The Reidemeister Spectra of Low Dimensional Almost-Crystallographic Groups

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ABSTRACT
We determine which non-crystallographic, almost-crystallographic groups of dimension-4 have the \(R_\infty\)-property. We then calculate the Reidemeister spectra of the 3-dimensional almost-crystallographic groups and the 4-dimensional almost-Bieberbach groups.

KEYWORDS
group theory; fixed point theory; almost-crystallographic group; automorphism; Nielsen fixed point theory

1. Introduction
Let \(G\) be any group and \(\varphi : G \to G\) an endomorphism of this group. Define an equivalence relation \(\sim_\varphi\) on \(G\) given by
\[\forall g, g' \in G : g \sim_\varphi g' \iff \exists h \in G : g = h g' (h)^{-1}.\]

An equivalence class \(\{g\}_\varphi\) is called a Reidemeister class of \(\varphi\) or \(\varphi\)-twisted conjugacy class. The Reidemeister number \(R(\varphi)\) is the number of Reidemeister classes of \(\varphi\) and is therefore always a positive integer or infinity. The Reidemeister spectrum of a group \(G\) is the set of all Reidemeister numbers when considering all possible automorphisms of that group
\[\text{Spec}_R(G) := \{R(\varphi) | \varphi \in \text{Aut}(G)\} .\]

If \(\text{Spec}_R(G) = \{\infty\}\) we say that \(G\) has the \(R_\infty\)-property.

Reidemeister numbers originate in Nielsen fixed point theory, where they are defined as the number of fixed point classes of a self-map of a topological space [Jiang 83], although they also yield applications in algebraic geometry and representation theory [Fel'shtyn and Troitsky 15].

It turns out that many (infinite) groups admit the \(R_\infty\)-property. This is also the case for most almost-crystallographic groups, e.g., in [Dekimpe and Penninckx 11] it was shown that 207 of the 219 three-dimensional crystallographic groups and 15 of the 17 families of 3-dimensional (non-crystallographic) almost-crystallographic groups all have the \(R_\infty\)-property. Furthermore, in [Dekimpe et al. 19a] it was shown that 4692 of the 4783 4-dimensional crystallographic groups admit the \(R_\infty\) property. Moreover, the Reidemeister spectra of all crystallographic groups of dimensions 1, 2, and 3 were calculated, as well as the spectra of the 4-dimensional Bieberbach groups. In this article, we extend these results by studying the 4-dimensional almost-crystallographic groups.

This article is structured as follows. In the next two sections, we provide the necessary preliminaries on Reidemeister numbers and almost-crystallographic groups. In Section 4, we determine which almost-crystallographic groups of dimension 4 possess the \(R_\infty\)-property. Sections 5 and 6 are devoted to calculating the Reidemeister spectra of the 3-dimensional almost-crystallographic groups and the 4-dimensional almost-Bieberbach groups, respectively. The final section summarizes the obtained results.

2. Reidemeister numbers and spectra
In this section, we introduce basic notions concerning the Reidemeister number. For a general reference on Reidemeister numbers and their connection to fixed point theory, we refer the reader to [Jiang 83].

The definitions of the Reidemeister number and Reidemeister spectrum were given in the introduction. However, nothing was said on how we actually determine whether a group has the \(R_\infty\)-property, and if
not, how we calculate its Reidemeister spectrum. The following lemma is an essential tool for the former.

**Lemma 2.1** (see [Fel’shtyn and Troitsky 15, Section 2.2], [Gonçalves and Wong 09, Lemma 1.1]). Let $N$ be a normal subgroup of a group $G$ and $\phi \in \text{Aut}(G)$ with $\phi(N) = N$. We denote the restriction of $\phi$ to $N$ by $\phi|_N$, and the induced automorphism on the quotient $G/N$ by $\phi'$. We then get the following commutative diagram with exact rows

$$
\begin{array}{cccc}
1 & \rightarrow & N & \rightarrow & G & \rightarrow & G/N & \rightarrow & 1 \\
\downarrow{\phi|_N} & & \downarrow{\phi} & & \downarrow{\phi'} & & \\
1 & \rightarrow & N & \rightarrow & G & \rightarrow & G/N & \rightarrow & 1
\end{array}
$$

We obtain the following properties:

1. $R(\phi) \geq R(\phi')$,
2. if $R(\phi') < \infty, R(\phi|_N) = \infty$ and $|\text{Fix}(\phi')| < \infty$, then $R(\phi) = \infty$.

A direct consequence for characteristic subgroups is the following:

**Corollary 2.2.** Let $N$ be a characteristic subgroup of $G$. If either

1. the quotient $G/N$ has the $R_\infty$-property, or
2. $N$ has finite index in $G$ and has the $R_\infty$-property, then $G$ has the $R_\infty$-property as well.

### 3. Almost-crystallographic groups

Let $G$ be a connected, simply connected, nilpotent Lie group with automorphism group $\text{Aut}(G)$. The affine group $\text{Aff}(G)$ is the semi-direct product $\text{Aff}(G) = G \rtimes \text{Aut}(G)$, where multiplication is defined by $(d_1, D_1)(d_2, D_2) = (d_1 D_1(d_2), D_1 D_2)$. If $C$ is a maximal compact subgroup of $\text{Aut}(G)$, then $G \rtimes C$ is a subgroup of $\text{Aff}(G)$. A cocompact discrete subgroup $\Gamma$ of $G \rtimes C$ is called an *almost-crystallographic group* modeled on the Lie group $G$. The dimension of $\Gamma$ is defined as the dimension of $G$.

If $\Gamma$ is torsion-free, then it is called an *almost-Bieberbach group*. If $G = \mathbb{R}^n$, then it is called a *crystallographic group*, or a *Bieberbach group* if it also torsion-free.

Crystallographic groups were historically studied first, and are well understood by the three Bieberbach theorems. These theorems have since been generalized to almost-crystallographic groups, which we will briefly discuss below. We refer to [Szczepeński 12] and [Dekimpe 96] for more information on the original and generalized theorems, respectively.

The generalized first Bieberbach theorem says that if $\Gamma \subseteq \text{Aff}(G)$ is an $n$-dimensional almost-crystallographic group, then its translation subgroup $N := \Gamma \cap G$ is a uniform lattice of $G$ and is of finite index in $\Gamma$. Moreover, $N$ is the unique maximal nilpotent normal subgroup of $\Gamma$, and is therefore characteristic in $\Gamma$. The quotient group $F := \Gamma/N$ is a finite group called the *holonomy group* of $\Gamma$. In fact $F = \{ A \in \text{Aut}(G) \mid \exists a \in G : (a, A) \in \Gamma \}$. If $\Gamma$ is crystallographic ($G = \mathbb{R}^n$), we may assume that $N = \mathbb{Z}^n$ and $F$ is a subgroup of $\text{GL}_n(\mathbb{Z})$.

The generalized second Bieberbach theorem tells us more about automorphisms of almost-crystallographic groups.

**Theorem 3.1** (generalized second Bieberbach theorem). Let $\phi : \Gamma \rightarrow \Gamma$ be an automorphism of an almost-crystallographic group $\Gamma \subseteq \text{Aff}(G)$ with holonomy group $F$. Then there exists a $(d, D) \in \text{Aff}(G)$ such that $\phi(\gamma) = (d, D) \circ \gamma \circ (d, D)^{-1}$ for all $\gamma \in \Gamma$. To shorten notation, we will write $\phi = \xi_{(d,D)}$.

An automorphism $\Phi : G \rightarrow G$ of a Lie group $G$ induces an automorphism $\Phi_\ast : g \mapsto \Phi$ of the associated Lie algebra $g$. We will henceforth always denote an induced automorphisms on a Lie algebra with a star (*), subscript, for example, $A_\ast$ is the Lie algebra automorphism induced by some $A \in F$ where $F \subseteq \text{Aut}(G)$ is the holonomy group of an almost-crystallographic group. In particular, an automorphism $\Phi_\ast = \xi_{(d,D)}$ of an almost-crystallographic group has an associated matrix $D_\ast$.

The generalized third Bieberbach theorem is less straightforward to generalize. Unlike for crystallographic groups, it is not true that there are only finitely many $n$-dimensional almost-crystallographic groups for a given dimension $n$. However, we can state that for a given finitely generated torsion-free nilpotent group $N$, there are (up to isomorphism) only finitely many almost-crystallographic groups $\Gamma$ such that the translation subgroup of $\Gamma$ is isomorphic to $N$.

In [Dekimpe 96, Section 2.5], this generalization is proved using the concept of an *isolator*, which shall prove useful to us as well.

**Definition 3.2.** Let $G$ be a group with subgroup $H$. The isolator of $H$ in $G$ is defined as

$$\sqrt{H} = \{ g \in G \mid g^k \in H \text{ for some } k \geq 1 \}.$$
Lemma 3.3 (see [Dekimpe 96, Lemma 2.4.2]). Let $\Gamma$ be an almost-crystallographic group with translation subgroup $N$ of nilpotency class $c$. Then the isolator $\bigvee_{\Gamma}^N(N) \leq Z(N)$ is a characteristic subgroup of $\Gamma$. Moreover, the quotient group $\Gamma/\bigvee_{\Gamma}^N(N)$ is an almost-crystallographic group whose translation subgroup $N/\bigvee_{\Gamma}^N(N)$ has nilpotency class $c$ if $c = 2$, and nilpotency class $2$ if $c = 2$. If $c = 2$, then this quotient is a crystallographic group.

We will now give the most important results for Reidemeister theory applied to almost-crystallographic groups. A first result allows us to easily determine whether an almost-crystallographic group admits the $R_\infty$-property or not.

Theorem 3.4 (see [Dekimpe and Penninckx 11, Corollary 3.10]). Let $\Gamma$ be an $n$-dimensional almost-crystallographic group with holonomy group $\mathcal{H} \subseteq \text{Aut}(G)$ and $\varphi = \xi_{(d, D)} \in \text{Aut}(\Gamma)$ (where we use the notation of Theorem 3.1). Then

$$R(\varphi) = \infty$$

$$\iff \exists A \in F \text{ such that } \det(I_n - A, D_\varphi) = 0$$

$$\iff \exists A \in F \text{ such that } A, D_\varphi \text{ has eigenvalue 1}.$$

The second result only holds for almost-Bieberbach groups, and allows for an easy computation of the Reidemeister number of an automorphism.

Theorem 3.5 (averaging formula, see [Ha et al. 12, Theorem 6.11] and [Lee and Lee 09, Theorem 4.3]). Let $\Gamma$ be an $n$-dimensional almost-Bieberbach group with holonomy group $\mathcal{H} \subseteq \text{Aut}(G)$, and $\varphi = \xi_{(d, D)} \in \text{Aut}(\Gamma)$ with $R(\varphi) < \infty$. Then

$$R(\varphi) = \frac{1}{#F} \sum_{A \in F} |\det(I_n - A, D_\varphi)|.$$ 

In general, this formula does not hold for automorphisms of almost-crystallographic groups, examples can be found in [Dekimpe et al. 19a] and later in this article. Therefore, the calculation of the Reidemeister spectra usually requires a deeper understanding of how the Reidemeister classes are formed in a specific group.

4. The $R_\infty$-property for 4-dimensional almost-crystallographic groups

Every almost-crystallographic group of dimension 1 or 2 is crystallographic. In [Dekimpe and Penninckx 11] it was determined which 3-dimensional almost-crystallographic groups admit the $R_\infty$-property. We extend these results to dimension 4. In this case the translation subgroup $N$ is a finitely generated, torsion-free, nilpotent group of rank 4 and nilpotency class at most 3. Nilpotency class 1 is of course the crystallographic case, which was done in [Dekimpe et al. 19a].

4.1. Nilpotency class 2

Let $\Gamma$ be an almost-crystallographic group whose translation subgroup $N$ is a nilpotent group of rank 4 and nilpotency class 2. In [Dekimpe 96] it was shown that $N$ can be given the following presentation

$$\begin{bmatrix}
[\epsilon_2, \epsilon_1] = 1 & [\epsilon_3, \epsilon_2] = \epsilon^1_i \\
[\epsilon_2, \epsilon_4, \epsilon_3] = 1 & [\epsilon_4, \epsilon_1] = 1 & [\epsilon_4, \epsilon_3] = \epsilon^1_i
\end{bmatrix}.$$ 

Moreover, let $G$ be the Lie group that $\Gamma$ is modeled on. By [Dekimpe 95, Theorem 4.1], there exists a faithful affine representation $\lambda : G \times \text{Aut}(G) \to \text{Aff}(\mathbb{R}^4)$ such that its restriction to $\Gamma$ is again a faithful affine representation. In particular,

$$\lambda(\epsilon_1) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$\lambda(\epsilon_2) = \begin{pmatrix}
1 & 0 & -\frac{l_1}{2} & -\frac{l_2}{2} & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$\lambda(\epsilon_3) = \begin{pmatrix}
1 & \frac{l_1}{2} & 0 & -\frac{l_3}{2} & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$\lambda(\epsilon_4) = \begin{pmatrix}
1 & \frac{l_2}{2} & \frac{l_3}{2} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

where the values of $l_1$, $l_2$, and $l_3$ are determined by the relations $[\epsilon_3, \epsilon_2] = \epsilon^1_i$, $[\epsilon_4, \epsilon_2] = \epsilon^1_i$ and $[\epsilon_4, \epsilon_3] = \epsilon^1_i$. Lemma 3.3 tells us that the subgroup $\langle \epsilon_1 \rangle = \bigvee_{\Gamma}^N(N)$ is characteristic and the quotient $\Gamma'$ :=
\( \Gamma / \langle e_1 \rangle \) is a 3-dimensional crystallographic group. Using Corollary 2.2, we know that if \( \Gamma' \) has the \( R_{\infty} \)-property, then so does \( \Gamma \). In [Dekimpe 96; Dekimpe and Eick 02] the almost-crystallographic groups were classified into families based on which crystallographic group \( \Gamma' \) is. Since only twelve 3-dimensional crystallographic groups do not have the \( R_{\infty} \)-property, we need only consider the corresponding twelve families of 4-dimensional almost-crystallographic groups.

Each of these families can be split in smaller subfamilies, determined by the action of \( F \) on \( \mathbb{Z}_2^4 / \langle N \rangle \): every \( A \in F \) acts on \( e_1 \) by \( A e_1 = \epsilon^a e_1 \) with \( \epsilon_A \in \{-1, 1\} \). The following proposition quickly deals with the subfamilies, where \( F \) does not act trivially on \( \mathbb{Z}_2^4 / \langle N \rangle \).

**Proposition 4.1.** Let \( \Gamma \) be an almost-crystallographic group with translation subgroup \( N \) of rank 4 and nilpotency class 2, and holonomy group \( F \). If \( F \) acts non-trivially on \( \mathbb{Z}_2^4 / \langle N \rangle \), then \( \Gamma \) has the \( R_{\infty} \)-property.

**Proof.** Let \( A \in F \) arbitrary and \( \varphi = \xi_{(d, D)} \in \text{Aut}(\Gamma) \). Since \( A \) acts on \( \langle e_1 \rangle = \mathbb{Z}_2^4 / \langle N \rangle \) by \( A e_1 = \epsilon^a e_1 \) with \( \epsilon_A \in \{-1, 1\} \) and \( \varphi(e_1) = e_1^\nu \) with \( \nu \in \{-1, 1\} \), \( A \) and \( D \) must have the following forms

\[
A = \begin{pmatrix}
\epsilon_A & * & * & * \\
0 & * & * & * \\
0 & * & * & * \\
0 & * & * & *
\end{pmatrix},

\[
D = \begin{pmatrix}
\nu & * & * & * \\
0 & * & * & * \\
0 & * & * & * \\
0 & * & * & * 
\end{pmatrix}.
\]

Thus, \( I_4 - A_D \) is of the form

\[
I_4 - A_D = \begin{pmatrix}
1 - \nu \epsilon_A & * & * & * \\
0 & * & * & * \\
0 & * & * & * \\
0 & * & * & *
\end{pmatrix}.
\]

Now let us look at specific \( A \in F \). First, let \( A \) be the neutral element of \( F \), which necessarily acts trivially on \( e_1 \). The above matrix then has upper left entry \( 1 - \nu \), hence \( \det(I_4 - A_D) \neq 0 \) if and only if \( \nu = -1 \).

Second, let \( A \) be an element of \( F \) for which \( \epsilon_A = -1 \). Such element exists since we assumed \( F \) acts non-trivially on \( \mathbb{Z}_2^4 / \langle N \rangle \). Then the matrix \( I_4 - A_D \) has upper left entry \( 1 + \nu \), and \( \det(I_4 - A_D) \neq 0 \) if and only if \( \nu = 1 \).

As \( \nu \) cannot be \(-1\) and \( 1 \) at the same time, we always have some \( A \in F \) for which \( \det(I_4 - A_D) = 0 \), and by Theorem 3.4 this means that \( R(\varphi) = \infty \). Since this holds for any automorphism, \( \Gamma \) has the \( R_{\infty} \)-property.

From the proof of the theorem above, we can also conclude the following:

**Table 1. Conjugacy matrices between representations.**

| Family | \( \delta \) |
|--------|---------------|
| 1, 2   | \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] |
| 3, 4   | \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] |
| 5      | \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] |
| 143    | \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\] |
| 146    | \[
\begin{pmatrix}
1 & -k_1 & k_2 + 2k_3 & -k_2 + k_3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] |

**Proposition 4.2.** Let \( \Gamma \) be an almost-crystallographic group with translation subgroup \( N \) of rank 4 and nilpotency class 2, and let \( e_1 \) be a generator of \( \mathbb{Z}_2^4 / \langle N \rangle \). If \( \varphi \in \text{Aut}(\Gamma) \) has finite Reidemeister number, then \( \varphi(e_1) = e_1^{-1} \).

We will number the 12 families under consideration according to the crystallographic group \( \Gamma / \mathbb{Z}_2^4 / \langle N \rangle \) using the classification in the International Tables in Crystallography [Aroyo 16]: they are families 1–5, 16, 19, 22–24, 143, and 146. When we write \( \Gamma_{n/m} \), we mean the \( n \)-dimensional crystallographic group with IT-number \( m \).

Using the techniques in [Dekimpe 96, Section 5.4], we find that for an almost-crystallographic group belonging to one of the families 16, 19, or 22–24, \( F \) acting trivially on \( \mathbb{Z}_2^4 / \langle N \rangle \) implies that the group is actually crystallographic. Therefore, we may omit these families and we are left with only seven families to study.

Note that the presentations given in this article may vary from those in [Dekimpe 96; Dekimpe and Eick 02]. Let \( \lambda \) and \( \mu \) denote a group and its faithful representation as given in this article, and let \( \lambda' \) and \( \mu' \) be the corresponding group and representation as given by [Dekimpe 96] or [Dekimpe and Eick 02].
hence $\lambda(\Gamma)$ and $\mu(\Gamma')$ are conjugate subgroups of $\text{Aff}(\mathbb{R}^4)$ and therefore $\Gamma$ and $\Gamma'$ are isomorphic.

**Family 1.** This family consists of the finitely generated, torsion-free, nilpotent groups of nilpotency class 2 and rank 4. It was shown in [Dekimpe et al. 19b, Section 3.2] that these groups do not have the $R_{\infty}$-property.

**Family 2.** Every group in this family has a presentation of the form

\[
eq \left\langle e_1, e_2, e_3, e_4, x \right\rangle
\]

and the faithful representation $\lambda$ is given by

\[
\lambda(x) = \begin{pmatrix}
1 & k_4 & k_5 & k_6 & k_7 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Set $k := \gcd(k_1, k_2, k_3)$ and $g := e_2^{k_1/k}e_3^{-k_1/k}e_4^{k_1/k}$, then the center $Z(N)$ of the translation subgroup is generated by $e_1$ and $g$. Let $\varphi : \Gamma \to \Gamma$ be any automorphism. Since $(e_1)$ and $Z(N)$ are both characteristic in $\Gamma$, we have that $\varphi(g) = g^{\epsilon}_m$ for some $\epsilon \in \{1, -1\}$ and $m \in \mathbb{Z}$. Consider the induced automorphism $\varphi' = \zeta_{(g, D)}$ on $\Gamma/(e_1) \cong \Gamma/3/2$. Then

\[
\varphi'(g(e_1)) = D'(g'(e_1)) = \varphi(g)(e_1) = g'\varphi(e_1).
\]

Depending on the value of $\epsilon$, $D'$ has either eigenvalue 1, in which case $\det(\mathbb{I}_3 - D') = 0$, or eigenvalue $-1$, in which case $\det(\mathbb{I}_3 + D') = 0$. Since the holonomy group of $\Gamma/3/2$ is $\{1, -1\}$, we obtain by Theorem 3.4 that $R(\varphi') = \infty$ and by Lemma 2.1 that $\varphi(e_1) = g(e_1)$. Since this holds for an arbitrary automorphism, $\Gamma$ has the $R_{\infty}$-property.

**Families 3, 4, and 5.** Every group in one of these families has a presentation of the form

\[
eq \left\langle e_1, e_2, e_3, e_4, x \right\rangle
\]

and the faithful representation $\lambda$ is given by

\[
\lambda(x) = \begin{pmatrix}
1 & 0 & k_2 & k_3 & k_4 \\
0 & 1 & -\nu & 0 & \frac{2}{\mu} \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Family 3 is given by $\mu, \nu = 0$, family 4 by $\mu = 1, \nu = 0$ and family 5 by $\mu = 0, \nu = 1$. Define an automorphism $\varphi = \zeta_{(d, D)}$ by

\[
\varphi(e_1) = e_1^{-1}, \\
\varphi(e_2) = e_2^{m_1}, \\
\varphi(e_3) = e_3^{k_1 - k_2 - k_3}e_4e_3, \\
\varphi(e_4) = e_4^{k_1 - k_2 - 2k_3}e_3e_4, \\
\varphi(x) = e_1^{-k_1}e_2^{-\nu}x,
\]

then $D_i$ is of the form

\[
D_i = \begin{pmatrix}
-1 & * & * & * \\
0 & -1 & * & * \\
0 & 0 & 1 & 2 \\
0 & 0 & 2 & 3
\end{pmatrix}.
\]

We can apply Theorem 3.4 to show that $R(\varphi) < \infty$ and hence $\Gamma$ does not have the $R_{\infty}$-property.

**Families 143 and 146.** Every group in one of these families has a presentation of the form

\[
eq \left\langle e_1, e_2, e_3, e_4, x \right\rangle
\]

and the faithful representation $\lambda$ is given by

\[
\lambda(x) = \begin{pmatrix}
1 & 0 & k_2 & -\frac{k_1}{2} + k_3 & k_4 \\
0 & 1 & 0 & \mu & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Family 143 is given by $\mu = 0$ and family 146 by $\mu = 1$. Using an argument identical to the proof of [Dekimpe and Penninckx 11, Theorem 4.4, family 13], we may conclude that all groups in these families have the $R_{\infty}$-property.

### 4.2. Nilpotency class 3

By an argument analogous to [Gonçalves and Wong 09, Example 5.2], a finitely-generated, torsion-free,
nilpotent group of nilpotency class 3 and rank 4 has the $R_{\infty}$-property. Applying Corollary 2.2 then proves that every 4-dimensional almost-crystallographic group with translation subgroup of nilpotency class 3 has the $R_{\infty}$-property.

5. The Reidemeister spectra of the 3-dimensional almost-crystallographic groups

Let $\Gamma$ be an almost-crystallographic group whose translation subgroup $N$ is a nilpotent group of rank 3 and nilpotency class 2. Such $N$ can be given the
following presentation
\[
\left\langle e_1, e_2, e_3 \mid [e_2, e_1] = 1, [e_3, e_1] = 1, [e_3, e_2] = e_1^l \right\rangle,
\]
with \(l > 0\). Moreover, let \(G\) be the Lie group that \(\Gamma\) is modeled on. By [Dekimpe 95, Theorem 4.1], there exists a faithful affine representation \(\lambda : G \rtimes \text{Aut}(G) \to \text{Aff}(\mathbb{R}^3)\) such that its restriction to \(\Gamma\) is again a faithful affine representation. In particular,

\[
\lambda(e_1) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
\lambda(e_2) = \begin{pmatrix} 1 & 0 & -\frac{l_1}{2} & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
\lambda(e_3) = \begin{pmatrix} 1 & l_1 & 0 & 0 \\ l_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},
\]

where the value of \(l_1\) is determined by the relation \([e_3, e_2] = e_1^l\). Like in Section 4.1, we have that the subgroup \((e_1) = N/\gamma_2(N)\) is characteristic in \(\Gamma\), and an automorphism \(\phi\) must satisfy \(\phi(e_1) = e_1^{-1}\) to have finite Reidemeister number.

As mentioned before, in [Dekimpe and Penninckx 11, Theorem 4.4] it was shown that there are only 2 families of almost-crystallographic groups that do not admit the \(R_{ac}\)-property. We again number these families according to the IT-number of the quotient \(\Gamma/N/\gamma_2(N)\).

| Table 5. Make List(0, 0, 1, 1). |
|---|---|---|
| \(M\) | \(d\) | \(R\) |
| (0 1) | 0 | 8N + 4 |
| (1 0) | 0 | 8N |
| (0 1) | 0 | 8N |
| (1 1) | 0 | 4N |
| (0 1) | 0 | 4N |
| (1 1) | 0 | 4N |
| (0 1) | 0 | 4N |

| Table 6. Make List(0, 1, 1, 0). |
|---|---|---|
| \(M\) | \(d\) | \(R\) |
| (0 1) | 0 | 4N + 4 |
| (1 0) | 0 | 4N + 4 |
| (1 0) | 0 | 4N + 4 |
| (0 1) | 0 | 8N + 4 |
| (0 1) | 0 | 8N |

| Table 7. Make List(0, 1, 1, 1). |
|---|---|---|
| \(M\) | \(d\) | \(R\) |
| (0 1) | 0 | 4N |
| (1 0) | 0 | 4N |
| (1 0) | 0 | 4N |
| (0 1) | 0 | 8N |
| (0 1) | 0 | 8N |

| Table 8. Make List(1, 0, 0, 0). |
|---|---|---|
| \(M\) | \(d\) | \(R\) |
| (0 1) | 0 | 4N + 2 |
| (1 0) | 0 | 4N + 2 |
| (0 1) | 0 | 4N |
| (1 0) | 0 | 4N |
| (0 1) | 0 | 4N |
| (1 0) | 0 | 4N |
| (0 1) | 0 | 4N |

| Table 9. Make List(1, 0, 0, 1). |
|---|---|---|
| \(M\) | \(d\) | \(R\) |
| (0 1) | 0 | 4N + 2 |
| (1 0) | 0 | 4N |
| (0 1) | 0 | 4N |
| (1 0) | 0 | 4N |
| (1 0) | 0 | 4N |
| (1 0) | 0 | 4N |
| (1 0) | 0 | 4N |

| Table 10. Make List(1, 0, 1, 0). |
|---|---|---|
| \(M\) | \(d\) | \(R\) |
| (0 1) | 0 | 4N + 2 |
| (1 0) | 0 | 4N + 2 |
| (0 1) | 0 | 4N + 2 |
| (1 0) | 0 | 4N + 2 |
| (1 0) | 0 | 4N + 2 |
| (1 0) | 0 | 4N + 2 |
| (1 0) | 0 | 4N + 2 |
Family 1. The groups in this family are exactly the finitely generated, torsion-free, nilpotent groups of nilpotency class 2 and rank 3. In [Romankov 11, Section 3] it was shown that these groups have Reidemeister spectrum $2\mathbb{N} \cup \{\infty\}$. This was shown specifically for the case $k_1 = 1$, but the argument holds for any $k_1 > 0$.

Family 2. Every group in this family has a presentation of the form

$$
e_{e_1, e_2, e_3, x}
e_{e_1} = e_1 x \quad e_{e_2} = e_1 x \quad e_{e_3} = e_1 e_2^{-1} x \quad e_{x^2} = e_1 e_3^{-1} x,$$

and the faithful representation $\lambda$ is given by

$$\lambda(x) = \begin{pmatrix} 1 & k_2 & k_3 & k_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Let $\varphi$ be an automorphism with finite Reidemeister number $R(\varphi)$. Under the representation $\lambda$, this automorphism will correspond to a matrix $\delta \in \text{Aff}(\mathbb{R}^4)$ such that

$$\lambda(\varphi(\gamma)) = \delta \lambda(\gamma) \delta^{-1}.$$ 

for all $\gamma \in \Gamma$. Since we assumed that $R(\varphi) < \infty$, we have that $\varphi(e_1) = e_1^{-1}$. Moreover, $\varphi$ induces an automorphism $\varphi'$ on $\Gamma' := \Gamma/\langle e_1 \rangle$. Thus, $\delta$ must be of the form

$$\delta = \begin{pmatrix} -1 & n_1 & n_2 & 0 \\ 0 & m_1 & m_2 & d_1/2 \\ 0 & m_2 & m_4 & d_2/2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the constants $m_1, d_1$ are integers, $m_1 m_4 - m_2 m_3 = -1$ and $n_1, n_2 \in \mathbb{R}$. Using a computer, one can calculate the (unique) values of $n_1, n_2$ and $l_1, l_2, l_3$ such that

Table 13. Make List$(1, 1, 1, 1)$.

| $M$ | $d$ | $R$ |
|-----|-----|-----|
| $(0, 1)$ | $(0)$ | $4N + 2$ |
| $(0, 1)$ | $(0)$ | $4N$ |
| $(0, 1)$ | $(0)$ | $8N + 2$ |
| $(0, 0, 0)$ | $(2m)$ | $4N$ |
| $(0, 0, 1)$ | $(1)$ | $4N$ |
| $(0, 1, 0)$ | $(1)$ | $4N$ |
| $(0, 1, 1)$ | $(2m)$ | $4N$ |
| $(0, 1, 1)$ | $(1)$ | $4N$ |
| $(0, 1, 1)$ | $(2m)$ | $4N$ |
| $(0, 1, 1)$ | $(2m)$ | $4N$ |
| $(0, 1, 1)$ | $(2m)$ | $4N$ |
| $(0, 1, 1)$ | $(2m)$ | $4N$ |
| $(0, 1, 1)$ | $(2m)$ | $4N$ |
| $(0, 1, 1)$ | $(2m)$ | $4N$ |

Table 14. Automorphisms and Reidemeister spectra for all $(k_1, k_2, k_3, k_4)$.

| $(k_1, k_2, k_3, k_4)$ | $M$ | $d$ | $R(\varphi)$ | Spec$_\Gamma(\Gamma')$ |
|------------------------|-----|-----|-------------|---------------------|
| $(0, 0, 0, 0)$ | $(0, 1)$ | $(0)$ | $4N$ |
| $(0, 0, 1)$ | $(1, 2m)$ | $(1)$ | $2N$ |
| $(0, 1, 0)$ | $(2m, 2m - 1)$ | $(1)$ | $4N$ |
| $(0, 1, 1)$ | $(2m, 2m - 1)$ | $(1)$ | $4N$ |
| $(0, 1, 1)$ | $(2m, 2m - 1)$ | $(1)$ | $4N$ |
| $(0, 1, 1)$ | $(2m, 2m - 1)$ | $(1)$ | $4N$ |
| $(0, 1, 1)$ | $(2m, 2m - 1)$ | $(1)$ | $4N$ |
| $(0, 1, 1)$ | $(2m, 2m - 1)$ | $(1)$ | $4N$ |
| $(0, 1, 1)$ | $(2m, 2m - 1)$ | $(1)$ | $4N$ |
| $(0, 1, 1)$ | $(2m, 2m - 1)$ | $(1)$ | $4N$ |

Table 12. Make List$(1, 1, 1, 0)$.

| $M$ | $d$ | $R$ |
|-----|-----|-----|
| $(0, 1)$ | $(0)$ | $4N + 2$ |
| $(0, 1)$ | $(0)$ | $4N$ |
| $(0, 1)$ | $(0)$ | $8N + 2$ |
| $(0, 1)$ | $(2m)$ | $4N$ |
| $(1, 0)$ | $(1)$ | $4N + 2$ |
| $(1, 0)$ | $(1)$ | $4N$ |
| $(1, 0)$ | $(0)$ | $4N$ |
| $(1, 0)$ | $(0)$ | $4N$ |
| $(1, 1)$ | $(0)$ | $4N + 2$ |
| $(1, 1)$ | $(0)$ | $4N$ |
| $(1, 1)$ | $(0)$ | $4N$ |
| $(1, 1)$ | $(0)$ | $4N$ |
\[ \delta \lambda(\varepsilon_2) \delta^{-1} = \lambda(\varepsilon_1) \delta \lambda(\varepsilon_2) \delta^{-1} = \lambda(\varepsilon_1) k \lambda(\varepsilon_2) \delta \lambda(\varepsilon_1) \delta^{-1} = \lambda(\varepsilon_1) \delta^i \lambda(\varepsilon_2)^i \delta(\varepsilon_1)^i. \]

From the obtained values of \( l_1, l_2, \) and \( l_3, \) we get
\[
\begin{align*}
\varphi(e_1) &= e_1^{-1}, \\
\varphi(e_2) &= e_2^{-1}(m_{11} m_{12} + m_{12} m_{21} - m_{21} m_{11}) - \frac{1}{2} m_{11} m_{12} + \frac{1}{2} m_{21} m_{11}, \\
\varphi(e_3) &= e_3^{-1}(m_{22} m_{12} + m_{22} m_{21} - m_{12} m_{21}) - \frac{1}{2} m_{22} m_{12} + \frac{1}{2} m_{22} m_{21}, \\
\varphi(x) &= e_x^{-1}(m_{11} m_{12} + m_{12} m_{21} - m_{21} m_{11}) - \frac{1}{2} m_{11} m_{12} + \frac{1}{2} m_{21} m_{11},
\end{align*}
\]

where all exponents must be integers. This places four conditions on the \( m_i \) and \( d_i \):

(a) \( k_1(m_1 m_2 + m_2 d_1) - k_2(m_1 + 1) - k_3 m_2 \equiv 0 \pmod{2}, \)
(b) \( k_1(m_1 m_4 + m_3 d_2 - m_4 d_1) - k_2 m_3 - k_3(m_4 + 1) \equiv 0 \pmod{2}, \)
(c) \( k_1 d_1 d_2 - k_2 d_1 - k_3 d_2 \equiv 0 \pmod{2}, \)
(d) \( m_1 m_4 - m_2 m_3 = -1. \)

For ease of notation, let us set
\[ M := \begin{pmatrix} m_1 & m_4 \\ m_2 & m_4 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}), \quad d := \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \mathbb{Z}^2. \]

We will determine \( R(\varphi) \) in a very similar way to the proof of [Dekimpe et al. 19b, Proposition 5.11]. Let \( [x]_\varphi \) be a Reidemeister class of \( \Gamma \), then for any \( k \in \mathbb{Z}, \)
\[ x = \left( e_1^{-k} \right) x e_1^{-k} \varphi(e_1)^{-k}, \]

therefore \( x \sim \varphi x \varphi^{-1} \) for all \( k \in \mathbb{Z}. \) Consider the quotient group \( \Gamma' = \Gamma(\varepsilon_1) \) and let \( \varphi' = \xi(\varepsilon_2) \) be the induced automorphism on this quotient. Since we assumed that \( R(\varphi) < \infty, \) we have that \( R(\varphi') < \infty \) as well. [Dekimpe et al. 19b, Proposition 5.10] tells us that \( R(\varphi') = |\text{tr}(M)| + \Omega(12 - d, M) \) with
\[ O(A, a) := \# \{ \bar{x} \in \mathbb{Z}^2 \mid A \bar{x} = \bar{a} \}, \]

where the bar-notation denotes the element-wise projection to \( \mathbb{Z}/2 \). A Reidemeister class \( [x(e_1)]_\varphi \) of \( \Gamma' \) will lift to at most 2 Reidemeister classes of \( \Gamma \): \( [x]_\varphi \) and \( [xe_1]_\varphi \); so the number of lifts is either 2 (when \( x \not\sim \varphi x e_1 \)) or 1 (when \( x \sim \varphi x e_1 \)). The latter happens if and only if
\[ \exists z \in \Gamma : xe_1 = zx \varphi(z)^{-1}. \]

Projecting this to the quotient \( \Gamma' \), we have
\[ \exists z \in \Gamma : x(e_1) = zx \varphi(z)^{-1}(e_1). \]

Since \( e_1 \) is central in \( \Gamma \) and \( x \) appears exactly once on each side of the equality sign in (5-1), the \( e_1 \)-component of \( x \) does not matter. Set \( x = e_1^h e_1^i x^h \) and \( z = e_1^h e_1^i e_1^j x^i \). Let us first assume that \( e_1^i = 0 \), then (5-2) is equivalent to
\[ \exists z_2, z_3 \in \mathbb{Z} : (I_2 - AM)(z_3)^{-1} = 0, \]

with A the holonomy part of \( x(e_1) \). As \( R(\varphi') < \infty, \) we must have \( z_2 = z_3 = 0 \). But then \( z = e_1^i \) and (5-1) then becomes \( xe_1 = xe_1^2 e_1^i \). As \( z_1 \) is an integer, this is impossible. So, we assume that \( e_1^i = 1 \). Writing out (5-1) component-wise, we find that this condition is equivalent to the following:

There exist \( z_1, z_2, z_3 \in \mathbb{Z} \) such that:

(i) \[ 2 \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} I_2 - ( -1)^e M \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = (-1)^e d, \]
(ii) \[ k_1 z_2 z_3 - k_2 z_2 - k_3 z_3 - k_4 + 1 = 2z_1. \]

Condition (i) is independent of the \( e_1 \)-components, and hence can be interpreted in terms of the quotient group \( \Gamma' \). In the proof of [Dekimpe et al. 19a, Proposition 5.11] it was shown that, for a fixed value of \( e_1^i, \) the number of Reidemeister classes \( [x(e_1)]_\varphi \) for which a pair \( (z_2, z_3) \) satisfying (i) exists is exactly \( O(12 - d, M) \), i.e., the number of solutions \( (z_2, z_3) \in \mathbb{Z}/2 \) of the linear system of equations
\[ \begin{pmatrix} i' \end{pmatrix} \begin{pmatrix} I_2 M \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} d \end{pmatrix}, \]

Note that the above equation is exactly condition (i) taken modulo 2.

Since \( e_1 \) can take two values (1 and \( -1 \)), there are in total \( 2O(12 - d, M) \) Reidemeister classes \( [x(e_1)]_\varphi \) satisfying condition (i). On the other hand, there are \( |\text{tr}(M)| - O(12 - d, M) \) Reidemeister classes of \( \Gamma' \) for which condition (i) does not hold (see [Dekimpe et al. 19a, Section 5]).

Recall that the variable \( z_1 \) appears only in condition (ii). If we have a Reidemeister class \( [x(e_1)]_\varphi \) and a pair \( (z_2, z_3) \) for which (i) holds, then we can find a \( z_1 \in \mathbb{Z} \) to make condition (ii) hold and only if
\[ k_1 z_2 z_3 - k_2 z_2 - k_3 z_3 - k_4 + 1 = 0, \]

which is exactly condition (ii) taken modulo 2.

We partition the solutions of (i') into those that do not satisfy condition (ii') and those that do. Let \( S \) be the number of the former and \( T \) the number of the latter, then \( S + T = O(12, M, d) \). Of the \( 2O(12 - d, M) \) Reidemeister classes \( [x(e_1)]_\varphi \) satisfying condition (i), \( 2S \) lift to two distinct Reidemeister classes \( [x(e_1)]_\varphi \) and \( [xe_1]_\varphi \), and \( 2T \) lift to a single Reidemeister class \( [x]_\varphi \). All together, we have
\[ R(\varphi) = 2(\text{tr}(M) - S - T) + 2(2S + 2T) \]

In particular, we get that \( R(\varphi) \in 2\mathbb{N} \). Taking the parity of \( \text{tr}(M) \) into account, we can further determine the possible Reidemeister numbers

\[ R(\varphi) \in \begin{cases} 4N + 2S & \text{if } \text{tr}(M) \equiv 0 \pmod{2}, \\ 4N + 2S - 2 & \text{if } \text{tr}(M) \equiv 1 \pmod{2}, \end{cases} \]

where

\[ S \leq O(12 - M, d) \leq \begin{cases} 4 & \text{if } \text{tr}(M) \equiv 0 \pmod{2}, \\ 1 & \text{if } \text{tr}(M) \equiv 1 \pmod{2}. \end{cases} \]

There is one special case, however. If \( M \equiv I_2 \mod 2 \) all entries of \( I_2 - M \) will be multiples of 2; so \( |\text{det}(I_2 - M)| = |\text{tr}(M)| \in 4\mathbb{N} \) and therefore \( R(\varphi) \in 8\mathbb{N} + 2S \).

For a fixed group \( \Gamma \) in this family (i.e., a fixed 4-tuple of parameters \( (k_1, k_2, k_3, k_4) \)), an automorphism \( \varphi \in \text{Aut}(\Gamma) \) is uniquely determined by the matrix \( M \in \text{GL}_2(\mathbb{Z}) \) and the vector \( d \in \mathbb{Z}^2 \). Our goal is to find out, for each group in the family (or equivalently, for each tuple \( (k_1, k_2, k_3, k_4) \)), which \( M \) and \( d \) satisfy conditions (a)–(d) and thus produce an automorphism.

Conditions (a)–(c) are actually conditions over \( \mathbb{Z}_2 \), and none of the parameters \( k_i \) appear in condition (d). Therefore, only the parity of the \( k_i \) will play a role, so we need to check 16 cases, each corresponding to an element of \( \mathbb{Z}_2^4 \). Furthermore, a group with parameters \( (k_1, k_2, k_3, k_4) \) is isomorphic to the group with parameters \( (-k_1, k_3, k_2, k_1) \), which allows us to omit the cases \( (0, 1, 0, 0), (0, 1, 0, 1), (1, 1, 0, 1) \) and \( (1, 1, 0, 1) \), leaving only 12 cases. Rather than trying to find all couples \( (M, d) \) (of which there are likely to be infinitely many), we can start by finding all couples \( (\tilde{M}, \tilde{d}) \in \text{GL}_2(\mathbb{Z}_2) \times \mathbb{Z}_2^2 \) satisfying conditions (a)–(c).

The function MakeList defined in Algorithm 1 does exactly this. Moreover, it assigns to every couple a set \( R \), which is the set of possible Reidemeister numbers the corresponding automorphisms can have. The results can be found in Tables 2–13. The Reidemeister spectrum of a group is a subset of (or the entirety of) the union of all these sets \( R \).

Next, for each quadruplet of parameters, we tried to find a family of automorphisms whose Reidemeister numbers produce the union of these sets \( R \). We succeeded in this for every choice of parameters, hence the Reidemeister spectrum always equals the union of the \( R \). These automorphisms and their Reidemeister spectra, for all \( (k_1, k_2, k_3, k_4) \), can be found in Table 14. For the sake of brevity, we omitted \( \infty \) from the spectra in this table.

We may thus conclude that, depending on the parity of the parameters \( k_1, k_2, k_3, k_4 \), the Reidemeister spectrum is \( 2\mathbb{N} \cup \{ \infty \} \), \( 4\mathbb{N} \cup \{ \infty \} \), \( (4\mathbb{N} - 2) \cup \{ \infty \} \) or \( (2\mathbb{N} + 2) \cup \{ \infty \} \). Note that all almost-Bieberbach groups have parameters with parities \( (0, 0, 0, 1) \) and therefore have spectrum \( 2\mathbb{N} \cup \{ \infty \} \).

Algorithm 1. MakeList function

1: function MakeList\((k_1, k_2, k_3, k_4)\)
2: \quad AutList := \emptyset 
3: \quad for \( M \in \text{GL}_2(\mathbb{Z}_2), d \in \mathbb{Z}_2^2 \) do
4: \quad \quad if \( \text{conditions (1), (2), (3) are met} \) then 
5: \quad \quad \quad \quad \quad S := 0 
6: \quad \quad \quad \quad \quad for \( \tilde{z} \in \mathbb{Z}_2^2 \) do
7: \quad \quad \quad \quad \quad \quad if \( \tilde{z} \) satisfies \((i')\) but not \((ii')\) then
8: \quad \quad \quad \quad \quad \quad \quad \quad \quad S := S + 1 
9: \quad \quad \quad \quad \quad \quad end if 
10: \quad \quad \quad \quad \quad end for 
11: \quad \quad \quad \quad \quad if \( \text{tr}(M) \equiv 0 \mod 2 \) then
12: \quad \quad \quad \quad \quad \quad if \( M \equiv 1_2 \mod 2 \) then
13: \quad \quad \quad \quad \quad \quad \quad \quad \quad R := 8N + 2S 
14: \quad \quad \quad \quad \quad \quad \quad \quad \quad else \quad R := 4N + 2S 
15: \quad \quad \quad \quad \quad \quad end if 
16: \quad \quad \quad \quad \quad end if 
17: \quad \quad \quad \quad \quad else 
18: \quad \quad \quad \quad \quad \quad R := 4N + 2S - 2 
19: \quad \quad \quad \quad \quad \quad end if 
20: \quad \quad \quad AutList := AutList \cup \{(\tilde{M}, \tilde{d}, R)\} 
21: \quad \quad \quad end if 
22: \quad \quad end for 
23: \quad return AutList 
24: end function

6. Spectra of 4D almost-Bieberbach groups

We already determined in Section 4 which families of four-dimensional almost-crystallographic groups do not have the \( R_\infty \)-property. In [Dekimpe 96] it is determined which groups among these families are almost-Bieberbach groups. We use the presentations from Section 4.

Family 1. Every group in this family is a finitely generated, torsion-free, nilpotent group of rank 4 and nilpotency class 2. In [Dekimpe et al. 19b, Section 3.2] it was shown that the Reidemeister spectrum of such group is always \( 4\mathbb{N} \cup \{ \infty \} \).

Family 3. The almost-Bieberbach groups in this family are those with \( (k_1, k_2, k_3, k_4) = (2k, 0, 0, 1) \) for some \( k \in \mathbb{N} \). An automorphism \( \varphi = \xi_{(d, D)} \) with
Using Theorem 3.5, we find that \( R(\varphi) = 8|m_1 + m_4| \in 8N \cup \{\infty\} \). Now, take the automorphism \( \varphi_m \) given by
\[
\varphi_m(\xi) = e^{km} e^m \xi, \\
\varphi_m(x) = e^{-1} x, \\
\varphi_m(x) = e^{-1} x,
\]
with \( m \in \mathbb{N} \). Then \( R(\varphi_m) = 8m \) and hence \( \text{Spec}_R(\Gamma) = 8N \cup \{\infty\} \).

7. Conclusion

We have determined which (non-crystallographic) almost-crystallographic groups of dimension 4 admit the \( R_\infty \) property, and calculated the Reidemeister spectra of the non-crystallographic 3-dimensional almost-crystallographic groups, as well as the spectra of the non-crystallographic 4-dimensional almost-Bieberbach groups. Together with the results of [Dekimpe et al. 19a], this completes the calculation of the Reidemeister spectra of the 3-dimensional almost-crystallographic groups and of the 4-dimensional almost-Bieberbach groups.

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