Confidence region of singular vectors for high-dimensional and low-rank matrix regression

Dong Xia
Hong Kong University of Science and Technology
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Abstract

Let $M \in \mathbb{R}^{m_1 \times m_2}$ be an unknown matrix with $r = \text{rank}(M) \ll \min(m_1, m_2)$ whose thin singular value decomposition is denoted by $M = U \Lambda V^\top$ where $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_r)$ contains its non-increasing singular values. Low rank matrix regression refers to instances of estimating $M$ from $n$ i.i.d. copies of random pair $\{(X, y)\}$ where $X \in \mathbb{R}^{m_1 \times m_2}$ is a random measurement matrix and $y \in \mathbb{R}$ is a noisy output satisfying $y = \text{tr}(M^\top X) + \xi$ with $\xi$ being stochastic error independent of $X$. The goal of this paper is to construct efficient estimator (denoted by $\hat{U}$ and $\hat{V}$) and confidence region of $U$ and $V$. In particular, we characterize the distribution of

$$\text{dist}^2[(\hat{U}, \hat{V}), (U, V)] = \|\hat{U} \hat{U}^\top - UU^\top\|_F^2 + \|\hat{V} \hat{V}^\top - VV^\top\|_F^2.$$  

We prove the asymptotic normality of properly centered and normalized $\text{dist}^2[(\hat{U}, \hat{V}), (U, V)]$ with data-dependent centering and normalization when $r^{5/2}(m_1 + m_2)^{3/2} = o(n/\log n)$, based on which confidence region of $U$ and $V$ is constructed achieving any pre-determined confidence level asymptotically.

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1 Introduction

The statistical inference of low-dimensional structure in (ambient) high-dimensional space, including the problems of sparse (vector) linear regression and low rank matrix regression, arise from diverse fields, such as gene expression analysis, network and graph analysis and statistical physics. Statistically efficient procedures are developed in the recent decades to handle the challenges posed by the high dimensionality such as the noise accumulation and algorithmic instability. Those statistically efficient procedures include the $\ell_1$-penalization for sparse linear regression, see [24], [30], [32], [6] and references therein, and matrix nuclear-norm penalization for low rank matrix regression, see [4], [21], [22], [17] and references therein. Under certain regularity conditions, those methods are guaranteed to be statistically efficient, meaning that minimax optimal rates of estimation error, usually relevant to the degrees of freedom, are attained. It becomes more subtle when the goal is to quantify the uncertainty or to construct confidence regions for the aforementioned estimators. In several recent papers [11], [31], [20], [3], a post-processing approach was proposed for debiasing the $\ell_1$-penalized least squares estimator for sparse linear regression. It was shown, in sparse linear regression, that uncertainty quantification usually requires conditions stronger than those required for guaranteed estimation optimality. The uncertainty quantification for low rank matrix regression is even more mysterious. One fundamental issue is on deciding which parameters for uncertainty quantification. For instance, confidence region with respect to matrix Frobenius norm is constructed in [7] and [8]. Note that matrix Frobenius norm is equivalent to the $\ell_2$-norm.
of vectorization of a matrix, where the matrix structure is not directly reflected. Similarly, the uncertainty quantification for individual matrix entries or linear forms of the unknown matrix, studied for sparse linear regression in [31] and [3], also neglects the matrix structure.

The objective of this paper is constructing confidence region of singular vectors for low rank matrix regression. The motivation for quantifying the uncertainty of singular vectors instead of other parameters is to take advantage of matrix structure. Since matrix regression is equivalent to vector regression when both the unknown matrix and measurement matrices are vectorized, it is crucial to dig out the fundamental difference between matrix regression and vector regression: the ultimate goal of matrix regression is to recover the column and row spaces, in addition to the signal strength (singular values). In addition to low rank matrix regression, there are recent papers investigating the asymptotic property of principle component analysis (PCA). For instance, the normal approximation of eigenvectors of sample covariance matrix is studied in [15] and [16], where data-dependent confidence region is constructed for individual eigenvectors. In both [15] and [16], data splitting technique is applied for estimating the bias of empirical eigenvectors, which is critical since the bias of empirical eigenvectors dominates its variance. In addition, a Bayesian approach for constructing confidence region of empirical eigenvectors of sample covariance matrix is studied in [23]. A more sophisticated bias reduction framework for spectral estimation is proposed in [13].

We propose a novel approach for estimating the singular vectors in low rank matrix regression. On a high level, the approach consists of two procedures. It begins with a statistically optimal estimator of the underlying low rank matrix, where, for instance, the nuclear-norm penalized least squares estimator is implemented. It is followed by a debiased process which outputs an unbiased estimator of the underlying low rank matrix, from which we compute the singular value decomposition and extract the corresponding left and right singular vectors to serve as the final estimator of singular vectors. We provide with a sharp characterization of the bias of estimating the singular vectors which depends on the sample size, ambient dimension and the inverse of true singular values, where explicit constant factors are developed. With near-optimal requirement on the sample size, we prove the asymptotical normality, when the bias is subtracted, for the proposed estimator of singular vectors. However, such result is insufficient for constructing confidence region of singular vectors since the bias is unknown. To ensure sharp estimation of the bias such that its error is dominated by the standard deviation of empirical singular vectors, it turns out that we require the sample size to be larger than the standard optimal sample size in low rank matrix estimation, but still much smaller than the ambient dimension of the underlying matrix. Analogous phenomenon also exists in the uncertainty quantification for sparse linear regression with unknown design, see [31] and [3].

Based upon the normal approximation of a novel data-dependent statistics, we construct an honest confidence region which achieves any pre-determined confidence level asymptotically.

The rest of the paper is organized as follows. In Section 2 we explain important notations and introduce the trace regression model for estimating low rank matrices, where basic assumptions are installed. An introduction of our main results is also provided in Section 2. The two-step procedure for estimating singular vectors is given in Section 3. We present the theoretical performance of the proposed estimator in Section 4 where both the concentration and normal approximation of empirical singular vectors are provided. The construction of data-dependent confidence region is introduced in Section 5. Numerical simulations are displayed in Section 6. In Section 7 we discuss about open problems and future possible directions. The proofs are packed in Section 8 and Section 9.
2 Problem setup and main results

2.1 Notations

We use boldfaced upper-case letters to denote matrices, and use the same letter in normal font with indices to denote its entries. For a matrix \( A \in \mathbb{R}^{m_1 \times m_2} \), denote by \( \|A\|_F \) its Frobenius norm and \( \|A\| \) its operator norm. The nuclear norm of \( A \) is denoted by \( \|A\|_\text{nuc} \), i.e., the sum of its singular values. Let \( \text{vec}(A) \in \mathbb{R}^{m_1 m_2} \) denote its vectorized version. Similarly, denote by \( M(\cdot) \) the inverse of \( \text{vec}(\cdot) \) such that \( M(\text{vec}(A)) = A \). Given \( B \in \mathbb{R}^{m_1 \times m_2} \), denote \( \langle A, B \rangle = \text{tr}(A^\top B) \). We use \( c_1, c_2, C_1, C_2, \cdots \) to denote absolute constants which might vary lines from lines during proof and statement of theorems. For two sequences of random variables \( \{A_n\}_n, \{B_n\}_n \) which are positive almost sure, we write \( A_n = O_P(B_n) \) to represent that there exists an absolute constant \( C_1 > 0 \) such that \( \lim_{n \to \infty} P(A_n/B_n \geq C_1) = 0 \).

2.2 Trace regression model and nuclear-norm penalized estimation

Let \( M \in \mathbb{R}^{m_1 \times m_2} \) be an unknown low rank matrix with \( r = \text{rank}(M) \ll \min(m_1, m_2) \) whose singular value decomposition is written as \( M = U \Lambda V^\top \) with \( U \in \mathbb{R}^{m_1 \times r}, V \in \mathbb{R}^{m_2 \times r} \) being \( M \)'s left and right singular vectors. The diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_r) \) where \( \lambda_1 \geq \cdots \geq \lambda_r > 0 \) are its corresponding singular values. The goal of matrix regression is to recover \( M \) from a set of measurements and noisy outcomes, which was intensively investigated in the last decade especially when \( M \) has low rank. See, e.g., [5], [17], [18], [21], [22], [10] and references therein. In particular, this paper is focused on the trace regression model which is characterized by a random pair \((X, y)\) such that

\[
y = \text{tr}(M^\top X) + \xi \tag{2.1}
\]

where the noise \( \xi \) is independent of \( X \) and \( \xi \sim \mathcal{N}(0, \sigma_\xi^2) \). Given i.i.d. copies \( \{(X_i, y_i)\}_{i=1}^{2n} \) satisfying (2.1), it was shown in [21] and [5] that the matrix nuclear norm penalized least squares estimator, see eq. (3.1), denoted as \( \hat{M}_\text{nuc} \), achieves statistically optimal convergence rate:

\[
\|\hat{M}_\text{nuc} - M\|_F^2 = O_P\left( \frac{\sigma_\xi^2 r(m_1 + m_2)}{n} \right) \tag{2.2}
\]

if the so-called restricted isometry property (RIP) or restricted strong convexity (RSC) hold. Specifically, if \( X \) has isotropic subgaussian distribution, it was shown in [5] and [21] that RIP and RSC hold with probability at least \( 1 - c_1 e^{-c_2 m} \) as long as \( n \geq C_1 r(m_1 + m_2) \log m \) where \( m = \max\{m_1, m_2\} \) and \( c_1, c_2, C_1 \) are absolute constants.

As explained in Section III the fundamental difference between matrix regression and vector regression is that, in matrix regression, the ultimate objective is to recover the unknown column space and row space of \( M \), in addition to \( \Lambda \), which is not directly reflected in the matrix Frobenius norm (2.2) since it neglects the matrix structure. If we apply the famous Wedin’s sin \( \Theta \) theorem [27] or Davis-Kahan theorem [9], by (2.2), a naive bound is

\[
\text{dist}^2\left( (\hat{U}_\text{nuc}, \hat{V}_\text{nuc}), (U, V) \right) = O_P\left( \frac{\sigma_\xi^2}{\lambda_r^2} \cdot \frac{r(m_1 + m_2)}{n} \right) \tag{2.3}
\]

where \( \hat{U}_\text{nuc} \) and \( \hat{V}_\text{nuc} \) are \( \hat{M}_\text{nuc} \)'s top-\( r \) left and right singular vectors and

\[
\text{dist}^2\left( (\hat{U}_\text{nuc}, \hat{V}_\text{nuc}), (U, V) \right) = \|\hat{U}_\text{nuc}(\hat{U}_\text{nuc})^\top - UV^\top\|_F^2 + \|\hat{V}_\text{nuc}(\hat{V}_\text{nuc})^\top - VV^\top\|_F^2. \tag{2.4}
\]
The naive bound (2.3) is sub-optimal especially when \( \lambda_1 \geq \cdots \geq \lambda_{r-1} \gg \lambda_r \) in which case the inhomogeneity of singular values is not reflected in (2.3). Moreover, bound (2.3) is insufficient for constructing confidence region of \( U \) and \( V \). We note that, from eq. (2.3), \( U^{\text{nuc}} \) and \( V^{\text{nuc}} \) are nontrivial if \( n \geq \frac{\sigma^2}{\lambda_r} \cdot r(m_1 + m_2) \). In light of the standard sample size requirement \( n \geq C_1 r(m_1 + m_2) \log \bar{m} \) for \( M^{\text{nuc}} \), it is therefore convenient for us to focus on the scenario that \( \frac{\sigma}{\lambda_r} = O(1) \) for simplicity. Otherwise, we shall adjust the baseline of sample size requirement accordingly involving \( \frac{\sigma}{\lambda_r} \).

### 2.3 Overview of our results on estimating singular vectors

To fill the void of confidence region for \( U \) and \( V \), we propose a two-step estimator for \( U \) and \( V \). In particular, we focus on the standard Gaussian design where \( X \) has i.i.d. standard Gaussian entries, i.e., \( X_{ij} \sim \mathcal{N}(0, 1) \) for all \( (i, j) \in [m_1] \times [m_2] \) where \([m] := \{1, \ldots, m\} \). The estimating procedure consists of two steps which starts with nuclear-norm penalized estimator \( M^{\text{nuc}} \) and is followed by a debiased process which produces a new estimator \( M \). Even though \( M \) loses the low rank property, it is an unbiased estimator of \( M \). Then, compute \( \hat{U} \) and \( \hat{V} \) as the top-\( r \) left and right singular vectors of \( M \) which serve as final estimators of \( U \) and \( V \).

We characterize the bias of \( \hat{U} \) and \( \hat{V} \). Specifically, we prove that, if \( n \geq C_1 \left[ \beta^2 \bar{m} + \bar{m} r \log^2 n \right] \) with \( \beta = \frac{\sigma}{\lambda_r} \) for a large enough constant \( C_1 > 0 \), then

\[
\mathbb{E} \text{dist}^2[(\hat{U}, \hat{V}), (U, V)] = \sigma^2 \| \Lambda^{-1} \|^2_F \frac{2m_*}{n} + O\left( \frac{\beta^2 r^2 \bar{m}^2 + \beta^3 r^3 \bar{m}^2 \log^{1/2} \bar{m} + \beta^4 r^4 \bar{m}^2}{n^2} \right) \tag{2.5}
\]

where \( m_* = m_1 + m_2 - 2r \). If \( n \gg r^2 \bar{m} \log^{1/4} \bar{m} \) when \( \beta = O(1) \), then the leading term in Eq. (2.5) becomes \( \| \Lambda^{-1} \|^2 \sigma^2 \cdot \frac{2m_*}{n} \) which is smaller than \( \lambda_r^{-2} \sigma^2 \cdot \frac{2r(m_1 + m_2)}{n} \) in Eq. (2.3). Moreover, the constant factor in (2.5) is explicitly developed.

In addition, we characterize the distribution of dist²[(\hat{U}, \hat{V}), (U, V)] which is written as

\[
\text{dist}^2[(\hat{U}, \hat{V}), (U, V)] \overset{\Delta}{=} \frac{2}{n^2} \sum_{i=1}^{r} \frac{\xi_i^2}{\lambda_i^2} \cdot \left( \sum_{k=1}^{r} \frac{z_k^2}{\lambda_k r} \right) + \varepsilon \tag{2.6}
\]

where \( \{z_k^2\}_{k=1}^{r} \) are i.i.d. and \( z_k^2 \sim \chi^2(m_*) \), i.e., chi-squared distribution with degrees of freedom \( m_* \) and

\[
\varepsilon = O_P \left( \frac{\beta^2 r \bar{m} \log^{3/2} n + \beta^3 r^2 \bar{m} \log^{3/2} \bar{m} + \beta^4 r^3 \bar{m}^2}{n^3/2} \right)
\]

implying that dist²[(\hat{U}, \hat{V}), (U, V)] is approximately a weighted sum of chi-squared random variables when \( n \gg r^3 \bar{m} \log n \) and \( \beta = O(1) \). Moreover, we show that the standard deviation of dist²[(\hat{U}, \hat{V}), (U, V)] is dominated by its expectation. To be exact, we prove that

\[
\left| \text{dist}^2[(\hat{U}, \hat{V}), (U, V)] - \mathbb{E} \text{dist}^2[(\hat{U}, \hat{V}), (U, V)] \right| = O_P \left( \frac{\beta^2 (r \bar{m} \log n)^{1/2}}{n} + \frac{\beta^3 r^2 \bar{m} \log^{1/2} n}{n^{3/2}} + \frac{\beta^4 r^3 \bar{m}^{3/2} \log n}{n^2} \right)
\]

which is much smaller than \( \mathbb{E} \text{dist}^2[(\hat{U}, \hat{V}), (U, V)] \) implying that in order to quantify the uncertainty of dist²[(\hat{U}, \hat{V}), (U, V)], it is necessary to develop sharp estimation of \( \mathbb{E} \text{dist}^2[(\hat{U}, \hat{V}), (U, V)] \).
In order to construct confidence region for \( U \) and \( V \), we investigate the normal approximation of \( \text{dist}^2[(\hat{U}, \hat{V}), (U, V)] \). We first show that

$$\sup_{x \in \mathbb{R}} \left\{ \mathbb{P} \left( \frac{\text{dist}^2[(\hat{U}, \hat{V}), (U, V)] - \mathbb{E}\text{dist}^2[(\hat{U}, \hat{V}), (U, V)]}{\sqrt{8\sigma^2_\xi \|\Lambda^{-2}\|_F \cdot \frac{m^{1/2}}{n}}} \leq x \right) - \Phi(x) \right\} \rightarrow 0 \quad (2.7)$$

as long as \( \bar{m}, n \rightarrow \infty \) and \( r^4 \bar{m} \log^{n_{m/2}} n \rightarrow 0 \) when \( \beta = O(1) \), where \( \Phi(x) \) represents the cumulative distribution function of standard normal distribution. Eq. (2.7) can not be directly used for constructing confidence region since the bias \( \mathbb{E}\text{dist}^2[(\hat{U}, \hat{V}), (U, V)] \) is unknown. By replacing \( \mathbb{E}\text{dist}^2[(\hat{U}, \hat{V}), (U, V)] \) with \( \sigma^2_\xi \|\Lambda^{-1}\|_F \cdot \frac{2m_{\alpha}}{n} \), we prove that

$$\sup_{x \in \mathbb{R}} \left\{ \mathbb{P} \left( \frac{\text{dist}^2[(\hat{U}, \hat{V}), (U, V)] - \sigma^2_\xi \|\Lambda^{-1}\|_F \cdot \frac{2m_{\alpha}}{n}}{\sqrt{8\sigma^2_\xi \|\Lambda^{-2}\|_F \cdot \frac{m^{1/2}}{n}}} \leq x \right) - \Phi(x) \right\} \rightarrow 0 \quad (2.8)$$

as long as \( \bar{m}, n \rightarrow \infty \) and \( r^{3/2} \bar{m}^{3/2} \log n \rightarrow 0 \) when \( \beta = O(1) \). Comparing (2.8) and (2.7), the sample size requirement for (2.8) turns out to be stronger than that for (2.7). It remains unclear whether this gap can be closed.

In view of (2.8), it suffices to estimate \( \|\Lambda^{-1}\|_F^2 \) and \( \|\Lambda^{-2}\|_F \) for which we propose the plug-in estimators

$$\hat{B}_n = \sum_{k=1}^r \hat{\lambda}_k^{-2} \quad \text{and} \quad \hat{V}_n^2 = \sum_{k=1}^r \hat{\lambda}_k^{-4}$$

where \( \hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_r \) are the top-\( r \) singular values of \( M \). Under the same conditions for (2.8), we prove

$$\sup_{x \in \mathbb{R}} \left\{ \mathbb{P} \left( \frac{\text{dist}^2[(\hat{U}, \hat{V}), (U, V)] - \sigma^2_\xi \hat{B}_n \cdot \frac{2m_{\alpha}}{n}}{\sqrt{8\sigma^2_\xi \hat{V}_n^{1/2} \cdot \frac{m^{1/2}}{n}}} \leq x \right) - \Phi(x) \right\} \rightarrow 0. \quad (2.9)$$

Clearly, confidence region of \( U \) and \( V \) can be constructed as

$$\text{CI}_\alpha := \left\{ (X, Y) \in (\mathbb{R}^{m_1 \times r}, \mathbb{R}^{m_2 \times r}) : X^\top X = Y^\top Y = I_r \text{ such that} \right\}$$

$$\text{dist}^2[(X, Y), (\hat{U}, \hat{V})] \leq \hat{B}_n \sigma^2_\xi \cdot \frac{2m_{\alpha}}{n} + \sqrt{8z_{\alpha} \hat{V}_n^{1/2} \sigma^2_\xi \cdot \frac{m^{1/2}}{n}} \right\}$$

where \( z_{\alpha} = \Phi^{-1}(1 - \alpha) \) for any \( \alpha \in (0, 1) \). According to (2.9), we get

$$\lim_{\bar{m}, n \rightarrow \infty} \mathbb{P}((U, V) \in \text{CI}_\alpha) = \alpha.$$

### 3 Two-step procedure for estimating singular vectors

Suppose that \( X \in \mathbb{R}^{m_1 \times m_2} \) satisfy the isotropic subgaussian design such that \( \mathbb{E}\text{vec}(X)\text{vec}^\top(X) = I_{m_1 m_2} \) and for any fixed matrix \( A \in \mathbb{R}^{m_1 \times m_2} \),

$$\mathbb{E} \exp\left\{ s \text{tr}(A^\top X) \right\} \leq \exp\{\|A\|^2_2 s^2 / 2\}, \quad \forall s \in \mathbb{R}.$$  

Suppose also that i.i.d. copies \( \{(X_i, y_i)\}_{i=1}^{2n_{\alpha}} \) satisfying (2.1) are available where the underlying matrix \( M = U A V^\top \) is unknown and has rank \( r = \text{rank}(M) \ll \min(m_1, m_2) \), our goal is to
construct estimators of \( U \) and \( V \). If \( n \gg \bar{m}r \) with \( \bar{m} = \max(m_1, m_2) \) and \( \lambda = C_1 \sigma_\xi \sqrt{\frac{m_2}{n}} \), it was shown in [5] and [21] that the nuclear-norm penalized estimator, using the first sample of data \( \{(X_i, y_i)\}_{i=1}^n \),

\[
\hat{M}^{\text{nuc}} := \arg\min_{\hat{M} \in \mathbb{R}^{m_1 \times m_2}} \frac{1}{n} \sum_{i=1}^n \left( y_i - \text{tr}(A^\top X_i) \right)^2 + \lambda \|A\|_*, \tag{3.1}
\]

achieves minimax optimal convergence rate in matrix Frobenius norm. Based on (3.1), we use the second sample of data \( \{(X_i, y_i)\}_{i=n+1}^{2n} \) to construct

\[
\hat{M} = \hat{M}^{\text{nuc}} + \frac{1}{n} \sum_{i=n+1}^{2n} \left( y_i - \text{tr}(X_i^\top \hat{M}^{\text{nuc}}) \right) X_i \tag{3.2}
\]

which is a debiased version of \( \hat{M}^{\text{nuc}} \). Indeed, it is straightforward to check \( \hat{M} = M \) even though \( \hat{M} \) has full rank almost surely. The idea of debiasing was initially proposed to conduct inference for sparse linear regression where sample splitting (2.1) serves as a simple approach when the design of \( X \) is known in advance. See, e.g., [11], [31], [20], [3] and references therein. Then, we compute the top-\( r \) left and right singular vectors, denoted by \( \hat{U} \) and \( \hat{V} \), of \( \hat{M} \), which are the final estimators of \( U \) and \( V \).

We note that \( \hat{M}^{\text{nuc}} \) is independent with \( \{(X_i, \xi_i)\}_{i=n+1}^{2n} \). Moreover, the initial low rank estimator is unnecessary to be always fixed to \( \hat{M}^{\text{nuc}} \). Actually, in the initial step, any estimator \( \hat{M}^{\text{init}} \), such as projection estimator [29], [12] and matrix Dantzig estimator [28], [5], achieving statistically optimal convergence rate can play the role of \( \hat{M}^{\text{nuc}} \).

### 4 Theoretic Properties

To characterize the empirical singular vectors \( \hat{U} \) and \( \hat{V} \), we assume that \( X \) is a standard Gaussian matrix such that its each entry has the standard normal distribution. Even though \( \hat{U} \) and \( \hat{V} \) are constructed from two-step estimators (3.1) and (3.2), it suffices to focus on analyzing the spectral properties of \( \hat{M} \). To this end, denote by \( \Delta = M - \hat{M}^{\text{nuc}} \). We write

\[
\hat{M} = M + \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i X_i + \left( \frac{1}{n} \sum_{i=n+1}^{2n} \text{tr}(\Delta^\top X_i) X_i - \Delta \right) \tag{4.1}
\]

where we note that \( \Delta, \{\xi_i\}_{i=n+1}^{2n} \) and \( \{X_i\}_{i=n+1}^{2n} \) are mutually independent but \( Z_1 \) and \( Z_2 \) are dependent. The following proposition is due to [5] and [21].

**Proposition 1.** ([21] Corollary 5] and [5] Theorem 2.7) If \( n \geq C_1 r \bar{m} \) and \( \lambda = C_2 \sigma_\xi \left( \frac{m_2}{n} \right)^{1/2} \) for some universal constants \( C_1, C_2 > 0 \), then with probability at least \( 1 - c_1 \exp(-c_2 \bar{m}) \),

\[
\|\Delta\|_F^2 \leq C_3 \sigma_\xi^2 \cdot \frac{r(m_1 + m_2)}{n} \tag{4.2}
\]

for some absolute constants \( c_1, c_2, C_3 > 0 \).

We apply the dilation operator to investigate the spectral of \( \hat{M} \), see [19] and [25]. For any matrix \( A \in \mathbb{R}^{m_1 \times m_2} \), define

\[
\mathcal{D}(A) = \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix} \in \mathbb{R}^{(m_1 + m_2) \times (m_1 + m_2)}
\]
which is a symmetric matrix. Then, write \( \hat{N} = \mathcal{D}(\hat{M}) \) and \( N = \mathcal{D}(M) \) where

\[
\hat{N} = N + E := N + E_1 + E_2
\]

with \( E_1 = \mathcal{D}(Z_1) \) and \( E_2 = \mathcal{D}(Z_2) \).

**Lemma 2.** There exist absolute constants \( C_1, C_2 > 0 \) such that

\[
E\|E_1\| \leq C_1\sigma_e \frac{\tilde{m}^{1/2}}{n^{1/2}},
\]

\[
E\|E_2\| \leq C_2\|\Delta\|_F \frac{\tilde{m}^{1/2} \log^{1/2} \tilde{m}}{n^{1/2}}
\]

There exist absolute constants \( C_3, C_4 > 0 \) such that for all \( t \geq 1 \), the following bound holds with probability at least \( 1 - 3e^{-t} - e^{-n} \),

\[
\|E_1\| - E\|E_1\| \leq C_3\sigma_e \cdot \left[ \frac{t^{1/2}}{n^{1/2}} + \frac{\tilde{m}^{1/2} t^{1/2}}{n} \right]
\]

\[
\|E_2\| - E\|E_2\| \leq C_4\|\Delta\|_F \cdot \left[ \frac{t^{1/2} + \log^{1/2} \tilde{m}}{n^{1/2}} + \frac{\tilde{m}^{1/2} t^{1/2} + t + \log \tilde{m}}{n} \right]
\]

### 4.1 Representation theory of empirical singular vectors

Since \( M = U\Lambda V^\top \) where \( U = (u_1, \ldots, u_r) \) and \( V = (v_1, \ldots, v_r) \) and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_r) \), it is easy to check that \( N \) has \( 2r \) non-zero eigenvalues which are \( \lambda_1 \geq \ldots \geq \lambda_r > 0 \geq \lambda_{-1} \geq \ldots \geq \lambda_{-r} \) where \( \lambda_{-k} = -\lambda_k \) for \( 1 \leq k \leq r \). The eigenvectors (which not might be unique) of \( M \) corresponding to eigenvalue \( \lambda_k \) and \( \lambda_{-k} \) can be written as

\[
\theta_k = \frac{1}{\sqrt{2}} \left( u_k^\top, v_k^\top \right)^\top \quad \text{and} \quad \theta_{-k} = \frac{1}{\sqrt{2}} \left( u_k^\top, -v_k^\top \right)^\top.
\]

The spectral projector corresponding of \( N \) is defined as

\[
\mathcal{P}_{U V} = \sum_{1 \leq |k| \leq r} \theta_k \theta_k^\top = \begin{pmatrix} UU^\top & 0 \\ 0 & VV^\top \end{pmatrix}.
\]

Let \( \{ \hat{\theta}_k \}_k \) and \( \{ \hat{\theta}_{-k} \}_k \) represent the eigenvectors of \( \hat{N} \) corresponding to the largest and smallest eigenvalues of \( \hat{N} \). Then,

\[
\mathcal{P}_{U V} = \sum_{1 \leq |k| \leq r} \hat{\theta}_k \hat{\theta}_k^\top = \begin{pmatrix} \hat{U} \hat{U}^\top & 0 \\ 0 & \hat{V} \hat{V}^\top \end{pmatrix}.
\]

By definition of \( \text{dist}^2([\hat{U}, \hat{V}), (U, V)] \) in eq. (2.4), we observe that

\[
\text{dist}^2([\hat{U}, \hat{V}), (U, V)] = \| \mathcal{P}_{UV} - \mathcal{P}_{U V} \|^2_F.
\]

We write the orthogonal projection onto the complement of the image space of \( \mathcal{P}_{U V} \) as \( \mathcal{P}_{U V}^\perp \) which is written as

\[
\mathcal{P}_{U V}^\perp = \begin{pmatrix} U_\perp U_\perp^\top & 0 \\ 0 & V_\perp V_\perp^\top \end{pmatrix}
\]

where \( U_\perp \) and \( V_\perp \) are chosen such that \( (U, U_\perp) \) and \( (V, V_\perp) \) are both orthonormal matrices. An additional operator shall be needed is

\[
C_{U V} = \sum_{1 \leq |k| \leq r} \frac{1}{\lambda_k} (\theta_k \theta_k^\top) = \begin{pmatrix} 0 & U\Lambda^{-1}V^\top \\ V\Lambda^{-1}U^\top & 0 \end{pmatrix}
\]
Lemma 3. The following decomposition of \( P_{UV} \) holds
\[
P_{UV} - P_{UV} = L_N(E) + S_N(E),
\]
where \( L_N(E) := P_{UV} E C_{UV} + C_{UV} E P_{UV} \) and
\[
\|L_N(E)\| \leq \frac{2\|E\|}{\lambda_r} \quad \text{and} \quad \|S_N(E)\| \leq 16(r + 1) \cdot \left( \frac{\|E\|}{\lambda_r} \right)^2.
\]

4.2 Concentration and normal approximation of singular vectors

By Lemma 3, we begin with the characterization of the linear term \( L_N(E) \) which will be the primary quantity determining the asymptotic normality of \( \|P_{UV} - P_{UV}\|_F^2 \).

Theorem 4. Denote by \( \beta = \frac{\sigma_{\xi}}{\lambda_r} \). Suppose that \( n \geq C \left( \beta^2 \tilde{m} + r \tilde{m} \log^2 n \right) \) and \( n \leq C^{-1} t \log m \) for a large enough constant \( C > 0 \). Then, there exist absolute constants \( c_1, c_2, C_5, C_6 > 0 \) such that with probability at least \( 1 - \frac{2n+5}{n^2} - 2e^{-n} - ne^{-m} - c_1 e^{-c_2 m} \),
\[
\|L_N(E)\|_F^2 - \mathbb{E}\|L_N(E)\|_F^2 \leq C_5 \sigma_{\xi}^2 \|\Lambda^{-2}\|_F \cdot \frac{\tilde{m}^{1/2} \log^{1/2} n}{n} + C_6 \sigma_{\xi}^2 \|\Lambda^{-1}\|_F^2 \cdot \frac{r \tilde{m}^{3/2} \log n}{n^2}
\]
and
\[
\mathbb{E}\|L_N(E)\|_F^2 = \sigma_{\xi}^2 \|\Lambda^{-1}\|_F^2 \cdot \frac{2m_*}{n} + O \left( \sigma_{\xi}^2 \|\Lambda^{-1}\|_F^2 \cdot \frac{2r \tilde{m}^{2}}{n^2} \right).
\]

Similarly, we establish the concentration of \( \|P_{UV} - P_{UV}\|_F^2 \) in Theorem 5 and obtain a sharp characterization of the bias \( \mathbb{E}\|P_{UV} - P_{UV}\|_F^2 \) in Corollary 9.

Theorem 5. Denote by \( \beta = \frac{\sigma_{\xi}}{\lambda_r} \). Suppose that \( n \geq C_6 \left( \beta^2 \tilde{m} + r \tilde{m} \log^2 n \right) \) some large enough absolute constant \( C_6 > 0 \). Then, there exist absolute constants \( c_1, c_2, C_7, C_8, C_9 > 0 \) such that with probability at least \( 1 - \frac{2n+5}{n^2} - 3e^{-n} - ne^{-m} - c_1 e^{-c_2 m} \),
\[
\|P_{UV} - P_{UV}\|_F^2 - \mathbb{E}\|P_{UV} - P_{UV}\|_F^2 \leq C_7 \sigma_{\xi}^2 \|\Lambda^{-2}\|_F \cdot \frac{\tilde{m}^{1/2} \log^{1/2} n}{n} + C_8 \left( \frac{\sigma_{\xi}}{\lambda_r} \right)^3 \cdot \frac{r^2 \tilde{m}^{3/2} \log^{1/2} n}{n^3/2}
\]
\[
+ C_9 \sigma_{\xi}^2 \|\Lambda^{-1}\|_F^2 \cdot \frac{r \tilde{m}^{3/2} \log n}{n^2}.
\]

Corollary 6. Suppose the conditions in Theorem 5 hold. Let \( \{z_k^2\}_{k=1}^r \) be i.i.d. chi-squared random variables with degrees of freedom \( m_* \). Then,
\[
\|P_{UV} - P_{UV}\|_F^2 \overset{d}{=} \frac{2 \sum_{i=n+1}^{n+1} z_i^2}{n^2} \cdot \left( \sum_{k=1}^r \frac{z_k^2}{\lambda_k^2} \right) + \varepsilon
\]
where with probability at least \( 1 - \frac{1}{n^2} - e^{-n} - c_1 e^{-c_2 m} \),
\[
|\varepsilon| \leq C_7 \sigma_{\xi}^2 \|\Lambda^{-2}\|_F \cdot \frac{r^{1/2} \tilde{m}^{3/2} \log^{3/2} n}{n^{3/2}} + C_8 \left( \frac{\sigma_{\xi}}{\lambda_r} \right)^3 \cdot \frac{r^2 \tilde{m}^{3/2}}{n^3/2}
\]
\[
+ C_9 \sigma_{\xi}^2 \|\Lambda^{-1}\|_F^2 \cdot \frac{r \tilde{m}^{3/2}}{n^2}.
\]
for some absolute constants $c_1, c_2, C_7, C_8, C_9 > 0$. In addition,
\[
E\|\mathcal{P}_{UV} - \mathcal{P}_{UV}\|^2_F = (\sigma_\xi^2\|\Lambda^{-1}\|^2_F) \cdot \frac{2m_*}{n}
\]
\[
+ O\left(\frac{\sigma_\xi^2\|\Lambda^{-1}\|^2_F}{n^2} \cdot \frac{2r\bar{m}^2}{n^2} + \frac{\lambda_r^3}{\lambda_r^2} \cdot \frac{r^{5/2}\bar{m}^2 \log^{1/2} n}{n^2} + \frac{\sigma_\xi^2}{\lambda_r^4} \cdot \frac{r^3\bar{m}^2}{n^2}\right).
\] (4.4)

Note from eq. (4.3) that dominating term of $|e|$ is in the order $\frac{\bar{m}^{3/2}}{n^{1/2}}$. However, from eq. (4.4), the leading term of $|Ee|$ is in the order $\frac{\bar{m}^2}{n}$. This fact is paramount for establishing the normal approximation in Corollary 9.

**Theorem 7.** Suppose the conditions in Theorem 5 hold and $n \geq C_1 r^2 \bar{m}$ and $n \leq C_1^{-1} e^\bar{m}$ for a large enough absolute constant $C_1 > 0$. Let $\Phi(\cdot)$ denote the cumulative distribution function of standard normal distribution. Then,
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{\frac{\mathcal{P}_{UV} - \mathcal{P}_{UV}}{\sqrt{8\sigma_\xi^2\|\Lambda^{-2}\|^2_F}} \cdot \frac{m_*^{1/2}}{n} \leq x\right\} - \Phi(x) \right|
\]
\[
\leq C_7 \frac{\sigma_\xi}{\lambda_r} \cdot \frac{r^2\bar{m}^{1/2} \log^{1/2} n}{n^{1/2}} + C_8 \frac{r^{1/2}\bar{m}^{1/2} \log^{3/2} n}{n^{1/2}}
\]
\[
+ 2e^{-n} + c_1(n + r)e^{-c_2\bar{m}^*} + \frac{2n + 7}{n^2} + \frac{C_9}{\bar{m}^{1/2}}.
\]
for absolute constants $c_1, c_2, C_7, C_8, C_9 > 0$.

**Remark 8.** Theorem 7 implies that if $\bar{m}, n \to \infty$ and $\frac{r^2\bar{m}^{3/2} n}{n} \to 0$ when $\beta = O(1)$, then
\[
\frac{\mathcal{P}_{UV} - \mathcal{P}_{UV}}{\sqrt{8\sigma_\xi^2\|\Lambda^{-2}\|^2_F}} \cdot \frac{m_*^{1/2}}{n} \overset{d}{\to} \mathcal{N}(0, 1).
\]

Next, we replace $E\|\mathcal{P}_{UV} - \mathcal{P}_{UV}\|^2_F$ with $\sigma_\xi^2\|\Lambda^{-1}\|^2_F \cdot \frac{2m_*}{n}$ and obtain the following corollary which can be immediately proved by Theorem 7 and the last claim of Corollary 6.

**Corollary 9.** Suppose the conditions in Theorem 5 hold and $n \geq C_1 r^2 \bar{m}$ and $n \leq C_1^{-1} e^\bar{m}$ for a large enough absolute constant $C_1 > 0$. Let $\Phi(\cdot)$ denote the cumulative distribution function of standard normal random variables. Then,
\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{\frac{\mathcal{P}_{UV} - \mathcal{P}_{UV}}{\sqrt{8\sigma_\xi^2\|\Lambda^{-2}\|^2_F}} \cdot \frac{m_*^{1/2}}{n} \leq x\right\} - \Phi(x) \right|
\]
\[
\leq C_7 \frac{\sigma_\xi}{\lambda_r} \cdot \frac{r^{5/2}\bar{m}^{3/2} \log^{1/2} n}{n} + C_8 \left(\frac{\sigma_\xi}{\lambda_r}\right)^2 \frac{r^{3/2}\bar{m}^{3/2}}{n} + C_9 \frac{r^{3/2}\bar{m}^{3/2}}{n}
\]
\[
+ 2e^{-n} + c_1(n + r)e^{-c_2\bar{m}^*} + \frac{3n + 6}{n^2} + \frac{C_{10}}{\bar{m}^{1/2}}.
\]
for absolute constants $c_1, c_2, C_7, C_8, C_9, C_{10} > 0$.

**Remark 10.** Corollary 9 implies that if $\bar{m}, n \to \infty$ and $\frac{r^{5/2}\bar{m}^{3/2} \log n}{n} \to 0$ when $\beta = O(1)$, then
\[
\frac{\mathcal{P}_{UV} - \mathcal{P}_{UV}}{\sqrt{8\sigma_\xi^2\|\Lambda^{-2}\|^2_F}} \cdot \frac{m_*^{1/2}}{n} \overset{d}{\to} \mathcal{N}(0, 1).
\]
5 Constructing confidence regions of $U$ and $V$

In this section, we apply the asymptotic normality established in Theorem 7 to construct confidence regions of $U$ and $V$. We assume $\sigma_\varepsilon$ is already known, see Section 7 for the discussion on estimating $\sigma_\varepsilon$ and $r$. In view of Theorem 7 it suffices to obtain $\|A^{-1}\|_F^2$ and $\|A^{-2}\|_F$. Recall that the singular values of $M$ are denoted by $\hat{\lambda}_k$. To this end, define

$$\hat{B}_n := \sum_{k=1}^r \hat{\lambda}_k^{-2}$$

(5.1)

, i.e., $\hat{B}_n$ serves as the plug-in estimator of $\|A^{-1}\|_F^2$. Similar, define the plug-in estimator of $\|A^{-2}\|_F^2$ as

$$\hat{V}_n = \sum_{k=1}^r \hat{\lambda}_k^{-4}.$$  

(5.2)

Lemma 11 provides the accuracy of $\hat{B}_n$ and $\hat{V}_n$. We note that sharper characterization of $\hat{B}_n$ and $\hat{V}_n$ might be possible, but the bounds in Lemma 11 are sufficient for objectives of the current paper.

**Lemma 11.** Denote by $\beta = \frac{\sigma_\varepsilon}{\lambda_r}$. Suppose that $n \geq C(r\tilde{m} + \beta^2 \tilde{m})$ for a large enough constant $C > 0$. Then, with probability at least $1 - 2e^{-n} - C_1 e^{-c_2 \tilde{m}}$,

$$|\hat{B}_n - \|A^{-1}\|_F^2| \leq C_6 \|A^{-1}\|_F^2 \left( \left( \frac{\sigma_\varepsilon}{\lambda_r} \right)^2 + \left( \frac{\sigma_\varepsilon}{\lambda_r} \right)^3 \right) \cdot \frac{r^2 \tilde{m}}{n}$$

and

$$|\hat{V}_n - \|A^{-2}\|_F^2| \leq C_6 \|A^{-2}\|_F^2 \left( \left( \frac{\sigma_\varepsilon}{\lambda_r} \right)^2 + \left( \frac{\sigma_\varepsilon}{\lambda_r} \right)^3 \right) \cdot \frac{r^2 \tilde{m}}{n}$$

for some absolute constant $c_1, c_2, C_6 > 0$.

Define a novel statistics:

$$\hat{T}_{UV} := \frac{\|P_{UV} - P_{UV} \|_F^2}{\sqrt{8 \tilde{m} / n} \cdot \bar{m} / n}.$$  

(5.3)

Theorem 12 shows the normal approximation of the statistics $\hat{T}_{UV}$.

**Theorem 12.** Denote by $\beta = \frac{\sigma_\varepsilon}{\lambda_r}$. Suppose that $n \geq C\left(r^{3/2} \tilde{m}^{3/2} + (\beta^2 + \beta^3) r^2 \tilde{m}\right)$ and $n \leq C^{-1} e^{\tilde{m}}$ for a large enough constant $C > 0$. Then,

$$\sup_x \left| \mathbb{P} \left\{ \hat{T}_{UV} \leq x \right\} - \Phi(x) \right| \leq C_7 (\beta + \beta^3) \cdot \frac{r \tilde{m}^{1/2} \log^{1/2} n}{n^{1/2}} + C_8 (\beta + \beta^2 + \beta^5 / 2 + \beta^3) \cdot \frac{r^{5/2} \tilde{m}^{3/2} \log^{1/2} n}{n}$$

$$+ C_9 \cdot \frac{r \tilde{m}^{3/2}}{n} + C_{10} (\beta^2 + \beta^5 / 2 + \beta^3 + \beta^7 / 2) \cdot \frac{r^4 \tilde{m}^{2} \log^{1/2} n}{n^{3/2}}$$

$$+ 6e^{-n} + (2n + r) e^{-m_*} + \frac{5n + 17}{n^2} + C_1 e^{-c_2 \tilde{m}} + \frac{C_{11}}{\tilde{m}}$$

for absolute constants $c_1, c_2, C_7, C_8, C_9, C_{10}, C_{11} > 0$. 

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Remark 13. In Theorem\cite{12} if $\beta = O(1)$ and
\[
\frac{r^2\hat{m}\log n + r^{5/2}\hat{m}^{3/2}\log^{1/2} n + r^{8/3}\hat{m}^{4/3}\log^{1/3} n}{n} \xrightarrow{m,n \to \infty} 0
\]
when $\beta = O(1)$, then $\hat{U}V \xrightarrow{d} \mathcal{N}(0, 1)$ as $m, n \to \infty$. In the case $r \ll \hat{m}$, it suffices to require $n \gg r^{5/2}\hat{m}^{3/2}\log^{1/2} n$.

We apply the normal approximation of $\hat{U}V$ to construct confidence regions of $U$ and $V$. The following corollary is an immediate result from Theorem\cite{12}

Corollary 14. Suppose the conditions of Theorem\cite{12} hold and suppose
\[
\lim_{\hat{m}, n \to \infty} \frac{r^{5/2}\hat{m}^{3/2}\log^{1/2} n}{n} = 0.
\]
For any $\alpha \in (0, 1)$, denote by $z_\alpha = \Phi^{-1}(1 - \alpha)$, construct a region
\[
\text{CI}_\alpha := \left\{ (X, Y) \in \mathbb{R}^{m_1 \times r}, \mathbb{R}^{m_2 \times r} : X^\top X = Y^\top Y = I_r \text{ such that} \right. \\
\left. \|XX^\top - \hat{U}\hat{U}^\top\|_F^2 + \|YY^\top - \hat{V}\hat{V}^\top\|_F^2 \leq \hat{B}_n\sigma_\xi^2 \cdot \frac{2m_\ast}{n} + \sqrt{8z_\alpha \hat{V}_n^1/2}\sigma_\xi^2 \cdot \frac{m_\ast}{n} \right\}
\]
where $\hat{B}_n$ and $\hat{V}_n$ are defined in \eqref{15} and \eqref{16}. If $\beta = O(1)$, then,
\[
\lim_{\hat{m}, n \to \infty} \mathbb{P}\left( (U, V) \in \text{CI}_\alpha \right) = \alpha.
\]

6 Simulations

We show some simulation results. In these simulations, the underlying low rank matrix $M \in \mathbb{R}^{m \times m}$ has $\text{rank}(M) = r$ and thin singular value decomposition $M = U\Lambda V^\top$ where $\lambda_k = 2^{r-k+1}$ for $1 \leq k \leq r$, i.e., the condition number of $M$ is $2^{r-1}$ growing fast with respect to $r$. The singular vectors $U$ and $V$ are generated from singular space of Gaussian random matrices. The initial estimator $\hat{M}^{\text{ms}}$ is solved by the famous alternating direction method of multipliers (ADMM) algorithm, see \cite{2}.

First, we confirm that $E\|P_{UV} - P_{\hat{U}\hat{V}}\|_F^2$ converges to $\sigma_\xi^2\|\Lambda^{-1}\|_F^2 \cdot 2m_\ast/n$ as $n$ grows. Two scenarios are implemented where $m = 50$, $r = 4$, $\sigma_\xi = 0.1$ and $m = 100$, $r = 2$, $\sigma_\xi = 0.5$. For each $n$, the algorithm is repeated for 10 times on independently sampled data and the average $E\|P_{UV} - P_{\hat{U}\hat{V}}\|_F^2$ is recorded. The empirical mean of $\|P_{UV} - P_{\hat{U}\hat{V}}\|_F^2$ and the theoretical bound $\sigma_\xi^2\|\Lambda^{-1}\|_F^2 \cdot 2m_\ast/n$ are displayed in Figure\eqref{fig1}. In Figure\eqref{fig1a}, $M$ has large condition number in which case there exists significant discrepancy between the empirical mean and theoretical bound when $n$ is small.

Then, we fix $m = 100$, $r = 4$, $\sigma_\xi = 0.4$ and show the normal approximation of $\sqrt{8\sigma_\xi^2\|\Lambda^{-1}\|_F^2}\cdot 2m_\ast/n$. For each $n = 4000, 8000, 12000, 16000$, we repeat the algorithm for 10000 times and take its average value as $E\|P_{UV} - P_{\hat{U}\hat{V}}\|_F^2$. The density histogram and the probability density function of standard normal distribution are displayed in Figure\eqref{fig2}.

We fix $m = 100$, $r = 4$, $\sigma_\xi = 0.4$ and show the normal approximation of $\sqrt{8\sigma_\xi^2\hat{V}_n^1/2}\hat{B}_n\cdot 2m_\ast/n$. where $\hat{B}_n$ and $\hat{V}_n$ are plug-in estimators as in \eqref{15} and \eqref{16}. The density histogram and the probability density function of standard normal distribution are displayed in Figure\eqref{fig3}. From Figure\eqref{fig3}, we clearly observe that there exists bias when estimating $\|\Lambda^{-1}\|_F^2$ by $\hat{B}_n$, which diminishes as $n$ grows.
Figure 1: Comparison of $E\|P_{UV} - \hat{P}_{UV}\|_F^2$ and $\sigma_\xi^2 \|\Lambda^{-1}\|_F^2 \cdot \frac{2m}{n}$ with respect to $n$.

(a) $m = 50, r = 4$ and $\sigma = 0.1$

(b) $m = 100, r = 2$ and $\sigma = 0.5$

Figure 2: Normal approximation of $\frac{\|P_{UV} - \hat{P}_{UV}\|_F^2 - E\|P_{UV} - \hat{P}_{UV}\|_F^2}{\sqrt{8\sigma_\xi^2 \|\Lambda^{-1}\|_F^2 \cdot \frac{2m}{n}}}$ with $m_1 = m_2 = 100, r = 4$ and $\sigma_\xi = 0.4$. For each $n$, the density histogram is based on 10000 repetitions whose average is used to estimate $E\|P_{UV} - \hat{P}_{UV}\|_F^2$. The red curve represents the probability density function of standard normal distribution.
The function of standard normal distribution. It clearly demonstrates that bias exists when using $B\sigma$ the settings of Section 5. One plug-in approach is

By Proposition 1, it is straightforward to show get that, if $m = m_2 = 100$, $r = 4$ and $\sigma_\xi = 0.4$. For each $n$, the density histogram is based on 10000 repetitions. The plug-in estimator $\hat{B}_n$ and $\hat{V}_n$ are defined as eq. (5.1) and eq. (5.2). The red curve represents the probability density function of standard normal distribution. It clearly demonstrates that bias exists when using $\hat{B}_n$ for estimating $\|A^{-1}\|_F^2$.

7 Discussion

For constructing the confidence region in Section 5, we assumed that the noise variance $\sigma_\xi^2$ is known in advance, which can be easily violated in practice. However, $\sigma_\xi^2$ can be sharply estimated under the settings of Section 5. One plug-in approach is

$$\hat{\sigma}_\xi^2 := \frac{1}{n} \sum_{i=n+1}^{2n} (y_i - \text{tr}(X_i^T \hat{M}^{mc}))^2$$

$$= \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i^2 + \frac{1}{n} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle^2 + \frac{2}{n} \sum_{i=n+1}^{2n} \xi_i \cdot \langle \Delta, X_i \rangle.$$

By Proposition 1 it is straightforward to show

$$|\frac{\hat{\sigma}_\xi^2 - \sigma_\xi^2}{\sigma_\xi^2}| = O_p \left( \frac{r^{1/2} m^{1/2}}{n^{1/2}} \right) = O_p \left( \frac{1}{r^{3/4} \bar{m}^{1/4}} \right)$$

where the last equality is due to the requirement $n \gg r^{5/2} \bar{m}^{3/2}$ in Corollary 14. Therefore, if $\bar{m} \to \infty$, we get $\frac{\hat{\sigma}_\xi^2}{\sigma_\xi^2} \to 1$ in probability.

In addition, the rank $r$ can be exactly estimated under similar settings. Indeed, by Lemma 2 we get that, if $n \geq r \bar{m} \log n$, then with probability at least $1 - e^{-n} - c_1 e^{-c_2 \bar{m}}$,

$$\sup_{1 \leq k \leq \min(m_1, m_2)} |\hat{\lambda}_k - \lambda_k| \leq C_1 \sigma_\xi \bar{m}^{1/2} / n^{1/2}.$$
for an absolute constant $C_1 > 0$ where $\lambda_k = 0$ for $k > r$. By choosing

$$\hat{r} = \text{Card}\left\{ \hat{\lambda}_k : \hat{\lambda}_k \geq 2C_1 \hat{\sigma}_\xi \cdot \hat{m}^{1/2} / n^{1/2}, 1 \leq k \leq \min(m_1, m_2) \right\}$$

then $P(\hat{r} = r) \geq 1 - e^{-n} - c_1 e^{-r2\hat{m}}$ as long as $n \geq 9C_1^2 \beta^2 \hat{m} + r \hat{m} \log n$.

To construct the unbiased estimator $\hat{M}$ as in eq. (4.2), our procedure splits the data $\{(X_i, y_i)\}_{i=1}^{2n}$ into two independent samples which might be inefficient when $n$ has a moderate size. We are curious about the performance if we reuse the same data to debias $\hat{M}^{\text{unc}}$, i.e.,

$$\hat{M} := \hat{M}^{\text{unc}} + \frac{1}{n} \sum_{i=1}^{n} (y_i - \text{tr}(X_i^\top \hat{M}^{\text{unc}}))X_i$$

$$= M + \frac{1}{n} \sum_{i=1}^{n} \xi_i X_i + \left[ \frac{1}{n} \sum_{i=1}^{\bar{n}} \text{tr}(\Delta X_i)X_i - \Delta \right]$$

where $\{\xi_i\}_i$ are independent with $\{X_i\}_i$. However, $\Delta$ depends on $\{\xi_i\}_i$ and $\{X_i\}_i$. By a covering argument, it is easy to show

$$\|Z_2\| = O_P\left(\|\Delta\|_F \cdot \frac{\hat{m}^{1/2} \text{rank}^{1/2}(\Delta)}{n^{1/2}}\right)$$

while

$$\|Z_1\| = O_P\left(\sigma_\xi \cdot \frac{\hat{m}^{1/2}}{n^{1/2}}\right).$$

If $n \geq r^{5/2} \hat{m}^{3/2}$ as required in Corollary [14] then $\|Z_2\|_F = O_P\left(\frac{\sigma_\xi}{\sqrt{m^{1/2} \text{rank}^{1/2}(\Delta)}} \cdot \frac{\hat{m}^{1/2} \text{rank}^{1/2}(\Delta)}{n^{1/2}}\right)$ implying that $\|Z_2\| \ll \|Z_1\|$ with high probability if $\text{rank}(\Delta) = O_P(r)$. Therefore, the dominating term of $\hat{M} - M$ is $Z_1$ where the technique tools in this paper is still applicable. However, it requires nontrivial effort to investigate the concentration of quantities involving $Z_2$.

8 Proofs

8.1 Proof of Theorem [4] Theorem [5] and Corollary [6]

8.1.1 Supporting lemmas

The proof of Theorem [4] and Theorem [5] involves several lemmas. Observe that $\| P_{\hat{U} \hat{V}} - P_{UV} \|^2_\mathcal{F} = \| P_{\hat{U} \hat{V}} \|^2_\mathcal{F} + \| P_{UV} \|^2_\mathcal{F} - 2 \langle P_{\hat{U} \hat{V}}, P_{UV} \rangle$. By definitions of $P_{\hat{U} \hat{V}}$ and $P_{UV}$, we have $\| P_{\hat{U} \hat{V}} \|^2_\mathcal{F} \equiv \| P_{UV} \|^2_\mathcal{F} = 2r$. Then,

$$\| P_{\hat{U} \hat{V}} - P_{UV} \|^2_\mathcal{F} - E\| P_{\hat{U} \hat{V}} - P_{UV} \|^2_\mathcal{F} = -2 \langle P_{\hat{U} \hat{V}}, P_{UV} \rangle + 2E\langle P_{\hat{U} \hat{V}}, P_{UV} \rangle$$

Recall the decomposition from Lemma [3] that

$$P_{UV} = P_{UV} + L_N(E) + S_N(E)$$

where $L_N(E) = P_{UV} \text{E}C_{UV} + C_{UV} \text{E}P_{UV}$. Therefore,

$$\langle E P_{\hat{U} \hat{V}} - P_{UV}, P_{UV} \rangle = \langle E S_N(E) - S_N(E) - L_N(E), P_{UV} \rangle$$
where we used the fact $\mathbb{E}_N(E) = \mathcal{P}_U^\perp(\mathbb{E}E)C_UV + C_{UV}(\mathbb{E}E)\mathcal{P}_U^\perp = 0$. Observe that
\[
\langle \mathcal{L}_N(E), \mathcal{P}_{UV} \rangle = \langle \mathcal{P}_U^\perp \mathbb{E}C_UV + C_{UV}\mathbb{E}\mathcal{P}_U^\perp, \mathcal{P}_{UV} \rangle = 0.
\]
We conclude that
\[
\|\mathcal{P}_{UV} - \mathcal{P}_{UV}\|_F^2 - \mathbb{E}\|\mathcal{P}_{UV} - \mathcal{P}_{UV}\|_F^2 = 2\langle \mathbb{E}S_N(E) - S_N(E), \mathcal{P}_{UV} \rangle.
\]

**Lemma 15.** Suppose that $\lambda_r \geq 5\mathbb{E}\|E\|$ and $n \geq \bar{m}$, the following bound holds with probability at least $1 - 3e^{-c_1\bar{m}} - e^{-n}$,
\[
\left|\left(\|\mathcal{P}_{UV} - \mathcal{P}_{UV}\|_F^2 - \mathbb{E}\|\mathcal{P}_{UV} - \mathcal{P}_{UV}\|_F^2\right) - \left(\|\mathcal{L}_N(E)\|_F^2 - \mathbb{E}\|\mathcal{L}_N(E)\|_F^2\right)\right| \
\leq C_7(\sigma_\xi + \|\Delta\|_F)\left(\frac{9\mathbb{E}\|E\|}{2\lambda_r}\right)^2 \cdot \frac{t^{1/2} + \log^{1/2}\bar{m}}{n^{1/2}}
\]
for an absolute constant $C_7 > 0$.

It is thus sufficient to investigate $\|\mathcal{L}_N(E)\|_F^2$. By the definition of $\mathcal{L}_N(E)$,
\[
\|\mathcal{L}_N(E)\|_F^2 = \|\mathcal{P}_U^\perp \mathbb{E}C_UV + C_{UV}\mathbb{E}\mathcal{P}_U^\perp\|_F^2 \
= \|\mathcal{P}_U^\perp \mathbb{E}C_UV\|_F^2 + 2\langle \mathcal{P}_U^\perp \mathbb{E}C_UV, C_{UV}\mathbb{E}\mathcal{P}_U^\perp \rangle + \|C_{UV}\mathbb{E}\mathcal{P}_U^\perp\|_F^2 \
= 2\|\mathcal{P}_U^\perp \mathbb{E}C_UV\|_F^2 + 2\|\mathcal{P}_U^\perp \mathbb{E}C_UV\|_F^2 + 4\langle \mathcal{P}_U^\perp \mathbb{E}C_UV, \mathbb{E}\mathcal{P}_U^\perp C_{UV} \rangle,
\]
where the third equality is due to the fact that $\mathcal{P}_U^\perp C_{UV} = 0$. Recall the definition of $\mathcal{P}_U^\perp$ and $C_{UV}$, we write $\mathcal{P}_U^\perp C_{UV}$ explicitly as
\[
\mathcal{P}_U^\perp C_{UV} = \left(\begin{array}{c|c}
\mathbb{I} & 0 \\
0 & \mathbb{V}_- \mathbb{V}_\perp \\
\end{array}\right) \left(\begin{array}{c|c}
0 & \mathbb{Z}_1 \\
\mathbb{V}_- \mathbb{V}_\perp & 0 \\
\end{array}\right) \left(\begin{array}{c|c}
0 & \mathbb{U} \mathbb{A}^{-1} \mathbb{V}^\perp \\
\mathbb{U} \mathbb{A}^{-1} \mathbb{V}^\perp & 0 \\
\end{array}\right) 
= \left(\begin{array}{c|c}
\mathbb{I} & \mathbb{Z}_1 \mathbb{V} \mathbb{A}^{-1} \mathbb{U}^\perp \\
0 & \mathbb{V}_- \mathbb{V}_\perp \mathbb{Z}_1 \mathbb{U} \mathbb{A}^{-1} \mathbb{V}^\perp \\
\end{array}\right)
\]
implying that
\[
\|\mathcal{P}_U^\perp C_{UV}\|_F^2 = \|\mathbb{I} \mathbb{V} \mathbb{A}^{-1} \mathbb{U}^\perp\|_F^2 + \|\mathbb{V}_- \mathbb{V}_\perp \mathbb{Z}_1 \mathbb{U} \mathbb{A}^{-1} \mathbb{V}^\perp\|_F^2.
\]

**Lemma 16.** Let $\{z_k^2\}_{k=1}^r$ be i.i.d. Chi-squared random variables with degrees of freedom $m_*$ where $m_* = m_1 + m_2 - 2r$. Then,
\[
\|\mathcal{P}_U^\perp C_{UV}\|_F^2 \xrightarrow{d} \frac{1}{n^2} \sum_{i=n+1}^{n+2r} \frac{\xi_i^2}{\lambda_i^2},
\]
Therefore, $\mathbb{E}\|\mathcal{P}_U^\perp C_{UV}\|_F^2 = \sigma_\xi^2 \left(\sum_{i=n+1}^{n+2r} \frac{\xi_i^2}{\lambda_i^2}\right)$.

Thus, for any $t \geq 1$ and an absolute and large constant $C_1 > 0$, with probability at least $1 - e^{-t} - e^{-n}$,
\[
\left(\|\mathcal{P}_U^\perp C_{UV}\|_F^2 - \mathbb{E}\|\mathcal{P}_U^\perp C_{UV}\|_F^2\right) \
\leq C_1 \frac{\sigma_\xi^2}{n} \cdot \max\left\{\|\mathbb{A}^{-1}\|_F m_*^{1/2} t^{1/2}, \frac{t}{\lambda_r^2}\right\}.
\]

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Lemma 17. Under the conditions of Lemma 15, the following bounds hold with probability at least $1 - 3e^{-t} - e^{-c_1m} - e^{-n}$,
\[
\left| E\| P_{UV} E_{2C_{UV}} \|_F^2 - E\| P_{UV} E_{2C_{UV}} \|_F^2 \right| 
\leq C_7\| \Delta \|_F^2 \| \Lambda^{-1} \|_F^2 \frac{\bar{m}^{1/2}(t^{1/2} + \log^{1/2} \bar{m}) \log^{1/2} \bar{m}}{n}
\]
for absolute constants $c_1, C_7 > 0$. Meanwhile,
\[
E\| P_{UV} E_{2C_{UV}} \|_F^2 \leq \frac{m_*}{n} \| \Delta \|_F^2 \| \Lambda^{-1} \|_F^2 + \frac{4}{n} \| \Delta \|_F^2.
\]
Moreover, we have $E\langle P_{UV} E_1 C_{UV}, P_{UV} E_{2C_{UV}} \rangle = 0$.

Lemma 18. The following bound holds with probability at least $1 - (2n + 1)e^{-t} - ne^{-m_*}$ for all $t \geq 2 \log n$,
\[
\left| E\| P_{UV} E_{1C_{UV}}, P_{UV} E_{2C_{UV}} \|_F^2 \right| 
\leq C_0 \sigma_\xi \| \Delta \|_F \| \Lambda^{-2} \|_F \cdot \frac{t^{3/2} m_*^{1/2}}{n} + C_7 \sigma_\xi \| \Delta \|_F \| \Lambda^{-1} \|_F \cdot \frac{t^{1/2} m_*}{n^{3/2}}
\]
for absolute constants $C_0, C_7 > 0$.

Combining Lemma 16, Lemma 17 and Lemma 18, we prove the concentration inequality of $\| \mathcal{L}(E) \|_F^2$ by choosing $t = 2 \log n$.

8.1.2 Proof of Theorem 4

By putting together the bounds in Lemma 16, Lemma 17 and Lemma 18, we immediately obtain, with probability at least $1 - \frac{2n+5}{n^2} - 2e^{-n} - ne^{-m_*}$, that
\[
\left| \| \mathcal{L}(E) \|_F^2 - E\| \mathcal{L}(E) \|_F^2 \right| 
\leq C_1 \sigma_\xi^2 \| \Lambda^{-2} \|_F \cdot \frac{m_*^{1/2} \log^{1/2} n}{n} + C_2 \| \Delta \|_F^2 \| \Lambda^{-1} \|_F^2 \cdot \frac{m_*^{1/2} \log n}{n}
\]
\[
+ C_3 \sigma_\xi \| \Delta \|_F \| \Lambda^{-2} \|_F \cdot \frac{m_*^{1/2} \log^{3/2} n}{n}
\]
\[
+ C_4 \sigma_\xi \| \Delta \|_F \| \Lambda^{-1} \|_F^3 \cdot \frac{m_* \log^{1/2} n}{n^{3/2}}.
\]
By Proposition 1, with probability at least $1 - c_1 e^{-c_2 m}$, (see 21)
\[
\| \Delta \|_F \leq C_2 \sigma_\xi \frac{(r \bar{m})^{1/2}}{n^{1/2}} \leq C_3 \sigma_\xi
\]
for absolute constants $C_2, C_3 > 0$. Therefore, simplifying the above bounds and we get
\[
\left| \| \mathcal{L}(E) \|_F^2 - E\| \mathcal{L}(E) \|_F^2 \right| 
\leq C_1 \sigma_\xi^2 \| \Lambda^{-2} \|_F \cdot \left[ \frac{\bar{m}^{1/2} \log^{1/2} n}{n} + \frac{r^{1/2} \bar{m} \log^{3/2} n}{n^{3/2}} \right]
\]
\[
+ C_2 \sigma_\xi^2 \| \Lambda^{-1} \|_F^2 \cdot \frac{r \bar{m}^{3/2} \log n}{n^2}
\]
\[
\leq C_1 \sigma_\xi^2 \| \Lambda^{-2} \|_F \cdot \frac{\bar{m}^{1/2} \log^{1/2} n}{n} + C_2 \sigma_\xi^2 \| \Lambda^{-1} \|_F^2 \cdot \frac{r \bar{m}^{3/2} \log n}{n^2}
\]
where the last inequality is due to $n \geq Cr \bar{m} \log^2 n$. Since $E\| \mathcal{L}(E) \|_F^2 = 2E\| P_{UV} E_1 C_{UV} \|_F^2 + 2E\| P_{UV} E_{2C_{UV}} \|_F^2$, we immediately obtain the second claim from Lemma 16 and Lemma 17.
8.1.3 Proof of Theorem 5

By Lemma 15 and setting $t = 2 \log n$, with probability at least $1 - \frac{4}{m^2} - e^{-n} - e^{-c_1 m}$,

$$
\left| \left( \| P_{UV} - P_{UV}\|_F^2 - \mathbb{E}\| P_{UV} - P_{UV}\|_F^2 \right) - \left( \| L_N(E)\|_F^2 - \mathbb{E}\| L_N(E)\|_F^2 \right) \right|
\leq C_7(\sigma_\xi + \| \Delta \|_F) \frac{9\| \mathbb{E}\|_F}{2\lambda_r} \cdot \frac{\log^{1/2} n}{n^{1/2}}
\leq C_7 \left( \frac{\sigma_\xi}{\lambda_r} \right)^3 \cdot \frac{r^2 m \log^{1/2} n}{n^{3/2}}.
$$

Together with the concentration of $\| L_N(E)\|_F^2 - \mathbb{E}\| L_N(E)\|_F^2$ in Theorem 4, we obtain the claimed bound.

8.1.4 Proof of Corollary 6

By Lemma 15 with probability at least $1 - e^{-n} - 3e^{-c_1 m}$,

$$
\| P_{UV} - P_{UV}\|_F^2 - \mathbb{E}\| L_N(E)\|_F^2 \leq C_1 \left( \frac{\sigma_\xi}{\lambda_r} \right)^3 \cdot \frac{r^2 m^{3/2}}{n^{3/2}}.
$$

Meanwhile,

$$
\| L_N(E)\|_F^2 = 2\| P_{UV}^1 E_1 C_{UV}\|_F^2 + 2\| P_{UV}^1 E_2 C_{UV}\|_F^2 + 4\langle P_{UV}^1 E_1 C_{UV}, P_{UV}^1 E_2 C_{UV} \rangle.
$$

Then, the first claim follows immediately from Lemma 16, Lemma 17 and Lemma 18.

To prove the last claim, recall that $P_{UV} - P_{UV} = L_N(E) + S_N(E)$. Therefore,

$$
\mathbb{E}\| P_{UV} - P_{UV}\|_F^2 = \mathbb{E}\| L_N(E)\|_F^2 + \mathbb{E}\| S_N(E)\|_F^2 + 2 \cdot \mathbb{E}\langle L_N(E), S_N(E) \rangle.
$$

By Theorem 4

$$
\mathbb{E}\| L_N(E)\|_F^2 = \sigma_\xi^2 \| \Lambda^{-1} \|_F^2 \cdot \frac{2m}{n} + O \left( \sigma_\xi^2 \| \Lambda^{-1} \|_F^2 \cdot \frac{2r^2 m^2}{n^2} \right).
$$

Since rank $\langle S_N(E) \rangle \leq \text{rank} (P_{UV} - P_{UV} - L_N(E)) \leq 8r$, by Lemma 3

$$
\mathbb{E}\| S_N(E)\|_F^2 \leq 8r \cdot \mathbb{E}\| S_N(E)\|_F^2 \leq C \cdot r^2 \| E \|_F^4 \frac{\sigma_\xi^4}{\lambda_r^4}
\leq C_1 \left( \frac{\sigma_\xi}{\lambda_r} \right)^4 \cdot \frac{r^3 m^2}{n^2}.
$$

The upper bound of $\mathbb{E}\langle L_N(E), S_N(E) \rangle$ requires more delicate treatments. To this end, denote the event $E_1 := \{ \| E \| \leq 2 \mathbb{E}\| E \| \} \text{ with } \mathbb{P}(E_1) \geq 1 - e^{-c_1 m}$ by the proof of Lemma 15.

Recall that $L_N(E) = P_{UV}^1 E C_{UV} + C_{UV} E P_{UV}^1$ and

$$
S_N(E)_{1, E_1} = \frac{1}{2\pi i} \left( \int_{\gamma_N^+} + \int_{\gamma_N^-} \right) \sum_{j \geq 2} (-1)^j \left[ R_N(\eta) E \right] \cdot \left[ R_N(\eta) E \right] \cdot \langle L_N(E), S_N(E) \rangle_{1, E_1} d\eta.
$$

Therefore,

$$
\mathbb{E}\langle L_N(E), S_N(E) \rangle_{1, E_1} = \frac{1}{2\pi i} \left( \int_{\gamma_N^+} + \int_{\gamma_N^-} \right) \sum_{j \geq 2} (-1)^j \mathbb{E}\langle \left[ R_N(\eta) E \right] \cdot \langle L_N(E), S_N(E) \rangle \rangle_{1, E_1} d\eta.
$$
We specifically consider \( j = 2 \) and recall \( E = E_1 + E_2 \) where \( E_1 = n^{-1} \sum \xi_i \mathcal{D}(X_i) \) and \( E_2 = n^{-1} \sum \langle \Delta, X_i \rangle \mathcal{D}(X_i) - \mathcal{D}(\Delta) \). Then,

\[
\mathbb{E} \left\langle \left[ \mathcal{R}_N(\eta) E \right]^2 \mathcal{R}_N(\eta), \mathcal{L}_N(E) \right\rangle = \mathbb{E} \text{ tr} \left( \left[ \mathcal{R}_N(\eta) E \right]^2 \mathcal{R}_N(\eta) \mathcal{L}_N(E) \right) \\
= \mathbb{E} \text{ tr} \left( \mathcal{R}_N(E_2) \mathcal{R}_N(\eta) E \mathcal{R}_N(\eta) \mathcal{L}_N(E) \right) + \mathbb{E} \text{ tr} \left( \mathcal{R}_N(E_1) \mathcal{R}_N(\eta) E_2 \mathcal{R}_N(\eta) \mathcal{L}_N(E) \right) + \text{ tr} \left( \mathcal{R}_N(E_1) \mathcal{R}_N(\eta) E_1 \mathcal{R}_N(\eta) \mathcal{L}_N(E_1) \right) \\
+ \text{ tr} \left( \mathcal{R}_N(E_1) \mathcal{R}_N(\eta) E_1 \mathcal{R}_N(\eta) \mathcal{L}_N(E_1) \right).
\]

Since \( \xi_i \sim \mathcal{N}(0, \sigma_\xi^2) \) are i.i.d., we have \( \mathbb{E} \text{ tr} \left( \mathcal{R}_N(E_1) \mathcal{R}_N(\eta) E_1 \mathcal{R}_N(\eta) \mathcal{L}_N(E_1) \right) = 0 \). Therefore, by Lemma 2,

\[
\left| \mathbb{E} \left\langle \left[ \mathcal{R}_N(\eta) E \right]^2 \mathcal{R}_N(\eta), \mathcal{L}_N(E) \right\rangle \right| \leq 4 \cdot 2 \cdot \mathbb{E} \frac{||E||^2 ||E_2||}{\lambda^r} + \mathbb{E} \frac{||E_2|| ||E||}{\lambda^r} \leq C \frac{\sigma_\xi^2}{\lambda^r} \cdot \frac{r^{3/2} \bar{m}^2 \log^{1/2} \bar{m}}{n^2}
\]

where we used the fact \( \| \Delta \|_F \leq C_1 \sigma_\xi^{(r\bar{m}^2)/n^{3/2}} \). Therefore,

\[
\left| \mathbb{E} \left\langle \left[ \mathcal{R}_N(\eta) E \right]^2 \mathcal{R}_N(\eta), \mathcal{L}_N(E) \right\rangle \right| \leq C \frac{\sigma_\xi^2}{\lambda^r} \cdot \frac{r^{3/2} \bar{m}^2 \log^{1/2} \bar{m}}{n^2} + C e^{-c_1 \bar{m}} \frac{\sigma_\xi^2}{\lambda^r} \cdot \frac{r \bar{m}^{3/2}}{n^{3/2}} \leq C \frac{\sigma_\xi^2}{\lambda^r} \cdot \frac{r^{3/2} \bar{m}^2 \log^{1/2} \bar{m}}{n^2}.
\]

Then,

\[
\mathbb{E} \left\langle \mathcal{L}_N(E), \mathcal{S}_N(E) \right\rangle \leq C \frac{\sigma_\xi^2}{\lambda^r} \cdot \frac{r^{5/2} \bar{m}^2 \log^{1/2} \bar{m}}{n^2} + \frac{1}{2 \pi} \left( \int_{\gamma_0^+} + \int_{\gamma_0^-} \right) \mathbb{E} \left| \sum_{j \geq 3} \left( -1 \right)^j \mathbb{E} \left\langle \left[ \mathcal{R}_N(\eta) E \right]^j \mathcal{R}_N(\eta), \mathcal{L}_N(E) \right\rangle \right| d\eta
\]

\[
\leq C \frac{\sigma_\xi^2}{\lambda^r} \cdot \frac{r^2 \bar{m}^2}{n^2} + C e^{-c_1 \bar{m}} \frac{\sigma_\xi^2}{\lambda^r} \cdot \frac{r^{5/2} \bar{m}^2 \log^{1/2} \bar{m}}{n^2}
\]

which concludes the proof. Similarly,

\[
\mathbb{E} \left\langle \mathcal{L}_N(E), \mathcal{S}_N(E) \right\rangle \leq C e^{-c_1 \bar{m}} \left( \frac{\sigma_\xi}{\lambda^r} \right)^3 \frac{r^2 \bar{m}^{3/2}}{n^{3/2}}.
\]

Therefore,

\[
\mathbb{E} \left\langle \mathcal{L}_N(E), \mathcal{S}_N(E) \right\rangle \leq C \frac{\sigma_\xi^4}{\lambda^4} \cdot \frac{r^2 \bar{m}^2}{n^2} + C \frac{\sigma_\xi^3}{\lambda^3} \cdot \frac{r^{5/2} \bar{m}^2 \log^{1/2} \bar{m}}{n^2}.
\]

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8.2 Proof of supporting lemmas

8.2.1 Proof of Lemma 3

Define the event $\mathcal{E}_0 := \{\|E\| \leq \frac{\lambda_r}{4}\}$. For each $1 \leq k \leq r$, denote by $\gamma_k^+$ the circle on the complex plane with center at $\lambda_k$ and radius $\frac{\lambda_r}{2}$. Similarly, denote by $\gamma_k^-$ the circle on the complex plane with center at $-\lambda_k$ and radius $\frac{\lambda_r}{2}$. By Weyl theorem,

$$|\hat{\lambda}_k - \lambda_k| \leq \|E\| \quad \text{and} \quad |\hat{\lambda}_{-k} - \lambda_{-k}| \leq \|E\|, \quad \forall 1 \leq |k| \leq r.$$ 

where $\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_k \geq \hat{\lambda}_{-k} \geq \ldots \geq \hat{\lambda}_{-1}$ are the top- $r$ largest and smallest eigenvalues of $\hat{\mathbf{N}}$. On event $\mathcal{E}_0$, we have

$$|\hat{\lambda}_k - \lambda_k| < \frac{\lambda_r}{2}, \quad \forall 1 \leq |k| \leq r.$$ 

Therefore, for each $1 \leq |k| \leq r$, $\hat{\lambda}_k$ is located inside the circle $\gamma^{|k|}_s$. Denote by

$$\gamma_+^N = \bigcup_{k=1}^{r} \gamma_k^+ \quad \text{and} \quad \gamma_-^N = \bigcup_{k=1}^{r} \gamma_k^-$$

where the union of $\gamma_+^{j_1}$ and $\gamma_+^{j_2}$ is illustrated in the Figure 4. By Riesz formula (see [15] and [19]), $\mathcal{P}_{\hat{U}\hat{V}}$ can be explicitly expressed as

$$\mathcal{P}_{\hat{U}\hat{V}} = -\frac{1}{2\pi i} \oint_{\gamma_+^N} (\hat{\mathbf{N}} - \eta \mathbf{I})^{-1} d\eta - \frac{1}{2\pi i} \oint_{\gamma_-^N} (\hat{\mathbf{N}} - \eta \mathbf{I})^{-1} d\eta.$$ 

Denote by $\mathcal{R}_N(\eta) = (\mathbf{N} - \eta \mathbf{I})^{-1}$. We write the Neumann series:

$$(\mathbf{N} - \eta \mathbf{I})^{-1} = (\mathbf{N} - \eta \mathbf{I} + E)^{-1} = (\mathbf{I} + \mathcal{R}_N(\eta)E)^{-1}\mathcal{R}_N(\eta)$$

$$= \sum_{j=0}^{\infty} (-1)^j [\mathcal{R}_N(\eta)E]^j \mathcal{R}_N(\eta)$$

$$= \mathcal{R}_N(\eta) + (-1)^j \mathcal{R}_N(\eta)E\mathcal{R}_N(\eta) + \sum_{j=2}^{\infty} (-1)^j [\mathcal{R}_N(\eta)E]^j \mathcal{R}_N(\eta)$$

By Cauchy integral formula,

$$\mathcal{P}_{\hat{U}\hat{V}} = -\frac{1}{2\pi i} \oint_{\gamma_+^N} \mathcal{R}_N(\eta) d\eta - \frac{1}{2\pi i} \oint_{\gamma_-^N} \mathcal{R}_N(\eta) d\eta.$$ 

Therefore,

$$\mathcal{P}_{\hat{U}\hat{V}} - \mathcal{P}_{UV} = \mathcal{L}_N(E) + \mathcal{S}_N(E)$$

where

$$\mathcal{L}_N(E) = \frac{1}{2\pi i} \oint_{\gamma_+^N} \mathcal{R}_N(\eta)E\mathcal{R}_N(\eta) d\eta + \frac{1}{2\pi i} \oint_{\gamma_-^N} \mathcal{R}_N(\eta)E\mathcal{R}_N(\eta) d\eta$$
and

\[ S_N(E) = -\frac{1}{2\pi i} \oint_{\gamma_N^+} (\sum_{j \geq 2} (-1)^j [R_N(\eta)E]^j R_N(\eta)) d\eta \]

\[ -\frac{1}{2\pi i} \oint_{\gamma_N^-} (\sum_{j \geq 2} (-1)^j [R_N(\eta)E]^j R_N(\eta)) d\eta. \]

To compute \( L_N(E) \), we write \( R_N(\eta) = (N - \eta I)^{-1} \) as

\[ R_N(\eta) = \sum_{1 \leq |k| \leq r} \frac{1}{\lambda_k - \eta} P_k + \frac{1}{-\eta} P_{UV}^\perp \]

where \( P_k = \theta_k \theta_k^\top \) for \( 1 \leq |k| \leq r \) and \( P_{UV}^\perp \) denotes the spectral projector of \( N \) corresponding to the eigenvalue 0 which can be explicitly written as

\[ P_{UV}^\perp = \begin{pmatrix} U_{\perp} \ U_{\perp}^\top \\ 0 \ V_{\perp} V_{\perp}^\top \end{pmatrix} \]

Recall that 0 and \( \{\lambda_{-k}\}_{k=1}^r \) are located outside the contour \( \gamma_N^+ \). We write

\[ \frac{1}{2\pi i} \oint_{\gamma_N^+} R_N(\eta)E R_N(\eta) d\eta \]

\[ = \frac{1}{2\pi i} \oint_{\gamma_N^+} P_{in}^+(\eta)E P_{in}^+(\eta) d\eta + \frac{1}{2\pi i} \oint_{\gamma_N^+} P_{out}^+(\eta)E P_{out}^+(\eta) d\eta \]

\[ + \frac{1}{2\pi i} \oint_{\gamma_N^+} P_{out}^+(\eta)E P_{in}^+(\eta) d\eta + \frac{1}{2\pi i} \oint_{\gamma_N^+} P_{out}^+(\eta)E P_{out}^+(\eta) d\eta. \]

where

\[ P_{in}^+(\eta) = \sum_{1 \leq k \leq r} \frac{1}{\lambda_k - \eta} P_k \quad \text{and} \quad P_{out}^+(\eta) = \sum_{1 \leq k \leq r} \frac{1}{\lambda_{-k} - \eta} P_{-k} + \frac{1}{-\eta} P_{UV}^\perp. \]

According to Cauchy integral formula, we immediately obtain

\[ \frac{1}{2\pi i} \oint_{\gamma_N^+} P_{out}^+(\eta)E P_{out}^+(\eta) d\eta = 0. \]

Also, by Cauchy integral formula,

\[ \frac{1}{2\pi i} \oint_{\gamma_N^+} P_{in}^+(\eta)E P_{out}^+(\eta) d\eta \]

\[ = \sum_{1 \leq k \leq r} \sum_{1 \leq k' \leq r} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{d\eta}{(\lambda_k - \eta)(\lambda_{-k'} - \eta)} P_k E P_{-k'} \]

\[ + \sum_{1 \leq k \leq r} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{d\eta}{(\lambda_k - \eta)(-\eta)} P_k E P_{UV}^\perp \]

\[ = \sum_{1 \leq k \leq r} \sum_{1 \leq k' \leq r} \frac{1}{\lambda_k - \lambda_{-k'}} P_k E P_{-k'} + \sum_{1 \leq k \leq r} \frac{1}{\lambda_k} P_k E P_{UV}^\perp. \]
By the same calculation, we get
\[
\frac{1}{2\pi i} \oint_{\gamma_N} P^+_{\text{out}}(\eta) E P^+_{\text{in}}(\eta) d\eta = \sum_{1 \leq k \leq r} \sum_{1 \leq k' \leq r} \frac{1}{\lambda_k - \lambda_{k'}} P_{-k} E P_k + \sum_{1 \leq k \leq r} \frac{1}{\lambda_k} P_{\text{UV}} E P_k.
\]
Meanwhile, by Cauchy integral formula and Cauchy-Goursat theorem,
\[
\frac{1}{2\pi i} \oint_{\gamma_N} P^-_{\text{in}}(\eta) E P^-_{\text{in}}(\eta) d\eta
\]
\[
= \sum_{1 \leq k \leq r} \sum_{1 \leq k' \leq r} \frac{1}{2\pi i} \oint_{\gamma_N} \frac{d\eta}{(\lambda_k - \eta)(\lambda_{k'} - \eta)} P_{k} E P_{k'}
\]
\[
= \sum_{1 \leq k \leq r} \frac{1}{2\pi i} \oint_{\gamma_N} \frac{d\eta}{(\lambda_k - \eta)^2} P_{k} E P_{k} + \sum_{1 \leq k \leq r} \frac{1}{2\pi i} \oint_{\gamma_N} \frac{d\eta}{(\lambda_k - \eta)(\lambda_{k'} - \eta)} P_{k} E P_{k'}
\]
\[
= 0.
\]
Therefore,
\[
\frac{1}{2\pi i} \oint_{\gamma_N} R_N(\eta) E R_N(\eta) d\eta = \sum_{1 \leq k \leq r} \sum_{1 \leq k' \leq r} \frac{1}{\lambda_k - \lambda_{k'}} (P_{k} E P_{-k'} + P_{-k'} E P_k)
\]
\[
+ \sum_{1 \leq k \leq r} \frac{1}{\lambda_k} (P_{\text{UV}} E P_k + P_k E P_{\text{UV}}).
\]
We then calculate \((2\pi i)^{-1} \oint_{\gamma_N} R_N(\eta) E R_N(\eta) d\eta\) and write
\[
\frac{1}{2\pi i} \oint_{\gamma_N} R_N(\eta) E R_N(\eta) d\eta
\]
\[
= \frac{1}{2\pi i} \oint_{\gamma_N} P^-_{\text{in}}(\eta) E P^-_{\text{in}}(\eta) d\eta + \frac{1}{2\pi i} \oint_{\gamma_N} P^-_{\text{out}}(\eta) E P^-_{\text{out}}(\eta) d\eta
\]
\[
+ \frac{1}{2\pi i} \oint_{\gamma_N} P^-_{\text{in}}(\eta) E P^-_{\text{out}}(\eta) d\eta + \frac{1}{2\pi i} \oint_{\gamma_N} P^-_{\text{out}}(\eta) E P^-_{\text{in}}(\eta) d\eta
\]
where
\[
P^-_{\text{in}} = \sum_{1 \leq k \leq r} \frac{1}{\lambda_k - \eta} P_{-k} \quad \text{and} \quad P^-_{\text{out}} = \sum_{1 \leq k \leq r} \frac{1}{\lambda_k - \eta} P_k + \frac{1}{-\eta} P_{\text{UV}}.
\]
Following an identical approach as above, we end up with
\[
\frac{1}{2\pi i} \oint_{\gamma_N} R_N(\eta) E R_N(\eta) d\eta = \sum_{1 \leq k \leq r} \sum_{1 \leq k' \leq r} \frac{1}{\lambda_{k} - \lambda_{k'}} (P_{-k} E P_{k'} + P_{k} E P_{-k})
\]
\[
+ \sum_{1 \leq k \leq r} \frac{1}{\lambda_{-k}} (P_{\text{UV}} E P_{-k} + P_{-k} E P_{\text{UV}}).
\]
Therefore,
\[
\mathcal{L}_N(E)
\]
\[
= \frac{1}{2\pi i} \oint_{\gamma_N} R_N(\eta) E R_N(\eta) d\eta + \frac{1}{2\pi i} \oint_{\gamma_N} R_N(\eta) E R_N(\eta) d\eta
\]
\[
= \sum_{1 \leq k \leq r} \frac{1}{\lambda_k} (P_{\text{UV}} E P_k + P_k E P_{\text{UV}}) + \sum_{1 \leq k \leq r} \frac{1}{\lambda_{-k}} (P_{\text{UV}} E P_{-k} + P_{-k} E P_{\text{UV}})
\]
\[
= P_{\text{UV}} E C_{\text{UV}} + C_{\text{UV}} E P_{\text{UV}}.
\]
with
\[ C_{UV} = \sum_{1 \leq |k| \leq r} \frac{1}{\lambda_k} P_k = \begin{pmatrix} 0 & U \Lambda^{-1} V^T \\ V \Lambda^{-1} U^T & 0 \end{pmatrix}. \]

Clearly,
\[ \|L_N(E)\| \leq 2 \|P_{UV}^\perp\| \|E\| \|C_{UV}\| \leq \frac{2\|E\|}{\lambda_r}. \]

We further upper bound \(\|S_N(E)\|\). By definition of \(S_N(E)\),
\[ \|S_N(E)\| \leq \sum_{j \geq 2} \frac{|\gamma_N^+| + |\gamma_N^-|}{2\pi} \|R_N(\eta)E\| \|R_N(\eta)\| \]
\[ \leq \left( \frac{|\gamma_N^+|}{2\pi} + \frac{|\gamma_N^-|}{2\pi} \right) \sum_{j \geq 2} \left( \frac{2\|E\|}{\lambda_r} \right)^j \frac{2}{\lambda_r}. \]

where \(|\gamma_N^+|\) and \(|\gamma_N^-|\) represent the circumference of contour \(\gamma_N^+\) and \(\gamma_N^-\). Under the event \(E_0\), we have \(\frac{2\|E\|}{\lambda_r} \leq \frac{1}{2}\), we conclude with
\[ \sum_{j \geq 2} \left( \frac{2\|E\|}{\lambda_r} \right)^j \leq 8\left( \frac{\|E\|}{\lambda_r} \right)^2. \]

Meanwhile, the circumference
\[ |\gamma_N^+| \leq r \cdot 2\pi \frac{\lambda_r}{2} \leq r\pi \lambda_r. \]

Therefore, under event \(E_0\),
\[ \|S_N(E)\| \leq (r + 1)\lambda_r \cdot \left( \frac{\|E\|}{\lambda_r} \right)^2 \frac{2}{\lambda_r} \]
\[ \leq 16(1 + r) \left( \frac{\|E\|}{\lambda_r} \right)^2. \quad (8.2) \]

In addition, on event \(E_0^c\) where \(\|E\| > 4\lambda_r\),
\[ \|S_N(E)\| \leq \|P_{UV}^\perp P_{UV}\| + \|L_N(E)\| \leq 1 + 2 \cdot \frac{\|E\|}{\lambda_r} \]
\[ \leq \frac{9}{4} \cdot \frac{\|E\|}{\lambda_r} \leq 9 \cdot \left( \frac{\|E\|}{4\lambda_r} \right)^2. \quad (8.3) \]

Combine (8.2) and (8.3), we obtain the claimed bound of \(\|S_N(E)\|\)

### 8.2.2 Proof of Lemma 15

Since \(n \geq \tilde{m}\) and let \(t \leq \tilde{m}\) in Lemma 2, we obtain that
\[ \mathbb{P}\left( \|E\| - \mathbb{E}[E] \geq C_3 \sigma_{\xi} \cdot \frac{t^{1/2}}{n^{1/2}} + C_4 \|\Delta\|_F \cdot \frac{t^{1/2} + \log^{1/2} \tilde{m}}{n^{1/2}} \right) \leq 1 - 3e^{-t} - e^{-n}. \]

By setting \(t = c_1 \tilde{m}\) with small enough absolute constant \(c_1 > 0\), we conclude that with probability at least \(1 - 3e^{-c_1 \tilde{m}} - e^{-n}\), \(\|E\| \leq \frac{3}{4} \mathbb{E}[\|E\|]. \)
Denote $\bar{\delta} = 2E\|E\|$ and the event $\mathcal{E}_1 := \{\|E\| \leq \frac{9}{4}E\|E\|\}$ on which $\frac{2\|E\|}{\lambda_1} \leq \frac{9}{2}E\|E\| \leq \frac{9}{10}$ and $\mathbb{P}(\mathcal{E}_1) \geq 1 - e^{-n} - e^{-cm}$. Define a Lipschitz function $\phi(\cdot)$ on $\mathbb{R}_+$ such that

$$\phi(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq 1, \\ 1 - 8(s - 1), & \text{if } 1 \leq s \leq \frac{9}{8}, \\ 0, & \text{if } s \geq \frac{9}{8}. \end{cases}$$

Clearly, $\phi(s)$ is Lipschitz with constant 8. By definition of $\phi(\cdot)$, we have $\phi(\frac{\|E\|}{\bar{\delta}}) = 1$ if $\|E\| \leq 2E\|E\|$. Observe that on event $\mathcal{E}_2 := \{\|E\| \leq 2E\|E\|\}$ with $\mathbb{P}(\mathcal{E}_2) \geq 1 - e^{-n} - e^{-cm}$,

$$\mathbb{E}[S_N(E) - S_N(E), \mathcal{P}_{UV}] - \left(\frac{\|L_N(E)\|_2^2}{2} - \frac{E\|L_N(E)\|_F^2}{2}\right) = \mathbb{E}[S_N(E) - S_N(E), \mathcal{P}_{UV}]\phi\left(\frac{\|E\|}{\delta}\right) - \left(\frac{\|L_N(E)\|_2^2}{2} - \frac{E\|L_N(E)\|_F^2}{2}\right)\phi\left(\frac{\|E\|}{\delta}\right).$$

Eq. (8.4) is equivalent to the concentration of $(\langle S_N(E), \mathcal{P}_{UV} \rangle + \frac{\|L_N(E)\|_2^2_2}{2})\phi\left(\frac{\|E\|}{\delta}\right)$ around its expectation $\mathbb{E}(\langle S_N(E), \mathcal{P}_{UV} \rangle + \frac{\|L_N(E)\|_2^2_2}{2})\phi\left(\frac{\|E\|}{\delta}\right)$, which is trivial on event $\mathcal{E}_3^c$. The following analysis shall be focused on event $\mathcal{E}_1$, on which the Riesz representation formula holds.

We shall use the Riesz formula developed in the proof of Lemma 3 where the contours $\gamma_N^+$ and $\gamma_N^-$ are defined. On event $\mathcal{E}_1$, the positive eigenvalues of $N$, i.e., $\{\lambda_k\}_{k=1}^r$ are inside $\gamma_N^+$, while the negative eigenvalues of $N$, i.e., $\{-\lambda_k\}_{k=1}^r$ are inside $\gamma_N^-$. We represent $S_N(E)$ as follows,

$$S_N(E) = -\frac{1}{2\pi i} \oint_{\gamma_N^+} \sum_{j \geq 2} (-1)^j [R_N(\eta)E]^j R_N(\eta) d\eta - \frac{1}{2\pi i} \oint_{\gamma_N^-} \sum_{j \geq 2} (-1)^j [R_N(\eta)E]^j R_N(\eta) d\eta = -\frac{1}{2\pi i} \oint_{\gamma_N} [R_N(\eta)E]^2 R_N(\eta) d\eta - \frac{1}{2\pi i} \oint_{\gamma_N} [R_N(\eta)E]^2 R_N(\eta) d\eta$$

$$-\frac{1}{2\pi i} \oint_{\gamma_N^+ + \gamma_N^-} \sum_{j \geq 3} (-1)^j [R_N(\eta)E]^j R_N(\eta) d\eta. \quad (8.6)$$

Therefore, $\langle S_N(E), \mathcal{P}_{UV} \rangle \phi\left(\frac{\|E\|}{\delta}\right) = \langle T_1, \mathcal{P}_{UV} \rangle \phi\left(\frac{\|E\|}{\delta}\right) + \langle T_2, \mathcal{P}_{UV} \rangle \phi\left(\frac{\|E\|}{\delta}\right)$ with $T_1$ and $T_2$ defined in (8.5) and (8.6). We write $\langle T_1, \mathcal{P}_{UV} \rangle \phi\left(\frac{\|E\|}{\delta}\right)$ as

$$\langle T_1, \mathcal{P}_{UV} \rangle \phi\left(\frac{\|E\|}{\delta}\right) = -\frac{1}{2\pi i} \oint_{\gamma_N} [R_N(\eta)E]^2 R_N(\eta) d\eta - \frac{1}{2\pi i} \oint_{\gamma_N} [R_N(\eta)E]^2 R_N(\eta) d\eta$$

Recall that

$$R_N(\eta) = \sum_{k=1}^r \frac{1}{\lambda_k - \eta} P_k + \sum_{k=1}^r \frac{1}{\lambda_k - \eta} P_{-k} + \frac{1}{1-\eta} \overline{P}_{UV}$$

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and

\[ \mathcal{P}_{UV} = \sum_{1 \leq |k| \leq r} \mathcal{P}_k. \]

Then,

\[
-\frac{1}{2\pi i} \oint_{\gamma_N^+} \langle [\mathcal{R}_N(\eta)\mathcal{E}]^2 \mathcal{R}_N(\eta), \mathcal{P}_{UV} \rangle \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right) d\eta = \sum_{1 \leq |k| \leq r} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{R}_N(\eta) \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right)
= \sum_{1 \leq |k|, |k'| \leq r} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2(\lambda_{k'} - \eta)} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{P}_k \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right)
+ \sum_{1 \leq |k| \leq r} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2(-\eta)} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right).
\]

By definition of contour \( \gamma_N^+ \) and Cauchy integral formula, if \( k, k' < 0 \), then \( \oint_{\gamma_N^+} (\lambda_k - \eta)^{-2}(\lambda_{k'} - \eta)^{-1} d\eta = 0 \) and \( \oint_{\gamma_N^+} (\lambda_k - \eta)^{-2}(-\eta)^{-1} d\eta = 0 \). Therefore,

\[
\sum_{1 \leq |k| \leq r} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{P}_k \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right)
= \sum_{k=1}^r \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2(-\eta)} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right)
= \sum_{k=1}^r \frac{1}{\lambda_k^2} \langle \mathcal{E} \mathcal{P}_{UV} \mathcal{E}, \mathcal{P}_k \rangle \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right).
\]

and since \( \oint_{\gamma_N^+} (\lambda_k - \eta)^{-3} d\eta = 0 \) for all \( 1 \leq |k| \leq r \), we obtain

\[
\sum_{1 \leq |k|, |k'| \leq r} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2(\lambda_{k'} - \eta)} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{P}_k \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right)
= \sum_{1 \leq k \neq k' \leq r} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2(\lambda_{k'} - \eta)} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{P}_k \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right)
+ \sum_{1 \leq k \leq r} \sum_{-r \leq k' \leq -1} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2(\lambda_{k'} - \eta)} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{P}_k \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right)
+ \sum_{1 \leq k' \leq r} \sum_{-r \leq k \leq -1} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2(\lambda_{k'} - \eta)} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{P}_k \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right).
\]

By Cauchy-Goursat theorem, It is easy to check \( \oint_{\gamma_N^+} (\lambda_k - \eta)^{-2}(\lambda_{k'} - \eta) d\eta = 0 \) for all \( 1 \leq k \neq k' \leq r \). Then,

\[
\sum_{1 \leq |k|, |k'| \leq r} \frac{1}{2\pi i} \oint_{\gamma_N^+} \frac{1}{(\lambda_k - \eta)^2(\lambda_{k'} - \eta)} \text{tr} (\mathcal{P}_k \mathcal{E} \mathcal{P}_k \mathcal{E} \mathcal{P}_k) d\eta \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right)
= - \sum_{1 \leq k, k' \leq r} \frac{1}{(\lambda_k - \lambda_{k'})^2} \text{tr}(\mathcal{E} \mathcal{P}_k \mathcal{E} \mathcal{P}_k) \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right)
+ \sum_{1 \leq k, k' \leq r} \frac{1}{(\lambda_k - \lambda_{k'})^2} \text{tr}(\mathcal{E} \mathcal{P}_k \mathcal{E} \mathcal{P}_k) \phi\left(\frac{\|\mathcal{E}\|}{\delta}\right) = 0.
\]
The above facts imply that
\[ -\frac{1}{2\pi i} \oint_{\gamma_N^+} \langle [\mathcal{R}_N(\eta)]^2 \mathcal{R}_N(\eta), \mathcal{P}_{UV} \rangle \phi \left( \frac{\|E\|}{\delta} \right) = - \sum_{k=1}^r \langle EP_{UV}^+ \mathcal{P}_{\lambda_k^2} \phi \left( \frac{\|E\|}{\delta} \right) \rangle. \]
Repeat the same analysis, we can show
\[ -\frac{1}{2\pi i} \oint_{\gamma_N^-} \langle [\mathcal{R}_N(\eta)]^2 \mathcal{R}_N(\eta), \mathcal{P}_{UV} \rangle \phi \left( \frac{\|E\|}{\delta} \right) = - \sum_{k=1}^r \langle EP_{UV}^+ \mathcal{P}_{\lambda_k^2} \phi \left( \frac{\|E\|}{\delta} \right) \rangle. \]
Therefore,
\[ \langle T_1, \mathcal{P}_{UV} \rangle \phi \left( \frac{\|E\|}{\delta} \right) = - \sum_{k=1}^r \langle EP_{UV}^+ \mathcal{P}_{\lambda_k} \phi \left( \frac{\|E\|}{\delta} \right) \rangle - \sum_{k=1}^r \langle EP_{UV}^+ \mathcal{P}_{\lambda_k^2} \phi \left( \frac{\|E\|}{\delta} \right) \rangle = - \langle EC_{UV} \mathcal{P}_{UV}^2 \rangle \phi \left( \frac{\|E\|}{\delta} \right) = - \langle C_{UV} \mathcal{P}_{UV} \phi \left( \frac{\|E\|}{\delta} \right) \rangle = -\frac{1}{2} \mathcal{L}_N(E) \|\|_F^2 \phi \left( \frac{\|E\|}{\delta} \right) \]
where we used the fact
\[ \|\mathcal{L}_N(E)\|_F^2 = \|\mathcal{P}_{UV}^2 EC_{UV} + C_{UV} \mathcal{P}_{UV}^2 \|_F^2 = 2 \|\mathcal{P}_{UV}^2 EC_{UV}\|_F^2. \]
To this end, we conclude with
\[ \langle \mathcal{S}_N(E), \mathcal{P}_{UV} \rangle + \frac{1}{2} \mathcal{L}_N(E) \|\|_F^2 \phi \left( \frac{\|E\|}{\delta} \right) \]
\[ = \langle T_2, \mathcal{P}_{UV} \rangle = \frac{1}{2\pi i} \left( \oint_{\gamma_N^+} + \oint_{\gamma_N^-} \right) \sum_{j \geq 3}(-1)^j \langle [\mathcal{R}_N(\eta)]^j \mathcal{R}_N(\eta), \mathcal{P}_{UV} \rangle \phi \left( \frac{\|E\|}{\delta} \right) d\eta. \]

**Proof of first claim**  By above representation, on event \( \mathcal{E}_1 \), we have
\[ \|\mathcal{P}_{UV} - \mathcal{P}_{UV} \|_F^2 = 4r - 2\langle \mathcal{P}_{UV}, \mathcal{P}_{UV} \rangle = -2\langle \mathcal{S}_N(E), \mathcal{P}_{UV} \rangle \]
\[ = \|\mathcal{L}_N(E)\|_F^2 \]
\[ + \frac{1}{\pi i} \left( \oint_{\gamma_N^+} + \oint_{\gamma_N^-} \right) \sum_{j \geq 3} (-1)^j \langle [\mathcal{R}_N(\eta)]^j \mathcal{R}_N(\eta), \mathcal{P}_{UV} \rangle \phi \left( \frac{\|E\|}{\delta} \right) d\eta. \]
Since \( \|\mathcal{R}_N(\eta)\| \leq \frac{\delta}{\lambda_r} \), on event \( \mathcal{E}_1 \),
\[ \|\mathcal{P}_{UV} - \mathcal{P}_{UV} \|_F^2 \leq \frac{1}{\pi} \sum_{j \geq 3} 2r \|\mathcal{R}_N(\eta)\|^{j+1} \|E\|^j \]
\[ \leq 8r^2 \left( \frac{9\delta}{4\lambda_r} \right)^3 \sum_{j \geq 0} \left( \frac{9}{10} \right)^j \leq 80r^2 \left( \frac{9\delta}{4\lambda_r} \right)^3. \]
Proof of second claim  It suffices to prove the concentration inequality for the following complex-valued functions,
\[ \varphi_{\delta,\eta}(E) := \sum_{j \geq 3} \varphi_{j,\delta,\eta}(E) \]
\[ \varphi_{j,\delta,\eta}(E) := (-1)^j \langle [R_N(\eta)]^j R_N(\eta), P_{uv} \rangle \phi\left(\frac{||E||}{\delta}\right). \]
We abuse the notation here such that \( E \) is viewed as a point in \( \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)} \).

Lemma 19. Under the conditions in Lemma 15 for any \( E, E' \in \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)} \), the following bounds hold for all \( j \geq 3 \),
\[ |\varphi_{j,\delta,\eta}(E) - \varphi_{j,\delta,\eta}(E')| \leq 8r : \frac{j + 72}{\lambda_r^2} \left(\frac{9\delta}{4\lambda_r}\right)^{j-1} ||E - E'|| \]
and
\[ |\varphi_{\delta,\eta}(E) - \varphi_{\delta,\eta}(E')| \leq C_5 r \left(\frac{9\delta}{4\lambda_r}\right)^2 ||E - E'|| \]
for an absolute constant \( C_5 > 0 \). In other words, functions \( \varphi_{\delta,\eta}(\cdot) \) and \( \varphi_{j,\delta,\eta}(\cdot) \) are Lipschitz functions on \( \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)} \).

We continue the proof of Lemma 15. Define
\[ \varphi_\delta(E) = \left(\langle S_N(E), P_{uv}\rangle + \frac{1}{2} ||L_N(E)||^2_F \right) \phi\left(\frac{||E||}{\delta}\right) \]
\[ = -\frac{1}{2\pi i} \left(\oint_{N^{-1}} + \oint_{N}^{-}\right) \varphi_{\delta,\eta}(E) d\eta. \]

Lemma 20. Under the assumptions in Lemma 19 the following bound holds
\[ |\varphi_\delta(E) - \varphi_\delta(E')| \leq C_6 r^2 \left(\frac{9\delta}{4\lambda_r}\right)^2 ||E - E'|| \]
for an absolute constant \( C_6 \).

According to Lemma 20 we write
\[ \left|\left(\langle S_N(E), P_{uv}\rangle + \frac{1}{2} ||L_N(E)||^2_F \right) \phi\left(\frac{||E||}{\delta}\right) - E\left(\langle S_N(E), P_{uv}\rangle + \frac{1}{2} ||L_N(E)||^2_F \right) \phi\left(\frac{||E||}{\delta}\right)\right| \]
\[ = \left|\varphi_\delta(E) - E\varphi_\delta(E)\right| \]
where the function \( \varphi_\delta(\cdot) \) is a Lipschitz function with respect \( E \) with constant \( C_6 r^2 \left(\frac{9\delta}{4\lambda_r}\right)^2 \). Since \( E \) is a function of \( \Delta, \{\xi_i\}, \{X_i\}, \) we shall apply the Gaussian concentration inequality, Lemma 24, where, however, \( E \) is not a Lipschitz function of \( \{X_i\} \). To this end, recall that \( E = E_1 + E_2 \) with
\[ E_1 = \mathcal{D}(Z_1) = \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i \mathcal{D}(X_i) \]
and
\[ E_2 = \mathcal{T}(Z_2) = \frac{1}{n} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle \mathcal{D}(X_i) - \mathcal{D}(\Delta). \]
Denote by $\text{vec}(\Delta)$ the vectorization of $\Delta$ and $\mathcal{M}(\cdot)$ the matricization of vectors such that $\mathcal{M}(\text{vec}(\Delta)) = \Delta$. Denote $\mathcal{P}_{\text{vec}(\Delta)}$ the orthogonal projection onto $\text{vec}(\Delta)$, i.e.,

$$\mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i) = \text{vec}(\Delta) \cdot \frac{\langle \Delta, X_i \rangle}{\|\Delta\|_F^2}.$$ 

For each $i$, write

$$\text{vec}(X_i) = \mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i) + \mathcal{P}^\perp_{\text{vec}(\Delta)} \text{vec}(X_i).$$

Clearly, $\mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i)$ is independent with $\mathcal{P}^\perp_{\text{vec}(\Delta)} \text{vec}(X_i)$. We view $\varphi_{\delta}(E)$ as a function of

$$\{\xi_i\}_{i=n+1}^{2n}, \quad \mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i) \}_{i=n+1}^{2n} \quad \text{and} \quad \mathcal{P}^\perp_{\text{vec}(\Delta)} \text{vec}(X_i) \}_{i=n+1}^{2n},$$

which are mutually independent. Conditioned on $\{\xi_i\}_{i}, \{\mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i)\}_{i}$, we have

$$\|E - E'\|_F \leq \left\| \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i \mathcal{D} \circ \mathcal{M}(\mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i) - \mathcal{P}^\perp_{\text{vec}(\Delta)} \text{vec}(X_i)) \right\|_F$$

$$\leq 2 \times n^{-1} \left( \sum_{i=n+1}^{2n} \xi_i^2 \right)^{1/2} \left( \sum_{i=n+1}^{2n} \left\| \mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i) - \mathcal{P}^\perp_{\text{vec}(\Delta)} \text{vec}(X_i) \right\|_F^2 \right)^{1/2}$$

implying that $\|E - E'\|_F$ is a Lipschitz function with respect to $\{\mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i)\}_{i=n+1}^{2n}$ with constant

$$\frac{2}{n} \left( \sum_i \xi_i^2 \right)^{1/2} + \left( \sum_i \left\langle \Delta, X_i \right\rangle^2 \right)^{1/2}.$$ 

Therefore, by Lemma 20,

$$|\varphi_{\delta}(E) - \varphi_{\delta}(E')| \leq \frac{C_{tr} \lambda r^2}{n \lambda r} \left( \frac{9 \delta}{4 \lambda r} \right)^2 \left[ \sum_i \xi_i^2 \right]^{1/2} + \left( \sum_i \left\langle \Delta, X_i \right\rangle^2 \right)^{1/2}$$

$$\times \left( \sum_{i=n+1}^{2n} \left\| \mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i) - \mathcal{P}^\perp_{\text{vec}(\Delta)} \text{vec}(X_i) \right\|_F^2 \right)^{1/2}. $$

By the Gaussian isoperimetric inequality in Lemma 24, conditioned on $\{\xi_i\}_{i}$ and $\{\mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i)\}_{i}$, with probability at least $1 - e^{-t}$ for all $t \geq 1$,

$$|\varphi_{\delta}(E) - \mathbb{E}_{\mathcal{P}_{\text{vec}(\Delta)} X} [\varphi_{\delta}(E)]| \leq \frac{C_{tr} \lambda r^2 t^{1/2}}{n \lambda r} \left( \frac{9 \delta}{4 \lambda r} \right)^2 \times \left[ \sum_i \xi_i^2 \right]^{1/2} + \left( \sum_i \left\langle \Delta, X_i \right\rangle^2 \right)^{1/2}.$$ 

Meanwhile, by concentration of sum of exponential random variables, with probability at least $1 - e^{-n}$,

$$\left( \sum_i \xi_i^2 \right)^{1/2} + \left( \sum_i \left\langle \Delta, X_i \right\rangle^2 \right)^{1/2} \leq C_{r} n^{1/2} (\sigma_{\xi} + \|\Delta\|_F).$$ 

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Therefore, with probability at least \(1 - e^{-t} - e^{-n}\),
\[
\left| \varphi_{\delta}(E) - \mathbb{E}_{\{P_{\text{vec}(\Delta)}^\perp x\}}[\varphi_{\delta}(E)] \right| \\
\leq C_T r^2 t^{1/2} \frac{C}{n^{1/2}} \left( \frac{9\delta}{4\lambda_r} \right)^2 \left( \sigma_\xi + \|\Delta\|_F \right).
\]

Next, we study the bound of \(\left| \mathbb{E}_{\{P_{\text{vec}(\Delta)}^\perp x\}}[\varphi_{\delta}(E)] - \mathbb{E}[\varphi_{\delta}(E)] \right|\). We apply the following lemma whose proof is postponed to the appendix.

**Lemma 21.** Under the assumptions of Lemma [20] and \(n \geq \bar{m}\), with probability at least \(1 - 2e^{-t}\) for all \(t \in [1, n]\),
\[
\left| \mathbb{E}_{\{P_{\text{vec}(\Delta)}^\perp x\}}[\varphi_{\delta}(E)] - \mathbb{E}[\varphi_{\delta}(E)] \right| \\
\leq C_T (\sigma_\xi + \|\Delta\|_F) \frac{r^2}{\lambda_r} \left( \frac{9\delta}{4\lambda_r} \right)^2 \cdot \frac{(t + \log \bar{m})^{1/2}}{n^{1/2}}
\]
for some absolute constant \(C_T > 0\).

We conclude that for all \(t \in [1, n]\), with probability at least \(1 - 3e^{-t} - e^{-n}\),
\[
\left| \varphi_{\delta}(E) - \mathbb{E}[\varphi_{\delta}(E)] \right| \\
\leq C_T (\sigma_\xi + \|\Delta\|_F) \frac{r^2}{\lambda_r} \left( \frac{9\delta}{4\lambda_r} \right)^2 \cdot \frac{(t + \log \bar{m})^{1/2}}{n^{1/2}}.
\]

Since \(\bar{\delta} = 2\|E\|\), on event \(\mathcal{E}_2 := \{\|E\| \leq 2\|E\|\} \) with \(P(\mathcal{E}_2) \geq 1 - e^{-n} - e^{-c_2m}\),
\[
\varphi_{\delta}(E) = \left( \langle S_{N}(E), P_{UV} \rangle + \frac{1}{2} \|L_{N}(E)\|_F^2 \right)
\]
which concludes the proof.

### 8.2.3 Proof of Lemma [16]

Recall that
\[
Z_1 = \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i X_i.
\]

Conditional on \(\{\xi_i\}_{i=n+1}^{2n}\), \(Z_1\) has the same distribution as
\[
Z_1 \overset{d}{=} \tau \cdot Z := \frac{1}{n} \left( \sum_{i=n+1}^{2n} \xi_i^2 \right)^{1/2} \cdot Z
\]
where \(Z \in \mathbb{R}^{m_1 \times m_2}\) has i.i.d. standard Gaussian entries. Then,
\[
\|P_{\text{vec}(\Delta)}^\perp E_i C_{UV}\|_F^2 \overset{d}{=} \tau^2 \cdot \left( \|U_{\perp} U_{\perp}^T Z V A^{-1} U^T\|_F^2 + \|V_{\perp} V_{\perp}^T Z U A^{-1} V^T\|_F^2 \right).
\]

Denote by \(z_1, \ldots, z_{m_2}\) the columns of \(Z\), i.e., \(z_j \in \mathcal{N}(0, I_{m_1})\) are i.i.d. standard Gaussian vectors. Write
\[
U_{\perp} U_{\perp}^T Z V A^{-1} U^T = \sum_{j=1}^{m_2} (U_{\perp} U_{\perp}^T z_j) \otimes (U A^{-1} V^T e_j)
\]

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where \( \{e_j\}_{j=1}^{m_2} \) denotes the standard basis vectors in \( \mathbb{R}^{m_2} \). In a similar fashion, write

\[
V_{\perp} V_{\perp}^T Z^T U A^{-1} V^T = \sum_{j=1}^{m_2} (V_{\perp} V_{\perp}^T e_j \otimes (V A^{-1} U^T z_j)).
\]

We claim that \( U_{\perp} U_{\perp}^T z_j \) is independent of \( V A^{-1} U^T z_j \). Indeed, the correlation

\[
E(\sum_{j=1}^m U_{\perp} U_{\perp}^T z_j) \otimes (V A^{-1} U^T z_j) = U_{\perp} U_{\perp}^T U A^{-1} V^T = 0.
\]

Since both vectors are Gaussian, we conclude that \( U_{\perp} U_{\perp}^T z_j \) is independent of \( V A^{-1} U^T z_j \) for all \( 1 \leq j \leq m_2 \). Therefore,

\[
\|U_{\perp} U_{\perp}^T Z V A^{-1} U^T\|_F^2 \text{ is independent of } \|V_{\perp} V_{\perp}^T Z^T U A^{-1} V^T\|_F^2.
\]

**Claim 1** Let \( z_k \in \mathbb{R}^{m_1-r} \) be i.i.d. standard Gaussian vector independent of \( Z \) for all \( k = 1, \ldots, r \). Then, we claim that

\[
\sum_{j=1}^{m_2} (U_{\perp} U_{\perp}^T z_j) \otimes (V A^{-1} U^T e_j) = \sum_{k=1}^r (U_{\perp} z_k) \otimes (\lambda_k^{-1} u_k)
\]

where \( \{u_1, \ldots, u_r\} \) are the columns of \( U \). To prove the claim, it suffices to check their covariance. To this end, define the following multilinear mapping:

\[
\mathcal{K}(u_1 \otimes u_2 \otimes u_3 \otimes u_4) = u_1 \otimes u_3 \otimes u_2 \otimes u_4, \quad \forall u_1, u_2, u_3, u_4 \in \mathbb{R}^m,
\]

a technique introduced in [15]. Then,

\[
\text{Cov}(U_{\perp} U_{\perp}^T Z V A^{-1} U^T) = E\left(U_{\perp} U_{\perp}^T Z V A^{-1} U^T \otimes (U_{\perp} U_{\perp}^T Z V A^{-1} U^T)\right)
\]

\[
= E \sum_{j=1}^{m_2} \mathcal{K}\left((U_{\perp} U_{\perp}^T z_j) \otimes (U_{\perp} U_{\perp}^T z_j) \otimes (V A^{-1} U^T e_j) \otimes (V A^{-1} U^T e_j)\right)
\]

\[
= \sum_{j=1}^{m_2} \mathcal{K}\left(U_{\perp} U_{\perp}^T \otimes (V A^{-1} U^T)\right).
\]

Similarly,

\[
\text{Cov}\left(\sum_{k=1}^r \left(U_{\perp} z_k \otimes (\lambda_k^{-1} u_k)\right)\right)
\]

\[
= E \sum_{k=1}^r \mathcal{K}\left((U_{\perp} z_k) \otimes (\lambda_k^{-1} u_k) \otimes (\lambda_k^{-1} u_k) \otimes (\lambda_k^{-1} u_k)\right)
\]

\[
= E \sum_{k=1}^r \mathcal{K}\left(U_{\perp} U_{\perp}^T \otimes (\lambda_k^{-1} u_k) \otimes (\lambda_k^{-1} u_k)\right) = \mathcal{K}\left(U_{\perp} U_{\perp}^T \otimes (V A^{-2} U^T)\right).
\]
Therefore, we conclude that $\mathbf{U}_{\perp} \mathbf{U}_{\perp} \mathbf{Z} \mathbf{V} \Lambda^{-1} \mathbf{U}^\top = \sum_{k=1}^{r} (\mathbf{U}_{\perp} \bar{z}_k) \otimes (\lambda_k^{-1} \mathbf{u}_k)$ which implies

$$
\| \mathbf{U}_{\perp} \mathbf{U}_{\perp} \mathbf{Z} \mathbf{V} \Lambda^{-1} \mathbf{U}^\top \|_F^2 = \sum_{k=1}^{r} \| \mathbf{U}_{\perp} \bar{z}_k \|_2^2 \lambda_k^{-2}
$$

where the last equality is due to the orthogonality of $\{ \mathbf{u}_k \}_{k=1}^{r}$. Clearly, $\| \mathbf{U}_{\perp} \bar{z}_k \|_2^2$ has a Chi-squared distribution with degrees of freedom $m_1 - r$. Therefore, we claim that

$$
\| \mathbf{U}_{\perp} \mathbf{U}_{\perp} \mathbf{Z} \mathbf{V} \Lambda^{-1} \mathbf{U}^\top \|_F^2 = \sum_{k=1}^{r} \frac{z_k^2}{\lambda_k^2}
$$

where $z_k^2 \sim \chi^2(m_1 - r)$ are i.i.d. for $k = 1, \ldots, r$.

**Claim 2** Let $\bar{z}_k \in \mathbb{R}^{m_2 - r}$ be i.i.d. standard Gaussian vector independent of $\mathbf{Z}$ for all $k = 1, \ldots, r$. We claim that

$$
\mathbf{V}_{\perp} \mathbf{V}_{\perp}^\top \mathbf{Z}^\top \mathbf{U} \Lambda^{-1} \mathbf{V}^\top = \sum_{k=1}^{r} (\mathbf{V}_{\perp} \bar{z}_k) \otimes (\lambda_k^{-1} \mathbf{v}_k)
$$

where $\{ \mathbf{v}_1, \ldots, \mathbf{v}_r \}$ are the columns of $\mathbf{V}$. Indeed, if we denote by $\bar{z}_j$, $1 \leq j \leq m_1$ the rows of $\mathbf{Z}$. Then, $\bar{z}_j \sim \mathcal{N}(0, \mathbf{I}_{m_2})$ are i.i.d. for all $1 \leq j \leq m_1$. We write

$$
\mathbf{V}_{\perp} \mathbf{V}_{\perp}^\top \mathbf{Z}^\top \mathbf{U} \Lambda^{-1} \mathbf{V}^\top = \sum_{j=1}^{m_1} (\mathbf{V}_{\perp} \bar{z}_j) \otimes (\mathbf{V} \Lambda^{-1} \mathbf{U}^\top \mathbf{e}_j)
$$

where $\{ \mathbf{e}_1, \ldots, \mathbf{e}_{m_1} \}$ denotes the standard basis vectors in $\mathbb{R}^{m_1}$. It is straightforward to check that

$$
\text{Cov} \left( \mathbf{V}_{\perp} \mathbf{V}_{\perp}^\top \mathbf{Z}^\top \mathbf{U} \Lambda^{-1} \mathbf{V}^\top \right) = \mathcal{K} \left( \left( \mathbf{V}_{\perp} \mathbf{V}_{\perp}^\top \right) \otimes \mathbf{V} \Lambda^{-2} \mathbf{V}^\top \right).
$$

Similarly, we obtain

$$
\text{Cov} \left( \sum_{k=1}^{r} (\mathbf{V}_{\perp} \bar{z}_k) \otimes (\lambda_k^{-1} \mathbf{v}_k) \right) = \mathcal{K} \left( \left( \mathbf{V}_{\perp} \mathbf{V}_{\perp}^\top \right) \otimes \mathbf{V} \Lambda^{-2} \mathbf{V}^\top \right)
$$

which proves the claim. Thus,

$$
\| \mathbf{V}_{\perp} \mathbf{V}_{\perp}^\top \mathbf{Z}^\top \mathbf{U} \Lambda^{-1} \mathbf{V}^\top \|_F^2 = \sum_{k=1}^{r} \| \mathbf{V}_{\perp} \bar{z}_k \|_2^2 \lambda_k^{-2} = \sum_{k=1}^{r} \frac{z_k^2}{\lambda_k^2} \lambda_k^{-2}
$$

where $z_k^2$ are i.i.d. and $z_k^2 \sim \chi^2(m_2 - r)$.

**Finalize the first claim of Lemma 16** By Claim 1 and Claim 2, we conclude that

$$
\| \mathbf{P}_{\mathbf{U} \mathbf{V}} \mathbf{E}_{\mathbf{U} \mathbf{V}} \mathbf{C}_{\mathbf{U} \mathbf{V}} \|_F^2 = \tau^2 \cdot \sum_{k=1}^{r} \frac{z_k^2}{\lambda_k^2}
$$

where $\{z_k^2\}_{k=1}^{r}$ are i.i.d. Chi-squared random variables with degrees of freedom $m_* = m_1 + m_2 - 2r$. 

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Proof of second claim of Lemma 16 Recall from above that
\[ \|P_{UV}E_1C_{UV}\|_F^2 = \frac{\tau^2}{n} \cdot \sum_{k=1}^{r} \frac{z_k^2}{\lambda_k^2}, \]
where \( \{z_k^2\}_{k=1}^{r} \) are i.i.d. Chi-squared random variables with degrees of freedom \( m_\star \). Therefore, \( \|P_{UV}E_1C_{UV}\|_F^2 \) is a sum of sub-exponential random variables. By the standard concentration inequality for sum of sub-exponential random variables (e.g. [26 Proposition 5.6]), with probability at least \( 1 - e^{-t} \) for all \( t \geq \log 2 \),
\[ \left\| \|P_{UV}E_1C_{UV}\|_F^2 - \mathbb{E}\|P_{UV}E_1C_{UV}\|_F^2 \right\| \leq C \tau^2 \cdot \max \left\{ \|\Delta^{-2}\|_F m_\star^{1/2} t^{1/2}, \frac{t}{\lambda_i^2} \right\} \]
which concludes the proof.

8.2.4 Proof of Lemma 18
Write
\[ \langle P_{UV}^1E_1C_{UV}, P_{UV}^1E_2C_{UV} \rangle = \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i \langle P_{UV}^1X_iC_{UV}, K \rangle \]
where
\[ K = \frac{1}{n} \sum_{i=n+1}^{2n} \left( \langle \Delta, X_i \rangle P_{UV}^1X_iC_{UV} - P_{UV}^1C_{UV} \right). \]
Conditional on \( \{X_i\}_{i=n+1}^{2n} \), with probability at least \( 1 - e^{-t} \) for all \( t \geq 1 \),
\[ \left| \langle P_{UV}^1E_1C_{UV}, P_{UV}^1E_2C_{UV} \rangle \right| \leq C_2 \sigma_\xi \frac{t^{1/2}}{n} \left( \sum_{i=n+1}^{2n} \langle P_{UV}^1X_iC_{UV}, K \rangle \right)^{1/2}. \]
For each \( n+1 \leq i \leq 2n \),
\[ \langle P_{UV}^1X_iC_{UV}, K \rangle \]
\[ = \frac{1}{n} \sum_{j=n+1}^{2n} \left( \langle \Delta, X_j \rangle \langle P_{UV}^1X_jC_{UV}, P_{UV}^1X_iC_{UV} \rangle - \langle P_{UV}^1\Delta C_{UV}, P_{UV}^1X_iC_{UV} \rangle \right) \]
\[ = \frac{1}{n} \left( \langle \Delta, X_i \rangle \|P_{UV}^1X_iC_{UV}\|_F^2 - \langle P_{UV}^1\Delta C_{UV}, P_{UV}^1X_iC_{UV} \rangle \right) \]
\[ + \frac{1}{n} \sum_{j \neq i} \left( \langle \Delta, X_j \rangle \langle P_{UV}^1X_jC_{UV}, P_{UV}^1X_iC_{UV} \rangle - \langle P_{UV}^1\Delta C_{UV}, P_{UV}^1X_iC_{UV} \rangle \right). \]
Conditioned on \( X_i \), we apply the concentration inequality to sum of sub-exponential random variables ([26]) and obtain, with probability at least \( 1 - e^{-t} \),
\[ \left| \frac{1}{n} \sum_{j \neq i} \left( \langle \Delta, X_j \rangle \langle P_{UV}^1X_jC_{UV}, P_{UV}^1X_iC_{UV} \rangle - \langle P_{UV}^1\Delta C_{UV}, P_{UV}^1X_iC_{UV} \rangle \right) \right| \]
\[ \leq C_1 \|\Delta\|_F \|P_{UV}^1X_iC_{UV}\|_F^2 \frac{t}{n^{1/2}}, \]
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Therefore, with probability at least $1 - e^{-t},$

$$\| \langle \mathcal{P}^\perp_{UV} X, C_{UV}, K \rangle \| \leq C_1 \| \Delta \|_F \| \mathcal{P}^\perp_{UV} X, C_{UV}^2 \|_F \frac{t}{n^{1/2}} + \frac{\| \Delta \|_F \| \mathcal{P}^\perp_{UV} X, C_{UV}^2 \|_F}{n}$$

and

implying that with probability at least $1 - ne^{-t},$

$$\sum_{i=n+1}^{2n} \| \langle \mathcal{P}^\perp_{UV} X, C_{UV}, K \rangle \|^2 \leq C_1 \| \Delta \|_F^2 \frac{t^2}{n} \sum_{i=n+1}^{2n} \| \mathcal{P}^\perp_{UV} X, C_{UV}^2 \|_F^2$$

$$+ C_2 \| \Delta \|_F^2 \sum_{i=n+1}^{2n} \| \mathcal{P}^\perp_{UV} X, C_{UV}^2 \|_F^2 + C_2 \frac{1}{n^2} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle^2 \| \mathcal{P}^\perp_{UV} X, C_{UV} \|_F^2.$$
By Berry-Esseen theorem, for any

where

and

By Lemma [15] Lemma [17] and Lemma [18] and setting \( t = 2\log n \), with probability at least \( 1 - 2e^{-n} - ne^{-c_1 m_*} - \frac{2n+6}{n^2} \),

\[
\| \hat{T}_1 \| \leq C_1 \frac{\sigma\xi}{\| \Lambda^{-2} \|_F \lambda^3_r} \cdot \frac{r^2\bar{m}^{1/2} \log^{1/2} n}{n^{1/2}} + C_2 \frac{\| \Lambda^{-1} \|_F^2}{\| \Lambda^{-2} \|_F} \cdot \frac{r\bar{m} \log^{1/2} n}{n} + C_3 \frac{r^{1/2} \bar{m}^{1/2} \log^{3/2} n}{n^{1/2}}
\leq C_1 \frac{\sigma\xi}{\| \Lambda^{-2} \|_F \lambda^3_r} \cdot \frac{r^2\bar{m}^{1/2} \log^{1/2} n}{n^{1/2}} + C_2 \frac{\| \Lambda^{-1} \|_F^2}{\| \Lambda^{-2} \|_F} \cdot \frac{r\bar{m} \log^{1/2} n}{n} + C_3 \frac{r^{1/2} \bar{m}^{1/2} \log^{3/2} n}{n^{1/2}}
\]

where we used the fact \( \| \Lambda^{-2} \|_F \geq \lambda_r^{-2} \) and \( \frac{\| \Lambda^{-1} \|_F^2}{\| \Lambda^{-2} \|_F} \leq r^{1/2} \) and \( n \geq C_1 r^2 \bar{m} \). By Lemma [16],

\[
\hat{T}_0 = \frac{d}{\bar{t}_0} \sum_{k=1}^r \frac{\lambda_k^{-2} \sum_{j_k=1}^{m_*} (z_{k,j_k}^2 - 1)}{\sqrt{2m_*^{1/2}} \| \Lambda^{-2} \|_F} + \frac{\sum_{k=1}^r \lambda_k^{-2} \sum_{j_k=1}^{m_*} z_{k,j_k}^2}{\sqrt{2\sigma\xi^2} \| \Lambda^{-2} \|_F \cdot \frac{m_*^{1/2}}{n}}
\]

where \( \{ z_{k,j_k} \}_{k \in [r]} \) are i.i.d. standard normal random variables. Observe that \( \{ \xi_i \} \) are independent with \( \{ z_{k,j_k} \} \). With probability at least \( 1 - \frac{1}{n} - re^{-m_*} \),

\[
| \hat{T}_0 | \leq C_1 \frac{\| \Lambda^{-1} \|_F^2}{\| \Lambda^{-2} \|_F} \cdot \frac{\bar{m}^{1/2} \log n}{n^{1/2}} \leq C_1 \cdot \frac{r^{1/2} \bar{m}^{1/2} \log n}{n^{1/2}}
\]

By Berry-Esseen theorem, for any \( x \in \mathbb{R} \),

\[
| \mathbb{P} \{ \hat{T}_{00} \leq x \} - \Phi(x) | \leq \frac{\| \Lambda^{-3} \|_F^2}{\| \Lambda^{-2} \|_F^{3/2}} \cdot \frac{C_2}{\bar{m}^{1/2}} \leq \frac{C_2}{\bar{m}^{1/2}}
\]

where we used the facts \( \| \Lambda^{-3} \|_F^2 \leq \| \Lambda^{-2} \|_F^3 \) and

\[
\mathbb{E} \sum_{k=1}^r \lambda_k^{-4} \sum_{j_k=1}^{m_*} (z_{k,j_k}^2 - 1)^2 = 2m_* \| \Lambda^{-2} \|_F^2
\]

and

\[
\mathbb{E} \sum_{k=1}^r \lambda_k^{-6} \sum_{j_k=1}^{m_*} (z_{k,j_k}^2 - 1)^3 \leq C_1 m_* \| \Lambda^{-3} \|_F^2.
\]

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Now, recall that \( \hat{T} = \hat{T}_{00} + \hat{T}_{01} + \hat{T}_1 \). Then,

\[
\mathbb{P}(\hat{T} \leq x) \leq \mathbb{P}(\hat{T}_{00} \leq x + C_1 \frac{\sigma_\xi}{\lambda_r} \cdot \frac{r^2 \bar{m}^{1/2} \log^{1/2} n}{n^{1/2}} + C_2 \frac{r^{1/2} \bar{m}^{1/2} \log^{3/2} n}{n^{1/2}})
+ 2e^{-n} + c_1(n + r)e^{-c_2m_\ast} + \frac{3n + 6}{n^2}
\leq \Phi\left(x + C_1 \frac{\sigma_\xi}{\lambda_r} \cdot \frac{r^2 \bar{m}^{1/2} \log^{1/2} n}{n^{1/2}} + C_2 \frac{r^{1/2} \bar{m}^{1/2} \log^{3/2} n}{n^{1/2}}\right)
+ 2e^{-n} + c_1(n + r)e^{-c_2m_\ast} + \frac{3n + 6}{n^2} + C_3 \frac{1}{m^{1/2}}
\leq \Phi(x) + C_1 \frac{\sigma_\xi}{\lambda_r} \cdot \frac{r^2 \bar{m}^{1/2} \log^{1/2} n}{n^{1/2}} + C_2 \frac{r^{1/2} \bar{m}^{1/2} \log^{3/2} n}{n^{1/2}}
+ 2e^{-n} + c_1(n + r)e^{-c_2m_\ast} + \frac{3n + 6}{n^2} + C_3 \frac{1}{m^{1/2}}
\]

where the last inequality is due to the Lipschitz property of function \( \Phi(\cdot) \). Similarly,

\[
\mathbb{P}(\hat{T} \leq x) \geq \mathbb{P}(\hat{T}_{00} \leq x - C_1 \frac{\sigma_\xi}{\lambda_r} \cdot \frac{r^2 \bar{m}^{1/2} \log^{1/2} n}{n^{1/2}} - C_2 \frac{r^{1/2} \bar{m}^{1/2} \log^{3/2} n}{n^{1/2}})
- 2e^{-n} - c_1(n + r)e^{-c_2m_\ast} - \frac{3n + 6}{n^2}
\geq \Phi\left(x - C_1 \frac{\sigma_\xi}{\lambda_r} \cdot \frac{r^2 \bar{m}^{1/2} \log^{1/2} n}{n^{1/2}} - C_2 \frac{r^{1/2} \bar{m}^{1/2} \log^{3/2} n}{n^{1/2}}\right)
- 2e^{-n} - c_1(n + r)e^{-c_2m_\ast} - \frac{3n + 6}{n^2} - C_3 \frac{1}{m^{1/2}}
\geq \Phi(x) - C_1 \frac{\sigma_\xi}{\lambda_r} \cdot \frac{r^2 \bar{m}^{1/2} \log^{1/2} n}{n^{1/2}} - C_2 \frac{r^{1/2} \bar{m}^{1/2} \log^{3/2} n}{n^{1/2}}
- 2e^{-n} - c_1(n + r)e^{-c_2m_\ast} - \frac{3n + 6}{n^2} - C_3 \frac{1}{m^{1/2}}.
\]

Combine the above two inequalities, we obtain the claimed bound.

### 8.2.6 Proof of Lemma II

Recall that \( \hat{N} = N + E \). It is straightforward to check that on event \( \mathcal{E}_1 := \{ \lambda_r \geq 3\|E\| \} \) with \( \mathbb{P}(\mathcal{E}_1) \geq 1 - e^{-n} - e^{-c_1m} \),

\[
2\hat{B}_n = \text{tr} \left( P_{UV}^\top \cdot \left( \int_{\gamma_N} + \int_{\gamma_N}^\circ \right) - \frac{1}{2\pi i} (\hat{N} - \eta I)^{-1} \frac{d\eta}{\eta^2} \right).
\]

Indeed, by Cauchy integral formula, we immediately obtain

\[
\left( \int_{\gamma_N} + \int_{\gamma_N}^\circ \right) - \frac{1}{2\pi i} (\hat{N} - \eta I)^{-1} \frac{d\eta}{\eta^2} = \sum_{1 \leq |k| \leq r} \hat{\lambda}_k^{-2} \hat{\theta}_k \hat{\theta}_k^\top = \begin{pmatrix} \hat{U} \hat{\Lambda}^{-2} \hat{U}^\top & 0 \\ 0 & \hat{V} \hat{\Lambda}^{-2} \hat{V}^\top \end{pmatrix}.
\]

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By Riesz formula, write
\[
\left( \oint_{\gamma_N^+} + \oint_{\gamma_N^-} \right) - \frac{1}{2\pi i} \left( \int_{\gamma_N^+} + \int_{\gamma_N^-} \right) \frac{d\eta}{\eta^2}
\]
\[
= -\frac{1}{2\pi i} \left( \int_{\gamma_N^+} + \int_{\gamma_N^-} \right) \mathcal{R}_N(\eta) \frac{d\eta}{\eta^2}
\]
\[
+ \frac{1}{2\pi i} \left( \int_{\gamma_N^+} + \int_{\gamma_N^-} \right) \mathcal{R}_N(\eta) E R_N(\eta) \frac{d\eta}{\eta^2}
\]
\[
+ \frac{1}{2\pi i} \left( \int_{\gamma_N^+} + \int_{\gamma_N^-} \right) \sum_{j \geq 2} (-1)^j [\mathcal{R}_N(\eta)]^j \mathcal{R}_N(\eta) \frac{d\eta}{\eta^2}.
\]

Therefore,
\[
2 \hat{B}_n = \text{tr} \left( \mathcal{P}^{\perp}_{UV} \cdot (T_0 + T_1 + T_2) \right).
\]

**Lemma 22.** Under the conditions of Lemma [17] the following facts are true:
\[
T_0 = C_{UV}^2 = \begin{pmatrix} U^\perp A^{-2} U^\top & 0 \\ 0 & V^\perp A^{-2} V^\top \end{pmatrix}
\]

and
\[
T_1 = \mathcal{P}^\perp_{UV} E C_{UV}^3 + C_{UV}^3 E \mathcal{P}^\perp_{UV}
\]

where $C_{UV}^3$ can be explicitly expressed as
\[
C_{UV}^3 = \begin{pmatrix} 0 & U A^{-3} V^\top \\ V A^{-3} U^\top & 0 \end{pmatrix}.
\]

Recall that $\mathcal{P}_{UV} = \mathcal{P}_{UV} + \mathcal{L}_N(E) + \mathcal{S}_N(E)$. Then,
\[
\text{tr} \left( \mathcal{P}^\top_{UV} T_0 \right) = 2 \| A^{-1} \|_F^2 + \langle \mathcal{S}_N(E), T_0 \rangle,
\]

where we used the fact $\text{tr} \left( \mathcal{L}_N(E)^\top T_0 \right) = 0$. Similarly, we write
\[
\text{tr} \left( \mathcal{P}^\top_{UV} T_1 \right) = \langle \mathcal{L}_N(E) + \mathcal{S}_N(E), T_1 \rangle.
\]

Therefore, together with Lemma [3],
\[
\left| 2 \hat{B}_n - 2 \| A^{-1} \|_F^2 \right| \leq \left| \langle \mathcal{S}_N(E), T_0 \rangle \right| + \left| \langle \mathcal{L}_N(E) + \mathcal{S}_N(E), T_1 \rangle \right|
\]
\[
+ \left| \langle \mathcal{P}_{UV}, T_2 \rangle \right|
\]
\[
\leq \| S_N(E) \| \cdot 2 \| A^{-1} \|_F^2 + \| L_N(E) + S_N(E) \| \cdot (4r)^{1/2} \| T_1 \|_F
\]
\[
+ \| T_2 \| \cdot 2r
\]
\[
\leq C_1 r \| A^{-1} \|_F^2 \cdot \frac{\| E \|_2^2}{\lambda^2} + C_2 r^{1/2} \| A^{-3} \|_F \cdot \frac{\| E \|_2^2}{\lambda r} + C_3 r^{3/2} \| A^{-3} \|_F \cdot \frac{\| E \|_3^2}{\lambda^2}
\]
\[
+ C_4 r^2 \cdot \frac{\| E \|_4^2}{\lambda^4}.
\]

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By Lemma 2 and the fact $\|A^{-3}\| \leq \lambda_r^{-1}\|A^{-2}\|_F \leq \lambda_r^{-1}\|A^{-1}\|^2_F$, with probability at least $1 - e^{-t} - 2 e^{-n} - c_1 e^{-c_2 m}$,

$$
|2\hat{B}_n - 2\|A^{-1}\|_F^2| \leq C_1\|A^{-1}\|_F^2 \left( \frac{\sigma_x}{\lambda_r} \right)^2 \cdot \frac{r(m + t)}{n} + C_2\lambda_r^{-2} \left( \frac{\sigma_x}{\lambda_r} \right)^2 \cdot \frac{r^2(m + t)}{n}
$$

$$
+ C_3\|A^{-1}\|_F^3 \left( \frac{\sigma_x}{\lambda_r} \right)^3 \cdot \frac{r^{3/2}(m^3/2 + t^{3/2})}{n^{3/2}}.
$$

Choose $t = \tilde{m}$ and observe $\lambda_r^{-2} \leq \|A^{-1}\|_F^2$ and $n \geq r\tilde{m}$, we conclude that with probability at least $1 - 2 e^{-n} - c_1 e^{-c_2 m}$,

$$
|\hat{B}_n - \|A^{-1}\|_F^2| \leq C_1\|A^{-1}\|_F^2 \left( \left( \frac{\sigma_x}{\lambda_r} \right)^2 + \left( \frac{\sigma_x}{\lambda_r} \right)^3 \right) \cdot \frac{r^2\tilde{m}}{n}
$$

which proves the first claim. To prove the second claim, we write

$$
2\hat{V}_n = \text{tr} \left( P^\top_{UV} \cdot \left( \oint_{\gamma_N} + \oint_{\gamma_N} \right) - \frac{1}{2\pi i} (\tilde{N} - \eta I)^{-1} \frac{d\eta}{\eta^4} \right)
$$

where we further write

$$
\left( \oint_{\gamma_N} + \oint_{\gamma_N} \right) = \frac{1}{2\pi i} (\tilde{N} - \eta I)^{-1} \frac{d\eta}{\eta^4}
$$

$$
= -\frac{1}{2\pi i} \left( \oint_{\gamma_N} + \oint_{\gamma_N} \right) R_N(\eta) \frac{d\eta}{\eta^4}
$$

$$
+ \frac{1}{2\pi i} \left( \oint_{\gamma_N} + \oint_{\gamma_N} \right) R_N(\eta) E R_N(\eta) \frac{d\eta}{\eta^4}
$$

$$
+ -\frac{1}{2\pi i} \left( \oint_{\gamma_N} + \oint_{\gamma_N} \right) \sum_{j \geq 2} (-1)^j [R_N(\eta) E]^j R_N(\eta) \frac{d\eta}{\eta^4}.
$$

Lemma 23. Under the conditions of Lemma [77] the following facts are true:

$$
R_0 = C_{UV}^1 = \begin{pmatrix} U A^{-4} U^\top & 0 \\ 0 & V A^{-4} V^\top \end{pmatrix}
$$

and

$$
R_1 = P_{UV}^1 E C_{UV}^5 + C_{UV}^5 E P_{UV}
$$

where $C_{UV}^5$ can be explicitly expressed as

$$
C_{UV}^5 = \begin{pmatrix} 0 & U A^{-5} V^\top \\ V A^{-5} U^\top & 0 \end{pmatrix}.
$$

Following an identical proof for $\hat{B}_n$ and by the fact $\|A^{-5}\|_F \leq \lambda_r^{-1}\|A^{-4}\|_F \leq \lambda_r^{-1}\|A^{-2}\|^2_F$, we obtain that with probability at least $1 - e^{-t} - e^{-n}$,

$$
|2\hat{V}_n - 2\|A^{-2}\|_F^2| \leq C_1\|A^{-2}\|_F^2 \left( \left( \frac{\sigma_x}{\lambda_r} \right)^2 + \left( \frac{\sigma_x}{\lambda_r} \right)^3 \right) \cdot \frac{r^2(m + t)}{n}
$$

$$
+ C_2\|A^{-2}\|_F^3 \left( \frac{\sigma_x}{\lambda_r} \right)^3 \cdot \frac{r^{3/2}(m^3/2 + t^{3/2})}{n^{3/2}} + C_3\lambda_r^{-4} \left( \frac{\sigma_x}{\lambda_r} \right)^2 \cdot \frac{r^2(m + t)}{n}.
$$
Choose $t = \tilde{m}$ and observe that $\lambda_r^{-4} \leq \|A^{-2}\|_F^2$ and $n \geq r \tilde{m}$, we conclude that with probability at least $1 - 2e^{-n} - c_1 e^{-c_2 \tilde{m}}$,

$$|\hat{V}_n - \|A^{-2}\|_F^2| \leq C_1 \|A^{-2}\|_F^2 \left[ \left( \frac{\sigma_\xi}{\lambda_r} \right)^2 + \left( \frac{\sigma_\xi}{\lambda_r} \right)^3 \right] \cdot \frac{r^2 \tilde{m}}{n}$$

which concludes the proof.

### 8.2.7 Proof of Theorem 12

By definition of $\hat{T}_{UV}$, we write

$$\hat{T}_{UV} := \frac{\|P_{UV} - P_{UV}\|_F^2 - \sigma_\xi^2 \|A\|^{-1}_F^2 \cdot \frac{2m_\star}{n}}{\sqrt{8} \sigma_\xi^2 \|A^{-2}\|_F \cdot \frac{m_\star^{1/2}}{n}} + \frac{\sigma_\xi^2 (\|A^{-1}\|_F^2 - \hat{B}_n) \cdot \frac{2m_\star}{n}}{\sqrt{8V_n^{-1/2} \sigma_\xi^2 \cdot \frac{m_\star^{1/2}}{n}}} \leq \gamma_1$$

$$+ \frac{\|P_{UV} - P_{UV}\|_F^2 - \sigma_\xi^2 \|A\|^{-1}_F^2 \cdot \frac{2m_\star}{n}}{\sqrt{8} \sigma_\xi^2 \|A^{-2}\|_F \cdot \frac{m_\star^{1/2}}{n}} \cdot \left[ \frac{\|A^{-2}\|_F}{\hat{V}_n} - 1 \right].$$

By Lemma 11 with probability at least $1 - 2e^{-n} - c_1 e^{-c_2 m}$, $\hat{V}_n \geq \frac{\|A^{-2}\|_F}{2}$ as long as $n \geq C_1 (\beta^2 + \beta^3) r^2 \tilde{m} \log n$ for large enough $C_1 > 0$. Therefore, by Lemma 11 with the same probability,

$$|\Xi_1| \leq C_6 \left[ \frac{\|A^{-1}\|_F^2}{\|A^{-2}\|_F} \left( \frac{\sigma_\xi}{\lambda_r} \right)^2 + \left( \frac{\sigma_\xi}{\lambda_r} \right)^3 \right] \cdot \frac{r^2 \tilde{m}^3/2}{n}$$

$$\leq C_6 \left[ \left( \frac{\sigma_\xi}{\lambda_r} \right)^2 + \left( \frac{\sigma_\xi}{\lambda_r} \right)^3 \right] \cdot \frac{r^5/2 \tilde{m}^3/2}{n},$$

where we used the fact $\|A^{-1}\|_F^2 \leq r^{1/2} \|A^{-2}\|_F$. By Lemma 11 with the same probability,

$$\left| \frac{\|A^{-2}\|_F}{\hat{V}_n} - 1 \right] \leq C_6 \left[ \left( \frac{\sigma_\xi}{\lambda_r} \right)^2 + \left( \frac{\sigma_\xi}{\lambda_r} \right)^3/2 \right] \cdot \frac{r \tilde{m}^{1/2}}{n^{1/2}}.$$

By Theorem 5 and Corollary 6 with probability at least $1 - \frac{2n+10}{n} - 3e^{-n} - n e^{-m_\star} - c_1 e^{-c_2 \tilde{m}}$,

$$\left| \|P_{UV} - P_{UV}\|_F^2 - \sigma_\xi^2 \|A\|^{-1}_F^2 \cdot \frac{2m_\star}{n} \right| \leq C_1 \frac{\sigma_\xi}{\|A^{-2}\|_F^{1/3} \lambda_r^3} \left( \frac{r^2 \tilde{m}^{1/2} \log^{1/2} n}{n^{1/2}} + \frac{r^5/2 \tilde{m}^{3/2} \log^{1/2} n}{n} \right)$$

$$+ C_2 \log^{1/2} n + C_3 \frac{r^{3/2} \tilde{m}^{3/2} + r^{3/2} \tilde{m} \log n}{n} + C_4 \left( \frac{\sigma_\xi}{\lambda_r} \right)^2 \cdot \frac{r^3 \tilde{m}^{3/2}}{n}.$$ 

Therefore, if $n \geq C_1 r^{3/2} \tilde{m}^{3/2}$ and $n \leq e^{\tilde{m}}$, with probability at least $1 - \frac{2n+11}{n} - 4e^{-n} - n e^{-m_\star} - c_1 e^{-c_2 \tilde{m}}$,

$$|\Xi_2| \leq C_7 (\beta + \beta^{3/2}) \cdot \frac{r \tilde{m}^{1/2} \log^{1/2} n}{n^{1/2}} + C_8 (\beta^2 + \beta^{5/2}) \cdot \frac{r^3 \tilde{m} \log^{1/2} n}{n}$$

$$+ C_9 (\beta^2 + \beta^{5/2} + \beta^3 + \beta^{7/2}) \cdot \frac{r^4 \tilde{m}^{3/2} \log^{1/2} n}{n^{3/2}}.$$
Together with Corollary 9 we obtain
\[
\sup_x \left| \mathbb{P}\{T_{UV} \leq x\} - \Phi(x) \right| 
\leq C_7 (\beta + \beta^3). \frac{r^\beta \log^{1/2} n}{n^{1/2}} + C_8 (\beta + \beta^2 + \beta^5/2 + \beta^3) \cdot \frac{r^5 \bar{m}^{3/2} \log^{1/2} n}{n} + C_9 \cdot \frac{r^3 \bar{m}^{3/2} + C_{10} (\beta^2 + \beta^{5/2} + \beta^3) \cdot \frac{r^4 \bar{m}^2 \log^{1/2} n}{n^{3/2}} + 6e^{-n} + (2n + r)e^{-mr}}{n^2} + \frac{5n + 17}{n^2} + c_1 e^{-c_2 \bar{m}}
\]
for absolute constants \( c_1, c_2, C_7, C_8, C_9, C_{10} > 0 \).

9 Proof of additional lemmas

The following lemma will be frequently used through our proof. Basically, the Gaussian isoperimetric inequality can provide us with tight concentration bound for Lipschitz functions.

Lemma 24. Let \( X_1, \ldots, X_n \in \mathbb{R}^m \) be i.i.d. centered Gaussian random vector with \( \Sigma = \Sigma XX^\top \). Let \( h(\cdot) \) be a function \( \mathbb{R}^{nm} \mapsto \mathbb{C} \) satisfying the following Lipschitz condition with some constant \( L > 0 \):
\[
|h(\{X_i\}_{i=1}^n) - h(\{X'_i\}_{i=1}^n)| \leq L \left( \sum_{i=1}^n \|X_i - X'_i\|^2 \right)^{1/2},
\]
\( \forall X_1, \ldots, X_n, X_1', \ldots, X_n' \in \mathbb{R}^m. \)

Then, there exists some constant \( C_1 > 0 \) such that for all \( t \geq 1 \),
\[
\mathbb{P} \left\{ \left| h(\{X_i\}_{i=1}^n) - \mathbb{E}f(\{X_i\}_{i=1}^n) \right| \geq C_1 L \|\Sigma\| t^{1/2} \right\} \leq e^{-t}.
\]

9.0.1 Proof of Lemma 2

Recall that \( \mathbf{E}_1 = \mathcal{D}(\mathbf{Z}_1) \) with \( \mathbf{Z}_1 = \frac{1}{n} \sum_{i=n+1}^{2n} \xi X_i \). Therefore, \( \|\mathbf{E}_1\| = \|\mathbf{Z}_1\| \). Meanwhile, conditional on \( \{\xi_i\}_{i=n+1}^{2n} \),
\[
\mathbf{Z}_1 \overset{d}{=} \mathbf{X} \cdot \sqrt{\frac{\sum_{i=n+1}^{2n} \xi_i^2}{n}}
\]
where \( \mathbf{X} \) has i.i.d. standard Gaussian entries. By [1],
\[
\mathbb{E}_{\mathbf{X}} \|\mathbf{Z}_1\| \leq C_1 \sqrt{\frac{\sum_{i=n+1}^{2n} \xi_i^2}{n}} \bar{m}^{1/2}.
\]

By Jensen’s inequality,
\[
\mathbb{E} \|\mathbf{E}_1\| = \mathbb{E}_{\xi} \mathbb{E}_{\mathbf{X}} \|\mathbf{Z}_1\| \leq C_1 \mathbb{E}_{\xi} \sqrt{\frac{\sum_{i=n+1}^{2n} \xi_i^2}{n}} \bar{m}^{1/2}
\]
\[
\leq C_1 \frac{\bar{m}^{1/2}}{n} \left( \mathbb{E} \sum_{i=n+1}^{2n} \xi_i^2 \right)^{1/2} = C_1 \sigma \xi \frac{\bar{m}^{1/2}}{n^{1/2}}.
\]
Conditional on \( \{\xi_i\}_{i=n+1}^{2n} \), we view \( \|Z_1\| \) as a function of \( \{X_i\}_{i=n+1}^{2n} \), i.e.,

\[
h(\{X_i\}_{i=n+1}^{2n}) = \|Z_1\| = \left\| \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i X_i \right\|.
\]

Clearly,

\[
\left| h(\{X_i\}_{i=n+1}^{2n}) - h(\{X'_i\}_{i=n+1}^{2n}) \right| \leq \left\| \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i (X_i - X'_i) \right\| \\
\leq \frac{1}{n} \left( \sum_{i=n+1}^{2n} \xi_i^2 \right)^{1/2} \left( \sum_{i=n+1}^{2n} \|X_i - X'_i\|^2 \right)^{1/2}
\]

implying that \( h(\cdot) \) is Lipschitz with constant \( n^{-1} \left( \sum_{i=n+1}^{2n} \xi_i^2 \right)^{1/2} \). By Lemma 24, with probability at least \( 1 - e^{-t} \) with \( t \geq 1 \),

\[
\left\| \|E_1\| - \mathbb{E}_X \|E_1\| \right\| \leq C_1 \sigma_\xi \left( \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i^2 \right)^{1/2} t^{1/2}
\]

Since \( \{\xi_i\}_i \) are i.i.d. Gaussian random variables, \( \mathbb{P} \left( \sum_{i=n+1}^{2n} \xi_i^2 \leq c_1 n \sigma_\xi^2 \right) \geq 1 - e^{-n} \) for some absolute constant \( c_1 > 0 \). We conclude that with probability at least \( 1 - e^{-t} - e^{-n} \),

\[
\left\| \|E_i\| - \mathbb{E}_X \|E_i\| \right\| \leq C_2 \sigma_\xi \frac{t^{1/2}}{\sqrt{n}}.
\] (9.1)

Now, we view \( \mathbb{E}_X \|E_1\| \) as a function \( \{\xi_i\}_i \), i.e.,

\[
h_1(\{\xi_i\}_{i=n+1}^{2n}) = \mathbb{E}_X \left\| \left( \frac{1}{n} \sum_{i=n+1}^{2n} \xi_i^2 \right)^{1/2} X \right\|.
\]

Then, by denoting \( \xi = (\xi_{n+1}, \cdots, \xi_{2n})^\top \in \mathbb{R}^n \),

\[
\left| h_1(\{\xi_i\}_{i=n+1}^{2n}) - h_1(\{\xi'_i\}_{i=n+1}^{2n}) \right| \leq \|\xi - \xi'\|_{\ell_2} \cdot \frac{\mathbb{E}_X \|X\|}{n} \leq C_1 \frac{\sqrt{m t^{1/2}}}{n} \cdot \|\xi - \xi'\|_{\ell_2}.
\]

By applying Lemma 24, with probability at least \( 1 - e^{-t} \),

\[
\left| \mathbb{E} \|E_1\| - \mathbb{E}_X \|E_1\| \right| \leq C_1 \sigma_\xi \cdot \frac{\sqrt{m t^{1/2} t^{1/2}}}{n}.
\] (9.2)

By (9.1) and (9.2), we conclude that with probability at least \( 1 - 2e^{-t} - e^{-n} \),

\[
\left\| \|E_1\| - \mathbb{E} \|E_1\| \right\| \leq C_1 \sigma_\xi \cdot \left[ \frac{t^{1/2}}{n^{1/2}} + \frac{\sqrt{m t^{1/2}}}{n} \right].
\]

We turn to the proof of \( \mathbb{E} \|E_2\| \). Recall that \( \|E_2\| = \|Z_2\| \) where \( Z_2 = n^{-1} \sum_{i=n+1}^{2n} (\Delta, X_i) X_i - \Delta \). We apply the unbounded version of matrix Bernstein inequality, [14]. Indeed,

\[
\|Z_2\|_{\psi_1} \leq \|\Delta\|_{\psi_2} \cdot \|X_i\|_{\psi_2} \leq \|\Delta\|_{\psi_2} \|X_i\|_{\psi_2} \leq \|\Delta\|_{\psi_2} \sqrt{m t^{1/2}}
\]
where the Orlicz $\varphi_\alpha$-norm, for $\alpha \in [1, 2]$, of a random variable $X$ is defined as
\[ \|X\|_{\varphi_\alpha} := \inf \{ u > 0 : \mathbb{E} \exp(|X|^\alpha/u^\alpha) \leq 2 \} . \]

By matrix Bernstein inequality [14], with probability at least $1 - e^{-t}$ for $t \geq 0$,
\[ \|Z_2\| \leq C_1\|\Delta\|_F \sqrt{\frac{\bar{m}(t + \log \bar{m})}{n}} + C_2\|\Delta\|_F \frac{\bar{m}^{1/2}(t + \log \bar{m})}{n}. \]

By integrating over $t$, as long as $n \geq \log \bar{m}$, we end up with
\[ \mathbb{E}\|E_2\| = \mathbb{E}\|Z_2\| \leq C_1\|\Delta\|_F \frac{m^{1/2}\log^{1/2} \bar{m}}{n^{1/2}}. \]

Denote by $\text{vec}(\Delta)$ the vectorization of $\Delta$ and $\mathcal{M}(\mathbf{v})$ the matricization of a vector $\mathbf{v} \in \mathbb{R}^{m_1m_2}$ such that $\mathcal{M}((\text{vec}(\Delta))) = \Delta$. Write $Z_2 = Z_{21} + Z_{22}$ such that
\[ Z_{21} := \frac{1}{n} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle \mathcal{M}(P_{\text{vec}(\Delta)} \text{vec}(X_i)) - \Delta \]
\[ Z_{22} := \frac{1}{n} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle \mathcal{M}(P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i)) \]

where $P_{\mathbf{v}}$ denotes the orthogonal projection onto $\mathbf{v}$, i.e., $P_{\mathbf{v}}(\mathbf{u}) = \mathbf{v} \cdot (\mathbf{v}^\top \mathbf{u})$. In particular,
\[ P_{\text{vec}(\Delta)} \text{vec}(X_i)) = \text{vec}(\Delta) \cdot \frac{\langle \Delta, X_i \rangle}{\|\Delta\|_F^2}. \]

Moreover, since $\langle \Delta, X_i \rangle$ and $\mathcal{M}(P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i))$ both have Gaussian distribution, we claim that $\langle \Delta, X_i \rangle$ is independent with $\mathcal{M}(P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i))$. Indeed, it is straightforward to verify that they are uncorrelated.

We view $\|Z_{22}\|$ as a function of $\{ \mathcal{M}(P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i)) \}_{i=n+1}^{2n}$, conditional on $\{ \langle \Delta, X_i \rangle \}_{i=n+1}^{2n}$, i.e.,
\[ h_2(\{ \mathcal{M}(P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i)) \}_{i=n+1}^{2n}) = \left\| \frac{1}{n} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle \mathcal{M}(P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i)) \right\|. \]

Observe that $h(\cdot)$ is a Lipschitz function with constant $n^{-1} \left( \sum_{i=1}^{n} \langle \Delta, X_i \rangle^2 \right)^{1/2}$. By Lemma 24, with probability at least $1 - e^{-t}$, for all $t \geq 1$,
\[ \|Z_{22}\| - \mathbb{E}_{P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i))} \|Z_{22}\| \leq C_1 t^{1/2} \frac{\left( \sum_{i=1}^{n} \langle \Delta, X_i \rangle \right)^{1/2}}{n} \]

and with probability at least $1 - e^{-t} - e^{-n}$,
\[ \|Z_{22}\| - \mathbb{E}_{P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i))} \|Z_{22}\| \leq C_1 \|\Delta\|_F \cdot \frac{t^{1/2}}{n^{1/2}}. \tag{9.3} \]

Following the same fashion, we now view $\mathbb{E}_{P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i))} \|Z_{22}\|$ as a function of $\{ \langle \Delta, X_i \rangle \}_{i=n+1}^{2n}$, i.e.,
\[ h_3(\{ \langle \Delta, X_i \rangle \}_{i=n+1}^{2n}) = \mathbb{E}_{P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i))} \left\| \frac{1}{n} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle \mathcal{M}(P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i)) \right\| \]
\[ \overset{d}{=} \mathbb{E}_{P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i))} \left\| \frac{1}{n} \left( \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle \right)^{1/2} \mathcal{M}(P^\perp_{\text{vec}(\Delta)} \text{vec}(X_i)) \right\|. \]
where $X$ is an independent copy of $\{X_i\}_i$. Denote the vector $x_\Delta = \left( \langle \Delta, X_i \rangle \right)_{i=n+1}^{2n} \in \mathbb{R}^n$. Then,

$$
\begin{align*}
\left| h_3 \left( \{ (\Delta, X_i) \}_{i=n+1}^{2n} \right) - h_3 \left( \{ (\Delta, X'_i) \}_{i=n+1}^{2n} \right) \right| \\
\leq \| x_\Delta - x'_\Delta \|_2 \cdot \frac{1}{n} E_{\mathcal{P}_{vec(\Delta)vec(X)}} \left\| \mathcal{M} \left( \mathcal{P}_{vec(\Delta)vec(X)} \right) \right\|
\leq C_1 \| x_\Delta - x'_\Delta \|_2 \cdot \frac{\bar{m}^{1/2}}{n}.
\end{align*}
$$

Therefore, by Lemma 24, with probability at least $1 - e^{-t}$,

$$
\left| E \| Z_{22} \| - E \| (\mathcal{P}_{vec(\Delta)vec(X)}) Z_{22} \| \right| \leq C_1 \| \Delta \|_F \cdot \frac{\bar{m}^{1/2} t^{1/2}}{n}. \tag{9.4}
$$

By (9.3) and (9.4), we conclude that with probability at least $1 - 2e^{-t} - e^{-n}$,

$$
\left| \| Z_{22} \| - E \| Z_{22} \| \right| \leq C_1 \| \Delta \|_F \cdot \left[ \frac{t^{1/2}}{n^{1/2}} + \frac{\bar{m}^{1/2} t^{1/2}}{n} \right].
$$

By applying matrix Bernstein inequality to $\| Z_{21} \|$, we conclude that, with probability at least $1 - e^{-t}$,

$$
\| Z_{21} \| \leq C_1 \| \Delta \|_F \left[ \frac{t + \log \bar{m}}{n^{1/2}} \right]
$$

and thus

$$
E \| Z_{21} \| \leq C_1 \| \Delta \|_F \leq C_1 \| \Delta \|_F \cdot \frac{\log^{1/2} \bar{m}}{n^{1/2}}.
$$

Put the above three bounds together and adjusting constants, we obtain

$$
\left| \| E_2 \| - E \| E_2 \| \right| \leq C_1 \| \Delta \|_F \cdot \left[ \frac{t^{1/2} + \log^{1/2} \bar{m}}{n^{1/2}} + \frac{\bar{m}^{1/2} t^{1/2} + t + \log \bar{m}}{n} \right]
$$

with probability at least $1 - 3e^{-t} - e^{-n}$ for all $t \geq 1$.

### 9.0.2 Proof of Lemma 19

Recall that

$$
\varphi_{j,\delta}(E) = (-1)^j \operatorname{tr} \left( \left[ R_N(\eta) E \right] \mathcal{P}_{UV} \right) \phi \left( \frac{\| E \|}{\delta} \right)
$$

**Case 1** If $\| E \|, \| E' \| \geq \frac{9}{8} \delta$, then $\phi (\| E \| / \delta) = \phi (\| E' \| / \delta) = 0$. Therefore, the first claim bound trivially holds.
Case 2  If \( \|E\|, \|E'\| \leq \frac{9}{8} \cdot \delta \), then

\[
\left| \operatorname{tr} \left( [R_N(\eta)E]^j R_N(\eta)P_{UV} \right) \phi \left( \frac{\|E\|}{\delta} \right) \right| \leq \sum_{s=0}^{j-1} \left| \operatorname{tr} \left( [R_N(\eta)E']^s R_N(\eta)(E - E') \right) \phi \left( \frac{\|E\|}{\delta} \right) \right|
\]

\[
= \sum_{s=0}^{j-1} \operatorname{tr} \left( [R_N(\eta)E']^s R_N(\eta)(E - E') \right) \phi \left( \frac{\|E\|}{\delta} \right)
\]

\[
+ \|R_N(\eta)\|^{j+1} \|E'\| \|E\|^{j-1-s} \|E - E'\| (2r) \phi \left( \frac{\|E\|}{\delta} \right)
\]

\[
\leq j (2r) \left( \frac{2}{\lambda_r} \right)^{j+1} \left( \frac{9\delta}{8} \right)^{j-1} \|E - E'\|
\]

where we used the fact \( \|R_N(\eta)\| \leq \frac{9}{8}, \|P_{UV}\| \leq (2r) \) and the last inequality is due to the Lipschitz property of function \( \phi(\cdot) \). Therefore,

\[
|\varphi_{j,\delta,\eta}(E) - \varphi_{j,\delta,\eta}(E')| \leq 8r \cdot \left( \frac{j + 72}{\lambda_r^2} \right) \left( \frac{9\delta}{4\lambda_r} \right)^{j-1} \|E - E'\|
\]

which proves the first claim.

Case 3  If \( \|E\| \leq \frac{9}{8} \cdot \delta \) and \( \|E'\| \geq \frac{9}{8} \cdot \delta \), then \( \phi \left( \|E'\|/\delta \right) = 0 \). We write

\[
\left| \operatorname{tr} \left( [R_N(\eta)E]^j R_N(\eta)P_{UV} \right) \phi \left( \frac{\|E\|}{\delta} \right) \right| \leq \left| \operatorname{tr} \left( [R_N(\eta)E]^j R_N(\eta)P_{UV} \phi \left( \frac{\|E\|}{\delta} \right) \right| \right| \cdot \phi \left( \frac{\|E\|}{\delta} \right) - \phi \left( \frac{\|E'\|}{\delta} \right)
\]

\[
\leq 16r \left( \frac{2}{\lambda_r} \right)^{j+1} \left( \frac{9\delta}{8} \right)^j \|E - E'\|
\]

Therefore,

\[
|\varphi_{j,\delta,\eta}(E) - \varphi_{j,\delta,\eta}(E')| \leq 72r \left( \frac{9\delta}{\lambda_r^2} \right)^{j-1} \|E - E'\|
\]

which also proves the first claim.

Case 4  If \( \|E'\| \leq \frac{9}{8} \cdot \delta \) and \( \|E\| \geq \frac{9}{8} \cdot \delta \). The proof is identical to Case 3.
\textbf{Proof of second claim} By first claim,
\[
\left| \varphi_{\delta,\eta}(E) - \varphi_{\delta,\eta}(E') \right| \leq \sum_{j \geq 3} \left| \varphi_{\delta,\eta}(E) - \varphi_{\delta,\eta}(E') \right|
\]
\[
\leq \frac{8\gamma}{\lambda_r^2} \sum_{j \geq 3} (j + 8) \left( \frac{9\delta}{4\lambda_r} \right)^{j-1} \|E - E'\| \leq \frac{C_5 r^2}{\lambda_r^2} \left( \frac{9\delta}{4\lambda_r} \right)^2 \|E - E'\|
\]
for an absolute constant $C_5 > 0$.

\textbf{9.0.3 Proof of Lemma 20} 

By definition, we obtain
\[
\varphi_{\delta}(E) - \varphi_{\delta}(E') = -\frac{1}{2\pi i} \left( \oint_{\gamma^+_N} + \oint_{\gamma^-_N} \right) (\varphi_{\delta,\eta}(E) - \varphi_{\delta,\eta}(E')) d\eta
\]
implying that
\[
\left| \varphi_{\delta}(E) - \varphi_{\delta}(E') \right| \\
\leq \frac{1}{2\pi} \left| \left( \oint_{\gamma^+_N} + \oint_{\gamma^-_N} \right) (\varphi_{\delta,\eta}(E) - \varphi_{\delta,\eta}(E')) d\eta \right| \\
\leq \frac{1}{2\pi} \left( \oint_{\gamma^+_N} + \oint_{\gamma^-_N} \right) \left| \varphi_{\delta,\eta}(E) - \varphi_{\delta,\eta}(E') \right| d\eta.
\]

Apply Lemma 19 and $|\gamma_N^\pm| \leq r \pi \lambda_r$, we get
\[
\left| \varphi_{\delta}(E) - \varphi_{\delta}(E') \right| \\
\leq \frac{2r \pi \lambda_r}{2\pi} \frac{C_5 r^2}{\lambda_r^2} \left( \frac{9\delta}{4\lambda_r} \right)^2 \|E - E'\| \leq \frac{C_6 r^2}{\lambda_r^2} \left( \frac{9\delta}{4\lambda_r} \right)^2 \|E - E'\|
\]
which concludes the proof.

\textbf{9.0.4 Proof of Lemma 21} 

To this end, define
\[
A = \frac{1}{\eta} \sum_{i = n+1}^{2n} \left( \xi_i + \langle \Delta, X_i \rangle \right) \mathcal{V} \circ \mathcal{M} \left( P_{\text{vec}(\Delta)} \text{vec}(X_i) \right) - \Delta
\]
and
\[
A^\perp = \frac{1}{\eta} \sum_{i = n+1}^{2n} \left( \xi_i + \langle \Delta, X_i \rangle \right) \mathcal{V} \circ \mathcal{M} \left( P_{\text{vec}(\Delta)}^{\perp} \text{vec}(X_i) \right)
\]
such that $E = A + A^\perp$. Then, write
\[
\left| \mathbb{E}_{\left( P_{\text{vec}(\Delta)} X \right)} \left[ \varphi_{\delta}(A + A^\perp) \right] - \mathbb{E}_{\left( P_{\text{vec}(\Delta)} X \right)} \left[ \varphi_{\delta}(A + A^\perp) \right] \right| \\
\leq \left| \mathbb{E}_{\left( P_{\text{vec}(\Delta)} X \right)} \left[ \varphi_{\delta}(A + A^\perp) \right] - \mathbb{E}_{\left( P_{\text{vec}(\Delta)} X \right)} \left[ \varphi_{\delta}(A^\perp) \right] \right| \\
+ \left| \mathbb{E}_{\left( P_{\text{vec}(\Delta)} X \right)} \left[ \varphi_{\delta}(A^\perp) \right] - \mathbb{E}_{\left( P_{\text{vec}(\Delta)} X \right)} \left[ \varphi_{\delta}(A^\perp) \right] \right| \\
+ \left| \mathbb{E}_{\left( P_{\text{vec}(\Delta)} X \right)} \left[ \varphi_{\delta}(A^\perp) \right] - \mathbb{E}_{\left( P_{\text{vec}(\Delta)} X \right)} \left[ \varphi_{\delta}(A + A^\perp) \right] \right|.
\]
By Lemma [20]

\[
\| \mathbb{E}_{P_{\text{vec}(\Delta)}^{\perp} X} [\varphi_{\delta}(A + A^{\perp})] - \mathbb{E}_{P_{\text{vec}(\Delta)}^{\perp} X} [\varphi_{\delta}(A^{\perp})] \| \leq C_{\delta} \frac{\sigma_{\delta}^2}{\lambda_r} \left( \frac{9\delta}{4\lambda_r} \right)^2 \| A \|.
\]

Following the proof of upper bound of \( \|Z_{11}\| \) in the proof of Lemma [2] we conclude that with probability at least 1 – \( e^{-t} \),

\[
\| A \| \leq C_{\delta} \left( \| \Delta \|_{F} + \sigma_{\delta} \right) \left( \frac{(t + \log \tilde{m})^{1/2}}{n^{1/2}} + \frac{t + \log \tilde{m}}{n} \right)
\]

where the first term dominate if \( t \leq n \) and \( n \geq \log \tilde{m} \). Therefore, with probability at least 1 – \( e^{-t} \) for \( 1 \leq t \leq n \),

\[
\| \mathbb{E}_{P_{\text{vec}(\Delta)}^{\perp} X} [\varphi_{\delta}(A + A^{\perp})] - \mathbb{E}_{P_{\text{vec}(\Delta)}^{\perp} X} [\varphi_{\delta}(A^{\perp})] \| \leq C_{\delta} \frac{\sigma_{\delta}^2}{\lambda_r} \left( \frac{9\delta}{4\lambda_r} \right)^2 \left( \| \Delta \|_{F} + \sigma_{\delta} \right) \cdot \frac{(t + \log \tilde{m})^{1/2}}{n^{1/2}}.
\]

Similarly, by integrating out \( t \),

\[
\| \mathbb{E} [\varphi_{\delta}(A^{\perp})] - \mathbb{E} [\varphi_{\delta}(A + A^{\perp})] \| \leq C_{\delta} \frac{\sigma_{\delta}^2}{\lambda_r} \left( \frac{9\delta}{4\lambda_r} \right)^2 \left( \| \Delta \|_{F} + \sigma_{\delta} \right) \cdot \frac{\log^{1/2} \tilde{m}}{n^{1/2}}.
\]

It remains to bound \( \| \mathbb{E}_{P_{\text{vec}(\Delta)}^{\perp} X} [\varphi_{\delta}(A^{\perp})] - \mathbb{E} [\varphi_{\delta}(A^{\perp})] \| \). Recall that \( \{\xi_{i}\}, \{\langle \Delta, X_{i}\rangle\}, \text{ and } \{P_{\text{vec}(\Delta)}^{\perp} \text{vec}(X_{i})\} \) are mutually independent. Therefore, conditional on \( \{\xi_{i}\} \) and \( \{\langle \Delta, X_{i}\rangle\} \),

\[
A^{\perp} = \frac{1}{n} \left( \sum_{i=n+1}^{2n} (\xi_{i} + \langle \Delta, X_{i}\rangle)^2 \right)^{1/2} \mathcal{D} \circ \mathcal{M}(P_{\text{vec}(\Delta)}^{\perp} \text{vec}(F))
\]

where \( F \) is a copy of \( X_{i} \) being independent with \( \{\xi_{i}\} \) and \( \{\langle \Delta, X_{i}\rangle\} \). Define function:

\[
h(\{\xi_{i}\}, \{\langle \Delta, X_{i}\rangle\}) = \mathbb{E}_{P_{\text{vec}(\Delta)}^{\perp} X} [\varphi_{\delta}(A^{\perp})] = \mathbb{E}_{F}[\varphi_{\delta} \left( \frac{1}{n} \left( \sum_{i=n+1}^{2n} (\xi_{i} + \langle \Delta, X_{i}\rangle)^2 \right)^{1/2} \mathcal{D} \circ \mathcal{M}(P_{\text{vec}(\Delta)}^{\perp} \text{vec}(F)) \right).
\]

By Lemma [20]

\[
\left| h(\{\xi_{i}\}, \{\langle \Delta, X_{i}\rangle\}) - h(\{\xi'_{i}\}, \{\langle \Delta, X'_{i}\rangle\}) \right| \leq C_{\delta} \frac{\sigma_{\delta}^2}{\lambda_r} \left( \frac{9\delta}{4\lambda_r} \right)^2 \mathbb{E}_{F} \left\| \mathcal{D} \circ \mathcal{M}(P_{\text{vec}(\Delta)}^{\perp} \text{vec}(F)) \right\|
\times \left| \frac{1}{n} \left( \sum_{i=n+1}^{2n} (\xi_{i} + \langle \Delta, X_{i}\rangle)^2 \right)^{1/2} - \frac{1}{n} \left( \sum_{i=n+1}^{2n} (\xi'_{i} + \langle \Delta, X'_{i}\rangle)^2 \right)^{1/2} \right|.
\]
Define the vectors $\xi = (\xi_i)_{i=1}^n$ and $x_\Delta = (\langle \Delta, X_i \rangle)_{i=1}^n$. Then,

$$
\left| \frac{1}{n} \left( \sum_{i=n+1}^{2n} (\xi_i + \langle \Delta, X_i \rangle)^2 \right)^{1/2} - \frac{1}{n} \left( \sum_{i=n+1}^{2n} (\xi'_i + \langle \Delta, X'_i \rangle)^2 \right)^{1/2} \right|
= \frac{1}{n} \left| \| \xi + x_\Delta \|_2 - \| \xi' + x'_\Delta \|_2 \right|
\leq \frac{1}{n} \left( \| \xi - \xi' \|_2 + \| x_\Delta - x'_\Delta \|_2 \right)
\leq \frac{1}{n} \left( \| \xi - \xi' \|_2 + \| x_\Delta - x'_\Delta \|_2 \right)^{1/2}.
$$

Meanwhile,

$$
\mathbb{E}_{F} \| D \circ \mathcal{M}(P_{\text{vec}(\Delta)} \text{vec}(F)) \|
\leq \mathbb{E}_{F} \| F \| + \mathbb{E}_{F} \| D \circ \mathcal{M}(P_{\text{vec}(\Delta)} \text{vec}(F)) \| \leq C_1 \bar{m}^{1/2}.
$$

Therefore, we conclude that $h(\cdot)$ is Lipschitz with respect to $\{\xi_i\}$ and $\{\langle \Delta, X_i \rangle\}$ with constant $C_0 \sigma^2 \left( \frac{9\delta}{4\lambda_r} \right)^2 \frac{\bar{m}^{1/2}}{n}$. Since $\xi \sim \mathcal{N}(0, \sigma^2)$ and $\langle \Delta, X \rangle \sim \mathcal{N}(0, \|\Delta\|^2_F)$, we apply Lemma 24 and conclude that with probability at least $1 - e^{-t}$,

$$
\left| \mathbb{E}_{\{P_{\text{vec}(\Delta)} X\} \left[ \varphi_{\beta} (A^\perp) \right]} - \mathbb{E} \left[ \varphi_{\beta} (A^\perp) \right] \right|
\leq C_7 (\sigma \xi + \|\Delta\|_F) \frac{t^{1/2}}{\lambda_r} \left( \frac{9\delta}{4\lambda_r} \right)^2 \frac{\bar{m}^{1/2}}{n}.
$$

Therefore, with probability at least $1 - 2e^{-t}$ for all $t \in [1, n]$,

$$
\left| \mathbb{E}_{\{P_{\text{vec}(\Delta)} X\} \left[ \varphi_{\beta} (A + A^\perp) \right]} - \mathbb{E} \left[ \varphi_{\beta} (A + A^\perp) \right] \right|
\leq C_7 (\sigma \xi + \|\Delta\|_F) \frac{t^{1/2}}{\lambda_r} \left( \frac{9\delta}{4\lambda_r} \right)^2 \frac{(t + \log \bar{m})^{1/2}}{n^{1/2}}
$$

where we used the fact $n \geq \bar{m}$.

### 9.0.5 Proof of Lemma 17

Recall that

$$
P_{UV} E_2 C_{UV} = \frac{1}{n} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle P_{UV} X_i C_{UV} - P_{UV} \Delta C_{UV}
$$

where $\mathbb{E} \langle \Delta, X_i \rangle P_{UV} X_i C_{UV} = P_{UV} \Delta C_{UV}$. Clearly,

$$
\mathbb{E} \left\| P_{UV} E_2 C_{UV} \right\|_F^2 = \frac{1}{n^2} \sum_{i=n+1}^{2n} \mathbb{E} \left\| \langle \Delta, X_i \rangle P_{UV} X_i C_{UV} - P_{UV} \Delta C_{UV} \right\|_F^2
= \frac{1}{n^2} \sum_{i=n+1}^{2n} \left( \mathbb{E} \|\langle \Delta, X_i \rangle P_{UV} X_i C_{UV}\|_F^2 - \|P_{UV} \Delta C_{UV}\|_F^2 \right)
\leq \frac{1}{n} \mathbb{E} \langle \Delta, X \rangle^2 \|P_{UV} X C_{UV}\|_F^2
$$

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Write
\[ P_{UV}^\perp X_{UV} = P_{UV}^\perp M(P_{vec(\Delta)}^\perp vec(X))C_{UV} + P_{UV}^\perp M(P_{vec(\Delta)}^\perp vec(X))C_{UV}. \]
Since \( P_{vec(\Delta)}^\perp vec(X) \) is independent with \( P_{vec(\Delta)}^\perp vec(X) \), we obtain
\[
\begin{align*}
\mathbb{E}\|P_{UV}^\perp E_2 C_{UV}\|_F^2 &\leq \frac{1}{n} \mathbb{E}\langle \Delta, X \rangle^2 \|P_{UV}^\perp X C_{UV}\|_F^2 \\
&\leq \frac{1}{n} \mathbb{E}\langle \Delta, X \rangle^2 \|P_{UV}^\perp M(P_{vec(\Delta)}^\perp vec(X))C_{UV}\|_F^2 \\
&\quad + \frac{1}{n} \mathbb{E}\langle \Delta, X \rangle^2 \|P_{UV}^\perp M(P_{vec(\Delta)}^\perp vec(X))C_{UV}\|_F^2 \\
&\leq \frac{4}{n} \frac{\|\Delta\|_{F^2}^2}{\lambda^2} + \frac{4}{n} \|P_{UV}^\perp XC_{UV}\|_F^2 \\
&= \frac{4}{n} \frac{\|\Delta\|_{F^2}^2}{\lambda^2} + \frac{m_s}{n} \cdot \|\Lambda^{-1}\|_{F}^2 \|\Delta\|_{F}^2
\end{align*}
\]
where the last inequality is due to the proof of Lemma [16] which proves the second claim. To prove the first claim, denote
\[ \tilde{\delta} = 2\mathbb{E}\|E_2\|. \]
Therefore, \( \mathbb{P}\left(\|E_2\| \geq \tilde{\delta}\right) \leq e^{-n} + e^{-c_1 m} \) for an absolute constant \( c_1 > 0 \). Let \( \phi(\cdot) \) be the Lipschitz function defined in the proof of Lemma [15]. Define the function,
\[ h_{\tilde{\delta}}(\{X_i\}_i) = \|P_{UV}^\perp E_2 C_{UV}\|_F^2 \phi\left(\frac{\|E_2\|}{\tilde{\delta}}\right). \]
Since \( \{\langle \Delta, X_i \rangle\}_i \) and \( \{P_{vec(\Delta)}^\perp vec(X_i)\}_i \) are independent, we view \( h_{\tilde{\delta}}(\{X_i\}_i) \) as a function \( h_{\tilde{\delta}}(\{\langle \Delta, X_i \rangle\}_i, \{P_{vec(\Delta)}^\perp vec(X_i)\}_i) \). Conditional on \( \{\langle \Delta, X_i \rangle\}_i \), similarly as the proof of Lipschitz property in proof of Lemma [19] by discussing cases \( \|E_2\| \geq \frac{2}{\delta} \cdot \tilde{\delta} \) and \( \|E_2\| \leq \frac{2}{\delta} \cdot \tilde{\delta} \), we are able to show
\[
\begin{align*}
\left|h_{\tilde{\delta}}(\{\langle \Delta, X_i \rangle\}_i, \{P_{vec(\Delta)}^\perp vec(X_i)\}_i) - h_{\tilde{\delta}}(\{\langle \Delta, X_i \rangle\}_i, \{P_{vec(\Delta)}^\perp vec(X_i)\}_i)\right| \\
\leq C_3 \|P_{UV}^\perp (E_2 - E_2) C_{UV}\|_F \cdot \|C_{UV}\|_F + C_4 \tilde{\delta}^2 \|C_{UV}\|_F^2 \cdot \frac{\|E_2 - E_2\|}{\tilde{\delta}} \\
\leq C_3 \tilde{\delta} \|\Lambda^{-1}\|_{F}^2 \cdot \left(\sum_i \langle \Delta, X_i \rangle^2\right)^{1/2} \cdot \left(\sum_i \|P_{vec(\Delta)}^\perp vec(X_i - X_i')\|_F^2\right)^{1/2}.\]
\]
Therefore, conditional on \( \{\langle \Delta, X_i \rangle\}_i \), \( h_{\tilde{\delta}}(\cdot) \) is Lipschitz with constant \( C_3 \tilde{\delta} \|\Lambda^{-1}\|_{F}^2 n^{-1} \cdot (\sum_i \langle \Delta, X_i \rangle^2)^{1/2} \).
By Gaussian Isoperimetric inequality Lemma [24] with probability at least \( 1 - e^{-t} \) for all \( t \geq 1 \),
\[
\begin{align*}
\|P_{UV}^\perp E_2 C_{UV}\|_F^2 \phi\left(\frac{\|E_2\|}{\tilde{\delta}}\right) - \mathbb{E}_{\{P_{vec(\Delta)}^\perp vec(X)\}} \|P_{UV}^\perp E_2 C_{UV}\|_F^2 \phi\left(\frac{\|E_2\|}{\tilde{\delta}}\right) \\
\leq C_3 t^{1/2} \tilde{\delta} \|\Lambda^{-1}\|_{F}^2 \cdot \left(\sum_i \langle \Delta, X_i \rangle^2\right)^{1/2}.\]
\]
Meanwhile, with probability at least $1 - e^{-n}$,
\[
C_3 t^{1/2} \delta \| \Lambda^{-1} \|_F^2 \cdot \left( \frac{\sum_i \langle \Delta, X_i \rangle^2}{n} \right)^{1/2} \leq \frac{C_4 t^{1/2}}{n^{1/2}} \cdot \delta \| \Lambda^{-1} \|_F^2 \| \Delta \|_F^2
\]
Write $E_2 = E_{21} + E_{22}$ where
\[
E_{21} = \frac{1}{n} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle \mathcal{M}(P_{\text{vec}(\Delta)} \text{vec}(X_i)) - \Delta
\]
and
\[
E_{22} = \frac{1}{n} \sum_{i=n+1}^{2n} \langle \Delta, X_i \rangle \mathcal{M}(P_{\text{vec}(\Delta)} \text{vec}(X_i)).
\]
Then,
\[
\left| \mathbb{E} \left[ P_{\text{vec}(\Delta)} \text{vec}(X_i) \right] \| \mathcal{P}_{UV} \|_2^2 \cdot \left( \frac{\| E_2 \|}{\delta} \right) - \mathbb{E} \left[ P_{\text{vec}(\Delta)} \text{vec}(X_i) \right] \| \mathcal{P}_{UV} \|_2^2 \cdot \left( \frac{\| E_2 \|}{\delta} \right) \right|
\leq \mathbb{E} \left[ P_{\text{vec}(\Delta)} \text{vec}(X_i) \right] \| \mathcal{P}_{UV} \|_2^2 \cdot \left( \frac{\| E_2 \|}{\delta} \right) - \mathbb{E} \left[ \mathcal{P}_{\text{vec}(\Delta)} \text{vec}(X_i) \right] \| \mathcal{P}_{UV} \|_2^2 \cdot \left( \frac{\| E_2 \|}{\delta} \right)
\]
Applying the identical analysis as in proof of Lemma [21], we show that the function $\mathbb{E} \left[ P_{\text{vec}(\Delta)} \text{vec}(X_i) \right] \| \mathcal{P}_{UV} \|_2$ is Lipschitz with respect to $\{\langle \Delta, X_i \rangle\}_1$ with constant $C_1 n^{-1/2} \bar{m}^{1/2} \cdot \| \Lambda^{-1} \|_F^2$. Then, with probability at least $1 - 2e^{-t}$,
\[
\mathbb{E} \left[ \| P_{\text{vec}(\Delta)} \text{vec}(X_i) \|_2^2 \cdot \left( \frac{\| E_2 \|}{\delta} \right) \right]
\leq C_1 \bar{\delta} \| \Delta \|_F \| \Lambda^{-1} \|_F^2 \cdot \frac{t^{1/2} + \log^{1/2} \bar{m}}{n^{1/2}} + C_2 \bar{\delta} \| \Lambda^{-1} \|_F^2 \| \Delta \|_F \cdot \frac{\bar{m}^{1/2} t^{1/2}}{n}
\]
for some absolute constants $C_1, C_2 > 0$ where the last inequality is due to $n \geq \bar{m}$. Therefore, we conclude that with probability at least $1 - 3e^{-t} - e^{-n}$,
\[
\mathbb{E} \left[ \| P_{\text{vec}(\Delta)} \text{vec}(X_i) \|_2^2 \cdot \left( \frac{\| E_2 \|}{\delta} \right) \right]
\leq C_1 \| \Delta \|_F \| \Lambda^{-1} \|_F^2 \cdot \frac{\bar{m}^{1/2} (t^{1/2} + \log^{1/2} \bar{m}) \log^{1/2} \bar{m}}{n}
\]
Since $\mathbb{P} \left( \| E_2 \| \geq \hat{\delta} \right) \leq e^{-n} + e^{-c_1 \bar{m}}$, we obtain
\[
\mathbb{E} \left[ \| P_{\text{vec}(\Delta)} \text{vec}(X_i) \|_2^2 \cdot \left( \frac{\| E_2 \|}{\delta} \right) \right]
\leq C_1 \| \Delta \|_F \| \Lambda^{-1} \|_F^2 \cdot \frac{\bar{m} \log \bar{m}}{n} \cdot e^{-c_1 \bar{m}/2}.
\]
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Therefore, with probability at least $1 - 3e^{-t} - e^{-c_1 \tilde{m}} - e^{-n}$ for $t \geq 1$,
\[
\left| \mathbb{E}\| \mathcal{P}_{\text{in}}^+ \mathcal{E}_{2 \text{in}} + \mathcal{P}_{\text{out}}^+ \mathcal{E}_{2 \text{in}} \|_F^2 - \mathbb{E}\| \mathcal{P}_{\text{in}}^+ \mathcal{E}_{2 \text{in}} \|_F^2 \right|
\leq C_1 \| \Delta \|_F \| \Lambda^{-1/2} \tilde{m}^{1/2} (t^{1/2} + \log^{1/2} \tilde{m}) \log^{1/2} \tilde{m} \|_n.
\]

**9.0.6 Proof of Lemma 22 and Lemma 23**

We only prove Lemma 22. The proof of Lemma 23 is identical. Recall that on event $E_1$,
\[
T_0 = -\frac{1}{2\pi i} \left( \oint_{\gamma^+} + \oint_{\gamma^-} \right) \mathcal{R}_N(\eta) \frac{d\eta}{\eta^2}
\]
with
\[
\mathcal{R}_N(\eta) = \sum_{1 \leq |k| \leq r} \frac{\eta^{-2} \text{d}n_k}{\lambda_k - \eta} \mathcal{P}_k + \frac{\eta^{-2} \text{d}n^*_{-k}}{-\eta} \mathcal{P}^*_\text{UV}
\]
where $\mathcal{P}_k$ is defined as in the proof of Lemma 3. By Cauchy integral formula, we immediately obtain
\[
T_0 = \sum_{1 \leq |k| \leq r} \lambda_k^{-2} \mathcal{P}_k \left( \begin{array}{cc} \mathcal{U}^{-1} & 0 \\ 0 & \mathcal{V}^{-1} \end{array} \right).
\]

On event $E_1$,
\[
T_1 = \frac{1}{2\pi i} \oint_{\gamma^+} \mathcal{R}_N(\eta) \mathcal{E} \mathcal{R}_N(\eta) \frac{d\eta}{\eta^2} + \frac{1}{2\pi i} \oint_{\gamma^-} \mathcal{R}_N(\eta) \mathcal{E} \mathcal{R}_N(\eta) \frac{d\eta}{\eta^2}.
\]

By introducing $\mathcal{P}_{\text{in}}^+(\eta)$ and $\mathcal{P}_{\text{out}}^+(\eta)$ and following the same approach as in the proof of Lemma 3, we can show that
\[
\frac{1}{2\pi i} \oint_{\gamma^+} \mathcal{R}_N(\eta) \mathcal{E} \mathcal{R}_N(\eta) \frac{d\eta}{\eta^2}
\]
\[
= \sum_{1 \leq k, k' \leq r} \frac{1}{(\lambda_k - \lambda_{-k'}) \lambda^2_{-k'}} \cdot (\mathcal{P}_k \mathcal{E} \mathcal{P}_{-k'} + \mathcal{P}_{-k'} \mathcal{E} \mathcal{P}_k)
\]
\[
+ \sum_{1 \leq k \leq r} \frac{1}{\lambda^3_k} \cdot (\mathcal{P}_k \mathcal{E} \mathcal{P}^*_\text{UV} + \mathcal{P}^*_\text{UV} \mathcal{E} \mathcal{P}_k).
\]

Similarly, we can show that
\[
\frac{1}{2\pi i} \oint_{\gamma^-} \mathcal{R}_N(\eta) \mathcal{E} \mathcal{R}_N(\eta) \frac{d\eta}{\eta^2}
\]
\[
= \sum_{1 \leq k, k' \leq r} \frac{1}{(\lambda_{-k'} - \lambda_k) \lambda^2_{-k'}} \cdot (\mathcal{P}_k \mathcal{E} \mathcal{P}_{-k'} + \mathcal{P}_{-k'} \mathcal{E} \mathcal{P}_k)
\]
\[
+ \sum_{1 \leq k \leq r} \frac{1}{\lambda^3_{-k}} \cdot (\mathcal{P}_{-k} \mathcal{E} \mathcal{P}^*_\text{UV} + \mathcal{P}^*_\text{UV} \mathcal{E} \mathcal{P}_{-k}).
\]
It is straightforward to check
\[
\sum_{1 \leq k, k' \leq r} \frac{1}{(\lambda_k - \lambda_{-k'})\lambda_k^2} \cdot (P_k EP_{-k'} + P_{-k} EP_k) + \sum_{1 \leq k, k' \leq r} \frac{1}{(\lambda_{-k'} - \lambda_k)\lambda_{-k'}^2} \cdot (P_k EP_{-k'} + P_{-k} EP_k) = 0
\]
and
\[
\sum_{1 \leq k \leq r} \frac{1}{\lambda_k^2} \cdot (P_k EP_{UV}^+ + P_{UV} EP_k) + \sum_{1 \leq k \leq r} \frac{1}{\lambda_{-k}^2} \cdot (P_{-k} EP_{UV}^+ + P_{UV} EP_{-k}) = P_{UV} EC_{UV}^3 + \sigma_{UV}^3 E P_{UV}^\perp
\]
which concludes the proof.

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