A NOTE ON THE FEKETE–SZEGÖ PROBLEM FOR CLOSE-TO-CONVEX FUNCTIONS WITH RESPECT TO CONVEX FUNCTIONS

Bogumila Kowalczyk, Adam Lecko, and H. M. Srivastava

Abstract. We discuss the sharpness of the bound of the Fekete–Szegö functional for close-to-convex functions with respect to convex functions. We also briefly consider other related developments involving the Fekete–Szegö functional \(|a_3 - \lambda a_2^2|\) \((0 \leq \lambda \leq 1)\) as well as the corresponding Hankel determinant for the Taylor–Maclaurin coefficients \(\{a_n\}_{n \in \mathbb{N} \setminus \{1\}}\) of normalized univalent functions in the open unit disk \(\mathbb{D}\), \(\mathbb{N}\) being the set of positive integers.

1. Introduction

A classical problem in geometric function theory of complex analysis, which was settled by Fekete and Szegö [4], is to find for each \(\lambda \in [0, 1]\) the maximum value of the coefficient functional \(\Phi_\lambda(f)\) given by

\[
\Phi_\lambda(f) := |a_3 - \lambda a_2^2|
\]

over the class \(\mathcal{S}\) of univalent functions \(f\) in the open unit disk

\[
\mathbb{D} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}
\]

of the following normalized form (see, for details, [5][22][24]):

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}).
\]

By applying the Loewner method, Fekete and Szegö [4] proved that

\[
\max_{f \in \mathcal{S}} \Phi_\lambda(f) = \begin{cases} 
1 + 2 \exp \left( -\frac{2\lambda}{1 - \lambda} \right) & (0 \leq \lambda < 1) \\
1 & (\lambda = 1).
\end{cases}
\]
For various compact subclasses $\mathcal{F}$ of the class $\mathcal{A}$ of all analytic functions $f$ in $\mathbb{D}$ of the form (1.2), as well as with $\lambda$ being an arbitrary real or complex number, many authors computed

\[
\max_{f \in \mathcal{F}} \Phi_\lambda(f)
\]

or calculated the upper bound of (1.3) (see, e.g., [2,8,11,21]).

Let $S^*$ denote the class of starlike functions, that is, $f \in S^*$ if

\[ f \in \mathcal{A} \quad \text{and} \quad \Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{D}). \]

Given $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $g \in S^*$, let $C_\delta(g)$ denote the class of functions called close-to-convex with argument $\delta$ with respect to $g$, that is, the class of all functions $f \in \mathcal{A}$ such that

\[
\Re \left( e^{i\delta} \frac{zf'(z)}{g(z)} \right) > 0 \quad (z \in \mathbb{D}).
\]

We also suppose that, given $g \in S^*$, $C(g) := \bigcup_{g \in S^*} C_\delta(g)$ and that, given $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $C := \bigcup_{g \in S^*} C_\delta(g)$. Let

\[
C := \bigcup_{\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})} \bigcup_{g \in S^*} C_\delta(g)
\]

denote the class of close-to-convex functions (see, for details, [20, pp.184–185], [6,10]).

For the whole class $C$, the sharp bound of the Fekete–Szegö coefficient functional $\Phi_\lambda$ for $\lambda \in [0,1]$, given by (1.1), was calculated by Koepf [13] who extended the earlier result for the class $C_0$ and for $\lambda \in \mathbb{R}$ due to Keogh and Merkes [11], namely, it holds

\[
\max_{f \in C} \Phi_\lambda(f) = \max_{f \in C_0} \Phi_\lambda(f) = \begin{cases} 
|3 - 4\lambda| & (\lambda \in (-\infty, \frac{1}{4}] \cup [1, \infty)) \\
\frac{3}{4} + \frac{\lambda}{1} & (\lambda \in [\frac{1}{4}, \frac{1}{2}]) \\
1 & (\lambda \in [\frac{2}{3}, 1]).
\end{cases}
\]

For various subclasses of the class of close-to-convex functions, the problem to estimate the coefficient functional $\Phi_\lambda$ is continued in several subsequent works (see, for details, [9,12,14,16]). Some interesting and important subclasses of the class $C$ are the classes $C_\delta$ and $C^c$, which are defined below.

Let $S^c$ denote the class of convex functions, that is, $f \in S^c$ if

\[ f \in \mathcal{A} \quad \text{and} \quad \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in \mathbb{D}). \]

Since $S^c \subsetneq S^*$, the class $C^c_\delta := \bigcup_{g \in S^c} C_\delta(g)$ is a proper subclass of the class $C_\delta$ and the class

\[
C^c := \bigcup_{\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})} \bigcup_{g \in S^c} C_\delta(g)
\]

is a proper subclass of the class $C$. 
A NOTE ON THE FEKETE–SZEGÖ PROBLEM

The class \( C^c_0 \) was defined by Abdel-Gawad and Thomas [1]. The class \( C^c \) of close-to-convex functions with respect to convex functions was introduced by Srivastava, Mishra and Das [23]. In both of these cited papers, the authors (Abdel-Gawad and Thomas [1] and Srivastava, Mishra and Das [23]) considered the coefficient functional \( \Phi_\lambda \) with \( \lambda \in [0, 1] \) also. In fact, in Srivastava, Mishra and Das [23] extended, for the class \( C^c \), the earlier result of Abdel-Gawad and Thomas [1] for the class \( C^c_0 \). However, in each of the above-cited papers, the proof for the sharpness of the bound in (1.3) for \( \lambda \in \left( \frac{2}{3}, 1 \right] \) was proposed incorrectly as \( \frac{5}{6} \).

This note is motivated essentially by the earlier papers [1] and [23]. The main purpose of our investigation here is to discuss such sharpness results for the bound in (1.3). We also provide a rather brief consideration of other related developments involving the Fekete–Szegö functional \(|a_3 - \lambda a_2^2| (0 \leq \lambda \leq 1)\) in (1.1) as well as the corresponding Hankel determinant for the Taylor–Maclaurin coefficients \( \{a_n\}_{n \in \mathbb{N} \setminus \{1\}} \) of normalized univalent functions of the form (1.2).

2. Main Observation

As we remarked in Section 1, in both of the afore cited papers [1,23], the upper bounds of the Fekete–Szegö coefficient functional \( \Phi_\lambda \) \( (0 \leq \lambda \leq 1) \) for the classes \( C^c_0 \) and \( C^c \), were computed. In fact, Theorems 5 and 6 of Srivastava, Mishra and Das [23] state that the following sharp inequality
\[
\max_{f \in C^c} \Phi_\lambda(f) \leq \frac{5}{6} \quad (\lambda \in \left[ \frac{2}{3}, 1 \right])
\]
holds true and that this result is the same as in [1] for the class \( C^c_0 \) (a part of Theorem 3). However, the assertion that the extremal function, for which the equality in (2.1) is satisfied when \( \lambda \in \left( \frac{2}{3}, 1 \right] \), belongs to \( C^c \) is incorrect. Indeed, here in this section, we note that the above-cited papers [1,23] contain a statement to the effect that the equality in (2.1) is attained by a function \( f \in A \) given by
\[
zf'(z) \quad h(z) = 1 + \omega(z) \quad (z \in \mathbb{D}),
\]
where \( h \in S^c \) is of the form
\[
h(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathbb{D}; \ b_2 = b_3 := 1)
\]
and \( \omega \) is a function of the form
\[
\omega(z) = \sum_{n=1}^{\infty} \beta_n z^n \quad (z \in \mathbb{D})
\]
with
\[
\beta_1 := \frac{2 - 3\lambda}{6\lambda} \pm i \sqrt{\frac{6\lambda - 4}{6\lambda}} \quad \text{and} \quad \beta_2 := 1 - \beta_1^2.
\]
Unfortunately, however, \( \omega \) is not a Schwarz function for \( \lambda \in \left( \frac{2}{3}, 1 \right] \). We recall here that a Schwarz function means an analytic self-mapping of \( \mathbb{D} \) with \( \omega(0) := 0 \). Let us
denote the class of Schwarz functions by $B_0$. In order to see that $\omega \notin B_0$, we verify (by straightforward computation) that, for $\lambda \in \left(\frac{3}{4}, 1\right]$, the following inequality:

\begin{equation}
|\beta_2| \leq 1 - |\beta_1|^2
\end{equation}

is false, so a necessary condition for $\omega$ to be in $B_0$ (see, for example, [5 Vol. II, p. 78]) does not hold true. Alternatively, in order to get a contradiction, we suppose that $\omega$ with its coefficients in (2.5) is a Schwarz function. Thus, clearly, (2.6) holds true. Hence we find from (2.5) that $1 - |\beta_1|^2 \geq |\beta_2| = |1 - \beta_1^2| \geq 1 - |\beta_1|^2$. Thus we have $|1 - \beta_1^2| = 1 - |\beta_1|^2$ and, therefore, $\beta_1 = |\beta_1|$ or $\beta_1 = -|\beta_1|$. This means that $\beta_1$ is a real number, which by (2.5) is possible only for $\lambda = \frac{3}{4}$. Consequently, for $\lambda \in \left(\frac{3}{4}, 1\right]$, the function $\omega$ with its coefficients in (2.5) does not belong to $B_0$. So, in light of (2.2), it does not follow that $f$ is in $C^0$ or in $C_0^1$.

Equivalently, let

\begin{equation}
p(z) := \frac{1 + \omega(z)}{1 - \omega(z)} \quad (z \in \mathbb{D}),
\end{equation}

where $\omega$ is as given above. Then

\begin{equation}
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}),
\end{equation}

where, in view of (2.7), (2.1) and (2.5), we have $c_1 = 2\beta_1$ and $c_2 = 2(\beta_2 + \beta_1^2) = 2$. We observe further that, for $\lambda \in \left(\frac{3}{4}, 1\right]$, the function $p$ does not belong to the Carathéodory class. We recall here that the Carathéodory class, denoted as $\mathcal{P}$, contains analytic functions $p$ of the form (2.3) with a positive real part. In order to see that $p \notin \mathcal{P}$, we verify for $\lambda \in \left(\frac{3}{4}, 1\right]$ that the inequality $|c_2 - c_1^2/2| \leq 2 - |c_1|^2/2$, is false, which happens to be a necessary condition for $p$ to be in the class $\mathcal{P}$ (see, for example, [22, p. 166]).

3. Concluding remarks and further developments

By means of Theorem 3 of Abdel-Gawad and Thomas [1], Theorems 1 to 4 of Srivastava, Mishra and Das [23], and in light of our observation in Section 2, we arrive at the following result.

**Theorem 1.** Each of the following assertions holds true:

\begin{equation}
\max_{f \in C^0} \Phi_\lambda(f) = \max_{f \in C_0^1} \Phi_\lambda(f) = \left\{ \begin{array}{ll}
\frac{\delta}{4} - \frac{\lambda}{4} & (\lambda \in \left[0, \frac{3}{4}\right]) \\
\frac{\delta}{4} + \frac{1}{16} & (\lambda \in \left[\frac{3}{4}, \frac{2}{3}\right])
\end{array} \right.
\end{equation}

\begin{equation}
\max_{f \in C^0} \Phi_\lambda(f) \leq \frac{\delta}{\delta} \quad (\lambda \in \left(\frac{2}{3}, 1\right]).
\end{equation}

**Remark 1.** The sharpness of the inequality in (3.2) for the classes $C^0$ and $C_0^1$ is an open problem.

We now note that, by Loewner Theorem (see, for example, [5 Vol. I, p. 1127]), the function $h \in S^\infty$ of the form (2.3) (with $b_2 = b_3 := 1$) is uniquely determined, that is, $h(z) = \frac{1}{1 - z} = \sum_{n=1}^{\infty} z^n \quad (z \in \mathbb{D})$. Then (1.4) with $g := h$ is of the form

\begin{equation}
\Re(e^{i\delta}(1 - z)f'(z)) > 0 \quad (z \in \mathbb{D})
\end{equation}
and defines the class \( C_{\delta}(h) \), and further the class \( C(h) \). For the first time, the inequality in (3.3), treated as the univalence criterion, was distinguished explicitly in [20] p. 185. For the class \( C(h) \), the upper bound of the Fekete–Szegö coefficient functional \( \Phi_\lambda \) for \( \lambda \in \mathbb{R} \) was recently obtained in [14], where the following result was proven.

**Theorem 2.** It is asserted that

\[
(3.4) \quad \max_{f \in \mathcal{C}(h)} \Phi_\lambda(f) \leq \begin{cases} \left| \frac{1}{2} - \frac{1}{2} \lambda \right| + \frac{5}{4^2} (2 - 3 \lambda) & (\lambda \in (-\infty, \frac{2}{3}) \cup \left[\frac{4}{9}, \infty\right)) \\
\frac{1}{4} \cdot \frac{(2 - 3 \lambda)^2}{2 - 2 - 3 \lambda} + \left| \frac{1}{2} - \frac{1}{4} \lambda \right| + \frac{3}{4} & (\lambda \in \left[\frac{2}{9}, \frac{10}{19}\right]).
\end{cases}
\]

For each \( \lambda \in (-\infty, \frac{2}{3}] \cup \left[\frac{4}{9}, \infty\right) \), the inequality is sharp and the equality in (2) is attained by a function in \( \mathcal{C}_0(h) \).

**Remark 2.** For \( \lambda \in (-\infty, \frac{2}{3}] \cup \left[\frac{4}{9}, \infty\right) \), we can rewrite (3.4) as the following corollary.

**Corollary 1.** The following assertion holds true:

\[
(3.5) \quad \max_{f \in \mathcal{C}(h)} \Phi_\lambda(f) = \begin{cases} \left| \frac{1}{2} - \frac{1}{2} \lambda \right| & (\lambda \in (-\infty, \frac{2}{3}) \cup \left[\frac{4}{9}, \infty\right)) \\
\frac{1}{4} \cdot \frac{(2 - 3 \lambda)^2}{2 - 2 - 3 \lambda} & (\lambda \in \left[\frac{2}{9}, \frac{10}{19}\right]).
\end{cases}
\]

**Remark 3.** For \( \lambda \in \left[0, \frac{2}{3}\right] \), the result (3.5) asserted by Corollary 3 coincides with (3.1). Thus, naturally, Theorem 1 and Theorem 2 yield Corollary 2 below.

**Corollary 2.** Each of the following assertions holds true:

\[
\max_{f \in \mathcal{C}(h)} \Phi_\lambda(f) = \max_{f \in \mathcal{C}_0} \Phi_\lambda(f) = \max_{f \in \mathcal{C}'} \Phi_\lambda(f) \quad (\lambda \in \left[0, \frac{2}{3}\right]),
\]

\[
\max_{f \in \mathcal{C}(h)} \Phi_\lambda(f) \leq \frac{9 \lambda^2 - 30 \lambda + 26}{6(4 - 3 \lambda)} \leq \frac{5}{6} \quad (\lambda \in \left(\frac{2}{9}, 1\right]).
\]

**Remark 4.** The maximum of \( \Phi_\lambda \) for \( \lambda \in \left[0, \frac{2}{3}\right] \), over the class \( \mathcal{C}' \) of close-to-convex functions with respect to convex functions and over its subclass \( \mathcal{C}(h) \) of close-to-convex functions with respect to convex function \( h \), are identical.

**Remark 5.** The sharpness of the inequality in (3.4) for \( \lambda \in \left(\frac{2}{9}, \frac{4}{9}\right) \) is an open problem.

**Remark 6.** We reiterate the fact that the Fekete–Szegö coefficient functional \( |a_3 - \lambda a_2^2| \) is well known for its rich history in geometric function theory. Its origin was in the disproof by Fekete and Szegö [4] of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see, for details, [4]). The \( \lambda \)-generalized Fekete–Szegö coefficient functional \( |a_3 - \lambda a_2^2| \) has since received great attention, particularly in connection with many subclasses of the class \( S \) of normalized analytic and univalent functions. On the other hand, in the year 1976, Noonan and Thomas [17] defined the \( q \)th Hankel determinant of
the function $f$ in (1.2) by
\[
H_q(n) = \begin{vmatrix}
  a_n & a_{n+1} & \cdots & a_{n+q-1} \\
  a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}
\quad (n, q \in \mathbb{N}; \ a_1 := 1).
\]
The determinant $H_q(n)$ has also been considered by several other authors. For example, Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f$ given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained in the recent works [7][18] for different classes of functions. We note, in particular, that
\[
H_2(1) = \frac{a_1}{a_2} \frac{a_2}{a_3} = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = \begin{vmatrix}
  a_2 & a_3 \\
  a_3 & a_4
\end{vmatrix} = a_2a_4 - a_3^2.
\]
The Hankel determinant $H_2(1) = a_3 - a_2^2$ is the classical Fekete–Szegö coefficient functional. The upper bounds of $H_2(2)$ for some specific analytic function classes were discussed quite recently by Deniz et al. [3] (see also [19]).

References

1. H. R. Abdel-Gawad, D. K. Thomas, A subclass of close-to-convex functions, Publ. Inst. Math., Nouv. Sér. 49(63) (1991), 61–66.
2. B. Bhowmik, S. Ponnusamy, K. J. Wirths, On the Fekete–Szegö problem for close-to-convex functions, J. Math. Anal. Appl. 373 (2011), 432–438.
3. E. Deniz, M. Çağlar, H. Orhan, Second Hankel determinant for starlike and close-to-convex functions of order $\beta$, Appl. Math. Comput. 271 (2015), 301–307.
4. M. Fekete, G. Szegö, Eine Bemerkung über ungerade schlichte Funktionen, J. Lond. Math. Soc. 8 (1933), 85–89.
5. A. W. Goodman, Univalent Functions, Mariner, Tampa, Florida, 1983.
6. A. W. Goodman, E. B. Saff, On the definition of close-to-convex function, Int. J. Math. Math. Sci. 1 (1978), 125–132.
7. T. Hayami, S. Owa, Generalized Hankel determinant for certain classes, Int. J. Math. Anal. 52 (2010), 2473–2585.
8. Z. J. Jakubowski, Sur le maximum de la fonctionnelle $|A_3 - \alpha A_2^2|$ ($0 \leq \alpha < 1$) dans la famille de fonctions $f \in F_1$, Bull. Soc. Sci. Lett. Lódź 13(1) (1962), 1–19.
9. S. Kanas, A. Lecko, On the Fekete–Szegö problem and the domain of convexity for a certain class of univalent functions, Zesz. Nauk. Politech. Rzeszowskiej, Folia Sci. Univ. Tech. Resoviensis 73 (1990), 49–57.
10. W. Kaplan, Close to convex schlicht functions, Mich. Math. J. 1 (1952), 169–185.
11. F. R. Keogh, E. P. Merkes, A coefficient inequality for close-to-convex functions, Proc. Am. Math. Soc. 20 (1969), 8–12.
12. Y. C. Kim, J. H. Choi, T. Sugawa, Coefficient bounds and convolution properties for certain classes of close-to-convex functions, Proc. Japan Acad., Ser. A 76(6) (2000), 95–98.
13. W. Koepf, On the Fekete–Szegö problem for close-to-convex functions, Proc. Am. Math. Soc. 101 (1987), 89–95.
14. B. Kowalczyk, A. Lecko, Fekete–Szegö problem for a certain subclass of close-to-convex functions, Bull. Malays. Math. Sci. Soc. (2) 38 (2015), 1303–1410.
15. , Fekete–Szegö problem for close-to-convex functions with respect to a certain convex function dependent on a real parameter, Front. Math. China 11 (2016), 1471–1500.
16. R. R. London, Fekete–Szegö inequalities for close-to-convex functions, Proc. Am. Math. Soc. 117 (1993), 947–950.
17. J. W. Noonan, D. K. Thomas, *On the second Hankel determinant of arccosly mean p-valent functions*, Trans. Am. Math. Soc. **223** (1976), 337–346.

18. K. I. Noor, *Hankel determinant problem for the class of functions with bounded boundary rotation*, Rev. Roum. Math. Pures Appl. **28** (1983), 731–739.

19. H. Orhan, N. Magesh, J. Yamini, *Bounds for the second Hankel determinant of certain bi-univalent functions*, Turk. J. Math. **40** (2016), 679–687.

20. S. Ozaki, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku, Sect. A **2** (1935), 167–188.

21. A. Pfluger, *The Fekete–Szegö inequality for complex parameter*, Complex Variables, Theory Appl. **7** (1986), 149–160.

22. Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.

23. H. M. Srivastava, A. K. Mishra, M. K. Das, *The Fekete–Szegö problem for a Subclass of Close-to-Convex Functions*, Complex Variables, Theory Appl. **44** (2001), 145–163.

24. H. M. Srivastava, S. Owa, *Current Topics in Analytic Function Theory*, World Scientific, Singapore, New Jersey, London and Hong Kong, 1992.