Representations of $SU(1, 1)$ in Non-commutative Space Generated by the Heisenberg Algebra

H. Ahmedov and I. H. Duru

1. Feza Gürsey Institute, P.O. Box 6, 81220, Çengelköy, Istanbul, Turkey.
2. Trakya University, Mathematics Department, P.O. Box 126, Edirne, Turkey.

Abstract: $SU(1, 1)$ is considered as the automorphism group of the Heisenberg algebra $H$. The basis in the Hilbert space $K$ of functions on $H$ on which the irreducible representations of the group are realized is explicitly constructed. The addition theorems are derived.

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1. Introduction

Investigating the properties of manifolds by means of the symmetries they admit has a long history. Non-commutative geometries have become the subject of similar studies in recent decades. For example there exists an extensive literature on the $q$-deformed groups $E_q(2)$ and $SU_q(2)$ which are the automorphism groups of the quantum plane $zz^* = qz^*z$ and the quantum sphere respectively [1]. Using group theoretical methods the invariant distance and the Green functions have also been written in these deformed spaces [2].

In the recent work we started to analyze yet another non-commutative space $[z, z^*] = 1$ (i.e. the space generated by the Heisenberg algebra) by means of its automorphism groups: We considered $E(2)$ group transformations in $z, z^*$ space; and constructed the basis (which are written in terms of the Kummer functions) in this space where the unitary irreducible representations of $E(2)$ are realized [3]. This analysis revealed a peculiar connection between the 2-dimensional Euclidean group and the Kummer functions.

In the present work we continue to study the same non-commutative space $[z, z^*] = 1$, this time by means of the other admissible automorphism group $SU(1, 1)$.

In Section 2 we define $SU(1, 1)$ in the Heisenberg algebra $H$ and construct the unitary representations of the group in the Hilbert space $X$ where $H$ is realized.

In Section 3 we classify the invariant subspaces in the space of the bounded functions on $H$ where the irreducible representations of $SU(1, 1)$ are realized.

In Section 4 we show that in the Hilbert space $K$ of the square integrable functions only principal series is unitary. We construct the orthonormal basis in $K$ which can be written in terms of the Jacobi functions.

\begin{footnote}{E-mail: hagi@gursey.gov.tr and duru@gursey.gov.tr}
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Section 5 is devoted to the addition theorems. These theorems provide a
group theoretical interpretation for the already existing identities involving
the hypergeometric functions which all are actually the Jacobi functions. They may also lead to new identities.

2. Weyl representations of $SU(1,1)$

The one dimensional Heisenberg algebra $H$ is the 3-dimensional vector space with the basis elements $\{z, z^*, 1\}$ and the bilinear antisymmetric product

$$[z, z^*] = 1.$$  \hfill (1)

The $*$-representation of $H$ in the suitable dense subspace of the Hilbert space $X$ with the complete orthonormal basis $\{|n\rangle, \, n = 0, 1, 2, \ldots \}$ is given by

$$z | n\rangle = \sqrt{n} | n-1\rangle, \quad z^* | n\rangle = \sqrt{n+1} | n+1\rangle.$$  \hfill (2)

Let us represent the pseudo-unitary group $SU(1,1)$ in the vector space $H$

$$g \begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} z \\ z^* \end{pmatrix}.$$  \hfill (3)

Due to

$$a\bar{a} - b\bar{b} = 1$$  \hfill (4)

the transformations (3) preserve the commutation relation

$$[gz, gz^*] = [z, z^*].$$  \hfill (5)

Therefore

$$gz = U(g)zU^{-1}(g), \quad gz^* = U(g)z^*U^{-1}(g)$$  \hfill (6)

where $U(g)$ is the unitary representation of $SU(1,1)$ in $X$:

$$U(g_1)U(g_2) = U(g_1g_2), \quad U^*(g) = U^{-1}(g) = U(g^{-1}).$$  \hfill (7)

The Cartan decomposition for the group reads

$$g = k(\phi)h(\alpha)k(\psi),$$  \hfill (8)

where

$$k(\psi) = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}, \quad h(\alpha) = \begin{pmatrix} \cosh \frac{\alpha}{2} & \sinh \frac{\alpha}{2} \\ \sinh \frac{\alpha}{2} & \cosh \frac{\alpha}{2} \end{pmatrix}.$$  \hfill (9)

For the subgroup $k(\psi)$ we have

$$U(k(\psi)) | n\rangle = e^{-i\frac{2\psi}{2}} | n\rangle.$$  \hfill (10)
Let us choose the following realizations for \(z\), \(z^*\) and \(X\):

\[
z = \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} \right), \quad z^* = \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right),
\]

\[
\langle x \mid n \rangle = \Psi_n(x), \quad \Psi_n(x) = \sqrt{\frac{e^{-x^2}}{2^n n! \sqrt{\pi}}} H_n(x),
\]

where \(H_n\) is the Hermite polynomial. From

\[
h(\alpha) z = \frac{1}{\sqrt{2}} \left( x e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}} \frac{d}{dx} \right)
\]

and

\[
\int_{-\infty}^{\infty} dx \Psi_m(x) \Psi_n(x) = \delta_{nm}
\]

we get

\[
U(h(\alpha)) \Psi_m(x) = e^{\frac{\alpha}{2}} \Psi_m(e^{\frac{\alpha}{2}} x).
\]

Matrix elements of \(U(h(\alpha))\) in the basis \(|n\rangle\) reads

\[
U_{mn}(h) \equiv \langle m \mid U(h(\alpha)) \mid n \rangle = e^{\frac{\alpha}{2}} \int_{-\infty}^{\infty} dx \Psi_m(x) \Psi_n(e^{\frac{\alpha}{2}} x).
\]

Evaluating this integral we get

\[
U_{mn}(h) = \frac{2^{\frac{m-n}{2}}}{(\frac{m-n}{2})!} \sqrt{\frac{n! \sinh^{n-m} \frac{\alpha}{2}}{m! \cosh^{n+m+1} \frac{\alpha}{2}}} F\left(-\frac{m}{2}, 1 - \frac{m}{2}; 1 + \frac{n - m}{2}; -\sinh^2 \frac{\alpha}{2}\right)
\]

if \(n \geq m\) and \(n + m\) is even and

\[
U_{mn}(h) = 0
\]

if \(n + m\) is odd. For \(m \geq n\) one has to replace \(m\), \(n\) and \(\alpha\) in the above formulas by \(n\), \(m\) and \(-\alpha\) respectively.

### 3. Irreducible representations of \(SU(1, 1)\) in \(H\)

The formula

\[
T(g) F(z) = F(gz)
\]

defines the representation of \(SU(1, 1)\) in the space \(K_0\) of bounded operators in the Hilbert space \(X\) representable as the finite sums

\[
F = \sum (f_n(\zeta) z^n + z^* n f_{-n}(\zeta)).
\]

Here \(f_n(\zeta)\) are functions of the self-adjoint operator \(\zeta = z^* z\). Using (19) we can rewrite (20) in the form

\[
T(g) F(z) = U(g) F(z) U^*(g)
\]
With the one parameter subgroups $g_1 = h(\varepsilon)$, $g_2 = k(\frac{\pi}{2})h(\varepsilon)k(-\frac{\pi}{2})$ and $g_3 = k(\varepsilon)$ of $SU(1, 1)$ we associate the linear operators $E_k : K_0 \rightarrow K_0$

$$E_k(F) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(T(g_k)F - F)$$  \hspace{1cm} (22)

with the limit being taken in the strong operator topology. Inserting (21) into (22) we get (with $H_{\pm} = -E_1 + iE_2, \quad H = iE_3$)

$$H_-(F) = \frac{1}{2}[F, z^2], \quad H_+(F) = \frac{1}{2}[z^2, F], \quad H(F) = \frac{1}{2}[\zeta, F],$$ \hspace{1cm} (23)

which implies the Lie algebra of $SU(1, 1)$

$$[H_+, H_-] = 2H, \quad [H, H_{\pm}] = \pm H_{\pm}. \hspace{1cm} (24)$$

The irreducible representations labelled by pair $(\tau, \varepsilon)$, $\tau \in C$ and $\varepsilon = 0, \frac{1}{2}$ are given by the formulas \[4\]

$$H_- D_k^{(\tau, \varepsilon)} = -(k + \tau + \varepsilon)D_{k-1}^{(\tau, \varepsilon)}, \hspace{1cm} (25)$$

$$H_+ D_k^{(\tau, \varepsilon)} = (k - \tau + \varepsilon)D_{k+1}^{(\tau, \varepsilon)}, \hspace{1cm} (26)$$

$$HD_k^{(\tau, \varepsilon)} = (k + \varepsilon)D_k^{(\tau, \varepsilon)}. \hspace{1cm} (27)$$

(23) and (27) imply

$$D_k^{(\tau, \varepsilon)} = z^{*2(k+\varepsilon)}f_k^{(\tau, \varepsilon)}(\zeta) \hspace{1cm} (28)$$

for $k \geq 0$ and

$$D_k^{(\tau, \varepsilon)} = f_k^{(\tau, \varepsilon)}(\zeta)z^{-2(k+\varepsilon)} \hspace{1cm} (29)$$

for $k < 0$. By substituting (28) in (23) and (26) with

$$f_k^{(\tau, \varepsilon)}(\zeta) = \sum_{n=0}^{\infty} \frac{(-)^n 2^{n+k+\varepsilon}}{n!}C_{kn}z^{*n}z^n \hspace{1cm} (30)$$

we get the recurrence relations

$$nC_{kn-1} + \frac{k + \varepsilon + \tau}{2k + 2\varepsilon + n - 1}C_{k-1n} - (2k + 2\varepsilon + n)C_{kn} = 0, \hspace{1cm} (31)$$

$$C_{kn+1} - C_{kn+2} - (k + \varepsilon - \tau)C_{k+1n} = 0, \hspace{1cm} (32)$$

which are solved by

$$C_{kn} = \frac{\Gamma(1 + \tau + \varepsilon + k + n)}{\Gamma(1 + 2\varepsilon + 2k + n)}. \hspace{1cm} (33)$$

Using

$$z^{*n}z^n = \zeta(\zeta - 1)\cdots(\zeta - n + 1) \hspace{1cm} (34)$$

for $k \geq 0$ we get

$$f_k^{(\tau, \varepsilon)}(\zeta) = (-2)^k \frac{\Gamma(1 + \tau + k')}{\Gamma(1 + 2k')} F(-\zeta, 1 + \tau + k'; 1 + 2k'; 2), \hspace{1cm} (35)$$
where \( k' = k + \epsilon \). The functions \( f_k^{(\tau, \epsilon)} \) for \( k < 0 \) is shown to be defined from the expression
\[
f_k^{(\tau, \epsilon)}(\zeta) = f_{-k}^{(\tau, -\epsilon)}(\zeta). \tag{36}
\]

From (25), (26) and (27) we conclude that \( SU(1, 1) \) admits the following irreducible representations:

i) \( T_{(\tau, \epsilon)} : (\tau + \epsilon) \in \mathbb{Z} \)

ii) \( T_{(\tau, \epsilon)}^\pm : (\tau + \epsilon) \in \mathbb{Z}, \tau - \epsilon < 0 \), that is \( \tau = -\frac{1}{2}, -1, -\frac{3}{2}, ... \)

iii) \( T_{(\tau, \epsilon)}^0 : (\tau + \epsilon) \in \mathbb{Z}, \tau - \epsilon \geq 0 \), that is \( \tau = 0, \frac{1}{2}, 1, \frac{3}{2}, ... \)

The corresponding invariant subspaces are:

i) \( V_{(\tau, \epsilon)} \) generated by \( \{D_{(\tau, \epsilon)}k\}_{k=\infty}^{-\infty} \)

ii) \( V^+_{(\tau, \epsilon)} \) and \( V^-_{(\tau, \epsilon)} \) generated by \( \{D_{(\tau, \epsilon)}k\}_{k=-\infty}^{\infty} \) and \( \{D_{(\tau, \epsilon)}k\}_{k=\infty}^{-\infty} \)

iii) \( V^0_{(\tau, \epsilon)} \) generated by \( \{D_{(\tau, \epsilon)}k\}_{k=-\tau-\epsilon}^{\infty} \)

4. Unitary irreducible representations of \( SU(1, 1) \) in \( H \)

We can define the norm in the subspace of \( K_0 \) with \( f_n(\zeta) \) in (24) being the functions with finite support in \( \text{Spect}(\zeta) = \{0, 1, 2, ...\} \) as
\[
\|F\| = \sqrt{\text{tr}(F^*F)}. \tag{37}
\]

Completion of this subspace leads to the Hilbert space \( K \) of the square integrable functions in the linear space \( H \) with the scalar product
\[
(F, G) = \text{tr}(F^*G). \tag{38}
\]

Using (21), the unitarity of \( U(g) \) and the property of the trace we conclude that the representation \( T(g) \) in \( K \) is unitary. (23) implies the real structure in the Lie algebra
\[
H^+_\mp = -H^-_\mp, \quad H^*_\mp = H. \tag{39}
\]

To investigate the unitarity of the irreducible representations in the Hilbert space \( K \) classified in the previous section we consider the orthogonality condition for the basis elements \( D_k^{(\tau, \epsilon)} \). Using (2) and (28) we get
\[
(D_k^{(\tau, \epsilon)} , D_m^{(\tau', \epsilon')}) = \delta_{mk}\delta_{\epsilon\epsilon'}\sum_{n=0}^{\infty} \frac{(n+2k+2\epsilon)!}{n!} f_k^{(\tau, \epsilon)}(-n) f_k^{(\tau', \epsilon)}(-n). \tag{40}
\]

Putting
\[
s = 1 - e^{-t}, \quad \lambda = 1 + 2(k + \epsilon) + \mu \tag{41}
\]
in the formula [3]
\[
\sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} s^n F(-n, a; \lambda; 2) F(-n, b; \lambda; 2) =
\]
\[
= (1-s)^{a+b-\lambda}(1+s)^{-a-b} F(a, b; \lambda; \frac{4s}{1+s^2}) \tag{42}
\]
and taking first the limit \( \mu \to +0 \) and then \( t \to \infty \) we obtain for \( \tau = -\frac{1}{2} + i\rho, \rho \in R \) the orthogonality relations
\[
(D_k^{(-\frac{1}{2}+i\rho,\epsilon)}, D_m^{(-\frac{1}{2}+i\rho',\epsilon')} ) = \delta_{nk} \delta_{\epsilon\epsilon'} \delta(\rho - \rho').
\] (43)

In the deriving of the above relation we used
\[
F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}
\] (44)

and the representation
\[
\lim_{t \to \infty} e^{-izt} e^{i0} = -2\pi i \delta(z)
\] (45)

for the Dirac delta function. For other values of \( \tau \) there is no orthogonality condition. Thus in \( K \) only the representation \( T_{(\tau,\epsilon)} \) with \( \tau = -\frac{1}{2} + i\rho \) of Section 3, which is the principal series is unitary.

5. The addition theorems

(i) Restriction of (19) on the subspace \( V_{(\tau,\epsilon)} \) reads:
\[
T(g) D_k^{(\tau,\epsilon)} = \sum_{n=-\infty}^{\infty} t_{nk}^{(\tau,\epsilon)}(g) D_n^{(\tau,\epsilon)}
\] (46)
or
\[
U(g) D_k^{(\tau,\epsilon)} U^*(g) = \sum_{n=-\infty}^{\infty} t_{nk}^{(\tau,\epsilon)}(g) D_n^{(\tau,\epsilon)}
\] (47)

where
\[
t_{kn}^{(\tau,\epsilon)}(g) = \frac{e^{-i(k+n)\phi-i(k+\epsilon)\psi}}{(k-n)!} \frac{\Gamma(1+\tau-\epsilon-n) \sinh^{k-n} \frac{\alpha}{2}}{\Gamma(1+\tau-\epsilon-k) \cosh^{k+n+2\epsilon} \frac{\alpha}{2}} \times F(-\tau-\epsilon-n, 1+\tau-\epsilon-n; 1+k-n; -\sinh^2 \frac{\alpha}{2})
\] (48)

are the matrix elements of the irreducible representations which are valid for \( k \geq n \). For \( k < n \) one has to replace \( k \) and \( n \) on the right hand side by \(-k\) and \(-n\) respectively.

(ii) Restriction of (19) on the subspaces \( V_{(\tau,\epsilon)}^+ \) and \( V_{(\tau,\epsilon)}^- \) gives the following addition theorems:
\[
U(g) D_k^{(\tau,\epsilon)} U^*(g) = \sum_{n=\infty}^{\tau-\epsilon} t_{nk}^{(\tau,\epsilon)}(g) D_n^{(\tau,\epsilon)}
\] (49)
and
\[
U(g) D_k^{(\tau,\epsilon)} U^*(g) = \sum_{n=-\infty}^{\infty} t_{nk}^{(\tau,\epsilon)}(g) D_n^{(\tau,\epsilon)}.
\] (50)
(iii) On the subspaces $V^0_{(\tau,\epsilon)}$ the addition theorem reads

$$U(g)D^{(\tau,\epsilon)}_kU^*(g) = \sum_{n=-\tau-\epsilon}^{\tau-\epsilon} t^{(\tau,\epsilon)}_{nk}(g)D^{(\tau,\epsilon)}_n.$$  \hfill (51)

Sandwiching both sides of (47), (49), (50) and (51) between the states $\langle l \mid$ and $\mid s \rangle$ we get

$$\sum_{m,t=0}^{\infty} U_{lm}(g)\overline{U_{nt}(g)}(D^{(\tau,\epsilon)}_k)_{mt} = \sum_{n} t^{(\tau,\epsilon)}_{nk}(g)(D^{(\tau,\epsilon)}_n)_{ts}$$  \hfill (52)

Multiplying (47), (49), (50) and (51) by $U(g)$ from the right and sandwiching them between the states $\langle l \mid$ and $\mid s \rangle$ we get

$$\sum_{m=0}^{\infty} U_{lm}(g)(D^{(\tau,\epsilon)}_k)_{ms} = \sum_{m=0}^{\infty} \sum_{n} t^{(\tau,\epsilon)}_{nk}(g)(D^{(\tau,\epsilon)}_n)_{tn}U_{ms}(g)$$  \hfill (53)

Multiplying (47), (49), (50) and (51) by $U^*(g)$ and $U(g)$ from the left and right respectively and sandwiching them between the states $\langle l \mid$ and $\mid s \rangle$ we get

$$(D^{(\tau,\epsilon)}_k)_{ts} = \sum_{m,t=0}^{\infty} \sum_{n} t^{(\tau,\epsilon)}_{kn}(g)U_{ts}(g)\overline{U_{mt}(g)}(D^{(\tau,\epsilon)}_n)_{mt}$$  \hfill (54)

where

$$(D^{(\tau,\epsilon)}_k)_{mt} = \sqrt{\frac{m!}{t!}} f^{(\tau,\epsilon)}_{k}(t)\delta_{m,t+2k+2\epsilon}$$  \hfill (55)

for $k \geq 0$ and

$$(D^{(\tau,\epsilon)}_k)_{mt} = \sqrt{\frac{t!}{m!}} f^{(\tau,\epsilon)}_{k}(m)\delta_{m,t+2k+2\epsilon}$$  \hfill (56)

for $k < 0$.

Finally we like to give two simple specific examples: Let $g = h(\alpha)$, $\epsilon, k = 0$ in (51) that is $\tau$ is positive integer. Taking $s, l = 0$ in (52) and (53) we get

$$P_{\tau}(\cosh \alpha) = \frac{1}{\sqrt{\pi} \cosh \frac{\tau}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n!} \tanh^{2n} \frac{\alpha}{2} F\left(-2n, 1 + \tau; 1 + n; -\sinh^{2} \frac{\alpha}{2}\right)$$  \hfill (57)

and

$$1 = \sum_{n=0}^{\tau} \frac{(-)^{n} (\tau + n)!}{(n!)^{2}} \frac{\alpha}{\tau - n)!} \tanh^{2n} \frac{\alpha}{2} F\left(-\tau, 1 + \tau; 1 + n; -\sinh^{2} \frac{\alpha}{2}\right)$$  \hfill (58)

respectively. $P_{\tau}$ in (57) is the Legendre function.
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