Superfield Description of Effective Theories on BPS Domain Walls

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Abstract

We derive the low-energy effective theory on the BPS domain wall in 4D $\mathcal{N} = 1$ global SUSY theories in terms of the 3D superfields. Our derivation makes the preserved SUSY by the wall manifest and the procedure for integrating out the massive modes easier. Our procedure clarifies how the 3D superfields are embedded into the 4D chiral and vector superfields. We also point out a shortcoming of the conventional procedure for deriving the effective theory on the wall.

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1 Introduction

Domain walls are inherent to the field theories with spontaneous breaking of a discrete symmetry. There are extensive researches on domain walls in many areas of physics, such as condensed matter physics, thermal evolution of the universe, chiral theories on the lattice, and so on.

In supersymmetric (SUSY) theories, the domain wall sector breaks not only the translational invariance but also supersymmetry of the original theory. Some domain walls, however, preserve part of the original supersymmetry. They saturate the Bogomol’nyi-Prasad-Sommerfield (BPS) bound \[1\], and are called BPS domain walls\(^1\). BPS saturated states like the BPS domain walls play a crucial role in the quantum field theories because their mass spectrum receives no quantum corrections. For instance, they can be used as powerful tools for the investigation of the vacuum structures at strong coupling regime. In addition, from the phenomenological point of view, the BPS domain walls are important in the brane-world scenario \[3, 4, 5\] because they are stable and can provide a natural realization of the partial SUSY breaking of the minimal five-dimensional (5D) SUSY (eight supercharges) to \(\mathcal{N} = 1\) SUSY (four supercharges), which is relevant to the phenomenology.

For the above reasons, the BPS domain wall is an intriguing subject for the study of the field theory. In particular, BPS domain walls in four-dimensional (4D) \(\mathcal{N} = 1\) theories are thoroughly researched in a number of papers because such theories are tractable and have various types of BPS domain walls with interesting features \[6, 7, 8, 9, 10\]. For example, the authors of Ref.\[6, 7\] discussed a BPS domain wall in a simple Wess-Zumino model in detail and derived the low-energy effective theory (LET). Surveying LET in the BPS wall background is a useful approach to investigate the quantum fluctuation of the BPS domain wall. Since the zero-modes are localized on the wall, LET becomes a theory on the wall. In other words, domain wall backgrounds gives rise to some kind of the dimensional reduction. The authors of Ref.\[6\] referred to it as “dynamical compactification”, in contrast to the naive Kaluza-Klein compactification\[11\]. Thus, studying LET on the wall background is also useful for the purpose of the model-building in the brane-world scenario. Of course, since our world is four-dimensional, we should discuss a domain wall in 5D theories for the realistic model-building. However, 5D SUSY theory is quite restrictive due to \(\mathcal{N} = 2\) SUSY, and difficult to handle. Hence, it is convenient and instructive to study the BPS walls in 4D \(\mathcal{N} = 1\) theories as a toy model.

In this paper, we will discuss LET on the BPS domain wall in 4D \(\mathcal{N} = 1\) theories. Since BPS walls preserve a half of the original SUSY, such LETs become 3D \(\mathcal{N} = 1\) theories. This means that LET on the BPS wall can be described in terms of the 3D superfields.

However, a conventional procedure for deriving LET on the BPS wall, which was discussed intensively in Ref.\[6, 7\], does not respect the preserved SUSY by the wall, and the resulting LET is described by the component fields of the 3D supermultiplets without the auxiliary fields. Although such a conventional procedure can be used in the

\(^1\)The first example of the BPS domain wall was given in Ref.\[2\].
computation of the mass spectrum in LET, a difficulty arises when we try to calculate interaction terms in LET. Namely, such a procedure is inconvenient for integrating out the massive modes in order to derive LET. We will explain this problem in the next section.

The above problem comes from the fact that the conventional procedure does not keep SUSY preserved by the wall to be manifest. So we need an alternative procedure for deriving LET on the BPS wall where the preserved SUSY is manifest. This is the purpose of the present paper.

Recall that the actions of 4D $\mathcal{N} = 1$ theories are expressed by the form of the integration over the 4D $\mathcal{N} = 1$ superspace $(x^m, x_2, \theta_1, \theta_2)$, where $x^m$ ($m = 0, 1, 3$) denote the 3D coordinates on the wall, $x_2$ is the coordinate of the extra dimension, $\theta_1$ and $\theta_2$ are the fermionic coordinates for the broken and unbroken SUSY, respectively. Since LET derived by the conventional procedure has the form of the integration over the 3D space-time, we can say that the conventional procedure corresponds to the execution of the explicit integrations in terms of $x_2, \theta_1$ and $\theta_2$. On the other hand, we want to leave the 3D superspace $(x^m, \theta_2)$ not to be integrated in order to make the unbroken SUSY manifest. Thus, our desirable procedure corresponds to the execution of the explicit integration in terms of only coordinates for the broken symmetries, that is, $x_2$ and $\theta_1$. However, due to the complexity of the dependence of the integrands on the fermionic coordinates for the broken and unbroken SUSYs, such integrations are not easy to be carried out. In this paper, we will perform such integrations systematically and derive LET on the BPS wall that is described by the 3D superfields.

The paper is organized as follows. In the next section, we will review the conventional procedure for the derivation of LET on a BPS wall, and point out its shortcoming. In Section 3, we will find how 3D scalar superfields are embedded into a 4D chiral superfield. Using the result of Section 3, LET on the BPS wall can be derived in the case of the generalized Wess-Zumino model. The detailed derivation is explained in Section 4. We will also discuss the model with gauge supermultiplets in Section 5. Section 6 is devoted to the summary and the discussion. Notations and some useful formulae are listed in the appendices.

2 Conventional derivation of LET on a BPS wall

To illustrate a shortcoming of the conventional procedure for deriving LETs on BPS walls, let us consider a simple Wess-Zumino model. The Lagrangian of the model is\(^2\)

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \bar{\Phi}\Phi + \int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}),$$  \tag{1}$$

where

$$W(\Phi) = \Lambda^2\Phi - \frac{g}{3}\Phi^3, \quad (\Lambda, g > 0)$$  \tag{2}$$

and

$$\Phi(y, \theta) = A(y) + \sqrt{2}\theta\Psi(y) + \theta^2 F(y),$$  \tag{3}$$

\(^2\text{Basically, we will follow the notations of Ref.\cite{12} throughout the paper.}\)
where $y^\mu = x^\mu + i\theta \sigma^\mu \bar{\theta}$.

After eliminating the auxiliary field $F$ by the equation of motion, the Lagrangian becomes

$$L = -\partial^\mu \bar{A} \partial_\mu A - i\bar{\Psi} \sigma^\mu \partial_\mu \Psi + gA \Psi^2 + g\bar{A} \bar{\Psi}^2 - |\Lambda^2 - gA^2|^2.$$  

(4)

This theory has the following classical field configuration as a solution of the equation of motion.

$$A_{\text{cl}}(x_2) = \frac{\Lambda}{\sqrt{g}} \tanh(\sqrt{g} \Lambda x_2),$$
$$\Psi_{\text{cl}}(x_2) = 0.$$  

(5)

This is the BPS domain wall configuration in this theory$^3$.

Next, we will consider the fluctuation fields around the above classical configuration $A_{\text{cl}}$ and $\Psi_{\text{cl}}$.

$$A(x) = A_{\text{cl}}(x_2) + \frac{1}{\sqrt{2}}(a(x) + ib(x)),$$
$$\Psi(x) = \frac{1}{\sqrt{2}}(\psi_1(x) + i\psi_2(x)).$$  

(6)

By substituting them into the equations of motion, and picking up only linear terms for the fluctuation fields, we obtain the linearized equations of motion for the fluctuation fields.

$$\{\partial^m \partial_m - O_2 O_1\} a = 0,$$
$$\{\partial^m \partial_m - O_1 O_2\} b = 0,$$
$$i\gamma^m_{(3)} \partial_m \psi_1 - O_2 \psi_2 = 0,$$
$$i\gamma^m_{(3)} \partial_m \psi_2 - O_1 \psi_1 = 0, \quad \left( \partial_m \equiv \frac{\partial}{\partial x^m} \right)$$  

(7)

where

$$O_1 \equiv -\partial_2 - 2gA_{\text{cl}}(x_2),$$
$$O_2 \equiv \partial_2 - 2gA_{\text{cl}}(x_2).$$  

(9)

Throughout this paper, the 3D Lorentz indices are denoted by $m$ ($m = 0, 1, 3$), while the Greek letters $\mu, \nu, \cdots$ are used as the 4D Lorentz indices. The matrices $\gamma^m_{(3)}$ are the 3D $\gamma$-matrices. (See Appendix A.)

$^3$From Eqs.(7) and (8), we can find the mode equations to be

$$O_1 c_n(x_2) = m_n d_n(x_2),$$
$$O_2 d_n(x_2) = m_n c_n(x_2).$$  

(10)

$^3$In this paper, we will choose the $x_2$-direction to be perpendicular to the wall.
where \( c_{(n)}(x_2) \) and \( d_{(n)}(x_2) \) are the eigenfunctions for the eigenvalues \( m_{(n)} \), and are called the mode functions.

By using these mode functions, we can expand the fluctuation fields as follows.

\[
A(x) = A_{(3)}(x_2) + \frac{1}{\sqrt{2}} \left\{ \sum_{n=0}^{\infty} c_{(n)}(x_2)a_{(n)}(x^m) + i \sum_{n=1}^{\infty} d_{(n)}(x_2)b_{(n)}(x^m) \right\},
\]

\[
\Psi^\alpha(x) = \frac{1}{\sqrt{2}} \left\{ \sum_{n=0}^{\infty} c_{(n)}(x_2)\psi_{1(n)}^\alpha(x^m) + i \sum_{n=1}^{\infty} d_{(n)}(x_2)\psi_{2(n)}^\alpha(x^m) \right\}.
\]  
(11)

Note that the operator \( \mathcal{O}_1 \) has a zero-mode \( c_{(0)}(x_2) \) while \( \mathcal{O}_2 \) does not.

Here, \( a_{(n)}(x^m) \), \( b_{(n)}(x^m) \) and \( \psi_{1(n)}^\alpha(x^m) \), \( \psi_{2(n)}^\alpha(x^m) \) become 3D real scalar and Majorana spinor fields in the resulting 3D effective theory with a common mass eigenvalue \( m_{(n)} \), respectively. Since \( a_{(n)} \) and \( \psi_{1(n)}^\alpha \), or \( b_{(n)} \) and \( \psi_{2(n)}^\alpha \) have a common mode function \( c_{(n)}(x_2) \), or \( d_{(n)}(x_2) \), they are supposed to form supermultiplets for 3D \( \mathcal{N} = 1 \) SUSY preserved by the wall.

We can obtain the 3D description of the original theory by substituting Eq.(11) into the Lagrangian Eq.(4) and performing the \( x_2 \)-integration.

To obtain LET for the zero-modes \( a_{(0)} \) and \( \psi_{(0)} \), the authors of Ref.[6] simply ignored all massive modes. As we will see below, however, this leads to a wrong result. The resulting 3D effective Lagrangian \( \mathcal{L}^{(3)} \) is

\[
\mathcal{L}^{(3)} = -\frac{1}{2} \partial^\alpha a_{(0)} \partial_\alpha a_{(0)} + \frac{i}{2} \psi_{(0)} \gamma^\alpha \partial_\alpha \psi_{(0)} - \frac{\lambda^2}{8} a_{(0)}^4,
\]  
(12)

where

\[
\lambda^2 = 2g^2 \int_{-\infty}^{\infty} dx_2 c_{(0)}^4(x_2) = \frac{9}{16} g^{5/2} \Lambda.
\]  
(13)

Eq.(12) is not supersymmetric! This wrong result stems from the fact that we have eliminated the auxiliary field \( F \) before dropping the massive modes.

For the purpose of understanding the situation, let us consider the following three-dimensional model.

\[
\mathcal{L}^{(3)} = \int d^2 \theta \left\{ \frac{1}{2} \mathcal{D}^\alpha \varphi_1 \mathcal{D}_\alpha \varphi_1 + \frac{1}{2} \mathcal{D}^\alpha \varphi_2 \mathcal{D}_\alpha \varphi_2 + m_2 \varphi_2^2 + g \varphi_1^2 \varphi_2 \right\},
\]  
(14)

where

\[
\varphi_i(x^m, \theta) = a_i(x^m) + \theta \psi_i(x^m) + \frac{1}{2} \theta^2 f_i(x^m) \quad (i = 1, 2)
\]  
(15)

are 3D scalar superfields, \( \theta \) is the 3D Majorana spinor coordinate, and \( \mathcal{D}_\alpha \) denotes the covariant derivative for 3D \( \mathcal{N} = 1 \) SUSY, which is defined by Eq.(170) in Appendix B.

In terms of the component fields, Eq.(14) is rewritten as follows.

\[
\mathcal{L}^{(3)} = -\frac{1}{2} \partial^\alpha a_1 \partial_\alpha a_1 - \frac{1}{2} \partial^\alpha a_2 \partial_\alpha a_2 + \frac{i}{2} \psi_1 \gamma_\alpha \partial_\alpha \psi_1 + \frac{i}{2} \psi_2 \gamma_\alpha \partial_\alpha \psi_2
\]

\[
+ \frac{1}{2} f_1^2 + \frac{1}{2} f_2^2 + m_2 \left( a_2 f_2 - \frac{1}{2} \psi_2^2 \right)
\]

\[
+ g \left\{ a_1 f_1 a_2 + \frac{1}{2} a_2^2 f_2 - \frac{1}{2} (2a_1 \psi_1 \psi_2 + a_2 \psi_1^2) \right\}.
\]  
(16)
From the equations of motion for the auxiliary fields,
\begin{align*}
f_1 &= -ga_1a_2, \\
f_2 &= -m_2a_2 - \frac{g}{2}a_1^2.
\end{align*}
Thus, by eliminating the auxiliary fields, we can obtain
\begin{align*}
\mathcal{L}^{(3)} &= -\frac{1}{2}\partial^m a_1 \partial_m a_1 - \frac{1}{2}\partial^m a_2 \partial_m a_2 + \frac{i}{2}\psi_1 \gamma^m(3) \partial_m \psi_1 + \frac{i}{2}\psi_2 \gamma^m(3) \partial_m \psi_2 \\
&\quad - \frac{1}{2}m_2^2 a_2 - \frac{1}{2}m_2^2 \psi_2^2 - ga_1 \psi_1 \psi_2 - \frac{g}{2}a_2 \psi_1^2 - \frac{1}{2}m_2 ga_1 a_2 - \frac{g^2}{2}a_1^2 a_2^2 - \frac{g^2}{8}a_1^4. \quad (17)
\end{align*}

Now we will derive LET whose cut-off scale is much smaller than the mass scale \( m_2 \). If we simply ignore the terms that involve massive modes \( a_2 \) and \( \psi_2 \), the effective Lagrangian becomes
\begin{align*}
\mathcal{L}_{\text{eff}}^{(3)} &= -\frac{1}{2}\partial^m a_1 \partial_m a_1 + \frac{i}{2}\psi_1 \gamma^m(3) \partial_m \psi_1 - \frac{g^2}{8}a_1^4. \quad (19)
\end{align*}
Evidently, this is not supersymmetric. Of course, this is the consequence of the inadequate procedure for integrating out the massive modes. The adequate way of integrating out is discussed in Ref.[13]. To integrate out the heavy modes, they expressed such modes as functions of the light modes by using the equations of motion for the heavy modes. Following their procedure, we can obtain the Lagrangian
\begin{align*}
\mathcal{L}_{\text{eff}}^{(3)} &= -\frac{1}{2}\partial^m a_1 \partial_m a_1 + \frac{i}{2}\psi_1 \gamma^m(3) \partial_m \psi_1 + \mathcal{O} \left( \frac{1}{m_2} \right). \quad (20)
\end{align*}
Then, there exists no quartic coupling of \( a_1 \) at the leading order of the \( 1/m_2 \)-expansion, and LET certainly becomes supersymmetric.

More explicitly, we can also see that the quartic coupling of \( a_1 \) vanishes at the leading order in the following way. Since the decoupling of \( a_2 \) and \( \psi_2 \) corresponds to taking the limit of \( m_2 \rightarrow \infty \), the cubic coupling of \( a_1^2 a_2 \) in Eq.(18) cannot be neglected. Thus a contribution from the tree-diagram in Fig.1 must be taken into account. Including this contribution, the tree-level four-point amplitude \( \mathcal{A}_{a_1}^{(4)} \) turn out to be zero.
\begin{align*}
\mathcal{A}_{a_1}^{(4)} &= -\frac{g^2}{8} \times 4! + \left( -\frac{m_2 g}{2} \right)^2 \frac{1}{m_2^2} \times 4C_2 \times 2 = 0. \quad (21)
\end{align*}
Here the first term corresponds to the contribution from the contact term in the Lagrangian and the second term is that from the diagram in Fig.1. The factor \( \frac{1}{m_2^2} \) corresponds to the propagator of the massive scalar field \( a_2 \), and \( 4! \) and \( 4C_2 \times 2 \) are the statistical factors. Therefore, no quartic coupling appears in LET.

The cause of the wrong result Eq.(12) is thought to be similar to that of the above illustrative model Eq.(14). Namely, we cannot simply ignore the massive modes to integrate out them if we have eliminated the auxiliary fields.

On the other hand, if we keep the superfield description during integrating out the massive modes, the situation changes. The authors of Ref.[14] demonstrated such a
procedure in four dimensions. They used the equations of motion for the heavy superfields and expressed them as functions of the light ones (and the spurion superfields) \(^4\). Thus the resulting LET is described by the superfields. They showed that the superfield formalism greatly simplifies the calculations. For instance, in our illustrative case, we can obtain the effective Lagrangian by their procedure,

\[
\mathcal{L}^{(3)} = \int \! d^2 \theta \left( \frac{1}{2} \partial^\alpha \varphi_1 \partial_\alpha \varphi_1 \right) + \mathcal{O} \left( \frac{1}{m_2^2} \right). \tag{22}
\]

This certainly reproduces the previous result Eq.(20). At the leading order, this has the form obtained by simply dropping the massive superfield \(\varphi_2\) from the original Lagrangian Eq.(14). Namely, we can drop the massive modes to integrate out if the original theory is described by the superfields.

Of course, we can obtain the correct LET Eq.(20) from the on-shell expression Eq.(18) if we follow the integrating-out procedure in Ref.[13] in our illustrative model. However, in our original case, such a procedure becomes terribly complicated task due to the infinite Kaluza-Klein modes. Therefore, in order to obtain the correct LET on the BPS wall, we need the 3D superfield description of the original theory, which corresponds to the expression Eq.(14) in the above example. We will provide such expressions in the rest of the paper.

### 3 Embedding 3D superfields into 4D superfield

In this section, we will find the way of embedding 3D superfields into a 4D chiral superfield.

The 4D \( \mathcal{N} = 1 \) SUSY algebra is

\[
\{ Q_\alpha, \bar{Q}_\beta \} = 2 \sigma^\mu_{\alpha \beta} P_\mu, \\
\{ Q_\alpha, Q_\beta \} = \{ \bar{Q}_\alpha, \bar{Q}_\beta \} = 0. \tag{23}
\]

\(^4\)Their purpose was the analysis of the soft SUSY breaking parameters appearing in LET of the Grand Unified Theory. So they performed the calculation including the soft SUSY breaking terms, in contrast to our exact SUSY case.
The SUSY transformation $\delta \xi$ of a chiral supermultiplet $(A, \Psi^\alpha, F)$ is defined by

\[ \delta \xi A = \sqrt{2} \xi \Psi, \]
\[ \delta \xi \Psi^\alpha = i\sqrt{2}(\sigma^\mu \bar{\xi})_\alpha \partial_\mu A + \sqrt{2} \xi_\alpha F, \]
\[ \delta \xi F = i\sqrt{2} \bar{\xi} \bar{\sigma}^\mu \partial_\mu \Psi. \]  

(24)

Define actions of the generators $P_\mu, Q_\alpha$ and $\bar{Q}^{\dot{\alpha}}$ on the fields $\phi = A, \Psi^\alpha, F$ as

\[ P_\mu \times \phi = -i \partial_\mu \phi, \]
\[ (\xi Q + \bar{\xi} \bar{Q}) \times \phi = \delta \xi \phi. \]  

(25)

Then, we can check that the SUSY transformation Eq.(24) is certainly a representation of the SUSY algebra Eq.(23).

Notice that a chiral superfield $\Phi$ can be written as

\[ \Phi(x, \theta, \bar{\theta}) = e^{\delta \theta} \times A(x) = \left( 1 + \delta \theta + \frac{1}{2} \delta^2 \theta \right) \times A(x). \]  

(26)

Then, we can express $\Phi$ as

\[ \Phi(x, \theta, \bar{\theta}) = \Omega \times A(0), \]

(27)

where

\[ \Omega \equiv e^{ix_\mu P_\mu + \theta Q + \bar{\theta} \bar{Q}}. \]  

(28)

Upon the chiral superfield $\Phi$, the generators $P_\mu, Q_\alpha$ and $\bar{Q}^{\dot{\alpha}}$ can be represented by the following differential operators.

\[ \hat{P}_\mu = -i \partial_\mu, \]
\[ \hat{Q}_\alpha = \partial_\alpha - i(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, \]
\[ \hat{\bar{Q}}^{\dot{\alpha}} = \partial^{\dot{\alpha}} - i(\bar{\sigma}^\mu \theta)^{\dot{\alpha}} \partial_\mu. \]  

(29)

where

\[ \partial_\alpha \equiv \frac{\partial}{\partial \theta^\alpha}, \quad \partial^{\dot{\alpha}} \equiv \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}. \]  

(30)

are left-derivatives.

Here, we will rewrite Eq.(24) to the form which is convenient for the following discussion. First, we redefine spinors as $\eta^\alpha \rightarrow \eta^\alpha$, $\eta_\alpha \rightarrow -i \eta_\alpha$, so that $\eta_\alpha = (\sigma^2)_{\alpha\beta} \eta^\beta$. Next, we express $\sigma^\mu$ and $\bar{\sigma}^\mu$ in terms of the 3D $\gamma$-matrices $\gamma^{\mu}_m$ through Eq.(147) in Appendix A. Furthermore, we decompose the transformation parameter $\xi$ as follows.

\[ \xi^\alpha = \frac{e^{i\theta/2}}{\sqrt{2}}(\xi^\alpha_1 + i \xi^\alpha_2), \]  

(31)

7
where $\xi_i^\alpha (i = 1, 2)$ are 3D Majorana spinors $((\xi_i^\alpha)^* = \xi_i^\alpha)$, and $\delta$ is a phase determined by the wall configuration. (See Eq.(69).) As a result, Eq.(24) is rewritten as

$$\delta \xi A = -ie^{i\delta/2} \xi_1 \Psi + e^{i\delta/2} \xi_2 \Psi,$$
$$\delta \xi \Psi_\alpha = -e^{i\delta/2} \left\{ (\gamma_3^m)_{\alpha \beta} \partial_m A - \xi_{1\alpha} \partial_2 A + e^{i\delta} \xi_{1\alpha} F \right\}$$
$$-ie^{i\delta/2} \left\{ (\gamma_3^m)_{\alpha \beta} \partial_m A - \xi_{2\alpha} \partial_2 A - e^{i\delta} \xi_{2\alpha} F \right\};$$
$$\delta \xi F = -ie^{i\delta/2} \left\{ \xi_1 \gamma_{(3)}^m \partial_m \Psi + \xi_1 \partial_2 \Psi \right\} - e^{-i\delta/2} \left\{ \xi_2 \gamma_{(3)}^m \partial_m \Psi + \xi_2 \partial_2 \Psi \right\}. \quad (32)$$

Corresponding to the decomposition Eq.(31), we also decompose the 4D supercharges as follows.

$$Q_\alpha = e^{i\delta/2} (Q_{1\alpha} - iQ_{2\alpha}), \quad Q_\bar{a} = -(Q_\alpha)^* = -e^{i\delta/2} (Q_{1\alpha} + iQ_{2\alpha}), \quad (33)$$

where $(Q_{ia})^* = Q_{ia}$ and $(Q_i^a)^* = -Q_i^a$. Then it follows that

$$\theta Q + \bar{\theta} \bar{Q} = \theta_1 Q_1 + \theta_2 Q_2. \quad (34)$$

By the above definition of $Q_1$ and $Q_2$, $Q_1$ becomes the broken supercharge and $Q_2$ is the unbroken supercharge by the wall.

Under the decomposition Eq.(33), the SUSY algebra Eq.(23) becomes

$$\{Q_{1\alpha}, Q_{1\beta}\} = \{Q_{2\alpha}, Q_{2\beta}\} = 2(\gamma_3^m \sigma_2)_{\alpha \beta} P_m,$$
$$\{Q_{1\alpha}, Q_{2\beta}\} = -\{Q_{2\alpha}, Q_{1\beta}\} = 2i(\sigma_2)_{\alpha \beta} P_2. \quad (35)$$

This can be interpreted as the (central extended) 3D $\mathcal{N} = 2$ SUSY algebra if we identify $P_2$ with the central charge.

Hence, $\Omega$ defined by Eq.(28) can be rewritten as

$$\Omega = e^{ix_2 P_2 + i(x_1 - \theta_2) Q_2} e^{\theta_1 Q_1}. \quad (36)$$

Now, define a group element,

$$\tilde{\Omega} \equiv e^{ix_2 P_2 + i(x_1 - \theta_2) Q_2} e^{\theta_1 Q_1}, \quad (37)$$

then we can show that

$$Q_{2a} \tilde{\Omega} = \left\{ \partial_{2a} + i(\gamma_3^m \theta_2)_a \partial_m \right\} \tilde{\Omega},$$
$$Q_{1\alpha} \tilde{\Omega} = \left\{ \partial_{1\alpha} + i(\gamma_3^m \theta_1)_a \partial_m - 2\theta_{2a} \partial_2 \right\} \tilde{\Omega}, \quad (38)$$

where

$$\partial_{1\alpha} \equiv \frac{\partial}{\partial \theta_1^\alpha}, \quad \partial_{2\alpha} \equiv \frac{\partial}{\partial \theta_2^\alpha}; \quad (39)$$

are left-derivatives.
Therefore, if we define
\[
\tilde{\Phi}(x^m, x_2, \theta_1, \theta_2) \equiv \Phi(x^m, x_2 + \theta_1 \theta_2, \theta_1, \theta_2) = \tilde{\Omega} \times A(0),
\]
(40)
the representation of each generator on \(\tilde{\Phi}\) is
\[
\begin{align*}
\hat{P}_m & = -i \partial_m, \\
\hat{P}_2 & = -i \partial_2, \\
\hat{Q}_{1a} & = \partial_{1a} + i (\gamma_3^m \theta_1)_a \partial_m - 2 \theta_{2a} \partial_2, \\
\hat{Q}_{2a} & = \partial_{2a} + i (\gamma_3^m \theta_2)_a \partial_m.
\end{align*}
\]
(41)
Namely, the unbroken SUSY \(Q_2\) is represented by the usual form of 3D \(\mathcal{N} = 1\) superspace on \(\tilde{\Phi}\). (See Eq.(165) in Appendix B.)

Then, we will decompose \(\tilde{\Phi}\) into 3D superfields. From Eq.(32),
\[
Q_{1a} \times A = -ie^{i \delta/2} \Psi_a = -i Q_{2a} \times A.
\]
(42)
Using this relation, we can convert \(Q_1\) acting on \(A\) into \(Q_2\) and \(P_2\).
\[
e^{\theta_1 \hat{Q}_1} \times A = e^{-i \theta_1 Q_2 - \theta_2^2 P_2} \times A.
\]
(43)
Thus,
\[
\tilde{\Phi} = \tilde{\Omega} \times A(0) = e^{ix^m P_m + ix_2 P_2 + \theta_2 Q_2} e^{-i \theta_1 Q_2 - \theta_2^2 P_2} \times A(0)
\]
\[
= e^{-i \theta_1 D_2 + i \theta_2 \partial_2} e^{ix^m P_m + ix_2 P_2 + \theta_2 Q_2} \times A(0).
\]
(44)
Here we have used the formula Eq.(175) in Appendix B.

Therefore, if we introduce a quantity
\[
\varphi(x^m, x_2, \theta_2) \equiv e^{ix^m P_m + ix_2 P_2 + \theta_2 Q_2} \times A(0).
\]
(45)
the following relation can be obtained.
\[
\Phi(x^m, x_2 + \theta_1 \theta_2, \theta_1, \theta_2) = e^{-i \theta_1 D_2 + i \theta_2 \partial_2} \varphi(x^m, x_2, \theta_2).
\]
(46)
Note that \(\varphi\) behaves like a 3D scalar superfield under \(Q_2\)-SUSY, though the component fields are still four-dimensional fields.

In fact, if we expand \(\varphi\) in terms of \(\theta_2\) as
\[
\varphi = a + \theta_2 \psi + \frac{1}{2} \theta_2^2 f,
\]
(47)
the transformation of the component fields is read off as
\[
\begin{align*}
\delta_{\xi_2} a & = \xi_2 \psi, \\
\delta_{\xi_2} \psi & = -i (\gamma_3^m \xi_2)_a \partial_m a + \xi_2 \partial_2 f, \\
\delta_{\xi_2} f & = -i \xi_2 \gamma_3^m \partial_m \psi.
\end{align*}
\]
(48)
By noticing $a = A$ and comparing Eq.(32) and Eq.(48), the relations of the component fields of $\varphi$ to the original fields are\(^5\)

\[
\begin{align*}
a &= A, \\
\psi^\alpha &= e^{i\delta/2}\Psi^\alpha, \\
f &= i(\partial_2 A + e^{i\delta} F).
\end{align*}
\]

The result of this section is Eq.(46). If $\varphi$ is mode-expanded, Eq.(46) will provide the relation between the 4D chiral superfield and the 3D scalar superfield. In order to carry out the mode-expansion, we need the mode-equation for $\varphi$. We will derive it in the next section.

### 4 Derivation of 3D effective theory

In this section, we will expand the 4D chiral superfield in terms of 3D superfields, and derive 3D effective theory, which is manifestly supersymmetric, by executing the $x_2$- and $\theta_1$-integrations.

Here we will consider the following generalized Wess-Zumino model as a four-dimensional bulk theory.

\[
L = \int d^2\theta d^2\bar{\theta} \, K(\Phi, \bar{\Phi}) + \int d^2\theta \, W(\Phi) + \int d^2\bar{\theta} \, \bar{W}(\bar{\Phi})
\]

\[
= K_{ij} \left\{ F^i \bar{F}^j - \partial^\mu A^i \partial_\mu A^j - i \bar{\Psi}^j \bar{\sigma}^\mu D_\mu \Psi^i \right\}
\]

\[
- \frac{1}{2} \bar{\Psi}^i \Psi^j K_{ik} \Gamma^l_{ij} \bar{F}^k - \frac{1}{2} \bar{\Psi}^i \Psi^j K_{ik} \Gamma^l_{ij} F^k + \frac{1}{4} K_{ijkl} \Psi^i \Psi^j \bar{\Psi}^k \bar{\Psi}^l
\]

\[
+ F^i W_i - \frac{1}{2} W_{ij} \bar{\Psi}^i \Psi^j + \bar{F}^i \bar{W}_i - \frac{1}{2} \bar{W}_{ij} \bar{\Psi}^i \bar{\Psi}^j.
\]

(50)

Lower indices denote derivatives in terms of corresponding chiral or anti-chiral superfields. For instance,

\[
K_{ij} \equiv \frac{\partial^2 K}{\partial \Phi^i \partial \Phi^j}.
\]

(51)

$\Gamma^l_{ij}$ and $\Gamma^l_{ij}$ are the connections on the Kähler manifold and defined by

\[
\Gamma^l_{ij} \equiv K^{kl} K_{ijk}, \quad \Gamma^l_{ij} \equiv K^{lk} K_{ijk}.
\]

(52)

where $K^{kl}$ is the inverse matrix of the Kähler metric $K_{ik}$. The definitions of integral measures $d\theta$ and $d\bar{\theta}$ are listed in Appendix A.

\(^5\)Note the redefinition of spinors mentioned above Eq.(31).
4.1 Equation of motion

The equation of motion of the above theory can be expressed by the superfields as follows.

$$-\frac{1}{4} \bar{D}^2 K_i + W_i = 0. \quad (53)$$

We will express this equation of motion in terms of $\varphi(x^m, x_2, \theta_2)$ defined by Eq.(45). For the second term in L.H.S. of Eq.(53), such rewriting can easily be done. Note that $W_i(\Phi)$ is a chiral superfield. Then, by repeating the procedure in the previous section with the replacement of $A$ with $W_i(A)$, we can obtain

$$W_i(\Phi) = e^{-\theta_1 \theta_2 \partial_2} e^{-i\theta_1 D_2 + i\theta_2^2 \partial_2} W_i(\varphi). \quad (54)$$

On the other hand, rewriting the first term in L.H.S. of Eq.(53) is somewhat complicated. Using the formula Eq.(174) in Appendix B,

$$\bar{D}^2 \Phi = D^2 (\Omega \times \bar{A}) = \Omega \times (Q^2 \times \bar{A}). \quad (55)$$

Since

$$\bar{Q}^2 = e^{i\delta} \left\{ Q_2 Q_1 + 2iP_2 - \frac{i}{2} (Q_1^2 - Q_2^2) \right\} \quad (56)$$

from Eq.(33) and Eq.(35), we can calculate $\bar{Q}^2 \times \bar{A}$ as

$$\bar{Q}^2 \times \bar{A} = 2ie^{i\delta}(Q_2^2 + 2P_2) \times \bar{A}. \quad (57)$$

Here we have used the complex conjugate of Eq.(43).

Since $\bar{D}^2 \Phi$ is a chiral superfield, it can be written in the similar form of Eq.(46). In this case, the quantity corresponding to $\varphi$ is

$$\chi \equiv e^{ix^mP_m + ix_2P_2 + \theta_2 Q_2} \times (Q^2 \times \bar{A}(0)) = 2ie^{i\delta} (D_2^2 - 2i\partial_2) \bar{\varphi}. \quad (58)$$

In the second equation, we have used Eq.(57) and the formula Eq.(175).

Then, Eq.(55) can be written as

$$\bar{D}^2 \Phi = e^{-i\theta_1 \theta_2 P_2} \bar{\Omega} \times (\bar{Q}^2 \times \bar{A}) = e^{-\theta_1 \theta_2 \partial_2} e^{-i\theta_1 D_2 + i\theta_2^2 \partial_2} \chi \times (\bar{Q}^2 \times \bar{A}) \quad (59)$$

Since a product of two chiral superfields $\Phi^1 = e^{-\theta_1 \theta_2 \partial_2} e^{-i\theta_1 D_2 + i\theta_2^2 \partial_2} \varphi$ and $\Phi^2 = e^{-\theta_1 \theta_2 \partial_2} e^{-i\theta_1 D_2 + i\theta_2^2 \partial_2} \varphi$ is also a chiral superfield, we can easily show that

$$\Phi^1 \Phi^2 = e^{-\theta_1 \theta_2 \partial_2} e^{-i\theta_1 D_2 + i\theta_2^2 \partial_2} \varphi^1 \varphi^2. \quad (60)$$

Thus, for

$$K_i = \sum \kappa_{i_1 \ldots i_n j_1 \ldots j_m} \Phi^{i_1} \ldots \Phi^{i_n} \bar{\Phi}^{j_1} \ldots \bar{\Phi}^{j_m}. \quad (61)$$
we can obtain the expression

\[
\bar{D}^2 K_i = \sum_{i_1 \cdots i_n j_1 \cdots j_m} \Phi^{i_1} \cdots \Phi^{i_n} \bar{D}^2 \left( \Phi^{j_1} \cdots \Phi^{j_m} \right) \\
= 2i e^{i\theta_2} e^{-i\theta_1} \sum_{i_1 \cdots i_n} \left( \phi^{i_1} \cdots \phi^{i_n} (D_2^2 - 2i\partial_2) (\bar{\phi}^{j_1} \cdots \bar{\phi}^{j_m}) \right) \\
= 2i e^{i\theta_2} e^{-i\theta_1} \left( D_2^2 (K_{ij}D_{2\alpha} \bar{\varphi}^j) - K_{ikj}D_2^2 \varphi^k D_{2\alpha} \bar{\varphi}^j - 2i K_{ij} \partial_2 \bar{\varphi}^j \right) \\
= 2i e^{i\theta_2} e^{-i\theta_1} \left( K_{ij}D_2^2 \bar{\varphi}^j + K_{ijk}D_2^2 \varphi^k \bar{\varphi}^j - 2i K_{ij} \partial_2 \bar{\varphi}^j \right).
\]

(62)

In the second equation, we have used Eq.(59).

As a result, the equations of motion Eq.(53) can be rewritten as

\[-\frac{i}{2} \left( K_{ij}D_2^2 \bar{\varphi}^j + K_{ijk}D_2^2 \varphi^k \bar{\varphi}^j \right) - K_{ij} \partial_2 \bar{\varphi}^j + e^{-i\delta} W_i = 0. \tag{63}\]

### 4.2 BPS equation

The BPS equation for the domain wall can be obtained from the minimal energy condition for the classical field configuration. The energy (per unit area) of the domain wall is expressed by

\[
E = \int \! dx_2 \left\{ K_{ij} \partial_2 A^i \partial_2 \bar{A}^j + K^{ij} W_i \bar{W}_j \right\} \\
= \int \! dx_2 \left\{ K_{ij} (\partial_2 A^i - e^{-i\delta} K^{ki} \bar{W}_k) (\partial_2 \bar{A}^j - e^{-i\delta} K^{ji} W_i) + e^{i\delta} \bar{W}_j \partial_2 \bar{A}^j + e^{-i\delta} W_i \partial_2 A^i \right\} \\
\geq \int \! dx_2 \left\{ e^{i\delta} \partial_2 \bar{W} + e^{-i\delta} \partial_2 W \right\} = 2 \int \! dx_2 \partial_2 \text{Re}(e^{-i\delta} W) = 2 \text{Re}(e^{-i\delta} \Delta W),
\]

(64)

where

\[
\Delta W \equiv \int \! dW
\]

(65)

and \(\Gamma\) is the orbit for the classical field configuration on the target space of the scalar fields.

In the case that the extra dimension (the \(x_2\)-direction) is non-compact, \(\Delta W\) depends only on the values of the superpotential at the end points of \(\Gamma\), that is,

\[
\Delta W = W(x_2 = \infty) - W(x_2 = -\infty).
\]

(66)

On the other hand, when the extra dimension is compactified on \(S^1\), \(\Gamma\) must be a non-contractible cycle in order for the field configuration to be topologically stable. Furthermore, \(W\) must be a multi-valued function\(^6\) because non-zero \(\Delta W\) is needed for the existence of the BPS field configuration\(^9\). In this case, the value of \(\Delta W\) is determined by a homotopy class where \(\Gamma\) belongs.

\(^6\)Of course, \(dW\) must be a single-valued function since it determines the scalar potential.
The equality in Eq.(64) holds when the scalar fields satisfy the equation
\[ \partial_2 A^i = e^{i\delta} K^{ji} W_j. \] (67)

From Eq.(64), the most stringent bound is
\[ E \geq 2|\Delta W|, \] (68)
which comes from the case that the phase \( \delta \) is chosen as
\[ \delta = \arg(\Delta W). \] (69)

Eq.(67) with this choice of \( \delta \) is called the BPS equation.

When the classical field configuration \( A^i_{\text{cl}}(x_2) \) satisfies Eq.(67), we can see that a half of the original supersymmetry, i.e. \( Q_2\text{-SUSY} \), is preserved from Eq.(32),
\[ \delta_\xi \phi = 0 \quad (\phi = A, \Psi, F) \] (70)
Here we have used the equation of motion for the auxiliary fields \( F^i \),
\[ F^i = \frac{1}{2} \Gamma^{ij}_{\;\;\;kl} \Psi^j \Psi^k - K^{ij} W_j, \] (71)
and \( \Psi^i_{\text{cl}} = 0 \).

For a solution of Eq.(67) \( A^i_{\text{cl}}(x_2) \), let us define a quantity
\[ \varphi^i_{\text{cl}} \equiv e^{ix^m P_m + ix_2 P_2 + \theta_2 Q_2} \times A^i_{\text{cl}}(0). \] (72)
Then, \( \varphi^i_{\text{cl}} \) becomes a solution of the equations of motion Eq.(63). In fact, since \( \psi^i_{\text{cl}} = 0 \) and \( f^i_{\text{cl}} = 0 \) from Eq.(49) and the BPS equations, we can see
\[ \varphi^i_{\text{cl}} = A^i_{\text{cl}}(x_2). \] (73)
Thus,
\[ D_{2\alpha} \tilde{\varphi}^i_{\text{cl}} = 0, \] (74)
and from Eq.(67),
\[ K_{ij} \partial_2 \tilde{\varphi}^j_{\text{cl}} - e^{-i\delta} W_i(\varphi_{\text{cl}}) = 0. \] (75)
Therefore, \( \varphi^i_{\text{cl}} \) certainly satisfy the equations of motion Eq.(63).

4.3 Mode expansion of the fluctuation fields

Next, we will consider the equations of motion for the fluctuation fields \( \tilde{\varphi}^i \) around the classical solution \( \varphi^i_{\text{cl}} \). Substituting \( \varphi^i = \varphi^i_{\text{cl}} + \tilde{\varphi}^i \) into Eq.(63), we obtain
\[ -\frac{i}{2} K_{ij}(\varphi_{\text{cl}}) D^2_{2\alpha} \tilde{\varphi}^j - \left\{ K_{ijk}(\varphi_{\text{cl}}) \tilde{\varphi}^k + K_{ijk}(\varphi_{\text{cl}}) \tilde{\varphi}^k \right\} \partial_2 \varphi^i_{\text{cl}} \tilde{\varphi}^j \]
\[ -K_{ij}(\varphi_{\text{cl}}) \partial_2 \tilde{\varphi}^j + e^{-i\delta} W_{ij}(\varphi_{\text{cl}}) \tilde{\varphi}^j + \cdots = 0, \] (76)
where ellipsis denotes the higher order terms for $\tilde{\varphi}$.

Using Eq.(75), the equations of motion become

$$\frac{1}{2}K_{ij}(\varphi_{cl})D_2^{\varphi_j} + i\left\{D_y\tilde{\varphi}_i - e^{-i\theta}D_i\tilde{\varphi}^j\right\} + \cdots = 0,$$  \hspace{1cm} (77)

where

$$\tilde{\varphi}_i \equiv K_{ij}\varphi^j, \quad D_y\tilde{\varphi}_i \equiv \partial_2\tilde{\varphi}_i - \Gamma^k_{ij}\partial_2\varphi^j\bar{\varphi}_k, \quad D_i\tilde{\varphi}^j \equiv \partial_i\bar{\varphi}^j - \Gamma^k_{ij}\bar{\varphi}_k.$$  \hspace{1cm} (78)

From Eq.(77), we can find the mode equation,

$$i\left\{D_y\tilde{u}(n)i - e^{-i\theta}D_i\tilde{u}(n)\right\} = m_{(n)}\tilde{u}(n)i,$$  \hspace{1cm} (79)

The eigenfunctions of this equation $\tilde{u}(n)i$ are called the mode functions.

Now we expand $\varphi^i$ by the mode functions $u^i_{(n)}(x_2)$ ($n = 0, 1, 2, \cdots$).

$$\varphi^i(x^m, x_2, \theta_2) = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty} u^i_{(n)}(x_2)\varphi_{(n)}(x^m, \theta_2).$$  \hspace{1cm} (80)

As we will see in the following, $\varphi_{(n)}$ becomes a 3D $\mathcal{N} = 1$ scalar superfield with the mass $m_{(n)}$. In particular, Eq.(79) has a zero-mode $u^i_{(0)}(x_2) = C\partial_2A^{(0)}_i(x_2)$ ($C$ is a real normalization factor), which corresponds to the Nambu-Goldstone mode for the translational invariance and $Q_1$-SUSY.

When the eigenvalues of Eq.(79) $m_{(n)}$ are all real, we can show the orthogonal relation of the mode functions $u^i_{(n)}(x_2)$, (See Appendix C.)

$$\text{Re}\left\{\int dx_2\tilde{u}_{(n)i}(x_2)u^i_{(n)}(x_2)\right\} = \delta_{nm}.$$  \hspace{1cm} (81)

In the following, the mode functions are supposed to be normalized.

### 4.4 3D effective theory

Now we will express the original theory Eq.(50) in terms of $\varphi^i$ and $\varphi^j$ defined by Eq.(45), and carry out the $\theta_1$- and $x_2$-integration in order to obtain LET on the wall.

First, we will express the Kähler potential term in terms of $\varphi^i$ and $\varphi^j$.

Under the $x$-integration, note that

$$\int d^2\theta d^2\bar{\theta} K(\bar{\Phi}, \Phi) = \int d^2\theta \left(-\frac{1}{4}D^2K\right).$$  \hspace{1cm} (82)

By the same procedure as that for the derivation of Eq.(62), we can show that

$$\bar{D}^2K = 2ie^{i\delta}e^{-\theta_1\theta_2}e^{-i\theta_2D_1+i\theta_1D_2}\left\{D_2^\alpha(K_jD_2\varphi^j) - K_{ij}D_2^\alpha\varphi^iD_2\varphi^j - 2iK_{j2}\partial_2\varphi^j\right\}.$$  \hspace{1cm} (83)

Thus, up to the total derivatives, we can obtain the expression

$$\int d^2\theta d^2\bar{\theta} K(\bar{\Phi}, \Phi) =$$

$$\int d^2\theta \left[-\frac{1}{2}e^{i\delta}e^{-i\theta_1D_2}\left\{D_2^\alpha(K_jD_2\varphi^j) - K_{ij}D_2^\alpha\varphi^iD_2\varphi^j - 2iK_{j2}\partial_2\varphi^j\right\}\right].$$  \hspace{1cm} (84)
Noting that (see Appendix A)

\[ d^2\theta d^2\bar{\theta} = -d^2\theta_1 d^2\theta_2, \quad \bar{\theta}^2 = e^{-i\delta} \left\{ \theta_1 \theta_2 + \frac{i}{2} (\theta_1^2 - \theta_2^2) \right\}, \quad (85) \]

it follows that

\[
\int d^2\theta e^{-i\delta_1 D_2} = \int d^2\theta d^2\bar{\theta} e^{-i\delta_1 D_2} e^{-i\delta_1 D_2} \\
= -\int d^2\theta_1 d^2\theta_2 e^{-i\delta} \left\{ \theta_1 \theta_2 + \frac{i}{2} (\theta_1^2 - \theta_2^2) \right\} e^{-i\delta_1 D_2} \\
= -\int d^2\theta_2 \frac{i}{2} e^{-i\delta} e^{i\theta_2 D_2}. \quad (86)
\]

The explicit appearance of \( \theta_2 \) in the integrand seems to break \( Q_2 \)-SUSY at first sight, but it can be absorbed into \( \varphi \) and \( \bar{\varphi} \) as follows.

Under the \( x \)-integration, it can be shown for an arbitrary function \( \mathcal{F} \) that

\[
\int d^2\theta e^{-i\delta_1 D_2} \mathcal{F}(\varphi, \bar{\varphi}) = \int d^2\theta_2 \left\{ -\frac{i}{2} e^{-i\delta} e^{i\theta_2 D_2} \mathcal{F}(\varphi, \bar{\varphi}) \right\} \\
= \int d^2\theta_2 \left\{ -\frac{i}{2} e^{-i\delta} e^{i\theta_2 D_2} e^{ixmP_{m+ix_2P_2+\theta_2Q_2}} \mathcal{F}(a(0), \bar{a}(0)) \right\} \\
= \int d^2\theta_2 \left\{ -\frac{i}{2} e^{-i\delta} e^{ixmP_{m+ix_2P_2+2\theta_2Q_2}} \mathcal{F}(a(0), \bar{a}(0)) \right\} \\
= \int d^2\theta_2 \left\{ -2ie^{-i\delta} \mathcal{F}(\varphi, \bar{\varphi}) \right\}. \quad (87)
\]

Here we have used the formula Eq.(175) in the third step, and changed the integration variable \( 2\theta_2 \rightarrow \theta_2 \) in the last step.

Using this formula, Eq.(84) can be expressed as

\[
\int d^2\theta d^2\bar{\theta} K(\bar{\Phi}, \Phi) = \int d^2\theta_2 \left\{ -\left\{ D_2^\alpha (K_j D_2^\alpha \varphi^j) - K_{ij} D_2^\alpha \varphi^i D_2^\alpha \varphi^j - 2iK_j \partial_2 \bar{\varphi}^j \right\} \right\} \\
= \int d^2\theta_2 \left\{ K_{ij} D_2^\alpha \varphi^i D_2^\alpha \varphi^j - 2iK_j \partial_2 \bar{\varphi}^i \right\}. \quad (88)
\]

Here we have dropped the total derivatives.

The superpotential terms can easily be rewritten in terms of \( \varphi^i \) by using the formula Eq.(87),

\[
\int d^2\theta W(\Phi) = \int d^2\theta e^{-i\theta_1 \theta_2} e^{-i\delta_1 D_2 + i\partial_2 \partial_2} W(\varphi) \\
= \int d^2\theta_2 \left\{ -2ie^{-i\delta} W(\varphi) \right\}. \quad (89)
\]

The total derivatives have been dropped again.

As a result, the action after the \( \theta_1 \)-integration is

\[
S = \int d^3x \int d^2\theta_2 \int dx_2 \left\{ K_{ij} D_2^\alpha \varphi^i D_2^\alpha \varphi^j - 2iK_j \partial_2 \varphi^i + 4\text{Im} \left( e^{-i\delta} W(\varphi) \right) \right\}. \quad (90)
\]
Substituting \( \varphi = \varphi_{\text{cl}} + \tilde{\varphi} \) into Eq.\((90)\), the effective Lagrangian becomes

\[
\mathcal{L}^{(3)} = \int d^2 \theta_2 \int dx_2 \left\{ K_{i\bar{j}}(\varphi_{\text{cl}}) D_2 \tilde{\varphi}^i D_{2\alpha} \tilde{\varphi}^\alpha \\
- i \tilde{\varphi}^i \left( K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 \varphi_{\text{cl}}^j \bar{\varphi}^k + K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 \varphi_{\text{cl}}^j \bar{\varphi}^k + K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 \bar{\varphi}^j \bar{\varphi}^k - e^{i\bar{\varphi}^j \varphi_{\text{cl}}^l} W_{i\bar{j}l}(\varphi_{\text{cl}}) \bar{\varphi}^k \right) \\
+ i \tilde{\varphi}^i \left( K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 \varphi_{\text{cl}}^j \bar{\varphi}^k + K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 \varphi_{\text{cl}}^j \bar{\varphi}^k + K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 \bar{\varphi}^j \bar{\varphi}^k - e^{-i\bar{\varphi}^j \varphi_{\text{cl}}^l} W_{i\bar{j}l}(\varphi_{\text{cl}}) \bar{\varphi}^k \right) \\
+ \cdots \right\},
\]

(91)

where the ellipsis denotes the higher terms for \( \tilde{\varphi} \) or \( \tilde{\varphi} \).

Then, we will expand \( \tilde{\varphi} \) as Eq.\((80)\), and use the mode equations Eq.\((79)\) and the orthonormalization of the mode functions Eq.\((81)\), so that we can obtain the desired 3D effective Lagrangian.

\[
\mathcal{L}^{(3)} = \int d^2 \theta_2 \left[ \sum_{n=0}^{\infty} \left\{ \frac{1}{2} (D_2 \varphi_{(n)})^2 + m_{(n)} \varphi_{(n)}^2 \right\} + \sum_{m,n,l} g_{m(n)} \varphi_{(m)} D_2 \varphi_{(n)} D_2 \varphi_{(l)} \\
- \sum_{m,n,l} \lambda_{m(n)l} \varphi_{(m)} \varphi_{(n)} \varphi_{(l)} + \cdots \right],
\]

(92)

where the complete symmetrization is supposed for indices in the parentheses, and

\[
g_{mnl} = \frac{1}{2 \sqrt{2}} \int dx_2 \left\{ K_{i\bar{j}k}(\varphi_{\text{cl}}) u^{k}_{(m)} u^{i}_{(n)} u^{\bar{j}}_{(l)} + K_{i\bar{j}k}(\varphi_{\text{cl}}) \bar{u}^{k}_{(m)} u^{i}_{(n)} u^{\bar{j}}_{(l)} \right\},
\]

(93)

\[
\lambda_{mnl} = \frac{1}{2 \sqrt{2}} \int dx_2 \text{Im} \left\{ K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 u^{i}_{(m)} u^{j}_{(n)} u^{k}_{(l)} + 2 K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 u^{i}_{(m)} u^{j}_{(n)} u^{k}_{(l)} \\
+ K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 u^{i}_{(m)} u^{j}_{(n)} u^{k}_{(l)} + \frac{1}{3} K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 u^{i}_{(m)} u^{j}_{(n)} u^{k}_{(l)} \\
+ K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 u^{i}_{(m)} u^{j}_{(n)} u^{k}_{(l)} + K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 u^{i}_{(m)} u^{j}_{(n)} u^{k}_{(l)} \\
+ \frac{1}{3} K_{i\bar{j}k}(\varphi_{\text{cl}}) \partial_2 u^{i}_{(m)} u^{j}_{(n)} u^{k}_{(l)} + \frac{2}{3} e^{-i\bar{\varphi}^j \varphi_{\text{cl}}^l} W_{i\bar{j}l}(\varphi_{\text{cl}}) u^{i}_{(m)} u^{j}_{(n)} u^{k}_{(l)} \right\}.
\]

(94)

### 4.5 Case of real wall-configurations

Before concluding this section, we will apply the above result to a simple case, where the 4D bulk theory involves only one chiral superfield, the Kähler potential is minimal, and all the parameters in the theory and the classical field configuration \( A_{\text{cl}}(x_2) \) are real.

In this case, the mode equation Eq.\((79)\) can be written as

\[
i \left\{ \partial_2 u_{(n)} - \frac{\partial^2 W}{\partial \varphi_{(n)}^2} (A_{\text{cl}}) u_{(n)} \right\} = m_{(n)} u_{(n)}.
\]

(95)
Taking the complex conjugation, we can obtain
\[ i \left\{ \partial_2 u(n) - \frac{\partial^2 W}{\partial \Phi^2} (A_{cl}) \bar{u}(n) \right\} = -m(n) u(n). \]  
(96)

These equations mean that when \( u(n) \) is the mode function with the eigenvalue \( m(n) \), \( \bar{u}(n) \) is also the mode function whose eigenvalue is \(-m(n)\). Namely, the mass spectrum in this case is doubly degenerate. (As we will see, the zero-mode is exceptional.) Considering this fact, the mode-expansion of \( \tilde{\varphi} \) is
\[ \tilde{\varphi} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \left\{ u(n) \varphi(\pm n) + \bar{u}(n) \varphi(-n) \right\}. \]  
(97)

Signs in the label of the mode functions denote those of the corresponding mass eigenvalues.
Here we decompose each mode function into the real and imaginary parts.
\[ u(n) = \frac{1}{\sqrt{2}} (u_{R(n)} + i u_{I(n)}). \]  
(98)

Then, the mode expansion Eq.(97) becomes
\[ \tilde{\varphi} = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} \left\{ u_{R(n)} \varphi_{R(n)} + i u_{I(n)} \varphi_{I(n)} \right\}, \]  
(99)

where
\[ \begin{pmatrix} \varphi_{I(n)} \\ \varphi_{R(n)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi(\pm n) \\ \varphi(-n) \end{pmatrix}. \]  
(100)

Therefore we can see that the decomposition Eq.(98) corresponds to the rotation between the degenerate modes.
In the basis of \( \varphi_{R(n)} \) and \( \varphi_{I(n)} \), the mode equation becomes the following combined equations.
\[ \begin{pmatrix} -\partial_2 + \frac{\partial^2 W}{\partial \Phi^2} \\ \partial_2 + \frac{\partial^2 W}{\partial \Phi^2} \end{pmatrix} u_{R(n)} = m(n) u_{I(n)}, \]  
\[ \begin{pmatrix} -\partial_2 + \frac{\partial^2 W}{\partial \Phi^2} \\ \partial_2 + \frac{\partial^2 W}{\partial \Phi^2} \end{pmatrix} u_{I(n)} = m(n) u_{R(n)}. \]  
(101)

These correspond to Eq.(10) in Section 2. From these equations, we can see that the zero-mode is non-degenerate. In fact, the zero-mode exists only in the first equation of Eq.(101).7 Namely,
\[ u(0) = \frac{1}{\sqrt{2}} u_{R(0)} = \bar{u}(0). \]  
(102)

7The zero-mode solution of the second equation of Eq.(101) diverges at \( x_2 \to \pm \infty \).
Therefore, Eq.(97) can be rewritten as
\[
\tilde{\varphi} = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} u(n) \varphi(n),
\] (103)

where each mode function satisfies the following constraint
\[
u(-n) = \bar{u}(n).
\] (104)

Then, the effective Lagrangian \(L^{(3)}\) is written as
\[
L^{(3)} = \int d^2 \theta \left[ \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{2} (D_2 \varphi(n))^2 + m(n) \varphi(n)^2 \right\} + \cdots \right]
\]
\[
= \int d^2 \theta \left[ \frac{1}{2} (D_2 \varphi(0))^2 + \sum_{n=1}^{\infty} \left\{ \frac{1}{2} (D_2 \varphi(n))^2 + \frac{1}{2} (D_2 \varphi(-n))^2 \right\}
\]
\[+ (\varphi(n), \varphi(-n)) \left( \begin{array}{c} m(n) \\ -m(n) \end{array} \right) \left( \begin{array}{c} \varphi(n) \\ \varphi(-n) \end{array} \right) \right] + \cdots.\] (105)

In the basis of \(\varphi_{R(n)}\) and \(\varphi_{I(n)}\), it can be rewritten as
\[
L^{(3)} = \int d^2 \theta \left[ \frac{1}{2} (D_2 \varphi(0))^2 + \sum_{n=1}^{\infty} \left\{ \frac{1}{2} (D_2 \varphi_{R(n)})^2 + \frac{1}{2} (D_2 \varphi_{I(n)})^2 \right\}
\]
\[+ (\varphi_{I(n)}, \varphi_{R(n)}) \left( \begin{array}{c} m(n) \\ m(n) \end{array} \right) \left( \begin{array}{c} \varphi_{I(n)} \\ \varphi_{R(n)} \end{array} \right) \right] + \cdots.\] (106)

Up to the quadratic terms, this expression coincides with the form derived in Ref.\[6, 7\] after eliminating the auxiliary fields.

The cubic couplings in Eqs.(93) and (94) are simplified in this case.
\[
g_{mn} = 0,
\]
\[
\lambda_{mn} = \lambda_{m} = \frac{1}{3\sqrt{2}} \int dx_2 \Im \left\{ \frac{\partial^3 W}{\partial \Phi^3 (A_{cl})} u(n) u(m) u(l) \right\}.\] (107)

Since the zero-mode function \(u(0)\) is real, we can see the zero mode \(\varphi(0)\) does not have cubic self-interactions. This is true for all higher order couplings. This result coincides with that of Ref.\[7\].

However, this is not the case when \(A_{cl}(x_2)\) is a complex field configuration.

## 5 Dimensional reduction of the gauge theories

So far, we have discussed the derivation of 3D LET from 4D theory that contains only chiral superfields. In this section, we will derive 3D LET including a vector supermultiplet. Here, we will suppose that the gauge supermultiplet does not couple to the scalar fields
that contribute to the wall configuration. So the gauge symmetry is unbroken by the wall, and the zero-mode of the 4D gauge field, which corresponds to the 3D gauge field in LET, lives in the bulk. Therefore, the extra dimension of the wall ($x_2$-direction) is supposed to be compactified on $S^1$ in this section\(^8\). Here we will concentrate ourselves to the case of the abelian gauge supermultiplet, for simplicity.

5.1 Supertransformation of the gauge supermultiplet

The 4D vector superfield $V(x, \theta, \bar{\theta})$ is written by

$$
V(x, \theta, \bar{\theta}) = C + i \theta \chi - i \bar{\theta} \bar{\chi} + i \theta^2 B - i i^2 B - \theta \sigma^\mu \bar{\theta} v_\mu + i \theta^2 \bar{\theta} \left( \tilde{\lambda} + \frac{i}{2} \sigma^\mu \partial_\mu \chi \right) - i \bar{\theta}^2 \theta \left( \lambda + \frac{i}{2} \sigma^\mu \partial_\mu \bar{\chi} \right) + \frac{1}{2} \theta^2 \bar{\theta}^2 \left( D + \frac{1}{2} \partial^\mu \partial_\mu C \right),
$$

where $C$ and $D$ are real scalars, $\chi$ and $\lambda$ are complex Weyl spinors, and $B$ is a complex scalar.

The SUSY transformations of the component fields are

$$
\delta \xi C = i (\xi \chi - \bar{\xi} \bar{\chi}),
\delta \xi \chi_\alpha = (\sigma^\mu \bar{\xi})_\alpha (\partial_\mu C + i v_\mu) + 2 \xi B,
\delta \xi B = \bar{\xi} \lambda + i \xi \bar{\sigma}^\mu \partial_\mu \chi,
\delta \xi v_\mu = i \xi \sigma^\mu \bar{\lambda} + i \bar{\xi} \bar{\sigma}^\mu \lambda + \xi \partial_\mu \chi + \bar{\xi} \partial_\mu \bar{\chi},
\delta \xi \lambda_\alpha = i \xi_\alpha D + (\sigma^\mu \xi)_\alpha v_\mu,
\delta \xi D = -\xi \sigma^\mu \partial_\mu \bar{\lambda} + \bar{\xi} \bar{\sigma}^\mu \partial_\mu \lambda,
$$

where $v_{\mu\nu} \equiv \partial_\mu v_\nu - \partial_\nu v_\mu$ is the field strength.

Now, we will again redefine spinors and rewrite $\sigma{}-, \bar{\sigma}{}-$matrices in terms of the 3D $\gamma$-matrices $\gamma^\mu_{(3)}$, as we did around Eq.(31). Furthermore, the following decompositions are performed.

$$
\xi^\alpha = \frac{e^{i \delta/2}}{\sqrt{2}} (\xi_1^\alpha + i \xi_2^\alpha),
\chi^\alpha = \frac{e^{-i \delta/2}}{\sqrt{2}} (\chi_1^\alpha + i \chi_2^\alpha),
\lambda^\alpha = \frac{e^{i \delta/2}}{\sqrt{2}} (\lambda_1^\alpha + i \lambda_2^\alpha),
B = \frac{e^{-i \delta}}{2} (M + i N),
$$

\(^8\text{BPS domain walls in such a case are discussed in Ref.}\,[9].\)
where $\delta$ is a phase defined in Eq.(69). Then, Eq.(109) is rewritten as
\[
\delta_x C = \xi_1 \chi_1 - \xi_2 \chi_2, \\
\delta_x \chi_{1\alpha} = -i (\gamma_{(3)}^m \xi_1)_{\alpha} \partial_m C - \xi_1 \alpha \nu_2 + \xi_1 \alpha M - i (\gamma_{(3)}^m \xi_2)_{\alpha} v_m + \xi_2 \alpha \partial_2 C - \xi_2 \alpha N, \\
\delta_x \chi_{2\alpha} = -i (\gamma_{(3)}^m \xi_1)_{\alpha} v_m + \xi_1 \alpha \partial_2 C + \xi_1 \alpha N + i (\gamma_{(3)}^m \xi_2)_{\alpha} \partial_m C + \xi_2 \alpha \nu_2 + \xi_2 \alpha M, \\
\delta_x \chi_{12} = \xi_1 \lambda_2 - i \xi_1 \gamma_{(3)} \partial_m \chi_1 + \xi_1 \nu_2 \nu_2 + \xi_2 \lambda_1 - i \xi_2 \gamma_{(3)} \partial_m \nu_2, \\
\delta_x \chi_{21} = \xi_1 \lambda_2 - i \xi_1 \gamma_{(3)} \partial_m \chi_2 - \xi_2 \lambda_1 - i \xi_2 \gamma_{(3)} \partial_m \nu_2, \\
\delta_x v_m = \xi_1 \gamma_{(3)} \partial_m \lambda_1 + \xi_1 \partial_m \chi_2 - i \xi_2 \gamma_{(3)} \partial_m \lambda_2 + \xi_2 \partial_m \chi_1, \\
\delta_x v_2 = \xi_1 \lambda_2 + \xi_1 \partial_2 \chi_2 - \xi_2 \lambda_1 + \xi_2 \partial_2 \chi_1, \\
\delta_x \chi_{1\alpha} = (\gamma_{(3)}^m \xi_1)_{\alpha} v_m - \xi_2 \alpha D + i (\gamma_{(3)}^m \xi_2)_{\alpha} v_m, \\
\delta_x \chi_{2\alpha} = \xi_1 \alpha D - i (\gamma_{(3)}^m \xi_1)_{\alpha} v_m + (\gamma_{(3)}^m \xi_2)_{\alpha} v_m, \\
\delta_x D = -i \xi_1 \gamma_{(3)} \partial_m \lambda_2 - \xi_1 \partial_2 \chi_1 + i \xi_2 \gamma_{(3)} \partial_m \lambda_1 - \xi_2 \partial_2 \lambda_2. \quad (111)
\]

Note that the vector superfield Eq.(108) can be expressed as
\[
V = e^{ix^m P_m + \theta Q + \bar{\theta} \bar{Q}} \times C(0) = e^{ix^m P_m + i(x_2 - \theta_1 \theta_2) P_2 + \theta_2 Q_2} e^{\theta_1 Q_1} \times C(0). \quad (112)
\]
Using Eq.(111),
\[
e^{\theta_1 Q_1} \times C = C + \theta_1 x_1 + \frac{1}{2} \theta_1^2 (-\nu_2 + M). \quad (113)
\]
Then, if we introduce the following quantities,
\[
\kappa(x^m, x_2, \theta_2) \equiv e^{ix^m P_m + i x_2 P_2 + \theta_2 Q_2} \times C(0), \\
\rho_\alpha(x^m, x_2, \theta_2) \equiv e^{ix^m P_m + i x_2 P_2 + \theta_2 Q_2} \times \chi_{1\alpha}(0), \\
\sigma(x^m, x_2, \theta_2) \equiv e^{ix^m P_m + i x_2 P_2 + \theta_2 Q_2} \times \frac{1}{2} (-\nu_2(0) + M(0)), \quad (114)
\]
we can express $V$ as
\[
V(x^m, x_2 + \theta_1 \theta_2, \theta_1, \theta_2) = \kappa(x^m, x_2, \theta_2) + \theta_1 \rho(x^m, x_2, \theta_2) + \theta_1^2 \sigma(x^m, x_2, \theta_2). \quad (115)
\]

### 5.2 Gauge transformation and gauge fixing

Next, we will discuss the gauge transformation. The gauge transformation for the vector superfield is
\[
V \rightarrow V + \Lambda + \bar{\Lambda}, \quad (116)
\]
where the transformation parameter $\Lambda$ is a chiral superfield, i.e. $D_\alpha \Lambda = 0$. From Eq.(46), $\Lambda$ is expressed in the following form.
\[
\Lambda(x^m, x_2 + \theta_1 \theta_2, \theta_1, \theta_2) = e^{-i \theta_1 D_2 + i \theta_2 \partial_2} \beta(x^m, x_2, \theta_2) = \beta - i \theta_1 D_2 \beta + \theta_1^2 (\frac{1}{4} D_2^2 \beta + i \partial_2 \beta), \quad (117)
\]

This $\theta_1$-expansion is analogous to the $\mathcal{N} = 1$ decomposition of the 3D $\mathcal{N} = 2$ vector superfield presented in Ref.[15].
where $\beta$ is a quantity defined by $\beta \equiv e^{ix^m P_m + i x^2 P_2 + \theta_2 Q_2} \times \Lambda (\alpha^\mu = \theta = \bar{\theta} = 0)$. Then, the transformation Eq.(116) is rewritten in terms of the quantities defined in Eq.(114) as

$$
\begin{align*}
\kappa & \rightarrow \kappa + \beta + \bar{\beta}, \\
\rho_\alpha & \rightarrow \rho_\alpha - i D_{2\alpha}(\beta - \bar{\beta}), \\
\sigma & \rightarrow \sigma + \frac{1}{4} D^2_2(\beta + \bar{\beta}) + i \partial_2(\beta - \bar{\beta}).
\end{align*}
$$

Using this gauge transformation, we can greatly simplify the expression of the vector superfield. First, $\kappa$ can be set to zero by choosing the gauge parameter $\beta$ as

$$
2 \text{Re} \beta = -\kappa.
$$

This means that $\kappa$ is a pure gauge degree of freedom. On the other hand, $\rho_\alpha$ and $\sigma$ correspond to a 3D $\mathcal{N} = 1$ vector and scalar superfield components of a 3D $\mathcal{N} = 2$ vector supermultiplet.

After eliminating $\kappa$ by the gauge transformation, there is still a gauge degree of freedom corresponding to the gauge transformation parameter $\text{Im} \beta$. From Eqs.(111) and (114), $\rho_\alpha$ is expanded as

$$
\rho_\alpha = \chi_{1\alpha} - i (\gamma_m^m \theta_2)_\alpha v_m + \theta_{2\alpha}(\partial_2 C - N) + \theta_2^2 \left( -\lambda_{2\alpha} + \frac{i}{2} (\gamma_m^m \partial_m \chi_1)_\alpha \right).
$$

Then, by choosing the components of the gauge parameter $2 \text{Im} \beta = a + \theta_2 \psi + \frac{1}{2} \theta_2^2 f$ as

$$
\psi^\alpha = -\chi_1^\alpha, \\
f = - (\partial_2 C - N),
$$

we can eliminate the auxiliary fields $\chi_1^\alpha$, $C$ and $N$, and obtain a simple expression,

$$
\rho_\alpha = -i (\gamma_m^m \theta_2)_\alpha v'_m - \theta_2^2 \lambda_{2\alpha},
$$

where

$$
v'_m \equiv v_m + \partial_m a.
$$

Here, Eq.(123) represents the usual 3D gauge transformation. In the choice of the gauge parameter Eqs.(119) and (121), the expression of $\sigma$ becomes

$$
\sigma = e^{\theta_2 Q_2} \times \frac{1}{2} (-v_2 + M) - \frac{1}{4} D_2^2 \left( e^{\theta_2 Q_2} \times C \right) - \partial_2 (2 \text{Im} \beta)
$$

$$
= e^{\theta_2 Q_2} \times (-v_2) - \partial_2 (2 \text{Im} \beta)
$$

$$
= -(v_2 + \partial_2 a) + \theta_2 \lambda_1 - \frac{1}{2} \theta_2^2 D.
$$

Namely, all the 4D auxiliary fields of $V$ is eliminated in this gauge. Hence, the gauge choice Eqs.(119) and (121) corresponds to the Wess-Zumino gauge in four dimensions. After these gauge fixings, there are still residual gauge degrees of freedom, which associate
with the parameter \( a(x) \). At first sight, it seems that the 3D scalar \( v_2 \) can completely be eliminated by choosing the gauge parameter \( a \) so that
\[
\partial_2 a = -v_2.
\] (125)

However, it must be noticed that \( a(x) \) does not contain the mode that is linear to \( x_2 \) because of the periodicity of \( a(x) \) along the \( x_2 \)-direction, i.e.
\[
a(x^m, x_2 + 2\pi R) = a(x^m, x_2).
\] (126)

Therefore the zero-mode \( v_{2(0)} \) cannot be gauged away by the gauge fixing, and is a physical mode. So Eq.(125) should be modified as
\[
\partial_2 a = -v_{2|nz},
\] (127)

where \( v_{2|nz} \equiv v_2 - v_{2(0)} \) contains only non-zero modes.

Note that the solution of Eq.(127) has an ambiguity of \( a(0)(x^m) \), which is independent of \( x_2 \). This means that there is a residual gauge symmetry whose transformation parameter is \( a(0)(x^m) \). This is the 3D gauge symmetry in LET.

In the following, we will take the gauge mentioned above. In this gauge, Eq.(115) is simplified as
\[
V(x^m, x_2 + \theta_1 \theta_2, \theta_1, \theta_2) = \theta_1 \rho(x^m, x_2, \theta_2) + \theta_2^2 \sigma(x^m, x_2, \theta_2),
\] (128)

where
\[
\rho_\alpha = -i(\gamma_m^{(3)} \theta_2)_\alpha v_m - \theta_2^2 \lambda_{2\alpha},
\]
\[
\sigma = -v_{2(0)} + \theta_2 \lambda_1 - \frac{1}{2} \theta_2^2 \lambda.
\] (129)

The gauge invariant quantities are
\[
w_\alpha \equiv \frac{1}{4} D^2 \rho_\alpha + i \frac{\gamma_m^{(3)} \partial_m \rho_\alpha}{2} = e^{\theta_2 Q_2} \times \lambda_{2\alpha},
\]
\[
u_\alpha \equiv D_{2\alpha} \sigma + \partial_2 \rho_\alpha = e^{\theta_2 Q_2} \times \lambda_{1\alpha}.
\] (130)

Here, \( w_\alpha \) is the 3D superfield strength. Eq.(130) is written in terms of the component fields as
\[
w_\alpha = \lambda_{2\alpha} + (\gamma_m^{(3)} \theta_2)_\alpha v_m - \frac{1}{2} \theta_2^2 (\gamma_m \partial_m \lambda_{2\alpha}),
\]
\[
u_\alpha = \lambda_{1\alpha} - \theta_2 \lambda \partial_2 + i(\gamma_m^{(3)} \theta_2)_\alpha (\partial_m v_{2(0)} - \partial_2 v_m) + \frac{\theta_2^2}{2} \left\{ \frac{i}{2} (\gamma_m \partial_m \lambda_1)_\alpha - \partial_2 \lambda_{2\alpha} \right\}.
\] (131)

These quantities are related to the 4D superfield strength \( W_\alpha \equiv -\frac{1}{4} \bar{D}^2 D_\alpha V \) through
\[
W_\alpha(x^m, x_2 + \theta_1 \theta_2, \theta_1, \theta_2) = -\frac{i}{\sqrt{2}} e^{i\beta/2} e^{-i\theta_1 D_2 + i\theta_2 \partial_2} \left\{ u_\alpha(x^m, x_2, \theta_2) + i w_\alpha(x^m, x_2, \theta_2) \right\}.
\] (132)

Since \( W_\alpha \) is a chiral superfield, this relation can also be obtained by the procedure discussed in the previous section.
5.3 3D effective theory

Now we will derive 3D LET by carrying out the integration in terms of \( \theta_1 \) and \( x_2 \).

The gauge kinetic term of the 4D theory is

\[
\mathcal{L}_{\text{gauge}} = \frac{1}{4} \int d^2 \theta f(\Phi) W^\alpha W_\alpha + \text{h.c.,}
\]

where \( f(\Phi) \) is a holomorphic function of \( \Phi \), called the gauge kinetic function, and \( f(0) = 1 \) from the requirement that \( W_\alpha \) is canonically normalized.

Substituting Eqs.(46) and (132) into Eq.(133) and following a similar procedure to that in the previous section, we can obtain the \( \theta_1 \)-integrated expression,

\[
\mathcal{L}_{\text{gauge}} = \int d^2 \theta_2 \left[ \frac{\text{Ref}(\varphi)}{2} \left\{ (D_2 \sigma)^2 + 2 D_2^2 \sigma \partial_2 \rho + (\partial_2 \rho)^2 - w^2 \right\} - \text{Im} f(\varphi) \{ (D_2^2 \sigma + \partial_2 \rho^0 \sigma_\alpha) \} \right].
\]

(134)

Since the gauge multiplet does not feel the existence of the domain wall at the classical level, the mode expansion of it is trivial, that is,

\[
\rho^\alpha(x^m, x_2, \theta_2) = \frac{1}{\sqrt{2\pi R}} \rho^\alpha(0)(x^m, \theta_2) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi R}} \left\{ \cos \frac{n x_2}{R} \rho^\alpha_+(x^m, \theta_2) + \sin \frac{n x_2}{R} \rho^\alpha_-(x^m, \theta_2) \right\},
\]

\[
\sigma(x^m, x_2, \theta_2) = \frac{1}{\sqrt{2\pi R}} \sigma(0)(x^m, \theta_2) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi R}} \left\{ \cos \frac{n x_2}{R} \sigma_+(x^m, \theta_2) + \sin \frac{n x_2}{R} \sigma_-(x^m, \theta_2) \right\},
\]

(135)

where signs in the label of the 3D superfields denote the parity change of the mode functions under \( x_2 \to -x_2 \). Thus, by carrying out the \( x_2 \)-integration, we can obtain the following effective Lagrangian.

\[
\mathcal{L}^{(3)}_{\text{gauge}} = \int d^2 \theta_2 \left[ \frac{1}{2} (D_2 \sigma(0))^2 - \frac{1}{2} w^2(0)
\right.

\left. \frac{1}{2} \sum_{n=1}^{\infty} \left\{ (D_2 \sigma_+(n))^2 + (D_2 \sigma_-(n))^2 + 2 \frac{n}{R} (D_2 \sigma_-(n) \rho_+(n) - D_2 \sigma_+(n) \rho_-(n))
\right.

\right.

\left. \left. + \frac{n^2}{R^2} (\rho_+(n) + \rho_-(n))^2 - w^2_+(n) - w^2_-(n) \right\} \right]

(136)

Here we used the assumption that scalar fields that couple to the gauge field do not have nontrivial background configurations, that is, \( f(\Phi_{\text{cl}}) = f(0) = 1 \).

The first two terms in Eq.(136) represent the kinetic terms of the 3D \( \mathcal{N} = 2 \) gauge multiplet, and the quadratic terms in the second and the third lines correspond to the Kaluza-Klein modes with masses \( m_{\text{K.K.}}^2 = n/R \).

Since we have already eliminated the non-zero modes of \( v_2 \), terms such as \( \partial_m v_2(n \pm) v^m_+(n \pm) \) are absent.
5.4 Couplings to the matter

Finally, we will write down the gauge couplings to the matter supermultiplet. For simplicity, we will consider the case that the Kähler potential is minimal and the chiral matter superfield $\Phi$ does not have a non-trivial classical configuration, i.e. $\Phi_{\text{cl}} = 0$.

Here, note that the abelian gauge symmetry should be represented as $O(2)$ symmetry in our case since 3D superfields are real. Thus, the vector superfield $V$ discussed so far should be understood as a $2 \times 2$ matrix

$$V = V_R \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

where $V_R$ is a real vector supermultiplet, and the matter superfield $\Phi$ should be understood as a 2-component column vector whose gauge transformation is

$$\Phi \to \exp \left\{-2g\Lambda \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right\} \Phi,$$

where $g$ is a gauge coupling constant.

Then, the 4D gauge coupling is written by

$$L_{\text{matter}} = \int d^2\theta d^2\bar{\theta} \Phi e^{2gV} \Phi$$

$$= - \int d^2\theta_1 d^2\theta_2 \left\{ \bar{\varphi} + i\theta_1 D_2 \bar{\varphi} + \theta_1^2 \left( \frac{1}{4} D_2^2 \bar{\varphi} - i\partial_2 \bar{\varphi} \right) \right\} \left\{ 1 + 2g\theta_1 \rho + \theta_1^2 (2g\sigma - g^2\rho^2) \right\} \times \left\{ \varphi - i\theta_1 D_2 \varphi + \theta_1^2 \left( \frac{1}{4} D_2^2 \varphi + i\partial_2 \varphi \right) \right\}$$

$$= \int d^2\theta_2 \left[ D_2^2 \bar{\varphi} D_2 \varphi + 2\text{Im} (\bar{\varphi} \partial_2 \varphi) + 2g \text{Im} (\bar{\varphi} \rho D_2 \varphi) - \bar{\varphi} (2g\sigma - g^2\rho^2) \varphi \right],$$

where $\varphi$ is a 2-component column vector, and $\rho_\alpha$ and $\sigma$ are $2 \times 2$ matrices. We can obtain the effective theory by expanding each superfield into the Kaluza-Klein modes and performing the $x_2$-integration.

$$L^{(3)}_{\text{matter}} = \int d^2\theta_2 \sum_{n=0}^{\infty} \left\{ \frac{1}{2} (D_2 \varphi(n))^2 + m(n) \varphi^2(n) + g(0) \text{Im} \left( \varphi(n) \rho(0) D_2 \varphi(n) \right) + \frac{1}{2} g(0)^2 \varphi(n) \rho(0) \varphi(n) \right\}$$

$$+ \sum_{n,m=0}^{\infty} \sum_{l=1}^{\infty} \sum_{s=\pm} g_{R(nm,ls)} \left\{ \text{Im} \left( \varphi(n) \rho(ls) D_2 \varphi(m) \right) + g(0) \varphi(n) \rho(ls) \rho(0) \varphi(m) \right\}$$

$$- \sum_{n,m=0}^{\infty} \left\{ g_{I(nm,0)} \varphi(n) \sigma(0) \varphi(m) + \sum_{l=1}^{\infty} \sum_{s=\pm} g_{I(nm,ls)} \varphi(n) \sigma(ls) \varphi(m) \right\}$$

$$+ \frac{1}{2} \sum_{n,m=0}^{\infty} \sum_{l,p=1}^{\infty} \sum_{s,t=\pm} \left( g^2 \right)_{(nm,lp,pt)} \varphi(n) \rho(ls) \rho(pt) \varphi(m),$$

(140)
where \( g(0) \equiv g/\sqrt{2\pi R} \) is the three-dimensional gauge coupling, and the other effective couplings are defined as follows.

\[
\begin{align*}
g_I(nm,0) &\equiv \frac{g}{\sqrt{2\pi R}} \int dx_2 \text{Im} \left( \bar{u}_n(x_2) u_m(x_2) \right), \\
g_R(nm,l+) &\equiv \frac{g}{\sqrt{\pi R}} \int dx_2 \text{Re} \left( \bar{u}_n(x_2) u_m(x_2) \right) \cos \frac{l x_2}{R}, \\
g_R(nm,l-) &\equiv \frac{g}{\sqrt{\pi R}} \int dx_2 \text{Re} \left( \bar{u}_n(x_2) u_m(x_2) \right) \sin \frac{l x_2}{R}, \\
g_I(nm,l+) &\equiv \frac{g}{\sqrt{\pi R}} \int dx_2 \text{Im} \left( \bar{u}_n(x_2) u_m(x_2) \right) \cos \frac{l x_2}{R}, \\
g_I(nm,l-) &\equiv \frac{g}{\sqrt{\pi R}} \int dx_2 \text{Im} \left( \bar{u}_n(x_2) u_m(x_2) \right) \sin \frac{l x_2}{R}, \\
(g^2)_{(nm,l+,p+)} &\equiv \frac{g^2}{\pi R} \int dx_2 \text{Re} \left( \bar{u}_n(x_2) u_m(x_2) \right) \cos \frac{l x_2}{R} \cdot \cos \frac{p x_2}{R}, \\
(g^2)_{(nm,l+,p-)} &\equiv \frac{g^2}{\pi R} \int dx_2 \text{Re} \left( \bar{u}_n(x_2) u_m(x_2) \right) \cos \frac{l x_2}{R} \cdot \sin \frac{p x_2}{R}, \\
(g^2)_{(nm,l-,p+)} &\equiv \frac{g^2}{\pi R} \int dx_2 \text{Re} \left( \bar{u}_n(x_2) u_m(x_2) \right) \sin \frac{l x_2}{R} \cdot \cos \frac{p x_2}{R}, \\
(g^2)_{(nm,l-,p-)} &\equiv \frac{g^2}{\pi R} \int dx_2 \text{Re} \left( \bar{u}_n(x_2) u_m(x_2) \right) \sin \frac{l x_2}{R} \cdot \sin \frac{p x_2}{R}.
\end{align*}
\]

The first line of Eq.(140) contains the minimal coupling of each Kaluza-Klein mode of the matter field \( \varphi_{(n)} \) and the gauge field \( v_{(0)}^m \) included in \( \rho(0) \).

6 Summary and discussion

We derived 3D effective theory on the BPS domain wall which is described in terms of the 3D superfields. Such a superfield description of LET on the BPS wall is useful not only because it makes the unbroken SUSY manifest, but also it greatly simplifies the procedure for integrating out the massive modes. So our procedure presented in the present paper should be used when we derive LET on the BPS wall including interaction terms.

The main obstacle in the derivation of LET is the execution of only the \( \theta_1 \)-integration while leaving \( \theta_2 \) unintegrated. Thus, our main results are Eqs.(90), (134) and (139). To obtain the 3D superfield description of the theory, we have to carry out the mode expansion of \( \varphi, \rho^a \) and \( \sigma \) defined by Eqs.(45) and (114). The mode expansion is trivial for the vector superfield and is given by Eq.(135). We derived the mode equation Eq.(79) in the generalized Wess-Zumino model including the case that the classical configuration \( A_3(x_2) \) is complex. In a simple case where \( A_3(x_2) \) is real, it is reduced to the familiar form Eq.(101) by some field rotations. If we can solve the mode equation, the desired 3D LET is obtained by substituting the mode-expanded expression

\[
\varphi^i(x^m, x_2, \theta_2) = A^i_3(x_2) + \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} u^i_{(n)}(x_2) \varphi_{(n)}(x^m, \theta_2)
\]

(142)
and Eq.(135) into the $\theta_1$-integrated actions Eqs.(90), (134) and (139), and performing the $x_2$-integration.

There is another approach to discuss LET on the wall. Since the BPS wall realizes the partial SUSY breaking of 3D $\mathcal{N} = 2$ to $\mathcal{N} = 1$, we can construct LET on the wall by the nonlinear realization approach [16]. This approach is useful because it does not involve the dimensional reduction and we do not have to suffer from integrating out the massive modes, since the discussion starts in three dimensions and only light modes are introduced from the beginning in this approach. However, since this approach uses only information about symmetries, we cannot obtain any information about the magnitude of various parameters in LET which reflect the wall structure. In order to discuss such parameters, we have to derive LET from the 4D bulk theory. In this sense, our derivation presented in this paper is a complementary approach to the nonlinear realization\textsuperscript{10}.

At the end of Section 4.5, we have mentioned that the zero-mode $\varphi(0)$ does not have self-interaction. By noting that $\varphi(0)$ in that case corresponds to the Nambu-Goldstone (NG) modes for the broken space-time symmetries, this fact can be interpreted as a consequence of the low-energy theorem. However, in the case that $A_{cl}(x_2)$ is complex, we can show that $\varphi(0)$ can have self-interaction. This seems inconsistent with the low-energy theorem. The cause of this contradiction is in the definition of the NG modes $\varphi(0)$. In order to define the NG modes that obey the low-energy theorem, they must be introduced as collective coordinates. The introduction of NG modes for the broken space-time symmetries in such a way leads to the clarification of the relation between LET on the BPS wall in our approach and the one obtained by the nonlinear realization or the one provided in Ref.[17]. The research along this direction is now in progress.

For other directions of research, we would like to discuss the case that a domain wall is saturated the BPS bound only \textit{approximately}. Such a situation must be considered when we try to construct a realistic model in the brane-world scenario, since our world has no exact SUSY. For example of such a wall configuration, the author has found with other collaborators a topologically stable non-BPS system that consists of an approximate BPS and anti-BPS domain walls in Ref.[18]. We would like to investigate whether there is a useful choice of the 3D superspace for the approximately preserved SUSY in the original 4D superspace in such a case. Expanding the discussion to the supergravity is also an interesting subject.

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\textsuperscript{10}The authors of Ref.[17] discussed LET on a structureless supersymmetric membrane. Their result corresponds to the thin-wall limit of ours.
A Notations

Basically, we follow the notations of Ref. [12] for the four-dimensional bulk theory.

A.1 Notations for 3D theories

The notations for the 3D theories are as follows.

We take the space-time metric as

$$\eta^{mn} = \text{diag}(-1, +1, +1).$$

(143)

The 3D $\gamma$-matrices, $(\gamma^m_{(3)})_{\alpha\beta}$, can be written by the Pauli matrices as

$$\gamma^0_{(3)} = \sigma^2, \quad \gamma^1_{(3)} = -i\sigma^3, \quad \gamma^3_{(3)} = i\sigma^1,$$

(144)

and these satisfy the 3D Clifford algebra,

$$\{\gamma^m_{(3)}, \gamma^n_{(3)}\} = -2\eta^{mn}.$$

(145)

The generators of the Lorentz group $Spin(1, 2)$ are

$$\gamma^m_{(3)} \equiv \frac{1}{4}[\gamma^m_{(3)}, \gamma^m_{(3)}].$$

(146)

The relations between the 4D $\sigma$-matrices and the above $\gamma^m_{(3)}$ are

$$\begin{align*}
(\sigma^{\mu})_{\alpha\beta} & = (\gamma^0_{(3)}, \gamma^1_{(3)}, -1, \gamma^3_{(3)})_{\alpha}(\sigma^2)(\gamma^\beta), \\
(\bar{\sigma}^{\mu})_{\dot{\alpha}\dot{\beta}} & = (-\sigma^2)^{\alpha\gamma}(\gamma^0_{(3)}, \gamma^1_{(3)}, 1, \gamma^3_{(3)})_{\gamma}(\gamma^\beta).
\end{align*}$$

(147)

$$\begin{align*}
(\sigma^{mn})_{\alpha} & = (\gamma^m_{(3)})_{\alpha}^\beta, \\
(\sigma^{m2})_{\alpha} & = \frac{1}{2}(\gamma^m_{(3)})_{\alpha}^\beta.
\end{align*}$$

(148)

Note that, in three dimensions, there is no discrimination between the dotted and undotted indices.

The spinor indices are raised and lowered by multiplying $\sigma^2$ from the left.

$$\psi_{\alpha} = (\sigma^2)_{\alpha\beta}\psi^\beta, \quad \psi^\alpha = (\sigma^2)^{\alpha\beta}\psi_{\beta}.$$  

(149)

We take the following convention of the contraction of spinor indices.

$$\psi_1^\alpha \psi_2^\beta \equiv \psi_1^\alpha \psi_2^\alpha = (\sigma^2)_{\alpha\beta}\psi^\alpha_1 \psi^\beta_2 = \psi_2 \psi_1.$$  

(150)

The relations between 4D Grassmannian coordinates $\theta, \bar{\theta}$ and 3D ones $\theta_1, \theta_2$ are

$$\begin{align*}
\theta^\alpha & = \frac{e^{i\theta/2}}{\sqrt{2}}(\theta_1^\alpha + i\theta_2^\alpha), \\
\bar{\theta}^{\dot{\alpha}} & = \frac{e^{-i\theta/2}}{\sqrt{2}}(\theta_1^{\dot{\alpha}} - i\theta_2^{\dot{\alpha}}),
\end{align*}$$

(151)
where $\delta$ is defined in Eq.(69). Then, it follows that

$$
\begin{align*}
\theta^2 &= e^{i\delta} \left\{ \theta_1 \theta_2 - \frac{i}{2} (\theta_1^2 - \theta_2^2) \right\}, \\
\bar{\theta}^2 &= e^{-i\delta} \left\{ \theta_1 \theta_2 + \frac{i}{2} (\theta_1^2 - \theta_2^2) \right\}, \\
\theta^2 \bar{\theta}^2 &= -\theta_1^2 \theta_2^2. 
\end{align*}
$$

(152)

The definitions of the integral measures are as follows.

$$
\int d^2 \theta \theta^2 = \int d^2 \bar{\theta} \bar{\theta}^2 = 1 
$$

(153)

and

$$
\int d^2 \theta_1 \theta_1^2 = \int d^2 \theta_2 \theta_2^2 = 1. 
$$

(154)

Then, using the relation Eq.(152), the following relation is obtained.

$$
d^2 \theta d^2 \bar{\theta} = -d^2 \theta_1 d^2 \theta_2. 
$$

(155)

A.2 Useful Formulae

$$
\begin{align*}
(\psi_1 \psi_2)^* &= \psi_1 \psi_2, \\
(\psi_1 \gamma^m (3) \psi_2)^* &= -\psi_1 \gamma^m (3) \psi_2, \\
(\psi_1 \gamma^m (3) \gamma^n (3) \psi_2)^* &= \psi_1 \gamma^m (3) \gamma^n (3) \psi_2. 
\end{align*}
$$

(156)

$$
\sigma^2 \gamma^m (3) \sigma^2 = -t \gamma^m (3). 
$$

(157)

$$
\theta^\alpha_1 \theta^\beta_1 = -\frac{1}{2} \theta^2_1 (\sigma^2)^{\alpha\beta}, \quad \theta_1^\alpha \theta_1^\beta = \frac{1}{2} \theta^2_1 (\sigma^2)^{\alpha\beta}. 
$$

(158)

$$
(\theta_1 \lambda)(\chi \psi) = -\frac{1}{2} \left\{ (\theta_1 \psi)(\chi \lambda) - (\theta_1 \gamma^m (3) \psi)(\chi \gamma^m (3) \eta \lambda) \right\} \quad \text{(Fierz transformation)} 
$$

(159)

$$
\begin{align*}
\theta_1 \gamma^m (3) \theta_2 &= -\theta_2 \gamma^m (3) \theta_1, \\
\theta_1 \gamma^m (3) \gamma^n (3) \theta_2 &= \theta_2 \gamma^n (3) \gamma^m (3) \theta_1. 
\end{align*}
$$

(160)

(161)

In particular,

$$
\begin{align*}
\theta_1 \gamma^m (3) \theta_1 &= 0, \\
\theta_1 \gamma^m (3) \gamma^n (3) \theta_1 &= -\theta^2_1 \eta^{mn}. 
\end{align*}
$$

(162)

(163)
B  SUSY algebra and the covariant derivatives

The 3D $\mathcal{N} = 1$ SUSY algebra is

$$\{Q^{(3)}_\alpha, Q^{(3)}_\beta\} = 2(\gamma^{m}_\alpha \sigma^2)_{\alpha\beta} P_m,$$  

(164)

where $Q^{(3)}_\alpha$ and $P_m$ denote the supercharge and the translational generators.

The representation of the generators on the 3D $\mathcal{N} = 1$ superspace $(x^m, \theta)$ is

$$\hat{P}_m = -i\partial_m,$$
$$\hat{Q}^{(3)}_\alpha = \frac{\partial}{\partial \theta^{\alpha}} + i(\gamma^{m}_\alpha \theta) \partial_m.$$  

(165)

For the group element $\Omega = e^{ix^m P_m + \theta Q^{(3)}_\alpha}$, the Cartan one-form is

$$\Omega^{-1} d\Omega = i(dx^m + i d\theta \gamma^{m}_\alpha \theta) P_m + d\theta^{\alpha} Q^{(3)}_\alpha$$
$$\equiv i \omega^m_m P_m + \omega^{\alpha}_\alpha Q^{(3)}_\alpha.$$  

(166)

For the superspace coordinate differentials $dX^M = (dx^m, d\theta^{\alpha})$, the supervielbein matrix $E^N_M$ is defined by

$$\omega^N = dX^M E^N_M.$$  

(167)

Then the covariant derivatives $\mathcal{D}_N$ can be obtained by

$$\mathcal{D}_N = (E^{-1})_M^N \partial_M.$$  

(168)

Namely,

$$\mathcal{D}_m = \partial_m,$$
$$\mathcal{D}_\alpha = \frac{\partial}{\partial \theta^{\alpha}} - i(\gamma^{m}_\alpha \theta) \partial_m.$$  

(169)  

(170)

By denoting the superspace generators as $\Gamma_M$ collectively, the Cartan one-form can be expressed as follows. (See Eq.(166).)

$$\Omega^{-1} d\Omega = i \omega^M \Gamma_M.$$  

(171)

Here $\Omega = e^{ix^M \Gamma_M}$. Then

$$d\Omega = i \omega^M \Omega \Gamma_M.$$  

(172)

Noticing $d = dX^M \partial_M = \omega^M \mathcal{D}_M$, Eq.(172) becomes

$$\omega^M \mathcal{D}_M \Omega = i \omega^M \Omega \Gamma_M,$$  

(173)

that is,

$$\mathcal{D}_M \Omega = \Omega(i \Gamma_M)$$  

(174)

Since $\Gamma_M = (P_m, -iQ^{(3)}_\alpha)$ in our case, it follows that

$$\mathcal{D}_\alpha \Omega = \Omega Q^{(3)}_\alpha.$$  

(175)

29
C Orthogonality of the mode functions

In this appendix, we will prove the orthogonality of the mode functions Eq.(81). Here we assume that the mass eigenvalues of the mode equation Eq.(79) are real.

We can easily show that
\[ \int dx_2 (\mathcal{D}_y \bar{u}_{(n)i}) u_{(l)i} = - \int dx_2 \bar{u}_{(n)i} \mathcal{D}_y u_{(l)i}. \] (176)

Thus,
\[ m_{(n)} \int dx_2 \bar{u}_{(n)i} u_{(l)i} = \int dx_2 i \left\{ \mathcal{D}_y \bar{u}_{(n)i} - e^{-i\delta} \mathcal{D}_i W^j u_{(n)}^j \right\} u_{(l)i} \]
\[ = \int dx_2 \left\{ -i \bar{u}_{(n)i} \mathcal{D}_y u_{(l)i} - ie^{-i\delta} \mathcal{D}_i W^j u_{(n)}^j u_{(l)i} \right\}. \] (177)

Using the conjugate of Eq.(79),
\[ -i \left\{ \mathcal{D}_y u_{(l)i} - e^{i\delta} \mathcal{D}_i W^j u_{(l)i} \right\} = m_{(l)} u_{(l)i}, \] (178)
we can obtain
\[ (m_{(n)} - m_{(l)}) \int dx_2 \bar{u}_{(n)i} u_{(l)i} = -2i \text{Re} \left\{ \int dx_2 e^{-i\delta} \mathcal{D}_i W^j u_{(n)}^j u_{(l)i} \right\}. \] (179)

Therefore, in the case that \( m_{(n)} \neq m_{(l)} \), we can show the orthogonality by taking the real part of the above equation.
\[ \text{Re} \left\{ \int dx_2 \bar{u}_{(n)i} u_{(l)i} \right\} = 0. \] (180)

In the case that the eigenvalues \( m_{(n)} \) and \( m_{(l)} \) are degenerate, we can redefine the corresponding mode functions so that they are orthogonal to each other. In fact, if we redefine the mode function \( u_{(l)i}(x_2) \) as
\[ \bar{u}_{(l)i}(x_2) \equiv \frac{u_{(l)i}(x_2) - C_{nl} u_{(n)}^i(x_2)}{1 - |C_{nl}|^2}, \] (181)
where
\[ C_{nl} \equiv \int dx_2 \bar{u}_{(n)i} u_{(l)i}, \] (182)
the functions \( u_{(n)}^i(x_2) \) and \( \bar{u}_{(l)i}(x_2) \) are certainly orthogonal.

As a result, by taking into account the normalization of each mode functions, we can obtain the desired relation,
\[ \text{Re} \left\{ \int dx_2 \bar{u}_{(n)i} u_{(l)i} \right\} = \delta_{nl}. \] (183)

In the above proof, we have used the assumption that all of the mass eigenvalues \( m_{(n)} \) are real. In the case that the Kähler potential is minimal, we can show the reality of \( m_{(n)} \).

The mode equation Eq.(79) can be rewritten as
\[ i \left\{ \partial_2 \bar{u} - \Gamma^k_{ij} (A_{cl}) \partial_2 A_{cl}^j \bar{u}_k - e^{-i\delta} \left( W_{ij} (A_{cl}) - \Gamma^k_{ij} (A_{cl}) W_k (A_{cl}) \right) K^{ji} (A_{cl}) u_l \right\} = m \bar{u}_i, \] (184)
In the case of the minimal Kähler potential,
\[ \Gamma^k_{ij} = 0, \quad K^{j\bar{l}} = \delta^{j\bar{l}}. \] (185)

So, denoting
\[ \bar{u} \equiv \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \end{pmatrix} = u_R - iu_I, \quad (\alpha_R + i\alpha_I)_{ij} \equiv e^{-i\delta}W_{ij}(A_d)K^{j\bar{l}}(A_d), \] (186)

Eq.(184) can be simplified as
\[ i \{ \partial_2 \bar{u} - (\alpha_R + i\alpha_I)u \} = m\bar{u}. \] (187)

This can also be written as
\[ \begin{pmatrix} \alpha_I & \partial_2 + \alpha_R \\ -\partial_2 + \alpha_R & -\alpha_I \end{pmatrix} \begin{pmatrix} u_R \\ u_I \end{pmatrix} = m \begin{pmatrix} u_R \\ u_I \end{pmatrix}. \] (188)

Since \( \alpha_R \) and \( \alpha_I \) are real symmetric matrices, the differential operator
\[ \mathcal{O} = \begin{pmatrix} \alpha_I & \partial_2 + \alpha_R \\ -\partial_2 + \alpha_R & -\alpha_I \end{pmatrix} \] (189)

is hermitian. Therefore the eigenvalues of \( \mathcal{O} \), \( m_{(n)} \), are real.

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31
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