The diplomat’s dilemma: Maximal power for minimal effort in social networks

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Closeness is a global measure of centrality in networks, and a proxy for how influential actors are in social networks. In most network models, and many empirical networks, closeness is strongly correlated with degree. However, in social networks there is a cost of maintaining social ties. This leads to a situation (that can occur in the professional social networks of executives, lobbyists, diplomats and so on) where agents have the conflicting objectives of aiming for centrality while simultaneously keeping the degree low. We investigate this situation in an adaptive network-evolution model where agents optimize their positions in the network following individual strategies, and using only local information. The strategies are also optimized, based on the success of the agent and its neighbors. We measure and describe the time evolution of the network and the agents’ strategies.

I. INTRODUCTION

To increase or maintain power, or position of influence, is a goal of many professionals. Many definitions of power recognize that it is not an inherent attribute of an actor,¹ but a result of the interaction between agents. One well-known definition by Max Weber reads (25):

‘Power’ is the probability that one actor within a social relationship will be in position to carry out his own will despite resistance, regardless of the basis on which this probability rests.

Definitions like this suggest that there is a link between the power of an actor and its position in the network of social relationships. Thus, by examining a social network, one should be able to say something about the power of the agents. A major theme in social network studies has been to infer the power structures in organizations based on the contact patterns of their members (14). In undirected networks of actors, coupled pairwise by their social ties, one idea of measuring, or defining power, is to say that an actor that is close to others has more power, than a more peripheral actor does (22). This can be turned into a network measure called closeness centrality (which will be defined explicitly in the next section). Naively, a way to achieve power would then be to position oneself as close to everyone else in the network as possible, i.e. to have a social tie to each one of the network’s actors. In practice, to make, and maintain, a social tie requires the actor to invest time and other resources. To have a direct tie to a significant fraction of the network is thus neither feasible, nor desirable. We call this situation of two contrasting interests—to maximize power (in terms of being central), while at the same time keeping the number of social ties to a minimum—the diplomat’s dilemma.

The diplomat’s dilemma can also be motivated from a more academic perspective. Fueled by the increased availability of large-scale network datasets, there is a wave of interest in analyzing and modeling systems as graphs. One theme within this field of complex-network theory (1, 4, 17) has been to study systems where the network is formed by strategic decisions by the agents (i.e., situations where the success of the agents depend on the choice of other agents). This problem has been traditionally been analyzed from a game theory perspective. Some of the most interesting game-theoretical problems have been inspired by situations where the agents have conflicting objectives. In, for example, the iterated prisoner’s dilemma (2), agents have to choose between trying to achieve short-time benefits by exploiting other agents, and trying to optimize their long-term profit by building a relationship of mutual trust, but at the same time making them vulnerable to exploitation. In most real complex networks, and network models, there is a strong positive correlation between different centrality measures (16) such as the local degree centrality (the number of neighbors of a vertex), and closeness centrality. However, one must note that the correlation between these quantities, though mathematically possible —high centrality and low degree (and vice versa)— is not strictly necessary. A potentially interesting question in the interface between complex networks and game theory would then be “How can agents simultaneously maximize their centrality and minimize their degree?”. Another interesting aspect of this problem, in a more model-theoretic sense, is that the success of the agents can be estimated from their network positions alone. In most models of adaptive, coevolutionary networks (8), the score of the agents is related to some additional traits of the agents themselves and their interaction. Our model differs from this approach in the sense that the success of agents can be measured from the topological features of the graph itself, rather than some extremal attribute artificially ascribed to the agents.

¹ A person, or other well-defined social unit, in the context of our model; we will use the term agent.
II. DEFINITION OF THE MODEL

A. Preliminaries

The framework of our study is a graph \( G(t) = (V(t), E(t)) \) of \( N \) vertices \( V \) and \( M(t) \) edges \( E(t) \). The vertex set \( V \) is fixed, but the edge set \( E(t) \) varies (both its content and size) with time. A vertex marks the position of an agent in a social network of edges representing social ties. We will henceforth also assume the graph to be simple, i.e. no multiple- or self-edges are allowed. Let \( d(i, j) \) denote the distance between \( i \) and \( j \). Technically we define \( d(i, j) \) as the smallest number of edges in any path (sequence of adjacent edges) connecting \( i \) and \( j \). Then, for a connected graph \( G \), the closeness centrality \( (2) \) is defined as:

\[
c_{c}(i) = \frac{N - 1}{\sum_{j \in c(i)} d(i, j)}.
\]

The score function, that the agents seek to optimize, should increase with closeness centrality and decrease with degree. A simple choice for such a function is \( c_{c}(i)/k_i \) (where \( k_i \) is the degree of \( i \)). However, we do not want to restrict ourselves to connected networks. If the network is disconnected, we make the assumption that being a part of a large component should contribute to a larger centrality. One way of modifying closeness centrality to incorporate both these aspects (short distances and being a part of a large component implies centrality), is to define the centrality \( c(i) \) as

\[
c(i) = \sum_{j \in H(i)(i)} \frac{1}{d(i, j)}.
\]

where \( H(i) \) is the connected subgraph \( i \) belongs to and \( d(i, j) \) is the graph distance between \( i \) and \( j \). The number of elements in the sum of Eq. \( (2) \) is proportional to the number of vertices of \( i \)’s connected component which gives a positive contribution from large components. To obtain this property, we use the average reciprocal distance, rather than the reciprocal average distance (as in the original definition of closeness centrality). This adjusted definition gives a higher weight on the count of closer vertices, but captures similar features as closeness does. With the definitions established above, we are now ready to state the score function:

\[
s(i) = \begin{cases} 
c(i)/k_i & \text{if } k_i > 0 \\
0 & \text{if } k_i = 0
\end{cases}.
\]

For the purpose of our simulations, the networks we consider will have a initial configuration similar to Erdős-Rényi networks \( (6) \) with \( M_0 \) number of edges. In other words, the network is generated by adding \( M_0 \) edges one-by-one to \( N \) (isolated) vertices such that no multiple- or self-edge is formed.

B. Moves

We have outlined so far the basic setup for the game—the underlying graph representing the actors and their social network, and the score function that the agents want to optimize. However, to go from this point to a sensible simulation scheme, we need to determine how an agent can update its connections. A first, very common assumption, is that the agents are myopic— that they can receive information from, and affect others in the network only within a certain radius from itself. This assumption lies behind so much of social network studies that one may argue that in situations where the myopic assumption is not needed, so agents can see, and manipulate the network at large distances, the representation of the social network as a simple graph is not appropriate. In our case, we assume that an agent \( i \) can change its connections (affect the network) within the second neighborhood \( \Gamma_2 = \{ j \in V : d(i, j) \leq 2 \} \), and that \( i \) can see the score \( s(j) \), centrality \( c(j) \) and degree \( k_j \) of vertices in \( \Gamma_2 \). (Since \( s \) and \( c \) are global quantities, some global information reach \( i \) indirectly. Nevertheless, since the actual contact network cannot be inferred from this information, we still consider the agents myopic.)

The simulations proceed iteratively where, each time step, every vertex can update its network position by adding an edge to a vertex in \( \Gamma_2 \) and delete an edge to a neighbor. An illustration of the possible moves can be found in Fig.\( (1) \).

C. Strategies

Ideally one would provide the agents with some intelligence and use no further restrictions for how they update their positions to increase their scores. This is not as easy one can imagine and, for simplification, one would like to reduce the capability of the agents further. To do this, we assume that an agent \( i \) updates its position (either by deleting or attaching an edge), by applying a sequence of tie-breaking actions.

- **MAXD** Choose vertices with maximal degree.
- **MIND** Choose vertices with minimal degree.
- **MAXC** Choose vertices with maximal centrality in the sense of Eq. \( (2) \).
- **MINC** Choose vertices with minimal centrality.
- **RND** Pick a vertex at random.
- **NO** Do not add (or remove) any edge.

The sequences of actions define the strategies of the agents. The strategy of an agent \( i \) can be stored in two six-tuples \( s_{\text{add}} = (s_{i,0}^{\text{add}}, \ldots, s_{i,6}^{\text{add}}) \) and \( s_{\text{del}} \) representing a priority ordering of the addition and deletion actions respectively. If \( s_{\text{add}}(i) = \text{(MAXD, MINC, NO, RND, MIND, MAXC)} \) then \( i \) tries at first to attach an edge to the vertex in \( \Gamma_2(i) \) with highest degree. If more than one vertex has the highest degree, then one of these is selected by the MINC strategy. If still no unique vertex is found, nothing is done (by application of the NO strategy). Note that such a vertex is always found after strategies NO or RND are applied. If \( X = \emptyset \) no edge is added (or deleted).
The agent knows the centrality and degree of the neighbors and their accumulated score the last time step. Based on this information the agents can, during a time step, based on their strategies, decide to delete the edge to a neighbor, and reconnect to a vertex two steps away.

When updating the strategy, $i$ copies the parts of $s_{\text{add}}(j)$ and $s_{\text{del}}(j)$ that $j$ used the last time step, and let the remaining actions come in the same order as the strategy vectors prior to the update. For the purposes of making the set of strategy vectors ergodic, driving the strategy optimization (15; 18), and modeling irrational moves by the agents (13): we swap, with probability $p_s$, two random elements of $s_{\text{add}}(j)$ and $s_{\text{del}}(j)$ every strategy vector update. In addition to the strategy space we also would like to impose ergodicity in the network space (i.e. the game can generate all $N$-vertex graphs from any initial configuration). In order to ensure this, disconnected clusters should have the ability to reconnect to the graph. We allow this by letting a vertex $i$ attach to any random vertex of $V$ with probability $p_r$, every $t_{\text{rand}}$'th time step. This is not unreasonable as even in real social systems, edges may form between agents out of sight from each other in the social network. In fact some authors have pointed out, that in addition to information spreading processes, there are other factors that lead to the evolution of the social networks (cf. Ref. (24)).

**D. Strategy updates and stochastic rewiring**

The strategy vectors are initialized to random permutations of the six actions. Every $t_{\text{strat}}$'th time step an agent $i$ updates its strategy vectors by finding the vertex in $\Gamma_i = \{j : d(i, j) \leq 1\}$ with highest accumulated score since the last strategy update. This practice of letting the agent mimic the best-performing neighbor is common in spatial games (19), and is closely related to the bounded rationality paradigm of economics (13).

**E. The entire algorithm**

To summarize, the algorithm works as follows:

1. Initialize the network to a Erdős-Rényi network with $N$ vertices and $M_0$ edges.
2. For all agents, start with random permutations of the six actions as strategy vectors $s_{\text{add}}$ and $s_{\text{del}}$.
3. Calculate the score for all agents.
4. Update the agents synchronously by adding and deleting edges as selected by the strategy vectors. With probability $p_s$, add an edge to a random vertex instead of a neighbor’s neighbor.
5. Every $t_{\text{strat}}$'th time step, update the strategy vectors. For each agent, with probability $p_r$, swap two elements in it’s strategy vector.
6. Increment the simulation time $t$. If $t < t_{\text{tot}}$, go to step 1.

The parameter $n_{\text{avg}}$, averages over different realizations of the algorithm are performed. We will primarily use the parameter values $M_0 = 3N/2$, $p_s = 0.005$, $t_{\text{strat}} = 10$, $t_{\text{tot}} = 10^5$ and $n_{\text{avg}} = 100$.

**III. NUMERICAL RESULTS**

**A. Time evolution**

To get a feeling for the time evolution, we start by plotting quantities characterizing the strategies of the agents and the network structure. The most important parts of the strategy vectors are the first positions $s_{\text{add}}^1$ and $s_{\text{del}}^1$. In practice, ~90% of the decisions whether or not to add (or delete) a specific edge do not pass this first tiebreaker. In Fig. 3(a) and (b) we can see how complex the time-evolution of $s_{\text{add}}^1$ and $s_{\text{del}}^1$ can
be. Each sector of the plot corresponds to a leading addition (or deletion) action, and they have a size in the y-direction proportional to the fraction of vertices having that leading action value. The time evolution is complex, having sudden cascades of strategy changes and quasi-stable periods. Cascades in the leading addition action seem to be accompanied by cascades in the leading deletion action. The particular time-window shown in Fig. 3 was chosen to highlight such cascades. For the parameter values of Fig. 3, cascades involving more than 75% of the vertices happens about once every 10^5 time steps.

In Fig. 3(c) we measure the average score function \( s \). Being a non-zero-sum game, the value of \( s \) can vary significantly, a fact which can be seen upon examining the figure. Most of the time, the system is close to the observed maximum \( s \approx 80 \). One reason for lower scores can be seen in Fig. 3(d) where we plot the average degree \( k \). For some time steps, the network becomes very dense with an average degree of almost 20. As high degree is not desirable, the average score is low during this period. This rise in degree has, naturally, a corresponding peak in the leading deletion action NO. Another reason of the occasional dips in the average score can be seen in Fig. 3(e) where we plot the fraction \( n_1 \) that belongs to the largest connected component. This quantity is usually close to one, meaning that all agents are connected (directly or indirectly), but sometimes this fraction becomes very low. It is harder (than for the high-degree peaks) to see the corresponding strategic cause for these fragmented states. There are usually peaks corresponding to NO as the leading addition action, but these are also accompanied by peaks corresponding to NO as the leading deletion action. As we will see, this feature becomes less pronounced as the system size increases.

![FIG. 3 Output from an example run of a 200 system with \( p_c = 0.012 \). (a) and (b) show the fraction of vertices having a certain leading action for addition \( \sigma_{\text{add}} \) and deletion \( \sigma_{\text{del}} \) respectively. (c) shows the average score \( \langle s \rangle \), (d) the average degree \( k \) and (e) the fraction of vertices in the largest connected component \( n_1 \).](image)

![FIG. 4 Four different example networks from a run with the same parameter values as in Fig. 3. The symbols indicate the leading addition action. (a) shows the common situation where MAXC is the leading addition action. \( \sigma_{\text{add}} \) is MAXC for almost all agents. (b) shows a transition stage between \( \sigma_{\text{add}} \) being mostly MAXC to \( \sigma_{\text{add}} \) being primarily MAXC. (c) shows another transient configuration where a large number of different addition strategies coexist. (d) shows the addition strategies in a fragmented state.](image)

### B. Example networks

In light of the complex time evolution of the system, it is not surprising that the system attains a great variety of network topologies as time progresses. In Fig. 4 we show four snapshots of the system for a run with the same parameter values as in Fig. 3. In Fig. 4(a) the network comes from the most common strategy configuration where both the leading deletion and addition actions are MAXC for a majority of the agents (in this situation, we call the actions dominating). In this configuration the network is centered around two indirectly connected hubs. The vertices between these two hubs have the highest centrality, and since they are within the second neighborhood of most vertices in the network, and most agents have \( \sigma_{\text{add}} = \text{MAXC} \), these vertices will get an edge from the majority of agents (thus becoming hubs in the next time-step). There are 18 isolates with \( \sigma_{\text{add}} = \text{NO} \). These will stay isolates until their strategy vectors are mutated, which occurs (on average) every \( t_{\text{mut}} = 2000 \)th time step. Fig. 4(b) shows a rather similar network topology with the difference that a majority of the vertices have \( \text{MAXD} \) as their leading addition action (almost all vertices have \( \sigma_{\text{del}} = \text{MAXC} \)). For this
configuration, the MAXC vertices will move their edges to
the most central vertices whereas the MAXD vertices will not
move their edge. In Fig. 4(c) we show a more rare, high-(k)
configuration (τ ≈ 273, 545 in Fig. 3). Here the leading dele-
tion action is NO for about one fourth of the vertices, and
the system is rapidly accumulating edges. In Fig. 3(d) we show
a fragmented state, where a number of vertices have the leading
addition action NO. The vertices with σ^add = NO that are not
isolates have σ^del = NO so they will not fragment the net-
work further. On the other hand, the vertices with σ^add,del = MAXC
and σ^add,del = MAXD can fragment the network.

C. Effects of strategies on the network topology

We are now in a position to examine in detail the net-
work topologies that arise from different dominating addition
and deletion actions. First, we plot histograms (rescaled to
show the probability density functions) of the network struc-
tural quantities shown in Figs. 3(c), (d) and (e)—see Fig. 5.
These diagrams all have two peaks—one with low ⟨s⟩, ⟨k⟩
and ⟨n⟩ values (where the network is fragmented, the num-
ber of edges small and the scores low), and another broader
peak corresponding to a connected network with higher scores
and more edges. Interestingly, the different leading actions
are not completely localized to different peaks but spread out
over the whole range. Another counter-intuitive observation
is that there seems to be more agents with σ^add = NO in the
more dense peaks. These vertices (with σ^add = NO) seem to
be primarily isolated and do not affect the majority of vertices
(connected in the largest component). They will therefore stay
isolated until their strategies have changed or they have been
connected to the rest of the network by random connections.
We also observe that there is a larger variety of leading addi-
tion actions than leading deletion actions. A possible inter-
pretation of this is that the fitness of agents is more depend-
ent on the leading addition action. This seems natural in a sit-
uation where it is disadvantageous to connect to a majority of
agents (so the choice of neighbor to disconnect is not impor-
tant), however it is beneficial to connect to to a minority of
well established agents.

One of the most widely studied and revealing metrics of
network structure, is the degree distribution—the probability
mass function of the degrees of vertices. In Fig. 6(a) we plot
the degree distribution for dominating actions σ^add = MAXC (a)
and σ^add = NO (b). The σ^add = MAXC graph has two high-k
peaks, corresponding to the hubs in the network. The exist-
ence of two broad peaks as opposed to only one is strange,
and the reasons for this is not immediately apparent. The
σ^add = NO graphs (whose averaged degree distribution are
shown in (b)) are more dense, as expected. However, they
also have a large-k peak, which is probably related to, either
the strategies of other agents, or a residue from the preceding
period (remember that the periods of dominating σ^add = NO
is very short compared with the σ^add = MAXC periods). This
implies that one can separate system-wide effects of some
strategy driving the decisions of the majority, but there will
also be other effects present in the network. Note that, while
many studies have focused on the emergent properties of de-
gree distributions as N → ∞, the interesting features of our
model occurs for smaller system sizes, consequently we be-
lieve this limit is not interesting or relevant to our study and
we do not consider it.

We now proceed to look at four other measures of differ-
ent network structures and how they depend on the domina-
ting addition and deletion actions. The first two measures we

![FIG. 5](image_url)

FIG. 5 The probability density function of average scores (a), (b),
average degrees (c), (d), and relative sizes of the largest connected
component (e), (f). The different fields represent different leading
addition actions (a), (c), (e), and different leading deletion actions
(b), (d), (f). The vertical size of a field gives the probability density
function conditioned to that leading action. The curves are averages
over ten runs of 10^5 timesteps with the same parameter values as in
Fig. 3. The color codes of the actions are the same as in Fig. 3.

![FIG. 6](image_url)

FIG. 6 The degree distribution for systems with the same parameter
values as in Fig. 3. Panel (a) shows the averaged degree distribution
when more than half of the agents have MAXC as their leading addi-
tion actions. Panel (b) displays the corresponding plot for the leading
addition action NO.
consider are the degree \( k \) and score \( s \). In Fig. 7(a) and (b) we plot the average values of these quantities (averaged over all vertices, regardless of strategy, and averaged over all samples with a particular dominating strategy). This plot is based on ten runs for 10^5 time steps, with network quantities measured every tenth time step. During these runs, the two leading actions—\( \sigma^{add}_1 = \text{MINC} \) and \( \sigma^{del}_1 = \text{MIND} \) were never employed. We note that the most common leading actions (for both addition and deletion) MAXC and MAXD gives the highest average score. This does not mean that all agents have a high score in these situations—from Figs. 3(a) and (b) we know that the score can differ much from one agent to another. The degrees are low for these strategies, which is a necessary (but not sufficient) condition for a low score. For \( \sigma^{add}_1 = \text{NO} \) the average degree is also low, but the score is much lower than for \( \sigma^{add}_1 = \text{MAXC} \) and \( \text{MAXD} \). The reason, as pointed out above, is that the network can become heavily fragmented for this leading action. The \( \sigma^{add}_1 \) corresponding to the highest degree is RND, this might seem strange, but during these runs (which is also visible in Fig. 3) \( \sigma^{add}_1 = \text{RND} \) is correlated with \( \sigma^{del}_1 = \text{NO} \) which is a state naturally leading to a comparatively dense network. The other leading actions \( \sigma^{add}_1 = \text{MINC} \) and \( \sigma^{del}_1 = \text{MINC} \) result in low scores and sparse networks.

The other two measures we examine are the assortativity and clustering coefficient. Before discussing our results, let us first define these quantities in detail. The average degree tells us if the network is sparse or dense. The degree distribution gives a more nuanced picture of how homogeneous the set of vertices are with respect to the number of neighbors. The next level of complexity in describing the network with respect to the agents’ degree, is to measure the correlations between the degrees of vertices at either side of an edge. In particular, one can determine if high-degree vertices are primarily connected to similar high degree vertices, or instead are linked to low-degree vertices. The assortativity \( r \) is a measure of vertices’ tendency to connect to other vertices of similar type, in this case those with similar degree \( \langle k \rangle \). In technical terms, \( r \) is the Pearson correlation coefficient of the degrees at either side of an edge. There is an additional caveat that we need to consider; since the edges in our networks are undirected, \( r \) has to be symmetric with respect to edge-reversal (i.e. replacing \( (i,j) \) by \( (j,i) \)). However the the standard definition of the Pearson correlation coefficient does not account for this symmetry. The way to fix this problem is to let one edge contribute twice to \( r \), i.e. to represent an undirected edge by two directed edges pointing in opposite directions. If one employs an edge list representation internally (i.e., if edges are stored in an array of ordered pairs \( (i_1, j_1), \cdots, (i_M, j_M) \)) then we can write the adjusted \( r \) as,

\[
r = \frac{4(k_1 k_2) - \langle k_1 + k_2 \rangle^2}{2(k_1^2 + k_2^2) - \langle k_1 + k_2 \rangle^2},
\]

where, for a given edge \( (i, j) \), \( k_1 \) is the degree of the first argument (i.e., the degree of \( i \)), \( k_2 \) is the degree of the second argument and the brackets \( \langle \cdots \rangle \) denote averaging. The range of \( r \) is \([-1, 1]\) where negative values indicate a preference for highly connected vertices to attach to low-degree vertices, and positive values imply that vertices tend to be attached to other vertices with degrees of similar magnitudes.

The clustering coefficient, on the other hand, is a measure of transitivity in the network. In other words it checks whether neighbors of a node are also connected to each other (thus forming triangles). It is a well known empirical fact that social acquaintance networks have a strong tendency to form triangles \( \langle 9 \rangle \) and it is therefore a worthwhile exercise to examine whether the networks generated by our model display this feature. There is in principle, more than one way to define the clustering coefficient. Here we employ the most commonly used one \( \langle 3 \rangle \),

\[
C = 3n_{\text{triangle}} / n_{\text{triple}},
\]

where \( n_{\text{triangle}} \) is the number of triangles and \( n_{\text{triple}} \) is the number of connected triples (subgraphs consisting of three vertices and two or three edges). The factor of three is included to normalize the quantity to the interval \([0, 1]\).

Now that we have defined these quantities we refer back to Fig. 7. We note that the most common leading actions \( \sigma^{add,del}_1 = \text{MAXC} \) and \( \text{MAXD} \) have the lowest \( C \) and \( r \) values. A possible explanation for this could be the following. Consider a triangle, a subgraph of three vertices connected by three edges. The graph will be connected even if one of these edges is deleted. In a situation where edges are expensive, this kind of redundancy is not desired. For this reason, it seems natural that, on average, the most successful strategies \( \text{MAXC} \) and \( \text{MAXD} \) have few triangles. The negative assortativity of these situations are also conspicuous features of

![Fig. 7](image-url)
the examples shown in Figs. 4(a) and (b) (most vertices there are only connected to the two hubs, but the hubs are not connected to each other). For networks with a broad spectrum of degrees, it is known that (C) and (r) are relatively strongly correlated [12]. This is also true in Figs. 7(c) and (d) where the relationship between (C) and (r) is monotonically increasing. The network configurations with highest (C) and (r) are the ones with $\sigma^{\text{add}}_1 = \text{MIND}$ and $\sigma^{\text{del}}_1 = \text{MINC}$. Since these networks are both sparse and fragmented, some components must have a large number of triangles (probably close to being fully connected). The denser states, with $\sigma^{\text{add}}_1 = \text{RND}$ and $\sigma^{\text{del}}_1 = \text{NO}$, have intermediate (C)- and (r)-values, meaning that the edges are more homogeneously spread out, similar to the network in Fig. 4(c).

D. Transition probabilities

From Fig. 3 it seems likely that the ability of one leading action to grow in the population depends on the other predominant strategies in the system. For example, $\sigma^{\text{add}}_1 = \text{RND}$ dominates after a period of many agents employing $\sigma^{\text{add}}_1 = \text{MIND}$ as the leading strategy. Consequently, it is worth asking the question: How does the probability of one leading action depend on the configuration at earlier time steps?

We investigate this qualitatively by calculating the “transition matrix” $T'$ with elements $T'(s_i, s'_j)$ giving the probability of a vertex with the leading action $s_i$ to have the leading action $s'_j$ at the next time step. However, note that the dynamics is not fully determined by $T'$, and is thus not a transition matrix in the sense of other physical models. If that were the case (i.e., the current strategy is independent of the strategy adopted in the previous time step) we would have the relation $T'_{ij} = \sqrt{T_{ij}T_{ji}}$. To study the deviation from this null-model, we assume the diagonal (i.e., the frequencies of the strategies) given, and calculate $T$ defined by,

$$T_{ij} = T'_{ij}/\sqrt{T'_{ij}T'_{ji}}. \quad (6)$$

The values of $T$ for the parameters defined in Fig. 3 are displayed in Tabs. II and III. The off-diagonal elements have much lower values than 1 (the average off-diagonal $\Theta$ values are 0.014 for addition strategies and 0.010 for deletion). This reflects the contiguous periods of one dominating action. Note that transitions between MAXC and RND are over-represented: $T^{\text{del}}_{\text{MAXC,RND}} \approx T^{\text{del}}_{\text{RND,MAXC}} \approx 0.027$, which is more than twice the value of any other off-diagonal element involving MAXC or RND. As another token of the problem’s complexity, the matrix is not completely symmetric $T^{\text{del}}_{\text{RND,NO}}$ is twice ($\sim 3$ s.d.) as large as $T^{\text{del}}_{\text{NO,RND}}$, meaning that it is easier for RND to invade a population with NO as a leading deletion action, than vice versa.

E. Dependence on system size and noise

So far we have focused on one set of parameter values. In this section we investigate how the system behavior depends on the number of agents and the noise level in the deletion and attachment mechanism. In Fig. 8 we tune the noise level (fraction of random attachments) $p_r$ for three system sizes. In panels (a)–(c) we show the fraction of leading actions among the agents $\langle \Sigma^{\text{add}}_N \rangle$ (averaged over $\sim 100$ runs and $10^6$ time steps). The quantities $\Sigma^{\text{add},\text{del}}$ denote the fraction of agents having a specific $\sigma^{\text{add},\text{del}}$. As observed in Fig. 8(a) the leading action is MAXC followed by MAXD and RND. The leading deletion actions, as seen in panels (d)–(f), are ranked similarly except that MAXD has a larger (and increasing) presence. If $p_r = 1$, then all actions are equally likely (they do not have any meaning—all strategies will result in random moves equal to $s^{\text{add}}_1 = s^{\text{del}}_1 = \text{RND}$). There are trends in the $p_r$-dependence of $\langle \sigma^{\text{add}}_1 \rangle$, but apparently no emerging discontinuity. This observation, (which also seems to hold for the $p_r$-scaling), that there is no phase transition for any parameter value governing the probability of random permutations in the strategy vectors, is an indication that the results above can be generalized to a large parameter range. We also note that, although the system has the opportunity to be passive (i.e., agents having $s^{\text{add}}_1 = s^{\text{del}}_1 = \text{NO}$), this does not happen. This situation is reminiscent of the “Red Queen hypothesis” of evolution [23]—organisms need to keep evolving to maintain their fitness.

Next we look at the dependence of the network structure on
the number of agents and the noise level. The average degree, plotted in Fig. 3(g), is monotonously increasing with $p_r$. There is, however, a qualitative difference in the size scaling—for $p_r \lessapprox 0.12$ the average degree increases with $N$, for $p_r \gtrapprox 0.12$ this situation is reversed. In Fig. 3(h) we plot the average largest-component size as a function of $p_r$ for different system sizes. The behavior is monotonous in both $p_r$ and $N$—larger $p_r$, or a larger system size, means higher $\langle n_1 \rangle$. In all network models we are aware of (allowing fragmented networks), a decreasing average degree implies a smaller giant component. For $p_r \gtrapprox 0.12$, in our model the picture is the opposite—as the system grows the giant component spans an increasing fraction of the network. This also means that the agents, on average, reach the twin goals of keeping the degree low and the graph connected.

### IV. DISCUSSION

We have presented a general game theoretic network problem, the diplomat’s dilemma—how can an agent in a network simultaneously maximize closeness centrality and minimize degree. The motivation for this problem comes in part from a type of social optimization situation where agents seek to gain power (via closeness centrality) and keep the cost (degree) low. It can also be motivated from a more academic point of view—interesting dynamics often comes from when agents simultaneously try to optimize conflicting objectives. The diplomat’s dilemma is one of the simplest such situations in a networked system, because the score function does not depend on any additional variable, or trait, of the vertices, only the vertex’ position in the network.

We devise an iterative simulation where at every time step, an agent can delete it’s connection to a neighbor and add an edge to a second neighbor, based on the information it possesses about the network characteristics of vertices within its local neighborhood (upto second neighbors). The agents use strategies that they update by imitating the best performing neighbor within this information horizon. The dynamics are driven by occasional random moves and random permutations of the vectors encoding the strategies of the agents. For the sake of flexibility, the definition of the problem as stated in this chapter, is deliberately vague. To turn it into a mathematically well-defined problem, one has to specify how the agents can affect their position in the network and what information they can use for this objective. There are of course many choices for how to do this. Although we believe our formulation is natural, it would be very interesting to rephrase these assumptions. A future enhancement would be to equip the agents with methods from the machine learning community to optimize their position, and to tune the amount of information accessible to the agents. A mathematical simplification of the problem would be to let all agents know the precise network topology at all times (this may however lead to some conceptual problems—if the information about the network is obtained via the network, it would be strange if the picture of the network close to an agent would not be more accurate than the picture of more remote sections of the network). Another interesting version of the problem would be to require an edge to represent an agreement between both vertices, so that an agent $i$ cannot add an edge $(i, j)$ unless $j$ finds this profitable.

Nevertheless, despite the simplicity of our model, the time evolution of the simulation is strikingly complex, with unstable states, trends, spikes and cascades of strategies among the agents. This complex dynamics is also captured in various metrics measuring different levels of network structure. Furthermore, the network structure and the agents’ strategies directly influence one another. If the agents stop deleting edges, the average degree of the network will grow rapidly.
which may benefit a strategy aiming to lower the degree of the agents. This feedback from network structure, to the agents and their decisions about how to update their networks is a central theme in the field of adaptive, coevolutionary networks \[8\]. We believe that all forms of social optimization involve such feedback loops, which is a strong motivation for studying adaptive networks. The complexity of the time-evolution, especially in the network structural dynamics is more striking for intermediate system sizes. Indeed, many interesting features of our simulation are not emergent in the large-system limit, but rather present only for small sizes.

Models in theoretical physics have traditionally focused on properties of the system as \(N \to \infty\). In models of social systems however, extrapolating to infinite size is not necessarily a natural limit in the same way (it will of course be interesting to examine the limiting behavior of such models). We believe this is a good example of the dangers of taking the large-size limit by routine—the most interesting relevant features of the model may be neglected.

In a majority of the cases in our simulations, most of the agents use a strategy where they both delete, and attach to vertices according to the MAXC action. This implies that the agent first deletes the edge to the most central vertex in the second neighborhood (in the sense of a modified closeness centrality), and then reattaches to the most central vertex two steps away (before the deletion). In practice this means that an agent typically transfers an edge from it’s most central immediate neighbor to it’s most central neighbor two steps away. This strategy makes the agent move towards the center without increasing its degree, which clearly seems like a reasonable procedure in the diplomat’s dilemma. However, this strategy is not evolutionary stable in the presence of noise (hence the complex time evolution). This strategy creates networks with low clustering coefficients, i.e., there are a comparatively small number of triangles. Since forming a triangle introduces an extra edge, which is expensive, without changing the size of connected component, one can understand why agents are reluctant to form these triangles \textit{per se} in our formulation of the problem.

Different strategies have different ability to invade one another. To test this we measure the deviation from random transitions from one dominating action to another (given the frequency of particular strategies), concluding for example that it is about twice as easy for RND to invade NO as a leading deletion action. Another interesting aspect is that (for some noise levels), as the system size increases, the network becomes both more connected (the relative fraction of vertices in the largest connected component increases), and more sparse (the average degree decreases). This is in sharp contrast to all other generative network models that we are aware of, but definitely consistent with the objectives of the general problem (where large connected components and low degrees are desired).

What does this result tell us about the real professional life of diplomats? Maybe that they can, by selfishly optimizing their positions in the network, self-organize to a connected business network where they need only a few business contacts, without knowing more about the network than the second neighborhood. However to make a stronger and more conclusive statement about the optimal strategy, more results are needed. This is something we hope to gather from future studies.

One of the problems facing this type of mechanic modeling of social information processes \[8\], \[9\], \[10\], \[11\], \[20\], \[21\], is that they are very hard to validate. Information spreading in social systems is neither routed from agent to agent like the information packets in the Internet, nor do they spread in the same fashion as epidemics. Instead the spreading dynamics is content dependent. Different types of information may be spreading over different social networks, following different dynamic rules. There are some promising datasets for studying social information spreading. For example, networks of blogs, Internet communities, or social networking sites generate large amounts of potentially valuable data, although these data sets are not necessarily conducive to the questions that adaptive models such as the one described in this chapter seek to address. In the near future, we hope mechanistic modeling of social information processes will be more data driven, asking questions that can actually be validated through empirical study.

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