Coarse compactifications of proper metric spaces

Elisa Hartmann*

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Abstract

This paper studies coarse compactifications and their boundary. We introduce two alternative descriptions to Roe’s original definition of coarse compactification. One approach uses bounded functions on $X$ that can be extended to the boundary. They satisfy the Higson property exactly when the compactification is coarse. The other approach defines a relation on subsets of $X$ which tells when two subsets closure meet on the boundary. A set of axioms characterizes when this relation defines a coarse compactification. Such a relation is called large-scale proximity.

Based on this foundational work we study examples for coarse compactifications Higson compactification, Freudenthal compactification and Gromov compactification. For each example we characterize the bounded functions which can be extended to the coarse compactification and the corresponding large-scale proximity relation.

We provide an alternative proof for the property that the Higson compactification is universal among coarse compactifications. Furthermore the Freudenthal compactification is universal among coarse compactifications with totally disconnected boundary. If $X$ is hyperbolic geodesic proper then there is a closed embedding $\nu(R_+) \times \partial X \to \nu(X)$. Its image is a retract of $\nu(X)$ if $X$ is a tree.

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*Department of Mathematics, Karlsruhe Institute of Technology
0 Introduction

This paper studies a class of compactifications of proper metric spaces which contains the Higson compactification, Gromov compactification and a coarse version of the Freudenthal compactification. With every such space $X$ one can associate a coarse structure $\mathcal{C}_p$. In [Roe03] Roe introduced a class of compactifications which are compatible with this structure. We provide an equivalent definition:

**Definition 1.** Let $X$ be a proper metric space and $\bar{X}$ a compactification of $X$. Then $\bar{X}$ is a coarse compactification if for every two nets $(x_i), (y_i) \subseteq X$, such that $(x_i, y_i)$ is an entourage in $X$, both nets have the same limit points on the boundary.

Instances of coarse compactifications have been studied by many authors, see e.g. [MY15, Kee94, Pro19, FS03, Pro11, PS15, Pro05, KB02, KH16, KH15, Cor19]. In this paper we give two new descriptions of coarse compactifications which are equivalent to the original one. One of the two equivalent definition starts with a relation on subsets of $X$, the other definition uses bounded functions on $X$ which can be extended to the compactification.

Every coarse compactification $\bar{X}$ of a proper metric space $X$ gives rise to a relation $r_{\bar{X}}$ on subsets of $X$ which tells when the closure of two sets meet on the boundary. Specifically denote by $\partial X$ the boundary $\bar{X} \setminus X$. If $A, B \subseteq X$ are subsets then

$$Ar_{\bar{X}}B \iff (\bar{A} \cap \partial X) \cap (\bar{B} \cap \partial X) \neq \emptyset.$$ 

A set of axioms tells if a relation on subsets of $X$ comes from a coarse compactification. Such a relation is then called large-scale proximity in the sense of the following definition.

**Definition 2.** If $X$ is a proper metric space a relation $r$ on subsets of $X$ is called large-scale proximity if

1. if $B \subseteq X$ then $BrB$ if and only if $B$ is bounded;
2. $ArB$ implies $BrA$ for every $A, B \subseteq X$;
3. if $A, A' \subseteq X$ are subsets and $E \subseteq X^2$ is an entourage with $E[A] \supseteq A', E[A'] \supseteq A$ then $ArB$ implies $A'rB$ for every $B \subseteq X$;
4. if $A, B, C \subseteq X$ then $(A \cup B)rC$ if and only if $(ArC$ or $BrC)$;
5. if $A, B \subseteq X$ with $ArB$ then there exist $C, D \subseteq X$ with $C \cup D = X$ and $C\overline{r}A, D\overline{r}B$.

Conversely given a large-scale proximity relation $r$ we can construct a coarse compactification $\bar{X}'$ which induces this relation on subsets of $X$. This is done in Definitions 23,29.

An entirely different approach characterizes coarse compactifications via the $C^*$-algebra $C_p(X)$ of bounded continuous functions on $X$ that can be extended to the boundary of the compactification. Every set of bounded continuous functions $A$ on $X$ generates the smallest compactification $\bar{X}^A\mathcal{A}$ such that functions in $A$ can be extended to the boundary. More specifically we introduce a property on bounded continuous functions:

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$^4$See Definition 0 which defines a coarse structure given a metric space
**Definition 3.** A bounded continuous function $\varphi : X \to \mathbb{R}$ is called *Higson* if for every entourage $E \subseteq X^2$ the map

$$d\varphi|_E : E \to \mathbb{R}, \quad (x, y) \mapsto \varphi(x) - \varphi(y)$$

vanishes at infinity.

Given a compactification $\tilde{X}$ denote by $C_{\tilde{X}}(X)$ the algebra of bounded continuous functions on $X$ that can be extended to $\tilde{X}$. They must be Higson if $\tilde{X}$ is coarse. Conversely if the algebra of bounded functions on $X$ that can be extended to the boundary of the compactification consists of Higson functions then $\tilde{X}$ is coarse.

We summarize those results in the following theorem:

**Theorem A.** If $X$ is a proper metric space and $\tilde{X}$ a compactification of $X$ the following statements are equivalent:

- The compactification $\tilde{X}$ is coarse.
- The relation $r_{\tilde{X}}$ on subsets of $X$ is a large-scale proximity relation.
- Every function in $C_{\tilde{X}}(X)$ is Higson.

Moreover given a large-scale proximity relation $r$ the compactification $\tilde{X}^r$ is coarse. If all functions in an algebra of bounded functions $\mathcal{A}$ on $X$ are Higson they generate a coarse compactification $\tilde{X}^\mathcal{A}$.

In Theorem 17 we translate Roe’s original definition of coarse compactification to a definition which is more suitable for us. The equivalence of statements 1, 2 in Theorem A is shown in Theorem 36. The equivalence of statements 1, 3 in Theorem A is shown in Theorem 37.

We investigate three specific examples: The Higson compactification, the Freudenthal compactification and the Gromov compactification.

**Example 4.** The Higson compactification $hX = \nu(X) \cup X$ of a proper metric space $X$ is characterized by the following large-scale proximity relation: Two subsets $A, B \subseteq X$ are called close, written $A \prec B$, if there exists an unbounded sequence $(a_i, b_i) \subseteq A \times B$ and some $R \geq 0$ such that $d(a_i, b_i) \leq R$ for every $i$.

Every Higson function on $X$ can be extended to the Higson corona $\nu(X)$.

**Example 5.** The Freudenthal compactification $\varepsilon X = \Omega X \cup X$ of a proper metric space $X$ is characterized by the following large-scale proximity relation $\prec_f$: Two subsets $A, B \subseteq X$ don’t have a same end, written $A \not\prec_f B$, if there exist $A' \supseteq A, B' \supseteq B$ with $A' \cup B' = X$ and $A' \not\prec_f B'$.

Let $x_0 \in X$ be a basepoint. A bounded continuous map $\varphi : X \to \mathbb{R}$ is called Freudenthal if for every $R \geq 0$ there exists $K \geq 0$ such that $d(x, y) \leq R, d(x_0, x) \geq K, d(x_0, y) \geq K$ implies $\varphi(x) = \varphi(y)$. We write $C_f(X)$ for the ring of Freudenthal functions on $X$. Every bounded function that can be extended to the Freudenthal compactification is Freudenthal and every Freudenthal function can be extended to the boundary of the Freudenthal compactification.

**Example 6.** If a metric space $X$ is hyperbolic, proper the Gromov compactification $\tilde{X} = \partial X \cup X$ is defined. The associated large-scale proximity relation $\prec_g$ is defined by $A \not\prec_g B$ if there are sequences $(a_i) \subseteq A, (b_i) \subseteq B$ such that

$$\lim_{i,j \to \infty} \inf(a_i | b_j) = \infty.$$
If $X$ is hyperbolic a continuous function $\varphi : X \to \mathbb{R}$ is called Gromov if for every $\varepsilon > 0$ there exists $K > 0$ such that

$$(x|y) > K \implies |\varphi(x) - \varphi(y)| < \varepsilon.$$ 

Every Gromov function can be extended to the Gromov boundary and every function that can be extended to the Gromov boundary is Gromov.

**Remark 7.** We establish functoriality in the following way. In Proposition 48 we show the association of a coarse compactification is in a way contravariant on coarse maps. We can always pull back a coarse compactification along a coarse map. The reverse direction push out is not always possible. We can glue coarse compactifications along a coarse cover though which is done in Proposition 39. Then Lemma 41 shows the poset of coarse compactifications is a sheaf on the Grothendieck topology of coarse covers on $X$.

We now describe the results on the specific examples in detail. In particular the boundary of the Higson compactification retains information about the coarse structure since the Higson corona is a faithful functor [Har19b]. This way not much information is lost if we restrict our attention to the boundary of a coarse compactification when studying coarse metric spaces. The Higson corona $\nu(X)$ of $X$ is connected if $X$ is one-ended. Aside from that and from being compact and Hausdorff the Higson corona does not have many nice property. The topology of the Higson corona does not have a countable base and is in fact is never metrizable [Roe03]. We provide a new proof that the Higson corona is universal among coarse compactifications. The original result can be found in [Roe03].

**Theorem B.** *(Roe)* If $X$ is a proper metric space the Higson compactification of $X$ is universal among coarse compactifications of $X$. This means a compactification of $X$ is a coarse compactification if and only if it is a quotient of the Higson compactification, where the quotient map restricts to the identity on $X$.

This in particular implies that the boundary of every coarse compactification of $X$ is connected if $X$ is one-ended.

The space of ends of a topological space is well known and dates back to Freudenthal’s works [Fre31, Hop44, Fre45]. We construct a version of Freudenthal compactification on coarse proper metric spaces given both descriptions via a large-scale proximity relation and via bounded functions. The Proposition 49 shows both the topological and the coarse version of Freudenthal compactification agree on proper geodesic metric spaces. The space of ends gives information about the number of ends of a coarse metric space [Har17]. It is both metrizable and totally disconnected.

For the space of ends we can obtain a similar result as in the case of the Higson corona regarding universality.

**Theorem C.** If $X$ is a proper metric space the boundary of the Freudenthal compactification $\Omega X = \varepsilon X \setminus X$ of $X$ is totally disconnected. If $(\bar{X}, X)$ is another coarse compactification whose boundary is totally disconnected then it factors through $\varepsilon X$. This means there is a surjective map $\varepsilon X \to \bar{X}$ which is continuous on the boundary and the identity on $X$.

The topology of the Gromov compactification is metrizable [KB02]. The usual description of its boundary is via geodesic rays or sequences that converge to infinity. We investigate in which way the Higson corona can be recovered from the Gromov boundary.

**Theorem D.** Let $X$ be a hyperbolic geodesic proper metric space. Then there is a closed embedding $\Phi : \nu(\mathbb{Z}_+) \times \partial X \to \nu(X)$. The image of $\Phi$ is a retract if $X$ is a tree.
All three instances of coarse compactification are functorial in that the boundary is a coarse invariant. The Higson corona and space of ends are even functors on coarse maps. This way we associate compact Hausdorff spaces to coarse proper metric spaces which serve in the classification of proper metric spaces according to their coarse geometry. This gives access to topological methods that can be used in the coarse setting.

1 Notions in coarse geometry

We consider metric spaces as coarse objects. The book [Roe03] introduces a more general notion of coarse spaces, which describes coarse structure in an abstract way. Since we only consider proper metric spaces as examples we do not need to do this here.

Definition 8. A metric space $X$ is proper if the closure $\bar{B}$ in $X$ of every bounded subset $B \subseteq X$ is compact.

Definition 9. Let $(X, d)$ be a metric space. Then the coarse structure associated to $d$ on $X$ consists of those subsets $E \subseteq X \times X$ for which
\[ \sup_{(x,y) \in E} d(x,y) < \infty. \]
We call an element of the coarse structure entourage. In what follows we assume the metric $d$ to be finite for every $(x, y) \in X \times X$.

Definition 10. A map $f : X \to Y$ between metric spaces is called

- coarsely uniform if $E \subseteq X^2$ being an entourage implies that $f \times^2(E)$ is an entourage ;
- coarsely proper if and if $A \subseteq Y$ is bounded then $f^{-1}(A)$ is bounded.
- coarse if it is both coarsely uniform and coarsely proper

Two maps $f, g : X \to Y$ between metric spaces are called close if
\[ f \times g(\Delta_X) \]
is an entourage in $Y$. Here $\Delta_X$ denotes the diagonal in $X \times X$.

If $S \subseteq X \times X, T \subseteq X$ are subsets of a set we write
\[ S[T] := \{x : \exists y \in T, (x, y) \in S\} \]
and $T^c = \{x \in X : x \not\in T\}$.

Notation 11. A map $f : X \to Y$ between metric spaces is called

- coarsely surjective if there is an entourage $E \subseteq Y \times Y$ such that
\[ E[\text{im } f] = Y; \]
- coarsely injective if for every entourage $F \subseteq Y^2$ the set $(f \times^2)^{-1}(F)$ is an entourage in $X$.

Two subsets $A, B \subseteq X$ are called not coarsely disjoint if there is an entourage $E \subseteq X^2$ such that the set
\[ E[A] \cap E[B] \]
is not bounded. We write $A \perp B$ in this case.
Remark 12. We study metric spaces up to coarse equivalence. For a coarse map \( f : X \to Y \) between metric spaces the following statements are equivalent:

- There is a coarse map \( g : Y \to X \) such that \( f \circ g \) is close to \( \text{id}_Y \) and \( g \circ f \) is close to \( \text{id}_X \).
- The map \( f \) is both coarsely injective and coarsely surjective.

We call \( f \) a coarse equivalence if one of the equivalent statements hold.

Notation 13. If \( X \) is a metric space and \( U_1, \ldots, U_n \subseteq X \) are subsets then \((U_i)_i\) are said to coarsely cover \( X \) if for every entourage \( E \subseteq X \times X \) the set

\[
E[U_1] \cap \cdots \cap E[U_n]
\]

is bounded.

2 The original definition

In this chapter we introduce the class of compactifications which are coarse.

Definition 14. A compactification of a proper metric space (or more generally a locally compact Hausdorff topological space) is an open embedding \( i : X \to \bar{X} \) such that \( i(X) \) is dense in \( X \). We identify \( X \) with the dense open set \( i(X) \subset \bar{X} \).

Now we define when a compactification is coarse. The original definition was given in [Roe03, Theorem 2.27, Definition 2.28, Definition 2.38]. We reproduce a slight modification of it.

Definition 15. Let \( X \) be a proper metric space. If \( \bar{X} \) is a compactification of \( X \) with boundary \( \partial X \) then the sets \( E \subseteq X \times X \) with

\[
E \cap (\partial X \times \bar{X} \cup \bar{X} \times \partial X) \subseteq \Delta_{\partial X}
\]

define the topological coarse structure associated to \( \bar{X} \).

A coarse compactification of \( X \) is a compactification whose topological coarse structure is finer than the originally given coarse structure on \( X \).

Note in [FOY18, Definition 1.1] a coarse compactification of a proper metric space has been defined as a metrizable compactification \( \bar{X} \) of \( X \) equipped with a continuous map \( f : hX \to X \) which is the identity on \( X \). This definition is different from our definition.

Definition 16. Let \( X \) be a metric space. Two subsets \( A, B \subseteq X \) are called close if there exists an unbounded sequence \((a_i, b_i)_i \subseteq A \times B\) and some \( R \geq 0 \) such that \( d(a_i, b_i) \leq R \) for every \( i \). We write \( A \bowtie B \) in this case. By [Har19a, Lemma 9, Proposition 10] the relation \( \bowtie \) is a large-scale proximity relation.

Theorem 17. Let \( X \) be a proper metric space and \( \bar{X} \) be a compactification of \( X \). Then \( \bar{X} \) is coarse if and only if for every two subsets \( A, B \subseteq X \) the relation \( A \bowtie B \) implies \( \bar{A} \cap \bar{B} \neq \emptyset \).

Proof. Suppose \( \bar{X} \) is coarse. Let \( A, B \subseteq X \) be two subsets with \( A \bowtie B \). Then there exist unbounded subsequences \((a_i)_i \subseteq A, (b_i)_i \subseteq B\) such that \((a_i, b_i)_i\) is an entourage. Then \((a_i, b_i)_i \cap (\delta X \times \bar{X} \cup \bar{X} \times \delta X) \subseteq \Delta_{\partial X}\). Thus if \( p \in \delta X \) is a limit point of \((a_i)_i\) then it is also a limit point of \((b_i)_i\). Note \((a_i)_i \cap (\delta X) \neq \emptyset \) since \((a_i)_i\) is unbounded. This way we have shown \((a_i)_i \cap (b_i)_i \neq \emptyset\). This implies \( A \cap B \neq \emptyset \).
Now suppose for every two subsets \( A, B \subseteq X \) the relation \( A \land B \) implies \( \bar{A} \cap \bar{B} \neq \emptyset \). Let \( E \subseteq X \times X \) be an entourage and \( (x_i, y_i)_i \subseteq E \) be a net such that \( (x_i)_i \to p \in \partial X \) and \( (y_i)_i \to q \in \bar{X} \). If \( q \in \bar{X} \) then there is an infinite subnet of \((y_i)_i \) contained in a ball around \( q \). Then an infinite subnet of \((x_i)_i \) is contained in a (larger) ball around \( q \), thus would have a limit point in this ball. This way we can conclude \( q \in \partial X \).

If \((x_i)_i \cap (y_i)_i \) is bounded then remove those finitely many elements in the intersection and obtain \((x_{i_k})_k \cap (y_{i_k})_k \) subnets with the same limit points. Now \((x_{i_k})_k \land (y_{i_k})_k \) which implies \((x_{i_k})_k \cap (y_{i_k})_k \neq \emptyset \). This implies \( p = q \).

This way we have shown \( E \cap (\partial X \times \bar{X} \cup \bar{X} \times \partial X) \subseteq \Delta_{\partial X} \). Thus \( \bar{X} \) is coarse.

\[ \square \]

### 3 Large-scale proximity relations

In this chapter we study a relation on subsets of a proper metric space \( X \) which induces the topology of a compactification \( \bar{X} \). The relation is large-scale proximity as defined below exactly when the compactification \( \bar{X} \) is coarse. Given a large-scale proximity relation \( r \) on \( X \) we are going to present two constructions of spaces \( \bar{r}_i X, \bar{r}_p X \) which happen to be boundaries of a coarse compactification \( \bar{X}' \) which induces the relation \( r \) on subsets of \( X \).

**Definition 18.** A relation \( r \) on subsets of a metric space is called large-scale proximity if

1. if \( B \subseteq X \) then \( BrB \) if and only if \( B \) is bounded;
2. \( ArB \) implies \( BrA \) for every \( A, B \subseteq X \);
3. if \( A, A' \subseteq X \) are subsets and \( E \subseteq X^2 \) is an entourage with \( E[A] \supseteq A', E[A'] \supseteq A \) then \( ArB \) implies \( A'rB \) for every \( B \subseteq X \);
4. if \( A, B, C \subseteq X \) then \( (A \cup B)rC \) if and only if \( (ArC \text{ or } BrC) \);
5. if \( A, B \subseteq X \) with \( ArB \) then there exist \( C, D \subseteq X \) with \( C \cup D = X \) and \( C \bar{r} A, D \bar{r} B \).

**Remark 19.** Note a large-scale proximity relation on a metric space is an instance of a coarse proximity relation as defined in [GS19, Definition 2.2]. Compare this notion with the notion of proximity relation [NW70, Wil70]. The axiom 3 of Definition 18 is the characteristic for our application on coarse metric spaces.

**Lemma 20.** Let \( X \) be a metric space. Every large-scale proximity relation on \( X \) is finer than the relation \( \land \).

**Proof.** Let \( r \) be a large-scale proximity relation on a metric space \( X \). If \( A \land B \) then there exist unbounded \((a_i, b_i)_i \subseteq X^2 \) and an entourage \( E \subseteq X^2 \) such that \( E[(a_i)_i] \supseteq (b_i)_i \) and \( E[(b_i)_i] \supseteq (a_i)_i \). This implies \((a_i)_i r (b_i)_i \) by axiom 3 of Definition 18. By axiom 4 of Definition 18 the relation \( ArB \) holds.

Now we construct a topological space \( \bar{r}_i X \) given a large-scale proximity relation \( r \). This will turn out to be the boundary of a coarse compactification. The topology on this construction is easier to describe than in the other equivalent definition which will follow below.

**Definition 21.** Let \( r \) be a large-scale proximity relation on a metric space \( X \). A system \( F \) of subsets of \( X \) is called an \( r \)-ultrafilter if
1. \(A, B \in \mathcal{F}\) implies \(ArB\);
2. if \(A, B \subseteq X\) are subsets with \(A \cup B \in \mathcal{F}\) then \(A \in \mathcal{F}\) or \(B \in \mathcal{F}\);
3. \(X \in \mathcal{F}\).

Denote by \(\delta_r X\) the set of \(r\)-ultrafilters. If \(A \subseteq X\) is a subset then define
\[
\text{cl}(A) := \{F \in \delta_r X : A \in \mathcal{F}\}.
\]

**Lemma 22.** If \(X\) is a metric space the \((\text{cl}(A)^c)_{A \subseteq X}\) constitute a base for a topology on \(\delta_r X\).

**Proof.** First we show the base elements cover \(\delta_r X\): Since \(\emptyset\) is bounded \(\emptyset \bar{r} X\). This implies \(\emptyset \notin \mathcal{F}\) for every \(r\)-ultrafilter \(F\). Thus \(\delta_r X = \text{cl}(\emptyset)^c\).

Now we show for every element in the intersection of two base elements there is a base element which contains the element and is contained in the intersection: Let \(A, B \subseteq X\) be two subsets. Let \(F \in \text{cl}(A)^c \cap \text{cl}(B)^c\) be an element. Then \(A \notin \mathcal{F}, B \notin \mathcal{F}\) thus \((A \cup B) \notin \mathcal{F}\). This implies \(F \in \text{cl}(A \cup B)^c \subseteq \text{cl}(A)^c \cap \text{cl}(B)^c\).

**Definition 23.** Define the topology on \(\delta_r X\) to be the topology generated by \((\text{cl}(A)^c)_{A \subseteq X}\).

Now define a relation \(\lambda_r\) on \(\delta_r X\): \(\mathcal{F} \lambda_r \mathcal{G}\) if \(A \in \mathcal{F}, B \in \mathcal{G}\) implies \(ArB\). The quotient by this equivalence relation \((\bar{\delta}_r X = \delta_r X / \lambda_r)\) is called \(r\)-boundary 1.

**Lemma 24.** The space \(\bar{\delta}_r(X)\) is a compact Hausdorff topological space.

**Proof.** The proof of [Har19c, Theorem 26] with \(\lambda\) replaced by \(r\) implies that \(\bar{\delta}_r(X)\) is compact.

Now we show \(\bar{\delta}_r(X)\) is Hausdorff. Let \(\mathcal{F}, \mathcal{G}\) be two \(r\)-ultrafilters with \(\mathcal{F} \lambda_r \mathcal{G}\). Thus there exist \(A \in \mathcal{F}, B \in \mathcal{G}\) with \(A \bar{r} B\). Then there exist \(C, D \subseteq X\) with \(C \cup D = X\) and \(C \bar{r} A, D \bar{r} B\). Then \(\mathcal{G} \in \text{cl}(D)^c, \mathcal{F} \in \text{cl}(C)^c\). Also:
\[
\text{cl}(C)^c \cap \text{cl}(D)^c = (\text{cl}(C) \cup \text{cl}(D))^c \\
= \text{cl}(X)^c \\
= \emptyset.
\]

This completes our discussion of \(\bar{\delta}_r(X)\). We now define another topological space \(\delta_r X\) given a large-scale proximity relation. This space is homeomorphic to \(\bar{\delta}_r X\). Compared with the previous model the points on \(\delta_r X\) are easier to describe.

Let \(R \geq 0\) be a real number. A metric space \(X\) is called \(R\)-discrete if \(d(x, y) \geq R\) for every \(x \neq y\). If \(X\) is a metric space an \(R\)-discrete for some \(R > 0\) subspace \(S \subseteq X\) is called a Delone set if the inclusion \(S \to X\) is coarsely surjective. Every metric space contains a Delone set.

**Definition 25.** Let \(r\) be a large-scale proximity relation on a proper metric space \(X\) and \(S \subseteq X\) a Delone subset. Denote by \(\hat{S}\) the set of nonprincipal ultrafilters on \(S\). If \(A \subseteq S\) is a subset define
\[
\text{cl}(A) := \{F \in \hat{S} : A \in \mathcal{F}\}.
\]

Then define a relation \(r\) on subsets of \(\hat{S}\): \(\pi_1 \pi_2\) if for every \(A, B \subseteq S\) the relations \(\pi_1 \subseteq \text{cl}(A), \pi_2 \subseteq \text{cl}(B)\) imply \(ArB\).

**Lemma 26.** The relation \(r\) on \(\hat{S}\) is a proximity relation.

**Proof.** The proof of [Har19c, Theorem 23] with \(r\) in place of \(\lambda\) applies.
As before we define a topology on $\hat{S}$:

**Definition 27.** The relation $r$ on subsets of $\hat{S}$ determines a Kuratowski closure operator

$$\tilde{\pi} = \{ F \in \hat{S} : \{ F \} \in r \}.$$ 

Now define a relation $\lambda_r$ on $\hat{S}$: $F \lambda_r G$ if $A \in F, B \in G$ implies $ArB$.

**Lemma 28.** The relation $\lambda_r$ is an equivalence relation on $\hat{S}$.

**Proof.** The relation is obviously symmetric and reflexive. We show transitivity. Let $F_1, F_2, F_3$ be nonprincipal ultrafilters on $X$ such that $F_1 \lambda_r F_2$ and $F_2 \lambda_r F_3$. We show $F_1 \lambda_r F_3$. Assume the opposite. There are $A \in F_1, B \in F_3$ with $ArB$. Then there exist $C, D \subseteq X$ with $C \cup D = X$ and $\partial C, \partial D \in B$. Now $C \in F_2$ or $D \in F_2$. If $C \in F_2$ this contradicts $F_2 \lambda_r F_1$ and if $D \in F_2$ this contradicts $F_2 \lambda_r F_3$.

**Definition 29.** Now the $r$-boundary is defined $\partial_r(X) = \hat{S}/\lambda_r$ as the quotient by $\lambda_r$.

We check this definition does not depend upon the choice of Delone set $S$:

**Lemma 30.** If $T \subseteq X$ is another Delone subset then $\hat{S}/\lambda_r = \hat{T}/\lambda_r$ are homeomorphic.

**Proof.** Suppose $S, T \subseteq X$ are two Delone sets. Without loss of generality assume $S \subseteq T$ is a subset. Then there exists a map $\varphi : T \to S$ with $H := \{(t, \varphi(t)) : t \in T\}$ an entourage and $\varphi \circ i = id_S$ where $i : S \to T$ is the inclusion. There is an induced map

$$\varphi_* : \hat{T} \to \hat{S},$$

$$\mathcal{F} \mapsto \{ A \subseteq S : \varphi^{-1}(A) \in \mathcal{F} \}.$$ 

We show $\varphi_*$ is continuous and respects $\lambda_r$. If $\pi_1, \pi_2 \subseteq \hat{T}$ are subsets with $\pi_1 r \pi_2$ and $A, B \subseteq S$ are subsets with $\varphi_* \pi_1 \subseteq \text{cl}(A), \varphi_* \pi_2 \subseteq \text{cl}(B)$ then $\pi_1 \subseteq \text{cl}(\varphi^{-1}(A)), \pi_2 \subseteq \text{cl}(\varphi^{-1}(B))$. Thus $\varphi^{-1}(A) \subseteq \varphi^{-1}(B)$. Since $H[\varphi^{-1}(A)] = A, H^{-1}[A] = \varphi^{-1}(A)$ and $H[\varphi^{-1}(B)] = B, H^{-1}[B] = \varphi^{-1}(B)$ this implies $ArB$. Thus $\varphi_* \pi_1 r \varphi_* \pi_2$.

Let $F, G \in \hat{T}$ be elements with $F \lambda_r G$. Let $A \in \varphi_* F, B \in \varphi_* G$ be elements. Then $\varphi^{-1}(A) \in F, \varphi^{-1}(B) \in G$. Thus $\varphi^{-1}(A) \subseteq \varphi^{-1}(B)$. This implies $ArB$ by the above.

Now the inclusion $i : \hat{S} \to \hat{T}$ in a similar way. We show just as for $\varphi_*$ that $i_*$ is continuous and respects $\lambda_r$. We have $\varphi_* \circ i_* (F) = F$ for every $F \in \hat{S}$ and $i_* \circ \varphi_*(G) \lambda_r G$ for every $G \in \hat{T}$ since $A, B \in G$ implies $ArB$ which implies $\varphi^{-1}(A \cap S)rB$.

Comparing the first and the second model, we can prove:

**Proposition 31.** If $X$ is a proper metric space the map

$$\Phi : \partial_r X \to \partial'_r X,$$

$$[\sigma] \mapsto \{ A \subseteq X : ArB, B \in \sigma \}$$

is a homeomorphism.

**Proof.** Let $S \subseteq X$ be a Delone subset. If $\sigma$ is an ultrafilter on $S$ then the collection $\{ A \subseteq X : ArB, B \in \sigma \}$ is an $r$-ultrafilter on $X$ by a proof similar to that of [Har19c, Theorem 17]. If $\sigma, \tau$ are two ultrafilters on $S$ with $\sigma \lambda_r \tau$ then $\sigma, \tau$ are in particular $r$-ultrafilters on $S$ and $\sigma \lambda_r (\Phi(\sigma))|_S, \tau \lambda_r (\Phi(\tau))|_S$. By transitivity of $\lambda_r$ we obtain $(\Phi(\sigma))|_S \lambda_r (\Phi(\tau))|_S$. This implies $\Phi(\sigma) \lambda_r \Phi(\tau)$. Thus $\Phi$ is well defined.
Now we show $\Phi$ is injective: Let $\sigma, \tau$ be nonprincipal ultrafilters on $S$ with $\Phi(\tau) \lambda, \Phi(\sigma)$. Then $\sigma \lambda, \tau$ since $\tau \subseteq \Phi(\tau), \sigma \subseteq \Phi(\sigma)$.

Now we show $\Phi$ is surjective: Let $\mathcal{F}$ be an $r$-ultrafilter on $X$. Without loss of generality assume $S \in \mathcal{F}$. Then by [NW70] Lemma 5.3 there exists an ultrafilter $\sigma$ on $X$ such that $S \in \sigma$ and $\sigma \subseteq \mathcal{F}$. Then $\sigma|S$ is mapped by $\Phi$ to the class of $\mathcal{F}$.

Proof. Assume without loss of generality that $\partial_r X$ is closed.

If $A \subseteq S$ is a subset then $\partial_r X$ is a closed subset of $\partial_r X$. Let $\sigma \in \check{S}$ be an element with $\sigma \lambda. \sigma$ in $S$. Then for every $B \in \sigma$ we obtain $BrA$. Thus $A \subseteq \Phi(\sigma)$. By [NW70] Lemma 5.7 there exists an ultrafilter $\tau$ on $S$ with $\tau \subseteq \Phi(\sigma)$ and $\tau \in \mathcal{F}$. This implies $\tau \lambda, \sigma$. Thus $[\sigma] \in \check{A}(\mathcal{F})$. In fact the $(\check{A}(\mathcal{F}))_{A \subseteq S}$ constitute a base for the topology on $\partial_r X$.

If $A \subseteq S$ is a subset then $\Phi(\check{A}(\mathcal{F})) = \check{A}(\mathcal{F})$. Thus $\Phi$ is a closed map.

Now we show $\Phi$ is surjective: Let $\Phi(\check{A}(\mathcal{F})) = \check{A}(\mathcal{F})$. Thus $\Phi$ is continuous.

Proposition 32. If $r, s$ are two large-scale proximity relations on a proper metric space $X$ and $s$ is finer than $r$ then there is a quotient map $\partial_r (X) \to \partial_s (X)$.

Proof. Let $S \subseteq X$ be a Delone subset.

If $\mathcal{F}, \mathcal{G}$ are nonprincipal ultrafilters on $S$ then $\mathcal{F} \lambda, \mathcal{G}$ implies $(A \in \mathcal{F}, B \in \mathcal{G}$ implies $ArB)$. Thus $AsB$ for every $A \in \mathcal{F}, B \in \mathcal{G}$ which implies $\mathcal{F} \lambda, \mathcal{G}$.

Now we show $id_{\check{S}} : (\check{S}, r) \to (\check{S}, s)$ is continuous. If $\pi_1, \pi_2 \subseteq \check{S}$ are subsets then $\pi_1 \pi_2$ implies $(A, B \subseteq S$ with $\pi_1 \subseteq \check{A}(\mathcal{F}), \pi_2 \subseteq \check{B}(\mathcal{F})$ implies $ArB)$. Then $AsB$ if $\pi_1 \subseteq \check{A}(\mathcal{F}), \pi_2 \subseteq \check{B}(\mathcal{F})$. Thus $\pi_1 * \pi_2$.

Since $id_{\check{S}}$ is surjective the induced map on quotients is surjective.

Now we produce the compactification of a proper metric space $X$ with the boundary $\partial_r X$ given a proximity relation $r$. Define $\check{X} = X \cup \partial_r X$ as a set. Closed sets on $\check{X}$ are generated by $(\check{A} \cup \check{A}(\mathcal{F}))_{A \subseteq X}$, where the closure $\check{A}$ of $A$ is taken in $X$.

Proposition 33. If $X$ is a proper metric space and $r$ a large-scale proximity relation on $X$ then $\check{X}$ is a compactification of $X$ with boundary $\partial_r X$.

Proof. This topology is compact by the first part of the proof of [Har19] Theorem 20] with $\lambda$ replaced by $r$. The spaces $\partial_r X, X$ appear as subspaces of $\check{X}$. The inclusion $X \to \check{X}$ is dense since $X \cup \check{A}(\mathcal{F}) = \check{X}$.

Remark 34. The statement in Proposition 32 can be strengthened: If $X$ is a proper metric space and $r, s$ are close relations on $X$ with $s$ finer than $r$ then there is a quotient map $\check{X} \to \check{X}$ which is the unique continuous map extending the identity on $X$.

Proof. Assume without loss of generality that $X$ is $R$-discrete for some $R > 0$. A net in $X$ converging to a point $p \in \check{X}$ can be written as a filter $\mathcal{F}$ on $X$. Then an ultrafilter $\sigma$ finer than $\mathcal{F}$ converges to the same point.

Let $\alpha : \check{X} \to \check{X}$ be a continuous map extending the identity on $X$. Then $\alpha, \sigma = \sigma$. Thus $\sigma$ converges to $\alpha(p)$. Now $p$ is represented by $\sigma$ in $\partial_r X$ and $\alpha(p)$ is also represented by $\sigma$ in $\partial_s X$. Thus $\alpha$ maps a point represented by $\sigma$ to a point represented by $\sigma$ and is thus uniquely determined. This implies $\alpha|_{\partial_r X}$ is the quotient map of Theorem 32.

3 LARGE-SCALE PROXIMITY RELATIONS
Remark 35. Remark [31] and Lemma [29] imply that the Higson compactification is universal among coarse compactifications. This recovers [Roe03, Proposition 2.39].

If $X$ is a coarse compactification of a proper metric space $X$, define a relation $r_X$ on subsets of $X$ as follows: For a subset $A \subseteq X$ define $\cl(A) = A \cap \partial X$. If $A, B \subseteq X$ are subsets then $Ar_XB$ if $\cl(A) \cap \cl(B) \neq \emptyset$.

**Theorem 36.** Let $X$ be a coarse compactification of a proper metric space $X$. Then $r_X$ is a large-scale proximity relation on $X$. There is a homeomorphism $\Phi : \bar{X} \to X^r$ extending the identity on $X$.

**Proof.** By [GS19] Theorem 6.7] the relation $r$ is a coarse proximity relation. Thus axioms 1, 2, 4, 5 of a large-scale proximity relation are satisfied. It remains to show $r$ satisfies axiom 3. Let $A, A', B \subseteq X$ be subsets and let $E \subseteq X \times X$ be an entourage with $E[A] \supseteq A', E[A'] \supseteq A$ and $Ar_B$. Then $\cl(A) \cap \cl(B) \neq \emptyset$. Since the compactification is coarse, $A, A'$ have the same limit points on $\partial X$. Thus $\cl(A) = \cl(A')$ which implies $\cl(A) \cap \cl(B) \neq \emptyset$. Thus $A'r_B$.

For the last statement we extend the proof of [GS19] Theorem 6.7]. If $x \in \bar{X} \setminus X$ is a point define $F_x := \{A \subseteq X : x \in \cl(A)\}$. Then we define

$$\Phi : \bar{X} \to X^r$$

$$\quad x \mapsto \begin{cases} x & x \in X \\ [F_x] & x \in \bar{X} \setminus X \end{cases}$$

Now [GS19] Theorem 6.7 showed $\Phi|_{\bar{X} \setminus X}$ is a homeomorphism. This implies in particular that $\Phi$ is a bijective map. We show $\Phi$ is continuous: Let $A \subseteq X$ be a subset. Then

$$\Phi^{-1}(\bar{A}^X \cup \cl(A)) = \bar{A}^X \cup \{x \in \bar{X} \setminus X : F_x \in \cl(A)\}$$

$$= \bar{A}^X \cup \{x \in \bar{X} \setminus X : A \in F_x\}$$

$$= \bar{A}^X \cup \cl(A)$$

is a closed set. Here $\bar{A}^X$ denotes the closure of $A$ in $X$.

\[ \square \]

4 Bounded functions

**Theorem 37.** Let $X$ be a proper metric space. A compactification $\bar{X}$ is coarse if and only if every bounded continuous function $\varphi : X \to \mathbb{R}$ that extends to $\bar{X}$ is Higson.

**Proof.** Suppose $\bar{X}$ is a coarse compactification and assume for contradiction there is a continuous function $\varphi : X \to \mathbb{R}$ such that $\varphi|_X$ is not Higson. Then there is an entourage $E \subseteq X^2$ and some $\varepsilon > 0$ and an unbounded sequence $(x_k, y_k)_k \subseteq E$ with

$$|\varphi(x_k) - \varphi(y_k)| > \varepsilon$$

Now $(x_k)_k \cap (y_k)_k \neq \emptyset$ since the compactification is coarse. This contradicts that $\varphi(x_k)_k, \varphi(y_k)_k$ have disjoint limit points in $\mathbb{R}$.

Now we give an alternative proof of this direction using that the Higson corona is universal among coarse compactifications. Suppose $\varphi : X \to \mathbb{R}$ is a bounded continuous function that extends a continuous function $\tilde{\varphi}$ on $X$. Denote by $q : hX \to \bar{X}$ the quotient map from the Higson corona. Then $\tilde{\varphi} \circ q : hX \to \mathbb{R}$ is an extension of $\varphi$ to $hX$. This implies $\tilde{\varphi}$ is Higson.

Now suppose every bounded continuous function $\varphi : X \to \mathbb{R}$ that extends to $\bar{X}$ is Higson. Let $A, B \subseteq X$ be subsets such that $A \cap B = \emptyset$. Since $\bar{X}$ is normal we can use Urysohn’s lemma:
there exists a bounded continuous function $\varphi : \tilde{X} \to \mathbb{R}$ with $\varphi|_\tilde{X} \equiv 0$ and $\varphi|_\tilde{X} \equiv 1$. Now $\varphi|_X$ is Higson. This implies for every entourage $E \subseteq X \times X$ there exists a bounded set $C \subseteq X$ with $E \cap (A \times B) \subseteq C \times C$.

This implies $A \not\sim B$.

Let $X$ be a proper metric space. To a coarse compactification $\tilde{X}$ we can associate a large-scale proximity relation $r$ on subsets of $X$ such that $\tilde{X} = \tilde{X}^r$. We can also associate to $\tilde{X}$ the set of bounded functions $C_r(X)$ that extend to $\tilde{X}^r$. They must be Higson. Note that $C_r(X)$ is a ring by pointwise addition and multiplication.

### 5 Functoriality

**Proposition 38.** Let $\alpha : X \to Y$ be a coarse map between proper metric spaces and let $\tilde{Y}$ be a coarse compactification. If $X$ is $R$-discrete for some $R > 0$ the functions $C_r(X) := \{\varphi \circ \alpha : \varphi \in C_r(Y)\}$ determine a coarse compactification on $X$. It is the same compactification which is induced by the relation $ArB$ if $\alpha(A)\alpha(B)$.

**Proof.** If $\varphi$ is a Higson function on $Y$ then $\varphi \circ \alpha$ is continuous and bounded since $\alpha$ is continuous and $\varphi \circ \alpha$ is continuous and bounded. Since $\alpha$ is a coarse map $\varphi \circ \alpha$ satisfies the Higson property. Thus $C_r(X)$ determines a compactification $\tilde{X}$ which is coarse. The set $C_r(X)$ is a ring by pointwise addition and multiplication and contains the constant functions. Thus $C_r(X)$ equals the bounded functions which can be extended to $\tilde{X}^r$.

Now we prove the relation $r$ defined on subsets of $X$ is a large-scale proximity relation. We check the axioms of a large-scale proximity relation:

1. If $B \subseteq X$ is bounded so is $\alpha(B) \subseteq Y$. Thus $\alpha(B)\alpha(B)$ which implies $B\alpha(B)$. If $A \subseteq X$ is unbounded then $\alpha(A) \subseteq Y$ is unbounded. Thus $\alpha(A)\alpha(A)$ which implies $A\alpha(A)$.

2. Symmetry is obvious.

3. Suppose $A, A', B \subseteq X$ are subsets and $E \subseteq X \times X$ an entourage with $E[A] \supseteq A', E[A'] \supseteq A$ and $ArB$. Then $\alpha^{\times 2}(E)[\alpha(A)] \supseteq \alpha(A')$ and $\alpha^{\times 2}(E)[\alpha(A')] \supseteq \alpha(A)$ and $\alpha(A)\alpha(A)$ then $\alpha(A')\alpha(A)$.

4. If $(A \cup B)\alpha C$ then $\alpha(A \cup B)\alpha(A)$. Now $\alpha(A \cup B) = \alpha(A) \cup \alpha(B)$ thus $\alpha(A)\alpha(C)$ or $\alpha(B)\alpha(C)$. This implies $\alpha(A)\alpha(B)$.

5. If $A\alpha B$ then $\alpha(A)\alpha(B)$. This implies there exist $C, D \subseteq Y$ with $C \cup D = Y$ and $C\alpha(A), D\alpha(B)$. Then $\alpha^{-1}(C) \cup \alpha^{-1}(D) = X$ and $\alpha^{-1}(C)\alpha^{-1}(D) = X$.

Now we define a map

$$
\Phi : \tilde{X}^r \to \mathbb{R}^{C_r(X)}
$$

$$
x \mapsto \begin{cases} 
(\varphi \circ \alpha(x))_{\varphi \circ \alpha} & x \in X \\
(x^{-}\lim \varphi \circ \alpha)_{\varphi \circ \alpha} & x \in \partial X
\end{cases}
$$

We show $\Phi$ is well-defined: If $F\lambda_r G$ then for every $A \in F, B \in G$ the relation $ArB$ holds. Then $\alpha(A)\alpha(B)$ which implies $\alpha, F\lambda_r \alpha, G$. Then

$$
F^{-}\lim \varphi \circ \alpha = \alpha^{-} \lambda^{-} \lim \varphi = \alpha^{-} \lambda^{-} \lim \varphi
$$

$$
= \alpha^{-} \lambda^{-} \lim \varphi = \lambda^{-} \lim \varphi \circ \alpha
$$
for every \( \varphi \in C_r(Y) \).

Now we show \( \Phi \) is injective: if \( \mathcal{F} \lambda, \mathcal{G} \) on \( X \) then there exist \( A \in \mathcal{F}, B \in \mathcal{G} \) with \( A \mathcal{F} B \). Thus \( \lambda(A) \mathcal{F} \lambda(B) \). This implies \( \lambda, \mathcal{F} \lambda \), \( \mathcal{G} \). Then there exists some bounded function \( \varphi \in C_r(Y) \) with
\[
\mathcal{F} \text{-lim} \varphi \circ \alpha = \mathcal{F} \text{-lim} \varphi \\
\neq \mathcal{G} \text{-lim} \varphi \\
= \mathcal{G} \text{-lim} \varphi \circ \alpha.
\]

Denote for \( Z = X, Y \) the evaluation map
\[
eq \theta \colon Z \to \mathbb{R}^{C_r(Z)} \\
z \mapsto (\varphi(z))_\varphi.
\]

Now the following diagram commutes
\[
\begin{array}{ccc}
\tilde{X}^r & \xrightarrow{\alpha_*} & \tilde{Y}^r \\
\phi \downarrow & & \downarrow \\
e(X) & \xrightarrow{\alpha_*} & e(Y)
\end{array}
\]

where the upper horizontal map maps \( \mathcal{F} \in \partial X \) to \( \alpha, \mathcal{F} \) and \( x \in X \) to \( \alpha(x) \) and the lower horizontal map maps \( (\varphi \circ \alpha(x))_C \) to \( (\varphi \circ \alpha(x))_C \).

Now both horizontal maps are continuous and open, thus \( \Phi \) is continuous and open. Since every ultrafilter on \( X \) induces an \( r \)-ultrafilter on \( X \) the map \( \Phi \) is surjective on \( e(X) \).

**Lemma 39.** Let \( \alpha : X \to Y \) be a coarse map between proper metric spaces and let \( \tilde{X}^r, \tilde{Y}^r \) be coarse compactifications of \( X, Y \), respectively. If \( r_1 \mathcal{F} r_2 \mathcal{B} \) implies \( \alpha(A) \mathcal{F} \alpha(B) \) then \( \alpha \) can be extended to a continuous map
\[
\alpha_* : \tilde{X}^r \to \tilde{Y}^r \\
x \mapsto \begin{cases} \\
\alpha(x) & x \in X \\
\alpha_* x & x \in \partial r_1 X.
\end{cases}
\]

If \( X \) is \( R \)-discrete for some \( R > 0 \) and \( C_{r_1} + \alpha \subseteq C_{r_1} \) then \( \alpha_* \) is the restriction of
\[
\mathbb{R}^{C_{r_1}(X)} \to \mathbb{R}^{C_{r_1}(Y)} \\
(\varphi(x))_\varphi \mapsto (\varphi \circ \alpha(x))_\varphi
\]
to \( e(X) \). Both descriptions of \( \alpha \) coincide.

**Proof.** Suppose \( r_1 \mathcal{F} r_2 \mathcal{B} \) implies \( \alpha(A) \mathcal{F} \alpha(B) \). Let \( \varphi \in C_{r_1}(Y) \) be a function and let \( \mathcal{F} \) be an \( r_1 \)-ultrafilter on \( X \). Then \( \alpha, \mathcal{F} \) is an \( r_2 \)-ultrafilter on \( Y \). Thus \( \alpha, F \)-lim \( \varphi \) exists. This point equals \( \mathcal{F} \)-lim \( \varphi \). Since \( \mathcal{F} \) was arbitrary the map \( \varphi \circ \alpha \) can be extended to \( \tilde{X}^r \). Thus \( \varphi \circ \alpha \in C_{r_1}(X) \).

Now suppose \( C_{r_2}(Y) \circ \alpha \subseteq C_{r_1}(X) \). Let \( A, B \subseteq Y \) be subsets with \( A \mathcal{F} B \). Then there exists \( \varphi \in C_{r_2}(Y) \) with \( \varphi|_A \equiv 1, \varphi|_B \equiv 0 \). Then \( \varphi \circ \alpha \mid_{\alpha^{-1}(A)} \equiv 1, \varphi \circ \alpha \mid_{\alpha^{-1}(B)} \equiv 0 \) and \( \varphi \circ \alpha \) can be extended to \( \tilde{X}^r \). Thus \( \alpha^{-1}(A) \mathcal{F} \alpha^{-1}(B) \).

Note the diagram
\[
\begin{array}{ccc}
\tilde{X}^r & \xrightarrow{\alpha_*} & \tilde{Y}^r \\
\phi \downarrow & & \downarrow \\
e(X) & \xrightarrow{\alpha_*} & e(Y)
\end{array}
\]
commutes.

**Proposition 40.** Let $X$ be a proper metric space. If subsets $U_1, \ldots, U_n$ coarsely cover $X$ and each $U_i$ is equipped with a large-scale proximity relation $r_i$ such that $r_i, r_j$ agree on $U_i \cap U_j$, then the relation $r$ on subsets of $X$ defined by $ArB$ if $(U_i \cap A)r_i(U_j \cap B)$ for some $i$ defines a large-scale proximity relation on $X$. If $X$ is $R$-discrete for some $R > 0$ and for every $i$ there is a ring $C_{s_i}(U_i)$ of Higson functions such that $C_{s_i}(U_i)|_{U_i} = C_{s_j}(U_j)|_{U_i}$, then the ring

$$C_s(X) = \{(\varphi_i)_i \in \prod_i C_{s_i}(U_i) : \varphi_i|_{U_i} = \varphi_j|_{U_i}\}$$

consists of Higson functions. If $r_i = s_i$ for every $i$ then the relation $r$ and the ring of bounded functions $C_s(X)$ describe the same compactification.

**Proof.** We show $r$ is a large-scale proximity relation on $X$:

1. If $B \subseteq X$ is bounded then $B \cap U_i$ is bounded for every $i$. Thus $B \bar{r} B$. If $A \subseteq X$ is not bounded then there exists some $i$ such that $A \cap U_i$ is not bounded. Then $(A \cap U_i)r_i(A \cap U_i)$ thus $ArA$.

2. Symmetry is obvious.

3. Without loss of generality assume $n = 2$. Let $A, A', B \subseteq X$ be subsets and let $E \subseteq X \times X$ be an entourage with $E[A] = A', E^{-1}[A'] = A$ and $ArB$. Then $U_1, U_2$ coarsely cover $X$ the relation $U_1 \cap U_2$ holds. Thus there exist $C, D \subseteq X$ with $C \cup D = X$ and $C \bar{r} U_1 \cap U_2$ and $D \bar{r} U_2$. Now $A = (A \cap C) \cup (A \cap D)$. Thus by axiom 4 $(A \cap C)rB$ or $(A \cap D)rB$. Suppose the former holds. Since $A \cap C \subseteq U_1 \cup B'$ where $B'$ is bounded we have $(A \cap C \cap U_1)\bar{r}_1(B \cap U_1)$. Now $E[A \cap C] \subseteq U_1 \cup B''$ where $B''$ is bounded. Then $(E[A \cap C] \cap U_1)\bar{r}_1(B \cap U_1)$. Thus $A'rB$ by axiom 4.

4. If $(A \cup B)\bar{r}C$ then $(A \cup B)r_iC$ for some $i$. Thus $ArC$ or $BrC$. This implies $ArC$ or $BrC$. If $ArC$ or $BrC$ then $(A \cap U_i)r_i(C \cap U_i)$ for some $i$ or $(B \cap U_j)r_j(C \cap U_j)$ for some $j$. This implies $((A \cup B) \cap U_i)r_i(C \cap U_i)$ or $(A \cup B) \cap U_j)\bar{r}_j(C \cap U_j)$. Thus $(A \cup B)\bar{r}C$.

5. Without loss of generality assume $n = 2$ and $U_1, U_2$ cover $X$ as sets. If $ArB$ then $(A \cap U_i)\bar{r}_i(B \cap U_i)$ for both $i$. Thus there exist $C_i, D_i \subseteq U_i$ with $C_i \bar{r}_i(A \cap U_i), D_i \bar{r}_i(B \cap U_i)$ and $C_i \cup D_i = U_i$ for $i = 1, 2$. Then $(A \cap U_1 \cap U_2)\bar{r}(C_1 \cup C_2), (A \cap U_1 \cap U_2)\bar{r}(C_1 \cup U_2), (A \cap U_2 \cap U_2)\bar{r}(C_2 \cup U_2)$ combine to $Ar(C_1 \cup C_2)$. Similarly we obtain $Br(D_1 \cup D_2)$. Now

$$X = U_1 \cup U_2 = C_1 \cup D_1 \cup C_2 \cup D_2.$$  

Now we show $C_s(X)$ consists of Higson functions. Suppose $\varphi_i \in C_{s_i}(U_i)$ for $i = 1, \ldots, n$ are elements with $\varphi_i|_{U_i} = \varphi_j|_{U_i}$. Then they can be glued to a bounded continuous function $\varphi : X \to \mathbb{R}$. Let $E \subseteq X \times X$ be an entourage. Then

$$E = (E \cap (U_1 \times U_1)) \cup \cdots \cup (E \cap (U_n \times U_n)) \cup A$$

where $A \subseteq B \times B$ with $B$ bounded in $X$. Now $(d\varphi)|_{E \cap (U_i \times U_i)} = (d\varphi|_{U_i})|E$ converges to zero at infinity for every $i$. This implies $(d\varphi)|_E$ converges to zero at infinity.

If $r_i = s_i$ for every $i$ then $U_i^r = \bar{e}(U_i)$ for every $i$. The $U_i^r$ glue to $X^r$ and the $e(U_i)$ glue to $e(X)$. The global axiom of Lemma implies uniqueness. Thus $X^r = e(X)$.

$\square$
Let $X$ be a proper $R$-discrete for some $R > 0$ metric space. For every subspace $A \subseteq X$ the poset of coarse compactifications on $A$ is called $CC(A)$. If $A \subseteq B$ is an inclusion of subspaces then there is a poset map $CC(B) \to CC(A)$ induced by the inclusion.

The Grothendieck topology determined by coarse covers on a metric space $X$ is called $X_{ct}$. A contravariant functor $\mathcal{F}$ on subsets of $X$ is a sheaf on $X_{ct}$ if for every coarse cover $U_1, U_2 \subseteq U$ of a subset of $X$ the following diagram is an equalizer

$$\mathcal{F}(U) \to \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \cong \mathcal{F}(U_1 \cap U_2).$$

Lemma 41. The functor $CC$ on subsets of $X$ is a sheaf on $X_{ct}$.

Proof. Note every subspace of $X$ is proper. If $A \subseteq X$ is a subset we define $\bar{A}^r \geq \bar{A}^s$ if $s$ is finer than $r$.

If $A \subseteq B$ is an inclusion of subspaces and $\bar{B}^r \in CC(B)$ then the restriction map associated to the inclusion $A \to B$ maps $\bar{B}^r \mapsto A^{\bar{r}}$. Here the relation $r|_A$ is defined as $Sr|_A$ if $SrT$. Then $r|_A$ is a large-scale proximity relation on $A$:

1. If $S \subseteq A$ is bounded, then $SrS$, thus $Sr|_A S$. If $S$ is unbounded then $SrS$ so $Sr|_A S$.
2. Symmetry is obvious.
3. If $S, S', T \subseteq A$ are subsets, $E \subseteq A^2$ is an entourage with $E[S] \supseteq S', E[S'] \supseteq S$ and $Sr|_A T$ then $E \subseteq B^2$ is an entourage in $B$. Thus $S'rT$ which implies $S'r|_A T$.
4. obvious.
5. If $S, T \subseteq A$ are subsets with $SrT$ then there exist subsets $C', D' \subseteq B$ with $C' \cup D' = B$ and $SrS, D'rT$. Then $C := C' \cap A, D := D' \cap A$ are subsets with $C \cup D = A, C'rS, D'rT$.

Note if a large scale proximity relation $s$ on $B$ is finer than another large-scale proximity relation $r$ on $B$ then $s|_A$ is finer than $r|_A$ on $A$. This makes $CC$ into a functor on the poset of subsets of $X$ to posets.

Now we check the global axiom: Let $(U_i)_i$ be a coarse cover of $X$ and let $r, s$ be close relations on $X$ with $r|_{U_i} = s|_{U_i}$. Two subsets $A, B \subseteq X$ satisfy $ArB$ if and only if $\bigvee \,(A \cap U_i)r(B \cap U_i)$ if and only if $\bigvee \,(A \cap U_i)s(B \cap U_i)$ if and only if $\bigvee \,(A \cap U_i)s(B \cap U_i)$ if and only if $AsB$.

Now we check the gluing axiom: Let $U_1, \ldots, U_n$ be a coarse cover of $X$ equipped with coarse compactifications $\bar{U}_i^r, \ldots, \bar{U}_n^r$ such that $\bar{r}|_{U_i} = \bar{r}|_{U_i}$ for every $ij$. Then the proof Proposition 40 implies the $\bar{U}_i^r$ glue to a coarse compactification $\bar{X}^r$ of $X$. 

6 Higson corona

This section is denoted to the Higson corona. We recall the original description.

Definition 42. (Higson corona) Let $X$ be a proper metric space. A bounded continuous function $\varphi : X \to \mathbb{R}$ is called Higson if for every entourage $E \subseteq X^2$ the map

$$d\varphi|_E : E \to \mathbb{R}$$

$$(x, y) \mapsto \varphi(x) - \varphi(y)$$

vanishes at infinity. Then the compactification $hX$ of $X$ generated by the Higson functions $C_h(X)$ is called the Higson compactification. The boundary of this compactification $\nu(X) = hX \setminus X$ is called the Higson corona.
The large-scale proximity relation induced on \( X \) is the close relation.

**Remark 43.** If \( X \) is a proper metric space then \( \partial_\lambda(X) = \nu(X) \).

**Proof.** This follows from by [Har19b, Theorem 20].

We provide an alternative proof: Let \( A, B \subseteq X \) be subsets. If \( A \not\close B \) then by [NW70, Theorem 5.14] there exists a \( \lambda \)-ultrafilter \( \mathcal{F} \) on \( X \) with \( A, B \in \mathcal{F} \). Thus \( \mathcal{F} \in \text{cl}(A) \cap \text{cl}(B) \) is not empty. If on the other hand \( A \not\close B \) then \( \mathcal{F} \notin \text{cl}(B) \). Thus \( \text{cl}(A) \cap \text{cl}(B) = \emptyset \) is empty. This way we have shown that \( \lambda \) is the unique relation on subsets of \( X \) that tells when the closure of two subsets meet on the boundary of the Higson compactification.

**Proposition 44.** If \( X \) is a one-ended proper metric space then \( \nu(X) \) is connected. This implies that every coarse compactification of \( X \) is connected.

**Proof.** Recall that a metric space has at most one end if for every \( A \subseteq X \) we have \( A \not\close A^c \) or one of \( A, A^c \) is bounded. Suppose \( \pi \subseteq \nu(X) \) is a clopen subset. Then \( \pi \not\close \pi^c \). Then there exist \( A, B \subseteq X \) with \( \pi \subseteq \text{cl}(A), \pi^c \subseteq \text{cl}(B) \) and \( A \not\close B \). By the proof of Theorem 52 the inclusion \( A \cup B \to X \) is coarsely surjective. Thus one of \( A, B \) is bounded which implies one of \( \pi, \pi^c \) is the empty set.

Now we select Higson functions which separate coarsely disjoint subsets of \( X \). A close examination shows they together with the constant functions already generate the Higson functions.

Let \( X \) be a proper metric space. For every two subsets \( A, B \subseteq X \) with \( A \not\close B \) we define

\[
\varphi_{A,B} : X \to \mathbb{R} \\
x \mapsto \frac{d(x, A)}{d(x, A) + d(x, B)}
\]

where we assume without loss of generality \( d(A, B) > 0 \). If \( F \subseteq C^*(X) \) is a subset \( \mathcal{A}(F) \) denotes the intersection of all algebras in \( C^*(X) \) which contain \( F \).

**Proposition 45.** There is an isomorphism of \( C^*\)-algebras

\[
\overline{\mathcal{A}((\varphi_{A,B})_{A \not\close B} \cup \{1\})} = C_b(X).
\]

Here the closure is in \( C^*(X) \) with the sup-metric.

**Proof.** Suppose \( A, B \subseteq X \) are subsets with \( A \not\close B \). By [DKU98, Lemma 2.2] the function \( \varphi_{A,B} \) is Higson. Thus we have shown \( (\varphi_{A,B})_{A \not\close B} \subseteq C_b(X) \).

Now we show \( (\tilde{\varphi}_{A,B})_{A \not\close B} \) separates points of \( \nu(X) \): Let \( \mathcal{F}, \mathcal{G} \in \nu(X) \) be points with \( \mathcal{F} \not\close \mathcal{G} \). Then there exist \( A \in \mathcal{F}, B \in \mathcal{G} \) with \( A \not\close B \). Then

\[
\tilde{\varphi}_{A,B}(\mathcal{F}) = \mathcal{F}\text{-lim } \varphi_{A,B} = 0 \\
\neq 1 \\
= \mathcal{G}\text{-lim } \varphi_{A,B} = \tilde{\varphi}_{A,B}(\mathcal{G}).
\]

Thus \( \tilde{\varphi}_{A,B} \) separates \( \mathcal{F}, \mathcal{G} \).

By [BYS82, Theorem 2.1] the \( (\varphi_{A,B})_{A \not\close B} \) generate the compactification \( hX \). Now we use [BYS82, Theorem 3.4] and obtain the result.
Call a metric space $W$ with $\{x, y \in W : d(x, y) \leq R, x \neq y\}$ finite for every $R > 0$ discrete coarse $[\text{Roe03} \text{ Example 2.7}].$

**Remark 46.** If $X$ is an unbounded proper metric space then it contains a sequence $(x_i)_i \subseteq X$ with $d(x_i, x_j) > i$ for $j < i$. Thus $(x_i)_i$ is discrete coarse. It is easy to check every bounded function on $(x_i)_i$ is Higson. Then $h((x_i)_i) = \beta(\mathbb{N})$ and $\nu((x_i)_i) = \beta(\mathbb{N}) \setminus \mathbb{N}$. Since $h, \nu$ preserves monomorphisms $h((x_i)_i), \nu((x_i)_i)$ arise as subspaces of $h(X), \nu(X)$. Thus $\nu(X)$ contains a copy of $\beta(\mathbb{N}) \setminus \mathbb{N}$ and $hX$ contains a copy of $\beta(\mathbb{N})$. This fact has already been proved in $[\text{Kee94} \text{ Theorem 3}].$

**Proposition 47.** If $X$ is a proper metric space then the union of $\mathfrak{cl}(W)$ over every discrete coarse subspace $W$ of $X$ is dense in $\nu(X)$.

**Proof.** Define

$$\Phi : \bigcup_{W \subseteq X \text{ discrete}} \nu(W) \to \nu(X)$$

where $i : W \to X$ is the inclusion. We show $\Phi^* : C(\nu(X)) \to C\left( \bigcup_{W \subseteq X \text{ discrete}} \nu(W) \right)$ is injective. Note $C(\nu(X)) = C_h(X)/C_0(X)$ and

$$C\left( \bigcup_{W} \nu(W) \right) = \prod_{W} C(W) = \prod_{W} C_h(W)/C_0(W).$$

Let $\varphi \in C_h(X)$ be a Higson function. We need to show if $(\varphi \circ \Phi)_W \in C_0(W)$ for every discrete subset $W \subseteq X$ then $\varphi \in C_0(X)$. Assume for contradiction that $\varphi$ does not converge to zero at infinity. Then there exists $\varepsilon > 0$ such that for every $i \in \mathbb{N}$ there is some $x_i \notin B(x_0, i)$ (Here $x_0 \in X$ is a fixed point and $B(x_0, i)$ denotes the ball of radius $i$ around $x_0$) with the property $|\varphi(x_i)| \geq \varepsilon$.

Now choose a subsequence $(x_{i_k})_k$ with $(x_{i_k} : k)$ discrete. Then $\varphi|_{\{x_{i_k} : k\}} \notin C_0(\{x_{i_k} : k\})$. Since bounded functions on $\nu(X)$ separate points from closed sets we have shown that the closure of $\text{im } \Phi$ is $\nu(X)$.

The closure of $\text{im } \Phi$ is $\nu(X)$ since $\bigcup_W \mathfrak{cl}(W) \subseteq \mathfrak{cl}(A)$ implies the inclusion $i : A \to X$ is coarsely surjective (Every unbounded subset of $X$ contains a discrete subset).

Suppose $X$ is $R$-discrete for some $R > 0$. If $X$ is not discrete there always exists an ultrafilter on $X$ which does not contain a discrete subspace. Define a filter

$$\mathcal{F} = \{X \setminus W : W \subseteq X \text{ discrete or finite}\}$$

Then $\mathcal{F}$ is a filter:

1. If $X \setminus W, X \setminus V \in \mathcal{F}$ then $W, V$ are discrete or finite. This implies $W \cup V$ is discrete or finite, thus $(X \setminus W) \cap (X \setminus V) = X \setminus (V \cup W) \in \mathcal{F}$.

2. If $X \setminus W \in \mathcal{F}$ and $X \setminus W \subseteq X \setminus V$ then $W$ is discrete or finite and $V \subseteq W$. This implies $V$ is discrete or finite. Thus $X \setminus V \in \mathcal{F}$.

If $X$ is not discrete or finite then $\mathcal{F}$ is a proper filter. Then there exists an ultrafilter finer than $\mathcal{F}$, it does not contain a discrete subspace. \qed


7 Space of ends

The space of ends $\Omega(X)$ of a topological space is the boundary of the Freudenthal compactification $\varepsilon(X)$. In this chapter we will study a coarse version of the Freudenthal compactification which coincides with the topological version of the Freudenthal compactification for a large class of proper metric spaces.

Recall [Wil70] Problem 41B:

**Definition 48. (Freudenthal compactification, topological version)** Let $X$ be a rim-compact Tychonoff space. Define a relation $\rho$ on subsets of $X$ by $\bar{A}\rho B$ for $A, B \subseteq X$ if there is a compact subset $K \subseteq X$ such that $X \setminus K = G \cup H$ is a disjoint union of two open subsets with $\bar{A} \subseteq G, B \subseteq H$.

The Smirnov compactification of the proximity space $(X, \rho)$ is rim-compact. Note every metric space is Tychonoff. Recall [Wil70, Problem 41B]:

**Proposition 49.** Let $X$ be a proper geodesic metric space. Then it is rim-compact Tychonoff. If $A, B \subseteq X$ are two subsets then $\bar{A}\rho B$ if and only if $\bar{A} \cap \bar{B} \neq \emptyset$ or $A \not\rho B$.

**Proof.** Since $X$ is a metric space every point $x \in X$ has a basis of open neighborhoods $\{B(x, \varepsilon) : \varepsilon > 0\}$, here $B(x, \varepsilon)$ denotes the open ball of radius $\varepsilon$ around $x$. Since $X$ is proper the the set

$$B(x, \varepsilon) \setminus B(x, \varepsilon) \subseteq \overline{B(x, \varepsilon)}$$

is compact. Thus $X$ is rim-compact. Note every metric space is Tychonoff.

Suppose $A, B \subseteq X$ are two subsets with $\bar{A}\rho B$. Then there exists a compact set $K \subseteq X$ such that $X \setminus K = G \cup H$ with appropriate properties. Let $R > 0$ be a number. If $g \in G, h \in H$ are points with $d(g, h) \leq R$ then there exists $k \in K$ with

$$d(g, k) + d(k, h) = d(g, h) \leq R.$$  

Now $K$ is bounded thus there exists $S \geq 0, x_0 \in X$ with $K \subseteq B(x_0, S)$. Then $g, h \in B(x_0, S + R)$. This proves $G \not\rho H$. Thus $A \not\rho B$. Since $\delta$ is compatible with the topology on $X$ the relation $\bar{A} \cap \bar{B} = \emptyset$ follows.

Suppose $A, B \subseteq X$ are two subsets with $A \not\rho B$ and $\bar{A} \cap \bar{B} \neq \emptyset$. The first relation implies there are $A' \supseteq A, B' \supseteq B$ with $X = A' \cup B', A' \not\rho B'$. Then there exists a bounded set $K' \subseteq X$ such that $d(A' \setminus K', B' \setminus K') > 1$. Define $A'' = \bigcup_{a \in A' \setminus K'} B(a, 1/4)$ and $B'' = \bigcup_{b \in B' \setminus K} B(b, 1/4)$. Now since $X$ is normal there exist open sets $U \supseteq \bar{A}, V \supseteq \bar{B}$ with $U \cap V = \emptyset$. The set

$$K := A'' \cap U \cap B'' \cap V \subseteq (A' \setminus K') \cap (B' \setminus K') = K'$$

is bounded and closed. Since $X$ is proper this set is compact. We define $G = A'' \cup U, H = B'' \cup V$. Then $G, H$ are open and disjoint. We have

$$X \setminus K = A'' \cup U \cup B'' \cup V = G \cup H$$

and $A \subseteq G, B \subseteq H$. Thus we have shown $\bar{A}\rho B$. 
\[\square\]
It is easy to see that $\lambda_f$ is a large-scale proximity relation. Thus $\tilde{X}^{\lambda_f}$ is a coarse compactification of $X$. By Proposition 49 the space is homeomorphic to $\varepsilon(X)$ if $X$ is proper geodesic metric. By slight abuse of notation we write $\Omega(X), \varepsilon(X)$ for the coarse versions of the space of ends, Freudenthal compactification as well.

**Definition 50.** Let $X$ be a metric space with basepoint $x_0 \in X$. A bounded continuous map $\varphi : X \to \mathbb{R}$ is called Freudenthal if for every $R > 0$ there exists $K \geq 0$ such that $d(x,y) \leq R, d(x_0, x) \leq K, d(x_0, y) \leq K$ implies $\varphi(x) = \varphi(y)$. We write $C_f(X)$ for the ring of Freudenthal functions on $X$.

**Lemma 51.** Let $X$ be a proper metric space. A bounded continuous function $\varphi : X \to \mathbb{R}$ is Freudenthal if and only if it can be extended to $X^f$.

**Proof.** Without loss of generality assume $X$ is $R$-discrete for some $R > 0$.

Suppose a bounded continuous function $\varphi : X \to \mathbb{R}$ is Freudenthal. If $\mathcal{F}$ is an ultrafilter on $X$ define $\bar{\varphi}(\mathcal{F}) = \mathcal{F}$-lim $\varphi$. We show $\bar{\varphi}$ is well defined: Let $\mathcal{F}, \mathcal{G}$ be ultrafilters on $X$ with $\mathcal{F}$-lim $\varphi \neq \mathcal{G}$-lim $\varphi$. Then $X = \varphi^{-1}((1, \infty, \varepsilon_{\mathcal{F}}(\mathcal{G}) \cdot \varepsilon_{\mathcal{F}}(\mathcal{G} \cdot \varepsilon_{\mathcal{F}}(\mathcal{G}))) \cup \varphi^{-1}((1, \varepsilon_{\mathcal{F}}(\mathcal{G}) \cdot \varepsilon_{\mathcal{F}}(\mathcal{G}))) \cup \varphi^{-1}((1, \varepsilon_{\mathcal{F}}(\mathcal{G})) \cdot \varepsilon_{\mathcal{F}}(\mathcal{G})))$. Thus $\bar{\varphi}(\mathcal{F}) \neq \bar{\varphi}(\mathcal{G})$.

We show $\bar{\varphi}$ is continuous: Choose an interval $I \subseteq \mathbb{R}$ such that $\mathrm{im} \varphi \subseteq I$ and consider $\varphi$ as a map $X \to I$. Let $S, T \subseteq I$ be subsets such that $S \cap T = \emptyset$. Then there is some subset $C \subseteq I$ with $S \subseteq C, T \subseteq C^c$ and $C \cap T = \emptyset, C^c \cap S = \emptyset$. Then we obtain $\varphi^{-1}(1) \supseteq \varphi^{-1}(S), \varphi^{-1}(C) \supseteq \varphi^{-1}(T)$ and $X = \varphi^{-1}(1) \cup \varphi^{-1}(C)$. Now let $R \geq 0$ be a number then there exists a bounded set $B \subseteq X$ such that $d(x,y) \leq R, \varphi(x) \neq \varphi(y)$ implies $x,y \in B$. Thus $x \in \varphi^{-1}(C), y \in \varphi^{-1}(C^c)$ and $d(x,y) \leq R$ then $x,y \in B$. This implies $\varphi^{-1}(C) \neq \varphi^{-1}(C^c)$. Thus $\varphi^{-1}(1) \neq \varphi^{-1}(C) \cup \varphi^{-1}(C^c)$. This shows $\bar{\varphi}(S) \cap \bar{\varphi}(T) = (\varphi^{-1}(S) \cup \varphi^{-1}(T)) \cap (\varphi^{-1}(S) \cup \varphi^{-1}(T)) = \emptyset$.

Thus $\bar{\varphi}$ is continuous.

Now we show $C_f(X)$ separates points of $\partial_f(X) = \tilde{X}^f \setminus X$. If $\mathcal{F}, \mathcal{G}$ are ultrafilters on $X$ with $\mathcal{F} \bar{\varphi} \mathcal{G}$ then there are $A \in \mathcal{F}, B \in \mathcal{G}$ with $A \neq \mathcal{F} B$. Then there exists $C \subseteq X$ with $A \subseteq C, B \subseteq C^c$ and $C \neq \mathcal{F} C^c$. Define

$$\varphi : X \to \mathbb{R}$$

$$x \mapsto \begin{cases} 1 & x \in C \\ 0 & x \in C^c. \end{cases}$$

Then $\varphi$ is a Freudenthal function. Now the extension $\bar{\varphi}$ of $\varphi$ separates $\mathcal{F}$ from $\mathcal{G}$. Then by [BYS82] the ring $C_f(X)$ determines the compactification $X^f$ of $X$. 

**Theorem 52.** Let $X$ be a proper metric space. The boundary of the Freudenthal compactification $\Omega X = \varepsilon X \setminus X$ of $X$ is totally disconnected. If $(\tilde{X}, X)$ is another coarse compactification whose boundary is totally disconnected then it factors through $\varepsilon X$. The association $\Omega$ is a functor that maps coarse maps modulo closed to continuous maps.

**Remark 53.** Compare this result with [Pes90, Theorem 1]. The Freudenthal compactification of a topological space with nice properties is universal among compactifications with totally disconnected boundary.

**Proof.** At first we show $\Omega X$ is totally disconnected. It is sufficient to show that there exists a basis consisting of clopen subsets in $\partial_{\lambda_f}(X)$. If $A \subseteq X$ has the property $A \neq A^c$ then $\varphi(A) = \varphi(A^c)$.
is both open and closed. Now we show \((\text{cl}(A))_{A \in \mathcal{A}}\) are a basis for the topology on \(\Omega X\). Note already \((\text{cl}(A))_{A \in \mathcal{A}}\) are a base for a topology on \(\Omega X\). Let \(A \subseteq X\) be a subset and \(F \in \text{cl}(A)^c\) be a \(\mathcal{F}\)-ultrafilter. Then there exists \(B \in F\) with \(B \not\subseteq A\). Thus there exists \(A' \supseteq A, B' \supseteq B\) with \(A' \cup B' = X\) and \(A' \not\subseteq B'\). This implies \(A' \not\subseteq B'\) thus \(F \in \text{cl}(A')^c \subseteq \text{cl}(A)^c\). Now \(A'\) is of the type \(A' \not\subseteq A\).

Suppose \(r\) is a close relation on \(X\) such that \(\partial_r X\) is totally disconnected. Then there exists a basis of clopen sets on \(\partial_r X\). Let \(\pi \subseteq \partial_r X\) be a clopen subset. Thus \(\pi \bar{\pi} \subseteq X\). This implies there exist \(A, B \subseteq X\) with \(A \not\subseteq B\) and \(\pi \subseteq \text{cl}(A), \bar{\pi} \subseteq \text{cl}(B)\). In particular \(A \not\subseteq B\) and

\[
\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B) \\
\supseteq \pi \cup \bar{\pi} \\
= \partial_r X.
\]

This implies the inclusion \(A \cup B \rightarrow X\) is coarsely surjective. Thus \(A \not\subseteq B\) which implies \(\pi \not\subseteq \bar{\pi}\).

Thus the unique map \(\varepsilon X \rightarrow \partial_r X\) extending the identity on \(X\) is well-defined and continuous.

Now we show \(\Omega\) is a functor. Let \(\varphi : X \rightarrow Y\) be a coarse map between metric spaces. It is sufficient to show that \(A \not\subseteq B\) implies \(\varphi(A) \not\subseteq \varphi(B)\). Suppose \(\varphi(A) \not\subseteq \varphi(B)\). Then there exist \(A' \supseteq \varphi(A), B' \supseteq \varphi(B)\) with \(A' \cup B' = Y\) and \(A' \not\subseteq B'\). This implies \(\varphi^{-1}(A') \supseteq A, \varphi^{-1}(B') \supseteq B, \varphi^{-1}(A') \cup \varphi^{-1}(B') = X\) and \(\varphi^{-1}(A') \not\subseteq \varphi^{-1}(B')\). Thus \(A \not\subseteq B\).

**Corollary 54. (Protasov)** Let \(X\) be a proper metric space. If \(\text{asdim}(X) = 0\) then \(\nu(X)\) and \(\Omega(X)\) coincide.

**Remark 55.** Compare this result with [Pro03, Lemma 4.3]. We prove the same result using universal properties.

**Proof.** Since \(\text{asdim}(X) = 0\) the space \(\nu(X)\) is zero dimensional by [DKU98, Dra00]. This implies \(\nu(X)\) is totally disconnected. By Theorem 52 there exists a unique surjective map \(h(X) \rightarrow \varepsilon(X)\) which extends the identity on \(X\). Now by Remark 54 there exists a unique surjective map \(h(X) \rightarrow \varepsilon(X)\). Since the composition of both maps \(h(X) \rightarrow h(X)\) and \(\varepsilon(X) \rightarrow \varepsilon(X)\) are unique surjective they agree with the identity. This proves the spaces \(\Omega(X), \nu(X)\) are homeomorphic.

## 8 Gromov boundary

The Gromov boundary is the last interesting example in this paper. There is a quotient map from the Higson compactification to the Gromov compactification since it is a coarse compactification. We are going to present in this chapter maps in the other direction.

If \(X\) is a metric space and \(x_0 \in X\) a fixed point then the **Gromov product** of two points \(x, y \in X\) is defined as

\[
(x|y) := 1/2(d(x,x_0) + d(y,x_0) - d(x,y)).
\]

**Definition 56. (Gromov boundary)** Let \(X\) be a proper geodesic hyperbolic metric space. A continuous function \(\varphi : X \rightarrow \mathbb{R}\) is called **Gromov** if for every \(\varepsilon > 0\) there exists \(K > 0\) such that

\[
(x|y) > K \rightarrow |\varphi(x) - \varphi(y)| < \varepsilon.
\]

The Gromov functions determine a compactification of \(X\) called the **Gromov compactification** \(gX\). The boundary \(\partial X = gX \setminus X\) is called the **Gromov boundary**.
Remark 57. Let \( X \) be a proper hyperbolic geodesic metric space. Two sequences \((a_i)_i, (b_i)_i \subseteq X\) converge to the same point on the Gromov boundary \( \partial X \) if and only if

\[
\liminf_{i,j \to \infty} |a_i - b_j| = \infty.
\]

If \( p \in \partial X \) define

\[
U_1(p, r) = \{ q \in \partial(X) : [(x_n)_n] = p, [(y_n)_n] = q, \liminf_{i,j \to \infty} |x_i - y_j| \geq r \}
\]

and

\[
U_2(p, r) = \{ y \in X : [(x_n)_n] = p, \liminf_{i,j \to \infty} |x_i - y| \geq r \}.
\]

Then \( \{U_1(p, r) \cup U_2(p, r) : r \geq 0\} \) is a neighborhood basis of \( p \) in \( \partial X \).

**Proof.** The first part is [FOY18 Proposition 4.3]. The second part is [KB02 Definition 2.13]. \( \square \)

Example 58. (**Gromov boundary**) Let \( A, B \subseteq \mathbb{R} \) be subsets of a hyperbolic proper metric space. Define \( A \upharpoonright B \) if there are sequences \((a_i)_i \subseteq A, (b_i)_i \subseteq B\) such that

\[
\liminf_{i,j \to \infty} |a_i - b_j| = \infty.
\]

If \( A \upharpoonright B \) then there exist unbounded sequences \((a_i)_i \subseteq A, (b_i)_i \subseteq B\) and some \( R \geq 0 \) such that \( d(a_i, b_i) \leq R \) for every \( i \). This implies \( \liminf_{i,j \to \infty} |a_i - b_j| = \infty \) thus \( A \upharpoonright B \). By [CS19 Proposition 9.8] the relation \( \upharpoonright \) is a coarse proximity relation, Thus axioms 1,2,4,5 of Definition [18] hold. Axiom 3 of Definition [18] holds trivially, thus \( \upharpoonright \) is a large-scale proximity relation.

Example 59. (**Gromov boundary**) Let \( X \) be a hyperbolic geodesic proper metric space. By Remark [37] we obtain \( \partial_{\upharpoonright} = \partial(X) \). Here the right side denotes the Gromov boundary of \( X \).

Remark 60. If \( X \) is a hyperbolic metric space and \( \gamma, \delta : \mathbb{Z}_+ \to X \) are quasigeodesic rays in \( X \) then \( \gamma(\mathbb{Z}_+) \upharpoonright \delta(\mathbb{Z}_+) \) implies there exists some entourage \( E \subseteq X \times X \) with \( E[\gamma(\mathbb{Z}_+)] \supseteq \delta(\mathbb{Z}_+) \) and \( E[\delta(\mathbb{Z}_+)] \supseteq \gamma(\mathbb{Z}_+) \).

**Proof.** By [Roe03 Definition 6.16] a map \( \gamma : \mathbb{Z}_+ \to X \) is a quasigeodesic ray if there are constants \( R > 0, S \geq 0 \) with

\[
R^{-1}|i - j| - S \leq d(\gamma(i), \gamma(j)) \leq R|i - j| + S
\]

for every \( i, j \in \mathbb{Z}_+ \). It follows from [Roe03 Theorem 6.17] that there exists some \( T \geq 0 \) such that \( d(\gamma(\mathbb{Z}_+), \delta(\mathbb{Z}_+)) \leq T \). \( \square \)

Remark 61. Let \( X \) be a geodesic metric space and \( \tilde{\gamma} : \mathbb{R}_+ \to X \) a geodesic ray. If \( \gamma : \mathbb{Z}_+ \to X \) is close to \( \tilde{\gamma} \) then it is coarsely injective coarse and the induced map \( \nu(\gamma) : \nu(\mathbb{Z}_+) \to \nu(X) \) is a closed embedding.

**Proof.** This is [Har19c Lemma 39]. \( \square \)

Theorem 62. Let \( X \) be a hyperbolic geodesic proper metric space. Then there is a closed embedding \( \Phi : \nu(\mathbb{Z}_+) \times \partial X \to \nu(X) \).
Remark 63. Compare this result with [BS07, Theorem 10.1.2] which states that for every proper geodesic hyperbolic metric space the inequality
\[ \text{asdim}(X) \geq \dim(\partial X) + 1 \]
holds. Note asdim\((Z_+) = 1\) and asdim\((X) = \dim(\nu(X))\) for every proper metric space [DKU98, [Dra00]. By [Mor77, Theorem 3] we obtain \(\dim(\nu(Z_+) \times \partial X) = \dim(\partial X) + 1\). Thus we obtain a new proof for the above inequality.

Proof. Let \(S \subseteq X\) be an \(R\)-discrete for some \(R > 0\) subspace such that the inclusion \(S \to X\) is coarsely surjective. Let \(p \in \partial(X)\) be a point. Then \(p\) is represented by an isometry \(\tilde{\gamma} : Z_+ \to X\).

Now choose \(\gamma : Z_+ \to S\) that \(\gamma\) point represented by \(\tilde{\gamma}\) onto its image. Thus we have shown \(\Phi(\gamma) : \nu(Z_+) \times \partial X \to \nu(X)\) of Theorem 32 restricted to the image of \(\Phi\).

We first show that \(\Phi(\gamma) : \nu(Z_+) \times \partial X \to \nu(X)\) is bijective and continuous.

Thus \(\Phi(\gamma) : \nu(Z_+) \times \partial X \to \nu(X)\) is bijective and continuous. Thus \(\Phi(\gamma) : \nu(Z_+) \times \partial X \to \nu(X)\) is bijective and continuous.

8 GROMOV BOUNDARY

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Proposition 64. If \(T\) is a tree then the space \(\Phi(\partial(T) \times \nu(R_+))\) is a retract of \(\nu(T)\).

Proof. We first show that \(\Phi(\partial(T) \times \nu(R_+))\) is a retract of \(\mathcal{F} := \bigcup_{A \in \xi, |A| = 1} \text{cl}(A) \subseteq \nu(T)\):

If \(F \subseteq \text{cl}(A)\) with \(|\partial(A)| = 1\) then there is a geodesic ray \(\gamma\) on \(T\) such that \(F\) converges to the point represented by \(\gamma\) in the Gromov compactification. Since \(T\) is CAT(0) and \(\gamma(R_+)\) is convex and complete [BH99, Proposition II.2.4] provides us with a projection map \(\pi : T \to \gamma(R_+)\) such that \(d(x, \pi(x)) = d(x, \gamma(R_+))\) for every \(x \in T\).

Then \(\{\pi(A) : A \in F, \partial(A) = \gamma\}\) define a base for a \(\lambda\)-ultrafilter \(F_1\) on \(T\):

1. If \(A, B \in F\) then \(A \wedge B\). Thus there exist unbounded sequences \((a_i) \subseteq A, (b_i) \subseteq B\) with \(d(a_i, b_i) \leq R\). Then \(\pi(a_i) \subseteq \pi(A), \pi(b_i) \subseteq \pi(B)\). By [BH99, Proposition II.2.4] the map \(\pi\) does not increase distances. Thus \(d(\pi(a_i), \pi(b_i)) \leq R\) for every \(i\). Since \(\lim_{i \to \infty} d(\pi(a_i), \pi(b_i)) = \infty\) the sequence \((\pi(a_i))\) is not bounded. This way we have shown that \(\pi(A) \wedge \pi(B)\).

2. If \(A \cup B \in F_1\) then \(A \cup B = \pi(C)\) for some \(C \subseteq F\). Define \(A' = \{a \in C : \pi(a) \in A\}\) and \(B' = \{b \in C : \pi(b) \in B\}\). Then \(\pi(A') = A, \pi(B') = B\) and \(A' \cup B' = C\). Now \(A' \subseteq F\) or \(B' \subseteq F\) which implies \(A \in F_1\) or \(B \in F_1\).
Now define a map
\[ r : \varpi \to \nu(T) \]
\[ F \mapsto F_1. \]

Let \( F, G \in \nu(T) \) be two elements with \( F \subseteq G \). If \( A \in F_1, B \in G_1 \) then there are \( A' \in F, B' \in G \) with \( \pi(A') = A, \pi(B') = B \). Now \( A' \cap B' = B \) implies \( \pi(A') \cap \pi(B') \) as above. This implies \( F \subseteq G \).

Now we show \( r \) is continuous on \( \varpi \): Suppose \( A, B \subseteq T \) are subsets with \( (\cl(A) \cap \varpi) \cap (\cl(B) \cap \varpi) \neq \emptyset \). Then
\[
\bigcap_{i \leq A, \|\theta(A)\|=1} \cl(A') \cap \bigcap_{i \leq A, \|\theta(A)\|=1} \cl(B')
\]
\[ = \bigcup_{i \leq A, \|\theta(A)\|=1} \cl(\pi(A')) \cap \bigcup_{i \leq A, \|\theta(A)\|=1} \cl(\pi(B'))
\]
\[ = \emptyset. \]

To see the last inequality choose \( A' := (a_i), b' :=(b_i) \) unbounded with \( d(a_i, b_i) \leq R \) for every \( i \) and some \( R \geq 0 \). If necessary we can choose a subsequence of \( (a_i) \), such that \( (a_i) \) converges to a point on the Gromov boundary. Then \( (b_i) \) converges to the same point. Thus we can assume \( \|\theta(A')\| = 1 = \|\theta(B')\| \). Then \( \pi(A') \cap \pi(B') \), in fact both sets are finite Hausdorff distance apart. Thus \( \cl(\pi(A')) \cap \cl(\pi(B')) = \emptyset \). We just showed \( r \) is uniformly continuous with regard to the unique uniformity on the compact space \( \nu(T) \).

Note that \( \bigcup_{A \subseteq T, \|\theta(A)\|=1} \cl(A) \) is dense in \( \nu(T) \): Consider the closure of \( \bigcup_{A \subseteq T, \|\theta(A)\|=1} \cl(A) \).

If \( B \subseteq T \) is a subset then there exists a sequence \( (b_i) \subseteq B \) such that \( (b_i) \) converges to a point \( \gamma \in \partial(X) \) in the Gromov compactification. This means \( \|\theta((b_i))\| = 1 \). Thus \( \bigcup_{A \subseteq T, \|\theta(A)\|=1} \cl(A) \subseteq \nu(T) \).

Then [Eng89, Theorem 8.3.10] implies the retract map \( r \) can be extended to \( \nu(T) \).

**Remark 65.** The results in Theorem 53 and Proposition 54 are functorial: If \( \alpha : T \to S \) is a coarse map between trees such that \( \alpha \circ \gamma \) is coarsely injective if \( \gamma : \mathbb{R}_+ \to X \) is coarsely injective coarse then there is a continuous map
\[
r \circ \nu(\alpha) \circ \Phi : \partial(T) \times \nu(\mathbb{R}_+) \to \partial(S) \times \nu(\mathbb{R}_+)
\]
\[ ((\gamma), [\mathcal{F}]) \mapsto ((\alpha \circ \gamma), [\mathcal{F}]). \]

If \( \alpha \) is a coarse equivalence then \( \partial(\alpha) : \partial(T) \to \partial(S) \) is a homeomorphism since the Gromov boundary is a functor on coarse equivalences. This implies \( r \circ \nu(\alpha) \circ \Phi \) is an isomorphism in the topological category.

### 9 Remarks

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