A connection is made between the random turns model of vicious walkers and random permutations indexed by their increasing subsequences. Consequently the scaled distribution of the maximum displacements in a particular asymmetric version of the model can be determined to be the same as the scaled distribution of the eigenvalues at the soft edge of the GUE. The scaling of the distribution gives the maximum mean displacement $\mu$ after $t$ time steps as $\mu = (2t)^{1/2}$ with standard deviation proportional to $\mu^{1/3}$. The exponent $1/3$ is typical of a large class of two-dimensional growth problems.

Non-intersecting (vicious) random walkers were introduced into statistical mechanics [6] as models of domain walls and wetting in two-dimensional lattice systems, and have also received attention as exactly solvable systems [7, 8, 9, 4, 3]. They can be viewed as directed lattice paths which start at sites say on the $x$-axis and finish on sites on the line $y = n$, with the additional constraint that the paths do not touch or overlap. Alternatively, vicious random walkers can be described as the stochastic evolution of particles on a one-dimensional lattice, which at each tick of the clock move to the left or to the right with a certain probability, subject to the constraint that no two particles can occupy the one lattice site.

Our interest is in the random turns vicious walker model [6, 9]. Here, in the stochastic evolution picture, at each time step $t$ ($t = 1, 2, \ldots$) one particle is selected at random and moved one lattice site to the left with probability $w_{-1}$, or one lattice site to the right with probability $w_1$ ($w_{-1} + w_1 = 1$). However, if the lattice site to the left (right) is already occupied, then the chosen walker moves to the right (left) with probability one unless this lattice site too is occupied. In the latter situation another walker is selected at random and the procedure repeated until one walker has been moved. The move of precisely one walker so determines the state at time interval $t$. An example of some typical configurations in the directed paths picture is given in Figure 1. We remark that the random turns vicious walker model can also be regarded as a particular asymmetric exclusion process [17].

Two aspects of the theory of the random turns vicious walker model are the subject of this Letter. The first concerns the number of configurations that have the $p$ walkers initially on adjacent lattice sites ($l = 1, \ldots, p$) on the one-dimensional line, and have the walkers returned to the same sites after $2n$ time steps (for an odd number of time steps this is not possible – thus the reason for $2n$). This will be shown to be simply related to the number of permutations of $\{1, 2, \ldots, n\}$ such that the maximum increasing subsequence has length no greater than $p$. Then known results concerning the distribution of the maximum increasing subsequence of a random permutation will be used to determine the distribution of the maximum displacement of the walkers starting near the $l = 1$ boundary of the initial positions for a variant of the random turns model. In this variant the number of walkers is greater than or equal to $n$, and all walkers must move to the left for time steps up to $n$, while they must move move to the right thereafter, returning to their starting points at time $2n$.

To count the configurations, label the particles by their initial location of the one-dimensional line $l = 1, \ldots, p$. At each time step one walker will move one lattice site to the left ($L$) or right ($R$), subject to the rule that no two particles can occupy the same lattice site (in the counting
Figure 1: A particular configuration of 3 random turns walkers performing 8 steps in the sequence $L^4R^4$. The walk can conveniently be represented diagramatically as done at right according to the rules specified in the text.

problem we take $w_1 = w_{-1}$). The constraint that the particles return to their initial positions after $2n$ steps requires that for each walker the number of steps to the left equals to the number of steps to the right after $2n$ steps, and that in total there are $n$ steps $L$ and $n$ steps $R$. This latter fact allows the walks to be partitioned according to the ordering of $L$’s and $R$’s, of which there are $\binom{2n}{n}$ possibilities. The simplest of these is $L^nR^n$, which means the first $n$ steps are all to the left, while the final $n$ steps are all to the right. We now pose the question: for a given combination of $\{L^n, R^n\}$ corresponding to a particular ordering of $L, R$ steps, how many distinct configurations of the $p$ random turns walkers are there?

Consider the particular ordering $L^nR^n$. We will represent each configuration diagramatically by placing a square marked $t$ in column $k$ ($k = 1, \ldots, p$) if walker $k$ moves at time step $t$. This square must be placed above any other squares in the column if the step is $L$, and below any other squares in the column if the step is $R$. If there are no other squares in the column (i.e. the walker $k$ is making its first move) then the square is placed immediately above the axis if the step is $L$, or immediately below the axis if the step is $R$. An example of such a diagramatical representation is given in Figure 1.

The problem now is to determine the number of distinct diagrams. Consider the region of the diagram above the axis. Here each column, of which there are no more than $p$, must be weakly decreasing in height. Each of the $n$ squares must be labelled by a different integer $1, \ldots, n$ with the numbers strictly increasing along rows and up columns. We recognize such a diagram as equivalent to a standard Young tableau (see e.g. [11]; the Young tableau results by interchanging rows and columns). Thus there is a bijection between the number of walks following the sequence $L^n$ and standard Young tableaux with entries $1, \ldots, n$ and no more than $p$ rows. We remark that this is not the first time a bijection between Young tableaux and vicious walker problems has been observed: in [12] a bijection between semi-standard tableaux and the configurations in a special case of the lock-step random walker model [6, 8] was identified.

Regarding the last $n$ walks, because each walker must return to its starting position after the $2n$ steps, the shape of the diagram below the $x$-axis must be the mirror image of the shape of the diagram below the $x$-axis. Furthermore, by pushing each column $k = 2, \ldots, p$ of this downwards so that they all align with the first entry of the first column, and then relabelling each box by $t \mapsto 2n + 1 - t$ we see that another standard Young tableau results. (This procedure is equivalent to constructing this diagram starting with the final step of the walk and working backwards.) These transformations are indicated in Figure 1.
Thus configurations of the \( p \) random turns walkers under consideration are in bijective correspondence with pairs of standard tableaux of \( n \)-boxes constrained so that there are no more than \( p \) rows (recall the role of rows and columns in Young tableaux and our diagrams is reversed). But such pairs of standard Young tableaux are well known (see e.g. [11]) to be in bijective correspondence with permutations of \( \{1, \ldots, n\} \) such that the length of each increasing subsequence is less than or equal to \( p \). Hence we have enumerated the number of walks in terms of such permutations.

**Proposition 1** Consider \( p \) random turns walkers, initially equally spaced one unit apart and returning to their initial position after \( 2n \) steps. Suppose the walkers make their steps in the sequence \( L^nR^n \). The total number of distinct configurations equals the number of permutations of \( \{1, \ldots, n\} \) such that the length of the maximum increasing subsequence is less than or equal to \( p \).

Consider now another sequence of \( n \) \( L \)'s and \( n \) \( R \)'s. This sequence can be transformed into the sequence \( L^nR^n \) by elementary transpositions \( s_i \) which interchange the \( i \)th and \((i+1)\)th members of the sequence, assumed to be \( R \) and \( L \) respectively. Likewise, we can define the corresponding action of \( s_i \) on a diagram (or equivalently the lattice paths) and so obtain a bijection between distinct configurations with walks following the sequence \( L^nR^n \), and distinct configurations with walks following some combination of the sequence \( L^nR^n \). We assume step \( i \) is opposite in direction to step \( i+1 \), and defined the action of \( s_i \) to first interchange the position of boxes \( i \) and \( i+1 \). At the level of the lattice paths this has the action depicted in Figure 3.

If the two boxes are in the same column, it may happen that the resulting lattice path is inadmissable, in that the new (local) configuration intersects with an existing path (note that this cannot happen in the first two cases of Figure 3). In such a circumstance we move the configuration to the left (right) in the second last (last) situation of Figure 3 until a permissable configuration is obtained (after so moving the left-right pairs two vertical lines, corresponding to a stationary walker, take its place). This is illustrated in Figure 3. Notice that in all cases \( s_i^2 = 1 \), so the procedure is invertable. Furthermore, the braid relations \( s_is_{i+1}s_is_{i+1}s_i = s_{i+1}s_is_{i+1}s_i \) are satisfied so the correspondence is independent of the order of application of the transpositions.

Let’s now describe this aspect of the definition of the action of \( s_i \) at the level of the diagram. For this purpose we must specify an admissable diagram. From the rules of the random turns model we see that an admissable diagram is constructed by putting boxes in the columns 1 to \( p \) subject to the conditions that box \( i \) must go above (below) the axis if the step is \( L \) (\( R \)), and that for each column the number of boxes above the axis minus the number below must be greater than or equal to the same quantity for the column on the right. Some examples of such diagrams and the corresponding combinations of \( \{L^n, R^n\} \) are given in Figure 3. So we want to describe the situation in which boxes \( i \) and \( i+1 \) are interchanged but an inadmissable diagram results. According to Figure 3 both boxes should be shifted to the left (right) if the new position of box \( i \) is below (above) the axis until an admissable diagram results. Also we adopt the convention that numbers in each column should be increasing above and below the axis, so we rearrange the boxes within a column above and below the axis accordingly.
Figure 3: In the first two cases the distinct walkers correspond to the columns of boxes \( i \) and \( i + 1 \), while in the last two cases boxes \( i \) and \( i + 1 \) are in the same column, and the resulting configuration assumed admissable

\[
\begin{align*}
S_i \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) &= \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) \\
S_i \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) &= \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) \\
S_i \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) &= \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) \\
S_i \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) &= \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right)
\end{align*}
\]

Figure 4: Here both box \( i \) and \( i + 1 \) are in the same column and the corresponding walker configurations going from step \( i \) to step \( i + 1 \) are as indicated. Because the rule of Figure 3 would lead to inadmissable configurations in each case, \( s_i \) acts by propagating the new configuration to the left and right in the two cases respectively until an admissable configuration is obtained.

\[
\begin{align*}
S_i \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) &= \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) \\
S_i \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right) &= \left( \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\end{array} \right)
\end{align*}
\]

Figure 5: An example of the action of the elementary transposition operator on some permissible diagrams in the case that both boxes are from the same column with the situation of Figure 3 applying. The corresponding sequences of \( L \)'s and \( R \)'s are also given.

\[
\begin{align*}
S_2 \left( \begin{array}{cccc}
\begin{array}{cccc}
5 & 1 & 2 & 8 \\
4 & 3 & 7 & 6 \\
\end{array}
\end{array} \right) &= \left( \begin{array}{cccc}
\begin{array}{cccc}
5 & 8 & 1 & 3 \\
4 & 2 & 6 & 7 \\
\end{array}
\end{array} \right) \\
S_4 \left( \begin{array}{cccc}
\begin{array}{cccc}
5 & 1 & 2 & 7 \\
8 & 4 & 3 & 6 \\
\end{array}
\end{array} \right) &= \left( \begin{array}{cccc}
\begin{array}{cccc}
4 & 1 & 2 & 7 \\
5 & 6 & 3 & 8 \\
\end{array}
\end{array} \right)
\end{align*}
\]

\[
\begin{align*}
S_2 \left( LRLRLRL \right) &= LRLRLRL \\
S_4 \left( LRLRLRL \right) &= LRLRLRL
\end{align*}
\]
The action of the elementary transpositions thus described give a bijection between lattice paths with steps in the sequence $L^n R^n$, and lattice paths with steps in a sequence of any particular combination of $\{L^n, R^n\}$. Thus by making use of Proposition 1 we can solve the enumeration problem for all such walks.

**Proposition 2** The result of Proposition 1 for the number of walks in the sequence $L^n R^n$ also applies for any combination of $\{L^n, R^n\}$.

It is of interest to note that there are multiple integral formulas for both the number of random permutations of $\{1, \ldots, n\}$ with at most $p$ increasing subsequences, and the total number of random turns paths with $p$ walkers starting at sites $l'_1, \ldots, l'_p$ and finishing at sites $l_1, \ldots, l_p$ in $2n$ steps. Let us denote these numbers by $f_{np}$ and $Z_{2n}(l'_1, \ldots, l'_p; l_1, \ldots, l_p)$ respectively. Then we have

\[
f_{np} = \frac{(n!)^2}{(2n)!} \frac{1}{p!} \frac{1}{(2\pi)^p} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_p \left( \sum_{j=1}^{p} 2 \cos \theta_j \right)^{2n} \prod_{1 \leq \alpha < \beta \leq p} |e^{i\theta_\alpha} - e^{i\theta_\beta}|^2
\]

and

\[
Z_{2n}(l'_1, \ldots, l'_p; l_1, \ldots, l_p) = \frac{1}{(2\pi)^p} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_p \left( \sum_{j=1}^{p} 2 \cos \theta_j \right)^{2n} \det \left[ e^{-i(l_\alpha - l'_\beta)\theta_\alpha} \right]_{\alpha, \beta = 1, \ldots, p}
\]

Let us specialize (3) to the situation of Propositions 1 and 2 by choosing $l_j = l'_j = j$ $(j = 1, \ldots, p)$. Now

\[
det[e^{-i(\alpha - \beta)\theta_\alpha}]_{\alpha, \beta = 1, \ldots, p} = \prod_{j=1}^{p} e^{-i(j-1)\theta_j} \det[e^{i(\beta-1)\theta_\alpha}]_{\alpha, \beta = 1, \ldots, p} = \prod_{j=1}^{p} e^{-i(j-1)\theta_j} \prod_{1 \leq \alpha < \beta \leq p} (e^{i\theta_\beta} - e^{i\theta_\alpha}),
\]

where the final equality follows from the Vandermonde formula. Since the product of differences is antisymmetric the non-symmetric factor in the integrand $\prod_{j=1}^{p} e^{-i(j-1)\theta_j}$ can be antisymmetrized, giving another Vandermonde product, provided we divide by $p!$. Thus we have

\[
Z_{2n}(\{l'_j = j\}_{j=1,\ldots,p}; \{l_k = k\}_{k=1,\ldots,p}) = \frac{1}{p!} \frac{1}{(2\pi)^p} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_p \left( \sum_{j=1}^{p} 2 \cos \theta_j \right)^{2n} \prod_{1 \leq \alpha < \beta \leq p} |e^{i\theta_\beta} - e^{i\theta_\alpha}|^2.
\]

Comparing (1) and (3) gives

\[
Z_{2n}(\{l'_j = j\}_{j=1,\ldots,p}; \{l_k = k\}_{k=1,\ldots,p}) = \binom{2n}{n} f_{np}
\]

which is of course also an immediate corollary of Proposition 1 and 2. However once having deduced (1), the formula (1) for $f_{np}$ follows as a special case of (3).

We know from the derivation of Proposition 1 that there is a bijection between configurations of $p$ random turns walkers performing $2n$ steps in the sequence $L^n R^n$ before returning to their initial positions of all one unit apart, and pairs of standard tableaux each of the same shape and consisting of $n$ boxes. Furthermore, in the bijection the length of row $j$ corresponds to the
maximum displacement of walker \( j \) to the left of its starting point (this occurs at step \( n \)). Thus if we choose \( p \geq n \) this same bijection holds but with no restriction on the number of rows. In such a situation the asymptotics of the row lengths are known precisely [1, 18, 2, 14]. We can therefore give these results an interpretation in the random walker setting.

**Proposition 3** Denote the positions of the walkers on the one-dimensional line in the above asymmetric random turns model by \( l_j \) where initially walker \( j \) is at position \( l_j = j \). Define the scaled displacements by

\[
\tilde{l}_j := n^{1/3} \left( \frac{l_j}{n^{1/2}} - 2 \right)
\]

and the corresponding scaled \( k \)-point distribution function by

\[
\rho_k(\tilde{l}_1, \ldots, \tilde{l}_k) := \lim_{n \to \infty} \left( \frac{1}{n^{1/6}} \right)^k \rho_k^{(n)}(\tilde{l}_1, \ldots, \tilde{l}_k)
\]

where \( \rho_k^{(n)} \) denotes the \( k \)-point distribution for the walker problem in the finite system. Then from the results of [18, 2, 14] for the tableau problem we have

\[
\rho_k(\tilde{l}_1, \ldots, \tilde{l}_k) = \det \left[ K(\tilde{l}_\alpha, \tilde{l}_\beta) \right]_{\alpha,\beta=1,\ldots,k}
\]

where

\[
K(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}
\]

The distribution (6) is precisely the scaled distribution of the eigenvalues at the edge of the spectrum for GUE random matrices [10, 20]. Perhaps more relevantly to the random walker problem, (5) coincides with the scaled distribution for free fermions on a line confined by a one-body harmonic potential, at the edge of the support of the density. The relevance is that there is a well known relationship between continuous models of non-intersecting walkers and free fermions (see e.g. [2]).

Regarding some physical features of Proposition 3, note from (5) that the average displacement is \( \mu = 2n^{1/2} \) (for a recent independent proof of this result see [13]), with standard deviation proportional to \( (4n)^{1/6} = \mu^\chi \), \( \chi = 1/3 \). As emphasized in [13], the exponent \( \chi = 1/3 \) is typical of two-dimensional growth models (it can be derived from the one-dimensional Burgers equation describing such processes [21]). On this point we recall that vicious walker paths fixed at the endpoints as in Proposition 3 form the well-known (see e.g. [13]) terrace-step-kink model of a crystal surface.

**Acknowledgement**

The financial support of the ARC, including funds to support the visit of G. Olshanski whose lectures benefitted the present work, are acknowledged. Also, the remarks of T.H. Baker on the original manuscript are appreciated.

**References**

[1] J. Baik, P. Dieft, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. [math.CO/9901118] 1998.
[2] A. Borodin, A. Okounkov, and G. Olshanski. Asymptotics of Plancherel measures for symmetric groups. preprint, 1999.

[3] R. Brak and A.L. Owczarek. A combinatorial interpretation of the free-fermion condition of the six vertex model. *J. Phys. A*, 32:3497–3503, 1999.

[4] R. Brak and A.L. Owczarek. Exact solution of $n$-directed non-intersecting walks interacting with one or two boundaries. *J. Phys. A*, 32:2921–2929, 1999.

[5] M. den Nijs. In C. Domb and J.L. Lebowitz, editors, *Phase Transitions and Critical Phenomena*, volume 12, page 219. Springer, Orlando, 1988.

[6] M.E. Fisher. Walks, walls and wetting. *J. Stat. Phys.*, 34:669, 1984.

[7] P.J. Forrester. Probability of survival for vicious walkers near a cliff. *J. Phys. A*, 22:L609–L613, 1989.

[8] P.J. Forrester. Exact solution of the lockstep model of vicious walkers. *J. Phys. A*, 23:1259–1273, 1990.

[9] P.J. Forrester. Exact results for vicious walker models of domain walls. *J. Phys. A*, 24:203–218, 1991.

[10] P.J. Forrester. The spectrum edge of random matrix ensembles. *Nucl. Phys. B*, 402:709–728, 1993.

[11] W. Fulton. *Young Tableaux*. London Mathematical Society Student Texts. CUP, Cambridge, 1997.

[12] A.J. Guttmann, A.L. Owczarek, and X.G. Viennot. Vicious walkers and Young tableaux I: without walls. *J. Phys. A*, 31:8123–8135, 1998.

[13] K. Johansson. The longest increasing subsequence in a random permutation and a unitary random matrix model. *Math. Research Lett.*, 5:63–82, 1998.

[14] K. Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *math.CO/9906120*, 1999.

[15] K. Johansson. Shape fluctuations and random matrices. *math.CO/9903134*, 1999.

[16] B. Joós, T.L. Einstein, and N.C. Bartelt. Distribution of terrace widths on a vicinal surface within the one-dimensional free-fermion model. *Phys. Rev. B*, 43:81, 1991.

[17] T.M. Liggett. *Interacting Particle Systems*. Springer-Verlag, New York, 1985.

[18] A. Okounkov. Random matrices and random permutations. *math.CO/9903170*, 1999.

[19] E.M. Rains. Increasing subsequences and the classical groups. *Elect. J. of Combinatorics*, 5:#R12, 1998.

[20] C.A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Commun. Math. Phys.*, 159:151–174, 1994.

[21] H. van Beijeren. Fluctuations in the motions of mass and of patterns in one-dimensional driven diffusion equations. *J. Stat. Phys.*, 63:47–57, 1991.