Novel integrable spin-particle models from
gauge theories on a cylinder

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Abstract

We find and solve a large class of integrable dynamical systems which includes Calogero-Sutherland models and various novel generalizations thereof. In general they describe $N$ interacting particles moving on a circle and coupled to an arbitrary number, $m$, of $\text{su}(N)$ spin degrees of freedom with interactions which depend on arbitrary real parameters $x_j$, $j = 1, 2, \ldots, m$. We derive these models from $\text{SU}(N)$ Yang-Mills gauge theory coupled to non-dynamic matter and on spacetime which is a cylinder. This relation to gauge theories is used to prove integrability, to construct conservation laws, and solve these models.

Integrable models have always played a central role in classical and quantum mechanics. Most prominent examples, like the Kepler problem, are systems with few ($\leq 3$) degrees of freedom. An important exception is a class of integrable $N$-particle models associated with the names Calogero, Moser and Sutherland [1, 2] (for review see [3]). These are models for identical particles moving on one dimensional space and interacting via certain repulsive two-body potentials $v(r)$. A well-known example is $v(r) \propto g^2 / \sin^2(gr)$ (which includes $v(r) \propto 1/r^2$ in the limit $g \to 0$), and we refer to the corresponding model as CS model. It is known that these models allow for interesting generalizations which also have dynamic spin degrees of freedom [4, 5]. The CS model and its generalizations have recently received quite some attention in different contexts. Here we only mention their relation to gauged matrix models [6] and gauge theories on a cylinder [7] which will be relevant for us.\footnote{the latter relation is implicit already in earlier work; see e.g. [8]}
In this article we find and solve a new class of integrable systems containing the CS model and their spin-generalizations as limiting cases. Our method is to extend and exploit the relation of the CS model to gauge theories on a cylinder, as will be explained in detail below. For simplicity our discussion here is restricted to classical models, and we only consider a special type of gauge theory. Our method is simple, and it should be possible to generalize it in several different directions: We conjecture that the corresponding quantum models are also integrable, and that our method to prove integrability should apply to the quantum case, too. (In this context it is worth noting that the quantum-analog of the gauge theory we consider is closely related to QCD on a cylinder and in the limit where the masses of the quarks becomes infinite; a good starting point to the literature on this is Ref. [9]). Moreover, it would be interesting to extend our method to other gauge theories (e.g. with supersymmetry and/or more general types of matter fields than the one considered here) and thus try to find and solve other integrable models.

The models we solve are given by a Hamiltonian

$$H = \sum_{\alpha=1}^{N} \frac{(p^\alpha)^2}{2} + \frac{1}{2} \sum_{\alpha,\beta=1}^{N} \sum_{j,k=1}^{m} v_{jk}(q^\alpha - q^\beta)\rho_k^{\alpha\beta} \rho_j^{\alpha\beta} + \frac{1}{2} \sum_{\alpha=1}^{N} \sum_{j,k=1}^{m} c_{jk} \rho_k^{\alpha\alpha} \rho_j^{\alpha\alpha}$$

where \(q^\alpha\) and \(p^\alpha\) are particle coordinates and momenta with the usual Poisson brackets \(\{q^\alpha, p^\beta\} = \delta_{\alpha\beta}\) etc., and \(\rho_j^{\alpha\beta} = \rho_j^{\beta\alpha}\) are complex valued \(\text{su}(N)\)-spins, i.e. \(\sum_{\alpha=0}^{N} \rho_j^{\alpha\alpha} = 0\) and

$$\{\rho_j^{\alpha\beta}, \rho_k^{\alpha'\beta'}\} = ig2\pi\delta_{jk} \left(\delta^{\alpha\alpha'} \rho_j^{\beta\beta'} - \delta^{\beta\beta'} \rho_j^{\alpha\alpha'}\right).$$

(The other Poisson brackets vanish.) The interaction potentials are given by

$$v_{jk}(r) = \frac{1}{4} e^{-i gr x_{jk}} \left(\frac{1}{\sin^2(\pi gr)} + \frac{i x_{jk}}{\pi \cot(\pi gr)} - \frac{|x_{jk}|}{\pi}\right)$$

where

$$x_{jk} = (x_j - x_k)_{2\pi}$$

with \(s_{2\pi} := s - 2\pi n\) for the integer \(n\) such that \(-\pi \leq s - 2\pi n < \pi\), and

$$c_{jk} = \frac{x_{jk}^2}{8\pi^2} - \frac{|x_{jk}|}{4\pi} + \frac{1}{12}.$$
The parameters $g$ (real positive), $N$ and $m$ (positive integers) are arbitrary, and $-\pi \leq x_1 \leq x_2 \leq \ldots \leq x_m < \pi$. Furthermore we have the following constraint on the possible initial conditions,

$$
\sum_{j=1}^{m} \rho_j^{\alpha} = 0 \quad \forall \alpha.
$$

(6)

Note that $v_{jk}(r) = \overline{v_{kj}(-r)} = v_{kj}(-r)$, which implies that the Hamiltonian is real. Moreover, since

$$
v_{jk}(r + \frac{n}{g}) = e^{-in(x_j - x_k)}v_{jk}(r),
$$

(7)

the Hamiltonian is invariant under the following transformations,

$$
q^\alpha \rightarrow q^\alpha + \frac{n^\alpha}{g}, \quad p^\alpha \rightarrow p^\alpha, \quad \rho_j^{\alpha\beta} \rightarrow \rho_j^{\alpha\beta} e^{-ix_j(n^\alpha - n^\beta)}
$$

(8)

for all integers $n^\alpha$. Thus these models describe particles moving on a circle of length $1/g$ and interacting with a potential whose strength depends on dynamic spins. We note that the particles repel each other, and we have a further natural restriction on phase space,

$$
q^\alpha \neq q^\beta \quad \forall \alpha \neq \beta.
$$

(9)

The main result of this article is a proof of integrability and the explicit solution of all these models.

Next the relation of our particle-spin models to gauge theories is discussed. It is known that the CS model can be obtained from a gauged one dimensional matrix model \[^6\]. More recently a relation to gauge theories in 1+1 dimensions was pointed out \[^7\]. In this article we explore this relation further and use it to find and solve new integrable models. We present a simple argument that SU($N$) Yang-Mills gauge theory on a cylinder coupled to certain non-dynamic matter is equivalent to a model of interacting particles and spins. We then show that this equivalence can be used as a powerful tool to analyze and solve these models: integrability is manifest, the construction of conservation laws trivial, and a simple solution method is obtained by exploiting gauge invariance, i.e. the possibility to change from the Weyl gauge.\[^b\]

\[^b\] $A_0$ and $A_1$ will be defined further below.
$A_0 = 0$ to what we call the *diagonal Coulomb gauge*, i.e. the condition that
the spatial component of the Yang-Mills field, $A_1(t, x)$, is independent of $x$
and diagonal in color space,

$$A_1(t, x) = Q(t) = \text{diag} \left( q^1(t), q^2(t), \ldots, q^N(t) \right) .$$

This is due to the fact that the gauge theory model in the Weyl gauge is
free and can be solved trivially, whereas in the diagonal Coulomb gauge the
time evolution equations are non-linear and, in a special case, equal to the
Hamilton equations of the model given by Eq. (1). To be more specific: We
restrict ourselves to gauge theory models with matter fields localized at a
finite number of points $x_j, j = 1, 2, \ldots m$ for simplicity (see Eq. (25) below).
We find that the dynamics of such a model in the diagonal Coulomb gauge
is governed by the equations of motion

$$\dot{X} = \{ X, H \}, \quad X = q^\alpha, p^\alpha, \rho_{\alpha\beta}^j,$$

which follow from the Hamiltonian Eq. (1) and the Poisson brackets $\{ \cdot, \cdot \}$ given
above. These observations allows us to derive the full solution of the initial
value problem for all these models generalizing the known solution of the CS
model [3].

We now consider 1+1 dimensional Yang-Mills theory, i.e. the differential
equations

$$\sum_{\mu=0,1} D_\mu F^{\mu\nu} = J^\nu \quad \text{where} \quad D_\mu = \partial_\mu + igA_\mu, \quad A_\mu \text{are the Yang-}
\text{Mills fields, } J^\mu \text{ the matter currents, } g \text{ the Yang-Mills coupling strength, } \partial_\mu = \partial/\partial x^\mu, \text{ and } \mu, \nu = 0, 1. \text{ Spacetime is a cylinder i.e. } t = x^0 \in \mathbb{R} \text{ is time, and } x = x^1 \in [-\pi, \pi] (= \text{circle}). \text{ Moreover, } F_{\mu\nu} = [D_\mu, D_\nu]/ig, \text{ and our metric}
tensor is } \text{diag}(1, -1). \text{ As gauge group we take } SU(N) \text{ in the fundamental}
representation.\text{[4]}

We restrict ourselves to *non-dynamic matter*, i.e. $J^1 = 0$, and $J^0 \equiv \rho$. We
denote $\rho$ as *charge*. Note that we have to impose $[D_0, \rho] = 0$ for consistency.

Setting $E := F_{01}$, we can write these equations as follows,

$$\begin{align*}
\partial_0 A_1 &= E + \partial_1 A_0 + ig[A_1, A_0] \quad (11) \\
\partial_0 E + ig[A_0, E] &= 0 \quad (12) \\
\partial_0 \rho + ig[A_0, \rho] &= 0 \quad (13) \\
\partial_1 E + ig[A_1, E] &= \rho . \quad (14)
\end{align*}
$$

Eq. (14) is called Gauss’ law and is a constraint on possible initial data for

---

$[a, b] := ab - ba$

$i.e. A_\mu, E \text{ and } \rho \text{ are functions with values in the traceless, complex } N \times N \text{ matrices}$
the system of time evolution equations (11)–(13). We now exploit gauge invariance: Eqs. (11)–(14) are obviously invariant under gauge transformations

\[ A_\mu \rightarrow U^{-1} A_\mu U + \frac{1}{ig} U^{-1} \partial_\mu U, \]

\[ E \rightarrow U^{-1} E U, \]

\[ \rho \rightarrow U^{-1} \rho U \]

(15)

where \( U = U(t, x) \) is an arbitrary differentiable SU(\( N \))-valued function on spacetime. To eliminate the gauge degrees of freedom one has to fix a gauge. One convenient choice is the Weyl gauge \( A_0(t, x) = 0 \).

Then the Eqs. (11)–(13) can be solved trivially:

\[ E(t, x) = E(0, x), \]

\[ A_1(t, x) = A_1(0, x) + E(0, x)t, \]

\[ \rho(t, x) = \rho(0, x) \]

(16)

with the initial data \( E(0, x), A_1(0, x) \) and \( \rho(0, x) \) satisfying the Gauss’ law Eq. (14) (note that our solution Eq. (16) satisfies Eq. (14) for all \( t \) if it satisfies it for \( t = 0 \)).

As mentioned, \( E, A_\mu \) and \( \rho \) are functions with values in the traceless \( N \times N \) matrices. In the following we write the matrix elements of \( M = E, A_\mu \) or \( \rho \) as \( M^{\alpha\beta} \), \( \alpha, \beta = 1, 2, \ldots N \). Note that, since \( \sum_{\alpha=0}^{N} M^{\alpha\alpha} \) is zero, the independent components are \( M^{\alpha\beta} \) for \( \alpha \neq \beta \), and \( M^{\alpha\alpha} = M^{\alpha+1,\alpha+1} \) for \( \alpha = 1, 2, \ldots N - 1 \).

We now show that one can also impose the diagonal Coulomb gauge (10), i.e. for each (generic) Yang-Mills configurations \( A_1(t, x) \) one can find a gauge transformation \( U \) such that \( A'_1 \equiv U^{-1} A_1 U + U^{-1} \partial_1 U/ig \) is a diagonal matrix \( Q \) independent of \( x \) [10]. For that we construct such a \( U \) explicitly. We first note that a solution to the equation \( \partial_1 S + igA_1 S = 0 \) with \( S(t, -\pi) = 1 \) is the parallel transporter

\[ S(t, x) = \mathcal{P} \exp \left( -ig \int_{-\pi}^{x} dy A_1(t, y) \right) \]

(17)

where \( \mathcal{P} \exp \) is the path ordered exponential. Note that \( S(t, x) \) is not a gauge transformation since it is not periodic in \( x \) (its values at \( x = -\pi \) and \( \pi \) are

\[ ^{\text{i.e. to consider the model in terms of the gauge transformed fields on the r.h.s. of Eq. (15), which by abuse of notation we denote by the same symbol, and with a gauge transformation } U \text{ which is a solution of } \partial_0 U + igA_0 U = 0. \]
different in general). To construct a gauge transformation, we introduce the SU($N$)-matrix $V(t)$ diagonalizing the SU($N$)-matrix $S(t, \pi)$,

$$V(t)^{-1}S(t, \pi)V(t) = e^{-ig2\pi Q(t)}$$  \hspace{1cm} (18)

for some diagonal matrix $Q(t)$. This implies that

$$U(t, x) = S(t, x)V(t)e^{ig(x+\pi)Q(t)}$$  \hspace{1cm} (19)

is periodic in $x$, and it satisfies $\partial_1 U + igA_1 U = igUQ$ equivalent to $A_1^U = Q$. Moreover, if $A_1(t, x)$ is a generic differentiable map on spacetime, then $Q(t)$ and $V(t)$ can be chosen to be differentiable in $t$ [11], and $U(t, x)$ Eq. (19) is indeed a differentiable function on space-time i.e. a gauge transformation. ‘Generic’ here means that the latter is only true if $q^\alpha(t) \neq q^\beta(t)$ for all $t$ and $\alpha \neq \beta$ since otherwise discontinuous functions $V(t)$ can occur [11]. Due to Eq. (9), gauge field configurations $A_1(t, x)$ where this condition fails are irrelevant for us. Note that our discussion here implies that the $q^\alpha(t)$ can be obtained as eigenvalues of the Wilson line $S(t, \pi)$. This observation will allow us to determine the explicit solution of the Hamilton eqs. following from Eqs. (1) and (3).

We now determine the time evolution equations for the variables $q^\alpha$ defined in Eq. (10). We use Fourier transformation,

$$\hat{E}^{\alpha\beta}(t, n) = \int_{-\pi}^{\pi} dx e^{-inx}E^{\alpha\beta}(t, x), \quad n \in \mathbb{Z}$$  \hspace{1cm} (20)

and similarly for $A_0$ and $\rho$. Then Eq. (11) gives

$$\partial_0 q^\alpha(t) = p^\alpha(t) \equiv \frac{\hat{E}^{\alpha\alpha}(t, 0)}{2\pi}.$$  \hspace{1cm} (21)

Note that this and the following equations all are consistent with $\sum_{\alpha=1}^{N} q^\alpha = \sum_{\alpha=1}^{N} p^\alpha = 0$ (this corresponds to translation invariance of the mechanical system defined in Eq. (11)). The time evolution of the $p^\alpha$ follows from Eq. (12),

$$\partial_0 p^\alpha(t) = -\frac{ig}{(2\pi)^2} \sum_{n \in \mathbb{Z}} \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{N} \left( \hat{A}^{\alpha\beta}_0(t, n)\hat{E}^{\beta\alpha}(t, -n) - \hat{E}^{\alpha\beta}(t, n)\hat{A}^{\beta\alpha}_0(t, -n) \right).$$  \hspace{1cm} (22)
The r.h.s. of this equation can be evaluated using Eqs. (11) and (14)

$$\begin{align*}
-i \left( n + g[q^\alpha(t) - q^\beta(t)] \right) \hat{A}_0^{\alpha\beta}(t, n) &= \hat{E}^{\alpha\beta}(t, n) \\
i \left( n + g[q^\alpha(t) - q^\beta(t)] \right) \hat{E}^{\alpha\beta}(t, n) &= \hat{\rho}^{\alpha\beta}(t, n)
\end{align*}$$

(23)

(note that this holds true even for $\alpha = \beta$ if $n \neq 0$). Inserting this we get

$$\partial_0 p^\alpha(t) = \frac{2g}{(2\pi)^2} \sum_{n} \frac{\hat{\rho}^{\alpha\beta}(t, n)\hat{\rho}^{\beta\alpha}(t, -n)}{(n + g[q^\alpha(t) - q^\beta(t)])^3}.$$  

(24)

Note that up to now no specific choice for the charges was made. To proceed, we restrict ourselves to charges of the following form for simplicity,

$$\rho^{\alpha\beta}(t, x) \equiv \sum_{j=1}^{m} \rho^{\alpha\beta}_j(t) \delta(x - x_j),$$

(25)

which describe matter localized at the points $x_j$, as mentioned above. Then

$$\hat{\rho}^{\alpha\beta}(t, n) = \sum_{k=1}^{m} \rho^{\alpha\beta}_k(t) e^{-inx_k},$$

(26)

and we can then write Eq. (24) as

$$\partial_0 p^\alpha(t) = -\frac{\partial}{\partial q^\alpha(t)} \sum_{\beta=1}^{N} \sum_{j,k=1}^{m} \rho^{\alpha\beta}_j(t) v_{jk} q^\alpha(t) - q^\beta(t)$$

(27)

with

$$v_{jk}(r) = \sum_{n} \frac{e^{inx_j - x_k}}{(2\pi)^2(n + gr)^2},$$

(28)

where the summation is over all $n \in \mathbb{Z}$. This equals Eq. (3), as can be seen by a simple computation using the identity

$$\sum_{n \in \mathbb{Z}} \frac{e^{ins}}{(n + r)^2} = e^{-irs_2 \pi} \left( \frac{\pi^2}{\sin^2(\pi r)} + i\pi s_2 \pi \cot(\pi r) - \pi |s_2| \right).$$

(29)

Note also that

$$\sum_{n \neq 0} \frac{e^{ins}}{n^2} = \frac{s_2^2 \pi}{2} - \pi |s_2| + \frac{\pi^2}{3}.$$  

(30)
Eqs. (21) and (27) are precisely the Hamilton equations \( \dot{q}^\alpha = \{q^\alpha, H\} \) and \( \dot{p}^\alpha = \{p^\alpha, H\} \). We are left to determine the time evolution of the \( \rho^{\alpha\beta} \). From Eq. (13) we get

\[
\partial_0 \rho^{\alpha\beta}(t) = -ig \sum_{\gamma=1}^{N} (A_0^{\alpha\gamma}(t, x_j) \rho^{\gamma\beta}_j(t) - \rho^{\alpha\gamma}_j(t) A_0^{\gamma\beta}(t, x_j)).
\] (31)

Moreover, we can compute \( A_0^{\alpha\beta}(t, x_j) = \frac{1}{2\pi} \sum_n \hat{A}_0^{\alpha\beta}(t, n) e^{inx_j} \) from Eqs. (23) and Eq. (26). Note that Eq. (23) also determines \( \hat{A}_0^{\alpha\alpha}(t, n) \) if \( n \) is non-zero, and we can set \( \hat{A}_0^{\alpha\alpha}(t, 0) = 0 \). Then a simple computation gives

\[
A_0^{\alpha\beta}(t, x_j) = 2\pi \sum_{k=1}^{m} v_{jk}(q^\alpha(t) - q^\beta(t)) \rho^{\alpha\beta}_k(t)
\] (32)

with \( v_{jk}(r) \) given by Eq. (3) and \( v_{jk}(0) = c_{jk} \) by Eq. (3). (We used Eqs. (28), (29) and (30)). Note that for \( \alpha = \beta \) the summation in Eq. (28) is restricted to the non-zero integers \( n \), which implies \( v_{jk}(0) = c_{jk} \). With that Eq. (31) becomes equal to the Hamilton eq. \( \dot{\rho}^{\alpha\alpha} = \{\rho^{\alpha\alpha}, H\} \) following from Eqs. (1) and (2).

We finally note that the \( n = 0 \) components of Gauss’ law Eq. (14) for \( \alpha = \beta \) reads

\[
\dot{\rho}^{\alpha\alpha}(t, 0) = 0 \quad \forall \alpha.
\] (33)

This is a consistency requirement. Our arguments above show that this condition is fulfilled for \( \rho \) in Eq. (23) if and only if Eq. (1) holds for all \( t \), which is true if it holds for \( t = 0 \).

We now solve the Eqs. (21), (27) and (31)–(32) with the initial conditions

\[
q^\alpha(0) = q_0^\alpha, \quad p^\alpha(0) = p_0^\alpha, \quad \rho^{\alpha\beta}_j(0) = \rho^{\alpha\beta}_{j,0}.
\] (34)

Our discussion above implies that we can obtain the solution \( q^\alpha(t) \) of this initial value problem by solving the Yang-Mills Eqs. (11)–(14) with the initial conditions

\[
A_1^{\alpha\beta}(t = 0, x) = \delta_{\alpha\beta} q_0^\alpha, \quad \int_{-\pi}^{\pi} dx 2\pi E^{\alpha\alpha}(t = 0, x) = p_0^\alpha
\]

\[
\rho^{\alpha\beta}(t = 0, x) = \sum_{j=1}^{N} \rho^{\alpha\beta}_{j,0} \delta(x - x_j).
\] (35)

\[\text{A non-zero } \hat{A}_0^{\alpha\alpha}(t) = 0 \text{ is irrelevant since it can be removed by a gauge transformation compatible with the DCG.}\]
We first have to determine $E^{\alpha\beta}(t = 0, x)$ for $\alpha \neq \beta$ from Gauss’ law Eq. (14). The solution $A_1(t, x)$ of the gauge theory in the Weyl gauge $A_0 = 0$ is then given in Eq. (16). To obtain the $q^\alpha(t)$, we only need to evaluate the corresponding parallel transporter $S(t, \pi)$ Eq. (17): as discussed, the eigenvalues of $S(t, \pi)$ are equal to $e^{-2\pi igq^\alpha(t)}$. Moreover,

$$
\rho_j(t) = U(t, x_j)^{-1}\rho_{j,0}U(t, x_j)
$$

with $U(t, x)$ given by Eq. (19). Here and in the following we use an obvious matrix notation.

For $t = 0$ we can write Gauss’ law Eq. (14) as follows

$$
\partial_1 \left( e^{i g Q_0 x} E(0, x) e^{-i g Q_0 x} \right) = e^{i g Q_0 x} \rho(0, x) e^{-i g Q_0 x}
$$

with $\rho(0, x) = \sum_{j=1}^m \rho_{j,0} \delta(x - x_j)$ and

$$
Q_0 = \text{diag} \left( q_0^1, q_0^2, \ldots, q_0^N \right).
$$

Since $\rho(0, x) = 0$ except for $x = x_j$, we obtain $E(0, x) = e^{-i g Q_0 x} B_j e^{i g Q_0 x}$, where $B_j$ is some constant matrix, for $x_j < x < x_{j+1}$, $j = 0, \ldots, m$, $x_0 = -\pi$ and $x_{m+1} = \pi$. To determine the matrices $B_j$ we integrate Eq. (37) from $x_j - 0^+$ to $x_j + 0^+$. This gives the recursion relations $B_j - B_{j-1} = e^{i g Q_0 x_j} \rho_{j,0} e^{-i g Q_0 x_j}$, and the condition $E(0, -\pi) = E(0, \pi)$ implies $e^{2i g Q_0 \pi} B_0 e^{-2i g Q_0 \pi} = B_m$. Putting this together and using the second relation in Eq. (35) we obtain after a straightforward calculation,

$$
B_{j}^{\alpha\beta} = \delta_{\alpha\beta} \left( \rho_0^\alpha + \sum_{\ell=1}^j \rho_{\ell,0}^\alpha - \sum_{i=1}^m \frac{x_{i+1} - x_i}{2\pi} \sum_{\ell=1}^i \rho_{\ell,0}^\alpha \right) +

(1 - \delta_{\alpha\beta}) \sum_{\ell=1}^m \rho_{\ell,0}^\alpha e^{ig\left(q_0^\alpha - q_0^\beta\right)} \frac{[x + \pi \text{sgn}(x-x_{\ell})]}{2i \sin(g\pi [q_0^\alpha - q_0^\beta])}
$$

with $\text{sgn}(x) = 1$ for $x \geq 0$ and $-1$ for $x < 0$. With that, Eq. (16) gives $A_1(t, x) = e^{-i g Q_0 x} (Q_0 + B_j t) e^{i g Q_0 x}$ for $x_j < x < x_{j+1}$. This is the solution of the Yang-Mills equations in the Weyl gauge.

We now solve $\partial_1 S(t, x) + ig A_1(t, x) S(t, x) = 0$ which is equivalent to

$$
\partial_1 \tilde{S}(t, x) + ig B_j t \tilde{S}(t, x) = 0 \quad \text{for} \quad x_j < x < x_{j+1}
$$

\footnote{Note that the same argument applies for times $t > 0$}
for $\tilde{S}(t, x) = e^{igQ_0x}S(t, x)$. This implies $\tilde{S}(t, x) = e^{-igB_jt(x-x_j)}\tilde{S}(t, x_j)$ for $x_j < x < x_{j+1}$, thus

$$S(t, x_{j+1}) = e^{-igQ_0x_{j+1}}e^{-igB_jt(x_{j+1}-x_j)}\tilde{S}(t, x_j) = \ldots = e^{-igQ_0x_j}e^{-igB_jt(x_j-x_{j-1})}\ldots \times e^{-igB_0t(x_1+\pi)}e^{-igQ_0\pi}$$

(41)

where we used $S(t, -\pi) = 1$. Especially (for $j = m$),

$$S(t, \pi) = e^{-igQ_0\pi}e^{-igB_m t(\pi-x_m)}\ldots \times e^{-igB_{m-1}t(x_{m-1}-x_{m-1})}\ldots e^{-igB_0t(x_1+\pi)}e^{-igQ_0\pi}. \quad (42)$$

We thus obtain our main

**Result:** The Hamiltonian equations of the dynamical system defined in Eqs. (1)–(6) are given in Eqs. (21), (27) and (31). The solutions of these equations with the initial conditions Eq. (34) can be obtained from the eigenvalues of the matrix $S(t, \pi)$ given in Eqs. (42) and (39) according to Eq. (18). Moreover, $\rho_j(t)$ is given by Eq. (36) with $U(t, x_j)$ defined in Eq. (19) and $S(t, x_j)$ in Eq. (41).

Note that for $m = 1$, the dynamics of the spin and the particles decouple, and our result reduces to the known solution of the CS model; see e.g. [3].

It is now also easy to construct conservation laws for our dynamical systems: Eq. (16) implies that $tr[E(t, x)^n]$, where $tr$ is the $N \times N$ matrix trace, is time independent for all $-\pi \leq x < \pi$ and all positive integers $n$. Since these quantities are gauge independent, they are time independent also in the diagonal Coulomb gauge. In this latter gauge, we can evaluate $E(t, x)$ as above and obtain $E(t, x) = e^{-igQ(t)x}B_j(t)e^{igQ(t)x}$ for $x_j < x < x_{j+1}$ where $Q(t)$ and $B_j(t)$ are as in Eqs. (38) and (39) but with $q_0^\alpha$, $p_0^\alpha$, and $\rho_{\alpha\beta}^j$ replaced by $q^\alpha(t)$, $p^\alpha(t)$, and $\rho_{\alpha\beta}^j(t)$, i.e. the solution of the initial value problem which we solved above. Using cyclicity of the trace, we conclude that $tr[B_j(t)^n]$ for an arbitrary positive integer $n$ and $j = 1, \ldots, m$ are time independent: Each of them is a conservation law. For $m = 1$ these are the known conservation laws for the CS model [3]. It is also worth noting the corresponding Lax-type equations $\partial_0 B_j(t) + ig[M_j(t), B_j(t)] = 0$ where

$$M_j(t) = e^{igQ(t)x_j}A_0(t, x_j)e^{-igQ(t)x_j} - x_j P(t) \quad (43)$$
which are obtained from eq. (12) setting $x = x_j$ and using $\partial_0 Q(t) = P(t) := \text{diag}(p_1(t), p_2(t), \ldots, p_N(t))$ ($A_0(t, x_j)$ is given by Eq. (52)).

As discussed above, the models we find generalize the CS models with the interaction potential $v(r) \propto g^2 / \sin^2(gr)$ which describe particle moving on a circle of length $1/g$. There is a an integrable CS-type model of particles moving on the real line and interacting with a potential $v(r) \propto g^2 / \sinh^2(gr)$, see e.g. [3]. The sinh-model and its solution can be obtained from the sin-model and its solution by replacing $q^\alpha \rightarrow iq^\alpha$ and $p^\alpha \rightarrow ip^\alpha$ [3]. This replacement in our Hamiltonian, Eq. (1) (together with $H \rightarrow -H$), leads to spin-generalizations of the sinh-model. It is natural to conjecture that this very replacement allows to obtain the solution of the latter from our solution of the former model.

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