ON THE MODULARITY OF SOLUTIONS TO CERTAIN DIFFERENTIAL EQUATIONS OF HYPERGEOMETRIC TYPE

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Abstract

We answer some questions in a paper by Kaneko and Koike ['On modular forms arising from a differential equation of hypergeometric type', *Ramanujan J.* 7(1–3) (2003), 145–164] about the modularity of the solutions of a certain differential equation. In particular, we provide a number-theoretic explanation of why the modularity of the solutions occurs in some cases and does not occur in others. This also proves their conjecture on the completeness of the list of modular solutions after adding some missing cases.

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1. Introduction

This paper deals with some questions and conjectures raised in the paper [2] regarding the solutions of the equation

\[ f'' - \frac{2\pi i (k + 1)}{6} E_2(\tau) f' + \frac{2\pi i k (k + 1)}{12} E'_2(\tau) f = 0, \]

where \( k \) is a rational number and \( E_2 \) is the weight 2 (quasi-modular) Eisenstein series. This differential equation appeared first in [3] for integers \( k \equiv 0, 4 \mod 6 \) in connection with the lifting of supersingular \( j \)-invariants of elliptic curves. In [2], modular solutions of the above differential equation are described explicitly when \( k \) is an integer or half an integer. In addition, the authors conjectured, based on numerical experiments, that their solutions exhaust all the modular solutions (at least when \( k \) is an integer or half an integer). Later Kaneko, in [1], extended this list by finding modular solutions when \( k = 6n/5 - 1 \), with \( n \) being an integer not divisible by 5.

The normal form of the above differential equation can be shown to be

\[ y'' + \pi^2 \left( \frac{k + 1}{6} \right)^2 E_4 y = 0, \]

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where $E_4$ is the weight 4 Eisenstein series. We studied this Fuchsian differential equation in a more general setting in [8] by seeking the solutions to the equation $y'' + \pi^2 r^2 E_4 y = 0$, where $r$ is a rational number. Our investigation relies on solving the Schwarzian equation $(h, \tau) = 2r^2 \pi^2 E_4$ where $(h, \tau)$ is the Schwarz derivative of $h$. We showed in [8] that the Schwarzian equation admits solutions that are modular functions if and only if $r = n/m$ where $2 \leq m \leq 5$ and $\gcd(m, n) = 1$. In these cases, the invariance group of the solution is $\Gamma(m)$. In [6], we explicitly solved the same Schwarzian equation for $r = k/6$ where $k \equiv 1 \mod 12$. Each solution is obtained from a system of algebraic equations. Finally, in [7], the same equation is solved for every positive integer $r$ by solving a certain eigenvector problem. Using these results, we shall provide in this paper, among other results, a number-theoretic explanation for the modularity or nonmodularity of the solutions. We also show that the cases for which all the solutions are modular, conjectured in [2], become exhaustive provided we include the solutions found by Kaneko in [1].

2. Automorphic Schwarzian equations

The Schwarz derivative of a meromorphic function $h$ on the upper half-plane $\mathbb{H} = \{ \tau \in \mathbb{C} \mid \Im \tau > 0 \}$ is defined by

$$(h, \tau) = \left( \frac{h''}{h'} \right)' - \frac{1}{2} \left( \frac{h''}{h'} \right)^2.$$ 

It is projectively invariant and $(h, \tau) = (g, \tau)$ if and only if $h$ is a linear fraction of $g$. Moreover, $(h, \tau) = 0$ if and only if $h$ is a linear fraction of $\tau$. There is a close connection between the Schwarz derivative and second-order ordinary differential equations. Indeed, if $R(\tau)$ is a meromorphic function on $\mathbb{H}$ and $y_1$ and $y_2$ are two linearly independent local solutions to $y'' + R(\tau)y = 0$, then locally $h = y_2/y_1$ is a solution to the Schwarzian differential equation $(h, \tau) = 2R(\tau)$. This connection yields most analytic properties of the Schwarz derivative.

If $w$ is a function of $\tau$, then we have the cocycle condition

$$(h, \tau)d\tau^2 = (h, w)dw^2 + (w, \tau)d\tau^2.$$ 

As a consequence, if $h$ is an automorphic function for a discrete subgroup $\Gamma$ of $\text{SL}_2(\mathbb{R})$, then $(h, \tau)$ is a weight 4 automorphic form for $\Gamma$ that has a double pole where $h'$ vanishes or where $h$ has at least a double pole, and it is holomorphic everywhere else including at the cusps of $\Gamma$ [4]. Conversely, if $(h, \tau)$ is a (meromorphic) automorphic form of weight 4 for $\Gamma$, then according to [8], there exists a two-dimensional complex representation $\rho$ of $\hat{\Gamma}$, and the image of $\Gamma$ is $\text{PSL}_2(\mathbb{R})$, such that for all $\tau \in \mathbb{H}$ and $\gamma \in \Gamma$,

$$h(\gamma \cdot \tau) = \rho(\gamma) \cdot h(\tau),$$

where the action on both sides is by linear fractional transformations. We call such an $h$ a $\rho$-equivariant function. The representation $\rho$ can be lifted naturally to $\Gamma$. If $\rho = 1$ is constant, then $h$ is an automorphic function, while if $\rho = \text{Id}$, then $h$ commutes with the action of $\Gamma$ and we simply say that $h$ is an equivariant function for $\Gamma$. For an arbitrary
Fuchsian group $\Gamma$ and for an arbitrary representation $\rho$ of $\Gamma$, $\rho$-equivariant functions always exist [5].

The following result completes the connection between Schwarzian equations and ordinary differential equations.

**Theorem 2.1** [8, Theorem 3.3]. Suppose that $f$ is a weight 4 automorphic form for $\Gamma$ that is holomorphic on $\mathbb{H}$.

1. If $y_1$ and $y_2$ are two linearly independent holomorphic solutions to $y'' + \frac{1}{2}fy = 0$ then $F = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$ is a weight $-1$ vector-valued automorphic form for some multiplier system $\rho$. Furthermore, $h = y_1 / y_2$ is a $\rho$-equivariant function with $\{h, \tau\} = f$.

2. If $h$ is a solution to $\{h, \tau\} = f$, then $h'$ does not vanish on $\mathbb{H}$ and $y_1 = h/\sqrt{h'}$ and $y_2 = 1/\sqrt{h'}$ are two linearly independent holomorphic solutions to $y'' + \frac{1}{2}fy = 0$.

We also introduce the main theorem from [8] which we will rely on to assert the modularity or nonmodularity of the solutions to the modular differential equations that we study.

**Theorem 2.2** [8, Theorem 8.3]. The Schwarzian equation $\{h, \tau\} = 2\pi^2 r^2 E_4$ admits a modular function as a solution if and only if $r = n/m$ with co-prime $m$ and $n$ and $2 \leq m \leq 5$, in which case the invariance group is $\Gamma(m)$.

### 3. Modular solutions

In this section we will give some preparatory results about the modularity of the solutions of the differential equations under consideration.

Let $h$ be a $\rho$-equivariant function on $\mathbb{H}$, that is,

$$h(\gamma \tau) = \rho(\gamma) h(\tau), \quad \tau \in \mathbb{H}, \gamma \in \text{SL}_2(\mathbb{Z}),$$

and let $\Gamma$ be a subgroup of $\text{SL}_2(\mathbb{Z})$ of finite index.

**Lemma 3.1.** Let $h$ be a nonconstant $\rho$-equivariant function where $\rho$ is a two-dimensional representation of the modular group. If there exists $k \in \mathbb{Z}$ such that $h$ is a weight $k$ modular form on a modular subgroup $\Gamma$ of finite index, then $k = 0$ and $\Gamma \subseteq \text{ker}(\rho)$.

**Proof.** Suppose that $h$ is a weight $k$ modular form for a modular subgroup $\Gamma$ of finite index. For $\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma$,

$$(c \gamma \tau + d \gamma)^k h(\tau) = h(\gamma \tau) = \rho(\gamma) h(\tau).$$

In other words,

$$(c \gamma \tau + d \gamma)^k = \frac{1}{h(\tau)} \rho(\gamma) h(\tau). \quad (3.1)$$

Since $\Gamma$ is a finite-index modular subgroup, there exists $\gamma \in \Gamma$ such that $c \gamma \neq 0$. Also, let $n_\Gamma$ be the cusp width at $\infty$ for $\Gamma$, that is, the least positive integer such
that \( \left( \begin{array}{cc} 1 & n_1 \\ 0 & 1 \end{array} \right) \in \Gamma \). Then \( h(\tau + n_1) = h(\tau) \) for all \( \tau \in \mathbb{H} \). Therefore, the right-hand side of (3.1) is \( n_1 \)-periodic while the left-hand side is not unless \( k = 0 \). Since \( h \) is a nonconstant meromorphic function on \( \mathbb{H} \), it assumes infinitely many values, and clearly \( \Gamma \subseteq \ker(\rho) \). \( \square \)

We now turn our attention to the main differential equation of hypergeometric type that was the object of study in [2, 3]:

\[
f'' - \frac{2\pi i(k + 1)}{6} E_2(\tau) f' + \frac{2\pi ik(k + 1)}{12} E_2'(\tau) f = 0. \tag{3.2}
\]

Here \( k \) is a rational number and \( E_2 \) and \( E_4 \) are the Eisenstein series given by their \( q \)-series where \( q = \exp(2\pi i\tau) \):

\[
E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n, \quad E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,
\]

with \( \sigma_m(n) = \sum_{d \geq 0, d \mid n} d^m \), for \( m \in \mathbb{N} \). Through the change of function \( f = \eta^{2(k + 1)} y \), where \( \eta \) is the Dedekind eta-function defined by

\[
\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n),
\]

and using the fact that \( E_2 = (12/\pi)\eta'/\eta \) and that \( (6/\pi)E'_2 = E_2^2 - E_4 \), we see that (3.2) is equivalent to

\[
y'' + \pi^2 \left( \frac{k + 1}{6} \right)^2 E_4(\tau) y = 0.
\]

**Theorem 3.2.** The differential equation (3.2) has two linearly independent modular solutions, for some finite-index subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \), if and only if there exist two integers \( n \) and \( m \) satisfying \( 2 \leq m \leq 5 \), \( \gcd(m, n) = 1 \), and

\[
\frac{k + 1}{6} = \frac{n}{m}.
\]

**Proof.** If the differential equation (3.2) has two linearly independent solutions, \( g_1 \) and \( g_2 \), which are modular for some finite-index subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \), then \( h = g_2/g_1 \) will be a solution to the Schwarzian equation

\[
\{h, \tau\} = 2\pi^2 \left( \frac{k + 1}{6} \right)^2 E_4(\tau).
\]

Therefore, \( h \) is \( \rho \)-equivariant for some two-dimensional representation \( \rho \) of the modular group. Since \( h \) is also a modular form for \( \Gamma \), then by Lemma 3.1, \( h \) must be a modular function for \( \Gamma \). In particular, \( g_1 \) and \( g_2 \) have the same weight. By Theorem 2.2,

\[
8\pi^2 \left( \frac{k + 1}{12} \right)^2 = 2\pi^2 \left( \frac{n}{m} \right)^2.
\]
with \( m \) and \( n_1 \) being two positive integers satisfying \( 2 \leq m \leq 5 \) and \( \gcd(m, n_1) = 1 \), which gives
\[
\frac{k + 1}{6} = \pm \frac{n_1}{m},
\]
and we take \( n = \pm n_1 \).

For the converse, by Theorem 2.2, the equation
\[
\{h, \tau\} = 2\pi^2 \left(\frac{k + 1}{6}\right)^2 E_4(\tau)
\]
has a \( \Gamma(m) \) modular function solution \( h \), and by Theorem 2.1, we see that \( y_1 = h/\sqrt{h'} \) and \( y_2 = 1/\sqrt{h'} \) are two linearly independent holomorphic solutions of
\[
y'' + \pi^2 \left(\frac{k + 1}{6}\right)^2 E_4(\tau)y = 0.
\]
It is clear that \( y_1 \) and \( y_2 \) are modular forms of weight \(-1\) and a character \( \chi \) for \( \Gamma(m) \), with \( \chi^2 = 1 \), hence they are also modular for the finite-index group \( \Gamma = \ker(\chi) \). The required solutions are then \( f_1 = \eta^{2(k+1)}y_1 \) and \( f_2 = \eta^{2(k+1)}y_2 \) which are well defined for any rational \( k \), since \( \eta \) does not vanish on \( \mathbb{H} \), and are both of weight \( k \). \hfill \Box

4. A second look at the solutions obtained by Kaneko and Koike

We now proceed to answer the question raised at the end of [2]. In particular, we provide a number-theoretic explanation for the modularity or nonmodularity of the solutions to (3.2) that appeared in Theorem 1 and we explain the quasi-modularity of the solution in [2]. Furthermore, we explain why these solutions do not exhaust all the modular solutions as was conjectured at the end of the paper and we show the completeness of the list of modular solutions after adding the solutions in [1].

When \( k \equiv 1, 2, 3 \mod 6 \), or when \( k \) is a half integer congruent to 1/2 modulo 3, then \((k + 1)/6\) is a rational number whose reduced form has a denominator between 2 and 4. Therefore, (3.2) has two linearly independent solutions, say \( f_1 \) and \( f_2 \), modular for certain congruence groups \( \Gamma_1 \) and \( \Gamma_2 \), respectively. Therefore any other solution \( f = af_1 + bf_2 \) is at least modular for \( \Gamma_1 \cap \Gamma_2 \). This explains the modularity of the solutions in [2, Theorem 1].

In case \( k \equiv 0, 4 \mod 6 \), there exists a one-dimensional modular solution space, say generated by \( g_1 \). If \( g_2 \) is another solution that is linearly independent with \( g_1 \), then \( g_2 \) cannot be modular. Otherwise, we will have
\[
\frac{k + 1}{6} = \frac{n}{m},
\]
with \( m \) and \( n \) two positive integers satisfying \( 2 \leq m \leq 5 \) and \( \gcd(m, n) = 1 \). Thus \( m(k + 1) = 6n \). As \( k + 1 \equiv \pm 1 \mod 6 \), we must have \( m \equiv 0 \mod 6 \) which is impossible since \( 2 \leq m \leq 5 \).

The case \( k \equiv 5 \mod 6 \) requires a different approach since \((k + 1)/6\) is now an integer. We use the following result from [7].
THEOREM 4.1 [7, Theorem 4.2]. Let \( r \) be a positive integer and let \( \Gamma = \text{SL}_2(\mathbb{Z}) \) if \( r \) is even and \( \Gamma = \text{SL}_2(\mathbb{Z})^2 \) if \( r \) is odd. Then there exist two linearly independent solutions \( y_1 \) and \( y_2 \) to \( y'' + \pi^2 r^2 E_4 y = 0 \) such that:

1. \( y_1 \) and \( y_2 \) have the \( q \)-expansions
   \[
   y_1(\tau) = q^{r/2} \sum_{n \geq 0} \alpha_n q^n \quad (\alpha_0 \neq 0), \quad y_2(\tau) = \tau y_1(\tau) + q^{-r/2} \sum_{n \geq 0} \beta_n q^n \quad (\beta_0 \neq 0);
   \]
2. \( y_1 \) is a quasi-modular form of weight 0 and depth 1 for \( \Gamma \) and \( y_2(\tau) - \tau y_1(\tau) \) is a modular form of weight \(-2\) for \( \Gamma \);
3. \( h_r = y_2/y_1 \) is equivariant for \( \Gamma \).

Here, \( \text{SL}_2(\mathbb{Z})^2 \) is the subgroup comprising the squares of elements of \( \text{SL}_2(\mathbb{Z}) \), and it is the unique subgroup of index 2 in \( \text{SL}_2(\mathbb{Z}) \).

It follows that if \( k \equiv 5 \mod 6 \), then we cannot have modular solutions. However, \( f = \eta^{2k+2} y_1 \) is a quasi-modular solution to (3.2) which has weight \( k + 1 \) and depth 1, in line with what is claimed in [2].

Thus, we have provided a number-theoretic justification why certain solutions to (3.2) are modular of weight \( k \), others are quasi-modular of weight \( k + 1 \), and some are neither. While these claims hold when \( k \) is an integer or a half integer congruent to \( 1/2 \) modulo 3, the list is not exhaustive as was conjectured in [2]. Indeed, what is missing and would make the list complete are the level 5 solutions found in [1]. It is clearly seen from Theorem 3.2 that one needs to include the rational numbers \( k \) such that \( (k + 1)/6 = n/5 \) for \( n \) not divisible by 5, that is, \( k = 6n/5 - 1 \). For example, for \( k = 1/5 \), the Hauptmodul for \( \Gamma(5) \) given by

\[
 t = q^{1/5} \prod_{n \geq 1} (1 - q^n)^{\frac{\phi}{5}},
\]

where \( (\cdot) \) is the Legendre symbol, is a solution to \( \{h, \tau\} = 2(1/5)^2 \pi^2 E_4 \) [8]. Thus, the modular solutions to (3.2) are \( f_1 = \eta^{2k+1} t^{r-1/2} \) and \( f_2 = \eta^{2k+1} t^{r-1/2} \) and both are of weight 1/5. In the cases \( k = 7/5 \) and 13/5, the solutions to the Schwarzian equation are given respectively by

\[
 \frac{t^2(t^5 - 7)}{7t^5 + 1} \quad \text{and} \quad \frac{t^3(t^{10} - 39t^5 - 26)}{26t^{10} - 39t^5 - 1},
\]

from which one can write down the modular solutions to (3.2). For more explicit examples, we refer the reader to the work of Kaneko in [1], where a systematic study of this missing case was made.

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