Influence of dark matter on black hole scalar hair

Bartłomiej Kiczek and Marek Rogatko
Institute of Physics, Maria Curie-Sklodowska University,
20-031 Lublin, pl. Marii Curie-Sklodowskiej 1, Poland
(Dated: April 15, 2020)

Searches for dark matter sector field imprints on the astrophysical phenomena are one of the most active branches of the current researches. Using numerical methods we elaborate the influence of dark matter on the emergence of black hole hair and formation of boson stars. We explore thermodynamics of different states of the system in Einstein-Maxwell-scalar dark matter theory with box boundary conditions. Finally we find that the presence of dark sector within the system diminishes a chance of formation of scalar hair around a black hole.

I. INTRODUCTION

The astrophysical evidence of the illusive ingredient of our Universe, dark matter, is overwhelming and authorizes the galaxy rotation curves, gravitational lensing, a thread-like structure (cosmic web) on which ordinary matter accumulates [1, 2]. On the contrary, the absence of the evidence of the most popular particle candidates for baryonic dark matter stipulates the necessity of diversifying experimental efforts [3]. Black holes and ultracompact horizonless objects being the ideal laboratories for dark matter studies, may help us to answer the tantalizing question of how dark matter sector leaves its imprint in the physics of these objects. However, it happens that Schwarzschild black hole has a negative specific heat and it cannot be in equilibrium with thermal radiation. To overcome this difficulty the idea of enclosing the black hole within a box was proposed [4, 5]. Einstein-Maxwell systems with box boundary conditions were elaborated in [6], where it was established that the phase structure of the models were similar to AdS gravity. Inclusion of the additional scalar field to the theory in question, envisages the correspondence of phase transitions in gravity in a box with s-wave holographic superconductor [7, 9]. The thermodynamical studies of Einstein-Maxwell scalar systems in the asymptotically flat spacetime with reflecting boundary conditions were conducted in [10]. A certain range of parameters allows to obtain stable black hole solution, giving a way to circumvent no-hair theorem.

The next compact objects studied in our paper, from the point of view of the influence of dark matter on their physics, are boson stars. Boson stars being a self-gravitating solution of massive scalar field with a potential coupled to gauge fields and gravity [11] are widely studied in literature [12–10], for a quite long period of time.

The purpose of our paper is to examine thermodynamical properties and stability of the black holes and horizonless objects-boson stars in Einstein-Maxwell-scalar system influenced by dark matter sector and envisage the role of the dark matter in the elaborated problems.

II. MODEL

The organization of the paper is as follows. In Sec. II we describe the basic features of the hidden sector model and derived the basic equations needed in what follows. Sec. III is devoted to the description of the obtained numerical results. In Sec. IV we concluded our researches.

We consider the spacetime manifold with time-like boundary ∂M, which will be referred as a box. The action for Einstein-Maxwell-scalar dark matter gravity is provided by

\[ S = \int d^4x\sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\alpha}{4}B_{\mu\nu}F^{\mu\nu}\right) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - |D\Psi|^2 - m^2|\Psi|^2 - \int_{\partial M} d^3x\sqrt{-\gamma}\mathcal{K}, \]

where \( F_{\mu\nu} \) is a Maxwell field strength tensor, \( B_{\mu\nu} \) is a strength tensor of a hidden sector vector boson. The complex scalar field \( \Psi = \psi e^{i\theta} \), where \( \theta \) denotes the phase, is coupled only to the ordinary electromagnetic field by the covariant derivative \( D_\mu = \nabla_\mu - iqA_\mu \). The theoretical justifications of the model in question originate from M/string theories, where such mixing portals coupling Maxwell and auxiliary gauge fields can be encountered [17]. The hidden sectors states are charged under their own groups and interact with the visible sector via gravitational interactions. The realistic string compactifications establish the range of values for \( \alpha \) between \( 10^{-2} \) to \( 10^{-10} \) [18–21]. It seems that astrophysical observations of gamma rays of energy 511 keV [22], positron excess in galaxies [23], and muon anomalous magnetic moment [24], argue for the aforementioned idea of coupling Maxwell field with dark matter sector. Recent experiments aimed at gamma rays emissions from dwarf galaxies [25], dilaton-like coupling to photons caused by ultra-light dark matter [26], oscillations of the fine structure constant [27], revisions of the constraints on dark photon 1987A supernova emission [28], measurements of excitation of electrons in CCD-like detector [29], as well as, the examinations in \( e^+e^- \) Earth colliders [30], give us some hints for the correctness of the proposed model. They and the future planned balloon d’essai will amelio-
rate the mass constraints on the hidden sector particles, especially for dark photons.

The second integral denotes the Gibbons-Hawking boundary term of our box with $\gamma$ metric on the three-dimensional hypersurface $(r = r_h)$, with the extrinsic curvature $\mathcal{K}$.

Varying the action \([1]\) we get the equations of motion of the forms

\[
\left( \nabla^\mu - iqA_\mu \right) \left( \nabla^\nu - iqA^\nu \right) \Psi - m^2 \Psi = 0, \quad \tag{2}
\]

\[
\bar{\alpha} \nabla^\mu F^{\mu\nu} = j^\nu, \quad \tag{3}
\]

where $\bar{\alpha} = 1 - \frac{4\pi}{3}$ and the current $j^\nu$ is provided by the relation

\[
j^\nu = iq \left[ \Psi^\dagger \left( \nabla^\nu - iqA^\nu \right) \Psi - \Psi \left( \nabla^\nu + iqA^\nu \right) \Psi^\dagger \right]. \quad \tag{4}
\]

In what follows we use a time independent spherically symmetric line element, with the metric coefficients being functions of $r$-coordinate

\[
ds^2 = -g(r)h(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad \tag{5}
\]

and the adequate components of the fields in the theory will constitute radial functions of the forms

\[
A_\mu dx^\mu = \phi(r)dt, \quad B_\mu dx^\mu = \chi(r)dt, \quad \Psi = \Psi(r). \quad \tag{6}
\]

In general the scalar field can have harmonic time dependence which can be absorbed by a redefinition of the gauge field function. Having this in mind it can be seen that the $r$-component of the equations of motion for the gauge and scalar fields leads the conclusion that $\Psi(r) = \psi(r)$. By virtue of this, the following equations of motion are provided:

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}, \quad \tag{7}
\]

\[
\nabla_\mu \nabla^\mu \psi - q^2 A_\mu A^\mu \psi - m^2 \psi = 0, \quad \tag{8}
\]

\[
\nabla_\mu F^{\mu\nu} + \frac{\alpha}{2} \nabla_\nu B^{\mu\nu} - 2q^2 A^\nu \psi^2 = 0, \quad \tag{9}
\]

\[
\nabla_\mu B^{\mu\nu} + \frac{\alpha}{2} \nabla^\mu F_{\mu\nu} = 0. \quad \tag{10}
\]

As in the case of the equation \([3]\), the last two equations can be rewritten as

\[
\bar{\alpha} \nabla_\mu F^{\mu\nu} - 2q^2 A^\nu \psi^2 = 0, \quad \tag{11}
\]

\[
\nabla_\mu B^{\mu\nu} + \frac{\alpha}{2} q^2 A^\nu \psi^2 = 0. \quad \tag{12}
\]

Consequently, the explicit forms of the equations of motion yield

\[
h' - rh \psi'^2 - \frac{q^2 r \phi^2 \psi^2}{g^2} = 0, \quad \tag{13}
\]

\[
g' + g \left( \frac{1}{r} + \frac{1}{2}r \phi'^2 \right) + \frac{q^2 r \phi^2 \psi^2}{2gh} - \frac{1}{r} \psi' = 0, \quad \tag{14}
\]

\[
\psi'' + \left( \frac{2}{r} - \frac{\hbar'}{2h} \right) \psi' + \frac{2q^2 \phi^2 \psi^2}{\alpha g} = 0, \quad \tag{15}
\]

\[
\psi'' + \left( \frac{2}{r} + \frac{\hbar'}{2h} + \frac{g}{g} \right) \psi' + \left( \frac{q^2 \phi^2}{g\hbar} - m^2 \right) \frac{\psi^3}{g} = 0, \quad \tag{16}
\]

\[
\chi'' + \left( \frac{2}{r} - \frac{\hbar'}{2h} \right) \chi' + \frac{\alpha q^3 \lambda^2 \psi^2}{\alpha g} = 0. \quad \tag{17}
\]

To solve the equations of the theory in question one has to provide adequate boundary conditions. Namely we can pick either a horizonless or a black hole solution. In case of a black hole we expand the underlying functions in a Taylor series around the horizon of radius $r_h$

\[
\psi = \psi_0 + \psi_1(r - r_h) + \psi_2(r - r_h)^2 + \mathcal{O}(r^3), \quad \tag{18}
\]

\[
\phi = \phi_1(r - r_h) + \phi_2(r - r_h)^2 + \mathcal{O}(r^3), \quad \tag{19}
\]

\[
g = g_1(r - r_h) + g_2(r - r_h)^2 + \mathcal{O}(r^3), \quad \tag{20}
\]

\[
h = 1 + h_1(r - r_h) + \mathcal{O}(r^2), \quad \tag{21}
\]

\[
\chi = \chi_1(r - r_h) + \chi_2(r - r_h)^2 + \mathcal{O}(r^3). \quad \tag{22}
\]

We set $g_0 = 0$, due to occurrence of the black hole event horizon. For the regularity of the $U(1)$-gauge fields on the event horizon, one also puts $\psi_0$ and $\chi_0$ equal to zero (in order to keep the terms with division by $g(r_h)$ in equations of motion finite). By implementing the expansions \([18]-[22]\) into the equations of motion, we find out that \{r_h, \psi_0, \phi_1, \chi_1, \alpha\} comprise free parameters of the theory in question, while the remaining ones can be expressed by them.

As far as the boson star scenario is concerned, we perform a similar expansion. However since the configuration in question is horizonless, the expansion accomplishes around the origin of the reference frame. At $r = 0$ we require that the derivatives of all the functions are set equal to zero, which ensures that there is no kink at this point. At $r = r_h$, we establish the Dirichlet boundary condition for the scalar field $\psi(r_h) = 0$ (the reflecting mirror-like boundary conditions).

Asymptotic analysis of matter fields, at the box boundary, enables us to write

\[
\psi \sim \psi^{(0)} + \psi^{(1)}(r_h - r) + \mathcal{O}(r^2), \quad \tag{23}
\]

\[
\phi \sim \phi^{(0)} + \phi^{(1)}(r_h - r) + \mathcal{O}(r^2), \quad \tag{24}
\]

\[
\chi \sim \chi^{(0)} + \chi^{(1)}(r_h - r) + \mathcal{O}(r^2). \quad \tag{25}
\]

As was proposed in Refs. \([10, 12]\), because of the fact that the scalar field satisfies the reflecting mirror-like boundary conditions $\psi(r_h) = 0$, one can fix $\psi^{(0)} = 0$ and the other parameter $\psi^{(1)}$ can be used for the phase transition description. This approach to the problem in
question is widely exploited in holographic studies of superconductors and superfluids.

For the gauge fields one has that $\phi^{(0)} = \mu$ and $\chi^{(0)} = \mu_d$ as chemical potentials for visible and hidden sector fields, treating the system as a grand canonical ensemble. In order to conduct the thermodynamical analysis we calculate the free energy of the system, to see which phase is thermodynamically preferable, for a fixed temperature. In the case of a hairless solution we take into account the classical formula $F = E - TS - \mu Q - \mu_d Q_d$, where $E$ is Brown-York quasilocal energy [4, 5]. Nevertheless this approach may cause problems in hairy solution analysis, being insufficient to capture the mass of the scalar field. Therefore we treat this problem evaluating the on-shell action in Euclidean signature $F = TS_{d}$, which enables to take into considerations the non-trivial profile of scalar field constituting the solution of the underlying system of differential equations.

Solution of the equations (13)-(17) with $\psi = 0$ can be achieved analytically, giving the Reissner-Nordstrom (RN) dark matter black object [31]. To proceed further and accomplish the complete numerical analysis of the underlying equations, we implement the shooting method, integrating the aforementioned relations from $r_h$ to $r_b$, using the fourth order Runge-Kutta method. From the set of free parameters we fix the scalar magnitude on the event horizon $\psi_0$ and pick $r_h$, $\phi_1$ and $\chi_1$ to be shooting parameters. Moreover we impose values on both chemical potentials, that serve as constrains in our shooting procedure for $\phi_1$ and $\chi_1$. We set a domain of shooting parameters from the series expansion of the solutions near the horizon, then by using the iterative bi-sections one finds a solution that meets constrains, with a desired tolerance. Therefore parameters $\{\mu_d, \alpha\}$, which are controlling respectively amount of dark charge and the coupling strength remain free, thus they can be varied to see their impact on the system in question. For the convenience let us refer to the parameter $\psi^{(1)}$, as a condensation, which serves as a handy analogy to holographic theory. As mentioned above, in our numerical scheme we treat $\mu_d$ as an input parameter in our code, however one might not be interested in expressing these relations in a language of chemical potentials. Therefore one might compute the total dark charge of the system

$$Q_d = \lim_{r\rightarrow r_0} \frac{1}{4\pi} \int_{S^2} B_{\mu\nu} t^\mu n^\nu \sqrt{-g} d^2 \theta,$$

where $t_\mu$ is a unit time-like vector and $n_\mu$ is a normal vector to the boundary. In the similar manner we compute electrical charge, for $F_{\mu\nu}$.

### III. RESULTS

We commence with the hairy black hole solution (HBH), i.e., a system with an event horizon and non-trivial scalar field profile. The parameter space of HBH can be illustrated on a plane of chemical potential and Hawking temperature ($\mu$-$T$) as a triangular shape. That region is bound between boson star phase from the left-hand side and generalized Reissner-Nordstrom (RN) solution from the right-hand side. A schematic phase diagram has been presented in the Fig. 1 where both mentioned lines are marked. Moreover the influence of the dark sector on phase boundaries is visualized by arrows, showing the trend of the flow by increasing the hidden sector chemical potential.

![FIG. 1. A scheme of phase diagram of the described system. Blue-yellow line indicates the border between boson star and hairy black hole parameter space, while the red line depicts hairy BH – generalized RN BH phase boundary. The arrows on the scheme show us the flow of phase boundaries driven by the chemical potential of dark matter. Lines have been split and labeled from A to C, with a point D being the center of rotation of left-hand side boundary.](image)

The hairy configuration can be achieved for a specific value of the chemical potential. Below the value $\mu_{RN}$ scalar can not condensate and we get RN-dark matter black hole. On the other hand, for the value greater than the critical one, $\mu_c$, the system becomes unstable. By stable hairy solution we mean a constrained solution of the equations of motion (13)-(17) that fulfils the boundary conditions with desired tolerance and its free energy is lower than the free energy of RN and BS, making it the ground state of the system. We can define $\mu_c$ as the chemical potential for which the phase transition driven by temperature is no longer of second order and the condensate collapses. In the range between $\mu_{RN}$ and $\mu_c$, we contend a typical second order phase transition, depicted in Fig. 2. In the vicinity of critical temperature the condensation can be described by a function $\psi^{(1)} \sim (T - T_c)^{1/2}$. It should also be noted that establishing a HBH solution requires relatively large value of the scalar charge. In our calculation we used $q = 100$ and a small mass of $m = 10^{-6}$.
FIG. 2. Condensation $\psi^{(1)}$ as a function of temperature, for the different values of $\mu_d$ and $\alpha = 10^{-3}$. For $\mu = 0.1$ a typical second order phase transition takes place, the dark matter presence influences the transition point and the condensation.

FIG. 3. Double valued profiles of the condensation as functions of Hawking temperature caused by increasing amount of dark matter in the system with $\alpha = 10^{-3}$ and $\mu = 0.14$. While the first transition for $\mu_d = 0.08$ might still be considered as regular the another strictly not – the value of condensation becomes double valued for some range of temperatures. Moreover these solutions obey the boundary conditions but their free energy is larger than both BS’s and RN’s, therefore they cannot be considered as thermodynamically preferred.

FIG. 4. Relative change of the critical temperature of hairy black hole – generalized RN black hole as a function of a ratio of chemical potentials. The temperature ratio has been normalized to the critical temperature of dark matter free solution, where $\mu_d = 0$.

Let us now discuss physical mechanisms behind the phase boundaries flow from the Fig. 1. When we cross the line of the critical chemical potential value $\mu_c$, one encounters the exotic phase, where for one value of temperature we have two values of the condensation parameter $\psi^{(1)}$. Moreover by evaluating its free energy we can find it is so high that the hairy state is no longer stable – our numerical method finds constrained solutions, but due to free energy leap, they are not thermodynamically preferred. The exotic phase effect occurs in case when scalar mass is close to or equal zero, for a mass away from this limit we do not obtain that phase. Instead we have a sharp crossing, from stable solutions below $\mu_c$ to the situation when the equations of motion do not provide solutions with condensed scalar above the $\mu_c$ threshold at all. It is worth mentioning that a similar condensation–temperature profile has been shown in the so-called vector p-wave holographic superfluids [32], but a first order phase transition was hidden behind it. However, it was revealed that for the real value of the vector field, the model in question gave us the same description as holographic s-wave model with dark matter sector [33]–[34].

Dark matter gauge field plays an interesting role in this transition, as it accelerates the appearance, let us say, the exotic phase. For a larger value of dark sector chemical potential, $\psi^{(1)}$ becomes double valued for the lower chemical potential, which is depicted in the Fig. 3 where $\mu_d$ has gradually increasing value. Moreover when system enters the exotic phase its free energy rises repeatedly and exceeds the free energy of RN black hole, so the hairy phase is no longer a preferred option. In this way, that effect restricts the range of chemical potential where the second order phase transition may occur, $\mu_c$ becomes a descending function of the dark charge (see curve (A) in the Fig. 1). However it cannot be increased without a limit. For every value of the electric charge there exists a certain limit of dark charge, below which a formation of scalar hair is possible. Above it, such condensation cannot take place and no stable solutions are found. This phenomenon adds up both gravitational influence of the charge on the metric and non-gravitational coupling between both gauge fields.

Now let us draw our attention to the HBH-RN BH (B) border. An interesting effect that hidden sector exerts on
the hairy black hole system is the shifting of the critical temperature of the phase transition. The larger growth of dark matter charge (and also $\mu_d$) we observe, the lower value of the transition temperature one achieves. Such effect has been depicted in the Fig. 3 where the critical temperature ratio described by the relation

$$\delta T_c = \frac{T_c(\mu_d)}{T_c(0)},$$

is shown as a function of the chemical potential of hidden sector normalized to the visible sector chemical potential. One can notice that the shift of the critical temperature is proportional to the square of $\mu_d$. Obviously it can not decrease as low as one wishes and a certain limit exist which has been discussed in analytic solution of dark matter charged RN-like black hole \[31\]. The descent of the critical temperature becomes steeper for larger values of the chemical potential. The points in question constitute the so-called isosbestic ones \[35\], where the curves dependent on temperature on condensation–Hawking temperature dependence present in Fig. 2. All the curves certainly cross each other in one point, located at $\psi^{(1)} \approx 0.345$ and $T \approx 0.3325$. This interesting phenomenon shows that while the dark sector may modify the phase structure of the system and has an imprint on its critical quantities, there exists a specific configuration of the system, that remains completely untouched.

To proceed further, we shed some light on the influence of dark matter sector on the black hole-boson star phase transition in the stable area of small values of the chemical potential (line (C) in Fig. 1). This process is depicted in Fig. 5 which asserts a phase diagram at the boundary between hairy black hole and boson star. While the boson star is a horizonless object its Hawking temperature remains undefined. However it is possible to calculate its characteristic – condensation and free energy as a function of chemical potential. Then to obtain the phase boundary curve, we start in the hairy black hole regime, then one moves towards lower Hawking temperatures and study the value of the free energy of the hairy black hole on the way. When it exceeds the free energy of a boson star, for the corresponding value of the chemical potential, the transition point is found. Both phases of the system are influenced by the hidden sector, nonetheless the free energy of a boson star is affected much less than black hole’s. Dark sector causes a significant drop of free energy of a hairy black hole. It means that the stability of a hairy solution is preserved for lower temperatures given the presence of the dark matter in the system. Such effect causes that hairy black hole solution is thermodynamically preferable for the lower Hawking temperatures and limits the emergence of boson star. The presence of $\alpha$-coupling constant slightly diminishes the space of parameters for which boson star can emerge.

At last it is sensible to mention some points that seem to be dark sector resistant. One of them appears on the phase boundary, labeled with (D) on the phase diagram scheme in Fig. 1. This point or rather its neighborhood does not seem to be susceptible on the dark charge presence in the system. For a particular numerical example, like in the Fig. 5, it is placed around $\mu \approx 0.1160942$ and $T \approx 0.2929537$. Another one can be noticed in the condensation–Hawking temperature dependence presented in Fig. 2. All the curves certainly cross each other in one point, located at $\psi^{(1)} \approx 0.345$ and $T \approx 0.3325$. This interesting phenomenon shows that while the dark sector may modify the phase structure of the system and has an imprint on its critical quantities, there exists a specific configuration of the system, that remains completely untouched.

The points in question constitute the so-called isosbestic ones \[35\], where the curves depend on temperature $T$ and parametrized by values of dark matter chemical potential, intersect. They illustrate the influence of temperature on condensation $\psi^{(1)}$ and chemical potential of visible sector. At this point we may perform a short analysis, which would reveal the leading order term of the dark sector influence. We take a following expansion of the condensate

$$\psi^{(1)}(T, \mu_d) = \psi^{(1)}(T, 0) + \mu_d^2 \psi_1^{(1)}(T) + \mathcal{O}(\mu_d^3). \quad (27)$$

Second order term takes the approximated form

$$\psi_1^{(1)}(T) = \frac{\psi^{(1)}(T, \mu_{d1}) - \psi^{(1)}(T, \mu_{d2})}{\mu_{d1}^2 - \mu_{d2}^2}, \quad (28)$$

where in a certain example of curves from Fig. 2 we took $\mu_{d1} = 0.12$ and $\mu_{d2} = 0.08$. The zero of this function refers to the isosbestic point, where the contribution of $\mu_d$ is by definition none. By calculating above function with help the leading order of the influence of the dark sector may be subtracted from the main function

$$\tilde{\psi}^{(1)}(T, \mu_d) = \psi^{(1)}(T, \mu_d) - \mu_d^2 \psi_1^{(1)}(T). \quad (29)$$

In the similar manner we can expand and analyze the chemical potential as a function of Hawking temperature, parametrized by $\mu_d$ from boson star – hairy black hole...
we define $\mu_1(T)$ analogically to (28) with $\mu_{d1} = 0.2$ and $\mu_{d2} = 0.08$ and perform the same transformation for $\mu(T, \mu_d)$ curve as for $\psi(T, \mu_d)$ in (29). The effect of these transformations is depicted in Fig. 4 where all curves tend to be much closer to each other than before. Obviously the total effect of $\mu_d$ is not ruled out completely, since it is much more complex than in the considered expansion.

\begin{equation}
\mu(T, \mu_d) = \mu(T, 0) + \mu_d^2 \mu_1(T) + O(\mu_d^3). \quad (30)
\end{equation}

We define $\mu_1(T)$ analogically to (28) with $\mu_{d1} = 0.2$ and $\mu_{d2} = 0.08$ and perform the same transformation for $\mu(T, \mu_d)$ curve as for $\psi(T, \mu_d)$ in (29). The effect of these transformations is depicted in Fig. 4 where all curves tend to be much closer to each other than before. Obviously the total effect of $\mu_d$ is not ruled out completely, since it is much more complex than in the considered expansion.

\begin{equation}
\mu(T, \mu_d) = \mu(T, 0) + \mu_d^2 \mu_1(T) + O(\mu_d^3). \quad (30)
\end{equation}

We define $\mu_1(T)$ analogically to (28) with $\mu_{d1} = 0.2$ and $\mu_{d2} = 0.08$ and perform the same transformation for $\mu(T, \mu_d)$ curve as for $\psi(T, \mu_d)$ in (29). The effect of these transformations is depicted in Fig. 4 where all curves tend to be much closer to each other than before. Obviously the total effect of $\mu_d$ is not ruled out completely, since it is much more complex than in the considered expansion.

\begin{equation}
\mu(T, \mu_d) = \mu(T, 0) + \mu_d^2 \mu_1(T) + O(\mu_d^3). \quad (30)
\end{equation}

We define $\mu_1(T)$ analogically to (28) with $\mu_{d1} = 0.2$ and $\mu_{d2} = 0.08$ and perform the same transformation for $\mu(T, \mu_d)$ curve as for $\psi(T, \mu_d)$ in (29). The effect of these transformations is depicted in Fig. 4 where all curves tend to be much closer to each other than before. Obviously the total effect of $\mu_d$ is not ruled out completely, since it is much more complex than in the considered expansion.

\begin{equation}
\mu(T, \mu_d) = \mu(T, 0) + \mu_d^2 \mu_1(T) + O(\mu_d^3). \quad (30)
\end{equation}

We define $\mu_1(T)$ analogically to (28) with $\mu_{d1} = 0.2$ and $\mu_{d2} = 0.08$ and perform the same transformation for $\mu(T, \mu_d)$ curve as for $\psi(T, \mu_d)$ in (29). The effect of these transformations is depicted in Fig. 4 where all curves tend to be much closer to each other than before. Obviously the total effect of $\mu_d$ is not ruled out completely, since it is much more complex than in the considered expansion.
boundary models of a hairy black hole and a boson star (the so-called small boson star). The tantalizing question can be asked about the different boson star configurations with additional fields and potentials. Further investigations in this direction will be published elsewhere.

ACKNOWLEDGMENTS

We acknowledge K. I. Wysokinski and N. Sedmayaír for fruitful discussions on various occasions.

[1] R. Massey et al., Dark matter maps reveal cosmic scaffolding, Nature 445, (2007) 286.
[2] J. Dietrich et al., A filament of dark matter between two cluster of galaxies, Nature 487, (2012) 202.
[3] G. Bertone and T.M.P. Tait, A new era in the search for dark matter, Nature 562, (2018) 51.
[4] J.W. York, Black-hole thermodynamics and the Euclidean Einstein action, Phys. Rev. D 33 (1986) 2092.
[5] H.W. Braden, J.D. Brown, F.B. Whiting, and J.W. York, Black holes in a box through a language similar to holographic superconductors, JHEP 07 (2017) 042.
[6] P. Basu, C. Krishnan, P.N. Bala Subramanian, Hairy black holes in a box, JHEP 11 (2016) 041.
[7] P. Jetzer, Boson stars, Phys. Reports 220 (1992) 163, S.L. Llibing, C. Palenzuela, Dynamical Boson Stars, Living Rev. Relativ. 15, 6 (2012).
[8] B. Kleihaus, J. Kunz, C. Lammerzahl, and M.List, Charged boson stars and black holes, Phys. Lett. B 675 (2009) 102.
[9] B. Hartmann, B. Kleihaus, J. Kunz, and I. Schaffer, Compact boson stars, Phys. Lett. B 714 (2012) 120.
[10] D. Pugliese, H. Quevedo, J.A. Rueda, R. Ruffini, Charged boson stars, Phys. Rev. D 88 (2013) 024053.
[11] S. Kumar, U. Kulshreshtha, D.S. Kulshreshta, S. Kahlen, and J.Kunz, Some new results on charged compact boson stars, Phys. Lett. B 772 (2017) 615.
[12] Y. Peng, Large regular reflecting stars have no scalar field hair, Eur. Phys. J. C 79 (2019) 309.
[13] B.S. Acharya, S.A.R. Ellis, G.L. Kane, B.D. Nelson and M.J. Perry, Lightest Visible-Sector Supersymmetric Particle is Likely Unstable, Phys. Rev. Lett. 117 (2016) 181802.
[14] S.A. Abel and B.W. Schofield, Brane-antibrane kinetic mixing, millicharged particles and SUSY breaking, Nucl. Phys. B 685 (2004) 150.
[15] S.A. Abel, J. Jaeckel, V.V. Khoe, and A. Ringwald, Illuminating the hidden sector of string theory by shining light through a magnetic field, Phys. Lett. B 666 (2008) 66.
[16] S.A. Abel, M.D. Goossens, J. Jaeckel, V.V. Khoe, and A. Ringwald, Kinetic mixing of the photon with hidden U(1)s in string phenomenology, JHEP 07 (2008) 124.