GRÖBNER BASES OF SYMMETRIC IDEALS

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Abstract. In this article we present two new algorithms to compute the Gröbner basis of an ideal that is invariant under certain permutations of the ring variables and which are both implemented in Singular (cf. [DGPS12]). The first and major algorithm is most performant over finite fields whereas the second algorithm is a probabilistic modification of the modular computation of Gröbner bases based on the articles by Arnold (cf. [A03]), Idrees, Pfister, Steidel (cf. [IPS11]) and Noro, Yokoyama (cf. [NY12], [Y12]). In fact, the first algorithm that mainly uses the given symmetry, improves the necessary modular calculations in positive characteristic in the second algorithm. Particularly, we could, for the first time even though probabilistic, compute the Gröbner basis of the famous ideal of cyclic 9-roots (cf. [BF91]) over the rationals with Singular.

1. Introduction

Computing the Gröbner basis of an ideal is a powerful tool in commutative algebra, with applications in algebraic geometry and singularity theory. The first general algorithm was proposed by Buchberger in 1965 (cf. [Bu65]).

There are previous works by Aschenbrenner and Hillar on symmetric Gröbner bases in infinite-dimensional rings (cf. [AH07], [AH08]) and Faugère and Rahmany using SAGBI-Gröbner bases for solving systems of polynomial equations with symmetries (cf. [FR09]).

Within this article we improve the computation of Gröbner bases in case that the input ideal has some special symmetry-character. Consider, for example, the ideal $I = \langle x^2y^2 - z, xy - 2y + 3z, xy - 2x + 3z \rangle \subseteq \mathbb{Q}[x,y,z]$. Then $I$ does not vary if one interchanges the variables $x$ and $y$, and we say that $I$ is symmetric with respect to the permutation $x \leftrightarrow y$. In the following we use this property to manipulate the ideal by an appropriate linear transformation and apply Buchberger’s algorithm subsequently.

We start in Section 2 by presenting some basic notations and definitions. In Section 3 we introduce the symmetric Gröbner basis algorithm, and state a theoretical result that justifies the impact of the symmetry in Proposition 3.8. Section 4 combines the symmetric algorithm of Section 3 with modular methods which results in a probabilistic but performant algorithm over the rationals. Moreover, examples and timings are provided in Section 3.2 and Section 4.2 respectively.

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2. Basic notations and definitions

Let $\sigma \in S_n := \text{Sym}(\{1, \ldots, n\})$ be a permutation. The order of $\sigma$ is the minimal natural number $k \in \mathbb{N}_{\geq 0}$ such that $\sigma^k = \text{id}$, in particular $\text{ord}(\sigma) := \#((\sigma)) < \infty$. In order to describe $\sigma$ properly, we make use of the following well-known result concerning the representation of permutations.

**Definition 2.1.** Let $\sigma \in S_n$ be a permutation. Then there exists a natural number $\vartheta(\sigma)$ and a finite disjoint partition $\{1, \ldots, n\} = \bigsqcup_{i=1}^{\vartheta(\sigma)} \{e_{i,1}, \ldots, e_{i,l_i}\}$ such that

$$\sigma = (e_{1,1} \ldots e_{1,l_1}) \cdots (e_{\vartheta(\sigma),1} \ldots e_{\vartheta(\sigma),l_{\vartheta(\sigma)}})$$

with $l_1 + \ldots + l_{\vartheta(\sigma)} = n$ and $0 \leq l_i \leq n$ for all $i \in \{1, \ldots, \vartheta(\sigma)\}$. The cycles $(e_{i,1} \ldots e_{i,l_i})$ are up to alignment uniquely determined, and we call this representation the cycle decomposition of $\sigma$. The tuple $(l_1, \ldots, l_{\vartheta(\sigma)})$ is called the cycle type of $\sigma$ if $l_1 \leq \ldots \leq l_{\vartheta(\sigma)}$.

Note that having the cycle decomposition of a permutation $\sigma$ it holds $\text{ord}(\sigma) = \text{lcm}(l_1, \ldots, l_{\vartheta(\sigma)})$. From now on we assume that all considered permutations $\sigma \in S_n$ are given in cycle decomposition.

Now let $K$ be a field and $X := \{x_1, \ldots, x_n\}$ be a set of indeterminates, then $\sigma \in S_n$ induces a canonical automorphism on the polynomial ring over $K$ in these indeterminates, $K[X]$, via $\varphi_\sigma : K[X] \longrightarrow K[X], \ x_i \longmapsto x_{\sigma(i)}$. By abuse of notation we always write $\sigma$ instead of $\varphi_\sigma$, i.e. we identify the group $S_n$ as a subgroup of the automorphism group $\text{Aut}(K[X])$.

**Definition 2.2.** Let $I \subseteq K[X]$ be an ideal and $\sigma \in \text{Aut}(K[X])$ be an automorphism. Then $I$ is called $\sigma$-symmetric if $\sigma(I) = I$. Moreover, let $S \subseteq \text{Aut}(K[X])$ be a subgroup then we call $I$ $S$-symmetric if it is $\sigma$-symmetric for all $\sigma \in S$.

Every subgroup of $S_n$ has only finitely many elements and is therefore finitely generated. Hence, let $S = \langle \sigma_1, \ldots, \sigma_m \rangle \subseteq S_n$ then an ideal $I \subseteq K[X]$ is $S$-symmetric if and only if it is $\sigma_i$-symmetric for all $i \in \{1, \ldots, m\}$. In particular, if an ideal is $\sigma$-symmetric then it is $\langle \sigma \rangle$-symmetric.

Moreover, given an ideal $I \subseteq K[X]$ we can always choose a finite set of polynomials $F_I = \{f_1, \ldots, f_s\}$ such that $I = \langle F_I \rangle$. Thus, if $I$ is $\sigma$-symmetric with $\sigma \in \text{Aut}(K[X])$ we even may assume that $\sigma(F_I) = F_I$ by possibly adding some polynomials to $F_I$.

**Example 2.3.** The ideal $I = \langle x^2y^2 - z, \ xy - 2y + 3z, \ xy - 2x + 3z \rangle \subseteq \mathbb{Q}[x, y, z]$ is obviously $\sigma$-symmetric for $\sigma = (12)(3) \in S_3$.

We denote by $\text{Mon}(X)$ the set of monomials. Moreover, if $>$ is a monomial ordering and $f \in K[X]$ a polynomial, then we denote by $\text{LC}(f)$ the leading coefficient of $f$, by $\text{LM}(f)$ the leading monomial of $f$, by $\text{LT}(f)$ the leading term (leading monomial with leading coefficient) of $f$, and by $\text{tail}(f) = f - \text{LT}(f)$ the tail of $f$ with respect to the ordering $>$. In particular, with our notation it holds $\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f)$.

**Convention 2.4.** In the following $>$ is a degree ordering, and we always consider reduced Gröbner bases $G$, that is $0 \notin G$, $\text{LM}(g) \nmid \text{LM}(f)$ for any two elements $f \neq g$ in $G$, and $\text{LC}(g) = 1$ respectively no monomial of $\text{tail}(g)$ is contained in the leading ideal of $G$ for any $g \in G$. 

3. Gröbner bases using symmetry

Within this section we describe how to achieve an improvement of the Gröbner basis computation of a \( \sigma \)-symmetric ideal \( I \subseteq K[X] \) by using its symmetric property. The basic idea is the construction and usage of an appropriate linear transformation \( \tau \in \text{Aut}(K[X]) \) which "diagonalises" \( \sigma \) and respects the \( \sigma \)-symmetry of \( I \). It turns out that in many cases the usual Gröbner basis computation on the transformed side is much faster than the computation on the original side. Since the pull back of this Gröbner basis is in general not a Gröbner basis anymore we have to add another Gröbner basis computation. Nevertheless, this indirection effects an enormous speed up compared to the usual Gröbner basis algorithm (cf. Section 3.2).

We assume that the tuple \((\sigma, K)\) with \( \text{ord}(\sigma) = k \in \mathbb{N} \) always satisfies \( \text{char}(K) \not| k \) and \( K \) has a \( k \)-th primitive root of unity \( \xi_k \).

**Remark 3.1.** We can always achieve this assumption by possibly adjoining \( \xi_k \). In particular, we can swap to \( K[\xi] \) by working over the field \( K[\sigma]/\Phi_k(\sigma) \) where \( \Phi_k(\sigma) \) is the \( k \)-th cyclotomic polynomial.

**3.1. The symmetric Gröbner basis algorithm.** We start by illuminating the basis for the symmetric Gröbner basis algorithm from a character theoretical point of view and in terms of linear algebra.

Therefore, we consider the \( n \)-dimensional \( K \)-subvector space \( V = \langle x_1, \ldots, x_n \rangle_K \) of the infinite-dimensional \( K \)-vector space \( K[X] \). Then due to our assumption that \( \text{char}(K) \not| \#(\langle \sigma \rangle) \), character theory guarantees that every representation of \( \langle \sigma \rangle \subseteq S_n \) is a direct sum of irreducible representations (cf. [Se96, Theorem 2]), and all irreducible representations of \( \langle \sigma \rangle \subseteq S_n \) have degree 1 since \( \langle \sigma \rangle \subseteq S_n \) is an abelian group (cf. [Se96] Theorem 9]). In particular, the representation \( \rho : \langle \sigma \rangle \rightarrow \text{Aut}(V) \) is diagonalisable, i.e. \( V = \bigoplus_{i=1}^{n} V_i \) with \( V_i = \langle y_i \rangle_K \) and \( \rho(\sigma)(y_i) = \xi_k^{\nu_i} \cdot y_i \) for some \( 0 \leq \nu_i \leq k - 1 \).

In terms of linear algebra we have the following quite simple proposition which, together with its proof, forms the basis of the symmetric Gröbner basis algorithm.

**Proposition 3.2.** Let \( \sigma \in S_n \) have cycle type \( (l_1, \ldots, l_{\vartheta(\sigma)}) \). Then \( \sigma \in \text{Aut}(V) \) is diagonalisable with eigenvalues \( \{\xi_k^{j\ell/m} | 1 \leq m \leq \vartheta(\sigma), 0 \leq j \leq \ell_m - 1\} \).

**Proof.** Let \( \sigma = (e_1 \ldots e_n) \in S_n \) with \( \{e_1, \ldots, e_n\} = \{1, \ldots, n\} \) be an \( n \)-cycle. The columns of the representation matrix \( M(\sigma, X) \) of \( \sigma \in \text{Aut}(V) \) with respect to the \( K \)-basis \( X = \{x_1, \ldots, x_n\} \) of \( V \) are just the permuted unit vectors of \( K^n \). Hence, \( M(\sigma, X) \) is a unitary matrix and in particular diagonalisable. Moreover, let \( C = tI_n - M(\sigma, X) \in \text{Mat}(n \times n, K[t]) \) then the characteristic polynomial of \( \sigma \in \text{Aut}(V) \) is

\[
\chi_{\sigma} = \det(C) = \sum_{\pi \in S_n} \text{sign}(\pi) \cdot c_{1\pi(1)} \cdots c_{n\pi(n)} = t^n + \text{sign}(\sigma) \cdot (-1)^n = t^n + (-1)^{n-1} \cdot (-1)^n = t^n - 1,
\]

and \( \{1, \xi_n, \xi_n^2, \ldots, \xi_n^{n-1}\} \) are exactly the eigenvalues of \( \sigma \). Now, consider the combinatorial set

\[
Y := \left\{ y_{e_i} = \sum_{j=1}^{n} \xi_n^{(i-1)(j-1)} \cdot x_{\sigma^{j-1}(e_i)} \mid 1 \leq i \leq n \right\}.
\]
Note that $Y$ is a $K$-basis of $V$ since the coefficients of each $y_{e_i}$ are just different powers of the primitive root of unity $\xi_n$, and $\{x_{\sigma^{-1}(e_i)} \mid 1 \leq j \leq n\} = X$ is a $K$-basis of $V$ itself.

Let $i \in \{1, \ldots, n\}$, then we easily compute that $\sigma(y_{e_i}) = \xi_n^{i-1} y_{e_i}$. Consequently, $y_i$ is the eigenvector corresponding to the eigenvalue $\xi_n^{i-1}$, and the representation matrix $M(\sigma, Y)$ of $\sigma \in \text{Aut}(V)$ with respect to $Y$ is diagonal.

Now, let $\sigma \in S_n$ have cycle type $(l_1, \ldots, l_{\varrho(\sigma)})$ with cycle decomposition $\sigma = \sigma_1 \cdots \sigma_{\varrho(\sigma)}$ and $\sigma_m = (e_1, \ldots, e_{m,l_m})$ for $1 \leq m \leq \varrho(\sigma)$. Then we have $\text{ord}(\sigma_m) = l_m$, $\text{ord}(\sigma) = k = \text{lcm}(l_1, \ldots, l_{\varrho(\sigma)})$, and

$$\xi_{l_m} = \xi_k^{k/l_m} \in K$$

is an $l_m$-th primitive root of unity. We set $X_m = \{x_{e_{m,1}}, \ldots, x_{e_{m,l_m}}\}$ such that $\sigma_m \in \text{Aut}(V_m)$ with $V_m = (X_m)_K$, and $X = X_1 \cup \ldots \cup X_{\varrho(\sigma)}$ is a $K$-basis of $V = V_1 \oplus \ldots \oplus V_{\varrho(\sigma)}$. Analogously to the $n$-cycle case we obtain the combinatorial sets

$$Y_m := \left\{ y_{e_{m,i}} = \sum_{j=1}^{l_m} \xi_{l_m}^{(i-1)(j-1)} \cdot x_{\sigma_{m,1}^{j-1}(e_{m,i})} \mid 1 \leq i \leq l_m \right\}$$

of eigenvectors of $\sigma_m \in \text{Aut}(V_m)$ so that the representation matrix $M(\sigma_m, Y_m)$ of $\sigma_m \in \text{Aut}(V_m)$ with respect to the $K$-basis $Y_m$ of $V_m$ is diagonal with eigenvalues $\{\xi_{l_m}^j, \xi_{l_m}^{j^2}, \ldots, \xi_{l_m}^{j^{l_m-1}}\}$. Hence, $Y = Y_1 \cup \ldots \cup Y_{\varrho(\sigma)}$ is a $K$-basis of $V$, consists of eigenvectors of $\sigma \in \text{Aut}(V)$, and the representation matrix

$$M(\sigma, Y) = \begin{pmatrix} M(\sigma_1, Y_1) & & & \\ & \ddots & & \\ & & M(\sigma_{\varrho(\sigma)}, Y_{\varrho(\sigma)}) \end{pmatrix} \in \text{Mat}(n \times n, K)$$

of $\sigma \in \text{Aut}(V)$ with respect to $Y$ is diagonal with eigenvalues $\{\xi_{l_m}^j \mid 1 \leq m \leq \varrho(\sigma), 0 \leq j \leq l_m - 1\} = \{\xi_{l_m}^{k/j} \mid 1 \leq m \leq \varrho(\sigma), 0 \leq j \leq l_m - 1\}$. \hfill \square

**Remark 3.3.** Due to the constructive proof of Proposition 3.2 the eigenvectors of $\sigma \in \text{Aut}(V)$ can be obtained by purely combinatorial methods which is quite profitable from the algorithmic point of view. Hence, let us define the ring homomorphism

$$\tau : K[X] \rightarrow K[X]$$

$$x_{e_{m,i}} \mapsto y_{e_{m,i}} = \sum_{j=1}^{l_m} \xi_{l_m}^{(i-1)(j-1)} \cdot x_{\sigma_{m,1}^{j-1}(e_{m,i})}$$

which maps the ring-variables onto the eigenvectors of $\sigma \in \text{Aut}(V)$. Consequently, we can define another ring homomorphism $\sigma_\tau$ induced by the ring automorphism $\sigma \in \text{Aut}(K[X])$, the linear transformation $\tau \in \text{Aut}(K[X])$ and the commutative diagram

$$\begin{array}{ccc}
K[X] & \xrightarrow{\sigma} & K[X] \\
\downarrow{\tau} & & \downarrow{\tau} \\
K[X] & \xrightarrow{\sigma_\tau} & K[X]
\end{array}$$

so that $\sigma_\tau = \tau \sigma \tau^{-1}$ satisfies the property that $\sigma_\tau(x_i) = \xi_{\nu_i}^\nu x_i$ for suitable exponents $0 \leq \nu_i \leq k - 1$ and all $1 \leq i \leq n$. 

Example 3.4. Let $\sigma = (12)(3) \in S_3$ with $\text{ord}(\sigma) = 2$. Then consider $\xi_2 = -1 \in \mathbb{Q}$ and construct

$$\tau : \mathbb{Q}[x, y, z] \longrightarrow \mathbb{Q}[x, y, z]$$

$$x \mapsto (-1)^0 x + (-1)^0 y = x + y,$n

$$y \mapsto (-1)^1 y + (-1)^1 x = y - x,$n

$$z \mapsto z$$

as in Remark 3.3. Hence, $\tau$ is bijective with inverse $\tau^{-1}$ defined by

$$\tau^{-1}(x) = \frac{x - y}{2}, \quad \tau^{-1}(y) = \frac{x + y}{2}, \quad \tau^{-1}(z) = z.$$

Referring to Remark 3.3, $\sigma_\tau$ is induced by $\sigma_\tau = \tau_\sigma^{-1}$ and thus it holds

$$\sigma_\tau(x) = \tau_\sigma^{-1}(x) = \tau_\sigma\left(\frac{x - y}{2}\right) = \frac{y - x - x + y}{2} = -x,$n

$$\sigma_\tau(y) = \tau_\sigma^{-1}(y) = \tau_\sigma\left(\frac{x + y}{2}\right) = \frac{x + y + (y - x)}{2} = y,$n

$$\sigma_\tau(z) = \tau_\sigma^{-1}(y) = \tau_\sigma(z) = \tau(z) = z.$$

Notation 3.5. For a better understanding we will index objects that live on the transformed side by $\tau$.

As aforementioned respectively proven above, the induced automorphism $\sigma_\tau$ has a nice multiplication property on the ring variables that, however, is a priori not a sufficient reason for a fast Gröbner basis computation. But, in addition, the linear transformation $\tau$ also respects the symmetry of the input ideal.

Proposition 3.6. If the ideal $I \subseteq K[x]$ is $\sigma$-symmetric, then the transformed ideal $I_\tau := \tau(I) \in K[x]$ is $\sigma_\tau$-symmetric.

Proof. Let $I = \langle f_1, \ldots, f_r \rangle$. By definition of $\sigma_\tau$ we obtain for $\sigma(f_i) = f_j$ that $\sigma_\tau(\tau(f_i)) = \tau(\sigma(f_i)) = \tau(f_j)$. Thus, the ideal $I_\tau := \tau(I) = \langle \tau(f_1), \ldots, \tau(f_r) \rangle$ is $\sigma_\tau$-symmetric.

Example 3.7. Let $\geq_{dp}$ be the degree reverse lexicographical ordering and $\sigma, \tau, \sigma_\tau$ as in Example 3.3. Now we consider the $\sigma$-symmetric ideal

$$I = \langle x^2 y^2 - z, xy - 2y + 3z, xy - 2x + 3z \rangle \subseteq \mathbb{Q}[x, y, z]$$

and obtain that the transformed ideal

$$I_\tau := \tau(I) = \langle x^4 - 2x^2 y^2 + y^4 - z, -x^2 + y^2 + 2x - 2y + 3z,$n

$$-x^2 + y^2 - 2y - 3z \rangle \subseteq \mathbb{Q}[x, y, z]$$

is $\sigma_\tau$-symmetric.

Due to Proposition 3.2 and Proposition 3.6, we see that transforming the original ideal via $\tau$ still respects some symmetry. In particular, the ideal $I_\tau$ is $\sigma_\tau$-symmetric and applying the automorphism $\sigma_\tau$ on any variable, respectively monomial, effects just a multiplication by a power of a primitive root of unity. The advantage of this circumstance is the fact that the symmetry propagates during the process of

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1Degree reverse lexicographical ordering: Let $X^\alpha, X^\beta \in \text{Mon}(X)$. $X^\alpha \geq_{dp} X^\beta :\iff \deg(X^\alpha) > \deg(X^\beta)$ or $\deg(X^\alpha) = \deg(X^\beta)$ and $\exists 1 \leq i \leq n : \alpha_i = \beta_i, \ldots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i$, where $\deg(X^\alpha) = \alpha_1 + \ldots + \alpha_n$; cf. [ADB97].
computing a Gröbner basis of $I_{\tau}$ which influences the performance in a positive way. More precisely, the following proposition holds.

**Proposition 3.8.** Let $I_{\tau}$ be $\sigma_{\tau}$-symmetric, then a Gröbner basis $G_{\tau}$ of $I_{\tau}$ satisfies $\sigma_{\tau}(g_{\tau}) = \xi_{k_{\tau}}^{\nu_{\tau}} \cdot g_{\tau}$ for all $g_{\tau} \in G_{\tau}$ with suitable $0 \leq \nu_{\tau} \leq k - 1$.

**Proof.** Let $I_{\tau} = \langle F_{\tau} \rangle$ and $f, g \in F_{\tau}$. Due to the property of $\sigma_{\tau}$ we can define $X^{\alpha} := \text{LM}(f) = \text{LM}(\sigma_{\tau}(f)), X^{\beta} := \text{LM}(g) = \text{LM}(\sigma_{\tau}(g))$, $X^{\gamma} := \text{lcm}(X^{\alpha}, X^{\beta})$ and $\sigma_{\tau}(X^{\alpha}) = \xi_{k_{\tau}}^{\nu_{\alpha}} X^{\alpha}, \sigma_{\tau}(X^{\beta}) = \xi_{k_{\tau}}^{\nu_{\beta}} X^{\beta}, \sigma_{\tau}(X^{\gamma}) = \xi_{k_{\tau}}^{\nu_{\gamma}} X^{\gamma}$ for suitable $\nu_{\alpha}, \nu_{\beta}, \nu_{\gamma} \in \{0, \ldots, k - 1\}$. Then

$$\text{spoly}(f, g) = X^{\gamma - \alpha} \cdot f - \frac{\text{LC}(f)}{\text{LC}(g)} \cdot X^{\gamma - \beta} \cdot g$$

and again by the property of $\sigma_{\tau}$ it holds $\text{LC}(\sigma_{\tau}(f)) = \xi_{k_{\tau}}^{\nu_{\alpha}} \cdot \text{LC}(f)$ respectively $\text{LC}(\sigma_{\tau}(g)) = \xi_{k_{\tau}}^{\nu_{\beta}} \cdot \text{LC}(g)$. Thus

$$\sigma_{\tau}(\text{spoly}(f, g)) = \sigma_{\tau}(X^{\gamma - \alpha}) \cdot \sigma_{\tau}(f) - \frac{\text{LC}(f)}{\text{LC}(g)} \cdot \sigma_{\tau}(X^{\gamma - \beta}) \cdot \sigma_{\tau}(g)$$

$$= \xi_{k_{\tau}}^{\nu_{\alpha}} \cdot X^{\gamma - \alpha} \cdot \sigma_{\tau}(f) - \frac{\text{LC}(f)}{\text{LC}(g)} \cdot \xi_{k_{\tau}}^{\nu_{\beta}} \cdot X^{\gamma - \beta} \cdot \sigma_{\tau}(g)$$

$$= \xi_{k_{\tau}}^{\nu_{\alpha}} \cdot \left( X^{\gamma - \alpha} \cdot \sigma_{\tau}(f) - \frac{\text{LC}(f)}{\text{LC}(g)} \cdot \xi_{k_{\tau}}^{\nu_{\beta}} \cdot X^{\gamma - \beta} \cdot \sigma_{\tau}(g) \right)$$

$$= \xi_{k_{\tau}}^{\nu_{\alpha}} \cdot \left( X^{\gamma - \alpha} \cdot \sigma_{\tau}(f) - \frac{\text{LC}(\sigma_{\tau}(f))}{\text{LC}(\sigma_{\tau}(g))} \cdot X^{\gamma - \beta} \cdot \sigma_{\tau}(g) \right)$$

$$= \xi_{k_{\tau}}^{\nu_{\alpha} - \nu_{\gamma}} \cdot \text{spoly}(\sigma_{\tau}(f), \sigma_{\tau}(g)).$$

Moreover, there are $a_{h}, r \in K[X]$ such that $\text{spoly}(f, g) = \sum_{h \in F_{\tau}} a_{h} h + r$. Due to the above computation it follows

$$\text{spoly}(\sigma_{\tau}(f), \sigma_{\tau}(g)) = \xi_{k_{\tau}}^{\nu_{\alpha} - \nu_{\gamma}} \cdot \sigma_{\tau} \left( \sum_{h \in F_{\tau}} a_{h} h + r \right) = \sum_{h \in F_{\tau}} b_{h} h + \xi_{k_{\tau}}^{\nu_{\alpha} - \nu_{\gamma}} \cdot \sigma_{\tau}(r),$$

for suitable $b_{h} = \xi_{k_{\tau}}^{\nu_{\alpha} - \nu_{\gamma}} \cdot a_{\sigma_{\tau}^{-1}(h)} \in K[X]$ since $I_{\tau}$ respectively $F_{\tau}$ is $\sigma_{\tau}$-symmetric and consequently

$$\text{NF} \left( \text{spoly}(\sigma_{\tau}(f), \sigma_{\tau}(g)), F_{\tau} \right) = \xi_{k_{\tau}}^{\nu_{\alpha} - \nu_{\gamma}} \cdot \sigma_{\tau} \left( \text{NF} \left( \text{spoly}(f, g), F_{\tau} \right) \right).$$

This property implies that the reduced Gröbner basis $G_{\tau} = \{g_{i}^{\tau}, \ldots, g_{s}^{\tau}\}$ of $I_{\tau}$ satisfies $\sigma_{\tau}(g_{j}^{\tau}) = \xi_{k_{\tau}}^{\nu_{i,j}} \cdot g_{j}^{\tau}$ for suitable $i, j \in \{1, \ldots, s\}$ and $\nu_{i,j} \in \{0, \ldots, k - 1\}$. Moreover, it follows $\text{LM}(g_{i}^{\tau}) = \text{LM}(\sigma_{\tau}(g_{j}^{\tau})) = \text{LM}(g_{j}^{\tau})$, but since $G_{\tau}$ is reduced we conclude $g_{i}^{\tau} = g_{j}^{\tau}$. Hence, we have $\sigma_{\tau}(g_{i}^{\tau}) = \xi_{k_{\tau}}^{\nu_{i}} \cdot g_{i}^{\tau}$ for all $i \in \{1, \ldots, s\}$ with suitable $\nu_{i} \in \{0, \ldots, k - 1\}$. 

**Example 3.9.** Let $I_{\tau} = \langle x^{4} - 2x^{2}y^{2} + y^{4} - z, -x^{2} + y^{2} + 2x - 2y + 3z, -x^{2} + y^{2} - 2x - 2y + 3z \rangle \subseteq \mathbb{Q}[x, y, z]$ and $\sigma_{\tau} \in \text{Aut}(\mathbb{Q}[x, y, z])$ as obtained in Example 3.7. Then $I_{\tau}$ is $\sigma_{\tau}$-symmetric, and its Gröbner basis

$$G_{\tau} = \{ x, 12yz - 9z^{2} - 8y + 13z, y^{2} - 2y + 3z, 81z^{3} + 36z^{2} - 56y + 11z \}.$$
satisfies $\sigma(g_r) = (-1)^{\nu_{g_r}} \cdot g_r$ for suitable $\nu_{g_r} \in \{1, 2\}$ and all $g_r \in G_r$. Now, the reverse transformation of $G_r$ yields the set

$$\tau^{-1}(G_r) = \{ \frac{1}{2}x - \frac{1}{2}y, 6xz + 6yz - 9z^2 - 4x - 4y + 13z, \frac{1}{4}x^2 + \frac{1}{2}xy + \frac{1}{4}y^2 - x - y + 3z, 81z^3 + 36z^2 - 28x - 28y + 115z \}.$$ 

Obviously, just pulling back the Gröbner basis $G_r$ via $\tau^{-1}$ does not lead to a Gröbner basis of the input ideal $I$. Thus, we have to compute a Gröbner basis of the ideal $\langle \tau^{-1}(G_r) \rangle$ as well. Nevertheless, the advantage of this computation is the fact that the achieved property as described in Proposition 3.8 is respected by applying $\tau^{-1}$ on $G_r$. More precisely, the following proposition holds.

**Proposition 3.10.** $\sigma(g) = \xi_k^{\nu_g} \cdot g$ for all $g \in \tau^{-1}(G_r)$ and suitable $0 \leq \nu_g \leq k - 1$.

**Proof.** Let $g \in \tau^{-1}(G_r)$, i.e. there is an $g_r \in G_r$ such that $g = \tau^{-1}(g_r)$. Then due to Proposition 3.2 and Proposition 3.8 we have

$$\tau(\sigma(g)) = \tau(\sigma(\tau^{-1}(g_r))) = \sigma(\tau^{-1}(g_r)) = \xi_k^{\nu_g} \cdot g_r$$

for some $\nu_{g_r} \in \{0, \ldots, k - 1\}$. Hence, we obtain

$$\sigma(g) = \tau^{-1}(\tau(\sigma(g))) = \tau^{-1}(\xi_k^{\nu_g} \cdot g_r) = \xi_k^{\nu_g} \cdot \tau^{-1}(g_r) = \xi_k^{\nu_g} \cdot g.$$

This proves the proposition. $\square$

**Example 3.11.** Let $\sigma = (12)(3) \in S_3$ with $\text{ord}(\sigma) = 2$, $\xi_2 = -1 \in \mathbb{Q}$ and $\tau^{-1}(G_r)$ as obtained in Example 3.9. Then we compute

$$\sigma(\frac{1}{2}x - \frac{1}{2}y) = -(\frac{1}{2}x - \frac{1}{2}y),$$

$$\sigma(6xz + 6yz - 9z^2 - 4x - 4y + 13z) = 6xz + 6yz - 9z^2 - 4x - 4y + 13z,$$

$$\sigma(\frac{1}{4}x^2 + \frac{1}{2}xy + \frac{1}{4}y^2 - x - y + 3z) = \frac{1}{4}x^2 + \frac{1}{2}xy + \frac{1}{4}y^2 - x - y + 3z,$$

$$\sigma(81z^3 + 36z^2 - 28x - 28y + 115z) = 81z^3 + 36z^2 - 28x - 28y + 115z,$$

as claimed in Proposition 3.10.

The following diagram summarizes and illustrates our way of improving the computation of a Gröbner basis $G$ of a $\sigma$-symmetric ideal $I$.

$$(I, \sigma) \xrightarrow{\tau} (I_r, \sigma_r) \xrightarrow{\text{std}} \langle \tau^{-1}(G_r) \rangle \xleftarrow{\tau^{-1}} G_r \xrightarrow{\text{std}} G$$

Note that the linear transformation $\tau$ is defined in Remark 3.3 and the procedure std is implemented in SINGULAR and computes a Gröbner basis (standard basis) of the input.

Algorithm \texttt{com} computes the Gröbner basis of a $\sigma$-symmetric ideal $I$.\footnote{The corresponding procedures are implemented in SINGULAR in the library symodstd.lib.}

**Theorem 3.12.** Algorithm \texttt{com} terminates and is correct, i.e. the output $G$ is a Gröbner basis of the input $I$.\footnote{The corresponding procedures are implemented in SINGULAR in the library symodstd.lib.}
Algorithm 1 Symmetric Gröbner Basis Computation (symmStd)

Assume that $>$ is a degree ordering.

Input: $I \subseteq K[X]$ and $\sigma \in S_n$, such that $I$ is $\sigma$-symmetric.

Output: $G \subseteq K[X]$, the Gröbner basis of $I$.

1: $k = \text{ord}(\sigma)$;
2: if $k \mod \text{char}(K) = 0$ then
3: print Warning, algorithm is not applicable.
4: return $\emptyset$;
5: if $k = 2$ or $(\text{char}(K) - 1) \mod k = 0$ then
6: compute $\xi_k \in K$;
7: else
8: $K = K[a]/\Phi_k(a)$;
9: $\xi_k := a$;
10: compute $\tau \in \text{Aut}(K[X])$;
11: $G_\tau = \text{std}(I_\tau)$;
12: $G = \text{std}(\tau^{-1}(G_\tau))$;
13: return $G$;

Proof. Termination is clear and for proving correctness it suffices to show that $I = \langle \tau^{-1}(G_\tau) \rangle$ since $G$ is by definition a Gröbner basis of $\langle \tau^{-1}(G_\tau) \rangle$. Let $f \in I$ and $G_\tau = \langle g_1^\tau, \ldots, g_s^\tau \rangle$. Then $\tau(f) \in \tau(I) = I_\tau$ and consequently there are $a_1, \ldots, a_s \in K[X]$ such that $\tau(f) = \sum_{i=1}^s a_i \cdot g_i^\tau$ since $G_\tau$ is a Gröbner basis of $I_\tau$. Hence, we obtain

$$f = \tau^{-1}(\tau(f)) = \sum_{i=1}^s \tau^{-1}(a_i) \cdot \tau^{-1}(g_i^\tau) \in \langle \tau^{-1}(G_\tau) \rangle.$$ 

For the other inclusion let $g \in \langle \tau^{-1}(G_\tau) \rangle$. It follows that $\tau(g) \in \langle G_\tau \rangle = I_\tau = \tau(I)$ and moreover $g \in I$ since $\tau$ is an automorphism.

For illustration of Algorithm 1 we combine all previous examples.

Example 3.13. Again, let $I = \langle x^2y^2 - z, xy - 2y + 3z, xy - 2x + 3z \rangle \subseteq Q[x, y, z]$ and $\sigma = (12)(3) \in S_3$. Referring to Examples 3.4, 3.7, 3.9 and 3.11 we already obtained

$$I_\tau := \tau(I) = \langle x^4 - 2x^2y^2 + y^4 - z, -x^2 + y^2 + 2x - 2y + 3z, -x^2 + y^2 - 2x - 2y + 3z \rangle,$$

and its Gröbner basis

$$G_\tau = \{x, 12yz - 9z^2 - 8y + 13z, y^2 - 2y + 3z, 81z^3 + 36z^2 - 56y + 115z\}$$

with

$$\tau^{-1}(G_\tau) = \{\frac{1}{2}x - \frac{1}{2}y, 6xz + 6yz - 9z^2 - 4x - 4y + 13z, \frac{1}{2}x^2 + \frac{1}{2}xy + \frac{1}{2}y^2 - x - y + 3z, 81z^3 + 36z^2 - 28x - 28y + 115z\}.$$ 

Finally, we compute

$$G = \{x - y, 12yz - 9z^2 - 8y + 13z, y^2 - 2y + 3z, 81z^3 + 36z^2 - 56y + 115z\},$$

the Gröbner basis of $\langle \tau^{-1}(G_\tau) \rangle$ respectively $I$. 

3.2. Examples and timings. In this section we provide examples on which we time the new algorithm \texttt{symmStd} (cf. Algorithm 1) as opposed to the algorithm \texttt{std} implemented in \textsc{Singular} (cf. [DGPS12]). Timings are conducted by using \textsc{Singular} 3-1-3 on an AMD Opteron 6174 machine with 48 CPUs, 2.2 GHz, and 128 GB of RAM running the Gentoo Linux operating system.

A more detailed description of the considered examples can be found in Section 4.2.

Example 3.14. Cyclic 7-roots, $\sigma_1 = (16)(25)(34) \in S_7$ with $\text{ord}(\sigma_1) = 2$, $\sigma_2 = (1234567) \in S_7$ with $\text{ord}(\sigma_2) = 7$.

| Algorithm | char($K$) | 127   | 30817 | 100003 | 2147483647 |
|-----------|----------|-------|-------|--------|------------|
| std [sec] |          | 2     | 2     | 3      | 11         |
| \texttt{symmStd}($\sigma_1$) [sec] |          | 1     | 2     | 2      | 7          |
| \texttt{symmStd}($\sigma_1$)/\texttt{std} |          | 0.50  | 1.00  | 0.67   | 0.64       |
| \texttt{symmStd}($\sigma_2$) [sec] |          | 5     | 5     | 7      | 20         |
| \texttt{symmStd}($\sigma_2$)/\texttt{std} |          | 2.50  | 2.50  | 2.33   | 1.82       |

Remark 3.15. Note that in case of Example 3.14 the pure \textsc{Gröbner} basis computation is comparably easy so that the symmetry based approach decelerates the whole computation when applying the permutation of order 7 so that the usage of the linear transformation $\tau$ (cf. Remark 3.3) dominates the process. This circumstance will partially also be transpired in the following examples.

Consequently, a permutation of higher order usually accelerates the \textsc{Gröbner} basis computation on the transformed side but, on the other hand, may also decelerate the whole algorithm because of an expensive application of the linear transformation.

However, summing up, we achieve an enormous advancement via \texttt{symmStd} (see the following examples) although we have to compute \textsc{Gröbner} bases internally twice on modified input ideals via \texttt{std}.

Example 3.16. Cyclic 8-roots, $\sigma_1 = (18)(27)(36)(45) \in S_8$ with $\text{ord}(\sigma_1) = 2$, $\sigma_2 = (1753)(2864) \in S_8$ with $\text{ord}(\sigma_2) = 4$, $\sigma_3 = (12345678) \in S_8$ with $\text{ord}(\sigma_3) = 8$.

| Algorithm | char($K$) | 137   | 30817 | 100049 | 2147483497 |
|-----------|----------|-------|-------|--------|------------|
| std [sec] |          | 69    | 79    | 104    | 125        |
| \texttt{symmStd}($\sigma_1$) [sec] |          | 49    | 59    | 76     | 93         |
| \texttt{symmStd}($\sigma_1$)/\texttt{std} |          | 0.71  | 0.75  | 0.73   | 0.74       |
| \texttt{symmStd}($\sigma_2$) [sec] |          | 32    | 36    | 46     | 55         |
| \texttt{symmStd}($\sigma_2$)/\texttt{std} |          | 0.46  | 0.46  | 0.44   | 0.44       |
| \texttt{symmStd}($\sigma_3$) [sec] |          | 54    | 57    | 70     | 82         |
| \texttt{symmStd}($\sigma_3$)/\texttt{std} |          | 0.78  | 0.72  | 0.67   | 0.66       |

Example 3.17. Cyclic 9-roots, $\sigma_1 = (18)(27)(36)(45) \in S_9$ with $\text{ord}(\sigma_1) = 2$, $\sigma_2 = (147)(258)(369) \in S_9$ with $\text{ord}(\sigma_2) = 3$, $\sigma_3 = (123456789) \in S_9$ with $\text{ord}(\sigma_3) = 9$. 
Example 3.18. 100 Swiss Francs Problem, $\sigma = (45)(89) \in S_9$ with $\text{ord}(\sigma) = 2$.

| Algorithm | $\text{char}(K)$ | 181 | 30817 | 100153 | 2147483647 |
|-----------|------------------|-----|-------|---------|-------------|
| $\text{std}$ [sec] | 16458 | 17312 | 21077 | 24697 |
| $\text{symmStd}(\_, \sigma_1)$ [sec] | 10655 | 9955 | 10881 | 13077 |
| $\text{symmStd}(\_, \sigma_1)/\text{std}$ | 0.65 | 0.58 | 0.52 | 0.53 |
| $\text{symmStd}(\_, \sigma_2)$ [sec] | 002 | 4554 | 5471 | 6419 |
| $\text{symmStd}(\_, \sigma_2)/\text{std}$ | 0.24 | 0.26 | 0.26 | 0.26 |
| $\text{symmStd}(\_, \sigma_3)$ [sec] | 4016 | 3756 | 4464 | 5272 |
| $\text{symmStd}(\_, \sigma_3)/\text{std}$ | 0.24 | 0.22 | 0.21 | 0.21 |

Example 3.19. $7 - 4.3^2 - 4.3^2$ for $S_{11}$, $\sigma = (15)(26)(37)(48) \in S_{10}$ with $\text{ord}(\sigma) = 2$.

| Algorithm | $\text{char}(K)$ | 181 | 30817 | 100153 | 2147483647 |
|-----------|------------------|-----|-------|---------|-------------|
| $\text{std}$ [sec] | 5 | 5 | 6 | 8 |
| $\text{symmStd}(\_, \sigma)$ [sec] | 2 | 3 | 4 | 5 |
| $\text{symmStd}(\_, \sigma)/\text{std}$ | 0.40 | 0.60 | 0.67 | 0.63 |

Example 3.20. $7 - 5.4 - 5.4$ for $S_{11}$, $\sigma = (15)(26)(37)(48) \in S_{10}$ with $\text{ord}(\sigma) = 2$.

| Algorithm | $\text{char}(K)$ | 181 | 30817 | 100153 | 2147483647 |
|-----------|------------------|-----|-------|---------|-------------|
| $\text{std}$ [sec] | 3 | 4 | 5 | 6 |
| $\text{symmStd}(\_, \sigma)$ [sec] | 1 | 2 | 2 | 2 |
| $\text{symmStd}(\_, \sigma)/\text{std}$ | 0.33 | 0.50 | 0.40 | 0.33 |

4. Gröbner bases using symmetry and modular methods

When applying Algorithm 1 on $\sigma$-symmetric ideals over the rationals so that $\text{ord}(\sigma) = k > 2$ we need to swap to $\mathbb{Q}[a]/\Phi_k(a)$ as explained in Remark 3.1. However, over fields $K$ of positive characteristic such that $k \mid (\text{char}(K) - 1)$ this can be omitted. Consequently, we use modular methods to improve Algorithm 1 applied on $\sigma$-symmetric ideals in the polynomial ring over the rationals. More precisely, we improve the modular Gröbner basis algorithm as introduced by Arnold (cf. [A03]), Idrees, Pfister, Steidel (cf. [IPS11]) and Noro, Yokoyama (cf. [NY12], [Y12]).

Remark 4.1. Noro and Yokoyama revealed that [IPS11, Theorem 2.4] for the inhomogeneous case is only correct with an additional assumption.
Let $I \subseteq \mathbb{Q}[X]$ be an ideal generated by a finite subset $F_I$. For homogenization we provide an extra variable $t$ and define $f^h := t^d \cdot f(x_1/t, \ldots, x_n/t)$ for $f \in \mathbb{Q}[X]$ where $d$ is the total degree of $f$ and $F_I^h := \{ f^h \mid f \in F_I \}$. Moreover, $>$ induces a monomial ordering $>_h$ such that $X^\alpha t^a >_h X^\beta t^b$ if and only if either $|\alpha| + a > |\beta| + b$ or $(|\alpha| + a = |\beta| + b$ and $X^\alpha > X^\beta$). Then Noro and Yokoyama advise to add the condition that $p$ is lucky for $\langle F^h_I \cup \{ t^m \} \rangle$ with respect to $>_h$ where $m$ is an integer such that $\langle \Phi_p(F^h_I) \rangle : t^m = \langle \Phi_p(F^h_I) \rangle : t^\infty$ and $\Phi_p$ denotes the canonical projection to $\mathbb{F}_p[X]$ for a prime $p$ (cf. [NY12], [Y12]).

4.1. The probabilistic symmetric modular Gröbner basis algorithm. Algorithm 2 combines the algorithms symmStd (cf. Algorithm 1) and a modification of modStd (cf. [IPS11] Algorithm 1).

Algorithm 2 Symmetric Modular Gröbner Basis Computation (syModStd)

Assume that $>$ is a degree ordering.

**Input:** $I \subseteq \mathbb{Q}[X]$ and $\sigma \in S_n$, such that $I$ is $\sigma$-symmetric.

**Output:** $G \subseteq \mathbb{Q}[X]$, the Gröbner basis of $I$.

1. $k = \text{ord}(\sigma)$;
2. choose $P$, a list of random primes such that $k \mid (p - 1)$ for all $p \in P$;
3. $GP = \emptyset$;
4. loop
5. for $p \in P$ do
6. $G_p = \text{symmStd}(I_p, \sigma)$;
7. $GP = GP \cup \{ G_p \}$;
8. $(GP, P) = \text{deleteUnluckyPrimesSB}(GP, P)$;
9. lift $(GP, P)$ to $G \subseteq \mathbb{Q}[X]$ by applying Chinese remainder algorithm and Farey rational map;
10. if $G$ passes finalVerificationTests then
11. return $G$;
12. enlarge $P$;

**Remark 4.2.** The essential differences of the algorithm symModStd compared to the algorithm modStd are the following:

1. The choice of the prime list $P$ has to be restricted. Every considered prime number $p \in P$ has to satisfy the condition $k \mid (p - 1)$ in order to assure that the coefficient field $\mathbb{F}_p$ has a $k$-th primitive root of unity.
2. The modular Gröbner bases $G_p$ are computed via symmStd instead of std. Similar to modStd we can parallelize Algorithm 2 by computing the modular Gröbner bases $G_p$ respectively performing the final tests in parallel.

**Theorem 4.3.** Algorithm 2 terminates and is correct, i.e. the output $G$ is a Gröbner basis of the input $I$.

**Proof.** Termination is clear and correctness follows directly from Theorem 3.12 and the improvement of [IPS11] Theorem 2.4 by Noro and Yokoyama (cf. [NY12], [Y12]).

---

3The corresponding procedures are implemented in SINGULAR in the library symodstd.lib.
The symmetric part of the symmetric modular Gröbner basis algorithm is not influenced by the additional verification test mentioned in Remark 4.1 and for this verification part the ideal \( \langle P_i^h \cup \{ t^m \} \rangle \) is homogeneous, and symmetric if \( I \) is so, such that Algorithm 2 can be directly applied to it without verifying the additional condition (cf. [A03]).

Nevertheless, the additional verification test, in general, decelerates the whole algorithm considerably. In order to emphasize the impact of Algorithm 1 we therefore just time a probabilistic variant (call it \( \text{syModStd}^* \) and \( \text{modStd}^* \), respectively) by skipping the additional verification due to Noro and Yokoyama.

4.2. Examples and timings. In this section we provide examples on which we time the new algorithms \( \text{symmStd} \) (cf. Algorithm 1) respectively \( \text{syModStd}^* \) (cf. Algorithm 2) and its parallelization as opposed to the former algorithms \( \text{std} \) respectively \( \text{modStd}^* \) implemented in SINGULAR (cf. [DGPS12]). Again, all timings are conducted by using SINGULAR 3-1-3 on an AMD Opteron 6174 machine with 48 CPUs, 2.2 GHz, and 128 GB of RAM running the Gentoo Linux operating system.

Example 4.4 (Cyclic \( n \)-roots (cf. [Bj85, Bj90, BF91])). The task to compute a Gröbner basis of the ideal in \( \mathbb{Q}[X] = \mathbb{Q}[x_1, \ldots, x_n] \) corresponding to the following system of polynomial equations

\[
\begin{align*}
  x_1 + \ldots + x_n &= 0, \\
  x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + xn_1 &= 0, \\
  &\vdots \\
  x_1x_2\cdots x_{n-1} + x_2x_3\cdots x_n + \ldots + x_{n-1}x_n\cdots x_{n-3} + x_nx_1\cdots x_{n-2} &= 0, \\
  x_1\cdots x_n - 1 &= 0
\end{align*}
\]

has become a benchmark problem for Gröbner basis techniques. We call this ideal cyclic(\( n \)), and its variety cyclic(\( n \)-roots) (cf. [Bj85]). The origin of the problem is related to Fourier analysis (cf. [Bj85, Bj90]). Obviously, the ideal cyclic(\( n \)) is by definition symmetric with respect to the \( n \)-cycle \( \sigma_n = (1\ldots n) \) such that we can apply the algorithms \( \text{symmStd} \) and \( \text{syModStd}^* \).

Until the end of 2009, SINGULAR was able to compute a Gröbner basis of cyclic(\( n \)) for \( n \leq 8 \). In April 2010, we could for the first time compute a Gröbner basis of cyclic(9) via a prototype of \( \text{syModStd}^* \) by using the 32-bit version of SINGULAR 3-1-1 on an Intel® Xeon® X5460 machine with 4 CPUs, 3.16 GHz each, and 64 GB of RAM under the Gentoo Linux operating system within 23 days.

Table 1 summarizes the present timings for computing a Gröbner basis of cyclic(\( n \)) for \( n = 7, 8, 9 \) with different numbers of cores \( \ell \) denoted by \( \text{modStd}^*(\ell) \) respectively \( \text{syModStd}^*(\ell) \), and different permutations \( \sigma \) where again \( k = \text{ord}(\sigma) \) denotes the order of \( \sigma \).

In these examples we used the permutations \((16)(25)(34), (1234567) \in S_7 \) for \( n = 7 \), the permutations \((18)(27)(36)(45), (1753)(2864), (12345678) \in S_8 \) for \( n = 8 \), and the permutation \((147)(258)(369) \in S_9 \) for \( n = 9 \).

Note that the symmetric approach in the modular version just influences the Gröbner basis computation in each prime characteristic. This means, on the one hand, that in most cases the use of parallelization decreasing the number of sequentially accomplished Gröbner basis computations is more decisive for higher efficiency than just applying the symmetry based approach. On the other hand, if
the calculations in positive characteristic are comparably easy as in the cyclic(7)-
case, then the symmetry based approach may even slow down the whole process
since the usage of the linear transformation \( \tau \) and the second Gröbner basis com-
putation in each prime characteristic overrun the original calculations of the purely
modular approach.

Moreover, note that the timings obtained by the modular versions are dependent
on the used permutation and especially on its order. In particular, a higher order \( k \),
that is a higher symmetry, speeds up the Gröbner basis computation on the trans-
formed side but in contrast slows down the application of the linear transformation
\( \tau \) (cf. Remark 3.3) and, in addition, allocates more memory since the support of a
ring variable’s image depends on the order \( k \) of the permutation. This circumstance
justifies that applying the symmetric modular algorithm for computing a Gröbner
basis of cyclic(8) is most performant when using the permutation \((1753)(2864) \in S_8\)
of order 4. Similarly, we make use of the permutation \((147)(258)(369) \in S_9\)
of order 3 for cyclic(9) since considering a permutation of order 9 implies substituting each
ring variable by a linear combination of 9 ring variables when applying the lin-
ear transformation \( \tau \), so that the parallel computation crashes because of memory
overflow.

Example 4.5 (100 Swiss Francs Problem (cf. [ZJG11], [St08])). Sturmfels offered
a cash prize of 100 Swiss Francs for the resolution of a very specific conjecture in
the Nachdiplomsvorlesung (postgraduate course) which he held at ETH Zürich in
the summer of 2005. Based on a concrete biological example proposed in [PS05,
Example 1.16] the problem arisen to maximize the likelihood function

\[
L(P) = \left( \prod_{i=1}^{4} p_{ii} \right)^4 \cdot \left( \prod_{i \neq j} p_{ij} \right)^2 \cdot \left( \sum_{i,j=1}^{4} p_{ij} \right)^{-40}
\]

everall (positive) \( 4 \times 4 \)-matrices \( P = (p_{ij})_{1 \leq i,j \leq 4} \) of rank at most two. Due to
numerical experiments by applying an expectation-maximization algorithm (EM
algorithm), B. Sturmfels conjectured that the matrix

\[
P = \frac{1}{40} \begin{pmatrix}
3 & 3 & 2 & 2 \\
3 & 3 & 2 & 2 \\
2 & 2 & 3 & 3 \\
2 & 2 & 3 & 3 \\
\end{pmatrix}
\]
is a global maximum of the likelihood function $L(P)$ (cf. [St08]).

The conjecture is positively confirmed in [ZJG11]. In their approach via Gröbner bases (cf. [ZJG11] Section 2.3) it is necessary to compute the Gröbner basis of the ideal $J$ defined by

$$I = \langle a_1 - b_1, \sum_{i=1}^{4} a_i, \sum_{i=1}^{4} b_i, f_1, \ldots, f_4, g_1, \ldots, g_4 \rangle \subseteq \mathbb{Q}[a_1, \ldots, a_4, b_1, \ldots, b_4]$$

with

$$f_i = \sum_{j=1}^{4} \left( b_j \cdot (1 + a_i b_i) \cdot \prod_{k \neq j} (1 + a_i b_k) \right) + b_i \cdot \prod_{k=1}^{4} (1 + a_i b_k),$$

$$g_i = \sum_{j=1}^{4} \left( a_j \cdot (1 + a_i b_i) \cdot \prod_{k \neq j} (1 + a_k b_i) \right) + a_i \cdot \prod_{k=1}^{4} (1 + a_k b_i)$$

for $1 \leq i \leq 4$, and

$$J = I + (1 - ua_1) \subseteq \mathbb{Q}[a_1, \ldots, a_4, b_1, \ldots, b_4, u]$$

with respect to an elimination ordering on the variable $u$. In a first approach we therefore applied $\text{modStd}$ using the lexicographical ordering $>_{lp}$ respectively the block ordering $>_{dp}(8), >_{lp}(1)$ to eliminate the variable $u$. It turned out that both variants are comparably slow so that we used in a second approach the degree reverse lexicographical ordering $>_{dp}$, and applied the FGLM-algorithm (cf. [FGLM93]) subsequently to obtain a Gröbner basis with respect to the block ordering $>_{dp}(8), >_{lp}(1)$. Since the ideal $J \subseteq \mathbb{Q}[a_1, \ldots, a_4, b_1, \ldots, b_4, u]$ is symmetric with respect to the permutation $(34)(78) \in S_9$ we could moreover apply the algorithm $\text{syModStd}^*$. The timings for the computations in SINGULAR are summarized in Table 2.

| Method                      | Running Time |
|-----------------------------|--------------|
| $\text{modStd}^*[>_{lp}]$   | 39919        |
| $\text{modStd}[>_{dp}(8),>_{lp}(1)]$ | 515       |
| $\text{syModStd}[>_{dp}(8),>_{lp}(1)]$ | 356       |
| $\text{modStd}[>_{dp}] - \text{fglm}[>_{dp}(8),>_{lp}(1)]$ | 375       |
| $\text{syModStd}[>_{dp}] - \text{fglm}[>_{dp}(8),>_{lp}(1)]$ | 284       |

**Table 2.** Total running times in seconds for computing the Gröbner basis of $J \subseteq \mathbb{Q}[a_1, \ldots, a_4, b_1, \ldots, b_4, u]$ with respect to an elimination ordering on the variable $u$ via different methods.

**Example 4.6** (Inverse Galois Problem (cf. [Mat87], [MM99])). A major topic in algebraic number theory is the inverse Galois problem over a field $K$, i.e. the question whether any finite group $G$ is the Galois group of a Galois extension of $K$. The most interesting case is $K = \mathbb{Q}$ which is still open in general. In contrast, the problem is known to be true for $K$ being a rational function field in one variable $t$ over an algebraically closed field of characteristic zero. In particular, it is true for $K = \mathbb{C}(t)$, and in this case it is solved via geometric field extensions (see for example [MM99], I, §I). Moreover, the same strategy applies to finite
field extensions of $\mathbb{Q}(t)$ ramified only over $\{0,1,\infty\}$ (see for example [MM99] I, §5). In this situation, for any triple $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ of elements generating a transitive subgroup $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subseteq S_n$ with $\sigma_1 \sigma_2 \sigma_3 = 1$ there exists a certain field extension $K_\sigma / \mathbb{Q}(t)$ of degree $n$, unramified outside $\{0,1,\infty\}$, and whose Galois group is isomorphic to $G$. In fact, any such extension $K_\sigma / \mathbb{Q}(t)$ is already defined over a number field $k_\sigma = \mathbb{Q}(\alpha_\sigma)$, the so-called field of definition of $K_\sigma / \mathbb{Q}(t)$, so that there exists a further field extension $K_\sigma / k_\sigma(t)$ which also has $G$ as its Galois group. The degree $[k_\sigma : \mathbb{Q}]$ is bounded from above by group theoretical information (see for example [Mal94] Proposition A)). In order to construct the extension $K_\sigma / \mathbb{Q}(t)$ it is necessary to solve a system of polynomial equations (see [MM99] I, §9). In case that, for example, $\sigma_1$ and $\sigma_2$ have the same cycle type, the defining ideal is symmetric with respect to a permutation of order 2 so that we can apply the algorithms $\text{symmStd}$ and $\text{syModStd}$ to compute a Gröbner basis of this system. In addition, choosing an elimination ordering for the last ring variable, the last polynomial $f$ of the Gröbner basis of this system of non-linear equations generates the field of definition $k_\sigma = \mathbb{Q}(\alpha_\sigma)$ insofar that $\alpha_\sigma$ is a zero of $f$. The irreducible factors of $f$ together with group theoretical information yield restrictions on $[k_\sigma : \mathbb{Q}]$. In case that $[k_\sigma : \mathbb{Q}] = 1$, the given group $G$ can even be realized over $\mathbb{Q}(t)$, and therefore also over $\mathbb{Q}$ by Hilbert’s irreducibility theorem.

In 1994, Malle collected computational data on several $k_\sigma$ of degree $[k_\sigma : \mathbb{Q}] \leq 13$ (cf. [Mal94]) with the intention to observe regularities and hints to decrease the group theoretical bound. Table 3 lists further examples in the spirit of this article and which could not be computed at that time.

| $n$ | $G$ | $C_\sigma$ | $[k_\sigma : \mathbb{Q}]$ | $\text{symmStd}$ | $\text{modStd}^*$ | $\text{syModStd}^*$ |
|-----|-----|------|----------------|----------------|----------------|----------------|
| 7   | $A_7$ | 4.2 - 4.2 - 4.2 | 12 | 24 | 19 | 19 |
| 9   | $S_9$ | 4.2^2 - 4.2^2 - 5.3 | 34 | 5 | 15 | 13 |
| 10  | $A_{10}$ | 5.2^2 - 5.2^2 - 7 | 37 | 977 | 186 | 107 |
| 11  | $A_{11}$ | 4.2 - 4.9 - 9 | 12 | 17 | 2 | 1 |
| 11  | $A_{11}$ | 4.2 - 4.2^3 - 4.2^3 | 8 | 34 | 3 | 2 |
| 11  | $A_{11}$ | 4.2 - 5.3^2 - 5.3^2 | 8 | 15 | 3 | 2 |
| 11  | $A_{11}$ | 5 - 8.2 - 8.2 | 11 | 265 | 17 | 8 |
| 11  | $A_{11}$ | 5 - 6.4 - 6.4 | 11 | 339 | 20 | 10 |
| 11  | $A_{11}$ | 5 - 7.3 - 7.3 | 11 | 292 | 16 | 8 |
| 11  | $S_{11}$ | 7 - 4.3^2 - 4.3^2 | 26 | 631245 | 2566 | 1493 |
| 11  | $S_{11}$ | 7 - 4.3.2^2 - 4.3.2^2 | 29 | - | 3039 | 1979 |
| 11  | $S_{11}$ | 7 - 7.2 - 7.2 | 29 | - | 1414 | 702 |
| 11  | $S_{11}$ | 7 - 5.4 - 5.4 | 26 | - | 1738 | 899 |

Table 3. Total running times in seconds for computing the defining Gröbner basis of the field extension $K_\sigma / \mathbb{Q}(t)$ having group $G$ and conjugacy class triple $C_\sigma$ (cf. [Mal94]) via $\text{symmStd}$, $\text{modStd}^*$, and $\text{syModStd}^*$. Here, $C_\sigma$ is a class of $G$ containing elements of the given cycle type. The symbol "-" indicates out of memory failures.

Note that all ideals belonging to the examples listed in Table 3 are zero-dimensional such that we can compute a Gröbner basis with respect to the degree reverse
lexicographical ordering, and obtain a lexicographical Gröbner basis by applying the FGLM-algorithm (cf. [FGLM93]) subsequently.

5. Conclusion

In all considered examples the symmetric (or equivalently the probabilistic symmetric modular) version of the Gröbner basis algorithm is the most performant one, and is, hence, a quite powerful tool if the input ideal is symmetric with respect to some permutation of the ring variables.

Although plenty of adaptive ideals are even symmetric under a whole permutation group the symmetry based approach presented in this article is designed for only a single permutation respectively cyclic subgroup of \( S_n \). As already mentioned in Remark 3.15 and at the end of Example 4.4 there is a particular conflict with respect to performance in the symmetric Gröbner basis algorithm between the Gröbner basis computations and the application of the linear transformation. Hence, a reasonable heuristic is to choose the applicable permutation \( \sigma \) of cycle type \( (l_1, \ldots, l_{\vartheta}(\sigma)) \) having maximal order \( \text{lcm}(l_1, \ldots, l_{\vartheta}(\sigma)) \) and minimal \( l_{\vartheta}(\sigma) \).

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