Study of polarization for even-denominator fractional quantum Hall states in SU(4) Graphene

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Abstract

We have focused on studying the even-denominator fractional quantum Hall (EDFQH) states observed in monolayer graphene. In this article, we have studied polarization mainly for the two EDFQH states at filling fractions $\nu = 1/2, 1/4$, which are observed in an experimental study [Nat. Phys. 14, 930 (2018)] a few years ago. We have applied Chern-Simon’s gauge field theory to explain the possible variational wave functions for different polarized states and calculated their ground state energies using the Coulomb potential. We have chosen the lowest energy states using suitable combinations of attached flux quanta to the electrons for different polarized states of those EDFQH states.

Introduction

Strongly correlated two-dimensional electron system under a strong perpendicular magnetic field shows a remarkable collective phenomenon, the fractional quantum Hall effect (FQHE) [1, 2]. Composite Fermion (CF) theory [3, 4] beautifully explains the fractional quantum Hall (FQH) states in detail qualitatively. CF is basically the bound state of an electron and an even number (2p) of quantized vortices that produce a Berry phase of 2p for a closed loop around it, such that the CFs obey the same statistics as that of electrons. So the CFs experience a reduced amount of magnetic field $B' = B - 2p\phi_0$, where $\rho$ is the electron density and $\phi_0$ is the flux quanta. The eigenstates of the Hamiltonian of a CF are called a levels, which are similar to the Landau levels (LLs) with reduced degeneracy. $n$ number of filled A levels of non-interacting CF's system (integer quantum Hall effect of CF) describes the FQHE of filling fraction $\nu = \frac{n}{2p+1}$ of a strongly interacting electronic system. This is the beauty of this mean-field theory. For many years, understanding the spin polarization of the various FQH states has become a topic of great interest among experimentalists and theorists. In sufficiently high magnetic fields, the low-energy states are of course fully spin-polarized. The partially polarized and unpolarized states occur due to the competition between the cyclotron energy and Zeeman energy. The magnetic field (Zeeman energy) dependency of the polarization has been reported by Kukushkin and co-authors [3] in detail. CF theory beautifully explains most of the polarized states, except for some partially polarized states such as 50% polarized state of filling fraction 2/5, 2/3 and 25%, 75% polarized state of filling fraction 4/7, 4/9 etc. Ganapathy Murthy explained the partially polarized state of 2/5 filling fraction as the density wave ordered state of CF [6]. Some of the partially polarized states are explained as mixture of different species of CF [7, 8].

Besides the two-dimensional (2D) semiconductor device, FQHE has been observed in monolayer [9, 10], and bilayer graphene [11, 12]. The effective low-energy theory of the Graphene system can be described in terms of Dirac-like quasi-particles [13], which endows the electron wavefunctions with an additional quantum number, termed valley isospin, that combined with the usual electron spin, yields four-fold degenerate LLs [14, 15]. This additional symmetry adds up some new incompressible FQH states [16–19] in graphene. Graphene has two types of Zeeman energies such as the spin Zeeman energy and valley Zeeman energy. In this SU(4) system CF theory has been used to study the polarization[20] and collective excitation[21] by the Jain group. But the CF theory failed to explain all the possible polarized states in Graphene. Moreover, it can not

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explain the SU(4) polarized states of $\nu = n/(2m+1)$ with $n$ less than 3. Goerbig and Regnault [8] applied
generalized Halperin’s wave function to study the ground state and excited states of $\nu = 2/3$ filling fraction
in this SU(4) symmetric state. The polarization of the FQHE in the SU(4) symmetric system has been addressed
by Mandal and coworkers [22] using Chern Simon’s (CS) gauge field theory [23].

When graphene is placed in a strong perpendicular magnetic field, a plethora of quantum Hall states are
observed. Particularly notable is the first observation of incompressible even-denominator fractional quantum
Hall (EDFQH) states at $\nu = \pm 1/2$ and $\pm 1/4$ [24]; although some EDFQH states were observed previously
in higher LLs of single-component systems at $\nu = 5/2$ in GaAs [25], and at $\nu = 3/2$ and 7/2 in ZnO [26]. The most
discussed EDFQH state in the higher LL occurs at the filling fraction $\nu = 5/2$, which can be described by the
Moore Read Pfaffain [27] state. Some EDFQH states are observed for the lowest LLs both in double quantum
wells [28] and wide single quantum wells at $\nu = 1/2$ [29, 30] and at $\nu = 1/4$ [31, 32]. The $\nu = 1/4$ state may be
interpreted as the pairing of CFs [33] as described for $\nu = 1/2$ state in a thick two-dimensional electron system
[34]. In literature, $\nu = 1/2$ state has been shown to have a large overlap with the so-called $3,3,1$ wave function
[35, 36], which was first introduced by Halperin [37]. Beyond this $3,3,1$ state this $\{m,m,n\}$ model can be
applied for $\nu = 1/4$ state considering two-component system. Possible wave function at $\nu = 1/4$ state is $\{5,5,3\}$,
$\{6,6,2\}$, $\{7,7,1\}$. Since in monolayer graphene there are 4-isospin configurations (two valleys and two spins)
[20, 38–41], we have many possibilities of polarized states such as spin polarization, valley polarization, and a
combination of spin and valley polarization, which we call mixed polarization. Golkmen et al [42] observed a
marked contrast between the spin and valley degrees of freedom. They found that an electron scattering from
one valley to another requires a large momentum transfer whereas an electron scattering from one spin to
another within the same valley requires a small momentum transfer. It is also confirmed by another study by
Papic [43]. Wu et al [44] proposed that $\nu = 1/2$ states can be understood using the two-component parton wave-
function. Recently collective excitation is studied by Sujit Narayanan [45] for various FQH states in monolayer
graphene. They have focused to study the chiral symmetry breaking orders i.e. anti-ferromagnetism and charge
density orders. We are interested here to study various possible wave functions for SU(4) graphene using CF or
CS theory of multicomponent FQH system. The various possibility of incompressible EDFQH states of filling
fractions $\nu = 1/2, 1/4$ in monolayer graphene was noted already in [22, 46]. Following their work, we have
investigated different polarized states of the filling fractions $\nu = 1/2, 1/4$ and then calculated the ground state
energies of those states using Coulomb potential. Although in Van der Waals heterostructures such as graphene,
the gate-screening effect is very significant but we haven’t included that in our calculation. We have restricted
our calculation in 2D monolayer graphene, the finite width correction in this calculation will give a more
accurate comparison with experiment although the basic discussions will be the same.

Calculation procedure

Following the Chern–Simon’s gauge field theory in SU(4) system, we have seen that the effective field depends on
the quantum numbers (Spin and Valley) carried by the quasiparticles. The quasiparticles of species $\alpha$ in CS
theory experiences a mean effective magnetic field, $B_\alpha$ [22]

$$B_\alpha = B - \phi_0 K_{\alpha \beta} \rho_\beta$$ (1)

where B is the applied actual magnetic field, $\phi_0$ is the flux quanta and $\rho_\beta$ represents density of the electrons for the
species $\beta$. We label the valley-spin configurations $\{(\uparrow, \uparrow), (\downarrow, \downarrow), (\uparrow, \downarrow), (\downarrow, \uparrow)\}$ by the index $\alpha = 1$to4 respectively.
The indices $\alpha, \beta$ runs from 1 to 4. We chose the symmetric $4 \times 4$ matrix $K_{\alpha \beta}$ as follows:

$$K = \begin{pmatrix}
k_1 & m_1 & n_1 & n_2 \\
m_1 & k_2 & n_3 & n_4 \\
n_1 & n_2 & k_3 & m_2 \\
n_3 & n_4 & m_2 & k_4
\end{pmatrix}$$ (2)

where $k$‘s, $m$‘s and $n$‘s are positive integers including zero. In [22] flux attachment scheme is represented
schematically. As suggested by them we have also simplified the flux attachment that $k_1 = k_2 = k_3 = k_4$,
$n_1 = n_2 = n_3 = n_4 = n$. Considering mean-field approximation, the electron filling factor $\nu$ and the effective
filling factor $\nu$, of CS quasiparticles are related by,

$$\rho_\alpha/\nu_\alpha = \rho/\nu = K_{\alpha \beta} \rho_\beta$$ (3)

Selection of number of particles:

We have considered that N number of correlated electrons resides on the spherical surface [4, 47] in presence of
radial magnetic field created by a Dirac monopole at the center of the sphere with monopole strength Q and

2
there are \( N_1, N_2, N_3, N_4 \) number of electrons in species 1, 2, 3, 4 respectively.

\[
\begin{align*}
N &= N_1 + N_2 + N_3 + N_4 \\
S &= N_1 + N_3 - N_2 - N_4 \\
V &= N_1 + N_2 + N_3 + N_4 \\
M &= N_1 + N_2 - N_3 - N_4 \\
\end{align*}
\]

We have chosen the values of \( N_1, N_2, N_3, N_4 \) such a way that it will fix the values of \( S, V, M \) in the following tables. The monopole strength corresponding to the effective field as seen by species \( \alpha \) can be written as,

\[
\begin{align*}
2q_{\alpha} &= 2Q - \sum_{\beta} (N_{\beta} - \delta_{\alpha\beta}) K_{\alpha\beta} \\
\sum_{\alpha} 2q_{\alpha} &= 8Q - \sum_{\alpha, \beta} (N_{\alpha} - \delta_{\alpha\beta}) K_{\alpha\beta} \\
\Rightarrow Q &= \frac{1}{8} \left( \sum_{\alpha} 2q_{\alpha} + \sum_{\alpha} K_{\alpha\alpha}(N_{\alpha} - 1) \right) \\
&\quad + \frac{1}{8} (m_1 N_1 + m_2 N_2 + m_3 N_3 + m_4 N_4 + 2n N) \\
2Q\phi_0 \text{ is the total flux through the spherical surface of radius } R, \text{ if } B \text{ is the corresponding magnetic filed then} \\
2Q\phi_0 &= 4\pi R^2 B \Rightarrow R^2 = \frac{2Q hc e^2}{4\pi e / \hbar} = Ql_0^2 \\
\text{So the radius of the sphere is } R = \sqrt{Q} \text{ in unit of magnetic length } l_0.
\end{align*}
\]

**Wave function**

Let us suppose that there are \( \nu_1 \) number of spin-up (+) valley \( \Lambda \)-level, \( \nu_2 \) number of spin-down (+) valley \( \Lambda \)-level, \( \nu_3 \) number of spin-up (−) valley \( \Lambda \)-level, \( \nu_4 \) number of spin-down (−) valley \( \Lambda \)-level. Following the route proposed by Mandal and co-workers the spin (S), the valley (V), and the mixed (M) polarization of these quasiparticles in a given FQH states is defined as,

\[
\begin{align*}
S &= (\rho_1 + \rho_3 - \rho_2 - \rho_4) / \rho, \\
V &= (\rho_1 + \rho_2 - \rho_3 - \rho_4) / \rho, \\
M &= (\rho_1 + \rho_4 - \rho_2 - \rho_3) / \rho, \\
\end{align*}
\]

So, the above equation (3) relates total filling factor \( \nu \) in terms of those three polarization \( S, V, M \) and the attached flux numbers \( k \)'s, \( m \)'s and \( n \)’s.

\[
\begin{align*}
\gamma_1(2k_1 + 1/\nu_1) + \gamma_2 m_1 + \eta_2 2n &= 4/\nu \\
\gamma_2(2k_2 + 1/\nu_2) + \gamma_1 m_1 + \eta_2 2n &= 4/\nu \\
\gamma_3(2k_3 + 1/\nu_3) + \gamma_4 m_2 + \eta_2 2n &= 4/\nu \\
\gamma_4(2k_4 + 1/\nu_4) + \gamma_3 m_2 + \eta_2 2n &= 4/\nu \\
\end{align*}
\]

where, \( \eta_{(12)} = 1 - (+) V, \delta_{(12)} = S - (+) M, \gamma_{(12)} = \eta_2 + (-) \delta_2, \) and \( \gamma_{N4} = \eta_4 + (-) \delta_4. \)

The five indices \( k_1, k_2, m_1, m_2, n \) mimic the interaction strength between different CF \( \Lambda \)-levels. From equation (6), it is very clear that the polarization index \( S, V, M \); four species CF \( \Lambda \)-level filling fractions \( \nu_1, \nu_2, \nu_3, \nu_4 \) and the interaction parameters \( k_1, k_2, m_1, m_2, n \) altogether fixes the total filling fraction \( \nu \) of the state and the particular state is symbolized as \((k_1 k_2 m_1 m_2 n)\).

**0.0.1. Interpretation of \((k_1 k_2 m_1 m_2 n)\)**

(i) \( 2k_1 \) number of flux attachment of CF in the (+) and (+) \( \Lambda \)-levels.

(ii) \( 2k_2 \) number of flux attachment of CF in the (−) and (−) \( \Lambda \)-levels.

(iii) \( m_1 \) flux attachment in the (+) level seen by the CFs in (+) \( \Lambda \)-level and vice versa.

(iv) \( m_2 \) flux attachment in the (−) level seen by the CFs in (−) \( \Lambda \)-level and vice versa.

(v) \( n \) flux attachment in the (+ & −) level seen by the CFs in (− & −) \( \Lambda \)-levels and vice versa.
The variational wave function for the state is proposed by Mandal and coworker [22] and is given by
\[
\Psi_{\{\nu_{\alpha}\}} = P_\Lambda \Phi_{\nu_{\alpha}}(\Omega_{\alpha_1}^1 \ldots \Omega_{\alpha_\nu_{\alpha}}^1) \Phi_{\nu_{\alpha}}(\Omega_{\alpha_1}^2 \ldots \Omega_{\alpha_\nu_{\alpha}}^2) \\
\times \Phi_{\nu_{\alpha}}(\Omega_{\alpha_1}^3 \ldots \Omega_{\alpha_\nu_{\alpha}}^3) \Phi_{\nu_{\alpha}}(\Omega_{\alpha_1}^4 \ldots \Omega_{\alpha_\nu_{\alpha}}^4) \\
\times J_{11} J_{12} J_{13} J_{14} J_{24} J_{34} J_{44}
\]
where, \( \Phi_{\nu_{\alpha}} \) is the Slater determinant of \( \nu_{\alpha} \) filled \( \Lambda \) level CFs, \( \Omega_{\alpha_i}^j \) are the positions of CFs on the spherical surface, upper index indicate the different species of CFs and the Jastrow factor is given by
\[
J_{\alpha\beta} = \prod_{i<j}^{N_{\nu_{\alpha}}} \left( u^{(\alpha)}_{ij} v^{(\beta)}_{ij} - u^{(\beta)}_{ij} v^{(\alpha)}_{ij} \right)^{\nu_{\alpha\beta}} \quad \text{if } \alpha \neq \beta
\]
\[
J_{\alpha\alpha} = \prod_{i<j}^{N_{\nu_{\alpha}}} \left( u^{(\alpha)}_{ij} v^{(\alpha)}_{ij} - u^{(\alpha)}_{ij} v^{(\alpha)}_{ij} \right)^{\nu_{\alpha\alpha}}
\]
The prefix within bracket represent the CFs of different \( \Lambda \) levels. Here we have used the spinor coordinates
\[
u(\Omega) = \cos(\theta/2) e^{-ib/2} \quad \text{and} \quad u(\Omega) = \sin(\theta/2) e^{ib/2}
\]
Some of the above wave functions are identical with the wave function proposed by Regnault and others using the plasma picture of the FQHE with internal SU(4) symmetry [19]. The ground state energy \( E_g \) per particle corresponding to a particular state with flux attachment \( (k_1 k_2 m_1 m_2) \) can be calculated by the Monte Carlo method as

| Table 1. Possible polarized states fitted with equation (6) for filling fraction \( \nu = 1/2 \) and their ground state energies with different parameters. |
|---|---|---|---|---|---|---|---|---|---|---|
| \( k_1 \) | \( k_2 \) | \( m_1 \) | \( m_2 \) | \( n \) | \( V \) | \( S \) | \( M \) | \( \nu_1 \) | \( \nu_2 \) | \( \nu_3 \) | \( \nu_4 \) | \( E_g(\epsilon_f/k_B) \) |
| 1 | 1 | 1 | 1 | 2 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | −0.4677 |
| 1 | 1 | 3 | 3 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | −0.46315 |
| 2 | 1 | 1 | 1 | 3 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | −0.45576 |
| 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | −0.44983 |
| 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | −0.46742 |
| 1 | 2 | 2 | 5 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | −0.46351 |
| 1 | 3 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | −0.46012 |
| 1 | 1 | 2 | 5 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | −0.45653 |

\[
E_g = \frac{\epsilon_f}{k_B} \left( \frac{\nu_{\alpha\beta}}{\nu_{\alpha\alpha}} \right)^{\nu_{\alpha\beta} - \nu_{\alpha\alpha}} \quad \text{if } \alpha \neq \beta
\]
\[
E_g = \left( \frac{\nu_{\alpha\alpha}}{\nu_{\alpha\alpha}^2} \right)^{\nu_{\alpha\alpha}}
\]
Here, $H = \sum_{i<j} \frac{e^2}{r_{ij}}$ is the Coulomb interaction, with $r_{ij}$ as the inter-electronic distance and $E_{bg} = -\frac{N e^2}{2 e R}$ term represents the background energy i.e. the interaction between the electrons and the background positive ions where $\epsilon$ is the dielectric constant of the background material. LL mixing parameter is independent of the applied magnetic field in the graphene system \cite{48}, so we do not need to include the LL mixing \cite{49, 50} in our calculation.

**Results & conclusion**

We observed that the EDFQH states do not correspond to the Fermi sea of CFs. According to the Chern-Simon’s theory, we have different degeneracy for four $\Lambda$-levels. Also in our earlier study of SU(2) system \cite{7}, we didn’t get the CF Fermi-sea using Chern-Simon’s theory. Zibrov et al \cite{24} proposed that the EDFQH states are associated with a phase transition from a partial sublattice polarized \cite{51} to a canted antiferromagnet phase \cite{52}. With this idea, Sujit et al \cite{46} concluded that there is a phase transition from a state with $V = 0$, $S = 0$, $M = 0$ to one with $V = 0$ and either $S$ or $M$ $\neq 0$. Sujit and their group claimed that if the flux attachment set for $\nu = 1/2$ is denoted by an index $j_1$, then for $\nu = 1/4$ the flux attachment set is found to have $j + 1$ i.e. each of the indices are increased by one. This observation also agrees with our result. In this article, we have accumulated all possible wave-functions for the recently observed EDFQH states at $\nu = 1/2$, $1/4$ and calculated their ground state energies which help us to find out the fitted wave functions for those states.

We have studied here the possible polarizations for the two EDFQH states ($\nu = 1/2$, $1/4$). We have considered different combinations of flux attachment ($k_1, k_3, m_1, m_2, n, n$) between the four species of electrons and calculated their ground state energies using the Coulomb potential for a finite number of particles. To get the thermodynamic limit, we have extrapolated the result in the $1/N \to 0$ region. The energy values for the filling fraction $\nu = 1/2$ and $1/4$ are listed in table 1 and table 3. Energy has been expressed in natural unit $e^2/\epsilon k_B$. The error of the Monte-Carlo integration is very less.

**Observations on filling fraction $\nu = 1/2$:** We have observed that different interaction strength between the four species can lead to different polarization. We can have the unpolarized state of $\nu = 1/2$ for different...
combinations of flux attachment i.e. different interaction parameters; keeping the four Λ-level filling fractions unity (ν₁ = ν₂ = ν₃ = ν₄ = 1). The ground state energies differ in each of the cases.

(i) We found that the state with interaction (11112) has the lowest energy (see figure 1 (a)). So this is the most stable state for the unpolarized condition. Other states having interaction parameters (11331), (21131), (22111) have higher energy compared to that. These states can be thought of as next stable states. By a slight change in the interaction parameters the system can suffer transition from one state to another.

(ii) After that, we have considered different valley polarized states. We have observed that also for V = 1/3 polarized state, with other two polarization index zero (S, M = 0), the state with interaction (11112) has the lowest energy (figure 1 (b)).

(iii) Then we have checked the energies for different valley polarizations for the combination (11112) keeping the spin and mixed polarization zero (S = M = 0). Energies of those valley polarized states (V = 0, 1/4, 1/3, 1/2, 2/3, ± 1) are slightly differ from each other. So there is a high chance for the phase transition depending upon the Zeeman energies between the four Λ-levels. We have noted that V = 0 state has the lowest energy and V = ± 1 state has the highest energy and intermediate states lie in between.

Table 2. Lowest energy states with different polarization indices for ν = 1/2.

| k₁ | k₃ | m₁ | m₂ | n | V | S | M | ν₁ | ν₂ | ν₃ | ν₄ |
|----|----|----|----|---|---|---|---|----|----|----|----|
| 1  | 1  | 1  | 1  | 2 | 0 | 0 | 0 | 1  | 1  | 1  | 1  |
|    |    |    |    |   | 1/3 | 0 | 0 | 1  | 1  | 1  | 1  |
| 1  | 2  | 1  | 1  | 1  | 1  | 0 | 0 | 1  | 1  | 0  | 0  |
| 2  | 1  | 1  | 1  | 1  | −1 | 0 | 0 | 0  | 0  | 1  | 1  |

Figure 2. Ground state energy per particle for different combinations of different interaction parameters (k₁, k₃, m₁, m₂, n) with different values of polarization indices V, S, M at filling fraction ν = 1/4.
Besides, we have checked whether there is any other state having lower energy for \( V = \pm 1 \) polarization with other two polarization index \((S, M)\) zero. We found that the states \((12111)\) and \((21111)\) have the lowest energy for the valley polarization \( V = 1 \) and \( V = -1 \) respectively; with other two polarization index \( S, M \) zero for both the two cases (see figure 1 (c)).

Next, we calculated the same taking the polarization \( S \) (or \( M \)) as \( \pm 1 \) and the other two polarization indices \( V, M \) (or \( V, S \)) as zero, as suggested by Sujit \textit{et al} \cite{46}. We observed that for the interaction \((11121)\) we get the lowest energy when the polarization \( S \) (or, \( M \)) is \( \pm 1 \) and the energy values are almost equal (see figure 1 (d)). All the results for the filling fraction \( \nu = 1/2 \) are shown in table 2.

\textbf{Observations on filling fraction }\nu = 1/4
Similarly, we have checked all the above possibilities for the filling fraction $\nu = 1/4$, the results are shown in figure 2 and in table 3.

(i) For the unpolarized state ($V = S = M = 0$), we have found that (22334), (22553) states have almost the same energy which is the lowest for this polarization (see figure 2 (a)). So the unpolarized state is doubly degenerate for $\nu = 1/4$.

(ii) We have found that for $V = 1$ polarized state with other two polarization zero ($S, M = 0$); (21311), (22311) and (22331) states have the same energy value (see figure 2 (c)). That is those three states are degenerate metastable states; so the system can change any one of those.

(iii) For the valley polarization $V = −1$ with $S, M = 0$; there are also three degenerate metastable states, which are (12131), (22131), and (22331) (see figure 2 (d)).

(iv) We have found that (22113) has the lowest energy when spin or mixed polarization $S$ (or, $M$) is $±1$ and valley polarization is zero (see figure 2 (b)). We also noticed that for same interaction parameters (22113), change in Zeeman energies between four $\Lambda$-levels can lead to different polarization $S = ±1, M = ±1$ states, those states have same energies i.e. those are the degenerate states for the filling fraction $\nu = 1/4$. The lowest energy states for the different polarized states of filling fraction $\nu = 1/4$ are enlisted in table 4.

In conclusion, we want to mention that we have confirmed the most stable ground state of a particular polarized state at filling fraction $\nu = 1/2$ and $\nu = 1/4$, and possible phase transition for different polarized states within the filling fraction, which are associated with the prediction of Sujit et al [46].

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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| $k_1$ | $k_3$ | $m_1$ | $m_2$ | $n$ | $V$ | $S$ | $M$ | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ |
|-------|-------|-------|-------|-----|-----|-----|-----|-------|-------|-------|-------|
| 2     | 2     | 3     | 3     | 4   | 0   | 0   | 0   | 1     | 1     | 1     | 1     |
| 2     | 2     | 5     | 5     | 3   |     |     |     |       |       |       |       |

| 2 | 1 | 3 | 1 | 1 |
| 2 | 3 | 1 | 1 | 1 |
| 2 | 2 | 3 | 3 | 1 |

| $k_1$ | $k_3$ | $m_1$ | $m_2$ | $n$ | $V$ | $S$ | $M$ | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ |
|-------|-------|-------|-------|-----|-----|-----|-----|-------|-------|-------|-------|
| 1     | 2     | 1     | 3     | 1   |     |     |     |       |       |       |       |
| 2     | 1     | 3     | 1     | −1  | 0   | 0   | 0   | 0     | 0     | 1     | 1     |
| 2     | 2     | 3     | 3     | 1   |     |     |     |       |       |       |       |

| $k_1$ | $k_3$ | $m_1$ | $m_2$ | $n$ | $V$ | $S$ | $M$ | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ |
|-------|-------|-------|-------|-----|-----|-----|-----|-------|-------|-------|-------|
| 2     | 2     | 1     | 3     | 1   |     |     |     |       |       |       |       |
| 2     | 1     | 1     | 3     | −1  | 0   | 0   | 0   | 0     | 0     | 1     | 1     |
| 2     | 2     | 3     | 3     | 1   |     |     |     |       |       |       |       |

| $k_1$ | $k_3$ | $m_1$ | $m_2$ | $n$ | $V$ | $S$ | $M$ | $\nu_1$ | $\nu_2$ | $\nu_3$ | $\nu_4$ |
|-------|-------|-------|-------|-----|-----|-----|-----|-------|-------|-------|-------|
| 2     | 2     | 1     | 1     | 3   | 0   | −1  | 0   | 0     | 1     | 0     | 1     |
| 2     | 1     | 1     | 1     | 0   | 0   | 1   | 0   | 0     | 0     | 1     |       |
| 2     | 2     | 3     | 3     | 0   | −1  | 0   | 1   | 1     | 1     |       |       |

Table 4. Lowest energy states of different polarization indices for $\nu = 1/4$. 

In conclusion, we want to mention that we have confirmed the most stable ground state of a particular polarized state at filling fraction $\nu = 1/2$ and $\nu = 1/4$, and possible phase transition for different polarized states within the filling fraction, which are associated with the prediction of Sujit et al [46].
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