An inverse boundary value problem for harmonic differential forms

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Abstract

We show that the full symbol of the Dirichlet to Neumann map of the $k$-form Laplace's equation on a Riemannian manifold (of dimension greater than 2) with boundary determines the full Taylor series, at the boundary, of the metric. This extends the result of Lee and Uhlmann for the case $k = 0$. The proof avoids the computation of the full symbol by using the calculus of pseudo-differential operators parametrized by a boundary normal coordinate and recursively calculating the principal symbol of the difference of boundary operators.
1 Introduction

While there are many results on the uniqueness of recovery of the coefficients of an elliptic partial differential equation from boundary data in the case of a single partial differential equation, there are few results for systems of PDEs. One might expect that the complete boundary data for a system might be sufficient to recover multiple coefficients and yet the results to date have been in essentially scalar cases. Lee and Uhlmann showed that the full Taylor series, at the boundary, of a metric can be obtained from the total symbol of the Dirichlet to Neumann map of the scalar Laplace’s equation. One might expect to obtain at least the same information from the Dirichlet to Neumann map associated with the Laplacian operator on $k$-forms. The 1-form, or vector Laplacian on 3-manifolds being the example with the most obvious applications. We will show here that the full symbol of the $k$-form Laplacian does indeed determine the Taylor series of the metric at the boundary, and hence under suitable assumptions an analytic metric can be recovered from this data. The method used, in common with [4], avoids the computation of the full symbol of the Dirichlet to Neumann map. Rather, by using the calculus of pseudo-differential operators parametrized by a boundary normal coordinate, we recursively calculate the principal symbol of the difference of boundary operators, checking that it vanishes to a suitable order. As well as being an natural extension of the scalar case and interesting in its own right, we hope that this paper will stimulate the use of the technique in other inverse boundary value problems for elliptic systems of equations of interest to applications.

The context for all will be a smooth compact orientable manifold with boundary $M$, equipped with a Riemannian metric $g$. We also assume $\dim M = n > 2$. The metric tensor induces a volume form $\mu \in \Omega^n(M)$ and Hodge star isomorphism $\ast : \Omega^k(M) \rightarrow \Omega^{n-k}(M)$ is defined by the property

$$\ast \omega \wedge \omega = g(\omega, \omega)\mu,$$

where the action of the metric is extended naturally to act on $k$-forms. We can consider the Hodge star on $k$-forms as a contraction of the tensor $g^{\sharp} \otimes^k \otimes \mu$. Here $g^{\sharp}$ is the covariant metric tensor.

The total symbol of an operator $P$ on functions on $\mathbb{R}^n$ is

$$p(x, \xi) = e^{-ix \cdot \xi} P(e^{ix \cdot \xi}).$$

A classical pseudo-differential operator of order $m$ has a full symbol which is an asymptotic sum of terms $p_{m-j}(x, \xi)$ which are smooth in $\xi \neq 0$ and for $\lambda > 0$ are homogeneous of degree $m - j$

$$p_{m-j}(x, \lambda \xi) = \lambda^{m-j} p_{m-j}(x, \xi).$$

The principal symbol is $p_m$ also denoted by $\sigma_m(P)$. The class of classical pseudo-differential operators is denoted by $\Psi DO^m_\text{cl}(\mathbb{R}^n)$. There are more general classes of pseudo-differential operators based on more general symbols, but we shall not need them here. These classes form a graded algebra under composition

$$\circ : \Psi DO^m_\text{cl} \times \Psi DO^{m'}_\text{cl} \rightarrow \Psi DO^{m+m'}_\text{cl}.$$

To obtain the principal symbol of the composite one just takes the product:

$$\sigma_{m+m'}(PQ) = \sigma_m(P) \sigma_{m'}(Q)$$

1
however the full symbol of the product is rather more complicated. Operators in
\( \Psi DO_{cl}^{\infty} = \bigcap_{m \in \mathbb{R}} \Psi DO_{cl}^m \) are called smoothing operators. The full symbol of a pseudo-
differential operator determines the operator modulo smoothing operators. For brief
introduction to pseudo-differential operators we recommend the notes \[ 3 \] and for more
detail Shubin \[ 10 \]. We note that the definition of pseudo-differential operators can be
extended to smooth manifolds using coordinate charts. Here the principal symbol is in-
variantly defined as a function on the cotangent bundle while the total symbol depends
on choice of coordinates.

Following \[ 4 \] we will consider pseudo-differential operators on a smooth manifold
\( Y \) depending smoothly on a parameter \( t \). For our purposes we will have \( Y = \partial M \) and \( t \) the normal distance from the boundary. We say that \( P \in \Psi DO^{m,r}(Y, \mathbb{R}^+) \) if it is a
family of pseudo-differential operators of order \( m \) on \( Y \), varying smoothly up to \( t = 0 \),
and such that
\[
P = \sum_{j=0}^r t^{r-j} P_j
\]
with \( P_j \) a smooth family of operators on \( Y \) of order \( m - j \). This definition extends
naturally to operators on bundles, in our case the bundle of \( k \)-forms being the important
element.

The symbol of \( P \in \Psi DO^{m,r}(Y, \mathbb{R}^+) \) is defined to be the vector
\[
(\sigma_{m-j}(P_j))'_{j=0}
\]
evaluated at \( t = 0 \). This is a vector of functions on the cotangent bundle of \( Y \). For the
case of an operator on a vector bundle, each of these functions is a field of endomor-
phisms on the fibres of the bundle.

Let \( u \) be a 1-form then the Bochner Laplacian (sometimes called the rough Laplacian)
is the operator expressed in coordinates as \( -\sum_{ij} g^{ij} u_{;ij} \). The principal symbol in
this case is \( gI \) where \( I \) is the identity on 1-forms. The formal adjoint with respect to a
metric of the exterior derivative on \( k \)-forms is \( \delta = (-1)^{(nk+n+1)} d* \). The Laplacian on
differential forms (or Hodge Laplacian) is then \( \Delta = d\delta + \delta d \). The principal symbol of
\( d \) is \( \sigma_d(\xi) = i\xi \wedge \omega \). Let \( X \wedge \omega \) denote the contraction of the differential form \( \omega \) with
respect to the vector field \( X \). We denote by \( \xi^* \) the vector field dual to the one form \( \xi \).
The principal symbol of \( \delta \) is then \( \sigma_{\delta}^1(\xi) = -i\xi^* \omega \). We conclude using that contraction
is an antiderivation on forms
\[
\sigma_{\delta}^1(\xi)\omega = \xi^* \omega + \xi \wedge (\xi^* \omega) = g(\xi, \xi)\omega.
\]
The connection between the Laplacian and the Bochner Laplacian, as well as an alter-
native way to calculate the principal symbol of the former, is given by the coordinate
expression for the Laplacian
\[
(\Delta u)_{i_1...i_k} = \sum_{ij} \left( -g^{ij} u_{i_1...i_k;i_j} + \sum_{\alpha=1}^k R^j_{i\alpha} u_{i_1...i_{\alpha-1}j i_{\alpha+1}...i_k} \right) + \frac{1}{2} \sum_{\alpha=1}^k \sum_{\beta=1}^k R^{i\beta}_{j\alpha} u_{i_1...i_{\alpha-1}j i_{\alpha+1}...i_k} u_{i_1...i_k}.
\]
Note that for a flat metric the Laplacian and Bochner Laplacian coincide. A differential form \( u \) satisfying Laplace’s equation \( \Delta u = 0 \) is called a harmonic form. On a
compact manifold without boundary, this is equivalent to the condition that the form is
a harmonic field, that is it is both exact, $du = 0$, and co-exact, $\delta u = 0$ as

$$
\langle u, \Delta u \rangle = ||du||^2 + ||\delta u||^2 + \int_{\partial M} \delta u \wedge *u + \int_{\partial M} u \wedge *du
$$

(1.2)

However on manifolds with boundary there can be harmonic forms which are not harmonic fields.

Closely related systems of elliptic partial differential equations occur in electromagnetics (the vector Helmholtz equation) and in linear elasticity.

In a linear elastic solid with metric tensor $g$ and with no body forces, the displacement field $u$ (a vector field) satisfies the equation $\text{Div}(CLu g) = 0$ where $u \mapsto \sum \partial_j u_{ij} g^{jk}$. (As usual a semi-colon indicates covariant differentiation with respect to following indices) and $\text{Div}$ is its formal adjoint $a_{ij} \mapsto \sum_{jk} a_{ij,k} g^{jk}$. (All sums will be indicated explicitly.) The elastic tensor $C$ is a field of automorphisms of the symmetric tensors on each fibre. The principal symbol of the elastic operator is $C$. For an isotropic solid $C = \lambda g \otimes g + \mu I$ where $I$ is the identity operator on symmetric tensor fields. The problem considered by \[7\] was the recovery of the Lamé parameters $\lambda$ and $\mu$ for an isotropic solid. They also considered a related anisotropic problem for two-dimensional elastic media.

Nakamura and Uhlmann \[8\] derive a factorisation for the anisotropic linear elasticity operator in boundary normal coordinates (for the flat metric). This allows them to recover the full Taylor series of the ‘surface impedance tensor’, which is a function of $C$, but not the complete Taylor series of $C$. For a special case, transversely isotropic media, $C$ can be recovered \[9\].

In electro-magnetic theory the electric field $E$ and magnetic filed $H$ are naturally defined as 1-forms, as to take measurements of these fields one must integrate over curves. The resulting electric and magnetic fluxes, $D$ and $B$ are naturally two forms as one must integrate them over surfaces to make a measurement. The material properties (for simplicity we consider a non-chiral, linear, insulating material) are the permittivity $\varepsilon$ and permeability $\mu$, these map one forms to two forms and are the Hodge star operators for an associated electric and magnetic Riemannian metric. Assuming all fields to be time harmonic with angular frequency $\omega$ and the electric charge density to be constant we have Maxwell’s equations

$$
\begin{align*}
\, dB &= 0, \, dD = 0 \\
\, dE &= -i\omega \mu H, \, D = \varepsilon E \\
\, dH &= i\omega \varepsilon E, \, B = \mu H
\end{align*}
$$

(1.3)

(1.4)

(1.5)

In a physically artificial situation where $\mu = \varepsilon = *$ (obviously after units have been scaled) we notice that $E$ and $H$ satisfy the vector Helmholtz equations $\Delta E = \omega^2 E$ and $\Delta H = \omega^2 H$.

The main result we prove is an extension to $k$-forms for $k > 0$, of the result of Lee and Uhlmann \[6\] using a similar factorization in boundary normal coordinates. Initially, our notion of a Dirichlet-to-Neumann map is non-standard for $k \neq 0$. Employing the multi-index notation $I = (i_1, \ldots, i_k)$ we write a $k$-form as $u = \sum_I u_I dx_I$ where $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$. Following \[6\] we use a coordinate chart $(x_1, \ldots, x_n) = (x', x_n)$
where \( x' = (x_1, \ldots, x_{n-1}) \) is a chart on the boundary, and \( x_n \) is the distance to the boundary. We denote by \( \partial_n \) both a unit vector field normal to the boundary and its associated derivation on functions. We extend this to \( k \) forms by its actions on the components as functions \( \partial_n f = \sum_{j} \partial_n x_j \). The operator \( \Lambda_g : u_{|\partial M} \mapsto (\partial_n u)_{|\partial M} \), where \( \Delta u = 0 \), is linear while somewhat unnatural. We will discuss the relationship to more natural Neumann data for Laplace’s equation in Section 2. Our main result is

**Theorem 1.1.** Let \( M \) be a smooth compact orientable Riemannian manifold with boundary, with \( \dim(M) = n > 2 \) and metric \( g \), and let \( k \) be an integer \( 0 \leq k \leq n \).

(i) The Dirichlet-to-Neumann map \( \Lambda_g : u_{|\partial M} \mapsto (\partial_n u)_{|\partial M} \) for the \( k \)-form Laplace’s equation \( \Delta u = 0 \) is a classical pseudo-differential operator of order one.

(ii) The Taylor series, at the boundary, of the metric in boundary normal coordinates is uniquely determined by the full symbol of \( \Lambda_g \). For \( 0 < k < n \) only one diagonal component of the full symbol is needed corresponding to \( dx_1 = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_i \) but for \( k = (n + 1)/2 \) the multi-index \( I = (i_1, \ldots, i_k) \) must exclude \( n \) and for \( k = (n - 1)/2 \), I must include \( n \).

Where our work differs from Lee and Uhlmann’s is in its use of families of operators parameterized by the normal distance. The case \( k = n \) is clearly equivalent to \( k = 0 \) so we need only consider the case \( 0 < k < n \). Lee and Uhlmann showed that the full Taylor series of a real analytic metric on a real analytic manifold, where the relative homotopy group of the boundary \( \pi(M, \partial M) \) is trivial, determines the metric, provided the manifold is strongly convex or the metric can be extended analytically to a larger manifold without boundary. Recent work of Lassas and Uhlmann 5 removes these geometric and topological hypotheses showing that an analytic metric is determined throughout any connected analytic manifold by the Dirichlet-to-Neumann map. It remains to be seen if the same techniques can be applied to the general \( k \)-form case.

Only a small modification of the argument is needed to prove a version of Theorem 1.1 for the equivalent \( k \)-form Helmholtz problem at fixed frequency.

## 2 Boundary Conditions

In a neighbourhood of the boundary where our boundary normal coordinates are defined, we can distinguish between tangential \( k \)-forms which have no \( dx_n \) in their coordinate expression, and normal forms which have a common factor \( dx_n \). Clearly the space of \( k \)-forms on this neighbourhood is the direct sum of the spaces of tangential and normal forms. The projection on to the tangential component is \( \pi_t(\omega) = \partial_n - (dx_n \wedge \omega) \) and on to the normal component \( \pi_n(\omega) = dx_n \wedge (\partial_n - \omega) \). From these expressions it is clear that \( *\pi_n = \pi_t * \) and that the Hodge star of a tangential form is normal and vice-versa.

Let \( i : \partial M \to M \) be the inclusion of the boundary. The tangential component of \( k \)-form \( u \in \Omega^k(M) \) at the boundary is then the pull-pack to the boundary \( i^* u \in \Omega^k(\partial M) \). The normal part of \( u \) at the boundary, can be determined uniquely from \( i^* u = *_\partial (\partial_n - u)_{|\partial M} \in \Omega^{n-k}(\partial M) \) where \( *_\partial \) is the induced Hodge-star on the boundary. Dirichlet data for harmonic \( k \)-forms consists of the both the tangential component and the normal component of the form at the boundary. Note a possible source of
confusion here. When considering forms on manifolds with a boundary \( i^* u \) is often thought of as the restriction of form to the boundary, and the pull back is often omitted, for example in Stoke’s theorem. In Theorem 1.1 our Dirichlet data \( u|_{\partial M} \) is the restriction to the boundary of a form, but in the sense of considering only base points on \( \partial M \) and no projection of the fibre on to the tangential component. This Dirichlet data together with the integrals of \( u \) on a basis of the relative homology group \( H_k(M, \partial M) \) determines a unique solution to \( \Delta u = 0 \). For simplicity we will assume that the said integrals are specified to be zero. We note that for the case \( k = 0 \) one simply specifies the integral of \( u \) on each connected component of \( M \) with a non-empty boundary. The natural Neumann data is \( i^* i^* du \) and \( i^* \delta u \). Note that as in the case \( k = 0 \) where Neumann data \( i^* du \) must have zero integral on the boundary, there are compatibility conditions for Neumann data. A natural Dirichlet-to-Neumann mapping is therefore \( \Pi_g : \Omega^k(\partial M) \times \Omega^{n-k}(\partial M) \rightarrow \Omega^{k-1}(\partial M) \times \Omega^{n-k-1}(\partial M) \) given by

\[
\Pi_g(f_\tau, f_\nu) = (i^* du, i^* \delta u)
\]

where \( \Delta u = 0 \), \( i^* u = f_\tau \), \( i^* i^* u = f_\nu \). Here we use \( \nu \) and \( \tau \) as labels for the normal and tangential components in the sense of the domain and range of \( \Pi_g \). We can now recast Theorem 1.1 in terms of this natural data.

**Corollary 2.1.** Let \( M \) be a smooth compact orientable Riemannian manifold with boundary, with \( \dim(M) = m > 2 \) and metric \( g \), and let \( k \) be an integer \( 0 \leq k \leq n \). (i) The natural Dirichlet-to-Neumann map \( \Pi_g \) for the \( k \)-form Laplace’s equation defined above is a classical pseudo-differential operator of order 1. (ii) The full symbol of \( \Pi_g \) determines the Taylor series (at the boundary) of the metric in boundary normal coordinates. Furthermore for \( k \not\in \left\{ (n-2)/2, (n-1)/2, n \right\} \) only the full symbol of the tangential part \( \Pi_g^{\tau \tau} \) is needed and for \( k \not\in \left\{ 0, (n+1)/2, (n+2)/2 \right\} \) only the full symbol of the normal part \( \Pi_g^{\nu \nu} \) is needed.

The proof of Corollary 2.1 follows the proof of Theorem 1.1 in Section 3.

In the case of electromagnetics note that when formulated in differential forms Maxwell’s equations (1.3)–(1.4) are independent of the ambient Euclidean metric and thus invariantly defined. In the inverse boundary value problems for time harmonic Maxwell’s equations one typically specifies the boundary data invariantly as the tangential component of \( E \) and \( H \). The isotropic case where both the electric and magnetic metrics are conformally flat, has been studied by \[11\] and \[4\].

By contrast in elasticity strain is a measure of the distortion of the Euclidean metric, and one seeks the elastic tensor with the ambient metric given. This problem is not diffeomorphism invariant.

## 3 Factorization and symbol calculation

We consider metrics to be equivalent if they are related by a diffeomorphism which fixes points on the boundary. Without loss of generality, therefore, we can assume that \( x_0 \) is the boundary normal coordinate for both metrics. Later we will make a more specific choice for the coordinate chart on the boundary.

We use notation from \[1\], in particular \( \Psi DO^{m,l} \) denotes families of pseudo-differential
operators, $P_{x_0}$, in $x'$ such that the $j$ term in the total symbol vanishes to order $l-j$ at $x_0 = 0$, and $\mathcal{DO}^{m,r}$ is the class of such differential operators.

Let $\Delta' = \sum_{i,j=1}^{n-1} h^{ij} D_i D_j$. We have that

$$\Delta = (D_{x_0}^2 + \Delta')I + ED_{x_0} + H(x, D_{x'}) ,$$

where $H$ is a first order system in $x'$ on the bundle of $k$-forms and $E$ is a smooth endomorphism of the bundle of $k$-forms. We use the notation $|\xi| = \sqrt{g(\xi, \xi)}$ for a covector $\xi$.

**Proposition 3.1.** There exists a $B(x, D_{x'}) \in \mathcal{PDO}_1^{\mathcal{M}}(\partial \mathcal{M}; \Omega^k(M))$ such that $\sigma_1(B) = |\xi'|_{x_0} I$ and

$$\Delta = (D_{x_0} I + E + iB)(D_{x_0} I - iB) ,$$

modulo smoothing and $B$ is unique modulo smoothing.

**Proof.** If we expand, we obtain

$$D_{x_0}^2 I + B^2 + ED_{x_0} - iEB + i[D_{x_0}, B].$$

Taking the principal symbol of $B$ as specified we obtain an error, in $\mathcal{PDO}_1^{\mathcal{M}}$. Now suppose we have chosen $B_j$ such that the error, $F_j$, is in $\mathcal{PDO}_1^{1-j}$. Let $B_{j+1} = B_j + C$ with $C$ in $\mathcal{PDO}_1^{1-j}$. Upon expanding we then obtain an extra term $CB_j + B_j C + E_j$ with $E_j$ of order $-j$. Taking $\sigma_{-j}(C) = -\frac{1}{2}(|\xi'|_{x_0}^{-1})^{1-j}(F_j)$. We have achieved an error one order better. Inducting and summing, we achieve an error in $\mathcal{PDO}_1^{\mathcal{M}}$.

As the choice at each stage was forced, $B$ is unique. $\square$

The importance of this factorization is that $B(0, D_{x'})$ is equal modulo smoothing terms to $\Lambda_x$. We will summarise the argument which is identical to that given by [3] for the 0-form case. Given a harmonic $k$-form $u$, let $\nu = (D_{x_0} I - iB)u$ so that $(D_{x_0} I + E + iB)\nu = 0$. These are both generalised heat equations, the second with ‘time’ reversed. As both are smoothing we see that $\partial_\nu u = Bu$ + smooth terms and hence $\Lambda_x = B \mod \mathcal{PDO}_1^{\mathcal{M}}$. This proves Theorem 1 part (i) and part (ii) will follow if we can show that two metrics $g_1, g_2$ with identical full symbols of $B$ at the boundary must agree to infinite order at the boundary. Rather than calculating the full symbol of $B$, we use the calculus of pseudo-differential operators parameterised by the normal distance. The advantage is that we need only calculate principal symbols.

We want to compare the Laplacians associated to two different metrics assumed equal up to order $l$ in the normal coordinate. Of course it is immediate from the principal symbol of $B$ that the metrics agree on the boundary so we can take $g_1 - g_2 = x_0^l k$ for some $l > 0$.

First we compare the Hodge star operators. By definition $\omega \wedge *\omega = g(\omega, \omega)\mu$ and we see that

$$*_1 = *_2 + x_0^l \alpha$$

where $\alpha$ is a smooth homomorphism from $\Omega^k$ to $\Omega^{n-k}$. 


Lemma 3.1. If \( \Delta_j \) is the Laplacian on k-forms associated with \( g_j \) then

\[
\Delta_2 - \Delta_1 = x_n^{l-1}FD_{x_n} + \text{A}
\]

where \( F \) is a smooth endomorphism and \( \text{A} \in \text{DO}^{2,l} \).

Proof. The Laplacian is defined by \( \ast d \ast d \) and \( d \ast d \ast \) where \( d \) is independent of the metric and \( \ast_2 = \ast_1 + x_n^l \alpha \).

The result follows simply by observing that in \( d \ast d \ast d \) and \( d \ast d \ast d \ast \) terms not in \( \Delta_1 \) will vanish to order \( l \) at \( x_n = 0 \) unless \( d \) is applied to the \( x_n^l \) term. If \( d \) is applied once to such a term we obtain a first order differential operator vanishing to order \( l - 1 \) and if twice a zeroth order operator vanishing to order \( l - 2 \). This is the result desired — we know there are no second order terms in \( D_{x_n} \) from our expression for the principal symbol.

\[\square\]

Lemma 3.2. Let \( \Delta_j \) be factored as in Prop 3.1 with \( E_j, B_j \) the corresponding terms. We then have that,

\[ B_2 - B_1 \in \Psi \text{DO}^{1,l}. \]

Proof. Let \( C = B_2 - B_1 \). As the principal symbols of \( B_1, B_2 \) agree at \( x_n = 0 \) so we have that \( C \) is in \( \Psi \text{DO}^{1,l} \).

Note that \( E_2 = E_1 + x_n^{l-1}F \). Expanding the factorizations for \( \Delta_2, \Delta_1 \) and subtracting we have that,

\[
\Delta_2 - \Delta_1 = i[\gamma, D_{x_n}] + B_1C + CB_1 + x_n^{l-1}FD_{x_n} - x_n^{l-1}iF(B + C) + C^2.
\]

After cancelling the \( x_n^{l-1}FD_{x_n} \), we have that

\[
i[\gamma, D_{x_n}] + B_1C + CB_1 - ix_n^{l-1}F(B + C) + C^2 \in \Psi \text{DO}^{2,l}.
\]

If \( C \in \Psi \text{DO}^{1,r} \) with \( 1 \leq r < l \) then we have that \( [\gamma, D_{x_n}] \in \Psi \text{DO}^{1,r-1}, B_1C + CB_1 \in \Psi \text{DO}^{2,r}, C^2 \in \Psi \text{DO}^{2,r}, x_n^{l-1}FB \in \Psi \text{DO}^{1,l-1} \) and \( x_n^{l-1}iFC \in \Psi \text{DO}^{1,r+l-1} \). Taking the residue modulo \( \Psi \text{DO}^{2,r} \) we have that \( i[\gamma, D_{x_n}] + B_1C + CB_1 \) is congruent to zero modulo \( \Psi \text{DO}^{2,r} \). Recall that

\[
C = \sum_{j \leq r} x_n^{j-1}C_j
\]

with \( C_j \in \Psi \text{DO}^{1,r+1} \). The only term of second order is \( C_l B + BC_1 \) so we deduce that the principal symbol of \( C_1 \) vanishes at \( x_n = 0 \). Let \( c_j \) denote the principal symbol of \( C_j \) at \( x_n = 0 \). We then have that

\[
(r - j)c_j + 2|\xi_j|c_{j-1} = 0
\]

for each \( j \). Iterating we conclude that \( c_j = 0 \) for each \( j \) which proves that \( C \in \Psi \text{DO}^{1,r+1} \).

Repeating, the result follows. \[\square\]

It follows from this Lemma that \( B_2 - B_1 \) restricted to \( x_n = 0 \) is \( C_l \) a pseudo-differential operator of order \( 1 - l \). Our main result will follow if we can compute the principal symbol of this operator and show that it determines \( k \), the lead term of \( g_1 - g_2 \).

Now

\[
\Delta_2 - \Delta_1 = x_n^{l_1}P_2 + x_n^{l_2}P_1 + x_n^{l_2-2}P_0 + x_n^{l_1}FD_{x_n}
\]
with $P_j$ a differential operator in $x'$ of order $j$ and we know from our principal symbol computation that $P_2$ is equal to $\sum \tilde{k}_{ij} D_{x_i} D_{x_j}$, where $\tilde{k} = -h^{-1} k h^{-1}$ where $g = dx^2 + h(x',d x') + O(x_n)$.

Arguing as above with $C = B_2 - B_1$ we have

$$i[C, D_{x_n}] + B_1 C + CB_1 - i x_n^{l-1} F (B + C) + C^2 = x_n^j P_2 + x_n^{l-1} P_1 + x_n^2 P_0. \quad (3.1)$$

So modulo $\Psi DO^{2-j}$ we have,

$$i[C, D_{x_n}] + B_1 C + CB_1 - i x_n^{l-1} F B_1 = x_n^j P_2 + x_n^{l-1} P_1 + x_n^2 P_0. \quad (3.2)$$

Let $C = \sum_{j=0}^l x_n^{l-j} C_j$ with $C_j \in \Psi DO_1^{1-j}$, let $c_j$ be the principal symbol of $C_j$. We have,

$$2|\xi'|_x c_0 = \sum_{i,j<n} \tilde{k}_{ij} \xi_i \xi_j,$$

$$2|\xi'|_x c_1 + i c_0 = \sigma_1(P_1)(\xi') + i|\xi'|_x F,$$

$$2|\xi'|_x c_2 + (l-1) c_1 = \sigma_0(P_0),$$

$$2|\xi'|_x c_{2+j} + (l-j-1) c_{1+j} = 0, \text{ for } 1 < j \leq l - 2.$$  

We therefore deduce that

$$c_l = K_l (|\xi'|_x)^{-l-1} \sum_{i,j<n} \tilde{k}_{ij} \xi_i \xi_j L + L_i |\xi'|^{-1} \sigma_1(P_1)(\xi') + i L_i |\xi'|^{-1} F + M_i \sigma_0(P_0) (|\xi'|_x)^{-l-1} \quad (3.3)$$

where $K_l, L_i, M_i$ are computable non-zero constants. (Of course, $M_0, M_1 = 0$.)

We want to show that $c_l = 0$ implies that $\tilde{k}_{ij} = 0$ for all $i, j$.

If the principal symbol $c_l = 0$, we have taking any component $rr$ of the symbol that

$$K_l \sum \sum \tilde{k}_{ij}(x) \xi_i \xi_j + L_i |\xi'| \sigma_1(P_1)_{rr}(x, \xi') + |\xi'|^2 (i L_i F_{rr} + c \sigma_0(P_0)_{rr}) = 0.$$  

As $P_1$ is a differential operator $\sigma_1(P_1)_{rr}(x, \xi')$ is linear in $\xi'$ and the final two terms are independent of $\xi'$. Since the middle term is not smooth, unless zero, as $\xi' \to 0$ and the other terms are smooth, we deduce that $\sigma_1(P_1) = 0$ and so we have,

$$K_l \sum \sum \tilde{k}_{ij}(x) \xi_i \xi_j + |\xi'|^2 (i L_i F_{rr} + c \sigma_0(P_0)_{rr}) = 0.$$  

This shows that $k_{ij}$ must be a scalar multiple of the identity matrix. Or more invariantly that $k(x)$ is a scalar multiple of $h(x)$. To see that $k$ is actually zero, we need to compute more precisely.

For convenience we now reduce to the case of a Euclidean background metric. We first prove

Lemma 3.3. Let

$$g_1 = h + x_n m,$$

$$g_2 = h + x_n m + x^j_n r,$$

and let $*_{j}$ be the Hodge * operator associated to $g_j$. We then have that $*_{1} - *_{2}$ modulo $O(x_n^{j+1})$ is independent of $m$.  

8
Proof. Let $\mu$ be a volume form of $h$ and $\mu_j$ a volume form for $g_j$. We have by definition $h(\mu, \mu) = 1$ and $g_j(\mu_j, \mu_j) = 1$. We then have that $\mu_2 = (1 + x_0 m(\mu, \mu) + x_1 r(\mu, \mu))^{-1/2} \mu$ and similarly for $g_1$. It is now clear that

$$\mu_2 = \frac{1}{\sqrt{1 + x_0 m(\mu, \mu) + x_1 r(\mu, \mu)}} \mu.$$ 

and

$$\mu_1 = \frac{1}{\sqrt{1 + x_0 m(\mu, \mu)}} \mu.$$ 

For any $v, \omega \in \Omega^k(M)$

$$v \wedge (\ast_2 - \ast_1) \omega = \frac{(h + x_0 m + x_1 r)(v, \omega)}{\sqrt{1 + x_0 m(\mu, \mu) + x_1 r(\mu, \mu)}} - \frac{(h + x_0 m)(v, \omega)}{\sqrt{1 + x_0 m(\mu, \mu)}}$$

$$= \left( x_1^j \left( r(v, \omega) - \frac{1}{2} r(\mu, \mu) h \right) + O(x_1^{j+1}) \right) \mu$$

which does not involve $m$. \hfill \Box

We also have,

Lemma 3.4. Let $g_2 = dx^2 + x_1 r + m$, and $g_1 = dx^2 + m$, where $m$ vanishes to second order at the origin. Let $\ast_j$ be the Hodge $\ast$ operator of $g_j$. We then have that $\ast_2 - \ast_1$ is independent of $m$ modulo terms of the form $x_1^j t + x_1^{j+1} w$ where $t$ vanishes to second order at the origin and $w$ is smooth.

Proof. Let $\mu$ be the volume form for $dx^2$. Arguing as above we have that,

$$v \wedge (\ast_2 - \ast_1) \omega = \frac{(dx^2 + m + x_1^j r)(v, \omega)}{\sqrt{1 + m(\mu, \mu) + x_1^j r(\mu, \mu)}} - \frac{(dx^2 + m)(v, \omega)}{\sqrt{1 + m(\mu, \mu)}}$$

which modulo terms vanishing appropriately at $x = 0$ equals the bilinear form

$$(dx^2 + m + x_1^j r)(1 - \frac{1}{2} (m(\mu, \mu) + x_1^j r(\mu, \mu))) - (dx^2 + m)(1 - \frac{1}{2} m(\mu, \mu)),$$  \hfill (3.4)

which upon expanding modulo appropriately vanishing terms equals,

$$x_1^j r - \frac{1}{2} x_1^j r(\mu, \mu) dx^2,$$  \hfill (3.5)

which does not involve $m$. \hfill \Box

Now fix a point $p$ where we will calculate the principal symbol of the difference of the Dirichlet to Neumann maps and show that it being zero implies that the next term of the difference of the metrics also vanishes there. We take geodesic normal coordinates about $p$ in the boundary and then extend normally with respect to $g_1$. Now we fix a point on the boundary $p$ where we will calculate the principal symbol of the difference of the Dirichlet to Neumann maps and show that it being zero implies that the next term of the difference of the metrics also vanishes there. We choose $x'$ to be a Riemann
normal coordinate system on the boundary and then extend normally with respect to $g_1$. In particular, we have
\[ g = dx^2_n + h(x, dx'), \]
and on the boundary $h((x', 0), dx') = dx'^2 + O(x'^2)$.

We then have, using Lemma 3.4 that,
\[ g_1 = dx^2 + m, \]
and
\[ g_2 = dx^2 + m + x'_r r \]
where $m = x'_r t$ with $t$ vanishing to second order at $x' = 0$. Let $g'_1 = dx^2$, and $g'_2 = dx^2 + x'_r r$. We then have that $* j - * j = x_n \alpha_j + \beta_j$ for $j = 1, 2$ with $\beta_j$ vanishing at $x' = 0$ to second order. We also have from our lemmas that
\[ (*_1 - *_2) - (*'_1 - *'_2) = x_n \gamma \]
with $\gamma$ vanishing to second order at $x' = 0$.

It is now clear that when computing the lead term of $d *_2 d(*_2 - *_1)$ and its appropriate permutations at the point $p$ that we can replace $* j$ by $* j'$ without changing the value. So to finish our theorem we take $g_1 = dx^2$ and $g_2 = dx^2_n + (1 + x'_n \lambda(x')) dx'^2$, with $\lambda$ a smooth function.

Let us consider the action of $*_2$ on normal and tangential forms. For a multi index $I = (i_1, i_2, \ldots, i_k)$ we use the convention $u_I dx_I = u_{i_1 i_2 \ldots i_k} dx_{i_1} \wedge dx_{i_2} \wedge \ldots dx_{i_k}$. We will denote by $I'$ the complimentary multi index with $I' = (j_1, \ldots, j_{n-k})$ where $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is an even permutation of $1, \ldots, n$.

For the metric $g_1, (\partial_1, \ldots, \partial_n)$ is an oriented orthonormal frame and $(dx_1, \ldots, dx_n)$ an oriented orthonormal co-frame. We get an orthonormal frame for $g_2$ by dividing each $\partial_i$ by $(1 + \lambda(x') x'_n)^{1/2}$ except for $\partial_n$. Similarly $\eta_i = (1 + \lambda(x') x'_n)^{1/2} dx_i$ for $i < n$ and $\eta_n = dx_n$ is an orthonormal co-frame for $g_2$. Applying $*_2$ to a typical normal basis $k$-form $dx_I$ (with $n \in I$), we have
\[ *_2(dx_I) = (1 + \lambda(x') x'_n)^{-\frac{n+1}{2}} *_2(\eta_I) \]
\[ = (1 + \lambda(x') x'_n)^{-\frac{n+1}{2}} (\eta_I) \]
\[ = (1 + \lambda(x') x'_n)^{\frac{n+1}{2} - k} (dx_{I'}). \]

For a typical basis element of $\Omega_I^k(\partial M)$, we have $n \notin I$ and
\[ *_2(dx_I) = (1 + \lambda(x') x'_n)^{-\frac{1}{2}} *_2(\eta_I) \]
\[ = (1 + \lambda(x') x'_n)^{-\frac{1}{2}} (\eta_I) \]
\[ = (1 + \lambda(x') x'_n)^{\frac{n+1}{2} - k} (dx_{I'}). \]
Modulo $x_n^{n+1}$, we have that on normal $k$-forms $(*_2 - *_1) = x_n^k \left( \frac{n+1}{2} - k \right) \lambda *_1$ and on tangential $k$-forms $(*_2 - *_1) = x_n^k \left( \frac{n+1}{2} - k \right) \lambda *_1$.

To prove the theorem we only have to consider the action of the difference of the operators on a particular $k$-form $u_I dx_I$. 

10
We first establish that there is no contribution from
\[ *_2 \ast_2 d \ast_2 d - *_1 \ast_1 d d = (\ast_2 - \ast_1) d \ast_2 d + \ast_1 d (\ast_2 - \ast_1) d. \]

Any first order term in the first term of the RHS will be in \( DO^{1,1} \) and therefore not contribute to the principal symbol. For the second term in the RHS, we have,
\[ \ast_1 d (\ast_2 - \ast_1) d (u_1 dx_l) = \ast_1 d \left( \left( \frac{n \pm 1}{2} - k \right) x_n^l \lambda (x') \ast_1 \sum_{j \not= l} \frac{\partial u_l}{\partial x_j} dx_j \land dx_l \right) \]
plus terms involving \( x_n^{l+1} \) which will not contribute. Now consider the second \( d \), if it applies to \( u_l \) we get a second order term which we already understand; if it applies to \( x_n \) we get a term involving \( dx_n \) which will then have no \( dx_n \) component on applying \( \ast_1 \) again. Thus we get no sufficiently low order contribution to either the zeroth order term or to the coefficient of \( D_{\ast n} \).

Moving on to
\[ d \ast_2 d \ast_2 - d \ast_1 d \ast_1 = d (\ast_2 - \ast_1) d \ast_2 + d \ast_1 d (\ast_2 - \ast_1), \]
looking at the first term on the RHS we can equally compute with \( d (\ast_2 - \ast_1) d \ast_1 \) as the difference will be in a non-contributory residue class.

First considering the normal case \( n \in I \) we compute
\[ d \ast_1 u = \sum_{j \not= l} \partial_j u_l dx_j \land dx_l. \]
Now consider
\[ d (\ast_2 - \ast_1) (\partial_j u_l dx_j \land dx_l) \]
\[ = \left( \frac{n \pm 1}{2} - k \right) d \left( x_n^l \lambda \partial_j u_l \ast_1 (dx_j \land dx_l) \right) \]
\[ = \left( \frac{n \pm 1}{2} - k \right) \left( x_n^{l-1} \lambda \partial_j u_l dx_n + x_n^l d (\lambda \partial_j u_l) \right) \land \ast_1 (dx_j \land dx_l) \]

Where the \( + \) holds for \( j = n \) and the \( - \) otherwise, but for this to have a \( dx_l \) component we must have \( j = n \). The second term in the final bracket has a coefficient \( x_n^l \) so can only contribute to the second order part which we already understand. We do have a contribution to \( F \) from
\[ \left( \frac{n \pm 1}{2} - k \right) x_n^{l-1} \lambda \partial_n u_l dx_l. \tag{3.7} \]

We are left with the contribution of \( d \ast_1 d (\ast_2 - \ast_1) \). If we apply \( d \) to \( (\ast_2 - \ast_1) (u_1 dx_l) \), we can drop the terms where \( d \) falls on a \( \lambda \) as these are in \( \Psi DO^{1,1} \). So on applying \( d (\ast_2 - \ast_1) \) to \( u_1 dx_l \) we are left with,
\[ \left( \frac{n \pm 1}{2} - k \right) x_n^l \lambda \sum_{j \not= l} \partial_j u_l dx_j \land dx_l + (l - 1) x_n^{l-1} \lambda u_l dx_n \land dx_l \].

On applying \( \ast_1 \), we get
\[ \left( \frac{n \pm 1}{2} - k \right) x_n^l \lambda \sum_{j \not= l} \partial_j u_l dx_j + (l - 1) x_n^{l-1} \lambda u_l dx_n \).
where $I_j$ is simply $I$ with $j$ deleted.

We now apply $d$ again to get the final contributions. We get another contribution to $F$ identical to Equation (3.7) so that

$$F = 2il\left(\frac{n+1}{2} - k\right)\lambda$$

and an $x^{(n-2)}u_I$ which gives

$$σ_0(P_0) = l(l-1)\left(\frac{n+1}{2} - k\right)\lambda$$

Substituting back into Equation (3.7) we see that if $c_l = 0$ then $k_{ij} = 0$ and Theorem 1.1 is proved for the case for $k \neq (n+1)/2$. Now we consider the case of tangential data that is $u_I dx_I$ with $n \notin I$.

A similar argument applies to a tangential form, with $k \neq (n-1)/2$ and Theorem 1.1 is proved.

**Proof of Corollary 2.1** First observe that given the induced metric on the boundary the data $(i^*u, i^*u)$ determines $u|_{\partial M}$. In a neighborhood’s of the boundary the $k$-form $u$ can be expressed as

$$u = \sum_{|l| = k, n \notin I} u_I dx_I + \sum_{|j| = k-1, n \notin J} u_{i,n,j} dx_n \wedge dx_J$$

so that

$$\partial_n \cdot du = \sum_{l| = k, n \notin I} \partial_n u_I dx_I$$

and

$$i^* du = *_\partial (\partial_n \cdot du)|_{\partial M} = *_{\partial} π_t \Lambda_x (u|_{\partial M})$$

This shows that $π_0 \Pi_x$ is a pseudo differential operator of order 1. Notice now that for a harmonic $k$-form $u$, $v = *u$ is a harmonic $n-k$ form for which both the tangential and normal parts of Dirichlet and Neumann data are exchanged $i^*v = i^*u$, $i^*v = \pm i^*u$, $i^* dv = \pm i^*d\delta u$ and $i^*d\nu = \pm i^*d\nu$. It follows that $π_\nu \Pi_x$ is also a pseudo differential operator of order 1. Thus we have proved part (i) of Corollary 2.1. For part (ii) notice that for $k = 0$ the result is proved by (6), and here $\partial u = 0$ identically so the normal part of the Neumann data gives us no information. Similarly for $k = n$ the tangential part of the Neumann data vanishes. For the case $0 < k < n$ and $k \neq (n-1)/2$ where any tangential-tangential diagonal component of $Λ_x$ determines the Taylor series we require, it is clear that the Taylor series is determined as long as we have $*_\partial$. We have the principal symbol $σ_1(\Pi_x ττ)(ξ) = *_\partial |ξ|_g$ and to finish the proof for this case we show that this determines $*_\partial$ at each point on the boundary.

Fix a point on $\partial M$ and choose any multi-indices $\xi_0$ and $\xi_0$ such that $g_0(ξ, ξ) := (σ_1(\Pi_x ττ)(ξ))_{h_0}^2$ is a non-zero quadratic function of $ξ$. Now $g(ξ, ξ) = α g_0(ξ, ξ)$ where $α = 1/(σ_1(\Pi_x ττ))$ is to be determined. Let $*_\partial$ be the Hodge star on $k$-forms on the boundary determined by $g_0$ then $*_\partial = α^{k-(n-1)/2}*_\partial$ and $(σ_1(\Pi_x ττ))_{h_0} = α^{k-(n-2)/2}*_\partial |ξ|_{h_0}$. As $g_0$ is known $α$ is determined provided $k \neq (n-2)/2$, hence we have $g$ at the boundary and $*_\partial$.

For the case $k \notin \{0, (n+1)/2, (n+2)/2\}$ we simply apply the above argument to boundary data for $*u$. As one of the conditions on $k$ must hold it is certainly true that the full symbol for the complete $Π_x$ determines the Taylor series, at the boundary, of $g$. 

12
References

[1] G.F.D. Duff and D.C. Spencer, Harmonic tensors on Riemannian manifolds with boundary, Ann. of Math, 56, 128–156, 1952

[2] P.B. Gilkey, Invariance theory, the heat equation and the Atiya-Singer index theorem, Publish or Perish, 1984, electronic reprint http://www.mi.sanu.ac.yu/EMIS/monographs/gilkey/index.html.

[3] M.S. Joshi, Introduction to Pseudo-differential operators, arXiv.org e-print math.AP/9906155 1999, http://arXiv.org/abs/math.AP/9906155.

[4] M.S. Joshi, S. McDowall, Total determination of material parameters from electromagnetic boundary information, Pacific J. Math, 193, 107–129, 2000

[5] M. Lassas, G. Uhlmann, On determining a Riemannian manifold from the Dirichlet-to-Neumann map, Ann. Sci. École Norm. Sup., 34, 771–787, 2001

[6] J. Lee, G. Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurements, Comm. Pure Appl. Math. 42, 1097–1112, 1989

[7] G. Nakamura and G. Uhlmann, Inverse problems at the boundary for an elastic medium, SIAM J. Math. Anal, 26, 263–279, 1995

[8] G. Nakamura and G. Uhlmann, A Layer Stripping Algorithm in Elastic Impedance Tomography, in Inverse Problems in Wave Propagation, IMA Vol. Math. Appl. 90, G. Chavent et al eds., Springer-Verlag, New York, 375–384, 1997.

[9] G. Nakamura, K. Tanuma and G. Uhlmann, Layer stripping for a transversely isotropic elastic medium, SIAM J. Appl. Math., 59, 1879–1891, 1999

[10] M.A. Shubin, Pseudo-differential Operators and Spectral Theory, Springer-Verlag, Berlin, 1985.

[11] P. Ola, L. Päivärinta, E. Somersalo An inverse boundary value problem in electrodynamics. Duke Math. J., 70, 617–653, 1993.