L²-ASYMPTOTIC STABILITY OF MILD SOLUTIONS TO NAVIER-STOKES SYSTEM IN $\mathbb{R}^3$

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ABSTRACT. We consider global-in-time small mild solutions of the initial value problem to the incompressible Navier-Stokes equations in $\mathbb{R}^3$. For such solutions, an asymptotic stability is established under arbitrarily large initial $L^2$-perturbations.

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1. INTRODUCTION

The classical theory of viscous incompressible fluid flow is governed by the celebrated Navier-Stokes equations

\begin{align}
    u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= F, \quad (x, t) \in \mathbb{R}^3 \times (0, \infty), \\
    \text{div } u &= 0, \\
    u(x, 0) &= u_0(x),
\end{align}

where $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ is the velocity of the fluid and $p = p(x, t)$ the scalar pressure. The functions $u_0 = u_0(x)$ and $F = F(x, t)$ denote a given initial velocity and an external force. A large area of modern research is devoted to deducing different qualitative properties of solutions for the incompressible Navier-Stokes equations. The work on this subject is too broad to attempt to give a complete list of references. We will limit ourselves to discussions directly connected to issues in this paper, specifically, questions on stability of solutions in the whole three dimensional space.

There are two main approaches for the construction of solutions to the initial value problem (1.1)–(1.3). In the pioneering paper by Leray [34], weak solutions to (1.1)–(1.3) are obtained for all divergence free initial data $u_0 \in L^2(\mathbb{R}^3)^3$ and $F = 0$. These solutions satisfy equations (1.1)–(1.2) in the distributional sense and fulfill a suitable energy inequality. Fundamental questions on regularity and uniqueness of the weak solutions to the 3D Navier-Stokes equations remain open, see e.g. the books [43, 31] and the review article [10] for an additional background and references.

The second approach leads to mild solutions. These solutions are given by an integral formulation using the Duhamel principle and are obtained by means of the Banach contraction principle. Specifically, mild solutions are known to exist for large initial conditions on a finite time interval. For sufficiently small data, in appropriate scale-invariant spaces, the corresponding mild solutions are global-in-time and their dependence on data is regular. We refer the reader to [9, 10, 31] as well as to Section 5 of our paper for a review of the theory on mild solutions to problem (1.1)–(1.3).

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The goal of this work is to describe a link between these two approaches. Our result can be summarized as follows. Assume that \( V = V(x, t) \) is a global-in-time mild solution of (1.1)–(1.3), small in some scale-invariant space. We show that problem (1.1)–(1.2) has a global-in-time weak solution in the sense of Leray corresponding to the initial datum \( V(x,0) \) perturbed by an arbitrarily large divergence free \( L^2 \)-vector field. Moreover, this weak solution converges in the energy norm as \( t \to \infty \) to the mild solution \( V(x,t) \).

In other words, we show that a sufficiently small mild solution \( V(x,t) \) of problem (1.1)–(1.2) is, in some sense, an asymptotically stable weak solution of this problem under all divergence free initial perturbations from \( L^2(\mathbb{R}^3) \).

This paper generalizes the recent work of the first two authors [22], where the result was restricted to some particular solutions. In [22], consideration was given to the explicit stationary Slezkin–Landau solutions \( V = V(x) \) of the system (1.1)–(1.2) [29, 30, 42], which are one-point singular and correspond to singular external forces. A similar technique to that one in [22] has been used in [20] to show the stability of the Ekman spiral, which is an explicit stationary solution to the three-dimensional Navier-Stokes equations with rotation in the half-space \( \mathbb{R}^3_+ \) subject to the Dirichlet boundary conditions.

The approach from [22] cannot be applied in a general case of a time dependent solution \( V = V(x,t) \) (see Remark 2.9 for more details). Here, we present a new method which allows to show the \( L^2 \)-stability of a large class of mild solutions including the Slezkin-Landau ones.

Our result generalizes also a series of papers on the \( L^2 \)-asymptotic stability either of the zero solution [41, 40] or nontrivial stationary solutions [5] to the Navier-Stokes system. We only give an incomplete list of reference papers since the entire list would be overwhelming. The method proposed here allows to show this type of asymptotic \( L^2 \)-stability of time-dependent solutions including time-periodic solutions (or almost periodic) and of self-similar solutions, see Remarks 2.10 and 2.11 below. Limiting our stability results to solutions satisfying the global Serrin criterion (see Remark 2.13 below), our result relates to the asymptotic stability of large solutions with large perturbations of the Navier-Stokes equations, obtained by Kozono [26].

Notation.

- We denote by \( \| \cdot \|_p \) the usual norm of the Lebesgue space \( L^p(\mathbb{R}^3) \) with \( p \in [1, \infty] \).
- In the case of all other Banach spaces \( X \) used in this work, the norm in \( X \) is denoted by \( \| \cdot \|_X \).
- For each space \( X \), we set \( X_\sigma = \{ u \in X^3 : \text{div} \ u = 0 \} \).
- \( C_\infty^\infty(\mathbb{R}^3) \) denotes the set of smooth and compactly supported functions.
- \( \mathcal{S}(\mathbb{R}^3) \) is the Schwartz class of smooth and rapidly decreasing functions.
- The Fourier transform of an integrable function \( f \) has the normalization
  \[
  \hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, dx,
  \]
  thus \( \| f \|_2 = \| \hat{f} \|_2 \) for every \( f \in L^2(\mathbb{R}^3) \).
- We use the Sobolev spaces \( H^s(\mathbb{R}^3) = \{ f \in L^2(\mathbb{R}^3) : |\xi|^s \hat{f} \in L^2(\mathbb{R}^3) \} \) and their homogeneous counterparts \( \dot{H}^s(\mathbb{R}^3) = \{ f \in \mathcal{S}'(\mathbb{R}^3) : |\xi|^s \hat{f} \in L^2(\mathbb{R}^3) \} \) supplemented with the usual norms.
- In the case of \( p = 2 \), the standard inner product in \( L^2_\sigma(\mathbb{R}^3) \) is given by \( \langle \cdot, \cdot \rangle \).
• Constants independent of solutions may change from line to line and will be denoted by $C$.

2. Statement of the problem and main results

In the sequel, we suppose that $V = V(x,t)$ is a global-in-time solution (in the sense of distributions) to the Navier-Stokes system (1.1)–(1.3) with an external force $F = F(x,t)$ and an initial datum $V(x,0) = V_0(x)$. We require that there exists a Banach space $(\mathcal{X}_\sigma, \| \cdot \|_{\mathcal{X}_\sigma})$ such that the solution $V$ satisfies the following properties:

i) $V = V(x,t)$ is bounded and weakly continuous in time as a function with values in $\mathcal{X}_\sigma$:

\begin{equation}
V \in C_w([0,\infty), \mathcal{X}_\sigma),
\end{equation}

that is $V \in L^\infty([0,\infty), \mathcal{X}_\sigma)$ and the function $(V(t), \varphi)$ is continuous with respect to $t \geq 0$ for all $\varphi \in \mathcal{X}_\sigma^*$.

ii) There exists a constant $K > 0$ such that

\begin{equation}
\left| \int_{\mathbb{R}^3} (g \cdot \nabla)h \cdot V(t) \, dx \right| \leq K \sup_{t>0} \| V(t) \|_{\mathcal{X}_\sigma} \| \nabla g \|_2 \| \nabla h \|_2,
\end{equation}

for each $t > 0$ and for all $g, h \in \dot{H}^1_{\sigma}(\mathbb{R}^3)$ (see Remark 2.1 below).

iii) The solution $V = V(x,t)$ is sufficiently small, in the sense that

\begin{equation}
K \sup_{t>0} \| V(t) \|_{\mathcal{X}_\sigma} < 1,
\end{equation}

where $K > 0$ is the constant from inequality (2.2).

In Section 3, we recall several classical results on existence of small, global in-time mild solutions to the Navier-Stokes system (1.1)–(1.2) satisfying assumptions (2.1)–(2.3). In particular, we show that the space $\mathcal{X}_\sigma$ can be chosen either as the Lebesgue space $L^3_{\sigma}(\mathbb{R}^3)$, the weak Lebesgue space $L^{3,\infty}_{\sigma}(\mathbb{R}^3)$, the Morrey space $\dot{M}^3_p(\mathbb{R}^3)$ for each $2 < p \leq 3$, or other scaling invariant spaces, see Theorem 3.1, below.

Remark 2.1. As it is standard in the study of the Navier-Stokes system (see e.g. [13]), we define the trilinear form

\begin{equation}
b(f,g,h) \equiv \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} f_i g_j^i h^j \, dx = \int_{\mathbb{R}^3} (f \cdot \nabla)g \cdot h \, dx = \int_{\mathbb{R}^3} (f \cdot \nabla)h \cdot g \, dx
\end{equation}

for all $f, g, h \in \mathcal{S}_{\sigma}(\mathbb{R}^3)$. All equalities in (2.4) can be established combining the integration by parts with the divergence free condition. In particular, equality in (2.4) with $g = h$, implies

\begin{equation}b(f,h,h) = 0 \quad \text{for all } f, h \in \mathcal{S}_{\sigma}(\mathbb{R}^3).\end{equation}

Due to (2.4), our standing assumption (2.2) can be rewritten either as the inequality

\begin{equation}|b(g,h,V)| \leq K \sup_{t>0} \| V(t) \|_{\mathcal{X}_\sigma} \| \nabla g \|_2 \| \nabla h \|_2
\end{equation}

or

\begin{equation}|b(g,V,h)| \leq K \sup_{t>0} \| V(t) \|_{\mathcal{X}_\sigma} \| \nabla g \|_2 \| \nabla h \|_2.
\end{equation}
for all \( g, h \in \dot{H}^1_\sigma(\mathbb{R}^3) \).

**Remark 2.2.** Notice that inequality (2.2) implies that the mapping \( g \mapsto b(g, h, V) \) is a bounded linear functional on \( \dot{H}^1_\sigma(\mathbb{R}^3) \) for every \( h \in \dot{H}^1_\sigma(\mathbb{R}^3) \) and every \( V \in \mathcal{X}_\sigma \). Thus, if \( g_n \rightarrow g \) weakly in \( \dot{H}^1_\sigma(\mathbb{R}^3) \), then \( b(g_n, h, V) \rightarrow b(g, h, V) \). This observation will allow us to pass to weak limits in the trilinear form \( b(\cdot, \cdot, \cdot) \).

**Remark 2.3.** Following the notation and the terminology from the monograph [31, Ch. 21], inequality (2.2) holds true if \( \mathcal{X}_\sigma \subset X_1(\mathbb{R}^3) \), where \( X_1(\mathbb{R}^3) \) is the set of pointwise multipliers from \( H^1_\sigma(\mathbb{R}^3) \) to \( L^2_\sigma(\mathbb{R}^3) \). The linear space \( X_1(\mathbb{R}^3) \) is a Banach space equipped with the norm \( \| f \|_{X_1} = \sup\{ \| fg \|_2 : \| g \|_{H^1} \leq 1 \} \). It is easy to show using a duality argument in the Hilbert space \( L^2(\mathbb{R}^3) \) (see e.g. [33, Ch. 2]) that the inequality

\[
\left| \int_{\mathbb{R}^3} W \cdot (g \cdot \nabla) h \, dx \right| \leq C \| \nabla g \|_2 \| \nabla h \|_2
\]

holds true for all vector fields \( g, h \in \dot{H}^1(\mathbb{R}^3) \) and a certain constant \( C = C(W) \) if and only if \( W \in X_1(\mathbb{R}^3) \). We refer the reader to [31, 33, 15, 18] for more properties of pointwise multipliers and, in particular, for explanations (as well as an extensive review and a complete bibliography) how inequality (2.2) is related to the so-called weak-strong uniqueness of solutions to the Navier-Stokes equations.

We now state the main results of this work: a type of asymptotic stability for global-in-time solutions \( V = V(x, t) \) under arbitrary large \( L^2(\mathbb{R}^3) \)-perturbations.

**Theorem 2.4** (Existence of weak solutions). Let \( V = V(x, t) \) be a global-in-time solution to the initial value problem (1.1)–(1.3) in \( C_w([0, \infty), \mathcal{X}_\sigma) \) satisfying properties (2.1)–(2.3). Denote \( V_0 = V(\cdot, 0) \) and let \( w_0 \in L^2_\sigma(\mathbb{R}^3) \) be arbitrary. Then, the Cauchy problem (1.1)–(1.3) with the initial condition \( u_0 = V_0 + w_0 \) and the same external force \( F \) has a global-in-time distributional solution \( u = u(x, t) \) of the form \( u(x, t) = V(x, t) + w(x, t) \), where \( w = w(x, t) \) is a weak solution of the corresponding perturbed problem (see (2.9)–(2.11) below) satisfying

\[
w \in X_T \equiv C_w([0, T], L^2_\sigma(\mathbb{R}^3)) \cap L^2([0, T], \dot{H}^1_\sigma(\mathbb{R}^3)) \quad \text{for each} \quad T > 0.
\]

**Theorem 2.5** (Asymptotic behavior of weak solutions). A solution \( u = u(x, t) \) of problem (1.1)–(1.3) considered in Theorem 2.4 can be constructed so that

\[
\| w(t) \|_2 = \| u(t) - V(t) \|_2 \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\]

For the proofs of Theorems 2.4 and 2.5, denote by \( u = u(x, t) \) a solution of the Navier–Stokes system (1.1)–(1.3) with external force \( F = F(x, t) \) and initial data \( u_0 = V_0 + w_0 \), where \( w_0 \in L^2_\sigma(\mathbb{R}^3) \). Then, the functions \( w(x, t) = u(x, t) - V(x, t) \) and \( \pi(x, t) = p(x, t) - p_V(x, t) \) (here, \( p_V \) is a pressure associated with the velocity field \( V \)) satisfy the perturbed initial value problem

\[
w_t - \Delta w + (w \cdot \nabla) w + (w \cdot \nabla)V + (V \cdot \nabla)w + \nabla \pi = 0,
\]

\[
div w = 0,
\]

\[
w(x, 0) = w_0(x).
\]

Thus, our main goal is to construct a weak solution \( w \) of problem (2.9)–(2.11) and show its \( L^2 \)-decay to zero as \( t \rightarrow \infty \). First, recall the following standard definition.
Definition 2.6. A vector field $w = w(x, t)$ is called a weak solution to problem\(^{(2.9)}-^{(2.11)}\) if it belongs to the classical energy space
\[
X_T = C_w^\infty([0, T], L^2_{\sigma}(\mathbb{R}^3)) \cap L^2([0, T], \dot{H}^1_{\sigma}(\mathbb{R}^3))
\]
and if
\[
\langle w(t), \varphi(t) \rangle + \int_s^t \left[ (\nabla w, \nabla \varphi) + \langle w \cdot \nabla w, \varphi \rangle - \langle (w \cdot \nabla) \varphi, V \rangle + \langle (V \cdot \nabla) w, \varphi \rangle \right] \, d\tau
\]
\[
= \langle w(s), \varphi(s) \rangle + \int_s^t \langle w, \varphi_\tau \rangle \, d\tau
\]
for all $t \geq s \geq 0$ and all $\varphi \in C([0, \infty), H^1_{\sigma}(\mathbb{R}^3)) \cap C^1([0, \infty), L^2_{\sigma}(\mathbb{R}^3))$, where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2_{\sigma}(\mathbb{R}^3)$.

Theorems 2.4 and 2.5 are immediate consequences of the following result.

Theorem 2.7. For every $w_0 \in L^2_{\sigma}(\mathbb{R}^3)$ and each $T > 0$, problem\(^{(2.9)}-^{(2.11)}\) has a weak solution $w \in X_T$ (cf.\(^{(2.12)}\)) for which the strong energy inequality
\[
\|w(t)\|_2^2 + 2(1 - K \sup_{t > 0} \|V(t)\|_{X_\sigma}) \int_s^t \|\nabla w(\tau)\|_2^2 \, d\tau \leq \|w(s)\|_2^2
\]
holds true for almost all $s \geq 0$, including $s = 0$ and all $t \geq s$ and, which satisfies
\[
\lim_{t \to \infty} \|w(t)\|_2 = 0.
\]

Remark 2.8. Assuming that $w_0 \in L^2_{\sigma}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with some $1 \leq p < 2$, we expect an algebraic decay rate of the quantity $\|w(t)\|_2$ as $t \to \infty$ as in the case of the $L^2$-decay of weak solutions to the Navier-Stokes equations (see e.g.\(^{[40, 4, 38]}\)). We do not attempt to make such improvements since they are more-or-less standard.

The proof of Theorem 2.7 will be split into two parts corresponding to Theorem 2.4 and Theorem 2.5 which will be developed in Sections 4 and 5 respectively. In Section 4 using Galerkin approximations, we show the existence of a solution to equation\(^{(2.13)}\) satisfying strong energy inequality\(^{(2.14)}\). Next, in Section 5, we show that this solution satisfies a class of general energy estimates. In Section 6, suitable test functions are used in generalized energy inequalities combined with a modified Fourier splitting technique\(^{[38]}\) to yield the convergence $\|w(t)\|_2 \to 0$ as $t \to \infty$.

First, however, we discuss some consequences of Theorems 2.4 and 2.5. In the following series of remarks we describe classes of global-in-time mild solutions, which are $L^2$-globally asymptotically stable in the sense discussed above.

Remark 2.9. If an external force $F$ in equations\(^{(1.1)}\) is independent of time, one may expect system\(^{(1.1)}-^{(1.2)}\) to have stationary solutions. This is indeed the case and there are several results on the existence of small stationary solutions in scaling invariant spaces, see e.g.\(^{[28, 44, 5, 11, 12, 22, 8]}\). By Theorems 2.4 and 2.5 if these stationary solutions belong to a Banach space $X_\sigma$ and satisfy our standing assumptions\(^{(2.2)}-^{(2.3)}\), they are asymptotically stable under arbitrary large $L^2$-perturbations.

In the case when $V$ is time-independent, the linearization at zero solution of the perturbed problem\(^{(2.9)}-^{(2.11)}\) generates an analytic semigroup of linear operators on $L^2(\mathbb{R}^3)$, see\(^{[22]}\ Ch. 4\) for an example of such a reasoning in the case of singular stationary solutions. This observation allows to apply ideas of Borchers and Miyakawa\(^{[4, 5]}\).
to show the decay of $\|w(t)\|_2$. This approach cannot be used in the case of time dependent coefficients in problem \((2.9)–(2.11)\), hence we needed to develop a new technique to overcome technical obstacles in the proof of Theorem \(2.5\).

**Remark 2.10.** If the external force $F = F(x,t)$ is small in a suitable sense and time-periodic (or almost periodic in $t$), then the Navier-Stokes problem \((1.1)–(1.3)\) has a unique global-in-time mild solution in the space $C_w([0,\infty),L^3_\sigma(\mathbb{R}^3))$. This solutions is time-periodic (almost periodic in $t$, respectively), see \([14\text{ Cor. 1.2}]\) and Subsection \(3.3\) for results and references on the existence of global-in-time solutions to the Navier-Stokes equations in the Marcinkiewicz space $L^3_\sigma(\mathbb{R}^3)$. Choosing $X_\sigma = L^3_\sigma(\mathbb{R}^3)$ in Theorems \(2.4\) and \(2.5\) we obtain that, for sufficiently small external forces, these solutions are also asymptotically stable under arbitrary large divergence free initial $L^2$-perturbations.

**Remark 2.11.** Suppose now that an initial datum $V_0 \in L^3_\sigma(\mathbb{R}^3)$ (or, more generally, in a suitable Morrey spaces) is sufficiently small and homogeneous of degree $-1$. Then, the Navier-Stokes problem \((1.1)–(1.3)\) has a global-in-time mild self-similar solution i.e. a solution of the form

$$V(x,t) = \frac{1}{\sqrt{t}}V\left(\frac{x}{\sqrt{t}}\right) \in C_w([0,\infty),L^3_\sigma(\mathbb{R}^3)),\,$$

see \([9, 10, 31]\) for a review of the theory of self-similar solutions to the Navier-Stokes equations. By Theorems \(2.4\) and \(2.5\) problem \((1.1)–(1.3)\) with the initial condition $u_0 = V_0 + w_0$, where $w_0 \in L^2_\sigma(\mathbb{R}^3)$ is arbitrary, and with a suitable and small external force has a global-in-time distributional solution of the form $u(x,t) = t^{-1/2}V(x/\sqrt{t}) + w(x,t)$, where $\|w(t)\|_2 \to 0$ as $t \to \infty$.

Our main result combined with Kato’s theorems \([23\text{ Thm. 4 and 4'}]\) yields the following corollary.

**Corollary 2.12.** Fix $p \in [2, 3)$. For every $u_0 \in L^p_\sigma(\mathbb{R}^3)$ and $F \equiv 0$, the Navier-Stokes problem \((1.1)–(1.3)\) has a global-in-time (distributional) solution $u(x,t)$. This solution can be written in the form $u(x,t) = V(x,t) + w(x,t)$ where $V \in C([0,\infty),L^3_\sigma(\mathbb{R}^3))$ is a mild solution of the Navier-Stokes problem \((1.1)–(1.3)\) and $w \in X_T$ for each $T > 0$ (see \((2.12)\) is a weak solution of the perturbed problem \((2.9)–(2.11)\). These two vector fields satisfy

$$(2.15)\quad \|V(t)\|_3 \to 0 \quad \text{and} \quad \|w(t)\|_2 \to 0 \quad \text{as} \quad t \to \infty.$$

The existence of global-in-time weak solutions to problem \((1.1)–(1.3)\), with $u_0 \in L^p_\sigma(\mathbb{R}^3)$ and $p \in [2, 3)$, was established by Calderón \([8]\). Here, we improve Calderón’s result by showing that these solutions decay as $t \to \infty$ in the sense expressed in \((2.13)\). The proof of this corollary is given in Subsection \(3.2\) after discussing Hardy type inequality \((2.2)\) for $L^p$-spaces.

**Remark 2.13.** Under the assumption that the solution $V = V(x,t)$ belongs to the Serrin class: $V \in L^\alpha(0,\infty;L^q(\mathbb{R}^3))$ for $2/\alpha + 3/q = 1$ with $3 < q \leq \infty$, its global asymptotic $L^2$-stability was shown by Kozono \([20]\). Corollary \(2.12\) can be considered as an extension of the Kozono result to the limit case $q = 3$ and $\alpha = \infty$.

In the next section, we explore examples of norm-scale-invariant Banach spaces $X_\sigma$, that is spaces satisfying $\|\lambda f(\lambda\cdot)\|_X = \|f\|_X$ for all $f \in X_\sigma$ and for all $\lambda > 0$, for which...
inequality (2.2) (or, equivalently, inequalities (2.6), and (2.7)) holds true. We also recall results on the global-in-time existence of mild solutions $V \in C_w([0, \infty), \mathcal{X}_\sigma)$ to problem (1.1)–(1.3) which satisfy our standing assumptions (2.1)–(2.3).

3. HARDY-TYPE INEQUALITIES

The following theorem contains a brief introduction to the theory of global-in-time mild solutions to the Navier-Stokes problem (1.1)–(1.3). These solutions satisfy assumptions (2.1)–(2.3), hence, are they globally asymptotically stable under arbitrary $L^2$-perturbation in the sense described in Theorems 2.4 and 2.5.

Theorem 3.1. There exists a constant $K > 0$ such that the following inequality holds true

$$
(3.1) \quad \left| \int_{\mathbb{R}^3} W \cdot (g \cdot \nabla) h \, dx \right| \leq K \|W\|_{\mathcal{X}_\sigma} \|\nabla g\|_2 \|\nabla h\|_2,
$$

for all vector fields $g, h \in \dot{H}^1(\mathbb{R}^3)$ and all $W \in \mathcal{X}_\sigma$, where $\mathcal{X}_\sigma$ is one of the Banach spaces

- $\mathcal{X}_\sigma = \dot{H}^{1/2}(\mathbb{R}^3)$ (the homogeneous Sobolev space),
- $\mathcal{X}_\sigma = L^3(\mathbb{R}^3)$ (the Lebesgue space),
- $\mathcal{X}_\sigma = \{ f \in L^\infty_{\text{loc}}(\mathbb{R}^3) : \sup_{x \in \mathcal{X}_\sigma} |x| |f(x)| < \infty \}$ (the weighted $L^\infty$-space),
- $\mathcal{X}_\sigma = \mathcal{P}M^2(\mathbb{R}^3)$ (the Le Jan-Sznitman space),
- $\mathcal{X}_\sigma = L^{3,\infty}(\mathbb{R}^3)$ (the Marcinkiewicz space),
- $\mathcal{X}_\sigma = \dot{M}^p_\sigma(\mathbb{R}^3)$ for each $2 < p \leq 3$ (the Morrey space).

In the case of each space $\mathcal{X}_\sigma$, there exist constants $\varepsilon > 0$ and $C > 0$ such that for all $V_0 \in \mathcal{X}_\sigma$ such that $\|V_0\|_{\mathcal{X}_\sigma} < \varepsilon$, the Navier-Stokes problem (1.1)–(1.3) with the initial datum $u_0 = V_0$ and the external force $F \equiv 0$ has a global-in-time solution in the space $C_w([0, \infty), \mathcal{X}_\sigma)$ which satisfies $\sup_{t > 0} \|V(t)\|_{\mathcal{X}_\sigma} \leq C \|V_0\|_{\mathcal{X}_\sigma}$.

We note that the proof of inequality (3.1) is more-or-less standard (see Remark 2.3) and we sketch it for the completeness of the exposition. Let us recall the well-known continuous embeddings

$$
\dot{H}^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3) \subset L^{3,\infty}(\mathbb{R}^3) \subset \dot{M}^3_\sigma(\mathbb{R}^3)
$$

as well as

$$
\{ f \in L^\infty_{\text{loc}}(\mathbb{R}^3) : \sup_{x \in \mathcal{X}_\sigma} |x| |f(x)| < \infty \} \subset L^{3,\infty}_\sigma(\mathbb{R}^3) \quad \text{and} \quad \mathcal{P}M^2(\mathbb{R}^3) \subset L^{3,\infty}_\sigma(\mathbb{R}^3),
$$

cf. Remark 3.2. Thus, it would suffice to prove inequality (3.1) in the case of the Morrey space $\dot{M}^3_\sigma(\mathbb{R}^3)$, solely. Nevertheless, we discuss separately each of the spaces embedded in the Morrey, because this approach emphasizes the relations of our theorems with classical results on the global-in-time existence of small mild solutions to the Navier-Stokes equations.

Here, for simplicity of the exposition, we consider the Navier-Stokes problem (1.1)–(1.3) with $F \equiv 0$, however, global-in-time small mild solutions exist also in the case of small non zero external forces, see e.g. the publications [14, 44, 11, 12] and references therein. In the following subsections, other references are provided for results on the existence of solutions to the Navier-Stokes problem (1.1)–(1.3) in $C_w([0, \infty), \mathcal{X}_\sigma)$, where $\mathcal{X}_\sigma$ is one of the spaces defined in Theorem 3.1. Our list of references is far from being complete since the existing literature is extensive.
3.1. Homogeneous Sobolev space. For \( \mathcal{X}_\sigma = \dot{H}^{1/2}_\sigma(\mathbb{R}^3) \), the proof of the inequality
\[
\left| \int_{\mathbb{R}^3} W \cdot (g \cdot \nabla)h \, dx \right| \leq K \| W \|_{\dot{H}^{1/2}_\sigma} \| \nabla g \|_2 \| \nabla h \|_2
\]
for all \( f, g \in \dot{H}^1(\mathbb{R}^3) \) and \( W \in \dot{H}^{1/2}(\mathbb{R}^3) \) can be found in a much more general setting e.g. in [33, Lemma 1]. Under suitable smallness assumptions on \( u_0 \in \dot{H}^{1/2}_\sigma(\mathbb{R}^3) \) with \( F \equiv 0 \), Fujita and Kato [16] obtained existence of solutions to the Navier-Stokes problem (1.1)–(1.3) in \( C([0, \infty), \dot{H}^{1/2}_\sigma(\mathbb{R}^3)) \), satisfying conditions (2.1)–(2.3).

3.2. Lebesgue space. In the case of the Lebesgue space \( \mathcal{X}_\sigma = L^3_\sigma(\mathbb{R}^3) \), the Hölder and the Sobolev inequalities yield
\[
\left| \int_{\mathbb{R}^3} W \cdot (g \cdot \nabla)h \, dx \right| \leq \| W \|_3 \| g \|_6 \| \nabla h \|_2 \leq K \| W \|_3 \| \nabla g \|_2 \| \nabla h \|_2,
\]
for all \( f, g \in \dot{H}^1(\mathbb{R}^3) \) and \( W \in L^3_\sigma(\mathbb{R}^3) \). The existence of a global-in-time solution to the Cauchy problem (1.1)–(1.3) with \( u_0 \in L^3_\sigma(\mathbb{R}^3) \) and \( F \equiv 0 \) was established by Kato [23]. Kato’s solutions satisfy suitable decay estimates, hence Theorems 2.4 and 2.5 yield our Corollary 2.12 which improves Calderón’s result [8]. The proof of Corollary 2.12 is as follows:

Proof of Corollary 2.12. Let \( p \in [2, 3) \) and \( u_0 \in L^p_\sigma(\mathbb{R}^3) \). For each constant \( R > 0 \) define
\[
u_{0,R}(x) = \begin{cases} u_0(x) & \text{if } |u_0(x)| \leq R \\ R & \text{if } |u_0(x)| > R \end{cases}
\]
and \( \nu_{0,R}(x) = \begin{cases} 0 & \text{if } |u_0(x)| \leq R \\ u_0(x) - R & \text{if } |u_0(x)| > R \end{cases} \)
Write \( u_0 = V_0 + w_0 \), where \( V_0 = \mathbb{P}(u_{0,R}) \) and \( w_0 = \mathbb{P}(u_{0,R}) \). Here, \( \mathbb{P} : L^2(\mathbb{R}^3)^3 \to L^3_\sigma(\mathbb{R}^3) \) is the Leray orthogonal projection on divergence free vector fields. This operator can be extended to a bounded operator on \( L^p(\mathbb{R}^3)^3 \) for each \( 1 < p < \infty \).

Notice that \( V_0 \in L^q_\sigma(\mathbb{R}^3) \) for each \( q \in [p, \infty) \). Apply Kato’s theorems [23, Thm. 4 and 4′] with sufficiently small \( R > 0 \) to obtain a global-in-time mild solution \( V \in C([0, \infty), L^q_\sigma(\mathbb{R}^3)) \) of (1.1)–(1.3). This solution satisfies the standing assumptions (2.1)–(2.3) with \( \mathcal{X}_\sigma = L^3_\sigma(\mathbb{R}^3) \) and decays according to the first relation in (2.14). Notice that the truncated vector field \( u_{0,R}^T \) is nonzero on a set of finite measure hence \( w_0 = \mathbb{P}(u_{0,R}^T) \in L^2(\mathbb{R}^3) \). The existence of a decaying solution \( w = w(x,t) \) of the perturbed problem (2.10)–(2.11) follows now by Theorems 2.4 and 2.5.

3.3. Weighted \( L^\infty \)-space. Let \( \mathcal{X}_\sigma = \{ f \in L^\infty_{loc}(\mathbb{R}^3) : \| f \|_{\mathcal{X}_\sigma} = \sup_{x \in \mathcal{X}_\sigma} |f(x)| < \infty \} \). Combine the Schwarz inequality, the definition of the space \( \mathcal{X}_\sigma \), and the following Hardy inequality (cf. [33, Eq. (1.14)])
\[
\int_{\mathbb{R}^3} \frac{|g(x)|^2}{|x|^2} \, dx \leq 4 \int_{\mathbb{R}^3} |\nabla g|^2 \, dx \quad \text{for all } g \in H^1(\mathbb{R}^3)
\]
to yield
\[
\left| \int_{\mathbb{R}^3} W \cdot (g \cdot \nabla)h \, dx \right| \leq \| Wg \|_2 \| \nabla h \|_2 \leq \| W \|_{\mathcal{X}_\sigma} \| \cdot^{-1}g \|_2 \| \nabla h \|_2 \leq K \| W \|_{\mathcal{X}_\sigma} \| \nabla g \|_2 \| \nabla h \|_2
\]
for all \( f, g \in \dot{H}^1(\mathbb{R}^3) \) and \( W \in \mathcal{X}_\sigma \). The existence of global-in-time small mild solutions to the Navier-Stokes problem (1.1)–(1.3) in \( \mathcal{X}_\sigma \) was established by Cannone [9, Sec. 4.3.1],
and Cannone and Planchon [13]. Improvements of this result may be found in [36, 37, 6]. Since \( \mathcal{X}_\sigma \subset L^{3,\infty}(\mathbb{R}^3) \), we postpone the further discussion of solutions to problem [11]–[13] in these spaces to Subsection 3.5.

### 3.4. Le Jan-Sznitman space

The existence of global-in-time small singular solutions to the incompressible Navier-Stokes system with singular external forces was established in [11]. In [11], the authors also studied the asymptotic stability of solutions in the Banach spaces

\[ \mathcal{P}\mathcal{M}^2 = \{ v \in \mathcal{S}'(\mathbb{R}^3) : \hat{v} \in L^1_{loc}(\mathbb{R}^3), \quad \| v \|_{\mathcal{P}\mathcal{M}^2} = \text{ess sup}_{\xi \in \mathbb{R}^3} |\xi|^2 |\hat{v}(\xi)| < \infty \}. \]

In particular, the Slezkin–Landau singular stationary solutions, which have been considered in [22] belong to this space.

Our results on the \( L^2 \)-global asymptotic stability can be applied to solutions solutions from \( C_w([0, \infty), \mathcal{P}\mathcal{M}^2) \), as well. Indeed, let us prove inequality [31] with \( \mathcal{X}_\sigma = \mathcal{P}\mathcal{M}^2(\mathbb{R}^3) \). Properties of the Fourier transform combined with the Hölder inequality in the Lorentz spaces (see e.g. [39]) yield

\[
\left| \int_{\mathbb{R}^3} W \cdot (g \cdot \nabla) h \, dx \right| \leq \| W \|_{\mathcal{P}\mathcal{M}^2} \int_{\mathbb{R}^3} |\xi|^{-2} \| g \cdot \nabla h(\xi) \| \, d\xi \\
\leq \| W \|_{\mathcal{P}\mathcal{M}^2} \| \cdot |^{-2} \|_{L^3} \| g \cdot \nabla h \|_{L^{3,1}}.
\]

It suffices to apply the Hausdorff-Young inequality in the Lorentz spaces [25]:

\[
\| \hat{f} \|_{L^{p',r}} \leq C \| f \|_{L^{p,r}}, \quad \text{where} \ 1 < p < 2, \ 1 < r < \infty, \ \text{and} \ \frac{1}{p'} + \frac{1}{p} = 1,
\]

together with the Hölder and Sobolev inequalities for Lorentz spaces. Hence it follows

\[
\int_{\mathbb{R}^3} W \cdot (g \cdot \nabla) h \, dx \leq \| W \|_{\mathcal{P}\mathcal{M}^2} \| g \cdot \nabla h \|_{L^{3,1}}
\]

for all \( g, h \in \dot{H}^1(\mathbb{R}^3) \) and \( W \in \mathcal{P}\mathcal{M}^2(\mathbb{R}^3) \).

**Remark 3.2.** As a byproduct of the estimates in this subsection, we obtain the following short proof of the embedding \( \mathcal{P}\mathcal{M}^2(\mathbb{R}^3) \subset L^{3,\infty}(\mathbb{R}^3) \). Repeating the reasoning which leads to the first inequality in (3.2) we find a constant \( C > 0 \) such that for all \( W \in \mathcal{P}\mathcal{M}^2(\mathbb{R}^3) \) and all \( \varphi \in L^{3/2,1}(\mathbb{R}^3) \) we have

\[
\left| \int_{\mathbb{R}^3} W \varphi \, dx \right| \leq C \| W \|_{\mathcal{P}\mathcal{M}^2} \| \varphi \|_{L^{3,2,1}}.
\]

Hence, the distribution \( W \) defines a continuous linear functional on \( L^{3/2,1}(\mathbb{R}^3) \) and, as a consequence, we have \( W \in L^{3,\infty}(\mathbb{R}^3) \).
3.5. Marcinkiewicz space. The proof of inequality (3.1) with $X_\nu = L_3^{3,\infty}(\mathbb{R}^3)$ involves the Hölder and Sobolev inequalities in the Lorentz $L^{p,q}$-spaces (see e.g. [39] and [7], resp.) as follows

$$\int_{\mathbb{R}^3} W \cdot (g \cdot \nabla)h \, dx \leq C \|Wg\|_2 \|\nabla h\|_2 \leq C \|Wg\|_{L^{2,2}} \|\nabla h\|_2$$

$$\leq C \|W\|_{L^{3,\infty}} \|g\|_{L^{6,2}} \|\nabla h\|_2$$

$$\leq C \|W\|_{L^{3,\infty}} \|\nabla g\|_2 \|\nabla h\|_2$$

for all $f, g \in \dot{H}^1_3(\mathbb{R}^3)$ and $W \in L_3^{3,\infty}(\mathbb{R}^3)$ (this proof may be also found e.g. in [15] Proposition 3.4 in the case $p = 3$).

Global-in-time mild solutions to the Navier-Stokes with small initial conditions in $L_3^{3,\infty}(\mathbb{R}^3)$ have been constructed by Kozono and Yamazaki [27] (see also Barraza [2]). These solutions are unique in the space $C_0([0, \infty), L_3^{3,\infty}(\mathbb{R}^3))$ intersected with a set of functions with appropriate decay in time. The construction of [27] was improved by Meyer [35], who applied the Banach contraction principle to obtain non-decaying solutions in the space $C_0([0, \infty), L_3^{3,\infty}(\mathbb{R}^3))$. An analogous argument was used by Yamazaki [44] to deal with the Navier-Stokes equations with time-dependent external forces in the whole space, the half-space and, exterior domains. Yamazaki formulated sufficient conditions on the initial conditions and external forces to insure the existence of unique small solutions bounded for all time in weak $L^3$-spaces. Stability properties of small mild global-in-time solutions to the Navier-Stokes problem with external forces has been also studied in [12].

3.6. Morrey space. Let $1 < p \leq q < \infty$ and $p' = p/(p - 1)$ and $q' = q/(q - 1)$. The homogeneous Morrey spaces are defined as

$$\dot{M}_p^q(\mathbb{R}^3) = \{ f \in L_p^q(\mathbb{R}^3) : \|f\|_{\dot{M}_p^q} = \sup_{R > 0, x \in \mathbb{R}^3} \left( \int_{B_R(x)} |f(y)|^q \, dy \right)^{1/q} < \infty \}.$$  

It is known (see [31] Lemma 21.1 and [15] Lemma 3.13) that this space is the dual space of $\dot{N}^{q'}_{p'}(\mathbb{R}^3)$ defined in the following way. For $1 < q' \leq p' < \infty$, we set

$$\dot{N}^{q'}_{p'}(\mathbb{R}^3) = \left\{ f \in L^{q'}(\mathbb{R}^3) : f = \sum_{k \in \mathbb{N}} g_k, \text{ where } \{g_k\} \subset L_{\text{comp}}^{p'}(\mathbb{R}^3) \text{ and } \right.$$

$$\left. \sum_{k \in \mathbb{N}} d_k^n \left( \frac{1}{q'} - \frac{1}{p'} \right) \|g_k\|_{L^{p'}} < \infty, \text{ where } \forall k \quad d_k = \text{diam}(\text{supp } g_k) < \infty \right\}$$

which is a Banach space equipped with the norm

$$\|f\|_{\dot{N}^{q'}_{p'}} = \inf \left\{ \sum_{k \in \mathbb{N}} d_k^n \left( \frac{1}{q'} - \frac{1}{p'} \right) \|g_k\|_{L^{p'}} \right\}.$$  

We also recall an estimate for the product functions in $\dot{N}^{q'}_{p'}(\mathbb{R}^3)$. This estimate will be essential for the proof of our of inequality (3.1) in $X_\nu = M_3^3(\mathbb{R}^3)$.

Lemma 3.3 ([31] Prop. 21.1 and [15] Lemma 3.14]). Let $1 \leq q' \leq p' \leq 2$. There exists a constant $C > 0$ such that for all $f \in L^2(\mathbb{R}^3)$ and $g \in H^{3/q}(\mathbb{R}^3)$,

$$\|fg\|_{\dot{N}^{q'}_{p'}} \leq C \|f\|_2 \|g\|_{H^{3/q}},$$

(3.3)
The proof of inequality (3.1) follows easily by inequality (3.3) with \( q = 3 \)
\[
\left| \int_{\mathbb{R}^3} W \cdot (g \cdot \nabla)h \, dx \right| \leq \left\| W \right\|_{\dot{M}^3_p} \left\| \nabla h \right\|_{l^2}^2 \leq K \left\| W \right\|_{\dot{M}^3_p} \left\| \nabla g \right\|_2 \left\| \nabla h \right\|_2,
\]
for all \( f, g \in H^1_\sigma(\mathbb{R}^3) \) and \( W \in \dot{M}^3_p(\mathbb{R}^3) \), where \( 2 < p \leq 3 \) and \( p \) and \( p' \) are conjugate exponents.

For constructions of global-in-time mild solutions to the Navier-Stokes problem (1.1)–(1.3) in Morrey spaces, we refer the reader to [19, 21] as well as to the book [31, Ch. 18] (and to references therein) For small initial data, these solutions satisfy our standing assumptions (2.1)–(2.3) with \( X_\sigma' = \dot{M}^3_p(\mathbb{R}^3) \), where \( 2 < p \leq 3 \).

4. EXISTENCE OF WEAK SOLUTIONS

In this section, we construct weak solutions to the perturbed initial value problem (2.9)–(2.11).

**Proof of Theorem 2.7. First part – Existence of solutions.** Step one is a construction of weak solutions to problem (2.9)–(2.11) satisfying the strong energy inequality (2.14). It follows a relatively standard Galerkin technique (see e.g. [13, Ch. III. Thm. 3.1]).

Let \( \{g_m\}_{m=1}^\infty \) be an orthonormal complete system in \( L^2_\sigma(\mathbb{R}^3) \) and assume that \( g_m \in H^1_\sigma(\mathbb{R}^3) \) for every \( m \in \mathbb{N} \). Let \( W_m \) be the linear space spanned by \( \{g_1, g_2, ..., g_m\} \) for each \( m = 1, 2, ... \). Define an approximate solution \( w_m : [0, T] \to W_m \) by
\[
w_m(t) = \sum_{i=1}^m d_{im}(t)g_i,
\]
where the coefficients \( d_{im} = d_{im}(t) \) satisfy
\[
\frac{d}{dt} d_{jm}(t) = \frac{d}{dt} \langle w_m(t), g_j \rangle = \langle \nabla w_m(t), \nabla g_j \rangle + b(w_m(t), w_m(t), g_j)
\]
\[
+ b(w_m(t), V(t), g_j) + b(V(t), w_m(t), g_j) = 0 \quad \text{for} \quad j = 1, ..., m,
\]
\[
w_m(0) = P_mw_0,
\]
with the orthogonal projection operator \( P_m : L^2_\sigma(\mathbb{R}^3) \to W_m \) is given by \( P_m(v) = \sum_{i=1}^m \langle v, g_i \rangle g_i \). Recall that the term corresponding to the pressure in (2.9) vanishes in (4.1) since \( \langle \nabla \pi, g_j \rangle = 0 \) as \( \text{div} \, g_j = 0 \).

Due to assumption (2.2) (cf. Remark 2.1) both terms in (4.1) containing the solution \( V(t) \) are convergent. Moreover, they are continuous in \( t \) due to the weak continuity of \( V \) with respect to time assumed in (2.1) and comments in Remark 2.2. Thus, the system of ordinary differential equations (4.1) has a unique local-in-time solution \( \{d_{im}(t)\}_{m=1}^\infty \). In view of the a priori estimates of the sequence \( \{w_m\}_{m=1}^\infty \) obtained below in (4.2), the solution \( d_{im}(t) \) is global-in-time.

Multiply equation (4.1) by \( d_{jm} \) and sum up the resulting equations for \( j = 1, 2, ..., m \). Using relation (2.5) we have \( b(w_m, w_m, w_m) = b(V, w_m, w_m) = 0 \). Consequently,
\[
\frac{1}{2} \frac{d}{dt} \|w_m(t)\|_2^2 + \|\nabla w_m(t)\|_2^2 + b(w_m(t), V(t), w_m(t)) = 0.
\]
Applying inequality (2.2) (or its equivalent version (2.7)) and integrating from \( s \) to \( t \) yields
\[
(4.2) \quad \|w_m(t)\|_2^2 + 2(1 - K \sup_{t > 0} \|V(t)\|_{X_w}) \int_s^t \|\nabla w_m(\tau)\|_2^2 \, d\tau \leq \|w_m(s)\|_2^2 \leq \|w_0\|_2^2.
\]
Recall that, by hypothesis (2.3), we have \( K \sup_{t > 0} \|V(t)\|_{X_w} < 1 \). Thus, we can extract a subsequence, also denoted by \( \{w_{m_k}\}_{k=1}^\infty \), converging towards a vector field \( w \in L^2([0, T], \dot{H}^1_w(\mathbb{R}^3)) \cap C_w([0, T], L^2_w(\mathbb{R}^3)) \) in the following sense
\[
(4.3) \quad w_{m_k} \to w \quad \text{in} \quad L^2([0, T], \dot{H}^1_w(\mathbb{R}^3)) \quad \text{weakly}
\]
\[
(4.4) \quad w_{m_k} \to w \quad \text{in} \quad L^\infty([0, T], L^2_w(\mathbb{R}^3)) \quad \text{weak} - \ast.
\]
By standard arguments, involving fractional derivatives in time, see e.g. [43, Ch. III. Thm. 3.1], it follows that there exists a subsequence denoted again by \( \{w_m\} \) such that
\[
(4.5) \quad w_m \to w \quad \text{in} \quad L^2([0, T], L^2_{loc}(\mathbb{R}^3)).
\]
The next step is to show that the limiting vector field \( w = w(x, t) \) satisfies equation (2.13). By (4.1), we have for all \( \varphi \in W_m \)
\[
(4.6) \quad \left\langle \frac{d}{dt} w_m(t), \varphi \right\rangle + \langle \nabla w_m(t), \nabla \varphi \rangle + b(w_m(t), w_m(t), \varphi)
+ b(w_m(t), V(t), \varphi) + b(V(t), w_m(t), \varphi) = 0,
\]
\[
w_m(0) = P_m w_0.
\]
Our goal now is to obtain (4.6) with a time dependent function \( \varphi \in C^1([0, \infty), W_m) \). The Leibniz formula yields
\[
(4.7) \quad \left\langle \frac{d}{dt} w_m(t), \varphi(t) \right\rangle = \frac{d}{dt} \left\langle w_m(t), \varphi(t) \right\rangle - \left\langle w_m(t), \frac{d}{dt} \varphi(t) \right\rangle.
\]
Combining (4.7) and (4.6) we obtain
\[
(4.8) \quad \frac{d}{dt} \left\langle w_m(t), \varphi(t) \right\rangle - \left\langle w_m(t), \frac{d}{dt} \varphi(t) \right\rangle + \langle \nabla w_m(t), \nabla \varphi(t) \rangle + b(w_m(t), w_m(t), \varphi(t))
+ b(w_m(t), V(t), \varphi(t)) + b(V(t), w_m(t), \varphi(t)) = 0,
\]
\[
w_m(0) = P_m w_0.
\]
Integration of (4.8) over \([s, t]\) gives for all \( \varphi \in C^1([0, \infty), W_m) \)
\[
(4.9) \quad \left\langle w_m(t), \varphi(t) \right\rangle - \left\langle w_m(s), \varphi(s) \right\rangle - \int_s^t \left\langle w_m(\tau), \frac{d}{d\tau} \varphi(\tau) \right\rangle \, d\tau
+ \int_s^t \langle \nabla w_m(\tau), \nabla \varphi(\tau) \rangle \, d\tau + \int_s^t b(w_m(\tau), w_m(\tau), \varphi(\tau)) \, d\tau
+ \int_s^t b(w_m(\tau), V(\tau), \varphi(\tau)) \, d\tau + \int_s^t b(V(\tau), w_m(\tau), \varphi(\tau)) \, d\tau = 0,
\]
\[
w_m(0) = P_m w_0.
\]
It suffices to pass to the limit in (4.9) using the convergence in (4.3)–(4.5). The first five terms in (4.9) are dealt as is standard when working with the classical Navier–Stokes
system, (see [43] Ch. III, the proof of Thm. 3.1] and in particular [43] Ch. III, the proof of Lemma 3.2] to pass to the limit in the nonlinear term). The convergence of

\[ \int_s^t b(w_m(\tau), V(\tau), \varphi(\tau)) \, d\tau \to \int_s^t b(w(\tau), V(\tau), \varphi(\tau)) \, d\tau \]

and

\[ \int_s^t b(V(\tau), w_m(\tau), \varphi(\tau)) \, d\tau \to \int_s^t b(V(\tau), w(\tau), \varphi(\tau)) \, d\tau. \]

follows by combining Remark 2.2, property (4.4), and the Lebesgue dominated convergence theorem.

The limit vector field \( w = w(x, t) \) satisfies equation (2.13) for all \( \varphi \in C^1([0, \infty), W_m) \) for each \( m \geq 1 \), and passing to the limit, for all \( \varphi \in C([0, \infty), H^1(\mathbb{R}^3)) \cap C^1([0, \infty), L^2(\mathbb{R}^3)) \).

Hence, \( w = w(x, t) \) is a weak solution of problem (2.9)–(2.11) in the energy space \( X_T \) defined in (2.12) and, by a classical reasoning (cf. [43] Ch. III), it satisfies strong energy inequality (2.14). \( \square \)

5. Generalized energy inequalities

The idea behind the \( L^2 \)-decay proof for the weak solution \( w = w(x, t) \) constructed in Theorem 2.7 is based on the work reported in [38], where the decay was shown for solutions to the Navier–Stokes system with slowly decaying external forces. The extra terms in equation (2.9) containing the solution \( V = V(x, t) \) cause difficulties which do not appear in [38]. To handle these terms, it is necessary to obtain a class of generalized energy inequalities (see (5.7) below). The proof of such inequalities requires stronger convergence of the approximations \( \{w_m\} \) than the one stated in (4.5). The following improvement of (4.5) seems to be well-known, however, for the completeness, we recall the proof.

Lemma 5.1. There exists a subsequence of the Galerkin approximations \( \{w_m\} \) considered in the proof of Theorem 2.7 which converges towards \( w = w(x, t) \) strongly in \( L^2([0, T], L^2(\mathbb{R}^3)) \), for every \( T > 0 \).

Corollary 5.2. The sequence of the Galerkin approximations \( \{w_m\} \) from Lemma 5.1 converges strongly in \( L^p([0, T], L^p_c(\mathbb{R}^3)) \) for every \( p \in [2, 6) \) and \( T > 0 \).

Proof of Corollary 5.2. This is an immediate consequence of the Hölder inequality, the Sobolev inequality \( \|w\|_a \leq \|\nabla w\|_2 \), Lemma 5.1 and estimate (4.2). \( \square \)

Proof of Lemma 5.1. Let \( \{w_m\} \) be a sequence of the Galerkin approximations which converges towards a weak solution \( w = w(x, t) \) of problem (2.9)–(2.11) in the local sense (4.5). For every \( R > 0 \), define the cut-off function \( \varphi_R(x) = \varphi_1(x/R) \), where

\[ \varphi_1 \in C^\infty(\mathbb{R}^3) \quad \text{and} \quad \varphi_1(x) = \begin{cases} 1 & \text{for} \ |x| \geq 1, \\ 0 & \text{for} \ |x| \leq 1/2. \end{cases} \]

Substitute the test function \( \varphi(t) = w_m(t)\varphi_R^2 \) into equation (4.8). Since the function \( \varphi_R \) does not depend on \( t \), it follows that

\[ \frac{d}{dt} \langle w_m(t), w_m(t)\varphi_R^2 \rangle - \langle w_m(t), \frac{d}{dt}(w_m(t)\varphi_R^2) \rangle = \frac{d}{dt} \|w_m(t)\varphi_R\|^2_2. \]
Thus, equation (4.8) yields

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|w_m(t) \varphi_R\|_2^2 + \langle \nabla w_m(t), \nabla (w_m(t) \varphi_R^2) \rangle + b(w_m(t), w_m(t), w_m(t) \varphi_R^2) \\
+ b(w_m(t), V(t), w_m(t) \varphi_R^2) + b(V(t), w_m(t), w_m(t) \varphi_R^2) = 0.
\end{equation}

By elementary calculations, we have

\[ \| \nabla (w_m(t) \varphi_R) \|_2^2 = \| (\varphi_R \nabla w_m(t)) \|_2^2 + \| \nabla w_m(t) \varphi_R \|_2^2 + \langle \nabla w_m(t), w_m(t) \nabla \varphi_R^2 \rangle. \]

The second term in (5.1) can be rewritten as

\[ \langle \nabla w_m(t), \nabla (w_m(t) \varphi_R^2) \rangle = \| \varphi_R \nabla w_m(t) \|_2^2 + \langle \nabla w_m(t), w_m(t) \nabla \varphi_R^2 \rangle. \]

Combining the last two equalities with (5.1) gives

\[ \frac{1}{2} \frac{d}{dt} \|w_m(t) \varphi_R\|_2^2 + \| \nabla (w_m(t) \varphi_R) \|_2^2 - \|w_m(t) \varphi_R \|_2^2 + b(w_m(t), w_m(t), w_m(t) \varphi_R^2) \\
+ b(w_m(t), V(t), w_m(t) \varphi_R^2) + b(V(t), w_m(t), w_m(t) \varphi_R^2) = 0. \]

Note that \( b(w_m(t), w_m(t), w_m(t) \varphi_R^2) = 0 \). Indeed, by the definition of the form \( b = b(\cdot, \cdot, \cdot) \), the divergence free condition of \( w_m(t) \), relation (2.5), and since \( \varphi_R \) is a scalar function, we get

\begin{equation}
b(w_m, w_m, w_m \varphi_R^2) = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} w_m^i(w_m^j)_{x_i}w_m^i \varphi_R^2_{x_j} dx = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} w_m^i \varphi_R^2(w_m^j)_{x_i}w_m^i dx \\
= b(w_m^2, w_m, w_m) = 0.
\end{equation}

Similarly, we have

\begin{equation}
b(V(t), w_m(t), w_m(t) \varphi_R^2) = b(\varphi_R^2 V, w_m, w_m) = 0.
\end{equation}

Using an analogous argument combined with inequality (2.7) yields

\[ \left| b(w_m(t), V(t), w_m(t) \varphi_R^2) \right| = \left| b(w_m(t) \varphi_R, V(t), w_m(t) \varphi_R) \right| \\
\leq K \left( \sup_{t>0} \|V(t)\|_{X_0} \right) \| \nabla (w_m \varphi_R) \|_2^2. \]

Applying relations (5.2)–(5.4) in equation (5.5), gives

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|w_m(t) \varphi_R\|_2^2 + (1 - K \sup_{t>0} \|V(t)\|_{X_0}) \| \nabla (w_m(t) \varphi_R) \|_2^2 \leq \|w_m(t) \nabla \varphi_R\|_2^2.
\end{equation}

Assumption (2.3) insures that the second term on the left-hand side of (5.5) is nonnegative. Thus, integration of (5.5) from 0 to \( t \) combined with the definition of the function \( \varphi_R \) yields

\[ \|w_m(t) \varphi_R\|_2^2 \leq \|w_m(0) \varphi_R\|_2^2 + R^{-2} \| \nabla \varphi_1 \|_{\infty} \int_0^t \|w_m(s)\|_2^2 ds \\
\leq \|w(0) \varphi_R\|_2^2 + R^{-2} T \| \nabla \varphi_1 \|_{\infty} \sup_{t \in [0,T]} \|w_m(t)\|_2^2. \]

By (4.2), we have \( \|w_m(t)\|_2^2 \leq \|w_m(0)\|_2^2 \leq \|w_0\|_2^2 \) and consequently, we obtain the inequality

\begin{equation}
\|w_m(t) \varphi_R\|_2^2 \leq \|w(0) \varphi_R\|_2^2 + CTR^{-2} \|w_0\|_2^2.
\end{equation}
Using the Cantor diagonal argument, we find a subsequence of the Galerkin approximations \( \{w_m\} \) constructed in Theorem 2.7 which converges towards a weak solution \( w \) in \( L^2([0,T],L^2(B_R)) \) for every ball \( B_R \) of radius \( R > 0 \). Since, the tail estimates (5.6) are independent of \( m \), this convergence holds true in the norm of the space \( L^2([0,T],L^2(\mathbb{R}^3)) \).

This completes the proof of the lemma. \( \square \)

We now prove a class of needed generalized energy inequalities for weak solutions to the perturbed problem (2.9)–(2.11). In the sequel, the following notation is introduced.

\( \psi \in C^1([0,\infty);S(\mathbb{R}^3)) \) be arbitrary functions. Then there exists a weak solution \( w \in X_T = C_w([0,T],L^2(\mathbb{R}^3)) \cap L^2([0,T],H^1(\mathbb{R}^3)) \) of the perturbed problem (2.9)–(2.11) satisfying the following generalized energy inequality

\[
E(t)\|\psi(t) + w(t)\|_2^2 \leq E(s)\|\psi(s) + w(s)\|_2^2 + \int_s^t E'(\tau)\|\psi(\tau) + w(\tau)\|_2^2 d\tau
\]

\[(5.7)\]

for almost all \( s \geq 0 \) including \( s = 0 \) and all \( t \geq s \geq 0 \).

Before proving Theorem 5.3, we note that the proof of the \( L^2 \)-decay of \( w = w(x,t) \) requires the following two corollaries, which are consequences of the generalized energy inequality (5.7).

**Corollary 5.4.** Let \( w \) be a weak solution to (2.9)–(2.11) satisfying the generalized energy inequality (5.7). Then for every \( \varphi \in S(\mathbb{R}^3) \), we have

\[
\|\varphi * w\|_2^2 \leq \|e^{(t-s)\Delta} \varphi \|_2^2 + 2 \int_s^t b(w,w,e^{2(t-\tau)\Delta} (\varphi * \varphi) w)(\tau) + b(V,w,e^{2(t-\tau)\Delta} (\varphi * \varphi) w)(\tau) \]

\[
+ b(w,V,e^{2(t-\tau)\Delta} (\varphi * \varphi) w)(\tau) \]

for almost all \( s \geq 0 \), including \( s = 0 \) and all \( t \geq s \).

**Proof.** Use the generalized energy inequality (5.7) with \( E(t) \equiv 1 \) and \( \psi(\tau) = e^{(t-\eta)\Delta} \varphi \), where \( \eta > 0 \).

Then, \( \psi(\tau) * w(\tau) = e^{(t+\eta)\Delta} \varphi * w(\tau) \) and \( \psi(\tau) * \psi(\tau) = e^{2(t+\eta)\Delta} \varphi * \varphi. \) A straightforward calculation involving properties of solutions to the heat equation shows

\[
\langle \psi'(\tau) * w(\tau), \psi(\tau) * w(\tau) \rangle - \|\nabla \psi(t) * w(t)\|_2^2 = 0
\]
The corollary follows by letting \( \eta \to 0 \).

**Corollary 5.5.** Let \( E \subset C^1[0, \infty) \) and \( E(t) \geq 0 \). Let \( w = w(x, t) \) be a weak solution constructed in Theorem \( 2.7 \) satisfying the generalized energy inequality \( (5.7) \). Then for every \( \varphi \in S(\mathbb{R}^3) \), the vector field \( w = w(x, t) \) fulfills the inequality

\[
E(t)\|w(t) - \varphi \ast w(t)\|_2^2 
\leq \int_s^t E'(\tau)\|w(\tau) - \varphi \ast w(\tau)\|_2^2 d\tau 
- 2\int_s^t E(\tau)\|\nabla w(\tau) - \varphi \ast \nabla w(\tau)\|_2^2 d\tau 
+ 2\int_s^t E(\tau) b(w, \varphi \ast w - 2\varphi \ast w)(\tau) d\tau 
+ 2\int_s^t E(\tau) b(V, w, \varphi \ast w - 2\varphi \ast w)(\tau) d\tau 
+ 2\int_s^t E(\tau) b(w, V, w - 2\varphi \ast w + \varphi \ast \varphi \ast w)(\tau) d\tau
\]

(5.8)

for almost all \( s \geq 0 \), including \( s = 0 \) and all \( t \geq s \).

**Proof.** Substitute in the generalized energy inequality \( (5.7) \) the function \( \psi(x, t) = \zeta_n(x) - \varphi(x) \), where \( \zeta_n(x) = n^{-3}\zeta(x/n) \) is a smooth and compactly supported approximation of the Dirac measure. The term in \( (5.7) \) containing \( \psi' \) is annihilated. Notice also that

\[
\psi \ast \psi \ast w = \zeta_n \ast \zeta_n \ast w - 2\zeta_n \ast \varphi \ast w + \varphi \ast \varphi \ast w.
\]

Since \( b(w, w, w) = b(V, w, w) = 0 \) for a divergence free vector field \( w \) (cf. Remark \( 2.4 \)), passing to the limit \( n \to \infty \), yields inequality \( (5.8) \). \( \square \)

**Proof of Theorem 5.3.** Let \( \{w_m\} \) be a sequence of Galerkin approximations converging towards a weak solution \( w \) of the perturbed problem \( (2.9) - (2.11) \) in the usual sense \((1.3) - (1.4)\) as well as in the global \( L^2 \)-sense established in Lemma \( 5.1 \). In the sequel we also assume that \( w_m \) is a finite linear combination of elements of an orthonormal basis \( \{g_m\} \) of \( L^2_s(\mathbb{R}^3) \) satisfying the properties stated in the following lemma.

**Lemma 5.6.** There exists an orthonormal basis \( \{g_m\}_{m=1}^\infty \) of \( L^2_s(\mathbb{R}^3) \) such that

- \( \{g_m\}_{m=1}^\infty \) is a Riesz basis of the Sobolev space \( W^{1,p}_\sigma(\mathbb{R}^3) \) for each \( 1 < p < \infty \);
- there exists \( C = C(p) > 0 \) such that for every \( v \in W^{1,p}_\sigma(\mathbb{R}^3) \) and every \( m \in \mathbb{N} \) we have

\[
\|P_m v\|_{W^{1,p}_\sigma} \leq C\|v\|_{W^{1,p}_\sigma},
\]

where \( P_m v = \sum_{k=1}^m \langle v, g_k \rangle g_k \) is the orthonormal \( L^2 \)-projection,

- for every \( v \in H^1_\sigma(\mathbb{R}^3) \)

\[
\sum_{k=1}^\infty \langle \nabla v, g_k \rangle g_k = \nabla v = \sum_{k=1}^\infty \langle v, g_k \rangle \nabla g_k,
\]

where the series converges strongly in \( L^2_s(\mathbb{R}^3) \).
Navier-Stokes System

Proof. These properties are satisfied by the divergence free vector wavelet basis introduced by Battle and Federbush [1]. See also [32, Ch. 3] for a review of properties of such bases. These wavelets decay exponentially. A standard application of the Calderón-Zygmund theory yields inequality (5.9) using the size and moments estimates for these wavelets. □

Corollary 5.7. Under the assumptions of Lemma 5.6, for every $v \in H^1_\sigma(\mathbb{R}^3)$ we have

\begin{equation}
\|P_m v - v\|_{H^1_\sigma} \to 0 \quad \text{as} \quad m \to \infty.
\end{equation}

Proof. Immediate properties of the $L^2$-projection $P_m$ give $\lim_{m \to \infty} \|P_m v - v\|_2 = 0$. Note now that we have $\nabla P_m v = \sum_{k=1}^m \langle v, g_k \rangle \nabla g_k$. Thus, by the second equality in (5.10) we obtain $\lim_{m \to \infty} \|\nabla P_m v - \nabla v\|_2 = 0$. □

We now return to the proof of Theorem 5.3. Use (4.9) with $\varphi(t)$ replaced by the following test function

\begin{equation}
E(t)\varphi_m(t) = E(t)P_m(w_m(t) * \psi(t) * \psi(t)),
\end{equation}

where $E \in C^1([0, \infty))$, $E(t) \geq 0$ and $\psi(t) \in C^1([0, \infty); \mathcal{S}(\mathbb{R}^3))$. Here, $P_m : L^2_\sigma(\mathbb{R}^3) \to W_m$ is the usual orthogonal projection (see the proof of the first part of Theorem 2.7), thus, $\varphi_m = P_m(w_m * \psi * \psi) \in C^1([0, \infty), W_m)$.

Using properties of the projection $P_m$ yields

\begin{equation}
\langle v_m, P_m(v) \rangle = \langle v_m, v \rangle \quad \text{for all} \quad v_m \in W_m \quad \text{and} \quad v \in L^2_\sigma(\mathbb{R}^3).
\end{equation}

By the Leibniz formula, we have

\begin{equation}
\frac{d}{dt} \left( E P_m(w_m * \psi * \psi) \right) = E' P_m(w_m * \psi * \psi) + 2 E P_m(w_m * \psi' * \psi) + E P_m \left( \left( \frac{d}{dt} w_m \right) * \psi * \psi \right).
\end{equation}

A combination of (5.13), (5.12), and properties of the convolution gives

\begin{equation}
\langle w_m, E P_m \left( \left( \frac{d}{dt} w_m \right) * \psi * \psi \right) \rangle = \langle w_m, E \left( \frac{d}{dt} w_m \right) * \psi * \psi \langle w_m, \nabla \varphi_m \rangle = \langle \frac{d}{dt} w_m, E w_m * \psi * \psi \rangle = \langle \frac{d}{dt} w_m, \varphi_m \rangle.
\end{equation}

Equality (4.16) can be rewritten as

\begin{equation}
\langle \frac{d}{dt} w_m, \varphi_m \rangle = - \langle \nabla w_m, \nabla \varphi_m \rangle - b(w_m, w_m, \varphi_m) - b(w_m, V, \varphi_m) - b(V, w_m, \varphi_m).
\end{equation}
After substituting $\varphi = E\varphi_m = EP_m(w_m * \psi * \psi)$ into (4.9) and using (5.13)–(5.15), we obtain the equality

$$E(t) \langle w_m(t), \varphi_m(t) \rangle = E(s) \langle w_m(s), \varphi_m(s) \rangle + \int_s^t E'(\tau) \langle w_m(\tau), \varphi_m(\tau) \rangle \, d\tau$$

$$+ 2 \int_s^t E(\tau) \langle w_m(\tau), w_m(\tau) * \psi'(\tau) * \psi(\tau) \rangle \, d\tau$$

$$- 2 \int_s^t E(\tau) \langle \nabla w_m(\tau), \nabla \varphi_m(\tau) \rangle \, d\tau$$

(5.17)

$$- 2 \int_s^t E(\tau) b(w_m(\tau), w_m(\tau), \varphi_m(\tau)) \, d\tau$$

$$- 2 \int_s^t E(\tau) b(V(\tau), w_m(\tau), \varphi_m(\tau)) \, d\tau$$

$$- 2 \int_s^t E(\tau) b(V(\tau), w_m(\tau), \varphi_m(\tau)) \, d\tau.$$  

Passing to the limit as $m \to \infty$ in (5.17) will yield the generalized inequality (5.7). Here, we adapt arguments from [38, Proof of Prop. 2.3] to our more general case. We need the following result on the convergence of the sequence $\varphi_m$.

**Lemma 5.8.** Under the above assumptions

(5.18) $\varphi_m = P_m(w_m * \psi * \psi) \to w * \psi * \psi$ as $m \to \infty$

in $L^2([0, T], H^s_\sigma(\mathbb{R}^3))$ for each $T > 0$. Moreover, for each $p \in [2, \infty)$ there exists a constant $C = C(p, \psi) > 0$ such that

(5.19) $\| \nabla \varphi_m(t) \|_p \leq C \| w_0 \|_2$ for all $t > 0$.

**Proof.** By Lemma 5.1 $w_m(t) \to w(t)$ strongly in $L^2_\sigma(\mathbb{R}^3)$ for almost all $t > 0$, hence, by properties of the convolution, we have

\[
\nabla \left( w_m * \psi * \psi \right)(t) = \left( w_m * \nabla \psi * \psi \right)(t) \to \left( w * \nabla \psi * \psi \right)(t) = \nabla \left( w * \psi * \psi \right)(t)
\]

strongly in $L^2_\sigma(\mathbb{R}^3)$ for almost all $t > 0$, as well. Thus, by Corollary 5.7 we obtain

(5.20) $P_m(w_m * \psi * \psi)(t) \to (w * \psi * \psi)(t)$ strongly in $H^s_\sigma(\mathbb{R}^3)$ for almost all $t > 0$. Finally, by (5.9) and by (4.2), we obtain

\[
\| P_m(w_m * \psi * \psi)(t) \|_{H^s_\sigma} \leq C \| w_m * \psi * \psi(t) \|_{H^s_\sigma} \leq C \| w_m(t) \|_2 \left( \| \psi * \psi(t) \|_1 + \| (\nabla \psi * \psi)(t) \|_1 \right) \leq C(\psi) \| w_0 \|_2.
\]

Hence, we may apply the Lebesgue dominated convergence theorem to complete the proof of the first part of Lemma 5.8.

To prove the second part, we use estimate (5.9) and (4.2) as well as the Young inequality for convolutions with exponents $p$ and $q$ satisfying $1/p + 1/q = 1/2 + 1/q$ to obtain

\[
\| \nabla \varphi_m(t) \|_p \leq C \| (w_m * \psi * \psi)(t) \|_{W^{1,p}} \leq C \| w_m(t) \|_2 \left( \| \psi * \psi \|_q + \| \nabla \psi * \psi \|_q \right) \leq C \| w_0 \|_2.
\]

$\square$
Now, we are in a position to pass to the limit in each term of (5.17). Since \(w_m(t) \to w(t)\) strongly in \(L^2_s(\mathbb{R}^3)\) for almost all \(t > 0\), we also have
\[
\varphi_m(t) = P_m(w_m \ast \psi \ast \psi)(t) \to (w \ast \psi \ast \psi)(t)
\]
strongly in \(L^2_s(\mathbb{R}^3)\). Consequently,
\[
E(t)\langle w_m(t), \varphi_m(t) \rangle \to E(t)\langle w(t), (w \ast \psi \ast \psi)(t) \rangle = E(t\|w(t) \ast \psi(t)\|_2^2
\]
almost everywhere in \(t\).

The convergence of the sequence \(\{w_m\}\) in \(L^2([0,T], L^2_s(\mathbb{R}^3))\) and Lemma 5.3 allows us to pass to the limit in the second and third term on the right-hand side of (5.17) and to obtain, as \(m \to \infty\),
\[
\int_s^t E'(\tau)\langle w_m(\tau), \varphi_m(\tau) \rangle \, d\tau \to \int_s^t E'(\tau)\langle w(\tau), w(\tau) \ast \psi(\tau) \ast \psi(\tau) \rangle \, d\tau
\]
\[
= \int_s^t E'(\tau)\|w(\tau) \ast \psi(\tau)\|_2^2 \, d\tau
\]
and
\[
\int_s^t E(\tau)\langle w_m(\tau), w_m(\tau) \ast \psi(\tau) \ast \psi(\tau) \rangle \, d\tau \to \int_s^t E(\tau)\langle w(\tau), w(\tau) \ast \psi(\tau) \ast \psi(\tau) \rangle \, d\tau.
\]

It follows from the weak convergence (4.3) that
\[
\liminf_{m \to \infty} \int_s^t E(\tau)\langle \nabla w_m(\tau), \nabla \varphi_m(\tau) \rangle \, d\tau \geq \int_s^t E(\tau)\langle \nabla w(\tau), \nabla (w(\tau) \ast \psi(\tau) \ast \psi(\tau)) \rangle \, d\tau
\]
\[
= \int_s^t E(\tau)\|\nabla w(\tau) \ast \psi(\tau)\|_2^2 \, d\tau.
\]

Using properties of the trilinear form \(b(\cdot, \cdot, \cdot)\) recalled in Remark 2.1, the nonlinear term (the fifth one on the right-hand side of (5.17)) is estimated as follows
\[
\left| \int_s^t E(\tau)b(w_m, w_m, \varphi_m)(\tau) \, d\tau - \int_s^t E(t)b(w, w, w \ast \psi \ast \psi)(\tau) \, d\tau \right|
\]
\[
\leq \left| \int_s^t E(\tau)\langle (w_m - w) \cdot \nabla \varphi_m, w_m \rangle(\tau) \right| \, d\tau
\]
\[
+ \left| \int_s^t E(\tau)\langle w \cdot \nabla \varphi_m, w_m - w \rangle(\tau) \right| \, d\tau
\]
\[
+ \left| \int_s^t E(\tau)\langle w \cdot \nabla (\varphi_m - w \ast \psi \ast \psi), w \rangle(\tau) \right| \, d\tau
\]
\[
\equiv I_1 + I_2 + I_3.
\]
To estimate the term $I_1$, we use the Hölder inequality, estimate (5.19) with $p = 4$, inequality (4.2), and Corollary 5.2 to obtain

$$I_1 \leq \int_s^t E(\tau)\|w_m(\tau) - w(\tau)\|_4\|\nabla \varphi_m(\tau)\|_4\|w_m(\tau)\|_2\,d\tau$$

$$\leq C\|E\|_{L^\infty(0,\infty)}\|w_0\|^2_2 \int_s^t \|w_m(\tau) - w(\tau)\|_4\,d\tau$$

$$\leq C(E, w_0)(t-s)^{1/2} \int_s^t \|w_m(\tau) - w(\tau)\|^2_4\,d\tau \to 0 \quad \text{as} \quad m \to \infty.$$ 

An analogous reasoning applies to the term $I_2$.

Estimates for $I_3$ are similar. By the Hölder inequality and the well-known inequality $\|w\|_4 \leq C\|w\|_2^{1/4}\|\nabla w\|_2^{3/4}$ we have

$$I_3 \leq \|E\|_{L^\infty(0,\infty)} \int_s^t \|w(\tau)\|^2_4\|\nabla (\varphi_m - w \ast \psi \ast \psi)(\tau)\|_2\,d\tau$$

$$\leq C(E) \int_s^t \|w(\tau)\|_2^{1/2}\|\nabla w(\tau)\|_2^{3/2}\|\nabla (\varphi_m - w \ast \psi \ast \psi)(\tau)\|_2\,d\tau$$

$$\leq C(E)\varepsilon \int_s^t \|w(\tau)\|_2^{3/2}\|\nabla w(\tau)\|_2\,d\tau$$

$$+ C(E, \varepsilon) \int_s^t \|\nabla (\varphi_m - w \ast \psi \ast \psi)(t)\|_2^4\,d\tau.$$ 

The first term is arbitrarily small with $\varepsilon > 0$ because by the energy inequality (2.14), the quantity $\int_s^t \|w(\tau)\|_2^{3/2}\|\nabla w(\tau)\|_2^{3/2}\,d\tau$ may be bounded by a multiple of $\|w_0\|_2^{2+2/3}$. The second term converges to zero as $m \to \infty$ which results from Lemma 5.8 and from the estimate

$$\|\nabla (\varphi_m - w \ast \psi \ast \psi)(t)\|_2 \leq \|\nabla \varphi_m\|_2 + \|w \ast \nabla \psi \ast \psi\|_2 \leq C\|w_0\|_2$$

being a direct consequence of estimate (5.19).

The final step is to deal with the last two terms on the right-hand-side of (5.17) which contain the solution $V = V(x, t)$. Using the standing assumptions (2.6) and (2.7) yields

$$\int_s^t E(\tau)\left|b(w_m, \varphi_m, V)(\tau) - b(w_m - w \ast \psi \ast \psi, V)(\tau)\right|\,d\tau$$

$$\leq \|E\|_{L^\infty(0,\infty)}K\sup_{t>0} \|V(t)\|_{x_e} \int_s^t \|\nabla w_m(\tau)\|_2\|\nabla (\varphi_m - w \ast \psi \ast \psi)(\tau)\|_2\,d\tau$$

$$+ \int_s^t E(\tau)\left|b((w_m - w \ast \psi \ast \psi, \psi)(\tau))\right|\,d\tau$$

$$\leq C\varepsilon \int_s^t \|\nabla w_m(\tau)\|^2_2\,d\tau + C(E) \int_s^t \|w_m(\tau) - w(\tau)\|^2_2\,d\tau$$

$$+ \|E\|_{L^\infty(0,\infty)} \int_s^t \left|b((w_m - w \ast \psi \ast \psi, V)(\tau))\right|\,d\tau.$$ 

The first term on the right-hand-side can be made arbitrarily small since $\varepsilon > 0$ is arbitrary and since $\int_s^t \|\nabla w_m(s)\|^2_2\,ds \leq \|w_0\|^2_2$ (see (4.2)). The second term converges to zero by
Lemma 5.8 and the third one converges to zero by Remark 2.2 combined with the Lebesgue dominated convergence theorem.

Analogously, we show that

\[
\int_s^t E(\tau) b(V, w_m, \varphi)(\tau) \, d\tau \to \int_s^t E(\tau) b(V, w, w * \psi)(\tau) \, d\tau \quad \text{as} \quad m \to \infty.
\]

This completes the proof of the generalized energy inequality (5.7). \qed

6. Asymptotic stability of weak solutions

The proof of the \( L^2 \)-decay of a solution to the perturbed problem (2.9)–(2.11) is somewhat challenging and we use the Fourier splitting, a technique that was introduced in [41, 40] and generalized in [38].

Proof of Theorem 2.7. Second part – decay of solutions. The \( L^2 \)-norm of the solution \( w = w(x, t) \) from Theorem 5.3 is decomposed into high and low frequency terms. To deal with these two terms, the generalized energy inequality (5.7) is used with suitable functions \( E(t) \) and \( \psi \), as were used in Corollaries 5.4 and 5.5. A modification of the Fourier splitting argument will be applied to estimate the term corresponding to high frequencies.

We begin by decomposing the \( L^2 \)-norm of the Fourier transform of \( w \) as follows

\[
(2\pi)^\frac{3}{2} \| \hat{\varphi}(t) \|^2 = \| \hat{\varphi} \hat{w}(t) \|^2 \leq \| \hat{\varphi} \hat{w}(t) \|^2 + \| (1 - \hat{\varphi}) \hat{w}(t) \|^2, \quad \text{where} \quad \hat{\varphi}(\xi) = e^{-|\xi|^2}.
\]

Each term on the right hand side of the last inequality will be estimated separately. Notice that \( \hat{\varphi} \) is the inverse Fourier transform of the function \( \varphi(x) = (4\pi)^{-3/2}e^{-|x|^2/4} \), which is the fundamental solution of the heat equation at \( t = 1 \).

Estimates of the low frequencies. Using the Plancherel identity and Corollary 5.4, we have

\[
(2\pi)^\frac{3}{2} \| \hat{\varphi} \hat{w}(t) \|^2 = \| \varphi * w(t) \|^2 \\
\leq \| e^{\langle x - s \rangle} \varphi * w(s) \|^2 \\
+ 2(2\pi)^{-\frac{3}{2}} \int_s^t | b(w, w, e^{2(t-\tau)} \Delta \varphi * \varphi * w)(\tau) | \, d\tau \\
+ 2(2\pi)^{-\frac{3}{2}} \int_s^t | b(V, w, e^{2(t-\tau)} \Delta \varphi * \varphi * w)(\tau) | \, d\tau \\
+ 2(2\pi)^{-\frac{3}{2}} \int_s^t | b(w, V, e^{2(t-\tau)} \Delta \varphi * \varphi * w)(\tau) | \, d\tau \\
= I_1(t, s) + 2(2\pi)^{-\frac{3}{2}} \int_s^t \left( I_2(\tau) + I_3(\tau) + I_4(\tau) \right) \, d\tau.
\]

Quantities \( I_i \) are going to be estimated separately for each \( i = 1, 2, 3, 4 \).

Note that \( \hat{\varphi}(s) \in L^2(\mathbb{R}^3) \). Hence, for each fixed \( s \geq 0 \), by the Plancherel identity and the Lebesgue dominated convergence theorem it follows that

\[
I_1(t, s) = \| e^{\langle x - s \rangle} \varphi * w(s) \|^2 = (2\pi)^3 \| e^{-|\xi|^2} \hat{\varphi} \hat{w}(s) \|^2 \to 0 \quad \text{as} \quad t \to \infty.
\]

Applying the Schwarz inequality, the well-known \( L^2 \)-estimate for the heat semigroup, the Hölder inequality, the Sobolev inequality and, finally, the energy inequality (2.14), we
where
\[ \| \varphi \ast (w \cdot \nabla w) \|_2 \leq C \| \varphi \|_2 \| w \cdot \nabla w \|_2 \]
\[ \leq C \| \varphi \|_2 \| w \|_6 \| \nabla w \|_2 \| w \|_2 \leq C \| \varphi \|_2 \| w \|_2 \| \nabla w \|_2^2. \]

Using the standing assumption (2.2) (formulated in Remark 2.1), properties of the heat semigroup and of the convolution, we get
\[ I_3 = |b(V, w, e^{2(t-\tau)\Delta} \varphi \ast w)| \]
\[ \leq K \| V \|_{x_r} \| \nabla w \|_2 \| \nabla e^{2(t-\tau)\Delta} \varphi \ast w \|_2 \]
\[ \leq K \| V \|_{x_r} \| \nabla w \|_2 \| \varphi \ast \nabla w \|_2 \leq K \| \varphi \|_1 \| \nabla w \|_2 \sup_{t>0} \| V(t) \|_{x_r}, \]

where \( K > 0 \) is the constant from inequality (2.2). By an analogous argument involving assumption (2.2), we have
\[ I_4 = |b(w, V, e^{2(t-\tau)\Delta} \varphi \ast w)| \]
\[ = |b(w, e^{2(t-\tau)\Delta} \varphi \ast \varphi, V)| \leq K \| \varphi \|_1 \| \nabla w \|_2 \sup_{t>0} \| V(t) \|_{x_r}. \]

Combining estimates (6.4)–(6.6), we obtain the following \( L^2 \)-bound of the low frequencies
\[ \| \hat{\varphi} \hat{w}(t) \|_2^2 \leq I_1(t, s) + C \left( \| w_0 \|_2 \sup_{t>0} \| V(t) \|_{x_r} \right) \int_s^t \| \nabla w(\tau) \|_2^2 \, d\tau, \]
where the constant \( C = C(K, \varphi) > 0 \) is independent of \( w \).

Let \( s > 0 \) be fixed and large on the right-hand side of inequality (6.7). The term \( I_1(t, s) \) tends to zero as \( t \to \infty \) by (6.3). Since \( \int_0^\infty \| \nabla w(\tau) \|_2^2 \, d\tau < \infty \), the quantity \( \int_s^\infty \| \nabla w(\tau) \|_2^2 \, d\tau \) can be made arbitrary small choosing \( s \) large enough. Hence, \( \| \hat{\varphi} \hat{w}(t) \|_2^2 \to 0 \) as \( t \to \infty \).

Estimates of the high frequencies. To deal with the term \( \| (1 - \hat{\varphi}) \hat{w}(t) \|_2 \) in the decomposition (6.1), we use Corollary 5.5 with the test function \( \varphi \) satisfying \( \hat{\varphi}(\xi) = \exp(-|\xi|^2) \) and a function \( E(t) > 0 \) to be determined below. We then apply the Fourier-splitting method to estimate each term on the right-hand side of (5.8).

For every \( G(t) \geq 0 \), the Plancherel formula applied to the second and third term on the right-hand side of (5.8) yields
\[ J_1 = \int_s^t E(\tau) \| (w(\tau) - \varphi \ast w(\tau)) \|_2^2 \, d\tau - 2 \int_s^t E(\tau) \| \nabla w(\tau) - \varphi \ast \nabla w(\tau) \|_2^2 \, d\tau \]
\[ = \int_s^t E(\tau) \int_{|\xi|>G(\tau)} \| (1 - \hat{\varphi}(\xi)) \hat{w}(\xi, \tau) \|_2^2 \, d\xi \, d\tau \]
\[ - 2 \int_s^t E(\tau) \int_{|\xi|>G(\tau)} \| \xi (1 - \hat{\varphi}(\xi)) \hat{w}(\xi, \tau) \|_2^2 \, d\xi \, d\tau \]
\[ + \int_s^t E(\tau) \int_{|\xi|<G(\tau)} \| (1 - \hat{\varphi}(\xi)) \hat{w}(\xi, \tau) \|_2^2 \, d\xi \, d\tau \]
\[ - 2 \int_s^t E(\tau) \int_{|\xi|<G(\tau)} \| \xi (1 - \hat{\varphi}(\xi)) \hat{w}(\xi, \tau) \|_2^2 \, d\xi \, d\tau. \]
Now, choose
\[(6.8) \quad E(t) = (1 + t)^\alpha \quad \text{and} \quad G^2(t) = \frac{\alpha}{2(t + 1)} \quad \text{with fixed} \quad \alpha > 0,\]
then \(E'(\tau) - 2E(\tau)G^2(\tau) = 0\). Thus,
\[
\int_s^t E'(\tau) \int_{|\xi| > G(\tau)} \left| (1 - \hat{\varphi}(\xi))\hat{w}(\xi, \tau) \right|^2 \, d\xi \, d\tau - 2 \int_s^t E(\tau) \int_{|\xi| > G(\tau)} \left| \xi \right|\left| (1 - \hat{\varphi}(\xi))\hat{w}(\xi, \tau) \right|^2 \, d\xi \, d\tau
\]
\[
\leq \int_s^t \left[ E'(\tau) - 2E(\tau)G^2(\tau) \right] \int_{|\xi| > G(\tau)} \left| (1 - \hat{\varphi}(\xi))\hat{w}(\tau) \right|^2 \, d\xi \, d\tau = 0.
\]
Moreover, for small \(|\xi|\), it follows that \(|1 - \hat{\varphi}(\xi)| = 1 - e^{-|\xi|^2} \leq |\xi|^2\). Hence, using (6.8), we obtain
\[
\int_s^t E'(\tau) \int_{|\xi| \leq G(\tau)} \left| (1 - \hat{\varphi}(\xi))\hat{w}(\xi, \tau) \right|^2 \, d\xi \, d\tau
\]
\[
= \int_s^t E'(\tau) \int_{|\xi| \leq G(\tau)} |(1 - \hat{\varphi}(\xi))\hat{w}(\xi, \tau)|^2 \, d\xi \, d\tau
\]
\[
\leq C\|w_0\|^2_2 \int_s^t E'(\tau)G^4(\tau) \, d\tau \leq C \int_s^t (1 + \tau)^{\alpha - 3} \, d\tau.
\]
Hence, since \(E(t) \geq 0\), we conclude that
\[(6.9) \quad J_1 \leq C \int_s^t (1 + \tau)^{\alpha - 3} \, d\tau.
\]
To deal with the other terms on the right-hand side of the inequality (5.8), set \(\eta = \varphi * \varphi - 2\varphi\) to simplify the notation. Combining the Hölder and the Young inequalities with the Sobolev inequality \(\|w\|_6 \leq \|\nabla w\|_2\) (as in the estimates of low frequencies in (6.5b)) yields
\[(6.10) \quad \int_s^t E(\tau)|b(w, w, \varphi * \varphi * w - 2\varphi * w)(\tau)| \, d\tau
\]
\[
= \int_s^t E(\tau)|b(w, w, \eta * w)(\tau)| \, d\tau \leq \int_s^t E(\tau)\|w(\tau)\|_6\|\nabla w(\tau)\|_2\|\eta * w(\tau)\|_3 \, d\tau
\]
\[
\leq \|\eta\|^2_6 \int_s^t E(\tau)\|w(\tau)\|_6\|\nabla w(\tau)\|_2 \|w(\tau)\|_2 \, d\tau \leq C\|\eta\|^2_6 \|w_0\|_2 \int_s^t E(\tau)\|\nabla w(\tau)\|_2^2 \, d\tau.
\]
The two terms on the right-hand side of (5.8) containing the solution \(V\) are estimated by the standing assumption (2.2), see Remark 2.1 Indeed, we have
\[(6.11) \quad \int_s^t E(\tau)|b(V, w, \varphi * \varphi * w - 2\varphi * w)(\tau)| \, d\tau
\]
\[
= \int_s^t E(\tau)|b(V, w, \eta * w)(\tau)| \, d\tau \leq C\|\eta\|^1_1 (\sup_{t > 0} \|V(t)\|_{x_0}) \int_s^t E(\tau)\|\nabla w(\tau)\|_2^2 \, d\tau.
\]
Proceeding in an analogous way, it follows that

$$\int_s^t E(\tau) |b(w, V, w - 2\varphi \ast w + \varphi \ast \varphi \ast w)(\tau)| \, d\tau$$

(6.12)

$$\leq C(1 + \|\eta\|_1) \left( \sup_{\tau > 0} \|V\|_{L^\infty} \right) \int_s^t E(\tau) \|\nabla w(\tau)\|^2 \, d\tau.$$  

Thus, dividing inequality (5.8) by $E(t)$ and using estimates (6.9), (6.10), (6.11) and (6.12), we get

$$\| (1 - \hat{\varphi}) \hat{w}(t) \|^2 \leq E(s) \| (1 - \hat{\varphi}) \hat{w}(s) \|^2$$

(6.13)

$$+ C \frac{1}{E(t)} \int_s^t (1 + \tau)^{\alpha - 3} \, d\tau + C \frac{1}{E(t)} \int_s^t E(\tau) \|\nabla w(\tau)\|^2 \, d\tau.$$  

For fixed $s > 0$, we compute the lim sup as $t \to \infty$ of both sides of (6.13). Since, $E(t) = (1 + t)\alpha$ with some $\alpha > 0$, the first term on the right-hand side of (6.13) tends to zero. By a direct calculation based on the de l’Hôpital rule, we obtain

$$\limsup_{t \to \infty} \frac{1}{(t + 1)^\alpha} \int_s^t (1 + \tau)^{\alpha - 3} \, d\tau = 0.$$  

Thus, using the inequality $\frac{E(\tau)}{E(t)} = (\frac{t}{t+\tau})^\alpha \leq 1$ for $\tau \in [0, t]$, it follows from estimate (6.13)

(6.14)

$$\limsup_{t \to \infty} \| (1 - \varphi) \hat{w}(t) \|^2 \leq C \left( \sup_{\tau > 0} \|V(\tau)\|_{L^\infty} \right) \int_s^\infty \|\nabla w(\tau)\|^2 \, d\tau.$$  

We conclude that $\limsup_{t \to \infty} \| (1 - \varphi(\xi)) \hat{w}(t) \|^2 = 0$, since the right-hand side of inequality (6.14) can be made arbitrarily small for sufficiently large $s > 0$. This completes the proof of Theorem 2.7.

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