Corrugation crack front waves

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Abstract

The paper presents a model of a dynamic crack with a wavy surface. So far, theoretical analysis of crack front waves has been performed only for in-plane perturbations of the crack front. In the present paper, generalisation is given to a more general three-dimensional perturbation, and equations that govern corrugation crack front waves are derived and analysed.

Keywords: Dynamic fracture, crack front waves, asymptotic analysis.

1 Introduction

The paper analyses singular fields around a dynamic crack whose surface is slightly perturbed from the original plane configuration. Crack front waves in the plane of the crack were discovered numerically by Morrissey and Rice in \cite{1}, and later confirmed analytically by Ramanathan and Fisher \cite{2}, using the results of the perturbation analysis of Willis and Movchan \cite{3}. Experimental observations of persistent crack front waves were reported by Sharon, Cohen and Fineberg \cite{4}. The more general development of Willis and Movchan \cite{5} and Woolfries \textit{et al.} \cite{6} extended the analysis to a crack propagating through a viscoelastic medium. The perturbation formulae for the stress intensity factors, specialised to a plane strain formulation, have been used by Obrezanova \textit{et al.} \cite{7} in the stability analysis of rectilinear propagation. A quasi-static advance of a tunnel crack under a mixed mode loading has been analysed by Lazarus and Leblond \cite{8}.

The aim of the present paper is to develop a model describing corrugation (out-of-plane) waves along the front of a moving crack. This work is based
on the ideas of the earlier publication by Willis [9]. The plan of the paper
is as follows. We begin, in Section 2, with the description of the geometry,
governing equations and perturbation functions. A summary of the first-
order approximations for the stress intensity factors is presented in Section
2.2. Section 3 includes the study of the corrugation waves in the first-order
asymptotic approximation for a basic Mode I loading. In Section 4, we
derive the dispersion equation for crack front waves in the mixed mode I-
III loading. The technical appendix contains an outline of the fundamental
integral identity, and the expressions for effective tractions.

2 Basic perturbation formulae

For a linearly elastic medium, we consider a semi-infinite crack with a slightly
perturbed surface. The unperturbed configuration of the crack at time $t$ is
defined by

$$S_0(t) = \{ x : -\infty < x_1 < Vt, -\infty < x_2 < \infty, x_3 = 0 \},$$  \hspace{1cm} (1)

where $V$ is a constant crack speed, which does not exceed the Rayleigh wave
speed. The perturbation is introduced through deviations of the crack front
in both in-plane and out-of-plane directions. The perturbed surface of the

crack at time $t$ is

$$S_\varepsilon(t) = \{ x : -\infty < x_1 < Vt + \varepsilon \varphi (x_2, t),
-\infty < x_2 < \infty, x_3 = \varepsilon \psi(x_1 - Vt, x_2) \}. \hspace{1cm} (2)$$

The functions $\varphi$ and $\psi$ are smooth and bounded, and $\varepsilon$ is a small non-
dimensional parameter, $0 \leq \varepsilon \ll 1$. It is helpful to use the moving-frame
coordinates, so that $X = x_1 - Vt$.

It is assumed that the medium is loaded so that a stress $\sigma^{nc}$ and a
displacement $u^{nc}$ would be generated in the absence of the crack. The crack
induces additional fields $\sigma$, $u$. They satisfy the equations of motion and the
traction boundary conditions on the crack faces:

$$\sigma_{ij,j} - \rho \ddot{u}_i = 0, \ i = 1, 2, 3, \ \text{outside the crack} \hspace{1cm} (3)$$

and

$$\sigma_{ij} n_j + \sigma_{ij}^{nc} n_j = 0, \ \text{on the crack faces}, \hspace{1cm} (4)$$

and correspond to waves outgoing from the crack as $x_3 \to \pm \infty$. 

2
2.1 Local coordinates and asymptotics for stresses

At a point \( \mathbf{x}^0 = (x_1^0, x_2^0, x_3^0) \), which is on the crack edge at time \( t \), so that
\[
x_1^0 = Vt + \varepsilon \varphi(x_2^0, t), \quad x_3^0 = \varepsilon \psi(x_1^0 - Vt, x_2^0),
\]
we define a coordinate system such that
\[
\mathbf{x} - \mathbf{x}^0 = \sum_{i=1}^{3} x_i' \mathbf{e}_i',
\]
where
\[
\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = \left\{ \mathbf{I} + \varepsilon \begin{pmatrix} 0 & -\varphi_2 & \psi_1^* \\ \varphi_2 & 0 & \psi_2^* \\ -\psi_1^* & -\psi_2^* & 0 \end{pmatrix} \right\} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}.
\]
Here \( \psi^* \) denotes \( \psi \) evaluated for \( x_1 = Vt \). The above transformation involves a shift to the crack edge and a further rotation of coordinate axes.

In the new frame, the stress components \( (\sigma_i'') \) have the asymptotic form
\[
\sigma_i''(x_1', x_2', 0) \sim (K_i''(0) + \varepsilon K_i''(1))/2\pi {x_1'}^{1/2} - (P_i''(0) + \Delta P_i) x_1' + (N_i''(0) + \Delta N_i)(x_1')^{3/2}, \quad i = 1, 2, 3.
\]

The first-order asymptotic approximation of stress-intensity factors was constructed and studied in [3], [10], [11], [12]. In Appendix we include a description of the fundamental identity, which is essential for this work. We also require the dynamic crack face weight function \( [U] \), as defined in Appendix. The field \( [U] \) has a singularity proportional to \( X^{-1/2}H(X)\delta(x_2)\delta(t) \) as \( X \to 0 \).

2.2 First-order perturbations of the stress intensity factors

We begin with the first-order approximation for the stress intensity factors, when
\[
K_j \sim K_j^{(0)} + \varepsilon K_j^{(1)}, \quad j = I, II, III.
\]
For the Mode-I unperturbed case, \( K_{II}^{(0)} = K_{III}^{(0)} = 0 \), and the perturbation terms are defined by (see [11], [12])
\[
K_{II}^{(1)} = -Q_{11} \psi^* \Theta_{13} K_I^{(0)} - \psi_1^* \omega_{13} K_I^{(0)} - \psi^* \left( \Sigma_{11} + \frac{V^2}{2b_2} \Sigma_{12} \right) A_3^{(0)} \sqrt{\pi/2}.
\]
\[ K_{III}^{(1)} = -Q_{12} * \psi^* \Theta_{13} K_I^{(0)} - \psi^* \omega_{23} K_I^{(0)} \]
\[ + [U]_{12} * \langle P_1^{(1)} \rangle + [U]_{22} * \langle P_2^{(1)} \rangle - \langle U \rangle_{32} * [P_3^{(1)}] \]
\[ K_I^{(1)} = Q_{33} * \varphi K_I^{(0)} + \left( \frac{\pi}{2} \right)^{1/2} \varphi A_3^{(0)} - \langle U \rangle_{13} * [P_1^{(1)}] \]
\[ - \langle U \rangle_{23} * [P_2^{(1)}] + [U]_{33} * \langle P_3^{(1)} \rangle \].

The matrix $Q$ is a block-diagonal matrix defined in [11]; other functions that appear in the above equations are

\[ \Theta_{13} = \Sigma_{11} + \frac{V^2}{2a^2} \Sigma_{12}, \quad \omega_{13} = \frac{\alpha - \beta}{R(V)} (1 + \beta^2)(\alpha + 2\beta) - 2, \]
\[ \omega_{23} = \frac{2\nu}{R(V)} (1 + \beta^2)(\alpha^2 - \beta^2) - 1, \]
\[ \Sigma_{11} = -\frac{4\alpha \beta - (1 + 2\alpha^2 - \beta^2)(1 + \beta^2)}{R(V)}, \quad \Sigma_{12} = -\frac{2(1 + \beta^2 - 2\alpha \beta)}{R(V)}, \]
\[ \alpha^2 = 1 - \frac{V^2}{a^2}, \quad \beta^2 = 1 - \frac{V^2}{b^2}, \quad R(V) = 4\alpha \beta - (1 + \beta^2)^2. \]  

Here, $a$ and $b$ denote the speeds of longitudinal and shear waves, respectively. The representations for the effective tractions $P_i^{(1)}, i = 1, 2, 3$, are given in Appendix.

### 2.3 Crack front waves confined to the plane $x_3 = 0$

Assuming that the out-of-plane deflection is not present ($\psi = 0$), we consider a first-order in-plane perturbation of the crack front and loading in Mode I, so that $\sigma_{13}^{inc} = \sigma_{23}^{inc} = 0$ on the plane $x_3 = 0$. In this special case, the only non-zero stress intensity factor is $K_I$, and the corresponding perturbation formula reduces to

\[ K_I^{(1)} = Q_{33} * \varphi K_I^{(0)} + \left( \frac{\pi}{2} \right)^{1/2} \varphi A_3^{(0)} \].

According to the Griffith energy balance equation, the energy flux $G$ into the crack edge is constant, denoted here by $G_c$:

\[ G \equiv \frac{1 - \nu^2}{E} f_1(v) K_I^2 = G_c. \]
Here, \( v \) is the local crack speed (to the first-order approximation, \( v = V + \varepsilon \dot{\varphi} \)) and \( f_I(v) \) is a known function (e.g., \[13\]):

\[
f_I(v) = \frac{v^2 \alpha(v)}{(1-\nu) b^2 R(v)}.
\] (14)

Expanding the Griffith energy balance equation (13) to order \( \varepsilon \), we obtain

\[
2Q_{33} \varphi + \frac{f_I'(V)}{f_I(V)} \dot{\varphi} + 2m \varphi = 0,
\] (15)

where \( m = (\pi/2)^{1/2} A_3^{(0)}/K_3^{(0)} \). Applying the Fourier transform with respect to \( t \) and \( x_2 \) we deduce that a non-zero solution is possible only if the dispersion relation

\[
2Q_{33}(\omega,k) - i\omega \frac{f_I'(V)}{f_I(V)} + 2m = 0
\] (16)

is satisfied. Here, the Fourier transform \( Q_{33} \) is a homogeneous function of degree 1 in \((\omega,k)\). At high frequency and large wavenumber, the third term in the above equation can be neglected. Such an equation can be solved for \( \omega/k \), and a real root represents a speed of wave propagating along the crack front. This computation was performed by Ramanathan and Fisher [2].

3 Corrugation waves for a Mode-I basic loading.

First-order analysis.

Can a Mode-I basic loading generate a corrugation wave propagating along the crack front? This case corresponds to a non-zero out-of-plane perturbation characterised by the function \( \psi(x_1 - Vt, x_2) \). Crack stability with respect to out-of-plane deflections can be studied, once a fracture criterion is identified.

If we suppose that \( K_{II} = 0 \) then, to lowest order, \( \psi \) must satisfy \( K_{II}^{(1)}(\psi) = 0 \), where \( K_{II}^{(1)} \) is given by (8). The proposition that the crack propagates so as to maintain \( K_{II} = 0 \) together with the Griffith energy balance has recently received theoretical support, on the basis of a version of Hamilton’s principle [14].

Assuming that the in-plane perturbation of the crack front equals zero, we look into stability against out-of-plane deflections. It is also assumed
that $\omega \equiv k_1 V$ and $k_2$ are large. The leading-order approximation of the stress intensity factor $K_{II}$ yields

$$K_{II}^{(1)} = \{ -Q_{11} \Theta_{13} + i(\omega/V)\omega_{13} \} K_{I}^{(0)} \psi^* = 0.$$  

This relation is homogeneous of degree 1 in $\omega$ and $k_2$, and so is non-dispersive.

The numerical study of equation (17) produced the following results.

- For crack speeds $V$ greater than a critical value $V_c$ (which is close to 0.6 of the Rayleigh wave speed) there is a value $\eta = \omega/|k_2|$ with small, negative, imaginary part that satisfies (17). The position of the root is shown in Figure 1; the calculation is produced for the case of $V/b = 0.69$, and the diagram shows the level curves of the modulus of the expression in the curly
brackets on the left side of (17).

Figure 1 is accompanied by a three dimensional surface plot, shown in Figure 2, of the function $W = |Q_{11} \Theta_{13} - i(\omega/V)\omega_{13}|$; the surface touches the $\eta$-plane at the point corresponding to the root of equation (17).

![Surface plot of the function $W = |Q_{11} \Theta_{13} - i(\omega/V)\omega_{13}|$, for $V/b = 0.69$ and $\nu = 0.3$.](image)

Figure 2: Surface plot of the function $W = |Q_{11} \Theta_{13} - i(\omega/V)\omega_{13}|$, for $V/b = 0.69$ and $\nu = 0.3$.

- The "corrugation wave" suffers slow attenuation as it propagates. The imaginary part of $\eta$, which characterises the rate of attenuation of the "corrugation wave", is shown in Fig. 3 for different values of the crack front velocity $V$, and it decreases with $V$. 

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**Figure 2: Surface plot of the function $W = |Q_{11} \Theta_{13} - i(\omega/V)\omega_{13}|$, for $V/b = 0.69$ and $\nu = 0.3$.**

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7
4 First-order coupling between in-plane and out-of-plane crack front perturbations for mixed Mode I-III loading

Here, we assume that $K_{II}^{(0)} = 0$, whereas $K_{I}^{(0)}$ and $K_{III}^{(0)}$ are non-zero for a half-plane crack propagating with constant speed $V$ (unperturbed configuration). To first order, the stress intensity factors are represented by the formulae (7), where the perturbation terms $K_{j}^{(1)}$, $j = I, II, III$, are defined by (see [11], [12])

$$K_{II}^{(1)} = -Q_{11} \psi^{*} \Theta_{13} K_{I}^{(0)} - \psi^{*}_{11} \omega_{13} K_{I}^{(0)} - \psi^{*} \left( \Sigma_{11} + \frac{V^2}{2b^2} \Sigma_{12} \right) A_{3}^{(0)} \sqrt{\frac{\pi}{2}}$$
+Q_{21}*(φK_{III}^{(0)}) - ϕ,2K_{III}^{(0)} + √(π/2)φA_1^{(0)}
+ [U]_{11} * ⟨P_1^{(1)}⟩ + [U]_{21} * ⟨P_2^{(1)}⟩ - ⟨U⟩_{31} * [P_3^{(1)}],
K_{III}^{(1)} = -Q_{12} * ψ^*Ω_{13}K_I^{(0)} - ψ^*ω_{23}K_I^{(0)} + Q_{22} * (φK_{III}^{(0)}) + √(π/2)φA_2^{(0)}
+ [U]_{12} * ⟨P_1^{(1)}⟩ + [U]_{22} * ⟨P_2^{(1)}⟩ - ⟨U⟩_{32} * [P_3^{(1)}],
K_I^{(0)} = Q_{33} * ϕK_I^{(0)} + (π/2)^{1/2}ϕA_3^{(0)} - ϕ^*(1 - V^2/2b^2 Σ_{12})A_1^{(0)} √(π/2)
- 2ψ^*K_{III}^{(0)} - ⟨U⟩_{13} * [P_1^{(1)}] - ⟨U⟩_{23} * [P_2^{(1)}] + [U]_{33} * (P_3^{(1)}).

We shall use the criterion of local symmetry $K_{II} = 0$, together with the Griffith energy balance equation

$G = (2μ)^{-1}f_I(v)K^2 + (2μ)^{-1}f_{III}(v)K_{III}^2 = G_c = \text{const.}$

(21)

Taking into account that, to first order, $v \sim V + εφ$, we deduce

$G = (2μ)^{-1}f_I(V)(K_I^{(0)})^2 + (2μ)^{-1}f_{III}(V)(K_{III}^{(0)})^2$

$+ ε(2μ)^{-1}(φf'_I(V)(K_I^{(0)})^2 + 2f_I(V)K_I^{(0)}K_I^{(1)})$

$+ φf'_{III}(V)(K_{III}^{(0)})^2 + 2f_{III}(V)K_{III}^{(0)}K_{III}^{(1)} + O(ε^2).$

(22)

It follows from (21), (22) and the local symmetry criterion $K_{II} = 0$ that

$φ(f'_I(V)(K_I^{(0)})^2 + f'_{III}(V)(K_{III}^{(0)})^2) + 2f_I(V)K_I^{(0)}K_I^{(1)}(φ,ψ)$

$+ 2f_{III}(V)K_{III}^{(0)}K_{III}^{(1)}(φ,ψ) = 0,$

(23)

$K_{III}^{(1)}(φ,ψ) = 0.$

(24)

The above equations define the coupling between the in-plane and out-of-plane perturbations of the crack front.

Applying the Fourier transform with respect to $t$ and $x_2$ and assuming that $ω = k_1 V$ and $k_2$ are large, we deduce

$\mathcal{F}\{\left(2f_I(V)Q_{33} - iωf'_I(V)\right)(K_I^{(0)})^2 + \left(2f_{III}(V)Q_{22} - iωf'_{III}(V)\right)(K_{III}^{(0)})^2\}$

$+ \mathcal{F}\{K_I^{(0)}K_{III}^{(0)}\left(2f_{III}(V)\left(-Q_{12}Ω_{13} + ik_2ω_{23}\right) + 4ik_2f_I(V)\right)\} = 0,$

(25)
The system (25), (26) is linear in \( \psi \) and \( \psi^* \), and it possesses a nontrivial solution if and only if the matrix of this system is degenerate. This yields the following dispersion relation:

\[
\{ -Q_{11}\Theta_{13} + i(\omega/V)\omega_{13} \} \overline{\psi} K_j^{(0)} + (Q_{21} + i k_2)\overline{\psi} K_{III}^{(0)} = 0. \tag{26}
\]

Here \( K_0 = K_{III}^{(0)}/K_{I}^{(0)} \). The above dispersion equation, connecting \( \omega \) and \( k_2 \), is to be analysed numerically to identify possible crack front waves associated with the external mixed mode I-III load.

**Appendix. Fundamental identity and effective tractions.**

Here, we briefly describe the method developed in [3], [10], [11]. We use the relation

\[
u = -G \ast \sigma, \tag{A1}
\]

where \( \mathbf{u} \) and \( \mathbf{\sigma} \) denote the values of the displacement vector \((u_i)\) and the traction vector \((\sigma_{ij})\) on the surface \( x_3 = 0 \) of the half-space \( x_3 > 0 \); \( G \) is the Green’s matrix function. The symbol \( \ast \) denotes convolution over \( x_1, x_2 \) and \( t \). It is assumed that all waves emanate from the surface \( x_3 = 0 \). A similar identity applies to the half-space \( x_3 < 0 \), with \( G \) being replaced by \( -G^T \).

Three column vectors like \( \mathbf{u} \) can be written side by side to form a matrix \( \mathbf{U}(+0) \), and similarly \( \mathbf{\Sigma}(+0) \) represents the matrix formed from the three corresponding vectors \( \mathbf{\sigma} \). Then

\[
\mathbf{U}(+0) = -G \ast \mathbf{\Sigma}(+0). \tag{A2}
\]

The argument \((+0)\) signifies values on the boundary of the upper half-space.

Applying similar reasoning to the identity for the lower half-space \( x_3 < 0 \) gives

\[
\mathbf{U}(-0) = G^T \ast \mathbf{\Sigma}(-0). \tag{A3}
\]

Next, we note that

\[
\{ \mathbf{U}(+0) \}^T \mathbf{\sigma}(-0) = -\{ \mathbf{\Sigma}(+0) \}^T G^T \mathbf{\sigma}(-0)
\]
\[
\begin{align*}
  \{\mathbf{U}(0)\}^T \ast \mathbf{\Sigma}(0) &= -\{\mathbf{u}(0)\}^T, \\
  \{\mathbf{U}(0)\}^T \ast \mathbf{\sigma}(0) &= \{\mathbf{\Sigma}(0)\}^T \ast \mathbf{G}^T \ast \mathbf{\sigma}(0)
\end{align*}
\]  
\begin{equation}
(A4)
\end{equation}

Subtracting the second line from the first and rearranging gives the identity
\[
\begin{align*}
  \{\mathbf{U}(0)\}^T \ast \mathbf{\Sigma}(0) &= \{\mathbf{u}(0)\}^T
\end{align*}
\]  
\begin{equation}
(A5)
\end{equation}

where \(f = \frac{1}{2}(f(+0) + f(-0))\) and \([f] = f(+0) - f(-0)\).

In the moving frame associated with the crack edge, we use the coordinate \(X = x_1 - Vt\). The operation of convolution survives, with functions regarded as functions of \(X, x_2, t\) and the convolutions taken over these new variables.

For the unperturbed crack problem,
\[
\begin{align*}
  [\mathbf{\sigma}] &\equiv 0, \quad [\mathbf{u}] = 0 \text{ when } X > 0, \quad \mathbf{\sigma} \equiv \langle \mathbf{\sigma} \rangle = -\mathbf{\sigma}^{nc} \text{ when } X < 0. \\
\end{align*}
\]  
\begin{equation}
(A7)
\end{equation}

We interpret equation (A6) relative to the moving frame, and perform factorizations of the Green’s function so that \(\mathbf{U}\) and \(\mathbf{\Sigma}\) display the related properties
\[
\begin{align*}
  [\mathbf{\Sigma}] &\equiv 0, \quad [\mathbf{U}] = 0 \text{ when } X < 0, \quad \mathbf{\Sigma} \equiv \langle \mathbf{\Sigma} \rangle = 0 \text{ when } X > 0.
\end{align*}
\]  
\begin{equation}
(A8)
\end{equation}

Equations (A2), (A3) yield
\[
\begin{align*}
  [\mathbf{U}] &= -(\mathbf{G} + \mathbf{G}^T) \ast \langle \mathbf{\Sigma} \rangle, \\
  \langle \mathbf{U} \rangle &= -\frac{i}{\pi}(\mathbf{G} - \mathbf{G}^T) \ast \langle \mathbf{\Sigma} \rangle.
\end{align*}
\]  
\begin{equation}
(A9)
\end{equation}

The first of these relations defines a Wiener–Hopf problem; the second then gives \(\langle \mathbf{U} \rangle\) directly. The Wiener–Hopf problem uncouples into two subproblems. One, associated with the opening mode I of the crack, is a scalar problem. It was solved in the case of elasticity in [3], and for a viscoelastic medium in [6]. The remaining problem involves modes II and III, coupled. It was solved in [10].

The field \([\mathbf{U}]\) has a singularity proportional to \(X^{-1/2} \mathcal{H}(X)\delta(x_2)\delta(t)\) as \(X \to 0\). With the constant of proportionality chosen as \((2/\pi)^{1/2}\mathbf{I}\), we call \([\mathbf{U}]\) the \textit{dynamic weight function} for the crack problem. With this choice, letting \(X \to +0\) in the identity (A6) generates
\[
\begin{align*}
  \mathbf{K} &= \lim_{X \to +0} \left\{ (\mathbf{U})^T \ast [\mathbf{\sigma}(0)] - [\mathbf{U}]^T \ast \langle \mathbf{\sigma}(0) \rangle \right\}, \\
\end{align*}
\]  
\begin{equation}
(A10)
\end{equation}
where \( \mathbf{K} \) denotes the vector of stress-intensity factors \((K_{II}, K_{III}, K_{I})^T\). The matrix function \( \langle \mathbf{U} \rangle \) represents a dynamical version of Bueckner’s non-symmetric weight function, as described in [15] and [10].

We assume that the unperturbed steady-state crack is subjected to a Mode-I loading, and the unperturbed displacement field is a vector function \( \mathbf{u}^{(0)} = \mathbf{u}^{(0)}(x_1 - Vt, x_2, x_3) \). We can write the resulting displacement field in the form

\[
\mathbf{u} \sim \mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)},
\]

where \( \varepsilon \) is a perturbation parameter.

The effective tractions \( P_i^{(1)} := -\sigma_{i3}(\mathbf{u}^{(1)})|_{x_3=0}, \ i = 1, 2, 3, \) have the form (see formula (4.11) of [11])

\[
P_i^{(1)} = -2 \sum_{k=1}^2 (\psi \sigma_{ik}^{(0)}/k) + \psi \left( \rho V^2 u_{i,11}^{(0)} - 2\rho V \frac{\partial^2 u_i^{(0)}}{\partial t \partial X} + \rho \frac{\partial^2 u_i^{(0)}}{\partial t^2} \right).
\]

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