Entangling power of holonomic gates in atom-based systems

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Received 21 September 2018, revised 29 September 2018
Accepted for publication 11 October 2018
Published 31 October 2018

Abstract
Entanglement is one of the main resources of quantum computation, and entangling power of a quantum system is a crucial element in the universality and efficiency of a proposed architecture for realization of quantum processing. Our goal here is to study the entangling power of holonomic gates in some particular systems. We explore the holonomy-induced entanglement, by means of nonadiabatic quantum holonomies, through different types of interactions in atom-based systems, namely, the tripod-type interaction induced by the quantum Zeno effect between three-level atoms, as well as the \( \Lambda \)-type interaction arising from dipole–dipole or van der Waals forces between high-lying states of two-level atoms in systems consisting of \( N \) optically trapped identical atoms. Our analysis shows that although the two schemes provide completely separate classes of entangling gates, both schemes permit for full entangling power and also in the sense of quantum efficiency both families of entanglers consist of holonomic gates that have the same efficiency in quantum algorithms. Besides, we observe that holonomy-induced entanglement characteristics remarkably depend on the interaction configuration of the system.

Keywords: entangling power, holonomic gates, quantum computation, holonomy-induced entanglement

(Some figures may appear in colour only in the online journal)
1. Introduction

Environment induced noise and decoherence is a great bane in realizing quantum computers. There are diverse theoretical proposals developed for physical implementations that try to avoid noise completely or at least protect against the effect of noise. Decoherence free subspaces [1–3], dynamical decoupling [4, 5], noiseless subsystems [6, 7], topological [8, 9] and holonomic [10–13] approaches, as well as quantum error-correction methods [14, 15] are among these proposals.

As one of the key approaches in achieving fault-tolerant quantum computation, holonomic quantum computation, i.e. the idea to use Abelian or non-Abelian geometric phases to implement quantum gates, has caught a great deal of interest in recent years. Holonomic quantum computation was initially introduced in the adiabatic regime [10, 16–20] and subsequently developed for nonadiabatic processes [11–13, 21–24], the latter being compatible with the short coherence time of quantum bits (qubits). Nonadiabatic holonomic gates have been experimentally implemented in various physical settings, such as NMR [25, 26], superconducting transmon [27–29], and NV centers in diamond [30–34]. Further feasible schemes have been established for nonadiabatic holonomic quantum processing with spin qubit systems [35], pseudo-spin charge qubits [36], Rydberg atoms [37], and NV center qubits interacting via cavities [38]. Moreover, nonadiabatic holonomic quantum computation has been combined with decoherence free subspaces [23, 39–45], noiseless subsystems [46], and dynamical decoupling [47].

Nonetheless, the design of externally controlled multipartite settings for holonomic quantum processing that can entangle qubits is one of the main challenges from a practical perspective. In fact, the entangling capability of quantum operations (gates) has always been one of the main concerns in quantum information science. Entangling ability of a gate can be quantified in terms of its entangling power [48]. This allows us to classifying two-qubit gates in terms of their entangling characteristics, namely, the local invariants [49, 50], as two-qubit entanglers play a fundamental role in realizing universal quantum information processing.

In this paper, we aim to establish a comparative study between nonadiabatic holonomic gates in two atom-based setups upon their entangling characteristics quantified by geometric local invariances and the entangling power. The latter is here taken to be a linear entropy based measure of the capability of quantum gate operations to create entanglement, being analogous to the quantification of entanglement in a given state by measures such as entanglement of formation [51]. First, we consider a system consisting of identical three-level atoms trapped along the symmetry axis of an optical cavity, where the quantum Zeno effect allows tripod-type interaction configuration between selected atom pairs. The second system is a set of identical two-level atoms trapped optically on a lattice, where dipole–dipole or van der Waals force can be used to introduce \(\Lambda\)-type interaction configuration between a selected pair of atoms. After summarizing how the two setups lead to nonadiabatic holonomic two-qubit gates, we turn to the key issue of this work, namely, to explore the entangling characteristics of the holonomic gates and discuss the results. We show that the two interaction configurations give rise to completely separate families of entanglers indicating that the holonomy-induced entanglement characteristics depend on the interaction configuration of the system, both systems allowing for full entangling power, and that the two-level system provides wider classes of perfect entanglers than the three-level system. In addition, our analyses reveal that both systems permit nonadiabatic holonomic special perfect entanglers, which have the same efficiency in quantum algorithms.
2. Quantum holonomy-induced entanglement

2.1. Three-level atomic system

As our first system, we consider the atom-cavity model developed by Beige et al [52]. We demonstrate that this model system, under certain experimentally feasible conditions, can be used to realize holonomic two-qubit gates and examine the entangling power of these gates.

The setup in [52] consists of $N$ identical atoms arranged in a line and trapped at fixed positions along the symmetry axis of an optical cavity so that each atom can be addressed individually. Each atom exhibits a three-level $\Lambda$-type structure, with the atomic ground states $|0\rangle$ and $|1\rangle$ coupled to an excited state $|e\rangle$ (see figure 1). The ground state levels $|0\rangle$ and $|1\rangle$ span a qubit state space. The atomic transition $|1\rangle \leftrightarrow |e\rangle$ is assumed to be in resonance with the field mode in the cavity. For simplicity, the atom-cavity coupling constant is taken to be the same for all atoms. With no photon in the cavity mode, one finds for any two atoms that the computational states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ and the maximally entangled trapped state $|\alpha\rangle = (|1e\rangle - |e1\rangle)/\sqrt{2}$ span a decoherence-free subspace (DFS) with respect to cavity emission.

In order to generate entanglement between a selected pair $i = 1, 2$ of atoms, a mechanism to manipulate the states inside the above DFS by means of quantum holonomies can be used. For this, resonant laser pulses address each atom individually. The Rabi frequencies of the laser pulses inducing $|0^{(i)}\rangle \leftrightarrow |e^{(i)}\rangle$ and $|1^{(i)}\rangle \leftrightarrow |e^{(i)}\rangle$ transitions in atom $i = 1, 2$, are taken to be $\Omega_{0}^{(i)}$ and $\Omega_{1}^{(i)}$, respectively. These frequencies are generally complex-valued. In the environment-induced quantum Zeno regime, where states outside the DFS are rapidly suppressed due to strong coupling to the environment as described in [52, 53], the system can be kept inside the DFS during the evolution of the system. The effective Hamiltonian in the DFS takes the following tripod configuration form (see figure 2)

$$H_{\text{eff}} = \mathbf{P} \left( \sum_{i=1}^{2} \sum_{j=0}^{1} \Omega_{j}^{(i)} |j^{(i)}\rangle\langle e^{(i)}| + \text{H.c.} \right) \mathbf{P}$$

$$= \frac{1}{\sqrt{2}} \left( - \Omega_{0}^{(1)} |01\rangle \langle \alpha| + \Omega_{0}^{(2)} |10\rangle \langle \alpha| + (\Omega_{1}^{(2)} - \Omega_{1}^{(1)}) |11\rangle \langle \alpha| + \text{H.c.} \right),$$

where we put $\hbar = 1$ from now on. $\mathbf{P}$ is the projection operator on the DFS and we have used the short-hand notation $|j^{(i)}k^{(i)}\rangle \equiv |jk\rangle$, $j, k = 0, 1$.

Solving the eigenvalue problem of $H_{\text{eff}}$, we obtain energy eigenstates

$$|D_{1}\rangle = e^{-i\phi_{1}} \cos \theta |01\rangle + e^{-i\phi_{1}} \sin \theta |10\rangle,$$

$$|D_{2}\rangle = e^{-i\phi_{1}} \cos \phi |x\rangle - \sin \phi |11\rangle,$$

$$|B_{\pm}\rangle = \frac{|y\rangle \pm |\alpha\rangle}{\sqrt{2}},$$

(2)
and corresponding eigenenergies $E_{D_1} = E_{D_2} = 0$, and $E_{B \pm} = \pm \frac{\omega}{\sqrt{2}}$. Here,

$$\omega = \sqrt{|\Omega_0^{(1)}|^2 + |\Omega_0^{(2)}|^2 + |\Omega_1^{(2)} - \Omega_1^{(1)}|^2},$$

$$\Omega_0^{(1)} = \omega e^{i\phi_0} \sin \varphi \cos \theta,$$

$$\Omega_0^{(2)} = \omega e^{i\phi_0} \sin \varphi \sin \theta,$$

$$\Omega_1^{(2)} - \Omega_1^{(1)} = \omega e^{i\phi_1} \cos \varphi,$$

$$|x\rangle = e^{i\phi_0} \sin \theta \ |10\rangle - e^{i\phi_0} \cos \theta \ |01\rangle,$$

$$|y\rangle = \sin \varphi \ |x\rangle + e^{i\phi_1} \cos \varphi \ |11\rangle.$$

(3)
The Rabi frequencies are kept constant for the duration \([0, \tau]\) of the laser pulses, resulting in the time evolution operator on the DFS

\[
U(\tau, 0) = e^{-i \int_0^\tau H_{\text{ad}} dt} = |D_1\rangle \langle D_1| + |D_2\rangle \langle D_2|
\]

\[
+ \cos a_\tau (|\gamma\rangle \langle \gamma| + |\alpha\rangle \langle \alpha|)
\]

\[
- i \sin a_\tau (|\gamma\rangle \langle \alpha| + |\alpha\rangle \langle \gamma|),
\]

where \(a_\tau = \frac{\omega \tau}{4}\) is the effective pulse area.

By choosing the run time \(\tau = \sqrt{2\pi/\omega}\) such that \(a_\tau = \pi\), the three-dimensional subspace \(\text{Span}\{|01\rangle, |10\rangle, |11\rangle\}\) undergoes cyclic evolution in the four-dimensional subspace \(\text{Span}\{|01\rangle, |10\rangle, |11\rangle, |\alpha\rangle\}\) of the DFS, while the remaining two-qubit state \(|00\rangle\) is fully decoupled. Along this evolution, we have

\[
U(t, 0)P_c U^\dagger(t, 0)H_{\text{eff}} U(t, 0)P_c U^\dagger(t, 0) = 0
\]

for the projection \(P_c = |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11|\). In other words, the three-dimensional subspace \(\text{Span}\{|01\rangle, |10\rangle, |11\rangle\}\) evolves along a loop \(C\) in the Grassmannian \(G(4, 3)\), i.e. the space of three-dimensional subspaces of the four-dimensional subspace \(\text{Span}\{|01\rangle, |10\rangle, |11\rangle, |\alpha\rangle\}\) of the DFS, along which the non-Abelian dynamical phase \([54]\) is trivial. It follows that

\[
U(C) = P_c U(t, 0)P_c
\]

is the nonadiabatic quantum holonomy of the loop \(C\) in the Grassmannian \(G(4, 3)\).

It is instructive to compare the underlying geometrical structure of the present scheme with that of the tripod-based architecture for adiabatic holonomic manipulation proposed in \([17]\). While the relevant loop in the present scheme resides in \(G(4, 3)\), the loop corresponding to each holonomic gate in \([17]\) resides in the Grassmannian \(G(3, 2)\), since the computational states are encoded in the two dark energy eigenstates states evolving adiabatically in the three-dimensional space \(\text{Span}\{|01\rangle, |10\rangle, |11\rangle\}\).

Since the computational basis state \(|00\rangle\) does not contribute to the dynamics described by the effective Hamiltonian in equation (1), it remains unchanged during the evolution. Therefore, the evolution results in the following two-qubit nonadiabatic holonomic gate

\[
U = |00\rangle \langle 00| + U(C),
\]

which takes the following form in the ordered computational basis \(\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}\)

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - 2 \sin^2 \phi \cos^2 \theta & e^{-i\phi_1} \sin 2\theta \sin^2 \phi & e^{-i\phi_1} \sin 2\phi \cos \theta \\
0 & e^{i\phi_1} \sin 2\theta \sin^2 \phi & 1 - 2 \sin^2 \phi \sin^2 \theta & -e^{-i\phi_1} \sin 2\phi \sin \theta \\
0 & e^{i\phi_1} \sin 2\phi \cos \theta & -e^{i\phi_1} \sin 2\phi \sin \theta & -\cos 2\phi
\end{pmatrix},
\]

where \(\phi_k = \phi_l - \phi_k, l, k = 1, 2, 3\).

An important feature of the above two-qubit gate \(U\) is that it provides holonomic gates with arbitrarily large entangling power. To see this, let us look at some entangling characteristics of \(U\). Evaluating the local invariances \([49]\), one obtains

\[
G_1 = -\sin^8 \phi \sin^4 2\theta,
\]

\[
G_2 = \cos 2\phi + 2 \sin^2 \phi \left(\cos^2 \phi + \cos 4\theta \sin^2 \phi\right),
\]

which consequently results in the entangling power \([48, 55]\)
$$e_p(U) = \frac{2}{9} (1 - |G_1|) = \frac{2}{9} (1 - \sin^8 \varphi \sin^4 2\theta).$$  \hspace{1cm} (10)$$

As shown in figure 3, a careful tuning of the laser pulses can provide any entangling power.

From equations (9) and (10), one may note that the entangling nature of the gate $U$ does not in general depend on the complex nature of the Rabi frequencies $\Omega^{(i)}$. Extracting the corresponding symmetry reduced coordinates $(c_1, c_2, c_3)$ of $U$ on the Weyl chamber [50], which classifies non-local two-qubit gates, we find

$$\langle c_1, c_2, c_3 \rangle = \left( \frac{\pi}{2}, c, c \right), \hspace{1cm} (11)$$

where

$$c = \arcsin \left( 2 \left| \frac{\Omega^{(1)}_0 \Omega^{(2)}_0}{\omega^2} \right| \right).$$  \hspace{1cm} (12)$$

This demonstrates that the holonomic gate $U$ covers the whole set of equivalence classes of two-qubit gates along the line segment connecting the equivalence class of special perfect entanglers [CNOT]$^4$, represented by the coordinates $(\frac{\pi}{2}, 0, 0)$, to the class of local gates represented by the coordinates $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ on the Weyl chamber [50]. Moreover, if the lasers are tuned so that

$$\left| \frac{\Omega^{(1)}_0 \Omega^{(2)}_0}{\omega^2} \right| = \sqrt{\frac{\omega^2}{2}},$$  \hspace{1cm} (13)$$

then the holonomic gate $U$ belongs to the equivalence class of perfect entanglers corresponding to the point $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ on the Weyl chamber. The entangling nature in fact depends on the absolute frequency ratios $\left| \frac{\Omega^{(2)}_0}{\Omega^{(1)}_0} \right|$ and $\left| \frac{\Omega^{(2)}_1 - \Omega^{(1)}_1}{\omega} \right|$. The gate $U$ tends to the equivalence class [CNOT] of special perfect entanglers with maximum entangling power of $\frac{\omega^2}{2}$, when $\left| \frac{\Omega^{(2)}_0}{\Omega^{(1)}_0} \right| \to 0, \infty$ or $\left| \frac{\Omega^{(2)}_1 - \Omega^{(1)}_1}{\omega} \right| \to 0$. Table 1 specifies some frequencies to achieve different classes of entangling gates.

Note that the tripod configuration in figure 2 reduces to a two-level interaction system for the three first upper cases in table 1. Therefore, in these cases, the loop $C$ would effectively reside in the Grassmannian $G(2, 1)$. Its corresponding nonadiabatic quantum holonomy would be Abelian and coincide with the Aharonov-Anandan geometric phase [56]. However, in the next three cases listed in the table, the tripod structure reduces to a three-level $\Lambda$ structure giving rise to non-Abelian quantum holonomies associated with loops in the Grassmannian $G(3, 2)$. In other words, table 1 shows that perfect entanglement can be induced by both Abelian and non-Abelian quantum holonomies in the above proposed interaction picture.

One may notice that the approach in [52] is a special case of row four of table 1. The present work, in other words, is an extension of the proposal in [52], introducing a wider class of entangling gates with more freedom in the choice of frequencies. Our analysis shows that nonadiabatic holonomies as realized in the atom-cavity system have full entangling power.

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$^4$The notation $[V]$ stands for the equivalence class of gates containing the gate $V$, i.e. the set of all gates, which have the same corresponding symmetry reduced coordinates as $V$. 

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2.2. Two-level atomic system

As our second system we consider two-level atoms trapped optically on a lattice. Each atom can be addressed individually by a resonant pulse to control the transition between its stable ground state $|0\rangle$ and long-lived excited state $|1\rangle$ (see figure 4). To achieve a holonomy-induced entanglement between atom pairs, we further assume atom-atom interaction mediated by dipole–dipole or van der Waals force between high-lying excited states, as shown in figure 4.

The conditional Hamiltonian describing the dynamics of a selected atom-pair may be written as

$$H_{12} = H_1 \otimes I_2 + I_1 \otimes H_2 + V |11\rangle \langle 11|,$$

where

$$H_1 = \hbar \omega_1 |1\rangle \langle 1| + \hbar \omega_0 |0\rangle \langle 0| + \hbar g_1 [\sigma_1^+ |1\rangle \langle 0| + |0\rangle \langle 1| \sigma_1^-],$$

$$H_2 = \hbar \omega_2 |1\rangle \langle 1| + \hbar \omega_0 |0\rangle \langle 0| + \hbar g_2 [\sigma_2^+ |1\rangle \langle 0| + |0\rangle \langle 1| \sigma_2^-],$$

and

$$V = \frac{1}{2} \hbar g_{12} [\sigma_1^+ \otimes \sigma_2^- + \sigma_1^- \otimes \sigma_2^+]$$

where $\sigma_i^\pm = \sigma_i^x \pm i \sigma_i^y$, $\sigma_i^x$, and $\sigma_i^y$ are the Pauli matrices for atom $i$. The resulting two-body Hamiltonian $H_{12}$ is the one responsible for the entanglement process.

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**Figure 3.** Entangling power, $e_p(U)$, of the two-qubit nonadiabatic holonomic gate $U$ as a function of the control parameters $\theta$ and $\varphi$ in a single period.

**Table 1.** Entanglement characteristics of the holonomic two-qubit entangling gate $U$ for some specific frequencies.

| #  | $(\Omega_0^{(1)}, \Omega_0^{(2)})$ | $\Omega_1^{(2)} - \Omega_1^{(1)}$ | $G_1$ | $G_2$ | $e_p(U)$ | Weyl chamber coordinates |
|----|-----------------------------------|-----------------------------------|-------|-------|----------|--------------------------|
| 1  | $(0, 0, \neq 0)$                  | 0                                 | 1     | 2/9    | $\pi/2, 0, 0$          | [CNOT]                   |
| 2  | $(0, \neq 0, 0)$                  | 0                                 | 1     | 2/9    | $\pi/2, 0, 0$          | [CNOT]                   |
| 3  | $(\neq 0, 0, 0)$                  | 0                                 | 1     | 2/9    | $\pi/2, 0, 0$          | [CNOT]                   |
| 4  | $(0, \neq 0, \neq 0)$            | 0                                 | 1     | 2/9    | $\pi/2, 0, 0$          | [CNOT]                   |
| 5  | $(\neq 0, 0, \neq 0)$            | 0                                 | 1     | 2/9    | $\pi/2, 0, 0$          | [CNOT]                   |
| 6  | $(\neq 0, \neq 0, 0)$            | $-\sin^4 2\varphi$               | $2\cos 4\theta - 1$ | $(2/9)(1 - \sin^4 2\theta)$ | $\pi/2, 0, 0$          | $0 < \theta \leq \frac{\pi}{2}$ |
| 7  | $(\neq 0, \neq 0, \neq 0)$       | $-\sin^4 \varphi$                | $1 - 4\sin^4 \varphi$ | $(2/9)(1 - \sin^8 \varphi)$ | $\pi/2, 0, 0$          | $0 < c = \arcsin(\sin^2 \varphi) \leq \pi/2$ |
\[ H_i = \Upsilon_i |0\rangle \langle 1| + \text{H.c.}, \ i = 1, 2, \tag{15} \]

are the single-atom Hamiltonians describing the atom interacting with laser pulse of Rabi frequency \( \Upsilon_i \). \( I_i \) denotes the identity operator on atom \( i \) and the third term in equation (14) describes the atom-atom interaction with strength \( V \).

Note that the type of interacting Hamiltonian in equation (14) has been considered previously in the study of Rydberg atoms as promising candidates for realization of quantum computing in general [57, 58] and holonomic quantum computation in particular [37].

By letting \( \Upsilon = \sqrt{|\Upsilon_1|^2 + |\Upsilon_2|^2} \), we may rewrite the Hamiltonian in equation (14) as

\[
H_{12} = \Upsilon \left( |B_0\rangle \langle 00| + |B_1\rangle \langle 11| + \text{H.c.} \right) \\
+ V |11\rangle \langle 11|, \tag{16}
\]

where

\[
\Upsilon_1 = e^{-i\phi_1} \sin \theta, \quad \Upsilon_2 = e^{-i\phi_2} \cos \theta, \\
|B_0\rangle = e^{i\phi_2} \cos \frac{\theta}{2} |01\rangle + e^{i\phi_1} \sin \frac{\theta}{2} |10\rangle, \\
|B_1\rangle = e^{-i\phi_1} \sin \frac{\theta}{2} |01\rangle + e^{-i\phi_2} \cos \frac{\theta}{2} |10\rangle. \tag{17}
\]

In the rotating frame associated with the transformation \( \mathcal{U} = e^{i\Omega |11\rangle \langle 11|} \), the two-atom Hamiltonian reads

\[
H_{12}^{\text{rot}} = \Upsilon \left( |B_0\rangle \langle 00| + |B_1\rangle \langle 11| e^{-i\Omega} + \text{H.c.} \right). \tag{18}
\]
In the case where $V \gg \Upsilon$, the off-resonant coupling terms to $|B_1\rangle$ can be neglected giving rise to the effective two-atom $\Lambda$-type Hamiltonian (see figure 5)

$$H_{12}^{\text{eff}} = \left( \Upsilon |B_0\rangle \langle 00| + \text{H.c.} \right)$$

$$= \left( \Upsilon_2 |00\rangle \langle 01| + \Upsilon_1 |00\rangle \langle 10| + \text{H.c.} \right).$$

(19)

By keeping the parameters $\theta$, $\phi_1$, and $\phi_2$ constant on $[0, \tau]$, one obtains the time evolution operator

$$U(\tau, 0) = e^{-i \int_0^\tau H_{12}^{\text{eff}} dt} = |11\rangle \langle 11| + |D_0\rangle \langle D_0|$$

$$+ \cos \tilde{\alpha}_\tau \left( |00\rangle \langle 00| + |B_0\rangle \langle B_0| \right)$$

$$- i \sin \tilde{\alpha}_\tau \left( |00\rangle \langle B_0| + |B_0\rangle \langle 00| \right),$$

(20)

where $\tilde{\alpha}_\tau = \int_0^\tau \Upsilon du$ and $|D_0\rangle = e^{i \phi_2} \sin \frac{\theta}{2} |01\rangle - e^{i \phi_1} \cos \frac{\theta}{2} |10\rangle$ is the dark state of the atom-pair system. $S_2 = \text{Span}\{|01\rangle, |10\rangle\} = \text{Span}\{|B_0\rangle, |D_0\rangle\}$ is a decoherence-free subspace with respect to collective dephasing [2, 3]. The time evolution operator in equation (20) follows from the fact that the states $|11\rangle$ and $|D_0\rangle$ do not contribute to the effective Hamiltonian in equation (19).

While the state $|11\rangle$ is kept unchanged during the evolution, equation (20) implies that if the operation time $\tau$ and the collective coupling strength $\Upsilon$ are chosen so that $\tilde{\alpha}_\tau = \pi$, then the subspaces $S_1 = \text{Span}\{|00\rangle\}$ and $S_2$ trace out loops $C_1$ and $C_2$ which reside in the Grassmannians $G(3, 1)$ and $G(3, 2)$, respectively. Moreover, along these loops we have

$$U(t, 0)P_1 U^\dagger(t, 0) H_{12}^{\text{eff}} U(t, 0) P_1 U^\dagger(t, 0) = 0,$$

$$U(t, 0)P_2 U^\dagger(t, 0) H_{12}^{\text{eff}} U(t, 0) P_2 U^\dagger(t, 0) = 0$$

(21)

for the projections $P_1 = |00\rangle \langle 00|$ and $P_2 = |01\rangle \langle 01| + |10\rangle \langle 10|$. This means that the subspaces $S_1$ and $S_2$ evolve without acquiring dynamical phases [54]. In other words,

Figure 5. The $\Lambda$ configuration resulting from strongly interacting pairs of two-level atoms exposed individually by resonant pulses of Rabi frequencies $\Upsilon_1$ and $\Upsilon_2$.

5 The condition $V \gg \Upsilon$ puts constraints on the strength of the atom transitions induced by the pulses. Since $V$ rapidly decreases with distance, the realization of the two-qubit gate for distant atoms would therefore require very long pulses. In other words, although the gate may in principle be implemented for any atom pair in the lattice, the scheme is likely to be effective in practice only for nearby atoms.
Figure 6. Entangling power $e_p(\tilde{U})$ of the two-qubit nonadiabatic holonomic gate $\tilde{U}$ as a function of the control parameter $\theta$ in a single period.

Figure 7. Tetrahedral $(OA_1A_2A_3)$ representation of non-local two-qubit operations, known as the Weyl chamber. The points $L$, $M$, $N$, $P$ and $Q$, respectively, are the midpoints of the line segments $A_1O, A_1A_2, A_1A_3, A_3O$, and $A_2O$ with $A_1 = (\pi, 0, 0)$, $A_2 = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$, $A_3 = (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$. Every point in the Weyl chamber corresponds to an equivalence class of two-qubit operations. The polyhedron $LMNPQA_2$ corresponds to perfect entanglers in the Weyl chamber. Line $LA_3$ identifies special perfect entanglers. The two-qubit gate $U$ in equation (8) belongs to the line segment $LA_3$ and the two-qubit gate $\tilde{U}$ in equation (24) covers the edge $OA_2$, respectively, illustrated in blue and red. $U$ is the special perfect entangler CNOT represented by the point $L$ on the Weyl chamber if $c = \arcsin \left( 2 \left| \Omega^{(1)}_0 \Omega^{(2)}_0 \right| \omega^{-2} \right) = 0$ and is a perfect entangler only when $LA_3$ cross $NP$, i.e. $c = \arcsin \left( 2 \left| \Omega^{(1)}_0 \Omega^{(2)}_0 \right| \omega^{-2} \right) = \frac{\pi}{4}$. However the gate $\tilde{U}$ covers a wider set of classes of perfect entanglers lying along the line segment $OA_2$, which here corresponds to $\frac{\pi}{4} \leq \theta = 2 \arctan \left| \frac{A_1}{A_2} \right| \leq \frac{\pi}{2}$. In particular, for $\theta = \pi/2$ the $\tilde{U}$ represents the point $A_2$ on the Weyl chamber, which is the DCNOT local equivalence class of special perfect entanglers.
$$U(C_i) = P_i U(\tau, 0) P_i, \quad i = 1, 2,$$

is the nonadiabatic quantum holonomy of the loop $C_i$ in the Grassmannian $G(3, i)$ [54]. Therefore, the evolution results in the following two-qubit nonadiabatic holonomic gate

$$\tilde{U} = U(C_1) \oplus U(C_2) \oplus 1,$$

which takes the matrix form

$$\tilde{U} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -\cos \theta & -e^{i\phi_1} \sin \theta & 0 \\
0 & -e^{-i\phi_1} \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

in the ordered computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

We may now examine the entangling characteristics of $\tilde{U}$ by extracting the local invariances [49]

$$G_1 = \cos^4 \theta$$

$$G_2 = 1 + 2 \cos 2\theta$$

yielding the entangling power [48, 55]

$$e_p(\tilde{U}) = \frac{2}{9} (1 - |G_1|) = \frac{2}{9} (1 - \cos^4 \theta),$$

as well as the corresponding symmetry reduced coordinates on the Weyl chamber [50]

$$(c_1, c_2, c_3) = (\theta, \theta, 0).$$

As illustrated in figure 6, the holonomic gate $\tilde{U}$ can achieve any entangling power by appropriate choice of laser pulses. The entangling nature of the holonomic gate $\tilde{U}$ is independent of the complex nature of the Rabi frequencies $\Upsilon_1$ and $\Upsilon_2$. Moreover, $\tilde{U}$ covers the whole set of equivalence classes of two-qubit gates along the line segment connecting the equivalence class of local gates, represented by the coordinates $(0, 0, 0)$, to the class of special perfect entangler [DCNOT], which is the equivalence class containing the double-CNOT gate [59], represented by the coordinates $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ on the Weyl chamber. In fact, the entangling nature of the $\tilde{U}$ depends on the absolute value of the ratio of the two frequencies, i.e.

$$\theta = 2 \arctan \left| \frac{\Upsilon_1}{\Upsilon_2} \right|.$$

This shows that the closer the strength of two lasers are, the stronger entangling power would be, while the entangling power is reduced when the strength of one the lasers is considerably larger than the strength of the other. For any $\frac{\pi}{4} \leq \theta = 2 \arctan \left| \frac{\Upsilon_1}{\Upsilon_2} \right| \leq \frac{3\pi}{4}$, the gate $\tilde{U}$ represents a perfect entangler. Note that $\frac{\pi}{4} \leq \theta = 2 \arctan \left| \frac{\Upsilon_1}{\Upsilon_2} \right| \leq \frac{3\pi}{4}$ produces a family of entanglers with the same entangling characteristics as the family of entanglers produced by $\frac{\pi}{4} \leq \theta = 2 \arctan \left| \frac{\Upsilon_1}{\Upsilon_2} \right| \leq \frac{5\pi}{4}$.

2.3. Discussion

Concerning the holonomy-induced entangling power of the two systems discussed above, one may notice that although both systems allow for full entangling power, they produce
separate classes of two-qubit entangling gates, as depicted in figure 7. This in a way implies that depending on the interaction configuration of the system, quantum holonomy produces entanglement of different characteristics.

Both systems are able to produce special perfect entanglers, i.e. CNOT and DCNOT, which are different but possess the same efficiency in quantum algorithms [60, 61]. While the three-level system produces only a single class of perfect entanglers, the two-level system permits implementation of infinitely many different classes of perfect entanglers (see figure 7).

3. Conclusions

In conclusion, we have analyzed entangling characteristics of nonadiabatic holonomies as realized in two different types of interaction configurations. We considered the tripod interaction configuration resulting from the quantum Zeno effect in the study of a chain of identical three-level atoms trapped in an optical cavity, and the $\Lambda$ interaction configuration established by dipole–dipole or van der Waals force between identical two-level atoms trapped optically on a lattice. In both cases, we have achieved arbitrary holonomy-induced entangling power through the nonadiabatic holonomic approach. However, the two interaction configurations provide completely separate classes of two-qubit entangling holonomic gates. Although different, both families of entanglers consist of perfect entanglers, special perfect entanglers as well as entanglers with the same efficiency in quantum algorithms. Moreover, we observe that the $\Lambda$ interaction configuration for the two-level atoms system allows for a wider set of classes of perfect entanglers than the tripod interaction configuration for the three-level atoms system.

Acknowledgments

This work was supported by the Department of Mathematics at the University of Isfahan (Iran). VAM acknowledges financial support from the Iran National Science Foundation (INSF) through Grant No. 96008297. ES acknowledges financial support from the Swedish Research Council (VR) through Grant No. 2017-03832.

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