ON THE PROBABILISTIC WELL-POSEDNESS OF THE TWO-DIMENSIONAL PERIODIC NONLINEAR SCHRÖDINGER EQUATION WITH THE QUADRATIC NONLINEARITY $|u|^2$

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ABSTRACT. We study the two-dimensional periodic nonlinear Schrödinger equation (NLS) with the quadratic nonlinearity $|u|^2$. In particular, we study the quadratic NLS with random initial data distributed according to a fractional derivative (of order $\alpha \geq 0$) of the Gaussian free field. After removing the singularity at the zeroth frequency, we prove that the quadratic NLS is almost surely locally well-posed for $\alpha < \frac{1}{2}$ and is probabilistically ill-posed for $\alpha \geq \frac{3}{4}$ in a suitable sense. The probabilistic ill-posedness result shows that in the case of rough random initial data and a quadratic nonlinearity, the standard probabilistic well-posedness theory for NLS breaks down before reaching the critical value $\alpha = 1$ predicted by the scaling analysis due to Deng, Nahmod, and Yu (2019), and thus this paper is a continuation of the work by Oh and Okamoto (2021) on stochastic nonlinear wave and heat equations by building an analogue for NLS.

RÉSUMÉ. Nous étudions l’équation de Schrödinger non linéaire (NLS) avec la non-linéarité quadratique $|u|^2$ sur un tore de dimension deux. En particulier, nous étudions NLS quadratique avec une donnée initiale aléatoire distribuée selon une dérivée fractionnaire (d’ordre $\alpha \geq 0$) du champ libre gaussien. Après suppression de la singularité à la fréquence zéro, nous prouvons que NLS quadratique est presque sûrement localement bien posé pour $\alpha < \frac{1}{2}$ et est mal posé pour $\alpha \geq \frac{3}{4}$ dans un sens probabiliste approprié. Le fait que NLS quadratique soit mal posé dans un sens probabiliste montre que dans le cas de données initiales aléatoires à basse regularité et d’une non-linéarité quadratique, la théorie de Cauchy probabiliste standard pour NLS perd sa validité avant d’atteindre le valeur critique $\alpha = 1$ prédite par l’analyse due à Deng, Nahmod et Yue (2019). Cet article est donc une continuation, dans le cas de NLS, des travaux de Oh et Okamoto (2021) sur les équations stochastiques non linéaires des ondes et du chaleur.

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1. Introduction

1.1. Quadratic NLS with random initial data. We consider the Cauchy problem for the following quadratic nonlinear Schrödinger equation (NLS) on the two-dimensional torus $T^2 = (\mathbb{R}/2\pi \mathbb{Z})^2$:

$$
\begin{cases}
  i\partial_t u + \Delta u = |u|^2 - f|u|^2 \\
u|_{t=0} = u_0^\omega.
\end{cases}
$$

(1.1)

Here, $f f(x) dx := \frac{1}{(2\pi)^2} \int_{T^2} f(x) dx$ and $u_0^\omega$ is the following Gaussian random initial data:

$$
u_0^\omega(x) = \sum_{n \in \mathbb{Z}^2} g_n(\omega) \langle n \rangle^{-\alpha} e^{in \cdot x},
$$

(1.2)

where $\alpha \in \mathbb{R}$ and $\{g_n\}_{n \in \mathbb{Z}^2}$ is a set of independent standard complex-valued Gaussian random variables with $Eg_0 = 0$ and $E|g_n|^2 = 1$. Note that when $\alpha = 0$, $u_0^\omega$ is the Gaussian random initial data distributed according to the massive Gaussian free field on $H^s(T^2)$, $s < 0$.

Over the past several decades, we have witnessed tremendous progress on well-posedness issues of NLS with various types of nonlinearities from both deterministic and probabilistic points of views. Let us first briefly mention the deterministic well-posedness results for NLS on periodic domains. In [2], Bourgain introduced the Fourier restriction norm method (see Subsection 2.2) and proved NLS with a gauge-invariant nonlinearity in the low regularity setting. In particular, he showed local well-posedness of the cubic NLS (i.e. with nonlinearity $|u|^2 u$) in $H^s(T^2)$ for any $s > 0$ by proving the following $L^4$-Strichartz estimate on $T^2$ (See also Lemma 2.2):

$$
\|e^{it\Delta} u\|_{L^4([-1,1];L^4(T^2))} \lesssim \|u\|_{H^s(T^2)},
$$

(1.3)

for any $s > 0$. We now focus on the following quadratic NLS:

$$
i\partial_t u + \Delta u = |u|^2.
$$

(1.4)

For (1.4), one can easily obtain local well-posedness in $H^s(T^2)$ for $s > 0$ by using the $L^3$-Strichartz estimate with a derivative loss, which follows from interpolating (1.3) and the trivial $L^2$ bound. In [23], Kishimoto proved ill-posedness of (1.4) in $H^s(T^2)$ for $s < 0$. In a recent preprint [25], the author and Oh proved local well-posedness of (1.4) in $H^0(T^2) = L^2(T^2)$, thus completing the deterministic well-posedness theory of (1.4).

We now turn our attention to NLS with rough random initial data. The idea of constructing local-in-time solutions of NLS using random initial data was first introduced by Bourgain in [4], where he proved almost sure local well-posedness of the (renormalized) cubic NLS on $T^2$ with random initial data (1.2) with $\alpha = 0$. See also [5, 9, 12, 13, 15]
for more results on almost sure local well-posedness of NLS with various types of nonlinearities on periodic domains with random initial data of the form (1.2). The almost sure local well-posedness results of NLS with a quadratic nonlinearity, to the best of the author’s knowledge, have not been explored yet. In this paper, we choose to work with the quadratic NLS (1.1) (see Remark 1.2 below for the necessity of removing the mean of the nonlinearity). Note that the initial data $u_0^\omega$ almost surely belongs to $H^{-\alpha-\varepsilon}(T^2) \setminus H^{-\alpha}(T^2)$ for any $\varepsilon > 0$. See Lemma B.1 in [8]. When $\alpha < 0$, the initial data $u_0^\omega$ almost surely belongs to $H^s(T^2)$ for sufficiently small $s = s(\alpha) > 0$, so that we can easily prove almost sure local well-posedness of (1.1) by using the $L^3$-Strichartz estimate mentioned above.

Our goal in this paper is to (i) obtain probabilistic local well-posedness of (1.1) with $\alpha \geq 0$ and (ii) identify bad behaviors of (1.1) when $\alpha$ gets too large. Specifically, we show that (1.1) is almost surely locally well-posed when $0 \leq \alpha < \frac{1}{2}$ (see Subsection 1.2 below) and is probabilistically ill-posed in a suitable sense when $\alpha \geq \frac{3}{4}$ (see Subsection 1.3 below).

1.2. Almost sure local well-posedness of the quadratic NLS. In this subsection, we state our almost sure local well-posedness theorem for the quadratic NLS (1.1) and describe our strategy for proving our result. We define

$$z(t) := z^\omega(t) = e^{it\Delta}u_0^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{-\varepsilon|n|^2 + in \cdot x}$$

as the solution to the linear Schrödinger equation with the random initial data $u_0^\omega$:

$$\begin{cases}
i \partial_t z + \Delta z = 0 \\ z|_{t=0} = u_0^\omega.
\end{cases}$$

The precise statement of our almost sure local well-posedness result reads as follows.

**Theorem 1.1.** Let $0 \leq \alpha < \frac{1}{2}$ and $s > 0$. Then, the quadratic NLS (1.1) is almost surely locally well-posed in the class $z + C([-T, T]; H^s(T^2))$. More precisely, there exist $T_0 > 0$ and constants $C, c, \theta > 0$ such that for all $0 < T \leq T_0$, there exists a set $\Omega_T \subset \Omega$ with the following properties:

1. $P(\Omega \setminus \Omega_T) \leq C \exp(-\frac{c}{T^\theta})$.
2. For each $\omega \in \Omega_T$, there exists a unique solution $u = u^\omega$ to (1.1) on $[-T, T]$ with $u|_{t=0} = u_0^\omega$ in the class $z + C([-T, T]; H^s(T^2))$.

We prove Theorem 1.1 by using the following first order expansion [26, 4, 10]:

$$u = z + v,$$

where the residual term $v$ satisfies the following equation:

$$\begin{cases}
i \partial_t v + \Delta v = |z + v|^2 - f |z + v|^2 \\ v|_{t=0} = 0.
\end{cases}$$

The uniqueness statement in Theorem 1.1 refers to the uniqueness of $v$ as a solution to this perturbed quadratic NLS (1.7) in an appropriate space (see the $X_T^{s,b}$-spaces in Subsection 2.2).

The well-posedness result of the perturbed quadratic NLS (1.7) follows from the corresponding bilinear estimates to the quadratic terms $|v|^2$, $v\overline{z}$, $z\overline{v}$, and $|z|^2$. To prove these bilinear estimates, we use the operator norm approach based on the random tensor theory.
Remark 1.2. Let us consider the following quadratic NLS:

\[
\begin{align*}
  i \partial_t u + \Delta u &= |u|^2 \\
  u|_{t=0} &= u_0^\omega,
\end{align*}
\]  

(1.8)

where \( u_0^\omega \) is the Gaussian random initial data as defined in (1.2). For \( N \in \mathbb{N} \), we let \( u_{0,N}^\omega \) be the truncation of \( u_0^\omega \) as defined in (1.5) to frequencies \( \{|n| \leq N\} \), and we define \( z_N(t) := e^{it\Delta} u_{0,N}^\omega \). One can easily check that the zeroth frequency of the following Picard second iterate

\[
\int_0^t e^{i(t-t')\Delta} (|z_N(t')|^2) dt'
\]

diverges almost surely when \( \alpha \geq 0 \) (see, for example, Subsection 4.4 in [29]). Thus, in order to make the almost sure local well-posedness problem non-trivial, we need to remove this singular behavior of (1.8) occurring at the zeroth frequency.

A more natural way of dealing with the above issue is to introduce the following renormalized quadratic NLS:

\[
\begin{align*}
  i \partial_t u_N + \Delta u_N &= |u_N|^2 - \sigma_N \\
  u_N|_{t=0} &= u_{0,N}^\omega,
\end{align*}
\]  

(1.9)

where \( \sigma_N = \mathbb{E}[|u_{0,N}^\omega|^2] \). However, due to the lack of the conservation of mass \( \int |u|^2 \), there seems to be no easy way to establish the equivalence between (1.9) and the quadratic NLS (1.1). One can compare this situation with the cubic NLS on \( T^2 \) in [4], where Bourgain used a Gauge transform \( u_N = \exp(2i(\int |u_N|^2 - \sigma_N) t) \cdot v_N \) to show the equivalence between

\[
  i \partial_t u_N + \Delta u_N = |u_N|^2 u_N - 2\sigma_N u_N
\]

and

\[
  i \partial_t v_N + \Delta v_N = \left( |v_N|^2 - 2 \int |v_N|^2 \right) v_N.
\]

Here in the case of the cubic NLS, the quantity \( \int |u_N|^2 - \sigma_N \) is time invariant and one can easily recover \( u_N \) from \( v_N \) by noticing that \( \int |u_N|^2 = \int |v_N|^2 \). Nevertheless, similar transforms do not seem to apply to the case of the renormalized quadratic NLS (1.9).

One of the problem with directly proceeding with (1.9) is that, by using the first order expansion \( u_N = z_N + v_N \) with \( z_N = e^{it\Delta} u_{0,N}^\omega \) and letting \( \mathcal{I} \) be the Duhamel operator, the zeroth frequencies of the bilinear terms \( \mathcal{I}(v_N z_N) \), \( \mathcal{I}(z_N v_N) \), and \( \mathcal{I}(z_N z_N) \) cannot be shown to converge when \( \alpha \geq 0 \) using our approach. Another problem is that the remainder term \( v_N \) is not necessarily of mean zero, which causes a trouble in estimating the bilinear term \( \mathcal{I}(z_N v_N) \) when \( \alpha \geq 0 \). See Proposition 3.2 and Remark 3.3 for more details. Therefore, in this paper, we choose to focus on the quadratic NLS (1.1) (i.e. with nonlinearity \( |u|^2 - \int |u|^2 \)).
Remark 1.3. Let \( \eta \in C(\mathbb{R}^2;[0,1]) \) be a mollification kernel such that \( \int \eta \, dx = 1 \) and \( \text{supp} \eta \subset (-1,1]^2 \simeq \mathbb{T}^2 \). For \( 0 < \varepsilon \leq 1 \), we define \( \eta_{\varepsilon}(x) = \varepsilon^{-2} \eta(\varepsilon^{-1} x) \), so that \( \{\eta_{\varepsilon}\}_{0 < \varepsilon \leq 1} \) forms an approximate identity on \( \mathbb{T}^2 \). With a slight modification of the proof of Theorem 1.1 we can show that when \( \alpha < \frac{1}{2} \), the solution \( u_\varepsilon \) to
\[
\begin{align*}
 i\partial_t u_\varepsilon + \Delta u_\varepsilon &= |u_\varepsilon|^2 - f |u_\varepsilon|^2 \\
 u_\varepsilon|_{t=0} &= \eta_\varepsilon * u_0^\omega
\end{align*}
\]
converges in probability to some (unique) limiting distribution \( u \) in \( C([-T_\omega, T_\omega]; H^{-\alpha}(\mathbb{T}^2)) \) with \( T_\omega > 0 \) almost surely. Here, the limiting distribution \( u \) is independent of the choice of the mollification kernel \( \eta \).

Remark 1.4. Let us also consider probabilistic well-posedness of NLS with other quadratic nonlinearities:
\[
\begin{align*}
 i\partial_t u + \Delta u &= \mathcal{N}(u) \\
 u|_{t=0} &= u_0^\omega
\end{align*}
\]
with \( \mathcal{N}(u) = u^2 \) or \( \overline{u}^2 \) and \( u_0^\omega \) as defined in (1.2). We first point out that these nonlinearities have different corresponding phase functions: \( -2n \cdot n_2 \) for \( |u|^2 \), \( -2n_1 \cdot n_2 \) for \( u^2 \), and \( |n|^2 + |n_1|^2 + |n_2|^2 \) for \( \overline{u}^2 \). Here, \( n_1 \) corresponds to the frequency of the first incoming wave, \( n_2 \) corresponds to the frequency of the second incoming wave, and \( n \) corresponds to the frequency of the outgoing wave.

For \( \mathcal{N}(u) = u^2 \), we can use a similar argument as in the proof of Theorem 1.1 to obtain almost sure local well-posedness of (1.10) when \( \alpha < \frac{1}{2} \). We point out that in this case, we do not need to remove any singularities as compared to the case of \( \mathcal{N}(u) = |u|^2 \).

For \( \mathcal{N}(u) = \overline{u}^2 \), due to the different nature of the corresponding phase function, we expect that one can go beyond the range \( \alpha < \frac{1}{2} \) established for the almost sure local well-posedness for NLS with nonlinearities \( |u|^2 \) and \( u^2 \). However, the method for proving Theorem 1.1 based on the first order expansion is not enough for this purpose, since the corresponding bilinear estimate involving the product of two random linear solutions (Proposition 3.2 (iii)) is still only valid when \( \alpha < \frac{1}{2} \). In this case, it may be possible to establish almost sure local well-posedness for some range of \( \alpha \geq \frac{1}{4} \) using higher order expansions as in [11, 30].

1.3. Probabilistic ill-posedness of the quadratic NLS. In this subsection, we discuss probabilistic ill-posedness issues of the quadratic NLS (1.1) for large values of \( \alpha \). Given \( N \in \mathbb{N} \), consider the following Picard second iterate:
\[
z_N^{(2)}(t) = \int_0^t e^{i(t-t')\Delta} \left( |z_N(t')|^2 - \int |z_N(t')|^2 \right) dt',
\]
where
\[
z_N(t) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{|n|^{1-\alpha}} e^{-it|n|^2+i \cdot x}
\]
is the truncation of the random linear solution \( z \) as defined in (1.5) to frequencies \( \{|n| \leq N\} \).

We now state the following proposition regarding the non-convergence of every non-zero Fourier coefficient of the Picard second iterate \( z_N^{(2)} \).

\[1. \] This includes the paracontrolled approach used in [17, 6, 7]
Proposition 1.5. Let \( n \neq 0 \) and \( t \neq 0 \). For \( \alpha \geq \frac{3}{4} \), the sequence \( \{E[|F_{x}z_{N}^{(2)}(t,n)|^{2}]\}_{N \in \mathbb{N}} \) goes to infinity as \( N \to \infty \). Consequently, any subsequence of the sequence of random variables \( \{F_{x}z_{N}^{(2)}(t,n)\}_{N \in \mathbb{N}} \) is not tight.

See Section 5 for the proof of Proposition 1.5.

Proposition 1.5 implies that when \( \alpha \geq \frac{3}{4} \), for every \( n \neq 0 \), any subsequence of \( \{F_{x}z_{N}^{(2)}(t,n)\}_{N \in \mathbb{N}} \) does not converge in law. This in particular implies that standard methods for establishing almost sure local well-posedness such as the first order expansion [26, 4, 10] or its higher order variants [1, 30, 17, 6, 7] do not work for \( \alpha \geq \frac{3}{4} \).

Bearing in mind the above discussion, we now briefly discuss the probabilistic scaling and the associated critical regularity introduced by Deng, Nahmod, and Yue in [12]. The notion of this probabilistic scaling is based on the observation that, if one wants to obtain local well-posedness, the Picard second iterate should not be rougher than the random linear solution. In [12], Deng, Nahmod, and Yue provided heuristics for one to compute the probabilistic scaling critical regularity without too much difficulty, and they conjectured in the paper that for NLS with nonlinearities \( |u|^{p-1}u \) (\( p \in 2\mathbb{N} + 1 \)), almost sure local well-posedness should hold for all subcritical regularities. Indeed, in [13], Deng, Nahmod, and Yue proved almost sure local well-posedness for NLS with nonlinearity \( |u|^{p-1}u \) (\( p \in 2\mathbb{N} + 1 \)) on \( \mathbb{T}^d \) (\( d \in \mathbb{N} \)) in the full subcritical range relative to the probabilistic scaling. We point out that for NLS with the quadratic nonlinearity \( |u|^2 \), however, the probabilistic scaling does not seem to provide a useful prediction for probabilistic well-posedness issues, as we shall see in the following.

Let us compute the probabilistic scaling critical regularity for the quadratic NLS with nonlinearity \( |u|^2 \). Let \( u_{0}^{\omega} \) be the random initial data as defined in (1.2). Let \( N \in \mathbb{N} \) be a dyadic number and consider the initial data \( u_{0}^{\omega} \) supported on frequencies \( \{|n| \sim N\} \):

\[
P_{N}u_{0}^{\omega} = \sum_{n \in \mathbb{Z}^2, |n| \sim N} \frac{g_{n}(\omega)}{\langle n \rangle^{1-\alpha}} e^{in \cdot x}.
\]

Note that \( \|P_{N}u_{0}^{\omega}\|_{H^{-\alpha}(\mathbb{T}^2)} \sim 1 \). Consider the following Picard second iterate term:

\[
u_{N}^{(2)}(t) = \int_{0}^{t} e^{it-t'} \Delta \left(|e^{it' \Delta} P_{N}u_{0}^{\omega}|^{2}\right) dt',
\]

whose \( n \)th Fourier coefficient can be computed as

\[
F_{x}u_{N}^{(2)}(t,n) = \int_{0}^{t} e^{-it|n|^{2}} \sum_{\substack{n_{1}, n_{2} \in \mathbb{Z}^2 \\sim N \\ n_{1} - n_{2} = n \\sim N \\sim N}} e^{it'|n_{1}|^{2}-|n_{1}|^{2}+|n_{2}|^{2}} \frac{g_{n_{1}}(\omega)g_{n_{2}}(\omega)}{\langle n_{1} \rangle^{1-\alpha} \langle n_{2} \rangle^{1-\alpha}} dt'.
\]

We restrict our attention to the frequency range \( \{|n| \sim N\} \) of \( u_{N}^{(2)}(t) \). Thus, by the Wiener chaos estimate (Lemma 2.8 below along with Chebyshev's inequality) and a counting

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2. Here, we do not need to subtract the zeroth frequency of the nonlinearity since later on we only focus on the case when \( |n| \sim N \).
estimate (see Lemma 2.6(i) below), we can estimate the $H^{-\alpha}(\mathbb{T}^2)$-norm of $u_N^{(2)}(t)$ as follows:

$$\|u_N^{(2)}(t)\|_{H^{-\alpha}(\mathbb{T}^2)} \lesssim_t \sum_{n \in \mathbb{Z}^2} \langle n \rangle^{-2\alpha} \left( \sum_{n_1, n_2 \in \mathbb{Z}^2 \atop |n_1|^{-N}, |n_2|^{-N}} \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{\langle |n|^2 - |n_1|^2 + |n_2|^2 \rangle^{1-\alpha} \langle n_1 \rangle^{1-\alpha} \langle n_2 \rangle^{1-\alpha}} \right)^2$$

$$\lesssim C_\omega \sum_{n, n_1, n_2 \in \mathbb{Z}^2 \atop |n_1|^{-N}, |n_2|^{-N}} \frac{\langle n \rangle^{-2\alpha}}{\langle |n|^2 - |n_1|^2 + |n_2|^2 \rangle^{2-2\alpha}}$$

$$\sim C_\omega N^{2\alpha-4} \sum_{n, n_1, n_2 \in \mathbb{Z}^2 \atop |n_1|^{-N}, |n_2|^{-N}} \frac{1}{\langle |n|^2 - |n_1|^2 + |n_2|^2 \rangle^{2}}$$

$$\lesssim C_\omega N^{2\alpha - 2 + \varepsilon} \quad (1.12)$$

for some $0 < C_\omega < \infty$ almost surely and $\varepsilon > 0$ arbitrarily small. In order to have $\|u_N^{(2)}\|_{H^{-\alpha}(\mathbb{T}^2)} \lesssim 1$, we need $2\alpha - 2 + \varepsilon \leq 0$, which is equivalent to $\alpha < 1$.

The above computation shows that the probabilistic scaling critical regularity is $\alpha_* = 1$. Proposition 1.5, however, shows that every non-zero Fourier coefficient of the Picard second iterate $z_N^{(2)}(t)$ diverges when $\alpha \geq \frac{5}{4}$ and $t \neq 0$, which happens before $\alpha$ reaches the critical value $\alpha_* = 1$. This shows that the probabilistic scaling introduced in [12] fails in the case of the quadratic nonlinearity $|u|^2$. We point out that this discrepancy is mainly due to the fact that the probabilistic scaling only considers the special case when all frequencies have comparable sizes, which oversimplifies the situation in the context of a quadratic nonlinearity. Also, this discrepancy is closely related to the fact that we are considering very rough random initial data (rougher than the Gaussian free field initial data), which is in particular relevant in studying NLS with a polynomial nonlinearity of low degree and in low dimensions. See Remark 1.7 below. Similar phenomena also occur in the contexts of wave equations and stochastic parabolic equations. See Remark 1.9 and Remark 1.10 for further details.

We finish this subsection by stating several remarks.

**Remark 1.6.** We would like to point out that there is a gap ($\frac{1}{2} \leq \alpha < \frac{3}{4}$) between our almost sure local well-posedness and probabilistic ill-posedness of the quadratic NLS (1.1). We would like to address this issue in a forthcoming work.

If some well-posedness results of the quadratic NLS (1.1) can be achieved in the range $\frac{1}{2} \leq \alpha < \frac{3}{4}$, this will imply that NLS behaves better than the nonlinear wave equation (NLW) in the quadratic case, which will be interesting because usually NLW behaves at least as well as NLS. See Remark 1.9 below or [29] for well-posedness issues of NLW with a quadratic nonlinearity.

**Remark 1.7.** The proof of Proposition 1.5, the probabilistic ill-posedness result of the quadratic NLS (1.1), can easily be adapted to general dimensions. Specifically, on $\mathbb{T}^d$ for $d \in \mathbb{N}$, when $\alpha \geq \frac{5}{4} - \frac{d}{4}$ and $n \neq 0$, any subsequence of $\{F_{x_1} z_N^{(2)}(t, n)\}_{N \in \mathbb{N}}$ is not tight.

The probabilistic scaling for the quadratic NLS with nonlinearity $|u|^2$ can also be easily computed on general $\mathbb{T}^d$, on which the probabilistic scaling critical regularity is $\alpha_* = 2 - \frac{d}{2}$. 


We note that when \( d = 1, 2 \), every non-zero Fourier coefficient of the Picard second iterate diverges before \( \alpha \) reaches the critical value \( \alpha_* \).

**Remark 1.8.** We can also address ill-posedness issues of the quadratic NLS with nonlinearity \( u^2 \) or \( \overline{u}^2 \) with random initial data (1.2) using a similar computation as in the case of \( |u|^2 \). Specifically, on \( \mathbb{T}^d \), with either nonlinearity \( u^2 \) or nonlinearity \( \overline{u}^2 \), every Fourier coefficient of the Picard second iterate diverges (in the same sense of that in Proposition 1.5) when \( \alpha \geq 2 - \frac{d}{4} \). The reason for this different range of \( \alpha \) from that in the context of nonlinearity \( |u|^2 \) is mainly due to the different phase functions corresponding to these nonlinearities (i.e. \(-2n \cdot n_2 \) for \( |u|^2 \), \(-2n_1 \cdot n_2 \) for \( u^2 \), and \(|n|^2 + |n_1|^2 + |n_2|^2 \) for \( \overline{u}^2 \)).

We can also use a similar procedure as in (1.12) to compute the probabilistic scaling for the quadratic NLS with nonlinearity \( u^2 \) or \( \overline{u}^2 \), each of which has the same critical regularity \( \alpha_* = 2 - \frac{d}{2} \). It is interesting to note that in the context of nonlinearity \( u^2 \) or \( \overline{u}^2 \), the non-convergence of the Picard second iterate does not happen before \( \alpha \) reaches the critical regularity.

**Remark 1.9.** In [29], Oh and Okamoto studied well-posedness issues of the stochastic nonlinear wave equation (NLW) with a quadratic nonlinearity on \( \mathbb{T}^d \). Let us compare the situations for the quadratic NLS (1.1) and the following quadratic NLW on \( \mathbb{T}^2 \):

\[
\begin{aligned}
&\partial_t^2 u + (1 - \Delta)u = u^2 \\
&(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega),
\end{aligned}
\tag{1.13}
\]

where

\[
(u_0^\omega, u_1^\omega) = \left( \sum_{n \in \mathbb{Z}^2} g_{0,n}^{\omega}(\omega) e^{in \cdot x}, \sum_{n \in \mathbb{Z}^2} \langle n \rangle^\alpha g_{1,n}^{\omega}(\omega) e^{in \cdot x} \right).
\]

Here, \( \alpha \in \mathbb{R} \) and \( \{g_{0,n}, g_{1,n}\}_{n \in \mathbb{Z}^2} \) is a sequence of independent standard complex Gaussian random variables conditioned that \( g_{j,-n} = \overline{g_{j,n}} \), for all \( n \in \mathbb{Z}^2 \), \( j = 0, 1 \). We point out that the probabilistic well-posedness and ill-posedness results in [29] for the quadratic SNLW also apply to (1.13) (with the standard Wick renormalization): (1.13) is almost surely locally well-posed when \( \alpha < \frac{1}{2} \) and is probabilistically ill-posed in the sense that every Fourier coefficient of the Picard second iterate diverges almost surely when \( \alpha \geq \frac{1}{2} \).

We note that both the quadratic NLS (1.1) and the quadratic NLW (1.13) are almost surely locally well-posed when \( \alpha < \frac{1}{2} \). Regarding the probabilistic ill-posedness, for the quadratic NLS (1.1), every non-zero frequency diverges when \( \alpha \geq \frac{3}{4} \); whereas for the quadratic NLW (1.13), every frequency of the Picard second iterate diverges when \( \alpha \geq \frac{1}{2} \), which also happens before reaching the critical regularity \( \alpha_* = 1 \) of (1.13). See Proposition 1.6 in [29] for more details. The difference of the pathological behaviors of the two equations is mainly due to the different structures of the corresponding Duhamel operators.

**Remark 1.10.** Let us also mention some failures of scaling analysis that happen in the context of parabolic equations forced by rough noises. In the past decade, there has been a huge progress in the study of stochastically forced parabolic equations using the theory of regularity structures introduced by Hairer [18, 19, 20, 21]. In particular, the theory of regularity structures is able to solve a wide range of parabolic equations with a space-time white noise forcing that are subcritical according to the notion of local subcriticality introduced...
by Hairer [19]. However, when the stochastic forcing is rougher than the space-time white noise, the scaling analysis may fail to provide a prediction for well-posedness issues. For example, in [22], Hoshino showed that for the KPZ equation driven by a fractional derivative of a space-time white noise, the standard solution theory breaks down before reaching the critical regularity. See also [29] for a similar phenomenon that occurs in the context of the stochastic nonlinear heat equation forced by a fractional derivative of a space-time white noise.

1.4. Organization of the paper. This paper is organized as follows. In Section 2, we introduce some notations, definitions, and preliminary lemmas. In Section 3, we establish a well-posedness result of (1.1) when $\alpha < \frac{1}{2}$. In Section 4, we prove Theorem 1.1, the almost sure local well-posedness result of (1.1) when $\alpha < \frac{1}{2}$. In Section 5, we prove Theorem 1.3, the probabilistic ill-posedness result of (1.1) for $\alpha > \frac{2}{3}$.

2. Notations and preliminary lemmas

In this section, we discuss some relevant notations and lemmas.

2.1. Notations. For a space-time distribution $u$ defined on $\mathbb{R} \times \mathbb{T}^2$, we write $\mathcal{F}_x u$ to denote the space Fourier transform of $u$ and we write $\tilde{u}$ to denote the space-time Fourier transform of $u$. We also define the following twisted space-time Fourier transform:

$$\tilde{u}(\tau, k) = \tilde{u}(\tau - |k|^2, k).$$

Given a dyadic number $N \in 2^{\mathbb{Z}_0}$, we let $P_N$ be the frequency projector onto the spatial frequencies $\{n \in \mathbb{Z}^2 : \frac{N}{2} < \langle n \rangle \leq N\}$, where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. For any subset $Q \subset \mathbb{Z}^2$, we let $P_Q$ be the frequency projector onto $Q$. Also, we use $P_{\neq 0}$ to denote the restriction to non-zero frequencies.

Let $\chi$ be a smooth cut-off function such that $\chi \equiv 1$ on $[-1,1]$ and $\chi \equiv 0$ outside of $[-2,2]$.

We use $A \lesssim B$ to denote $A \leq CB$ for some constant $C > 0$, and we write $A \sim B$ if $A \lessapprox B$ and $B \lessapprox A$. Also, we write $A \ll B$ if $A \leq cB$ for some sufficiently small $c > 0$. In addition, we use $a+$ and $a-$ to denote $a + \varepsilon$ and $a - \varepsilon$, respectively, for sufficiently small $\varepsilon > 0$.

2.2. Fourier restriction norm method. In this subsection, we introduce definitions and lemmas of $X^{s,b}$-spaces, also called the Bourgain spaces, due to Klainerman-Machedon [24] and Bourgain [2]. Given $s, b \in \mathbb{R}$, we define the $X^{s,b} = X^{s,b}(\mathbb{R} \times \mathbb{T}^2)$ norm as

$$\|u\|_{X^{s,b}} := \|\langle n \rangle^s \langle \tau + |n|^2 \rangle^b \hat{u}(\tau, n)\|_{L^2_x L^1_t(\mathbb{R} \times \mathbb{Z}^2)}.$$

The space $X^{s,b}$ is then defined by the completion of functions that are $C^\infty$ in space and Schwartz in time with respect to this norm. For $T > 0$, we define the space $X_T^{s,b}$ by the restriction of distributions in $X^{s,b}$ onto the time interval $[-T, T]$ via the norm

$$\|u\|_{X_T^{s,b}} := \inf \{ \|v\|_{X^{s,b}(\mathbb{R} \times \mathbb{T}^2)} : v|_{[-T,T]} = u \}.$$

For any $s \in \mathbb{R}$ and $b > \frac{1}{2}$, we have $X_T^{s,b} \subset C([-T,T]; H^s(\mathbb{T}^2))$, where $H^s(\mathbb{T}^2)$ is the $L^2$-based Sobolev space on $\mathbb{T}^2$ with regularity $s$. 
We define the truncated Duhamel operator as
\[
\mathcal{I}_x F(t) = \chi(t) \int_0^t \chi(t') e^{i(t-t')\Delta} F(t') \, dt'.
\] (2.1)

We first recall the following linear estimates. See [2, 16, 34].

Lemma 2.1. Let \( s \in \mathbb{R} \) and \( b > \frac{3}{2} \). Then, we have
\[
\|\mathcal{I}_x F\|_{X^{s,b}} \lesssim_b \|F\|_{X^{s,b-1}}.
\]

Next, we recall the following time localization estimate. For a proof, see Proposition 2.7 in [12].

Lemma 2.2. Let \( Q \) be a spatial frequency ball of radius \( N \) (not necessarily centered at the origin). Then, we have
\[
\|P_Q u\|_{L^2_t([-1,1] \times \mathbb{T}^2)} \lesssim N^{0+} \|u\|_{X^{0\frac{1}{2}}}.
\]

We also recall the following time localization estimate. For a proof, see Lemma 3.1 in [11].

Lemma 2.3. Let \( \varphi \) be a Schwartz function, and let \( \varphi_T(t) = \varphi(t/T) \) for \( 0 < T \leq 1 \). Let \( s \in \mathbb{R} \) and \( \frac{7}{8} < b \leq b_1 < 1 \). Then, for any space-time function \( u \) that satisfies \( u(0,x) = 0 \) for all \( x \in \mathbb{T}^2 \), we have
\[
\|\varphi_T \cdot u\|_{X^{s,b}} \lesssim T^{b_1-b} \|u\|_{X^{s,b_1}}.
\]

Finally, we record the following lemma. For a proof, see Lemma 3.1 in [11].

Lemma 2.4. For all \( \tau \in \mathbb{R} \) and \( n \in \mathbb{Z}^2 \), we have the formula
\[
\hat{\mathcal{I}_x} F(\tau, n) = \int_{\mathbb{R}} K(\tau, \tau') \tilde{F}(\tau', n) \, d\tau',
\]
where the kernel \( K \) satisfies
\[
|K(\tau, \tau')| \lesssim \left( \frac{1}{\langle \tau \rangle^3} + \frac{1}{\langle \tau - \tau' \rangle^3} \right) \frac{1}{\langle \tau' \rangle} \lesssim \frac{1}{\langle \tau \rangle \langle \tau - \tau' \rangle}.
\]

2.3. Counting estimates and a convolution lemma. In this subsection, we recall some counting estimates and a convolution lemma. We first record the following fact from number theory. For a proof, see Lemma 4.3 in [12].

Lemma 2.5. Let \( a_0, b_0 \in \mathbb{C} \). Let \( m \in \mathbb{Z}[i] \) be such that \( m \neq 0 \). Let \( M_1, M_2 > 0 \). Then, the number of tuples \( (a,b) \in (\mathbb{Z}[i])^2 \) that satisfies
\[
ab = m, |a - a_0| \leq M_1, |b - b_0| \leq M_2
\]
is \( O(M_1^2 M_2^2) \) for any small \( \varepsilon > 0 \), where the constant depends only on \( \varepsilon \).

We now show the following counting estimates.

Lemma 2.6. Let \( N, N_1, N_2 \geq 1 \) be dyadic numbers. Let \( n, n_1, n_2 \in \mathbb{Z}^2 \) be such that \( n \) lies in a ball of radius \( N \), \( n_1 \) lies in a ball of radius \( N_1 \), \( n_2 \) lies in a ball of radius \( N_2 \), \( n - n_1 + n_2 \neq 0 \), and \( |n|^2 - |n_1|^2 + |n_2|^2 = m \) for some fixed \( m \in \mathbb{Z} \).

(i) The number of tuples \( (n, n_1, n_2) \in (\mathbb{Z}^2)^3 \) that satisfy the above conditions is \( O(N_1 N_2 \max\{N_1^\varepsilon, N_2^\varepsilon\}) \) for any small \( \varepsilon > 0 \), where the constant depends only on \( \varepsilon \).
(ii) If \( n_1 \) is fixed, then the number of tuples \((n, n_2) \in (\mathbb{Z}^2)^2\) that satisfy the above conditions is \(O(\max\{N^\epsilon, N_2^\epsilon\})\) for any small \( \epsilon > 0 \), where the constant depends only on \( \epsilon \).

(iii) If \( n_2 \) is fixed and \( n_2 \neq 0 \), then the number of tuples \((n, n_1) \in (\mathbb{Z}^2)^2\) that satisfy the above conditions is \(O(\min\{N, N_1\})\).

(iv) If \( n \) is fixed and \( n \neq 0 \), then the number of tuples \((n_1, n_2) \in (\mathbb{Z}^2)^2\) that satisfy the above conditions is \(O(\min\{N_1, N_2\})\).

Proof. (i) See Lemma 4.3 in [12] for the proof of this part.

(ii) Since \( n_1 \) is fixed, we know that \( n + n_2 = n_1 \) is fixed. Let \( k = (k_1, k_2) = n - n_2 \), so that we have that

\[(k_1 + ik_2)(k_1 - ik_2) = |k|^2 = 2|n|^2 + 2|n_2|^2 - |n + n_2|^2 = 2m + |n_1|^2\]

is fixed. Since \( k = n - n_2 \) lies in a ball of radius \( N + N_2 \), by Lemma 2.3, we know that the number of choices for \( k \) is \(O(\max\{N^\epsilon, N_2^\epsilon\})\) for any small \( \epsilon > 0 \). Thus, the number of choices for \((n, n_2)\) is \(O(\max\{N^\epsilon, N_2^\epsilon\})\) for any small \( \epsilon > 0 \).

(iii) Note that since \( n = n_1 - n_2 \), we have

\[m = |n_1 - n_2|^2 - |n_1|^2 + |n_2|^2 = -2n_1 \cdot n_2 + 2|n_2|^2.\]

This shows that \( n_1 \cdot n_2 \) is fixed, which means that \( n_1 \) is restricted to a line. Also, we have

\[m = |n|^2 - |n + n_2|^2 + |n_2|^2 = -2n \cdot n_2.\]

This shows that \( n \cdot n_2 \) is fixed, which means that \( n \) is restricted to a line. Thus, the number of choices for \((n, n_1)\) is \(O(\min\{N, N_1\})\).

(iv) The proof of this part is the same as that in part (iii). Thus, we omit details. \(\square\)

We end this subsection by recording the following convolution inequality. For a proof, see Lemma 4.2 in [16].

Lemma 2.7. Let \( 0 \leq \beta \leq \gamma \) with \( \gamma > 1 \). Then, for any \( a \in \mathbb{R} \), we have

\[\int_{\mathbb{R}} \frac{1}{\langle x \rangle^\beta} \langle x - a \rangle^\gamma \lesssim \frac{1}{\langle a \rangle^\beta}.\]

2.4. Tools from stochastic analysis. In this subsection, we recall the Wiener chaos estimate. Let \((H, B, \mu)\) be an abstract Wiener space, where \( \mu \) is a Gaussian measure on a separable Banach space \( B \) and \( H \subset B \) is its Cameron-Martin space. Let \( \{e_j\}_{j \in \mathbb{N}} \subset B \) be an orthonormal system of \( H^* = H \). We define a polynomial chaos of order \( k \) as an element of the form \( \prod_{j=1}^k H_{k_j}(\langle x, e_j \rangle) \). Here, \( x \in B, k_j \neq 0 \) for finitely many \( j \)'s, \( k = \sum_{j=1}^\infty k_j \), \( H_{k_j} \) is the Hermite polynomial of degree \( k_j \), and \( \langle \cdot, \cdot \rangle =_B \langle \cdot, \cdot \rangle_{B^*} \) denotes the \( B - B^* \) duality pairing. We denote the closure of all polynomial chaoses of order \( k \) under \( L^2(B, \mu) \) by \( H_k \), whose elements are called homogeneous Wiener chaos of order \( k \). We also denote

\[\mathcal{H}_{\leq k} = \bigoplus_{j=0}^k \mathcal{H}_j \quad (2.2)\]

for \( k \in \mathbb{N} \).

Let \( L \) be the Ornstein-Uhlenbeck operator. It is known that any element in \( H_k \) is an eigenfunction of \( L \) with eigenvalue \(-k\). Then, we have the following Wiener chaos estimate
Theorem I.22] as a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup $U(t) = e^{tL}$ due to Nelson [27].

**Lemma 2.8.** Let $k \in \mathbb{N}$. Then, for any $p \geq 2$ and $X \in \mathcal{H}_{\leq k}$, we have

$$\mathbb{E}[|X|^p]^\frac{1}{p} \leq (p - 1)^\frac{1}{p} \mathbb{E}[|X|^2]^\frac{1}{2}.$$ 

2.5. **Random tensor and deterministic tensor estimates.** In this subsection, we recall some useful results of random tensor estimates developed in [13] and also prove some deterministic tensor estimates.

Let us first recall the definition of (random) tensors. Let $A$ be a finite index set. We denote $n_A$ as the tuple $(n_j : j \in A)$. A tensor $h = h_{n_A}$ is a function from $(\mathbb{Z}^2)^A$ to $\mathbb{C}$ with $n_A$ being the input variables. The support of $h$ is the set of $n_A$ such that $h_{n_A} \neq 0$. Note that $h$ may also depend on $\omega \in \Omega$, in which case $h$ is called a random tensor.

Given a finite index set $A$, we define the norm $\| \cdot \|_{n_A}$ by

$$\|h\|_{n_A} = \|h\|_{\ell^2_A} = \left( \sum_{n_A \in (\mathbb{Z}^2)^A} |h_{n_A}|^2 \right)^{1/2}.$$ 

For any partition $(B, C)$ of $A$, i.e. $B \cup C = A$ and $B \cap C = \emptyset$, we define the norm $\| \cdot \|_{n_B \to n_C}$ by

$$\|h\|^2_{n_B \to n_C} = \sup \left\{ \sum_{n_C \in (\mathbb{Z}^2)^C} \left| \sum_{n_B \in (\mathbb{Z}^2)^B} h_{n_A} \cdot f_{n_B} \right|^2 : \sum_{n_B \in (\mathbb{Z}^2)^B} |f_{n_B}|^2 = 1 \right\}.$$ 

For any tensor $h$, by duality, we have $\|h\|_{n_B \to n_C} = \|h\|_{n_C \to n_B} = \|\tilde{h}\|_{n_B \to n_C}$. If either $B = \emptyset$ or $C = \emptyset$, we have $\|h\|_{n_B \to n_C} = \|h\|_{n_A}$.

We also need the following definitions to state the random tensor estimate. For a complex number $a$, we define $a^+ = a$ and $a^- = \bar{a}$. Let $A$ be a finite index set. For each $j \in A$, we associate $j$ with a sign $\zeta_j \in \{\pm\}$. For $j_1, j_2 \in A$, we say that $(n_{j_1}, n_{j_2})$ is a pairing if $n_{j_1} = n_{j_2}$ and $\zeta_{j_1} = -\zeta_{j_2}$. Also, recall that $\{g_n\}_{n \in \mathbb{Z}^2}$ is a set of independent standard complex-valued Gaussian random variables. For each $n \in \mathbb{Z}^2$, we can write

$$g_n(\omega) = \rho_n(\omega) \eta_n(\omega),$$

where $\rho_n = |g_n|$ and $\eta_n = \rho_n^{-1}g_n$ are independent. Note that each $\eta_n$ is uniformly distributed on the unit circle of $\mathbb{C}$.

We now record the following random tensor estimate. For the proof, see Proposition 4.14 in [13].

**Lemma 2.9.** Let $0 < T \leq 1$. Let $h_{a_1a_2n_A} = h_{a_1a_2n_A}(\omega)$ be a random tensor, where each $n_j \in \mathbb{Z}^2$ and $(a_1, a_2) \in (\mathbb{Z}^2)^q$ for some integer $q \geq 2$. Given a dyadic number $M \geq 1$, we assume that $(a_1), (a_2) \lesssim M$ and $(n_j) \lesssim M$ for all $j \in A$. We also assume that in the support of $h_{a_1a_2n_A}$, there is no pairing in $n_A$. Moreover, we assume that $\{h_{a_1a_2n_A}\}$ is independent with $\{\eta_n\}_{n \in \mathbb{Z}^2}$. Define the tensor

$$H_{a_1a_2} = \sum_{n_A} h_{a_1a_2n_A} \prod_{j \in A} \zeta_{n_{j}}^{\eta_{n_j}}.$$
Then, there exists constants $C, c > 0$ such that outside an exceptional set of probability $\leq C \exp(-cM^{10})$ with $\theta > 0$, we have

$$\|H_{a_1 a_2}\|_{a_1 \to a_2} \lesssim T^{-\theta} M^\theta \cdot \max_{(A_1, A_2)} \|h\|_{a_1 n_{A_1} \to a_2 n_{A_2}},$$

where $(A_1, A_2)$ runs over all partitions of $A$.

We also record the following variant of Lemma 2.9. For the proof, see Proposition 4.15 in [13].

**Lemma 2.10.** Consider the same setting as in Lemma 2.9 with the following differences:

1. We only restrict $\langle n_j \rangle \lesssim M$ for all $j \in A$ but do not impose any condition on $\langle a_1 \rangle$ or $\langle a_2 \rangle$.

2. We assume that $a_1, a_2 \in \mathbb{Z}^2$ and that in the support of the random tensor $h_{a_1 a_2 n A}$ we have $|a_1 - \zeta a_2| \lesssim M$ where $\zeta \in \{\pm\}$.

3. The random tensor $h_{a_1 a_2 n A}$ only depends on $a_1 - \zeta a_2, |a_1|^2 - \zeta |a_2|^2, \text{ and } n_A,$ and is supported in the set where $|a_1|^2 - \zeta |a_2|^2 \lesssim M^{10}$.

Then, there exists constants $C, c > 0$ such that outside an exceptional set of probability $\leq C \exp(-cM^{10})$ with $\theta > 0$, we have

$$\|H_{a_1 a_2}\|_{a_1 \to a_2} \lesssim T^{-\theta} M^\theta \cdot \max_{(A_1, A_2)} \|h\|_{a_1 n_{A_1} \to a_2 n_{A_2}},$$

where $(A_1, A_2)$ runs over all partitions of $A$.

We now turn our attention to some deterministic tensor estimates. Given $m \in \mathbb{Z}$, we define the base tensor $h_{mn_{1}n_{2}}^m$ as

$$h_{mn_{1}n_{2}}^m = 1_{n-n_{1}+n_{2}=0} 1_{|n|^{2}-|n_{1}|^{2}+|n_{2}|^{2}=m}. \quad (2.3)$$

We now show the following estimates regarding the base tensor $h_{mn_{1}n_{2}}^m$.

**Lemma 2.11.** Let $N, N_1, N_2 \geq 1$ be dyadic numbers and let $\varepsilon > 0$ be arbitrarily close to 0. Let $J$ be a ball of radius $\sim N$, $J_1$ be a ball of radius $\sim N_1$, and $J_2$ be a ball of radius $\sim N_2$. We define

$$S := \{(n, n_1, n_2) \in (\mathbb{Z}^2)^3 : n \in J, n_1 \in J_1, n_2 \in J_2\}.$$

Thus, we have the following estimates:

$$\|h_{mn_{1}n_{2}}^m \cdot 1_S\|_{n_{1}n_{2}} \lesssim N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \max\{N_1^{\varepsilon}, N_2^{\varepsilon}\}, \quad (2.4)$$

$$\|h_{mn_{1}n_{2}}^m \cdot 1_S\|_{n_{1} \to n_{2}} \lesssim \max\{N_1^{\varepsilon}, N_2^{\varepsilon}\}, \quad (2.5)$$

$$\|h_{mn_{1}n_{2}}^m \cdot 1_{S} \cdot 1_{n_{2} \neq 0}\|_{n_{2} \to n_{1}} \lesssim \min\{N_1^{\frac{1}{2}}, N_2^{\frac{1}{2}}\}, \quad (2.6)$$

$$\|h_{mn_{1}n_{2}}^m \cdot 1_{S} \cdot 1_{n_{1} \neq 0}\|_{n_{1} \to n_{2}} \lesssim \min\{N_1^{\frac{1}{2}}, N_2^{\frac{1}{2}}\}. \quad (2.7)$$

**Proof.** For (2.4), we use Lemma 2.6 (i) to obtain

$$\|h_{mn_{1}n_{2}}^m \cdot 1_S\|_{n_{1}n_{2}} \lesssim N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} \max\{N_1^{\varepsilon}, N_2^{\varepsilon}\}.$$
For (2.5), we use Schur’s test and Lemma 2.6 (ii) to obtain
\[ \| h_{nn_1n_2}^m \cdot 1_S \|_{n_1 \to n_2} \leq \left( \sup_{n_1, n_2} \sum_{n_1, n_2} h_{nn_1n_2}^m \cdot 1_S \right)^{1/2} \left( \sup_{n_1, n_2} \sum_{n_2} h_{nn_1n_2}^m \cdot 1_S \right)^{1/2} \]
\[ \lesssim \max\{N^{s}, N^{\frac{1}{2}}_2\}. \]

For (2.6), we use Schur’s test and Lemma 2.6 (iii) to obtain
\[ \| h_{nn_1n_2}^m \cdot 1_S \cdot 1_{n_2 \neq 0} \|_{n_2 \to n_1} \leq \left( \sup_{n_2 \neq 0} \sum_{n_1, n_2} h_{nn_1n_2}^m \cdot 1_S \right)^{1/2} \left( \sup_{n_1, n_2} \sum_{n} h_{nn_1n_2}^m \cdot 1_S \right)^{1/2} \]
\[ \lesssim \min\{N^{1/2}, N^{1/4}_2\}. \]

For (2.7), we use Schur’s test and Lemma 2.6 (iv) to obtain
\[ \| h_{nn_1n_2}^m \cdot 1_S \cdot 1_{n \neq 0} \|_{n \to n_1} \leq \left( \sup_{n \neq 0} \sum_{n_1, n_2} h_{nn_1n_2}^m \cdot 1_S \right)^{1/2} \left( \sup_{n_1, n_2} \sum_{n} h_{nn_1n_2}^m \cdot 1_S \right)^{1/2} \]
\[ \lesssim \min\{N^{1/4}, N^{1/2}_2\}. \]

We thus finish our proof. □

Remark 2.12. The condition \( n_2 \neq 0 \) in the estimate (2.6) is necessary in view of the restriction \( n_2 \neq 0 \) in Lemma 2.6 (iii). Similarly, the condition \( n \neq 0 \) in the estimate (2.7) is necessary in view of the restriction \( n \neq 0 \) in Lemma 2.6 (iv).

3. Bilinear estimates

In this section, we establish several bilinear estimates that are crucial for proving Theorem 1.1, the almost sure local well-posedness result of the quadratic NLS (1.1). Specifically, we need to estimate the following term
\[ \| \varphi_T \cdot \mathcal{I}_\chi(v^{(1)}v^{(2)}) \|_{X_s^{s, \rac{1}{2} + \delta}}, \]
where \( s, \delta > 0 \) are sufficiently small, \( \varphi_T(t) = \varphi(t/T) \) with \( \varphi \) being a Schwartz function and \( 0 < T \leq 1 \), and \( \mathcal{I}_\chi \) is the truncated Duhamel operator as defined in (2.1) with \( \chi \) being a smooth cut-off function such that \( \chi \equiv 1 \) on \([-1, 1]\) and \( \chi \equiv 0 \) outside of \([-2, 2]\). Here, each of \( v^{(1)} \) and \( v^{(2)} \) is either an arbitrary space-time function on \( \mathbb{R} \times \mathbb{T}^2 \) or the random linear solution with a time cut-off \( \chi \cdot z \), where \( z \) is as defined in (1.5).

We first consider the case when neither \( v^{(1)} \) nor \( v^{(2)} \) is \( \chi \cdot z \). Specifically, we show the following bilinear estimate.

Proposition 3.1. Let \( s > 0 \) and let \( \delta > 0 \) be sufficiently small. Let \( 0 < T \leq 1 \). Then, we have
\[ \| \varphi_T \cdot \mathcal{I}_\chi(v^{(1)})v^{(2)} \|_{X_s^{s, \rac{1}{2} + \delta}} \lesssim T^\delta \| v^{(1)} \|_{X_s^{s, \rac{1}{2} + \delta}} \| v^{(2)} \|_{X_s^{s, \rac{1}{2} + \delta}}. \]

Proof. By Lemma 2.3 and Lemma 2.1, we have
\[ \| \varphi_T \cdot \mathcal{I}_\chi(v^{(1)}v^{(2)}) \|_{X_s^{s, \rac{1}{2} + \delta}} \lesssim T^\delta \| \mathcal{I}_\chi(v^{(1)}v^{(2)}) \|_{X_s^{s, \rac{1}{2} + 2\delta}} \lesssim T^\delta \| v^{(1)}v^{(2)} \|_{X_s^{s, \rac{1}{2} + 2\delta}}. \]  (3.1)
In this case, we have $N$. This leads us to the following three cases.

**Case 3:**

Combining (3.1), (3.2), (3.3) and summing over dyadic balls of radius $J$, and the Cauchy-Schwarz inequality in $L^4_{t,x}$. We denote the set of these balls as $J$. Note that for each fixed $J_1 \in \mathcal{J}_1$, the product $1_{J_1} (n_1) \cdot 1_{J}(n)$ is non-zero for at most a fixed constant number of $J \in \mathcal{J}$, and we denote the set of these $J$’s as $\mathcal{J}(J_1)$. Thus, by Hölder’s inequality, Lemma 2.2 and the Cauchy-Schwarz inequality in $J_1$, we have

$$
\left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} P_{N_2} v^{(2)}) P_N w \, dx dt \right|
\lesssim \sum_{J_1 \in \mathcal{J}_1} \left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} P_{N_2} v^{(2)}) P_N w \, dx dt \right|
\lesssim \sum_{J_1 \in \mathcal{J}_1} \left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} P_{N_2} v^{(2)}) P_N w \, dx dt \right|
\lesssim \sum_{J_1 \in \mathcal{J}_1} \left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} P_{N_2} v^{(2)}) P_N w \, dx dt \right|
$$

Let $n_1, n_2, n$ be the frequencies corresponding to the three terms $P_{N_1} v^{(1)}$, $P_{N_2} v^{(2)}$, $P_N w$, respectively. In order for the above integral on $T^2$ to be non-zero, we must have $n_1 - n_2 - n = 0$. This leads us to the following three cases.

**Case 1:** $N_1 \sim N_2$.

In this case, we have $N \lesssim N_1 \sim N_2$. By Hölder’s inequality and Lemma 2.2, we have

$$
\left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} P_{N_2} v^{(2)}) P_N w \, dx dt \right|
\lesssim N^s \| P_{N_1} v^{(1)} \|_{L^4_{t,x}} \| P_{N_2} v^{(2)} \|_{L^4_{t,x}} \| P_N w \|_{L^2_{t,x}}
\lesssim N^0 \| P_{N_1} v^{(1)} \|_{X^{0, \frac12 \delta}} \| P_{N_2} v^{(2)} \|_{X^{0, \frac12 \delta}} \| P_N w \|_{X^{0, \frac12 \delta}}
\lesssim N^0 \| P_{N_1} v^{(1)} \|_{X^{0, \frac12 \delta}} \| P_{N_2} v^{(2)} \|_{X^{0, \frac12 \delta}} \| P_N w \|_{X^{0, \frac12 \delta}}
$$

Combining (3.1), (3.2), (3.3) and summing over $N_1 \sim N_2 \gtrsim N$, we obtain the desired estimate.

**Case 2:** $N_1 \gg N_2$.

In this case, we have $N \sim N_1 \gg N_2$. We partition the annulus $\{|n_1| \sim N_1\}$ into balls of radius $J \sim N_2$ and denote the set of these balls as $\mathcal{J}_1$, and we partition the annulus $\{|n| \sim N\}$ into balls of radius $J \sim N_2$ and denote the set of these balls as $\mathcal{J}$. Note that for each fixed $J_1 \in \mathcal{J}_1$, the product $1_{J_1} (n_1) \cdot 1_{J}(n)$ is non-zero for at most a fixed constant number of $J \in \mathcal{J}$, and we denote the set of these $J$’s as $\mathcal{J}(J_1)$. Thus, by Hölder’s inequality, Lemma 2.2 and the Cauchy-Schwarz inequality in $J_1$, we have

$$
\left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} P_{N_2} v^{(2)}) P_N w \, dx dt \right|
\lesssim \sum_{J_1 \in \mathcal{J}_1} \left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} P_{N_2} v^{(2)}) P_N w \, dx dt \right|
\lesssim \sum_{J_1 \in \mathcal{J}_1} \left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} P_{N_2} v^{(2)}) P_N w \, dx dt \right|
\lesssim \sum_{J_1 \in \mathcal{J}_1} \left| \int_{\mathbb{R}} \int_{T^2} \langle \nabla \rangle^s (P_{N_1} v^{(1)} P_{N_2} v^{(2)}) P_N w \, dx dt \right|
$$

Combining (3.1), (3.2), (3.3), applying the Cauchy-Schwarz inequality in $N_1 \sim N$, and summing over $N_1 \sim N \gg N_2$, we obtain the desired estimate.

**Case 3:** $N_1 \ll N_2$. 

By Lemma 2.3, Lemma 2.4, duality, and dyadic decomposition, we have

\[ \text{Proposition 3.2.} \]

We now consider the case when at least one of \( v^{(1)} \) and \( v^{(2)} \) is the random linear solution with a time cut-off \( \chi \cdot z \). Our goal is to prove the following estimates. The idea of the computations in the proof comes from [36].

\[ \text{Proposition 3.2.} \quad \text{Let } \alpha < \frac{1}{2}. \text{ Let } s, \delta > 0 \text{ be sufficiently small. Let } 0 < T \leq 1. \]

(i) We have

\[ \| P_{\neq 0} (\varphi_T \cdot \mathcal{I}_\chi (v \cdot \chi \cdot z)) \|_{X^{s, \frac{1}{2} + \delta}} \lesssim T^{\delta - 2\theta} \| v \|_{X^{s, \frac{1}{2} + \delta}} \]  

outside an exceptional set of probability \( \leq C \exp(-\frac{c}{T}) \) with \( C, c > 0 \) being constants and \( 0 < \theta \ll \delta \).

(ii) If \( v \) has mean zero (i.e. has no zeroth frequency term), we have

\[ \| P_{\neq 0} (\varphi_T \cdot \mathcal{I}_\chi (\chi \cdot z \cdot \tau)) \|_{X^{s, \frac{1}{2} + \delta}} \lesssim T^{\delta - 2\theta} \| v \|_{X^{s, \frac{1}{2} + \delta}} \]  

outside an exceptional set of probability \( \leq C \exp(-\frac{c}{T}) \) with \( C, c > 0 \) being constants and \( 0 < \theta \ll \delta \).

(iii) We have

\[ \| P_{\neq 0} (\varphi_T \cdot \mathcal{I}_\chi (\chi \cdot z \cdot \tau)) \|_{X^{s, \frac{1}{2} + \delta}} \lesssim T^{\delta - 2\theta} \]  

outside an exceptional set of probability \( \leq C \exp(-\frac{c}{T}) \) with \( C, c > 0 \) being constants and \( 0 < \theta \ll \delta \).

\[ \text{Proof.} \quad \text{We first do the following general setup. Let } v^{(1)} \text{ and } v^{(2)} \text{ be two space-time functions. By Lemma 2.3 Lemma 2.4 duality, and dyadic decomposition, we have} \]

\[ \| P_{\neq 0} (\varphi_T \cdot \mathcal{I}_\chi (v^{(1)} v^{(2)})) \|_{X^{s, \frac{1}{2} + \delta}} \lesssim T^\delta \| P_{\neq 0} (\mathcal{I}_\chi (v^{(1)} v^{(2)})) \|_{X^{s, \frac{1}{2} + \delta}} \]

\[ = T^\delta \left| \sum_{n \neq 0} \langle n \rangle^s \int \int K(\tau, |n|^2 + (\tau_1 - |n_1|^2) - (\tau_2 - |n_2|^2)) \right| \]

\[ \times \left| \sum_{n_1, n_2 \in \mathbb{Z}^2 \atop n_1 - n_2 = n \neq 0} \langle n \rangle^s \int \int K(\tau, |n|^2 + (\tau_1 - |n_1|^2) - (\tau_2 - |n_2|^2)) \right| \]

\[ \lesssim T^\delta \sum_{n \neq 0} \sum_{n_1, n_2 \in \mathbb{Z}^2 \atop n_1 - n_2 = n \neq 0} \langle n \rangle^s \]

\[ \times \int \int K(\tau, |n|^2 + (\tau_1 - |n_1|^2) - (\tau_2 - |n_2|^2))(\tau)_s^{\frac{1}{2} + 2\delta} \]

\[ \times P_{N_1} v^{(1)}(\tau_1, n_1) P_{N_2} v^{(2)}(\tau_2, n_2) P_N w(\tau, n) d\tau d\tau_1 d\tau_2, \]  

(3.8)
where the kernel $K$ satisfies
\[
|K(\tau, \tau')| \lesssim \frac{1}{(\tau) (\tau - \tau')}.
\]
(3.9)

We now separately discuss the three situations (i), (ii), and (iii).

(i) We consider the following two cases.

Case 1: $\langle \tau \rangle \gg N_2^{10}$.

In this case, by Hölder’s inequality in $n_2$, (3.9), the Cauchy-Schwarz inequalities in $\tau_1$, $\tau_2$ and $n$, and Lemma 2.7, we have
\[
(3.8) \leq T^\delta \sup_{\|\tilde{w}\|_{L^2_{t,x}} \leq 1} \sum_{N,N_1,N_2 \geq 1 \text{ dyadic}} N^s N_2^{-5+8\delta} N_2^{-1+\alpha}
\]
\[
\times \sup_{n_2 \in \mathbb{Z}^2} \sum_{n_2 \in \mathbb{Z}^2 \atop \langle n_2 \rangle \sim N_2} \int \int \int (|\tau| - |n|^2) - (\tau_1 - |n_1|^2) + (\tau_2 - |n_2|^2) \rangle^{-1}
\]
\[
\times \left| \tilde{P}_{N_1} v(\tau_1, n + n_2) \right| |g_{n_2}(\omega)\tilde{\chi}(\tau_2)| \left| \tilde{P}_N w(\tau, n) \right| d\tau_1 d\tau_2 d\tau
\]
\[
\leq T^\delta \sup_{\|\tilde{w}\|_{L^2_{t,x}} \leq 1} \sum_{N,N_1,N_2 \geq 1 \text{ dyadic}} N^s N_2^{-4+8\delta+\alpha} \sup_{n_2 \in \mathbb{Z}^2 \atop \langle n_2 \rangle \sim N_2} |g_{n_2}(\omega)|
\]
\[
\times \left\| \langle \tau_1 \rangle^{1/2+\delta} \tilde{P}_{N_1} v(\tau_1, n_1) \right\|_{\ell^2_{n_1} L^2_t} \left\| \tilde{P}_N w(\tau, n) \right\|_{\ell^2_{\mathbb{Z}^2}}
\]
\[
\leq T^\delta \sup_{\|w\|_{L^2_{t,x}} \leq 1} \sum_{N,N_1,N_2 \geq 1 \text{ dyadic}} N^s N_1^{-s} N_2^{-4+8\delta+\alpha} \sup_{n_2 \in \mathbb{Z}^2 \atop \langle n_2 \rangle \sim N_2} |g_{n_2}(\omega)|
\]
\[
\times \left\| P_{N_1} v \right\|_{X^{s, \frac{1}{2} + \delta}} \left\| P_N w \right\|_{L^2_{t,x}}.
\]
(3.10)

Note that we have the following Gaussian tail bound:
\[
\sum_{n_2 \in \mathbb{Z}^2 \atop \langle n_2 \rangle \sim N_2} P(|g_{n_2}| > T^{-\theta} N_2^{\delta}) < C \exp \left( -c \frac{N_2^{\delta}}{T^\theta} \right)
\]
(3.11)

for some constants $C, c > 0$ and $0 < \theta \ll \delta$, so that (3.10) gives
\[
(3.8) \leq T^{\delta - \theta} \sup_{\|w\|_{L^2_{t,x}} \leq 1} \sum_{N,N_1,N_2 \geq 1 \text{ dyadic}} N^s N_1^{-s} N_2^{-4+2\delta+\alpha} \left\| P_{N_1} v \right\|_{X^{s, \frac{1}{2} + \delta}} \left\| P_N w \right\|_{L^2_{t,x}}
\]
(3.12)

outside an exceptional set of probability $\leq C \exp(-c N_2^{\delta}/T^\theta)$. Recall that $\delta$ and $s$ can be made sufficiently small and $\alpha < \frac{1}{2}$. If $N \gg N_1$, we have $N \sim N_2$, so that we can use $N^s \sim N_1^{-s} N_2^{s+}$ and sum up dyadic $N, N_1, N_2$ in (3.12) to obtain (3.5). If $N \ll N_1$, we have $N_1 \sim N_2$, so that we can use $N^s \ll N_1^{-s} N_2^{s+}$ and sum up dyadic $N, N_1, N_2$ in (3.12) to obtain (3.5). If $N \sim N_1$, we can use the Cauchy-Schwarz inequality in $N \sim N_1$ and sum up dyadic $N, N_1, N_2 \geq 1$ in (3.12) to obtain (3.5).

Case 2: $\langle \tau \rangle \lesssim N_2^{10}$.

We further split this case into two subcases.

Subcase 2.1: $N \lesssim N_2$. 

In this case, we have $N_1 \lesssim N_2$. By the Cauchy-Schwarz inequalities in $\tau$ and $n$, (3.9), and Minkowski’s inequality, we have

\[ T^\delta \lesssim \sum_{N, N_1, N_2 \geq 1} N^s N_2^{3\delta} \]

\[ \times \left| \sum_{n, n_1, n_2 \in \mathbb{Z}^2, n_1 - n_2 = n \neq 0} \int \int \int K(\tau, |n|^2 + (\tau_1 - |n_1|^2) - (\tau_2 - |n_2|^2)) \langle \tau \rangle^{\frac{1}{2} - \delta} \right| \]

\[ \lesssim T^\delta \sum_{N, N_1, N_2 \geq 1} N^s N_2^{3\delta} \left[ \int \langle \tau \rangle^{-1 - 2\delta} \right] \]

\[ \times \left( \sum_{m \in \mathbb{Z}} \int \int \langle \tau - \tau_1 + \tau_2 - m \rangle^{-1} \langle \tau_1 \rangle^{-\frac{1}{2} - \delta} \chi(\tau_2) \right) \]

\[ \times \left| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{m, n_1, n_2}^\prime S_{1} \cdot \frac{g_{n_2}(\omega)}{(n_2)^{1-\alpha}} \langle \tau_1 \rangle^{\frac{1}{2} + \delta} \widetilde{P}_{N_1} v(n_1, n_1) \right| \left\| d\tau_1 d\tau_2 \right\|^{1/2}, \tag{3.13} \]

where $h_{m, n_1, n_2}^\prime$ is the base tensor as defined in (2.3) and $S_1$ is a set defined by

\[ S_1 := S_1(N, N_1, N_2) \]

\[ = \{(n, n_1, n_2) \in (\mathbb{Z}^2)^3 : n \neq 0, |n| \sim N, |n_1| \sim N_1, |n_2| \sim N_2 \}. \tag{3.14} \]

Note that for $(n, n_1, n_2)$ restricted in $S_1$, we have $\lesssim N_2^2$ choices for the value

\[ m = |n|^2 - |n_1|^2 + |n_2|^2, \]

which implies that

\[ \sum_{m \in \mathbb{Z}} \langle \tau - \tau_1 + \tau_2 - m \rangle^{-1} \lesssim \log(1 + N_2^2) \lesssim N_2^\delta. \tag{3.15} \]

Thus, continuing with (3.13), by Hölder’s inequality in $m$, (3.15), the Cauchy-Schwarz inequality in $\tau_1$, we obtain

\[ T^\delta \lesssim \sum_{N, N_1, N_2 \geq 1} N^s N_2^{3\delta} \left/ \sum_{m \in \mathbb{Z}} \left\| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{m, n_1, n_2}^\prime S_{1} \cdot \frac{g_{n_2}(\omega)}{(n_2)^{1-\alpha}} \langle \tau_1 \rangle^{\frac{1}{2} + \delta} \widetilde{P}_{N_1} v(n_1, n_1) \right\|_{n \to n_1} \right\|_{X^s, \frac{1}{2} + \delta} \tag{3.16} \]

By Lemma 2.9 the Gaussian tail bound (3.11), and Lemma 2.11 we have

\[ \left\| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{m, n_1, n_2}^\prime S_{1} \cdot \frac{g_{n_2}(\omega)}{(n_2)^{1-\alpha}} \right\|_{n \to n_1} \]

\[ \lesssim T^{-2\theta} N_2^{-1 + 2\delta + \alpha} \max \left\{ \| h_{m, n_1, n_2}^\prime S_{1} \|_{n_2 \to n_1}, \| h_{m, n_1, n_2}^\prime S_{1} \|_{n \to n_1, n_2} \right\} \]

\[ \lesssim T^{-2\theta} N_2^{-\frac{1}{2} + 2\delta + \alpha} \tag{3.17} \]

outside an exceptional set of probability $\leq C \exp(-c N_2^\delta / T^\theta)$ for some universal constants $C, c > 0$. Thus, combining (3.16) and (3.17), using the fact that $\alpha < \frac{1}{2}, N \lesssim N_2, N_1 \lesssim N_2,
$\delta, s > 0$ are sufficiently small, and summing over dyadic $N, N_1, N_2 \geq 1$, we obtain the desired inequality (3.5).

**Subcase 2.2: $N \gg N_2$.**

In this subcase, note that due to (3.9), we can assume that $\langle \tau - |n|^2 \rangle - (\tau_1 - |n_1|^2) + (\tau_2 - |n_2|^2) \lesssim N_2^{10}$, since otherwise we can conclude by using similar steps as in Case 1. Similarly, we can assume that $\langle \tau_1 \rangle \lesssim N_2^{10}$ and also $\langle \tau_2 \rangle \lesssim N_2^{10}$. Thus, we have $\|n|^2 - |n_1|^2 + |n_2|^2 \lesssim N_2^{10}$, so that $\|n|^2 - |n_1|^2 \lesssim N_2^{10}$.

We now perform an orthogonality argument. Note that we have $N_1 \sim N \gg N_2$ in this subcase. Decompose the set $\{|n| \sim N\}$ into balls of radius $\sim N_2$ and denote the set of these balls as $\mathcal{J}_1$, and we decompose the set $\{|n_1| \sim N_1\}$ into balls of radius $\sim N_2$ and denote the set of these balls as $\mathcal{J}_1$. Note that for each fixed $J \in \mathcal{J}$, the product $1_J(n) \cdot 1_{J_1}(n_1)$ is non-zero for at most a fixed constant number of $J_1 \in \mathcal{J}_1$, and we denote the set of these $J_1$’s as $\mathcal{J}_1(J)$. Thus, by the Cauchy-Schwarz inequalities in $\tau$ and $n$, (3.9), and Minkowski’s inequality, we have

\[ \lesssim T^\delta \sup_{\|\tilde{u}\|_{L^2}} \sum_{N, N_1, N_2 \geq 1} \mathcal{J} \in \mathcal{J} \sum_{J_1 \in \mathcal{J}_1(J)} N^{s} N_2^{30\delta} \]

\[ \times \left| \sum_{n, n_1, n_2 \in \mathbb{Z}^2} \int \int K(\tau, |n|^2, (\tau_1 - |n_1|^2) - (\tau_2 - |n_2|^2)) \langle \tau \rangle^{1/2 - \delta} \]

\[ \times \overline{P_J v(\tau_1, n_1)} \frac{\gamma_{n_2}(\omega)}{\langle n_2 \rangle^{1-\alpha}} \overline{\lambda(\tau_2)} \overline{P_J w(\tau, n)} \, d\tau_1 d\tau_2 \]

\[ \lesssim T^\delta \sup_{\|\tilde{u}\|_{L^2}} \sum_{N, N_1, N_2 \geq 1} \mathcal{J} \in \mathcal{J} \sum_{J_1 \in \mathcal{J}_1(J)} N^{s} N_2^{30\delta} \|P_J w\|_{L^2} \]

\[ \times \left( \int \langle \tau \rangle^{-1/2 - 2\delta} \left( \sum_{m \in \mathbb{Z}} \int \int \langle \tau - \tau_1 + \tau_2 - m \rangle^{-1} \langle \tau_1 \rangle^{-1 - \delta} \overline{\lambda(\tau_2)} \right) \right. \]

\[ \times \left| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{n_1 n_2}^{m} \mathcal{S}_2 \cdot \frac{\gamma_{n_2}(\omega)}{\langle n_2 \rangle^{1-\alpha}} \langle \tau_1 \rangle^{1/2 + \delta} \overline{P_J v(\tau_1, n_1)} \right| \left. \| \frac{d\tau_1 d\tau_2}{\langle \tau_1 \rangle^{1/2}} \right|^2 \]

\[ \lesssim N_2^{10} \]

where $h_{n_1 n_2}^m$ is the base tensor as defined in (2.3) and $\mathcal{S}_2$ is a set defined by

$$ S_2 := S_2(N_2, J, J_1) = \{ (n, n_1, n_2) \in (\mathbb{Z}^2)^3 : n \neq 0, |n|^2 - |n_1|^2 \lesssim N_2^{10}, n \in J, n_1 \in J_1, |n_2| \sim N_2 \}. $$

Note that for $(n, n_1, n_2)$ restricted in $S_2$, we have $\lesssim N_2^{10}$ choices for the value

$$ m = |n|^2 - |n_1|^2 - |n_2|^2, $$

which implies that

$$ \sum_{m \in \mathbb{Z}} (\tau - \tau_1 + \tau_2 - m)^{-1} \lesssim \log(1 + N_2^{10}) \lesssim N_2^\delta. \quad (3.19) $$
Thus, continuing with (3.18), by using $N_1 \sim N$, Hölder’s inequalities in $m$, $J$, and $J_1$, (3.19), and the Cauchy-Schwarz inequalities in $\tau_1$, $J$, and $N_1 \sim N$, we obtain

\[
(3.18) \lesssim T^\delta \sum_{N,N_1,N_2 \geq 1} \sum_{J \in J} \sum_{J_1 \in J_1(J)} N_2^{31\delta} \| P_J w \|_{\ell_2^c L_2^J}^4 \times \sup_{m \in \mathbb{Z}} \left\| \sum_{n_2 \in \mathbb{Z}^2} \frac{g_{m_2}(\omega)}{(n_2)^{1-alpha}} \|_{n \to n_1} \| v \|_{X^{s,1/2+\delta}} \right\|
\]

By Lemma 2.10 the Gaussian tail bound (3.11), and Lemma 2.11 we have

\[
\left\| \sum_{n_2 \in \mathbb{Z}^2} \frac{g_{m_2}(\omega)}{(n_2)^{1-alpha}} \|_{n \to n_1} \| v \|_{X^{s,1/2+\delta}} \right\| \lesssim T^{-2\theta} N_2^{-1+2\delta+\alpha} \max \left\{ \| h_{m_1 n_2}^m 1_{S_2} \|_{n_2 \to n_1}, \| h_{m_1 n_2}^m 1_{S_2} \|_{n \to n_1 n_2} \right\} \lesssim T^{-2\theta} N_2^{-1/2+2\delta+\alpha}
\]

outside an exceptional set of probability $\leq C \exp(-cN_2^3/T^\theta)$ for some universal constants $C, c > 0$. Thus, combining (3.20) and (3.21), using the fact that $\alpha < 1/2$ and $\delta, s > 0$ are sufficiently small, and summing over dyadic $N_2 \geq 1$, we obtain the desired inequality (3.5).

(ii) This part follows similarly from part (i), so that we will be brief here. Using similar steps as in Case 1 of part (i), we can assume that $\langle \tau \rangle \lesssim N_1^{10}$.

When $N \lesssim N_1$, we use the Cauchy-Schwarz inequalities in $\tau$ and $n$, (3.9), Minkowski’s inequality, and Hölder’s inequality in $m$ to obtain

\[
(3.18) \lesssim T^\delta \sum_{N,N_1,N_2 \geq 1} N_1^{31\delta} \left[ \int \langle \tau \rangle^{-1-2\delta} \left( \int \langle \tau_1 \rangle^{-\frac{1}{2}+\delta} \hat{\chi}(\tau_1) \right) \right] \times \sup_{m \in \mathbb{Z}} \left( \sum_{n_1,n_2 \in \mathbb{Z}^2} \frac{g_{n_1}(\omega)}{(n_1)^{1-alpha}} \langle \tau \rangle^{\frac{1}{2}+\delta} \| P_{N_2} v(\tau_2, n_2) \|_{\ell_2^c}^2 \right)^{1/2},
\]

where $h_{mn_1 n_2}^m$ is the base tensor as defined in (2.3) and $S_3$ is a set defined by

\[
S_3 := S_3(N, N_1, N_2) = \{(n, n_1, n_2) \in (\mathbb{Z}^2)^3 : n \neq 0, n_2 \neq 0, |n| \sim N, |n_1| \sim N_1, |n_2| \sim N_2 \}.
\]
Then, by (3.22), the Cauchy-Schwarz inequality in $\tau_2$, Lemma 2.9, the Gaussian tail bound, and Lemma 2.11 we obtain

\[ \sum_{N,N_1,N_2 \geq 1, \text{dyadic}} N^s N_1^{1+\delta} \sup_{m \in \mathbb{Z}} \left\| \sum_{n_1 \in \mathbb{Z}^2} h_{n_1}^{m} 1_{S_3} \cdot \frac{g_{n_1}(\omega)}{(n_1)_{1-\alpha}} \right\|_{n \to n_2} \| P_{n_2} v \|_{X^{s,\frac{3}{4}+\delta}} \]

\[ \lesssim T^{\delta - \frac{27}{40}} \sum_{N,N_1,N_2 \geq 1, \text{dyadic}} N^s N_1^{-1+3\delta+\alpha} \sup_{m \in \mathbb{Z}} \left\{ \left\| h_{n_1}^{m} 1_{S_3} \right\|_{n \to n_2}, \left\| h_{n_1}^{m} 1_{S_3} \right\|_{n \to n_1 n_2} \right\} \]

outside an exceptional set of probability $\lesssim C \exp(-c N_1^{10}/T^{\delta})$ for some universal constants $C,c > 0$. Thus, since $\alpha < \frac{1}{2}$, $N \lesssim N_1$, $N_2 \lesssim N_1$, and $\delta,s > 0$ are sufficiently small, we can sum over dyadic $N, N_1, N_2 \geq 1$ to obtain the desired inequality (3.6).

When $N \gg N_1$, as in Subcase 2.2 in part (i), we can assume that $\langle \tau - |n|^2 \rangle - \tau_1 - |n_1|^2 + \tau_2 - |n_2|^2 \rangle \lesssim N_1^{10}, \langle \tau_1 \rangle \lesssim N_1^{10},$ and $\langle \tau_2 \rangle \lesssim N_1^{10},$ so that $|n|^2 + |n_2|^2 \lesssim N_1^{10}$. We perform an orthogonality argument as in Subcase 2.2 in part (i) to decompose $|n| \sim N$ into a set of balls (denoted as $J$) of radius $\sim N_1$ and decompose $|n_2| \sim N_2$ into a set of balls (denoted as $J_2$) of radius $\sim N_1$. For each $J \in J, 1_J(n) \cdot 1_{J_2}(n)$ is non-zero for at most a fixed constant number of $J_2 \in J_2$, and we denote the set of these $J_2$’s as $J_2(J)$. By the Cauchy-Schwarz inequalities in $\tau$ and $n$, (3.9), Minkowski’s inequality, Hölder’s inequalities in $m,J,\text{ and the Cauchy-Schwarz inequalities in } \tau_2, J, \text{ and } N_2 \sim N,$ we have

\[ \sum_{N \geq 1, \text{dyadic}} N^{\frac{1}{2}+\delta} \sup_{\| \hat{w} \|_{|2,|} \leq 1} \sum_{J \in J, J_2 \in J_2(J)} \sum_{n_1, n_2} N^s N_1^{1+\delta} \| P_J w \|_{\ell^2 L^2_t} \]

\[ \times \left[ \int \langle \tau \rangle^{-1-2\delta} \left( \sum_{m \in \mathbb{Z}} \int \langle \tau - \tau_1 + \tau_2 - m \rangle^{-\frac{1}{2}} \langle \tau_1 \rangle^{-\frac{1}{2}} \hat{X}(\tau) \right) \right] \]

\[ \times \left| \sum_{n_1, n_2} h_{n_1}^m 1_{S_4} \cdot \frac{g_{n_1}(\omega)}{(n_1)_{1-\alpha}} \langle \tau_1 \rangle^{\frac{1}{2}+\delta} \frac{P_{n_2} v(\tau_2, n_2)}{n} \right|^2 \left[ d\tau_1 d\tau_2 \right]^{\frac{1}{2}} \]

\[ \lesssim T^{\delta} \sum_{N_1 \geq 1, \text{dyadic}} N_1^{1+\delta} \sup_{\| \hat{w} \|_{|2,|} \leq 1} \sum_{J \in J, J_2 \in J_2(J)} \sum_{n_1, n_2} N^s N_1^{1+\delta} \| h_{n_1}^m 1_{S_4} \cdot \frac{g_{n_1}(\omega)}{(n_1)_{1-\alpha}} \|_{n \to n_2} \| v \|_{X^{s,\frac{3}{4}+\delta}}, \quad (3.23) \]

where $h_{n_1}^m$ is the base tensor as defined in (2.3) and $S_4$ is a set defined by

\[ S_4 := S_4(N_1, J, J_2) \]

\[ = \{(n, n_1, n_2) \in (\mathbb{Z}^2)^3 : n \neq 0, n_2 \neq 0, |n|^2 + |n_2|^2 \lesssim N_1^{10}, n \in J, |n_1| \sim N_1, n_2 \in J_2\} \]
By Lemma 2.10, the Gaussian tail bound, and Lemma 2.11, we have

\[
\left\| \sum_{n_1 \in \mathbb{Z}^2} h_{n_1,n_2}^m s_{n_1} \cdot g_{n_1}(\omega) \right\|_{n \to n_2} \leq T^{-2\theta} N_1^{-1+2\delta+\alpha} \max \left\{ \| h_{n_1,n_2}^m s_{n_1} \|_{n_1 \to n_2}, \| h_{n_1,n_2}^m s_{n_1} \|_{n \to n_1,n_2} \right\}
\]

\[
\lesssim T^{-2\theta} N_1^{-\frac{1}{2}+2\delta+\alpha}
\]

outside an exceptional set of probability \( \leq C \exp(-cN_1^\delta/T^\theta) \) for some universal constants \( C, c > 0 \). Thus, since \( \alpha < \frac{1}{2} \) and \( \delta, s > 0 \) are sufficiently small, we can combine (3.23) and (3.24) and sum over dyadic \( N_1 \geq 1 \) to obtain the desired inequality (3.6).

(iii) We consider the following two cases.

**Case 1:** \( \langle \tau \rangle \gg N_1^{10} N_2^{10} \).

In this case, by H"older's inequalities in \( n_1 \) and \( n_2 \), (3.9), and the Cauchy-Schwarz inequality in \( \tau \), we have

\[
(3.8) \lesssim T^\delta \sup_{\| \omega \|^{\theta} \lesssim 1} \sum_{N,N_1\geq 1} N^s N_1^{-10} N_2^{-10} N_1^2 N_2^{-1+\alpha} N_1^{-1+\alpha} \cdot \sum_{n_1 \in \mathbb{Z}^2} n_1 \sum_{n_2 \in \mathbb{Z}^2} n_2 \sup_{n_1 \sim N_1, n_2 \sim N_2} |g_{n_1}(\omega)||g_{n_2}(\omega)| \int \frac{|\tilde{P}_{N\omega}(n_1 - n_2, \tau)|}{\langle \tau - |n_1 - n_2|^2 + |n_1|^2 - |n_2|^2 \rangle} d\tau
\]

Without loss of generality, we can assume that \( N_1 \leq N_2 \). By using the following Gaussian tail bounds:

\[
\sum_{n_1 \in \mathbb{Z}^2, n_1 \sim N_1} P(|g_{n_1}| > T^{-\theta} N_2^\delta) < C \exp \left( -\frac{N_2^\delta}{T^\theta} \right), \quad (3.25)
\]

\[
\sum_{n_2 \in \mathbb{Z}^2, n_2 \sim N_2} P(|g_{n_2}| > T^{-\theta} N_2^\delta) < C \exp \left( -\frac{N_2^\delta}{T^\theta} \right), \quad (3.26)
\]

we obtain

\[
(3.8) \lesssim T^{\delta-\theta} \sum_{N,N_1,N_2 \geq 1} N^s N_1^{-9+\alpha} N_2^{-9+2\delta+\alpha}.
\]

outside an exceptional set of probability \( \leq C \exp(-cN_2^\delta/T^\theta) \). Note that we have \( N \lesssim N_2 \). Thus, since \( \alpha < \frac{1}{2} \) and \( \delta, s > 0 \) are sufficiently small, we can sum over dyadic \( N, N_1, N_2 \geq 1 \) to obtain (3.7).

**Case 2:** \( \langle \tau \rangle \lesssim N_1^{10} N_2^{10} \).
In this case, by the Cauchy-Schwarz inequalities in \( \tau \) and \( n \), (3.3), and Minkowski’s inequality, we have

\[
\| \tilde{w} \|_{L^2_t L^4_x} \lesssim T^\delta \sup_{N, N_1, N_2 \geq 1 \text{ dyadic}} \sum_{n, n_1, n_2 \in \mathbb{Z}^2} N^s N_1^{3\delta} N_2^{3\delta} \times \left| \sum_{n_1, n_2 \in \mathbb{Z}^2} \int \int \int K(\tau, |n|^2 + |\tau_1|^2 - |\tau_2|^2) \langle \tau \rangle^{1/2 - \alpha} \right|
\]

\[
\times \left| \sum_{n_1, n_2 \in \mathbb{Z}^2} \int \langle \tau - \tau_1 + \tau_2 - m \rangle^{-1} \tilde{\chi}(\tau_1) \tilde{\chi}(\tau_2) \right|
\]

\[
\times \left| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{n_1 n_2}^m 1_{S_1} \cdot \frac{g_{n_1}(\omega)}{(n_1)^{1-\alpha}} \frac{g_{n_2}(\omega)}{(n_2)^{1-\alpha}} \langle \tau \rangle \langle \tau_1 \rangle \langle \tau_2 \rangle \left( \tau_1 \tau_2 \right) \right|^{1/2}, \quad (3.27)
\]

where \( h_{n_1 n_2}^m \) is the base tensor as defined in (2.3) and \( S_1 \) is as defined in (3.14). Note that for \( (n, n_1, n_2) \) restricted in \( S_1 \), we have \( \lesssim \max\{N_1^2, N_2^2\} \) choices for the value

\[
m = |n|^2 - |n_1|^2 + |n_2|^2,
\]

which implies that

\[
\sum_{m \in \mathbb{Z}} \langle \tau - \tau_1 + \tau_2 - m \rangle^{-1} \lesssim \log(1 + N_1^2 N_2^2) \lesssim N_1^\delta N_2^\delta. \quad (3.28)
\]

Again, we can assume without loss of generality that \( N_1 \leq N_2 \). Thus, continuing with (3.27), by Hölder’s inequality in \( m \), (3.28), Lemma 2.9 the Gaussian tail bounds (3.25) and (3.26), and Lemma 2.11 we have

\[
\lesssim T^\delta \sum_{N, N_1, N_2 \geq 1 \text{ dyadic}} N^s N_1^{3\delta} N_2^{3\delta} \sup_{m \in \mathbb{Z}} \left| \sum_{n_1, n_2 \in \mathbb{Z}^2} h_{n_1 n_2}^m 1_{S_1} \cdot \frac{g_{n_1}(\omega)}{(n_1)^{1-\alpha}} \frac{g_{n_2}(\omega)}{(n_2)^{1-\alpha}} \right|_n
\]

\[
\lesssim T^{\delta - 2\theta} \sum_{N, N_1, N_2 \geq 1 \text{ dyadic}} N^s N_1^{-1 + 3\delta + \alpha} N_2^{-1 + 3\delta + \alpha} \sup_{m \in \mathbb{Z}} \left| h_{n_1 n_2}^m 1_{S_1} \right|_{n_1 n_2}
\]

\[
\lesssim T^{\delta - 2\theta} \sum_{N, N_1, N_2 \geq 1 \text{ dyadic}} N^s N_1^{-\frac{1}{2} + 3\delta + \alpha} N_2^{-\frac{1}{2} + 3\delta + \alpha}
\]

outside an exceptional set of probability \( \lesssim C \exp(-cN_2^\delta/T^\theta) \) for some universal constants \( C, c > 0 \). Note that we have \( N \lesssim N_2 \). Thus, since \( \alpha < \frac{1}{2} \) and \( \delta, s > 0 \) are sufficiently small, we can sum over dyadic \( N, N_1, N_2 \geq 1 \) to obtain the desired inequality (3.7).

\( \square \)

**Remark 3.3.** The frequency projections \( P_{\neq 0} \) in all three parts of Proposition 3.2 are necessary in our approach. For (3.5) and (3.6), we need to avoid the zeroth frequencies.
due to the necessity of the condition \( n \neq 0 \) in the base tensor (2.8) (see also Remark 2.12).

For (3.7), without the frequency projection \( P_{\neq 0} \), one can show that the zeroth frequency diverges almost surely when \( \alpha \geq 0 \) using the argument as in the proof of Proposition 1.6 in [29], so that we need to remove the zeroth frequency.

Furthermore, in part (ii) of Proposition 3.2 the assumption that \( v \) has mean zero is important for us to obtain the desired estimate. Without this assumption, i.e. when \( v \) is allowed to be a non-zero constant, the LHS of (3.6) essentially becomes \( \|\chi \cdot z\|_{X^{s,\frac{1}{2}+\delta}} \), which is equal to infinity almost surely when \( \alpha \geq 0 \).

4. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1 the almost sure local well-posedness result of the quadratic NLS (1.1). We fix \( \alpha < \frac{1}{2} \) throughout this section.

We recall from (1.6) the following first order expansion:

\[
\Gamma \equiv z + v.
\]

Here, \( z \) is the random linear solution as in (1.5) and \( v \) is the remainder term that satisfies (1.7), which we can write in the following Duhamel formulation:

\[
v(t) = \Gamma[v](t) := -i\mathcal{I}_\chi \left(|z + v|^2 - \int |z + v|^2\right)(t),
\]

where \( 0 < t \leq 1 \) and \( \mathcal{I}_\chi \) is the Duhamel operator as defined in (2.1) with \( \chi \) being a smooth cut-off function such that \( \chi \equiv 1 \) on \([-1, 1]\) and \( \chi \equiv 0 \) outside of \([-2, 2]\). We note from (4.1) that \( v \) has mean zero (i.e. has no zeroth frequency term). We show that \( \Gamma \) is a contraction map on a ball of the space \( X^{s,b}_T \subset C([-T, T]; H^s(\mathbb{T}^2)) \) for some \( s > 0 \) and \( b > \frac{1}{2} \) outside an exceptional set of exponentially small probability.

Let \( s, \delta > 0 \) be sufficiently small. Let \( \varphi \) be an arbitrary smooth function with \( \varphi \equiv 1 \) on \([-1, 1]\) and \( \varphi \equiv 0 \) outside of \([-2, 2]\), and let \( \varphi_T(t) = \varphi(t/T) \) for \( 0 < T \leq 1 \). By the definition of \( X^{s,b}_T \)-norm, (1.1), Proposition 3.1 and Proposition 3.2, we have that for every \( 0 < T \leq 1 \),

\[
\|\Gamma[v]\|_{X^{s,\frac{1}{2}+\delta}} \leq \| P_{\neq 0} (\varphi_T \cdot \mathcal{I}_\chi (|\chi \cdot z + v|^2)) \|_{X^{s,\frac{1}{2}+\delta}}
\]

\[
\leq \| P_{\neq 0} (\varphi_T \cdot \mathcal{I}_\chi (|z + v|^2)) \|_{X^{s,\frac{1}{2}+\delta}} + \| P_{\neq 0} (\varphi_T \cdot \mathcal{I}_\chi (v \cdot \mathcal{I}_z)) \|_{X^{s,\frac{1}{2}+\delta}}
\]

\[
+ \| P_{\neq 0} (\varphi_T \cdot \mathcal{I}_\chi (\chi \cdot z \cdot \overline{z})) \|_{X^{s,\frac{1}{2}+\delta}} + \| P_{\neq 0} (\varphi_T \cdot \mathcal{I}_\chi (|\chi \cdot z|^2)) \|_{X^{s,\frac{1}{2}+\delta}}
\]

\[
\lesssim T^{\delta-\theta} \left( \|v\|_{X^{s,\frac{1}{2}+\delta}}^2 + 2 \|v\|_{X^{s,\frac{1}{2}+\delta}} + 1 \right),
\]

outside an exceptional set of probability \( \leq C \exp(-\frac{T}{\theta^2}) \) with \( C, c > 0 \) being constants and \( 0 < \theta \ll \delta \). Taking the infimum over all extensions of \( v \) outside the time interval \([-T, T]\), we obtain

\[
\|\Gamma[v]\|_{X^{s,\frac{1}{2}+\delta}} \lesssim T^\frac{\delta}{2} \left( \|v\|_{X^{s,\frac{1}{2}+\delta}}^2 + 1 \right)^2.
\]

Similarly, we obtain the following difference estimate outside an exceptional set of probability \( \leq C \exp(-\frac{T}{\theta^2})\):

\[
\|\Gamma[v_1] - \Gamma[v_2]\|_{X^{s,\frac{1}{2}+\delta}} \lesssim T^\frac{\delta}{2} \|v_1 - v_2\|_{X^{s,\frac{1}{2}+\delta}} \left( \|v_1\|_{X^{s,\frac{1}{2}+\delta}} + \|v_2\|_{X^{s,\frac{1}{2}+\delta}} + 1 \right).
\]
Therefore, for a fixed $R > 0$, by choosing $T = T(R) > 0$ sufficiently small, we obtain that $\Gamma$ is a contraction on the ball $B_R \subset X_T^{s,\frac{1}{2}+\delta}$ of radius $R$ outside an exceptional set of probability $\leq C \exp(-\frac{s}{R^\gamma})$. This finishes the proof of Theorem 1.1.

5. PROOF OF PROPOSITION 1.5

In this section, we prove Proposition 1.5, the non-convergence of the Picard second iterate $z_N^{(2)}$ as defined in (1.11).

We fix $n \neq 0, t \neq 0$, and $\alpha \geq \frac{3}{4}$. Let us first show that $\lim_{N \to \infty} \mathbb{E}[|F_N z_N^{(2)}(t,n)|^2] = \infty$.

A direct computation yields

$$
F_N z_N^{(2)}(t,n) = \int_0^t e^{-i(t-t')|n|^2} \sum_{k \in \mathbb{Z}^2 \atop |k| \leq N} e^{-i|n+k|^2 + i|k|^2} \frac{g_{n+k}(\omega)g_k(\omega)}{(n+k)^{1-\alpha}(k)^{1-\alpha}} dt'.
$$

By independence, we can compute that

$$
\mathbb{E}[|F_N z_N^{(2)}(t,n)|^2] = \sum_{k \in \mathbb{Z}^2 \atop |k| \leq N} \frac{1}{(n+k)^{2-2\alpha}(k)^{2-2\alpha}} \frac{2\sin (tn \cdot k)^2}{|n \cdot k|^2}. 
$$

We focus on the case when $n \cdot k = 0$, so that (5.2) is bounded from below (up to some constant depending only on $n$ and $t$) by

$$
\sum_{k \in \mathbb{Z}^2 \atop n \cdot k = 0} \frac{1}{(k)^{4-4\alpha}}. 
$$

We write $n = (n_1, n_2)$. Note that if either $n_1 = 0$ or $n_2 = 0$, then we can easily see that (5.3) diverges as $N \to \infty$ when $\alpha \geq \frac{5}{4}$. If $n_1 \neq 0$ and $n_2 \neq 0$, we note that all $k$’s that satisfy $n \cdot k = 0$ are of the form $k = ak'$, where $a \in \mathbb{Z}$ and

$$
k' = \left( -\frac{n_2}{\gcd(n_1, n_2)}, \frac{n_1}{\gcd(n_1, n_2)} \right).
$$

Thus, (5.3) is bounded from below by

$$
\sum_{a \in \mathbb{Z} \atop 0 < |a| \leq N/k|} \frac{1}{|a|^{4-4\alpha}(k')^{4-4\alpha}},
$$

which increases to infinity as $N \to \infty$ when $\alpha \geq \frac{3}{4}$. This shows that

$$
\mathbb{E}[|F_N z_N^{(2)}(t,n)|^2] \to \infty
$$

as $N \to \infty$.

We now show that, for any sequence $\{N_\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{N}$, the sequence of random variables $\{F_N z_N^{(2)}(t,n)\}_{\ell \in \mathbb{N}}$ is not tight. Assume for the sake of contradiction that $\{F_N z_N^{(2)}(t,n)\}_{\ell \in \mathbb{N}}$ is
tight. Using the explicit formula of $F\xi z_N^{(2)}(t,n)$ in (5.1), we can write $F\xi z_N^{(2)}(t,n) = X_\ell + iY_\ell$, where $X_\ell, Y_\ell \in \mathcal{H}_{\leq 2}$ are real-valued. Here, we recall that the space $\mathcal{H}_{\leq 2}$ is as defined in (2.2). By Lemma 2.8 we have

$$E[|F\xi z_N^{(2)}(t,n)|^4]^{\frac{1}{4}} \leq E[|X_\ell|^4]^{\frac{1}{4}} + E[|Y_\ell|^4]^{\frac{1}{4}} \leq 3E[|X_\ell|^2]^{\frac{1}{2}} + 3E[|Y_\ell|^2]^{\frac{1}{2}} \leq 3\sqrt{2}E[|F\xi z_N^{(2)}(t,n)|^2]^{\frac{1}{2}}.$$  \hspace{1cm} (5.5)

By the Paley-Zygmund inequality and (5.5), we have

$$P\left(|F\xi z_N^{(2)}(t,n)|^2 > \frac{E[|F\xi z_N^{(2)}(t,n)|^2]}{2}\right) \geq \frac{1}{4} \left(\frac{E[|F\xi z_N^{(2)}(t,n)|^2]}{E[|F\xi z_N^{(2)}(t,n)|^4]}\right)^{\frac{1}{2}} \geq \frac{1}{1296}.$$  \hspace{1cm} (5.6)

By tightness, we know that there exists a constant $A > 0$ such that for all $\ell \in \mathbb{N},$

$$P(|F\xi z_N^{(2)}(t,n)| > A) < \frac{1}{1296}.$$  \hspace{1cm} (5.7)

Due to (5.6) and (5.7), we must have $E[|F\xi z_N^{(2)}(t,n)|^2] \leq 2A^2$ for all $\ell \in \mathbb{N}$, which is a contradiction to (5.3). Therefore, the sequence $\{F\xi z_N^{(2)}(t,n)\}_{t \in \mathbb{N}}$ is not tight. This finishes the proof of Proposition 1.5.

**Remark 5.1.** In the proof above, although we only considered the case when $n \cdot k = 0$, we point out that the range $\alpha \geq \frac{3}{4}$ for the divergence of $E[|F\xi z_N^{(2)}(t,n)|^2]$ is sharp. More precisely, suppose that we have $\alpha < \frac{3}{4}$. Note that the RHS of (5.2) converges as $N \to \infty$ if and only if the following integral converges:

$$\int_{\{x \in \mathbb{R}^2 : |x| \leq N\}} \frac{1}{\langle x \rangle^{1-4\alpha}} \sin(tn \cdot x)^2 \frac{1}{|n \cdot x|^2} dx.$$  \hspace{1cm} (5.8)

By using a change of variable, we note that the convergence of (5.8) is equivalent to the convergence of the following term:

$$\int_0^N \int_0^N \frac{1}{(1 + |y_1|^2 + |y_2|^2)^{2-2\alpha}} \frac{\sin(ty_1)^2}{|y_1|^2} dy_1 dy_2,$$

which can easily be seen to converge when $\alpha < \frac{4}{3}$.

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