Analytical solution for the correlator with Gribov propagators

V. Šauli

1Department of Theoretical Physics, Institute of Nuclear Physics Rez near Prague, CAS, Czech Republic

The propagators approximated by a meromorphic functions with complex conjugated poles are widely used to model infrared behavior of colored excitations. In this paper, the analytical solution for a correlators made out of such meromorphic functions has been obtained in Minkowski space for the first time. The scalar correlator is represented by a sum of \( \tan^{-1} \) of a complex argument and it receives a nontrivial momentum dependent phase at the both -the timelike and the spacelike- domain of square of momenta.

PACS numbers: 11.10.St, 11.15.Tk

I. INTRODUCTION

The explanation of hadron properties in terms of QCD degrees of freedom represents hard non-perturbative task, especially when the energy of a process does not comply with the asymptotic freedom and particle like description of hadron constituents. Chirall symmetry breaking and confinement are the main phenomena beyond the applicability of perturbation theory, which, by its definition, deals with a free propagators of quarks and gluons. That confinement should be naturally encoded in an analytical properties of QCD Green’s functions is an old-fashionable conjecture [1–7]. In practice, most of QCD calculations to date are based on the lattice or Schwinger-Dyson equations studies, where an Euclidean metric is used as a defining one. Obviously, physics happens in Minkowski space. The problem is whether an analytical continuation of Euclidean solutions can give a definite answer, especially if one is interested in the sector of excitations, which are not directly physically observable, to name the most important examples: the quarks and the gluons.

If there were the exact duality between the particles and the quantum fields in the nature, then related two point functions would satisfy Lehmann representation. As a consequence, the only non-analyticities associated with the propagator functions were located at the real positive axis of momenta, the lowest point would be called a threshold or a simple pole and there would be no confinement of quarks and gluons in such a world. The lost of reflection positivity is believed to be associated with confinement, while silently assuming that some form of dispersion relation with a positive threshold and sign changing “generalized spectral” function exists. This, one step aside out of the usual, can be nicely check by a fitting a lattice data against the form of Stijltjes representation [8, 9] or by solving SDEs system [10, 11] in Minkowski space within a conventional assumption of perturbation theory analyticity. While the lattice data are improving from year to year, the fit made in [8] is certainly not ultimate one. However, when comparing a toy Breyght-Wigner model there, already now one sees that the lattice gluon propagator does not fit spectral integral with vanishing difference. Also the DSEs solutions comply with analyticity only bellow some critical coupling. It has been shown in the paper [12] that chiral symmetry breaking in QCD can be hardly achieved within the use of Lehmann representation for the quark propagator. In fact, all these studies represent an indirect evidence that the analytical structure of Greens functions describing confined objects changes. Not surprisingly, a complex conjugated poles or singularities unavoidably arises in the Bethe-Salpeter SDEs meson studies [13, 14]. It is plain to say that, if there is no other real pole, this is a must, since the only sum complex conjugated functions, involving any kind of singularities (not necessarily a poles), which should be located at timelike (right half of complex plane) square of momenta can ensure the reality of correlators at spacelike (\( \Re p^2 < 0 \) within our metric convention). To get rid of reality of an Euclidean data, slightly generalized Gribov form of propagators, i.e. the sum of the functions with complex conjugated poles have been widely considered in the literature. For instance, the quark propagator in the models [13, 14] shows up three pairs of complex conjugated poles with a complex conjugated residue as well. Recall, most of a recent BSE/SDEs studies represent an UV improved variants of the original model introduced in the paper [17], where a complex conjugated singularities were briefly mentioned in the context of quark SDE for the first time. Actually, the scenario of analytical confinement is undertaken seriously at these days [20].

Here we would like to show, that one can deal with such propagator analytically in Minkowski space (the same cane be shown for the Euclidean space correlator, however we do not wish to compare apples and pies in this paper.
and do not publish the result here). Let us consider the following two point correlators

$$\Pi(p) = i \int \frac{d^4l}{(2\pi)^4} \Gamma G_a(l) \Gamma G_b(l - p)$$  \hspace{1cm} (1.1)$$
defined in the momentum Minkowski space rather then in the Euclidean one. In rel. (1.1) $\Gamma$ is a matrix with possible Lorentz, Dirac and any discrete group indices suppressed for a while. The function $G$ stand for the Gribov gluon (or quark) propagator approximated by a sum of simple complex conjugated poles. Depending on explicit indices, the relation (1.1) represents skeleton graph contribution to either gluon polarization function, quark selfenergy or it can stand for $V - V$ or $A - A$ colorless correlators celebrated in history of QCD Sum Rules \cite{18, 19}. The implicit summation over the group indices as well as other tensorial structure is relevant, however it is largely unimportant for the following discussion.

In what follow we restrict to a simple case of the scalar correlator ($\Gamma = 1$) with a confining infrared form of the propagators

$$G^{-1}(l) = (l^2 - a)^2 + b^2,$$  \hspace{1cm} (1.2)$$
noting that (1.2) ensures UV 3+1d finiteness of (1.1) as $G$ falls like $1/l^4$ at UV asymptotic and a usual renormalization procedure should be in the work otherwise.

The scalar correlator under consideration

$$\Pi(p^2) = iI(p^2) = i \int \frac{d^4l}{(2\pi)^4} G(l) G(l - p),$$  \hspace{1cm} (1.3)$$
is thus the convolution of two identical meromorphic functions, each with a pair of complex conjugated poles. In accordance to our choice, it is obvious that $I$ has positive definite and regular integrand, which is real valued function of $l$ and $p$. Without any effort one sees, that the results must be definitively real for $I$ and completely imaginary for $\Pi$. Recall for clarity here, that contrary to this, the $\Pi_E$ defined through the Euclidean theory is a real quantity for Euclidean (spacelike) momenta. Obviously $\Pi(p^2)$ is not an analytical continuation of its counter-partner defined $\Pi_E$ since they already differs at spacelike domain (eg. at the domain where the continuation to the rest of complex plane should start).

### II. MINKOWSKI SPACE RESULT FOR $\Pi$

Similarly to the perturbation theory treatment it is advantageous to rewrite "propagators" in a way we can utilize the translation invariance explicitly during the integration. The Feynman parameterizations represent a useful trick and for this purpose let as rewrite the propagator in the following way:

$$\frac{1}{(p^2 - a)^2 + b^2} = \frac{1}{p^2 - a + ib} \frac{1}{p^2 - a - ib} = \int_0^1 dx \frac{1}{[p^2 - a + ib - 2ibx]^2}. \hspace{1cm} (2.1)$$

Performing the the same for the second propagator, the product of two propagators in (1.3) can be written as

$$\int_0^1 dx_1 dx_2 \frac{1}{[l^2 - a + 2ibx_1]^2} \frac{1}{[(l - p)^2 - a + 2ibx_2]^2} \frac{1}{[2y + (l - p)^2(1 - y) - a + ib - 2ib(x_1y + x_2(1 - y))]^4 \Gamma(4)} \hspace{1cm} (2.2)$$

Shifting momentum $l$ one gets for polarization:

$$\Pi(p^2) = i \int \frac{d^4l}{(2\pi)^4} \int_0^1 dy dx_1 dx_2 \frac{y(1 - y) \Gamma(4)}{[l^2y + (l - p)^2(1 - y) - a + ib - 2ib(x_1y + x_2(1 - y))]^4},$$

$$\Omega = a + ib - 2ib(x_1y + x_2(1 - y)). \hspace{1cm} (2.3)$$

Depending on the parameters the pole is located in the upper or bellow part of complex plane of square of momentum $l^2$. For such $\Omega$ for which the pole is located down, one can use the Wick rotation in variable $l_o$, while for opposite case i.e. $\Im \Omega < 0$ it is advantageous to use the integration contour, which is just mirror symmetric of mentioned Wick rotation contour \cite{21}. In both cases the inner parts are thus free of singularities and the use of Cauchy lemma formally.
switches to the Euclidean metric for spacelike external momenta. Actually, one gets the following prescription for the integral

\[ i \int \frac{d^4l}{(2\pi)^4} f(l, p) \rightarrow -\int \frac{d^4l_E}{(2\pi)^4} \Theta(3\Omega)f(l_E, p_E) + \int \frac{d^4l_E}{(2\pi)^4} \Theta(-3\Omega)f(l_E, p_E), \]

wherein the subscript \( E \) implies the arguments of \( f_E \) uses an Euclidean metric \( l_E^2 = l_x^2 + l_y^2 + l_z^2 + l_t^2 \) since Minkowski component \( l_t \) is replaced by the Euclidean \( i l_x \) variable in both cases and the signs follow from a mutually backward orientation of the integration contour lines. In addition and in accordance with causality, we will assumed \( a > 0 \) in order to avoid poles at the spacelike region of momenta. Integrating over the momentum \( l \) one gets

\[ \Pi(p^2) = \int_0^1 dy dx_1 dx_2 \frac{y(1 - y)[ -\Theta(3\Omega) + \Theta( -3\Omega)]}{(4\pi)^2 |p^2 y(1 - y) - \Omega|^2}, \]

where \( p \) is Minkowski momentum again, and the function \( \Pi(p^2 > 0) \) is an analytical continuation of \( \Pi(p^2 < 0) \).

Performing the substitution \( z = 1 - 2(x_1 y + x_2 (1 - y)) \) for variable \( x_1 \), one gets

\[ \Pi(p^2) = \int_0^1 dy dx_2 \int_{1 - 2(y + x_2 (1 - y))}^{1 - 2x_2 (1 - y)} dz \frac{(1 - y) [ -\Theta(z) + \Theta(1 - z)]}{(4\pi)^2 |p^2 y(1 - y) - a + ibz|^2}, \]

For purpose of the integration over the variable \( z \) let us distinguish three cases. The first case is defined such that upper and down \( z \)-integral boundaries are both located above zero. This enables us to consider only the mirror Wick rotation, i.e. when the variable \( z \) is negative. Integrating over the variable \( z \) leads to the following formula:

\[ \frac{-1}{2ib} \int_0^1 dy_1 \int dx_2 \frac{(1 - y) \Theta(z_1) \Theta(z_2)}{(4\pi)^2} \left[ \frac{2}{p^2 y(1 - y) - a - \Sigma_{i=1,2}} \right] - \frac{1}{p^2 y(1 - y) - a + ibz_i} \]

with two \( z_i \) defined previously.

The third and the ultimate case corresponds to the condition \( z_1 > z_2 > 0 \), for which one gets

\[ \frac{-1}{2ib} \int dy \int dx_2 \frac{(1 - y) \Theta(z_2)}{(4\pi)^2} \Sigma_{i=1,2} (-1)^i \frac{1}{p^2 y(1 - y) - a + ibz_i}, \]

The scalar polarization correlator \( \Pi \) is then equal to the sum of expressions (2.7), (2.8) and (2.9).

Integrating over the variable \( x_2 \) gives for (2.8)

\[ \frac{1}{(2b)^2 (4\pi)^2} \left[ \int_0^{1/2} dy \ln \frac{R - 2iby}{R + 2iby} + \int_{1/2}^1 dy \ln \frac{R + ib(2y - 1)}{R + ib(1 - 2y)} + \int_{1/2}^1 dy \ln \frac{R + ib}{R - ib} \right] \]

\[ + \frac{i}{2b} \frac{1}{(4\pi)^2} \left[ \int_0^{1/2} dy \frac{-y}{R} + \int_{1/2}^1 dy \frac{1}{R} \right], \]

where we have defined \( R = sy(1 - y) - a \) and \( s = p^2 \) for purpose of brevity.

Using the identity

\[ \ln \frac{R - 2iby}{R + 2iby} = 2i \tan^{-1} \frac{2by}{p^2 y(1 - y) - a} \]

for the first and similarly for other terms in (2.10), one immediately sees that the result for (2.8) is purely imaginary.
Further, summing (2.7) and (2.9) together and integrating over $x_2$ gives after some trivial algebra the following formula:

$$\frac{1}{(-2i\beta)^2 (4\pi^2)} \int_0^{1/2} dy \left[ -\ln(R + 2ib) + \ln(R + ib) - \ln(R + ib(1 - 2y)) + \ln(R) + c.c. \right], \quad (2.12)$$

where c.c. stands for complex conjugated term. Recall, as was discussed in the beginning, the total result must be completely imaginary and as (2.8) already is and rel. (2.12) is purely real, the later must be exactly zero for all $p$ and the total contribution is solely given by the expression (2.10).

The all integrals in (2.10) can be integrated analytically, providing the following final result:

$$\Pi = \frac{i}{2b(4\pi)^2} \left\{ \frac{1}{2s} \ln \left( \frac{(a + s/4)^2 + b^2}{a^2 + b^2} \right) - \frac{1}{4s} \ln \left( \frac{(a - s/4)^2}{a^2} \right) + \frac{1}{2b} \tan^{-1} \frac{b}{a} - \frac{1}{s} \sqrt{\frac{4a^2 - s}{4a - s}} \sqrt{\frac{s}{D}} \right\}$$

$$- \frac{1}{4b^2(4\pi)^2} \left\{ \frac{\sqrt{D}}{s} \tan^{-1} \frac{-2ib + s}{\sqrt{D}} - \frac{\sqrt{D}}{s} \tan^{-1} \frac{-2ib}{\sqrt{D}} - c.c. \right.$$  

$$+ \frac{\sqrt{D_1}}{s} \tan^{-1} \frac{2ib}{\sqrt{D_1}} - \frac{\sqrt{D_1}}{s} \tan^{-1} \frac{2ib - s}{\sqrt{D_1}} - c.c. \right.$$  

$$- \frac{\sqrt{D_2}}{s} \tan^{-1} \sqrt{\frac{s}{D_2}} - c.c. \right\}, \quad (2.13)$$

where we have used the following abbreviations

$$D = s(4a - s) + 4b^2; \quad D_1 = s(4a - s) + 4b^2 + 4ibs; \quad D_2 = 4a + 4ibs - s. \quad (2.14)$$

The inverse tangent function of complex argument is defined through the multi-valuable complex logarithms, the same branches of square roots should be considered in the Eq. (2.13). As one can see the singularity structure is quite complicated, noting that apart complex branch points there is a real branch point presented as well.

### III. Conclusion

The propagators approximated by a meromorphic functions with complex conjugated poles are widely used to model infrared behavior of colored excitations. In this paper, the analytical solution for a correlators made out of such meromorphic functions has been obtained in Minkowski space for the first time. Contrary to study performed in the Euclidean space [7], the correlator receives nontrivial phase at the both -the timelike and the spacelike- domain of square of momenta.

However, the complex conjugated poles or branch points unavoidably arise in the calculations performed in the Euclidean space and affect the property of bound states solutions in complex Euclidean space, the structure of propagator can be different in Minkowski space. In principle, a slightly accommodated method can be very useful when evaluating nonperturbative skeleton Feynman diagram in Minkowski space, e.g. in a cases where an intended integration contour of some generalized integral representation does not correspond with a real spectral quantity. Furthermore, even as a toy Minkowski space integral, the obtained result can be useful for numerical studies, which are much difficult and cumbersome when performed directly in Minkowski space. Recall here, that a recent numerical studies of a certain gap equations as well as the first combined Schwinger-Dyson and Bethe-Salpeter equation already exist [22, 23]. Actually, a check against analytical results obtained here can optimized a numerical integration method used in Minkowski space.

It is less plain, what could be consequence for the renormalization if one gets nontrivial imaginary pieces in Green’s functions at spacelike domain in Minkowski space. Recall trivially here, that for instance a choice of gluon propagator:

$$G(p^2) = \frac{p^2}{p^4 + 4\mu^4} \quad (3.1)$$

proposed at [2] and discussed in [20] is an exemplar which can provide complex infinites in gluon correlators, e.g. gluon polarization function, if calculated and considered in the Minkowski space instead of Euclidean one. The issues
for QCD renormalization is beyond scope of presented paper and it remains an open question for a future study.

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