ON MONODROMIES OF A DEGENERATION OF IRREDUCIBLE SYMPLECTIC KÄHLER MANIFOLDS

YASUNARI NAGAI

Abstract. We study the monodromy operators on the Betti cohomologies associated to a good degeneration of irreducible symplectic manifold and we show that the unipotency of the monodromy operator on the middle cohomology is at least the half of the dimension. This implies that the “mildest” singular fiber of a good degeneration with non-trivial monodromy of irreducible symplectic manifolds is quite different from the generic degeneration of abelian varieties or Calabi-Yau manifolds.

Introduction

For the study of smooth algebraic (or analytic) varieties, it is often important and useful to consider its degenerations. One can see this principle, for example, by remembering the role played by the singular fibers in the theory of elliptic surfaces due to Kodaira.

The Kodaira singular fibers are completely described in terms of the periods and monodromies around the singular fiber. The study of the periods and degenerations of abelian varieties is one of the most direct generalizations of the theory of elliptic surfaces to higher dimensions. The theory of periods is generalized by Griffiths and many other contributors to the general situation using the variation of Hodge structures. Since then, many significant results are proved using this mechanism. In some cases, the Hodge structure on the middle cohomology, i.e. $H^n(X;\mathbb{C})$ for $X$ of dimension $n$, plays an important role. As an example, we can recall the classification of Kulikov models for degeneration of K3 surfaces [Ku, PP] and the proof of global Torelli theorem for K3 surfaces via Kulikov models by Friedman [Fr2].

An irreducible symplectic Kähler manifold is a generalization of a K3 surface to higher dimension. As is well known, the local Torelli theorem holds for irreducible symplectic manifolds on the second cohomology group (not only on the middle cohomology). From this result, one can easily suppose some strong similarities between the irreducible symplectic manifolds and K3 surfaces. There is no reason that prevents us from studying degenerations of irreducible symplectic manifolds.

But if we consider the period and monodromy only on the second cohomology as invariants of the degeneration of irreducible symplectic manifolds, it seems

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that one misses some important information. Since the weight of the Hodge structure on the second cohomology is 2, the index of unipotency of the monodromy is either 0, 1, or 2. The monodromy operator should have some information about the “complexity” of the degenerate fiber. In the case of Kulikov models of K3 surfaces or toroidal degenerations of abelian varieties, the unipotency of the monodromy on the middle cohomology certainly corresponds to the combinatorial complexity of the singular fiber. There are \((n+1)\) patterns of local combinations of the components of the degenerate variety with normal crossings of dimension \(n\), so the unipotency of the monodromy on the second cohomology, \textit{a priori}, may not fully reflect the combinatorial information of the degenerate fiber.

In this article, we examine the relation between the monodromy operators on the cohomologies, in particular the relation between the monodromies on the second cohomology and the middle cohomology.

To consider this problem in a general situation, we assume a semi-stable model which is called a \textit{good degeneration} (see Definition 4.2). In fact, we can construct an example of a good degeneration (Theorem 4.3). Our main result is the following.

\textbf{Theorem} (Theorem 5.3). Let \(\pi : \mathcal{X} \rightarrow \Delta\) be a good degeneration of irreducible symplectic 2n-folds. Let \(T_{2n}\) be the monodromy operator on \(H^{2n}(\mathcal{X}_t, \mathbb{C}) \not\equiv 0\) associated to the family \(\pi\), and \(N_{2n} = \log T_{2n}\). Assume \(N^n_{2n} = 0\), then \(N_{2n} = 0\).

The maximal \(l\) with \(N^l_{2n} \not\equiv 0\) is called the unipotency of \(T_{2n}\). This theorem asserts that the unipotency of the monodromy on the middle cohomology associated to a good degeneration of irreducible symplectic manifold is 0 or not less than \(n\). This is quite different from the situation one can expect in the case of general semi-stable degeneration. For example, we can easily show the following corollary.

\textbf{Corollary} (See Corollary 5.7). Consider a good degeneration of irreducible symplectic 2n-folds with non-trivial monodromy on the middle cohomology, and let \(\Gamma\) be the dual graph of the configuration of the irreducible components of the singular fiber. Then the dimension of the topological realization \(\mathcal{X}_\ast\) is at least \(n\).

This phenomenon can be seen as an aspect of the general principle that the geometry of irreducible symplectic manifold is very restrictive.

\textit{Plan of the article.} This article consists of five sections: In the first section, we review some necessary definitions about symplectic Kähler manifolds, degenerations and monodromy associated to them. In \(\S 2\), we consider an example of degeneration of irreducible symplectic manifold arising from a family of K3 surfaces by the Hilbert scheme construction and compute the monodromies associated to it. In fact, this example motivates this research. The next section is devoted to the study of a family of generalized Kummer varieties and its associated monodromy. In \(\S 4\), we define a notion of a good degeneration of symplectic
Kähler manifold. In the last section, we state a conjecture based on the examples in $\times 2,3$, and get a partial answer to the conjecture for good degenerations of irreducible symplectic manifold.

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1. Definitions and Notations

In this section, we collect some basic definitions and notations which are needed in this article.

Symplectic Kähler manifold. Let us start with a review of some basics of compact symplectic Kähler manifold. A fundamental reference is [Be]. See also [Huy1, Huy2].

Definition 1.1. Let $X$ be a compact Kähler manifold of dimension $2n$. A holomorphic 2-form $\sigma \in H^0(X;\Omega^2_X)$ on $X$ is a (holomorphic) symplectic form if its top exterior power $\sigma ^* \in H^0(X;\Omega^2_X)$ is nowhere vanishing. The pair $(X;\sigma_X)$ is called a holomorphic symplectic Kähler manifold. $X$ is said to be an irreducible symplectic if the following conditions are satisfied.

(i) There exists a holomorphic symplectic form $\sigma$.
(ii) The space of global holomorphic 2-forms $H^0(X;\Omega^2_X)$ is spanned by the symplectic form $\sigma$.
(iii) $X$ is simply connected.

Given an irreducible symplectic manifold $X$, we have a non-degenerate primitive quadratic form $q_X$ on $H^2(X;\mathbb{Z})$, which is called the Beauville-Bogomolov form.

Famous decomposition theorem for compact Kähler manifolds with trivial canonical bundle implies that a compact symplectic Kähler manifold is an étale quotient of a product of a complex torus and irreducible symplectic manifolds. There are few known examples of irreducible symplectic Kähler manifold. Here we describe two families of examples, which are classically known.

Example 1.2 ([Be]). Let $S$ be a K3 surface and $\text{Hilb}^n(S)$ the Hilbert scheme (or Douady space) of 0-dimensional sub-schemes of length $n$ on $S$. Then, $\text{Hilb}^n(S)$ is an irreducible symplectic Kähler manifold of dimension $2n$. For the second Betti cohomology of $\text{Hilb}^n(S)$, we have an isomorphism

\[ H^2(\text{Hilb}^n(S);\mathbb{C}) = H^2(S;\mathbb{C}) \]
where $E$ is an irreducible exceptional divisor of the birational morphism
\[ \text{Hilb}^n(S) \to \text{Sym}^n(S); \]
which is called the Hilbert–Chow morphism ([Be]) and decomposition in (1) is orthogonal with respect to the Beauville–Bogomolov form.

More generally, a compact connected component $M$ of the moduli space of stable sheaves on a K3 surface $S$ is known to be an irreducible symplectic manifold which is deformation equivalent to $\text{Hilb}^n(S)$ of appropriate dimension. An irreducible symplectic manifold $X$ is said to be of Hilbert type if $X$ is deformation equivalent to some $\text{Hilb}^n(S)$.

**Example 1.3 ([Be]).** Let $A$ be a complex torus and consider $\text{Hilb}^{n+1}(A)$ and its Albanese morphism $\alpha : \text{Hilb}^{n+1}(A) \to A$, which is a locally trivial fiber bundle. Take its fiber $\text{Kum}^n(A) = \alpha^{-1}(0)$. Then this is an irreducible symplectic Kähler manifold of dimension $2n$. We can regard $\text{Kum}^n(A)$ as a resolution of $\mathbb{A}^n = \mathbb{S}_{n+1}$. Let $E$ be the exceptional divisor. Then $E$ is irreducible and we have
\[ H^2(\text{Kum}^n(A); \mathbb{C}) \cong H^2(A; \mathbb{C}) \oplus [E] \]
and this is also orthogonal with respect to the Beauville-Bogomolov form.

More recently, O’Grady constructed two sporadic examples of irreducible symplectic manifold [OG1, OG2]. It is noteworthy that these four types of examples are all of the known examples for the moment.

**Degeneration and Monodromy.**

**Definition 1.4.** Let $Y$ be a compact Kähler manifold and $\Delta$ a unit disk in $\mathbb{C}$. A flat proper morphism $\pi : \mathcal{X} \to \Delta$ of a normal complex Kähler space $\mathcal{X}$ is said to be a degeneration or degenerating family of $Y$ if $\pi$ is smooth over $\Delta \setminus \{0\}$ and $\mathcal{X}_t = \pi^{-1}(t)$ is a resolution of $\mathbb{C}^n = \mathbb{S}_{n+1}$. The fiber $X = \mathcal{X}_0 = \pi^{-1}(0)$ is said to be the singular fiber if $\pi$ is not smooth.

Given a degenerating family $\mathcal{X}$, we have the monodromy operators on the cohomologies of $\mathcal{X}_t \setminus \{0\}$.

**Definition 1.5.** Let $\pi : \mathcal{X} \to \Delta$ be a degenerating family. Then parallel displacement on the local system $R^m(\pi_0) \otimes \mathcal{O}_\Delta$ induces a homomorphism
\[ T_m : \pi_1(\Delta \setminus \{0\}) \to \text{GL}(H^m(\mathcal{X}_t; \mathbb{C})); \]
the monodromy representation. We denote also by $T_m$ the image of a generator under the monodromy representation, which is called the monodromy operator.

By the monodromy theorem, $T_m$ is quasi-unipotent, i.e., $(T_m^k I)^N = 0$ for some $k \in \mathbb{N}$. Therefore, changing the base by a cyclic cover $\Delta \to \Delta; t \mapsto t^k$, we can always make the monodromy operator $T_m$ unipotent, i.e., $k = 1$. 
**Definition 1.6.** Let $T$ be an unipotent automorphism of a finite dimensional vector space over a field of characteristic zero. The logarithm of $T$ is defined by

$$N = \log T = (T - I) (T - I)^2 + \ldots + (T - I)^n + :$$

Note that the right hand side is a finite sum because $T$ is unipotent and the logarithm $N$ is a nilpotent endomorphism. Of course, we can reconstruct $T$ from $N$ by the exponential:

$$T = \exp(N) = I + \frac{N}{1!} + \frac{N^2}{2!} + \frac{N^n}{n!} + :$$

We define the index of nilpotency of $N$ by

$$\text{nilp}(N) = \max \{ j \mid N^j \not= 0 \} :$$

We mean by the index of unipotency of a unipotent $T$ the index of nilpotency of $N = \log T$.

If $T_m$ is the unipotent monodromy operator on $H^m(\mathcal{X}_t; \mathbb{C})$ associated to a degeneration $\pi: \mathcal{X} \to \Delta$, then by the theory of variation of Hodge structures, more precisely, by $SL_2$-orbit theorem \cite{Sc} for example, we know that the index of unipotency of $T_m$ is at most $m$, i.e.

(2) $\text{nilp}(N_m) = 0$, $1$, or $2$;

for $N_m = \log T_m$ (see also a lecture note by Griffiths \cite{Topics}, Chapter IV).

2. **Example: the case of Hilbert type**

In this section, we consider degenerations of irreducible symplectic manifolds of Hilbert type, i.e. irreducible symplectic manifolds which is deformation equivalent to $\text{Hilb}^n(S)$ for some K3 surface $S$.

Let us consider an easy example.

**Example 2.1.** Let $p: \mathcal{X} \to \Delta$ be a projective degeneration of K3 surfaces. Consider the Hilbert scheme $\mathcal{H}_n = \text{Hilb}^n(\mathcal{X} = \Delta)$ of 0-dimensional sub-schemes relative to $p$, and take the normalization $\mathcal{X}_n \to \mathcal{X}_n$. Then, the natural morphism $\pi_n: \mathcal{X}_n \to \Delta$ is projective and $\mathcal{X}_n$ is a normal Kähler space after shrinking $\Delta$ if necessary. Therefore, $\pi_n$ is a degeneration of irreducible symplectic manifolds, whose general fiber $(\mathcal{X}_n)$ is isomorphic to $\text{Hilb}^n(\mathcal{X})$.

Consider the monodromy operator on the cohomologies associated to this degeneration. Let us assume that the monodromy operator $T^0$ on the second cohomology group $H^2(\mathcal{X}_t; \mathbb{C})$ is unipotent and let $N^0 = \log T^0$. Then by (2), we have

$$\text{nilp}(N^0) = 0$, $1$, or $2$:}$$
Proposition 2.3. Let \( \mathfrak{g} \) on the general fiber and the class \( \mathcal{E}_i \) is clearly invariant under the action of the monodromy \( T_2 \) on the second cohomology \( H^2(\text{Hilb}^n(\mathcal{S});\mathbb{C}) \), we have

\[
T_2 = T^0 \quad \text{id};
\]

in particular, \( T_2 \) is unipotent. In fact we can say more about the monodromy operator \( T_m \) on \( H^m((\mathcal{S}_n);\mathbb{C}) \):

**Proposition 2.2.** Notation as above. Then for \( m \neq 2n \),

(i) \( T_m \) is unipotent.

(ii) Let \( N_{2k} = \log T_{2k} \). Then, nilp \( \langle N_{2k} \rangle = k \quad \text{nilp} \langle \mathfrak{g} \rangle \) for \( k \neq n \). In particular, nilp \( \langle N_{2k} \rangle \) is nilpotent. In fact we can say more about the monodromy on \( \mathfrak{g} \).

Let \( SH(\text{Hilb}^n(\mathcal{S});\mathbb{C}) \) be the sub-algebra of \( H(\text{Hilb}^n(\mathcal{S});\mathbb{C}) \) generated by \( H^2(\text{Hilb}^n(\mathcal{S});\mathbb{C}) \). For \( SH(\text{Hilb}^n(\mathcal{S});\mathbb{C}) \), we have the following general result.

**Proposition 2.3** (Verbitsky, [Bo]. See also [Huy2] §24). Let \( X \) be an irreducible symplectic manifold of dimension \( 2n \) and let \( SH(\mathcal{X};\mathbb{C}) \) be the sub-algebra of \( H(\mathcal{X};\mathbb{C}) \) generated by \( H^2(\mathcal{X};\mathbb{C}) \). Then

\[
SH(\mathcal{X};\mathbb{C}) = \text{Sym}^k H^2(\mathcal{X};\mathbb{C}) = \mathbb{C} \alpha^{n+1} \quad \text{where} \ q_\mathcal{X}(\alpha) = 0 \]

where \( q_\mathcal{X} \) is the Beauville–Bogomolov form (Definition 1.7).

By this proposition, we have a natural injection

\[
\text{Sym}^k H^2(\text{Hilb}^n(\mathcal{S});\mathbb{C}) \rightarrow H^2(\text{Hilb}^n(\mathcal{S});\mathbb{C})
\]

for \( k \neq n \). The easiest case to prove the proposition is the case where \( n = 2 \), i.e., the case of \( \text{Hilb}^2(\mathcal{S}) \). In this case, one can easily see that \( H(\text{Hilb}^2(\mathcal{S})) = SH(\text{Hilb}^2(\mathcal{S})) \). This implies that \( H^4(\text{Hilb}^2(\mathcal{S})) = \text{Sym}^2 H^2(\text{Hilb}^2(\mathcal{S})) \). Therefore, the proposition is just a consequence of following lemma.

**Lemma 2.4.** Let \( V_1, V_2 \) be finite dimensional vector spaces over a field of characteristic zero, \( T_i \) be a unipotent automorphism on \( V_i \), and \( N_i = \log T_i \). Then \( T_1 T_2 \) is also unipotent and

\[
\log (T_1 T_2) = N_1 I + I N_2
\]

on \( V_1 \) \( V_2 \). Moreover, we have

\[
\text{nilp } \log (T_1 T_2) = \text{nilp } \langle N_1 \rangle + \text{nilp } \langle V_2 \rangle.
\]

In particular, \( \text{Sym}^k T_1 \) on \( \text{Sym}^k V_1 \) is unipotent and

\[
\text{nilp } \langle \text{Sym}^k T_1 \rangle = k \quad \text{nilp } \langle \mathfrak{g} \rangle.
\]

**Proof.** The first assertion is just a property of exponentials and logarithms of the matrices. The second equality follows from

\[
\langle N_1 I + I N_2 \rangle^k = \sum_{i=0}^{k} k_i N_1^i \langle \psi_1 \rangle N_2^i \langle \psi_2 \rangle.
\]
Since $\text{Sym}^k V_1 \, V_1^ k$, we have

$$\text{nilp} \, \log(\text{Sym}^k T_1)) \in \text{nilp} \, \log T_1^k) = k \, \text{nilp} \, \mathcal{Y}$$

On the other hand, there exists $v_1 \in V_1$ such that $N_1^l(v_1) \equiv 0; N_1^{l+1}(v_1) = 0$ for $l = \text{nilp} \, \mathcal{N}_1$. Therefore

$$\log(\text{Sym}^k T_1) = \sum_i \alpha_i \mathcal{N}_i(v_1)^k = M_{k, \mathcal{Y}}$$

for some positive integer $M_{k, \mathcal{Y}}$, which shows the last claim. Q.E.D.

Since $H^{2k}(\text{Hilb}^n(S))$ is not generated by $H^2(\text{Hilb}^n(S))$ for $n > 2$, we need some more argument to prove Proposition 2.2 for $n > 2$. This can be done by applying Göttsche–Soergel formula.

**Theorem 2.5** (Göttsche–Soergel [GS], Theorem 2). Let $S$ be a smooth algebraic surface and fix a natural number $n$. Consider the set $P(n)$ of partitions of $n$, i.e.,

$$P(n) = \{ \alpha : \alpha = (\alpha_1, \alpha_2) \, n \mathcal{O} \} \, 1 + \alpha \, 2 + n + \alpha = n \mathcal{O};$$

and put $\mathcal{O} = \sum \alpha_i$. Then, there is a canonical isomorphism

$$(3) \quad H^{i+2n}(\text{Hilb}^n(S)) = \bigoplus_{\alpha \in P(n)} H^{i+2 \mathcal{O} \alpha}(S^{(\alpha)})$$

where $S^{(\alpha)} = \text{Sym}^n(S)$ for a positive integer $n$ and $S^{(\alpha)} = S^{(\alpha_1)}$ for $\alpha = (\alpha_1; \ldots, \alpha_n)$. We regard $S^{(0)} = \text{Sym}^0(S)$ as a one point set).

**Remark 2.6.** By this theorem, we know that $H^m(X; \mathbb{C}) = 0$ for odd $m$ if $X$ is deformation equivalent to some Hilb$^n(K3)$. Therefore, we have nothing to do with the monodromy operators $T_m$ for odd $m$ in this section.

**Proof of Proposition 2.2.** Apply Theorem 2.5 to the case $S$ is a projective K3 surface. Since $H^1(S) = H^3(S) = 0$, the theorem and Küneth formula implies that the cohomology group $H^{2k}(\text{Hilb}^n(S))$ is a direct sum of tensor products of several symmetric products of $H^2(S)$. Göttsche–Soergel decomposition is invariant under the monodromy operator of the family Hilb$^n(S)$! Because the decomposition is induced by the action of the symmetric group on $S^n$. This proves (i). Since the weights of the components of the Göttsche–Soergel decomposition of $H^{2k}(\text{Hilb}^n(S))$ do not exceed $2k$, (ii) follows from Lemma 2.4. Q.E.D.

Under the assumption that the monodromy operators $T_m$ on $H^m(\text{Hilb}^n(S); \mathbb{C})$ are unipotent, one can generalize (ii) in Proposition 2.2 to the following theorem.

**Theorem 2.7.** Let $\pi : \mathcal{X} \to \Delta$ be a degenerating family of irreducible symplectic manifolds such that $\mathcal{X}_t = \text{Hilb}^n(S)$ for some $t \in \Delta$ and a projective K3 surface $S$. Let $T_m$ be the monodromy operator on $H^m(\mathcal{X}; \mathbb{C})$ associated to this family and assume that $T_{2k}$ is unipotent for $k \leq n$. Then, $\text{nilp} \, \mathcal{N}_{2k} = k \, \text{nilp} \, \mathcal{N}$, where $N_{2k} = \log T_{2k}$. In particular $\text{nilp} \, \mathcal{N}_{2k} \geq k_0 + k \mathcal{O}; 2k \mathcal{O}$. Q.E.D.
In fact, the monodromy action on the cohomology ring of an irreducible symplectic manifolds of Hilbert type is extensively studied by Markman \[\textit{[M1, M2]}\]. He constructed generators of the cohomology ring using the universal family and clarified the action of the group of all possible monodromy operators which need not be unipotent. Markman proved the existence of the following monodromy invariant decomposition which is more or less analogous to the Götsche–Soergel decomposition in the case of Proposition 2.2.

**Theorem 2.8** (Markman, \[\textit{[M2]}, \textit{Corollary 4.6 and Lemma 4.8}\].

(i) Let \(X\) be an irreducible symplectic manifold, \(A_1\) the sub-algebra of \(H^1(X; \mathbb{C})\) generated by \(\bigoplus_{i=0}^l H^i(X; \mathbb{C})\) and \(\{A_i\} = A_1 \setminus H^j(X; \mathbb{C})\). Then, we have a monodromy invariant decomposition

\[
H^j(X; \mathbb{C}) = \bigoplus C_j
\]

for some subspace \(C_j\).

(ii) If \(X = \text{Hilb}^n(S)\), \(C_{2k}\) has a monodromy invariant decomposition

\[
C_{2k} = C_{2k}^0 C_{2k}^{\text{tr}}
\]

such that \(C_{2k}^0\) is a one dimensional character and \(C_{2k}^{\text{tr}}\) is \(H^2(\text{Hilb}^n(S))\) \(\chi^{\text{tr}}\) where \(\chi^{\text{tr}}\) is a one dimensional character with values \(f_1 g\) as representations of monodromy, unless \(C_{2k}^0\) or \(C_{2k}^{\text{tr}}\) does not vanish.

We should note that the part (i) of this theorem is based on the result of Verbitsky and Looijenga–Lunts \[\textit{[V1, V2, LL]}\] about the action of \(\mathfrak{so}(4; b_2 \ 2)\) (where \(b_2\) is the second Betti number) on the cohomology ring arising from the existence of the hyper-Kähler metric.

Theorem 2.7 easily follows from this theorem.

**Proof of Theorem 2.7** We proceed by induction on \(k\). Assume that \(\text{nilp}(\mathcal{N}_{2l}) = l \ \text{nilp}(\mathcal{N})\) for \(l < k\). Then we have \(\text{nilp}(\mathcal{N}_{2k} \mathfrak{a}_{2k} \mathfrak{p}_k) = k \ \text{nilp}(\mathcal{N})\). Moreover, \(\text{Sym}^k H^2(X; \mathbb{C})\) injects to \(\mathfrak{a}_{2k} \mathfrak{p}_k\) (Proposition 2.3). This implies that \(\text{nilp}(\mathcal{N}_{2k} \mathfrak{a}_{2k} \mathfrak{p}_k) = k \ \text{nilp}(\mathcal{N})\) by Lemma 2.4. On the other hand, (ii) of Theorem 2.8 implies that the nilpotency of \(N_{2k} \mathcal{C}_{2k}\) is at most \(\text{nilp}(\mathcal{N}_{2k})\). Since the decomposition of Theorem 2.8 (i) is monodromy invariant, we conclude that \(\text{nilp}(\mathcal{N}_{2k}) = k \ \text{nilp}(\mathcal{N})\).

Q.E.D.

3. **Example: A family of generalized Kummer varieties**

From the results of Proposition 2.2 and Theorem 2.7, it is natural to ask whether there is some restriction on the index of unipotency of the monodromy operator for more general case. The life in general is not as simple as in the case of Hilbert type, since not only the cohomology ring of a general irreducible symplectic Kähler manifold is not generated by the second degree part, but also it can have non-zero odd degree, so some mysterious thing may happen in the
higher degree cohomologies. To catch a glimpse of these general cases, we may use the family of generalized Kummer varieties as a test case.

**Definition 3.1.** Let $p : \mathcal{A} \to \Delta = \Delta_0 = \emptyset$ be a smooth projective family of abelian surfaces with 0-section and consider its relative Hilbert scheme of $(n + 1)$ points $\bar{\pi} : \text{Hilb}^{n+1}(\mathcal{A} = \Delta) \to \Delta$. Then we have the commutative diagram

$$
\begin{array}{ccc}
\text{Hilb}^{n+1}(\mathcal{A} = \Delta) & \xrightarrow{\alpha} & \mathcal{A} \\
\pi \downarrow & & \downarrow p \\
\Delta & & \\
\end{array}
$$

where $\alpha$ is the Albanese morphism over $\Delta$, and let $\text{Kum}^n(\mathcal{A} = \Delta)$ be the inverse image of the 0-section of $p$ by $\alpha$. Then $\pi : \text{Kum}^n(\mathcal{A} = \Delta) \to \Delta$ is a family of generalized Kummer varieties of dimension $2n$ (cf. Example 1.3).

**Remark 3.2.** We should be able to compactify $\pi : \text{Kum}^n(\mathcal{A} = \Delta) \to \Delta$ by allowing some mild singular fiber of $p$ over the origin as in the case of Hilbert schemes in $\mathbb{P}^2$. Or we can obtain some compactification using the embedding of $\pi$ to the projective space over $\Delta$ in an obvious way. But we do not discuss about the singular fiber here because we do not use any geometric information of the singular fiber in the sequel.

Let $p : \mathcal{A} \to \Delta$ be the family of abelian surfaces in Definition 3.1. Let $T_m$ be the associated monodromy operator on $H^m(\mathcal{A}_t)$ and assume that $T_1$ is unipotent. Then one can easily see that all $T_m$'s are unipotent. We denote $N_m = \log T_m$. Of course $\text{nilp}(N_1) \neq 1$ and equality holds if $T_1$ is non-trivial. Assume $\text{nilp}(N_1) = 1$ and let $l = \text{rank}N_1 (= 1; 2)$. Then we know that

$$
\text{nilp}(N_2) = l; \quad \text{nilp}(N_3) = 1; \quad \text{nilp}(N_4) = 0:
$$

(4)

Take the family of generalized Kummer $2n$-folds $\pi : \text{Kum}^n(\mathcal{A} = \Delta) \to \Delta$ and let $T_m$ be the monodromy operator on $H^m(\text{Kum}^n(\mathcal{A}_t))$ and $N_m = \log T_m$. As in $\mathbb{P}^2$, $N_2$ can be expressed by

$$
N_2 = N_2 \quad \text{id}
$$

under $H^2(\text{Kum}^n(\mathcal{A}_t); \mathcal{C}) = H^2(\mathcal{A}; \mathcal{C}) \subset \mathcal{C}$. We calculate $\text{nilp}(N_{2k})$ for $1 < k \leq n$ below.

We prepare a lemma.

**Lemma 3.3.** Let $\pi : \mathcal{X} \to \Delta$ be a degeneration of irreducible symplectic manifolds, and $T_m$ the associated monodromy operator on $H^m(\mathcal{X}_t; \mathcal{C})$. Assume $T_2$ and $T_{2k}$ are unipotent and take $N_m = \log T_m$ ($m = 2; 2k$). Then, $\text{nilp}(N_{2k}) > k \quad \text{nilp}(N_2)$. In particular $N_{2k}^{1k} = 0 \quad N_2^l = 0$.

**Proof.** We again use the result of Verbitsky (Proposition 2.3). We have an injective homomorphism $S^k = \text{Sym}^k H^2(\mathcal{A}_t; \mathcal{C}) \to H^{2k}(\mathcal{A}_t)$ and we have

$$
\text{nilp}(N_{2k}) > \text{nilp}(N_{2k}S^k) = \text{nilp}(\log \text{Sym}^k T_2) = k \quad \text{nilp}(N_2).$$
by Lemma [2,4] Q.E.D.

The main tool of our calculation is again the theorem of Göttsche–Soergel:

**Theorem 3.4** (Göttsche–Soergel [GS], Theorem 7). Let $A$ be an abelian surface and fix a natural number $n$. Then, there is a canonical isomorphism

$$(5) \quad H^{i+2n} (A \ \text{Kum}^n (\mathcal{A})) = \bigoplus_{\alpha \in \mathbb{P}^{n+1}} H^{i+2\alpha \cdot \Delta (\mathcal{A})} \text{gcd} (\alpha) \mathcal{A}$$

where the notations $\mathbb{P}^{n+1}$, $\alpha \cdot \Delta$ and $A (\alpha)$ are the same as in Theorem 2.5 and $\text{gcd} (\alpha) = \gcd (\alpha_k) = 0$.

Let us consider $\mathcal{A} \rightarrow \text{Kum}^n (\mathcal{A}) = \Delta$, which is in fact a degree $(n + 1)^4$ étale cover of $\text{Hilb}^{n+1} (\mathcal{A} = \Delta)$ over $\Delta$, and the associated monodromy operator $\tilde{T}_m$ on $H^m (\mathcal{A}, \text{Kum}^n (\mathcal{A}))$. Since Göttsche–Soergel decomposition (5) is induced by the action of symmetric group $S_n$ on $\mathcal{A} / \Delta$ corresponding to $\text{Hilb}^{n+1} (\mathcal{A} = \Delta)$, the monodromy operator $\tilde{T}_m$ respects the decomposition (5).

**Example 3.5.** Consider the case where $n = 2$, i.e., the monodromies on the cohomologies $H^m (\text{Kum}^2 (\mathcal{A}))$ associated to $\pi : \text{Kum}^2 (\mathcal{A} = \Delta)$! $\Delta$. We shall write $A = \mathcal{A}_i$ below for simplicity. By the Göttsche–Soergel formula (Theorem 3.4), we have

$$H^4 (\mathcal{A}, \text{Kum}^2 (\mathcal{A})) = H^0 (\mathcal{A}) \ 3^4 \ H^2 (\mathcal{A}) \ A \ H^4 (\mathcal{A}^3)$$

and one can easily check

$$H^2 (\mathcal{A}) = H^2 (\mathcal{A})^2 \ H^1 (\mathcal{A}) \ H^1 (\mathcal{A})$$

$$H^4 (\mathcal{A}^3) = H^1 (\mathcal{A}) \ H^3 (\mathcal{A}) \ H^1 (\mathcal{A}) \ \text{Sym}^2 H^2 (\mathcal{A}) \ H^2 (\mathcal{A}) \ \times 2H^1 (\mathcal{A})$$

In particular, $\tilde{T}_m$ is unipotent. Take $\tilde{N}_m = \log \tilde{T}_m$. Using Lemma 2.4 and (4), we know that $\text{nilp} (\mathcal{N}_4) = 2l$, where $l = \text{nilp} (\mathcal{N}_2) > 1$. On the other hand, the Küneth formula

$$H^4 (\mathcal{A}, \text{Kum}^2 (\mathcal{A})) = H^4 (\text{Kum}^2 (\mathcal{A})) \ H^3 (\text{Kum}^2 (\mathcal{A})) \ H^1 (\mathcal{A}) \ H^2 (\text{Kum}^2 (\mathcal{A})) \ H^2 (\mathcal{A}) \ H^4 (\mathcal{A})$$

infers that $T_m$ is also unipotent and

$$\text{nilp} (\mathcal{N}_4) = \max \text{nilp} (\mathcal{N}_4), \text{nilp} (\mathcal{N}_3) + 1 \geq 2l \geq 2l$$

if $l > 0$, $\text{nilp} (\mathcal{N}_1) = 1$. Therefore

$$2l = \text{nilp} (\mathcal{N}_4) > \text{nilp} (\mathcal{N}_4) > 2l$$

where the last inequality is due to Lemma 3.3. Note that this holds also for $l = \text{rank} \mathcal{N}_1 = 0$. In summary, we have

$$(6) \quad \text{nilp} (\mathcal{N}_4) = 2 \ \text{nilp} (\mathcal{Y})$$

Generalizing this method of calculation, we can show the following.
Theorem 3.6. Take $n > 2$ and $2 \leq k \leq n$. Let $T_m$ be the monodromy operator on $H^m(\text{Kum}^n(\mathcal{A}_t); \mathbb{C})$ associated with the family $\pi : \text{Kum}^n(\mathcal{A}_t) \to \Delta$ in Definition 3.1. Then, $T_m$ is also unipotent and

\begin{equation}
\text{nilp} (N_{2k}) = kl; \quad \text{nilp} (N_{2k+1}) \equiv kl + 1;
\end{equation}

for $N_m = \log T_m$. In particular $\text{nilp} (N_{2k}) \approx 0; k; 2k\mathfrak{g}$.

We prove this theorem via several steps. We keep the notation $A = \mathcal{A}_t$.

Lemma 3.7. Let $a$ be a positive integer and $N(\mu; a)$ the logarithm of the induced monodromy operator on $H^m(A^{(\mu)})$. Then,

\begin{align*}
\text{nilp} (N(\mathfrak{g}M; a)) &= \begin{cases} 
(\mathfrak{g}a + M)l & (a < M \leq 2a); \\
0 & (M > 2a)
\end{cases} \\
\text{nilp} (N(\mathfrak{g}M + 1; a)) &= \begin{cases} 
(\mathfrak{g}a + M + 1)l + 1 & (a < M \leq 2a); \\
0 & (M > 2a)
\end{cases}
\end{align*}

Proof. By the Poincaré duality, we have only to consider the case $M \leq a$. Let $\mu = (\mu_1; \ldots; \mu_4)$ be the partition of $m = 2M$ or $2M + 1$,

\begin{align*}
m &= \mu_1 \quad 1 + \mu_2 + 2 + \mu_3 + 3 + \mu_4 + 4
\end{align*}

under $\mathfrak{g}\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4 \leq a$. Then we have

\begin{align*}
H^m(A^{(\mu)}) &= \bigwedge^\mu H^1(A) \quad \text{Sym}^{\mu_2} H^2(A) \quad \bigwedge^\mu H^3(A) \quad \text{Sym}^{\mu_4} H^4(A);
\end{align*}

where

\begin{align*}
H^{(\mu)}(A^{(\mu)}) &= \bigwedge^\mu H^1(A) \quad \text{Sym}^{\mu_2} H^2(A) \quad \bigwedge^\mu H^3(A) \quad \text{Sym}^{\mu_4} H^4(A);
\end{align*}

Let $N(\mu; a)$ be the logarithm of the induced monodromy operator on $H^{(\mu)}(A^{(\mu)})$, then $\text{nilp} (N(\mu; a)) = \text{max} \{\text{nilp} (N(\mu; a))\}$. One can easily see that $\text{nilp} N(\mu; a)$ attains the maximum at

\begin{align*}
\mu &= (0; M; 0; 0) \quad (n = 2M); \\
\mu &= (1; M; 0; 0) \quad (n = 2M + 1);
\end{align*}

and the corresponding maximum values are $Ml$ and $Ml + 1$, respectively. Q.E.D.

Lemma 3.8. Notation as above. $\text{nilp} (\mathfrak{g}N_{2k}) = kl$.

Proof. By the Götsche–Soergel formula,

\begin{align*}
H^{2k}(A^{(\mathcal{A})}) &= \bigwedge^\mu H^{m_k(\mathcal{A})} A^{(\mathcal{A})} \quad (\mathcal{A})^4
\end{align*}
where $m_k(\alpha) = 2 \gamma j \cdot 2 \eta \cdot k + 1$. Let $\tilde{N}_{2k}(\alpha)$ be the logarithm of the induced monodromy on $H^{m_k(\alpha)}(A(\alpha))$ and $\nu(\alpha) = \# \{ j \alpha_i \in 0 g \mid 1 \}$. Then, by the Künneth formula and Lemma [3.7], we have

$$\text{nilp}(\tilde{N}_{2k}(\alpha)) = \max \left( \frac{m_k(\alpha)}{2}, l; \frac{m_k(\alpha)}{2}, l + 2; \frac{m_k(\alpha)}{2}, \nu(\alpha), l + 2 \nu(\alpha) \right).$$

As we have an obvious inequality $\gamma j + \nu(\alpha) \leq n + 1$, which is immediate from $\alpha \leq P \eta + 1$, we have

$$\frac{m_k(\alpha)}{2}, \nu(\alpha), l + 2 \nu(\alpha) = (\gamma j, n + k, \nu(\alpha), 1)$$

even if $l = 1$. Therefore, we get

$$\text{nilp}(\tilde{N}_{2k}) = \max \text{nilp} (\tilde{N}_{2k}(\alpha)) \text{ j} P \eta + 1) \in \text{a} \text{ kl}.$$ 

But $\text{nilp}(\tilde{N}_{2k}(\alpha))$ attains the maximum $kl$ at $\alpha = (\eta + 1; 0; \ldots)$. This completes the proof of the lemma. Q.E.D.

**Proof of Theorem 3.6** By the Künneth formula

$$H^{2k}(A, \text{Kum}^n(A)) = \bigoplus_{i=0}^{M} H^i(A) \cdot H^{2k-i}(\text{Kum}^n(A));$$

we have

$$(8) \text{ nilp}(\tilde{N}_{2k}) = \max \text{nilp} (\tilde{N}_{2k}), \text{nilp}(\tilde{N}_{2k-1}) \text{ nilp}(\tilde{N}_{2k-2}) + l; \text{nilp}(\tilde{N}_{2k-3}) + 1; \text{nilp}(\tilde{N}_{2k-4}) \in \text{a}.$$

In particular, $T_m$ is unipotent and

$$\text{nilp}(\tilde{N}_{2k}) \in \text{a} \text{ kl} \text{ nilp}(\tilde{N}_{2k-1}) \in \text{a} \text{ kl} \text{ \{1}$$

by Lemma [3.8]. On the other hand, we have $\text{nilp}(\tilde{N}_{2k}) > kl$ by Lemma [3.3], so this proves the theorem. Q.E.D.

**Remark 3.9.** We should note that the proof of the equality (7) is limited to the case of the family of generalized Kummer varieties arising from a degeneration of abelian surfaces, unlike the case of Hilbert type (cf. Theorem [2.7]).

4. Good Degeneration of a Compact Symplectic Kähler Manifold

From what we have seen in the previous sections, we can easily pose the following question: *Do the monodromy operators on the cohomologies associated with the degeneration of an irreducible symplectic manifold have some special property?* To study the question in a somewhat general situation, it is certainly one way to consider the relation between the monodromies and the geometry of the singular fiber. Along this direction, we have the powerful theory of the limit mixed Hodge structure in the case of semi-stable degenerations.
Definition 4.1. Let $M$ be a complex manifold. A divisor $D$ on $M$ is a simple normal crossing divisor (SNC in short) if $D = \sum D_i$ is reduced, every irreducible component $D_i$ is smooth, and for any point $p \notin D$, the local equation of $D$ in $M$ is given by $x_0^r = 0$ for some $r$. A degeneration $\pi : \mathcal{X} \to \Delta$ is semi-stable if the total space $\mathcal{X}$ is smooth and $X = \mathcal{X}_0 = \pi^{-1}(0)$ is a SNC divisor as scheme theoretic fiber.

By the semi-stable reduction theorem, a degenerating family is always birational to a semi-stable one after taking some cyclic base change. It is also known that the monodromies $T_\ell$ on $H^m(\mathcal{X}_t; \mathbb{C})$ is unipotent if the degeneration $\pi : \mathcal{X} \to \Delta$ is semi-stable. In this sense, we can consider a semi-stable degeneration as a geometric counterpart of the concept of “unipotent monodromy”.

But one should note that there are many semi-stable models for a given degenerating family since one can operate birational modifications keeping the family semi-stable. To carry out some geometric arguments on the singular fiber, it is desirable to have a kind of “minimality” of the family. As that kind of thing, we propose the following definition of a good degeneration of a compact symplectic Kähler manifold.

Definition 4.2. A good degeneration of compact symplectic Kähler manifold is a degeneration $\pi : \mathcal{X} \to \Delta$ of relative dimension $2n$ satisfying

(i) $\pi$ is semi-stable.
(ii) There exists a relative logarithmic 2-form $\sigma_\pi \in H^0(\mathcal{X}; \Omega^2_{\mathcal{X}/\Delta})$ such that $\wedge^n \sigma_\pi \in H^0(\mathcal{X}; K_{\mathcal{X}/\Delta})$ is nowhere vanishing (see, for example, [St] for the definition of the logarithmic differential forms).

Note that the condition (ii) implies that $K_{\mathcal{X}/\Delta}$ is trivial. In particular the definition above agrees with the definition of good degeneration of K3 surface, so-called Kulikov model, by Kulikov–Persson–Pinkham [Ku, PP] if $n = 1$.

Let us construct an example of a good degeneration of an irreducible symplectic manifold using the degenerating family of the Hilbert schemes on K3 surfaces in $\mathbb{P}^2$.

Theorem 4.3. Let $p : \mathcal{X} \to \Delta$ be a projective type II degeneration of K3 surface, i.e., $p$ is a projective good degeneration of K3 surface with the singular fiber $\mathcal{X}_0 = S_0 \cup S_1 \cup \cdots \cup S_k$ where $S_0$ and $S_k$ are rational surfaces, $S_i (0 < i < k)$ are elliptic ruled surfaces and $S_i$ meets only $S_{i-1}$ in smooth elliptic curves $C_i = S_i \setminus S_{i+1}$ $i = 0; \cdots; k$. Consider the Hilbert scheme $\rho : \mathcal{Y} = \text{Hilb}(\mathcal{X}; \Delta)$ of relative sub-schemes of length 2. Then there exists a projective birational morphism $\mu : \mathcal{X} \to \mathcal{Y}$ such that $\pi = \rho \circ \mu : \mathcal{X} \to \Delta$ is a good degeneration of compact symplectic Kähler manifold.
In the proof of the theorem given below, it is essential the condition that the length of sub-schemes in question is 2. It is natural to ask either the same conclusion holds for the larger length sub-schemes. To answer this question, it is likely that a more intrinsic interpretation of the resolved space $\mathcal{X}$ is needed.

**Proof.** Note that we can assume that $Y$ is a Kähler space (Example 2.1) so that $X$ in the theorem is automatically a Kähler manifold as far as $\mu$ is a projective resolution.

The family $p : \mathcal{S} \to \Delta$ induces the morphism

$$f : \text{Hilb}^2 (\mathcal{S}) \to \text{Sym}^2 (\Delta).$$

We can consider $\text{Hilb}^2 (\mathcal{S})$ as a closed sub-scheme of $\text{Hilb}^2 (\mathcal{S})$. Moreover if we define $d : \Delta \to \text{Sym}^2 (\Delta)$ by $z \mapsto 2z$, then we have the commutative diagram

$$\begin{array}{ccc}
\text{Hilb}^2 (\mathcal{S}) & \xrightarrow{\rho} & \text{Sym}^2 (\Delta) \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{d} & \text{Sym}^2 (\Delta).
\end{array}$$

A singular sub-scheme of length 2 is given as the base point plus tangent direction at that point and any sub-scheme of length 2 is given as a limit of smooth sub-schemes (i.e., sub-schemes consisting of different 2 points). Therefore

$$\text{Hilb}^2 (\mathcal{S}) = \text{Bl}_D (\mathcal{S}) = S_2$$

where $D$ is the diagonal, $\text{Bl}_D$ stands for the blowing up along $D$ and $S_2$ acts as the permutation of components. Moreover $\text{Hilb}^2 (\mathcal{S})$ is the closure of the open subset of $\text{Hilb}^2 (\mathcal{S})$ consisting of points corresponding to the smooth sub-schemes. Thus we have the following commutative diagram

$$\begin{array}{ccc}
\text{Bl}_D (\mathcal{S}) & \xrightarrow{g} & \mathcal{S} \\
\downarrow & \xrightarrow{h} & \Delta \\
\text{Hilb}^2 (\mathcal{S}) & \xrightarrow{\rho} & \Delta
\end{array}$$

Let $W$ be the strict transform on $\text{Bl}_D (\mathcal{S})$ of the inverse image by $h$ of the diagonal of $\Delta$. Then we have the commutative diagram

$$\begin{array}{ccc}
W & \xrightarrow{g_W} & \Delta \\
\downarrow & & \downarrow \\
\mathcal{Y} = \text{Hilb} (\mathcal{S} = \Delta) & \xrightarrow{\rho} & \Delta
\end{array}$$

where the first vertical arrow is the quotient map by the action of $S_2$. 
Now, since $\mathcal{S}$ is a type II degeneration of K3 surfaces, components of $\mathcal{Y}_0 = \rho^{-1}(0)$ consists of $Y_{ij}$ ($0 \leq i \leq j \leq k$) where $Y_{ii} = \text{Hilb}^2(S_i)$ and $Y_{ij}$ is isomorphic to $S_i \times S_j$ if $j > i + 1$. $Y_{i,i+1}$ is a singular variety given by identifying the points in $C_i \times C_i \times S_j \times S_{i+1}$ by the natural action of $S_2$ on $C_i \times C_i$. The configuration of these components is as follows.

\[
\begin{array}{ccc}
Y_{00} & & \\
Y_{01} & Y_{11} & \\
\vdots & \ddots & \\
Y_{0k} & Y_{1k} & \cdots & Y_{kk}
\end{array}
\]

Now we define $\mu : \mathcal{X} \rightarrow \mathcal{Y}$ as the blowing up along $C = \bigcup_{0 \leq i \leq j \leq k} Y_{ij}$ and show that $\mu$ is a small resolution and $\pi = \rho \circ \mu : \mathcal{X} \rightarrow \mathcal{Y}$ is semi-stable.

Noting that $\rho$ is smooth over $\Delta = \Delta \cap 0_3$, we have only to look at the singular fiber. Let $Z$ be a sub-scheme of length 2 on $\mathcal{X}_0$. If $\text{supp}(Z)$ is contained in the smooth locus of $\mathcal{X}_0$, $\rho$ is obviously smooth at the point $[Z]_2 \mathcal{Y}$. Suppose that exactly two components, say $Y_{ij}$ and $Y_{ij+1}$, meet at $[Z]$. Then $Z = q_1 + q_2$ where $q_1 \geq S_i \cap \{k \leq j \leq k\}$ and $q_2 \geq S_j \setminus S_{j+1}$. Locally at $[Z]_1$, $\text{Hilb}^2(\mathcal{X})$ is a product of open neighborhoods of $q_1$ and $q_2$ since $[Z]$ is away from diagonal. Hence, $\rho$ is given by

\[
(z_1 \llbracket \ldots \llbracket z_5) \mapsto (z_1 z_4 z_5)
\]

for some appropriate coordinate. $\mathcal{Y}$ is defined by $z_1 z_4 z_5 = 0$ and $\mathcal{Y}_0$ is given by $z_1 = z_4 z_5 = 0$. Therefore $\mathcal{Y}$ have a coordinate $(z_2 \llbracket \ldots \llbracket z_6)$ at $[Z]$ such that $\rho$ is given by $(z_2 \llbracket \ldots \llbracket z_6) \mapsto z_4 z_5$. This shows that $\mathcal{Y}$ is non-singular and $\rho$ is semi-stable at $[Z]$. Therefore, if $\rho$ is not semi-stable at $[Z]$ the_supp$(Z)$ must be contained in $\text{Sing}(\mathcal{Y}_0)$. In other words, the points where $\rho$ is not semi-stable are contained in the locus of points on $\mathcal{Y}_0$ where at least 3 components of $\mathcal{Y}_0$ intersect.

Now assume $\text{supp}(Z) \cap \text{Sing}(\mathcal{Y}_0)$. Away from the diagonal, $\text{Hilb}^2(\mathcal{X})$ is locally the direct product of open neighborhoods of $\mathcal{X}$ and $\rho$ is given by

\[
(z_1 \llbracket \ldots \llbracket z_6) \mapsto (z_1 z_2 ; z_4 z_5)
\]

with respect to some coordinate at $[Z]$ so that total space of $\rho$ is defined by $z_1 z_2 = z_4 z_5 = 0$ and the central fiber of $\rho$ is defined by $z_1 z_2 = z_4 z_5 = 0$. As the center $C$ of $\mu$ is defined by, say, $z_1 = z_4 = 0$, the defining equation of $\mathcal{X}$ is given by $z_1^0 z_2^0 = z_5^0 = 0$ and $\pi = \rho \circ \mu$ is described by

\[
(z_1^0 \llbracket \ldots \llbracket z_6^0) \mapsto z_1^0 z_2^0 = z_5^0
\]
This shows $\mathcal{X}$ is smooth and $X = \pi^{-1}(\emptyset)$ is normal crossing divisor in this coordinate neighborhood.

Next we consider local description of $[Z]_2 \rho^{-1}(\emptyset)$ with $\text{supp} Z = \mathcal{C} q \rho_2 \text{Sing}(\mathcal{S})$. Take a local coordinate $(x_1; \ldots; x_6) \in (q \rho q) 2 \mathcal{S}$ $\mathcal{X}$. The defining equation of diagonal $D$ is $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = 0$. Put

\[
\begin{align*}
x_1 &= z_1 + z_4; \quad x_3 = z_2 + z_5; \quad x_5 = z_3 + z_6; \\
x_2 &= z_1; \quad z_4; \quad x_4 = z_2; \quad z_5; \quad x_6 = z_3; \quad z_6;
\end{align*}
\]

The map $h$ is given by

\[
(x_1; \ldots; x_6) \mapsto (x_1 + x_2; x_4 + x_3; x_1; x_2; y; x_4):
\]

The defining ideal of $D$ is $(x_2; x_4; x_6)$ and a piece of the blowing up $g$ is locally described by

\[
\begin{align*}
x_1 &= y_1; \quad x_2 = y_2 y_6; \quad x_3 = y_3; \\
x_4 &= y_4 y_6; \quad x_5 = y_5; \quad x_6 = y_6;
\end{align*}
\]

and $h \cdot g$ is given by

\[
(y_1; \ldots; y_6) \mapsto (y_1 + y_2 y_6; y_3 + y_4 y_6; y_1; y_2 y_6; y_3; y_4 y_6):
\]

Restricting on $W$, i.e., imposing the condition

\[
y_2 y_3 + y_1 y_4 = 0;
\]

$h \cdot g$ is given by

\[
(y_1; \ldots; y_6) \mapsto y_1 y_3 + y_2 y_4 y_6^2:
\]

Noting that the action of $S_2$ is $\text{diag}(1; 1; 1; 1; 1; 1)$, $\rho$ is described by

\[
(w_1; \ldots; w_6) \mapsto w_1 w_3 + w_2 w_4 w_6
\]

under the condition $w_2 w_3 + w_1 w_4 = 0$. In this coordinate, the equation of the center $C$ of $\mu$ is, say, $w_1 = w_2 = 0$. Therefore the equation of $\mathcal{X}$ is given by $w_3^0 + w_1^0 w_4^0 = 0$ and $\pi = \rho \quad \mu$ is described by

\[
(w_1^0; \ldots; w_6^0) \mapsto w_1^0 w_2^0 w_3^0 + w_2^0 w_4^0 w_6^0:
\]

The fiber $\pi^{-1}(\emptyset)$ is $w_2^0 w_4^0 (w_6^0 w_1^0) = t$ in the coordinate $(w_1^0; w_2^0; w_3^0; w_4^0; w_5^0; w_6^0)$. These calculations show that $\pi$ is semi-stable also in this coordinate neighborhood.

In summary, we proved that $\mu$ is a small resolution and $\rho$ is everywhere semi-stable. We remark that the configuration of components of $\mathcal{X}_0 = \pi^{-1}(\emptyset)$ is as follows.
It remains to show that there exists a relative logarithmic symplectic form $\sigma_\pi \in H^0(\mathcal{S}; \Omega_\mathcal{S}^2_{\mathcal{D}=\Delta}(\log X))$. The original family $p : \mathcal{S} \to \Delta$ have nowhere vanishing relative logarithmic 2-form $\omega$. We claim that the relative 2-form $\vartheta = \text{pr}_1 \omega + \text{pr}_2 \omega$ on $\mathcal{S}$ induces $\sigma_\pi$. The restriction of $\vartheta$ to the inverse image by $h$ of the diagonal of $\Delta$ is invariant under the action of $S_2$, therefore $(g_{\vartheta})_W$ descends to a relative log 2-form $\varphi$ on $\text{Hilb}^2(\mathcal{S}=\Delta)$. Let $\sigma_\pi = \mu \varphi$. As $\varphi$ is non-degenerate outside the critical points of $\pi$ by the argument same as in [Be], we know that $K_{\mathcal{S}} = 0$. By the theory of limiting Hodge structure induced by a semi-stable degeneration [Si], the direct image sheaf $\pi_\ast \Omega_\mathcal{S}^2_{\mathcal{D}=\Delta}(\log X)$ is locally free. Since $\mu : \mathcal{S} \to \mathcal{Y}$ is a small resolution, $K_{\mathcal{S}}$ is trivial. Therefore we see that $\sigma_\pi$ above defines everywhere non-degenerate section of the invertible sheaf $\pi_\ast \Omega_\mathcal{S}^2_{\mathcal{D}=\Delta}(\log X)$. This shows that $\sigma_\pi$ satisfies the condition of Definition 4.2 (ii). Q.E.D.

Remark 4.4. We may construct similar example from type III degenerations of K3 surfaces. But in these cases, the combinatorics of the components becomes more complicated and the choice of the center of a resolution should be subtle.

We should also remark that our definition of good degeneration may be “too good” in general. In view of the situation of the complexity of the minimal models in higher dimensions, it is too optimistic to expect a good degeneration model for a given degeneration of irreducible symplectic Kähler manifolds. For example, Kulikov–Persson–Pinkham model can be seen as a relatively minimal 3-fold model over the base and their construction of the good model is already quite complicated.

5. MONODROMY OF A GOOD DEGENERATION OF AN IRREDUCIBLE SYMPLECTIC MANIFOLD

Now we consider the behavior of the monodromy operators on the cohomologies associated to a good degeneration of compact irreducible symplectic Kähler manifolds.

Inspired by Theorems 2.7 and 3.6 we propose the following conjecture.
Conjecture 5.1. Let $\pi : \mathcal{X} \to \Delta$ be a degeneration of an irreducible symplectic $2n$-fold, $T_m$ the associated monodromy operator on $H^m(\mathcal{X}_t; \mathbb{C})$. Put $N_m = \log T_m$ and assume $T_{2k}$ is unipotent for $k \leq n$. Then $\text{nilp}(N_{2k}) = k \text{ nilp}(\mathcal{Y})$ for $k \leq n$.

We can also consider a weak version of this conjecture.

Conjecture 5.2. Under the assumption and notation as in Conjecture 5.1, we have $\text{nilp}(N_{2k}) \leq f_0; k; k + 1; 2k; g$. Of course, Conjecture 5.1 implies Conjecture 5.2.

These conjectures could be too naive for higher dimensions. The author suspects that the conjectures may be true at least for lower dimensions, for example $2n = 4$ and 6.

Under the assumption of the existence of a good degeneration model, we can easily prove the following theorem, which is a partial answer to Conjecture 5.2.

Theorem 5.3. Let $\pi : \mathcal{X} \to \Delta$ be a good degeneration of irreducible symplectic $2n$-folds. Let $H^m_t = H^m(\mathcal{X}_t; \mathbb{C})$ for $t \not\equiv 0$, $T_m$ be the monodromy operator on $H^m_t$ associated to the family $\pi$, and $N_m = \log T_m$. Take $k \leq n$ and assume $N_{2k}' = 0$, then $N_{2k} = 0$. In other words, $\text{nilp}(N_{2k}) \leq f_0; k; k + 1; 2k; g$.

First we prepare a basic lemma.

Lemma 5.4. Let $\pi : \mathcal{X} \to \Delta$ be a semi-stable degeneration, $X = \mathcal{X}_0 = \pi^{-1}(0)$ the singular fiber, $H^m = H^m(X; \mathbb{C})$ and $H^m_t = H^m(\mathcal{X}_t; \mathbb{C})$ for $t \not\equiv 0$. Consider the sheaf of torsion free differentials $\hat{\Omega}^m_X = \Omega^m_X = (\text{Torsion})$ and the sheaf of logarithmic differentials $\Omega^m_X(\log) = \Omega^m_X(\log X)$, $\Omega^m_{\mathcal{X}}(\log)$. Then the $F^m$-part

$$F^m\Psi : F^mH^m \to F^mH^m$$

of the natural morphism $\Psi : H^m \to H^m_t$ of the mixed Hodge structures is nothing but a map $H^0(X; \hat{\Omega}^m_X) \to H^0(X; \Omega^m_{\mathcal{X}})$ induced by the natural inclusion $\hat{\Omega}^m_X \to \Omega^m_{\mathcal{X}}$.

Proof. First, we recall the fact that there is a resolution $C_X \to \hat{\Omega}^m_X$ of the constant sheaf and the induced Hodge spectral sequence

$$E_1^{pq} = H^q(X; \hat{\Omega}^p_X) = H^{p+q}(X; \mathbb{C})$$

is $E_1$-degenerate ([Fr1], Proposition 1.5). Moreover the associated filtration is the Hodge filtration of the standard mixed Hodge structure on $H^m(X; \mathbb{C})$. 


Next, let us consider the real blow-up $\hat{\rho} : \mathfrak{F} ! X$ as in [KN] \#4. Let $d$ be the dimension of $X$ and $V_{rд} = \subset (\xi_0; \ldots; \xi_r; \zeta^{d+1}) \cdot \sum_{i=0}^{d+1} j_{x_0} \cdot \sum_{i=0}^{d} = 0$. Define the real blow-up $\rho : \mathfrak{V}_{rд} ! V_{rд}$ by

$$
\mathfrak{V}_{rд} = \subset \times \ldots \times \xi_0; \ldots; \xi_r; \zeta^{d+1} \cdot \sum_{i=0}^{d+1} j_{x_0} \cdot \sum_{i=0}^{d} = 0 \cdot \sum_{i=0}^{d} = 0_g
$$

and the obvious relation $z_j = s_j e^{-\psi_{\theta_j}(0 \leq j \leq r)}$. One can easily check that this local construction is compatible with coordinate change so that we have the global real blow-up $\rho : \mathfrak{F} ! X$ ([KN], p.404-405).

We can regard $\rho$ homotopically as the collapsing map $C_{t : \mathcal{X} ! X}$, which is the restriction of the Clemens’ retraction map $C : \mathcal{X} ! X$ defined by Theorem 6.9 in [C] (reader can find a more readable and short summary in [LTY], \#4). The local description of the collapsing map $C_{t}$ is given as following; first we define a continuous map

$$
\text{Re} C_{t} : \mathfrak{X}_0 = \sum_{i=0}^{d+1} j_{x_0} \cdot \sum_{i=0}^{d} = 0 \cdot \sum_{i=0}^{d} = 0_g
$$

for small enough real number $t \notin 0$, which is a homotopy equivalence (for the construction, see [LTY], \#4, or the original [C], \#6). Then, define

$$
C_t : \mathfrak{X}_0 = \sum_{i=0}^{d+1} j_{x_0} \cdot \sum_{i=0}^{d} = 0 \cdot \sum_{i=0}^{d} = 0_g
$$

by

$$
\begin{align*}
(\text{Re} C_t (s_i^0)_{0} & e^{-\psi_{\theta_i}(0 \leq j \leq r)} ; (\Omega_X)_{r} e^{-\psi_{\theta_i}; w_{r+1} ; d}) \ \\
(\text{Re} C_t (s_i^0)_{0} & e^{-\psi_{\theta_i}(0 \leq j \leq r)} ; (\Omega_X)_{r} e^{-\psi_{\theta_i}; w_{r+1} ; d})
\end{align*}
$$

where $w_j = s_j^0 e^{-\psi_{\theta_j}(0 \leq j \leq r)}$. But this map factors as $\mathcal{X}_t \mathfrak{X}_t ! \mathfrak{X}_t \mathfrak{X}_t ! X$ by

$$
\begin{align*}
\mathfrak{V}_0 & \mathfrak{V}_{rд} \cdot \sum_{i=0}^{d+1} j_{x_0} \cdot \sum_{i=0}^{d} = 0 \cdot \sum_{i=0}^{d} = 0_g
\end{align*}
$$

The map $\nu$ is obviously a homotopy equivalence so that it induces an isomorphism $H^m (\mathfrak{X}_t ; \mathfrak{C}) ! H^m$. Using this factorization, we get a decomposition of $\Psi$ as

$$
H^m \overset{\hat{\rho}_!}{\longrightarrow} H^m (\mathfrak{X}_t ; \mathfrak{C}) \overset{\nu!}{\longrightarrow} H^m
$$

where the second arrow is an isomorphism.

Moreover, we have a quasi-isomorphism $\tilde{\rho} : \mathfrak{E} ! X$ and $\Omega_X (\log)$ ([KN], p.405) and the induced Hodge spectral sequence

$$
E_{1}^{pq} = H^q (X ; \Omega_X^p (\log)) = H^{p+q} (\mathfrak{X}_t ; \mathfrak{C}) = H_{t}^{p+q}
$$

is $E_1$-degenerate ([KN], Lemma 4.1). The associated filtration is the Hodge filtration of the limit Hodge structure on $H^m_t$ (See [St]. A reader can find a readable
survey by Zucker in [Topics], Chapter VII). From the construction of $\check{e}$, we have the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}_X & \xrightarrow{j} & \mathbb{R} \mathcal{P} \mathbb{C}_{\check{X}} \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} \\
\check{\Omega}_X & \xrightarrow{i} & \Omega_X \left(\log\right)
\end{array}
\]

where $i$ and $j$ are the natural morphisms. The lemma follows from the fact that the map $\Psi : H^m \to H^0_t$ is the same thing as the morphism induced by $j$ and $\Psi$ is a morphism of mixed Hodge structures of weight $(0;0)$. Q.E.D.

**Remark 5.5.** We can actually modify the definition of the collapsing map $C_t$ in a way that $C_t$ is a homeomorphism as is explained in [P].

**Proof of the Theorem.** Let $H^m = H^m(X;\mathbb{C})$ be the cohomology of the singular fiber $X = \mathscr{X}_0$ as in the previous lemma. Consider the Clemens–Schmid exact sequence (a convenient reference is Morrison’s lecture note, [Topics], Chapter VI):

(9) \[ F^m \text{Gr}_m^W \to H^m \cong F^m \text{Gr}_m^W H^m \xrightarrow{N^m} F^m \text{Gr}_m^W 2H^m \]

The last term must be zero because $\text{Gr}_m^W 2H^m$ is pure Hodge structure of weight $(m - 2)$.

As $\mathscr{X}_t$ is an irreducible symplectic manifold, we have $h^2(\mathscr{X}_t) = 1$. The condition $N_{2k}^k = 0$ implies $N_2 = 0$ by Lemma [3.3] hence $\text{Gr}_4^W H^2_i = \text{Gr}_4^W H^2_{t_i} = 0$. Therefore $F^2 \text{Gr}_4^W H^2_i$ is isomorphic to $F^2 H^2_i = H^0(\mathcal{C}_X \Omega^2_X \langle \log \rangle)$. By Lemma [5.4] we have $F^2 H^2 = H^0(\mathcal{C}_X \Omega^2_X \langle \log \rangle)$ and $F^2 \Psi$ is identified with the natural morphism $H^0(\mathcal{C}_X \Omega^2_X \langle \log \rangle)$. Since we assumed that $\pi : \mathscr{X} \to \Delta$ is a good degeneration, we have a logarithmic symplectic form $\sigma \in H^0(\mathcal{C}_X \Omega^2_X \langle \log \rangle)$. Since $\Psi$ is surjective, this $\sigma$ lifts to an element of $H^0(\mathcal{C}_X \Omega^2_X)$. Let $X^{p} = \bigcap_{i_0 \neq i} X_{i_0} \setminus \bigcap_{i \neq i_0} X_i$, the $p$-fold intersection of the components of the singular fiber $X = \sum X_i$. Then, the weight spectral sequence

(10) \[ wE_1^{pq} = H^q(X^{p};\mathbb{C}) \quad E_2^{pq} = H^{q+p}(X;\mathbb{C}) \]

is $E_2$-degenerate. Assume $N_{2k}^k \neq 0$. Then $X^{p} \neq \emptyset$ by (10) and Clemens–Schmid exact sequence. $\sigma \in H^0(\mathcal{C}_X \Omega^2_X)$ implies that $\sigma^{n}$ vanishes as a section of the canonical sheaf $\mathcal{O}_X$ at the generic points of the image of $X^{p}$ on $X$. This contradicts to non-degeneracy of the log symplectic form $\sigma$. Q.E.D.

**Remark 5.6.** The proof shows in fact that $N_2 = 0$, $N_{2k} = 0$. Therefore, this theorem is also a partial answer to Conjecture [5.1].

**Corollary 5.7.** Notation as above. For a good degeneration of irreducible symplectic $2n$-folds with non-trivial monodromy on the middle cohomology, we have $X^{p} \neq \emptyset$ for $p \leq n$. In other words, for the dual graph $\Gamma$ of the configuration of the
irreducible components of the singular fiber $X$, the dimension of the topological realization $\Gamma_j$ is at least $n$.

**Proof.** According to the weight spectral sequence (10), we have

$$\text{Gr}_q^W H^{2n} = \frac{\text{Ker} \ H^q(\mathcal{X})}{\text{Im} \ H^q(\mathcal{X})} \bigg|_{q+1} \bigg( H^q(\mathcal{X}) \bigg)$$

By the Clemens–Schmid exact sequence, we also have

$$\text{Gr}_q^W H^{2n} = \text{Gr}_q^W \text{Ker} \ H^{2n}$$

for $q < 2n$. If we have $X^{h^1} = 0$, (11) implies $\text{Gr}_q^W H^{2n} = 0$ for $q \geq n$, therefore

$$\text{Gr}_q^W H^{2n} = \text{Gr}_q^W \text{Ker} \ H^{2n}$$

for $q < n$, i.e., $N_{2n}^q = 0$. By Theorem 5.3, we get $N_{2n} = 0$, which is a contradiction. Q.E.D.

This corollary means in particular that there is no chain degeneration nor cycle degeneration of irreducible symplectic manifold, i.e. no good degeneration such that the dual graph of the singular fiber is as following:

![Diagram of dual graph](image)

It is remarkable to compare this to the fact that we have cycle degenerations for generic degeneration of abelian varieties and one can also generically expect chain degenerations for Calabi-Yau manifolds. We can regard the situation in the example of Theorem 4.3 is the least degenerate case of the degeneration of symplectic manifolds. We should also note that we can expect some special property of the period map on the middle cohomology of irreducible symplectic manifolds.

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**Korea Institute for Advanced Study (KIAS), 207-43 Cheongnyangni 2-dong, Dongdaemun-gu, Seoul 130-722, Korea**

*E-mail address*: nagai@kias.re.kr