Higher order symmetries for linear and nonlinear Schrödinger equations

A. G. Nikitin

Institute of Mathematics, National Academy of Sciences of Ukraine,
3 Tereshchenkivs'ka Street, Kyiv-4, Ukraine, 01601

Abstract

We study arbitrary order symmetry operators for the linear Schrödinger equations with arbitrary number of spatial variables. We deduce determining equations for coefficient functions of such operators and consider in detail some cases when these equations can be explicitly solved. In addition, the complete group classification of the nonlinear Schrödinger equation is presented.

1E-mail: nikitin@imath.kiev.ua
1 Introduction

Higher order symmetry operators present powerful tools for investigations of partial differential equations. They can be used to construct constants of motion for evolution equations, to search for coordinate systems in which solutions with separated variables exist, and also for investigations of superintegrable systems whose existence was predicted by Arnold Sommerfeld in 1923 [1] while the systematical inquiry into the superintegrability problem was started by P. Winternitz and co-authors [2].

We remind that a nice kind of symmetries called supersymmetry is also connected with existence of higher order symmetry operators.

Investigations of symmetries of the Schrödinger equation (SE) started with papers of Niederer and Boyer [3] who presented group classification of linear SEs with potential terms. Such classification is connected with description of first order symmetry operators for the related equation.

In fundamental paper [4] the complete description of second order symmetry operators for the 1+2 dimensional SE was presented. Applications of these operators to the separation of variables are discussed in the Miller book [5].

The problem of description of arbitrary order symmetry operators for the Schrödinger equation was formulated in papers [6] and [7]. In paper [8] the complete set of such operators was found for the case of trivial potential. In [9] the third order symmetry operators for 1+1 dimensional SE with arbitrary potential were described and used to construct exact solutions. Moreover, a rather nontrivial connections between the existence of the third order symmetry operators for the linear SE and exact solutions of the related nonlinear equation was established in [9].

The very existence of many of exact solutions for the nonlinear SE can be understood starting with its symmetries with respect to continuous groups of transformations. The group analysis of the nonlinear SE was carried out by a number of investigators, refer, e.g., to [11] and references cited therein. However, in fact the related results were valid for ad hoc restricted classes of the nonlinear SEs. The completed group classification of this equation was presented in the recent paper [12], the related problem for systems of reaction-diffusion equations was solved in [13].

In the present paper we deduce the determining equations for coefficients of arbitrary order symmetry operators for the 1+m dimensional SE with arbitrary number \( m \) of spatial variables. We consider some cases when this system can be completely solved, and discuss the related symmetry operators and their possible applications. In addition, we present the complete group classification of the nonlinear SE.

2 Symmetry operators of arbitrary order

2.1 Determining equations

Consider the SE with an arbitrary number \( m \) of spatial variables

\[
L \Psi(t, x) \equiv \left( \frac{i}{\hbar} \frac{\partial}{\partial t} - \frac{p^2}{2M} - V(x) \right) \Psi(t, x) = 0. \tag{1}
\]

Here \(- p^2 = \Delta_m\) is the Laplace operator in space of \( m \) variables, \( X = (x_1, x_2, \ldots x_m) \), \( \Psi(t, x) \) is a complex wave function.
For simplicity we restrict ourselves to such solutions of (1) which are defined on an open set $D$ of the $m + 1$-dimensional manifold $R$ consisting of points with coordinates $(t, x_1, x_2, \ldots, x_m)$ and are analytic in variables $t, x_1, \ldots, x_m$. The set of such solutions forms a complex vector space $F_0$. Fixing $D$ (e.q., supposing that $D$ coincides with $R$) we come to space $F_0$ of solutions of the SE.

Let us denote by $F$ the vector space of complex valued functions defined and real-analytic on $D$ and by $L$ the linear operator (1) defined on $F$. Then $L \in \Psi$ is $\Psi \in F$ and $F_0$ is the subspace of $F$ which coincides with the zero-space of $L$.

Let $Q$ be a $n$-order differential operator defined on $L$:

$$Q_j = h^{a_1, a_2, \ldots, a_j} \frac{\partial^j}{\partial x_{a_1} \partial x_{a_2} \ldots \partial x_{a_j}}. \quad (2)$$

Here and in the following the summation over the repeating indices is imposed over the values $1, 2, \ldots, m$.

We say $Q$ is a symmetry operators for SE (1) provided

$$[L, Q] = \alpha Q L \quad (3)$$

where $[,]$ denotes commutator and $\alpha Q$ is a differential operator of order $n - 1$.

It is convenient to represent $Q$ as a sum of $j$-multiple anticommutators [9]

$$Q = \sum_{j=0}^{n} Q_j, \quad Q_j = \left[\cdots \left[K^{a_1 a_2 \ldots a_j} \frac{\partial}{\partial x_{a_1}} + \frac{\partial}{\partial x_{a_2}} \right] \cdots \frac{\partial}{\partial x_{a_j}} \right]_{+} \quad (4)$$

where $K^{a_1 a_2 \ldots a_j}$ are symmetric tensors depending on $t$ and $x$ and $[K^*, \frac{\partial}{\partial x_{a_j}}] = K^* \frac{\partial}{\partial x_{a_j}} + \frac{\partial}{\partial x_{a_j}} K^*$.

Commuting (4) with the Laplace operator we automatically come to a sum of $j + 1$ multiple anticommutators, and it is the reason why just this representation of $Q$ leads to the most simple system of determining equations.

It is possible *ad hoc* set in (3) $\alpha Q = 0$ in as much as the term $\frac{\partial}{\partial t}$ cannot appear in the l.h.s. of this equation. Substituting (4) into (3) and equating the coefficients for all powers of differential operators we come to the following system of determining equations

$$\begin{align*}
\partial^{(n+1)} & K^{a_1 a_2 \ldots a_n} = 0, \\
2K^{a_1 a_2 \ldots a_2} & + \frac{1}{M} \partial^{(a_2)} K^{a_1 a_2 \ldots a_{2+1}} \\
& + \sum_{k=s}^{[(n-1)/2]} (-1)^{s+k+1} \frac{2(2k+1)!}{(2k-2s+1)! (2s)!} U_k^{a_1 a_2 \ldots a_{2s}} = 0, \\
2K^{a_1 a_2 \ldots a_{2l+1}} & + \frac{1}{M} \partial^{(a_{2l+1})} K^{a_1 a_2 \ldots a_{2l}} \\
& + \sum_{k=l+1}^{[n/2]} (-1)^{k+l} \frac{2(2k)!}{(2k-2l+1)! (2l+1)!} W_k^{a_1 a_2 \ldots a_{2l+1}} = 0
\end{align*} \quad (5)$$

where the dot denotes the derivative with respect to $t$, i.e., $K^{\cdots} = \frac{\partial K^{\cdots}}{\partial t}$,

$$s = 0, 1, \ldots, [n/2], \quad l = 0, 1, \ldots, [(n - 1)/2],$$

$$U_k^{a_1 a_2 \ldots a_{2s}} = K^{a_1 a_2 \ldots a_2 b_1 b_2 \ldots b_{2s-2s+1} \partial_{b_1} \partial_{b_2} \ldots \partial_{b_{2s-2s+1}}},$$

$$W_k^{a_1 a_2 \ldots a_{2l+1}} = K^{a_1 a_2 \ldots a_{2l+1} b_1 b_2 \ldots b_{2l-2l+1} \partial_{b_1} \partial_{b_2} \ldots \partial_{b_{2l-2l+1}}}. \quad (6)$$
and the complete symmetrization is imposed over the indices in brackets.

By definition, the terms with negative values of index numbers are identically zeros, while the index with number zero is forbidden. For example, it is the case for $W_k^{a_1a_2...a_{2k+1}}$ if $k = l$, i.e., $W_k^{a_1a_2...a_{2k+1}} = 0$, (see (6)), and $K^{a_1a_2...a_s} = \hat{K}$ if $s = 0$.

The system (5) describes the coefficients of $n$-order symmetry operator for non-stationary Schrödinger equations. However, it is valid also for the stationary equations provided functions $K^{a_1a_2...a_j}$, $j = 1, 2, ..., n$ are time independent. It is interesting to note that in this case the system (5) is decoupled to two independent subsystems one of which includes tensors $K^{a_1a_2...a_j}$ with odd number of indices while the other is defined for even number of indices. Thus the $n$-order integral of motion can be represented as a sum of two independent operators (4) of orders $n$ and $n - 1$. The first of them includes terms with $j = n$, $n - 2$, $n - 4$, ... while for the second one we have expression (4) with $j = n - 1$, $n - 3$, $n - 5$, ..... We see that integrals of motions with even and odd $d$ are linearly independent and can be considered separately. For even $n$ the determining equations (5) are reduced to the following form:

$$\frac{\partial^{(a_{n+1})}K^{a_1a_2...a_n}}{\partial t} = 0, \quad n = 2r,$$

$$\frac{\partial^{(a_{2l+1})}K^{a_1a_2...a_{2l+1}}}{\partial t} + M \sum_{k=l+1}^{r} (-1)^{k-l} \frac{2(2k)!}{(2k-2l-1)!(2l+1)!} W_k^{a_1a_2...a_{2l+1}} = 0,$$  

$$l = 0, 1, 2, ..., r - 1$$

while for $n$ odd we have:

$$\frac{\partial^{(a_{n+1})}K^{a_1a_2...a_n}}{\partial t} = 0, \quad n = 2r + 1,$$

$$\frac{\partial^{(a_{2s})}K^{a_1a_2...a_{2s-1}}}{\partial t} + M \sum_{k=s}^{r} (-1)^{s+k+1} \frac{2(2k+1)!}{(2k-2s+1)!(2s)!} U_k^{a_1a_2...a_{2s}} = 0,$$  

$$s = 0, 1, 2, ..., r.$$ 

Let us stress that formulae (7) and (8) together with definitions (6) present explicitly the determining equations for potentials and the related coefficients of integrals of motion of arbitrary order $n$ and for arbitrary number $m$ of spatial variables. We believe that the direct use of these equations could be convenient in studies of higher (and even arbitrary) order integrals of motion for quantum mechanical systems, which are rather popular now, see survey [14] and paper [15]. Indeed, the derivation of the determining equations for higher order symmetry operators with fixed $n$ and $m$ looks as unnecessary waste of time and energy since this job has been already done.

2.2 Complete set of symmetry operators of arbitrary order

For the case of ad hoc fixed potential $V(t, x)$ we can try to find exact solutions of (5) for arbitrary $n$.

If $V(x) = 0$ than the system of equations (5) is significantly simplified and reduced to the following form

$$\frac{\partial^{(a_{j+1})}K^{a_1a_2...a_j}}{\partial t} = -2M \dot{K}^{a_1a_2...a_{j+1}}, \quad j = 0, 1, ..., n - 1,$$

$$\frac{\partial^{(a_{n+1})}K^{a_1a_2...a_n}}{\partial t} = 0,$$

$$\dot{K} = 0, \quad j = 0.$$
Equations \((9)\) can be solved explicitly for arbitrary \(n\). A differential consequence of \((9)\) is

\[ \partial^{(a_j+1} \partial^{a_{j+2} ...} \partial^{a_{j+s}} K^{a_1 a_2 ... a_j}) = 0, \quad s = n - j + 1. \]  

(10)

Solutions of \((10)\) are the generalized Killing tensors of order \(s\) and rank \(j\) whose explicit form is presented in \([10]\). The first of relations \((9)\) reduces to the first order ordinary equations for arbitrary parameters defining the generalized Killing tensors which are easily integrated. The number \(N_n\) of linearly independent \(n\)-order symmetries is given by the following formula

\[ N_n = \frac{1}{4!} (n + 1) (n + 2) (n + 3) \]

and all these symmetries belong to the enveloping algebra of the Lie algebra of the Schrödinger group \([8]\).

2.3 Third order symmetry operators for 1 + 1-dimensional Schrödinger equation

One more simplified version of the determining equations \((5)\) corresponds to a non-trivial potential in the important case of the only spatial variable. Then the first and second order symmetry operators reduce on the set of solutions of the related SE to generators of a Lie group.

Let us consider the case \(n = 3\), which corresponds to the simplest non-Lie symmetry, in more detail. For simplicity we set \(m = 1\) and \(V = \frac{U}{x}\). The related tensors \(K^{a_1 a_2 ... a_j}\) in \((4)\) reduce to scalars and so

\[ Q = \left[ \left[ h_3, \frac{\partial}{\partial x} \right] + \frac{\partial}{\partial x} \right] + \left[ \left[ h_2, \frac{\partial}{\partial x} \right] + \frac{\partial}{\partial x} \right] + \left[ h_1, \frac{\partial}{\partial x} \right] + h_0. \]  

(11)

The corresponding system \((5)\) reduces to

\[ \begin{align*}
    h'_3 &= 0, \\
    h'_2 + 2h_3 &= 0, \\
    2h_2 + h'_1 - 6h_3 U &= 0,
\end{align*} \]

(12)

\[ \begin{align*}
    2h_1 + h'_0 - 4h_2 U' &= 0, \\
    h_0 - h_1 U' + h_3 U''' &= 0,
\end{align*} \]

(13)

where the dots and primes denote derivatives w.r.t. \(t\) and \(x\) respectively.

Excluding \(h_0\) from \((13)\) and using \((12)\) we arrive at the following equation

\[ aU''' - (2\ddot{a}x^2 + 6aU + c - 2\ddot{b}x)U'' - 6(2\ddot{a}x + aU' - \ddot{b})U' - 12\dddot{a}U - 2(2\partial^2_{\alpha} ax^2 - 2 \dddot{b} x + c) = 0 \]

(14)

where \(a, b, c\) are arbitrary functions of \(t\). Equation \((14)\) is nothing but the compatibility condition for system \((12), (13)\). If the potential \(U\) satisfies \((14)\) then the corresponding coefficients of the \(SO\) have the form

\[ \begin{align*}
    h_3 &= a, \\
    h_2 &= -2\ddot{a}x + b, \\
    h_1 &= g_1 + 6aU, \\
    h_0 &= -\frac{1}{3} \dddot{a} x^3 + 2\dddot{b}x^2 - 2\dddot{c}x - 4a\dddot{\varphi} + 4(b - 2\ddot{a}x)U + d
\end{align*} \]

(15)
where
\[ g_1 = 2\ddot{x}x^2 - 2\dot{x}x + c, \quad \varphi = \int U dx, \quad u = \varphi', \quad d = d(t). \]

Separating variables in (14) we conclude that up to equivalence, this equation can be reduced to one of the following forms:

\[ U'' - 3U^2 + 3\omega_1 = 0, \quad (16) \]
\[ U'' - 3U^2 - 8\omega x = 0, \quad (17) \]
\[ (U'' - 3U^2)' - 2\omega_3 (xU' + 2U) = 0, \quad (18) \]
\[ \varphi'' - 3(\varphi')^2 - 2\omega_4 (x^2 \varphi)' = \frac{1}{4}\omega_4^2 x^4 + \omega_5, \quad U = \varphi' \quad (19) \]

where \( \omega_1, \ldots, \omega_5 \) are arbitrary constants.

From (11), (16)-(19) we find the corresponding symmetry operators

\[ Q = p^3 + \frac{3}{4}\{U, p\} \equiv 2pH + \frac{1}{2}Up + \frac{1}{4}U', \quad (20) \]
\[ Q = p^3 + \frac{3}{4}\{U, p\} - \omega_2 t, \quad (21) \]
\[ Q = p^3 + \frac{3}{4}\{U, p\} + \omega_3 \left(tH - \frac{1}{4}\{x, p\}\right), \quad (22) \]
\[ Q_{\pm} = \frac{1}{\sqrt{24}} \left[p^3 \pm \frac{3}{4}\omega\{\{x, p\}, p\} + \frac{1}{4}\{3\varphi' - \omega^2 x^2, p\} \pm \frac{1}{4}\omega \left(\varphi + 2x\varphi' - \frac{\omega^2}{6} x^3\right)\right] \exp(\pm i\omega t), \quad \omega = \sqrt{-\omega_4} \quad (23) \]

where \( U \) are solutions of (16)-(19) and \( H = \frac{1}{2}\left(-\frac{\partial}{\partial x} + U(x)\right) \).

Thus, the Schrödinger equation (2.1) admits a third-order SO if potential \( U \) satisfies one of the equations (16)-(19). The explicit form of the corresponding SOs is present in (20)-(23).

Formula (16) presents the Weierstrass equation while relation (17) defines the first Painlevé transcendent. Using generalized Miura ansatz equations (18) and (19) can be reduced to the second and forth Painlevé transcendents respectively or to special Riccati equations [9].

Operators (20)-(23) together with Hamiltonian \( H \) form rather interesting algebras whose analysis lies out of frames of the present paper.

3 Group classification of the nonlinear SE

It was shown in [9] that if the linear one dimensional SE (11) admits the third order symmetry operator (19) than there exists a wide class of solutions for (19) which solve also the nonlinear SE with cubic nonlinearity. Thus there exist rather non-trivial connections between the third order symmetries for the linear SE and symmetries of the nonlinear SE which cause the existence of exact solutions.
Here we present a complete description of classical Lie symmetries of the non-linear Schrödinger equation
\[
\left( \frac{\partial}{\partial t} + \Delta \right) \Psi + F(\Psi, \Psi^*) = 0 \tag{24}
\]
where \( \Psi = \Psi(t, x), x \in \mathbb{R}^m \), \( F \) is an arbitrary function of two variables: \( \Psi \) and complex conjugated function \( \Psi^* \).

Classical Lie symmetries of some special classes of equations \((24)\) were investigated in numerous papers, refer, e.g., to \([11]\) and references cited therein. Here we present the results of complete group classification for all nonequivalent nonlinearities \( F \), based on results of papers \([12]\) and \([13]\).

To describe symmetries of the SE with non-fixed nonlinearity in the l.h.s. we can use either the classical Lie algorithm or its specific simplified version \([13]\) which presupposes solution of the following operator equation (compare with \((1)\))
\[
[L, Q] = \alpha Q L + \phi(t, x)
\]
where \( Q \) is a first order differential operator, \( \phi \) is unknown function. We will not present here details of calculations but present the final result of group classification of \((24)\).

For arbitrary \( F \) equation \((24)\) is invariant with respect the Euclid group \( E(1, m) \) and so admits \( 1 + \frac{m(m+1)}{2} \) symmetry operators of first order
\[
P_0 = \frac{\partial}{\partial t}, \ P_a = \frac{\partial}{\partial x_a}, \ J_{ab} = x_a p_b - x_b p_a, \ a, b = 1, 2, \ldots, m.
\tag{25}
\]

All cases when the basic invariance group \( E(1, m) \) can be extended are enumerated in the following tables.

| Table 1. Nonlinearities dependent on arbitrary function \( f(\Omega) \). | \( F \) | \( \Omega \) | Additional symmetries |
|---|---|---|---|
| 1.1 \( f(\Omega) | \psi^\gamma e^{\gamma_2 \varphi} \psi, \ \gamma_1^2 + \gamma_2^2 \neq 0 \) | \( |\psi^\gamma e^{-\gamma_2 \varphi} \) | \( (\gamma_1^2 + \gamma_2^2) D - \gamma_1 I - \gamma_2 M \) |
| 1.2 \( f(\Omega) + (\gamma - i) \delta \ln |\psi| \) | \( |\psi| e^{-\varphi} \) | \( e^{\delta t} (I + \gamma M) \) |
| 1.3 \( f(\Omega) + \delta \varphi \psi, \ \delta \neq 0 \) | \( |\psi| \) | \( e^{\delta t} M, \ e^{\delta t} (\partial_a + \frac{1}{2} \delta x_a M) \) |
| 1.4 \( f(\Omega) \psi \) | \( |\psi| \) | \( M, \ G_a \) |
| 1.5 \( f(\Omega) e^{i \psi} \) | \( \Re \psi \) | \( D + i(\partial_\psi - \partial_{\psi^*}) \) |
| 1.6 \( f(\Omega) + i(\delta_1 + i\delta_2) \psi \) | \( \Re \psi \) | \( i e^{-\delta_1 \psi} (\partial_\psi - \partial_{\psi^*}) \) |

Here \( \gamma, \gamma_1, \gamma_2, \delta, \delta_1, \delta_2 \) are real numbers, \( \theta = \theta(x) \in \mathbb{R} \) is a solution of equation \( \Delta \theta = \delta_2 \theta \). We present our results in terms of amplitude \( \rho = |\psi| \) and phase \( \varphi = \frac{i}{2} \ln \frac{\psi^*}{\psi} \) of function \( \Psi \) and
use the following notations
\[ I := \Psi \partial \psi + \Psi^* \partial \psi^* = \rho \partial \rho, \quad M := i(\Psi \partial \psi - \Psi^* \partial \psi^*) = \partial \varphi, \]
\[ D := t \partial_t + \frac{1}{2} x_a \partial_a, \quad G_a := t \partial_a + \frac{1}{2} x_a M, \]
\[ \Pi := t^2 \partial_t + t x_a \partial_a - \frac{n}{2} t I + \frac{1}{4} x_a x_a M. \]

In the following page we present non-linearities which are defined up to arbitrary parameters.

**Table 2.** Nonlinearities depending on arbitrary parameters.

| $F$ | Additional symmetries |
|-----|------------------------|
| 2.1 | $0$ | $G_a, I, M, D, \Pi, \eta^0 \partial \psi + \eta^0 \partial \psi^*$ |
| 2.2 | $\gamma \psi + \psi^*$ | $I, \eta^0 \partial \psi + \eta^0 \partial \psi^*$ |
| 2.3 | $\sigma |\mathcal{R} \psi|^\gamma, \gamma \neq 0, 1$ | $I + (1 - \gamma) D, i \theta(x) (\partial \psi - \partial \psi^*)$ |
| 2.4 | $\sigma \ln |\mathcal{R} \psi|$ | $I + D - i(t |\mathcal{R} \sigma + \frac{1}{2\eta} x_a x_a \delta \sigma| (\partial \psi - \partial \psi^*), i \theta(x) (\partial \psi - \partial \psi^*)$ |
| 2.5 | $\sigma e^{\mathcal{R} \psi}$ | $D - \partial \psi - \partial \psi^*, i \theta(x) (\partial \psi - \partial \psi^*)$ |
| 2.6 | $\sigma |\psi|^{\gamma_1} e^{\gamma_2 \varphi}, \gamma_2 \neq 0$ | $M - \gamma_2 D, \gamma_2 I - \gamma_1 M$ |
| 2.7 | $\sigma |\psi|^{\gamma_3} \psi, \gamma_3 \neq 0, \frac{4}{n}$ | $G_a, M, I - \gamma D$ |
| 2.8 | $\sigma |\psi|^{1/n} \psi,$ | $G_a, M, I - \frac{4}{n} D, \Pi$ |

In the following $F = (-\delta_1 + i \delta_2) \ln |\psi| + (\delta_3 - i \delta_4) \varphi) \psi, \Delta = (\delta_2 - \delta_3)^2 - 4 \delta_1 \delta_4$

| 2.9 | $\delta_4 = 0, \delta_3 \neq 0, \delta_2 \neq \delta_3$ | $e^{\delta_1 t} M, e^{\delta_1 t} (\partial_a + \frac{1}{2} \delta_3 x_a M), e^{\delta_1 t} (I - \frac{\delta_1}{\delta_2 - \delta_3} - M)$ |
| 2.10 | $\delta_4 = 0, \delta_3 \neq 0, \delta_2 = \delta_3$ | $e^{\delta_1 t} M, e^{\delta_1 t} (\partial_a + \frac{1}{2} \delta_3 x_a M), e^{\delta_1 t} (I - \delta_1 t M)$ |
| 2.11 | $\delta_4 = 0, \delta_3 = 0, \delta_2 \neq 0$ | $M, G_a, e^{\delta_1 t} (\delta_2 I - \delta_1 M)$ |
| 2.12 | $\delta_4 = 0, \delta_3 = 0, \delta_2 = 0, \delta_1 \neq 0$ | $M, G_a, I - \delta_1 t M$ |
| 2.13 | $\delta_4 \neq 0, \Delta > 0$ | $e^{\lambda_i t} (\delta_4 I + (\lambda_i - \delta_2) M), i = 1, 2, \lambda_1 = \frac{1}{2} (\delta_2 + \delta_3 - \sqrt{\Delta}), \lambda_2 = \frac{1}{2} (\delta_2 + \delta_3 + \sqrt{\Delta})$ |
| 2.14 | $\delta_4 \neq 0, \Delta < 0$ | $e^{\mu t} (\delta_4 \cos \nu t I + ((\mu - \delta_2) \cos \nu t \sin \nu t) M), e^{\mu t} (\delta_4 \sin \nu t I + ((\mu - \delta_2) \sin \nu t \sin \nu t) M), \mu = \frac{1}{2} (\delta_2 + \delta_3), \nu = \frac{1}{2} \sqrt{-\Delta}$ |
| 2.15 | $\delta_4 \neq 0, \Delta = 0$ | $e^{\mu t} (\delta_4 I + \frac{1}{2} (\delta_2 - \delta_3) t M + M), e^{\mu t} (\delta_4 I + \frac{1}{2} (\delta_3 - \delta_2) M), \mu = \frac{1}{2} (\delta_2 + \delta_3)$ |

Here $\gamma, \gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3, \delta_4$ are real numbers, $\sigma \neq 0; \eta^0$ is an arbitrary solution of the initial equation, $\theta = \theta(x) \in R$ is a solution of the Laplace equation $\Delta \theta = 0$.  

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Thus we present the results of group classification of the nonlinear Shrödinger equation. We will not discuss possible applications of the obtained results which consist in description all possible models with non-trivial extension of the basic invariance with respect to the group $E(1,n)$.

We stress that our classification is valid for nonlinear SE with arbitrary number of spatial variables. Rather surprisingly, increasing of the number of these variables do not lead to increasing of calculations difficulties.

4 Note added to this preprint version

This preprint includes the contribution to the CRM Proceedings [16] which is slightly corrected and added by the text placed in the first half of Page 3. The reasons of its publication is caused by the current interest in higher and arbitrary order integrals of motion, see, e.g., refs. [14] and [15]. I believe that the determining equations for such integrals, valid for arbitrary number of independent variables, could be interesting for researcher working in this field. Just these equations were presented in [16] for more general case of non-stationary quantum mechanical systems. And the additional text on Page 3 presents these equations reduced to the case of stationary systems.

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