The model analytical finite rod-type element for static and dynamic analysis

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Abstract. The paper presents a mathematical model for the calculation of rod structures based on analytical solutions of static and dynamic equations of state for rod or beam elements. Analytical solutions of the equations of state are used in the work and the matrix of influence is constructed from them. The model underlying the calculation allows the use of a real diagram of stretching and compression of structural materials. For a mathematical model, analytical solutions are formed in matrix form. This approach allows us to use the known algorithms of the finite-element and super-elements method for numerical studies and to obtain the result with high accuracy.

1. Introduction
For the optimal calculation of the construction, you must create the appropriate model that can be analyzed using a computer. These models are, as a rule, formalized mathematical descriptions, reflecting with the required accuracy the processes occurring in the object under study [1]-[5]. It should be attributed the normalized impacts, of different the static and dynamic loads, as well as the need to take into account the physical condition of the investigated object.

In the proposed article when building a mathematical model for the calculation of one element, the principle that is the basis of the super-element method (SEM) is used – the super-element has no internal nodes [6]-[11]. We emphasize that the most important values for the SE model are unknown functions (motion and temperature) in the boundary nodes. They completely determine the state of everything the element, in this case – flexure, stress, deformation, temperature, and thus allow us to judge the strength of the structural element under external influences. As a result of sufficient to obtain a relationship between the condition in the object of study and the state in the boundary nodes in the form of analytical expression. So are formed of the analytical finite element (AFE) for a rod or beam that does not contain internal nodes and is built on analytical solutions. The adduced formulas are used to create static and dynamic models of the rods, which are the physical model of the building structural elements. On their basis, it is proposed to form matrix expressions and perform calculation on the structure of the finite element method (FEM) [12], [13]. The results of test calculations are presented [14].

2. Finding of basic equations
The initial state of the element – are straight uniform bars regardless of cross-section properties. The deformation of the rod is considered to be small depending on the operating conditions. Then
Bernoulli hypotheses can be used to model the rod [15]. It should be noted that not all the conditions of deformation the flat hypothesis sections are fair [16].

If we identify the angles of rotation section about the axes y and z from the transverse displacement along the longitudinal coordinate, then the hypothesis of plane sections provides linear distribution of longitudinal displacement on the cross-sectional area:

\[ u(x, y, z) = u(x) - y \frac{dv}{dx}(x) - z \frac{dw}{dx}(x) = u(x) - y \theta_y(x) - z \theta_z(x) \]  

(1)

Here, \( u, v, w \) are the components of the displacement vector in the coordinate system \((x, y, z)\).

The latter assumption limits the applicability of this theory with deflections, exceeding a half of the cross-sectional size, which corresponds to the conditions of building structures normal use.

Longitudinal force.

\[ \sigma = \frac{N}{A} \]

(2)

Torque

\[ M_y = \int_{A} \tau \sqrt{y^2 + z^2} \, dA = GJ_y \frac{\partial \theta_y}{\partial x} \]

(3)

Bending moments

\[ M_z = -\int_{A} \sigma y dA = -EJ_z \frac{\partial^2 v}{\partial x^2} + EJ_y \alpha \int_{A} y \cdot \Delta T dA = -EJ_z \frac{\partial \theta_y}{\partial x} + m_{z,T} \]

\[ M_y = -\int_{A} \sigma z dA = -EJ_y \frac{\partial^2 w}{\partial x^2} + EJ_z \alpha \int_{A} z \cdot \Delta T dA = -EJ_y \frac{\partial \theta_z}{\partial x} + m_{y,T} \]

(4)

Temperature loads \( q_{x,T}, m_{y,T}, m_{z,T} \) at a given temperature distribution over the cross section are also known functions of the longitudinal coordinate \( x \) (and possibly time).

In the equations of motion, takes the solidification hypothesis, i.e. ignore the changes in the shape and size of the rod elementary section. The inertia of the cross-section rotation about the axes \( y \) and \( z \) of ignore. Then the differential equations of motion take the form:

\[ \frac{\partial N}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2} - q_x; \quad \frac{\partial M_z}{\partial x} = \rho J_z \frac{\partial^2 \theta_z}{\partial t^2} - m_z; \quad \frac{\partial M_y}{\partial x} = Q_y - m_y \]

\[ \frac{\partial M_z}{\partial x} = -Q_z - m_z; \quad \frac{\partial Q_y}{\partial x} = \rho A \frac{\partial^2 v}{\partial t^2} - q_y; \quad \frac{\partial Q_z}{\partial x} = \rho A \frac{\partial^2 w}{\partial t^2} - q_z \]

(5)

Where \( t \) is time, \( Q_x, Q_y, Q_z \) are lateral forces, \( q_x, q_y, q_z, m_z, m_y, m_z \) are components of the forces and moments distributed along the length, those we calculate with the known functions of \( x \) coordinate and time. Of course, \( q_x, m_z, m_z \) also include and thermal components.

All expressions of this subparagraph are recorded in the local coordinate system of \( x, y, z \).

Equations of state we obtain by combining (1) – (5) into a single system:
To this system it is necessary to add boundary and initial conditions. The initial conditions must be specified by the displacements and their velocities:

\[ \begin{aligned}
&u(x,0) = u_1(x); \quad v(x,0) = v_1(x); \quad w(x,0) = w_1(x) \\
&\theta_x(x,0) = \theta_{1x}(x); \quad \theta_y(x,0) = \theta_{1y}(x); \quad \theta_z(x,0) = \theta_{1z}(x) \\
&\frac{\partial u}{\partial t}(x,0) = u_2(x); \quad \frac{\partial v}{\partial t}(x,0) = v_2(x); \quad \frac{\partial w}{\partial t}(x,0) = w_2(x) \\
&\frac{\partial \theta_x}{\partial t}(x,0) = \theta_{2x}(x); \quad \frac{\partial \theta_y}{\partial t}(x,0) = \theta_{2y}(x); \quad \frac{\partial \theta_z}{\partial t}(x,0) = \theta_{2z}(x)
\end{aligned} \]  

(7)

The boundary conditions are set on both ends of the rod:

**Kinematic conditions:**

\[ \begin{aligned}
&u(0,t) = u_5(t); \quad v(0,t) = v_5(t); \quad w(0,t) = w_5(t) \\
&\theta_x(0,t) = \theta_{5x}(t); \quad \theta_y(0,t) = \theta_{5y}(t); \quad \theta_z(0,t) = \theta_{5z}(t) \\
&u(L,t) = u_4(t); \quad v(L,t) = v_4(t); \quad w(L,t) = w_4(t) \\
&\theta_x(L,t) = \theta_{4x}(t); \quad \theta_y(L,t) = \theta_{4y}(t); \quad \theta_z(L,t) = \theta_{4z}(t)
\end{aligned} \]  

(8)

**Power conditions:**

\[ \begin{aligned}
&N(0,t) = N_5(t); \quad Q_x(0,t) = Q_{5x}(t); \quad Q_y(0,t) = Q_{5y}(t) \\
&M_x(0,t) = M_{5x}(t); \quad M_y(0,t) = M_{5y}(t); \quad M_z(0,t) = M_{5z}(t) \\
&M_x(L,t) = M_{4x}(t); \quad M_y(L,t) = M_{4y}(t); \quad M_z(L,t) = M_{4z}(t)
\end{aligned} \]  

(9)

The either end the mixed conditions can be set. In determining the mixed conditions at one end cannot be set simultaneously kinematic and force values that form a pair of "generalized force" – "generalized displacement".

The system (6) is linear in the made assumptions. She is convenient to rewrite it in a matrix form to execute a programmed calculation. Introduce the state vector that combines kinematic and force factors:

\[ \mathbf{y} = \{u \quad v \quad w \quad \theta_x \quad \theta_y \quad \theta_z \quad M_x \quad M_y \quad M_z \quad N \quad Q_x \quad Q_y \quad Q_z\}(x,t) \]  

(10)

Then system (6) can be rewritten as:

\[ \frac{\partial \mathbf{y}(x,t)}{\partial x} = \mathbf{A}\mathbf{y}(x,t) + M \frac{\partial^2 \mathbf{y}(x,t)}{\partial t^2} - \mathbf{q}(x,t) \]  

(11)

The matrix \( \mathbf{A} \) captures the essence of geometrical and static hypotheses adopted in the derivation.
of the basic equations and the matrix $M$ – hypothesis adopted in determining the inertial forces.

Vector of the external distributed loads is:

$$\mathbf{q}(x,t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & m_x & m_y & m_z & q_x & q_y & q_z \end{bmatrix}^T(x,t)$$  \hspace{1cm} (12)

The boundary and initial conditions are similarly converted.

For a static task it is enough in (11) to take that the matrix $M$ is equal to zero.

3. The equations of state in statics

The system of equations of state (11) is linear inhomogeneous. For static tasks (for example, the effect of its own weight and other static load), system contains no time derivatives. Then we obtain an inhomogeneous system of ordinary differential equations from the system of equations of mathematical physics of hyperbolic type [17]. She links the instantaneous values of the load on the rod with the instantaneous state – displacements and internal forces:

$$\frac{\partial^2 \mathbf{y}(x)}{\partial x^2} = A \mathbf{y}(x) - \mathbf{q}(x)$$  \hspace{1cm} (13)

The solution of such a system should consist of the fundamental part – the solution of the accompanying (13) homogeneous system and the particular solution corresponding to the right part – the vector $\mathbf{q}(x)$. To determine the fundamental solutions, we use the Laplace transform [18]-[20] on the coordinate $x$. Denote the parameter of Laplace transformation $p$. Then (13) takes the form:

$$(pI - A)\mathbf{y}^\prime(p) = \mathbf{y}(0) - \mathbf{q}'(p)$$  \hspace{1cm} (14)

Here, $I$ – the identity matrix, $\mathbf{y}(0)$ the state vector at the origin, the symbol ($'$) denotes the image by Laplace. Then we obtain the analytical expression for the vector of state:

$$\mathbf{y}'(p) = (pI - A)^{-1}\mathbf{y}(0) - (pI - A)^{-1}\mathbf{q}'(p)$$  \hspace{1cm} (15)

We introduce the designation

$$V'(p) = (pI - A)^{-1}$$  \hspace{1cm} (16)

and we call the original (16) the matrix of influence of initial parameters or simply the matrix of influence. The matrix of influence is a property of the rod that does not depend either on external loads or on the conditions of fastening the rod. Then from (15) we obtain, using the convolution theorem [21], the original solution-the state vector at any point of the rod:

$$\mathbf{y}(x) = V(x)\mathbf{y}(0) - \int_0^x V(x-z)\mathbf{q}(z)dz = V(x)\mathbf{y}(0) - F_y(x)$$  \hspace{1cm} (17)

It is obvious from the solution (17) that the matrix of influence has an indispensable property:

$$V(0) = I$$  \hspace{1cm} (18)

and, thus, it is a normalized matrix of fundamental solutions. The second term of the formula (17) is a particular solution corresponding to the given right part; let's call it the vector of influence of the distributed load. It is clear that this vector should be calculated for each specific distribution of linear loads again. The whole solution (17) is a solution of the Cauchy problem in the form of the Duhamel integral [21]. Analytical expressions for the matrix of influence in static problems are obtained elementary (example [11]).

4. The equations of state in dynamics

The problem of free oscillations, which provides a functional basis for the decomposition of
inhomogeneous task into a generalized Fourier series, is defining for solving the task of rod dynamics [22], [23]. The equation of state will be linear homogeneous, in this case, contains the first derivatives of the coordinates and the second derivative by time. The material of the rod should be linearly-elastic. Then we can assume that the free oscillations are made according to the harmonic law:

\[ y(x,t) = Y(x)e^{i\omega t} \]  \hspace{1cm} (19)

Here, \( Y(x) \) – forms of free oscillations, \( \omega \) – the frequency of free oscillations, \( i \) – imaginary unit.

Then the equation for the forms of free oscillations will take the form of a linear homogeneous equation and contains a free parameter \( \omega \):

\[ Y' = \left( A - \omega^2 M \right) Y \]  \hspace{1cm} (20)

This equation has only the fundamental solution, which can also be found using Laplace transform [24]

\[ \left( pI - A + \omega^2 M \right) Y'(p) = Y(0) \]  \hspace{1cm} (21)

The result:

\[ Y'(p) = V^*(p)Y(0); \quad V'(p) = \left( pI - A + \omega^2 M \right)^{-1} \]  \hspace{1cm} (22)

The original matrix \( V^*(p) \) is again called the matrix of influence. Finally, we present the solution in the form of:

\[ Y(x,\omega) = V(x,\omega)Y(0,\omega) \]  \hspace{1cm} (23)

We note that in contrast to (17), the solution depends on an arbitrary parameter \( \omega \) - the frequency of free oscillations.

To determine this parameter, to find the eigenfunctions of the linear operator (20). Recall that the six initial parameters are known from the conditions at the beginning of the rod, at \( x=0 \). To determine the remaining six parameters, writing (23) for \( x=L \), we obtain a system of homogeneous linear equations of the sixth order. She has a nontrivial solution in the case when its main determinant is zero. The latter condition gives a transcendental frequency equation for determining the parameter \( \omega \).

The matrix of influence in a dynamic task is different from the static matrix of influence. Application of the Laplace transform leads to a completed matrix of influence, moreover, in contrast to the statics of its coefficients are defined by functions of Krylov A. [25] [26].

The dynamic matrix of influence has the same property (18) as the static one; she meaning is the same as before – it is a normalized matrix of fundamental solutions.

5. Analytical model finite element of the rod

We will obtain expressions for analytical finite rod-type element. To do this it is necessary to express the state at any point from through the displacement of the boundary nodes. Kinematic and force factors form a state vector:

\[ y_c = \{u \, v \, w \, \theta_x \, \theta_y \, \theta_z \} \]
\[ y_f = \{M_x \, M_y \, M_z \, \theta_x \, \theta_y \, N \, Q \, Q_z \} \]
\[ y = \{y_c \, y_f \} \]  \hspace{1cm} (24)

Next, we write the matrix of influence and the influence vector:
Whereupon (17) can be written in the following form:

\[ y_c(x) = V_{cc}(x)y_c(0) + V_{cf}(x)y_f(0) - F_{qc}(x) \]
\[ y_f(x) = V_{fc}(x)y_c(0) + V_{ff}(x)y_f(0) - F_{qf}(x) \]  

Then we obtain:

\[
\begin{align*}
  y(x) &= K_{Nod}(x)U + F_{Nod}(x) \\
  K_{Nod}(x) &= \begin{bmatrix}
  V_{cc}(x) - V_{cf}(x)V_{cf}(L)^{-1}V_{cc}(L) & V_{cf}(x)V_{cf}(L)^{-1} \\
  V_{fc}(x) - V_{ff}(x)V_{ff}(L)^{-1}V_{fc}(L) & V_{ff}(x)V_{ff}(L)^{-1}
  \end{bmatrix} \\
  F_{Nod}(x) &= \begin{bmatrix}
  V_{cf}(x)V_{cf}(L)^{-1}F_{qc}(L) - F_{qc}(x) \\
  V_{ff}(x)V_{ff}(L)^{-1}F_{qf}(L) - F_{qf}(x)
  \end{bmatrix} \\
  U &= \{y_c(0) \quad y_c(L)\}
\end{align*}
\]

As a result, we obtain a state at any point of the rod through the displacement of the boundary nodes \( U \).

Further expression for a vector of nodal forces is given (for boundary clusters):

\[
\begin{align*}
  F &= KU + R \\
  K &= \begin{bmatrix}
  -V_{cf}(L)^{-1}V_{cc}(L) & V_{cf}(L)^{-1} \\
  V_{fc}(L) - V_{ff}(L)V_{ff}(L)^{-1}V_{fc}(L) & V_{ff}(L)V_{ff}(L)^{-1}
  \end{bmatrix} \\
  R &= \begin{bmatrix}
  V_{cf}(L)^{-1}F_{qc}(L) \\
  V_{ff}(L)V_{ff}(L)^{-1}F_{qf}(L) - F_{qf}(L)
  \end{bmatrix}
\end{align*}
\]

Thus, analytical expressions for the rod model in the form of an analytical finite element expressing the state at any point through the displacements of the boundary nodes (as in the SEM) are obtained for the static problems. The form of expression (28) does not differ from the same expression in the classical formulation of the FEM [27]. The physical meaning of the matrix \( K \) in this expression is the stiffness matrix, as it allows to determine the nodal forces through nodal displacements. Vector \( R \) has the physical meaning of nodal forces, equivalent to uniformly distributed load per unit length.

We will note that ratios (17) and (23) differ from each other only in designation of state vector and absence in (23) constant terms, passing to a ratio’s formulation for an analytical finite element in dynamics. Therefore, all reasoning (24)-(28) can be repeated also in dynamics; formulas (36) for calculation of a stiffness matrix remain. But then we come to expression for a solve analytical finite element:

\[
F(\omega) = K_d(\omega)U(\omega)
\]  

in which the internal forces acting on the nodes of the element are expressed through the nodal displacement and the free parameter \( \omega \) (frequency of free oscillations). We will further call the matrix \( K_d(\omega) \) a dynamic stiffness matrix. Note that the form of expression (28) and (29) coincide, although they describe qualitatively different processes. But this similarity allows using the same algorithm to form the stiffness matrix of the finite element ensemble – the model of the building frame.

Naturally, for the use of (23) it is necessary to solve a finite set of tasks for analytical finite element in accordance with the number of eigenfrequencies held in the decomposition (23).
6. Conclusion
In this paper propose a new approach to solving static and dynamic problems for spatial rod systems. State equations based on analytical expressions are used to determine the state of an element as a result moving its nodes. The formed matrices are used for final calculations as in the finite element method (FEM). The considered method differs from the FEM in that the stiffness matrix of the element is composed of nodal forces. Solutions of the equations of motion for the rod are based on rigorous analytical expressions. The proposed method is the basis of the algorithm of the computer program for the calculation of rod systems with high accuracy of the result and for developing interactive graphics applications [28].

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