Regularization and the BV Formalism

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Abstract

We summarize the application of the field–antifield formalism to the quantization of gauge field theories. After the gauge fixing, the main issues are the regularization and anomalies. We illustrate this for chiral $W_3$ gravity. We discuss also gauge theories with a non-local action, induced by a matter system with an anomaly. As an illustration we use induced WZW models, including a discussion of non–local regularization, the importance of the measure and regularization of multiple loops. In these theories antifields become propagating — one of many roles assigned to them in the BV formalism.

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1. Introduction

During a dinner a few weeks ago, Mario Tonin reminded us that Cardinal Bellarmino was right in the argument with Galileo. Bellarmino agreed that indeed the picture of the planets going around the earth on circles with epicycles could be simplified by putting the sun in the center. But he insisted that there is not more truth in the sun being the center of the universe rather than the earth.

Ten minutes later, Pietro Fré asked what he could expect from the Batalin–Vilkovisky (BV) formalism, if he knows already the BRST framework. We made the comparison with the above situation. One may learn first the BRST formalism for usual Lie algebras. Then one may extend it for the case of structure functions. Learning how to treat open algebras is a further epicycle, and the next one may be the reducible algebras, ... . In this way, one does not need the BV formalism. But the latter puts the sun in the center.

We consider the BV formalism for Lagrangian gauge theories. Our main subject will be, how to treat regularization in this framework, but we will first recall the classical theory, in particular how to obtain, from the classical action, a gauge–fixed action including ghosts. This text is complementary to [1]. A longer text is in preparation [2].

First we will indicate the main concepts: antifields, antibrackets, and the extended classical action. This is sufficient for the classical Lagrangian theory. We will use chiral \( W_3 \) gravity [3] as an example, of which a detailed treatment in the present context can be found in [4]. Then we will show how the gauge fixing procedure in this formalism just amounts to a canonical transformation [5]. The fourth section will contain some generalities about the quantum theory, and explain why we use mainly Pauli–Villars (PV) regularization. Anomalies are the central issue of the fifth section. There we will explain how a cohomological theorem leads to drastic simplifications in the calculation of the general form (i.e. including the antifield dependence) of anomalies. Also the cancellation of some anomalies by background charges will be incorporated. In the final part, we will see that there is another role for the antifields, when considering anomaly–induced actions, as bona fide physical fields that correspond to the additional degree of freedom (because the anomalies do not cancel) of the quantum theory. We will illustrate this in Wess–Zumino–Witten (WZW) models. We will treat multiple loops using higher derivative regularization, and will also take into account the issue of locality [6].

Since our main example is \( W_3 \) gravity in the chiral gauge, let us recapitulate the (well known) elements that we need (see [3]). We will use only the realization in terms of free bosonic fields. The classical fields are \( \phi^i = \{ X^\mu, h, B \} \), and their action is

\[
S^0 = \int d^2 x \left\{ -\frac{1}{2}(\partial X^\mu)(\partial X^\mu) + \frac{1}{4}h(\partial X^\mu)(\partial X^\mu) + \frac{1}{4}Bd_{\mu\nu\rho}(\partial X^\mu)(\partial X^\nu)(\partial X^\rho) \right\}
\]

with \( h \) and \( B \) Lagrange multipliers, imposing two constraints on the classical level.
In order to close the algebra of these constraints, we have to impose
\[ d_{\mu} (\nu \sigma \delta)_{\tau \mu} = \delta (\nu \sigma \delta)_{\tau \mu} . \] (2)

There are two gauge symmetries: the remnant of general coordinate and dilatational symmetries, and the spin–3 symmetry. These imply that the path integral
\[ Z(J_i) = \int D\phi^i \exp \left[ \frac{i}{\hbar} S_0(\phi) + J_i \phi^i \right] \] (3)

(where \( \phi \) denotes \( X, h \) and \( B \)) is not well defined, some propagators are singular. This problem is characteristic for gauge theories, and solved by gauge fixing.

2. Antifields, antibrackets, and the extended action

The BV formalism for a Lagrangian gauge theory starts by introducing a ghost field for every symmetry, and then an antifield for every field. This leads to Table 1. Note that the ghosts are fermionic (statistics opposite to the symmetry), and that the antifields have statistics opposite to that of the fields. The classical fields have ghost number 0, and the ghosts have ghost number 1. The ghost numbers of a field and its antifield always add up to –1. In the space of functionals of fields and antifields, one introduces a Poisson bracket–like structure (called antibrackets), where the antifields take over the role of the canonical momenta. Thus for example,

\[ (X^\mu(x), X^\nu_*(y)) = \delta^\mu_\nu \delta(x - y) = -(X^\nu_*(x), X^\mu(y)) . \] (4)

The new feature is that this antibracket is odd, i.e. the Grassmann parity of \((A, B)\) is opposite to that of the product \(AB\). Equipped with this structure, the BV formalism is analogous to the canonical formalism of classical mechanics in many respects: brackets, canonical transformations, . . . . Note however that we will stay within the Lagrangian framework, thus sticking to the covariant formulation. For the Hamiltonian approach, we refer to the literature.

The extended classical action is a function of fields and antifields: \( S(\Phi^A, \Phi^*_A) \). The part of the extended action which does not contain antifields is the classical action:

\[ S(\Phi^A, \Phi^*_A = 0) = S_0(\phi^i) . \] (5)
In our example it is Eq. (I). This classical action depends only on the classical fields \( \phi^i \), which in our example (with \( \mu \) running over \( D \) values) means that it depends on \( D + 2 \) fields. The problem of singular propagators for the quantum theory can be restated as

\[
\text{rank} \ (\partial_\mu \partial_\nu S^0) = D < D + 2 .
\] (6)

To remedy this problem, we add a term

\[
S^1 = \phi^*_i R^i_a c^a ,
\] (7)

where \( R^i_a \) is the variation of \( \phi \) for the symmetry transformation \( a \). In \( W_3 \) this is

\[
S^1 = X^*_\mu [(\partial X^\mu)c + d_{\mu\nu\rho} (\partial X^\nu)(\partial X^\rho)u] + h^* \left[ \nabla^{(-1)}c + (\partial X^\mu)[(\partial X^\mu)(u\partial B - B\partial u)] \right] + B^* \left[ \nabla^{(-2)}u - 2B\partial c + c\partial B \right] \] (8)

with \( \nabla^{(j)} = \partial - h\partial - j(\partial h) \). Note that for each symmetry this gives an extra contribution 2 to the rank of \( \partial_\alpha \partial_\beta S \). (To be more precise, we look at this matrix at the surface where all fields of non-negative ghost number vanish, and those of ghost number 0 satisfy the equations of motion from \( S^0 \)). In our example this contributes 4, and the total rank is \( D + 4 \), while introducing the 2 ghosts and all the antifields, raised the dimension of this matrix to \( 2(D + 4) \). The rank is thus half the dimension, which turns out to be the maximum possible, due to the master equation, Eq. (II).

If the rank would still be lower, this would signal that the transformations were dependent. This would be remedied by introducing a term \( c^*_a Z^a_{\alpha_1} c^{\alpha_1} \) where \( Z \) stands for the dependence relations between the symmetries and \( c^{\alpha_1} \) would be new ghosts (so-called ghosts for ghosts, having ghost number 2).

The statement that \( S^0 \) is invariant under the transformations defined by \( R^i_a \) is neatly expressed with the help of antibrackets: \( (S^0,S^1) = 0 \). This is a special consequence of imposing the more general relation

\[
(S,S) = 0 \] (9)

which contains, apart form the invariance of the action, also the commutators, non-closure relations, (modified) Jacobi identities and any other relevant data of the symmetry algebra. This relation is called the (classical) master equation, and is a cornerstone of the formalism. The extended action that satisfies this relation will in general have, apart from \( S^0 \) and \( S^1 \), other pieces. Also necessary are the following terms (written symbolically):

\[
S^2 = (-)^{i+a} \frac{1}{2} c^*_a T^{ab} c^b + (-)^{i+a} \frac{1}{2} \phi^*_i \phi^*_j E^{ij}_{\alpha\beta} c^b c^a ,
\] (10)

where \( T \) are structure functions and \( E \) appear in the non-closure relations. For \( W_3 \) for example, the master equation is solved by

\[
S^2 = c^* [(\partial c)c + (1 - \alpha)(\partial X^\mu)(\partial X^\nu)(\partial u)u] + u^* [2(\partial c)u - c(\partial u)] - 2oh^* (-2\partial B\partial^* - 3B^*\partial B + \nabla^2 h^*)(\partial u)u - 2(\alpha + 1) X^*_\mu h^* (\partial u)u(\partial X^\mu) ,
\] (11)

*We omit \( \int d^2 x \), here and in the sequel, for actions and anomaly expressions.
where $\alpha$ is a free parameter. In general one can prove that a solution (of ghost number 0) of the master equation always exists (possibly containing terms with three or more antifields). This was proven in \cite{7, 8} under some regularity conditions. In \cite{2} it will be proven that there is always a \textit{local} solution (local in space-time). The regularity conditions for this theorem have been weakened: previous versions were not applicable to, e.g., $W_2$ (chiral gravity) and chiral $W_3$. The theorem says moreover that the solution of the master equation is unique up to a canonical transformation. In \cite{4} it is shown explicitly that indeed the parameter $\alpha$ in Eq. (11) can be removed by a canonical transformation.

For $W_3$ one finds that $S = S^0 + S^1 + S^2$ is the full solution of the master equation. It satisfies the three requirements: master equation, properness (rank of the matrix of second derivatives is on–shell equal to the number of fields) and classical limit (for vanishing antifields), which were discussed also in section 3.1 of \cite{1}.

In this section we used a basis where the fields have $gh(\Phi) \geq 0$, while the antifields satisfy $gh(\Phi^*) < 0$. Such a basis always exists, and will be called the \textquote{classical basis}. We define the antifield number ($\text{afn}$), as zero for the fields in this basis, and equal to minus the ghost number for the antifields. It is often convenient, as for example in the construction of the extended action above, to work perturbatively in antifield number.

3. Gauge fixing

The extended action has been constructed such that the rank of $\partial_\alpha \partial_\beta S$ is half its dimension, i.e. equal to the number of fields. Now imagine making a canonical transformation between fields and antifields such that in the new basis (to be called the \textit{gauge–fixed} basis) the submatrix $\partial_A \partial_B S$ (derivatives w.r.t. fields only) is invertible (on shell). The path integral with the new fields as integration variables no longer suffers from the gauge problem above: the propagators of the new \textquote{fields} have no zero–modes. This canonical transformation is the gist of the gauge fixing procedure in the BV framework.

In our example the canonical transformation needed is extremely simple: we just have to interchange the names field and antifield for $h$ and $B$ (like changing a momentum into a coordinate and vice versa in the canonical formalism). Then $b \equiv h^*$ and $v \equiv B^*$ are fields and $b^* = -h, v^* = -B$ are antifields. The fields $\Phi^A$ of the gauge–fixed basis are then $\{X^\mu, v = B^*, b = h^*, c, u\}$. For the extended action this implies

$$S = -\frac{1}{2}(\partial X^\mu)(\bar{\partial} X^\mu) + b \nabla (-1)c + v \nabla (-2)u + \text{antifield dependent terms} \quad (12)$$

and in the antifield independent terms one recognizes the gauge fixed action. This has no gauge symmetries left.

It may often be preferable (for example, to keep explicit Lorentz covariance) to introduce extra fields to fix the gauge (\textquote{non–minimal} solutions of the master equation), and to perform the canonical transformation with the help of a generating
function called the gauge fermion $\Psi(\Phi)$:

\begin{equation}
\Phi' = \Phi ; \quad \Phi'_{\Lambda} = \Phi'_{\Lambda} + \frac{\partial}{\partial \Phi_{\Lambda}} \Psi.
\end{equation}

For the examples we consider, this is a detour, which leads back to the same gauge fixed action after eliminating auxiliary fields.

4. Quantum theory

4.1. Measure and counterterms

The quantum theory is not determined by the classical action alone. First, there is the question what is the measure of the path integral, or equivalently one needs a regularization. Secondly, what is the quantum action? The (extended) action $S$ may have terms of order $\hbar$. For now, we will use the following expansion (later we will see that half-integer powers of $\hbar$ may also be needed):

\[ W = S + \hbar M_1 + \hbar^2 M_2 + \ldots, \]

where $M_1, \ldots$ are integrals of local functionals. The extra terms may have infinite coefficients related to the regularization, or may be finite. The ‘counterterms’ $M_i$ are restricted by extra requirements, e.g. rigid symmetries, renormalization conditions, or experimental numbers. Symmetry usually fixes the structure of these terms, leaving only some undetermined constants.

The two questions above are connected. A change in the regularization can be compensated by a change of the $M_i$. E.g. the regularization scheme which will be introduced below depends on a mass matrix, and there is a formula giving the change of $M_i$ which is connected to a change of the mass matrix.

4.2. Regularization procedure

The quantum theory starts from the path integral

\[ Z(J, \Phi^*) = \int D\Phi \exp \left[ \frac{i}{\hbar} S(\Phi, \Phi^*) + J\Phi \right]. \]

For a variety of reasons it is advisable not to put the antifields equal to zero (or, as is usually done, to put $\Phi^* = \partial \Psi / \partial \Phi$, which is the same as putting them to zero after a gauge–fixing canonical transformation). We recall that this antifield dependence was already used, in the form of dependence on sources for the BRST transformations, in \[\text{[10]}\], to study the renormalization properties of ordinary non–abelian gauge theories (where $S$ is linear in $\Phi^*$). For anomalous theories this dependence will describe additional degrees of freedom.

For the regularized theory, the path integral is considered to be the limit when some cutoff $M$ is removed from a well–defined path integral, $Z_R = \lim_{M \to \infty} Z_M$. One might read this $M$ as $1/(n - 4)$ in dimensional regularisation, but it is hard to see how to make sense of $Z_M$ as a path integral based on an action for $n$ not an integer. Another proposal would be a point–splitting or lattice regularization,
but as we try to work in the context of local field theories, we prefer not to use these. We will use a two-step procedure, first modifying propagators with higher derivative terms, and then applying the method of Pauli and Villars (PV). For the present formalism, both steps have the advantage that they can be incorporated in a Lagrangian framework. The higher derivatives serve to regularize all the diagrams apart from the one–loop contributions, while PV should regularize the remaining one–loop divergences. When we only consider the quantum corrections of order $\hbar$ (one–loop diagrams), then PV alone suffices.

4.3. PV action

To determine the PV action, we introduce for every field $\Phi^A$, another field $\chi^A$ as PV–partner, with statistics such that a minus sign is introduced in each loop. For a more detailed description, see section 5.1 of [1]. Each PV-field comes with its own antifield. We will collectively denote the original fields and antifields as $z^\alpha$, the PV partners as $w^\alpha$: $z^\alpha = \{\Phi^A, \Phi^*_A\}$; $w^\alpha = \{\chi^A, \chi^*_A\}$. Then suppose that we have found an action with higher derivatives which regularizes the higher loops. If $S_\Lambda$ is this extended action, (when considering only one loop, one may replace $S_\Lambda$ with $S$), then we propose for the full action

$$S_M = S_\Lambda(z) + w^\alpha \left( \frac{\partial}{\partial z^\alpha} S_\Lambda \frac{\partial}{\partial z^\beta} \right) w^\beta - \frac{1}{2} M^2 \chi^A T_{AB} \chi^B.$$  \hfill (14)

Here, $T_{AB}$ is an matrix that is arbitrary but invertible for the propagating fields. Note that in the mass term only the PV fields, and not the PV antifields appear. On the other hand, $T$ may depend on ordinary fields and antifields. The general prescription given here is certainly not the most general one, but it satisfies some nice properties: to one–loop order it commutes with canonical transformations, satisfies the master equation automatically, and corresponds to the addition (classically) of a cohomologically trivial system when $M \to \infty$.

5. Anomalies

5.1. General theory

Since we keep the antifield dependence, all the usual generating functionals will generically depend on $\Phi^*$ also. The one–particle irreducible functional, defined as usual, is then a functional both of some classical fields $\Phi_{cl}$ and on $\Phi^*$. The Ward identities, as derived by Zinn–Justin [10] for the usual non–abelian gauge theories, find a very natural expression in this framework. The general formal derivation is simply a matter of filling in the definitions and doing a partial integration:

$$(\Gamma, \Gamma) = \Delta \cdot \Gamma ,$$  \hfill (15)

where $\Gamma$ is the effective action, and the antibracket is now with respect to $\{\Phi_{cl}, \Phi^*\}$. The right hand side symbolizes an operator insertion which computes the eventual
anomalies, its path integral expression is
\[(\Delta \cdot \Gamma) = \frac{-2i\bar{\hbar}}{Z} \int D\Phi \ A(\Phi, \Phi^*) \ \exp \frac{i}{\hbar} (W + J\Phi) .\]

Using formal manipulations on the path integral, the anomaly \(A(\Phi, \Phi^*)\) takes the form
\[A = \Delta W + \frac{i}{2\bar{\hbar}} (W, W) ,\]
where
\[\Delta = \frac{\partial}{\partial \Phi_A} \frac{\partial}{\partial \Phi_A^*} .\]

However, for a local action \(\Delta S\) gives an expression proportional to \(\delta(0)\), and is thus ill–defined, indicating where regularization necessarily enters. Continuing anyway formally for a moment, we expand \(A = 0\) in \(\bar{\hbar}\), obtaining the equations
\[(S, S) = 0 ; \quad \Delta S + i(S, M_1) = 0 ; \ldots .\]

Now we consider this in a regularized version, using the PV framework and including only single loops. For the regularized path integral we take
\[Z_R(J, \Phi^*) = \exp -\frac{i}{\hbar} W_c(J, \Phi^*) = \lim_{M \to \infty} \int D\Phi D\chi \ \exp \left[ \frac{i}{\hbar} W_M(z, w) + J\Phi \right] \bigg|_{\chi^* = 0} ,\]
and \(\Gamma\) introduced above is of course the Legendre transform of \(W_c\). The measure we define so that \(\Delta W_M\) vanishes (up to terms quadratic in PV (anti)fields). Indeed, the contributions of the ordinary fields and the PV fields in Eq. (17) cancel (again see section 5.1 of [1] for more details). The anomaly arises from the fact that \((S_M, S_M) \neq 0\) generically, due to the non–invariance of the PV mass term. The integral over the PV fields gives for the terms quadratic in PV fields a propagator, which is of order \(\bar{\hbar}\), and we obtain
\[A = Tr \left[ J \exp(\mathcal{R}/M^2) \right] + i(S, M_1) + \mathcal{O}(\hbar) .\]
where \(J\) is like the jacobian matrix of the BRST transformations and \(\mathcal{R}\) is a regulator constructed from the action. The full expressions are
\[J^A_B = \frac{\tilde{\partial}}{\partial \Phi_A} S \frac{\tilde{\partial}}{\partial \Phi_B} + \frac{1}{2} (T^{-1})^{AC} (T_{CB}, S) (-)^B ; \quad \mathcal{R}^A_B = (T^{-1})^{AC} \left( \frac{\tilde{\partial}}{\partial \Phi^C} S \frac{\tilde{\partial}}{\partial \Phi^B} \right) .\]

Comparing Eq. (19) with Eq. (16), we have effectively obtained a one–loop regularized definition of \(\Delta S\). Its value depends on the matrix \(T\). Different choices of \(T\) give values of \(\Delta S\) which differ by \((S, G)\) for some local function \(G\). In practice, Gilkey’s formulas for the heat kernel [12, 3] are very useful.

Both for the formal expression and for the regularized one, one can prove that
\[(S, \mathcal{A}) = 0 .\]
This is the expression in BV language of the Wess–Zumino consistency conditions. Thus we automatically obtain the ‘consistent’ anomalies. This was to be expected, since these conditions are satisfied for finite values of $M$ also.

5.2. Simplifications using a theorem on antibracket cohomology

The calculations of the one–loop anomalies for chiral $W_3$ has been greatly simplified [4] by using a theorem on the cohomology of the nilpotent operator $SF \equiv (F, S)$. Whereas in [8] this was discussed in general terms, in [4] it has been shown to be applicable also within the set of local functions.

The theorem (see [4], theorem 3.1) states that the cohomology of $\mathcal{S}$ in the set of local functions of fields and antifields is equivalent to the ‘weak’ cohomology of an operator $D^0$ which acts in the set of local functions of fields (or antifields) with non–negative ghost number. Thus it is most useful in the classical basis, where $D^0$ acts on fields only: $D^0 F^0 = (F^0, S)|_{\Phi^* = 0}$. ‘Weak’ cohomology means that field equations of $S^0$ (i.e. the part of the action which depends on fields of ghost number 0) may be used freely.

The theorem implies that it is sufficient to calculate the part of the possible anomaly which depends only on the fields of the classical basis, call it $\mathcal{A}^0$. Given $\mathcal{A}^0$, the WZ consistency condition Eq. (20) implies that $\mathcal{A}$ is then fixed up to terms $(S, M)$, with $M$ the integral of a local function. Such terms are removable anyway by including $M$ as a local counterterm.

This way, the same results are obtained as in [13]. Removing all dependence on the antifields of the classical basis (such as the antighosts $b = h^*$ and $v = B^*$) one finds

$$\mathcal{A}^0 \approx -\frac{1}{24\pi} (c\delta^\mu_\nu + 2ud_{\mu\nu\rho}(\partial X^\rho)) \partial^3 (b^* \delta_{\mu\nu} + 2d_{\mu\nu\rho}v^* \partial X^\rho)$$

$$+ \frac{100}{24\pi} c \partial^3 v^* + \frac{\kappa}{2\pi} (u\partial v^* - v^* \partial u)(\partial^3 X^\mu)(\partial X^\nu),$$

where $\approx$ means ‘up to field equations’(the complete result is in [4]). We call attention to the fact that the anomaly depends on $b^*$ and $v^*$, which are antifields in the gauge–fixed basis. In other treatments one has to include background fields in the gauge fixing to see the anomalies (for example, in the gauge $h = 0$ the anomaly would not show up). In the BV formalism, this task is taken over by the antifields, without such extra ingredients.

5.3. Background charges

It is well known that the anomalies of the chiral $W_3$ model of Eq. (1) can be cancelled by introducing background charges. In our framework these correspond to terms with $\sqrt{h}$ in the quantum action, which takes the form

$$W = S + \sqrt{h} M_{1/2} + h M_1 + \ldots .$$

(21)

Then the master equation $\mathcal{A} = 0$ expands as

$$(S, S) = (S, M_{1/2}) = 0 ; \quad (S, M_1) = i\Delta S - \frac{1}{2}(M_{1/2}, M_{1/2}) .$$

(22)
In view of the cohomology theorem, $M_{1/2}$ is fixed when the terms independent of the antifields (of the classical basis) are given. These are

$$M_{1/2}^{0} = a_{\mu} h(\partial^{2} X^{\mu}) + e_{\mu\nu} B(\partial X^{\mu})(\partial X^{\nu}) ,$$

where $a_{\mu}$ and $e_{\mu\nu}$ are constants (background charges). The theorem says that $(S,M_{1/2}) = 0$ has a solution for $M_{1/2}$ if and only if $D^{0}M_{1/2} \approx 0$. This gives some conditions on the background charges \[14\]. Taking into account Eq. (2), there is a unique solution for $e_{\mu\nu}$ for each value of $D$ (the range $\mu$), and $a_{\mu}$ is still arbitrary. The latter is also fixed by requiring that Eq. (24) has a solution for $M_{1}$, so that we have an anomaly–free action of the form Eq. (21). The simplification brought about by the theorem is that the analysis could be performed in the restricted space of fields of non–negative ghost number.

We have used PV regularization for one loop, to be combined with higher derivatives for more loops as in the next section. The reason is that these regularizations are possible at the level of the Lagrangian, thus leaving the BV–setup intact. If the regularization does not break symmetries, then there are no anomalies. In the usual cases the higher derivatives can be taken to be covariant, so that anomalies only come from the PV mass term, and are genuine one–loop anomalies. For $W_{3}$ it is not known how to define fully covariant derivatives (with a finite number of terms) and this clarifies the existence of anomalies beyond 1 loop in these theories.

6. Induced theories

Now we will consider theories where the anomalies are not cancelled, i.e. $(\Gamma, \Gamma) \neq 0$, implying that $\Gamma(\Phi_{cl}, \Phi^{*})$ describes new quantum degrees of freedom in addition to the classical ones. They manifest themselves as propagating antifields. For conformal matter coupled to 2–d gravity, there is generically an anomaly proportional to $A = c \partial^{3} h$. The $h$–dependence of effective action obeying Eq. (15) is proportional to the non–local expression:

$$\Gamma \sim \partial^{2} h \frac{1}{\partial f - \partial h \partial} \partial^{2} h .$$

It was noticed by Polyakov that this is local in $f$, where $h = \frac{\partial f}{\partial \phi}$. This fact can be understood \[15\] by first expressing the original action in $f$ and performing the regularization using as a mass term $M^{2} X^{\mu} X^{\nu}(\partial f)$. This mass term is invariant under $\delta X = \epsilon \partial X, \delta f = \epsilon \partial f$, there is no anomaly and zero induced action for $h$. However, we really wanted a regularisation that is local in $h$, not just in $f$, which can be done using the mass term $M^{2} X^{\mu} X^{\nu}$. This is just a change of regularization, and, as mentioned before, this amounts to changing the theory with a local counterterm, local in $f$ that is. Actually, there is another integral left, over a parameter interpolating between the regularizations, but this is trivial here. Therefore, the final induced action is local in $f$.

A similar reasoning can be given for the WZW model, to which we now turn. This will again illustrate some aspects of the quantization procedure, like
the importance of the definition of the measure, non–local regularization including also higher loops\textsuperscript{[6]}. This model was also discussed, from a different point of view, at this conference by P. van Nieuwenhuizen \textsuperscript{[16]}. The WZW action arises as an induced action from

\[ S = \psi^i (\bar{\partial} - A) \psi - \psi^* c \psi + A^* \bar{D}(A) c - c^* cc \ , \]  

with \( \bar{D}(A)c \equiv \bar{\partial}c - [A, c] \), where the fields \( A \) and \( c \) are in the adjoint representation of a Lie–algebra, and \( \psi \) is in an arbitrary representation. The gauge fixing of this model can be done by considering \( A^* \) as a field (antighost), and \( A \) is then an antifield. Integrating out the fermions, the anomaly implies that the resulting action is a non–local action for the ‘antifield’ \( A \):

\[ \Gamma[A] = \frac{k}{2\pi x} \left\{ \frac{1}{2} A \frac{\partial}{\partial A} - \frac{1}{3} \frac{1}{A} \frac{\partial}{\partial A} [A, \bar{\partial} A] + \ldots \right\} \]  

(24)

Again it is possible to compare different regularizations \textsuperscript{[15]} to arrive at the conclusion that this action is (almost) local when written in terms of a group element \( g \). Almost, since this time the integral over the interpolating parameter remains (as the third space coordinate below):

\[ \Gamma[A = \bar{\partial} gg^{-1}] = k \Gamma^0[A] = -k \ S^+(g) \ , \]  

(25)

with

\[ S^+(g) = \frac{1}{4\pi x} \left( \int d^2 x \ (\partial g^{-1} \bar{\partial} g) + \frac{1}{3} \int d^3 x \ \epsilon^{\alpha\beta\gamma} \ (g_{,\alpha} g^{-1} g_{,\beta} g^{-1} g_{,\gamma} g^{-1}) \right) \ . \]  

(26)

To investigate the antifield degree of freedom, one takes the action Eq. (24) as a starting point and quantizes. The generating functional of interest is then

\[ Z[u] = e^{-W[u]} = \int \mathcal{D}A \ e^{-\Gamma[A]} + \frac{1}{2\pi x} \{ u A \} \]  

(27)

It can be argued in different ways (\textsuperscript{[17, 18]}, see \textsuperscript{[6]} for a review) that it enjoys the remarkable renormalization property

\[ W[u] = Z_W W^0[Z_u u] \ , \]  

(28)

where the Legendre transform of \( \Gamma^0[A] \) defines \( W^0[u_0] \):

\[ W^0[u_0] = \min_{\{ A \}} \left( \Gamma^0[A] - \frac{1}{2\pi x} \{ u A \} \right) \ . \]  

(29)

Different formal arguments lead to \( Z_W = k + 2\tilde{h} \), but the values obtained for \( Z_u \) vary. We will discuss now a regularized argument, which makes use of the antifields in the regularization. To start \textsuperscript{[18]}, consider the non-chiral induced action depending on the gauge fields \( A \) and \( \bar{A} \):

\[ \Gamma^v[A, \bar{A}] = \Gamma[A] + \bar{\Gamma}[\bar{A}] - \frac{k}{2\pi x} \{ A \bar{A} \} \ . \]  

(30)
It can be viewed as arising from two chirally conjugated matter systems like Eq. (23). If one separately regulates both chiral parts (a separate mass term for each chiral fermion) then one obtains of course just the first two terms in Eq. (30). Regulating in a vector invariant way (a vector invariant mass term for the fermions), the result differs from the previous one by a finite local counterterm, which is the third term in Eq. (30). This term can be computed explicitly, example 4.1 in [9] does it in 4 dimensions. To quantize this action, one has to take the vector invariance into account, with corresponding ghost $c$, and furthermore the antifields $A^\ast$, $\bar{A}^\ast$, and $c^\ast$. The gauge fixing is done by considering $\bar{A}$ as an antifield, and $A^\ast$ as a field. We thus consider the path integral

$$Z_{nc}(\bar{A}) \equiv \int DAD\bar{A}^\ast Dc e^{\Gamma_v(A, \bar{A}) + \bar{A}^\ast D(\bar{A})c}.$$ (31)

For our present purpose it is sufficient to take this path integral at $A^\ast = c^\ast = 0$, and not to introduce sources for the fields $A$, $A^\ast$ and $c$. The vector gauge invariance allows us to fix $A$ to an arbitrary value, and if there is no anomaly then $Z_{nc}$ will not depend on it. Thus by investigating the vector anomaly we obtain the dependence on $\bar{A}$ (an antifield from the present point of view). For a one–loop calculation, we introduce PV fields with the mass terms [3]

$$M^2 \frac{1}{\Lambda^2} \partial^2 \bar{A} + M \bar{A}^\ast c,$$

where the underscore denotes PV–partners. These mass terms are not invariant, but the anomalies cancel. In [3] also more loops were considered. In the spirit of the method of higher derivative regularization, the added term was taken to be the vector–covariant

$$\frac{1}{\Lambda^2} (\partial A - \bar{\partial} \bar{A} + [A, \bar{A}])^2.$$ (32)

Note the $\bar{A}$–dependence, which from the present point of view is really an antifield–dependence. This term regularizes all but the one–loop diagrams. Then one could add the PV sector as in Eq. (34), but it is shown in [3] that a simpler action regularizes all the diagrams. It turns out that also for multiple loops there is no anomaly, and thus Eq. (31) is independent of $\bar{A}$. One can therefore normalize $Z_{nc}(\bar{A}) = 1$. We can now compare Eq. (31) with Eq. (27). Identifying $\bar{A}$ with the source $u$,

$$W[u] = (k + 2\tilde{h})W^0 \left[ \frac{1}{k} u \right].$$ (33)

The above amounts to a specific method to fix what we mean by the functional integral measure $DA$. The result for the field renormalization factor is valid for this specific regularization, but not necessarily for others. To develop some understanding for other approaches, one may relate this measure to an integration measure over group variables by the usual formula:

$$DA = Dg e^{2\tilde{h} S^+[g]}.$$ (34)
Checking the consequences for this $\mathcal{D}g$, one finds that it is not Haar invariant:

$$[\mathcal{D}(hg)(hg)^{-1}] \neq [\mathcal{D}g \ g^{-1}] .$$  \hspace{1cm} (35)

It differs from the Haar invariant measure only by a counterterm that is local in $g$ however. If one uses that Haar invariant measure, then for the new path integrals one finds

$$W_{\text{Haar}}(u) = (k + 2\tilde{h})W^0\left(\frac{1}{k + 2\tilde{h}}u\right),$$

differing from Eq. (33) in the field renormalization factor. This shows once more the importance of the definition of the measure.

7. Conclusions

We have seen that the BV formalism unifies many aspects of the quantization in one formalism. It is not just that it offers the possibility to treat all sorts of gauge theories, but also it phrases Ward identities, WZ consistency conditions, induced actions, ... in a very natural language. Another indication that the sun is in the center is the multiple role played by the antifields, viz. first to generate the (Faddeev–Popov) antighosts, then also as sources for BRST transformations, as background fields in general gauges, and finally to describe propagating quantum degrees of freedom appearing in induced actions arising from antifield–dependent effective actions.

8. Acknowledgments

We thank the organizers of the conference for providing a stimulating atmosphere. Also discussions with Joaquim Gomis, Alexander Sevrin, Ruud Siebelink and Stefan Vandoren are gratefully acknowledged.

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