Photonic Galton board in the synthetic dimension for tera-sample-per-second arbitrary waveform generation

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Abstract

We propose and experimentally demonstrate a photonic Galton board (PGB) in the temporal synthetic dimension for ultra-high speed arbitrary waveform generation (AWG). The proposed PGB consists of two mutually coupled fiber loops with different time delays and each loop having an acousto-optic modulator, to have a synthesized one-dimensional (1D) temporal photonic system. By controlling the gain or loss of the optical pulse recirculating in the loop and tuning the time delay difference between the two loops, arbitrary waveforms with a sampling interval equal to the time delay difference between the two fiber loops are generated. Since the time delay difference can be controlled ultra-small, the equivalent sampling rate is ultra-high. The approach is studied theoretically and evaluated experimentally. The experimental results show for an input ultra-short optical pulse with a minimum temporal width of 1.07 ps, arbitrary waveforms with a sampling rate up to 341.53 GSa/s are generated. The sampling rate can be increased to the terahertz range if the temporal width of the recirculating optical pulse is reduced to less than 1 ps.

1. Introduction

Microwave arbitrary waveforms with a wide bandwidth have been widely used in modern radar and microwave imaging systems to increase the range and imaging resolution¹⁻⁴. Due to the limited sampling rate of an electronic analog-to-digital converter (ADC), a state-of-the-art electronic arbitrary waveform generator using an electronic ADC can operate with a sampling rate up to 128 GSa/s and a maximum analog bandwidth of 65 GHz⁵. On the other hand, a photonic-assisted microwave arbitrary waveform generator (AWG) can operate at a speed far beyond 128 GSa/s thanks to the inherent high speed and broad bandwidth offered by modern photonics. Photonic-assisted microwave AWGs can be generally classified into two categories,
arbitrary waveform generation based on optical pulse shaping\cite{2,6-10} and arbitrary waveform generation based on photonic digital-to-analog conversion\cite{11-18}. In an optical pulse shaping system, a spatial light modulator (SLM)\cite{2,6,7}, metasurface\cite{8,9} or other spatial mask\cite{10} is used to shape the optical spectrum of an ultra-short optical pulse, to tailor the magnitude and phase of the spectrum corresponding to the temporal waveform to be generated. However, those systems are implemented based on free-space optics, making the systems bulky, costly, and vulnerable to environmental disturbances. In recent years, benefiting from the rapid progress in microwave photonics\cite{19-24}, microwave arbitrary waveform generation with lower loss and smaller size based on optical pulse shaping implemented by fiber-optic or integrated devices in the temporal or frequency domain has been reported\cite{3,25-28}. However, the bandwidth of a microwave waveform is still small, limited by the bandwidths of electro-optical modulators and photodetectors. Compared with an optical pulse shaping system, an optical digital-to-analog converter (DAC) based AWG can have a better waveform reconfigurability, but the bandwidth of a generated microwave waveform is still limited\cite{16-18}. Thus, a new approach must be discovered to generate arbitrary microwave waveforms at a much higher sampling rate with a much wider bandwidth, and at the same time, better reconfigurability.

Recently, synthetic dimension in photonics has been extensively investigated to use physical systems with lower dimensions to explore higher dimensional physical phenomena\cite{29,30}. Numerous photonic systems based on photonic synthetic dimension have been proposed for the implementations of optical quantum walks\cite{31,32}, band structure\cite{33,34}, topological photonics\cite{35-37}, and parity-time symmetry\cite{38-40}.

On the other hand, the Galton board invented by Galton for the demonstration of the central limit theorem in 1889, has been introduced to the optics community for studying spectral diffusion of a light wave\cite{41} and quantum phenomenon in nonlinear optics\cite{42-44}. Here, we show that an optical Galton board can also be employed in a photonic system with synthetic dimension, to generate ultra-fast microwave arbitrary waveforms.

In this paper, a photonic Galton board (PGB) in the temporal synthetic dimension for the generation of ultra-high speed arbitrary waveforms with a sampling rate up to terahertz range is proposed. Specifically, the PGB is implemented based on two coupled fiber loops with different time delays and one acousto-optic modulator (AOM) incorporated in each of the two loops, to control the gain or loss of the optical pulse recirculating in the loop, thus a synthesized one-dimensional (1D) temporal photonic system is implemented. The operation of the 1D temporal photonic system is identical to that of a Galton board in which an ultra-short pulse containing multiple photons (beads) is inputted from the top of the PGB with the photons bounced to the left or right by a beam splitter (pegs). At the bottom of the PGB, a sampled temporal waveform with its shape determined by the time delays of the two loops and the modulation functions to the AOMs is generated. The sampling interval is equal to the time delay difference between the two loops which can be made ultra-small, thus the equivalent sampling rate is ultra-high. The approach is studied theoretically. For an input pulse with a pulse duration of 1 ps, and the time delay difference of the two loops of 1 ps, an arbitrary
waveform with a sampling rate of a THz can be generated. Then, an experiment is performed to evaluate the operation of the system. The experimental results show for an input ultra-short optical pulse with a minimum temporal width of 1.07 ps, an arbitrary waveform with a sampling rate up to 341.53 GSa/s is generated. The sampling rate can be increased to the terahertz range if the round-trip dispersion is fully compensated, and the temporal width of the recirculating optical pulse is reduced to less than 1 ps.

2. Principle

A traditional Galton board has multiple rows of pegs equally separated by a given interval. When a bead reaches a peg, it will be bounced to the right or left with a fixed probability of 50%. At the bottom of the Galton board, a Gaussian distribution of the beads is produced. In a PGB, as shown in Figure 1, multiple pegs (gray rounded rectangles) with each having a tunable coupling ratio (the number in the gray rounded rectangles, corresponding to the probability) are used to bounce a photon to the left or right. $T_1$ and $T_2$ are the time delays of the left and the right paths, respectively. At the bottom of the PGB, a temporal distribution of the photons is produced which is a temporally sampled arbitrary waveform with its shape determined by the coupling ratio and path length difference. The time delay difference between the adjacent pulses is given by $T_2 - T_1$, thus by controlling the time delay difference between the two loops, the sampling rate can be controlled.

![Fig.1 Schematic diagram of the proposed PGB. The pegs (gray rounded rectangles) of each row can be made to control the number of photons (beads, orange circles) going to the left or right. At the bottom of the PGB, a temporal distribution of the photons is formed which is a temporally sampled arbitrary waveform.](image-url)
The PGB can be implemented based on fiber optics. Figure 2(a) shows a dual-loop fiber-optic system consisting of two coupled loops having different loop lengths with an AOM incorporated in each loop for the implementation of the PGB in the temporal synthetic dimension. An ultra-short pulse containing multiple photons (beads) from a mode locked laser (MLL) with a temporal width \( d \) is inputted into the long loop (blue line) through an optical coupler (OC). The photons are split into two parts by a beam splitter (BS). The BS has a fixed splitting ratio, but the joint operation of the BS and the AOMs corresponding to a tunable BS with the splitting ratio controlled by the gain and loss of the AOMs. Then, the two groups of photons, i.e., two pulses, return to the BS and are split into four parts corresponding to four pulses. As the input pulse recirculates in the two loops of different lengths for more round trips, at the output of each loop, a sequence of pulses is generated. By detecting the optical pulse sequences from the two loops at two photodetectors (PDs) and combining the detected electrical pulse sequences, a temporal distribution of the photons is produced which is a temporally sampled arbitrary waveform with its shape determined by the coupling ratio and path length difference. The time interval between adjacent pulses in the generated pulse sequence is the time delay difference between the two loops. The equivalence of the fiber optic system in Fig. 2(a) to a PGB is shown in Fig. 2(b), in which the blue and red arrowed lines indicate the paths by which the pulses traveling in the long and short loop, respectively. The BS in the long and short loop is depicted as the gray rectangles and the AOMs in the long and short loop are depicted as the orange and green circles, respectively. The \( n \)-th pulse in the \( m \)-th round trip after being modulated by the AOM with an electric field \( u_{m,n} \) in the long loop and \( v_{m,n} \) in the short loop are coupled to its nearest neighbor sites \( u(\hat{v})_{m+1,n+1} \) and \( v(\hat{u})_{m+1,n} \) in the next round trip, respectively. Thus, a synthetic temporal dimension, pulse position \( n \), is synthesized along the round-trip times \( m \).

Figure 2 A dual-loop fiber-optic system for the implementation of the proposed PGB. (a) The long loop (blue line) and the short one (red line) are mutually coupled by a beam splitter (BS). Each loop contains an acousto-optic modulator (AOM) for controlling the gain and loss of the pulses recirculating in the loops. (b) The equivalence of the fiber optic system in (a) to the proposed PGB. The blue and red arrowed lines indicate the pulses traveling in the long and short loop, respectively. The BS is depicted as the gray rectangle and the AOMs in the long and short loop are depicted as orange and green circles, respectively. At the bottom of the PGB, a temporally sampled arbitrary waveform is generated. MLL: mode locked laser; OC: optical coupler; BS: beam splitter, AOM: acousto-optic modulator; PD: photodetector.
The electric field of the optical pulse at the output of the two loops can be expressed by

\[ u_{m+1,n} = G_{u,m+1} \left( \frac{\sqrt{2}}{2} u_{m,n-1} + i \frac{\sqrt{2}}{2} v_{m,n} \right) \]  
\[ v_{m+1,n} = G_{v,m+1} \left( i \frac{\sqrt{2}}{2} u_{m,n-1} + \frac{\sqrt{2}}{2} v_{m,n} \right) \]  

(1.1)

(1.2)

where \( G_{u,m+1} \) and \( G_{v,m+1} \) are the gain/loss of the AOMs in the long and short loop for the \((m+1)\)-th round trip, respectively. Thus, as the input pulse recirculating in the two loops of different lengths for more round trips, a temporally sampled arbitrary waveform \( s[n] \) is generated at the bottom of the PGB, which can be expressed by

\[ s[n] = |u_{m,n}|^2 + |v_{m,n+1}|^2, \quad 0 \leq n \leq m \]  

(2)

where \( u_{m,0} = 0 \) and \( v_{m,m+1} = 0 \) (see Supplementary Material 1). The sampling interval is equal to the time delay difference between the two loops which can be made ultra-small. The smallest sampling interval is only limited by the temporal width of the recirculating pulse at the \(m\)-th round trip, which is broadened by the chromatic dispersion in the loops and given by

\[ d' = D \times \Delta \lambda \times m \times l \]  

(3)

where \( D \) is the dispersion coefficient of the loops, \( \Delta \lambda \) is the spectrum width of the input pulse and \( l \) is the length of the long loop. Thus, the sampling rate \( f_s \) of the generated temporally sampled waveform can be expressed by

\[ f_s = \frac{1}{T_2 - T_1} = \frac{1}{\Delta t} \leq \frac{1}{d'} \]  

(4)

where \( d' \) is the temporal width of the recirculating pulse. Eq. (4) indicates that the sampling rate of the generated waveform from the PGB can be ultra-high when the time delay difference is ultra-small and the maximum sampling rate is only limited by the temporal width of the recirculating pulse.

3. Numerical and experimental results

A numerical study is performed to evaluate operation of the proposed PGB for the generation of arbitrary waveforms. We set a constant loss, \( G_{v,m} = 0.5 \), in the short loop, and a variable loss, \( G_{u,m} \in (0,1] \), in the long loop for different round trips. We develop an Auto-Fit system with a backpropagation algorithm which is used to calculate \( G_{u,m} \) for the target waveform (see Supplementary Material 2). The generation of six waveforms with 11 sampling points and two waveforms with 31 sampling points is presented. Figure 3(a1)-(h1) shows the eight target waveforms. By calculating \( G_{u,m} \) of the target waveforms and applying them to the AOM in the long loop, the desired waveforms are generated, which are shown in Fig. 3(a2)-(h2). The sampling interval between two adjacent sampling points equals to the time delay difference \( \Delta t \) between the two loops. If the temporal duration of every sampling point and the time delay difference \( \Delta t \) are 1 ps, the generated arbitrary waveform can have a sampling rate of a THz.
Note that the burst waveform shown in Fig. 3(g2) can be a linearly chirped microwave waveform after passing through a bandpass filter3.

Fig. 3 Target waveforms and the corresponding simulation results. The target waveforms with 11 sampling points: (a1) binary coded, (b1) burst, (c1) PAM-4, and their simulated results in (a2), (b2) and (c2), respectively. The target waveforms with 11 sampling points: (d1) triangular, (e1) rectangular and (f1) sawtooth waveforms and their simulated results in (d2), (e2) and (f2), respectively. The target waveforms with 31 sampling points: (g1) burst and (h1) Gaussian waveforms and their simulated results in (g2) and (h2), respectively.
To validate the numerical results, an experiment is performed. The experimental setup is shown in Figure 4. An optical pulse train with a repetition rate of 20 MHz is generated by a mode locked laser (MLL). Before being injected into the dual-loop fiber optic system, the repetition rate of the pulse train is reduced to 50 kHz through a Mach-Zehnder modulator (MZM) to make two adjacent pulses in the pulse train have sufficiently large time spacing to allow a pulse to recirculate in the loops for multiple times without overlapping with a second pulse from the pulse train. A polarization controller (PC1) is employed before the MZM to minimize the polarization dependent loss. The temporal width is broadened by using a single-mode fiber (SMF) of 330 m, to make the pulse to have a temporal width that can be fully sampled by our oscilloscope. In this dual-loop system, the short loop has a length of 34.6 m, and a tunable delay line (TDL) is incorporated in the long loop, which is used to tune the time delay difference between the two loops and hence the sampling rate of the generated waveforms. A PC (PC2 or PC3) is connected before the AOM in each loop to control the polarization state of the recirculating pulse in order to minimize the polarization dependent loss at the AOMs. Another PC (PC4) is incorporated in the short loop before the BS to ensure two pulse sequences combine with the same polarization state. In each loop, an erbium-doped fiber amplifier (EDFA) and a tunable optical filter (TOF) are used to compensate for the round-trip loss and to filter out the amplified spontaneous (ASE) noise from the EDFA outside the signal frequency band. Finally, an optical coupler (OC) is used in each loop to direct the pulse sequence to a detection unit (DU), which consists of an AOM employed to apply a designated time window, a photodetector (PD) for optical-to-electrical conversion, and a high-speed oscilloscope (OSC). A temporally sampled arbitrary waveform is generated by combining the detected electrical pulse sequences from the two DUs.

**Fig. 4** The experimental setup of the PGB for the generation of ultra-high speed arbitrary waveforms. MLL: mode locked laser; AWG: arbitrary waveform generator; PC: polarization controller; MZM: Mach-Zehnder modulator; SMF: single mode fiber; OC: optical coupler; BS: bean splitter; TDL: tunable delay line; AOM: acousto-optic modulator; EDFA: Erbium-doped fiber amplifier; TOF: tunable optical filter; DU: detection unit; PD: photodetector; OSC: high-speed oscilloscope.
A femtosecond optical pulse from a repetition-rate-reduced pulse train is broadened to have a 3-dB temporal width of 31 ps or a 10 dB width of 56 ps through the SMF which has a length of 330 m (see Supplementary Material 3). Then, it is injected into the long loop. We keep $G_{s,m} = 0.5$ by fixing the driving signal applied to the AOM in the short loop to be a sinusoidal signal at 200 MHz with a fixed amplitude. At the same time, a second sinusoidal signal with a carrier frequency of 80 MHz amplitude modulated by $G_{s,m}$ at 5.78 MHz is applied to the AOM as a driving signal in the long loop (see Supplementary Material 3). After the input pulse recirculating in the proposed PGB for 10 round trips, waveforms with 11 sampling points are generated, including a binary coded, a burst, a PAM-4, a triangular, a rectangular, and a sawtooth waveform (see Supplementary Material 3). The time delay difference between the two loops is 149.93 ps, corresponding to a sampling rate of 6.67 GSa/s. A burst and a Gaussian waveform with 31 sampling points are also generated after the 30th round trip, as shown in Fig. 5(g) and (h), respectively. In these cases, the time delay difference between the two loops is 168.07 ps, which results in a sampling rate of 5.95 GSa/s.

We calculate the root mean square error (RMSE) of the generated waveforms to evaluate the performance of the PGB for the generation of waveforms, which shows an average RMSE of 0.1820 (see Supplementary Material 4), confirming good fidelity of the generated waveforms.

![Fig. 5 Experimentally generated arbitrary waveforms (solid blue lines) with 11 sampling points: (a) binary coded, (b) burst, (c) PAM-4, (d) triangular, (e) rectangular, and (f) sawtooth waveforms. The time delay difference between the two loops is 149.93 ps, corresponding to a sampling rate of 6.67 GSa/s. Experimentally generated waveforms (solid blue line) with 31 sampling points: (g) burst and (h) Gaussian waveforms. The time delay difference is 168.07 ps, corresponding to a sampling rate of 5.95 GSa/s. The target waveforms are also shown in (d), (e), (f) and (h) as dashed red lines for comparison.](image-url)
In order to further explore the capability of the system in performing faster-speed waveform generation, a second experiment is performed. In this case, a pulse after the MZM is directly injected into the long loop without passing through the SMF. To eliminating the chromatic dispersion in the two loops, a dispersion compensating fiber (DCF) of about 7 m is employed in each of the two loops. Then, we tune the time delay difference between the two loops from 46.08 to 12.50 ps, corresponding to a tunable sampling rate from 21.7 to 80.0 GSa/s. Four analog waveforms are generated and shown in Figure 6, in which the dashed red lines and solid blue lines are the target waveforms and the experimentally generated waveforms, respectively. The corresponding RMSE is calculated to have an average value of 0.0798 (see Supplementary Material 4), again confirming good fidelity of the generated waveforms.

Fig. 6 Experimentally generated arbitrary waveforms (solid blue lines) with a high sampling rate tunable from 21.7 to 80.0 GSa/s. The generated waveforms with 11 sampling points at different sampling rates: (a) triangular at 21.7 GSa/s, (b) rectangular at 48.0 GSa/s, sawtooth waveforms at 31.0 GSa/s, (d) triangular at 44.1 GSa/s, (e) rectangular at 64.0 GSa/s, sawtooth waveforms at 51.2 GSa/s. The generated Gaussian waveforms (solid blue lines) with 31 sampling points: (g) Gaussian at 52.2 GSa/s and (h) Gaussian at 80.0 GSa/s. The target waveforms are also shown as dashed red lines for comparison.
Fig. 7 Experimentally generated arbitrary waveforms with 31-sampling-points at different sampling rates.

(a) The evolution of an experimentally generated Gaussian waveform when the time delay difference between the two loops is tuned from +100 to -100 ps. The dashed white lines show the evolution of the waveform width to indicate the variation of the sampling rate. At the area near the intersection of the two dashed white lines, the sampling rate can reach a terahertz or higher when the time delay difference is within +1 and -1 ps. (b) The evolution of fifteen generated Gaussian waveforms in (a) shown in 3D plot. (c) A generated Gaussian waveform in solid blue lines at 10.42 GSa/s. A target waveform in solid orange lines is also shown for comparison. A zoom-in view of an individual pulse in the target waveform is given in the inset by in which the 3 dB temporal width is measured to be 23.8 ps. (d) A generated Gaussian waveform in solid blue line at 341.56 GSa/s. The dashed orange line shows the target waveform, which is generated by an ideal pulse burst shown in solid orange lines by passing the pulse burst through a lowpass filter. The insert is a zoom-in view of an individual pulse with a 3 dB temporal width of 2.86 ps in the ideal pulse burst. (e)-(h) show the generation of a rectangular waveform at different sampling rates.
Then, the evolution of an experimentally generated waveforms when the time delay difference is tuned is studied. Figure 7(a) shows the evolution of an experimentally generated Gaussian waveform when the time delay difference is tuned from +100 to -100 ps. Note the long loop will become a short loop if the time delay difference becomes negative. The sampling rate is increased from 10.00 GSa/s to infinity and then decreased from infinity back to 10.00 GSa/s. The dashed white lines in Fig 7(a) show the evolution of an experimentally generated Gaussian waveform when the time delay difference between the two loops is tuned from +100 to -100 ps. The dashed white lines show the evolution of the temporal width to indicate the variation of the sampling rate. As can be seen, if the time delay difference is reduced to 1 ps or less, the sampling rate can reach a terahertz or higher. Fig. 7(b) shows the evolution of the generated Gaussian waveforms in Fig. 7(a) in 3D plot, where fifteen Gaussian waveforms with different sampling rates are shown. Fig. 7(c) shows a generated Gaussian waveform in solid blue lines at 10.42 GSa/s. A target waveform in solid orange lines is also shown for comparison. A zoom-in view of an individual pulse in the target waveform is given in the inset by in which the 3 dB temporal width is measured to be 23.8 ps. Fig. 7(d) shows a generated Gaussian waveform in solid blue line at 341.56 GSa/s. The dashed orange line shows the target waveform, which is generated by passing an ideal pulse burst shown in solid orange lines through a lowpass filter, which is a PD (due to its finite bandwidth) in the experiment. The insert is a zoom-in view of an individual pulse with a 3 dB temporal width of 2.86 ps in the ideal pulse burst. The generation of a rectangular waveform with 31 sampling points at different sampling rates is also performed and the results are shown in Fig 7(e)-(h). These experimental results are all achieved by using a PD with a bandwidth of 50 GHz and a high-speed oscilloscope with a bandwidth of 80 GHz, a sampling rate of 256 GSa/s and minimum rise/fall time of 5.6 ps (10-90%) and 3.9 ps (20-80%).

Noted that, the sampling rate is determined by the time delay difference and is limited by the temporal width of the pulse recirculating in the loops. If the dispersion in the two loops is fully compensated, then the sampling rate is limited by the temporal width of the input pulse. In the experiment, although chromatic dispersion is compensated, the two loops still have residual dispersion, thus the pulse recirculating in the two loops will be broadened, making the sampling rate limited. In addition, the bandwidth of the PDs will also limit the maximum sampling rate. In order to capture a generated waveform with a sampling rate of 1 TSa/s or higher, the bandwidth of the PDs needs to be a terahertz. To monitor a generated waveform, a high-speed oscilloscope is used. The sampling rate of the oscilloscope must be high enough to effectively sample the generated waveforms. In our experiment, the oscilloscope has a bandwidth of 80 GHz and a sampling rate of 256 GSa/s, thus the maximum sampling rate of the waveforms that we can capture are limited with a few hundreds of gigahertz, but this does not affect the potential of the system to operate at terahertz range.
6. Conclusion

We have proposed and experimentally demonstrated a PGB in the temporal synthetic dimension for the generation of ultra-high speed arbitrary waveforms with a sampling rate up to 341.53 GSa/s. The sampling rate of the generated arbitrary waveform is determined by the time delay difference between the two loops in the PGB. Thus, by reducing the time delay difference, the sampling rate can be increased. On the other hand, the maximum sampling rate is also limited by the temporal width of the recirculating optical pulse, which is affected by the temporal width of the input pulse and the dispersion in the loops. Thus, if the input pulse is controlled to have a temporal width of a few femtoseconds and the dispersion in the system is fully compensated, the sampling rate can be increased to a terahertz.

The concept of ultra-high speed arbitrary waveform generation by the PGB in the temporal synthetic dimension opens a fresh prospect for ultra-high-speed signal generating and processing. Synthesizing additional dimensions in physical systems with lower dimensions may provide more opportunities and facets for solving complex problems in future works.

7. Methods

Implementation of the PGB in the synthetic dimension

The PGB in the temporal synthetic dimension for the generation of ultra-high speed arbitrary waveforms is implemented using commercial off-the-shelf optical and optoelectronic components. The MLL is a femtosecond laser source (CALMAR OPTCOM FPL-03CFFJNU). The MZM (FUJITSU H74M-5208-J048) for reducing the repetition rate of the pulse train has an optical bandwidth of 10 GHz. The TDL (General Photonics MDL-002) with a maximum time delay of 560 ps is used for controlling the time delay difference between the two loops. The AOM1 (CETC SGTF80-1550-1) with a shifting frequency of 80 MHz, the EDFA1 (Max-Ray Photonics EDFA-BA-20-B) bumped at about 75 mA, and the TOF1 (santec OTF-350) with a center wavelength of 1553.11 nm and passband of 3.94 nm are incorporated in the long loop. In the short loop, the AOM2 (Brimrose TEM-210-50-10-1550-2FP) has a shifting frequency of 200 MHz, the EDFA2 (PYOE-EDFA-C) is bumped at about 66 mA and the TOF2 (Alnair Labs BVF-300CL) is also working at a center wavelength of 1553.11 nm and passband of 3.94 nm. The signals applied to the MZM and the AOM3 in the DUs are generated by a two-channel arbitrary waveform generator (RIGOL DG822). We adjust the PC, EDFA in each loop to ensure the pulses recirculating in each loop for enough round trips.

Generation of arbitrary waveforms

The modulation signal $G_{\text{am}}$ applied to the AOM1 in the long loop is generated by another arbitrary waveform generator (Tektronix AWG70002A). The output optical pulse sequence from each loop is converted into electrical signal by the PDs (u’t XPDV21x0RA), which have a bandwidth of 50 GHz. A high-speed oscilloscope (Teledyne LabMaster 10-36Zi) with a
bandwidth of 36 GHz and sampling rate of 80 GSa/s is used to monitor the low-speed generated waveforms in Fig. 5. Another high-speed oscilloscope (Keysight UXR 0804A) with a bandwidth of 80 GHz and sampling rate of 256 GSa/s is used in the case of monitoring faster-speed waveforms in Fig. 6 and Fig. 7.

The Sync port of the MLL (CALMAR OPTCOM FPL-03CFFJNU) is connected to the Trigger port of the two-channel arbitrary waveform generator (RIGOL DG822) for time synchronization. At the same time, the Sync port of this AWG is connected to another AWG (Tektronix AWG70002A), which generates modulation signal applied to the AOM1 in the long loop.

8. Author contributions

Y. G. conceived the idea, designed the theoretical simulation, conducted the experiment, analyzed the data and wrote the paper; J. Z. and J. Y. conceived the idea, directed the theoretical simulation, guided the experiment and wrote the paper; L. L., R. C., G. W. and J. C. contributed to the experiment.

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Supplementary Materials

Supplementary Material 1: transfer matrix

Figure S1(a) shows the photonic Galton board (PGB) in which an input pulse is propagating in the first three round trips, and the generation of a waveform with four sampling points is shown in Figure S1(b). In Figure S1(a), the arrowed blue and red lines indicate the paths by which the pulses traveling in the long and short loops, respectively. The beam splitters (BSs) are depicted as gray rectangles. The orange and green circles are the acousto-optic modulators (AOMs) in the long and short loops, respectively. In Figure S1(b), the orange lines show the temporally sampled waveform with four sampling points generated after the 3rd roundtrip.

A $2 \times 2$ BS, in free space optics form and fiber optics form shown in Figure S2(a) and (b), respectively, has a transfer matrix given by

\[
\begin{pmatrix}
  b_1 \\
  b_2 \\
\end{pmatrix} = \begin{pmatrix}
  \frac{\sqrt{2}}{2} & i\frac{\sqrt{2}}{2} \\
  i\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
\end{pmatrix} \begin{pmatrix}
  a_1 \\
  a_2 \\
\end{pmatrix} \tag{S1}
\]

where the imaginary symbol $i$ indicates a $\pi/2$ phase change when the optical pulse from one path propagates into the other path. Thus, the electric field of the optical pulse at the output of the two loops can be expressed by

\[
u_{m+1,n} = G_{u,m+1} \left( \frac{\sqrt{2}}{2} u_{m,n} + i \frac{\sqrt{2}}{2} \nu_{m,n} \right) + \nu_{m+1,n} = G_{v,m+1} \left( i \frac{\sqrt{2}}{2} u_{m,n} + \frac{\sqrt{2}}{2} \nu_{m,n} \right) \tag{S2.1, S2.2}
\]

where $G_{u,m+1}$ and $G_{v,m+1}$ are the gain/loss of the AOMs in the long and short loops for the $(m+1)$-th round trip, respectively. The output pulse sequence of the two loops at the $m$-th round trip can be expressed in vectors with $m$ elements, given by

\[
U_{m} = [u_{m,1}, u_{m,2}, u_{m,3}, \ldots, u_{m,m}]^T \tag{S3.1}
\]

\[
V_{m} = [v_{m,1}, v_{m,2}, v_{m,3}, \ldots, v_{m,m}]^T \tag{S3.2}
\]

If we combine $U_{m}$ and $V_{m}$, and define another vector $P_{m}$ with $2m$ elements, which contains all the pulses in the two loops, we have
\[ P_m = [v_{m,1}, u_{m,1}, v_{m,2}, u_{m,2}, \ldots, v_{m,m}, u_{m,m}]^T \]  

(S4)

\( P_3^T \) is shown as the grey vector at the bottom of Fig. S1(a)). Then, \( P_{m+1} \) can be calculated from \( P_m \) by a transfer matrix \( T_m^{m+1} \) with a scale of \( 2(m+1) \times 2m \), which can be expressed by

\[
T_m^{m+1} = A_m^{m+1} \cdot B_m^{m+1}
\]

where \( A_m^{m+1} \) and \( B_m^{m+1} \) are the transfer matrices of the gain/loss of the AOMs and the BS from the \( m \)-th to the \((m+1)\)-th, respectively. For \( m = 0 \), we have

\[ T_0^1 = A_0^1 \cdot B_0^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i \sigma_z/2 \\ \sigma_z/2 \end{pmatrix} \]

(S6)

Thus, all the pulses in the two loops at the \( m \)-th round trip can be recursively calculated from the input pulse. By digitally combining the pulses at the same time slot, as shown in Fig. S1(b), a temporally sampled arbitrary waveform \( s[n] \) is generated, as shown at the bottom of the PGB, which can be expressed by

\[ s[n] = |u_{m,n}|^2 + |v_{m,n+1}|^2, \quad 0 \leq n \leq m \]

(S7)

where \( u_{m,0} = 0 \) and \( v_{m,m+1} = 0 \). The sampling interval is equal to the time delay difference between the two loops.
Fig. S1 (a) An input pulse propagating in the photonic Galton board (PGB) in the first three round trips and (b) the generation of a waveform with four sampling points. In (a), the arrowed blue and red lines indicate the pulses traveling in the long and short loops, respectively. The BSs are depicted as gray rectangles. The AOMs in the long and short loop are depicted as orange and green circles, respectively. In (b), a temporally sampled waveform with four sampling points output from the 3rd round trip is shown in the orange lines. The sampling interval is equal to the time delay difference between the two loops.

Supplementary Material 2: calculating the optimal $G_{u,m}$

Five steps are involved for the calculation of $G_{u,m}$, as shown in Figure S3(a). Firstly, in Step 1, we define a target waveform, $s[n]$, with $j$ sampling points and randomly generate 10,000 different $G_u^{(k)}$, each of which is a vector consisting of the $G_{u,m}$ for different round trips. It is shown as

$$G_u^{(k)} = \left[ G_{u,1}^{(k)}, G_{u,2}^{(k)}, G_{u,3}^{(k)}, \cdots, G_{u,j-1}^{(k)} \right]^T, 1 \leq k \leq 10000$$

(S8)

The distribution of all the $G_{u,m}^{(k)} (1 \leq m \leq j-1, 1 \leq k \leq 10000)$ is shown in Figure S4. We use...
these \( G_u^{(k)} \) as the inputs for the numerical model of the PGB to calculate the corresponding waveforms \( s[n]^{(k)} \), as shown in Fig. S3(b). These 10,000 pairs of \( G_u^{(k)} \) and \( s[n]^{(k)} \) form a database. In Step 2, we compare the target waveform with the database. In Step 3, by calculating the root mean square error (RMSE): \( R \) of every \( s[n]^{(k)} \) in the database with the target waveform, we find a \( s[n]^{(c)} \), which has the smallest RMSE (marked it as \( R_s \)), and the corresponding \( G_u^{(c)} \). After that, in Step 4, \( G_u^{(c)} \), \( R_s \), and \( s[n] \) are sent into an Auto-Fit system with a backpropagation algorithm, with the principle illustrated in Fig. S3(c).

In the Auto-Fit operation, we firstly assign \( R \) to an initial \( R \), then set \( m = 1 \) and add a detune \( \delta \) to \( G_u^{(c)} \). Next, we send the new \( G_u^{(c)} \) to the numerical model of the PGB to get the newly generated waveform and calculate the new RMSE: \( R \) between the newly generated and the target waveform. If \( R \) is smaller than the previous \( R \), adding 1 to \( m \) and doing the same operation on \( G_{u,m}^{(c)} \) until \( m \) is larger than \( a \). If not, \( R \) becomes larger, taking 2\( \delta \) from \( G_{u,1}^{(c)} \) and sending the new \( G_u^{(c)} \) to the PGB again to get a newly generated waveform, then calculate the new RMSE: \( R \) between the newly generated and the target waveform. If this new \( R \) is smaller than the previous \( R \), adding 1 to \( m \) and doing the same operation on \( G_{u,m}^{(c)} \) until \( m \) is larger than \( a \). If \( R \) still becomes larger than previous, adding \( \delta \) to \( G_{u,1}^{(c)} \) and refreshing \( m \) by \( m = m + 1 \), then doing the same operation on \( G_{u,m}^{(c)} \) until \( m \) is larger than \( a \).

When \( m \) is larger than \( a \), if the present \( R \) does not equal to \( R_s \), we set \( R \) as a new \( R_s \) and reset \( m \) to 1, then repeat the fitting operation from \( m = 1 \) to \( m = a \) again. If the present \( R \) equals to \( R_s \), we refresh \( \delta \) to be smaller by a coefficient \( \Omega \) for more delicate adjustment of \( G_{u,m}^{(c)} \) in the next fitting operation until \( \delta \) is smaller than a predefined threshold. Finally, all the elements in \( G_u^{(c)} \) are the optimal \( G_{u,m}^{(c)} \) for the generation of the target waveform.

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Fig. S3 The diagram to show the five steps for the calculation of \( G_{u,m} \). (a) The five steps. Start at defining the target waveform and database generating and end up with the optimal \( G_{u,m} \). (b) The illustration of database generation. (c) The principle of the Auto-Fit system with backpropagation algorithm. Y: yes, N: no, ↓: smaller than previous.
Supplementary Material 3: input pulse, modulation signals and pulse sequences

In the experimental results in Fig. 5 of the Main text, a femtosecond optical pulse from a repetition-rate-reduced pulse train is broadened to have a 3 dB temporal width of 31 ps or a 10 dB width of 56 ps through the SMF with a length of 330 m, as shown in Figure S5(a). For the faster-speed waveform generation shown in Fig. 6 of the Main text, the femtosecond optical pulse after the MZM is injected into the long loop as an input pulse without broadening, as shown in Fig S5(b), which shows a 3 dB temporal width of 11 ps or a 10 dB width of 19 ps on the OSC.

In our experimental setup, we set a constant loss, $G_{r,m} = 0.5$, in the short loop, by fixing the driving signal applied to AOM2 in the short loop to be a sinusoidal signal at 200 MHz with fix amplitude. At the same time, a variable loss is applied to the long loop for different round trips by driving the AOM1 in the long loop with a second sinusoidal signal, which has a carrier frequency of 80 MHz. It is amplitude modulated by $G_{u,m}$ at 5.78 MHz, as shown in Figure S6(a)-(h).

Figure S7 shows the pulse sequences in the two loops in the case of generating a burst waveform with 31 sampling points. There is a round-trip time delay of about 173 ns. The insert figures show the pulse sequences at the 30th roundtrip of the long and short loops. A designated time window is applied by the AOM3 in the DU to exclude the input pulse and the pulse sequences after the 43rd or the 44th round trips.

Noted that, a target waveform with $j$ sampling points is generated after the $(j-1)$-th round trip, which establish time $t_e$ is given by
\[ t_c = \left( j-1 \right) \times \frac{l}{v} \tag{S9} \]

where \( l \) and \( v \) are the length of the long loop and the velocity of light in the fiber, respectively.

Fig. S5 Input pulse. (a) The femtosecond pulse, after broadening through the SMF with a length of 330 m, has a 3 dB temporal width of 31 ps or a 10 dB width of 56 ps, which is captured on an OSC with a sampling rate of 80 GSa/s. (b) The femtosecond pulse generated from the MLL without broadening shown on another OSC, which has a higher sampling rate of 256 GSa/s. Each hollow square is a sampling point of the OSC.

Fig. S6 Normalized driving signal applied to AOM1 in the long loop. Sinusoidal signals amplitude modulated by \( G_{m,m} \) at 5.78 MHz for the generated waveforms: (a) binary coded, (b) burst, (c) PAM-4, (d) triangular, (e) rectangular, (f) sawtooth, (g) burst and (h) Gaussian. Different amplitudes correspond to the different \( G_{m,m} \) for every round trips.
Supplementary Material 4: performance evaluation

We calculate the RMSE of the generated waveforms to evaluate the performance of the PGB for the generation of arbitrary waveforms. The RMSE is given by

$$\text{RMSE} = \sqrt{\frac{\sum_{n=1}^{j} (s[n] - s[n])^2}{j}} \tag{S10}$$

where $s[n]$ and $s[n]$ are the target waveform and the generated waveform, respectively. $j$ is the number of sampling points in the target waveform.

Table 1 shows the RMSEs of the generated waveforms in Fig. 5 of the Main text, which shows an average value of 0.1820, confirming good fidelity of the generated waveforms. Note that, the RMSEs in Tab. 1 are calculated by extracting the peak values of the sampled pulses in the generated waveforms, and comparing them with the target values of the sampling points in the target waveforms. There are two error symbols in the 31-sampling-point waveforms, which are the largest RMSE among those waveforms. This is caused by the polarization-induced difference of the modulation efficiency of the AOMs and the phase detuning of the optical pulses recirculating in the system.
Table 1. The RMSE of the generated waveforms in Fig. 5 of the Main text

| Waveform                  | RMSE  |
|---------------------------|-------|
| (a) Binary coded          | 0.2425|
| (b) 11-sampling-points burst | 0.2477|
| (c) PAM-4                 | 0.1170|
| (d) Triangular            | 0.1228|
| (e) Rectangular           | 0.1524|
| (f) Sawtooth              | 0.1197|
| (g) 31-sampling-points burst | 0.3793|
| (h) Gaussian              | 0.0742|

As for the faster-speed waveforms with tunable sampling rates from 21.7 to 80.0 GSa/s in Fig. 6 of the Main text, we put the target waveforms into a linear-time-invariant system, which has a passband of 50 GHz and a sampling rate of 256 GSa/s, to emulate the PD and OSC in our experimental setup. Then we use those processed target waveforms to evaluate the performance of the generated waveforms. The corresponding RMSEs are shown in Table 2, which shows an average value of 0.0798, again confirming good fidelity of the generated waveforms.

Table 2. The RMSE of the generated waveforms in Fig. 6 of the Main text

| Waveform – Sampling rate      | RMSE  |
|-------------------------------|-------|
| (a) Triangular - 21.7 GSa/s   | 0.0691|
| (b) Rectangular - 48.0 GSa/s  | 0.1071|
| (c) Sawtooth - 31.0 GSa/s     | 0.0819|
| (d) Triangular - 44.1 GSa/s   | 0.0999|
| (e) Rectangular - 64.0 GSa/s  | 0.1008|
| (f) Sawtooth - 51.2 GSa/s     | 0.0830|
| (g) Gaussian - 52.2 GSa/s     | 0.0534|
| (h) Gaussian - 80.0 GSa/s     | 0.0427|

The RMSEs of the rectangular waveforms are a little bit larger than those of the other waveforms, which, to some extent, are caused by the speed of rise/fall slopes of our OSC. In addition, the deviation of all generated waveforms is also caused by the noise figure of our EDFAs in the two loops, the polarization-induced difference of the modulation efficiency during every round trip, the optical nonlinear effect, phase detuning of the optical pulses recirculating in the system and etc.