{array} \right.
\]

(11c)
Then,
\[
Z_{t|k}^T \begin{bmatrix}
c_{t|k} - C_{t|k} \hat{x}_{t+1|k} \\
\hat{c}_{t|k} - \hat{E}_{t|k} \hat{x}_{t+1|k} \\
\hat{f}_{t|k} - \hat{F}_{t|k} u_{t|k}
\end{bmatrix} = \begin{bmatrix}
d_{t|k} - \hat{D}_{t|k}^x x_{t|k} - \hat{D}_{t|k}^u u_{t|k} \\
\hat{c}_{t-1|k} + C_{t-1|k} x_{t|k}
\end{bmatrix}.
\] (11d)

Constraints \(c_{t|k} - C_{t|k} \hat{x}_{t+1|k} = 0\), \(\hat{c}_{t|k} - \hat{E}_{t|k} \hat{x}_{t+1|k} = 0\) and \(\hat{f}_{t|k} - \hat{F}_{t|k} u_{t|k} = 0\) are transformed into
\[
d_{t|k} - \hat{D}_{t|k}^x \hat{x}_{t+1|k} - \hat{D}_{t|k}^u u_{t|k} = 0 \quad \text{and} \quad \hat{c}_{t-1|k} + C_{t-1|k} x_{t|k} = 0.
\]

Let \(\hat{\xi}_{t|k}\) and \(-\zeta_{t-1|k}\) be, respectively, the Lagrange multipliers associated with these two transformed constraints. Assume that
\[
\begin{bmatrix}
\hat{\xi}_{t|k} \\
\hat{\eta}_{t|k}
\end{bmatrix} = Z_{t|k} \begin{bmatrix}
\hat{\xi}_{t|k} \\
-\zeta_{t-1|k}
\end{bmatrix}.
\] (11e)

It is possible that this transformation produces a state constraint \(\hat{c}_{t-1|k} + C_{t-1|k} x_{t|k} = 0\), which can be combined with constraint (8a) for the maximization problem at the previous stage \(t - 1\).

Then, with the transformation matrices \(\hat{Z}_{t|k}\) and \(Z_{0|k}\) defined previously, these constraint transformations are carried out from the last stage \(t = N - 1\) to the first stage \(t = 0\). Note that there are no constraints \(\hat{c}_{N-1|k} + C_{N-1|k} x_{N|k} = 0\) and \(c_{N-1|k} - C_{N-1|k} \hat{x}_{N|k} = 0\), and \(\hat{Z}_{N-1|k}\) is an identity matrix. In contrast to [4], \(\hat{c}_{t-1|k} + C_{t-1|k} x_{t|k} = 0\), for \(t \in \mathbb{N}_{N-1}\), and \(c_{t|k} - C_{t|k} \hat{x}_{t+1|k} = 0\), for \(t \in \mathbb{N}_{N-2}\), are not introduced artificially, but are instead derived from (11) and (10) respectively.

By performing the two transformations backwards, if the matrix
\[
\begin{bmatrix}
(C_{0|k} B^u)^T, (\hat{E}_{0|k} B^u)^T, \hat{F}_{0|k}^T
\end{bmatrix}^T
\]
is not of full row rank, then there will be an equality constraint on the initial state, i.e., \(\hat{c}_{-1|k} + C_{-1|k} x_{0|k} = 0\) with its Lagrange multiplier \(-\zeta_{-1|k}\). Let \(\alpha^d_k\) denote the dimension of \(\zeta_{-1|k}\). The variable \(\alpha^d_k\) is important in this paper; it will be explained in a different way in the sequel.

For \(k \in \mathbb{N}\), define the following block upper-triangular matrix:
\[
\Lambda_k \triangleq \begin{bmatrix}
A_{N-1|k} & A_{N-1|k}^b & 0 & \cdots & 0 \\
0 & A_{N-2|k} & A_{N-2|k}^b & 0 & \cdots \\
0 & 0 & A_{N-3|k} & A_{N-3|k}^b & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 & A_{1|k}^a \\
0 & \cdots & \cdots & 0 & 0 & A_{0|k}^a
\end{bmatrix},
\] (12a)

where, for \(t \in \mathbb{N}_{N-1}\),
\[
\Lambda_{t|k}^a \triangleq \begin{bmatrix}
-I_{n_x} & B^u & I_{n_x} & 0 \\
0 & \hat{G}_{t|k} & 0 & 0 \\
0 & 0 & -I_{n_x} & B^u \\
0 & 0 & 0 & \hat{E}_{t|k} B^u \\
0 & 0 & 0 & \hat{F}_{t|k}
\end{bmatrix}
\] and \(\Lambda_{t|k}^b \triangleq \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
A & 0 & 0 & 0 \\
\hat{E}_{t|k} A & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.\] (12b)
In (12), $I_{n_x} \in \mathbb{R}^{n_x \times n_x}$ denotes an identity matrix. Let $n^x_{t,k}$ be the number of rows of $\Lambda^x_{t,k}$. $\Lambda$ is related to transformations (10) and (11) through the following lemma.

**Lemma 3.2.** The Lagrange multiplier $\zeta_{-1|k}$ has the same dimension as that of $(\Lambda^x_{k})_\perp$. Specifically,

$$\sigma^x_k = -\text{rank}(\Lambda_k) + \sum_{t \in \mathbb{N}_{N-1}} n^x_{t,k}. \quad (13)$$

Problem (2) is degenerate (of the first type) [2] if and only if $\sigma^x_k > 0$.

**Proof:** For $k \in \mathbb{N}$, by carrying out the aforementioned transformations with $t$ ranging from $N - 1$ to 0, and performing row permutation to the rows of $\Lambda_k$, we can transform $\Lambda_k$ to another block upper-triangular matrix

$$\tilde{\Lambda}_k \triangleq \begin{bmatrix} \bar{\Lambda}^a_{N-1|k} & \bar{\Lambda}^b_{N-1|k} & 0 & \cdots & 0 \\ \bar{\Lambda}^a_{N-2|k} & \bar{\Lambda}^b_{N-2|k} & 0 & \cdots & 0 \\ 0 & 0 & \bar{\Lambda}^a_{N-3|k} & \bar{\Lambda}^b_{N-3|k} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \bar{\Lambda}^a_{1|k} \bar{\Lambda}^b_{1|k} \\ 0 & \cdots & 0 & 0 & \bar{\Lambda}^a_{0|k} \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}. \quad (14a)$$

For $t \in \mathbb{N}_{N-1}$, $\bar{\Lambda}^a_{t|k}$ and $\bar{\Lambda}^b_{t|k}$ in (14a) are defined, respectively, by

$$\bar{\Lambda}^a_{t|k} \triangleq \begin{bmatrix} -I_{n_x} & B^w & I_{n_x} & 0 \\ 0 & D^w_{t|k} & D^x_{t|k} & 0 \\ 0 & 0 & -I_{n_x} & B^u \\ 0 & 0 & 0 & D^u_{t|k} \end{bmatrix}, \quad \bar{\Lambda}^b_{t|k} \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (14b)$$

The last block row (the $(N + 1)$-th block row) of $\tilde{\Lambda}_k$ has the same number of rows as $\tilde{C}_{-1|k}$. Since both $D^w_{t|k}$ and $D^u_{t|k}$, for $t \in \mathbb{N}_{N-1}$, have full row rank, it follows that the dimension of $(\Lambda^x_{k})_\perp$ equals to $\sigma^x_k$.

Then, let

$$z_k \triangleq \begin{bmatrix} x^T_{N|k} & w^T_{N-1|k} & \hat{x}^T_{N|k} & u^T_{N-1|k} & \cdots & x^T_{1|k} & w^T_{0|k} & \hat{x}^T_{1|k} & u^T_{0|k} \end{bmatrix}^T$$

and

$$\theta_k \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (-A x_{0|k})^T & (-\tilde{E}_{0|k} x_{0|k})^T & 0 \end{bmatrix}^T + \begin{bmatrix} 0 & \bar{g}^T_{N-1|k} & \bar{c}^T_{N-1|k} & \tilde{f}^T_{N-1|k} & \cdots & 0 & \bar{g}^T_{0|k} & \bar{c}^T_{0|k} & \tilde{f}^T_{0|k} \end{bmatrix}^T.$$
In the latter case, the active set is degenerate (of the first type).

**Definition 3.4.** If the LICQ holds, or if the matrix $[\tilde{c}_{-1|k}, \tilde{C}_{-1|k}]$ has full row rank when the LICQ is violated, then we say this set of active constraints contains no redundant constraints.

In [2, 17], there is another type (the second type) of degeneracy, i.e., the case where there are active constraints associated with zero multipliers. This concept is not needed in this paper. Thus, if we say in the following that Problem (2), or just an active set, is degenerate, then we mean that it is degenerate of the first type.

**Remark 2.** Suppose, at one stage, the maximum number of active constraints for the maximization and minimization problems are, respectively, $n_{A_w}$ and $n_{A_u}$. Then, the computation for the constraint transformations per iteration requires $O\left(\frac{1}{2}(n_{A_w}n_w+n_{\tilde{A}_w}n_\tilde{w})N\right)$ operations for QR factorization [14] and $O((n_wn_{A_w}+n_{\tilde{A}_w}n_\tilde{w})n_\tilde{w}+n_{A_w}n_\tilde{w}^2)N)$ for matrix multiplication. However, if we compute rank($\Lambda_k$) through QR factorization directly, the number of operations is increased to $O\left(\frac{1}{2}(2n_\tilde{w}+n_{A_w}+n_{\tilde{A}_w})^2(2n_\tilde{w}+n_w+n_\tilde{w})N^3\right)$.

### 3.2. Riccati recursion.

For a given $x_{0|k}$, with the inequality constraints in (7b), (7d) and (7e) replaced by their active parts (8a), (8b) and (8c) corresponding to $A^*(x_{0|k})$, computing the solution to the KKT system (7) becomes a two-point boundary value problem, which can be solved by the backward sweep method [3].

Let $\{P_{t|k}, q_{t|k}, \hat{P}_{t|k}, \hat{q}_{t|k}\}$, for $t \in \mathbb{N}_{N-1}$, be solutions of the following Riccati equations:

\[
\begin{bmatrix}
\hat{P}_{t|k} & \hat{q}_{t|k}
\end{bmatrix}
= \begin{bmatrix}
P_{t|k} & q_{t|k}
\end{bmatrix}
+ \begin{bmatrix}
P_{t|k}B^w & -(D^w_{t|k})^T
\end{bmatrix}
\begin{bmatrix}
M^w_{t|k} & m^w_{t|k}
M^\xi_{t|k} & m^\xi_{t|k}
\end{bmatrix},
\quad t \in \mathbb{N}_{N-1},
\]

and

\[
\begin{bmatrix}
P_{t|k} & q_{t|k}
\end{bmatrix}
= \begin{bmatrix}
Q + A^T\hat{P}_{t+1|k}A & A^T\hat{q}_{t+1|k}
\end{bmatrix}
+ \begin{bmatrix}
A^T\hat{P}_{t+1|k}B^u & (\hat{D}^u_{t+1|k})^T
\end{bmatrix}
\begin{bmatrix}
L^u_{t+1|k} & l^u_{t+1|k}
L^\xi_{t+1|k} & l^\xi_{t+1|k}
\end{bmatrix},
\quad t \in \mathbb{N}_{N-2},
\]

with boundary condition

\[P_{N-1|k} = P \quad \text{and} \quad q_{N-1|k} = 0,
\]

where, for $t \in \mathbb{N}_{N-1}$,

\[
\begin{bmatrix}
M^w_{t|k} & m^w_{t|k}
M^\xi_{t|k} & m^\xi_{t|k}
\end{bmatrix}
= \begin{bmatrix}
\gamma^2 I_{n_w} - (B^w)^TP_{t|k}B^w & (D^w_{t|k})^T
D^w_{t|k} & 0
\end{bmatrix}^{-1}
\]

and

\[
\begin{bmatrix}
L^u_{t|k} & l^u_{t|k}
L^\xi_{t|k} & l^\xi_{t|k}
\end{bmatrix}
= \begin{bmatrix}
R + (B^u)^T\hat{P}_{t|k}B^u & (\hat{D}^u_{t|k})^T
\hat{D}^u_{t|k} & 0
\end{bmatrix}^{-1}
\]

and

\[
\begin{bmatrix}
\hat{P}_{t|k} & \hat{q}_{t|k}
\end{bmatrix}
= \begin{bmatrix}
R + (B^u)^T\hat{P}_{t|k}B^u & (\hat{D}^u_{t|k})^T
\hat{D}^u_{t|k} & 0
\end{bmatrix}^{-1}
\]

\[
\begin{bmatrix}
\hat{P}_{t|k} & \hat{q}_{t|k}
\end{bmatrix}
= \begin{bmatrix}
R + (B^u)^T\hat{P}_{t|k}B^u & (\hat{D}^u_{t|k})^T
\hat{D}^u_{t|k} & 0
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
R + (B^u)^T\hat{P}_{t|k}B^u & (\hat{D}^u_{t|k})^T
\hat{D}^u_{t|k} & 0
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
R + (B^u)^T\hat{P}_{t|k}B^u & (\hat{D}^u_{t|k})^T
\hat{D}^u_{t|k} & 0
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
R + (B^u)^T\hat{P}_{t|k}B^u & (\hat{D}^u_{t|k})^T
\hat{D}^u_{t|k} & 0
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
R + (B^u)^T\hat{P}_{t|k}B^u & (\hat{D}^u_{t|k})^T
\hat{D}^u_{t|k} & 0
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
R + (B^u)^T\hat{P}_{t|k}B^u & (\hat{D}^u_{t|k})^T
\hat{D}^u_{t|k} & 0
\end{bmatrix}^{-1}
\]
From the solution of (15), construct the following feedback control, for \( t \in \mathbb{N}_{N-1} \),

\[
 u_{t|k} = u^*_{t|k}(x_{t|k}) \triangleq L^u_{t|k}x_{t|k} + I^u_{t|k},
\]  

(16a)

with

\[
 w_{t|k} = w^*_{t|k}(x_{t|k}) \triangleq M^w_{t|k}(Ax_{t|k} + B^u(L^u_{t|k}x_{t|k} + I^u_{t|k})) + m^w_{t|k}.
\]  

(16b)

Let, for \( t \in \mathbb{N}_{N-1} \),

\[
 \eta_{t|k} = \eta^*_{t|k}(x_{t|k}, \xi_{t|k}) \triangleq Z^2_{t|k} \left( M^\xi_{t|k} \left( Ax_{t|k} + B^u(L^u_{t|k}x_{t|k} + I^u_{t|k}) \right) + m^\xi_{t|k} \right) - \hat{Z}^2_{t|k} \xi_{t|k},
\]  

(17a)

\[
 \hat{\eta}_{t|k} = \hat{\eta}^*_{t|k}(x_{t|k}, \xi_{t|k}) \triangleq Z^2_{t|k} \left( L^\xi_{t|k}x_{t|k} + t^\xi_{t|k} \right) - Z^2_{t|k} \xi_{t|k},
\]  

(17b)

\[
 \tilde{\eta}_{t|k} = \tilde{\eta}^*_{t|k}(x_{t|k}, \xi_{t|k}) \triangleq Z^2_{t|k} \left( L^\xi_{t|k}x_{t|k} + t^\xi_{t|k} \right) - Z^2_{t|k} \xi_{t|k},
\]  

(17c)

with \( \xi_{t|k} \) and \( \xi_{t|k} \), for \( t \in \mathbb{N}_{N-2} \), satisfying

\[
 \begin{cases}
 \xi_{t|k} = Z^2_{t|k} \left( M^\xi_{t|k} \left( Ax_{t|k} + B^u(L^u_{t|k}x_{t|k} + I^u_{t|k}) \right) + m^\xi_{t|k} \right) - \hat{Z}^2_{t|k} \xi_{t|k}, \\
 \xi_{t|k} = Z^2_{t|k} \left( L^\xi_{t|k}x_{t|k} + t^\xi_{t|k} \right) - Z^2_{t|k} \xi_{t|k}.
\end{cases}
\]  

(18)

Now, from (15) we can compute \( \{ \tilde{P}_{t|k}, \tilde{Q}_{t|k}, \tilde{P}_{t|k}, \tilde{Q}_{t|k} \} \) backward with \( t \) ranging from \( N-1 \) to \( 0 \) together with coefficient matrices in (16), (17) and (18). Meanwhile, from (1) and (16), we deduce, for a given \( x_{0|k} \) and \( A^*(x_{0|k}) \), that

\[
 x_{t|k} = \hat{\Phi}_{t|k}x_{0|k} + \phi_{t|k}, \quad t \in \mathbb{N}_{N-1},
\]  

(19a)

with \( \hat{\Phi}_{t|k} \) and \( \phi_{t|k} \) constructed recursively by

\[
 \begin{cases}
 \hat{\Phi}_{t+1|k} = (I_{n_x} + B^w M^w_{t|k})(A + B^u L^u_{t|k})\hat{\Phi}_{t|k}, \\
 \phi_{t+1|k} = (I_{n_x} + B^w M^w_{t|k}) \left( A + B^u(L^u_{t|k})\phi_{t|k} + B^w m^w_{t|k} \right) + B^w m^w_{t|k},
\end{cases}
\]  

(19b)

for \( t \in \mathbb{N}_{N-2} \) and initial conditions

\[
 \hat{\Phi}_{0|k} = I_{n_x} \quad \text{and} \quad \phi_{0|k} = 0.
\]  

(19c)

From (19), (18) can be rearranged into:

\[
 \begin{cases}
 \xi_{t|k} = \tilde{\Phi}_{t|k}x_{0|k} + \tilde{\Phi}_{t|k} \xi_{t-1|k} + \tilde{\psi}_{t|k}, \\
 \xi_{t|k} = \tilde{\Phi}_{t|k}x_{0|k} + \tilde{\Phi}_{t|k} \xi_{t-1|k} + \tilde{\psi}_{t|k},
\end{cases}
\]  

(20a)

with

\[
 \begin{cases}
 \tilde{\Phi}_{t|k} = -Z^2_{t|k} \tilde{\Phi}_{t|k} + Z^2_{t|k} M^\xi_{t|k} \left( A + B^u L^u_{t|k} \right) \hat{\Phi}_{t|k}, \\
 \tilde{\psi}_{t|k} = -Z^2_{t|k} \tilde{\psi}_{t|k} + Z^2_{t|k} \tilde{\psi}_{t-1|k} \hat{\Phi}_{t|k},
\end{cases}
\]  

(20b)

\[
 \begin{cases}
 \tilde{\psi}_{t|k} = -Z^2_{t|k} \tilde{\psi}_{t|k}, \\
 \tilde{\psi}_{t|k} = -Z^2_{t|k} \tilde{\psi}_{t-1|k},
\end{cases}
\]  

(20c)

\[
 \begin{cases}
 \tilde{\psi}_{t|k} = -Z^2_{t|k} \tilde{\psi}_{t|k} + Z^2_{t|k} \left( M^\xi_{t|k} \left( A + B^u L^u_{t|k} \right) \phi_{t|k} + B^w m^w_{t|k} \right), \\
 \psi_{t|k} = -Z^2_{t|k} \psi_{t|k} + Z^2_{t|k} \left( L^\xi_{t|k} \phi_{t|k} + t^\xi_{t|k} \right),
\end{cases}
\]  

(20d)

for \( t \in \mathbb{N}_{N-2} \), and initial conditions

\[
 \tilde{\psi}_{-1|k} = 0, \quad \tilde{\psi}_{-1|k} = I_{n_x} \quad \text{and} \quad \tilde{\psi}_{-1|k} = 0.
\]  

(20e)

We now present the following lemma.
Lemma 3.5. Consider an initial state $x_{0|k}$ and its optimal active set $A^*(x_{0|k})$. Let
\[
\begin{align*}
\mathbf{s}^*_t(k) &= \mathbf{u}^*_t(k) + \mathbf{w}^*_t(k) + \mathbf{\tilde{\gamma}}^*_t(k), \\
\mathbf{v}^*_t(k) &= \mathbf{\tilde{\gamma}}^*_t(k) + \mathbf{\tilde{\mu}}^*_t(k),
\end{align*}
\] with $\mathbf{u}^*_t(k)$ and $\mathbf{w}^*_t(k)$ constructed by (16), and $\mathbf{\tilde{\gamma}}^*_t(k)$, $\mathbf{\tilde{\mu}}^*_t(k)$ constructed by (17). Then, $\mathbf{s}^*_t(k)$ is a local solution to Problem (2) if and only if
\[
\begin{align*}
\mathbf{\tilde{g}}_t(k) &\geq \mathbf{G}_{t|k} M_{t|k}^* (A + B^u L_{t|k}^n) \Phi_{t|k} x_{0|k} \\
&\quad + \mathbf{\tilde{G}}_{t|k} \left( M_{t|k}^* \left( (A + B^u L_{t|k}^n) \phi_{t|k} + B^u l_{t|k}^u \right) + m_{t|k}^w \right), \\
\mathbf{\tilde{f}}_t(k) &\geq \mathbf{\tilde{F}}_{t|k} L_{t|k}^n \Phi_{t|k} x_{0|k} + \mathbf{\tilde{F}}_{t|k} (L_{t|k}^n \phi_{t|k} + l_{t|k}^u), \\
\mathbf{\tilde{e}}_t(k) &\geq \mathbf{\tilde{E}}_{t|k} (A + B^u L_{t|k}^n) \Phi_{t|k} x_{0|k} + \mathbf{\tilde{E}}_{t|k} \left( (A + B^u L_{t|k}^n) \phi_{t|k} + B^u l_{t|k}^u \right),
\end{align*}
\]
and
\[
\begin{align*}
0 &\leq \left( Z_{t|k}^{21} M_{t|k}^* (A + B^u L_{t|k}^n) \Phi_{t|k} - Z_{t|k}^{22} \Psi_{t|k} \right) x_{0|k} - Z_{t|k}^{22} \Psi_{t|k} \zeta_{t-1|k} \\
&\quad + Z_{t|k}^{21} M_{t|k}^* \left( (A + B^u L_{t|k}^n) \phi_{t|k} + B^u l_{t|k}^u \right) + m_{t|k}^w - Z_{t|k}^{22} \Psi_{t|k} \zeta_{t-1|k}, \\
0 &\leq \left( Z_{t|k}^{21} L_{t|k}^n \Phi_{t|k} - Z_{t|k}^{22} \Psi_{t|k} \right) x_{0|k} - Z_{t|k}^{22} \Psi_{t|k} \zeta_{t-1|k} \\
&\quad + Z_{t|k}^{21} \left( L_{t|k}^n \phi_{t|k} + l_{t|k}^u \right) - Z_{t|k}^{22} \Psi_{t|k} \zeta_{t-1|k}, \\
0 &\leq \left( Z_{t|k}^{21} L_{t|k}^n \Phi_{t|k} - Z_{t|k}^{22} \Psi_{t|k} \right) x_{0|k} - Z_{t|k}^{22} \Psi_{t|k} \zeta_{t-1|k} \\
&\quad + Z_{t|k}^{21} \left( L_{t|k}^n \phi_{t|k} + l_{t|k}^u \right) - Z_{t|k}^{22} \Psi_{t|k} \zeta_{t-1|k},
\end{align*}
\]
for $t \in \mathbb{N}_{N-1}$, are satisfied, and the second-order sufficient condition
\[
\begin{align*}
0 &\prec (D_{t|k}^u)^T \left( \gamma^2 I_{m} - (B^w)^T P_{N-1|k} B^w \right) (D_{t|k}^u)^T, \\
0 &\prec (\tilde{D}_{t|k}^u)^T \left( R + (B^u)^T \tilde{P}_{t|k} B^u \right) (\tilde{D}_{t|k}^u)^T,
\end{align*}
\]
for $t \in \mathbb{N}_{N-1}$, holds for the active set $A^*(x_{0|k})$, where $(\cdot)_{\perp}$ denotes a matrix whose columns form a basis for the null-space of matrix $(\cdot)$.

Proof: We prove this lemma by induction. At stage $t = N - 1$, assume from (7i) and (15c) that
\[
\lambda_{N-1|k} = P_{N-1|k} x_{N|k} + q_{N-1|k}.
\]
From (7g), equalities (7a) and (8a) ensure that
\[
\begin{align*}
\begin{bmatrix}
\gamma^2 I_{m} - (B^w)^T P_{N-1|k} B^w \\
\tilde{G}_{N-1|k}
\end{bmatrix}
\begin{bmatrix}
w_{N-1|k} \\
\tilde{\eta}_{N-1|k}
\end{bmatrix}
&= \begin{bmatrix}
(B^w)^T P_{N-1|k} \tilde{x}_{N|k} + (B^w)^T q_{N-1|k} \\
\tilde{g}_{N-1|k}
\end{bmatrix},
\end{align*}
\]
which according to (10f) and (15d) has a solution in the form,
\[
\begin{align*}
\begin{bmatrix}
w_{N-1|k} \\
\tilde{\eta}_{N-1|k}
\end{bmatrix}
&= \begin{bmatrix}
M_{N-1|k}^w \\
M_{N-1|k}^\tilde{\eta}
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_{N|k} \\
\tilde{\eta}_{N-1|k}
\end{bmatrix}
+ \begin{bmatrix}
m_{N-1|k}^w \\
m_{N-1|k}^\tilde{\eta}
\end{bmatrix},
\end{align*}
\]
if Assumption 1 and the first condition in (23) are satisfied. Then, from (7f), (24), (7g), (10f) and (15a), we have
\[
\lambda_{N-1|k} = \tilde{P}_{N-1|k} \tilde{x}_{N|k} + \tilde{q}_{N-1|k} + \tilde{E}_{N-1|k} \tilde{\eta}_{N-1|k}.
\]
If (11e) holds, then we have from (11d) that
\[
(\tilde{E}_{N-1|k}B^u)^T \tilde{v}_{N-1|k} + \tilde{F}_{N-1|k} \tilde{p}_{N-1|k} = (\tilde{D}_{N-1|k}^u)^T \xi_{N-1|k}
\] (27a)
and
\[
(\tilde{E}_{N-1|k}A)^T \tilde{v}_{N-1|k} = (\tilde{D}_{N-1|k}^x)^T \xi_{N-1|k} + \tilde{C}_{N-2|k}^T \xi_{N-2|k}.
\] (27b)
Then, using \(\hat{\lambda}_{N-1|k}\) defined in (26), it is clear from (7c), (7g), (27a) and (11d) that
\[
\begin{bmatrix}
R + (B^u)^T \tilde{P}_{N-1|k} B^u & (\tilde{D}_{N-1|k}^u)^T \\
\tilde{D}_{N-1|k}^x & 0
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
u_{N-1|k} \\
\xi_{N-1|k}
\end{bmatrix} \\
\end{bmatrix}
= -\begin{bmatrix}
(B^u)^T \tilde{P}_{N-1|k} A \\
\tilde{D}_{N-1|k}^x
\end{bmatrix} x_{N-1|k} + \begin{bmatrix}
-(B^u)^T \tilde{q}_{N-1|k} \\
\tilde{d}_{N-1|k}
\end{bmatrix},
\]
which, from Assumption 1, the second condition in (23) and (15e), has a solution satisfying
\[
\begin{bmatrix}
u_{N-1|k} \\
\xi_{N-1|k}
\end{bmatrix} = \begin{bmatrix}
L_{N-1|k}^u \\
L_{N-1|k}^x
\end{bmatrix} x_{N-1|k} + \begin{bmatrix}
u_{N-1|k} \\
\xi_{N-1|k}
\end{bmatrix}.
\] (28)
From (7f), (26), (7g), (27b) and (15b), we have
\[
\lambda_{N-2|k} = P_{N-2|k} x_{N-1|k} + \xi_{N-2|k} + \tilde{C}_{N-2|k}^T \xi_{N-2|k}.
\] (29)
Next, we consider stage \(t \in \mathbb{N}_{N-2}\). Assume from (29) that
\[
\lambda_{t|k} = P_{t|k} x_{t+1|k} + \xi_{t|k} + \tilde{C}_{t|k}^T \xi_{t|k}.
\] (30)
If (10e) holds, we deduce from (10d) that
\[
- (\tilde{C}_{t|k} B^w)^T \xi_{t|k} + \tilde{G}_{t|k}^T \tilde{q}_{t|k} = (D_{w|t|k}^u)^T \xi_{t|k}
\] (31a)
and
\[
- \tilde{C}_{t|k}^T \xi_{t|k} = (D_{t|k}^u)^T \xi_{t|k} - \tilde{C}_{t|k}^T \xi_{t|k}.
\] (31b)
With \(\lambda_{t|k}\) assumed in (30), equations (7a), (7g), (31a) and (10d) yield
\[
\gamma_2 I_{n_w} - (B^w)^T P_{t|k} B^w 
\begin{bmatrix}
u_{t|k} \\
\xi_{t|k}
\end{bmatrix} = \begin{bmatrix}
(B^w)^T P_{t|k} \\
-D_{t|k}^x
\end{bmatrix} \tilde{x}_{t+1|k} + \begin{bmatrix}
(B^w)^T q_{t|k} \\
d_{t|k}
\end{bmatrix},
\]
which, from Assumption 1, the first condition in (23) and (15d), has a solution satisfying
\[
\begin{bmatrix}
u_{t|k} \\
\xi_{t|k}
\end{bmatrix} = \begin{bmatrix}
M_{t|k}^w \\
M_{t|k}^x
\end{bmatrix} \tilde{x}_{t+1|k} + \begin{bmatrix}
m_{t|k}^w \\
m_{t|k}^x
\end{bmatrix}.
\] (32)
From (7f), (30), (7g), (31b) and (15a), we have
\[
\hat{\lambda}_{t|k} = \tilde{P}_{t|k} \tilde{x}_{t+1|k} + \tilde{q}_{t|k} + \tilde{E}_{t|k}^T \tilde{v}_{t|k} + \tilde{C}_{t|k}^T \xi_{t|k}.
\] (33)
If (11e) holds, it follows from (11d) that
\[
(C_{t|k} B^w)^T \xi_{t|k} + (\tilde{E}_{t|k} B^w)^T \tilde{v}_{t|k} + \tilde{F}_{t|k} P_{t|k} = (\tilde{D}_{t|k}^w)^T \xi_{t|k}
\] (34a)
and
\[
(C_{t|k} A)^T \xi_{t|k} + (\tilde{E}_{t|k} A)^T \tilde{v}_{t|k} = (\tilde{D}_{t|k}^x)^T \xi_{t|k} + \tilde{C}_{t-1|k}^T \xi_{t-1|k}.
\] (34b)
From (7c), (7g), (34a) and (11d), we have

\[
\begin{bmatrix}
R + (B^u)^T \hat{P}_{t|k} B^u & (D^u_{t|k})^T \\
\hat{D}^u_{t|k} & 0
\end{bmatrix}
\begin{bmatrix}
u_{t|k} \\
\xi_{t|k}
\end{bmatrix}
= -
\begin{bmatrix}
(B^u)^T \hat{P}_{t|k} A \\
\hat{D}^e_{t|k}
\end{bmatrix} x_{t|k}
- \begin{bmatrix}
(B^u)^T \hat{q}_{t|k} \\
-\hat{d}_{t|k}
\end{bmatrix},
\]

which, from Assumption 1, the second condition in (23) and (15e), has a solution satisfying

\[
\begin{bmatrix}
u_{t|k} \\
\xi_{t|k}
\end{bmatrix} = \begin{bmatrix}
L^u_{t|k} \\
L^\xi_{t|k}
\end{bmatrix} x_{t|k} + \begin{bmatrix}
l^u_{t|k} \\
l^\xi_{t|k}
\end{bmatrix}.
\]

Finally, the validity of (30) follows from (7f), (33), (7g), (34b) and (15b). From (19) and (9), we have (22a). Similarly, (22b) is derived by substituting (19) and (20) into (17). Local solution (21) can be derived if conditions (22) and (23) hold. \(\square\)

**Remark 3.** In the presence of degeneracy (of the first type), the active inequality constraint \(d_{t|k} - D^x_{t|k} \hat{x}_{t+1|k} - D^w_{t|k} w_{t|k} = 0\) in the maximization problem may depend on \(\hat{x}_{t+1|k}\) explicitly. Then, for a matrix \(P_{t|k} > 0\), \(\hat{P}_{t|k}\) derived by the Riccati equation (15a) and (15d) may lose its positive definiteness when \(\gamma > 0\) increases. This is different to Proposition 2 in [13]. In [13], constraint transformations are not used and it follows that \(D^x_{t|k} = 0\) and \(D^w_{t|k} = \hat{G}_{t|k}\). In that situation, there exists a \(\gamma^* > 0\) such that, for all \(\gamma \geq \gamma^*\), \(\hat{P}_{t|k} > 0\) if \(P_{t|k} > 0\).

Note that, if a multiplier involved in (22b) depends on \(\zeta_{-1|k} \in \mathbb{R}^{d_0}\), then, for a fixed \(x_{0|k}\), the value of this multiplier can be zero or positive under different \(\zeta_{-1|k}\). By checking whether these multipliers depend on \(\zeta_{-1|k}\), we can partition them into two sets, namely, free multiplier set and fixed multiplier set. Let \(0 < n^f_k \in \mathbb{N}\) denote the number of free multipliers. It is clear that \(n^f_k < n^e_k < \infty\).

**Remark 4.** Suppose \(d_{t|k} > 0\) (LICQ violated), matrix \([\hat{c}_{-1|k}, \hat{C}_{-1|k}]\) has full row rank, and the set of \((x_{0|k}, \zeta_{-1|k})\) described by (22) is non-empty. All the fixed multipliers can be positive under some \(x_{0|k}\) except when only one \(x_{0|k}\) is feasible under the non-negative requirement. This exception rarely happens as it requires special structure of the multiplier inequalities. Then, from the complementary conditions (7b), (7d) and (7e), all the corresponding active constraints can be satisfied simultaneously. Moreover, the rows in \(\Lambda_k\) (12) corresponding to these active constraints are linearly independent as \([\hat{c}_{-1|k}, \hat{C}_{-1|k}]\) has full row rank. Since fixed multiplier has a unique value for each \(x_{0|k}\), we know that the rows in \(\Lambda_k\) for constraints corresponding to free multipliers are linearly dependent when \(d_{t|k} > 0\). Thus, for a fixed \(x_{0|k}\), there is at least one free multiplier equal to zero.

If \(A^*(x_{0|k})\) is known for a given \(x_{0|k}\), then we can compute \(x_{t|k}\) by forward simulation in (19) with \(t\) ranging from 1 to \(N\). From (19) and (16), it is clear that the \(x_k \triangleq \{x_{t|k} : t \in \mathbb{N}_{[1,N-1]}\}\) obtained is an affine function of \(x_{0|k}\), and so are \(u_k \triangleq \{u_{t|k} : t \in \mathbb{N}_{-1}\}\) and \(w_k \triangleq \{w_{t|k} : t \in \mathbb{N}_{-1}\}\). Denote these primal solutions as \(x_k(x_{0|k}, A^*), u_k(x_{0|k}, A^*)\) and \(w_k(x_{0|k}, A^*)\), respectively.

However, as illustrated by (20), \(\hat{c}_k \triangleq \{\hat{c}_{t|k} : t \in \mathbb{N}_{-2}\}\) and \(\hat{C}_k \triangleq \{\hat{C}_{t|k} : t \in \mathbb{N}_{-2}\}\) are affine functions of \(x_{0|k}\) and \(\zeta_{-1|k}\). Thus, from (17), \(\hat{\eta}_k \triangleq \{\hat{\eta}_{t|k} : t \in \mathbb{N}_{-1}\}\), \(\hat{\nu}_k \triangleq \{\hat{\nu}_{t|k} : t \in \mathbb{N}_{-1}\}\) and \(\hat{\mu}_k \triangleq \{\hat{\mu}_{t|k} : t \in \mathbb{N}_{-1}\}\) are all affine.
Lemma 3.8. Let $x_{0|k}$ and $\zeta_{-1|k}$. Correspondingly, we rewrite them as $\tilde{\zeta}(x_{0|k}, \zeta_{-1|k}, A^*)$, $\tilde{\zeta}(x_{0|k}, \zeta_{-1|k}, A^*)$, $\tilde{\eta}(x_{0|k}, \zeta_{-1|k}, A^*)$ and $\tilde{\mu}(x_{0|k}, \zeta_{-1|k}, A^*)$.

Then, the solution to Problem (2) can be represented by

$$s_k(x_{0|k}, \zeta_{-1|k}, A^*) \triangleq \{ u_k(x_{0|k}, A^*), w_k(x_{0|k}, A^*), \tilde{\eta}(x_{0|k}, \zeta_{-1|k}, A^*), \tilde{\mu}(x_{0|k}, \zeta_{-1|k}, A^*) \}$$

in the presence of degeneracy, or

$$s_k(x_{0|k}, A^*) \triangleq \{ u_k(x_{0|k}, A^*), w_k(x_{0|k}, A^*), \tilde{\eta}(x_{0|k}, A^*), \tilde{\mu}(x_{0|k}, A^*) \}$$

otherwise.

3.3. Active set solver. For a selected active set $A$, let $\mathcal{X}(A) \subset \mathbb{R}^{n_x}$ denote the set of feasible initial states for Problem (2) within which the solution $s_k(x_{0|k}, \zeta_{-1|k}, A)$ satisfies the optimality condition in Lemma 3.5 for some $\zeta_{-1|k}$. Specifically,

$$\mathcal{X}(A) \triangleq \left\{ x_{0|k} \in \mathcal{X}_0 : A^*(x_{0|k}) = A \text{ for some } \zeta_{-1|k} \in \mathbb{R}^{d_k} \right\}. \quad (37)$$

By linearity of (22) and boundedness of $\mathcal{X}_0$, $\mathcal{X}(A)$ is a convex polytope. $\mathcal{X}(A)$ is called the critical region (CR) in [1], which represents the largest subset of $\mathcal{X}_0$ such that the active set $A$ at the optimal solution remains optimal. If $\mathcal{X}(A) = \emptyset$, then $A$ is infeasible. In the following, we only consider feasible active sets. Let $\Omega$ denote the set of all feasible active sets. Then, $\bigcup_{A \in \Omega} \mathcal{X}(A) = \mathcal{X}_0$. Let $\partial \mathcal{X}(A)$ and $\partial \mathcal{X}_0$ denote, respectively, the boundaries of $\mathcal{X}(A)$ and $\mathcal{X}_0$.

If $A$ is degenerate, then, for a fixed $x_{0|k} \in \mathcal{X}(A)$, the inequalities in the free multiplier set form a convex polyhedron $\mathcal{Z}(A, x_{0|k}) \subset \mathbb{R}^{d_k}$ of dimension $d_k$. Assume the active constraints indexed by $A$ are non-redundant. Then, $[\hat{c}_{-1|k}, \hat{c}_{1|k}]$ has full row rank and $0 \leq d_k \leq n_x$.

Before proceeding, we need the following definitions [19].

Definition 3.6. For a convex polyhedron $P \subset \mathbb{R}^n$, we say that $F$ is a face of $P$ if there exists a hyperplane $\{ z \in \mathbb{R}^n : a^T z = b \}$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, such that $F = P \cap \{ z \in \mathbb{R}^n : a^T z = b \}$ and $a^T z \leq b$ for all $z \in P$.

Definition 3.7. A vertex of $P$ is a 0-dimensional face $F$ of $P$.

From Remark 4, the feasible values of $\zeta_{-1|k}$ with respect to $A$ and a fixed $x_{0|k} \in \mathcal{X}(A)$ form a subset of $\partial \mathcal{Z}(A, x_{0|k})$. At each vertex of $\mathcal{Z}(A, x_{0|k})$, there are at least $d_k$ multipliers in the free multiplier set equal to zero. Denoting $\zeta_{-1|k}$ as such a vertex, we represent $A_v(x_{0|k}, \zeta_{-1|k}, A)$ as the index set of active constraints whose associated multipliers are zero at $\zeta_{-1|k}$. In the presence of degeneracy, if we revise $A$ by removing all the indices in $A_v(x_{0|k}, \zeta_{-1|k}, A)$, the reduced active set $A \setminus A_v$ is possibly non-degenerate.

Our active set solver is based on this intuitive idea. Before presenting the algorithm, we give the following lemma, which characterizes the continuity properties of the feasible set $\mathcal{X}(A)$, for a feasible $A \in \Omega$, and the solution $s_k(x_{0|k}, \zeta_{-1|k}, A)$ to Problem (2).

Lemma 3.8. Let $A_1 \in \Omega$ be a feasible active set, which indexes non-redundant constraints. If $A_1$ is degenerate with degeneracy degree $1 \leq d_k \leq n_x$, then the following properties hold.

1) $\mathcal{X}(A_1) \subset \mathbb{R}^{n_x}$ is contained in an affine subspace of dimension $(n_x - d_k)$.
2) For any $x_{0|k} \in \mathcal{X}(A_1)$, either $x_{0|k} \in \partial \mathcal{X}_0$ or $x_{0|k} \notin \partial \mathcal{X}_0$; and in both cases, $x_{0|k} \in \bigcap_{i \in I_1, I_2} \partial \mathcal{X}(A_i)$, where $I_1$ and $I_2$ are subsets of $\Omega$ such that $A_i$ for $i \in I_1$ are non-degenerate, and $A_i$ for $i \in I_2$ are degenerate.

3) For each $x_{0|k} \in \mathcal{X}(A_1)$, $s_k(x_{0|k}, \zeta_{-1|k}, A_1)$ satisfies the optimality conditions in Lemma 3.5 for some $\zeta_{-1|k} \in \partial \mathcal{Z}(A_1, x_{0|k})$, where $\mathcal{Z}(A_1, x_{0|k}) \subset \mathbb{R}^{d_k}$ is a convex polyhedron of dimension $d_k$.

4) For each $x_{0|k} \in \mathcal{X}(A_1)$, $\mathcal{Z}(A_1, x_{0|k})$ has a finite number of vertices. For each $i \in \bar{I}_1$, there exists a vertex $\zeta_{-1|k}^v$ of $\mathcal{Z}(A_1, x_{0|k})$ such that $A_i = A_1 \setminus A_v(x_{0|k}, \zeta_{-1|k}^v, A_1)$ and $s_k(x_{0|k}, \zeta_{-1|k}^v, A_1) = s_k(x_{0|k}, A_1)$. For a vertex $\zeta_{-1|k}^v$ of $\mathcal{Z}(A_1, x_{0|k})$, the active set formed by $A_i \triangleq A_1 \setminus A_v(x_{0|k}, \zeta_{-1|k}^v, A_1)$ is non-degenerate, i.e. there exists a vertex $v \in \mathcal{I}_1$, and $s_k(x_{0|k}, \zeta_{-1|k}^v, A_1) = s_k(x_{0|k}, A_1)$ if the matrix formed by removing from $\Lambda_k(12)$ all rows corresponding to active constraints indexed by $A_v(x_{0|k}, \zeta_{-1|k}^v, A_1)$ has full row rank.

5) In the special case of $d_k = 1$, for each $x_{0|k} \in \mathcal{X}(A_1)$, if $x_{0|k} \in \partial \mathcal{X}_0$, then $\mathcal{Z}(A_1, x_{0|k})$ has one vertex and $\mathcal{I}_1$ has one member; and, if $x_{0|k} \notin \partial \mathcal{X}_0$, then $\mathcal{Z}(A_1, x_{0|k})$ is a convex polytope with two vertices and $\mathcal{I}_1$ has two members. In both cases, the vertices of $\mathcal{Z}(A_1, x_{0|k})$ correspond to the active sets in $\mathcal{I}_1$ one-to-one.

On the other hand, if $A_1 \in \Omega$ is non-degenerate, the following properties hold.

6) For each $x_{0|k} \in \partial \mathcal{X}(A_1)$, $x_{0|k} \in \bigcap_{i \in \mathcal{I}} \partial \mathcal{X}(A_i)$ holds, where $\mathcal{I}$ is a subset of $\Omega$.

7) For each $i \in \mathcal{I}$, if $A_i$ is non-degenerate, then $s_k(x_{0|k}, A_1) = s_k(x_{0|k}, A_i)$; otherwise, there is a vertex $\zeta_{-1|k}^v$ of $\mathcal{Z}(A_1, x_{0|k})$ such that $s_k(x_{0|k}, \zeta_{-1|k}^v, A_1) = s_k(x_{0|k}, \zeta_{-1|k}^v, A_i)$.

Proof: From (16), (17), (19) and (20), the solution $s_k(x_{0|k}, \zeta_{-1|k}, A_1)$ for $A_1 \in \Omega$ to Problem (2) is an affine function of $(x_{0|k}, \zeta_{-1|k})$ when $A_1$ is degenerate or an affine function of $x_{0|k}$ only otherwise. If $A_1 \in \Omega$ is degenerate with $1 \leq d_k \leq n_x$, there exist $d_k$ equality constraints on $x_{0|k}$, i.e. $\hat{c}_{-1|k} + \hat{C}_{-1|k} x_{0|k} = \mathbf{0}$. $\hat{c}_{-1|k} + \hat{C}_{-1|k}$ is of full row-rank under the assumption that $A_1$ indexes non-redundant constraints. This implies 1). Then, for $A_1 \in \Omega$ and $A_2 \in \Omega$, which index two sets of non-redundant active constraints, we have $\mathcal{X}(A_1) \cap \mathcal{X}(A_2) = \partial \mathcal{X}(A_1) \cap \partial \mathcal{X}(A_2)$. Hence, 2) and 3) hold.

Suppose $A_1 \in \Omega$ is degenerate. For each $x_{0|k} \in \mathcal{X}(A_1)$, $\mathcal{Z}(A_1, x_{0|k})$ has a finite number of vertices from Remark 4. For a vertex $\zeta_{-1|k}^v$ among them, we can obtain a new active set by removing from $A_1$ all the indices in $A_v(x_{0|k}, \zeta_{-1|k}^v, A_1)$. From Lemma 3.2, the derived active set $A_1 \setminus A_v$ is non-degenerate if the matrix formed by removing from $\Lambda_k(12)$ all the rows corresponding to the constraints indexed by $A_v(x_{0|k}, \zeta_{-1|k}^v, A_1)$ is of full row rank. Since the active sets derived by removing less than $d_k$ active constraints are still degenerate, for each $i \in \mathcal{I}_1$, where $\mathcal{I}_1$ indexes all the neighbouring active sets (with respect to $x_{0|k}$) that are feasible and non-degenerate, there exists a vertex $\zeta_{-1|k}^v$ of $\mathcal{Z}(A_1, x_{0|k})$ such that $A_i = A_1 \setminus A_v$. Then, 4) is proved.

In the special case of $d_k = 1$, $\mathcal{X}(A_1)$ becomes a convex polytope of dimension $(n_x - 1)$, which is either a $(n_x - 1)$-dimensional face of a $n_x$-dimensional CR in the case of $x_{0|k} \in \partial \mathcal{X}_0$, or the common $(n_x - 1)$-dimensional face of two $n_x$-dimensional CRs otherwise. From 1), the active sets of these $n_x$-dimensional CRs are non-degenerate. Then, 4) indicates that $\mathcal{Z}(A_1, x_{0|k})$ either has one vertex when
\( x_{0|k} \in \partial X_0 \) or two otherwise. Thus, 5) holds true considering that the polyhedron \( \mathcal{Z}(A_1, x_{0|k}) \) has at most two vertices.

Conclusions 6) and 7) for a non-degenerate \( A_1 \) can be derived similarly. This completes the proof. \( \Box \)

**Remark 5.** For a degenerate \( A_1 \in \Omega \), if the active constraints in \( A_1 \) are redundant, \([\hat{c}_{-1|k}, \hat{C}_{-1|k}]\) is row-rank deficient. Then, \( \mathcal{X}(A_1) \) may intersect its neighbouring full-dimensional CRs in their interiors. This complexity can be removed by slightly perturbing the weight matrices and the constraints [17], which will not significantly influence the closed-loop behavior.

Based on the properties presented in Lemma 3.8, we now propose a vertex exploration strategy to deal with degeneracy. In the general situation, there may be too many active inequality constraints for a degenerate active set \( A \) such that not all their associated multipliers can be determined uniquely [1, 2]. For our problem, the inequalities in (22b) with a fixed initial state \( x_{0|k} \in \mathcal{X}(A) \) form a convex polyhedron \( \mathcal{Z}(x_{0|k}, A) \) of dimension \( \alpha_k^d \). By exploring \( \zeta_{-1|k} \) in this polyhedron, we can make some of the multipliers (inequalities) in (22b) become zero (equalities). Specifically, at each vertex of \( \mathcal{Z}(x_{0|k}, A) \), no less than \( \alpha_k^d \) of multipliers (inequalities) become zero (equalities).

From Lemma 3.8, we only need to search among these vertices, and remove all the indices in \( A_v(x_{0|k}, \zeta_{-1|k}, A) \) from \( A \), corresponding to a vertex \( \zeta_{-1|k} \), such that the derived active set \( A \setminus A_v \) is non-degenerate and different from all the active sets explored before. In this way, degeneracy will be avoided for the new active set, and cycling will not occur in the search process.

**Remark 6.** It was assumed in \([4, 6]\) that only a single constraint could become active or inactive at each iteration. A real scalar \( \beta_0 \) was then explored to find a non-degenerate active set. This assumed situation corresponds to the particular situation 5), where \( \alpha_k^d = 1 \), in Lemma 3.8.

Following the idea in [5, 4], our active set solution method for Problem (2) is proposed as follows. Let \( x_k \) be the state measured at sampling time \( k \). Starting from an initial state \( x_{0|k} \neq x_k \) for which the optimal active set \( \mathcal{A}^*(x_{0|k}) \) is known, this method uses a line search in the space of \( x_{0|k} \) based on the continuity properties presented in Lemma 3.8 to update the active set until \( x_{0|k} = x_k \).

**Algorithm 1.**

1) Initialize with \( x_{0|k}^{(0)} \) and an active set \( A^{(0)} \) such that \( x_{0|k}^{(0)} \in \mathcal{X}(A^{(0)}) \). Set \( \Omega_s \leftarrow \{A^{(0)}\} \) and \( i \leftarrow 0 \).

2) With respect to the active set \( A^{(i)} \), compute recursively, with \( t \) from \( N - 1 \) to 0, \( \{d_{i|k}, D_{i|k}, D_{i|k}, c_{i|k}, C_{i|k}\}^{(i)} \) and \( \hat{Z}_{i|k}^{(i)} \) for the transformed constraints (10c), and \( \{\hat{d}_{i|k}, \hat{D}_{i|k}, \hat{D}_{i|k}, \hat{c}_{i|k}, \hat{C}_{i|k}\}^{(i)} \) and \( Z_{i|k}^{(i)} \) for the transformed constraints (11c) by, respectively, (10) and (11). If \( \alpha_k^d > 0 \), go to Step 8). Otherwise, proceed to the next step.

3) Then, with respect to \( A^{(i)} \), compute \( \{P_{i|k}, q_{i|k}, \hat{P}_{i|k}, \hat{q}_{i|k}\}^{(i)} \) with \( \{L_{i|k}, L_{i|k}\}^{(i)}, \{L_{i|k}, L_{i|k}\}^{(i)} \), \( \{M_{i|k}, M_{i|k}\}^{(i)} \) and \( \{M_{i|k}, M_{i|k}\}^{(i)} \) backward with \( t \) from \( N - 1 \) to 0 through Riccati recursion (15).

4) With respect to \( A^{(i)} \), compute \( \{\Phi_{i|k} x_{0|k}, \Phi_{i|k} x_{0|k}, \phi_{i|k}\}^{(i)} \) forward with \( t \) from 0 to \( N - 1 \) by (19), where \( x_k^{(i)} \). Then, compute \( \{\hat{\Phi}_{i|k} x_{0|k}, \hat{\Phi}_{i|k} x_{0|k}, \hat{\phi}_{i|k}\}^{(i)} \)
5) Let

\[
\alpha (i) \triangleq \max_{0 < \alpha \leq 1} \left\{ \alpha : x_{0|k}^{(i)} + \alpha (x_{k} - x_{0|k}^{(i)}) \in \mathcal{X}(A(i)) \right\}.
\]  

(38)

6) If \( \alpha (i) = 1 \), go to the next step. Otherwise, set

\[
x_{0|k}^{(i+1)} = x_{0|k}^{(i)} + \alpha(i)(x_{k} - x_{0|k}^{(i)}).
\]  

(39)

Set \( i \leftarrow i + 1 \). Update \( A(i) \) by adding the indices of the new active constraints from condition (22a) and discard the indices of the zero multipliers from condition (22b), and set \( \Omega_s \leftarrow \Omega_s \cup \{A(i)\} \). Then, return to Step 2).

7) Set \( A^* = A(0) \), compute \( u^{(0)}_{N|k}(x_k) \) and stop.

8) With \( x_{0|k}^{(i)} \) fixed, form \( \mathcal{Z}(x_{0|k}^{(i)}, A(i)) \) by (22b). Select a vertex \( \zeta_{\nu(i)}^{u(i)} \) of \( \mathcal{Z}(x_{0|k}^{(i)}, A(i)) \) such that the reduced active set \( A_r = A(i) \setminus A_r(x_{0|k}^{(i)}, \zeta_{\nu(i)}^{u(i)} \}) \) satisfies \( A_r \notin \Omega_s \) and is non-degenerate (i.e. \( \alpha_r = 0 \) for \( A_r(i) \)). Let \( x_{0|k}^{(i+1)} = x_{0|k}^{(i)} \), \( A^{(i+1)} = A_r \), and \( \Omega_s \leftarrow \Omega_s \cup \{A_r(i)\} \). Set \( i \leftarrow i + 1 \), and return to Step 2).

In Algorithm 1, there are two possible initializations: 1) \( x_{0|k}^{(0)} = 0 \) and \( A(0) = \emptyset \) for cold start; and 2) \( x_{0|k}^{(0)} = x_{k-1} \) and \( A(0) = A^*(x_{k-1}) \) for warm start at time step \( k > 0 \).

Remark 7. Assuming that the CRs are in their minimum representations and the active inequality constraints are not weakly active, a linear programming method was proposed in [17] to find a non-degenerate active set. However, this method is limited to the case of 1-degree degeneracy.

Since the number of active sets in \( \Omega \) is finite and the active sets in the searching sequence cannot repeat, we have the following result similar to Theorem 5 in [4].

Theorem 3.9. For \( x_k \in \mathcal{X}_0 \), Algorithm 1 terminates after a finite number of iterations at \( A = A^*(x_k) \).

From Remark 2, \( \sigma_k^d \) in Steps 2) and 8) of Algorithm 1 is not computed by (13), but instead through constraint transformations. At iteration \( i \), for \( i \in \mathbb{N} \), the case of \( x_{0|k}^{(i)} \in \partial \mathcal{X}_0 \) happens only if \( x_k \in \partial \mathcal{X}_0 \cup (\mathbb{R}^n \setminus \mathcal{X}_0) \).

Now, let us revisit the vertex exploration procedure. The number of vertices of \( \mathcal{Z}(x_{0|k}, A) \) is about \( (n_k^d)!/(\sigma_k^d)!/(n_k^d - \sigma_k^d)! \). In the implementation of Step 8), we can first exclude vertices that correspond to active sets already explored, then search a vertex that will not cause degeneracy. When \( n_k^d \) and \( \sigma_k^d \) are not large, the vertex exploration strategy in Step 8) will not require too much computational effort.

From the definition (2f) and Assumption 3, stability of the closed-loop system can be guaranteed if \( x_0 \in \mathcal{X}_0 \). The following theorem [4] is recalled here for completeness.

Theorem 3.10. If \( x_0 \in \mathcal{X}_0 \), then for any disturbance sequence \( \{w_k \in \mathcal{W} : k \in \mathbb{N}\} \), the state of (1) with \( u_k = u_{0|k}^x(x_k) \) satisfies \( x_k \in \mathcal{X}_0 \) for all \( k \in \mathbb{N} \) and

\[
\sum_{k=0}^{\infty} \|y_k\|^2 \leq \gamma^2 \sum_{k=0}^{\infty} \|w_k\|^2 + 2J_0^\circ(x_0),
\]  

(40)

where \( y_k \) is defined by (3).
Table 1. Computational complexity of Algorithm 1

| Step | Number of operations |
|------|---------------------|
| 2)   | \( O\left( (\frac{4}{3}n_w^2 n_u + \frac{4}{3}n_w^2 n_u \right) + (n_w n_{A_u} + n_w n_{A_u} + n_w n_{A_u} + n_w n_{A_u}) n_x + n_{A_u} n_u^2 N) \) |
| 3)   | \( O\left( a_1 (n_{A_u} n_w^2 + n_{A_u} n_u^2 + a_2 (n_w^3 + n_u^3) + a_3 (n_w^2 + n_u^2) n_x \right) + (n_w + n_{A_u} + n_u + n_{A_u}) n_w^3 + 2n_u^2) N) \) |
| 4)   | \( O\left( (b_1 n_x^2 + b_2 (o_k^2)^3) N \right) \) |
| 8)   | \( O\left( n_u^2 (\frac{4}{3}n_{A_u} n_w + n_{A_u} n_u) + (n_w n_{A_u} + n_w n_{A_u} + n_{A_u} n_u + n_{A_u} n_u^2) n_x + n_{A_u} n_u^2 N) \right) \) |

4. **Computational complexity.** Suppose that (15d) and (15e) are solved by the null-space method [14] involving Cholesky and QR factorizations. Then, the number of operations required by Riccati recursion (15) is

\[
O\left( (a_1 (n_{A_u} n_w^2 + n_{A_u} n_u^2 + a_2 (n_w^3 + n_u^3) + a_3 (n_w^2 + n_u^2) n_x \right) + (n_w + n_{A_u} + n_u + n_{A_u}) n_w^3 + 2n_u^2) N) \right),
\]

where \( 0 < a_j \in \mathbb{R} \), for \( j \in \mathbb{N}_{1,3} \), depend on the implementation of the null-space method.

Since the line search involves computing \( \{ \Phi_{\ell \mid k}, \Phi_{\ell \mid k}(x_k - x_0) \}^{(i)} \), \( \{ \Psi_{\ell \mid k}, \Psi_{\ell \mid k}(x_k - x_0) \}^{(i)} \), \( \{ \hat{\Psi}_{\ell \mid k}, \hat{\Psi}_{\ell \mid k}(x_k - x_0) \}^{(i)} \), \( \{ \tilde{\Psi}_{\ell \mid k}, \tilde{\Psi}_{\ell \mid k}(x_k - x_0) \}^{(i)} \), and the vertex exploration involves computing \( \{ \Psi_{\ell \mid k}, \Psi_{\ell \mid k} \}^{(i)} \), the computational complexity of forward simulation (19) and (20) with constructions of \( \mathcal{X}(A) \) by (22) and \( \mathcal{Z}(x_0, A) \) by (22b) is

\[
O\left( (b_1 n_x^2 + b_2 (o_k^2)^3) N \right),
\]

where \( 0 < b_j \in \mathbb{R} \), \( j \in \{ 1, 2 \} \).

Let \( x_0 \in \mathcal{X}(A_1) \), where \( A_1 \) is degenerate. Suppose the number of degenerate active sets indexed by \( I_k \) (refer to 2 of Lemma 3.8) is \( n_k^d \). Then, the maximal number of the operations required by vertex exploration is

\[
O\left( n_k^d \left( \frac{4}{3}n_{A_u} n_w + n_{A_u} n_u + (n_w n_{A_u} + n_{A_u} n_u + n_{A_u} n_u^2) n_x + n_{A_u} n_u^2 N) \right) \right).
\]

Compared to the other steps, the computational efforts of line search in Algorithm 1 can be neglected. Then, from the preceding discussion and Remark 2, the computational complexity of Algorithm 1 can be summarized as in Table 1.

Then, it can be concluded that the computational complexity per iteration of Algorithm 1 is \( O(N) \). However, the number of iterations may be large [5]. It is expected that the iteration number can be reduced by using the warm start strategy.

5. **Numerical simulations.** Consider the first example presented in [4]. Specifically, for a system (1) with coefficient matrices

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B^u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad B^w = I_2,
\]

and constraint sets \( U \triangleq \{ u \in \mathbb{R} : \pm u \leq 1 \} \) and \( W \triangleq \{ w \in \mathbb{R}^2 : \pm w \leq [0.3, 0.3]^T \} \), the \( H_\infty \) optimal control problem (2) is to be solved with cost weights \( Q = \text{diag}(1, 0) \) and \( R = 0.8 \). By solving an unconstrained \( H_\infty \) optimal control problem following
the method in [10], we can derive the control laws \( u_k = u^f_{\infty}(x_k) \triangleq K_u x_k \) and \( w_k = w^f_{\infty}(x_k, u_k) \triangleq K_w (A x_k + B u_k) \) with

\[
P = \begin{bmatrix} 2.6956 & 2.1679 \\ 2.1679 & 3.3164 \end{bmatrix},
\]

\[
K_u = \begin{bmatrix} -0.5903 & -1.4356 \end{bmatrix} \quad \text{and} \quad K_w = \begin{bmatrix} 0.1749 & 0.1527 \\ 0.1527 & 0.2186 \end{bmatrix}.
\]

Different from [4], the state in this paper is confined to set \( \mathcal{X} \triangleq \{ x \in \mathbb{R}^2 : \pm x \leq [10, 10]^T \} \).

Letting \( \mathcal{X}_t \subseteq \{ x \in \mathbb{R}^2 : \pm x \leq [1.5, 1.5]^T \} \), we can compute \( \mathcal{X}_f \) with \( u_k = K_u x_k \) using the algorithm in [11]. Then, \( \mathcal{X}_t \) for \( t \in \mathbb{N}_{[0,15]} \) can be computed in a similar way.

Algorithm 1 is implemented in Matlab. Figure 1 shows the CPU time per iteration (with circle markers) and the maximum number of iterations (with triangle markers) with respect to the prediction horizon length \( N \). Each value in this figure is the averaged one of 50 simulations with randomly selected initial state \( x_0 \in \mathcal{X}_t \setminus \mathcal{X}_{t+1} \) for \( t \in \mathbb{N}_{[0,15]} \). Cold start is selected in the algorithm. We can see that the CPU time per iteration is an approximately linear function of \( N \). The “spike” in the number of iterations for \( N = 7 \) is due to degeneracy.

Next, we consider the second example in [4]. This example involves the linearized pitch dynamics of an aircraft with state \( x \triangleq [\alpha, \dot{\alpha}, h, \dot{h}]^T \), where \( \alpha \) and \( h \) are, respectively, angle of attack (degree) and altitude (meter). The coefficient matrices
in (1) are
\[
A = \begin{bmatrix}
0.9384 & 0.1341 & 0 & 0  \\
-0.5363 & 0.4022 & 0 & 0  \\
0.1186 & 0.0066 & 1 & 0.2  \\
1.1737 & 0.0923 & 0 & 1  \\
\end{bmatrix}, \quad B^u = \begin{bmatrix}
0.0462  \\
0.4022  \\
-0.0190  \\
-0.1803  \\
\end{bmatrix}, \quad \text{and} \quad B^w = \begin{bmatrix}
0 & 0  \\
1 & 0  \\
0 & 0  \\
0 & 1  \\
\end{bmatrix}.
\]

The constraint sets on control, disturbance and state are defined, respectively, as
\[U \triangleq \{ u \in \mathbb{R} : \pm u \leq 15 \}, \quad W \triangleq \{ w \in \mathbb{R}^2 : \pm w \leq \begin{bmatrix} 0.25, 0.25 \end{bmatrix}^T \} \quad \text{and} \quad \mathcal{X} \triangleq \{ x \in \mathbb{R}^4 : \pm x \leq [4, 10, 10, 10]^T \}.
\]
Note that the constraint on control is tighter than that in [4].

\(Q, R\) and \(\gamma^2\) in the cost are selected as \(Q = \text{diag}\{0, 0, 0.04, 0\}\), \(R = 2.5 \times 10^{-4}\) and \(\gamma^2 = 5\). \(P, K_u\) and \(K_w\) can be derived as
\[
P = \begin{bmatrix}
0.7166 & 0.1188 & 0.3678 & 0.2951  \\
0.1188 & 0.0197 & 0.0599 & 0.0486  \\
0.3678 & 0.0599 & 0.2762 & 0.1609  \\
0.2951 & 0.0486 & 0.1609 & 0.1238  \\
\end{bmatrix},
\]
\[
K_u = \begin{bmatrix}
-25.6170 & -4.6166 & -6.4619 & -8.5778  \\
\end{bmatrix},
\]
and
\[
K_w = \begin{bmatrix}
0.0244 & 0.0041 & 0.0124 & 0.0100  \\
0.0608 & 0.0100 & 0.0331 & 0.0255  \\
\end{bmatrix}
\]
by solving an unconstrained \(H_\infty\) optimal control problem. Let \(T = 20\) and the length of the prediction horizon \(N = 4\). Figure 2 illustrates the trajectories of the pitch dynamics under our control law. In this simulation, the initial state is selected randomly as \(x_0 = [-0.9306, -7.2551, 2.9303, 2.0384]^T \in X_0 \setminus \mathcal{X}_1\) and the disturbance is set as
\[
w_k = \begin{bmatrix}
0.25 \cos \left( \frac{16k\pi}{T} \right)  \\
0.25 \cos \left( \frac{16k\pi}{T} + \frac{5\pi}{6} \right)  \\
\end{bmatrix}.
\]
From Figure 2, we can see our \(H_\infty\) optimal control is effective for this system. Both the constraints on the state and the control are satisfied as expected.

6. Conclusion. In this paper, we have extended the active set solver in [7, 5, 4, 6] to the \(H_\infty\) min-max optimal control problem for linear discrete-time systems with linear state constraints. Line search based mp-QP is utilized to seek an optimal active set for the sampled state. Constraint transformations are introduced at each stage of the dynamic programming process such that any surplus equality constraints on the state, when the gradients of the active constraints are linearly dependent, can be determined and moved together with their multipliers to the previous stage for handling. In the presence of a degenerate active set, i.e. LICQ violations, if all the active constraints indexed by it are non-redundant, a vertex exploration strategy is proposed to find a non-degenerate active set based on the continuity properties of polyhedral partitions of state space. The computational complexity per iteration of this solver depends linearly on the length of prediction horizon owing to the integrated Riccati recursion. However, due to the incorporated state constraints, the positive definiteness of cost matrix \(\hat{P}_{i,k} (15a)\) derived in the maximization procedure cannot be guaranteed by selecting a sufficiently large \(\gamma^2\) as in [13].
Figure 2. The controlled pitch dynamics

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