A Complex of Incompressible Surfaces for handlebodies and the Mapping Class Group.

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Abstract

For a genus $g$ handlebody $H_g$ a simplicial complex, with vertices being isotopy classes of certain incompressible surfaces in $H_g$, is constructed and several properties are established. In particular, this complex naturally contains, as a subcomplex, the complex of curves of the surface $\partial H_g$. As in the classical theory, the group of automorphisms of this complex is identified with the mapping class group of the handlebody.

1 Definitions and statements of results

For a compact surface $F$, the complex of curves $\mathcal{C}(F)$, introduced by Harvey in [6], has vertices the isotopy classes of essential, non-boundary-parallel simple closed curves in $F$. A collection of vertices spans a simplex exactly when any two of them may be represented by disjoint curves, or equivalently when there is a collection of representatives for all of them, any two of which are disjoint. Analogously, for a 3−manifold $M$, the disk complex $\mathcal{D}(M)$ is defined by using the proper isotopy classes of compressing disks for $M$ as the vertices. It was introduced in [12], where it was used in the study of mapping class groups of 3−manifolds. In [11], it was shown to be a quasi-convex subset of $\mathcal{C}(\partial M)$.

By $H_g$ we denote the 3−dimensional handlebody of genus $g \geq 2$. Recall that a compact connected surface $S \subset H_g$ with boundary is properly embedded if $S \cap \partial H_g = \partial S$ and $S$ is transverse to $\partial H_g$. A compressing disk for $S$ is a properly embedded disk $D$ such that $\partial D$ is essential in $S$. A properly embedded surface $S \subset H_g$ is incompressible if there are no compressing disks for $S$. Recall also that a

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map \( F : S \times [0, 1] \to H_g \) is a proper isotopy if for all \( t \in [0, 1] \), \( F \mid_{S \times \{t\}} \) is a proper embedding. In this case we will be saying that \( F (S \times \{0\}) \) and \( F (S \times \{1\}) \) are properly isotopic in \( H_g \) and we will use the symbol \( \simeq \) to indicate isotopy in all cases (curves, surfaces etc).

**Definition 1** Let \( \mathcal{I} (H_g) \) be the simplicial complex whose vertices are the proper isotopy classes of compressing disks for \( \partial H_g \) and of properly imbedded boundary-parallel incompressible annuli and pairs of pants in \( H_g \). For a vertex \([S]\) which is not a class of compressing disks, it is also required that \( S \) is isotopic to a surface \( \overline{S} \) embedded in \( \partial H_g \) via an isotopy

\[
F : S \times [0, 1] \to H_g
\]

with \( F (S \times \{0\}) = S, F (S \times \{1\}) = \overline{S} \) and \( F \) being proper when restricted to \([0, 1)\). A collection of vertices spans a simplex in \( \mathcal{I} (H_g) \) when any two of them may be represented by disjoint surfaces in \( H_g \).

Note that the class of properly embedded incompressible surfaces in \( H_g \) is very rich. For example, it contains surfaces of arbitrarily high genus (see [13], [3]) which are not included as vertices in the complex \( \mathcal{I} (H_g) \) defined above. Also observe that there exist properly embedded annuli and pairs of pants which are not isotopic to a surface entirely contained in \( \partial H_g \). The isotopy classes of such surfaces are also excluded from the vertex set of \( \mathcal{I} (H_g) \).

Note that we may regard \( \mathcal{D} (H_g) \) as a subcomplex of \( \mathcal{I} (H_g) \) or, by taking boundaries of the representative disks, of \( \mathcal{C} (\partial H_g) \). Note also that the vertices of \( \mathcal{I} (H_g) \) represented by annuli correspond exactly to the vertices of \( \mathcal{C} (\partial H_g) \) represented by curves that are essential in \( \partial H_g \) but are not meridian boundaries. We define the complex of annuli \( \mathcal{A} (H_g) \) to be the subcomplex of \( \mathcal{I} (H_g) \) spanned by these vertices. Together, the vertices of \( \mathcal{D} (H_g) \cup \mathcal{A} (H_g) \) span a copy of \( \mathcal{C} (\partial H_g) \) in \( \mathcal{I} (H_g) \), and we regard \( \mathcal{C} (\partial H_g) \) as a subcomplex of \( \mathcal{I} (H_g) \).

Our goal is to show that for a handlebody \( H_g \) of genus \( g \geq 2 \) the automorphisms of the complex \( \mathcal{I} (H_g) \) are all geometric, that is, they are induced by homeomorphisms of \( H_g \). This can be rephrased by saying that the map

\[
A : \mathcal{MCG} (H_g) \to \text{Aut} (\mathcal{I} (H_g))
\]

is an onto map, where \( \text{Aut} (\mathcal{I} (H_g)) \) is the group of automorphisms of the complex \( \mathcal{I} (H_g) \) and \( \mathcal{MCG} (H_g) \) is the (extended) mapping class group of \( H_g \), i.e. the group of isotopy classes of self-homeomorphisms of \( H_g \). Moreover, we will show (see Theorem 7 below) that the map \( A \) is 1-1 except when \( H_g \) is the handlebody of genus 2 in which case a \( \mathbb{Z}_2 \) kernel is present generated by the hyper-elliptic involution.

For the proof of this result we perform a close examination of links of vertices in \( \mathcal{I} (H_g) \). This examination establishes that an automorphism \( f \) of \( \mathcal{I} (H_g) \) must
map each vertex \( v \) in \( \mathcal{I}(H_g) \) to a vertex \( f(v) \) consisting of surfaces of the same topological type as those in \( v \). In particular, \( f \) induces an automorphism of the subcomplex \( \mathcal{C}(\partial H_g) \) which permits the use of the corresponding result for surfaces (see [7], [9]).

It is a well known result that for genus \( \geq 2 \) the complex of curves \( \mathcal{C}(\partial H_g) \) is a \( \delta \)-hyperbolic metric space in the sense of Gromov (see [10],[2]). In the last section we deduce that the complex \( \mathcal{I}(M) \) is itself a \( \delta \)-hyperbolic metric space in the sense of Gromov. Moreover, it follows that \( \text{Aut}(\mathcal{I}(H_g)) \) does not contain parabolic elements and the hyperbolic isometries of \( \mathcal{I}(M) \) correspond to pseudo-Anosov elements of \( \mathcal{MCG}(H_g) \).

In a recent preprint of M. Korkmaz and S. Schleimer (see [8]), it was shown, in a more general context, that \( \mathcal{MCG}(H_g) \) and \( \text{Aut}(\mathcal{D}(H_g)) \) are isomorphic. Apart from this isomorphism, our motivation for constructing the complex \( \mathcal{I}(H_g) \) is the study of the mapping class group of a Heegaard splitting in a 3-manifold \( M \). This group (originally defined for \( S^3 \) and often called the Goeritz mapping class group) consists of the isotopy classes of orientation preserving homeomorphisms of \( M \) that preserve the Heegaard splitting. The mapping class group of a Heegaard splitting is known to be finitely presented (see [1], [4], [14]) only for \( M = S^3 \) and for a genus 2 Heegaard splitting. We aim to examine the corresponding open questions for \( M = S^3 \) and Heegaard splittings of genus \( \geq 3 \) as well as for certain classes of hyperbolic 3-manifolds. For these purposes, the complex \( \mathcal{I}(H_g) \) is a suitable building block for defining a complex encoding the complexity of the Goeritz mapping class group, because \( \mathcal{I}(H_g) \) contains a copy of the curve complex of the boundary surface \( \partial H_g \).

### 1.1 Notation and terminology

A 3-dimensional handlebody \( H_g \) of genus \( g \) can be represented as the union of a handle of index 0 (i.e. a 3-ball) with \( g \) handles of index 1 (i.e. \( g \) copies of
D^2 \times [0, 1]).

For an essential simple closed curve \( \alpha \) in \( \partial H_g \) we will be writing \([\alpha]\) for its isotopy class and the corresponding vertex in \( C(\partial H_g) \). We will be writing \([S_\alpha]\) for the corresponding vertex in \( A(H_g) \) where \( S_\alpha \) is the annulus corresponding to the curve \( \alpha \), provided that \( \alpha \) is not a meridian boundary. We will be saying that \([S_\alpha]\) is an annular vertex. If \( \alpha \) is a meridian boundary we will be writing \([D_\alpha]\) for the corresponding vertex in \( D(H_g) \). We will be saying that \([D_\alpha]\) is a meridian vertex and \( \alpha \) a meridian curve. A vertex in \( I(H_g) \ \backslash (D(H_g) \cup A(H_g)) \) will be called a pants vertex.

By writing \([\alpha] \cap [\beta] = \emptyset\) for non-isotopic curves \( \alpha, \beta \) we mean that there exist curves \( \alpha' \in [\alpha] \) and \( \beta' \in [\beta] \) such that \( \alpha' \cap \beta' = \emptyset \). By writing \([\alpha] \cap [\beta] \neq \emptyset\) we mean that for any \( \alpha' \in [\alpha] \) and \( \beta' \in [\beta] \), \( \alpha' \cap \beta' \neq \emptyset \). By saying that the class \([\alpha]\) intersects the class \([\beta]\) at one point we mean that, in addition to \([\alpha] \cap [\beta] \neq \emptyset\), there exist curves \( \alpha' \in [\alpha] \) and \( \beta' \in [\beta] \) which intersect at exactly one point.

The above notation with square brackets will be similarly used for surfaces. If \( S \) is an incompressible surface we will denote by \( Lk([S]) \) the link of the vertex \([S]\) in \( I(H_g) \), namely, for each simplex \( \sigma \) containing \([S]\) consider the faces of \( \sigma \) not containing \([S]\) and take the union over all such \( \sigma \). We will use the notation \( \not\cong \) to declare that two links are not isomorphic as complexes.

We will also use the classical notation \( \Sigma_{n,b} \) to denote a surface of genus \( n \) with \( b \) boundary components.

## 2 Properties of the complex \( I(H_g) \)

In this section we will show that every automorphism of \( I(M) \) must preserve the subcomplexes \( A(H_g) \) and \( D(H_g) \). In particular, we will show that for \([S] \in I(H_g)\), the topological type of the surface \( S \) determines the link of \([S]\) in \( I(H_g) \) and vice-versa. To do this we will find topological properties for the link of each
It is well known that a pants decomposition for $\partial H_g$ is a collection $\alpha_1, \ldots, \alpha_{3g-3}$ of $3g-3$ essential, non-parallel, simple closed curves such that the closure of each component of the complement of these curves is a pair of pants. The number of pairs of pants is $2g-2$. Thus, the maximal number of vertices in a simplex of $\mathcal{I}(H_g)$ is $5g-5$. In other words the dimension of $\mathcal{I}(H_g)$ is $\leq 5g-6$. To see that simplices of dimension $5g-6$ actually exist, observe that there exists a pants decomposition $\alpha_1, \ldots, \alpha_{3g-3}$ so that each $\alpha_i$ is a non-separating, non-meridian curve for all $i$. This is displayed in Figure 1 for $g \geq 3$ and for $g = 2$ see Remark 6 below. For such a choice of $\alpha_i$’s, all $2g-2$ pairs of pants formed by $\alpha_1, \ldots, \alpha_{3g-3}$ are incompressible surfaces. Apparently, all such pairs of pants give rise to distinct elements in $\mathcal{I}(H_g)$. Thus, a pants decomposition $\alpha_1, \ldots, \alpha_{3g-3}$ with all $\alpha_i$’s being non-meridian curves gives rise to $3g-3$ annular surfaces $S_{\alpha_1}, \ldots, S_{\alpha_{3g-3}}$. These surfaces along with the $2g-2$ pairs of pants formed by $\alpha_1, \ldots, \alpha_{3g-3}$ give rise to a simplex in $\mathcal{I}(H_g)$ containing $5g-5$ vertices. We have established the following

**Proposition 2** The dimension of the complex $\mathcal{I}(H_g)$ is $5g-6$.

We next examine the dimension of $Lk([D])$ when $D$ is a meridian and of $Lk([S_\alpha])$ when $S_\alpha$ is an annular surface.

**Lemma 3** If $S_\alpha$ is an annular (incompressible) surface then the link of the vertex $[S_\alpha]$ in $\mathcal{I}(H_g)$ has dimension $5g-7$.

**Proof.** We first assume that $\alpha$ is a separating curve. Then $\alpha$ decomposes $\partial H_g$ into surfaces $\Sigma_{n,1}$ and $\Sigma_{m,1}$ with $m+n=g$ and $m,n \geq 1$ with $\alpha$ being isotopic to the boundary of $\Sigma_{n,1}$ as well as to the boundary of $\Sigma_{m,1}$. To complete the proof in this case, it suffices to find a pants decomposition for $\partial H_g$ consisting
of non-meridian curves and containing the curve \( \alpha \). For the latter, it suffices to show the following

**Claim** \( \Sigma_{n,1} \) can be decomposed into \( 2n - 1 \) pairs of pants so that the boundary curves of each are non-meridian when viewed as curves in \( \partial H_g \).

The first step is to find pair-wise disjoint non-separating curves \( \alpha_1, \ldots, \alpha_n \) in \( \Sigma_{n,1} \) such that \( \alpha_i \) does not bound a disk in \( H_g \) for all \( i \). To see this, let \( \alpha_1, \alpha_1' \) be two simple non-separating curves in \( \partial H_g \) such that the curves \( \alpha, \alpha_1, \alpha_1' \) bound a pair of pants in \( \partial H_g \). As \( \alpha \) is not the boundary of a meridian in \( H_g \), it is clear that \( \alpha_1, \alpha_1' \) cannot both be meridian boundaries in \( H_g \). Assuming \( \alpha_1 \) is not meridian boundary, we may cut \( \Sigma_{n,1} \) along \( \alpha_1 \) to obtain a surface \( \Sigma_{n-1,3} \). By the same argument, we may find a non-separating curve \( \alpha_i \) in \( \Sigma_{n-(i-1),2i-1}, \ i = 2, \ldots, n \) which is not meridian boundary.

Apparently, cutting \( \Sigma_{n,1} \) along \( \alpha_1, \ldots, \alpha_n \) we obtain a sphere \( \Sigma_{0,1+2n} \) with 1 + 2\( n \) holes, such that the boundary components of \( \Sigma_{0,1+2n} \) do not bound disks when viewed as curves in \( \partial H_g \). We now claim that we may find pair-wise disjoint curves \( \beta_1, \ldots, \beta_{2n-2} \) such that \( \beta_j \) does not bound a disk in \( H_g \) for all \( j = 1, \ldots, 2n - 2 \).

To see this, let \( \beta_1, \beta_1' \) be two simple closed curves in \( \Sigma_{0,1+2n} \) such that the curves \( \alpha_1, \alpha_2, \alpha_3, \beta_1 \) bound a pair of pants and the curves \( \alpha_1, \alpha_3, \beta_1' \) bound a pair of pants as shown in Figure 3. If both \( \beta_1, \beta_1' \) bound properly embedded disks in \( H_g \), say \( D_{\beta_1}, D_{\beta_1'} \) respectively, then \( D_{\beta_1} \cap D_{\beta_1'} \) is a properly embedded arc in \( H_g \) which separates \( D_{\beta_1} \) into two half-disks. Similarly for \( D_{\beta_1'} \). Appropriate unions of these half-disks along \( D_{\beta_1} \cap D_{\beta_1'} \) establish a contradiction since none of \( \alpha_1, \alpha_2, \alpha_3 \) is a meridian boundary. Thus, at least one of \( \beta_1, \beta_1' \), say \( \beta_1 \), does not bound a disk.

Cutting \( \Sigma_{0,1+2n} \) along \( \beta_1 \) we obtain a pair of pants and a surface \( \Sigma_{0,1+2n-1} \) which has the same property as \( \Sigma_{0,1+2n} \), namely, all boundary components of \( \Sigma_{0,1+2n-1} \) do not bound disks when viewed as curves in \( \partial H_g \). By applying the same argument repeatedly, we may find the desired collection of curves \( \beta_1, \ldots, \beta_{2n-2} \) none of which is a meridian boundary. Apparently, the collection of curves \( \beta_1, \ldots, \beta_{2n-2} \) decomposes \( \Sigma_{0,1+2n} \) into 2\( n - 1 \) pairs of pants as required. This completes the proof of the Claim and the proof of the lemma in the case \( \alpha \) is separating.

Assume now that \( \alpha \) is non-separating. Using two copies of \( \alpha \) and a simple arc joining them we may construct a separating curve \( \beta \) which decomposes \( \partial H_g \) into surfaces \( \Sigma_{g-1,1} \) and \( \Sigma_{1,1} \) with \( \beta \) being isotopic to the boundary of \( \Sigma_{g-1,1} \) as well as to the boundary of \( \Sigma_{1,1} \). Note that \( \Sigma_{1,1} \) contains \( \alpha \). Then by the above claim we have that \( \Sigma_{g-1,1} \) can be decomposed into \( 2(g - 1) - 1 \) (incompressible) pairs of pants by using non-meridian curves \( \alpha_i, \ i = 1, \ldots, 3g - 5 \) contained in \( \Sigma_{g-1,1} \) together with the curve \( \beta \). By adding the curve \( \alpha \) we obtain a pants decomposition \( \alpha_1, \ldots, \alpha_3g-5, \beta, \alpha \) with all curves being non-meridian. Hence, \( [S_\alpha] \) is contained in a simplex of maximum dimension, namely, of dimension \( 5g - 6 \) which shows that the dimension of \( Lk ([S_\alpha]) \) is \( 5g - 7 \).
Lemma 4 If $D$ is a meridian then the link of the vertex $[D]$ in $\mathcal{I}(H_g)$ has dimension $5g - 9$.

Proof. First assume that $[D]$ is non-separating. We may find a pants decomposition $\alpha_1, \ldots, \alpha_{3g-4}, \alpha_{3g-3} = \partial D$ for $\partial H_g$ such that $\alpha_i$ is non-meridian for all $i = 1, \ldots, 3g - 4$ (see Figure 2). This collection of curves decomposes $\partial H_g$ into $2g - 2$ pairs of pants such that exactly two of these have $\partial D$ as boundary component and, hence, they are compressible surfaces. Thus, a non-separating meridian $[D]$ is contained in a simplex with $3g - 3 + 2g - 4$ vertices and, hence, the dimension of $Lk([D])$ is $\geq 5g - 9$. Let now $\alpha'_1, \ldots, \alpha'_{3g-4}, \alpha'_{3g-3} = \partial D$ be any pants decomposition with corresponding pairs of pants $P_1, \ldots, P_{2g-2}$ such that one of them, say $P_1$, has two boundary components isotopic to $\partial D$. Then the third boundary component of $P_1$ will also be a meridian, thus, another pair of pants distinct from $P_1$ will also be compressible. This shows that a class $[D]$ with $D$ non-separating meridian cannot be contained in a simplex of more that $5g - 7$ vertices and, hence, $Lk([D])$ is equal to $5g - 9$.

If $[D]$ is separating, it is clear that any decomposition $\alpha_1, \ldots, \alpha_{3g-4}, \alpha_{3g-3} = \partial D$ for $\partial H_g$ with $\alpha_i$ being non-meridian for all $i = 1, \ldots, 3g - 4$ has the property that exactly two of the corresponding pairs of pants are compressible and we work similarly. \(\blacksquare\)

Proposition 5 Let $[D]$ be a meridian vertex, $[S_\alpha]$ an annular vertex and $[P]$ a pants vertex. Then the links $Lk([D])$, $Lk([S_\alpha])$ and $Lk([P])$ are pair-wise non-isomorphic as complexes.

Proof. By the previous two Lemmata, the links of the vertices $[D]$ and $[S_\alpha]$ have distinct dimensions, hence, it is clear that $Lk([D]) \not\cong Lk([S_\alpha])$. It remains to distinguish $Lk([P])$ from $Lk([D])$ and $Lk([S_\alpha])$.

Let $[P]$ be a vertex in $\mathcal{I}(M)$ such that $P$ is a pair of pants with boundary components $\beta, \gamma, \delta$. The vertices in $Lk([P])$ form a cone graph, that is, the vertex $[S_\beta]$ belongs to $Lk([P])$ and is connected by an edge with any vertex in $Lk([P])$. We will reach a contradiction by showing that

$$\forall [Q] \in Lk([D]), \exists [R] \in Lk([D]) : [Q] \cap [R] \neq \emptyset \tag{\ast}$$

and similarly for $Lk([S_\alpha])$. For, if $\beta_Q$ is a boundary component of a surface representing $[Q] \in Lk([D])$ then there exists a curve $\gamma$ such that $\partial D \cap \gamma = \emptyset$ and $\gamma \cap \beta_Q \neq \emptyset$. Let $[R]$ be the vertex represented by $S_\gamma$ if $\gamma$ is non-meridian and by $D_\gamma$ if $\gamma$ is a meridian boundary. Then $[R] \in Lk([D])$ is the required vertex which is not connected by an edge with $[Q]$, thus $Lk([D])$ satisfies property (\ast). Similarly, we show that $Lk([S_\alpha])$ also satisfies property (\ast). \(\blacksquare\)

Remark 6 Let $\alpha, \beta, \gamma$ be non-separating curves in $\partial H_2$ decomposing $\partial H_2$ into two components which we denote by $P, P'$. Note that $P, P'$ may not be isotopic. To
see this, denote by \( f_1, f_2 \) the generators of \( \pi_1 (H_2) \) corresponding to the longitudes of \( H_2 \). We may choose non-separating curves \( \alpha, \beta \) on \( \partial H_2 \) which represent the second powers \( f_1^2, f_2^2 \) up to conjugacy. Choose an essential non-separating curve \( \gamma \) such that \( \alpha, \beta, \gamma \) are mutually disjoint and non isotopic. These curves separate \( \partial H_2 \) into two components (pairs of pants) \( P \) and \( P' \). If \( P, P' \) were isotopic then \( H_2 \) would be homeomorphic to the product \( P \times [0, 1] \) and any two of the boundary components of \( P \) would give rise to generators for \( \pi_1 (H_2) \). Since neither \( \alpha \simeq f_1^2 \) nor \( \beta \simeq f_2^2 \) are generators for the free group on \( f_1, f_2 \) it follows that, for this particular choice of \( \alpha, \beta, \gamma \), the surfaces \( P, P' \) are not isotopic.

3 Proof of the Main Theorem

Let

\[
A : \text{MCG} (H_g) \to \text{Aut} (\mathcal{I}(H_g))
\]

be the map sending a mapping class \( F \) to the automorphism it induces on \( \mathcal{I}(H_g) \), that is, \( A(F) \) is given by

\[
A(F) [S] := [F(S)].
\]

**Theorem 7** The map \( A : \text{MCG} (H_g) \to \text{Aut} (\mathcal{I}(H_g)) \) is onto for \( g \geq 2 \) and injective for \( g \geq 3 \). For \( g = 2 \), \( A \) has a \( \mathbb{Z}_2 \)−kernel generated by the hyper-elliptic involution.

We will use the following immediate Corollary of Proposition 5.

**Corollary 8** Automorphisms of \( \mathcal{I}(H_g) \) preserve all types (meridian, annular and pants) of vertices.

We will also need the following

**Lemma 9** If \( f \in \text{Aut} (\mathcal{I}(H_g)) \) and \( f|_{\partial(D(H_g))} = \text{id}_{\partial(D(H_g))} \) then \( f([S]) = [S] \) for any vertex \([S] \in \mathcal{I}(M)\) except in the case mentioned in Remark 7, namely, if \( g = 2 \) and \( P \) is a pair of pants with all boundary components of \( \partial P \) being separating curves decomposing \( \partial H_2 \) into 2 components \( P, P' \), then either, \( f([P]) = [P] \) or, \( f([P]) = [P'] \).

**Proof.** We have to show that \( f \in \text{Aut} (\mathcal{I}(H_g)) \) fixes every vertex \([P]\) where \( P \) is a pair of pants. Let \([P]\) be such a vertex in \( \mathcal{I}(H_g) \). By Corollary 8 it is clear that \( f([P]) \) is a vertex \([P']\) with \( P' \) being a pair of pants. Denote by \( \alpha_1, \alpha_2, \alpha_3 \) the boundary components of \( P \) and, similarly, \( \alpha'_1, \alpha'_2, \alpha'_3 \) for \( P' \). If \([\alpha_i_0] \cap [\alpha'_j_0] \neq \emptyset \) for some \( i_0, j_0 \in \{1, 2, 3\} \) then the vertex \([S_{\alpha_i_0}]\) is connected by an edge with \([P]\)
and is not connected by an edge with \([P']\). As \([S_{\alpha_0}]\) is fixed by \(f\), it follows that 
\(f([P])\) cannot be equal to \([P']\). Thus, we may assume that

\[\alpha_i \cap [\alpha'_j] = \emptyset \text{ for all } i, j = 1, 2, 3. \quad (**)\]

Consider the following property:

Up to change of enumeration, \(\alpha_i \simeq \alpha'_i \) for \(i = 1, 2, 3\). \quad (***)

If property (*** ) holds then \(P \simeq P'\) unless \(g = 2\) and \(\alpha_1, \alpha_2, \alpha_3\) are all non-separating curves which decompose \(\partial H_2\) into 2 pairs of pants (cf. Remark 6) which may or may not be isotopic. Thus, if property (*** ) holds then either \(f([P]) = [P]\) or the exception in the statement of the lemma occurs.

We examine now the case where \(g \geq 3\) and property (*** ) does not hold. By assumption (**), we may cut \(\partial H_g\) along \(\alpha_1, \alpha_2, \alpha_3\) to obtain either

- the surface \(P\) and a surface \(\Sigma_{g-2,3}\) (if all \(\alpha_1, \alpha_2, \alpha_3\) are non-separating) or,
- the surface \(P\), a surface \(\Sigma_{g_1,1}\) and a surface \(\Sigma_{g-g_1-1,2}\) for some \(0 < g_1 < g\) (if exactly one of \(\alpha_1, \alpha_2, \alpha_3\) is separating and the other two curves are non-isotopic) or,
- the surface \(P\) and a surface \(\Sigma_{g-1,1}\) (if exactly one of \(\alpha_1, \alpha_2, \alpha_3\) is separating and the other two curves are isotopic) or,
- the surface \(P\) and surfaces \(\Sigma_{g_1,1}, \Sigma_{g_2,1}, \Sigma_{g_3,1}\) for some \(g_1, g_2, g_3 \geq 1\) with \(g_1 + g_2 + g_3 = g\) (if all \(\alpha_1, \alpha_2, \alpha_3\) are separating)

Note that if \(P\) is a pair of pants, it is impossible to have exactly two of its boundary curves \(\alpha_1, \alpha_2, \alpha_3\) being separating. In all cases, \(P'\) is contained in a surface of the form \(\Sigma_{g',b}\) for some \(g' \in \{1, \ldots, g - 1\}\) and \(b \in \{1, 2, 3\}\) mentioned above. Thus, we may find a non-meridian curve \(\alpha\) in \(\partial H_g\) such that

\[\alpha \cap \alpha_i = \emptyset, \forall i = 1, 2, 3\text{ and } [\alpha] \cap [\alpha'_{j_0}] \neq \emptyset\text{ for some } j_0 \in \{1, 2, 3\}.\]

Then, for the annular surface \(S_\alpha\) we have that \([S_\alpha]\) is connected by an edge with \([P]\) and is not connected by an edge with \([P']\). As \([S_\alpha]\) is fixed by \(f\), it follows that \(f([P])\) cannot be equal to \([P']\). This completes the proof of the lemma. \(\blacksquare\)

**Proof of Theorem 7**. We will use the corresponding result for surfaces, see \([7, 9]\), which applies to the boundary of the handlebody \(\partial H_g\).

We first show that every \(f \in Aut(\mathcal{I}(H_g))\) is geometric. By Proposition 5 we know that \(f(\mathcal{A}(H_g)) = \mathcal{A}(H_g)\) and \(f(\mathcal{D}(H_g)) = \mathcal{D}(H_g)\). In particular, \(f(\mathcal{C}(\partial H_g)) = \mathcal{C}(\partial H_g)\). The restriction \(f|_{\mathcal{C}(\partial H_g)}\) of \(f\) on \(\mathcal{C}(\partial H_g)\) induces an automorphism of \(\mathcal{C}(\partial H_g)\) which by the analogous result for surfaces (see \([7, 9]\)) is geometric, that is, there exists a homeomorphism

\[F_{\partial H_g} : \partial H_g \rightarrow \partial H_g\]
such that $A(F_{\partial H_g}) = f|_{\partial(H_g)}$. As $f|_{\partial(H_g)}$ maps $D(M)$ to $D(M)$, $F_{\partial H_g}$ sends meridian boundaries to meridian boundaries. It follows that $F_{\partial H_g}$ extends to a homeomorphism $F : H_g \to H_g$. We know that $A(F) = f$ on $C(\partial H_g)$ and we must show that $A(F) = f$ on $I(H_g)$. This follows from Lemma 9 which completes the proof that every $f \in Aut(I(H_g))$ is geometric.

Let $f \in Aut(I(H_g))$. Since $A$ is shown to be onto, there exists a homeomorphism $F : H_g \to H_g$ such that $A([F]) = f$. This implies that $f(D(H_g)) = D(H_g)$ and $f(A(H_g)) = A(H_g)$. In particular, $f$ restricted to $C(\partial H_g) = D(H_g) \cup A(H_g)$ induces an automorphism $\bar{f}$ of the complex of curves $C(\partial H_g)$. By [7], [9], there exists a homeomorphism $F_{\partial H_g} : \partial H_g \to \partial H_g$ such that $A(F_{\partial H_g}) = \bar{f}$. Such a homeomorphism is unique unless $g = 2$ in which case the map

$$\mathcal{MCG}(\partial H_2) \to Aut(C(\partial H_2))$$

has a $\mathbb{Z}_2$-kernel generated by an involution of $\partial H_2$. However, any homeomorphism of $\partial H_g$ which extends to $H_g$ does so uniquely (see, for example, [5 Theorem 3.7 p.94]), and therefore the map

$$\mathcal{MCG}(H_g) \to Aut(I(H_g))$$

is injective unless $g = 2$ in which case it has a $\mathbb{Z}_2$-kernel.

\section{Applications}

We first establish hyperbolicity for $I(H_g)$.

\begin{proposition}
The complex $I(H_g)$ is $\delta$-hyperbolic in the sense of Gromov.
\end{proposition}

\begin{proof}
As far as hyperbolicity is concerned, the 1-skeleton $I(H_g)^{(1)}$ of $I(H_g)$ is relevant. $I(H_g)^{(1)}$ is endowed with the combinatorial metric so that each edge has length 1. Apparently, we have an embedding

$$i : C(\partial H_g)^{(1)} \hookrightarrow I(H_g)^{(1)}$$

with $i\big(C(\partial H_g)^{(1)}\big) = D(H_g)^{(1)} \cup A(H_g)^{(1)}$ where the superscript $^{(1)}$ always denotes 1-skeleton. We claim that this embedding is isometric. Indeed, if $[\alpha_1], [\alpha_2]$ are distinct vertices with distance $d_C([\alpha_1], [\alpha_2])$ in $C(\partial H_g)^{(1)}$ then the distance $d_I(i([\alpha_1]), i([\alpha_2]))$ cannot be smaller. For, if $[S_0] = i([\alpha_1]), [S_1], \ldots, [S_k] = i([\alpha_2])$ is a sequence of vertices which gives rise to a geodesic in $I(M)^{(1)}$ of length less than $d_C([\alpha_1], [\alpha_2])$, equivalently,

$$d_I(i([\alpha_1]), i([\alpha_2])) = k < d_C([\alpha_1], [\alpha_2])$$

10
then for each $j = 1, 2, \ldots, k - 1$ consider $\beta_j$ to be any boundary component of $S_j$. It is clear that $\beta_j$ is disjoint from $\beta_{j-1}$ and $\beta_{j+1}$. Therefore, the sequence $[\alpha_1], [\beta_1], \ldots, [\beta_{k-1}], [\alpha_2]$ is a segment in $C(\partial H_g)^{(1)}$ of length $k$ with $k < d_C([\alpha_1],[\alpha_2])$, a contradiction.

For any vertex $[P]$ in $I(H_g)^{(1)} \setminus D(H_g)^{(1)} \cup A(H_g)^{(1)}$ we may find an annular vertex, namely, $[S_{\partial P}]$ where $\partial P$ is any component of the boundary of $P$, which is connected by an edge with $[P]$. Thus, $I(H_g)^{(1)}$ is within bounded distance from $i(C(\partial H_g)^{(1)})$. Since $C(\partial H_g)^{(1)}$ is $\delta$–hyperbolic in the sense of Gromov, so is $I(H_g)^{(1)}$. \qed

An element $F \in \mathcal{MCG}(H_g)$ is called pseudo-Anosov when it restricts to a pseudo-Anosov homeomorphism on $\partial H_g$. The proof of the following proposition is immediate from the corresponding result for surfaces (see [10, Prop. 4.6]) along with the above mentioned fact that $C(\partial H_g)^{(1)}$ is cobounded in $I(H_g)^{(1)}$.

**Proposition 11** For any $g \geq 2$, there exists a $c > 0$ such that any pseudo-Anosov $F \in \mathcal{MCG}(H_g)$, any vertex $v \in I(H_g)$ and any $n \in \mathbb{Z}$,

$$d_I(F^n(v), v) \geq c|n|.$$  

Thus, pseudo-Anosov elements in $\mathcal{MCG}(H_g)$ correspond to hyperbolic isometries of $I(H_g)$ and there are no parabolic isometries for $I(H_g)$.

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