Deterministic coding theorems for blind sensing: optimal measurement rate and fractal dimension

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Abstract—Completely blind sensing is the problem of recovering bandlimited signals from measurements, without any spectral information beside an upper bound on the measure of the whole support set in the frequency domain. Determining the number of measurements necessary and sufficient for reconstruction has been an open problem, and usually partially blind sensing is performed, assuming to have some partial spectral information available a priori. In this paper, the minimum number of measurements that guarantees perfect recovery in the absence of measurement error, and robust recovery in the presence of measurement error, is determined in a completely blind setting. Results show that a factor of two in the measurement rate is the price pay for blindness, compared to reconstruction with full spectral knowledge. The minimum number of measurements is also related to the fractal (Minkowski-Bouligand) dimension of a discrete approximating set, defined in terms of the Kolmogorov “entropy.” These results are analogous to a deterministic coding theorem, where an operational quantity defined in terms of minimum measurement rate is shown to be equal to an information-theoretic one. A comparison with parallel results in compressed sensing is illustrated, where the relevant dimensionality notion in a stochastic setting is the information (Rényi) dimension, defined in terms of the Shannon entropy.

I. INTRODUCTION

A. Problem set-up

Let \( f : \mathbb{R} \to \mathbb{R} \) be square-integrable and such that

\[
\mathcal{F} f(\omega) = 0, \quad \text{for } \omega \notin \mathcal{Q},
\]

where \( \mathcal{F} \) indicates Fourier transform, \( \omega \) indicates angular frequency, and \( \mathcal{Q} \) is a subset of the interval \([ -\Omega, \Omega ]\) of measure

\[
m(\mathcal{Q}) \leq 2\Omega'.
\]

A typical example occurs when \( \mathcal{Q} \) is the union of a finite number of disjoint sub-intervals of \([ -\Omega, \Omega ]\) and \( \Omega' \ll \Omega \), see Figure 1. These kind of signals arise in many applications, ranging from radio, to audio, and biological communication and sensing systems. A natural question is what is the minimum number of measurements that can be performed over a given time interval and that guarantees reconstruction with a minimum amount of error.

To address this question, we consider a measurement vector \( y \in \mathbb{R}^M \)

\[
y = \mathcal{M} f(t) + e,
\]

where \( \mathcal{M} \) is an operator from multi-band signals to \( M \)-dimensional vectors and \( e \in \mathbb{R}^M \) is the measurement error.

We assume each measurement \( y_n \in y \) results from observing the signal over the interval \([ -T/2, T/2 ]\) through the inner product with a bandlimited kernel, plus an error term.

**Definition 1.** (Measurements) For all \( n \in \{ 1, \ldots, M \} \), we have

\[
y_n = \int_{-T/2}^{T/2} f(t) \varphi_n(t) dt + e_n,
\]

where

\[
\mathcal{F} \varphi_n(\omega) = 0 \quad \text{for } \omega \notin [ -\Omega, \Omega ].
\]

This set-up covers a wide range of real measurements. Possible bandlimited kernels that fall in this framework include the Shannon cardinal basis \( \text{sinc}(\cdot) \) functions [1], the Slepian prolate spheroidal wave functions (PSWF) [2], as well as other bandlimited functions of practical interest, such as wavelets, and splines. The measurements are functionals of the signal over the entire observation interval, but in some cases they can reduce to the sampled signal values. For example, for the cardinal basis the measurements in (4) also correspond to low-pass filtering and sampling, and the signal can be recovered by low-pass filtering the sampled signal values [3]. This special case is illustrated in Figure 2. The general case is illustrated in Figure 3.

In the general setting, our aim is to determine the smallest measurement rate

\[
\bar{M} = \lim_{T \to \infty} \frac{M}{T}
\]

for which it is possible to obtain an approximation \( f_M \) of \( f \) from \( y \), such that the energy of the reconstruction error is at most proportional to the energy of the measurement error,
as the size of the observation interval $T \to \infty$. This corresponds to determining the scaling of the minimum number of measurements $M = M(T)$ that guarantees robust recovery of any multi-band signal, namely a small perturbation in the measurement does not lead to a large reconstruction error.

**Definition 2.** (Robust recovery). There exists a universal constant $c \geq 0$, such that for $T$ large enough

$$
\|f - f_M\|^2 = \int_{-T/2}^{T/2} [f(t) - f_M(t)]^2 dt \\
\leq c \sum_{n=1}^{M} e_n^2.
$$

When the measurement error tends zero, robust recovery reduces to perfect recovery of the signal. Namely,

**Definition 3.** (Perfect recovery).

$$
\lim_{T \to \infty} \|f - f_M\|^2 = 0.
$$

**B. Bandlimited signals**

Since our signals are assumed to be bandlimited to $\Omega$, one may readily observe that in the absence of measurement error they can be perfectly recovered from a number of measurements slightly above the Nyquist number

$$
N_0 = \Omega T / \pi.
$$

For any $f$ satisfying (1) and (2), and $\nu > 0$, we can construct an approximation $f_N$ of $f$ from a measurement vector $y$ of size

$$
N = (1 + \nu)\Omega T / \pi,
$$

and such that

$$
\lim_{T \to \infty} \|f - f_N\|^2 = 0.
$$

This classic result is equivalent to stating that a measurement rate strictly above $\Omega / \pi$ is sufficient for reconstruction of any bandlimited signal, and constitutes one of the milestones of electrical and communication engineering.

For bandlimited signals, the rate $\Omega / \pi$ is also optimal, in the following approximation-theoretic sense. Consider performing signal reconstruction by a linear interpolation of a number $N > 0$ of orthogonal basis functions

$$
f_N(t) = \sum_{n=1}^{N} y_n \varphi_n(t),
$$

and let the Kolmogorov $N$-width be the smallest approximation error achievable for all signals in the space, over all possible choices of basis sets. This minimum error is achieved by measurements that provide the coefficients of the interpolation through the integrals

$$
y_n = \frac{1}{\lambda_n} \int_{-T/2}^{T/2} f(t) \varphi_n(t) dt, \quad n \in \{1, \ldots, N\},
$$

where $\{\lambda_n\}$ are the eigenvalues of a Fredholm integral equation of the second kind arising from Slepian’s concentration problem [2], and the basis functions $\{\varphi_n\}$ are the corresponding eigenfunctions, called PWSF [4]. The measurement rate $\Omega / \pi$ corresponds to the critical threshold at which the Kolmogorov $N$-width transitions from strictly positive values to zero, as $T \to \infty$ [5]. This phase transition behavior of the approximation error is illustrated in Figure 4. With a number of measurements $(1 + \nu)\Omega T$ the error tends to zero as $T \to \infty$, while with a number of measurements $\Omega T / \pi + o(T)$ the error remains positive as $T \to \infty$.

**C. Multi-band signals**

For bandlimited signals that are supported over disjoint sub-bands, an important extension of the results above, due to Landau and Widom [6], states that if we have a priori knowledge of the size and positions of all the sub-bands, then signal reconstruction with vanishing error as $T \to \infty$ is also possible using the smaller number of measurements

$$
S = (1 + \nu)\Omega' T / \pi.
$$
A simple way to achieve this result is to demodulate each sub-band down to baseband, isolate it through low-pass filtering, and then sample each sub-band separately. The key contribution of Landau and Widom is to consider the optimal subspace approximation, and show a phase transition of the error expressed in terms of Kolmogorov N-width. As in the single-band case, a subspace approximation with vanishing error for all multi-band signals of a given frequency allocation is obtained with a number of measurements \((1 + \nu)\Omega T/\pi\), while a subspace approximation with vanishing error is not possible for all multi-band signals using a number of measurements \(\Omega T/\pi + o(T)\), and the value of the error is controlled by the pre-constant in the \(o(T)\) term. It follows that for multi-band signals the Nyquist number \(N_0 = \Omega T/\pi\) can be replaced by the “sparsity number”

\[
S_0 = \Omega T/\pi, \quad (15)
\]

and the occupied portion of the frequency bandwidth determines the critical measurement rate \(\Omega T/\pi\) required for reconstruction. In the case of sampling measurements, Landau also showed that a rate \(\Omega T/\pi\) is necessary for reconstruction, regardless of the reconstruction strategy being linear or not.

The results of above rely on two critical assumptions. First, they need a priori knowledge of the spectral occupation, since the eigenvalues and the optimal eigenfunctions used for reconstruction are solutions of an integral equation that depends on the spectral support set. In practice, it might be difficult to know the exact number of sub-bands, their location, and their widths prior to the measurements. A second critical assumption is the absence of measurement error. In practice, the measurement process always carries a certain amount of error and its impact on the reconstruction error should be taken into account.

**D. Completely blind sensing**

In this paper, we consider robust signal reconstruction in the presence of measurement error and without any a priori knowledge of the sub-bands beside an upper bound on the measure of the whole support set of the signal in the frequency domain. We call this robust, completely blind sensing. The blindness requirement is important when detecting the sub-bands is impossible or too expensive to implement. The robustness requirement is important to guarantee stability in the reconstruction process.

Partially blind sensing, where some partial spectral information is assumed, has been studied extensively. First key results were given in a series of papers by Bresler and co-authors [8], [9], [10]. Later extensions [11], [12] reduced the number of a priori assumptions, but still require knowledge of the number of sub-bands, and of their widths. The same assumptions are made in [13], [14], [15]. The main result in this setting is that the price to pay for partial blindness is a factor of two in the measurement rate. Several reconstruction strategies have been proposed using a measurement rate above \(2\Omega T/\pi\), all assuming some partial spectral knowledge, and lacking an information-theoretic converse. We remove these assumptions, show that a measurement rate \(2\Omega T/\pi\) is sufficient for robust reconstruction in a completely blind setting, and provide a tight converse result. We also provide a deterministic coding theorem for continuous analog sources, giving an interpretation of the minimum number of measurements in terms of the “effective” Minkowski-Boulingand dimension of the infinite-dimensional set of multi-band signals, expressed in terms of the Kolmogorov \(\varepsilon\)-entropy. This is compared with an analogous interpretation arising in the framework of compressed sensing, where the objective is the lossless source coding of a discrete, analog, stochastic process [16], [17]. In that case, an analogous coding theorem has been given in terms of the Rényi dimension, expressed in terms of the Shannon entropy.

Finally, we remark that while in the case of multi-band signals of a given sub-band allocation the results of Landau and Widom provide an optimal subspace approximation in terms of a linear interpolation of eigenfunctions supported over multiple sub-bands, and having the highest energy concentration over the observation domain, our results only provide an answer to the question of whether recovery is possible or not, without giving an explicit approximation procedure. In our case, the discrete-to-continuous block in Figure 3 remains unknown. Nevertheless, from an information-theoretic perspective one is primarily interested in the possibility of recovery using any discrete to continuous transformation, and does not wish to restrict reconstruction to a linear approximation strategy. The explicit construction of practical blind recovery strategies is certainly of interest, and these should be compared with the information-theoretic optimum determined here.

The rest of the paper is organized as follows: In section II we describe our results. In section III we compare our results with compressed sensing and illustrate coding theorems in deterministic and stochastic settings. In section IV we provide some definitions and preliminaries that are useful for our derivation. Proofs are given in section V and VI. Section VII draws conclusions and discusses future work.

**II. DESCRIPTION OF THE RESULTS**

**A. Noiseless Case**

**Theorem 1. (Direct).** *In the absence of measurement error, we can perfectly recover any signal* \(f\) *satisfying (1) and (2) using a measurement rate*

\[
\tilde{M} > \frac{2\Omega'}{\pi}. \quad (16)
\]

**Theorem 2. (Converse).** *In the absence of measurement error, we cannot perfectly recover all signals* \(f\) *satisfying (1) and (2) using a measurement rate*

\[
\bar{M} \leq \frac{2\Omega'}{\pi}. \quad (17)
\]

These results can interpreted in terms of the effective dimensionality of the signals’ space, leading to a coding theorem. For bandlimited signals, the effective number of dimensions can be identified with the Nyquist number \(N_0 = \Omega T/\pi\). For multi-band signals for which the location and widths of all the sub-bands is fixed a priori, as in the Landau-Widom case, it can be identified with the sparsity number \(S_0 = \Omega T/\pi\). On
the other hand, without any a priori knowledge, we need to account for the additional degrees of freedom of allocating the sub-bands in the frequency domain, and our results indicate that the effective dimensionality increases to $2S_0$.

To make these considerations precise, we consider an information-theoretic quantity that measures the dimensionality of a set in metric space, namely its fractal (Minkowski-Bouligand) dimension, which corresponds to the rate of growth of the Kolmogorov $\epsilon$-entropy of successively finer discretizations of the space, and represents the degree of fractality of the set [18].

**Definition 4.** (Fractal dimension). For any subset $X$ of a metric space, the fractal dimension is

$$\dim_F(X) = \lim_{\epsilon \to 0} \frac{H_{\epsilon}(X)}{-\log \epsilon},$$

where $H_{\epsilon}$ is the Kolmogorov $\epsilon$-entropy [19].

If this limit does not exist, then the corresponding upper and lower fractal dimensions are defined using $\limsup$ and $\liminf$, respectively.

We also define the dilation

**Definition 5.** (Minkowski sum).

$$X \oplus X = \{ x_1 + x_2 : x_1, x_2 \in X \}.$$  

Consider now the set of all bandlimited signals whose energy is at most one. These signals can be approximated by an infinite set $X_B$ of vectors, each containing $N = (1 + \nu)\Omega T/\pi$ real coefficients. Using the PSWF as a basis for interpolation, every assignment of coefficients satisfying the given energy constraint approximates, with vanishing error as $T \to \infty$, a bandlimited signal. In the appendix, we show that

$$\dim_F(X_B) = \dim_F(X_B \oplus X_B),$$

and letting the fractal dimension rate of the approximating set be

$$R_F(X_B) = \lim_{T \to \infty} \frac{\dim_F(X_B)}{T},$$

we have

$$R_F(X_B) = \Omega/\pi,$$

which coincides with the measurement rate needed for reconstruction.

Next, we quantize the bandwidth at level $\Delta > 0$ and let

$$J = \{-\Omega, -\Omega + \Delta, -\Omega + 2\Delta, \cdots, \Omega\}.$$  

We consider the subset of all multi-band signals of a given sub-band allocation, whose energy is at most one, and such that the extremal points of all sub-bands belong to $J$. This subset of signals approximates, with vanishing energy error as $\Delta \to 0$, the one of all multi-band signals of a given sub-band allocation and of energy at most one. It can also be approximated, with vanishing error as $T \to \infty$, by an infinite set $\lambda_{MB}(\Delta)$ of vectors, each containing $N = (1 + \nu)\Omega T/\pi$ real coefficients of a PSWF interpolation. Compared to the previous case, the choice of the coefficients is now restricted by the given sub-band allocation, so that we have

$$\lambda_{MB}(\Delta) \subset X_B.$$  

Following the same argument used to derive (20), we obtain

$$\dim_F[\lambda_{MB}(\Delta)] = \dim_F[\lambda_{MB}(\Delta) \oplus \lambda_{MB}(\Delta)].$$

In this case, however, the $N$-dimensional prolate spheroidal approximation is somewhat redundant, and following the same argument used to derive (22), we obtain

$$\lim_{\Delta \to 0} R_F[\lambda_{MB}(\Delta)] = \Omega'/\pi,$$

which coincides with the Landau-Widom rate [6, 7] needed for reconstruction.

Finally, consider the subset of all multi-band signals whose energy is at most one, having an arbitrary sub-band allocation of measure at most $2\Omega'$, and such that the extremal points of all sub-bands belong to $J$. These signals can be approximated, as $T \to \infty$, by an infinite set $X(\Delta)$ of vectors, each containing $N = (1 + \nu)\Omega T/\pi$ real coefficients of a PSWF interpolation. The choice of the coefficients is now restricted only by the measure of the occupied portion of the spectrum and not by a specific sub-band allocation, and we have

$$\lambda_{MB}(\Delta) \subset X(\Delta) \subset X_B.$$  

By combining Theorems 1 and 2 with Theorems 3 and 4 below, we obtain

$$\lim_{\Delta \to 0} R_F[\lambda(X)] = \Omega'/\pi.$$  

**Theorem 3.** (Direct). In the absence of measurement error, we can perfectly recover any signal $f$ satisfying (7) and (8) using a measurement rate

$$M > 2 \lim_{\Delta \to 0} R_F[\lambda(X)].$$

**Theorem 4.** (Converse). In the absence of measurement error, we cannot perfectly recover all signals $f$ satisfying (7) and (8) using a measurements rate

$$M \leq 2 \lim_{\Delta \to 0} R_F[\lambda(X)].$$

In section VI, we also show that

$$R_F[\lambda(X) \oplus \lambda(X)] = 2R_F[\lambda(X)],$$

which also implies

$$\lim_{T \to \infty} \frac{\dim_F[\lambda(X) \oplus \lambda(X)]}{\dim_F[\lambda(X)]} = 2.$$  

We now give a geometric interpretation of these results. The set of all multi-band signals is the union of infinitely many subsets, each corresponding to the multi-band signals of a given sub-band allocation. The Minkowski sum in (19) takes into account the additional degrees of freedom of allocating the sub-bands in the frequency domain. Within any subset, any multi-band signal is specified by essentially $\dim_F[\lambda(X)]$ coordinates, but when considering the union of all subsets, it is specified by essentially $2\dim_F[\lambda(X)]$ coordinates. By (31) it then follows that the relevant information-theoretic quantity that characterizes the possibility of reconstruction is the fractal dimension rate of the dilation, rather than the fractal dimension rate of the set itself.
Finally, it is useful introduce the *sparsity fraction* as the ratio of the fractal dimension of the approximating set and its ambient dimension:

**Definition 6. (Sparsity fraction).**

\[
\sigma = \inf_{\nu > 0} \lim_{\Delta \to 0} \lim_{T \to \infty} \frac{\dim_F [X(\Delta)]}{N}. \tag{33}
\]

By the results above, it is easy to see that the sparsity fraction is equal to the fraction of occupied bandwidth, namely substituting \( N = (1 + \nu)\Omega T/\pi \) into (33) we get

\[
\sigma = \inf_{\nu > 0} \lim_{\Delta \to 0} \lim_{T \to \infty} \frac{R_F[X(\Delta)]}{\Omega} \frac{\pi}{(1 + \nu)} = \frac{\Omega'}{\Omega}, \tag{34}
\]

and twice the sparsity fraction corresponds to the critical number of measurements per unit ambient dimension necessary and sufficient for reconstruction.

**B. General Case**

Results generalize to the noisy case. The critical threshold for the number of measurements is not affected by the presence of a measurement error, provided that we ask for robust, rather than perfect reconstruction.

**Theorem 5. (Direct).** We can robustly recover all signals \( f \) satisfying (7) and (2) using a measurements rate

\[
\bar{M} > 2 \lim_{\Delta \to 0} R_F[X(\Delta)] = \frac{2\Omega'}{\pi}. \tag{35}
\]

**Theorem 6. (Converse).** We cannot robustly recover all signals \( f \) satisfying (7) and (2) using a measurements rate

\[
\bar{M} \leq 2 \lim_{\Delta \to 0} R_F[X(\Delta)] = \frac{2\Omega'}{\pi}. \tag{36}
\]

A factor of two is the price to pay for blindness for both robust recovery and perfect recovery of multi-band signals, and in virtue of (31) the relevant dimensionality notion is the one associated to the dilation of the set.

**III. COMPARISON WITH COMPRESSED SENSING**

There are analogies between our results and the ones in compressed sensing. We illustrate similarities and differences in deterministic and stochastic settings. For simplicity, we only consider the case of zero measurement error, but the same considerations apply to the case of non-zero measurement error.

**A. Deterministic setting**

Consider an \( N \)-dimensional vector \( x \) such that

\[
x = \Phi x, \tag{37}
\]

where \( \Phi \) is an \( N \times N \) orthogonal matrix and \( X \) has at most \( S \) non-zero elements. If \( S \ll N \) we say that \( X \) is a sparse representation of \( x \). An example is illustrated in Figure 5.

We define a measurement vector

\[
y = Ax, \tag{38}
\]

where \( A \) is an \( M \times N \) matrix, and \( M \) is the number of measurements. Clearly, \( x \) can be recovered from \( N \) measurements by observing all the elements of \( x \). In this case, the \( N \times N \) measurement matrix \( A \) is diagonal. If we know the position of the nonzero elements of \( X \), then \( S \) measurements are also enough to perfectly reconstruct \( x \). In this case, each measurement extracts the \( n \)th coefficient of \( X \) from \( \Phi^{-1}x \), and the signal is recovered by performing a final multiplication by \( \Phi \). However, if we only know that \( x \) has a sparse representation, but we do not know the positions of the nonzero elements of \( X \), without further investigation we can only conclude that that the minimum number \( M \) of measurements sufficient for reconstruction is \( S \leq M \leq N \). The objective of compressed sensing is to reconstruct any sparse, discrete signal \( x \) using \( M \ll N \) measurements.

Without worrying about an explicit reconstruction procedure, a simple linear algebra argument [16] Remark 2]. [20] Section 2.2] shows that the necessary and sufficient number of measurements for reconstruction is \( 2S \). It follows that in both the continuous and discrete settings, the number of linear measurements necessary and sufficient for reconstruction is equal to twice the sparsity level of the signal. The main differences between the two settings are as follows: the compressed sensing formulation assumes knowledge of the matrix \( \Phi \), corresponding to the basis where the discrete signal is sparse. In the case of blind sensing, it is only assumed that the signal does not occupy the whole frequency spectrum, but the discrete basis set required for the optimal representation is unknown a priori. A more extreme situation is the blind compressed sensing set-up [21], [22], where there is a complete lack of knowledge about the signal. In this case, the basis must either be learned from data, or selected from a restricted set. Finally, in blind sensing the reconstruction error tends to zero as \( T \to \infty \), while in compressed sensing perfect reconstruction is possible for all \( N \).

**B. Stochastic setting**

The problem of compressed sensing can also be formulated in a probabilistic setting. In this case, the discrete signal to be recovered is modeled as a stochastic process and the objective is to reconstruct the signal with arbitrarily small probability of error, given a sufficiently long observation sequence. Viewing the measurement operator as an encoder and the reconstruction operator as a decoder acting on a sequence of independent, identically distributed (i.i.d.), real-valued random variables, the compressed sensing set-up corresponds to lossless source coding of analog memoryless sources when the encoding operation \( C : \mathbb{R}^N \to \mathbb{R}^M \) is the multiplication by a real-valued matrix, see Figure 6. Compared to the deterministic setting, where reconstruction is required for all possible source signals, here the performance is measured on a probabilistic basis by considering long block lengths and averaging with
Fig. 6. Source coding view of compressed sensing.

We can also give an information-theoretic definition of the sparsity fraction in the stochastic setting that is analogous to Definition 6.

**Definition 8.** (Sparsity fraction—stochastic setting).

\[
\gamma = \frac{\dim\{X_1, X_2, \ldots, X_N\}}{N}. \tag{42}
\]

For a mixture distribution such as \(\|X\| < \infty\), Rényi showed that

\[
\dim\{X\} = \gamma. \tag{43}
\]

Combining this result with (41) it follows that the sparsity fraction is also equal to \(\gamma\), and the fraction of non-zero elements of the signal coincides with the information dimension per unit ambient dimension. In the analogous deterministic setting, the fraction of occupied bandwidth plays the role of the fraction of non-zero elements of the discrete-time signal, and this coincides with the fractal dimension per unit ambient dimension of its prolate spheroidal approximation.

**C. Coding theorems**

The results of Wu and Verdú combined with Rényi’s one in (43) yield the following general coding theorem:

**Theorem 7.** (Coding theorem—stochastic setting).

The minimum number of measurement per unit dimension sufficient for reconstruction with vanishing probability of error of an analog, \(\gamma\)-sparse, memoryless, discrete-time process coincides with the information dimension per unit ambient dimension of the space, which is equal to \(\gamma\).

The analogous deterministic coding theorem in our continuous setting is obtained by combining Theorems 3 and 4 and using Definition 6.

**Theorem 8.** (Coding theorem—deterministic setting).

The minimum number of measurement per unit dimension sufficient for reconstruction with vanishing error of any \(\sigma\)-sparse, continuous-time signal coincides with twice the fractal dimension per unit ambient dimension of its prolate spheroidal approximation, which is equal to \(2\sigma\).

A factor of two appears in the deterministic formulation, due to the worst case reconstruction scenario.

**IV. TECHNICAL PRELIMINARIES**

**A. Metric spaces**

We begin our proofs by defining the metric spaces associated to the bandlimited and multi-band signals satisfying (1) and (2). Let \(f \in L^2(-\infty, \infty), 2\Omega' \subset \Omega\), and

\[
B_\Omega = \{f(t) : \mathcal{F}f(\omega) = 0, \text{ for } |\omega| > \Omega\}, \tag{44}
\]

\[
B_\mathcal{Q} = \{f(t) : \mathcal{F}f(\omega) = 0, \omega \notin \mathcal{Q}\}, \tag{45}
\]

\[
\mathcal{Q}' = \{\mathcal{Q} : \mathcal{Q} \subset [-\Omega, \Omega] \text{ and } m(\mathcal{Q}) \leq 2\Omega'\}, \tag{46}
\]

\[
B_{\mathcal{Q}'} = \bigcup_{\mathcal{Q} \in \mathcal{Q}'} B_{\mathcal{Q}}. \tag{47}
\]
It follows that $B_{Q^c} \subset B_{\Omega}$. We equip $B_{\Omega}$ and $B_{Q^c}$ with the $L^2[-T/2, T/2]$ norm

$$
\| f \| = \left( \int_{-T/2}^{T/2} f^2(t) dt \right)^{1/2} .
$$

(48)

It follows that $(B_{\Omega}, \| \cdot \|)$ and $(B_{Q^c}, \| \cdot \|)$ are metric spaces, whose elements are square-integrable, real, bandlimited or multi-band signals, of infinite duration and observed over the finite interval $[-T/2, T/2]$.

\section*{B. Optimal representations}

Let $Q$ be a measurable subset of $\mathbb{R}$ and $T = [-T/2, T/2]$. We define the following time-limiting and band-limiting operators

\begin{align*}
\mathcal{T}_T f(t) &= \mathbb{1}_T f(t) \\
\mathcal{B}_Q f(t) &= \mathcal{F}^{-1} \mathcal{Q} \mathcal{F} f(t),
\end{align*}

(49)

(50)

where $\mathbb{1}_{(\cdot)}$ is the indicator function. We consider the following eigenvalues equation

$$
\mathcal{T}_T \mathcal{B}_Q \mathcal{T}_T \psi(t) = \lambda^Q \psi(t). 
$$

(51)

There exists a countably infinite set of real functions \{\psi_n^Q(t)\}_{n=1}^\infty and a set of real positive numbers $1 > \lambda_1^Q > \lambda_2^Q > \cdots > 0$ with the following properties, see \cite{Slepian1978}.

\begin{enumerate}
\item Property 1. The elements of \{\psi_n^Q(t)\} and \{\psi_n^Q(t)\} are solutions of (51).
\item Property 2. The elements of \{\psi_n^Q(t)\} are in $B_Q$.
\item Property 3. \{\psi_n^Q(t)\} is complete in $B_Q$.
\item Property 4. The elements of \{\psi_n^Q(t)\} are orthonormal in $(-\infty, \infty)$.
\item Property 5. The elements of \{\psi_n^Q(t)\} are orthogonal in $(-T/2, T/2)$
\end{enumerate}

$$
\int_{-T/2}^{T/2} \psi_n^Q(t) \psi_m^Q(t) dt = \begin{cases} 
\lambda_n^Q & n = m, \\
0 & \text{otherwise}.
\end{cases}
$$

(52)

We write $\psi(t)$ and $\lambda$ instead of $\psi^Q(t)$ and $\lambda^Q$ when $Q = [-\Omega, \Omega]$. In this special case, the eigenfunctions \{\psi_n(t)\} are the prolate spheroidal wave functions (PSWF) \cite{Landau1967}.

\begin{lemma} \text{(Slepian \cite{Slepian1978}).} \label{lemma1}
For any $\nu > 0$, $N = (1 + \nu)\Omega T/\pi$, and $f \in B_{\Omega}$, there exist real coefficients \{\{x_n\}\}, such that the approximation

$$
f_N(t) = \sum_{n=1}^N x_n \psi_n(t)
$$

(53)

has vanishing error norm $\| f - f_N \|$, as $T \to \infty$.
\end{lemma}

\begin{lemma} \text{(Landau and Widom \cite{Landau1967}).} \label{lemma2}
For any $\nu > 0$, $S = (1 + \nu)\Omega T/\pi$, and $f \in B_{Q^c}$, there exist real coefficients \{\alpha_n\}, such that the approximation

$$
f_S(t) = \sum_{n=1}^S \alpha_n \psi_n^Q(t),
$$

(54)

has vanishing error norm $\| f - f_S \|$, as $T \to \infty$.
\end{lemma}

\section*{C. Measurement vector}

We consider the measurements of $f(t) \in B_{Q^c} \subset B_{\Omega}$

$$
y_n = \int_{-T/2}^{T/2} f(t) \varphi_n(t) dt + e_n, \quad n \in \{1, \ldots, M\},
$$

(55)

where $e_n$ is the measurement error and each measurement kernel $\varphi_n$ is a bandlimited function. Since $\varphi_n$ is bandlimited, this can be represented by a linear combination of the “canonical” PSWF basis of $B_{\Omega}$, namely

$$
\varphi_n(t) = \sum_{k=1}^\infty \alpha_{nk} \psi_k(t).
$$

(56)

Using the completeness of the $\{\psi_n(t)\}$ in $B_{\Omega}$, and their orthogonality property, it follows that the $n$-th measurement can also be expressed as

$$
y_n = \int_{-T/2}^{T/2} f(t) \varphi_n(t) dt + e_n = \int_{-T/2}^{T/2} f(t) \sum_{k=1}^\infty \alpha_{nk} \psi_k(t) dt + e_n
$$

$$
= \sum_{j=1}^N a_{nj} x_j \sqrt{\lambda_j} + \sum_{j=N+1}^{\infty} a_{nj} x_j \sqrt{\lambda_j} + e_n
$$

(57)

Letting $N = (1 + \nu)\Omega T/\pi$, we have

$$
\lim_{T \to \infty} \sum_{j=N+1}^{\infty} a_{nj} x_j \sqrt{\lambda_j} = 0.
$$

(58)

It follows that as $T \to \infty$ the measurements become

$$
y_n = \sum_{j=1}^N a_{nj} x_j \sqrt{\lambda_j} + e_n + o(1).
$$

(59)

Letting $y = (y_1, \ldots, y_M)$, $x = (x_1 \sqrt{\lambda_1}, \ldots, x_N \sqrt{\lambda_N})$, and $A$ be an $M \times N$ matrix such that $[A]_{nj} = a_{nj}$, we define

$$
y = Ax + e,
$$

(60)

and consider the set

$$
\mathcal{X} = \left\{ x : x = \left( x_1 \sqrt{\lambda_1}, \ldots, x_N \sqrt{\lambda_N} \right) \right\}.
$$

(61)

In virtue of Lemma\ref{lemma1} there exists a one-to-one correspondence between $B_{Q^c}$ and $\mathcal{X}$, as $T \to \infty$. By \cite{Ong2012} it then follows that to complete our proofs we can derive lower and upper bounds on the number of rows of $A$ required to recover $x \in \mathcal{X}$ from $y = Ax + e$ in \cite{Ong2012}, and then evaluate their order of growth as $T \to \infty$.

\section*{V. PROOFS OF THEOREMS \cite{Proctor2012} AND \cite{Proctor2013}}

We consider a function $\zeta(t)$ such that

$$
f_N(t) = f_S(t) + \zeta(t),
$$

(62)

where $f_N(t)$ and $f_S(t)$ are given in \cite{Proctor2012} and \cite{Proctor2013}, and let

$$
\zeta_k = \int_{-T/2}^{T/2} \zeta(t) \psi_k(t) dt.
$$

(63)
It follows that for all $1 \leq k \leq N$, we have

$$\sqrt{\lambda_k} x_k = \int_{-T/2}^{T/2} \sum_{n=1}^{S} \alpha_n \psi_n(t) \psi_k(t) dt + \zeta_k$$

which implies $\text{rank}(A\Phi') = r$. We can then conclude that $A\Phi$ contains at least $r$ independent columns, which implies $\text{rank}(A\Phi) \geq r = \text{rank}(\Phi)$.

Lemma 4. A number of measurements

$$m \geq \max_{\Phi_1, \Phi_2 \in \mathcal{D}} \left( \text{rank}[\Phi_1, \Phi_2] \right),$$

is sufficient to recover all the elements of $\mathcal{X}$.

Proof: From Lemma 3 it follows that for all $\Phi_1, \Phi_2 \in \mathcal{D}$ there exists an $m \times N$ matrix $A$ such that $\text{rank}(A[\Phi_1, \Phi_2]) = \text{rank}[\Phi_1, \Phi_2]$. Let us assume $Ax_1 = Ax_2$ where $x_1 = \Phi_1\alpha_1$ and $x_2 = \Phi_2\alpha_2$. The expression $Ax_1 = Ax_2$, can be rewritten as

$$A[\Phi_1, \Phi_2] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 0.$$

namely $[\alpha_1, \alpha_2]^T$ belongs to the null space of $A[\Phi_1, \Phi_2]$. Since $\text{rank}(A[\Phi_1, \Phi_2]) = \text{rank}[\Phi_1, \Phi_2]$, the null space of $A[\Phi_1, \Phi_2]$ is the same as the null space of $[\Phi_1, \Phi_2]$. It follows that $[\alpha_1, \alpha_2]^T$ belongs to the null space of $[\Phi_1, \Phi_2]$, or equivalently $[\Phi_1, \Phi_2] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = 0.$

This means $\Phi_1\alpha_1 = \Phi_2\alpha_2$, namely $x_1 = x_2$. Therefore, $A$ is one-to-one on $\mathcal{X}$, which implies that the elements of $\mathcal{X}$ can be recovered.

Lemma 5. A number of measurements

$$m < \max_{\Phi_1, \Phi_2 \in \mathcal{D}} \left( \text{rank}[\Phi_1, \Phi_2] \right)$$

is not sufficient to recover all the elements of $\mathcal{X}$.

Proof: It is enough to show that for all $\Phi_1, \Phi_2 \in \mathcal{D}$, if $A$ is an i.i.d Gaussian random matrix of size $m \times N$, then $\text{rank}(A[\Phi_1, \Phi_2]) = \text{rank}[\Phi_1, \Phi_2]$ with probability 1. Since $\text{rank}(A[\Phi_1, \Phi_2]) \leq \text{rank}([\Phi_1, \Phi_2])$, it is enough to show that $\text{rank}(A[\Phi_1, \Phi_2]) \geq \text{rank}([\Phi_1, \Phi_2])$. For convenience, we let $[\Phi_1, \Phi_2] = \Phi$ and we will show $\text{rank}(A\Phi) \geq \text{rank}(\Phi)$.

Note that $\Phi$ is an $N \times 2S$ matrix with $\text{rank}(\Phi) = r \leq m$. Collect $r$ independent columns of $\Phi$ and compose an $N \times r$ matrix $\Phi'$. Using the Gram-Schmidt process, we can transform $\Phi'$ into $\Phi_G$, an $N \times r$ matrix, whose columns are orthonormal. By adding redundant $N-r$ orthonormal columns followed by the original $r$ columns of $\Phi_G$, we obtain an $N \times N$ orthogonal matrix $\Phi_G$.

Let us define $\sigma(X)$ as the smallest number of linearly dependent columns of a matrix $X$. It is well known that, if $A$ is an i.i.d Gaussian random matrix of size $m \times N$, where $m < N$, then $\sigma(AP) = m + 1$ with probability 1 for any fixed orthogonal matrix $P$, see for example [21 Proposition 1] for a proof. Therefore, the first $r$ columns of $A\Phi_G$ are independent. Thus, we have $\text{rank}(A\Phi_G) = r$.

Lemma 6. We have

$$\lim_{T \to \infty} \frac{\max_{\Phi_1, \Phi_2 \in \mathcal{D}} \left( \text{rank}[\Phi_1, \Phi_2] \right)}{2S} = 1.$$
Proof: Let \( Q_1, Q_2 \in Q' \), and consider the multi-band signals
\[
\begin{align*}
f_1(t) &= \sum_{n=1}^{\infty} \alpha_n \psi_{Q_1}^n(t), \quad (81) \\
f_2(t) &= \sum_{n=1}^{\infty} \beta_n \psi_{Q_2}^n(t), \quad (82)
\end{align*}
\]
and
\[
f_S(t) = f_1(t) + f_2(t). \quad (83)
\]
Consider the \( N \)-dimensional vector
\[
z = [\Phi_{Q_1}, \Phi_{Q_2}] \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_S \\ \beta_1 \\ \vdots \\ \beta_S \end{bmatrix},
\]
whose elements, by (65) and (68), and then using (81), (82), (83), are
\[
z_n = \sum_{j=1}^{S} \alpha_j \varphi_{Q_1,n,j} + \sum_{j=1}^{S} \beta_j \varphi_{Q_2,n,j}
\]
\[
= \sum_{j=1}^{S} \alpha_j \int_{-T/2}^{T/2} \psi_{Q_1,n}^j(t) \varphi_{n,j}(t) \, dt \\
+ \sum_{j=1}^{S} \beta_j \int_{-T/2}^{T/2} \psi_{Q_2,n}^j(t) \varphi_{n,j}(t) \, dt
\]
\[
= \int_{-T/2}^{T/2} f_1(t) \varphi_{n,j}(t) \, dt + \int_{-T/2}^{T/2} f_2(t) \varphi_{n,j}(t) \, dt
\]
\[
= \int_{-T/2}^{T/2} f_S(t) \varphi_{n,j}(t) \, dt. \quad (85)
\]
We consider the case when \( z \) is the all zero vector. In this case, since by (65) the elements \( \{z_n\} \) are also the PSWF coefficients of \( f_S(t) \), it follows that
\[
\lim_{T \to \infty} f_S(t) = 0. \quad (69)
\]
We now choose \( Q_1 \) and \( Q_2 \) such that \( Q_1 \cap Q_2 = \emptyset \), so that (86) implies
\[
\lim_{T \to \infty} f_1(t) = \lim_{T \to \infty} f_2(t) = 0. \quad (87)
\]
It follows that all coefficients \( \{\alpha_n\} \) and \( \{\beta_n\} \) in (81) and (82) must tend to zero as \( T \to \infty \), the columns of \( [\Phi_{Q_1}, \Phi_{Q_2}] \), become independent, and we have
\[
\lim_{T \to \infty} \frac{\text{rank}[\Phi_{Q_1}, \Phi_{Q_2}]}{2S} = 1. \quad (88)
\]
On the other hand, \( \text{rank}[\Phi_1, \Phi_2] \leq 2S \) for all \( \Phi_1, \Phi_2 \in \mathcal{D} \) because the number of columns of \( [\Phi_1, \Phi_2] \) is \( 2S \). It follows that our choice \( \Phi_1 = \Phi_{Q_1} \) and \( \Phi_2 = \Phi_{Q_2} \) achieves the maximum rank and the result follows.

By combining Lemmas 4 and 6 it follows that with \( 2S = 2(1 + \nu)\Omega T/\pi \) measurements we can recover any vector \( x \) in (69) with vanishing error as \( T \to \infty \), and since the vector \( \zeta \) tends zero we can also recover any vector \( x \) in (67). It follows that we can recover the coefficients representing any signal in \( B_{Q'} \) with vanishing error using a measurement rate
\[
\bar{M} = \frac{2\Omega'}{\pi} + \frac{2\Omega}{\pi} > \frac{2\Omega'}{\pi}, \quad (89)
\]
and the proof of Theorem 1 is complete.

On the other hand, by combining Lemmas 5 and 6 it follows that with less than \( 2S = 2(1 + \nu)\Omega T/\pi \) measurements we cannot recover all possible vectors \( x \) in (69) with vanishing error as \( T \to \infty \). This also means that we cannot recover all possible vectors \( x \) in (67). It follows that with a number of measurements \( M = 2\Omega T/\pi + o(T) \), and hence a measurement rate
\[
\bar{M} = \frac{2\Omega'}{\pi} \quad (90)
\]
we cannot recover all signals in \( B_{Q'} \), and the proof of Theorem 2 is also complete.

VI. PROOFS OF THEOREMS 3, 4

In the following, we use \( \| \cdot \| \) to denote the Euclidean norm for vectors in \( \mathbb{R}^N \)
\[
\| x \| = \sqrt{\sum_{n=1}^{N} x_n^2}, \quad (91)
\]
and the spectral norm for matrices
\[
\| A \| = \sup_{\| x \| = 0} \frac{\| Ax \|}{\| x \|}. \quad (92)
\]
We also use the usual notation for signals defined in (48).

A. The key lemmas

Let \( \mathcal{B}_\Delta \) be the collection of all elements in \( B_{Q'} \) such that the extremal points of all sub-bands belong to the discrete set \( \mathcal{J} \) defined in (23). For any signal \( f \in B_{Q'} \), let \( f_\Delta \in \mathcal{B}_\Delta \) such that
\[
\bar{r}_\Delta = \arg \min_{f' \in \mathcal{B}_\Delta} \| f - f' \|. \quad (93)
\]
Since all \( f \in B_{Q'} \) are square-integrable, it follows that
\[
\lim_{T \to 0} \| f - f_\Delta \| = 0. \quad (94)
\]
Hence, if \( f_\Delta \) can be recovered using a measurement rate \( \bar{M}_\Delta \), then \( f \) can be recovered using a measurement rate
\[
\bar{M} = \lim_{\Delta \to 0} \bar{M}_\Delta. \quad (95)
\]
Consider now the set \( \mathcal{X}(\Delta) \) of vectors of \( N = (1 + \nu)\Omega T/\pi \) real coefficients, such that every element of \( \mathcal{B}_\Delta \) is approximated, with vanishing error as \( T \to \infty \), by an element of \( \mathcal{X}(\Delta) \). We also consider \( \mathcal{X}(\Delta) \subset \mathcal{X}(\Delta) \) containing all elements of \( \mathcal{X}(\Delta) \) that have norm at most one. To prove Theorems 3, 4 it is enough to prove following two lemmas.

Lemma 7. We can robustly recover all signals \( f \in \mathcal{B}_\Delta \), using a measurements rate
\[
\bar{M}_\Delta > R_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)]. \quad (96)
\]
Lemma 8. In the absence of measurement error, we cannot perfectly recover all signals \( f \in \mathcal{B}_\Delta \) using a measurement rate

\[
\bar{M}_\Delta < 2R_F[\mathcal{X}(\Delta)].
\]  

To see that Theorems 3 follow from these two lemmas, first note that the lemmas imply

\[ R_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)] \geq 2R_F[\mathcal{X}(\Delta)], \]

on the other hand, we have

\[ \dim_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)] \leq 2\dim_F[\mathcal{X}(\Delta)], \]

which implies

\[ R_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)] \leq 2R_F[\mathcal{X}(\Delta)]. \]

Combining (98) and (100) it follows that

\[ R_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)] = 2R_F[\mathcal{X}(\Delta)]. \]

Theorem 4 now follows from Lemma 7 and (101) by taking the limit for \( \Delta \to 0 \) with a measurement rate

\[ \bar{M} < \lim_{\Delta \to 0} 2R_F[\mathcal{X}(\Delta)] \]

we cannot perfectly recover all signals \( f \in \mathcal{B}_\Delta \). As for the equality, combining this result with Theorems 1 and 2 we conclude that

\[ \lim_{\Delta \to 0} 2R_F[\mathcal{X}(\Delta)] = \frac{2\gamma^\circ}{\pi}, \]

which completes the proof of Theorem 4. Theorem 5 follows directly from Theorem 4.

B. Proof of Lemma 7

Definition 9. (Inverse Lipschitz condition.) A matrix \( A \) satisfies the inverse Lipschitz condition on a set \( U \) if there exists a constant \( \beta > 0 \) such that for all \( u_1, u_2 \in U \), we have

\[ \beta \|u_1 - u_2\| \leq \|Au_1 - Au_2\|. \]

We claim that if \( A \) satisfies the inverse Lipschitz condition on \( \mathcal{X}(\Delta) \), then every \( x \in \mathcal{X}(\Delta) \) can be robustly recovered from \( y = Ax + \epsilon \). To prove this claim, consider the following two cases: (a) \( y \in A\mathcal{X}(\Delta) \), where \( A\mathcal{X}(\Delta) \) is the set \( \{Ax : x \in \mathcal{X}(\Delta)\} \), and (b) \( y \notin A\mathcal{X}(\Delta) \). In the first case, let \( x' \) be a solution of \( y = Ax' \) and let \( x'' \) be the vector used to recover \( x \). Then, the recovery error is bounded as

\[ \beta \|x - x'\| \leq \|Ax - Ax'\| = \|\epsilon\|, \]

which guarantees robust recovery. On the other hand, if \( y \notin A\mathcal{X}(\Delta) \), let \( x'' \in A\mathcal{X}(\Delta) \) such that \( Ax'' \) is the closest to \( y \) among all the elements of \( A\mathcal{X}(\Delta) \). By letting \( x'' \) be the vector used to recover \( x \), we can bound the recovery error as

\[ \beta \|x - x''\| \leq \|Ax - Ax''\| \leq \|Ax - y\| + \|y - Ax''\| \leq 2\|\epsilon\|, \]

which guarantees robust recovery. The claim now follows and we can proceed to derive a sufficient condition that ensures \( A \) satisfies the inverse Lipschitz condition on the set \( \mathcal{X}(\Delta) \).

By letting \( \mathcal{Z}(\Delta) = \mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta) \), the inverse Lipschitz condition is equivalent to stating that for all \( z \in \mathcal{Z}(\Delta) \)

\[ \beta \|z\| \leq \|Az\|. \]

Consider the normalized set \( \mathcal{Z}'(\Delta) \subset \mathcal{Z}(\Delta) \) containing all the elements of \( \mathcal{Z}(\Delta) \) that are vectors of unit norm, and let \( k' = \dim_F[\mathcal{Z}'(\Delta)] \). If (109) holds for all \( z \in \mathcal{Z}(\Delta) \), then it also holds for all \( z \in \mathcal{Z}'(\Delta) \), and vice versa. In the following, we consider \( \mathcal{Z}'(\Delta) \) instead of \( \mathcal{Z}(\Delta) \).

Let \( L_z[\mathcal{Z}'(\Delta)] \) be a minimal \( \epsilon \)-covering set of \( \mathcal{Z}'(\Delta) \), namely a minimum cardinality set such that any point in \( \mathcal{Z}'(\Delta) \) is within distance \( \epsilon \) from at least one point of \( L_z[\mathcal{Z}'(\Delta)] \). Let \( L_z[\mathcal{Z}'(\Delta)] = \{ \xi : \xi \in \mathcal{Z}'(\Delta), \|\xi\| \leq \epsilon \} \).

We need the following preliminary results.

Lemma 9. [26 Fact 2.1.] \( \dim_F[\mathcal{Z}'(\Delta)] = \inf \{ d : \forall \epsilon \in (0, 1) \exists \gamma > 0 : \}

\[ L_z[\mathcal{Z}'(\Delta)] \leq \gamma \left( \frac{1}{\epsilon} \right)^d \} \]

Let \( \mathcal{G} \) be the space of all orthogonal projections in \( \mathbb{R}^N \) of rank \( m \), and \( \mu \) be the invariant measure on \( \mathcal{G} \) with respect to orthogonal transformations.

Definition 10. (Shadow of a set.) The shadow of a set \( B \) in \( \mathbb{R}^N \) is

\[ S(B) = \{ P \in \mathcal{G} : 0 \in PB \}. \]

Lemma 10. [26 Theorem 5.1.] The measure of the shadow of a \( \rho \)-ball \( B \) centered at a distance \( \gamma \) from the origin is bounded as

\[ \mu(S(B)) \leq \delta \left( \frac{\rho}{\gamma} \right)^m, \]

where \( \delta \) is a positive constant.

We now provide a key lemma.

Lemma 11. For almost every projection \( P \) of rank \( m > k' \), there exists a constant \( \epsilon > 0 \) such that, for all \( z \in \mathcal{Z}'(\Delta) \)

\[ \|Pz\| > \epsilon \|z\|. \]

Proof: From Lemma 9 it follows that for any \( 0 < \epsilon < 1 \) there exists a constant \( \gamma > 0 \) such that

\[ L_z[\mathcal{Z}'(\Delta)] \leq \gamma \left( \frac{1}{\epsilon} \right)^k \].

By definition of \( \epsilon \)-covering, for any \( z \in \mathcal{Z}'(\Delta) \), there exists a vector \( \xi \in L_z[\mathcal{Z}'(\Delta)] \) such that

\[ \|z - \xi\| \leq \epsilon. \]

Letting \( v = z - \xi \), we have

\[ \|Pz\| = \|P(1 + v)\| \geq \|Pv\| \geq \|P1\| - \epsilon, \]

which completes the proof.
where the last inequality follows from
\[ ||Pv|| \leq ||P|| ||v|| \]
\[ = ||v|| \leq \epsilon. \]  
(117)

From [116] we have that if for all \( I \in L_c[\mathcal{Z}^\prime(\Delta)] \) we have \( ||P_I|| > 2\epsilon \), then we also have \( ||Pz|| > \epsilon = \epsilon ||z|| \), and letting \( c = \epsilon \) the result follows. What remains to be shown then, is that for almost every projection \( P \) of rank \( m \), and for all \( I \in L_c[\mathcal{Z}^\prime(\Delta)] \), we have \( ||P_I|| > 2\epsilon \).

We let
\[ L_c[\mathcal{Z}^\prime(\Delta)] = \{I_1, \ldots, I_L\}, \]
where \( L = L_c[\mathcal{Z}^\prime(\Delta)] \), and for all \( 1 \leq i \leq L \) we define
\[ \mathcal{H}_i = \{P \in \mathcal{G} : ||P_I|| \leq 2\epsilon\}. \]
We also let
\[ \mathcal{H} = \bigcup_{i=1}^L \mathcal{H}_i, \]
so that
\[ \mu(\mathcal{H}) = \mu \left( \bigcup_{i=1}^L \mathcal{H}_i \right) \leq \sum_{i=1}^L \mu(\mathcal{H}_i). \]
(121)

We claim that if \( ||P_I|| \leq 2\epsilon \), then \( 0 \in P \mathbb{B}^1_{2\epsilon} \), where \( \mathbb{B}^1_{2\epsilon} \) is a \( 2\epsilon \)-ball whose center is \( I \). This can be shown as follows: let \( b = I - P \), then \( b \in \mathbb{B}^1_{2\epsilon} \) and \( Pb = P(I - P) = 0 \). It follows that
\[ \mu(\mathcal{H}_i) \leq \mu \left( \{P \in \mathcal{G} : 0 \in P \mathbb{B}^1_{2\epsilon}\} \right) = \mu(S(\mathbb{B}^1_{2\epsilon})) \leq \delta (2\epsilon)^m, \]
(122)
where the last inequality follows from Lemma 10. We now have
\[ \mu(\mathcal{H}) \leq \sum_{i=1}^L \mu(\mathcal{H}_i) \leq L \delta (2\epsilon)^m \]
\[ \leq \gamma \delta^2 m e^{m-k'}, \]
(123)
where the last inequality follows from [114]. By taking a sufficiently small \( \epsilon \), we can now make \( \mu(\mathcal{H}) \) arbitrary close to 0, and the proof is complete.

By Lemma 11 there exists a projection \( P \) of rank \( m \) such that for all \( z \in \mathcal{Z}(\Delta) \) we have \( ||Pz|| > c ||z|| \). By applying Gaussian elimination to such a projection and selecting the non-zero rows of it, we obtain an \( m \times N \) matrix \( A \). Since \( ||Pz|| = ||Az|| \), it follows that any \( x \in \mathcal{X}(\Delta) \) can be robustly recovered from \( y = Ax + e \) with a number of measurements larger than \( k' \).

What remains to be done is to show that \( k' = \dim_F[\mathcal{Z}(\Delta)] \leq \dim_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)] \). Let \( \mathcal{Z}(\Delta) \subseteq \mathcal{Z}(\Delta) \) containing all elements of \( \mathcal{Z}(\Delta) \) that have norm at most one. Since \( \mathcal{Z}(\Delta) \subseteq \mathcal{Z}(\Delta) \), we have \( k' \leq \dim_F[\mathcal{Z}(\Delta)] \). It is then enough to show that \( \dim_F[\mathcal{Z}(\Delta)] = \dim_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)] \).

**Lemma 12.** We have
\[ \dim_F[\mathcal{Z}(\Delta)] = \dim_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)] \]  
(124)

**Proof:** Let \( z \in \mathcal{Z}(\Delta) \) be a vector of coefficients of a multi-band function \( f_z \) whose spectral support is bounded by \( 4\Omega' \) and whose energy is bounded by one. It follows that \( f_z \) can be represented as
\[ f_z = f_{x_1} + f_{x_2} \]
(125)
where \( f_{x_i}, i \in \{1,2\} \) is a multi-band signal whose spectral support is bounded by \( 2\Omega' \) and whose energy is bounded by one. Let \( x_i \) be a vector of coefficients for \( f_{x_i}, i \in \{1,2\} \). Then, we have
\[ z = x_1 + x_2 \]
(126)
where \( x_i \in \mathcal{X}(\Delta) \). Since \( \mathcal{Z}(\Delta) \subseteq \mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta) \), we conclude that
\[ \dim_F[\mathcal{Z}(\Delta)] \leq \dim_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)]. \]
(127)
Conversely, let us consider \( x_1, x_2 \in \mathcal{X}(\Delta) \). Then, we have
\[ \frac{x_1 + x_2}{2} \in \mathcal{Z}(\Delta), \]
(128)
which implies \( \mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta) \subseteq 2\mathcal{Z}(\Delta) \), where \( 2\mathcal{Z}(\Delta) \) indicates the set \( \{2z : z \in \mathcal{Z}(\Delta)\} \). Therefore, we conclude that
\[ \dim_F[\mathcal{Z}(\Delta)] \geq \dim_F[\mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta)]. \]
(129)
By combining (127) and (129), we obtain the desired result.

**C. Proof of Lemma 8**

If all vectors \( x \in \mathcal{X}(\Delta) \) can be recovered from \( y = Ax \), then all vectors \( x \in \mathcal{X}(\Delta) \) can also be recovered from \( y = Ax \), and vice versa. In the following, we consider \( \mathcal{X}(\Delta) \) rather than \( \mathcal{X}(\Delta) \).

In order to prove Lemma 8 it is enough to show that a number of measurements
\[ m < 2 \dim_F[\mathcal{X}(\Delta)] \]
(130)
is not sufficient to recover all the elements of \( \mathcal{X}(\Delta) \) as \( T \to \infty \).

Let us define the set \( \mathcal{W}(\Delta) = \mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta) \). If all \( x \in \mathcal{X}(\Delta) \) can be recovered from \( y \), then \( A \) is a one-to-one map on \( \mathcal{X}(\Delta) \), and vice versa. Also, if \( A \) is a one-to-one map on \( \mathcal{X}(\Delta) \), then
\[ \ker(A) \cap \mathcal{W}(\Delta) = \{0\}, \]
(131)
and vice versa, where \( \ker(A) \) indicates the kernel of \( A \). We then need to show that (130) violates (131). For convenience of notation, we define \( k = 2 \dim_F[\mathcal{X}(\Delta)] \).

Let us assume that \( \mathcal{W}(\Delta) \) contains a \( k \)-dimensional Euclidean ball. Note that (130) implies \( \dim_F[\mathcal{X}(\Delta)] < k \). Since \( \dim_F[\mathcal{X}(\Delta)] + \dim_F[\mathcal{W}(\Delta)] = N \), it follows that \( \dim_F[\mathcal{W}(\Delta)] = N - k \). This means that the dimension of \( \ker(A) \) is larger than \( N - k \), which violates (131) because \( \mathcal{W}(\Delta) \) contains a \( k \)-dimensional Euclidean ball.

It follows that in order to prove Lemma 8 it is enough to show that \( \mathcal{W}(\Delta) \) contains a \( k \)-dimensional Euclidean ball. We will show that this is the case when \( T \to \infty \).
We need some additional definitions, followed by a preliminary result.

**Definition 11.** (Diameter). For any $S \subset \mathbb{R}^N$, we let
\[
\text{diam}(S) = \sup_{x, y \in S} \|x - y\|. \quad (132)
\]

**Definition 12.** (Hausdorff measure). Let $U \subset \mathbb{R}^N$ and $\{S_i\}$ be a cover of $U$ formed by balls of radius $r < \mu$. We let
\[
\zeta^s_\mu(U) = \inf_{\{S_i\}} \sum_i (\text{diam}(S_i))^s. \quad (133)
\]

The $s$-dimensional Hausdorff measure of $U$ is given by the limit
\[
\zeta^s(U) = \lim_{\mu \to 0} \zeta^s_\mu(U). \quad (134)
\]

**Definition 13.** (Hausdorff dimension). For any $U \subset \mathbb{R}^N$, the Hausdorff dimension of $U$ is
\[
\dim_H(U) = \sup\{s \geq 0 : \zeta^s(U) = \infty\}. \quad (135)
\]

The Hausdorff dimension has the following two important properties, see [13].

Property 1. (Unit ball). For any integer $d$ such that $0 \leq d \leq N$, the Hausdorff dimension of the unit ball $B^d(0, 1) \subset \mathbb{R}^d \subset \mathbb{R}^N$ is $d$.

Property 2. (Countable stability). Let $U_i \subset \mathbb{R}^N$. Then,
\[
\dim_H(\bigcup_{i=1}^\infty U_i) = \sup\{\dim_H(U_i)\} \quad (136)
\]

From these definitions it follows that
\[
\dim_H(U) \leq \dim_F(U). \quad (137)
\]

However, by Lemma [13] below, if a set satisfies a quasi self-similar property, then the Hausdorff dimension is equal to the fractal dimension.

**Definition 14.** (Quasi self-similarity) Let $U \subset \mathbb{R}^N$. If for all $x, y \in U \cap B$, there exist $a, r_0 > 0$ such that for any ball $B$ with radius $r < r_0$, there is a mapping $\phi : U \cap B \to U$ satisfying
\[
a \cdot \|x - y\| \leq r \cdot \|\phi(x) - \phi(y)\|, \quad (138)
\]

then, we say that $U$ is quasi self-similar.

**Lemma 13.** [25] Theorem 3] Let $U$ be a nonempty compact subset of $\mathbb{R}^N$ that is quasi self-similar. Then,
\[
\dim_H(U) = \dim_F(U). \quad (139)
\]

We are now ready to show our final step.

**Lemma 14.** For sufficiently large $T$, the set $\mathcal{W}(\Delta) = \mathcal{X}(\Delta) \oplus \mathcal{X}(\Delta')$ contains a $k$-dimensional Euclidean ball.

**Proof:** We have
\[
\mathcal{X}(\Delta) = \bigcup_i \mathcal{X}_i \quad (140)
\]

where $\mathcal{X}_i$ is the set of coefficient vectors of all multi-band signals of a fixed sub-band allocation of measure at most $2\delta'$ and norm at most one. Since $\mathcal{X}(\Delta)$ is a countable union, by Property 2 of the Hausdorff dimension we have
\[
\dim_H[\mathcal{X}(\Delta)] = \sup_i \{\dim_H(\mathcal{X}_i)\}. \quad (141)
\]

Since the Hausdorff dimension of $\mathcal{X}_i$ does not depend on $i$, we also have that for all $i$
\[
\dim_H[\mathcal{X}(\Delta)] = \dim_H(\mathcal{X}_i). \quad (142)
\]

Next, we consider two sets of coefficient vectors $\mathcal{X}_1$ and $\mathcal{X}_2$, whose sub-bands do not have any intersection. We have
\[
\mathcal{X}_i = \{x : x = \Phi_i\alpha \text{ where } \|x\| \leq 1 \text{ and } \alpha \in \mathbb{R}^s\}, \quad (143)
\]

for $i = 1, 2$. By the same argument used in the proof of Lemma 6, it follows that for $T$ large enough the columns of $\Phi_1$ and $\Phi_2$ are independent. Also, note that $\mathcal{X}_i$ is an Euclidean ball and by Property one of the Hausdorff dimension it follows that $\mathcal{X}_i$ is a $\dim_H(\mathcal{X}_i)$-dimensional Euclidean ball. Now, by definition, $\mathcal{W}(\Delta)$ includes $\mathcal{X}_1 \oplus \mathcal{X}_2$, and since $\mathcal{X}_1$ and $\mathcal{X}_2$ are $\dim_H(\mathcal{X}_i)$-dimensional Euclidean balls and the columns of $\Phi_1$ and $\Phi_2$ are independent, it follows that $\mathcal{W}(\Delta)$ contains a $2\dim_H(\mathcal{X}_i)$-dimensional Euclidean ball. Using (141) and (142), it follows that for $T$ large enough $\mathcal{W}(\Delta)$ contains a $2\dim_F[\mathcal{X}(\Delta)]$ Euclidean ball, or equivalently, a $k$-dimensional Euclidean ball.

VII. CONCLUSION

We have investigated the phase-transition threshold of the minimum measurement rate sufficient for completely blind reconstruction of any multi-band signal of given spectral support measure. This threshold has been shown to coincide with twice the fractal dimension per unit ambient dimension of the space spanned by the optimal approximation for bandlimited signals. This result provides an operational characterization of the fractal dimension, and parallels an analogous coding theorem for the compression of discrete-time, analog, i.i.d. sources, where the critical threshold is shown to be equal to the information dimension per unit ambient dimension of the source [10, 11]. Advantages of the deterministic approach include being oblivious to a priori assumptions on the source distribution, and providing recovery guarantees for all signals, rather than for a large fraction of them. In both cases, fundamental limits apply to the asymptotic regime of large signal dimension. In the stochastic case, probabilistic concentration is achieved exploiting the ergodicity of the process, while in the deterministic case vanishing error energy is achieved exploiting spectral concentration. Despite both results can be viewed at the high level as an instance of dimensional reduction due to regularity constraints, the tools required in the deterministic setting are quite different from those used in traditional information theory, and include machinery from approximation theory, and geometry of functional spaces. The systematic study of these techniques is clearly desirable, and this recommendation dates back to Kolmogorov [19]. Exploiting some of our recent results [28], we have shown that the price to pay to obtain deterministic guarantees of
reconstruction for all signals is only a factor of two in the measurement rate, compared to probabilistic reconstruction. It is also the case that the absence of additional spectral information such as the one assumed in [11], [12], [15], does not lead to any penalty in the measurement rate.

Practical achievability schemes for blind reconstruction of continuous signals that come close to the information-theoretic optimum remain an open problem, while much progress has been made for both discrete-time and continuous-time settings, under various assumptions on what information about the signal is available a priori [11], [12], [15], [16], [17], [29]. Another interesting open question is the determination of the critical threshold for linear approximation schemes. In this case, without any knowledge of the spectral support it is not possible to set-up the eigenvalue equation leading to the optimal subspace approximation [6], and the challenge is to infer the basis functions directly from the measurements.

Investigation of sampling schemes for blind reconstruction is also of interest, due to their relevance for practical applications. Our results provide an information-theoretic baseline for performance assessment in all of these cases. Finally, extensions to signals of multiple variables would be of interest in various settings, for example in the context of remote sensing. In this case, a desirable outcome would be the computation of the fractal dimension of signals radiated from a bounded domain, generalizing the notion of number of degrees of freedom for bandlimited signals studied in [30], to signals that are sparse in both the frequency and the wavenumber spectrum.

APPENDIX

A. Proofs of (20) and (22)

First let us consider (20). Since \( \mathcal{X}_B \subseteq \mathcal{X}_B \oplus \mathcal{X}_B \), we have

\[
\dim_F(\mathcal{X}_B) \leq \dim_F(\mathcal{X}_B \oplus \mathcal{X}_B). \tag{144}
\]

For any \( x \in \mathcal{X}_B \oplus \mathcal{X}_B \), we have \( x/2 \in \mathcal{X}_B \), or equivalently \( x \in 2\mathcal{X}_B \), where \( 2\mathcal{X}_B \) indicates the set \( \{2x : x \in \mathcal{X}_B\} \). This implies \( \mathcal{X}_B \oplus \mathcal{X}_B \subseteq 2\mathcal{X}_B \), and we have

\[
\dim_F(\mathcal{X}_B) \geq \dim_F(\mathcal{X}_B \oplus \mathcal{X}_B). \tag{145}
\]

Combining (144) and (145), we obtain (20).

Next, we consider (22). We let \( X'_B \) be a set of vectors such that for any \( x = (x_1, \ldots, x_N) \in \mathcal{X}_B \), \( x' \in X' \) is the vector of its first \( N' \) components, namely \( x' = (x_1, \ldots, x_{N'}) \) where \( N' = \Omega T/\pi + o(T) \). From inequality (137) in Theorem 6 of [28], we have

\[
H_\epsilon(\mathcal{X}_B) \geq H_\epsilon(X'_B). \tag{146}
\]

By inequality (99) of Theorem 3 in [28], we have

\[
H_\epsilon(X'_B) \geq N' \left[ \log \left( \frac{\zeta(N')}{\epsilon} \right) \right], \tag{147}
\]

where \( \zeta(N') \) is independent of \( \epsilon \). Combining (146) and (147), we obtain

\[
H_\epsilon(X_B) \geq N' \left[ \log \left( \frac{\zeta(N')}{\epsilon} \right) \right]. \tag{148}
\]

Similarly, by inequality (138) of Theorem 6 in [28], we have

\[
H_\epsilon(X_B) \leq H_{\epsilon-\mu}(X'_B). \tag{149}
\]

and using inequality (100) of Theorem 3 in [28], we have

\[
H_{\epsilon-\mu}(X'_B) \leq N' \left[ \log \left( \frac{1}{\epsilon - \mu} \right) \right] + \eta(N'), \tag{150}
\]

where \( 0 < \mu < \epsilon \), and \( \eta(N') \) is independent of \( \epsilon \). Combining (149) and (150), we obtain

\[
H_\epsilon(X_B) \leq N' \left[ \log \left( \frac{1}{\epsilon - \mu} \right) \right] + \eta(N'). \tag{151}
\]

Since \( \mu \) can be arbitrarily small and the logarithm is a continuous function, it follows that

\[
H_\epsilon(X_B) \leq N' \log (1/\epsilon) + \eta(N'). \tag{152}
\]

Putting together (148) and (152), we finally obtain

\[
\begin{cases}
H_\epsilon(X_B) \geq N' \log \left( \zeta(N')/\epsilon \right),
H_\epsilon(X_B) \leq N' \log (1/\epsilon) + \eta(N').
\end{cases} \tag{153}
\]

Dividing both sides of (153) by \( -\log \epsilon \) and taking the limit for \( \epsilon \to 0 \), we have

\[
\dim_F(X_B) = N', \tag{154}
\]

so that

\[
\lim_{T \to \infty} \frac{\dim_F(X_B)}{T} = \Omega/\pi. \tag{155}
\]

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