A note on the lattice Dirac-Kähler equation

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Abstract

A lattice version of the Dirac-Kähler equation (DKE) describing fermions was discussed in articles by Becher and Joos. The decomposition of lattice Dirac-Kähler fields (inhomogeneous cochains) to lattice Dirac fields remained as an open problem. I show that it is possible to extract Dirac fields from the DKE and discuss the resulting lattice Dirac equation.

1 Introduction

In 1962 E. Kähler proposed an alternative to the Dirac equation describing fermionic fields [3]. In his framework the fermionic fields are inhomogeneous differential forms

\[ \phi = \phi^0(x) + \phi^i_1(x) dx^i + \phi^i_2(x) dx^i dx^j + \phi^i_3(x) dx^i dx^j dx^k + \phi^i_4(x) dx^i dx^j dx^k dx^l \]  

obeying the Dirac-Kähler equation (DKE)

\[ (d - \delta + m)\phi = 0. \]  

Here \( d \) is the ordinary exterior derivative and \( \delta \) is the coderivative \( \delta = -*^{-1}d* \) (where * is the Hodge-operator).

Similar to the ordinary Dirac-operator acting on spinors the operator \( d - \delta \) is a root of the Laplace (or Laplace-Beltrami) operator acting on inhomogeneous differential forms.

\[ (d - \delta)^2 = -d\delta - \delta d = \partial_i \partial^i. \]  

Here and in the following I use the summation convention on space-time indices. I work in a four-dimensional Euclidean space-time, but changing the signature to Minkowski and going to other space-time dimensions is unproblematic.

By introducing a suitable basis of the space of inhomogeneous differential forms the DKE can be seen to be equivalent to four degenerate Dirac equations [1].

Becher and Joos use a correspondence between the continuum and the lattice which has its origin in algebraic topology (see for example Nakahara [4]) to discretize the DKE on a hypercubic lattice. In this correspondence differential forms are related to cochains (functions on the points, links, plaquettes, cubes and hypercubes of the lattice). Thus the lattice Dirac-Kähler field is an inhomogeneous cochain. The lattice DKE reads

\[ (\tilde{\Delta} - \tilde{\nabla} + m)\phi = 0 \]  

where \( \tilde{\Delta} \) denotes the dual boundary and \( \tilde{\nabla} \) the dual coboundary operator. The lattice DKE shares the property of the continuum DKE of being a root of the (lattice) Klein-Gordon equation. The procedure to read off Dirac components and a Dirac equation from
the DKE does not extend directly to the lattice. Finding a lattice version of this procedure remained an open problem (see the corresponding remark in [2]). The reduction presented in [1] works only in momentum space.

The inhomogeneous cochains $\phi$ can be expanded in the cochains $d^{x,H}$

$$\phi = \sum_{x,H} \varphi(x, H) d^{x,H},$$

where the $H$ are sets $\{i_1 \ldots i_p\}$ with $p = 1, \ldots, 4$ and $x$ is a point of the lattice. The index sets $(x, H)$ label the points, links, plaquettes, cubes and hypercubes of the lattice. The $d^{x,H}$ are antisymmetric in their indices $d^{x,i_1\ldots i_k+1\ldots i_p} = -d^{x,i_1\ldots i_k i_{k+1}\ldots i_p}$. The subset of the $d^{x,H}$ where $H$ is an ordered set is a basis of the space of cochains.

We can sum up the $d^{x,H}$ over the lattice to get constant cochains $d^H$.

$$d^H = \sum_x d^{x,H}$$

From these the original $d^{x,H}$ can be recovered by

$$d^{x,H} = \chi(x, H)d^H$$

where $\chi$ is the characteristic function for a lattice with spacing $a$

$$\chi(x, H)(y) = \begin{cases} 1/a^p & \text{if } y \in (x, H) \\ 0 & \text{otherwise.} \end{cases}$$

A general cochain $\phi$ can then be written as

$$\phi = \sum_{x,H} \varphi(x, H) d^{x,H} = \sum_{x,H} \varphi(x, H)\chi(x, H)d^H = \sum_H \left( \sum_x \varphi(x, H)\chi(x, H) \right) d^H = \sum_H \varphi_H(x) d^H.$$  

The dual boundary and coboundary operator act on a p-cochain

$$\omega = \frac{1}{p!} \sum_{i_1\ldots i_p} \omega_{i_1\ldots i_p}(x) d^{i_1\ldots i_p}$$

via

$$\Delta \omega = \frac{1}{p!} \sum_{j_1\ldots j_p} \partial_{+j} \omega_{j_1\ldots j_p}(x) d^{j_1\ldots j_p} \quad \text{and} \quad \nabla \omega = \frac{1}{(p-1)!} \sum_{j_2\ldots j_p} \partial_{-j} \omega_{j_2\ldots j_p}(x) d^{j_2\ldots j_p}$$

where $\partial_{+i}$ and $\partial_{-i}$ are the lattice (nearest neighbor) derivatives

$$\partial_{+i}f(x) = \frac{1}{a}(f(x + a^i) - f(x)) \quad \partial_{-i}f(x) = \frac{1}{a}(f(x) - f(x - a^i))$$

with $(x + a^i)^j = x^j + \delta^{ij}a$. 

2
2 Dirac-components for the lattice DKE

To read off Dirac-components from the lattice DKE I have to introduce an algebra of matrices \( C^i \), \( i = 1, \ldots, 4 \), replacing the Dirac-\( \gamma \)-matrices on the lattice. The \( C^i \) are a representation of the anti-commutation relations
\[
\{ C^i , C^j \} = 0 \quad \{ C^i , C^{j\dagger} \} = \delta^{ij}.
\]
The lattice Dirac equation involving the \( C^i \) reads
\[
(C^i \partial_{+i} + C^{i\dagger} \partial_{-i} + m I) \psi = 0.
\]
The usual lattice Klein-Gordon equation \([6]\) is obtained by multiplying \((14)\) with \( C^i \partial_{+i} + C^{i\dagger} \partial_{-i} - m \) just as in the continuum \([1]\). An equivalent version of the lattice Dirac equation was first discussed by Becher \([5]\). He shows that this equation does not lead to the well known fermion doubling problem from additional zeros in the inverse propagator \([6]\).

A drawback of this treatment of fermions on the lattice appears if we try to find a representation of the algebra \((13)\). The eight combinations
\[
\xi^i = C^i + C^{i\dagger}, \quad \xi^{4+i} = i \left( C^{i\dagger} - C^i \right)
\]
fulfill the defining relations of an 8 generator Clifford algebra
\[
\{ \xi^i , \xi^j \} = 2\delta^{ij}.
\]
Therefore the smallest representation of the \( C^i \) is 16-dimensional \([7]\). For any representation \( \gamma^i \) of the Dirac algebra the matrices \( \xi^i = I \otimes \gamma^i \), \( \xi^{4+i} = \gamma^i \otimes \gamma^5 \) are a representation of \((16)\). In this representation it is particularly easy to see what happens in the continuum limit. For the lattice derivatives we have \( \partial_{+i} \longrightarrow \partial_i \) and \( \partial_{-i} \longrightarrow \partial_i \). For the Dirac operator this leads to
\[
(C^i \partial_{+i} + C^{i\dagger} \partial_{-i} + m) \longrightarrow \xi^i \partial_i + m.
\]
Obviously our lattice Dirac equation is equivalent to four independent Dirac equations in the continuum limit. Thus the fermion doubling that was absent in the spectrum partially comes back with the size of the representation. This seems strange if we are interested in discretizing the ordinary Dirac equation, but it is natural if we think of the Dirac-Kähler equation, because as mentioned above this equation describes four independent fermions in the first place.

To read off Dirac components from the lattice DKE a suitable basis of the space of inhomogeneous cochains has to be introduced. To do so I notice that the algebra \((13)\) is the algebra of fermionic creation and annihilation operators. Therefore a vector \( \Omega \) with \( C^{i\dagger} \Omega = 0 \) for all \( C^{i\dagger} \) exists. With the help of this let us define
\[
\theta = \Omega^\dagger \left( I + d^i C^{i\dagger} + \frac{1}{2!} d^{ij} C^{j\dagger} C^{i\dagger} + \frac{1}{3!} d^{ijk} C^{k\dagger} C^{j\dagger} C^{i\dagger} + \frac{1}{4!} d^{ijkl} C^{l\dagger} C^{k\dagger} C^{j\dagger} C^{i\dagger} \right).
\]
This equation is to be read as a 16-dimensional vector equation. To prove that the components of \( \theta \) are a basis of the space of inhomogeneous differential forms I give the inverse of this formula,
\[
d^H = \theta C^H \Omega \]

\(^1\)This is not the case for the discretizations of the Dirac equation in which the \( \gamma^i \) appear.
with \( C^H = C^{i_1} \cdots C^{i_p} \) for \( H = \{ i_1 \ldots i_p \} \). (Here I assume that \( \Omega \) is normalized \( \Omega^\dag \Omega = 1 \). )

A solution of the lattice DKE can be expanded in terms of this basis (in vector notation)

\[
\phi = \theta \cdot \varphi(x) \tag{20}
\]

A somewhat lengthy calculation reveals that DKE in the basis \( \theta \) reads

\[
(d - \delta + m) \theta \cdot \varphi(x) = \theta \cdot (C^i \partial_i + C^{i_\dagger} \partial_{-i} + m) \varphi(x) \tag{21}
\]

So we see that the components \( \varphi \) are Dirac components for the lattice DKE. They obey Becher’s version of the lattice Dirac equation. In the continuum limit we get four independent Dirac equations.

### 3 Summary

I have demonstrated that it is possible to read off Dirac components from the lattice DKE. In the continuum case the equation for the 16 Dirac components of the Dirac-Kähler field can be reduced to 4 independent 4-component Dirac equations. This is not possible on the lattice. For nonzero lattice spacing we get one irreducible 16 component Dirac equation. Only in the continuum limit the continuum result is reproduced and we get four independent equations.

The resulting lattice Dirac equation is discussed by Becher in his paper [5]. Here I only want to point out that there is no further fermion-doubling from the spectrum of the discrete Dirac operator and that there is a discrete \( \gamma^5 \) invariance (for the question of chiral symmetry see also [8]).

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