Triangle (Causal) Distributions in the Causal Approach

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Abstract

The tensor Feynman amplitudes are reduced to scalar integrals by a procedure of Passarino and Veltman. We provide an alternative approach based on the causal formalism.
1 Introduction

One way to arrive at the Bogoliubov axioms of perturbative quantum field theory (pQFT) is by analogy with non-relativistic quantum mechanics [6], [12]; a discussion on this point can also be found in [11]. We give the main ideas. Suppose that we have a time-dependent interaction potential \( V \). Then one goes to the interaction picture and the time evolution is governed by the evolution equation:

\[
\frac{d}{dt} U(t, s) = -i V_{\text{int}}(t) U(t, s); \quad U(s, s) = I. \tag{1.1}
\]

This equation can be solved in some cases by a perturbative method, namely the series

\[
U(t, s) \equiv \sum \frac{(-i)^n}{n!} \int_{\mathbb{R}^n} dt_1 \cdots dt_n T(t_1, \ldots, t_n) \tag{1.2}
\]

makes sense. The operators \( T_n(t_1, \ldots, t_n) \) are called chronological products; \( n \) is called the order of the perturbation theory. They verify a number of properties spelled in detail in the references from above. Basically they are unitarity and causality; the causality property means:

\[
T_n(t_1, \ldots, t_n) = T_m(t_1, \ldots, t_m) T_{n-m}(t_{m+1}, \ldots, t_n),
\]

for \( t_j > t_k, \quad j = 1, \ldots, m; k = m + 1, \ldots, n. \tag{1.3}\]

An explicit formula is available (see the references above).

The purpose is to generalize this idea in the relativistic context especially the causality property. Essentially we try to substitute \( t \in \mathbb{R} \) by a Minkowski variable \( x \in \mathbb{R}^4 \). The chronological operators will be some operators \( T(x_1, \ldots, x_n) \) and all the axioms from the non-relativistic case can be easily generalized rather naturally. The causally axiom is more subtle. We have to replace temporal succession \( t_1 > t_2 \) by causal succession \( x_1 \succ x_2 \) which means that \( x_1 \) should not be in the past causal shadow of \( x_2 \) i.e. \( x_2 \cap (x_1 + V^+) = \emptyset \). In formulas: if \( x_i > x_j, \quad \forall i \leq k, \quad j \geq k + 1 \) then we have:

\[
T(x_1, \ldots, x_n) = T(x_1, \ldots, x_k) T(x_{k+1}, \ldots, x_n). \tag{1.4}\]

From here it follows that the “initial condition” \( T(x) \) should satisfy

\[
[T(x), T(y)] = 0, \quad (x - y)^2 < 0 \tag{1.5}\]

where for the Minkowski product we use the convention \( 1, -1, -1, -1 \). It a difficult problem to obtain solutions of the preceding equation. The solutions for pQFT are distribution-valued operators, (Wick monomials) and act in some Fock space where we can describe scattering processes with creation and annihilation of particles. According to Epstein and Glaser, we should solve directly the axioms of pQFT in an recursive way.

So we start from Bogoliubov axioms [1], [4] as presented in [3], [2]; for every set of Wick polynomials \( A_1(x_1), \ldots, A_n(x_n) \) acting in some Fock space \( \mathcal{H} \) one associates the operator-valued
distributions $T^{A_1,\ldots,A_n}(x_1,\ldots,x_n)$ called chronological products; it will be convenient to use another notation: $T(A_1(x_1),\ldots,A_n(x_n))$ and we should require skew-symmetry in all arguments: for arbitrary $A_1(x_1),\ldots,A_n(x_n)$ we should have

$$T(\ldots,A_i(x_i),A_{i+1}(x_{i+1}),\ldots) = (-1)^{f_i}f_{i+1}T(\ldots,A_{i+1}(x_{i+1}),A_i(x_i),\ldots)$$

(1.6)

where $f_i$ is the number of Fermi fields appearing in the Wick monomial $A_i$.

There are a number of rigorous ways to construct the chronological products: (a) **Hepp axioms** [12] (one rewrites the axioms in terms of vacuum averages of chronological products); (b) **Polchinski flow equations** [14], [16] (one considers an ultra-violet cut-off for the Feynman amplitudes and establishes some differential equations in this parameter); (c) **The causal approach** due to Epstein and Glaser [4], [6]: is a recursive procedure for the basic objects $T(A_1(x_1),\ldots,A_n(x_n))$ and reduces the induction procedure to a distribution splitting of some distributions with causal support, or to the process of extension of distributions [15]. An equivalent point of view uses retarded products [19]. The causal method is the most elementary one from the point of view of conceptual clarity and also for practical computations. It is a very good approach for the study of gauge models [17], [18].

The basic recursive idea of Epstein and Glaser starts from the chronological products

$$T(A_1(x_1),\ldots,A_m(x_m)) \quad m = 1, 2, \ldots$$

up to order $n-1$ and constructs a causal commutator in order $n$. For instance for $n = 2$ the causal commutator according to:

$$D(A(x),B(y)) = A(x)B(y) - (-1)^{|A||B|}B(y)A(x)$$

(1.7)

and after the operation of causal splitting one can obtain the second order chronological products. Generalizations of this formula are available for higher orders of the perturbation theory. In particular we have in the third order

$$D(A(x),B(y);C(z)) \equiv -[T(A(x),B(y)),C(z)]$$

$$+(-1)^{|B||C|}[T(A(x),C(z)),B(y)] + (-1)^{|A||B|+|C|}[T(B(y),C(z)),A(x)]$$

(1.8)

where all commutators are understood to be graded. The causal commutators (1.7) and (1.8) have the generic structure

$$D = \sum d_j(X)W_j(X)$$

(1.9)

where $d_j(X)$ are numerical distributions with causal support and $W_j(X)$ are Wick monomials. The numerical distributions $d_j$ have various Lorentz indexes, so to compute them we need some sort of procedure which reduces everything to a certain master scalar causal distribution. To obtain the corresponding chronological products one has to causally split only the master distribution.

A more popular approach is the so-called functional formalism; here one computes the chronological products making sense of Feynman amplitudes. They are expressions of the type:

$$I_N \sim \int \frac{d^4l}{(2\pi)^4} \frac{\mathcal{N}(l)}{\prod_{j=1}^N[(l + q_{j-1})^2 - m_j]}$$

(1.10)
which are associated to one-loop Feynman graphs [5]. Here \( N \) is the number of external particles and the denominator \( \mathcal{N}(l) \) collects kinematic factors coming from vector and spinor propagators. Only the cases \( N \leq 4 \) then one is faced with the problem of computing integrals of the type can produce ultra-violet divergences and a regularization is needed (usually the dimensional regularization.)

In the particular case of a triangle graph one needs to consider the regularized integrals of type \( C \) (rel. (2.9) of [5]). The idea is to use Lorentz covariance and express everything in terms of some scalar integrals. A recursive procedure due to Passarino and Veltman [13] is used. In this procedure a singular region appears due to the annihilation of a certain Gram determinant. The procedure to circumvent this singularity is to use different variables. For the general case more sophisticated methods are available [5]. The avoidance of the infra-red singularities is rather complicated in this approach.

The purpose of this paper is to present how the computations are done in the framework of the causal approach. The idea is to compute some expressions with causal support properties called in [4] causal commutators. We will consider only the second and third order of perturbation theory. There causal commutators are sums of products between numerical distributions with causal support and Wick monomials. The numerical distributions are similar to the type \( C \) Feynman amplitudes from [13], but no regularization procedure is needed. Also infra-red divergences do not appear because the chronological products do not have such divergences: they appear only if we do the adiabatic limit. Finally, the treatment of the singularity region associated to the Gram determinant seems to be easier.

We will present the computation of one-loop contributions in second and third order of perturbation theory in Sections 2 and 3.
2 Second Order Distributions with Causal Support

In second order we have some typical distributions. We remind the fact that the Pauli-Villars distribution is defined by

\[ D_m(x) = D_m^+(x) + D_m^-(x) \]  

(2.1)

where

\[ D_m^{(\pm)}(x) = \pm \frac{i}{(2\pi)^3} \int dp e^{ip\cdot x} \theta(\pm p_0) \delta(p^2 - m^2) \]  

(2.2)

such that

\[ D_m^-(x) = -D_m^+(x). \]  

(2.3)

This distribution has causal support. In fact, it can be causally split (uniquely) into an advanced and a retarded part:

\[ D = D^{\text{adv}} - D^{\text{ret}} \]  

(2.4)

and then we can define the Feynman propagator and anti-propagator

\[ D^F = D^{\text{ret}} + D^{(+)}, \quad \bar{D}^F = D^{(+) - D^{\text{adv}}}. \]  

(2.5)

All these distributions have singularity order \( \omega(D) = -2 \).

These distributions do appear in the tree contributions to the chronological products. For one-loop contributions in the second order we need the basic distributions

\[ d_{D_1,D_2}(x) \equiv d_{D_1,D_2}^{(+)}, \quad d_{D_1,D_2}^{(-)}, \quad d_{D_1,D_2}^{(\pm)}(x) \equiv \pm \frac{1}{2} D_1^{(\pm)}(x) D_2^{(\pm)}(x) \]  

(2.6)

(where \( D_j \equiv D_{m_j} \)) with causal support also. This expression is linear in \( D_1 \) and \( D_2 \). We will also use the notation

\[ d_{12} \equiv d(D_1, D_2) \equiv d_{D_1,D_2} \]  

(2.7)

and when no confusion about the distributions \( D_j = D_{m_j} \) can appear, we skip all indexes altogether. The causal split

\[ d_{12} = d_{12}^{\text{adv}} - d_{12}^{\text{ret}} \]  

(2.8)

is not unique because \( \omega(d_{12}) = 0 \) so we make the redefinitions

\[ d_{12}^{\text{adv(ret)}}(x) \to d_{12}^{\text{adv(ret)}}(x) + c \delta(x) \]  

(2.9)

without affecting the support properties and the order of singularity. The corresponding Feynman propagators can be defined as above and will be denoted as \( d_{12}^F \).

In [7] one can find the expressions of the dominant one-loop contributions from the chronological products. It is necessary to consider the case \( D_1 = D_2 = D_m \) and determine its Fourier transform. By direct computations it can be obtained that

\[ \tilde{d}_{m,m}(k) = \frac{1}{(2\pi)^4} \int dx \, e^{ik\cdot x} d_{m,m}(x) = \frac{1}{8(2\pi)^3} \epsilon(k_0) \theta(k^2 - 4m^2) \sqrt{1 - \frac{4m^2}{k^2}}. \]  

(2.10)

We can consider associated causal distributions substituting in (2.6) \( D_j \to \partial_\alpha D_j \) etc. It can be proved that we can reduce such causal distributed to some polynomials in partial derivatives applied to \( d_{12} \). Detailed examples are provided in [10].
3 Third Order Causal Distributions of Triangle Type

First, we take $D_j = D_{m_j}$, $j = 1, 2, 3$ and define

$$
d_{D_1,D_2,D_3}(x,y,z) \equiv \tilde{D}_1^+(z-x)D_1^+(y-z) - D_2^+(z-x)D_2^+(y-z) + D_3^+(z-x)D_3^+(y-z)
$$

which are with causal support $[11]$. These distributions have the singularity order $\omega(d_{D_1,D_2,D_3}) = -2$. As in the previous Section we use the alternative notation

$$d_{123} \equiv d(D_1, D_2, D_3) \equiv d_{D_1,D_2,D_3}$$

and when there is no ambiguity about the distributions $D_j$ we simply denote $d = d_{123}$. There are some associated distributions obtained from $d_{D_1,D_2,D_3}(x,y,z)$ applying derivatives on the factors $D_j = D_{m_j}$, $j = 1, 2, 3$ for instance

$$\mathcal{D}_1^\mu d_{D_1,D_2,D_3} \equiv d_{\partial^\mu D_1,D_2,D_3}, \quad \mathcal{D}_2^\mu d_{D_1,D_2,D_3} \equiv d_{D_1,\partial^\mu D_2,D_3}, \quad \mathcal{D}_3^\mu d_{D_1,D_2,D_3} \equiv d_{D_1,D_2,\partial^\mu D_3},$$

and so on for more derivatives $\partial_\alpha$ distributed on the factors $D_j = D_{m_j}$, $j = 1, 2, 3$.

It is known that these distributions can be causally split in such a way that the order of singularity, translation invariance and Lorentz covariance are preserved. The same will be true for the corresponding Feynman distributions. Because $\omega(d_{123}) = -2$ and $\omega(\mathcal{D}_1^\mu d_{123}) = -1$ the corresponding advanced, retarded and Feynman distributions are unique. For more derivatives we have some freedom of redefinition.

As in the previous Section, let us consider the case $D_1 = D_2 = D_3 = D_m$, $m > 0$ and study the corresponding distribution $d_{m,m,m}$. We consider it as distribution in two variables $X \equiv x - z$, $Y \equiv y - z$ and we will need its Fourier transform which we define by

$$\tilde{d}(p,q) \equiv \frac{1}{(2\pi)^4} \int e^{i(p \cdot x + q \cdot y)} \, d(X,Y).$$

We will also need the distributions with causal support

$$f_1(x,y,z) = \delta(y-z) \, d_{m,m}(x-y), \quad f_2(x,y,z) = \delta(z-x) \, d_{m,m}(y-z), \quad f_3(x,y,z) = \delta(x-y) \, d_{m,m}(y-z)$$

with

$$\omega(f_j) = 0$$

and the Fourier transforms are:

$$\tilde{f}_1(p,q) = \frac{1}{(2\pi)^2} \tilde{d}_{m,m}(p), \quad \tilde{f}_2(p,q) = \frac{1}{(2\pi)^2} \tilde{d}_{m,m}(q), \quad \tilde{f}_3(p,q) = \frac{1}{(2\pi)^2} \tilde{d}_{m,m}(P)$$

with $P = p + q$. 

5
Theorem 3.1 The following formula is valid:

\[
\tilde{d}_{m,m}(p,q) = \frac{1}{8(2\pi)^5} \frac{1}{\sqrt{N}} [\epsilon(p_0)\theta(p^2 - 4m^2) \ln_1 + \epsilon(q_0)\theta(q^2 - 4m^2) \ln_2 + \epsilon(P_0)\theta(P^2 - 4m^2) \ln_3]
\]

where

\[
\begin{align*}
\ln_1 & \equiv \ln \left( \frac{P \cdot q + \sqrt{N(1 - 4m^2/p^2)}}{P \cdot q - \sqrt{N(1 - 4m^2/p^2)}} \right) \\
\ln_2 & \equiv \ln \left( \frac{P \cdot p + \sqrt{N(1 - 4m^2/q^2)}}{P \cdot p - \sqrt{N(1 - 4m^2/q^2)}} \right) \\
\ln_3 & \equiv \ln \left( \frac{-p \cdot q + \sqrt{N(1 - 4m^2/P^2)}}{-p \cdot q - \sqrt{N(1 - 4m^2/P^2)}} \right)
\end{align*}
\]

(3.8)

with the notations \( P = p + q \) and \( N \equiv (p \cdot q)^2 - p^2 q^2 \).

The previous expression is continuous in the limit \( N \to 0 \) (\( \Leftrightarrow p \parallel q \)) and it is

\[
\tilde{d}_{m,m}(p,q) = 2(F_1 + F_2 + F_3)
\]

(3.10)

where

\[
F_1 \equiv \frac{1}{P \cdot q} \tilde{f}_1, \quad F_2 \equiv \frac{1}{P \cdot p} \tilde{f}_2, \quad F_3 \equiv \frac{1}{p \cdot q} \tilde{f}_3.
\]

(3.11)

Proof: (i) From the definition, it follows that we have six contributions:

\[
d(X,Y) = \sum_{j=1}^{6} d^{(j)}(X,Y)
\]

(3.12)

of the form

\[
d^{(j)}(X,Y) = d_3^{(j)}(X - Y) \ d_2^{(j)}(-X) \ d_1^{(j)}(Y), \quad j = 1, \ldots, 6
\]

(3.13)

If we substitute

\[
d^{(j)}(X) = \frac{1}{(2\pi)^2} \int e^{-ik \cdot X} \ \tilde{d}^{(j)}(k)
\]

(3.14)

we get

\[
\tilde{d}^{(j)}(p,q) = \frac{1}{(2\pi)^2} \int dk \ \tilde{d}_3^{(j)}(k) \ \tilde{d}_2^{(j)}(k - p) \ \tilde{d}_1^{(j)}(k + q)
\]

(3.15)

We consider for illustration the case \( j = 1 \) for which

\[
\tilde{d}_3^{(1)}(k) = \frac{1}{(2\pi)^2} \ \frac{1}{k^2 - m^2 - i \ 0},
\]

\[
\tilde{d}_2^{(1)}(k) = - \frac{i}{2\pi} \theta(-k_0) \ \delta(k^2 - m^2), \quad \tilde{d}_1^{(1)}(k) = \frac{i}{2\pi} \theta(k_0) \ \delta(k^2 - m^2).
\]

(3.16)
We substitute in the previous formula and obtain

\[ \tilde{d}^{(1)}(p, q) = \frac{1}{(2\pi)^6} \int \frac{1}{k^2 - m^2 - i 0} \, \theta(p_0 - k_0) \, \delta((p - k)^2 - m^2) \, \theta(k_0 + q_0) \, \delta((k + q)^2 - m^2) \]

(3.17)

We make the change of variables \( k \to k + p \) leading to

\[ \tilde{d}^{(1)}(p, q) = \frac{1}{(2\pi)^6} \int \frac{1}{(k + p)^2 - m^2 - i 0} \, \theta(-k_0) \, \delta(k^2 - m^2) \, \theta(k_0 + P_0) \, \delta((k + P)^2 - m^2) \]

and afterwards we use the distribution \( \delta(k^2 - m^2) \) to integrate over \( k_0 \). The result is

\[ \tilde{d}^{(1)}(p, q) = \frac{1}{(2\pi)^6} \int_{\omega_k \leq P_0} \frac{dk}{2\omega_k} \, \delta(P^2 - 2P_0\omega_k - 2P \cdot k) \, (p^2 - 2p_0\omega_k - 2p \cdot k - i 0)^{-1} \]

(3.19)

where we have defined \( \omega_k \equiv \sqrt{k^2 + m^2} \).

This expression is Lorentz invariant. We can use this fact to prove that the integral is zero in the cases \( P^2 \leq 0 \) and \( P^2 > 0, P_0 < 0 \). We are left with the case \( P^2 = M^2 (M > 0), P_0 \geq 0 \) so we can evaluate it in a frame where \( P = (M, 0) \). In this frame we get

\[ \tilde{d}^{(1)}(p, q) = \frac{1}{(2\pi)^6} \int_{\omega_k \leq M} \frac{dk}{2M^2} \, \delta(\omega_k - \frac{M}{2}) \, (p^2 - Mp_0 - 2p \cdot k - i 0)^{-1} \]

(3.20)

It is obvious that we must consider two cases: \( p \neq 0 \) and \( p = 0 \).

(ii) We first consider the case \( p \neq 0 \). We perform the integration in spherical coordinates \((r, \theta, \phi)\) with the third axis \( e_3 \parallel p \). The integrals over \( \phi \) and \( r \) are elementary. In particular we find out that the integral is non-zero only if \( M \geq 2m \) and we are left with

\[ \tilde{d}^{(1)}(p, q) = \theta(P_0) \, \theta(P^2 - 4m^2) \, \frac{r_0}{4(2\pi)^5 M} \int d\theta sin\theta \, (p^2 - Mp_0 - 2|p|r_0 cos\theta - i 0)^{-1} \]

(3.21)

where \( r_0 \equiv \sqrt{\frac{M^2}{4} - m^2} \). With the new variable \( z = cos\theta \) we get

\[ \tilde{d}^{(1)}(p, q) = \theta(P_0) \, \theta(P^2 - 4m^2) \, \frac{r_0}{4(2\pi)^5 M} \, I_0(A, B) \]

(3.22)

where

\[ I_0(A, B) \equiv \int_{-1}^{1} \frac{dz}{A - Bz} \]

(3.23)

and

\[ A = p^2 - Mp_0 - i 0, \quad B = 2|p|r_0. \]

(3.24)

The integral is elementary

\[ I_0(A, B) = \frac{1}{B} \ln\left(\frac{A + B}{A - B}\right). \]

(3.25)
Now we want to rewrite the expression $\tilde{d}^{(1)}(p, q)$ in covariant coordinates. We will use the invariant $N$ defined in the statement of the theorem and also $I = P \cdot p$. In the particular frame we have used we have $I = M p_0, \quad N = M^2 p^2$ so it follows that we also have in this frame $A = -p \cdot q, r_0 = \sqrt{\frac{P^2}{4} - m^2}, \frac{r_0}{B} = \sqrt{\frac{P^2}{N}}$. So, the formula

$$
\tilde{d}^{(1)}(p, q) = \theta(P_0) \theta(P^2 - 4m^2) \frac{1}{8(2\pi)^5} \frac{1}{\sqrt{N}} \ln 3
$$

(3.26)

is valid in the particular frame and, because of Lorentz invariance, it is valid in general.

Next we use the relation

$$
\tilde{d}^{(2)}(p, q) = -\tilde{d}^{(1)}(-q, -p)
$$

(3.27)

and the obtain the other piece proportional to $\ln 3$.

In a similar way we obtain

$$
\tilde{d}^{(3)}(p, q) = -\tilde{d}^{(1)}(q, -P)^*
$$

(3.28)

$$
\tilde{d}^{(4)}(p, q) = -\tilde{d}^{(3)}(-p, -q)
$$

(3.29)

and these relations lead to the $\ln 1$ contribution. Finally

$$
\tilde{d}^{(5)}(p, q) = \tilde{d}^{(3)}(q, p)
$$

(3.30)

$$
\tilde{d}^{(6)}(p, q) = \tilde{d}^{(4)}(q, p)
$$

(3.31)

and these relations lead to the $\ln 2$ contribution.

(iii) We consider now the case $p = 0$. We return to (3.20) which is in this case

$$
\tilde{d}^{(1)}(p, q) = \frac{1}{(2\pi)^6} \int_{\omega_k \leq M} \frac{d \mathbf{k}}{2M^2} \delta \left( \omega_k - \frac{M}{2} \right) (p^2 - M p_0 - i 0)^{-1}
$$

(3.32)

We also perform the integration in spherical coordinates, but now we can chose the axis $e_3$ at will. The result is similar to (3.22):

$$
\tilde{d}^{(1)}(p, q) = \theta(P_0) \theta(P^2 - 4m^2) \frac{r_0}{2(2\pi)^5 M A}.
$$

(3.33)

(iv) We prove now that the expression (3.22) is continuous in the limit $p \to 0$ and gives us the preceding formula. This is in fact, equivalent to

$$
lim_{B \to 0} I_0(A, B) = \frac{2}{A}
$$

(3.34)

and this is elementary. Lastly, we give the covariant form of (3.33). As in the previous case we have:

$$
\tilde{d}^{(1)}(p, q) = \frac{2}{(2\pi)^2} \frac{1}{p \cdot q} \tilde{d}^{(+)}_{m,m}(P)
$$

(3.35)

where the expression $\tilde{d}_{m,m}$ was defined in the previous section. We obtain the formula from the statement. ■
We proceed in the same way for the distributions
\[ d_i^\mu \equiv \mathcal{D}_i^\mu d \]
and we have
\[ \omega(d_j^\mu) = -1 \]
and the result is

**Theorem 3.2** For \( N \neq 0 \) the following formula is true:
\[ \tilde{d}_3^\mu(p, q) = i (\mathcal{A}_1^\mu \tilde{d} + \mathcal{A}_2^\mu \tilde{f}_3 + \mathcal{A}_3^\mu \tilde{f}_1 + \mathcal{A}_4^\mu \tilde{f}_2) \]
where
\[ \mathcal{A}_j^\mu(p, q) = p^\mu a_j + q^\mu b_j, \quad j = 1, \ldots, 4 \]
and
\[ a_1 = \frac{q^2(p \cdot P)}{2N}, \quad b_1 = -\frac{p^2(q \cdot P)}{2N} \]
\[ a_2 = -\frac{q \cdot P}{N}, \quad b_2 = \frac{p \cdot P}{N} \]
\[ a_3 = \frac{p \cdot q}{N}, \quad b_3 = -\frac{p^2}{N} \]
\[ a_4 = \frac{q^2}{N}, \quad b_4 = -\frac{p \cdot q}{N}. \]

In the limit \( N \to 0 \) the previous expression is continuous and we have
\[ \tilde{d}_3(p, q) = -i (p - q)^\mu F_3 + i P^\mu (F_1 + F_2). \]

**Proof:** As in the previous Theorem, we obtain the first of the six contributions:
\[ \tilde{d}_3^{\mu(1)}(p, q) = -\frac{i}{(2\pi)^6} \int \frac{d^4k}{k^2 - m^2 - i \epsilon} (p_0 - k_0) \delta((k - p)^2 - m^2) \delta(k_0 + q_0) \delta((k + q)^2 - m^2). \]
If we make the change of variables \( k \to k + p \) we obtain
\[ \tilde{d}_3^{\mu(1)}(p, q) = -i [p^\mu \tilde{d}^{(1)}(p, q) + e^\mu(p, q)] \]
where
\[ e^\mu(p, q) = \frac{1}{(2\pi)^6} \int \frac{d^4k}{(k + p)^2 - m^2 - i \epsilon} (k_0) \delta(k^2 - m^2) \delta(k_0 + P_0) \delta((k + P)^2 - m^2). \]
We proceed as in the previous theorem and obtain as in (3.19)
\[ e^\mu(p, q) = \frac{1}{(2\pi)^6} \int_{\omega_k \leq P_0} \frac{d^4k}{2\omega_k} \tau^\mu(k) \delta(P^2 - 2P_0\omega_k - 2P \cdot k) (p^2 - 2p_0\omega_k - 2p \cdot k - i \epsilon)^{-1}. \]
where \( \tau^\mu(k) = (-\omega_k, k) \). Next, we use Lorentz covariance and do the computations in the particular frame we have used above; the result is (for \( P^2 > 0, \ P^0 \geq 0 \)):

\[
e^\mu(p, q) = \frac{1}{(2\pi)^6} \int_{\omega_k \leq M} \frac{dk}{2M^2} \left( \frac{\omega_k - M}{2} \right) \delta \left( \omega_k - \frac{M}{2} \right) \left( p^2 - Mp_0 - 2p \cdot k - i0 \right)^{-1} \tag{3.46}
\]

We consider the case \( p \neq 0 \) and treat separately the cases \( \mu = 0 \) and \( \mu \neq 0 \). The first case is easy:

\[
e^0(p, q) = -\frac{1}{2} M \tilde{d}^{(1)}(p, q). \tag{3.47}
\]

We also have

\[
e^1 = e^2 = 0. \tag{3.48}
\]

The remaining case can be treated as in the preceding theorem;

\[
e^3(p, q) = \theta(P_0) \theta(P^2 - 4m^2) \frac{r_0^2}{4(2\pi)^5 M} I_1(A, B) \tag{3.49}
\]

where

\[
I_1(A, B) \equiv \int_{-1}^{1} \frac{dz}{A - Bz} \tag{3.50}
\]

and \( A \) and \( B \) have the same values as before: \( A = p^2 - Mp_0 - i0, \ B = 2|p|r_0 \). The integral is elementary:

\[
I_1(A, B) = \frac{1}{B} \left[ -2 + \frac{A}{B} \ln \left( \frac{A + B}{A - B} \right) \right]. \tag{3.51}
\]

In the case \( |p| = 0 \) we easily obtain

\[
e^3(p, q) = 0. \tag{3.52}
\]

Again, as in the previous theorem, we obtain that the limit \( |p| \to 0 \) of (3.49) exists and is 0. It remains to go to an arbitrary frame. After a tedious computation we obtain for \( N \neq 0 \)

\[
\tilde{d}_3^{(1)}(p, q) = i \left( A_1^\mu \tilde{d}^{(1)} + A_2^\mu \tilde{f}^{(+)} \right) \tag{3.53}
\]

where the expressions \( A_j, j = 1, 2 \) are those from the statement. For \( N = 0 \) we get

\[
\tilde{d}_3^{\mu(1)}(p, q) = -\frac{i}{p \cdot q} (p - q)^\mu \tilde{f}_3^{(+)} \tag{3.54}
\]

If we use now relations similar to (3.27) - (3.31) we get the other five contributions and the relation from the statement follows. \( \blacksquare \)

The expression \( \tilde{d}_1^\mu, \tilde{d}_2^\mu \) can be obtained from \( \tilde{d}_3^\mu \) by clever changes of variables, as in [7]. We note that for \( N \neq 0 \) the expressions \( \tilde{d}_3^\mu \) obtained above are identical to those from [7] where the derivation was made by another method.
Finally we define
\[ d_{jk}^{\mu\nu} \equiv D_j^{\mu} D_k^{\nu} d \]  
and we have the following orders of singularity:
\[ \omega(d_{jk}^{\mu\nu}) = 0. \]  
We will first consider the case \( d_{33} \). The result is

**Theorem 3.3** For \( N \neq 0 \) the following formula is true:
\[ \tilde{d}_{33}^{\mu\nu}(p, q) = A_1^{\mu\nu} \tilde{d} + A_2^{\mu\nu} \tilde{f}_3 + A_3^{\mu\nu} \tilde{f}_1 + A_4^{\mu\nu} \tilde{f}_2 \]  
where
\[ A_j^{\mu\nu}(p, q) = -[p^\mu p^\nu \alpha_j + q^\mu q^\nu \beta_j + (p^\mu q^\nu + p^\nu q^\nu) \gamma_j + \eta^{\mu\nu} \delta_j], \quad j = 1, \ldots, 4 \]  
and
\[ \alpha_1 = \frac{3P^2 p^2(q^2)^2}{8N^2} + \frac{(q^2)^2}{4N} + \frac{m^2}{2N}, \quad \beta_1 = \frac{3P^2 q^2(p^2)^2}{8N^2} + \frac{(p^2)^2}{4N} + \frac{m^2}{2N}, \quad \gamma_1 = -\frac{3P^2 p^2 q^2(p \cdot q)}{8N^2} + \frac{p^2 q^2}{4N} - \frac{m^2(p \cdot q)}{2N}, \quad \delta_1 = \frac{3P^2 q^2(p \cdot q)}{8N} + \frac{m^2}{2} \]  
\[ \alpha_2 = -\frac{3(P \cdot q)^2(p \cdot q)}{4N^2} + \frac{4P \cdot q + p \cdot q}{4N}, \quad \beta_2 = -\frac{3(P \cdot p)^2(p \cdot q)}{4N^2} + \frac{4P \cdot p + p \cdot q}{4N}, \quad \gamma_2 = \frac{3(P \cdot p)(P \cdot q)(p \cdot q)}{4N^2} - \frac{2P^2 - p \cdot q}{4N}, \quad \delta_2 = -\frac{3P^2(p \cdot q)}{4N} \]  
\[ \alpha_3 = \frac{3(p \cdot q)^2(P \cdot q)}{4N^2} - \frac{4p \cdot q + P \cdot q}{4N}, \quad \beta_3 = \frac{3(P \cdot q)(p \cdot q)^2}{4N^2}, \quad \gamma_3 = -\frac{3(P \cdot q)(p \cdot q)p^2}{4N^2} + \frac{p^2}{2N}, \quad \delta_3 = \frac{p^2(P \cdot q)}{4N} \]  
and the expressions \( \alpha_4, \ldots, \delta_4 \) are obtained from \( \alpha_3, \ldots, \delta_3 \) making \( p \leftrightarrow q \). In the limit \( N \to 0 \) the previous expression is continuous and we have
\[ \tilde{d}_{33}^{\mu\nu}(p, q) = -[\alpha_{33}(p, q) P^\mu P^\nu + \eta^{\mu\nu} \beta_{33}(p, q)] F_3 \]
\[ -[\alpha_{33}(q, -P)p^\mu p^\nu + \eta^{\mu\nu} \beta_{33}(q, -P)] F_1 - [\alpha_{33}(-p, P)q^\mu q^\nu + \eta^{\mu\nu} \beta_{33}(-p, P)] F_2 \]  
where
\[ \alpha_{33}(p, q) = \frac{1}{6} \left[ 4 - \frac{m^2}{4P^2} - 12 \frac{(P \cdot p)(P \cdot q)}{(P^2)^2} \right] \]
\[ \beta_{33}(p, q) = \frac{P^2}{6} \left( 1 - \frac{m^2}{4P^2} \right) \]  

\[ (3.55) \]  
\[ (3.56) \]  
\[ (3.57) \]  
\[ (3.58) \]  
\[ (3.59) \]  
\[ (3.60) \]  
\[ (3.61) \]
Proof: As in the previous Theorems, we obtain the first of the six contributions:

\[ \tilde{d}^{\mu\nu(1)}(p, q) = -\frac{1}{(2\pi)^6} \int dk \frac{k^\mu k^\nu}{k^2 - m^2 - i 0} \theta(p_0 - k_0) \delta((p - k)^2 - m^2) \theta(k_0 + q_0) \delta((k + q)^2 - m^2). \]  

(3.62)

If we make the change of variables \( k \to k + p \) we obtain

\[ \tilde{d}^{\mu\nu(1)}(p, q) = -p^\mu p^\nu \tilde{d}^{(1)}(p, q) - [p^\mu e^\nu(p, q) + p^\nu e^\mu(p, q)] - e^{\mu\nu}(p, q) \]  

(3.63)

where the expressions \( e^\mu(p, q) \) have been defined before - rel (3.44) and

\[ e^{\mu\nu}(p, q) = \frac{1}{(2\pi)^6} \int dk \frac{k^\mu k^\nu}{(k + p)^2 - m^2 - i 0} \theta(-k_0) \delta(k^2 - m^2) \theta(k_0 + P_0) \delta((k + P)^2 - m^2). \]  

(3.64)

We proceed as in the previous theorem and obtain as in (3.19)

\[ e^{\mu\nu}(p, q) = \frac{1}{(2\pi)^6} \int \frac{dk}{2\omega_k} \tau^\mu(k) \tau^\nu(k) \delta(P^2 - 2P_0\omega_k - 2P \cdot k) (p^2 - 2p_0\omega_k - 2p \cdot k - i 0)^{-1} \]  

(3.65)

where \( \tau^\mu(k) = (-\omega_k, k) \). Next, we use Lorentz covariance and do the computations in the particular frame we have used above; the result is:

\[ e^{\mu\nu}(p, q) = \frac{1}{(2\pi)^6} \int_{\omega_k \leq M} \frac{dk}{2M^2} \tau^\mu(k) \tau^\nu(k) \delta(\omega_k - \frac{M}{2}) (p^2 - Mp_0 - 2p \cdot k - i 0)^{-1} \]  

(3.66)

We consider the case \( p \neq 0 \). We easily obtain

\[ e^{00}(p, q) = \frac{1}{4} M^2 \tilde{d}^{(1)}(p, q), \quad e^{\mu0}(p, q) = -\frac{1}{2} M e^\mu(p, q) \]  

(3.67)

We also have

\[ e^{jk} = 0, \quad j, k = 1, 2, 3, \quad j \neq k. \]  

(3.68)

Next

\[ e^{33}(p, q) = \theta(P_0) \theta(P^2 - 4m^2) \frac{r_0^3}{4(2\pi)^6 M} I_2(A, B) \]  

(3.69)

where

\[ I_2(A, B) = \int_{-1}^{1} \frac{dz z^2}{A - Bz} \]  

(3.70)

and \( A \) and \( B \) have the same values as before: \( A = p^2 - M p_0 - i 0, \quad B = 2|p|r_0 \). The integral is elementary:

\[ I_2(A, B) = \frac{A}{B} I_1(A, B) \]  

(3.71)

In the case \( |p| = 0 \) the expression \( e^{33}(p, q) \) is the limit \( |p| \to 0 \) of the previous expression. The expressions \( e^{11} \) and \( e^{22} \) can be obtained similarly. It remains to to an arbitrary frame. After a tedious computation we obtain for \( N \neq 0 \)

\[ \tilde{d}_3^{\mu\nu(1)}(p, q) = A_1^{\mu\nu} \tilde{d}^{(1)} + A_2^{\mu\nu} f_3^{(+)} \]  

(3.72)
where the expressions \( A_j, j = 1, 2 \) are those from the statement. For \( N = 0 \) we get

\[
\tilde{d}^{\mu
u(1)}_3(p, q) = -(p^\mu P^\nu \alpha + \eta^{\mu\nu} \beta)
\]  

\hspace{1cm} (3.73)

where

\[
\alpha = -\frac{1}{6p \cdot q} \left[ 4 - \frac{m^2}{4P^2} - 12 \frac{(P \cdot p)(P \cdot q)}{(P^2)^2} \right] f_3^{(+)}
\]

\[
\beta = -\frac{P^2}{6p \cdot q} \left( 1 - \frac{m^2}{4P^2} \right) f_3^{(+)}
\]  

\hspace{1cm} (3.74)

If we use now relations similar to \((3.27) - (3.31)\) we get the other five contributions and the relation from the statement follows. \(\blacksquare\)

The expression \( \tilde{d}^{\mu}_1, \tilde{d}^{\mu}_2 \) can be obtained from \( \tilde{d}^{\mu}_3 \) by clever changes of variables, as in \([7]\). We note that for \( N \neq 0 \) the expressions \( \tilde{d}^{\mu}_{ij} \) obtained above are identical to those from \([7]\) where the derivation was made by another method.

We still have to consider the case \( \tilde{d}^{\mu
u}_{12} \). The result is

**Theorem 3.4** For \( N \neq 0 \) the following formula is true:

\[
\tilde{d}^{\mu
u}_{12}(p, q) = B^{\mu
u}_1 \tilde{d} + B^{\mu
u}_2 \tilde{f}_3 + B^{\mu
u}_3 \tilde{f}_1 + B^{\mu
u}_4 \tilde{f}_2
\]  

\hspace{1cm} (3.75)

where

\[
B^{\mu
u}_j(p, q) = p^\mu p^\nu A_j + q^\mu q^\nu B_j + p^\mu q^\nu C^{(1)}_j + p^\nu q^\mu C^{(2)}_j + \eta^{\mu\nu} D_j, \quad j = 1, \ldots, 4
\]  

\hspace{1cm} (3.76)

and

\[
A_1 = \frac{3P^2 p^2 q^2(q^2)^2}{8N^2} + \frac{(q^2)^2}{4N} + \frac{q^2(P \cdot p)}{4N} + \frac{m^2 q^2}{2N},
\]

\[
B_1 = \frac{3P^2 q^2(p^2)^2}{8N^2} + \frac{(p^2)^2}{4N} + \frac{P^2(p \cdot q)}{4N} + \frac{m^2 p^2}{2N},
\]

\[
C^{(1)}_1 = -\frac{3P^2 p^2 q^2(p \cdot q)}{8N^2} + \frac{p^2 q^2}{4N} - \frac{m^2 p \cdot q}{2N},
\]

\[
D_1 = \frac{P^2 p^2 q^2}{8N} + \frac{m^2}{2},
\]

\[
A_2 = -\frac{3(P \cdot q)^2(p \cdot q)}{4N^2} + \frac{p \cdot q}{N}, \quad B_2 = -\frac{3(P \cdot p)^2(p \cdot q)}{4N^2} + \frac{p \cdot q}{N},
\]

\[
C^{(2)}_1 = \frac{3(P \cdot p)(P \cdot q)(p \cdot q)}{4N^2} - \frac{P^2}{2N} + \frac{p \cdot q}{4N},
\]

\[
D_2 = -\frac{P^2(p \cdot q)}{4N},
\]

\[
A_3 = \frac{3(p \cdot q)^2(P \cdot q)}{4N^2} - \frac{P \cdot q}{4N}, \quad B_3 = \frac{3(p^2)^2(P \cdot q)}{4N^2} + \frac{p^2}{N},
\]

\[
C^{(1)}_3 = -\frac{3(p \cdot q)(P \cdot q)p^2}{4N^2} + \frac{p^2}{2N}.
\]
we have to multiply by $N$ derivatives acting on distributions with causal support. The same idea is valid for (3.57), but in the coordinate space, we will have in both hands some polynomials in the partial distribution from the preceding theorems by a simple operation, namely the causal splitting of a master [17] and [18]. In fact, if we want to split causally (3.38) it is better to multiply it by $\tilde{D}^0$ where the derivation was made by another method.

In the limit $N \to 0$ the previous expression is continuous and we have

$$d_{12}^{\mu\nu}(p, q) = [\alpha_{12}(p, q)P^\mu P^\nu + \eta^{\mu\nu} \beta_{12}(p, q)] F_3 + [\alpha_{12}(q, -P)p^\mu p^\nu + \eta^{\mu\nu} \beta_{12}(q, -P)] F_1 + [\alpha_{12}(-p, P)q^\mu q^\nu + \eta^{\mu\nu} \beta_{12}(-p, P)] F_2$$

(3.78)

where

$$\alpha_{12}(p, q) = -\frac{1}{6} \left( 2 + \frac{m^2}{4p^2} \right), \quad \beta_{12}(p, q) = \frac{p^2}{6} \left( 1 - \frac{m^2}{4p^2} \right).$$

(3.79)

Proof: As in the previous Theorems, we obtain the first of the six contributions:

$$d_{12}^{\mu(1)}(p, q) = -\frac{1}{(2\pi)^6} \int dk \frac{(k + q)^\mu(k - p)^\nu}{k^2 - m^2 - i0} \theta(p_0 - k_0) \delta((p - k)^2 - m^2) \theta(k_0 + q_0) \delta((k + q)^2 - m^2).$$

(3.80)

If we make the change of variables $k \to k + p$ we obtain

$$d_{12}^{\mu(1)}(p, q) = P^\mu e^\nu(p, q) + e^{\mu\nu}(p, q)$$

(3.81)

where the expressions $e^\mu(p, q)$ and $e^{\mu\nu}$ have been defined before - rel (3.44) and (3.64). Proceeding as before we get the formulas from the statement.

The expression $d_{23}^{\mu}, d_{31}^{\mu}$ can be obtained from $d_{12}^{\mu}$ by clever changes of variables, as in [7].

We note that for $N \neq 0$ the expressions $d_{jk}^{\mu}, j \neq k$ obtained above are identical to those from [7] where the derivation was made by another method.

One can obtain in the same way the expressions

$$d_{jkl}^{\mu\nu} \equiv D_j^{\mu} D_k^{\nu} D_l^{\rho} d.$$

(3.82)

We only emphasize in the end the main idea: the chronological products can be obtained from the preceding theorems by a simple operation, namely the causal splitting of a master distribution $d$ given by (3.1). An explicit procedure to do this causal splitting can be found in [17] and [18]. In fact, if we want to split causally (3.38) it is better to multiply it by $N$ so, if we go in the coordinate space, we will have in both hands some polynomials in the partial derivatives acting on distributions with causal support. The same idea is valid for (3.57), but we have to multiply by $N^2$. 
References

[1] N. N. Bogoliubov, D. Shirkov, “Introduction to the Theory of Quantized Fields”, John Wiley and Sons, 1976 (3rd edition)

[2] M. Dütsch, F. M. Boas, “The Master Ward Identity”, Rev. Math. Phys 14 (2002) 9771049

[3] M. Dütsch, K. Fredenhagen, “A Local (Perturbative) Construction of Observables in Gauge Theories: the Example of QED”, Commun. Math. Phys. 203 (1999) 71-105

[4] H. Epstein, V. Glaser, “The Rôle of Locality in Perturbation Theory”, Ann. Inst. H. Poincaré 19 A (1973) 211-295

[5] R. Keith Ellis, Z. Kunszt, K. Melnikov, G. Zanderighi, “One-Loop Calculations in Quantum Field Theory: from Feynman Diagrams to Unitarity Cuts”, hep-ph/1105.4319, Physics Reports 518 (2012) 141 - 250

[6] V. Glaser, “Electrodynamique Quantique”, L’enseignement du 3e cycle de la physique en Suisse Romande (CICP), Semestre d’hiver 1972/73

[7] D. R. Grigore, “Loop Anomalies in the Causal Approach”, hep-th/1302.1692, Int. J. of Geometric Methods in Modern Physics 12 No. 2 (2015) 1550026 (38 pages)

[8] D. R. Grigore, “A Generalization of Gauge Invariance”, hep-th/1612.04998, Journal of Mathematical Physics 58 (2017) 082303

[9] D. R. Grigore, “Anomaly-Free Gauge Models: A Causal Approach”, hep-th/1804.08276, Romanian Journ. Phys. 64 (2019) 102

[10] D. R. Grigore, “On the Super-Renormalizablity of Quantum Gravity in the Linear Approximation”, hep-th/1905.05410, Romanian Journ. Phys. 65 (2020) 101

[11] D. R. Grigore, “Third Order Anomalies in the Causal Approach”, hep-th/1910.10640, to appear in Romanian Journ. Phys.

[12] K. Hepp, “Renormalization Theory”, in “Statistical Mechanics and Quantum Field Theory” pp. 429 - 500, (Les Houches 1970), C. DeWitt-Morette, Raymond Stora (eds.), Gordon and Breach 1971

[13] G. Passarino, M. J. G. Veltman, “One loop corrections for $e^+e^-$ annihilation into $\mu^+\mu$ in the Weinberg model”, Nuclear Phys. B 160 (1979) 151 - 207

[14] J. Polchinski, “Renormalization and Effective Lagrangians”, Nucl. Phys. B 231 (1984) 269 - 295

[15] G. Popineau, R. Stora, “A Pedagogical Remark on the Main Theorem of Perturbative Renormalization Theory”, Nuclear Physics B 912 (2016) 70 - 78
[16] M. Salmhofer, “Renormalization: An Introduction”, (Theoretical and Mathematical Physics) Springer 1999

[17] G. Scharf, “Finite Quantum Electrodynamics: The Causal Approach”, (second edition) Springer, 1995

[18] G. Scharf, “Quantum Gauge Theories: A True Ghost Story”, John Wiley, 2001 and “Quantum Gauge Theories - Spin One and Two”, Google books, 2010

[19] O. Steinmann, “Perturbation Expansions in Axiomatic Field Theory”, Lect. Notes in Phys. 11, Springer, 1971