Positive trigonometric polynomials

Tkachev V.G.

Abstract. We study the boundary of the nonnegative trigonometric polynomials from the algebraic point of view. In particularly, we show that it is a subset of an irreducible algebraic hypersurface and established its explicit form in terms of resultants.

Introduction

Let \( n \geq 1 \) be an integer and

\[
T(t) = \text{Re} \sum_{k=0}^{n} y_k e^{ikt} = \sum_{k=0}^{n} (a_k \cos kt + b_k \sin kt)
\]

be a trigonometric polynomial of degree \( n \) with real coefficients \( a_k = \text{Re} y_k, b_k = -\text{Im} y_k \) where \( b_0 = 0 \) and \( y_0 = a_0 \) is a real positive number. We say that \( T(t) \) is nonnegative if the inequality \( T(t) \geq 0 \) holds for all \( t \in \mathbb{R} \).

The problem of deciding whether given a trigonometric polynomial is nonnegative has long and reach history. We refer to recent papers [12], [10] for more details and circumstance. We mention that various reasons for the interest in the problem of constructing nonnegative trigonometric polynomials are: the Gibbs phenomenon, univalent functions and polynomials, positive Jacobi polynomial sums [4], orthogonal polynomials on the unit circle [5], zero-free regions for the Riemann zeta-function [2], [3] just to mention a few.

The well-known fact of complex analysis [1] states that an equivalent criterion is to check that the Hermitian Toeplitz matrices \( T_l = [c_{i-j}; 0 \leq i, j \leq l; c_l = 0 \text{ for } l > k] \) are nonnegative definite for all positive integers \( l \) (here \( c_k = y_k \) and \( c_{-k} = \overline{y_k} \) for all \( k \geq 0 \)). But this condition involves infinitely many inequalities and does not give exact information about the structure of nonnegative polynomials.

Our interest in this subject comes from the theory of univalent algebraic polynomials in the unit disk. The simplest class of the univalent polynomials is the body of all starlike polynomials in the unit disk \( U \). It is well-known fact that an algebraic polynomial \( P(z) \) with \( P(0) = 0 \) is star-like univalent if and only if \( \text{Re} (P'(z)z/P(z)) \geq 0 \) in \( U \). The last can be reformulated as \( P(z)/z \) has no roots in \( U \) and the trigonometric polynomial \( \text{Re} T_P(e^{it}) \) where

\[
T_P(z) = P'(z) \overline{P(z)}
\]
is nonnegative. So, an arbitrary starlike polynomial one can associate with a certain positive trigonometric polynomial.

The main purpose of this article is translation the initial problem into the algebraic context and our point of view of the problem is studying of the image of $\mathbb{R}^+ \times \mathbb{C}^n$ under the associated polymorphism $P \to T_P$. The associated morphism is the special quadratic and maps $\mathbb{R}^+ \times \mathbb{C}^n$ onto certain cone in the same target space.

This problem was initially studied in the lower dimensional case by Brannan [6]–[8]. In particularly, it was proved that the boundary of three dimensional body (the case $\deg P = 3$) is an real algebraic manifold of degree $d(3) \leq 28$. Brannan also established the explicit representation of the boundary of the body.

In his paper [15] Quine was showed that the boundary of the body of all univalent polynomials is also an real algebraic manifold of degree at most $q(n) = 4(2n^2 - 4n + 1)(n - 1)$. We notice that this estimate is too far from to be sharp. Even in the Brannan’s case $n = 3$ we have the value $q(3) = 56 > d(3)$.

Acknowledgement: We wish to thank all of staff of Mittag-Leffler Institute for heart hospitality during the visit of the author Mittag-Leffler Institute in 2001.

1. Quadratic map

1.1. Let $T(t)$ is given by the representation (1). Then by the well-known result due to L. Fejér [13] $T(t)$ is a nonnegative trigonometric polynomial if and only if there exists an algebraic polynomial $X(z) = x_0 + x_1 z + \ldots + x_n z^n$ with complex coefficients $x_k$ such that

$$T(t) = |X(e^{it})|^2. \quad (2)$$

Moreover, such representation with normalization $x_0 > 0$ is unique. In sequel we fix this normalization.

But still this characterization does not make it easy to decide if a given trigonometric polynomial is nonnegative. We consider the representation in more details.

We use the following notations. Given an algebraic polynomial $P(z)$ of the $n$-th degree let us denote by $P^*(z)$ its reciprocal double

$$P^*(z) \equiv z^n \overline{P(1/z)},$$

where $\overline{P(\zeta)} = \overline{P(\overline{\zeta})}$ is the conjugate polynomial to $P(z)$. By $\text{Res} (P; Q)$ we denote the resultant of two corresponding polynomials. We also usually identify a polynomial $P$ and the vector in $\mathbb{R}^+ \times \mathbb{C}^n$ (or in $\mathbb{R}^{2n+1}$) which consists of the coefficients of $P$.

Now we can reformulate the initial problem (2) in more appropriate terms. Namely, we associate the trigonometric polynomial $T(t)$ in (1) with the unique algebraic polynomial $Y(z)$ of the form $\sum_{k=0}^{n} y_k z^k$ with $y_0$ to be a positive real. Then the problem (2) is equivalent to following: Given a vector $y \in \mathbb{R}^+ \times \mathbb{C}^n$ find a vector $x \in \mathbb{R}^+ \times \mathbb{C}^n$
such that
\[
y_0 = 2\Phi_0(x_0, x_1, x_1, \ldots, x_n, \bar{x}_n) = \frac{1}{2} \sum_{j=0}^{n} x_j x_j,
\]
\[
y_m = \Phi_m(x_0, x_1, x_1, \ldots, x_n, \bar{x}_n) = \sum_{k=0}^{n-m} \bar{x}_k x_{k+m}, \quad m = 0, 1, \ldots, n.
\]
If such a vector \(x\) exists then we say that \(y\) is a nonnegative vector of \(\mathbb{R}^+ \times \mathbb{C}^n\).

It follows from formulae (3) that
\[
X(z)X^*(z) = \Phi_n(x) + \Phi_{n-1}(x)z + \ldots + 2\Phi_0(x)z^n + \Phi_1(x)z^{n+1} + \ldots + \Phi_n(x)z^{2n},
\]
or in the other form,
\[
X(z)X^*(z) = Y^*(z) + z^nY(z).
\]

Remark 1. The mentioned above problem has a clear geometric interpretation. Really, the set of all nonnegative vectors forms a cone \(K_n\) in \(\mathbb{R}^+ \times \mathbb{C}^n\) which is strongly contained in \(\mathbb{R}^{2n+1}\). Moreover, it follows that a point \(y\) is an inner point of \(K_n\) if and only if there exists a point \(x \in \mathbb{R}^+ \times \mathbb{C}^n\) such that the mapping \(\Phi\) has the highest rank at \(x\). Other words,
\[
y \in \text{int} K_n \iff \max_{x \in \Phi^{-1}(y)} \text{rank} d_\mathbb{R} \Phi(x) = 2n + 1.
\]

As a consequence, the boundary \(\partial K_n\) is contained in the image \(\Phi(V_n)\) where \(V_n\) is zero-set of the Jacobian \(d_\mathbb{R} \Phi(x)\). Nevertheless, in general case only the strong inclusion \(\partial K_n \not\subseteq \Phi(V_n)\) can be possible.

1.2. Our first step is to describe the structure of the image \(\Phi(V_n)\). To do this we need some auxiliary assertions.

Lemma 1. We have
\[
\det d_\mathbb{R} \Phi(x) = \det \frac{\partial(\Phi_0, \Phi_1, \Phi_1, \ldots, \Phi_n, \Phi_n)}{\partial(x_0, x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n)} = 2x_0 \text{Res}(X, X^*).
\]

Proof. Let us denote by \(R(z)\) the polynomial \(X(z)X^*(z)\). Then we have from (3) the expressions for the derivatives
\[
R_j \equiv \frac{\partial R}{x_j} = \frac{\partial X}{x_j} X^* = z^j X^*,
\]
\[
R_{-j} \equiv \frac{\partial R}{\bar{x}_j} = X \frac{\partial X^*}{\bar{x}_j} = z^{n-j} X,
\]
\[
R_0 \equiv \frac{\partial R}{x_0} = \frac{\partial X}{x_0} X^* + X \frac{\partial X^*}{x_0} = X^* + z^n X.
\]
On the other hand, the coefficients of polynomials \( R_j, R_{-j} \) and \( R_0 \) produce the corresponding strings of the Jacobian \( d_R \Phi(x) \). Moreover, we claim that
\[
X(z)(z^n - Q_{n-1}(z)) = P_n(z)X^*(z),
\]
where \( Q_{n-1} \) and \( P_n \) are some polynomials of degrees \((n-1)\) and \( n \) respectively. Indeed, we can assume that
\[
Q_{n-1} \equiv z^n - \frac{1}{x_0}X^*(z), \quad P_n \equiv \frac{1}{x_0}X(z).
\]

Moreover, the last expressions imply that \( P_n(0) = X(0)/x_0 = 1 \) and it follows from (9) that
\[
X(z)z^n = \left(g_0 + q_1 z + \ldots + q_{n-1}z^{n-1}\right)X(z) + \left(1 + p_1 z + \ldots + p_n z^n\right)X^*(z).
\]

Here \( q_k \) and \( p_k \) are the coefficients of the corresponding capitals polynomials.

By virtue of (8) we obtain that
\[
X(z)z^n - X^*(z) = q_0 R_{-n} + q_1 R_{-n+1} + \ldots + q_{n-1} R_{-1} + p_1 R_1 + \ldots + p_n R_n,
\]
and
\[
2X(z)z^n \equiv \left(X(z)z^n - X^*(z)\right) + \left(X(z)z^n + X^*(z)\right) = \left(X(z)z^n - X^*(z)\right) + R_0(z).
\]

By (11) the last equality can be interpreted as the fact that \( 2X(z)z^n \) is a linear combination of \( R_0 \) and other strings of the Jacobian matrix \( d_R \Phi(x) \) (here and further on we use the mentioned above equivalence between the polynomial and vectors).

Now we recall that by Sylvester’s formula for the resultant (see [17])
\[
det d_R \Phi(x) = 2\det[z^nX, z^{n-1}X^*, z^{n-1}X, \ldots, z^jX^*, z^{n-j}X, \ldots, z^nX^*, X] = 2(-1)^\sigma \det[X, zX, \ldots, z^n X^*, X^*, zX^*, zX^*, \ldots, z^n X^*] = 2(-1)^\sigma \text{Res}(zX^*, X).
\]

Here by \( \det[h_1, \ldots, h_k] \) we denote the determinant of the matrix with strings \( h_k \). The symbol \( \sigma \) is equal to the corresponding permutation number between to matrices. It is easy to compute that \( \sigma = n^2 + n \equiv 0 \mod 2 \) and by the multiplication property of resultant we arrive at
\[
det d_R \Phi(x) = 2\text{Res}(zX^*, X) = 2\text{Res}(z, X)\text{Res}(X^*, X) = 2x_0 \text{Res}(X^*, X)
\]
and the lemma is proved completely. \( \square \)

1.3. We introduce the special notation for the characteristic \( \text{Res}(X, X^*) \). Namely, we call this quantity the M"obius discriminant of \( X(z) \) and will denote it by \( V_n(X) \). The direct consequence of its definition and Sylvester’s formula is that \( V_n(X) \) is a homogeneous form of \( X \) (or it the same as the form of variables \( x_k \) and \( x_k \)) of degree \( \deg V_n = 2n \). Moreover,
\[
V_n(X) = (-1)^n V_n(X^*),
\]
because \( \text{Res}(P, Q) = (-1)^{\deg P\deg Q} \text{Res}(Q, P) \)
Further we use the fact that $V_n(X)$ is actually a real valued form. To see that we notice that by the decomposition property of resultant and relation between the roots of $X$ and $X^*$

$$V_n(X) = |x_n|^{2n} \prod_{j,k=1}^{n} (z_jz_k - 1), \quad (13)$$

where $z_j$ are the roots of $X(z)$.

Really, we recall that for arbitrary polynomials $P(z)$ and $Q(z)$ of $n$ and $m$ degrees one holds the Cayley formula for its resultant

$$\text{Res } (P; Q) = \frac{p^m q^m n!}{n! m!} \prod_{j=1}^{n} \prod_{k=1}^{m} (a_j - b_k), \quad (14)$$

where $a_j$ and $b_k$ are the roots of $P(z)$ and $Q(z)$ respectively.

So, if we have decomposition $X(z) = x_0 + x_1 z + \ldots + x_n z^n = x_n \prod_{j=1}^{n} (z - z_j)$, then by the definition of $X^*$

$$X^*(z) = x_n + x_{n-1} z + \ldots + x_0 z^n = z^n X(1/z) = x_n \prod_{j=1}^{n} (1 - z z_j).$$

It follows from (14) and Viète formula that

$$V(X) \equiv \text{Res } (X; X^*) = x_n^n \prod_{j=1}^{n} \prod_{k=1}^{n} (z_j - 1/z_k) = |x_n|^{2n} \prod_{j,k=1}^{n} (z_jz_k - 1), \quad (15)$$

and (13) is proved.

1.4. To obtain the explicit representation for the boundary $\partial K_n$ we use arguments of Remark 1. We need a suitable generalization of discriminant. Let $Y(z)$ be an arbitrary polynomial of degree $n$ with real initial coefficient $y_0$. Let $\text{Dis}(Y)$ is the discriminant of $Y$ (see [17, § 33]) and

$$\text{Dis}_2(Y) = \text{Dis}(Y^*(z) + z^n Y(z)).$$

The last expression can be regarded as certain $A$-discriminant in terminology of [11, c. 271]. It is the direct corollary of homogeneity of $\text{Dis}(Y)$ that the 2nd discriminant $\text{Dis}_2(Y)$ has $(4 \deg Y - 2)$ degree.

**Lemma 2.** If $Y = \Phi(X)$ then

$$\text{Dis}_2(Y) = |\text{Dis}(X)|^2 V^2(X). \quad (16)$$

**Proof.** First we notice that by the discriminant multiplicative rule (see. [16, Theorem 3.4]) we have

$$\text{Dis}(PQ) = \text{Dis}(P) \text{Dis}(Q) \text{Res}^2(P; Q), \quad (17)$$

which yields from the definition of the Möbius discriminant and identity (5) that

$$\text{Dis}_2(Y) = \text{Dis}(X) \text{Dis}(X^*) V^2(X).$$
On the other, it is easy to see by the definition that the reciprocal polynomial $X^*$ has conjugate to $X$ discriminant which completes the proof.

We denote by $V_n$ and $D_n$ the zero-sets $V(X) = 0$ and $\text{Dis}(X) = 0$ respectively. Then the following assertion shows the shadow-character property of $D_n$ with respect to mapping $\Phi$.

**Lemma 3.** We have $\Phi(D_n) \subset \Phi(V_n)$.

**Proof.** Due to (4) it is sufficient to prove that given a polynomial $X \in D_n$ there exists a polynomial $Q(z)$ with positive initial coefficient which is in $V_n$ and the equality $XX^* = QQ^*$ holds.

Let $X$ be arbitrary polynomial of $n$-th degree regarded as a point in $D_n$. Then $\text{Dis}(X) = 0$. It follows from the definition of discriminant that there exist a multiple root $z_p$ of $X$. We can assume that $p = n$ and therefore

$$X(z) = x_n(z - z_n)^2 \prod_{j=1}^{n-2} (z - z_j).$$

We consider the following polynomial

$$Q(z) = \lambda x_n(z - z_n)(z - 1/\overline{z}_n) \prod_{j=1}^{n-2} (z - z_j) = q_0 + q_1z + \ldots + q_nz^n,$$

where $\lambda$ is some positive real number which is specified later. Then from its definition $V(Q) = 0$ and hence $Q \in V_n$. Moreover,

$$q_0 = Q(0) = \lambda X(0)/|z_n|^2 = \lambda x_0/|z_n|^2$$

is a positive real and therefore $Q(z)$ is admissible.

On the other hand

$$Q^*(z) = z^n\overline{Q}(1/z) = \lambda \overline{x}_n(1 - z\overline{z}_n)(1 - z/z_n) \prod_{j=1}^{n-2} (1 - z\overline{z}_j),$$

and it is easy to compute that

$$Q(z)Q^*(z) = \frac{\lambda^2}{|z_n|^2}XX^*(z).$$

Consequently, the choice $\lambda = |z_n|$ lead us to required equality $QQ^* = XX^*$ and the lemma is proved.

Now the main result of this section follows from (6) and previous lemmas:

**Theorem 1.** The set $K_n$ of nonnegative polynomials $Y$ is a closed cone in $\mathbb{R}^{2n+1}$ (with nonempty interior) which boundary $\partial K_n$ is contained in the algebraic hypersurface $\text{Dis}_2(Y) = 0$. 
We emphasize that the theorem gives the sharp description of $K_n$ because the boundary $K_n$ turns out to be a part of the irreducible hypersurface (see the next section).

**Example 1.** Let $n = 1$ and we use the previous notations. Then it is easy to compute that $V(X) \equiv V(x_0, x_1) = x_0^2 - |x_1|^2$. In this case we have the very similar expression $\text{Dis}_2(Y) \equiv \text{Dis}_2(y_0, y_1) = 4(y_0^2 - |y_1|^2)$.

**Example 2.** In the two-dimensional case $n = 2$ we have

$$V(x_0, x_1, x_2) = (x_0^2 - |x_1|^2)^2 - (x_0x_1 - x_1x_2)(x_0x_1 - x_2x_1),$$

and

$$\text{Dis}_2(Y) = 36y_1^3y_0y_2 - 320y_2y_1y_1y_2^2 - 4y_1^3y_1^3 + 256y_2y_0y_1^2 - 32y_2y_0y_1^2 - 27y_1y_2^2 - 512y_2^2y_0^2 + 288y_2y_0y_2^2 + 36y_2y_1y_1y_0 + 4y_0^2y_0^2y_2 - 32y_2y_0y_1^2 - 192y_2y_2y_0y_1^2 - 6y_2y_2y_0^2y_1^2 + 288y_2y_0y_2^2 + 256y_2y_2^2 - 27y_2y_1^2.

**2. Irreducibility**

**Theorem 2.** The form $\text{Dis}_2(Y) = \text{Dis}_2(y_0, y_1, \overline{y}_1, \ldots, y_n, \overline{y}_n)$ as well as its real representative $\text{Dis}_{2, \mathbb{R}}(a_0, a_1, b_1, \ldots, a_n, b_n)$ are irreducible over $\mathbb{C}$.

The following auxiliary assertion seems to be known but we can not arrange any citation on it in the wide literature and because we give its proof for completeness.

**Lemma 4.** The form $\text{Dis}(p_0, p_1, \ldots, p_m) \equiv \text{Dis}(P)$ is irreducible in $\mathbb{C}[p_0, p_1, \ldots, p_m]$.

**Proof.** Let us suppose that there exists nontrivial factorization

$$\text{Dis}(P) = A(P)B(P)$$

where $A(P)$ and $B(P)$ are different from constants. Because $\text{Dis}(P)$ is homogeneous then $A(P)$ and $B(P)$ are homogeneous too. Moreover, $\min\{\deg_p A, \deg_p B\} \geq 1$, where $\deg_p$ is the degree with respect to $\mathbb{C}[p_0, p_1, \ldots, p_m]$.

We consider factorization $P(z) = p_m \prod_{k=1}^m (z - z_k)$ where $z_k$ are the roots of $P(z)$ with their multiplicity. Then every fraction $p_k/p_m$, $k = 1, \ldots, m - 1$ is the $k$-th elementary symmetric function of $z_k$. Other words,

$$(-1)^{k-1} \frac{p_k}{p_m} = \sum_{1 \leq i_1 < \cdots < i_k \leq m} z_{i_1} \cdots z_{i_k} \equiv \sigma_k(z_1, \ldots, z_m),$$

(19)
and it follows from (19) that

\[ A(P) = p_m^{\deg p} A(\sigma_1, \ldots, \sigma_{m-1}, 1) \equiv p_m^{\deg p} A_1(z_1, \ldots, z_m), \]

\[ B(P) = p_m^{\deg p} B(\sigma_1, \ldots, \sigma_{m-1}, 1) \equiv p_m^{\deg p} B_1(z_1, \ldots, z_m). \]

(20)

Obviously, both of \( A_1(z_1, \ldots, z_m) \) and \( B_1(z_1, \ldots, z_m) \) are symmetric polynomials by their appearance. On the other hand, \( \deg p \text{Di} = \deg p A + \deg p B \) and it follows

\[ \text{Di}(P) = p_m^{2n-2} \prod_{m \geq i > j \geq 1} (z_i - z_j)^2. \]

(21)

Hence we have from (20) and (21) that

\[ \prod_{m \geq i > j \geq 1} (z_i - z_j)^2 = A_1(z_1, \ldots, z_m)B_1(z_1, \ldots, z_m). \]

(22)

But the left-hand side of (22) does not admit any nontrivial decomposition in \( \mathbb{C}[z_1, \ldots, z_m] \) on two symmetric polynomial multipliers (see [14, Chapt. V]).

It follows that \( A_1(z_1, \ldots, z_m) \) (or \( B_1(z_1, \ldots, z_m) \)) is a constant. This can occur if and only if \( A(P) = c_1 p_m^{\deg p} \) for some \( c_1 \neq 0 \). But this means that \( p_m \) divides \( \text{Di}(P) \), and, in particular,

\[ \text{Di}(p_0, p_1, \ldots, p_{m-1}, 0) \equiv 0. \]

But the last property is not the case because by the elementary discriminant property

\[ \text{Di}(p_0, p_1, \ldots, p_{m-1}, 0) = \pm p_m^{2} \text{Di}(p_0, p_1, \ldots, p_{m-1}) \neq 0. \]

The contradiction obtained proves the lemma.

\[ \square \]

**Proof of Theorem 2.** First we notice that the forms are irreducible or not simultaneously because there exists \( \mathbb{C} \)-linear isomorphism between \( \{y_0, \ldots, y_n\} \) and \( \{a_0, a_1, b_1, \ldots, b_n\} \). So it sufficient to check irreducibility of the first form only.

We notice that \( \text{Di}_2(Y) = \text{Di}(\overline{y}_n, \overline{y}_{n-1}, \ldots, \overline{y}_1, 2y_0, y_1, \ldots, y_n) \) and it follows from previous lemma that \( \text{Di}(Y) \) is irreducible in \( \mathbb{C}[y_0, y_1, \ldots, y_n, \overline{y}_1, \ldots, \overline{y}_n] \), as well as \( \text{Di}_2(Y) \) and the theorem is proved.

\[ \square \]

**References**

[1] Akhiezer N.I., The classical moment problem. Oliver & Boyd, London, 1965.

[2] Arestov V. V., On extremal properties of the nonnegative trigonometric polynomials // Trans. Inst. Mat. Mekh. (Ekaterinburg), 1992. V. 1. C. 50-70.

[3] Arestov V. V., Kondrat’ev V. P., On an extremal problem for nonnegative trigonometric polynomials // Mat. Zametki, 1990. V. 47. C. 15-28.

[4] Askey R., Gasper G., Inequalities for polynomials // In: The Bieberbach Conjecture: Proceedings of the Symposium on the Occasion of Proof (A. Baernstein II et al., eds.). 1986. Providence, RI: American Mathematical Society, C. 7-32.

[5] Assche W. Van, Orthogonal polynomials in the complex plane and on the real line // Fields Inst. Comm., 1997. V. 14. C. 211-245.

[6] Brannan A., Coefficient regions for univalent polynomials of small degree // Mathematika. 1967, V. 14. C. 165-169.

[7] Brannan A., On univalent polynomials // Glasgow Math. J., 1970. V. 11, C. 102-107.
[8] Brannan A., Brickman L., Coefficient regions for starlike polynomials // Ann. Univ. Mariae-Curie Składowska, Sect. A. 1975. V. 29, C. 15-21 (1977).
[9] Burnside W.S., Panton A.W., Theory of equations. Vol. 1. 6th. Edt. Dublin Univ. Press, 1916.
[10] Dimitrov D.K., Merlo C.A., Nonnegative trigonometric polynomials // Constr. Approx. 2002. V. 18. C. 117-143.
[11] Gelfand I.M., Kapranov M.M., Zelevinsky A.V., Descriminants, resultants and multidimensional determinants. Birkhäuser. Boston, 1994.
[12] Gluchoff A., Hartmann F., Univalent polynomials and non-negative trigonometric sums // Amer. Math. Month. 1998
[13] Fejér L., Uber trigonometrice Polynome // J. Reine Angew. Math., 1915. V. 146. C. 53-82.
[14] Littlewood D.E., Heinmann W., A university algebra. Melbourn - London - Toronto, 1950.
[15] Quine J.R., On univalent polynomials // Proc. Amer. Math. Soc. 1976. V. 57, N. 1. C. 75-78.
[16] Praslov V.V., Polynomials. 2nd Edit. Moscow, Nauka, 2001.
[17] Van Der Warden B.L., Modern algebra. Vol. 1. Springer. Berlin, 1930.

Volgograd State University

E-mail address: Vladimir.Tkachev@volsu.ru