PARTIAL ORTHOGONAL SPREADS OVER $\mathbb{F}_2$ INVARIANT UNDER THE SYMMETRIC AND ALTERNATING GROUPS

ROD GOW

Abstract. Let $m \geq 3$ be an integer and let $V$ be a vector space of dimension $2^m$ over $\mathbb{F}_2$. Let $Q$ be a non-degenerate quadratic form of maximal Witt index $2^{m-1}$ defined on $V$. We show that the symmetric group $\Sigma_{2m+1}$ acts on $V$ as a group of isometries of $Q$ and leaves invariant a partial orthogonal spread of size $2m+1$. This is enough to show that any group of even order $2^m$ or odd order $2m+1$, $m \geq 3$, acts transitively and regularly on a partial orthogonal spread in $V$. We construct the $\Sigma_{2m+1}$ linear action on $V$ by means of the spin representation of the symmetric group. Furthermore, when $m \equiv 3 \mod 4$, we show that the partial spread can be extended by two further maximal totally singular subspaces that are interchanged by $\Sigma_{2m+1}$. We also show that the alternating group $A_9$ acts in a natural manner on a complete spread of size 9 defined on a vector space of dimension 8 over $\mathbb{F}_2$.

1. Introduction

Let $K$ be a finite field of characteristic 2, with $|K| = q$. Let $V$ be a vector space of even dimension $2r$ over $K$. A quadratic form defined on $V$ is a function $Q : V \to K$ that satisfies

\[
Q(\lambda u) = \lambda^2 Q(u)
\]

\[
Q(u + v) = Q(u) + Q(v) + f(u, v)
\]

for all $\lambda$ in $K$ and all $u$ and $v$ in $V$. In this formula, $f$ is an alternating bilinear form, called the polarization of $Q$. We say that $Q$ is a non-degenerate quadratic form if $f$ is a non-degenerate bilinear form.

We say that a vector $v \in V$ is singular with respect to $Q$ if $Q(v) = 0$. We say that a subspace $U$ of $V$ is totally isotropic with respect to $f$ if $f(u, w) = 0$ for all $u$ and $w$ in $U$. We also say that $U$ is totally singular with respect to $Q$ if all the elements of $U$ are singular. It should be clear that if $U$ is totally singular with respect to $Q$, then it is totally isotropic with respect to $f$, but the converse is not necessarily true.

We shall assume henceforth that $Q$ is non-degenerate. In this case, the maximum dimension of a totally singular subspace of $V$ is $r$. We say that $Q$ has index $r$ if there is at least one $r$-dimensional totally singular subspace of $V$. We shall also assume henceforth that $Q$ has index $r$.

We recall that an isometry of $Q$ is a $K$-linear automorphism, $\sigma$, say, of $V$ that satisfies

\[
Q(\sigma v) = Q(v)
\]
for all $v \in V$. The set of all isometries of $Q$ forms a group under composition, called the orthogonal group of $Q$. This group is determined up to isomorphism by stipulating, as we have done, that $Q$ is non-degenerate and has index $r$. We denote this orthogonal group by $O_{2^r}(K)$, or $O^{+}_{2^r}(q)$.

We call a set of $r$-dimensional totally singular subspaces of $V$ a partial orthogonal spread for $V$ and $Q$ if the subspaces intersect trivially in pairs. Since $V$ contains precisely $(q^{r-1} + 1)(q^r - 1)$ non-zero singular vectors, it follows that a partial orthogonal spread contains at most $q^{r-1} + 1$ subspaces. We refer to a partial orthogonal spread of size $q^{r-1} + 1$ as a complete orthogonal spread.

It is a non-trivial fact that $V$ contains a complete orthogonal spread if and only if $r$ is even. On the other hand, if $r$ is odd, the maximum size of a partial orthogonal spread in $V$ is two.

It is clear that if $\sigma$ is an element of $O_{2^r}(K)$, and $U$ is a totally singular subspace of $V$, $\sigma(U)$ is also a totally singular subspace. Thus, $O_{2^r}(K)$ permutes the maximal ($r$-dimensional) totally singular subspaces, and it is known that this permutation action is transitive. It is a remarkable fact, proved by Kantor and Williams, that provided $r$ is even, $V$ has a complete orthogonal spread that consists of a regular orbit of a single maximal totally singular subspace under the action of a cyclic group of order $q^{r-1} + 1$. See Theorem 6.5 and the proof of Theorem 3.3 (i) in [4].

The main purpose of this paper is to construct a partial orthogonal spread of size $2m + 1$ in $V$ invariant under the action of the symmetric group $\Sigma_{2m+1}$, when $\dim V = 2^m$, $m \geq 3$ is an integer, and $K = \mathbb{F}_2$. This enables us to prove the following result. Let $G$ be a finite group with $|G| = 2m + \epsilon$, where $\epsilon = 0$ or 1. Then if $m \geq 3$, $G$ acts transitively and regularly on a partial orthogonal spread in $V$. We also show that the alternating group $A_9$ acts in a natural manner on a complete spread of size 9 defined on a vector space of dimension 8 over $\mathbb{F}_2$.

2. Modules for $\Sigma_n$ in Characteristic 2

The irreducible representations of the symmetric group $\Sigma_n$ over a field of characteristic 2 may all be realized in $\mathbb{F}_2$. They are labelled by partitions of $n$ no whose two parts are equal (so-called 2-regular partitions). We let $D^\lambda$ denote the irreducible $\mathbb{F}_2\Sigma_n$-module labelled by the 2-regular partition $\lambda$.

There is one partition $\lambda$ of $n$ for which the corresponding module $D^\lambda$ has remarkable properties, which we wish to describe. Suppose that $n \geq 3$ is odd, and set $n = 2m + 1$. Consider the partition $(m+1, m)$ of $n$. It is known that
\[
\dim D^{(m+1, m)} = 2^m.
\]

Similarly, suppose that $n \geq 4$ is even and write $n = 2m$. Then
\[
\dim D^{(m+1, m-1)} = 2^{m-1}
\]
in this case.

Given an $\mathbb{F}_2\Sigma_n$-module, $D$, say, let $D \downarrow_{\Sigma_{n-1}}$ denote its restriction to $\Sigma_{n-1}$. From what we have described, $D \downarrow_{\Sigma_{n-1}}$ has a composition series consisting of irreducible modules of the form $D^\mu$, where $\mu$ is a 2-regular partition of $n - 1$.

When $n = 2m + 1$ is odd, $D^{(m+1, m)} \downarrow_{\Sigma_{2m}}$ is indecomposable and has a composition series consisting of two copies of $D^{(m+1, m-1)}$. (When $m = 1$, $D^{(m+1, m-1)} =$
\[ D^{(2)} \] is the trivial module of \( \Sigma_2 \). When \( n = 2m \) is even,
\[
D^{(m+1,m-1)} \downarrow_{\Sigma_{2m-1}} = D^{(m,m-1)}
\]
is irreducible. These results are proved in the main theorem of [7].

We shall call these irreducible modules \( D^{(m+1,m)} \) and \( D^{(m+1,m-1)} \) spin modules for \( \Sigma_{2m+1}, \Sigma_{2m} \), respectively, over \( \mathbb{F}_2 \).

We say that an irreducible \( \mathbb{F}_2 \Sigma_n \)-module \( D \) is of quadratic type if there is a non-degenerate \( \Sigma_n \)-invariant quadratic form \( Q \), say, defined on \( D \) (thus \( \Sigma_n \) acts as isometries of \( Q \)).

We do not know necessary and sufficient conditions on a 2-regular partition \( \lambda \) that tell us if \( D^{(\lambda)} \) is of quadratic type or not, except in the case that \( \lambda \) has only two parts. This enables us to show that the spin modules are of quadratic type provided \( n \geq 7 \).

We quote the following lemma, proved in [5], Theorem 1.

**Lemma 1.** Let \( \lambda \) be a 2-regular partition of \( n \) into two parts. Then \( D^{(\lambda)} \) is not of quadratic type precisely when the smaller part of \( \lambda \) is a power of 2, say \( 2^r \), where \( r \geq 0 \), and \( n \equiv k \mod 2^{r+2} \), where \( k \) is one of the \( 2^r \) consecutive integers \( 2^{r+1} + 2^r - 1, \ldots, 2^{r+2} - 2 \).

We can now deal with the quadratic type of the spin module.

**Lemma 2.** The spin module of \( \Sigma_n \) over \( \mathbb{F}_2 \) is of quadratic type except when \( n = 5 \) or 6.

**Proof.** Suppose that \( n = 2m \) is even and the spin module \( D^{(m+1,m-1)} \) is not of quadratic type. Then Lemma 1 implies that \( m-1 = 2^r \), for some integer \( r \geq 0 \), and hence \( n = 2^{r+1} + 2 \). We must also have
\[
\begin{align*}
n \equiv k \mod 2^{r+2},
\end{align*}
\]
where \( k \) is one of \( 2^{r+1} + 2^r - 1, \ldots, 2^{r+2} - 2 \).

It follows certainly that \( 2^{r+1} \) divides \( k - 2 \), but \( 2^{r+2} \) does not divide \( k - 2 \). Clearly, then, \( k \) cannot equal \( 2 \), nor can it equal \( 2^{r+2} + 2 \), and the possibility that \( k - 2 \geq 3 \times 2^{r+1} \) is excluded by the inequality \( k \leq 2^{r+2} - 2 \). Thus we must have \( k = n = 2 + 2^{r+1} \).

When we take account of the inequality
\[
2^{r+1} + 2^r - 1 \leq k = 2^{r+1} + 2,
\]
we deduce that \( 2^r \leq 3 \) and hence \( r = 0 \) or 1.

It is impossible that \( r = 0 \), since it implies \( k = 4 \), which is inconsistent with the inequality \( k \leq 2^{r+2} - 2 = 2 \). Thus, \( r = 1 \), \( k = 6 \), and it is the case that \( D^{(4,2)} \) is not of quadratic type. This deals with the case that \( n \) is even.

Suppose next that \( n = 2m + 1 \) is odd. We have noted above that
\[
D^{(m+2,m)} \downarrow_{\Sigma_{2m+1}} = D^{(m+1,m)}.
\]
Since \( D^{(m+2,m)} \) is of quadratic type for \( m \geq 3 \) by what we have proved above, we deduce that \( D^{(m+1,m)} \) is also of quadratic type for \( m \geq 3 \). However, \( D^{(3,2)} \) is not of quadratic type, by the criterion of Lemma 1. This completes the analysis. \( \square \)
Lemma 3. Let $Q$ be a non-degenerate $\Sigma_{2m+1}$-invariant quadratic form defined on the spin module $D^{(m+1,m)}$. Let $U$ be the socle of $D^{(m+1,m)}\downarrow\Sigma_{2m}$. Then, $U$ is irreducible, has dimension $2^{m-1}$ and is totally singular with respect to $Q$.

Proof. We have remarked that $D^{(m+1,m)}\downarrow\Sigma_{2m}$ is indecomposable, having exactly two composition factors, both isomorphic to $D^{(m+1,m-1)}$. It follows that the socle $U$ of $D^{(m+1,m)}\downarrow\Sigma_{2m}$ is irreducible, isomorphic to $D^{(m+1,m-1)}$, and hence has dimension $2^{m-1}$.

Let $f$ be the polarization of $Q$ and let $U^\perp$ be the perpendicular subspace of $U$ with respect to $f$. Since $U$ and $f$ are invariant under $\Sigma_{2m}$, $U^\perp$ is also invariant under the group. It follows that $U\cap U^\perp$ is a subspace of $U$ that is also $\Sigma_{2m}$-invariant.

The irreducibility of $U$ implies that $U\cap U^\perp = 0$ or $U$.

Now if $U\cap U^\perp = 0$, the general equality that

$$
\dim U + \dim U^\perp = \dim D^{(m+1,m)}
$$

implies that $D^{(m+1,m)}\downarrow\Sigma_{2m}$ is the direct sum of the two invariant submodules $U$ and $U^\perp$. This contradicts the fact that $U$ is the socle.

We deduce that $U\cap U^\perp = U$, and hence $U$ is totally isotropic with respect to $f$.

This implies that

$$
Q(u + w) = Q(u) + Q(w)
$$

for all elements $u$ and $w$ in $U$, and hence $Q$ is a linear mapping from $U$ to $\mathbb{F}_2$. The subset of all singular vectors in $U$ is then a subspace of $U$ of codimension at most one in $U$, and this subspace is certainly $\Sigma_{2m}$-invariant. The irreducibility of $U$ forces the conclusion that $Q$ is identically zero on $U$, and hence $U$ is totally singular.

\[\square\]

3. Construction of an Invariant Partial Orthogonal Spread

We have compiled enough information about the spin module to prove the following result relating to a partial orthogonal spread invariant under symmetric group action.

Theorem 1. Suppose that $m \geq 3$. Let $V$ be a vector space of dimension $2^m$ over $\mathbb{F}_2$ that defines the spin module for $\Sigma_{2m+1}$ and let $U$ denote the socle of $V\downarrow\Sigma_{2m}$. Let $g_1, \ldots, g_{2m+1}$ be a set of coset representatives for $\Sigma_{2m}$ in $\Sigma_{2m+1}$. Then the $2m+1$ subspaces $U = g_1U, g_2U, \ldots, g_{2m+1}U$ form a partial orthogonal spread in $V$, permuted by $\Sigma_{2m+1}$ according to its natural action.

Proof. We first note that, since $m \geq 3$, $V$ is indeed a module of quadratic type, by Lemma 2 and Lemma 3 implies that $U$ is totally singular with respect to the invariant non-degenerate quadratic form. Let $g$ be any element of $\Sigma_{2m+1}$ not in $\Sigma_{2m}$. Consider the subspace $gU$, which is also totally singular.

We claim that $gU \neq U$. For if $gU = U$, $U$ is invariant under the subgroup of $\Sigma_{2m+1}$ generated by $g$ and $\Sigma_{2m}$. Since $\Sigma_{2m}$ is a maximal subgroup of $\Sigma_{2m+1}$, this subgroup is all of $\Sigma_{2m+1}$. But as $V$ is irreducible for $\Sigma_{2m+1}$, $U$ cannot be invariant under $\Sigma_{2m+1}$. We deduce that $gU \neq U$, as claimed.

Consider now the subspace $U \cap gU$, which we have just shown is not the whole of $U$. It is easy to see that $U \cap gU$ is invariant under the subgroup $\Sigma_{2m} \cap (g\Sigma_{2m}g^{-1})$ of $\Sigma_{2m}$. If we assume, as we may, that $\Sigma_{2m}$ is the subgroup of $\Sigma_{2m+1}$ fixing 1 in the natural representation of $\Sigma_{2m+1}$ on the numbers $\{1, 2, \ldots, 2m+1\}$, we see that
\( \Sigma_{2m} \cap (g \Sigma_{2m} g^{-1}) \) is the subgroup fixing 1 and \( g(1) \). We may then identify this subgroup unambiguously as \( \Sigma_{2m-1} \), since the subgroups of \( \Sigma_{2m+1} \) fixing 1 and a different number are conjugate.

We know that \( U \) affords the spin representation of \( \Sigma_{2m} \), and its restriction to \( \Sigma_{2m-1} \) is irreducible. Thus the only subspaces of \( U \) that are invariant under \( \Sigma_{2m-1} \) are \( U \) and 0. We deduce that \( U \cap gU = 0 \), as required, and the rest of the theorem follows from this argument.

Corollary 1. Let \( G \) be a finite group of order at least 6. Then \( G \) acts in a regular transitive manner on a partial orthogonal spread of size \( |G| \) defined on a quadratic space of dimension \( 2^m \) over \( \mathbb{F}_2 \), where \( m \) is the integer part of \( \frac{|G|}{2} \).

Proof. Suppose first that \( |G| = 2m + 1 \), where \( m \geq 3 \). We may embed \( G \) into \( \Sigma_{2m+1} \) by means of its regular representation. Then \( G \) permutes the subspaces in the partial orthogonal spread described in Theorem 1 in a regular transitive manner, as required.

Suppose next that \( |G| = 2m \) is even, with \( m \geq 3 \). We may embed \( G \) into \( \Sigma_{2m+1} \) in such a way that \( G \) fixes one point and permutes the remaining \( 2m \) points regularly. Then, in the action on the partial orthogonal spread described in Theorem 1, \( G \) clearly fixes one subspace and permutes the other \( 2m \) subspaces regularly.

Note. If \( |G| = 3, 4 \) or 5, we may embed \( G \) into \( \Sigma_7 \) and then show that \( G \) acts transitively on a partial orthogonal spread of size \( |G| \) defined on a quadratic space of dimension 8 over \( \mathbb{F}_2 \). However, a group of order 4 or 5 does not act transitively on a partial orthogonal spread of size 4 or 5 on a space of dimension 4 over \( \mathbb{F}_2 \), since a complete spread only contains three subspaces in such a case. A similar remark holds for a group of order 3 acting on a two-dimensional space over \( \mathbb{F}_2 \).

4. Extension of the partial spread when \( m \equiv 3 \mod 4 \)

When we take into account the influence of the alternating subgroup \( A_{2m+1} \) of \( \Sigma_{2m+1} \), we shall show that if \( m \equiv 3 \mod 4 \), the partial orthogonal spread of size \( 2m + 1 \) just described can be extended by two more maximal totally singular subspaces to give a partial orthogonal spread of size \( 2m + 3 \). The two additional subspaces are invariant under the alternating group \( A_{2m+1} \) and are interchanged by any odd permutation in \( \Sigma_{2m+1} \).

In order to find these additional subspaces, it is necessary to describe how \( A_{2m+1} \) acts on the spin module. The results we need to know are quite sensitive to properties of the integer \( m \), and require careful explanation.

Lemma 4. The spin module \( D^{(m+1,m)} \) over \( \mathbb{F}_2 \) splits as a direct sum of two non-isomorphic irreducible \( \mathbb{F}_2 A_{2m+1} \)-modules if \( m \equiv 0 \mod 4 \) or if \( m \equiv 3 \mod 4 \). The two \( A_{2m+1} \)-modules are conjugate under the action of \( \Sigma_{2m+1} \), as described by Clifford’s theorem.

Proof. This follows from Theorem 6.1 of [1].

The next question we need to address is whether or not the two irreducible \( \mathbb{F}_2 A_{2m+1} \)-modules described in Lemma 1 are self-dual. Here again, the answer is not obvious, and depends on the residue of \( m \) modulo 4. It seems that to establish what we want to know, we must invoke an alternative construction of the spin module over \( \mathbb{F}_2 \).
Let $\Gamma_{2m+1}$ denote either of the two non-isomorphic double covers of $\Sigma_{2m+1}$. The commutator subgroup of $\Gamma_{2m+1}$ has index 2 in $\Gamma_{2m+1}$ and is a double cover of $A_{2m+1}$, which we shall denote by $\tilde{A}_{2m+1}$. Since $\tilde{A}_{2m+1}$ is an extension of a central subgroup of order 2 by $A_{2m+1}$, given any element of odd order in $A_{2m+1}$, there is a unique element of the same order that projects onto it under the canonical homomorphism from $\tilde{A}_{2m+1}$ onto $A_{2m+1}$. We shall refer to this element of $\tilde{A}_{2m+1}$ as the canonical inverse image of the given element of odd order in $A_{2m+1}$.

$\Gamma_{2m+1}$ has a faithful irreducible complex representation of degree $2^m$, known as the basic spin representation (it is an example of a so-called projective representation of $\Sigma_{2m+1}$). Let $\theta$ denote the character of the basic spin representation. Schur shows in [8], Formula VII*, p. 205, that $\theta$ is rational-valued. Furthermore, it follows Theorem 7.7 of [8] that $\theta$ defines an absolutely irreducible Brauer character modulo the prime 2. Corollary 9.4 of Chapter IV of [2] implies then that $\theta$ has Schur index one over the field $\mathbb{Q}_2$ of 2-adic numbers. Thus, since $\theta$ certainly takes values in $\mathbb{Q}_2$, we deduce that the basic spin representation may be realized over $\mathbb{Q}_2$.

Let $\mathbb{Z}_2$ denote the ring of 2-adic integers in $\mathbb{Q}_2$. $R$ is a principal ideal domain and it follows that there is a $\Gamma_{2m+1}$-invariant $\mathbb{Z}_2$-lattice $L$, say, of rank $2^m$ which affords the basic spin representation. The quotient $L/2L$ is then a vector space, $\mathbb{L}$, say, of dimension $2^m$ over $\mathbb{F}_2$. Since the central involution of $\Gamma_{2m+1}$ acts as $-I$ on $L$, this involution acts trivially on $\mathbb{L}$, and thus $\mathbb{L}$ is naturally an $\mathbb{F}_2 \Sigma_{2m+1}$-module, which it turns out is isomorphic to the spin module $D^{(m+1,m)}$ we have been considering.

Working over the algebraic closure of $\mathbb{Q}_2$, the basic spin module is reducible on restriction to $\tilde{A}_{2m+1}$. This splitting does not necessarily occur over $\mathbb{Q}_2$, since we need a square root of $(-1)^m(2m+1)$ for it to take place. We refer to [6], Formula VII*, p. 205, for this theory. Schur shows that $\theta$ splits into two different irreducible characters of $\tilde{A}_{2m+1}$, $\theta_1$ and $\theta_2$, say. These characters $\theta_1$ and $\theta_2$ are real-valued if and only if $m$ is even. Furthermore, $\theta_1$ and $\theta_2$ differ on the canonical inverse image of a $2m + 1$-cycle. In particular, if $m$ is odd, $\theta_1$ and $\theta_2$ take non-real values on the canonical inverse image of a $2m + 1$-cycle.

Now $\theta$ restricted to elements of odd order is the Brauer character of the spin module $D^{(m+1,m)}$ of $\Sigma_{2m+1}$. Since we know that $D^{(m+1,m)}$ is reducible on restriction to $\tilde{A}_{2m+1}$, the Brauer characters of the irreducible constituents are $\theta_1$ and $\theta_2$, again restricted to elements of odd order. Finally, since $\theta_1$ and $\theta_2$ are not real-valued on the canonical inverse image of a $2m + 1$-cycle, the Brauer characters defined by $\theta_1$ and $\theta_2$ are not real-valued, and consequently, the two irreducible constituents of $D^{(m+1,m)}$ are not self-dual.

When we collect the information we have derived from the work of Schur, we have proved the following important result.

**Lemma 5.** The spin module $D^{m+1,m}$ of $\Sigma_{2m+1}$ over $\mathbb{F}_2$ splits as a direct sum of two non-isomorphic irreducible $\mathbb{F}_2 \tilde{A}_{2m+1}$-modules if $m \equiv 3 \mod 4$. The two $\mathbb{F}_2 \tilde{A}_{2m+1}$-modules are not self-dual in this case.

We proceed to extend the partial orthogonal spread when $m \equiv 3 \mod 4$.

**Theorem 2.** Suppose that $m \equiv 3 \mod 4$. Let $V$ be a vector space of dimension $2^m$ over $\mathbb{F}_2$ that defines the spin module for $\Sigma_{2m+1}$ and let $U$ denote the socle of $V \downarrow_{\Sigma_{2m}}$. Let $V \downarrow_{\tilde{A}_{2m+1}} = U_1 \oplus U_2$, where $U_1$ and $U_2$ are non-isomorphic irreducible $\mathbb{F}_2 \tilde{A}_{2m+1}$-modules. Then $U_1$ and $U_2$ are both totally singular.
Furthermore, let \( g_1 = 1, \ldots, g_{2m+1} \) be a set of coset representatives for \( \Sigma_2m \) in \( \Sigma_{2m+1} \). Then \( U = g_1U, g_2U, \ldots, g_{2m+1}U \), \( U_1 \) and \( U_2 \) form a partial orthogonal spread in \( V \) consisting of \( 2m + 3 \) subspaces.

**Proof.** We note that all \( 2m + 3 \) subspaces described above have dimension \( 2^{m-1} \). We also proved in Lemma 5 that \( U_1 \) and \( U_2 \) are not self-dual. Since they are both irreducible under the action of \( A_{2m+1} \), it follows in a straightforward way (as in the proof of Lemma 3) that \( U_1 \) and \( U_2 \) are both totally singular.

We claim that \( U \neq U_1 \). For, \( U \) is invariant under \( \Sigma_2m \), whereas \( U_1 \) is invariant under \( A_{2m+1} \) but not under \( \Sigma_{2m+1} \). Let \( g \) be any odd permutation in \( \Sigma_{2m} \). Then \( g \notin A_{2m+1} \) also, and hence \( gU_1 = U_2 \). Now if \( U = U_1 \), then \( U = gU = gU_1 = U_2 \). This is clearly a contradiction.

We deduce that \( U \cap U_1 \) is a proper subspace of \( U \), since \( \dim U = \dim U_1 \). \( U \cap U_1 \) is also invariant under \( \Sigma_2m \cap A_{2m+1} = A_{2m} \). We also know from the previous section of this paper that \( U \) is an irreducible \( \mathbb{F}_2 \Sigma_2m \)-module, isomorphic to the spin module \( D((m+1,m-1)) \). Since \( m \) is odd by assumption, \( D((m+1,m-1)) \downarrow_{A_{2m}} \) is irreducible, by Theorem 1.1 of [1]. Thus the only \( A_{2m} \)-submodules of \( U \) are \( U \) and \( 0 \). Since \( U \cap U_1 \) is an \( A_{2m} \)-submodule, and not equal to \( U \), it must be \( 0 \). A similar argument proves that \( U \cap U_2 = 0 \) also.

Finally, let \( h \) be any element of \( \Sigma_{2m+1} \) not in \( \Sigma_{2m} \), and consider \((hU) \cap U_1 \). Since \( h^{-1}U_1 = U_1 \) or \( U_2 \), we see that
\[
(hU) \cap U_1 = h(U \cap h^{-1}U_1) = 0,
\]
since we have proved that \( U \cap U_1 = U \cap U_2 = 0 \). Similarly, \((hU) \cap U_2 = 0 \) also.

This completes the proof. \( \square \)

This argument shows, for example, that the partial orthogonal spread of seven subspaces in an 8-dimensional space over \( \mathbb{F}_2 \), invariant under the action of \( \Sigma_7 \), can be extended to a complete spread of nine subspaces, also invariant under \( \Sigma_7 \).

5. **Action of \( A_9 \) on a Complete Spread in 8 Dimensions**

We have just shown that \( \Sigma_7 \) acts on a complete spread of nine subspaces in an 8-dimensional space over \( \mathbb{F}_2 \). We intend to give another explanation of this fact by showing that \( A_9 \) acts on a complete spread of nine subspaces in an 8-dimensional space over \( \mathbb{F}_2 \) and then observing that \( \Sigma_7 \) is a subgroup of \( A_9 \).

We take as our starting point the data that \( A_9 \) has three inequivalent irreducible representations of degree 8 over \( \mathbb{F}_2 \). We need to exclude one of these representations from consideration, and this is the so-called deleted permutation module, which arises from the natural permutation action of \( A_9 \) on nine points. The restriction of the deleted permutation module to \( A_8 \) has a composition series consisting of an irreducible module of dimension 6 and two copies of the trivial module, and this is not what we want.

The modules that we require arise from the restriction of the 16-dimensional spin module \( D^{(5,4)} \) of \( \Sigma_9 \) to \( A_9 \). We remarked already that Theorem 6.1 of [1] implies that \( D^{(5,4)} \downarrow_{A_9} \) is the direct sum of two non-isomorphic \( \mathbb{F}_2 A_9 \)-modules. These two 8-dimensional modules are both self-dual. By way of proof, albeit not a self-contained one, we can refer to p.85 of [3], where we see that all three inequivalent irreducible representations of \( A_9 \) of degree 8 in characteristic 2 support \( A_9 \)-invariant quadratic forms.
We can now proceed to fashion this information into a statement about the action of $A_9$ on a complete orthogonal spread.

**Theorem 3.** Let $V$ be an $F_2 A_9$-module of dimension 8 defined as an irreducible constituent of the restriction of the spin module $D^{(5,4)}$ of $\Sigma_9$. Then $V$ is a module of quadratic type. Let $U$ be an irreducible $F_2 A_8$-submodule of $V$. Then $U$ is 4-dimensional and is totally singular. Moreover, if $g_1, \ldots, g_9$ are a set of coset representatives for $A_8$ in $A_9$, the nine subspaces $g_i U$, $1 \leq i \leq 9$, form a complete orthogonal spread in $V$.

**Proof.** We first note that we can take the Brauer character of $A_9$ acting on $V$ to be that denoted by $\phi_3$ in the table found on p.85 of [1]. (The character $\phi_3$ has the same properties as $\phi_3$, and is conjugate to $\phi_3$ under the action of $\Sigma_9$.) As we noted above, $V$ is of quadratic type. (This also follows from the fact that $D^{(5,4)}$ is of quadratic type and $V$ is self-dual, since its Brauer character is real-valued.)

Reference to the table on p.48 of [2] shows that the restriction of $\phi_3$ to $A_8$ consists of two different irreducible Brauer characters of degree 4, one being the complex conjugate of the other. Now it is a fact that $V$ is reducible as an $F_2 A_8$-module. To see this, we note that $A_8$ is isomorphic to the general linear group $GL_4(F_2)$. The Brauer characters that occur in the restriction of $\phi_3$ to $A_8$ are those of the natural 4-dimensional module for $GL_4(F_2)$ over $F_2$ and its contragredient (or dual). If $V$ were irreducible as an $F_2 A_8$-module, it would follow that the two non-isomorphic irreducible modules for $GL_4(F_2)$ were Galois-conjugate over $F_4$, which is not the case, as they are defined over $F_2$.

This argument establishes that $U$ is 4-dimensional and furthermore, it is not self-dual, since its Brauer character is not real-valued. This then implies that $U$ is totally singular.

To complete the proof, we imitate the proof of Theorem 1.

Let $g$ be any element of $A_9$ not in $A_8$. Consider the subspace $gU$, which is also totally singular. We claim that $gU \neq U$. For if $gU = U$, $U$ is invariant under the subgroup of $A_9$ generated by $g$ and $A_8$. Since $A_8$ is a maximal subgroup of $A_9$, this subgroup is all of $A_9$. But as $V$ is irreducible for $A_9$, $U$ cannot be invariant under $A_9$. We deduce that $gU \neq U$, as claimed.

Consider now the subspace $U \cap gU$, which we have just shown is not the whole of $U$. It is easy to see that $U \cap gU$ is invariant under the subgroup $A_8 \cap (gA_9g^{-1})$ of $A_9$. We may take $A_8$ to be the subgroup of $A_9$ fixing 1 in the natural representation of $A_9$ on the numbers $\{1, 2, \ldots, 9\}$. Then we see that $A_8 \cap (gA_9g^{-1})$ is the subgroup fixing 1 and $g(1)$, which is isomorphic to $A_7$.

We know that $U$ affords an irreducible representation of $A_8$, and its restriction to $A_7$ is irreducible. Thus the only subspaces of $U$ that are invariant under $A_7$ are $U$ and 0. We deduce that $U \cap gU = 0$, as required, and the rest of the theorem follows from this argument.

We note that any group $G$ of order 9 may be embedded into $A_9$ by its regular representation. It follows that $G$ acts in a regular transitive manner on a complete orthogonal spread of size 9 defined on a quadratic space of dimension 8 over $F_2$. This is a consequence of the theorem of Kantor and Williams already described when $G$ is cyclic, but seems to be a new observation when $G$ is elementary abelian. Similarly, any non-cyclic group $G$ of order 8 may be embedded into $A_8$ by its regular representation, and then into $A_9$. It follows that $G$ acts in a regular transitive
manner on a partial orthogonal spread of size 8 defined on a quadratic space of dimension 8 over $\mathbb{F}_2$.

There are 135 non-zero singular vectors in the 8-dimensional quadratic space of index 4 over $\mathbb{F}_2$. These vectors are permuted transitively by $A_9$. The action is imprimitive, there being nine blocks of imprimitivity, namely, the non-zero vectors in each of the 4-dimensional subspaces that constitute the invariant complete spread. The stabilizer of a block is isomorphic to $A_8$, and it acts doubly transitively on the 15 non-zero vectors in the block.

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