Specific heat and entropy of tachyon Fermi gas

Ernst Trojan

Moscow Institute of Physics and Technology

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Abstract

We consider an ideal Fermi gas of tachyons and derive a low temperature expansion of its thermodynamical functions. The tachyonic specific heat is linear dependent on temperature $C_V = \varepsilon_F k_F T$ and formally coincides with the specific heat of electron gas if the tachyon Fermi energy is defined as $\varepsilon_F = \sqrt{k_F - m^2}$.

1 Introduction

The concept of tachyon fields plays significant role in the modern research, and tachyons are considered as candidates for the dark matter and dark energy, they often appear in the brane theories and cosmological models. Tachyons, are commonly known as instabilities with energy spectrum

$$\varepsilon_k = \sqrt{k^2 - m^2} \quad k > m \quad (1)$$

where $m$ is the tachyon mass and relativistic units $c = \hbar = 1$ are used.

A system of many tachyons can be studied in the frames of statistical mechanics \[1\] \[2\], and thermodynamical functions of ideal tachyon Fermi and Bose gases are calculated \[3\]. We have recently studied the equation of state (EOS) and acoustic properties of the cold tachyon Fermi \[4\] and Fermi gas of tachyonic thermal excitations \[5\].

In the present paper we consider a Fermi gas of tachyons at finite but low temperature. When the temperature $T$ is much lower than the Fermi energy $\varepsilon_F$ of cold tachyon gas at zero temperature, all thermodynamical functions are expanded into a series of $T/\varepsilon_F$. We want to find such important quantities
as the entropy and specific heat (heat capacity) of tachyon Fermi gas. This problem has been already initiated, however, for the limiting nonrelativistic and ultrarelativistic cases \[6\], and we proceed with its general solution for arbitrary range of parameters.

## 2 Tachyon Fermi gas

Consider a system of free tachyons with the energy spectrum \( \varepsilon_k \) that obey the Fermi statistics. Its energy density \( E \) and particle number density \( n \) are defined by standard formulas \[7, 4\]:

\[
E = \frac{\gamma}{2\pi^2} \int_{m}^{\infty} \varepsilon_k f_k k^2 dk
\]

\[
n = \frac{\gamma}{2\pi^2} \int_{m}^{\infty} f_k k^2 dk
\]

where \( \gamma \) is the degeneracy factor, and

\[
f_k = \frac{1}{\exp \left[ (\varepsilon_k - \mu) / T \right] + 1}
\]

is the distribution function, while \( \mu \) is the chemical potential of tachyon Fermi gas at temperature \( T \). The specific heat is also determined by standard formula \[8\]

\[
C_V = T \frac{\partial S}{\partial T} = \frac{\partial E}{\partial T}
\]

where \( S \) is the entropy density.

If we introduce dimensionless variables

\[
x = \frac{\varepsilon_k}{T} \quad \beta = \frac{m}{T} \quad \lambda = \frac{\mu}{T}
\]

the thermodynamical functions of tachyons \[2\)-\[3\] will be written so

\[
E = \frac{\gamma T^4}{2\pi^2} \int_{0}^{\infty} \frac{\sqrt{x^2 + \beta^2} x^2 dx}{\exp (x - \lambda) + 1}
\]
\[ n = \frac{\gamma T^3}{2\pi^2} \int_0^\infty \frac{\sqrt{x^2 + \beta^2} dx}{\exp(x - \lambda) + 1} \] (8)

Both integrals can be also presented in the following universal form
\[ Q = \sigma(T) J(\lambda) \] (9)

where \( \sigma(T) \) is a function of temperature, while integral
\[ J(\lambda) = \int_0^\infty g(x) f_k(x, \lambda) dx \] (10)

includes the distribution function \( f_k \) and function
\[ g(x) \] (11)

each taken for the energy density and particle number density.

At zero temperature the chemical potential \( \mu \rightarrow \varepsilon_F \) tends to the Fermi energy
\[ \varepsilon_F = \sqrt{k_F^2 - m^2} \] (12)

corresponding to the Fermi momentum \( k_F \). The distribution function of fermions \( f_k \) degenerates into the Heaviside step-function
\[ f_k = \Theta(\varepsilon_F - \varepsilon_k) \] (13)

Then, the energy and particle number density of tachyon Fermi gas are immediately calculated \( [4] \)
\[ E_0 = \frac{\gamma}{2\pi^2} \int_m^{k_F} \sqrt{k^2 - m^2} k^2 dk = \frac{\gamma}{8\pi^2} \left[k_F^3 \varepsilon_F - \frac{1}{2} m^2 \left(k_F \varepsilon_F + m^2 \ln \frac{k_F + \varepsilon_F}{m}\right)\right] \] (14)
\[ n = \frac{\gamma}{2\pi^2} \int_m^{k_F} \sqrt{k^2 - m^2} k dk = \frac{\gamma}{6\pi^2} \left(k_F^3 - m^3\right) \] (15)

The latter formula determines the Fermi momentum of tachyons at zero temperature
\[ k_F = \left(\frac{6\pi^2 n}{\gamma} + m^3\right)^{1/3} \] (16)

Now we are looking for low-temperature corrections to formulas \( [4] \)-\( [15] \).
3 Low temperature expansion

At low temperature

\[ T \ll \varepsilon_F \]  

(17)

the chemical potential \( \mu \) is close to the Fermi energy \( \varepsilon_F \) \([12]\). However, the thermodynamical functions depend on the temperature so that the Fermi gas has finite entropy. In order to calculate the specific heat of tachyon Fermi gas \([5]\) we need a low-temperature expansion of thermodynamical functions \([9, 10]\).

Integrating \((10)\) by parts, we have

\[ J(\lambda) = G(x)f_k(x)|_0^\infty - \int_0^\infty G(x)f'_k(x) \, dx \]  

(18)

where

\[ f'_k(x) = \frac{\partial f_k(x)}{\partial x} = -\frac{\exp(x - \lambda)}{[\exp(x - \lambda) + 1]^2} \]  

(19)

and

\[ G(x) = \int g(x) \, dx \]  

(20)

According to \((4)\), the distribution function has the following asymptotic behavior

\[ f_k(0) = 1 \lim_{x \to \infty} f_k(x) \sim \lim_{x \to \infty} \exp(\lambda - x) = 0 \]  

(21)

Hence

\[ J_0 = G(x)f_k(x)|_0^\infty = \lim_{x \to \infty} [G(x)f_k(x)] - \lim_{x \to 0} [G(x)f_k(x)] = -G(0) \]  

(22)

Therefore, integral \((18)\) is immediately written in the form

\[ J(\lambda) = -G(0) - \int_0^\infty G(x)f'_p(x) \, dx \]  

(23)

Let us expand function \( G(x) \) in a Taylor series

\[ G(x) = G(\lambda) + \sum_{k=1}^{\infty} \frac{g^{(k-1)}(\lambda)}{k!} (x - \lambda)^k \]  

(24)
where
\[ g^{(k)}(x) = \frac{\partial^k g(x)}{\partial x^k} \] (25)

Substituting (24) in (23) we have
\[ J(\lambda) = -G(0) - G(\lambda) \int_0^\infty f'_k(x) \, dx - \sum_{k=1}^{\infty} \frac{g^{(k-1)}(\lambda)}{k!} \int_0^\infty (x - \lambda)^k f'_k(x) \, dx \] (26)

In the light of (21), the first term in (26) is simplified so
\[ -G(\lambda) \int_0^\infty f'_k(x) \, dx = -G(\lambda) f_k(x)|_0^\infty = G(\lambda) \] (27)

Hence
\[ J(\lambda) = -G(0) + G(\lambda) - \sum_{k=1}^{\infty} \frac{g^{(k-1)}(\lambda)}{k!} \int_0^\infty (x - \lambda)^k f'_k(x) \, dx \] (28)

where \( f'_k(x) \) is determined by (19). At low temperature \((\lambda \gg 1)\) integral (28) is approximated by formula
\[ J(\lambda) \approx G(\lambda) - G(0) + \sum_{k=1}^{k=\infty} g^{(k)}(\lambda) C_k \] (29)

with coefficients
\[ C_k = \frac{1}{k!} \int_0^\infty (x - \lambda)^{2k} \frac{\exp [(x - \lambda)]}{\{\exp [(x - \lambda)] + 1\}^2} \, dx = 0 \] (30)

where all odd coefficients (30) are tending to zero
\[ C_{2k+1} \to 0 \] (31)

Integral (29) can be written in the explicit form
\[ J(\lambda) = G(\lambda) - G(0) + g'(\lambda) \frac{\pi^2}{6} + g''(\lambda) \frac{7\pi^4}{360} + \ldots \] (32)

Formula (32) determines a low temperature expansion of thermodynamical quantity (9) corresponding to function \( g(11) \).

5
4 Thermodynamical functions

Let us find a low-temperature expansion of the particle number density \( \sigma \), whose presentation according to (10) includes

\[
\sigma_n(T) = \frac{\gamma T^3}{2\pi^2} \tag{33}
\]

\[
g_n(x) = x^{1/2} \tag{34}
\]

so that the relevant function \( G_n \) will be

\[
G_n(x) = \frac{1}{3} \sqrt{(x^2 + \beta^2)^3} \tag{35}
\]

Substituting function (34) and (35) in integral (32), we obtain, up to the second-order terms:

\[
J_n(\lambda) = \frac{1}{3} \sqrt{(\lambda^2 + \beta^2)^3} - \frac{1}{3} \beta^3 + \frac{\pi^2}{6} \frac{2\lambda^2 + \beta^2}{\sqrt{\lambda^2 + \beta^2}} \tag{36}
\]

Substituting (33) and (36) in (10), we find the particle number density

\[
n = \frac{\gamma}{6\pi^2} (q^3 - m^3) + \frac{\gamma}{12} \frac{2q^2 - m^2}{q} T^2 \tag{37}
\]

where

\[
q = \sqrt{\mu^2 + m^2} > m \tag{38}
\]

is the Fermi momentum at low temperature.

At zero temperature limits \( \mu \to \varepsilon_F \) and \( q \to k_F \) takes place, and formula (37) is reduced to (15). Hence, the Fermi momentum at low temperature is approximated by formula

\[
q = k_F \left( 1 - \frac{\pi^2 k_F^2 + \varepsilon_F^2}{6 k_F^4 T^2} \right) \tag{39}
\]

and the Fermi level is

\[
\mu = \varepsilon_F \left( 1 - \frac{\pi^2 k_F^2 + \varepsilon_F^2}{6 k_F^4 \varepsilon_F^2 T^2} \right) \tag{40}
\]
Now let us find a low-temperature expansion of the energy density \( E \), which is presented in the form of (10) with

\[
\sigma_E (T) = \frac{\gamma T^4}{2\pi^2}
\]

(41)

\[
g_E (x) = x^2 \sqrt{x^2 + \beta^2}
\]

so that the relevant function (20) will be

\[
G_E (x) = \frac{1}{4} x^3 \sqrt{x^2 + \beta^2} + \frac{1}{8} \beta^2 x \sqrt{x^2 + \beta^2} - \frac{1}{8} \beta^4 \ln \left( x + \sqrt{x^2 + \beta^2} \right)
\]

(43)

Substituting function (42)-(43) in integral (32), we obtain

\[
J_E (\lambda) = \frac{1}{4} \lambda^3 \sqrt{\lambda^2 + \beta^2} + \frac{1}{8} \beta^2 \lambda \sqrt{\lambda^2 + \beta^2} - \frac{1}{8} \beta^4 \frac{\lambda + \sqrt{\lambda^2 + \beta^2}}{\beta} + \frac{\pi^2}{6} \lambda \frac{3\lambda^2 + 2\beta^2}{\sqrt{\lambda^2 + \beta^2}}
\]

(44)

Substituting (41) and (44) in (10) we define the energy density

\[
E = \frac{\gamma}{8\pi^2} q^3 \mu - \frac{\gamma}{16\pi^2} m^2 \left( \mu q + m^2 \ln \frac{\mu + q}{m} \right) + \frac{\gamma}{12} \mu \frac{\mu^2 + 2q^2}{q} T^2
\]

(45)

Substituting (39) and (40) in (45) we find

\[
E = E_0 + \frac{\gamma}{12} \varepsilon_F k_F T^2
\]

(46)

where \( E_0 \) is the energy density of tachyon Fermi gas at zero temperature.

Thus, according to (5) and (46), the tachyonic specific heat is

\[
C_V = \frac{\gamma}{6} \varepsilon_F k_F T
\]

(47)

and, according to formulas (5) and (47) the entropy density of tachyons is

\[
S = \frac{\gamma}{6} \varepsilon_F k_F T
\]

(48)
5 Conclusion

The energy density of tachyon Fermi gas at low temperature $T \ll \varepsilon_F$ is determined by formula (46). The specific heat of tachyons (47) and the tachyonic entropy (47) are linear dependent on temperature that bear resemblance with ordinary Fermi gas of subluminal particles.

Moreover, expression for tachyonic specific heat (47) formally satisfies the general formula for nonrelativistic electronic specific heat [11]

$$C_V = \frac{\pi^2}{3} N(\varepsilon_F) T$$

where the density of states

$$N(\varepsilon_k) = \frac{\gamma}{2\pi^2} \sqrt{\varepsilon_k^2 + m^2 \varepsilon_k}$$

is incorporated in formula

$$n = \int_0^\infty \frac{N(\varepsilon_k) d\varepsilon_k}{\exp[(\varepsilon_k - \mu)/T] + 1}$$

The Fermi level of tachyon gas at finite temperature is shifted with respect to the zero-temperature level (40). This formula for ultrarelativistic tachyons at large particle number density $n$ ($k_F \gg m$) implies

$$\mu \cong \varepsilon_F \left( 1 - \frac{\pi^2 T^2}{3 \varepsilon_F^2} \right)$$

while for non-relativistic tachyons at small density $n$ ($k_F \to m$) it will be

$$\mu \cong \varepsilon_F \left( 1 - \frac{\pi^2 T^2}{6 \varepsilon_F^2} \right)$$

where the Fermi energy of non-relativistic tachyons, according to (12) and (16), is defined so

$$\varepsilon_F = \sqrt{\frac{4\pi^2 n}{\gamma}} \ll m$$

Formulas (52) and (53) coincide with the relevant expressions obtained in the earlier research [6]. Formula (40) is the general expression for arbitrary
relation between \( k_F \) and \( m \), while formulas (53)-(54) are applied only at low density when \( k_F \rightarrow m \).

However, we should not forget that tachyon Fermi gas at low density is unstable to the causality \([4]\), and its Fermi momentum must exceed critical value

\[
k_F > k_T = \sqrt{\frac{3}{2}} m
\]

(55)

that corresponds to

\[
\varepsilon_F > \varepsilon_T = \frac{m}{\sqrt{2}}
\]

(56)

It is the minimum possible Fermi energy of stable tachyon gas at zero temperature. At finite temperature, according to (40), it will be

\[
\mu_T = \varepsilon_T \left( 1 - \frac{2\pi^2 T^2}{9 \varepsilon_T^2} \right)
\]

(57)

that also much more sufficient than the non-relativistic energy shift (53).

Since the Fermi level of stable tachyon gas must be always higher than \( \varepsilon_T \) (56), we can warrant that temperature

\[
T < \frac{\varepsilon_T}{3} \sim \frac{m}{4}
\]

(58)

should be regarded as definitely small at all \( k_F > k_T \).

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References

[1] St. Mrówczyński, Nuovo Cim. B 81, 179 (1984).
[2] R. L. Dawe, K. C. Hines and S. J. Robinson, Nuovo Cim. A 101, 163 (1989).
[3] K. Kowalski, J. Rembielinski, and K.A. Smolinski, Phys.Rev. D 76, 045018 (2007). arXiv:0712.2725 [hep-th]
[4] E. Trojan and G. V. Vlasov, Phys. Rev. D 83, 124013 (2011). arXiv:1103.2276 [hep-ph]
[5] E. Trojan and G. V. Vlasov, Tachyonic thermal excitations and causality. arXiv:1106.5857 [hep-ph]
[6] K. Kowalski, J. Rembielinski, and K.A. Smolinski, Phys.Rev. D 76, 127701 (2007). arXiv:0712.2728

[7] J. I. Kapusta and C. Gale, *Finite-temperature field theory, Principles and Applications*, 2nd ed. (Cambridge Univ. Press, Cambridge, 2006), p. 8.

[8] L. D. Landau and E. M. Lifshitz, *Statistical physics, Part I*, 3rd ed. (Pergamon Press, Oxford, 1980), p. 47.

[9] J. M. Ziman, *Principles of the Theory of Solids* (Cambridge Univ. Press, Cambridge, 1972), p. 137.

[10] E. Trojan and G. V. Vlasov, Thermodynamics of exotic matter with constant $w=P/E$. arXiv:1108.0824 [cond-mat.stat-mech]

[11] J. M. Ziman, *Principles of the Theory of Solids* (Cambridge Univ. Press, Cambridge, 1972), p. 163.