Copula–Induced Measures of Concordance

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Abstract: We study measures of concordance for multivariate copulas and copulas that induce measures of concordance. To this end, for a copula $A$, we consider the maps $C \mapsto \psi_{\Lambda}(C, A) - [\Pi, A]$ resp. $C \mapsto \Lambda(C, A) - [\Pi, \Pi]$ where $C$ denotes the collection of all $d$–dimensional copulas, $M$ is the Fréchet–Hoeffding upper bound, $\Pi$ is the product copula, $[\cdot, \cdot] : C \times C \to \mathbb{R}$ is the biconvex form given by $[C, D] := \int_{[0,1]^d} C(u) \, dQ^D(u)$ with the probability measure $Q^D$ associated with the copula $D$, and $\psi_{\Lambda} : C \to C$ is a transformation of copulas. We present conditions on $\psi_{\Lambda}$ and on $A$ under which these maps are measures of concordance. The resulting class of measures of concordance is rich and includes the well–known examples Spearman’s rho and Gini’s gamma.

Keywords: copulas, transformations of copulas, measures of concordance

MSC: 62H20

1 Introduction

In the present paper we study measures of concordance for copulas (of fixed but arbitrary dimension) and copulas that induce measures of concordance.

The literature on measures of concordance for multivariate copulas is rich. It provides a huge number of multivariate generalizations of well–known bivariate measures of concordance (see e.g. [2, 11–13, 16, 19]), and various axiomatic definitions of a measure of concordance (see e.g. [3, 8, 16]).

We here employ the quite general definition proposed in [8]. Besides the usual normalization with respect to the Fréchet–Hoeffding upper bound $M$, this definition includes two axioms which are formulated in terms of certain subgroups of the group $\Gamma$ of transformations on copulas discussed in [10] for the bivariate case and in [8] for the general case. These axioms provide an easy access to the investigation of invariance properties of a measure of concordance. In particular, it turns out that for copulas which are invariant under a specific subgroup of $\Gamma$ the value of every measure of concordance is equal to zero. Since the product copula $\Pi$ satisfies this invariance property, we obtain the usual normalization with respect to $\Pi$.

We study measures of concordance which are induced by a fixed copula. Throughout this paper, let $\mathcal{C}$ denote the collection of all $d$–dimensional copulas and consider the biconvex form $[\cdot, \cdot] : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ studied in [9], which is given by $[C, D] := \int_{[0,1]^d} C(u) \, dQ^D(u)$ where $Q^D$ is the probability measure associated with the copula $D$. Consider now a copula $A \in \mathcal{C}$ and a subgroup $\Lambda$ of $\Gamma$ and let $\psi_{\Lambda} : \mathcal{C} \to \mathcal{C}$ denote the map which turns every copula $C$ to the arithmetic mean of the
orbit \( \{ \gamma(C) | \gamma \in A \} \) of \( C \) under \( A \). Since \([M, A] \neq [II, A]\), by Lemma A.3 (1), the map \( \kappa^*_A : \mathcal{C} \to \mathbb{R} \) given by

\[
\kappa^*_A(C) := \frac{[\psi_A(C), A] - [II, A]}{[M, A] - [II, A]}
\]

is well–defined. In the case where \([M, A] \neq [II, II]\), we also define the map \( \kappa^*_A : \mathcal{C} \to \mathbb{R} \) given by

\[
\kappa^*_A(C) := \frac{[\psi_A(C), A] - [II, II]}{[M, A] - [II, II]}
\]

We note that \([M, A] \neq [II, II]\) is fulfilled whenever \( d \geq 3 \); see Lemma A.3 (2).

In what follows we present conditions on \( A \) under which these maps are measures of concordance. Thereby, we improve results given in [3, 16] for the general case, and for the bivariate case we extend results from [1, 5, 6, 10]. In addition, for a subclass of such copula–induced measures of concordance, we present lower bounds and show that these lower bounds tend to 0 when the dimension of the copulas tends to infinity.

This paper is organized as follows: We first recall essential definitions and results concerning the group \( \Gamma \) of transformations on \( \mathcal{C} \) discussed in [8] and the biconvex form \([\ldots, \ldots]\) for copulas introduced in [9] (Section 2). We then study measures of concordance for copulas (Section 3) and copulas that induce measures of concordance (Section 4), and we present some results on lower bounds for such measures of concordance (Section 5). Auxiliary results needed for the proofs in Sections 3 and 4 are presented in the appendix (Section A).

## 2 Preliminaries

Let \( I := [0, 1] \) and let \( d \geq 2 \) be an integer which will be kept fix throughout this paper. For the sake of a concise definition of a copula we consider, for \( L \subseteq \{1, \ldots, d\} \), the map \( \eta_L : I^d \times I^d \to I^d \) given coordinatewise by

\[
\eta_L(u, v)_l := \begin{cases} u_l & l \in \{1, \ldots, d\} \setminus L \\ v_l & l \in L \end{cases}
\]

We denote by \( \mathbf{0} \) the vector with entries 0 and by \( \mathbf{1} \) the vector with entries 1. A **copula** is a function \( C : I^d \to I \) satisfying the following conditions:

(i) The inequality

\[
\sum_{L \subseteq \{1, \ldots, d\}} (-1)^{|L|} C(\eta_L(u, v)) \geq 0
\]

holds for all \( u, v \in I^d \) such that \( u \leq v \).

(ii) The identity \( C(\eta_i(u, \mathbf{0})) = 0 \) holds for all \( u \in I^d \) and all \( i \in \{1, \ldots, d\} \).

(iii) The identity \( C(\eta_{\{i\}}(\mathbf{1}, u)) = u_i \) holds for all \( u \in I^d \) and all \( i \in \{1, \ldots, d\} \).

Note that, for \( u \leq v \), the family \( \{ \eta_L(u, v) \}_{L \subseteq \{1, \ldots, d\}} \) consists of all vertices of the interval \([u, v]\). Thus, this definition of a copula is appropriate and in accordance with the literature; see [4, 14]. The collection \( \mathcal{C} \) of all copulas is convex.

### The Group \( \Gamma \)

A map \( \varphi : \mathcal{C} \to \mathcal{C} \) is said to be a **transformation**. We denote by \( \Phi \) the collection of all transformations and define the composition \( \circ : \Phi \times \Phi \to \Phi \) by letting \( (\varphi_1 \circ \varphi_2)(C) = \varphi_1(\varphi_2(C)) \). For the composition of \( n \in \mathbb{N}_0 \) transformations \( \varphi_m \in \Phi, m \in \{1, \ldots, n\} \), we write

\[
\bigcirc_{m=1}^n \varphi_m := \begin{cases} i & n = 0 \\ \varphi_n \circ \bigcirc_{m=1}^{n-1} \varphi_m & \text{otherwise} \end{cases}
\]

where \( i \) denotes the identity on \( \mathcal{C} \). \((\Phi, \circ)\) is a semigroup with neutral element \( i \).
We now introduce two elementary transformations: For \( i, j, k \in \{1, \ldots, d\} \) with \( i \neq j \) we define the maps \( \pi_{i,j}, \nu_k : \mathcal{C} \to \mathcal{C} \) by letting
\[
\begin{align*}
(\pi_{i,j}(C))&(\mathbf{u}) := C(\eta_{(i,j)}(\mathbf{u}, u_i \mathbf{e}_i + u_j \mathbf{e}_j)) \\
(\nu_k(C))&(\mathbf{u}) := C(\eta_{(k)}(\mathbf{u}, 1)) - C(\eta_{(k)}(\mathbf{u}, 1 - \mathbf{u}))
\end{align*}
\]
where \( \mathbf{e}_i \) denotes the \( i \)-th unit vector. \( \pi_{i,j} \) is called a \textit{transposition}, and \( \nu_k \) is called a \textit{partial reflection}. Both, \( \pi_{i,j} \) and \( \nu_k \), are involutions. There exists a smallest subgroup \( (\Gamma, \circ) \) of \( \Phi \) containing all transpositions and all partial reflections. This group \( \Gamma \) is a representation of the hyperoctahedral group with \( \left( 2^d - 1 \right) \) elements.

A transformation is called a \textit{permutation} if it can be expressed as a finite composition of transpositions, and it is called a \textit{reflection} if it can be expressed as a finite composition of partial reflections. We denote by \( \Gamma^\pi \) the set of all permutations and by \( \Gamma^r \) the set of all reflections. Then \( \Gamma^\pi \) and \( \Gamma^r \) are subgroups of \( \Gamma \), \( \Gamma^r \) is commutative while \( \Gamma^\pi \) is not, and every transformation in \( \Gamma \) can be expressed as a composition of a permutation and a reflection. Due to its particular interest we emphasize the reflection \( \tau := \Omega_{k=1}^d \nu_k \), an involution called \textit{total reflection}. The total reflection \( \tau \) transforms every copula into its survival copula. We set \( \Gamma^r := \{ \iota, \tau \} \) and \( \Gamma^r = \{ \gamma \in \Gamma \mid \gamma = \pi \circ \varphi \text{ for some } \pi \in \Gamma^\pi \text{ and some } \varphi \in \Gamma^r \} \). Then \( \Gamma^r \) is the center of \( \Gamma \), and \( \Gamma^r = \Gamma^r \) is a subgroup of \( \Gamma \).

For a subgroup \( \Delta \) of \( \Gamma \), a copula \( C \) is called \( \Delta \)-\textit{invariant} if it satisfies \( \gamma(C) = C \) for every \( \gamma \in \Delta \). The copula \( M \) is \( \Gamma^r \)-invariant and the copula \( II \) is \( \Gamma \)-invariant. For the ease of notation, we put
\[
C_\Delta := \psi_\Delta(C) = \frac{1}{|\Delta|} \sum_{\gamma \in \Delta} \gamma(C)
\]
Then \( C_\Delta \) is \( \Delta \)-invariant.

### The Biconvex Form

For a convenient study of measures of concordance, we use the map \( [\ldots] : \mathcal{C} \times \mathcal{C} \to \mathbb{R} \) given by
\[
[C, D] := \int_{\mathcal{T}} C(\mathbf{u}) \, dQ^D(\mathbf{u})
\]
where \( Q^D \) denotes the probability measure associated with the copula \( D \). The map \( [\ldots] \) is linear with respect to convex combinations in both arguments and is therefore called a \textit{biconvex form}. Moreover, \([\ldots]\) is monotonically increasing in both arguments with regard to the concordance order \( \preceq \) on \( \mathcal{C} \) (i.e. \( C \preceq D \) and \( \tau(C) \preceq \tau(D) \)), and it satisfies \( 0 \leq [C, D] \leq [M, M] = 1/2 \) for all \( C, D \in \mathcal{C} \) (note that \( C \preceq M \) for all \( C \in \mathcal{C} \)); we also note that \([II, II]\) = \( 1/2^d \).

### 3 Measures of Concordance

In this section, we study measures of concordance for copulas. We employ the quite general definition of a measure of concordance proposed in \[8\], in which the axioms are formulated in terms of two particular subgroups of the group \( \Gamma \), and we discuss invariance properties following from these axioms.

A map \( \kappa : \mathcal{C} \to \mathbb{R} \) is said to be a \textit{measure of concordance} if it satisfies the following axioms:

(i) \( \kappa(M) = 1 \).

(ii) The identity \( \kappa(\gamma(C)) = \kappa(C) \) holds for all \( \gamma \in \Gamma^r \) and all \( C \in \mathcal{C} \).

(iii) The identity
\[
\sum_{\nu \in \Gamma_r} \kappa(\nu(C)) = 0
\]
holds for all \( C \in \mathcal{C} \).
For the case $d = 2$, this definition is in accordance with [5–7, 10, 15].

Axiom (ii) implies that every transformation in $\Gamma^{n, r}$ preserves the value of every measure of concordance. Moreover, it turns out that the transformations in $\Gamma^{n, r}$ are in fact the only transformations in $\Gamma$ having this property; see [8, Theorem 6.2.]:

**Proposition 3.1.** Let $\kappa$ be a measure of concordance and $\gamma \in \Gamma$. Then the identity $\kappa(\gamma(C)) = \kappa(C)$ holds for every $C \in \mathcal{C}$ if, and only if, $\gamma \in \Gamma^{n, r}$.

Axiom (iii) provides a simple test on copulas under which every measure of concordance of a copula is equal to zero:

**Corollary 3.2.** Let $\kappa$ be a measure of concordance. Then the identity $\kappa(C) = 0$ holds for every $\Gamma^n$–invariant copula $C \in \mathcal{C}$. In particular, $\kappa(\Pi) = 0$.

Thus, every measure of concordance is normalized by the copulas $M$ and $\Pi$.

Except for these basic axioms of a measure of concordance which are essential for our investigation, many authors like [3, 5–7, 15–18] require additional properties: A map $\kappa : \mathcal{C} \to \mathbb{R}$ is said to be

- convex if the identity $\kappa(C + (1 - a)D) = a \kappa(C) + (1 - a) \kappa(D)$ holds for every $C, D \in \mathcal{C}$ and every $a \in (0, 1)$.
- concordance order preserving if, for any $C, D \in \mathcal{C}$, $C \preceq D$ implies $\kappa(C) \leq \kappa(D)$.
- continuous if, for any sequence $(C_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}$ and any copula $C \in \mathcal{C}$, uniform convergence $\lim_{n \to \infty} C_n = C$ implies $\lim_{n \to \infty} \kappa(C_n) = \kappa(C)$.

We note that, for every convex map $\kappa : \mathcal{C} \to \mathbb{R}$ satisfying axioms (i) and (ii), $\kappa$ is a measure of concordance if, and only if, the identity $\kappa(C_n) = 0$ holds for all $C \in \mathcal{C}$. We also note that every concordance order preserving measure of concordance satisfies $\kappa(C) \leq \kappa(M) = 1$.

## 4. Copula–Induced Measures of Concordance

In the present section, we discuss a class of measures of concordance which are defined in terms of the bi-convex form $[\cdot, \cdot]$, a subgroup $A$ of $\Gamma$ and a fixed copula $A \in \mathcal{C}$. More precisely, we consider the map $\kappa_{A, A}^{\ast}$; we note that $[M, A] \neq [\Pi, A]$, by Lemma A.3 (1), and we recall that $\kappa_{A, A}^{\ast}$ satisfies

$$
\kappa_{A, A}^{\ast}(C) = \frac{[C, A] - [\Pi, A]}{[M, A] - [\Pi, A]}
$$

for all $C \in \mathcal{C}$. In the case where $[M, A] \neq [\Pi, \Pi]$, we also consider the map $\kappa_{A, A}^{\circ}$; we note that $[M, A] \neq [\Pi, \Pi]$ is fulfilled whenever $d \geq 3$, by Lemma A.3 (2), and we recall that $\kappa_{A, A}^{\circ}$ satisfies

$$
\kappa_{A, A}^{\circ}(C) = \frac{[C, A] - [\Pi, \Pi]}{[M, A] - [\Pi, \Pi]}
$$

for all $C \in \mathcal{C}$. We also note that, if $A \subseteq \Gamma^{n, r}$, both maps satisfy axiom (i) of a measure of concordance due to the fact that $M$ is $\Gamma^{n, r}$–invariant. Without any assumptions on $A$ we obtain the following result whose proof is straightforward:

**Lemma 4.1.** (1) The map $\kappa_{A, A}^{\ast}$ is convex, continuous and, if $A \subseteq \Gamma^{n, r}$, also concordance order preserving.

(2) Assume that $[M, A] \neq [\Pi, \Pi]$. Then the map $\kappa_{A, A}^{\circ}$ is convex, continuous and, if $A \subseteq \Gamma^{n, r}$, also concordance order preserving.

In what follows we present conditions on $A$ under which $\kappa_{A, A}^{\ast}$ and $\kappa_{A, A}^{\circ}$ are measures of concordance. The following proposition summarizes the results for the bivariate case which are essentially due to [5, 6]; see also [10]. Note that $C_{(\cdot)} = C$ for all $C \in \mathcal{C}$.
 Proposition 4.2. Assume that \( d = 2 \).

1. \( \Lambda \) is \( \Gamma \)-invariant if, and only if, \( \kappa_{A,\{1\}}^\bullet \) is a measure of concordance.
2. \( \Lambda \) is \( \Gamma \)-invariant if, and only if, \( [M, \Lambda] \neq [\Pi, \Pi] \) and \( \kappa_{\Lambda,\{1\}}^\circ \) is a measure of concordance.

By contrast, for \( d \geq 3 \) and although the copula \( \Pi \) is \( \Gamma \)-invariant, \( \kappa_{\Pi,\{1\}}^\bullet \) and \( \kappa_{\Pi,\{1\}}^\circ \) fail to satisfy axiom (ii) of a measure of concordance. Thus, Proposition 4.2 cannot be extended to the general case:

Example 4.3. Consider \( d \geq 3 \). Then there exists some \( C \in \mathcal{C} \) satisfying

\[
\kappa_{\Pi,\{1\}}^\bullet (\tau(C)) \neq \kappa_{\Pi,\{1\}}^\circ (C) \quad \text{and} \quad \kappa_{\Pi,\{1\}}^\bullet (\tau(C)) \neq \kappa_{\Pi,\{1\}}^\circ (C)
\]

which contradicts axiom (ii) of a measure of concordance. Indeed, if \( d \) is odd, then the copula \( C \) with

\[
C(u) := \Pi(u) + \sum_{i=1}^{d} u_i (1 - u_i)
\]

discussed in [9, Examples 3.4] satisfies \( \tau(C), \Pi] = 1/2^d - 1/6^d \neq 1/2^d + 1/6^d = [C, \Pi] \) and hence \( \kappa_{\Pi,\{1\}}^\bullet (\tau(C)) \neq \kappa_{\Pi,\{1\}}^\circ (C) \) and \( \kappa_{\Pi,\{1\}}^\bullet (\tau(C)) \neq \kappa_{\Pi,\{1\}}^\circ (C) \); if \( d \) is even, then the copula \( C \) with

\[
C(u) := \Pi(u) + u_1 \sum_{i=2}^{d} u_i (1 - u_i)
\]

discussed in [9, Examples 3.4] satisfies \( \tau(C), \Pi] = 1/2^d - 3/6^d \neq 1/2^d + 3/6^d = [C, \Pi] \) and hence \( \kappa_{\Pi,\{1\}}^\bullet (\tau(C)) \neq \kappa_{\Pi,\{1\}}^\circ (C) \) and \( \kappa_{\Pi,\{1\}}^\bullet (\tau(C)) \neq \kappa_{\Pi,\{1\}}^\circ (C) \).

To obtain measures of concordance in the general case, one may replace the trivial subgroup \( \{1\} \) by an appropriate subgroup \( \Lambda \) of \( \Gamma \) with \( \Lambda \subseteq \Gamma^{\text{inv}} \) and impose a condition on \( A \). For example, one has the following result which is essentially due to [3, 16]:

Proposition 4.4. Assume that \( A \) is \( \Gamma \)-invariant. Then each of the maps \( \kappa_{A,\Pi}^\bullet \) and \( \kappa_{A,\Pi}^\circ \) is a measure of concordance.

The following examples are multivariate generalizations of Gini’s gamma and Spearman’s rho:

Example 4.5. (1) The map \( \kappa_{M,\Pi}^\bullet \) is called Gini’s gamma, and from \( [M, M_\Pi] = 1/4 + 1/2^d + 1 \) and \( \Pi, M_\Pi] = 1/2^d \) we obtain

\[
\kappa_{M,\Pi}^\bullet (C) = \frac{[C, M_\Pi] - [\Pi, M_\Pi]}{[M, M_\Pi] - [\Pi, M_\Pi]} = \frac{2^d + 1}{2^d - 1} \left( \frac{[C, M_\Pi] - 1/2^d}{[M, M_\Pi] - [\Pi, M_\Pi]} \right)
\]

for all \( C \in \mathcal{C} \). Since \( M_\Pi \) is \( \Gamma \)-invariant, Proposition 4.4 implies that Gini’s gamma is a measure of concordance.

(2) The map \( \kappa_{\Pi,\Pi}^\circ \) is called Spearman’s rho, and from \( [M, \Pi] = 1/(d + 1) \) and \( \Pi, \Pi] = 1/2^d \) we obtain

\[
\kappa_{\Pi,\Pi}^\circ (C) = \frac{[C, \Pi] - [\Pi, \Pi]}{[M, \Pi] - [\Pi, \Pi]} = \frac{2^d}{2^d - (d + 1)} \left( \frac{[C, \Pi] - 1/2^d}{[M, \Pi] - [\Pi, \Pi]} \right)
\]

for all \( C \in \mathcal{C} \). Since \( \Pi \) is \( \Gamma \)-invariant, Proposition 4.4 implies that Spearman’s rho is a measure of concordance.

Since \( \Gamma \)-invariance of a copula is a quite strong property, one would like to relax the assumption on \( A \). To this end, we replace the arithmetic mean under \( \Gamma^r \) by the arithmetic mean under the larger subgroup \( \Gamma^{\text{inv}} \) used in axiom (ii) and Proposition 3.1. Thus, we consider the map \( \kappa_{A,\Pi}^\bullet \) and, if \( [M, A] \neq [\Pi, \Pi] \), the map \( \kappa_{A,\Pi}^\circ \); we recall that \( \kappa_{A,\Pi}^\bullet \) and \( \kappa_{A,\Pi}^\circ \) satisfy

\[
\kappa_{A,\Pi}^\bullet (C) = \frac{[C, A] - [\Pi, A]}{M, A] - [\Pi, A]} \quad \text{resp.} \quad \kappa_{A,\Pi}^\circ (C) = \frac{[C, A] - [\Pi, A]}{M, A] - [\Pi, A]}
\]

Since

\[
\kappa_{A,\Pi}^\bullet (C) = \frac{[C, A] - [\Pi, A]}{M, A] - [\Pi, A]} \quad \text{resp.} \quad \kappa_{A,\Pi}^\circ (C) = \frac{[C, A] - [\Pi, A]}{M, A] - [\Pi, A]}
\]
The identity of concordance:$$\sum_{v \in F_r} \left[ v(C) \right]_{F_n, A} = \left[ \sum_{v \in F_r} (v(C))_{F_n, A} \right] = \left[ \sum_{v \in F_r} (v(C))_{F_n, A} \right]$$

\[ \text{for all } C \in \mathcal{C} \]. Moreover, we recall that $\kappa_{A,F_n,r}$ and $\kappa_{A,F_n,r}^*$ satisfy axiom (i) of a measure of concordance, and it follows from Lemma A.1 that they satisfy axiom (ii) as well. We are hence interested in conditions on $A$ under which each of these maps also satisfies axiom (iii) of a measure of concordance. We start our investigation with a quite abstract characterization:

**Theorem 4.6.** (1) The identity $[C_{F_n}, A_{F_n}] = [II, A]$ holds for all $C \in \mathcal{C}$ if, and only if, $\kappa_{A,F_n,r}$ is a measure of concordance.

(2) The identity $[C_{F_n}, A_{F_n}] = [II, II]$ holds for all $C \in \mathcal{C}$ if, and only if, $[M, A] \neq [II, A]$ and $\kappa_{A,F_n,r}^*$ is a measure of concordance.

(3) If $\kappa_{A,F_n,r}$ is a measure of concordance, then $\kappa_{A,F_n,r}^*$ is a measure of concordance as well.

**Proof.** Lemma A.2 and Lemma A.4 first imply

$$\frac{1}{2d} \sum_{v \in F_r} \left[ v(C) \right]_{F_n, A} = \left[ \frac{1}{2d} \sum_{v \in F_r} (v(C))_{F_n, A} \right] = \left[ \sum_{v \in F_r} (v(C))_{F_n, A} \right] = [C_{F_n}, A_{F_n}]$$

for all $C \in \mathcal{C}$.

We now prove (1). Lemma A.3 (1) yields $[M, A] \neq [II, A]$ and the above identity then implies

$$\left( [M, A] - [II, A] \right) \frac{1}{2d} \sum_{v \in F_r} \kappa_{A,F_n,r}(v(C)) = \frac{1}{2d} \sum_{v \in F_r} \left[ (v(C))_{F_n, A} - [II, A] \right] = [C_{F_n}, A_{F_n}] - [II, A]$$

for all $C \in \mathcal{C}$. Thus, the identity $[C_{F_n}, A_{F_n}] = [II, A]$ holds for all $C \in \mathcal{C}$ if, and only if, $\kappa_{A,F_n,r}$ satisfies axiom (iii) of a measure of concordance. This proves (1).

We now prove (2). To this end, assume that the identity $[C_{F_n}, A_{F_n}] = [II, II]$ holds for all $C \in \mathcal{C}$. The fact that $II = II_{F_n} = II_{F_n}$ together with Lemma A.3 (1) and Lemma A.4 yields $[M, A] \neq [II, A] = [II_{F_n}, A] = [II_{F_n}, A]$ and the above identity then implies

$$\sum_{v \in F_r} \kappa_{A,F_n,r}(v(C)) = \frac{\sum_{v \in F_r} \left[ (v(C))_{F_n, A} - [II, II] \right]}{[M, A] - [II, II]} = \frac{2d ([C_{F_n}, A_{F_n}] - [II, II])}{[M, A] - [II, II]} = 0$$

for all $C \in \mathcal{C}$. Thus, $\kappa_{A,F_n,r}^*$ satisfies axiom (iii) of a measure of concordance. Assume now that $[M, A] \neq [II, II]$ and that $\kappa_{A,F_n,r}^*$ is a measure of concordance. Then the above identity implies

$$[C_{F_n}, A_{F_n}] - [II, II] = \frac{1}{2d} \sum_{v \in F_r} \left[ (v(C))_{F_n, A} - [II, II] \right] = \left( [M, A] - [II, II] \right) \frac{1}{2d} \sum_{v \in F_r} \kappa_{A,F_n,r}(v(C)) = 0$$

for all $C \in \mathcal{C}$. Thus, the identity $[C_{F_n}, A_{F_n}] = [II, II]$ holds for all $C \in \mathcal{C}$. This proves (2).

Finally, let $\kappa_{A,F_n,r}^*$ be a measure of concordance. Then (2) and Lemma A.4 imply $[C_{F_n}, A_{F_n}] = [II, II] = [II_{F_n}, A_{F_n}] = [II_{F_n}, A]$ and the above identity then implies

$$[C_{F_n}, A_{F_n}] - [II, II] = \frac{1}{2d} \sum_{v \in F_r} \left[ (v(C))_{F_n, A} - [II, II] \right] \left( [M, A] - [II, II] \right) \frac{1}{2d} \sum_{v \in F_r} \kappa_{A,F_n,r}(v(C)) = 0$$

for all $C \in \mathcal{C}$. Thus, the identity $[C_{F_n}, A_{F_n}] = [II, II]$ holds for all $C \in \mathcal{C}$. This proves (3).

The following result provides a condition on $A$ which ensures that the maps $\kappa_{A,F_n,r}^*$ and $\kappa_{A,F_n,r}$ are measures of concordance:

**Theorem 4.7.** Assume that $A_{F_n}$ is $F_n$-invariant. Then $[M, A] \neq [II, II]$ and each of the maps $\kappa_{A,F_n,r}$ and $\kappa_{A,F_n,r}^*$ is a measure of concordance. Moreover, $\kappa_{A,F_n,r}^* = \kappa_{A,F_n,r}^*$.  

**Proof.** [9, Theorem 5.7] implies $[C_{F_n}, A_{F_n}] = 1/2d = [II, II]$ for all $C \in \mathcal{C}$. It hence follows from Theorem 4.6 that $[M, A] \neq [II, II]$ and that each of the maps $\kappa_{A,F_n,r}$ and $\kappa_{A,F_n,r}^*$ is a measure of concordance. Lemma A.4 further implies $[II, A] = [II_{F_n}, A] = [II_{F_n}, A]$ which yields $\kappa_{A,F_n,r}^* = \kappa_{A,F_n,r}^*$.  

The next example provides a copula $A$ for which $A_{F_n}$ is $F_n$-invariant:
Example 4.8. The function $A : I^d \to I$ given by

$$A(u) := \Pi(u) + \frac{1}{2} (u_1 - u_2) \prod_{i=1}^d u_i (1 - u_i)$$

is a copula for which $A_{\tau^n} = \Pi$. Therefore, $A_{\tau^n}$ is $I^\nu$–invariant and $\kappa^*_{A,\tau^n} = \kappa^0_{A,\tau^n}$ is a measure of concordance.

At this point, we briefly discuss the condition on $A$ used in Theorem 4.7; the proof of Lemma 4.9 is straightforward:

Lemma 4.9. (1) If $A$ is $I^\nu$–invariant, then $A_{\tau^n}$ is $I^\nu$–invariant.

(2) If $A$ is $I$–invariant, then $A$ is $I^\nu$–invariant.

The converse implication of Lemma 4.9 (1) is not true in general; see Example 4.8 ($A \neq \nu_1(A)$). The following example shows that also the converse implication of Lemma 4.9 (2) is not true, in general:

Example 4.10. The function $A : I^d \to I$ given by

$$A(u) := \Pi(u) + \frac{1}{4} (1 - u_1)(1 - 2u_1)^3 (1 - u_2)(1 - 2u_2) \prod_{i=1}^d u_i$$

is a copula which is $I^\nu$–invariant. However, $A$ is not $I$–invariant ($A \neq \pi_{1,2}(A)$).

We recall that the copula $A_{\tau^n}$ is $I^\nu$–invariant, and it follows from Lemma 4.9 that the copula $(A_{\tau^n})_{\tau^n}$ is $I^\nu$–invariant. Therefore, Theorem 4.7 implies that, via a suitable transformation, even the copula $A$ induces a measure of concordance.

For the case where $A$ is $I$–invariant, we obtain the following improvement of the assertions of Proposition 4.4 and Theorem 4.7:

Corollary 4.11. Assume that $A$ is $I$–invariant. Then $\kappa^*_{A,\tau^n} = \kappa^*_{A,\tau^n} = \kappa^0_{A,\tau^n}$.

Proof. Lemma A.2 and Lemma A.4 first imply $[C_{\tau^n}, A] = [C_{\tau^n}, A] = [C_{\tau^n}, A] = [C_{\tau^n}, A]$ for all $C \in \mathcal{C}$, and thus, $\kappa^*_{A,\tau^n} = \kappa^*_{A,\tau^n}$ and $\kappa^0_{A,\tau^n} = \kappa^0_{A,\tau^n}$. The assertion then follows from Lemma 4.9 and Theorem 4.7.

We conclude our study considering the case $d = 2$. Proposition 4.2 implies the following result which relaxes the condition on $A$ to induce a measure of concordance:

Corollary 4.12. Assume that $d = 2$ and let $A \subseteq \Gamma^{\tau^T}$ be a subgroup of $\Gamma$.

(1) $A_A$ is $I$–invariant if, and only if, $\kappa^*_{A_A}$ is a measure of concordance.

(2) $A_A$ is $I$–invariant if, and only if, $[M, A] \neq \Pi, \Pi$ and $\kappa^0_{A_A}$ is a measure of concordance.

Proof. It is evident that $M_A = M$ and $\Pi_A = \Pi$. We first prove (1). Lemma A.4 (1) implies

$$\kappa^*_{A_A}(C) = \frac{[C_A, A] - [\Pi, A]}{[M_A, A] - [\Pi, A]} = \frac{[C_A, A] - [\Pi, A]}{[M_A, A] - [\Pi, A]} = \frac{[C, A_A] - [\Pi, A_A]}{[M, A_A] - [\Pi, A_A]} = \kappa^*_{A_A}(C)$$

for all $C \in \mathcal{C}$, and thus, by Proposition 4.2, $A_A$ is $I$–invariant if, and only if, $\kappa^*_{A_A}$ is a measure of concordance.

This proves (1).

We now prove (2). To this end, assume that $A_A$ is $I$–invariant. Then Lemma A.4 (1) and Proposition 4.2 imply $[M, A] = [M_A, A] = [M_A, A] \neq [\Pi, \Pi]$ and

$$\kappa^0_{A_A}(C) = \frac{[C_A, A] - [\Pi, \Pi]}{[M_A, A] - [\Pi, \Pi]} = \frac{[C_A, A] - [\Pi, \Pi]}{[M_A, A] - [\Pi, \Pi]} = \frac{[C, A_A] - [\Pi, \Pi]}{[M, A_A] - [\Pi, \Pi]} = \kappa^0_{A_A}(C)$$

for all $C \in \mathcal{C}$. Thus, by Proposition 4.2, the map $\kappa^0_{A_A}$ is a measure of concordance. Assume now that $[M, A] \neq [\Pi, \Pi]$ and that $\kappa^0_{A_A}$ is a measure of concordance. Then Lemma A.4 (1) implies $[M_A, A] = [M_A, A]$
Theorem 5.1. Assume that \( [M, A] \neq [II, II] \) and \( \kappa_{\hat{A}, \{t\}}^C = \kappa_{\hat{A}, A}^C \). Thus, \( \kappa_{\hat{A}, \{t\}}^C \) is a measure of concordance and, by Proposition 4.2, \( A_{\hat{A}} \) is \( \Gamma \)-invariant. This proves (2). \( \square \)

Note that, if \( A_{\hat{r}} \) is \( \Gamma \)-invariant, we have \( A_{\hat{r}} \) is \( \Gamma \)-invariant and hence \( A_{\hat{r}, r} \) is \( \Gamma \)-invariant. Thus, Corollary 4.12 improves the result given in Theorem 4.7 for the case \( d = 2 \).

5 Some Results on Lower Bounds

In the present section, we discuss lower bounds for measures of concordance. As stated in Section 3, every concordance order preserving measure of concordance is bounded by 1 from above. Moreover, for \( d = 2 \), these measures of concordance are also bounded by \( -1 \) from below and the value \( -1 \) is attainable. In general, however, a lower bound is not known. Nevertheless, for some of the discussed copula–induced measures of concordance a lower bound can be determined:

**Theorem 5.1.** Assume that \( A_{\hat{r}} \) is \( \Gamma \)-invariant. Then the inequality

\[
\kappa_{\hat{A}, \{r\}}^C(C) = \kappa_{\hat{A}, A}^C(C) \geq \frac{1}{1 - 2^d [M, A]}
\]

holds for every \( C \in \mathcal{C} \).

**Proof.** Since \( [D, A] \geq 0 \) for all \( D \in \mathcal{C} \), we obtain

\[
\kappa_{\hat{A}, \{r\}}^C(C) = \frac{[C_{\hat{r}, r}, A] - [II, II]}{[M, A] - [II, II]} - [II, II] \geq \frac{[M, A] - [II, II]}{[M, A] - [II, II]} - 1/2^d = \frac{1}{1 - 2^d [M, A]}
\]

for every \( C \in \mathcal{C} \). \( \square \)

**Corollary 5.2.** Assume that \( A_{\hat{r}} \) is \( \Gamma \)-invariant and that \( [M, A] \geq 1/(d + 1) = [M, II] \). Then the inequality

\[
\kappa_{\hat{A}, \{r\}}^C(C) = \kappa_{\hat{A}, A}^C(C) \geq \max \left\{ \frac{d + 1}{d + 1 - 2^d}, -1 \right\}
\]

holds for every \( C \in \mathcal{C} \) and the lower bound increases to zero when the dimension \( d \) tends to infinity.

**Proof.** By Theorem 5.1, we obtain

\[
\kappa_{\hat{A}, \{r\}}^C(C) \geq \frac{1}{1 - 2^d [M, A]} \geq \frac{1}{1 - 2^d/(d + 1)} = \frac{d + 1}{d + 1 - 2^d}
\]

for every \( C \in \mathcal{C} \). This proves the assertion for \( d \geq 3 \), and for \( d = 2 \) we know that \( \kappa_{\hat{A}, \{r\}}^C(C) \geq -1 \) for every \( C \in \mathcal{C} \). \( \square \)

The condition \( [M, A] \geq 1/(d + 1) = [M, II] \) is satisfied, for instance, by the copulas \( II \) and \( M_{\hat{r}} \) and by the copulas considered in Examples 4.8 and 4.10.

A Appendix

In the appendix we provide all necessary results for the proofs in Sections 3 and 4 concerning the group \( \Gamma \) discussed in [8] and the biconvex form \([, , ,]\) introduced in [9]. The following result is straightforward:

**Lemma A.1.** Let \( A \subseteq \Gamma \) be a subgroup and \( \gamma \in A \). Then the identity \((\gamma(C))_A = C_A\) holds for all \( C \in \mathcal{C} \).

The next result deals with specific subgroups of \( \Gamma \):
Lemma A.2. The identities \( C_{F^n,\tau} = (C_{F^n})_{\tau} \) and \( 1/2^d \sum_{v \in F^n} (v(C))_{F^n,\tau} = (C_{F^n})_{\tau} \) hold for all \( C \in \mathfrak{C} \).

Proof. For every \( \gamma \in I^{n,\tau} \) there exist unique \( \pi \in I^n \) and \( \varphi \in I^\tau \) such that \( \gamma = \pi \circ \varphi \) (see [8, Lemma 3.10]). \[8, \text{Lemma 3.10.}\] then yields

\[
C_{F^n,\tau} = \frac{1}{2d!} \left( \sum_{\pi \in I^n} \pi(C) + \sum_{\pi \in I^n} (\pi \circ \tau)(C) \right) = \frac{1}{d!} \sum_{\pi \in I^n} \frac{1}{2} \pi(C) + \frac{1}{2} (\pi \circ \tau)(C) = \frac{1}{d!} \sum_{\pi \in I^n} \pi \left( \frac{1}{2} C + \frac{1}{2} \tau(C) \right) = (C_{F^n})_{\tau}
\]

for all \( C \in \mathfrak{C} \). This first identity together with the fact that \( \tau \) is in the center of \( I \), \[8, \text{Lemma 3.10.}\] and Lemma A.1 further yields

\[
\frac{1}{2^d} \sum_{v \in I^\tau} (v(C))_{F^n,\tau} = \frac{1}{2^d} \sum_{v \in I^\tau} \frac{1}{d!} \sum_{\pi \in I^n} \pi \left( \frac{1}{2} v(C) + \frac{1}{2} (\tau \circ v)(C) \right) = \frac{1}{d!} \sum_{\pi \in I^n} \frac{1}{2^d} \sum_{v \in I^\tau} \pi \left( \frac{1}{2} v(C) + \frac{1}{2} (\tau \circ v)(C) \right) = \frac{1}{d!} \sum_{\pi \in I^n} \pi \left( \frac{1}{2} C_{F^n} + \frac{1}{2} (\tau(C))_{\tau} \right) = \frac{1}{d!} \sum_{\pi \in I^n} \pi \left( \frac{1}{2} C_{F^n} + \frac{1}{2} C_{\tau} \right) = (C_{F^n})_{\tau}
\]

for all \( C \in \mathfrak{C} \). This proves the result.

We now present results concerning the biconvex form \([\cdot, \cdot, \cdot]\):

Lemma A.3. (1) The inequality \([M, C] \succ [II, C]\) holds for all \( C \in \mathfrak{C} \).
(2) The inequality \([M, C] \succ [II, II]\) holds for all \( C \in \mathfrak{C} \) if, and only if, \( d \geq 3 \).

Proof. The proof of assertion (1) is straightforward. We now prove (2). To this end, assume that \( d \geq 3 \). It follows from [4, Theorem 2.6.1] that the diagonal of each copula is Lipschitz continuous with Lipschitz constant \( d \), and thus, \( D(u 1) \geq \chi_{([d-1]/d, 1]}(1 - d + du) \) for all \( u \in I \) and all \( D \in \mathfrak{C} \). \[9, \text{Lemma 4.1.}\] and \[19, \text{Lemma 4.1.}\] together with the above inequality then imply

\[
[M, C] = [\tau(C), \tau(M)] = [\tau(C), M] = \int_{[1]} (\tau(C)(u 1)) dA(u) \geq \int_{[d-1]/d, 1]} 1 - d + du \ dA(u) = \frac{1}{2d} > \frac{1}{2^d} = [II, II]
\]

for all \( C \in \mathfrak{C} \). Assume now that \( d = 2 \). Then it follows from \[9, \text{Corollary 5.5.}\] and [9, Theorem 3.3] that the copula \( \nu_1(M) \) satisfies \([M, \nu_1(M)] = 1/2 - [M, \nu_1(M)] = 1/2 - [M, \nu_1(M)] \) and hence \([M, \nu_1(M)] = 1/4 = [II, II]\). This proves (2).

Lemma A.4. (1) Let \( d = 2 \) and \( \Lambda \subseteq I^{n,\tau} \) be a subgroup. Then the identity \([C_{\Lambda}, D] = [C, D_\Lambda]\) holds for all \( C, D \in \mathfrak{C} \).
(2) Let \( d \geq 3 \) and \( \Lambda \subseteq I^n \) be a subgroup. Then the identity \([C_{\Lambda}, D] = [C, D_\Lambda]\) holds for all \( C, D \in \mathfrak{C} \).
Proof. \cite[Theorem 5.2 (2,3)]{9} yields
\[
[C, D] = \frac{1}{|A|} \sum_{\gamma \in A} [\gamma(C), D] = \frac{1}{|A|} \sum_{\gamma \in A} [C, \gamma^{-1}(D)] = \frac{1}{|A|} \sum_{\gamma \in A} [C, \gamma(D)] = [C, D_A]
\]
for all \(C, D \in \mathcal{C}\). This proves the assertion. \(\square\)

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