A Latent Variable Model for Relational Events with Multiple Receivers

Joris Mulder & Peter D. Hoff

Abstract

Directional relational event data, such as email data, often include multiple receivers for each event. Statistical methods for adequately modeling such data are limited however. In this article, a multiplicative latent factor model is proposed for relational event data with multiple receivers. For a given event (or message) all potential receiver actors are given a suitability score. When this score exceeds a sender-specific threshold value, the actor is added to the receiver set. The suitability score of a receiver actor for a given message can depend on observed sender and receiver specific characteristics, and on the latent variables of the sender, of the receiver, and of the message. One way to view these latent variables as the degree of specific unobserved topics on which an actor can be active as sender, as receiver, or that are relevant for a given message. Bayesian estimation of the model is relatively straightforward due to the Gaussian distribution of the latent suitability scale. The applicability of the model is illustrated on simulated data and on Enron email data for which about a third of the messages have at least two receivers.

Keywords: Relational events, multiple receivers, latent factor modeling.

1 Introduction

Social behavior between individuals in a network can often be characterized by short communication messages, or events, of one actor towards another actor or several other actors. In an information-sharing network of employees in an organization, a relational event can be an email message sent by an
employee to one or several fellow employees (Mulder & Leenders, 2019), in a classroom a relational event can be a teacher hushing one or several students (DuBois, Butts, McFarland, & Smyth, 2013), and in a military convoy a relational event can be a soldier giving a command to one or more fellow soldiers (Leenders et al., 2016). Such relational event data are becoming increasingly available due to the technical innovations of email, instant messaging apps, or radio communication in today’s social and work life (Butts, 2008; Brandes et al., 2009; Quintane et al., 2014; Stadtfeld & Block, 2017).

Despite the ubiquity of relational data with multiple receivers, statistical models that explicitly capture this property of the data remain limited, with notable exceptions by Perry & Wolfe (2013) and Shafiei & Chipman (2010). Other than in these works, generally ad hoc solutions are considered such as splitting an event with multiple receivers in multiple dyadic events of the sender and the separate receiver actors (e.g. Mulder & Leenders, 2019). This approach however artificially elevates the number of events, resulting in an overestimation of the certainty when making statistical inferences, and possibly resulting in misleading statistical inference. Another ad hoc solution is to add new actors to the network based on the observed receiver sets in the data so that the event towards these receiver sets can be modeled as a dyadic events (e.g. DuBois, Butts, McFarland, & Smyth, 2013). This would however artificially elevate the number actors in the network, and complicate the interpretation of endogenous effects such as inertia or reciprocity.

Instead of relying on ad hoc solutions yielding various implementation and statistical issues, a statistical framework is needed that can capture such relational events in a natural and direct manner. This paper proposes a generative model for this purpose which builds on the well-developed probit regression model. To model the receiver set of a message of a given sender, all potential receivers are placed on a latent suitability scale for a given message. Next a threshold parameter is drawn on this scale such such that suitability scores that fall on the right side of the threshold become part of the receiver set and scores that fall on the left side do not become part of the receiver set. This model may be more appropriate for modeling these data than existing approaches. For example, the model of Shafiei & Chipman (2010), which assumes independent inclusion probabilities for the potential receivers to become part of the receiver set, has a strictly positive probability that the receiver set can be empty. This property conflicts with the observed data where all messages contain at least one receiver. Alternatively, the proposal of Perry & Wolfe (2013) conditions of the size of the receiver sets in the observed
data. Therefore there is no generative model for the number of receivers for a message using their proposal. Also note that it seems somewhat unnatural to condition on the number of receivers as this implies that a sender first decides how many actors will receive the message, and then decide which actors will be included in the receiver set. As will be shown the proposed model does not have these shortcomings.

To capture unobserved heterogeneity in relational event data, latent variables have been proven useful. Popular latent variable approaches in the social network literature include the stochastic block model (Nowicki & Snijders, 2001) and the latent distance model (Hoff et al., 2002). The first approach, which has also been extended to relational event data (DuBois, Butts, & Smyth, 2013), assumes that actors can be divided across different latent classes with equivalent stochastic interaction behavior. The second approach assumes that actors can be positioned in a latent space where the latent distance between actors affects their tendency to interact with each other, which captures latent homophily. The two models capture fundamentally different interaction behavior (Hoff, 2008). A latent multiplicative model has been proposed which is able to capture the induced interaction behavior of both approaches (Hoff, 2008, 2009). A general multiplicative latent factor model will therefore be implemented in the proposed relational event model for multiple receivers. Given the directional nature of the data actors are allowed to have different latent variables as a receiver and as a sender. Moreover each message also contains latent variables which capture the dependence between the inclusion probabilities of the potential receivers.

The paper is organized as follows. Section 2 introduces some notation and discusses two other approaches for modeling relational event data with multiple receivers. Section 3 presents the Bayesian multiplicative latent factor model for relational data with multiple receivers, including prior specification and posterior predictive checks for model assessment. Next, Section 4 describes the application of the model on simulated data and on the empirical Enron email data. We end the paper with a discussion in Section 5.

2 Notation and related approaches

We consider a social network where the set of actors is denoted by $\mathcal{A}$ and is of size $|\mathcal{A}| = A$. Let $y_{si}$ be the binary vector of length $A$ that indicates the receivers of the $i$th message sent by sender $s \in \mathcal{A}$, so that $y_{sir} = 1$ if actor $r$ is
a receiver of message $i$, and is zero otherwise. We assume that an actor does not send a message to one’s self, and thus the $s$th element of $y_{si}$ is empty. Let $n_s$ be the total number of messages sent by actor $s$, so that $n = \sum_{s=1}^{A} n_s$ is the total number of messages sent across all actors. We define the receiver set $R_{si}$ of message $i$ of sender $s$ to be the subset of $A$ that received this message, so that $R_{si} = \{r \in A : y_{sir} = 1\}$. We denote the cardinality of the receiver set to be $|R_{si}| = m_{si}$. We note that the receiver set of any message is non-empty, i.e., $m_{si} > 0$.

We deem it important that a statistical model for such relational event data has the following properties: First the probabilistic model for the receiver set for a given sender should follow an intuitive rational. Second the model should be able to capture certain key characteristics of the data. Third the model should be computationally feasible. The methods that have been proposed for analyzing relational data with multiple receivers may not necessarily satisfy all three conditions. Specifically, Shafiei & Chipman (2010) proposed an independent Bernoulli model for the receiver set where the inclusion probability of actor $r$ to be in the receiver set $R_{si}$ of message $i$ sent by $s$ is denoted by $\pi_{si} = (\pi_{si1}, \ldots, \pi_{siA})'$. The probabilistic model for the receiver set is then given by

$$\Pr(R_{si}|\pi_{si}) = \prod_{r \in R_s} \pi_{sir}^{y_{sir}} (1 - \pi_{sir})^{1-y_{sir}}.$$  (1)

A mixed membership stochastic block model was then used where actors can belong to different latent classes across messages resulting in a flexible statistical model for multicast messages for various empirical applications. An unrealistic assumption however may be that the model uses independent inclusion probabilities given the latent class memberships of the actors. Furthermore the model considers a strictly positive probability for an empty receiver set, i.e., $\Pr(R_{si} = \emptyset|\pi_{si}) > 0$. Because all messages in the observed data typically contain at least one receiver, the model would not be able to capture this key characteristic of the data.

Perry & Wolfe (2013) extended the Cox proportional hazard model (Cox, 1972, 1975) for event history (or survival) analysis to the relational event framework. A log linear model was considered where endogenous and exogenous statistics capture the intensity (or rate) for actor $r$ to become a receiver of a message sent by actor $s$. Their multinomial probability model for the receiver set $R_{si}$ conditions on the cardinality of the observed receiver
set according to

$$\Pr(R_{si}|\beta, x_{sir}, m_{si}) = \frac{\exp \left\{ \sum_{r \in R_{si}} \beta^\top x_{sir} \right\}}{\sum_{R' \subset R_{si}, |R'|=m_{si}} \exp \left\{ \sum_{r \in R'} \beta^\top x_{sir} \right\}},$$

(2)

where $x_{sir}$ is a vector containing $K$ endogenous and exogenous statistics for message $i$ sent by $s$ received by $r$. By conditioning on the cardinality of the receiver set, the model implicitly assumes that the sender first chooses the number of receivers that the actor will send the message to. Then the probabilities of all possible receiver sets of this magnitude are evaluated. This underlying process does not seem realistic, for example when an employee decides to send an email to fellow employees. In this case, the number of receivers for a given message is more likely to be a consequence of the decision process when a sender chooses who to include as recipient, instead of being part of initialization of the decision process. Moreover, there is also a potential computational problem as the number of possible receiver sets of size $m_{si}$, i.e., $\binom{A}{m_{si}}$, can become computationally infeasible. To address this computational issue, Perry & Wolfe (2013) considered an approximation where the induced bias was corrected using bootstrapping. Finally, another potential limitation of the model is that it does not provide a generative model for a receiver set for a given sender, as the size of the receiver set needs to be conditioned upon.

3 A multiplicative latent factor model for relational events with multiple receivers

We propose the following generative model for the binary receiver vector $y_{si}$ for message $i$ sent by actor $s$,

$$z_{si,−s} \sim N(\theta_{si,−s}, I_{A−1}),$$

$$c_{si}|z_{si,−s} \sim TN(\mu_c, \sigma_c^2, −\infty, z_{si}(A)),$$

$$y_{sir} = \begin{cases} 1 & \text{if } z_{sir} > c_{si} \\ 0 & \text{elsewhere}, \end{cases}$$

(3)

where the latent variable $z_{sir}$ quantifies the suitability of message $i$ for potential receiver actor $r$ sent by actor $s$ on a latent suitability scale, $\theta_{sir}$ is the expected suitability, $c_{si}$ is a threshold parameter for message $i$ sent by $s$. 
which follows a truncated normal distribution with mean $\mu_c$ and variance $\sigma^2_c$, restricted to the set $(-\infty, z_{si}(A))$, and where $z_{si}(A)$ is the $A$-th ordered value of the latent suitability scores, i.e., $z_{si}(A) = \max_r \{z_{sir}\}$. By using a truncated distribution for $c_{si}$ with upper bound $z_{si}(A)$ the size of the receiver set will be at least 1. The mean $\mu_c$ is set to 0 to ensure identification. The variance of the threshold parameters, $\sigma^2_c$, control the distribution of the number of receivers across messages.

A latent factor model is considered for the mean suitability score according to

\[
\theta_{sir} = x_{sir}^\top \beta + b_r + u_r^\top (v_s + w_{si}),
\]

\[
b_r \sim N(0, \sigma^2_b)
\]

\[
u_r, v_s, w_{si} \sim N(0, I_Q),
\]

where $x_{sir}$ is a factor of observed predictor variables (e.g., exogenous or endogenous), $\beta$ contains the unknown coefficients that quantify the relative importance of these predictor variables, the random effect $b_r$ quantifies the average popularity of actor $r$ as a receiver, $u_r$ is a vector containing the $Q$ latent variables of actor $r$ as a receiver, $v_s$ is a vector containing the $Q$ latent variables of actor $s$ as sender, and $w_{si}$ is a vector containing the $Q$ latent variables of message $i$ by sender $s$. The three types of latent variables are assumed to be independent a priori.

Each latent variable can be viewed as a theme or topic on which an actor can be (in)active as a sender or as a receiver, or which is (not) important in a message. If a latent variable is large in absolute value and of the same (opposite) sign for a sender actor and a potential receiver actor, it becomes likely (unlikely) for the sender to include this potential receiver in the receiver set in a message. Furthermore the inner product of the latent vectors imply that the multiplicative function is added for the different latent dimensions, i.e., $u_r^\top v_s = u_{r1} v_{s1} + \ldots + u_{rQ} v_{sQ}$. A similar rational applies to the latent variable of a potential receiver $u_r$ and the latent variable of a message $w_{si}$. For example, if a latent variable of an actor as a receiver is large in absolute value and of the same (opposite) sign as the latent variable of the sent message, then it becomes likely (unlikely) for the actor to become a receiver of the message.

By also including a latent variable for each message, the suitability scores of the potential receivers of belonging to the receiver set are not conditionally independent given the latent variables of the sender and the potential
receivers, the fixed effects, and the random popularity effects. For example, if two actors have large latent variables and of the same (opposite) sign, we would expect that there is a positive (negative) dependency between the latent suitability scores of these two actors. Specifically, a positive dependency (as a result of equal signs) implies that if the message is (not) suitable for one actor, the message should also (not) be suitable for the other actor. A negative dependency (as a result of opposite signs) implies that if a message is suitable for one actor the message is not suitable for the other actor, and vice versa.

To see that this dependency is also present in the proposed model, we can integrate out the latent variable $w_{si}$ for each messages, so that the suitability scores of the potential receiver actors for a given sender $i$ follow a multivariate normal distribution given by

$$z_{si,-s} | \beta, b, v_s, U_{-s} \sim \mathcal{N}(X_{si,-s} \beta + b_{-s} + U_{-s} v_s, I_\Lambda^{-1} + U_{-s} U_{-s}^\top),$$

(7)

where $X_{si,-s}$ and $U_{-s}$ are the stacked matrices of the observed and latent predictor variables of the potential receivers (excluding the sender actor $s$). Thus, the suitability scores of potential receivers are positively (negatively) correlated if their latent receiver variables are large and of the same (opposite) sign, as described above.

In sum the model builds on the following intuition: When an actor decides to initiate an event (e.g., sent a message), the sender actor determines the suitability for all potential receivers for the given message. The suitability is based on the popularity of actors as receiver, observed predictor variables (e.g., observed past interactions between actors or common locations where the actors work), and a multiplicative relation of the latent variables between the sender and the receiver, i.e., $X_{si,-s} \beta + b_{-s} + U_{-s} v_s$, as well as the implied covariance structure between the potential receivers based on their latent receiver scores, i.e. $I_\Lambda^{-1} + U_{-s} U_{-s}^\top$. Next a threshold value is drawn from a normal distribution truncated in an interval with an upper bound equal to the largest suitability score. This ensures that the receiver set will always contain at least one actor, and the actor with the largest suitability score will always be included (Figure 1). Receivers are included in the receiver set whose suitability score exceeds the threshold value. Given the suitability scores of potential receivers, $z_{si,-s}$, the inclusion probability for actor $r$ in
Figure 1: Graphical representation of the inclusion of the actors 3, 5, and 7 in the receiver set for a message send by actor 2 based on the random threshold value $c$. The index of the sender actor $s$ and message $i$ are omitted.

The receiver set $R_{s,i}$ is then given by

$$
\Pr(r \in R_{s,i}|z_{s,i,-s}, \mu_c, \sigma_c) = \frac{\Phi\left(\frac{z_{sir} - \mu_c}{\sigma_c}\right)}{\Phi\left(\frac{z_{s(A)} - \mu_c}{\sigma_c}\right)},
$$

where $\Phi(\cdot)$ denotes the standard normal cdf. Note that the inclusion indicators of the separate actors are not conditionally independent, as every message has at least one recipient.

### 3.1 Prior specification and model estimation

As the model build of the well-established Bayesian probit model (Albert & Chib, 1993; Chib & Greenberg, 1998), Bayesian estimation of the model is relatively straightforward using the following priors:

$$
\begin{align*}
\beta & \sim \mathcal{N}(\beta_0, \Psi_0) \\
\sigma_b^2 & \sim \text{IG}(\alpha_b, \gamma_b) \\
\sigma_c^2 & \sim \text{IG}(\alpha_c, \gamma_c),
\end{align*}
$$

where the hyperparameters can be specified using prior knowledge or heuristic considerations. By setting $\beta_0 = 0$ and $\Psi_0 = g(X'X)^{-1}$, where $g$ is
equal to the number of messages and \( X \) is the stacked matrix of \( X_{si} \), a unit information \( g \) prior would be specified (Zellner, 1986; Kass & Wasserman, 1995). Furthermore the mean of the threshold parameter \( \mu_c \) is fixed to fix the location of the latent suitability scale.

The conditional posterior distributions for \( \beta, b, u, v, \) and \( w_{si} \) then follow multivariate normal distributions, the random effects variance \( \sigma_b^2 \) follows an inverse gamma distribution, and the threshold parameters \( c_{si} \) follow truncated normal distributions \( N(\mu_c, \sigma_c^2) \) in the interval \( (\max_{r:y_{sir}=0}\{z_{sir}\}, \min_{r:y_{sir}=1}\{z_{sir}\}) \). As the posterior for the variance for the threshold parameter \( \sigma_c^2 \) does not have a known form, a random walk can be used. Finally, the conditional posterior for the latent suitability scores is proportional to

\[
\pi(z_{si,-r} | \theta_{si}, y_{si,-r}, c_{si}) \propto N(\theta_{si,-r}, I_{A-1}) \times 
\prod_{r \neq s} 1(z_{sir} < c_{si})^{1-y_{sir}} 1(z_{sir} > c_{si})^{y_{sir}} \Phi \left( \frac{z_{si(A)} - \mu_c}{\sigma_c} \right)^{-1}.
\]

To sample \( z_{si,-r} \), first a candidate \( z_{si,-r}^* \) is drawn from a multivariate truncated normal \( N(\theta_{si,-r}, I_{A-1}) \prod_{r \neq s} 1(z_{sir} < c_{si})^{1-y_{sir}} 1(z_{sir} > c_{si})^{y_{sir}} \), and the draw is accepted with probability \( \min \left( \frac{\Phi((z_{si,-r}^{(l-1)} - \mu_c)/\sigma_c)}{\Phi((z_{si,-r}^* - \mu_c)/\sigma_c)}, 1 \right) \), where \( z_{si}^{(l-1)} \) is the previous \( (l-1\text{-th}) \) draw, using a Metropolis-Hastings step. These conditional distributions can be used to obtain a MCMC algorithm from which it is relatively straightforward to sample from, yielding posterior draws from the fitted model.

### 3.2 Posterior predictive checks for model assessment and latent dimension selection

Posterior predictive checks are a natural choice to assess whether a model captures certain key characteristics of the data (Meng, 1994; Gelman et al., 2004; Van Kollenburg et al., 2015). These checks can only be performed when there is a generative model for the data. Moreover in the presence of latent variables, posterior predictive checks are generally easier to apply (and perhaps easier to interpret) than traditional model fit indices, such as the BIC. For this reason posterior predictive checks are used for evaluating the fit of the proposed multiplicative latent factor model for multiple receivers and for selecting the number of latent dimensions.
A key ability of the proposed model is to allow messages to have multiple receivers via the threshold parameters. To assess whether this is properly done for a given dataset at hand, the distribution of the number of receivers in the data can be compared with the distribution based on replicated data sampled from the posterior draws of the unknown model parameters during the MCMC sampler. As a test statistics we can therefore look at the relative number of messages with \( m \) receivers,

\[
t_{1,m}(Y) = N^{-1} \sum_{s,i} 1(|R_{si}| = m).
\]

for \( m = 1, 2, \text{etc.} \), in the observed data, and compare this with replicated data.

The inclusion of latent variables affects the relative popularity of an actor to be receiver for a given sender actor. Therefore it is useful to look at the relative number of messages received by a potential receiver \( r \) given a sender \( s \), i.e.,

\[
t_{2,sr}(Y) = n_s^{-1} \sum_{i=1}^{n_s} y_{sir}.
\]

Finally multiplicative latent factor models have the ability to capture higher-order dependency structures in the data, such as transitivity and clusterability (Hoff, 2005, 2009). For the current paper we consider a posterior predictive check for transitivity which captures the tendency of actor \( s \) to send messages to actor \( r \) as a function of the number of messages sent by \( s \) to other actors than \( r \) and the number of messages sent by these other actors to \( r \). Transitivity is often phrased as “the friends of my friends are my friends”, an example of structural balance theory (Heider, 1946; Cartwright & Harary, 1956). Transitivity can also be seen as a form of “broker-skipping” as it captures the tendency of actors to skip intermediate actors in a communication network (Leenders et al., 2016). Transitivity can be captured by the following statistic,

\[
t_3(Y) = \sum_{s \neq r} \sum_{j \in A \setminus \{s,r\}} e_{sj} e_{jr} e_{sr},
\]

where \( e_{sr} = \sum_i y_{sir} - \frac{1}{A-1} \sum_{i,r} y_{sir} \), i.e., the difference between the total number of messages sent by \( s \) to \( r \) receiver and the average number of receivers of messages sent by \( s \). Note that \( e_{sr} \) is centered around zero receiver actors \( r \).
slightly alternative form is obtained when instead considering the tendency of \( r \) to send messages to \( s \) as a function of the number of messages that \( s \) sent to \( r \) via another actor, i.e.,

\[
t_4(Y) = \sum_{s \neq r} \sum_{j \in A \setminus \{s, r\}} e_{sj} e_{jr} e_{rs}.
\]

4 Numerical examples

4.1 Example analysis of simulated relational event data

To see how the posterior predictive checks can be used to assess the fit and to determine the number of latent dimensions for the model for a given data set, a relational event data set was simulated for a network of \( A = 50 \) actors. Three fixed effects were considered of size \( \beta = (0.25, 0, 0.25)' \) for covariates that were generated from a multivariate normal distribution with zero means, variances of 1, and correlations of .3. The mean and variance of the random popularity effect were set to \( \mu_b = -4.5 \) and \( \sigma^2_b = .25 \). A 1-dimensional latent variable was considered with a mean of 0, a variance of 1, and a correlation between the latent receiver effect \( u_j \) and latent sender effect \( v_j \) for actor \( j \) of .7, as actors who are (in)active on a latent dimension as a sender are also likely to be (in)active on this latent dimension as a receiver. The mean and variance of the threshold parameters were set to \( \mu_c = 0 \) and \( \sigma^2_c = 0.16 \). The number of messages send by each actor were drawn from a Poisson distribution with a rate parameter of 40. The simulated data were fit using a model of 0, 1, and 2 dimensions for the latent variables.

Figure 2 displays the estimated posterior distributions of the fixed effects using the R package bayesplot (Gabry et al., 2019) for \( Q = 0, 1, \) and 2 latent dimensions. As can be seen the posteriors for the fixed effects do not vary substantively when increasing the dimension of the latent variables.

Figure 3 displays the posterior predictive distribution of the number of receivers over all messages (grey lines) together with the empirical distribution of the number of receivers over all messages (black circles) (first row), the posterior predictive distribution of the relative popularity of potential receiver actors when actor 1 sends a message (second row) followed by the relative popularity of potential receiver actors when actor 3 sends a message (third row) (ordered by relative popularity in observed data), and finally a histogram of the posterior predictive distribution of transitivity together
with the transitivity in the observed data (vertical black line) (fourth row), for a model with \( Q = 0 \) (left column), \( Q = 1 \) (middle column), and \( Q = 2 \) (right column) latent dimensions. The posterior predictive distribution of the number of receivers seems to be accurately captured by for the three models. The prior for the variance of the threshold parameters was tuned so that the fit was acceptable using \( \sigma_c^2 \sim IG(20, 3) \) for all three models. The relative popularity of potential receiver actors shows that the fit does not improve much when actor 1 sends a message when increasing the number of latent dimensions from \( Q = 0 \) to 1 or 2. When actor 3 sends a message the fit improves considerably when increasing the number latent variables from \( Q = 0 \) to 1, while the fit does not increase substantially when increasing the latent dimension from \( Q = 1 \) to 2. The nonincrease in fit for actor 1 as sender and increase in fit for actor 3 as sender can be explained from the 95% posterior credibility intervals of the latent sender effect of actor 1 and 3 which were equal to \((-0.225, 0.615)\) and \((1.12, 2.00)\) under the model with \( Q = 1 \), respectively. These intervals indicate that actor 3 has a clear positive latent sender score while actor 1 has an average latent sender score. Finally the results of the posterior predictive check for transitivity \((t_3)\) indicate that the model with \( Q = 0 \) does not accurately capture the transitivity in the data while the models with \( Q = 1 \) and \( Q = 2 \) capture the transitivity in the data accurately. The results looked similar for \( t_4 \) and are therefore omitted in the figure.

In sum, due to the misfit for the model with \( Q = 0 \) regarding the receiver popularity of sender actor 3 (these checks for the other actors looked similar to the results for actor 1 or 3) and the misfit for capturing transitivity, and the model with a 2-dimensional latent variable does not result in a substantial increase in fit relative to the model with a 1-dimensional latent variable, the model with \( Q = 1 \) is preferred. This results was expected as model 1 also generated the data. In practice it may sometimes be more difficult to ‘choose’ the number of latent dimensions. Generally the choice should depend on the gain in fit when increasing the number of latent dimensions relative to the loss in interpretability and stability of the results when fitting a more complex model. This is illustrated in the next empirical data set.

4.2 Empirical analysis of the Enron email data

To illustrate the applicability of the proposed model on an empirical data set we consider the data from the Enron e-mail corpus which was also con-
Figure 2: Estimated posterior distributions of the fixed effects for $Q = 0, 1,$ and 2 dimensional latent variables.

Considered by Perry & Wolfe (2013), and originally compiled by Zhou et al. (2007). The data consist of 21635 messages sent between 156 employees between November 13, 1998, and June 21, 2002. 69% of these messages had one receiver and 31% of the messages had more than one receiver. The employees’ genders (male, female), seniority (junior, senior), their type of work (“Legal”, “Trading”, “Other”), title (“Specialist”, “Administrator”, etc.) and their department (“ENA Gas Financial”, “Energy Operations”, etc.) were also available. Following Perry & Wolfe (2013) we considered 7 different inertia and reciprocity statistics for 7 different time intervals, dummy covariates for all 16 combinations of senders and receivers working in “Legal” (1=yes, 0=no), working in “Trading” (1=yes, 0=no), being “junior” (1=yes, 0=no), being female (1=yes, 0=no). We also added two dummy covariates of whether sender and receiver have the same title and whether they work in the same department. In total this resulted in 32 covariates. The data were analyzed by models with latent variables of dimension 2, 1, and 0. The R code for the analyses can be found on [www.github.com/jomulder/relationaldata_multiplereceivers](http://www.github.com/jomulder/relationaldata_multiplereceivers).

The posterior distributions for the fixed effects under the model with 0 latent variables can be found in Figure 4. The distributions were similar for the model with 1 and 2 latent variables, and therefore omitted to keep the presentation concise. The results of the posterior predictive checks can be found in Figures 5 and 6. The posterior predictive checks for the relative popularity of the potential receivers show that a higher dimension of the latent variables results in a better fit in general (Figure 5). Furthermore, we see that all models are able to capture the distribution of the number of
receivers over all messages quite well (Figure 6, first row). This was achieved by setting an appropriate prior for the variance of the distribution of the threshold parameters, \( \sigma^2 \). Furthermore, the model with a 2-dimensional latent variable captures the degree of transitivity in the data best. Based on these results we can conclude that a model with a 2-dimensional latent variable results in an acceptable fit to these data.

5 Discussion

A generative model was proposed for analyzing relational events with multiple receivers. For a message by a given sender, the model places all potential receivers on a latent suitability scale. An actor is added to the receiver set when the suitability score exceeds the latent threshold value of the message. For the mean suitability, a multiplicative latent factor structure was considered where the actors in the network have a unique latent variable as a sender, as a receiver, and where each message also receives a unique latent variable. The latent dimensions capture unobserved themes or topics which may affect the behavior of actors to interact with each other. A multiplicative function was considered as it generalizes the popular latent class and the latent distance models. Posterior predictive checks can be used to assess model fit and to determine the number of latent dimensions.

Several extensions would be interesting to consider for future research. First as certain actors may be more inclined to send a message to many actors than other actors, it would be interesting to consider different distributions of the threshold value for different actors. Additionally, instead of a probit model with Gaussian latent variables, a logistic model could also be considered. A possible advantage of the logistic model would be the interpretation of the model parameters. Bayesian computation would become less efficient due to the nonconjugacy of the logistic model however.

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Figure 3: First row: Posterior predictive distributions of the number of receivers (grey lines) and the observed distribution of the number of receivers (black circles). Second row: Posterior predictive distribution of the relative popularity of the actors when actor 1 is sender, ordered by relative popularity in observed data. Third row: Same as second row but for actor 3 as the sender. Fourth row posterior predictive distribution of transitivity ($t_3$), and the transitivity in the observed data (vertical line). The results are shown for models with $Q = 1, 2, \text{ and } 3 \text{ dimensional latent variables.}$
Figure 4: Estimated posterior distributions of the fixed effects for the 32 covariates under a model with 0 latent variables.
Figure 5: Posterior predictive checks for the relative popularity of receiver actors for four different senders (actor 4, 5, 6, and 27 who sent 125, 318, 73, and 150 messages, respective) for a model with 0 (left), 1 (middle), and 2 (right) latent variables.
Figure 6: Posterior predictive checks for the distribution of the number of receivers over all messages (top row), transitivity $t_3$ (middle row), and transitivity $t_4$ (bottom row), for a model with 0 (left), 1 (middle), and 2 (right) latent variables.