CONVERGENCE PROPERTIES OF KEMP’S $q$-BINOMIAL DISTRIBUTION

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Abstract. We consider Kemp’s $q$-analogue of the binomial distribution. Several convergence results involving the classical binomial, the Heine, the discrete normal, and the Poisson distribution are established. Some of them are $q$-analogues of classical convergence properties. Besides elementary estimates, we apply Mellin transform asymptotics.

1. Introduction

Kemp [4] introduced the following $q$-analogue $KB(n, \theta, q)$ of the binomial distribution:
\[
P(X_{KB} = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \frac{\theta^x q^{(x-1)/2}}{\theta^x} \frac{q^x}{(\theta, q)_n}, \quad 0 \leq x \leq n, \ 0 < \theta,
\]
where
\[
\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q, q)_n}{(q, q)_k (q, q)_{n-k}} \quad \text{and} \quad (z, q)_n = \prod_{i=0}^{n-1} (1 - z q^i)
\]
are the $q$-binomial coefficient and the $q$-shifted factorial. See [3, 5, 6] for properties and applications of this distribution; for an introduction to the $q$-calculus see [2].

For $q \to 1$, this distribution tends to a binomial distribution:
\[
KB(n, \theta, q) \to B\left(n, \frac{\theta}{1 + \theta}\right).
\]

For $n \to \infty$, it tends to the Heine distribution $H(\theta)$:
\[
P(X_H = x) = \frac{q^{(x-1)/2} \theta^x}{(q, q)_x} e_q(-\theta), \quad x \geq 0,
\]
where
\[
e_q(z) = \frac{1}{(z, q)_\infty}, \quad z \in \mathbb{C} \setminus \{q^{-i} : \ i = 1, 2, \ldots\}
\]
is a $q$-analogue of the exponential function, since $e_q((1 - q)z) \to e^z$. The Heine distribution is a $q$-analogue of the Poisson distribution, since $H((1 - q)\theta) \to P(\theta)$.

Moreover, we need a second $q$-analogue of the exponential function, the function $E_q(z) = (-z, q)_\infty$. Note that we have $e_q(z)E_q(-z) = 1$.

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The random variable $X_{KB}$ can be written as the sum of independent Bernoulli random variables \cite{6}, which leads to the expressions

\[
\mu = \sum_{i=0}^{n-1} \frac{\theta q^i}{1 + \theta q^i} \quad \text{and} \quad \sigma^2 = \sum_{i=0}^{n-1} \frac{\theta q^i}{(1 + \theta q^i)^2}
\]

for the mean and variance. We are now interested in sequences of random variables $X_n$ with $X_n \sim KB(n, \theta_n, q)$, in particular we show that there are analogues to the convergence of the classical binomial distribution to the Poisson distribution and the Normal distribution, and that the limits $q \to 1$ and $n \to \infty$ can be exchanged.

Section 2 deals with two cases of convergent parameter $\theta_n$, in particular with the case of constant mean. In Section 3 we show that, if $\theta_n$ grows sub-exponentially, the normalized $X_n$ converge to a discrete normal distribution. In Section 4 we examine the case of an exponentially growing parameter sequence $\theta_n$.

### 2. Convergent Parameter

As noted above we consider sequences of random variables $X_n$ with $X_n \sim X_{KB}(n, \theta_n(q), q)$. In the present section we will provide convergence results for two different sequences $\theta_n(q)$ which both tend to a limit as $n \to \infty$. In the following we need

**Lemma 2.1.** Let $(\theta_n)$ be a sequence of real numbers with limit $\theta \geq 0$. Then

\[
\lim_{n \to \infty} \prod_{i=0}^{n-1} (1 + \theta_n q^i) = E_q(\theta).
\]

**Proof.** For small $\epsilon > 0$ and $n$ large enough, we have

\[
\prod_{i=0}^{n-1} (1 + (\theta - \epsilon) q^i) \leq \prod_{i=0}^{n-1} (1 + \theta_n q^i) \leq \prod_{i=0}^{n-1} (1 + (\theta + \epsilon) q^i),
\]

hence

\[
E_q(\theta - \epsilon) = \lim_{n \to \infty} \prod_{i=0}^{n-1} (1 + (\theta - \epsilon) q^i) \leq \liminf_{n \to \infty} \prod_{i=0}^{n-1} (1 + \theta_n q^i)
\]

\[
\leq \limsup_{n \to \infty} \prod_{i=0}^{n-1} (1 + \theta_n q^i) \leq \lim_{n \to \infty} \prod_{i=0}^{n-1} (1 + (\theta + \epsilon) q^i) = E_q(\theta + \epsilon).
\]

By continuity of $E_q$, the lemma follows. \qed

The $q$-number $[x]_q$ is defined as

\[
[x]_q := \frac{1 - q^x}{1 - q}.
\]

for $q \to 1$, we have $[x]_q \to x$. Now we can establish our first convergence result.

**Theorem 2.2.**

\[
X_{KB}(n, \theta_n(q), q) \xrightarrow{n \to \infty} H((1 - q)\lambda) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{with } \theta_n(q) = \lambda / [n - \lambda]_q.
\]
Proof. We only have to show $X_{KB}(n, \theta_n(q), q) \to H((1-q)\lambda)$. Note that

$$
P(X_n = x) = \binom{n}{x} \frac{\lambda^x (1-\lambda)^{n-x}}{\binom{n}{x} q^x (1-q)^{n-x}} \prod_{i=0}^{n-1} \left( 1 + \frac{\lambda (1-q)}{1-q - q^i} \right).
$$

Moreover, first we check that for given $X_n$ of $f_n(x) = 0$. Moreover, suppose that for each $n$ there is a unique sequence $x_n$ of $f_n(x) = 0$. Then $(x_n)_{n \in \mathbb{N}}$ converges to $\hat{x}$.

Therefore $x_n$ converges to a limit $\hat{x} \geq \tilde{x}$. Moreover,

$$0 = \lim_{n \to \infty} f_n(x_n) \leq \lim_{n \to \infty} f_n(\tilde{x}) = f(\tilde{x}),$$

thus $\tilde{x} \leq \hat{x}$.

Our second convergence result is analogous to the classical convergence of the binomial distribution with constant mean to the Poisson distribution.

**Theorem 2.4.** Fix $\mu > 0$. Then we have

$$X_{KB}(n, \theta_n(q), q) \xrightarrow{n \to \infty} H(\theta(q)) \xrightarrow{q \to 1} \mu \xrightarrow{n \to \infty} B(n, \frac{\mu}{n}) \xrightarrow{n \to \infty} P(\mu).$$

with $\theta_n(q)$ such that $\mu_n = \mu$ and $\theta(q) = \lim_{n \to \infty} \theta_n(q)$.

**Proof.** First we check that for given $\mu > 0$ and fixed $q$ there is a unique sequence $\theta_n(q)_{n \geq N}$, such that $\mu_n(\theta_n(q), q) = \mu$. The function $\mu_n(\theta, q)$ is strictly increasing in $\theta$ and $\mu_n(0, q) = 0$. Since

$$\mu_n(q^{-n+1}, q) \geq \sum_{i=0}^{n-1} \frac{q^{i-n+1}}{2q^{i-n+1}} = \frac{n}{2},$$

and $\mu_n(\theta, q)$ is continuous in $\theta$, there exists a unique solution $\theta_n(q)$ of $\mu_n(\theta, q) = \mu$ for each $n \geq 2\mu$. As $\mu_n(\theta, q)$ is increasing in $n$, we can apply Lemma 2.3 to obtain $\lim_{n \to \infty} \theta_n = \theta(\mu)$, with $\theta(q)$ the unique solution of $\mu(\theta, q) = \mu$. Thus $X_{KB}(n, \theta_n(q), q) \to H(\theta(q))$ by Lemma 2.1.

The function $\mu_n(\theta, q)$ is also increasing in $q$, so we get $\theta_n(q) \to \frac{\mu_n(q)}{n-q}$ (or equivalently $\theta_n(q) \to \frac{\mu}{n}$) for $q \to 1$ by Lemma 2.3. So $X_{KB}(n, \theta_n(q), q) \to B(n, \frac{\mu}{n})$.

It remains to check that $\theta(q)/(1-q)$ converges to $\mu$ for $q \to 1$ (then $H(\theta(q))$ converges to $P(\mu)$). The value $\theta(q)/(1-q)$ is the unique solution of $\mu(\theta, q) = \mu$. Moreover, $\mu(\theta, q)$ is increasing in $x$ and $\theta$ and $\lim_{q \to 1} \mu(\theta, q) = \theta$ (because $H((1-q)\theta) \to P(\theta)$). Thus we can again apply Lemma 2.3. □
3. Sub-Exponentially Increasing Parameter

Now we consider parameter sequences $\theta_n = q^{-f(n)}$ with $f(n) \to \infty$ and $n - f(n) \to \infty$ for $n \to \infty$. These assumptions on $f(n)$ will be in force throughout the section. Theorems 3.2 and 3.3 and Lemmas 3.4–3.6 are devoted to the asymptotic behavior of the sequence $(\mu_n)$ of means. As they tend to infinity, we will normalize our sequence of random variables to $(X_n - \mu_n)/\sigma_n$. Still, this sequence does not converge in distribution without further assumptions on $f(n)$. A fruitful way to proceed is to pick subsequences along which the fractional part $\{f(n)\}$ is constant.

Theorem 3.7 shows that this induces convergence to discrete normal distributions.

To investigate the sequence of means, we begin by providing an elementary estimate for the variance.

Lemma 3.1. If $\theta_n = q^{-f(n)}$ with the above assumptions on $f(n)$, then the sequence of variances satisfies $\sigma_n^2 \leq \frac{2}{1 - q}$.

Proof. By (1), the variance $\sigma_n^2$ equals

$$\sum_{i=0}^{n} \frac{q^i - f(n)}{(1 + q^i - f(n))^2} = \sum_{i=0}^{[f(n)]} \frac{q^i - f(n)}{(1 + q^i - f(n))^2} + \sum_{i=[f(n)]+1}^{n} \frac{q^i - f(n)}{(1 + q^i - f(n))^2}$$

$$\leq \sum_{i=0}^{\infty} \frac{1}{q^{i-f(n)-1}} + \sum_{i=0}^{\infty} q^{i+1-f(n)}$$

$$\sum_{i=0}^{\infty} q^i + \sum_{i=0}^{\infty} q^i = \frac{2}{1 - q}. \quad \Box$$

The following result about the sequence of means does not reveal the structure of the $O(1)$ term, but will be useful later on (Lemma 3.5).

Theorem 3.2. Let $X_n \sim KB(n, \theta_n, q)$ with $\theta_n = q^{-f(n)}$. Then, for $n \to \infty$,

$$\mu_n = f(n) + c([f(n)], q) + o(1),$$

where

$$c([f(n)], q) := 1 - \frac{1}{1 + q^{-f(n)}} - \sum_{\ell \geq 0} \frac{1}{1 + q^{-\ell - f(n) - 1}} + \sum_{\ell \geq 0} \frac{1}{1 + q^{-\ell + f(n) - 1}} = O(1).$$

Proof. We start from

$$\mu_n = \sum_{i=0}^{n-1} \frac{q^i - f(n)}{1 + q^i - f(n)} = \sum_{i=0}^{n-1} \frac{1}{1 + q^{i-f(n)}}$$

(3)
and split the sum into two parts (w.l.o.g. \( f(n) < n \)):

\[
\sum_{i=0}^{[f(n)]-1} \frac{1}{1 + q^{f(n)-i}} = \sum_{i=0}^{[f(n)]-1} \sum_{\ell \geq 0} (-1)^\ell q^{\ell f(n)-i}
\]

\[
= \sum_{\ell \geq 0} (-1)^\ell q^{\ell f(n)} \sum_{i=0}^{[f(n)]-1} q^{-\ell i}
\]

\[
= [f(n)] + \sum_{\ell \geq 1} (-1)^\ell q^{\ell f(n)} \frac{1 - q^{-\ell [f(n)]}}{1 - q^{-\ell}}
\]

\[
= [f(n)] - \sum_{\ell \geq 1} (-1)^\ell q^{\ell f(n)} \frac{1 - q^{-\ell i}}{1 - q^{-\ell}} + O(q^{f(n)\ell})
\]

\[
= [f(n)] + \sum_{\ell \geq 1} q^{\ell} (-1)^\ell q^{\ell f(n)} \frac{1 - q^{-\ell i}}{1 - q^{\ell}} + O(q^{f(n)\ell})
\]

\[
= [f(n)] + \sum_{\ell \geq 1} \sum_{j \geq 0} (-q^{j+1} f(n))^{\ell} q^{\ell j} + O(q^{f(n)\ell})
\]

\[
= [f(n)] + \sum_{j \geq 0} \sum_{\ell \geq 1} (-q^{j+1} f(n)) \ell q^{\ell j+1} + O(q^{f(n)\ell})
\]

\[
= [f(n)] - \sum_{j \geq 0} \frac{1}{1 + q^{-j-1} f(n)} + O(q^{f(n)})
\]

For the upper portion of the sum, we find

\[
\sum_{i=[f(n)]+1}^{n-1} \frac{1}{1 + q^{f(n)-i}} = \sum_{i=[f(n)]+1}^{\infty} \frac{1}{1 + q^{f(n)-i}} + O\left( q^{n-f(n)} \right)
\]

\[
= \sum_{i=0}^{\infty} \frac{1}{1 + q^{f(n)-i-1}} + O\left( q^{n-f(n)} \right),
\]

since

\[
\sum_{i=n}^{\infty} \frac{1}{1 + q^{f(n)-i}} = \sum_{i=0}^{\infty} \frac{1}{1 + q^{f(n)-n-i}} \leq \sum_{i=0}^{\infty} \frac{1}{q^{f(n)-n-i}} = q^{-f(n)} \frac{1}{1 - q^{-1}}.
\]

\(\square\)

In the limit \( q \to 1 \), the term \( c(\{f(n)\}, q) \) tends to \( \frac{1}{2} \). To see this, apply the Euler-Maclaurin formula to

\[
f(x) = \frac{1}{1 + q^{-x-b}}
\]

with \( b > 0 \), which yields

\[\begin{align*}
\sum_{\ell \geq 0} f(\ell) &= \int_0^\infty f(x)dx + \frac{f(0)}{2} + \frac{1}{12} f''(x)|_{x=0} + R_2
\end{align*}\]

with

\[
R_2 = -\frac{1}{2} \int_0^\infty B_2(\{x\}) f''(x)dx.
\]
\[ f''(x) = \frac{(\log q)^2 q^{x+b} (1 - q^{x+b})}{(1 + q^{x+b})^3} \]

does not change sign, we have

\[ |R_2| \leq \frac{1}{12} \int_0^\infty |f''(x)| \, dx = \frac{1}{12} \int_0^\infty f''(x) \, dx = - (\log q)q^{-b} (1 + q^{-b})^{-2} = o(1), \quad q \to 1. \]

The first integral in (4) is

\[ \int_0^\infty f(x) \, dx = -\frac{\log(1 + q^b)}{\log q} \frac{\log 2}{1 - q} - \frac{\log 2 + b}{2} + O \left( \frac{1}{q}, \quad q \to 1. \right) \]

So we have

\[ \sum_{\ell \geq 0} f(\ell) = \frac{\log 2}{1 - q} - \frac{\log 2 + b}{2} + \frac{1}{4} + o(1), \quad q \to 1. \]

Application to the sums appearing in \( c(\{f(n)\}, q) \) gives

\[ c(\{f(n)\}, q) = \frac{1}{2} - \{f(n)\} + \{f(n)\} + o(1). \]

Note that for \( q \to 1 \) the error term in the representation for \( \mu_n \) increases. This is why the limits for \( q \to 1 \) and \( n \to \infty \) can’t be exchanged (\( \mu_n \) tends to \( n/2 \) for \( q \to 1 \)).

The following theorem provides a different representation of the \( O(1) \) term from Theorem 3.2, which shows that it is a \( \frac{1}{2} \)-periodic function of \( f(n) \).

**Theorem 3.3.** Let \( X_n \sim KB(n, \theta_n, q) \) with \( \theta_n = q^{-f(n)} \). Then, as \( n \to \infty \),

\[ \mu_n = f(n) + \frac{1}{2} + \sum_{k>0} 2\pi \frac{\sin(2k f(n) \pi)}{\log q} + O \left( q^{\min(f(n)/2, n-f(n))} \right). \]

**Proof.** We write

\[ \mu_n = \sum_{i=0}^{n} \frac{1}{1 + q^{f(n)-i}} = \sum_{i=0}^{\infty} \frac{1}{1 + q^{f(n)-i}} + O \left( q^{n-f(n)} \right) \]

and apply the Mellin transformation \( \mathcal{M} \) to

\[ h(t) = \sum_{i=0}^{\infty} \frac{1}{1 + i q^{-t}}. \]

By the linearity of the Mellin transformation \( \mathcal{M} \) and the properties \( \mathcal{M} \left( \frac{1}{1+t} \right) = \frac{\pi}{\sin \pi s} \) and \( \mathcal{M}(\alpha t)(s) = \alpha^{-s} \mathcal{M}(h)(s) \), we see that

\[ \mathcal{M}(h)(s) = \sum_{i=0}^{\infty} (q^{-i})^{-s} \frac{\pi}{\sin \pi s} = \frac{1}{1 - q^s} \frac{\pi}{\sin \pi s}. \]

Exchanging \( \mathcal{M} \) and the sum is permitted by the monotone convergence theorem. From the inverse transformation formula we get

\[ h \left( q^{f(n)} \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q^{f(n)s} \frac{1}{1 - q^s} \frac{\pi}{\sin \pi s} \, ds \]

for $c \in (0, 1)$. To evaluate this integral, we choose the integration contour $\gamma_k = \gamma_{k,1} \cup \gamma_{k,2} \cup \gamma_{k,3} \cup \gamma_{k,4}$ with

$$
\gamma_{k,1} = \left\{ s \mid s = \frac{1}{2} + iv : -T_k \leq v \leq T_k \right\}
$$
$$
\gamma_{k,2} = \left\{ s \mid s = u + iT_k : -\frac{1}{2} \leq u \leq \frac{1}{2} \right\}
$$
$$
\gamma_{k,3} = \left\{ s \mid s = -\frac{1}{2} + iv : -T_k \leq v \leq T_k \right\}
$$
$$
\gamma_{k,4} = \left\{ s \mid s = u - iT_k : -\frac{1}{2} \leq u \leq \frac{1}{2} \right\}
$$

where $T_k = \frac{2\pi}{\log q} (k + \frac{1}{4})$. Then

$$
h(q^{f(n)}) = \lim_{k \to \infty} \frac{1}{2\pi i} \int_{\gamma_{k,1}} - \lim_{k \to \infty} \left( \frac{1}{2\pi i} \int_{\gamma_{k,2}} + \frac{1}{2\pi i} \int_{\gamma_{k,3}} + \frac{1}{2\pi i} \int_{\gamma_{k,4}} + \sum \text{residues} \right),
$$

since the integral on the left side exists. Now we estimate the integrals on the right side.

$$
\left| \int_{\gamma_{k,3}} q^{-f(n)s} \frac{1}{1 - q^s \sin \pi s} ds \right| = \left| \int_{-T_k}^{T_k} q^{-f(n)(-\frac{1}{2} + iv)} \frac{1}{1 - q^{-\frac{1}{2} + iv} \sin(\pi(-\frac{1}{2} + iv))} dv \right|
$$

$$
\leq \pi q^{\frac{f(n)}{2}} \int_{-\infty}^{\infty} \frac{1}{1 - q^{-\frac{1}{2} + iv}} \left| \sin(\pi(-\frac{1}{2} + iv)) \right| dv
$$

$$
\leq \pi q^{\frac{f(n)}{2}} \frac{1}{1 - q^{-\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sin^2 \frac{\pi}{2} + \sinh^2 \pi v}} dv
$$

$$
= q^{\frac{f(n)}{2}} \frac{\pi}{1 - q^{-\frac{1}{2}}}
$$

$$
\left| \int_{\gamma_{k,2}} q^{-f(n)s} \frac{1}{1 - q^s \sin \pi s} ds \right| = \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} q^{-f(n)(u + iT_k)} \frac{1}{1 - q^{u + iT_k} \sin(\pi(u + iT_k))} du \right|
$$

$$
\leq \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} q^{-f(n)u} \frac{1}{|1 - q^{u + iT_k}| \sqrt{\sin^2 \pi u + \sinh^2 \pi T_k}} du
$$

$$
\leq \pi q^{\frac{f(n)}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} q^{u} \frac{1}{|\sin(T_k \log q)| \sqrt{\sinh^2 \pi T_k}} du
$$

$$
\leq \pi q^{\frac{f(n)}{2}} \frac{1}{\sinh \pi T_k} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{q^{u}} du \sim 0
$$

The integral over $\gamma_{k,4}$ is treated similarly. Now let us compute the residues: $\frac{1}{1 - q^s}$ has simple poles at $z_k := \frac{2\pi k}{\log q}$, and $\frac{1}{\sin \pi s}$ has a simple pole at 0. First we consider
Lemma 3.4. If we choose a subsequence \((n_k)\) such that \(\{f(n_k)\} = \beta\) constant, then:

(a) For \(k \to \infty\)

\[
\mu_{n_k} = f(n_k) + c(\beta, q) + o(1),
\]

with \(c(\beta, q)\) is a constant depending on \(\beta\) and \(q\).

(b) (i) \(c(0, q) = c(1/2, q) = 1/2\)

(ii) \(c(\beta, q) + c(-\beta, q) = 1\)

Proof. Use [5] and simple properties of \(\sin\).
Lemma 3.5. Set $\beta = \{f(n)\}$. Then
\[ [c(\beta, q) + \beta] = \begin{cases} 0 & 0 \leq \beta < 1/2 \\ 1 & 1/2 \leq \beta < 1 \end{cases} \]

Proof. We define
\[ \hat{c}(\{f(n)\}, q) := c(\{f(n)\}, q) - 1 + \{f(n)\}. \]
By Theorem 3.2, $\hat{c}(\beta, q)$ is strictly increasing in $\beta$. Therefore we have for $0 \leq \beta < 1/2$
\[ \hat{c}(0, q) = -\frac{1}{2} \leq \hat{c}(\beta, q) < \hat{c}(1/2, q) = -\frac{1}{1 - q^{-1/2}} + \frac{1}{1 - q^{-1/2}} = 0. \]
Thus
\[ (7) \quad \frac{1}{2} - \beta \leq c(\beta, q) < 1 - \beta \quad \text{and} \quad \frac{1}{2} \leq c(\beta, q) + \beta < 1. \]
Similarly, we get for $1/2 \leq \beta < 1$
\[ (8) \quad 1 - \beta \leq c(\beta, q) < \frac{1}{2} \quad \text{and} \quad 1 \leq c(\beta, q) + \beta < \frac{1}{2} + \beta < \frac{3}{2}. \]

Lemma 3.6.

(i) If $\beta \neq \frac{1}{2}$, then $f(n) + c(\beta, q) \notin \mathbb{Z}$. Thus
\[ [\mu_n] = [f(n) + c(\beta, q)] = [f(n)] + [\beta + c(\beta, q)]. \]
(ii) For $\beta = \frac{1}{2}$,
\[ \mu_n > f(n) + \frac{1}{2} \quad \text{if} \quad 2f(n) \leq n - 1 \quad \text{and} \quad \mu_n < f(n) + \frac{1}{2} \quad \text{if} \quad 2f(n) \geq n. \]
Thus
\[ [\mu_n] = f(n) + \frac{1}{2} \quad \text{if} \quad 2f(n) \leq n - 1 \quad \text{and} \quad [\mu_n] = f(n) + \frac{1}{2} \quad \text{if} \quad 2f(n) \geq n. \]

Proof. (i): From (7) we get for $0 \leq \beta < 1/2$
\[ f(n) + \frac{1}{2} - \beta < f(n) + c(\beta, q) < f(n) + 1 - \beta \]
and therefore
\[ [f(n)] + \frac{1}{2} < f(n) + c(\beta, q) < [f(n)] + 1. \]
Similarly, from (8) we get for $1/2 < \beta < 1$
\[ [f(n)] + 1 < f(n) + c(\beta, q) < [f(n)] + \frac{3}{2}. \]

(ii): Assume $2f(n) \leq n - 1$. Then
\[
\sum_{i=0}^{n-1} \frac{q^{i-f(n)}}{1 + q^{i-f(n)}} = \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{i-f(n)}}{1 + q^{i-f(n)}} + \sum_{i=f(n)+\frac{1}{2}}^{2f(n)} \frac{q^{i-f(n)}}{1 + q^{i-f(n)}} + \sum_{i=2f(n)+1}^{n-1} \frac{q^{i-f(n)}}{1 + q^{i-f(n)}} \\
= \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{i-\frac{1}{2}}}{1 + q^{i-\frac{1}{2}}} + \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{i+\frac{1}{2}}}{1 + q^{i+\frac{1}{2}}} + o(1) \\
= f(n) + \frac{1}{2} + o(1). \]
We used $\frac{q^{1/2}}{1+q^{1/2}} + \frac{q^{-1/2}}{1+q^{-1/2}} = 1$; the $o(1)$-term is non-negative (and vanishes only for $2f(n) = n - 1$). If $2f(n) \geq n$, then the third sum vanishes and the second sum just runs up to $n - 1 < 2f(n)$, so $\mu_n < f(n) + \frac{1}{2}$. \qed
Note that similarly to the proof of (ii) we can prove the properties of \( c(\beta, q) \) in Lemma 3.4 (b). Especially one can directly show the following theorem for \( \beta = \frac{1}{2} \) and \( \beta = 0 \).

**Theorem 3.7.** Let \((n_k)_{k \in \mathbb{N}}\) be an increasing sequence of natural numbers and \( X_{n_k} \sim KB(n_k, \theta_{n_k}, q) \) with \( \theta_{n_k} = q^{-f(n_k)} \) and \( \{f(n_k)\} = \beta \) constant. Recall that we always assume \( f(n) \to \infty \) and \( n - f(n) \to \infty \). Then \( (X_{n_k} - \mu_{n_k})/\sigma_{n_k} \) converges for \( k \to \infty \) to a limit \( X \), with

\[
\mathbb{P}
\left(
X = - (\beta + c) \frac{1}{\sigma} + \frac{1}{\sigma} x\right) = e_q(q)q(-q^{1-\beta})q\left((x+1)(x-2\beta)^2\right), \quad x \in \mathbb{Z},
\]

if \( \beta < 1/2 \),

\[
\mathbb{P}
\left(
X = - (\beta + c - 1) \frac{1}{\sigma} + \frac{1}{\sigma} x\right) = e_q(q)q(-q^{1-\beta})q\left((-x)(x-2\beta)^2\right), \quad x \in \mathbb{Z},
\]

if \( \beta > 1/2 \) and

\[
\mathbb{P}
\left(
X = \frac{1}{\sigma} x\right) = e_q(q)q(-q^{1/2})q^{x^2/2}, \quad x \in \mathbb{Z},
\]

if \( \beta = 1/2 \), where \( c = c(\beta, q) \) is the constant from Lemma 3.4 and \( \sigma = \lim_{k \to \infty} \sigma_{n_k} \).

The distribution of \( X \) is symmetric iff \( \beta = 0 \) or \( \beta = 1/2 \).

**Proof.** For simplicity we write in the following \( n \) instead of \( n_k \). From (2) one gets that the \( \sigma_n \) converge. Consider the case \( \beta \neq 1/2 \):

\[
\mathbb{P}(X_n = [\mu_n] + x) = \binom{n}{[\mu_n] + x} \frac{\theta^{[\mu_n] + x} q^{([\mu_n] + x)([\mu_n] + x - 1)/2}}{\prod_{i=0}^{n-1} (1 + \theta q^{i})} \prod_{i=[f(n)]+1}^{n-1} \left(1 + \frac{q^i}{q^{[f(n)]}}\right).
\]

The product in the denominator equals

\[
\prod_{i=0}^{n-1} \left(1 + \frac{q^i}{q^{[f(n)]}}\right) = \prod_{i=0}^{[f(n)]} \left(1 + \frac{q^i}{q^{[f(n)]}}\right) \prod_{i=[f(n)]+1}^{n-1} \left(1 + \frac{q^i}{q^{[f(n)]}}\right)
\]

\[
= q^{-f(n)([f(n)]+1)+([f(n)]+1)[f(n)]/2} \prod_{i=0}^{[f(n)]} \left(q^{f(n)-i} + 1\right)
\]

\[
\times \prod_{i=0}^{n-[f(n)]-2} \left(1 + q^i q^{[f(n)]-f(n)+1}\right)
\]

\[
= q^{-f(n)([f(n)]+1)+([f(n)]+1)[f(n)]/2} \prod_{i=0}^{[f(n)]} \left(q^{f(n)-[f(n)]-q[f(n)]-1} + 1\right)
\]

\[
\times \prod_{i=0}^{n-[f(n)]-2} \left(1 + q^i q^{[f(n)]-f(n)+1}\right)
\]

\[
= q^{-f(n)([f(n)]+1)+([f(n)]+1)[f(n)]/2} (-q^{\beta}; q)_{[f(n)]+1} (-q^{\beta+1}; q)_{n-[f(n)]-2}.
\]

The second equality uses the easy relation

\[
\prod_{i=0}^{n-1} (1 + q^{i}) = q^{n(n-1)/2} \prod_{i=0}^{n-1} (1 + (q^{i})^{-1}).
\]
The last two terms in (13) tend to \( e_q(-q^2) \) and \( e_q(-q^{-\beta+1}) \). The \( q \)-binomial coefficient in (12) tends to \( e_q(q) \). The exponent of \( q \) resulting from (12) and (13) is

\[
-((\mu_n) + x) f(n) + ((\mu_n) + x) [(\mu_n) + x - 1]/2 + f(n) [(f(n) + 1 - ((f(n) + 1) (f(n)))/2
\]

\[
= ([f(n)] + [\beta + c] + x) ([f(n)] + [\beta + c] - 1 + x)/2 - ([f(n)] + [\beta + c] + x) f(n)
\]

\[
+ f(n) ([f(n)] + 1) - ([f(n)] + 1) [f(n)]/2
\]

\[
= 1/2 (x - 1 + \delta)(\delta - 2f(n) + 2[f(n)] + x)
\]

\[
= 1/2 (x - 1 + \delta)(\delta - 2\beta + x),
\]

where \( c = c(\beta, q) \) and

\[
\delta = [\beta + c] = \begin{cases} 0 & \beta < 1/2 \\ 1 & \beta > 1/2 \end{cases}
\]

by Lemma 3.5 Putting things together, we obtain

\[
P(X_n = [\mu_n] + x) \rightarrow e_q(q)e_q(-q^2) e_q(-q^{-\beta+1}) q^{(x(n+1)(n-x-2))/2}.
\]

By normalizing \( X_n \) we get (9) and (10). The distribution of \( X \) is symmetric iff

\[
(\beta + c - [\beta + c]) = -(\beta + c - [\beta + c]) + 1
\]

\[
\iff \beta + c - [\beta + c] = 1/2.
\]

This is true for \( \beta = 0 \) by Lemma 3.4 (b) (i). For \( 0 < \beta < 1/2 \) we have \( [\beta + c] = 0 \) by Lemma 3.5. But then we must have \( \beta + c = 1/2 \), which would contradict (7) (since equality only holds for \( \beta = 0 \)). For \( \beta > 1/2 \) we must have \( \beta + c = 3/2 \) by Lemma 3.5, but this would be a contradiction to (8).

For \( \beta = 1/2 \) define

\[
H(\mu_n) := \begin{cases} [\mu_n] & \text{if} \ 2f(n) \leq n - 1 \\ [\mu_n] & \text{if} \ 2f(n) \geq n \end{cases}
\]

Then

\[
P(X_n = H(\mu_n) + x) = \frac{n}{H(\mu_n) + x} q^{-(H(\mu_n) + x)f(n) + (H(\mu_n) + x)(H(\mu_n) + x - 1)/2} \prod_{i=0}^{n-1} \left( 1 + \frac{q_i}{q^{H(n)}} \right).
\]

The \( q \)-binomial-coefficient tends to \( e_q(q) \), and the product can be transformed as above. This time the exponent of \( q \) equals

\[
-H(\mu_n) + x) f(n) + (H(\mu_n) + x) (H(\mu_n) + x - 1)/2 + f(n) ([f(n)] + 1)
\]

\[
-(f(n) + 1)/2 = (f(n) + 1/2 + x) f(n) + (f(n) + 1/2 + x) (f(n) - 1/2 + x)/2
\]

\[
+f(n) \left( f(n) - 1/2 + 1 \right) - \left( f(n) - 1/2 + 1 \right) \left( f(n) - 1/2 \right)/2
\]

\[
= \frac{x^2}{2}.
\]

So we have

\[
P(X_n = H(\mu_n) + x) \rightarrow e_q(q)e_q\left(-q^{1/2}\right) q^{x^2/2}.
\]

By normalizing \( X_n \) we get (11). □

The discrete normal distribution is defined by

\[
P(X = x) = \sum_{k=-\infty}^{\infty} q^{-\alpha q x^2/2} \quad \alpha \in \mathbb{R}, \quad x \in \mathbb{Z}.
\]
So the limit distributions in the preceding theorem are normalized discrete normal distributions with parameters
\[
\alpha = \begin{cases} 
\frac{1}{2} + \beta & \text{if } \beta < \frac{1}{2} \\
\frac{1}{2} - \beta & \text{if } \beta > \frac{1}{2} \\
0 & \text{if } \beta = \frac{1}{2} 
\end{cases}
\]

For \( q \to 1 \), they converge to the standard normal distribution, see [7].

4. Exponentially Growing Parameter

**Theorem 4.1.** If \( X_n \sim KB(n, \theta q^{-n}, q) \), then for \( n \to \infty \),

\[
X_n \to H \left( \frac{q}{\theta} \right)
\]

*Proof.* Define \( Y_n = n - X_n \). Then, by (14) with \( z = \theta q^{-n} \),

\[
\mathbb{P}(Y_n = x) = \left[ \begin{array}{c} n \\ x \end{array} \right] q^{-n(n-x)}q^{(n-x)(n-x-1)/2n} \prod_{i=0}^{n-1} \left( 1 + \theta q^{-i} \right)
\]

\[
= \left[ \begin{array}{c} n \\ x \end{array} \right] q^{-n(n-1)/2}q^{-n^2} \prod_{i=0}^{n-1} (1 + q^{n-i} \theta^{-1})
\]

\[
= \left[ \begin{array}{c} n \\ x \end{array} \right] q^{x(x+1)/2} \theta^{-x} \prod_{i=1}^{n} (1 + q^{i} \theta^{-1})
\]

Therefore

\[
\mathbb{P}(Y_n = x) \to \frac{q^{x(x-1)/2}}{(q,q)_x} \left( \frac{q}{\theta} \right)^x e_q \left( -\frac{q}{\theta} \right).
\]

\( \square \)

If the parameter grows only slightly faster than in Theorem 4.1, then the limit distribution is degenerate.

**Theorem 4.2.** If \( X_n \sim KB(n, q^{-n-f(n)}, q) \) with \( f(n) \to \infty \) for \( n \to \infty \), then for \( n \to \infty \),

\[
X_n \to \delta_0
\]

where \( \delta_0 \) denotes the point measure in 0.

*Proof.* Define \( Y_n = n - X_n \). Then

\[
\mathbb{P}(Y_n = x) = \left[ \begin{array}{c} n \\ x \end{array} \right] q^{-(n-f(n))(n-x)}q^{(n-x)(n-x-1)/2} \prod_{i=0}^{n-1} (1 + q^{i-n-f(n)})
\]

\[
= \left[ \begin{array}{c} n \\ x \end{array} \right] q^{-(n^2+2nf(n)+n-xf(n))-(n-x)^2/2} \prod_{i=0}^{n-1} (1 + q^{i-n-f(n)})
\]

It suffices to prove

\[
\lim_{n \to \infty} \frac{q^{-(n^2+2nf(n)+n)/2} \prod_{i=0}^{n-1} (1 + q^{i-n-f(n)})}{\prod_{i=0}^{n-1} (1 + q^{i-n-f(n)})} = 1,
\]

which is equivalent to

\[
\lim_{n \to \infty} q^{-(n^2+2nf(n)+n)/2} \prod_{i=0}^{n-1} (1 + q^{i-n-f(n)}) = 1.
\]
By (14)
\[ q^{\left(n^2+2nf(n)+n\right)/2} \prod_{i=0}^{n-1} \left(1 + q^{i-n-f(n)}\right) = \prod_{i=0}^{n-1} \left(1 + q^{n+f(n)-i}\right) = \prod_{i=0}^{n-1} \left(1 + q^{f(n)+1+i}\right). \]

This tends to 1 as \( n \to \infty \) by Lemma 2.1.

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