Dynamical Scaling in Time dependence of Correlation Length in Non-equilibrium Critical Relaxation of Pure and Site-diluted 2D XY-model

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Abstract. Structural disorder in two-dimensional XY-model creates an anomalous slowing down in the process of critical relaxation. Dynamic dependences of the correlation length \( \xi(t) \) show slow growth with an “effective” dynamical critical exponent \( z > 2 \). For a pure system, it has been shown that a detailed consideration of the vortex dynamics leads to deviations from the known dependence \( \xi(t) \sim (t/\ln t)^{1/2} \).

The study of critical behavior of disordered systems causes considerable fundamental and applied scientific interest [1,2]. The critical phenomena are characterized by a strong fluctuation behavior. The low-dimensional systems and the systems with continuous symmetry of the ground state are usually characterized by much stronger fluctuation effects than bulk systems and systems with discrete symmetry. Therefore, low-dimensional systems with continuous symmetry are characterized by a significant contribution of fluctuations to critical properties [3]. In two-dimensional systems with continuous symmetry, the long-range order (LRO) is destroyed by anomalously strong transverse fluctuations of the spin density at all non-zero temperatures. However, in such systems there are more exotic forms of ordering, which are associated with the existence of topological excitations in them, such as vortices [4–6].

A special place among two-dimensional systems with continuous symmetry is occupied by a two-dimensional XY-model [3–5]. In the two-dimensional XY-model take place a Berezinskii-Kosterlitz-Thouless (BKT) topological phase transition at temperature \( T = T_{BKT} \) and a low-temperature Berezinskii phase \( T < T_{BKT} \), where each temperature \( T \) is a critical point (there is a continuous set of fixed points of the renormalization group transformation) and a quasi-long-range order (QLRO) occurs. This makes it possible to study the critical behavior of the system not only in one isolated point \( T_{C} \) (as, for example, in the Ising model), but in the whole temperature range \( T \leq T_{BKT} \). The phase transition is physically related to dissociation at the transition point of the coupled vortex pairs. A two-dimensional XY-model is used to describe the critical properties of a wide range of real physical systems [3], such as critical properties of ultra-thin magnetic films [7] and extensive class of easy plane planar magnets [8–13]. The singularities in the critical properties of superconducting [14,15] and superfluid helium [16–19] thin films are described by a two-dimensional XY-model. The arrays of Josephson [3,20,21] and SFS-junctions [22–24]; two-dimensional crystals [5] and smectic liquid crystals [25–29]; some...
correlation properties of two-dimensional turbulence [30]; melting of several layers of sorbed xenon in single-crystal graphite [31]; the process of sorbing hydrogen on tungsten [32]; and properties of many other physical systems [3,33] are described using a two-dimensional XY-model. Also, to the phenomena described by the XY-model, it is possible to relate dynamic properties of such exotic systems as collective behavior of living organisms [34–36].

The non-equilibrium critical relaxation of a two-dimensional XY-model can be a vortex and spin-wave depending on the initial non-equilibrium state [37–41]. The low-temperature initial state is prepared in contact with a thermostat at zero temperature $T_0 = 0$ and the relaxation dynamics will be spin-wave [37,38,40]. The high-temperature initial state is prepared at a high temperature $T \gg T_{BKT}$ thermostat, and the relaxation is accompanied by vortex dynamics [37–39]. Inclusion of the structural disorder in the model leads to a significant change in critical behavior of a two-dimensional XY-model [39–41]. The effect of structure defects on spin-wave relaxation is not considered in this work, details are given in our other work [40]. The effect of structure defects on the non-equilibrium critical vortex dynamics is accompanied by the appearance of non-equilibrium pinning of vortices on defects [39,41].

The relaxation dynamics of the two-dimensional XY-model is accompanied by slow dynamics and aging effects [2,37–40]. It was shown in the papers [42,43] that the vortex relaxation of the diluted model occurs extremely slowly. This particularly applies to the vortex concentration and geometric characteristics of QLRO-areas. In the works [39,40] it was shown that the inclusion of structural disorder into the system leads to changes in the aging effects. For spin-wave relaxation, structural disorder leads to the violation of the canonical dynamic scaling of the autocorrelation function $C(t,t_w)$ and to the occurrence of superaging and subaging effects [40].

Previously, such properties were observed only in complex disordered systems, such as spin glasses [44]. However, in recent years, similar behavior has been found in the critical dynamics of the diluted three-dimensional Ising model [45–49] in the process of relaxation from the initial low-temperature state. It also recently emerged that superaging occurs in the critical relaxation of the two-dimensional diluted Ising model [50]. It is clear that the two-dimensional diluted XY-model is no exception [40], especially considering the much slower dynamics of this system compared with the diluted Ising model.

The dynamic scaling was considered based on the dynamic dependence of the correlation

![Figure 1.](image-url)
length $\xi(t)$ of the system, that was used in all the above works, for both two-dimensional XY-model [2, 37–40] and the Ising model [45–50]. Thus, the dynamic dependence of the correlation length $\xi(t)$ becomes the “stumbling block” (or “hot spot”) of all studies of slow dynamics and aging in non-equilibrium critical relaxation of systems in the critical state. The dynamic dependence of two-dimensional XY-model of the correlation length $\xi(t)$ in the spin-wave relaxation has the form $\xi(t) \sim t^{1/z}$, where $z = 2$ is the dynamical critical exponent, following from the considerations of dimension and dynamic scale invariance. In the work [51] it was shown that the vortex dynamics is characterized by a logarithmic correction to the dynamic dependence $\xi(t) \sim (t/\ln t)^{1/z}$. However, this result is not based on direct calculation of the dynamic dependence of the correlation length $\xi(t)$, but only on the collapse of the dynamic dependences of the cumulants of the magnetization of the system and is also explained by solving the equation of motion of a vortex under the action of effective friction in the long-time mode. However, this dependence was obtained and, mostly, well-proven for studying processes in a pure system. In this paper, a direct calculation of the dynamic dependence of the correlation length $\xi(t)$ of a pure and diluted two-dimensional XY-model was performed.

The Hamiltonian of the two-dimensional XY-model in this work was chosen as

$$H[p, S] = -\frac{1}{2} \sum_{\langle i,j \rangle} p_i p_j S_i S_j = -\frac{1}{2} \sum_{\langle i,j \rangle} p_i p_j \cos (\varphi_i - \varphi_j),$$

where $S = \{S_i\}$ and $p = \{p_i\}$ are the spin and defect lattice fields; $S_i = (S_{i,x}, S_{i,y}) \equiv (\cos \varphi_i, \sin \varphi_i)$ is a classical planar spin which is associated with $i$-node of square lattice with the linear size $L$; $\varphi_i$ is a phase of spin $S_i$; $p_i$ is an occupation number of $i$-node: if $p_i = 1$ then $i$-node is occupied by spin, else if $p_i = 0$ by defect; $\sum_{\langle i,j \rangle}$ is the summation over all pairs of the nearest neighbors. Defects on the lattice are distributed uniformly at time $t = 0$ with spin concentration $p$, i.e. $c_{\text{imp}} = 1 - p$ is the concentration of impurity.

Simulation was carried out in the low-temperature Berezinskii phase at “freezing” temperatures $T \leq T_{\text{BKT}}(p)$ by using Metropolis algorithm, as it adequately sets the dynamic properties of this system [52]. The study was carried out for the spin concentrations $p = 1.0$ (pure system), 0.9, 0.8 and 0.7 (diluted systems). The values of the phase transition $T_{\text{BKT}}(p)$ temperature were chosen from the works [41, 53].

The correlation length was calculated from the Ornstein-Zernike relation (see, for example, [54])

$$\xi(t) = \frac{1}{2 \sin(\pi/L)} \sqrt{\frac{\chi(t)}{\Phi(t)}} - 1,$$

where $\chi(t) = \sum_i \langle |p_i S_i^2| \rangle$ and $\Phi(t)$ is the structure factor:

$$\Phi(t) = \frac{1}{2} \left\langle \left| \sum_{n=1,2} \sum_i p_i S_i \exp \left( i \frac{2\pi x_n}{L} \right) \right|^2 \right\rangle;$$

where the square brackets $[\ldots]$ denote averaging over the initial configurations of the distribution of impurities in the system and the angle brackets $\langle \ldots \rangle$ denote averaging over various initial spin configurations within the created configuration of structural disorder. The dynamic scaling were determined from the considerations of the collapse of the dynamic dependences of the Binder

![Figure 2. Parametric scaling dependences of the fourth-order magnetization cumulant $U_4(t)$ on $\sqrt{t/\ln t}/L$. There is good agreement with the results of the work [51].](image)
magnetization cumulants $U_2(t) = \langle (\mathbf{M}^2) \rangle / \langle \mathbf{M} \rangle^2 - 1$ and $U_4(t) = 2 - \langle (\mathbf{M}^4) \rangle / \langle \mathbf{M}^2 \rangle^2$, where $\mathbf{M} = \sum_i p_i \mathbf{S}_i$ is the magnetization of system, in reduced variables $\xi(t)/L$ for various linear sizes $L$.

The averaging of the calculated values for diluted system was carried out over 50000 impurity configurations and 25 statistical passes for each impurity configuration. A large amount of statistical sampling is associated with a bad convergence of dynamic dependences with averaging for diluted system and with the slowness of the relaxation process, which required the study of large time scales. This allowed the use of linear sizes $L = 16, 20, 24$ and 32 to study the non-equilibrium critical relaxation of the diluted system, and the observation time was chosen equal to 20000 MCS/s.

The obtained parametric dependences of the fourth-order magnetization cumulant $U_4$ on $\xi(t)/L$ (see Fig. 1) for a diluted system demonstrate the collapse of curves for different linear sizes $L$. The results for a pure system show excellent performance of finite-size dynamic scaling dependence $U_4(t) = U_4(\xi(t)/L)$ for all considered “freezing” temperatures up to the phase transition point $T_{\text{BKT}}$. The insertion of disorder leads to a slower relaxation of the system, while during the observation time of 20000 MCS/s, the quantities of $U_4(t)$ do not reach the asymptotic value $U_4(t \to \infty) \approx 1$. Parametric dependences $U_4$ on $\xi(t)/L$ do not have a power growth range for pure system ($p = 1.0$); and the curves are clearly separated for different temperatures $T$. With the insertion of disorder, a power stage of growth occurs $U_4 \sim (\xi(t)/L)^\alpha$. The duration of this power stage increases with increasing impurity concentration $c_{\text{imp}}$, and the separation of the curves for different temperatures $T$ disappears.

It should be noted that the parametric dependence for a pure system is somewhat different from that obtained in the work [51]. To verify the results obtained, curves were constructed for an explicitly defined dynamic dependence $\xi(t) \sim (t/\ln t)^{1/2}$ (Fig. 2). This demonstrates good agreement with the results [51]. However, the results in Fig. 1 for a pure system ($p = 1.0$) demonstrate a significantly better collapse of the curves $U_4(\xi(t)/L)$ than those shown in Fig. 2 and in work [51]. The obtained parametric dependences of the second-order cumulant $U_2(t)$ on $\xi(t)/L$ (see Fig. 1) for a diluted system also demonstrate the collapse of curves for different linear sizes $L$.

The obtained dynamic dependences of the correlation length $\xi(t)$ are shown in Fig. 3. For a pure system, one can immediately note that the dependence $\xi(t)$ explicitly calculated by the formula (2) cannot be correctly approximated by dependence $(t/\ln t)^{1/2}$ at all time scales. If we conditionally introduce the dependence $\xi(t) \sim t^{b(t)}$ or $\xi(t) \sim (t/\ln t)^{b(t)}$ (qualitative aspect of the case does not change the selection) and the “effective exponent” $b(t)$ of the slope of the curve, then we get $db(t)/dt > 0$. In this case, starting from a certain time $t$ (separately selected for each temperature $T$), we can get $b(t) > 1/2$, i.e. “effective” dynamical critical exponent $z < 2$. This immediately raises many questions about the singularity of the kinetic coefficient. The obtained dynamic dependence $\xi(t)$ conveys in detail the features of ordering in the system, while for the qualitative performance of the collapse (Fig. 2), a rather simple
relation $\xi(t) \sim (t/\ln t)^{1/2}$. In this case, the actual values $z < 2$ may not occur. This is due to the detail of the description embedded in the dependence according to the formula (2). In the work [51], the dynamic dependence $\xi(t) \sim (t/\ln t)^{1/2}$ is obtained as a solution to the equation $d\xi/dt = \rho_s F/\xi \ln(\xi/a)$ on time $t \gg a^2/(4\rho_s \Gamma)$. This equation is obtained from considering the motion of a free vortex in the presence of friction, which only effectively includes the processes of interaction of vortices, such as annihilation. However, it is clear that this approach does not take into account the whole picture of the dynamics of vortex relaxation, and the dependences obtained in this work take into account (of course, within the framework of the designated model representations). We associate the features of the deviation of dynamical dependence $\xi(t)$ from the dependence introduced in the work [51] with a detailed account of the full picture of the non-equilibrium vortex dynamics at different stages of the relaxation dynamics, and not only the effective friction of an isolated free vortex in the process of movement.

The insertion of disorder into the system leads to a significant slowing down of the process of critical relaxation. For illustration in Fig. 3 curves are shown corresponding to dynamic dependencies $t^{1/2}$ and $(t/\ln t)^{1/2}$. It is clearly seen that with increasing impurity concentration $c_{\text{imp}}$, the “effective exponent” $b(t)$ (see above) of the curves decreases, and $b(t) < 1/2$. This may indicate that the “effective” dynamical critical exponent $z > 2$. The reliability of the obtained dynamic dependencies can be ensured by the quality of the collapse of dimensional dynamic scaling curves (Fig. 1). An anomalous slowing down in the dynamics of relaxation can be associated with pinning of the vortices on the defects, which indirectly affects the process of coarsening of the quasi-long-order areas [42, 43]. It was shown [43] that the characteristic times of relaxation processes in a two-dimensional diluted XY-model can increase almost unlimitedly even for small linear sizes of the system ($L \approx 256$).

Conclusion

The insertion of disorder into the system leads to an anomalous slowing down of the process of critical relaxation. The resulting dynamic dependences of correlation length $\xi(t)$ show slow growth with an “effective” dynamical critical exponent $z > 2$, but this does not necessarily apply to the true value of the dynamic critical exponent $z$. For a pure system, it has been shown that a detailed consideration of the vortex dynamics leads to deviations from the known dependence $\xi(t) \sim (t/\ln t)^{1/2}$.

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References

[1] U.C. Tauber, Critical Dynamics: A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior, (Cambridge University Press, Cambridge, 2014).
[2] V.V. Prudnikov, P.V. Prudnikov, M.V. Mamonova, Phys. Usp. 60, 762 (2017).
[3] S.E. Korshunov, Phys. Usp. 49, 225 (2006).
[4] V.L. Berezinskii, Sov. Phys. JETP 32, 493 (1971).
[5] V.L. Berezinskii, Low-Temperature Properties of Two-Dimensional Systems (Fizmatlit, Moscow, 2007) [in Russian].
[6] I.S. Popov, P.V. Prudnikov, A.N. Ignatenko, A.A. Katanin, Phys. Rev. B, 95, 134437 (2017).
[7] C.A.F. Vaz, J.A.C. Bland, G. Lauhoff, Rep. Progr. Phys. 71, 056501 (2008).
[8] C. Kawabat, A.R. Bishop, Solid State Commun. 60, 167 (1986).
[9] H.-J. Elmers, J. Hauschild, G.H. Liu, U. Gradmann, J. Appl. Phys. 79, 4984 (1996).
[10] J. Als-Nielsen, S.T. Bramwell, M.T. Hutchings, G.J. McIntyre, D. Visser, J. Phys.: Condens. Matter. 5, 7871 (1993).
[11] C. Bellitto, P. Filaci, S. Patrizio, Inorg. Chem. 26, 191 (1987).
[12] A. Paduan-Filho, C.C. Becerra, J. Appl. Phys. 91, 8294 (2002).
[13] F.L. Pratt, P.M. Zdetsis, M. Balanda, R. Podgajny, T. Wasiyutyński, B. Sieklucka, J. Phys.: Condens. Matter. 19, 456208 (2007).
[14] R. Ganguly, D. Chaudhuri, P. Raychaudhuri, L. Benfatto, Phys. Rev. B. 91, 054514 (2015).
[15] H.N. Wolfgang, Y.K. Na, G. Roumpos, C. Schneider, M. Kamp, S. Höfling, A. Forchel, Y. Yamamoto, Phys. Rev. B. 90, 205430 (2014).
[16] Y.S. Karimov, Y.N. Novikov, Sov. Phys. – JETP Lett. 19, 159 (1974).
[17] D.J. Bishop, J.D. Reppy, Phys. Rev. Lett. 40, 1727 (1978).
[18] D.J. Bishop, J.D. Reppy, Phys. Rev. B. 22, 5171 (1980).
[19] S. Misra, L. Urban, M. Kim, G. Sambandamurthy, A. Yazdani, Phys. Rev. Lett. 110, 037002 (2013).
[20] M.R. Beasley, J.E. Mooij, T.P. Orlando, Phys. Rev. Lett. 41, 1165 (1979).
[21] A.F. Hebard, A.T. Fiory, Phys. Rev. Lett. 44, 291 (1980).
[22] L.N. Bulaevskii, V.V. Kuzii, A.A. Sobyanin, JETP Lett. 25, 314 (1977).
[23] A.I. Buzdin, L.N. Bulaevskii, S.V. Panyukov, JETP Lett. 35, 147 (1982).
[24] A.I. Buzdin, B. Bujić, M.Yu. Kupriyanov, Zh. Eksp. Teor. Fiz. 101, 231 (1992).
[25] A.P. Solon, H. Chaté, J. Tailleur, Phys. Rev. Lett. 114, 068101 (2015).
[26] L. Berthier, P.C.W. Holdsworth, M. Sellitto, J. Phys. A. 34, 1805 (2001).
[27] S. Abriet, D. Karevski, Euro. Phys. J. B. 36, 1 (2002).
[28] V.V. Prudnikov, P.V. Prudnikov, I.S. Popov, JETP Lett. 101, 596 (2015).
[29] V.V. Prudnikov, P.V. Prudnikov, I.S. Popov, JETP Lett. 126, 369 (2018).
[30] P.V. Prudnikov, I.S. Popov, J. Phys.: Conf. Series 510, 012014 (2014).
[31] I.S. Popov, P.V. Prudnikov, Solid state phenomena 233-234, 8 (2015).
[32] I.S. Popov, P.V. Prudnikov, V.V. Prudnikov, J. Phys.: Conf. Series 681, 012015 (2016).
[33] M. Henkel, M. Pleimling, Non-Equilibrium Phase Transitions, vol. 2, (Springer Netherlands, 2010).
[34] V.V. Prudnikov, P.V. Prudnikov, E.A. Pospelov, P.N. Malyarenko, JETP Lett. 102, 167 (2015).
[35] P.V. Prudnikov, E.A. Pospelov, P.N. Malyarenko, A.N. Vakilov, Progr. Theor. Exp. Phys. 2015 053A0120 (2015).
[36] A.J. Bray, Adv. Phys. 34, 357 (1994).
[37] A.J. Bray, Adv. Phys. 51, 481 (2002).
[38] P. Tabeling, Phys. Rep. 362, 1 (2002).
[39] W.J. Nuttall, D.Y. Noh, B.O. Wells, R.J. Birgeneau, J. Phys.: Condens. Matter. 7, 4337 (1995).
[40] I.F. Lyubskiyutov, A.G. Fedorus, Sov. Phys. JETP. 53, 1317 (1981).
[41] A. Taroni, S.T. Bramwell, P.C.W. Holdsworth, J. Phys.: Condens. Matter. 20, 275233 (2008).
[42] Y. Tu, J. Tonner, Phys. Rev. Lett. 75, 4326 (1995).
[43] A. Cavagna, I. Giardina, T.S. Grigera, A. Jelic, D. Levine, S. Ramaswamy, M. Viale, Phys. Rev. Lett. 114, 218101 (2015).
[44] A.P. Solon, H. Chaté, J. Tailleur, Phys. Rev. Lett. 114, 068101 (2015).
[45] L. Berthier, P.C.W. Holdsworth, M. Sellitto, J. Phys. A. 34, 1805 (2001).
[46] S. Abriet, D. Karevski, Euro. Phys. J. B. 37, 47 (2004).
[47] P.V. Prudnikov, V.V. Prudnikov, I.S. Popov, JETP Lett. 101, 596 (2015).
[48] V.V. Prudnikov, P.V. Prudnikov, I.S. Popov, JETP Lett. 126, 369 (2018).
[49] P.V. Prudnikov, I.S. Popov, J. Phys.: Conf. Series 510, 012014 (2014).
[50] I.S. Popov, P.V. Prudnikov, Solid state phenomena 233-234, 8 (2015).
[51] I.S. Popov, P.V. Prudnikov, V.V. Prudnikov, J. Phys.: Conf. Series 681, 012015 (2016).
[52] M. Henkel, M. Pleimling, Non-Equilibrium Phase Transitions, vol. 2, (Springer Netherlands, 2010).
[53] V.V. Prudnikov, P.V. Prudnikov, E.A. Pospelov, P.N. Malyarenko, JETP Lett. 102, 167 (2015).
[54] V.V. Prudnikov, E.A. Pospelov, P.N. Malyarenko, A.N. Vakilov, Progr. Theor. Exp. Phys. 2015 053A0120 (2015).
[55] P.V. Prudnikov, V.V. Prudnikov, E.A. Pospelov, P.N. Malyarenko, Mat. Science Forum. 845 101 (2016).
[56] P.V. Prudnikov, V.V. Prudnikov, E.A. Pospelov, J. Stat. Mech. 043303 (2016).
[57] P.V. Prudnikov, V.V. Prudnikov, P.N. Malyarenko, JETP 125 1102 (2017).
[58] V.V. Prudnikov, P.V. Prudnikov, E.A. Pospelov, P.N. Malyarenko, JETP Lett. 107, 569 (2018).
[59] A.J. Bray, A.J. Briant, and D.K. Jervis, Phys. Rev. Lett. 84, 1503 (2000).
[60] V.V. Prudnikov, P.V. Prudnikov, S.V. Alekseev, I.S. Popov, Phys. Met. Metalogr. 115, 1186 (2014).
[61] B. Berche, A.I. Farihías-Sánchez, Yu. Holovatch, and R. Paredes V., Eur. Phys. J. B 36, 91 (2003).
[62] M. Winten, H.U. Everts, W. Apel, Phys. Rev. B 52, 13480 (1995).