Effects of backreaction and exponential nonlinear electrodynamics on the holographic superconductors

A. Sheykhi$^{1,2}$, F. Shaker$^1$

$^1$ Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran
$^2$ Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran

We analytically study the properties of a (2 + 1)-dimensional s-wave holographic superconductor in the presence of exponential nonlinear electrodynamics. We consider the case in which the scalar and gauge fields back react on the background metric. Employing the analytical Sturm-Liouville method, we find that in the black hole background, the nonlinear electrodynamics correction will affect the properties of the holographic superconductors. We find that with increasing both backreaction and nonlinear parameters, the scalar hair condensation on the boundary will develop more difficult. We obtain the relation connecting the critical temperature with the charge density. Our analytical results support that, even in the presence of the nonlinear electrodynamics and backreaction, the phase transition for the holographic superconductor still belongs to the second order and the critical exponent of the system always takes the mean-field value $1/2$.

I. INTRODUCTION

The AdS/CFT correspondence is an equivalence between a conformal field theory (CFT) in $d$ spacetime dimensions, and a theory of gravity in $(d+1)$-dimensional anti-de Sitter (AdS) spacetime [1-3]. The $d$-dimensional theory does not have a gravitational force, and is to be viewed as a hologram of the $(d+1)$-dimensional theory. The AdS/CFT correspondence is a well-known approach to explore strongly coupled field theories in which certain questions become computationally smooth and conceptually more explicit. The AdS/CFT correspondence can be applied to condensed matter phenomena. In condensed matter physics, there are many strongly coupled systems such as superconductors. In this regards, it was recently argued that it is quite possible to shed some light on the problem of understanding the mechanism of the high temperature superconductors in condensed matter physics, by studying a classical general relativity in one higher dimensional spacetime [4, 5]. The holographic superconductivity is a phenomenon associated with asymptotic

* asheykhi@shirazu.ac.ir
AdS black holes. The studies on the holographic superconductors have received a lot of attentions [6–10].

Most studies on the holographic superconductors are focused on the cases where the gauge field is in the form of the linear Maxwell field. But nonlinear electrodynamics is constructed by the desire to find non-singular field theories. One may consider nonlinear electrodynamics as a possible mechanism for avoiding the singularity of the point-like charged particle at the origin. The nonlinear extension of the original Maxwell electrodynamics in the context of holographic superconductors have arisen intensive investigations [11–18]. In particular, in order to see what difference will appear for holographic superconductor in the presence of Born-Infeld (BI) nonlinear electrodynamics, compared with the case of linear Maxwell electrodynamics, the authors of Ref. [19] have studied condensation and critical phenomena of the holographic superconductors with BI electrodynamics in $d$-dimensional spacetime. Their analytical results indicate that the nonlinear BI electrodynamics decreases the critical temperature of the holographic superconductor. It was observed that the higher BI corrections make it harder for the condensation to form but do not affect the critical phenomena of the system [19].

It is also interesting to investigate the effects of gauge and scalar fields of the holographic superconductor on the background geometry. Although if we ignore this backreaction, the problem is simplified, but retains most of the interesting physics since the nonlinear interactions are retained. Indeed, considering the holographic superconductor model away from the probe limit may bring rich physics. Therefore, many authors have tried to study the holographic superconductors away from the probe limit [20–26]. Employing the analytical Sturm-Liouville method, the effects of both backreaction and BI nonlinear parameter on the critical temperature as well as scalar condensation were explored in Ref. [27]. Furthermore, the relation between the critical temperature and charge density was established [27]. It was shown that it is more difficult to have scalar condensation in BI electrodynamics when the backreaction is taken into account [27].

In the present work we would like to extend our analytical study on the backreacting holographic superconductors by considering another form of the higher order corrections to the gauge field, i.e., the exponential form of nonlinear electrodynamics. It was shown that when the backreaction is taken into account, even the uncharged scalar field can form a condensation in the $(2 + 1)$-dimensional holographic superconductor model [5]. Numerical studies on the holographic superconductors with exponential nonlinear (EN) electrodynamics are carried out in the probe limit [28]. It was shown that the higher nonlinear electrodynamics corrections makes the condensation harder to form [28]. As far as we know, analytical study on the holographic superconductor
in the presence of EN electrodynamics and away from the probe limit has not been done. Considering exponential form of the higher corrections to the gauge field, we shall analytically investigate the properties of the holographic superconductors when the gauge and scalar field do back react on the metric background. We shall use the analytical Sturm-Liouville eigenvalue problem. We will also compare our results with those for the holographic superconductors with BI nonlinear electrodynamics with backreaction given in [27].

This paper is outlined as follows. In section II we introduce the action and basic field equations of the (2 + 1)-holographic superconductor with EN electrodynamics with backreaction. In section III we compute the critical temperature in terms of the charge density and disclose its dependence on the both nonlinear and backreaction parameters. Section IV includes step-by-step computations for obtaining the critical exponent and the condensation values of the holographic superconductor and provides explanations about them. Section V will help us to collect the obtained results briefly.

II. BASIC EQUATIONS OF HOLOGRAPHIC SUPERCONDUCTORS WITH BACKREACTIONS

The action of Einstein gravity coupled to a charged complex scalar field in the presence of nonlinear electrodynamics is described by

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} (R - 2\Lambda) + \mathcal{L}(\mathcal{F}) - |\nabla \psi - iqA\psi|^2 - m^2|\psi|^2 \right] , \] (1)

where \( \kappa \) is the usual four dimensional gravitational constant, \( \kappa^2 = 8\pi G_4 \), \( \Lambda = -3/L^2 \) is the cosmological constant, where \( L \) is the AdS radius which will be scaled unity in our calculations. \( R \) and \( g \) are, respectively, representing the Ricci scalar and the determinant of the metric. \( A \) is the gauge field and \( \psi \) represents a scalar field with charge \( q \) and mass \( m \). \( \mathcal{L}(\mathcal{F}) \) is a simple generalization of Maxwell Lagrangian in a exponential form [29]

\[ \mathcal{L}(\mathcal{F}) = \frac{1}{4b} \left( e^{-b\mathcal{F}} - 1 \right) , \] (2)

where \( b \) is the nonlinear parameter, \( \mathcal{F} = F_{\mu\nu}F^{\mu\nu} \) and \( F^{\mu\nu} \) is the electromagnetic field tensor. Expanding this nonlinear Lagrangian for small \( b \), the leading order term is the linear Maxwell theory, \( \mathcal{L}(\mathcal{F}) = -\mathcal{F}/4 + O(b) \). The plane-symmetric black hole solution with an asymptotically AdS behavior including the backreaction is described by the metric,

\[ ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2(da^2 + dy^2) \] (3)
We adopt the following gauge choices for the vector field and the scalar field,
\[ A_\mu = (\phi(r), 0, 0, 0), \quad \psi = \psi(r), \]
with these functions being real-valued. Then, we need to establish the Einstein equations by varying action (1) with respect to the metric. We find
\[ R^{\mu\nu} - \frac{g^{\mu\nu}}{2} R - \frac{3}{L^2} g^{\mu\nu} = \kappa^2 T^{\mu\nu}, \]
where the energy momentum tensor is given by
\[ T^{\mu\nu} = \frac{1}{4b} g^{\mu\nu} \left( e^{-bF} - 1 \right) + e^{-bF} F^\mu_\sigma F^{\sigma\nu} - m^2 g^{\mu\nu} |\psi|^2 - g^{\mu\nu} |\nabla \psi - iqA\psi|^2 + [(\nabla^\nu + iqA^\nu)\psi^*(\nabla^\mu - iqA^\mu)\psi + \mu \leftrightarrow \nu]. \]
Variation with respect to the scalar field yields
\[ (\nabla^\mu - iqA^\mu)(\nabla^\mu - iqA^\mu)\psi - m^2 \psi = 0, \]
while the electrodynamic equation,
\[ \nabla^\mu \left( F^{\mu\nu} e^{-bF} \right) = iq \left[ \psi^*(\nabla^\mu - iqA^\mu)\psi - \psi(\nabla^\mu + iqA^\mu)\psi^* \right], \]
is obtained by varying action (1) with respect to the gauge field. These equations can easily reduce to those of the holographic superconductor in Maxwell theory [5], provided \( b \to 0 \). Calculations of the Einstein, scalar and electrodynamic field equations, with respect to the metric (3), yield the following expressions,
\[ \chi' + 2r\kappa^2 \left( \psi'^2 + \frac{q^2 e^\chi \phi^2 \psi^2}{f^2} \right) = 0, \]
\[ f' - \left( \frac{3r}{L^2} - \frac{f}{r} \right) - \frac{\chi'}{2} f + r\kappa^2 \left[ m^2 \psi^2 + \frac{1}{4b} \left( 1 - e^{2b\phi^2 r}x \right) + \phi'^2 e^{\chi + 2b\phi^2 r}x \right] = 0, \]
\[ \phi''(1 + 4be^{\chi \phi^2}) + \frac{2}{r} \phi'(1 + rbe^{\chi \phi^2}) + \frac{\chi' \phi'}{2} - 2q^2 \phi \psi^2 \frac{e^{-2be^x \phi^2}}{f} = 0, \]
\[ \psi'' + \left( \frac{f'}{f} + \frac{2}{r} - \frac{\chi'}{2} \right) \psi' + \left( \frac{q^2 e^\chi \phi^2}{f^2} - \frac{m^2}{f} \right) \psi = 0, \]
where the prime denotes derivative with respect to \( r \). We further assume there exists an event horizon \( r_+ \) for which \( f(r_+) = 0 \), and thus the corresponding Hawking temperature of the black hole reads
\[ T = \frac{f'(r_+) e^{-\chi(r_+)/2}}{4\pi}. \]
For the case with $b \rightarrow 0$, Eqs. (9)-(12) coincide with their corresponding equations presented in [20]. Also in the probe limit where $\kappa = 0$, Eqs. (11) and (12) go back to the $(2 + 1)$-dimensional holographic superconductor model studied in [28]. In this case the solution of Eq. (10) is

$$f(r) = \frac{r^2}{L^2} \left( 1 - \frac{r_+^3}{r^3} \right).$$ (14)

It should be noted that we can set the charge parameter, $q$, as unity and keep $\kappa^2$ finite when the backreaction is taken into account by adopting the scaling symmetry [30]. When the Hawking temperature is above the critical temperature $T > T_c$, the system leads to the well-known exact black holes as $b \rightarrow 0$ with the metric coefficient and the potential function given by

$$f(r) = \frac{r^2}{L^2} - \frac{1}{r} \left( \frac{r_+^3}{L^2} + \frac{\kappa^2 \rho^2}{2r_+^2} \right) + \frac{\kappa^2 \rho^2}{2r^2}, \quad \phi \approx \mu - \frac{\rho}{r}. \quad (15)$$

On the dual side, $\mu$ and $\rho$ are, respectively, the chemical potential and charge density of the holographic superconductor. When $\kappa = 0$, the metric coefficient $f(r)$ recovers the case of Schwarzschild AdS black holes (14). For investigating the properties of dual model in superconducting phase, i.e., $\psi(r) \neq 0$, we need the suitable boundary conditions. Examining the behavior of the fields near the horizon, we find the suitable boundary conditions as

$$\phi(r_+) = 0, \quad \psi(r_+) = \frac{f'(r_+) \psi'(r_+)}{m^2},$$ (16)

and hence the metric functions $\chi$ and $f(r)$ satisfy

$$\chi'(r_+) = -2r_+ \kappa^2 \left( \psi^2(r_+) + \frac{\rho^2 e^{\chi(r_+)} \phi^2(r_+) \psi^2(r_+)}{f'^2(r_+)} \right),$$

$$f'(r_+) = \frac{3r_+}{L^2} - r_+ \kappa^2 \left[ m^2 \psi^2(r_+) + \frac{1}{4b} \left( 1 - e^{2b \rho^2(r_+)} e^{\chi(r_+)} \right) + \phi'^2(r_+) e^{\chi(r_+)} + 2b \rho^2(r_+) e^{\chi(r_+)} \right].$$ (17)

The asymptotic behavior of the fields, corresponding to the solution of Eqs. (11) and (12) in the limit $r \rightarrow \infty$, are given by

$$\phi \approx \mu - \frac{\rho}{r},$$

$$\psi \approx \frac{\psi_+}{r^{-\Delta_-}} + \frac{\psi_-}{r^{\Delta_+}},$$ (19)

where

$$\Delta_\pm = \frac{3}{2} \pm \frac{\sqrt{9 + 4m^2}}{2},$$ (20)

is the conformal dimension of the dual operator $O_\pm$ in the boundary field theory. Here $\psi_+$ and $\psi_-$ can be considered as the source and the vacuum expectation values of the dual operator. Hereafter, we set $\psi_+ = 0$ and investigate the condensation of $\psi_- = \langle O_- \rangle$, analytically. In what follows we choose the scalar to have $m^2 = -2$, and hence the corresponding dual operator has mass dimension $\Delta_- = 1$.  

III. ANALYTICAL STUDY AND CRITICAL TEMPERATURE

In this section, we investigate the analytical properties of a \((2 + 1)\)-holographic superconductor in the framework of EN electrodynamics. We study the problem by taking the backreaction into account. We find the critical temperature \(T_c\) via the Sturm-Liouville variational approach. Further, we obtain a relationship between the critical temperature and the charge density and investigate the effects of both backreaction and EN parameter on the critical temperature. In order to get the solutions in superconducting phase, we can define a new variable \(z = r_+ / r\). Then, the equations of motion can be rewritten as

\[
\chi' - 2\kappa^2 \left( z \psi'^2 + \frac{r_+^2}{z^3 f^2} e^\chi \phi'^2 \psi^2 \right) = 0,
\]

(22)

\[
f' - \frac{f}{z} + \frac{3r_+^2}{z^3} - \frac{\chi' f}{2} - \frac{\kappa^2 r_+^2}{z^3} \left[ m^2 \psi'^2 + \frac{1}{46} \left( 1 - e^{\frac{2b z^4}{r_+^2}} \phi'^2 e^\chi \right) + \frac{z^4}{r_+^2} \phi'^2 e^{\chi + \frac{2b z^4}{r_+^2}} \phi'^2 \right] = 0,
\]

(23)

\[
\phi'' \left( 1 + \frac{4b z^4}{r_+^2} e^\chi \phi'^2 \right) + \frac{8b z^3}{r_+^2} e^\chi \phi'^3 + \frac{2b z^4}{r_+^2} e^\chi \phi'^3 \chi + \frac{\phi' \chi'}{2} - \frac{2r_+^2 \psi^2}{f z^4} e^{-\frac{2b z^4}{r_+^2}} \phi'^2 \phi = 0,
\]

(24)

\[
\psi'' - \left( \frac{\chi'}{2} - \frac{f'}{f} \right) \psi' - \frac{r_+^2}{z^4} \left( m^2 \frac{\phi'^2}{f^2} - \frac{e^\chi \phi'^2}{f^2} \right) \psi = 0,
\]

(25)

where now the prime denotes derivative with respect to \(z\). When \(b \to 0\), the above equations restore the corresponding equations in Ref. [20], while in the absence of the backreaction, Eqs. (24) and (25) reduce to their corresponding equations in Ref. [28]. Following the perturbation scheme, since close to the critical point, the value of the scalar operator is small, it can be introduced as an expansion parameter

\[
\epsilon \equiv \langle \mathcal{O}_i \rangle,
\]

(26)

with \(i = + \) or \(i = -\). Besides, near the critical point the scalar and gauge fields are small and therefore we can expand the gauge field \(\phi\), the scalar field \(\psi\), and the metric functions \(f(z), \chi(z)\) as

\[
\psi = \epsilon \psi_1 + \epsilon^3 \psi_3 + \epsilon^5 \psi_5 + \ldots,
\]

(27)

\[
\phi = \phi_0 + \epsilon^2 \phi_2 + \epsilon^4 \phi_4 + \ldots,
\]

(28)

\[
f = f_0 + \epsilon^2 f_2 + \epsilon^4 f_4 + \ldots,
\]

(29)
\[ \chi = \epsilon^2 \chi_2 + \epsilon^4 \chi_4 + \ldots, \quad (30) \]

where the metric function \( f(z) \) and \( \chi(z) \) are expanded around the Reissner-Nordström AdS spacetime. Also, the chemical potential \( \mu \) may be expanded as

\[ \mu = \mu_0 + \epsilon^2 \delta \mu_2 + \ldots, \quad (31) \]

where \( \delta \mu_2 \) is positive. Thus, near the phase transition, the order parameter as a function of the chemical potential has the form

\[ \epsilon \approx \left( \frac{\mu - \mu_0}{\delta \mu_2} \right)^{1/2}. \quad (32) \]

whose critical exponent \( \beta = 1/2 \) is the same as in the Ginzburg-Landau mean field theory. The phase transition can take place when \( \mu \to \mu_0 \). In this case the critical value of the chemical potential is given by \( \mu_c = \mu_0 \).

From Eq. (24) the equation for \( \phi \) is obtained at zeroth order as

\[ \phi''(z) \left( 1 + 4b \frac{z^4}{r^2 + c} \phi'^2 \right) + \frac{8b^3}{r^2 + c} \phi'^3(z) = 0, \quad (33) \]

which admits the following solutions for the gauge field

\[ \phi(z) = \int_1^z dz \frac{r + c}{2z^2} \sqrt{L_W \left( \frac{4b^4 \beta^2}{r^2 + c} \right)} . \quad (34) \]

In the above expression \( \beta \) is an integration constant and \( L_W(x) = LambertW(x) \) is the Lambert function which satisfies

\[ L_W(x)e^{L_W(x)} = x, \quad (35) \]

and has the following series expansion

\[ L_W(x) = x - x^2 + \frac{3}{2} x^3 - \frac{8}{3} x^4 + \ldots \quad (36) \]

Obviously, the series (36) converges provided \( |x| < 1 \). If we expand the solution (34) for small \( b \) and keep the only linear terms in \( b \), we arrive at

\[ \phi(z) = -\beta(1 - z) + \frac{2 \beta^3 b}{5r^2 + c} (1 - z^5) + \mathcal{O}(b^2) . \quad (37) \]

Differentiating Eqs. (19) and (37) with respect to \( z \) and equating them at \( z = 0 \), we find \( \beta = -\rho/\rho_{++} \). Rearranging Eq. (37) and using the relation \( \beta \), we arrive at

\[ \phi_0(z) = \lambda r_{++}(1 - z) \left\{ 1 - \frac{2}{5} b \lambda^2 (1 + z + z^2 + z^3 + z^4) \right\}, \quad b \lambda^2 < 1, \quad (38) \]
where
\[ \lambda = \frac{\rho}{r_+^2} , \] (39)
and we have neglected \( O(b^2) \). Thus, to zeroth order the equation for \( f \) is solved as
\[ f_0(z) = r_+^2 g(z) = r_+^2 \left[ \frac{1}{z^2} - z - \frac{\kappa^2 \lambda^2}{2} z(1-z) + \frac{b}{10} \kappa^2 \lambda^4 z (1-z^5) \right] . \] (40)

At the first order, the behavior of \( \psi \) at the asymptotic AdS boundary is given by
\[ \psi_1 \approx \frac{\psi_0}{r_+^{\Delta}} z^{\Delta} + \psi_0' \frac{1}{r_+^{\Delta}} z^{\Delta}. \] (41)

Next, we introduce a variational trial function \( F(z) \) near the boundary
\[ \psi_1(z) = \frac{\mathcal{O}_i}{\sqrt{2 r_+^{\Delta}}} z^{\Delta_i} F(z) , \] (42)
with the boundary condition \( F(0) = 1 \) and \( F'(0) = 0 \). Then, we can obtain the equation of motion for \( F(z) \) by substituting (42) into Eq. (25). We find
\[ F''(z) + \left[ \frac{2\Delta}{z} + \frac{g'}{g} \right] F'(z) + \left[ \frac{\Delta}{z} \left( \frac{\Delta - 1}{z} + \frac{g'}{g} \right) - \frac{m^2}{z^4 g} \right] F(z) + \frac{\lambda^2 (1-z)^2}{z^4 g} \left[ 1 - \frac{4}{5} b \lambda^2 \left( 1 + z + z^2 + z^3 + z^4 \right) \right] F(z) = 0. \] (43)

Defining the new functions
\[ T(z) = z^{2\Delta_i+1} \left[ 2(z^{-3} - 1) - \kappa^2 \lambda^2 (1-z) + \frac{b}{5} \kappa^2 \lambda^4 (1-z^5) \right] , \] (44)
\[ P(z) = \frac{\Delta_i}{z} \left( \frac{\Delta_i - 1}{z} + \frac{g'}{g} \right) - \frac{m^2}{z^4 g} , \] (45)
\[ Q(z) = \frac{(1-z)^2}{z^4 g^2} \left[ 1 - \frac{4}{5} b \lambda^2 \left( 1 + z + z^2 + z^3 + z^4 \right) \right] . \] (46)
we can rewrite Eq. (43) as
\[ TF'' + T'F' + PF + \lambda^2 QF = 0. \] (47)
According to the Sturm-Liouville eigenvalue problem \[32\], the eigenvalue \( \lambda^2 \) can be obtained by minimizing the expression
\[ \lambda^2 = \frac{\int_0^1 T \left( F'^2 - PF^2 \right) \, dz}{\int_0^1 TQF^2 \, dz} , \] (48)
where we have chosen the trial function in the form $F(z) = 1 - \alpha z^2$. In order to simplify our calculations, we express the backreaction parameter as

$$\kappa_n = n\Delta\kappa, \quad n = 0, 1, 2, ...$$

(49)

where $\Delta\kappa = \kappa_{n+1} - \kappa_n$ is the step size of our iterative procedure. The main purpose is to work in the small backreaction approximation so that all the functions can be expanded by $\kappa^2$ and the $\kappa^4$ term can be neglected. Furthermore, we retain the terms that are linear in nonlinear parameter $b$ and keep terms up to $O(b)$. So we use the following relations

$$\kappa^2\lambda^2 = \kappa_n^2\lambda^2 = \kappa_n^2(\lambda^2|_{\kappa_{n-1}}) + O[(\Delta\kappa)^4],$$

(50)

$$b\lambda^2 = b(\lambda^2|_{b=0}) + O(b^2),$$

(51)

and

$$b\kappa^2\lambda^4 = b\kappa_n^2(\lambda^4|_{\kappa_{n-1}, b=0}) + O(b^2) + O[(\Delta\kappa)^4],$$

(52)

where we have assumed $\kappa_{-1} = 0$, $\lambda^2|_{\kappa_{-1}} = 0$ and $\lambda^2|_{b=0}$ is the value of $\lambda^2$ for $b = 0$. Now we are going to compute the critical temperature $T_c$. First of all, we start with the following equation

$$T_c = \frac{f'(r_{+c})}{4\pi}.$$  

(53)

From Eq. (18), $f'(r_{+c})$ is expressed as

$$f'(r_{+c}) = 3r_{+c} - \kappa^2 r_{+c} \left[ \frac{\phi_0 t^2(r_{+c})}{2} + \frac{3}{2} b\phi_0' t^4(r_{+c}) \right].$$  

(54)

Substituting Eq. (18) in the above equation, and then inserting the result back into Eq. (53), we arrive at the following expression for the critical temperature,

$$T_c = \frac{1}{4\pi} \sqrt{\frac{\pi}{\lambda}} \left[ 3 - \frac{\kappa_n^2(\lambda^2|_{\kappa_{n-1}})}{2} + \frac{1}{2} b\kappa_n^2(\lambda^4|_{\kappa_{n-1}, b=0}) \right].$$  

(55)

With these obtained computations out of the analytical approach at hand, we are in a position to present the results of the critical temperature $T_c$ for a $(2 + 1)$-dimensional holographic superconductors in the presence of both EN electrodynamics as well as backreaction. To do this, we assume the nonlinear parameter $b$ is small, by choosing it as $b = 0, 0.1, 0.2, 0.3$. We also get the values $m^2 = -2$, $\Delta_i = \Delta_\omega = 1$ and $\Delta\kappa = 0.05$. As an example, we bring the details of our calculations for the case of $n = 4$ and summarize all results in table 1.
For $b = 0$, we obtain $\lambda^2$ from Eq. (48) as

$$\lambda^2 = \frac{-8.279205\alpha^2 + 4.924220\alpha - 4.957900}{-3.579048 + 0.878281\alpha - 0.153258\alpha^2}.$$  

which has a minimum value $\lambda^2_{\text{min}} = 1.4757$ at $\alpha = 0.2417$, and we can get the critical temperature $T_c = 0.2147\sqrt{\rho}$.

For $b = 0.1$, we find

$$\lambda^2 = \frac{-4.95286 + 4.91624\alpha - 8.27411\alpha^2}{-3.035 + 0.673477\alpha - 0.109325\alpha^2},$$

which has a minimum value $\lambda^2_{\text{min}} = 1.4757$ at $\alpha = 0.2417$, and we can get the critical temperature $T_c = 0.2147\sqrt{\rho}$.

For $b = 0.2$, we arrive at

$$\lambda^2 = \frac{-4.94387 + 4.90127\alpha - 8.26415\alpha^2}{-2.49088 + 0.46851\alpha - 0.0652154\alpha^2},$$

whose minimum is $\lambda^2_{\text{min}} = 1.7811$ at $\alpha = 0.24955$ and the critical temperature becomes $T_c = 0.2046\sqrt{\rho}$.

For $b = 0.3$, we have

$$\lambda^2 = \frac{-4.93051 + 4.87819\alpha - 8.24877\alpha^2}{-1.94661 + 0.263502\alpha - 0.021046\alpha^2},$$

which attains its minimum $\lambda^2_{\text{min}} = 2.2451$ at $\alpha = 0.26133$ and the critical temperature reads $T_c = 0.1927\sqrt{\rho}$. We summarize our results for the critical temperature in cases of different values of nonlinear and backreaction parameters in Table 1. From this table, we see that, for fixed value of the backreaction parameter, with the nonlinear parameter $b$ getting stronger, the critical temperature decreases. Similarly, for a fixed value of the nonlinear parameter $b$, the critical temperature drops as the backreaction parameter increases. Thus, we conclude that the critical temperature becomes smaller and so, make the condensation harder when we increase the values of both backreaction and nonlinear parameters. These features were also observed in the study of a $(2 + 1)$-dimensional holographic superconductors with backreaction when the gauge field is in the form of BI nonlinear electrodynamics. Comparing the results obtained here with those of [27], we observe that the effect of the EN corrections on the condensation with respect to the BI nonlinear one is stronger when the backreaction is taken into account in both cases. In other words, the formation of scalar hair in the presence of EN electrodynamics is harder compared to the case of BI nonlinear electrodynamics. Obviously, our analytic results back up the findings in other articles.
of $b = \kappa = 0$, we observe that the analytic results for the critical temperature are consistent with both the analytical results of Ref. [33] as well as the numerical result of Ref. [5]. Also, we confirm the numerical result found in Ref. [28] when the backreaction parameter $\kappa$ is equal to zero. On the other hand, for $b = 0$ the data obtained for the critical temperature, is analogous to those reported for the holographic superconductors with backreaction in Maxwell theory [20].

| $n$ | $b=0$ | $b=0.1$ | $b=0.2$ | $b=0.3$ |
|-----|-------|---------|---------|---------|
|     | BI    | EN      | BI      | EN      |
| 0   | 0.2250| 0.2250  | 0.2161  | 0.2206  |
| 1   | 0.2249| 0.2267  | 0.2160  | 0.2204  |
| 2   | 0.2246| 0.2257  | 0.2160  | 0.2203  |
| 3   | 0.2241| 0.2220  | 0.2148  | 0.2199  |
| 4   | 0.2235| 0.2214  | 0.2147  | 0.2192  |
| 5   | 0.2226| 0.2208  | 0.2141  | 0.2184  |
| 6   | 0.2216| 0.2196  | 0.2130  | 0.2174  |

Table 1: The critical temperature $T_c/\sqrt{\rho}$ for holographic superconductors in the presence of BI and EN electrodynamics. Here we have taken $\kappa_n = n\Delta \kappa$ where $\Delta \kappa = 0.05$. The results for BI case are invoked from Ref. [27].

IV. CRITICAL EXPONENT AND THE CONDENSATION OF THE SCALAR OPERATOR

We use the Sturm-Liouville method to analytically examine the scalar condensation and the order of the phase transition with backreactions near the critical temperature. With the help of Eq. (42), when $T$ is close to $T_c$, the equation of motion (24) can be rewritten as

$$\phi'' \left(1 + \frac{4bz^4}{r_+^2}\phi'^2\right) + \frac{8bz^3}{r_+^2}\phi'^3 = \frac{(\mathcal{O})^2}{r_+^2}\mathcal{B}(z)\phi(z),$$

(60)

$$\mathcal{B}(z) = \frac{F^2(z)}{1 - z^3} \left(1 - \frac{2bz^4}{r_+^2}\phi'^2(z)\right) \left[1 + \frac{\kappa_2 z^3}{1 + z + z^2} \left(\frac{\lambda^2}{2} - \frac{b\lambda^4}{10}\xi(z)\right)\right],$$

(61)

where $\xi(z) = 1 + z + z^2 + z^3 + z^4$. Since the parameter $(\mathcal{O})^2/r_+^2$ is very small, we can expand $\phi(z)$ as

$$\frac{\phi(z)}{r_+} = \lambda (1 - z) \left(1 - \frac{2}{5}b\lambda^2 \xi(z)\right) + \frac{(\mathcal{O})^2}{r_+^2}\chi(z).$$

(62)
Substituting Eq. (62) into Eq. (60), we can obtain the equation of motion for \( \chi(z) \) as

\[
\left[ K(z)\chi'(z) \right]' = \left(1 + 4b\lambda^2 z^4 \right)^{1/2} \frac{\lambda F^2}{1 + z + z^2} \times \left\{ 1 - \frac{2}{5} b\lambda^2 (\xi(z) + 5z^4) + \frac{z^3}{1 + z + z^2} \left( \frac{\kappa^2 \lambda^2}{2} - \frac{b\kappa^2 \lambda^4}{10} (3\xi(z) + 10z^4) \right) \right\}. \tag{63}
\]

with \( \chi(1) = 0 = \chi'(1) \) and we have defined

\[
K(z) = \left(1 + 4b\lambda^2 z^4 \right)^{3/2}. \tag{64}
\]

Integrating both sides of Eq. (63) between \( z = 0 \) to \( z = 1 \), we reach

\[
\chi'(0) = -\lambda \int_0^1 dz \left(1 + 4b\lambda^2 z^4 \right)^{1/2} \frac{F^2}{1 + z + z^2} \times \left\{ 1 - \frac{2}{5} b\lambda^2 (\xi(z) + 5z^4) + \frac{z^3}{1 + z + z^2} \left( \frac{\kappa^2 \lambda^2}{2} - \frac{b\kappa^2 \lambda^4}{10} (3\xi(z) + 10z^4) \right) \right\}. \tag{65}
\]

Equating \( \phi(z) \) from Eqs. (19) and (62), we arrive at

\[
\frac{\mu}{r_+} - \frac{\rho}{r_+} = \lambda (1 - z) \left\{ 1 - \frac{2}{5} b\lambda^2 \xi(z) \right\} + \frac{\langle \mathcal{O} \rangle^2}{r_+^2} \chi(z) = \lambda (1 - z) \left\{ 1 - \frac{2}{5} b\lambda^2 \xi(z) \right\} + \frac{\langle \mathcal{O} \rangle^2}{r_+^2} \left( \chi(0) + z\chi'(0) + \ldots \right), \tag{66}
\]

where in the last step we have expanded \( \chi(z) \) around \( z = 0 \). Considering the coefficients of \( z \) term in both sides of Eq. (66), we find that

\[
\frac{\rho}{r_+^2} = \lambda - \frac{\langle \mathcal{O} \rangle^2}{r_+^2} \chi'(0). \tag{67}
\]

Substituting \( \chi'(0) \) from Eq. (65) in the above relation, we get

\[
\frac{\rho}{r_+^2} = \lambda \left\{ 1 + \frac{\langle \mathcal{O} \rangle^2}{r_+^2} A \right\}, \tag{68}
\]

where

\[
A = \int_0^1 dz \left(1 + 4b\lambda^2 z^4 \right)^{1/2} \frac{F^2}{1 + z + z^2} \times \left\{ 1 - \frac{2}{5} b\lambda^2 (\xi(z) + 5z^4) + \frac{z^3}{1 + z + z^2} \left( \frac{\kappa^2 \lambda^2}{2} - \frac{b\kappa^2 \lambda^4}{10} (3\xi(z) + 10z^4) \right) \right\}. \tag{69}
\]

Using Eqs. (13), (18) and (38), and taking into account the fact that \( T \) is very close to \( T_c \), we can deduce

\[
r_+ = \frac{4\pi T}{\left[ 3 - \frac{\kappa^2 \lambda^2}{2} + \frac{b}{2} \kappa^2 \lambda^4 \right]^{1/2}}. \tag{70}
\]
Eqs. (39) and (70) show that Eq. (68) can be rewritten as

\[ T_c^2 - T^2 = \langle O \rangle^2 \left( \frac{A}{4\pi} \right)^2 \left[ 3 - \frac{\kappa^2 \lambda^2}{2} + \frac{b}{2}\kappa^2 \lambda^4 \right]^2. \]  

(71)

Thus, we find the expectation value \( \langle O \rangle \) near the critical point as

\[ \langle O \rangle = \gamma T_c \sqrt{1 - \frac{T}{T_c}}, \]  

(72)

where \( \gamma \) is the condensation parameter of the system which is given by

\[ \gamma = \frac{4\pi \sqrt{2}}{\sqrt{A}} \left[ 3 - \frac{\kappa^2 \lambda^2}{2} + \frac{b}{2}\kappa^2 \lambda^4 \right]^{-1}. \]  

(73)

The relation obtained in Eq. (72) is valid for small nonlinear coupling and backreaction parameters and satisfies \( \langle O \rangle \sim \sqrt{1 - \frac{T}{T_c}} \). Therefore, the analytical result supports that the phase transition for the superconductor belongs to the second order and the critical exponent of the system takes the mean-field value 1/2. This implies that considering nonlinear coupling and backreaction parameters the value of the critical exponent will not be altered. As we see in table 2, condensation values \( \gamma \) increases with increasing the nonlinear parameter \( b \) for the fixed parameter \( \kappa \). Also, we see the same behavior between the condensation values \( \gamma \) and the backreaction parameter with a fixed value of the nonlinear parameter \( b \). This means that the condensation becomes harder to be formed by considering both the nonlinear corrections to the gauge field and taking the backreactions into account. It should be noted that, at a temperature slightly below \( T_c \) for the \((2 + 1)\)-dimensional holographic superconductors with backreaction, condensation values for both BI and EN holographic superconductors have the same behaviour, as we see in table 2. Also, because of the larger parameter \( \gamma \), effect of the EN electrodynamics on the condensation of the scalar operators is bigger than that of BI case. This implies that the scalar hair is more difficult to be developed in the holographic superconductors with EN electrodynamics.

| n | b=0 | b=0.1 | b=0.2 | b=0.3 |
|---|-----|-------|-------|-------|
|   | BI  | EN    | BI    | EN    | BI    | EN    |
| 0 | 8.07| 8.07  | 8.1801| 8.5298| 8.3094| 9.1579| 8.4696| 10.0355|
| 1 | 8.09| 8.09  | 8.1869| 8.5331| 8.3212| 9.1616| 8.4890| 10.0399|
| 2 | 8.11| 8.11  | 8.1943| 8.5443| 8.3237| 9.1742| 8.4893| 10.0565|
| 3 | 8.115| 8.115| 8.2121| 8.5630| 8.3417| 9.1951| 8.5023| 10.0818|
| 4 | 8.13| 8.13  | 8.2370| 8.5889| 8.3669| 9.2241| 8.5277| 10.1014|
| 5 | 8.16| 8.16  | 8.2909| 8.6425| 8.3994| 9.2615| 8.5606| 10.1639|
| 6 | 8.20| 8.20  | 8.3079| 8.6617| 8.4391| 9.3070| 8.6007| 10.2193|
Table 2: The values of the condensation parameter $\gamma$ for holographic superconductors in the presence of EN electrodynamics. Here we have taken $\kappa_n = n\Delta\kappa$ where $\Delta\kappa = 0.05$. We have also provided the results for BI holographic superconductor from Ref. [27], for comparison.

V. CONCLUSIONS

We have introduced a different type of gravity dual models, i.e., the charged AdS black holes in the context of Einstein-nonlinear electrodynamics with a scalar field. We have assumed the EN electrodynamics as the gauge field, and analytically investigated the behavior of the $(2+1)$-dimensional holographic superconductors. We have worked in a limit in which the scalar and gauge fields backreact on the background metric. We have employed the Sturm-Liouville analytic method to explore the problem. We have found the influence of the nonlinear corrections to the gauge filed as well as the backreaction effects on the critical temperature and the process of the scalar field condensation. We observed that the formation of the scalar hair condensation on the boundary becomes harder in the presence of nonlinear electrodynamics. This is mainly caused by the decreasing of the critical temperature when the both nonlinear and backreaction parameters become stronger. This phenomenon was also obtained in the study of the effect of the BI and backreaction parameters in the $(2+1)$-dimensional holographic superconductors [27]. Comparing these different models show that for a specific $b$ the critical temperature $T_c$ becomes larger for a holographic superconductor with BI nonlinear electrodynamics comparing to the case with EN electrodynamics. This implies that the scalar hair is more difficult to develop in the latter case than the former one. We have also given the critical exponent for the EN holographic superconductor model with backreaction, which still takes the mean-field value $1/2$. We found out that the condensation parameter $\gamma$ in Eq. (72) increases with (i) increasing the nonlinear parameter $b$ with a fixed backreaction parameter $\kappa$, (ii) increasing the backreaction parameter with a fixed value of the nonlinear parameter $b$. This implies that both the nonlinear corrections to the gauge field as well as backreaction, cause the formation of condensation harder.
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