Multiscaling Anderson localization of cosmic electromagnetic fields

A. Bershadskii

November 5, 2018

ICAR, P.O. Box 31155, Jerusalem 91000, Israel

Abstract

Multiscaling properties of the Anderson localization of the cosmic electromagnetic fields before the recombination time are studied and results of a numerical simulation for a random banded matrix ensemble are found to be in good agreement with the large-scale cosmic data.

PACS numbers: 52.40_Db, 71.55_Jv, 98.80_Cq, 98.70_Vc
Key words: Anderson localization, cosmic electromagnetic fields.
1 Introduction

Large-scale electromagnetic fields are expected to be generated by the cosmological phase transitions (electroweak and QCD) [11]-[12] before the recombination time. On the other hand, before the recombination time the cosmic plasma is strongly fluctuating at the large scales due to the gravitational density perturbation. These plasma density fluctuations can play a role of randomly distributed scatterers for the electromagnetic fields and make it impossible for the electromagnetic fields to propagate if the wavelength is longer than the elastic mean free path of electromagnetic wave by the scatterers. The localized modes are thermally excited and the electric field is more or less aligned in the localization region. The localization phenomenon by random fields was first proposed by Anderson [13] to explain the behavior of the electric resistivity of metals in the presence of impurities and the theories were developed by himself and his collaborators [14], [15]. The localization phenomena were then observed in various physical systems. Probably the first paper which studied the possible localization of electromagnetic field in plasma is the Ref. [16] (see also [17]). Roughly speaking, the Anderson localization phenomenon occurs because of the destructive interference of the incident wave and the scattered waves from the randomly distributed sources. It is known that the localization occurs if the strength of the randomness is larger than a critical value. For a sufficiently large random potential, only the modes of the wave length shorter than the mean free path can propagate, while the other modes are localized. The authors of Ref. [18] argue that this localization phenomenon should occur in the matter dominated era (before the recombination time) in the standard cosmology. The authors of [18] took into account a single (Jeans) scale $l_c$ for the primordial perturbation for simplicity. Inclusion of a scale invariant spectrum for scales larger than $l_c$ produces a hierarchy of structures, which needs in a multifractal description. During the past decade, multifractality of critical eigenfunctions at the Anderson transition has been a subject of intensive analytical and numerical studies (see Refs. [19]-[27] and references therein). A remarkable virtue of the Anderson localization mechanism is that the photon diffusion does not occur to the localized mode. This makes the localized structures to be imprinted in the photon gas after the decoupling of the photon gas (so-called cosmic microwave background, or CMB) from the baryonic matter. In the present paper we will show that results of the numerical simulations of multifractal Anderson localization are consistent with the CMB data.
2 Anderson localization of cosmic electromagnetic fields

Let us, following the paper [18], consider Anderson localization of cosmic electromagnetic fields for the spatially flat Friedmann-Robertson-Walker Universe. We assume that the charged particles are non-relativistic. So the four velocity of the charged particles: \( u^\alpha = (a^{-1}, v^i) \), with \( v^i \) being small compared with unity. Equation for the vector potential \( A \) in terms of the conformal comoving coordinates \( (\lambda, x) \) in this case is

\[
\frac{\partial^2 A}{\partial \lambda^2} + \frac{\partial^2 A}{\partial x^2} = \frac{e^2 n a^2}{m} A
\]  

(1)

where \( \lambda \) is the conformal time defined by \( \lambda = \int^t \frac{dt}{a} \), \( m \) is the electron mass and \( e \) is the charge, \( n \) is the density.

Split the plasma density \( n \) into the homogeneous part \( \overline{n} \) and the space dependent random part \( \delta n \);

\[
n = \overline{n} + \delta n
\]  

(2)

Then the right hand side of the plasma equation becomes

\[
\left[ \frac{\omega_{rec}^2}{a} + \frac{\omega_{rec}^2}{a} \frac{\delta n}{\overline{n}} \right] A
\]  

(3)

where \( \omega_{rec}^2 = \sqrt{\frac{e^2 n_{rec} m_e}{m_e}} \). Here we have used the conservation law: \( \overline{n} = n_{rec}/a^3 \) before the recombination time, with the scale factor \( a \) normalized at the recombination time. \( n_{rec} \) is the electron density (therefore the baryon density) at the recombination time. The first term in the bracket is spatially constant but a decreasing function of the conformal time \( \lambda \). The second term represents the fluctuating random environment induced by the baryonic density perturbation \( \frac{\delta \rho_b}{\rho_b} \) which is equal to \( \frac{\delta n}{\overline{n}} \) in our non-relativistic case. Knowing the standard theory of linear perturbation in cosmology we see that \( \frac{\delta n}{\overline{n}} = \frac{\delta \rho_b}{\rho_b} \propto a \) in the matter dominated era, so that the second term is independent of the conformal time in that era. We write the random potential as \( V(x) \equiv \frac{\omega_{rec}^2}{a} \frac{\delta n}{\overline{n}} \), which is a function of comoving coordinates \( x \) only. So far we have discussed
the period after the equal time\(^1\) and before the recombination time and found that the random potential induced by the density perturbation is constant in time. As we shall see this is important for the stability of the Anderson localization. For the matter dominated universe, the scale factor is given by 

\[ a(t) = \left( \frac{t}{t_{\text{rec}}} \right)^{2/3} = \left( \frac{\lambda}{3t_{\text{rec}}} \right)^2 \]  

with \( \lambda = \int \frac{dt}{a(t)} = t_{\text{rec}}^{2/3}t^{1/3} \). One can reduce the Maxwell equations to the following wave equation as

\[
\frac{\partial^2 A}{\partial \lambda^2} + \frac{\partial^2 A}{\partial x^2} = \left[ Q^2 / \lambda^2 + V(x) \right] A
\]  

(4)

Here \( Q^2 = 9t_{\text{rec}}^2 \omega_p^2 \). We expand the vector potential \( A \) in terms of the complete orthonormal set \( \{ \psi_n \} \) as

\[
A = \sum a_n f_n(\lambda) \psi_n(x) + c.c.
\]  

(5)

with \( a_n \) being arbitrary coefficients. Here \( \psi_n(x) \) is a normalized solution of the eigenvalue equation:

\[
[-\Delta + V(x)] \psi_n(x) = E_n^2 \psi_n(x)
\]  

(6)

Localization means that a number of modes \( \{ \psi_n, n \leq n_0 \} \) are bound states exponentially falling off at infinity. The marginal state \( n_0 \) corresponds to the so-called mobility edge as we shall explain further.

Here we only consider the adiabatic baryonic density perturbation for simplicity, the spectrum of which is roughly like the picture below. To simplify our discussion we assume in this section that the spectrum is sharply peaked around the Jeans length \( l_c \) at the equal time and the distribution of perturbation is Gaussian. Inclusion of the scale invariant spectrum for scales larger than \( l_c \) produces a hierarchy of structures (see next section).

To understand the Anderson localization in the Universe before the recombination time, we have to evaluate the mobility edge of the wave number, which is roughly the inverse of the mean free path \( l_{mf} \) of the electromagnetic wave by the random potential. Consider a Schrödinger equation for the spatial part of the electromagnetic field:

\[
[-\Delta + V(x)] \psi = E^2 \psi
\]  

(7)

---

\(^1\)The equal time \( t_{eq} \) is the time when the matter and radiation energy density are equal. This value depends on the value of the present baryon density. We take \( \frac{a(t_{eq})}{a(t_{eq})} = (\frac{l_{eq}}{t_{eq}})^{2/3} = 6 \) as a typical value for \( \Omega_0 = 0.2 \).
with $x$ being the comoving coordinates. Recall that the random potential $V(x) = \omega_p^2 \delta n/n$ does not depend on the conformal time in the matter-dominated era. The characteristic length scale $l_c$ of the density perturbation is given by the Jeans length at the equal time, which is roughly the same as the horizon scale at that time. After that time the length scale $l_c$ develops as $\propto a$. This gives $l_c \approx 10^{23} \text{cm}$ at the recombination time, while the horizon is $\approx 2 \times 10^{25} \text{cm}$. Then the mean free path is given by $l_{mf} = \frac{1}{n \sigma}$ where $\sigma$ is the elastic scattering cross section of the electromagnetic wave by a single potential and $n_c \approx \frac{3}{4\pi} l_c^{-3}$ is the density of "impurities". The elastic cross section is estimated roughly as $\approx 4\pi l_c^2$ by the standard wave mechanics, because the "energy" is low and is roughly given by $E^2 l_c^2 \approx 1$, while the potential is effectively gigantic $V \times l_c^2 \approx \omega_p^2 \delta n l_c^2 \approx 5 \times 10^{32}$. Here we have taken $\delta n \approx 10^{-5}$ because the assumed baryonic perturbation can be comparable to the temperature fluctuation of CMB observed by COBE (see [28] and next section). However, the localization length and other results in the present consideration are insensitive to the magnitude of the density perturbation. Therefore the localization should occur, since the critical value of $V l_c^2$ will be of order one. Then the mean free path is given by $l_{mf} \approx l_c/3$ and the mobility edge is $E^* \approx \frac{12\pi}{l_c}$, which is consistent with our low energy picture.

Fortunately in the present case, the randomness is huge so that an estimate of the localization length is available [30]. We have $\xi = l_c/\log(|V|/E^2)$, where $l_c$ is the minimum length scale in which range the potential can be considered as a constant. For the adiabatic baryonic perturbation, it will be reasonable to take the Jeans length at the equal time $t_{eq}$ as the small scale cut-off $l_c$, which is $\approx 10^{23} \text{cm}$ at the recombination time. In the present case, $|V|/E^2 \approx 5 \times 10^{32}$ so that $\xi \approx 10^{21} \text{cm}$ at the recombination time.

One can estimate the magnitude of the thermally excited localized electric fields as $\tilde{E}_{loc}^2 \approx T/\xi^3$ in a localization region of the size $\xi$. Here the contribution from a single localized mode only has been taken into account, because the peaks of the other localized modes are randomly distributed and will be located outside of the region under consideration. The estimate of $\tilde{E}_{loc}^2$ can be derived, for instance, by the equal partition law. Note that the propagating modes ($n > n_0$) contribute to the Stefan-Boltzmann law. At this stage we realize that approximately $T/\omega_p$ photons occupies a localized state. As far as $T/\omega_p >> 1$, a coherent localized state of the size $\xi$ is a good picture.
so that we can approximately describe it by a classical field:

\[ E_{\text{loc}} = -\sum_{n} \sqrt{T/\omega_{n}(\lambda)} e_{n}(x)\psi_{n}(x) + c.c. \]  

(8)

where \( e_{n}(x) \) is the polarization vector, which is perpendicular to the gradient of the wave function \( \psi_{n}(x) \). Therefore, in a localization domain of the size \( \xi \), we have aligned electric and magnetic fields, which are perpendicular to each other and oscillate almost at the plasma frequency. One might worry about the damping of the electromagnetic structure of the size \( \xi \) by photon diffusion. However, the diffusion does not occur to the localized modes. This is one of the virtues of the Anderson localization mechanism. Since the localization is a delicate interference phenomenon, the random potential \( V(x) \) should vary slowly so that the motion of wave packet is not disturbed [16]. This condition is met before the recombination time. In the previous consideration we, following to [18], only took into account a single scale \( l_{c} \) for the primordial perturbation for simplicity. The contribution from the larger scales to the localization phenomenon is interesting in the sense that it may account for the hierarchy of the structures.

3 Multifractal Anderson localization

Even arbitrary close to the mobility edge a state should occupy only an infinitesimal fraction of space if it is to be labeled a localized state. On the other side of the mobility edge the states should extend through the whole volume. Both characteristics can be accommodated at the mobility edge if one assumes a fractal wave function with filamentary structure likes a net over the whole volume, as suggested originally by Aoki [29]. Observations of the scaling properties of different characteristics of the eigenstates show that the wave functions cannot be adequately treated as a simple fractal. Rather, the more general concept of multifractality has to be employed, yielding a set of generalized dimensions.

The multifractal fluctuations of eigenfunctions can be characterized by a set of inverse participation ratios

\[ P_{p} = \int d^{d}r|\psi(r)|^{2p} \]  

(9)

with an anomalous scaling with respect to system size \( L \)

\[ P_{p} \sim L^{-\tau_{p}}, \quad \tau_{p} = D_{p}(p - 1) \]  

(10)
The scaling (10) characterized by an infinite set of generalized dimensions $D_p$ implies that the critical eigenfunction represents a multifractal distribution. During the past decade, multifractality of critical eigenfunctions at the Anderson transition has been a subject of intensive numerical studies see Refs. [19]–[27] and references therein. In recent paper [27] the authors explore the fluctuations at criticality for the power-law random banded matrix (PRBM) ensemble. The model is defined as the ensemble of random Hermitian $N \times N$ matrices. The matrix elements $H_{ij}$ are independently distributed Gaussian variables with zero mean $\langle H_{ij} \rangle = 0$ and the variance

$$\langle |H_{ij}|^2 \rangle = a^2(|i - j|),$$  \hspace{1cm} (11)

where $a(r)$ is given by

$$a^2(r) = \frac{1}{1 + (r/b)^{2\alpha}} \quad \text{ (12)}$$

At $\alpha = 1$ the model undergoes an Anderson transition from the localized ($\alpha > 1$) to the delocalized ($\alpha < 1$) phase. At $\alpha = 1$ the PRBM model was found to be critical for arbitrary value of $b$; it shows all the key features of the Anderson critical point, including multifractality of eigenfunctions. In a straightforward interpretation, the PRBM model describes an one-dimensional sample with random long-range hopping, the hopping amplitude decaying as $1/r^\alpha$ with the length of the hop. Such a random matrix ensemble arises in various contexts in the theory of quantum chaos [33], [34] and disordered systems [35], [36], [37]. In the paper [27] the authors studied the periodic (critical) generalization of (12)

$$a^{-2}(r) = 1 + \frac{1}{b^2} \frac{\sin^2(\pi r/N)}{(\pi/N)^2}$$  \hspace{1cm} (13)

Indeed, for $N \gg r$ Eq. (13) reduces to Eq. (12) at the critical case. The generalization (13) allows diminish finite-size effects (an analog of periodic boundary conditions). At $b = 1$ a limiting distribution can be reached most easily and, therefore, the finite-size effects are minimal [27]. This value of parameter $b$ corresponds to crossover from weak ($b \gg 1$) to strong ($1 \gg b$) multifractality. Below we will compare results of the numerical simulations performed for $b = 1$ with the large-scale CMB data.
Temperature \((T)\) dissipation rate of the photon (CMB) gas can be characterized by a "gradient" space measure (\cite{38}, p. 381):

\[
\chi_r = \frac{\int_{v_r} (\nabla T)^2 dv}{v_r} \tag{14}
\]

where \(v_r\) is a subvolume with space-scale \(r\). Scaling law of this measure moments,

\[
\frac{\langle \chi_r^p \rangle}{\langle \chi_r \rangle^p} \sim r^{-\mu_p} \tag{15}
\]

(where \(\langle ... \rangle\) means an average) is an important characteristic of the dissipation rate \cite{38,39}. The exponents \(\mu_p\) can be related to correspondent generalized dimensions by equation \cite{39}:

\[
\mu_p = (d - D_p)(p - 1) \tag{16}
\]

The COBE satellite was launched on 1989 into 900 km altitude sun-synchronous orbit. The Differential Microwave Radiometers (DMR) operated for four years of the COBE mission and mapped the full sky. The instrument consists of six differential microwave radiometers, two nearly independent channels that operate at each of three frequencies: 31.5, 53 and 90 GHz. Each differential radiometer measures the difference in power received from two directions in the sky separated by 60°, using a pair of horn antennas. Each antenna has a 7° beam. In this investigation we use a DMR data map of the cosmic microwave radiation temperature. The COBE-DMR team made an effort to remove the most of the Galactic microwave emission (i.e. thermal emission of cosmic dust, free-free emission and synchrotron emission) from the map using so-called "combination technique" in which a linear combination of the DMR maps is used to cancel the Galactic emission. This was made in order to extract the cosmic microwave background radiation.

To organize the DMR data in the temperature maps the sky was divided into 6144 equal area pixels. These are formed by constructing a cube with each face divided into 32 x 32 = 1024 squares, projected onto a celestial sphere in elliptic coordinates. The projection is adjusted to form equal area pixels having a solid angle of \(4\pi/6144\) sr or 6.7 square degrees. Because the 7° beamwidth of the sky horn is greater than separation between pixels (2.6° average), this binning oversamples the sky.
Using the CMB pixel map, we will calculate the CMB temperature gradient measure using summation over pixel sets instead of integration over subvolumes $v_r$. So that multiscaling (15) (if exists) will be written as

$$\frac{\langle \chi^p_s \rangle}{\langle \chi_s \rangle^p} \sim s^{-\mu_p}$$

(17)

where metric scale $r$ (from (15)) is replaced by number of the pixels, $s$, characterizing size of the summation set. The $\chi_s$ is a one-dimensional surrogate of the real dissipation rate $\chi_r$. It is believed that the surrogates can reproduce quantitative multiscaling properties of the dissipation rate [39]. Since in our case $\langle \chi_s \rangle$ is independent on $s$, we will calculate the exponents $\mu_p$ directly from scaling of $\langle \chi^p_s \rangle$.

Figure 1 shows scaling of the CMB temperature dissipation rate moments $\langle \chi^p_s \rangle$ calculated for the DMR map. The straight lines (the best fit) are drawn to indicate the scaling in the log-log scales.

Figure 2 shows the generalized dimensions $D_p$ extracted from figure 1 (circles) using Eq. (16). Crosses in this figure correspond to the generalized dimensions calculated for the above described PRBM model [27] of the Anderson localization.

4 Discussion

The Anderson localization is an unique linear phenomenon which exhibits profound multifractality. This allows explain the observed multifractality of the CMB dissipation in the frames of standard (linear) approach to the CMB, which seems to be the most compatible with the small magnitude of the CMB fluctuations. Generation of the cosmic electromagnetic fields in the electroweak or QCD phase transitions occurring in the particle physics scales needs in a non-linear (turbulent) mechanism for reasonable space scale expansion of the fields [11]-[12]. Then the Anderson localization of the large-scale electromagnetic fields can be considered as an effective linear mechanism of the final CMB fractalization. This is because the photon diffusion, which can effectively suppress the non-linear fluctuations just before the recombination time, does not occur to the localized modes.
The author is grateful to C.H. Gibson and to K.R. Sreenivasan for discussions, and to A.D. Mirlin, to F. Evers and to the NASA Goddard Space Flight Center for providing the data.
References

[1] A. Brandenburg, K. Enqvist and P. Olesen, Phys. Rev. D, **54**, 1291 (1996).

[2] J.D. Barrow, P.G. Ferreira and J. Silk, Phys. Rev. Lett., **78**, 3610 (1997).

[3] K. Subramanian and J.D. Barrow, Phys. Rev. Lett., **81**, 3575 (1998).

[4] K. Subramanian and J.D. Barrow, MNRAS, **335**, 3 (2002).

[5] R. Durrer, P.G. Ferreira and T. Kahniashvili, Phys. Rev. D, **61**, 043001 (2000).

[6] A. Kosowsky, A. Mack, and T. Kahniashvili, Phys. Rev. D, **66**, 024030 (2002).

[7] A.D. Dolgov and D. Grasso, Phys. Rev. Lett., **88**, 011301 (2002).

[8] A.D. Dolgov, D. Grasso and A. Nicolis, Phys. Rev. D, **66**, 103505 (2002).

[9] M. M. Forbes and A. Zhitnitsky, Phys. Rev. Lett., **85**, 5268 (2000).

[10] M. M. Forbes and A. Zhitnitsky, Primordial galactic magnetic fields: An application of QCD domain walls, hep-ph/0102158 (2001).

[11] D.T. Son, Phys. Rev. D, **59**, 063008 (1999).

[12] J.M. Cornwall, Phys. Rev. D, **56**, 6146 (1997).

[13] P.W. Anderson, Phys. Rev., **109** (1958) 1492.

[14] E. Abrahams, P.W. Consequence, D.C. Liccaiardello and T.V. Ramarkrishnan, Phys. Rev. Lett., **42** (1979) 673.

[15] D.J. Thouless, Phys. Rep., **13** (1974) 93.

[16] D.F. Escande and B. Souillard, Phys. Rev. Lett., **52** (1984) 1296.

[17] F. Doveil, Y. Vsluisant, S.I. Tsunoda, Phys. Rev. Lett., **69** (1992) 2074.

[18] A. Hosoya and S. Kobayashi, Phys. Rev. D, **54** (1996) 4738.

[19] C. Castellani and L. Peliti, J. Phys. A: Math. Gen., **19** (1986) L429.
[20] M. Schreiber and H. Grussbach, Phys. Rev. Lett., 67, (1991) 607.
[21] W. Pook and M. Janssen, Z. Phys. B, 82 (1991) 295.
[22] B. Huckestein, B. Kramer, and L. Schweitzer, Surf. Sci., 263 (1992) 125.
[23] M. Janssen, Int. J. Mod. Phys., B, 8, (1994) 943.
[24] B. Huckestein, Rev. Mod. Phys., 67 (1995) 357.
[25] A. Bershadskii, J. of Stat. Phys., 94 (1999) 725.
[26] A. D. Mirlin, Phys. Rep., 326 (2000) 259.
[27] A.D. Mirlin and F. Evers, Phys. Rev. B, 62 (2000) 7920.
[28] G.F. Smoot et al., Ap.J., 396 (1992) L1.
[29] H. Aoki, J. Phys. C, 16 (1983) L205.
[30] M. Aizenman and S. Molchanov Commun. Math. Phys., 157 (1993) 245.
[31] E. M. Lifshitz and L. P. Pitaevskii, Physical Kinetics (Pergamon Press, Oxford, 1981)
[32] T. Tajima, S. Cable, K. Shibata, and R. M. Kulsrud, Ap.J., 390 (1992) 309.
[33] J.V. Jose and R. Cordery, Phys. Rev. Lett., 56 (1986) 290.
[34] B.L. Altshuler and L.S. Levitov, Phys. Rep., 288 (1997) 487.
[35] A.V. Balatsky and M.I. Salkola, Phys. Rev. Lett., 76, (1996) 2386.
[36] I.V. Ponomarev and P.G. Silvestrov, Phys. Rev. B, 56, (1997) 3742.
[37] M.A. Skvortsov, V.E. Kravtsov, and M.V. Feigelman, JETP Lett., 68 (1998) 84.
[38] A.C Monin and A.M. Yaglom, 1975, Statistical fluid mechanics, Vol. 2 (MIT Press, Cambridge).
[39] K.R. Sreenivasan, Annu. Rev. Fluid Mech., 23 (1991) 539.
[40] K.R. Sreenivasan and R.A. Antonia, 1997, Annu. Rev. Fluid Mech., 29 (1997) 435.
Figure Captions

Figure 1. Scaling of the CMB temperature dissipation rate moments \( \langle \chi_p \rangle \) calculated for the four-year DMR map. The straight lines (the best fit) are drawn to indicate the scaling in the log-log scales.

Figure 2. Generalized dimensions \( D_p \) extracted from figure 1 (circles) using Eq. (16). The crosses correspond to the generalized dimensions calculated for the PRBM model [27] of the Anderson localization.
\[ \ln \langle \chi_s^p \rangle \]

\[ \ln s \]

- \( p=5 \)
- \( p=4 \)
- \( p=3 \)
- \( p=2 \)
\[ D_p \]

\[ p \]

- \( \bigcirc \) CMB dissipation 31.5, 53 and 90 GHz
- \( \bigtimes \) Anderson localization