Resonant behavior of the generalized Langevin system with tempered Mittag–Leffler memory kernel

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Abstract
The generalized Langevin equation describes anomalous dynamics. Noise is not only the origin of uncertainty but also plays a positive role in helping to detect signals with information, termed stochastic resonance (SR). This paper analyzes the anomalous resonant behaviors of the generalized Langevin system with a multiplicative dichotomous noise and an internal tempered Mittag–Leffler noise. For a system with a fluctuating harmonic potential, we obtain the exact expressions of several types of SR such as the first moment, the amplitude and autocorrelation function for the output signal as well as the signal–noise ratio. We analyze the influence of the tempering parameter and memory exponent on the bona fide SR and the general SR. Moreover, it is detected that the critical memory exponent changes regularly with the increase of the tempering parameter. Almost all the theoretical results are validated by numerical simulations.

Keywords: generalized Langevin equation, tempered Mittag–Leffler friction memory kernel, dichotomous noise, stochastic resonance

(Some figures may appear in colour only in the online journal)

1. Introduction
The fundamental equation for approximating the dynamics of a nonequilibrium system is the (generalized) Langevin equation, which contains both frictional and random forces, related by the fluctuation–dissipation theorem [1]. Extending the frictional term of the standard Langevin equation so that it has a memory kernel leads to the generalized Langevin equation (GLE),
\[
\ddot{x}(t) + \int_0^t \eta(t-t') \dot{x}(t') dt' = \xi(t),
\] (1.1)

where \(x(t)\) is the displacement and \(\eta(t)\) is the dissipation memory kernel linking the autocorrelation function of the internal noise \(\xi(t)\) through the second fluctuation–dissipation theorem \([2]\): \(\langle \xi(t_1) \xi(t_2) \rangle = k_B T \eta(t_1 - t_2)\) with the Boltzmann constant \(k_B\) and the absolute temperature \(T\).

The GLE with the power law memory kernel describes the anomalous dynamics \([3–5]\), mainly focusing on subdiffusion. A Mittag–Leffler correlated random force is introduced to the GLE in \([6]\), which can describe both subdiffusion and superdiffusion, the autocorrelation function of which is nonsingular at origin and behaves as a power law for large \(t\). More specifically, the Mittag–Leffler memory kernel is defined as

\[
\eta(t) = \gamma \tau^\alpha E_\alpha \left( -\frac{t^\alpha}{\tau^\alpha} \right), \quad t \geq 0,
\] (1.2)

where \(E_\alpha(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(\alpha k + 1)}\) denotes the Mittag–Leffler function \([7]\). In (1.2), \(0 < \alpha < 2\) is the memory exponent, \(\tau\) is the characteristic memory time, and \(\gamma\) is the friction coefficient. Taking \(\alpha = 1\), (1.2) becomes an exponential form

\[
\eta(t) = \gamma \tau \exp \left( -\frac{t}{\tau} \right).
\] (1.3)

In this case, (1.1) describes the process driven by the Ornstein–Uhlenbruck noise \([8]\). Moreover, in the limit \(\tau \rightarrow 0\), the colored noise (1.3) reduces to Gaussian white noise \(\eta(t) = 2\gamma \delta(t)\) \([9]\). On the other hand, for \(\alpha \neq 1\), we can write (1.2) as the following asymptotic forms \([4, 10]\):

\[
\eta(t) \approx \frac{\gamma}{\tau^\alpha} \exp \left( -\frac{(t/\tau)^\alpha}{\Gamma(\alpha + 1)} \right), \quad t \ll \tau,
\] (1.4)

and

\[
\eta(t) \approx \frac{\gamma}{\tau^\alpha} \frac{1}{\Gamma(1 - \alpha)} \left( \frac{t}{\tau} \right)^{-\alpha}, \quad t \gg \tau,
\] (1.5)

which means that the correlation behaves as a power law in the asymptotic limit at a short memory time \(\tau\). In the case of (1.5), the system (1.1) displays subdiffusion for \(0 < \alpha < 1\) and superdiffusion for \(1 < \alpha < 2\) \([6]\). Besides this, there are other more generalized Mittag–Leffler-type kernels that have been introduced to the analysis of the generalized Langevin equation, such as the two-parameter Mittag–Leffler memory kernel in \([11]\) and three-parameter Mittag–Leffler memory kernel in \([12]\).

Noise is an extremely important concept, and means a lot of things in science and engineering. One issue that naturally comes to our mind is the extraction of the desired information (signal) from a background of unwanted noise in an ultrasound machine. Sometimes, noise is no trouble at all; on the contrary, it can help to amplify a weak useful signal, termed stochastic resonance (SR). SR was originally coined by Benzi and collaborators to explain the periodic recurrence of the ice ages \([13, 14]\). This very meaningful discovery was raised independently by Nicolis in \([15]\) at the same time. SR is a common phenomenon in electronic and magnetic systems \([16]\) as well as in biological systems (sensory neurons) \([17]\). The review paper \([16]\) pointed out that SR has three basic ingredients: a form of threshold, a weak coherent input and a source of noise inherent in the system or being added to the coherent input. The original understanding of SR is that it can only occur in nonlinear systems, but in recent years, the
SR has been found in linear systems with multiplicative noise [18, 19]. It is often observed that when a physical system is displaced from its equilibrium position, the it experiences a restoring force; if the force is proportional to the displacement, the system is called a harmonic oscillator, which has been widely studied in physics [20, 21]. The anomalous diffusive behavior of (1.1) with the harmonic oscillator was discussed in [22] for the power law noise and in [9] for Mittag–Leffler noise. In particular, in [9, 12, 23], it was found that the relaxation function, which is closely related to the diffusion behavior of the particles, exhibits more oscillations in the generalized Langevin system with Mittag–Leffler noise than the system with a pure power law noise.

Adding a periodic input signal (external periodic driving force) $A_0\cos(\Omega t)$ and a fluctuating harmonic potential $V(x, t) = (\omega^2 + z(t))\frac{x^2}{2}$ to (1.1) leads to the type of system we consider in this paper [24, 25]:

$$\ddot{x}(t) + \int_0^t \eta(t-t')\dot{x}(t')dt' + \omega^2 x(t) + z(t)x(t) = A_0\cos(\Omega t) + \xi(t),$$  \hspace{1cm} (1.6)

where $A_0$ is the amplitude, $\Omega$ is the driving angular frequency, $\omega$ is the undamped angular frequency, and $z(t)$ is a multiplicative noise [26–28], playing the role of fluctuating control parameter. In fact, multiplicative fluctuations are usual in nature, especially in biological systems [29]. The model (1.6) involving the power law or Mittag–Leffler memory kernel is discussed in [24, 25, 30, 31], with concern on how the behavior of the response function (output amplitude) depends on the system parameters. Under some circumstances, for more accurately/reasonably approximating physical problems, e.g. considering the finite lifespan of the particles and the restricted moving space, tempered anomalous dynamics are widely considered [32–37]. More precisely, we know that colored fluctuations and viscoelasticity—implying that the friction has a long time memory—are inherent in living cells. However, considering the finite lifespan of the particles, one should truncate the long memory time reasonably. Thus, this paper discusses the SR phenomenon of the model (1.6) with the tempered Mittag–Leffler memory kernel defined in (2.1), and uncovers how the behavior of the output amplitude depends on the system parameters and the influence of the exponential truncation on the resonance behavior. We also analyze the signal–noise ratio (SNR), its dependence on the system parameters, and the influence of the tempering parameter. Obtaining the theoretical results requires complex analysis and sophisticated mathematical tools; to verify their correctness, the numerical simulations are performed by a kinetic Monte Carlo method with the algorithms presented in the appendix.

The outline of this paper is as follows. In section 2, we present the model which describes the oscillator system with the tempered Mittag–Leffler frictional memory kernel, and obtain the first moment of the output signal as well as its amplitude. In section 3, the dependence of the output signal’s amplitude on the system parameters, such as the characteristic memory time $\tau$, the driving frequency $\Omega$, and the amplitude $a$ of the multiplicative noise, is presented. Besides this, we also give the conditions that the friction coefficient $\gamma$ and the memory exponent $\alpha$ ($0 < \alpha < 2$) should satisfy for the appearance of the resonance phenomenon; moreover, the exponential tempering parameter affects the critical memory exponent and reduces the resonance region in the parameter space $(\gamma, \alpha)$. In section 4, we calculate the correlation function of the output signal and the SNR of the oscillator dynamic system, and consider the SNR versus driving frequency $\Omega$ for different truncation parameters and memory exponents. Finally, in section 5, some conclusions are presented.
2. Model

In dynamic oscillatory systems (1.6), which are susceptible to noisy environments, we take into account the tempered Mittag–Leffler friction memory kernel, i.e. a modification to the Mittag–Leffler memory kernel by an exponential truncation:

$$\eta(t) = \frac{\gamma}{\tau} e^{-bt} E_{\alpha} \left( -\frac{R^\alpha}{\tau^\alpha} \right), \quad t \geq 0, \ 0 < \alpha < 2, \ b \geq 0, \ \gamma > 0, \ \tau > 0. \quad (2.1)$$

Here, $b$ is the truncation parameter, and the other parameters are the same as the ones in (1.2). It should be noted that (2.1) is a special case involving a truncated three-parameter Mittag–Leffler kernel in [37, 38] with $\beta = \delta = 1$. In figure 1, the graphical representation of the tempered Mittag–Leffler kernel (2.1) is presented and it can also be found that for different $\alpha$ and $b$, $\eta(t)$ may be positive or negative, but eventually tends to zero. The noise spectral density $\langle |\xi(k)|^2 \rangle$, which is defined as the Fourier transform of the noise correlation function $k_T \eta(t)$, can be derived by the Fourier–Laplace transform technique in [10]:

$$\langle |\xi(k)|^2 \rangle = \frac{2k_T \gamma}{\pi} \left( \frac{b^\alpha + R^{1-\alpha} \cos(\vartheta(1-\alpha))}{R^{1-\alpha}} \right)^{-1} e^{-k^\alpha} \frac{1}{1 + (k|\tau|)^{2\alpha} + 2(k|\tau|)^\alpha \cos \left( \frac{\pi}{2\alpha} \right)}, \quad (2.2)$$

for any real $k$, $R = \sqrt{b^2 + R^2}$ and $\vartheta = \arctan \left( \frac{b}{R} \right)$. Here, we define the Fourier transform of $f(x)$ as $\tilde{f}(k) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$. Letting $b = 0$, the noise spectral density becomes

$$\langle |\xi(k)|^2 \rangle = 2k_T \gamma \left( \frac{R^{1-\alpha}}{R^{1-\alpha}} \right)^{-1} \sin \left( \frac{\pi}{2\alpha} \right) \left( 1 + (k|\tau|)^{2\alpha} + 2(k|\tau|)^\alpha \cos \left( \frac{\pi}{2\alpha} \right) \right)^{-1} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx,$$

which recovers the result in [10, 31]. Letting $\tau \to 0$, the noise spectral density becomes

$$\langle |\xi(k)|^2 \rangle = 2k_T \gamma \frac{\cos(\vartheta(1-\alpha))}{R^{1-\alpha}},$$

When $\alpha = 1$, (2.2) becomes

$$\langle |\xi(k)|^2 \rangle = 2k_T \gamma \frac{b}{R^2} \frac{b + 1}{(b^2 + R^2)^{3/2}},$$

which indicates that the noise spectral density is monotone decreasing in the case where the noise correlation function is an exponential form. The noise spectral density (2.2) is shown in figure 2. It can be seen that for small $b$, the frequency spectrum (2.2) is monotone decreasing.
for $\alpha \leq 1$ and nonmonotonic for $\alpha > 1$. For large $b$, the exponential form comes into prominence in (2.1) so the corresponding spectral density is monotone decreasing for $0 < \alpha < 2$ (see the dashed-dotted line ($b = 1.3$) in figures 2(b) and (c)).

Furthermore, we consider a dichotomically perturbed oscillator added to the harmonic potential

$$V(x) = \omega^2 \frac{x^2}{2},$$

which is determined by the dichotomous noise $z(t)$ [39] consisting of jumps between two values: $z_1 = a$ and $z_2 = -a$ with $a > 0$; the value of the jump occurs with the stationary probabilities $P_i(a) = P_i(-a) = \frac{1}{2}$ and follows the pattern of a Poisson process in time. In fact, the dichotomous noise $z(t)$ can be completely characterized by the following transition probability because of its time-homogeneous Markovian property:

$$P_{ij}(t) = \frac{1}{2} \begin{pmatrix} 1 + e^{-\nu t} & 1 - e^{-\nu t} \\ 1 - e^{-\nu t} & 1 + e^{-\nu t} \end{pmatrix}, \quad i, j \in \{-a, a\},$$

with the switching rate $\nu > 0$. Following the transition probability, it can be calculated that the mean value of dichotomous noise $\langle z(t) \rangle = 0$ and the autocorrelation function $\langle z(t + \tau)z(t) \rangle = a^2 e^{-\nu |\tau|}$. Moreover, we assume that $z(t)$ and $\xi(t)$ are independent with $\langle z(t)\xi(t) \rangle = 0$.

To study the SR phenomenon, we mainly discuss the behavior of the first moment $\langle x(t) \rangle$ of the system (1.6). The usual approach of getting the exact expression of $\langle x(t) \rangle$ is to use the Shapiro–Loginov formula [40]:

$$\frac{d}{dt} \langle x(t)g(z) \rangle = \left\langle z(t) \frac{d}{dt} g(z) \right\rangle - \nu \langle z(t)g(z) \rangle,$$

(2.3)

where $g(z)$ is an arbitrary functional of the noise $z(t)$. More precisely, define $X_1 = \langle x(t) \rangle$, $X_2 = \langle \dot{x}(t) \rangle$, $X_3 = \langle z(t)x(t) \rangle$, and $X_4 = \langle z(t)\dot{x}(t) \rangle$. Then from (1.6) and (2.1), there exist

$$\begin{align*}
X_1 &= X_2, \\
X_2 &= -\omega^2 X_1 - X_3 - \int_0^t \eta(t - t')X_2(t')dt' + A_0 \cos(\Omega t), \\
X_3 &= X_4 - \nu X_3, \\
X_4 &= -a^2 X_1 - \omega^2 X_3 - \nu X_4 - e^{-\nu t} \int_0^t \eta(t - t')X_4(t')e^{\nu t'} dt',
\end{align*}$$

(2.4)

where use has been made of the fact that $\langle z^2(t)x(t) \rangle = a^2 \langle x(t) \rangle$ and $\langle \dot{x}(t)z(t) \rangle = \langle \dot{x}(t) \rangle z(t) e^{-\nu(t-t')}$. Using the Laplace transform technique, we obtain

\[ \boxed{ } \]
\[ sX_1(s) = X_2(s) + X_1(0), \]
\[ (s + \eta(s))X_2(s) = -\omega^2X_1(s) - X_3(s) + A_0\mathcal{L}[\cos(\Omega t)] + X_2(0), \]
\[ (s + \nu)X_3(s) = X_4(s) + X_3(0), \]
\[ (s + \nu + \eta(s + \nu))X_4(s) = -a^2X_1(s) - \omega^2X_3(s) + X_4(0), \]
where \( X_i(s) \) is the Laplace transform of \( X_i(t) \), \( i = 1, 2, 3, 4 \), i.e. \( X_i(s) = \int_0^\infty e^{-st} X_i(t) \, dt \).

Solving the above equations leads to
\[ X_1 = \sum_{k=1}^{4} H_{1k}(t) X_k(0) + A_0 \int_0^t H_{12}(t - t') \cos(\Omega t') \, dt', \]
and
\[ X_3 = \sum_{k=1}^{4} H_{3k}(t) X_k(0) + A_0 \int_0^t H_{32}(t - t') \cos(\Omega t') \, dt' \]
with
\[ H_{11}(s) = [\omega^2 + (s + \nu)^2 + (s + \nu)\eta(s + \nu)] \frac{\mathcal{L}[\cos(\Omega t)] + \eta(s)}{D(s)}, \]
\[ H_{12}(s) = \frac{\omega^2 + (s + \nu)^2 + (s + \nu)\eta(s + \nu)}{D(s)}, \]
\[ H_{13}(s) = -s + \nu + \eta(s + \nu) \frac{1}{D(s)}, \]
\[ H_{14}(s) = -a^2 \frac{s + \eta(s)}{D(s)}, \]
\[ H_{31}(s) = -a^2 \frac{s + \eta(s)}{D(s)}, \]
\[ H_{32}(s) = -a^2 \frac{s + \eta(s)}{D(s)}, \]
\[ H_{33}(s) = [\omega^2 + s^2 + s\eta(s)] \frac{s + \nu + \eta(s + \nu)}{D(s)}, \]
\[ H_{34}(s) = \frac{\omega^2 + s^2 + s\eta(s)}{D(s)}, \]
\[ D(s) = [\omega^2 + s^2 + s\eta(s)] [\omega^2 + (s + \nu)^2 + (s + \nu)\eta(s + \nu)] - a^2, \]
\[ \eta(s) = \frac{\gamma}{\tau^\alpha (s + \nu)^\alpha + \tau^{-\alpha}}. \]

In order to ensure the stability of the solutions (2.5) and (2.6), the roots of \( D(s) = 0 \) should not have a positive real part. To meet this condition,
\[ a^2 < a^2_{cr} = \omega^2 \left[ \omega^2 + \nu^2 + \frac{\gamma \nu (\nu + b)^{\alpha - 1}}{\tau^\alpha (\nu + b)^\alpha + 1} \right] \]

or \( a^2 = a^2_{cr} \). Assuming the inequality (2.7) is fulfilled, in the long time limit, the influence of the initial conditions can be ignored, i.e.
\[ \langle x(t) \rangle_{st} = A_0 \int_0^t H_{12}(t - t') \cos(\Omega t') \, dt' \]
and
\[ \langle z(t)x(t) \rangle_{as} = A_0 \int_0^t H_{32}(t-t') \cos(\Omega t') dt' . \] \hspace{1cm} (2.9)

Finally, using the techniques in \([16, 41−43]\), the asymptotic expressions of the first moments (2.8) and (2.9) read
\[ \langle x(t) \rangle_{as} = \text{sgn}(\chi'(\Omega)) A \cos(\Omega t + \Psi), \] \hspace{1cm} (2.10)
\[ \langle z(t)x(t) \rangle_{as} = \text{sgn}(\chi'_1(\Omega)) A_1 \cos(\Omega t + \Psi_1), \] \hspace{1cm} (2.11)

where the variables above will be explicitly explained in the following. In (2.10), the output amplitude \( A = A_0 |\chi(\Omega)| \), the phase shift \( \Psi = \arctan(−\frac{\chi''}{\chi'}) \), the sign function depends on the choice of \( \Psi \), and the complex susceptibility \( \chi(\Omega) \) of the dynamical oscillator system is
\[ \chi(\Omega) = \chi'(\Omega) + i\chi''(\Omega) = H_{12}(-i\Omega) = \int_0^\infty e^{it\Omega} H_{12}(t) dt. \]

Here
\[ \chi(\Omega) = \frac{m_1(m^2 + n^2) - a^2m}{(m_1m - n_1n - a^2)^2 + (m_1n + n_1m)^2} \]
\[ + i\frac{m_1(m^2 + n^2) + a^2n}{(m_1m - n_1n - a^2)^2 + (m_1n + n_1m)^2}. \]
\[ A = A_0 \frac{(m^2 + n^2)^{\frac{1}{2}}}{((m_1m - n_1n - a^2)^2 + (m_1n + n_1m)^2)^{\frac{1}{2}}}, \] \hspace{1cm} (2.12)

and
\[ \Psi = \arctan \left( \frac{a^2n + n_1(m^2 + n^2)}{a^2m - m_1(m^2 + n^2)} \right), \]

with
\[ \varphi = \arctan \left( \frac{\Omega}{\nu + b} \right), \quad \theta = \arctan \left( \frac{\Omega}{b} \right), \quad \phi = \arctan \left( \frac{\Omega}{\nu} \right), \]
\[ f = \frac{(\Omega^2 + (\nu + b)^2)^{\frac{1}{2}} \cos(\varphi - \varphi) + \cos(\varphi(\alpha - 1) + \phi)}{[\cos(\varphi(\alpha))] + \tau^\alpha (\Omega^2 + (\nu + b)^2)^{\frac{1}{2}} + \sin^2(\varphi(\alpha))}, \]
\[ g = \frac{(\Omega^2 + (\nu + b)^2)^{\frac{1}{2}} \sin(\varphi - \varphi) + \sin(\varphi(\alpha - 1) + \phi)}{[\cos(\varphi(\alpha))] + \tau^\alpha (\Omega^2 + (\nu + b)^2)^{\frac{1}{2}} + \sin^2(\varphi(\alpha))}, \]
\[ f_1 = \frac{(\Omega^2 + b^2)^{\frac{1}{2}} \sin(\theta(\alpha - 1))}{[\cos(\theta(\alpha))] + \tau^\alpha (\Omega^2 + b^2)^{\frac{1}{2}} + \sin^2(\theta(\alpha))}, \]
\[ g_1 = \frac{(\Omega^2 + b^2)^{\frac{1}{2}} \cos(\theta(\alpha) + \cos(\theta(\alpha - 1))}{[\cos(\theta(\alpha))] + \tau^\alpha (\Omega^2 + b^2)^{\frac{1}{2}} + \sin^2(\theta(\alpha))}, \]
\[ m = \omega^2 + \nu^2 - \Omega^2 + (\nu^2 + \Omega^2)^{\frac{1}{2}} (\Omega^2 + (\nu + b)^2)^{n} \gamma f, \]
\[ n = 2\Omega \nu + (\nu^2 + \Omega^2)^{\frac{1}{2}} (\Omega^2 + (\nu + b)^2)^{n} \gamma g, \]
\[ m_1 = \omega^2 - \Omega^2 + \Omega (\Omega^2 + b^2)^{n_1} \gamma f_1, \]
\[ n_1 = \Omega (\Omega^2 + b^2)^{n_1} \gamma g_1. \]
It seems that the output amplitude $A$ not only depends on the periodic driving force, but also the multiplicative and additive noise. In the case $b = 0$, these results recover the ones in [30]. As to (2.11), similarly, one can obtain
\[ H_{32}(-i\Omega) = \chi_1'(\Omega) + i\chi_1''(\Omega) \]
with
\[ \chi_1'(\Omega) = -\frac{a^2(m_1m - n_1n - a^2)}{(m_1m - n_1n - a^2)^2 + (m_1n + n_1m)^2} \]
and
\[ \chi_1''(\Omega) = -\frac{a^2(m_1n + n_1m)}{(m_1m - n_1n - a^2)^2 + (m_1n + n_1m)^2}. \]
Besides this, we have
\[ A_1 = A_0 \frac{a^2}{((m_1m - n_1n - a^2)^2 + (m_1n + n_1m)^2)^{\frac{1}{2}}} \]
and
\[ \Psi_1 = \arctan \left( -\frac{m_1n + n_1m}{m_1m - n_1n - a^2} \right). \]
To sum up the previous formulae, $\langle x(t) \rangle_\omega$ in (2.10) and its associated terms will be used in section 3, while the formulae associated with $\langle z(t)x(t) \rangle_\omega$ in (2.11) will be used in section 4.

3. Results

In this section, using the expression of amplitude $A$ in (2.12), we will analyze the behavior of $A^2$ for the different system parameters. In the first part, the behavior of $A^2$ versus the system parameters $\tau$, $\Omega$, $a$ is mainly discussed. In the second part, we study the influence of exponential tempering on the critical memory exponent $\alpha_{\text{cr}}$ and the resonance regions where SR versus the noise amplitude $a$ is possible.

3.1. Output amplitude versus system parameters

We analyze the behavior of the squared output amplitude $A^2$ versus the characteristic memory time $\tau$ and discuss the SR phenomenon, as understood in the wide sense, i.e. the nonmonotonic behavior of the output signal or some functions of it (moments, autocorrelation functions, SNR) with respect to the noise and system parameters [44]. Here, if the curve of the response function $A^2(\tau)$ is nonmonotonic, we say that the SR phenomenon occurs.

Figure 3 presents the squared output amplitude $A^2$ versus the characteristic memory time $\tau$ with a different memory exponent $\alpha$ and tempering parameter $b$. It can be noted that for a fixed $\alpha$, with the increase of tempering parameter $b$, the resonance peak decreases and moves slightly to the left. In other words, with the increase of $b$, the appearance of the SR phenomenon needs a shorter characteristic memory time, which, in some sense, can be explained by the increase of the tempering parameter $b$, reducing the memory effect of friction, so a shorter characteristic memory time $\tau$ can balance this effect. For a fixed $\alpha < 1$ (see figure 3(a)), with the increase of $b$, the resonance peak keeps decreasing, until it disappears. But for $\alpha > 1$
(see figure 3(c)), the SR phenomenon still exists, although the tempering parameter $b$ is very large, and can even reach a figure of several hundred. More precisely, there exists a peak and a valley in the curves of $A^2(\tau)$ for a small $b$. However, there is only a peak for a large $b$ and the value of the peak does not change with the increase of $b$. Next, we turn to the horizontal comparison with the fixed tempering parameter $b$. When $b$ is small, with the increase of $\alpha$, it changes from having a peak to having a peak as well as a valley; for a large $b$, the behavior of $A^2(\tau)$ is primarily monotonic (no SR) and then becomes nonmonotonic (SR appears) with the increase of $\alpha$.

Another interesting issue can be found relating to the bona fide SR, whereby the squared output amplitude $A^2$ versus the driving frequency $\Omega$ exhibits a peak. In figure 4, we show the dynamical behavior of $A^2(\Omega)$ with respect to the tempering parameter $b$, the memory exponent $\alpha$, and the switching rate $\nu$, which uncovers the bona fide SR phenomenon. Figure 4(a) ($\alpha = 0.2$) displays a double-peak phenomenon, which has already been observed in the literature [5, 22, 37, 45, 46]. Besides this, with the increase of the tempering parameter $b$, the resonance peak is primarily restrained and then increases and moves slightly to the left. Figure 4(b) ($\alpha = 1$) shows that with the increase of the tempering parameter $b$, $A^2(\Omega)$ is primarily monotonous ($b < 0.3$); then a resonance peak appears and increases gradually. Finally, with the sustained increase of $b$, a double-peak phenomenon appears. In figure 4(c) ($\alpha = 1.6$), with the increase of the tempering parameter $b$, the resonance peak is primarily restrained ($A^2(\Omega)$ is monotonous when $b = 2.3$), then increases and finally a double-peak phenomenon appears.

The above phenomena imply that the exponentially tempered Mittag–Leffler dissipation memory kernel can be taken as exponent-form one for a sufficiently large tempering parameter $b$ (obviously, to get similar dynamics, in the case of $\alpha > 1$, the tempering parameter $b$ needs to be larger than the case of $\alpha < 1$). Thus, with the increase of the tempering parameter $b$, the dynamic behaviors of $A^2(\Omega)$ in figures 4(a) and (c) are finally similar to the case $\alpha = 1$ in figure 4(b). More precisely, the output amplitude finally increases with the increase of $b$ and a double-peak phenomenon appears; see the three representations $b = 2.3$, $b = 8$ and $b = 35$ in figures 4(a)–(c) respectively. In addition, the positions of the peaks in figures 4(a)–(c) are almost the same for the respective large $b$.

When the noise switching rate $\nu$ increases, the double-peak phenomena cannot be observed (see figures 4(d)–(f)). The reason is that the dynamic system (1.6) tends to a deterministic system with a harmonic potential $V(x) = \omega^2 x^2/2$ [25]. But the movement and the rising or
suppression of the resonance peak are similar to the case of small $\nu$. Besides this, with the increase of the noise switching rate $\nu$, the driving frequency $\Omega$ needed for the maximum peak becomes larger. Next, we pay attention to the effect of frequency $\omega$ on the amplitude $A$ of the output signal. As shown in figures 5(a) and (b), with the increase of frequency $\omega$, the value of the resonance peak increases and the position of the peak moves slightly to the right. But when $\omega$ increases to 1.5, there is no resonance phenomenon. In figures 5(c) and (d), with the increase of frequency $\omega$, the value of the resonance peak finally decreases. It can be noticed that in the case $\omega = 0$, there exists a resonance-like maximum, but not in the stability region due to condition (2.7).

We further examine the dependence of the squared response function $A^2$ on the amplitude $a$ of the multiplicative symmetric dichotomous noise. In figure 6, we display the curves of $A^2(a)$ with respect to the tempering parameter $b$ and the memory exponent $\alpha$. There exist peaks in figures 6(a)–(c), which means that the typical SR phenomena appear. From figure 6(a) ($\alpha = 0.4$), it can be noted that the increase of the tempering parameter $b$ first suppresses the resonance peak, and then enhances the SR; the position of the peak moves slightly to the left, then remains unchanged. Figure 6(b) ($\alpha = 1$) shows that the resonance peak increases with the increase of the tempering parameter $b$, and the position of the peak remains unchanged. In figure 6(c) ($\alpha = 1.6$), the resonance peak is first restrained and then increases with the increase of the tempering parameter $b$. But in contrast with figure 6(a), the position of the peak moves slightly to the right, then remains unchanged. The final positions of the resonance peaks are almost the same in figures 6(a)–(c), and this phenomenon is also detected in figure 4 and explained there. Besides this, the parameter values in figure 6(d) are consistent with those in figure 9(h), in which the reason for there being no SR phenomenon is uncovered, i.e. the

**Figure 4.** The squared output amplitude $A^2$ versus the angular frequency $\Omega$ of the harmonic driving force. The solid lines are the analytical results and the marks are the computer simulations with the time $T = 100$ and the number of sampled trajectories 5000. Parameter values: $A_0 = \omega = 1, a^2 = 0.2, \tau = 0.2, \gamma = 1$; (a): $\alpha = 0.2, \nu = 0.1$; (b): $\alpha = 1, \nu = 0.1$; (c): $\alpha = 1.6, \nu = 0.1$; (d): $\alpha = 0.2, \nu = 1$; (e): $\alpha = 1, \nu = 1$; (f): $\alpha = 1.6, \nu = 1$. 
frictional coefficient $\gamma$ is not small enough. This implies that the existence of SR strongly depends on the system parameter.

3.2. SR regions in the parameter space ($\gamma, \alpha$)

In this part, the relation between the friction constant $\gamma$ and the memory exponent $\alpha$ ($0 < \alpha < 2$), as well as the influence of the tempering parameter $b$, for the emergence of SR versus noise amplitude $a$, is studied. From (2.12), we know that the response $A(a)$ reaches its maximum at $a_{max}^2 = m_1 m - n_1 n$. Then, because of the stability condition (2.7) $0 < a < \alpha_{cr}$, the nonmonotonic behavior of $A(a)$ in the region $[0, a_{cr}]$ can be guaranteed if $0 < m_1 m - n_1 n < a^2_{cr}$. One of our tasks is to discuss the adiabatic multiplicative noise, i.e. $\nu \to 0$; figure 7(a) is related to this case. In this case, the boundary lines between the regions where SR (versus $a$) occurs or not in the parameter space ($\gamma, \alpha$) are as follows:

$$\gamma_{1,2}(\alpha) = \frac{G_1}{G_2 \pm G_3},$$

where $0 < \alpha < 2$ and

$$G_1 = (\omega^2 - \Omega^2) \left[ \cos(\theta \alpha) + \tau^\alpha \left( \Omega^2 + b^2 \right)^{\alpha/2} \sin^2(\theta \alpha) \right],$$

$$G_2 = -(\Omega^2 + b^2)^{(2\alpha-1)/2} \Omega \tau^\alpha \sin(\theta) + \left( \Omega^2 + b^2 \right)^{(\alpha-1)/2} \Omega \sin(\theta \alpha - \theta),$$

$$G_3 = (\Omega^2 + b^2)^{(2\alpha-1)/2} \Omega \tau^\alpha \cos(\theta) + \left( \Omega^2 + b^2 \right)^{(\alpha-1)/2} \Omega \cos(\theta \alpha - \theta).$$

After reasonably choosing some parameters, there will be a critical memory exponent $\alpha_{cr}$, where a sharp transition in the dynamical behaviors of the system happens; at $\alpha_{cr}$, one of the boundaries $\gamma_{1,2}(\alpha)$ between the resonance and nonresonance regions tends to infinity. Taking $G_2 \pm G_3 = 0$, we obtain that $\alpha_{cr}$ satisfies the following expressions:

$$-(\Omega^2 + b^2)^{\alpha/2} \tau^\alpha \alpha_{cr} = \frac{\sin(\frac{\pi}{4} - (1 - \alpha_{cr})\theta)}{\sin(\frac{\pi}{4} - \theta)}, \quad 0 < \alpha_{cr} < 2,$$

$$-(\Omega^2 + b^2)^{\alpha/2} \tau^\alpha \alpha_{cr} = \frac{\sin(\frac{\pi}{4} + (1 - \alpha_{cr})\theta)}{\sin(\frac{\pi}{4} + \theta)}, \quad 1 < \alpha_{cr} < 2,$$

where $\theta \in (0, \frac{\pi}{4})$. It is obvious that the value of $\alpha_{cr}$ depends on the driving frequency $\Omega$, tempering parameter $b$, and characteristic memory time $\tau$. Moreover, if $b \geq \Omega$, i.e. $\theta \in (0, \frac{\pi}{4})$, there is no root for (3.2) and (3.3), which means that no $\alpha_{cr}$ induces $\gamma_{1,2}(\alpha_{cr}) \to \infty$. If $b < \Omega$, i.e. $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, from (3.2), we have

(i) $\alpha_{cr} \geq \alpha_{cr_{min}} = 1 - \frac{\pi}{2b} \in (0, 1/2]$, and $\alpha_{cr_{min}}$ decreases as the tempering parameter $b$ increases;

(ii) If $b = 0$, one has $\alpha_{cr} \geq \frac{1}{2}$, which recovers the result in [30];

(iii) If $\tau \to 0$, (2.1) can be treated as the tempered power law memory kernel. In this case, (3.2) becomes $\alpha_{cr} = 1 - \frac{\pi}{2b}$. Taking $b = 0$ again, one has $\alpha_{cr} = \frac{1}{2}$, which recovers the conclusion in [25].

Besides this, from (3.3), we have

(i) $2 > \alpha_{cr} \geq \alpha_{cr_{min}} = 1 + \frac{\pi}{2b} \in [3/2, 2)$, and $\alpha_{cr_{min}}$ increases as the tempering parameter $b$ increases;

(ii) If $b = 0$, one has $\alpha_{cr} \geq \frac{3}{2}$;
If $\tau \to 0$, (2.1) can be treated as the tempered power law memory kernel and it is nonsingular at origin. In this case, (3.3) becomes $\alpha_{\text{cr}} = 1 + \pi \frac{\theta}{4}$. Taking $b = 0$ again, one has $\alpha_{\text{cr}} = \frac{3}{2}$.

Then, we consider the number of critical memory exponents $\alpha_{\text{cr}}$ for any given characteristic memory time $\tau$. The relationship between $\tau$ and $\alpha$ is shown in figure 7(a), and it can be found that

(i) If $0 \leq \tau \leq \tau_1$, there exist $\alpha_{\text{cr}1}$ and $\alpha_{\text{cr}2}$, such that (s.t.) $\gamma_1(\alpha_{\text{cr}1}) \to \infty$ and $\gamma_2(\alpha_{\text{cr}2}) \to \infty$;
(ii) If $\tau_2 \leq \tau < \tau_{\text{max}}$, there exist $\alpha_{\text{cr}11}$ and $\alpha_{\text{cr}12}$, s.t., $\gamma_1(\alpha_{\text{cr}11}) \to \infty$ and $\gamma_1(\alpha_{\text{cr}12}) \to \infty$;
(iii) If $\tau_1 < \tau < \tau_2$, or $\tau = \tau_{\text{max}}$, there exists $\alpha_{\text{cr}1}$, s.t., $\gamma_1(\alpha_{\text{cr}1}) \to \infty$;
(iv) If $\tau > \tau_{\text{max}}$, there is no $\alpha_{\text{cr}}$, s.t., $\gamma_1(\alpha_{\text{cr}}) \to \infty$ or $\gamma_2(\alpha_{\text{cr}}) \to \infty$.

Here $\tau_1 = \left(\frac{\Omega - b}{(\Omega + b)(\Omega - b)^2}\right)^{1/2}$, $\tau_2 = \left(\frac{\Omega + b}{(\Omega - b)(\Omega + b)^2}\right)^{1/2}$ and $\tau$ reaches its maximum value $\tau_{\text{max}}$ when $\alpha = 25.4357 \cos(0.3735 b \frac{\pi}{11} - 0.0414) - 24.0431$ by the fitting method. In addition, for $b = 0$, one has $\tau_1 = \tau_2 = \frac{1}{\Omega}$; see the dashed lines in figure 7(a).

The required conditions of the appearance of the critical memory exponent $\alpha_{\text{cr}}$ have been obtained from the above discussions, i.e. $b < \Omega$ and $\tau \leq \tau_{\text{max}}$. Figure 8 intuitively displays the resonance and nonresonance regions in the parameter space $(\gamma, \alpha)$. The dark grey domains

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{The squared output amplitude $A^2$ versus the characteristic memory time $\tau$ (a), (b) and angular frequency $\Omega$ (c), (d). The solid/dashed lines are the analytical results and the marks are the computer simulations with time $T = 200$ and the number of sampled trajectories 5000. Parameter values: $\nu = 0.1, A_0 = 1, \gamma = 0.7, b = 0.3, a^2 = 0.05$; (a): $\alpha = 0.7, \Omega = 1$; (b): $\alpha = 1.6, \Omega = 1$; (c): $\alpha = 0.7, \tau = 0.2$; (d): $\alpha = 1.6, \tau = 0.2$. We emphasize that these parameters satisfy the stability condition (2.7) except the case $\omega = 0$.}
\end{figure}
represent the regions of the parameters $\gamma$ and $\alpha$, where SR versus $a$ is possible in the stability region; i.e. in those regions, the parameter values satisfy $0 < m_1 m - n_1 n < \Delta_c^2$. The light grey domains are the regions where the response $A(a)$ formally also exhibits a resonance-like maximum, but in those regions the mean value $\langle x(t) \rangle$ is unstable because $a_{\text{max}}^2 > \Delta_c^2$. If the

**Figure 6.** The squared output amplitude $A^2$ versus the dichotomous noise amplitude $a$. Parameter values: $\nu = 1, \Omega_0 = \omega = 1$; (a): $\Omega = 0.6$, $\tau = 0.8$, $\gamma = 0.3$, $\alpha = 0.4$; (b): $\Omega = 0.6$, $\tau = 0.8$, $\gamma = 0.3$, $\alpha = 1$; (c): $\Omega = 0.6$, $\tau = 0.8$, $\gamma = 0.3$, $\alpha = 1.6$; (d): $\Omega = 1.8$, $\tau = 0.65$, $\gamma = 0.6$, $\alpha = 1.6$. We emphasize that these parameters satisfy the stability condition (2.7). In addition, if we keep the parameters the same as above except $\nu = 0.1$, the SR phenomena of $A(a)$ are similar.

**Figure 7.** Characteristic memory time $\tau$ versus memory exponent $\alpha$. Parameter values: (a): $\nu = 0$, $\Omega = 1.8$, $b = 0.3$ (solid line), $b = 0$ (dashed line); (b): $\nu = 1$, $\Omega = 1.8$, $b = 0.3$ (solid line), $b = 0$ (dashed line).
critical memory exponent \( \alpha_{cr} \) exists, it can be noted that with the increase of the tempering parameter \( b \) \((b < \Omega)\), the critical memory exponent \( \alpha_{cr1} \) (or \( \alpha_{cr11} \)) moves left slightly and \( \alpha_{cr2} \) (or \( \alpha_{cr12} \)) moves right slightly; see figures 8(a) \(\rightarrow\) (d) and (c) \(\rightarrow\) (f). This conclusion can also be obtained from figure 7(a). Next, the solid lines in figure 8 are for \( \gamma_1(\alpha) \) and \( \gamma_2(\alpha) \). By some calculations, we can obtain that if both \( \gamma_1(0) \) and \( \gamma_2(0) \) exist \((\Omega > \omega)\), then

\[
\gamma_1(0) = \frac{\sqrt{3}(\Omega^2 - \omega^2)}{(\Omega^2 + b^2) + \Omega \sin(\theta - \frac{\pi}{4})} \geq \gamma_1(0)|_{b=0} = 2(\Omega^2 - \omega^2), \quad (3.4)
\]

\[
\gamma_2(0) = \frac{\sqrt{2}(\Omega^2 - \omega^2)}{(\Omega^2 + b^2) + \Omega \sin(\theta + \frac{\pi}{4})} \leq \gamma_2(0)|_{b=0} = 2(\Omega^2 - \omega^2),
\]

and \( \gamma_1(0) - \gamma_2(0) \) increases with the increase of the tempering parameter \( b \). If both \( \gamma_1(2) \) and \( \gamma_2(2) \) exist (see figures 8(b) and (c)), there exist

\[
\gamma_1(2) = \frac{(\omega^2 - \Omega^2) (1 + 2\tau^2 (b^2 - \Omega^2)) + \tau^4 (b^2 + \Omega^2)^2)}{(\Omega^2 + b^2) \Omega \tau^2 (b - \Omega) + \Omega (b + \Omega)},
\]

\[
\gamma_2(2) = \frac{(\omega^2 - \Omega^2) (1 + 2\tau^2 (b^2 - \Omega^2)) + \tau^4 (b^2 + \Omega^2)^2)}{-(\Omega^2 + b^2) \Omega \tau^2 (b - \Omega) + \Omega (b + \Omega)},
\]

\[
\gamma_{1,2}(2)|_{b=0} = \frac{(\omega^2 - \Omega^2) (1 - \tau^2 \Omega^2)}{\Omega^2},
\]

and \( |\gamma_1(2) - \gamma_2(2)| \) increases with the increase of the tempering parameter \( b \). Besides this, if both \( \gamma_1(1) \) and \( \gamma_2(1) \) exist \((\omega < \Omega, (\Omega - b) > \frac{1}{\tau})\) (see figures 8(b) and (e)), there exist

\[
\gamma_1(1) = \frac{(\Omega^2 - \omega^2) (1 + 2\tau \Omega + \Omega^2 (b^2 + \Omega^2))}{\Omega^2 \tau - \Omega + \Omega \tau b},
\]

\[
\gamma_2(1) = \frac{(\Omega^2 - \omega^2) (1 + 2\tau \Omega + \Omega^2 (b^2 + \Omega^2))}{\Omega^2 \tau + \Omega + \Omega \tau b},
\]

and \( \gamma_1(1) - \gamma_2(1) \) increases with the increase of the tempering parameter \( b \). All in all, with the increase of the tempering parameter \( b \), the resonance region in the parameter space \((\gamma, \alpha)\) shrinks and the nonresonance region expands.

Next, we pay attention to a more general case, \( \nu \neq 0 \); figure 7(b) is related to this case. Now, the boundary lines between the regions where SR (versus \( a \)) appears or not in the parameter space \((\gamma, \alpha)\) are

\[
\gamma_{1,2}(\alpha) = \frac{2C}{-B \pm \sqrt{B^2 - 4AC}} \quad (3.5)
\]

with

\[A = fsf_4 - f_5f_6,\]

\[B = fsf_4(\omega^2 - \Omega^2) + f_5f_3(\omega^2 - \Omega^2 + \nu^2) - 2\Omega \nu f_2f_5,\]

\[C = (\omega^2 - \Omega^2)(\omega^2 - \Omega^2 + \nu^2)f_1f_2,\]
and

\[ f_1 = (\cos(\theta \alpha) + \tau \alpha (\Omega^2 + b^2)^{\alpha/2})^2 \sin^2(\theta \alpha), \]
\[ f_2 = (\cos(\phi \alpha) + \tau \alpha (\Omega^2 + (\nu + b)^2)^{\alpha/2})^2 \sin^2(\phi \alpha), \]
\[ f_3 = (\Omega^2 + b^2)^{\alpha-1} \Omega^2 \tau \alpha + (\Omega^2 + b^2)^{\alpha/2-1} (\Omega^2 \cos(\theta \alpha) - \Omega b \sin(\theta \alpha)), \]
\[ f_4 = (\nu(\nu + b) + \Omega^2)(\Omega^2 + (\nu + b)^2)^{\alpha-1} \tau \alpha \]
\[ + (\Omega^2 + (\nu + b)^2)^{\alpha/2-1} ((\nu(\nu + b) + \Omega^2) \cos(\phi \alpha) - b \Omega \sin(\phi \alpha)), \]
\[ f_5 = (\Omega^2 + b^2)^{\alpha-1} \Omega b \tau \alpha + (\Omega^2 + b^2)^{\alpha/2-1} (\Omega b \cos(\theta \alpha) + \Omega^2 \sin(\theta \alpha)), \]
\[ f_6 = (\Omega^2 + (\nu + b)^2)^{\alpha-1} \tau \alpha \]
\[ + (\Omega^2 + (\nu + b)^2)^{\alpha/2-1} ((\nu(\nu + b) + \Omega^2) \sin(\phi \alpha) + b \Omega \cos(\phi \alpha)). \]

Similar to the case \( \nu = 0 \), the expression of the critical memory exponent \( \alpha_{cr} \) is

\[ f_3 f_4 - f_5 f_6 = 0. \]  \hspace{1cm} (3.6)

After some calculations, then

\[ \alpha_{cr} \geq \alpha_{cr1\text{min}} = (\theta + \phi)^{-1} \arctan \left( \frac{\Omega(\nu(\nu + b) + \Omega^2 - b^2)}{b(2\Omega^2 + \nu(\nu + b))} \right) \in (0, 1), \]  \hspace{1cm} (3.7)

\[ 2 > \alpha_{cr2} \geq \alpha_{cr2\text{min}} = (\theta + \phi)^{-1} \left( \arctan \left( \frac{\Omega(\nu(\nu + b) + \Omega^2 - b^2)}{b(2\Omega^2 + \nu(\nu + b))} \right) + \pi \right) > 1. \]  \hspace{1cm} (3.8)

The analyses for (3.7) and (3.8) are as follows:
(i) $\alpha_{cr1\text{min}} \to 1 - \frac{\pi}{2\theta}$ and $\alpha_{cr2\text{min}} \to 1 + \frac{\pi}{2\theta}$ as the switching rate $\nu \to 0$, which recovers the adiabatic noise case;
(ii) In the very fast flipping case, $\nu \to \infty$, we have $\alpha_{cr1\text{min}} \to 1$ and $\alpha_{cr2\text{min}} \to 1 + \frac{\pi}{2} > 2$; that is to say, in the system with harmonic potential $V(x) = \omega^2x^2$, there only exists $\alpha_{cr1}$, s.t., $\gamma_1(\alpha_{cr1}) \to \infty$;
(iii) $\alpha_{cr1\text{min}}$ ($\alpha_{cr2\text{min}}$) decreases (increases) with the increase of the tempering parameter $b$; moreover, when $b = 0$, there is $\alpha_{cr1\text{min}} \to \pi \nu_\tau$ $\to \infty$, which recovers the conclusion in [30], and $\alpha_{cr2\min} \to \pi + 2\arctan(\Omega/\gamma)$;
(iv) $\alpha_{cr1} \to \alpha_{cr1\text{min}}$ in (3.7) and $\alpha_{cr2} \to \alpha_{cr2\text{min}}$ in (3.8) when $\tau \to 0$.

The number of the critical memory exponent $\alpha_{cr}$ also depends on the characteristic memory time $\tau$ when $\nu \neq 0$. We present this relationship in figure 7(b), and find that

(i) If $0 \leq \tau \leq \tau_1$, there exist $\alpha_{cr1}$ and $\alpha_{cr2}$, s.t., $\gamma_1(\alpha_{cr1}) \to \infty$ and $\gamma_2(\alpha_{cr2}) \to \infty$;
(ii) If $\tau_2 \leq \tau < \tau_{max}$, there exist $\alpha_{cr11}$ and $\alpha_{cr12}$, s.t., $\gamma_1(\alpha_{cr11}) \to \infty$ and $\gamma_1(\alpha_{cr12}) \to \infty$;
(iii) If $\tau_1 < \tau < \tau_2$, or $\tau = \tau_{max}$, there exists $\alpha_{cr1}$, s.t., $\gamma_1(\alpha_{cr1}) \to \infty$;
(iv) If $\tau > \tau_{max}$, there is no $\alpha_{cr}$, s.t., $\gamma_1(\alpha_{cr}) \to \infty$ or $\gamma_2(\alpha_{cr}) \to \infty$,

where

$$
\tau_1 = \left( \frac{-b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1} \right)^{\frac{1}{2}}, \quad \tau_2 = \left( \frac{-b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1} \right)^{\frac{1}{2}},
$$

$$
a_1 = (\Omega^2 + b^2)(\Omega^2 + (\nu + b)^2)(\Omega^2 + \nu b + \Omega^2 - b^2),
$$

$$
b_1 = (\Omega^2 + b^2)\Omega^2(\nu^2 - \nu b - \Omega^2 - b^2) - (\Omega^2 + (\nu + b)^2)(\nu^2 + \nu b + \Omega^2 + b^2),
$$

$$
c_1 = \Omega^2(\Omega^2 - \nu^2 - 3\nu b - b^2).
$$

One has $\tau_2 \to \left( \frac{\Omega + b}{(\Omega - b)(\Omega^2 + b^2)} \right)^{\frac{1}{2}}, \tau_1 \to \left( \frac{\Omega - b}{(\Omega + b)(\Omega^2 + b^2)} \right)^{\frac{1}{2}}$ as the switching rate $\nu \to 0$, which is consistent with the adiabatic noise case. In addition, there is a trend for the lines in the figure 7 to move right with the increase of the switching rate $\nu$.

In figure 9, we depict the resonance and nonresonance regions in the parameter space $(\gamma, \alpha)$. As they are the same as figure 8, the dark grey shaded domains correspond to the regions where SR versus $a$ is possible in the stability region. The light grey domains represent the unstable regions but the response $A(a)$ formally also exhibits a resonance-like maximum. With the increase of the tempering parameter $b$, the critical memory exponent $\alpha_{cr1}$ (or $\alpha_{cr11}$) moves left slightly and $\alpha_{cr2}$ (or $\alpha_{cr12}$) moves right slightly. A similar conclusion can also be drawn from figure 7(b). Similar to the case $\nu = 0$, the resonance region shrinks and the nonresonance region expands with the increase of the tempering parameter $b$. Moreover, when the noise switching rate $\nu$ is large and the characteristic memory time $\tau$ is small, system (1.6)—without the influence of multiplicative noise—behaves as a fractional oscillatory system. In this case, there only exists one $\alpha_{cr1} \in (0, 1)$, s.t., $\gamma_1(\alpha_{cr1}) \to \infty$. On the left side of $\alpha_{cr1}$, i.e. $\alpha \to 0$, for a large friction force $\gamma$, the resonance phenomenon still occurs. However, when $\alpha \to 1$, the emergence of the resonance phenomenon needs a small friction force $\gamma$. This is the cage effect [5], i.e. for a small $\alpha$, the friction force does not just slow down the particle but also causes it to develop a rattling motion.
4. Signal-to-noise ratio

Another statistical quantity—the signal-to-noise ratio (SNR)—can also describe the SR phenomenon. In order to obtain the expression of SNR, we define $X_5 = \langle x(t)x'(t) \rangle$, $X_6 = \langle x(t)x''(t') \rangle$, $X_7 = \langle z(t)x(t)x'(t') \rangle$, $X_8 = \langle z(t)x(t)x'(t') \rangle$. The following expressions are obtained by the same method used in section 2:

\[
\begin{align*}
\dot{X}_5 &= X_6, \\
\dot{X}_6 &= -\omega^2 X_5 - X_7 - \int_0^{t'} \eta(t-u) X_6(u) du + A_0 \cos(\Omega t) \langle x'(t) \rangle, \\
\dot{X}_7 &= -\nu X_7 + X_8, \\
\dot{X}_8 &= -a^2 X_5 - \omega^2 X_7 - \nu X_8 - e^{-\nu t} \int_0^{t'} \eta(t-u) X_6(u) e^{\nu u} du + A_0 \cos(\Omega t) \langle x'(t)z(t) \rangle. 
\end{align*}
\]

To solve these equations, we use the Laplace transform technique and obtain that

\[
\begin{align*}
X_5(s) &= \frac{(s + \nu)^2 + (s + \nu) \eta(s + \nu) + \omega^2}{D(s)} \frac{A_0 s}{s^2 + \Omega^2} \langle x'(t) \rangle, \\
&- \frac{A_0 L[\cos(\Omega t) \langle z(t)x(t') \rangle]}{D(s)} + \sum_{i=5}^{8} L_i(s) X_i(0), \\
(4.1)
\end{align*}
\]

with

\[
\begin{align*}
L_5(s) &= \frac{[(s + \nu)^2 + (s + \nu) \eta(s + \nu) + \omega^2][s + \eta(s)]}{D(s)}, \\
L_6(s) &= \frac{(s + \nu)^2 + (s + \nu) \eta(s + \nu) + \omega^2}{D(s)}, \\
L_7(s) &= -\frac{s + \nu + \eta(s + \nu)}{D(s)}, \\
L_8(s) &= -\frac{1}{D(s)}. \\
\end{align*}
\]

By the inverse Laplace transform, the asymptotic expression of $\langle x(t)x'(t) \rangle$ in the long time limit is

\[
\begin{align*}
\langle x(t)x'(t) \rangle_{\text{as}} &= \text{sgn}(\chi_1^I(\Omega)) \text{sgn}(\chi_2^I(\Omega)) \frac{A_1 A_2}{a^2} \cos(\Omega t + \Psi_2) \cos(\Omega t' + \Psi_1) e^{-\nu |t-t'|} \\
&+ A^2 \cos(\Omega t + \Psi) \cos(\Omega t' + \Psi), \\
(4.3)
\end{align*}
\]

where

\[
\begin{align*}
\chi_2^I(\Omega) + i \chi_2''(\Omega) &= H_{32}(-i\Omega - \nu), \\
A_2 &= A_0 \frac{a^2}{(m_1 m_2 - n_1 n_2 - a^2)^2 + (m_1 n_2 + n_1 m_2)^2} \\
&\text{and} \\
\end{align*}
\]
Figure 9. The friction constant $\gamma$ versus the memory exponent $\alpha$ with parameters $A_0 = \omega = 1$, $\nu = 1$, $\Omega = 1.8$. (a): $\tau = 0.25$, $b = 0$, $\alpha_{st} = 0.879$, $\alpha_{cr} = 1.910$; (b): $\tau = 1$, $b = 0$; (c): $\tau = 0.55$, $b = 0$, $\alpha_{st} = 1.231$; (d): $\tau = 0.6$, $b = 0$, $\alpha_{st} = 1.360$, $\alpha_{cr} = 1.867$; (e): $\tau = 0.25$, $b = 0.3$, $\alpha_{st} = 0.835$; (f): $\tau = 1$, $b = 0.3$; (g): $\tau = 0.55$, $b = 0.3$, $\alpha_{st} = 1.157$; (h): $\tau = 0.65$, $b = 0.3$, $\alpha_{st} = 1.426$, $\alpha_{cr} = 1.851$.

and

$$\Psi_2 = \arctan \left( -\frac{m_1 n_2 + n_4 m_2}{m_1 m_2 - n_1 n_2 - a^2} \right),$$

with

$$\varphi_2 = \arctan \left( \frac{\Omega}{b - \nu} \right), \quad \phi_2 = \arctan \left( \frac{\Omega}{\nu} \right),$$

$$\chi_2' (\Omega) = -\frac{\nu^2 (m_1 m_2 - n_1 n_2 - a^2)}{(m_1 m_2 - n_1 n_2 - a^2)^2 + (m_1 n_2 + n_1 m_2)^2},$$

$$\chi_2'' (\Omega) = -\frac{\nu^2 (m_1 m_2 - n_1 n_2 - a^2)^2 + (m_1 n_2 + n_1 m_2)^2}{(m_1 m_2 - n_1 n_2 - a^2)^2 + (m_1 n_2 + n_1 m_2)^2},$$

$$f_2 = \frac{\Omega^2 + (b - \nu)^2}{\cos(\varphi_2 - \phi_2) \tau^\alpha + \cos(\varphi_2 (\alpha - 1) + \phi_2)},$$

$$g_2 = \frac{\Omega^2 + (b - \nu)^2}{\sin(\varphi_2 - \phi_2) \tau^\alpha + \sin(\varphi_2 (\alpha - 1) + \phi_2)},$$

$$m_2 = \omega^2 + \nu^2 - \Omega^2 + (\nu^2 + \Omega^2) \frac{2}{(\Omega^2 + (b - \nu)^2)^{\frac{1}{4}}} \gamma f_2,$$

$$n_2 = -2\Omega \nu + (\nu^2 + \Omega^2)^{\frac{1}{2}} (\Omega^2 + (b - \nu)^2)^{\frac{1}{4}} \gamma g_2.$$
\[ C(\epsilon) := \frac{1}{T} \int_0^T \text{d}t \langle x(t)x(t+\epsilon) \rangle_{\text{as}} \]

\[ = F_1 \cos(\Omega \epsilon) e^{-\nu \epsilon} + F_2 \sin(\Omega \epsilon) e^{-\nu \epsilon} + \frac{A^2}{2} \cos(\Omega \epsilon), \quad (4.4) \]

where

\[ F_1 = \frac{A^2 a^2}{2} \left[ \frac{(m_1 n - n_1 n - a_1^2)(m_1 m_2 - n_1 n_2 - a_2^2) + (m_1 n + n_1 m)(m_1 n_2 + n_1 m_2)}{(m_1 m - n_1 n - a_1^2)^2 + (m_1 n + n_1 m)^2} \right], \]

\[ F_2 = \frac{A^2 a^2}{2} \left[ \frac{(m_1 n + n_1 m)(m_1 m_2 - n_1 n_2 - a_2^2) - (m_1 m - n_1 n - a_1^2)(m_1 n_2 + n_1 m_2)}{(m_1 m - n_1 n - a_1^2)^2 + (m_1 n + n_1 m)^2} \right]. \]

Taking advantage of the conclusions in [16, 47, 48], we can rewrite (4.4) as the sum of the coherent part \( C_{\text{coh}}(\epsilon) \) and the incoherent part \( C_{\text{incoh}}(\epsilon) \), i.e.

\[ C(\epsilon) = C_{\text{coh}}(\epsilon) + C_{\text{incoh}}(\epsilon), \quad (4.5) \]

where

\[ C_{\text{coh}}(\epsilon) := \frac{1}{T} \int_0^T \text{d}t \langle x(t) \rangle_{\text{as}} \langle x(t+\epsilon) \rangle_{\text{as}} = \frac{A^2}{2} \cos(\Omega \epsilon) \quad (4.6) \]

and

\[ C_{\text{incoh}}(\epsilon) := \frac{1}{T} \int_0^T \text{d}t \langle x(t)x(t+\epsilon) \rangle_{\text{as}} - \langle x(t) \rangle_{\text{as}} \langle x(t+\epsilon) \rangle_{\text{as}} \]

\[ = F_1 \cos(\Omega \epsilon) e^{-\nu \epsilon} + F_2 \sin(\Omega \epsilon) e^{-\nu \epsilon}. \]

The output SNR is defined as [16, 46, 48, 49]

\[ \text{SNR} := \frac{\lim_{\epsilon \to 0^+} \int_{-\Omega-\epsilon}^{\Omega+\epsilon} \text{d}k \tilde{C}(k)}{C_{\text{incoh}}(\Omega)} = \frac{\Gamma_1}{\Gamma_2}, \quad (4.7) \]

where \( \tilde{C}(k) := \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-ik\epsilon} C(\epsilon) \text{d}\epsilon \) is the Fourier transform of \( C(\epsilon) \). Using the Fourier cosine transform, the expressions of \( \Gamma_1 \) and \( \Gamma_2 \) are as follows:

\[ \Gamma_1 = \frac{2}{\pi} \int_0^T \text{d} \epsilon C_{\text{coh}}(\epsilon) \cos(\Omega \epsilon) = \frac{A^2}{2}, \quad (4.8) \]

\[ \Gamma_2 = \frac{2}{\pi} \int_0^\infty \text{d} \epsilon C_{\text{coh}}(\epsilon) \cos(\Omega \epsilon) = \frac{2}{\pi} \left( \frac{F_1 (\nu^2 + 2\Omega^2) + F_2 \nu \Omega}{\nu(\nu^2 + 4\Omega^2)} \right). \quad (4.9) \]

Finally, the output SNR at the driving frequency \( \Omega \) is

\[ \text{SNR} = \frac{A^2 \pi \nu(\nu^2 + 4\Omega^2)}{4(F_1 (\nu^2 + 2\Omega^2) + F_2 \nu \Omega)}. \quad (4.10) \]

The SNR versus the driving angular frequency \( \Omega \), for the different tempering parameter \( b \) and memory exponent \( \alpha \), is shown in figure 10, the nonmonotonic dependence of which is the so-called bona fide resonance. In figure 10(a) \( (\alpha = 0.7) \), with the increase of the tempering parameter \( b \), the resonance peak and the resonance valley move left slightly. The value of the resonance peak decreases, while the value of the resonance valley is almost unaffected by the tempering parameter \( b \). In figure 10(b) \( (\alpha = 1.6) \), with the increase of the tempering parameter
b, the resonance peak suffers a suppression, while the resonance valley rises. Finally, we have
the monotonic behavior of the SNR versus the driving frequency $\Omega$.

5. Conclusion

Stochastic resonance has big potential in applications, e.g. detecting signals with information from noisy environments. This paper discusses the SR in a generalized Langevin system with a tempered Mittag–Leffler memory kernel. Besides the internal noise, the influence of the fluctuating environment is modeled by the multiplicative dichotomous noise. Using the Shapiro–Loginov formula, we get the exact expressions of the first moment and the correlation function of the output signal as well as the SNR of the stochastic oscillator system. The obtained results for this oscillator dynamical system with the tempered Mittag–Leffler memory kernel for the friction term can recover the ones from the system with a Mittag–Leffler memory kernel, power-law memory kernel, and exponent-form memory kernel. We analyze the SR phenomena, which is the nonmonotonic behavior of the output amplitude and SNR, and find that the exponential tempering exerts an influence on the critical memory exponent and shrinks the resonance region in the parameter space $(\gamma, \alpha)$. The extensively performed numerical simulations verify the theoretical results.

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Appendix. Algorithms for numerical simulations

There are many numerical simulations provided above, which boil down to two parts: i.e. the asymptotic expression of the first moment of position $\langle x(t) \rangle_{\text{as}}$ and the SNR. For the first part, it is sufficient to generate the trajectories of $x(t)$ in (1.6). To do this, we must deal with the noise $z(t)$ and $\xi(t)$ first. Note that the additive noise $\xi(t)$ is independent from $z(t)$ and has a zero mean. In this work, since we are only concerned about the behavior of the first moment $\langle x(t) \rangle_{\text{as}}$, the term $\xi(t)$ can be omitted in numerical simulations to reduce the computation cost.
As for the dichotomous noise \( z(t) \), it can only take values between \( a \) and \(-a\), and is completely characterized by the transition probability

\[
P_{ij}(\tau) = \frac{1}{2} \left( 1 + e^{-\nu \tau} \right), \quad i,j \in \{-a,a\},
\]

where \( \tau \) is the time step. Assume that the initial distribution of \( z(t) \) is uniform, i.e. \( P(z(t_0) = a) = P(z(t_0) = -a) = \frac{1}{2} \). From the transition probability, we find that \( z(t) \) has the probability \( \left( 1 + e^{-\nu \tau} \right)/2 \) to keep its value and the probability \( \left( 1 - e^{-\nu \tau} \right)/2 \) to change its value at each step. Therefore, at the \( j \)th step, we can generate a random number \( r_j \) with uniform distribution in \([0, 1]\), and then \( z(t_{j+1}) = z(t_j) \cdot \text{sgn}(r_j - (1 - e^{-\nu \tau})/2) \). Step by step, the dichotomous noises \( z(t_0), z(t_1), \cdots \), are obtained.

Then we solve (1.6) with the scheme

\[
\begin{align*}
x(t_{j+1}) &= x(t_j) + v(t_j) \cdot \tau, \\
\frac{x(t_{j+1} + r_j \tau) - x(t_j)}{\tau} &= - \int_0^{r_j \tau} \eta(t_{j+1} - \tau') v(\tau') d\tau' - w^2 x(t_{j+1}) - z(t_j) x(t_{j+1}) + A_0 \cos(\Omega t_j),
\end{align*}
\]

where

\[
\int_0^{r_j \tau} \eta(t_{j+1} - \tau') v(\tau') d\tau' = \frac{\tau}{2} (\eta(t_{j+1}) v(t_0) + \eta(t_0) v(t_{j+1})) + \tau \sum_{k=1}^{j} \eta(t_k) v(t_{j+1-k}). \tag{A.1}
\]

By this scheme, we can generate \( N \) trajectories of \( x(t)_n \), \( i = 1, \cdots, N \), and then obtain

\[
\langle x(t_n) \rangle \approx \frac{1}{N} \sum_{i=1}^{N} x(t_n^i).
\]

Next, we use \( x(t)_n \) obtained above to compute SNR. We can calculate the formula (4.5) and (4.6), i.e.

\[
\begin{align*}
C(\epsilon) &= \frac{1}{T} \int_0^T \langle x(t + \epsilon) x(t) \rangle_{\text{as}} dt, \\
C_{\text{coh}}(\epsilon) &= \frac{1}{T} \int_0^T \langle x(t + \epsilon) \rangle_{\text{as}} \langle x(t) \rangle_{\text{as}} dt, \\
C_{\text{incoh}}(\epsilon) &= C(\epsilon) - C_{\text{coh}}(\epsilon),
\end{align*}
\]

by

\[
\begin{align*}
\langle x(t_n + \epsilon) x(t_n) \rangle &\approx \frac{1}{N} \sum_{i=1}^{N} x(t_n^i + \epsilon) \cdot x(t_n^i), \\
\langle x(t_n + \epsilon) \rangle \langle x(t_n) \rangle &\approx \left( \frac{1}{N} \sum_{i=1}^{N} x(t_n^i + \epsilon) \right) \cdot \left( \frac{1}{N} \sum_{i=1}^{N} x(t_n^i) \right).
\end{align*}
\]

The notation \( \langle \cdot \rangle_{\text{as}} \) denotes the asymptotic behavior of \( \langle x(t) \rangle \), so we only use the later parts of \( x(t_n^i) \) to obtain the corresponding results. Then we use the trapezoidal formula to compute \( C(\epsilon), C_{\text{coh}}(\epsilon) \) like (A.1), where for convenience we take \( \epsilon \) to be an integer multiple of time step \( \tau \). Following this, the SNR can be obtained by the formulae (4.7) and (4.8), i.e.
\[ \Gamma_1 = \frac{2}{\pi} \int_0^\infty C_{\text{coh}}(\epsilon) \cos(\Omega \epsilon) \, d\epsilon \]
\[ \Gamma_2 = \frac{2}{\pi} \int_0^\infty C_{\text{inc}}(\epsilon) \cos(\Omega \epsilon) \, d\epsilon \]
\[ \text{SNR} = \frac{\Gamma_1}{\Gamma_2}. \]

Since \( C_{\text{inc}}(\epsilon) \) decays to 0 for large values of \( \epsilon \), the integral interval of \( \Gamma_2 \) needs to be truncated first. Then the integrals of \( \Gamma_1 \) and \( \Gamma_2 \) can be numerically computed by the trapezoidal formula.

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