On Strong Continuity of Weak Solutions to the Compressible Euler System

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Abstract
Let \( S = \{ \tau_n \}_{n=1}^{\infty} \subset (0, T) \) be an arbitrary countable (dense) set. We show that for any given initial density and momentum, the compressible Euler system admits (infinitely many) admissible weak solutions that are not strongly continuous at each \( \tau_n, n = 1, 2, \ldots \). The proof is based on a refined version of the oscillatory lemma of De Lellis and Székelyhidi with coefficients that may be discontinuous on a set of zero Lebesgue measure.

Keywords
Compressible Euler system · Weak solution · Convex integration · Oscillatory lemma

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1 Introduction

We consider the Euler system describing the time evolution of the mass density $\rho = \rho(t, x)$ and the momentum $m = m(t, x)$ of a barotropic inviscid fluid:

$$
\begin{align*}
\partial_t \rho + \text{div}_x m &= 0, \\
\partial_t m + \text{div}_x \left( \frac{m \otimes m}{\rho} \right) + \nabla_x p(\rho) &= 0,
\end{align*}
$$

(1.1)

where $t \in (0, T)$, and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded domain. The problem is supplemented by the impermeability condition

$$
m \cdot n|_{\partial \Omega} = 0,
$$

(1.2)

and the initial conditions

$$
\rho(0, \cdot) = \rho_0, \quad m(0, \cdot) = m_0.
$$

(1.3)

As is well known, problem (1.1)–(1.3) is locally well posed for sufficiently regular initial data, however, the smooth solutions blow up in a finite time. The weak solution exists globally in time, however, the problem is essentially ill-posed even in the class of admissible weak solutions satisfying the energy inequality

$$
\int_\Omega \left[ \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho) \right](t) \, dx \leq \int_\Omega \left[ \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho) \right](s) \, dx, \quad P'(\rho)\rho - P(\rho) = p(\rho)
$$

(1.4)

for a.a. $s$, including $s = 0$, and any $t$, $0 \leq s \leq t \leq T$. First examples of non-uniqueness were obtained in the seminal paper by De Lellis and Székelyhidi (2010) and later extended by Chiodaroli (2014) and Feireisl (2016), Luo et al. (2016) to a rather general class of initial data.

The key tool for using the convex integration machinery of De Lellis and Székelyhidi (2010), developed originally for the incompressible fluids, is a suitable adaptation of the so-called Oscillatory Lemma, proved originally in De Lellis and Székelyhidi (2010) and extended to “variable coefficients” in Chiodaroli (2014). Probably the most general version including “non-local coefficients” can be found in Feireisl (2016). The limitation of this approach is due to the fact that certain quantities, in particular the initial density and the desired energy profile, must enjoy some degree of smoothness to transform the problem to its basic form handled in De Lellis and Székelyhidi (2010). The largest possible class used so far is that of piecewise continuous functions, cf. Feireisl (2016), Luo et al. (2016).

A closer inspection of the problem reveals apparent similarity between the regularity properties required for the coefficients in Oscillatory Lemma and their integrability in the Riemann sense. Our goal is to extend validity of Oscillatory Lemma to the case of
Riemann integrable coefficients, specifically belonging to the class:

$$\mathcal{R}(Q) \equiv \left\{ v: Q \to \mathbb{R} \mid \text{meas} \left\{ y \in Q \mid v \text{ is not continuous at } y \right\} = 0 \right\}$$

where the symbol “meas” stands for the Lebesgue measure. Such an extension allows us to show the existence of weak solutions to the Euler system with a given total energy profile belonging to $\mathcal{R}$. In particular, as the weak solutions $[\varrho, \mathbf{m}]: t \mapsto L^1(\Omega) \times L^1(\Omega; \mathbb{R}^d)$ are strongly continuous at a time $t$ if and only if the total energy is continuous at $t$, we obtain the existence of an admissible weak solution that is not strongly continuous at an arbitrary given countable dense set of times $S = \{\tau_n\}_{n=1}^{\infty} \subset (0, T)$. As already pointed out, the result is valid in the multidimensional case due to the convex integration method. Its extension to the case $d = 1$ remains an open problem.

The paper is organized as follows. In Sect. 2, we collect the preliminary material and state our main results. In Sect. 3, we show a version of Oscillatory Lemma with coefficients belonging to $\mathcal{R}$. Applications, including the proofs of the main results, are discussed in Sect. 4.

## 2 Preliminaries, Main Results

We say that the functions

$$\varrho \in C_{\text{weak}}([0, T]; L^2(\Omega)), \quad \mathbf{m} \in C_{\text{weak}}([0, T]; L^2(\Omega; \mathbb{R}^d))$$

represent weak solution to the Euler problem (1.1)–1.3 if:

- $\varrho \geq 0$, $p(\varrho) \in L^1((0, T) \times \Omega)$;
- the equation of continuity

$$\int_0^T \int_{\Omega} \left[ \varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \, dx \, dt = - \int_{\Omega} \varrho_0 \varphi(0, \cdot) \, dx \quad (2.1)$$

holds for any $\varphi \in C^1_{\text{loc}}((0, T) \times \overline{\Omega})$;
- the momentum equation

$$\int_0^T \int_{\Omega} \left[ \mathbf{m} \cdot \partial_t \varphi + \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) : \nabla_x \varphi + p(\varrho) \text{div}_x \varphi \right] \, dx \, dt = - \int_{\Omega} \mathbf{m}_0 \varphi(0, \cdot) \, dx \quad (2.2)$$

holds for any $\varphi \in C^1_{\text{loc}}((0, T) \times \overline{\Omega}; \mathbb{R}^d)$, $\varphi \cdot n|_{\partial \Omega} = 0$.

A weak solution $[\varrho, \mathbf{m}]$ is admissible if it satisfies the energy inequality (1.4) for any $t \in (0, T)$ and a.a. $s \in (0, T)$, $0 \leq s < t$.

### 2.1 Main Results, Solutions with Arbitrary Energy Profile

We are ready to state our first result.
Theorem 2.1 Let \( \Omega \subset \mathbb{R}^d, d = 2, 3 \), be a bounded domain with \( C^2 \) boundary. Let the initial data \( \varrho_0, m_0 \) be given,

\[
0 < \varrho \leq \varrho_0(x) \leq \bar{\varrho} \text{ for all } x \in \bar{\Omega}, \quad \varrho_0 \in \mathcal{R}(\bar{\Omega}),
\]

\[
m_0 \in \mathcal{R}(\bar{\Omega}; \mathbb{R}^d), \quad \text{div}_x m_0 \in \mathcal{R}(\bar{\Omega}), \quad m_0 \cdot n|_{\partial \Omega} = 0.
\]

Let \( E(t) \) be an arbitrary function satisfying

\[
0 \leq E(t) \leq \bar{E} \text{ for all } t \in [0, T], \quad E \in \mathcal{R}[0, T].
\]

Then, there exist \( E_0 \geq 0 \) such that the Euler system (2.1), (2.2) admits infinitely many solutions \([\varrho, m]\) in \((0, T) \times \Omega\) satisfying

\[
\int_{\Omega} \left[ \frac{1}{2} \frac{|m|^2}{\varrho} + P(\varrho) \right] \tau, \cdot \, d\varrho + E_0 + E(\tau) \text{ for a.a. } \tau \in (0, T).
\]

(2.3)

Remark 2.2 It will be clear from the proof given below that the density profile can be taken \( \varrho = \varrho_0(x) \) as soon as \( \text{div}_x m_0 = 0 \). In such a case, we may consider \( \varrho_0 \equiv 1 \) obtaining the same conclusion for the incompressible Euler system. Moreover, the result holds for any bounded domain, no smoothness of the boundary is necessary.

Remark 2.3 To avoid any confusion, we point out the \( E(\tau) \) in (2.3) is a given function and \( E_0 \) is a constant. In particular, \( E_0 \) should not be confused with the initial data energy.

Solutions satisfying strict energy inequality cannot be regular, cf., e.g., Constantin et al. (1994) or Feireisl et al. (2017). Similarly to other “wild” solutions produced by the method of convex integration, the solutions may experience the initial energy jump, meaning the energy inequality (1.4) may not hold for \( s = 0 \). However, as there is definitely a sequence of times \( \tau_n \to 0 \) for which

\[
\int_{\Omega} \left[ \frac{1}{2} \frac{|m|^2}{\varrho} + P(\varrho) \right] (\tau_n, \cdot) \, d\varrho = E_0 + E(\tau_n).
\]

One could also deduce the existence of infinitely many solutions with the energy continuous at the initial time, performing the procedure described, e.g., in De Lellis and Székelyhidi (2010). We leave the details to the interested reader.

2.2 Strong Continuity in Time

We say that a weak solution \([\varrho, m]\) of the Euler system is strongly continuous at a time \( \tau \in (0, T) \) if

\[
\varrho(t, \cdot) \to \varrho(\tau, \cdot) \text{ in } L^1(\Omega), \quad m(t, \cdot) \to m(\tau, \cdot) \text{ in } L^1(\Omega; \mathbb{R}^d) \text{ for } t \to \tau.
\]
Theorem 2.4 Let \( \Omega \subset \mathbb{R}^{d} \), \( d = 2, 3 \), be a bounded domain with \( C^{2} \) boundary. Let the initial data \( \varrho_{0}, m_{0} \) be given,

\[
0 < \underline{\varrho} \leq \varrho_{0}(x) \leq \overline{\varrho} \text{ for all } x \in \Omega, \quad \varrho_{0} \in \mathcal{R}(\overline{\Omega}),
\]

\[
m_{0} \in \mathcal{R}(\overline{\Omega} ; \mathbb{R}^{d}), \quad \text{div}_{x} m_{0} \in \mathcal{R}(\overline{\Omega}), \quad m_{0} \cdot n|_{\partial \Omega} = 0.
\]

Let \( S = \{ \tau_{n} \}_{n=1}^{\infty} \subset (0, T) \) be an arbitrary (countable) set of times.

Then, the Euler system admits infinitely many admissible weak solutions that are not strongly continuous at any \( \tau_{n}, n = 1, 2, \ldots \)

Here again admissible means the total energy is equal to a non-increasing function for a.a. time. In particular, the solutions need not be strongly continuous at \( t = 0 \).

3 Oscillatory Lemma

The proof of our main results depends on a generalized version of Oscillatory Lemma of De Lellis and Székelyhidi (2010). Our starting point is its most elementary version showed in De Lellis and Székelyhidi (2010, Proposition 3):

Lemma 3.1 (Oscillatory Lemma, basic form) Let \( Q = (0, 1) \times (0, 1)^{d} \), \( d = 2, 3 \). Suppose that \( v \in \mathbb{R}^{d}, U \in \mathbb{R}^{d \times d}_{0, \text{sym}}, e \leq \overline{\varrho} \) are given constant quantities satisfying

\[
\frac{d}{2} \lambda_{\text{max}} [v \otimes v - U] < e.
\]

Then, there are a constant \( c = c(d, \overline{\varrho}) \) and sequences of vector functions \( \{ w_{n} \}_{n=1}^{\infty}, \{ \nabla v_{n} \}_{n=1}^{\infty} \),

\[
w_{n} \in C^{\infty}_{c}(Q; \mathbb{R}^{d}), \quad \nabla v_{n} \in C^{\infty}_{c}(Q; \mathbb{R}^{d \times d}_{0, \text{sym}})
\]

satisfying

\[
\partial_{t} w_{n} + \text{div}_{x} \nabla v_{n} = 0, \quad \text{div}_{x} w_{n} = 0 \text{ in } Q,
\]

\[
\frac{d}{2} \lambda_{\text{max}} [(v + w_{n}) \otimes (v + w_{n}) - (U + \nabla v_{n})] < e \text{ in } Q \text{ for all } n = 1, 2, \ldots,
\]

\[
w_{n} \to 0 \text{ in } C_{\text{weak}}([0, 1]; L^{2}((0, 1)^{d}; \mathbb{R}^{d})) \text{ as } n \to \infty,
\]

\[
\liminf_{n \to \infty} \int_{Q} |w_{n}|^{2} \, dx \, dt \geq c(d, \overline{\varrho}) \int_{Q} \left( e - \frac{1}{2} |v|^{2} \right)^{2} \, dx \, dt.
\]

\(^{1}\) \( \mathbb{R}^{d \times d}_{0, \text{sym}} \) denotes the space of real symmetric matrices with zero trace, while \( \lambda_{\text{max}}[\cdot] \) is the maximum eigenvalue.
3.1 Extension by Scaling

We say that $\mathcal{Q} \subset [0, T] \times \mathbb{R}^d$ is a block, if

$$\mathcal{Q} = (t_1, t_2) \times \prod_{i=1}^d (a_i, b_i), \ t_1 < t_2, \ a_i < b_i, \ i = 1, \ldots, d.$$ 

The following can be easily deduced from Lemma 3.1 by a scaling argument, see, e.g., Chiodaroli (2014, Section 6, formula (6.9)).

**Lemma 3.2** (Oscillatory Lemma, scaled form) Let

$$\mathcal{Q} = (t_1, t_2) \times \prod_{i=1}^d (a_i, b_i), \ t_1 < t_2, \ a_i < b_i, \ i = 1, \ldots, d,$$

be a block. Suppose that $v \in \mathbb{R}^d$, $U \in \mathbb{R}^{d \times d}_{0, \text{sym}}$, $e \leq \overline{e}$, and $r > 0$ are given constant quantities satisfying

$$\frac{d}{2} \lambda_{\text{max}} \left[ \frac{v \otimes v}{r} - U \right] < e.$$ 

Then, there are a constant $c = c(d, \overline{e})$ and sequences of vector functions $(w_n)_{n=1}^{\infty}$, $(V_n)_{n=1}^{\infty}$,

$$w_n \in C_c^\infty(\mathcal{Q}; \mathbb{R}^d), \ V_n \in C_c^\infty \left( \mathcal{Q}; \mathbb{R}^{d \times d}_{0, \text{sym}} \right)$$

satisfying

$$\partial_t w_n + \nabla_x V_n = 0, \ \nabla_x w_n = 0 \text{ in } \mathcal{Q},$$

$$\frac{d}{2} \lambda_{\text{max}} \left[ \frac{(v + w_n) \otimes (v + w_n)}{r} - (U + V_n) \right] < e \text{ in } \mathcal{Q} \text{ for all } n = 1, 2, \ldots,$$

$$w_n \to 0 \text{ in } C_{\text{weak}}([t_1, t_2]; L^2(\prod_{i=1}^d (a_i, b_i); \mathbb{R}^d)) \text{ as } n \to \infty,$$

$$\liminf_{n \to \infty} \int_{\mathcal{Q}} \frac{|w_n|^2}{r} \, dx \, dt \geq c(d, \overline{e}) \int_{\mathcal{Q}} \left( e - \frac{1}{2} \frac{|v|^2}{r} \right)^2 \, dx \, dt.$$ 

3.2 Oscillatory Lemma for Riemann Integrable Coefficients

Our main goal is to show the following extension of Oscillatory Lemma.

**Lemma 3.3** (Oscillatory Lemma, general coefficients) Let

$$\mathcal{Q} = (t_1, t_2) \times \prod_{i=1}^d (a_i, b_i), \ t_1 < t_2, \ a_i < b_i, \ i = 1, \ldots, d,$$

be a block. Suppose that

$$v \in \mathcal{R}(\mathcal{Q}; \mathbb{R}^d), \ U \in \mathcal{R} \left( \mathcal{Q}; \mathbb{R}^{d \times d}_{0, \text{sym}} \right), \ e \in \mathcal{R}(\mathcal{Q}), \ r \in \mathcal{R}(\mathcal{Q})$$

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be given such that
\begin{equation}
0 < r \leq r(t, x) \leq \bar{r}, \ e(t, x) \leq \bar{e} \text{ for all } (t, x) \in \overline{Q},
\end{equation}
\begin{equation}
\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max} \left[ \frac{v \otimes v}{r} - \frac{1}{r} \right] < \inf_{\overline{Q}} e.
\end{equation}

Then, there are a constant $c = c(d, \bar{e})$ and sequences of vector functions \( \{w_n\}_{n=1}^{\infty} \), \( \{V_n\}_{n=1}^{\infty} \),

\begin{align*}
& w_n \in C_c^\infty(Q; \mathbb{R}^d), \quad V_n \in C_c^\infty(Q; \mathbb{R}^{d \times d}) \\
& \partial_t w_n + \text{div}_x V_n = 0, \quad \text{div}_x w_n = 0 \text{ in } Q,
\end{align*}

satisfying
\begin{equation}
\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max} \left[ \frac{(v + w_n) \otimes (v + w_n)}{r} - \frac{1}{r} \right] < \inf_{\overline{Q}} e \text{ for all } n = 1, 2, \ldots,
\end{equation}

\begin{equation}
w_n \to 0 \text{ in } C_{\text{weak}}([t_1, t_2]; L^2(\prod_{i=1}^{d}(a_i, b_i); \mathbb{R}^d)) \text{ as } n \to \infty,
\end{equation}

\begin{equation}
\liminf_{n \to \infty} \int_Q \frac{|w_n|^2}{r} \, dx \, dt \geq c(d, \bar{e}) \int_Q \left( e - \frac{1}{2} \frac{|v|^2}{r} \right)^2 \, dx \, dt.
\end{equation}

The remaining part of this section will be devoted to the proof of Lemma 3.3.

### 3.2.1 Basic Properties of Riemann Integrable Functions

The leading idea is to approximate the coefficients $v$, $U$, $e$, and $r$ by piecewise constant functions and use Lemma 3.2. The following is standard and may be found, e.g., in the textbook by Zorich (2016, Chapter 11).

For a real valued function $v: \overline{Q} \to \mathbb{R}$ we introduce:

\[
\text{osc}[v](t, x) = \lim_{s \to 0} \left[ \sup_{B((t, x), s) \cap \overline{Q}} v - \inf_{B((t, x), s) \cap \overline{Q}} v \right],
\]

where $B((t, x), s)$ denotes the ball of radius $s$ centered at $(t, x)$. It holds:

- \[ A_\eta = \left\{ (t, x) \in \overline{Q} \mid \text{osc}[v](t, x) \geq \eta \right\} \text{ is closed} \quad (3.4) \]
- for any $v \in \mathcal{R}(\overline{Q})$ and $\eta > 0$, the set $A_\eta$ is of zero content, meaning for any $\delta > 0$, there exist a finite number of (open) boxes $Q_i$ such that

\[
A_\eta \subset \bigcup_i Q_i, \quad \sum_i |Q_i| < \delta.
\]
3.2.2 Continuity of Eigenvalues

We recall the algebraic inequalities (see, e.g., De Lellis and Székelyhidi 2010)

\[
\frac{1}{2} \frac{|v|^2}{r} \leq \frac{d}{2} \lambda_{\text{max}} \left[ \frac{v \otimes v}{r} - U \right], \quad \|U\|_{\infty} \leq \frac{d}{2} \lambda_{\text{max}} \left[ \frac{v \otimes v}{r} - U \right]
\]

(3.5)

for any \( v \in \mathbb{R}^d, r > 0, U \in \mathbb{R}^{d \times d}_{0, \text{sym}} \), where \( \|U\|_{\infty} \) denotes the operator norm of the matrix.

Consider the set

\[
K = \left\{ r \in (0, \infty), v \in \mathbb{R}^d, U \in \mathbb{R}^{d \times d}_{0, \text{sym}} \left| \frac{r}{L} \leq r \leq \frac{r}{R}, \frac{d}{2} \lambda_{\text{max}} \left[ \frac{v \otimes v}{r} - U \right] \leq \bar{e} \right\}.
\]

In view of (3.5), \( K \) is a compact subset of \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^{d \times d}_{0, \text{sym}}\). Moreover, as shown in De Lellis and Székelyhidi (2010), the function

\[
[w, U] \mapsto \frac{d}{2} \lambda_{\text{max}} [w \otimes w - U]
\]

is convex. As convex functions are Lipschitz continuous on compact subsets of their domain, we deduce there is a constant \( L \) such that

\[
\lambda_{\text{max}} \left[ \frac{v_1 \otimes v_1}{r_1} - U_1 \right] - \lambda_{\text{max}} \left[ \frac{v_2 \otimes v_2}{r_2} - U_2 \right] \leq L \left( |r_1 - r_2| + |v_1 - v_2| + |U_1 - U_2| \right)
\]

for any \((r_i, v_i, U_i) \in K, i = 1, 2\).

(3.6)

3.2.3 Domain Decomposition

Suppose \( v, r, U, r, e \) satisfy (3.1), (3.2). It follows from (3.2) that there exists \( \varepsilon_0 > 0 \) such that

\[
\frac{d}{2} \lambda_{\text{max}} \left\{ \frac{v \otimes v}{r} - U \right\} < e - \varepsilon_0 \leq \bar{e} \text{ in } \overline{Q}.
\]

In particular, \((r, v, U)(t, x) \in K \) for any \((t, x) \in \overline{Q} \). Thus, for any \( 0 < \varepsilon \leq \varepsilon_0 \)

\[
\frac{d}{2} \lambda_{\text{max}} \left\{ \frac{v \otimes v}{r} - U \right\} < e - \varepsilon \leq \bar{e} \text{ in } \overline{Q}.
\]

(3.7)

For \( \eta > 0 \) consider the set

\[
A_\eta = A_\eta[v] \cup A_\eta[r] \cup A_\eta[U] \cup A_\eta[e],
\]

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cf. (3.4). In accordance with our hypotheses, this is a set of zero content, meaning there are a finite number of (open) boxes $Q^i_\eta(\eta)$ such that

$$A_\eta \subset \bigcup_i Q^i_\eta(\eta), \quad \sum_i |Q^i_\eta(\eta)| < \varepsilon \text{ for any given } \eta > 0.$$ 

The complement $\overline{Q} \setminus \bigcup_i Q^i_\eta(\eta)$ is compact. Moreover, each point $y \in \overline{Q} \setminus \bigcup_i Q^i_\eta$ has an open neighborhood $U(y)$ such that

$$|r(y_1) - r(y_2)| < 2\eta, \quad |v(y_1) - v(y_2)| < 2\eta, \quad |U(y_1) - U(y_2)| < 2\eta, \quad |e(y_1) - e(y_2)| < 2\eta$$

whenever $y_1, y_2 \in U(y) \cap \overline{Q}.$

As the set $\overline{Q} \setminus \bigcup_i Q^i_\eta(\eta)$ is compact and there are a finite number of $Q^i_\eta$, we may infer that for any given $\varepsilon > 0, \eta > 0$, there exists a decomposition of $\overline{Q}$ into a finite number of blocks:

$$\overline{Q} = (\bigcup_i Q^i_\eta(\eta)) \cup (\bigcup_j Q^j_\eta(\eta)), \quad Q^j_\eta \cap Q^k_\eta = \emptyset \text{ for } j \neq k,$$

such that

$$\sum_i |Q^i_\eta(\eta)| < \varepsilon$$

and

$$|r(y_1) - r(y_2)| < 2\eta, \quad |v(y_1) - v(y_2)| < 2\eta, \quad |U(y_1) - U(y_2)| < 2\eta, \quad |e(y_1) - e(y_2)| < 2\eta$$

for any $y_1, y_2 \in \overline{Q}_j$, $j = 1, 2, \ldots$

(3.9)

### 3.2.4 Localization

Given $0 < \varepsilon \leq \varepsilon_0, \eta > 0$, consider the decomposition of $\overline{Q}$ given by (3.9). Choosing $y_j \in Q^j_\eta$ we fix

$$\tilde{r} = r(y_j), \quad \tilde{v} = v(y_j), \quad \tilde{U} = U(y_j), \quad \text{and } \tilde{e} = e(y_j).$$

Applying the constant coefficient version of Oscillatory Lemma (Lemma 3.2) on each $Q^j_\eta$ we get a sequence of functions $w^j_n, \nabla^j_n$, smooth and compactly supported in $Q^j_\eta \equiv (s_1, s_2) \times O^j_\eta$, such that

$$\partial_t w^j_n + \text{div}_x \nabla^j_n = 0, \quad \text{div}_x w^j_n = 0 \text{ in } Q^j_\eta,$$

(3.10)

$$w^j_n \to 0 \text{ in } C_{\text{weak}}([s_1, s_2]; L^2(O^j_\eta; \mathbb{R}^d)),$$

(3.11)

$$\frac{d}{2} \lambda_{\text{max}} \left\{ \frac{\tilde{v} + w^j_n}{\tilde{r}} \otimes \frac{\tilde{v} + w^j_n}{\tilde{r}} - (\tilde{U} + \nabla^j_n) \right\} + \varepsilon < \tilde{e} \text{ in } Q^j_\eta,$$

(3.12)
and
\[
\liminf_{n \to \infty} \int_{Q_j^r} \frac{|w_n^j|^2}{r} \, dx \, dt \geq c(d, \bar{\varepsilon}) \int_{Q_j^r} \left( \bar{\varepsilon} - \frac{1}{2} \frac{|\tilde{v}|^2}{r} \right)^2 \, dx \, dt. \tag{3.13}
\]

In view of the Lipschitz continuity of the eigenvalues established in (3.6), and in accordance with (3.9), we may choose \( \eta = \eta(\varepsilon) \) small enough so that
\[
\frac{d}{2} \lambda_{\text{max}} \left\{ \frac{(v + w_n^j) \otimes (v + w_n^j)}{r} - (U + \nabla w_n^j) \right\} + \frac{e}{2} < e \text{ in } Q_j^r. \tag{3.14}
\]

By the same token, we get
\[
\liminf_{n \to \infty} \int_{Q_j^r} \frac{|w_n^j|^2}{r} \, dx \, dt \geq c(d, \bar{\varepsilon}) \int_{Q_j^r} \left( e - \frac{1}{2} \frac{|\tilde{v}|^2}{r} \right)^2 \, dx \, dt - \varepsilon |Q_j^r|. \tag{3.15}
\]

Finally, setting \( w_n^i = \nabla w_n^j = 0 \text{ in } Q_j^i \) and summing up over all boxes, we obtain sequences defined on \( Q \) satisfying
\[
\liminf_{n \to \infty} \int_Q \frac{|w_n|^2}{r} \, dx \, dt \geq c(d, \bar{\varepsilon}) \sum_j \int_{Q_j^r} \left( e - \frac{1}{2} \frac{|\tilde{v}|^2}{r} \right)^2 \, dx \, dt - \varepsilon |Q|
\]
\[\geq c(d, \bar{\varepsilon}) \int_Q \left( e - \frac{1}{2} \frac{|\tilde{v}|^2}{r} \right)^2 \, dx \, dt - \varepsilon \left( |Q| + \bar{\varepsilon}^2 \right), \tag{3.16}\]

and
\[
w_n \to 0 \text{ in } C_{\text{weak}}([t_1, t_2]; L^2(Pi_1^d(a_i, b_i); \mathbb{R}^d)). \tag{3.17}\]

As pointed out, the oscillatory perturbations can be constructed for any \( 0 < \varepsilon < \varepsilon_0 \).

### 3.2.5 Diagonalization Argument

To complete the proof of Lemma 3.3, it remains to get rid of the \( \varepsilon \)-dependent term in (3.16). This can be achieved by a simple diagonalization argument. By the previous subsection, for any \( \varepsilon > 0 \) there exists \( \{w_n^k\}_{n \in \mathbb{N}} \) such that (3.16) and (3.17) hold. Combining (3.16) and a basic property of the liminf, we get that there exists \( n_{0,\varepsilon} \) such that for all \( n \geq n_{0,\varepsilon} \) it holds
\[
\int_Q \frac{|w_n^k|^2}{r} + \varepsilon (|Q| + \bar{\varepsilon}^2) \, dx \, dt \geq c(d, \bar{\varepsilon}) \int_Q \left( e - \frac{1}{2} \frac{|\tilde{v}|^2}{r} \right)^2 \, dx \, dt. \tag{3.18}\]

In addition, we can fix \( n_{0,\varepsilon} \) in such a way that
\[
d(w_n^k, 0) < \varepsilon \text{ for all } n \geq n_{0,\varepsilon} \tag{3.19}\]
where \( d(\cdot, \cdot) \) is the metric defined as
\[
d(\cdot, \cdot) = \sup_{t \in [0,T]} d_B(\cdot, \cdot)
\]
and \( d_B(\cdot, \cdot) \) is the metric induced by the weak topology on bounded sets of the Hilbert space \( L^2(\prod_{i=1}^d (a_i, b_i); \mathbb{R}^d) \). For any \( k \in \mathbb{N} \), let us choose \( \varepsilon = \frac{\varepsilon_0}{k} \), then there exists a sequence \( \{w_n^k\}_{n \in \mathbb{N}} \), which fulfills (3.18) and (3.19) definitely. We do not relabel such subsequence. Thus, we get an infinite matrix
\[
\left( \begin{array}{cccccc}
w_1^1 & w_2^1 & \cdots & w_k^1 & \cdots & w_n^1 \\
w_1^{1/2} & w_2^{1/2} & \cdots & w_k^{1/2} & \cdots & w_n^{1/2} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
 w_1^{1/k} & w_2^{1/k} & \cdots & w_k^{1/k} & \cdots & w_n^{1/k} \\
 \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
 \end{array} \right)
\]
where the \( k \)th row corresponds to a sequence fulfilling (3.18) and (3.19) with \( \varepsilon = \frac{\varepsilon_0}{k} \). Consider the sequence \( \{w_k^{1/k}\}_k \), which corresponds to the diagonal of the matrix above, it enjoys
\[
\int_Q \frac{|w_k^{1/k}|^2}{r} + \frac{1}{k}(|Q| + \varepsilon^2) \, dx \, dt \geq c(d, e) \int_Q \left( e - \frac{1}{2} \frac{|v|^2}{r} \right)^2 dx \, dt \tag{3.20}
\]
and
\[
d(w_k^{1/k}, 0) < \frac{\varepsilon_0}{k} \tag{3.21}
\]
Taking, respectively, the liminf and the limit as \( k \to +\infty \), we conclude that
\[
\liminf_{k \to +\infty} \int_Q \frac{|w_k^{1/k}|^2}{r} \, dx \, dt \geq c(d, e) \int_Q \left( e - \frac{1}{2} \frac{|v|^2}{r} \right)^2 dx \, dt \tag{3.22}
\]
and
\[
w_k^{1/k} \to 0 \text{ in } C_{\text{weak}}([t_1, t_2]; L^2(\prod_{i=1}^d (a_i, b_i); \mathbb{R}^d)) \tag{3.23}
\]

\textbf{Remark 3.4} The conclusion of Lemma 3.3 holds if \( Q \) is a bounded open set. Indeed \( Q \) can be covered by a countable number of blocks on each of which we may apply the previous arguments. The relevant result is provided by Whitney decomposition lemma (Stein 1970), see Donatelli et al. (2015, Section 4.4) for details.

\section*{4 Applications}

Our ultimate goal is to apply the general version of Oscillatory Lemma to show existence of weak solutions to the compressible Euler system with given energy.
4.1 Rewriting the Euler System as an Abstract Problem

Following Feireisl (2016), we write the initial momentum in the form of its Helmholtz decomposition,

$$\mathbf{m}_0 = \mathbf{v}_0 + \nabla_x \Phi_0,$$

where

$$\Delta_x \Phi_0 = \text{div}_x \mathbf{m}_0 \text{ in } \Omega, \quad \nabla_x \Phi_0 \cdot \mathbf{n}|_{\partial \Omega} = 0.$$

As the boundary $\partial \Omega$ is of class $C^2$, the standard elliptic estimates imply $\nabla_x \Phi_0 \in \mathcal{W}^{1,p}(\Omega; \mathbb{R}^d)$, in particular $\nabla_x \Phi_0 \in C(\overline{\Omega}; \mathbb{R}^d)$, see e.g. Agmon et al. (1959).

Next, we fix the density profile

$$\varrho(t, x) = \varrho_0 + h(t) \Delta_x \Phi_0, \quad h \in C^\infty\mathbb{R}, \quad h(0) = 0, \quad h'(0) = -1.$$

We look for solutions in the form

$$\mathbf{m} = \mathbf{v} - h'(t) \nabla_x \Phi_0, \quad \text{div}_x \mathbf{v} = 0, \quad \mathbf{v} \cdot \mathbf{n}|_{\partial \Omega} = 0.$$

Seeing that

$$\partial_t \varrho = h'(t) \Delta_x \Phi_0 = -\text{div}_x \mathbf{m},$$

we can adjust $h$ in such a way that

$$0 < \frac{1}{2} \underline{\varrho} \leq \varrho(t, x) \leq 2 \overline{\varrho} \text{ for all } t \geq 0, \quad x \in \overline{\Omega}$$

provided the initial density is uniformly bounded below and above. In addition, for $\underline{\varrho} \in \mathcal{R}(\overline{\Omega})$, we have

$$\varrho \in \mathcal{R}([0, T] \times \overline{\Omega}).$$

Accordingly, we look for a vector field $\mathbf{v}$ solving the following problem:

$$\text{div}_x \mathbf{v} = 0, \quad \mathbf{v} \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \mathbf{v}(0, \cdot) = \mathbf{v}_0,$$

$$\partial_t \mathbf{v} - h''(t) \nabla_x \Phi_0 + \text{div}_x \left( \frac{(\mathbf{v} - h'(t) \nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t) \nabla_x \Phi_0)}{\varrho} - \frac{1}{d} \frac{|\mathbf{v} - h'(t) \nabla_x \Phi_0|^2}{\varrho} \right) = 0,$$

with prescribed kinetic energy

$$\frac{1}{2} \frac{\mathbf{v} - h'(t) \nabla_x \Phi_0|^2}{\varrho} = \Lambda(t) - \frac{d}{2} p(\varrho) + \frac{d}{2} h''(t) \Phi_0$$ (4.2)
where \( \Lambda = \Lambda(t) \) is a spatially homogeneous function to be chosen below.

Plugging (4.2) in (4.1) and using the fact that \( \text{div}_x \Lambda(t) = 0 \) we observe that any weak solution \( v \) of (4.1), (4.2) gives rise to a weak solution \([\varrho, m = v - h'(t)\nabla_x \Phi_0]\) of the Euler system (2.1), (2.2), with the total energy

\[
\int_\Omega \left[ \frac{1}{2} \frac{|m|^2}{\varrho} + P(\varrho) \right] (\tau, \cdot) \, dx = \Lambda(\tau)|\Omega| + \int_\Omega \left[ P(\varrho) - \frac{d}{2} p(\varrho) + \frac{d}{2} h''(\tau)\Phi_0 \right] \, dx
\]

for a.a. \( \tau \in (0, T) \),

(4.3)

cf. Chiodaroli (2014) and Feireisl (2016).

Evoking the notation of Theorem 2.1, we set

\[
\Lambda(\tau) = \frac{E(\tau)}{|\Omega|} + \Lambda_0(\tau), \quad E_0 = \Lambda_0(\tau)|\Omega| + \int_\Omega \left[ P(\varrho) - \frac{d}{2} p(\varrho) + h''(\tau)\Phi_0 \right] \, dx.
\]

Thus, the proof of Theorem 2.1 consists in showing that for given \( \varrho_0 \) and \( E \), there exists \( E_0 \) large enough so that the problem (4.1), (4.2) admits (infinitely many) weak solutions.

### 4.2 Subsolutions

We start by fixing the energy profile

\[
e = e(t, x) = \frac{E(t)}{|\Omega|} + \Lambda_0(t) - \frac{d}{2} p(\varrho) + \frac{d}{2} h''(t)\Phi_0, \quad e \in \mathcal{R}([0, T] \times \overline{\Omega}).
\]

Similar to De Lellis and Székelyhidi (2010), we introduce the space of subsolutions,

\[
X_0 = \left\{ (v - v_0) \in C^1([0, T] \times \overline{\Omega}) \mid v(0, \cdot) = v_0, \quad v \cdot n|_{\partial\Omega} = 0, \quad \text{div}_x v = 0, \quad \partial_t v + \text{div}_x U = 0 \text{ for some } U \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^{d\times d}) \right\}.
\]

The functions \( E \) and \( m_0 \) given, we fix \( \Lambda_0 \), together with the constant \( E_0 \), so that the set \( X_0 \) is non-empty. This can be achieved by considering \( v = v_0, U = 0 \) and fixing \( \Lambda_0 \) appropriately. Finally, we set

\[
\overline{e} = \sup_{[0, T] \times \overline{\Omega}} e(t, x) < \infty.
\]

Thus, by virtue of (3.5), the set \( X_0 \) is bounded in \( L^\infty((0, T) \times \Omega; \mathbb{R}^d) \); whence metrizable in the topology of \( C_{\text{weak}}([0, T]; L^2(\Omega; \mathbb{R}^d)) \). We denote by \( X \) its closure in the corresponding metric \( d \).
4.3 Critical Points of the Energy Functional

Following De Lellis and Székelyhidi (2010), we introduce the functional

\[ I[v] = \int_0^T \int_\Omega \left( \frac{1}{2} \frac{|v - h'(t)\nabla_x \Phi_0|^2}{\varrho} - e \right) \, dx \, dt \text{ for } v \in X. \]

The functional \( I \) is convex lower-semicontinuous on the complete metric space \( X \). By Baire category argument we conclude that the points of continuity must form a dense set in \( X \).

The second observation is that

\[ I[v] = 0 \Rightarrow v \text{ is a weak solution of the problem (4.1), (4.2)}. \]

Indeed, from convexity of the function

\[
[v; U] \mapsto \frac{d}{2} \lambda_{\max} \left[ \frac{(v - h'(t)\nabla_x \Phi_0) \otimes (v - h'(t)\nabla_x \Phi_0)}{\varrho} - U \right],
\]

we deduce that for any \( v \in X \) there is \( U \in L^\infty((0, T) \times \Omega; \mathbb{R}^{d \times d}) \)

\[
\partial_t v + \text{div}_x U = 0 \text{ in } \mathcal{D}'((0, T) \times \Omega), \quad \frac{1}{2} \frac{|v - h'(t)\nabla_x \Phi_0|^2}{\varrho} \leq \frac{d}{2} \lambda_{\max} \left[ \frac{(v - h'(t)\nabla_x \Phi_0) \otimes (v - h'(t)\nabla_x \Phi_0)}{\varrho} - U \right] \leq e \text{ a.e. in } (0, T) \times \Omega.
\]

Consequently, \( I \leq 0 \) on \( X \); while \( I[v] = 0 \) implies the desired relations (cf. De Lellis and Székelyhidi 2010)

\[
\frac{1}{2} \frac{|v - h'(t)\nabla_x \Phi_0|^2}{\varrho} = e, \quad U = \frac{(v - h'(t)\nabla_x \Phi_0) \otimes (v - h'(t)\nabla_x \Phi_0)}{\varrho} - \frac{d}{2} \frac{|v - h'(t)\nabla_x \Phi_0|^2}{\varrho} \mathbb{1} \text{ a.e. in } (0, T) \times \Omega.
\]

Thus, similar to the arguments used in De Lellis and Székelyhidi (2010), it remains to observe:

\[ v \text{ a point of continuity of } I \text{ on } X \Rightarrow I[v] = 0. \quad (4.4) \]

To show (4.4), we argue by contradiction. Assuming

\[ I[v] = \underline{I} < 0 \]

we construct a sequence of functions

\[ v_m \in X_0 \text{ with the corresponding fields } U_m \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^{d \times d}). \]
such that

\[ v_m \to v \text{ in } X, \ I[v_m] \to I < 0 \text{ as } m \to \infty. \]

For fixed \( m \), we apply Oscillatory Lemma (Lemma 3.3) for \( v = v_m - h'(t)v_x \Phi_0, \ U = U_m, r = \rho_0, \) and \( e \). We obtain sequences \[ w_{m,n} \to v \]

\[ u_m = U_m, \ \rho = \rho_0, \ e \]. We obtain sequences \( \{w_{m,n}\}_{n=1}^{\infty}, \ {v_{m,n}}_{n=1}^{\infty} \) satisfying:

- \( v_m + w_{m,n} \in X_0 \) with the associated fields \( U_m + V_{m,n} \) for any \( m, n \);
- \( v_m + w_{m,n} \to v \) in \( X \) as \( n \to \infty \) for any fixed \( m \);

(4.5)

\[
\lim_{n \to \infty} \int_0^T \int_\Omega \frac{1}{2} \frac{|v_m + w_{m,n} - h'(t)v_x \Phi_0|^2}{\rho} \, dx \, dt \\
= \int_0^T \int_\Omega \frac{1}{2} \frac{|v_m - h'(t)v_x \Phi_0|^2}{\rho} \, dx \\
+ \lim_{n \to \infty} \int_0^T \int_\Omega \frac{1}{2} \frac{|w_{m,n}|^2}{\rho} \, dx \, dt \\
\geq \int_0^T \int_\Omega \frac{1}{2} \frac{|v_m - h'(t)v_x \Phi_0|^2}{\rho} \, dx \\
+ c(d, \overline{\varepsilon}) \int_0^T \int_\Omega \left( \frac{1}{2} \frac{|v_m - h'(t)v_x \Phi_0|^2}{\rho} - e \right)^2 \, dx \, dt \\
\geq \int_0^T \int_\Omega \frac{1}{2} \frac{|v_m - h'(t)v_x \Phi_0|^2}{\rho} \, dx + c(d, \overline{\varepsilon}) \frac{T}{\Omega} (I[v_m])^2; \]

(4.6)

where we have used Jensen’s inequality in (4.6). Relation (4.6) rewritten as

\[
\liminf_{n \to \infty} I[v_m + w_{m,n}] \geq I[v_m] + \frac{c(d, \overline{\varepsilon})}{T \Omega} (I[v_m])^2 \]

for any \( m \)

implies that \( v \) cannot be a point of continuity of \( I \) unless \( I[v] = 0 \).

We have proved Theorem 2.1.

4.4 Points of Strong Continuity

We show how Theorem 2.4 follows from Theorem 2.1. Given the set \( \{\tau_n\}_{n=1}^{\infty} \) it is a routine matter to construct a function \( E: [0, T] \to \infty, \)

\[
0 \leq E(t) \leq E \quad \text{for all } t \in [0, T], \ E \text{ strictly decreasing in } [0, T], \\
\lim_{t \to \tau_n^-} E(t) > \lim_{t \to \tau_n^+} E(t) \text{ for any } \tau_n, \ n = 1, 2, \ldots
\]
Consider the solutions $[\rho, m]$, the existence of which is guaranteed by Theorem 2.1 with the energy profile

$$
\int_{\Omega} \left( \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho) \right) (\tau, \cdot) \, dx = E_0 + E(\tau) \text{ for a.a. } \tau \in (0, T).
$$

As $\rho, m$ is uniformly bounded and $\rho$ bounded below away from zero, the energy

$$
\tau \mapsto \int_{\Omega} \left( \frac{1}{2} \frac{|m|^2}{\rho} + P(\rho) \right) (\tau, \cdot) \, dx
$$

must be continuous at any point of strong continuity of $[\rho, m]$. Consequently, $\tau_n$ cannot be points of strong continuity of $[\rho, m]$.

We have shown Theorem 2.4.

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