Superpixel Based Graph Laplacian Regularization for Sparse Hyperspectral Unmixing

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Abstract—An efficient spatial regularization method using superpixel segmentation and graph Laplacian regularization is proposed for sparse hyperspectral unmixing method. A superpixel is defined as a group of structured neighboring pixels which constitutes a homogeneous region. First, we segment the hyperspectral image into many superpixels. Then, a weighted graph in each superpixel is constructed. Each node in the graph represents the spectrum of a pixel and edges connect the similar pixels inside the superpixel. The spatial similarity is investigated in each superpixel using graph Laplacian regularization. A weighted sparsity promoting norm is included in the formulation to sparsify the abundance matrix. Experimental results on simulated and real data sets show the superiority of the proposed algorithm over the well-known algorithms in the literature.

Index Terms—Sparse unmixing, graph Laplacian, abundance estimation, superpixel.

I. INTRODUCTION

HYPERSPECTRAL imaging used in remote sensing allows the identification of the substances in the scene by measuring the light spectrum over hundreds of contiguous bands. However, low spatial resolution of hyperspectral sensor and combination of different materials in the homogeneous mixtures cause mixed pixels. Decomposition of a mixed pixel into spectral signatures (endmembers) with corresponding fractions (abundances) is known as spectral unmixing [1].

Spectral unmixing methods mainly use linear mixture model (LMM) in which the observed spectra is a linear combination of the endmembers with corresponding abundances. Linear spectral unmixing (LSU) methods are simple and has tractable solutions. In LSU, an endmember extraction step is applied and then abundance value of each pixel is estimated. There are many algorithms for endmember extraction such as N-FINDR [2], pixel purity index (PPI) [3] and vertex component analysis (VCA) [4]. These algorithms require pure pixel assumption and it is not always satisfied due to the spatial resolution. One way of solving this problem is to use ground spectral libraries and then obtaining the abundance value of each pixel using this large spectral library. Generally, the number of endmembers in the scene are small compared to the number of endmembers in the spectral library. This means that only small number of endmembers contribute the mixed pixel. Therefore, abundance vector of mixed pixel is expected to be sparse. Estimating the sparse abundance vector using a priori available spectral library is known as sparse unmixing (SU) [5]. Sparse unmixing by variable splitting and augmented Lagrangian (SUnSAL) [5] solves an $l_1$ minimization problem satisfying abundance non-negativity constraint (ANC) and abundance sum constraint (ASC). Collaborative SUnSAL (CLSUnSAL) [6] solves an $l_{2,1}$ norm optimization problem to promote the row-sparse structure of the abundance matrix. Local collaborative sparse unmixing (LCSU) [7] estimates the sparse abundance matrix by solving CLSUnSAL in a neighborhood of pixels to obtain more accurate abundance values. Iterative reweighted sparse unmixing (IRWSU) [8] use a weighting strategy in the formulation which has a better abundance estimation compared to CLSUnSAL.

Furthermore, spatial-contextual information of the abundance map is exploited in many works [9]–[12] by considering the piecewise smoothness of the abundance map. Total variation (TV) regularization [13] is used in SUnSAL-TV [9] which minimizes the fractional abundance of neighboring pixels. It provides smooth abundance map, however it does not take into account the local changes in the abundance map. A spatial discontinuity weight strategy is developed to preserve the details in the abundance map better [10] using the idea that smooth abundance map condition is not hold in real scenarios. A double reweighted sparse unmixing and TV (DRSU-TV) [11] improves the sparsity of the abundance matrix by using a double reweighting strategy which is applied in both spectral and spatial domains. Spectral-spatial weighted sparse unmixing ($S^2$WSU) method is presented in [12] which includes a single regularizer with spectral and spatial weighting matrices in the proposed formulation to improve the abundance estimation.

It is known that similar pixels in a local region are likely to have similar abundances, graph based approaches are developed for hyperspectral unmixing [14]–[16]. A hypergraph-regularized sparse nonnegative matrix factorization (NMF) [14] based unmixing approach employs a hypergraph structure where each pixel is taken as a vertex and the pixels in the neighborhood of that pixel form a hypergraph. Therefore, similar pixels having similar abundances are found which leads to obtain better unmixing results. In a same manner, graph Laplacian regularization is used in [15] to promote the smoothness of the abundance map in sparse regression framework. Recently, spatial-contextual information is exploited using hypergraph learning [16] to extract the similarity between the pixels in a small spatial neighborhood.

Recently, superpixel segmentation is investigated in several works [17]–[21] in hyperspectral imaging. Fang et al. [17] propose a hyperspectral classification method based on superpixel segmentation. The pixels in each superpixel are jointly repre-
sented by a set of common atoms. [13] employs multiscale superpixels to extract the local information for hyperspectral image (HSI) classification. A superpixel weighting strategy is used in [19] to include the spatial correlation. A fast multiscale spatial regularization based on simple linear iterative clustering (SLIC) [22] is proposed in [20]. Superpixel-based reweighted low-rank and total variation (SUSRL2-TV) [21] minimizes the rank of the abundance matrix in each superpixel and promote the smoothness of the abundance map using TV.

In this paper, we propose a superpixel based graph Laplacian for sparse unmixing (SBGLSU). First, HSI is segmented into many superpixels using SLIC. A weighted graph for each superpixel is constructed, where each node represents the neighboring pixels in the superpixel. Although, superpixels are homogeneous shape adaptive spatial-neighboring pixels, we include a weighted graph regularization to measure the similarity of the pixels in each superpixel. In this manner, spatial correlation among the neighboring pixels inside the superpixel is better extracted. The sparsity of the abundance matrix is satisfied using an $l_1$ norm regularizer with a weighting strategy that promotes the joint-sparsity of the abundance matrix.

The rest of the paper is organized as follows. Section II explains the proposed method. The simulated and real data experiments are given in Section III. Finally, Section IV concludes the paper.

II. Superpixel Based Graph Laplacian Regularization for Sparse Unmixing (SBGLSU)

LMM assumes that endmembers are linearly combined to form the measured spectrum of a pixel. It can be modeled as

$$Y = AS + N$$

where $Y = [y_1, \ldots, y_n] \in \mathbb{R}^{L \times n}$ is the spectrum of $n$ pixels where each $y_i$ ($i = 1, 2, \ldots, n$) represents the spectrum of $i$th pixel in the HSI, $A \in \mathbb{R}^{L \times m}$ is the mixing matrix containing $m$ endmembers, $S = [s_1, \ldots, s_n] \in \mathbb{R}^{m \times n}$ fractional abundance matrix where each $s_i$ ($i = 1, 2, \ldots, n$) represents the fractional abundance vector of $i$th pixel and $N \in \mathbb{R}^{L \times n}$ models the error in the measurements. If the number of active endmembers is much lower than the number of the endmembers in spectral library $A$, then abundance matrix $S$ is expected to be sparse.

Furthermore, a spatial similarity exists between neighboring pixels in a HSI which leads to abundance similarity of neighboring pixels. Therefore, we first construct a weighted graph $G = (V, E)$ [23] where $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_n\}$ denote the vertex set and weighted edge set, respectively. A weighted adjacency matrix $W \in \mathbb{R}^{n \times n}$ is constructed where each entry $W_{ij}$ defines the degree of similarity between the spectrum of pixels $y_i$ and $y_j$. If $y_i$ and $y_j$ are similar then a large positive weight is assigned to $W_{ij}$. If they are not similar, a small positive value is assigned. There are different choice of selecting adjacency matrix to construct similarity graphs. We use Gaussian heat kernel which is defined as

$$W_{ij} = \exp\left(-\frac{||y_i - y_j||^2}{2\sigma^2}\right)$$

where $\sigma$ controls the width of neighborhood.

When constructing a similarity graph for hyperspectral data using weighted adjacency matrix, it is likely to find two similar pixels in different regions of the HSI. However, local regions tend to have similar pixels leading to similar abundances. Therefore, we extract the homogeneous regions using a segmentation algorithm. Generally, $k$-means algorithm is used to extract the local regions, however $k$-means search the whole image to find the similar pixels. For this reason, our purpose is to search a local area which have spatially similar regions. We resort the SLIC to segment the HSI into homogeneous regions. SLIC is a variant of $k$-means clustering but it searches a limited region and it takes into account the image boundaries. It is also easy to use, fast and memory efficient and it requires little number of parameters.

Therefore, we first segment the HSI into superpixels and then construct graph Laplacian for each superpixel. We can express the abundance similarity in each superpixel as

$$\frac{1}{2} \sum_{g=1}^{n_g} \sum_{(i,j) \in \varepsilon_g} W_{gij} ||s_i - s_j||^2 = \sum_{g=1}^{n_g} \text{Tr}(S_g L_g S_g^T)$$

Here, $n_g$ denotes the number of superpixels in the image, $\varepsilon_g$ is the neighborhood of each superpixel, and $\text{Tr}(\cdot)$ denotes the trace of a matrix. $S_g$ is the abundance matrix of the $g$th superpixel, $L_g = D_g - W_g$ is the graph Laplacian matrix of $g$th superpixel where $W_g$ is the adjacency matrix of $g$th superpixel, $D_g$ is a diagonal matrix which is calculated as $D_{gij} = \sum_{j=1}^{n} W_{gij}$ where $W_{gij}$ denotes the each entry of $W_g$.

After defining the abundance similarity measure in each superpixel, SBGLSU is proposed as

$$\min_{S} \frac{1}{2} ||Y - AS||_F^2 + \lambda_s ||W_s \odot S||_1 + \lambda_g \sum_{g=1}^{n_g} \text{Tr}(S_g L_g S_g^T)$$

$$+ t_{R_k}(S)$$

where $\cdot||_F$ and $\cdot||_1$ denote the Frobenius norm and $l_1$ norm, respectively. $\lambda_s$ and $\lambda_g$ are regularization parameters, $W_s$ is the weight matrix to promote the sparsity of $S$, $\odot$ denotes Hadamard product. $t_{R_k}(S)$ is indicator function that is equal to zero if $s \geq 0$ and $+\infty$ otherwise.

We split the optimization problem into subproblems using alternating direction method of multipliers (ADMM) [24] to solve alternately. The optimization problem (3) can be written in a compact form as

$$\min_{V, S, \Lambda} g(V) \quad \text{subject to} \quad GS + BV = 0$$

where $g(V) = \frac{1}{2} ||Y - V_1||_F^2 + \lambda_s ||W_s \odot V_2||_1 + \lambda_g \sum_{g=1}^{n_g} \text{Tr}(V_3 g L_g V_3^T) + t_{R_k}(V_4)$, $V = (V_1, V_2, V_3, V_4)$, $G = [A, I, I, 0]^T$ and $B = \text{diag}[-I]$. The augmented lagrangian formulation of (4) is

$$\mathcal{L}(V, S, \Lambda) = g(V) + \frac{\mu}{2} ||GS + BV - A||_F^2$$

where $\mu > 0$ is a penalty parameter and $\Lambda/\mu$ denotes the Lagrange multipliers.

The algorithm of SBGLSU is shown in Algorithm [1] SBGLSU includes a weighting strategy to promote the row-sparsity of the abundance matrix. However, ADMM requires...
that all functions should be closed, proper and convex to guarantee convergence. Therefore, we use inner and outer loops in Algorithm 1 to make the convergence of the algorithm better. In simulation section, the maximum iteration number of outer and inner loops are set to $l = 60$ and $t = 6$, respectively.

Algorithm 1 Pseudocode of the proposed SBGLSU

Input: $Y$, $A$, $\lambda_s$, $\lambda_g$, $\mu > 0$, $\epsilon$, SLIC parameters

Initialization: $l = 0$, $t = 0$, $S^{(0)}$, $V_1^{(0)}$, $V_2^{(0)}$, $V_3^{(0)}$, $V_4^{(0)}$, $\Lambda_1^{(0)}$, $\Lambda_2^{(0)}$, $\Lambda_3^{(0)}$, $\Lambda_4^{(0)}$

1: for $g = 1$ to $n_g$
2: $L_g = D_g - W_g$
3: end for
4: repeat
5: $W_s(:, i)^{(l)} = \left[\left[\frac{1}{\left\|\left(S^{(l)} - \Lambda_2^{(l)}\right)(m, :)\right\|_2 + \epsilon} \right] \left[\frac{1}{\left\|\left(S^{(l)} - \Lambda_2^{(l)}\right)(m, :)\right\|_2 + \epsilon} \right] \right] i = 1, 2, \ldots, n$
6: repeat
7: $S^{(l+1)} = (A^T A + 3 I)^{-1} \left[ A^T V_1^{(l)} + \Lambda_1^{(l)} \right] + \left[ (V_2^{(l)} + \Lambda_2^{(l)}) + (V_3^{(l)} + \Lambda_3^{(l)}) + (V_4^{(l)} + \Lambda_4^{(l)}) \right]$
8: $V_1^{(l+1)} = \frac{1}{1 + \mu} (Y + \mu (AS^{(l+1)} - \Lambda_1^{(l)}))$
9: $V_2^{(l+1)} = \text{soft}(S^{(l+1)} - \Lambda_2^{(l)}, (\lambda_s/\mu)W_s^{(l+1)}))$
10: end for
11: $V_3^{(l+1)} = \mu (S^{(l+1)} - \Lambda_3^{(l)}) (2 \lambda_g L_g + \mu I)^{-1}$
12: $\Lambda_4^{(l+1)} = \max(\Lambda_4^{(l)} - S^{(l+1)} - V_1^{(l+1)})$
13: $\Lambda_1^{(l+1)} = \Lambda_1^{(l)} - \Lambda_4^{(l+1)} - V_1^{(l+1)}$
14: $\Lambda_2^{(l+1)} = \Lambda_2^{(l)} - S^{(l+1)} - V_2^{(l+1)}$
15: $\Lambda_3^{(l+1)} = \Lambda_3^{(l)} - S^{(l+1)} - V_3^{(l+1)}$
16: $\Lambda_4^{(l+1)} = \Lambda_4^{(l)} - S^{(l+1)} - V_4^{(l+1)}$
17: Update iteration: $t \leftarrow t + 1$
18: $S^{(l+1)} \leftarrow S^{(t+1)}$
19: $\Lambda_2^{(l+1)} \leftarrow \Lambda_2^{(t+1)}$
20: Update iteration: $l \leftarrow l + 1$
21: until some stopping criteria is satisfied.

II. SIMULATED AND REAL DATA EXPERIMENTS

In this section, we demonstrate the performance of the proposed method by using synthetic hyperspectral data sets. We perform two synthetic data experiments to demonstrate the effectiveness of SBGLSU. We compare the results of SBGLSU with SUNSAL-TV [9], S2WSU [12], MUASLIC [20] and SUSRLR-TV [21]. Signal to reconstruction error (SRE) is used to measure the quality of the unmixing results. It is defined as $SRE = 10 \log_{10}(\left\|S\right\|_F^2 / \left\|S - \hat{S}\right\|_F^2)$, where $S$ represents the ground truth abundance map and $\hat{S}$ is the estimated abundance map.

A. Simulated Data Sets

In the synthetic data experiments, we create a spectral library $A$ by selecting 240 signatures randomly from digital spectral library (splib06) [25] obtained from the U.S. Geological Survey (USGS) which contains the spectra of 498 materials measured in 224 spectral bands distributed uniformly in the interval 0.4 and 2.5 $\mu$m. We generate two simulated data sets satisfying ASC and ANC. Simulated data cube 1 (DC1) is created by selecting five spectral signatures randomly from library $A$ as active endmembers and using the corresponding fractional abundance maps having size of $75 \times 75$. For simulated data cube 2 (DC2), we select nine spectral signatures randomly from $A$ as active endmembers and using the corresponding fractional abundance maps having size of $100 \times 100$. Simulated data sets DC1 and DC2 is then contaminated with Gaussian noise of signal-to-noise ratio (SNR) with SNR = 20, 30 and 40 dB, respectively.

SLIC requires two parameters. These are superpixel size and regularization parameter. We set superpixel size to 5 and 6 for DC1 and DC2, respectively. SLIC regularization parameter is set to 0.00125 for both datasets. In order to find the optimal $\lambda_s$ and $\lambda_g$ for two datasets. We vary $\lambda_s$ and $\lambda_g$ and obtain the best regularization parameter set for SNR levels 20, 30 and 40 dB. SRE (dB) values as a function of parameters $\lambda_s$ and $\lambda_g$ for SNR = 30 dB for both datasets are shown in Fig. 1.

B. Comparison to Other Unmixing Methods

We now compare the results of SBGLSU with SUNSAL-TV, S2WSU, MUASLIC, SUSRLR-TV. Optimal regularization parameters of all algorithms are found by varying the regularization parameters in a suitable range. Table I reports the SRE values along with optimal regularization parameters obtained by different algorithms for DC1 and DC2 for SNR values 20, 30 and 40 dB. We can see clearly that the SBGLSU performs best in all noise levels for DC1 and DC2. SUNSAL-TV has lowest SRE values in all SNR values so it is not reported in Table I. Furthermore, we compare the unmixing results visually for individual endmember. Fig. 2 shows the estimated abundance map obtained by different unmixing algorithms for endmember #5 in DC1 with SNR = 20 dB. It can be seen clearly that SBGLSU is able to recover the details much better than the other algorithms. Similar conclusions can be made for DC2. Fig. 3 shows the estimated abundance map obtained by different unmixing algorithms for endmember #1 in DC1 with SNR = 20 dB.

![Fig. 1. SRE(dB) as a function of parameters $\lambda_s$ and $\lambda_g$ for DC1 and DC2 for SNR = 30 dB (left column: DC1, right column: DC2)](image)

![Fig. 2. Estimated abundance map for endmember #5 in DC1 with SNR = 20 dB (left column: SUNSAL-TV, right column: SBGLSU)](image)

![Fig. 3. Estimated abundance map for endmember #1 in DC1 with SNR = 20 dB (left column: SUNSAL-TV, right column: SBGLSU)](image)
TABLE I
SRE VALUES OF DIFFERENT ALGORITHMS

|          | S3WSU | MUA SLIC | SUSRLR-TV | SBGLSU |
|----------|-------|----------|-----------|--------|
| DC1      |       |          |           |        |
| SNR      |       |          |           |        |
| 20       | 7.68  | 11.34    | 14.38     | 17.07  |
|          | \(\lambda = 1e - 1\) | \(\lambda_1 = 3e - 2\) | \(\rho = 1e - 1\) | \(\lambda_s = 5e - 2\) |
|          | \(\lambda_2 = 1e - 1\) | \(\lambda_{TV} = 5e - 2\) | \(\lambda_g = 1e - 3\) |        |
| 30       | 15.48 | 15.73    | 25.29     | 28.98  |
|          | \(\lambda = 5e - 3\) | \(\lambda_1 = 7e - 3\) | \(\rho = 5e - 2\) | \(\lambda_s = 1e - 2\) |
|          | \(\lambda_2 = 5e - 2\) | \(\lambda_{TV} = 1e - 2\) | \(\lambda_g = 1e - 3\) |        |
| 40       | 28.23 | 22.34    | 38.72     | 41.26  |
|          | \(\lambda = 1e - 3\) | \(\lambda_1 = 1e - 3\) | \(\rho = 1e - 2\) | \(\lambda_s = 5e - 3\) |
|          | \(\lambda_2 = 1e - 2\) | \(\lambda_{TV} = 5e - 4\) | \(\lambda_g = 1e - 3\) |        |

C. Real Data Experiment

In this section we present the performance of the proposed method on the real data set. The real data set used in the experiment is Cuprite dataset\(^1\) which is used frequently to compare the performance of the unmixing algorithms. It contains 224 bands with a wavelength range of 0.4-2.5 \(\mu\)m. However, some bands of the Cuprite data set have low-SNR and water absorption. Therefore, we removed the bands 1-2, 105-115, 150-170, and 223-224 prior to analysis. The spatial size of the data used in the experiment is 250 \(\times\) 191. We use the spectral library of 498 minerals from the USGS library.

We compare the unmixing results of SBGLSU with SUnSAL-TV, S3WSU, MUA SLIC, SUSRLR-TV. We also use Tetracorder 4.4 \(^{26}\) algorithm in order to compare the unmixing results qualitatively. Fig. 3 shows a qualitative comparison among the classification maps obtained by Tetracorder 4.4 \(^{26}\) algorithm and abundance maps obtained by SBGLSU, SUnSAL-TV, S3WSU, MUA SLIC and SUSRLR-TV. Three minerals in Cuprite district namely Alunite, Buddington and Chalcedony are considered in the comparison. It can be concluded that SBGLSU is a valid unmixing algorithm for real hyperspectral data.

The computation times of all algorithms under comparison are reported in Table II. MUA SLIC is the fastest unmixing algorithm under comparison. SBGLSU computation time is comparably much better than other algorithms except MUA SLIC.

IV. CONCLUSION

In this paper, we have developed a novel graph Laplacian regularized sparse hyperspectral unmixing method based on superpixel segmentation. Superpixel segmentation extracts the spatially homogeneous regions and graph Laplacian regularization minimizes the abundance similarity of each superpixel. A sparsity inducing norm with a weighting strategy is included in the formulation to sparsify the abundance matrix better. The proposed method is solved using a variable splitting approach which includes inner and outer loops to converge better. Experimental results on both simulated and real data sets have shown that the proposed method is a very effective sparse unmixing method compared to other state-of-the-art sparse unmixing methods in the literature.

REFERENCES

[1] N. Keshava and J. F. Mustard, “Spectral unmixing,” IEEE Signal Process. Mag., vol. 19, no. 1, pp. 44–57, Jan 2002.

[2] M. E. Winter, “N-findr: an algorithm for fast autonomous spectral endmember determination in hyperspectral data,” vol. 3753, 1999, pp. 266–275.

\(^1\)http://aviris.jpl.nasa.gov/html/aviris.freedata.html
Fig. 4. Abundance maps obtained by different algorithms. From top to bottom: Abundance maps corresponding to Alunite, Buddingtonite and Chalcedony, respectively.

[3] J. W. Boardman, F. A. Kruse, and R. O. Green, “Mapping target signatures via partial unmixing of AVIRIS data,” in *Fifth JPL Airborne Earth Science Workshop*, vol. 95. JPL Publication, 1995, pp. 23–26.

[4] J. M. P. Nascimento and J. M. B. Dias, “Vertex component analysis: a fast algorithm to unmix hyperspectral data,” *IEEE Trans. Geosci. Remote Sens.*, vol. 43, no. 4, pp. 898–910, April 2005.

[5] J. M. Bioucas-Dias and M. A. T. Figueiredo, “Alternating direction algorithms for constrained sparse regression: Application to hyperspectral unmixing,” in *Proc. 2nd Workshop Hyperspectral Image Signal Process., Evol. Remote Sens. (WHISPERS)*, June 2010, pp. 1–4.

[6] M. D. Iordache, J. M. Bioucas-Dias, and A. Plaza, “Collaborative sparse regression for hyperspectral unmixing,” *IEEE Trans. Geosci. Remote Sens.*, vol. 52, no. 1, pp. 341–354, Jan 2014.

[7] S. Zhang, J. Li, K. Liu, C. Deng, L. Liu, and A. Plaza, “Hyperspectral unmixing based on local collaborative sparse regression,” *IEEE Geosci. Remote Sens. Lett.*, vol. 13, no. 5, pp. 631–635, May 2016.

[8] C. Y. Zheng, H. Li, Q. Wang, and C. L. Philip Chen, “Reweighted sparse regression for hyperspectral unmixing,” *IEEE Trans. Geosci. Remote Sens.*, vol. 54, no. 1, pp. 479–488, Jan 2016.

[9] M. D. Iordache, J. M. Bioucas-Dias, and A. Plaza, “Total variation spatial regularization for sparse hyperspectral unmixing,” *IEEE Trans. Geosci. Remote Sens.*, vol. 50, no. 11, pp. 4484–4502, Nov 2012.

[10] S. Zhang, J. Li, Z. Wu, and A. Plaza, “Spatial discontinuity-weighted sparse unmixing of hyperspectral images,” *IEEE Trans. Geosci. Remote Sens.*, vol. 56, no. 10, pp. 5767–5779, Oct 2018.

[11] R. Wang, H. Li, A. Pizurica, J. Li, A. Plaza, and W. J. Emery, “Hyperspectral unmixing using double reweighted sparse regression and total variation,” *IEEE Geosci. Remote Sens. Lett.*, vol. 14, no. 7, pp. 1146–1150, July 2017.

[12] S. Zhang, J. Li, H. Li, C. Deng, and A. Plaza, “Spectral-spatial weighted sparse regression for hyperspectral image unmixing,” *IEEE Trans. Geosci. Remote Sens.*, vol. 56, no. 6, pp. 3265–3276, June 2018.

[13] L. I. Rudin, S. Osher, and E. Fatemi, “Nonlinear total variation based noise removal algorithms,” *Phys. D*, vol. 60, no. 1-4, pp. 259–268, Nov. 1992.

[14] W. Wang, Y. Qian, and Y. Y. Tang, “Hypergraph-regularized sparse nmf for hyperspectral unmixing,” *IEEE J. Sel. Topics Appl. Earth Observ. Remote Sens.*, vol. 9, no. 2, pp. 681–694, 2016.

[15] R. Ammanouil, A. Ferrari, and C. Richard, “A graph laplacian regularization for hyperspectral data unmixing,” in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process. (ICASSP)*, Apr. 2015, pp. 1637–1641.

[16] P. Jia, M. Zhang, and Y. Shen, “Hypergraph learning and reweighted l1-norm minimization for hyperspectral unmixing,” *IEEE J. Sel. Topics Appl. Earth Observ. Remote Sens.*, vol. 12, no. 6, pp. 1898–1904, 2019.

[17] L. Fang, S. Li, X. Kang, and J. A. Benediktsson, “Spectral-spatial classification of hyperspectral images with a superpixel-based discriminative sparse model,” *IEEE Trans. Geosci. Remote Sens.*, vol. 53, no. 8, pp. 4186–4201, Aug 2015.

[18] T. Dundar and T. Ince, “Sparse representation-based hyperspectral image classification using multiscale superpixels and guided filter,” *IEEE Geosci. Remote Sens. Lett.*, vol. 16, no. 2, pp. 246–250, Feb 2019.

[19] S. Zhang, C. Deng, J. Li, S. Wang, F. Li, C. Xu, and A. Plaza, “Superpixel-guided sparse unmixing for remotely sensed hyperspectral imagery,” in *Proc. IEEE Int. Geosci. Remote Sens. Symp.*, July 2019, pp. 2155–2158.

[20] R. A. Borsoi, T. Imbiriba, J. C. M. Bermudez, and C. Richard, “A fast multiscale spatial regularization for sparse hyperspectral unmixing,” *IEEE Geosci. Remote Sens. Lett.*, vol. 16, no. 4, pp. 598–602, 2019.

[21] H. Li, R. Feng, L. Wang, Y. Zhong, and L. Zhang, “Superpixel-based reweighted low-rank and total variation sparse unmixing for hyperspectral remote sensing imaging,” *IEEE Trans. Geosci. Remote Sens.*, pp. 1–19, 2020.

[22] R. Achanta, A. Shaji, K. Smith, A. Lucchi, P. Fua, and S. Süssstrunk, “Slic superpixels compared to state-of-the-art superpixel methods,” *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 34, no. 11, pp. 2274–2282, 2012.

[23] U. Von Luxburg, “A tutorial on spectral clustering,” *Statistics and computing*, vol. 17, no. 4, pp. 395–416, 2007.

[24] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, “Distributed optimization and statistical learning via the alternating direction method of multipliers,” *Found. Trends Mach. Learn.*, vol. 3, no. 1, pp. 1–122, 2011.

[25] R. N. Clark et al., *USGS digital spectral library splib06a*. U.S. Geological Survey Denver, CO, 2007.

[26] ——, “Imaging spectroscopy: Earth and planetary remote sensing with the USGS Tetracorder and expert systems,” *J. Geophys. Res.*, vol. 108, no. E12, p. 5131, Dec. 2003.