Solutions of Word Equations over Partially Commutative Structures

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Abstract. We give $\text{NSPACE}(n \log n)$ algorithms solving the following decision problems. Satisfiability: Is the given equation over a free partially commutative monoid with involution (resp. a free partially commutative group) solvable? Finiteness: Are there only finitely many solutions of such an equation? $\text{PSPACE}$ algorithms with worse complexities for the first problem are known, but so far, a $\text{PSPACE}$ algorithm for the second problem was out of reach. Our results are much stronger: Given such an equation, its solutions form an EDT0L language effectively representable in $\text{NSPACE}(n \log n)$. In particular, we give an effective description of the set of all solutions for equations with constraints in free partially commutative monoids and groups.

1. Introduction

Free partially commutative monoids (a.k.a. trace monoids) and groups (a.k.a. RAAGs: right-angled Artin groups) are well-studied objects, both in computer science (latest since [18]) and in mathematics (with increasing impact since [24]). For years, decidability of the satisfiability problem (i.e., the problem whether a given equation is solvable) over these structures was open. A positive solution for trace monoids was obtained by Matiyasevich [17] and for RAAGs by Diekert and Muscholl [8]. The known techniques did not cope with the finiteness problem (i.e., the problem whether a given equation has only finitely many solutions). Decidability of finiteness for trace monoids was wide open, whereas for RAAGs a sophisticated generalization of Razborov-Makanin diagrams and geometric methods, available for groups, yielded decidability [8], but without any complexity estimation.

We give a simple (almost trivial) and effective method to describe the set of all solutions for equations with constraints in free partially commutative monoids and groups; the correctness proof is mathematically challenging. Once the correctness is established, the simplicity is also reflected in a surprisingly low complexity for this kind of result. We give an upper bound of $\text{NSPACE}(n \log n)$ for both satisfiability and finiteness—for both problems for trace monoids as well as for RAAGs. Even for satisfiability this complexity improves the previously known upper bounds. On the other hand these problems are NP-hard. It remains open whether $\text{NSPACE}(n \log n)$ is optimal.

To obtain these results we apply a recent recompression technique [12], which was used as a simple method to solve word equations. It applies simple compression operations: compress $ab$ into a letter $c$; and modify the equation so that such operations are sound. An algebraic setting of the current paper enables a shift of perspective: the inverse operation, replacing $c$ by $ab$, is an endomorphism. Thus, the set of all solutions of an equation (solvable or not) can be represented as a graph, whose nodes are labeled with equations and edges by endomorphisms of free monoids. This graph can also be seen as a nondeterministic finite automaton (NFA) that accepts a rational set of endomorphisms over a free monoid. (Recall that a subset in a monoid $M$ is rational if it is accepted by some NFA whose transitions have labels from $M$.) It is known that applying a rational set of endomorphisms to a letter yields an EDT0L language [1], and our construction guarantees that the obtained EDT0L language describes exactly the set of all solution.

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of the given equation. Moreover, as usual in automata theory, the structure of the NFA reflects whether the solution set is finite. Last not least, our method is conceptually simpler than all previously known approaches to solving equations over partially commutative structures.

Related work. Studying word equations is part of combinatorics on words for more than half a century [2]. From the very beginning, motivation came partly from group theory: the goal was to understand and parametrize solutions for equations in free groups. For example, Lyndon and Schützenberger needed sophisticated combinatorial arguments to give a parametrized solution to the equation $a^n = b^r c^s$ in a free group [14]. On the other hand, it is known that a parametric description of the solution set is not always possible [10]. The satisfiability of word equations in free monoids and free groups became a main open problem due to its connection with Hilbert’s tenth problem. The problem was solved affirmative by Makanin in his seminal papers [15, 16]. His algorithms became famous also due to the difficulty of the termination proof and the extremely high complexity. A breakthrough to lower the complexity was initiated by Plandowski and Rytter [21], who were the first to apply compression techniques on word equations. More precisely, they showed that every solution is highly compressible. Since this result, compression is a key tool for solving word equations. Indeed compression was essential in showing that the satisfiability of word equations is in PSPACE [19]. This approach was further developed [12] using the “recompression technique”, which simplified all existing proofs for solving word equations; in particular, it provided an effective description of all solutions; a similar representation was given earlier by Plandowski [20]. In free groups, an algorithmic description of all solutions was known much earlier due to Razborov [22]. His description became known as a Makanin-Razborov diagram, a major tool in the positive solution of Tarski’s conjectures about the elementary theory in free groups [13, 23]. None of these results provided a structural result on the set of all solutions, interest in such results was explicitly expressed [11] by asking whether it is an “indexed language” language. Apparently, this question was posed without too much hope that a positive answer is within reach. However, the answer was positive for quadratic equations [9] (which is a severe restriction); the general case was established in [4]. Actually, a stronger result holds: the set of all solutions for equations in free monoids (as well as in free groups) is an EDT0L language, which is a proper subclass of indexed languages. The closest results on word equations with partial commutation are in [8], but techniques used there do not apply here as they boil down to a purely combinatorial construction of a normal form and ignore the algebraic structure as well as the set of all solutions.

2. Main result

Resource monoids and groups. Given a finite alphabet $\Gamma$, the free monoid $\Gamma^*$ is the set of all finite words over $\Gamma$ with concatenation. The empty word is denoted by 1. The length of a word $w$ is denoted by $|w|$; by $|w|^R$ we count how often the letter $a$ appears in $w$. A resource function $\rho : \Gamma \rightarrow 2^{\mathbb{R}}$ maps elements of $\Gamma$ to subsets of a finite set of resources $\mathcal{R}$. We assume that $\mathcal{R}$ is of constant size. The pair $(\Gamma, \rho)$ is called a resource alphabet. If $\rho(a) = S$, then $a$ is called an $S$-constant; a nonempty sequence of $S$-constants is an $S$-run.

A resource monoid $M(\Gamma, \rho)$ is the quotient of all finite words $\Gamma^*$ by a partial commutation: $M(\Gamma, \rho) = \Gamma^*/\{ab = ba \mid \rho(a) \cap \rho(b) = \emptyset\}$, i.e., letters $a \neq b$ commute if and only if they do not share a resource. Resource monoids can equivalently be seen as free partially commutative monoids or trace monoids. We choose the resource-based approach as it best suits our purposes. Elements of a resource monoid are called traces. The natural projection $\pi$ maps elements of the free monoid $\Gamma^*$ to traces in $M(\Gamma, \rho)$; this is not a bijection and we view $w \in \Gamma^*$ as a word representation of the trace $\pi(w)$. In a monoid, an element $u$ is a factor of $w$ if $w = pq$ for some $p, q$. We assume that the monoid $M(\Gamma, \rho)$ is equipped with an involution, that is, a bijection $x \mapsto \overline{x}$ on $M(\Gamma, \rho)$ such that $\overline{\overline{x}} = x$, $\overline{xy} = \overline{y}\overline{x}$ for all $x, y \in M(\Gamma, \rho)$. To make the definition well defined, we require that $\rho(x) = \rho(\overline{x})$ for $x \in \Gamma$. In the following, a trace monoid means a resource monoid with involution. A morphism $\varphi : M \rightarrow M'$ between monoids with involution is a homomorphism additionally respecting the involution. If $\Delta$ is a subset of $M$, then we often denote the restriction of $\varphi$ to $\Delta$ by $\varphi_\Delta$. If $\varphi(x) = d$ for all $d \in \Delta$, then $\varphi$ is a $\Delta$-morphism.

If there is no letter $a \in \Gamma$ with $a = \overline{a}$, then, by adding defining relations $a\overline{a} = 1$ for all $a \in \Gamma$, we obtain the free partially commutative group $G(\Gamma, \rho)$. Free partially commutative groups are also known as right-angled Artin groups or RAAGs for short. As a set, we can identify a RAAG $G(\Gamma, \rho)$ with the subset
traces of the trace monoid $M(Γ, ρ)$ without factors $σ^k$. Such traces are called reduced. We take inversion on groups as involution; the canonical projection of the monoid $M(Γ, ρ)$ onto the group $G(Γ, ρ)$ respects the involution.

Equations with constraints. Let $(Γ, ρ)$ be a resource alphabet. An equation is a pair of words $(U, V)$ over an alphabet $Γ = A ∪ X$ has a partition into constants $A$ and variables $X$, both sets are closed under involution. A constraint is a morphism $μ : M(Γ, ρ) → N$, where $N$ is a finite monoid with involution. For our purposes, it is enough to consider constraints such that the elements of $N$ can be represented by $O(\log |Γ|)$ bits, and that all necessary computations in $N$ (multiplication, involution, etc.) can be performed in space $O(\log |Γ|)$ and the specification of these operations requires $O(|Γ| \log |Γ|)$ space. If $(U, V)$ is an equation over $(Γ, ρ)$, then we define the input size of an equation with constraints as $n = |UV| + |Γ|$.

We write $(U, V, μ)$ for an equation $(U, V)$ with constraints $μ$. A solution of $(U, V, μ)$ over $M(A, ρ)$ is an $A$-morphism $σ : M(A ∪ X, ρ) → M(A, ρ)$ such that $σ(U) = σ(V)$ and $μσ(X) = μ(X)$ for all $X \in X$. If the equation is over $G(A, ρ)$, then instead of $σ(U) = σ(V)$ we require $πσ(U) = πσ(V)$ for the canonical projection $π : M(A, ρ) → G(A, ρ)$. We also say that $σ$ solves $(U, V, μ)$ in $M(A, ρ)$ (resp. in $G(A, ρ)$). For equations over $G(A, ρ)$ we only allow solutions where the trace $σ(X)$ is reduced for all $X \in X$. The main result of this paper is that the set of all solutions of a trace equation (resp. an equation in a RAAG) with rational constraints is an effectively computable EDT0L language, and the underlying automaton reflects whether there are infinitely many solutions.

Theorem 1. [Monoid version] There is an NSPACE$(n \log n)$ algorithm for the following task. The input is a resource alphabet $(A ∪ X, ρ)$ with involution and a trace equation $(U, V, μ)$ with constraints $μ$ in constants $A$ and variables $X = \{X_1, ..., X_k\}$. The algorithm computes an alphabet $A ⊇ C$ of size $O(n)$, constants $a_1, ..., a_k ∈ C$, and an NFA $A$ accepting a rational set $R$ of $A$-endomorphisms on $C^*$ such that: $h(C^*) ⊆ A^*$ for all $h \in R$ and under the canonical projection $π : A^* → M(A, ρ)$ we have

$$\{(σh(a_1), ..., σh(a_k)) | h ∈ R\} = \{(σ(X_1), ..., σ(X_k)) | σ \text{ solves } (U, V, μ) \text{ in } M(A, ρ)\}.$$

Thus, the set of all solutions is an effectively computable EDT0L language. Furthermore, $(U, V, μ)$ has a solution if and only if $A$ accepts a nonempty set; $(U, V, μ)$ has infinitely many solutions if and only if $A$ has a directed cycle. These conditions can be tested in NSPACE$(n \log n)$.

[Group version] The same, but solutions $σ$ satisfy $σ(U) = σ(V)$ in the RAAG $G(A, ρ)$ and for a variable $X$ the solution $σ(X)$ is restricted to be a reduced trace.

Theorem 1 generalizes to systems of equations. Another generalization are finitely generated graph products with involution over free monoids, free groups, and finite groups. See [7] for a definition and the known results concerning solvability of equations in graph products. This generalization is rather technical but does not reveal new ideas; it is done elsewhere.

3. Basic concepts

Groups via monoids. Equations in RAAGs can be reduced to equations in resource monoids [8], such an approach is standard since its introduction for free groups [6], which are reduced to free monoids. In essence, the reduction simulates the inverse operation by involution and it enforces that the solution in the monoids is in the reduced form by (additional) constraints. We employ a similar approach; thus, our presentation focuses on the equations over resource monoids.

Simplifications. Using a standard technique one can ensure that there are no self-involving constants in the initial equation [8]. This step is not needed for RAAGs as $a = π = a^{-1}$ implies $a^2 = 1$ in groups, but RAAGs are torsion-free. For technical reasons we introduce a new special symbol $#$, which serves as a marker and becomes the only self-involving constant; set $ρ(♯) = R$ and $∅ ≠ ρ(♯) ⊆ R$ for all other constants. We let

$$W_{init} = ∗X_1#...X_k#U#V#U#V#X_k#...X_1#.$$

During the process, the $#$’s will not be touched, so we keep control over the prefix corresponding to $X_1#...X_k#$ which encodes the tuples $(σ(X_1), ..., σ(X_k))$. Moreover, we have $σ(U) = σ(V)$ if and
only if \( \sigma(W) = \sigma(W') \). Thus, we can treat a single trace as an equation. Solutions become \( A \)-morphisms \( \sigma \) satisfying \( \sigma(W) = \sigma(W') \).

**Monoids and equations.** The equations that we consider are over a more general structure than a trace monoid. To simplify the notation, we denote the equation and a monoid over which it is by a tuple \((W, B, X, \rho, \theta, \mu)\), where \( W \in (B \cup X)^+ \) is the “equation” with constants \( B \) and variables \( X \), the mapping \( \rho : B \cup X \to \mathbb{N}^+ \) is the resource function, and \( \mu : M(B \cup X, \rho) \to N \) represents the constraints (given by the mapping \( \mu : B \cup X \to N \)). The symbol \( \theta \) refers to a “type” which adds partial commutation. A type is given by a list of certain pairs \((x, y) \in (B \cup X)^+ \times B^+ \); and each such pair yields a defining equation \( xy = yx \). For example, we typically have \( (X, y) \in \theta \) when considering a solution \( \sigma \) with \( \sigma(X) \in y^+ \). Then \( \rho(X) = \rho(y) \), but we wish to use the fact that \( \sigma(X)y = y\sigma(X) \). This is the purpose of a type. We only use types in the subprocedures of block and quasi-block compression, see Section 4.2.1. Such a monoid is denoted as \( M(B \cup X, \rho, \theta, \mu) \). In most cases, \( \theta \) is empty. Then we use \((W, B, X, \rho, \mu)\) as an abbreviation of \((W, B, X, \rho, \emptyset, \mu)\) and \( M(B \cup X, \rho, \theta, \mu) \) as an abbreviation of \( M(B \cup X, \rho, \mu) \).

A \( B \)-solution of \((W, B, X, \rho, \mu)\) is a \( B \)-morphism \( \sigma : M(B \cup X, \rho, \mu) \to M(B, \rho, \mu) \) such that \( \sigma(W) = \sigma(W') \) (i.e., it solves the equation) and \( \mu(\sigma(X)) = \mu(X) \) (i.e., it satisfies the constraints).

**Increasing the resources.** During the algorithm we “increase the resources” of constants. It is useful to assume that for every constant \( a \in A \) and every set of resources \( S \) with \( \rho(a) \subseteq S \) the alphabet \( A \) has a corresponding constant with set of resources \( S \). We denote such a constant by \((a, S)\), the involution on it is defined by \((a, S) = ([a], S)\). We naturally identify \( a \) with \((a, \rho(a))\). We assume that the initial alphabet \( A \) is closed under taking such constants, i.e., if \( a \in A \) and \( \rho(a) \subseteq S \), then \([a], S) \in A \).

In some cases, when we “increase the resources”, we prefer to use a fresh constant of appropriate resources: For a constant \( a \) with \( \rho(a) \subseteq S \), by \([a], S)\) we denote a “fresh” \( S \)-constant outside \( A \) such that \( \rho([a], S) = S \), \( \mu([a], S) = \mu(a) \) and \([a], S) = ([a], S)\); replacing \( a \) with \([a], S)\) is called lifting.

**Operations, involution and constraints.** During the algorithm we perform various operations on variables and constants. As a rule, whenever we perform such an operation, we perform a symmetric action on the involuted constants/variables. That is, whenever we replace \( X \) by \( aX \), we replace \([X] \) by \([X] \); and when we replace \( ab \) by \( c \), then we also replace \( [a] \) by \([c] \). This simplifies the description, as actions performed on “the right side” of \( X \) are unaffected (except for lifting, in which case we explicitly change the set of resources); if we replace a trace \( W \) by a trace \( W' \), then (if not explicitly stated otherwise) we ensure that \( \rho(W) = \rho(W') \) and \( \mu(W) = \mu(W') \). For instance, when replacing \( X \) by \( aX' \), we set \( \mu(X') \) so that \( \mu(aX') = \mu(X) \). The same applies to \( \rho \). Similarly, when replacing \( ab \) by \( c \), we set \( \rho(c) = \rho(ab) \) and \( \mu(c) = \mu(ab) \). In particular, we do not mention in the description of the procedures that we perform such operations.

**Hasse diagrams and arcs.** A trace \( a_1a_2 \cdots a_n \) has many word representations and we would like to formalize a notion that some constants occur before others (in all word representations). To this end consider a set of positions \( \{1, 2, \ldots, n\} \) and the smallest partial order \( \preceq \) such that \( i \preceq j \) if both \( i \leq j \) and \( \rho(a_i) \cap \rho(a_j) \neq \emptyset \). A Hasse diagram \( H(W) \) of a trace \( W = a_1a_2 \cdots a_n \) is a graph with a set of nodes \( \{1, 2, \ldots, n\} \), labeled with \( a_1, a_2, \ldots, a_n \). It contains (directed) edges between immediate successors, i.e., \((i, j)\) is an edge if \( i < j \) and \( i \preceq k \preceq j \) implies \( k \in \{i, j\} \). By a standard result in trace theory [15], we have \( W = W' \) in \( \Gamma(\rho, \mu) \) if and only if \( H(W) \) and \( H(W') \) are isomorphic as abstract node-labeled directed graphs. When considering traces we usually work with their Hasse diagrams. If this causes no confusion, we identify \( W = a_1 \cdots a_n \) with \( H(W) \) and refer to labels \( a_1, a_2, \ldots \) rather than to node names.

A constant \( a \in A \) is minimal in a trace \( W \) if it is minimal in its Hasse diagram, which means that \( W = aY \) for some trace \( Y \). We denote the set of minimal elements of \( W \) by \( \text{min}(W) \). Maximal elements are left-right dual; they are denoted by \( \text{max}(W) \).

An arc \( a \to b \) is an \( S \)-arc if \( S \in \{\rho(a), \rho(b)\} \); it is balanced if \( \rho(a) = \rho(b) \), unbalanced otherwise.

4. NFA recognising the set of all solutions

In this section we define the NFA \( \mathcal{A} \) that recognises the set of all solutions of a trace equation, treated as a set of endomorphisms \( \text{End}(C^*) \) over an alphabet \( C \supseteq A \).
4.1. The automaton

**Alphabet.** We first fix an alphabet of constants $C \supseteq A$ of size $\kappa n$, where $\kappa$ is a suitable constant (which depends on the number of resources $|\mathcal{R}|$, viewed as $\mathcal{O}(1)$) and a set of variables $\Omega$ with $|\Omega| \leq |C|$. Henceforth, we assume $A \subseteq B \subseteq C$ and $X \subseteq \Omega$.

**States.** The states of the automaton $\mathcal{A}$ are equations of the form $(W, B, X, \rho, \theta, \mu)$. Each state $V = (W, B, X, \rho, \theta, \mu)$ has a weight $\|V\|$ which is 5-tuple of natural numbers:

$$\|V\| = ([W], \omega, \omega', |W| - |\theta|, |B|) \in \mathbb{N}^5$$

with $\omega = \sum_{a \in B} (|\mathcal{R}| - |\rho(a)|) \cdot |W|_a$ and $\omega' = |W| - |\{a \in B \mid |W|_a \geq 1\}|$. We order tuples in $\mathbb{N}^5$ lexicographically. The NFA contains only states $V$ whose max-norm $\|V\|_\infty = \max \{|W|, \omega, \omega', |W| - |\theta|, |B|\} \in \mathbb{N}$ is at most $\kappa' n$ for a suitable constant $\kappa'$.

The initial state is $(W_{\text{init}}, A_{\text{init}}, X_{\text{init}}, \rho_{\text{init}}, \mu_{\text{init}})$, it corresponds to the input equation. A state $(W, B, \emptyset, \rho, \mu)$ without variables is final if $W = \overline{W}$ and $W$ has the prefix $\# c_1 \# \cdots \# c_k \#$, where $c_1, \ldots, c_k$ are the distinguished constants. We require that the initial state has no incoming and the final states no outgoing transitions.

**Transitions.** The transitions (say between the states $V = (W, B, X, \rho, \theta, \mu)$ and $V' = (W', B', X', \rho', \theta', \mu')$) are labeled by $A$-morphisms and they either affect the variables (substitution transitions), or the monoid (compression transitions). The former is formalized using a $B$-morphism $\tau : M(B \cup X, \rho, \theta, \mu) \to M(B' \cup X', \rho', \theta', \mu')$. In this case we put several requirements on the equations: the new equation should be obtained by substitution $\tau(X)$ for each $X$, there are no new constants, resources and constraints of $X$ and $\tau(X)$ should be the same; this is formalized as

$$W' = \tau(W), \quad B' = B, \quad \rho = \rho \tau, \quad \mu = \mu \tau. \quad (1)$$

Moreover, $\tau(X)$ is either the empty word (it removes $X$ from $W$) or $\tau(X) \in X^* B^* X^*$ (at least one constant pops up in the substituted variable). Note that the requirement $W' = \tau(W)$ implicitly upper-bounds the size $\|\tau\|$, defined as $\sum_{a \in B \cup X} |\tau(a)|$, to be linear.

Furthermore, as $B' = B$ we have a natural identity morphism from $M(B', \rho', \theta', \mu')$ to $M(B, \rho, \theta, \mu)$, call it the associated morphism and denote it by $\epsilon$. This morphism labels the transition, its direction is opposite of the transition and $\tau$; we denote the transitions from $V$ to $V'$ with a corresponding morphism $h$ by $V \xrightarrow{h} V'$.

A compression transition leaves the variables invariant and so it is defined by an $(A \cup X)$-morphism $h : M(B' \cup X, \theta', \rho') \to M(B \cup X, \theta, \rho)$, note that it could be that $\theta \neq \theta'$ which corresponds to a type introduction or removal; this is the associated morphism in this case. A morphism $h$ defined by, say $h(c) = ab$, represents a compression of a factor $ab$ into a single letter $c$. For its properties, $W$ is obtained by decompression of new constants, and the resources and constraints are preserved:

$$W = h(W'), \quad \rho' = \rho h, \quad \mu' = \mu h. \quad (2)$$

As in the case of substitutions, the assumption that $W = h(W')$ implicitly upper-bounds $\|h\|$ to linear values.

The transition in the NFA is in the direction of the compression which is opposite to direction of the morphism $h$. Note that $W = h(W')$ implies $\|V'| < \|V\|$. For technical reasons we do not allow compression transitions which introduce self-involuting letters (such as $c \mapsto a \overline{a}$); we never compress the marker symbol $\#$. Moreover, following the last compression transition to final states, the restriction $\|V'\| < \|V\|$ is not applied.

**Trimming.** So far the defined NFA can have many useless states, so as a last step we trim the automaton, i.e., we remove all vertices not appearing on some accepting path.

**Conclusion.** The algorithmic part is finished: $\mathcal{A}$ can be constructed using standard arguments in $\text{NSPACE}(n \log n)$.

**Recognised language.** By the usual definition, the recognized language $\mathcal{R}$ consists of all $A$-morphisms $h_1 h_2 \cdots h_k$, where $h_1, h_2, \ldots, h_k$ are consecutive labels on an accepting path. We claim that the set of all solutions is exactly $(\{\pi(h(c_1)), \ldots, \pi(h(c_k)) \mid h \in \mathcal{R}\})$ where $\pi : A^* \to M(A, \rho)$ is the natural projection.
The correctness proof boils down to show that we can calculate the exact constants $\kappa, \kappa'$ (depending on $R$ but not on $n$) and to prove soundness and completeness, i.e., that $h \in R$ yields a solution and that every solution can be obtained in this way. Out of those, soundness is relatively easy to show, see Section 4.1.1; the completeness argument spans over Sections 4.2–4.3. Those arguments also show the other claims on the automaton (conditions for emptiness and acyclicity).

4.1.1. Soundness

As the final states have only one solution (identity), using an induction on the following Lemma, any accepting path labeled with $h_1, \ldots, h_k$ yields a solution $\pi h_1 \cdots h_k$, which shows soundness.

**Lemma 2.** Given two states $V = (W, B, \mathcal{X}, \rho, \theta, \mu)$ and $V' = (W', B', \mathcal{X}', \rho', \theta', \mu')$, if $V \xrightarrow{h} V'$ and $V'$ has a $B'$-solution $\sigma'$ then $V$ has a $B$-solution $\sigma = h\sigma'$.

The proof follows by mechanical application of (1) or (2).

4.2. On-the-fly construction of the NFA

While we described the NFA recognizing all solutions, we did not discuss how to find the appropriate constants $\kappa, \kappa'$ nor how to show completeness. For this it is easier to first describe the construction as an “on-the-fly” algorithm, that is, given an equation $(W, B, \mathcal{X}, \rho, \theta, \mu)$ (= current state $V$ of the NFA) and its $B$-solution $\sigma$ we will transform it into a different equation $(W', B', \mathcal{X}', \rho', \theta', \mu')$ (= next state $V'$ of the NFA) and a corresponding $B'$-solution $\sigma'$, where $V \xrightarrow{h} V'$ and $\sigma = h\sigma'$. Thus we moved from one state of the NFA to the other, without the knowledge of the full NFA. Note that the solutions are not given explicitly, but they are “used” in the nondeterministic choices of the algorithm.

For a fixed set of resources $S$ traces consisting only of $S$-constants and variables behave as words and we apply to them the known recompression approach: we iteratively apply compression operations to $S$-runs (so we replace $S$-runs by new single $S$-constants). Those operations are applied on constants in the equation, but conceptually we apply them to a solution of the trace equation. To make this approach sound, we also modify the variables, by popping $S$-constants from them. We apply these operations until each $S$-run (in the solution) is reduced to a single $S$-constant; when the compressions and popping operations are applied in appropriate order, the size of the trace equation remains linear.

Compression of $S$-runs alone is not enough, as there are constants of different resources in the solution of the trace equation. To remedy this, we gradually linearize the solution. This is done by increasing the size of the set of resources of particular constants: when we compressed each $S$-run to a single constant, we lift all $S$-constants, so that all $S$-constants and $S$-variables are eliminated from the equation. To make the whole approach work, we define an order $\preceq$ on sets of resources: it is any linear order that extends the partial ordering by the size, i.e., $|S| \leq |T|$ implies $S \preceq T$. A set of resources $S$ is called *minimal* (for a solution $\sigma$), if it is minimal according to $\preceq$ in the set $\{T \mid$ there is a $T$-constant in $\sigma(W)\}$. We process the sets of resources according to $\preceq$, each time treating a minimal set of resources.

4.2.1. Fixed resources

We define the actions of the algorithm eliminating the $S$-constants for a fixed minimal set of resources $S$. To this end, we need some notions of “easy” and “difficult” factors of $\sigma(W)$.

**Definition 3.** Let $(W, B, \mathcal{X}, \rho, \mu)$ be a state and $\sigma$ its $B$-solution. A factor $\nu$ of $\sigma(W)$ is visible if for some occurrence of $\alpha \psi$ at least one of its positions is obtained form a position labeled by a constant in $W$; a factor is invisible if it is not visible. A trace $v$ is crossing if for some occurrence of $\nu$ in $\sigma(W)$ some but not all positions belong to the substitution of a variable $X$ by $\sigma(X)$; and this occurrence is visibly crossing. A trace is noncrossing if it is not crossing.

The factors that we typically consider are pairs, i.e., $ab$ where $a \neq b \neq \pi$, $a$-blocks, i.e., a maximal factor of the form $\alpha^t$ (this occurrence of $\alpha^t$ if not part of a factor $\alpha^{t+1}$), and $a$-quasi-blocks, i.e., $(\alpha\pi)^t$ that is not part of a factor $(\alpha\pi)^{t+1}$. In the latter case, $\alpha\pi$ is called a *quasi-letter*. The intuitive meaning of a quasi-letter is that we cannot compress $\alpha\pi$ into a single constant as it is would be self-involuting, hence we treat those two letters as if they were a single constant.
Given a subalphabet $S_{\pm}$, we consider an involuting partition $(S_+, S_-)$ that satisfies the conditions $S_+ = S_-$, $S_+ \cap S_- = \emptyset$ and $S_+ \cup S_- = S_{\pm}$. Such a partition is crossing if at least one pair $ab \in S_+ S_-$ is; it has crossing quasi-blocks if there is $a \in S_+$ that has crossing quasi-blocks. Lastly, $S_{\pm}$ has crossing blocks if there is $a \in S_{\pm}$ that has crossing blocks.

**Pair compression.** Pair compression is implemented essentially in the same way as in the case of word equations. Given a pair $ab$ with $a \neq b$ we want to replace each factor $ab$ in $\sigma(W)$ with a fresh constant $c$. This is easy, when $ab$ is noncrossing: it is enough to perform this operation on $W$ and each $\sigma(X)$, the latter is done implicitly and we obtain a different solution $\sigma'$ in this way. We also set $\rho$ and $\mu$ for $c$ appropriately: $\rho(c) = \rho(ab)$ and $\mu(c) = \mu(ab)$. Performing several such compressions is possible for $ab \in S_+ S_-$, where $(S_+, S_-)$ is a noncrossing involuting partition, as for each constant in $\sigma(W)$ we can uniquely determine to which replaced pair it belongs (if any). We do not compress pairs $ab$, though, as this would create a self-involuting letter.

We need to ensure that indeed $(S_+, S_-)$ is noncrossing. A pair $ab \in S_+ S_-$ is crossing if $aX$ is a factor of $W$ and $b \in \min(\sigma(X))$. The other option is $Xb$ is a factor and $a \in \max(\sigma(X))$; it is taken care by considering $bX$ and the pair $\overline{ab}$. Then we replace $X$ with $bX$. After doing this, for all variables, the partition $(S_+, S_-)$ is noncrossing and so we can compress pairs in this partition.

**Block compression.** Pair compression cannot be applied to $aa$, as it makes the compression of longer blocks ambiguous. However, when $a$ has no crossing block, (in several steps) we replace each $a$-block $a^\ell$ by $c_\ell a$. Similarly as in the case of pair compression, we can compress blocks of several letters in parallel, as blocks of different letters do not overlap.

Again, to apply this subprocess we need to ensure that each $a \in S_{\pm}$ has no crossing blocks. Given a visibly crossing block $a^\ell$, popping one node may not be enough as this block may still be crossing. Thus for each variable $X$ we pop its whole $a$-prefix whenever $a \in \min(X) \cap S$, where $a^\ell$ is the $a$-prefix of a trace $V$ when $\ell$ is maximal with $V = a^\ell V'$.

**Quasi-block compression.** We do not apply the pair compression to $a\overline{a}$ as this introduces self-involuting letters. Instead, we perform a variant of block compression on them: the quasi-block compression. We replace each $a$-quasi-block $(a\overline{a})^\ell$ with $c_\ell a \overline{a}$; note that we treat $a$ and $\overline{a}$ symmetrically. We again perform this operation in parallel (in several steps), for all $a \in S_+$, where $(S_+, S_-)$ is an involuting partition.

For uncrossing of quasi-blocks we act the same as for uncrossing of blocks, but we pop the whole $(a\overline{a})$-prefix when $a \in S_+$; the $a\overline{a}$ prefix of $V$ is the longest factor $V' \in \overline{a}(a\overline{a})^* \cup (a\overline{a})^*$ such that $V = V'V''$.

**Algorithm for a fixed resource set.** Using those operations we can process a minimal set of resources $S$: We iterate the following operations as long as something changes in the equation. For each variable we guess whether it has a minimal S-letter and if so we pop this letter. Then we compute the set $S_{\pm}$ of visible S-constants. We uncross blocks from $S_{\pm}$ and then compress blocks of $S_{\pm}$. We then arbitrarily partition $S_+$ into an involuting partition $(S_+, S_-)$. Then we uncross quasi-blocks for $S_+$ and then compress quasi-blocks from $S_+$. We again partition $S_{\pm}$ into an involuting partition $(S_+, S_-)$; the partition is chosen so that there are many occurrences of pairs in $S_+ S_-$ in the equation, see the appendix. Finally, we uncross $(S_+, S_-)$ for pair compression and perform the pair compression for $(S_+, S_-)$.

Using similar arguments as in the case of word equations, one can show that FixedResources$(S)$ uses linear space. Concerning the $S$-runs after FixedResources$(S)$, ideally all $S$-runs are of length 1 and are either visible or invisible. This is not entirely true, as $a\overline{a}$ cannot be compressed, but those are the longest visible $S$-runs that can prevail.

**Lemma 4.** Let $S$ be minimal. The length of the equation during FixedResources$(S)$ is linear. After FixedResources$(S)$ there are no crossing $S$-runs, no $S$-variables. Furthermore, visible $S$-runs have length at most 2.

**4.2.2. Lifting arcs**

Compression of $S$-runs alone is not enough, as there are runs for different sets of resources. To remedy this we linearize the trace, for technical reasons it is easier to lift whole Hasse arcs rather than individual nodes.

To lift a Hasse arc $e = (a \rightarrow b)$ we want to relabel its ends by $[a, \rho(a) \cup \rho(b)]$ and $[b, \rho(a) \cup \rho(b)]$, i.e., by fresh $(\rho(a) \cup \rho(b))$-constants. For correctness reasons we need to also lift the edges that “correspond” to
e; moreover, as in the case of compression, lifting may be difficult when an arc connects constants in the equation with constants in the substitution for a variable. Those notions are formalized below.

**Definition 5.** Let \((W, B, X, \rho, \mu)\) be a state and \(\sigma\) its \(B\)-solution. A Hasse-arc \(a \to b\) in \(\sigma(W)\) is visible (invisible, visibly crossing) if the corresponding factor \(ab\) in \(\sigma(W)\) has this property. Let \(\sim\) be the smallest equivalence relation which satisfies the following conditions:

- If \(e = (a \to b)\) in \(\sigma(W)\) and \(f = (b \to \overline{a})\) is the corresponding arc in \(\sigma(\overline{W})\), then \(e \sim f\).
- If \(e\) is invisible and inside some \(\sigma(X)\) where \(X \in X\) and \(f\) is a corresponding arc in some different \(\sigma(X)\), then \(e \sim f\).

We say that \(e\) is crossing if there exists a visibly crossing \(f\) with \(f \sim e\); \(e\) is free otherwise.

Note that for arcs the notion of crossing/free is finer than for traces: since it is possible that \(e \not\sim e'\) while both are of the form \((a \to b)\), in particular \(e\) could be free and \(e'\) crossing.

When \(e = (a \to b)\) is a free unbalanced arc, the promised linearization of traces is done through lifting: let \(S = \rho(a) \cup \rho(b)\), then for \(f \sim e\) we change the label on each of its ends from \(c \in \{a, b, \pi, \overline{\pi}\}\) to \([c, S]\). Note that this balances \(f\). To make this operation well defined, we partially linearizes a trace: each position that was before (after) any of relabeled \(a, b\) is now before (after) both of \([a, S], [b, S]\) (the same is done for arc \(b \to \overline{a}\)).

We can lift free arcs “for free”, but some \(S\)-arcs may be crossing. Freeing them is similar to uncrossing factors, but we need to take into the account that \(\rho(a) \neq \rho(b)\). Thus \(ab\) could be a crossing arc in \(aX\) and \(b\) is not a minimal element of \(\sigma(X)\), so it cannot be popped. Freeing is done in two stages: first we deal with the case when \(b\) is an \(S\)-letter. Then for \(\sigma(X) = P\rho Q\), such that \(S \neq \rho(P) \subseteq \rho(X)\) we pop the whole \(P\), which is done by introducing a fresh variable, i.e., we substitute \(X \mapsto X'bX\). The new solution is \(S'(X') = P\) and \(\sigma'(X) = Q\). Then we deal with the case when \(a\) is an \(S\)-letter (and \(b\) not). Thus for \(\sigma(X) = P\rho Q\), where \(\rho(a) \cap \rho(P) = \emptyset\), we substitute \(X \mapsto X'bX\). The new solution is \(S'(X') = P\) and \(\sigma'(X) = Q\). Those operations are called splitting of variables. Observe that the first splitting can be done for any set of \(S\)-constants and all variables in parallel, while the second can be performed in parallel for all variables and any set of constants that is a subset of \([b \mid \rho(b) \cap S \neq \emptyset]\).

We want to lift all unbalanced \(S\) arcs, but this is not possible for all such arcs in parallel due to involution: for an \(S\)-letter \(a\) and a trace \(bac\) we have to choose which arc, \(b \to a\) or \(a \to c\), we lift. But it can be done in stages: let \((S_+, S_-)\) and \((T_+, T_-)\) be involuting partitions of all \(S\)-constants and all constants having a common resource with \(S\), i.e., \(\{a \mid \rho(a) \cap S \neq \emptyset\}\). Then we process all \(S\) arcs in four groups \(S_+T_+, S_-T_+, S_+T_-\) and \(S_-T_-\); processing of each one is similar, we describe processing of one — \(S_+T_+\). We first split the variables for \(S_+\) and then for \(T_+\), as described above. Then each arc \((a \to b)\) with \(ab \in S_+T_+\) is free, thus we lift those arcs. We continue with groups \(S_-T_+, S_+T_-\) and \(S_-T_-\). Note that the processing may introduce new crossing arcs, but it can be shown that they are always in next groups. Afterwards, there are no \(S\) arcs.

It is easy to show that after \text{Remove}(S) all \(S\)-constants and \(S\)-variables are eliminated.

**Lemma 6.** After \text{Remove}(S) there are no \(S\)-constants nor \(S\)-variables in \(\sigma(X)\).

### 4.2.3. The algorithm

TrEqSat considers possible sets of resources \(S\) in order \(\leq\) on them. For a fixed \(S\) it first runs \text{FixedResources}(S) and then \text{Remove}(S).

### 4.3. Analysis

We begin with estimating the space usage. Firstly we upper-bound the number of introduced variables: they are introduced only during splitting of variables, which happens \(O(1)\) times per resource set, and each variable introduces \(O(1)\) variables, which have less resources; this yields that the number of occurrences of variables is linear in the size of the input equation.

We then estimate the length of the equation, which is also linear in the size of the input equation: Here the estimations are similar as in the case of word equations. For a fixed resource set \(S\) we claim that
the number of $S$-constants in the equation stays linear and that processing $S$ introduces in total $O(1)$
constants per variable. Together with the estimation on the number of variables this yields a bound on
the size of the equation.

This guarantees that our algorithm does not exceed a space limit, but may loop forever. Thus we want
to show that solutions in consecutive steps get “smaller”. Unfortunately, the length of $\sigma(W)$ is not good
enough for our purposes, but we can define the weight of the solutions (for an equation) and indeed show
that our subprocedures decrease it. This guarantees termination.

We then move to the correctness of the algorithm, i.e., we show how the algorithm transforms the
solutions between different equations obtained on the way. In a first step we equip each solution with a
function that tells us, what solution of the input equation it represents. Then we show that if subprocedure
transforms one equation into the other, then the morphism associated with this transition transforms the
solution of the latter equation to a solution of the former, so that they they represent the same solution
of the input equation.

4.3.1. Space usage

The below estimations of space usage do not depend on the nondeterministic choices, they apply to all
executions of the algorithm.

Number of variables. Comparing to the algorithms in the free monoid case, the main difference is that
our algorithm introduces new variables to the equation. This is potentially a problem, as the whole
recompression is based on the assumption that the number of constants is not altered. However, we can
still bound the number of introduced variables.

\textbf{Lemma 7.} During $\text{TrEqSat}$ there are $O(n)$ occurrences of variables in the trace equation.

Fix a variable $X$ for which initially $T = \rho(X)$. Observe that $\rho(X)$ cannot increase, though it can
decrease: resources increase by lifting arcs and we only lift free arcs, thus, each resource of the new
constant was present on one of the ends of the arc. On the other hand, popping constants as well as
splitting may decrease the resources of a variable.

We say that $X$ \textit{directly created} an occurrence of $X'$ when $X'$ was created in Split when it considered
$X;X$ created $X'$ when there is a sequence $X = X_1, X_2, \ldots, X_k = X'$ such that $X_i$ directly created $X_{i+1}$.\n
Consider a variable $X$, it can be split at most eight times during lifting of crossing arcs when we consider
$T' \subseteq R$. This gives all variables that are directly created by $X$. Note, that each of the directly generated
variable has less resources than $X$: when we replace $X$ with $X'bX$, then we require that
$\rho(X') \subseteq \rho(X'bX)$.

Let $f(k)$ be the maximal number of occurrences of variables that can be created by a variable with
at most $k$ resources. Using the above analysis we can write a recursive formula for $f; as the number of
resources is a constant, this yields the bound.

Length of the equation. We show that during $\text{TrEqSat}$ the length of the trace equation is linear in the
size of the variables, this is similar as in the case of word equations and in fact the proof proceeds using
similar steps. First, we focus on $\text{FixedResources}$ and its processing of a fixed set of resources $S$. In each
application of the while loop we introduce $O(1)$ $S$-constants per variable (in case of block and quasi-block
compression we may introduce long blocks but they are replaced with $O(1)$ constants afterwards). On the
other hand, using standard expected value argument we can show that compression of a randomly chosen
partition results in removal of a constant fraction of $S$-constants from the equation.

Comparing the number of constants in the equation before and after processing $S$, it increases only by
the $S$-factors that were popped from variables. There are $O(1)$ such factors for a variable and each is of
length at most 2. Thus for a fixed set of resources the size of the equation increases by $O(n)$. Summing
over possible sets of resources (which is of constant size) yields the claim.

\textbf{Lemma 8.} During $\text{TrEqSat}$ the length of the trace equation is $O(n)$.

4.3.2. Weight of solutions

To guarantee the termination, we show that all subprocedures decrease the (appropriately defined) weight
of a solution. This weight is in fact defined with respect to the original solution: The $B$-solution $\sigma$
corresponds to some solution of the input equation, as letters of \( B \) correspond to some traces in the original equation. To keep track of those traces we use an \( A \)-morphism \( \alpha : M(B, \rho, \theta, \mu) \rightarrow M(A, \rho_0, \mu_0) \); the idea is that \( c \in B \) represents a trace \( \alpha(c) \) in \( M(A, \rho_0, \mu_0) \). Conceptually, \( \alpha(\sigma(W)) \) is the corresponding solution of the input equation. We call a pair \((\sigma, \alpha)\) a solution at \((W, B, X, \rho, \mu)\), where \( \sigma \) is a \( B \)-solution. Note that this morphism is a tool of analysis and proof, it is neither computed nor stored anywhere by the algorithm.

Using the morphism we can define a weight \( \|\alpha, \sigma\| \) of a solution \((\alpha, \sigma)\) of an equation \( W \):

\[
\|\alpha, \sigma\| = \sum_{X \in V} |\alpha \sigma(X)|.
\]

All subprocedures performed by our algorithm do not increase the weight. In order to ensure that they all decrease some “weight”, we take into the account also the weight of the equations and define a weight of a solution \((\alpha, \sigma)\) at a state \( V \) as \((\|\alpha, \sigma\|, \|V\|)\) which is evaluated in lexicographic order. All subprocedures decrease such defined weight. Thus, the path in NFA for a fixed solution is finite and terminates in a final state.

### 4.3.3. Internal operations

So far all the described operations were performed on the equation and had some influence also on the solutions. However, there are also operations that are needed for the proof but are performed either on the monoid or on the solutions alone, hence they do not affect the equation at all. For this reason we call them internal. In essence, we apply them to the equation whenever this is possible.

**Useless constants and variables.** A constant \( a \in B \setminus A \) is useless if it does not occur in \( \sigma(W) \); it is useful otherwise; useless constants are invisible. A variable is useless if it does not occur in \( W \). We remove from the monoid all useless constants and variables.

**Invisible constants.** Due to compression we can be left with invisible but useful constants, i.e., such that they occur in \( \sigma(W) \) but not in \( W \).

We cannot remove such constants from \( B \), as we deal with all solutions. However, we can replace them with corresponding traces over \( M(A, \rho_0, \mu_0) \). The idea is that we replaced \( \alpha(c) \) with \( c \) too eagerly. We revert this compression. We do not revert the linearization of the trace, though. Thus we lift each letter in \( \alpha(c) \) so that it has the same resources as \( c \): we replace every invisible letter \( c \) with \((a_1, \rho(c))(a_2, \rho(c)) \cdots (a_t, \rho(c))\), where \( \alpha(c) = a_1a_2 \cdots a_t \), i.e., with a chain of letters corresponding to the trace compressed into \( c \) but lifted into current resources of \( c \). Note that we use letters from \( A \subseteq B \), so the procedure is not applicable to letters that it just introduced.

### 4.3.4. Completeness

The last step to show completeness is an observation that each given subprocedure corresponds to a composition of finitely many substitution and compression transitions: indeed, this is done by mechanical verification.

The completeness, formulated below, easily follows: given an equation with a solution \((\alpha, \sigma)\) we apply the subprocedures that lead to a final state. By observation above each subprocedure corresponds to a short path in the NFA. The guarantee on the size of the states follows from Lemma 9. Finally, we cannot iterate forever, as each subprocedure decreases the weight of the solution at a state.

**Lemma 9.** There is constant \( \kappa'' \geq 1 \) (depending on \( \mathcal{R} \) but independent of \( n \)) such that for all states \( V \), if \( \|V\| \leq \kappa'' \cdot n \) and \( V \) has a solution \((\alpha, \sigma)\), then there exists a path to final state labeled with \( h_1, h_2, \ldots, h_k \) such that

\[
\sigma = h_1 h_2 \cdots h_k.
\]
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Appendix

How to read the appendix. There is a major difference in the presentation between the extended abstract and this appendix. The main text should put the reader into a position to “understand” on a high level how the procedures lead to the assertions in Theorem 4. For such a high-level understanding the technical details are not necessary, in fact they may obfuscate the big picture, so they are omitted. The appendix has a different purpose. It should put the reader in a position to verify the mathematical statements, including the technical details. As a result, the content of the appendix is given in the inverse order to the one in the extended abstract. Moreover, the appendix contains redundancy: we repeat some statements and definitions from the extended abstract.

A. Reductions and simplifications

We give some general properties which streamline the argument, as they allow avoiding considerations of several special cases later on.

A.1. The reduction from RAAGs to trace monoids

This subsection contains material expanding the Groups via monoids in Section 3.

This section reduces the case of RAAGs (=free partially commutative monoids) to the case of trace monoids (=free partially commutative monoids). Readers interested mainly in trace monoids can skip this section.

The reduction is fairly standard and it is also used to ensure that solutions of equations over RAAGs are in a reduced normal form. The reduction does not create any self-involuting letters, but our statement concerning trace monoids is more general.

In principle, we can use the same reduction as in [8], but we additionally require that the solutions are reduced. The set of all reduced traces can be defined by a finite list $\mathcal{L}$ of forbidden factors:

$$\mathcal{L} = \{ a\overline{\rho} \mid a \in A \}.$$ 

For computational complexity results it is enough to observe that $|\mathcal{L}| \in \mathcal{O}(|A|)$. (As a matter of fact, a quadratic bound on $|\mathcal{L}|$ would not suffice.)

We ensure that factors from $\mathcal{L}$ do not occur in the solution by defining an additional constraint: Consider a finite monoid with involution $N_L$ whose elements form a subset of $2^A \times 2^A \times 2^A$ and it has an additional zero element “0”, for which $0 \cdot x = x \cdot 0 = 0$ for all $x \in N_L$. Intuitively, a triple $(P, S, R)$ represents traces $V$ such that $\min(V) = P$, $\max(V) = R$ and $\rho(V) = S$ and $V$ does not have a factor from $\mathcal{L}$. In particular, $N_L$ contains only those triples, for which $\rho(P \cup R) \subseteq S$ and each of $P, R$ contains pairwise independent letters, only (recall that $u, v$ are independent if $\rho(u) \cap \rho(v) = \emptyset$).

The multiplication $(P, S, R)$ and $(P', S', R')$ is defined so that it satisfies this intuitive property: if $RP'$ and $L$ have a common element, then the result is 0, as it corresponds to a non-reduced trace. Otherwise it is $(P'', S'', R'')$, where $P''$ contains elements in $P$ and those elements in $P'$ that do not share a resource with $S$ (note that those correspond to minimal elements of a trace), $S'' = S \cup S'$ (the resources of concatenation is the union of the resources) and $R''$ contains elements in $R'$ and those elements in $R$ that do not share a resource with $S'$. Formally:

$$(P, S, R) \cdot (P', S', R') = \begin{cases} (P \cup \{ a \in P' \mid \rho(a) \cap S = \emptyset \}, S \cup S', R' \cup \{ a \in R \mid \rho(a) \cap S' = \emptyset \}) & \text{if } RP' \cap \mathcal{L} = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The involution is defined by $\overline{G} = 0$ and $(P, S, R) = (\overline{R}, S, \overline{P})$. There is canonical morphism $\mu_L : M(A, \rho) \to N_L$ defined by $\mu_L(a) = (\{ a \}, \rho(a), \{ a \})$ for $a \in A$.

It is easy to see that $g = M(A, \rho)$ is in reduced normal form if and only if $\mu(g) \neq 0$. By replacing a given constraint monoid $N$ by the direct product $N \times N_L$ we may henceforth assume that $\mu(x) = \mu(y)$ implies $\mu_L(x) = \mu_L(y)$. Moreover, we may assume that $N$ contains a zero 0 and $\mu(X) \neq 0$ for all $X \in X$. Note
that $N$ still satisfies our assumption from Section\ref{sec:preliminaries} the description of element of $N$ needs $O(\log |A|)$ bits and all operations in $N$ can be performed in space $O(\log |A|)$.

The next steps are well-known and standard, for details we refer to \cite{8}. These steps transform (in nondeterministic linear space) an equation $(U, V)$ over $G(A, \rho)$ into a system of equations over the trace monoid $M(A, \rho)$ with the help of additional variables. In order to come back to a single equation we introduce a marker symbol $\#$. It is convenient and non-restrictive to assume that

$$\# = \overline{\#}, \mu(\#) = 0, \text{ and } \rho(\#) = \mathcal{R}.$$ 

Thus, in nondeterministic linear space a system of equations $\{V_i\}_{i=1}^k$ with constraints over a RAAG $G(A, \rho)$ can be transformed into a single equation $V'$ over $M(A, \rho)$. Due to nondeterminism several outcomes (i.e., equations $V'$) are possible. But the reduction never transforms an unsolvable equation into a solvable one and every solution of $V'$ has a corresponding solution of at least one possible outcome $V'$, see \cite{8}. As a conclusion and for a later reference we state the following lemma.

**Lemma 10.** The proof of Theorem\ref{thm:main} follows from its monoid version.

### A.2. Resources

This subsection contains material expanding the Resource monoids and groups in Section\ref{sec:preliminaries}. By increasing the set of resources $\mathcal{R}$ by one element, we can assume that

$$\rho(X) \neq \mathcal{R} = \rho(\#) \text{ for every variable } X,$$

where $\#$ is a special symbol in the alphabet.

This has the advantage that in a solution $\sigma(X)$ cannot use any letter $a$ with $\rho(a) = \mathcal{R}$ since a solution must satisfy $\rho\sigma(X) \subseteq \rho(X)$. In particular, $\#$ cannot occur in $\sigma(X)$.

Moreover, we assume that every constant $a$ uses some resource. If this is not the case, then letters from $D = \{a \mid \rho(a) = \emptyset\}$ commute with everything, and it is easy to show that our theorem holds in such a setting as well: essentially we can move all those letters to the left, popping them from variables if needed. As there are no dependency between them, the equation on them reduces to linear Diophantine equations, which can be solved in NP and also in nondeterministic linear space \cite{12}. Moreover, all solutions of systems of linear Diophantine can be easily represented in the desired form as an EDT0L language \cite{5}. Since linear Diophantine equations are not our focus, we henceforth assume (for simplicity of the presentation) that

$$\rho(a) \neq \emptyset \text{ for all constants.}$$

### A.3. Self-involuting letters

This subsection contains material expanding the Simplifications in Section\ref{sec:application}. Our results hold in the context of free partially commutative monoids in the original definition of Mazurkiewicz, as they are frequently used in computer science. However, this setting originally does not use involution. To make the results formally applicable one can take involution as an identity. Hence all letters are self-involuting, but the involution on a trace means to read them from right-to left. A self-involuting trace becomes a palindrome. This opens an interesting avenue: we can solve trace equations that use the palindrome predicate.

The reduction from a RAAGs to trace monoids did not introduce any self-involuting letter other than $\#$; and the convention $\mu(\#) = 0$ ensures that $\#$ does not appear in any solution obeying the constraint. However, Theorem\ref{thm:main} allows a resource monoid to have self-involuting letters. (This is necessary because we wish that the identity to be a possible involution on the alphabet $A$.) As in Section\ref{sec:preliminaries} the technique to cope with that problem has been described in \cite{8}. However, in \cite{8} the constraints are represented by Boolean matrices, which causes a quadratic increase on space. As we aim at $O(n \log n)$ space, we need to improve on that.

This involves a simple construction on monoids. If $N$ is a monoid, then we define its dual monoid $N^T$ to have the same carrier set $N^T = N$, but $N^T$ is equipped with a new multiplication $x \circ y = yx$. In order to indicate whether we view an element in the monoid $N$ or $N^T$, we use a flag: for $x \in N$ we
write $x^T$ to indicate the same element in $N^T$. Thus, we can suppress the symbol $\circ$ and we simply write $x^T y^T = (yx)^T$; the notation is intended to mimic transposition in matrix calculus. Similarly, we write $1$ instead of $I^T$ which is true for the identity matrix as well. The direct product $N \times N^T$ becomes a monoid with involution by letting $(x, y^T) = (y, x^T)$. Indeed,

$$\begin{align*}
(x_1, y_1^T) \cdot (x_2, y_2^T) &= (y_2 y_1, (x_1 x_2)^T) = (y_2, x_2^T) \cdot (y_1, x_1^T) = (x_2, y_2^T) \cdot (x_1, y_1^T).
\end{align*}$$

Clearly, if $N$ is finite then $N \times N^T$ is finite, too. The projection $\pi_1 : N \times N^T \rightarrow N$ onto the first component defines a homomorphism of monoids, thus:

**Remark 11.** Let $N$ be any monoid (with or without involution) and let $M$ be a monoid with involution. If $\nu : M \rightarrow N$ is a homomorphism of monoids, then there is unique morphism $\mu : M \rightarrow N \times N^T$ of monoids with involution such that $\nu = \pi_1 \mu$. Indeed, it is sufficient and necessary to define $\mu(x) = (\nu(x), \nu(\overline{x})^T)$.

We are now ready to eliminate the self-involuting letters from the alphabet.

**Lemma 12.** The proof of Theorem 4 follows from its monoid version where $\#$ is the only self-involuting letter of $A$ and where the constraint $\mu$ guarantees that $\#$ cannot appear in any solution $\sigma(X)$ of a variable $X$.

**Proof.** Due to Lemma 10 we can start with the monoid version with a constraint $\mu : M(A, \rho) \rightarrow N$. Define

$$A_\# = \{ a \in A \mid \# \neq a \in \overline{A} \}. $$

If $A_\# = \emptyset$, then we are done. Otherwise define two disjoint copies: $A_+ = \{ a_+ \mid a \in A_\# \}$ and $A_- = \{ a_- \mid a \in A_\# \}$. We may assume that $A_+ \cap A_- = \emptyset$. We let $\overline{A}_+ = a_-$ and $\overline{A}_- = a_+$. In this way, $A_\# = A_+ \cup A_- \text{ becomes an alphabet without self-involuting letters.}$ Moreover, we now replace the equation $(U, V)$ by the new equation $(\iota(U), \iota(V))$. Define a homomorphism of monoids $\nu : M(A' \cup X, \rho) \rightarrow N$ by $\nu(x) = \mu(x)$ for $x \in X \cup A \setminus A_\#$, $\nu(a_+) = \mu(a)$ for $a \in A_+$, and $\nu(a_-) = 1$ for all $a \in A_-$. Note that $\nu$ never respects the involution, unless $\mu$ is trivial. However, we have $\nu(\overline{a}) = \nu(a)$ for all $a \in A$.

Define the constraint by a morphism $\mu' : M(A', \rho) \rightarrow N \times N^T$ as in Remark 10. Thus, we have $\mu'(a) = (\nu(a), \nu(\overline{a})^T)$ for $a \in A'$. We can guarantee $\sigma(X) \in \iota(M(A, \rho))$.

Finally, add an additional constraint which makes sure that for a variable $X$ and a solution $\sigma$ of $(\iota(U), \iota(V), \mu')$ we can guarantee $\sigma(X) \in \iota(M(A, \rho))$.

Let us comment, what we have done. The morphism $\iota$ embeds $M(A \cup X, \rho)$ into $M(A' \cup X, \rho)$. In $A' \cup X$ there is no other self-involuting letter than $\#$. Every solution $\sigma : M(A \cup X, \rho) \rightarrow M(A, \rho)$ yields a solution $\sigma' : M(A' \cup X, \rho) \rightarrow M(A', \rho)$ of $(\iota(U), \iota(V), \mu')$ such that $\sigma'(X) \in \iota(M(A, \rho))$ for all $X \in X$ (take $\sigma' = \iota \sigma$).

It remains to show that if $\sigma'$ is a solution of $(\iota(U), \iota(V), \mu')$ such that $\sigma'(X) \in \iota(M(A, \rho))$ for all $X \in X$ then we can find a corresponding solution of $(U, V, \mu)$. Define a homomorphism $\eta : M(A', \rho) \rightarrow M(A, \rho)$ by $\eta(a) = a$ for $a \in A' \setminus A_\#$, $\eta(a_+) = a$ for $a_+ \in A_+$, and $\eta(a_-) = 1$ for all $a_- \in A_-$. Then $\eta \sigma' : M(A' \cup X, \rho) \rightarrow M(A', \rho)$ is a solution of $(U, V, \mu)$. Without loss of generality we may assume that $A \cup A' \subseteq C$ and $\eta$ is the endomorphism of $C^*$ which fixes all letters outside $A_\#$ and sends $A_-$ to the empty word $\varepsilon$. We may also assume that all $A-$endomorphisms of $C^*$ under consideration fix $A'$ with the single exception of $\eta$. This will also be the only endomorphism of $C^*$ which sends some letters to $1$, and which does not respect the involution. Nevertheless, if we can prove Theorem 4 in the better setting without self-involuting letters (up to $\#$), then we can apply in a final step the endomorphism $\eta$ to the EDT0L set of solutions and we have the statement of Theorem 4 in the general form. One piece is still missing, we need to explain how to realize the constraint that a solution $\sigma'$ satisfies $\sigma'(X) \in \iota(M(A, \rho))$ for all $X \in X$. Fortunately, this is
easy and similar to the monoid construction which guarantees solution in reduced traces. We work again with a list of forbidden factors \( \mathcal{L} \). We do not need the list in explicit form (this might be too long): we content ourselves that we can list all pairs \((a_+, a_-)\) for \(a \in A\) in linear space. Each time see an Hasse arc \(a \to b\) with \(a \in A_+\) or \(b \in A_-\), then we must have \(a = a_+\) and \(b = a_-\) for some \(a \in A\). Actually, this is not enough. We must also make sure that no \(a_-\) appears as first letter and no \(a_+\) appears as last letter in any \(\sigma(X)\). Again, this is standard and left to the reader.

A.4. The initial trace \( W_{\text{init}} \) as an equation

This subsection contains material expanding the Simplifications in Section 3.

Note that the previous reduction introduced additional variables, but the original variables \(X_i\) for \(1 \leq i \leq k\) are still present in \(X\). Intuitively, if \(X_1, X_2, \ldots, X_k, X_\bar{k}\) is the set of all variables occurring in the word \(UV\bar{V}\), then the set of all solutions of an equation \((U, V)\) with constraints \(\mu\) is a set of tuples

\[
S_{(U,V,\mu)} = \{(\sigma(X_1), \ldots, \sigma(X_k)) \mid \sigma \text{ solves } (U, V) \text{ in } M(A, \rho) \text{ and satisfies } \mu\}.
\]

In the following it is more convenient to think of a solution as a single trace. Thus, we encode the values of the variables in the equation. Therefore, we define \(W_{\text{init}} \in M(A \cup X)\) as follows:

\[
W_{\text{init}} = \#X_1 \# \cdots \#X_k \#UV \#\bar{V}\#X_\bar{k} \# \cdots \#X_1 \#.
\]

A solution of \(W_{\text{init}}\) is an \(A\)-morphism \(\sigma : M(A \cup X, \rho) \to M(A, \rho)\) such that \(\sigma(W_{\text{init}}) = \sigma(\bar{W}_{\text{init}})\) and \(\mu \sigma(X) = \mu(X)\) for all \(X \in \Omega\). Thus

\[
S_{(U,V,\mu)} = \{\sigma(X_1), \ldots, \sigma(X_k) \mid \sigma \text{ solves } W_{\text{init}} \text{ in } M(A, \rho)\}.
\]

Consequently, we can also write \(S_{(U,V,\mu)} = S_{(W_{\text{init}}, \mu)}\).

In order to prove Theorem 11, it is enough to show the following statement. The main difference is that we removed self-involuting letters. Other than that, the next theorem is a stronger and more precise version of Theorem 13.

**Theorem 13.** There is an \(\text{NSPACE}(n \log n)\) algorithm for the following task. On input a trace \(W_{\text{init}}\) as in Equation 2, constraints given by a morphism \(\mu : M(A \cup X, \rho)\) over constants \(A\) and variables \(X\) (with involutions where \# is the only self-involuting letter in \(A\) and the constraint \(\mu\) does not allow to use \# in any equation), and a list \(X_1, \ldots, X_k\) of variables in \(X\), the algorithm computes an alphabet \(C\) of size \(O(n)\) with \(A \subseteq C\), constants \(c_1, \ldots, c_k \in C\), and an NFA \(A\) accepting a rational set \(R\) of \(A\)-endomorphisms of \(C^*\) such that \(S = \{h(c_1), \ldots, h(c_k) \mid h \in R\}\) is a subset of \((A^*)^k\). The set \(S\) is mapped under the canonical projection \(\pi_A : A^* \to M(A, \rho)\) onto the set \(S_{(W_{\text{init}}, \mu)} = \{\sigma(X_1), \ldots, \sigma(X_k) \mid \sigma \text{ solves } (W_{\text{init}}, \mu) \text{ in } M(A, \rho)\}\).

Thus:

\[
\{(\pi_A(h(c_1)), \ldots, \pi_A(h(c_k)) \mid h \in R\} = \{\sigma(X_1), \ldots, \sigma(X_k) \mid \sigma \text{ solves } (W_{\text{init}}, \mu) \text{ in } M(A, \rho)\}.
\]

The solution set is an effectively computable \(\text{EDT0L}\) language. Furthermore, the NFA \(A\) accepts a nonempty set if and only if \((U, V, \mu)\) has some solution; \(A\) has a directed cycle if and only if \(R\) is infinite if and only if there are infinitely many solutions. These conditions can be tested \(\text{NSPACE}(n \log n)\).

**B. Framework (monoids)**

This subsection contains material expanding Sections 2 and 3.

Let us repeat and introduce some technical notation which will be used throughout the proof. First of all, in order to use the notation \(A, X, \rho, \mu\) in a more flexible way, we replace the statement Theorem 13 by using \(A_{\text{init}}, X_{\text{init}}, \rho_{\text{init}}, \mu_{\text{init}}\). Thus, the initial situation is given by the tuple

\[
(W_{\text{init}}, A_{\text{init}}, X_{\text{init}}, \rho_{\text{init}}, \mu_{\text{init}}).
\]

Replace \(A_{\text{init}}\) by some larger alphabet \(A\). This is additional material to increasing the resources in Section 3. As we increase the set of resources of constants, it is better to use a larger alphabet from the
very beginning which allows to increase the set of resources without creating a fresh letter. Remember that we have \# ∈ A_{init} and no other symbol in A_{init} ∪ X_{init} is self-involuting. We let

\[ A = \{(a, S) ∈ A_{init} × 2^{|S|} \mid ρ_{init}(a) ⊆ S\} . \]

This is a resource alphabet with involution by letting ρ₀(a, S) = S and \((a, S) = (\#, S)\). In the following we will use A as the basic alphabet and we fix this notation. We use the convention \((a, S), T) = (a, T)\) for ρ(a) ⊆ S ⊆ T.

We have a canonical embedding \(M(A_{init}, ρ_{init}) ⊆ M(A, ρ₀)\) by \(a ↦ (a, ρ₀(a))\). Moreover, the free monoid \(A_{init}^*\) with involution embeds into \(M(A, ρ₀)\) by \(a ↦ (a, ϱ)\). We use both embedding throughout. For \(a ∈ A\) we write \(a = (a, ρ(a))\). Note that \# = (#, ϱ) is the only self-involuting letter in A. We define \(µ₀(a, S) = µ(a)\) for \(a ∈ A\); and we adopt the notation for variables \(X ∈ X_{init}\): we let \(X₀ = X_{init}, µ₀(X) = ρ_{init}(X),\) and \(ρ₀(X) = ρ_{init}(X)\). The projection on the first component induces a canonical length-preserving projection:

\[ π₀ : M(A, ρ₀) → M(A_{init}, ρ_{init}), \quad π₀(a, S) = a \text{ for } a ∈ A_{init}. \]

Let \(V_{init} = (W_{init}, A_{init}, X_{init}, µ_{init}, ρ_{init})\) be any equation with constraints which is the input to Theorem \[13\]. Then we can switch to \(V₀ = (W_{init}, A, X₀, µ₀, ρ₀)\) over the larger alphabet of constants A. If \(σ : M(A_{init} ∪ X₀, ρ_{init}) → M(A_{init}, ρ_{init})\) is a solution of \(V_{init}\), then \(σ\) is also a solution of \(V₀\). Vice versa, if \(σ' : M(A ∪ X₀, ρ₀) → M(A, ρ₀)\) is a solution of \(V₀\), then \(σ = π_A σ'\) is a solution of \(V_{init}\). Moreover, \(V₀\) has infinitely many solutions if and only if \(V₀\) has infinitely many solutions: in both cases there are infinitely many solutions if and only if \(V₀\) has arbitrary long solutions. So, we can start right away with the state \(V₀ = (W_{init}, A, X₀, µ₀, ρ₀)\).

The index 0 is used because we wish to use the notation \(X, µ, ρ\) freely in intermediate steps of the procedure. We define the input size of \(V_{init}\) and \(V₀\) by the same number:

\[ n = |W_{init}| + |A| + |X₀|. \]

The notation for the alphabets \(A_{init},\) A, and the projection \(π₀\) according to \[9\] and the corresponding input size \(n\) in Equation \[7\] remain fixed. Let us summarize the effect of replacing \(A_{init}\) by the larger alphabet A.

**Lemma 14.** The change of \(V_{init}\) to \(V₀\) allows to prove Theorem \[13\] in a more specific setting. Instead of using the morphism \(π_A : A^* → M(A, ρ₀)\) we can use the composition

\[ π₀ : A^* \overset{π_A}{→} M(A, ρ₀) \overset{π₀}{→} M(A_{init}, ρ_{init}). \]

Moreover, instead of showing Equation \[8\] it is enough to show the weaker assertion.

\[ \{(π₀ h(c₁), \ldots, π₀ h(cₖ)) \mid h ∈ Ρ\} = \{(π₀ σ(X₁), \ldots, π₀ σ(Xₖ)) \mid σ \text{ solves } (W_{init}, µ) \text{ in } M(A, ρ)\}. \]

In other words, it is enough to show that the rational set of endomorphisms \(Ρ = L(A) \subseteq A^*\) is mapped under the canonical projection \(π₀\) onto \(π₀(S(W_{init}, µ₀))\).

**Proof.** This is clear from the construction above: indeed if \(σ\) solves \((W_{init}, µ)\) in \(M(A, ρ)\), then replacing \(σ(X)\) by \(π₀ σ(X)\) defines another solution since \(π₀(W_{init}) = W_{init}\).

We use \[9\] in order to focus on Equation \[8\]. This will become important when, later, we introduce a notion of “forward property” in Definition \[25\].

**Choosing the frame.** This part extends the material presented in *Groups via monoids* in Section \[5\].

We let \(κ ∈ \mathbb{N}\) be a large enough constant (which actually depends exponentially on \(|Ω|\)). A suitable upper bound on \(κ\) can be calculated easily from the exposition and the explicit calculation is omitted. Next, we define an alphabet of constants \(C\) and a disjoint alphabet of variables \(Ω\) such that \(A ⊆ C, X₀ ⊆ Ω\), and \(|C| = |Ω| = κ \cdot n ∈ \mathcal{O}(n)\). We assume that \(C\) and \(Ω\) have involutions and we let \(Σ = C ∪ Ω\). We assume that \# ∈ A is the only self-involuting symbol in \(Σ\). This fixes a frame \(Σ\) of linear size. Henceforth, all constants and variables are in \(Σ\).
We reserve the notation $B$ and $\mathcal{X}$ to denote sets of constants and variables such that $A \subseteq B \subseteq C$ and $\mathcal{X} \subseteq \Omega$. We assume that $B$ and $\mathcal{X}$ are closed under involution. Moreover, $\mu$ and $\rho$ refer to morphisms $\mu : B \cup \mathcal{X} \rightarrow N$ and $\rho : B \cup \mathcal{X} \rightarrow 2^{\mathcal{X}}$. This applies to \textquotedblleft primed symbols\textquotedblright{} $B'$ and $\mathcal{X}'$, too. If $B$ and $\mathcal{X}$ are given, then a \textit{fresh constant/variable} refers to an element $x \in \Sigma \setminus (B \cup \mathcal{X})$. Another convention: Small letters $a, b, c, a', b', c', \ldots$ refer to constants, capital letters $X, Y, X', Y', \ldots$ refer to variables.

If a function $f$ is defined on a domain $\Delta'$, then we typically use the same symbol $f$ to denote its restriction to a subset $\Delta$ of $\Delta'$.

**Endomorphisms.** An endomorphism of a free monoid $\Delta^*$ with involution is given by mapping $h : S \rightarrow \Delta^*$ where $S \subseteq \Delta$. The convention is that $h$ is extended to $h : \Delta^* \rightarrow \Delta^*$ by letting $h(c) = c$ if $c, \bar{c} \notin S$ and $h(c) = h(\bar{c})$ for $c \in S \setminus \mathcal{S}$. Thus we typically assume $S \cap \mathcal{S} = \emptyset$, which means that the definition of $h$ respects the involution by the convention above. If $S = \emptyset$, then $h = \text{id}_{\Delta^*}$ is the identity. If we have $S \subseteq \Delta'$ and $h(S) \subseteq \Delta'^*$, then we also view $h$ as a homomorphism $h : \Delta'^* \rightarrow \Delta'^*$.

**B.1. $\Sigma$-homogeneous monoids, resource monoids with types, structured monoids**

This Section extends the material presented in \textit{Monoids and equations} in Section 3.

While we are interested in word equations over trace monoids, in its intermediate steps our algorithm encounters also word equations in more general structures, which are described in this Section.

**B.1.1. $\Sigma$-homogeneous monoids**

In algebra, the notion of \textit{homogeneous monoid} refers to a finitely presented monoids where the two words in each defining relation have the same length, for instance, resource monoids are of this type. In our paper, a monoid $M$ is called \textit{$\Sigma$-homogeneous} if it is a monoid with involution, generated by a subset $\Delta \subseteq \Sigma$ and a finite set of defining relations $S$ such that $(\ell, r) \in S$ implies $|\ell| = |r|$ for letters $x \in \Sigma$. For a $\Sigma$-homogeneous monoid $M$ we can write $M = \Delta^*/S$ where $\Delta = \overline{\Delta}$ and that $(\ell, r) \in S$ implies both, $|\ell| = |r|$ and $(\ell, r) \in S$.

Whenever $\Delta' \subseteq \Delta$ and $S' = S \cap \Delta'^* \times \Delta'^*$, then $\Delta'^*/S'$ naturally embeds into $\Delta^*/S$.

During our algorithm we are often tasked with a question, whether a trace $V$ is a factor of a trace $W$. In general, we need to cope with such question also in the more general framework of $\Sigma$-homogeneous monoids.

Formally, the \textit{uniform factor problems in $\Sigma$-homogeneous monoids} is given as:

- **Input.** The alphabet $\Sigma$, a pair $(\Delta, S)$ describing a $\Sigma$-homogeneous monoid $M = \Delta^*/S$, and a pair of words $U, V \in \Delta^*$ such that

  $$|\Sigma| + |UV| + \sum_{(\ell, r) \in S} |\ell r| \in \mathcal{O}(n).$$

- **Question.** Are there $p, q \in \Delta^*$ such that $pUq = V$ holds in $M$?

**Lemma 15.** The uniform factor problem in $\Sigma$-homogeneous monoids is decidable in $\text{NSPACE}(n \log n)$.

**Proof.** The encoding of $U$ and $V$ uses at most $\mathcal{O}(n \log n)$ bits. Scanning $V$ from left to right and guessing for each letter of $V$ whether it belongs to $p$ or to $U$ or to $q$, we compute nondeterministically the words $p$ and $q$; and we write $W = pUq$ to the work space with $\mathcal{O}(n \log n)$ bits. Finally, we modify $W$ nondeterministically by applying defining relations $\ell = r$ where $(\ell, r) \in S$. Each time before applying a relation we may guess to stop this process. We accept $U$ as a factor of $V$ if the current $W$ coincides with $V$ as a word in $\Delta^*$. \hfill \Box

**Resource monoids over $\Sigma$ with types** We consider a subclass of $\Sigma$-homogeneous monoids, which extend the framework of partial commutation by using types. Let us fix the notation: $(\Gamma, \rho)$ is a resource alphabet and $\overline{\Gamma} = \Gamma = B \cup \mathcal{X}$.

The idea of a type comes from a different approach to block compression. Instead replacing the whole $a^\lambda$ with a fresh constant $c_\lambda$ in one go we make the compression in stages. We replace a block $a^\lambda$, which we intend to compress, first by factor $c_\lambda a^{\lambda-1}$ where $c_\lambda$ and $c$ are fresh letters; this is just a renaming at certain positions labeled by $a$. (The notation $c_\lambda$ is purely formal, we do not write down the natural
number \( \lambda \) which can be arbitrary large.) Then \( c_\lambda \) should commute with \( c \), as it indeed represents a single \( c \), even though \( c_\lambda \) and \( c \) have the same resources. Moreover, a variable \( X \) may also commute with \( c \), when in the solution it represents a block of \( c \). We want to formalise this by giving \( c, c_\lambda \) and \( X \) the same type \( \theta(c) = \theta(c_\lambda) = \theta(X) = c \) and allow a commutation \( xy = yx \) when \( \theta(x) = y \).

A similar approach is applied also to quasi-block compression, this time though the type represents quasi-blocks of \( \mathbf{a} \).

The idea of types is formalised as follows: Formally, a type \( \theta \) for \( (\Gamma, \rho) \) is a relation \( \theta \subseteq (B \cup B^2 \cup X) \times (B \cup B^2) \) such that following conditions are satisfied.

- \((x, y) \in \theta \) implies \((\mathbf{t}, \mathbf{t}) \in \theta \) (intuitively: when \( x \) is a block of \( y \) then \( \mathbf{t} \) should be a block of \( \mathbf{t} \))
- \((x, y) \in \theta \) and \((x, y') \in \theta \) implies \( y = y' \) (intuitively: \( x \) can represent a block of only one letter).
- \((u, y) \in \theta \) and \( u \in B^+ \) implies \(|u| = |y| \) and \( u \in \{a, \mathbf{t}, \mathbf{t} \mathbf{t} \} \) for some \( a \in B \setminus A \) (intuitively: types are defined only for \( x = c \) or \( x = \mathbf{a} \) or \( x \) being a variable, in two first cases the type is of the same length as its argument).

Note that we can read \( \theta \) as a partially defined function where the domain is a subset of variables, letters, and quasi-letters. If \( \theta(x) \) is defined, then \( \theta(x) \) is also called the type of \( x \). Moreover, there is a letter \( c \in B \setminus A \) such that either \( \{\theta(x), \theta(\mathbf{t})\} = \{c, \mathbf{t}\} \) or \( \theta(x) = \theta(\mathbf{t}) = \mathbf{a} \).

Each tuple \((\Gamma, \rho, \theta)\) defines a \( \Sigma \)-homogeneous monoid with involution \( M(\Gamma, \rho, \theta) \) as follows:

\[
M(\Gamma, \rho, \theta) = \Gamma^* / \{xy = yx \mid (x, y) \in \theta \lor \rho(x) \cap \rho(y) = \emptyset \}. \tag{9}
\]

Since the range of \( \theta \) does not involve any symbol from \( A \cup X \) we obtain a canonical embedding of the resource monoid \( M(A \cup X, \rho) \) into \( M(\Gamma, \rho, \theta) \). The monoid \( M(\Gamma, \rho, \theta) \) is called a resource monoid with a type.

### B.1.2. Structured monoids and their morphisms

We now give the definition of monoids used in the automaton from Theorem 13; this extends the material presented in Equations during the algorithm in Section 3.

**Definition 16** (Structured monoid). A resource monoid \( M(B \cup X, \rho, \theta) \) with type \( \theta \) and constraints \( \mu : M(B \cup X, \rho, \theta) \to \text{finite monoid } N \) is a structured monoid, if

- \( A \subseteq B = \overline{B} \subseteq C \) and \( X = \overline{X} \subseteq \Omega \).
- \( A \subseteq B = \overline{B} \subseteq C \) and \( X = \overline{X} \subseteq \Omega \).
- As a resource monoid with type we have \( M(B, X, \rho, \theta, \mu) = M(\Delta, \rho, \theta) \) for \( \Delta = B \cup X \).
- \( \rho : \Delta \to 2^\Omega \) with \( \rho(a) = \rho_\theta(a) \) for \( a \in A \).
- \( \mu : \Delta \to N \) is a morphism with \( \mu(a) = \mu_\theta(a) \) for \( a \in A \).
- The morphism \( \mu \) induces a morphism of monoids with involution \( \mu : M(\Delta, \rho, \theta) \to N \).

We denote this structured monoid by \( M(B, X, \rho, \theta, \mu) \).

We use \( M(B, \rho, \theta, \mu) \) as an abbreviation for the submonoid of \( M(B, X, \rho, \theta, \mu) \) which is generated by the constants from \( B \). The inclusion \( M(B, \rho, \theta, \mu) \subseteq M(B, X, \rho, \theta, \mu) \) is a morphism. Note that the earlier notation \( M(B, X, \rho, \mu) \) without types is now replaced by \( M(B, X, \rho, \emptyset, \emptyset, \mu) \). Thus, we have shorthands:

\[
M(B, \rho, \emptyset, \mu) = M(B, \emptyset, \rho, \emptyset, \mu) = M(B, \emptyset, \rho, \emptyset, \emptyset, \mu),
M(B, X, \rho, \emptyset, \mu) = M(B, X, \rho, \emptyset, \emptyset, \mu).
\]

**Definition 17** (Morphism of structured monoids). A morphism between structured monoids \( M(B, X, \rho, \theta, \mu) \) and \( M(B', X', \rho', \theta', \mu') \) is an \( A \)-morphism \( \varphi \) between \( M(B \cup X, \rho, \theta) \) and \( M(B' \cup X', \rho', \theta') \) such that

\[
\forall x : \rho' \varphi(x) \subseteq \rho(x), \quad \mu = \mu' \varphi. \tag{10a}
\]

\[
\mu' \subseteq \mu \tag{10b}
\]
A morphism $\varphi$ always respects the constraints but it may “forget” resources, thus, a morphism can introduce additional partial commutation.  
The norm $\|\varphi\|$ of a morphism $\varphi$ is
\[ \|\varphi\| = \sum_{x \in B \cup X} |\varphi(x)|. \]  
(11)  
We are interested only in morphisms $\varphi$ where $\|\varphi\| \in \mathcal{O}(n)$ because such a $\varphi$ can be specified by $\mathcal{O}(n \log n)$ bits.

C. Extended equations

As explained, in Section A.4 it is enough to consider a single word $W$ as an equation and in this setting $\sigma$ is a solution of $W$ with constraints $\mu$ if and only if $\sigma(W) = \sigma(W)$ and $\mu_0 \sigma = \mu_0$. So elements in structured monoids $M(B, X, \rho, \theta, \mu)$ represent equations. We use the shorthand $M$ to denote the initial structured monoid without variables and types:

\[ M = M(A, 0, 0, 0). \]

We now define the elements of the structured monoids that represent equations.

**Definition 18.** An element $W \in M(B, X, \rho, \theta, \mu)$ is well-formed if the following conditions hold.

- $|W| < |X|$ and $|W|_# = |W_{\text{init}}|_#$. Moreover, $#$ appears as the first and last symbol in $W$.
- Every factor $x \leq W$ and every $x \in B \cup X$ satisfies: $\mu(x) \neq 0 \iff |x|_# = 0$.
- If $x \leq W$ is a factor with $|x|_# = 0$, then $x \leq W$, too.

An extended equation is a tuple $(W, B, X, \rho, \theta, \mu)$, where $W$ is well-formed and $M(B, X, \rho, \theta, \mu)$ is a structured monoid.

The intuition behind the conditions is as follows: the first ensures that the equation is short enough and that it can be separated into parts that encode the substitutions for variables and equations. The second ensures that no $#$ was compressed into the letters in $B$. The third holds initially and guarantees that we can consider a factor together with its inversion at the same time.

For example, $(W_{\text{init}}, A, X_0, \rho_0, 0, \mu_0)$ is an extended equation.

Let us define the notion of solution to an extended equation. We define the notions of $B$-solution and a refined notion of solution.

**Definition 19 (cf. 4.3.2).** Let $V = (W, B, X, \rho, \theta, \mu)$ be an extended equation.

- A $B$-solution at $V$ is a $B$-morphism $\sigma : M(B, X, \rho, \theta, \mu) \to M(B, \rho, \theta, \mu)$ such that $\sigma(W) = \sigma(W)$ and $\sigma(X) \in y^*$ whenever $(X, y) \in \theta$.
- A solution at $V$ is is a pair $(\alpha, \sigma)$ where $\alpha : M(B, \rho, \theta, \mu) \to M$ is an $A$-morphism and $\sigma$ is a $B$-solution.

Due to the notion of morphisms between structured monoids we have $\mu_0 \alpha = \mu$ for all solutions $(\alpha, \sigma)$. Moreover, $\alpha$ and $\sigma$ leave letters from $A$ invariant.

Recall that if $\sigma$ is a $B$-solution at $V$, then we can write $W = x_1 \cdots x_t$ and $\sigma(W) = a_1 \cdots a_m$ where $x_i \in B \cup X$ and $a_j \in B$. Each position $i$ where $x_i \in B$ can be uniquely identified with a corresponding position $j \in \{1, \ldots, m\}$ with $x_i = a_j$. These positions $j$ and the corresponding letter $a_j$ are called visible in $W$.

D. From the nondeterministic “on-the-fly” construction to an NFA

This Section extends the material in Section 4.

In the Section 1.2 of the extended abstract we described a nondeterministic algorithm (running in space $\mathcal{O}(n \log n)$) that given an equation constructs “on-the-fly” a path in the NFA recognizing the set of all solutions; each transition of this NFA is labeled with an endomorphism of $C$.

In this Section we will give a precise definition of this NFA and show two its main properties:
**Soundness:** Given any path from the initial state to a final state the composition of the endomorphisms of the transitions yields a solution of the initial equation.

**Completeness:** Every solution can be obtained in the way described above (for some path from the initial state to a final state).

We first construct an NFA, see Section D.1 in particular we define all technician conditions that should be satisfied by its transitions and labels of those transitions. Then we argue that every $A$-morphism recognised by it is a solution of the original equation (soundness), see Section D.2. Then we show that indeed every solution is recognised by this NFA (completeness), see Section E. Finally, the constructed NFA does not satisfy all the conditions required by Theorem 13: it may have directed cycles even though it has only finitely many solutions. This is due to the fact that some of the morphisms labeling the edges do not affect the solution but rather only permute the letters. We identify such labels and modify the NFA so that they are not used, essentially we compute a transitive closure and then delete those problematic edges, see Section D.3.2.

### D.1. The ambient NFA $F$

In this section we define an NFA $F$ of singly exponential size in $n$ which accepts some rational set of $A$-endomorphisms of $\text{End}(C^*)$.

#### D.1.1. States

The states of $F$ are extended equations $(W, B, X, \rho, \theta, \mu)$ according to Definition 18. Observe that there are no more than $2^{O(n)}$ states in $F$.

The initial state is $(W_{\text{init}}, A, X_0, \rho_0, \emptyset, \mu_0)$. Final states are without any variables and they are defined w.r.t. some distinguished letters $c_1, \ldots, c_k \in C$. Formally, a state $(W, B, \emptyset, \rho, \emptyset, \mu)$ is final if

1. $W = W'$,
2. The element $W \in M(B, \rho, \emptyset, \mu)$ has a prefix of the form $\#c_1\# \cdots \#c_k\#$, where $c_1, \ldots, c_k$ are distinguished letters.

**D.1.2. Transitions**

This extends the material in Transitions in Section 4. Transitions between states in $F$ are labeled transitions where the label is an $A$-endomorphism of $C^*$. The initial state does not have any incoming transition and final states do not have outgoing transitions.

There are two types of transitions: those which transform variables (substitution transitions) and those which transform constants (compression transitions).

**D.1.3. Substitution transitions**

This extends the definition of the substitution transitions as given in [1]. Substitution transitions transform the variables and do not affect the constants. For an equation $(W, B, X, \rho, \theta, \mu)$ such a transition is defined by a $B$-morphism $\tau$

$$\tau : (W, B, X, \rho, \theta, \mu) \rightarrow (W', B', X', \rho', \theta', \mu').$$

Intuitively, $\tau(X)$ is the substitution for $X$. Thus $\tau$ should satisfy the following properties

- $W'$ is obtained by substituting variables;
- $\tau$ is non-trivial, i.e., it substitutes something;
- the alphabet is not modified;
- the resources of each letter are the same and the resources of $X$ before the substitution and $\tau(X)$ after the substitution are the same;
Lemma 20. Let $V = \alpha, \sigma \rightarrow (W', B', \rho', \theta', \mu') = V'$ be a substitution transition. Let $\alpha : M(B', \rho', \theta', \mu') \rightarrow M(B, \rho, \theta, \mu)$ be an $A$-morphism at state $V$ and let $\sigma'$ be a $B$-solution to $V'$. Define a $B$-morphism $\sigma : M(B, X, \rho, \theta, \mu) \rightarrow M(B', \rho', \theta', \mu')$ by $\sigma(X) = \sigma'(X)$. Then $(\alpha, \sigma)$ is a solution at $V$ and $(\alpha, \sigma')$ is a solution at $V'$. Moreover, $\alpha \sigma(W) = \alpha \varepsilon \sigma'(W')$ where $\varepsilon$ is the identity morphism from $M(B, \rho', \theta', \mu')$ to $M(B, \rho, \theta, \mu)$.

Proof. Since $W' = \tau(W)$ and $\sigma'$ is a $B$-solution to $V'$ we have

\[
\sigma(W) = \sigma'(\tau(W)) = \sigma'(W') = \sigma'(W) = \sigma'(W) = \sigma'(W) = \sigma'(W) = \sigma(W) = \sigma(W) = \sigma(W)
\]

by definition of $\sigma$

by (12)

as $\sigma'$ is a $B$'-solution of $W'$

as $\sigma'$ is morphism of monoids with involution

by (12)

as $\sigma$ is a morphism of monoids with involution.

So $\sigma$ is a $B$-solution and hence $(\alpha, \sigma)$ is a solution at $W$. Since $M(B, \rho, \theta, \mu) = M(B', \rho', \theta', \mu')$ we see that $(\alpha, \sigma')$ is a solution at $V'$. The assertion $\alpha \sigma(W) = \alpha \varepsilon \sigma'(W')$ is trivial.

D.1.4. Compression transitions

We now generalise the properties of compression transitions, as defined in (2).

These transitions transform the constants, but do not affect the variables. For states $(W, B, X, \rho, \theta, \mu)$ and $(W', B', X', \rho', \theta', \mu')$ such a transition is defined by a $A$-morphism $h : M(B', \rho', \theta', \mu') \rightarrow M(B, \rho, \theta, \mu)$. Intuitively, it should satisfy the following constraints:

- $W$ is obtained by applying $h$ to $W'$;
- $h$ is non-trivial and it is not length-decreasing;
- it does not affect variables: types, resources and constraints of variables are preserved.

Additionally, there are two technical conditions, all those conditions are formalised as:

\[
W = h(W'), B \neq B' \text{ or } \exists a \in B h(a) \neq a, B \subseteq B' \text{ or } B' \subseteq B, \forall a \in B' |h(a)| \geq 1, X = X', \\
\rho(X) = \rho'(X), \theta(X) = \theta'(X), \mu(X) = \mu'(X), \exists X h(X) = c \implies h(y) \in y^*, ||h|| \in O(n) \tag{13}
\]
Note that condition “$h(y) \in y^*$ whenever $\theta'(X) = y$ and $X \in X^0$” implies that $h$ can be lifted to an $A \cup X$-morphism $h : M(B', B', \rho', \theta', \mu') \rightarrow M(B, X, \rho, \theta, \mu)$. Indeed, in $M(B', X, \rho', \theta', \mu')$ the defining relations in between variables and constants have the form $yX = Xy$ for $\theta'(X) = y$. Due to $\theta(X) = \theta'(X)$ this implies $y \in (B' \cap B)^*$ and hence, $h(yX) = h(y)X = Xh(y) = h(Xy)$ in $M(B, X, \rho, \theta, \mu)$ because $h(y) \in y^*$.

If (13) is satisfied, then there is a compression transition

$$(W, B, \chi, \rho, \theta, \mu) \xrightarrow{h} (W', B', \chi, \rho', \theta', \mu').$$

The direction of the morphism $h$ is again opposite to that of the transition.

As for substitution transition, if the target equation has a solution, then the source equation has a corresponding solution, which is essentially obtained by applying $h$ on the target solution.

**Lemma 21.** Let $V = (W, B, \chi, \rho, \theta, \mu) \xrightarrow{h} (W', B', \chi, \rho', \theta', \mu') = V'$ be a compression transition. Let $\alpha : M(B, \rho, \theta, \mu) \rightarrow M$ be an $A$-morphism at the state $V$ and let $\sigma'$ be a $B'$-solution at $V'$. Define $\sigma : M(B, \chi, \rho, \theta, \mu) \rightarrow M(B, \chi, \rho, \theta, \mu)$ by $\sigma(X) = h\sigma'(X)$ for $X \in X$ and $\sigma(a) = a$ for $a \in B$.

Then $\sigma$ is a well-defined $B$-morphism, $(\alpha h, \sigma')$ is a solution at $V'$, and $\sigma h = h\sigma'$. In particular, $\sigma(W) = \alpha h\sigma'(W)$ and $(\alpha, \sigma)$ is a solution at $V$.

**Proof.** In order to see that $\sigma$ is a well-defined $B$-morphism, it is enough to show that if $xy = yx$ in $M(B, \chi, \rho, \theta, \mu)$, then $\sigma(xy) = \sigma(yx)$ in $M(B, \rho, \theta, \mu)$. This is clear for $x, y \in B^*$. Thus, we may assume that $x = X \in X$. For $y \in X$ this is also clear, because $\sigma(Z) = h\sigma'(Z)$ for all $Z \in X$ and $h\sigma'$ is a morphism. It remains to consider $x = X \in X$ and $y \in B^*$. There are two options: the commutation may be due to resources or due to types. If $\rho(X) \cap \rho(y) = \emptyset$, then $\rho(\sigma(X)) \subseteq \rho(\sigma'(X)) = \rho(X)$ and hence, $\rho(\sigma(X)) \cap \rho(y) = \emptyset$, too. Finally, if $\theta(X) = y$, then $\sigma'(X) \in y^*$. Hence $\sigma(X) = h\sigma'(X) \in y^*$ (due to $h(y) \in y^*$) and therefore

$$\sigma(Xy) = \sigma(X)\sigma(y) = \sigma(Xy) = y\sigma(X) = \sigma(yX).$$

By definition, $\mu h = \mu'$ and $\mu \alpha = \mu$. Hence $(\alpha h, \sigma')$ is a solution at $V'$. Now, $h(X) = X$ for all $X \in X^0$. Hence, $\sigma(h(X)) = \sigma(X) = h\sigma'(X)$. For $b' \in B'$ we obtain $\sigma h(b') = h(b') = h\sigma'(b')$ since $\sigma$ and $\sigma'$ are the identity on $B'$ and $B$ respectively. It follows $\sigma h = h\sigma'$ and hence, $\sigma(W) = \alpha h\sigma'(W')$. Next,

$$\sigma(W) = \sigma(h(W')) = h(\sigma'(W')) = h(h\sigma'(W)) = \sigma(h(W)) = \sigma(W).$$

Thus, $\sigma$ is a $B$-solution to $V$ and, consequently, $(\alpha, \sigma)$ solves $V$. 

**D.1.5. Initial transitions**

At initial states we allow only outgoing transitions, all of them are substitution transitions. To be more precise, we allow a transition

$$(W_{\text{init}}, A, \chi_{\text{init}}, \rho_{\text{init}}, \theta_{\text{init}}, \mu_{\text{init}}) \xrightarrow{h} (W', A, \chi', \rho', \theta', \mu')$$

only if is it defined by some $\tau$ such that for each $X$ we have $\tau(X) = u$ or $\tau(X) = uX$ where $0 \leq |u| \leq |X|$ and $X \subseteq X$. 

The idea is as follows. If $\sigma$ is an $A$-solution for $V_{\text{init}}$ then we can follow an outgoing transition to put us in some convenient situation. We can remove all $X$ with $\sigma(X) = 1$. We can adjust $\rho(X) = \rho\sigma(X)$. No crossing Hasse arc involves the special letter $\#$.

**D.1.6. Final transitions**

For final states $(W, B, \emptyset, \rho, \emptyset, \mu)$ we need additional incoming “singular” compression transitions, in general. They are singular in the sense that the labeling $h$ is allowed to send letters to the empty word. This corresponds to the case that the initial solutions substitutes a variable to the empty word. More precisely, we add transitions

$$(h(W), B', \emptyset, \rho, \emptyset, \mu) \xrightarrow{h} (W, B, \emptyset, \rho, \emptyset, \mu)$$

whenever $(W, B, \emptyset, \rho, \emptyset, \mu)$ is final, $B' = B \setminus \{c_1, \overline{c_1}, \ldots, c_k, \overline{c_k}\}$, and the transitions satisfies the condition of a compression transition with the relaxation that we may have $h(c_i) = 1$ for some $c_i$. 

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D.2. Soundness

This extends the material in Section 4.1.1.

In this section we prove that by following the transitions in the NFA $F$ from an initial to a final state and applying the corresponding endomorphisms in reverse order gives a solution to the equation $W_{\text{init}}$.

Recall that we have chosen distinguished letters $c_1, \ldots, c_k \in C$ and that if $(W, B, \emptyset, \rho, \emptyset, \mu)$ is a final state, then $W = \overline{W}$ and $W \in \#c_1 \cdots \#c_k \#B^*$.

**Proposition 22.** Let $V_0 \xrightarrow{h_1} \cdots V_t$ be a path in $F$, where $V_0 = (W_{\text{init}}, A, X_0, \rho_0, \emptyset, \mu_0)$ is the initial and $V_t = (W, B, \emptyset, \rho, \emptyset, \mu)$ is a final state. Then $V_0$ has a solution $(id_M, \sigma)$ with $\sigma(W_{\text{init}}) = h_1 \cdots h_t(W)$.

Moreover, for $1 \leq i \leq k$ we have $\sigma(X_i) = h_1 \cdots h_t(c_i)$.

**Proof.** For all $0 \leq i \leq t$ let $\alpha_i = id \# h_1 \cdots h_i$ and $V_i = (W_i, B_i, X_i, \rho_i, \emptyset, \mu_i)$. For $0 \leq s \leq t$ consider the prefix $V_0 \xrightarrow{h_1} \cdots \xrightarrow{h_s} V_s$. If $V_s$ has a $B_s$-solution $\sigma_s$, then define for all $0 \leq i \leq s$:

$$\sigma_i = h_{i+1} \cdots h_s \sigma_s.$$

Then for each $i$ the $(\alpha_i, \sigma_i)$ is a solution at $V_i$. In particular, $V_s$ has a solution $(id_M, h_1 \cdots h_s, \sigma_s)$ and $V_0$ has some solution $(id_M, \sigma)$. Moreover, we claim that we have for all $0 \leq s \leq t$:

$$\sigma(W_{\text{init}}) = h_1 \cdots h_s \sigma_s(W).$$  \hspace{1cm} (14)

Claim (14) is trivial for $s = 0$. For larger $s$ it follows by induction using Lemma 20 or Lemma 21 depending on whether $h_s$ is a substitution transition or a compression transition; note that in order to use the induction we need that $\alpha_i$ is an $A$-morphism to $M$, but this holds as all $h_i$ are $A$-morphisms and the image of $h_i$ is within $M(B_{t-i}, \rho_{t-i}, \emptyset, \mu_{t-i})$ and $M(B_0, \rho_0, \emptyset, \mu_0) = M$.

Consider $s = t$. We have $W = \overline{W}$ by the definition of a final state. Since no variables occur in $W$, $\sigma_t = id_{\overline{W}}$ is the (unique) $B$-solution of $W$, so $\sigma(W_{\text{init}}) = h_1 \cdots h_t(W)$.

By definition $\#X_1 \# \cdots \#X_k \#$ is a prefix of $W_{\text{init}}$ and $\#c_1 \# \cdots \#c_k \#$ is a prefix of $W$ for the final state $V_t$, but $h = h_1 \cdots h_t$ is an $A$-morphism from $B^*$ to $M$ with $|h(c)|_{\overline{c}} = 0$ for all $c \in B$. This implies

$$\sigma(\#X_1 \# \cdots \#X_k \#) = h(\#c_1 \# \cdots \#c_k \#).$$

In particular, $\sigma(X_i) = h_1 \cdots h_t(c_i)$ holds in the trace monoid $M$ for $1 \leq i \leq k$. \hfill \Box

The soundness of the constructed NFA $F$ follows from Proposition 22 every endomorphism recognized by $F$ is a solution of the input equation.

**Corollary 23.** Define $R = L(F)$ to be the accepted set of endomorphisms in $\text{End}(C^*)$ and let $\pi_0 : A^* \to M(A_{\text{init}}, \rho_{\text{init}})$ be the canonical projection. Then we have $h(c_i) \in A^*$ for all $1 \leq i \leq k$ (hence, $\pi_0 h(c_i)$ is defined) and

$$\{(\pi_0 h(c_1), \ldots, \pi_0 h(c_k)) \in C^* \times \cdots \times C^* \mid h \in R\} \subseteq \{((\pi_0 \sigma(X_1), \ldots, \pi_0 \sigma(X_k))) \in M(A_{\text{init}}, \rho_{\text{init}})^k \mid \sigma(W_{\text{init}}) = (\overline{W_{\text{init}}})\}.$$

Here, $\sigma$ runs over all morphisms $\sigma : M(A, X_0, \rho_0, \emptyset, \mu_0) \to M$ of structured monoids.

D.3. The trimmed NFA

The NFA $F$ is sound: every path from the initial to some final state witnesses some solution $\sigma$ and reading the labels of that path yields the value for $\sigma(X)$; this is what Corollary 23 says. The automaton is however not good enough for our purposes. It has many states and transitions which lead to nowhere as they do not belong to any accepting path. Moreover, it may be that there are cycles on accepting path although there are finitely many solutions. These problems are resolved by removing unnecessary transitions and by “trimming”. We are defining the NFA $A$ a subautomaton of $F$ in Section D.3.2. Before we do so, we need a refined notion of weight.
D.3.1. Weights and the max-norm at states

In the construction of the NFA $A$ we want to guarantee that for a given solution $(\alpha, \sigma)$ the path on which we obtain this solution is of bounded (in terms of $\alpha, \sigma$ and initial equation) length. The weight of a solution $(\alpha, \sigma)$ is defined as

$$\|\alpha, \sigma\| = \sum_{X \in \mathcal{X}} |\alpha \sigma(X)|.$$ 

We need to take into the account also the state, as our algorithm sometimes changes the state without modifying the solution. The weight of a state $V = (W, B, \mathcal{X}, \rho, \theta, \mu)$ is given as a tuple:

$$\|V\| = (|W|, \omega, \omega', |W| - |\theta|, |B|) \in \mathbb{N}^5.$$ 

We also define the derives notion of max-norm

$$\|V\|_\infty = \|(|W|, \omega, \omega', |W| - |\theta|, |B|)\|_\infty = \max \{|W|, \omega, \omega', |W| - |\theta|, |B|\} \in \mathbb{N}.$$

We order tuples over linear orders lexicographically; thus, for example $(1,1,1,42,1) < (1,1,2,0,0)$, but $\|(1,1,1,42,1)\|_\infty = 42 > \||(1,1,2,0,0)\|_\infty$. Recall that the lexicographic order on $\mathbb{N}^k$ is a well-founded order, i.e., there are no infinite descending chains.

The values $\omega$ and $\omega'$ are defined as follows.

- $\omega = \sum_{a \in B} (|\rho(a)| \cdot |W|)$. Thus, the more resources are put on constants the better if the length of $W$ is not changed.
- $\omega' = |W| - \{a \in B \mid |W|_{a} \geq 1\}$. Thus, the more labels of constants are used in $W$ the better, if $|W|$ and $\omega$ did not change.

Consequently, the weight of state $V$ with a solution $(\alpha, \sigma)$ is

$$\|\alpha, \sigma, V\| = (\|\alpha, \sigma\|, \|V\|) \in \mathbb{N}^6.$$ 

Again, the tuples are ordered lexicographically.

Finally we say that a transition $V \xrightarrow{h} V'$ is weight reducing if either $V \xrightarrow{h} V'$ is a substitution transition or $V \xrightarrow{h} V'$ is a compression transition with $\|V'\| < \|V\|$. The rational behind this definition is as follows: Assume that we start at a state $V = (W, B, \mathcal{X}, \rho, \theta, \mu)$ with a solution $(\alpha, \sigma)$ and if we follow a weight-reducing transition in the NFA $F$, then we arrive at $V' = (W', B', \mathcal{X}', \rho', \theta', \mu')$ with solution $(\alpha', \sigma')$. If the transition is the substitution transition then $|\alpha \sigma(X)| \geq |\alpha' \sigma'(X)|$ and the strict inequality holds for at least one variable. Thus $\|\alpha', \sigma', V'\| < \|\alpha, \sigma, V\|$. On the other hand, if the transition is the compression transition then for each variable $|\alpha \sigma(X)| = |\alpha' \sigma'(X)|$ and so $\|V'\| < \|V\|$ implies $\|\alpha', \sigma', V'\| < \|\alpha, \sigma, V\|$. 

D.3.2. The trimmed NFA $A$

This section defines the NFA $A$ which is used in Theorem 13. We define it inside $F$ using the notion of weight from Section D.3.1. In a first step we keep all substitution transitions, but we keep only those compression transitions $V \xrightarrow{h} V'$ where $\|V\| > \|V'\|$. In other words we keep only those transitions which are weight reducing. Let us call the new (intermediate) NFA $F'$. The final step is to remove all states and the adjacent transitions which do not belong to some accepting path.; this procedure is called trimming and the resulting automaton $A$ is trim. By definition it contains weight reducing transitions, only. From now on we work in the NFA $A$.

Proposition 24. The trim NFA $A$ can be constructed in $\text{NSPACE}(n \log n)$. If it is nonempty, then the original equation is solvable. If it contains a directed cycle, then the original equation has infinitely many solutions.

Proof. We show that each step can be performed in $\text{NSPACE}(n \log n)$. In $\text{NSPACE}(n \log n) we can list all candidates for transitions $V \rightarrow V'$ and decide, whether $\|V\|, \|V'\|$ are small enough and $h$ is weight reducing.
To perform the trimming, again we can $\text{NSPACE}(n \log n)$ list all $V \rightarrow V'$ and check that they survived the first step. If so, we run a standard nondeterministic graph reachability procedure to check whether $V$ is reachable within $A$ from the initial state and that from $V'$ there is a path in $F'$ to a final state. This shows that $A$ can be constructed in $\text{NSPACE}(n \log n)$.

Now, if $A$ is a nonempty NFA, then there exists at least one accepting path as $A$ is trim: thus there exists a solution $\sigma$.

Finally, assume that $A$ contains a directed cycle. Since the compression transitions reduce the weight of the equation, this cycle has to include at least one substitution transition. Let $V \xrightarrow{h} V'$ be transition in $A$ and $(\alpha', \sigma')$ be a solution at $V'$. If this is a substitution transition then Lemma 20 shows how to construct a solution $(\alpha, \sigma)$ at $V$ with $\|\alpha, \sigma\| > \|\alpha', \sigma'\|$. If this is a compression transition then Lemma 21 shows how to construct a solution $(\alpha, \sigma)$ at $V$ with $\|\alpha, \sigma\| \geq \|\alpha, \sigma\|$. Thus, the weight of the solution decreases along a cycle.

Fix a cycle in $A$ and consider the paths $P_1, P_2, \ldots$ such that $i$-th of them goes $i$ times through this cycle. Consider also the solutions at the initial state constructed according to those paths. Then by the observation above those solutions have arbitrary large weights, thus there are infinitely many of them. □

E. Completeness

The main task at hand is to show “completeness”: starting with any solution $\sigma$ at the initial state we can reconstruct it by some path in $F$ such that $\sigma$ is obtained by composing endomorphisms on this path.

Ideally, we shall construct such a path step by step, which leads to the following notion.

**Definition 25.** Let $V = (W, B, X, \rho, \theta, \mu)$ $\xrightarrow{h}$ $(W', B', X', \rho', \theta', \mu') = V'$ be a transition in $F$ and let $(\alpha, \sigma)$ be a solution at $V$. We say that $V \xrightarrow{h} V'$ satisfies the forward-property with respect to $(\alpha, \sigma)$ if $V'$ has a solution $(\alpha h, \sigma')$ such that

$$\pi_0 \alpha \sigma(W) = \pi_0 \alpha h \sigma'(W').$$

The mapping $\pi_0$ is applied because it may be that some constants in $\alpha \sigma(W)$ had their resources increased and so the corresponding constants in $\alpha h \sigma'(W')$ may have more resources and so be different. But projecting by $\pi_0$ from $(A)$ to $A_{\text{init}}$ resolves this problem.

The notion is also used for paths of transitions satisfying the forward property. As for every $A$-morphism $h$ we have

$$\pi_0 \pi_0 = \pi_0 \text{ and } \pi_0 h = h \pi_0,$$

then for such a path it is enough to apply $\pi_0$ once at the end.

By abuse of language: if $V \xrightarrow{h} V'$ is a transition in $F$ and if the solution $(\alpha, \sigma)$ at the source $V$ is clear from the context, then we also say that the transition $V \xrightarrow{h} V'$ satisfies the forward property. Moreover, we implicitly assume that we continue with the solution $(\alpha h, \sigma')$ at $V'$. The same convention holds for paths. Consider a state $V$ with a solution $(\alpha, \sigma)$ and assume that we start at $V$ a path $V \xrightarrow{h_1} V_1 \xrightarrow{h_2} V_2$ of transitions satisfying the forward property. Then we arrive at state $V_2 = (W, B, X', \rho', \theta', \mu')$ with a solution $(\alpha', \sigma')$ such that $\pi_0 \alpha h_1 \cdots h_2 \sigma'(W') = \pi_0 \alpha \sigma(W)$.

In particular, assume that the initial state $V_0 = (W_{\text{init}}, A, X_0, \rho_0, \emptyset, \mu_0)$ has a solution $(\text{id}_M, \sigma)$. Then $(\text{id}_M, \sigma')$ is another solution where $\sigma'$ is defined by $\sigma'(\cdot) = \pi_0 \sigma(\cdot)$. Follow a path $V_0 \xrightarrow{h_1} V_1 \xrightarrow{h_2} \cdots \xrightarrow{h_i} V_i$ where $V_i = (W, B, \emptyset, \emptyset, \emptyset, \mu_i)$ is final, then $V_i$ has a solution of the form $(h_1 \cdots h_i, \text{id}_M)$. We obtain $\pi_0 \sigma(W_{\text{init}}) = \pi_0 h_1 \cdots h_i (W)$ and $h_1 \cdots h_i \in L(F)$, which shows the completeness of our construction.

We cannot guarantee that for each state and each solution there is a forward transition. It is enough to show that for some class of vertices, that include the initial state, for each solution there is a forward path. Formally, a state $V = (W, B, X, \rho, \theta, \mu)$ is a standard state if there is no type: $\theta = \emptyset$; it is small, if

$$\|V\|_\infty \leq \frac{|C|}{100}$$
It will be enough to consider only paths in $F$ which start and end in small standard vertices, but in between we need vertices which do have types and/or which are not small. The main technical result in the paper is to show that for every solution at $(W_{\text{init}}, A, X_0, \rho_0, \theta, \mu_0)$ there exists a path in $F$ to some final state where all intermediate transitions satisfy the forward property.

**Proposition 26.** Let $V_s = (W_s, B_s, X_s, \rho_s, \emptyset, \mu_s)$ be a small standard state with a solution $(\alpha_s, \sigma_s)$. Then the NFA $A$ contains a path from $V_s$ to some final state satisfying the forward property.

The rest of the paper is mostly devoted to a proof of Proposition 26.

As induction base we use the case where there are no variables: $X_s = \emptyset$. Since there are no variables we have $\sigma(W) = W = \overline{W}$ and $W$ has a prefix of the form $\#u_1\# \cdots \#u_k\#$ where $u_i \in B^*$. Let $c_1, \ldots, c_k \in C$ be the distinguished letters, which are not allowed to appear in nonfinal vertices, and $\rho(c_i) = \emptyset$. We let $\mu(c_i) = \mu(u_i)$. Define an endomorphism $h_t$ of $C^*$ by $h_t(c_i) = u_i$ and $h_t(c) = 1$ for $C \setminus B$. Then we have $\|h_t\| \leq |W_s|$ and $W_s = h_t(W^*)$ for some final state $(W, B', \emptyset, \rho', \theta', \mu')$. Clearly, this transition satisfies the forward property.

Thus, we need only to consider the case where $X_s \neq \emptyset$. Due to the definition of the outgoing transitions at the initial state in Section D.1.5 no crossing transition involves $\#$ and $\rho(X) = \rho\sigma(X)$ for all $X$. Henceforth we keep this as an invariant.

**The running theme.** Choose a linear order $\leq$ on $2^R$ such that $S \leq T$ implies $S \subseteq T$. Let $S$ be minimal in that order such that $S = \rho(a)$ for some $a \in B_s$. If $S = \emptyset$, then we back in the situation that there are no more variables, so we are done. If $S \neq \emptyset$ it is enough to find a path to some standard state $V' = (W', B', X', \rho', \emptyset, \mu')$ such that

$$\|V'\| \leq \|V_s\| + O(n)$$

and for all $a' \in B'$: either $S < \rho'(a')$ for each $a' \in B'$ or $X' = \emptyset$. Indeed, if we reach $X' = \emptyset$ we are done. If $S < \rho'(a')$, then we cannot have $\rho'(X) = S$ for any variable, because the forward property guarantees a solution at $V'$. Thus, $S < \rho'(X)$, too. So assume that removing one set $S$ can be afforded for an increase in the weight by $O(n)$. The number of subsets of $R$ is bounded by $2^{2^R}$ which in turn is constant. Thus, starting with larger sets $C$ and $\Omega$ (still of linear size) is enough to conclude Proposition 26.

**Basic operations.** We now describe some basic operations, for which it is easy to show that they are weight reducing and have the forward property.

Let $V = (W, B, X, \rho, \theta, \mu)$ be a state with a given solution $(\alpha, \sigma)$ and $X \neq \emptyset$. We define and name the following rules.

1. **Substitution.** If there exists a variable with $\sigma(X) \neq 1$, then we substitute $\tau(X) = yX$. We distinguish two cases: if $\theta(X) = \emptyset$ then we allow $y$ to be any constant or variable; if $\theta(X) \neq \emptyset$, then $y = \theta(X)$, in particular, it is a constant. The corresponding substitution transition is defined by $\tau$. If we had $\sigma(X) = \epsilon w$ and $\sigma(y) = \nu$ then following this transition, the new solution becomes $\sigma'(X) = w$, and if $\theta(X) = y$ then $w \in y^*$. In the latter case we need to verify that $\sigma'$ is a morphism: indeed, if $\theta(X) = y$ and $\sigma(X) \neq 1$, then $w \in y^*$. Hence,

$$\sigma'(Xy) = wy = yw = \sigma'(yX).$$

For the weight observe that $|\alpha\sigma(X)|$ drops and so does the weight of the solution.

2. **Remove “useless” letters.** Consider a state $V = (W, B, X, \rho, \theta, \mu)$ such that $B$ contains a letter $a \in B \setminus A$ but $a$ does not appear in $\sigma(W)$. Such a letter is called useless and can be removed. More precisely, let $B' = A \cup \{c \in B \mid c \text{ appears in } \sigma(W)\}$ and suppose $B' \neq B$. Then we have $W \in M(B', X, \rho, \theta, \mu)$ and $\sigma(W) \in M(B', X, \rho, \theta, \mu)$; and we can view $\sigma$ as a morphism $\sigma'$ from $M(B', X, \rho, \theta, \mu)$ to $M(B', X, \rho, \theta, \mu)$. Let $\alpha'$ be the restriction of $\alpha$ to $M(B', X, \rho, \theta, \mu)$. Then we have $\alpha\sigma W = \alpha'\sigma' W$; and we obtain a compression transition $(W, B, X, \rho, \theta, \mu) \xrightarrow{\epsilon} (W, B', X, \rho, \theta, \mu)$. Since this is a compression transition, it is enough to show that $\|V\| < \|V'\|$. This is true as the size of $B$ is smaller, and so the last component of the weight goes down (and it is easy to see that other cannot increase).

3. **Remove “invisible” letters at standard vertices.** We assume that there is no type and thus, $V$ has the form $V = (W, B, X, \rho, \emptyset, \mu)$. We want to make the alphabet $B$ as small as possible. This
a non-trivial issue because the solution \( \sigma \) might use letters from \( B \) which are neither visible in \( W \) nor belong to \( A \) (hence they are useful). Let \( c \) be such a constant. Define \( B' = B \setminus \{c, \overline{c}\} \). Consider the trace

\[
\alpha(c) = a_1 \cdots a_k \in M(A, \rho_0), \text{ where } a_1, \ldots, a_k \in A.
\]

Defines a trace

\[
\beta(c) = (a_1, \rho(c)) \cdots (a_k, \rho(c)) \in M(A, \rho_0)
\]

and observe that

\[
\pi_0 \beta(c) = \pi_0 \alpha(c) \in M(A_{\text{init}}, \rho_{\text{init}}).
\]

Indeed, \( \pi_0(a, S) = \pi_0(a) \) for all \( (a, S) \in A \). Since \( \alpha \) is a morphism, we have \( \rho(a_i) \subseteq \rho(c) \) for all \( a_i \); and we have \( \mu(c) = \mu \alpha(c) = \mu \beta(c) \). We extend \( \beta \) to a morphism \( \beta : M(W, B, X, \rho, \emptyset, \mu) \to M(W, B', X, \rho, \emptyset, \mu) \) by leaving all symbols in \( B' \cup X \) invariant. This is morphism because there are no types: It is enough to verify that \( \rho(x) \cap \rho(y) = \emptyset \) implies \( \beta(xy) = \beta(yx) \) (which in turn is trivial). It is clear that \( \sigma' \) defines a \( B' \)-solution at \( V' = (W, B', X, \rho, \emptyset, \mu) \). Moreover, for all \( x \in B' \cup X \) we have

\[
\pi_0 \alpha \sigma(x) = \pi_0 \alpha \sigma'(x) = \pi_0 \alpha \sigma'(x).
\]

Thus, if \( \alpha' \) denotes the restriction of \( \alpha \) to \( B'^* \), then we obtain \( \pi_0 \alpha' \sigma(W) = \pi_0 \alpha \sigma(W) \). As a consequence, the transition

\[
(W, B, X, \rho, \emptyset, \mu) \xrightarrow{\varepsilon} (W, B', X, \rho, \emptyset, \mu)
\]

satisfies the forward property. Note that the transition is labeled \( \varepsilon = \text{id}_{C^*} \). Any composition of such transitions is a transition in \( F \); clearly if the composition changes, then one more letter is removed from the alphabet and there are only \( O(n) \) such letters. Clearly the removal of invisible letters satisfies the forward property. Hence, we can remove all invisible letters using a single transition. (The reader might also notice that Equation \( \text{[15]} \) needs the prefix \( \pi_0 \), otherwise it would not hold, in general.)

For the weight reduction observe that this is a compression transition, we should show that the weight of the equation decreases. This is the case, as we decrease the size of the alphabet, so the last component of the weight decreases and it is easy to see that other cannot increase.

4. **Introduce letters via renaming.** The idea is to introduce a letter from \( B' \setminus B \) such that it becomes visible \( W \) by using a renaming arc \( V \xrightarrow{h} (W', B', X, \rho', \theta', \mu') \). Suppose that the morphism \( h \) is defined by \( h(c) = a \) and \( W = h(W') \neq W' \). Since \( |W| = |W'| \) and \( |B| \leq |B'| \) to ensure that this is weight-decreasing we replacement of visible (in \( W' \)) letter \( c \) only if \( c \) does not appear in \( W \) and, in addition, one of the following holds:

a) \( \rho(a) \not\subseteq \rho'(c) \).

b) \( \theta'(c) = a \).

c) The letter \( c \) appears in the range of a type, but \( a \) does not.

It is easy to see that each of those cases indeed corresponds to a decrease of weight of the equation.

5. **Compression of visible letters.** Let \( V = (W, B, X, \rho, \theta, \mu) \) be a state and let \( B \subseteq B' \). Consider a word \( ab \in (B \setminus \{\#\})^* \) where \( a, b \) are letters such that \( ab \) occurs as a factor in \( W \). We assume \( \overline{a} \neq b \). Choosing any letter \( c \in B' \setminus B \) we can define a compression arc \( V \xrightarrow{h} (W', B', X, \rho', \theta', \mu') \) where \( h \) is defined by \( h(c) = ab \). Suppose that \( W = h(W') \neq W' \), then we have \( |W| < |W'| \) and hence the weight decreases as this is a compression operation and the length of the equation drops.

**Lemma 27.** Let \( V = (W, B, X, \rho, \theta, \mu) \) be a state with a solution \((\alpha, \sigma)\). All basic operations above define transitions \( V \xrightarrow{h} V' \) which satisfy the forward property. Moreover, if \((\alpha', \sigma')\) is the corresponding solution at state \( V' \), then we have

\[
\|\alpha', \sigma', V'\| < \|\alpha, \sigma, V\|.
\]

**Proof.** The assertion about the weight was already addressed in the construction. The forward property has to verified. This is easy and left to the reader: The most involved case is the removal of invisible letters, which was analyzed in detail above.
E.1. Processing a minimal set of resources

This section contains additional material to Section 4.2.1.

E.1.1. Lifting free unbalanced arcs at standard vertices

Recall the definition of a free Hasse arc $e = (a \to b)$ in $\sigma(W)$, which was given in Definition 5. Our primary target are the free unbalanced Hasse arcs. Consider a free Hasse arc $e = (a \to b)$ with $\rho(a) \neq \rho(b)$.

**First case.** There exists some Hasse arc $f$ in $\sigma(W)$ with $e \sim f$ where one position of the arc is visible. In this case, as $f$ is free, both positions of $f$ are visible. We may assume that $f = (a \to b)$ and $|\rho(a)| \leq |\rho(b)|$. In particular, $a \neq \#$. We choose a fresh letter $c$ with $\rho(c) = \rho(a) \cup \rho(b)$; and we replace every Hasse arc $g = (a \to b) \sim f$ by $c$ and every Hasse arc $g = (\overline{b} \to \overline{a}) \sim f$ by $\overline{c}$. This can be realized using a compression transition defined by $h(c) = ab$.

**Second case.** For all Hasse arcs $f$ in $\sigma(W)$ with $e \sim f$ no position of $f$ is visible. Let $T = \rho(a) \cup \rho(b)$. We can write $\alpha(ab) = a_1 \cdots a_k$ where $a_i \in A$. Define

$$\alpha_T(ab) = (a_1, T) \cdots (a_k, T).$$

Replacing all $f = (a \to b)$ in $\sigma(W)$ with $e \sim f$ by the word $\alpha_T(ab)$ and all $f = b \to a$ in $\sigma(W)$ with $e \sim f$ by the word $\alpha_T(ab)$, we obtain a new solution $(\alpha, \sigma')$ at the very same state $V$. Note that $ab \neq \alpha_T(ab) \neq (a, \sigma')$ since $a \to b$ is unbalanced. However, $\pi_0\alpha(ab) = \pi_0\alpha_T(ab)$. We also have $\pi_0\sigma = \pi_0\sigma'$.

**Lemma 28.** Let $V = (W, B, X, \rho, \emptyset, \mu)$ be a standard state with solution $(\alpha, \sigma)$ such that $|B \setminus A| \leq |W| \leq |C|/2$. Then following compression transitions, we can move to a state $V' = (W', B', X, \rho, \emptyset, \mu)$ with a solution $(\alpha', \sigma')$ such that:

1. $B'$ does not have useless or invisible letters.
2. $|W'| \leq |W|$.
3. $\pi_0\alpha\sigma(W) = \pi_0\alpha'\sigma'(W)$.
4. Every unbalanced arc in $\sigma'(W')$ is crossing.

In particular, the switch from $V$ to $V'$ follows a path satisfying the forward property.

**Proof.** First of all, if there is a useless or invisible letter, we may remove it by following appropriate compression transitions that satisfy the forward property by Lemma 27. This does not affect 2 (as the equation is unchanged), nor 3. For 4 observe that removal of useless letters does not influence it and for invisible letters, we may introduce only balanced arcs and unbalanced crossing arcs remain crossing. Thus, enforcing 1 can be done at any moment, also after ensuring 2.

If every unbalanced arc in $\sigma(W)$ crossing, there is nothing more to do. Thus, we may assume that there is at least one unbalanced free arc $e = (a \to b)$. We shall eliminate such arcs one by one, each elimination will not increase the size of the equation. So when we finish, all claims of the Lemma will hold.

Take any free arc $e$. If there exists some $f$ with $e \sim f$ where $f$ has a visible position, then we are in the first case above. We follow a compression transition defined by $h(c) = ab$ where $c$ is a fresh letter. It satisfies the forward property by Lemma 27 and so it satisfies 4.

In the second case we can replace every $f = (a \to b)$ with $e \sim f$ by $\alpha_T(ab)$ defining thereby a new $B$-solution $\sigma'$. This happens without leaving the state or following any path: we only switch to the solution $(\alpha, \sigma')$. Furthermore,

$$\pi_0\alpha\sigma(W) = \alpha\pi_0\sigma(W) = \alpha\pi_0\sigma'(W) = \pi_0\alpha\sigma'(W).$$

so 4 still holds. This ends the proof. □
E.1.2. Strategy: if $S = \rho(x)$ is minimal, then remove all $S$-letters

We consider a standard state $V = (W, B, \mathcal{X}, \rho, \emptyset, \mu)$ where $\mathcal{X} \neq \emptyset$ with a solution $(\alpha, \sigma)$. Recall our general assumption made at the very beginning that $\rho(X) \neq \emptyset$ for all $X$. As a consequence, let $S \subseteq \emptyset$ be of minimal size such that $S = \rho(x)$ for some $x \in B \cup \mathcal{X}$. Then we have $S \neq \emptyset$.

The idea is to remove in $B$ all $S$-constants without introduction of $T$-constants, where $T < S$. If we succeed to do so by keeping the length of $W$ in $O(n)$, then we are done: after at most $2|\mathcal{R}|$ steps we have $\rho(a) = \emptyset$ for all $a \in B$ but still $\rho(X) \neq \emptyset$ for the remaining variables, which simply means there are no variables in any solvable equation. We are done because we view $|\mathcal{R}|$ as a constant. In the following we fix a minimal $S$.

E.1.3. Visibility of maximal $S$-runs

We want to ensure that every maximal $S$-run has an occurrence with at least one visible position. The intuition is as follows: Let $b$ be a first $S$-letter in a $\sigma(X)$. Then there is a constant $a$ such that the arc $a \rightarrow b$ is in $\sigma(w)$ and by Lemma 28 it is crossing. Ideally we would like to substitute $X$ with $bX$, claiming that $b$ is minimal, but this is not necessarily the case.

In general, an $S$-factorisation of $\sigma(X)$ (if it exists) is defined as $uvb$, where $\rho(b) = S$ and $S \nsubseteq \rho(u)$, to make the factorisation unique we additionally assume that $|u|$ is minimal. Observe that if $a \rightarrow b$ is a crossing arc in $aX$ (for a solution $\sigma$) then $\sigma(X)$ has an $S$-factorisation $uvb$ for some $u, v$.

We iterate over the list of all variables and for each $X$ we guess, whether it has an $S$-factorisation. If so, then we follow a substitution transition $X \mapsto X' bX$. We define (resp. redefine):

- $\rho(X') = \rho(u)$ and $\rho(X) = \rho(v)$,
- $\mu(X') = \mu(u)$ and $\mu(X) = \mu(v)$.

Lifting all free unbalanced arcs followed by the above procedure establishes the following lemma.

Lemma 29. Let $V = (W, B, \mathcal{X}, \rho, \emptyset, \mu)$ be a standard state and $(\alpha, \sigma)$ be a solution. Increasing the length of $W$ by at most $O(n)$, we may assume that every maximal $S$-run has an occurrence with at least one visible position.

Proof. We use Lemma 28 to convert the state, by lifting free unbalanced arcs, to some state where all unbalanced $S$-arcs are crossing. Consider a maximal $S$-run $b_1 b_2 \cdots b_m$. Take any arcs $a \rightarrow b_1$ and $b_m \rightarrow c$, they are unbalanced, as $a, c$ are not $S$-constants. Thus by Lemma 28 they are crossing.

As a first step, we want to show that for every maximal $S$-run there is one of its occurrences such that one of the $S$-letters is part of a visibly crossing arc.

We extend the relation $\sim$ from arcs to sequences of arcs (of which we think as factors, as we in fact apply this notion only to $ab_1 b_2 \cdots b_m e$) if $e_1, e_2, \ldots, e_k$ are consecutive arcs in a trace then $\sim$ is the smallest equivalence relation satisfying

- If $e_1, e_2, \ldots, e_k$ and $f_1, f_2, \ldots, f_k$ are the corresponding arcs in $\sigma(W)$ then $e_1 e_2 \cdots e_k \sim f_1 f_2 \cdots f_k$.
- If all $e_1, e_2, \ldots, e_k$ are invisible within the same occurrence of a variable $\sigma(X)$ and $f_1, f_2, \ldots, f_k$ are the corresponding arcs in some different occurrence of $\sigma(X)$, then $e_1 e_2 \cdots e_k \sim f_1 f_2 \cdots f_k$.

We know that $e = (a \rightarrow b_1)$ is crossing, thus there is a visible crossing $e' = (a \rightarrow b)$ (or $e' = (b \rightarrow \bar{b})$, but we consider only the first for simplicity of presentation) such that $e \sim e'$. Consider, whether our $a \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_m \rightarrow c$ has a sequence of arcs in $\sim$ beginning with $e'$. If so then this is a maximal $S$-run with a letter in a visibly crossing arc: $b_1$. If not, then let us retrace the sequence of arcs $e = e_1 \sim e_2 \sim \cdots \sim e_k = e'$ and check, what is the last such edge $e_i$ which begins a sequence of arcs in $\sim$ with $a \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_m \rightarrow c$.

It cannot be that $e_{i+1}$ is the corresponding involuted arc (of $e_i$), as then the same would apply also to the arcs in the rest of $a \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_m \rightarrow c$. So $e_i$ is in $\sigma(X)$ for some variable $X$ and $e_{i+1}$ is the corresponding arc in a different occurrence of $\sigma(X)$. Since this does not apply to arcs in $a \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_m \rightarrow c$, at least one of them is not in this occurrence of $\sigma(X)$, and so some arc in $a \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots \rightarrow b_m \rightarrow c$ is visibly crossing.
We first uncross all arcs from $H\sigma$ variables), we can follow transitions satisfying the forward property such that we arrive at a standard state $V$. Let $X \rightarrow \rightarrow a$ be a part of a crossing arc. By symmetry we may assume that $a \rightarrow b$, where $b$ is a $S$-letter and $a$ may but not have to be an $S$-letter, crosses the left border at the occurrence of some $\sigma(X)$, which means that $\sigma(X)$ has an $S$-factorisation.

We iterate over the list of all variables, for each one of them we guess, whether it has an $S$ factorisation. If so we follow the substitution transition $X \rightarrow \rightarrow X' b X$, which makes the position of $b$ visible. After doing this for every variable, every maximal $S$-run has an occurrence with a visible position.

Note that every $X$ created at most two variables, which gives the claim on the size of the equation. □

We keep the property in Lemma 29 from now on as an invariant.

### E.2. How to finish if all maximal $S$-runs are short

After iterating over the $S$-run compression procedure in Section E.3 we arrive eventually at a standard state $V = (W, B, X, \rho, \emptyset, \mu)$ with given a solution $(\alpha, \sigma)$ where all maximal $S$-runs $s$ are short, meaning here $|s| \leq 2$.

There will be no more compression, we remove the remaining $S$-constants by lifting $S$-arcs.

**Lemma 30.** Let $V = (W, B, X, \rho, \emptyset, \mu)$ be a standard state with solution $(\alpha, \sigma)$ where all $S$-runs have length at most two. By increasing the length of $W$ by at most $O(n)$ (and, hence, with at most $O(n)$ fresh variables), we can follow transitions satisfying the forward property such that we arrive at a standard state $V' = (W', B', X', \rho', \emptyset, \mu')$ where we have $\rho(b) > S$ for all $b \in B$.

**Proof.** We begin with a preprocessing. Consider, one after another, all variables $X$ where $\sigma(X)$ has an occurrence with a visibly crossing Hasse arc $a \rightarrow b$ at the left border with $\rho(b) = S$. In such a case we can factorize $\sigma(X) = abb'v$ where $\rho(a) \cap \rho(u) = \emptyset$ and $bb'$ is a maximal $S$-run of length one or two. In particular, $S \neq \rho(u) \subseteq \rho(X)$. For each such an $X$, we follow a substitution transition induced by $X \rightarrow X' bb' X$ where $X'$ is a fresh variable. We define (resp. redefine):

- $\rho(X') = \rho(u)$ and $\rho(X) = \rho(v)$,
- $\mu(X') = \mu(u)$ and $\mu(X) = \mu(v)$.

Since all maximal $S$-runs have length at most two, after this substitution it holds that

- **S1** If for some $b$ and $X$ we have $\sigma(X) \in bB^* \cup B^*b$, then $\rho(b) \neq S$.
- **S2** If in any occurrence of some $S$-run one position is visible, then all positions are visible. In particular, if an $S$-arc $a \rightarrow b$ is crossing, then it is unbalanced.
- **S3** If an unbalanced Hasse arc $a \rightarrow b$ visibly crosses an occurrence of $\sigma(X)$ on the left (resp. right), then $\rho(b) \neq S$ (resp. $\rho(a) \neq S$).

We lift all free unbalanced arcs one by one, and after that remove useless and invisible letters; note that this preserves $\{S1\}$-$\{S3\}$. In particular, we may assume that there are no Hasse arcs $\# \rightarrow b$ where $\rho(b) = S$. Let us choose four disjoint subsets of $B \setminus \{\#\}$:

- Let $(S_+, S_-)$ be an involuting partition of all $S$-letters. Define $S_{\pm} = S_+ \cup S_-.$
- Let $(T_+, T_-)$ be an involuting partition of $\{b \in B \mid \rho(b) \neq S\}$. Define $T_{\pm} = T_+ \cup T_-.$

The choices of $S_+$ and $T_+$ give rise to four disjoint sets $H_1, \ldots, H_4$ of Hasse arcs in $\sigma(W)$.

$$
H_1 = \{a \rightarrow b \mid ab \in S_+ T_+ \cup T_- S_-\} \\
H_2 = \{a \rightarrow b \mid ab \in S_- T_+ \cup T_- S_+\} \\
H_3 = \{a \rightarrow b \mid ab \in S_+ T_- \cup T_+ S_-\} \\
H_4 = \{a \rightarrow b \mid ab \in S_- T_- \cup T_+ S_+\}
$$

We first uncross all arcs from $H_1$ that occur in $\sigma(W)$, then arcs from $H_2$, then $H_3$ and finally $H_4$. We investigate what happens for an occurrence of $\sigma(X)$ on the left border. (This is the same as to investigate
an occurrence of $\sigma(\overline{X})$ on the right border. Since in each round we consider $X$ as well as $\overline{X}$ (in some order) we investigate in fact the situation at both borders.)

Consider any unbalanced $S$-arc $(a \to b) \in H_1 \cup H_2$ which visibly crosses some occurrence of $\sigma(X)$ on the left border. Then $\rho(b) \neq S$ by (S3). We can factorise $\sigma(X) = uvbc$, where $\rho(u) \cap S = \emptyset$, as otherwise $a \to b$ is not an arc. We follow a substitution transition $X \mapsto X' b X$ and we change $\rho, \mu, \sigma$ accordingly. Afterwards, all arcs in $\sigma(W)$ from $H_1 \cup H_2$ that were crossing are free.

Note that $\rho(\sigma(X')) \cap S = \emptyset$, so no unbalanced $S$-arc can start or end in any occurrence of $\sigma(X')$. However, due to the splitting, some new unbalanced $S$-arc might be visibly crossing on the left for $X$ (or, equivalently, right of $\overline{X}$). The crucial observation is that they all are in $H_3 \cup H_4$: by the construction we have that $b \in T_+$ and so an $S$-arc $b \to a'$ is in $H_3 \cup H_4$. Thus, all arcs from $\sigma(W)$ that are in $H_1 \cup H_2$ are free and so they can be lifted, one after another.

Note that when we lift $a$ to $a'$ then we may assign it to $T_-$ or $T_+$. The rule is that if $a \in T_+$ ($T_-$) then $a' \in T_+$ ($T_-$, respectively); if $a \in S_{\pm}$ then $a'$ is not assigned to neither $T_2$ nor to $S_{\pm}$. As a result, all unbalanced $S$-arcs are in $H_3 \cup H_4$: we lifted existing $H_1 \cup H_2$ arcs, no new $H_1 \cup H_2$ arc was introduced and lifting the non-$S$ end of a Hasse arc does to change to which $H_i$ this arc belongs.

Thus, in $\sigma(W)$ contains only $S$-arcs from $H_3 \cup H_4$. We treat them in the same way as we treated arcs from $H_1 \cup H_2$: first we perform the preprocessing that ensures (S1)–(S3). Then we free all those arcs and finally we lift them. The analysis is the same as in the case of arcs from $H_1 \cup H_2$. In particular, all arcs from $H_3 \cup H_4$ are freed and only arcs in $H_1 \cup H_2$ could be made crossing; but as there are none, this does not happen. Arcs in $H_3 \cup H_4$ are lifted one by one and in the end there are no $S$-arcs, so no position with $S$-constants.

\subsection{S-run compression}

This section contains three procedures: $S$-block compression, $S$-quasi-block compression, and $S$-pair compression. They are repeated, in this order, until all maximal $S$-runs have length at most 2. The $S$-run compression is always applied to a minimal set of resources, i.e., such that $\rho(a) = S$ for some constant $a \in \sigma(W)$ and for each $T < S$ there is no $T$-constant in $\sigma(W)$ (recall that “<” is an arbitrary, but fixed, linear order on the sets of resources that extends the ordering by size, i.e., $|S| > |T|$ implies $S > T$).

For the space requirements during the cycle we use the following well-known fact.

\begin{lemma}
Let $p$ be any positive real number with $p < 1$ and let $s_0, s_1, \ldots$ be sequence of natural number such that $s_0 \leq s \cdot n$ for some positive constant $s$ and such that for all $i \geq 1$ we have

$$s_i \leq p \cdot s_{i-1} + s \cdot n.$$ 

Then there is a constant $q$ such that $s_i \leq q \cdot n$ for all $i$.
\end{lemma}

\subsection{S-block compression}

Block compression as employed here was described in \[4\] in the setting of free monoids. Since $S$-block compression applies only to factors of constants with the same resources, this method can be applied with virtually no changes, as long as we apply it for a fixed set $S \subseteq \mathcal{R}$ of resources.

We begin at a standard state $V = (W, B, X, \rho, \emptyset, \mu)$ with a solution $(\alpha, \sigma)$ where $\sigma(X) \neq 1$ for at least one variable. Without restriction we assume that there are no invisible letters and that all maximal $S$-runs have at least one occurrence with a visible position, see Lemma 20. Recall that $S$ is a minimal set of resources. $W$ contains some factor $aa$ where $a \in B$ and $\rho(a) = S$. The purpose of this phase is to remove all visible factors of the form $aa$, where $\rho(a) = S$. Note that after the $S$-block compression we do allow that there is such a factor in $\sigma(W)$, but it cannot be visible. During the procedure we will increase the length of the equation by $\mathcal{O}(n)$. Moreover, after the $S$-block compression the set of variables $\mathcal{X}'$ satisfies $\mathcal{X}' \subseteq \mathcal{X}$, i.e., we have only removed variables.

The first step is to create a list $\mathcal{L}$ of all pairs $(a, \lambda) \in B \times \mathbb{N}$ such that $\sigma(W)$ contains a sequence of Hasse arcs

$$d \to a \to a \longrightarrow \cdots \longrightarrow a \to e \quad \text{($\lambda$ times)}$$

(16)
where $\rho(a) = S$, $\lambda \geq 2$, $d \neq a \neq e$ and at least one position labeled with an $S$-constant in the sequence above is visible. (Thus, it could be $d$ or $e$, but in such a case we have $\rho(d) = S$ or $\rho(e) = S$.) The inner factor $a^\lambda$ in such a sequence is called a maximal visible $S$-block. Since such a block can be associated with a visible position in $W$ and at most 2 such blocks can be associated with one position of $S$-letter, $|L| \leq 2 |W| \in O(n)$.

For each $a \in B$, where some $(a, \lambda) \in L$, we take a fresh letter $c_a$ and for each $(a, \lambda) \in L$ a fresh letter $c_{a,\lambda}$. The notation is purely symbolic, so we do not write $\lambda$ as natural number. Moreover, we define a type by $\theta(c_{a,\lambda}) = c_a$.

The number of fresh letters is at most $2 |W| + |B|$. As usual, we let $\overline{c_{a,\lambda}} = \overline{c_{a,\lambda}}$ and $\overline{c_a} = \overline{c_a}$. We split the list $L$ into sublists $L_a$ where for each $a \in B$ 

$$L_a = \{(a, \lambda) \mid (a, \lambda) \in L \}.$$ 

Each $L_a$ defines a set of variables 

$$\mathcal{X}_a = \{X \in \mathcal{X} \mid \sigma(X) \in aB^* \text{ for some } (a, \lambda) \in L_a \}.$$ 

Note that $\mathcal{X}_a \cap \mathcal{X}_b \neq \emptyset$ implies $a = b$. We treat one $\mathcal{X}_a \cup \mathcal{X}_b$ after another. For simplification of notation we fix $a$ and we rename $c = c_a$ and $c_{\lambda} = c_{a,\lambda}$. Thus, the range of $\theta$ is just the set $\{c, \overline{c} \}$, but for the moment we are still at the standard state $V = (W, B, X, \rho, \emptyset, \mu)$ with its solution $(a, \sigma)$.

When we change $B$ to $B' = B \cup \{c, \overline{c}, c_{\lambda}, \overline{c}_{\lambda} \}$ and $(a, \lambda) \in L_a$ we introduce a type. We define $W'$ by replacing all maximal visible $S$-blocks $a^\lambda$ by $c_{\lambda}$. We can write $W = h(W')$ where $h$ is defined by $h(c) = h(c_{\lambda}) = a$. Thus, we follow a renaming transition 

$$V \xrightarrow{h} (W', B', X', \rho', \emptyset', \mu') = V'.$$ 

Note that $\|V'\| < \|V\|$, due to introduction of type. It is obvious how to define $(a', \sigma')$ at $V'$. Next, we consider all variables $X \in \mathcal{X}_a \cup \mathcal{X}_b$ one after another in any order. For each $X$ exactly one of the following rules is applied.

- If $\sigma(X) \in cB^*$, then choose $m \in \mathbb{N}$ maximal such that $\sigma(X) = c^m w$ and follow a substitution transition defined by $\tau(X) = cX'X$ where $X'$ is a fresh variable. We let $\sigma(X') = c^{m-1}$ and $\sigma(X) = w$. We adjust $\rho(X') = \rho(c^{m-1})$, $\mu(X') = \mu(c^{m-1})$, $\rho(X) = \rho(w)$, and $\mu(X) = \mu(w)$. We remove $X'$ if $m = 1$ and otherwise we give a type $\theta(X') = c$.

- If $\sigma(X) \in \overline{c}B^*$, then we do the analogous change as just above with $\overline{c}$ instead of $c$.

As usual, whenever we reach a situation with $\sigma(X) = 1$, then we remove the variable $X$. We scan $\sigma(W)$ from left to right and we stop at each occurrence of a maximal $c$-block $c^\lambda$ with $\lambda \in \Lambda$. If this occurrence contains a visible position, choose any of them and replace one $c$ by $c_{\lambda}$ at such a visible position. In the other case, if no position is visible replace the occurrence of the $c$-block $c^\lambda$ by $c_{\lambda}c^{\lambda-1}$. After this first scan we repeat the procedure with $\overline{c}$ instead of $c$.

Note that at this point, all constants $c_{\lambda}$ are visible: we create $c_{\lambda}$ only for maximal visible blocks, thus there was $a^\lambda$ as in (16) such that some $S$-constant was visible. This was not necessarily $a$, but if this was not $a$ then $a$ is a minimal/maximal constant in $\sigma(X)$. Thus after replacing $a$ with $c$ and the substitution we introduced at least one $c$ from this maximal visible block to the equation. And so it was chosen as the position, in which $c$ was replaced with $c_{\lambda}$.

Furthermore, for every $\tilde{c} \in \{c, \overline{c}\}$ and every occurrence of a maximal $\overline{c}$-block $c^{\lambda}$ with $\lambda \in \Lambda$ has been replaced by some block $\overline{c}^{\lambda}c_{\lambda}\overline{c}^{\lambda}$ with $\ell_1 + \ell_2 = \lambda - 1$. We mark in each such block, the positions of the $\tilde{c}$’s with the label $(\tilde{c}, \lambda)$.

Before we start the loop observe that for all $\ell_1, \ell_2 \in \mathbb{N}$ we have 

$$\overline{c}^{\lambda}c_{\lambda}\overline{c}^{\lambda} = \overline{c}^{\lambda}c_{\lambda}^{\ell_1 + \ell_2}.$$ 

This holds because $\overline{c}_{\lambda}$ and $\overline{c}$ commute. The following loop is performed until there are no more marked positions:

1. Use substitution transitions in order to guarantee for all variables: if $\theta(X) = c$, then $|\sigma(X)|$ is even.
2. Scan $\sigma(W)$ from left to right. Stop at each marked position. If the label is $(\tilde{c}, \lambda)$, then choose $\ell$ maximal such that this position occurs in a factor $\tilde{c}^\ell$. If $\ell$ is even do nothing. Otherwise, $\ell$ is odd. Then use a compression transition defined $h(c_\lambda) = c c_\lambda$. Note that this is possible, since all $|\sigma(X)|$ are even when $\theta(X) = c$, there is marked position available which is not covered by any variable.

3. Use a compression transition defined $h(c) = c^2$. This halves the number of marked positions.

4. Remove all variables with $\sigma(X) = 1$; and if there remains a variable with $\theta(X) = c$, then use a substitution transition which is defined by $\tau(X) = cX$.

It is clear that the loop eventually removes all marked positions, in particular every typed variable was removed, as it had only marked positions.

After that $c$ is useless: the last compression transition compressed factors $c_\lambda c$ into $c_\lambda$. We can extend this transition so that it also removes the type and the letter $c$, it is still weight-decreasing, as it reduces the size of the equation.

Hence, we are back at a standard state with empty type and without invisible letters. There are no invisible letters because all $c_\lambda$ are visible in $W$.

For the space analysis we must show that we do not run out of space. This is clear by Lemma 31. But actually we need more.

A simple analysis shows that the total length increase by this phase is bounded by

$$\sum_{\{a \in B \mid \rho(a) = S\}} 2 \cdot |X_a| \leq 2 \cdot |X| \leq 2 \cdot |\Omega| \in \mathcal{O}(n).$$

**Lemma 32.** After $S$-block compression there is no visible factor $aa$ for an $S$-letter $a$ in the corresponding solution $\sigma$. There are no invisible nor useless letters in the obtained equation.

The size of the equation increases by $\mathcal{O}(m)$, where $m$ is the number of occurrences of variables; all new introduced symbols are constants.

**E.3.2. $S$-quasi-block compression**

Quasi-block compression has been described in the arXiv version of [4] in the setting of free monoids. We therefore content ourselves to give a sketch which highlights the differences between an $S$-quasi-block and an $S$-block compression.

$S$-quasi-block compression is similar to $S$-block compression, but there are subtle differences. We will introduce new constants $c_\lambda$ and $c$ as above, but as we compress blocks of quasi-letters we need $\theta(c_\lambda c_\lambda) = \alpha \tau$. We would like to define a morphism satisfying $c_\lambda \tau_\lambda \mapsto c_\lambda \tau_\lambda \alpha \tau$, but such morphism is impossible. What is possible is a morphism with $c_\lambda \tau_\lambda \mapsto c_\lambda \tau_\lambda (\alpha \tau)^2$, see below. Such a morphism is enough for our purposes.

More precisely, we start at a standard state $V = (\tilde{W}, B, X, \rho, \emptyset, \mu)$ with a solution $(\alpha, \sigma)$ where $\sigma(X) \neq 1$ for at least one variable. Without restriction we assume that there are no invisible letters and, by Lemma 30, that all maximal $S$-runs have at least one occurrence with a visible position. Due to Lemma 32 there is no visible factor $aa$, where $\rho(a) = S$. But there can be quasi-letters $a \alpha \tau$. The purpose of presented subprocedure is to remove all factors $a \alpha \tau a$ from $W$, where $\rho(a) = S$. We may assume that there is at least one such factor, as otherwise there is nothing to do.

The first step is an arbitrary choice of an involuting partition

$$S_+ \cup S_- = \{a \in B \mid \rho(a) = S\}.$$  

We say that a letter $a$ and a quasi-letter $a \alpha \tau$ is positive, if $a \in S_+$, it is negative otherwise. We concentrate on positive quasi-letters; and we ignore the others. For all $X$ in any order we do: if $\sigma(X) \in \alpha B^+$ and $a \in S_+$, then follow a substitution transition defined by $X \mapsto \alpha X$. After that there are no more crossing arcs $a \mapsto \alpha \tau$ where $a \in S_+$.

The Hasse arcs $a \mapsto \alpha \tau$ and the factors $a \alpha \tau$ are self-involuting; and we are not allowed to compress them into a single letter. Imagine that we nevertheless do that. This means we take a fresh self-involuting letter $\tilde{a}$ and compress a positive quasi-letters $a \alpha \tau$ into $\tilde{a}$. Afterwards we would like to perform the $\tilde{a}$ block compression. As noted, we cannot compress $a \alpha \tau$ to $\tilde{a}$. Instead, we simulate the desired block compression of $(\tilde{a})^3$ by a quasi-block compression of $(a \alpha \tau)^3$.
Similarly as in the case of block compression, we first create a list $L$ of all pairs $(a\overline{a}, \lambda) \in S_+S_- \times \mathbb{N}$ when $\sigma(W)$ contains a sequence of Hasse arcs

$$d \rightarrow (a \rightarrow \overline{a}) \rightarrow \cdots \rightarrow (a \rightarrow \overline{a}) \rightarrow e$$

$\lambda$ times

such that $\lambda \geq 1$, and at least one position in the sequence above is visible where the resource is equal to $S$. (Thus, it could be $d$ or $e$, but then $\rho(d) = S$ or $\rho(e) = S$.) Note that we allow $\lambda = 1$. We might have $d = \overline{a}$ ($e = a$), but in such a case there is no Hasse arc $a \rightarrow d$ ($e \rightarrow \overline{a}$). The inner factor $(a\overline{a})^{\lambda}$ in such a sequence is called a maximal visible $S$-quasi-block. Note that $W$ contains a sequence of Hasse arcs $a \rightarrow \overline{a} \rightarrow a$ and $\overline{a} \rightarrow a \rightarrow \overline{a}$, and if we have $(a\overline{a}, \lambda) \in L$ for some $\lambda \geq 2$, then there is a self-involuting sequence $a \rightarrow \overline{a} \rightarrow a \rightarrow \overline{a}$.

For each $a \in S_+$, where some $(a\overline{a}, \lambda)$ occurs in $L$, we create a fresh letter $c_a$ and for each pair $(a\overline{a}, \lambda) \in L$ we create a fresh letter $c_{a,\lambda}$ with $a \in S_+$. As in the case of block compression, we content ourselves to assume $S_+ = \{a\}$ for a single letter $a$. Thus, we abbreviate $c_a = c$ and $c_{a,\lambda} = c$. Clearly, these letters are not self-involuting and we can use renaming, defined by $h(c) = h(c_{a,\lambda}) = a$ to replace maximal visible $S$-quasi-block $(a\overline{a})^{\lambda}$ by $(c\overline{c})^{\lambda}$. The type $\theta$ is now between quasi-letters; and introducing the type reduced the weight at the new state. We define

$$\theta(c_{a,\lambda}) = c\overline{c}.$$ 

Note that we have $c_{c_{a,\lambda}} = c\overline{c}_a c\overline{c}$, but we do not have any commutation between $c$ (or $\overline{c}$) and $c\overline{c}_a c\overline{c}$. In particular, $c_{a,\lambda} c\overline{c}_a c\overline{c} \neq c\overline{c}_a c\overline{c}_a c\overline{c}\lambda$.

The next step introduces fresh variables. For each $X$ such that $\sigma(X) \in c\overline{c}B^*$ let $m$ be maximal such that $\sigma(X) = (c\overline{c})^{m+1}w$. We follow a substitution transition defined by $X \rightarrow c\overline{c}X'X$, where $\sigma(X') = (c\overline{c})^m$.

For every variable $X$ with $\sigma(X) = (c\overline{c})^m$ for some $m \in \mathbb{N}$, we define a type:

$$\theta(X) = c\overline{c}.$$ 

The type says $Xc = c\overline{c}X$. But, nevertheless $Xc \neq cX$ and $X\overline{c} \neq \overline{c}X$. If additionally the $m$ such that $\sigma(X') = (c\overline{c})^m$ is odd, we follow a substitution transition $X' \rightarrow c\overline{c}X'$, so that $|\sigma(X')|$ is divisible by 4. Thus, $m$ becomes even. (If we have $m = 0$, we remove $X'$.)

We first show the main compression step and then explain how to ensure that its assumption holds. Assuming, that every $\lambda$-block $(c\overline{c})^\ell$ we have $\ell \equiv 0 \mod 4$ (thus, the length is divisible by 8), we follow a compression transition defined by $h(c) = c\overline{c}$ (halving).

After each halving we follow for each $X$ such that $\theta(X) = c\overline{c}$ a substitution transition: either defined by $\tau(X) = c\overline{c}X$ or defined by $\tau(X) = c\overline{c}\overline{c}X$. The choice is done to keep the invariant that $|\sigma(X)|$ is divisible by 4.

The additional difficulty for quasi-block compression (as compared to the block compression) arise from halving the number of $(c\overline{c})^\ell$s. Consider in $\sigma(W)$ a maximal visible quasi-block $(c\overline{c})^\ell$, suppose that it originated from a block $(a\overline{a})^{\lambda}$. Halving it leads to a first situation where this block is compressed into some $(c\overline{c})^{\ell+1}$ where $\ell + 1$ is odd. So, the length of $(c\overline{c})^\ell$ is divisible by 2 but not by 4. At this moment we introduce $c_{\lambda}\overline{c}_a$. If there is a visible position in this occurrence then we replace $c\overline{c}$ by $c\overline{c}_a c\overline{c}_a$. If no position is visible, we choose any position of a $c\overline{c}$ which we replace by $c_{a,\lambda} c\overline{c}_a$. This done by a renaming transition defined by $c_{a,\lambda} \rightarrow c$. Note that each time we use this transition, the letter $c_{a,\lambda}$ becomes visible in the equation; the proof of this fact is similar as in the case of blocks compression.

So, a maximal $c\overline{c}$-quasi-block $(c\overline{c})^{\ell+1}$ is turned into a block $c\overline{c}_a c\overline{c}_a (c\overline{c})^\ell$. To have a name we say that $c\overline{c}_a c\overline{c}_a (c\overline{c})^\ell$ is maximal $\lambda$-block. The point is that having an even exponent $\ell$ in a maximal $\lambda$-block $c\overline{c}_a c\overline{c}_a (c\overline{c})^\ell$ is not good enough. We need that before compressing a letter $c$ into $c\overline{c}$ that $\ell$ is divisible by 4. The solution for a maximal $\lambda$-block $c\overline{c}_a c\overline{c}_a$ with $\ell \equiv 2 \mod 4$ is to follow a compression transition with label $h$ defined by $h(c_{\lambda}) = c\overline{c} c_{\lambda}$. Note that

$$h(c_{\lambda}) = c\overline{c} c_{\lambda} (c\overline{c})^2 c\overline{c}_a c\overline{c}_a \lambda.$$ 

After that the remaining process becomes essentially as before in block compression. The remaining difference is that we distinguish in the exponent of powers of quasi-letters between 2 mod 4 and 0 mod 4 rather than between odd and even.
To summarize, if we started with
\[ d \to \tilde{a} \to \cdots \to \tilde{a} \to e = d \to (a \to \bar{\pi}) \to \cdots \to (a \to \bar{\pi}) \to e, \]
then we end up with
\[ d \to c_{\lambda} \to \bar{\pi} \to e. \]
As in case of block compression, we remove all \( X \) as soon as \( \sigma(X) = 1 \). We clean-up by removing invisible letters, to make this operation weight-reducing we perform it when the last \( c_\lambda \) is compressed with \( c_\lambda \bar{\pi} \), as in the case of block compression.

**Lemma 33.** After \( S \)-quasi block compression there is no visible factor \( aa, a\bar{\pi}a \) for an \( S \)-letter \( a \) in the corresponding solution \( \sigma \). There are no invisible nor useless letters in the obtained equation.

The size of the equation increases by \( O(m) \), where \( m \) is the number of occurrences of variables; all new introduced symbols are constants.

**E.3.3. \( S \)-pair compression**

The \( S \)-pair compression is used for \( S \)-block compression followed by an \( S \)-quasi-block compression. It is essentially the same method as in [12], so we focus only on the difference to the case of free monoids.

We begin at a standard state \( V = (W, B, X, \rho, \emptyset, \mu) \) with a solution \((\alpha, \sigma)\). Without restriction we may assume that the following holds.

- For some \( a \in B \) we have \( \rho(a) = S \).
- \( \sigma(X) \neq 1 \) for at least one variable.
- There are no invisible letters.
- All maximal \( S \)-runs have at least one occurrence with a visible position (Lemma 29). Hence, some constant \( a \in B \) with \( \rho(a) = S \) is visible in \( W \).
- Let \( a \in B \) with \( \rho(a) = S \). Then there is no visible factor from the set \( \{a^2, a\bar{\pi}a\} \). (Due to the previous block and quasi-block compressions, see Lemma 32 and 33.)

If all maximal \( S \)-runs have length at most 2, then we are done. Otherwise we choose an involuting partition
\[ S_+ \cup S_- = \{a \in B \mid \rho(a) = S\} = S_{\pm}. \]

The choice is done to maximize the number of visible occurrences from
\[ \{ab \in S_+S_- \mid a \neq \bar{\pi}\}. \]

A simple probabilistic argument shows that a constant fraction of \( S \)-letters occurring in \( W \) can be covered for some choice of a partition.

**Lemma 34.** Let \( V = (W, B, X, \rho, \emptyset, \mu) \) be a standard vertex and \( \sigma \) its B-solution such that there are no visible factors \( aa, a\bar{\pi}a \) for each \( S \)-letter \( a \). Let the number of occurrences of \( S \)-letters that are part of visible \( S \)-factors of length at least 3 be \( k \). Then there exists a partition \((S_+, S_-)\) of all \( S \)-letters such that the number of occurrences of factors from \( S_+S_- \) in \( W \) is at least \( k/16 \).

**Proof.** Take the random partition, every pair of \( S \)-letters \( a, \bar{\pi} \) being assigned to either \((S_+, S_-)\) or \((S_-, S_+)\) with the same probability. Consider any \( S \)-factor of length \( \ell > 2 \) and look at each of its \( \ell - 1 \) factors of length two. At least half (rounding down) of them are of the form \( ab \) where \( b \neq \bar{\pi} \) and all of them are not of the form \( aa \). Thus the chance for those two constants that \( a \in S_+ \) and \( b \in S_- \) is \( 1/4 \). It remains to verify that the expected value is at least \( \left\lfloor \frac{k}{2} \right\rfloor \cdot \frac{1}{4} \geq \frac{k}{16} \). Summing over all factors yields the claim. \( \square \)
We create a list
\[ \mathcal{L} = \{ab \in S_+S_- \mid ab \text{ is visible}\}. \]
We compress all factors \(ab \in \mathcal{L}\). For all \(X \in \mathcal{X}\) we do the following: if \(\sigma(X) \in S_-B^*\), then write \(\sigma(X) = bw\) with \(b \in S_-\); and follow a substitution transition \(X \mapsto bX\). After such a substitution there is no crossing factor from \(\mathcal{L}\). This increments the length of \(W\) by at most 2 \(S\)-letters per variable occurrence, so by \(O(n)\). Note that \(ab \in S_+S_-\) implies \(\mathcal{L}\). Thus, for each \(ab \in \mathcal{L}\) we choose a fresh letter \(c \in C\) and compress all factors \(ab\) in \(\sigma(W)\) into a single letter \(c\). This can be realized by a compression transition defined by \(h(c) = ab\).

Once, the list \(\mathcal{L}\) is empty, we remove invisible and useless letters. (We are not allowed to do it earlier, since all Hasse arcs \(a \to b\) with \(ab \in \mathcal{L}\) must remain free.) We also remove all variables with \(\sigma(X) = 1\).

As we started \(S\)-pair compression with an \(S\)-letter which was visible in \(W\), we are sure to end with an equation with a visible \(S\)-letter.

**Lemma 35.** After \(S\)-pair compression There are no invisible nor useless letters in the obtained equation. The size of the equation increases by \(O(m)\), where \(m\) is the number of occurrences of variables; all new introduced symbols are constants.

Let the number of occurrences of \(S\)-letters that are part of visible \(S\)-factors of length at least 3 be \(k\). Then we also decrease the size of the equation by \(k/16\).

### E.3.4. Space requirements for \(S\)-run compression

First observe that during the \(S\)-run compression the only new occurrences of variables are those inside the \(S\)-block compression and \(S\)-quasi block compression. Those occurrences are removed till the end of the respective subprocedure. Otherwise, the occurrences of variables are introduced only due to lifting of crossing arcs. We show that number of such occurrences can be bounded and so all states encountered on a forward path from a small standard state to a final state have \(O(n)\) occurrences of variables.

**Lemma 36.** The number of occurrences of variables in equation on a forward path from a small standard state is \(O(n)\).

**Proof.** We say that \(X\) directly created an occurrence of \(X'\) when \(X'\) was created during lifting of crossing arcs and \(X'\) was introduced in a substitution for \(X\); \(X\) created \(X'\) when there is a sequence \(X = X_1, X_2, \ldots, X_k = X'\) such that \(X_i\) directly created \(X_{i+1}\). Consider a variable \(X\); it can be split during lifting of crossing arcs at most 4 times for a fixed set of resources. This gives all variables that are directly created by \(X\). Note, that each of the directly generated variable has less resources than \(X\): when we replace \(X\) with \(X'bX\); then we require that \(\rho(X')\) is strictly smaller than \(\rho(X'bX)\).

Let \(f(k)\) be the maximum number of variables that can be created by a variable with \(k\) resources, over the whole algorithm (including itself). Using the previous discussion we get a recursive estimation the number of variables introduced by \(X\) for which initially \(|\rho(X)| = k\) and

\[
1 + \sum_{S \subseteq \mathcal{R}} 4f(k - 1) = 1 + 2^{k+2}f(k - 1)
= 1 + 2^{4+5+\cdots+k+2}f(1)
= 1 + 2^{(k+6)(k-1)/2}f(1)
\]

Since the initial equation contains at most \(n\) variables and each of them has at most \(|\mathcal{R}|\) resources and \(f(1) = 1\) (such a variable cannot introduce anything) the maximal number of occurrences of variables in the equations on a forward path is at most \(nf(n) < n(1 + 2(|\mathcal{R}|+6)(|\mathcal{R}|-1)/2)\). As we treat \(|\mathcal{R}|\) as a constant, we get the desired linear bound.

When a bound on occurrences of variables is established, we can give a bound on the number of letters introduced during the \(S\)-run compression.

**Lemma 37.** Given a state \(V\) with equation \(W\) and solution \((\alpha, \sigma)\), the whole forward path induced by the \(S\)-run compression goes only through states of size \(|W| + O(n)\).
Proof. First of all, we know that each maximal $S$-run has a visible position, so there are at most $|W|$ many of them and at most $2m$ of them have a letter that is not visible, where $m$ is the number of occurrences of variables. Hence we can disregard maximal $S$-runs that are reduced to 2 or less letters, as in total they can have at most $|W| + 2m$ letters, i.e., the claimed $|W| + O(n)$.

The number of new constants introduced during one iteration of $S$-run compression is linear in terms of number of occurrences of variables, so linear in $n$. On the other hand, we show that a constant fraction of $S$-letters in runs of length at least 3 present at the beginning of the iteration is removed from the equation till the end of this iteration. Clearly, $S$-block compression and $S$-quasi block compression can only lower this amount from some $k$ to $k'$ (and perhaps introduce some other letters). Then the pair compression removes a constant fraction of those $k'$ letters, see Lemma 35. Thus in the end we lose at least a constant fraction of those original $k$ letters.

Using Lemma 31 we get that the amount of $S$-letters is linear during all iterations of $S$-run compression.

E.4. Termination and space requirement for a forward path

The forward path for any solution is necessarily finite: by definition it decreases the weight of a solution at a state and so it needs to terminate.

It is left to show that a forward path exists in the sense that we do not consider too large equations.

Lemma 38. Given a small standard state $V$ its forward path to a final state passes only through states with equations of size additively larger by $O(n)$ than the size of the equation of $V$.

Proof. For each resource set $S$ the size of the equation is increased by $O(n)$ during the lifting of crossing arcs and by $O(n)$ during the $S$-run compression. Since the number of resources is constant, this shows that the total increase is at most $O(n)$. 

Now, the proof of Proposition 26 follows by Lemma 38, an appropriate choice of the constants, and by the observation that every forward path terminates.