Message Transmission and Common Randomness Generation Over MIMO Slow Fading Channels With Arbitrary Channel State Distribution

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Abstract—We investigate the problem of message transmission and the problem of common randomness (CR) generation over single-user multiple-input multiple-output (MIMO) slow fading channels with average input power constraint, additive white Gaussian noise (AWGN), arbitrary state distribution and with complete channel state information available at the receiver side (CSI). We derive a lower and an upper bound on the outage transmission capacity of MIMO slow fading channels for arbitrary state distribution and show that the bounds coincide except possibly at points of discontinuity of the outage transmission capacity, of which there are, at most, countably many. Such discontinuity issues might occur because the channel state distribution is arbitrary. We also establish the capacity of a specific compound MIMO Gaussian channel in order to prove the lower bound on the outage transmission capacity. Furthermore, we define the outage CR capacity for a two-source model with unidirectional communication over a MIMO slow fading channel with arbitrary state distribution and establish a lower and an upper bound on it using our bounds on the outage transmission capacity of the MIMO slow fading channel.

Index Terms—Common randomness, outage transmission capacity, MIMO slow fading channels, compound MIMO Gaussian channels.

I. INTRODUCTION

MOTIVATED by its striking applications in the theory of identification, Ahlswede and Csiszár introduced the concept of generation of non-secret common randomness (CR) in [1]. The identification scheme is an approach in communications developed by Ahlswede and Dueck [2] in 1989. In the identification framework, the decoder is not interested in knowing what the received message is. He rather wants to know if a specific message of special interest to him has been sent or not. Naturally, the sender has no knowledge of that specific message, otherwise, the problem would be trivial. It turns out that CR may allow a significant increase in the identification capacity of channels [1], [3], [4]. While the number of identification messages (also called identities) increases exponentially with the block-length in the deterministic identification scheme for discrete memoryless channels (DMCs), the size of the identification code increases doubly exponentially with the block-length when CR is used as a resource. The identification scheme is more suitable than the classical transmission scheme proposed by Shannon [5] in many practical applications which require robust and ultra-reliable low latency information exchange including several machine-to-machine and human-to-machine systems [6], industry 4.0 [7] and 6G communication systems [8]. It is therefore expected that CR will be an important resource for future communication systems [8], [9] and, in particular, that resilience requirements [8] and security requirements [10] can also be met on the basis of CR. These requirements are again of particular importance for achieving trustworthiness, which is a key challenge for future communication systems due to modern applications [11]. For this reason, CR generation for future communication networks is a central research question in large 6G research projects [12], [13].

The applications of CR generation are not restricted to the identification scheme. The availability of CR as a resource plays in general a key role in distributed settings [14]. It allows
to design correlated random protocols that often perform faster and more efficiently than the deterministic ones or the ones using independent randomization. Further examples of the applications of CR include correlated random coding over arbitrarily varying channels (AVCs) [15] and oblivious transfer and bit commitment schemes [16], [17]. CR is also of high relevance in the key generation problem. Indeed, under additional secrecy constraints, the generated CR can be used as secret keys, as shown in the fundamental two papers [18], [19]. The generated secret keys can be used to perform cryptographic tasks including secure message transmission and message authentication. In our work, however, we will not impose any secrecy requirements.

We study the problem of CR generation in the basic two-party communication setting in which Alice and Bob aim to agree on a common random variable with high probability by observing independent and identically distributed (i.i.d.) samples of correlated discrete sources and while communicating as little as possible. Ahlswede and Csiszár initially introduced the problem of CR generation from discrete correlated sources where the communication was over perfect channels with limited capacity [1]. A single-letter characterization of the CR capacity for this model was established in [1]. CR capacity refers to the maximum rate of CR that Alice and Bob can generate using the resources available in the model. The results on CR capacity were later extended to single-input single-output (SISO) and multiple-antenna Gaussian channels in [20] for their practical relevance in many communication situations such as wired and wireless communications, satellite and deep space communication links, etc. The results on CR capacity over Gaussian channels have been used to establish a lower-bound on the corresponding correlation-assisted secure identification capacity in the log-log scale in [20]. This lower bound can already exceed the secure identification capacity over Gaussian channels with randomized encoding established in [21].

In our work, we consider the problem of CR generation over single-user multiple-input multiple-output (MIMO) slow fading channels with complete channel state information available at the receiver side (CSIR). The focus is on the MIMO setting since multiple-antenna systems present considerable practical benefits including increased capacity, reliability and spectrum efficiency. This is due to a combination of both diversity and spatial multiplexing gains [22]. In particular, a practically relevant model in wireless communications is the slow fading model with additive white Gaussian noise (AWGN) [22], [23], [24], [25]. In the multiple-antenna slow fading scenario, the channel state, represented by the channel matrix, is random but remains constant during the codeword transmission. Therefore, channel fades cannot be averaged out and ensuring reliable communication is consequently challenging.

A commonly used concept to assess the performance in slow fading environments is the $\eta$-outage transmission capacity defined to be the supremum of all rates for which the outage probability is lower than or equal to $\eta$ [22], [23]. From the channel transmission perspective and for a given coding scheme, outage occurs when the instantaneous channel state is so poor that that coding scheme is not able to establish reliable communication over the channel. The capacity versus outage approach was initially proposed in [24] for fading channels. Later, this approach was applied to multi-antenna channels in [26], where the analysis was restricted to MIMO Rayleigh fading channels. To the best of our knowledge, neither an operational definition of the outage transmission capacity of SISO and MIMO slow fading channels with arbitrary state distribution nor a closed-form formula and a proof of it is provided in the literature. For instance, the capacity formula provided in [22] for the SISO case is not valid when the distribution function of the square of the absolute value of the state is discontinuous.

The first contribution of this paper lies in providing an operational definition of an achievable outage transmission rate and the outage transmission capacity as well as deriving a lower and an upper bound on the $\eta$-outage transmission capacity of MIMO slow fading channels with average input power constraint, AWGN and arbitrary state distribution. Since the state distribution is arbitrary, we will show that the bounds coincide except possibly at possible points of discontinuity of the outage transmission capacity, of which there are, at most, countably many. Furthermore, we will show that when the state has a density, which is positive except on a set with Lebesgue measure equal to zero, then the bounds on the $\eta$-outage transmission capacity coincide for all possible values of $\eta$, including the ones at which the outage transmission capacity is discontinuous. To prove the lower bound on the outage transmission capacity, we will also establish the capacity of a compound MIMO Gaussian channel corresponding to the set of MIMO Gaussian channels for which the channel matrix of each of these channels is an element of an arbitrary compact set. We will additionally establish the $\eta$-outage transmission capacity of single-input multiple-output (SIMO) slow fading channels and provide an alternative proof of the outage transmission capacity for the SISO case based on the degradedness of SISO Gaussian channels as well as the strong converse for this type of channels. It is here worth-mentioning that for the SISO and the SIMO case, the $\eta$-outage transmission capacity is exactly known at points of discontinuity. The outage transmission capacity formula that we prove for the SISO case is an extension of the formula presented in the literature to arbitrary state distribution.

The second contribution of this paper lies in introducing the concept of outage in the CR generation framework as well as deriving a lower and an upper bound on the $\eta$-outage CR capacity for a two-source model with one-way communication over MIMO slow fading channels with AWGN and arbitrary state distribution. In the CR generation framework, outage occurs when the channel state is so poor that Alice and Bob cannot agree on a common random variable with high probability. The $\eta$-outage CR capacity is defined to be the maximum of all achievable CR rates for which the outage probability from the CR generation perspective does not exceed $\eta$. In the proof of the bounds on the $\eta$-outage CR capacity, we will use our bounds on the $\eta$-outage transmission capacity of MIMO slow fading channels. Since the state distribution is arbitrary, the derived bounds on the outage CR capacity coincide except at the points where our bounds on the
outage transmission capacity do not coincide, of which there are, at most, countably many. In [1], no discontinuity issues for the CR capacity occur because it is assumed that Alice and Bob communicate over rate-limited perfect channels. It is also worth mentioning that with the standard techniques proposed in [1], it is not possible to prove the upper-bound on the \(\eta\)-outage CR capacity. We have to use a change of measure argument to prove this upper-bound.

**Paper Outline:** Section II describes the system model and provides the key definitions as well as the main and auxiliary results. In Section III, we derive a lower and an upper bound on the \(\eta\)-outage transmission capacity of MIMO slow fading channels with average input power constraint, AWGN and with arbitrary state distribution. In Section IV, we establish the \(\eta\)-outage transmission capacity for the SIMO case and provide an alternative proof of it for the SISO case. Section V is devoted to the derivation of a lower and an upper bound on the \(\eta\)-outage transmission capacity for the SIMO case and provide an alternative proof of it for the SISO case. In Section VI, we establish the capacity of a specific compound MIMO complex Gaussian channel. Section VII contains concluding remarks and proposes potential future research in this field.

**Notation:** \(\mathbb{C}\) denotes the set of complex numbers and \(\mathbb{R}\) denotes the set of real numbers; \(H(\cdot)\) and \(h(\cdot)\) correspond to the entropy and the differential entropy function, respectively; \(I(\cdot;\cdot)\) denotes the mutual information between two random variables. All information quantities are taken to base 2. Throughout the paper, \(\log\) is taken to base 2. The natural exponential and the natural logarithm are denoted by \(\exp\) and \(\ln\), respectively. For any random variables \(X\), \(Y\) and \(Z\), we use the notation \(X\circ Y\circ Z\) to indicate a Markov chain. \(T_{n}^{U}\) denotes the set of \(U\)-typical sequences of block-length \(n\) and of type \(P_{U}\). For any \(x \in \mathbb{C}\), \(\text{Re}(x)\) refers to its real part. For any matrix \(A\), \(\text{tr}(A)\) refers to the trace of \(A\), \(\|A\|\) stands for the operator norm of \(A\) with respect to the Euclidean norm, \(A^{H}\) stands for the standard Hermitian transpose of \(A\) and \(A^{-1}\) refers to the matrix inverse of \(A\), \(\lambda_{\min}(A)\) refers to the minimum eigenvalue of \(A\) and \(\lambda_{\max}(A)\) refers to the maximum eigenvalue of \(A\). For any random matrix \(A \in \mathbb{C}^{m \times n}\) with entries \(A_{i,j}\), \(i = 1, \ldots, m, j = 1, \ldots, n\), we define

\[
E[A] = \begin{bmatrix}
E[A_{11}] & E[A_{12}] & \cdots \\
\vdots & \ddots & \vdots \\
E[A_{m1}] & E[A_{m2}] & \cdots
\end{bmatrix}.
\]

For any vector \(X\), \(X^{T}\) refers to its transpose. For any random vector \(X\), \(\text{cov}(X)\) refers to its covariance matrix. For any set \(\mathcal{E}\), \(\mathcal{E}^{c}\) refers to its complement and \(|\mathcal{E}|\) refers to its cardinality.

**II. System Model, Definitions and Results**

**A. System Model**

Let a MIMO slow fading channel \(W_{G}\) be given. First, we define the MIMO slow fading channel \(W_{G}\). Suppose that one terminal called Terminal \(A\) wants to transmit a message to another terminal called Terminal \(B\) by sending, for arbitrary \(n > 0\), an input sequence \(t^{n} = (t_{1}, \ldots, t_{n}) \in \mathbb{C}^{nT \times n}\) of block-length \(n\) over the MIMO slow fading channel. Terminal \(B\) observes the output sequence \(z^{n} = (z_{1}, \ldots, z_{n}) \in \mathbb{C}^{nR \times n}\) of block-length \(n\) such that

\[
z_{i} = G t_{i} + \xi_{i}, \quad i = 1, \ldots, n.
\]

Here, \(N_{T}\) and \(N_{R}\) refer to the number of transmit and receive antennas, respectively. \(G \in \mathbb{C}^{nR \times nT}\) models the random complex gain, where we assume that both terminals \(A\) and \(B\) know the distribution of the gain \(G\) and that the actual realization of the gain is known by Terminal \(B\) only. \(\xi^{n} = (\xi_{1}, \ldots, \xi_{n}) \in \mathbb{C}^{nR \times n}\) models the noise sequence. We assume that the \(\xi\)’s are i.i.d. such that \(\xi_{i} \sim N_{C}(0_{nR}, \sigma^{2} I_{nR})\), \(i = 1, \ldots, n\). We further assume that \(G\) and \(\xi^{n}\) are mutually independent.

We are interested in the problem of common randomness (CR) generation over \(W_{G}\). Let a discrete memoryless multiple source (DMMS) \(P_{XY}\) with two components, with generic variables \(X\) and \(Y\) on alphabets \(\mathcal{X}\) and \(\mathcal{Y}\), respectively, be given. The DMMS emits i.i.d. samples of \((X,Y)\). Suppose that the outputs of \(X\) are observed only by Terminal \(A\) and those of \(Y\) only by Terminal \(B\). We further assume that the joint distribution of \((X,Y)\) is known to both terminals. Terminal \(A\) can communicate with Terminal \(B\) over the MIMO slow fading channel \(W_{G}\). We also assume that \((X^n,Y^n)\) is independent of \((G,\xi^n)\). There are no other resources available to any of the terminals.

**Definition 1:** A CR-generation protocol of block-length \(n\) consists of:

1) A function \(\Phi\) that maps \(X^n\) into a random variable \(K\) with alphabet \(\mathcal{K}\) generated by Terminal \(A\).

2) A function \(\Lambda\) that maps \(X^n\) into the channel input sequence \(T^n \in \mathbb{C}^{nR \times n}\) satisfying the power constraint

\[
\frac{1}{n} \sum_{i=1}^{n} T_{i}^H T_{i} \leq P, \quad \text{almost surely.} \quad (1)
\]

3) A function \(\Psi\) that maps \(Y^n\) and the channel output sequence \(Z^n = (Z_1, \ldots, Z_n) \in \mathbb{C}^{nR \times n}\) into a random variable \(L\) with alphabet \(\mathcal{K}\) generated by Terminal \(B\).

Such a protocol induces a pair of random variables \((K, L)\) whose joint distribution is determined by \(P_{XY}\) and by the channel \(W_{G}\). Such a pair of random variables \((K, L)\) is called permissible. This is illustrated in Fig. 1.

We define first an achievable \(\eta\)-outage CR rate and the \(\eta\)-outage CR capacity for the model presented above. This is an extension of the definition of an achievable CR rate and of the CR capacity over rate-limited discrete noiseless channels introduced in [1].

**Definition 2:** Fix a non-negative constant \(\eta < 1\). A number \(H\) is called an achievable \(\eta\)-outage CR rate if there exists a non-negative constant \(c\) such that for every \(\alpha > 0\) and \(\delta > 0\) and for sufficiently large \(n\) there exists a permissible pair of random variables \((K, L)\) such that

\[
\mathbb{P}[\mathbb{P}[K \neq L|G] \leq \alpha] \geq 1 - \eta, \quad (2)
\]

\[
|K| \leq 2^n c, \quad (3)
\]

\[
\frac{1}{n} H(K) > H - \delta, \quad (4)
\]
where the constant $0 \leq \eta < 1$ and the constant $\alpha > 0$ in (2) correspond to an upper-bound on the outage probability and to an upper-bound on the error probability, from the common randomness generation perspective, respectively, and where the outer probability in (2) is with respect to $G$.

**Remark 1:** Together with (2), the technical condition (3) ensures for every $\epsilon > 0$ and sufficiently large block-length $n$ that $\mathbb{P}\left[ G \in A^{(n, \epsilon)} \right] \geq 1 - \eta$, where

$$ A^{(n, \epsilon)} = \left\{ G \in \mathbb{C}_{N_R \times N_T}^{N_R \times N_T} : \frac{H(K|G=g) - H(L|G=g)}{n} \leq \epsilon \right\}. $$

This follows from the analogous statement in [1].

**Remark 2:** The most convenient form of CR is uniform CR, i.e., $K$ and $L$ are uniform (or nearly uniform) random variables [1]. In Section V-A, we will provide a scheme for generation of nearly uniform random variables that coincide with high probability when the system is not in outage from the CR generation perspective.

**Definition 3:** The $\eta$-outage CR capacity $C_{\eta, CR}^{X,Y}(P, W_G)$ is the maximum achievable $\eta$-outage CR rate defined according to Definition 2.

Next, we define an achievable $\eta$-outage transmission rate for the MIMO slow fading channel $W_G$ and the corresponding $\eta$-outage transmission capacity. For this purpose, we begin by providing the definition of a transmission-code for $W_G$.

**Definition 4:** A transmission-code $\Gamma$ of block-length $n$ and size $|\Gamma|$ and with average power constraint $P$ for the MIMO channel $W_G$ is a family of pairs of codewords and decoding regions

$$ \left\{ (t_\ell, D_\ell^{(g)}) : g \in \mathbb{C}_{N_R \times N_T}^{N_R \times N_T}, \ell = 1, \ldots, |\Gamma| \right\} $$

such that for all $\ell, j \in \{1, \ldots, |\Gamma|\}$ and all $g \in \mathbb{C}_{N_R \times N_T}^{N_R \times N_T}$:

$$ t_\ell \in \mathbb{C}_{N_R \times N_T}, \quad D_\ell^{(g)} \subseteq \mathbb{C}_{N_R \times N_T}, $$

$$ \frac{1}{n} \sum_{i=1}^{n} t_\ell^H t_\ell,i \leq P, \quad t_\ell = (t_{\ell,1}, \ldots, t_{\ell,n}), $$

$$ D_\ell^{(g)} \cap D_j^{(g)} = \emptyset, \quad \ell \neq j. $$

The maximum error probability for gain $g$ is expressed as

$$ e(\Gamma, g) = \max_{\ell \in \{1, \ldots, |\Gamma|\}} W_g(D_\ell^{(g)} | t_\ell). $$

**Remark 3:** Since we do not assume any channel state information at the transmitter side, the codewords $t_\ell$, $\ell = 1, \ldots, |\Gamma|$, do not depend on the gain.

**Remark 4:** Throughout the paper, we consider the maximum error probability criterion. However, due to the converse, the rate and capacity expressions hold also for the average error probability criterion.

**Definition 5:** Let $0 \leq \eta < 1$. A real number $R$ is called an achievable $\eta$-outage transmission rate of the channel $W_G$ if for every $\theta, \delta > 0$ there exists a code sequence $(\Gamma_n)_{n=1}^\infty$, where each code $\Gamma_n$ of block-length $n$ is defined according to Definition 4, such that

$$ \log \| \Gamma_n \| / n \geq R - \delta $$

and

$$ \mathbb{P}[e(\Gamma_n, G) \leq \theta] \geq 1 - \eta $$

for sufficiently large $n$, where the probability in (5) is with respect to $G$.

**Definition 6:** The $\eta$-outage transmission capacity of the channel $W_G$ is the supremum of all achievable $\eta$-outage transmission rates defined according to Definition 5 and it is denoted by $C_\eta(P, W_G)$.

### B. Main Results

**Theorem 1:** Let $Q_P$ be the set of complex positive semidefinite $N_T \times N_T$ matrices whose trace is smaller than or equal to $P$. For any $g \in \mathbb{C}_{N_R \times N_T}^{N_R \times N_T}$ and any $Q \in Q_P$, we define

$$ f(g, Q) = \log \det( I_{N_R} + \frac{1}{g^2} gQg^H ). \quad (6) $$

Let $G \in \mathbb{C}_{N_R \times N_T}^{N_R \times N_T}$ be any random matrix. Then, the $\eta$-outage transmission capacity of the channel $W_G$ satisfies

$$ C_\eta(P, W_G) \geq l(\eta) $$

and

$$ C_\eta(P, W_G) \leq u(\eta), $$

where

$$ l(\eta) = \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P}[f(G, Q) < R] < \eta \right\} $$

and

$$ u(\eta) = \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P}[f(G, Q) < R] \leq \eta \right\}. $$

The lower and upper bound in (7) and (8) coincide possibly at the points of discontinuity of $\eta \rightarrow C_\eta(P, W_G)$, of which there are, at most, countably many. Furthermore, if $G$ has a density, which is positive except on a set with Lebesgue measure equal to zero, then the bounds in (7) and (8) coincide
for all possible values of \( \eta \), including the values of \( \eta \) at which the outage capacity is discontinuous and it holds that

\[
C_\eta(P, W_G) = \sup \left\{ R : \min_{Q \in \mathcal{Q}_P} \mathbb{P} \left[ f(G, Q) < R \right] \leq \eta \right\}.
\] (11)

The proof of the Theorem 1 is provided in Section III.

Remark 5: For MIMO slow Rayleigh fading channels, one can show that the outage capacity formula in (11) coincides the one provided in [22].

Next, we give a single-letter formula of the \( \eta \)-outage transmission capacity, for the SISO and the SIMO case, respectively. It is here worth mentioning that for both cases, the outage transmission capacity is exactly known at points of discontinuity.

Theorem 2: If \( N_T = 1 \), then the \( \eta \)-outage transmission capacity of the SIMO slow fading channel \( W_G \) is equal to

\[
C_\eta(P, W_G) = \log \left( 1 + \frac{P \gamma_0}{\sigma^2} \right),
\] (12)

where

\[
\gamma_0 = \sup\{ \gamma : \mathbb{P} \left[ |G|^2 < \gamma \right] \leq \eta \}
\] (13)

is the generalized inverse of \( \gamma \rightarrow \mathbb{P} \left[ |G|^2 < \gamma \right] \).

The proof of Theorem 2 is provided in Section IV.

Remark 6: If the cumulative distribution function of \( |G|^2 \) is continuous and strictly increasing, then the generalized inverse in (13) coincides with the normal inverse and the outage transmission capacity formula in (12) coincides with the one provided in [22].

Note that the function \( W_G \rightarrow C_\eta(P, W_G) \) is in general discontinuous for \( N_T = N_R = 1 \). To see this, consider the following example:

Example 1: We define a probability distribution for \( G \) which we will show to be a point of discontinuity of the function \( W_G \rightarrow C_\eta(P, W_G) \). The distribution we define is discrete. The channel gains that we consider will be from the set \( \{0, 1\} \) such that \( P_G(0) = P_G(1) = \frac{1}{2} \). If we set \( \eta = \frac{1}{2} \), then we have

\[
\gamma_0 = \sup\{ \gamma : \mathbb{P} \left[ |G|^2 < \gamma \right] = \frac{1}{2} \},
\]

because

\[
\mathbb{P} \left[ |G|^2 \leq 1 \right] = \mathbb{P} \left[ |G|^2 = 0 \right] + \mathbb{P} \left[ |G|^2 = 1 \right] = \frac{1}{2},
\]

for any \( 0 < \delta < 1 \),

\[
\mathbb{P} \left[ |G|^2 < \delta \right] = \mathbb{P} \left[ |G|^2 = 0 \right] = \frac{1}{2}.
\]

Now, define for any \( \epsilon > 0 \) a random variable \( G_\epsilon \) with values in \( \{0, 1\} \) such that \( P_{G_\epsilon}(0) = \frac{1}{2} + \epsilon \) and \( P_{G_\epsilon}(1) = \frac{1}{2} - \epsilon \). Then again, with \( \eta = \frac{1}{2} \),

\[
\gamma_\epsilon = \sup\{ \gamma : \mathbb{P} \left[ |G_\epsilon|^2 < \gamma \right] \leq \eta \} = 0,
\]

because \( \mathbb{P} \left[ |G_\epsilon|^2 < 0 \right] = 0 \) and for any \( 0 < \delta < 1 \),

\[
\mathbb{P} \left[ |G_\epsilon|^2 < \delta \right] = \mathbb{P} \left[ |G_\epsilon|^2 = 0 \right] = \frac{1}{2} + \epsilon.
\]

Obviously, \( P_{G_\epsilon} \searrow P_G \) in total variation distance as \( \epsilon \to 0 \). However, \( \gamma_\epsilon \) does not tend to \( \gamma_0 \) as \( \epsilon \to 0 \), so we have discontinuity here and this carries over the outage capacity function. Note that one can use the same argument to prove the discontinuity of \( W_G \rightarrow C_\eta(P, W_G) \), for each gain \( G \) for which the function \( \gamma \rightarrow \mathbb{P} \left[ |G|^2 < \gamma \right] \) is not strictly increasing.

Theorem 3: For the model described in Section II-A, the \( \eta \)-outage CR capacity satisfies

\[
C_{\eta,CR}^X(Y, W_G) \geq \max_{I(U;X) - I(U;Y) \leq l(\eta)} I(U;X)
\] (14)

and

\[
C_{\eta,CR}^X(Y, W_G) \leq \max_{I(U;X) - I(U;Y) \leq u(\eta)} I(U;X),
\] (15)

where \( l(\eta) \) and \( u(\eta) \) are defined in (9) and (10), respectively. The lower and upper bound in (14) and (15) coincide except at the points where \( l(\eta) \) and \( u(\eta) \) do not coincide, of which there are, at most, countably many.

The proof of Theorem 3 is provided in Section V.

C. Auxiliary Result

For the proof of the lower bound in Theorem 1, we require the following result on the capacity of a compound MIMO complex Gaussian channel corresponding to the set of MIMO complex Gaussian channels with a fixed noise covariance matrix equal to \( \sigma^2 I_{N_R} \) and with \( N_R \times N_T \) channel matrix, which is element of an arbitrary compact set \( \mathcal{G} \subset \mathbb{C}^{N_R \times N_T} \).

We define the compound channel as follows

\[
\mathcal{C} = \{ W_g : g \in \mathcal{G} \}.
\]

We define next a transmission code, an achievable transmission rate and the transmission capacity for the compound channel \( \mathcal{C} \).

Definition 7: A transmission-code \( \Gamma \) of block-length \( n \) and size\(^2 \| \Gamma \| \) and with average power constraint \( P \) for the compound channel \( \mathcal{C} \) is a family of pairs of codewords and decoding regions \( \{(t_\ell, D_\ell) : , \ell = 1, \ldots, \| \Gamma \| \} \) such that for all \( \ell, j \in \{1, \ldots, \| \Gamma \| \} \),

\[
t_\ell \in \mathbb{C}^{N_T \times n}, \quad D_\ell \subset \mathbb{C}^{N_R \times n}, \quad \frac{1}{n} \sum_{i=1}^n t_{\ell,i}^H t_{\ell,i} \leq P, \quad t_\ell = (t_{\ell,1}, \ldots, t_{\ell,n}),
\]

\[
D_\ell \cap D_j = \emptyset, \quad \ell \neq j.
\]

\(^2\)This is the same notation used in [27].
For any \( g \in \mathbb{C}^{N_R \times N_T} \), the maximum error probability for gain \( g \) is expressed as

\[
e_\epsilon(\Gamma, g) = \max_{\ell \in \{1, \ldots, ||\Gamma||\}} W_\epsilon(D_{\ell} | t_\ell).
\]

**Definition 8:** A real number \( R \) is called an achievable rate for the compound channel \( C = \{ W_\epsilon : g \in G \} \) if for every \( \theta, \delta > 0 \) and all \( g \in G \) there exists a code sequence \( (\Gamma_n)_{n=1}^\infty \), where each code \( \Gamma_n \) of block-length \( n \) is defined according to Definition 7, such that for sufficiently large \( n \),

\[
\frac{\log ||\Gamma_n||}{n} \geq R - \delta
\]

and for all \( g \in G \)

\[
e_\epsilon(\Gamma_n, g) \leq \theta,
\]

where \( e_\epsilon(\Gamma_n, g) \) is defined in Definition 7.

**Definition 9:** The compound capacity of \( C \) is the supremum of all achievable rates for \( C \) defined according to Definition 8.

**Theorem 4:** The compound capacity of \( C \) is equal to

\[
\max_{Q \in Q_P} \min_{g \in G} \log \det(I_{N_R} + \frac{1}{\sigma^2} gQg^H).
\]

The proof of Theorem 4 is provided in Section VI.

### III. PROOF OF THEOREM 1

**A. Proof of the Lower Bound on the Outage Transmission Capacity**

Under the assumption of the validity of Theorem 4, which will be proved in Section VI, we will show that

\[
C_\theta(P, W_\epsilon) \geq l(\eta) - \mu \epsilon,
\]

for some \( 1 \leq \mu \leq 2 \), where \( \epsilon \) is an arbitrarily small positive constant and where

\[
l(\eta) = \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P}[f(G, Q) < R] < \eta \right\}.
\]

Clearly, from the definition of \( l(\eta) \), it holds that

\[
P_{\inf}(\epsilon) = \inf_{Q \in Q_P} \mathbb{P}[f(G, Q) < l(\eta) - \epsilon] < \eta.
\]

We fix \( \alpha_1 > 0 \) to be sufficiently small such that \( P_{\inf}(\epsilon) + \alpha_1 \leq \eta \). We will show in the following lemma that for sufficiently large \( n \), we can choose a non-singular \( Q \in Q_P \) such that for some \( 1 \leq \mu \leq 2 \),

\[
\mathbb{P}[f(G, \hat{Q}) < l(\eta) - \mu \epsilon] \leq \eta.
\]

**Lemma 1:** For sufficiently large \( n \), there exists a non-singular \( Q \in Q_P \) satisfying for some \( 1 \leq \mu \leq 2 \)

\[
\mathbb{P}[f(G, \hat{Q}) < l(\eta) - \mu \epsilon] \leq \eta.
\]

**Proof:** Notice first that from the definition of \( P_{\inf}(\epsilon) \), there exists a \( Q \in Q_P \) such that

\[
\mathbb{P}[f(G, Q) < l(\eta) - \epsilon] \leq P_{\inf}(\epsilon) + \frac{\alpha_1}{2}.
\]

Notice also that there exist at most countably many discontinuity points in the distribution function of \( f(G, Q) \). Therefore, one can always find a \( 1 \leq \mu \leq 2 \) such that either \( l(\eta) - \mu \epsilon \) is not element of the support of \( f(G, Q) \) or \( l(\eta) - \mu \epsilon \) is element of the support of \( f(G, Q) \) and the distribution function of \( f(G, Q) \) is continuous at \( l(\eta) - \mu \epsilon \). In both cases, it holds that

\[
\mathbb{P}[f(G, \hat{Q}) = l(\eta) - \mu \epsilon] = 0.
\]

Then, we have

\[
\mathbb{P}[f(G, \hat{Q}) < l(\eta) - \mu \epsilon] \leq \mathbb{P}[f(G, \hat{Q}) < l(\eta) - \epsilon] \leq P_{\inf}(\epsilon) + \frac{\alpha_1}{2}.
\]

It is known that for all \( g \in \mathbb{C}^{N_R \times N_T} \) there exists a sequence of non-singular \( (Q_n)_{n=1}^\infty \), with each \( Q_n \in Q_P \), converging in the operator norm with respect to any norm \( (L^1, \text{the Euclidean norm}, L^\infty, \ldots) \) to \( Q \), regardless of whether \( Q \) is singular or not. It follows from (16) that \( (I_{1 \leq i \leq N_R \times N_T} : f(g, Q_n) < l(\eta) - \mu \epsilon) \) converges to \( I_{f(g, \hat{Q}) < l(\eta) - \mu \epsilon} \) almost surely, where \( I_{\{\cdot\}} \) is the indicator function. Therefore, it follows using the Lebesgue’s dominated convergence theorem that for sufficiently large \( n \),

\[
\mathbb{P}[f(G, Q_n) < l(\eta) - \mu \epsilon]
\]

\[
= \int f(G, Q_n) < l(\eta) - \mu \epsilon) d\mathbb{P}
\]

\[
\leq \int f(G, Q) < l(\eta) - \mu \epsilon) d\mathbb{P} + \frac{\alpha_1}{2}
\]

\[
= \mathbb{P}[f(G, \hat{Q}) < l(\eta) - \mu \epsilon] + \frac{\alpha_1}{2}
\]

\[
\leq P_{\inf}(\epsilon) + \frac{\alpha_1}{2}
\]

\[
\leq \eta.
\]

Therefore, one can find for sufficiently large \( n \) a non-singular \( Q \in Q_P \) such that

\[
\mathbb{P}[f(G, \hat{Q}) < l(\eta) - \mu \epsilon] \leq \eta.
\]

**Lemma 2:** It holds that

\[
\lim_{a \to \infty} \min_{g \in Q} f(g, Q) = \infty.
\]

**Proof:** Consider any \( g \in \mathbb{C}^{N_R \times N_T} \) satisfying \( ||g|| = a \). Let \( \lambda_i(g^Hg) \) and \( \lambda_i(Q) \), \( i = 1, \ldots, N_T \) be the eigenvalues of \( g^Hg \) and \( Q \), respectively, in decreasing order. Furthermore, we denote the eigenvalues of \( gQg^H \) by \( \lambda_i(gQg^H) \), \( i = 1, \ldots, N_R \).

We have

\[
f(g, Q) = \log \det \left( I_{N_R} + \frac{1}{\sigma^2} gQ g^H \right)
\]

\[
= \log \prod_{i=1}^{N_R} \left( 1 + \frac{1}{\sigma^2} \lambda_i(gQg^H) \right)
\]
holds because $\mathbf{g}^H \mathbf{Q} \mathbf{g}^H$ is a positive semi-definite Hermitian matrix, (b) follows because the trace is invariant under cyclic permutations, (c) follows from [28, Theorem H.1.1.] by noticing that $\mathbf{g}^H \mathbf{g}$ is a $N_T \times N_T$ positive semi-definite Hermitian matrix and that $\mathbf{Q}$ is a $N_T \times N_T$ positive definite Hermitian matrix and (d) follows because $\lambda_1(\mathbf{g}^H \mathbf{g}) = \lambda_{\text{max}}(\mathbf{g}^H \mathbf{g}) = \|\mathbf{g}\|^2$. As a result, we have

$$\log \left( 1 + \frac{\sigma^2}{\lambda_{\text{max}}(\mathbf{Q})} \right) \leq \min_{\|\mathbf{g}\|=a} f(\mathbf{g}, \mathbf{\hat{Q}}).$$

Now notice that $\lim_{a \to \infty} \log \left( 1 + \frac{\sigma^2}{\lambda_{\text{max}}(\mathbf{Q})} \right) = \infty$. This holds because $\lambda_{\text{max}}(\mathbf{Q}) = \lambda_{\text{min}}(\mathbf{Q}) > 0$ since $\mathbf{Q}$ is non-singular. It follows that

$$\lim_{a \to \infty} \min_{\|\mathbf{g}\|=a} f(\mathbf{g}, \mathbf{\hat{Q}}) = \infty. \quad \square$$

Now that we proved the lemma, consider the sets

$$\mathcal{B}_a = \{ \mathbf{g} \in \mathbb{C}^{N_R \times N_T} : \|\mathbf{g}\| \leq a \}$$

and

$$\mathcal{\hat{G}}_a = \{ \mathbf{g} \in \mathbb{C}^{N_R \times N_T} : f(\mathbf{g}, \mathbf{\hat{Q}}) \geq l(\eta) - \mu \epsilon \}$$

for some $a > 0$ chosen sufficiently large such that

$$\left\{ \mathbf{g} \in \mathbb{C}^{N_R \times N_T} : \|\mathbf{g}\| = a \right\} \subseteq \mathcal{\hat{G}}_a. \quad (18)$$

From (17), we know the existence of an $a > 0$ satisfying (18).

Now, notice that $\mathcal{\hat{G}}_a$ is a compact set. By applying Theorem 4, it follows that the compound capacity of $\mathcal{C} = \{ W : \mathbf{g} \in \mathcal{\hat{G}}_a \}$ is equal to

$$\max_{\mathbf{Q} \in \mathcal{Q}_{\mathbf{g} \in \mathcal{\hat{G}}_a}} \min_{\mathbf{g} \in \mathcal{\hat{G}}_a} f(\mathbf{g}, \mathbf{Q}).$$

Since $\mathbf{\hat{Q}} \in \mathcal{Q}_P$, it follows that

$$\min_{\mathbf{g} \in \mathcal{\hat{G}}_a} f(\mathbf{g}, \mathbf{\hat{Q}})$$

is an achievable rate for $\mathcal{\hat{C}}$.

Let $\theta, \delta > 0$. Since $l(\eta) - \mu \epsilon \leq \min_{\mathbf{g} \in \mathcal{\hat{G}}_a} f(\mathbf{g}, \mathbf{\hat{Q}})$, there exists a code sequence $(\mathbf{\Gamma}_{\mathbf{\hat{a}}, n})_{n=1}^\infty$ and a block-length $n_0$ such that

$$\frac{\log \|\mathbf{\Gamma}_{\mathbf{\hat{a}}, n}\|}{n} \geq l(\eta) - \mu \epsilon - \delta$$

and such that

$$\mathbf{g} \in \mathcal{\hat{G}}_a \implies e_a(\mathbf{\Gamma}_{\mathbf{\hat{a}}, n}, \mathbf{g}) \leq \theta$$

for $n \geq n_0$.

Next, we will prove the following lemma:

Lemma 3: For $n \geq n_0$

$$\mathbf{g} \in \mathcal{B}_a \implies e(\mathbf{g}) \leq \theta,$$

where $\mathbf{g}$ is some code with block-length $n$, and with the same size and the same encoder as $\mathcal{\hat{G}}_a$.

Proof: Suppose first that for a Gaussian channel $W_{\mathbf{g}_1}$, a code $\mathbf{\Gamma}_1$ satisfies $e(\mathbf{\Gamma}_1) \leq \theta$. Then, it can be shown that there exists a code $\mathbf{\Gamma}_2$ for a Gaussian channel $W_{\mathbf{g}_2}$, where $W_{\mathbf{g}_2}$ is a degraded version of $W_{\mathbf{g}_2}$, such that $e(\mathbf{\Gamma}_2) \leq \theta$. The code $\mathbf{\Gamma}_2$ has the same encoder as $\mathbf{\Gamma}_1$ but has possibly a different decoder. The analogous statement for DMCs is a special case of the statement provided in [29].

Now, let $\mathbf{g}$ with $\|\mathbf{g}\| > a$ be fixed arbitrarily. We recall that $\mathbf{g}$ satisfies (18). Then, the channel $W_{\mathbf{g}}$ with $\mathbf{g}^* = \frac{\mathbf{g}}{\|\mathbf{g}\|} \mathbf{g} \in \mathcal{\hat{G}}_a$ is a degraded version of the channel $W_{\mathbf{g}}$. It follows that there exists a code sequence $(\mathbf{\Gamma}_{\mathbf{\hat{g}}, n})_{n=1}^\infty$ for $W_{\mathbf{g}}$ such that each code $\mathbf{\Gamma}_{\mathbf{\hat{g}}, n}$ of block-length $n$ has the same encoder and the same size as the code $\mathbf{\Gamma}_{\mathbf{\hat{g}}, n}$ of block-length $n$ but a different decoder adjusted to $\mathbf{g}$ and such that for $n \geq n_0$, $e(\mathbf{\Gamma}_{\mathbf{\hat{g}}, n}, \mathbf{g}) \leq \theta$.

Here, we require channel state information at the receiver side (CSIR) so that the decoder can adjust its decoding strategy according to the channel state.

So far, we have proved the existence of a block-length $n_0$ and of a code sequence $(\mathbf{\Gamma}_n)_{n=1}^\infty$, where each code $\mathbf{\Gamma}_n$ of block-length $n$ has the same size and the same encoder as the code $\mathbf{\Gamma}_{\mathbf{\hat{g}}, n}$ of block-length $n$ and a decoder adjusted to the actual gain $\mathbf{g}$, such that

$$\frac{\log \|\mathbf{\Gamma}_n\|}{n} \geq l(\eta) - \mu \epsilon - \delta$$

and such that

$$\mathbf{g} \in \mathcal{\hat{G}}_a \cup \mathcal{B}_a \implies e(\mathbf{\Gamma}_n, \mathbf{g}) \leq \theta$$

for $n \geq n_0$.

Now, we have for $n \geq n_0$

$$\mathbb{P}[e(\mathbf{\Gamma}_n, \mathbf{g}) \leq \theta] \geq \mathbb{P}[\mathbf{g} \in \mathcal{\hat{G}}_a \cup \mathcal{B}_a] \geq \mathbb{P}[\mathbf{g} \in \mathcal{\hat{G}}_a] + \mathbb{P}[\mathbf{g} \in \mathcal{B}_a] \geq 1 - \eta,$$

where (a) follows from the choice of the constant $\alpha$. This completes the proof of the lower-bound on the $\eta$-outage transmission capacity.

### B. Proof of the Upper Bound on the Outage Transmission Capacity

We will show that

$$C_\eta(P, W_{\mathbf{\Gamma}}) \leq u(\eta), \quad (19)$$
where
\[ u(\eta) = \sup \left\{ R : \inf_{Q \in \mathcal{Q}_P} \mathbb{P}[f(G, Q) < R] \leq \eta \right\}. \]

The weak converse for compound channels does not guarantee that the error probability cannot be made arbitrarily small for all possible states when the target rate exceeds the compound capacity. Therefore, we cannot use the weak converse theorem of compound channels to prove the upper bound in (19). We will proceed differently. Suppose (19) were not true. Then there exists an \( \epsilon > 0 \) such that \( u(\eta) + \epsilon \) is an achievable \( \eta \)-outage transmission rate for \( W_G \). The goal is to find a contradiction. Choose \( \theta > 0 \) so small that
\[ \frac{(1 - \theta)\epsilon}{2} - \theta u(\eta) > \epsilon/4. \]

Due to the achievability of \( u(\eta) + \epsilon \), there exists a code sequence \( (\Gamma_n)_{n=1}^\infty \) such that
\[ \frac{\log \|\Gamma_n\|}{n} \geq u(\eta) + \frac{\epsilon}{2} \quad (20) \]
and
\[ \mathbb{P}[\epsilon(\Gamma_n, G) > \theta] \leq \eta \quad (21) \]
for sufficiently large \( n \). Choose an \( n \) for which the above holds and which satisfies
\[ \frac{1}{n} \leq \frac{\epsilon}{8}. \quad (22) \]

The uniformly-distributed message \( W \) is mapped to the random input sequence \( T^n = (T_1, \ldots, T_n) \) of \( W_G \). We fix the covariance matrices \( Q_1, \ldots, Q_n \) of the random inputs \( T_1, \ldots, T_n \), respectively and let \( Q^* = \frac{1}{n} \sum_{i=1}^n Q_i \). Furthermore, we let
\[ \epsilon' = \frac{\epsilon}{8}. \]

We consider the following set:
\[ \mathcal{G}_\theta = \{ g \in \mathbb{C}^{Nn \times N_T} : f(g, Q^*) < u(\eta) + \epsilon' \text{ and } \epsilon(\Gamma_n, g) \leq \theta \}. \]

To complete the proof of the upper-bound in (19) by contradiction, the next step is to show that the set \( \mathcal{G}_\theta \) is non-empty. For this purpose, we will prove that \( \mathbb{P}[f(G, Q^*) < u(\eta) + \epsilon'] > \eta \) in what follows:

**Lemma 4:**
\[ \mathbb{P}[f(G, Q^*) < u(\eta) + \epsilon'] > \eta. \]

**Proof:** By Lemma 5 below, we know that \( \text{tr}(Q^*) \leq P \) and therefore \( Q^* \in \mathcal{Q}_P \). By Lemma 6 below, it follows that
\[ R(Q^*) = \sup \left\{ R : \mathbb{P}[f(G, Q^*) < R] \leq \eta \right\} \leq u(\eta). \]

This yields
\[ \mathbb{P}[f(G, Q^*) < u(\eta) + \epsilon'] \geq \mathbb{P}[f(G, Q^*) < R(Q^*) + \epsilon'] > \eta. \]

**Lemma 5:**
\[ \text{tr}(Q^*) \leq P. \]

**Proof:** From (1), it holds that
\[ \frac{1}{n} \sum_{i=1}^n T_i^H T_i \leq P, \quad \text{almost surely.} \]

This implies that
\[ \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n T_i^H T_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E} [T_i^H T_i] \leq P. \]

This yields
\[ \text{tr} \left[ Q^* \right] = \text{tr} \left[ \frac{1}{n} \sum_{i=1}^n Q_i \right] \leq \frac{1}{n} \sum_{i=1}^n \text{tr} [Q_i] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ T_i^H T_i \right] \leq P, \]
where we used \( r = \text{tr}(r) \) for scalar \( r \), \( \text{tr}(AB) = \text{tr}(BA) \) and the linearity of the expectation and of the trace operators. □

**Lemma 6:** For any \( Q \in \mathcal{Q}_P \), it holds that
\[ \sup \left\{ R : \mathbb{P}[f(G, Q) < R] \leq \eta \right\} \leq \sup \left\{ R : \inf_{Q \in \mathcal{Q}_P} \mathbb{P}[f(G, Q) < R] \leq \eta \right\}. \]

**Proof:** For any \( Q \in \mathcal{Q}_P \), we have
\[ \left\{ R : \mathbb{P}[f(G, Q) < R] \leq \eta \right\} \subseteq \left\{ R : \inf_{Q \in \mathcal{Q}_P} \mathbb{P}[f(G, Q) < R] \leq \eta \right\}. \]

As a result:
\[ \sup \left\{ R : \mathbb{P}[f(G, Q) < R] \leq \eta \right\} \leq \sup \left\{ R : \inf_{Q \in \mathcal{Q}_P} \mathbb{P}[f(G, Q) < R] \leq \eta \right\}. \]

Now, we can prove that the set \( \mathcal{G}_\theta \) is non-empty in what follows:

**Lemma 7:** \( \mathcal{G}_\theta \) is a non-empty set.

**Proof:** By Lemma 4, we have
\[ \eta < \mathbb{P}[f(G, Q^*) < u(\eta) + \epsilon'] \]

□
\[
\begin{align*}
&= \mathbb{P} [f(G, Q^*) < u(\eta) + \epsilon'] e(\Gamma_n, G) \leq \theta] \\
&+ \mathbb{P} [e(\Gamma_n, G) > \theta] \\
\leq \mathbb{P} [f(G, Q^*) < u(\eta) + \epsilon'] e(\Gamma_n, G) \leq \theta] \\
&+ \mathbb{P} [e(\Gamma_n, G) > \theta] \\
\leq \mathbb{P} [f(G, Q^*) < u(\eta) + \epsilon'] e(\Gamma_n, G) \leq \theta] + \eta,
\end{align*}
\]

where we used (21) in the last step. This implies that
\[
\mathbb{P} [f(G, Q^*) < u(\eta) + \epsilon'] e(\Gamma_n, G) \leq \theta] > 0.
\]

Furthermore, since \( \eta < 1 \), it follows that
\[
\mathbb{P} [e(\Gamma_n, G) \leq \theta] \geq 1 - \eta > 0.
\]

As a result, we have
\[
\mathbb{P} [f(G, Q^*) < u(\eta) + \epsilon', e(\Gamma_n, G) \leq \theta] > 0,
\]

which means that
\[
\mathbb{P} [G \in G_\theta] > 0
\]

and therefore \( G_\theta \) is a non-empty set. \( \square \)

Pick a \( g \in G_\theta \) and consider the channel
\[
z_i = gt_i + \xi_i, \quad i = 1, \ldots, n. \tag{23}
\]

The uniformly-distributed message \( W \) is mapped to the random input sequence \( T^n = (T_1, \ldots, T_n) \) of the channel in (23). We model the random output sequence of the channel in (23) by \( Z^n = (Z_1, \ldots, Z_n) \). We model the random decoded message by \( \hat{W} \). The set of messages is denoted by \( \mathcal{W} \). We use \( \Gamma_n \) as a transmission-code for the channel in (23) with the fixed block-length \( n \) satisfying (22). Since \( g \in G_\theta \), it follows that
\[
\mathbb{P} [W \neq \hat{W}] \leq e(\Gamma_n, g) \leq \theta.
\]

We have
\[
H(W) = \log |\mathcal{W}| = \log \|\Gamma_n\| \geq n \left( u(\eta) + \frac{\epsilon}{2} \right), \tag{24}
\]

where we used (20) in the last step. By applying Fano’s inequality, we obtain
\[
H(W|\hat{W}) \leq 1 + \mathbb{P} [W \neq \hat{W}] \log |\mathcal{W}|
\leq 1 + \theta \log |\mathcal{W}|
= 1 + \theta H(W).
\]

Now, on the one hand, it holds that
\[
I(W; \hat{W}) = H(W) - H(W|\hat{W}) \geq (1 - \theta) H(W) - 1,
\]

which yields
\[
H(W) \leq \frac{1 + I(W; \hat{W})}{1 - \theta}. \tag{25}
\]

On the other hand, we have
\[
\frac{1}{n} I(W; \hat{W}) \leq \frac{1}{n} I(T^n; Z^n) \leq \frac{1}{n} \sum_{i=1}^n h(Z_i|T^{i-1}) = \frac{1}{n} \sum_{i=1}^n h(Z_i|T^n, Z^{i-1}) = \frac{1}{n} \sum_{i=1}^n h(Z_i) - h(Z_i|T_i)
\]

\[
\leq \frac{1}{n} \sum_{i=1}^n I(T_i, Z_i)
\leq \frac{1}{n} \log \det \left( \frac{1}{\sigma^2} gQ \right)
\leq \log \det \left( \frac{1}{\sigma^2} gQ \right) + \frac{1}{n} \log \det \left( \frac{1}{\sigma^2} gQ \right) = \log \det \left( \frac{1}{\sigma^2} gQ \right), \tag{26}
\]

where (a) follows from the Data Processing Inequality because \( W o T^n o Z^n o W \) forms a Markov chain, (b) follows from the chain rule of mutual information, (c) follows because \( T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n, Z^{i-1} o T_i o Z_i \) forms a Markov chain, (d) follows because conditioning does not increase entropy and (e) follows because \( \log \det \) is concave on the set of Hermitian positive semi-definite matrices. This yields
\[
H(W) \leq \frac{1 + n \log \det(I_{N^2} + \frac{1}{\sigma^2} gQ^* g)}{1 - \theta}. \tag{27}
\]

The inequalities (24) and (27) imply that
\[
n \left( u(\eta) + \frac{\epsilon}{2} \right) \leq \frac{1 + n \log \det(I_{N^2} + \frac{1}{\sigma^2} gQ^* g)}{1 - \theta} < \frac{1 + n(u(\eta) + \epsilon')}{1 - \theta}, \tag{28}
\]

where we used that \( g \in G_\theta \). The inequality (28) is equivalent to
\[
-\theta u(\eta) + (1 - \theta) \frac{\epsilon}{2} - \frac{1}{n} < \epsilon'.
\]

However, by the choice of \( \theta \) and \( n \), the left-hand side of this inequality is strictly larger than \( \frac{\epsilon}{2} \), whereas \( \epsilon' = \frac{\epsilon}{2} \). This is a contradiction. Thus (19) must be true. This completes the proof of the upper-bound on the \( \eta \)-outage transmission capacity.

**C. Equality of the Bounds at the Points of Continuity of \( \eta \mapsto C_\eta(P, W_G) \)**

We will show that the bounds in (7) and in (8) are tight except at the points of discontinuity of \( \eta \mapsto C_\eta(P, W_G) \).
Notice first that \( u : \eta \mapsto \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R] \leq \eta \right\} \) is non-decreasing. Therefore, the set \( \mathcal{D} \subset [0,1) \) of \( \eta \), at which it is discontinuous, is at most countable. We will prove next the following lemma.

**Lemma 8:** The function

\[
g_{\text{inf}} : R \mapsto \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R] \tag{29}
\]

is non-decreasing.

**Proof:** Let \( 0 \leq R_1 \leq R_2 \). For any \( Q \in Q_P \), it holds that

\[
\inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R_1] \leq \mathbb{P} [f(G, Q) < R_1]. \tag{30}
\]

Clearly, the function \( s_Q : R \mapsto \mathbb{P} [f(G, Q) < R] \) is non-decreasing for any \( Q \in Q_P \). Therefore, it follows that for any \( Q \in Q_P \)

\[
\mathbb{P} [f(G, Q) < R_1] \leq \mathbb{P} [f(G, Q) < R_2]. \tag{31}
\]

It follows from (30) and (31) that for all \( Q \in Q_P \)

\[
\inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R_1] \leq \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R_2].
\]

This yields

\[
\inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R_1] \leq \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R_2].
\]

This proves that

\[
g_{\text{inf}}(R_1) \leq g_{\text{inf}}(R_2).
\]

We deduce that the function in (29) is non-decreasing.

Select now any \( \eta^* \in [0,1) \setminus \mathcal{D} \) and a strictly increasing sequence \( (\eta^{(n)})_{n=1}^{\infty} \) in \([0,1)\) converging to \( \eta^* \). One can show analogously to the proof of Lemma 10 below and using Lemma 8 that

\[
\sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R] < \eta^* \right\} = \lim_{n \to \infty} \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R] \leq \eta^{(n)} \right\}.
\]

It follows that

\[
l(\eta^*) = \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R] < \eta^* \right\} = \lim_{n \to \infty} \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R] \leq \eta^{(n)} \right\} = \lim_{n \to \infty} u(\eta^{(n)}) \tag{a}
\]

where (a) follows because

\[
u : \eta \mapsto \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R] \leq \eta \right\}
\]
is continuous non-decreasing at \( \eta = \eta^* \). So far, we know that \( u \) has at most countably many points of discontinuity and that

\[
l : \eta \mapsto \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R] < \eta \right\}
\]

and \( u : \eta \mapsto \sup \left\{ R : \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R] \leq \eta \right\} \) coincide in points of continuity of \( u \), and in particular, they are equal to \( \eta \mapsto C_{\eta}(P, W_G) \) in these points.

Now assume that \( l(\eta_0) \neq u(\eta_0) \) in some point \( \eta_0 \). We are going to show that \( \eta \mapsto C_{\eta}(P, W_G) \) is not continuous at \( \eta_0 \). By assumption, \( l(\eta_0) < u(\eta_0) \). Let \( (\eta^{(n)}_n) \) be a sequence of points of continuity of \( u \) converging to \( \eta_0 \) from above, and let \( (\eta^{-n}_n) \) be a sequence of points of continuity of \( u \) converging to \( \eta_0 \) from below. Then by Lemma 1,

\[
l(\eta^{(n)}_n) = C^{(n)}_{\eta^-}(P, W_G), \quad u(\eta^{(n)}_n) = C^{(n)}_{\eta^+}(P, W_G)
\]

for all \( n \). In particular,

\[
\lim_{n \to \infty} C^{(n)}_{\eta^-}(P, W_G) = \lim_{n \to \infty} C^{(n)}_{\eta^+}(P, W_G) = \lim_{n \to \infty} l(\eta^{(n)}_n) = l(\eta_0).
\]

Hence, \( \eta \mapsto C_{\eta}(P, W_G) \) is not continuous at \( \eta = \eta_0 \).

**D. Equality of the Bounds in (9) and (10) When \( G \) Has a Positive Density Except on a Set With Lebesgue Measure Equal to Zero**

Let us first introduce and prove the following lemma.

**Lemma 9:** When \( G \) has a positive density except on a set with Lebesgue measure equal to zero, the function

\[
g_{\text{inf}} : R \mapsto \inf_{Q \in Q_P} \mathbb{P} [f(G, Q) < R]. \tag{32}
\]

is strictly increasing in \([0, \infty)\).

**Proof:** We introduce and prove first the following claims:

**Claim 1:** For any \( R > 0 \) and any \( Q \in Q_P \setminus \{0_N\} \), consider \( f_Q : g \mapsto f(g, Q) \). Let \( g_0 \in \mathbb{C}^{N_R \times N_T} \), such that \( f_Q(g_0) = R \). Then, it holds that

\[
\nabla_g f_Q(g)|_{g=g_0} \neq 0_{N_R \times N_T}.
\]

**Proof of Claim 1:** Let \( g_0 \in \mathbb{C}^{N_R \times N_T} \), such that \( f_Q(g_0) = R \). Now, it holds that

\[
0 < f_Q(g_0) = \log \det \left( I_{N_R} + \frac{1}{\sigma^2} g_0 Q g_0^H \right) \leq \frac{1}{\ln(2)} \text{tr} \left( \frac{1}{\sigma^2} g_0 Q g_0^H \right),
\]

where we used that \( \log(x) = \frac{\ln(x)}{\ln(2)} \) and that for any Hermitian positive definite matrix \( A \in \mathbb{C}^{N_R \times N_R} \), \( \ln \det(A) \leq \text{tr} \left( A - I_{N_R} \right) \). Therefore, it follows that

\[
\text{tr} \left( \frac{1}{\sigma^2} g_0 Q g_0^H \right) > 0. \tag{33}
\]

We consider now the gradient of \( f_Q : g \mapsto f(g, Q) \) along a specific direction given by \( g_0 \). For \( A_t = g + t g_0, t \in \mathbb{R} \), we have

\[
\frac{d}{dt} f_Q(A_t) = 2 \text{Re} \left[ \text{tr} \left( \nabla_A, f_Q(A_t) \times \frac{d}{dt} A_t \right) \right], \tag{34}
\]

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where $\nabla_{A_t} f_Q(A_t) \times \frac{d}{dt} A_t$ refers to the product of the matrix $\nabla_{A_t} f_Q(A_t)$ and the matrix $\frac{d}{dt} A_t$. On the one hand, we have

$$f_Q(A_t) = \log \det \left( I_{N_R} + \frac{1}{\sigma^2} A_t Q A_t^H \right),$$

which implies using [30, eq. (82)] that

$$\nabla_{A_t} f_Q(A_t) = \frac{1}{\sigma^2} Q A_t^H \left( I_{N_R} + \frac{1}{\sigma^2} A_t Q A_t^H \right)^{-1}.$$

(35)

On the other hand, it holds that

$$\frac{d}{dt} A_t = g_0.$$  

(36)

(34), (35) and (36) yield

$$\frac{d}{dt} f_Q(A_t) = 2 \Re \left[ \text{tr} \left( \frac{1}{\sigma^2} Q A_t^H \left( I_{N_R} + \frac{1}{\sigma^2} A_t Q A_t^H \right)^{-1} g_0 \right) \right].$$

Thus, we have

$$\left. \frac{d}{dt} f_Q(A_t) \right|_{t=0} = 2 \Re \left[ \text{tr} \left( \frac{1}{\sigma^2} Q g_0^H \left( I_{N_R} + \frac{1}{\sigma^2} g_0 Q g_0^H \right)^{-1} g_0 \right) \right].$$

Now, let $g = g_0$. Then, we have

$$\left. \frac{d}{dt} f_Q(A_t) \right|_{t=0} = 2 \Re \left[ \text{tr} \left( \frac{1}{\sigma^2} Q g_0^H \left( I_{N_R} + \frac{1}{\sigma^2} g_0 Q g_0^H \right)^{-1} g_0 \right) \right].$$

(37)

where (a) follows because the trace is invariant under cyclic permutations, (b) follows because $\frac{1}{\sigma^2} Q g_0^H \left( I_{N_R} + \frac{1}{\sigma^2} g_0 Q g_0^H \right)^{-1} g_0 Q^\perp$ is a Hermitian matrix, (c) follows because the trace is invariant under cyclic permutations, (d) follows from [28, Theorem H.1.1] by noticing that $\frac{1}{\sigma^2} g_0 Q g_0^H$ is a Hermitian positive semi-definite matrix and that $\left( I_{N_R} + \frac{1}{\sigma^2} g_0 Q g_0^H \right)^{-1}$ is a Hermitian positive definite matrix and (e) follows from (33) and from the fact that $\left( I_{N_R} + \frac{1}{\sigma^2} g_0 Q g_0^H \right)^{-1}$ is a Hermitian positive definite matrix. (37) implies that $\nabla_{g_0} f_Q(g)|_{g=g_0} \neq 0_{N_R \times N_T}$. This completes the proof of claim 1.

Claim 2: For any $R > 0$ and any $Q \in Q_P$, we have

$$\mathbb{P} \left[ f(G, Q) = R \right] = 0.$$  

Proof of Claim 2: Let $R > 0$. Clearly, the claim is valid for $Q = 0_{N_T}$. Now, let $Q \in Q_P \setminus \{0_{N_T}\}$ be fixed arbitrarily. It follows from Claim 1 that all $g_0 \in \mathbb{C}^{N_R \times N_T}$ satisfying $f_Q(g_0) = R$ are regular points. It follows from the Regular Level Set Theorem (see, e.g., [31, Corollary 5.14]) that $S = \{g_0 \in \mathbb{C}^{N_R \times N_T} : f_Q(g_0) = R\}$ is a properly embedded sub-manifold with dimension $\dim \mathbb{R}(S) = 2N_R N_T - 1$. Now, we know that $\mathbb{C}^{N_R N_T}$ is a smooth manifold with dimension $\dim \mathbb{R}(S) = 2N_R N_T$ and that

$$S = \{g_0 \in \mathbb{C}^{N_R N_T} : f_Q(g_0) = R\} \subset \mathbb{C}^{N_R N_T}$$

is an immersed sub-manifold satisfying $\dim \mathbb{R}(S) < 2N_R N_T$. It is then well-known (see, e.g., [31, Corollary 6.12]) that $S$ has Lebesgue measure equal to zero. Since $G$ has a density, it holds then that $\mathbb{P} \left[ f(G, Q) = R \right] = 0$. This completes the proof of Claim 2.

Claim 3: The infimum in (32) is a minimum and for any $R > 0$, it holds that $g_{\inf}(R) = \min_{Q \in Q_P \setminus \{0_{N_T}\}} \mathbb{P} \left[ f(G, Q) < R \right]$.

Proof of Claim 3: Clearly, the claim is valid for $R = 0$. Now, let $R > 0$ be fixed arbitrarily. We will show first that the function $s_R : Q \mapsto \mathbb{P} \left[ f(G, Q) < R \right]$ is a continuous function on $Q_P$. Let $(Q_n)_{n=1}^\infty$ with each $Q_n \in Q_P$ such that $\lim_{n \to \infty} Q_n = Q_0$, for some $Q_0 \in Q_P$. Thus, it follows using Claim 2 that $I_{\{f(G, Q_0) < R\}}$ converges to $I_{\{f(G, Q) < R\}}$ almost surely, where $I_{\{\cdot\}}$ refers to the indicator function. Thus, we obtain using the dominated convergence theorem

$$\lim_{n \to \infty} \mathbb{P} \left[ f(G, Q_n) < R \right] = \lim_{n \to \infty} \int_{\{f(G, Q_n) < R\}} p_G(g) dg$$

$$= \int_{\{f(G, Q_0) < R\}} p_G(g) dg$$

$$= \mathbb{P} \left[ f(G, Q_0) < R \right],$$

where $p_G(g)$ is the density of $G$. This shows that the function $s_R$ is continuous on $Q_P$. Therefore, it attains its minimum on the compact set $Q_P$. As a result, the infimum in (32) is actually a minimum. This implies that for any $R > 0$,

$$g_{\inf}(R) = \min_{Q \in Q_P} \mathbb{P} \left[ f(G, Q) < R \right].$$

Now, suppose that $g_{\inf}(R) = \mathbb{P} \left[ f(G, 0_{N_T}) < R \right]$, for any $R > 0$. Then, we have $\min_{Q \in Q_P} \mathbb{P} \left[ f(G, Q) < R \right] = 1$ for any $R > 0$. Consequently, for any $R > 0$ and for any $Q \in Q_P$, it holds that

$$1 = \min_{Q \in Q_P} \mathbb{P} \left[ f(G, Q) < R \right] \leq \mathbb{P} \left[ f(G, Q) < R \right].$$

Thus, for any $R > 0$ and for any $Q \in Q_P$, $\mathbb{P} \left[ f(G, Q) < R \right] = 1$.

This is in contradiction with the fact that $G$ has a positive density except on a set with Lebesgue measure equal to zero. Therefore, for any $R > 0$,

$$g_{\inf}(R) \neq \mathbb{P} \left[ f(G, 0_{N_T}) < R \right].$$

As a result, it follows that for any $R > 0$,

$$g_{\inf}(R) = \min_{Q \in Q_P \setminus \{0_{N_T}\}} \mathbb{P} \left[ f(G, Q) < R \right].$$

This completes the proof of Claim 3.
Claim 4: For any $R \geq 0$ and any $Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}$, the function $s_Q: R \mapsto P\{f(G, Q) < R\}$ is strictly increasing.

Proof of Claim 4: Notice first that for any $R \geq 0$ and any $Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}$, the function $f_Q: g \mapsto f(g, Q)$ is continuous. Let $0 \leq R_1 < R_2$. Consider the open interval $(R_1, R_2)$. Denote the inverse image of $(R_1, R_2)$ under $f_Q$ by $f_Q^{-1}(R_1, R_2)$. It is known that for any $Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}$, there exist some $g_1 \in \mathbb{C}^{N_R \times N_T}$ and $g_2 \in \mathbb{C}^{N_R \times N_T}$ such that $f_Q(g_1) = R_1$ and $f_Q(g_2) = R_2$. It follows from the continuity of $f_Q$ that $f_Q^{-1}(R_1, R_2)$ is a non-empty open set and from the intermediate value theorem that $f_Q^{-1}(R_1, R_2)$ is a non-empty open set.

This yields

$$P\{R_1 \leq f(G, Q) < R_2\} \geq P\{G \in f_Q^{-1}(R_1, R_2)\} > 0,$$

where we used the fact that $G$ has a positive density except on a set with Lebesgue measure equal to zero. Therefore, the function $s_Q: R \mapsto P\{f(G, Q) < R\}$ is strictly increasing for $R \geq 0$ and for any $Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}$ and any $R \geq 0$. This completes the proof of Claim 4.

Now that we proved the claims, we let $0 \leq R_1 < R_2$. For any $Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}$, it holds that

$$\min_{Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}} P\{f(G, Q) < R_1\} \leq P\{f(G, Q) < R_2\}. \quad (38)$$

From Claim 4, we know that the function $s_Q: R \mapsto P\{f(G, Q) < R\}$ is strictly increasing for $R \geq 0$ and for any $Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}$. This implies that for any $Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}$

$$P\{f(G, Q) < R_1\} < P\{f(G, Q) < R_2\}. \quad (39)$$

It follows from (38) and (39) that for all $Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}$

$$\min_{Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}} P\{f(G, Q) < R_1\} < P\{f(G, Q) < R_2\}.$$ 

This implies, using Claim 3,

$$\min_{Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}} P\{f(G, Q) < R_1\} \leq \min_{Q \in \mathbb{Q}_P \setminus \{0_{N_T}\}} P\{f(G, Q) < R_2\}.$$

It follows using Claim 3 that

$$g_{\text{inf}}(R_1) < g_{\text{inf}}(R_2).$$

We deduce that the function in (32) is strictly increasing. This completes the proof of Lemma 9. \hfill \Box

Now that we proved Lemma 9, suppose that $l(\eta) \neq u(\eta)$. Then, for any $0 \leq \eta < R < u(\eta)$, it follows from the strict monotonicity of $g_{\text{inf}}$ that

$$g_{\text{inf}}(l(\eta)) < g_{\text{inf}}(R) < g_{\text{inf}}(u(\eta)),$$

where $g_{\text{inf}}(u(\eta)) \leq \eta$ and since $R > l(\eta)$, it follows that

$$g_{\text{inf}}(R) \geq \eta.$$ 

Therefore, we have $g_{\text{inf}}(R) < g_{\text{inf}}(u(\eta)) \leq g_{\text{inf}}(R)$, which is a contradiction. Therefore, $l(\eta)$ and $u(\eta)$ must be equal.

IV. PROOF OF THEOREM 2

A. Proof of the Outage Transmission Capacity for $N_T = 1$

1) Direct Proof: Under the assumption of the validity of Theorem 4, which will be proved in Section VI, we will show that for $N_T = 1$

$$C_\eta(P, W_G) \geq R_{\eta, \sup},$$

where

$$R_{\eta, \sup} = \sup \left\{ R : P\left[ \log \det\left( I_{N_R} + \frac{P}{\sigma^2} G G^H \right) < R \right] \leq \eta \right\}. \quad (40)$$

We first show that the supremum in (40) is actually a maximum.

Lemma 10:

$$P\left[ \log \det\left( I_{N_R} + \frac{P}{\sigma^2} G G^H \right) < R_{\eta, \sup} \right] \leq \eta$$

so the supremum in (40) is actually a maximum.

Proof: Let $R_n \rightarrow R_{\eta, \sup}$ be a sequence converging to $R_{\eta, \sup}$ from the left. Then

$$\{ R \in \mathbb{R} : R < R_{\eta, \sup} \} = \bigcup_{n=1}^{\infty} \{ R \in \mathbb{R} : R < R_n \}.$$

From the sigma-continuity of probability measures, it follows that

$$P\left[ \log \det\left( I_{N_R} + \frac{P}{\sigma^2} G G^H \right) < R_{n, \sup} \right] \rightarrow \lim_{n \rightarrow \infty} P\left[ \log \det\left( I_{N_R} + \frac{P}{\sigma^2} G G^H \right) < R_n \right] \leq \eta.$$

Now, consider the set

$$\tilde{G}_a = \{ g \in \mathbb{C}^{N_R \times 1} : R_{\eta, \sup} \leq \log \det\left( I_{N_R} + \frac{P}{\sigma^2} g g^H \right) \text{ and } \|g\| \leq a \} \quad (41)$$

for some $a > 0$ chosen sufficiently large such that

$$\{ g \in \mathbb{C}^{N_R \times 1} : \|g\| = a \} \subseteq \tilde{G}_a.$$

Such an $a > 0$ exists because

$$\lim_{a \rightarrow \infty} \min_{\|g\|=a} \log \det\left( I_{N_R} + \frac{P}{\sigma^2} g g^H \right) = \infty.$$ 

Since the set

$$\{ g \in \mathbb{C}^{N_R \times 1} : R_{\eta, \sup} \leq \log \det\left( I_{N_R} + \frac{P}{\sigma^2} g g^H \right) \}$$

is closed, it follows that $\tilde{G}_a$ is a closed subset of

$$B_a = \{ g \in \mathbb{C}^{N_R \times 1} : \|g\| \leq a \}.$$

By applying Theorem 4 for $N_T = 1$, it follows that the compound capacity of $\tilde{C} = \{ W_g : g \in \tilde{G}_a \}$ is equal to

$$\min_{g \in \tilde{G}_a} \log \det\left( I_{N_R} + \frac{P}{\sigma^2} g g^H \right).$$

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Let $\theta, \delta > 0$. One can now use the same argument as in the MIMO case to prove the existence of a block-length $n_0$ and of a code sequence $(\Gamma_n)_{n=1}^{\infty}$, where each code $\Gamma_n$ of block-length $n$ has the same size and the same encoder as the code $\tilde{\Gamma}_{g,n}$ of block-length $n$ and a decoder adjusted to the actual gain $g$, such that
\[
\frac{\log \|\Gamma_n\|}{n} \geq R_{\eta,\sup} - \delta
\]
and such that
\[
g \in \tilde{\Gamma}_{g} \cap B_{\eta}^{c} \implies e(\Gamma_n, g) \leq \theta
\]
for $n \geq n_0$.

We have for $n \geq n_0$
\[
\mathbb{P}[e(\Gamma_n, G) \leq \theta] \geq \mathbb{P}[G \in \tilde{\Gamma}_{g} \cap B_{\eta}^{c}] \geq \mathbb{P}[\log \det(I_{N_R} + \frac{P}{\sigma^2}GG^H) \geq R_{\eta,\sup}] \geq 1 - \eta.
\]
This completes the direct proof of the $\eta$-outage transmission capacity for $N_T = 1$.

2) Converse Proof: We are going to show that for $N_T = 1$,
\[
C_\eta(P,W_G) \leq \sup \left\{ R : \mathbb{P}\left[ \log \det(I_{N_R} + \frac{P}{\sigma^2}GG^H) < R \right] \leq \eta \right\}.
\]
For this purpose, we introduce and prove the following lemma:

**Lemma 11:** For $N_T = 1$, it holds that
\[
u(\eta) = \sup \left\{ R : \inf_{Q \in \mathcal{Q}_P} \mathbb{P}[f(G,Q) < R] \leq \eta \right\}
\]
\[
\leq \sup \left\{ R : \mathbb{P}\left[ \log \det(I_{N_R} + \frac{P}{\sigma^2}GG^H) < R \right] \leq \eta \right\}.
\]

**Proof:** Notice first that for $N_T = 1$, it holds that for any $Q \in \mathcal{Q}_P$ and any $g \in \mathbb{C}^{N_R \times 1}$
\[
f(g,Q) = \log \det(I_{N_R} + \frac{1}{\sigma^2}gQg^H) \leq \log \det(I_{N_R} + \frac{P}{\sigma^2}gg^H).
\]
Therefore, for any $Q \in \mathcal{Q}_P$ and any $R \in \mathbb{R}$, we have
\[
\mathbb{P}\left[ \log \det(I_{N_R} + \frac{P}{\sigma^2}GG^H) < R \right] \leq \mathbb{P}[f(G,Q) < R].
\]
This implies that for any $Q \in \mathcal{Q}_P$ and any $R \in \mathbb{R}$
\[
\mathbb{P}\left[ \log \det(I_{N_R} + \frac{P}{\sigma^2}GG^H) < R \right] \leq \inf_{Q \in \mathcal{Q}_P} \mathbb{P}[f(G,Q) < R].
\]
It follows that
\[
\left\{ R : \inf_{Q \in \mathcal{Q}_P} \mathbb{P}[f(G,Q) < R] \leq \eta \right\} \subseteq \left\{ R : \mathbb{P}\left[ \log \det(I_{N_R} + \frac{P}{\sigma^2}GG^H) < R \right] \leq \eta \right\}.
\]
This implies that
\[
u(\eta) \leq \sup \left\{ R : \mathbb{P}\left[ \log \det(I_{N_R} + \frac{P}{\sigma^2}GG^H) < R \right] \leq \eta \right\}.
\]
Now, from Theorem 1, we know that
\[
C_\eta(P,W_G) \leq \nu(\eta).
\]
By Lemma 11, it follows that for $N_T = 1$
\[
C_\eta(P,W_G) \leq \sup \left\{ R : \mathbb{P}\left[ \log \det(I_{N_R} + \frac{P}{\sigma^2}GG^H) < R \right] \leq \eta \right\}.
\]
This completes the converse proof of the $\eta$-outage transmission capacity for $N_T = 1$.

B. Alternative Proof of the Outage Transmission Capacity for $N_T = N_R = 1$

In this section, we will show that the $\eta$-outage transmission capacity for the SISO case is equal to
\[
C_\eta(P,W_G) = \log \left( 1 + \frac{P\gamma_0}{\sigma^2} \right),
\]
where
\[
\gamma_0 = \sup \{ \gamma : \mathbb{P}[|G|^2 < \gamma] \leq \eta \}.
\]

Analogously to the proof of Lemma 10, one can first show that for $N_T = N_R = 1$
\[
\mathbb{P}[|G|^2 \geq \gamma_0] \geq 1 - \eta.
\]

1) Direct Proof: We will show that for $N_T = N_R = 1$
\[
C_\eta(P,W_G) \geq \log \left( 1 + \frac{P\gamma_0}{\sigma^2} \right).
\]
Let $s \in \mathbb{C}$ such that $|s|^2 = \gamma_0$ and let $\theta, \delta > 0$. It is well-known that there exists a code sequence $(\Gamma_{s,n})_{n=1}^{\infty}$ for the channel $W_s$ and a block-length $n_0$ such that for $n \geq n_0$, the rate of each code $\Gamma_{s,n}$ of block-length $n$ satisfies
\[
\frac{\log \|\Gamma_{s,n}\|}{n} \geq \log \left( 1 + \frac{P\gamma_0}{\sigma^2} \right) - \delta
\]
and such that
\[
e(\Gamma_{s,n},s) \leq \theta.
\]
For any $g$ with $|g|^2 \geq \gamma_0$, the SISO Gaussian channel $W_s$ is degraded from the SISO Gaussian channel $W_g$. Analogously to the MIMO case, it follows that there exists a code sequence $(\Gamma_{g,n})_{n=1}^{\infty}$ for $W_g$ such that each code $\Gamma_{g,n}$ of block-length $n$ has the same encoder and the same size as $\Gamma_{s,n}$ but a different decoder adjusted to $g$ and such that for $n \geq n_0$, $e(\Gamma_{g,n},g) \leq \theta$. Here, we require channel state information at the receiver side (CSIR) so that the decoder can adjust its decoding strategy according to the channel state. So far, we have proved the
existence of a code sequence \((\Gamma_n)^\infty_{n=1}\) and a block-length \(n_0\) such that
\[
\frac{\log||\Gamma_n||}{n} \geq \log \left(1 + \frac{P\gamma_0}{\sigma^2}\right) - \delta
\]
and such that
\[
|\gamma|^2 \geq \gamma_0 \implies e(\Gamma_n, g) \leq \theta
\]
for \(n \geq n_0\). Now, for \(n \geq n_0\), we have
\[
P[e(\Gamma_n, G) \leq \theta] \geq \mathbb{P}[|G|^2 \geq \gamma_0] \geq 1 - \eta.
\]
This implies (42) and completes the direct proof.

2) Converse Proof: We will show that for \(N_T = N_R = 1\)
\[
C_0(P, W_G) \leq \log \left(1 + \frac{P\gamma_0}{\sigma^2}\right).
\]
Suppose this were not true. Then there exists an \(\varepsilon > 0\) such that for all \(\theta, \delta > 0\) there exists a code sequence \((\Gamma_n)^\infty_{n=1}\) satisfying
\[
\frac{\log||\Gamma_n||}{n} \geq \log \left(1 + \frac{P(\gamma_0 + \varepsilon)}{\sigma^2}\right) - \delta
\]
and
\[
P[e(\Gamma_n, G) \leq \theta] \geq 1 - \eta
\]
for sufficiently large \(n\). Since \(\delta\) may be arbitrary, we may choose it in such a way that the right-hand side of (44) is strictly larger than \(\log(1 + (P\gamma_0)/\sigma^2)\). We define \(\gamma_1\) to be the solution of the equation
\[
\log(1 + (P\gamma_1)/\sigma^2) = \log \left(1 + \frac{P(\gamma_0 + \varepsilon)}{\sigma^2}\right) - \delta.
\]
\(\gamma_1\) is chosen such that the rate of the code sequence is greater than the capacity of the channel \(W_g\) when \(|\gamma|^2 < \gamma_1\). Therefore, even under the CSIR assumption, the strong converse for SISO Gaussian channels implies that for large \(n\), the error probability is greater than \(\theta\) when \(|\gamma|^2 < \gamma_1\). It follows that
\[
P[e(\Gamma_n, G) > \theta] \geq \mathbb{P}[|G|^2 < \gamma_1] > \eta,
\]
by the definition of \(\gamma_0\), where we used that \(\gamma_1 > \gamma_0\) from the choice of \(\delta\). This is a contradiction to (45), and so (43) must be true. This completes the converse proof.

V. PROOF OF THEOREM 3

A. Proof of the Lower Bound on the Outage CR Capacity

1) If \(l(\eta) = 0\): It is shown in [1] that when the terminals do not communicate over the channel, the CR capacity defined in [1] is equal to
\[
H_0 = \max_{U \in \mathcal{U}_0} \max_{X \in \mathcal{X}} \max_{Y \in \mathcal{Y}} I(U; X), I(U; X') - I(U; Y) \leq 0
\]
Hence, when the terminals do not communicate over the MIMO slow fading channel \(W_G\), \(H_0\) is also an achievable \(\eta\)-outage CR rate. Therefore, we have
\[
C_{n,C,R}(P, W_G) \geq \max_{U \in \mathcal{U}_0} \max_{X \in \mathcal{X}} \max_{Y \in \mathcal{Y}} I(U; X), I(U; X') - I(U; Y) \leq 0
\]
2) If \(l(\eta) > 0\): We extend the coding scheme provided in [1] to MIMO slow fading channels. By continuity, it suffices to show that
\[
\max_{U \in \mathcal{U}_0} \max_{X \in \mathcal{X}} \max_{Y \in \mathcal{Y}} I(U; X), I(U; X') - I(U; Y) \leq l(\eta)
\]
is an achievable \(\eta\)-outage CR rate for every \(R' < l(\eta)\). Let \(U\) be a random variable satisfying \(U \rightarrow X \rightarrow Y\) and \(I(U; X) - I(U; Y) \leq R'\). Let the upper-bound \(0 \leq \eta < 1\) on the outage probability, from the CR generation perspective, be fixed arbitrarily. We are going to show that \(H = I(U; X)\) is an achievable \(\eta\)-outage CR rate. Let \(\alpha, \delta > 0\). Without loss of generality, assume that the distribution of \(U\) is a possible type for block-length \(n\). For any \(\mu > 0\), we let
\[
N_1 = \left|2^n[I(U; X) - I(U; Y) + 3\mu]\right|
\]
and
\[
N_2 = \left|2^n[I(U; Y) - 2\mu]\right|
\]
For each pair \((i, j)\) with \(1 \leq i \leq N_1\) and \(1 \leq j \leq N_2\), we define a random sequence \(U_{i,j} \in \mathcal{U}^n\) of type \(P_U\). Let \(M = U_1, \ldots, U_{N_1, N_2}\) be the joint random variable of all \(U_{i,j}\). We define \(\Phi_M\) as follows: Let \(\Phi_M(X^n) = U_{i,j}\), if \(U_{i,j}\) is jointly \(UX\)-typical with \(X^n\) (either one if there are several). If no such \(U_{i,j}\) exists, then \(\Phi_M(X^n)\) is set to a constant sequence \(u_0\) different from all the \(U_{i,j}\)s, jointly \(UX\)-typical with none of the realizations of \(X^n\) and known to both terminals.

We further define the following two sets which depend on \(M\):
\[
S_1(M) = \{(x, y): (\Phi_M(x), x, y) \in T_{U,X,Y}^n\}
\]
and
\[
S_2(M) = \left\{(x, y): (x, y) \notin S_1(M) \text{ s.t. } U_{i,j} = \Phi_M(x) \text{ and } \exists U_{i',j'} \neq U_{i,j} \text{ jointly } UY\text{-typical with } y \right\}.
\]
It is proved in [1] that
\[
\mathbb{E}_M \left[\mathbb{P}(X^n, Y^n) \notin S_1(M)\right] + \mathbb{P}(X^n, Y^n) \in S_2(M)\right) \leq \beta(n),
\]
where \(\beta(n) \leq \frac{n}{2}\) for sufficiently large \(n\). We choose a realization \(m = u_1, \ldots, u_{N_1, N_2}\) satisfying:
\[
\mathbb{P}(X^n, Y^n) \notin S_1(m)\right) + \mathbb{P}(X^n, Y^n) \in S_2(m)\right) \leq \beta(n).
\]
From (46), we know that such a realization exists. We denote \( \Phi_m \) by \( \Phi \). We assume that each \( u_{i,j}, i = 1, \ldots, N_1, j = 1, \ldots, N_2 \), is known to both terminals. This means that \( N_1 \) codebooks \( C_i, 1 \leq i \leq N_1 \), are known to both terminals, where each codebook contains \( N_2 \) sequences, \( u_{i,j}, j = 1, \ldots, N_2 \).

Let \( x \) be any realization of \( X^n \) and \( y \) be any realization of \( Y^n \). Let \( f_i(x) = i \) if \( \Phi(x) = u_{i,j} \). Otherwise, if \( \Phi(x) = u_{i,j} \), then \( f_i(x) = N_1 + 1 \). Since \( R' < l(\eta) \), we choose \( \mu \) to be sufficiently small such that

\[
\frac{\log ||f_i||}{n} = \frac{\log(N_1 + 1)}{n} \leq l(\eta) - \mu', \tag{47}
\]

for some \( \mu' > 0 \). The message \( i^* = f_i(x) \), with \( i^* \in \{1, \ldots, N_1 + 1\} \), is encoded to a sequence \( t \) using a code sequence \( (\Gamma_n^*)^\infty_{n=1} \), where each code \( \Gamma_n^* \) of block-length \( n \) is defined according to Definition 4, with rate \( \frac{\log ||\Gamma_n^*||}{n} \equiv \frac{\log ||f_i||}{n} \) satisfying (47) and with error probability \( e(\Gamma_n^*, G) \) satisfying for sufficiently large \( n \)

\[
P[e(\Gamma_n^*, G) \leq \theta] \geq 1 - \eta, \tag{48}
\]

where \( \theta \) is a positive constant satisfying \( \theta \leq \frac{\eta}{2} \). Here, \( ||f_i|| \) refers to the cardinality of the set of messages \( \{i^*: i^* \in \{1, \ldots, N_1 + 1\}\} \). Since \( l(\eta) \) is an achievable \( \eta \)-outage transmission rate, we know that such a code sequence exists. The sequence \( t \) is sent over the MIMO slow fading channel. Let \( z \) be the corresponding channel output sequence. Terminal \( B \) decodes the message \( i^* \) from the knowledge of \( z \). Let \( \Psi(y, z) = u_{i,j} \) if \( u_{i,j} \) and \( y \) are jointly \(UY\)-typical. If there is no such \( u_{i,j} \) or there are several, we set \( \Psi(y, z) = u_0 \) (since \( K \) and \( L \) must have the same alphabet). Now, we are going to show that the requirements in (2), (3) and (4) are satisfied. Clearly, (3) is satisfied for \( c = H(X) + \mu + 1 \) because

\[
|K| = N_1N_2 + 1 \leq 2^{n[I(U;X) + \mu]} + 1 \leq 2^{n[H(X) + \mu + 1]}.
\]

We define next for any \((i, j) \in \{1, \ldots, N_1\} \times \{1, \ldots, N_2\}\) the set

\[
S = \{x \in X^n \text{ s.t. } (u_{i,j}, x) \text{ jointly } UX\text{-typical}\}.
\]

Then, it holds that

\[
P[K = u_{i,j}] = \sum_{x \in S} P[K = u_{i,j} | X^n = x] P_X^n(x) + \sum_{x \in S^c} P[K = u_{i,j} | X^n = x] P_X^n(x)
\]

\[
\overset{(a)}{=} \sum_{x \in S} P[K = u_{i,j} | X^n = x] P_X^n(x) \leq \sum_{x \in S} P_X^n(x) = P_X^n(\{x : (u_{i,j}, x) \text{ jointly } UX\text{-typical}\}) = 2^{-n[I(U;X) - \kappa(n)],}
\]

for some \( \kappa(n) > 0 \) with \( \lim_{n \to \infty} \frac{\kappa(n)}{n} = 0 \), where (a) follows because for \((u_{i,j}, x)\) being not jointly \(UX\)-typical, we have

\[
P[K = u_{i,j} | X^n = x] = 0.
\]

This yields

\[
H(K) \geq nI(U; X) - \kappa'(n)
\]

for some \( \kappa'(n) > 0 \) with \( \lim_{n \to \infty} \frac{\kappa'(n)}{n} = 0 \). Therefore, for sufficiently large \( n \), it holds that

\[
\frac{H(K)}{n} > H - \delta.
\]

Thus, (4) is satisfied.

**Remark 7:** It is to notice that for sufficiently large \( n \)

\[
H(K) \approx \log |K| \approx nI(U; X).
\]

Therefore the random variable \( K \) is nearly uniform for sufficiently large \( n \). It follows from Remark 1 that, for sufficiently large \( n \), the random variable \( L \) is also nearly uniform when the system is not in outage from the CR generation perspective. As a result, when the system is not in outage and for sufficiently large \( n \), \((K, L)\) is a pair of nearly uniform random variables. This is the most convenient form of CR, as already mentioned in Remark 2.

Now, it remains to prove that (2) is satisfied. For this purpose, we define the following event:

\[
D_m = \{\Phi(X^n) \text{ is equal to none of the } u_{i,j}s\}.
\]

We denote its complement by \( D_m^c \). We further define \( I' = f_i(X^n) \) to be the random message generated by Terminal \( A \) and \( \tilde{I} \) to be the random message decoded by Terminal \( B \). We have

\[
P[K \neq L| G] = P[K \neq L| G, I' = \tilde{I'}] P[I' = \tilde{I'}| G] + P[K \neq L| G, I' \neq \tilde{I'}] P[I' \neq \tilde{I'}| G]
\]

\[
\leq P[K \neq L| G, \tilde{I'} = I'] + P[I' \neq \tilde{I'}| G].
\]

Here,

\[
P[K \neq L| G, I' = \tilde{I'}] = P[K \neq L| G, I' = \tilde{I'}, D_m] P[D_m| G, I' = \tilde{I'}]
\]

\[
+ P[K \neq L| G, I' = \tilde{I'}, D_m^c] P[D_m^c| G, I' = \tilde{I'}]
\]

\[\overset{(a)}{=} P[K \neq L| G, I' = \tilde{I'}, D_m^c] P[D_m^c| G, I' = \tilde{I'}]
\]

\[
\leq P[K \neq L| G, I' = \tilde{I'}, D_m^c],
\]

where (a) follows from \( P[K \neq L| G, I' = \tilde{I'}, D_m] = 0 \), since conditioned on \( G \), \( I' = \tilde{I'} \) and \( D_m \), we know that \( K \) and \( L \) are both equal to \( u_0 \). It follows that

\[
P[K \neq L| G] \leq P[K \neq L| G, I' = \tilde{I'}] + P[I' \neq \tilde{I'}| G]
\]

\[
\leq P[(X^n, Y^n) \in S_1(m) \cup S_2(m)] + P[I' \neq \tilde{I'}| G]
\]

\[\overset{(a)}{=} P[(X^n, Y^n) \notin S_1(m)] + P[(X^n, Y^n) \in S_2(m)] + P[I' \neq \tilde{I'}| G]
\]

\[
\leq \beta(n) + P[I' \neq \tilde{I'}| G],
\]
where (a) follows because $S_1^c(m)$ and $S_2(m)$ are disjoint. It holds that

$$\mathbb{P}[I^* \neq \hat{I}^* | G] \leq \theta \implies \mathbb{P}[K \neq L | G] \leq \beta(n) + \theta.$$ 

Since, for sufficiently large $n$, $\beta(n) + \theta \leq \alpha$, it follows that

$$\mathbb{P}[I^* \neq \hat{I}^* | G] \leq \theta \implies \mathbb{P}[K \neq L | G] \leq \alpha.$$ 

From (48), we know that

$$\mathbb{P}[\mathbb{P}[K \neq L | G] \leq \alpha] \geq \mathbb{P}[\mathbb{P}[I^* \neq \hat{I}^* | G] \leq \theta] \geq 1 - \eta.$$ 

Thus

$$\mathbb{P}[\mathbb{P}[K \neq L | G] \leq \alpha] \geq 1 - \eta.$$ 

This completes the proof of the lower-bound on the $\eta$-outage CR capacity.

**B. Proof of the Upper Bound on the Outage CR Capacity**

Let $0 \leq \eta < 1$. Let $H$ be any achievable $\eta$-outage CR rate. So, there exists a non-negative constant $c$ such that for every $\alpha > 0$ and $\delta > 0$ and for sufficiently large $n$, there exists a permissible pair of random variables $(K,L)$ according to a fixed CR-generation protocol of block-length $n$ as introduced in Section II-A such that

$$\mathbb{P}[\mathbb{P}[K \neq L | G] \leq \alpha] \geq 1 - \eta,$$ 

(49)

$$\frac{1}{n} H(K) > H - \delta.$$ 

(50)

We recall that the CR generation protocol consists of:

1) A function $\Phi$ that maps $X^n$ into a random variable $K$ with alphabet $\mathcal{K}$ generated by Terminal $A$.

2) A function $\Lambda$ that maps $X^n$ into the input sequence $T^n \in \mathbb{C}^{N_H \times n}$ satisfying the following power constraint

$$\frac{1}{n} \sum_{i=1}^{n} T_i H T_i \leq P,$$ 

almost surely.

3) A function $\Psi$ that maps $Y^n$ and the output sequence $Z^n \in \mathbb{C}^{N_H \times n}$ into a random variable $L$ with alphabet $\mathcal{K}$ generated by Terminal $B$.

We are going to show that for any $\epsilon > 0$

$$\frac{H(K)}{n} \leq \max_{U: \mathbb{P}[\mathbb{P}[U \neq X] \leq \epsilon]} I(U; X),$$ 

$$I(U; X) - I(U; Y) \leq u(\eta) + \zeta(n, \alpha, \epsilon),$$ 

where $u(\eta)$ is defined in (10) and where $\zeta(n, \alpha, \epsilon) = \frac{1}{n} + \alpha c + \epsilon$. In our proof, we will use the following lemma:

**Lemma 12:** (Lemma 17.12 in [27]) For arbitrary random variables $S$ and $R$ and sequences of random variables $X^n$ and $Y^n$, it holds that

$$I(S; X^n | R) - I(S; Y^n | R) = \sum_{i=1}^{n} I(S; X_i | X_1, \ldots, X_{i-1}, Y_{i+1}, \ldots, Y_n, R)$$ 

$$= n [I(S; X_j | V) - I(S; Y_j | V)],$$ 

where $V = (X_1, \ldots, X_{j-1}, Y_{j+1}, \ldots, Y_n, R, J)$, with $J$ being a random variable independent of $R, S$, $X^n$ and $Y^n$ and uniformly distributed on $\{1, \ldots, n\}$. Let $J$ be a random variable uniformly distributed on $\{1, \ldots, n\}$ and independent of $K, X^n$ and $Y^n$. We further define $U = (K, X_1, \ldots, X_{j-1}, Y_{j+1}, \ldots, Y_n, J)$. It holds that $U \in (X_j, Y_j, J)$.

Notice that

$$H(K) \equiv H(K) - H(K|X^n)$$ 

$$= n [I(K; X_j | V) - I(K; Y_j | V)],$$ 

(51)

where (a) follows because $K = \Phi(X^n)$ and (b) and (c) follow from the chain rule for mutual information.

We will show next that

$$I(U; X_j) - I(U; Y_j) \leq u(\eta) + \zeta(n, \alpha, \epsilon),$$ 

(52)

where $\zeta(n, \alpha, \epsilon) = \frac{1}{n} + \alpha c + \epsilon$. Applying Lemma 12 for $S = K$, $R = \emptyset$ with $V = (X_1, \ldots, X_{j-1}, Y_{j+1}, \ldots, Y_n, J)$ yields

$$I(K; X^n) - I(K; Y^n) = n[I(K; X_j | V) - I(K; Y_j | V)]$$ 

$$= n[I(KV; X_j) - I(V; X_j) - I(KV; Y_j) + I(V; Y_j)]$$ 

(53)

where (a) follows from the chain rule for mutual information and from the fact that $V$ is independent of $(X_j, Y_j)$ and (b) follows from $U = (K, V)$. It results using (53) that

$$n[I(U; X_j) - I(U; Y_j)] = I(K; X^n) - I(K; Y^n) = H(K) - I(K; Y^n) = H(K|Y^n).$$ 

(54)

Next, to prove (52), we will show that

$$\frac{H(K|Y^n)}{n} \leq u(\eta) + \zeta(n, \alpha, \epsilon).$$ 

(55)

In order to prove (55), we will use a change of measure argument. To prepare this, we need some technicalities. Let $\text{cov}(T_i) = Q_i$ for $i = 1, \ldots, n$, where $T_i \in \mathbb{C}^{N_H}, i = 1, \ldots, n$. We define

$$Q^* = \frac{1}{n} \sum_{i=1}^{n} Q_i.$$ 

By Lemma 5, we know that $\text{tr}(Q^*) \leq P$ and therefore $Q^* \in Q_P$. Let

$$R(Q^*) = \sup \left\{ R : \mathbb{P}[f(G, Q^*) < R] \leq \eta \right\}.$$ 

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We recall that the function $f$ is defined in (6). Since $Q^* \in Q_P$, Lemma 6 implies that
\[ R(Q^*) \leq u(\eta). \] (56)

We consider for any $\epsilon > 0$ the set
\[ \Omega = \left\{ g \in \mathbb{C}^{N_R \times N_T} : P[K \neq L|G = g] \leq \alpha \right\}, \]
and define $\hat{G}$ to be a random matrix, independent of $X^n, Y^n$ and $\xi^N$, with alphabet $\Omega$ such that for every Borel set $A \subseteq \mathbb{C}^{N_R \times N_T}$, it holds that
\[ P[\hat{G} \in A] = P[G \in A|G \in \Omega]. \]

In order to prove that such a $\hat{G}$ is well-defined, it suffices show that $P[G \in \Omega] > 0$. This is proved in what follows:

**Lemma 13:**
\[ P[G \in \Omega] > 0. \]

**Proof:** From the definition of $R(Q^*)$, we have
\[ \eta \leq P[f(G, Q^*) \leq R(Q^*) + \epsilon] \leq P[f(G, Q^*) \leq R(Q^*) + \epsilon]. \]

Then, it holds that
\[ P[f(G, Q^*) \leq R(Q^*) + \epsilon] = \eta_1, \]
where $0 \leq \eta \leq \eta_1 \leq 1$.

It follows using (49) that
\[ \frac{1-\eta}{1-\alpha} \leq P[K \neq L|G] \leq \alpha \]
\[ = P[K \neq L|G] \leq \alpha \]
\[ \leq f(G, Q^*) \leq R(Q^*) + \epsilon \]
\[ \leq P[f(G, Q^*) \leq R(Q^*) + \epsilon] \]
\[ \leq 1 + \alpha \]
\[ \leq 1 + \alpha \]
\[ \leq 1 + \alpha. \]

where we used that $1 - \eta_1 < 1 - \eta$. This means that
\[ P[K \neq L|G] \leq \alpha \]
\[ f(G, Q^*) \leq R(Q^*) + \epsilon > 0. \]

In addition, since $\eta_1 > 0$, we have
\[ P[K \neq L|G] \leq \alpha, f(G, Q^*) \leq R(Q^*) + \epsilon > 0. \]

Thus
\[ P[G \in \Omega] > 0. \]

Next, we fix the CR generation protocol and change the state distribution of the slow fading channel. We obtain the following new MIMO channel:
\[ Z_i = G_i \xi_i + \xi_i, \quad i = 1, \ldots, n, \]
where $\hat{Z}^n$ is the new output sequence. We further define $\hat{L}$ such that
\[ \hat{L} = \Psi(Y^n, \hat{Z}^n). \]

Clearly, it holds for any $g \in \Omega$ that
\[ P[K \neq \hat{L}|\hat{G} = g] \leq \alpha \] (57)
and that
\[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} gQ^* g^H \right) \leq R(Q^*) + \epsilon. \] (58)

Furthermore, since $\xi_i \sim N_0 \sigma^2 I_{N_R}$, $i = 1, \ldots, n$, it follows for $i = 1, \ldots, n$ that
\[ I(T_i; \hat{Z}_i|G = g) \leq \log \det \left( I_{N_R} + \frac{1}{\sigma^2} gQ_i g_i^H \right) \forall g \in \Omega. \] (59)

We recall that the goal is to prove that
\[ \frac{H(K|Y^n)}{n} \leq u(\eta) + \zeta(n, \alpha, \epsilon). \] (60)

Now, we have
\[ \frac{1}{n} H(K|Y^n) = \frac{1}{n} H(K|\hat{G}, Y^n) \]
\[ = \frac{1}{n} H(K|\hat{G}, Y^n, \hat{Z}^n) + \frac{1}{n} I(K; \hat{Z}^n|\hat{G}, Y^n), \]
where we used that $\hat{G}$ is independent of $(K, Y^n)$. On the one hand, we have
\[ \frac{1}{n} H \left( K|\hat{Z}^n, \hat{G}, Y^n \right) \leq \frac{1}{n} H \left( \hat{L}|\hat{G} \right) + \frac{1}{n} I \left( T_i; \hat{Z}_i|\hat{G}, Y^n \right) \]
\[ \leq \frac{1}{n} H \left( \hat{L}|\hat{G} \right) \]
\[ \leq \frac{1}{n} + \alpha \log |K| \]
\[ \leq \frac{1}{n} + \alpha \epsilon, \]
where (a) follows from $\hat{L} = \Psi(Y^n, \hat{Z}^n)$, (b) follows from Fano’s Inequality, (c) follows from (57) and (d) follows from \( \log |K| \leq cn \) in (50). On the other hand, we have
\[ \frac{1}{n} I(K; \hat{Z}^n|\hat{G}, Y^n) \]
\[ \leq \frac{1}{n} I(X^n, K; \hat{Z}^n|\hat{G}, Y^n) \]
\[ \leq \frac{1}{n} I(T^n; \hat{Z}^n|\hat{G}, Y^n) \]
\[ \leq \frac{1}{n} \left[ h(\hat{Z}^n|\hat{G}, Y^n) - h(\hat{Z}^n|T^n, \hat{G}, Y^n) \right] \]
\[ \leq \frac{1}{n} \left[ h(\hat{Z}^n|\hat{G}, Y^n) - h(\hat{Z}^n|\hat{G}, T^n) \right] \]
since on the set of Hermitian positive semi-definite matrices and a Markov chain, 

\[ \sum_{i=1}^{n} \frac{1}{n} I(\hat{Z}; T^n | \hat{G}, \hat{Z}^{i-1}) \]

\[ = \sum_{i=1}^{n} h(\hat{Z}; T^n | \hat{G}, \hat{Z}^{i-1}) - h(\hat{Z}; T^n, \hat{Z}^{i-1}) \]

\[ \sum_{i=1}^{n} h(\hat{Z}; T^n | \hat{G}, \hat{Z}^{i-1}) - h(\hat{Z}; T^n, \hat{Z}^{i-1}) \]

\[ \leq \sum_{i=1}^{n} h(\hat{Z}^{i} | \hat{G}) - h(\hat{Z}^{i} | \hat{G}, T_i) \]

\[ = \sum_{i=1}^{n} I(T_i; \hat{Z}^{i}) \]

\[ \leq \sum_{i=1}^{n} E \left[ \log \det(I_{N_R} + \frac{1}{\sigma^2} \hat{G} Q_i \hat{G}^H) \right] \]

\[ \leq \sum_{i=1}^{n} \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \hat{G} Q_i \hat{G}^H \right) \]

\[ \leq \sum_{i=1}^{n} \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \hat{G} \left( \sum_{i=1}^{n} Q_i \right) \hat{G}^H \right) \]

\[ \leq \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \hat{G} \hat{G}^H \right) \]

\[ \leq R(Q^*) + \epsilon \]

\[ \leq u(\eta) + \epsilon, \]

where (a) follows from the Data Processing Inequality because \( Y^n \circ X^n \circ K \circ G T^n \circ Z^n \) forms a Markov chain, (b) follows because \( Y^n \circ X^n \circ K \circ G T^n \circ Z^n \) forms a Markov chain, (c) (f) follow because conditioning does not increase entropy, (d) follows from the chain rule for mutual information, (e) follows because \( T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n, \hat{Z}^{i-1} \circ \hat{G}, T_i \circ \hat{Z}_i \) forms a Markov chain, (g) follows from (59), (h) follows from Jensen’s Inequality since the function \( \log \circ \det \) is concave on the set of Hermitian positive semi-definite matrices and since \( I_{N_R} + \frac{1}{\sigma^2} \hat{G} Q_i \hat{G}^H \) is Hermitian positive semi-definite for \( i = 1, \ldots, n \), (i) follows from (58) and (j) follows from (56). This proves that for \( 0 \leq \eta < 1 \), (60) is satisfied for \( \zeta(n, \alpha, \epsilon) = \frac{1}{\epsilon} + \alpha \epsilon + \epsilon > 0 \).

From (54) and (60), we deduce that for \( 0 \leq \eta < 1 \)

\[ I(U; X_J) - I(U; Y_J) \leq u(\eta) + \zeta(n, \alpha, \epsilon), \]

where \( U \circ X_J \circ Y_J \).

Since the joint distribution of \( X_J \) and \( Y_J \) is equal to \( P_{XY}, \)

\[ H(\hat{K}) \]

is upper-bounded by \( I(U; X) \) subject to \( I(U; X) - I(U; Y) \leq u(\eta) + \zeta(n, \alpha, \epsilon) \) with \( U \) satisfying \( U \circ X \circ Y \).

As a result, it holds using (51) that for sufficiently large \( n \) and for every \( \alpha, \delta, \epsilon > 0 \), any achievable \( \eta \)-outage CR rate \( H \)

satisfies

\[ H < \max_{U \circ X \circ Y} I(U; X) + \delta. \]

It follows that

\[ H \leq \inf_{\alpha, \delta, \epsilon > 0} \lim_{n \to \infty} \max_{U \circ X \circ Y} \left[ I(U; X) + \delta \right] \]

\[ = \max_{U \circ X \circ Y} \left[ I(U; X) \right] \]

This completes the proof of the upper-bound on the \( \eta \)-outage CR capacity.

VI. PROOF OF THEOREM 4

The goal is to prove that the capacity of \( C = \{ W_g : g \in G \} \)

is

\[ \max_{Q \in Q^*} \min_{g \in G} \log \det(I_{N_R} + \frac{1}{\sigma^2} g Q g^H), \]

where \( G \subset \mathbb{C}^{N_R \times N_T} \) is an arbitrary compact set. In our proof, we follow the strategy of [32].

A. Direct Proof of Theorem 4 for Finite \( G \)

We prove first the direct part of Theorem 4 for finite \( G \). This result will be later extended for infinite \( G \) using an approximation inequality.

Theorem 5: Let \( G \subset \mathbb{C}^{N_R \times N_T} \) be any finite set. We define the compound channel

\[ C' = \{ W_g : g \in G \}. \]

An achievable rate for \( C' \) is

\[ \max_{Q \in Q^*} \min_{g \in G} \log \det(I_{N_R} + \frac{1}{\sigma^2} g Q g^H). \]

1) Auxiliary Lemmas: In order to prove Theorem 5, we introduce the following lemmas first.

Lemma 14 (Feinstein’s Lemma With Input Constraint):

For any channel \( W \) with input set \( T \) and output set \( Z \), with random input \( T \) distributed according to \( p(t) \) and with corresponding random channel output \( Z \) distributed according to \( q(z) \) and for any integer \( \tau \geq 1 \), real number \( \alpha > 0 \), and measurable subset \( E \) of \( T \), there exists a code with size \( \tau \), maximum error probability \( \epsilon \) and block-length \( n = 1 \), whose codewords are contained in the set \( E \), where \( \epsilon \) satisfies

\[ \epsilon = \tau 2^{-\alpha} + P \left[ i(T, Z) \leq \alpha \right] + P \left[ T \notin E \right], \]

where

\[ i(T, Z) = \log \frac{W(Z | T)}{q(Z)}. \]

3In [32], the focus was on compound real Gaussian channels with square channel matrix whose operator norm is upper-bounded by \( a > 0 \) and with noise covariance matrix satisfying further conditions.
Proof: As stated in [32], the proof is the same as the one for Theorem 2 in [33] or Lemma 8.2.1 in [34]. □

For any $g \in G$, we assume that the random input sequence $T^n$ of $W_{g}$ is distributed according to $p(T^n)$ and that the corresponding random channel output sequence $Z^n$ is distributed according to $q(z^n)$. We define for any $t^n \in \mathbb{C}^{N_T \times n}$, any $z^n \in \mathbb{C}^{N_R \times n}$ and any $g \in G$

$$i_g(t^n, z^n) = \log \frac{W_{g}(z^n | t^n)}{q(z^n)}.$$

Lemma 15: For any real numbers $\alpha > 0$, $\delta > 0$, and any integer $\tau \geq 1$, there exists a code $\Gamma_n$ for $C'$ with size $|\Gamma_n| = \tau$, block-length $n$ and with codewords contained in

$$E_n = \{ t^n = (t_1, \ldots, t_n) \in \mathbb{C}^{N_T \times n} : \frac{1}{n} \sum_{i=1}^{n} ||t_i||^2 \leq P \}$$

such that for all $g \in G$

$$e_c(\Gamma_n, g) \leq |G|\tau^{-\alpha} + |G|^2 \delta + |G|^2 ||P[T^n \notin E_n] + \sum_{g \in G} ||i_g(T^n, Z^n) \leq \alpha + \delta \}.$$

Proof: The proof is a simple modification of that of Lemma 3 in [35]. It is based on an application of Feinstein’s lemma.

Lemma 16: Let $W_{g}$ be a fixed channel with $g \in \mathbb{C}^{N_R \times N_T}$. Let $T^n \in \mathbb{C}^{N_T \times n}$ and $Z^n \in \mathbb{C}^{N_R \times n}$ be the random input and output sequence, respectively. We further assume that the $T_i$s are i.i.d., where each $T_i \in \mathbb{C}^{N_T}$ is Gaussian distributed with mean $0_{N_T}$ and with a non-singular covariance matrix $Q$. Then for any $\delta > 0$

$$P[i_g(T^n, Z^n) \leq E[i_g(T^n, Z^n)] - n\delta] \leq \frac{1}{2} \left\{ \frac{1 + \frac{\ln(2)}{2\sigma^2}}{1 + \frac{\ln(2)}{N_T}} \right\}^{\frac{n}{\sigma^2}}.$$

Proof: Since $(T_i, Z_i), i = 1, \ldots, n$, are i.i.d., we introduce $(T, Z)$ such that $(T, Z)$ has the same joint distribution as each of the $(T_i, Z_i)$. Now

$$E[i_g(T^n, Z^n)] = nE[i_g(T, Z)] = n \log \det \left( I_{N_R} + \frac{1}{\sigma^2} gQg^H \right).$$

Let

$$\Theta = QgQ^H + \sigma^2 I_{N_R}$$

be the covariance matrix of $Z$. Here, $\Theta$ is positive definite and therefore non-singular. We further define

$$\phi_i = -\frac{1}{\sigma^2} (Z_i - gT_i)^H (Z_i - gT_i) + Z_i^H \Theta^{-1} Z_i.$$

Since the $\phi_i$s are i.i.d., we define $\phi$ to be a random variable with the same distribution as each of the $\phi_i$ as follows:

$$\phi = -\frac{1}{\sigma^2} (Z - gT)^H (Z - gT) + Z^H \Theta^{-1} Z.$$

Since $i_g(T_i, Z_i) = \log \det \left( I_{N_R} + \frac{1}{\sigma^2} gQg^H \right) + \frac{\phi}{\ln(2)}, i = 1, \ldots, n$, it follows that

$$P[i_g(T^n, Z^n) \leq E[i_g(T^n, Z^n)] - n\delta]$$

$$= P \left[ \sum_{i=1}^{n} \phi_i \leq -n\delta \right]$$

$$= P \left[ -\ln(2) n\delta + \sum_{i=1}^{n} \phi_i \geq 0 \right]$$

$$\leq E \left[ \exp(-\beta (\ln(2)n\delta + \sum_{i=1}^{n} \phi_i)) \right]$$

$$= \exp(-\beta n(\ln(2)) (E[\exp(-\beta \phi)])^n \forall \beta \geq 0.$$
Now, let \( M(\beta) = \mathbf{O}^{-1} + \beta(\Lambda - \Phi) \in \mathbb{C}^{(N_T+N_R) \times (N_T+N_R)} \). It follows that
\[
\zeta(\beta) = \mathbb{E}\left[ \exp(-\beta \Phi) \right] = \int_{\mathbb{C}^{N_T+N_R}} \exp(-w^H \mathbf{O}^{-1} w) \times \exp\left[ -\beta \left( w^H \Lambda w - w^H \Phi w \right) \right] dw = \frac{1}{\pi^{N_T+N_R} \det(\mathbf{O})} \int_{\mathbb{C}^{N_T+N_R}} \exp(-w^H M(\beta) w) dw \]
where the integral is a \((N_T+N_R)\)-fold integral over \( \mathbb{C}^{N_T+N_R} \). Here, \( M(\beta) \) is positive definite for \( 0 \leq \beta < \beta_0 \) for some \( \beta_0 \geq 1 \). Indeed, for \( \beta \geq 0 \), it holds that
\[
M(\beta) = \left( \begin{array}{cc} Q^{-1} + \frac{1}{\sigma^2} (1 - \beta) g^H g & -\frac{1}{\sigma^2} (1 - \beta) g^H \\ -\frac{1}{\sigma^2} (1 - \beta) g & \beta \Theta^{-1} + \frac{1}{\sigma^2} (1 - \beta) I_{N_R} \end{array} \right).
\]
Notice that
\[
M(\beta) = \beta M(1) + (1 - \beta) M(0)
\]
and that \( M(0) \) and \( M(1) \) are both positive definite. From the convexity of the set of positive definite Hermitian matrices, it follows that \( \beta M(1) + (1 - \beta) M(0) \) is positive definite for all \( \beta \in (0, 1) \). This proves that \( M(\beta) \) is positive definite for \( 0 \leq \beta \leq 1 \).

By substituting \( \Lambda, \Phi \) and \( \mathbf{O} \), and by using the fact that \( \Theta = g Q g^H + \sigma^2 I_{N_R} \), we obtain
\[
M(\beta) \mathbf{O} = I_{N_T+N_R} + \beta(\Lambda - \Phi) \mathbf{O}
\]

\[
= I_{N_T+N_R} + \beta \left( \begin{array}{cc} \frac{1}{\sigma^2} g^H g & -\frac{1}{\sigma^2} g^H \\ -\frac{1}{\sigma^2} g & \Theta^{-1} \end{array} \right) \left( \begin{array}{cc} Q & Q^H \\ g Q & \Theta \end{array} \right) = I_{N_T+N_R} + \beta \left( \begin{array}{cc} -g^H g Q + g^H g Q^H & -g^H g Q g Q^H + g^H \Theta \\ g Q + \sigma^2 \Theta^{-1} g - g Q & g Q^H + \sigma^2 I_{N_R} - \Theta \end{array} \right) \]

\[
= \left( \begin{array}{cc} I_{N_T} & \beta g^H \\ \beta \Theta^{-1} g & I_{N_R} \end{array} \right).
\]

As a result, we obtain using the determinant rule for block-matrices
\[
\det(M(\beta) \mathbf{O}) = \det(I_{N_T} - \beta^2 \Theta^{-1} g Q g^H) = \det(\Theta^{-1}) \det(\Theta - \beta^2 g Q g^H) \]
\[
= \det(\Theta^{-1}) \det(\sigma^2 I_{N_R} + (1 - \beta^2) g Q g^H) = \sigma^{2N_R} \det(I_{\mathbf{N}_R} + (1 - \beta^2) \frac{1}{\sigma^2} g Q g^H),
\]
where
\[
\det(\Theta) = \sigma^{2N_R} \det(I_{\mathbf{N}_R} + \frac{1}{\sigma^2} g Q g^H).
\]

We define \( \lambda_1, \ldots, \lambda_{N_R} \) to be the eigenvalues of the positive semi-definite matrix \( \frac{1}{\sigma^2} g Q g^H \). Then it holds that
\[
\det(I_{\mathbf{N}_R} + \frac{1}{\sigma^2} g Q g^H) = \prod_{i=1}^{N_R} (1 + \lambda_i)
\]
and
\[
\det(I_{N_R} + (1 - \beta^2) \frac{1}{\sigma^2} g Q g^H) = \prod_{i=1}^{N_R} (1 + (1 - \beta^2) \lambda_i).
\]
This yields
\[
\det(M(\beta) \mathbf{O}) = \prod_{i=1}^{N_R} \frac{1 + (1 - \beta^2) \lambda_i}{1 + \lambda_i} = \prod_{i=1}^{N_R} \left( \frac{1 - \beta^2}{1 + \lambda_i} \right),
\]
such that
\[
\zeta(\beta) = \prod_{i=1}^{N_R} \left( 1 - \beta^2 \lambda_i \right)^{-1}, \quad 0 \leq \beta < \beta_0.
\]
Then, we have
\[
\zeta(\beta) \leq \frac{1}{(1 - \beta^2)^{N_R}}, \quad 0 \leq \beta < \beta_0.
\]
and hence
\[
\exp\left(-\frac{(\ln(2)) \beta}{N_R} \right) \leq \frac{1}{(1 - \beta^2)^{N_R}}, \quad 0 \leq \beta < \beta_0.
\]
Now if we put
\[
\beta = \frac{N_R}{\ln(2)} \left[ 1 + \left( 1 + \frac{(\ln(2))^2}{N_R^2} \right)^{1/2} \right],
\]

which follows that \( 0 < \beta < 1 \) and it holds that
\[
\exp\left(-\frac{(\ln(2)) \beta}{N_R} \right) = \exp\left[-1 + \left( 1 + \frac{(\ln(2))^2}{N_R^2} \right)^{1/2} \right]
\]
and that
\[
\frac{1}{1 - \beta^2} = \frac{1}{1 - \left( \frac{N_R}{\ln(2)} \right)^2 \left[ 1 - 2 \sqrt{1 + \left( \frac{(\ln(2))^2}{N_R^2} \right)^2} + 1 + \left( \frac{(\ln(2))^2}{N_R^2} \right)^2 \right]}
\]
\[
= \frac{1}{1 - \left( \frac{N_R}{\ln(2)} \right)^2 \left[ 1 - 2 \sqrt{1 + \left( \frac{(\ln(2))^2}{N_R^2} \right)^2} + 1 + \left( \frac{(\ln(2))^2}{N_R^2} \right)^2 \right]}
\]
\[
= \frac{\ln(2)^2}{N_R^2} \left[ 1 + \left( \frac{(\ln(2))^2}{N_R^2} \right)^2 \right]
\]
\[
= \left( 1 + \frac{1}{2} \left[ 1 - \left( 1 + \frac{(\ln(2))^2}{N_R^2} \right)^{1/2} \right] \right).
\]

This implies that
\[
(1 - \beta^2)^{-1} \exp\left(-\frac{(\ln(2)) \beta}{N_R} \right)
\]
\[
= \left( 1 + \frac{1}{2} \left[ 1 - \left( 1 + \frac{(\ln(2))^2}{N_R^2} \right)^{1/2} \right] \right)
\times \exp\left(-1 + \left( 1 + \frac{(\ln(2))^2}{N_R^2} \right)^{1/2} \right).
\]

Since \( (1 + \frac{1}{2} x) \exp(-x) \leq \exp(-\frac{x}{2}) \) for \( x \geq 0 \), we have
\[
\exp\left(-\frac{(\ln(2)) \beta}{N_R} \right) \zeta(\beta)
\]
whose first \( r \) diagonal elements are positive and where the remaining diagonal elements are equal to zero. Next, we let \( \mathbf{V}^* = S_{\mathbf{O}}^{*} \mathbf{A}^{*} \) and remove the \( N - r \) last columns of \( \mathbf{V}^* \), which are null vectors to obtain the matrix \( \mathbf{V} \). Then, it can be verified that \( \mathbf{O} = \mathbf{V} \mathbf{V}^H \). We can write \( \mathbf{X} = \mathbf{V} \mathbf{U}^* \) where \( \mathbf{U} \sim \mathcal{C}(0, \mathbf{I}_r) \). As a result:

\[
\mathbf{X}^H \mathbf{X} = (\mathbf{U}^*)^H \mathbf{V} \mathbf{V}^H \mathbf{U}^*.
\]

Let \( \mathbf{S} \) be a unitary matrix which diagonalizes \( \mathbf{V} \mathbf{V}^H \) such that \( \mathbf{S} \mathbf{H} \mathbf{V} \mathbf{V}^H \mathbf{S} = \text{Diag}(\mu_1, \ldots, \mu_r) \) with \( \mu_1, \ldots, \mu_r \) being the positive eigenvalues of \( \mathbf{O} = \mathbf{V} \mathbf{V}^H \) in decreasing order, as mentioned above. One defines \( \mathbf{U} = \mathbf{S} \mathbf{H} \mathbf{U}^* \). We have

\[
\text{cov}(\mathbf{U}) = \mathbf{S} \text{cov}(\mathbf{U}^* \mathbf{S}) = \mathbf{S} \mathbf{H} \mathbf{S} = \mathbf{I}_r.
\]

Therefore, it holds that \( \mathbf{U} = (U_1, \ldots, U_r)^T \sim \mathcal{C}(0, \mathbf{I}_r) \). Since \( \mathbf{S} \) is unitary, it follows that

\[
\mathbf{X}^H \mathbf{X} = ((\mathbf{S}^H)^{-1} \mathbf{U})^H \mathbf{V} \mathbf{V}^H \mathbf{S} \mathbf{H}^{-1} \mathbf{U} = \mathbf{U}^H \mathbf{S} \mathbf{H} \mathbf{V} \mathbf{S} \mathbf{H}^{-1} \mathbf{U} = \mathbf{U}^H \text{Diag}(\mu_1, \ldots, \mu_r) \mathbf{U} = \sum_{j=1}^{r} \mu_j |U_j|^2.
\]

Then, we have

\[
\mathbb{E} \left[ \exp(\beta \|\mathbf{X}\|^2) \right] = \mathbb{E} \left[ \prod_{j=1}^{r} \exp \left( \frac{1}{2} \beta \mu_j |U_j|^2 \right) \right] = \prod_{j=1}^{r} \mathbb{E} \left[ \exp \left( \frac{1}{2} \beta \mu_j |U_j|^2 \right) \right] = \prod_{j=1}^{N} \left( 1 - \beta \mu_j \right)^{-1},
\]

where we used that all the \( U_j \)'s are independent, that \( \forall j \in \{1, \ldots, r\} \), \( 2|U_j|^2 \) is chi-square distributed with \( k = 2 \) degrees of freedom and with moment generating function equal to

\[
\mathbb{E} \left[ \exp(2t|U_j|^2) \right] = (1 - 2t)^{-k/2} \text{ for } t < \frac{1}{2} \text{ and that } \forall j \in \{1, \ldots, r\} \text{ and for } \beta < \beta_0, \frac{1}{2} \mu_j < \frac{1}{2}. \]

This completes the proof of (63).

Now, it holds that

\[
\prod_{i=1}^{N} (1 - \beta \mu_i) \geq 1 - \beta (\mu_1 + \ldots + \mu_N) \geq 1 - \beta M.
\]

This yields

\[
\exp(-\mu_1) \mathbb{E} \left[ \exp(\beta \|\mathbf{X}\|^2) \right] \leq \frac{\exp(-M + \beta \mu_1)}{1 - \beta M},
\]

where \( 0 < \beta < \frac{1}{\mathbb{M}} = \beta_0 \). Putting \( \beta = \frac{\delta}{\mathbb{M}(\beta + \mathbb{M})} < \frac{1}{\mathbb{M}} \) yields

\[
\exp(-M + \delta) \mathbb{E} \left[ \exp(\beta \|\mathbf{X}\|^2) \right] \leq (1 + \frac{\delta \mu_1}{2}) \exp(-\delta \mathbb{M}).
\]
which combined with (62) proves the lemma. □

Lemma 18: Let \( \epsilon > 0 \) be fixed arbitrarily. Let \( S \subset \mathbb{C}^{N_X \times N_T} \) an arbitrary compact set. Then, there exists a non-singular \( Q \in \mathcal{Q}_P \) such that

1) \( \text{tr}(Q) < P \)
2) \( \log \det(I_{N_R} + gQg^H) \geq \max_{Q \in \mathcal{Q}_P} \min_{g \in S} \log \det(I_{N_R} + \frac{1}{\sigma^2}gQg^H) - \epsilon \) \( \forall g \in S \).

Proof: Notice first that the set \( \mathcal{L} = S \times \mathcal{Q}_P \) is a compact set, because the conditions on the matrices \( g \in S \) and on the positive semi-definite matrices \( Q \in \mathcal{Q}_P \) guarantee that \( \mathcal{L} \) is bounded and closed in \( \mathbb{C}^{N_R \times N_T} \times \mathbb{C}^{N_T \times N_T} \). Now the function \( f(g, Q) = \log \det(I_{N_R} + \frac{1}{\sigma^2}gQg^H) \) is uniformly continuous on \( \mathcal{L} \). One can find a non-singular \( Q_0 \in \mathcal{Q}_P \) such that

\[
\log \det(I_{N_R} + \frac{1}{\sigma^2}gQ_0g^H) \geq \max_{Q \in \mathcal{Q}_P} \min_{g \in S} \log \det(I_{N_R} + \frac{1}{\sigma^2}gQg^H) - \epsilon \ \forall g \in S.
\]

If \( \text{tr}(Q_0) < P \), the proof is complete. If \( \text{tr}(Q_0) = P \), we can find, by the uniform continuity of \( f \) on \( \mathcal{L} \), a number \( \delta > 0 \) such that \( |f(g, Q) - f(g, Q_0)| \leq \frac{\epsilon}{2} \) for all \( g \) if \( \|Q - Q_0\| \leq \delta \).

We can then change \( Q_0 \) into a non-singular \( Q_1 \) in such a way that \( \|Q_1 - Q_0\| \leq \delta \) and \( \text{tr}(Q_1) < \text{tr}(Q_0) = P \). \( Q_1 \) satisfies the conditions of the lemma. This completes the proof of the lemma. □

2) Proof of Theorem 5: Proof: Now that we have proved the lemma, we fix \( R \) to be any positive number strictly less than \( \min_{Q \in \mathcal{Q}_P} \log \det(I_{N_R} + \frac{1}{\sigma^2}gQg^H) \) and put

\[
2\theta = \min_{Q \in \mathcal{Q}_P} \log \det(I_{N_R} + \frac{1}{\sigma^2}gQg^H) - R.
\]

By Lemma 18, one can find a non-singular \( Q_1 \in \mathcal{Q}_P \) such that \( \text{tr}(Q_1) = P - \beta, \beta > 0 \)

\[
\mathbb{E}[\|g(T, Z)^n\|_2] = \log \det(I_{N_R} + \frac{1}{\sigma^2}gQ_1g^H) \geq R + \frac{\theta}{2} \ \forall g \in \mathcal{G},
\]

where \( T \) and \( Z \) represent the random input and output of \( W \), respectively, and where \( T \sim \mathcal{N}(0_{N_R}, Q_1) \). Let \( E_n \) be the set of all input sequences \( t^n \) satisfying \( \frac{1}{n} \sum_{i=1}^n \|t_i\|^2 \leq P \). For any \( g \in \mathcal{G} \), we define \( T_i, i = 1, \ldots, n \), to be the i.i.d. random inputs of \( W \), each normally distributed with mean \( 0_{N_T} \) and covariance matrix \( Q_1 \). Let \( \hat{P} = P - \beta \) and \( \hat{\beta} = \frac{P - \beta}{\ln(2)} - \log(1 + \frac{\beta}{P}) > 0 \). Then, by Lemma 17, it holds that

\[
\mathbb{P} \left[ \sum_{i=1}^n \|T_i\|^2 \geq n(\hat{P} + \hat{\beta}) \right] \leq \left( 1 + \frac{\beta}{P} \right)^2 \left( \frac{\beta}{\ln(2)} \right)^n \leq 2^{n\beta/\ln(2)} \leq 2^{n}\beta.
\]

As a result, we have

\[
\mathbb{P} \left[ T^n \notin E_n \right] = \mathbb{P} \left[ \sum_{i=1}^n \|T_i\|^2 > nP \right] \leq \mathbb{P} \left[ \sum_{i=1}^n \|T_i\|^2 \geq n(\hat{P} + \beta) \right]
\]

\[
\leq 2^{n\beta}.
\]

Now define \( \tau = (2R)^n \), \( \alpha = n(R + \frac{\theta}{4}) \) and \( \delta = \frac{n\theta}{P} \). It follows from Lemma 15 that there exists a code \( \Gamma_n \) for \( C' \) with size \( |\Gamma_n| = \tau \) and block-length \( n \) such that for all \( g \in \mathcal{G} \)

\[
e_{c}(\Gamma_n, g) \leq |\mathcal{G}|^{2nR^2} - n(R + \frac{\theta}{4}) + |\mathcal{G}|^{2} - n\beta + |\mathcal{G}|^{2} - n\beta
\]

\[
+ \sum_{g \in \mathcal{G}} \mathbb{P} \left[ i_g(T^n, Z^n) \leq n(R + \frac{\theta}{4}) \right].
\]
where we used that
\[
\|z_i - gt_i\|^2 - \|\hat{g}t_i\|^2 \leq \frac{1}{\sigma^2} \left[ \|z_i - gt_i\|^2 - \|\hat{g}t_i\|^2 \right]
\]
\[
\leq \frac{1}{\sigma^2} \left[ \|z_i - gt_i\|^2 - \|\hat{g}t_i\|^2 \right]
\]
follows because we require that
\[
\leq \frac{1}{\sigma^2} \left[ \|z_i - gt_i\|^2 - \|\hat{g}t_i\|^2 \right]
\]
where we used Lemma 17 in the last step. This completes the proof of the lemma. □

2) Direct Proof of Theorem 4: Now that we proved the lemmas, we fix \(R\) to be any positive number strictly less than \(\min \log \det(I_{N_R} + \frac{1}{\sigma^2} gQg^H)\) and put \(\beta = \max \log \det(I_{N_R} + \frac{1}{\sigma^2} gQg^H) - R\). By Lemma 18, one can find a non-singular \(Q_1 \in Q_P\) such that \(\text{tr}(Q_1) = P - \beta, \beta > 0,\) and

where \(T\) and \(Z\) being the random input and output of \(W_g\), respectively, where \(T \sim N_C(0, N_{Q1})\). We now pick a finite subset \(G'\) of \(G\) such that for every \(g \in G\), there is a \(\hat{g} \in G'\) satisfying \(\|g - \hat{g}\| \leq \nu\). This can be done because the set \(G\) is compact. By inequality (66) and since

\[
\max \min \log \det(I_{N_R} + \frac{1}{\sigma^2} gQg^H) \geq \max \min \log \det(I_{N_R} + \frac{1}{\sigma^2} gQg^H),
\]

it follows that

The calculations of Theorem 5 imply that there exists a code \(C'\) with block-length \(n\), size \(\|\Gamma_n\| = [2^{nR}]\) such
that the codewords \( \mathbf{t}^n = (t_1, \ldots, t_n) \) satisfy \( \frac{1}{n} \sum_{i=1}^{n} \| t_i \|^2 \leq P \) and such that for all \( \hat{g} \in \mathcal{G}' \)

\[
e_{e}(\Gamma_n, \hat{g}) \leq (|G'| + |G'|^2)2^{-\frac{n}{2} \beta + |G'|^2 - n \beta} + |G'|^2 \left(1 + \frac{1}{2nR(2R)} \right)^{\frac{1}{2} - 1}, \tag{67}
\]
where \( \hat{\beta} = \frac{\beta}{2 \ln(2)} - \log(1 + \frac{1}{2 \beta}) \) and where \( \beta \) is independent of \( n \).

We now consider the use of codewords and decoding sets belonging to the code \( \Gamma_n \) for \( \mathcal{C} \) with the larger compound channel \( \mathcal{C} \). Let \( g \in \mathcal{G} \) and \( \hat{g} \in \mathcal{G}' \) such that \( |g - \hat{g}| \leq \nu. \) Let \( \mathbf{t}^n \) be any codeword of \( \Gamma_n \) and \( B \) the corresponding decoding set. Let \( F = \{ z^n = (z_1, \ldots, z_n) : \frac{1}{n} \sum_{i=1}^{n} \| z_i \|^2 \leq \rho \} \), where \( \rho = 2a^2 + 2nR^2 + 2 \). Then

\[
W_g(B^c|F)^n = W_g((B^c \cap F) \cup (B^c \cap F^c)|t^n) \\ &\leq W_g(B^c|F|t^n) + W_g(F^c|t^n).
\]

By Lemma 20, it holds that

\[
W_g(F^n|t^n) \leq \left[ 1 + \frac{1}{N_R \sigma^2} \right] 2^{- \frac{1}{2} \ln(2) N_R \sigma^2} \log \left( 1 + \frac{1}{N_R \sigma^2} \right)^n
\]

By Lemma 19, it holds that

\[
W_g(B^c \cap F|t^n) \leq 2^{\frac{2n}{2} \ln(2) \sqrt{P_\rho + aP}} W_g(B^c|F|t^n).
\]

Now

\[
W_g(B^c \cap F|t^n) \leq W_g(B^c|t^n) \leq e_{e}(\Gamma_n, \hat{g}).
\]

This implies using (67) that for all \( g \in \mathcal{G} \)

\[
e_{e}(\Gamma_n, g) \leq 2^{-n \left( \frac{1}{2} \ln(2) \theta^2 \right)} \left( 1 + \frac{1}{2nR} \right)^{\frac{1}{2} - 1} + (|G'| + |G'|^2)2^{-n \left( \frac{1}{2} \ln(2) \theta^2 \right) \sqrt{P_\rho + aP}} \log \left( 1 + \frac{1}{2nR} \right)^n + |G'|^2 - n \beta - \log(1 + \frac{1}{2nR}) \log \left( 1 + \frac{1}{2nR} \right)
\]

where

\[
c_1 = \frac{N_R}{2 \ln(2)} \left[ 1 + \frac{1}{2nR} \right]^{\frac{1}{2} - 1}
\]
and

\[
c_2 = \frac{2}{2nR \sigma^2} \left[ \sqrt{P_\rho + aP} \right].
\]

The exponentials in (68) are all of the form \( 2^{-n(K_1 - K_2\nu)} \) where \( K_1 \) and \( K_2 \) do not depend on \( n \) and where \( K_1 \) is positive and \( K_2 \) is non-negative. For \( \nu \) sufficiently small, it holds that \( K_1 - K_2\nu > 0 \), which yields \( \lim_n e_{e}(\Gamma_n, g) = 0 \).

This proves that \( \max_{Q \in \mathcal{Q}_p, g \in \mathcal{G}} \min_{Q \in \mathcal{Q}_p, g \in \mathcal{G}} \log |\Gamma_n| \geq R - \delta \) is an achievable rate for \( \mathcal{C} \). This completes the direct proof of Theorem 4.

### C. Converse Proof of Theorem 4

Let \( R \) be any achievable rate for \( C = \{ W_g : g \in \mathcal{G} \} \). So, for every \( \theta, \delta > 0 \), there exists a code sequence \( (\Gamma_n)_{n=1}^{\infty} \) such that for all \( g \in \mathcal{G} \)

\[
\frac{\log |\Gamma_n|}{n} \geq R - \delta
\]

and

\[
e_{e}(\Gamma_n, g) \leq \theta,
\]

for sufficiently large \( n \). Notice that from (69), it follows that the average error probability is also bounded from above by \( \theta \). The uniformly-distributed message is modeled by \( W \) and the random decoded message is modeled by \( W \). The set of messages is denoted by \( W \). For any \( g \in \mathcal{G} \), the uniformly-distributed message \( W \) is mapped to the random input sequence of the channel \( W_g \), denoted by \( T^n = (T_1, \ldots, T_n) \). The corresponding random output sequence is denoted by \( Z^n = (Z_1, \ldots, Z_n) \). The covariance matrix of each input \( T_i \) is denoted by \( Q_i \). We define \( Q^* \) such that \( Q^* = \frac{1}{n} \sum_{i=1}^{n} Q_i \).

By using \( \Gamma_n \) as a transmission-code for \( \mathcal{C} \), it follows that

\[
\Pr \left[ W \neq \hat{W} \right] \leq \theta.
\]

We have

\[
H(W) = \log |W| \\ = \log |\Gamma_n| \\ \geq n(R - \delta).
\]

On the one hand, as shown in (25), we obtain by applying Fano’s inequality

\[
H(W) \leq \frac{1 + I(W; \hat{W})}{1 - \theta}.
\]

On the other hand, as shown in (26), it holds that

\[
\frac{1}{n} I(W; \hat{W}) \leq \log \det(I_{NR} + \frac{1}{\sigma^2} gQ^*H).
\]

As a result, it follows from (70), (71) and (72) that for every \( g \in \mathcal{G} \)

\[
n(R - \delta) \leq \frac{n \log \det(I_{NR} + \frac{1}{\sigma^2} gQ^*H) + 1}{1 - \theta}.
\]

Hence,

\[
n(R - \delta) \leq \frac{n \min_{Q \in \mathcal{Q}_p, g \in \mathcal{G}} \log \det(I_{NR} + \frac{1}{\sigma^2} gQ^*H) + 1}{1 - \theta}.
\]

Since \( Q^* \in \mathcal{Q}_p \) (see Lemma 5), it follows that

\[
n(R - \delta) \leq \frac{n \max_{Q \in \mathcal{Q}_p, g \in \mathcal{G}} \log \det(I_{NR} + \frac{1}{\sigma^2} gQ^*H) + 1}{1 - \theta}.
\]

This implies that for sufficiently large \( n \) and for every \( \theta, \delta > 0 \), we have

\[
R \leq \inf_{\delta, \theta > 0} \lim_{n \to \infty} \frac{n \max_{Q \in \mathcal{Q}_p, g \in \mathcal{G}} \log \det(I_{NR} + \frac{1}{\sigma^2} gQ^*H) + 1}{1 - \theta} + \delta.
\]

It follows that

\[
R \leq \inf_{\delta, \theta > 0} \lim_{n \to \infty} \frac{n \max_{Q \in \mathcal{Q}_p, g \in \mathcal{G}} \log \det(I_{NR} + \frac{1}{\sigma^2} gQ^*H) + 1}{1 - \theta}.
\]
This completes the converse proof of Theorem 4.

VII. CONCLUSION

In this paper, we considered the problem of message transmission and the problem of CR generation over point-to-point MIMO slow fading channels. The first goal of this paper was to derive a lower and an upper bound on the outage transmission capacity of single-user MIMO slow fading channels with average input power constraint, AWGN and with arbitrary state distribution under the assumption of CSIR and to show that our bounds coincide except possibly at points of discontinuity of the outage transmission capacity, of which there are, at most, countably many. Such discontinuity issues might occur because the channel state distribution is arbitrary. The second goal was to establish a lower and an upper bound on the outage CR capacity of a two-source model with unidirectional communication over the MIMO slow fading channel with AWGN and with arbitrary state distribution using our bounds on the outage transmission capacity of the MIMO slow fading channel. The obtained results are particularly relevant in the problem of correlation-assisted identification over MIMO slow fading channels, where Alice and Bob have access to a correlated source. This is an extension to the work done in [37], where the focus is on deterministic identification over fading channels. As a future work, it would be interesting to study the problem of CR generation in fast fading environments, where the channel state varies over the time scale of transmission.

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