Three-loop electroweak corrections to the $W$-boson mass and $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ in the large Higgs mass limit

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Abstract

We present an analytical calculation of the leading three-loop radiative correction to the $S$-parameter in the Standard Model in the large Higgs mass limit. Numerically, $S^{(3)} = 1.1105 \times g^4/(1024\pi^3) \times m_H^4/M_W^4$. When combined with the corresponding three-loop correction to the $\rho$-parameter, this leads to shifts of $\Delta^{(3)} \sin^2 \theta_{\text{eff}}^{\text{lept}} = 4.6 \times 10^{-9} \times m_H^4/M_W^4$ in the effective weak mixing angle and $\Delta^{(3)} M_W = -6.3 \times 10^{-4} \text{MeV} \times m_H^4/M_W^4$ in the $W$ boson mass. For both of these observables, the sign of the three-loop correction is equal to that of the one-loop correction.

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1 Introduction

Radiative corrections to four-fermion processes are sensitive to the presence of heavy particles in the standard model. This allows one to draw conclusions about the mass of the Higgs boson from fits of precise electroweak measurements to theoretical predictions of the standard model [1]. For a light Higgs boson, the Higgs mass dependence of the theoretical predictions is mainly due to one-loop radiative corrections to the gauge boson propagators. These one-loop corrections depend on $m_H$ logarithmically [2]. However, because the Higgs self-interaction $\lambda$ is proportional to $m_H^2$, there are higher order radiative corrections which grow like powers of $m_H$ in the limit $m_H \to \infty$, eventually overcoming their relative suppression by powers of $\alpha$. Such higher order corrections could become important if the Higgs boson is very heavy. At the two-loop level, the leading corrections are proportional to $m_H^2$, but the numerical coefficient of these terms turns out to be very small [3, 4, 5], and therefore, they are not important for $m_H$ less than a few TeV. However, it has been suggested that the smallness of the two-loop corrections may be somewhat accidental [6]. If this is true, then one may expect larger corrections to appear at the three loop level.

An explicit calculation of the leading three-loop correction to the electroweak $\rho$-parameter in the large Higgs mass limit [7] has shown that, for this observable, the numerical coefficient of the three-loop correction is indeed larger than the two-loop coefficient, and that the leading three-loop correction $\Delta \rho^{(3)}$ is already equal in magnitude to the two-loop correction $\Delta \rho^{(2)}$ at a Higgs mass of approximately 480 GeV. The purpose of the present paper is to extend our previous investigation to other electroweak observables, in particular the sine of the effective leptonic weak mixing angle $\sin^2 \theta^\text{eff}_{\text{lep}}$, defined in terms of the couplings of the $Z$-boson to leptons, and the mass of the $W$ boson. At the two-loop level, the complete electroweak fermionic corrections to $\sin^2 \theta^\text{eff}_{\text{lep}}$ are known [8], and for the $W$ mass, both the fermionic and the bosonic corrections have been calculated [9]. Here, we are concerned with the leading three-loop bosonic corrections, which grow like $m_H^4$ in the large Higgs mass limit.

We will present our results in terms of the $S$, $T$ and $U$ parameters introduced by Peskin and Takeuchi [10] to describe the effects of heavy particles that enter only through corrections to the gauge boson propagators (oblique corrections). They are defined in terms of the transverse gauge boson self-energies at zero momentum $\Sigma_X^T \equiv \Sigma_X^T (p^2) |_{p^2=0}$ and their first derivative $\Sigma_X'^T \equiv \frac{\partial}{\partial p^2} \Sigma_X^T (p^2) |_{p^2=0}$ as

\begin{align}
S & \equiv \frac{4s_W^2 - c_W^2}{\alpha} \left( \frac{\Sigma_{ZZ}^T}{c_W s_W} - \frac{\Sigma_{AZ}^T}{c_W s_W} \right) \quad (1)
\end{align}

\begin{align}
T & \equiv \frac{1}{\alpha M_W^2} \left( c_W \Sigma_{ZZ}^T - \Sigma_{WW}^T \right) \quad (2)
\end{align}

\begin{align}
U & \equiv \frac{4s_W^2}{\alpha} \left( \Sigma_{WW}^T - c_W \Sigma_{ZZ}^T - 2c_W s_W \Sigma_{AZ}^T - s_W^2 \Sigma_{AA}^T \right) \quad (3)
\end{align}
where \( c_W = \cos \theta_W, s_W = \sin \theta_W \). The leading corrections to four-fermion processes in the heavy Higgs mass limit can be treated in this framework provided that there are no further contributions from vertex and box diagrams. In the renormalization scheme we are using, where \( m_H \) and \( M_W \) are defined by subtractions at \( p^2 = -m_H^2 \) and \( p^2 = 0 \), respectively (which is equivalent to the on-shell scheme if one is only interested in the leading term in the large Higgs mass limit), this condition is satisfied [3, 11].

2 Calculation

All our diagrams are generated by the program QGRAF [12]. The rest of the calculation is done mainly in FORM [13]. First of all, we need the bare gauge boson self-energies, which we decompose into their transverse and longitudinal components as

\[
\Sigma^{X}_{\mu\nu}(p) = \left( g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right) \Sigma^{X}_{T}(p^2) + \frac{p_{\mu}p_{\nu}}{p^2} \Sigma^{X}_{L}(p^2),
\]

where \( X = AA, AZ, ZZ, WW \). The scalar functions \( \Sigma^{T}_{X}(p^2) \) and \( \Sigma^{L}_{X}(p^2) \) are extracted from \( \Sigma^{X}_{\mu\nu}(p) \) by means of the projectors \( P_T = \frac{1}{d-1} \left( g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right) \) and \( P_L = \frac{p_{\mu}p_{\nu}}{p^2} \) respectively, where \( d = 4 - \varepsilon \) is the space-time dimension, and expanded in a Taylor series up to order \( p^2 \). The coefficients are vacuum integrals, which, in general, depend on three different non-zero masses: \( m_H, M_W \) and \( M_Z \).

We use the method of expansion by regions [14] to expand them in powers of \( m_H \) for \( m_H \gg M_W, M_Z \), keeping all terms proportional to \( m_H^4 \) or higher in the three-loop gauge boson self-energies. As discussed in more detail in ref. [7], the expansion by regions produces:

(a) three-loop single-scale vacuum integrals, where the mass scale is \( m_H \) (from the region where all loop momenta are large);

(b) products of one- (two-) loop vacuum integrals depending on \( m_H \) and two- (one-) loop vacuum integrals depending on the small scales \( M_W \) and \( M_Z \) (from the regions where some of the loop momenta are small and others are large);

(c) and finally three-loop vacuum integrals depending on \( M_W \) and \( M_Z \) (from the region where all loop momenta are small).

Using integration-by-parts identities, we reduce all of these vacuum integrals to master integrals. For cases (a) and (c), we use the automatic integral reduction package AIR [15]. The integrals that occur in case (a) have \( \mathcal{N}_+ \leq 5 \) and \( \mathcal{N}_- \leq 2 \), where \( \mathcal{N}_+ \) is equal to the sum of the positive indices, minus the number of positive indices, and \( \mathcal{N}_- \) denotes the absolute value of the sum of the negative indices. The three-loop single-scale master integrals appearing in case (a) can be found as expansions in \( \varepsilon \) in ref. [16, 17].

In case (c), we have to reduce two-scale integrals with \( \mathcal{N}_+ \leq 4 \) and \( \mathcal{N}_- \leq 2 \). However, once we sum over all diagrams, in all the three-loop gauge boson self-energies, the non-factorizable three-loop master integrals occurring in case (c) cancel, so that explicit formulae for such master integrals are not needed.
The longitudinal parts of the gauge boson self-energies are related to the self-energies of the scalars and the mixings between the scalars and the gauge bosons by a set of Ward identities, which are shown explicitly in appendix A. We have verified that these Ward identities are satisfied by the full (ie., including tadpole contributions), unrenormalized self-energies (up to order $p^2$).

3 Renormalization

We work with the Lagrangian, $\mathcal{L} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{FP}}$, with the invariant part

$$\mathcal{L}_{\text{inv}} = -\frac{1}{4} W_\mu^\alpha W^{\alpha,\mu} - \frac{1}{4} B_\mu B^{\mu} - (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{1}{2} \lambda (\Phi^\dagger \Phi)^2 - \mu \Phi^\dagger \Phi,$$  

with the Higgs doublet

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} H + \sqrt{2}v + i\phi^0 \\ i\phi^1 - \phi^2 \end{pmatrix},$$

and the gauge fixing term

$$\mathcal{L}_{\text{fix}} = -C^+ C^- - \frac{1}{2} (C^Z)^2 - \frac{1}{2} (C^A)^2,$$

with

$$C^\pm = -\partial_\mu W^{\pm,\mu} + M_W \phi^\pm$$

$$C^Z = -\partial_\mu Z^\mu + M_Z \phi^0$$

$$C^A = -\partial_\mu A^\mu.$$  

The parameters $v$, $\lambda$ and $\mu$ are given by:

$$v = \sqrt{2} \frac{M_W}{g},$$

$$\lambda = g^2 \frac{m_H^2}{4M_W^2},$$

$$\mu = \beta - \frac{1}{2} m_H^2.$$  

Giving $\beta$ a non-zero value produces a term in the Lagrangian that is linear in the Higgs field $H$. We adjust this parameter order by order to make the renormalized Higgs tadpole vanish. We introduce two further renormalization constants, $Z_H$ and $Z_{m_H}$, by making the following substitutions in $\mathcal{L}_{\text{inv}}$ (but not in $\mathcal{L}_{\text{fix}}$ or $\mathcal{L}_{\text{FP}}$):

$$m_H \rightarrow Z_{m_H} m_H$$

$$M_W \rightarrow Z_H M_W$$
\[ H \rightarrow Z_H H \] (16)
\[ \phi^\pm \rightarrow Z_H \phi^\pm \] (17)
\[ \phi^0 \rightarrow Z_H \phi^0 . \] (18)

The renormalization constants are fixed by imposing the following conditions on the renormalized Higgs tadpole \( \Gamma_{H,\text{ren}} \) and on the renormalized \( \phi \) and \( H \) self-energies:

\[ \Gamma^{(1),H,\text{ren}} = \mathcal{O}(m_H^0) \] (19)
\[ \Gamma^{(2),H,\text{ren}} = \mathcal{O}(m_H^2) \] (20)
\[ \Sigma'^{(1),\phi\phi,\text{ren}}|_{p^2=0} = \mathcal{O}(m_H^0) \] (21)
\[ \Sigma'^{(2),\phi\phi,\text{ren}}|_{p^2=0} = \mathcal{O}(m_H^2) \] (22)
\[ \text{Re } \Sigma^{(1),HH,\text{ren}}|_{p^2+m_H^2=0} = \mathcal{O}(m_H^2) \] (23)
\[ \text{Re } \Sigma^{(2),HH,\text{ren}}|_{p^2+m_H^2=0} = \mathcal{O}(m_H^4) \] (24)

These renormalizations remove all the terms of order \( m_H^2 \) and of order \( m_H^4 \) from the one- and two-loop gauge boson self-energies, the \( \phi \) self-energies, and the mixings between \( \phi \)'s and gauge bosons. This ensures that no two- or three-loop vertex or box graphs containing such self-energies as subgraphs can give corrections that grow like \( m_H^2 \) or \( m_H^4 \) in the large Higgs mass limit, and allows us to restrict our attention to the corrections that come from the gauge boson self-energies. The renormalization procedure used here is slightly different from the one used in ref. [7], where the parameter \( \beta \) was not used and therefore, tadpole contributions had to be taken into account explicitly. In particular, the constant \( Z_{m_H} \) used here is different from \( Z_m \) of ref. [7]. However, the two procedures are equivalent to each other. Explicit formulae for the renormalization constants are listed in appendix B.

We have verified that the renormalized longitudinal photon self-energy and photon-Z mixing are zero, as they should.

### 4 Results and conclusion

Here, we collect the leading one-loop [18, 19] and two-loop [4, 5] contributions and the new three-loop contribution to the \( S \)-parameter in the heavy Higgs limit:

\[ S^{(1)} = \frac{1}{12\pi} \log \left( \frac{m_H^2}{M_W^2} \right), \] (25)

\[ S^{(2)} = \frac{1}{4\pi} \left( \frac{g^2}{16\pi^2} \right) \frac{m_H^2}{M_W^2} \left( -\frac{35}{72} - \frac{1}{8} \pi \sqrt{3} + \frac{7}{54} \pi^2 \right) \]
\[ = \frac{1}{4\pi} \left( \frac{g^2}{16\pi^2} \right) \frac{m_H^2}{M_W^2} (0.1131), \] (26)
\[ S^{(3)} = \frac{1}{4\pi} \left( \frac{g^2}{16\pi^2} \right)^2 \frac{m_H^4}{M_W^4} \left( \frac{1153}{576} - \frac{19}{48} \pi \sqrt{3} + \frac{13}{16} \pi C + \frac{2753}{10368} \pi^2 - \frac{109}{432} \pi^3 \sqrt{3} \right) \]
\[ \quad - \frac{7199}{155520} \pi^4 + \frac{7}{4} \sqrt{3} C \log 3 - \frac{21}{8} \sqrt{3} Ls_3 \left( \frac{2\pi}{3} \right) \]
\[ \quad - \frac{105}{16} \sqrt{3} C + \frac{38525}{3456} \zeta(3) - \frac{25}{24} C^2 - \frac{17}{18} U_{3,1} - 2 V_{3,1} \] 
\[ = \frac{1}{4\pi} \left( \frac{g^2}{16\pi^2} \right)^2 \frac{m_H^4}{M_W^4} (1.1105) . \] (27)

The constants appearing in these expressions are [16, 17]:

\[ U_{3,1} = \frac{1}{2} \zeta(4) + \frac{1}{2} \zeta(2) \log^2 2 - \frac{1}{12} \log^4 2 - \text{Li}_4 \left( \frac{1}{2} \right) = -0.11787599965 \] (28)

\[ V_{3,1} = \sum_{m>n>0} (-1)^m \frac{\cos(2\pi n/3)}{m^3 n} = -0.03901272636, \] (29)

\[ C = \text{Cl}_2 (\pi/3) = 1.0149416064, \] (30)

\[ Ls_3 \left( \frac{2\pi}{3} \right) = - \int_0^{2\pi/3} d\phi \log^2 \left| 2 \sin \frac{\phi}{2} \right| = -2.1447672126. \] (31)

The \( T \)-parameter is related to the \( \rho \)-parameter by \( \Delta \rho = \alpha T \), so that, from refs. [20, 3, 7],

\[ T^{(1)} = -\frac{3}{16\pi c_W^2} \log \left( \frac{m_H^2}{M_W^2} \right) , \] (32)

\[ T^{(2)} = \frac{1}{4\pi c_W^2} \left( \frac{g^2}{16\pi^2} \right) \frac{m_H^2}{M_W^2} (0.1499) \] , (33)

\[ T^{(3)} = \frac{1}{4\pi c_W^2} \left( \frac{g^2}{16\pi^2} \right)^2 \frac{m_H^4}{M_W^4} (-1.7282) . \] (34)

In contrast to \( S \), the parameter \( U \) can only be different from zero if custodial symmetry is broken, and as a result, it should be suppressed compared to \( S \) [5]. In the approximation where we keep only logarithmic terms in \( m_H \) at the one-loop level, quadratic terms at the two-loop level and quartic terms at the three-loop level, we find that \( U \) vanishes. This provides a very useful check on the calculation.

Knowing \( S, T, \) and \( U \), one can easily obtain the corresponding corrections to various electroweak precision observables [10, 21, 22]. We illustrate this by two examples. The effective weak mixing angle is shifted relative to its tree level value, expressed in terms of \( \alpha, G_F \) and \( M_Z \), by

\[ \sin^2 \theta_{\text{lept}}^\text{eff} = \Delta \sin^2 \theta_{\text{lept}}^\text{eff} + \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\pi \alpha}{\sqrt{2} G_F M_Z^2}} , \] (35)
With
\[ \Delta \sin^2 \theta^{\text{lept}}_{\text{eff}} = \frac{\alpha}{c_W^2 - s_W^2} \left( \frac{1}{4} S - s_W^2 c_W^2 T \right) \]  
(36)

Similarly, one finds a shift in the W-mass,
\[ M_W = \Delta M_W + M_Z \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\pi \alpha}{\sqrt{2} G_F M_Z^2}}} \]  
(37)

with
\[ \Delta M_W = \frac{\alpha M_W}{2(c_W^2 - s_W^2)} \left( -\frac{1}{2} S + c_W^2 T + \frac{e_W^2 - s_W^2}{4 s_W^2} U \right) \]  
(38)

Numerical values of these shifts, calculated using
\[ g^2 = \frac{e^2}{s_W^2} = \frac{4\pi \alpha}{s_W^2} \]  
(39)

for the weak coupling constant, with \( \alpha = 1/137 \) and \( s_W^2 = 0.23 \), are shown in Tables 1 and 2 and in Figures 1 and 2. The two-loop corrections are extremely small. This is partly due to a cancellation between \( S^{(2)} \) and \( T^{(2)} \) in eqs. (36) and (38). At the three-loop level, there is no such cancellation between \( S^{(3)} \) and \( T^{(3)} \), since they have

| \( m_H/M_W \) | \( \Delta^{(1)} \sin^2 \theta^{\text{lept}}_{\text{eff}} \) | \( \Delta^{(2)} \sin^2 \theta^{\text{lept}}_{\text{eff}} \) | \( \Delta^{(3)} \sin^2 \theta^{\text{lept}}_{\text{eff}} \) |
|-----------------|-----------------|-----------------|-----------------|
| 2               | 3.8 \times 10^{-4} | -6.7 \times 10^{-8} | 7.4 \times 10^{-8} |
| 5               | 8.9 \times 10^{-4} | -4.2 \times 10^{-7} | 2.9 \times 10^{-6} |
| 10              | 1.3 \times 10^{-3} | -1.7 \times 10^{-6} | 4.6 \times 10^{-5} |
| 15              | 1.5 \times 10^{-3} | -3.8 \times 10^{-6} | 2.3 \times 10^{-4} |
| 20              | 1.6 \times 10^{-3} | -6.7 \times 10^{-6} | 7.4 \times 10^{-4} |
| 25              | 1.8 \times 10^{-3} | -1.1 \times 10^{-5} | 1.8 \times 10^{-3} |

Table 1: Corrections to \( \sin^2 \theta^{\text{lept}}_{\text{eff}} \) as a function of \( m_H/M_W \).

| \( m_H/M_W \) | \( \Delta^{(1)} M_W \) | \( \Delta^{(2)} M_W \) | \( \Delta^{(3)} M_W \) |
|-----------------|-----------------|-----------------|-----------------|
| 2               | -0.055 | 0.00041 | -0.00010 |
| 5               | -0.13 | 0.00025 | -0.00039 |
| 10              | -0.18 | 0.0010 | -0.0063 |
| 15              | -0.21 | 0.0023 | -0.032 |
| 20              | -0.24 | 0.0041 | -0.10 |
| 25              | -0.26 | 0.0064 | -0.25 |

Table 2: Corrections to \( M_W \) in GeV as a function of \( m_H/M_W \).
opposite signs. For both observables, the three-loop correction becomes equal to the one-loop correction at \( m_H \approx 2 \text{ TeV} \). Since the three-loop term has the same sign as the one-loop term in both cases, it is clearly not possible for the radiative corrections due to a heavy Higgs boson to mimic those of a light Higgs boson.

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A Ward identities

The Ward identities for the self-energies can be derived by expanding the following identities for the dressed propagators \( G \):

\[
1 = p^\mu p^\nu G^{AA}_{\mu\nu}(p) \tag{40}
\]

\[
0 = p^\mu p^\nu G^{AZ}_{\mu\nu}(p) + ip^\mu M_Z G_\phi^{\phi A}(p) \tag{41}
\]

\[
1 = p^\mu p^\nu G^{ZZ}_{\mu\nu}(p) + 2ip^\mu M_Z G_\phi^{\phi Z}(p) + M_Z^2 G_\phi^{\phi\phi}(p) \tag{42}
\]

\[
1 = p^\mu p^\nu G^{WW}_{\mu\nu}(p) + 2ip^\mu M_W G_\phi^{\phi W}(p) + M_W^2 G_\phi^{\phi\phi}(p) \tag{43}
\]

Expanding the propagators \( G \) in eqs. (40)–(43) in terms of the self-energies \( \Sigma \), one obtains a set of Ward identities. For the mixings between Goldstone and gauge
bosons, we write
\[ \Sigma^X_\mu(p) = p_\mu \Sigma^X(p^2) \]  
where \( X = \phi^0 A, \phi^0 Z, \phi W \).

Up to three loop order, we find the following relations.

\[ 0 = \Sigma^{(1)}_{Z Z} \]  
\[ 0 = \Sigma^{(1)}_{W W} \]  
\[ 0 = \Sigma^{(1)}_{A A} \]  
\[ 0 = \Sigma^{(2)}_{A A} \]  
\[ 0 = \Sigma^{(1)}_{A Z} \]  
\[ 0 = \Sigma^{(2)}_{A Z} \]  
\[ 0 = \Sigma^{(1)}_{Z W} \]  
\[ 0 = \Sigma^{(2)}_{Z W} \]  
\[ 0 = \Sigma^{(1)}_{W W} \]

Figure 2: Shifts in \( M_W \) as a function of \( m_H/M_W \).

\[ + M_W^2 \left( \frac{1}{p^2} \left( \Sigma^{(1),\phi^0} \right)^2 + \Sigma^{(2),\phi^0} \right) \]

\[
0 = \Sigma^{(3),AA}_L - \frac{1}{p^2} \Sigma^{(1),\phi^0 \phi^0} \left( \Sigma^{(1),\phi^0 A} \right)^2 - 2 \Sigma^{(1),\phi^0 A} \Sigma^{(2),\phi^0 A} \]

\[
0 = \Sigma^{(3),AZ}_L - \frac{1}{p^2} \Sigma^{(1),\phi^0 \phi^0} \Sigma^{(1),\phi^0 A} \Sigma^{(2),\phi^0 Z} - \Sigma^{(1),\phi^0 A} \Sigma^{(2),\phi^0 Z} - \Sigma^{(1),\phi^0 Z} \Sigma^{(2),\phi^0 A} + \Sigma^{(3),\phi^0 A} \]

\[
0 = p^2 \Sigma^{(3),ZZ}_L - \Sigma^{(1),\phi^0 \phi^0} \left( \Sigma^{(1),\phi^0 Z} \right)^2 - 2 p^2 \Sigma^{(1),\phi^0 Z} \Sigma^{(2),\phi^0 Z} \]

\[
+ 2 i M_Z \left( \frac{1}{(p^2)^2} \left( \Sigma^{(1),\phi^0 \phi^0} \right)^2 \Sigma^{(1),\phi^0 A} + \frac{1}{p^2} \Sigma^{(1),\phi^0 \phi^0} \Sigma^{(2),\phi^0 A} + \frac{1}{p^2} \Sigma^{(2),\phi^0 \phi^0} \Sigma^{(1),\phi^0 A} \Sigma^{(3),\phi^0 A} \right) \]

\[
+ M_Z^2 \left( \frac{1}{(p^2)^2} \left( \Sigma^{(1),\phi^0 \phi^0} \right)^3 + \frac{2}{p^2} \Sigma^{(1),\phi^0 \phi^0} \Sigma^{(2),\phi^0 \phi^0} + \Sigma^{(3),\phi^0 \phi^0} \right) \]

\[
0 = p^2 \Sigma^{(3),WW}_L - \Sigma^{(1),\phi^0} \left( \Sigma^{(1),\phi W} \right)^2 - 2 p^2 \Sigma^{(1),\phi W} \Sigma^{(2),\phi W} \]

\[
+ 2 i M_W \left( \frac{1}{(p^2)^2} \left( \Sigma^{(1),\phi^0} \right)^2 \Sigma^{(1),\phi W} + \Sigma^{(1),\phi^0} \Sigma^{(2),\phi W} + \Sigma^{(2),\phi W} \Sigma^{(1),\phi W} \right) \]

\[
+ M_W^2 \left( \frac{1}{(p^2)^2} \left( \Sigma^{(1),\phi^0} \right)^3 + \frac{2}{p^2} \Sigma^{(1),\phi^0} \Sigma^{(2),\phi^0} + \Sigma^{(3),\phi^0} \right) \]  

B  Renormalization constants

In \( d = 4 - \varepsilon \) dimensions, the renormalization constants \( Z_H \) and \( \beta \) are given by \( Z_H = 1 - \delta_H^{(1)} - \delta_H^{(2)} \) and \( \beta = m_H^2 \left( \beta^{(1)} + \beta^{(2)} \right) \), with

\[
\delta_H^{(1)} = \frac{1}{i(2\pi)^d} \frac{g^2}{M_W^2} I_1(m_H^2; 1) \frac{\varepsilon}{8 (\varepsilon - 4)} \]

\[
\delta_H^{(2)} = \left[ \frac{1}{i(2\pi)^d} \frac{g^2}{M_W^2} \right]^2 \left\{ I_1(m_H^2; 1)^2 \left( \frac{1}{128} - \frac{3}{16 (\varepsilon - 4)} + \frac{1}{8 (\varepsilon - 4)^2} \right) \right\}
\]
\[ + m_{H}^2 I_2(m_{H}^2, m_{H}^2, m_{H}^2; 1, 1, 1) \left( \frac{3 \varepsilon}{64} + \frac{9}{64} + \frac{3}{8(\varepsilon - 4)} \right) \]
\[ + m_{H}^2 I_2(m_{H}^2, 0, 0; 1, 1, 1) \left( \frac{\varepsilon}{64} + \frac{11}{64} + \frac{3}{4(\varepsilon - 4)} \right) \}
\[ + \frac{1}{i(2\pi)^d M_W^2} I_1(m_{H}^2; 1) \left( \frac{1}{4} + \frac{\varepsilon}{8} + \frac{1}{(\varepsilon - 4)} \right) \delta_{m_H}^{(1)} \] (58)

\[ \beta^{(1)} = \frac{1}{i(2\pi)^d M_W^2} \left\{ -\frac{3}{8} I_1(m_{H}^2; 1) - \frac{1}{4} I_1(M_W^2; 1) - \frac{1}{8} I_1(M_Z^2; 1) \right\} \] (59)
\[ \beta^{(2)} = \left[ \frac{1}{i(2\pi)^d M_W^2} \right]^2 \left\{ I_1(m_{H}^2; 1)^2 \left( -\frac{3}{32} - \frac{3}{8(\varepsilon - 4)} \right) \right. \]
\[ + m_{H}^2 I_2(m_{H}^2, m_{H}^2, m_{H}^2; 1, 1, 1) \left( \frac{15}{64} - \frac{9 \varepsilon}{64} \right) \]
\[ + I_2(m_{H}^2, 0, 0; 1, 1, 1) \left( \frac{m_{H}^2 9 - 3 \varepsilon}{64} - M_W^2 \frac{2 c_W^2 + 1)(\varepsilon - 2)(\varepsilon - 9)}{32 c_W^2} \right) \]
\[ + I_1(m_{H}^2; 1)I_1(M_W^2; 1) \left( \frac{3 \varepsilon}{32} - \frac{3}{16} \right) + I_1(m_{H}^2; 1)I_1(M_Z^2; 1) \left( \frac{3 \varepsilon}{64} - \frac{3}{32} \right) \}
\[ + \frac{1}{i(2\pi)^d M_W^2} \left\{ I_1(m_{H}^2; 1) \left( \frac{3}{2} - \frac{3 \varepsilon}{8} \right) + \frac{1}{2} I_1(M_W^2; 1) + \frac{1}{4} I_1(M_Z^2; 1) \right\} \delta_{m_H}^{(1)} \] (60)

The one- and two-loop scalar integrals used in these expressions are defined by
\[ I_2(m_1^2, m_2^2, m_3^2; n_1, n_2, n_3) = \]
\[ \int d^d k_1 d^d k_2 P(k_1; m_1)^{n_1} P(k_2; m_2)^{n_2} P(k_1 + k_2; m_3)^{n_3} \] (61)
\[ I_1(m_1^2; n_1) = \]
\[ \int d^d k_1 P(k_1; m_1)^{n_1} \] (62)

with
\[ P(k; m) = \frac{1}{k^2 + m^2}. \] (63)

Note that in \( \beta^{(1)} \) and \( \beta^{(2)} \), it is necessary to keep not only the leading power of \( m_H \), but also the next-to-leading power.
The Higgs mass renormalization constant $Z_{m_H} = 1 - \delta_{m_H}^{(1)} - \delta_{m_H}^{(2)}$.

$$\delta_{m_H}^{(1)} = \left( \frac{g^2}{16\pi^2} \right) \left( \frac{m_H^2}{4\pi} \right)^{-\frac{3}{2}} \Gamma \left( 1 + \frac{\varepsilon}{2} \right) \frac{m_H^2}{M_W^2} \left\{ -\frac{3}{2\varepsilon} - \frac{3}{16} \pi \sqrt{3} ight. +$$

$$+ \varepsilon \left( -\frac{3}{32} \pi \sqrt{3} \log 3 + \frac{3}{16} \pi \sqrt{3} + \frac{1}{16} \pi^2 + \frac{3}{8} \sqrt{3} C - \frac{3}{2} \right) \right\}, \quad (64)$$

$$\delta_{m_H}^{(2)} = \left( \frac{g^2}{16\pi^2} \right)^2 \left( \frac{m_H^2}{4\pi} \right)^{-\varepsilon} \Gamma^2 \left( 1 + \frac{\varepsilon}{2} \right) \frac{m_H^4}{M_W^4} \left\{ -\frac{27}{8\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{27}{32} \pi \sqrt{3} - \frac{363}{64} \right) ight.$$

$$- \frac{1575}{256} - \frac{27}{64} \pi \sqrt{3} \log 3 + \frac{291}{128} \pi \sqrt{3} - \frac{39}{32} \pi C - \frac{177}{512} \pi^2 + \frac{3}{32} \sqrt{3} C + \frac{63}{32} \zeta(3) \right\}. \quad (65)$$

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