On Equivalence of Duffin-Kemmer-Petiau and Klein-Gordon Equations

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Abstract
A strict proof of equivalence between Duffin-Kemmer-Petiau (DKP) and Klein-Gordon (KG) theories is presented for physical S-matrix elements in the case of charged scalar particles interacting in minimal way with an external or quantized electromagnetic field. First, Hamiltonian canonical approach to DKP theory is developed in both component and matrix form. The theory is then quantized through the construction of the generating functional for Green functions (GF) and the physical matrix elements of S-matrix are proved to be relativistic invariants. The equivalence between both theories is then proved using the connection between GF and the elements of S-matrix, including the case of only many photons states, and for more general conditions - so called reduction formulas of Lehmann, Symanzik, Zimmermann.

1 Introduction
More than 60 years ago G. Petiau, R. Duffin and N. Kemmer proposed the first order equation (DKP equation) for description of spin 0 and 1 particles. This period of time is conventionally divided in three periods: the first one from 1939 until, aproximately 1970; the second from 1970 to 1980 and the last one from 1980 on.

During the first period the majority of the papers about DKP equation was devoted to the development of DKP formalism and to the investigation of DKP charged particles interaction with electromagnetic field (EM field). For many classes of processes (such as QE of spin 0 mesons, meso-atom and

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others) calculations based on DKP and KG equations yield identical results \(^\text{1}\), including one-loop corrections \(^\text{2}, \text{3}, \text{4}\).

A very important contribution to the understanding of this question was made by A. Wightman in his paper \(^\text{10}\) devoted to the 70th birthday of P. A. M. Dirac in 1971. He showed that, for minimal coupling with EM field \((\sim \bar{\psi} \gamma^\mu \beta \psi A^\mu)\) in DKP theory, one has stability of DKP equation for spin-0 particle under smooth local perturbation of external field or, expressing in another way, that the retarded character of DKP equation solutions is conserved under such perturbation as well as in Dirac equation without anomalous magnetic moment.

The instability of such sort has been discussed by G. Velo and D. Zwanziger in \(^\text{11}\), where they showed that it destroys causality in Rarita-Swinger equation for spin 3/2 particles (in external field).

The central point of the proof by A. Wightman is: the retarded character of solution is connected with the renormalizability of the theory in the case of interaction with a quantized EM field.

The second period can be characterized as by some disappointments and hesitations. By this time two great discoveries had been made: parity violation and creation of unified theory of electro-weak interaction (Weinberg-Salam theory or Standard Model). The question about the equivalence of both DKP and KG theories arises again at the attempts to describe new processes. Many works (see references in \(^\text{8}\)) have been made applying DKP formalism to decays of \(K\) and other unstable mesons and to strong interaction. The conclusion presented in reference \(^\text{8}\) was not optimistic: DKP formalism in some cases yield different results from a second order formalism.

The third period goes under the sign of uncertainty: are both DKP and KG equivalent or not? Not so many papers have been published on this theme. In our opinion one of main reasons for the decrease of interest in DKP formalism in the last period is the conclusion about nonequivalence between DKP and KG theories.

We believe that the equivalence between these theories in the case of nonstable particles can be proved as well as for all processes which are

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\(^{1}\) A rich list of references with historical comments can be found in reference \(^\text{8}\). Unfortunately in this work there are no references to the works by I. Gelfand and A. Yaglom, who obtained the first order equation for particles with fixed arbitrary spin. For references to these and others works see \(^\text{1}\).

\(^{2}\) Moreover, in work \(^\text{12}\) it is affirmed that DKP theory gives for \(K\) meson decay qualitatively different results when compared to KG-formalism.

\(^{3}\) See section 5; Conclusion, point 5.
described by renormalizable theories. This question, however, goes beyond the scope of this paper.

As we know there are no strict proof of the equivalence between DKP and KG theories in Quantum Electrodynamics of spin 0 particles, too. Coincident results have been obtained for many processes in first order perturbation theory, one loop corrections and the infrared approximation [1, 2, 3].

The main goal of this paper is to give a strict proof of the equivalence of DKP and KG theories for charged scalar particles interacting with an external or quantized EM field in minimal way for physical matrix elements in any order of perturbation theory.

In section 2 Hamiltonian canonical approach to DKP theory is developed in component and matrix forms. The construction of generating functional of Green function (GF) of DKP theory is used for quantization of the theory and the physical matrix elements of S-matrix are proved to be relativistic invariants.

In section 3 the equivalence of DKP and KG theories is proved for physical matrix elements utilizing the connection between GF and the elements of S-matrix, including the case of only many photons states. In section 4 the equivalence of both theories is shown in the more general framework condition - so called reduction formulas of Lehmann, Symanzik, Zimmermann [13].

In section 5 we shortly discuss the basic results and questions about construction of renormalizable DKP theory for spin 0 particles.

The appendix contains some proofs.

2 Canonical Quantization

2.1 Hamiltonian approach in component form

Our aim is to construct the Hamiltonian for DKP theory which is one with constraints due to degeneration of $\beta$ matrices first we will work in component representation. The Lagrangian density is

$$\mathcal{L} = \bar{\psi} (i \beta_\mu D^\mu - m) \psi$$  \hspace{1cm} (1)$$

where $D^\mu = \partial^\mu - ie A^\mu$; $\partial^\mu = \frac{\partial}{\partial \varphi^\mu}$; $g_{\mu\nu} = \text{diag} \{1, -1, -1, -1\}$. Primarily one considers $A^\mu$ as an external EM field. We choose the $\beta_\mu$ matrices in the
following form:

\[
\begin{align*}
\beta_0 &= \begin{vmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}; \\
\beta_1 &= \begin{vmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \\
\beta_2 &= \begin{vmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}; \\
\beta_3 &= \begin{vmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}
\end{align*}
\]

(2)

\[
\begin{align*}
\psi_\alpha &= (\psi^* \eta)_\alpha = (\varphi^*, \varphi^*0, -\varphi^*1, -\varphi^*2, -\varphi^*3); \\
\quad \eta = 2(\beta_0)^2 - 1 \\
\psi_\alpha &= (\varphi, \varphi^0, \varphi^1, \varphi^2, \varphi^3).
\end{align*}
\]

(3)

In component form $L$ is then equal to

\[
L = -\varphi^* D_{\mu} \varphi^\mu + \varphi^*\mu D_{\mu} \varphi - m (\varphi^* \varphi + \varphi^*\mu \varphi_{\mu})
\]

(4)

From equation (4) one finds all the 1st stage constraints:

\[
\begin{align*}
p &= \frac{\partial L}{\partial \varphi} = \varphi^*0, \\
p^* &= \frac{\partial L}{\partial \varphi^*} = 0 \\
p_0 &= \frac{\partial L}{\partial \varphi^0} = -\varphi^*, \\
p^*_0 &= \frac{\partial L}{\partial \varphi^{*0}} = 0 \\
p_i &= \frac{\partial L}{\partial \varphi^i} = 0, \\
p^*_i &= \frac{\partial L}{\partial \varphi^{*i}} = 0, \\
& \quad i = 1, 2, 3.
\end{align*}
\]

(5) \hspace{1cm} (6) \hspace{1cm} (7)

Now we construct the initial Hamiltonian $H$:

\[
H = \int d^3x \left\{ p_a \dot{\varphi}^a - L \right\} = \int d^3x \left\{ \varphi^* \partial_i \varphi^i - \varphi^*\bar{i} \partial_i \varphi + m (\varphi^* \varphi + \varphi^*\mu \varphi_{\mu}) + i \varepsilon (\varphi^*\mu A_{\mu} \varphi - \varphi^* A_{\mu} \varphi^\mu) \right\}
\]

(8)

Here $p_a = \{ p, p^*, p_0, p^*_0, p_i, p^*_i \}$ and we used equations (5) to (7).

From equations (5) to (7) we get the 1st stage constraints:

\[
\theta = p - \varphi^*0, \quad \theta^* = p^*, \quad \theta_0 = p_0 + \varphi^*, \quad \theta^*_0 = p^*_0
\]

(9)

\footnote{We follow terminology of the book [14].}
\[ \theta_i = p_i, \; \theta^*_i = p^*_i. \] (10)

Thus one can speak that the initial \( H \) is the Hamiltonian on the constraint surface:

\[ H = \int d^3x \left\{ p_a \dot{\varphi}^a - \mathcal{L} \right\}_{\theta_a = 0}. \] (11)

It is easy to check that the Poisson brackets for constraints \( \theta_a \) are\(^5\):

\[ \{ \theta, \theta^*_0 \} = -1; \; \{ \theta^*, \theta_0 \} = -1. \] (12)

All others brackets equal zero. Constraints (11) are second-class constraints\(^6\) as they do not disappear on the constraint surface \( \theta_\alpha = 0 \).

From equation (12) it follows that

\[ \text{rank} \| \theta_a, \theta_b \| = 4. \] (13)

This means that if we introduce new hamiltonian off constraint surface for definition Lagrangian multipliers \( \lambda_a \)

\[ H^{(1)}_2 = H + \int d^3x \lambda_a (x) \theta^a (x), \] (14)

and demanding that all constraints conserve in time

\[ \dot{\theta} = \left\{ \theta^a, H^{(1)}_2 \right\} = 0, \] (15)

we can not define all \( \lambda_a \), but only \( \lambda, \lambda^*, \lambda_0, \lambda^*_0 \).

Six constraints from (15) give the six 2\textsuperscript{nd} stage ones that are:

\[ \dot{\theta}_i = \left\{ \theta_i, H^{(1)}_2 \right\} = \left\{ \theta_i, H \right\} = - \frac{\partial H}{\partial \dot{\varphi}^i} = D_i \dot{\varphi}^* - \varphi_i \equiv \theta^2_i, \] (16)

\[ \dot{\theta^*_i} = \left\{ \theta^*_i, H^{(1)}_2 \right\} = - \frac{\partial H}{\partial \varphi^* i} = D_i \dot{\varphi} - \varphi_i \equiv \theta^*_2i. \] (17)

Including all constraints (ten 1\textsuperscript{st} stage and six 2\textsuperscript{nd} stage) in total system, where now

\[ \theta_a = \{ \theta, \theta^*, \theta_0, \theta^*_0, \theta_i, \theta^*_i, \theta^2_i, \theta^*_2i \}, \] (18)

we can check that

\[ \text{rank} \| \theta_a, \theta_b \| = 16 \] (19)

\(^5\)All variables: \( p_a, \varphi_a \) depend on \( x \). Therefore the r.h.s. of equation (12) is \( \sim \delta (x - y) \) which we omitted.

\(^6\)For simplicity we will not use the Dirac brackets.
and
\[ \det \| \theta_a, \theta_b \| = 1. \] (20)

All these constraints are second class ones. According to the general scheme [14] we can find all \( \lambda_a \) from the conditions:
\[ \dot{\theta}_a = \{ \theta_a, H_2^{(1)} \} = 0. \] (21)

From equations (1), (10), (14), (17), (18), and (21) one gets:
\[ \lambda = m \varphi_0 + ieA_0 \varphi; \quad \lambda^* = m \varphi_0^* - ieA_0 \varphi^*; \] (22)
\[ \lambda_0 = -\frac{1}{m} (D_i D^i \varphi + m^2 \varphi) + ie \varphi_0 A_0; \quad \lambda_0^* = -\frac{1}{m} (D_i^* D^i \varphi^* + m^2 \varphi^*) - ie \varphi_0^* A_0; \] (23)
\[ \lambda_i = -\frac{1}{m} D_i (m \varphi_0 + ieA_0 \varphi); \quad \lambda_i^* = -\frac{1}{m} D_i^* (m \varphi_0^* + ieA_0 \varphi^*). \] (24)

Equation (24) is a consequence of conservation of the 2nd stage constraints:
\[ \theta^2 \equiv \dot{\theta}_i = \{ \theta_i, H_2^{(1)} \} = 0; \quad \theta_i^* \equiv \dot{\theta}_i^* = \{ \theta_i, H_2^{(1)} \}. \] (25)

The main criterium of the correctness of the canonical or \( H \)-approach to the \( L \)-theory is the coincidence of the Lagrangian and Hamiltonian equations of motion.

To get \( H \)-equations of motion we must introduce solutions (22) to (24) in equation (14) and to consider the following equations:
\[ \dot{\varphi} = \{ \varphi(x), H_2^{(1)} \} = \int d^3 y \{ \varphi(x), \lambda^a(y) \theta_a(y) \} = \int d^3 y \lambda(y) \{ \varphi(x), \theta(y) \} = \lambda(x) = m \varphi_0 + ieA_0 \varphi \] (26)
or
\[ D_0 \varphi - m \varphi_0 = 0, \] (27)
and analogously
\[ D_0^* \varphi^* - m \varphi_0^* = 0. \] (28)

Conservation of constraint \( \theta_i; \ \dot{\theta}_i = \{ \theta_i, H_2^{(1)} \} = 0; \ \text{gives:} \)
\[ D_i \varphi - m \varphi_i = 0; \quad D_i^* \varphi^* - m \varphi_i^* = 0. \] (29)
Collecting equations (27), (28) and (29) one gives
\[ D^\mu \varphi - m\varphi^\mu = 0; \quad D^{*\mu} \varphi^* - m\varphi^{*\mu} = 0. \] (30)

Consider
\[ \dot{\varphi}_0 (x) = \{ \varphi_0, H_2^{(1)} \} = \int d^3 y \lambda^0 (y) \{ \varphi_0 (x), \theta_0 (y) \} = \lambda^0 (x) = -\frac{1}{m} (D_i \varphi + m^2 \varphi) + ie\varphi_0 A^0 = -\frac{1}{m} D^i (D_i \varphi + m\varphi_i) - (D_i \varphi^i + m\varphi) + ie\varphi_0 A^0 = - (D_i \varphi^i + m\varphi) + ie\varphi_0 A^0 \]

being the first term equal to zero due to equation (29). Or
\[ D_\mu \varphi^\mu + m\varphi = 0. \] (31)

We obtain all \( L \)-equations, which follow from equation (4).
Thus we get by canonical way, from equations (30) and (31), on the classical level, the KG equation for \( \varphi \)-field:
\[ D_\mu D^\mu \varphi + m\varphi = 0. \] (32)

Obviously, this result follows in a very simple way from component form of \( L \), equations (4), too.

2.2 Quantization. Generating Functional

For canonical quantization of the theory with second class constraints\[14, 15\] one chooses the new canonical variables:
\[ \omega^{(i)} = \varphi - p\Omega^{(1)} = -\theta = \varphi^0 - p_0\Omega_0^{(1)} = -\theta^{(2)} = \varphi_i - \frac{D_i \varphi}{m}, \]
\[ \omega_0^{(i)} = \varphi - p\Omega^{(1)} = \theta_0 = \varphi^* + p_0\Omega_0^{(2)} = \theta^* = \varphi^i - \frac{D^i \varphi^*}{m}, \]
\[ \Omega_i^{(1)} = \theta_i = p_i \]
\[ \Omega_i^{(1)} = \theta_i^* = p_i^* \] (33)

\(^7\)Quantization of theories with the 2nd -class constraints devoted many papers, beginning with classical work of Dirac [15], see also [16, 17].
Here we have four physical variables, $\omega^{(k)}$ and $\omega^{(k)}_0$, $k = 1, 2$, and sixteen constraints $\Omega^{(k)}_a = 0$, which allow to express the rest variables through $\omega^{(k)}, \omega^{(k)}_0$.

Now the rules of quantization are very simple [14, 15]:

\[
\begin{align*}
\left[ \hat{\varphi} (x), \hat{p} (y) \right] &= i \{ \varphi (x), p (y) \} = i \delta (x - y) \\
\left[ \hat{\varphi}_0 (x), \hat{p}_0 (y) \right] &= i \{ \varphi_0 (x), p_0 (y) \} = i \delta (x - y) \\
\end{align*}
\]

\[\hat{\Omega}^{(k)}_a = 0. \tag{34}\]

In terms of $\hat{p}, \hat{\varphi}, \hat{p}_0, \hat{\varphi}_0$ the Dirac brackets coincide with the Poisson ones.

Then we get for Heisenberg operators $\hat{\varphi}$ the equation (32) and for $H$ the same expression as in KG theory. However, for proof of equivalence of the physical matrix elements of S-matrix for scalar particles and of many photons GF in both theories it is more simple to start not from $\hat{H}$, but from generating functional for GF of DKP theory, which follows from canonical quantization, too.

For generating functional of DKP theory in component form we get

\[
Z (\mathcal{J}, \mathcal{J}^*, \mathcal{J}_\mu, \mathcal{J}_\mu^*) = Z^{-1}_0 \int \mathcal{D}_a \mathcal{D}_\mu \mu (\theta^a) \\
\times \exp \left\{ i \int d^4 x \left( p_a \hat{\varphi}_a - \mathcal{H}_2^{(1)} + \varphi_0 \mathcal{J}^a \right) \right\}, \tag{35}\]

where

\[
\mu (\theta^a) = \prod_a \delta (\theta_a) (\det ||\theta_a, \theta_b||)^{1/2}, \quad Z_0 = Z (0, 0, 0, 0), \quad \mathcal{H}_2^{(1)} = \text{density of } \mathcal{H}_2^{(1)}. \tag{36}\]

Taking into account equations (36, 10, 14, 16, 17, 36) and integrating over all momentum $p_a$ we get

\[
Z (\mathcal{J}, \mathcal{J}^*, \mathcal{J}_\mu, \mathcal{J}_\mu^*) = Z^{-1}_0 \prod_a \int \mathcal{D}_a \frac{3}{2} \delta (D^i \varphi - m \varphi^i) \delta (D^{*i} \varphi^* - m \varphi^{*i}) \\
\times \exp \left\{ i \int d^4 x \left( \varphi^{*\mu} D_\mu \varphi - \varphi^{*} D_\mu \varphi^* - m (\varphi^{*} \varphi + \varphi^{*\mu} \varphi_\mu) \right) \right. \\
\left. + \mathcal{J}^* \varphi + \mathcal{J} \varphi^* + \mathcal{J}_\mu \varphi^{*\mu} + \mathcal{J}_\mu^* \varphi^\mu \right\}. \tag{37}\]
The difference between $H$-quantization of DKP-theory and formal $L$-quantization consists in appearance in equation (37) of two functional $\delta$-functions, which reflect the existence of the 2nd stage constraints (16) and (17) in $H$- approach to DKP theory.

After integration in equation (37) over $\varphi_a$ and utilizing the $\delta$-function we get:

$$Z (\mathcal{J}, \mathcal{J}^*, \mathcal{J}_\mu, \mathcal{J}^*_\mu) = \exp \left\{ i \int d^4x d^4y \left( m \mathcal{J}^*(x) G(x,y) \mathcal{J}(y) - \mathcal{J}^*(x) G(x,y) D_\mu \mathcal{J}^\mu (y) - \mathcal{J}^\mu (x) D_\mu G(x,y) D_\nu \mathcal{J}^\nu (y) - \frac{1}{m} \mathcal{J}_0 (x) \delta^4 (x-y) \mathcal{J}_0 (y) \right) \right\}. \tag{38}$$

Here

$$G(x,y) = (D_\mu D^\mu + m^2)^{-1} \delta^4 (x-y) \tag{39}$$

is the GF of the scalar charged field $\varphi$ in external EM field.

If one formally makes $\mathcal{J}_\mu = \mathcal{J}^*_\mu = 0$ we obtain from equation (38) the generating functional GF in KG theory.

In the general case, as it will be shown bellow (see equation (51)), the generating functional (38) exactly coincides with that calculated in matrix form, where

$$\mathcal{I}_\alpha = (\mathcal{J}^*, \mathcal{J}_0^0, -\mathcal{J}^*1, -\mathcal{J}^*2, -\mathcal{J}^*3); \quad \mathcal{I}_\alpha = (\mathcal{J}, \mathcal{J}_0, \mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3). \tag{40}$$

We want stress that $\det (D_\mu D^\mu + m^2)^{-1}$ does not appear in equation (38), see point 2 after equation (52).

### 2.3 $H$-approach. Matrix form

Starting from equation (1) we can define the momenta

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \psi_\alpha} = i (\psi^* \beta_0)_\alpha, \quad p^*_\alpha = \frac{\partial \mathcal{L}}{\partial \psi^*_\alpha} = 0, \quad p_\alpha = \frac{\partial \mathcal{L}}{\partial \psi^*_\alpha} = 0, \quad p^*_\alpha = \frac{\partial \mathcal{L}}{\partial \psi_\alpha} = 0,$$  \tag{41}

for $\alpha = 1, 2, 3 \equiv i$.  \tag{42}
The initial Hamiltonian \( H \) is equal to:

\[
H = \int d^3x \left\{ p^a \dot{\psi}_a + p^{*a} \dot{\psi}^*_a - \mathcal{L} \right\} = \\
= \int d^3x \left( -i\overline{\psi} \beta_k D^k \psi + \overline{\psi} m \psi - e\overline{\psi} \beta_0 A^0 \psi \right) \\
= \int d^3x \left( -i\overline{\psi} \beta_k \partial^k \psi + m \overline{\psi} \psi - e\overline{\psi} \beta_m A^m \psi \right), \quad k = 1, 2, 3 \quad (43)
\]

This is exactly equation (8), if we write equation (43) in component form.

Here we write down all the 1\(^{st}\) stage and the 2\(^{nd}\) stage constraints and Lagrangian multipliers in matrix form, omitting calculations:

- 1\(^{st}\) stage constraints
  \[
  \theta_\alpha = p_\alpha - i (\overline{\psi} \beta_0)_\alpha \\
  \theta^*_\alpha = p^{*\alpha}_\alpha
  \]

- 2\(^{nd}\) stage constraints
  \[
  \theta_2^\alpha = \left[ \left( 1 - (\beta_0)^2 \right) (i\beta_k D^k - m) \psi \right]_\alpha \\
  \theta^{*2}_\alpha = \left[ \overline{\psi} \left( i\beta_k \overline{D}^* k + m \right) \left( 1 - (\beta_0)^2 \right) \right]_\alpha
  \]

- Lagrangian multipliers

\[
(\beta_0^2 \lambda)_\alpha = \left[ i\beta_0 \left( i\beta_k D^k - m \right) \psi \right]_\alpha; \quad (\lambda^{*2} \beta_0^2)_\alpha = - \left[ \overline{\psi} \left( i\beta_k \overline{D}^* k + m \right) \beta_0 \right]_\alpha
\]

\[
\left[ \left( 1 - (\beta_0)^2 \right) (m - i\beta_k D^k) \eta \right]_{\alpha \beta} \lambda^\beta = 0, \\
\lambda^{*\beta} \left[ \eta \left( i\beta_k \overline{D}^* k + m \right) \left( 1 - (\beta_0)^2 \right) \right]_{\beta \alpha} = 0.
\]

All these \( \lambda_\alpha, \lambda^*_\alpha \) expressed through the components coincide with the corresponding \( \lambda_a \) in equations (22) to (24).

### 2.4 Quantization. Generating Functional.Matrix Form

As well as in component form in equation (33) we get, for generating functional in external EM field, the following expression:

\[
Z \left( I, \overline{I} \right) = Z_0^{-1} \int \mathcal{D} \psi \mathcal{D} \overline{\psi} \delta \left( \left( 1 - (\beta_0)^2 \right) \left( i\beta_k D^k - m \right) \psi \right)
\]
\[ \times \delta \left( \overline{\psi} \left( i \beta_k \overrightarrow{D^k} + m \right) \left( 1 - (\beta_0)^2 \right) \right) \]

\[ \times \exp \left\{ i \int d^4x \left( \overline{\psi} \left( i \beta_\mu D^\mu - m \right) \psi + \overline{\psi} I \psi + \psi I \right) \right\} \quad (49) \]

Introducing the auxiliary fields \( C \) and \( \tilde{C} \) instead functional \( \delta \)– function in equation (49) we get

\[ Z \left( I, \tilde{I} \right) = Z_0^{-1} \int \mathcal{D} \psi \mathcal{D} \overline{\psi} \mathcal{D} C \mathcal{D} \tilde{C} \times \exp \left\{ i \int d^4x \left( \overline{\psi} \left( i \beta_\mu D^\mu - m \right) \psi \right. \right. \]

\[ \left. + \left( 1 - (\beta_0)^2 \right) \left( i \beta_\mu D^\mu - m \right) \psi \right\} \]

\[ \left\{ \overline{\psi} \left( i \beta_\mu \overrightarrow{D^\mu} + m \right) \left( 1 - (\beta_0)^2 \right) C + \overline{\psi} I \psi + \psi I \right\}, \quad (50) \]

where one used that \( \beta_0 \left( 1 - (\beta_0)^2 \right) = 0 \).

Integrating over all fields \( \psi, \overline{\psi}, C \) and \( \tilde{C} \) we finally obtain:

\[ Z \left( I, \tilde{I} \right) = \exp \left\{ -i \int d^4x d^4y I (x) \left( S (x, y, A) \right. \right. \]

\[ \left. + \left( 1 - (\beta_0)^2 \right) \delta^4 (x - y) \right) I (y) \right\}, \quad (51) \]

where we have introduced the total GF of DK particle in external EM field \( A_\mu \):

\[ S (x, y, A) = (i \beta_\mu D^\mu - m)^{-1} \delta^4 (x - y) \quad (52) \]

One can make some important comments about expression (52).

1) If one writes down equation (52) in component form we get equation (38).

2) When we integrate over \( \psi, \overline{\psi}, C \) and \( \tilde{C} \) divergences appear and infinite expression for \( \det \left( i \beta_\mu D^\mu - m \right)^{-1} \). All this multipliers also arise in \( Z_0 \) and disappear from the final equation (51).

3) Nonrelativistic invariant term \( \sim \left( 1 - (\beta_0)^2 \right) \) in equation (51) arises at excluding nonphysical component \( \psi \), due to the second stage constraints \( (\theta_i^2, \theta_i^* \theta_i) \) in equations (16) and (17).

We will show that this term (which does not depend on charge) does not contribute to physical matrix elements of \( S \)-matrix.(see Appendix, point 1).

4) If one generalizes equations (49) to (52) to the case of interaction of DK particles with quantized EM fields we get the following expression for
generating functional for all GF of the theory (in \( \alpha \)-gauge):

\[
Z \left( I, \tilde{I}, \mathcal{J}_\mu \right) = Z_0^{-1} \int DA_\mu \exp \left\{ -i \int d^4 x \left( Tr \ln \frac{S(x, x, A)}{S(x, x, 0)} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right. \\
- \mathcal{J}_\mu A^\mu - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 - \int d^4 y \tilde{I}(x) (S(x, y, A) \\
+ \frac{1}{m} \left( 1 - (\beta_0)^2 \right) \delta^4(x - y) I(y) \right\},
\]

(53)

Here we insert in denominator and in \( Z_0 \) infinite constant \((\det S(x, x, 0))^{-1}\). As it is well known, the term \( \sim Tr \ln S(x, x, A) \) in equation (53) is responsible for appearance of all vacuum polarizations diagrams (see Section 3).

3 Equivalence matrix elements of S-matrix for physical states in DKP and KG Theories

1) From the beginning we write down the physical operator’s solution of DK equation for free particle; which will be needed for the proof of equivalence of the both theories.

This solution can be written in the following form (see [4])

\[
\psi_\alpha^{(0)}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 p \left\{ a^- (p) e^{-ipx} \psi^a_\alpha(p) + b^+ (p) e^{ipx} \psi^b_\alpha(p) \right\},
\]

(54)

\[
\bar{\psi}_\alpha^{(0)}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 p \left\{ a^+ (p) e^{ipx} \bar{\psi}^a_\alpha(p) + b^- (p) e^{-ipx} \bar{\psi}^b_\alpha(p) \right\}.
\]

(55)

Here

\[
\begin{aligned}
\left( \hat{p} - m \right) \psi_{a,b} = 0; \quad \bar{\psi}_{a,b} \left( \hat{p} - m \right) = 0; \quad \hat{p} \equiv \beta_\mu p^\mu \\
(i\beta_\mu \partial^\mu - m) \psi^0(x) = 0; \quad p_0 = \omega(p) = + (p^2 + m^2)^{1/2} = \omega
\end{aligned}
\]

(56)

Operators \( a^+ \), \( a^- \), \( b^+ \) and \( b^- \) satisfy to the usual commutation relations;

\[
\begin{aligned}
\psi^a_\alpha(p) &= \sqrt{\frac{\pi}{2\omega}} \left( 1, \frac{-i\omega}{m}, \frac{-ip^1}{m}, \frac{-ip^2}{m}, \frac{-ip^3}{m} \right) \\
\psi^b_\alpha(p) &= \sqrt{\frac{\pi}{2\omega}} \left( 1, \frac{i\omega}{m}, \frac{ip^1}{m}, \frac{-ip^2}{m}, \frac{ip^3}{m} \right)
\end{aligned}
\]

(57)
It is easy to check the scalar products

\[
\begin{align*}
(\psi^a(p), \psi^a(p)) &= \psi_{\alpha}^a(p) (\beta_0)_{\alpha\beta} \psi_\beta^a(p) = \overline{\psi}^a(p) \beta_0 \psi^a(p) = 1 \\
(\psi^b(p), \psi^b(p)) &= \psi_{\alpha}^b(p) (\beta_0)_{\alpha\beta} \psi_\beta^b(p) = \overline{\psi}^b(p) \beta_0 \psi^b(p) = -1 \\
(\psi^a(p), \psi^b(p)) &= \psi_{\alpha}^a(p) (\beta_0)_{\alpha\beta} \psi_\beta^b(p) = 0
\end{align*}
\]

(58)

Thus, there are only two linearly independent (physical) solutions of free DK equation.

One introduce

\[
\psi_p^{a,b}(x) = \pm (2\pi)^{-3/2} e^{\pm ipx} \psi^{a,b}(p), \quad p_0 = \omega.
\]

(59)

Then, utilizing equations (54) to (58) we get

\[
a^-(p) = \int d^3x \psi_p^{a}(x) \beta_0 \psi^0(x); \quad b^+(p) = \int d^3x \psi_p^{b}(x) \beta_0 \psi^0(x),
\]

(60)

and \( a^+ = (a^-)^*, \; b^- = (b^+)^* \).

2) Now we write the general connection between GF and matrix elements of S-matrix for physical states in DKP theory.

By definition \( n \)-particles GF of DKP-particles without external photon’s states equal, from equation (52), to

\[
G_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = (-1)^n \frac{\delta^{2n} \ln \mathcal{Z} (I, \vec{I}, \mathcal{J}_\mu = 0)}{\delta I(x_1) \ldots \delta I(x_n) \delta I(y_1) \ldots \delta I(y_n)} \bigg|_{I, I = 0} = \langle 0 | T \psi(x_1) \ldots \psi(x_n) \overline{\psi}(y_1) \ldots \overline{\psi}(y_n) | 0 \rangle,
\]

(61)

where we omit nonconnected GF.

On the other hand, GF (61) can be (formally) expressed through matrix element of S-matrix in the following way:

\[
G_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = \prod_{i=1}^n \int dx_i^x dy_i^y S(x_i - x_i^x) \\
\times \Gamma_n(x_1, \ldots, x_n; y_1, \ldots, y_n) S(y_i^y - y_i)
\]

(62)

Where \( S(x - y) \) is the total one particle GF and the so-called \( \Gamma_n \) functions which are equal, in momentum representation, to \( S_n(p_1, \ldots, p_n; q_1, \ldots, q_n) \), \( n \)-particles matrix element of S-matrix for physical states on the mass-shell

\[
\Gamma_n(p_1, \ldots, p_n; q_1, \ldots, q_n) = S_n(p_1, \ldots, p_n; q_1, \ldots, q_n) \\
p_{i0} = \left( (p_i)^2 + m^2 \right)^{1/2}; \quad q_{i0} = \left( (q_i)^2 + m^2 \right)^{1/2}; \quad \sum_{i=1}^n (q_i - p_i) = 0
\]

(63)
It is well known that one particle GF after renormalization has a pole \( \sim (m - \hat{p} - i\varepsilon)^{-1} \) with residue equal to one and \( \delta^4(x - y) \) peculiarity (see equation (51)). Thus, if we consider the following physical matrix element

\[
A \equiv \lim_{p_0 \to ((p_i)^2 + m^2)^{1/2}} \prod_{i=1}^{n} \bar{\psi}^{a,b}(p_i) \left( \frac{\hat{p}_i - m}{(\hat{p}_i^2 - m^2 + i\varepsilon)}^{-1} \right) G_n(p_1, \ldots, p_n; q_1, \ldots, q_n) \psi^{a,b}(q_i),
\]

and use equation (63) we get \( n \)-particle matrix element of \( S \)-matrix for physical states on mass-shell

\[
A \equiv \prod_{i=1}^{n} \bar{\psi}^{a,b}(p_i) S^{\alpha_1 \ldots \alpha_n; \beta_1 \ldots \beta_n}(p_1, \ldots, p_n; q_1, \ldots, q_n) \psi^{a,b}(q_i),
\]

where: \( p_0^0 = \left((p_i)^2 + m^2\right)^{1/2} \) and \( q_0^i = \left((q_i)^2 + m^2\right)^{1/2} \).

In Appendix, point 1, we show that all terms in \( S(x - y) \sim \delta^4(x - y) \left(1 - (\beta_0)^2\right) \) do not contribute to matrix elements on mass-shell.

The same situation arises if one considers matrix elements for physical states with many photons in the presence of DKP-particles.

Thus one goes to a very important conclusion: we can start from \( L \)-formulation of DKP-theory forgetting about constraints and to consider instead equation (53) the following expression for generating functional:

\[
Z(I, \bar{I}, J_\mu) = Z_0^{-1} \int D\psi D\bar{\psi} D A_\mu \exp \left\{ -i \int d^4x \left( \bar{\psi} i \beta_\mu D^\mu - m \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (A_\mu^\mu)^2 + \bar{\psi} I + \bar{\psi} J_\mu A_\mu \right\}
\]

and to get for physical matrix element of \( S \)-matrix the correct expression.

Expressed in another way we can use equation (91) for \( S \)-matrix with relativistic invariant definition of T-product of operators \( \psi(x) \) and \( \bar{\psi}(y) \) in interaction representation [5, 6]:

\[
\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \frac{i}{(2\pi)^4} \int d^4x e^{-ipx} d^4p \left\{ \frac{\hat{p}}{p^2 - m^2 + i\varepsilon} \right\}
\]

\[
\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \frac{i}{(2\pi)^4} \int d^4x e^{-ipx} d^4p \left\{ \frac{\hat{p}}{p^2 - m^2 + i\varepsilon} \right\}
\]

For simplicity we do not write down symmetrized expressions for all matrix elements due to identity of particles.
3) We prove equality between equation (64) and corresponding matrix element $S$-matrix in KG-theory.

Starting from definition (61) for GF in DKP-theory one write equation (64) in the form (we omit non-essential multipliers)

$$A \equiv \lim_{p_0 \to (p^2 + m^2)^{1/2}} \int \frac{d^n p}{(2\pi)^n} \left( \frac{1}{2} \int dy \frac{\overline{\psi}_p (y_i)}{\psi_p (x_i)} \left( i\beta^\mu_\mu \overline{\partial}_\mu - m \right) \right)\right) \times \langle 0 | T \psi (x_1) \cdots \psi (x_n) \psi (y_1) \cdots \psi (y_n) | 0 \rangle \left( i\beta^\mu_\mu \partial_j - m \right) \overline{\psi}_q (y_j) , \quad (68)$$

where $\psi (x_j)$ and $\overline{\psi}_q (y_j)$ are Heisenberg operators; $\psi^a_p (x)$ and $\overline{\psi}_q^b (y)$ are defined by equation (59).

Utilizing equations (3) and (67) for expression of operator $\psi (x)$ and $\overline{\psi}_q^b (y)$ in component form, consider the following term:

$$\int e^{ipx} d^4 x \overline{\psi}_p (x) \left( i\beta^\mu_\mu \partial_j - m \right) \psi (x) = \int e^{ipx} d^4 x \left[ \left( -m + \frac{i}{m} p^\mu \partial_\mu \right) \varphi (x) - \left( \partial_\mu + ip^\mu \right) \varphi_\mu \right]$$

As $ip^\mu e^{ipx} = \frac{\partial}{\partial x^\mu} e^{ipx}$ we can rewrite equation (69) in the form

$$\int e^{ipx} d^4 x \overline{\psi}_p (x) \left( i\beta^\mu_\mu \partial_j - m \right) \psi (x) = \int e^{ipx} d^4 x \left[ \left( \partial_\mu + \frac{1}{m} \right) \varphi (x) \right] + \int d^4 x \frac{\partial}{\partial x^\mu} \left[ e^{ipx} \left( \frac{\partial_\mu}{m} \varphi - \varphi_\mu \right) \right]$$

(70)

In the appendix one shows that the second term in equation (70) is equal to zero and thus physical matrix elements of scattering scalar charged particles coincide in DKP and KG theories.

Now we will prove the equality in both theories of many photons GF.

It is easy to show that generating functional of GF in KG theory has the form:

$$Z (\mathcal{J}^* , \mathcal{J} , \mathcal{J}_\mu ) = Z_0^{-1} \int D A \exp \left\{ -i \int d^4 x Tr \left( \ln G(x,x,A) - F_{\mu\nu} F^{\mu\nu} \right) + \frac{1}{2\alpha} \left( \partial_\mu A^\mu \right)^2 + \mathcal{J}_\mu A^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} , \quad (71)$$

---

The case of scattering charged particles by external EM field is a particular one and the equivalence of the both theories follows the general formula (71) in Appendix.
where GF $G(x, y, A)$ is defined in equation (39).

To get the generating functional of GF only for photons we have to put $J^* = J = 0$ in equation (71) and $I = \overline{I} = 0$ in equation (53). Equality of these equations will be established if we prove that

$$Z_A \equiv \det \frac{S(x, y, A)}{S(x, y, 0)} = \exp \text{Tr} \ln \frac{S(x, x, A)}{S(x, x, 0)} = \det \frac{G(x, y, A)}{G(x, y, 0)}.$$ \hspace{1cm} (72)

On the other hand

$$Z_A = Z_0^{-1} \int \mathcal{D} \psi \overline{\psi} \exp \left\{ i \int d^4 x \overline{\psi} \left( i \hat{D} - m \right) \psi \right\},$$ \hspace{1cm} (73)

where

$$Z_0 = \int \mathcal{D} \psi \overline{\psi} \exp \left\{ i \int d^4 x \overline{\psi} \left( i \hat{D} - m \right) \psi \right\}.$$ \hspace{1cm} (74)

In component form expression (73) equals to:

$$Z_A = Z_0^{-1} \int \mathcal{D} \varphi \mathcal{D} \varphi^* \mathcal{D} \varphi^* \mathcal{D} \varphi^* \exp \left\{ i \int d^4 x (\varphi^* \partial^\mu \varphi - \varphi^* \partial^\mu \varphi^* \varphi^\mu - m (\varphi^* \varphi + \varphi^* \varphi^\mu \varphi^\mu)) \right\},$$

After integration over $\varphi^* \varphi^\mu$ and $\varphi^\mu$ we get

$$Z_A = \tilde{Z}_0^{-1} \int \mathcal{D} \varphi \mathcal{D} \varphi^* \exp \left\{ -\frac{i}{m} \int d^4 x \varphi^* (D^\mu D_\mu + m^2) \varphi \right\},$$ \hspace{1cm} (75)

where now

$$\tilde{Z}_0^{-1} = \int \mathcal{D} \varphi \mathcal{D} \varphi^* \exp \left\{ -\frac{i}{m} \int d^4 x \varphi^* (\partial^\mu \partial_\mu + m^2) \varphi \right\}.$$ \hspace{1cm} (76)

Doing substitution the $\varphi \rightarrow \varphi^\mu \rightarrow \varphi$ we see that the determinant (73) is equal to the right hand side of equation (72).

The equivalence was proved.

\section{Reduction formulas in DKP formalism and equivalence with the KG Theory}

One writes down Yang-Feldman equations for Heisenberg operators in DKP formalism.

$$\psi(x) = \psi_{in}(x) + \int S_R(x - y) j(y) d^4 y$$ \hspace{1cm} (77)
\[ \psi (x) = \psi_{\text{out}} (x) + \int S_A (x - y) j (y) d^4 y \] (78)

Here \( \psi_{\text{in}} (x) \) and \( \psi_{\text{out}} (x) \) satisfy the free DKP equation:

\[ (i \beta_\mu \partial_\mu - m) \psi_{\text{in,out}} (x) = 0 \] (79)

\( S_{R,A} \) are the retarded and advanced GF of the free equation

\[ (i \beta_\mu \partial_\mu - m) S_{R,A} (x) = \delta^4 (x), \] (80)

\[ S_{R,A} (x) = \frac{1}{(2\pi)^4 m} \int e^{-ipx} \left( \frac{\hat{p} (\hat{p} + m)}{p^2 - m^2 \pm i\varepsilon p_0} - 1 \right). \] (81)

\( S_{R,A} \) is not equal to zero in up and down light cones. From equations (77) to (81) we obtain:

\[ (i \beta_\mu \partial_\mu - m) \psi (x) = j (x) . \] (82)

Choosing for \( \psi_{\text{in,out}} \) (\( \overline{\psi}_{\text{in,out}} \)) two linearly independent solutions \((54)\) and \((55)\) we can write (we omit index \( x \) and equation for \( \psi_{\text{in,out}} \))

\[ \psi_{\text{in,out}} = \frac{1}{(2\pi)^{3/2}} \int d^3 p \left\{ a_{\text{in,out}}^{-} (p) e^{-ipx} \psi^a (p) + b_{\text{in,out}}^{+} (p) e^{ipx} \psi^b (p) \right\} \] (83)

Analogously we can use equations \((59)\) and \((60)\) to express \( a_{\text{in,out}}^\pm (p) \) and \( b_{\text{in,out}}^\pm (p) \) through the scalar product

\[ a_{\text{in,out}}^{-} (p) = \int d^3 x \psi^{*a} (x) \beta_0 \psi_{\text{in,out}} (x) , \] (84)

and so on. By definition

\[ a^\pm_{\text{out}} = S^\dagger a^\pm_{\text{in}} S . \] (85)

We also can define Hisenberg operators of creation and annihilation

\[ a^- (p, x_0) = \int d^3 x \psi^{*a} (x) \beta_0 \psi (x) , \] (86)

\[ b^+ (p, x_0) = \int d^3 x \psi^{*b} (x) \beta_0 \psi (x) , \] (87)

and others; where \( \psi (x) \) is the solution of equation \((82)\).
Now it is possible to prove (13), starting from equations (77), (78), (86) and (87), that operators $a^\pm(p, x_0)$ and $b^\pm(p, x_0)$ have the limits (in weak sense) for any matrix elements over total system physical states, $|n_{in}\rangle |m_{out}\rangle$,

$$\lim_{x_0 \to \pm\infty} \langle m_{out} | a^\pm(p, x_0) | n_{in}\rangle = \langle m_{out} | a^\pm_{out,in} | n_{in}\rangle,$$

(88)

and analogously for $b^\pm(p, x_0)$.

For simplicity we carry out the proof of equivalence for matrix elements of scattering scalar particle (with positive charge from arbitrary initial state $|n_{in}\rangle$ to final one $\langle m_{out}|$. Using equation (88) we have

$$\langle m_{out} | a^-_{out}(q) a^+_{in}(p) | n_{in}\rangle = \langle m_{out} | a^-_{in} S a^+_{in} | n_{in}\rangle =$$

$$= \lim_{x_0 \to -\infty} \langle m_{out} | a^-_{out}(q) \int d^2 x \psi^*_\alpha(x) \beta_0 \psi(x) | n_{in}\rangle$$

$$= \langle m_{out} | a^-_{out}(q) (-i) \int d^2 x dx_0 i \frac{\partial}{\partial x_0} \left( \psi^\alpha(x) \beta_0 \psi(x) \right) | n_{in}\rangle$$

$$= \frac{1}{i(2\pi)^{3/2}} \int d^4 x e^{ipx} \psi^\alpha_{\alpha}(p) \left( i\beta_\mu \frac{\partial}{\partial x} - m \right)_{\alpha\beta} \langle m_{out} | a^-_{out}(q) \psi_\beta(x) | n_{in}\rangle$$

$$= -\frac{1}{(2\pi)^3} \int d^4 x d^4 y e^{ipx} \psi^\alpha_{\alpha}(p) \left( i\beta_\mu \frac{\partial}{\partial x} - m \right)_{\alpha\beta} \langle m_{out} | T(\psi_\alpha(x) \psi_\beta(y)) | n_{in}\rangle$$

$$\times \left( i\beta_\mu \frac{\partial}{\partial y} + m \right)_{\delta\gamma} \psi^b_\gamma(q) e^{-ipy} \equiv \langle m_{out}, q | p, n_{in}\rangle$$

(89)

If we utilize the LSZ method for all particles states in $|n_{in}\rangle$ and $\langle m_{out}|$ we get the equation (88).

Further, the proof of equivalence between DKP and KG goes in the same way as in the end of Section 3 (see equations (70) to (75) and Appendix).

5 Conclusions

1) Starting from canonical approach to DKP theory interacting with quantized EM field and constructing the generating functional for GF of the theory we strictly proved total equivalence between physical matrix elements of $S$-matrix in DKP and KG theories and between many photons GF in both theories.

The proof of equivalence between both theories have been carried out utilizing the more general approach of Lehmann, Symanzik and Zimmermann by reduction formalism.
We also proved the equivalence of the both theories, starting from Lagrangian approach to generating functional in DKP theory (see equation (66)) and forgetting about constraints.

2) In principle, the DKP as well as KG theories are nonrenormalizable ones even for scalar particles due to the logarithmical divergence of one loop diagrams of scattering two particles with exchange of two photons [5].

As it is well known that KG theory becomes renormalizable if we introduce a self interaction term $\sim \lambda (\varphi^* \varphi)^2$. This problem can be solved in DKP theory in the same way: it is necessary to add to $L$ in equation (1) terms

$$\lambda (\overline{\psi} P \psi)^2 = \lambda (\varphi^* \varphi)^2,$$

(90)

where $P = \prod \beta_{\mu}^2$ is the projector on the scalar part of $\psi$-function; $P$ is pseudoscalar.

3) In the framework the same method (sections 3,4) formally it is possible to prove equivalence between DKP and Proca equation for spin one particles, destructing from nonrenormalizability of these theories.

4) We would like to stress that DKP theory until now did not find wilder application although this theory has some advantages just due to the degeneration of $\beta_{\mu}$ matrices (one very simple to calculate trace that of) and due to minimal character of interaction with EM fields. One compares expressions for $S$-matrix in both theories:

$$S_{DKP} = T \exp \left\{ i \int \overline{e \psi} \beta_{\mu} A^\mu \psi d^4x \right\}$$

(91)

$$S_{KG} = T \exp \left\{ i \int i e (\varphi^* \partial_{\mu} \varphi - \partial_{\mu} \varphi^* \varphi - e \varphi^* A_{\mu} \varphi) A_{\mu} d^4x \right\}$$

(92)

In the last case the interaction contains proportional terms to $e$ and $e^2$. Due to this in higher (two and more loops) approximations combinatorial coefficients given to order $e^2$ before having a complicated form.

5) About equivalence of DKP and KG for description of unstable particles we would like to note that if we can apply conception of asymptotic states to some such particle and utilize for physical matrix elements of $S$-matrix the same method which have been used in Sections 3 and 4, then the proof of equivalence is obvious: for instance, for decay of $K_l^- mesons$ we can calculate the imaginary part of the GF of $K_l^- meson$ and get equivalence with exactness redefining the $\varphi$ component of DKP $\psi$ function: $i \varphi_{KG} = \varphi_{DKP} / m$, see equation (70).
6 Appendix

1) One proves that all terms \( \sim \left( 1 - (\beta_0)^2 \right) \delta^4 (x - y) \) in GF of scalar particles, equation (53), do not contribute to physical matrix elements (see phrase after equation (65)). First, consider the simplest case: GF in external EM field \( A_\mu \), which is equal

\[
i \langle 0 | T\psi(x) \bar{\psi}(y) | 0 \rangle = S(x, y, A) + \frac{1}{m} \left( 1 - (\beta_0)^2 \right) \delta^4 (x - y), \quad (93)\]

where \( S(x, y, A) = (i\beta_\mu D_\mu - m)^{-1} \delta^4 (x - y) \).

By definition, matrix elements of scattering particles with positive charge by external EM field is:

\[
\lim_{p_0 \to \sqrt{(p)^2 + m^2}} \lim_{q_0 \to \sqrt{(q)^2 + m^2}} \frac{(-1)}{(2\pi)^3} \int d^4x d^4y \bar{\psi}_p(x) \left( i\beta_\mu \frac{\bar{\psi}_p(x)}{\partial_x} - m \right) \left( i\beta_\mu \frac{\psi_q(y)}{\partial_y} + m \right) \psi_q(y) \quad (94)
\]

Inserting equation (93) in equation (94) we see that the term \( \sim \delta^4 (x - y) \) does not contribute to result since, by definition, \( \delta^- \) function we must change directions of arrows on the inverse and get zero:

\[
\frac{1}{m} \left( 1 - (\beta_0)^2 \right) \delta^4 (x_i - y_j) = 0, \quad \left( i\beta_\mu \frac{\psi_q(y)}{\partial_y} + m \right) \psi_q(y) = 0 \quad (95)
\]

In the case of quantized EM field we have to start from equation (53). So far as external photons do not influence on the appearance of terms \( \sim \delta^4 (x - y) \) it is enough to consider the case \( J_\mu = 0 \) in equation (93).

Thus any GF of scalar particles expressed through symmetrized product of one particle GF (93) under integral over \( A_\mu \) in equation (71)

\[
\langle 0 | T\psi(x_1) \ldots \psi(x_n) \bar{\psi}(y_1) \ldots \bar{\psi}(y_n) | 0 \rangle =
\]

\[
= \mathcal{Z}^{-1} \int D A_\mu \exp \left\{ -i \int d^4x \left( \det \ln \frac{S(x, x, A)}{S(x, x, 0)} - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \right) \right\}
\]

\[
\times \left( \sum_{i, j=1}^n \left( S(x_i, y_j, A) - \frac{1}{m} \left( 1 - (\beta_0)^2 \right) \delta^4 (x_i - y_j) \right) \right), \quad (96)
\]

\[\text{This case is considered in equations (71) to (76).}\]
where \( \sum_{p_{i,j}} \) means summation over all permutations \( x_i \) or \( y_j \).

All \( \delta^4 (x_i - y_j) \) we can wear out from sign of \( \int D\!A_\mu \).

After transition to matrix elements of \( S \)-matrix we can apply to these terms the same procedure which we used for equation (94) and get zero.

Thus we proved that terms \( \sim (1 - (\beta_0)^2) \delta^4 (x_i - y_j) \) do not contribute to physical matrix elements of \( S \)-matrix and do not violate relativistic invariance and microcausality for physical observable values.

2) Proof that the last term under total derivative in equation (70) which can contain quasilocal one is equal zero.

One writes down all terms for matrix element (89) in component form, utilizing equation (70) we get

\[
\langle m_{\text{out}}, q | p, n_{\text{in}} \rangle \approx \int d^4 x d^4 y \langle m_{\text{out}} | \exp \{ i (p x - q y) \} (\Box_x + m^2) (\Box_y + m^2) \times T (\varphi (x) \varphi^* (y)) - \partial_x \mu \left[ e^{i p x} (\Box_y + m^2) \right] e^{-i q y} + \partial_y \mu \left[ e^{-i q y} (\Box_x + m^2) \right] e^{i p x} + \partial_x \mu \partial_y \nu \left[ e^{i p x - i q y} (T (\varphi_x \varphi^y (y)) \right] \right] \bigg| \rightarrow (97) \]

The quasilocal term containing \( \delta^4 (x - y) \) arises only from the seventh one:

\[
\partial_x \mu \partial_y \nu T (\varphi (x) \varphi^* (y)) = T (\partial_x \mu \varphi (x) \partial_y \nu \varphi^* (y)) - i g^0 \mu g^{0 \nu} \delta^4 (x - y). \]

\(^2\)No essential multipliers are omitted
Here we used that $\delta^4(x - y) \{ \varphi(x), \varphi^*(y) \} = -i\delta^4(x - y)$. Thus,

$$\int d^4xd^4y \frac{\partial^\mu \partial^\nu}{m^2} T(\varphi(x) \varphi^*(y)) = $$

$$= - \frac{i}{m^2} (2\pi)^3 \delta(p - q) \int dx_0 dy_0 \partial_x \partial_{x_0} (\exp(ip_0x_0 - iq_0y_0) \delta(x_0 - y_0)) = 0.$$

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