On-line estimation of the Caputo fractional
derivatives with application to $PI^\mu D^\nu$ control *

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Abstract: This paper proposes new procedures for calculation of the Caputo derivative
of model-free measured signals. The evaluation of the non-integer derivative is realized by
integrating a set of ordinary differential equations and convolution. The derivative of order
$\nu$ ($0 < \nu < 2$) is seen as an output of a linear-time-varying system driven by a time-
dependent known signal. Two procedures are proposed depending on the variation range of
the non-integer differentiation order. The proposed formulations facilitate the estimation of
the fractional derivatives when they are associated to dynamical systems represented by integer-
order differential equations. The efficiency of the developed numerical procedures are validated
and compared to exact fractional derivatives for different values of $\nu$. It is shown that $PI^\mu D^\nu$
controllers can be easily realized by system augmentation and convolution. The advantages, the
straightforwardness and the main features of the proposed design are given.

Keywords: Fractional-order derivatives; Estimation; Time-varying Systems; System Theory,
$PI^\mu D^\nu$ control.

1. INTRODUCTION

Fractional calculus is recognized as a branch of mathematical
analysis that exposes various possibilities of defining
the powers of the differentiation operator. The generalization
of the derivative notion consists of letting the power of
the derivative take a real or a complex value. Extensive
results and definitions were exposed in Zwillinger (2014) and the reader can find a good introduction
to the subject in Annaby and Mansour (2012), Oustaloup
(2014), Oustaloup (1995), Sabatier et al. (2007), Samko
et al. (1987). The author is also referred to the references
Farges et al. (2010), Trigeassou et al. (2011), Sabatier et al.
(2007), Sabatier et al. (2015), Li et al. (2010), Ibrir (2017),
Ibrir and Bettayeb (2015) for some results on control of
fractional-order linear systems of commensurate type.

The $\nu$-derivative of a given signal has a local property
when the derivative of order $\nu$ is a positive integer number.
Actually, when the order $\nu$ takes non-integer values, the
$\nu$-derivative of a signal becomes dependent on the signal
past values; which means that, the non-integer derivative
has a memory or it has simply a non-local property. In
fact, many real systems are better modeled with fractional
differential equations like systems with long-term “mem-
ory” and systems exhibiting chaotic behaviors. Referring
to the literature, it was found that fractional-order systems
are suitable for characterization of the anomalous behavior
of dynamical systems and more representative of complex
dynamics that are slower or faster than exponential func-
tions. In those cases, the solutions are best represented
by Mittag-Leffler functions. Moreover, the utilization of
fractional-order derivatives and integrals in closed-loop
control systems has shown outstanding performances, see
e.g., Luo and Chen (2009), Sabatier et al. (2015), Dumont
(2012).

There are numerous attempts to realise and implement
fractional-order differentiators. The design of limited-
bandwidth fractional-order differentiators has been inves-
tigated in Serrier et al. (2007). In Liu et al. (2017) and
Wei et al. (2019), the authors propose fractional-order
differentiator to estimate the Riemann–Liouville fractional
derivatives of the system output in discrete noisy environ-
ment. Fractional-order differentiation through polynomial
integration has been proposed in Liu et al. (2015). In Tolba
et al. (2019) the authors present an implementation of the
Gr"{u}nwald-Letnikov differentiator on Field-Programmable
Gate Arrays (FPGAs). Roughly speaking, the methods
to evaluate fractional-order signals are basically classified
into three main groups. The first group gathers computa-
tional methods based on the analytic definitions of the
different types of fractional derivatives. These methods
necessitate the well knowledge of the explicit forms of the
signal and its derivatives which are often difficult to obtain
except for some types of signals that are of polynomial
shape, see for example Samadi et al. (2004). In the second
group, we find all types of fractional-derivative approxi-
mations by rational functions or transfer functions de-
scribing frequency-domain realizations of the operator $D^\nu$
in continuous time, see e.g., Oustaloup (1983), Oustaloup
(1995), Oustaloup (2014). Numerical methods based on
approximation of the fractional-derivative operators by
discrete transfer functions constitute the third group of

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methods; however, these techniques are essentially based upon finite series approximation which makes the derivative very sensible to the signal frequencies. Moreover, if the series has a large number of coefficients, more processing time is required to implement the procedures in real time.

In this paper, new computational procedures for estimation of the Caputo fractional-order derivatives are proposed. The proposed methods consist of transforming the fractional order operator into the solution of a set of ordinary differential equations followed by a convolution operation. The overall design is seen as an output of a Linear-Time-Varying (LTV) system that can be easily integrated/solved in real time. The speed of convergence of to the exact fractional derivative is controlled by a positive design parameter. The new proposed methods are essentially based upon finite series approximation which makes the derivative very sensible to the signal frequencies. Moreover, if the series has a large number of coefficients, more processing time is required to implement the procedures in real time.

For all $t \geq 0$, the solution of (7) is explicitly given by:

$$I_\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \quad t > 0$$

where $\mu \in \mathbb{R}_{>0}$ stands for the fractional-integration order. In fact, the evaluation of the integral necessitates the knowledge of the function $f(t)$ which is not always available as an explicit function of time. Additionally, even the function $f(t)$ is known, the convolution term that involves the fractional derivatives (i.e., the Caputo and the Riemann Liouville) are not always easy to find except for some functions like polynomial functions. When the function $f(t)$ is given numerically as a set of values in $\mathbb{R}$, the formulas (1) and (3) shall be discretized to have an estimate of the fractional derivatives. The objective of this section is to provide the Caputo derivatives through the solution of an ordinary differential equations when the function or the signal $f(t) \in \mathbb{R}$ is measured in continuous or discrete manners. More importantly, it is assumed that the model that generates $f(t)$ is not necessarily known.

2.2 The first procedure: case $0 < \nu < 1$

In this part of the paper, the numerical procedure is limited to the calculation of the Caputo fractional derivatives in the particular case where $0 < \nu < 1$. Notice that the Caputo derivative is exactly the convolution of the signal $g(t) = t^{\nu-1} \frac{d^{\nu}}{dt^{\nu}}$ and the first derivative of $f(t)$; $\nu \in \mathbb{R}_{>0}$.

Having the exact first derivative of $f(t)$ for all $t > 0$ renders the evaluation of the Caputo fractional derivative a simple convolution operation of two time-dependent functions. Based on this fact, and using some preliminary results given in Ibir (2003), the procedure is divided into two steps: the first step consists of evaluating the exact time derivative of $f(t)$ and in the second step, the convolution function $g(t) + \frac{df}{dt}(t)$ is calculated over the time interval $(a, t]$.

Theorem 1. Let $f(t) \in C^2(\mathbb{R}_{\geq 0})$ be a continuously measured signal. Define $x_1(t)$ and $x_2(t)$ as the trajectories of the following ordinary differential equations:

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = -\beta^2 t^2 \left( x_1(t) - \tan^{-1}(f(t)) \right) - 2\beta t x_2(t),$$

$$\beta > 0, x_2(0) = 0,$$

$$y = \frac{t^{-\nu}}{\Gamma(1-\nu)} \ast \left( (1 + f^2(t))x_2(t) \right), \quad t > a,$$

$$y(s) = 0 \quad \text{for } s \leq a.$$  

Then, for sufficiently large value of $a > 0$,

$$\lim_{t \to +\infty} \frac{C}{a} D_t^{\nu} f(t) = \lim_{t \to +\infty} y(t), \quad 0 < \nu < 1.$$  

Proof. Let $z(t) = x_1(t)$ for all time. To prove the result of Theorem 1, it is sufficient to prove that $z(t)$ will converge to the signal $u(t) = \tan^{-1}(f(t))$. The $z$ variable verifies the following differential equation:

$$\ddot{z} + 2\beta t \dot{z} + t^2 \beta^2 (z-u) = 0, \quad \forall t > 0, z(0) = z_0, \dot{z}(0) = 0.$$  

For all $t \geq 0$, the solution of (7) is explicitly given by:

\[ I_\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \quad t > 0 \]
\[ z(t) = \left( \frac{1}{2} e^{-\frac{1}{2}t(\beta t - 2 \sqrt{\beta})} + \frac{1}{2} e^{-\frac{1}{2}t(\beta t + 2 \sqrt{\beta})} \right) z_0 + \frac{1}{2} \beta^2 \left( e^{2t \sqrt{\beta}} \int_0^t s^2 u(s) e^{\frac{1}{2} (\beta s - 2 \sqrt{\beta})} ds \right) e^{-\frac{1}{2}(\beta t + 2 \sqrt{\beta})}. \] (8)

In Eq. (8), the term involving \( z_0 \) vanishes to zero when time tends to infinity; therefore, the behavior of \( z(t) \) at infinity is only dependent on the remaining terms, i.e.,
\[
\lim_{t \to +\infty} z(t) =
\lim_{t \to +\infty} \frac{1}{2} \beta^2 \left( e^{2t \sqrt{\beta}} \int_0^t s^2 u(s) e^{\frac{1}{2} (\beta s - 2 \sqrt{\beta})} ds \right) e^{-\frac{1}{2}(\beta t + 2 \sqrt{\beta})}. \] (9)

Define
\[
V_1(t) = \int_0^t s^2 u(s) e^{\frac{1}{2} (\beta s - 2 \sqrt{\beta})} ds,
\] (10)
\[
V_2(t) = \int_0^t s^2 u(s) e^{\frac{1}{2} (\beta s + 2 \sqrt{\beta})} ds.
\]

Using integration by parts, we have
\[
V_1(t) = \left( \frac{t \beta^2 + \beta}{\beta^2} e^{-\frac{1}{2}(\beta t + 2 \sqrt{\beta})} u(t) - \frac{u(0)}{\beta^2} \right) e^{-\frac{1}{2}(\beta t + 2 \sqrt{\beta})}, \] (11)

and
\[
V_2(t) = \left( \frac{t \beta^2 - \beta}{\beta^2} e^{\frac{1}{2}(\beta t + 2 \sqrt{\beta})} u(t) + \frac{u(0)}{\beta^2} \right) e^{\frac{1}{2}(\beta t + 2 \sqrt{\beta})} - \int_0^t \left( \frac{t \beta^2 + \beta}{\beta^2} \right) e^{-\frac{1}{2}(\beta s + 2 \sqrt{\beta})} \dot{u}(s) ds. \] (12)

This gives,
\[
\lim_{t \to +\infty} z(t) =
\lim_{t \to +\infty} \frac{1}{2} \beta^2 \left( V_1(t) e^{2 \sqrt{\beta}t} - V_2(t) \right) e^{-\frac{1}{2}t(\beta t + 2 \sqrt{\beta})} + \frac{1}{2} e^{-\frac{1}{2}t(\beta t + 2 \sqrt{\beta})} \int_0^t \left( -\frac{1}{2} e^{-\frac{1}{2}t(\beta t + 2 \sqrt{\beta})} - \frac{1}{2} e^{-\frac{1}{2}t(\beta t + 2 \sqrt{\beta})} \right) u(t) ds
\]
\[\quad - \frac{1}{2} e^{-\frac{1}{2}t(\beta t + 2 \sqrt{\beta})} \int_0^t \left( \sqrt{\beta s + 1} \right) e^{\frac{1}{2}(\beta s - 2 \sqrt{\beta})} \dot{u}(s) ds + \frac{1}{2} e^{-\frac{1}{2}(\beta t + 2 \sqrt{\beta})} \int_0^t \left( \sqrt{\beta s + 1} \right) e^{\frac{1}{2}(\beta s + 2 \sqrt{\beta})} \dot{u}(s) ds. \] (13)

This implies that
\[
\lim_{t \to +\infty} z(t) = \lim_{t \to +\infty} \left[ u(t) - \frac{1}{2} e^{-\frac{1}{2}t(\beta t + 2 \sqrt{\beta})} \int_0^t \left( \sqrt{\beta s + 1} \right) e^{\frac{1}{2}(\beta s - 2 \sqrt{\beta})} \dot{u}(s) ds + \frac{1}{2} e^{-\frac{1}{2}(\beta t + 2 \sqrt{\beta})} \int_0^t \left( \sqrt{\beta s - 1} \right) e^{\frac{1}{2}(\beta s + 2 \sqrt{\beta})} \dot{u}(s) ds \right]. \] (14)

Let
\[
V_3(t) = \int_0^t \left( \sqrt{\beta s + 1} \right) e^{\frac{1}{2}(\beta s - 2 \sqrt{\beta})} \dot{u}(s) ds,
\]
\[
V_4(t) = \int_0^t \left( \sqrt{\beta s - 1} \right) e^{\frac{1}{2}(\beta s + 2 \sqrt{\beta})} \dot{u}(s) ds.
\]

Using integration by parts, \( V_3(t) \) takes the form (16) (see the top of the next page) where \( \beta \) is the pure complex number verifying \( \beta^2 = -1 \). Similarly, integrating \( V_4(t) \) by parts gives (17), see the top of the next page, where
\[
\text{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2} dt, \quad -i \text{erf}(i s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{t^2} dt. \] (18)

According to (16) and (17), one can write
\[
\lim_{t \to +\infty} z(t) = \lim_{t \to +\infty} u(t) - \frac{1}{2} \lim_{t \to +\infty} e^{-\frac{1}{2}t^2(\beta t + 2 \sqrt{\beta})} V_3(t) + \frac{1}{2} \lim_{t \to +\infty} e^{-\frac{1}{2}t^2(\beta t + 2 \sqrt{\beta})} V_4(t). \] (19)

Since \( u(t) \) and \( \dot{u}(t) \) are bounded for all time \( t \geq 0 \) with
\[
\lim_{t \to +\infty} e^{-t^2} \int_0^t e^{t^2} dt = 0 \quad \text{and} \quad \lim_{t \to +\infty} e^{-t^2} \int_0^t e^{t^2} dt = 0 \quad \text{then,}
\]
\[
- \frac{1}{2} \lim_{t \to +\infty} e^{-\frac{1}{2}t^2(\beta t + 2 \sqrt{\beta})} V_3(t) + \frac{1}{2} \lim_{t \to +\infty} e^{-\frac{1}{2}t^2(\beta t + 2 \sqrt{\beta})} V_4(t) = 0. \] (20)

Consequently,
\[
\lim_{t \to +\infty} z(t) = \lim_{t \to +\infty} u(t), \quad \lim_{t \to +\infty} x_2(t) = \lim_{t \to +\infty} \dot{u}(t). \] (21)

As a result; \( \lim_{t \to +\infty} f(t) = \lim_{t \to +\infty} \dot{u}(t)(1 + f^2(t)) \). For a and \( \beta \) sufficiently large, the system output \( y(t) \) will converge to the exact fractional-order derivative of \( f(t) \). Remark that the transient of the differentiator is fast when \( \beta \) is large enough. This means that the rate of convergence is dependent on the free parameter \( \beta \). This ends the proof.

Remark that system (5) is able to produce the first derivative of \( \tan^{-1}(f) \) whatever the nature of \( f \); i.e., being bounded or not bounded. Hence, the fractional-order derivative of order \( \nu \) can be always estimated once \( f \in C^2(\mathbb{R}_0^+) \). The coefficient \( \beta \) in system (5) regulates the transient behavior of the fractional-order derivative estimates. Obviously, fast transient behaviors are seen when \( \beta \) is large.

2.3 The second procedure: case \( 1 < \nu < 2 \)

When the non-integer differentiation order increases, the fractional derivatives will be dependent on the higher-order derivatives of the signal \( f(t) \). In this subsection, the second algorithm deals with the case of \( \nu \) between one and
\[ V_3(t) = \left( -ie^{-\frac{i}{2}} \text{erf} \left( \frac{i}{\sqrt{2}} \sqrt{\beta} \right) \right) \hat{u}(0) \sqrt{\pi \sqrt{\beta}} + \left( -ie^{-\frac{i}{2}} \sqrt{2} \text{erf} \left( \frac{i}{2} \sqrt{\beta} s - 1 \right) \right) \hat{u}(s) ds \]

\[ V_4(t) = \left( -ie^{-\frac{i}{2}} \text{erf} \left( \frac{i}{\sqrt{2}} \sqrt{\beta} \right) \right) \hat{u}(0) \sqrt{\pi \sqrt{\beta}} + \left( -ie^{-\frac{i}{2}} \sqrt{2} \text{erf} \left( \frac{i}{2} \sqrt{\beta} + 1 \right) \right) \hat{u}(s) ds \]

Theorem 2. Let \( f(t) \in \mathcal{C}^3(\mathbb{R}_{\geq 0}) \) be a continuously measured signal. Define \( x_1(t), x_2(t), \) and \( x_3(t) \) as the trajectories of the following ordinary differential equations for all \( t \geq 0 \):

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= x_3(t), \\
\dot{x}_3(t) &= -\beta^3 t^3 \left( x_1(t) - \tan^{-1}(f(t)) \right) - 3\beta^2 t^2 x_2(t) - 3\beta t x_3(t), \quad \beta > 0,
\end{align*}
\]

with \( y(s) = 0 \) for \( s \leq a \), \( x_2(0) = x_3(0) = 0 \). Then, for sufficiently large value of \( a > 0 \), we have

\[
\lim_{t \to +\infty} \frac{C}{t} f(t) = \lim_{t \to +\infty} y(t), \quad 1 < \nu < 2.
\]

Proof. The proof is omitted for space limitation.

3. DISCRETE-TIME IMPLEMENTATION AND ROBUSTNESS AGAINST ADDITIVE NOISE

3.1 Numerical discretization

The dynamics of systems (5) and (22) are sets of ordinary differential equations that can be solved by any numerical method like the Runge-Kutta, Dormand-Prince, Adams-Bashforth, or the predictor-corrector numerical procedures. However, the outputs of these systems can be approximated by the discrete convolution formulae that provides an integration error of order \( h \), where \( h \) is the time sampling period. For \( 0 < \nu < 1 \) and \( a = 0 \), the discrete output of the fractional derivative is approximated by:

\[
y[k] = y(k h) = h \sum_{j=1}^{k-1} \phi[j] \varphi[k-j], \quad y[0] = 0.
\]

where

\[
\phi[j] = \frac{\Gamma(1-\nu)}{\Gamma(2-\nu)}, \quad \varphi[j] = (1 + f^2[j]) x_2[j], \quad t_j = j h.
\]
sufficient to estimate the fractional-order derivatives with bounded errors. The parameter $\beta$ can be also tuned automatically to achieve this objective. The procedure of tuning the parameter $\beta$ is not considered herein due to space limitation.

4. NUMERICAL SIMULATIONS

To illustrate the efficacy of the proposed algorithm, we have compared the exact fractional-order derivative of the sine wave with its counterpart given by numerical integration of the system of equations (5). By taking $\nu = \frac{1}{2}$, $a = 0$, and $f(t) = \sin(t)$, we have

$$C_a D^\nu_{a} f(t) = \sqrt{2} \mathcal{F}_c \left( \frac{\sqrt{2} \sqrt{t}}{\pi} \right) \cos(t) + \sqrt{2} \mathcal{F}_s \left( \frac{\sqrt{2} \sqrt{t}}{\pi} \right) \sin(t),$$

(29)

where $\mathcal{F}_c(\tau)$ and $\mathcal{F}_s(\tau)$ are respectively the Fresnel cosine integral and the Fresnel sine integral defined by:

$$\mathcal{F}_c(\tau) = \int_0^\tau \cos \left( \frac{\pi t^2}{2} \right) dt,$$

$$\mathcal{F}_s(\tau) = \int_0^\tau \sin \left( \frac{\pi t^2}{2} \right) dt.$$  

(30)

The system of equations (5) is integrated by the Dormand-Prince algorithm of order five with a uniform sampling period $\Delta t = 10^{-3}$ (sec), $\beta = 30$, $a = 0$. The system output of system (5) is approximated by the discrete convolution formulae and compared to the exact derivative (29), see Fig. 1. Actually, the value of the convolution term, provided by the system output $y$, could be improved by minimizing the sampling period $\Delta t$ or by selecting another numerical scheme of signal integration.

Now, we shall assess the quality of estimation of the fractional derivative for $\nu = \frac{3}{2}$ and select the signal $f(t)$ as $\frac{1}{\pi} t \sin(t)$. The exact fractional derivative of $f$ is

$$C_0 D^\frac{3}{2}_{0} f(t) = \frac{1}{10\sqrt{\pi}} \left[ -\sqrt{2\pi} \left( t \sin(t) - \frac{3}{2} \cos(t) \right) \mathcal{F}_c \left( \frac{\sqrt{2t}}{\pi} \right) + \sqrt{2\pi} \left( \cos(t) t + \frac{3}{2} \sin(t) \right) \mathcal{F}_s \left( \frac{\sqrt{2t}}{\pi} \right) \right].$$

(31)

The result of Theorem 2 is used to evaluate the instantaneous value of the fractional derivative as shown in Fig. 2 where $\beta = 20$, $a = 0$. System (22) is integrated by the Dormand-Prince method of order five with a sampling period $\Delta t = 10^{-3}$ (sec). It is noticed that even $a = 0$, the quality of the estimates of the fractional derivatives are quite good.

The second differentiator is tested again for a noisy unbounded signal $f(t) = \frac{1}{\pi} t \sin(t) + \eta(t)$ where $\eta(t)$ is a white noise, see Fig. 3. For $a = 0$, $\Delta t = 1$ ms, the fractional derivative of order $\frac{3}{2}$ is represented in Fig. 4. In this simulation, $\beta = 3$ while the time “$\nu$” in system (22) is saturated at 5 (sec). Notice that the output of system (22) is able to reproduce the exact derivative with a bounded error.
Fig. 4. The fractional derivatives of noisy unbounded function, \( \nu = \frac{3}{2}, f(t) = \frac{1}{10} t \sin(t) + n(t) \)

5. CONCLUSIONS

The Caputo derivative of a measured signal is transformed into a simple convolution of a time-dependent function with the solutions of a set of ordinary differential equations. This formulation favors the realization of non-integer-order derivatives when combined with other system dynamics. The LTV-system approach to the calculation of the fractional derivatives is not dependent on the signal frequency and its form (bounded or not bounded). Moreover, the rate of convergence to the true derivatives is easily tunable by increasing or decreasing the value of a design parameter.

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REFERENCES

Annaby, M.H. and Mansour, Z.S. (2012). q-fractional calculus and equations. Lecture Notes in Mathematics. Springer, Berlin.

Duma, R. (2012). Embedded application of fractional order control. Electronics Letters, 48, 1526–1528.

Farges, C., Moze, M., and Sabatier, J. (2010). Pseudo-state feedback stabilization of commensurate fractional-order systems. Automatica, 46(10), 1730–1734.

Ibrir, S. (2003). Online exact differentiation and notion of asymptotic algebraic observers. IEEE Transactions on Automatic Control, 48(11), 2055–2060.

Ibrir, S. (2017). Sufficient conditions for domain stabilizability of uncertain fractional-order systems under static-output feedbacks. IET Control Theory and Applications, 11(12), 2004–2011.

Ibrir, S. and Bettayeb, M. (2015). New sufficient conditions for observer-based control of fractional-order uncertain systems. Automatica, 59, 216 – 223.

Li, Y., Chen, Y., and Podlubny, I. (2010). Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability. Computers and Mathematics with Applications, 59(5), 1810–1821.

Liu, D.Y., Gibaru, O., Perruquet, W., and Laég-Kirati, T.M. (2015). Fractional order differentiation by integration and error analysis in noisy environment. IEEE Transactions on Automatic Control, 60(11), 2945–2959.

Liu, D.Y., Zheng, G., Boutat, D., and Liu, H.R. (2017). Non-asymptotic fractional order differentiator for a class of fractional order linear systems. Automatica, 78, 61–71.

Luo, Y. and Chen, Y.Q. (2009). Fractional order [proportional derivative] controller for a class of fractional order systems. Automatica, 45(10), 2446 – 2450.

Oustaloup, A. (1983). Systèmes asservis linéaires d’ordre fractionnaire. Paris.

Oustaloup, A. (1995). La dérivation non entière: Théorie, synthèse et applications. Paris.

Oustaloup, A. (2014). Diversity and non-integer differentiation for system dynamics.

Sabatier, J., Agrawal, O.P., and Machado, J.A.T. (eds.) (2007). Advances in fractional calculus: Theoretical developments and applications in Physics and Engineering. Springer, ebook.

Sabatier, J., Lanusse, P., Melchior, P., and Oustaloup, A. (eds.) (2015). Fractional Order Differentiation and Robust Control Design: CRONE, H-infinity and Motion Control. Intelligent Systems, Control and Automation: Science and Engineering. Springer.

Samadi, S., Ahmad, M.O., and Swamy, M.N.S. (2004). Exact fractional-order differentiators for polynomial signals. IEEE Signal Processing Letters, 11(6), 529–532.

Samko, S.G., Kilbas, A.A., and Marichev, O.I. (1987). Fractional integrals and derivatives: Theory and Applications. Gordon and Breach Science Publishers.

Serrier, P., Moreau, X., and Oustaloup, A. (2007). Limited-bandwidth fractional differentiator: synthesis and application in vibration isolation. In A.O. Sabatier J. and M. J.A.T. (eds.), Advances in fractional calculus. Springer.

Tolba, M.F., AboAlNaga, B.M., Said, L.A., Madian, A.H., and Radwan, A.G. (2019). Fractional order integrator/differentiator: FPGA implementation and FOPID controller application. AEU - International Journal of Electronics and Communications, 98, 220–229.

Trigeassou, J.C., Maamri, N., Sabatier, J., and Oustaloup, A. (2011). A Lyapunov approach to the stability of fractional-differential equations. Signal Processing, 91(3), 437–445.

Wei, Y.Q., Liu, D.Y., and Boutat, D. (2019). Innovative fractional derivative estimation of the pseudo-state for a class of fractional order linear systems. Automatica, 99, 157–166. doi: https://doi.org/10.1016/j.automatica.2018.10.028.

Zwillinger, D. (2014). Handbook of Differential Equations. Elsevier Science.