On graded $P$-compactly packed modules

Abstract: Let $G$ be a group with identity $e$. Let $R$ be a $G$-graded commutative ring and $M$ a graded $R$-module. In this paper, we introduce the concept of graded $P$-compactly packed modules and we give a number of results concerning such graded modules. In fact, our objective is to investigate graded $P$-compactly packed modules and examine in particular when graded $R$-modules are $P$-compactly packed. Finally, we introduce the concept of graded finitely $P$-compactly packed modules and give a number of its properties.

Keywords: Graded primary submodules, Graded $P$-compactly packed modules, Graded finitely $P$-compactly packed modules

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Let \( M = \bigoplus_{g \in G} M_g \) be a graded \( R \)-module and \( N \) a submodule of \( M \). Then \( N \) is called a graded submodule of \( M \) if \( N = \bigoplus_{g \in G} N_g \) where \( N_g = N \cap M_g \) for \( g \in G \). In this case, \( N_g \) is called the \( g \)-component of \( N \) (see [6].)

Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. The graded radical of a graded ideal \( I \), denoted by \( Gr(I) \), is the set of all \( x = \sum_{g \in G} x_g \in R \) such that for each \( g \in G \) there exists \( n_g > 0 \) with \( x_g^{n_g} \in I \). Note that, if \( r \) is a homogeneous element, then \( r \in Gr(I) \) if and only if \( r^n \in I \) for some \( n \in \mathbb{N} \). A proper graded ideal \( P \) of \( R \) is said to be graded primary ideal if whenever \( r, s \in h(R) \) with \( rs \in P \), then either \( r \in P \) or \( s \in Gr(P) \) (see [8]). A proper graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be graded prime submodule if whenever \( r \in h(R) \) and \( m \in h(M) \) with \( rm \in N \), then either \( r \in (N :_RM) = \{ r \in R : rM \subseteq N \} \) or \( m \in N \). A proper graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be graded submodule if whenever \( r \in h(R) \) and \( m \in h(M) \) with \( rm \in N \), then either \( m \in N \) or \( r \in Gr((N :_RM)) \) (see [7]).

The graded primary and prime submodules are different concepts (see [8, Example 1.6].) The graded radical of a graded submodule \( N \) of a graded \( R \)-module \( M \), denoted by \( Gr_M(N) \), is defined to be the intersection of all graded prime submodules of \( M \) containing \( N \). If \( N \) is not contained in any graded primary submodule of \( M \), then \( Gr_M(N) = N \) (see [7].) A graded \( R \)-module \( M \) is called graded finitely generated if there exist \( x_{g_1}, x_{g_2}, \ldots, x_{g_n} \in h(M) \) such that \( M = Rx_{g_1} + \cdots + Rx_{g_n} \). A graded \( R \)-module \( M \) is called graded cyclic if \( M = Rm_g \) where \( m_g \in h(M) \).

## 2 Graded \( P \)-compactly packed modules

In this section, we define the graded \( P \)-compactly packed modules and give a number of its properties. We also find the necessary and sufficient conditions for any graded \( R \)-module \( M \) to be graded \( P \)-compactly packed.

**Definition 2.1.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded \( R \)-module and \( N \) a proper graded submodule of \( M \). \( N \) is called graded \( P \)-compactly packed if whenever \( N \) is contained in the union of a family of graded primary submodules of \( M \), \( N \) is contained in one of the graded primary submodules of the family. \( M \) is called graded \( P \)-compactly packed if every proper graded submodule of \( M \) is graded \( P \)-compactly packed.

**Lemma 2.2** ([4, Lemma 2.1]). Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. Then the following hold:

(i) If \( I \) and \( J \) are graded ideals of \( R \), then \( I + J \) and \( I \cap J \) are graded ideals.

(ii) If \( N \) is a graded submodule of \( M \), \( r \in h(R) \), \( x \in h(M) \) and \( I \) is a graded ideal of \( R \), then \( Rx, IN \) and \( rN \) are graded submodules of \( M \).

(iii) If \( N \) and \( K \) are graded submodules of \( M \), then \( N + K \) and \( N \cap K \) are also graded submodules of \( M \) and \( (N :_RM) = \{ r \in R : rM \subseteq N \} \) is a graded ideal of \( R \).

(iv) Let \( \{N_\lambda\} \) be a collection of graded submodules of \( M \). Then \( \sum_\lambda N_\lambda \) and \( \cap_\lambda N_\lambda \) are graded submodules of \( M \).

The graded primary radical of a graded submodule \( N \) of a graded \( R \)-module \( M \), denoted by \( P-Gr_M(N) \), is defined to be the intersection of all graded primary submodules of \( M \) containing \( N \). Should there be no graded primary submodule of \( M \) containing \( N \), then we put \( P-Gr_M(N) = M \). It is easy to see that \( P-Gr_M(N) \) is a graded submodule of \( M \) containing \( N \). We say \( N \) is a graded primary radical submodule if \( P-Gr_M(N) = N \) (see [1, Definition 2.2].)

**Theorem 2.3.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. Then the following statements are equivalent:

(i) \( M \) is a graded \( P \)-compactly packed.

(ii) For each proper graded submodule \( N \) of \( M \), there exists \( n_g \in N \cap h(M) \) such that \( P-Gr_M(N) = P-Gr_M(Rn_g) \).

(iii) For each proper graded submodule \( N \) of \( M \), if \( \{P_\alpha\}_{\alpha \in \Delta} \) is a family of graded submodules of \( M \) and \( N \subseteq \cup_{\alpha \in \Delta} P_\alpha \), then \( N \subseteq P-Gr_M(P_\beta) \) for some \( \beta \in \Delta \).

(iv) For each proper graded submodule \( N \) of \( M \), if \( \{P_\alpha\}_{\alpha \in \Delta} \) is a family of graded primary radical submodules of \( M \) and \( N \subseteq \cup_{\alpha \in \Delta} P_\alpha \), then \( N \subseteq P_\beta \) for some \( \beta \in \Delta \).
Proof. (i)⇒(ii) Assume (i) holds and let $N$ be a proper graded submodule of $M$. By [1, Theorem 2.4], $P_{Gr_M}(Rn_g) \subseteq P_{Gr_M}(N)$ for each $n_g \in N \cap h(M)$. Now, suppose that $P_{Gr_M}(N) \not\subseteq P_{Gr_M}(Rn_g)$ for each $n_g \in N \cap h(M)$. Then for each $n_g \in N \cap h(M)$ there exists a graded primary submodule $P_{n_g}$ for which $Rn_g \subseteq P_{n_g}$ and $N \not\subseteq P_{n_g}$. But $N = \cup_{n_g \in N} Rn_g \subseteq \cup_{n_g \in N} P_{n_g}$, that is $M$ is not $P$-compactly packed, a contradiction.

(ii)⇒(iii) Assume (ii) holds. Let $N$ be a proper graded submodule of $M$ and let $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded submodules of $M$ such that $N \subseteq \cup_{\alpha \in \Delta} P_\alpha$ by (ii) there exists $n_g \in N \cap h(M)$ such that $P_{Gr_M}(N) = P_{Gr_M}(Rn_g)$. Hence $n_g \in \cup_{\alpha \in \Delta} P_\alpha$ and so $n_g \in P_\beta$ for some $\beta \in \Delta$. Hence $Rn_g \subseteq P_\beta$ and by [1, Theorem 2.4], we conclude that $N \subseteq P_{Gr_M}(N) = P_{Gr_M}(Rn_g) \subseteq P_{Gr_M}(P_\beta)$.

(iii)⇒(iv) Assume (iii) holds. Let $N$ be a proper graded submodule of $M$ and let $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded primary radical submodules of $M$ such that $N \subseteq \cup_{\alpha \in \Delta} P_\alpha$ by (iii) there exists $\beta \in \Delta$ such that $N \subseteq P_{Gr_M}(P_\beta)$. Since $P_\beta$ is graded primary radical submodule, $P_\beta = P_{Gr_M}(P_\beta)$. Thus $N \subseteq P_\beta$.

(iv)⇒(i) Assume (iv) holds. Let $N$ be a proper graded submodule of $M$ and let $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded primary radical submodules of $M$ such that $N \subseteq \cup_{\alpha \in \Delta} P_\alpha$. Since $P_\alpha$ is graded primary for each $\alpha \in \Delta$, we have $P_\beta = P_{Gr_M}(P_\beta)$. Hence $N \subseteq \cup_{\alpha \in \Delta} P_\alpha = \cup_{\alpha \in \Delta} P_{Gr_M}(P_\beta)$. By (iv), there exists $\beta \in \Delta$ such that $N \subseteq P_{Gr_M}(P_\beta) = P_\beta$. Therefore, $M$ is graded $P$-compactly packed.

Lemma 2.4. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. If every proper graded submodule of $M$ is graded cyclic, then $M$ is graded $P$-compactly packed.

Proof. Let $N$ be a proper graded submodule of $M$ and let $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded primary submodules of $M$ such that $N \subseteq \cup_{\alpha \in \Delta} P_\alpha$. Since $N$ is a graded cyclic, $N = Rn_G$ for some $n_G \in N \cap h(M)$. Since $n_G \in N \subseteq \cup_{\alpha \in \Delta} P_\alpha$, $n_G \in P_\beta$ for some $\beta \in \Delta$ it follows that $N = Rn_G \subseteq P_\beta$. Therefore $M$ is graded $P$-compactly packed.

A graded $R$-module $M$ is said to be with graded primary decomposition if each of its proper graded submodules is an intersection, possibly infinite, of graded primary submodules of $M$.

Lemma 2.5. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. $M$ is a graded module with graded primary decomposition if and only if $P_{Gr_M}(N) = N$ for all graded submodules $N$ of $M$.

Proof. Let $N$ be a proper graded submodule of $M$, then $N$ has a graded primary decomposition $N = \cap_{\alpha \in \Delta} P_\alpha$. Each of $P_\alpha$ is containing $N$. Since $P_{Gr_M}(N)$ is the intersection of all graded primary submodules containing $N$, $P_{Gr_M}(N) \subseteq N$ and it is clear that $N \subseteq P_{Gr_M}(N)$. Thus $P_{Gr_M}(N) = N$. Conversely, assume that $P_{Gr_M}(N) = N$ for all graded submodules $N$ of $M$. Then every proper graded submodule of $M$ is an intersection of graded primary submodules of $M$. Hence $M$ is a graded module with graded primary decomposition.

Theorem 2.6. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module with graded primary decomposition. Then the following statements are equivalent:

(i) $M$ is a graded $P$-compactly packed.

(ii) Every proper graded submodule of $M$ is graded cyclic.

Proof. (i)⇒(ii) Assume (i) holds and let $N$ be a proper graded submodule of $M$. By Theorem 2.3, there exists $n_g \in N \cap h(M)$ such that $P_{Gr}(N) = P_{Gr}(Rn_g)$ but $M$ is graded module with graded primary decomposition, then by previous Lemma $N = Rn_g$. Thus $N$ is graded cyclic.

(ii)⇒(i) Lemma 2.4.

Recall that a proper graded submodule $N$ of a graded $R$-module $M$ is said to be graded maximal submodule if there is no graded submodule $K$ of $M$ such that $N \subsetneq K \subsetneq M$ (see [2].)

Theorem 2.7. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. If $M$ is graded $P$-compactly packed which has at least one graded maximal submodule, then $M$ satisfies the ascending chain condition on graded primary radical submodules.
Proof. Let $P_1 \subseteq P_2 \subseteq P_3 \subseteq \cdots$ be an ascending chain of graded primary radical submodules of $M$. If $P_k = M$ for some $k$, then the result follows immediately, so assume that none of $P_k$’s is $M$ and let $P = \bigcup_{i=1}^{\infty} P_i$. We claim that $P$ is a proper graded submodule of $M$. Assume on contrary that $P = M$ and let $L$ be a graded maximal submodule of $M$. Then $L \subseteq \bigcup_{i=1}^{\infty} P_i$. Since $M$ is graded $P$-compactly packed, by Theorem 2.3 $L \subseteq P_k$ for some $k$. Hence $L = P_k$ and so $P_k$ is graded maximal. Hence $P_k = P_i$ for all $i \geq k$ it follows that $P_k = \bigcup_{i=1}^{\infty} P_i = M$, which is impossible. Thus $P$ is a proper graded submodule of $M$. Since $M$ is graded $P$-compactly packed, by Theorem 2.3 $P \subseteq P_s$ for some $s$ and hence $P_s = P_i$ for all $i \geq s$. Therefore the ascending chain condition is satisfied on graded primary radical submodules. 

By [2, Lemma 2.7], every graded finitely generated module over graded ring has a graded proper maximal submodule. Then we have the following Corollary.

**Corollary 2.8.** Let $R$ be a $G$-graded ring and $M$ a graded finitely generated $R$-module. If $M$ is graded $P$-compactly packed, then $M$ satisfies the ascending chain condition on graded primary radical submodules.

**Lemma 2.9.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. If $M$ satisfies the ascending chain condition on graded primary radical submodules, then every graded primary radical submodule is the graded primary radical of a graded finitely generated submodule.

Proof. Assume that there exists a graded primary radical $P$ which is not graded primary radical of a graded finitely generated submodule. Let $n_1 \in P \cap h(M)$ and let $P_1 = P \cdot \text{Gr}_M(Rn_1)$. Then $P_1 \subsetneq P$. Hence there exists $n_2 \in (P \cap h(M)) - P_1$. Let $P_2 = P \cdot \text{Gr}_M(Rn_1 - Rn_2)$. Then $P_1 \subsetneq P_2 \subsetneq P$ and hence there exists $n_3 \in (P \cap h(M)) - P_2$ etc. This gives an ascending chain of graded primary radical submodules $P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq \cdots$ which is a contradiction.

**Theorem 2.10.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module such that every graded finitely generated submodule of $M$ is graded cyclic. If $M$ satisfies the ascending chain condition on graded primary radical submodules, then $M$ is a graded $P$-compactly packed.

Proof. Let $N$ be a proper graded submodule of $M$. By Lemma 2.9, there exists a graded finitely generated submodule $P$ of $M$ such that $P \cdot \text{Gr}_M(N) = P \cdot \text{Gr}_M(P)$. By our assumption we conclude that $P$ is a graded cyclic, it follows that there exists $n_\ell \in N \cap h(M)$ such that $P = Rn_\ell$. By Theorem 2.3, $M$ is a graded $P$-compactly packed.

Let $M$ and $M'$ be two graded $R$-modules. A homomorphism of graded $R$-modules $\varphi : M \rightarrow M'$ is a homomorphism of $R$-modules verifying $\varphi(M_\ell) \subseteq M'_\ell$ for every $g \in G$.

**Lemma 2.11.** Let $R$ be a $G$-graded ring and $M$, $M'$ be two graded $R$-modules and $\varphi : M \rightarrow M'$ be an epimorphism of graded modules. If $M$ is a graded $P$-compactly packed, then so is $M'$.

Proof. Assume that $M$ is a graded $P$-compactly packed. Let $N'$ be a proper graded submodule of $M'$ and let $\{P'_\alpha\}_{\alpha \in \Delta}$ be a family of graded primary submodules of $M'$ such that $N' \subseteq \bigcup_{\alpha \in \Delta} P'_\alpha$. Since $\varphi$ is an epimorphism of graded modules, $\varphi^{-1}(N') \subseteq \bigcup_{\alpha \in \Delta} \varphi^{-1}(P'_\alpha)$. Hence $\varphi^{-1}(N') \subseteq \bigcup_{\alpha \in \Delta} \varphi^{-1}(P'_\alpha)$. By [1, Lemma 2.14], $\varphi^{-1}(P'_\alpha)$ is a graded primary submodule of $M$ for each $\alpha \in \Delta$. Since $M$ is a graded $P$-compactly packed, there exists $\beta \in \Delta$ such that $\varphi^{-1}(N') \subseteq \varphi^{-1}(P'_\beta)$. Thus $N' \subseteq P'_\beta$ for some $\beta \in \Delta$. Therefore $M'$ is a graded $P$-compactly packed.

**Theorem 2.12.** Let $R$ be a $G$-graded ring and $M$, $M'$ be two graded $R$-modules and $\varphi : M \rightarrow M'$ be an epimorphism of graded modules such that $\text{Ker}(\varphi) \subseteq P \cdot \text{Gr}_M(\{0\})$. Then $M$ is a graded $P$-compactly packed if and only if $M'$ is a graded $P$-compactly packed.

Proof. $(\Rightarrow)$ Lemma 2.11.

$(\Leftarrow)$ Assume that $M'$ is a graded $P$-compactly packed. Let $N$ be a proper graded submodule of $M$ and let $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded primary submodules of $M$ such that $N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. Then $\varphi(N) \subseteq \varphi(\bigcup_{\alpha \in \Delta} P_\alpha)$
and hence \( \varphi(N) \subseteq \bigcup_{\alpha \in \Delta} \varphi(P_{\alpha}). \) Since \( \text{Ker}(\varphi) \subseteq P_{\alpha} \) for each \( \alpha \in \Delta, \) by [1, Lemma 2.15], \( \varphi(P_{\alpha}) \) is a graded primary submodule of \( M'. \) Since \( M' \) is a graded \( P \)-compactly packed, \( \varphi(N) \subseteq \varphi(P_{\beta}) \) for some \( \beta \in \Delta. \) Now, we show that \( N \subseteq P_{\beta}. \) Let \( n = \sum_{g \in G} n_{g} \in N. \) For \( g \in G, n_{g} \in N \) and so \( \varphi(n_{g}) \in \varphi(N) \subseteq \varphi(P_{\beta}). \) Hence there exists \( t \in P_{\beta} \cap h(M) \) such that \( \varphi(n_{g}) = \varphi(t). \) Hence \( n_{g} - t \in \text{Ker}(\varphi) \subseteq P_{\beta}, \) it follows that \( n_{g} \in P_{\beta}. \) So \( N \subseteq P_{\beta}. \) Therefore \( M \) is a graded \( P \)-compactly packed.

Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module and \( S \subseteq h(R) \) a multiplicatively closed subset of \( R. \) A non empty subset \( S^* \) of \( h(M) \) is said to be graded \( S \)-closed if \( \forall s \in S \) and \( e \in S^* \) (see [4, Definition 2.11]).

**Lemma 2.13.** Let \( S \subseteq h(R) \) be a multiplicatively closed subset of graded ring \( R \) and \( S^* \subseteq h(M) \) be a graded \( S \)-closed of a graded \( R \)-module \( M. \) If \( N \) is a graded submodule of \( M \) contained in \( M - S^*, \) then \( \text{Gr}((N : R M)) \cap S = \emptyset. \)

**Proof.** Assume that \( \text{Gr}((N : R M)) \cap S \neq \emptyset \) and let \( r_{g} \in \text{Gr}((N : R M)) \cap S. \) Then \( r_{g}^{k} M \subseteq \text{N} \) for some \( k \in \mathbb{N} \) and for any \( e \in S^*, r_{g}^{k} e \in S^* \cap \text{N}, \) which is contradiction with \( N \subseteq M - S^*. \)

Recall that a graded \( R \)-module \( M \) is called graded multiplication if for each graded submodule \( N \) of \( M, N = IM \) for some graded ideal \( I \) of \( R. \) One can easily show that if \( N \) is a graded submodule of a graded multiplication module \( M, \) then \( N = (N : R M), \) (see [7, Definition 2.]) Also, a proper graded ideal \( P \) of a \( G \)-graded ring \( R \) is graded \( P \)-compactly packed if whenever \( P \) is contained in the union of a family of graded primary ideals of \( R, \) \( P \) is contained in one of the graded primary ideals of the family. A graded ring \( R \) is said to be graded \( P \)-compactly packed if every proper graded ideals of \( R \) is graded \( P \)-compactly packed.

**Theorem 2.14.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded multiplication \( R \)-module such that \( \text{Gr}_{M}(N) = N \) for all graded submodules \( N \) of \( M. \) If \( R \) is a graded \( P \)-compactly packed and \( M \neq \bigcup_{\alpha \in \Delta} P \) for each family \( \{P_{\alpha}\}_{\alpha \in \Delta} \) of graded primary submodules of \( M, \) then \( M \) is graded \( P \)-compactly packed.

**Proof.** Let \( N \) be a proper graded submodule of \( M \) and let \( \{P_{\alpha}\}_{\alpha \in \Delta} \) be a family of graded primary radical submodules of \( M \) such that \( N \subseteq \bigcup_{\alpha \in \Delta} P_{\alpha}. \) Put \( S^* = h(M) - \bigcup_{\alpha \in \Delta} P_{\alpha}. \) Then \( S^* \) is graded \( S \)-closed of \( M \) where \( S = h(R) - \bigcup_{\alpha \in \Delta} \text{Gr}((P_{\alpha} : R M)). \) Since \( \emptyset \cap S^* = \emptyset, \) by Lemma 2.13 \( \text{Gr}((N : R M)) \cap S = \emptyset. \) Hence \( \text{Gr}((N : R M)) \subseteq \bigcup_{\alpha \in \Delta} \text{Gr}((P_{\alpha} : R M)). \) By [1, Lemma 2.7], \( \text{Gr}((P_{\alpha} : R M)) \) is graded primary ideals of \( R \) for all \( \alpha. \) Since \( R \) is a graded \( P \)-compactly packed, \( \text{Gr}((N : R M)) \subseteq \text{Gr}((P_{\beta} : R M)) \) for some \( \beta. \) Therefore, \( M \) is graded \( P \)-compactly packed.

## 3 Graded finitely \( P \)-compactly packed modules

In this section, we define the graded finitely \( P \)-compactly packed modules and give a number of its properties. Also, we find the conditions that make graded finitely \( P \)-compactly packed modules graded \( P \)-compactly packed.

**Definition 3.1.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. A proper graded submodule \( N \) of \( M \) is called graded finitely \( \bigcup_{\alpha \in \Delta} P_{\alpha} \) of graded primary submodules of \( M \) with \( N \subseteq \bigcup_{\alpha \in \Delta} P_{\alpha}, \) there exist \( \alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \in \Delta \) such that \( N \subseteq \bigcup_{i=1}^{n} P_{\alpha_{i}}. \) A graded module \( M \) is called graded finitely \( P \)-compactly packed if every proper graded submodule of \( M \) is graded finitely \( P \)-compactly packed.

It is clear that if \( M \) is graded \( P \)-compactly packed, then \( M \) is graded finitely \( P \)-compactly packed.

**Theorem 3.2.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module in which every finite family of graded primary submodules of \( M \) is totally ordered by inclusion. If \( M \) is graded finitely \( P \)-compactly packed, then \( M \) is graded \( P \)-compactly packed.
Theorem 3.3. Let $R$ be a $G$-graded ring and $M$ a graded multiplication $R$-module such that $Gr_M(N) = N$ for all graded submodules $N$ of $M$. If $M$ is graded finitely $P$-compactly packed, then $M$ is graded $P$-compactly packed.

Proof. Let $N$ be a proper graded submodule of $M$ and let $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded primary submodules of $M$ such that $N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha$. Since $M$ is graded finitely $P$-compactly packed, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $N \subseteq \bigcup_{i=1}^n P_{\alpha_i}$. Since $\{P_{\alpha_i}\}_{i=1}^n$ is totally ordered by inclusion, there exists $\beta \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $\bigcup_{i=1}^n P_{\alpha_i} = P_\beta$. Thus $M$ is graded $P$-compactly packed.

Let $N_1, N_2, \ldots, N_n$ be graded submodules of a graded $R$-module $M$. We call a covering $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ efficient if $N$ is not contained in the union of any $n - 1$ of the graded submodules $N_1, N_2, \ldots, N_n$. Any covering of a union of graded submodules can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms, (see [3].)

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