NAVIGATING AROUND THE ALGEBRAIC JUNGLE OF QCD:
EFFICIENT EVALUATION OF LOOP HELICITY AMPLITUDES

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Abstract

A method is developed whereby spinor helicity techniques can be used to simplify
the calculation of loop amplitudes. This is achieved by using the Feynman-parameter
representation where the offending off-shell loop momenta do not appear. Background
Feynman gauge also helps to simplify the calculations. This method is applicable to any
Feynman diagram with any number of loops as long as the external masses can be ignored,
and it is at least as efficient as the string technique in the special circumstances when
the latter can be used. In order to minimize the very considerable algebra encountered in
non-abelian gauge theories, graphical methods are developed for most of the calculations.

This enables the large number of terms encountered to be organized visually in the
Feynman diagram without the necessity of having to write down any of them algebraically.
A one-loop four-gluon amplitude in a particular helicity configuration is computed explicity
to illustrate the method.

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1. Introduction

The number of diagrams and the number of terms in a QCD calculation increase dramatically with the multiplicity of the external particles as well as the number of loops. Even in the absence of quarks, a pure QCD tree process describing the production of six gluon jets from a glue-glue collision is given by the sum of some 34,000 diagrams, and roughly half a billion terms. A one-loop pure QCD glue-glue elastic scattering amplitude has 39 diagrams and some ten thousand terms. The large number of diagrams is due to the large number of ways triple and quadruple gluon (and ghost) vertices can be assembled together, and the large number of terms is due to the presence of six terms at each vertex. The necessity of having to sum over intermediate color indices makes it more complicated; if loops are present loop integrations must be done and the problem gets worse. Similar complexity occurs in electroweak computations. These difficulties are not something that one can ignore in practice because a large number of jets is present at high energies, and because loop calculations are increasingly demanded for precision comparisons with the Standard Model. To make progress one must find a way around this algebraic jungle.

Unnecessary algebraic complications are already present in QED bremsstrahlung calculations as is evidenced by the fact that simple results emerge from complicated covariant technique calculations [1]. It was later discovered that the use of the spinor helicity technique enables one to obtain the final result directly in a much simpler way [1,2]. Over the last ten years or so this technique has been further developed and applied to various QCD and electroweak processes in the tree approximation [1–18]. It leads to a tremendous simplification in the calculations, reducing impossibly large number of terms into manageable sizes. As a result of these techniques, many tree amplitudes which are too complicated to calculate by ordinary means have been successfully computed. See Ref. [16] for an excellent review of the techniques and the results.

The basic idea of this technique is that quark masses are negligible at high energies. If they are neglected, chirality is conserved, and this conservation can be exploited to simplify the calculations. For example, the trace $\text{tr}(\gamma p_1 \cdots \gamma p_{2n})$ which according to the usual formula is given by the sum of $(2n)!/2^n n!$ terms, can be written using this technique as a sum of merely two terms if all $p_i^2 = 0$ (see eq. (39) below). This simplification is equally applicable to processes involving gluons and photons if their wavefunctions are written in a multispinor basis. Moreover, gauge freedom allows one to choose these wavefunctions to be orthogonal to any massless ‘reference momentum’, thereby further simplifying the calculations by rendering many terms zero. On top of that, recursion [4] and supersymmetry [17,12] relations may be exploited to further simplify the calculations.

Unfortunately it is difficult to apply this beautiful technique to the computation of loop diagrams. The internal loop momenta are offshell; chirality is not conserved and massless spinor methods are not useful for these momenta. In an era when precision experiments are increasingly called for this is a serious handicap and it is important to find a way around this obstacle. This turns out to be possible and this is the subject matter of the present paper.

The idea is very simple, though the detailed implementation of the idea is far from being trivial. If a representation of the scattering amplitude can be found where the internal loop momenta do not appear, then every momentum in the problem is a linear
A combination of the external massless momenta and the spinor helicity technique can be
used.

This happens to be true for certain one-loop amplitudes which have a string extension. The point is that a one-loop string scattering amplitude can be written as an integral over the Koba-Nielsen variables, without the explicit presence of any internal momenta. By taking the string tension to infinity, one obtains a formula for the scattering of the massless particles in the string in which no internal momentum is present and the spinor helicity technique can be used. The simplification thus achieved is very considerable [19–21].

This ingenious method has its limitations. It is difficult to find a simple string formula beyond one loop, and in any case the technique is not applicable if the corresponding field theory has no string extension, or that the one-loop formula for the desired external particle is difficult to write down, as is the case for external fermions. Moreover, one cannot help but feel that there must be a purely field-theoretical way of calculating a field-theoretical scattering amplitude, without having to resort to the artificial, though ingenious, means of creating an intermediate string and then destroying it again by taking the infinite tension limit.

There is indeed a well-known and purely field-theoretical way of getting rid of the internal loop momenta: by introducing the Feynman parameters to combine the propagators, the loop-momentum integrations can be explicitly carried out. The result is again a formula in which the only momenta present are the external massless momenta, and thus the spinor helicity technique can again be used. It is this route that we would like to explore here. Unlike the string technique, Feynman parameter representations are available for any Feynman diagram with any number of loops, so this method can be used for all processes subject only to the validity of ignoring the external masses. As a matter of fact, in the known cases [19–21], the Koba-Nielsen parameters reduce themselves to the Feynman parameters in the infinite tension limit of the string, thus suggesting the close connection of the two methods.

This simple idea must be supplemented by a number of other developments to make it useful, for otherwise the amount of algebra necessary to carry out the calculations using elementary means is too unmanageable. These means are available; they will be mentioned below and discussed in much more detail in the next two sections.

First of all, one needs a set of rules analogous to the usual momentum-space Feynman rules to write down the Feynman parameter representation directly from the Feynman diagram. Otherwise if we had to do the internal loop integrations explicitly every time then the task would be too complicated. These Feynman-parameter rules are available [22] and will be reviewed in Sec. 2.

The number of terms in the Feynman-parameter representation is even larger than the number of terms in the conventional momentum representation. This would not have represented progress towards simplification except for the fact that the spinor helicity technique is now available to render many terms zero. But to be able to do that one must first find a way to organize the large number of terms in a simple and systematic way so that one can recognize beforehand which terms to discard in the calculation. The way to do that is to reorganize these terms into gauge-invariant color subamplitudes. It is known how this can be done algebraically in tree and one-loop processes [16,19]. We shall discuss
in Sec. 3 how this can be one for any number of loops graphically by introducing color-oriented Feynman diagrams. In this graphical language the different terms in the scattering amplitude correspond to different covering paths. The use of this graphical language does not in any way reduce the number of terms, but it gives a way to organize them in a visual way without the necessity of writing down anything algebraically. This graphical organization can be used in the usual momentum-space representation of a scattering amplitude, as well as the Feynman-parameter space representation discussed in Sec. 2.

This graphical technique is particularly useful when it is combined with the spinor helicity technique, which as a result of chirality conservation renders many terms zero. In graphical language this means that paths of certain topologies lead to vanishing results and do not have to be included. We shall also find that the use of Feynman gauge in the background field method reduces further the amount of labour of calculation by rendering more terms zero. A review of the spinor helicity technique and how this can be implemented graphically will be discussed in Sec. 4.

We choose to illustrate the present method in Sec. 5 by computing the one-loop gluon-gluon elastic scattering amplitude in a particular helicity configuration. This amplitude in the absence of quark loops has already been computed in the string method [19–21]; we choose it to illustrate our method so that the efficiency of the two techniques can be compared. We shall find that within the present framework there are two ways to compute this amplitude. The direct way yields a result as efficient as the string method; the indirect way making use of supersymmetry is even simpler and the result can be obtained in only a few lines. This is to be compared with the ordinary Feynman diagram calculations where some ten thousand terms appear.

It should be emphasized that chirality conservation affects only the spin flows, viz., the derivative couplings and the numerators of the propagators in the usual momentum-space representation. The denominators of the propagators may remain massive without in any way affecting the effectiveness of these techniques. This means that while the external particles must remain massless, exchange and internal particles may often be massive, as is the case for the Z and W bosons. Thus the present technique can be used to compute heavy particle productions if their subsequent decays into light particles are also incorporated into the diagrams.

### 2. Feynman-parameter Rules

Consider a Feynman diagram in \(d\)-dimensional spacetime, with \(N\) internal lines and \(\ell\) loops. Let \(p_A\) be the external outgoing momenta, \(q_r\) (\(r = 1, \cdots, N\)) be the momentum of the \(r\)th propagator, and \(m_r\) be the mass of the particle being propagated. Every \(q_r\) is given by a linear combination of \(p_A\) and the \(\ell\) loop-momenta \(k_a\), the specific combination depends on the topology of the diagram.

The scattering amplitude corresponding to a Feynman diagram expressed in momentum-space representation is of the form

\[
\mathcal{M} = \left(\frac{-i}{(2\pi)^d}\right)^{\ell} \prod_{a=1}^{\ell} (d^dk_a) \frac{S(q,p)}{\prod_{r=1}^{N} (-q_r^2 + m_r^2 - i\epsilon)},
\]

where

\(\mathcal{M}\) is the scattering amplitude,

\(S(q,p)\) is the propagator function,

\(d\) is the number of spacetime dimensions,

\(\ell\) is the number of loops,

\(q_r\) is the momentum of the \(r\)th propagator,

\(m_r\) is the mass of the particle being propagated.
where \( S(q,p) \) receives its contributions from the vertices and the numerators of propagators. It also contains the symmetry factor and the minus sign for each fermion loop. By introducing the Feynman parameters \( \alpha_r \) and carrying out the integrations over \( k_a \), it can be shown \[22\] that

\[
\mathcal{M} = \left( \frac{\pi^{d/2-4}}{16} \right)^{\ell} \sum_{k=0}^{\infty} \Gamma \left( N - \frac{d \ell}{2} - k \right) \int \frac{D^N \alpha}{\Delta(\alpha)^{d/2}} \frac{S_k(J,p)}{D(\alpha,p)^{N-d \ell/2-k}} \equiv \sum_{k=0}^{\infty} \mathcal{M}_k, \tag{2}
\]

where

\[
\mathcal{D}^N \alpha = \prod_{r=1}^{N} d\alpha_r \delta(\sum_{r=1}^{N} \alpha_r - 1), \tag{3}
\]

\[
\Delta(\alpha) = \sum_{T_1} (\prod_{r=1}^{\ell} \alpha_r), \tag{4}
\]

\[
D(\alpha,p) = \sum_{r=1}^{N} \alpha_r m_r^2 - P(\alpha,p), \tag{5}
\]

\[
P(\alpha,p) = \Delta(\alpha)^{-1} \sum_{T_2} (\prod_{r=1}^{\ell+1} \alpha_r)(\sum_{r=1}^{\ell} p)^2, \tag{6}
\]

\[
J_r = \Delta(\alpha)^{-1} \sum_{T_2(r)} \alpha_r^{-1} \prod_{r=1}^{\ell+1} \alpha_r(\sum_{r=1}^{\ell} p), \tag{7}
\]

\[
S_0(J,p) = S(J,p). \tag{8}
\]

The quantities appeared in (2) have a very simple physical interpretation. If we consider the Feynman diagram as an electrical circuit, with the external momenta \( p_A \) as the external currents, and the Feynman parameters \( \alpha_r \) as the resistance of the \( r \)th line, then \( J_r \) is simply the current flowing through the \( r \)th line, and \( P(\alpha,p) \), which can be proven to be equal to \( \sum_{r=1}^{N} J_r^2 \alpha_r \), is just the power dissipated in the circuit. The cryptic formulas (3)–(8) offers a simple and practical way to compute these currents and the power directly from the Feynman diagram.

We shall now elaborate on these cryptic formulas. A connected diagram with \( \ell \) loops can be made into a connected tree diagram if \( \ell \) internal lines are cut. There are many ways to do this, each resulting in a different \( \text{one-} \)tree \( T_1 \). The sum in (4) is taken over all such one-trees \( T_1 \), with each term in the sum equal to the product of all the Feynman parameters \( \alpha_r \) of the cut lines. As a result, \( \Delta(\alpha) \) is of a homogeneous degree \( \ell \) in the \( \alpha \)'s.

Similarly, \( \ell + 1 \) cuts can bring the diagram into two connected trees that are disjoint, or a ‘two-tree’ \( T_2 \). The sum in (6) is over the set of all such two-trees \( T_2 \). This time each term consists of the product of the \( \ell + 1 \) Feynman parameters of the cut lines, times the square of the sum of all the external momentum \( p_A \) attached to one of the two trees. It does not matter which tree we choose to compute the momentum sum because of conservation of momentum.
Now we come to the numerator $S_k(J, p)$ in (2), for $k = 0, 1, 2, \ldots$. The first term $S_0(J, p)$ is just the numerator factor $S(q, p)$ in (1) with each $q_r$ replaced by $J_r$. The rule for computing the current $J_r$ is given in (7), where the sum is over the set of all two-trees $T_2(r)$ obtained by having the line $r$ always cut. The summand consists of the product of the $\alpha$’s of the cut lines except the $r$th (so it is of homogeneous degree $\ell$ in $\alpha$), times a momentum factor given by the sum of all the external momenta attached to one of the two resulting trees. If the momentum $q_r$ flows from tree 1 to tree 2, then it is the sum of $p_A$ attached to tree 2, or minus the sum of $p_A$ attached to tree 1, that should be used in the sum. This convention presumes that all the external momenta $p_A$ are outgoing. In other words, the sign is such that the direction of the flow of the current $J_r$ must match that of the external currents $p_A$.

We are now in a position to describe $S_k(J, p)$ for $k > 0$. It is obtained from $S_0(J, p)$ by contracting $k$ pairs of $J$’s in all possible ways, and summing over all such contractions. If no contractions are possible then $S_k = 0$. For each pair $J_r, J_s$ in the contraction, one makes the replacement

\[ J_r^\mu J_s^\nu \rightarrow -\frac{1}{2} g^{\mu\nu} H_{rs}, \]

and the factor $H_{rs}$ is given by

\begin{align}
H_{rr} &= -\Delta(\alpha)^{-1} \partial \Delta(\alpha) / \partial \alpha_r, \\
H_{rs} &= \pm \Delta(\alpha)^{-1} \sum_{T_2(rs)} (\alpha_r \alpha_s)^{-1} (\prod \alpha), \quad (r \neq s). 
\end{align}

This time the sum in (11) is over the set of all two-trees $T_2(rs)$ in which lines $r$ and $s$ must have been cut, and each term in the sum is a product of the $\alpha$’s of the cut lines except the $r$th and the $s$th. The sign in front is $+1$ if both $q_r$ and $q_s$ flow from tree 1 to tree 2, and $-1$ otherwise.

This concludes the description of the quantities in (2). We shall now illustrate these rules with one-loop diagrams. See Ref. [22] for an illustration of these rules for a two-loop diagram.

A tree is obtained from a one-loop diagram by cutting any of its $N$ internal lines. Thus for any one-loop diagram, (4) with (3) yields

\[ \Delta(\alpha) = \sum_{r=1}^{N} \alpha_i = 1 \]

and (10) gives

\[ H_{rr} = -1. \]

Now specialize to a box diagram (Fig. 1(a)) and a vertex diagram (Fig. 1(b)). Using (6), (7), (11) and (12), one gets for the box diagram

\[ P(\alpha, p) = \alpha_1 \alpha_3 (p_1 + p_2)^2 + \alpha_2 \alpha_4 (p_1 + p_4)^2 + \alpha_1 \alpha_2 p_1^2 + \alpha_2 \alpha_3 p_2^2 + \alpha_3 \alpha_4 p_3^2 + \alpha_4 \alpha_1 p_4^2, \]

\[ J_1 = \alpha_2 p_1 + \alpha_3 (p_1 + p_2) + \alpha_4 (p_1 + p_2 + p_3), \]
\[ J_2 = \alpha_3 p_2 + \alpha_4 (p_2 + p_3) + \alpha_1 (p_2 + p_3 + p_4), \]
\[ J_3 = \alpha_4 p_3 + \alpha_1 (p_3 + p_4) + \alpha_2 (p_3 + p_4 + p_1), \]
\[ J_4 = \alpha_1 p_4 + \alpha_2 (p_4 + p_1) + \alpha_3 (p_4 + p_1 + p_2), \]
\[ H_{rs} = -1 \ (r \neq s), \tag{14} \]

and for the vertex diagram
\[ P(\alpha, p) = \alpha_1 \alpha_2 (p_2 + p_3)^2 + \alpha_2 \alpha_3 p_2^2 + \alpha_1 \alpha_3 p_3^2, \]
\[ J_1 = -\alpha_3 p_3 - \alpha_2 (p_2 + p_3), \]
\[ J_2 = \alpha_3 p_2 + \alpha_1 (p_2 + p_3), \]
\[ J_3 = \alpha_1 p_3 - \alpha_2 p_2, \]
\[ H_{rs} = -1 \ (r \neq s). \tag{15} \]

Before leaving this section let us say a word about renormalization. For theories that are no more than logarithmically divergent, primitive ultraviolet divergence, if any, comes from the term with the highest power of \( q \) in the numerator of (1), and this power is \( 2(k_{\text{max}}) = 2(N - 2\ell) \). In the language of (2), this occurs only in the term \( S_{k_{\text{max}}}(J, p) \) with the maximum number of contractions. If we let \( d = 4 + 2\epsilon \), then this term is
\[ M_{k_{\text{max}}} = \left( \frac{\pi^{2+\epsilon}}{16} \right)^{\ell} \Gamma(\ell) \int \frac{\mathcal{D}^N \alpha}{\Delta(\alpha)^{2+\epsilon}} \frac{S_{k_{\text{max}}}(J, p)}{D(\alpha, p)\ell^\epsilon}. \tag{16} \]

Renormalization in the MS or \( \overline{\text{MS}} \) scheme is therefore easy to carry out.

### 3. Color Decomposition and Spin Flow

We specialize now to QED and \( SU(N) \) QCD. With simple modifications this can also be applied to the electroweak processes. The quark \((q)\) belongs to the fundamental representation and both the gluon \((g)\) and the Fedeev-Popov ghost \((G)\) belong to the adjoint representation. For the purpose of discussing color decomposition, there is no need to distinguish ‘\( g \)’ and ‘\( G \)’ so we shall collectively refer to them as ‘\( G \)’.

The factor \( S(q,p) \) in (1) is composed of vertex contributions and the numerators of propagators. As such it contains information on both spin and color. It is this factor that contains the large number of terms mentioned in the Introduction. The purpose of this section is to discuss how this quantity (and the corresponding scattering amplitude \( \mathcal{M} \)) can be reorganized to simplify calculations. Specifically, the factor \( S(q,p) \) (the amplitude \( M \)) is to be decomposed into the form \( \sum_i \mathcal{C}_i m_i \), where \( \mathcal{C}_i \) is the color factor, which consists of products of the \((S)U(N)\) generators and their traces, and \( m_i \) carries spin and momentum but no color information. We shall refer to \( m_i \) in the decomposition of \( S(q,p) \) as the spin factor (other than some trivial factors of the coupling constant \( g \) to be discussed later), and \( m_i \) in the decomposition of \( \mathcal{M} \) as the color subamplitude. There are at least three advantages for such a decomposition. First of all, many \( \mathcal{C}_i \)’s differ from one another
only by permutations of the color indices. Consequently only one of these \(m_i\)'s has to be computed explicitly; the rest of them can be obtained by similar permutations. Secondly, the color factors \(C_i\) are independent. As a result each spin factor and color subamplitude are invariant under an arbitrary gauge change of an external polarization vector. This aspect of it will be particularly useful in the spinor helicity technique because we can choose the reference momenta independently for each \(m_i\). Thirdly, the \(m_i\)'s satisfy other important identities like the ‘dual Ward’s identity’ which can be utilised in practical computations.

Color decomposition has been carried out algebraically for tree and one-loop diagrams \([16,19]\); we shall do it to all loops and do it graphically in order to minimize the algebra. For that purpose we will introduce color-oriented Feynman diagrams, from which color factors as well as spin factors can be read off directly.

The discussion in this section is independent of the last section. Hence the results are equally applicable to momentum-space representations as well as Feynman-parameter representations.

It is convenient to extend the gauge theory by an extra \(U(1)\) factor to complete it to an \(U(N) = U(1) \times SU(N)\) gauge theory. This simplifies the algebraic manipulation without losing any information, for as we shall see QCD \((SU(N))\) expressions can be read off from the simpler results of an \(U(N)\) gauge theory.

Let \(T^A (A = 0, a; a = 1, \cdots, N^2 - 1)\) be the \(U(N) = U(1) \times SU(N)\) generators in the fundamental representation. \(T^0 = (1/\sqrt{N})1\) is the \(U(1)\) generator and \(T^a\) are the \(SU(N)\) generators. The normalization of the \(U(1)\) factor is chosen to satisfy the normalization

\[
\text{Tr}(T^A T^B) = \delta^{AB}. \tag{17}
\]

The structure constant \(f^{ABC}\) can be obtained from the commutation relation \([T^A, T^B] = if^{ABC}T^C\) by the formula

\[
f^{ABC} = -i\{\text{Tr}(T^A T^B T^C) - \text{Tr}(T^A T^C T^B)\}, \tag{18}\]

and is seen to be antisymmetric in its \(U(N)\) indices. Note from this that \(f^{0BC} = 0\), which reflects the physics that the \(SU(N)\) and the \(U(1)\) gauge bosons do not interact directly with each other. This is an important fact which will allow us to project out the \(U(1)\) bosons to regain QCD.

The completeness relation dual to (17) is

\[
(T^A)_{ij}(T^A)_{kl} = \delta_{jk}\delta_{il}, \tag{19}
\]

where summation over the \(N^2\) repeated indices \(A\) is understood. From this one obtains

\[
f^{ABE} f^{ECD} = (-i)^2\{\text{Tr}(ABCD) - \text{Tr}(BACD) - \text{Tr}(ABDC) + \text{Tr}(BADC)\}. \tag{20}\]

For brevity, we have chosen to write \(\text{Tr}(T^A T^B T^C T^D)\) simply as \(\text{Tr}(ABCD)\), and we shall often use this same abbreviation of replacing \(T^A\) simply by \(A\) in the rest of this paper.

The vertices for QCD and \(U(N)\) gauge theory in the background Feynman gauge \([23]\) are exhibited in Fig. 2. A thin solid line stands for a gluon, a dashed line stands for the
In that case the gluon lines have appeared explicitly in a diagram in Fig. 2 in which a circled ‘A’ is present. As an internal line or an external line, unless such a combination of external and internal where no circled ‘A’ appear, each of the gluon lines in the diagram may be taken either where no circled ‘A’ appear, each of the gluon lines in the diagram may be taken either as an internal line or an external line, unless such a combination of external and internal gluon lines have appeared explicitly in a diagram in Fig. 2 in which a circled ‘A’ is present. In that case the Feynman rule (see eq. (21) below) for the diagram with circled ‘A’s should be used.

The discussion in this section can be applied to any gauge. However for the sake of application in the next two sections we shall use explicitly the background Feynman rules for these vertices in the background Feynman gauge. Although there are more vertices (those with circled ‘A’) in the background gauge than the usual covariant gauges, nevertheless we shall see in the next section that the use of background gauge along with the spinor helicity basis simplifies enormously the calculations.

All the momenta in Fig. 2 are understood to be pointing outwards. It is also understood that the line labelled by 1 carries an outgoing momentum $p_1$, a color index $a$, and a Lorentz index $\alpha$. Similarly, line 2 carries the quantum numbers $(p_2, b, \beta)$, etc. The Feynman rules for these vertices in the background Feynman gauge are given below, with the equation numbers corresponding to the diagrams in Fig. 2. For example, eq. (21a) is the Feynman rule for the vertex in Fig. 2(a).

\begin{align*}
\bullet & \quad ig f^{abc} \{ g_{\alpha \beta} (p_1 - p_2)_{\gamma} + g_{\beta \gamma} (p_2 - p_3)_{\alpha} + g_{\gamma \alpha} (p_3 - p_1)_{\beta} \}, \\
\bullet & \quad ig f^{abc} \{ g_{\alpha \beta} (2p_1)_{\gamma} + g_{\beta \gamma} (p_2 - p_3)_{\alpha} + g_{\gamma \alpha} (-2p_1)_{\beta} \}, \\
\bullet & \quad - g^2 f^{abc} f^{eecd} (g_{\alpha \gamma} g_{\beta \delta} - g_{\alpha \delta} g_{\beta \gamma}) - g^2 f^{abe} f^{febd} (g_{\alpha \beta} g_{\gamma \delta} - g_{\alpha \delta} g_{\beta \gamma}) \\
& \quad - g^2 f^{bce} f^{fed} (g_{\beta \alpha} g_{\gamma \delta} - g_{\delta \alpha} g_{\gamma \beta}), \\
\bullet & \quad - g^2 f^{abe} f^{ecd} (g_{\alpha \beta} g_{\gamma \delta} - g_{\delta \alpha} g_{\beta \gamma} + g_{\alpha \beta} g_{\gamma \delta}) - g^2 f^{ace} f^{fbd} (g_{\alpha \beta} g_{\gamma \delta} - g_{\delta \alpha} g_{\beta \gamma}) \\
& \quad - g^2 f^{bce} f^{fed} (g_{\beta \alpha} g_{\gamma \delta} - g_{\delta \alpha} g_{\beta \gamma} - g_{\beta \gamma} g_{\delta \alpha}), \\
\bullet & \quad - ig f^{abc} (-p_3)_{\alpha}, \\
\bullet & \quad - ig f^{abc} (p_2 - p_3)_{\alpha}, \\
\bullet & \quad - g^2 f^{abe} f^{ecd} g_{\alpha \beta}, \\
\bullet & \quad - g^2 (f^{abe} f^{ecd} + f^{ace} f^{fbd}) g_{\alpha \beta}, \\
\bullet & \quad g T^{a} \gamma_{\alpha}.
\end{align*}

The $U(N)$ Feynman rules are the same except that the $SU(N)$ color indices $a, b, c, d$ should be replaced by the $U(N)$ indices $A, B, C, D$.

We shall use (18) and (20) to replace the factors $f^{ABC}$ and $f^{ABE} f^{ECD}$ in (21), and then proceed to group the terms with the same $U(N)$ traces. Each of these terms defines a color-oriented vertex in which the $U(N)$ indices of the external lines read clockwise coincides with the indices in the trace read from left to right.

Each color-oriented vertex factor is a product of three quantities: the coupling constant factor, the color factor, and the spin factor. The coupling constant factor will be taken to
be $g$ for cubic vertices and $g^2$ for quartic vertices. The color factors will be taken to be $T^A$ for a $qqg$ vertex, to be $Tr(ABC)$ for a $GGG$ vertex, and to be $Tr(ABCD)$ for a $GGGG$ vertex. The rest of the vertex factor will be defined to be the spin factor, the details of which are exhibited in Fig. 3.

Fig. 3 should be read in the following way. A gluon line continuing through the vertex indicates a factor of a metric tensor in spacetime. A dot represents the vector written below the diagram; other numerical factors are also written below the diagram. For example, the color factor for diagram (a) is $Tr(ABC)$, its coupling-constant factor is $g$, and its spin factor is $g_{\beta\gamma}(p_2 - p_3)_{\alpha}$. The color factor for diagram (e) is $Tr(ABCD)$, its coupling-constant factor is $g^2$, and its spin factor is $2g_{\alpha\gamma}g_{\beta\delta}$.

The Feynman diagrams assembled from color-oriented vertices will be called color-oriented Feynman diagrams, or just an oriented diagrams for short. A color-oriented diagram can be obtained from an ordinary Feynman diagram by flipping any number of external gluon lines each about the $G$ propagator it emerges from, by interchanging two identical external $G$ lines emerging from the same vertex, by interchanging two identical internal $G$ lines if this does not alter the topology of the diagram, or a combination of these. In general, an ordinary Feynman diagram leads to many color-oriented Feynman diagrams. The total contribution to a scattering amplitude is the sum of the contributions from all the color-oriented diagrams.

If the Feynman diagram in question can be obtained from the infinite tension limit of a string diagram, then flipping of the gluon line corresponds to a twisting of the string. The color factors for the whole diagram to be discussed below are nothing but the Chan-Paton factors [24].

The total color factor of an oriented diagram is the product of the color factors of its oriented vertices, summed over intermediate color indices. Eq. (19) can be used to carry out these sums; the result of which gratifyingly can be read off once again directly from the color-oriented Feynman diagram.

Let us start with a tree diagram having $n$ external $G$ particles and no fermion anywhere in the diagram. The color factor of this tree turns out to be

$$Tr(C_1C_2\cdots C_n),$$

(22)

where $C_1, C_2, \cdots, C_n$ are the $U(N)$ color indices of the oriented Feynman diagram read clockwise around the whole tree. For example, the color factor for Fig. 4 is $Tr[(1)(2)(3)(4)\cdots(14)(15)(16)]$.

From now on, we shall use capital letters near the end of the alphabets to denote products of $U(N)$ generators, e.g., $X = C_1C_2\cdots C_p$.

Eq. (22) can be proven by induction. By definition, a color-oriented diagram with a single $GGG$ or a single $GGGG$ vertex is already of the form of (22). Suppose now we have two trees, the color factor of each is of the form (22). This is illustrated in Fig. 5, where the color factors of the two trees are respectively $Tr(XA)$ and $Tr(AY)$. When we sew these two trees together at index ‘A’ to obtain a bigger tree, the resulting color factor, using (19), is indeed $Tr(XY)$, which can be read out directly from Fig. 5 using the rules of (22). This completes the induction proof of (22).
The result for $SU(N)$ QCD is equally simple and the color factor is again given by (22), but with the upper case $U(N)$ indices $C_i$ replaced by the corresponding lower case $SU(N)$ indices $c_i$. This is so because of the absence of coupling between the $U(1)$ and the $SU(N)$ gauge bosons, viz., $f^{ABC} = 0$. Therefore as long as the external lines of a connected tree carry an $SU(N)$ indices, the $U(1)$ gluon is decoupled and will never makes its appearance in the internal line either.

Next, consider tree diagrams in which a single fermion line is present. The color factor for a qqg vertex is $T^A$ if $A$ is the $U(N)$ color index of the gluon. If a whole tree of $G$ particles with the color factor $Tr(XA)$ is planted at this vertex, then the combined color factor is obtained from (19) to be $X$. If $A_1, A_2, \cdots, A_n$ are the successive qqg vertices as we go along a fermion line, and if $G$-trees with color factors $Tr(X_iA_i)$ are planted at these vertices, then the combined color factor would be

$$X_1X_2\cdots X_n. \quad (23)$$

Graphically, this is simply the multiplication of all the $U(N)$ generators $T$ in clockwise order around the whole tree, as shown in Fig. 6.

Note the difference between a ggg and a qqg vertex. The former is oriented, in the sense that there are two oriented vertices associated with one ordinary Feynman vertex, but the oriented vertex in the latter case is the same as the ordinary vertex. In the ggg vertex, there is no further color specification other than the indices of the external $G$-lines, but in the qqg vertex, the color state of the initial and the final quarks must still be specified. The result of these differences is that as we traverse clockwise around an oriented diagram to read out its color factor, we must cover both sides of every $G$-line, and that the trace of the product of generators must be taken. On the other hand, we should traverse only through the top side of a fermion line and no trace of the product of generators is to be taken. If we should find it convenient to draw a $G$-tree downward from a fermion line, as is the case of the $X_2$-tree in Fig. 6, then clockwise order must still be maintained in the way indicated in the figure. In other words, since we only follow the top and not the bottom of the fermion line, the color factor in Fig. 6 is $X_1X_2X_3$, and not $X_1X_3X_2$ as we might think if we were to follow both sides of the fermion line.

The same result (23) is again true if we consider only $SU(N)$ QCD. Once again this is due to the lack of coupling between the $U(1)$ and the $SU(N)$ gluons.

The situation of having a $G$-tree connecting two separate quark lines can be obtained similarly from (19), but the paths along which the color generators are multiplied together now cross over from one fermion line to another, as in Fig. 7. This is so because

$$Tr(AYBV)XAW \otimes UBZ = XYZ \otimes UVW. \quad (24)$$

If on the way one encounters another $G$-tree connecting to a third fermion line, then one must cross over to the third tree at that point, etc.

This formula is still valid in $SU(N)$ QCD as long as either $Y$ or $V$ factor appearing in the tree $Tr(AYBV)$ connecting the two fermion lines contains at least one external $SU(N)$ gluon. In that case, as before, decoupling prevents the $U(1)$ gluon to appear even in the internal lines. The situation is different if the tree is simply $Tr(AB)$. In that case an $U(1)$
The gluon connecting the two fermion lines is present, and its effect must be subtracted away when $SU(N)$ is considered. The result is then

$$XY \otimes UW - (1/N)XW \otimes' UZ.$$ (25)

The second term follows the original fermion lines all the way without a cross-over, and this distinction from the first term is indicated in the formula by using $\otimes'$ rather than $\otimes$.

Loop diagrams are obtained by joining ends of tree diagrams. Consider first the cases when ends of fermions are joined into fermion loops. In the presence of a single fermion line, or whenever such a fermion line is not attached to another fermion line by gluons, then all we have to do is to take the trace over (23). If two fermion lines are present, as in Fig. 7, and if the two ends of the top fermion line are joined together to form a fermion loop, then the color factor for an $U(N)$ gauge theory can be obtained from (24) to be

$$XYZUVW.$$ (26)

Note that this can be read off directly from Fig. 7 as long as we remember to cross over at the $G$-tree. Other cases involving more fermions can be obtained similarly.

Consider next a fermion-$G$ loop, as in Fig. 8, obtained by attaching a $G$-tree of color factor $Tr(AYBU)$ to a fermion line with color factor $XAVBZ$ at points $A$ and $B$. Again first imagine point $A$ to have been attached but not point $B$. Then the color factor of the combined tree is $XYBUVZ$, summed over $B$. This yields

$$XYZTr(UV).$$ (27)

Note that this factor can again be read off directly from Fig. 8: $XVZ$ is the multiplication of the color generators clockwise order following the outside path of the loop, and $Tr(VY)$ is the trace factor corresponding to the lines inside of the loop. This feature about tracing the outside of a loop and the inside separately will occur again when we discuss $G$-loops. Note that the outside of the loop, like the original trees, is followed clockwise, whereas the inside of the loop is followed counter-clockwise.

As a check, note that Fig. 8 can also be obtained from Fig. 7 by joining the right end of the bottom fermion line to the left end of the top fermion line. In this way one again obtains (26).

Lastly, we will consider sewing ends of a $G$-tree together to form a loop, as in Fig. 9. Before we fuse it together at the point $A$, the color factor for the tree is $Tr(RSAXYATU)$. Summing over $A$ yields

$$Tr(RSTU)Tr(XY).$$ (28)

The result can be read again directly from the graph. The first trace is taken in clockwise order along a closed path around the whole diagram passing through the outside of the loop; and the second trace is taken in counter-clockwise order along the closed path passing through the inside of the loop.

One can apply (19) to more complicated diagrams with any number of loops. The result in the case of an $U(N)$ gauge theory can always be read off simply from the color-oriented diagram. The general rule for the color factor $C_i$ is the following. Circle around
the diagram with continuous ‘color paths’ of the following kind. These paths may start
from one end of a fermion line and end at another end (of possibly another fermion line), or
else they must be closed. The upper side of every fermion line and both sides of every gluon
and ghost line must be covered once and only once by these paths. Associate each external
gluon with $U(N)$ color index $A$ the generator $T^A$. Go along the path in clockwise order
(counter-clockwise order if it is inside a loop) and multiply these generators successively
from left to right. If the path is an open path, this product is the color factor associated
with the path. If the path is closed, then a trace should be taken. The total color factor
for the color-oriented connected diagram is the product of these individual color factors.

See Figs. 4–9 for illustrations.

The rules for $SU(N)$ can be obtained from the $U(N)$ rules by subtracting out the
$U(1)$ gluons which remains coupled in the diagram.

Having thus a graphical way to read out the total color factor for an oriented diagram,
the next task is to find an equally simple and general graphical method to read out the
total spin factor of the oriented diagram. This can be done very simply, and the notation
adopted in Fig. 3 is actually designed with this in mind.

To do so, cover the maximal gluon subdiagram of the oriented diagram in question
with ‘spin-flow paths’. A spin-flow path is different from a color path discussed above in
that it stays right on the gluon lines of the diagram and not above or below them. A
spin-flow path is meaningful only for a gluon line, internal or external, and it is simply a
continuous path tracing through a portion of the gluon subdiagram. Such a path may be a
closed path, or an open path. If it is an open path, it must end at an external gluon line,
or a cubic vertex. Conversely, there must be one and only one path ending at each cubic
vertex. See Figs. 12 and 13 for examples of these paths.

The spin factor associated with a closed path is $g^\mu_\nu = d$, and the spin factor associated
with an open path is the dot product of the vectors at the two ends. The vector at a cubic
vertex is given in Fig. 3, and the vector associated with an external gluon line is simply
its polarization vector $\epsilon$. The total spin factor of the oriented diagram is the product of
the spin factors of all the paths, times whatever extra numerical factors appearing at the
quartic vertices in Fig. 3, times products of the numerators of fermion propagators $(\gamma \cdot q)$ if
present, summed over all possible spin-flow path coverings of the maximal gluon subgraph.

The coupling constant factor for an oriented diagram is simply the product of the
coupling constant factors of all its vertices.

The numerator factor $S(q, p)$ in (1) for a Feynman diagram is then the product of the
color factor, the coupling-constant factor, and the spin factor, summed over all spin-flow
paths and all color-oriented diagrams. Extra factors such as the minus sign associated with
each closed fermion loop and the symmetry factor will be absorbed into $S(q, p)$ as well.
To be sure, there are many terms present for a complex diagram corresponding to many
spin-paths and many color-oriented diagrams. In fact, all that we have done up to this
point is to give a graphical interpretation of every term that appears in $S(q, p)$. We have
not reduced the number of terms there in any way. However, this graphical approach helps
to organize the terms mentally without having to write down a single algebraic formula,
so it helps to keep us away from the algebraic jungle. The real simplification comes in only
when we start using the spinor helicity technique and the background gauge in the next
Simplifications can also result from supersymmetry. Consider for example an oriented diagram containing a quark line. The spin factor is essentially the same when the quark is replaced by a gluino. Under such a replacement, the color factor changes simply by having traces taken over the original color factor. So the colored subamplitudes of a quark diagram is the same as that for a gluino diagram. On the other hand, the colored subamplitude of a gluino diagram is related to that for a pure gluon diagram by supersymmetry. This chain of reasoning makes it possible to relate pure gluon amplitudes with those with a quark line in it. Such supersymmetry relations [16,17] have been used in tree processes to simplify calculations, and as we shall see in Sec. 5, it can be used to simplify calculations for loop amplitudes as well.

4. Spinor helicity basis

We have discussed how to organize graphically the numerator factor $S(q, p)$ in the last section. Nevertheless, there are many terms involved, corresponding to the many color-oriented diagrams and the many spin-flow paths for a given diagram. A clever choice of gauge and polarization vectors at this point can render many of the terms zero, making it unnecessary to consider some path coverings and/or color-oriented diagrams, and thereby reduce the labour of computation enormously. We shall see that the use of background gauge together with polarization vectors chosen in the helicity spin basis will accomplish this purpose.

We shall first summarize the known results of the spinor helicity basis taken from Ref. [16], and then go on to discuss further simplifications brought about by the use of the background gauge.

Let $|p\pm\rangle$ be the incoming wave function of a massless fermion with momentum $p$ and chirality $\pm 1$, normalized in such a way that

$$\langle p\pm|\gamma_{\mu}|p\pm\rangle = 2p_{\mu}. \quad (29)$$

From chirality conservation, one gets

$$\langle p\pm|q\pm\rangle = 0 \quad (30)$$

valid for any other massless momentum $q$. This is the central relation that leads to much of the simplifications. Let

$$\langle pq \rangle = \langle p - |q+\rangle = -\langle q - |p+\rangle = -\langle qp \rangle,$$

$$[pq] = \langle p + |q-\rangle = -\langle q + |p-\rangle = -\langle qp \rangle. \quad (31)$$

Then

$$\langle pq \rangle^* = sign(p \cdot q)[qp]; \quad (32)$$

$$\langle pq \rangle [qp] = 2(p \cdot q), \quad (33)$$
\[ \langle p \pm |\gamma_\mu_1 \cdots \gamma_\mu_{2n+1}|q\rangle = \langle q \mp |\gamma_\mu_{2n+1} \cdots \gamma_\mu_1|p\mp \rangle, \quad (34) \]
\[ \langle p \pm |\gamma_\mu_1 \cdots \gamma_\mu_{2n}|q\rangle = -\langle q \pm |\gamma_\mu_{2n} \cdots \gamma_\mu_1|p\mp \rangle, \quad (35) \]
\[ \langle AD\rangle\langle CD\rangle = \langle AD\rangle\langle CB\rangle + \langle AC\rangle\langle BD\rangle, \quad (36) \]
\[ \langle A + |\gamma_\mu|B\rangle\langle C - |\gamma'|D\rangle = 2\langle AD\rangle\langle CB\rangle, \quad (37) \]
\[ \gamma p = |p+\rangle\langle p+| + |p-\rangle\langle p-|. \quad (38) \]

To illustrate how massless momenta and the ensuing chirality conservation can simplify calculations, consider the calculation of \( Tr[(\gamma p_1)(\gamma p_2) \cdots (\gamma p_{2n-1})(\gamma p_{2n-1})] \), where every \( p_i \) is massless. Using usual formulas, this is given by a sum of \( (2n)!/2^n n! \) terms, each containing a product of \( n \) pairs of momentum dot products. Using (38), (30) and (31), this can be reduced to just a sum of two terms:
\[ \langle p_1p_2\rangle[p_2p_3] \cdots [p_{2n-1}p_{2n}]\langle p_{2n}p_1 \rangle + [p_1p_2]\langle p_2p_3 \rangle \cdots [p_{2n-1}p_{2n}]\langle p_{2n}p_1 \rangle. \quad (39) \]

The polarization vector for an outgoing photon or gluon with momentum \( p \) and helicity \( \pm 1 \) can chosen in a multispinor basis to be
\[ \epsilon_\mu^\pm(p, k) = \pm \frac{\langle p \pm |\gamma_\mu|k\rangle}{\sqrt{2}\langle k \mp |p\rangle}, \quad (40) \]
where the reference momentum \( k \) in (40) is massless but otherwise arbitrary. The choice of different \( k \) corresponds to the choice of a different gauge, and these different choices are related by
\[ \epsilon_\mu^+(p, k) \rightarrow \epsilon_\mu^+(p, k') - \sqrt{2} \frac{\langle kk'\rangle}{\langle kp\rangle\langle k'p\rangle} p_\mu. \quad (41) \]

These polarization vectors satisfy the following identities:
\[ \epsilon_\mu^\pm(p, k) = (\epsilon_\mu^- (p, k))^*, \quad (42) \]
\[ \epsilon^\pm(p, k) \cdot p = \epsilon^\pm(p, k) \cdot k = 0, \quad (43) \]
\[ \epsilon^\pm(p, k) \cdot \epsilon^\pm(p, k') = 0, \quad (44) \]
\[ \epsilon^\pm(p, k) \cdot \epsilon^\mp(p, k') = -1, \quad (45) \]
\[ \epsilon^\pm(p, k) \cdot \epsilon^\pm(p', k) = 0, \quad (46) \]
\[ \epsilon^\pm(p, k) \cdot \epsilon^\mp(k, k') = 0, \quad (47) \]
\[ \epsilon_\mu^+(p, k)\epsilon_\nu^-(p, k) + \epsilon_\mu^-(p, k)\epsilon_\nu^+(p, k) = -g_{\mu\nu} + \frac{p_\mu k_\nu + p_\nu k_\mu}{p \cdot k}, \quad (48) \]
\[ \gamma \cdot \epsilon^\pm(p, k) = \pm \frac{\sqrt{2}}{\langle k \mp |p\rangle} (|p\mp\rangle\langle k \mp | + |k\pm\rangle\langle p \pm |). \quad (49) \]

This completes the summary of the properties of the spin-helicity basis. The vanishing dot products (43), (44), (46), (47) are what make this basis particularly useful.

Background gauge is convenient for loop calculations because (43) – (47) can be used to eliminate many terms in this gauge. In this gauge, two of the three terms in the ggg
vertex with an external line (see Fig. 3) involves only the momentum of the external line. This enables many terms to vanish as we shall see in the following illustration.

Consider an $n$-gluon color-oriented diagram where either all the gluons have the same helicity, or all but one have the same helicity. Let us consider the latter, and assume gluon 1 to have a negative helicity while all the other gluons have positive helicities. Let us choose the reference vectors $k_i$ for the polarization vector $\epsilon(p_i, k_i)$ to be $k_1 = p_2$ and $k_i = p_1$ for all $i \neq 1$. This choice is designed so that (43) to (47) can be used to show that

$$\epsilon_i \cdot \epsilon_j = 0, \quad (\forall i, j), \quad (50)$$
$$p_i \cdot \epsilon_i = p_1 \cdot \epsilon_i = 0 \quad (\forall i), \quad (51)$$
$$p_2 \cdot \epsilon_1 = 0. \quad (52)$$

Eq. (50) is particularly useful. The spin factor consists of products of dot products of the form $\epsilon \cdot \epsilon'$, $\epsilon \cdot q$, $q \cdot q'$, of degree $n$ in the polarization vectors and of degree $m$ in the momenta if there are $m$ GGG vertices. For tree diagrams $m \leq n - 2$, so at least one $\epsilon \cdot \epsilon'$ must be present in every term. Because of (50) the tree amplitude with this helicity configuration must vanish [16]. For one loop diagrams, $m = n$ only if no quartic vertices are present. Otherwise $m < n$, and the corresponding contribution again vanishes on account of $\epsilon \cdot \epsilon'$. This greatly simplifies calculations because no gggg nor ggGG vertices need ever be considered. Moreover, in the Feynman-parameter representation, contraction (9) again leads to the presence of $\epsilon \cdot \epsilon'$ so current contractions never have to be considered. As a result, all $S_k(J, p) = 0$ except $S_0(J, p) = S(J, p)$.

Other simplifications can be seen in Fig. 10. Paths A and B vanish in the background Feynman gauge because of (51) and Figs. 3(c,d). Path C vanishes because of (50). As a corollary paths between two dots like $C'$ are also forbidden because a path C must then be present to take up the leftover $\epsilon$ factors. Path D vanishes because of (52).

5. One-loop four-gluon amplitude

To illustrate techniques developed in the last three sections, we compute in this section a one-loop four-gluon amplitude in which three of the four gluons have the same helicity. We will first compute the case when quarks are absent because that pure gluon amplitude has been computed with the string technique [19-21], so a comparison of the efficiency of the two methods can be made. We find that the present method is every bit as efficient as the string technique.

Next, we shall compute the same four-gluon amplitude with an internal quark loop. Thanks to chirality conservation the calculation is even simpler than the pure gluon case. Using supersymmetry arguments, this amplitude can be related to the pure gluonic amplitude, thereby providing a second method to compute the pure QCD four-gluon amplitude. The result agrees with the first calculation, but the number of steps needed to reach the result is now even smaller.

The reference momenta for the polarization vectors will be chosen as in the last section, so that eqs. (50) – (52) can be used. As discussed there, quartic vertices do not contribute, and current contractions cannot occur so $S_k(J, p) = 0$ in (2) for $k > 0$. 

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There is a further simplification when four-point amplitudes are considered. Each polarization vector is perpendicular to two momenta: its own gluon momentum and its reference momentum. That means that its dot product with the other two external momenta are equal and opposite, thereby resulting in only one independent dot product per polarization vector. Let

\[
A_1 = \epsilon_1 \cdot p_3 = -\epsilon_1 \cdot p_4 = -\frac{\langle 13 \rangle [32]}{\sqrt{2} [21]},
\]
\[
A_2 = \epsilon_2 \cdot p_3 = -\epsilon_2 \cdot p_4 = -\frac{[24] [41]}{\sqrt{2} [12]},
\]
\[
A_3 = \epsilon_3 \cdot p_2 = -\epsilon_3 \cdot p_4 = -\frac{[34] [41]}{\sqrt{2} [13]},
\]
\[
A_4 = \epsilon_4 \cdot p_2 = -\epsilon_4 \cdot p_3 = +\frac{[42] [21]}{\sqrt{2} [14]}.
\]

Then the numerator \( S_0(J, p) = S(J, p) \) in (2) is proportional to \( A_1 A_2 A_3 A_4 \).

With quartic vertices out of the way, the color-oriented diagrams contributing to this process are the box diagrams, the vertex insertion diagrams, and the self-energy insertion diagrams. We shall see that the self-energy diagrams are zero, and three out of the vertex-insertion diagrams also do not contribute.

To see that, consider the spin-flow path of Fig. 11(a) originating from gluon 4. Because of eq. (50) this path cannot end at another external gluon line. It cannot end at the vertex joining lines 2 and 3 either for then one is forced to have the factor \( \epsilon_2 \cdot \epsilon_3 \), which is zero. The only other possibility then is for the spin-flow to end at a vertex within the loop, in which case a factor \( \epsilon_4 \cdot J \) will result, where \( J \) is some combination of the currents flowing through the loop. From (7), we see that \( J \) is a linear combination of \( p_1 \) and \( p_4 \). Since \( \epsilon_4 \cdot p_1 = \epsilon_4 \cdot p_4 = 0 \), these paths are not allowed either. Consequently diagram 11(a) makes no contribution. The same argument will hold if instead of 4 and 1 it is gluons 1 and 2 which are attached to the loop. Consider now diagram 11(b), which we claim also makes no contribution. To see that, consider a spin-flow path that starts from gluon 1. This path cannot flow on to gluon 2, so it either ends at its own vertex, or goes beyond. In the former case the contribution vanishes because \( \epsilon_1 \cdot p_1 = \epsilon_1 \cdot p_2 = 0 \). In the latter case, the path starting from gluon 2 must end at its own vertex, and this vanishes because \( \epsilon_2 \cdot p_1 = \epsilon_2 \cdot p_2 = 0 \). Consequently, three of the four vertex-insertion diagrams associated with the color factor \( \text{Tr}(abcd) \) gives no contributions.

Essentially the same argument also shows that none of the self-energy insertion graphs like 11(c) makes any contribution. This leaves only the box graph Fig. 12 and the vertex insertion graph Fig. 13.

Let us consider the allowed spin-flow paths in the box diagrams, Fig. 12, remembering that paths A, B, C, C’, D of Fig. 10 may not be present. This means the path starting from gluon 1 must end at the same vertex, or else there must be another path ending at vertex 1 giving rise to some \( \epsilon_i \cdot p_1 = 0 \). There are actually altogether nine possible spin flows in an internal gluon loop and two more in a ghost loop, as shown in Fig. 12. Using (2) and (14),
the amplitude from the box diagram contributing to the color factor term \( Tr(abcd)Tr(1) \) is
\[
\mathcal{M}^B = \frac{1}{16\pi^2} \int D^4\alpha \frac{S^B(J,p)}{(\alpha_2\alpha_4 s + \alpha_1\alpha_3 t)^2},
\] (54)
with \( t = (p_1 + p_2)^2 \), \( s = (p_1 + p_4)^2 \), and \( u = (p_1 + p_3)^2 \). The numerator is given by \( S^B(J,p) = Tr(abcd)Tr(1)g^4B \), with \( B \) being the spin factor from all the box diagrams. Using the spin factors for the color-oriented vertices in Fig. 2, and the expression for the currents in eq. (14), one gets
\[
B_1 = g_\mu^\nu [\epsilon \cdot (J_2 + J_1)][\epsilon \cdot (J_3 + J_2)][\epsilon \cdot (J_4 + J_3)][\epsilon \cdot (J_1 + J_4)]/A_1A_2A_3A_4
\]
\[
= 4(2\alpha_4)(2\alpha_4)(-2\alpha_1 - 2\alpha_2)(2\alpha_3) = -64\alpha_4^2\alpha_3(\alpha_1 + \alpha_2),
\]
\[
B_2 = [\epsilon \cdot (J_2 + J_1)][\epsilon \cdot (-2p_4)][\epsilon \cdot (-2p_2)][\epsilon \cdot (-2p_3)]/A_1A_2A_3A_4
\]
\[
= (2\alpha_4)(2)(-2)(2) = -16\alpha_4.
\]

Similar calculations show that
\[
B_3 = B_2,
\]
\[
B_4 = B_5 = 16\alpha_4^2,
\]
\[
B_6 = B_7 = 16\alpha_4(\alpha_1 + \alpha_2),
\]
\[
B_8 = B_9 = 16\alpha_4\alpha_3,
\]
\[
B_{10} = B_{11} = -B_1/4. \quad (55)
\]

The sum is
\[
B = \sum_{i=1}^{11} B_i = -32(\alpha_1 + \alpha_2)\alpha_3\alpha_4^2. \quad (56)
\]

Consider now the vertex insertion graphs, Fig. 13. Using (2) and (15), the amplitude from the box diagram contributing to the color factor term \( Tr(abcd)Tr(1) \) is
\[
\mathcal{M}^V = \frac{1}{16\pi^2} \int D^3\alpha \frac{S^V(J,p)}{\alpha_1\alpha_2 s^2},
\] (57)
with the numerator given by \( S^V(J,p) = Tr(abcd)Tr(1)g^4V \), and \( V \) to be the spin factor contribution from all the vertex insertion diagrams. Using the spin factors for the color-oriented vertices in Fig. 2, and the expression for the currents in eq. (15), one gets
\[
V_1 = -64\alpha_1\alpha_2\alpha_3,
\]
\[
V_2 = V_3 = 16\alpha_3,
\]
\[
V_4 = -16(\frac{1}{2}\alpha_1 + \alpha_2 + \alpha_3),
\]
\[
V_5 = 16\alpha_2(\frac{1}{2}\alpha_1 + \alpha_2 + \alpha_3),
\]
\[
V_6 = V_9 = V_{12} = V_{13} = 8\alpha_1\alpha_2\alpha_3,
\]
\[ V_7 = -16(\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3), \]
\[ V_8 = 16\alpha_1(\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3), \]
\[ V_{10} = 16\alpha_1(\frac{1}{2}\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3), \]
\[ V_{11} = 16\alpha_2(\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3). \] (58)

The sum is
\[ V = \sum_{i=1}^{13} V_i = -32\alpha_1\alpha_2\alpha_3. \] (59)

The final result is
\[ M^B = \text{Tr}(abcd)\text{Tr}(1)\frac{g^4}{12\pi^2}s\frac{[24]^2}{[12](23)(34)[41]}, \] (60)
\[ M^V = \text{Tr}(abcd)\text{Tr}(1)\frac{g^4}{12\pi^2}t\frac{[24]^2}{[12](23)(34)[41]}, \] (61)
\[ M = M^B + M^V = \text{Tr}(abcd)\text{Tr}(1)\frac{g^4}{12\pi^2}(-u)\frac{[24]^2}{[12](23)(34)[41]}. \] (62)

This result agrees with the result obtained by the string method [19–21]. The vanishing of the diagrams in Fig. 11 is also a feature shared by the string method. In fact, it has been observed [21] that the string expression for box diagram corresponds to a calculation in the background Feynman gauge, though a mixture of the background gauge and the Neveu-Gervais gauge seem to be required for the vertex-insertion diagram. In the present case we use the background gauge throughout. The total number of terms in the box diagram (11) and the vertex-insertion diagram (13) is quite comparable with that using the string method [20] (2 × 14 each) as well. We conclude therefore that in most respects this method is just as efficient as the string method.

There is actually another similarity which is telling. One notes from (55) and (56) that \( B \) is proportional to \( B_1 \), so that many of the 11 terms in (55) combine to cancel one another. The same happens in the vertex-insertion diagrams, and the same happens in the string approach. This strongly suggests that as simple as the computation is, all together 24 non-vanishing terms rather than something of the order of \( 10^4 \) which one has in the ordinary approach, there must be even simpler way of calculating things where one can avoid writing down terms that eventually cancel one another. The following calculation shows how this can be attained.

We turn now to the computation of the box diagram with an internal quark loop, Fig. 14(a,b). The result per flavor is obtained from (2) to be
\[ M'^B = \frac{1}{16\pi^2} \int D^4\alpha \frac{S^B(J,p)}{(\alpha_2\alpha_4s + \alpha_1\alpha_3t)^2}, \] (63)
where

\[ S'^B(J, p) = -Tr(abcd)g^4\{tr[(\gamma_1)(\gamma J_1)(\gamma_4)(\gamma J_4)(\gamma_3)(\gamma J_3)(\gamma_2)(\gamma J_2)]
\quad + tr[(\gamma_1)(\gamma J_2)(\gamma_3)(\gamma J_3)(\gamma_4)(\gamma J_4)(\gamma_1)(\gamma J_1)]\} \]  

(64)

Like (39), these traces are easily computable using (14), (38), and (49). The first trace is

\[ tr[(\gamma_1)(\gamma J_1)(\gamma_4)(\gamma J_4)(\gamma_3)(\gamma J_3)(\gamma_2)(\gamma J_2)] =
-4(\alpha_1 + \alpha_2)\alpha_3\alpha_4^2 \frac{[12][24][14][43][13][32][13][32] + [23][31][42][21][34][41][23][31]}{[21][12][13][14]} =
8(\alpha_1 + \alpha_2)\alpha_3\alpha_4^2 s^2 t \frac{[24]^2}{[12][23][34][41]} \]  

(65)

The second trace is equal to the first trace. Therefore,

\[ M'^B = -2M^B / Tr(1) \]  

(66)

Similarly, one can compute Fig. 14(c,d) to get

\[ M'^V = -2M^V / Tr(1) \]  

(67)

The equalities in (66) and (67) are easy to understand by using supersymmetry arguments[12,16,17,25]. Quarks and gluinos have the same spacetime coupling with the gluon, though they carry different colors. If we should replace the quark loop by a gluino loop, the only change after we replace the color factor \( Tr(abcd) \) in the quark loop by the color factor \( Tr(abcd)Tr(1) \) in the gluino loop would be an extra factor of 1/2, reflecting the majorana nature of the gluino. Since the four-gluon amplitude in a pure supersymmetric QCD theory (gluinos are present but not quarks) is zero, the gluon/ghost loop contribution is equal and opposite to the gluino loop contribution. Putting these two facts together, the equality of (66) and (67) are obtained.

These arguments can be reversed and to be used to compute \( M^B \) and \( M^V \) from \( M'^B \) and \( M'^V \). This simplifies the calculation of the pure gluon amplitudes because the quark loop box diagram contains only four terms, which are equal, instead of the 11 terms in Fig. 12. This also explains why many of these terms in (55) (similarly (58)) add up to give zero.

6. Conclusions

For high energy scatterings lepton and light-quark masses can be ignored. Chirality is then conserved and tremendous simplifications in the calculations of these amplitudes can be obtained by the use of the spinor-helicity techniques [1–21]. With one exception [19–21], this technique was used only to calculate the tree amplitudes [1–18], because loop graphs contain off-shell momenta where chirality is not conserved and this technique cannot be applied. The exception [19–21] makes use of the string theory and is applicable to certain
one-loop processes. We have developed in this paper a technique, making use of the Feynman-parameter representation of a scattering amplitude to avoid the off-shell internal momenta, to enable the spinor-helicity method to be used for any Feynman diagram with any number of loops.

Graphical methods are used throughout to organize the terms and to avoid treading into the algebraic tangle. The method was applied to a one-loop four-gluon amplitude to show that the present method is at least as efficient as the string technique. Application of the method to the calculation of other processes is underway.

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Figure Captions

[Fig. 1] Diagrams used to illustrate the Feynman-parameter rules. (a) a box diagram; (b) a vertex diagram.

[Fig. 2] Vertices for QCD in the background Feynman gauge. A thin solid line represents a gluon, a thick solid line represents a quark, and a dashed line represents a Fadeev-Popov ghost. Gluon lines with a circled ‘A’ are external lines; those in the same diagram without a circled ‘A’ are internal lines. Each gluon line in a diagram without any circled ‘A’s present can be taken either as an external or an internal line, provided such a combination of external and internal lines has not appeared already in diagrams where explicit circled ‘A’s appear.

[Fig. 3] Color-oriented vertices and their spin factors. A line continuing through the vertex represents a $g_{\rho\sigma}$ factor; a line terminated at a heavy dot at the vertex represents a vector written below the diagram. Other numerical factors for the vertex also appear below the diagrams. The line labelled ‘1’ carries a momentum $p_1$, a spacetime index $\alpha$, and a color index $a$. Similarly, a line labelled ‘2’ carries a momentum $p_2$, a spacetime index $\beta$, and a color index $b$, etc.

For example, the spin factor for diagram (a) is $(p_2 - p_3)_{\alpha}g_{\beta\gamma}$; the spin factor for diagram (e) is $+2g_{\alpha\gamma}g_{\beta\delta}$.

[Fig. 4] The color factor $C$ for a gluon tree diagram.

[Fig. 5] A gluon tree diagram and its color factor used to illustrate the proof of eq. (22).

[Fig. 6] The color factor $C$ for a tree diagram containing a number of gluon trees attached to a quark line.

[Fig. 7] The $U(N)$ color factor $C$ for a tree diagrams with two quark lines.

[Fig. 8] The color factor $C$ for a one-loop diagram with a quark line.

[Fig. 9] The color factor $C$ for a one-loop diagram without a quark line.

[Fig. 10] One-gluon-loop diagrams and the vanishing spin-flow paths A,B,C,D. Background Feynman gauge is used; the helicity configurations as well as the choice of the reference momenta are discussed in the text.

[Fig. 11] These diagrams make no contributions to the process calculated in Sec. 5.

[Fig. 12] Non-vanishing spin-flow paths (diagrams 1 to 11) for the one-gluon-loop box diagram calculated in Sec. 5.

[Fig. 13] Non-vanishing spin-flow paths (diagrams 1 to 13) for the one-gluon-loop vertex-insertion diagram calculated in Sec. 5.

[Fig. 14] One-fermion-loop diagrams for the processes calculated in Sec. 5.