EXACT SOLUTIONS OF A BLACK-SCHOLES MODEL WITH TIME-DEPENDENT PARAMETERS BY UTILIZING POTENTIAL SYMMETRIES

Rehana Naz\textsuperscript{a} and Imran Naeem\textsuperscript{b}

\textsuperscript{a} Centre for Mathematics and Statistical Sciences
Lahore School of Economics
Lahore, 53200, Pakistan
\textsuperscript{b} Department of Mathematics
School of Science and Engineering
Lahore University of Management Sciences
LUMS, Lahore Cantt 54792, Pakistan

Abstract. We analyze the local conservation laws, auxiliary (potential) systems, potential symmetries and a class of new exact solutions for the Black-Scholes model time-dependent parameters (BST model). First, we utilize the computer package GeM to construct local conservation laws of the BST model for three different forms of multipliers. We obtain two conserved vectors for the second-order multipliers of form \( \Lambda(x, u, u_x, u_{xx}) \). We define two potential variables \( v \) and \( w \) corresponding to the conserved vectors. We construct two singlet potential systems involving a single potential variable \( v \) or \( w \) and one couplet potential system involving both potential variables \( v \) and \( w \). Moreover, a spectral potential system is constructed by introducing a new potential variable \( p_\alpha \) which is a linear combination of potential variables \( v \) and \( w \). The potential symmetries of BST model are derived by computing the point symmetries of its potential systems. Both singlet potential systems provide three potential symmetries. The couplet potential system yields three potential symmetries and no potential symmetries exist for the spectral potential system. We utilize the potential symmetries of singlet potential systems to construct three new solutions of BST model.

1. Introduction. The Black-Scholes model is one of fundamental model of mathematical finance [21, 22, 3]. This model is expressed as a linear evolutionary partial differential equation (PDE) having variable coefficients. Gazizov and Ibragimov [15] studied this model in Lie symmetry classification perspective and converted it to the heat equation. Edelstein and Govinder [12] established the conservation laws. They also computed potential symmetries and then constructed exact solutions by utilizing the potential symmetries. Tamizhmani et al. [35] analyzed different forms of the Black-Scholes models subject to the terminal condition \( u(x, T) = U \) in Lie symmetry perspective.

The BST model [33] for the value of an option satisfies the following PDE

\[
 u_t + \frac{1}{2} \theta(t)^2 x^2 u_{xx} + r(t) \left( xu_x - u \right) = 0.
\]

1. Introduction. The Black-Scholes model is one of fundamental model of mathematical finance [21, 22, 3]. This model is expressed as a linear evolutionary partial differential equation (PDE) having variable coefficients. Gazizov and Ibragimov [15] studied this model in Lie symmetry classification perspective and converted it to the heat equation. Edelstein and Govinder [12] established the conservation laws. They also computed potential symmetries and then constructed exact solutions by utilizing the potential symmetries. Tamizhmani et al. [35] analyzed different forms of the Black-Scholes models subject to the terminal condition \( u(x, T) = U \) in Lie symmetry perspective.

The BST model [33] for the value of an option satisfies the following PDE

\[
 u_t + \frac{1}{2} \theta(t)^2 x^2 u_{xx} + r(t) \left( xu_x - u \right) = 0.
\]
Naz and Johnpillai [23], used the invariant technique [20] for the scalar linear 
(1 + 1) parabolic PDE and attained two different sets of equivalence transfor-
mations. They have shown that with the use of equivalence transformations, the 
BST model reduced to the heat equation provided \( r(t) = \frac{1}{2} \theta(t)^2, \) \( \theta(t) = \frac{1}{t} \). For \( r(t) = \frac{1}{2} \theta(t)^2, \) \( \theta(t) = \frac{1}{t} \), the Black Scholes equation (1) takes the following form:

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \theta(t)^2 (x^2 u_{xx} + x u_x - u) = 0. \quad (2)
\]

In this article, we will establish conservation laws, potential symmetries and exact 
solutions via potential symmetries for the BST model given in equation (2).

Several systematic approaches have been developed for the construction of conser-
vation laws/first integrals (see, e.g., [4, 16, 24, 25] and references therein). Recently, 
Cheviakov and Naz [5] provided a recursion formula to establish the local conserva-
tion laws of differential equations. One can also utilize the computer programmes 
in REDUCE by Wolf [37, 38], GeM on Maple by Cheviakov [10, 11] and SADE 
on Maple by Filho [32] for computation of conservation laws/first integrals. Re-
cently, an elegant approach known as partial Hamiltonian approach [26, 27, 28, 29] 
has been developed for derivation of first integrals of optimal control problems and 
dynamical systems.

The Lie symmetries [19] are important to construct solutions of partial differen-
tial equations (PDEs). New solutions of existing PDEs can be found by utilizing the 
non-local symmetries. Krasil’Shchik and Vinogradov [17, 18] introduced non-local 
symmetries through their works on “covering” systems. The non-local symmetries 
of a PDE are local symmetries of a system of PDEs which “covers” the given PDE or 
auxiliary “covering” system [13]. Bluman et al. [6] and Bluman [7] developed 
an algorithm to derive a new class of solutions for a given PDE via non-local sym-
metries. This class of new solutions contains the invariant solutions obtained by 
utilizing point symmetries of the auxiliary system [31]. The auxiliary system was 
constructed by introducing a variable which is now known as a potential variable. 
For the heuristic approaches, the algorithmic framework for the construction of non-
local symmetries and non-local conservation laws was developed by Akhatov et al. 
[1]. The elegant generation theorem for the non-local symmetries was developed by 
Sjöberg and Mahomed [34]. Gandarias [14] studied new potential symmetries for 
some evolution equations. For different types of diffusion-convection equations, the 
(hierarchical) trees of inequivalent potential systems were constructed by Popovych 
and Ivanova [30]. An algorithmic framework for potential systems and non-local 
symmetries is provided in [8, 9].

The non-trivial conservation laws of a PDE can be utilized to construct the 
potential systems. Consider a second-order partial differential equation (PDE) (See 
e.g. [6, 14])

\[
F(t, x, U, U_t, U_x, U_{tt}, U_{xx}, U_{tx}) = 0, \quad (3)
\]

where \( t, x \) are independent variables and the dependent variable is denoted by \( u \). 
Suppose system (3) has \( n \) local conservation laws of form

\[
D_t T^t_i (t, x, U, U_t, U_x) + D_x T^x_i (t, x, U, U_t, U_x) = 0, \quad (4)
\]

where \( T^t_i \) and \( T^x_i \) are differentiable functions of \( t, x, U, U_t, U_x \). The total derivative 
operators \( D_t \) and \( D_x \) are defined by

\[
D_t = \frac{\partial}{\partial t} + U_t \frac{\partial}{\partial U_t} + U_{tt} \frac{\partial}{\partial U_t} + U_{tx} \frac{\partial}{\partial U_x} + \cdots,
\]
D_x = \frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + U_{xx} \frac{\partial}{\partial U_x} + U_{tx} \frac{\partial}{\partial U_t} + \cdots.

One can define potential variables $V_i$ corresponding to each conservation law (4) of the PDE (3) and thus arrive at the potential equations of following form:

$$(V_i)_x = T^i_t (t, x, U, U_t, U_x)$$

$$(V_i)_t = -T^i_t (t, x, U, U_t, U_x).$$

The potential equations (5) associated with each conservation law (4) can be adjoined to the PDE (3) to obtain a non-locally related potential system [8, 9].

The paper is organized in the following manner: the Lie symmetries of BST model are derived in Section 2. The conservation laws and potential systems are constructed in Section 3. The potential symmetries are computed in Section 4. Exact solutions of BST model via potential symmetries are constructed in Section 5. The conclusions are given in Section 6.

2. The Lie symmetries of BST model. In this section, we compute the Lie symmetries of BST model. Equation (2) can be re-written in extended Kovalevskaya form with respect to $u_t$ as follows:

$$U\{t, x; u\} : u_t = \frac{1}{2t^2} (u - x^2 u_{xx} - xu_x).$$

The determining equations for point symmetries of Eq. (6) are obtained by utilizing Maple based program DESOLVII [36]:

$$\xi^1_x = 0, \xi^1_u = 0, \xi^2_u = 0, \eta_{uu} = 0,$$

$$\xi^2 + x^2 \xi^2_{xx} + 2t^2 \xi^2_t - x^2 \eta_{xu} = 0,$$

$$(4t^2 x \xi^2_x - 4t^2 \xi^2 - 2t^2 x \xi^1_x - x^3 \xi^1_{xx} + 4tx \xi^1 - x^2 \xi^1_x = 0),$$

$$-2xu \xi^2_x + x^3 \eta_{xx} + xu \eta_u + x^2 \eta_x + 2u \xi^2 - x \eta^1 + 2t^2 x \eta_t = 0,$$

where $\xi^1, \xi^2$ and $\eta$ are functions of variables $t, x, u$.

The solution of undetermined system of PDEs (7) yields the following Lie point symmetries:

$$X_1 = t^2 \frac{\partial}{\partial t},$$

$$X_2 = u \frac{\partial}{\partial u},$$

$$X_3 = x \frac{\partial}{\partial x},$$

$$X_4 = x \frac{\partial}{\partial x} - u \ln(x) \frac{\partial}{\partial u},$$

$$X_5 = 2 \frac{\partial}{\partial t} - \frac{2x \ln(x)}{t} \frac{\partial}{\partial x} + \frac{tu + u + t^2 u \ln(x)^2}{t^2} \frac{\partial}{\partial u},$$

$$X_6 = 2t \frac{\partial}{\partial t} - x \ln(x) \frac{\partial}{\partial x} + \frac{u}{t} \frac{\partial}{\partial u},$$

$$X_\infty = F_1(t, x) \frac{\partial}{\partial u},$$

where $F_1(t, x)$ satisfies

$$-F_1 + x^2 F_{1xx} + xF_{1x} + 2t^2 F_{1t} = 0.$$
3. Conservation laws and potential systems of BST model. In this section, we derive the conservation laws of (6) utilizing the computer package GeM \cite{10, 11} and then we utilize these to construct the potential systems. Without loss of generality, the leading derivative \( u_t \) of (6) and its differential consequences can be omitted from the multiplier dependence (see Lemma 3 \cite{2}). We assume second-order multiplier of form \( \Lambda(t, x, u, u_x, u_{xx}) \). The most efficient Maple-based package GeM \cite{10, 11} is utilized to compute the conservations laws of (6) with a multiplier of form \( \Lambda(t, x, u, u_x, u_{xx}) \). We have the following result:

**Proposition 1.** The linear space of inequivalent non-trivial local conservation laws of the PDE (6) arising from

(a) the second-order multipliers \( \Lambda(t, x, u, u_x, u_{xx}) \) is spanned by infinite many conserved vectors with infinite many multipliers \( \Lambda = F(t, x) \) satisfying

\[
F_{xx} + \frac{3xF_x - 2t^2F_t}{x^2} = 0 \tag{9}
\]

(b) the second-order multipliers \( \Lambda(x, u, u_x, u_{xx}) \) is spanned by the two conserved vectors

\[
T^t = u, \quad T^x = -\frac{1}{2t^2}x(u - xu_x), \tag{10}
\]
\[
T^t = \frac{u}{x^2}, \quad T^x = \frac{u + xu_x}{2xt^2}, \tag{11}
\]

corresponding to zeroth-order multipliers

\[
\Lambda^1 = 1, \quad \Lambda^2 = \frac{1}{x^2}. \tag{12}
\]

(c) the second-order multipliers \( \Lambda(t, u, u_x, u_{xx}) \) is spanned by the only one conserved vector (10) corresponding to multiplier \( \Lambda = 1 \).

***Proof.*** The results follow by direct computations on Maple-based package GeM \cite{10, 11}.

One can introduce two potential variables and construct potential systems, as follows. We introduce the first potential variable \( v \) corresponding to the conserved vector (10), and arrive at a singlet potential system:

\[
\text{UV}(t, x; u, v) : \begin{cases} v_x = u, \\ v_t = -\frac{x^2}{4t^2}u_x + \frac{x}{2t}u. \end{cases} \tag{13}
\]

Defining the potential variable \( w \) corresponding to conserved vector given in equation (11) to obtain following second singlet potential system:

\[
\text{UW}(t, x; u, w) : \begin{cases} w_x = \frac{u}{x^2}, \\ w_t = -\frac{u + xu_x}{2xt^2}. \end{cases} \tag{14}
\]

The singlet potential systems (13) and (14) involving a single non-local variable \( v \) or \( w \) are non-locally related to the BST model (6). With the aid of both potentials \( v \) and \( w \), we obtain a couplet potential system

\[
\text{UVW}(t, x; u, v, w) : \begin{cases} \begin{align*}
v_x &= u, \\
v_t &= -\frac{x^2}{4t^2}u_x + \frac{x}{2t}u, \\
w_x &= \frac{u}{x^2}, \\
w_t &= -\frac{u + xu_x}{2xt^2}. \end{align*} \end{cases} \tag{15}
\]
The couplet potential system (15) involving both non-local variables \( v \) and \( w \) is non-locally related to the BST model (6). We take the linear combination of the conserved vectors (10) and (11) with factors 1 and \( \alpha \in \mathbb{R} \setminus \{0\} \)

\[
T^t = u + \alpha \frac{u}{x^2}, \quad T^x = -\frac{1}{2t^2} x(u - xu_x) + \alpha \left( \frac{u + xu_x}{2xt^2} \right),
\]

to construct a spectral potential system

\[
\mathbf{UP}_\alpha \{t, x; u, p_\alpha\} : \begin{cases}
(p_\alpha)_x &= u + \alpha \frac{u}{x^2}, \\
(p_\alpha)_t &= -\frac{x^2}{2t^2} u_x + \frac{x}{2t^2} u - \alpha \left( u + xu_x \right),
\end{cases}
\]

the variable \( p_\alpha \) is a local function of potential variables \( v \) and \( w \):

\[
p_\alpha = v + \alpha w + a,
\]

where \( a \) is arbitrary constant. It is important to mention here that the potential variable \( p_\alpha \) is not a non-local variable on solutions of the couplet potential system (15). The spectral potential system (17) involving non-local variable \( p_\alpha \) is non-locally related to the BST model (6).

4. Potential symmetries of BST model. In this section, we compute the potential symmetries of BST model (6). The potential symmetries of BST model (6) can be established by computing point symmetries of its potential systems. The MAPLE package DESOLVII [36] is utilized for computation of point symmetries of the potential systems.

4.1. Potential symmetries of potential system \( \mathbf{UV}\{t, x; u, v\} \). The Maple based program DESOLVII [36] yields the following Lie symmetries of the potential system (13):

\[
\begin{align*}
X_1 &= t^2 \frac{\partial}{\partial t}, \\
X_2 &= x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \\
X_3 &= u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \\
X_4 &= \frac{x}{t} \frac{\partial}{\partial x} - (u \ln(x)) \frac{\partial}{\partial u} + (v \frac{\partial}{\partial v} - v \frac{\partial}{\partial t}), \\
X_5 &= 2t \frac{\partial}{\partial t} - \frac{2x \ln(x)}{t} \frac{\partial}{\partial x} + (u \ln^2(x) + \frac{2v \ln(x)}{x} + \frac{3u}{t} \ln(x) + \frac{2v}{tx} + \frac{u}{t^2}) \frac{\partial}{\partial u} \\
&\quad + (v \ln^2(x) - \frac{2v \ln(x)}{t} + \ln^2(x) + \frac{v}{t} \ln(x) + \frac{v}{t^2}) \frac{\partial}{\partial v}, \\
X_6 &= 2t \frac{\partial}{\partial t} - x \ln(x) \frac{\partial}{\partial x} + (u + \frac{u}{t} - \frac{v}{x} \frac{\partial}{\partial u} + (v \frac{\partial}{\partial v} + \frac{v}{t} \frac{\partial}{\partial t}), \\
X_7 &= \frac{F_x(t, x)}{t^2} \frac{\partial}{\partial u} + \frac{F(t, x)}{t^2} \frac{\partial}{\partial v}.
\end{align*}
\]

In (19), the symmetries \( X_4-X_6 \) are the Lie symmetries of potential system (13) but they become potential symmetries of the BST model (6) as the coefficient of original generator involve the potential variable \( v \).
4.2. Potential symmetries of potential system \(UW\{t,x;u,w\}\). We obtain following Lie symmetries of potential system (14) with the aid of Maple based program DESOLVII [36]:

\[
Y_1 = t^2 \frac{\partial}{\partial t},
Y_2 = x \frac{\partial}{\partial x} + \frac{\partial}{\partial u},
Y_3 = u \frac{\partial}{\partial u} + w \frac{\partial}{\partial w},
Y_4 = \frac{x}{t} \frac{\partial}{\partial x} - \left[u \ln(x) + wx\right] \frac{\partial}{\partial u} - \left[w \ln(x) + \frac{w}{t} \frac{\partial}{\partial w}\right],
Y_5 = 2 \frac{\partial}{\partial t} - \frac{2x \ln(x)}{t} \frac{\partial}{\partial x} + \left(u \ln^2(x) + 2xw \ln(x) + \frac{2xw}{t} + \frac{3u}{t} + \frac{u}{t^2} \frac{\partial}{\partial u}\right)\frac{\partial}{\partial u} - \left(w \ln^2(x) + \frac{2w \ln(x)}{t} + \frac{w}{t} + \frac{w}{t^2} \frac{\partial}{\partial w}\right)\frac{\partial}{\partial w},
Y_6 = 2 \frac{\partial}{\partial t} - x \ln(x) \frac{\partial}{\partial x} + \left(u + xw + \frac{u}{t}\right) \frac{\partial}{\partial u} + \left(w \ln(x) + \frac{w}{t}\right) \frac{\partial}{\partial w},
Y_7 = \frac{x^2}{t^2} F_s(t,x) \frac{\partial}{\partial u} + \frac{1}{t^2} F(t,x) \frac{\partial}{\partial w}.
\]

In (20), the symmetries \(Y_4 - Y_6\) are the Lie symmetries of potential system (14) but they are potential symmetries of the BST model (6) as the coefficient of original generator involve the potential variables \(w\).

4.3. Potential symmetries of potential system \(UVW\{t,x;u,v,w\}\). Next, we derive the potential symmetries of potential system \(UVW\{t,x;u,v,w\}\) (14). The Lie point symmetries of potential system (15) are obtained by utilizing Maple based program DESOLVII [36]:

\[
Z_1 = t^2 \frac{\partial}{\partial t},
Z_2 = x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} - 2w \frac{\partial}{\partial w},
Z_3 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w},
Z_4 = -2t \frac{\partial}{\partial t} + \left[x \ln(x) + \frac{x}{t} \frac{\partial}{\partial x}\right] - \left[u \ln(x) + 2wx + 2u + \frac{u}{t} \frac{\partial}{\partial u}\right] - \left[2w \ln(x) + w + \frac{2w}{t} \frac{\partial}{\partial w}\right],
Z_5 = (-4t + x \ln(x) - \frac{2x \ln(x)}{t} \frac{\partial}{\partial x} + \left[u \ln^2(x) + 2xw \ln(x) - xw + \frac{2xw}{t} - 3u + \frac{2v \ln(x)}{x} + \frac{3u}{t} + \frac{v}{x} + \frac{2v}{tx} + \frac{u}{t^2} \frac{\partial}{\partial u}\right] \frac{\partial}{\partial u} + \left[w \ln^2(x) - \frac{2w \ln(x)}{t} + \frac{w}{t} + \frac{w}{t^2} \frac{\partial}{\partial w}\right] \frac{\partial}{\partial w},
Z_6 = -4t \frac{\partial}{\partial t} + 2x \ln(x) \frac{\partial}{\partial x} + \left[-3u - 2xw - \frac{2u}{t} + \frac{2v}{x} \frac{\partial}{\partial u}\right].
\]
\[ +[-wx^2 + 2v \ln(x) - \frac{2v}{t} \frac{\partial}{\partial v} - [2w \ln(x) + \frac{2w}{t} + \frac{v}{x^2}] \frac{\partial}{\partial w}, \]

\[ Z_7 = F_x(t, x) \frac{\partial}{\partial u} + F(t, x) \frac{\partial}{\partial v}. \]

In (21), the symmetries \( Z_4 - Z_6 \) are the Lie symmetries of potential system (15) but they are potential symmetries of the BST model (6) as the coefficient of original generator involve the potential variables \( v \) and \( w \).

4.4. **Potential symmetries of potential system** \( \text{UP}_\alpha \{t, x ; u, p_\alpha\} \). All symmetries of the spectral potential system \( \text{UP}_\alpha \{t, x ; u, p_\alpha\} \) project on the point symmetries of the BST model (6).

5. **Exact solutions of BST model via potential symmetries.** Now, using potential symmetries result from potential system \( \text{UV}_\{t, x ; u, v\} \) (13) and potential system \( \text{UW}_{\{t, x ; u, w\}} \) (14), we compute new exact solutions of BST model (6).

5.1. **Exact solutions via potential symmetries of potential system** \( \text{UV}_\{t, x ; u, v\} \). We utilize potential symmetries \( X_4, X_5 \) and \( X_6 \) of potential system (13) to establish exact solutions of BST model (6).

5.1.1. **Exact solutions via potential symmetry** \( X_4 \). The similarity variables corresponding to the potential symmetry \( X_4 \) give rise to

\[ t = z, x = r, \]

\[ u = [-zk_1(z) \ln(r) + k_2(z)]e^{-\frac{1}{2} \ln^2(r)}, \]

\[ v = k_1(z)re^{-\frac{1}{2} \ln^2(r)}. \]  

Using the change of variables (22), and after some simplification, Eq. (6) reduces to

\[ z^3 \left( \frac{d}{dz} k_1 \right) \ln(r) - \left( \frac{d}{dz} k_2 \right) z^2 - \frac{1}{2} (z + 1) (zk_1 \ln(r) - k_2) = 0. \]  

The separation of Eq. (23) with respect to \( \ln(r) \) yields

\[ z^3 \frac{d k_1}{dz} - \frac{1}{2} z (z + 1) k_1 = 0, \]

\[ z^2 \frac{d k_2}{dz} - \frac{1}{2} (z + 1) k_2 = 0. \]  

The solution of system (24) provides

\[ k_1(z) = c_1 \sqrt{z} e^{-\frac{z}{4}}, \]

\[ k_2(z) = c_2 \sqrt{z} e^{-\frac{z}{4}}. \]  

Substituting the values of \( k_1(z) \) and \( k_2(z) \) in (22) and using \( z = t \) and \( r = x \), we obtain the following exact solution of Eq. (6)

\[ u(t, x) = -\sqrt{t} e^{-\frac{t}{4}} (tc_1 \ln(x) - c_2) e^{-\frac{1}{2} t \ln(x)^2}. \]
5.1.2. Exact solutions via potential symmetry $X_5$. The similarity transformations for the potential symmetry $X_5$ are

\[ t = z \ln(r), \quad x = r, \]
\[ u = \left[ -zk_1(z)\ln(r) + k_1(z)\ln(r) + k_2(z) \right] r^{-\frac{1}{2} z - \frac{1}{2}}, \]
\[ (\ln(r))^\frac{1}{2}, \]
\[ v = k_1(z) r^{-\frac{1}{2} z + 1 - \frac{d}{r}}, \]
\[ \sqrt{\ln(r)}, \]
\[ (27) \]

and the closed form solution of (6) is

\[ u(t, x) = -\sqrt{t \ln(x) x^{-\frac{1}{4} t \ln(x) - \frac{1}{2} \ln(r)}} (3c_4 + 3t \ln(x) c_3 + 3t^2 \ln^2(x)c_2 \]
\[ + t^3 \ln^3(x)c_1 - 3t^2 \ln(x)c_1 - 3tc_2). \]
\[ (28) \]

5.1.3. Exact solutions via potential symmetry $X_6$. In a similar fashion, one can use the potential symmetry $X_6$ to compute the similarity transformations

\[ t = \frac{z}{\ln^2(r)}, \quad x = r, \]
\[ u = \left[ k_1(z)\ln(r) + k_2(z) \right] e^{-\frac{\ln^2(r)}{2}}, \]
\[ v = k_1(z) r^{-\frac{1}{2} z + \frac{1}{4} \ln^2(r) - \frac{1}{4} \ln^2(r), \]
\[ \sqrt{\ln(r)}, \]
\[ (29) \]

With the aid of similarity transformations (29), Eq. (6) transforms to

\[ \left[ \left( \frac{d k_1}{dz} \right) z + \left( \frac{d k_1}{dz} \right) z^2 + 2 \left( \frac{d^2 k_1}{dz^2} \right) z^2 \right] \ln(r) - \left( \frac{d k_2}{dz} \right) z \]
\[ + \left( \frac{d k_2}{dz} \right) z^2 + k_2 + 2 \left( \frac{d^2 k_2}{dz^2} \right) z^2 = 0. \]
\[ (30) \]

Since $k_1$ and $k_2$ are functions of $z$, hence, separation of Eq. (30) with respect to $\ln(r)$ gives

\[ \left( \frac{d k_1}{dz} \right) z + \left( \frac{d k_1}{dz} \right) z^2 + 2 \left( \frac{d^2 k_1}{dz^2} \right) z^2 = 0, \]
\[ - \left( \frac{d k_2}{dz} \right) z + \left( \frac{d k_2}{dz} \right) z^2 + k_2 + 2 \left( \frac{d^2 k_2}{dz^2} \right) z^2 = 0. \]
\[ (31) \]

The solution of system (31) results in

\[ k_1(z) = c_1 + \text{erf} \left( \frac{1}{2} \sqrt{2z} \right) c_2, \]
\[ k_2(z) = c_2 \sqrt{\pi} \text{MeijerG} \left( \frac{1}{4}, 1; \frac{1}{2}, \frac{1}{2}; z \right) e^{-\frac{1}{2} z} + c_2 \sqrt{\pi} e^{-\frac{1}{2} z}. \]
\[ (32) \]

Replacing the values of $k_1(z)$ and $k_2(z)$ from (32) in (29) yields another exact solution of the BST model (6).
\[ u(t, x) = e^{-\frac{t}{\ln(x)}} \left( c_1 + erf(\sqrt{\frac{t \ln^2(x)}{2}}) c_2 \right) \ln(x) + c_2 (t \ln^2(x))^\frac{1}{4} \times M \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{2} t \ln^2(x) \right) e^{-\frac{1}{2} t \ln^2(x)} + c_2 \left( t \ln^2(x) e^{-\frac{1}{4} t \ln^2(x)} \right). \]  

(33)

We have established three new solutions given in (26), (28), (33) for the BST model (6) by utilizing the potential symmetries. These solutions are not reported in Naz and Johnpillai [23].

5.2. Exact solutions via potential symmetries of potential system \( U_W \{ t, x; u, w \} \). Now, we compute the exact solutions of BST model by utilizing the potential symmetries \( Y_4 - Y_6 \) of potential system \( U_W \{ t, x; u, w \} \) (14). The similarity variables corresponding to the potential symmetry \( Y_4 \) give rise to

\[ t = z, \quad x = r, \]

\[ u = [-zk_1(z) \ln(r) + k_2(z)] e^{-\frac{1}{4} \ln^2(r)}, \]

\[ w = \frac{k_1(z)}{r} e^{-\frac{1}{4} \ln^2(r)}. \]  

(34)

Using similarity transformations (34), Eq. (6) is reduced to second order ODE which upon solving yields \( k_1(z) \) and \( k_2(z) \). The substitution of \( k_1(z) \) and \( k_2(z) \) in (34) result in the exact solution of (6) as

\[ u(t, x) = -\sqrt{t} e^{-\frac{t}{\ln(x)}} \left( t c_1 \ln(x) - c_2 \right) e^{-\frac{1}{4} t (\ln(x))^2}. \]  

(35)

Note that the solution obtained in (35) is exactly the same solution obtained in (26). Similarly, using potential symmetries \( Y_5 \) and \( Y_6 \) from (20), we find same solutions presented in (28) and (33).

6. Conclusions. We derived the local conservation laws, potential systems, potential symmetries and solutions of the BST model for the value of an option. The computer package GeM gave two local conservation laws of BST model. We defined two potential variables \( v \) and \( w \) corresponding to first and second conserved vectors. We established two singlet potential systems involving a single potential variable \( v \) or \( w \) and one couplet potential system involving both potential variables \( v \) and \( w \). Moreover, a spectral potential system was constructed by introducing a new potential variable \( p_\alpha \) which was a linear combination of potential variables \( v \) and \( w \). The potential symmetries of BST model were derived by computing the point symmetries of its potential systems. The singlet potential system potential system \( U_V \{ t, x; u, v \} \) provided three potential symmetries. The second singlet potential system potential system \( U_W \{ t, x; u, w \} \) also provided three potential symmetries. The couplet potential system \( U_{WW} \{ t, x; u, v, w \} \) yields three potential symmetries. All symmetries of the spectral potential system \( U_{P_\alpha} \{ t, x; u, p_\alpha \} \) project on the point symmetries of the BST model. We utilize the potential symmetries of potential system \( U_V \{ t, x; u, v \} \) to construct three new solutions of BST model. The potential symmetries of potential system \( U_W \{ t, x; u, w \} \) yield same three solutions as established by utilizing potential system \( U_V \{ t, x; u, v \} \).
REFERENCES

[1] I. S. Akhatov, R. K. Gazizov and N. K. Ibragimov, Nonlocal symmetries: A heuristic approach, *Journal of Soviet Mathematics*, 55 (1991), 1401–1450.

[2] L. Martín Almoro, On the Noether map, *Letters in Mathematical Physics*, 3 (1979), 419–424.

[3] F. Black and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy*, 81 (1973), 637–654.

[4] G. W. Bluman, A. F. Cheviakov and S. C. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Applied Mathematical Sciences, 168. Springer, New York, 2010.

[5] A. F. Cheviakov and R. Naz, A recursion formula for the construction of local conservation laws of differential equations, *Journal of Mathematical Analysis and Applications*, 448 (2017), 198–212.

[6] G. W. Bluman, G. J. Reid and S. Kumei, New classes of symmetries for partial differential equations, *Journal of Mathematical Physics*, 29 (1988), 806–811.

[7] G. Bluman, Use and construction of potential symmetries, *Mathematical and Computer Modelling*, 18 (1993), 1–14.

[8] G. Bluman and A. F. Cheviakov, Framework for potential systems and non-local symmetries: Algorithmic approach, *Journal of Mathematical Physics*, 46 (2005), 123506, 19 pp.

[9] G. Bluman, A. F. Cheviakov and N. M. Ivanova, Framework for nonlocally related partial differential equation systems and nonlocal symmetries: Extension, simplification, and examples, *Journal of Mathematical Physics*, 47 (2006), 113505, 23 pp.

[10] A. F. Cheviakov, GeM software package for computation of symmetries and conservation laws of differential equations, *Computer Physics Communications*, 176 (2007), 48–61.

[11] A. F. Cheviakov, Computation of fluxes of conservation laws, *Journal of Engineering Mathematics*, 66 (2010), 153–173.

[12] R. M. Edelstein and K. S. Govinder, Conservation laws for the Black-Scholes equation, *Nonlinear Analysis: Real World Applications*, 9 (2008), 412–417.

[13] I. S. Krasil’shchik and A. M. Vinogradov, Nonlocal symmetries and the theory of coverings: An addendum to Vinogradov’s “Local symmetries and conservation laws”, *Acta Applicandae Mathematica*, 2 (1984), 79–96.

[14] I. S. Krasil’shchik and A. M. Vinogradov, Nonlocal trends in the geometry of differential equations: Symmetries, conservation laws, and Bäklund transformations, *Acta Applicandae Mathematica*, 15 (1989), 161–209.

[15] S. Lie, On integration of a class of linear partial differential equations by means of definite integrals Archiv for Matematik og Naturvidenskab, *Skrifter Meddelelser*, 6 (1881), 328–368.

[16] F. M. Mahomed, Complete invariant characterization of scalar linear (1 + 1) parabolic equations, *J. Nonlinear Math. Phys.*, 15 (2008), 112–123.

[17] R. C. Merton, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory*, 3 (1971), 373–413.

[18] R. C. Merton, Theory of rational option pricing, *The Bell Journal of Economics and Management Science*, 4 (1973), 141–183.

[19] R. Naz and A. G. Johnpillai, Exact solutions via invariant approach for Black-Scholes model with time-dependent parameters, *Mathematical Methods in the Applied Sciences*, 41 (2018), 4417–4427.

[20] R. Naz, F. M. Mahomed and D. P. Mason, Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics, *Applied Mathematics and Computation*, 205 (2008), 212–230.
R. Naz, I. L. Freire and I. Naeem, Comparison of different approaches to construct first integrals for ordinary differential equations, *Abstract and Applied Analysis*, (2014), Art. ID 978636, 15 pp.

R. Naz, F. M. Mahomed and A. Chaudhry, A partial Hamiltonian approach for current value Hamiltonian systems, *Communications in Nonlinear Science and Numerical Simulation*, 19 (2014), 3600–3610.

R. Naz, A. Chaudhry and F. M. Mahomed, Closed-form solutions for the Lucas zawa model of economic growth via the partial Hamiltonian approach, *Communications in Nonlinear Science and Numerical Simulation*, 30 (2016), 299–306.

R. Naz, The applications of the partial Hamiltonian approach to mechanics and other areas, *International Journal of Non-Linear Mechanics*, 86 (2016), 1–6.

R. Naz and I. Naeem, The artificial hamiltonian, first integrals, and closed-form solutions of dynamical systems for epidemics, *Zeitschrift f Naturforschung A*, 73 (2018), 323–330.

R. O. Popovych and N. M. Ivanova, Hierarchy of conservation laws of diffusion-convection equations, *Journal of Mathematical Physics*, 46 (2005), 043502, 22 pp.

E. Pucci and G. Saccomandi, Potential symmetries and solutions by reduction of partial differential equations, *Journal of Physics A: Mathematical and General*, 26 (1993), 681–690.

T. M. Rocha Filho and A. Figueiredo, [SADE] a Maple package for the symmetry analysis of differential equations, *Computer Physics Communications*, 182 (2011), 467–476.

M. R. Rodrigo and R. S. Mamon, An alternative approach to solving the Black choles equation with time-varying parameters, *Applied Mathematics Letters*, 19 (2006), 398–402.

A. Sjöberg and F. M. Mahomed, Non-local symmetries and conservation laws for one-dimensional gas dynamics equations, *Applied Mathematics and Computation*, 150 (2004), 379–397.

K. M. Tamizhmami, K. Krishnakumar and P. G. L. Leach, Algebraic resolution of equations of the Black-Scholes type with arbitrary time-dependent parameters, *Applied Mathematics and Computation*, 247 (2014), 115–124

K. T. Vu, G. F. Jefferson and J. Carminati, Finding higher symmetries of differential equations using the MAPLE package DESOLVII, *Computer Physics Communications*, 183 (2012), 1044–1054.

T. Wolf, A comparison of four approaches to the calculation of conservation laws, *European Journal of Applied Mathematics*, 13 (2002), 129–152.

T. Wolf, A. Brand and M. Mohammadzadeh, Computer algebra algorithms and routines for the computation of conservation laws and fixing of gauge in differential expressions, *Journal of Symbolic Computation*, 27 (1999), 221–238.

Received January 2019; revised May 2019.

E-mail address: drrehana@lahoreschool.edu.pk

E-mail address: Imran.naeem@lums.edu.pk