A LIE GROUP WITHOUT UNIVERSAL COVERING

JÖRG WINKELMANN

Abstract. We present an example of a disconnected Lie group for which there is no universal covering (as Lie group).

A connected and locally pathwise connected topological space $X$ with a base point $x$ admits a universal covering $\tilde{X}$ which is constructed as space of equivalence classes of curves starting at the given base point. The constant curve with value $x$ defines a base point $\tilde{x} \in \tilde{X}$. Continuous maps between two such spaces mapping the base point of the source space to the base point of the target space lift uniquely to a continuous map between the respective universal coverings mapping basepoint to basepoint.

Using this, for every connected Lie group the universal covering admits a structure as Lie group, because the maps defining the group structure lift uniquely. The neutral element of the Lie group is the canonical choice for the basepoint.

However, the pictures changes if we regard disconnected spaces. In order to construct a universal covering for a disconnected locally pathwise connected topological space we need a basepoint for each connected component, because we need to treat each connected component separately. But for a Lie group only the connected component containing the neutral element $e$ has a canonical choice of a basepoint, namely $e$. For the other connected components there is no natural choice. As a consequence, we may construct a universal covering as a manifold for a disconnected Lie group, but there is no natural way to put a group structure on this manifold, because there is no canonical way to lift the map defining the group structure.

In fact, there is an example, where there does not exist any compatible group structure.

Proposition 1. Let $\tilde{X} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{R}$ and $X = \mathbb{Z}_2 \times \mathbb{Z}_2 \times (\mathbb{R}/\mathbb{Z})$ as manifolds and $\pi: \tilde{X} \to X$ the natural projection induced by the natural map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. (Here $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.)
Let $X$ be endowed with the following group structure:

$$(a, b, v) \cdot (c, d, w) = (a + c, b + d, v + w + \frac{1}{2}(ad))$$

with $a, b, c, d \in \mathbb{Z}_2$ and $v, w \in \mathbb{R}/\mathbb{Z}$.

Then there does not exist any Lie group structure on $	ilde{X}$ for which $\pi$ is a group homomorphism.

**Proof.** Assume the contrary. We begin by observing that the connected component $X^0$ of $X$ containing $e$ is central in $X$. It follows that $ghg^{-1}h^{-1} \in \pi^{-1}(e)$ for all $g \in X$, $h \in (\tilde{X})^0$. Since $\zeta : (g, h) \rightarrow ghg^{-1}h^{-1}$ is continuous, $(\tilde{X})^0$ is connected, $\pi^{-1}(e)$ is discrete and $\zeta(g, e) = e$ it follows that $\zeta(g, h) = e$ for all $g \in X$, $h \in (\tilde{X})^0$. In other words, $(\tilde{X})^0$ is central.

Now let $g \in \tilde{X} \setminus (\tilde{X})^0$. Then $g^2 \in (\tilde{X})^0$, because $\tilde{X}/(\tilde{X})^0 \simeq X/X^0 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $(\tilde{X})^0$ is central and $(\tilde{X})^0 \simeq (\mathbb{R}, +)$, it follows that every connected component of $\tilde{X}$ contains a unique element of order two.

Let $g \in \tilde{X}$. Then conjugation by $g$ stabilizes each of the connected components, because $\tilde{X}/(\tilde{X})^0$ is commutative. Therefore conjugation by $g$ must stabilize the unique element of order two for every connected component. Furthermore, conjugation by $g$ acts trivially on the central subgroup $(\tilde{X})^0$. The elements of order two together $(\tilde{X})^0$ generate the whole group $\tilde{X}$. Hence conjugation by $g$ is entirely trivial. Since $g$ was arbitrary, we have deduced that $\tilde{X}$ is commutative. But this is in contradiction to the fact that $X$ is not commutative. \hfill \Box