DERIVATION RELATION FOR FINITE MULTIPLE ZETA VALUES IN $\hat{\mathcal{A}}$

HIDEKI MURAHARA AND TOMOKAZU ONOZUKA

Abstract. Ihara, Kaneko, and Zagier proved the derivation relation for multiple zeta values. The first named author obtained its counterpart for finite multiple zeta values in $\mathcal{A}$. In this paper, we present its generalization in $\hat{\mathcal{A}}$.

1. Introduction

For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$ with $k_r \geq 2$, the multiple zeta values (MZVs) is defined by

$$\zeta(k_1, \ldots, k_r) = \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

For $n \in \mathbb{Z}_{\geq 1}$, we define the $\mathbb{Q}$-algebra $\mathcal{A}_n$ by

$$\mathcal{A}_n := \left( \bigotimes_p \mathbb{Z}/p^n\mathbb{Z} \right) / \left( \bigoplus_p \mathbb{Z}/p^n\mathbb{Z} \right) = \{ (a_p)_p \mid a_p \in \mathbb{Z}/p^n\mathbb{Z} \} / \sim,$$

where $(a_p)_p \sim (b_p)_p$ are identified if and only if $a_p = b_p$ for all but finitely many primes $p$. For $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 1}$ and $n \in \mathbb{Z}_{\geq 1}$, the finite multiple zeta values (FMZVs) in $\mathcal{A}_n$ is defined by

$$\zeta_{\mathcal{A}_n}(k_1, \ldots, k_r) := \left( \sum_{1 \leq n_1 < \cdots < n_r \leq p-1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \mod p^n \right)_p \in \mathcal{A}_n.$$

Recently, Rosen \cite{Rosen} introduced the $\mathbb{Q}$-algebra $\hat{\mathcal{A}}$. By natural projections $\mathcal{A}_n \to \mathcal{A}_{n-1}$, we define $\hat{\mathcal{A}} := \lim_{\leftarrow} \mathcal{A}_n$, where we put the discrete topology on each $\mathcal{A}_n$. We also define the natural projections $\pi : \prod_p \mathbb{Z}_p \to \hat{\mathcal{A}}$ and $\pi_n : \hat{\mathcal{A}} \to \mathcal{A}_n$ for each $n$, where $\mathbb{Z}_p$ is the ring of $p$-adic integers. Then FMZVs in $\hat{\mathcal{A}}$ is given by

$$\zeta_{\hat{\mathcal{A}}}(k_1, \ldots, k_r) := \pi \left( \sum_{1 \leq n_1 < \cdots < n_r \leq p-1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \right)_p \in \hat{\mathcal{A}}.$$

We can easily check that $\pi_n(\zeta_{\hat{\mathcal{A}}}(k)) = \zeta_{\mathcal{A}_n}(k)$ for $k \in \mathbb{Z}_{\geq 1}$. Furthermore, we define $p := \pi((p)_p) \in \hat{\mathcal{A}}$ and we also use the notation $p := \pi_n \circ \pi((p)_p) \in \mathcal{A}_n$ (for details, see Rosen \cite{Rosen} and Seki \cite{Seki}).

2010 Mathematics Subject Classification. Primary 11M32.

Key words and phrases. Multiple zeta values, Finite multiple zeta values, Derivation relation.
We recall Hoffman’s algebraic setup with a slightly different convention (see Hoffman [2]). Let $\mathfrak{H} := \mathbb{Q}(x, y)$ be the noncommutative polynomial ring in two indeterminates $x$, $y$, and $\mathfrak{H}^1$ (resp. $\mathfrak{H}^0$) its subring $\mathbb{Q} + y\mathfrak{H}$ (resp. $\mathbb{Q} + y\mathfrak{H} x$). Set $z_k := yx^{k-1}$ ($k \in \mathbb{Z}_{\geq 1}$). We define the $\mathbb{Q}$-linear map $Z : \mathfrak{H}^0 \to \mathbb{R}$ by $Z(1) := 1$, $Z(z_1 \cdots z_n) := \zeta(k_1, \ldots, k_r)$.

A derivation $\partial$ on $\mathfrak{H}$ is a $\mathbb{Q}$-linear map $\partial : \mathfrak{H} \to \mathfrak{H}$ satisfying Leibniz’s rule $\partial(wv') = \partial(w)v + w\partial(v')$. Such a derivation is uniquely determined by its images of generators $x$ and $y$. Set $z := x + y$. For each $l \in \mathbb{Z}_{\geq 1}$, the derivation $\partial_l$ on $\mathfrak{H}$ is defined by $\partial_l(x) := yz^{l-1}x$ and $\partial_l(y) := -yz^{l-1}x$. We note that $\partial_l(1) = 0$ and $\partial_l(z) = 0$. In addition, $R_x$ is the $\mathbb{Q}$-linear map given by $R_x(w) := wx$ for any $w \in \mathfrak{H}$.

**Theorem 1.1 (Derivation relation for MZVs; Ihara-Kaneko-Zagier [5, Theorem 3]).** For $l \in \mathbb{Z}_{\geq 1}$ and $w \in \mathfrak{H}^0$, we have

$$Z(\partial_l(w)) = 0.$$  

Similar to the definition of $Z$, we define two $\mathbb{Q}$-linear maps $Z_{\mathfrak{H}_n} : \mathfrak{H}^1 \to \mathfrak{A}_n$ and $Z_{\hat{\mathfrak{A}}} : \mathfrak{H}^1 \to \hat{\mathfrak{A}}$ by $Z_{\mathfrak{H}_n}(1) = Z_{\hat{\mathfrak{A}}}(1) := 1$, $Z_{\mathfrak{H}_n}(z_1 \cdots z_n) := \zeta_{\mathfrak{H}_n}(k_1, \ldots, k_r)$, and $Z_{\hat{\mathfrak{A}}}(z_1 \cdots z_n) := \zeta_{\hat{\mathfrak{A}}}(k_1, \ldots, k_r)$. We write $\mathfrak{A} := \mathfrak{A}_1$. The derivation relation for FMZVs in $\mathfrak{A}$ was conjectured by Oyama and proved by the first named author.

**Theorem 1.2 (Derivation relation for FMZVs in $\mathfrak{A}$; Murahara [7, Theorem 2.1]).** For $l \in \mathbb{Z}_{\geq 1}$ and $w \in y\mathfrak{H} x$, we have

$$Z_{\mathfrak{A}}(R_x^{-1}\partial_l(w)) = 0$$  

in the ring $\mathfrak{A}$.

In this paper, we prove a generalization of the above theorem in the ring $\hat{\mathfrak{A}}$. For non-negative integers $m$ and $n$, we define $\beta_{m,n} : \mathbb{Q}(\langle x,y \rangle)[[u,v]] \to \mathfrak{H}$ by setting $\beta_{m,n}(w)$ to be the coefficients of $u^m v^n$ in $w$.

**Theorem 1.3 (Main theorem).** For $m \in \mathbb{Z}_{\geq 0}$ and $w \in y\mathfrak{H}$, we have

$$\sum_{n=0}^{\infty} Z_{\hat{\mathfrak{A}}} \left( \beta_{m,n} R_x^{-1} \Delta_u R_x \left( w - wyu \frac{1}{1+xu} - xv \frac{1}{1-xv} \right) \right) \frac{p^n}{1-p^n}$$

$$= Z_{\mathfrak{A}}(w) Z_{\hat{\mathfrak{A}}} \left( \beta_{m,0} \left( \frac{1}{1-yu} \right) \right)$$

in the ring $\hat{\mathfrak{A}}$, where $\Delta_u$ is an automorphism on $\mathbb{Q}(\langle x,y \rangle)[[u,v]]$ given by

$$\Delta_u := \exp \left( \sum_{l=1}^{\infty} \frac{l}{l!} (-u)^l \right).$$

**Remark 1.4.** Since $Z_{\mathfrak{A}}(1, \ldots, 1) = 0$ (see, for example, Hoffman [3 eq.(15)]), Theorem 1.3 is a generalization of the equality

$$Z_{\mathfrak{A}} \left( \beta_{m,0} R_x^{-1} \Delta_u R_x(w) \right) = 0$$
for \( m \in \mathbb{Z}_{\geq 0} \), which was obtained by Ihara (see Horikawa-Murahara-Oyama [4, Section 5.3]). We note that this is equivalent to Theorem 1.2.

As a corollary of our main theorem, we have Hoffman’s relation (see Hoffman [11, Theorem 5.1] for original formula) for FMZVs in \( \hat{A} \).

**Corollary 1.5.** For \( w \in y\mathcal{H} \), we have

\[
Z_{\hat{A}}(R_1^{-1} \partial_1(wx)) = -\sum_{n=1}^{\infty} Z_{\hat{A}}(wyx^n) p^n - Z_{\hat{A}}(w)Z_{\hat{A}}(y)
\]

in the ring \( \hat{A} \).

2. **Proof of the main theorem**

2.1. **Notation.** The harmonic product \( \star \) and the shuffle product \( \mathfrak{m} \) on \( \mathcal{H}_1 \) are defined by

\[
1 \star w = w \star 1 := w,
\]

\[
z_kw_1 \star z_lw_2 := z_k (w_1 \star z_lw_2) + z_l (z_kw_1 \star w_2) + z_{k+l} (w_1 \star w_2),
\]

\[
1 \mathfrak{m} w = w 1 \mathfrak{m} := w,
\]

\[
uw_1 \mathfrak{m} vw_2 := u (w_1 \mathfrak{m} vw_2) + v (uw_1 \mathfrak{m} w_2)
\]

\((k, l \in \mathbb{Z}_{\geq 1}, u, v \in \{x, y\} \) and \( w, w_1, w_2 \) are words in \( \mathcal{H}_1 \)), together with \( \mathbb{Q} \)-bilinearity. The harmonic product \( \star \) and the shuffle product \( \mathfrak{m} \) are commutative and associative, therefore \( \mathcal{H}_1 \) is a \( \mathbb{Q} \)-commutative algebra with respect to \( \star \) and \( \mathfrak{m} \), respectively (see Hoffman [2]).

2.2. **Propositions and lemmas.** In this subsection, we prepare some propositions which will be used later.

**Proposition 2.1.** For \( w_1, w_2 \in \mathcal{H}_1 \), we have

\[
Z_{\hat{A}}(w_1 \star w_2) = Z_{\hat{A}}(w_1)Z_{\hat{A}}(w_2)
\]

in the ring \( \hat{A} \).

**Proof.** This is obtained by the definition of the harmonic product. \( \square \)

**Proposition 2.2** (Jarossay [4], Seki [9, Theorem 6.4]). For \( w_1, w_2 \in \mathcal{H}_1 \) with \( w_2 = z_{k_1} \cdots z_{k_r} \), we have

\[
Z_{\hat{A}}(w_1 \mathfrak{m} w_2) = (-1)^{k_1 + \cdots + k_r} \sum_{l_1, \ldots, l_r \in \mathbb{Z}_{\geq 0}} \left[ \prod_{i=1}^{r} \left( \frac{k_i + l_i}{l_i} - 1 \right) \right] Z_{\hat{A}}(w_1 z_{k_1 + l_1} \cdots z_{k_r + l_r}) p^{l_1 + \cdots + l_r}
\]

in the ring \( \hat{A} \).
Proposition 2.3 (Ihara-Kaneko-Zagier [5, Corollary 3]). For \( w \in H^1 \), we have
\[
\frac{1}{1 - yu} * w = \frac{1}{1 - yu} \Delta_u(w).
\]

2.3. Proof of Theorem 1.3. By Proposition 2.1, we have
\[
Z_\hat{A} \left( \beta_{m,0} \left( \frac{1}{1 - yu} * w \right) \right) = Z_\hat{A} \left( \beta_{m,0} \left( \frac{1}{1 - yu} \right) \right) Z_\hat{A}(w).
\]
On the other hand, we have
\[
Z_\hat{A}(w \overline{w} y^r) = \sum_{n=0}^\infty Z_\hat{A} \left( \beta_{0,n} \left( w \left( -yu \frac{1}{1 - xv} \right)^r \right) \right) p^n
\]
holds by Proposition 2.2. Then we find
\[
Z_\hat{A} \left( \beta_{m,0} \left( \Delta_u(w) \overline{w} \Delta_u(w) \right) \right) = Z_\hat{A} \left( \beta_{m,0} \left( \Delta_u(w) \overline{w} \sum_{i=0}^{\infty} y^i u^i \right) \right)
\]
\[
= \sum_{n=0}^\infty Z_\hat{A} \left( \beta_{m,n} \left( \sum_{i=0}^{\infty} \Delta_u(w) \left( -yu \frac{1}{1 - xv} \right)^i \right) \right) p^n
\]
\[
= \sum_{n=0}^\infty Z_\hat{A} \left( \beta_{m,n} \left( \Delta_u(w) \frac{1}{1 + yu(1 - xv)^{-1}} \right) \right) p^n.
\]
From the direct calculation, we have
\[
\{1 + yu(1 - xv)^{-1}\}^{-1}
\]
\[
= \{(1 - xv + yu)(1 - xv)^{-1}\}^{-1}
\]
\[
= (1 - xv)(1 - xv + yu)^{-1}
\]
\[
= (1 - xv)\{(1 + yu)(1 - (1 + yu)^{-1}xv)\}^{-1}
\]
\[
= (1 - xv)(1 - (1 + yu)^{-1}xv)^{-1}(1 + yu)^{-1}
\]
\[
= (1 - xv) \sum_{i=0}^{\infty} \left( \frac{1}{1 + yu} \right)^i \left( 1 + yu \right)^{-1}.
\]
Since \( \Delta_u(x) = (1 + yu)^{-1}x \), we have
\[
\{1 + yu(1 - xv)^{-1}\}^{-1}
\]
\[
= (1 - xv) \sum_{i=0}^{\infty} \Delta_u(x) v^i \left( 1 + yu \right)^{-1}
\]
\[
= (1 + yu)^{-1} + \sum_{i=1}^{\infty} \left( \Delta_u(x) - x \right) \left( \Delta_u(x) \right)^{i-1} v^i \left( 1 + yu \right)^{-1}.
\]
Since \( x = \Delta_u(x + y(1 + xu)^{-1}xu) \), we have
\[
\{1 + yu(1 - xv)^{-1}\}^{-1} = (1 + yu)^{-1} - \sum_{i=1}^{\infty} \Delta_u \left( \frac{xu}{1 + xu} \right) \Delta_u \left( x^{i-1} \right) v^{i}(1 + yu)^{-1}
\]
\[= R_x^{-1} \Delta_u R_x \left( 1 - yu \frac{1}{1 + xu} \cdot \frac{xv}{1 - xv} \right).\]

Hence we get
\[
Z_{\tilde{A}} \left( \beta_{m,0} \left( \frac{1}{1 - yu} \Delta_u(w) \right) \right)
\]
\[= \sum_{n=0}^{\infty} Z_{\tilde{A}} \left( \beta_{m,n} R_x^{-1} \Delta_u R_x \left( w - wyu \frac{1}{1 + xu} \cdot \frac{xv}{1 - xv} \right) \right) p^n.\]

By (1), (2), and Proposition 2.3 we finally obtain the theorem.

2.4. Proof of Corollary 1.5. When \( m = 1 \) in Theorem 1.3, we have
\[
\sum_{n=0}^{\infty} Z_{\tilde{A}} \left( \beta_{1,n} R_x^{-1} \Delta_u R_x \left( w - wyu \frac{1}{1 + xu} \cdot \frac{xv}{1 - xv} \right) \right) p^n = Z_{\tilde{A}}(w) Z_{\tilde{A}}(y).
\]

Since
\[
\beta_{1,n} R_x^{-1} \Delta_u R_x \left( w - wyu \frac{1}{1 + xu} \cdot \frac{xv}{1 - xv} \right)
\]
\[= \begin{cases} -wyx^n & \text{if } n \geq 1, \\ -R_x^{-1} \partial_1 R_x(w) & \text{if } n = 0, \end{cases}
\]
we get the result.

Acknowledgement

The authors would like to express their sincere gratitude to Doctor Minoru Hirose for valuable comments.

References

[1] M. E. Hoffman, Multiple harmonic series, Pacific J. Math. 152 (1992), 275–290.
[2] M. E. Hoffman, The algebra of multiple harmonic series, J. Algebra 194 (1997), 477–495.
[3] M. E. Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, Kyushu J. Math. 69 (2015), 345–366.
[4] Y. Horikawa, K. Oyama, and H. Murahara, A note on derivation relations for multiple zeta values and finite multiple zeta values, arXiv:1809.08389.
[5] K. Ihara, M. Kaneko, and D. Zagier, Derivation and double shuffle relations for multiple zeta values, Compositio Math. 142 (2006), 307–338.
[6] D. Jarossay, An explicit theory of \( \pi^{un,crys}([\mathbb{P}^1 - \{0, \mu_N, \infty\}] \), arXiv:1412.5099
[7] H. Murahara, Derivation relations for finite multiple zeta values, Int. J. Number Theory 13 (2017), 419–427.
[8] J. Rosen, Asymptotic relation for truncated multiple zeta values, J. Lond. Math. Soc. (2) 91 (2015), 554–572.
[9] S. Seki, Finite multiple polylogarithms, doctoral dissertation.
[10] S. Seki, The $p$-adic duality for the finite star-multiple polylogarithms, Tohoku Math J. (2) 71 (2019), 111–122.

(Hideki Murahara) Nakamura Gakuen University Graduate School, 5-7-1, Befu, Jonan-ku, Fukuoka, 814-0198, Japan
E-mail address: hmurahara@nakamura-u.ac.jp

(Tomokazu Onozuka) Multiple Zeta Research Center, Kyushu University 744, Motooka, Nishi-ku, Fukuoka, 819-0395, Japan
E-mail address: t-onozuka@math.kyushu-u.ac.jp