Vanishing theorems on \((\ell|k)\)-strong Kähler manifolds with torsion

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Abstract

We derive sufficient conditions for the vanishing of plurigenera, \(p_m(J), m > 0\), on compact \((\ell|k)\)-strong, \(\omega^\ell \wedge \partial \bar{\partial} \omega^k = 0\), Kähler manifolds with torsion. In particular, we show that the plurigenera of closed \((\ell|k)\)-strong manifolds, \(k < n - 1\), for which \(\text{hol}(\tilde{\nabla}) \subseteq SU(n)\) vanish, where \(\tilde{\nabla}\) is the Hermitian connection with skew-symmetric torsion. As a consequence all generalized k-Gauduchon manifolds for which \(\text{hol}(\tilde{\nabla}) \subseteq SU(n)\) do not admit holomorphic \((n,0)\) forms. Furthermore we show that all conformally balanced, \((\ell|k)\)-strong Kähler manifolds with torsion, \(k \neq n - 1\), are Kähler. We also give several examples of \((\ell|k)\)-strong Kähler and Calabi-Yau manifolds with torsion.
1 Introduction

Hermitian manifolds have widespread applications in both physics and differential geometry. These are complex manifolds equipped with a metric \( g \), \( g(JX, JY) = g(X, Y) \), and a Hermitian form \( \omega(X, Y) = g(X, JY) \) which is (1,1) with respect to the complex structure \( J \). There are many examples of Hermitian manifolds as every complex manifold admits a Hermitian structure. In many applications, Hermitian manifolds have additional properties which are expressed as either a condition on \( \omega \) or as a restriction on the holonomy of one of the Hermitian connections. A condition on the Hermitian form of a 2n-dimensional manifold is

\[
\omega^\ell \wedge \partial \bar{\partial} \omega^k = 0 , \quad 1 \leq k + \ell \leq n - 1 .
\]

(1.1)

An alternative way to write this condition is

\[
\omega^\ell \wedge d(\omega^{k-1} \wedge H) = 0 ,
\]

(1.2)

where the 3-form \( H = -i(\partial - \bar{\partial})\omega \) is the torsion of \( \hat{\nabla} \), and \( \hat{\nabla} \) is the unique Hermitian connection with skew-symmetric torsion. There are some advantages of writing (1.1) as (1.2) as the latter can be generalized to all G-structures which admit a compatible connection with skew-symmetric torsion. These include for example \( U(n) \), \( SU(n) \), \( Sp(n) \), \( Sp(n) \cdot Sp(1) \), \( G_2 \) and \( Spin(7) \) structures [15].

Special cases of (1.1) and (1.2) conditions have appeared before in the literature. First take \( \ell = 0 \), and so (1.1) and (1.2) can be rewritten as

\[
\partial \bar{\partial} \omega^k = 0 , \quad d(\omega^{k-1} \wedge H) = 0 , \quad 1 \leq k \leq n - 1 ,
\]

(1.3)

respectively. For \( k = n - 1 \), the above conditions coincide with the Gauduchon structure on Hermitian manifolds [13] which is usually written as \( \delta \theta = 0 \), where \( \theta = \delta \omega \circ J \) is the Lee form of the Hermitian form. As a consequence of the Gauduchon’s theorem in the every conformal class, there is a Hermitian metric which satisfies the Gauduchon condition. Thus every Hermitian manifold admits a Gauduchon structure. Furthermore for \( k = n - 2 \), the (1.3) condition has been called astheno-Kähler [23] and it has been studied in the context of harmonic maps and in connection with the extension of Siu’s rigidity theorem to non-Kähler complex manifolds. Recently, examples of such manifolds have been given in [7].

Another special case that has been extensively investigated for many years is (1.3) for \( k = 1, \; n > 2 \), or equivalently \( dH = 0 \). This coincides with the strong structure on Hermitian manifolds [19] and has found many applications in both physics, see eg [12, 20, 25] and geometry, see eg [21, 22, 5, 26, 9, 10, 6, 24]. For example in type II string theory, \( H \) is identified with the 3-form field strength. This is required by construction to satisfy \( dH = 0 \). Recently Streets and Tian [24] introduced a hermitian Ricci flow under which the pluriclosed or equivalently strong KT structure is preserved. Viewing (1.3) as a generalization of the strong condition on a Hermitian manifold, we shall refer to it uniformly as \( k \)-strong condition and the associated Hermitian manifolds as \( k \)-strong Kähler with torsion or k-SKT for short. Similarly, we shall refer to (1.1),
or equivalently to (1.2), as the \((\ell|k)\)-strong condition and to the associated Hermitian manifolds as admitting a \((\ell|k)\)-strong Kähler with torsion structure or \((\ell|k)\)-SKT for short.

More recently, (1.1) for \(\ell = n - k - 1\), the generalized \(k\)-Gauduchon condition, has been used by Fu, Wang and Wu \[11\] to prove a generalization of the Gauduchon theorem. Examples of manifolds which satisfy the 1-Gauduchon condition have been given in \[11\] and \[8\].

Apart from the condition (1.3) above, Hermitian manifolds can also be restricted by a holonomy condition. This is usually expressed as the requirement that one of the Hermitian connections has reduced holonomy \(G \subset U(n)\). In many investigations, the holonomy condition is imposed in addition to conditions like (1.2) on the Hermitian form. In many applications, see eg \[25, 2, 21, 22, 14, 9, 10, 4, 17, 18, 15\], the holonomy condition is imposed on the Hermitian connection with skew-symmetric torsion \(\hat{\nabla}\). Because of this, we say that a 2\(n\)-dimensional Hermitian manifold is \((\ell|k)\)-strong Calabi-Yau with torsion, or equivalently \((\ell|k)\)-SCYT, iff it is \((\ell|k)\)-SKT and

\[
\text{hol}(\hat{\nabla}) \subseteq SU(n).
\]

If \(\ell = 0\), we simply refer to such manifolds as k-SCYT. It is clear from this that the Ricci form \(\hat{\rho}\) of \(\hat{\nabla}\) on \((\ell|k)\)-SCYT manifolds must vanish, \(\hat{\rho} = 0\), and consequently, the \(\omega\)-trace \(\hat{b}\) of \(\hat{\rho}\) is also zero, \(\hat{b} = 0\). Such manifolds have appeared before in the literature. In particular, it has been shown in \[15\] that the supersymmetric IIB black hole horizons are 8-dimensional 2-SCYT manifolds and some examples have been constructed.

The purpose of this paper is twofold. First, we shall prove some vanishing results for the Dolbeault cohomology of \((\ell|k)\)-SKT and \((\ell|k)\)-SCYT manifolds. Then we shall give some examples of manifolds with k-SKT and k-SCYT structures. One of our main results is a vanishing theorem on the plurigenera,

\[
\text{hol}(\nabla) \subseteq SU(n),
\]

which is the dimension of the number of holomorphic sections of the m-th power of the canonical bundle. In particular, one has the following.

**Theorem 1.1.** Let \(M\) be a compact 2\(n\)-dimensional \((\ell|k)\)-SKT manifold satisfying the condition

\[
\hat{b} + \frac{n - k - 1}{3(n - 2)} ||H||^2 + \frac{2(k - 1)}{n - 2} ||\theta||^2 > 0, \quad n \neq 2.
\]

Then

\[
\text{p}_m(J) = 0, m > 0.
\]

The proof of this result is based on an inequality derived in \[22\] for the vanishing of plurigenera for KT manifolds.

A consequence of this is the following.

**Theorem 1.2.** Let \(M\) be a compact 2\(n\)-dimensional \((\ell|k)\)-SCYT non Calabi-Yau manifold and \(k < n - 1, n > 2\). Then the plurigenera \(\text{p}_m(J) = 0, m > 0\), and so \(M\) does not admit a holomorphic \((n,0)\)-form.

In particular, \(\text{p}_m(J) = 0, m > 0\) for all generalized \(k\)-Gauduchon manifolds, \(k < n - 1\), for which hol(\(\nabla\)) \(\subseteq SU(n)\).
This theorem generalizes the results obtained in [21, 22, 8] for 1-SCYT and \((n - 2)1\) -SCYT manifolds and that obtained in [7] for \((n - 2)\)-SCYT manifolds.

An immediate application of theorem 1.2 is that every conformally balanced, \(\theta = d\phi\), \(\phi\) a function on \(M\), compact \((\ell|k)\)-SCYT manifold, \(k < n - 1\), \(n > 2\), is Calabi-Yau. This is similar to the result originally proved for the special case of conformally balanced 1-SCYT and \((n - 2)1\)-SCYT manifolds in [21, 22, 8] and adapted to \((n - 2)\)-SCYT manifolds in [7]. However, one can generalize these results using the work of [11] on generalized k-Gauduchon manifolds. In particular, one has the following.

**Theorem 1.3.** Every compact, conformally balanced, \((\ell|k)\)-SKT manifold, \(k < n - 1\), \(n > 2\), is Kähler.

One can also consider Hermitian manifolds admitting a generalized \((\ell|k)\)-SKT structure given by

\[
2k_i \omega^l \wedge \partial \bar{\partial} \omega^k \equiv d(\omega^k \wedge H) \wedge \omega^l = \frac{1}{(k + \ell + 1)!} \alpha_{k,\ell} \omega^{k+\ell+1}, \quad 1 \leq k + \ell \leq n - 1, \tag{1.4}
\]

where \(\alpha_{k,\ell}\) is a function on \(M^{2n}\) which depends on \(\omega\). Our results in theorems (1.1) and (1.2) generalize to this case provided that \(\alpha_{k,\ell} > 0\).

Some of our results also apply to \((k_1, k_2, k_3)\)-strong hyper-Kähler manifolds with torsion \(((k_1, k_2, k_3)\)-SHKT) [15]. In particular, one can show that for all \((\ell, 1, 1|k, 1, 1)\)-SHKT manifolds with \(k < n - 1\) and \(n > 2\), \(p_m(I) = 0, m > 0,\) and cyclicly for \(J\) and \(K\). Furthermore if a \((\ell, 1, 1|k, 1, 1)\)-SHKT manifold \(M^{4n}\) is conformally balanced, \(\theta_{\omega_j} = 2d\phi, \phi\) a function on \(M^{4n}\), then \(M^{4n}\) is hyper-Kähler. The latter also applies cyclicly for \(J\) and \(K\). These statements follow because these \((k_1, k_2, k_3)\)-SHKT structures are special cases of the \((\ell|k)\)-SCYT structures that appear in theorems (1.1) and (1.2). Because of this, we shall not elaborate further.

We shall construct several examples of k-SKT. Some examples of 2-SKT and 2-SCYT manifolds have already been given in [15]. Here we shall extend a method initially used by Swann [26] to construct examples of 1-SKT and HKT manifolds to give new examples of k-SKT and k-SCYT manifolds. In particular, we shall construct several simply connected examples.

## 2 Vanishing theorems for 2-SKT and 2-SCYT manifolds

It is instructive to first prove theorems (1.1) and (1.2) for 2-SKT and 2-SCYT manifolds and then extend the results to the most general case. In particular, this will establish the results of theorems (1.1) and (1.2) for 2-Gauduchon manifolds. As it has been mentioned in the introduction the theorems (1.1) and (1.2) have already been proven for 1-SKT and 1-SCYT manifolds, respectively [21, 22]. Essentially the proof extends to 1-Gauduchon manifolds, see also [8]. We shall demonstrate the proof of theorem (1.3) after those for the theorems (1.1) and (1.2) for \((\ell|k)\)-SKT and \((\ell|k)\)-SCYT manifolds. Before we proceed with this, we establish our conventions.
2.1 Conventions and preliminaries

Let \((M, J, g)\) a Hermitian manifold of dimension \(2n\). Then, the Hermitian form\(^1\) is defined as \(\omega(X,Y) := g(X, JY)\) or equivalently in components

\[\omega_{ij} = g_{ik}J^k_j.\]

The torsion \(H\) of \(\hat{\nabla}\) is \(H = -i(\partial - \bar{\partial})\omega\) or equivalently

\[H(X,Y,Z) = d\omega(X,Y,Z) = Jd\omega(X,Y,Z) = -d\omega(JX,JY,JZ),\]

where we have use that \(JF(X_1, \ldots, X_r) := (-1)^rF(JX_1, \ldots, JX_r)\) for a \(r\)-form \(F\).

For the curvature we use the convention \(\hat{R} = [\hat{\nabla}, \hat{\nabla}] - \nabla[^\parallel]\). Consequently, \(\hat{\rho}(X,Y) = \hat{R}(X,Y,e_i,Je_i), b = \rho(Je_i,e_i),\) where we use Einstein summation conventions, i.e. repeated indices are summed over.

The Lee form \(\theta := \delta\omega \circ J\) of the Hermitian manifold is given in terms of \(H\) as

\[\theta(X) = -\frac{1}{2}H(JX, Je_i) = \frac{1}{2}g(H(JX),\omega) = \frac{1}{2} (\omega \omega H(JX)), \quad \theta_1 = \frac{1}{2} J^k_i H_{kj} \omega^{j\ell}.\]

Moreover, we define the \((1,1)\) form\(^2\)

\[\lambda(X,Y) := d\omega(X,Y, e_i, Je_i) = -g(d\omega(X,Y), \omega) = -(\omega \omega d\omega(X,Y)),\]

i.e. \(\lambda_{ij} = -dH_{ijkl}\omega^{kl}\). We also write

\[||H||^2 = H(e_i,e_j)H(e_i,e_j,e_k) = H_{ijk} H^{ijk}.\]

As a volume form, we use \(dvol(M) = \frac{1}{n!} \omega^n\), where \(\omega^p = \wedge^p \omega\). In particular \(\frac{1}{p!} \star \omega^p = \frac{1}{(n-p)!} \omega^{n-p}\), where \(\star\) is the Hodge star operator.

2.2 The \(\alpha_2\) function

On a \(2n\) dimensional hermitian manifolds \((M,g,J)\), we define the function \(\alpha_k(\omega)\) by

\[\alpha_k(\omega) = 2k \star (i\partial \bar{\partial} \omega^k \wedge \omega^{n-k-1}) = \star(d(\omega^{k-1} \wedge H) \wedge \omega^{n-k-1}).\] (2.1)

Clearly, \(\alpha_k(\omega) = 0\) provided \((M,g,J)\) admits a \((\ell|k)\)-SKT structure.

Using \(\omega^{n-k-1} = \frac{(n-k-1)!}{(k+1)!} \star \omega^{k+1}\), we have the expression

\[
\alpha_k(\omega) = \frac{(n-k-1)!}{(k+1)!} \star (d(\omega^k \wedge H) \wedge \star \omega^{k+1}) = g(d(\omega^{k-1} \wedge H), \omega^{k+1}) \\
= (-1)^{k+1} \frac{(n-k-1)!}{(k+1)!} 2^{k+1} d(\omega^{k-1} \wedge H)(e_{i_1}, Je_{i_1}, \ldots, e_{i_{k+1}}, Je_{i_{k+1}}) (2.2)
\]

First we calculate \(\alpha_2\). For this let us consider the following.

\(^1\)There is a sign difference from the definition of \(\omega\) given in [11] which is important in the proof of theorem (1.3).
Lemma 2.1. On a 2n-dimensional hermitian manifold we have

\[-\omega \cdot d(\omega \wedge H) = (8 - 2n) dH + \omega \wedge \lambda + 2(JH \wedge J\theta) + 2(H \wedge \theta) + 2J(e_i \wedge H) \wedge (e_i \wedge H). \tag{2.3}\]

In particular, on a 2-SKT manifold of dimension 2n we have:

\[(8 - 2n) dH + \omega \wedge \lambda + 2(JH \wedge J\theta) + 2(H \wedge \theta) + 2J(e_i \wedge H) \wedge (e_i \wedge H) = 0. \tag{2.4}\]

Proof. The identity \(d(\omega \wedge H) = d\omega \wedge H + \omega \wedge dH\) also reads

\[d(\omega \wedge H) = -JH \wedge H + \omega \wedge dH. \tag{2.5}\]

A straightforward calculation using our conventions reveals that

\[
\begin{align*}
(-JH \wedge H)(X, Y, Z, U, e_i, J e_i) &= [2JH \wedge J\theta + 2H \wedge \theta \\
&\quad + [2J(e_i \wedge H) \wedge (e_i \wedge H)](X, Y, Z, U); \tag{2.6} \end{align*}
\]

\[
(\omega \wedge dH)(X, Y, Z, U, e_i, J e_i) = [\omega \wedge \lambda + (8 - 2n) dH](X, Y, Z, U). \tag{2.7}
\]

The last two equalities together with (2.5) imply (2.3).

The 2-SKT condition \(d(\omega \wedge H) = 0\) and (2.3) give (2.4). Q.E.D.

Corollary 2.2. On a 2n-dimensional Hermitian manifold, we have

\[
\omega \cdot (d(\omega \wedge H)) = (12 - 4n) \lambda + \lambda(e_i, J e_i) \omega + 8 \theta \wedge J \theta \\
- 8(J \theta) \wedge H - 8J((J \theta) \wedge H) + 4J(e_i e_j \wedge H) \wedge (e_i e_j \wedge H). \tag{2.8}
\]

In particular, on a 2-SKT manifold of dimension 2n we have:

\[(4n - 12) \lambda = \lambda(e_i, J e_i) \omega + 8 \theta \wedge J \theta - 8(J \theta) \wedge H - 8J((J \theta) \wedge H) + 4J(e_i e_j \wedge H) \wedge (e_i e_j \wedge H). \tag{2.9}\]

Proof. Taking the traces in (2.3), we get

\[
\begin{align*}
d(\omega \wedge H)(e_i, J e_j, e_j, J e_j, X, Y) &= (12 - 4n) \lambda(X, Y) + \lambda(e_i, J e_i) \omega(X, Y) \\
&\quad - 8\theta(X) \theta(JY) + 8\theta(JX) \theta(Y) - 8H(X, Y, J \theta) - 8H(JX, JY, J \theta) \\
&\quad - 4H(JX, e_i, e_j) H(Y, e_i, e_j) + 4H(X, e_i, e_j) H(JY, e_i, e_j),
\end{align*}
\]

which proves the assertion. Q.E.D.

Proposition 2.3. On a 2n-dimensional Hermitian manifold the function \(\alpha_2\) is given by

\[
\alpha_2(\omega) = (n - 3)! \left( (n - 2) \delta \theta + (n - 3) \left[ ||\theta||^2 - \frac{1}{6} ||H||^2 \right] \right) \tag{2.9}
\]

In particular, on a 2-nd Gauduchon manifold as well as on a 2-SKT manifold we have

\[(n - 2) \delta \theta + (n - 3) \left[ ||\theta||^2 - \frac{1}{6} ||H||^2 \right] = 0. \tag{2.10}\]

Proof. The trace in (2.8) together with (2.2) gives
Lemma 2.4. On a 2n-dimensional hermitian manifold the function $\alpha_2(\omega)$ is given by

$$\alpha_2(\omega) = \frac{(n-3)!}{2^3} \left[ (n-2)\lambda(e_i, Je_i) - 8||\theta||^2 + \frac{4}{3}||H||^2 \right]$$ \hspace{1cm} (2.11)

On a 2-SKT manifold of dimension 2n we have:

$$(n - 2)\lambda(e_i, Je_i) = 8||\theta||^2 - \frac{4}{3}||H||^2 = 0.$$ \hspace{1cm} (2.12)

To complete the proof of the proposition, we use the identity

$$\lambda(e_i, Je_i) = 8\delta\theta + 8||\theta||^2 - \frac{4}{3}||H||^2,$$ \hspace{1cm} (2.13)

established in [1, 21] in the context of KT manifolds. Combining (2.13) with (2.11), it is straightforward to prove (2.9).

Q.E.D.

Corollary 2.5. On a 2n-dimensional 2-SKT manifold, one has

$$(n - 3)\lambda(e_i, Je_i) = -8\delta\theta ,$$ \hspace{1cm} (2.14)

and

$$(n - 2)\lambda(e_i, Je_i) = 8||\theta||^2 - \frac{4}{3}||H||^2 .$$ \hspace{1cm} (2.15)

Proof. The proof of the above two equations follows from (2.12) and (2.13). Q.E.D.

2.3 Proof of theorems (1.1) and (1.2)

Proof of theorem (1.1): Now let us turn to the proof of theorem (1.1) for 2-SKT manifolds. For this, we use the result in [22, Theorem 4.1] that the plurigenera, $p_m(J), m > 0$, of a KT manifold vanish provided that

$$\hat{b} + ||C||^2 + \frac{1}{4} \sum_{i=1}^{2n} \lambda(e_i, Je_i) > 0 ,$$ \hspace{1cm} (2.16)

where $\hat{b}$ is the $\omega$-trace of the Ricci form $\hat{\rho}$ of $\tilde{\nabla}$ and $C$ is the torsion of the Chern connection. The fact that $H$ is of type (1,2)+(2,1) implies

$$H(Je_k, Je_i, e_j)(e_k, e_i, e_j) = \frac{1}{3}||H||^2.$$ \hspace{1cm} (2.17)

We recall that the torsion $C$ of the Chern connection of a KT manifold $(M, g, J)$ is expressed in terms of $H$ as,

$$g(C(X, Y), Z) = \frac{1}{2}H(X, JY, JZ) + \frac{1}{2}H(JX, Y, JZ),$$

see e.g. [21]. Using this, (2.17) and that $H$ is a (1,2)+(2,1)-form, one finds that

$$||C||^2 = \frac{1}{3}||H||^2.$$ \hspace{1cm} (2.18)
Next using (2.18) and (2.15), one has that
\[
\hat{b} + ||C||^2 + \frac{1}{4} \lambda(e_i, J e_i) = \hat{b} + \frac{1}{3}||H||^2 + \frac{2}{n-2}||\theta||^2 - \frac{1}{3(n-2)}||H||^2 \\
= \hat{b} + \frac{1}{3}(1 - \frac{1}{n-2})||H||^2 + \frac{2}{n-2}||\theta||^2 > 0 \tag{2.19}
\]
which is positive for \(n > 2\) according to the condition of theorem (1.1). This establishes theorem (1.1) for \(k = 2\). \(Q.E.D.\)

Proof of theorem (1.2): Now, let us turn to theorem (1.2) for 2-SCYT manifolds. It readily follows from theorem (1.1). Since the holonomy of the connection with skew-symmetric torsion \(\hat{\nabla}\) is in \(SU(n)\), \(\hat{b} = 0\), and the inequality (2.19) is always satisfied provided that \(H\) does not vanish. Clearly the above statement also holds under the weaker assumption that \(\hat{b} = 0\). \(Q.E.D.\)

Corollary 2.6. A compact, conformally balanced, 2-SCYT manifold is Calabi-Yau.

Proof: This is a special case of theorem (1.3) which we shall demonstrate later. This is also an extension of a similar theorem proved in [21] for conformally balanced 1-SCYT manifolds. It follows from [25] that a 2n-dimensional conformally balanced CYT manifold admits a holomorphic \((n,0)\)-form. Combining this with the statement of theorem (1.2) for 2-SCYT manifolds, one concludes that \(H = 0\), and so \(M\) is Calabi-Yau. \(Q.E.D.\)

3 Vanishing theorems for 2n-dimensional k-SKT and k-SCYT manifolds

3.1 The \(\alpha_k\) function

We have shown theorems (1.1) and (1.2) for k-SKT and k-SCYT manifolds for \(k = 1, 2\). It remains to extend these to all \((\ell|k)\)-SKT and \((\ell|k)\)-SCYT manifolds for \(k > 2, \ell > 0\). Instrumental in this is the generalization of (2.9) and (2.10) for \(k > 2\).

Proposition 3.1. On a 2n-dimensional Hermitian manifold the function \(\alpha_k\) is given by
\[
\alpha_k(\omega) = (n-3)![(n-2)\delta \theta + (n-k-1)][||\theta||^2 - \frac{1}{6}||H||^2] \tag{3.1}
\]
In particular, a 2n-dimensional Hermitian manifold is generalized k-Gauduchon, if and only if, the next identity holds
\[
(n-2)\delta \theta + (n-k-1)[||\theta||^2 - \frac{1}{6}||H||^2] = 0. \tag{3.2}
\]

Proof. First we show

Lemma 3.2. Let \(M^{2n}\) be a Hermitian manifold. Then
\[
\alpha_k(\omega) = \frac{(n-3)!}{2^k}[(n-2)^{2n}\sum_{i=1}^{2n} \lambda(e_i, J e_i) - 8(k-1)||\theta||^2 + \frac{4}{3}(k-1)||H||^2] \tag{3.3}
\]
To prove the lemma, we write (2.1) as
\[
\alpha_k(\omega) = *(d(\omega^{k-1} \wedge H) \wedge \omega^{n-k-1}) = *\left( [(k-1)\omega^{k-2} \wedge d\omega \wedge H + \omega^{k-1} \wedge dH] \wedge \omega^{n-k-1} \right)
\]
\[
= *\left( \omega^{n-3} \wedge [(k-1)d\omega \wedge H + \omega \wedge dH] \right) = *\left( \omega^{n-3} \wedge [- (k-1)JH \wedge H + \omega \wedge dH] \right)
\]
\[
= \frac{(n-3)!}{3!} * \left( \omega^3 \wedge [- (k-1)JH \wedge H + \omega \wedge dH] \right)
\]
\[
= g(\omega^3, [- (k-1)JH \wedge H + \omega \wedge dH])
\]
\[
= -\frac{(n-3)!}{3! \cdot 2^3} \left( - (k-1)JH \wedge H + \omega \wedge dH \right) (e_i, Je_i, e_j, Je_j, e_k, Je_k)
\]
\[
= \frac{(n-3)!}{2^3} \left( (n-2) \sum_{i=1}^{2n} \lambda(e_i, Je_i) - 8(k-1)||\theta||^2 + \frac{4}{3}(k-1)||H||^2 \right),
\]
where we used the (2.6) and (2.7) and their traces. This completes the proof of the lemma.

Next to prove the proposition, substitute (2.13) into (3.3) to get (3.1). This completes the proof.

Q.E.D.

A generalization of (2.14) and (2.15) is as follows.

Corollary 3.3. Let \( M \) be a 2n-dimensional \((\ell|k)\)-SKT manifold, then

\[
(n - k - 1) \sum_{i=1}^{2n} \lambda(e_i, Je_i) = -8(k-1)\delta\theta
\]

and

\[
(n - 2) \sum_{i=1}^{2n} \lambda(e_i, Je_i) = 8(k-1)||\theta||^2 - \frac{4}{3}(k-1)||H||^2.
\]

Proof. It follows immediately as an application of (2.13), (3.3) and the fact that \( \alpha_k(\omega) = 0 \) for all \((\ell|k)\)-SKT manifolds.

Q.E.D.

Integrate (3.2) over a compact \( M \) observing that

\[
||\delta\omega||^2 = ||J\delta\omega||^2 = ||\delta\omega||^2 \quad \text{and} \quad ||H||^2 = \frac{1}{4}||d\omega||^2
to obtain

Corollary 3.4. Let \((M, \omega)\) be a compact 2n-dimensional \((\ell|k)\)-SKT manifold, then for \( k < 1 < n - 1 \) we have

\[
\int_M ||\delta\omega||^2 dvol(M) = \frac{1}{6}||d\omega||^2 dvol(M).
\]

3.2 Proof of Theorems (1.1) and (1.2)

Proof of theorem (1.1): To show this for all \((\ell|k)\)-SKT manifolds, we apply again the inequality (2.16) established in \[22\, Theorem 4.1\] as a condition for the vanishing of plurigenera for KT manifolds and use (3.6). One finds that

\[
\hat{b} + ||C||^2 + \frac{1}{4} \lambda(e_i, Je_i) = \hat{b} + \frac{1}{3}||H||^2 + \frac{2(k-1)}{n-2}||\theta||^2 - \frac{k-1}{3(n-2)}||H||^2
\]
\[ \hat{b} + \frac{n - k - 1}{3(n - 2)} \|H\|^2 + \frac{2(k - 1)}{n - 2} \|\theta\|^2 > 0. \tag{3.7} \]

which is positive for \( n > 2 \) according to the condition of theorem (1.1).

\textit{Proof of theorem (1.2):} Now if \( M \) is \((\ell|k)\)-SCYT, then one has that \( \hat{b} = 0 \). This follows from the requirement that the holonomy of the Hermitian connection with skew-symmetric torsion, \( \hat{\nabla} \), is contained in \( SU(n) \). It is clear then that the inequality (3.7) always holds and so \( p_m(J) = 0 \), \( m > 0 \) for all \((\ell|k)\)-SCYT manifolds.

Q.E.D.

Theorems (1.1) and (1.2) can be extended to the generalized \((\ell|k)\)-SKT and \((\ell|k)\)-SCYT manifolds as follows.

\textbf{Corollary 3.5.} \textit{Let } \( M^{2n} \text{ be a non Kähler generalized } (\ell|k)\text{-SKT manifold, then } p_m(J) = 0, m > 0 \text{, provided that} \}

\[ \hat{b} + \frac{n - k - 1}{3(n - 2)} \|H\|^2 + \frac{2(k - 1)}{n - 2} \|\theta\|^2 + \frac{2n(n - 1)}{(k + \ell + 1)!} \alpha_{k,\ell} > 0, \quad n \neq 2, \]

\textit{where } \( \alpha_{k,\ell} \) \textit{is given in (1.4). In particular, the plurigenera vanish for every generalized } \((\ell|k)\text{-SCYT manifold for which } \alpha_{k,\ell} \geq 0. \}

The proof of this follows immediately from those of theorems (1.1) and (1.2) above.

Note that \( \alpha_k = \frac{n!}{(k + \ell + 1)!} \alpha_{k,\ell}. \)

\textbf{Corollary 3.6.} \textit{A compact, conformally balanced, } \((\ell|k)\text{-SCYT manifold is Calabi-Yau.} \}

\textit{Proof :} This is a special case of theorem (1.3) and it follows directly from the results of [21] together with theorems (1.1) and (1.2). The proof is similar to that given as for the case of conformally balanced 2-SCYT manifolds in section 2. \textit{Q.E.D.} \]

\subsection{3.3 Proof of Theorem (1.3)}

It has been shown in [11] that on a compact Hermitian manifold there is a unique constant \( \gamma_k(\omega) \) invariant under biholomorphisms which depends smoothly on \( \omega \) such that the \( k \)-generalized Gauduchon equation \[^1\]

\[ \frac{i}{2} e^{-u} \partial \bar{\partial}(e^u \omega^k) \wedge \omega^{n-k-1} = -\gamma_k(\omega) \omega^n, \tag{3.8} \]

has a solution \( u \), where \( u \) is uniquely determined up to a constant. In particular, a Hermitian manifold \( M \) admits a generalized \( k \)-Gauduchon metric in the conformal class of \( \omega \), if and only if \( \gamma_k = 0 \) [11, Proposition 8].

The existence of generalized \( k \)-Gauduchon metrics depends crucially on the sign of \( \gamma_k \). It is also shown [11, Proposition 11] that the sign of \( \gamma_k(\omega) \) remains constant in the conformal class of \( \omega \). Moreover, [11, Proposition 12], in our notations, tells us that \( \gamma_k(\omega) > 0 (= 0, or < 0) \) if there exists a hermitian form \( \omega' \) in the conformal class of \( \omega \) such that \( \alpha_k(\omega') < 0 (= 0, or > 0) \), respectively.

\[^1\]The sign difference in (3.8) from that in [11] is conventional and it is due to a sign difference in the definition of Hermitian form \( \omega \).
Suppose now that $\omega$ is conformally balanced. In such case, there is a function $f$ on $M$ specified up to a constant such that $\tilde{\omega} = e^f \omega$ is balanced, i.e., the corresponding Lee form $\tilde{\theta} = 0$. The next lemma makes [11, Lemma 16] more precise and proofs our Theorem 1.3.

**Lemma 3.7.** On a compact balanced non-Kähler Hermitian manifold $(X, \tilde{\omega})$ the constant

$$\gamma_k(\tilde{\omega}) > 0, \quad \text{for} \quad 1 \leq k \leq n-2$$

**Proof.** To proof the lemma substitute $\tilde{\theta} = 0$ into (3.1) to conclude

$$\alpha_k(\tilde{\omega}) = -(n-3)! \frac{n-k-1}{6} ||H(\tilde{\omega})||^2.$$

Therefore for $k \neq n-1$, $\alpha_k(\tilde{\omega}) < 0$, provided that $H(\tilde{\omega}) \neq 0$. Hence $\gamma_k(\tilde{\omega}) > 0$ and the lemma follows. Q.E.D.

To complete the proof of the theorem 1.3 recall that the sign of $\gamma_k$ does not depend on the conformal class of $\omega$ and if $H(\tilde{\omega}) \neq 0$, then also $\gamma_k(\omega) > 0$. Now from the assumptions of theorem (1.3), $\omega$ is $(\ell|k)$-SKT and therefore generalized k-Gauduchon which requires that $\gamma_k(\omega) = 0$. This leads to a contradiction unless $H(\tilde{\omega}) = 0$ and so $\tilde{\omega}$ is Kähler which completes the proof of the theorem. Q.E.D.

### 3.4 Locally conformally Kähler manifolds

It is observed in [1] that 1-SKT locally conformally Kähler manifold must be Kähler. Recently, it is shown in [11] that the standard hermitian structure on $S^5 \times S^1$ which is locally conformally Kähler has $\gamma_1 < 0$. We show that this is true in general.

We recall that a Hermitian manifold $(X, \omega)$ is **locally conformally Kähler** if there locally exists a conformal metric which is Kähler and this is not true globally. For $n > 2$ this condition is equivalent to the equation $d\omega = \frac{1}{n-1} \theta \wedge \omega$ which, in terms of $H$, reads

$$H = \frac{1}{n-1} J\theta \wedge \omega$$

We have

**Proposition 3.8.** On a compact locally conformally Kähler $2n$-manifold $(X, \omega)$ the constant $\gamma_k(\omega)$ is negative for $1 \leq k \leq n-2$,

$$\gamma_k(\omega) < 0, \quad \text{for} \quad 1 \leq k \leq n-2.$$

In particular, compact locally conformally Kähler structure does not admit $(\ell|k)$-SKT structure.

**Proof.** Let $\tilde{\omega}$ be the Gauduchon structure globally conformal to $\omega$, which, in particular is locally conformally Kähler and not Kähler. Then we have

$$\tilde{\delta}\tilde{\theta} = 0, \quad \tilde{H} = \frac{1}{n-1} J\tilde{\theta} \wedge \tilde{\omega}, \quad ||\tilde{\theta}||^2 \neq 0,$$

(3.9)
where $\tilde{\theta}$ and $\tilde{H}$ are the Lee form and 3-form torsion associated to $\tilde{\omega}$, respectively. A straightforward calculation yields

$$||\tilde{H}||^2 = \frac{6}{n-1}||\tilde{\theta}||^2.$$  

(3.10)

To prove the assertion substitute (3.9) and (3.10) into (3.1) to conclude

$$\alpha_k(\tilde{\omega}) = (n-3)!(n-k-1)
\left[||\bar{\theta}||^2 - \frac{1}{6}||H(\tilde{\omega})||^2\right] = (n-3)!(n-k-1)\frac{n}{n-1}||\tilde{\theta}||^2.\quad (\tilde{\theta})$$

Therefore for $k \neq n-1$, $\alpha_k(\tilde{\omega}) > 0$. Consequently, $\gamma_k(\tilde{\omega}) < 0$. 

Q.E.D.

4 Fibrations and k-SKT structures

4.1 k-SKT structures on product manifolds

We shall focus on the construction of k-SKT and k-SCYT structures as they are more restrictive than $(\ell|k)$-SKT and $(\ell|k)$-SCYT, respectively. In particular, if a Hermitian manifold admits a k-SKT or k-SCYT structure, then it also admits a $(\ell|k)$-SKT or $(\ell|k)$-SCYT for all $\ell$. It is straightforward to construct k-SKT structures on products of manifolds. In particular one has the following.

**Proposition 4.1.** The product $M^{2m} \times N^4$ where $M^{2m}$ is a Kähler manifold and $N^4$ is Hermitian 4-manifold admits a k-SKT structure for all $k$.

**Proof.** Let $\omega(4)$ be the Hermitian form of a Gauduchon structure on $N^4$. Then $N$ is an 1-SKT manifold with respect to $\omega(4)$, i.e. $dH(4) = 0$ as this coincides with the co-closure of the Lee form. If $\omega(2m)$ is the Kähler form on $M^{2m}$, then

$$d((\omega(2m) + \omega(4))^k \wedge H(4)) = d(\omega(2m)^k \wedge H(4)) = \omega(2m)^k \wedge dH(4) = 0.$$

This proves the proposition. Q.E.D.

For an explicit example one can take $N^4 = S^1 \times S^3$ and $M^{2m} = \mathbb{C}P^m$.

Similarly, it is straightforward to see the following.

**Proposition 4.2.** Let $N$ be a k-SKT manifold for $k \leq \ell$. Then the product $M^{2m} \times N$, where $M^{2m}$ is a Kähler manifold, is also k-SKT manifold for all $k \leq \ell$.

4.2 The Swann twist

KT and CYT manifolds can be constructed using torus fibrations, see [3, 14, 16]. These provide a large class of examples and so some of them may admit the more restrictive k-SKT and k-SCYT structures. Although this can be done directly by consider torus fibrations over suitable base spaces, it is advantageous to use a construction proposed by Swann [26] to find 1-SKT and (1,1,1)-SHKT metrics. This will be adapted to give new examples k-SKT and k-SCYT manifolds. We begin with a summary of the Swann’s twist construction.
Let $M^{2n}$ be a Hermitian manifold equipped with a $T^p$ torus action $A_M$ which preserves the Hermitian structure. Denote the Lie algebra of the group $T^p$ acting on $M^{2n}$ with $a_M$. In addition, let $P$ be a $T^p$ principal bundle over $M^{2n}$ equipped with a connection $\lambda$. Clearly $\lambda \in \Omega^1(P, a_P)$, where $a_P$ is the Lie algebra of $T^p$ which acts on $P$ from the right. Suppose now that the $A_M$ group action on $M^{2n}$ can be lifted to and a $T^p$ action $A_P$ on $P$. If $\xi$'s are the vector fields generated by the action of $A_M$ on $M^{2n}$, then the $A_P$ action on $P$ is generated by the vector fields

$$\hat{\xi} = \tilde{\xi} + \beta \rho ,$$

where $\tilde{\xi}$ is the horizontal lift of $\xi$ with respect to $\lambda$, i.e. $\lambda(\tilde{\xi}) = 0$, $\rho$ are the vectors generated by the right action of $T^p$ on $P$ and $\beta \in \Omega^0(P, a_P \otimes a_M^*)$. Necessary conditions for the $T^p$ action on $M^{2n}$ to lift to $P$ are

$$\mathcal{L}_\xi F = 0, \quad i_\xi F = d\beta, \quad i_\xi i_\xi F = 0, \quad (4.1)$$

and $\beta = \pi^* \beta$, where $\beta \in \Omega^0(M, a_P \otimes a_M^*)$ with $\mathcal{L}_\xi \beta = 0$, $\pi$ is the projection of $P$ onto $M^{2n}$ and $\pi^* F = d\lambda$ is the curvature of $\lambda$.

Provided that (4.1) holds, there is a lift of the $A_M$ action to $P$ which covers $A_M$ and commutes with the right action on $P$. This lift is not unique because $\beta$ is determined up to a constant $\nu$. For every choice $\nu \in \Omega^0(M, a_P \otimes a_M^*)$ $\otimes \mathbb{Z}$, one finds another lift of the $A_M$ action. All these lifts are free provided that $A_M$ group action on $M^{2n}$ is free.

The Swann twist is a new fibration which is constructed by taking the quotient of $P$ with respect to $A_P$ of $T^p$. If $A_P$ is a free action, then $W = P/A_P$ is a manifold. Otherwise, it may have orbifold singularities. For the explicit examples we consider, $A_P$ is a free action. Under certain conditions, the Hermitian structure on $M^{2n}$ can be inherited on $W^{2n}$. For this, one should induce a metric and a Hermitian form on $W^{2n}$ from those on $M^{2n}$. Let us first begin with forms. Given a form $\tau \in \Omega^\ell(M)$, one can define $\pi^* \tau \in \Omega^\ell(P)$. The aim is to construct a new form $\hat{\tau} \in \Omega^\ell(P)$ such that $\hat{\tau} = \pi_W^* \tau_W$, where $\pi_W$ is the projection of $P$ onto $W^{2n}$. For this assume that $\beta$ is invertible and take

$$\hat{\tau} = \pi^* \tau - \lambda_{\beta^{-1}}^A \wedge \pi^* (i_{\xi_A} \tau) - \cdots (-1)^{p(\ell)} \frac{1}{\ell!} \lambda_{\beta^{-1}}^{A_\ell} \wedge \cdots \wedge \lambda_{\beta^{-1}}^{A_1} \pi^* (i_{\xi_{A_1}} \cdots i_{\xi_A} \tau) , \quad (4.2)$$

where $\lambda_{\beta^{-1}} = \beta^{-1} \lambda \in \Omega^1(P, a_M)$, and $p(\ell) = 1$ if $[\ell/4] = 1, 2$ and $p(\ell) = 0$ if $[\ell/4] = 3, 0$. One can verify that $i_\xi \hat{\tau} = 0$ and $\mathcal{L}_\xi \hat{\tau} = 0$ provided that $L_\xi \tau = 0$. Therefore $\hat{\tau}$ projects down onto $W^{2n}$, i.e. there is a $\tau_W$ such that $\hat{\tau} = \pi_W^* \tau_W$.

Observe that to determine $\tau_W$ it suffices to know $\hat{\tau}$ up to $\lambda$-terms. This is because all the components of $\hat{\tau}$ proportional to $\lambda$'s are determined by the $\lambda$ independent term and the vector fields $\xi$. In [26], this is referred as $\mathcal{H}$-relation or equivalence, where $\mathcal{H} = \text{Ker} \lambda$. Because of this, it suffices to establish the various relations up to $\mathcal{H}$-equivalence. Suppose now that $\pi^*_M \tau = \mathcal{H} \chi$, then a direct application of (4.2) reveals that

$$\pi^*_M d\tau = \mathcal{H} d\chi - F^A \wedge \pi^* (i_{\xi_A} \tau) , \quad (4.3)$$

where $F = \beta^{-1} F \in \Omega^2(M, a_M)$.

Using this construction, we can lift to $P$ both the metric and Hermitian form of $M^{2n}$ as

$$\hat{g} = \pi^* g - 2 \lambda_{\beta^{-1}}^A \otimes \pi^* \eta_A + g(\xi_A, \xi_B) \lambda_{\beta^{-1}}^A \otimes \lambda_{\beta^{-1}}^B ,$$
where $\eta_A(X) = g(\xi_A, X)$. Provided that both $g$ and $\omega$ are invariant under $A_M$, these can be projected down to $W^{2n}$ to define an almost Hermitian structure on $W^{2n}$. Using the $\mathcal{H}$-equivalence equivalence one can write

$$\pi^*_W g_W = \mathcal{H} \pi^* g, \quad \pi^*_W \omega_W = \mathcal{H} \pi^* \omega,$$

It remains to find the conditions for the almost complex structure on $W^{2n}$ to be integrable. For this let $\mathcal{A}$ the set of all Killing vector fields in $M^{2n}$ and $\mathcal{A}_I = \mathcal{A} \cap I \mathcal{A}$. Clearly one can find basis $e_1, \ldots, e_{2s}$, $e_{2s+1}, \ldots, e_k$ in $\mathcal{A}$ which is an extension of the basis $e_1, \ldots, e_2s$ of $\mathcal{A}_I$ with $I(e_{2j-1}) = e_{2j}$. Next choose a basis $e^\alpha$ of (1,0)-forms in $M^{2n}$, lift them to $P$ and define $\tilde{e}^\alpha$. Then

$$d\tilde{e}^\alpha = \pi^* d\epsilon^\alpha - \mathcal{F} \pi^* \epsilon^\alpha.$$

The complex structure on $W^{2n}$ is integrable iff the (0,2)-part of the above 2-form vanishes. The (0,2) component of $d\epsilon^\alpha$ vanishes as consequence of the integrability of the complex structure on $M^{2n}$. In addition the (0,2) component of the term involving $F$ also vanishes provided that

$$(\mathcal{F}_{2j-1} + i\mathcal{F}_{2j})^{0,2} = 0, \quad j = 1, \ldots, s,$$

$$\mathcal{F}_{2r}^{2,0} = \mathcal{F}_{r}^{0,2} = 0, \quad r = 2s + 1, \ldots, k,$$

where $F_i = \mathcal{F}(e_i)$. In particular the complex structure on $W^{2n}$ is always integrable if $F$ is a (1,1)-form on $M^{2n}$.

It remains to determine the torsion $H_W$. For this observe that

$$\pi^*_W H_W = \mathcal{H} \pi^* H + i_I \mathcal{F}^A \wedge \pi^* (i_{\xi_A} \omega) - \mathcal{F}^A \wedge \pi^* \eta_A,$$

where again $\eta_A(X) = g(\xi_A, X)$. This follows directly from (4.4) and (4.3) using $H = -i(\partial - \bar{\partial})\omega$. In particular, if $F$ is a (1,1)-form, this simplifies to

$$\pi^*_W H_W = \mathcal{H} \pi^* H - \mathcal{F}^A \wedge \pi^* \eta_A.$$

### 4.3 k-SKT structures from Kähler manifolds

As a starting point let us take $X$ to be a Kähler manifold with Kähler form $\omega_X$ and take $M = X \times T^{2m}$. Assuming that $T^{2m}$ is also Kähler with respect to the standard flat metric and complex structure, clearly $M$ is a Kähler manifold with Kähler form $\omega_X + \omega_T$, where $\omega_T$ is the Kähler form of $T^{2m}$. Next assume that the $A_M$ action on $M$ is the standard one, where the vector fields $\xi$ generate the standard basis in the homology of $T^{2m}$. Choose now a principal $T^{2m}$ bundle over $X$ with $F$ a (1,1)-form on $X$. By construction $i_{\xi} F = 0$, and so $\beta$ is a constant matrix. Furthermore, the condition $\beta \in \Omega^0(M, a_P \otimes a^*_M) \otimes \mathbb{Z}$ implies that $\beta \in (\Lambda_P \otimes \Lambda^*_M) \otimes \mathbb{Z}$, where $\Lambda_P$ and $\Lambda_M$ are the lattices used to construct the tori of the typical fibre of $P$ and that of the torus action on $M$. 
Next using some constant invertible matrix $\beta$, let us perform a Swann twist to find

$$\pi_W^* H_W = -\mathcal{F}^A \wedge \pi^* \eta_A.$$ 

The k-SKT condition on $W$ is satisfied provided that

$$\mathcal{F}^A \wedge \mathcal{F}^B \wedge i_{\xi_A} (\omega_X + \omega_T)^{k-1} \wedge \pi^* \eta_B + \mathcal{F}^A \wedge \mathcal{F}^B \wedge (\omega_X + \omega_T)^{k-1} g_T(\xi_A, \xi_B) = 0. \quad (4.5)$$

For $k = 1$, this becomes

$$\mathcal{F}^A \wedge \mathcal{F}^B g_T(\xi_A, \xi_B) = 0,$$

which is the SKT condition derived in [26].

### 4.3.1 2-SKT manifolds

Let us now consider the $k = 2$ case. The condition \[(4.5)\] becomes

$$g_T(\xi_A, \xi_B) \mathcal{F}^A \wedge \mathcal{F}^B \wedge \omega_X = 0,$$

$$\mathcal{F}^A \wedge \mathcal{F}^B \wedge i_{\xi_A} \omega_T \wedge \pi^* \eta_B + \mathcal{F}^A \wedge \mathcal{F}^B \wedge \omega_T \ g_T(\xi_A, \xi_B) = 0. \quad (4.6)$$

**Proposition 4.3.** If $M = X \times T^2$ and $g_T(\xi_A, \xi_B) = \delta_{AB}$, then $W$ is 2-SKT if and only if

$$\delta_{AB} \mathcal{F}^A \wedge \mathcal{F}^B \wedge \omega_X = 0,$$

**Proof:** It is easily seen that the second condition \[(4.6)\] is automatically satisfied. The first condition in \[(4.6)\] gives the restriction stated above. Q.E.D.

It is therefore clear that if $W$ is SKT, then it is also 2-SKT. However, there are 2-SKT structures which are not induced from an 1-SKT one. To find one such example, take $X^6 = X^4 \times X^2$ and $m = 2$, where $X^4$ and $X^2$ are Kähler manifolds with Kähler forms $\omega_{(4)}$ and $\omega_{(2)}$, respectively. Moreover, we choose $g_T(\xi_A, \xi_B) = \delta_{AB}$ as it is required in above proposition but take

$$\beta^{-1} = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}, \quad (4.7)$$

and $F^1$ to have support over $X^4$ while $F^2$ to have support over $X^2$ with $[F^1] \in H^2(X^4, \mathbb{Z})$ and $[F^2] \in H^2(X^2, \mathbb{Z})$. The 2-SKT condition then becomes

$$(p_1^2 + p_2^2) F^1 \wedge F^1 \wedge \omega_{(2)} + 2(p_1 q_1 + p_2 q_2) F^2 \wedge F^1 \wedge \omega_{(4)} = 0. \quad (4.8)$$

The condition that $\beta \in (\Lambda_P \otimes \Lambda_M^*) \otimes \mathbb{Z}$ implies that $p_1, p_2, q_1, q_2$ are integers up to possibly multiplying them with $\det \beta$.

**Example 1:** There are many solutions to this equation. First, suppose that $F^1$ can be chosen such that $F^1 \wedge F^1 = 0$. Such classes exist on any complex manifold $N$ which admits a non-trivial holomorphic map $\Phi : N \to \mathbb{C}P^1$. Then $F^1 = \Phi^* \zeta$, where $\zeta \in H^2(\mathbb{C}P^1, \mathbb{Z})$.

---

\[The \]inner derivation $i_L \chi$, of a vector $k$-form $L$ with a $p$-form $\chi$ is defined as $i_L \chi = \frac{1}{k! (p-\ell)!} L^j_{i_1 \cdots i_k} \chi_{jk_{k+1} \cdots j_{k+p-k-1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{p-k-1}}.$
In particular, $K_3$ admits two such representatives in the second cohomology. For this one uses the Weierstrass $\wp$-function. Other examples include any 4-dimensional Kähler manifold which arises as a blow up at the intersection points of an algebraic 4-dimensional Kähler manifold with a complex co-dimension $r - 2$ hyperplane in $\mathbb{CP}^r$, see [26] for further explanation. In such a case, the condition \((4.8)\) reduces to

\[
(p_1q_1 + p_2q_2) = 0.
\]

Using the scale invariance of the equation, set $p_1 = 1$. Then $q_1 = -p_2q_2$ where $p_2, q_2$ are any integers. The only additional requirement is that $p_1q_2 - p_2q_1 \neq 0$ for $\beta$ to be invertible.

For an explicit example set $X^4 = K_3$ and $X^2 = \mathbb{CP}^1$ with $F^1 = \wp^*\zeta$ and $F^2 = \omega_{\mathbb{CP}^1}$. Then $W$ which is identified as a $T^2$ bundle over $K_3 \times \mathbb{CP}^1$ admits a 2-SKT structure.

**Example 2:** For another example assume that $[\omega(2)] \in H^2(X^2, \mathbb{Z})$ and $[\omega(4)] \in H^2(X^4, \mathbb{Z})$. Then set $F^1 = \omega(4)$ and $F^2 = \omega(2)$. The resulting equation reads

\[
p_1^2 + p_2^2 + 2(p_1q_1 + p_2q_2) = 0.
\]

One solution to the above equation is $p_1 = 0$ and $p_2 + 2q_2 = 0$. Then $q_1 \neq 0$ can be arbitrary. The only additional condition is that $p_2 \neq 0$ which is required for $\beta$ to be invertible.

Clearly there are many explicit examples by taking $X^2$ to be $\mathbb{CP}^1$ and $X^4$ to a Kähler 4-dimensional manifold like $\mathbb{CP}^2$. The resulting 8-dimensional 2-SKT manifold $W$ has at most finite fundamental group. So its universal cover $\tilde{W}$ will provide an example of a simply connected compact 2-SKT manifold. In particular, $S^3 \times S^5$ admits a 2-SKT structure.

### 4.3.2 k-SKT manifolds

The results described in the previous section can be generalized to k-SKT manifolds.

**Proposition 4.4.** If $M = X \times T^2$ and $g_T(\xi_A, \xi_B) = \delta_{AB}$, then $W$ is k-SKT iff

\[
\delta_{AB} \mathcal{F}^A \wedge \mathcal{F}^B \wedge \omega_X^{k-1} = 0,
\]

The proof of this is similar to that given for 2-SKT manifolds.

To find examples take $X = X^{2k} \times X^2$ with Kähler forms $\omega(k)$ and $\omega(2)$, respectively. Take again a $T^2$ bundle over $X$ with curvature $(F^1, F^2)$ which has support on $X^{2k}$ and $X^2$, respectively. Then the 2-SKT condition reads

\[
(k-1)(p_1^2 + p_2^2)F^1 \wedge F^1 \wedge \omega_{2k}^{k-2} \wedge \omega(2) + 2(p_1q_1 + p_2q_2)F^1 \wedge F^2 \wedge \omega_{2k}^{k-1} = 0,
\]

where we have chosen $\beta$ as in \((4.7)\).

**Example 1:** Clearly if $F^1 \wedge F^1 = 0$, as in the example given for the 2-SKT case in the previous section, the above condition reduces to $(p_1q_1 + p_2q_2) = 0$. This is again solved as in the 2-SKT case.

**Example 2:** Another possibility is to assume that $[\omega(2)] \in H^2(X^2, \mathbb{Z})$ and $[\omega(2k)] \in H^2(X^{2k}, \mathbb{Z})$, and set $F^1 = \omega(2k)$ and $F^2 = \omega(2)$, then one finds that \((4.9)\) reduces to

\[
(k-1)(p_1^2 + p_2^2) + 2p_1q_1 + 2p_2q_2 = 0.
\]
A solution to this equation is $p_1 = 0$ and $(k - 1)p_2 + 2q_2 = 0$ for arbitrary $q_1$. There are many solutions to these equations in $\mathbb{Z}$ for which $p_2 \neq 0$ which is required for $\beta$ to be invertible. Taking $X^{2k} = \mathbb{C}P^k$ and $X^2 = \mathbb{C}P^1$, one can show that $S^{2k+1} \times S^3$ admits a k-SKT structure.

To give more examples take $M = X \times T^2$ with $g_T(\xi_A, \xi_B) = \delta_{AB}$ as before but now $X = X^{2k} \times X^4$, where $X^{2k}$ and $X^4$ are Kähler manifolds with Kähler forms $\omega(4)$ and $\omega(2k)$, respectively. Furthermore assume that the $T^2$ fibration over $X \times T^2$ has curvature $(F^1, F^2)$, where $F^1$ and $F^2$ have support on $X^{2k}$ and $X^4$, respectively.

**Proposition 4.5.** $W$ admits a (k+1)-SKT condition provided

$$\frac{k(k - 1)}{2}(p_1^2 + p_2^2)F^1 \wedge F^1 \wedge \omega_{(2k)}^{k-2} \wedge \omega_{(4)}^2 + 2k(p_1q_1 + p_2q_2)F^1 \wedge F^2 \wedge \omega_{(2k)}^{k-1} \wedge \omega_{(4)}^2 + (q_1^2 + q_2^2)F^2 \wedge F^2 \wedge \omega_{(2k)}^k = 0, \quad (4.11)$$

where $\beta$ is chosen as in (4.7).

**Example 3:** To find solutions to (4.11) suppose that $F^1 \wedge F^1 = F^2 \wedge F^2 = 0$. Then the condition reduces to requiring that $p_1q_1 + p_2q_2 = 0$ which can be solved as in the 2-SKT case. For an explicit example take $M = K_3 \times K_3 \times T^2$ and $F^1 = \varphi_1^1\zeta$ and $F^2 = \varphi_2^1\zeta$, where $\varphi_1$ and $\varphi_2$ are the Weierstrass functions of the first and second $K_3$ subspaces in $M$, respectively. This will give 3-SKT structures on $T^2$ bundles over $K_3 \times K_3$.

**Example 4:** Next assume that $[\omega(4)] \in H^2(X^4, \mathbb{Z})$ and $[\omega(2k)] \in H^2(X^{2k}, \mathbb{Z})$, and set $F^1 = \omega(2k)$ and $F^2 = \omega(2)$. Then substituting in (4.11), one finds that $W$ admits a (k+1)-SKT condition if

$$\frac{k(k - 1)}{2}(p_1^2 + p_2^2) + 2k(p_1q_1 + p_2q_2) + (q_1^2 + q_2^2) = 0.$$

To find a solution set $p_1 = 0$ and observe that the above equation can be rewritten as

$$q_1^2 + (kp_2 + q_2)^2 = \frac{k(k + 1)}{2}p_2^2.$$

This has solutions, eg $k = 4$, $p_2 = 2$, $q_1 = 2$ and $q_2 = -2$.

### 4.4 k-SCYT structures from Kähler-Einstein manifolds

Examples of 2-SCYT manifolds have been constructed in [15] in the context of IIB black hole horizons. Some of the k-SKT manifolds we have constructed also admit a k-SCYT structure. For this, we shall find the conditions such that the Ricci form, $\rho_W$, of the connection $\nabla_W$ with skew-symmetric torsion on $W$ vanishes, $\rho_W = 0$. This condition is equivalent to requiring that the reduced holonomy of $\nabla_W$ is included in $SU(n)$.

We shall not investigate the general case, instead we shall take $M = X \times T^n$ with metric $g = g_X + g_T$ and Hermitian form $\omega = \omega_X + \omega_T$ and assume that $X$ and $T^{2m}$ equipped with $(g_X, \omega_X)$ and $(g_T, \omega_T)$, respectively, are Hermitian manifolds. Next we apply a Swann twist associated with a $T^{2m}$ principal bundle over $M$ with connection $\lambda$
and curvature $F$ supported on $X$. Then $W$ is a $T^{2m}$ fibration over $X$ with metric $g_W$ and Hermitian form $\omega_W$ given by

$$\pi_W^* g_W = h_{ab} \lambda^a \otimes \lambda^b + \pi^* g_X$$

$$\pi_W^* \omega_W = \frac{1}{2} J_{ab} \lambda^a \wedge \lambda^b + \pi^* \omega_X$$

where $\lambda(\ell) = 0$, $d\lambda = F$, $h = g_T(\beta^{-1}\xi, \beta^{-1}\xi)$ and $J = \omega_T(\beta^{-1}\xi, \beta^{-1}\xi)$. In such a case, one can show, see [16, 15], that $\hat{\rho}_W = 0$, provided that

$$\hat{\rho}_X = -\kappa h_{ab} F^b,$$

$$\omega_X \cdot F^a = \kappa^a,$$

where $\hat{\rho}_X$ is the Ricci form of the Hermitian connection with skew-symmetric torsion on $X$, $\kappa$ is constant and $\omega_X \cdot F^a$ is the inner product of $\omega_X$ and $F$. Observe that if $F$ is Hermitian-Einstein, ie $\kappa = 0$, then $\hat{\rho}_X = 0$ and so $X$ is CYT.

To find examples, let us take $M = X \times T^2$ with $X = X^{2k} \times X^2$, where both $X^{2k} \times X^2$ are Kähler-Einstein spaces with cosmological constants $\ell_1$ and $\ell_2$, respectively. In such case, the Ricci forms of the Kähler metrics satisfy $\rho_{X^{2k}} = \ell_1 \omega_{(2k)}$ and $\rho_{X^2} = \ell_2 \omega_{(2)}$. Assuming that $\omega_{(2k)} \in H^2(X^{2k}, \mathbb{Z})$ and $\omega_{(2)} \in H^2(X^2, \mathbb{Z})$ and focusing on the k-SKT examples for which $F^1 = \omega_{(2k)}$ and $F^2 = \omega_{(2)}$. Then $\kappa = (k, 1)$, and so on finds that

$$\ell_1 = -kh_{11} - h_{12}, \quad \ell_2 = -kh_{12} - h_{22}.$$  

Next consider the examples of k-SKT manifolds with $g_T(\xi_A, \xi_B) = \delta_{AB}$ and $\beta$ given in (4.7). Then one can show that the above two conditions become

$$\ell_1 = -k(p_1^2 + p_2^2) - (p_1 q_1 + p_2 q_2) = \frac{k + 1}{2} (p_1^2 + p_2^2),$$

$$\ell_2 = -k(p_1 q_1 + p_2 q_2) - (q_1^2 + q_2^2),$$

where we have also used (4.10).

Next consider the k-SKT manifolds constructed from $M = X \times T^2$ with $X = X^{2k} \times X^4$. Assuming that both $X^{2k}$ and $X^4$ are Kähler-Einstein, one finds that

$$\ell_1 = -k(p_1^2 + p_2^2) - 2(p_1 q_1 + p_2 q_2),$$

$$\ell_2 = -k(p_1 q_1 + p_2 q_2) - 2(q_1^2 + q_2^2),$$

where $\ell_1$ and $\ell_2$ are the cosmological constants of $X^{2k}$ and $X^4$, respectively.

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References

[1] B. Alexandrov, S. Ivanov, *Vanishing theorems on hermitian manifolds*, Diff. Geom. Appl. 14 (2001), 251-265.

[2] J. M. Bismut, *A local index theorem of non-Kahler manifolds*, Math. Ann. 284 (1989), 681699.

[3] K. Dasgupta, G. Rajesh and S. Sethi, *M theory, orientifolds and G-flux*, JHEP 081999023; [hep-th/9908088].

[4] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa, *Non-Kaehler Heterotic String Compactifications with non-zero fluxes and constant dilaton*, Comm. Math. Phys. 288 (2009), 677-697.

[5] A. Fino, M. Parton, S. Salamon, *Families of strong KT structures in six dimensions*, Comment. Math. Helv. 79 (2004), 317-340; [arXiv:math/0209259]

[6] A. Fino, A. Tomassini, *Blow-ups and resolutions of strong Kahler with torsion metrics*, Adv. Math. 221 (2009), 914935.

[7] A. Fino, A. Tomassini, *On astheno-Kaehler metrics*, [math.DG/arXiv:0806.0735], J. Lond. Math. Soc, in press.

[8] A. Fino, L. Ugarte, *On generalized gauduchon metrics*, [math.DG/arXiv:1103.1033].

[9] J. X. Fu and S. T. Yau, *Existence of supersymmetric Hermitian metrics with torsion on non-Kaehler manifolds*, [arXiv:hep-th/0509028]

[10] J. X. Fu and S. T. Yau, *The theory of superstring with flux on non-Kahler manifolds and the complex Monge-Ampere equation*, J. Diff. Geom. 78 (2009), 369–428. [arXiv:hep-th/0604063]

[11] J. Fu, Z. Wang, and D. Wu, *Semilinear equations, the $\gamma_k$ function, and generalized Gauduchon metrics*, [math.DG/arXiv:1010.2013].

[12] S. J. Gates, Jr., C. M. Hull and M. Rocek, *Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models*, Nucl. Phys. B 248 (1984) 157.

[13] P. Gauduchon, *Le théorème de l’ excentricité nulle*, C.R. Acad. Sci. Paris, 285, (1977) 387-390.

[14] E. Goldstein and S. Prokushkin, *Geometric model for complex non-Kaehler manifolds with SU(3) structure*, Commun. Math. Phys. 251 (2004) 65 [arXiv:hep-th/0212307].

[15] U. Gran, J. Gutowski and G. Papadopoulos, *IIB black hole horizons with five-form flux and KT geometry*, JHEP 1105 (2011) 050 [arXiv:1101.1247 [hep-th]].

[16] D. Grantcharov, G. Grantcharov and Y. S. Poon, *Calabi-Yau Connections with Torsion on Toric Bundles*, J. Diff. Geom. 78 (2008), 13-32. [arXiv:math/0306207]
[17] J. Gutowski and G. Papadopoulos, *Heterotic Black Horizons*, JHEP 1007 (2010) 011 [arXiv:0912.3472 [hep-th]].

[18] J. Gutowski and G. Papadopoulos, *Heterotic horizons, Monge-Ampere equation and del Pezzo surfaces*, JHEP 1010 (2010) 084 [arXiv:1003.2864 [hep-th]].

[19] P. S. Howe and G. Papadopoulos, *Twistor spaces for HKT manifolds*, Phys. Lett. B 379 (1996) 80 [hep-th/9602108].

[20] P. S. Howe and G. Sierra, *Two-dimensional Supersymmetric Nonlinear Sigma Models With Torsion*, Phys. Lett. B 148 (1984) 451.

[21] S. Ivanov and G. Papadopoulos, *A no go theorem for string warped compactifications*, Phys. Lett. B 497 (2001) 309; [hep-th/0008232].

[22] S. Ivanov and G. Papadopoulos, *Vanishing theorems and string backgrounds*, Class. Quant. Grav. 18 (2001) 1089; [math.DG/0010038].

[23] J. Jost and S.T. Yau, *A non-linear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry*, Acta Math 170 (1993) 221; Corrigendum Acta Math 177 (1994) 307.

[24] J. Streets, G. Tian, *A parabolic flow of pluriclosed metrics*, Int. Math. Res. Not. IMRN 16 (2010), 3101-3133.

[25] A. Strominger, *Superstrings With Torsion*, Nucl. Phys. B 274, 253 (1986).

[26] A. Swann, *Twisting hermitian and hypercomplex geometries*, Duke Math. J. 155 (2010), 403-431; arXiv:math/0812.2780.