THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION ON THE INTERVAL

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Abstract. We use the Fokas method to analyze the derivative nonlinear Schrödinger (DNLS) equation $iq_t(x,t) = -q_{xx}(x,t) + (rq^2)_x$ on the interval $[0, L]$. Assuming that the solution $q(x,t)$ exists, we show that it can be represented in terms of the solution of a matrix Riemann-Hilbert problem formulated in the plane of the complex spectral parameter $\xi$. This problem has explicit $(x,t)$ dependence, and it has jumps across $\{ \xi \in \mathbb{C} | \text{Im} \xi^4 = 0\}$. The relevant jump matrices are explicitly given in terms of the spectral functions $\{a(\xi), b(\xi)\}$, $\{A(\xi), B(\xi)\}$, and $\{A(\xi), B(\xi)\}$, which in turn are defined in terms of the initial data $q_0(x) = q(x,0)$, the boundary data $g_0(t) = q(0,t)$, $g_1(t) = q_x(0,t)$, and another boundary values $f_0(t) = q(L,t)$, $f_1(t) = q_x(L,t)$. The spectral functions are not independent, but related by a compatibility condition, the so-called global relation.

1. INTRODUCTION

The inverse scattering transformation is an important method for solving initial value problems of complete integrable equations, but it is currently unknown for the case of initial-boundary value problems. A new method based on the Riemann-Hilbert factorization problem to solve initial-boundary value problems for nonlinear integrable systems was presented by Fokas [9, 10], and later it was further developed by several authors [11, 12, 20, 21]. The Fokas method is based on reducing the initial-boundary value problems to the Riemann-Hilbert...
problem on the complex plane of the spectral parameter. But in the analysis of the initial-boundary value problem, we face the problem that for the construction of the associated Riemann-Hilbert problem, more boundary values are needed than a well-posed initial-boundary value problem, since the boundary values are dependent. In [8], Fokas and Its extended the initial-boundary value problem for the nonlinear Schrödinger equation on the half-line to the case of initial-boundary value on the finite interval.

In this paper, we analyze the Dirichlet initial-boundary value problem for the DNLS equation on a finite interval

\[ \begin{align*}
    i q_t &= -q_{xx} + (r q^2)_x, \quad x \in (0, L), t \in (0, T), \\
    q(x, 0) &= q_0(x), x \in (0, L), \\
    q(0, t) &= g_0(t), q(L, t) &= f_0(t), t \in (0, T),
\end{align*} \]  

where \( r = \pm \bar{q} \), and \( \bar{q} \) denotes complex conjugate of \( q \), the subscripts denote differentiation with respect to the corresponding variables. \( L \) and \( T \) are positive constants, and \( q_0, g_0, f_0 \) are smooth functions compatible at \( x = t = 0 \) and at \( x = L, t = 0 \), i.e. \( q_0(0) = g_0(0), q_0(L) = f_0(0) \).

The DNLS equation (1.1) is also called Kaup-Newell equation [13]. We just consider \( r = \bar{q} \), because the two equations

\[ i q_t(x, t)q + q_{xx}(x, t) = \pm(|q|^2q)_x \]

can be transformed into each other by replacing \( x \rightarrow -x \) [4].

The DNLS Eq. (1.1) has several applications in plasma physics. In plasma physics, it is a model for Alfvén waves propagating parallel to the ambient magnetic field, \( q \) being the transverse magnetic field perturbation and \( x \) and \( t \) being space and time coordinates, respectively [1]. For more physical meaning of Eq. (1.1), we refer to [2, 3, 4], and the references therein. Being integrable, Eq. (1.1) admits an infinite number of conservation laws and can be analyzed by means of inverse scattering techniques both in the case of vanishing and nonvanishing boundary conditions [13, 14]. A tri-Hamiltonian structure of Eq. (1.1)
was put forward in [17]. The Darboux transformation and soliton solutions have been investigated in [15, 16, 19, 18]. Recently, Lenells analyzed its Riemann-Hilbert problem associated with initial-boundary value problem of the DNLS Eq.(1.1) on the half-line [4].

In this paper, we extend Lenells’s result to the initial-boundary value on the finite interval (1.1)-(1.3) following the Fokas and Its idea [8]. We will show that the dependence of Riemann-Hilbert problem associated with initial-boundary values can be characterized in terms of spectral functions. The spectral functions associated with initial and boundary values of a solution for the DNLS equation must satisfy certain global relations. In the following section 2, we derive the spectral analysis of the Lax pair for Eq.(1.1). In section 3, we investigate the spectral functions $a(\xi), b(\xi); A(\xi), B(\xi); A(\xi), B(\xi)$. Then Riemann-Hilbert problem associated with the initial-boundary value (1.1)-(1.3) is further presented.

2. Spectral analysis under the assumption of existence

The DNLS equation admits the Lax pair formulation [13, 27]

$$
v_{1x} + i\xi^2 v_1 = q\xi v_2, \quad v_{2x} - i\xi^2 v_2 = r\xi v_1,  
iv_{1t} = Av_1 + Bv_2, \quad iv_{2t} = Cv_1 - Av_2,
$$

(2.1)

where $\xi \in \mathbb{C}$ is the spectral parameter, and

$$
A = 2\xi^4 + \xi^2 rq, \quad B = 2i\xi^3q - \xi q_x + i\xi rq^2,  
C = 2i\xi^3r - \xi r_x + i\xi r^2 q.
$$

By introducing

$$
\psi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},  
$$

we can rewrite the Lax pair (2.1) in a matrix form.
\[
\psi_x + i\xi^2 \sigma_3 \psi = \xi Q \psi,
\]
\[
\psi_t + 2i\xi^4 \sigma_3 \psi = (-i\xi^2 Q^2 \sigma_3 + 2\xi^3 Q - i\xi Q_x \sigma_3 + \xi Q^3) \psi,
\] (2.2)

Extending the column vector \(\psi\) to a \(2 \times 2\) matrix and letting
\[
\Psi = \psi e^{i(\xi^2 x + 2\xi^4 t)\sigma_3},
\]
we obtain the equivalent Lax pair
\[
\Psi_x + i\xi^2 [\sigma_3, \Psi] = \xi Q \Psi,
\]
\[
\Psi_t + 2i\xi^4 [\sigma_3, \Psi] = (-i\xi^2 Q^2 \sigma_3 + 2\xi^3 Q - i\xi Q_x \sigma_3 + \xi Q^3) \Psi,(2.3)
\]
which can be written in full derivative form
\[
d(e^{i(\xi^2 x + 2\xi^4 t)\sigma_3} \Psi(x, t, \xi)) = e^{i(\xi^2 x + 2\xi^4 t)\sigma_3} U(x, t, \xi) \Psi,
\] (2.4)

where
\[
U = U_1 dx + U_2 dt = \xi Q dx + (-i\xi^2 Q^2 \sigma_3 + 2\xi^3 Q - i\xi Q_x \sigma_3 + \xi Q^3) dt. \ (2.5)
\]

In order to formulate a Riemann-Hilbert problem for the solution of the inverse spectral problem, we seek solutions of the spectral problem which approach the \(2 \times 2\) identity matrix as \(\xi \to \infty\). It turns out that solutions of Eq. (2.4) do not exhibit this property, hence we use Lenell’s method in Ref. [4] to transform the solution \(\Psi\) of Eq. (2.4) into the desired asymptotic behavior. We write the process of the transformation as follows.

2.1 Asymptotic analysis

Consider a solution of Eq. (2.4) of the form
\[
\Psi = D + \frac{\Psi_1}{\xi} + \frac{\Psi_2}{\xi^2} + \frac{\Psi_3}{\xi^3} + O\left(\frac{1}{\xi^4}\right), \quad \xi \to \infty
\]
where \(D, \Psi_1, \Psi_2, \Psi_3\) are independent of \(\xi\). Substituting the above expansion into the first equation of (2.3) and comparing the same order of \(\xi\)’s frequency, it follows from the \(O(\xi^2)\) terms that \(D\) is a diagonal
matrix. Furthermore, one finds the following equations for the \(O(\xi)\) and the diagonal part of the \(O(1)\) terms

\[
O(\xi) : i[\sigma_3, \Psi_1] = QD, \quad \text{i.e.} \quad \Psi_1^{(o)} = \frac{i}{2} QD \sigma_3,
\]

with \(\Psi_1^{(o)}\) being the off-diagonal part of \(\Psi_1\), and

\[
O(1) : D_x = Q\Psi_1^{(o)},
\]

i.e.

\[
D_x = \frac{i}{2} Q^2 \sigma_3 D. \quad (2.6)
\]

On the other hand, substituting the above expansion into the second equation of (2.3), one obtains from that

\[
O(\xi^3) : 2i[\sigma_3, \Psi_1] = 2QD, \quad \text{i.e.} \quad \Psi_1^{(o)} = \frac{i}{2} QD \sigma_3; \quad (2.7)
\]

and

\[
O(\xi) : 2i[\sigma_3, \Psi_3] = -iQ^2 \sigma_3 \Psi_1^{(o)} + 2Q\Psi_2^{(d)} - iQ_x \sigma_3 D + Q^3 D, \quad (2.8)
\]

i.e.

\[
- iQ^2 \sigma_3 \Psi_2^{(d)} + 2Q\Psi_3^{(o)} = -\frac{1}{2} Q^3 \Psi_1^{(o)} + \frac{1}{2} QQ_x D + \frac{i}{2} Q^4 \sigma_3 D, \quad (2.9)
\]

where \(\Psi_2^{(d)}\) denotes the diagonal part of \(\Psi_2\); and for the diagonal part of the \(O(1)\) terms

\[
O(1) : D_t = -iQ^2 \sigma_3 \Psi_2^{(d)} + 2Q\Psi_3^{(o)} - iQ_x \sigma_3 \Psi_1^{(o)} + Q^3 \Psi_1^{(o)},
\]

again, using (2.7) and (2.9), we have

\[
D_t = (\frac{3i}{4} Q^4 \sigma_3 + \frac{1}{2} [Q, Q_x]) D,
\]

which can be written in terms of \(q\) and \(r\) as

\[
D_t = (\frac{3i}{4} r^2 q^2 + \frac{1}{2} (r_x q - r q_x)) \sigma_3 D. \quad (2.10)
\]

### 2.2 Desired Lax pair

We note that Eq. (1.1) admits the conservation law

\[
(\frac{1}{2} r q)_t = (\frac{3i}{4} r^2 q^2 + \frac{1}{2} (r_x q - r q_x))_x.
\]
Consequently, the two Eqs. (2.6) and (2.10) for $D$ are consistent and are both satisfied if we define

$$D(x,t) = e^{i \int_{(0,0)}^{(x,t)} \Delta \sigma_3},$$

where $\Delta$ is the closed real-valued one-form

$$\Delta(x,t) = \frac{1}{2} dx + \left( \frac{3}{4} r^2 q^2 - \frac{i}{2} (r_x q - r q_x) \right) dt.$$

Noting that the integral in (2.11) is independent of the path of integration and the $\Delta$ is independent of $\xi$, then we introduce a new function $\mu$ by

$$\Psi(x,t,\xi) = e^{i \int_{(0,0)}^{(x,t)} \Delta \hat{\sigma}_3 \mu(x,t,\xi)} \hat{\sigma}_3 D(x,t),$$

Thus, we have

$$\mu = I + O(\frac{1}{\xi}), \quad \xi \to \infty,$$

and the Lax pair of Eq. (2.4) becomes

$$d(e^{i(\xi^2 x + 2 \xi^4 t) \hat{\sigma}_3 \mu(x,t,\xi)}) = W(x,t,\xi),$$

where

$$W(x,t,\xi) = e^{i(\xi^2 x + 2 \xi^4 t) \hat{\sigma}_3 V(x,t,\xi)} \mu,$$

Taking into account the definition of $U$ and $\Delta$, we find that

$$V = V_1 dx + V_2 dt = e^{i \int_{(0,0)}^{(x,t)} \Delta \hat{\sigma}_3 (U - i \Delta \sigma_3)}.$$

Then Eq. (2.15) for $\mu$ can be written as

$$\mu_x + i \xi^2 [\sigma_3, \mu] = V_1 \mu,$$
$$\mu_t + 2i \xi^4 [\sigma_3, \mu] = V_2 \mu.$$
Throughout this section we assume that \( q(x, t) \) is sufficiently smooth, in
\[
\Omega = \{0 < x < L, 0 < t < T\}
\]
where \( T \leq \infty \) is a given positive constant; unless otherwise specified, we suppose that \( T < \infty \).

Following the idea in Ref. [24], we define four solutions of Eq. (2.15) by
\[
\mu_j(x, t, \xi) = 1 + \int_{(x_j, t_j)}^{(x, t)} e^{-i(\xi^2x + 2\xi^4t)\sigma_3} W(y, \tau, \xi), \quad j = 1, 2, 3, 4,
\]
where \((x_1, t_1) = (0, T), (x_2, t_2) = (0, 0), (x_3, t_3) = (L, 0),\) and \((x_4, t_4) = (L, T)\), see Figure 1.

Since the one-form \( W \) is exact, the integral on the righthand side of (2.19) is independent of the path of integration. We choose the particular contours shown in Figure 2. By splitting the line integrals into integrals parallel to the \( t \) and the \( x \) axis we find
\[
\mu_1(x, t, \xi) = 1 + \int_0^x e^{i\xi^2(y-x)\sigma_3} (V_1\mu_1)(y, t, \xi) dy \\
- e^{-i\xi^2x\sigma_3} \int_0^T e^{2i\xi^4(\tau-t)\sigma_3} (V_2\mu_1)(0, \tau, \xi) d\tau,
\]
\[
\mu_2(x, t, \xi) = 1 + \int_0^x e^{i\xi^2(y-x)\sigma_3} (V_1\mu_2)(y, t, \xi) dy \\
+ e^{-i\xi^2x\sigma_3} \int_0^t e^{2i\xi^4(\tau-t)\sigma_3} (V_2\mu_2)(0, \tau, \xi) d\tau,
\]
\[
\mu_3(x, t, \xi) = 1 - \int_x^L e^{i\xi^2(y-x)\sigma_3} (V_1\mu_3)(y, t, \xi) dy \\
+ e^{-i\xi^2(L-x)\sigma_3} \int_0^t e^{2i\xi^4(\tau-t)\sigma_3} (V_2\mu_3)(L, \tau, \xi) d\tau,
\]
\[
\mu_4(x, t, \xi) = 1 - \int_x^L e^{i\xi^2(y-x)\sigma_3} (V_1\mu_4)(y, t, \xi) dy \\
- e^{-i\xi^2(L-x)\sigma_3} \int_t^T e^{2i\xi^4(\tau-t)\sigma_3} (V_2\mu_4)(L, \tau, \xi) d\tau.
\]
And we note that this choice implies the following inequalities on the contours,
\[(x_1, t_1) \rightarrow (x, t) : \ y - x \leq 0, \ \tau - t \geq 0\]
\[(x_2, t_2) \rightarrow (x, t) : \ y - x \leq 0, \ \tau - t \leq 0\]
\[(x_3, t_3) \rightarrow (x, t) : \ y - x \geq 0, \ \tau - t \leq 0\]
\[(x_4, t_4) \rightarrow (x, t) : \ y - x \geq 0, \ \tau - t \geq 0\]

We find that the second column of the matrix equation (2.19) involves \(e^{2i(\xi^2(y-x) + 2\xi^4(\tau-t))}\), and using the above inequalities it implies that the exponential term of \(\mu_j\) is bounded in the following regions of the complex \(\xi\)-plane,
\[(x_1, t_1) \rightarrow (x, t) : \ \{\text{Im}\xi^2 \leq 0\} \cap \{\text{Im}\xi^4 \geq 0\},\]
\[(x_2, t_2) \rightarrow (x, t) : \ \{\text{Im}\xi^2 \leq 0\} \cap \{\text{Im}\xi^4 \leq 0\},\]
\[(x_3, t_3) \rightarrow (x, t) : \ \{\text{Im}\xi^2 \geq 0\} \cap \{\text{Im}\xi^4 \leq 0\},\]
\[(x_4, t_4) \rightarrow (x, t) : \ \{\text{Im}\xi^2 \geq 0\} \cap \{\text{Im}\xi^4 \geq 0\}.\]

Thus, we have

\[\mu_1 = (\mu_1^{(2)}, \mu_1^{(3)}), \ \mu_2 = (\mu_2^{(1)}, \mu_2^{(4)}), \ \mu_3 = (\mu_3^{(3)}, \mu_3^{(2)}), \ \mu_4 = (\mu_4^{(4)}, \mu_4^{(1)}), \quad (2.24)\]

where \(\mu_j^{(i)}\) denotes \(\mu_j\) is bounded and analytic for \(\xi \in D_i\), and \(D_i = \omega_i \cup (-\omega_i), -\omega_i = \{ -\xi \in \mathbb{C} | \xi \in \omega_i \}, \omega_i = \{ \xi \in \mathbb{C} | \frac{i\pi}{4} < \xi < \frac{3\pi}{4} \}\), see Figure 3.

But the functions \(\mu_1(0, t, \xi), \mu_2(0, t, \xi), \mu_3(x, 0, \xi), \mu_3(L, t, \xi), \mu_4(L, t, \xi)\) are bounded in larger domains:

\[\mu_1(0, t, \xi) = (\mu_1^{(24)}(0, t, \xi), \mu_1^{(13)}(0, t, \xi)), \]
\[\mu_2(0, t, \xi) = (\mu_2^{(13)}(0, t, \xi), \mu_2^{(24)}(0, t, \xi)), \]
\[\mu_3(x, 0, \xi) = (\mu_3^{(34)}(x, 0, \xi), \mu_3^{(12)}(x, 0, \xi)), \quad (2.25)\]
\[\mu_4(L, t, \xi) = (\mu_4^{(13)}(L, t, \xi), \mu_4^{(24)}(L, t, \xi)), \]
\[\mu_4(L, t, \xi) = (\mu_4^{(24)}(L, t, \xi), \mu_4^{(13)}(L, t, \xi)).\]

By (2.14), it holds that

\[\mu_j(x, t, \xi) = I + O\left(\frac{1}{\xi}\right), \quad \xi \to \infty, j = 1, 2, 3, 4. \quad (2.26)\]
The $\mu_j$ are the fundamental eigenfunctions needed for the formulation of a Riemann-Hilbert problem in the complex $\xi$-plane.

In order to derive a Riemann-Hilbert problem, we have to compute the jumps across the boundaries of the $D_j$'s. It turns out that the relevant jump matrices can be uniquely defined in terms of three $2 \times 2$-matrix valued spectral functions $s(\xi), S(\xi)$ and $S_L(\xi)$ defined as follows.

Assuming $\mu$ and $\tilde{\mu}$ are the solutions of Eq. (?) then the two solutions are related by

$$\mu(x, t, \xi) = \tilde{\mu}(x, t, \xi)e^{-i(\xi^2 x + 2\xi^4 t)\hat{\sigma}_3}C_0(\xi),$$  \hspace{1cm} (2.27)

where $C_0(\xi)$ is a $2 \times 2$ matrix independent of $x$ and $t$. Let $\psi$ and $\tilde{\psi}$ be the solutions of Eq. (2.22) corresponding to $\mu$ and $\tilde{\mu}$ according to

$$\psi(x, t, \xi) = e^{i \int_{(0,0)}^{(x,t)} D(x, t) e^{-i(\xi^2 x + 2\xi^4 t)\hat{\sigma}_3}}$$  \hspace{1cm} (2.28)

but we note that there exists a $2 \times 2$ matrix $C_1(\xi)$ independent of $x$ and $t$ such that

$$\psi(x, t, \xi) = \tilde{\psi}(x, t, \xi)C_1(\xi).$$  \hspace{1cm} (2.29)

By using (2.27), (2.28) and (2.29), we have

$$C_0(\xi) = e^{-i \int_{(0,0)}^{(L,0)} \Delta \hat{\sigma}_3} C_1(\xi).$$  \hspace{1cm} (2.30)

From (2.27), the functions $\mu_j$ are related by the equations

$$\mu_3(x, t, \xi) = \mu_2(x, t, \xi)e^{-i(\xi^2 x + 2\xi^4 t)\hat{\sigma}_3}s(\xi),$$  \hspace{1cm} (2.31)

$$\mu_1(x, t, \xi) = \mu_2(x, t, \xi)e^{-i(\xi^2 x + 2\xi^4 t)\hat{\sigma}_3}S(\xi),$$  \hspace{1cm} (2.32)

$$\mu_4(x, t, \xi) = \mu_2(x, t, \xi)e^{-i(\xi^2 x + 2\xi^4 t)\hat{\sigma}_3}S_L(\xi).$$  \hspace{1cm} (2.33)

Evaluating equation (2.31) at $(x, t) = (0, 0)$, implies

$$s(\xi) = \mu_3(0, 0, \xi).$$  \hspace{1cm} (2.34)

Evaluating equation (2.32) at $(x, t) = (0, 0)$, gives

$$S(\xi) = \mu_1(0, 0, \xi).$$  \hspace{1cm} (2.35)

Evaluating equation (2.32) at $(x, t) = (0, T)$, yields

$$S(\xi) = (e^{2i\xi^4 T\hat{\sigma}_3} \mu_2(0, T, \xi))^{-1}.$$  \hspace{1cm} (2.36)
Evaluating equation (2.33) at \((x, t) = (L, 0)\), we have
\[
S_L(\xi) = \mu_4(L, 0, \xi).
\] (2.37)
Evaluating equation (2.33) at \((x, t) = (L, T)\), we get
\[
S_L(\xi) = (e^{2i\xi T\phi_3} \mu_3(L, T, \xi))^{-1}.
\] (2.38)
Eq. (2.31) and (2.33) imply
\[
\mu_4(x, t, \xi) = \mu_2(x, t, \xi) e^{-i(\xi^2 x + 2i\xi t)\phi_3} [s(\xi) e^{i\xi^2 L_\phi_3 S_L(\xi)}],
\] (2.39)
which will lead to the global relation.

Hence, the function \(s(\xi)\) can be obtained from the evaluations at \(x = 0\) of the function \(\mu_3(x, 0, \xi)\); \(S(\xi)\) can be obtained from the evaluations at \(t = T\) of the function \(\mu_2(0, t, \xi)\) and \(s_L(\xi)\) can be obtained from the evaluations at \(t = T\) of the function \(\mu_4(L, t, \xi)\). And these functions about \(\mu_j\) satisfy the linear integral equations
\[
\begin{align*}
\mu_1(0, t, \xi) &= \mathbb{I} - \int_t^T e^{2i\xi^2 (\tau - t)\phi_3} \langle V_2 \mu_1 \rangle(0, \tau, \xi) d\tau, \\
\mu_2(0, t, \xi) &= \mathbb{I} + \int_0^t e^{2i\xi^2 (\tau - t)\phi_3} \langle V_2 \mu_2 \rangle(0, \tau, \xi) d\tau, \\
\mu_3(x, 0, \xi) &= \mathbb{I} - \int_x^L e^{i\xi^2 (y - x)\phi_3} \langle V_1 \mu_3 \rangle(y, 0, \xi) dy, \\
\mu_3(L, t, \xi) &= \mathbb{I} + \int_0^t e^{2i\xi^2 (\tau - t)\phi_3} \langle V_2 \mu_3 \rangle(L, \tau, \xi) d\tau, \\
\mu_4(L, t, \xi) &= \mathbb{I} - \int_t^T e^{2i\xi^2 (\tau - t)\phi_3} \langle V_2 \mu_4 \rangle(L, \tau, \xi) d\tau.
\end{align*}
\] (2.40)
(2.41)
(2.42)
(2.43)
(2.44)
By evaluating the equations (2.16) at \(t = 0\) and (2.17) at \(x = 0, x = L\), we find the equations
\[
\begin{align*}
V_1(x, 0, \xi) &= \begin{pmatrix}
-\frac{i}{2} |q_0|^2 & \xi q_0 e^{-i \int_0^x |q_0|^2 dy} \\
\xi q_0 e^{i \int_0^x |q_0|^2 dy} & -\frac{i}{2} |q_0|^2
\end{pmatrix}, \\
V_2(0, t, \xi) &= \begin{pmatrix}
-\frac{i\xi^2 |q_0|^2}{2} & -\frac{3i}{4} |q_0|^4 - \frac{i}{2}(\bar{g}_1 g_0 - g_0 g_1) \\
(2\xi^3 g_0 - i\xi g_1 + \xi g_0 |g_0|^2) e^{i\int_0^x |g_0|^2 dy} & (2\xi^3 g_0 + i\xi g_1 + \xi g_0 |g_0|^2) e^{-i\int_0^x |g_0|^2 dy} + \frac{1}{4} |g_0|^4 + \frac{1}{2}(g_1 g_0 - g_0 g_1)
\end{pmatrix}.
\end{align*}
\] (2.45)
(2.46)
\[ V_2(L, t, \xi) = \left( \begin{array}{c} -i\xi^2 |f_0|^2 - \frac{g_0}{4} |f_0|^4 - \frac{1}{2} (f_1 f_0 - \bar{f}_0 f_1) \\ (2\xi^3 f_0 - i\xi f_1 + \xi f_0 |f_0|^2) e^{2i f_0^* \Delta_2(L, r) dr} \end{array} \right). \]

where \( q_0(x) = q(0, 0), g_0(t) = q(0, t), g_1(t) = q_s(0, t), f_0(t) = q(L, t) \) and \( f_1(t) = q_s(L, t) \) are the initial and boundary values of \( q(x, t) \), and

\[ \Delta_2(0, t) = \frac{3}{4} |g_0|^4 - i \left( \frac{1}{2} \bar{g}_1 g_0 - \bar{g}_0 g_1 \right). \] (2.48)

\[ \Delta_2(L, t) = \frac{3}{4} |f_0|^4 - i \left( \frac{1}{2} \bar{f}_1 f_0 - \bar{f}_0 f_1 \right). \] (2.49)

These expressions for \( V_1(x, 0, \xi), V_2(0, t, \xi) \) and \( V_2(L, t, \xi) \) contain only \( q_0(x), \{g_0(t), g_1(t)\} \) and \( \{f_0(t), f_1(t)\} \), respectively. Therefore, the integral equation (2.42) determining \( s(\xi) \) is defined in terms of the initial data \( q_0(x) \), the integral equation (2.41) determining \( S(\xi) \) is defined in terms of the initial data \( \{g_0(t), g_1(t)\} \) and the integral equation (2.43) determining \( S_L(\xi) \) is defined in terms of the initial data \( \{f_0(t), f_1(t)\} \).

Let us show the function \( \mu(x, t, \xi) \) satisfy the symmetry relations.

**Theorem 2.1.** For \( j = 1, 2, 3, 4 \), the function \( \mu(x, t, \xi) = \mu_j(x, t, \xi) \) satisfies the symmetry relations

\[ \mu_{11}(x, t, \xi) = \mu_{22}(x, t, \xi), \]

\[ \mu_{21}(x, t, \xi) = \mu_{12}(x, t, \xi), \] (2.50)

as well as

\[ \mu_{11}(x, t, -\xi) = \mu_{11}(x, t, \xi), \]

\[ \mu_{12}(x, t, -\xi) = -\mu_{12}(x, t, \xi), \]

\[ \mu_{21}(x, t, -\xi) = -\mu_{21}(x, t, \xi), \]

\[ \mu_{22}(x, t, -\xi) = \mu_{22}(x, t, \xi). \] (2.51)

**Proof.** Following the proposition 2.1’s proof in [4]. \( \square \)

If \( \psi(x, t) \) satisfies (2.22), it follows that \( \det(\psi) \) is independent of \( x \) and \( t \). Hence, since \( \det D(x, t) = 1 \), the determinant of the function \( \mu \) corresponding to \( \psi \) according to (2.28) is also independent of \( x \) and
In particular, for $\mu_j$, $j = 1, 2, 3, 4$, evaluation of $\det(\mu_j)$ at $(x_j, t_j)$ shows that

$$\det(\mu_j) = 1, \quad j = 1, 2, 3, 4,$$  \hspace{1cm} (2.52)

In particular,

$$dets(\xi) = detS(\xi) = detS_L(\xi) = 1.$$

It follows from (2.50) that

$$s_{11}(\xi) = s_{22}(\xi), \quad s_{21}(\xi) = s_{12}(\xi),$$

$$S_{11}(\xi) = S_{22}(\xi), \quad S_{21}(\xi) = S_{12}(\xi),$$

so that we use the following notations for $s(\xi), S(\xi)$ and $S_L(\xi)$.

\[
s(\xi) = \begin{pmatrix} a(\xi) & b(\xi) \\ b(\xi) & a(\xi) \end{pmatrix}, \quad S(\xi) = \begin{pmatrix} A(\xi) & B(\xi) \\ B(\xi) & A(\xi) \end{pmatrix},
\]

$$S_L(\xi) = \begin{pmatrix} A(\xi) & B(\xi) \\ B(\xi) & A(\xi) \end{pmatrix}.$$  \hspace{1cm} (2.53)

The relations in (2.51) imply that $a(\xi), A(\xi)$ and $A(\xi)$ are even functions of $\xi$, whereas $b(\xi), B(\xi)$ and $B(\xi)$ are odd functions of $\xi$, that is,

$$a(-\xi) = a(\xi), \quad b(-x) = -b(\xi),$$

$$A(-\xi) = A(\xi), \quad B(-x) = -B(\xi),$$

$$A(-\xi) = A(\xi), \quad B(-x) = -B(\xi).$$  \hspace{1cm} (2.54)

The definitions of $\mu_3(0, 0, \xi), \mu_2(0, T, \xi), \mu_3(0, L, 0, \xi)$ imply

$$s(\xi) = \mu_3(0, 0, \xi) = I - \int_0^L e^{2\xi^2(y-x)^3(V_1\mu_3)}(y, 0, \xi)dy,$$  \hspace{1cm} (2.55)

$$S^{-1}(\xi) = e^{2\xi^4T\delta_3} \mu_2(0, T, \xi) = I + \int_0^T e^{2\xi^4(\tau-t)\delta_3(V_2\mu_2)}(0, \tau, T, \xi)d\tau,$$  \hspace{1cm} (2.56)

$$S_L^{-1}(\xi) = e^{2\xi^4T\delta_3} \mu_3(L, T, \xi) = I + \int_0^T e^{2\xi^4(\tau-t)\delta_3(V_2\mu_3)}(L, \tau, T, \xi)d\tau.$$  \hspace{1cm} (2.57)

Equations (2.25), the determinant conditions (2.52), and the large $\xi$ behavior of $\mu_j$ imply the following properties

$$a(\xi), b(\xi)$$
• $a(\xi), b(\xi)$ are defined for $\{\xi \in \mathbb{C}| \text{Im} \xi^2 \geq 0\}$ and analytic for $\{\xi \in \mathbb{C}| \text{Im} \xi^2 > 0\}$.
• $a(\xi)a(\xi) - b(\xi)b(\xi) = 1, \xi^2 \in \mathbb{R}$.
• $a(\xi) = 1 + O(\frac{1}{\xi}), b(\xi) = O(\frac{1}{\xi}), \xi \to \infty, \text{Im} \xi^2 \geq 0$.

$A(\xi), B(\xi)$

• $A(\xi), B(\xi)$ are defined for $\{\xi \in \mathbb{C}| \text{Im} \xi^4 \geq 0\}$ and analytic for $\{\xi \in \mathbb{C}| \text{Im} \xi^4 > 0\}$.
• $A(\xi)A(\xi) - B(\xi)B(\xi) = 1, \xi^4 \in \mathbb{R}$.
• $A(\xi) = 1 + O(\frac{1}{\xi}), B(\xi) = O(\frac{1}{\xi}), \xi \to \infty, \text{Im} \xi^4 \geq 0$.

2.4 The global relation

We now show that the spectral functions are not independent but they satisfy an important global relation.

**Theorem 2.2.** Let the spectral functions $a(\xi), b(\xi), A(\xi), B(\xi), A(\xi), B(\xi)$ be defined in equations (2.53), where $s(\xi), S(\xi), S_L(\xi)$ are defined by equations (2.34), (2.36), (2.38), and $\mu_2, \mu_4$ are defined by equations (2.21), (2.22). Then these spectral functions are not independent but they satisfy an important global relation

$$(a(\xi)A(\xi) + b(\xi)B(\xi))B(\xi) - (b(\xi)A(\xi) + a(\xi)B(\xi))A(\xi) = e^{4i\xi^4T}c^+(\xi),$$

(2.58)

where $c^+(\xi)$ denotes the $(12)$ element of $-\int_0^L (e^{i\xi^2 y\hat{\sigma}_3})(V_4\mu_4)(y, T, \xi)dy$, and $\mu_4$ is defined by (2.23).

**Proof.** We just evaluating equation (2.39) at $(x, t) = (0, T)$. \qed

2.5 The jump conditions
Let \( M(x, y, \xi) \) be defined by

\[
M_+ = (\mu_1^{(1)}, \mu_2^{(1)}), \xi \in D_1, \quad M_- = (\mu_2^{(2)}, \mu_3^{(2)}), \xi \in D_2,
\]
\[
M_+ = (\mu_4^{(3)}, \mu_3^{(3)}), \xi \in D_3, \quad M_- = (\mu_4^{(4)}, \mu_4^{(4)}), \xi \in D_4,
\]  

(2.59)

where the scalars \( \alpha(\xi) \) and \( d(\xi) \) are defined below

\[
\alpha(\xi) = a(\xi)A(\xi) + b(\bar{\xi})e^{2\xi^2L}B(\xi),
\]  

(2.60)

\[
d(\xi) = a(\xi)A(\xi) - b(\xi)B(\xi).
\]  

(2.61)

These definitions imply

\[
\det M(x, t, \xi) = 1,
\]  

(2.62)

and

\[
M(x, t, \xi) = \mathbb{I} + O\left(\frac{1}{\xi}\right), \quad \xi \to \infty.
\]  

(2.63)

**Theorem 2.3.** Let \( M(x, t, \xi) \) be defined by equation (2.59), where \( \mu_1(x, t, \xi), \mu_2(x, t, \xi) \) and \( \mu_3(x, t, \xi), \mu_4(x, t, \xi) \) are defined by equations (2.20), (2.21) and (2.22), (2.23), and \( q(x, t) \) is a smooth function. Then \( M \) satisfies the jump condition

\[
M_+(x, t, \xi) = M_-(x, t, \xi)J(x, t, \xi), \quad \xi^4 \in \mathbb{R},
\]  

(2.64)

where the \( 2 \times 2 \) matrix \( J \) is defined by

\[
J = \begin{cases}
J_1, & \text{arg}\xi^2 = 0, \\
J_2, & \text{arg}\xi^2 = \frac{\pi}{2}, \\
J_3 = J_2J_1^{-1}J_4, & \text{arg}\xi^2 = \pi, \\
J_4, & \text{arg}\xi^2 = \frac{3}{2}\pi.
\end{cases}
\]  

(2.65)

and

\[
J_1 = \begin{pmatrix}
\frac{1}{\alpha(\xi)\alpha(\xi)} & \frac{\beta(\xi)}{\alpha(\xi)}e^{-2i\theta(\xi)} \\
-\frac{\beta(\xi)}{\alpha(\xi)}e^{2i\theta(\xi)} & 1
\end{pmatrix},
\]

\[
J_2 = \begin{pmatrix}
\frac{a(\xi)}{\alpha(\xi)} & B(\xi)e^{-2i\theta(\xi)}e^{2i\xi^2L} \\
-\frac{d(\xi)}{B(\xi)}e^{2i\theta(\xi)} & \frac{\delta(\xi)}{B(\xi)}
\end{pmatrix},
\]

\[
J_4 = \begin{pmatrix}
\frac{a(\xi)}{\alpha(\xi)} & B(\xi)e^{-2i\theta(\xi)}e^{-2i\xi^2L} \\
-\frac{d(\xi)}{B(\xi)}e^{2i\theta(\xi)} & \frac{\delta(\xi)}{B(\xi)}
\end{pmatrix},
\]

\[
\theta(\xi) = \xi^2x + 2\xi^4t,
\]
\[ \alpha(\xi) = a(\xi)A(\xi) + b(\xi)e^{2i\xi^2 L}B(\xi), \quad \beta(\xi) = b(\xi)A(\xi) + a(\xi)e^{2i\xi^2 L}B(\xi), \]  
(2.66)

\[ \delta(\xi) = \alpha(\xi)A(\xi) - \beta(\xi)B(\xi), \quad d(\xi) = a(\xi)A(\xi) - b(\xi)B(\xi). \]  
(2.67)

**Proof.** We can following the method of Proposition 2.2’s proof in [8]. □

The matrix \( M(x, t, \xi) \) defined in (2.59) is in general a meromorphic function of \( \xi \) in \( \mathbb{C}\{\xi^4 \in \mathbb{R}\} \). The possible poles of \( M \) are generated by the zeros of \( \alpha(\xi), d(\xi) \) and by the complex conjugates of these zeros. Since \( a(\xi), A(\xi) \) are even functions and \( b(\xi), B(\xi) \) are odd functions, \( \alpha(\xi) \) is even function. That means each zero \( \xi_j \) of \( \alpha(\xi) \) is accompanied by another zero at \( -\xi_j \). Similarly, each zero \( \lambda_j \) of \( d(\xi) \) is accompanied by a zero at \( -\lambda_j \). In particular, both \( \alpha(\xi) \) and \( d(\xi) \) have even number of zeros.

**Hypothesis H.2.4.** We assume that

- \( a(\xi) \) has \( 2\Lambda \) simple zeros \( \{k_j\}_{j=1}^{2\Lambda}, 2\Lambda = 2\Lambda_1 + 2\Lambda_2, \) such that \( k_j, j = 1, \ldots, 2\Lambda_1, \) lie in \( D_1 \) and \( \xi_j, j = 2\Lambda_1 + 1, \ldots, 2\Lambda \) lie in \( D_2 \).
- \( \alpha(\xi) \) has \( 2n \) simple zeros \( \{\xi_j\}_{j=1}^{2n}, 2n = 2n_1 + 2n_2, \) such that \( \xi_j, j = 1, \ldots, 2n, \) lie in \( D_1 \).
- \( d(\xi) \) has \( 2N \) simple zeros \( \{\lambda_j\}_{j=1}^{2N}, 2N = 2N_1 + 2N_2, \) such that \( \lambda_j, j = 1, \ldots, 2N, \) lie in \( D_2 \).
- None of the zeros of \( \alpha(\xi) \) coincides with any of the zeros of \( a(\xi) \).
- None of the zeros of \( d(\xi) \) coincides with any of the zeros of \( a(\xi) \).

According to the 2.4, we can evaluate the associated residues of \( M \). We introduce the notation \([A]_1([A]_2)\) for the first(second) column of a \( 2 \times 2 \) matrix \( A \) and we also write \( \dot{a}(\xi) = \frac{da}{d\xi} \). Then we get the following proposition

**Proposition 2.5.**

\[ \text{Res}_{\xi = \xi_j}[M(x, t, \xi)] = c_j^{(1)} e^{2i(\xi_j^2 x + 2\xi_j^4 t)} [M(x, t, \xi_j)]_2, \]  
(2.68)
\[ R_{\xi=\xi_j}[M(x,t,\xi)]_2 = c_j^{(1)} e^{2i(\xi_j^2 x + 2\xi_j^4 t)} [M(x,t,\xi_j)]_1, \]
\[ R_{\xi=\lambda_j}[M(x,t,\xi)]_1 = c_j^{(2)} e^{2i(\lambda_j^2 x + 2\lambda_j^4 t)} [M(x,t,\lambda_j)]_2, \]
\[ R_{\xi=\bar{\lambda}_j}[M(x,t,\xi)]_2 = c_j^{(2)} e^{2i(\bar{\lambda}_j^2 x + 2\bar{\lambda}_j^4 t)} [M(x,t,\bar{\lambda}_j)]_1, \]

where
\[ c_j^{(1)} = \frac{1}{a(\xi_j)b(\xi_j)} \frac{a(\xi_j)}{e^{2i\xi_j^2 t} \delta(\xi_j)}, \]
\[ c_j^{(2)} = \frac{B(\lambda_j)}{a(\lambda_j)d(\lambda_j)}. \]

**Proof.** Following [8]. \( \square \)

### 2.6 The inverse problem

The inverse problem involves reconstructing the potential \( q(x,t) \) from the eigenfunctions \( \mu_j(x,t,\xi), j = 1, 2, 3, 4 \). We follow the steps of [4]. That means we want to reconstruct the potential \( q(x,t) \), then the first step is using any of the four eigenfunctions \( \mu_j, j = 1, 2, 3, 4 \), to compute \( m(x,t) \) according to

\[ m(x,t) = \lim_{\xi \to \infty} (\xi \mu_j(x,t,\xi))_{12}. \]

The second step is determining \( \Delta(x,t) \) by

\[ rq = 4|m|^2, \]
\[ r_xq - r q_x = 4(\bar{m}_x m - m_x \bar{m}) - 32i|m|^4, \]
\[ \Delta = 2|m|^2 dx - (4|m|^4 + 2i(\bar{m}_x m - m_x \bar{m})) dt. \]

finally, \( q(x,t) \) is given by

\[ q(x,t) = 2im(x,t)e^{2i \int_{(0,0)}^{(x,t)} \Delta}. \]

### 3. The definition of spectral functions and The Riemann-Hilbert Problem

#### 3.1 The definition of spectral functions

The analysis of section 2 motivates the following definitions for the spectral functions.
Definition 3.1. (The spectral functions \(a(\xi)\) and \(b(\xi)\)) Given the smooth function \(q_0(x)\), we define the map

\[ S : \{q_0(x)\} \rightarrow \{a(\xi), b(\xi)\} \]

with

\[
\begin{pmatrix}
  b(\xi) \\
  a(\xi)
\end{pmatrix} = [\mu_3(0, \xi)]_2, \quad \text{Im} \xi^2 \geq 0.
\]

where \(\mu_3(x, \xi)\) is the unique solution of the Volterra linear integral equation

\[
\mu_3(x, \xi) = \mathbb{I} - \int_x^L e^{i\xi^2(y-x)x} (V_1 \mu_3)(y, 0, \xi) \, dy,
\]

and \(V_1(x, 0, \xi)\) is given in terms of \(q_0(x)\) by (2.45).

Properties of \(a(\xi), b(\xi)\)

- \(a(\xi), b(\xi)\) are defined for \(\{\xi \in \mathbb{C} | \text{Im} \xi^2 \geq 0\}\) and analytic for \(\{\xi \in \mathbb{C} | \text{Im} \xi^2 > 0\}\);
- \(a(\xi)a(\bar{\xi}) - b(\xi)b(\bar{\xi}) = 1, \xi^2 \in \mathbb{R}\),
- \(a(\xi) = 1 + O(\frac{1}{\xi}), b(\xi) = O(\frac{1}{\xi}), \xi \rightarrow \infty, \text{Im} \xi^2 \geq 0\),
  in particular,
  \(a(\xi), b(\xi), a(\xi)e^{2i\xi^2L}, b(\xi)e^{2i\xi^2L}\) are bounded for \(\text{Im} \xi^2 \geq 0\).

Remark 3.1. The definition 3.1 gives rise to the map,

\[ S : \{q_0(x)\} \rightarrow \{a(\xi), b(\xi)\}. \]

The inverse of this map,

\[ Q : \{a(\xi), b(\xi)\} \rightarrow \{q_0(x)\}, \]

can be defined as follows:

\[
q_0(x) = 2i \text{Im}(x, t) e^{4i \int_0^x |m(y)|^2 \, dy}, \\
m(x) = \lim_{\xi \rightarrow \infty} (\xi M(x, \xi))_{12}, \tag{3.1}
\]

where \(M(x, \xi)\) is the unique solution of the following Riemann-Hilbert problem:
\[ M(x, \xi) = \begin{cases} 
M(x, \xi) \quad \text{Im} \xi^2 \leq 0, \\
M_+(x, \xi) \quad \text{Im} \xi^2 \geq 0. 
\end{cases} \]

is a sectionally meromorphic function.

\[ M_+(x, \xi) = M_-(x, \xi)J(x, \xi), \quad \xi^2 \in \mathbb{R}, \]

where

\[
J(x, \xi) = \begin{pmatrix} 
\frac{1}{a(\xi)} & b(\xi) e^{-2i\xi^2 x} \\
-b(\xi) & a(\xi) e^{2i\xi^2 x} 
\end{pmatrix}, \quad \xi^2 \in \mathbb{R}.
\] (3.2)

\[ M(x, \xi) = I + O\left(\frac{1}{\xi}\right), \quad \xi \to \infty. \]

\[ a(\xi) \text{ has } 2\Lambda \text{ simple zeros } \{k_j\}_{j=1}^{2\Lambda}, \quad 2\Lambda = 2\Lambda_1 + 2\Lambda_2, \text{ such that } \]

\[ k_j, j = 1, \ldots, 2\Lambda_1, \text{ lie in } D_1, \text{ and } k_j, j = 2\Lambda_1 + 1, \ldots, 2\Lambda \text{ lie in } D_2. \]

The first column of \( M_+ \) has simple poles at \( \xi = k_j, j = 1, \ldots, 2\Lambda_1 \), and the second column of \( M_- \) has simple poles at \( \xi = \bar{k}_j, j = 1, \ldots, 2\Lambda_1 \). The associated residues are given by

\[
\text{Res}_{\xi=k_j}[M(x, \xi)]_1 = \frac{1}{a(k_j) b(k_j)} e^{2ik_j^2 x} [M(x, k_j)]_2, \quad j = 1, \ldots, 2\Lambda. \] (3.3)

\[
\text{Res}_{\xi=\bar{k}_j}[M(x, \xi)]_2 = \frac{1}{a(\bar{k}_j) b(\bar{k}_j)} e^{-2i\bar{k}_j^2 x} [M(x, \bar{k}_j)]_1, \quad j = 1, \ldots, 2\Lambda. \] (3.4)

**Definition 3.2.** (The spectral functions \( A(\xi) \) and \( B(\xi) \)) Given the smooth function \( g_0(t), g_1(t) \), we define the map

\[ S^{(0)} : \{g_0(t), g_1(t)\} \to \{A(\xi), B(\xi)\} \]

with

\[ \begin{pmatrix} 
B(\xi) \\
A(\xi) 
\end{pmatrix} = [\mu_1(0, \xi)]_2, \quad \text{Im} \xi^4 \geq 0. \]

where \( \mu_1(t, \xi) \) is the unique solution of the Volterra linear integral equation

\[ \mu_1(x, \xi) = I - \int_t^T e^{2i\xi^4(\tau-t)} \sigma_3(V_2 \mu_1)(0, \tau, \xi) d\tau, \]

and \( V_2(0, t, \xi) \) is given in terms of \( g_0(t), g_1(t) \) by (2.46).

Properties of \( A(\xi), B(\xi) \)
A(ξ), B(ξ) are defined for \( \{ ξ ∈ \mathbb{C} | \text{Im} ξ^4 ≥ 0 \} \) and analytic for \( \{ ξ ∈ \mathbb{C} | \text{Im} ξ^4 > 0 \} \),

\[ A(ξ)A(¯ξ) − B(ξ)B(¯ξ) = 1, ξ^4 ∈ R, \]

\[ A(ξ) = 1 + O(\frac{1}{ξ}), B(ξ) = O(\frac{1}{ξ}), ξ → ∞, \text{Im} ξ^4 ≥ 0, \text{in particular,} \]

\[ A(ξ), B(ξ) \text{ are bounded for } ξ ∈ D_1 ∪ D_2. \]

**Remark 3.2.** The definition 3.2 gives rise to the map,

\[ S^{(0)} : \{ g_0(x), g_1(x) \} → \{ A(ξ), B(ξ) \} \]

The inverse of this map,

\[ Q^{(0)} : \{ A(ξ), B(ξ) \} → \{ g_0(x), g_1(x) \}, \]

can be defined as follows:

\[ g_0(t) = 2im_{12}^{(1)}(t)e^{2i∫_0^t Δ_2(τ)dτ}, \]

\[ g_1(t) = (4m_{12}^{(3)}(t) + |g_0(t)|^2m_{12}^{(1)}(t))e^{2i∫_0^t Δ_2(τ)dτ} + ig_0(t)(2m_{22}^{(2)}(t) + |g_0(t)|^2), \]

(3.5)

where

\[ Δ_2(t) = 4|m_{12}^{(1)}|^4 + 8(\text{Re}[m_{12}^{(1)}m_{12}^{(3)}] − |m_{12}^{(1)}|^2\text{Re}[m_{22}^{(2)}]). \]

The functions \( m^{(1)}(t), m^{(2)}(t), m^{(3)}(t) \) are determined by the asymptotic expansion

\[ M^{(t)}(t, ξ) = Π + \frac{m^{(1)}(t)}{ξ} + \frac{m^{(2)}(t)}{ξ^2} + \frac{m^{(3)}(t)}{ξ^3} + O\left(\frac{1}{ξ^4}\right), \quad ξ → ∞, \]

where \( M^{(t)}(t, ξ) \) is the unique solution of the following Riemann-Hilbert problem:

- \( M^{(t)}(t, ξ) = \begin{cases} M_+^{(t)}(t, ξ) & \text{Im} ξ^4 ≤ 0, \\ M_−^{(t)}(t, ξ) & \text{Im} ξ^4 ≥ 0. \end{cases} \)

is a sectionally meromorphic function.

- \( M_+^{(t)}(t, ξ) = M_{−}^{(t)}(t, ξ)J^{(+0)}(t, ξ), \quad ξ^4 ∈ R, \)

where

\[ J^{(t,0)}(t, ξ) = \begin{pmatrix} \frac{1}{A(ξ)A(¯ξ)} & \frac{B(ξ)}{A(ξ)} e^{-4iξ^4t} \\ \frac{B(ξ)}{A(ξ)} e^{4iξ^4t} & 1 \end{pmatrix}, \quad ξ^4 ∈ R. \]

(3.6)
\[ M(t, \xi) = I + O\left(\frac{1}{\xi}\right), \quad \xi \to \infty. \]

- \( A(\xi) \) has 2A simple zeros \( \{K_j\}_{j=1}^{2A}, 2A = 2A_1 + 2A_2 \), such that \( K_j, j = 1, \cdots, 2A_1 \), lie in \( D_1 \), and \( K_j, j = 2A_1 + 1, \cdots, 2A \) lie in \( D_3 \).

- The first column of \( M(t) \) has simple poles at \( \xi = K_j, j = 1, \cdots, 2A \), and the second column of \( M(t) \) has simple poles at \( \xi = \bar{K}_j, j = 1, \cdots, 2A \). The associated residues are given by

\[
\text{Res}_{\xi = K_j}[M(t)(t, \xi)]_1 = \frac{1}{A(K_j)B(K_j)} e^{4i\xi_4 t}[M(t)(t, K_j)]_2, \quad j = 1, \cdots, 2A.
\]

(3.7)

\[
\text{Res}_{\xi = \bar{K}_j}[M(t)(t, \xi)]_2 = \frac{1}{A(K_j)B(K_j)} e^{-4i\xi_4 t}[M(t)(t, \bar{K}_j)]_1, \quad j = 1, \cdots, 2A.
\]

(3.8)

**Definition 3.3.** (The spectral functions \( A(\xi) \) and \( B(\xi) \)) Given the smooth function \( f_0(t), f_1(t) \), we define the map

\[
S^{(L)} : \{f_0(t), f_1(t)\} \to \{A(\xi), B(\xi)\}
\]

with

\[
\begin{pmatrix}
B(\xi) \\
A(\xi)
\end{pmatrix} = [\mu_4(0, \xi)]_2, \quad \text{Im} \xi^4 \geq 0.
\]

where \( \mu_4(t, \xi) \) is the unique solution of the Volterra linear integral equation

\[
\mu_4(x, \xi) = I - \int_t^T e^{2i\xi_4(\tau-t)}\mu_4(L, \tau, \xi) d\tau,
\]

and \( V_2(L, t, \xi) \) is given in terms of \( f_0(t), f_1(t) \) by (2.47).

**Properties of \( A(\xi), B(\xi) \)**

- \( A(\xi), B(\xi) \) are defined for \( \{\xi \in \mathbb{C} | \text{Im} \xi^4 \geq 0\} \) and analytic for \( \{\xi \in \mathbb{C} | \text{Im} \xi^4 > 0\} \).
- \( A(\xi)A(\xi) - B(\xi)B(\xi) = 1, \xi^4 \in \mathbb{R}, \)
- \( A(\xi) = 1 + O\left(\frac{1}{\xi}\right), B(\xi) = O\left(\frac{1}{\xi}\right), \xi \to \infty, \text{Im} \xi^4 \geq 0. \)

**Remark 3.3.** The definition 3.3 gives rise to the map,

\[
S^{(L)} : \{f_0(x), f_1(x)\} \to \{A(\xi), B(\xi)\}.
\]
The inverse of this map, 

\[ Q^{(L)} : \{A(\xi), B(\xi)\} \to \{f_0(x), f_1(x)\}, \]

can be defined as follows

\[
\begin{align*}
    f_0(t) &= 2im_{12}^{(1)}(t)e^{2i \int_0^t \Delta_2^L(r)dr}, \\
    f_1(t) &= (4m_{12}^{(3)}(t) + |g_0(t)|^2m_{12}^{(1)}(t))e^{2i \int_0^t \Delta_2^L(r)dr} + ig_0(t)(2m_{22}^{(2)}(t) + |g_0(t)|^2),
\end{align*}
\]

where

\[
\Delta_2^L(t) = 4|m_{12}^{(1)}|^4 + 8(\text{Re}[m_{12}^{(1)} \bar{m}_{12}^{(3)}] - |m_{12}^{(1)}|^2 \text{Re}[m_{22}^{(2)}]),
\]

and the functions \(m^{(1)}(t), m^{(2)}(t), m^{(3)}(t)\) are determined by the asymptotic expansion

\[
M(t, \xi) = I + \frac{m^{(1)}(t)}{\xi} + \frac{m^{(2)}(t)}{\xi^2} + \frac{m^{(3)}(t)}{\xi^3} + O\left(\frac{1}{\xi^4}\right), \quad \xi \to \infty,
\]

where \(M(t, \xi)\) is the unique solution of the following Riemann-Hilbert problem:

- \(M(t, \xi) = \begin{cases} M_-^{(t)}(t, \xi) & \text{Im} \xi^4 \leq 0, \\ M_+^{(t)}(t, \xi) & \text{Im} \xi^4 \geq 0. \end{cases}\)
  is a sectionally meromorphic function.
- \(M_+^{(t)}(t, \xi) = M_-^{(t)}(t, \xi)J^{(t,L)}(t, \xi), \quad \xi^4 \in \mathbb{R},\)
  where

\[
J^{(t,L)}(t, \xi) = \begin{pmatrix}
\frac{1}{\mathcal{A}(\xi)\mathcal{A}(\xi)} & \frac{\mathcal{B}(\xi)}{\mathcal{A}(\xi)}e^{-4i\xi^4t} \\
\frac{\mathcal{B}(\xi)}{\mathcal{A}(\xi)}e^{4i\xi^4t} & 1
\end{pmatrix}, \quad \xi^4 \in \mathbb{R}. \tag{3.10}
\]

- \(M(t, \xi) = I + O\left(\frac{1}{\xi}\right), \quad \xi \to \infty.\)
- \(\mathcal{A}(\xi)\) has 2\(\mathcal{A}\) simple zeros \(\{\mathcal{K}_j\}_{j=1}^{2\mathcal{A}}, 2\mathcal{A} = 2\mathcal{A}_1 + 2\mathcal{A}_2,\) such that \(\mathcal{K}_j, j = 1, \cdots, 2\mathcal{A}_1,\) lie in \(D_1,\) and \(\mathcal{K}_j, j = 2\mathcal{A}_1 + 1, \cdots, 2\mathcal{A}\) lie in \(D_3.\)
- The first column of \(M_+^{(t)}\) has simple poles at \(\xi = \mathcal{K}_j, j = 1, \cdots, 2\mathcal{A},\) and the second column of \(M_-^{(t)}\) has simple poles at
\[ \xi = K_j, j = 1, \ldots, 2A. \] The associated residues are given by
\[
\text{Res}_{\xi = K_j} [M(t)(t, \xi)]_1 = \frac{1}{\mathcal{A}(K_j)\mathcal{B}(K_j)} e^{4iK_j^2t}[M(t)(t, K_j)]_2, \quad j = 1, \ldots, 2A. \tag{3.11}
\]
\[
\text{Res}_{\xi = K_j} [M(t)(t, \xi)]_2 = \frac{1}{\mathcal{A}(K_j)\mathcal{B}(K_j)} e^{-4iK_j^4t}[M(t)(t, \bar{K}_j)]_1, \quad j = 1, \ldots, 2A. \tag{3.12}
\]

**Definition 3.4.** (An admissible set). Given the smooth function \( q_0(x) \) define \( a(\xi), b(\xi) \) according to definition 3.1. Suppose that there exist smooth functions \( g_0(t), g_1(t), f_0(t), f_1(t) \), such that

- The associated \( A(\xi), B(\xi), A(\xi), B(\xi) \), defined according to definition 3.2 and 3.3, satisfy the relation
  \[
  (a(\xi)A(\xi) + b(\xi)e^{2i\xi^2L}B(\xi))B(\xi) - (b(\xi)A(\xi) + a(\xi)e^{2i\xi^2L}B(\xi))A(\xi) = e^{4i\xi^4T}c^+(\xi), \quad \xi \in \mathbb{C},
  \]
  where \( c^+(\xi) \) is an entire function, which is bounded for \( \text{Im} \xi^2 \geq 0 \) and \( c^+(\xi) = O(\frac{1}{\xi^4}) \), as \( \xi \to \infty \).
- \( g_0(0) = q_0(0), g_1(0) = q_0'(0), f_0(0) = q_0(L), f_1(0) = q_0'(L). \)

Then we call the functions \( g_0(t), g_1(t), f_0(t), f_1(t) \), an admissible set of functions with respect to \( q_0(x) \).

### 3.2 The Riemann-Hilbert problem

**Theorem 3.4.** Let \( q_0(x) \) be a smooth function. Suppose that the set of functions \( g_0(t), g_1(t), f_0(t), f_1(t) \), are admissible with respect to \( q_0(x) \). Define the spectral functions \( a(\xi), b(\xi), A(\xi), B(\xi), A(\xi), B(\xi) \), in terms of \( q_0(x), g_0(t), g_1(t), f_0(t), f_1(t) \). According to the (2.4), Define \( M(x, t, \xi) \) as the solution of the following 2 \times 2 matrix Riemann-Hilbert problem

- \( M \) is sectionally meromorphic in \( \mathbb{C}\backslash\{\xi^4 \in \mathbb{R}\} \), and has unit determinant.
- \( M \) satisfies the jump condition
  \[
  M_+(x, t, \xi) = M_-(x, t, \xi) J(x, t, \xi), \quad \xi^4 \in \mathbb{R}.
  \]
where \( M \) is \( M_+ \) for \( \text{Im} \xi^4 \geq 0 \), \( M_- \) for \( \text{Im} \xi^4 \leq 0 \), and \( J \) is defined in terms of \( a, b, A, B, A, B \) by Eq. (2.66).

- \( M(x, t, \xi) = \mathbb{I} + O(\frac{1}{\xi}), \xi \to \infty. \)

- Residue conditions (2.68)-(2.71).

Then \( M(x, t, \xi) \) exists and is unique.

Define \( q(x, t) \) in terms of \( M(x, t, \xi) \) by

\[
q(x, t) = 2im(x,t)e^{2i\int_{(0,0)}^{(x,t)} \Delta},
\]

\[
m(x, t) = \lim_{\xi \to \infty} (\xi \mu_j(x, t, \xi))_{12},
\]

\[
\Delta = 2|m|^2 dx - (4|m|^4 + 2i(\tilde{m}_x m - m_x \tilde{m}))dt.
\]

Then \( q(x, t) \) solves the DNLS equation (1.1) with

\[
q(x, 0) = q_0(x), q(0, t) = g_0(t), q_x(0, t) = g_1(t), q(L, t) = f_0(t), q_x(L, t) = f_1(t).
\]

Proof. If \( \alpha(\xi) \) and \( d(\xi) \) have no zeros for \( \xi \in D_1 \) and for \( \xi \in D_2 \) respectively, then the function \( M(x, t, \xi) \) satisfies a non-singular Riemann-Hilbert problem. Using the fact that the jump matrix \( J \) satisfies appropriate symmetry conditions it is possible to show that this problem has a unique global solution [25]. The case that \( \alpha(\xi) \) and \( d(\xi) \) have a finite number of zeros can be mapped to the case of no zeros supplemented by an algebraic system of equations which is always uniquely solvable [25].

Proof that \( q(x, t) \) satisfies the DNLS equation

Using arguments of the dressing method [26], it can be verified directly that if \( M(x, t, \xi) \) is defined as the unique solution of the above Riemann-Hilbert problem, and if \( q(x, t) \) is defined in terms of \( M \) by equation (3.14), then \( q \) and \( M \) satisfy both parts of the Lax pair, hence \( q \) solves the DNLS equation.

Proof that \( q(x, 0) = q_0(x) \).

Evaluating the equation (2.65) at \( t = 0 \), we can divide the jump matrix into product of \( 2 \times 2 \) matrix. By changing the problem into the
Thus we can verify that:

$$J_1 = \begin{pmatrix}
\frac{1}{\alpha(\xi)\alpha(\xi)} & \frac{\beta(\xi)}{\alpha(\xi)} e^{-2i\xi^2 x} \\
-\frac{\beta(\xi)}{\alpha(\xi)} e^{2i\xi^2 x} & 1
\end{pmatrix},$$  \hspace{1cm} (3.15)

$$J_2 = \begin{pmatrix}
\frac{\alpha(\xi)}{\alpha(\xi)} & B(\xi) e^{-2i\xi^2 x} e^{2i\xi^2 L} \\
\frac{\delta(\xi)}{d(\xi)} & \frac{\beta(\xi)}{d(\xi)} e^{2i\xi^2 x}
\end{pmatrix},$$  \hspace{1cm} (3.16)

$$J_4 = \begin{pmatrix}
\alpha(\xi) & -\overline{B}(\xi) e^{2i\xi^2 x} e^{-2i\xi^2 L} \\
\alpha(\xi) & \frac{\beta(\xi)}{d(\xi)} e^{2i\xi^2 x}
\end{pmatrix}. $$  \hspace{1cm} (3.17)

And then we introduce some 2 × 2 matrix

$$J_1^{(\infty)} = \begin{pmatrix}
\frac{1}{\alpha(\xi)\alpha(\xi)} & \frac{\beta(\xi)}{\alpha(\xi)} e^{-2i\xi^2 x} \\
-\frac{\beta(\xi)}{\alpha(\xi)} e^{2i\xi^2 x} & 1
\end{pmatrix},$$

$$J_2^{(\infty)} = \begin{pmatrix}
1 & 0 \\
\frac{\beta(\xi)}{d(\xi)} e^{2i\xi^2 x} & 1
\end{pmatrix},$$

$$J_4^{(\infty)} = \begin{pmatrix}
1 & \frac{\beta(\xi)}{d(\xi)} e^{-2i\xi^2 x} \\
0 & 1
\end{pmatrix},$$

$$J_1 = \begin{pmatrix}
\frac{\alpha(\xi)}{\alpha(\xi)} & B(\xi) e^{2i\xi^2 x} e^{-2i\xi^2 x} \\
0 & \frac{\alpha(\xi)}{\alpha(\xi)}
\end{pmatrix},$$

$$J_4 = \begin{pmatrix}
\frac{\alpha(\xi)}{\alpha(\xi)} & -\overline{B}(\xi) e^{-2i\xi^2 x} e^{2i\xi^2 x} \\
0 & \frac{\alpha(\xi)}{\alpha(\xi)}
\end{pmatrix}. $$

Thus we can verify that:

$$J_1(x, 0, \xi) = \hat{J}_4 J_1^{(\infty)} \hat{J}_1, \quad J_2(x, 0, \xi) = J_2^{(\infty)} J_1, \quad J_3(x, 0, \xi) = J_2^{(\infty)} (J_1^{(\infty)})^{-1} J_4^{(\infty)}, \quad J_4(x, 0, \xi) = \hat{J}_4 J_4^{(\infty)}. $$  \hspace{1cm} (3.19)

Let $M^{(1)}(x, t, \xi), M^{(2)}(x, t, \xi), M^{(3)}(x, t, \xi), M^{(4)}(x, t, \xi)$ denote $M(x, t, \xi)$ for $\xi \in D_1, \xi \in D_2, \xi \in D_3, \xi \in D_4$. Then the jump condition (2.64) becomes

$$M^{(1)} = M^{(4)} J_1, M^{(1)} = M^{(2)} J_2, M^{(3)} = M^{(2)} J_3, M^{(3)} = M^{(4)} J_4. $$  \hspace{1cm} (3.20)
Using equations (3.19), we find
\begin{align*}
M^{(1)}(x, 0, \xi) &= M^{(4)}(x, 0, \xi)\hat{J}_4 J^{(\infty)}_4, \\
M^{(1)}(x, 0, \xi) &= M^{(2)}(x, 0, \xi)\hat{J}_2 J^{(\infty)}_2, \\
M^{(3)}(x, 0, \xi) &= M^{(2)}(x, 0, \xi)\hat{J}_2 J^{(\infty)}_2 (J^{(\infty)}_1)^{-1} J^{(\infty)}_4, \\
M^{(3)}(x, 0, \xi) &= M^{(4)}(x, 0, \xi)\hat{J}_4 J^{(\infty)}_4.
\end{align*}
(3.21)

Defining $M_j^{(\infty)}$, $j = 1, 2, 3, 4$, by
\begin{align*}
M_1^{(\infty)} &= M^{(1)}(x, 0, \xi)(\hat{J}_1)^{-1}, & M_2^{(\infty)} &= M^{(2)}(x, 0, \xi), \\
M_3^{(\infty)} &= M^{(3)}(x, 0, \xi), & M_4^{(\infty)} &= M^{(4)}(x, 0, \xi)\hat{J}_4.
\end{align*}
(3.22)
then we find that the sectionally holomorphic function $M^{(\infty)}(x, \xi)$ satisfies the jump conditions
\begin{align*}
M_1^{(\infty)} &= M_4^{(\infty)} J_1^{(\infty)}, & M_1^{(\infty)} &= M_2^{(\infty)} J_2^{(\infty)}, \\
M_3^{(\infty)} &= M_2^{(\infty)} J_3^{(\infty)}, & M_3^{(\infty)} &= M_4^{(\infty)} J_4^{(\infty)}.
\end{align*}
(3.23)

These conditions are precisely the jump conditions satisfied by the unique solution of the Riemann-Hilbert problem associated with DNLS for $0 < x < \infty$, $0 < t < T$. Also $detM^{(\infty)} = 1$ and $M^{(\infty)} = \mathbb{I} + O(\xi)$, $\xi \to \infty$. Moreover, by a straightforward calculation one can verify that the associated residue conditions change into the proper residue conditions. Therefore, $M^{(\infty)}(x, \xi)$ satisfies the same Riemann-Hilbert problem as the Riemann-Hilbert problem associated with the half-line evaluated at $t = 0$. Hence, $q(x, 0) = q_0(x)$.

Proof that $q(0, t) = g_0(t), q_x(0, t) = g_1(t)$

Let $M^{(t,0)}(t, \xi)$ be defined by
\begin{align*}
M^{(t,0)}(t, \xi) &= M(0, t, \xi)G(t, \xi),
\end{align*}
(3.24)
where $G$ is given by $G^{(1)}, G^{(2)}, v, G^{(3)}, G^{(4)}$, for $\xi \in D_1, \xi \in D_2, \xi \in D_3, \xi \in D_4$. Suppose we can find matrices $G^{(j)}$ which are holomorphic, tend to $\mathbb{I}$ as $\xi \to \infty$, and satisfy
\begin{align*}
J_2(0, t, \xi)G^{(1)}(t, \xi) &= G^{(2)}(t, \xi)J^{(t,0)}(t, \xi), \\
J_1(0, t, \xi)G^{(1)}(t, \xi) &= G^{(4)}(t, \xi)J^{(t,0)}(t, \xi), \\
J_4(0, t, \xi)G^{(3)}(t, \xi) &= G^{(4)}(t, \xi)J^{(t,0)}(t, \xi),
\end{align*}
(3.25)
where $J^{(t,0)}(\xi)$ is defined in (3.6). Then equation (3.25) yield $J_3(0, t, \xi)G^{(3)}(t, \xi) = G^{(2)}(t, \xi)J^{(t,0)}(\xi)$, and equations (3.20), (3.24) imply that $M^{(t,0)}(t, \xi)$ satisfies the Riemann-Hilbert problem defined in Remark 3.2. Then the Remark 3.2 implies the desired result.

We will show that such $G^{(j)}$ matrices exist and can be written as

$$G^{(1)}(t, \xi) = \begin{pmatrix} \frac{\alpha(\xi)}{\Lambda(\xi)} & c^+(\xi)e^{4\xi^4(T-t)} \\ 0 & \Lambda(\xi) \end{pmatrix},$$

$$G^{(2)}(t, \xi) = \begin{pmatrix} d(\xi) & \frac{\beta(\xi)}{\Lambda(\xi)}e^{-4\xi^4t} \\ 0 & \frac{1}{d(\xi)} \end{pmatrix},$$

$$G^{(3)}(t, \xi) = \begin{pmatrix} \frac{1}{d(\xi)} & 0 \\ -\frac{\beta(\xi)}{\Lambda(\xi)}e^{4\xi^4t} & \frac{1}{d(\xi)} \end{pmatrix},$$

$$G^{(4)}(t, \xi) = \begin{pmatrix} \frac{\Lambda(\xi)}{\alpha(\xi)} & 0 \\ c^+(\xi)e^{-4\xi^4(T-t)} & \frac{\alpha(\xi)}{\Lambda(\xi)} \end{pmatrix}. \tag{3.26}$$

We use straight forward calculation to verify these $G^{(j)}$ matrices satisfy the conditions (3.25). Similar to the proof of the equation $q(x, 0) = q_0(x)$, it can be verified that the transformation (3.24) replaces the residue conditions (2.68) -(2.71) by the residue conditions of Remark 3.2.

**Proof that** $q(L, t) = f_0(t), q_x(L, t) = f_1(t)$

Following the arguments similar to the prove above we must show that the matrices $F^{(j)}(t, \xi)$ such that

$$J_2(L, t, \xi)F^{(1)}(t, \xi) = F^{(2)}(t, \xi)J^{(t,L)}(\xi),$$

$$J_1(L, t, \xi)F^{(1)}(t, \xi) = F^{(4)}(t, \xi)J^{(t,L)}(\xi),$$

$$J_4(L, t, \xi)F^{(3)}(t, \xi) = F^{(4)}(t, \xi)J^{(t,L)}(\xi). \tag{3.27}$$
We will show that such $F^{(j)}$ matrices are

\[
F^{(1)}(t, \xi) = \begin{pmatrix}
-1 & 0 \\
\frac{-b(\xi)e^{4(\xi^4 t + 2\xi^2 L)}}{\alpha(\xi)\tilde{A}(\xi)} & -1
\end{pmatrix},
\]

\[
F^{(2)}(t, \xi) = \begin{pmatrix}
-\tilde{A}(\xi) & 0 \\
\frac{-1}{\tilde{A}(\xi)} & \frac{c^+(\xi)e^{4(\xi^4 (T-t) + 2\xi^2 L)}}{d(\xi)}
\end{pmatrix},
\]

\[
F^{(3)}(t, \xi) = \begin{pmatrix}
-\frac{1}{\tilde{A}(\xi)} & 0 \\
\frac{c^+(\xi)e^{-4(\xi^4 (T-t) - 2\xi^2 L)}}{d(\xi)} & -\tilde{A}(\xi)
\end{pmatrix},
\]

\[
F^{(4)}(t, \xi) = \begin{pmatrix}
-1 & 0 \\
\frac{-b(\xi)e^{-4(\xi^4 t - 2\xi^2 L)}}{\alpha(\xi)\tilde{A}(\xi)} & -1
\end{pmatrix}.
\]

Similar to the previous case, the transformation

\[M(L, t, \xi) \to M^{(t,L)}(t, \xi) = M(L, t, \xi)F(t, \xi)\] (3.29)

maps the Riemann-Hilbert problem of Theorem 3.4 to the Riemann-Hilbert problem of Remark 3.3.

**Remark 3.5.** It is well-known that there are three kinds of celebrated DNLS equations, including Kaup-Newell equation (i.e. Eq. (1.1)), Chen-Lee-Liu equation \(28\)

\[iq_t + q_{xx} + i|q|^2 q_x = 0,\] (3.30)

and Gerdjikov-Ivanov equation \(29\)

\[iq_t + q_{xx} - iq^2 \bar{q}_x + \frac{1}{2}|q|^4 \bar{q} = 0.\] (3.31)

It has been found that they may be transformed into each other by gauge transformations \(29, 30\). We can show that, for the decaying and smooth potential $q$, the Jost functions associated with spectral problems for these three kinds of DNLS equations have the same asymptotic behavior as $|x| \to \infty$, so the Chen-Lee-Liu equation (3.30) and Gerdjikov-Ivanov equation (3.31) have the same type Riemann-Hilbert problem 3.4 with the DNLS equation (1.1).
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