Currently there are no reasons to doubt the Riemann Hypothesis

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Abstract

We examine published arguments which suggest that the Riemann Hypothesis may not be true. In each case we provide evidence to explain why the claimed argument does not provide a good reason to doubt the Riemann Hypothesis. The evidence we cite involves a mixture of theorems in analytic number theory, theorems in random matrix theory, and illustrative examples involving the characteristic polynomials of random unitary matrices. Similar evidence is provided for four mistaken notions which appear repeatedly in the literature concerning computations of the zeta-function. A fundamental question which underlies some of the arguments is: what does the graph of the Riemann zeta-function look like in a neighborhood of its largest values? We explore that question in detail and provide a survey of results on the relationship between L-functions and the characteristic polynomials of random matrices. We highlight the key role played by the emergent phenomenon of carrier waves, which arise from fluctuations in the density of zeros. ¹

1 Introduction

Should one believe the Riemann Hypothesis (RH)? Since it is a conjecture with no proposed roadmap to prove it, one point of view is that it should neither be believed nor disbelieved. Yet many mathematicians have an opinion, presumably backed up by logical reasoning.

Here we consider all published arguments for doubting RH. A paper of Ivić [53, 54] lists 4 reasons, and a paper of Blanc [10] provides a 5th reason. Three of those reasons involve speculation about the distribution of zeros and their relationship to the value distribution of the \( \zeta \)-function. A fundamental question is: what does the \( \zeta \)-function look like in a neighborhood of its largest values? The majority of this paper is a survey of prior results, which lead to an answer to that question.

We do not include any detailed proofs. The reason is that our goal is to provide intuition and to be persuasive. The arguments against RH generally take the form “It would be surprising if X.” So, the burden we bear in refuting that argument is to give good reasons why X is not surprising. To crystallize our main points we present 26 Principles which we hope are also useful for future reference.

Is the purpose of this paper to persuade that RH is true? Certainly not. But perhaps those who continue to doubt RH will realize that their belief is not based on good evidence. As for those who believe RH: perhaps someone will write a companion paper: Currently there are no good reasons to believe the Riemann Hypothesis. Note the subtle difference from the opposite of the

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title to this paper. Also useful would be a paper explaining why every currently known equivalence to RH is unlikely to be helpful for proving RH (as in [33]).

The \( \zeta \)-function is the simplest example of an \textit{L-function}. All L-functions have properties similar to the \( \zeta \)-function, and all L-functions have an analogue of the Riemann Hypothesis. As much as possible we try to discuss the \( \zeta \)-function in isolation, but in a few places it is necessary to expand our perspective. We attempt to keep this paper self-contained, providing definitions and background as needed.

In Section 2 we provide basic definitions and background. In Section 3 we describe four Mistaken Notions which appear repeatedly in discussions of computations of the \( \zeta \)-function, some of which play an important role in the claimed reasons to doubt RH. In Section 4 we describe the connection between the distribution of zeros and the size of the \( \zeta \)-function, introducing carrier waves as a way to separate local from long-range behavior. In Section 5 we briefly describe the connection between the \( \zeta \)-function and unitary polynomials, and in Section 6 we provide an historical account of the connections to Random Matrix Theory. This leads to Section 7, where we use large unitary matrices to illustrate phenomena which occur far outside the range in which we can compute the \( \zeta \)-function. By the end of Section 7 we have a good understanding of the “typical” large values of the \( \zeta \)-function, and the relationship between the carrier wave, the density wave, and \( S(t) \), but it is not until Section 8 that we address the most extreme values. Finally in Section 9 we use information from the prior sections to refute the three arguments against RH based on the distribution of zeros and values of the \( \zeta \)-function, and for completeness in Section 10 we cite recent results to refute the other two arguments against RH. Finally, in Section 11 we use the Principles to explain why the Mistaken Notions of Section 3 are, in fact, mistaken.

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2 Background on the \( \zeta \)- function

The Riemann zeta-function (which we will call the \( \zeta \)-function) is defined by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]  

(2.1)

for \( \sigma > 1 \), where \( s = \sigma + it \) is a complex variable. The \( \zeta \)-function has a meromorphic continuation to the complex plane, with a simple pole with residue 1 at \( s = 1 \).

The \( \zeta \)-function has a symmetry, known as the \textit{functional equation}, which can be expressed in several ways:

\[
\zeta(s) = X(s)\zeta(1 - s), \quad \text{where} \quad X(s) := \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}
\]  

(2.2)

or equivalently

\[
\xi(s) := \frac{1}{2}s(s - 1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)
\]  

(2.3)
or equivalently
\[
Z(t) := X \left( \frac{1}{2} + it \right)^{-\frac{1}{2}} \zeta \left( \frac{1}{2} + it \right) \quad \text{is real if} \quad t \in \mathbb{R}.
\] (2.5)

Here \( \Gamma \) is the Euler Gamma-function. The factor \( \frac{1}{2} s (s - 1) \) in (2.3) is irrelevant to the invariance under \( s \leftrightarrow 1 - s \), but it is traditionally included so that \( \xi(s) \) is an entire function of \( s \). In (2.5) the square root is chosen so that \( Z(0) = \zeta(\frac{1}{2}) \approx -1.46 \) and \( Z(t) \) is analytic for \( |\Im(t)| < \frac{1}{2} \). It is common to refer to \( t, \) the imaginary part of \( s = \sigma + it \), as the \textbf{height} when referring to the behavior of \( \zeta(s) \) or \( Z(t) \) in a particular region.

The \textbf{Hardy Z-function} (2.5) is useful because it can be graphed, and it tells us essentially everything we might want to know about the \( \zeta \)-function because

\[
|Z(t)| = |\zeta(\frac{1}{2} + it)| \quad \text{if} \quad t \in \mathbb{R}.
\]

Figure 2.1 shows \( Z(t) \), \( \log |Z(t)| \), and \( S(t) \) near the 5000th zero \( \gamma_{5000} \approx 5447.86 \).

The critical facts (pun intended) about \( Z(t) \) are that it is smooth, and if \( t \) is real then \( Z(t) \) is real and \( |Z(t)| = |\zeta(\frac{1}{2} + it)| \). Those facts do not uniquely determine \( Z(t) \): there is still a global choice of \( \pm 1 \). One might ask whether the choice matters, and if it does, is the standard choice for the square-root in (2.5) the correct option?

**Principle 2.2** \( Z(t) \) is statistically indistinguishable from \(-Z(t)\).

Thus, the global choice of sign for \( Z(t) \) does not matter. Indeed, \( \xi(\frac{1}{2} + it) \) is also real for \( t \in \mathbb{R} \), and has the same critical zeros as \( Z(t) \), but it has the opposite sign of \( Z(t) \). So, the standard normalizations for those functions disagree with each other. A consequence of **Principle 2.2**, which could be considered an equivalent statement, is that if \( k \) is a non-negative integer then

\[
\int_T^{2T} Z(t)^{2k+1} dt = o(T),
\] (2.6)

and as \( T \to \infty \) that integral changes sign infinitely many times. Equation (2.6) is a theorem for \( k = 0 \) and a conjecture for larger \( k \).

**Principle 2.2** is a stronger statement than asserting that to leading order the value distribution of \( Z(t) \) is symmetric. In contrast, the average value of

\[
-\xi(1 - s),
\] (2.4)
We will see that the function \( \zeta(\frac{1}{2} + it) \) is 1 in a very strong sense: if \( k \) is a non-negative integer then

\[
\int_T^{2T} \zeta(\frac{1}{2} + it)^k \, dt \sim T. \tag{2.7}
\]

Such small biases are the underlying cause of many misconceptions arising from numerical computations, see (3.4) and Principle 3.2.

### 2.1 We care about zeros because we care about primes

Riemann’s great insight was that the zeros of the \( \zeta \)-function encode information about the primes. Based at least partially on numerical computation, he formulated the conjecture which is now known as the Riemann Hypothesis:

**Conjecture 2.3 The Riemann Hypothesis (RH).** The zeros of \( \zeta(s) \) with \( 0 < \sigma < 1 \) lie on the line \( \sigma = \frac{1}{2} \), equivalently, all zeros of \( \zeta(s) \) lie on the line \( \sigma = \frac{1}{2} \), equivalently, all zeros of \( Z(t) \) with \( |\Im(t)| < \frac{1}{2} \) are real.

Riemann’s original formulation involved yet another function: all the zeros of \( \Xi(t) := \zeta(\frac{1}{2} + it) \) are real. RH has been rigorously verified for \( t < 3 \times 10^{12} \), involving more than \( 12.3 \times 10^{12} \) zeros \([69]\).

The zeros of \( \zeta(s) \) with \( 0 < \sigma < 1 \) are traditionally denoted by \( \rho = \beta + i\gamma \). By the functional equation and the fact that \( \zeta(s) \) is real if \( s \) is real, if \( \rho = \beta + i\gamma \) is a zero of \( \zeta(s) \) then so is \( \rho = 1 - \beta + i\gamma \). So either \( \beta = \frac{1}{2} \), as predicted by RH, or there is a pair of zeros of \( \zeta(s) \) (or \( \xi(s) \)) located symmetrically around the critical line \( \sigma = \frac{1}{2} \). Equivalently, a failure of RH corresponds to a pair of complex conjugate zeros of \( Z(t) \). Assuming RH, the zeros of \( \zeta(s) \) with positive imaginary part are denoted \( \frac{1}{2} + i\gamma_n \) with \( 0 < \gamma_1 < \gamma_2 < \cdots \). That notation would break down if there were repeated zeros, but no plausible reason has been given to expect a multiple zero.

There are 41 zeros of the \( \zeta \)-function visible in Figure 2.1. RH implies that (for \( |t| > 3 \)) all local maxima of \( Z(t) \) (or \( \xi(\frac{1}{2} + it) \)) are positive and all local minima are negative, a condition which is visible in the first two graphs in Figure 2.1. The converse is not necessarily true, although that is mistakenly asserted in the official statement of the Riemann Hypothesis Millennium Problem, see [34]. So, one cannot trivially deduce from the graph of \( Z(t) \) or \( \log(|Z(t)|) \) in Figure 2.1 that RH holds for \( 5430 < t < 5448 \).

It is possible to verify RH on an interval by a computer calculation, based on two properties of the \( \zeta \)-function. One property we have already seen: \( Z(t) \) is real when \( t \) is real, so one can count critical zeros by looking for sign changes. The other is that the zeros of the \( \zeta \)-function have a nice counting function with a small and computable error term. Let \( N(T) \) be the number of zeros of \( \zeta(s) \) with \( 0 < \gamma_n \leq T \). We have \([83]\)

\[
N(T) = \frac{1}{2\pi} T \log T - \frac{\log 2\pi e}{2\pi} T + \frac{7}{8} + S(T) + O(T^{-1}) \tag{2.8}
\]

where

\[
S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT) = \frac{1}{\pi} \Im \log \zeta(\frac{1}{2} + iT). \tag{2.9}
\]

We will see that the function \( S(t) \) grows very slowly. The third plot in Figure 2.1 illustrates the basic properties of \( S(t) \): it has a jump discontinuity (of height 1) at a critical zero, and where it is continuous it is approximately linear with slope \( -\frac{1}{2} \log t \).

The argument in (2.9) is determined by continuous variation along the line \( 2 + it \) for \( 0 \leq t \leq T \), and then along the line \( \sigma + iT \) for \( 2 \geq \sigma \geq \frac{1}{2} \). One can
take either (2.8) or (2.9) as the definition, with the other as a theorem. Those formulas assume $T$ is not the imaginary part of a zero of $\zeta(s)$.

By (2.8) and (2.9) one can rigorously prove that RH holds on an interval: use sign changes to count real zeros of $Z(t)$, then compute the change in $S(t)$ to determine the change in $N(t)$ and thus find the total number of zeros. Then check if those two quantities are equal. This is known as Turing’s method.

In Figure 2.1 a failure of RH would correspond to a jump discontinuity in $S(t)$ of height 2, at a point where $Z(t)$ does not have a zero. Thus, from the graph of either $Z(t)$ or $\log |Z(t)|$, coupled with the graph of $S(t)$, one can “see” a proof in Figure 2.1 that RH is true for $5430 < t < 5448$.

2.2 Unfolding the zeros

By (2.8) the zeros at larger height are on average closer together. Specifically, at height $T$ the average gap between zeros is $2\pi/\log T$, and $\gamma_n \approx 2\pi n/\log n$.

When discussing the statistics of the zeros, in particular the gaps between zeros, it is helpful to use the normalized or unfolded zeros $\tilde{\gamma}_n = 1/2\pi \gamma_n \log \gamma_n$. Note that $\tilde{\gamma}_n \sim n$ and $\tilde{\gamma}_{n+1} - \tilde{\gamma}_n$ equals 1 on average.

When it is necessary to be more precise than leading-order behavior, we set $\tilde{\gamma}_n = \tilde{N}(\gamma_n)$, where

$$\tilde{N}(T) = \frac{1}{2\pi} T \log T - \frac{\log 2\pi e}{2\pi} T + \frac{7}{8}. \quad (2.10)$$

We leave it as an exercise to determine the average value of $\tilde{\gamma}_n - n$ as $n \to \infty$. The answer is in Subsection 7.6. That answer also explains the usual definition of $S(t)$ in the case $t$ is the imaginary part of a zero of the $\zeta$-function.

2.3 The size of $Z(t)$ and $S(t)$

A surprising fact about $S(t)$ is that it grows very slowly. Assuming RH, Littlewood [83] showed that $S(T) = O(\log T/\log \log T)$. It is conjectured [36] that

$$|S(T)| \leq (1 + o(1)) \frac{1}{\pi} \sqrt{\frac{\log T \log \log T}{2}} \quad (2.11)$$

and that bound is sharp. The results for $Z(t)$ are the same. On RH, we have [83] $\log |Z(t)| = O(\log t/\log \log t)$, and the conjecture (with sharp constant) is

$$\log |Z(t)| \leq (1 + o(1)) \sqrt{\frac{1}{2} \log T \log \log T}. \quad (2.12)$$

In the other direction [14] there exists $C > 0$ such that there exist arbitrarily large $T$ with:

$$\log |Z(T)| > C \sqrt{\frac{\log T \log \log T}{\log \log T}}. \quad (2.13)$$

The typical size of $Z(t)$ is much smaller. Selberg [77] proved that if $t$ is chosen uniformly at random from $[T, 2T]$, then

$$\frac{\log \zeta(\frac{1}{2} + it)}{\sqrt{\frac{1}{2} \log \log T}} \to N(0, 1) \quad \text{as} \quad T \to \infty \quad (2.14)$$

where $N(0, 1)$ is the standard (complex) Gaussian, and (2.14) indicates convergence in distribution. In particular, $\log |Z(t)|$ and $S(t)$ each have a (real)
Gaussian distribution, which are different only because the definition of $S(t)$ contains a factor of $1/\pi$, and those distributions are independent.

In numerical computations of the $\zeta$-function the scale factor in Selberg’s theorem is practically irrelevant: at $T_{BH} = 10^{33}$, which is approximately the largest height where the $\zeta$-function has been calculated [13], the typical values are of size $e^{\sqrt{\frac{3}{2} \log \log T_{BH}}} < e^{1.5} < 4.5$. The conjectured extreme values are larger, $\sqrt{\frac{3}{2} \log \log T_{BH}} \log \log T_{BH} \approx 12.8$, so within the realm of current computation one might expect to see $|Z(t)|$ larger than 300,000. Unfortunately, the extreme values are rare, and possibly the predicted largest values do not occur until significantly greater heights. The largest value of $Z(t)$ found in [13] is approximately 16244 near $3.92 \times 10^{31}$, which is 3.7% less than the largest computed value [82]. The largest calculated value of $S(t)$ is 3.345 near $t = 7.7573 \times 10^{27}$.

Figure 2.4 shows $Z(t)$, $\log |Z(t)|$, and $S(t)$ in a neighborhood of the largest value of $Z(t)$ found in [13]. (Note that there is no good linear scale on which to plot $Z(t)$ in that region.) The author thanks Jonathan Bober and Ghaith Hiary for providing open access to their extensive data [48].

![Figure 2.4 Plots of $Z(T_{\text{max}} + t)$, $\log |Z(T_{\text{max}} + t)|$, and $S(T_{\text{max}} + t)$ where $T_{\text{max}} = 39246764589894309155251169284084 \approx 3.9 \times 10^{31}$.](image_url)

In Figure 2.4 we see that the large value of $Z(t)$ arises from a large gap between neighboring zeros. The large zero gap also contributes to the large value of $S(t)$: a zero gap of $K$ times the local average spacing must be accompanied by a value $|S(t)| \geq K/2$. The large zero gap in Figure 2.4 is 5.93 times the local average. The relationship between the size of an isolated large zero gap and the local maximum of $Z(t)$ is subtle: see Principle 8.2. The relationship between a large gap and $S(t)$ is almost trivial, but we record it for later use.

**Principle 2.5** If $\gamma_{j+1} - \gamma_j = K$, then $S(\gamma_{j+1}) - S(\gamma_j) = K$. So in particular either $|S(\gamma_{j+1})| \geq K/2$ or $|S(\gamma_j)| \geq K/2$. Thus, an upper bound on $|S(t)|$ implies a comparable upper bound on the size of the normalized zero gaps.

Note that Principle 2.5 does not say that a large value of $S(t)$ must be accompanied by a large gap between zeros.
As will be explored in detail, Figure 2.4 does not illustrate the typical behavior for the large values of $Z(t)$ and $S(t)$. One way to see this is from Selberg’s theorem [77] that $\log |Z(t)|$ and $S(t)$ are independently distributed. One of those being particularly large should have no effect on the other. Figure 2.4 illustrates that a large zero gap causes large values of $Z(t)$ and $S(t)$ to occur in close proximity, therefore that cannot be the typical behavior near a large value.

In Section 3 we briefly explore the history of finding large values of $Z(t)$ and explain why that work has inadvertently led to a mistaken impression of what the graph of $Z(t)$ looks like in a neighborhood of its largest values. In Section 4 we introduce carrier waves, which are the actual cause of the largest values of $Z(t)$.

## 3 Misleading ideas about large values

Computation has been an important tool for studying the $\zeta$-function ever since Riemann calculated the first few zeros by hand. Computers have enabled large-scale computations: large on the human scale but minuscule on an absolute scale. In reference to whether existing computations should be seen as evidence for RH, Andrew Odlyzko [66] has sounded a cautionary note: the true nature of the $\zeta$-function is unlikely to be revealed until we reach regions where $S(t)$ is routinely over 100. Since (by Selberg’s theorem) $S(t)$ is typically of size $\sqrt{\frac{1}{2} \log \log t}$, such regions will be inaccessible for a long time.

Despite Odlyzko’s warning, there are certain aspects of the $\zeta$-function which appear in computations and have influenced the direction of research, but which do not accurately portray the true nature of the $\zeta$ function. These misconceptions are partially based on the way the $\zeta$-function is computed, which we describe next.

### 3.1 Computations of $Z(t)$

The earliest large-scale computations of the $\zeta$-function calculated $Z(t)$ using the Riemann-Siegel formula:

$$Z(t) = 2 \sum_{n<\sqrt{t/2\pi}} n^{-\frac{1}{2}} \cos(\theta(t) - t \log n) \ + \text{remainder}$$

where

$$\theta(t) = \arg \left( \pi^{-it/2} \Gamma \left( \frac{1}{4} + \frac{it}{2} \right) \right)$$

$$= \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O(t^{-1}).$$

The function $\theta(t)$ arises in an alternate expression for $Z(t)$:

$$Z(t) = e^{i\theta(t)} \zeta \left( \frac{1}{2} + it \right).$$

The $n = 1$ term in (3.1) is the largest, and if $t$ is small then that term has a strong influence on the overall sum. The points where $\cos(\theta(t)) = \pm 1$ are known as Gram points, numbered so that $\theta(g_m) = m \pi$. Gram [43] noted that the first several zeros of $Z(t)$ lie between consecutive Gram points. In other words, when $t$ is small the other terms are insufficient to flip the sign of the first term at a Gram point.
In the course verifying RH for the first 75 million zeros, Brent [17] found that $Z(t)$ was unusually large, more than 79.6, at the 70354406th Gram point. Furthermore, at that point the first 72 terms in the Riemann-Siegel formula (3.1) were positive. Figure 3.1 shows a graph of $Z(t)$ near that Gram point.

![Figure 3.1 Plot of $Z(g_{70354406} + t)$, where $g_{70354406} \approx 30694257.761$ is a Gram point.](image)

At a Gram point the first term in the Riemann-Siegel formula is 1 or $-1$, and the other terms have no particular bias, so it is not surprising that [83]

$$\sum_{n \leq N, \, n \text{ even}} Z(g_n) \sim N \quad \text{and} \quad \sum_{n \leq N, \, n \text{ odd}} Z(g_n) \sim -N. \quad (3.4)$$

Those averages are similar to the fact that the average value of $\zeta\left(\frac{1}{2} + it\right)$ is 1, see (2.7). Such biases, combined with Selberg’s central limit theorem, are major contributors to:

**Principle 3.2** Biases of order $O(1)$ can have a significant, and unwelcome, influence on the interpretation of numerical computations because the true nature of the $\zeta$-function reveals itself on the scale of $\sqrt{\log \log T}$.

Principle 3.2, and other sources for misinterpreting the data, are explored throughout this paper.

### 3.2 Three unfounded inferences

Brent’s observations have re-appeared in subsequent numerical computations. The result is that reinforcing those observations has been (at least partially) a goal of recent computational work, particularly as it relates to understanding the largest values of the $\zeta$-function. That is unfortunate because the largest values as they appear in computations are not representative of the largest values at greater height. Thus, the impressions one has from those data are not helping to build intuition for the true nature of the $\zeta$-function. Quite the opposite: those numerical examples tempt one into mistaken notions.

The notions we refer to are:

**Mistaken Notion 3.3** The largest values of the zeta-function occur when there is a particularly large gap between zeros.

**Mistaken Notion 3.4** The largest values of the zeta-function occur when a large number of initial terms in the Riemann-Siegel formula (3.1) have the same sign.

**Mistaken Notion 3.5** Counterexamples to RH are more likely to occur near an unusually large gap between zeros.

We do not attribute those Notions as conjectures due to a specific person. Indeed, in many cases where those Notions appear in the literature (implicitly or explicitly), the author acknowledges the Notion without necessarily endorsing it. For example, concerning Mistaken Notion 3.4, Brent [17] says “This suggests that ‘interesting’ regions might be predicted by finding values of $t$ such that the first few terms in the Riemann-Siegel sum reinforce each other.”
“interesting” behavior of an unusually large gap does indeed occur when the initial terms reinforce, at least in the range accessible by computers. That Notion is not Mistaken in the limited context of computing the \( \zeta \)-function at moderate height.

The even more “interesting” behavior of a possible counterexample to RH requires a second step to get to Mistaken Notion 3.5. Odlyzko [68] explains it this way: “One reason for the interest in large values of \( \zeta(1/2 + it) \) is that one could think of a large peak as ‘pushing aside’ the zeros that would normally lie in that area, and if these zeros were pushed off of the critical line, one would find a counterexample to the RH.” Odlyzko is not endorsing Mistaken Notion 3.5, merely noting the reasoning which might have led to its formulation.

Brent’s and Odlyzko’s use of “scare quotes” is a further indication that they are not endorsing those Notions as fundamental principles. Indeed, it may be that individually the vast majority of experts understand the limitations of what can be learned from existing computations. But the frequency with which those Notions have been repeated, not always with qualifiers such as “it has been said that...”, indicates the need for clarification. In Section 11 we revisit these Notions in the context of the Principles discussed in this paper, justifying the claim that the Notions are indeed “Mistaken”.

3.3 Gram’s law

If \( (g_n) \) are the Gram points for the Z-function, then we say Gram’s law holds for the Gram interval \( (g_m, g_{m+1}] \) if that interval contains exactly one zero of the Z-function. Gram noted that this law holds for the first 15 zeros, and Hutchinson [51] found that it holds for the first 126 zeros, and he also coined the term “Gram’s law”. Subsequent work found that it holds for more than 91% of the first 1000 zeros, and most papers on zero calculations report statistics on Gram’s law. Such numerical evidence, and repeated attention, leads to the archetypal mistaken notion concerning zeros:

Mistaken Notion 3.6 Gram points are special, and Gram’s law is helpful for developing intuition about the location of zeros of the \( \zeta \)-function.

It has been known for a long time that Gram’s law fails infinitely often [83] and in fact fails a positive proportion of the time (although there are good reasons to believe it is true more than 66% of the time, see [84, 45]). Gram’s law is not mistaken merely because it is not always true: it is mistaken because it paints a picture of the zeros which inhibits one from gaining a proper intuition about the behavior of the \( \zeta \) function at large height. Also, as we will explain in Subsection 11.4, if Gram’s law is stated properly, it is true 0% of the time.

4 Separating out the local zero spacing

Figure 2.1 and Figure 2.4 show that the local spacing of the zeros has an influence on the size of the \( \zeta \)-function: when there is a large gap the function is larger, and when there is a small gap the function stays small. Unfortunately, those graphs are misleading unless you ignore the vertical scale. The specific spacing of nearby zeros strongly influences the relative sizes of the nearby maxima and minima. But the local spacings have very little to do with whether or not the actual function values are particularly large or small, compared to what one would expect for the \( \zeta \)-function in that region.

Indeed, by the end of this section we will see that the local spacing of zeros is not the leading order contribution to the size of the \( \zeta \)-function.
In general, the size of the $\zeta$-function is controlled by the $\zeta$-function’s **carrier wave**. The terminology is due to Hejhal [47], with detailed discussion by Bombieri and Hejhal [15]. The idea of carrier waves is based on some unpublished speculations of H.L. Montgomery. The main idea is that, on a logarithmic scale, the $\zeta$-function changes its size slowly. If it is large, then outside a set with small measure, it usually stays large for a while. (The “set with small measure” is small neighborhoods of the zeros.) If it is small, it usually stays small for a while. Here “a while” can be interpreted as “on an interval containing a large number of zeros”. In particular, the typical large values occur as clusters of large local maxima and minima, not as a single isolated large maximum. That is contrary to what we see in graphs of $Z(t)$, which is why we have Mistaken Notion 3.3. That is why for many people, this idea is in the “I find that hard to believe” category. Indeed, the phenomenon of carrier waves is not visible in any numeric computation of the $\zeta$-function, because the scale of the carrier waves, just like the scale of $S(t)$, grows so slowly that it appears bounded within the range we can compute. However, we can take a first-principles approach to defining what we mean by “carrier wave”, and then build intuition by looking at illustrative examples.

4.1 The wave as a local constant factor

Suppose $f(t)$ is a high degree polynomial, with zeros $\gamma_1 < \ldots < \gamma_M$, and suppose we want to understand the graph of $f$ near $t_0$. Further suppose that $t_0$ is roughly near the middle of the $\gamma_j$. We can write

$$f(t) = a_0 \prod_{\gamma_j \text{ near } t_0} \left(1 - \frac{t}{\gamma_j}ight) \prod_{\gamma_j \text{ far from } t_0} \left(1 - \frac{t}{\gamma_j}\right)$$

$$\approx A_0 \prod_{\gamma_j \text{ near } t_0} \left(1 - \frac{t}{\gamma_j}\right) \quad \text{for } t \text{ very close to } t_0.$$ (4.1) (4.2)

The approximation in the second line above will be valid for $t$ in a neighborhood of $t_0$ if the zeros far from $t_0$ are balanced on either side of $t_0$, meaning that the product over “$\gamma_j$ far from $t_0$” is approximately constant near $t_0$.

$L$-functions are much like high degree polynomials, and the global spacing of their zeros is very regular (as can be seen from the small error term in the zero counting function $N(T)$, see (2.8)). Thus we have:

**Principle 4.1** The behavior of $Z(t)$ near $t_0$ depends on three things: a global factor independent of $t_0$, the arrangement of the zeros near $t_0$, and a scale factor which depends on the zeros far from $t_0$ and which does not change too quickly as a function of $t_0$.

Principle 4.1 does not specify which of the two factors that depend on $t_0$ are most relevant to the size of $Z(t)$. Any graph of $Z(t)$ in the range accessible by current computers, such as in Figure 2.4, makes it appear that the arrangement of the zeros is more important. We will see that those examples are misleading.

4.2 Carrier waves

The scale factor $A_0 = A_0(t_0)$ has been termed the **carrier wave** by Hejhal [47] and Bombieri and Hejhal [15], making rigorous a speculation of Montgomery. Montgomery’s preliminary calculations suggested that the carrier wave for the Riemann zeta-function should not vary too much over a window of width
\[
\exp(\delta_T \log \log T) / \log T, \text{ for some function } \delta_T \to 0. \text{ Bombieri and Hejhal [15] show that, for most } T, \text{ the carrier wave does not vary significantly over a window of width } M / \log T, \text{ for any fixed } M > 0 \text{ as } T \to \infty. \text{ That is, across a span of } M \text{ consecutive zeros, the size of } \log |\zeta(\frac{1}{2} + it)| \text{ usually varies very little. (The proof in [15] might actually show that one can take } M = \log \log(T)^\kappa \text{ for any } \kappa < \frac{1}{4}, \text{ with (6.21) in that paper providing the limiting constraint.)}
\]

Bombieri and Hejhal used carrier waves as the key idea toward their proof of the following surprising theorem: if \(L_1(s)\) and \(L_2(s)\) are L-functions which individually satisfy RH and have the same functional equation, and \(\alpha \in \mathbb{R}\), then \(L_1(s) + \alpha L_2(s)\) has 100\% of its zeros on the critical line (there are technical conditions which we have omitted). The proof is: the \(\log(L_j(\frac{1}{2} + it))\) are independently and normally distributed, so most of the time one of the \(L_j\) is significantly larger than the other, and furthermore (because of the carrier wave) it stays larger across an arbitrarily large number of zeros. Therefore in that region the zeros of the linear combination are very close to the zeros of the larger L-function, and that accounts for most of the zeros.

Note that Selberg’s theorem on the normal distribution of \(L(\frac{1}{2} + it)\) is not sufficient: one needs the additional fact that the carrier wave causes the larger L-function to stay large over a significant range.

Thus we have:

**Principle 4.2** The carrier wave is responsible for the bulk of the value distribution of an L-function, with the variation due to the local zero distribution playing a secondary role. In particular, it is the carrier wave which obeys Selberg’s central limit theorem.

Principle 4.2 explains why, in the range accessible by current computers, the observed value distribution of \(\log |Z(t)|\) departs significantly from normal. The local zero spacing contributes a lesser (typically, bounded) amount. But a bounded amount is significant in the range where the carrier waves are very small.

### 4.3 Measuring the wave

We now describe a way to measure and observe the carrier wave. The goal is to isolate the contribution of the nearby zeros, as suggested in (4.2). The idea is to think of the zeros as parameters which can change: we can slide the zeros side-to-side, and this will cause a change in the graph of \(Z(t)\). If we slide the zeros apart, making a large zero gap, then the function will acquire a large local maximum. If we slide the zeros so they become more equally spaced, then the maxima of the function will will be approximately the same size. In the extreme case of moving the zeros to be equally spaced, the result will be the (scaled and shifted) cosine function.

It is not possible to achieve a globally equal spacing for the zeros, because the local average spacing of the zeros is not constant. Instead we focus on “nearby” zeros, where the average spacing is close to constant. Suppose \(t_0 \in [\gamma_M, \gamma_{M+1}]\). If \(K\) is large enough and we slide the zeros \(\gamma_{M-K}, \ldots, \gamma_{M+K+1}\) to be equally spaced, then near \(t_0\) the resulting function will look like \(a \cos(b(x + c))\) for some \(a, b, c \in \mathbb{R}\).

Running the above process in reverse,

\[
Z(t) \approx A_Z a \cos(b(t + c)) \prod_{j=M-K}^{M+K+1} \frac{t - \gamma_j}{t - g_j}, \quad (4.3)
\]

where the \(g_j\) are zeros of \(\cos(b(t+c))\), the parameters \(a, b, c\) depend on \(t_0\) and \(K\), \(A_Z\) is a normalization factor, and the approximation is good if the zeros near \(t_0\)
are well spaced (so that $t_0$ is near the middle of the interval $[\gamma_{M-K}, \gamma_{M+K+1}]$). We apply this idea to $Z(t)$ from Figure 2.1, with

$$
t_0 = 5447.86
M = 5000
K = 10
A_Z = 2
a = 1.0991
b = 3.3825
(4.4)
c = 0.17008.
$$

The result is shown in the top plot in Figure 4.3, where we superimpose $Z(t)$ and its approximation based at $t_0 = 5447.86$. The bottom plot shows the same idea based at $t_0 = \gamma_{7010} \approx 7273.70$, which has scale factor $a = 0.9130$. In both cases one sees that the behavior near $t_0$ is determined by the nearby zeros and a local scale factor (which depends on $t_0$).

In the top plot in Figure 4.3 we can interpret the scale factor 1.0991 as the magnitude of the carrier wave at $t_0 = 5447.86$, and similarly for the scale factor 0.9130 at $t_0 = 7273.70$. Those scale factors are different, which shows that the magnitude of $Z(t)$ is not solely due to the local zero spacing and a single global factor.

4.4 Some caveats

The scale factors in Figure 4.3 are not the carrier wave in the sense of Bombieri-Hejhal, because Montgomery’s heuristic calculation suggests that at the small height of that example the carrier wave should only be approximately constant in a very narrow window. In Figure 4.3 we have fit to the zeros in a much wider window. Nevertheless, those plots illustrate that the nearby zeros and a single local scale factor determine the local behavior of $Z(t)$.

**Note 4.4** The scale factor at a point is not well-defined: it depends on $K$, the number of zeros, or more precisely the width of the window, over which one has done the fit. If the window covers an enormous number of zeros then the local scale factor will just be 1. If the function changes scale in the window, i.e., the carrier wave is not approximately constant in the window, then the scale factor is not providing useful information. This is discussed further in **Note 7.6**.
It remains to justify that the global scale factor (independent of $t_0$) is $A_Z = 2$. That is the overall factor of 2 in the Riemann-Siegel formula (3.1). Indeed, one could argue that a more natural way to define the carrier wave is not to start with the cosine function in (4.3), but rather with the first term in the Riemann-Siegel formula:

$$2 \cos(\theta(t)).$$

That approach would avoid the parameters $b$ and $c$ in (4.3), indeed $\theta'(5447.86) \approx 3.38255$, which is the value for $b$ in (4.4). Using $2 \cos(\theta(t))$ in (4.3) would not change the point illustrated Figure 4.3.

In order to really “see” the carrier wave, one must go to enormously larger values of $t$. That is not computationally feasible in the L-function world, but it is easy in the random matrix world. In Section 5 and Section 6 we review the connections between the $\zeta$-function and the characteristic polynomials of random unitary matrices, returning to carrier waves in Section 7.

## 5 A random model for the $\zeta$-function

A graph of $Z(t)$ gives the impression of randomness: the function wiggles, and there is no apparent pattern to those wiggles. Obviously $Z(t)$ is not random, because it is a specific function which has no randomness in its definition. But we can make a random function in the following way. Fix an interval $\mathcal{Y} = [t_0, t_1] \subset \mathbb{R}$. If $T \in \mathbb{R}$ is random, then $Z_T(t) := Z(T + t)$ is a random function on $\mathcal{Y}$. (If it is bothersome that the probability distribution of $T$ has not been specified, it can be taken to be uniform on $[T_0, 2T_0]$ for some $T_0 > 0$.)

If we had another set of random functions $Z_T$ on $\mathcal{Y}$, and it was possible to prove theorems about $Z_T$, and if furthermore we had reason to believe that $Z_T$ and $Z_T$ had similar properties, then we could turn theorems about $Z_T$ into conjectures about $Z_T$. That would be illuminating, particularly if precise conjectures about $Z_T$ were in short supply. That is how random matrix theory made fundamental contributions to the study of L-functions.

### 5.1 Self-reciprocal polynomials and the functional equation

Before getting into the details, let’s consider a certain class of polynomials which have constant term 1:

$$f(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_{N-1} z^{N-1} + a_N z^N. \quad (5.1)$$

The polynomial $f$ is self-reciprocal if

$$f(re^{i\theta}) = 0 \quad \text{implies} \quad f(r^{-1}e^{i\theta}) = 0, \quad (5.2)$$

or equivalently,

$$a_j = a_N a_{N-j} \quad \text{for} \quad 0 \leq j \leq N \quad (5.3)$$

or equivalently

$$f(z) = X(z) \overline{f}(z^{-1}) \quad \text{where} \quad X(z) = a_N z^N, \quad (5.4)$$

or equivalently

$$Z_f(\theta) = X(e^{i\theta})^{-\frac{1}{2}} f(e^{i\theta}) \quad \text{is real if} \quad \theta \in \mathbb{R}. \quad (5.5)$$


In (5.4), \( \overline{f} \) is the **Schwarz reflection** of \( f \): \( \overline{f}(z) := \overline{f(\overline{z})} \). The parallel with the functional equation for the \( \zeta \)-function is evident if one compares (5.4) to (2.2), and (5.5) to (2.5). At the risk of belaboring the point, in both cases there is an involution of the plane, the set of zeros is fixed by the involution, the **Riemann Hypothesis** is the assertion that every zero is fixed by the involution, and there is a Z-function which is real on the set of points which are fixed by the involution. The unit circle is the analogue of the critical line for self-reciprocal polynomials. Thus, if a set of polynomials is claimed to be a random model for the \( \zeta \)-function, those polynomials should be self-reciprocal.

### 5.2 Random polynomials and trigonometric polynomials

We seek random self-reciprocal polynomials which can serve as a model for \( \zeta(s) \) or equivalently \( Z(t) \). How to choose the randomness? There are three reasonable options.

The first option is to choose the coefficients of \( f(z) \) so that they satisfy (5.3). The second is to recognize that \( Z_f \) is a trigonometric polynomial (i.e. a finite Fourier series), so its coefficients can be chosen randomly with only the requirement that they are real. The third option is to choose the zeros so that they satisfy (5.2). We will argue that only the third of those options is reasonable.

Figure 5.1 shows \( Z(t) \) at height \( 1.5 \times 10^{30} \), in a region where nothing special is happening. (Data courtesy of Bober and Hiary [13][48].)

![Figure 5.1](image-url)

**Figure 5.1** \( Z(t) \) on an interval of width \( 2\pi \) near \( 1.5 \times 10^{30} \)

We wish create random polynomials which capture the type of randomness visible in Figure 5.1. But what does that mean? The following features are worthy of attention.

(A) Do all the zeros appear to be real?

(B) How often does the graph get big? Almost equivalently, how often are there large gaps between zeros?

(C) How often are pairs of zeros close together? How often are there clusters of close together zeros?

(D) Are the maxima and minima generally around the same size, with only occasional much larger or smaller local extrema?

**Item D** is obviously a leading question. The answer is “yes”.

**Item A** refers to RH. We can’t escape the fact that RH is true within the realm where we can experiment, so the random polynomials we seek must satisfy RH. That is, we seek **unitary polynomials**, which is standard terminology for polynomials with all zeros on \( |z| = 1 \).

The requirement of generating unitary polynomials effectively rules out the first two options for choosing random polynomials. If the coefficients are chosen randomly, then one can choose small coefficients to obtain a perturbation of \( z^N - 1 \). The zeros will be on the unit circle, but they will be close to regularly
spaced and the function will not have randomness similar to Figure 5.1. If the coefficients are random and large, then the polynomial will usually not be unitary (although it may have a large proportion of its zeros on the unit circle). Choosing the coefficients randomly and then conditioning on having all zeros on the unit circle turns out to produce the wrong type of randomness [37].

By process of elimination we conclude that the polynomials we seek must arise by having their zeros randomly generated, and on the unit circle. Such polynomials have arisen in mathematical physics, which we describe next.

**Note 5.2** It would be interesting to define a natural-looking ensemble of random self-reciprocal polynomials which are unitary most of time, but not always. If such an ensemble was consistent with RH being true within the realm of current computation, and it predicted a height at which RH would start to fail, that could throw doubt on RH. Indeed, one of the great shortcomings of RH skepticism is the lack of conjectures for: the height at which RH fails, how often zeros would be off the line, and how their real parts would be distributed.

5.3 The circular $\beta$-ensembles

The **circular $\beta$-ensemble** consists of $N$ random points $e^{i\theta_1}, \ldots, e^{i\theta_N}$ on the unit circle, equipped with the joint probability density function

$$\frac{\Gamma(1 + \beta/2)^N}{\Gamma(1 + N\beta/2)(2\pi)^N} \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^\beta. \quad (5.6)$$

This probability space is denoted $C\beta E(N)$. We use $\langle f(\theta_1, \ldots, \theta_N) \rangle$ for the expected value of $f$ averaged over $C\beta E(N)$.

The $C\beta E(N)$ is defined for any $\beta \geq 0$ and any positive integer $N$. We can obtain a random unitary (hence, self-reciprocal) polynomial by taking the points to be the zeros of a polynomial with constant term 1, as in (5.1).

If we write the $\theta_j$ in increasing order, then it is clear that $\langle \theta_{j+1} - \theta_j \rangle = 2\pi/N$. Almost equally clear is that the PDF of $\theta_{j+1} - \theta_j$ vanishes to order $\beta$ at 0. For that reason $\beta$ is referred to as the **degree of repulsion** between the points.

As $\beta \to 0$ the repulsion disappears and the points become independently uniformly distributed on the circle. As $\beta \to \infty$ the points approach equal spacing (known as the **picket fence** distribution), with the only randomness being the angle of an initial point which determines the location of all the others. Thus, as $\beta \to 0$ the particles behave like a gas, and as $\beta \to \infty$ they crystallize. For this reason, $\beta$ is sometimes referred to as **inverse temperature**.

Between those temperature extremes there are three values which have been studied extensively in the mathematical physics literature: $\beta = 1, 2, \text{ and } 4$. Those are more commonly known as the COE, CUE, and CSE, where the O/U/S stand respectively for Orthogonal/Unitary/Symplectic. Examples of random polynomials $Z(\theta)$ from each of those ensembles are shown in Figure 5.3.

Note that the graphs are all have the same vertical scale, ranging from $-6$ to $6$. The graphs show the effect of $\beta$ on the frequency of small zero gaps, large zero gaps, and large values. Those examples have $N = 68$, that choice ensuring the polynomials have the same number of zeros in a span of $2\pi$ as the $Z$-function shown in Figure 5.1. Our next goal is to persuade that one of those ensembles provides a good model for the $\zeta$-function.
Figure 5.3 Example random polynomials from $C\beta E(68)$ with, reading top to bottom, $\beta = 1, 2, \text{ and } 4$.

Hopefully one of the plots in Figure 5.3 looks like it has similar randomness to the $Z$-function in Figure 5.1, because otherwise we are out of ideas. The bottom plot, $\beta = 4$, is ruled out in multiple ways. The 4th order repulsion causes the zeros to be too rigidly spaced, which in turn prevents the function from having sufficiently many large or small values. Polynomials from the CSE are pleasant but rather boring, lacking the pizzazz of the Riemann $\zeta$-function.

The top plot in Figure 5.3 can be ruled out by carefully examining the features of the function. The most obvious property is that the COE example is large much more often: consider the width of the cutoff at the large maxima. The COE example also has more small zero gaps, both individually (small local extrema) and in clusters.

The CSE was too rigid to serve as a model for the $\zeta$-function, and the COE was too flexible. The Goldilocks zone is occupied by the CUE which is conjectured to provide, with appropriate adjustments and caveats, a model for the $\zeta$ function. We clarify the connection, and provide more terminology and historical details, in Section 6.

5.4 Where does the CUE come from?

To do calculations involving the CUE, such as computing an expectation $\langle f \rangle$, all one needs is the joint law of the zeros: (5.6) with $\beta = 2$. That is, it does not matter how the points are generated. But to construct an example, such as the plots in Figure 5.3, one needs a way to produce actual sets of points with the appropriate distribution. Here are some ways:

1. Brownian motion.

Have $N$ points start at the origin and undergo non-intersecting Brownian motion until they hit the unit circle. Those points will be distributed according to (5.6). (There are Brownian motion models for the CUE and other $\beta$-ensembles. See [30].)

2. A product of tridiagonal matrices.
Let $\Xi_k$ be certain independent random $2 \times 2$ matrices, depending on a parameter $\beta$, defined in [60]. Let

$$L = \text{diag}(\Xi_0, \Xi_2, \ldots, \Xi_{\lfloor N/2 \rfloor}) \quad \text{and} \quad M = \text{diag}(\Xi_{-1}, \Xi_1, \ldots, \Xi_{\lfloor N/2 \rfloor - 1}).$$

The eigenvalues of the tridiagonal matrix $LM$, equivalently $ML$, will be distributed according to the $\text{C}\beta\text{E}(N)$.

3. The unitary group $U(N)$.

Let $A \in U(N)$ be chosen randomly with respect to Haar measure. The eigenvalues of $A$ will be distributed according to (5.6) with $\beta = 2$.

The final option is the easiest to implement, but see Mezzadri’s paper [63] which describes possible pitfalls.

To summarize:

**Principle 5.4** The connection between random matrices and the $\zeta$-function is not actually about random matrices: all that really matters is the $\beta = 2$ repulsion (5.6) on the zeros/eigenvalues. Random matrices, either Haar-random from $U(N)$ or products of tridiagonal matrices [60], just happen to be convenient ways to generate such distributions.

However, for historical reasons, and for lack of an alternate term which applies more generally, we have:

**Principle 5.5** It is common to refer to the $N$ points in the $\text{C}\beta\text{E}(N)$, as “eigenvalues”, even if they arise in another way.

The description we have given is incomplete and our claim of a connection between the CUE and the $\zeta$-function is perhaps not persuasive, so in Section 6 we provide more details about random matrix eigenvalues, random unitary polynomials, and the connection with L-functions. Then in Section 7 we return to our main theme and explore carrier waves in random polynomials from the CUE.

6 RMT and L-functions: terminology and history

The C$\beta$E were not how Random Matrix Theory (RMT) first arose in mathematical physics. RMT came about in the 1950s as a way to understand complicated quantum mechanical systems. For example: the energy levels of a large atomic nucleus. The justification is that the Hamiltonian of the system is a large and complicated matrix with certain symmetries, and so the eigenvalues of the Hamiltonian should have similar statistical properties to a large random matrix with the same symmetries.

The connection with L-functions arose in three phases.

1. The GUE era.

The original formulation in the early 1970’s concerned the limiting local statistics of eigenvalues/zeros. By limiting we refer to the rescaled eigenvalues in the large matrix limit. The matrices are Hermitian so the eigenvalues are real. See Subsection 6.1 for more details.

2. The classical compact group era.

In the late 1990’s it was realized that certain compact unitary groups of matrices could be associated with families of L-functions. The size of the matrices is a natural function of the conductor of the L-functions (so one
could model at finite height), the particular matrix group is determined from properties of the L-functions, and the characteristic polynomial can be used to model the values of the L-functions. See Subsection 6.3 for more details.

3. The recipe era.

In the early 2000’s a heuristic, which was inspired RMT but does not actually make use of random matrices, was developed which reproduces virtually every conjecture arising from RMT. Furthermore, the heuristic produces the complete main term and not just the leading order behavior. The heuristic goes by the name “the recipe”. See the end of Subsection 6.5 for a brief mention.

Despite the fact that the recipe has largely replaced RMT as a tool for making conjectures about L-functions, random matrices are essential for our discussion of carrier waves, because the characteristic polynomials provide concrete examples which we can visualize.

6.1 The GUE era

The matrices in this setting are Hermitian, so the eigenvalues are real. The random entries are real, complex, or quaternionic depending on the type of physical system being modeled, and the randomness is Gaussian. As in the circular ensembles, the three Gaussian ensembles are named GOE, GUE, and GSE, where again O/U/S stands for Orthogonal/Unitary/Symplectic. Again there is a single parameter \( \beta \) which describes the eigenvalue repulsion, with \( \beta = 1, 2, \) or \( 4 \) for each of those cases, respectively. And again the way the eigenvalues (or energy levels) are generated could be Brownian motion [30], or tridiagonal matrices [31], or the original formulation of random Hermitian matrices with a given symmetry. The only concern for our purposes is that what one has is \( N \) points on the real line with joint probability density function:

\[
\frac{\Gamma(1 + \beta/2)^N}{(2\pi)^N \prod_{j=1}^N \Gamma(1 + j\beta/2)} \exp\left(-\frac{1}{2} \sum_{j=1}^N \lambda_j^2\right) \prod_{1 \leq j < k \leq N} |\lambda_j - \lambda_k|^\beta.
\] (6.1)

A similarity with the C\( \beta \)E is the order \( \beta \) repulsion between points. A difference is that the points are on a line, with the factor \( \exp\left(-\frac{1}{2} \lambda_j^2\right) \) keeping more of the points near the origin and preventing the points from wandering too far away. There is a tension between the factor keeping the eigenvalues close to the origin and the order \( \beta \) repulsion which tries to keep them apart. The result is that \( |\lambda_j| < \sqrt{2\beta N} \) for most \( j \) and the rescaled points \( \lambda_j / \sqrt{\beta N} \) are distributed according to the Wigner semicircle law \( \pi^{-1} \sqrt{2 - x^2} \).

The similarities outweigh the differences. The G\( \beta \)E and C\( \beta \)E belong to the same universality class which is characterized only by order \( \beta \) repulsion between points. Let’s be specific about which properties are universal. Define the rescaled eigenvalues \( \tilde{\lambda} \) so that \( \langle \tilde{\lambda}_{j+k} - \tilde{\lambda}_j \rangle = k \) for all fixed \( k \). Consider any local statistic, that is, a statistic only involving a finite number of differences of rescaled eigenvalues. Examples of local statistics are the normalized nearest neighbor spacing

\[
p_2(x) = \lim_{\Delta x \to 0} \left\langle \frac{\# \{ \tilde{\lambda}_{j+1} - \tilde{\lambda}_j \in [x, x + \Delta x] \} }{N} \right\rangle,
\] (6.2)

the pair correlation function

\[
R_2(x) = \lim_{\Delta x \to 0} \left\langle \frac{\# \{ \tilde{\lambda}' - \tilde{\lambda} \in [x, x + \Delta x] \}}{N} \right\rangle,
\] (6.3)
or more generally the joint distribution of \((\lambda_{j_1} - \lambda_{j_0}, \ldots, \lambda_{j_k} - \lambda_{j_0})\) for any \(j_0\) and \(j_1 < \ldots < j_k\). Note that the focus is on the relationship between an eigenvalue and its neighbors, not on their absolute location.

The quantities above depend on \(N\), but we suppress that notation because the leading order behavior is independent of \(N\).

**Principle 6.1 Universality.** For each \(\beta\), all \(\beta\)-ensembles have the same limiting local eigenvalue statistics as \(N \to \infty\).

Principle 6.1 comes from the fact that the \(\beta\)-repulsion term is the most important feature of the measure. Here is a less hand-wavy explanation in the case of the \(\text{CUE}(N)\) and the \(\text{GUE}(N)\). Both ensembles are examples of a determinantal point process. This means there is a kernel function \(K_N(x, y)\) such that the \(n\)-correlation function of the eigenvalues can be expressed as the determinant of an \(n \times n\) matrix with entries involving \(K_N(x, y)\). For example, the pair correlation function is given by

\[
R_{2,N}(x - y) = \left| \begin{array}{cc} 1 & K_N(x, y) \\ K_N(y, x) & 1 \end{array} \right|. \quad (6.4)
\]

Thus, the kernel function determines all the correlation functions, so it also determines all the local statistics (because those can be expressed in terms of the correlation functions, possibly involving a complicated inclusion-exclusion argument).

For \(\text{GUE}(N)\) the kernel function is

\[
K_{\text{GUE}(N)}(x, y) = N \frac{\psi_n(x \sqrt{n}) \psi_{n-1}(y \sqrt{n}) - \psi_{n-1}(x \sqrt{n}) \psi_n(y \sqrt{n})}{x - y}, \quad (6.5)
\]

with \(\psi_n(x) = H_n(x) e^{-x^2/4} \sqrt{\frac{2}{\pi n}}\), where \(H_n(x)\) is a Hermite polynomial.

For \(\text{CUE}(N)\) the kernel function is

\[
K_{\text{CUE}(N)}(\theta, \phi) = \frac{1}{2\pi} \frac{\sin(N(\theta - \phi)/2)}{\sin((\theta - \phi)/2)}. \quad (6.6)
\]

To leading order those kernels are the same, because by properties of Hermite polynomials, \(\lim_{N \to \infty} N^{-1}K_{\text{GUE}(N)}(0, x/N) = \sin(\pi x)/(\pi x)\). The same limit (with a simple change of variables) holds for \(K_{\text{CUE}(N)}\). The functional form (6.5) might suggest that their difference between the \(\text{GUE}(N)\) and \(\text{CUE}(N)\) is of order \(1/\sqrt{N}\), but in fact the difference is of size \(O(1/N)\). Figure 6.2 shows the pair correlation for each ensemble for \(N = 50\), and their difference (with the \(\text{GUE}(N)\) pair correlation being larger in a neighborhood of 0).

![Figure 6.2](image)

**Figure 6.2** Pair correlation for \(\text{GUE}(50)\) and \(\text{CUE}(50)\), and their difference.

Note that by (6.4) and the limiting values of the kernel functions for the \(\text{GUE}\) and \(\text{CUE}\), we find that both have the limiting normalized pair correlation function:

\[
R_2(x) = 1 - \frac{\sin^2(\pi x)}{\pi x^2}. \quad (6.7)
\]

That pair correlation function was the key to discovering the connection between random matrix eigenvalues and the zeros of the \(\zeta\)-function.
6.2 Zero statistics for L-functions

The RMT revolution in number theory began when Montgomery [64] determined partial information about the pair correlation of the zeros of the $\zeta$-function. The theorem he proved, combined with some heuristics about the prime numbers, led him to conjecture that the pair correlation function was

$$R_{2,\zeta}(x) = 1 - \frac{\sin^2(\pi x)}{(\pi x)^2}.$$  \hfill (6.8)

That conjecture became significant when Montgomery met Freeman Dyson, who informed him that (6.8) was the limiting pair correlation of eigenvalues of the GUE, as shown in (6.7). Combined with extensive computations by Odlyzko [68], this established:

**Principle 6.3 The GUE Hypothesis, aka the Montgomery-Odlyzko law.** *The limiting local statistics of the zeros of the $\zeta$-function are the same as the limiting local statistics of the GUE.*

The “limiting statistics” above refer to $T \to \infty$ and $N \to \infty$.

**Consequences of the GUE Hypothesis.** The GUE Hypothesis has many useful consequences:

1. Montgomery’s pair correlation conjecture (6.8).

2. 100% of the zeros of the $\zeta$-function are simple.

3. The PDF $p_2(x)$, see (6.2), of the nearest neighbor spacing $\tilde{\gamma}_{j+1} - \tilde{\gamma}_j$ is given by an explicit expression, which is well-approximated in the bulk by the **Wigner surmise**

$$\frac{32}{\pi^2} x^2 e^{-\frac{4}{3} x^2},$$ \hfill (6.9)

which is the exact expression for GUE(2). The limiting behavior of $p_2(x)$ for GUE($n$) as $n \to \infty$ is $\pi^2 x^2/3$ as $x \to 0$ and $\exp(-\pi^2 x^2/8)$ as $x \to \infty$.

The next two items are consequences of those limiting behaviors.

4. The smallest gaps between consecutive zeros satisfy

$$\tilde{\gamma}_{j+1} - \tilde{\gamma}_j \sim \gamma_j^{-\frac{1}{3} + o(1)},$$ \hfill (6.10)

that is,

$$\liminf_{j \to \infty} \frac{\log(\gamma_{j+1} - \gamma_j)}{\log \gamma_j} = -\frac{1}{3}.$$ \hfill (6.11)

It is possible to make a more precise statement. See Subsection 7.7.

5. The largest gaps satisfy

$$\tilde{\gamma}_{j+1} - \tilde{\gamma}_j \sim c_{\text{large}} \sqrt{\log \gamma_j},$$ \hfill (6.12)

where perhaps $c_{\text{large}} = 1/\sqrt{32}$ [6]. Or perhaps $c_{\text{large}}$ is a bit smaller. See Subsection 7.7.

Information about small gaps between zeros has implications for the class number problem [25], which was Montgomery’s original motivation for studying the pair correlation function [64].
Shortcomings of the GUE Hypothesis. The predictions of the GUE Hypothesis are spectacular, but it was clear from the beginning that only part of the story was being revealed. Odlyzko’s computations showed a discrepancy with the predictions. The discrepancy decreased at larger heights, as would be expected since the GUE Hypothesis only refers to the limiting behavior, but clearly there was a need for a more precise model.

Another shortcoming is that the prime numbers are nowhere to be seen in the GUE Hypothesis. How can it be that the truth about the \( \zeta \)-function has nothing to do with the prime numbers?

Finally, and most relevant to this present work, the GUE Hypothesis says little or nothing about the actual values of the \( \zeta \) function. This appeared in Odlyzko’s calculations (which computed the values of the \( \zeta \)-function and not just the zeros), which found a large discrepancy between the computed value distribution and the limiting Gaussian distribution from Selberg’s central limit theorem. The convergence to Gaussian is slow, so it was not surprising to see a discrepancy. But the nature of the discrepancy is not explained by the GUE Hypothesis. It took until the year 2000 to address that shortcoming.

6.3 After 30 years of GUE: classical compact groups and the Keating-Snaith Law

In the late 1990’s two innovations increased the influence of RMT on number theory. The first was due to Katz and Sarnak [56] who found that naturally arising collections of L-functions have a symmetry type which is given by a classical compact matrix group. The possible symmetry types are Unitary, Symplectic, and Orthogonal, with Orthogonal splitting into two cases depending on whether the sign of the functional equation is always +1 or equally likely +1 or −1. The random matrices in this context are elements of the compact unitary groups \( U(N) \), \( Sp(N) \), and \( O(N) \), where the randomness is uniform with respect to Haar measure. Note that \( U(N) \) with Haar measure is the same as the CUE(\( N \)), but the others are completely different than their mathematical physics counterparts having similar names. In fact, all of those classical groups have the same bulk eigenvalue statistics: the only difference arises with the eigenvalues near 1. Numerical evidence [73], and much subsequent work, supports the conjecture that the symmetry type of the family of L-functions determines the distribution of zeros near the critical point.

Keating and Snaith [57][58] made the second key leap when they recognized that the characteristic polynomial could be used to model the L-function itself. Here the characteristic polynomial of the \( N \times N \) matrix \( A \) is written in the slightly nonstandard form

\[
A_A(z) = \det(I-zA)
= \prod_{1 \leq j \leq N}(1-z\epsilon^{i\theta_j}),
\]

which is in keeping with our normalization (5.1) for self-reciprocal polynomials. Here the \( \epsilon^{i\theta_j} \) are the eigenvalues of \( A \). The analogue of the \( Z \)-function is

\[
Z_A(\theta) = ((-1)^N \det A)^{-\frac{1}{2}}e^{-i\theta/2} \det(I-zA)
= ((-1)^N \det A e^{-i\theta})^{-\frac{1}{2}}A_A(e^{i\theta}).
\]

We write \( S_A \) for the error term in the zero counting function:

\[
\#\{\theta_j \in [0,X]\} = \frac{1}{2\pi}X + S_A(X) + C_A
\]
where \( C_A \) is chosen so that \( S_A \) is 0 on average.

We confine our discussion to the case of the \( \zeta \)-function, which constitutes a unitary family, meaning that \( \zeta(\frac{1}{2} + it) \) is modeled by the characteristic polynomial of Haar-random matrices from \( U(N) \).

**Principle 6.4  The Keating-Snaith Law.** \( \zeta(\frac{1}{2} + it) \) for \( t \approx T \) is modeled by \( \Lambda_A(e^{it}) \) for Haar-random \( A \in U(N) \), where \( N = \log(T/2\pi) \).

A key component of Principle 6.4 is the identification \( N = \log(T/2\pi) \), where "\( \approx \)" means "is an integer close to". This is justified by "equating the density of zeros". When calculating asymptotics, where the only realistic hope is to make leading-order predictions, one can take \( N = \log T \). But for numerical comparison involving small numbers, the more precise choice of \( N \) is helpful. See Subsection 7.5 for further discussion of \( N = \log T \).

The first great success of Principle 6.4 was explaining the apparent discrepancy between numerical data and Selberg’s central limit theorem. Keating and Snaith proved a central limit theorem for \( \Lambda_A(e^{it}) \) for Haar-random \( A \in U(42) \) gives a close fit with the \( \zeta \)-function data. The "42" comes from \( \log(1.52 \times 10^{19}/(2\pi)) \approx 42.33 \). (Analogous to Principle 4.2, it is the carrier wave which obeys the Keating-Snaith central limit theorem.)

The second great success came from predicting the mysterious factor \( g_k \) in the conjectured moments \( (10.2) \) of the \( \zeta \)-function. Previously it was known that \( g_1 = 1 \) and \( g_2 = 2 \). It had been conjectured \([22, 24]\) that \( g_3 = 42 \) and \( g_4 = 24024 \). What Keating and Snaith found was (they compute an exact expression, but only the asymptotic is relevant for this discussion):

\[
\langle |\Lambda_A(e^{it})|^{2k} \rangle_{U(N)} \sim g_{U,k} N^2, \tag{6.15}
\]

as \( N \to \infty \), where

\[
g_{U,k} = k! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}. \tag{6.16}
\]

They found the beautiful equality \( g_{U,k} = g_k \) for the known and conjectured values for \( k = 1, 2, 3, 4 \).

We make explicit some of the assumptions in Principle 6.4.

**Principle 6.5  The Keating-Snaith Law, part 2.** To model \( \zeta(\frac{1}{2} + it) \) throughout the interval \( [T, 2T] \), choose \( e^N \) Haar-random characteristic polynomials from \( U(N) \), where \( N = \log(T/2\pi) \).

In other words, the \( \zeta \)-function is treated as being independent on each interval of length \( 2\pi \). One could argue that \( e^N/2\pi \) random matrices is a better choice, but that makes no difference in practice. Principle 6.5 is one of the many ways to obtain the conjectured maximum values of \( Z(t) \) and \( S(t) \) \([36]\).

The success of the Keating-Snaith Law is our justification to use characteristic polynomials from \( U(N) \), equivalently CUE\( (N) \), to explore the behavior of \( Z(t) \) beyond the realm accessible by direct computation. We address some natural questions concerning the connection between RMT and the \( \zeta \)-function before examining carrier waves in characteristic polynomials in Section 7.

### 6.4  Modeling \( Z(t) \) vs. modeling \( \zeta(s) \)

Our original motivation was to model \( Z(t) \), but our description of the Keating-Snaith law referred to modeling \( \zeta(s) \). To clarify:
**Principle 6.6** Model $Z(t)$ for $t \in \mathbb{R}$ by $Z_A(\theta)$. Model $\zeta(\frac{1}{2} + a + it)$ for $a \in \mathbb{C}$ by $\Lambda(e^{\theta - a})$.

Note that the CβE measure is invariant under rotation, so the expected value of expressions like $Z_A(\theta)Z_A(\theta + \alpha)$ or $|\Lambda_A(e^{i\theta})|^k$ are independent of $\theta$, so one typically sets $\theta = 0$. See for example (6.17) and (6.18).

Modeling $Z(t)$ by $Z_A(\theta)$ should seem natural: both are real-valued functions on $\mathbb{R}$, and (with $N \approx \log t/2\pi$) have the same average spacing between their zeros.

Modeling $\zeta(s)$ by $\Lambda_A(z)$ is slightly more subtle: one is a function which (conjecturally) has its zeros on the line $\sigma = \frac{1}{2}$, and the other has its zeros on the circle $|z| = 1$. How to translate between those worlds? The answer is to identify the half-plane $\sigma > \frac{1}{2}$ with the interior of the unit disc $|z| < 1$. That choice makes sense for several reasons; two of the author’s favorites are: $\lim_{z \to \infty} \zeta(s) = 1$ and $\lim_{z \to 0} \Lambda_A(z) = 1$; and all zeros of $\zeta(s)$ lying on $\sigma = \frac{1}{2}$ implies [80] all zeros of $\zeta'(s)$ lie in $\sigma > \frac{1}{2}$, while all zeros of $\Lambda_A(z)$ lying on $|z| = 1$ implies (by Gauss-Lucas) all zeros of $\Lambda'_A(z)$ lie in $|z| < 1$.

The most natural mapping from $\sigma > \frac{1}{2}$ to $|z| < 1$ is $s \mapsto e^{\frac{1}{2} - s}$, as in Principle 6.6.

### 6.5 Analogies, and lack thereof

We have seen that, conjecturally, to leading order the zeros of the $\zeta$-function have the same limiting local statistics as the GUE or CUE. When modeling the $\zeta$-function at height $T$ we use Haar-random characteristic polynomials from $U(N)$ under the matching $N \leftrightarrow \log T/2\pi$. One of three things can happen.

**Case 1. Agreement to leading order.** For example, the author conjectured [32]

$$\frac{1}{T} \int_0^T \frac{\zeta(\frac{1}{2} + a + it)\zeta(\frac{1}{2} + b - it)}{\zeta(\frac{1}{2} + c + it)\zeta(\frac{1}{2} + d - it)} \, dt \sim \frac{(a + d)(b + c)}{(a + b)(c + d)} - \frac{(a - c)(d - b)}{(a + b)(c + d)},$$

as $T \to \infty$. For characteristic polynomials we have the theorem [49][20]

$$\left\langle \frac{\Lambda(e^{-a})\Lambda(e^{-b})}{\Lambda(e^{-c})\Lambda(e^{-d})} \right\rangle_{U(N)} \sim \frac{(a + d)(b + c)}{(a + b)(c + d)} - e^{-N(a+b)(c - a)(d - b)} \frac{(a - c)(d - b)}{(a + b)(c + d)},$$

as $N \to \infty$. Those expressions fit the analogy in Principle 6.6 and are identical under $N \leftrightarrow \log T$. Both expressions can be made more precise; they have different lower order terms. (See [26], Section 4.)

In Section 4 we described the result of Bombieri and Hejhal [15] that if a set of Z-functions satisfy RH, then a linear combination has 100% of its zeros on the critical line. The same holds for characteristic polynomials [5]: if $a_1, \ldots, a_n$ are nonzero real numbers and $A_1, \ldots, A_n$ are chosen independently and Haar-random from $U(N)$, then the expected number of zeros of $\sum a_nZ_{A_n}$ on the unit circle is $N - o(N)$.

**Case 2. Agreement to leading order, after inserting an arithmetic factor.** This situation appears in the moments of L-functions, as illustrated in (10.2) and (6.15)

**Case 3. Failure of the analogy.** For L-functions, RH implies the Lindelöf Hypothesis (LH), see Conjecture 8.8. But for unitary polynomials the analogue
of RH is true but the analogue of LH can fail. For example, the unitary polynomial \((z + 1)^N\) achieves the value \(2^N\) on the unit circle, which is not \(\ll e^{\varepsilon N}\) for all \(\varepsilon > 0\).

A less extreme example is a degree \(N\) polynomial with all zeros in half of the unit circle: it takes values as large as \(\sqrt{2}^N\). Somewhat less obviously: LH fails for a degree \(N\) unitary polynomial which has no zeros in an arc of a fixed length. The relationship between large zero gaps and the size of the function is central to the main theme of this paper, and is the topic of Section 8. A related situation is key to one of our refutations, see Subsection 9.3.

**How to have confidence in the predictions?** Given the range of possibilities above, how can one to give credence to any specific prediction?

The 3rd case, complete failure, may seem alarming but in fact it is of little concern. The goal is to model statistically. By Principle 6.4, at height \(T\) the \(\zeta\)-function is modeled by Haar-random characteristic polynomials from \(U(N)\). More specifically, by Principle 6.5, \(e^N\) random matrices are chosen from \(U(N)\). That may seem like a large number, but one would never expect to encounter a polynomial with all its zeros in half of the circle. Examining the CUE(\(N\)) measure, (5.6) with \(\beta = 2\), we see that a polynomial with all zeros in half of the circle would have many of the terms in that product be less than half the average. Therefore such a polynomial is \(e^{-cN^2}\) times less likely than typical, so it has little chance of appearing in a sample of size \(e^N\) if \(N\) is large. The same argument applies in the case of polynomials with no zeros in an arc of fixed width.

Thus, combining both parts of the Keating-Snaith law finds that the Lindelöf Hypothesis is predicted by random matrix theory.

Cases 1 and 2 are actually the same: the arithmetic factor in the first case happens to be 1. Those arithmetic factors are well-understood: those factors arose long before RMT entered the picture. In any specific case one can identify the arithmetic factor.

Note that this discussion is mostly moot, because tools like the recipe [19] and the ratios conjecture [20] directly produce expressions which include any arithmetic factors and furthermore include all lower-order main terms. However, RMT remains useful for the purpose of this paper because it enables us to generate examples for building intuition about \(Z(t)\) for large \(t\).

### 7 Waves in a unitary polynomial

As defined in (6.14), the rotated characteristic polynomial \(Z_A(\theta)\) for Haar-random \(A \in U(N)\) is a model for \(Z(t)\) for \(t\) of size \(e^N\). If we let \(N = 1000\), we obtain a model for the zeta function at height \(e^{1000} \approx 10^{434}\). That height is sufficient to “see” the carrier waves in the characteristic polynomial. The analogy between L-functions and characteristic polynomials suggests that carrier waves in the \(\zeta\)-function at that height should be comparable.

We will illustrate carrier waves by picking one large random matrix and exploring its characteristic polynomial in detail. In particular, we will see that carrier waves occur within each individual random unitary polynomial.

#### 7.1 An example of a carrier wave

Suppose \(B \in U(1000)\) is a specific (randomly chosen) matrix, which is fixed and will be used in all the illustrations below. The function \(Z_B(\theta)\) is real-valued, periodic with period \(2\pi\), and in principle it can be graphed. However, that
A graph would convey little useful information because 1000 wiggles across the width of a piece of paper is beyond the resolution of what can be printed or seen. Instead, we offer two other ways to graph that function and see the carrier wave.

Figure 7.1 shows graphs of $Z_B(\theta)$ over two intervals of width $2\pi \times 30/1000$. Note the vertical scales in the graphs.

Figure 7.1 illustrate the effect of the carrier wave. Both graphs are part of the same polynomial, so the global scale factors are equal. The variation in the local zero spacing has a similar effect in both regions, yet there is a factor of 500 difference in the graphs. That factor of 500 is due to the carrier wave.

We can use the idea of (4.3) to measure the carrier wave in $Z_B(\theta)$, with the obvious modification of replacing $\cos(x)$ by a function with equally spaced zeros on the unit circle: $c(\theta) = z^{N/2} - z^{-N/2}$, where $z = e^{i\theta}$. At a given point $e^{i\theta_0}$, we move the nearby zeros of $c(\theta)$ to match the zeros of the polynomial, and then choose a scale factor so the functions are equal at $e^{i\theta_0}$. That scale factor is the carrier wave at $e^{i\theta_0}$. The resulting carrier wave for our example function $Z_B(\theta)$, sampled at a large number of points around the circle, is shown in Figure 7.2. In the notation of (4.3), we set $K = 10$, meaning that 10 zeros on either side of a given point were used to calculate the carrier wave. Note: the vertical axis is on a logarithmic scale.

Figure 7.2 The carrier wave for $Z_B(\theta)$, on a logarithmic scale, calculated using $K = 10$ in (4.3).

Figure 7.2 indicates that near $e^{1.2i}$ the polynomial is wiggling with small amplitude, and near $e^{6.1i}$ it is wiggling with large amplitude. That is how the examples were chosen for Figure 7.1.

The carrier wave is the primary factor in determining the large values, but the local spacing will still have an effect. If one focuses on the maximum value
for a specific characteristic polynomial, the variation in the local spacing will cause that maximum to occur within a larger than average gap. This can be seen in the lower plot in Figure 7.1, where the maximum near 6.083 occurs in a gap of 1.6 times the average. See Figure 5 in [41] for a detailed numerical study of this phenomenon. At $N = 1000$ the carrier wave is more important than the local variation. The polynomial in Figure 7.1 has 20 gaps which are more than twice the average, the largest of which is more than 2.45 times the average. But the local maximum in those gaps is smaller than in the 1.6 gap in the lower graph in Figure 7.1 because the carrier waves in those regions are smaller.

7.2 The density of zeros is more important than individual gaps

A close examination of the two plots in Figure 7.1 reveals another key fact. The two plots cover equal widths, namely 30 times the average gap between zeros. But the first plot has 33 zeros, while the second plot has only 29 zeros. What we have observed is:

**Principle 7.3** The main contribution to the carrier wave is the difference between the density of nearby zeros and the average density: the local density is negatively correlated with the size of the function.

Thus, it is the gaps between zeros which determine the size of the function, but it is collections of nearby gaps which are the dominant factor: not individual gaps.

Principle 7.3 would be more useful if we had a definition of “local density” and a way to measure it. We will develop an expression based on the gaps between widely spaced zeros.

In the top graph of Figure 7.1 the function is small over a wide region. Subsection 4.1 described the scale of the graph as arising from the zeros outside that region. Since the zeros in that region are closer together than average, one can view this as being caused by the zeros on either side pushing in towards that region, compressing those zeros. Equivalently, one expects the gap between a zero to the left of that region and a zero to the right of that region to be smaller than average. If $\gamma, \gamma'$ are zeros, and

$$\gamma' - \gamma = j + \delta(\gamma, \gamma')$$

(7.1)

where $j$ is the expected normalized gap between $\gamma$ and $\gamma'$, then we will call $\delta(\gamma, \gamma')$ the $j$th neighbor discrepancy. Thus, when the zeros are more dense in a region, the $j$th neighbor discrepancy for pairs of zeros either side of that region, should tend to be negative.

To turn the neighbor discrepancy into a measure of the density, we need to determine the relative weights to give to all the neighbor gaps spanning the region. Consider a function $F(t)$ in a region centered on the point $t_0$, and renumber the zeros of $F$ as $\cdots \leq \gamma_{-2} \leq \gamma_{-1} < t_0 < \gamma_1 \leq \gamma_2 \leq \cdots$.

**Principle 7.4** The logarithm of the carrier wave is given to leading order by the density wave, expressed as a weighted sum of the $j$th neighbor discrepancies by

$$\sum_{j \geq J} \frac{\delta(\gamma_{-j}, \gamma_j)}{j}.$$  

(7.2)

The sum (7.2) converges, by Dirichlet’s test.

Figure 7.5 compares the density wave of zeros for the same degree 1000 polynomial used in the above examples, to the logarithm of the carrier wave from Figure 7.2. Here the density wave is measured by (7.2) with $J = 11$. Note
that the thin blue graph was calculated using 20 zeros centered around a given point, while the thicker red graph was calculated using the complementary set of zeros.

Figure 7.5 The thicker red graph is the local density of zeros (the density wave) measured by (7.2) with \( J = 11 \), for the degree 1000 polynomial \( Z_B(\theta) \). The thinner blue graph is the (logarithm of the) carrier wave for the same polynomial, calculated with \( K = 10 \). The correlation of those functions illustrates Principle 7.3 and Principle 7.4.

Note 7.6 The formula in Principle 7.4 depends on \( J \), so the density of the zeros at a point is not uniquely defined; the same shortcoming applies to the local scale factor (i.e., the carrier wave) which depends on the parameter \( K \), as observed in Note 4.4. However, those are the same shortcomings: an interval of zeros is used to determine the value of the carrier wave. If the zeros used to calculate the pressure wave are the complement of the zeros chosen to calculate the carrier wave, as in Figure 7.5, then it is perhaps not surprising in retrospect that those neighbor gaps provide an accurate measure of the density of zeros they contain, and so are highly correlated with the carrier wave. An unresolved issue is how to identify the appropriate window. In Principle 8.7 we suggest that the carrier wave for degree \( N \) polynomials covers around \( \log N \) zeros, which is around 7 for \( N = 1000 \), so the size window we are using in these examples is not unreasonable. Larger examples will require a wider window.

Justification of Principle 7.4. We now justify (7.2). Let \( \kappa \) be the expected density of zeros near \( t_0 \). Near \( t_0 \) we have

\[
F(t/\kappa) = \prod_{j=1}^{\infty} \left(1 - \frac{t}{\gamma_j} \right) \left(1 - \frac{t}{\gamma_{-j}} \right).
\]  

(7.3)

In (7.3) write \( \gamma_j = j \pm \frac{1}{2} + \delta_j \), where \( \pm \frac{1}{2} \) has the opposite sign of \( j \). That choice assures that \( \delta_j \) is 0 on average. We have

\[
F(t/\kappa) = \prod_{j=1}^{\infty} \left(1 - \frac{t}{j - \frac{1}{2} + \delta_j} \right) \left(1 - \frac{t}{-j + \frac{1}{2} + \delta_{-j}} \right)
\]

(7.4)

\[
= \prod_{j=1}^{\infty} \left(1 - \frac{t}{j - \frac{1}{2}} + \frac{t\delta_j}{(j - \frac{1}{2})^2} - \frac{t\delta^2_j}{(j - \frac{1}{2})^3} + \cdots \right)
\]

(7.5)

\[
\times \left(1 + \frac{t}{j - \frac{1}{2}} + \frac{t\delta_{-j}}{(j - \frac{1}{2})^2} + \frac{t\delta^2_{-j}}{(j - \frac{1}{2})^3} + \cdots \right)
\]

(7.6)

\[
= \prod_{j=1}^{\infty} \left(1 - \frac{t^2 + t(\delta_j + \delta_{-j})}{(j - \frac{1}{2})^2} + \frac{t^2(\delta_j - \delta_{-j}) - t(\delta^2_j - \delta^2_{-j})}{(j - \frac{1}{2})^3} + \cdots \right)
\]

(7.7)

\[
= \prod_{j=1}^{\infty} \left(1 - \frac{t^2}{(j - \frac{1}{2})^2} + \frac{t^2(\delta_j - \delta_{-j})}{(j - \frac{1}{2})^3} + \cdots \right)
\]

(7.8)
\[
= \prod_{j=1}^{\infty} \left(1 - \frac{t^2}{(j - \frac{1}{2})^2} \left(1 + \frac{\delta_j - \delta_{-j}}{j - \frac{1}{2}} \right) + \cdots \right).\tag{7.9}
\]

On the next-to-last line we moved terms involving \(\delta_j + \delta_{-j}\) and \(\delta_j^2 - \delta_{-j}^2\) into the lower order terms because those expressions do not detect departures from the average zero spacing. Now recognize \(\prod_{j \geq 1} (1 - t^2/(j - \frac{1}{2})^2)\) as \(\cos(\pi t)\) and factor it out of the above expression. The terms with \(j \geq J\) give the carrier wave. Taking the logarithm, expanding, and replacing \(j - \frac{1}{2}\) by \(j\), because \(j \geq J\) is large, we have (7.2) because \(\delta_j - \delta_{-j} = \delta(\gamma_j, \gamma_{-j})\).

### 7.3 The carrier wave and \(S(t)\)

The relationship between zero density and the size of the carrier wave can also be expressed in terms of \(S(t)\). Since \(S(t)\) is the error term in the zero counting function, we see that \(S(t)\) will be generally increasing in a region where the zeros are more dense than the local average, and vice-versa. By “generally” we mean when averaged across a range of several zeros, because of course \(S(t)\) is decreasing wherever it is continuous, and jumps by 1 unit at each critical zero. Thus we have:

**Principle 7.7** In regions where \(Z(t)\) is particularly large, \(S(t)\) will on average be decreasing. In regions where \(Z(t)\) is particularly small, \(S(t)\) will on average be increasing.

Figure 7.8 shows \(S_B(\theta)\), superimposed on the logarithm of the carrier wave, for \(Z_B(\theta)\).

![Figure 7.8](image)

Figure 7.8 The thinner blue graph is the carrier wave of \(Z_B(\theta)\) from Figure 7.2 and the thicker red graph is the \(S_B(\theta)\) for \(Z_B(\theta)\). The graphs illustrate Principle 7.7: \(S_B(\theta)\) is increasing/decreasing in regions where \(Z_B(\theta)\) is particularly small/large.

Principle 7.7 explains why carrier waves are not observed in plots of \(Z(t)\): \(S(t)\) grows so slowly that in the realm accessible by computers it does not have the opportunity to decrease enough to create a visible carrier wave.

Note that the relationship described in Principle 7.7 does not contradict Selberg’s result that \(\log|Z(t)|\) and \(S(t)\) are statistically independent. However, it does suggest correlations in lower order terms and in the tails.

Principle 7.3 also explains why carrier waves have not been observed. We cloak this in fancy language in the next subsection.

### 7.4 Carrier waves are an emergent phenomenon

The conclusion to all the discussion in this section is that carrier waves are an **emergent phenomenon**. That is defined as an observable property of a system which arises from the interaction of many components of the system, but is not a property of those individual components. In the case at hand, the components are the zeros and the interaction is the repulsion between the zeros. The emergent property is variations in the density of zeros.

28
An emergent property cannot be understood merely by examining the components and their interactions. The study of microscopic water droplets interacting with gaseous nitrogen and oxygen does not reveal the different types of clouds, and cannot tell you that a particular cumulus cloud is shaped like a duck.

Emergent phenomena can only exist in a large system with many interacting components. Points made earlier in this section explain why carrier waves can only occur in high degree polynomials. By Principle 7.3, the carrier wave arises from fluctuations in the density of the zeros. The concept of density does not make sense unless one has more than just a few points. Once one has enough points to meaningfully talk about their density, those points must exist in a larger system in order to allow the local density to be significantly different than average. And that system must be yet larger in order for the repulsion between points to allow enough flexibility that significant fluctuations in density are likely. Thus, carrier waves can only exist within a large system.

The explanation above describes why carrier waves do not appear in unitary polynomials until the degree is large. But what about the $\zeta$-function, which has infinitely many zeros? Isn’t that a large system, even if one restricts to the range accessible by computers?

The explanation lies in the rigidity of the zero spacing, i.e., the slow growth of $S(t)$, interpreted as the error term in the zero counting function. Since the zeros must stay close to their expected location, there is limited opportunity to create a region where there are significantly more, or significantly fewer zeros. Not only does $S(t)$ grow slowly, it also changes sign frequently. It is as if the $\zeta$-function not only resets itself after every interval of length $2\pi$, there is an anti-correlation between extreme values on one interval and the next. Thus, a sequence of independent characteristic polynomials slightly overestimates the frequency of large values. In other words, the arithmetic factors used to adjust random matrix predictions, such as $a_k$ in the $2k$th moment of the $\zeta$-function, see (10.2), tend to be very small.

7.5 A hidden choice in the Keating-Snaith law

The matching $N = \log(T/2\pi)$ is designed to allow the random characteristic polynomial $Z_A(\theta)$ with $A \in U(N)$ to model $Z(t)$ for $t \approx T$. The justification is “matching the density of zeros”, but that justification contains a hidden choice. If instead we had $A \in U(\kappa N)$ and considered $Z_A(\theta/\kappa)$, we again would have two functions with the same average zero gap. Maybe there is a better choice than $\kappa = 1$? It is tempting to think that a larger value of $\kappa$ would be better, because the standard choice imposes the absurdity $Z(t) = Z(t + 2\pi)$.

Choosing a different value for $\kappa$ does not work, and carrier waves are the reason. A rescaled characteristic polynomial $Z_A(\theta/10)$ for $A \in U(10N)$ does not look like the concatenation of 10 different $Z_A(\theta)$ for $A \in U(N)$. The rescaled polynomial will have much larger carrier waves, because the system is larger so there is more scope for the zeros to slosh around and create regions of higher and lower density. A side-effect will be that moments of the polynomials will not accurately predict moments of the $\zeta$-function: the maximum from a $10 \times$ larger system will typically be much larger than the maximum among 10 different smaller systems.

Thus, in some ways (such as the lack of large carrier waves at low height), the $\zeta$-function behaves like a sequence of small systems, not a single large system. But actually it is specific properties of the large $\zeta$-system which is the underlying reason the carrier waves start out small and are confined to short intervals. This is explained in the next section.
7.6 Spectral rigidity

Another way to see that the \( \zeta \)-function cannot be modeled by a small number of large characteristic polynomials involves the analogue of what is called **spectral rigidity** \[8\] in physics. For the \( \zeta \)-function, that terminology is just another way of saying that \( S(T) \), the error term in the zero counting function \( N(T) \), grows very slowly. We examine this in the context of the neighbor spacing between zeros, in particular the normalized \( j \)th nearest neighbor spacing \( \tilde{\gamma}_n + j - \tilde{\gamma}_n \). Figure 7.9 shows the distribution of the 1st, 3rd, and 10th normalized neighbor spacings for 10000 consecutive zeros at height \( 10^{12} \). Data from \([67]\), which is the same data used in Berry’s original study \([8]\) of spectral rigidity in the context of the \( \zeta \)-function.

![Figure 7.9](image)

The histograms in Figure 7.9 are on the same scale, so one can see (even without the calculated variances) that the variance of the \( n \)th neighbor spacing is growing for \( n \leq 10 \). However, it is clear that the variance cannot continue to grow at that rate, because the variance of the neighbor spacing is bounded by a small multiple of the variance \( \langle S(T)^2 \rangle = \log \log T \). Figure 7.10 shows the variance of the \( j \)th nearest-neighbor spacing for \( 1 \leq j \leq 200 \) for the same set of zeros. Note that the phenomenon illustrated in Figure 7.10 is usually expressed in terms of the number variance \([8, 9]\),

\[
V_T(L) = \text{Var}(N(T + L) - N(T)) = \text{Var}(S(T + L) - S(T)),
\]

which is easier to handle theoretically but perhaps more difficult to grasp intuitively.

![Figure 7.10](image)

The dots show the variance of the \( j \)th nearest-neighbor spacing, \( 0 \leq j \leq 200 \), for zeros \( \gamma_n \) with \( 10^{12} \leq n < 10^{12} + 10000 \). The height of the horizontal red dashed line is twice the average variance of the samples, which would be the variance of the neighbor gaps if the zeros were distributed independently. The curve is the variance of the \( j \)th nearest-neighbor spacing, \( 0 \leq j \leq 24 \), for eigenvalues of random matrices in \( U(24) \).

Note that the average value of \( \tilde{\gamma}_n - n \) is \( \frac{1}{2} \), and in the sample used in the above figures, has variance 0.15005. If the the \( j \)th nearest neighbors were distributed independently, then the \( j \)th nearest neighbor gaps would have variance \( 2 \times 0.15005 \), which is the horizontal line in Figure 7.10. Thus, the relative displacement of neighboring zeros (relative to their expected location) is sometimes positively correlated, and sometimes negatively correlated, across
a span of many zeros. Furthermore, the peaks in Figure 7.10 occur near \( c \gamma_n \) for \( n = 1, 2, \ldots \), where \( c = 3.894 \) is the density of those zeros (which are at height \( 2.67 \times 10^{11} \)). The resurgence of the first few zeros occurs in many zero statistics [9, 26], typically manifested via the influence of the zeros on the size of \( \zeta(1 + it) \).

The universality of Principle 6.1 is illustrated in Figure 7.10 by the approximate agreement, for small gaps, between the \( \zeta \)-zeros and eigenvalues variances. This is referred to as the universal regime. In the universal regime, no details about the \( \zeta \)-function are relevant to the leading-order behavior. This idea appears again in Principle 7.11. In the non-universal regime, the prime numbers (equivalently, the low-lying zeros) have a strong effect. Only recently has this effect been treated rigorously [61].

The slow growth of \( S(t) \) prevents the zeros from moving far from their expected location, which prevents large regions with a high or low density of zeros, which prevents large carrier waves at low height.

7.7 Extreme gaps

In Subsection 6.2 we described the GUE predictions for the largest and smallest gaps between zeros. We now elaborate on how those predictions are made, and indicate some issues which impact those predictions.

Small gaps. Subsection 6.1 mentioned the universality of the \( \beta \)-ensembles: the local statistics of the eigenvalues only depend (to leading order) on the degree of repulsion. This has implications for the neighbor spacing. From the CUE measure, (5.6) with \( \beta = 2 \), it follows immediately that the distribution of nearest-neighbor spacing vanishes to order 2, and the next-nearest-neighbor spacing vanishes to order 7. Thus, to leading order the pair correlation for the CUE, (6.8), describes the distribution of small gaps between eigenvalues, and that prediction is not sensitive to any details about the particular system being discussed. That is,

Principle 7.11 The PDF \( p_2(x) \) of the normalized gaps between zeros of the \( \zeta \)-function, or any other L-function, decays like \( \left( \frac{\pi^2}{3} \right) x^2 \) as \( x \to 0 \). That prediction is very likely to be accurate, with the prime numbers and other details particular to any specific L-function having little influence on the smallest gaps.

Any skepticism about Principle 7.11 should be alleviated by Figure 7.12, which shows the distribution of the smallest normalized gaps among the first \( 10^{13} \) zeros of the \( \zeta \)-function, compared to the \( \left( \frac{\pi^2}{3} \right) x^2 \) prediction. Note that Figure 7.12 concerns the universal regime, so there is no contradiction with our repeated assertion that numerical calculations of the \( \zeta \)-function give a misleading impression of the behavior at large height.
To estimate the size of the normalized smallest gaps among a sample of $X$ gaps, calculate the gap size which has a 50% chance of appearing:

$$\text{solve } X \int_0^{\xi_{\text{min}}} Ax^2 \, dx = \frac{1}{2} \text{ to get } \xi_{\text{min}} = \left( \frac{AX}{3} \right)^{-\frac{1}{3}}, \quad (7.11)$$

where $A = \frac{\pi^2}{3}$. So, among the gaps up to height $T$, one expects the smallest normalized gaps to be of size $T^{-\frac{1}{3}} + o(1)$. One can try to be more precise, but there probably is no meaning in the specific constant or power of log $T$. There is meaning, however, to the exponent $-\frac{1}{3}$. A small gap between two zeros forces a small value for the derivative of the $\zeta$-function at both of those zeros. An implication [50] is that the predicted discrete mean values

$$\frac{1}{N(T)} \sum_{0 < \gamma_j \leq T} |\zeta'(\frac{1}{2} + i\gamma_j)|^{2k} \sim c_k T^{2k(k+2)}, \quad (7.12)$$

conjectured to hold for $k > -\frac{3}{2}$, cannot hold for $k < -\frac{3}{2}$ because (due to the smallest neighbor gaps) there are individual terms which are very large.

**Large gaps.** The situation for large gaps is likely to be significantly more complicated.

For large random unitary matrices we have $p_2(x) \sim e^{-\pi^2 x^2 / 8}$ as $x \to \infty$. For an individual matrix, the neighbor gaps are not independent because adjacent neighbor gaps are negatively correlated, and also they average to exactly 1. But, if one pretends those gaps are independent, then the expected maximum among $M$ normalized gaps would be $\sqrt{8 \log M} / \pi$. Setting $M = N$ gives the correct answer for the expected maximum among the gaps of an individual random matrix [6]. Thus, there is some justification to using the tail of the CUE neighbor spacing to conjecture the maximum gap over longer ranges, and Figure 1 in [6] shows that such a prediction is supported by data involving $2 \times 10^9$ zeros of the $\zeta$-function.

As illustrated in Figure 7.10, for more widely spaced zeros, where “widely” might only mean a few times the average zero spacing, the distribution of $n$th neighbor gaps for the $\zeta$-function departs significantly from that of random unitary matrices: the $\zeta$ gaps are more constrained. Therefore, it would not be surprising if the largest normalized gaps between zeros of the $\zeta$-function were smaller than the analogous normalized gaps between eigenvalues of random unitary matrices. It particular, it would not be surprising if the largest normalized neighbor gaps of the $\zeta$-function (either as conjectured in [6], or from a naive interpretation of the tail of $p_2(x)$) were actually smaller than $\sqrt{8 \log T} / \pi$. 

**Figure 7.12** Histogram of the 4065 normalized smallest neighbor gaps among the first $10^{13}$ zeros of the $\zeta$ function, compared to the predicted density $\frac{\pi^2}{3} x^2$. Data from Xavier Gourdon [44].
Resolving such an issue with data is difficult because the distribution of large gaps decays rapidly. In this paper we only make use of the fact that the largest normalized gaps are likely to be $O(\sqrt{\log T})$.

8 Extreme values

So far we have considered typical large and small values. That is, values within the bulk of the distribution of $\log |Z(t)|$. The conclusion was that those large values are due to the carrier wave. Since the carrier wave arises from variations in the local density of zeros, once the wave is large it must stay large over a span of several zeros. Thus, the graph of the $Z$-function when it is particularly large looks the same as it does anywhere else — if you ignore the scale on the axes.

However, we have not yet ruled out the possibility that even larger values might arise from very rare events, such as an extremely large gap between zeros. The effect might look similar to what appears in the data of Bober and Hiary [13], as shown in Figure 2.4. We will explain why the largest neighbor gaps cannot cause the function reach the largest values. First a trivial observation, following immediately from Principle 2.5 and (6.12):

Principle 8.1 Isolated large gaps between zeros contribute $O(\sqrt{\log T})$ to $S(t)$.

It takes more effort to establish the analogous result for $Z(t)$:

Principle 8.2 Isolated large gaps between zeros contribute $O(\sqrt{\log T})$ to $\log |Z(t)|$.

Since the largest values of $\log |Z(t)|$ are conjectured to be $\gg \sqrt{\log(t) \log \log(t)}$, Principle 8.2 tells us that the largest neighbor gaps are not responsible for the extreme values.

By Principle 7.4, a weighted sum of neighbor gaps (the density wave) is the leading factor in the size of $\log |Z(t)|$. It would be interesting to analyze that weighted sum to possibly give another way of predicting the extreme values. Because of spectral rigidity, Subsection 7.6, each neighbor discrepancy $\delta(\gamma_j - \gamma_k)$ has a similar distribution. Can one recover the conjectured extreme values of $\log |Z(t)|$? When that sum is large, is it due to a handful of very large terms, or a large number of moderately sized terms? Analysis will be complicated by the fact that $\delta(\gamma_j - \gamma_k)$ and $\delta(\gamma_j - \gamma_k + \delta)$ are positively correlated if $k$ is small.

Principle 8.2 follows immediately from (6.12), see also Subsection 7.7, combined with:

Principle 8.3 An individual large gap of $g$ times the average zero spacing contributes $O(g)$ to the local maximum of $\log |Z(t)|$ within that large gap.

Note that Principle 8.3 and Principle 7.4 both indicate that when zeros move apart by $g$ units on the scale of the average zero spacing, the effect is a factor $e^{O(g)}$ on the size of the function. However, the justification in the case of a large nearest neighbor gap is quite different than the case of neighbor discrepancy for distant gaps. We have not been successful at handling the two cases in a unified way.

Subsection 8.1 is devoted to justifying Principle 8.3.

8.1 Zeros near a large gap

To understand the effect of a large gap on the size of the function, we need information about the gaps adjacent to the large gap. From Haar measure (5.6) on $U(N)$ we see that the expected size of the immediately adjacent gaps is smaller than average. Bober and Hiary [13] expressed it as: $S(t)$ tends to be
increasing immediately before and immediately after a large gap. To isolate
the effect of only the single large gap, we propose to focus on the case where
the nearby zeros are in their most likely configuration. We will do this for zeros
on the unit circle.

Determining the most likely configuration is an easy computer experiment.
Fix two zeros with a chosen gap, and distribute the other \(N - 2\) zeros on
the circle anywhere outside that gap. Repeatedly perturb the \(N - 2\) zeros to
increase the measure (5.6), until the process stabilizes. The result will be a
good approximation to the most likely configuration.

With \(N = 74\) and a gap of 6 times the average, which is the random
matrix analogue of \(Z(t)\) in Figure 2.4, the most likely configuration is shown in
Figure 8.4.

![Figure 8.4](image_url)

**Figure 8.4** Graphs of \(Z(\theta)\) and \(S(\theta)\) for a degree 74 polynomial with a zero
gap of 6 times the average and all other zeros in their most likely configuration
with respect to Haar measure (5.6).

A similarity with Figure 2.4 is the general shape of \(S(t)\) near the large gap.
A difference is that the maximum in Figure 8.4 is smaller, which is due to
the fact that the immediate neighbor gaps in Figure 2.4 are wider than the
most likely configuration. The wider neighboring gaps in Figure 2.4 are not
surprising, because the large gap in Figure 8.4 is not as large as possible in
that range. That means there is room for more variation in the nearby zeros,
so less likely configurations can occur.

We see that a large gap causes nearby gaps to be smaller. The key issue
for justifying Principle 8.3 is: how far does that influence extend? That is, at
what distance from the large gap do the neighbor gaps (approximately) return
to their average? The answer is easy to express if we measure on the scale of
the average gap between zeros.

**Principle 8.5** The normalized range of influence of a large gap is proportional
to the normalized width of the gap.

So, doubling the width of a large gap will double the range of its influence
on nearby neighbor gaps.

To see why Principle 8.5 is true, consider the following thought experiment.
Fix a large gap, and then distribute the other gaps in their most likely configu-
ration. It is helpful to think of the zeros as particles with equal charge, obeying
a force law such that the particles are in equilibrium when they are in their
most likely configuration. Now double all the charges. The particles will still be
in equilibrium. Then split every doubled charge into two equal charges, leave one in place, and move the other halfway toward the neighboring charge, with the movement away from the large gap. The new configuration will not be in equilibrium, but it will be nearly so. As the particles settle into equilibrium (holding fixed the two next to the large gap), the particles will move very little, because they were already close to equilibrium. The result is a new system where the large gap is the same on an absolute scale, as is the range of influence of that large gap. But in the new system, the large gap and its influence are twice as large on the scale of the average spacing. Thus, Principle 8.5.

Figure 8.6 provides another way to understand Principle 8.5. For 230 zeros on the circle, which corresponds to $Z(t)$ near $10^{100}$, we used the procedure described at the start of this section to determine the most likely configuration with a gap of 5, 10, or 15 times the average. The first plot in Figure 8.6 shows the successive neighbor gaps in each of the three cases. As expected, the immediate neighbor gap is small, and is smaller when the large gap is bigger. When the large gap is 5 times the average, the immediate neighbor gap is approximately 0.32 times the average. When the large gap is 15 times the average, the immediate neighbor gap is approximately 0.1 times the average. The gaps grow, approaching 1, the average gap, when farther away from the large gap. And as expected, when the large gap is larger, it takes longer for the neighbor gaps to approach 1.

The second plot in Figure 8.6 is made from the same data, but this time the horizontal axis is scaled by the size of the large gap. We see that the three graphs are virtually indistinguishable. In other words, measured on the scale of the large gap, the decay of the influence of that gap is independent of the size of the gap. By 3 times the width of the large gap, there is very little influence on the neighbor gaps.

![Figure 8.6](image)

**Figure 8.6** The typical normalized gap between successive zeros adjacent to a large gap of 5, 10, or 15 times the average spacing. In the plot on the left the horizontal scale is the index of the gap. On the right the horizontal scale is the index divided by the size of the large gap.

We use Principle 8.5 to justify Principle 8.3. The idea is similar to (4.3). Start with a function having equally spaced zeros, and then move some of those zeros to a new location. In this case it is easiest to start with $f_0(z) = z^N + 1$ and then move the zeros with $-\pi g < \theta_j < \pi g$ to create a normalized gap of size $g$, placing those zeros in their most likely location. Denote the new value of $\theta_j$ by $\hat{\theta}_j$, and call the resulting function $f_g(z)$. We have

$$f_g(z) = (z^N + 1) \prod_{-\pi g \leq j \leq \pi g} \frac{1 - e^i(\hat{\theta}_j)}{1 - e^i(\theta_j)}.$$  

By Principle 8.5, the zeros which were moved will have $|\hat{\theta}_j| \leq Kg/N$ where $K$ is some absolute constant, independent of $g$ and $N$. Therefore

$$f_g(1) \ll \prod_{-\pi g \leq j \leq \pi g} \frac{2\pi Kg}{2\pi(2j-1)}.$$  

35
\[
\leq \frac{K^2 g^{2\pi g}}{(\pi g)!^2} = e^{O(g)},
\]
the last step following from Stirling’s formula. That establishes Principle 8.3.

We have seen that a large gap has essentially the same effect on \( S(t) \) and on \( \log |Z(t)| \). Yet, it was almost trivial to handle \( S(t) \), and fairly complicated to handle \( \log |Z(t)| \). Why is it simple to understand the effect of large gaps on the imaginary part of \( \log \zeta(\frac{1}{2} + it) \) but not on the real part? Is there a simple way to deal with both cases? Perhaps not: the discussion at the end of Section 1.2 of [61] describes some essential differences between \( S(t) \) and \( \log |Z(t)| \).

8.2 More intuition for the need for carrier waves

We describe yet another way to see that carrier waves are primarily responsible for the local size of \( Z(t) \).

Suppose the typical large value of \( Z(t) \) were primarily due to large gaps, such as in Figure 2.4. Combining Selberg’s theorem on the distribution of \( \log |Z(t)| \) with Principle 8.3 would imply that a positive proportion of \( t \) occur in a gap of relative size \( \sqrt{\log \log(t)} \). In other words, the probability that a gap is larger than \( \sqrt{\log \log(t)} \) times the average, must be \( \gg \frac{1}{\sqrt{\log \log(t)}} \). This would correspond to the distribution of nearest-neighbor spacings decreasing much more slowly than conjectured. That is contrary to our observations and not consistent with the conjectures on neighbor spacing arising from RMT.

The situation becomes more clear when we consider small values. Obviously the zeros on the critical line cause very small values for \( \log |Z(t)| \). It is not difficult to see (consider \( \log |\sin x| \), for example) that the neighborhoods of the zeros cannot account for \( \log |Z(t)| \) being \( \ll \sqrt{\log \log(t)} \) for a positive proportion of \( t \). Therefore the typical small values must arise from the function staying small between neighboring zeros.

Suppose two neighbor zeros are separated by \( g \) times the average spacing, with \( g \) very small. By considering \( x(g - x) \) or \( \sin(x) - 1 + g^2 \), one expects the local maximum between those zeros to scale as \( g^2 \). Thus, if the local zero spacing is the main cause of those small values, then a positive proportion of \( t \) must occur within a gap of size \( o(1) \) times the local average. That is absurd because it requires more than 100% of the gaps to be very small! Thus, the large proportion of small values required by Selberg’s result must come from a conspiracy involving many gaps, with most of those gaps not particularly small.

8.3 Carrier waves and short-range maximum values

A major theme of this paper is the nature of the extreme values of the \( \zeta \)-function. That is, the largest values on a long interval \([T, 2T]\). A related question was considered by Fyodorov, Hiary, and Keating [42] and Fyodorov and Keating [40], who considered

\[
\zeta_{\text{max}}(L; T) = \max_{T \leq t \leq T + L} \log |\zeta(\frac{1}{2} + it)|.
\]

For comparison with characteristic polynomials, the interesting case is \( L = 2\pi \). The FHK conjecture [42] is

\[
\zeta_{\text{max}}(2\pi; T) \sim \log \log(T/2\pi) - C \log \log \log(T/2\pi) + X_T,
\]

where
\[ C = \frac{3}{4}, \]

- \( X_T \) is a random variable which has a limiting distribution, \( X \), as \( T \to \infty \), and
- The PDF of \( X \) decays like \( xe^{-x} \) as \( x \to \infty \).

The conjecture relates to several areas of mathematical physics, and has an analogous statement for the CUE(\( N \)) under the usual identification \( N = \log(T/2\pi) \). Substantial progress has been made on the conjecture: see Section 8 of [79] and Section 2 of [4] and references therein. We limit our discussion to the relationship between the FHK conjecture (8.2) and carrier waves.

**Why the FHK conjecture is surprising.** Much of the discussion below follows Section 8 of Soundararajan [79].

Consider the random matrix analogue of \( \zeta_{\text{max}}(2\pi; T) \):

\[ Z_{\text{max}}(U) = \max_{\theta \in [0, 2\pi]} \log |Z_U(e^{i\theta})|. \]  

(8.3)

One can view \( Z_{\text{max}}(U) \) as arising from \( M \) random \( \theta \) samples of the random variable \( \log |Z_U(e^{i\theta})| \), for some sample size \( M \). If those samples were independent, and we use the fact that \( \log |Z_U(e^{i\theta})| \) has a Gaussian distribution with variance \( \frac{1}{2} \log N \) (Keating and Snaith’s analogue of Selberg’s theorem), then \( Z_{\text{max}}(U) \) should typically be around

\[ \sqrt{\frac{1}{2} \log N} \left( \sqrt{2 \log M} - \log \frac{4\pi \log M}{\sqrt{2 \log M}} \right). \]  

(8.4)

That leaves us to decide on the number of samples \( M \). Clearly we cannot choose \( M > N \) because \( Z_U \) is determined by the \( N \) eigenvalues of \( U \). If we choose \( M = N \), a seemingly reasonable choice because \( |Z_U(e^{i\theta})| \) has \( N \) local maxima and we want to know the largest, we find that (8.4) equals

\[ \log N - \frac{1}{4} \log \log N. \]  

(8.5)

Setting \( N = \log T/2\pi \) we contradict the \( C = \frac{3}{4} \) part of conjecture (8.2). That is the first way in which the FHK conjecture is surprising. In particular, their conjecture says that the short range maximum values tend to be slightly smaller than one would expect from a naive view of Selberg’s theorem.

The second surprising aspect is the size of the tail of the random variable \( X \). The maximum among independent choices from a Gaussian follow the Gumbel distribution, which decays like \( e^{-x} \), whereas the FHK conjecture posits decay like \( xe^{-x} \).

The conclusion is that the \( N \) choices from \( |Z_U(e^{i\theta})| \) are not independent, which Fyodorov-Hiary-Keating [42, 40] phrase as long-range correlations. That is another way of describing the carrier waves, and we now make that connection more explicit.

**Implications for carrier waves.** Our discussion of carrier waves has not indicated how large they are. That is, over what span of zeros does the logarithm of the function typically change very little? We will use the FHK conjecture (8.2) to obtain:

**Principle 8.7** At height \( T \), the carrier wave for the \( \zeta \)-function typically covers a span of approximately \( \log \log T \) zeros. For the characteristic polynomial of a random matrix in \( U(N) \), the carrier wave typically covers a span of
approximately $\log N$ zeros.

In other words, if $a = g(t) \log \log t$, then $\log |Z(t)|$ and $\log |Z(t+a)|$ are highly correlated if $g(t) \ll 1$ and are uncorrelated if $g(t) \to \infty$ sufficiently fast.

We obtain Principle 8.7 from (8.4). Suppose $Y$ is the span over which the (logarithm of) the function typically changes very little, measured on the scale of the average zero spacing. Independent choices from $Z_U(e^{i\theta})$ must have $\theta$ separated by $Y/N$, so can cover the circle with $M = N/Y$ independent choices. Making that substitution in (8.4) gives main terms

$$\log N - \frac{2 \log Y + \log \log N}{4}. \quad (8.6)$$

Equating to (8.2) with $C = \frac{3}{4}$ and $N = \log T/2\pi$ we find $Y = \log \log(T/2\pi)$, as claimed.

An independent argument for the $\log \log T$ scale of the carrier waves could provide an alternate route to the $C = \frac{3}{4}$ part of conjecture (8.2).

**Non-implications for extreme values.** We have seen that the carrier wave has an influence on the maximum of the $\zeta$-function over short intervals, through its effect on the parameter $C$ in (8.2). The simple explanation is that the carrier wave is the difference between sampling $N$ points and sampling $N/\log N$ points. For the extreme values, the conjectured maximum (2.12) can be obtained from Principle 6.5 and the tail of the value distribution of characteristic polynomials. This involves sampling $e^N$, or maybe $Ne^N$, characteristic polynomials. Decreasing the sample size by a factor of $\log N$ makes much less difference in this case, so the carrier waves have no impact on the most extreme values.

It is tempting to use the FHK conjecture (8.2) as a step toward a conjecture for the extreme values, by invoking Principle 6.5 and using the conjectured tail $xe^{-x}$ for $X$. That would predict much larger values for the $\zeta$-function, but the reasoning is flawed because of an implicit switching of limits. The limiting distribution of $X_T$ as $T \to \infty$ may decrease like $xe^{-x}$, but the distribution of $X_T$ has an additional factor $e^{-x^2/\log \log T}$. See [1]. That factor dominates when the sample size is $e^N = T$.

### 8.4 The Lindelöf Hypothesis and self-reciprocal polynomials

As mentioned in Subsection 6.5, an apparent failure in the analogy between the $\zeta$-function and random unitary polynomials is the Lindelöf Hypothesis:

**Conjecture 8.8** The Lindelöf Hypothesis (LH). *If $\varepsilon > 0$ then*

$$Z(t) = O_{\varepsilon}(t^\varepsilon) \quad (8.7)$$

*as $t \to \infty$, or in other notation,*

$$Z(t) = e^{o(\log t)}. \quad (8.8)$$

The analogue for a unitary matrix $A \in U(N)$ is

$$\max_{|z|=1} Z_A(z) = e^{o(N)}. \quad (8.9)$$

That estimate need not hold: if all of the eigenvalues of $A$ are in $\Re z \geq 0$ then $|Z_A(-1)| \geq \sqrt{2}^N$.

We suggest that when one is discussing LH (therefore implicitly not assuming RH because RH implies LH), the comparison should be with self-reciprocal
polynomials — which are not necessarily unitary. With that in mind, consider Backlund’s [3] equivalence (see [83] Section 13.5). Let

\[ N(\sigma, T) = \# \{ \rho = \beta + i\gamma : \zeta(\rho) = 0, \ \beta > \sigma, \ 0 < t \leq T \}. \quad (8.10) \]

**Theorem 8.9** Backlund [3]. *The Lindelöf Hypothesis true if and only if*

\[ N(\sigma, T + 1) - N(\sigma, T) = o(\sigma (\log T)) \quad (8.11) \]

for all \( \sigma > \frac{1}{2} \).

Backlund’s equivalence (8.11) says that for any fixed \( A > 0 \) and any fixed-width strip around the critical line, a negligible proportion of the zeros with \( T \leq \gamma \leq T + A \) lie outside that strip.

One of the implications in Theorem 8.9 holds for self-reciprocal polynomials. If \( P \) is a polynomial, let

\[ N(\delta, P) = \# \{ \rho = re^{i\theta} : P(\rho) = 0, \ r < e^{-\delta} \}. \quad (8.12) \]

**Proposition 8.10** Suppose \( \{ P_j \} \) is a sequence of self-reciprocal polynomials with

\[ \max_{|z|=1} |P_j(z)| = e^{o(\deg P_j)}. \quad (8.13) \]

Then

\[ N(\delta, P_j) = o_\delta(\deg P_j) \quad (8.14) \]

for all \( \delta > 0 \). In other words, a negligible proportion of the zeros lie outside any fixed-width annulus around \( |z| = 1 \).

**Proof.** Suppose \( \delta > 0 \) and factor \( P_j \) as

\[ P_j(z) = Q_\delta(z) Q_\delta^*(z) \quad (8.15) \]

where the zeros of \( Q_\delta \) are the zeros of \( P_j \) in the annulus \( e^{-\delta} < |z| < e^\delta \). Suppose the degree of \( Q_\delta \) is \( 2M \). The maximum of \( Q_\delta(z) \) on \( |z| = 1 \) is decreased if the zeros of \( Q_\delta \) are moved to have equally spaced arguments and all have absolute value \( e^{-\delta} \) or \( e^\delta \). Thus, it is sufficient to consider the case that

\[ Q_\delta(z) = (z^M - e^{-\delta M})(z^M - e^{\delta M}) = z^{2M} - (e^{-\delta M} + e^{\delta M})z^M + 1. \quad (8.16) \]

By (8.16) we have \( |Q_\delta(z)| \geq e^{\delta M} - 3 \) on all of \( |z| = 1 \). Since \( Q_\delta(z) \) cannot be small on all of \( |z| = 1 \) (because it is 1 on average), by (8.13) we have \( M = o(\deg P_j) \), as claimed. \( \blacksquare \)

The converse of Proposition 8.10 is not true, and adapting the above proof fails because \( Q_\delta(z) \) in (8.15) could be large on \( |z| = 1 \). What we have seen is:

**Principle 8.11** There are two different ways a self-reciprocal polynomial can be very large on \( |z| = 1 \): there may be a large number of zeros far from the unit circle, or there may be a large gap (or other irregularity) in the arguments of the zeros.

For the \( \zeta \)-function, the second option does not occur, because the slow growth of \( S(t) \) prevents extreme irregularities in the distribution of zeros.

By “very large” we mean a violation of (the polynomial analogue of) LH.

In Subsection 9.3 we suggest that one of the arguments against RH is actually an argument for the possibility of improving bounds on the size of extreme gaps between zeros of the \( \zeta \)-function, based on bounds for the size of \( Z(t) \) and its derivatives.
9.1 Reason 1: Lehmer’s phenomenon

This reason for doubting RH comes from Section 2 of [54].

Computing zeros has been a fundamental part of exploring the ζ-function, going back to Riemann who calculated approximations to the first few zeros (by hand). When D.H. Lehmer computed the first 15,000 zeros in 1956, he found that \( \gamma_{6710} - \gamma_{6709} < 0.054 \). Those zeros are unusually close, in a sense which can be made precise. Such zeros are now called a Lehmer pair.

The existence of Lehmer pairs is a claimed reason to doubt RH. The justification is that if a real holomorphic function \( f(t) \) has two real zeros very close together, then \( f(t) + \varepsilon \) will have a pair of complex zeros for some small \( \varepsilon \). For example, near the Lehmer pair noted above, \( Z(t) - 0.004 \) has a pair of complex zeros. So, in a sense, the Lehmer pairs suggest:

Principle 9.2 If the Riemann Hypothesis is true, it is “barely true”.

The claimed argument is: Lehmer pairs indicate that RH is barely true, therefore we should be skeptical of RH.

The above argument contradicts Principle 9.1. The gaps between neighboring zeros of random unitary polynomials can be arbitrarily small. That is, arbitrarily close Lehmer pairs exist for random unitary polynomials: the normalized nearest-neighbor spacing is supported on all of \((0, \infty)\). Furthermore, the calculations of Odlyzko [68] and subsequent computations indicate that the
distribution of small gaps between zeros of the \( \zeta \)-function closely match the (suitably scaled) distribution of small gaps between random unitary eigenvalues. Thus, the data on Lehmer pairs support the connection between \( \zeta \)-zeros and the GUE. But by Principle 9.1 the data provide no reason to doubt RH.

**Other ways in which RH is barely true.** There are other ways in which RH is “barely true”. The function \( Z(t) + \delta \) has complex zeros for any nonzero (positive or negative) \( \delta \in \mathbb{R} \), and the reason has nothing to do with Lehmer pairs. Carrier waves (the small values, not the large values, see Subsection 8.2) cause \( Z(t) \) to stay small on intervals containing many zeros, without those zeros being close together. Thus, for any \( \delta > 0 \) there will exist intervals \( I \) containing arbitrarily many zeros such that \( |Z(t)| < \delta \) on \( I \). Therefore \( Z(t) + \delta \) and \( Z(t) - \delta \) will have many positive local minima (respectively, negative local maxima) on \( I \), and so will have many pairs of non-real zeros.

Here is yet another way. RH is equivalent to \( \Lambda \leq 0 \), where \( \Lambda \) is the de Bruijn-Newman constant [65]. Rodgers and Tao [71] showed that \( \Lambda \geq 0 \), extended by Dobner [29] to all L-functions. So \( \Lambda = 0 \) is the only possibility consistent with RH.

Thus, RH is at best “barely true” in multiple ways. That is just a fact one has to live with. It does not tip the scales for or against RH.

“**The Riemann Hypothesis is not an analysis problem**”. That is a quote from Brian Conrey, although he suggests that others have expressed a similar sentiment. There are several ways to understand its meaning, one of which comes from the fact that RH is at best barely true. There are many functions which are known to have all (or almost all) of their zeros on the real line or the unit circle. Examples are sine and cosine, Bessel and other special functions, Eisenstein series, period polynomials, and many families of orthogonal polynomials. What all those examples have in common is that the techniques come from analysis and not number theory, and all the zeros are asymptotically close to equally spaced. In other words, the analogue of RH is not barely true: it remains true under a small perturbation. That is a feature of analysis techniques, and that is why those techniques are not going to prove RH.

We are not saying analysis techniques are not important or useful. Indeed, some of the refutations of reasons to doubt RH are purely analysis arguments. But if there is a proof of RH, the deep facts in that proof will come from number theory, not analysis.

**9.2 Reason 2: Large values of \( Z(t) \)**

In Section 5 of [54] the claimed reason to doubt RH directly addresses carrier waves. Specifically, it is noted that RH combined with conjectures about the maximum size of \( Z(t) \) indicate that (this is quoted from [54]).

... the graph of \( Z(t) \) will consist of tightly packed spikes, which will be more and more condensed as \( t \) increases, with larger and larger oscillations. This I find hardly conceivable.

The author agrees that it is hard to believe, and the difficulty of believing it was specifically mentioned in Section 4. But believe it we must, because that is how carrier waves work.

Thus, this reason for doubting RH is not valid. Indeed, a significant portion of the zeros are known to be on the critical line, so the “tightly packed spikes” must exist whether or not RH is true. The appearance of the graph is merely
a matter of perspective: you can either see a pleasantly undulating curve, or tightly packed spikes, depending on the choice of scales on the axes.

9.3 Reason 3: Derivatives near a large maximum

The approach of Blanc [10] is based on the following interesting formula [11]. Suppose $f$ is a smooth function on an open interval containing $[-a, a]$ for some $a > 0$, and suppose $x = \{x_0, \ldots, x_n\}$ are distinct zeros of $f$ in that interval. Then

$$
\sum_{k=1}^{r} \Psi_{2k-1}(x, a)f^{(2k-1)}(a) - \sum_{k=1}^{r} \Psi_{2k-1}(x, -a)f^{(2k-1)}(-a) = \int_{-a}^{a} \Psi_{2r-1}(x, t)f^{(2r)}(t) dt.
$$

(9.1)

The exact form of $\Psi_\ell$ is tangential to our discussion, but we include it for completeness:

$$
\Psi_\ell(x, t) = \frac{(4a)^\ell}{(\ell + 1)!} \sum_{k=0}^{n} \mu_k(x) \left( B_{\ell-1}\left( \frac{1}{2} + \frac{x + x_k}{4a} \right) + B_{\ell-1}\left( \frac{x - x_k}{4a} \right) \right)
$$

where

$$
\mu_k(x) = 2^{-n} \prod_{0 \leq j \leq n \atop j \neq k} \left( \sin\left( \frac{\pi x_k}{2a} \right) - \sin\left( \frac{\pi x_j}{2a} \right) \right)^{-1}.
$$

Here $B_\ell$ is the Bernoulli polynomial and $\{x\} = x - [x]$ is the fractional part of $x$. The relevant fact about $\mu_k$ is that $\sum k \mu_k = 0$.

Blanc’s approach involves applying (9.1) in the context shown in Figure 9.3, where we have repurposed data from Bober and Hiary [13] which previously appeared in Figure 2.4.

Figure 9.3 The function $\log |Z(t)|$ near a local extremum caused by a large zero gap, and $S(t)$ in the same region. The parameters $T$ and $a$ are chosen so that $Z'(T + a)$ is a (large) local extremum, $Z'(T - a)$ is a (presumably small) local extremum, and $S(T + a)$ and $S(T - a)$ have opposite sign.

Choose $T$ and $a$ so that $T + a$ lies in a large zero gap, $|Z(T + t)|$ has a very large maximum a bit to the right of $a$, and $T + a$ is a local extremum of $Z'(T + a)$. (Those choices are illustrated in Figure 9.3, although it is difficult
to see an inflection point of \( Z(t) \) on the graph of \( \log |Z(t)| \). Note how the large zero gap causes the maximum of \( Z'(t) \) to shift toward the maximum of \( Z(t) \).

Furthermore assume \( S(T + t) \) changes sign in \((-a, a)\). Blanc also chooses \( a \) so that \( Z'(T - a) \) is a local extremum, although that does not seem to be essential.

In the original formulation there was a presumption that \( Z(t) \) achieved its largest possible value at the chosen local extremum. That is unlikely because by Principle 8.2 it is the carrier waves, not the largest gaps, which lead to the largest values. However, the underlying ideas are equally valid if one considers the local maxima caused by the largest possible zero gaps.

The setup above lets us describe the main idea. With \( f(t) = Z(T + t) \), the quantity \( f^{(2k-1)}(a) \) in (9.1) will be large for small \( k \). For larger \( k \) there is no reason for \( f^{(2k-1)}(a) \) to be large, and for all \( k \) there is no reason for \( f^{(2k-1)}(-a) \) to be large. Thus, some of the quantities on the left side of (9.1) will be very large. Table 9.4 lists the quantities in (9.1) in the context of Figure 9.3. Similar to Table of [10], we write

\[
\beta^\pm_\ell = (-1)^{(\ell-1)/2} \Psi_{\ell}(x, T \pm a) \theta'(T)^\ell \\
d^\pm_\ell = (-1)^{(\ell-1)/2} Z^\ell(T \pm a)/\theta'(T)^\ell,
\]

so the left side of (9.1) is \( \sum \beta^+_\ell d^+_\ell - \sum \beta^-_\ell d^-_\ell \).

Table 9.4 Quantities appearing in (9.1) using data from Bober and Hiary [13] as shown in Figure 9.3

| \( \ell \) | \( \beta^+_\ell \) | \( d^+_\ell \) | \( \beta^+_\ell d^+_\ell \) | \( \beta^-_\ell \) | \( d^-_\ell \) | \( \beta^-_\ell d^-_\ell \) |
|---|---|---|---|---|---|---|
| 1 | 0.000896 | 3209.249 | 2.877 | -0.00561 | 0.24181 | -0.001 |
| 3 | 0.004657 | 625.137 | 2.911 | -0.01944 | 0.37736 | -0.007 |
| 5 | 0.014213 | 182.535 | 2.549 | -0.04274 | 0.52865 | -0.022 |
| 7 | 0.033762 | 67.231 | 2.296 | -0.07822 | 0.70713 | -0.055 |
| 9 | 0.069663 | 28.657 | 1.996 | -0.13156 | 0.91859 | -0.120 |
| 11 | 0.132016 | 13.376 | 1.765 | -0.21285 | 1.17131 | -0.249 |

We can obtain a contradiction if we can show that the left side of (9.1) is large and the right side is small. For the data from Figure 9.3, the right side of (9.1) is approximately 13.95852, which is consistent with Table 9.4. That merely serves as a check on the numerical computations, because RH is true in the range of those data.

If we were only considering characteristic polynomials, then by Principle 9.1 we cannot obtain such a contradiction. Therefore we need another ingredient which is specific to L-functions. This is provided by another interesting result of Blanc [10].

Recall that by (3.2) we have \( \theta'(t) \sim \frac{\ell}{T} \log \frac{T}{\theta'(T)} \).

**Theorem 9.5 Blanc [10].** Suppose \( T \) is sufficiently large, \( \delta > \frac{1}{2} \), and \( a = o(T/\theta'(T)) \). If

\[
\delta \theta'(T) \leq 2K \leq 2\theta'(T)^2
\]

then

\[
|Z^{(2K)}(t)| \leq 2\zeta(\frac{1}{2} + \delta)\theta'(T)^{2K}
\]

for \( t \in [T - a, T + a] \).

Blanc [10] states the case \( \delta = \frac{3}{2} \) and estimates \( 2\zeta(2) < 4 \). It is worth investigating what choice of \( \delta \) is optimal for applications of Theorem 9.5.

By Theorem 9.5 we can choose \( r \) so that the right side of (9.1) is small. Therefore, if the setup described above makes the left side of (9.1) large, we have obtained a contradiction.
The analogue of Theorem 9.5 is not true for unitary polynomials, so we explore the implications of (9.5) in some detail before returning to Blanc’s argument.

Note that (9.5) is not true for any fixed $K$ as $T \to \infty$, because that would contradict known $\Omega$-results and conjectures for $Z^{(n)}(t)$. We now explain why, more generally, the bound (9.5) is not true for $K$ far outside the range (9.4). We do this by exploiting properties of repeated differentiation.

**Everyone wants to be cosine.** Differentiating a function like $Z(t)$ can be viewed as an operation on its set of zeros, replacing the zeros of $Z(t)$ by the zeros of $Z'(t)$. The relevant fact is that differentiation causes the zeros to become more evenly spaced [39][59], with large gaps becoming smaller and small gaps becoming larger. Thus, differentiation damps the irregularities in the size of the function: very large maxima (measured relative to the size of nearby maxima) become less extreme. Under repeated differentiation the zero gaps are becoming more regular, eventually approaching equality. For sufficiently large $k$ the $k$th derivative is locally approximately a function with equally spaced zeros. In other words, for $t$ in any fixed interval, if $k$ is sufficiently large, then

$$Z^{(k)}(t) \approx A_k \cos(B_k t + C_k)$$

throughout that interval. Berry [7] termed this phenomenon cosine is a universal attractor.

If the number of derivatives is too small for the zeros to be approximately equally spaced, then the variation in the zero spacing will prevent the bound (9.5) from holding. So (9.5) will not hold if $K$ is much smaller than the lower bound in (9.4).

For a moderate number of derivatives, the parameter $B_k$ in (9.6) directly relates to the local density of zeros. Since the local density of zeros of $Z(t)$ varies slowly, when the $k$th derivative initially approaches cosine we have $B_k \approx \theta'(T)$, as we previously saw in (4.4). That fact, and the chain rule, explains why powers of $\theta'(T)$ appear on the right side of (9.5), and the bound is optimal [12].

Finally, if we take “too many” derivatives then the local density of zeros begins increasing, so $B_k$ starts to increase and therefore (9.5) will not hold if $K$ is much larger than the upper bound in (9.4).

**Some unitary polynomials approach cosine more slowly.** We have already alluded to the fact that Theorem 9.5 does not hold for all unitary polynomials. More precisely, the analogue of the bound (9.5) does hold, but the number of derivatives needed to realize that bound can be much larger than the range (9.4). To see this, it is sufficient to consider self-reciprocal polynomials in polar form:

$$Z(z) = z^N + a_{N-1}z^{N-1} + \cdots + a_{N-1}z^{N-1} + \cdots + a_{N-1}z^{N-1}$$

with differentiation operator $D = z \frac{d}{dz}$. Note also that the analogue of $\theta'(T)$ is $N/2$.

We see that

$$D^{2K}Z(z) = \left(\frac{N}{2}\right)^{2K} z^{N} + \left(\frac{N-1}{2}\right)^{2K} a_{N-1}z^{N-1} + \cdots + \left(\frac{N-1}{2}\right)^{2K} a_{N-1}z^{N-1} + \left(\frac{N}{2}\right)^{2K} z^{-N}$$

$$= \left(\frac{N}{2}\right)^{2K} z^{-N} \left(z^N + (1-N^{-1})^{2K}a_{N-1}z^{N-1}+ \right)$$

44
\[ \cdots + (1 - 2N^{-1})^{2K} \pi_{N-2} z^2 + (1 - N^{-1})^{2K} \pi_{N-1} z + 1 \]

\[ = \left( \frac{N}{2} \right)^{2K} z^{-N} \left( z^N + 1 + o(1) \right) \]  

(9.8)
as \( K \to \infty \). By (9.8) the roots of \( D^{2K}Z(z) \) are approaching the \( N \)th roots of 1, so in particular they are approaching equal spacing. It was not even necessary to assume the original self-reciprocal polynomial had all its zeros on the unit circle. Yet the analogue of Theorem 9.5 fails without further restrictions on the polynomials because the required number of derivatives may be much larger than the analogue of the lower bound in (9.4). One can see this from (9.8) because \( (1 - N^{-1})^{2K} \) is not small if \( K = O(N) \).

For unitary polynomials the analogue of Theorem 9.5 has counterexamples similar to the failure of the analogue of the Lindelöf hypothesis for some unitary polynomials. Consider \( z^{-N/2}(z + 1)^N \), with differentiation operator \( D = z^2 \pi^2 \).

If we take \( \kappa N D \)-derivatives, with \( \kappa < 1 \), the polynomial will still have a high multiplicity zero at \( z = -1 \). That will force the other zeros to be spread out, and so the function will be large. Even if we take \( N \) derivatives, so all the zeros are simple, it will still take many more derivatives for the zeros to become approximately equally spaced. In particular, by considering only the \( N - \sqrt{N} \)th coefficient of \( (z + 1)^N \), we see that \( D^N z^{-N/2}(z + 1)^N \) is large at \( z = 1 \).

From the above discussion we see that the analogue of Theorem 9.5 does not hold for characteristic polynomials, so Blanc’s argument, which we analyze next, does not fall victim to Principle 9.1.

**Analysis of Blanc’s argument.** Summarizing the ideas so far: we want to assume a very large gap between zeros, causing a very large isolated local maximum of \( Z(t) \). With the choices of \( T \) and \( a \) as in Figure 9.3, the quantity \( Z^{(2k-1)}(a) \) in (9.1) will be very large for small \( k \). This might lead to a contradiction if the parameter \( r \) allows Theorem 9.5 to apply, because then the right side of (9.1) will be small. If the left side of (9.1) is large and the right side is small, then we have obtained a contradiction.

Given a large gap between zeros, and choosing \( r \) so that Theorem 9.5 applies, how can having \( Z^{(2k-1)}(a) \) be very large for small \( k \) fail to lead to a contradiction? There are two ways.

1. \( \Psi_1(x, a) \) might be very small, so actually the left side of (9.1) is small. (For example, see Table 9.4)

2. Some other terms on the left side might be large and of the opposite sign, causing cancellation.

If the above possibilities do not apply (so we do obtain a contradiction), what has been contradicted? Have we contradicted RH, or something else? We will argue that such a contradiction, if it happens, has nothing to do with RH but rather it comes from assuming a particularly large gap between zeros. In other words, the theorems of Blanc may actually contain hidden information which might be used to provide an improved upper bound on gaps between zeros of the \( \zeta \)-function.

Blanc addresses Item 2 above by noting that the setup in Figure 9.3 specifically avoids the likelihood that other terms are large. Figure 2 in [10] provides good evidence for the intuition that \( \beta^\ell_{10} \) should be small for large \( \ell \).

To investigate the size of \( \Psi_{\ell}(x, a) \), we consider scenario similar to Figure 9.3 at height \( 10^{100} \), which Figure 2 in [10] suggests is sufficient to test the method. Since \( \log 10^{100} \approx 230.26 \), we can use degree 230 unitary polynomials. Given
a large zero gap, we can place the other zeros in their most likely location as described in Subsection 8.1. An example of this, with a gap 6 times the average in degree 74 polynomial, is shown in Figure 8.4. Since everything is explicit, we can evaluate every quantity in (9.1). Table 9.6 lists the terms in (9.2) and (9.3) for degree 230 polynomials with a large gap of 5, 10 and 15 times the average spacing, with all other zeros close to their most likely configuration.

Table 9.6

| ℓ | \( \beta^+ \) (5) | \( d^+ \) (5) | \( \beta^+ \) (10) | \( d^+ \) (10) | \( \beta^+ \) (15) | \( d^+ \) (15) |
|---|---|---|---|---|---|---|
| 1 | 0.00099 | 363.5 | 2.5 \times 10^{-6} | 446988.0 | 1.9 \times 10^{-8} | 4.9 \times 10^{8} |
| 3 | 0.00329 | 95.2 | 0.000016 | 57835.6 | 1.9 \times 10^{-7} | 4.2 \times 10^{7} |
| 5 | 0.00654 | 37.7 | 0.000057 | 11436.1 | 1.0 \times 10^{-6} | 5.6 \times 10^{6} |

The main takeaway from Table 9.6 is that \( \beta^+ \) is small for small \( ℓ \) when \( d^+ \) is large. Thus, we must question the possibility of obtaining a contradiction from (9.1). Specifically, we suggest that there is no evidence to support Blanc’s assertion (bottom of first column on page 5 of [10]) “the sum of the first \( \beta^+ \) \( 2^{\ell-1} d^{+} \) is probably large”.

The comments above do not address a key element of Blanc’s argument, based on (9.5), for which the analogous result for unitary polynomials is not true. Translating (9.5) to the realm of self-reciprocal polynomials gives the statement: in the context of the polar form (9.7), with \( Z_2 K(z) := D_2 K Z(z) \),

\[
\text{if } K \geq \frac{3}{8} N \text{ then } \max_{|z|=1} |Z_2^K(z)| \leq 3.29 \left( \frac{N}{2} \right)^{2K}.
\]

That bound on \( |Z_2^K(z)| \) holds if \( K \) is sufficiently large, but not in general for \( K \ll N \). In particular, for the degree 230 polynomial with a zero gap 15 times the average used in Table 9.6, one must take 197 derivatives in order to achieve the estimate in (9.9). The polynomial with gap 10 times the average requires 123 derivatives. This suggests to the author that the ingredients in Blanc’s argument do not throw doubt on RH, but instead point to a new method for bounding the size of large gaps between zeros of the \( \zeta \)-function. We propose:

**Problem 9.7** Suppose \( Z \) is a polar unitary polynomial of degree \( N \) with a zero gap of \( \kappa \) times the average. For \( C > 2 \), determine a lower bound on \( \eta = \eta(\kappa, C) \) such that if \( 2K < \eta N \) then

\[
\max_{|z|=1} |Z^{(2K)}(z)| > C \left( \frac{N}{2} \right)^{2K}.
\]

The techniques used to solve that problem might translate into a method for improving the upper bound on gaps between zeros of the \( \zeta \)-function.

**Further reflections on characteristic polynomials.** There is an apparent discrepancy in our discussion of large gaps between zeros. On the one hand we have described how random matrix theory provides a precise conjecture on the maximum gap size. On the other hand, we have described properties of \( Z(t) \) which suggest constraints on large gaps between zeros, and those constraints do not apply to characteristic polynomials.

These apparent discrepancies might be resolved by examining in more detail how characteristic polynomials are used to model \( Z(t) \). By Principle 6.5, to model the largest gaps one chooses around \( e^N \) independent random matrices in \( U(N) \), where \( N \approx \log T \). Based on the enormous success of using RMT to shed light on L-functions, it is plausible that the analogue of Theorem 9.5 holds
almost surely (in the probabilistic sense) for $e^N$ Haar-random matrices chosen from $U(N)$.

10 Claimed reasons to doubt RH: other properties

The final two claims we consider involve averages of high powers of $Z(t)$, and properties of Dirichlet series which are not L-functions.

10.1 Reason 4: Error terms in moment formulas, and maybe the Lindelöf Hypothesis is also false

The argument in Section 4 of [54] is based on the meager knowledge we have about moments of the $\zeta$-function:

$$
\int_0^T Z(t)^2 \, dt = T \log T + (2\gamma - 1 - \log 2\pi)T + O(T^{E_1+\varepsilon})
$$

$$
\int_0^T Z(t)^4 \, dt = TP_2(\log T) + O(T^{E_2+\varepsilon})
$$

where $P_2$ is a certain degree 4 polynomial. The conjectured optimal error terms are $E_1 = \frac{1}{4}$ and $E_2 = \frac{1}{2}$.

For higher moments, the conjecture [21] is:

$$
\int_0^T Z(t)^{2k} \, dt = TP_k(\log T) + O(T^{E_k+\varepsilon})
$$

(10.1)

$$
\sim g_k a_k T \log^k T,
$$

(10.2)

as $T \to \infty$, where $P_k(x)$ is a polynomial of degree $k^2$ with leading coefficient $g_k a_k$, where $a_k$ is given explicitly, and $g_k$ is an integer. That conjecture is open for $k \geq 3$.

Why moments? Moments of the $\zeta$-function were introduced by Hardy and Littlewood [46] as an approach toward the Lindelöf Hypothesis 8.8.

RH implies LH, so it is natural to attack LH as an “easier” problem than RH. LH is equivalent to

$$
\int_0^T Z(t)^{2k} \, dt \ll_k T^{1+\varepsilon}
$$

(10.3)

for all integer $k > 0$, which is why Hardy and Littlewood considered moments.

Bounding such moments proved to be difficult: 100 years after Hardy and Littlewood we only know (10.3) for $k = 1$ and 2. But instead of becoming a dead-end in terms of proving LH, moments became a topic of independent interest. Other “families” of L-functions were introduced, with various types of moments. The focus shifted to finding precise formulas for the main terms in moments, not just on estimating the order of magnitude. Random matrix theory played a major role in understanding the leading order behavior and the structure of those moments, although ultimately number-theoretic heuristics provided the most precise conjectures. See [57][18][58][19][28].
**Analysis of the argument.** The claimed argument against RH arises from
the error term in the $2k$th moment. Based on the conjectured best possible
values $E_1 = \frac{1}{4}$ and $E_2 = \frac{1}{2}$ Ivić suggests $[53][54]$ that the best possible error term
in the $2k$th moment is $O(T^{k/4+\varepsilon})$. There is slightly more to Ivić’s reasoning,
such as the apparent inability of available techniques provide bounds on high
moments, but the main input is the conjectured values for $E_1$ and $E_2$.

If the error term (10.1) truly is of size $T^{k/4+\varepsilon}$, meaning that (10.1) is false
with $k/4$ replaced by a smaller number, then (10.3) is also false. Therefore LH
is false, so RH is false. That is the essence of that argument against RH.

Basing an argument on such limited data for small $k$ is problematic, and
the conjecture of $E_k = k/4$ is not based on any underlying principles. Suppose
someone instead conjectured $E_k = \frac{3}{4} - 2^{-k}$ for $k \geq 1$. That also is a simple
formula which fits the limited data, and it has the added benefit of implying LH.
But without any good evidence, that conjecture should also not be taken seri-
ously. Thus, this proposed reason to doubt RH is not based on a mathematical
foundation, but on naive pattern matching.

Before the reader criticizes the author for being unkind, let us note that
this author fully recognizes the folly of speculating on the size of error terms,
and indeed is guilty of similarly unwise speculation. In a foundational paper
on moments of L-functions $[19]$ this author and coauthors conjecture that in
great generality the error term in any conjectured higher moment is $E_k = \frac{1}{2}$.
The conjecture of Ivić is as pessimistic as possible, and the conjecture in $[19]$ is
as naively optimistic as possible. Unfortunately, that optimistic error term
is not always correct. Specifically, in some cases $[27][28]$ there is a secondary
main term of size $X^{\frac{3}{4}}$. It may even be that similar secondary main terms are
common. The lesson is that speculating on the error term in moment
conjectures is difficult, and it is folly to draw conclusions from such error term
conjectures.

**The structure of moments.** Immediately after making a poorly founded
conjecture about error terms, Ivić goes on $[53][54]$ to make the perceptive
speculation:

the shape of the asymptotic formula...changes when $k = 4$.

That speculation has been borne out by subsequent work on the structure of
moments.

The $2k$th moment (10.1) is only known for $k = 0$, $1$, and $2$. Conjectures for
larger moments are a fairly recent development. One reason those moments are
difficult is that the natural way to proceed is via an “approximate functional
equation” for $Z(t)^k$, which is a sum of two Dirichlet polynomials of length $t^{k/2}$.
(The Riemann-Siegel formula (3.1) can be viewed as an example: there are
$\approx t^{k/2}$ terms in the sum.) When used to evaluate $\int_0^T Z(t)^{2k} dt$ the approximate
functional equation will have length $T^{k/2}$. The problem is the lack of tools to
handle Dirichlet polynomials where the length of the polynomial is greater than
the length of the integral.

Conrey and Ghosh $[22]$ developed a heuristic approach which gave a conjecture
when $k = 3$, and Conrey and Gonek $[24]$ pushed the method to its limit to
make a conjecture when $k = 4$. That truly is the limit of the method, because
when applied to the case $k = 5$ it produces an answer which is negative! Thus,
Ivić was correct: some new phenomenon appears, beginning at $k = 4$. But that
new phenomenon has more implications for that main term than for the error
term.
10.2 Reason 5: The Deuring-Heilbronn phenomenon

The Riemann ζ-function is the simplest of an infinite family of functions known as “L-functions”. The next simplest examples are the Dirichlet L-functions, \( L(s, \chi) \), where \( \chi \) is a primitive Dirichlet character. RH for all \( L(s, \chi) \) is known as the **Generalized Riemann Hypothesis**.

Informally, by **L-function** we mean a Dirichlet series with a functional equation and an Euler product. To give a more precise definition, there are two primary approaches. The **axiomatic approach** considers L-functions as analytic functions with certain properties. Within the axiomatic approach at one extreme we have the Selberg class [78][23], which has a minimal set of general axioms. At the other extreme are the tempered balanced analytic L-functions [38] which have a large number of precise axioms.

The **structural approach** describes L-functions as arising from arithmetic or automorphic objects. For the perspective in this paper, the important fact is that conjecturally all approaches describe the same set of functions, and **all such L-functions have an analogue of the Riemann Hypothesis**. The collection of all such Riemann Hypotheses is called the **grand Riemann Hypothesis**.

Under any of the above definitions, each L-function has a functional equation analogous to (2.2), a \( \xi_L \)-function analogous to (2.3), and a \( Z_L \)-function analogous to (2.5). Each \( Z_L \)-function is a real valued function which wiggles with the same type of randomness as \( Z(t) \). It also has carrier waves as described in the first part of this paper.

**Linear combinations.** The Deuring-Heilbronn phenomenon concerns functions which are not L-functions, but rather are linear combinations of L-functions. As we will see in **Principle 10.2**, a nontrivial linear combination of L-functions never satisfies RH.

Consider two L-functions which are being added, and focus on a small interval of the real line (say, where one expects to see a few dozen zeros of each \( Z \)-function). If one of the \( Z \)-functions happens to be much larger than the other over that entire interval, then the sum looks just like the larger \( Z \)-function. In particular, if the larger \( Z \)-function satisfies RH, then their sum satisfies RH on that interval. That is the typical situation and this was the motivation for developing the idea of carrier waves: the logarithm of each \( Z \)-function has a Gaussian distribution, and because of the carrier waves its size does not change too rapidly. Thus, 100% of the time one of them is much larger than the other and stays larger over a span of many zeros. Under some mild additional hypotheses, that is how Bombieri and Hejhal [15] proved that if two L-functions have the same functional equation and each satisfies RH, then their sum has 100% of its zeros on the critical line.

But, 100% is not everything. As the two \( Z \)-functions take turns being the largest, occasionally they are about the same size. When that happens, and we are adding two different functions that look like the top graph in **Figure 2.1**, then not all of the zeros of the sum will be real. In other words:

**Principle 10.1** A nontrivial linear combination of L-functions will not satisfy the Riemann Hypothesis.

**Naturally occurring examples.** The above discussion makes it seem obvious that a nontrivial linear combination of L-functions will never satisfy RH. But the situation was not always so clear. Let \( \chi_{5,2} \) be the Dirichlet character mod 5 with \( \chi(2) = i \), and define \( \theta \) by \( \tan \theta = (\sqrt{10} - 2\sqrt{5} - 2)/(\sqrt{5} - 1) \). The
The Deuring-Heilbronn function is given by
\[ F_{DH}(s) = \frac{1}{2} \sec \theta \left( e^{-i\theta} L(s, \chi_{5,2}) + e^{i\theta} L(s, \overline{\chi}_{5,2}) \right) \] (10.4)

which satisfies the functional equation
\[ \xi_{DH}(s) := 5^{\frac{s}{2}} \pi^{-\frac{s}{2}} \Gamma(\frac{1}{2}s + \frac{1}{2}) F_{DH}(s) = \xi_{DH}(1-s). \] (10.5)

Except for the sign, that is the same functional equation satisfied by \( L(s, \chi_{5,2}) \). However \( F_{DH}(s) \) does not satisfy RH: its first zero off the critical line is near 0.8085 + 85.699i. Furthermore [83] it has infinitely many zeros in \( \sigma > 1 \), although it does not seem that an explicit example has ever been computed.

We have two Dirichlet series, \( F_{DH}(s) \) and \( L(s, \chi_{5,2}) \), which satisfy similar functional equations. We know that one of them does not satisfy RH, so why should the other? That is the essence of the argument in Section 3 of [54]: why should we believe RH for one function when we know it is false for another, and those functions (quoting from [54]) “share many common properties”?

The Euler product. The standard answer to the above question (as noted in [54]) is that the functions are fundamentally different because \( L(s, \chi_{5,2}) \) is an L-function, but \( F_{DH}(s) \) is not — because it does not have an Euler product. It is an exercise to check that every axiom in the Selberg class [78] is essential: omit any one of them and there will be functions which do not satisfy RH. The Euler product condition is not more important than the other axioms, but it is important, and without it one has no expectation of RH.

The Deuring-Heilbronn function is one of many examples constructed by taking linear combinations of L-functions. It is an exercise to show that a nontrivial linear combination of Euler products is not an Euler product. Thus, any such linear combination is not expected to satisfy RH, even if the constituent functions do satisfy RH. This explains why examples like \( F_{DH}(s) \) do not cast doubt on RH. However, there is a bit more to be said if we consider the locations of the zeros of those functions.

Zeros outside the critical strip. The Deuring-Heilbronn function failed RH in the worst possible way: it has zeros in the region where the Dirichlet series converges absolutely. In fact it has \( \gg T \) zeros in \( \sigma > 1 \) up to height \( T \), but that is not any extra information because Dirichlet series are almost periodic functions (of \( t \)) in \( \sigma > 1 \). Those zeros in \( \sigma > 1 \) are a general phenomenon, so we obtain a refinement of Principle 10.1:

Principle 10.2 A nontrivial linear combination of L-functions will have infinitely many zeros in \( \sigma > 1 \).

Principle 10.2 is a theorem for Dirichlet L-functions [76], for automorphic L-functions [16], and in an axiomatic setting [72]. Those results are interesting because one has a vector space of Dirichlet series, all of which satisfy the same functional equation. Conjecturally only the functions in the span of individual basis elements (i.e. the actual L-functions) satisfy RH, yet every other element of the space fails RH in the worst possible way. This highlights the key role of the Euler product.

Why exactly is the Euler product so important? Maybe it isn’t. In terms of the analytic properties of the L-function, the most obvious consequence is that the Euler product prevents the L-function from having any zeros in \( \sigma > 1 \). So, is it the Euler product that matters, or the lack of zeros in \( \sigma > 1 \)?
11 Discussion of the Mistaken Notions

We revisit the topics in Section 3 in the context of the Principles.

11.1 Large values and large gaps

Mistaken Notion 3.3, equating the largest values of the \( \zeta \)-function with the largest gaps, is the core idea for all three Notions. As has already been adequately addressed by the discussion of carrier waves, see Principle 4.2, that Notion does not accurately describe the typical large values, nor the truly largest values.

However, Mistaken Notion 3.3 does describe the behavior in the range of current computations, and sometimes a wrong idea can lead to the right answer. Kotnik [55] sought computational evidence that the known \( \Omega \)-result (2.13) was not the true order of growth. The method was to find the sequence of largest values of \( |\zeta(\frac{1}{2} + it)| \) in the initial range. Those computations did make it appear plausible that the true order of growth is faster than the \( \Omega \)-result. Since those large values arise from large gaps, the computations were not revealing the true order of growth. But by Principle 8.2, the maxima in the large gaps should be \( \gg e^{\sqrt{\log t}} \), which is larger than the current best \( \Omega \)-result.

11.2 Finding large values

Mistaken Notion 3.4 is the source of all known large values of the \( \zeta \)-function. The Notion makes sense: the first few terms are the largest, so if you want the entire sum to be large, it is efficient to focus on the initial terms. That argument ignores the fact that there are \( \gg t^{\frac{1}{2}} \) terms of size \( \gg t^{-\frac{1}{4}} \). Even a small bias in such a large number of terms could contribute much more than the initial terms. Indeed, one of the arguments for the conjectured maximum size of the \( \zeta \)-function is based on a random model for the tail of that sum, which is larger than one can obtain from a strong bias in the initial terms.

Nevertheless, Mistaken Notion 3.4 is how the largest known values of the \( \zeta \)-function have been found [13][82]. It is possible to take that idea too far. Consider

\[ F(t, g(t)) = 2 \sum_{n < g(t)} n^{-\frac{1}{2}} \cos(\theta(t) - t \log n). \]  

(11.1)

The main expression in the Riemann-Siegel formula has \( g(t) = \sqrt{t/2\pi} \). One might expect that \( F(t, t^\epsilon) \) is a good approximation for the large values of \( Z(t) \) because (assuming the Lindelöf Hypothesis) a similarly short sum is a good approximation to \( \zeta(s) \) in \( \sigma > \frac{1}{2} \). Kotnik [55] considered \( F(t, t^\delta) \) for \( \delta = \frac{1}{7} \) and \( \delta = \frac{1}{4} \). Those functions are faster to evaluate than the Riemann-Siegel formula, and this sped up the computations by more quickly eliminating regions where \( Z(t) \) was less likely to be large.

Tihanyi [81] took it a step too far by considering \( F(t, \log t / 2\pi) \). Since \( F(t, g(t)) \ll \sqrt{g(t)} \), Tihanyi’s approximation has no hope of capturing the largest values. Yet, using \( F(t, \log t / 2\pi) \) as a filter (the same strategy as Kotnik [55]) turned out to be useful for finding many points where \( Z(t) \) was larger than 1000, and in subsequent work [82], found a value of \( Z(t) \) even larger than that found by Bober and Hiary [13]. So, a Notion may be wrong or misleading, yet still serve as the seed for a new discovery.

Although it is out of reach to compute in regions where the carrier wave, and not the local zero gaps, provide the main contribution to the size of \( Z(t) \),

51
it would still be of theoretical interest to find heuristics for locating truly large values.

11.3 Trying to refute RH

If RH were not true, how would one go about searching for a pair of zeros off the line? Mistaken Notion 3.5 suggests that a large gap might “push some zeros off the line”. That is contrary to the idea behind Principle 9.1: random unitary polynomials can have large zero gaps — conjecturally with the same size and frequency as the $\zeta$-function — which influence the nearby gaps but do not destroy the fact that the polynomial is unitary.

We offer two more arguments against Mistaken Notion 3.5. The first is similar to an idea used in the discussion of carrier waves in Section 4: run the argument in reverse. Suppose there is a pair of zeros off the line. Would one expect there to be a particularly large gap nearby? Surely not. That pair off the line, if they were not too far from the line, would only cause a small gap in zeros on the critical line, and furthermore the function would be small (not large) within that zero gap.

The second argument concerns the fact that no plausible suggestion has been made for how RH would fail if it were false. Maybe the zeros in $\sigma > \frac{1}{2}$, if they exist, all lie on $\sigma = \frac{3}{4}$? Is that a particularly absurd suggestion? Zeros that far from the line have a very small influence on the behavior of $Z(t)$ — in particular there is no associated large value or large zero gap. It is perfectly reasonable to seek fame and fortune by trying to computationally disprove RH. But searching for zeros off the line should be accompanied by, and tailored to, a model for how the failure will occur. A scenario in which RH fails, and those failures are typically associated to a large zero gap, is arguably even less plausible than having all non-critical zeros on $\sigma = \frac{3}{4}$.

The carrier wave frequently causes the $Z$-function to stay very small over a wide span of zeros. In that region a slight perturbation would cause many zeros to fall off the critical line. Maybe attempts to computationally disprove RH should begin by looking for regions where the initial terms in the Riemann-Siegel formula mostly cancel to nothing?

11.4 Gram’s law, as it was originally formulated

Mistaken Notion 3.6 contains two statements. Here we address Gram’s law. For reasons why Gram points are not special, see [35].

Gram’s law, as usually stated, misses a key point of Gram’s original observation. What Gram noted is that the zeros and the Gram points “separate each other”. In other words, in the range computed by Gram not only does the $n$th Gram interval $(g_n, g_{n+1}]$ contain exactly one zero, that zero is the $n + 2$nd zero. (The $+2$ arises because the definition of Gram point has $g_0 \approx 17.84$.)

We propose that the precise Gram’s law holds for the Gram interval $(g_m, g_{m+1}]$ if that interval contains the $m + 2$nd zero of the $Z$-function and no other zeros. Gram’s observation was based on the first 15 zeros, but note that he suggested the interlacing would not continue to hold indefinitely. In fact, Gram’s law (either version) fails after the 126th zero — because there is a large zero gap and the 127th zero falls in the 126th Gram interval. The Gram interval $[g_{3357}, g_{3358}]$ contains exactly one zero, but the precise Gram’s law fails because the previous interval contains two zeros. That is the first time there is a difference between the two versions of the law. There are 16 differences among the first 10,000 zeros.
The precise Gram’s law holds initially, for familiar reasons: initially $S(t)$ is small, so all the zeros must stay close to their expected locations. But at very large height, the fact that $|S(t)|$ typically is large implies that the precise Gram’s law holds rarely. To say it another way: a necessary (but not sufficient) condition for the precise Gram’s law to hold on a Gram interval $[g_n, g_{n+1}]$ is that $|S(g_n)| \leq 1$. The frequency of that occurrence can be estimated using Selberg’s central limit theorem. The slow growth of $S(t)$ makes this effect difficult to observe: even for characteristic polynomials of random matrices in $U(1000)$, as in the examples in Section 7, the precise Gram’s law still holds more than 40% of the time.

The fact that Gram’s law holds around 66% of the time should be viewed as an aspect of the random matrix model of zeros of the $\zeta$-function: the normalized nearest neighbor spacing of zeros/eigenvalues, see the leftmost plot in Figure 7.9, is concentrated around 1. So it is reasonably likely that any interval of normalized length 1 contains exactly one eigenvalue (although at low height a Gram interval is more likely than a random interval, see [45]).

The precise formulation of Gram’s law describes a rigidity which only holds initially. At larger heights the zeros slosh around. That term is meant to evoke what happens when one tries to walk while carrying a wide but shallow pan of water. Large groups of zeros are shifted to the left or the right of their expected location, even though locally all sets of zeros behave similarly. Because of the sloshing, $S(t)$ is large (in absolute value) most of the time. Long runs of zeros miss their intended Gram intervals: many in a row falling to the right, and later, many in a row falling to the left. (But locally the zeros are still obeying random matrix statistics, so a Gram interval (or any other interval of that length) will contain exactly one zero more than 66% of the time.) The $Z$-function is wiggling with great amplitude for a while, and then wiggling with small amplitude. The size of the carrier wave is independent of whether the nearby zeros arrive before or after their Gram interval, that is, independent of the size or the sign of $S(t)$. All of these phenomena are manifestations of the density wave, a wave which cannot exist if the precise Gram’s law were true too often.

11.5 The chicken or the egg?

Which came first: the $\zeta$-function or its zeros? That sounds like a meaningless question, but we suggest that the answer factors into whether or not one is skeptical of RH.

Suppose you believe the function is what matters, either the $\zeta$-function given by its Dirichlet series or some other expression, or the $Z$-function given by the Riemann-Siegel formula. From that perspective, the zeros are determined by the function, and it would take an unlikely conspiracy for RH to be true. It might still bother you that RH is “barely true”. The suggestion at the end of Subsection 11.3 might be viewed as an intriguing idea worth exploring. A failure of RH could occur for accidental reasons.

Now suppose you believe it is the zeros which matter. The zeros determine the function, with the density wave setting the scale and the local arrangement providing lower-order adjustments. Every zero is important and meaningful: a purported list of zeros containing one tiny error would be detected (by Weil’s explicit formula or some other means). If there were zeros off the line, they would be there to serve a specific purpose and their location would be meaningful. The suggestion at the end of Subsection 11.3 is just silly: the $Z$-function stays small over a wide region only because the density of zeros is higher.
It is hoped that this paper helps cultivate an appreciation for the explanatory power which comes from viewing the zeros as the basic object.

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