On a Class of Polynomials with Integer Coefficients

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Abstract
A class \( P_{n,m,p}(x) \) of polynomials is defined. The combinatorial meaning of its coefficients is given. Chebyshev polynomials are the special cases of \( P_{n,m,p}(x) \). It is first shown that \( P_{n,m,p}(x) \) may be expressed in terms of \( P_{n,0,p}(x) \). From this we derive that \( P_{n,2,2}(x) \) may be obtain in terms of trigonometric functions, from which we obtain some of its important properties.

Some questions about orthogonality are also concerned.

Furthermore, it is shown that \( P_{n,2,2}(x) \) fulfills the same three terms recurrence as Chebyshev polynomials. Some others recurrences for \( P_{n,m,p}(x) \) and its coefficients are also obtained.

At the end a formula for coefficients of Chebyshev polynomials of the second kind is derived.

1 Introduction

In the paper [1] the following result is proved.

Theorem A. If a finite set \( X \) consists of \( n \) blocks of the size \( p \) and an additional block of the size \( m \), then, for \( n \geq 0, \ k \geq 0 \), the number \( f(n,k,m,p) \) of \( n + k \)-subsets of \( X \) intersecting each block of the size \( p \) is

\[
f(n,k,m,p) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{np + m - ip}{n + k}.
\]

The following relations for the function \( f \) are also proved in [1]:

\[
f(n,k,m,p) = \sum_{i=0}^{m} \binom{m}{i} f(n,k-i,0,p), \tag{1}
\]

\[
f(n,k,m,p) = \sum_{i=0}^{t} (-1)^i \binom{t}{i} f(n,k+t,m+t-i,p), \tag{2}
\]

\[
f(n,k,m,p) = \sum_{i=1}^{p} \binom{p}{i} f(n-1,k-i+1,m,p-1), \tag{3}
\]

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\[
 f(n, k, m, p) = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} f(n-j, k-i+j, m, p-1). \quad (4)
\]

Furthermore, it is shown that \((-1)^k f(n, k, 0, 2)\) is the coefficient of Chebyshev polynomial \(U_{n+k}(x)\) by \(x^{n-k}\), and that \((-1)^k f(n, k, 1, 2)\) is the coefficient of Chebyshev polynomial \(T_{n+k-1}(x)\) by \(x^{n-k+1}\).

**Definition 1.1** We define the set of coefficients
\[
\{c(n, k, m, p) : n = m, m+1, \ldots ; k = 0, 1, \ldots, n\}
\]
such that
\[
c(n, k, m, p) = (-1)^{\frac{n-k}{2}} \binom{n+k-2m}{n-k} f\left(\frac{n+k-2m}{2}, \frac{n-k}{2}, m, p\right),
\]
if \(n\) and \(k\) are of the same parity, and \(c(n, k, m, p) = 0\) otherwise. Polynomials \(P_{n,m,p}(x)\) are defined to be
\[
P_{n,m,p}(x) = \sum_{k=0}^{n} c(n, k, m, p)x^k.
\]

**Remark 1.1** Chebyshev polynomials are particular cases of \(P_{n,m,p}(x)\), obtained for \(m = 1, p = 2\) and \(m = 0, p = 2\), that is,
\[
U_n(x) = P_{n,0,2}, \quad T_n(x) = P_{n,1,2}.
\]

The polynomial \(P_{n,2,2}(x)\) is the closest to Chebyshev polynomials, and will be denoted simply by \(P_n(x)\).

In the next table we state the first few of \(P_n(x)\).

\[
\begin{align*}
x^2 & \\
2x^3 - 2x & \\
4x^4 - 5x^2 + 1 & \\
8x^5 - 12x^3 + 4x & \\
16x^6 - 28x^4 + 13x - 1 & \\
32x^7 - 64x^5 + 38x^3 - 6x. & 
\end{align*}
\]

Among coefficients of the above polynomials the following sequences from [2] appear: A024623, A049611, A055585, A001844, A035597.

Triangles of coefficients for \(P_{n,m,2}(x)\), \(m = 2, 3, 4, 5, 6\) are given in A136388, A136389, A136390, A136397, A136398 respectively.

2 **Reduction to the case** \(m = 0\).

We shall first prove an analog of the formula (1) for polynomials.
Theorem 2.1 The following equation is fulfilled:

\[ P_{n,m,p}(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{m-i} P_{n-m-i,0,p}(x). \]

Proof. It holds

\[ P_{n,m,p}(x) = \sum_{k=0}^{n} \sum_{i=0}^{m} (-1)^{n+k-2m} \binom{m}{i} f\left(\frac{n+k-2m}{2}, \frac{n-k}{2}, m, p\right) x^k. \]

Using (1) one obtains

\[ P_{n,m,p}(x) = \sum_{k=0}^{m} \sum_{i=0}^{m} (-1)^{n+k} \binom{m}{i} f\left(r, s, 0, p\right) x^k. \]

where

\[ r = \frac{n + k - 2m}{2}, \quad s = \frac{n-k}{2} - i. \]

Changing the order of summation yields

\[ P_{n,m,p}(x) = \sum_{i=0}^{m} \binom{m}{i} x^{m-i} \sum_{k=0}^{n} (-1)^{n+k} f\left(r, s, 0, p\right) x^{k-m+i}. \]

Terms in the sum on the right side of the preceding equation produce nonzero coefficients only in the case \(0 \leq s \leq r\), that is,

\[ m - i \leq k \leq n - 2i. \]

It follows that

\[ P_{n,m,p}(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{m-i} \sum_{k=m-i}^{n-2i} (-1)^{n-2i-k} f\left(r, s, 0, p\right) x^{k-m+i}. \]

Denoting \( k - m + i = j \) we obtain

\[ P_{n,m,p}(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{m-i} \sum_{j=0}^{n-i} c(n-i,j,0,p) x^j, \]

which means that

\[ P_{n,m,p}(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{m-i} P(n-m-i,0,p)(x), \]

and the theorem is proved.

According the the preceding theorem we may express \( P_n(x) \) in terms of Chebyshev polynomials of the second kind. Namely, for \( m = 2, n \geq 4 \) holds

\[ P_n(x) = x^2 U_{n-2}(x) - 2x U_{n-3}(x) + U_{n-4}(x). \]

This allow us to express \( P_n(x) \) in terms of trigonometric functions.
Theorem 2.2. For each \( n \geq 3 \) holds
\[
P_n(\cos \theta) = -\sin \theta \sin(n-1)\theta. \tag{6}
\]

Proof. According to (5) and well-known property of Chebyshev polynomials we obtain
\[
\sin \theta P_n(\cos \theta) = \cos^2 \theta \sin(n-1)\theta - 2 \cos \theta \sin(n-2)\theta + \sin(n-3)\theta.
\]
From the identity
\[
2 \cos \theta \sin(n-2)\theta = \sin(n-1)\theta + \sin(n-3)\theta
\]
follows
\[
\sin \theta P_n(\cos \theta) = \cos^2 \theta \sin(n-1)\theta - \sin(n-1)\theta = -\sin^2 \theta \sin(n-1)\theta.
\]
Dividing by \( \sin \theta \neq 0 \) we prove the theorem.

Note that this proof is valid for \( n \geq 4 \). The case \( n = 3 \) may be checked directly.

In the following theorem we prove that \( P_n(x) \) have the same important property concerning zeroes as Chebyshev polynomials do.

Theorem 2.3. For \( n \geq 3 \), the polynomial \( P_n(x) \) has all simple zeroes lying in the segment \([-1, 1]\).

Proof. Since
\[
U_n(1) = n + 1, \quad U_n(-1) = (-1)^n(n + 1)
\]
the equation (5) implies
\[
P_n(1) = U_{n-2}(1) - 2U_{n-3}(1) + U_{n-4}(1) = n - 1 - 2(n-2) + n - 3 = 0,
\]
and
\[
P_n(-1) = U_{n-2}(-1) + 2U_{n-3}(-1) + U_{n-4}(-1) = (-1)^{n-2}(n - 1) + 2(-1)^{n-3}(n - 2) + (-1)^{n-4}(n - 3) = 0.
\]
Thus, \( x = -1 \) and \( x = 1 \) are zeroes of \( P_n(x) \). The remaining \( n - 2 \) zeroes are obtained from the equation
\[
\sin(n-1)\theta = 0,
\]
and they are
\[
x_k = \cos \frac{k\pi}{n-1}, \quad (k = 1, 2, \ldots, n-2).
\]

We shall now state an immediate consequence of (6) which shows that values of \( P_n(x), \; (x \in [-1,1]) \) lie inside the unit circle.
Corollary 2.1 For $n \geq 3$ and $x \in [-1, 1]$ we have
\[ P_n(x)^2 + x^2 \leq 1. \]

Remark 2.1 Dividing $P_n(x)$ by $2^{n-2}$ we obtain a polynomial with the leading coefficient 1. Thus, its supremum norm on $[-1, 1]$ is $\leq \frac{1}{2n-2}$, which means that $\frac{1}{2n-2}P_n(x)$ has at most 2 times greater supremum norm, comparing with the supremum norm of $T_n(x)$, that is minimal.

Taking derivative in the equation (6) we obtain the following equation for extreme points of $P_n(x)$:
\[ (n-1)\tan \theta + \tan(n-1)\theta = 0. \]

The values $\theta = 0$, and $\theta = \pi$ obviously satisfied this equation, which implies that endpoints $x = -1$ and $x = 1$ are extreme points. The remaining extreme points of $P_3(x)$ are $x = \arctan \sqrt{2}$ and $x = -\arctan \sqrt{2}$.

3 Orthogonality

In this section we investigate the set $\{ P_n(x) : n = 2, 3, 4, \ldots \}$ concerning to the problem of orthogonality, with respect to some standard Jacobi’s weights.

The first result is for the weight $\frac{1}{\sqrt{1-x^2}}$ of Chebyshev polynomials of the first kind.

Theorem 3.1 It holds
\[ \int_{-1}^{1} \frac{P_n(x)P_m(x)}{\sqrt{1-x^2}} \, dx = \begin{cases} \frac{\pi}{4} & m = n \\ -\frac{\pi}{8} & |n-m| = 2 \\ 0 & \text{otherwise}. \end{cases} \]

Proof. Putting $x = \cos \theta$ implies
\[ I = \int_{-1}^{1} \frac{P_n(x)P_m(x)}{\sqrt{1-x^2}} \, dx = \int_{0}^{\pi} P_n(\cos \theta)P_m(\cos \theta) \, d\theta. \]

Using (6) we obtain
\[ I = \int_{0}^{\pi} \sin^2 \theta \sin(n-1)\theta \sin(m-1)\theta \, d\theta. \]

Transforming the integrating function we obtain
\[ \sin^2 \theta \sin(n-1)\theta \sin(m-1)\theta = \frac{1}{4} \cos(n-m)\theta - \frac{1}{4} \cos(n+m-2)\theta - \frac{1}{8} \cos(n-m-2)\theta - \frac{1}{8} \cos(n+m-2)\theta + \frac{1}{8} \cos(n+m)\theta. \]

Taking into account that $m, n \geq 3$ we conclude that integrals of the terms on the right side of the preceding equation are zero if $n \neq m$ and $|n-m| \neq 2$. If $n = m$ we obtain $I = \frac{\pi}{4}$, and $I = -\frac{\pi}{8}$ if $|n-m| = 2$, and the theorem is proved.
Corollary 3.1 Each subset of the set \( \{ P_n(x) : n \geq 3 \} \), not containing polynomials \( P_k(x) \) and \( P_m(x) \) such that \( |k - m| = 2 \), is orthogonal.

The next result concerns the weight \( \sqrt{1 - x^2} \) of Chebyshev polynomials of the second kind. The result is similar to the result of the preceding theorem.

Theorem 3.2 It holds

\[
\int_{-1}^{1} \sqrt{1 - x^2} P_n(x)P_m(x)dx = \begin{cases} 
\frac{3\pi}{16} & m = n \\
\frac{-\pi}{8} & |n - m| = 2 \\
\frac{-\pi}{12} & |n - m| = 4 \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. In this case we have

\[
\int_{-1}^{1} \sqrt{1 - x^2} P_n(x)P_m(x)dx = \int_{0}^{\pi} \sin^2 \theta P_n(\cos \theta)P_m(\cos \theta)d\theta.
\]

We therefore need to calculate the integral

\[
\int_{0}^{\pi} \sin^4 \theta \sin(n - 1)\theta \sin(m - 1)\theta d\theta.
\]

In this case we have

\[
\sin^4 \theta \sin(n - 1)\theta \sin(m - 1)\theta = \frac{3}{16} \cos(n - m)\theta - \frac{3}{16} \cos(n + m - 2)\theta +
\]

\[
+ \frac{1}{32} \cos(n - m - 4)\theta + \frac{1}{32} \cos(n - m + 4)\theta - \frac{1}{32} \cos(n + m + 4)\theta - \frac{1}{32} \cos(n + m - 6)\theta + \frac{1}{32} \cos(n + m - 2)\theta -
\]

\[
- \frac{1}{8} \cos(n - m - 2)\theta - \frac{1}{8} \cos(n - m + 2)\theta + \frac{1}{8} \cos(n + m - 4)\theta + \frac{1}{8} \cos(n + m)\theta.
\]

The integral of each term on the right side with \( m \neq n, |m-n| \neq 2, |n-m| \neq 4 \) is zero.

For these particular values we easily obtain the desired result, and the theorem is proved.

Taking, for instance, the weight \((1 - x^2)^{\frac{3}{2}}\) in the similar way one obtains

\[
\int_{-1}^{1} (1 - x^2)^{\frac{3}{2}} P_n(x)P_m(x)dx = \begin{cases} 
\frac{5\pi}{16} & m = n \\
\frac{-\pi}{14} & |n - m| = 2 \\
\frac{-\pi}{10} & |n - m| = 4 \\
\frac{-\pi}{12} & |n - m| = 6 \\
0 & \text{otherwise.}
\end{cases}
\]

Considering the weight 1 leads to the following result:

Theorem 3.3 If \( m \) and \( n \) are of different parity then

\[
\int_{-1}^{1} P_n(x)P_m(x)dx = 0.
\]
Proof. In this case we need to calculate the integral
\[ \int_0^\pi \sin^3 \theta \sin(n-1)\theta \sin(m-1)d\theta. \]

We have
\[
\sin^3 \theta \sin(n-1)\theta \sin(m-1)\theta = \frac{1}{16} \sin(n-m+3)\theta + \frac{1}{16} \sin(n-m-3)\theta + \frac{1}{16} \sin(n+m+1)\theta - \frac{3}{16} \sin(n+m-1)\theta + \frac{3}{16} \sin(n-m-1)\theta + \frac{1}{16} \sin(n+m-3)\theta.
\]

Since \(m\) and \(n\) are of different parity each function on the right is of the form \(\sin(2k+1)\theta\), which implies that its integral is zero, and the theorem is proved.

### 4 Some recurrence relations

In this section we prove some recurrence relation for \(P_{n,m,p}(x)\) as well as some recurrence relations for their coefficients.

**Theorem 4.1** For each integer \(t \geq 0\) holds
\[
P_{n,m,p}(x) = \sum_{i=0}^{t} (-1)^{t-i} \binom{t}{i} x^i P_{n+2t-i,m+t-i,p}(x).
\]

**Proof.** Translating (2) into the equation for coefficients we obtain
\[
c(n, k, m, p) = \sum_{i=0}^{t} (-1)^{i+t} \binom{t}{i} c(n + 2t - i, k - i, m + t - i, p).
\]

Multiplying by \(x^k\) yields
\[
c(n, k, m, p)x^k = \sum_{i=0}^{t} (-1)^{i+t} \binom{t}{i} x^i c(n + 2t - i, k - i, m + t - i, p)x^k - i,
\]

which easily implies the claim of the theorem.

In the case \(t = 1, m = 1, p = 2\) we obtain the following formula, expressing \(P_n(x)\) in terms of Chebyshev polynomials of the first kind:
\[
P_n(x) = xT_{n-1}(x) - T_{n-2}(x).
\]

From this we easily conclude that \(P_n(x)\) satisfies the same three term recurrence as Chebyshev polynomials.
Corollary 4.1 The polynomials $P_{n,m,2}(x)$ satisfy the following equation:

$$P_{n,m,2}(x) = 2xP_{n-1,m,2}(x) - P_{n-2,m,2}(x),$$

with initial conditions

$$P_{0,m,2}(x) = x^m, \quad P_{1,m,2}(x) = 2x^{m+1} - mx^m - 1.$$ Combining the equations (1) and (4) we obtain

$$f(n, k, m, p) = \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{t=0}^{m} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} m \\ t \end{array} \right) \left( \begin{array}{c} i \\ j \end{array} \right) f(n - j, k - i + j - t, 0, p - 1).$$

Translating this equation into the equation for coefficients we obtain

$$c(n, k, m, p) = \sum_{i=0}^{n} \sum_{j=0}^{i} \sum_{t=0}^{m} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} m \\ t \end{array} \right) \left( \begin{array}{c} i \\ j \end{array} \right) (-1)^{i-j+t} c(n-i-t, k+i-2j+t, 0, p-1).$$

Applying the preceding equation several times we obtain the following:

Corollary 4.2 Coefficients of $P_{n,m,p}(x)$ may be obtained as a function of coefficients of Chebyshev polynomials of the second kind.

Converting (3) into the equation for coefficients we obtain

$$c(n, k, m, p) = \sum_{i=1}^{p} (-1)^{i} \binom{p}{i} c_{n-i,k+i-2,m,p}.$$ This implies the following:

Corollary 4.3 Coefficients of $P_{n,m,p}(x)$ may be expressed in terms of coefficients of polynomials $P_{n',m,p}(x)$, where $n' < n$.

We shall finish the paper with a formula for coefficients of Chebyshev polynomials of the second kind. Taking $p = 2$ in (4) we obtain

$$f(n, k, m, 2) = \sum_{i=0}^{n} \sum_{j=0}^{i} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} i \\ j \end{array} \right) f(n - j, k - i + j, m, 1).$$

Since $f(r, s, m, 1) = \binom{m}{s}$ we have

$$f(n, k, m, 2) = \sum_{i=0}^{n} \sum_{j=0}^{i} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} i \\ j \end{array} \right) \binom{m}{k - i + j}.$$ For $m = 0$, in the sum on the right side of this equation only terms with $k = i - j$ remains. We thus obtain

$$f(n, k, 0, 2) = \sum_{s=0}^{n-k} \binom{n}{s} \binom{n-s}{k}.$$ Accordingly, the following formula follows
Corollary 4.4 For coefficients $c(n, k)$ of Chebyshev polynomial $U_n(x)$ hold

$$c(n, k) = (-1)^{n-k} \sum_{i=0}^{k} \binom{n+k}{i} \binom{n+k}{n-k-i},$$

if $n \equiv k \pmod{2}$ and $c(n, k) = 0$ otherwise.

References

[1] M. Janjic, *An Enumerative Function*, arXiv:0801.1976v2

[2] N. J. Sloane, *The Encyclopedia of Integer Sequences*, electronically published at www.research.att.com/~njas/sequences/