Zitterbewegung with spin–orbit coupled ultracold atoms in a fluctuating optical lattice

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Abstract
The dynamics of non-interacting ultracold atoms with artificial spin–orbit coupling is considered. Spin–orbit coupling is created using two moving optical lattices with orthogonal polarizations. Our main goal is to study influence of lattice noise on Rabi oscillations. Special attention is paid to the phenomenon of the Zitterbewegung being trembling motion caused by Rabi transitions between states with different velocities. Phase and amplitude fluctuations of lattices are modelled by means of the two-dimensional stochastic Ornstein-Uhlenbeck process, also known as harmonic noise. It is shown that lattice noise significantly extends duration of the Zitterbewegung as compared to the noiseless case. This effect originates from noise-induced decoherence of Rabi oscillations.

Keywords: ultracold atoms, optical lattice, spin–orbit coupling, decoherence, Zitterbewegung, harmonic noise

1. Introduction

In recent years, enormous progress has been achieved in the experimental realization of artificial gauge fields with ultracold atoms \cite{1, 2}. It gives rise to the new era in quantum simulation of solid-state phenomena, such as the quantum Hall effect \cite{3–5}, spin–orbit coupling \cite{6, 7}, Majorana fermions \cite{8}, to name a few. This breakthrough is mainly conditioned by high degree of atom state controllability, allowing for reduced undesirable concomitant factors which lead to fast decoherence of quantum states in solid-state experiments. However, we have to account for mechanisms of decoherence which are relevant for ultracold atoms, like optical lattice amplitude noise \cite{9, 10} or spontaneous emission \cite{11–13}.

In the this work we consider the dynamics of ultracold atoms with artificial spin–orbit coupling in the presence of optical lattice fluctuations. Attention is focused on influence of noise on Rabi inter-level oscillations. It is well-known that Rabi coupling for spin–orbit coupled states leads to the onset of the Zitterbewegung (ZB) oscillations. The term ZB means specific trembling motion originally predicted by Schrödinger for relativistic Dirac electrons \cite{14}. Basically, ZB occurs as a consequence of coupling between states with different velocities. The phenomenon of the ZB with ultracold atoms was considered, for example, in \cite{15–17}. We are interested in the influence of lattice amplitude fluctuations on ZB oscillations. We consider the experimental setup where spin–orbit coupling is created by means of the Raman dressing scheme \cite{6}. Lattice fluctuations are introduced using the harmonic noise \cite{18, 19}. In the present work we restrict ourselves by the case of non-interacting atoms.

The paper is organized as follows. In section 2 we introduce the model under consideration. Section 3 represents the analytical solution for Rabi oscillations in the absence of noise. Noise influence on Rabi oscillations is studied in section 4. The effect of the Zitterbewegung is considered in section 5. In section 6 we summarize the results obtained and outline possible ways for further research.

2. Model

Consider gas of ultracold non-interacting \textsuperscript{87}Rb atoms with mass $M$ and momentum $p$ moving in an external magnetic field and an optical field of two Raman lasers with orthogonal linear polarizations. Frequency difference between the two...
lasers is $\omega_L$ and the $x$-projection of wave vector difference is $k_L$. Magnetic field results in Zeeman splitting of the triplet state $F = 1$. The laser field is able to induce transitions between three hyperfine states $m_F = -1$, $m_F = 0$, $m_F = 1$. We set laser frequency difference $\omega_1$ to be close to the Zeeman frequency splitting $\omega_2$ between $m_F = -1$ and $m_F = 0$ states. If $\omega_1 = g_\nu H_B$ ($g_\nu$ is the Lande g-factor, $\mu_B$ is the Bohr magneton, $B$ is external magnetic field) is sufficiently large, we can neglect the level $m_F = 1$ due to the quadratic Zeeman effect. Elimination of $m_F = 1$ was successfully realized in the experiment [6] with $\omega_2$ of approximately 4.81 MHz. Thus we have an effective two-level $(|m_F = -1), |m_F = 0\rangle$ system with the Hamiltonian [20]

$$
\hat{H} = \frac{\hat{p}^2}{2M} + \frac{\hbar \Omega}{2} [\hat{\sigma}_1 \hat{U}_1(x, t) + \hat{\sigma}_2 \hat{U}_2(x, t)] - \frac{\hbar \omega_2}{2} \hat{\sigma}_3, \quad (1)
$$

where $\hat{\sigma}_{1,2,3}$ are the Pauli matrices and $\Omega$ is the Rabi frequency (fixed by the intensity of the lasers). The Hamiltonian (1) is characterized by equal contributions of Rashba and Dresselhaus couplings. In the absence of lattice fluctuations, the terms $U_1$ and $U_2$ read

$$
U_1(x, t) = \cos(2k_L x - \omega_L t),
U_2(x, t) = \sin(2k_L x - \omega_L t), \quad (2)
$$

Lattice fluctuations can be incorporated into the Hamiltonian by means of the replacement [21–23]

$$
U_1(x, t) = f(t) \cos(2k_L x) - sf(t + \tau) \sin(2k_L x),
U_2(x, t) = f(t) \sin(2k_L x) + sf(t + \tau) \cos(2k_L x), \quad (3)
$$

where $f(t)$ is the two-dimensional Ornstein-Uhlenbeck process, also called harmonic noise. Harmonic noise is described by the coupled stochastic differential equations

$$
f = \dot{y}, \quad \dot{y} = -\Gamma y - \omega_L^2 y + \sqrt{2\Gamma} \xi(t), \quad (4)
$$

where $\Gamma$ is the harmonic noise parameter being a positive constant, and $\xi(t)$ is Gaussian white noise. It should be emphasized that terms $f(t)$ and $f(t + \tau)$ have to correspond to the same realization of noise $\xi$ in order to provide wavelike form of $U_1$ and $U_2$ [21].

Power spectrum of harmonic noise has an unique peak at frequency

$$
\omega_p = \sqrt{\omega_1^2 - \Gamma^2/4}. \quad (5)
$$

Width of the peak is given by

$$
\Delta \omega = \sqrt{\omega_p^2 + \Gamma^2} - \sqrt{\omega_p^2 - \Gamma^2}, \quad (6)
$$

where

$$
\omega' = \sqrt{\omega_1^2 - \Gamma^2/4}. \quad (7)
$$

If $\Gamma \ll \omega_L$, then $\omega_0 \simeq \omega' \simeq \omega_1$, and (6) reduces to

$$
\Delta \omega \approx \Gamma. \quad (7)
$$

It is important to note that elimination of the hyperfine level $m_F = 1$ requires

$$
|\omega_L - \omega_1| \gg \Delta \omega. \quad (8)
$$

In the present work we consider values of $\Gamma/\omega_L$ ranging from $10^{-3}$ to $10^{-1}$. So, if we set $\omega_L = 4.81$ MHz, as in [6], the corresponding laser bandwidth should be approximately in the range from 5 to 500 kHz.

In the deterministic limit $\Gamma \to 0$ we have

$$
f(t) \to \cos(\omega_L t + \phi_0). \quad (9)
$$

Choosing initial conditions in (4) as $f(0) = 1$, $y(0) = 0$, we specify $\phi_0 = 0$. Then setting

$$
\tau = \frac{\pi}{2\omega_L}, \quad (9)
$$

one reproduces (2) if $\Gamma = 0$. Hence, it turns out that $U_1(x, t)$ and $U_2(x, t)$ for $\Gamma > 0$ behave as fluctuating plane waves. Let us denote

$$
f(t) + if(t + \tau) = W(t)e^{-i\omega(t)}. \quad (10)
$$

In the deterministic case $\Gamma = 0$ we have $\varphi = \omega_L t$

After the replacement (10), the Schrödinger equation can be represented as a pair of coupled equations

$$
i\hbar \frac{\partial \psi_a}{\partial t} = \left(\frac{\hat{p}^2}{2M} - \frac{\hbar \omega_a}{2}\right) \psi_a + \frac{\hbar \Omega}{2} e^{i(\varphi - 2k_L x)} \psi_b, \quad (11)
$$

and

$$
i\hbar \frac{\partial \psi_b}{\partial t} = \left(\frac{\hat{p}^2}{2M} + \frac{\hbar \omega_b}{2}\right) \psi_b + \frac{\hbar \Omega}{2} e^{-i(\varphi - 2k_L x)} \psi_a, \quad (11)
$$

where $\psi_a$ and $\psi_b$ are wave functions of the $|m_F = -1\rangle$ and $|m_F = 0\rangle$ states, respectively.

Assume that dynamics is restricted by the spatial domain of length $L$ being integer multiple $N$ of the lattice period, i.e.

$$
L = \frac{\pi N}{k_L}, \quad N \gg 1. \quad (12)
$$

Imposing periodic boundary conditions at the ends of the domain, we can define discrete set of momentum eigenfunctions

$$
\psi_n = \frac{1}{\sqrt{L}} \exp(-ip_n x/h), \quad (13)
$$

where $p_n = n/hk_0$, $n$ is integer, $k_0 = 2\pi/L$. Expanding $\psi_a$ and $\psi_b$ over momentum eigenfunctions

$$
\psi_a = \sum_n \alpha_n(t) \psi_n, \quad \psi_b = \sum_n \beta_n(t) \psi_n, \quad (14)
$$

we can significantly simplify the problem by reducing (11) to pairs of coupled ODE

$$
i\frac{d\alpha_n}{dt} = \frac{\Omega}{2} \exp[i(\varphi - \nu_a t)] \beta_{n+1}, \quad (15)
$$

and

$$
i\frac{d\beta_{n+1}}{dt} = \frac{\Omega}{2} \exp[-i(\varphi - \nu_b t)] \alpha_n, \quad (15)
$$

where

$$
\nu_n = E_b(n + l) - E_a(n), \quad (16)
$$

and

$$
E_{a,b}(n) = \frac{p_n^2}{2M} + \frac{\hbar \omega_a}{2}, \quad l = \frac{2k_L}{k_0}. \quad (16)
$$

For simplicity, we use scaling of variables corresponding to $M = 1$. 


3. Noiseless case—analytical solution

In the purely deterministic case $\Gamma = 0$ solution of the equations (15) is given by

$$a_n = \sqrt{\rho_n} (c_{n1} e^{i\Omega t_1} + c_{n2} e^{-i\Omega t_1}) e^{i\chi_n},$$
$$b_{n+1} = \sqrt{\rho_n} (c_{n3} e^{-i\Omega t_2} + c_{n4} e^{i\Omega t_2}) e^{i\chi_n},$$

where $\chi_n$ is phase of a corresponding term in the expansion (14), $c_{nj}$ are real-valued coefficients,

$$\Omega_n^\pm \equiv \frac{1}{2} (\chi_n \pm \Omega_n), \quad \Omega_n \equiv \sqrt{\chi_n^2 + \Omega^2},$$

$$\chi_n \equiv \omega_L - \nu_n.$$

Constants $c_{nj}$ obey the following relation:

$$c_{n1}^2 + c_{n2}^2 + c_{n3}^2 + c_{n4}^2 = 1.$$  

Amplitudes $\alpha_n$ and $\beta_{n+1}$ satisfy the conservation law

$$|\alpha_n|^2 + |\beta_{n+1}|^2 = \rho_n.$$  

Substituting (17) into (15), we find the relations

$$c_{n1} = -\frac{\Omega_n^-}{\Omega_n}, \quad c_{n2} = \frac{\Omega_n^+}{\Omega_n},$$

$$c_{n3} = -\frac{\Omega}{2\Omega_n} = -c_{n4}.$$  

We use the initial condition of a form

$$|\alpha_n(t = 0)| = \sqrt{\rho_n}, \quad |\beta_n(t = 0)| = 0$$

for all $n$. Then we find

$$c_{n1} = -\frac{\Omega_n^-}{\Omega_n}, \quad c_{n2} = \frac{\Omega_n^+}{\Omega_n},$$

$$c_{n3} = -\frac{\Omega}{2\Omega_n} = -c_{n4}.$$  

The case of

$$\chi_n = 0$$

corresponds to the onset of resonance in equations (15), when $c_{n1} = c_{n2} = -c_{n3} = c_{n4} = 1/2$, therefore, $\alpha_n$ and $\beta_{n+1}$ oscillate with frequency $\Omega/2$. Accordingly, we can regard the parameter $\chi_n$ as detuning of resonance between coupled momentum states. Increasing of $\chi_n$ results in growing of $|c_{nj}|$, while $|c_{n1}|, |c_{n3}|$ and $|c_{n4}|$ decrease.

Link between $\chi_n$ and corresponding momentum value can be easily derived from (16) and (19):

$$\chi_n = \omega_L - \nu_n - 2k_L p_n - 2\hbar k_L^2.$$  

Any localized quantum state has finite width in the momentum space and therefore consists of many momentum states. In particular, we consider the initial state with Gaussian-like distribution

$$\rho_n = \frac{e^{-(p_n - \nu_n)^2/2\sigma^2_p}}{\sum_n e^{-(p_n - \nu_n)^2/2\sigma^2_p}}.$$  

Owing to the linear dependence of $\chi_n$ on $p_n$, only one of the momentum states may satisfy the condition (25), other ones correspond to off-resonant transitions with nonzero values of $\chi_n$.

4. Phase space representation of Rabi oscillations

Rabi oscillations between resonantly coupled momentum components of $|m_r = 1\rangle$ and $|m_r = 0\rangle$ states can be readily described by means of normalized population imbalance for an $n$-th momentum state

$$Z(n, t) = \frac{|\alpha_n(t)|^2 - |\beta_{n+1}(t)|^2}{\rho_n}.$$  

After some simple algebra with the usage of (17) and (24), we find the solution for $Z(n, t)$ in the absence of noise

$$Z = \frac{1}{\Omega_n} (\chi_n^2 + \Omega^2 \cos \Omega t).$$

Further we omit the subscript $n$. From equation (29) it is clear that quantity $\Omega$ stands for effective Rabi frequency taking into account momentum-dependent detuning from the resonance (25).

In the presence of noise, dynamics of $Z$ is governed by the equation that comes from (15) and looks as

$$\frac{dZ}{dt} = -\Omega W(t) \sqrt{1 - Z^2} \sin \theta,$$

where

$$\theta = \arg \alpha - \arg \beta - \varphi + \nu t.$$  

Equation (30) is complemented by equation

$$\frac{d\theta}{dt} = \Omega W(t) \frac{Z}{\sqrt{1 - Z^2}} \cos \theta - \chi - \eta \chi(t).$$

where $\eta \chi(t) = \varphi(t) - \omega_L$ is a fluctuating part of the frequency difference between the Raman lasers. Equations (30) and (32) can be rewritten in the Hamiltonian form

$$\frac{dZ}{dt} = -\frac{\partial H}{\partial \theta}, \quad \frac{d\theta}{dt} = \frac{\partial H}{\partial Z},$$

with the Hamiltonian

$$H = -\Omega (1 + \eta \chi) \sqrt{1 - Z^2} \cos \theta - (\chi + \eta \chi) Z,$$

where we used the replacement $W(t) = 1 + \eta \chi(t)$. Form of the Hamiltonian (34) is generic for problems studying population dynamics. For example, a similar expression is used for describing population dynamics in external [24] and internal Bose-Josephson junctions [25, 26].

Stationary points of the Hamiltonian (34) in the absence of noise $\eta = \eta \chi = 0$ can be found by solving the equations

$$\frac{dZ}{dt} = \frac{d\theta}{dt} = 0.$$

This gives

$$\theta_{st} = \pi (2k + 1), \quad Z_{st} = -\frac{\chi}{\Omega},$$

and

$$\theta_{st} = 2\pi k, \quad Z_{st} = \frac{\chi}{\Omega}, \quad k \in \mathbb{Z}.$$
Equations (36) and (37) determine equilibrium points for Rabi oscillations with the dominance of the $|m_F = 0\rangle$ or $|m_F = -1\rangle$ states, depending on the sign of $\chi$. Figure 1 illustrates phase portraits for Rabi oscillations with various values of $\chi$. Equilibrium points are placed within the island-like phase space regions where phase $\theta$ is trapped. Despite the solution (17) goes by the trapping regions, the corresponding trajectory lies predominantly in the upper half-plane, inferring prevalence of the state $|m_F = -1\rangle$. The only exception is the case of exact resonance $\chi = 0$ (not shown in the figure).

As the noise is turned on, a phase space trajectory corresponding to Rabi oscillations is able to diffuse in the phase space. So, it can drift to a region with different population dynamics, or even enter a trapping region. This leads to intermittency in Rabi oscillations. Phase space diffusion tends to mix the regimes where one of the state dominates, therefore, one may expect that time- and ensemble-averaged population imbalance

$$Z = \frac{1}{N_T} \sum_{j=1}^{N_T} \int_{t=0}^{T} Z^{(j)}(t) \, dt, \quad (38)$$

should be closer to zero in the presence of noise than in the noiseless case. In equation (38), superscript $j$ labels the realizations of harmonic noise, and $N_T$ is total number of realizations. We calculate $Z$ for initial conditions (23). In the absence of noise, it can be easily found from (28) that

$$Z = \frac{\chi^2}{\chi^2 + \Omega^2}. \quad (39)$$

The corresponding curve is shown in figure 2 by the bold line. As $\chi$ approaches to resonance, $Z$ decreases from 1 to 0 at $\chi = 0$. Thus, we can regard function $Z(\chi)$ as resonance curve describing the process of the photon absorption by a two-level system. Numerical simulation in the presence of noise shows that intermittency reduces values of the time-averaged population imbalance as compared to the noiseless case. This
leads to remarkable broadening of the resonance absorption peak, as $\Gamma$ grows. This is especially pronounced in the case of $\Gamma = 0.1 \Omega$.

5. Zitterbewegung

Population imbalance is related to a momentum expectation value $\langle p(t) \rangle = \langle \psi_{\text{g}} | \hat{p} | \psi_{\text{g}} \rangle$, by means of the formula

$$p(t) = p(t = 0) + \frac{\hbar k_L}{1 - \sum_n \rho_n Z(n, t)},$$

(40)

where it is taken into account that $p_{n+1} - p_n = 2k_L$. As we use scaling corresponding to $M = 1$, $p$ is equal to velocity of center-of-mass motion. As it follows from (40), Rabi inter-level transitions result in oscillations of $p$, hence, there occurs trembling motion referred to as the Zitterbewegung (or shortly ZB). Form of ZB oscillations depends on the initial momentum distribution (27). Firstly, it depends on the width of the distribution, as interference of terms with different $n$ in (40) should damp ZB. In numerical simulation, we set $\sigma_p = 0.01/k_L$. Owing to the Heisenberg uncertainty relation, it corresponds to atomic wavepacket width $\sigma_z = \hbar/2\sigma_p \approx 8\lambda_L$, where $\lambda_L$ is laser wavelength. The center of the momentum distribution $p_c$ is taken of 0.

Secondly, oscillations of $p$ depend on the detuning of the initial state from the resonance (25). Let us introduce mean detuning as

$$\chi_c = \omega_L - \omega_z - 2k_L p_c - 2/k_L^2.$$

(41)

Figure 3 represents ZB oscillations in the absence of noise for $\chi_c = 0$ (upper panel) and $\chi_c = 0.5\Omega$ (lower panel). Notably, ZB exposes much faster damping in the case of $\chi_c = 0.5\Omega$ than in the case of $\chi_c = 0$ corresponding to the strongest influence of resonance.

Now let’s consider effect of noise on ZB oscillations. In the presence of noise, strength of the ZB phenomenon can be quantified by amplitude of momentum oscillations. Therefore, it is reasonable to consider standard deviation of $p$ within a temporal interval of length $\Delta t$:

$$\Lambda_p(t) = \frac{1}{\hbar k_L} \sqrt{\bar{p}^2 - \bar{p}^2},$$

$$\bar{p}^2(t) \equiv \frac{1}{\Delta t} \int_{t - \Delta t/2}^{t + \Delta t/2} \bar{p}^2(t') dt', \quad k = 1, 2.$$  

(42)

The length of the temporal interval $\Delta t$ has to be much larger than characteristic period of inter-level transitions $T \approx 2\pi/\Omega$. In numerical simulation we set $\Delta t = 10\pi/\Omega$. Data for $\Lambda_p$ is averaged over 100 realizations of harmonic noise.

Evidently, lattice noise should bring stochasticity into Rabi oscillations. One may expect that noise-induced fluctuations of Rabi oscillations can reduce phase coherence that is responsible for the ZB damping in the noiseless case, in analogy with noise-induced destruction of the Anderson localization [22, 31, 32]. This expectation is fully confirmed by results of numerical simulation presented in figure 4. The case of $\chi_c = 0.5\Omega$ is considered. After decreasing within the initial stage, ensemble-averaged value of $\Lambda_p$ achieves a plateau for all nonzero values of $\Gamma$, indicating persistence of ZB. Although increasing of $\Gamma$ leads to higher values of $\Lambda_p$, effect of noise-induced ZB persistence is apparent even in the case of very weak noise $\Gamma = 10^{-3}\omega_L$. To understand the origin of the plateau, let’s consider a typical realization of $p(t)$ for

![Figure 3. Quantum expectation value of momentum as function of time for $\chi_c = 0$ (upper panel) and $\chi_c = 0.5\Omega$ (lower panel).](image)

![Figure 4. Standard deviation of momentum within a temporal interval of length $\Delta t = 10\pi/\Omega$. Values of other parameters: $\omega_L = 10\Omega$, $\chi_c = 0.5\Omega$.](image)
\[ \Gamma = 10^{-3} \omega_L. \] Such an example is presented in figure 5. One can see that ZB oscillations firstly demonstrate damping, as in the noiseless case, but then there arises a burst with growing amplitude, as phase coherence of Rabi oscillations for different momentum eigenstates pairs is destructed.

6. Summary and conclusion

In the present work we consider the dynamics of non-interacting ultracold atoms with artificial spin–orbit coupling imposed. Spin–orbit coupling is realized by means of the Raman dressing. Atoms move in a superposition of two moving optical lattices which experience random amplitude and phase fluctuations. Our main concern is to study the influence of laser fluctuations on inter-level Rabi oscillations, with particular emphasis on the phenomenon of the Zitterbewegung (ZB). We show that in the noiseless case dynamics of Rabi oscillations can be represented by a autonomous Hamiltonian system, and find its analytical solution. As detuning from resonance (25) is significantly deviated from zero, there occurs the effect of coherent population trapping.

Noise results in the onset of intermittency in Rabi oscillations. It anticipates enhancing of transitions between different spin states. The main result of the paper is the noise-induced elongation of ZB oscillations that is observed even if noise is weak. We link this effect with destruction of coherence in Rabi oscillations for different momentum components. It is worth to note that noise-induced persistence of Rabi oscillations is observed with laser bandwidth values of few kHz.

We model the fluctuations of a laser field by means of the harmonic noise. It implies that they are caused by some uncontrollable factors reducing laser coherence. However, the same role can be played by properly chosen broadband modulation of the laser wave. We expect that the effect of noise-induced persistence for ZB oscillations can be observed with broadband deterministic signals as well. It is reasonable to expect that a deterministic signal should influence ZB oscillations in a predictable way. As long as inter-level transitions are revealed in transport properties of an atomic ensemble, control of ZB oscillations can be an important step to engineering spin-dependent atomtronic devices [33, 34]. Also, it is very interesting how this effect will change in the presence of interaction between atoms, for example, within a mean-field picture. We intend to address these issues in the forthcoming works.

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