Geometry of principal stress trajectories for a Tresca material under axial symmetry

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Abstract. In the case of Tresca’s yield criterion it is usually convenient to subdivide all possible deformations into two subclasses: deformations corresponding to a face of the yield surface and deformation corresponding to an edge of the yield surface. The present paper deals with the second subclass of deformations under conditions of axial symmetry. In this case the stress equations comprising the equilibrium equation and yield criterion can be investigated independently of any flow rule and the final result is also independent of whether elastic strains are included. It is shown that it is always possible to introduce such a principal line coordinate system (i.e. the coordinate system whose coordinate curves coincide with principal stress trajectories) that the product of the scale factors is equal to unity. Using this result the system of equations for mapping between the principal line coordinate and cylindrical coordinate systems is derived and it is shown that this system of equations is hyperbolic.

1. Introduction
A remarkable property of principal stress trajectories in plane strain plasticity of materials obeying the Tresca yield criterion is that it is always possible to choose such a principal line coordinate system (i.e. the coordinate system whose coordinate curves coincide with principal stress trajectories) that the product of the scale factors is equal to unity [1]. This result has been extended to materials obeying the Mohr-Coulomb yield criterion in [2]. In this case the product of one of the scale factors and a power of the other scale factor is equal to unity. A similar result has been found for both Tresca and Mohr-Coulomb yield criteria under plane stress in [3]. These properties of the scale factors can be used for developing an efficient method of constructing principal stress nets, which is equivalent to finding stress solutions. The practical usefulness of principal line coordinates has also been noted in [4]. In the present paper, a geometric property of principal stress trajectories similar those reported in [1-3] is proven for the Tresca yield criterion under axial symmetry. Edge regimes are considered. Therefore, the system of equations comprises the two equilibrium equations and two equations that represent the yield criterion for four non-zero stress components. The results presented here for this system are consequently independent of any flow rule that may be chosen to calculate the deformation and also independent of whether elastic strains are included. Other general properties of the system of equations under consideration have been studied in [5, 6].
2. Geometry of principal stress trajectories

Let \((r, \theta, z)\) be a cylindrical coordinate system and \((\xi, \eta)\) be a principal stress coordinate system in \(rz\) - planes. The states of stress considered in the present paper are independent of \(\theta\). The angle between the \(\xi\) - lines and the \(r\) - axis measured from the axis anticlockwise is denoted by \(\psi\) (figure 1).

![Figure 1. Cylindrical \((r, z)\) and principal line \((\xi, \eta)\) coordinate systems.](image)

The normal stresses in the \((\xi, \theta, \eta)\) coordinate system are the principal stresses. These stresses are denoted as \(\sigma_\xi\), \(\sigma_\theta\), and \(\sigma_\eta\). Without a loss of generality, it is assumed that \(\sigma_\xi \geq \sigma_\eta\). The yield locus corresponding to the Tresca yield criterion is shown in figure 2. Since \(\sigma_\xi \geq \sigma_\eta\), the two edge regimes of practical importance are those denoted by \(A\) and \(B\). The yield criterion is represented as

\[
\sigma_\xi - \sigma_\eta = 2k \quad \text{and} \quad \sigma_\theta - \sigma_\eta = 2k
\]

in regime \(A\) and

\[
\sigma_\xi - \sigma_\theta = 2k \quad \text{and} \quad \sigma_\eta - \sigma_\theta = 2k
\]

in regime \(B\). In equation (1) and (2), \(k\) is the shear yield stress, a material constant. The equilibrium equations in the \((\xi, \theta, \eta)\) coordinate system can be written as [7]

\[
\frac{\partial (r h_\eta \sigma_\xi)}{\partial \xi} - \sigma_\xi \frac{\partial h_\eta}{\partial \xi} - \sigma_\eta r \frac{\partial h_\eta}{\partial \xi} = 0, \quad \frac{\partial (r h_\xi \sigma_\eta)}{\partial \eta} - \sigma_\xi r \frac{\partial h_\xi}{\partial \eta} - \sigma_\eta h_\xi \frac{\partial r}{\partial \xi} = 0.
\]

Here \(h_\xi\) and \(h_\eta\) are the scale factors of the \(\xi\) - and \(\eta\) - lines, respectively.
Figure 2. Tresca’s yield locus.

Equation (3) can be transformed to

\[
\left( \sigma_z - \sigma_\theta \right) \frac{\partial \ln r}{\partial \xi} + \left( \sigma_z - \sigma_\eta \right) \frac{\partial \ln h_\eta}{\partial \xi} + \frac{\partial \sigma_z}{\partial \xi} = 0, \\
\left( \sigma_\eta - \sigma_\theta \right) \frac{\partial \ln r}{\partial \eta} - \left( \sigma_z - \sigma_\eta \right) \frac{\partial \ln h_\xi}{\partial \eta} + \frac{\partial \sigma_\eta}{\partial \eta} = 0. 
\]

(4)

Consider the regime of flow corresponding to point A (figure 2). It follows from (1) that \( \sigma_z = \sigma_\theta \).

Substituting this equation and (1) into (4) leads to

\[
2k \frac{\partial \ln h_\eta}{\partial \xi} + \frac{\partial \sigma_z}{\partial \xi} = 0, \\
2k \frac{\partial \ln r}{\partial \eta} + 2k \frac{\partial \ln h_\xi}{\partial \eta} - \frac{\partial \sigma_\eta}{\partial \eta} = 0. 
\]

(5)

Each of these equations can be immediately integrated to give

\[
2k \ln h_\eta + \sigma_z = 2k + 2k \ln \left[ C_1(\eta) \right], \\
2k \ln \left( r h_\xi \right) - \sigma_\eta = 2k \ln \left[ C_2(\xi) \right]. 
\]

(6)

Here \( C_1(\eta) \) is an arbitrary function of \( \eta \) and \( C_2(\xi) \) is an arbitrary function of \( \xi \). Summing the equations in (6) and eliminating \( \sigma_z - \sigma_\eta \) by means of the first equation in (1) results in

\[
rh_\xi h_\eta = C_1(\eta) C_2(\xi). 
\]

(7)

Different choices of the functions \( C_1(\eta) \) and \( C_2(\xi) \) merely change the scale of the \( \eta - \) and \( \xi - \) curves, respectively. Therefore, without loss of generality it is possible to choose \( C_1(\eta) = C_2(\xi) = 1 \).

Then, equation (7) becomes

\[
rh_\xi h_\eta = 1 
\]

(8)

and equation (6) becomes

\[
\sigma_z = 2k - 2k \ln h_\eta, \\
\sigma_\eta = 2k \ln \left( r h_\xi \right). 
\]

(9)
Consider the regime of flow corresponding to point B (figure 2). It follows from (2) that \( \sigma_{\eta} = \sigma_{\dot{\eta}} \). Substituting this equation and (2) into (4) leads to
\[
2k \frac{\partial \ln r}{\partial \xi} + 2k \frac{\partial \ln h_{z}}{\partial \xi} + \frac{\partial \sigma_{\eta}}{\partial \xi} = 0, \quad 2k \frac{\partial \ln h_{z}}{\partial \eta} - \frac{\partial \sigma_{\eta}}{\partial \eta} = 0. \quad (10)
\]
Each of these equations can be immediately integrated to give
\[
2k \ln \left( r h_{z} \right) + \sigma_{\eta} = 2k + 2k \ln \left[ C_{3}(\eta) \right], \quad 2k \ln h_{z} - \sigma_{\eta} = 2k \ln \left[ C_{4}(\xi) \right]. \quad (11)
\]
Here \( C_{3}(\eta) \) is an arbitrary function of \( \eta \) and \( C_{4}(\xi) \) is an arbitrary function of \( \xi \). Summing the equations in (11) and eliminating \( \sigma_{\eta} \) by means of the first equation in (2) results in
\[
r h_{z} = C_{3}(\eta) C_{4}(\xi). \quad (12)
\]
Different choices of the functions \( C_{3}(\eta) \) and \( C_{4}(\xi) \) merely change the scale of the \( \eta \) – and \( \xi \) – curves, respectively. Therefore, without loss of generality it is possible to choose \( C_{3}(\eta) = C_{4}(\xi) = 1 \). Then, equation (12) transforms to (8). Thus, the latter is valid for both A and B regimes (figure 2). Since \( C_{3}(\eta) = C_{4}(\xi) = 1 \), equation (11) becomes
\[
\sigma_{\eta} = 2k - 2k \ln \left( r h_{z} \right), \quad \sigma_{\eta} = 2k \ln h_{z}. \quad (13)
\]
3. Mapping between the cylindrical and principal stress coordinate systems

It follows from the geometry of figure 1 that
\[
\frac{\partial r}{\partial \xi} = h_{z} \cos \psi, \quad \frac{\partial r}{\partial \eta} = -h_{\eta} \sin \psi, \quad \frac{\partial z}{\partial \xi} = h_{z} \sin \psi, \quad \frac{\partial z}{\partial \eta} = h_{\eta} \cos \psi. \quad (14)
\]
Using (8) these equations can be rewritten as
\[
\frac{\partial r}{\partial \xi} = h \cos \psi, \quad \frac{\partial r}{\partial \eta} = -\sin \psi, \quad \frac{\partial z}{\partial \xi} = h \sin \psi, \quad \frac{\partial z}{\partial \eta} = \cos \psi. \quad (15)
\]
Here \( h \equiv h_{z} \). The compatibility equations are
\[
\frac{\partial^{2} r}{\partial \eta^{2} \xi} = \frac{\partial^{2} r}{\partial \xi^{2} \eta}, \quad \frac{\partial^{2} z}{\partial \eta^{2} \xi} = \frac{\partial^{2} z}{\partial \xi^{2} \eta}. \quad (16)
\]
Substituting (15) into (16) gives
\[
\cos \psi \frac{\partial h}{\partial \eta} - h \sin \psi \frac{\partial \psi}{\partial \eta} + \cos \psi \frac{\partial \psi}{\partial \xi} - \sin \psi \frac{\partial h}{\partial \xi} - \frac{\partial \psi}{\partial \eta} - \frac{\partial \psi}{\partial \xi} - \frac{\partial r}{\partial \eta} = 0, \quad (17)
\]
Eliminating in these equations the derivative \( \partial r/\partial \xi \) by means of (15) yields
Using a standard technique it is possible to show that this system of equations is hyperbolic. The equations of characteristics are

\[ \begin{align*}
\cos \psi \frac{dh}{d\eta} - h \sin \psi \frac{d\psi}{d\eta} + \frac{\cos \psi \frac{d\psi}{d\xi}}{rh} &= 0, \\
\sin \psi \frac{dh}{d\eta} + h \cos \psi \frac{d\psi}{d\eta} + \frac{\sin \psi \frac{d\psi}{d\xi}}{rh} &= 0.
\end{align*} \tag{18} \]

The first equation determines \( \alpha \) – characteristic lines and the second equation \( \beta \) – characteristic lines. The principal line, characteristic and cylindrical coordinate systems are illustrated in figure 3.

**Figure 3.** Cylindrical \((r, z)\), principal line \((\xi, \eta)\) and characteristic \((\alpha, \beta)\) coordinate systems.

It is evident from (8) and (19) that the characteristic curves are orthogonal. In particular, the orientation of the \( \alpha \) – lines relative to the \( r \) – axis is \( \psi = \pi/4 \) and the orientation of the \( \beta \) – lines relative to the \( r \) – axis is \( \psi + \pi/4 \). The characteristic relations are

\[ \begin{align*}
\frac{d\xi}{d\eta} &= -\frac{d\eta}{rh^2} \quad \text{and} \quad \frac{d\xi}{d\eta} = \frac{d\eta}{rh^2}. \tag{19}
\end{align*} \]

This system of equations should be solved numerically in conjunction with (15) and (19). Numerical methods for solving this type of equations have been developed, for example, in [8]. Once the system of equations (20) has been solved, the principal stresses are found from (1) and (9) or (2) and (13). In particular, using (8)

\[ \begin{align*}
\sigma_z &= \sigma_\theta = 2k + 2k \ln \left( rh \right), \quad \sigma_\eta = 2k \ln \left( rh \right) \tag{21}
\end{align*} \]

if regime \( A \) is operative and

\[ \begin{align*}
\sigma_z &= 2k + 2k \ln h, \quad \sigma_\eta = \sigma_\theta = 2k \ln h. \tag{22}
\end{align*} \]
if regime $B$ is operative. The components of the stress tensor in the cylindrical coordinate system can be found by means of the standard transformation equations for stress components since the principal stresses in $rz$–planes and the angle $\psi$ have been determined.

4. Conclusions
The system of equations comprising the equilibrium equations and the equations corresponding to regimes $A$ and $B$ (figure 2) of the Tresca yield criterion have been considered under conditions of axial symmetry. It has been shown that it is always possible to choose such a principal line coordinate system that its scale factors satisfy (8). This result generalises the results derived in [1-3] under plane strain and plane stress conditions. Using (8) the hyperbolic system of equations (16) has been derived for finding mapping between the cylindrical and principal line coordinate systems. Numerical methods for solving this type of systems are available in the literature [8].

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