Abstract

We study the exclusive production of $\pi\pi$ and $\rho\pi$ in hard $\gamma^*\gamma$ scattering in the forward kinematical region where the virtuality of one photon provides us with a hard scale in the process. The newly introduced concept of Transition Distribution Amplitudes (TDA) is used to perform a QCD calculation of these reactions thanks to two simple models for TDAs. Cross sections for $\rho\pi$ and $\pi\pi$ production are evaluated and compared to the possible background from the Bremsstrahlung process. This picture may be tested at intense electron-positron colliders such as CLEO and $B$ factories. The cross section $\gamma \gamma \rightarrow e\pi^0\pi^0$ is finally shown to provide a possible determination of the $\pi^0$ axial form factor, $F_{A_{\pi^0}}$, at small $t$, which seems not to be measurable elsewhere.

1 Introduction

In a recent paper \cite{1}, we have advocated that factorisation theorems \cite{2} for exclusive processes may be extended to the case of the reaction $\pi^-\pi^+ \rightarrow \gamma^*\gamma$ in the kinematical regime where
the virtual photon is highly virtual (of the order of the energy squared of the reaction) but
the momentum transfer $t$ is small. This enlarges the successful description of deep exclusive
reactions in terms of distribution amplitudes [3] and/or generalised parton distributions [4, 5]
on the one side and perturbatively calculable coefficient functions describing hard scattering
at the partonic level on the other side. We want here to describe along the same lines the
(crossed) reactions $\gamma^* \to AB$:

$$\gamma^*_L \gamma \to \rho^\pm \pi^\mp, \quad \gamma^*_L \gamma \to \pi^\pm \pi^\mp, \quad \gamma^*_L \gamma \to \pi^0 \pi^0, \quad (1)$$

in the near forward region and for large virtual photon invariant mass $Q$, which may be
studied in detail at intense electron colliders such as those which are mostly used as $B$
factories.

![Figure 1: The factorised amplitude for $\gamma^* \gamma \to A\pi$ at small transfer momentum.](image)

Let us recall the main ingredients of the analysis developed in Ref. [1, 6].

With the kinematics described in Fig. 1 and introducing light-cone coordinates $v^\pm = (v^0 \mp v^3)/\sqrt{2}$
and transverse components $v_T = (v^1, v^2)$ for any four-vector $v$, we define the $\gamma \to \pi$ transition
distribution amplitudes (TDAs) $T(x, \xi, t)$ as the Fourier transform of matrix elements
$\langle \pi(p_\pi) | O^\mu | \gamma(p_\gamma) \rangle$ where $O^\mu = \bar{\psi}(z) \gamma^\mu \psi(z)$ with $\Gamma^\mu = \gamma^\mu, \gamma^\mu \gamma^5, \sigma^{\mu\nu}$. We then
factorise the amplitude of the process $\gamma^*_L \gamma \to A\pi$ as

$$M(Q^2, \xi, t) \propto \int dx dz \Phi_A(z) M_h(z, x, \xi) T(x, \xi, t), \quad (2)$$

with a hard amplitude $M_h(z, x, \xi)$ and $\Phi_A(z)$ is the hadron distribution amplitude (DA).

The variable $z$ is as usual the light-cone momentum fraction carried by the quark entering
the meson $A$, $x+\xi$ (resp. $x-\xi$) is the corresponding one for the quark leaving (resp. entering)
the TDA. The skewness variable $\xi$ describes the loss of light-cone momentum of the incident
photon and is connected to the Bjorken variable $x_B$ (see section 3 for detailed kinematics).

Contrarily to the case of generalised parton distributions (GPD) where the forward limit
is related to the conventional parton distributions measured in the deep inelastic scattering

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$^3$The $\gamma^*_T$ case is more difficult to analyse since the leading twist amplitude vanishes.
(DIS), there is no such interesting constraints for the new TDAs. The only constraints are sum rules obtained by taking the local limit of the corresponding operators and possibly soft limits when the produced-meson momentum vanishes. Lacking any non-perturbative calculations of matrix element defining TDAs we are forced to build toy models to get estimates for the cross sections, to be compared with future experimental data. In particular, Tiburzi [7] has recently developed a model for both these axial and vector TDAs, which we shall use in a simplified version to make a comparison with the results obtained from our model. Also some other approaches used to model GPD in the pion case [8] could be applied in this context.

2 The $\gamma \to \pi$ TDAs

Let us first stress an obvious point. The $\gamma \to \pi^\pm$ TDA involve of course only quark correlators, and so does the $\gamma \to \pi^0$ TDA since the charge conjugation property of the $\pi^0$ forbids the leading twist gluonic TDA. Therefore, in the following, we only need to take into account quark correlators. For definiteness, let us consider the $\gamma \to \pi^-$ TDAs which are given by [1] ($P = \frac{p_{\pi^-} + p_\gamma}{2}$, $\Delta = p_{\pi^-} - p_\gamma$):

$$\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle \bar{\pi}^- (p_{\pi^-}) | \bar{d}(-\frac{z}{2}) \left[ - \frac{z}{2}; \frac{z}{2} \right] \gamma^\mu u(\frac{z}{2}) | \gamma(p_\gamma, \varepsilon) \rangle \bigg|_{z^+ = 0, z^- = 0} = \frac{1}{P^+} e^{\mu P^+ \Delta} V_{\pi^-}^\pi(x, \xi, t),$$

$$\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle \bar{\pi}^- (p_{\pi^-}) | \bar{d}(-\frac{z}{2}) \left[ - \frac{z}{2}; \frac{z}{2} \right] \gamma^\mu \gamma^5 u(\frac{z}{2}) | \gamma(p_\gamma, \varepsilon) \rangle \bigg|_{z^+ = 0, z^- = 0} = \frac{1}{P^+} e^{\mu P^+ (\varepsilon \cdot \Delta)} A_{\pi^-}^\pi(x, \xi, t),$$

$$\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle \bar{\pi}^- (p_{\pi^-}) | \bar{d}(-\frac{z}{2}) \left[ - \frac{z}{2}; \frac{z}{2} \right] \sigma^{\mu\nu} u(\frac{z}{2}) | \gamma(p_\gamma, \varepsilon) \rangle \bigg|_{z^+ = 0, z^- = 0} = \frac{e}{P^+} \epsilon^{\mu\nu\rho\sigma} P_\sigma \left[ \varepsilon_{\mu T_1^\pi^-} (x, \xi, t) - \frac{1}{f_\pi} (\varepsilon \cdot \Delta) \Delta_{\perp} T_2^\pi^- (x, \xi, t) \right],$$

(3)

where the first two TDAs, $V_{\pi^-}^\pi(x, \xi, t)$ and $A_{\pi^-}^\pi(x, \xi, t)$ (commonly labelled by $G(x, \xi, t)$ in the following) are chiral even and the latter ones, $T_i^\pi^- (x, \xi, t), i = 1, 2$, are chiral odd. In the TDA Eq. (3), we include the Wilson line $[y; z] = \exp [ig(y - z) \int_0^1 dt \, n_\mu A_\mu (ty + (1 - t)z)]$, which provides the QCD-gauge invariance for non local operators and equals unity in a light-like (axial) gauge. On the other hand, we do not write the electromagnetic Wilson line caused by the presence of the photon, since we choose an electromagnetic axial gauge for the photon.

The TDAs $V_{\pi^-}^\pi(x, \xi, t)$, $A_{\pi^-}^\pi(x, \xi, t)$, $T_1^\pi^- (x, \xi, t)$ and $T_2^\pi^- (x, \xi, t)$ are dimensionless quantities; $f_\pi$ is the pion decay constant ($f_\pi = 131$ MeV). These four leading twist TDAs are in fact linear combinations of the four independent helicity amplitudes for the process $q\gamma \to q\pi^-$. 

3
2.1 Constraints on $\gamma \rightarrow \pi$ TDAs

Sum rules may be derived for the photon to meson TDAs. Since the local matrix elements appear in radiative weak decays, we can relate them to the vector and axial form factors $F_V$ and $F_A$.

For $\pi^\pm$, for which one has [9]

$$F_{\pi^\pm}^V = 0.017 \pm 0.008 \text{ and } F_{\pi^\pm}^A = 0.0116 \pm 0.0016,$$

we get

$$\int_0^1 dx G_{\pi^\pm}(x, \xi, t) = \frac{f_\pi}{m_\pi} F_G^{\pi^\pm}(t) \quad G = (V, A).$$

To what concerns the neutral $\pi^0$, one can constrain the vector TDA thanks to the electromagnetic (transition) form factor $F_{\pi^0 \gamma^* \gamma}$ which is known at small $t$ [9]. Indeed, one has

$$\int_0^1 dx \left( Q_u V_{\pi^0 u}(x, \xi, t) + Q_d V_{\pi^0 d}(x, \xi, t) \right) = f_\pi F_{\pi^0 \gamma^* \gamma}(t),$$

where $Q_u = 2/3$, $Q_d = -1/3$ and $V_{\pi^0 q}(x, \xi, t)$ is the TDA related to the operator built from the quark $q$. Current algebra fixes the value of the right hand side at $t = 0$ since $F_{\pi^0 \gamma^* \gamma}(t = 0) = \frac{\sqrt{2}}{4\pi f_\pi}$ [10].

These sum rules\(^6\) constrain possible parametrisations of the TDAs. Note, in particular, the $\xi$-independence of the right hand side of the relations.

2.2 Toy models for $\gamma \rightarrow \pi$ axial and vector TDA

Since we want to respect the $\xi$ independence of the first moment in $x$ of the TDA leading to the sum rules (Eq. (5) and Eq. (6)), we shall use the analogy with the construction of GPDs through the double distributions [11]. We consider here the case of the $\gamma \rightarrow \pi^-$ TDA.

2.2.1 $t$-independent double distributions

In this section, we start from double distribution for $t = 0$ [11] and we omit the D-term which being isoscalar does not exist for the $\gamma \rightarrow \pi^\pm$ TDA. A D-term may exist for the $\gamma \rightarrow \pi^0$ TDA, but we shall not include it for simplicity.

We start from the representation of the $x$ and $\xi$ dependence of the TDA in the form

$$G^{(0)}(x, \xi) = \int_{-1}^1 d\beta \int_{-1-|\beta|}^{1-|\beta|} d\alpha \delta(x - \beta - \xi \alpha) f(\beta, \alpha),$$

\(^4\)In Eq. (5), we use the same definition as in PDG [9], $F_G$ being a dimensionless quantity. According to these conventions, the $F_G$ defined in [1] should be divided by $m_\pi$.

\(^5\)This sum rule is slightly different from the one written in [1] where we had assumed a quark model decomposition of the $\pi^0$.

\(^6\)In principle, one can also derive analogous sum rules for the chiral-odd TDAs ($T_1$ and $T_2$ in Eq. (3)) but we do not see an obvious way to relate them to experimentally interesting observables, see the discussion at the end of section 3.
with
\[ f(\beta, \alpha) = q(\beta)h(\beta, \alpha), \] (8)
where \( q(\beta) \) is analogous to the forward quark distribution in the GPD case. \( h(\beta, \alpha) \) is a profile function usually parametrised as \[ h_{(b, \alpha)}(\beta, \alpha) = \frac{\Gamma(2b + 2)}{2^{2b+1}\Gamma^2(b + 1)} \left[ (1 - |\beta|)^2 - \alpha^2 \right]^b, \] (9)
where the parameter \( b \) characterises the strength of the \( \xi \)-dependence. As a first guess we assume that the \( \beta \)-dependence of \( q \) is given by a simple linear law
\[ q(\beta) = 2(1 - \beta)\theta(\beta). \] (10)
As usually done for GPDs, we assume a mild \( \xi \) dependence as given by \( b = 1 \) and implement the normalisation (with \( \int \,dxG^{(0)}(x, \xi) = 1 \)) and \( t \)-dependence of the TDA through the axial or vector form factor:
\[ G(x, \xi, t) = G^{(0)}(x, \xi) \cdot \frac{f_\pi}{m_\pi} F_G(t). \] (11)
The \( t \)-dependence of these form factors has been studied in chiral perturbation theory \[ [12] \] and turned out to be weak, so we shall neglect it in this model and we shall use the measured values of Eq. (1).
We get
\[
G^{(0)}(x, \xi) = \int_0^1 \,d\beta \int_{-1+\beta}^{1-\beta} \,d\alpha \frac{1}{\xi} \delta \left( \alpha - \frac{x - \beta}{\xi} \right) \frac{3}{2} \left[ 1 - \frac{\alpha^2}{(1 - \beta)^2} \right] \\
= \theta(x + \xi)\theta(\xi - x) \int_0^{\frac{x + \xi}{1 - \xi}} \,d\beta \frac{3}{2\xi} \left[ 1 - \frac{(x - \beta)^2}{\xi^2(1 - \beta)^2} \right] + \\
(1 - \theta(\xi - x)) \int_{\frac{x + \xi}{1 - \xi}}^{\frac{x + \xi}{1 + \xi}} \,d\beta \frac{3}{2\xi} \left[ 1 - \frac{(x - \beta)^2}{\xi^2(1 - \beta)^2} \right] \\
= \frac{3}{2\xi^3} \left[ \theta(x + \xi)\theta(\xi - x) \left( (x + \xi - 2)(x + \xi) + 2(x - 1)\log \left( \frac{1 - x}{1 + \xi} \right) \right) \\
+ (1 - \theta(\xi - x))2(x - 1) \left( 2\xi + \log \left( \frac{1 - \xi}{1 + \xi} \right) \right) \right]. \] (12)
We do not include any QCD evolution to our TDAs, the effects of which are supposedly much less important than the uncertainty of our modelling. The resulting \( \gamma \to \pi^- \) vector TDA is plotted on Fig. 2(a). The axial one and the ones for \( \pi^0 \) have the same shape with a different normalisation. In the following, we shall refer to this approach as Model 1.
2.2.2 \( t \)-dependent double distributions

We shall use here the initial \( t \)-dependent double distribution of Tiburzi [7]. Explicitly, we have:

\[
V(x, \xi, t) = \frac{f_{\pi}}{m_\pi} \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \, \delta(x - \beta - \xi \alpha) W(\beta, \alpha; t), \tag{13}
\]

\[
A(x, \xi, t) = \frac{f_{\pi}}{m_\pi} \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \, \delta(x - \beta - \xi \alpha) B(\beta, \alpha; t), \tag{14}
\]

with

\[
W(\beta, \alpha; t) = \frac{m^2}{4\pi^2} \left\{ m^2 + \frac{1}{2} \beta(\alpha + \beta - 1)m^2 - [1 - \alpha^2 - \beta(2 - \beta)] \frac{t}{4} \right\}^{-1}, \tag{15}
\]

\[
B(\beta, \alpha; t) = \frac{3m^2(\alpha + \beta)}{4\pi^2} \left\{ m^2 + \frac{1}{2} \beta(\alpha + \beta - 1)m^2 - [1 - \alpha^2 - \beta(2 - \beta)] \frac{t}{4} \right\}^{-1}, \tag{16}
\]

where \( m \) is set to 0.18 GeV [7]. The sum rule for \( V \) is satisfied for \( t = -0.5 \) GeV\(^2\). We shall keep \( t \) fixed in the following to enable comparison with Model 1. In the following, we shall refer to this approach as Model 2. We see that these two models give quite different results and we shall use both in section 3 to estimate the sensitivity of the cross sections to TDA models.

Figure 2: The \( \gamma \to \pi^- \) vector transition distribution amplitude \( V(x, \xi, t) \) in Model 1 and in Model 2 (for \( t = -0.5 \) GeV\(^2\)).
3 $e\gamma \to e\rho L \pi$ cross-section estimates

We first consider the $\rho L \pi$ production case and more precisely when the $\rho$ flies in the direction of the virtual photon and the $\pi$ enters the TDA. It is cleaner than the $\pi^\pm \pi^\mp$ production case. Indeed, the reaction $\gamma^* \gamma \to AB$ is accessible only in an $e\gamma$ reaction\(^7\) and thus must compete with the Bremsstrahlung process where the meson pair $A\pi$ comes from a virtual photon emitted by the lepton line. In the $\rho L \pi$ case, this Bremsstrahlung process is much suppressed because of the vector and axial-vector character of the $\rho L$ and $\pi$ leading-twist DAs; this leads to a vanishing leading-twist contribution to the form factor $F_{\rho\pi}$ at high transfer momentum. Moreover, in the neutral $\rho^0 \pi^0$ case, the TDA process is forbidden by $C$-conjugation.

3.1 Kinematics

The momenta in the process $e\gamma \to e\rho^\pm \pi^\mp$ are defined as shown on Fig. 3 in the CMS of the $\rho^\pm$ and $\pi^\mp$ (or equally of the $\gamma$ and $\gamma^*$). We choose the $x-z$ plane to be the one of the collisions $\gamma^* \gamma \to \rho \pi$, which we call hadronic plane. The leptonic one, i.e. where the $\gamma^*$ is emitted by the electron is at an angle $\varphi$ of the hadronic one. We also have $P = \frac{p_{\rho\pi} + p_{\gamma}}{2}$, $\Delta = p_{\rho\pi} - p_{\gamma}$ and $q = p_e - p'_e$, $q$ being the momentum of the $\gamma^*$.

![Figure 3: The kinematics of $e(p_e) + \gamma(p_\gamma) \to e(p'_{e}) + \rho(p_\rho) + \pi(p_\pi)$ in the center of mass of the meson pair.](image)

Then we define the light-cone vectors $p$ and $n$ ($p^2 = n^2 = 0$) such that $2p.n = 1$ and the invariants in terms of scalar products of the momenta:

\[
Q^2 = -q^2 = -(p_e - p'_e)^2, \quad \xi = -\frac{\Delta.n}{2P.n},
\]

\[
t = (p_{\rho\pi} - p_{\gamma})^2 = \Delta^2, \quad s_{e\gamma} = (p_e + p_\gamma)^2;
\]

\[
W^2 = (q + p_{\gamma})^2, \quad y = \frac{2q.p_{\gamma}}{p_{\gamma}.p_e} = \frac{(\xi + 1)Q^2}{2s_{e\gamma}} = \frac{Q^2 + W^2}{s_{e\gamma}}.
\]

\(^7\)We do not consider the complete case of $e^+ e^-$ collisions which may as usual be rewritten in the equivalent photon approximation in terms of the $e\gamma$ reaction and the quasi real photon flux.
For definiteness, we choose, in the CMS of the meson pair, \( p = \frac{Q^2 + W^2}{2(1 + \xi)W}(1, 0, 0, -1) \) and \( n = \frac{(1 + \xi)W}{2(Q^2 + W^2)}(1, 0, 0, 1) \) and we can then express the momenta through their Sudakov decomposition:

\[
\begin{align*}
p_\gamma &= (1 + \xi)p, \\
p_\pi^- &= (1 - \xi)p - \frac{\Delta^2}{1 - \xi}n + \Delta_T,
\end{align*}
\]

\( q = \frac{Q^2 + W^2}{1 + \xi}n - \frac{Q^2}{Q^2 + W^2}(1 + \xi)p, \)

\( \Delta^2_T = \frac{1 - \xi}{1 + \xi}t \)

We can see that \( \xi \) is determined by the external kinematics of \( \gamma^* \gamma \to \rho \pi \) through \( \xi \simeq \frac{Q^2}{Q^2 + W^2} \). Similarly to \( x_B = \frac{Q^2}{W^2} \), to which it is linked via the simple relation \( \xi \simeq \frac{x_B}{2 - x_B} \).

Since we want to focus on the study of the TDA behaviour, we decide to choose \( Q^2, t, \xi \) and \( \varphi \) as our kinematical variables. The differential cross section thus reads

\[
\frac{d\sigma^{\gamma^* \gamma \to \rho^0 \pi^+}}{dQ^2 dt d\xi d\varphi} = \frac{1}{32(2\pi)^4 s^{1/2}_\gamma} \frac{1}{\xi(\xi + 1)} |M^{\gamma^* \gamma \to \rho^0 \pi^+}|^2. \tag{17}
\]

### 3.2 The \( \gamma^*_L \gamma \to \rho_L^\pm \pi^\mp \) amplitude

Let us first consider the longitudinally polarised \( \rho \) meson, described by its twist-2 DA. Now the vector character of this DA selects the vector TDA and the amplitude for the reaction \( \gamma^*_L \gamma \to \rho_L^+ \pi^- \) becomes proportional to \( V(x, \xi, t) \). It reads

\[
M_{\gamma^*_L \gamma}^{TDA}(Q^2, \xi, t) = - \int_{-1}^{1} dx \int_{0}^{1} dz f_\rho(z) M_h(z, x, \xi) V(x, \xi, t), \tag{18}
\]

where the hard amplitude is (\( \bar{z} = 1 - z \))

\[
M_h(z, x, \xi) = \frac{8 \pi^2 \alpha_{em} \alpha_s C_F}{N_C Q} \frac{1}{z \bar{z}} \left( \frac{Q_u}{x - \xi + i\epsilon} + \frac{Q_d}{x + \xi - i\epsilon} \right) \epsilon_{\mu
u\rho\sigma} n_\mu \varepsilon_\nu p_\rho \Delta_\sigma. \tag{19}
\]

In the hard subprocess, the quark momenta have been as usual restricted to their component collinear w.r.t. the associated meson, i.e. for the quark leaving (resp. entering) the TDA, \( k \simeq (x + \xi)p \) (resp. \( k' \simeq (x - \xi)p \)) and the quark (resp. antiquark) entering the \( \rho \) meson, \( \ell \simeq z p_\rho \) (resp. \( \ell' \simeq \bar{z} p_\rho \)).

We choose \( f_\rho(z) = 6z\bar{z} \) as the asymptotic normalised meson distribution amplitude, \( f_\rho = 0.216 \text{ GeV}^{-1} \). After separating the real and imaginary part of the amplitude, the \( x \)-integration gives:

\[
I^V_x = \int_{-1}^{1} dx \left( \frac{Q_u}{x - \xi + i\epsilon} + \frac{Q_d}{x + \xi - i\epsilon} \right) V(x, \xi, t)
\]

\[
= Q_u \int_{-1}^{1} dx \frac{V(x, \xi, t) - V(\xi, \xi, t)}{x - \xi} + Q_d \int_{-1}^{1} dx \frac{V(x, \xi, t) - V(-\xi, \xi, t)}{x + \xi}
\]

\[
+ Q_u V(\xi, \xi, t)(\log \frac{1 - \xi}{1 + \xi} - i\pi) + Q_d V(-\xi, \xi, t)(\log \frac{1 + \xi}{1 - \xi} + i\pi). \tag{20}
\]

\( \Delta_T^2 < 0. \)
The scaling law for the amplitude is

$$\mathcal{M}_{\gamma^*\gamma}^{TDA}(Q^2, \xi, t) \sim \frac{\alpha_s \sqrt{-t}}{Q},$$

up to logarithmic corrections due to the anomalous dimension of the TDA.

3.3 The $\gamma_L^*\gamma \to \rho_L^\pm \pi^\mp$ contribution to $e\gamma \to e\rho_L^\pm \pi^\mp$

The squared amplitude for $e\gamma \to e\rho_L^\pm \pi^\mp$ from the subprocess $\gamma_L^*\gamma \to \rho_L^\pm \pi^\mp$ is obtained from $\mathcal{M}_{\gamma^*\gamma}^{TDA}(Q^2, \xi)$ and from the contribution of the fermionic line for the emission of a longitudinal virtual photon

$$|\mathcal{M}_{e\gamma}^{TDA}|^2 = \frac{4\pi \alpha_{em}}{2Q^4} \text{Tr}(p_e^\dagger \gamma_{\mu\nu} p_{\rho\sigma} \Delta_{\mu\nu})$$

(22)

Figure 4: $\xi$-dependence of the ratio of the real part to the modulus of the TDA amplitude for $t = -0.5$ GeV$^2$ and $Q^2 = 4$ GeV$^2$.

Averaging over the real photon polarisation and integrating over $\varphi$ thanks to the $\varphi$-independence of the TDA process, we eventually obtain the differential cross section:

$$\frac{d\sigma_{e\gamma \to e\rho^\pm \pi^-}^{TDA}}{dQ^2 dt d\xi} = \frac{128\pi \alpha_{em}^3 \alpha_s^2 2\pi}{9(\xi + 1)^4 Q^8 s_{e\gamma}} \frac{f_{\rho}^2}{f_D^2} (-t)(2s_{e\gamma} \xi - (\xi + 1)Q^2)(1 - \xi)(\text{Re}^2(I^V_x) + \text{Im}^2(I^V_x)).$$

(23)

Figure 4: $\xi$-dependence of the ratio of the real part to the modulus of the TDA amplitude for $t = -0.5$ GeV$^2$ and $Q^2 = 4$ GeV$^2$.

Both real and imaginary part of the amplitude contribute significantly to the cross section, which is reasonable at these moderate energies. To quantify this statement, we plot on Fig.
the relative contribution of the real part of the amplitude, as a function of the skewness $\xi$. It is independent of the other variables.

The phenomenological analysis of the pion form factor \cite{14} seems to indicate that a rather large ($\sim 1$) value of $\alpha_s$ should be used together with the asymptotic DA. In our plots, we therefore put $\alpha_s = 1$. Our upcoming conclusions would not be strongly affected by a different choice.

Figure 5: $e\gamma \to e'\rho_T^+\pi^-$ differential cross section plotted as a function of $\xi$ (a) and $Q^2$ (b) for $s_{e\gamma} = 40$ GeV$^2$, $t = -0.5$ GeV$^2$ and respectively $Q^2 = 4$ GeV$^2$ for (a) and $\xi = 0.2$ for (b).

In Fig. 5 (a), we plot the cross section vs $\xi$ and in Fig. 5 (b) vs $Q^2$ for both models of the vector TDA. The $\xi$-dependence can be generically understood at large $\xi$ by the factor $(1 - \xi)$ and at small values by the limitation on the phase space by the kinematical factor previously discussed. The behaviour for intermediate values of $\xi$ is sensitive to specific models for the TDA. As shown in Fig. 5 (b), the $Q^2$-behaviour is model independent and thus constitutes a crucial test of the validity of our approach.

Let us finally add a remark about the case of transversally polarised $\rho$, which could have been quite interesting since one may naively try to write the amplitude of the TDA under the form

$$M^{\rho_T\pi}(Q^2, \xi) \propto \int dz \Phi^{\rho_T}(z) \Phi^{\rho_T}(z, x, \xi) \rho_T(x, \xi, t),$$

with the chiral-odd DA for $\rho_T$ \cite{14} and the tensor chiral-odd TDAs since chiral-odd quantities should appear in pairs in a physical amplitude. However, a straightforward calculation shows that such an amplitude vanishes at leading order due to the identity $\gamma^a \sigma^{\mu\nu} \gamma_\alpha = 0$. This is reminiscent of the analysis of transversally-polarised vector-meson electroproduction in the forward region, where it has been shown \cite{16} that chiral-odd GPD contributions vanish at all orders and the measurement of these GPDs needs another hard subprocess \cite{17}.  

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4 \quad e\gamma \rightarrow e\pi \pi \text{ cross-section estimates}

4.1 \quad \gamma^*_L e \rightarrow \pi^+ \pi^- \text{ amplitude}

The momenta in the process $e\gamma \rightarrow e\pi^+\pi^-$ are defined as in the $e\gamma \rightarrow e\rho^+\pi^-$ case discussed above, where the $\pi^+$ meson flies in the direction of the virtual photon and the $\pi^-$ meson in the direction of the real one.

The main difference with the previous case is that the pion DA replaces the $\rho$ one and that the axial $\gamma \rightarrow \pi$ TDA $A(x,\xi,\ell)$ defined in Eq. (3) replaces the vector one. The amplitude thus reads:

$$M_{TDA}^{\gamma^*_L e}(Q^2,\xi) = \int_{-1}^{1} dx \int_{0}^{1} dz \phi_{\pi}(z) M_h(z, x, \xi) A(x, \xi, t),$$

where the hard amplitude is

$$M_h(z, x, \xi) = \frac{4 \pi^2 \alpha_{em} \alpha_s C_F}{N_C Q} \frac{1}{z \bar{z}} \left( \frac{Q_u}{x - \xi + i\epsilon} + \frac{Q_d}{x + \xi - i\epsilon} \right) \varepsilon \cdot \Delta,$$

and where $\phi(z)$ is the normalised meson distribution amplitude, $\bar{z} = 1 - z$. The factor $f_{\pi}$ cancels with the one from the TDA definition. Now if we choose $\phi_{\pi}(z) = \frac{6}{z \bar{z}}$, the $z$-integration is readily carried out and the result has the form of Eq. (20) with obvious replacement of TDAs $V \rightarrow A$. The scaling law for the amplitude is similar to the one for the $\rho\pi$ case.

4.2 The $e\gamma \rightarrow e\pi^+ \pi^-$ cross section

As said above, the contribution of the subprocess $\gamma^*_L e \rightarrow \pi^+ \pi^-$ to the $e\gamma \rightarrow e\pi^+\pi^-$ cross section is obtained from $M_{TDA}^{\gamma^*_L e}(Q^2, \xi)$ with the addition of the fermionic line for the emission of a longitudinal virtual photon. The cross section is obtained from Eq. (23) by the substitutions $I^V \rightarrow I^A$ and $f_{\rho} \rightarrow f_{\pi}$.

The Bremsstrahlung process where the $\pi^+ \pi^-$ pair is produced by a photon radiated from the leptonic line (see the graphs of Fig. 6) has the following squared amplitude (omitting for now the interference terms)

$$\left| M_{e\gamma \rightarrow e\pi^+ \pi^-}^{B} \right|^2 = \frac{64(4\pi \alpha_{em})^3 \xi^2}{Q^2(\xi + 1)^2} |F_{\pi}(W^2)|^2 \left[ 2 + \sqrt{-t} \frac{A}{Q^2} + \sqrt{-t} \frac{B}{Q^2} \right]\left[ 2 + \sqrt{-t} \frac{C}{Q^2} + \sqrt{-t} \frac{D}{Q^4} \right]$$

with

$$A = -\cos \varphi \frac{2(3\xi - 1)(y - 2)}{\sqrt{1 + \xi} \sqrt{1 - \xi} \sqrt{1 - y}}$$

$$B = \frac{\left[ \xi(5\xi - 2) + 1 \right] y^2 - 2(y - 1)[(1 - 3\xi)^2 + 4(\xi - 1)\xi \cos 2\varphi]}{(\xi - 1)(\xi + 1)(y - 1)}$$

$$C = \cos \varphi \frac{4\xi(3\xi - 1)(y - 2)}{(1 + \xi) \sqrt{1 + \xi} \sqrt{1 - \xi} \sqrt{1 - y}}$$

$$D = \frac{8\xi^2}{(\xi + 1)^2}$$
Figure 6: Feynman graphs for the Bremsstrahlung contribution to $e\gamma \rightarrow e\pi^+\pi^-$ at LO.

The factor used at the $\pi^+\pi^-$ vertex is $ie(p_{\pi^-} - p_{\pi^+})^\mu F_\pi((q + p_\gamma)^2)$ where $F_\pi(W^2)$ is the pion form factor which is well measured for $W^2$ up to a few GeV$^2$ and is described in a satisfactorily way at large space-like values by perturbative QCD [3], while the time-like region is less understood [19] but is measured [20].

In the following estimates, and to be consistent with our previous choice of the asymptotic pion DA in the TDA subprocess amplitude, we describe the pion form factor $F_\pi(W^2)$ with the asymptotic distribution amplitude and a large value of $\alpha_s$. This choice will not affect the main conclusions of this section, and the inclusion of a fit to the available data would give the same order of magnitude for the Bremsstrahlung contribution, which will remain mostly negligible in all the kinematical domain considered.

The use of the asymptotic form for the pion DA has been criticised and other inputs have been proposed on the basis of QCD sum rules [21]. If one uses the C-Z parametrisation and choose $\alpha_s = 0.6$, the amplitude remains unchanged. This has to be paralleled with the form factor analysis, where the use of the C-Z DA is sometimes accompanied by the use of a lower value for the strong coupling constant $\alpha_s$ of the order of 0.4 [15].

We show in Fig. 7 the relative contributions of the Bremsstrahlung and TDA subprocesses (omitting interference terms) to the differential cross section integrated over $\varphi$, as a function of $s_{e\gamma}$, $\xi$ and $Q^2$. Except for small values of $s_{e\gamma}$, the TDA subprocess clearly dominates for all reasonable values of $\xi$. We also show the $\varphi$-dependence of the Bremsstrahlung contribution, which may be turned into a positive check of the dominance of the $\varphi$ independent TDA subprocess, much in the manner of the successful test of the GPD framework in deeply virtual Compton scattering [22].

Since the Bremsstrahlung contribution is quite small, we have not given the expression of the interference terms which may be anyhow cancelled out if one considers the charge-averaged cross section $d\sigma(\pi^+\pi^-) + d\sigma(\pi^-\pi^+)$ since the TDA and Bremsstrahlung amplitudes produce hadronic states with opposite charge conjugations. On the other hand, this latter property may be used to separate the interference contribution by measuring the charge asymmetric quantity $d\sigma(\pi^+\pi^-) - d\sigma(\pi^-\pi^+)$, and thus analysing the TDA subprocess at the amplitude level [23].
Figure 7: Differential cross sections for the TDA and the Bremsstrahlung subprocesses as a function of $s_{e\gamma}$, $\xi$ and $Q^2$, and for the sole Bremsstrahlung subprocess as a function of $\cos \varphi$ (lower right plot). Except for the variable studied, $Q^2 = 4$ GeV$^2$, $\xi = 0.2$, $t = -0.5$ GeV$^2$ and $s_{e\gamma} = 40$ GeV$^2$.

4.3 The neutral case: $e\gamma \rightarrow e\pi^0\pi^0$

In this case, $C$ invariance forbids any contribution from the Bremsstrahlung subprocess. We emphasise here that the measurement of the cross section for $e\gamma \rightarrow e\pi^0\pi^0$ in the forward region would provide one of the sole possible determination of the axial form factor $F_A$ of the $\pi^0$ at small $t$. Indeed, its measurement is not feasible in $\pi^0 \rightarrow \nu\bar{\nu}\gamma$ and it is completely drowned in the electromagnetic background in $\pi^0 \rightarrow e^+e^-\gamma$.

Therefore, any measurement of the cross section for $e\gamma \rightarrow e\pi^0\pi^0$ would constrain the value of $F_A^{\pi^0}$ via the sum rule for the axial TDAs $A_q^{\pi^0}(x,\xi,t)$ similar to the one linking $F_{\pi^0\gamma^*\gamma}$ to $V_q^{\pi^0}(x,\xi,t)$ (see Eq. (6)). We expect the cross section $\gamma^*\gamma \rightarrow \pi^0\pi^0$ to be of the order of the one for $\gamma^*\gamma \rightarrow \pi^\pm\pi^\mp$ depicted in Fig. 7.
5 Conclusion

In summary, we have shown how to test experimentally the new factorised QCD approach to forward hard exclusive scattering. We believe that our models for the photon to meson transition distribution amplitudes are sufficiently constrained to give reasonable orders of magnitude for the estimated cross sections. The good news is that the hard hadronic process dominates the Bremsstrahlung contribution in the kinematics which are accessible in existing $e^+e^-$ colliders. The goal is now to test our approach, in particular by verifying the scaling of the cross sections, and then to measure these new hadronic matrix elements. Cross sections are large enough for quantitative studies to be performed. On the other hand, lattice studies could calculate these TDAs, at least within some approximations. Chiral perturbation theory may even be used to improve the extrapolation from a large quark mass to a realistic model. Higher-order corrections to the hard scattering process should be studied. The long history of the improvements of the QCD understanding of form factors at large momentum transfers \[24\] should give us a way to better include the effects of e.g. end point regions and their partial Sudakov suppressions. As in all studies of this type, possible higher-twist contributions may be relevant at measurable values of $Q^2$. Some studies have been done in related cases in particular models \[25\]. Experimental verification of the scaling laws are the first tests to be applied to our description. Moreover, the kinematical domain of applicability of the present description (and in particular the small transfer, large energy, requirements) may not be clear cut in real experiments using existing accelerators. An interesting further study should be to try to understand the transition regions between the kinematics where our framework should apply on the one hand and on the other hand the other existing QCD descriptions of the same $\gamma^*\gamma \to AB$ reactions at small energy \[26\] or at very large energy \[27\].

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