Penrose Dodecahedron, Witting Configuration and Quantum Entanglement

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Abstract

A model with two entangled spin-3/2 particles based on geometry of dodecahedron was suggested by Roger Penrose for formulation of analogue of Bell theorem 'without probabilities.' The model was later reformulated using so-called Witting configuration with 40 rays in 4D Hilbert space. However, such reformulation needs for some subtleties related with entanglement of two such configurations essential for consideration of non-locality and some other questions. Two entangled systems with quantum states described by Witting configurations are discussed in presented work. Duplication of points with respect to vertices of dodecahedron produces rather significant increase with number of symmetries in \( \frac{25920}{60} = 432 \) times. Quantum circuits model is a natural language for description of operations with different states and measurements of such systems.

1 Introduction

Specific configuration of 40 rays based on geometry of dodecahedron was suggested by Roger Penrose in [1, 2] for demonstration of non-probabilistic version of Bell theorem. The construction was later popularized in his book Shadow of the Mind [3].

The Majorana map [4] between \( n \)-dimensional complex space and set with \( n-1 \) points on Riemann sphere was used in suggested approach. Thus, three points can be used for description of state of spin-3/2 particle in 4D Hilbert space.

Positions of some points in the triple may coincide and the order is not prescribed, e.g., in suggested approach for any vertex \( V \) of dodecahedron a
triple mapped into one of 20 ‘explicit’ rays was defined as \( \{V, V, -V\} \), where \( -V \) is opposite vertex. The less straightforward construction of yet another 20 ‘implicit’ rays is obtained as complements to full bases of some triples of orthogonal ‘explicit’ rays.

Symmetries of whole set with 40 rays in 4D Hilbert spaces considered as 40-point configuration in complex projective space \( \mathbb{CP}^3 \) was also discussed in [2]. It was mentioned an existence of “transitive group of 25920 unitary projective transformations of \( \mathbb{CP}^3 \) sending the configuration of 40 points into itself” [2]. The group is denoted further as \( \mathcal{W} \).

The model was analyzed later by P. K. Aravind et al [5, 6] and it was found, that all 40 rays can be represented in alternative way without Majorana map using so-called Witting configuration. However, entanglement of two such configurations essential for discussions about non-locality was not considered.

The presented paper is devoted to consideration of some basic properties of Witting configuration necessary for work with two entangled systems. Any such system can be represented as pair of qubits or ‘ququart’. Some useful properties and symmetries of single Witting configuration are recollected in Section 2. Entanglement of two such configurations is discussed in Section 3. Most analytic calculations are performed using GAP software [7].

## 2 Witting configuration and polytope

### 2.1 Complex Witting polytope

Witting polytope was introduced by Coxeter [8, 9]. It has 240 vertexes in complex space

\[
\begin{align*}
(0, \pm \omega^\mu, \mp \omega^\nu, \pm \omega^\lambda), & \quad (\mp \omega^\mu, 0, \pm \omega^\nu, \pm \omega^\lambda), \\
(\pm \omega^\mu, \mp \omega^\nu, 0, \pm \omega^\lambda), & \quad (\mp \omega^\mu, \mp \omega^\nu, \mp \omega^\lambda, 0), \\
(\pm i\omega^\lambda \sqrt{3}, 0, 0, 0), & \quad (0, \pm i\omega^\lambda \sqrt{3}, 0, 0), \\
(0, 0, \pm i\omega^\lambda \sqrt{3}, 0), & \quad (0, 0, 0, \pm i\omega^\lambda \sqrt{3}),
\end{align*}
\]

where \( \omega = (-1 + i\sqrt{3})/2 \), \( \omega^3 = 1 \) and \( \lambda, \mu, \nu \in \{0, 1, 2\} \) are independent numbers. The polytope is named after configuration of 40 points in projective spaces suggested by Alexander Witting [10]. The Witting configuration with
nonzero coordinate. It coincides with convention used in [6].

After normalization of Eq. (1) there are 40 rays described by unit vectors

\[ \frac{1}{\sqrt{3}}(0, 1, -\omega^\mu, \omega^\nu), \quad \frac{1}{\sqrt{3}}(1, 0, -\omega^\mu, -\omega^\nu), \]

\[ \frac{1}{\sqrt{3}}(-1, -\omega^\mu, 0, \omega^\nu), \quad \frac{1}{\sqrt{3}}(1, \omega^\mu, \omega^\nu, 0), \]

(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) \hspace{1cm} (2a)

with \( \mu, \nu \in \{0, 1, 2\} \). Each vector in Eq. (2) corresponds to six points of complex Witting polytope in Eq. (1) due to ‘phase multipliers’. In comparison with [8] the Eq. (2) use slightly different notation with normalization on the first nonzero coordinate. It coincides with convention used in [6].

Possible indexing for the states is shown in Table 1. Here \( |\phi_0^l\rangle = |l\rangle \), \( l = 0, \ldots, 3 \) is a ‘computational’ basis Eq. (2b) and the normalization \( 1/\sqrt{3} \) from Eq. (2a) for 36 states \( |\phi_n^l\rangle \) with \( n \neq 0 \) is omitted in Table 1 due to typographical and some other reasons. A correspondence of such indexing with implicit and explicit rays and vertices of dodecahedron can be found in [6], but it is rather excessive for presented work.

| \( n \) | \( |\phi_0^0\rangle \) | \( |\phi_1^1\rangle \) | \( |\phi_2^2\rangle \) | \( |\phi_3^3\rangle \) |
|---|---|---|---|---|
| 0 | (1, 0, 0, 0) | (0, 1, 0, 0) | (0, 0, 1, 0) | (0, 0, 0, 1) |
| 1 | (0, 1, -1, 1) | (1, 0, -1, -1) | (1, -1, 0, 1) | (1, 0, 1, 0) |
| 2 | (0, 1, -\omega, \overline{\omega}) | (1, 0, -\omega, -\overline{\omega}) | (1, -\omega, 0, \overline{\omega}) | (1, \omega, \overline{\omega}, 0) |
| 3 | (0, 1, -\overline{\omega}, \omega) | (1, 0, -\overline{\omega}, -\omega) | (1, -\overline{\omega}, 0, \omega) | (1, \overline{\omega}, \omega, 0) |
| 4 | (0, 1, -\omega, 1) | (1, 0, -1, -\omega) | (1, -\omega, 0, \omega) | (1, \omega, 1, 0) |
| 5 | (0, 1, -\overline{\omega}, \overline{\omega}) | (1, 0, -\overline{\omega}, -\overline{\omega}) | (1, -\overline{\omega}, 0, \omega) | (1, \overline{\omega}, \omega, 0) |
| 6 | (0, 1, -\omega, \overline{\omega}) | (1, 0, -\omega, -\overline{\omega}) | (1, -\omega, 0, \overline{\omega}) | (1, \omega, 0, \overline{\omega}) |
| 7 | (0, 1, -\overline{\omega}, 1) | (1, 0, -1, -\overline{\omega}) | (1, -\overline{\omega}, 0, \omega) | (1, \overline{\omega}, 1, 0) |
| 8 | (0, 1, -\overline{\omega}, \omega) | (1, 0, -\omega, -\overline{\omega}) | (1, -\omega, 0, \overline{\omega}) | (1, \omega, 0, \overline{\omega}) |
| 9 | (0, 1, -\omega, \overline{\omega}) | (1, 0, -\overline{\omega}, -1) | (1, -1, 0, \overline{\omega}) | (1, \overline{\omega}, 0, \overline{\omega}) |

Table 1: Indexes for 40 states (up to normalization).

Let us note that four triples of states

\[ |\phi_0^0\rangle, |\phi_0^1\rangle, |\phi_0^2\rangle; \quad |\phi_1^1\rangle, |\phi_2^2\rangle, |\phi_3^3\rangle; \quad |\phi_1^3\rangle, |\phi_2^3\rangle, |\phi_3^3\rangle; \quad |\phi_2^3\rangle, |\phi_3^3\rangle, |\phi_3^3\rangle \]
are mutually unbiased bases (MUB) for 3D subspace of vectors with zero last coordinate. Other states in the Table 1 are obtained by cyclic shift of coordinates and appropriate change of signs in agreement with Eq. (2a). Each state $|\phi_k^n\rangle$ is orthogonal with 12 states from 4 bases in Table 1 and ‘unbiased’ with other 27 states, i.e., $|\langle \phi_n^k | \phi_m^l \rangle|^2$ is either 0 or 1/3.

2.2 Triflections

Witting polytope Eq. (1) was described by H. Coxeter using four ‘unitary reflection’ matrices $R_k$ with property $R_3^k = 1$

\[
R_1 = \begin{pmatrix}
\omega & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad R_2 = \frac{-i\omega}{\sqrt{3}} \begin{pmatrix}
\overline{\omega} & 1 & 1 & 0 \\
1 & \overline{\omega} & 1 & 0 \\
1 & 1 & \overline{\omega} & 0 \\
0 & 0 & 0 & i\omega \sqrt{3}
\end{pmatrix},
\]

\[
R_3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \omega & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad R_4 = \frac{-i\omega}{\sqrt{3}} \begin{pmatrix}
i\omega \sqrt{3} & 0 & 0 & 0 \\
0 & \overline{\omega} & -1 & 1 \\
0 & -1 & \overline{\omega} & -1 \\
0 & 1 & -1 & \overline{\omega}
\end{pmatrix}.
\] (3)

Group of symmetries of Witting polytope with 155520 elements is generated by matrices $R_k$ [8, 9]. Each complex ray Eq. (2) or state $|\phi_n^k\rangle$ corresponds to six vertexes of the polytope with different phase multipliers $\pm \omega^\mu$.

The matrices Eq. (3) are complex reflections [8] of order $k = 3$ and for some ray $|\phi\rangle$ and root of unity $\zeta^k = 1$ of arbitrary order

\[
R_\phi = 1 + (\zeta - 1)|\phi\rangle\langle \phi|.
\] (4)

In Eq. (3) $\zeta = \omega$ and four rays are $|\phi_0^0\rangle, |\phi_1^2\rangle, |\phi_0^3\rangle, |\phi_1^0\rangle$ respectively.

The determinants of all matrices Eq. (3) are $\omega$. Let us introduce matrices

\[
r_k = \overline{\omega} R_k, \quad k = 1, \ldots, 4
\] (5)

with unit determinants, $(\overline{\omega})^4 \omega = 1$. Such matrices generate group denoted further as $\tilde{W}$ with 51840 elements.
2.3 Symmetries of Witting configuration

Group of projective transformations of $\mathbb{CP}^3$ mentioned in [2] is factor group $\mathcal{W} = \tilde{\mathcal{W}}/\{+1, -1\}$ with $51840/2 = 25920$ elements. It is isomorphic with finite group $\mathcal{U}_4(2)$ of unitary $4 \times 4$ matrices over Galois field $\mathbb{F}_4$ [13, p. 26].

Formally, matrix $-1 \in \tilde{\mathcal{W}}$ converts a ray or state into an equivalent one and so, group $\mathcal{W}$ could be formally treated as a more proper way to represent symmetries of Witting configuration.

Alternative description of Witting configuration with finite field $\mathbb{F}_4$ regarded as quadratic extension $\mathbb{F}_2$ can be found in [8]. With some abuse of notation four elements of $\mathbb{F}_4$ can be written as $0, 1, \omega, \omega^2 = \overline{\omega}, \omega^3 = 1$ with addition and multiplication tables

| + | 0 | 1 | $\omega$ | $\overline{\omega}$ |
|---|---|---|---|---|
| 0 | 0 | 1 | $\omega$ | $\overline{\omega}$ |
| 1 | 1 | 0 | $\omega$ | $\overline{\omega}$ |
| $\omega$ | $\omega$ | $\overline{\omega}$ | 0 | 1 |
| $\overline{\omega}$ | $\overline{\omega}$ | $\omega$ | 1 | 0 |

Here Galois conjugation is introduced as $\overline{a} = a^2$ and for the field of the characteristic two $a + a = 0$, $a = -a$. Thus, 40 points representing Witting configuration in finite projective space $PG(3, 2^2)$ can be obtained directly from Table 1 by dropping all minus signs.

The construction of representation $\mathcal{W} \cong \mathcal{U}_4(2)$ is rather straightforward in such a model. The $4 \times 4$ matrices with coefficients from $\mathbb{F}_4$ generating the group can be written using analogue of Eq. (4) for appropriate rays to obtain matrices similar with Eq. (3) or Eq. (5) with extra $\overline{\omega}$ multiplier to set determinant equivalent to unit in field $\mathbb{F}_4$

$r_1^{(\mathbb{F}_4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \overline{\omega} & 0 & 0 \\ 0 & 0 & \overline{\omega} & 0 \\ 0 & 0 & 0 & \overline{\omega} \end{pmatrix}$, $r_2^{(\mathbb{F}_4)} = \begin{pmatrix} 1 & \omega & \omega & 0 \\ \omega & 1 & \omega & 0 \\ \omega & \omega & 1 & 0 \\ 0 & 0 & 0 & \overline{\omega} \end{pmatrix}$

$r_3^{(\mathbb{F}_4)} = \begin{pmatrix} \overline{\omega} & 0 & 0 & 0 \\ 0 & \overline{\omega} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \overline{\omega} \end{pmatrix}$, $r_4^{(\mathbb{F}_4)} = \begin{pmatrix} \overline{\omega} & 0 & 0 & 0 \\ 0 & 1 & \omega & \omega \\ 0 & \omega & 1 & \omega \\ 0 & \omega & \omega & 1 \end{pmatrix}$

Due to Galois conjugation Hermitian form $\sum_k u_k \overline{u}_k$ can be written as a cubic $\sum_k u_k^3$ in $\mathbb{F}_4$. Such representation provides some useful links with prop-
erties of cubic surfaces [8], but such relations are not discussed in presented work.

The bigger group $\tilde{W}$ can be more appropriate here due to existence of representation by unitary operators in 4D Hilbert space and some other reasons. The group is isomorphic with finite group sometimes denoted as $2.U_4(2)$ [13]. The full group of symmetries of Witting polytope with 155520 elements is denoted $3x2.U_4(2)$ in [13, p.26] and complex reflections Eq. (4) of third order such as $R_k$ Eq. (3) are called triflections.

2.4 Bases and contextuality

Witting configuration with 40 states from Table 1 can be arranged into 40 orthogonal tetrads (bases). They are listed in Table 2 in lexicographic order with respect to indexes $(l, n)$ of $|\phi_n^l\rangle$.

| Base | States |
|------|--------|
| 1    | $\{ |\phi_0^0\rangle, |\phi_1^0\rangle, |\phi_2^0\rangle, |\phi_3^0\rangle \}$ |
| 2    | $\{ |\phi_0^1\rangle, |\phi_1^1\rangle, |\phi_2^1\rangle, |\phi_3^1\rangle \}$ |
| 3    | $\{ |\phi_0^2\rangle, |\phi_1^2\rangle, |\phi_2^2\rangle, |\phi_3^2\rangle \}$ |
| 4    | $\{ |\phi_0^3\rangle, |\phi_1^3\rangle, |\phi_2^3\rangle, |\phi_3^3\rangle \}$ |

Each state belongs to four different bases including all 12 states orthogonal to given one. The diagram with vertices corresponding to rays and edges connecting orthogonal rays is called Kochen-Specker graph (diagram)
and the 40 bases represent maximal 4-cliques \[2\]. Such graphs are commonly used in discussions about contextuality in quantum mechanics \[1, 2, 5, 6\].

Formally, any element of \(\tilde{\mathcal{W}}\) up to permutations and multiplication on phases can be associated with some basis represented by rows (or columns) of the unitary 4 \(\times\) 4 matrix. There are \(3^3 = 27\) diagonal matrices with coefficients \(\omega^\nu\) and unit determinant. The exchange of basic vectors (with possible change of directions) corresponds to subgroup of \(\tilde{\mathcal{W}}\) with \(2 \cdot 4! = 48\) elements generated by three matrices \((R_k/R_{k+1})^2, k = 1,\ldots, 3\)

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\]  

Thus, all \(40 \cdot 27 \cdot 48 = 51840\) elements of \(\tilde{\mathcal{W}}\) can be represented in such a way.

Due to principles of quantum mechanics measurement of a state \(\phi\) in any orthogonal basis should produce result corresponding to one of elements of this basis. Structure of 40 vectors in Witting configuration makes impossible to suggest non-contextual deterministic classical model of measurement corresponding to map of 40 states into set \(\{0, 1\}\) with requirement respecting a principle: one and only one vector of any basis maps into 1.

Such property follows from a correspondence with Penrose model with dodecahedra discussed earlier, but here is useful to reintroduce some ideas directly from structure of Witting configuration. Indeed, let us consider set of 10 bases underlined in Table 2. The bases include all 40 rays of Witting configuration and thus, precisely 10 vectors should be mapped into 1. On the other hand, these 10 states may not include orthogonal vectors, because any such pair could be complemented to full basis with at least two vectors mapped into 1.

Thus, necessary requirement could not be satisfied if it is not possible to find 10 nonorthogonal rays in Witting configuration. This question also could be reformulated using cliques, but instead of Kochen-Specker diagram mentioned above it should be used dual graph with edges connecting non-orthogonal vertices. Straightforward test using GAP software \[7\] provides maximal size of cliques with non-orthogonal vectors is equal to seven. \(\square\)

In particular, two kinds of the non-orthogonal maximal cliques could be considered. There are 2880 different sets with seven non-orthogonal vectors mentioned above. Symmetry group \(\mathcal{W}\) acts transitively on the cliques and
any such set can be used to construct others, e.g.

\[ |\phi_0\rangle, |\phi_1\rangle, |\phi_4\rangle, |\phi_7\rangle, |\phi_3\rangle, |\phi_5\rangle, |\phi_8\rangle. \]  

(9)

Second kind with 90 maximal non-orthogonal cliques has only four states. They are also can be obtained by application of elements of \( \tilde{W} \) to one of them, e.g.,

\[ |\phi_0\rangle, |\phi_2\rangle, |\phi_5\rangle, |\phi_8\rangle. \]  

(10)

Any such tetrad belong to some plane in 4D Hilbert space corresponding to a line in complex projective space \( \mathbb{CP}^3 \). Thus, 40 points of Witting configuration lie by four on 90 projective lines \([2]\).

2.5 Measurements with single system

Measurement of two entangled systems is discussed further in Section 3.2. For simplicity let us first consider two basic measurement schemes with single Witting configuration adapted for possible implementation by quantum circuits. First one is measurement of a state in one of 40 bases.

Let us consider an unitary matrix \( U_B \) made up of four vectors in a basis \( B \). For convenience it may be chosen from symmetry group of Witting configuration \( U_B \in \tilde{W} \), yet there are still 51840/40 = 1296 such matrices for each basis \( B \) due to phases and permutations Eq. (8). A smaller \( \tilde{W}_B \subset \tilde{W} \) also can be used for the same purpose. The subgroup may be generated by \( r_1r_3 \) and \( r_3r_2r_1r_4 \) together with \( J_a \) defined later in Eq. (19). The subgroup has 1920 elements and less redundant for representation of bases, 1920/40 = 48.

Any basic state up to insignificant phase can be represented as

\[ |\phi(B,k)\rangle = U_B|k\rangle, \quad k = 0, \ldots, 3. \]  

(11)

The quantum circuit with two qubits for such a scheme should use decomposition of matrix \( U_B^{-1} \) into set of elementary quantum gates to transform given state for final measurement in a computational basis.

Let us consider quantum system in one of 40 states \( |\phi'_n\rangle \) corresponding to Witting configuration. A measurement without disturbance for given state could be performed using one of four bases including of the state. Such a scheme is ‘contextual,’ because construction of each basis used for measurement should include three extra states together with initial one. Such a method is also has a problem with initial idea introduced by R. Penrose.
with possibility of few steps, when tetrad of orthogonal states should not be chosen from very beginning.

Such a measurement scheme for some state \( |\phi\rangle \) can be implemented using and auxiliary qubit and unitary operator

\[
CM_\phi = P_\phi \otimes X + (1 - P_\phi) \otimes 1, \quad P_\phi = |\phi\rangle \langle \phi|.
\] (12)

Here operator \( X \) denotes NOT gate for an auxiliary qubit controlled by state of other two qubits representing ququart. The Toffoli gate is a particular case with \( |\phi_3^0\rangle = |3\rangle = |1\rangle|1\rangle \). Such a method is not ‘contextual,’ because it does not suppose description of three extra states together with given \( |\phi\rangle \). However contextuality can be also included in such approach if to use consequent application of different operators Eq. (12) with orthogonal states.

### 3 Entanglement of Witting configurations

#### 3.1 An extension of model with dodecahedra

Essential property of model discussed in \([1, 2, 3]\) is consideration of two entangled systems. In initial model with two spin-3/2 particles it was suggested to consider so-called ‘singlet state’ \([1]\) corresponding to entangled pair with total spin zero \([3]\)

\[
|\Omega\rangle = |↑↑↑⟩|↓↓↓⟩ - |↓↑↑⟩|↑↓↓⟩ + |↓↓↑⟩|↑↑↓⟩ - |↓↓↓⟩|↑↑↑⟩.
\] (13)

The state Eq. (13) is invariant with respect to transformation of two spin-3/2 particles due to spatial rotations, i.e., representation of SU(2) as some subgroup in SU(4). Due to such representation binary dodecahedral (or binary icosahedral) group with 120 elements (corresponding to symmetries of dodecahedron or icosahedron) can be mapped into transformations of Witting configuration discussed earlier in \([5, 6]\). The group is double cover of usual icosahedral group in SO(3) with 60 spatial rotations.

However such group describes directly only transformation of 20 states and more symmetries can be used instead. An entangled state of two systems can be described using some matrix \( J \)

\[
|\Omega_J\rangle = \mu_J \sum_{jk} J_{jk} |j\rangle |k\rangle.
\] (14)
where \( \mu_J = \text{Tr}(J^*)^{-1/2} \) is a multiplier for normalization. Let us apply the same unitary transformation \( A \) to both systems

\[
|j'\rangle = \sum_{lj} A_{lj} |j\rangle, \quad |k'\rangle = \sum_{mk} A_{mk} |k\rangle.
\]

(15)

In such a case Eq. (14) is transformed as

\[
|\Omega'_{j}\rangle = \mu_J \sum_{jklm} J_{jk} A_{lj} A_{mk} |j\rangle |k\rangle
\]

(16)

and \( |\Omega'_{j}\rangle = |\Omega_{j}\rangle \) if

\[
J = A J A^T \implies A J = J A,
\]

(17)

where \( A^T \) is transposed matrix and \( A = (A^T)^{-1} \) for unitary \( A \).

The simplest analogue of Eq. (13) is ‘antisymmetric’ entangled state

\[
|\Omega_a\rangle = \frac{1}{2} (|0\rangle|3\rangle - |1\rangle|2\rangle + |2\rangle|1\rangle - |3\rangle|0\rangle)
\]

(18)

with matrix

\[
J_a = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

(19)

Two alternative examples with matrices \( J_1 \) Eq. (35) and \( J_2 \) Eq. (36) are discussed later.

Similar condition Eq. (17) with slightly different matrix

\[
J = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]

(20)

was also used in [14] for identification of Spin(5) group with Sp(2). The matrix \( J \) would correspond to entangled state

\[
\frac{1}{2} (|0\rangle|3\rangle + |1\rangle|2\rangle - |2\rangle|1\rangle - |3\rangle|0\rangle)
\]

(21)

that can be converted into \( |\Omega_a\rangle \) by swap of two basic states \( |1\rangle \leftrightarrow |2\rangle \). However, such a swap formally did not saves Witting configuration with chosen positions of signs for some coordinates.
Anyway, subgroup of transformations respecting ‘antisymmetric’ state $|\Omega_a\rangle$ is also isomorphic with Spin(5) group described by 10 real parameters. On the other hand, ‘symmetric’ state

$$\frac{1}{2}(|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle + |3\rangle|3\rangle)$$

(22)

is invariant with respect to group SO(4) with only 6 real parameters. It may justify consideration of models with antisymmetric entangled states despite of certain complication.

One subtlety is an analogy of ‘an opposite state’ used in models with two dodecahedra for description of joint measurement. Let us prove that for entangled state Eq. (14) with matrix such as $J_a$ in Eq. (19) the ‘$J$-opposite’ state can be expressed by *anti-unitary map*

$$|\psi_J\rangle \simeq J|\psi\rangle.$$  

(23)

An equivalence in Eq. (23) is correctly defined up to unessential phase multiplier. For basic states $|k\rangle$, $k = 0, \ldots, 3$ the Eq. (23) is valid for $J$ or $J_a$ up to unessential signs. Let us consider some transformation $A_J$ satisfying Eq. (17) for given $J$

$$|\psi'\rangle = A_J|\psi\rangle.$$  

In such a case for $J$-opposite state Eq. (23)

$$A_J|\psi_J\rangle \simeq A_J J|\psi\rangle = J\overline{A_J} |\psi\rangle = J|\overline{\psi}\rangle \simeq |\psi_J'\rangle$$

(24)

and so, $J$-opposite states $|\psi\rangle$ and $|\psi_J\rangle$ are transformed by operator $A$ into $J$-opposite states $|\psi'\rangle$ and $|\psi_J'\rangle$. The expression Eq. (24) also illustrate that map Eq. (23) should be anti-unitary due to structure of Eq. (17).

List of $J_a$-opposite states for Witting configuration with indexing from Table 1 is shown in Table 3.

Let us note, that Eq. (17) is true for any element of some group if it is satisfied for generators. An example is a subgroup of $\hat{W}$ with 720 elements generated by two matrices

$$r_1 r_3 = \begin{pmatrix} \overline{\omega} & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \overline{\omega} & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \quad r_2 r_4^{-1} = \frac{-i}{\sqrt{3}} \begin{pmatrix} \omega & \overline{\omega} & \overline{\omega} & 0 \\ \overline{\omega} & 0 & -1 & -\omega \\ \overline{\omega} & -1 & 0 & \omega \\ 0 & -\omega & \omega & -\overline{\omega} \end{pmatrix}. $$

(25)
The group is denoted further as $\tilde{W}_H$. It is isomorphic with finite group $\text{SL}_2(\mathbb{F}_9)$ also known as double cover of $A_6$ (even permutations on six elements). Let us point some analogy with mentioned earlier binary dodecahedral group as a double cover of $A_5$ (even permutations on five elements).

The condition Eq. (37) can be extended due to insignificance of phase multipliers, $|\Omega'_J\rangle = e^{i\phi}J|\Omega_J\rangle$. The simplest case $|\Omega'_J\rangle = \pm|\Omega_J\rangle$ corresponds to generalization of Eq. (37)

\[ J = \pm AJA^T \Rightarrow \tilde{A}J = \pm JA. \tag{26} \]

For example, yet another group $\tilde{W}_{2H} \cong \text{SL}_2(\mathbb{F}_9) \times C_2$ with 1440 elements can be constructed using two generators from Eq. (25) together with third generator $J_2$, see Eq. (36). Such a matrix is satisfying Eq. (26) with minus sign.

Let us note, that measurements of entangled state in any pair of bases obtained by transformation $A_J$ from such $J_a$-invariant groups as $\tilde{W}_H$ or $\tilde{W}_{2H}$ should produce a pair of $J_a$-opposite states and so such bases should contain two pairs of $J_a$-opposite states from Table 3. It may be checked directly, that only 10 underlined bases in Table 1 have such properties.

The more general example can be obtained, if for given $J$ to apply arbitrary unitary transformation $A$ to first system and for second one to find $B$ saving state $|\Omega_J\rangle$ Eq. (14) invariant. Thus, instead of Eq. (16) an expression
for two different transformation should be written

$$|\Omega'_j\rangle = \mu \mu J \sum_{jklm} J_{jk} A_{lj} B_{mk} |j\rangle |k\rangle = |\Omega_j\rangle$$  \hspace{1cm} (27)

and equations with $B$ similar with Eq. (17) can be simply obtained. For invertible matrices $J$ it can be also written

$$J = A J B^T \implies B J = J A, \quad B = J A J^{-1}$$  \hspace{1cm} (28)

and for real matrices $J = \overline{J}$ the expressions can be simplified

$$B J = J A, \quad B = J A J^{-1}.$$  \hspace{1cm} (29)

The Eq. (23) for $J$-opposite state is not changed in such a case, but now pair of states are transformed by operators $A_J$ and $B_J$ respectively, cf. Eq. (24)

$$|\psi_J\rangle = A_J |\psi\rangle, \quad |\psi'_J\rangle \simeq J |\overline{\psi}\rangle = \overline{J A_J |\overline{\psi}\rangle} = B_J |\overline{\psi}\rangle \simeq B_J |\psi_J\rangle.$$  \hspace{1cm} (30)

For 30 bases between 40 a pair representing tetrad with $J_a$-opposite states does not coincide with initial base. Thus a different matrices $A$ an $B$ should be used for transformations of such bases. For Table 2 the indexes for all 40 pairs of bases are listed in Table 4.

| 1 : 38 | 2 : 40 | 3 : 39 | 4 : 4 | 5 : 5 |
| 6 : 28 | 7 : 21 | 8 : 30 | 9 : 14 | 10 : 10 |
| 11 : 19 | 12 : 12 | 13 : 23 | 14 : 9 | 15 : 29 |
| 16 : 16 | 17 : 25 | 18 : 18 | 19 : 11 | 20 : 20 |
| 21 : 7 | 22 : 27 | 23 : 13 | 24 : 24 | 25 : 17 |
| 26 : 26 | 27 : 22 | 28 : 6 | 29 : 15 | 30 : 8 |
| 31 : 31 | 32 : 35 | 33 : 37 | 34 : 36 | 35 : 32 |
| 36 : 34 | 37 : 33 | 38 : 1 | 39 : 3 | 40 : 2 |

Table 4: Indexes of bases with ‘$J_a$-opposite’ states in Table 2

### 3.2 Measurements of two entangled systems

Schemes of measurement adapted for single system of states from Witting configuration were outlined in Section 2.5. A measurement of two such systems with the same setup is appropriate for 10 bases underlined in Table 2.
corresponding to symmetries from subgroups $\tilde{W}_H$ or $\tilde{W}_{2H}$. The bases contain all 40 states and could be used for simple demonstration of basic property of measurements with entangled state $|\Omega_a\rangle$ Eq. (18): if one system after measurement is found in some state $|\phi'_n\rangle$ the second one has $J_a$-opposite state from Table 3.

However, 10 bases are not enough for formulation of some problems related with contextuality. More general setup suggests possibility to select some state $|\phi'_n\rangle$ for first system using scheme with $CM_\phi$ operator Eq. (12) together with $J_a$-opposite state for second system. There are four bases (between 40) containing any state $|\phi'_n\rangle$, but in subset with 10 bases mentioned above there is only one basis with any given state.

Only consideration of all 40 bases let us choose for any state $|\phi'_n\rangle$ one between four bases for first system together with corresponding basis with $J_a$-opposite states for second one, see Table 4. Such approach is more close analogue of ‘multi-step’ scheme of measurements suggested for the model with two dodecahedra in [1, 2].

### 3.3 Other entangled states

The entangled state $|\Omega_a\rangle$ Eq. (18) is not an only possible analogue of state Eq. (13) used in a model with two dodecahedra [1, 2]. Let us consider some symmetry $S \in \tilde{W}$ of Witting configuration

$$|\psi\rangle \mapsto |\psi^S\rangle = S|\psi\rangle. \quad (31)$$

For a subgroup such as $\tilde{W}_{2H}$ isomorphic subgroups $\tilde{W}^S_{2H}$ can be constructed using conjugations of all elements

$$H \mapsto H^S = SHS^{-1} \quad (32)$$

and there are $|\tilde{W} : \tilde{W}_{2H}| = 51840/1440 = 36$ such subgroups with $S$ from different cosets $S\tilde{W}_{2H}$. For $H \in \tilde{W}_{2H}$ and Eq. (17) can be rewritten as

$$\bar{H} = J^{-1}HJ \quad (33)$$

and due to Eq. (32)

$$\bar{S}^{-1}\bar{H}^S\bar{S} = J^{-1}S^{-1}H^SJ$$

and thus instead of $J$ for group $\tilde{W}^S_{2H}$ should be used matrix

$$J^S = SJ\bar{S}^{-1} = SJS^T. \quad (34)$$
Let us consider simplest cases with \( J^S \) are matrices with coefficients 0 and ±1. Together with initial example with \( S = 1 \) and \( J_a \) Eq. (19) only two cases between 35 satisfy such conditions.

\[
J_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix} \tag{35}
\]

and

\[
J_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}. \tag{36}
\]

Entangled states for such matrices are

\[
|\Omega_1\rangle = \frac{1}{2} (|0\rangle|2\rangle - |2\rangle|0\rangle + |1\rangle|3\rangle - |3\rangle|1\rangle) \tag{37}
\]

and

\[
|\Omega_2\rangle = \frac{1}{2} (|0\rangle|1\rangle - |1\rangle|0\rangle + |3\rangle|2\rangle - |2\rangle|3\rangle). \tag{38}
\]

As transformations \( S \) used for construction of \( \tilde{W}_{2H}^S \) can be chosen

\[
S_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad S_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}. \tag{39}
\]

List of \( J_1 \)- and \( J_2 \)-opposite states for Witting configuration with indexing from Table 1 are shown in Table 5 and Table 6 respectively. Analogues of Table 4 with pairs of bases written earlier for \( J_a \) are Table 7 for \( J_1 \) and Table 8 for \( J_2 \). Schemes of measurements discussed in Section 3.2 can be extended accordingly for entangled states \(|\Omega_1\rangle\) Eq. (37) and \(|\Omega_2\rangle\) Eq. (38).
| $\phi_0^0 \leftrightarrow \phi_0^1$ | $\phi_0^1 \leftrightarrow \phi_0^0$ | $\phi_0^2 \leftrightarrow \phi_0^3$ | $\phi_0^3 \leftrightarrow \phi_0^2$ | $\phi_0^4 \leftrightarrow \phi_0^5$ | $\phi_0^5 \leftrightarrow \phi_0^4$ |
| $\phi_1^0 \leftrightarrow \phi_1^1$ | $\phi_1^1 \leftrightarrow \phi_1^0$ | $\phi_1^2 \leftrightarrow \phi_1^3$ | $\phi_1^3 \leftrightarrow \phi_1^2$ | $\phi_1^4 \leftrightarrow \phi_1^5$ | $\phi_1^5 \leftrightarrow \phi_1^4$ |
| $\phi_2^0 \leftrightarrow \phi_2^1$ | $\phi_2^1 \leftrightarrow \phi_2^0$ | $\phi_2^2 \leftrightarrow \phi_2^3$ | $\phi_2^3 \leftrightarrow \phi_2^2$ | $\phi_2^4 \leftrightarrow \phi_2^5$ | $\phi_2^5 \leftrightarrow \phi_2^4$ |
| $\phi_3^0 \leftrightarrow \phi_3^1$ | $\phi_3^1 \leftrightarrow \phi_3^0$ | $\phi_3^2 \leftrightarrow \phi_3^3$ | $\phi_3^3 \leftrightarrow \phi_3^2$ | $\phi_3^4 \leftrightarrow \phi_3^5$ | $\phi_3^5 \leftrightarrow \phi_3^4$ |
| $\phi_4^0 \leftrightarrow \phi_4^1$ | $\phi_4^1 \leftrightarrow \phi_4^0$ | $\phi_4^2 \leftrightarrow \phi_4^3$ | $\phi_4^3 \leftrightarrow \phi_4^2$ | $\phi_4^4 \leftrightarrow \phi_4^5$ | $\phi_4^5 \leftrightarrow \phi_4^4$ |
| $\phi_5^0 \leftrightarrow \phi_5^1$ | $\phi_5^1 \leftrightarrow \phi_5^0$ | $\phi_5^2 \leftrightarrow \phi_5^3$ | $\phi_5^3 \leftrightarrow \phi_5^2$ | $\phi_5^4 \leftrightarrow \phi_5^5$ | $\phi_5^5 \leftrightarrow \phi_5^4$ |

Table 5: Pairs of ‘$J_1$-opposite’ states.

| $\phi_0^0 \leftrightarrow \phi_1^1$ | $\phi_0^1 \leftrightarrow \phi_1^0$ | $\phi_0^2 \leftrightarrow \phi_1^3$ | $\phi_0^3 \leftrightarrow \phi_1^2$ | $\phi_0^4 \leftrightarrow \phi_1^5$ | $\phi_0^5 \leftrightarrow \phi_1^4$ |
| $\phi_1^0 \leftrightarrow \phi_2^1$ | $\phi_1^1 \leftrightarrow \phi_2^0$ | $\phi_1^2 \leftrightarrow \phi_2^3$ | $\phi_1^3 \leftrightarrow \phi_2^2$ | $\phi_1^4 \leftrightarrow \phi_2^5$ | $\phi_1^5 \leftrightarrow \phi_2^4$ |
| $\phi_2^0 \leftrightarrow \phi_3^1$ | $\phi_2^1 \leftrightarrow \phi_3^0$ | $\phi_2^2 \leftrightarrow \phi_3^3$ | $\phi_2^3 \leftrightarrow \phi_3^2$ | $\phi_2^4 \leftrightarrow \phi_3^5$ | $\phi_2^5 \leftrightarrow \phi_3^4$ |
| $\phi_3^0 \leftrightarrow \phi_4^1$ | $\phi_3^1 \leftrightarrow \phi_4^0$ | $\phi_3^2 \leftrightarrow \phi_4^3$ | $\phi_3^3 \leftrightarrow \phi_4^2$ | $\phi_3^4 \leftrightarrow \phi_4^5$ | $\phi_3^5 \leftrightarrow \phi_4^4$ |
| $\phi_4^0 \leftrightarrow \phi_5^1$ | $\phi_4^1 \leftrightarrow \phi_5^0$ | $\phi_4^2 \leftrightarrow \phi_5^3$ | $\phi_4^3 \leftrightarrow \phi_5^2$ | $\phi_4^4 \leftrightarrow \phi_5^5$ | $\phi_4^5 \leftrightarrow \phi_5^4$ |
| $\phi_5^0 \leftrightarrow \phi_0^1$ | $\phi_5^1 \leftrightarrow \phi_0^0$ | $\phi_5^2 \leftrightarrow \phi_0^3$ | $\phi_5^3 \leftrightarrow \phi_0^2$ | $\phi_5^4 \leftrightarrow \phi_0^5$ | $\phi_5^5 \leftrightarrow \phi_0^4$ |

Table 6: Pairs of ‘$J_2$-opposite’ states.
(1 : 35), (2 : 37), (3 : 36), (4 : 4), (5 : 5),
(6 : 25), (7 : 15), (8 : 19), (9 : 9), (10 : 26),
(11 : 30), (12 : 20), (13 : 13), (14 : 14), (15 : 7),
(16 : 24), (17 : 28), (18 : 18), (19 : 8), (20 : 12),
(21 : 20), (22 : 22), (23 : 23), (24 : 16), (25 : 6),
(26 : 10), (27 : 27), (28 : 17), (29 : 21), (30 : 11),
(31 : 31), (32 : 38), (33 : 40), (34 : 39), (35 : 1),
(36 : 3), (37 : 2), (38 : 32), (39 : 34), (40 : 33).

Table 7: Indexes of bases with ‘J1-opposite’ states in Table 2

(1 : 32), (2 : 34), (3 : 33), (4 : 4), (5 : 5),
(6 : 29), (7 : 17), (8 : 8), (9 : 23), (10 : 20),
(11 : 11), (12 : 26), (13 : 14), (14 : 13), (15 : 28),
(16 : 16), (17 : 7), (18 : 31), (19 : 19), (20 : 10),
(21 : 25), (22 : 22), (23 : 9), (24 : 24), (25 : 21),
(26 : 12), (27 : 27), (28 : 15), (29 : 6), (30 : 30),
(31 : 18), (32 : 1), (33 : 3), (34 : 2), (35 : 38),
(36 : 40), (37 : 39), (38 : 35), (39 : 37), (40 : 36).

Table 8: Indexes of bases with ‘J2-opposite’ states in Table 2

4 Conclusion

Entanglement of two quantum systems described by Witting configurations with 40 states in 4D Hilbert space is discussed in presented work. The model is originated from system of two entangled particles with spin-3/2 suggested by Roger Penrose for formulation of analogue of Bell theorem [1, 2, 3]. The initial model is known also as ‘Penrose dodecahedron,’ because of a model with measurements is associated with vertices of two regular dodecahedra.

Here the description of measurements briefly outlined in Section 2.5 and Section 3.2 is reformulated for more convenient implementation with quantum circuit model. The structure of 40 rays in Witting configuration looks much more complicated in comparison with initial set with 20 rays. It was mentioned already in [2] that such configuration has full symmetry \( W \) group with 25920 transformations. Such group can be represented as discrete subgroup in group of unitary projective transformations of 4D complex space or
as finite group $U_4(2)$ [13, p. 26]. More convenient choice is double cover $\tilde{\mathcal{W}}$ with implementation by 51840 unitary $4 \times 4$ matrices discussed in this paper. The $\mathcal{W}$ can be naturally considered as factor group of $\tilde{\mathcal{W}}$ by the center with two matrices $\pm 1$ representing trivial transformation of projective space.

It can be mentioned for comparison, what group of symmetries of dodecahedron has only 60 transformations and double cover can be represented by 120 unitary $2 \times 2$ matrices for particle with spin-1/2 and representation with $4 \times 4$ unitary matrices used by R. Penrose correspond to spin-3/2 particle. Initial model based on geometry of dodecahedron uses only symmetries induced by rotations in 3D space. Approach suggested in presented work is more general with consideration of full group of symmetries of 4D Hilbert space $U(4)$ and discrete subgroup $\tilde{\mathcal{W}} \cong 2U_4(2)$ of symmetries of Witting configuration.

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