Analysis of Undergraduate Students’ Mathematical Understanding Ability of the Limit of Function Based on APOS Theory Perspective

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Abstract. The aim of this study was to know the level of undergraduate students’ mathematical understanding ability based on APOS theory perspective. The APOS theory provides an evaluation framework to describe the level of students’ understanding and mental structure about their conception to a mathematics concept. The levels of understanding in APOS theory are action, process, object, and schema conception. The subjects were 59 students of mathematics education whom had attended a class of the limit of function at a university in Palembang. The method was qualitative descriptive with 4 test items. The result showed that most of students were still at the level of action conception. They could calculate and use procedure precisely to the mathematics objects that was given, but could not reach the higher conception yet.

1. Introduction

Principal assignment of instruction in school covered planning, ‘presentation and observation’, and ‘reflection and evaluation’. The assignments form a continued cyclic. Planning involves a teacher decision about lesson purpose, context, and learning activity that is provided attain the objective effectively. Presentation and observation involve decision that is made by the teacher concerning lesson development during taking place. Reflection and evaluation involve the teacher decision after a lesson is finished and include a planning for next learning activity. Reflection and evaluation after a lesson is needed to fix and improve learning quality. Two principal assignments that are involved in this activity, that are, firstly, consider whether a lesson has successful and determine the next implication of teaching practice, and secondly, assess and record students’ progress. The assessment of a successful lesson engages comprehensive review of subject matter attention [1].

The subject matter attention in learning mathematics assessment is the ability of thinking mathematically. There are five main competencies concerning thinking mathematically [2], that are, 1) mathematical understanding, 2) mathematical problem solving, 3) mathematical reasoning, 4) mathematical connection, and 5) mathematical communication. The competency of mathematical understanding is needed to understand mathematics. This is a knowledge with abstract object. Individual that learns it is an uneasy matter. This showed that a teacher needs to analyze students’ mathematical understanding to get information about their understanding. The information can be used to find about pedagogic approach that can help students to understand mathematics and improve it. The data about students’ mathematical understanding and their error when answering mathematics problems can be used to improve mathematics learning quality[3], which is mathematical
understanding is a cognition process in absorb a mathematics concept indirectly where individual with this ability can apply the concept in mathematics problem situation[4]. The identification of errors may also help instructor to develop pedagogic approach and students’ learning difficulties [5].

The level of individual mathematical understanding ability can be seen from APOS (Actions, Processes, Objects, and Schemas) theory perspective. The APOS theory provides a framework to describe the students’ level understanding and their mental structure in learning about a mathematics concept [6]. Hartati [7] also explained that the success or failure of student in solving mathematics problem can be identified with mental construction that he/she achieved. Action in APOS theory is a transformation from mathematics objects that carry out by an individual through step by step procedure and operation. When an action is repeated and the individual reflects to the action, then he/she can form internal mental construction. This calls as Process. An Object is constructed from a process when the individual realized that process as a totality and could transform on it. Finally, schema is sets of Actions, Processes, Objects, and other schemas that individual have relate on mathematics problem situation involve the concept [8]. Actions, Processes, Objects, and Schemas Conception can be designed in test item. The examples of test item about the limit of function that connect to Action, Processes, and Objects Conception, that are, as follows

1. The test item for action conception
   Let \( f(x) = \begin{cases} x + 2, & x \leq 3 \\ 6 - x, & x > 3 \end{cases} \)
   Determine \( \lim_{x\to 3} f(x) \). [9]
   Let \( f(x) = \begin{cases} 2x + 1, & x > 5 \\ 3x - 6, & x \leq 5 \end{cases} \)
   Determine \( \lim_{x\to 5} f(x) \). [10]
   If a student can only determine left or right limit, but he/she cannot determine whether the limit exists or not, then the students’ understanding was indicated in action conception

2. The test item for object conception
   Find \( \lim_{x\to 6} \sqrt{x} \). [9]
   Find \( \lim_{x\to 5} \sqrt{x} \). [10]
   If students answer the limit of \( f(x) \) is 0, then he/she was indicated in action conception. Furthermore, if students give an answer perfectly, then he/she has conceptualized the limit as an object and has schemas of algebra manipulation.

3. The test item for schema conception
   Let \( g(x) = \begin{cases} \frac{2x^2 - x - 15}{x+2}, & x \neq 3 \\ kx - 1, & x = 3 \end{cases} \)
   Determine \( k \) so that \( g(x) \) is continue.[9]
   Let \( f(x) = \begin{cases} \frac{3x^2 + 5x - 2}{x+2}, & x \neq 2 \\ mx + 1, & x = -2 \end{cases} \)
   Determine \( m \) so that \( f(x) \) is continue. [10]
   If students can answer it perfectly, then he/she has developed schema to elaborate that mathematics problem.

From the examples of test item, it showed an image about problems that can be used to observe the level of students’ mathematical understanding from APOS perspective. This study used 4 test items that are representing action, process, object, and schema conception. The research problem was how the level of undergraduate students’ mathematical understanding ability of the limit of function based on APOS theory perspective?. Therefore, the aim of this study was to know the level of undergraduate students’ mathematical understanding ability of the limit of function based on APOS theory perspective.

2. Method
The subjects of this study were 59 undergraduate students of mathematics education that had attend calculus of differential class at one of universities in Palembang. The research method that was used in this study is qualitative descriptive. The data that was collected in this study is concerning undergraduate students’ mathematical understanding ability after they finished a lecture with expository method. The technique of data collection was 4 essay tests. The test items were designed
based on mathematical understanding ability in perspective of APOS theory. The data was obtained by evaluating students answer with descriptor of four conceptions based on this theory, and then it was analyzed descriptively to know the attainment of undergraduate students’ mathematical understanding ability.

3. Result and Discussion

In this study, the test items that were used to analyze students’ mathematical understanding represent four conceptions in APOS theory perspective, that are, Action, Process, Object, and Schema conceptions. The first item was action conception, that is, as follows:

Determine a) \( \lim_{x \to 0} \frac{x^2+4}{x+3} \), b) \( \lim_{x \to 1} \sqrt{x-1} \). To answer the test item of 1a, students have to make calculation with procedure correctly to the mathematics objects. In this theory perspective, the objects are \( \lim_{x \to 0} \frac{x^2+4}{x+3} \) and \( \lim_{x \to 1} \sqrt{x-1} \). The result of this study showed 10.17% students made error in calculation. At 1b, students could not substitute \( x = 1 \) to \( f(x) = \sqrt{x} \) directly, because it will effect \( f(x) \) undefined. Firstly, they must manipulate algebra on \( f(x) \) so that \( f(x) = \frac{1}{\sqrt{x+1}} \), then substitute \( x = 1 \) to \( f(x) \), but, the result of this study was found that 69.49% students made that error by substitute the value of \( x \) to \( f(x) \), and the rest of them did not, they did the procedure properly. Students could not do algebra manipulations or algorithms are called procedural errors[5].

The second item was process conception, that is, as follows:

For \( f(x) = 2x + 1, 0 \leq x \leq 1 \). Whether \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 1} f(x) \) exist? Explain it why!. To answer the test item, students have to make action from left and right limit. After that, they process that action to show whether the limit exist or not. First, they must identify the domain of \( f(x) \). It is definite from 0 to 1. This mean that if \( x \) is outside in interval \([0, 1]\), then \( f(x) \) is undefined. It shows that \( x \) can only closed to 0 from right. Let \( x = 0.0001, 0.001, 0.01, 0.1 \), \( x \) with that values can be processed into \( f(x) \). Furthermore, \( x \) cannot close to 0 from left, because \( x < 0 \) is not in the domain. Let \( x = -0.1, -0.01, -0.001, -0.0001 \), \( x \) with that values cannot be processed into \( f(x) \). Hence, \( \lim_{x \to 0^+} f(x) \) exist, but \( \lim_{x \to 0^-} f(x) \) vice versa. For the object of \( \lim_{x \to 1^+} f(x) \), \( x \) can only close to 1 from left. Let \( x = 0.9, 0.99, 0.999 \), \( x \) with that values can be processed into \( f(x) \). Besides that, \( x \) cannot close to 1 from right, because \( x > 0 \) is not in the domain. Let \( x = 1.001, 1.01, 1.1 \), \( x \) with that values cannot be processed into \( f(x) \). So, \( \lim_{x \to 1^+} f(x) \) exist, but \( \lim_{x \to 1^-} f(x) \) vice versa. The result of this study showed that 8.47% students could indentify that \( x \) can close to 0 from right and \( x \) can close to 1 from left, but they could not explain why \( x \) cannot close 0 from left and \( x \) cannot close 1 from right, so that they gave irrelevant conclusion. The rest of them gave error identification.

The third test item was object conception, that is, as follows:

Sketch the graph of \( f \) function that appropriates all this condition:

a) The domain is \([0, 6]\)

b) \( f(0) = f(2) = f(4) = f(6) = 2 \)

c) \( f \) is continue function except at \( x = 2 \)

d) \( \lim_{x \to 2^-} f(x) = 1 \) and \( \lim_{x \to 2^+} f(x) = 3 \)

To answer the test item, students have to make actions by indentifying all that condition and process them to form the graph of \( f \) function as an object. To form that object, they have to realize that graphic is plotted in interval 0 to 6. For \( x = 0, 2, 4, \) and 6, then \( f(x) = 2 \). If it is depicted, then it is like a solid circle. The third condition mean \( x \) is discontinue when \( x = 2 \) at interval \([0, 6] \). When \( x \) close to 2 from left, then \( f(x) \) close to 1. Because \( f \) discontinue at \( x = 2 \), then when \( x \) close to 2 from right and \( f(x) \) can closed to 1, even though \( f(2) = 2 \). This mean \( \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \) which \( f \) does not appropriate continuity condition at \( x = 2 \). Besides that, when \( x \) close to 2 from right, then \( f(x) \) can be made close to 2 so that \( \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \). This mean \( \lim_{x \to 2^-} f(x) \) does not exist which \( f \) discontinue at \( x = 2 \). Afterward, last condition showed that \( \lim_{x \to a^-} f(x) = 3 \) and \( f \) only discontinue at \( a = 2 \). This must be \( \lim_{x \to a^+} f(x) = 3 \).
and \( f(5) \) has value in interval \([0, 6]\). The value of \( f(5) \) must be 3, so that \( \lim_{x \to 5} f(x) \) exist and \( \lim_{x \to 5} f(x) = f(5) \). This show that \( f \) continue at \( x = 5 \). The curve of this graph can be depicted as bend or linear through the points among 0 to 6 without ignore all condition. The result of this study showed that 98.31% students represent error graph which is irrelevant with the conditions. This result corresponds with the study of Ningsih [11]. It showed that the ability to represent a mathematics concept in another mathematics representation form, such as graphic is very low category.

The last test item was schema conception, that is, as follows:

Let \( f(x), g(x) = 1 \), for all \( x \) and \( \lim_{x \to a} g(x) = 0 \). Proof that \( \lim_{x \to a} f(x) \) does not exist. To answer the test item, students have to make actions and process every mathematics object that is given and then cheme them in one set to proof the statement validity. They have to realize that \( f(x), g(x) = 1 \) for all \( x \) means if \( x = a \), then \( f(a), g(a) \) is also 1. So, it can be form \( \lim_{x \to a} f(x)g(x) = \lim_{x \to a} 1 \) as an object which is \( \lim_{x \to a} f(x)g(x) = f(a)g(a) \). This means that \( f(x), g(x) \) is a continue function for all \( x \). By using product and constant rule of limit, then \( \lim_{x \to a} f(x) \lim_{x \to a} g(x) = 1 \), and because \( \lim_{x \to a} g(x) = 0 \), then there is no \( \lim_{x \to a} f(x) \) that appropriate the validity of the statement so that \( \lim_{x \to a} f(x) \lim_{x \to a} g(x) = 1 \). The result of this study showed that 83.05% students could not generate a schema that is needed to prove the statement validity of the limit of function. It is because the main schema is a concept of the continuity of function, the product, quotient, and constant rule of limit. Overall description about the finding of this study can be seen on this table below.

### Table 1. Percentage of Students’ Conception of Mathematical Understanding in APOS Theory

| Number of test item | Mathematical Understanding Category | Descriptor                                      | Percentage |
|---------------------|-----------------------------------|-------------------------------------------------|------------|
| 1a                  | Action conception                 | Do not give an answer                           | 0.00%      |
|                     |                                   | Give an answer but error in calculation         | 10.17%     |
|                     |                                   | Can calculate with appropriate procedure correctly | 89.83%   |
|                     |                                   | Do not give an answer                           | 0.00%      |
| 1b                  | Action conception                 | Can calculate but error in procedure            | 69.49%     |
|                     |                                   | Can calculate with appropriate procedure correctly | 30.51%   |
|                     |                                   | Do not give an answer or error in identification | 91.53%    |
|                     |                                   | Can identify but cannot explain                 | 8.47%      |
|                     |                                   | Can identify and explain completely             | 0.00%      |
|                     |                                   | Do not give an answer                           | 0.00%      |
| 2                   | Process conception                | Represent with irrelevant condition              | 98.31%     |
|                     |                                   | Represent with appropriate condition            | 1.69%      |
| 3                   | Object Conception                 | Do not give an answer                           | 0.00%      |
| 4                   | Schema Conception                 | Give an answer but the arguments of proof is irrelevant | 83.05%  |
|                     |                                   | Can prove with argument logically                | 16.95%     |

From the results of this study that have been described above, it showed that most of the students had still in action conception which is they could do calculation with procedure correctly to the mathematics object, but could not reach next conception yet. Based on Arnawa [12] analysis technique, the result of this study showed that there are no students achieved process or object or schema conception perfectly, because there are no students achieved a process conception even though there are students can answer the 3rd and 4th of test items correctly but the 2nd is wrong. This can be seen in table 1 that no one of students can answer the 2nd of test item are correct. This is equivalent with the observation results of Maharaj [9], ‘Aydin and Mutlu’ [10], and Mrdja, et al. [13].
One of the causal factors is expository method was applied in teaching and learning activity before the students were given test. This method does not give student opportunity widely to construct her/his mathematics knowledge. Whereas, Students must actively relate new knowledge to what they are already know through construction process, not passive reception [14]. This means that an individual construct knowledge and understanding from his/her experiences. The theory that suggested it is Constructivism [15]. Therefore, to improve mathematical understanding based on Constructivism, APOS theory was developed from it and suggested ADE (Activity, Discussion, Exercise) teaching cyclic as pedagogic approach [8]. At activity stage, the students learn with computer programming application to solve a mathematics task. They are divided into a few groups. The choice of program application is appropriated with subject matter characteristic. Subject such as calculus consist algebra and geometry objects that need to be visualized. Maple is a computer programming that can manipulate symbols, create and manipulate algebra, and geometry objects. According to Pavel and Tatiana, Maple can assist students to understand mathematics, obtain knowledge about computer, and develop programming skill [16]. Applying Maple in Calculus lecture also give students opportunity to improve conceptual and procedural mathematical understanding [17]. This lecture is designed to help the development of students’ mental construction. The main purpose is to give students an experience than guide them to check an answer. Through this activity, students can feel mathematics problem and then develop it into class activity. At class discussion stage, the students learn to solve mathematics task which is continue and connect to the activity in computer laboratory. Discussion is organized using cooperative learning model. The purpose of this model, according to Terwel [18], is to accommodate individual differences between students. Instructor guide group discussion that is designed to give the students opportunity to reflect the result of their activity in laboratory and review it in class discussion. The final stage is exercise. In this stage, students are given assignment to solve a set of task. They work together in a group and solve it outside laboratory and class meeting. This treatment is provided to complete the cyclic. The aim of this treatment is to consolidate students idea that have been constructed, to apply mathematics concepts that have been studied by them, and continue to thought another mathematics situation.

4. Conclusion

From this study, we have succeeded to know the level of undergraduate students’ mathematical understanding ability of mathematics education based on APOS theory perspective at one of universities in Palembang. We used 4 item tests that are different from the studies before. The complexity of the test item is more multiple diverse. The finding of this study confirmed that 10,17% students made error in calculation, 69,49% students could not manipulate algebra, 91,53% students could not identify the existence of the limit of function, 98,31% students represented error graph which is irrelevant with the given conditions, and 83,05% students could not generate a schema that is needed to prove the statement validity of the limit of function. This showed that we could conclude most of the undergraduate students of mathematics education had still in action conception which is they could calculate with procedure correctly to the mathematics object, but could not reach next conception yet. We also found that there are students can answer the 3rd and 4th of test items correctly but the 2nd is wrong. This phenomenon needed to analyze furthermore whether the students achieved object or schema conception or not. Based on the result of this study, to improve mathematical understanding based on APOS theory, pedagogic approach that was suggested is ADE (Activity, Discussion, Exercise) teaching cyclic. Instructional design of that approach is managed into several sections in one week. One section is held in computer laboratory and others in regular class without computer. Besides that, home work is given too as exercise to complete the cyclic and reinforce students’ mathematical understanding.

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References
[1] Kyriacou, C., (2009). Effective Teaching in Schools: Theory and Practice. 3rd Ed. Nelson Thornes Ltd. United Kingdom.
[2] Sumarmo, U., (2006). Pembelajaran untuk Mengembangkan Kemampuan Berpikir Matematik. Paper was presented at Seminar Nasional Pendidikan MIPA, FPMIPA, Universitas Pendidikan Indonesia.
[3] Rosita, C. D., Laelasari, dan Noto, M. S., (2014). Analisis Kemampuan Pemahaman Matematis Mahasiswa pada Mata Kuliah Aljabar Linear 1. Jurnal Euclid, Vol. 1, No. 2, pp. 60 – 69. Prodi Pendidikan Matematika Unswagati Cirebon.
[4] Ompusunggu, V. D. K., (2014). Peningkatan Kemampuan Pemahaman Matematik dan Sikap Positif Terhadap Matematika Siswa SMP Nasrani 2 Medan melalui Pendekatan Problem Posing. Jurnal Saintech, Vol. 06, No. 04, pp. 93 – 105.
[5] Siyepu, S. W., (2015). Analysis of Errors in Derivatives of Trigonometris Functions. International Journal of STEM Education 2:16, a SpringerOpen Journal.
[6] Arnon, I., Cottrill, J., Dubinsky, E., Oktac, A., Fuenstes, S. R., Trigueros, M., and Weller, K., (2014). APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education. New York: Springer Science+Business Media.
[7] Hartati, S. J., (2014). Design of Learning Model of Logic and Algorithms Based on APOS Theory. International Journal of Evaluation and Research in Education (IJERE), Vol. 3, No. 2, pp. 109 – 118.
[8] Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., and Thomas, K., (1996). A framework for research and curriculum development in undergraduate mathematics education, Research in Collegiate Mathematics Education II, CBMS Issues in Mathematics Education, 6, 1 – 32.
[9] Maharaj, A., (2010). An APOS Analysis of Students’ Understanding of the Concept of A Limit of A Function. Pythagoras, 71, 4 – 25.
[10] Aydin, S. and Mutlu, C., (2013). Students’ Understanding of The Concept of Limit of A Function in Vocational High School Mathematics. The Online Journal of Science and Technology, Vol. 3, Issue 1, pp. 145 – 152.
[11] Ningsih, Y. L., (2016). Kemampuan Pemahaman Konsep Matematika Mahasiswa Melalui Penerapan Lembar Aktivitas Mahasiswa (LAM) Berbasis Teori APOS pada Materi Turunan. Edumatica, Vol. 6, No. 1, pp. 1 – 8.
[12] Arnawa, I. M., (2007). Mengembangkan Kualitas Pemahaman dalam Aljabar Abstrak melalui Pembelajaran Berdasarkan Teori APOS. Jurnal Pendidikan dan Kebudayaan, No. 068, Tahun Ke-13. pp. 809 – 826.
[13] Mrdja, M., Romano, D. A., and Zubac, M., (2015). Analysis of Students’ Mental Structures When Incorrectly Calculating the Limit of Functions. IMVI Open Mathematical Education Notes, Vol. 5, pp. 101 – 113.
[14] Pundir, R. and Surana, A., (2016). Constructivism Learning: A Way to Make Knowledge Construction. The International Journal of Indian Psychology, Vol. 3, Issue 2, No. 10, pp. 158 - 162.
[15] Bada, S. O., (2015). Constructivism Learning Theory: A Paradigm for Teaching and Learning. IOSR Journal of Research & Method in Education, Vol. 5, Issue 6, Ver. 1, pp. 66 – 70.
[16] Pavel, P., and Tatiana, G., (2012). Animation Of Essential Calculus Concepts in Maple. Proceedings of the11th International Conference APLIMAT 2012, Section: New Trends in Mathematics Education. Faculty of Mechanical Engineering – Slovak University of Technology in Bratislava.
[17] Salleh, T. S. A., and Zakaria, E., (2013). Enhancing Students’ Understanding in Integral Calculus through the Integration of Maple in Learning. Procedia – Social and Behavioral Sciences, 102, pp. 204 – 211. Published by Elsevier Ltd.
[18] Terwel, J., (2011). Cooperative Learning and Mathematics Education: A Happy Marriage?. Paper was presented at the OECD/France Workshop, Education for Innovation: The Role of Arts and STEM Education, Paris.