THE \((u, v)\)-CALKIN-WILF FOREST

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Abstract. In this paper we consider a refinement, due to Nathanson, of 
the Calkin-Wilf tree. In particular, we study the properties of such trees 
associated with the matrices \(L_u = \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix}\) and \(R_v = \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}\), where 
\(u\) and \(v\) are nonnegative integers. We extend several known results 
of the original Calkin-Wilf tree, including the symmetry, numerator-
denominator, and successor formulas, to this new setting. Additionally, 
we study the ancestry of a rational number appearing in a generalized 
Calkin-Wilf tree.

Keywords: Calkin-Wilf tree, continued fractions

1. Introduction

The Calkin-Wilf tree \([4]\) is an infinite binary tree generated by two rules. 
The number 1, represented as 1/1, is the root of the tree and each vertex 
\(a/b\) has two children: the left one is \(a/(a + b)\) and the right one is \((a + b)/b\) 
(see Figure 1).

![Figure 1. The first four rows of the Calkin-Wilf tree.](image)

By following the breadth-first order, this tree provides an enumeration of 
positive rational numbers:

\[1, \frac{1}{2}, 1, \frac{3}{2}, 1, 4, 3, 5, 2, 5, 3, 4, \ldots\]

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In fact, Calkin and Wilf [4] showed that every reduced positive rational number appears in this list exactly once.

In addition to enumerating the positive rationals, the Calkin-Wilf tree has many interesting properties and generalizations that have been explored by various researchers (for example, [3 4 5 6 8 9 10 12 13 14 15]). In particular, as in [12], we highlight the following four properties. We denote by \( c(n, i) \) the vertex in the \( i \)th position (from left to right) of the \( n \)th row.

**Property 1** (Successor formula, Newman [14]). For every nonnegative integer \( n \) and \( i = 1, \ldots, 2^n - 1 \), we have

\[
\frac{1}{2\lfloor c(n, i) \rfloor + 1 - c(n, i)}
\]

where \([x]\) denotes the integer part of \( x \).

**Property 2** (Denominator-numerator formula, Calkin and Wilf [4]). For every nonnegative integer \( n \) and \( i = 1, \ldots, 2^n - 1 \), the denominator of \( c(n, i) \) is equal to the numerator of \( c(n, i + 1) \).

**Property 3** (Symmetry formula, [12]). For every nonnegative integer \( n \) and \( i = 1, \ldots, 2^n \), we have \( c(n, i) \cdot c(n, 2^n - i + 1) = 1 \).

**Property 4** (Depth formula, [7]). Let \( a/b \) be a positive reduced rational number. Let \( n \) and \( i \) be the unique pair such that \( c(n, i) = a/b \). If

\[
\frac{a}{b} = a_0 + \frac{1}{a_1 + \cdots + \frac{1}{a_k}} = [a_0, a_1, \ldots, a_{k-1}, a_k]
\]

is the finite continued fraction representation\(^3\) of \( a/b \), then

\[
n = a_0 + a_1 + \cdots + a_{k-1} + a_k - 1.
\]

In other words, the sum of the coefficients of the continued fraction representation encodes the row number where \( a/b \) appears, i.e. the depth, in the Calkin-Wilf tree.

Let \( z \) be a variable. In [12], Nathanson considers the infinite binary tree \( \mathcal{T}(z) \), whose root is \( z \), where each vertex \( w \) has two children: the left child is \( w/(w + 1) \), and the right child is \( w + 1 \) (see Figure 2).

The original Calkin-Wilf tree is clearly the special case of \( z = 1 \). For general \( z \), Properties [14] of the Calkin-Wilf tree extend\(^3\) to \( \mathcal{T}(z) \).

We can associate each vertex in \( \mathcal{T}(z) \) with a column vector as in Figure 3.

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1Our convention is that the row containing the root is the zero row. So, for example, \( c(2, 3) = 2/3 \).

2For a rational number not equal to 1, we always take the shorter continued fraction representation where \( a_k \neq 1 \).

3Of independent interest, the generalization of Property 4 requires an appropriate definition of a continued fraction representation for linear fractional transformations.
Figure 2. The first four rows of $\mathcal{T}(z)$.

Figure 3. Association between rational numbers and vectors.

Letting $L_1 := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $R_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we see that subsequent vertices in $\mathcal{T}(z)$ can be obtained by matrix multiplication. A vertex $\begin{bmatrix} a \\ b \end{bmatrix}$ has left child

$$(2) \quad L_1 \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a+b \end{bmatrix}$$

and right child

$$(3) \quad R_1 \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ b \end{bmatrix}.$$

In particular, every vertex in $\mathcal{T}(z)$ is obtained by multiplying a matrix generated freely by the set $\{L_1, R_1\}$ with the vector associated with $z$. In this way, we can label the vertices of $\mathcal{T}(z)$ with matrices in $SL_2(\mathbb{N}_0) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{N}_0 \text{ and } ad - bc = 1 \right\}$ acting on $z$ (see Figure 4). For ease of notation, we denote the left and right child of $w$ by $L_1(w)$ and $R_1(w)$, respectively.

With this perspective in mind, it is natural to consider an analogous infinite binary tree generated by other pairs of matrices in $SL_2(\mathbb{N}_0)$. Let $u$ and $v$ be integers such that $u, v \geq 2$,

$L_u := \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix}$ and $R_v := \begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}.$
Nathanson [11, 12] proposed to investigate the infinite binary tree associated to \( \{L_u, R_v\} \) obtained by replacing \( L_1 \) in (2) and \( R_1 \) in (3) by \( L_u \) and \( R_v \), respectively (see Figure 5 for the generation rule).

We refer to this generalization as a \((u, v)\)-Calkin-Wilf tree and denote it by \( T^{(u,v)}(z) \), where \( z \) is the root (see Figure 6). Note that by setting \( u = v = 1 \) and \( z = 1 \), we obtain the original Calkin-Wilf tree, \( T(1) \). From now on, we assume that \( u \) and \( v \) are integers such that \( u, v \geq 1 \), and so \( T^{(1,1)}(1) \) is \( T(1) \).

As an example, consider the tree \( T^{(2,3)}(5/2) \) (see Figure 7). One can immediately notice that the denominator-numerator and the symmetry formulas (Properties 2 and 3) do not hold in \( T^{(2,3)}(5/2) \). Furthermore, many rational numbers appearing in \( T(1) \) seem to be missing in \( T^{(2,3)}(5/2) \). In fact, it is not too difficult to show that 1 does not appear in any tree \( T^{(u,v)}(z) \) unless \( z = 1 \). In the next section we will address this issue, and define the \((u, v)\)-Calkin-Wilf forest which will enumerate positive rational numbers.
The first four rows of $T^{(2,3)}(5/2)$.

We have already shown by example that Properties 1-4 do not, in general, hold for a $(u, v)$-Calkin-Wilf tree. However, $(u, v)$-Calkin-Wilf trees share enough of a similar structure with $T(1)$ that we are able to provide some appropriate, universal generalizations (see Theorem 1 and Corollary 3, for example). In other cases, we will show that some of Properties 1-4 completely characterize the Calkin-Wilf tree (see Proposition 4 and Corollary 1, for example).

2. Global Properties

For a fixed $u$ and $v$, consider the set of all positive reduced rational numbers that are not the children of any rational number appearing in any $(u, v)$-Calkin-Wilf tree. We refer to such numbers as $(u, v)$-orphans (when the context is clear, we may refer to such numbers simply as orphans). A straightforward proof shows that the set of $(u, v)$-orphans is

$$\left\{ \frac{a}{b} : \frac{1}{u} \leq \frac{a}{b} \leq \frac{v}{u} \right\}$$

(see [12]). It follows that the set of $(u, v)$-orphans is finite if and only if $u = v = 1$. Furthermore, it can be seen that every left child in a $(u, v)$-Calkin-Wilf tree is strictly bounded above by $1/u$ and every right child is strictly bounded below by $v$. In the case of the original Calkin-Wilf tree $1$ is the only orphan. In $T^{(2,3)}(5/2)$, the vertex $5/2$ satisfies the condition $1/2 \leq 5/2 \leq 3$, and so it is one of the many $(2,3)$-orphans.

Lemma 1. Let $z$ and $z'$ be distinct $(u, v)$-orphans. Then the vertices of $T^{(u,v)}(z)$ and $T^{(u,v)}(z')$ form disjoint sets.

Proof. Suppose that $w$ is a rational number that appears as a vertex in both $T^{(u,v)}(z)$ and $T^{(u,v)}(z')$. Without loss of generality, we can assume that $w$ is such that no other ancestor of it (in either tree) holds this property. It follows $w$ is not a root and must be the child of vertices in both trees. Furthermore, $w$ cannot be a left child (right child, resp.) in both $T^{(u,v)}(z)$ and $T^{(u,v)}(z')$. So $w$ is a left child in, say, $T^{(u,v)}(z)$ and a right child in $T^{(u,v)}(z')$. This implies that $w < 1/u \leq 1$ and $w > v \geq 1$, a contradiction. □
Since every positive reduced rational number is either a \((u, v)\)-orphan or the descendant of a \((u, v)\)-orphan, Lemma 1 shows that the set of \((u, v)\)-orphans enumerates a forest of trees that partitions the set of positive rational numbers, the \((u, v)\)-Calkin-Wilf forest.

**Lemma 2.** Let \(u\) and \(v\) be positive integers. Then \(L_u = L_1^u\) and \(R_v = R_1^v\).

**Proof.** We show that \(L_u = L_1^u\) by induction on \(u\). This is clearly true when \(u = 1\). Suppose it is true for \(u \geq 1\). Then

\[
L_{u+1} = L_u \cdot L_1 = [1 \ 0] \cdot [1 \ 0] = \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} = L_{u+1}.
\]

A similar argument shows that \(R_v = R_1^v\). \(\square\)

When comparing \(T^{(u,v)}(z)\) to \(T(1)\), one can see from Lemma 2 that the vertices of \(T^{(u,v)}(z)\) can be obtained by starting with the vertex \(z\) in \(T(1)\) and skipping over \(u - 1\) generations of left children or \(v - 1\) generations of right children to arrive at the children of \(z\) in \(T^{(u,v)}(z)\). For example, compare Figure 4 and Figure 8 in the case where \(u = 2\) and \(v = 3\). In other words, the vertex set of \(T^{(u,v)}(z)\) is a submonoid of \(\mathbb{Q}\) or, equivalently, the vertex set of \(T(1)\). More generally, we have the following two results as other immediate consequences of Lemma 2.

**Proposition 1.** The vertex set of \(T^{(u,v)}(z)\) is a submonoid of the vertex set of \(T^{(u',v')}(z')\) if and only if \(z \in T^{(u',v')}(z')\), \(u' \mid u\), and \(v' \mid v\).

**Proposition 2.** Let \(U\) and \(V\) be finite sets of nonnegative integers. Set \(u := \text{lcm}\{u' : u' \in U\}\) and \(v := \text{lcm}\{v' : v' \in V\}\). Then

\[
T^{(u,v)}(z) = \bigcap_{u' \in U, v' \in V} T^{(u',v')}(z).
\]

Lemma 1 and Lemma 2 show that the \((u, v)\)-orphans partition the set of positive rational numbers into a collection of trees with a similar structure derived from \(T(1)\). This idea serves as the main motivation for this paper.

### 3. The Successor and the Numerator-Denominator Formulas

We begin this section by establishing some immediate properties of \((u, v)\)-Calkin-Wilf trees related to Properties 1-2. We denote by \(c^{(u,v)}_z(n, i)\) the \(i\)th element, from left to right, in the \(n\)th row of the \((u, v)\)-Calkin-Wilf tree
whose root is $z$. For any integer $n \geq 0$, the first and the last elements of the $n$th row with root $z$ are readily seen to be

\[ c_z^{(u,v)}(n,1) = \frac{z}{nuz + 1} \quad \text{and} \quad c_z^{(u,v)}(n,2^n) = z + nv, \]

respectively. Furthermore, since $z$ is assumed to be in reduced form, then all vertices of $T_z^{(u,v)}(z)$ are also in reduced form.

**Proposition 3 (Generalized successor formula).** Consider the $(u,v)$-Calkin-Wilf tree with root $z$. For every nonnegative integer $n$ and $i = 1, \ldots, 2^n - 1$, let $\alpha_i = c_z^{(u,v)}(n,i)$. Then we have

\[ \alpha_{i+1} = \frac{v\alpha_i + v^2(1 - u\alpha_i)}{u[\alpha_i](\{\alpha_i\} + v(1 - u\alpha_i)) + v(1 - u\alpha_i)} \]

where $[x]$ and $\{x\}$ denote the integer and fractional parts of the real number $x$, respectively.

**Proof.** Our proof is a generalization of an argument by Newman \[1\] in the case where $u = v = 1$.

If $\alpha_i$ and $\alpha_{i+1}$ are adjacent siblings in a $(u,v)$-Calkin-Wilf tree, then they share a common ancestor $w$ (see Figure 9) such that, for some $k \geq 0$, $\alpha_i$ is the $k$th right child of the left child of $w$ and $\alpha_{i+1}$ is the $k$th left child of the right child of $w$. (This is not a feature that is unique to $(u,v)$-Calkin-Wilf trees; it is common to all full binary trees.) It follows that $\alpha_i = \frac{w}{kw+1} + kw$ and $\alpha_{i+1} = \frac{w+v}{kw(w+v)+1}$. Note that since $\frac{w}{kw+1} < 1$, then $\{\alpha_i\} = \frac{w}{kw+1}$ and $[\alpha_i] = kw$.

![Figure 9. Successors in a $(u,v)$-Calkin-Wilf tree with common ancestor $w$.](image)

In order to complete the proof, we must eliminate the dependence of $\alpha_{i+1}$ on $k$ and $w$. This can be accomplished by taking the formula for $\{\alpha_i\}$ and solving for $w$. This gives that

\[ w = \frac{\{\alpha_i\}}{1 - u\{\alpha_i\}}. \]
It follows that

$$\alpha_{i+1} = \frac{w + v}{ku(w + v) + 1} = \frac{w + v}{kv\left(\frac{wu}{v} + u\right) + 1} = \frac{w + v}{\alpha_i(\frac{wu}{v} + u) + 1}. \quad (6)$$

Inserting (5) into the right-hand side of (6) and simplifying gives the desired result. \qed

While (4) does collapse down to (1) when \(u = v = 1\), something is lost in this generalization. Iterating (1) not only gives successive elements in a fixed row of the Calkin-Wilf tree. When it is applied to the rightmost element of a row, it returns the leftmost element of the next row. The same is not true of (4).

It follows from Proposition 3 that if we consider successive terms in each row of a \((u, v)\)-Calkin-Wilf tree, the denominator-numerator formula (Property 2) holds only in the original Calkin-Wilf tree.

**Proposition 4.** The denominator-numerator formula holds if and only if \(u = v = 1\).

**Proof.** Using the same notation in the proof of Proposition 3, for a common ancestor \(w\),

$$\alpha_i = \frac{w' + kv(uw' + w'')}{uw' + w''} \quad \text{and} \quad \alpha_{i+1} = \frac{w' + vw''}{ku(w' + vw'') + w''},$$

where \(w = w'/w''\) is in lowest terms. It is easy to see that the above representations of \(\alpha_i\) and \(\alpha_{i+1}\) are also in lowest terms. So we can let \(d_i = uw' + w''\) be the denominator of \(\alpha_i\) and \(n_{i+1} = w' + vw''\) be the numerator of \(\alpha_{i+1}\). It quickly follows that

$$vd_i + (1 - uv)w' = n_{i+1}. \quad (7)$$

\((\Leftarrow)\) If \(u = v = 1\), then \(d_i = n_{i+1}\) follows from (7).

\((\rightarrow)\) If \(d_i = n_{i+1}\), then it follows from (7) that

$$(uv - 1)w' = (v - 1)n_{i+1}$$

$$= (v - 1)(w' + vw'').$$

Collecting like terms on either side of the equality shows that

$$(u - 1)w' = (v - 1)w''.$$ 

If \(u = 1\) and \(v \neq 1\), then \(w'' = 0\), a contradiction. A similar argument works for the case where \(u \neq 1\) and \(v = 1\). If \(u, v \neq 1\), then \(w = w'/w'' = (v - 1)/(u - 1)\). This would imply that \(w\) is fixed for all pairs of successors, another contradiction. Therefore, \(u = v = 1\). \qed

We see from (7) that the relationship between successive denominators and numerators in a row of a \((u, v)\)-Calkin-Wilf tree is significantly more complicated than in the statement of Property 2. In order to generalize the denominator-numerator formula, one would need to know more about the common ancestors of successive terms. At this time, no clear generalization of Property 2 is evident.
4. Symmetry Properties

In this section, we study symmetry properties of \((u, v)\)-Calkin-Wilf trees closely related to Property 3. As in the previous section, we are able to find some appropriate generalizations, in some sense, while showing that Property 3 completely characterizes \(T(1)\). We begin with a lemma which will be used in the theorems that follow.

Lemma 3. For every vertex in the \((u, v)\)-Calkin-Wilf tree with root \(z\) there are nonnegative integers \(a, b, c,\) and \(d\) with \(ad - bc = 1\) such that the vertex is represented as

\[ \frac{az + b}{cz + d}. \]

Proof. The statement follows from induction on the row number of the \((u, v)\)-Calkin-Wilf tree \(T^{(u,v)}(z)\) (see Figure 6). □

Note that the integers \(a, b, c,\) and \(d\) in Lemma 3 depend on \(u, v,\) and the position of the vertex in the tree. See (14) in Section 5 for an example on how to compute \(a, b, c,\) and \(d\) for \(2147/620\) in \(T^{(2,3)}(5/2)\). Furthermore, Lemma 3 shows that every vertex in a \((u, v)\)-Calkin-Wilf tree can be written as some linear fractional transformation of the root (see Figures 6 and 11).

Theorem 1 (General symmetry formula). For every nonnegative integer \(n\) and \(i = 1, 2, \ldots, 2^n\), if \(c^{(u,v)}_z(n, i) = \frac{az + b}{cz + d}\) where \(a, b, c, d\) are nonnegative integers, then

\[ c^{(u,v)}_z(n, 2^n + 1 - i) = \frac{dz + cv}{bu} \frac{u}{z + a}. \]

Proof. The proof is by induction on the row number \(n\). Since \(c^{(u,v)}_z(1, 1) = \frac{z}{uz + 1}\), we have that \(c^{(u,v)}_z(1, 1) = \frac{az + b}{cz + d}\) with \(a = 1, b = 0, c = u,\) and \(d = 1,\) and so

\[ \frac{dz + cv}{bu} \frac{u}{z + a} = z + v = c^{(u,v)}_z(1, 2). \]

On the other hand, starting from \(c^{(u,v)}_z(1, 2) = z + v,\) we get that \(c^{(u,v)}_z(1, 2) = \frac{az + b}{cz + d}\) with \(a = 1, b = v, c = 0,\) and \(d = 1.\) Hence

\[ \frac{dz + cv}{bu} \frac{u}{z + a} = \frac{z}{uz + 1} = c^{(u,v)}_z(1, 1). \]

This shows that the statement is true when \(n = 1.\) Suppose that the theorem is true for some row \(n \geq 1.\) An element in the row \(n + 1\) is either of the
form $c_z^{(u,v)}(n+1,2i-1)$ or $c_z^{(u,v)}(n+1,2i)$ for some integer $i$, $1 \leq i \leq 2^n$. If $c_z^{(u,v)}(n+1,2i-1) = \frac{az + b}{cz + d}$ (we know that such a representation exists by Lemma 3) then it is the left child of

$$c_z^{(u,v)}(n,i) = \frac{az + b}{(c - ua)z + (d - ub)}.$$ 

Thus, by using the symmetry on row $n$, we obtain

$$c_z^{(u,v)}(n+1,2^{n+1} + 2 - 2i) = R_v \left( c_z^{(u,v)}(n,2^n + 1 - i) \right)$$

$$= R_v \left( \frac{(d - ub)z + (c - ua)v}{bu/v z + a} \right) = \frac{dz + cv}{bu/v z + a}.$$ 

Similarly, if $c_z^{(u,v)}(n+1,2i) = \frac{az + b}{cz + d}$ then it is the right child of

$$c_z^{(u,v)}(n,i) = \frac{(a - cv)z + (b - vd)}{cz + d}.$$ 

Hence

$$c_z^{(u,v)}(n+1,2^{n+1} + 1 - 2i) = L_u \left( c_z^{(u,v)}(n,2^n + 1 - i) \right)$$

$$= L_u \left( \frac{dz + cv}{(b - vd)u/v z + (a - cv)} \right) = \frac{dz + cv}{bu/v z + a}.$$ 

As a consequence, we obtain necessary and sufficient conditions for the symmetry formula (Property 3) to hold in a $(u,v)$-Calkin-Wilf tree.

**Corollary 1** (Symmetry formula). The symmetry formula,

$$c_z^{(u,v)}(n,i) \cdot c_z^{(u,v)}(n,2^n + 1 - i) = 1,$$

holds if and only if $u = v$ and $z = 1$.

**Proof.** Suppose, using Lemma 3, that $c_z^{(u,v)}(n,i) = \frac{az + b}{cz + d}$ where $a, b, c, d$ are nonnegative integers and $ad - bc = 1$. By Theorem 4 we obtain that (5) is equivalent to

$$\left(\frac{az + b}{cz + d}\right) \cdot \left(\frac{dz + cv}{bu/v z + a}\right) = 1,$$

or

$$\left( ad - \frac{bcu}{v} \right) z^2 + \left[ bd \left( 1 - \frac{u}{v} \right) - ac \left( 1 - \frac{v}{u} \right) \right] z + \left( \frac{bcv}{u} - ad \right) = 0$$
It follows that \( ad - \frac{bcu}{v} = 0 \) and \( bcv - ad = 0 \), from which we get
\[
\frac{v}{u} = \frac{bc}{ad} \quad \text{and} \quad \frac{v}{u} = \frac{ad}{bc},
\]
thus \( v^2 = u^2 \). Since \( u, v > 0 \), this implies that \( u = v \). By substituting \( u = v \) into (10), we obtain that
\[
(ad - bc)(z^2 - 1) = 0.
\]
Since \( ad - bc = 1 \) and \( z > 0 \), we conclude that \( z = 1 \). \( \square \)

We remark that Corollary 1 can be also proved using induction on the row number. The result explains why the symmetry formula does not hold in \( T^{(2,3)}(5/2) \) (see Figure 7), as we had observed earlier.

**Corollary 2** (Skew symmetry). Using the same hypothesis as Theorem 1, it follows that
\[
c_{z}^{(u,v)}(n,i) \cdot c_{z-1}^{(v,u)}(n,2^n+1-i) = \frac{v}{u}.
\]

**Proof.** Suppose, using Lemma 3, that \( c_{z}^{(u,v)}(n,i) = \frac{a(z+b)}{cz+d} \) where \( a, b, c, d \) are nonnegative integers. Replacing \( z \) by \( \frac{v}{u} \) in (8) yields that
\[
c_{z}^{(u,v)}(n,2^n+1-i) = \frac{v}{u} \left( \frac{b + az}{cz} \right) = \frac{v}{u} \cdot \frac{cz + d}{az + b},
\]
which is equivalent to the desired result. \( \square \)

Corollary 1 shows that the symmetry formula does not hold for \((u, v)\)-Calkin-Wilf trees in general. However, Corollary 2 (above) and Theorem 2 (below) show that other symmetry formulas do hold when comparing either pairs of \((u, v)\)-Calkin-Wilf trees or \((u, v)\)- and \((v, u)\)-Calkin-Wilf trees, respectively. For examples, see Table 1.

| Row 2 of \( T^{(2,3)}(5/2) \) | 5/22 | 41/12 | 11/24 | 17/2 |
| Row 2 of \( T^{(2,3)}(3/5) \) | 3/17 | 36/11 | 18/41 | 33/5 |
| Row 2 of \( T^{(3,2)}(2/5) \) | 2/17 | 24/11 | 12/41 | 22/5 |

**Table 1.** Examples of Corollary 2 and Theorem 2

**Theorem 2** (Nathanson’s symmetry, \([13]\)). Let \( z \) be a variable, and let \( u \) and \( v \) be positive integers. For all nonnegative integers \( n \) and \( i = 1, 2, \ldots, 2^n \),
\[
c_{z}^{(u,v)}(n,i) \cdot c_{z-1}^{(v,u)}(n,2^n+1-i) = 1.
\]
If \( u = v \geq 1 \), then Theorem 2 gives one of the directions of Corollary 1. If \( u = v = 1 \), then this is the familiar symmetry of the Calkin-Wilf tree.

Nathanson's symmetry was proved in [13] using induction on the row number. We conclude this section with two alternative proofs of Theorem 2. The first one is a consequence of Theorem 1, and only holds when \( u = v \).

First Proof of Theorem 2 when \( u = v \). By Lemma 3, let \( c_z^{(u,u)}(n,i) = az + b \) for some nonnegative integers \( a, b, c, \) and \( d \). By Theorem 1, we have

\[
 c_z^{(u,u)}(n, 2^n + 1 - i) = \frac{dz + c}{bz + a} = \frac{cz + d}{az + b},
\]

which is the reciprocal of \( c_z^{(u,u)}(n,i) \). \( \square \)

The identity presented in Theorem 1 only holds in a \((u,v)\)-Calkin-Wilf tree where \( u \) and \( v \) are fixed. Therefore we cannot use it to derive Nathanson's symmetry in the case \( u \neq v \). In order to show the desired relationship between \((u,v)\)- and \((v,u)\)-Calkin-Wilf trees, we will use a lemma that shows the following:

![Figure 10. Relating \( T^{(v,u)}(z) \) and \( T^{(u,v)}(z^{-1}) \).](image)

**Lemma 4.** Let \( \sigma : \mathbb{Q}^* \to \mathbb{Q}^* \) be defined by \( \sigma(x) = x^{-1} \). Then

(a) \( \sigma \circ L_u \circ \sigma = R_u \)

(b) \( \sigma \circ R_u \circ \sigma = L_u \)

**Proof.** Part (a) of the lemma follows from the following straightforward computation:

\[
(\sigma \circ L_u \circ \sigma)(x) = \sigma \left( \frac{x^{-1}}{ux^{-1} + 1} \right) = \sigma \left( \frac{1}{ux} \right) = x + u = R_u(x).
\]

Part (b) follows from (a) since \( \sigma^2 = id \) \( \square \)

Comparing Figures 6 and 11 we can see that if we view \( c_z^{(u,v)}(n,i) \) as the result of a (unique) word \( w(L_u, R_v) \) on two letters acting on \( z \), then \( c_z^{(v,u)}(n, 2^n + 1 - i) = w(R_u, L_v) \). Specifically, the vertex \( c_z^{(u,v)}(n,i) \) is \( w(L_u, R_v)(z) \) where \( w \) is the \( i \)th word of length \( n \) on the letters \( R_u \) and \( L_v \) in the reverse lexicographic order. We will use this approach to prove Nathanson’s symmetry in its general form.
Second Proof of Theorem 2. Let $c_z^{(u,v)}(n, i) = w(L_u, R_v)(z)$ where $w$ is the $i$th word of length $n$ on the letters $R_v$ and $L_v$ in the reverse lexicographic order. For $\sigma(z) = z^{-1}$, it follows from Lemma 4 that

$$\sigma \circ w(L_u, R_v) \circ \sigma = w(\sigma \circ L_u \circ \sigma, \sigma \circ R_v \circ \sigma) = w(R_u, L_v).$$

Therefore $\sigma \circ w(L_u, R_v) = w(R_u, L_v) \circ \sigma$, which means that $(c_z^{(u,v)}(n, i))^{-1} = c_z^{(v,u)}(n, 2^n + 1 - i)$. □

5. The Descendant Conditions and the Depth Formula

In a full binary tree, each vertex can be assigned a binary representation by enumerating the vertices in a breadth-first order. For example, the root of the tree is assigned the number 1; its left child is 2 and right child is 3, or $10_2$ and $11_2$ in their respective binary representations. In the next row, the vertices are 4, 5, 6, 7, or $100_2$, $101_2$, $110_2$, $111_2$, in binary representation form (See Figure 12).

The parent-child relation is clearly demonstrated by the binary representation. Each left child is represented by the binary representation of its parent followed by a 0, while each right child is represented by the binary representation of its parent followed by a 1. Moreover, for each vertex, its binary representation encodes the binary representations of all of its ancestors back to the root.
We construct a 1-1 correspondence between the binary representations of the vertices in a full-binary tree and the words associated with each vertex in the Calkin-Wilf tree. Begin with the binary representation of a vertex. Truncate the leftmost 1 digit (all such representations begin with a 1), reverse the order of the string and map 0 → \( L_1 \) and 1 → \( R_1 \). For example, the vertex in position 1100_2 corresponds to the word \( L_2^3 R_1 \), which corresponds to the number \( 2/5 = L_2^3 R_1(1) \) in the Calkin-Wilf tree.

In the \((u, v)\)-Calkin-Wilf tree, if we use the same binary representation as those in the original Calkin-Wilf tree, we can easily see that the left child is represented by the binary representation of its parent followed by \( u \) consecutive 0s and the right child is represented by the binary representation of its parent followed by \( v \) consecutive 1s. Let \( B \) be the binary representation of the position of \( w \) in the original Calkin-Wilf tree. Figure 13 shows the first three rows of \( T^{(2,3)}(w) \) in binary form.

\[
\begin{array}{c}
B \\
B00_2 \quad B11_2 \\
B0000_2 \quad B0011_2 \quad B1100_2 \quad B111111_2
\end{array}
\]

**Figure 13.** Binary representation tree for \( T^{(2,3)}(w) \).

The \((u, v)\)-ancestor-descendant relation is clearly demonstrated by the sequence of \( u \) consecutive 0s or \( v \) consecutive 1s. We give a few examples related to \((2, 3)\)-Calkin-Wilf trees:

- We have that \( 2/5 \mapsto 1100_2 \), which is the left child of \( 11_2 \mapsto 2 \). Incidentally, 2 is an orphan root in the \((2, 3)\)-Calkin-Wilf forest.
- The rational number corresponding to \( 11001110000_2 \) in the Calkin-Wilf tree is a descendant of the orphan root \( 110_2 \). One can trace from the right, a sequence of four 0s, three 1s, two 0s, and then it offers neither two consecutive 0s nor three consecutive 1s.
- The rational number corresponding to the position \( 110001110001_2 \) in the Calkin-Wilf tree is an orphan in the \((2, 3)\)-Calkin-Wilf forest.

The following result formalizes the above criterion for an element to be an orphan or a child of a \((u, v)\)-Calkin-Wilf tree.

**Proposition 5.** Let \( w \) be a vertex of a \((u, v)\)-Calkin-Wilf tree, and \( B(w) \) be the binary representation of its corresponding position in the original Calkin-Wilf tree.

(a) Suppose that \( B(w) = B_1 0 \ldots 0_2 \), i.e., the binary representation \( B(w) \) ends in exactly \( i \) 0s. If \( i \geq u \), then \( w \) is the left child of the vertex whose position is \( B_1 0 \ldots 0_{i-u} \). Otherwise, \( w \) is an orphan.
Suppose that \( B(w) = B_0 1 \ldots 1 \), i.e., the binary representation \( B(w) \) ends in exactly \( j \) 1s. If \( j \ge v \), then \( w \) is the right child of the vertex whose position is \( B_0 1 \ldots 1 \). Otherwise, \( w \) is an orphan.

Another viewpoint for understanding the relationship between descendants in a \((u, v)\)-Calkin-Wilf tree is via continued fractions. We begin the study of the relationship between continued fractions and \((u, v)\)-Calkin-Wilf trees with the following useful lemma (see [2] for the case \( u = v = 1 \)).

**Lemma 5** (Continued fraction relationship). Let \( \frac{a}{b} \) be a positive rational number with continued fraction representation \( \frac{a}{b} = [q_0, q_1, \ldots, q_r] \). It follows that

(a) if \( q_0 = 0 \), then \( \frac{a}{ua+b} = [0, u+q_1, \ldots, q_r] \);
(b) if \( q_0 \neq 0 \), then \( \frac{a}{ua+b} = [0, u, q_0, q_1, \ldots, q_r] \);
(c) and \( \frac{a+vb}{b} = [v + q_0, q_1, \ldots, q_r] \).

**Proof.** Let

\[
\frac{a}{b} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots + \cfrac{1}{q_r}}}.
\]

Note that \( \frac{a}{ua+b} = (u + \frac{b}{a})^{-1} \), so

\[
\frac{a}{ua+b} = \cfrac{1}{u + \cfrac{1}{q_0 + \cfrac{1}{q_1 + \cdots + \cfrac{1}{q_r}}}.
\]

By considering the cases when \( q_0 = 0 \) and \( q_0 \neq 0 \), we get (a) and (b). The remaining case follows from the fact that \( \frac{a+vb}{b} = \frac{a}{b} + v \). \( \square \)

Lemma 5 shows that the continued fraction representations of rationals appearing in a \((u, v)\)-Calkin-Wilf tree follow a nice pattern. In fact, in the case where \( u = v = 1 \), we can recover several of the properties of the original Calkin-Wilf tree listed in Section 1.

The next theorem gives more insight into the properties of coefficients in the continued fraction representation of rational numbers appearing in a \((u, v)\)-Calkin-Wilf tree.

**Theorem 3** (Descendant conditions). Suppose that \( w \) and \( w' \) are positive rational numbers with continued fraction representations \( w = [q_0, q_1, \ldots, q_r] \)
and \( w' = [p_0, p_1, \ldots, p_s] \). Then \( w' \) is a descendant of \( w \) in the \((u,v)\)-Calkin-Wilf tree with root \( w \) if and only if the following conditions all hold:

(a) \( s \geq r \) and \( 2 \mid (s-r) \);
(b) for \( 0 \leq j \leq s-r-1 \), \( v \mid p_j \) when \( j \) is even and \( u \mid p_j \) when \( j \) is odd;
(c) for \( 2 \leq i \leq r \), \( p_{s-r+i} = q_i \);
(d) and
   (i) if \( q_0 \neq 0 \), then \( p_{s-r} \geq q_0 \), \( v \mid (p_{s-r} - q_0) \) and \( p_{s-r+1} = q_1 \);
   (ii) otherwise, if \( q_0 = 0 \), then \( v \mid p_{s-r}, p_{s-r+1} \geq q_1, \) and \( u \mid (p_{s-r+1} - q_1) \).

Proof. (\( \Rightarrow \)) We prove the first direction by induction. Note that (a) holds by Lemma 5, so our main concern will involve the remaining conditions.

Let \( A_n \) be the set of descendants of \( w \) of depth \( n \). Then \( A_1 \) consists of both children of \( w \). If \( w' \) is the left child of \( w \) and \( q_0 = 0 \), then, by Lemma 6, \( w' \) has a continued fraction representation \( w' = [0, u + q_1, \ldots, q_r] \). In this case, \( s = r \), so (b) is vacuously true and (c) immediately holds. (Note that (c) is also vacuously true if \( r = 1 \).) Since \( s = r = 0 \), it follows that \( p_{s-r} = p_0 = q_0 \), which implies that \( v \mid (p_{s-r} - q_0) \). Also, it is clear that \( p_{s-r+1} \geq q_1 \) and \( u \mid (p_{s-r+1} - q_1) \) since \( p_{s-r+1} = u + q_1 \). This shows that part (ii) of condition (d) holds. The two remaining cases, where \( w' \) is a left child of \( w \) with \( q_0 \neq 0 \) and where \( w' \) is a right child of \( w \), can be handled in a similar way using Lemma 5. This shows that the theorem holds for \( A_1 \).

Now suppose that the desired result holds for \( A_k \) for some \( k \geq 1 \) and assume that \( w' \in A_{k+1} \). Furthermore, assume that \( w' \) is the left child of some \( w'' \in A_k \), where \( w'' \) has a continued fraction representation \( w'' = [d_0, d_1, \ldots, d_t] \). By Lemma 5 if \( d_0 = 0 \), then \( s = t \) and \( w' = [0, u + d_1, \ldots, d_t] \). Since \( p_k = d_k \) for \( 0 \leq k \leq t \) with \( k \neq 1 \), then, with the exception of one coefficient, the result holds. For the case \( k = 1 \), notice that if \( t > r \), then \( u \mid d_1, \) so \( u \mid (u + d_1) \). If \( t = r \), then \( u + d_1 - q_1 > d_1 - q_1 \geq 0 \) and \( u \mid (d_1 - q_1) \), so \( u \mid (u + d_1 - q_1) \). This implies the desired result.

As was the case with \( A_1 \), there are two remaining cases to handle. The proofs of the statement when \( d_0 \neq 0 \) and when \( w' \) is the right child of some \( w'' \in A_k \) are both similar to the argument presented above. We omit the details.

(\( \Leftarrow \)) Using Lemma 5, a simple computation shows that when \( q_0 \neq 0 \),

\[
(12) \quad w' = P_{-r}^{p_0/u} L_u^{p_1/u} \cdots R_{-r}^{p_{s-r-2}/u} L_u^{p_{s-r-1}/u} R_{-r}^{p_{s-r+1}/u}(w).
\]

A similar formula gives the desired result when \( q_0 = 0 \). \( \square \)
Corollary 3 (Depth formula). Using the same hypothesis as Theorem 3, if \( n \) is the depth of \( w' \), then

\[
(13) \quad n = \frac{1}{v} \left( \sum_{0 \leq j \leq s-r-1 \text{ even}} p_j + \sum_{0 \leq i \leq r \text{ even}} (p_{s-r+i} - q_i) \right) + \frac{1}{u} \left( \sum_{0 \leq j \leq s-r-1 \text{ odd}} p_j + \sum_{0 \leq i \leq r \text{ odd}} (p_{s-r+i} - q_i) \right).
\]

The proof of Corollary 3 follows from Theorem 3 by induction. Note that the majority of the terms in the sum (13) are actually zero. In the case where \( u = v = 1 \), Corollary 3 recovers the formula from Property 4.

From Lemma 5 and Theorem 3, we can construct a recursive algorithm that determines the orphan ancestor of \( w' \) in the \((u, v)\)-Calkin-Wilf that contains it. The algorithm makes heavy use of the continued fraction representation of \( w' \).

**Algorithm 1** \((u, v)\)-Calkin-Wilf tree orphan ancestor

1: procedure ANCESTOR([\( p_0, p_1, \ldots, p_s \]), u, v)
2: \hspace{1em} if \( s = 0 \) then
3: \hspace{2em} if \( p_0 \leq v \) then return [\( p_0 \)]
4: \hspace{2em} else return ANCESTOR([\( p_0 - v \)], u, v)
5: \hspace{1em} else if \( s = 1 \) then
6: \hspace{2em} if \( 0 < p_0 < v \) then return [\( p_0, p_1 \)]
7: \hspace{2em} else if \( p_0 > v \) then
8: \hspace{2em} \hspace{1em} return ANCESTOR([\( p_0 - v, p_1 \)], u, v)
9: \hspace{2em} else if \( p_0 = 0 \) and \( p_1 \leq u \) then return [\( 0, p_1 \)]
10: \hspace{2em} else return ANCESTOR([\( 0, p_1 - u \)], u, v)
11: \hspace{1em} else
12: \hspace{2em} if \( p_0 < v \) then return [\( p_0, p_1, \ldots, p_s \)]
13: \hspace{2em} else if \( p_0 \geq v \) then
14: \hspace{2em} \hspace{1em} return ANCESTOR([\( p_0 - v, p_1, \ldots, p_s \)], u, v)
15: \hspace{2em} else if \( p_0 = 0 \) and \( 0 < p_1 < u \) then
16: \hspace{2em} \hspace{1em} return ANCESTOR([\( 0, p_1, \ldots, p_s \)], u, v)
17: \hspace{2em} else if \( p_0 = 0 \) and \( p_1 > u \) then
18: \hspace{2em} \hspace{1em} return ANCESTOR([\( 0, p_1 - u, \ldots, p_s \)], u, v)
19: \hspace{2em} else return ANCESTOR([\( p_2, \ldots, p_s \)], u, v)

For example, let \( u = 2 \) and \( v = 3 \). The continued fraction representation of \( 2147/620 \) is given by \([3, 2, 6, 4, 5, 2]\). Using the above algorithm, we can compute the list of ancestors of \( 2147/620 \) as: \( 287/620 = [0, 2, 6, 4, 5, 2] \), \( 287/46 = [6, 4, 5, 2] \), \( 149/46 = [3, 4, 5, 2] \), \( 11/46 = [0, 4, 5, 2] \), \( 11/24 = [0, 2, 5, 2] \), \( 11/2 = [5, 2] \), and \( 5/2 = [2, 2] \). Since \( 1/2 \leq 5/2 \leq 3 \), then \( 5/2 \) is the orphan ancestor of \( 2147/620 \).
By [12], we see that the coefficients of the continued fraction of 2147/620 encode the path taken from the orphan 5/2 to the descendant 2147/620. This can be computed as follows. Consider the continued fraction representation \[ 3, 2, 6, 4, 5, 2 \] as a row vector. Extend the continued fraction representation of 5/2 to a row vector of the same length by adding zeros at the front, \[ 0, 0, 0, 0, 2, 2 \]. Take the difference between both vectors, \[ 3, 2, 6, 4, 5, 2 \]. Divide the even-indexed (note that the leading term is indexed by 0) terms by 3 and the odd-indexed terms by 2, \[ 1, 3, 2, 2, 1, 0 \]. Corollary 3 states that the sum of the terms in this vector gives the depth of 2147/620. The terms also show that 2147/620 = \( R_u L_u R_v^2 L_v^2 R_v(5/2) \). In particular, since
\[
R_u L_u R_v^2 L_v^2 R_v = \begin{bmatrix} 187 & 606 \\ 54 & 175 \end{bmatrix},
\]
then \( a = 187 \), \( b = 606 \), \( c = 54 \), and \( d = 175 \) in Lemma 3 for this case.

When \( u = v = 1 \), the above discussion shows that every positive rational number appears in the original Calkin-Wilf tree (see [4]).

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