Signed permutations and the four color theorem

Shalom Eliahou and Cédric Lecouvey
Laboratoire de Mathématiques Pures et Appliquées
Université du Littoral Côte d’Opale
50 rue F. Buisson, B.P. 699
62228 Calais Cedex, France

Abstract

To each permutation $\sigma$ in $S_n$ we associate a triangulation of a fixed $(n+2)$-gon. We then determine the fibers of this association and show that they coincide with the sylvester classes depicted in [6]. A signed version of this construction allows us to reformulate the four color theorem in terms of the existence of a signable path between any two permutations in the Cayley graph of the symmetric group $S_n$.

1 Introduction

In this paper, we obtain a reformulation of the four color theorem in terms of signed permutations (Theorem 4.2.1). Signed permutations are standard words (that is with no letter repeated) $w$ of length $n$ on the alphabet $\{\bar{k}, \ldots, \bar{1}, 1, \ldots, n\}$ which do not contain any pair $(k, \bar{k})$. The barred (resp. unbarred) letters are interpreted as negative (resp. positive) integers and we write $|w|$ for the word on $\{1, \ldots, n\}$ obtained by erasing the bars which appear on the letters of $w$. We first describe a combinatorial procedure which attaches to each permutation $\sigma \in S_n$ (considered as a standard word on a given totally ordered alphabet $X_n$) a triangulation $\varphi(\sigma)$ of a fixed $(n+2)$-gon $P$. The fibers of this association coincide with the sylvester classes defined in [6]. Our construction is close to that used by Reading in [8] and the sylvester congruence on standard words is in fact a special case of cambrian congruence. Nevertheless, we have chosen to give an independent exposition of the occurrence of the sylvester relations in the context of triangulations. This makes the paper self-contained and permits us to expose precisely the results which are required in our reformulation of the four color theorem in terms of signed permutations. This connection with the results of [8] motivates an additional reformulation of the four color theorem in terms of the geometry of the associahedron.

We also introduce colored triangulations obtained by associating a color (considered as a letter of a totally ordered alphabet $C$ with $p$ colors) to each face (or to each vertex) of a given
triangulation of $P$. The map $\varphi$ admits a natural extension $\Phi$ defined from the set of words $w$ with letters in $C$ to a particular subset of colored triangulations we have called simple. We show that the fiber $\Phi^{-1}\{w\}$ coincides with the sylvester class of $w$ (Theorem 3.4.3). This implies that the simple colored triangulations can be regarded as combinatorial objects analogous to binary search trees. The map $\Phi$ can be interpreted as an insertion scheme for simple triangulations. This gives an alternative to Knuth’s insertion algorithm on binary search trees.

Signed triangulations are colored triangulations with $C = \{\pm 1\}$. To each triangulation $T$ of $P$ corresponds the sylvester class $\varphi^{-1}(T)$. This permits us to associate to each signed triangulation $T_\varepsilon$, defined as a signing $\varepsilon$ of $T$, the set $[T]_s$ of signed permutation words $w$ such that $|w| \in \varphi^{-1}(T)$. To obtain our reformulation of the four color theorem, we establish that for each signed flip (defined as a particular diagonal flip) in $T_\varepsilon$ one can find two words $w_1$ and $w_2$ in $[T]_s$ such that

$$w_1 = u \alpha \gamma v \quad w_2 = u \gamma \alpha v$$

where $\alpha, \gamma$ are letters with the same sign, $u, v$ are factors of $w_1$ and $v$ does not contain any letter $\beta$ such that $|\alpha| < |\beta| < |\gamma|$.

We also study the combinatorial problem of describing the graph obtained from a colored triangulation $T_\varepsilon$ by computing successive flips. When no condition is imposed on the flip, it is well known that the flip graph of the (ordinary) triangulations $T$ is connected and contains all the triangulations of $P$. In addition to signed flips, we also consider in this paper switched flips. They are flips which preserve the coloring and for which a flip operation is authorized when the two faces considered have distinct colors. For any $\mu \in \mathbb{N}_p$ such that $\mu_1 + \cdots + \mu_p = n$, let $S_\mu$ be the Frobenius subgroup of $S_n$ defined by $\mu$. We prove that the map $\Phi$ is a morphism from the Cayley graph of $S_n/S_\mu$ to the graph whose vertices are the simple colored triangulations associated to $\mu$ connected by an edge when they differ by a switched flip. So the graph defined from a simple colored triangulation by applying switched flips contains all the simple colored triangulations which have the same coloring and thus, is connected.

The paper is organized as follows. Section 2 is devoted to the combinatorial background on triangulations, colored triangulations and flip operations we need in the sequel. In section 3, we introduce the maps $\varphi$ and $\Phi$ and prove that their fibers coincide with sylvester classes. The reformulations of the four color theorem using respectively the signed permutations and the geometry of the associahedron are given in Section 4. Finally, we study in section 5 the flip graphs generated by colored triangulations when restrictive conditions are imposed on the flips.
2 Triangulations and flips

2.1 Triangulations of an \((n + 2)\)-gon

Consider an integer \(n \geq 0\) and \(X = \{x_1 < \ldots < x_n\}\) a subset of \(\mathbb{N} \setminus \{0\}\) of size \(n\). We denote by \(\hat{X}\) the augmented set \(\hat{X} = X \cup \{0, \infty\}\), totally ordered as follows:

\[
\hat{X} = \{0 < x_1 < \ldots < x_n < \infty\}.
\]

Each integer \(x \in X\) has a predecessor and a successor in \(\hat{X}\) that we denote \(\text{pred}_{\hat{X}}(x)\) and \(\text{succ}_{\hat{X}}(x)\) (or \(\text{pred}(x)\) and \(\text{succ}(x)\), for short), respectively. Thus,

\[
\text{pred}(x) < x < \text{succ}(x)
\]

is a 3-element interval in \(\hat{X}\). Let \(P = P_n\) be a convex \((n + 2)\)-gon, with vertices labelled by the elements of \(\hat{X}\) in a clockwise increasing way. By a triangulation \(T\) of \(P\), we mean a plane graph with the \(n + 2\) vertices and \(n + 2\) edges of \(P\), and with \(n - 1\) additional edges, called diagonals, which subdivide the inner face of \(P\) into \(n\) triangles, called the faces of \(T\).

We denote by \(T_n\) the set of triangulations of the polygon \(P = P_n\). It is well known that the cardinality of \(T_n\) is equal to the Catalan number \(c_n = \frac{1}{2n+1} \binom{2n}{n}\). By a slight abuse of notation, we shall make no distinction between a vertex \(v\) of \(P\) and its label in \(\hat{X}\). In other words, we consider the polygon \(P\) and its triangulations \(T\) as graphs with vertex set \(\hat{X}\).

Recall that the degree of a vertex \(v\) in a graph \(G\) is the number of edges of \(G\) which are incident to \(v\). Now, in any triangulation \(T\) of the polygon \(P\), as above, the degree of each vertex \(v\) is at least 2, because of the two edges of \(P\) which are incident to \(v\). An ear in \(T\) is a vertex of degree exactly 2, i.e. a vertex incident to zero diagonals of \(T\). Thus, an ear \(e\) belongs to exactly one face of \(T\), whose three vertices are \(\{\text{pred}(e), e, \text{succ}(e)\}\). (More generally, a vertex of degree \(d\) in \(T\) belongs to exactly \(d - 1\) faces of \(T\).) It is easy to see that every triangulation of \(P\) contains at least two ears, and that no two ears are adjacent if \(n \geq 2\).

By convention, we shall label each face \(F\) of a triangulation \(T\) by its middle vertex. In other words, if \(x < y < z \in \hat{X}\) are the three vertices of the face \(F\), we shall label \(F\) by \(y\). This labelling gives a bijection between the faces of \(T\) and the set \(X\); there are no faces labelled 0 of \(\infty\).

![Figure 1: Labelling of the faces](image_url)
2.2 Colored triangulations

Consider now a totally ordered set $\mathcal{C} = \{c_1 < \ldots < c_p\}$, which will be referred to in the sequel as the set of colors. Given a triangulation $T$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{C}^n$ a $n$-tuple of colors, we call colored triangulation the triangulation $T_\varepsilon$ obtained by replacing in $T$ each label $x_i \in X$ by the pair $(x_i, \varepsilon_i)$. The faces of $T_\varepsilon$ are colored by associating the color $\varepsilon_i$ to the face labelled $x_i$ by the previous procedure. The vertex $x_i$ and the face $T_i$ in the triangulation $T_\varepsilon$ are hence colored by the color $\varepsilon_i \in \mathcal{C}$. Note that the additional vertices labelled by 0 and $\infty$ are not colored. We write simply $T$ for the underlying triangulation associated to $T_\varepsilon$.

Denote by $T_n(\varepsilon)$ the set of triangulations of $P$ colored by $\varepsilon$. When $p = 2$, we set $\mathcal{C} = \{-, +\}$ (with $- < +$) and call signed triangulations the colored triangulations on $\mathcal{C}$. We will say that $\varepsilon$ is an increasing coloring when $\varepsilon_1 \leq \cdots \leq \varepsilon_n$.

**Definition 2.2.1** The colored triangulation $T_\varepsilon$ is called simple if it verifies the following conditions:

- $\varepsilon$ is an increasing coloring;
- there are no inner diagonals in $T_\varepsilon$ connecting two vertices with the same color;
- when two consecutive vertices $x_i$ and $x_{i+1}$ are colored with the same color, the third vertex $t_i$ of the face they define verifies $t_i < x_i$.

**Remarks.**

(i) In a simple colored triangulation, there is no face with three vertices of the same color. Similarly, if a face has two vertices colored with the same color, these vertices must be consecutive.

(ii) In a colored triangulation, the coloring of the vertices determines that of the faces. Conversely, the coloring of the faces gives the coloring of the vertices since each face of a triangulation has the color of its middle vertex. So the colored triangulation $T_\varepsilon$ is characterized by $T$ and one of these two colorings.

![Figure 2: Non simple and simple colored triangulations for $\mathcal{C} = \{a < b < c\}$](1)

The set of simple triangulations colored by $\varepsilon$ will be denoted by $\mathcal{ST}_n(\varepsilon)$. 

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2.3 Flips and restricted flips on colored triangulations

Consider $T \in \mathcal{T}_n$ a triangulation of the polygon $P = P_n$. The triangulation $T$ may be transformed into another one by a diagonal flip. The diagonal flip, or flip for short, of a diagonal $d$ in $T$ is the following operation: in the quadrilateral $Q = v_1v_2v_3v_4$ formed by the two faces of $T$ adjacent to $d = v_2v_4$, remove $d$ and replace it by the opposite diagonal $d' = v_1v_3$. The result is a new triangulation $T'$ of $P$ (see figure below). It is well known that given any triangulations $T_1$ and $T_2$ of $P$, there exists a sequence of diagonal flips transforming $T_1$ into $T_2$. The graph on $\mathcal{T}_n$ obtained by joining two triangulations which differ by exactly one flip is called the flip graph (see [1]). We will denote it by $\mathcal{F}_n$. There are many other labellings of the flip graph by objects enumerated by the Catalan numbers, such as binary trees, parenthesizations, etc.

We now introduce various types of colored flips on colored triangulations of $P$, by imposing special constraints on the two faces adjacent to the diagonal being flipped. The notion of signed flips plays a key role in our reformulation of the four color theorem.

- A signed flip in a signed triangulation is a diagonal flip such that the two faces adjacent to the flipped diagonal have equal signs, which are changed after the flip.

- A homogeneous flip in a colored triangulation is a diagonal flip such that the two faces adjacent to the flipped diagonal have the same color, which are preserved after the flip.

- A switched flip is a diagonal flip which preserves the vertex colors and such that the two faces adjacent to the flipped diagonal have different colors.

Such flips will be referred to in the sequel as restricted flips.
3 Triangulations and sylvester relations

3.1 The triangulation associated to a permutation

As above, let $X = \{x_1 < x_2 < \ldots < x_n\}$ be a subset of $\mathbb{N} \setminus \{0\}$. As usual, we denote by $S_n$ the symmetric group of rank $n$. By realizing $S_n$ as the permutation group of $X$, one can identify each $\sigma \in S_n$ with a standard word $\sigma = x_{i_1} \ldots x_{i_n}$ on $X$. We shall now define a map

$$\varphi : S_n \rightarrow \mathcal{T}_n$$

which, being surjective, will allow us to represent each triangulation of $P$ by a suitable word in $S_n$. 

Figure 3
Consider $\sigma = x_{i_1} \ldots x_{i_n}$ a permutation of $S_n$. The associated triangulation $\varphi(\sigma)$ of $P$ is defined by adding $n - 1$ noncrossing diagonals to $P$ with the following algorithm:

1. In $P^{(1)} = P$, join the two neighbors of $x_{i_1}$, i.e. $\text{pred}(x_{i_1})$ and $\text{succ}(x_{i_1})$, by a diagonal.

2. For each integer $2 \leq k \leq n - 1$, consider the polygon $P^{(k)}$ obtained from $P^{(k-1)}$ by deleting the vertex $x_{i_{k-1}}$ and the edges connected to it, then joining the two neighbors of $x_{i_k}$ in the polygon $P^{(k)}$ by a diagonal.

One easily verifies that the plane graph obtained when the procedure terminates is a triangulation $T$ of $P$.

Remark. In [8], the author uses a similar combinatorial map, denoted $\eta$, from the symmetric group to the set of triangulations. The definitions of the maps $\varphi$ and $\eta$ are quite different, but one can verify that for any $\sigma = x_{i_1} \cdots x_{i_n} \in S_n$, one has

$$\eta(\sigma) = \varphi(x_{i_n} \cdots x_{i_1}).$$

We shall now prove that each triangulation $T \in \mathcal{T}_n$ can be represented by a suitable word $\sigma \in S_n$.

Lemma 3.1.1 The map $\varphi : S_n \to \mathcal{T}_n$ is surjective.
The words in $S \in \{k\}$ on the alphabet $C$.

From the previous lemma we have $\phi$ of $T$ with the greatest label is deleted at each step. This is equivalent to say that the readings of $T$ are allowed to be repeated. Consider a word $w$ of length $n$. The lemma is trivial for $n = 1$. Assume $n \geq 2$ and the statement true for $n - 1$. Let $X = \{x_1 < x_2 < \ldots < x_n\} \subset \mathbb{N} \setminus \{0\}$ and let $T$ be a triangulation of the $(n + 2)$-gon $P$ with vertex set $\hat{X} = \{0, x_1, x_2, \ldots, x_n, \infty\}$. Since $T$ contains at least two non-adjacent ears, some vertex $x_i \in X$ must be an ear. Denote by $T'$ the triangulation obtained by cutting the ear $x_i$ in $T$, i.e. by deleting in $T$ the unique face containing the ear $x_i$. Then $T'$ is a triangulation of a convex $(n+1)$-gon $P'$ on the vertex set $\hat{X} \setminus \{x_i\}$, so that $T' \in \mathcal{T}_{n-1}$. By the induction hypothesis, $T' = \varphi(\sigma')$ where $\sigma' = x_{i_2} \ldots x_{i_n}$ is a permutation of the set $X \setminus \{x_i\}$. Denoting $\sigma = x_1 x_{i_2} \ldots x_{i_n} \in \mathcal{S}_n$, we have $\varphi(\sigma) = T$ by construction. 

We shall refer to the procedure used in the above proof as the cutting ear procedure. The words in $\mathcal{S}_n$ obtained with this procedure will be called the readings of the triangulation $T$. From the previous lemma we have $\varphi(\sigma) = T$ for any reading $\sigma$ of $T$. The canonical reading of $T$ is the reading obtained by considering the cutting ear procedure for which the vertex with the greatest label is deleted at each step. This is equivalent to say that the readings of $T$ are the words in $\varphi^{-1}(T)$ and the canonical reading of $T$ is the greatest reading for the lexicographic order (see Example 3.2.3).

### 3.2 The simple colored triangulation associated to a word

We shall now slightly generalize the preceding construction, by associating a suitably colored triangulation to a word where letters are allowed to be repeated. Consider a word $w$ of length $n$ on the alphabet $C$. The evaluation of $w$ is the $p$-uple $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{N}^p$ where for any $k \in \{1, \ldots, p\}$, $\mu_k$ is the number of occurrences of the color $c_k$ in $w$. We write for short $\text{eval}(w) = \mu$. We denote by $\mathcal{C}_{n, \mu}$ the set of words of length $n$ and evaluation $\mu$ on the alphabet $C$. The standardization of $w$ on $X$ will be denoted by $\text{std}(w)$. Recall that $\text{std}(w)$ is obtained by labelling from $x_1$ to $x_{\mu}$ the occurrences of the color $c_1$ reading from left to right, then from $x_{\mu_1+1}$ to $x_{\mu_1+\mu_2}$ the occurrences of $c_2$, and so on. (See the example below.)

For any $\sigma \in \mathcal{S}_n$ considered as a word on $X = \{x_1 < \cdots < x_n\}$ of length $n$, one associates the sequence $\Delta(\sigma) = (\delta_1, \ldots, \delta_p)$ where $\delta_1$ is the longest increasing sequence in $\sigma$ starting at $x_1$ of successive letters read from left to right, $\delta_2$ the longest increasing sequence in $\sigma$ starting at $y_1 = \max(\delta_1)$ of successive letters and so on. Set $L(\sigma) = \{\ell_1, \ldots, \ell_p\}$, where $\ell_k$ is the length of $\delta_k$ for any $k = 1, \ldots, p$, and set $S_{n}^{(\mu)} = \{\sigma \in \mathcal{S}_n \mid L(\sigma) = \mu\}$. Then, for any word $w \in \mathcal{C}_{n, \mu}$, its standardization $\text{std}(w)$ belongs to $S_{n}^{(\mu)}$, and the standardization map

$$ \text{std} : \mathcal{C}_{n, \mu} \rightarrow S_{n}^{(\mu)} $$

is a bijection. The corresponding inverse map is called the destandardization and denoted by $\text{dstd}_\mu$.

**Example 3.2.1** Suppose $X = \{1, \ldots, 8\}$ and $C = \{a, b, c, d\}$. For $w = bacbbacd$ we have

$$ \begin{aligned}
\text{eval}(w) &= (2, 3, 2, 1), \quad \text{std}(bacbbacd) = 31645278, \\
\Delta(31645278) &= (12, 345, 678), \quad L(\sigma) = (2, 3, 3).
\end{aligned} $$
In particular $\sigma = 31672485 \in S_n^{(\mu)}$ and $\text{dstd}_\mu(\sigma) = \text{baccadbl}$.

To each $n$-tuple $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{N}^p$ we associate the coloring

$$\varepsilon_\mu = (c_1, \ldots, c_1, c_2, \ldots, c_2, \ldots, c_p, \ldots, c_p) \in \mathbb{N}^n. \quad (2)$$

Our purpose now is to define a map

$$\Phi : \mathcal{C}_{n,\mu} \rightarrow \mathcal{ST}_n(\varepsilon_\mu),$$

where $\mathcal{ST}_n(\varepsilon_\mu)$ is the set of simple colored triangulations of $P$ with coloring $\varepsilon_\mu$. To the word $w \in \mathcal{C}_{n,\mu}$ we associate the colored triangulation $\Phi(w) = T_{\varepsilon_\mu}$, where $T = \varphi(\text{std}(w))$.

**Lemma 3.2.2** Let $\sigma$ be a word in $S_n$ on the alphabet $X = \{x_1 < \ldots < x_n\}$ and let $T = \varphi(\sigma)$ be the associated triangulation of the polygon $P$ with vertex set $\widehat{X} = \{0 < x_1 < \ldots < x_n < \infty\}$. For each $i = 1, \ldots, n - 1$, denote by $t_i \in \widehat{X}$ the third vertex of the unique face of $T$ containing the edge $\{x_i, x_{i+1}\}$. Then

1. The letter $x_i$ is on the left of the letter $x_{i+1}$ in $\sigma$ if and only if $t_i < x_i$.
2. If $\sigma$ contains the increasing sequence $x_i, x_{i+1}, \ldots, x_j$ with $j \geq i + 2$ in its left to right reading, then $t_{j-1} \leq t_{j-2} \leq \ldots \leq t_i < x_i$.
3. For any word $w$ on $\mathcal{C}$, $\Phi(w)$ is a simple colored triangulation.

**Proof.**

(1) Suppose that $x_i$ is on the left of the letter $x_{i+1}$ in $\sigma = y_1y_2 \cdots y_n$. Assume $x_i = y_k$. Then, when constructing $T = \varphi(\sigma)$ step-by-step, the vertex $x_i$ is an ear in the subtriangulation $T' = \varphi(y_k \cdots y_n)$, and its successor in $Y = \widehat{X} \setminus \{y_1, \ldots, y_{k-1}\}$ is $x_{i+1}$. Since the unique face of $T'$ containing $x_i$ is $\{\text{pred}_Y(x_i), x_i, x_{i+1}\}$, we must have $t_i = \text{pred}_Y(x_i) < x_i$. Conversely, assume that $x_i$ is on the right of $x_{i+1}$ in $\sigma$. Then $x_{i+1}$ is an ear in some subtriangulation $T'$ of $T$, and its predecessor in the vertex set $Y$ of $T'$ must be $x_i$. Thus $t_i$ is the successor of $x_{i+1}$ in $Y$, and therefore $x_i < x_{i+1} < t_i$. This proves assertion (1).

(2) We know by (1) that $t_i < x_i$. By reasoning inductively on $j$, it suffices to show that $t_{i+1} \leq t_i$. As above, the vertex $x_i$ is an ear in some subtriangulation $T'$ of $T$ with vertex set $Y \subset \widehat{X}$, and $t_i$ is the predecessor of $x_i$ in $Y$. Since $t_{i+1}$ is the predecessor of $x_{i+1}$ in some subset of $Y$, and since $t_{i+1} \neq x_i$ (as there is an edge $\{x_{i+2}, t_{i+1}\}$ but no edge $\{x_{i+2}, x_i\}$), we must have $t_{i+1} \leq t_i$ as claimed.

(3) Note first that by definition of $\Phi$, $\Phi(w)$ is an increasing colored triangulation. Set $\sigma = \text{std}(w)$ and $T = \varphi(\sigma)$. Consider a face $F$ of $\Phi(w)$ with two vertices colored by $c$. In $T$, $F$ has two vertices labelled by $x_r$ and $x_s$ such that $x_r < x_s$. All the letters $y \in X$ with $x_r \leq y \leq x_s$
label vertices colored by $c$ in $\Phi(w)$. Thus they correspond to letters $c$ in $w$. Since $\sigma = \text{std}(w)$, they must appear in increasing order in the left to right reading of $\sigma$. Suppose $r - s > 1$, that is $x_r x_s$ is an inner diagonal of $T$. Then $x_{r+1}$ appears in $\sigma$ on the right of $x_r$. When $\varphi(\sigma)$ is constructed, a diagonal joining the vertex labelled $x_{r+1}$ to a vertex labelled by $t < x_{r+1}$ is drawn. Since there are no intersections between the inner diagonals of $\varphi(\sigma)$, this gives a contradiction for $F$ is a face with an edge joining $x_r$ and $x_s$ and $s > r + 1$. This means that $x_s = x_{r+1}$. By (1), we obtain that the third face of $F$ is labelled by $t_r < x_r$. Thus $\Phi(w)$ is a simple colored triangulation. $lacksquare$

Consider $T_{\varepsilon \mu} \in ST_n(\varepsilon \mu)$ (see (2)). The readings of the simple colored triangulation $T_{\varepsilon \mu}$ are the words obtained by applying cutting ear procedures on $T$ and by forming the words of $C_{n, \mu}$ obtained from the successive colors of the ears deleted instead of their labels. Since the cutting ear procedure depends only of the inner diagonals of the triangulation $T$ and not on the labels or colorings of its faces, there is a one-to-one correspondence between the readings of $T_{\varepsilon \mu}$ and those of $T$. Moreover by definition of $\Phi$, the readings of $T$ coincide with the standardized of the readings of $T_{\varepsilon \mu}$. The canonical reading of $T_{\varepsilon \mu}$ is the greatest reading of $T_{\varepsilon \mu}$ for the lexicographic order. By proceeding as in Lemma 3.1.1, we prove that for any reading $w$ of $T_{\varepsilon \mu}$, we have $\Phi(w) = T_{\varepsilon \mu}$. In particular the map $\Phi$ is surjective.

**Example 3.2.3** The readings of the triangulation obtained in Figure 4 are the permutations \{235461, 253461, 523461\}. Thus the readings of the colored triangulation obtained in the above figure are the words \{bbcbca, bcbbca, cbbcbca\}.

The map $\Phi$ can alternatively be thought of as an insertion scheme for the simple colored triangulations. Indeed consider a simple colored triangulation $T_{\varepsilon}$. Suppose that the vertices
of $T$ are indexed by the integers of the set $\hat{X} = \{0 < x_1 < \cdots < x_n < \infty\}$. Let $x$ be a positive integer such that $x \notin X$ and $x_1 < x < x_n$. Consider the pair $(x, c)$ where $c \in C$. There exists a unique vertex $v$ in $T_\varepsilon$ which is not colored by $c$ (thus is colored by $b < c$ since $\varepsilon$ is an increasing coloring) and such that $v' = \text{succ}(v)$ is colored by $c$. The insertion of $(x, c)$ in $T_\varepsilon$ (denoted $(x, c) \rightarrow T_\varepsilon$ for short) is defined as follows:

- Add a vertex $v^*$ indexed by $(x, c)$ between the vertices $v$ and $v'$ in $T_\varepsilon$.
- Draw the two edges joining $v^*$ to $v$ and $v^*$ to $v'$.
- Color the face defined by $v, v^*$ and $v'$ with $c$.

**Proposition 3.2.4**

1. The output of the insertion $(x, c) \rightarrow T_\varepsilon$ is a simple colored triangulation.

2. For any word $w = c_1 \cdots c_n$ on $C_{n,\mu}$, we have

$$\Phi(w) = (x_1, c_1) \rightarrow (x_2, c_2) \rightarrow \cdots \rightarrow (x_n, c_n) \rightarrow T_0$$

where $\sigma = \text{sd}(w)$ and $T_0$ is the triangulation with one edge joining the vertices 0 and $\infty$.

**Proof.**

(1) If $F$ is a face of $(x, c) \rightarrow T_\varepsilon$ with two vertices colored by the same color $a$, $F$ is face of $T_\varepsilon$ or $F = vv^*v'$. In the first case, the third vertex of $F$ is labelled by 0 or colored by $a' < a$ since $T_\varepsilon'$ is simple. In the second case, the third vertex of $F$ is $v'$ which is labelled by 0 or colored by $b' < c$.

(2) We proceed by induction on $n$. When $n = 1$, the assertion is true since $\Phi(w)$ has only one face. Suppose that $\Box$ holds for $n - 1$ and consider $w \in C_{n,\mu}$. Write $w = t_1w$ where $w' = c_2 \cdots c_n$. Then we have by induction $\Phi(w') = (x_2, c_2) \rightarrow \cdots \rightarrow (x_n, c_n) \rightarrow T_0$. Thus it suffices to show that $(x_1, c_1) \rightarrow \Phi(\sigma') = \Phi(\sigma)$ which immediately follows from the definition of $\Phi$ and the description of the insertion algorithm given above. □

**Remark.** Binary trees and triangulations are known to be formally equivalent combinatorial objects. This analogy can be extended to binary search trees which corresponds to simple colored triangulations. The previous insertion scheme can be then regarded as an analogue of Knuth’s insertion algorithm on binary search trees [7].
3.3 The Cayley graph of $S_n$ and the flip graph $F_n$

In this paragraph we prove that the above map $\varphi : S_n \to T_n$ is in fact a surjective morphism from the Cayley graph of $S_n$, Cay$(S_n)$, to the flip graph $F_n$. Here Cay$(S_n)$ is the graph on the vertex set $S_n$, with edges all pairs $\{\sigma, \sigma'\}$ such that $\sigma, \sigma' \in S_n$ differ by an elementary transposition. Since we have identified the permutations of $X = \{x_1 < \ldots < x_n\}$ with the standard words of length $n$ with letters in $X$, this is equivalent to say that $\sigma'$ is obtained by switching two consecutive letters of $\sigma$, that is

\[
\sigma = u x z v \\
\sigma' = u z x v
\]

where $x, z$ are letters in $X$ and $u, v$ standard words on $X$. By symmetry, we can suppose $x < z$. 

Lemma 3.3.1 Let $\sigma = uxzv$ and $\sigma' = uzxv$, where as above $x < z \in X$ and $u, v$ are words on $X$.

1. Suppose that $v$ does not contain any letter $y \in X$ such that $x < y < z$. Then the associated triangulations $\varphi(\sigma)$ and $\varphi(\sigma')$ differ by a diagonal flip. In other words, the edge $\{\sigma, \sigma'\}$ in Cay($S_n$) is mapped by $\varphi$ to an edge in the flip graph $F_n$.

2. Suppose that $v$ contains a letter $y$ such that $x < y < z$. Then $\varphi(\sigma) = \varphi(\sigma')$. Here, the edge $\{\sigma, \sigma'\}$ in Cay($S_n$) is contracted by $\varphi$ to a single vertex in $F_n$.

Proof. (1) Since $\sigma$ is a standard word, the letters $y$ such that $x < y < z$, if any, belong to $u$. Set $Y = X - \{u\}$. Then $\text{pred}_{\hat{Y}}(z) = x$. Set $x' = \text{pred}_{\hat{Y}}(x)$ and $z' = \text{succ}_{\hat{Y}}(z)$. Then $\varphi(uxz)$ is obtained from $\varphi(uzx)$ by flipping the diagonal $x'z$ in the quadrilateral $x'xz'z$. Thus $\varphi(\sigma)$ and $\varphi(\sigma')$ also differ by a diagonal flip in $F_n$.

(2) This times there exists a letter $y$ such that $x < y < z$ in $Y$. Hence $\text{pred}_{\hat{Y}}(z) \neq x$. This implies immediately that $\varphi(uxz) = \varphi(uxz)$ and thus $\varphi(\sigma) = \varphi(\sigma')$. ■

Proposition 3.3.2 The map $\varphi$ is a surjective morphism of graphs from Cay($S_n$) to $F_n$.

Proof. We have already obtained that $\varphi$ is a surjective map. Now suppose that the permutation $\sigma$ and $\sigma'$ belongs to the same edge in Cay($S_n$). Then they can be written as in $\square$. The previous lemma implies that we will either have $\varphi(\sigma) = \varphi(\sigma')$ or the pair $\varphi(\sigma), \varphi(\sigma')$ differs by exactly one diagonal flip. Thus $\varphi$ is a morphism of graphs, as claimed. ■

3.4 Sylvester relations

In this section we shall determine the fibers of the morphism $\varphi : S_n \rightarrow T_n$. It turns out that they can be characterized in terms of the sylvester relations introduced by Hivert, Novelli and Thibon in the context of binary search trees $[6]$. Two words $w_1$ and $w_2$ on a totally ordered alphabet $A$ are said to be sylvester adjacent if there exist three words $u, u', u'' \in A^*$ and three letters $x \leq y < z \in A$ such that

$$w_1 = uxzuyu''$$
$$w_2 = uzxuyu''.$$ 

(5)

Two words $w$ and $w'$ are sylvester congruent if there exists a chain of words

$$w = w_1, w_2, \ldots, w_k = w'$$

such that for any $i = 1, \ldots, k-1$, the words $w_i$ and $w_{i+1}$ are sylvester adjacent. The sylvester congruence is a congruence denoted by $\equiv$ on the free monoid $A^*$ on $A$. We write $[w]$ for the sylvester class of the word $w \in A^*$.

The sylvester congruence has the following remarkable property.

---

1By analogy with the french word sylvestre which means forestal in english
Lemma 3.4.1

1. Let \( w_1 \) and \( w_2 \) two words on \( A \) such that \( w_1 \equiv w_2 \). Then we have \( \text{std}(w_1) \equiv \text{std}(w_2) \).

2. Conversely if \( w_1 \) and \( w_2 \) have the same evaluation \( \mu \) and verify \( \text{std}(w_1) \equiv \text{std}(w_2) \), we have \( w_1 \equiv w_2 \).

Proof. The statement follows immediately from the definition of the standardization map and the definition \([3]\) of sylvester adjacency.

We now turn to the description of the fibers of \( \varphi : S_n \to T_n \) in terms of the sylvester relations.

Proposition 3.4.2 Consider \( \sigma_1 \) and \( \sigma_2 \) two permutations in \( S_n \). Then

\[ \varphi(\sigma_1) = \varphi(\sigma_2) \iff \sigma_1 \equiv \sigma_2. \]

In other words, two permutations are sylvester congruent if and only if they are readings of the same triangulation.

Proof. The proposition is immediate for \( n \leq 2 \), and so we assume now \( n \geq 3 \).

Suppose that \( \sigma_1 \equiv \sigma_2 \). By induction, it is enough to prove that \( \varphi(\sigma_1) = \varphi(\sigma_2) \) when \( \sigma_1 \) and \( \sigma_2 \) are sylvester adjacent. This directly follows from \((2)\) of Lemma 3.3.1. This shows in particular that for any triangulation \( T \), the fiber \( \varphi^{-1}(T) \) is a nonempty disjoint union of sylvester classes.

To obtain the left to right part of the proposition, observe first that all the words belonging to the same sylvester class have the same last letter. By an easy induction, this implies that each sylvester class contains words \( w \) which verify the separation property

\[ w = uvy \quad (6) \]

where \( y \) is a letter and \( u, v \) are words either empty or containing letters greater than \( y \) and smaller than \( y \), respectively. Such a word \( w \) is not unique. To obtain a normal form it suffices to impose that \( u \) and \( v \) are empty or verify themselves the separation property. Denote by \( f_k \) the number of such normal forms for the permutations of \( S_k \). Then the separation property \([3]\) gives the following recurrence formula for the numbers \( f_k \):

\[ f_n = \sum_{i=1}^{n} f_{i-1} f_{n-i}. \]

Moreover we have \( f_0 = 1 \) and \( f_1 = 1 \). Thus the above recurrence formula is the recurrence formula for the Catalan numbers \( c_n \). This shows that the number of sylvester classes is equal to \( c_n \). Since \( \varphi \) is surjective, there are exactly \( c_n \) nonempty fibers \( \varphi^{-1}(T) \) which all contains at least a sylvester class. This means that each fiber is a sylvester class and immediately yields the desired implication \( \varphi(\sigma_1) = \varphi(\sigma_2) \implies \sigma_1 \equiv \sigma_2 \).

The previous result easily extends to the setting of colored triangulations.
Theorem 3.4.3 Consider \( \mu = (\mu_1, \ldots, \mu_p) \in \mathbb{N}^p \) such that \( \mu_1 + \cdots + \mu_p = n \) and \( w_1, w_2 \) two words of evaluation \( \mu \) on the alphabet \( \mathcal{C} \). Then
\[
\Phi(w_1) = \Phi(w_2) \iff w_1 \equiv w_2.
\]

Two words on \( \mathcal{C} \) are sylvester congruent if and only if they are readings of the same simple colored triangulation.

**Proof.** Since \( w_1 \) and \( w_2 \) have the same evaluation \( \mu \), we deduce from Lemma 3.4.1 and Proposition 3.4.2 the equivalencies:
\[
\Phi(w_1) = \Phi(w_2) \iff \varphi(\text{std}(w_1)) = \varphi(\text{std}(w_2)) \iff \text{std}(w_1) \equiv \text{std}(w_2) \iff w_1 \equiv w_2.
\]

Remarks.
(i) The interpretation of the sylvester congruence we use in this paper is not that originally given in [6] where the sylvester classes are defined as the fibers of the map which associated to each word the binary search tree obtained via Knuth’s insertion algorithm on binary trees. This means that we have chosen to parametrize the sylvester classes by simple colored triangulations rather than binary search trees. To obtain the binary tree corresponding to a simple signed triangulation of canonical reading \( w \), one applies the insertion algorithm on binary trees starting from \( w \). Conversely, the triangulation associated to a binary tree with right to left postfix reading \( w \), is the triangulation \( \Phi(w) \). Note that \( w \) is then the canonical reading of \( \Phi(w) \).
(ii) It is very easy to obtain the sylvester class of a word \( w \) from its associated colored triangulation \( \Phi(w) \). Indeed, \( [w] \) is simply the set of readings of \( \Phi(w) \) defined in 3.2.
(iii) For any simple triangulation \( T_e \), we know by Theorem 3.4.3 that the set \([T_e]\) of all the different readings of \( T_e \) is a plactic class. The same property holds for the set of readings \([T]\) of \( T \). Then the map \( \text{std} \) is a one-to-one correspondence between \([T_e]\) and \([T]\).

Proposition 3.4.4 Two triangulations \( T_1 \) and \( T_2 \) differ by a diagonal flip if and only if there exist a reading \( w_1 \) of \( T_1 \) and a reading \( w_2 \) of \( T_2 \) such that
\[
\begin{align*}
w_1 &= uxzv \\
w_2 &= uzxv
\end{align*}
\]
where \( x, z \) belong to \( X \) and where \( u, v \) are words such that \( v \) contains no letter \( y \in X \) satisfying \( x < y < z \) or \( z < y < x \).

**Proof.** If \( w_1 \) and \( w_2 \) are readings respectively of \( T_1 \) and \( T_2 \) verifying (7), they differ by a diagonal flip by (1) of Lemma 3.3.1 Conversely suppose that \( T_1 \) and \( T_2 \) differ by a diagonal flip. Denote by \( Q = x'xzz' \) the quadrilateral in which the diagonal flip happens.
One can suppose that $x'xzz'$ is the clockwise reading of the vertices of $Q$ and consider that
the diagonal $x'z$ is flipped in $T_1$ to give the diagonal $xz'$ in $T_2$. Consider the triangulations
$K, L$ and $M$ whose vertices are respectively indexed by the integers $k, l$ and $m$ in $\widehat{X}$ such that
\[ x' \leq k \leq x, \ x \leq l \leq z \text{ and } z \leq m \leq z'. \]
In $T_1$ and $T_2$, the vertices $x'$ and $x$ are connected to edges which are not edges of $K$. We have
the same property for the vertices $z$ and $x$ in $L$, and for the vectors $z$ and $z'$ in $M$. Thus there is at least a reading of $T_1$ and $T_2$ which starts with the vertices of $K, L, M$ distinct
of $x', x, z$ and $z'$. Choose one of these readings and denote by $u$ the word obtained. Then $u$
contains all the vertices labelled by an integer $y$ such that $x < y < z$. Moreover, we can go
on the lecture of $T_1$ by reading successively $x, z$ and next choose a reading $v$ of the remaining
triangulation. Similarly, we can go on the lecture of $T_2$ by reading successively the vertices
$z, x$ and next form the word $v$ as previously. Finally the readings we obtain for $T_1$ and $T_2$
are respectively $uxzv$ and $uzxv$ and the integers $y$ such that $x < y < z$ are in $u$. ■

Remark. This proof shows also that the letters $x$ and $z$ appearing in (7) are the labels of
the faces of the quadrilateral of $T_1$ in which the flip happens.

4 Reformulations of the four color theorem

It is well known that, considered as a statement of graph theory, the four color theorem is
equivalent to say that every simple finite planar graph admits a proper four coloring of its
vertices. In [2], the first author obtains a reformulation of this theorem in terms of signed
paths between triangulations of polygons. Let us recall briefly the main ideas which permit
this reformulation. Tutte has proved in [11] that every 4-connected finite planar graph is
hamiltonian (i.e. admits a cycle which visits each vertex exactly once). This implies that it
suffices to prove the four color theorem for hamiltonian planar triangulations of the sphere
$S^2$. For such a triangulation $\mathcal{X}$, it is natural to confine the hamiltonian path $P$ on the equator
of $S^2$. The path $P$ can be then regarded as a polygon and one defines two triangulations
$T_1$ and $T_2$ of $P$ by considering the sub-graphs of $\mathcal{X}$ lying respectively in the northern and
southern hemispheres of $S^2$. Conversely, two triangulations $T_1$ and $T_2$ of the same polygon $P$
define a triangulation $\mathcal{X} = T_1 \cup T_2$ of $S^2$ obtained by embedding them in the two hemispheres
of $S^2$ and gluing them along their common boundary $P$ confined on the equator. We denote
by $F(\mathcal{X})$ the set of faces of $\mathcal{X}$ and by $F_v$ the subset of faces incident to some vertex $v$. A
signing of $\mathcal{X}$ is a map $\varepsilon : F(\mathcal{X}) \to \{ \pm 1 \}$ which associates to each face of $\mathcal{X}$ one of the integers
$1$ or $-1$. We then denote by $\mathcal{X}_\varepsilon$ the signed triangulation obtained. Given $v$ a vertex of $\mathcal{X}_\varepsilon$
write
\[
\varepsilon_v(v) = \sum_{F \in F(\mathcal{X})} \varepsilon(F)
\]
for the sum of signs of the faces $F$ incident to $v$. The signing $\varepsilon$ is a Heawood signing if at each vertex $v$ of $\Sigma_\varepsilon$, one has
\[ s_\varepsilon(v) \equiv 0 \mod 3. \]

We will then say that $\Sigma_\varepsilon$ is a Heawood signed triangulation. At the end of the 19-th century, Heawood [5] has proved that $\Sigma$ admits a proper four coloring of its vertices if and only if there exists a Heawood signing on its faces. Now consider a triangulation $T$ of the polygon $P$. As observed in [11] there is a very simple way to obtain from $T$ a triangulation of the sphere which admits a Heawood signing. Indeed, for $\Sigma = T \cup T$, every northern face has a corresponding southern face having the same three vertices. So its suffices to define $\varepsilon$ such that these faces have opposite signs to obtain a Heawood signing. The diagonal flip operations on the triangulation $\Sigma$ are natural geometrical transformations which yields new triangulations from $\Sigma$. Two adjacent faces $F$ and $F'$ in the same plane triangulation $T_1$ or $T_2$ defining $\Sigma$ form a quadrilateral $Q$ and the bound between $F$ and $F'$ coincide with a diagonal $D$ of $Q$. The diagonal flip operation in $Q$ delete the diagonal $D$ and replace it by the opposite diagonal of $Q$. To obtain diagonal flips operations on Heawood signed triangulations (that is which preserve the Heawood property), one has to restrict the authorized diagonals flips to what we call signed flips, defined as flips for which the signs of the two faces of $Q$ are the same and are changed into their opposite during the flip operation. Consider $\Sigma_\varepsilon$ a hamiltonian planar triangulation of the sphere and denote by $T_1, \varepsilon_1$ and $T_2, \varepsilon_1$ the two signed plane triangulations such that $\Sigma_\varepsilon = T_1, \varepsilon_1 \cup T_2, \varepsilon_1$. Suppose that there exists a sequence of signed flips from $T_1, \varepsilon_1$ to $T_2, \varepsilon_1$. Since $\Sigma_{(\varepsilon_1, -\varepsilon_1)} = T_1, \varepsilon_1 \cup T_1, -\varepsilon_1$ is a Heawood signed triangulation and by using that the signed flips preserve the Heawood property, one then obtains that $\Sigma_\varepsilon$ is a Heawood triangulation of the sphere. Thus the existence of a signed flip sequence between any two triangulations of a polygon implies the four color theorem. This is the result obtained in [2]. The converse is true as proved by Gravier and Payan in [4].

### 4.1 Signed flips and the four color theorem

Consider a triangulation $T$ of a convex $(n + 2)$-gon $P$. For any $\varepsilon \in \{-, +\}^n$, denote by $T_\varepsilon$ the signed triangulation obtaining by signing $T$ following $\varepsilon$. Note that $T_\varepsilon$ is not simple in general. Let $\Sigma(T_\varepsilon)$ be the set of all signed triangulations obtained by applying a sequence of signed flips starting from $T_\varepsilon$. For any signed triangulation $U_\varepsilon$, belonging to $\Sigma(T_\varepsilon)$, we will say that there exists a signed path between $T_\varepsilon$ and $U_\varepsilon$.

**Lemma 4.1.1** Suppose that $\Sigma(T_\varepsilon)$ contains signed triangulations $U_\varepsilon'$ and $U_\varepsilon''$ with the same underlying triangulation $U$. Then $\varepsilon' = \varepsilon''$.

**Proof.** Since the signed sphere triangulation $T_\varepsilon \cup_P T_{-\varepsilon}$ has the Heawood property, and since signed flips preserve this property, it follows that the signed sphere triangulation $U_\varepsilon' \cup_P U_{-\varepsilon''}$ also has the Heawood property. We now deduce from this that $\varepsilon' = \varepsilon''$ by
induction on the number $n + 2$ of vertices. The statement is trivial for $n = 1$, as there is only one face in $U$. Assume $n \geq 2$, and let $v$ be an ear in $U$. Thus $v$ is contained in a unique face $F$ of $U$, and therefore is contained in exactly two faces of $U \cup P \cup U$, namely one copy of $F$ on each hemisphere. Since the signs of these two faces must sum up to 0 mod 3 in $U \cup P \cup U - v$, by the Heawood property, it follows that $\varepsilon'(F) = \varepsilon''(F)$. Let $Q$ denote the polygon obtained by contracting one of the two edges of $P$ containing $v$, let $V$ denote the triangulation of $Q$ obtained by cutting the ear $v$ in $U$, and let $\mu'$, respectively $\mu''$, denote the restrictions of $\varepsilon'$ and $\varepsilon''$ to the faces of $V$. Then the signed sphere triangulation $V \cup Q \cup V - \mu''$ still has the Heawood property, as easily seen. It follows by the induction hypothesis that $\mu' = \mu''$. Therefore, $\varepsilon' = \varepsilon''$ as claimed.

Remark. By the previous lemma, there exists a signed path between the signed triangulations $T_\varepsilon$ and $U_\varepsilon$ only if there exists a path without loop between their underlying triangulations $T$ and $U$ in $F_n$.

Write $|\Sigma(T_\varepsilon)|$ for the set of triangulations obtained by deleting the signs $-$ and $+$ in the signed triangulations of $\Sigma(T_\varepsilon)$. We deduce from [2] and [3] the following reformulation of the four color theorem.

**Theorem 4.1.2** The four color theorem is equivalent to the following statement. For any triangulation $T \in T_n$, we have

$$\bigcup_{\varepsilon \in \{-, +\}^n} |\Sigma(T_\varepsilon)| = T_n.$$  

(8)

In other words, for any pair $T, T'$ of triangulations in $T_n$, there exist $\varepsilon, \varepsilon' \in \{-, +\}^n$ and a sequence of signed flips from $T_\varepsilon$ to $T'_{\varepsilon'}$.

### 4.2 Signed permutations and the four color theorem

Representing triangulations by permutations via the map $\varphi : S_n \to T_n$, and using the preceding theorem, we obtain in this section a reformulation of the four color theorem using now signed permutations.

Consider the alphabet $I = \{\bar{\pi}, \ldots, \bar{1}, 1, \ldots, n\}$ and set $I_- = \{\bar{\pi}, \ldots, \bar{1}\}, I_+ = \{1, \ldots, n\}$. The letters of $I_-$ (resp. $I_+$) are said negative (resp. positive). Define the bar involution on the letters $\beta$ of $I_n$ by

$$\bar{\beta} = k \text{ if } \beta = \bar{k} \in I_- \text{ and } \bar{\beta} = \bar{k} \text{ if } \beta = k \in I_+.$$  

Write $|\beta| = \beta$ if $\beta \in I_+$ and $|\beta| = \bar{\beta}$ if $\beta \in I_-$. For any word $w = \beta_1 \cdots \beta_n$ on $I_n$, set $|w| = |\beta_1| \cdots |\beta_n|$. The set of signed permutations on $I_n$ is defined by

$$SP(n) = \{w = \beta_1 \cdots \beta_n \mid |\beta_i| \neq |\beta_j| \text{ for any } i \neq j\}.$$  

Two words $w_1$ and $w_2$ of $SP(n)$ differ by an authorized transposition if one of the two situations happens:
1. \(|w_1|\) and \(|w_2|\) are sylvester adjacent, that is there exist letters \(\alpha, \beta, \gamma\) such that \(|\alpha| < |\beta| < |\gamma|\) and

\[
\begin{align*}
w_1 &= u \alpha \gamma \nu \beta w \\
w_2 &= u \gamma \alpha \nu \beta w
\end{align*}
\]  
(9)

where \(u, v, w\) are factors of \(w_1\).

2. There exist letters \(\alpha, \gamma\) with the same sign such that

\[
\begin{align*}
w_1 &= u \alpha \gamma v \\
w_2 &= u \gamma \alpha v
\end{align*}
\]  
(10)

where \(u, v\) are factors of \(w_1\) and \(v\) does not contain any letter \(\beta\) such that \(|\alpha| < |\beta| < |\gamma|\) (i.e. \(|w_1|\) and \(|w_2|\) are not sylvester adjacent).

Consider \(\sigma\) and \(\sigma'\) two permutations of \(S_n\). We will say that there exists a signed path between \(\sigma\) and \(\sigma'\) if one can find two words \(w, w'\) in \(SP(n)\) such that \(|w| = \sigma, |w'| = \sigma'\) and \(w'\) can be obtained from \(w\) by applying successive authorized transpositions. From Proposition 3.4.2 and Theorem 4.1.2 we deduce the following reformulation of the four color theorem:

**Theorem 4.2.1** The four color theorem is equivalent to the following statement:
For any positive integer \(n\), there exists at least a signable path joining two permutations of \(S_n\).

**Proof.** To each signed triangulation \(T_\varepsilon\) with \(\varepsilon \in \{-, +\}^n\), we associate the set of signed permutations

\[
[T_\varepsilon]_s = \{w = \beta_1 \cdots \beta_n \mid |w| \in [T] \text{ and for each } \Diamond \in \{-, +\}, \beta_i \in I_\Diamond \iff \varepsilon_i = \Diamond\}
\]

that is, \([T_\varepsilon]_s\) is the set of signed permutations obtained by signing the words contained in the sylvester class of \(T\) so that the sign associated to each \(x_i \in X\) is equal to \(\varepsilon_i\). By Proposition 3.4.2, two signed permutations \(w'\) and \(w\) belong to \([T_\varepsilon]_s\) if and only if they differ by successive transpositions of kind \((9)\).

Given two triangulations \(T\) and \(T'\) in \(T_n\), we said that there exists a signable flip path between \(T\) and \(T'\) if \(T'\) belongs to \(\bigcup_{\varepsilon \in \{-, +\}^n} |\Sigma(T_\varepsilon)|\) (with the notation of \(4.1\)). We deduce from Proposition 3.4.4 that the signed triangulations \(T_\varepsilon\) and \(T'_\varepsilon\) differ by a signed flip if and only if there exist a signed word in \([T_\varepsilon]_s\) and a signed word in \([T'_\varepsilon]_s\) which differ by a transposition of kind \((10)\). Thus, \(T_\varepsilon\) and \(T'_\varepsilon\) differ by a signed flip if and only if each signed word of \([T'_\varepsilon]_s\) can be obtained by applying successive transpositions of kind \((9)\) or \((10)\) from any signed word of \([T_\varepsilon]_s\). By transitivity, we deduce that the existence of a signable flip path between \(T\) and \(T'\) is equivalent to that of a signable path between each reading of \(T\) and each reading of \(T'\).

Now, by Theorem 4.1.2 the four color theorem is equivalent to the statement:
For any positive integer $n$, there exists at least a signable flip path joining two triangulations of $\mathcal{T}_n$. By the previous argument, this implies our theorem.

**Example.** Consider

$$\sigma_1 = 324156,$$
$$\sigma_2 = 453126.$$ 

Here is a signed path between suitable signings of $\sigma_1$ and $\sigma_2$, where $\rightarrow$ indicates a signed flip and $\equiv$ a sylvester adjacency:

$$324156 \rightarrow 324516 \equiv 352416 \rightarrow 354216 \equiv 534216 \rightarrow 543216 \rightarrow 453126.$$ 

This signed path produces an explicit Heawood signing, and hence an explicit proper four vertex-coloring, of the sphere triangulation obtained by gluing the octogon triangulations $\varphi(324156)$ and $\varphi(453126)$ along their boundary.

**Remark.** There exists a simple procedure deciding whether a given path $(\sigma_1, \ldots, \sigma_r)$ in $\text{Cay}(S_n)$ is signable. The path $(\sigma_1, \ldots, \sigma_r)$ is signable if one can compute a sequence $w_2, \ldots, w_r$ of signed permutations such that $|w_i| = \sigma_i$ for any $i = 2, \ldots, r$ by the following procedure. First, the permutations $\sigma_1$ and $\sigma_2$ differ by a transposition, thus can be written

$$\sigma_1 = u a c v, \quad \sigma_2 = u c a v.$$ 

where $a, c \in \{1, \ldots, n\}$. If $\sigma_1$ and $\sigma_2$ differ by a sylvester relation, set $w_2 = \sigma_2$. Otherwise set $w_2 = u \overline{c} \overline{a} v$. Now suppose we have obtained the signed permutations $w_2, \ldots, w_i$ from $\sigma_2, \ldots, \sigma_i$. Since $\sigma_i$ and $\sigma_{i+1}$ differ by a transposition we have

$$w_i = U_i \alpha_i \gamma_i V_i, \quad \sigma_{i+1} = u_i c_i a_i v_i,$$

where $|\alpha_i| = a_i, |\gamma_i| = c_i$ and $|U_i| = u_i, |V_i| = v_i$. If $\sigma_i$ and $\sigma_{i+1}$ differ by a sylvester relation, set $w_{i+1} = w_i$. Otherwise two situations can happen.

- When $\alpha_i$ and $\gamma_i$ have opposite signs the algorithm stops and the path is not signable.
- When $\alpha_i$ and $\gamma_i$ have the same sign, set $w_{i+1} = U_i \overline{\gamma_i} \overline{\alpha_i} V_i$.

**Further remarks.**

(i) To decide if there exists a signed path between the two permutations $\sigma$ and $\sigma'$, it is sufficient by Lemma 4.1.1 to test the paths joining these vertices in $\text{Cay}(S_n)$ by restricting to the paths with no loop.

(ii) If $(\sigma_1, \ldots, \sigma_r)$ is signable in $\text{Cay}(S_n)$, then $(T_1, \ldots, T_r)$ where for any $i = 1, \ldots, r$, $T_i = [\sigma_i]$ is a signable path in $\mathcal{F}_n$.  

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4.3 Diagonal signings

We are going to describe an alternative formulation of this theorem using signings of the diagonals in the triangulations, rather than of the faces, to be used later in subsection 4.4.

Consider a signed triangulation $T_\varepsilon$ and $d$ a diagonal of $T$. Let $\varepsilon_i$ and $\varepsilon_j$ be the signs of the two faces of $T_\varepsilon$ adjacent to $d$. Then label the diagonal $d$ by the product $\varepsilon_i \varepsilon_j \in \{-, +\}$. By proceeding similarly for all the diagonals of $T$, we obtain a triangulation with signed diagonals $T_\delta$. Note that the signing $T_\delta$ of the diagonals of $T$ determines the signing of its faces up to an overall inversion of signs. Indeed, the signing of the faces of $T$ can be obtained from the sign of one face and the signs of the diagonals of $T$.

Suppose that the diagonal $d$ in $T$ is flipped in $d'$ and write $Q$ for the quadrilateral whose diagonals are $d$ and $d'$. The signed flip of the diagonal $d$ is defined in $T_\delta$ when the sign attached to $d$ is $+$. In this case, this yields a triangulation with signed diagonals defined by the three followings requirements:

1. $d'$ is signed by $+$
2. the signs of the diagonals of $T_\delta$ which are edges of $Q$ are changed into their opposite
3. the signs of the remaining diagonals are unchanged.

Note that the signed flip is not defined when the sign of $d$ is $-$. Denote by $\Sigma(T_\delta)$ the subgraph of the flip graph generated from $T_\delta$ by applying signed flips and write $|\Sigma(T_\delta)|$ for the set of triangulations obtained by deleting the signs $-$ and $+$ in the triangulations with signed diagonals belonging to $\Sigma(T_\delta)$. By definition of the signed flips on triangulations with signed diagonals, we have $|\Sigma(T_\varepsilon)| = |\Sigma(T_\delta)|$ and Theorem 4.1.2 is equivalent to the assertion: For any triangulation $T \in T_n$,

$$\bigcup_{\delta \in \{-, +\}^{n-1}} |\Sigma(T_\delta)| = T_n$$

where $\delta$ yields a signing $T_\delta$ of the $n - 1$ diagonals of $T$.

We suppose in the sequel that we have chosen a labelling of the $\frac{n(n+1)}{2}$ diagonals of $P$ (for example we can consider the labelling by roots belonging to the root system of type $A_{n-1}$ depicted in [3]). Consider a path $\mathcal{P} = (T_1, \ldots, T_r)$ in $F_n$ where for $i = 1, \ldots, r - 1$, the triangulations $T_i$ and $T_{i+1}$ differs by a diagonal flip. The path $\mathcal{P}$ is said signable if for each $i \in \{1, \ldots, r\}$, there exists a signing $T_{\delta(i)}$ of the diagonals in $T_i$ such that the transformation $T_{\delta(i)} \rightarrow T_{\delta(i+1)}$ is a signed flip (that is, Conditions 1, 2 and 3 above are verified). In fact it is rather easy to determinate if a path is signable or not. For any $i = 2, \ldots, r$, write $N_i$ for the set of diagonals in $T_i$ which has appeared after a flip $T_k \rightarrow T_{k+1}$, $1 \leq k \leq i - 1$. This means that $N_i$ is the set of diagonals of $T_i$ which result of a flip operation at one step of the path $T_1, \ldots, T_i$. 

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Consider first the flip $T_1 \rightarrow T_2$ of the diagonal $d$ into $d'$. We have $N_2 = \{d'\}$ and the sign of $d'$ in $T_2$ must be $+$. Now suppose by induction that we have determined the signs of the diagonals of $N_i \ i \geq 2$ so that at each step $T_k \rightarrow T_{k+1}, \ 1 \leq k \leq i - 1$, the signs obtained are compatible with a signed flip (that is with Conditions 1, 2 and 3 above). Let $d_i$ be the diagonal flipped in $T_i$ and $d'_i$ the new diagonal obtained in $T_{i+1}$. When $d_i$ has sign $-$, the path is not signable. When $d_i$ is not signed or is signed by $+$, we have $N_{i+1} = N_i + \{d'_i\} - \{d_i\}$. In $T_{i+1}$, we sign $d'_i$ with $+$ and we change the signs of the edges of the quadrilateral associated to $d_i$ and $d'_i$ which belong to $N_i$.

The path $P$ is signable if the previous algorithm does not stop until the last flip $T_{r-1} \rightarrow T_r$ has been considered. In this case, it becomes immediate to complete in each $T_i$ the signs of the diagonals which are not in $N_i$. Indeed, its suffices to sign arbitrary the unsigned diagonals of $T_r$. The complete signings in $T_1, \ldots, T_{r-1}$ are then determined by conditions 1, 2 and 3 above.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Signable and nonsignable paths in $F_n$}
\end{figure}

Given a path $P = (T_1, \ldots, T_r)$, we denote by $D_P$ the subset of diagonals on $P$ which appear in the triangulations $T_1, \ldots, T_r$. We endow $D_P$ with the structure of an oriented graph by drawing an edge $d \rightarrow d'$ between the diagonals $d$ and $d'$ if there exists $i \in \{1, \ldots, r-1\}$ such that $T_{i+1}$ is obtained by flipping $d \in T_i$ into $d'$.

\begin{itemize}
\item[(i)] If $P =(T_1, \ldots, T_r)$ is a signable path, the paths $P =(T_p, \ldots, T_q), \ 2 \leq p < q \leq r$ are also signable.
\item[(ii)] If all the chains in $D_P$ have length less than or equal to 2 (i.e. each diagonal is flipped at most on time), the path $P =(T_1, \ldots, T_r)$ is signable. Indeed $r = 1$ or for any $i \in \{2, \ldots, r\}$, the diagonal $d_i$ in the above procedure is never signed.
\item[(iii)] Suppose that $T$ and $U$ are two triangulations of $F_n$. By Lemma 4.1.1 the problem of determining whether there exists a signable diagonal path between $T$ and $U$ can be solved
\end{itemize}
by considering only paths without loop between $T$ and $U$ in $\mathcal{F}_n$. There is a finite number of such paths and one can apply to each of them the previous procedure.

> From the above arguments and Theorem 4.1.2 we obtain the following reformulation of the four color theorem:

**Corollary 4.3.1 (of Theorem 4.1.2)**
The four color theorem is equivalent to the following statement:
For any positive integer $n$, there exists at least a signable path between two triangulations of the $(n + 2)$-gon.

### 4.4 Signed walks on the associahedron

The flip graph $\mathcal{F}_n$ is the 1-skeleton of a convex polytope called the $n$-dimensional associahedron $\mathfrak{A}_n$ (see [3]). The vertices of the associahedron can be identified with the triangulations of the $(n + 2)$-gon $P$ and its facets with the $\frac{n(n+1)}{2}$ diagonals of $P$. Its edges correspond to partial triangulations of $P$. Given a star $S$, write $v_S$ for the vertex of maximal order in $S$.

This yields a natural one-to-one correspondence between the stars and the vertices of $P$. The vertices of $\mathfrak{A}_n$ belonging to the facet $F_d$ corresponding to the diagonal $d$ coincide with the triangulations which contains $d$. This implies that each facet $F_d$ contains exactly two stars. More precisely we have:

**Lemma 4.4.1** Let $S_1$ and $S_2$ be two stars considered as vertices of $\mathfrak{A}_n$. Then we have the following equivalences:

1. $S_1$ and $S_2$ belong to the same facet $F_d$ if and only if $v_{S_1}v_{S_2} = d$.

2. $S_1$ and $S_2$ do not belong to the same facet $F_d$ if and only if $v_{S_1}v_{S_2}$ are two consecutive vertices of $P$ (that is form an edge of $P$).

Consider a triangulation $T$ and a diagonal $d$ in $T$. Denote by $T'$ the triangulation obtained by flipping $d$ in $T$ and by $d'$ the diagonal of $T'$ such that $T' = T - \{d\} + \{d'\}$. The diagonal flip $T \rightarrow T'$ can be interpreted as a move from $F_d$ to $F_{d'}$ in $\mathfrak{A}_n$ along the edge defined by the partial triangulation $T - \{d,d'\}$. In this case, the facets $F_d$ and $F_{d'}$ are disjoint. Indeed the diagonals $d$ and $d'$ are secant in $P$, thus one cannot find a triangulation $T$ belonging to $F_d$ and $F_{d'}$.

An edge $E$ of $\mathfrak{A}_n$ is contained in $n - 2$ facets. Suppose that $E = TT'$ where $T$ and $T'$ are triangulations belonging respectively to $F_d$ and $F_{d'}$. Then $T'$ is obtained from $T$ by flipping $d$ into $d'$. Denote by $Q$ the quadrilateral in $P$ whose diagonals are $d$ and $d'$. The edges of $Q$ are either diagonals either edges of $P$. Moreover, for $n \geq 2$, the number of edges of $Q$ which are diagonals of $P$ belongs to $\{1, 2, 3, 4\}$. To distinguish the facets corresponding to diagonals of $P$ among the facets containing $E$, it suffices to consider the four stars $S_1, S_2$ and $S_3, S_4$.
which appear respectively in $F_d$ and $F_{d'}$. Then the facets corresponding to diagonals of $P$ are the diagonals which can be obtained by connecting two vertices $S_i S_j$, $i \neq j$. By Lemma 4.4.1 this is equivalent to find the pairs of vertices $(S_i, S_j)$ in $\mathfrak{A}_n$ which belongs to the same facet. The pairs $(S_1, S_2)$ and $(S_3, S_4)$ correspond respectively to $d$ and $d'$. In the sequel we will denote by $\Delta_E$ the set of facets corresponding to the edges of $Q$ which are diagonals of $P$ distinct of $d$ and $d'$. By the previous arguments $\Delta_E$ coincide with the facets containing a pair $(S_i, S_j)$ distinct of $(S_1, S_2)$ and $(S_3, S_4)$ where $S_1, S_2, S_3, S_4$ are the stars appearing in $F_d$ and $F_{d'}$ the two facets of $\mathfrak{A}_n$ connected by $E$.

Our aim is now to obtain a reformulation of the four color theorem using only the geometry of the associahedron and the distinguish subset $\mathfrak{S}_n$ of its vertices which correspond to stars. A walk $W$ of length $r$ on $\mathfrak{A}_n$ is determined by $r$ vertices $V_1, \ldots, V_r$ such that for any $i = 1, \ldots, r - 1$, $V_i V_{i+1}$ is an edge of $\mathfrak{A}_n$. In this case write $W = (V_1, \ldots, V_r)$. For each edge $V_i V_{i+1}$, denote by $(F_i, F'_i)$ the unique pair of facets in $\mathfrak{A}_n$ such that $V_i \in F_i$, $V_{i+1} \in F'_i$ and $F_i \cap F'_i = \emptyset$. Set $M_{i+1} = \{F'_1, \ldots, F'_i\}$, $i = 1, \ldots, r - 1$. Note that $F'_i \neq F_{i+1}$ in general (see example below).

The walk $W$ is signable if the facets of $M_r$ can be signed by the following recursive procedure. First sign the face $F'_1$ with $+$. Suppose that the facets of $M_i$ are signed. If $F_i \in M_i$ is signed by a $-$, the algorithm stops. Otherwise, consider the edge $V_i V_{i+1}$ and the set of facets $\Delta_{i+1} = \Delta_{V_i V_{i+1}}$. Then sign the face $F'_i$ with a $+$ and change the sign of the facets of $\Delta_{i+1} \cap M_i$.

From Corollary 4.3.1 and the above arguments, we derive the following reformulation of the four color theorem in terms of the geometry of the associahedron:

**Theorem 4.4.2** The four color theorem is equivalent to the following statement:
For any positive integer $n$, there is a least a signable walk between two vertices of the associahedron $\mathfrak{A}_n$.

**Remarks.**
(i) By Lemma 4.1.1 the problem of finding a signed walk between two vertices $V$ and $V'$ of the associahedron can be solved by applying the previous procedure to the walks joining $V$ to $V'$ in which each vertex is attained at more one time (that is by excluding the walks with loops).
(ii) At each step of the above procedure, the set $\Delta_{i+1} = \Delta_{V_i V_{i+1}}$ is determined only by the vertices of $F_i$ and $F_{i+1}$ which belong to $\mathfrak{S}_n$.

**Example 4.4.3** The signable path of Figure 7 is equivalent to the signed walk on $\mathfrak{A}_4$ given
Figure 8

where the solid (resp. dashed) signs belong to apparent (resp. non apparent) facets.

5 Flip graph generated by a colored triangulation

5.1 A combinatorial problem
The set $\mathcal{T}_n$ can be identified with the set of simple colored triangulations $\mathcal{ST}_n(\varepsilon)$ with $\varepsilon = (1, \ldots, 1) \in \mathbb{N}^n$. The flip graph can be generated starting from any triangulation, by applying diagonal flips. This yields to the following natural problem:

**Problem 5.1.1** What is the graph generated from a colored triangulation by applying successive restrictive flips?
5.2 Homogeneous flips case

Consider a colored triangulation $T_\varepsilon$ and denote by $\mathcal{H}_F(T_\varepsilon)$ the subgraph of the flip graph generated from $T_\varepsilon$ by applying homogeneous flips. The coloring $\varepsilon$ defines subtriangulations in $T_\varepsilon$ obtained by gluing together the adjacent faces which have the same color. These subtriangulations will be called the connected components of $T_\varepsilon$. Denote them by $T_1, \ldots, T_r$. For any $i = 1, \ldots, r$, let $\nu_i$ be the number of faces in $T_i$. Since the homogeneous flips stabilize the connected components $T_i$, we obtain:

**Proposition 5.2.1** The subgraph $\mathcal{H}_F(T_\varepsilon)$ is isomorphic to the direct product of flip graphs $\mathcal{F}_{\nu_1} \times \cdots \times \mathcal{F}_{\nu_r}$.

The above isomorphism can be explicitized by associating to each colored triangulation of $\mathcal{H}_F(T_\varepsilon)$ the $r$-uple of triangulations defined from its connected components $T_1, \ldots, T_r$ as pictured in the figure below. This answers Problem 5.1.1.

![Figure 9](image)

5.3 Switched flips case

Consider $\mu \in \mathbb{N}^d$ and $T_{\varepsilon, \mu}$ a simple colored triangulation of $\mathcal{T}_n(\varepsilon_\mu)$. Denote by $\mathcal{S}_F(T_\varepsilon)$ the subgraph of the flip graph generated from $T_\varepsilon$ by applying switched flips. Write $\mathcal{S}_F(T_\varepsilon)$ for
the graph obtained by drawing an edge between two simple triangulations of \( \mathcal{F}_n(\varepsilon_\mu) \) when they differ by a switched flip. We are going to show that \( \mathcal{SF}_n(\varepsilon_\mu) \) is connected this will imply the equality \( \mathcal{SF}(\varepsilon_\mu) = \mathcal{SF}(T_\varepsilon) \) for any simple colored triangulation such that \( \varepsilon = \varepsilon_\mu \).

Let \( S_\mu \) be the Frobenius subgroup of \( S_n \) defined by \( \mu \), that is the subgroup of permutations which stabilize the intervals \( I_{\mu_1} = \{1, \ldots, \mu_1\} \) and \( I_{\mu_s} = \{\mu_{s-1} + 1, \ldots, \mu_s\} \) for \( s = 2, \ldots, d \). The elements of the coset \( S_n/S_\mu \) will be identified with the words of length \( n \) and evaluation \( \mu \) on the totally ordered alphabet \( \mathcal{C} = \{c_1 < \cdots < c_d\} \), that is we set \( S_n/S_\mu = \mathcal{C}_{n,\mu} \).

Denote by \( \text{Cay}(\mathcal{C}_{n,\mu}) \) the Cayley graph of \( \mathcal{C}_{n,\mu} \). This means that the vertices of \( \mathcal{C}_{n,\mu} \) are the words of length \( n \) and evaluation \( \mu \) and there is an edge between \( w \) and \( w' \) if and only if \( w' \) is obtained by switching two adjacent letters of \( w \). Denote by \( \text{STD} \) the standardization map on simple colored triangulations defined by \( \text{STD}(T_{\varepsilon_\mu}) = T \). From the definition of the standardization map and since switched flips are particular cases of flips, we have:

**Lemma 5.3.1**

1. The standardization map \( \text{std} \) on words is an injective morphism of graphs from \( \text{Cay}(\mathcal{C}_{n,\mu}) \) to \( \text{Cay}(S_n) \).

2. The standardization map \( \text{STD} \) is an injective morphism of graphs from \( \mathcal{SF}_n(\varepsilon_\mu) \) to \( \mathcal{F}_n \).

**Proposition 5.3.2** Let \( T_{\varepsilon_\mu} \) and \( T'_{\varepsilon_\mu} \) two triangulations in \( \mathcal{SF}_n(\varepsilon_\mu) \). Then \( T_{\varepsilon_\mu} \) and \( T'_{\varepsilon_\mu} \) differ by a switched flip if and only if there exist a reading \( w \) of \( T_{\varepsilon_\mu} \) and a reading \( w' \) of \( T'_{\varepsilon_\mu} \) of the form

\[
\begin{align*}
w &= u x z v \\
w' &= u z x v
\end{align*}
\]

(11)

where \( x, z \) are letters of \( \mathcal{C} \) and \( u, v \) words on \( \mathcal{C} \) such that \( v \) does not contain any letter \( y \) verifying \( x \leq y < z \) or \( z \leq y < x \).

**Proof.** By symmetry we only consider the case \( x \leq y < z \). Suppose that \( w \) and \( w' \) are readings of \( T_{\varepsilon_\mu} \) and \( T'_{\varepsilon_\mu} \) verifying \( w = u x z v \) and \( w' = u z x v \) as in the theorem. Then \( \text{std}(w) = u_x x_z x_v \) and \( \text{std}(w') = u_z z_x z_v \) where \( x_x, z_z \) are letters of \( X \) and \( u_x, v_z \) words on \( X \) such that \( v_z \) does not contain any letter \( y \) verifying \( x < y < z \). Thus by applying 1 of Proposition 3.4.4, \( T \) and \( T' \) differs by a diagonal flip and the faces corresponding to its flip are labelled by \( x_x \) and \( z_z \). This implies that \( T_{\varepsilon_\mu} \) and \( T'_{\varepsilon_\mu} \) differ by a flip corresponding to faces colored by \( x \) and \( z \), hence by a switched flip.

Conversely, suppose that \( T_{\varepsilon_\mu} \) and \( T'_{\varepsilon_\mu} \) differ by a switched flip. Then \( T \) and \( T' \) differ by a flip and by 2 of Proposition 3.4.4 we have readings \( \sigma = u_x x_z x_v \) and \( \sigma' = u_x z_x z_v \) respectively of \( T \) and \( T' \) where \( x_x, z_z \) are letters of \( X \) and \( u_x, v_z \) words on \( X \) such that \( v_z \) does not contain any letter \( y \) verifying \( x < y < z \). Hence by applying the destandardization procedure (see 4.4) to \( \sigma \) and \( \sigma' \), one can find readings \( w = u x z v \) and \( w' = u z x v \) respectively of \( T_{\varepsilon_\mu} \) and \( T'_{\varepsilon_\mu} \) with \( x \leq y < z \).
Theorem 5.3.3

1. The following diagram commutes:

\[
\begin{array}{cc}
\text{Cay}(C_{n,\mu}) \rightarrow & \text{Cay}(S_n) \\
\Phi \downarrow & \downarrow \varphi \\
\mathcal{S}F_n(\varepsilon_{\mu}) & \rightarrow \\
\leftarrow \text{STD} & \mathcal{F}_n
\end{array}
\]

where \(\varphi, \Phi, \text{std} \) and \(\text{STD}\) are morphisms of graphs.

2. \(\mathcal{S}F_n(\varepsilon_{\mu})\) is connected.

Proof.

1 : We have already seen that \(\Phi\) is a surjective map. Suppose that \(w\) and \(w'\) are words in \(\text{Cay}(C_{n,\mu})\) which differ by the transposition of two consecutive letters. If \(w\) and \(w'\) are sylvester adjacent, \(\text{std}(w)\) and \(\text{std}(w')\) are also sylvester adjacent, thus \(\varphi(\text{std}(w)) = \varphi(\text{std}(w'))\). This implies that \(\Phi(w) = \Phi(w')\) since \(w\) and \(w'\) have the same evaluation. If \(w\) and \(w'\) are not sylvester adjacent, they verify and by Proposition 5.3.2 we know that \(\Phi(w)\) and \(\Phi(w')\) differ by a switched flip in \(\mathcal{S}F_n(\varepsilon_{\mu})\). This proves that \(\Phi\) is a morphism of graphs. Now for any word \(w\) in \(\text{Cay}(C_{n,\mu})\), \(\text{STD} \circ \Phi(w) = \varphi \circ \text{std}(w)\) by definition of the map \(\Phi\). Thus by Lemma 5.3.1 the above diagram is a commuting diagram of morphisms.

2 : Since \(\text{Cay}(C_{n,\mu})\) is connected and \(\Phi\) is a surjective morphism of graphs, we obtain immediately that \(\mathcal{S}F_n(\varepsilon_{\mu})\) is connected. ■

Remark. The theorem implies in particular that \(\mathcal{S}F(T_\varepsilon) = \mathcal{S}F_n(\varepsilon_{\mu})\) for any simple colored triangulation \(T_\varepsilon \in \mathcal{T}_n(\varepsilon_{\mu})\). Thus it answers to Problem 5.1.1 when the colored triangulation \(T_\varepsilon\) is simple. When \(T_\varepsilon\) is not simple, we have find no algebraic interpretation of the graph \(\mathcal{S}F(T_\varepsilon)\).

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