Quantum kinematics on q-deformed quantum spaces I

Mathematical framework

Hartmut Wachter*

Max-Planck-Institute
for Mathematics in the Sciences
Inselstr. 22, D-04103 Leipzig

Arnold-Sommerfeld-Center
Ludwig-Maximilians-Universität
Theresienstr. 37, D-80333 München

Abstract

The aim of these two papers (I and II) is to try to give fundamental concepts of quantum kinematics to q-deformed quantum spaces. Paper I introduces the relevant mathematical concepts. A short review of the basic ideas of q-deformed analysis is given. These considerations are continued by introducing q-deformed analogs of Fourier transformations and delta functions. Their properties are discussed in detail. Furthermore, q-deformed versions of sesquilinear forms are defined, their basic properties are derived, and q-analogs of the Fourier-Plancherel identity are proved. In paper II these reasonings are applied to wave functions on position and momentum space.

Keywords: Space-Time-Symmetries, Non-Commutative Geometry, Quantum Groups

*e-mail: Hartmut.Wachter@physik.uni-muenchen.de
# 1 Introduction

In Refs. [1–3] it was outlined that deformation of classical spacetime symmetries can lead to discretizations of the spectra of spacetime observables. This observation nourishes the hope for a new method to regularize quantum field theories [4–9]. Let us recall that the concept of deforming spacetime symmetries is heavily based on the Gelfand-Naimark theorem [10], which tells us that Lie groups can be naturally embedded in the category of algebras. Realizing that spacetime symmetries are usually described by Lie groups the utility of this interrelation lies in formulating the geometrical structure of Lie groups in terms of a Hopf structure [11]. The point is that during
the last two decades generic methods have been discovered for continuously deforming matrix groups and Lie algebras within the category of Hopf algebras. This development finally led to the arrival of quantum groups and quantum spaces [12–18].

From a physical point of view the most realistic and interesting deformations are given by q-deformed versions of Minkowski space and Euclidean spaces as well as their corresponding symmetries, i.e. respectively Lorentz symmetry and rotational symmetry [19–23]. Further studies even allowed to establish differential calculi on these q-deformed quantum spaces [24–27] representing nothing other than q-analogs of classical translational symmetry. In this sense we can say that q-deformations of Euclidean and Poincaré symmetries are now available [28].

The aim of our previous work [29–36] was to give a q-deformed version of analysis to quantum spaces of physical interest, i.e. Manin plane, q-deformed Euclidean space in three or four dimensions, and q-deformed Minkowski space. In this respect, attention was focused on explicit formulae realizing the elements of q-analysis on commutative coordinate algebras. In doing so, we obtained expressions for calculating star products, operator representations, q-integrals, q-exponentials, q-translations, and braided products on the quantum spaces under consideration. In this manner we can say that our results established multi-dimensional versions of the well-known q-calculus [37].

In Ref. [38] we continued these considerations, but stress was taken on a presentation that reveals the properties of the elements of q-deformed analysis. It should be mentioned that in some sense our examinations are based on the general ideas of Shahn Majid [39–42], but the considerations in Refs. [43] and [44] go into the same direction. The key idea of this approach is that all the quantum spaces to a given quantum symmetry form a braided tensor category. Consequently, operations and objects concerning quantum spaces must rely on this framework of a braided tensor category, in order to guarantee their well-defined behavior under quantum group transformations. This so-called principle of covariance can be seen as the essential guideline for constructing a consistent theory.

In this article we collect the mathematical concepts to formulate quantum kinematics within the framework of q-deformed quantum spaces. In Sec. 2 we first give a short review of the present development of q-analysis. For the details we refer the reader to Ref. [38]. In Sec. 3 we continue this work by introducing q-analogs of Fourier transformations and delta functions, discuss different possibilities for their definition, and exhibit their basic properties. In doing so, we adjust the general ideas exposed in Ref. [45] to
our formalism and special needs. Section 4 is devoted to sesquilinear forms on quantum spaces. We first concern ourselves with their definition and derive the adjoint of certain operators. Then we prove that the sesquilinear forms are invariant under symmetry transformations and derive q-analogs of the Fourier-Plancherel identity. Section 5 closes our considerations by a short conclusion. For reference and for the purpose of introducing consistent and convenient notation, we provide a review of key notation and results in App. A. Last but not least App. B contains some proofs that are less interesting or more complicated.

2 Review of q-deformed analysis

In this section, we collect definitions and basic constructions that will be needed throughout the article. First of all, let us recall that q-analysis can be regarded as a non-commutative analysis formulated within the framework of quantum spaces [46]. Quantum spaces are defined as comodule algebras of quantum groups and can be interpreted as deformations of ordinary coordinate algebras. For our purposes, it is at first sufficient to consider a quantum space as an algebra $A_q$ of formal power series in non-commuting coordinates $X^1, X^2, \ldots, X^n$, i.e.

$$A_q = \mathbb{C}[[X^1, \ldots, X^n]] / \mathcal{I},$$

(1)

where $\mathcal{I}$ denotes the ideal generated by the relations of the non-commuting coordinates.

The two-dimensional Manin plane is one of the simplest examples for a quantum space [47]. It consists of all the power series in two coordinates $X^1$ and $X^2$ being subject to the commutation relations

$$X^1 X^2 = q X^2 X^1, \quad q > 1.$$  

(2)

We can think of $q$ as a deformation parameter measuring the coupling among different spatial degrees of freedom. In the classical case, i.e. if $q$ becomes 1 we regain commutative coordinates.

Next, we would like to focus our attention on the question how to perform calculations on an algebra of quantum space coordinates. This can be accomplished by a kind of pullback transforming operations on non-commutative coordinate algebras to those on commutative ones. For this to become more clear, we have to realize that the non-commutative algebras we are dealing with satisfy the Poicaré-Birkhoff-Witt property. It tells
us that the dimension of a subspace of homogenous polynomials should be the same as for commuting coordinates. This property is the deeper reason why monomials of a given normal ordering constitute a basis of $A_q$. Due to this fact, we can establish a vector space isomorphism between $A_q$ and a commutative algebra $A$ generated by ordinary coordinates $x^1, x^2, \ldots, x^n$:

$$W : A \rightarrow A_q,$$

$$W((x^1)^{i_1} \cdots (x^n)^{i_n}) \equiv (X^1)^{i_1} \cdots (X^n)^{i_n}. \quad (3)$$

This vector space isomorphism can even be extended to an algebra isomorphism by introducing a non-commutative product in $A$, the so-called star product [48–50]. This product is defined via the relation

$$W(f \star g) = W(f) \cdot W(g), \quad (4)$$

being tantamount to

$$f \star g \equiv W^{-1}(W(f) \cdot W(g)), \quad (5)$$

where $f$ and $g$ are formal power series in $A$. In the case of the Manin plane, the star product is of the well-known form

$$f(x^i) \star g(x^j) = q^{-\hat{n}_x \hat{n}_y} \left[ f(x^i) g(y^j) \right]_{y \rightarrow x}$$

$$= f(x^i) g(x^j) + O(h), \quad \text{with} \quad q = e^h, \quad (6)$$

where we have introduced the operators

$$\hat{n}_x^i \equiv x^i \frac{\partial}{\partial x^i}, \quad i = 1, 2. \quad (7)$$

From the last equality in (6) we see that star products on quantum spaces lead to modifications of commutative products. Evidently, these modifications vanish in the classical limit $q \rightarrow 1$.

Next, we would like to deal with tensor products of quantum spaces. To this end, we have to specify the commutation relations between generators of distinct quantum spaces. These relations are determined by the requirement of being invariant under the action of a Hopf algebra $\mathcal{H}$ describing the symmetry of the quantum spaces. It is well-known that for the case of $\mathcal{H}$ being quasitriangular the commutation relations between two quantum
space generators $X^i \in \mathcal{A}_q$ and $Y^j \in \mathcal{A}'_q$ have to be of the form

$$X^i Y^j = (\mathcal{R}[2] \triangleright Y^j) (\mathcal{R}[1] \triangleright X^i) = (Y^j \triangleleft \mathcal{R}[2]) (X^i \triangleleft \mathcal{R}[1]) = k \hat{R}^{ij}_{kl} Y^k X^l,$$

(8)

where $\mathcal{R} = \mathcal{R}[1] \otimes \mathcal{R}[2] \in \mathcal{H} \otimes \mathcal{H}$ denotes the universal R-matrix of $\mathcal{H}$ and $k$ stands for a complex number. Alternatively, we can also take the transposed inverse of $\mathcal{R}$, i.e. $\tau \circ \mathcal{R}^{-1} = \mathcal{R}^{-1}[2] \otimes \mathcal{R}^{-1}[1]$, giving us

$$X^i Y^j = (\mathcal{R}^{-1}[1] \triangleright Y^j) (\mathcal{R}^{-1}[2] \triangleright X^i) = (Y^j \triangleleft \mathcal{R}^{-1}[1]) (X^i \triangleleft \mathcal{R}^{-1}[2]) = k^{-1}(\hat{R}^{-1})^{ij}_{kl} Y^k X^l.$$

(9)

It should be obvious that from relations (8) and (9) it also follows how arbitrary elements of distinct quantum spaces commute with each other. By virtue of the algebra isomorphism $W$ we are able to realize this process of commutation on commutative coordinate algebras. To this end we introduce the operations

$$f(x^i)^{x|y}_L g(y^j) \equiv W^{-1}(\mathcal{R}^{-1}[1] \triangleright W(g)) \otimes W^{-1}(\mathcal{R}^{-1}[2] \triangleright W(f)),$$

$$f(x^i)^{x|y}_R g(y^j) \equiv W^{-1}(W(g) \triangleleft \mathcal{R}^{-1}[1]) \otimes W^{-1}(W(f) \triangleleft \mathcal{R}^{-1}[2]),$$

(10)

and

$$f(x^i)^{x|y}_L g(y^j) \equiv W^{-1}(\mathcal{R}[2] \triangleright W(g)) \otimes W^{-1}(\mathcal{R}[1] \triangleright W(f)),$$

$$f(x^i)^{x|y}_R g(y^j) \equiv W^{-1}(W(g) \triangleleft \mathcal{R}[2]) \otimes W^{-1}(W(f) \triangleleft \mathcal{R}[1]),$$

(11)

where $f$ and $g$ denote formal power series in the commutative coordinate algebras $\mathcal{A}_x$ and $\mathcal{A}_y$, respectively. The operations in (10) and (11) are referred to as braided products.

Using the explicit form of the identities in (8) and (9) we derived in Ref. [34] commutation relations between normally ordered monomials. With these results at hand we were able to write down explicit formulae for computing braided products. As an example we give the expression we obtained for the braided product of the Manin plane:

$$f(x^1, x^2)^{x|y}_L g(y^1, y^2)$$
\[
\sum_{i=0}^{\infty} q^{i^2} (-\lambda)^i i! (y^2)^i \otimes (x^1)^i \frac{q^{-\hat{n}_{y_1} \otimes \hat{n}_{x_2} - 2\hat{n}_{y_2} \otimes \hat{n}_{x_2} - 2\hat{n}_{y_1} \otimes \hat{n}_{x_1}}{[i]_{q^{-2}}} \times (D_{q^{-2}}^1)^i \frac{g(q^{-y_1^1}y_2^1 q^{-2i}y_2^2) \otimes (D_{q^{-2}}^2)^i f(q^{-2i}x_1, q^{-i}x_2),}{(1 - q^a)x_i}
\]

where we introduced \( \lambda \equiv q - q^{-1} \) and the so-called Jackson derivatives being defined by \([37, 51]\)

\[
D_{q^a}^i f(x^i) \equiv \frac{f(x^i) - f(q^a x^i)}{(1 - q^a)x_i}, \quad a \in \mathbb{C}.
\]

Now, we are in a position to introduce the tensor product of quantum spaces. It is equipped with a multiplication being determined by

\[
(a \otimes a')(b \otimes b') = (a(R_{[2]} \triangleright b)) \otimes ((R_{[1]} \triangleright a')b'),
\]

or

\[
(a \otimes a')(b \otimes b') = (a(R_{[1]}^{-1} \triangleright b)) \otimes ((R_{[2]}^{-1} \triangleright a')b'),
\]

where \( a, b \in A_q \) and \( a', b' \in A_q' \). We see that multiplication on a tensor product requires to know the commutation relations between the elements of the two tensor factors. Essentially for us is the fact that the algebra isomorphism \( \mathcal{W} \) allows us to represent the tensor product of quantum spaces on a tensor product of commutative coordinate algebras. To be more specific, this can be achieved by extending the braided products in \([10]\) and \([11]\) as follows:

\[
(f(x^i) \otimes f'(y^j)) \overset{y|x}{\circ \bar{L}} (g(x^k) \otimes g'(y^l)) \equiv \left[ f \overset{x}{\circ \bar{L}} \mathcal{W}^{-1}(R_{[1]}^{-1} \triangleright \mathcal{W}(g)) \right] \otimes \left[ \mathcal{W}^{-1}(R_{[2]}^{-1} \triangleright \mathcal{W}(f')) \overset{y}{\circ \bar{L}} g' \right],
\]

and similarly

\[
(f(x^i) \otimes f'(y^j)) \overset{y|x}{\circ \bar{L}} (g(x^k) \otimes g'(y^l)) \
\equiv \left[ f \overset{x}{\circ \bar{L}} \mathcal{W}^{-1}(R_{[2]} \triangleright \mathcal{W}(g)) \right] \otimes \left[ \mathcal{W}^{-1}(R_{[1]} \triangleright \mathcal{W}(f')) \overset{y}{\circ \bar{L}} g' \right],
\]

Now, we come to q-deformed analogs of partial derivatives, which act upon the algebra of quantum space coordinates \([24–26]\). In analogy to \([5]\) and \([9]\) there are several possibilities for commutation relations between partial derivatives and quantum space coordinates. However, the action

\[
\frac{\partial f}{\partial x^i} = D_{q^a}^i f(x^i) = \frac{f(x^i) - f(q^a x^i)}{(1 - q^a)x_i},
\]

where \( a \in \mathbb{C} \).

\[
\frac{\partial f}{\partial y^j} = D_{q^a}^j f(y^j) = \frac{f(y^j) - f(q^a y^j)}{(1 - q^a)y_j}, \quad a \in \mathbb{C}.
\]

\[
\frac{\partial f}{\partial x^i} = D_{q^a}^i f(x^i) = \frac{f(x^i) - f(q^a x^i)}{(1 - q^a)x_i},
\]

where \( a \in \mathbb{C} \).

\[
\frac{\partial f}{\partial y^j} = D_{q^a}^j f(y^j) = \frac{f(y^j) - f(q^a y^j)}{(1 - q^a)y_j}, \quad a \in \mathbb{C}.
\]
of partial derivatives on quantum space coordinates requires to modify the
commutation relations in (8) and (9) in such a way that they take the form

\[ \partial^i X^j = g^{ij} + k(\hat{R}^{-1})_{kl}^{ij} X^k \partial^l, \]
\[ \hat{\partial}^i X^j = \hat{g}^{ij} + k^{-1} \hat{R}^{ij}_{kl} X^k \hat{\partial}^l, \]
\[ X^i \partial^j = -g^{ij} + k(\hat{R}^{-1})_{kl}^{ij} \partial^k X^l, \]
\[ X^i \hat{\partial}^j = -\hat{g}^{ij} + k^{-1} \hat{R}^{ij}_{kl} \hat{\partial}^k X^l, \] (18)

where we introduced a conjugate quantum metric \( \hat{g}^{ij} \). Notice that \( \partial^i \) and \( \hat{\partial}^i \) differ from each other by a normalization factor, only (see the discussion in Ref. [38]).

From the q-deformed Leibniz rules in (18) and (19) we can derive left and right actions of partial derivatives on quantum spaces, respectively. To this end, we repeatedly apply the Leibniz rules in (18) to the product of a partial derivative with a normally ordered monomial of quantum space coordinates, until all partial derivatives stand to the right of all quantum space coordinates. In the expression obtained this way we pick out the summands without a partial derivative and bring them to normal ordering. This method finally yields left actions of partial derivatives on normally ordered monomials. Right actions of partial derivatives can be calculated in a similar way. The only difference is that we start from a partial derivative standing to the right of a normally ordered monomial and commute it to the left of all quantum space coordinates.

The algebra isomorphism \( \mathcal{W} \) allows us to introduce q-deformed derivatives that act upon commutative functions. With the help of the relations

\[ \mathcal{W}(\partial^i \triangleright f) = \partial^i \triangleright \mathcal{W}(f), \quad f \in \mathcal{A}, \]
\[ \mathcal{W}(f \triangleright \partial^i) = \mathcal{W}(f) \triangleright \partial^i, \] (20)
or

\[ \partial^i \triangleright f \equiv \mathcal{W}^{-1} (\partial^i \triangleright \mathcal{W}(f)), \]
\[ f \triangleright \partial^i \equiv \mathcal{W}^{-1} (\mathcal{W}(f) \triangleright \partial^i), \] (21)

the actions of partial derivatives on the quantum space algebra \( \mathcal{A}_q \) carry over to the corresponding commutative algebra \( \mathcal{A} \). It should be obvious that each Leibniz rule in (18) and (19) leads to its own q-derivative:

\[ \partial^i X^j = g^{ij} + k(\hat{R}^{-1})_{kl}^{ij} X^k \partial^l \]

\[ \Rightarrow \quad \partial^i \triangleright f, \]
\[
\begin{align*}
\partial^i X^j &= g^{ij} + k^{-1} \hat{R}_{kl}^{ij} X^k \partial^j \quad \Rightarrow \quad \partial^i \triangleright f, \\
X^i \partial^j &= -g^{ij} + k(\hat{R}^{-1})_{kl}^{ij} \partial^k X^l \quad \Rightarrow \quad f \triangleleft \partial^i, \\
X^i \hat{\partial}^j &= -g^{ij} + k^{-1} \hat{R}_{kl}^{ij} \hat{\partial}^k X^l \quad \Rightarrow \quad f \triangleleft \hat{\partial}^i.
\end{align*}
\] (22)

(23)

In the work of Ref. [30] we derived operator representations for q-deformed partial derivatives by applying these ideas. Our results can be viewed as multi-dimensional versions of the celebrated Jackson derivative [51]. As an example we write down the left representations for partial derivatives on the two-dimensional quantum plane:

\[
\begin{align*}
\partial^1 \triangleright f(x^1, x^2) &= -q^{-1/2} D^2_q f(qx^1, x^2), \\
\partial^2 \triangleright f(x^1, x^2) &= q^{1/2} D^1_q f(x^1, q^2 x^2).
\end{align*}
\] (24)

Next, we wish to introduce q-translations. Before doing so, it is useful to make contact with the notion of a cross product algebra [52–54]. It is well-known that we can combine a Hopf algebra \( H \) with its representation space \( A_q \) to form a left cross product algebra \( A_q \triangleleft H \) built on \( A_q \otimes H \) with product

\[
(a \otimes h)(b \otimes g) = a(h^{(1)} \triangleright b) \otimes h^{(2)} g, \quad a, b \in A_q, \quad h, g \in H,
\] (25)

where the coproduct of \( h \) is written in the Sweedler notation, i.e. \( \Delta(h) = h^{(1)} \otimes h^{(2)} \). There is also a right-handed version of this notion called a right cross product algebra \( H \triangleright A \) and built on \( H \otimes A \) with product

\[
(h \otimes a)(g \otimes b) = h g^{(1)} \otimes (a \triangleleft g^{(2)}) b.
\] (26)

When \( A_q \) is a q-deformed quantum space the cross product algebras have a Hopf structure. On quantum space generators the corresponding coproduct, antipode, and counit take the form [27,28]

\[
\begin{align*}
\Delta_L(X^i) &= X^i \otimes 1 + (\hat{\mathcal{L}}_x)^i_j \otimes X^j, \\
\Delta_L(X^i) &= X^i \otimes 1 + (\mathcal{L}_x)^i_j \otimes X^j, \\
S_L(X^i) &= -S(\hat{\mathcal{L}}_x)^i_j X^j, \\
S_L(X^i) &= -S(\mathcal{L}_x)^i_j X^j, \\
\epsilon_L(X^i) &= \epsilon_L(X^i) = 0.
\end{align*}
\] (27)
where $\mathcal{L}_x$ and $\bar{\mathcal{L}}_x$ stand for the so-called L-matrix and its conjugate. The entries of the L-matrices are elements of the Hopf algebra $\mathcal{H}$, so $S$ stands for the antipode of $\mathcal{H}$. One should also notice that the above Hopf structures are related to opposite Hopf structures via

$$
\Delta_{\bar{R}/R} = \tau \circ \Delta_{L/L}, \quad S_{\bar{R}/R} = S_{L/L}^{-1}, \quad \epsilon_{\bar{R}/R} = \epsilon_{L/L},
$$

where $\tau$ shall denote the usual transposition of tensor factors.

An essential observation is that coproducts of coordinates imply their translations [28, 33, 36, 55–57]. This can be seen as follows. The coproduct on coordinates is an algebra homomorphism. If the coordinates constitute a module coalgebra then the behavior of the quantum space coordinates $X^i$ under symmetry transformations carries over to their coproducts $\Delta_A(X^i)$.

More formally, we have

$$
\Delta_A(X^iX^j) = \Delta_A(X^i)\Delta_A(X^j), \quad \Delta_A(hX^i) = \Delta(h)\Delta_A(X^i),
$$

where $h \in \mathcal{H}$. Due to this fact we can think of (27) as nothing other than an addition law for q-deformed vector components.

To proceed any further we have to realize that our algebra morphism $W^{-1}$ can be extended by

$$
W^{-1}_L : \mathcal{A}_q \times \mathcal{H} \to \mathcal{A},
$$

$$
W^{-1}_L((X^1)^{i_1} \ldots (X^n)^{i_n} \otimes h) = W^{-1}((X^1)^{i_1} \ldots (X^n)^{i_n}) \epsilon(h),
$$

or

$$
W^{-1}_R : \mathcal{H} \times \mathcal{A}_q \to \mathcal{A},
$$

$$
W^{-1}_R(h \otimes (X^1)^{i_1} \ldots (X^n)^{i_n}) = \epsilon(h)W^{-1}((X^1)^{i_1} \ldots (X^n)^{i_n}),
$$

with $\epsilon$ being the counit of the Hopf algebra $\mathcal{H}$. With these mappings at hand we are able to introduce q-deformed translations:

$$
f(x^i \oplus_{L/L} y^i) \equiv ((W^{-1}_L \otimes W^{-1}_L) \circ \Delta_{L/L})(W(f)),
$$

$$
f(x^i \oplus_{R/R} y^i) \equiv ((W^{-1}_R \otimes W^{-1}_R) \circ \Delta_{R/R})(W(f)),
$$

$$
f(\ominus_{L/L} x^i) \equiv (W^{-1}_L \circ S_{L/L})(W(f)),
$$

$$
f(\ominus_{R/R} x^i) \equiv (W^{-1}_L \circ S_{R/R})(W(f)).
$$

Let us note that these operations make the algebra $\mathcal{A}$ equipped with a star
product into a braided Hopf algebra. For the details we refer the reader to Refs. [40–42, 52]. In the work of Ref. [33] we found formulae that enable us to compute the results of the above operations for arbitrary functions. The results for the Manin plane, for example, reads as

\[ f(x^i \oplus_L y^j) = \sum_{k_1, k_2=0}^{\infty} \frac{(x^1)^{k_1} (x^2)^{k_2}}{[[k_1]] q^{-2}![[k_2]] q^{-2}!} ((D_{q^{-2}}^1)^{k_1} (D_{q^{-2}}^2)^{k_2} f)(q^{-k_2} y^1), \] (36)

\[ f(\ominus_L x^i) = q^{-(\hat{n}_{x^1})^2 - (\hat{n}_{x^2})^2 - 2\hat{n}_{x^1}\hat{n}_{x^2}} f(-qx^1, -qx^2). \] (37)

It should be mentioned that Eq. (36) can be seen as q-deformed version of the Taylor rule in two dimensions.

Important for us is the fact that q-deformed translations show a number of properties that are very similar to those fulfilled by classical translations:

\[ f((x^i \oplus_A y^j) \oplus_A z^k) = f(x^i \oplus_A (y^j \oplus_A z^k)), \]
\[ f(0 \oplus_A x^i) = f(x^i \oplus_A 0) = f(x^i), \]
\[ f((\ominus_A x^i) \oplus_A x^j) = f(x^i \oplus_A (\ominus_A x^j)) = f(0). \] (38)

For a correct understanding of the relations in (38) one has to realize that

\[ f(0) \equiv \epsilon(\mathcal{W}(f)) = f(x^i)\bigg|_{x^i=0}. \] (39)

Furthermore, we took the convention that

\[ f(x^i \oplus_A x^j) = f(1) \otimes f(2), \] (40)

i.e. tensor factors which are addressed by the same coordinates have to be multiplied via the star product. The relations in (38) can be interpreted as follows. The first relation is nothing else than the law of associativity, whereas the second and third relation concern the existence of the identity and that of an inverse, respectively.

Let us make contact with another important ingredient of q-analysis, i.e. dual pairings between the algebra of quantum space coordinates and that of the corresponding partial derivatives. In Ref. [55] it was shown that these pairings are given by

\[ \langle \cdot, \cdot \rangle : A_q^* \otimes A_q \to \mathbb{C} \quad \text{with} \quad \langle f(\partial^i), g(X^j) \rangle \equiv \epsilon(f(\partial^i) \triangleright g(X^j)), \] (41)
or

\[ \langle \ldots \rangle' : A_q \otimes A_q^* \rightarrow \mathbb{C} \quad \text{with} \quad \langle f(X^i), g(\partial^j) \rangle' = \epsilon(f(X^i) \circ g(\partial^j)). \quad (42) \]

As usual the above pairings carry over to commutative algebras by means of the algebra isomorphism \( W \), i.e.

\[ \langle \ldots \rangle : A^* \otimes A \rightarrow \mathbb{C} \quad \text{with} \quad \langle f(\partial^j), g(x^i) \rangle \equiv \langle W(f(\partial^j)), W(g(x^i)) \rangle, \quad (43) \]

and likewise

\[ \langle \ldots \rangle' : A \otimes A^* \rightarrow \mathbb{C} \quad \text{with} \quad \langle f(x^i), g(\partial^j) \rangle' \equiv \langle W(f(x^i)), W(g(\partial^j)) \rangle'. \quad (44) \]

Since partial derivatives can act on coordinates in four different ways, there are four possibilities for defining a pairing between coordinates and derivatives. More concretely, we have

\[ \langle f(\partial^j), g(x^i) \rangle_{L,R} \equiv (f(\partial^j) \triangleright g(x^i))|_{x^i = 0} = (f(\partial^j) \triangleright g(x^i))|_{\partial^j = 0}, \]

\[ \langle f(\hat{\partial}^j), g(x^i) \rangle_{L,R} \equiv (f(\hat{\partial}^j) \triangleright g(x^i))|_{x^i = 0} = (f(\hat{\partial}^j) \triangleright g(x^i))|_{\hat{\partial}^j = 0}. \quad (45) \]

\[ \langle f(x^i), g(\partial^j) \rangle_{L,R} \equiv (f(x^i) \triangleright g(\partial^j))|_{x^i = 0} = (f(x^i) \triangleright g(\partial^j))|_{\partial^j = 0}, \]

\[ \langle f(x^i), g(\hat{\partial}^j) \rangle_{L,R} \equiv (f(x^i) \triangleright g(\hat{\partial}^j))|_{x^i = 0} = (f(x^i) \triangleright g(\hat{\partial}^j))|_{\hat{\partial}^j = 0}. \quad (46) \]

Notice that in (45) and (46) labels were introduced to characterize the different pairings. Their meaning can be understood in the following way. The left and right indices refer to the left and right arguments of the dual pairings, respectively. If an argument of a given dual pairing acts on the other argument via the conjugate representation the index for the acting argument is overlined. One should also notice that each pairing can be calculated in two different ways. This observation is a consequence of the fact that in our approach derivatives and coordinates play symmetrical roles. As an example we give the explicit form of a pairing for the Manin plane. In this case the first pairing in (45) reads on normally ordered monomials as follows:

\[ \langle (\partial_1)^{n_1} (\partial_2)^{n_2}, (x^2)^{m_2} (x^1)^{m_1} \rangle_{L,R} = \delta_{m_1,n_1} \delta_{m_2,n_2} [][n_1][n_2][][q^2]!. \quad (47) \]

Now, we are ready to turn to a short discussion of \( q \)-deformed exponentials. From an abstract point of view an exponential is nothing other than an object whose dualization is given by one of the pairings in (45) or (46).
Thus, the exponential can be introduced as the mapping

\[ \exp : \mathbb{C} \rightarrow \mathcal{A}_q \otimes \mathcal{A}_q^* \]

with \( \exp(\alpha) = \alpha \sum_a e_a \otimes f^a \), \( \text{or} \)

\[ \exp' : \mathbb{C} \rightarrow \mathcal{A}_q^* \otimes \mathcal{A}_q, \]

with \( \exp'(\alpha) = \alpha \sum_a f_a \otimes e_a \), (48)

where \( \{ e_a \} \) is a basis in \( \mathcal{A}_q \) and \( \{ f^b \} \) a dual basis in \( \mathcal{A}_q^* \), i.e. it holds

\[ \langle e_a, f^b \rangle = \delta_a^b, \quad \langle f_b, e_a \rangle' = \delta_a^b. \] (50)

It should be obvious that the algebra isomorphism \( W \) enables us to introduce q-deformed exponentials that live on a tensor product of commutative algebras. More concretely, this is achieved by the expressions

\[ \exp(x^i|\partial^j) \equiv \sum_a W(e_a) \otimes W(f^a), \] (51)

and

\[ \exp'(\partial^i|x^j) \equiv \sum_a W(f^a) \otimes W(e_a). \] (52)

If we want to derive explicit formulae for q-deformed exponentials it is our task to determine a basis of the coordinate algebra \( \mathcal{A}_q \) being dual to a given one of the derivative algebra \( \mathcal{A}_q^* \). Inserting the elements of the two bases into the formulae of (51) and (52) will then provide us with expressions for q-deformed exponentials. It should also be mentioned that in general the two bases being dually paired depend on the choice of the pairing. Thus, each pairing in (45) and (46) leads to its own q-exponential:

\[ \langle f(\partial^i), g(x^j) \rangle_{L,R} \Rightarrow \exp(x^i|\partial^j)_{\bar{R},L}, \]

\[ \langle f(\hat{\partial}^i), g(x^j) \rangle_{L,R} \Rightarrow \exp(x^i|\hat{\partial}^j)_{R,L}, \] (53)

\[ \langle f(x^i), g(\partial^j) \rangle_{L,R} \Rightarrow \exp(\partial^i|x^j)_{\bar{R},L}, \]

\[ \langle f(x^i), g(\hat{\partial}^j) \rangle_{\bar{L},R} \Rightarrow \exp(\hat{\partial}^i|x^j)_{R,\bar{L}}. \] (54)

In Ref. [32] we presented explicit formulae for q-deformed exponentials. Once again, we would like to give a result from the two-dimensional quantum
plane. For the second exponential in (53) we found the expression

\[
\exp(x^i \hat{\partial}^j)_{R,L} = \sum_{n_1, n_2 = 0}^{\infty} \frac{(x^1)^{n_1} (x^2)^{n_2} \otimes (\hat{\partial}_2)^{n_2} (\hat{\partial}_1)^{n_1}}{[[n_1]]_{q-2}! [[n_2]]_{q-2}!}.
\]  

(55)

There are two properties of q-exponentials worth recording here. First, q-exponentials are subject to the addition laws

\[
\exp(x^i \oplus_L y^j | \hat{\partial}^k)_{R,L} = \exp(y^j | \hat{\partial}^k)_{R,L} \circ_R \exp(x^i | \hat{\partial}^j)_{R,L},
\]

(56)

\[
\exp(x^i \oplus_R y^j | \hat{\partial}^k)_{R,L} = \exp(y^j | \hat{\partial}^k)_{R,L} \circ_R \exp(x^i | \hat{\partial}^j)_{R,L},
\]

(57)

Second, they are in some sense eigenfunctions of partial derivatives, since they obey [32, 44, 55]

\[
\hat{\partial}^i \triangleleft \exp(x^i | \hat{\partial}^j)_{R,L} = \exp(x^k | \hat{\partial}^j)_{R,L} \circ_R \hat{\partial}^i,
\]

(58)

\[
\hat{\partial}^i \triangleright \exp(x^i | \hat{\partial}^j)_{R,L} = \exp(x^k | \hat{\partial}^j)_{R,L} \circ_R \hat{\partial}^i.
\]

(59)

Lastly, we wish to introduce q-deformed integrals as operations being inverse to partial derivatives [31]. (For other concepts of integration on quantum spaces see Ref. [45,58–60].) To this end, we first extend the algebra of partial derivatives by inverse elements. In the case of the two-dimensional quantum plane the algebra of partial derivatives is now characterized by the following relations:

\[
\hat{\partial}^2 \hat{\partial}^1 = q^{-1} \hat{\partial}^1 \hat{\partial}^2,
\]

(60)

\[
\hat{\partial}^i (\hat{\partial}^i)^{-1} = (\hat{\partial}^i)^{-1} \hat{\partial}^i = 1, \quad i = 1, 2,
\]

\[
\hat{\partial}^2 (\hat{\partial}^1)^{-1} = q(\hat{\partial}^1)^{-1} \hat{\partial}^2,
\]

\[
\hat{\partial}^1 (\hat{\partial}^2)^{-1} = q(\hat{\partial}^2)^{-1} \hat{\partial}^1,
\]

(61)
\[(\partial^2)^{-1}(\partial^1)^{-1} = q^{-1}(\partial^1)^{-1}(\partial^2)^{-1}. \quad (62)\]

As next step we have to find representations for the inverse partial derivatives. This can be achieved in the following way. In Ref. [30] it was shown that according to
\[\partial^i \triangleright F = (\partial^i_{\text{cl}} + \partial^i_{\text{cor}}) F \quad (63)\]
representations of our partial derivatives can be divided up into a classical part and corrections vanishing in the undeformed limit \(q \to 1\). For this reason a solution to the difference equation
\[\partial^i \triangleright F = f \quad (64)\]
for given \(f\) can be written in the form
\[F = (\partial^i)^{-1} \triangleright f = \frac{1}{\partial^i_{\text{cl}} + \partial^i_{\text{cor}}} f = \sum_{k=0}^{\infty} (-1)^k \left[ (\partial^i_{\text{cl}})^{-1} \partial^i_{\text{cor}} \right]^k (\partial^i_{\text{cl}})^{-1} f. \quad (65)\]

To apply this formula, we need to identify the contributions \(\partial^i_{\text{cl}}\) and \(\partial^i_{\text{cor}}\) in the representations of partial derivatives. In the two-dimensional case, for example, we can read off from the expressions in (24) that
\[(\partial^i_{\text{cl}}) f = -q^{-1/2} D_{q^2}^2 f(q x^1), \quad (\partial^i_{\text{cor}}) f = 0,
(\partial^2_{\text{cl}}) f = q^{1/2} D_{q^2}^1 f(q^2 x^2), \quad (\partial^2_{\text{cor}}) f = 0. \quad (66)\]

Plugging this into formula (65) leaves us with
\[(\partial^1)^{-1}|_{-\infty}^{x^2} \triangleright f = (\partial^1_{\text{cl}})^{-1}|_{-\infty}^{x^2} \triangleright f = -q^{1/2} (D_{q^2}^2)^{-1}|_{-\infty}^{x^2} f(q^{-1} x^1),
(\partial^2)^{-1}|_{-\infty}^{x^1} \triangleright f = (\partial^2_{\text{cl}})^{-1}|_{-\infty}^{x^1} \triangleright f = q^{-1/2} (D_{q^2}^1)^{-1}|_{-\infty}^{x^2} f(q^{-2} x^2), \quad (67)\]
where \((D_{q^a}^i)^{-1}|_{-\infty}^{x^i}\) denotes the Jackson integral operator [61] over the interval \([-\infty, x^i]\).

What we have done so far applies to each of the representations in (22) and (23). In this manner, q-deformed integrals can be placed into four categories:
\[\partial^i \triangleright f \quad \Rightarrow \quad \int_{-\infty}^{x^i} dL_y \, f(y^i) \equiv (\partial^i)^{-1}|_{-\infty}^{x^i} \triangleright f,\]
\[ \hat{\partial}^i \triangleright f \quad \Rightarrow \quad \int_{-\infty}^{x^i} d_L y^j f(y^j) \equiv (\hat{\partial}^i)^{-1}|_{-\infty}^{x^i} \triangleright f, \quad (68) \]

\[ f \triangleright \hat{\partial}^i \quad \Rightarrow \quad \int_{-\infty}^{x^i} d_R y^j f(y^j) \equiv f \triangleright (\hat{\partial}^i)^{-1}|_{-\infty}^{x^i}, \]

\[ f \triangleright \partial^i \quad \Rightarrow \quad \int_{-\infty}^{x^i} d_R y^j f(y^j) \equiv f \triangleright (\hat{\partial}^i)^{-1}|_{-\infty}^{x^i}. \quad (69) \]

Notice that \( \bar{i} \) denotes the so-called conjugate index of \( i \). For its definition see Ref. [38].

If we want to have integrals over the entire space we can apply one-dimensional integrals in succession, i.e., for example,

\[ \int d^n_L x f \equiv \int d_L x^1 \int d_L x^2 \int d_L x^3 f(x^i) \]

\[ = \lim_{x^1, \ldots, x^n \to \infty} ((\hat{\partial}^n)^{-1}|_{-\infty}^{x^n} \ldots (\hat{\partial}^1)^{-1}|_{-\infty}^{x^1} \triangleright f), \quad (70) \]

\[ \int d^n_R x f \equiv \int d_R x^1 \int d_R x^2 \int d_R x^3 f(x^i) \]

\[ = \lim_{x^1, \ldots, x^n \to \infty} (f \triangleright (\hat{\partial}^1)^{-1}|_{-\infty}^{x^1} \ldots (\hat{\partial}^n)^{-1}|_{-\infty}^{x^n}). \quad (71) \]

For the quantum spaces we are interested in these integrals take the form [38]

(i) (quantum plane)

\[ \int_{-\infty}^{+\infty} d^n_L x f(x^1, x^2) = -\frac{q}{16}(D_{q^{1/2}}^1)^{-1}|_{-\infty}^{+\infty} (D_{q^{1/2}}^2)^{-1}|_{-\infty}^{+\infty} f, \quad (72) \]

(ii) (three-dimensional Euclidean space)

\[ \int_{-\infty}^{+\infty} d^n_L x f(x^+, x^3, x^-) = \]

\[ \frac{q^{-6}}{4}(D_{q^2}^+)^{-1}|_{-\infty}^{+\infty} (D_{q^2}^3)^{-1}|_{-\infty}^{+\infty} (D_{q^2}^-)^{-1}|_{-\infty}^{+\infty} f, \quad (73) \]
(iii) (four-dimensional Euclidean space)

\[
\int_{-\infty}^{+\infty} d^4x \, f(x^4, x^3, x^2, x^1) = \frac{1}{16} (D^4_q)^{-1} \int_{-\infty}^{+\infty} (D^2_q)^{-1} \int_{-\infty}^{+\infty} (D^3_q)^{-1} \int_{-\infty}^{+\infty} (D^4_q)^{-1} \int_{-\infty}^{+\infty} f, \tag{74}
\]

(iv) (q-deformed Minkowski space)

\[
\int_{-\infty}^{+\infty} d^4x \, f(r^2, x^-, x^{3/0}, x^+) = -\frac{1}{16} (q\lambda_+)^{-3} (D^{r^2}_{q^{-1}})^{-1} \int_{-\infty}^{+\infty} (D^+_{q^{-1}})^{-1} \int_{-\infty}^{+\infty} \times (D^{3/0}_{q^{-1}})^{-1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (D^-_{q^{-1}})^{-1} \int_{-\infty}^{+\infty} f, \tag{75}
\]

where \((q > 1, a > 0)\)

\[
(D^i_{q \pm a})^{-1} \int_{-\infty}^{+\infty} f \equiv \mp (1 - q^{\pm a}) \sum_{k=-\infty}^{\infty} q^{ak} \left( f(q^{ak}) + f(-q^{ak}) \right). \tag{76}
\]

The expressions in (72)-(75) behave like scalars, which implies their trivial braiding with space and momentum coordinates, i.e.

\[
f(x^i) \xrightarrow{\partial y} \int_{-\infty}^{+\infty} d^m_A y \, g(y^j) = \int_{-\infty}^{+\infty} d^m_A y \, f(\kappa_C x^i) \xrightarrow{\partial y} \int_{-\infty}^{+\infty} d^m_A y \, g(y^j) = \int_{-\infty}^{+\infty} d^m_A y \, g(y^j) \otimes f(x^i), \tag{77}
\]

and

\[
f(p^j) \xrightarrow{\partial y} \int_{-\infty}^{+\infty} d^n_A y \, g(y^j) = \int_{-\infty}^{+\infty} d^n_A y \, f((\kappa_C)^{-1} p^j) \xrightarrow{\partial y} \int_{-\infty}^{+\infty} d^n_A y \, g(y^j) = \int_{-\infty}^{+\infty} d^n_A y \, g(y^j) \otimes f(p^j), \tag{78}
\]

where \(A, C \in \{L, \bar{L}, R, \bar{R}\}\). Notice that the scalings of position and momentum coordinates, which in the above formulae are given by the constants \(\kappa_C\) and \((\kappa_C)^{-1}\), respectively, result from the braiding properties of the volume.
elements. With our conventions the values of the constant $\kappa_C$ are determined as follows:

(i) (quantum plane)
\[ \kappa = \kappa_L = \kappa_R = (\kappa_L)^{-1} = (\kappa_R)^{-1} = q^3, \]  

(ii) (three-dimensional q-deformed Euclidean space)
\[ \kappa = \kappa_L = \kappa_R = (\kappa_L)^{-1} = (\kappa_R)^{-1} = q^6, \]

(iii) (four-dimensional q-deformed Euclidean space)
\[ \kappa = \kappa_L = \kappa_R = (\kappa_L)^{-1} = (\kappa_R)^{-1} = q^4, \]

(iv) (q-deformed Minkowski space)
\[ \kappa = \kappa_L = \kappa_R = (\kappa_L)^{-1} = (\kappa_R)^{-1} = q^{-4}. \]

Important for us is the fact that q-deformed integrals over the whole space are invariant under translations in the sense that
\[
\int_{-\infty}^{\infty} d^n_A x (\partial^i \triangleright f) = \int_{-\infty}^{\infty} d^n_A x (\partial^i \triangleright f) = \int_{-\infty}^{\infty} d^n_A x (f \triangleright \partial^i) = \int_{-\infty}^{\infty} d^n_A x (f \triangleright \partial^i) = 0,
\]
where $A \in \{L, \bar{L}, R, \bar{R}\}$. Finally, it should be mentioned that the identities in (83) are equivalent to

\[
\int_{-\infty}^{+\infty} d^n_A x^n f(x^i) = \int_{-\infty}^{+\infty} d^n_A x^n f(y^j \oplus_L x^i) = \int_{-\infty}^{+\infty} d^n_A x^n f(\bar{y}^j \oplus_L x^i) = \int_{-\infty}^{+\infty} d^n_A x^n f(x^i \oplus_R \bar{y}^j) = \int_{-\infty}^{+\infty} d^n_A x^n f(x^i \oplus_R y^j).
\]

3 Fourier transformations on quantum spaces

Fourier transformations play a very important role in quantum physics, since they allow decomposition of wave-functions into plane-waves. In this section
we present a detailed discussion of q-deformed Fourier transformations. In doing so, we will follow the ideas exposed in Ref. [45], as it is our aim to include them into our framework.

First, we use the objects of q-analysis to write down q-analogs of Fourier transformations and delta functions. Then, it is shown that q-deformed Fourier transformations intertwine actions of partial derivatives with star multiplication by coordinates. In addition to this we give the proof that q-deformed Fourier transformations are in some sense invertible. Towards this end we introduce a second set of q-deformed Fourier transformations and discuss their basic properties. For later purpose and to make our considerations more concrete we calculate the Fourier transforms of q-exponentials and q-deformed delta functions. For the sake of completeness a subsection on convolution products has been added, but in this article we do not make any further use of this notion. Lastly, we examine how Fourier transformations behave under conjugation.

3.1 Definition of Fourier transformations and related objects

Since we have different versions of q-deformed integrals and q-deformed exponentials, there are the following possibilities for defining q-deformed Fourier transformations:

\[ F_L(f)(p^k) \equiv \int_{-\infty}^{+\infty} d^n_L x \, f(x) \otimes \exp(x^j | i^{-1} p^k)_{R,L}, \]

\[ F^*_L(f)(p^k) \equiv \int_{-\infty}^{+\infty} d^n_L x \, f(x) \otimes \exp(x^j | i^{-1} p^k)_{R,L}, \]  \hspace{1cm} (85)

\[ F_R(f)(p^k) \equiv \int_{-\infty}^{+\infty} d^n_R x \, \exp(i^{-1} p^k | x^j)_{R,L} \otimes f(x), \]

\[ F^*_R(f)(p^k) \equiv \int_{-\infty}^{+\infty} d^n_R x \, \exp(i^{-1} p^k | x^j)_{R,L} \otimes f(x), \]  \hspace{1cm} (86)

where we applied the substitutions \( \partial^k = i^{-1} p^k \) and \( \hat{\partial}^k = i^{-1} p^k \) to our q-exponentials.

In complete analogy to the classical case we introduce q-deformed versions of the delta function by

\[ \delta_L^n(p^k) \equiv F_L(1)(p^k) = \int_{-\infty}^{+\infty} d^n_L x \exp(x^j | i^{-1} p^k)_{R,L}, \]
\[
\delta^n_L(p^k) \equiv \mathcal{F}_L(1)(p^k) = \int_{-\infty}^{+\infty} d^n_L x \exp(x^j |i^{-1}p^k)_{R,L}, \quad (87)
\]

\[
\delta^n_R(p^k) \equiv \mathcal{F}_R(1)(p^k) = \int_{-\infty}^{+\infty} d^n_R x \exp(i^{-1}p^k|x^j)_{R,L},
\]

\[
\delta^n_{\bar{R}}(p^k) \equiv \mathcal{F}_{\bar{R}}(1)(p^k) = \int_{-\infty}^{+\infty} d^n_{\bar{R}} x \exp(i^{-1}p^k|x^j)_{\bar{R},L}. \quad (88)
\]

As a consequence the q-deformed volume elements are then given by the expressions

\[
\text{vol}_L \equiv \int_{-\infty}^{+\infty} d^n_R p \delta^n_L(p^k) = \int_{-\infty}^{+\infty} d^n_L x \int_{-\infty}^{+\infty} d^n_R p \exp(x^j |i^{-1}p^k)_{R,L},
\]

\[
\text{vol}_L \equiv \int_{-\infty}^{+\infty} d^n_R p \delta^n_L(p^k) = \int_{-\infty}^{+\infty} d^n_L x \int_{-\infty}^{+\infty} d^n_R p \exp(x^j |i^{-1}p^k)_{R,L}, \quad (89)
\]

\[
\text{vol}_R \equiv \int_{-\infty}^{+\infty} d^n_L p \delta^n_R(p^k) = \int_{-\infty}^{+\infty} d^n_R x \int_{-\infty}^{+\infty} d^n_L p \exp(i^{-1}p^k|x^j)_{R,L},
\]

\[
\text{vol}_R \equiv \int_{-\infty}^{+\infty} d^n_L p \delta^n_R(p^k) = \int_{-\infty}^{+\infty} d^n_L p \int_{-\infty}^{+\infty} d^n_R x \exp(i^{-1}p^k|x^j)_{R,L}. \quad (90)
\]

As in the classical case q-deformed Fourier transformations can be viewed as mappings between position and momentum space. In our approach there is a complete symmetry between position and momentum variables - apart from occasional minus signs (see the discussion in Ref. [38]). Thus, it is always possible to interchange the roles of \(x\) and \(p\) in the above definitions. Especially, this observation implies the identities

\[
\text{vol}_L = \text{vol}_R \quad \text{and} \quad \text{vol}_L = \text{vol}_R. \quad (91)
\]

At this point, we should also make some comments on the convergence of the integral expressions in (87)-(90). In general, we cannot expect that a q-deformed integral of a q-exponential takes on a finite value if the integral is taken over the whole space. The reason for this lies in the fact that q-exponentials do not decrease rapidly enough at infinity. To circumvent this problem we can try to modify q-exponentials in such a way that they become functions with the necessary boundary conditions. In complete analogy to the classical case this can be achieved by the requirement that the values of momentum variables contain small imaginary parts. From a physical point of view this means we deal with wave-packets instead of plane-waves and
the imaginary parts in the values of the momentum variables describe the extension of the wave-packets in space and time.

With this modification our expressions in (87)-(90) should be well-defined. However, there is a price we have to pay, since our results now depend on additional parameters. Thus, at the end of our calculations we have to take the limit in which these parameters vanish. The discrete structure of our formulae gives rise to the hope that such a method can lead to finite expressions, but we do not want to investigate this problem here.

3.2 Elementary properties of Fourier transformations

From classical Fourier theory we know that Fourier transformations interwine the action of partial derivatives with multiplication by coordinates. For q-deformed Fourier transformations a similar statement holds, since we have

\[ \mathcal{F}_L(f \odot x^j)(p^k) = \mathcal{F}_L(f)(p^k) \odot (i^{-1} p^j), \]
\[ \mathcal{F}_L(f \odot \hat{\partial}^j)(p^k) = \mathcal{F}_L(f)(p^k) \odot (i^{-1} p^j), \]  \hspace{1cm} (92)
\[ \mathcal{F}_R(\hat{\partial}^j \odot f)(p^k) = i^{-1} p^j \odot \mathcal{F}_R(f)(p^k), \]
\[ \mathcal{F}_R(\partial^j \odot f)(p^k) = i^{-1} p^j \odot \mathcal{F}_R(f)(p^k), \]  \hspace{1cm} (93)

and

\[ \mathcal{F}_L(f \odot x^j)(p^k) = i \mathcal{F}_L(f)(p^k) \odot \hat{\partial}^j, \]
\[ \mathcal{F}_L(f \odot \hat{\partial}^j)(p^k) = i \mathcal{F}_L(f)(p^k) \odot \hat{\partial}^j, \]  \hspace{1cm} (94)
\[ \mathcal{F}_R(x^j \odot f)(p^k) = i \hat{\partial}^j \odot \mathcal{F}_R(f)(p^k), \]
\[ \mathcal{F}_R(x^j \odot \hat{\partial})(p^k) = i \hat{\partial}^j \odot \mathcal{F}_R(f)(p^k). \]  \hspace{1cm} (95)

Notice that the variable on top of the symbol for the action indicates upon which space the partial derivatives shall act.

The above identities follow from the property of q-exponentials to be eigenfunctions of q-deformed partial derivatives [cf. the identities in (58) and (59)], as can be seen by the following calculation:

\[ \mathcal{F}_L(f \odot \hat{\partial}^j)(p^k) = \int_{-\infty}^{+\infty} d_L x (f \odot \hat{\partial}^j) \odot \exp(x^j |i^{-1} p^k) \hat{R}. \]
\[
\begin{align*}
&= \int_{-\infty}^{+\infty} d^n_L x f \otimes \partial^j x \exp(x^l i^{-1} p^k)_{R,L} \\
&= \int_{-\infty}^{+\infty} d^n_L x f \otimes \exp(x^l i^{-1} p^k)_{R,L} \otimes (i^{-1} p^j) \\
&= \mathcal{F}_L(f)(p^k) \otimes (i^{-1} p^j),
\end{align*}
\] 

and

\[
\begin{align*}
\mathcal{F}_L(f \otimes x^j)(p^k) &= \int_{-\infty}^{+\infty} d^n_L x (f \otimes x^j) \otimes \exp(x^l i^{-1} p^k)_{R,L} \\
&= \int_{-\infty}^{+\infty} d^n_L x f \otimes (x^j \otimes \exp(x^l i^{-1} p^k)_{R,L}) \\
&= i \int_{-\infty}^{+\infty} d^n_L x f \otimes \exp(x^l i^{-1} p^k)_{R,L} \otimes \partial^j \\
&= i \mathcal{F}_L(f)(p^k) \otimes \partial^j.
\end{align*}
\] 

Notice that the second equality in (96) can be recognized as integration by parts. The other identities in (92)-(95) can be proven in very much the same way.

### 3.3 Invertibility of Fourier transformations

In this section we would like to show that q-deformed Fourier transformations are invertible in a certain sense. Towards this end, it is convenient to introduce another type of q-deformed Fourier transformations:

\[
\begin{align*}
\mathcal{F}^*_L(f)(x^k) &\equiv \frac{1}{\text{vol}_L} \int_{-\infty}^{+\infty} d^n_L p \exp(i^{-1} p^l \otimes_L x^k)_{R,L} \otimes_L f(p^j), \\
\mathcal{F}^*_L(f)(x^k) &\equiv \frac{1}{\text{vol}_L} \int_{-\infty}^{+\infty} d^n_L p \exp(i^{-1} p^l \otimes_L x^k)_{R,L} \otimes_L f(p^j), \\
\mathcal{F}^*_R(f)(x^k) &\equiv \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^{n_R} p f(p^j) \otimes_R \exp(\otimes_R x^k i^{-1} p^l)_{R,L}, \\
\mathcal{F}^*_R(f)(x^k) &\equiv \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^{n_R} p f(p^j) \otimes_R \exp(\otimes_R x^k i^{-1} p^l)_{R,L}.
\end{align*}
\]
Essentially for us is the fact that this second set of Fourier transformations enables us to invert the mappings in (85)-(86). Concretely, we have

\[(F_R \circ F_L)(f)(x^k) = f(\kappa x^k),\]  
\[(F_R \circ F_L)(f)(x^k) = f(\kappa^{-1} x^k),\]  
\[(F_L \circ F_R)(f)(x^k) = f(\kappa^{-1} x^k),\]  
\[(F_L \circ F_R)(f)(x^k) = f(\kappa x^k),\]  

and

\[(F_L \circ F_R^*)(f)(x^k) = \kappa^{-n} f(\kappa^{-1} x^k),\]  
\[(F_L \circ F_R^*)(f)(x^k) = \kappa^n f(\kappa x^k),\]  
\[(F_R \circ F_L^*)(f)(x^k) = \kappa^n f(\kappa x^k),\]  
\[(F_R \circ F_L^*)(f)(x^k) = \kappa^{-n} f(\kappa^{-1} x^k).\]  

Next, we would like to prove the relations in (100)-(103). To reach this goal, we need some useful identities which we now collect. First of all, we have

\[\int_{-\infty}^{+\infty} d_A y^i f(y^i) \otimes_B g(x^j \oplus_L y^k) = \int_{-\infty}^{+\infty} d_A y^i f(\ominus_B x^m \oplus_B x^l \oplus_B y^i) \otimes_B g(x^j \oplus_B y^k) = \int_{-\infty}^{+\infty} d_A y^i f((\ominus_B x^m) \oplus_B (x^l \oplus_B y^i)) \otimes_B g(x^j \oplus_B y^k) = \int_{-\infty}^{+\infty} d_A y^i f((\ominus_B x^j) \oplus_B y^i) \otimes_B g(y^k),\]  

where \(A, B \in \{L, \bar{L}, R, \bar{R}\}\). The first and second equality are applications of the axioms of q-translations [cf. the identities in (38)]. The third equality uses the fact that q-deformed integrals over the whole space are invariant under q-translations [cf. the identities in (84)]. For the sake of completeness, it should be noted that by a slight modification of these arguments we can also verify that

\[\int_{-\infty}^{+\infty} d_A y^i g(y^i \ominus_B x^k) \otimes_B f(y^j) = \int_{-\infty}^{+\infty} d_A y^i g(y^i) \otimes_B f(y^j \ominus_B x^k).\]
However, the proof of the identities in (100)-(103) also requires the relations

\[
\int_{-\infty}^{+\infty} d^n x \ g(x^i) \ast \delta^n_R(x^j) = g(0) \text{vol}_L,
\]
\[
\int_{-\infty}^{+\infty} d^n x \ g(x^i) \ast \delta^n_L(x^j) = g(0) \text{vol}_L.
\]  \hspace{1cm} (106)

Both relations follow from the same reasonings. Thus, it suffices to restrict attention to the first identity in (106), which can be derived as follows:

\[
\int_{-\infty}^{+\infty} d^n x \ g(x^i) \ast \delta^n_R(x^j)
= \int_{-\infty}^{+\infty} d^n x \int_{-\infty}^{+\infty} d^n p \ g(x^i) \ast \exp(x^j|^{-1} p^k)_R,L
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d^n x \ g(x^i) \ast e_a(x,\bar{R}) \otimes e_a(R,x)
\]
\[
\ast \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d^n p \ e_a(p,L)
= \int_{-\infty}^{+\infty} d^n x \ \langle e^a(R,1), g(x^i) \rangle_{L,R} e^a(x,\bar{R})
\]
\[
\otimes \int_{-\infty}^{+\infty} d^n p \ e_0(p,L)
= \int_{-\infty}^{+\infty} d^n x \ \langle 1, g(x^i) \rangle_{L,R} e^a(x,\bar{R})
\]
\[
\otimes \int_{-\infty}^{+\infty} d^n p \ e_0(p,L)
= g(0) \int_{-\infty}^{+\infty} d^n x \ int_{-\infty}^{+\infty} d^n p \ \exp(x^j|^{-1} p^k)_R,L = g(0) \text{vol}_L. \hspace{1cm} (107)
\]

The first equality is the definition of the delta function in position space. For the second equality the q-deformed exponential was rewritten as

\[
\exp(x^j|^{-1} p^k)_R,L = \sum_a e^a(x,\bar{R}) \otimes e^a(p,L). \hspace{1cm} (108)
\]

Notice that for notational convenience the summation symbols were skipped in the expressions of (107). For the third equality we applied the complete-
ness relation

\[ \text{id} = \left( \ldots,_{L,R} \otimes \text{id} \right) \circ \left( \text{id} \otimes \exp(x^i \mid \! i^{-1} p^k)_{R,L} \right). \]  

(109)

The fourth equality follows from the addition law for q-exponentials written in the form

\[ \sum_{a,b} e^a_{(x,R)} \otimes e^b_{(x,R)} \otimes e^b_{(p,L)} \otimes e^a_{(p,L)} = \sum_a e^a_{(x,R)} \otimes (e^a_{(p,L)})(R,1) \otimes (e^a_{(p,L)})(R,2), \]  

(110)

For the fifth equality we used translation invariance of the integral over the whole momentum space and the sixth equality is a consequence of the identities

\[ g(0) \equiv g(x^i)\bigg|_{x^i = 0} = \epsilon(W(g)) = \langle 1, g(x^i) \rangle_{L,R}. \]  

(111)

The relations in (106) are formulated with left integrals and the delta functions are multiplied from the right. However, we can write down a variant of each relation in (106) using right integrals and multiplying delta functions from the left:

\[ \int_{-\infty}^{+\infty} d^n_R x \delta^a_L(x^j) \otimes g(x^i) = g(0) \text{vol}_R, \]

\[ \int_{-\infty}^{+\infty} d^n_R x \delta^a_L(x^j) \otimes g(x^i) = g(0) \text{vol}_R. \]  

(112)

These identities can be verified in a similar fashion as those in (106). We wish to illustrate this by the following calculation:

\[ \int_{-\infty}^{+\infty} d^n_R x \delta^a_L(x^j) \otimes g(x^i) \]

\[ = \int_{-\infty}^{+\infty} d^n_R x \int_{-\infty}^{+\infty} d^n_L p \exp(i^{-1} p^k x^j)_{R,L} \overset{x}{\otimes} g(x^i) \]

\[ = \int_{-\infty}^{+\infty} d^n_L p e^a_{(p,R)} \otimes \int_{-\infty}^{+\infty} d^n_R x e^a_{(x,L)} \overset{x}{\otimes} g(x^i) \]

\[ = \int_{-\infty}^{+\infty} d^n_L p e^a_{(p,R)} \otimes \int_{-\infty}^{+\infty} d^n_R x e^a_{(x,L)} \overset{x}{\otimes} e^b_{(x,L)} \langle g(x^i), e^b_{(p,R)} \rangle_{L,R} \]

\[ = \int_{-\infty}^{+\infty} d^n_L p (e^a_{(p,R)})(L,1) \otimes \int_{-\infty}^{+\infty} d^n_R x e^a_{(x,L)} \langle g(x^i), (e^a_{(p,R)})(L,2) \rangle_{L,R} \]

25
\[ \int_{-\infty}^{+\infty} d_L^p e^{a_{(p,R)}} \otimes \int_{-\infty}^{+\infty} d_R^p x \langle g(x^i), 1 \rangle_{L,R} e^{a_{(x,L)}} \]

\[ = g(0) \int_{-\infty}^{+\infty} d_L^p \int_{-\infty}^{+\infty} d_R^p x \exp(\mathrm{i} p^k |x^j)_{R,L} = g(0) \text{ vol}_R. \quad (113) \]

Remarkably, the arguments leading to the identities in (106) and (112) do not really depend on the realizations of q-deformed delta function and q-integral over the whole space. For this reason, one can even show that we have

\[ \int_{-\infty}^{+\infty} d_A^p g(x^i) x^j \otimes \delta_B^p(x^j) = g(0) \text{ vol}_{A,B}, \]

\[ \int_{-\infty}^{+\infty} d_A^p \delta_B^p(x^j) \otimes g(x^i) = g(0) \text{ vol}_{A,B}, \quad (114) \]

if we introduce as q-deformed volume elements

\[ \text{vol}_{A,B} \equiv \int_{-\infty}^{+\infty} d_A^p \delta_B^p(x^j), \quad A, B \in \{L, \bar{L}, R, \bar{R}\}. \quad (115) \]

From what we have done so far we see that the integral of the product between a given function and a q-deformed delta function is proportional to the value of the given function at zero. This observation is in complete analogy to the classical case and can be extended in the following way:

\[ \int_{-\infty}^{+\infty} d_L^p y f(y^i) y^j \otimes_L \delta_R^p((\otimes_R x^j) \oplus_L y^k) = \text{ vol}_L f(x^j), \]

\[ \int_{-\infty}^{+\infty} d_L^p y f(y^i) y^j \otimes_L \delta_R^p((\otimes_R x^j) \oplus_L y^k) = \text{ vol}_L f(x^j), \quad (116) \]

\[ \int_{-\infty}^{+\infty} d_R^p y \delta_L^p(y^k \otimes_R (\otimes_L x^j) \oplus_R y^j) = \text{ vol}_R f(x^j), \]

\[ \int_{-\infty}^{+\infty} d_R^p y \delta_L^p(y^k \otimes_R (\otimes_L x^j) \oplus_R y^j) = \text{ vol}_R f(x^j). \quad (117) \]

The above identities are verified by rather simple calculations. Using (104) together with the axioms for q-translations we have, for example,

\[ \int_{-\infty}^{+\infty} d_L^p y f(y^i) y^j \otimes_L \delta_R^p((\otimes_R x^j) \oplus_L y^k) \]

26
\[
\int_{-\infty}^{+\infty} d^2 y \, f((\ominus_L (\ominus_R x^j)) \oplus_L y^i) \overset{y}{\oplus} \delta^0_R(y^k)
= \text{vol}_L f((\ominus_L (\ominus_R x^j)) \oplus_L 0) = \text{vol}_L f(x^j).
\] (118)

Finally, let us note that there exist generalizations of the relations in (116) and (117), which take the form

\[
\int_{-\infty}^{+\infty} d^2 y \, f(y^j) \overset{y}{\oplus} \delta^0_B(x^j \oplus_B y^k) = \text{vol}_{A,C} f(\ominus_B x^j),
\]

\[
\int_{-\infty}^{+\infty} d^2 y \, \delta^0_C(y^k \oplus_B x^j) \overset{x^j}{\oplus} \delta^0_B(y^i) = \text{vol}_{A,C} f(\ominus_B x^j).
\] (119)

For the sake of completeness we would like to give the identities

\[
\int_{-\infty}^{+\infty} d^2 y \, f(y^j) \overset{y}{\oplus} \delta^0_B(y^j \oplus_C (\ominus_C x^k))
= \int_{-\infty}^{+\infty} d^2 y \, f(y^i \oplus_C x^k) \overset{x^j}{\oplus} \delta^0_C(y^j)
= \text{vol}_{A,B} f(y^i \oplus_C ((\kappa_C)^{-1} x^k))|_{y^i=0} = \text{vol}_{A,B} f((\kappa_C)^{-1} x^k),
\] (120)

and

\[
\int_{-\infty}^{+\infty} d^2 y \, \delta^0_B((\ominus_C x^k \oplus_C y^j) \overset{y}{\oplus} f(y^i)
= \int_{-\infty}^{+\infty} d^2 y \, \delta^0_B(y^j) \overset{y}{\oplus} f(x^k \oplus_C y^i)
= \text{vol}_{A,B} f(((\kappa_C)^{-1} x^k) \oplus_C y^j)|_{y^i=0} = \text{vol}_{A,B} f((\kappa_C)^{-1} x^k).
\] (121)

Notice that for the second equality in (120) and (121) we exploited the identities (104) and (105), respectively. The third equality in (120) as well as in (121) results from (114) and the braiding of q-deformed delta functions which is characterized by the relations

\[
f(x^i) \overset{x^j}{\oplus} \delta_B(y^j) = \delta_B(y^j) \overset{y^i}{\oplus} f((\kappa_A)^{-1} x^i),
\]

\[
\delta_B(y^j) \overset{y^i}{\oplus} f(x^i) = f((\kappa_A)^{-1} x^i) \overset{x^j}{\oplus} \delta_B(y^j).
\] (122)

A short glance at the definitions in (87) and (88) tells us that q-deformed delta functions are made up out of q-integrals over the entire space and q-
deformed exponentials. The relations in (122) follow rather easily from the braiding properties of these objects.

Finally, we would like to mention that the functions \( \delta_n^A(\ominus_B x^i) \) give equally good delta functions, as they satisfy

\[
\int_{-\infty}^{+\infty} d^n_A x g(x^i) \overset{\times}{\oplus} \delta_B^n(\ominus_C x^j) = g(0) \operatorname{vol}_{A,B} \times \begin{cases} (\kappa_C)^n & \text{if } B \in \{R, \bar{R}\}, \\ (\kappa_C)^{-n} & \text{if } B \in \{L, \bar{L}\}, \end{cases}
\]

(123)

The above relations are needed to check the identities in (102) and (103). For a proof of the relations in (123) we refer the reader to Appendix B.

Now, we have everything together to prove the identities in (100)-(103). In this section we give a proof of the relations in (100) and (101), only. To check the relations in (102) and (103) is a little bit harder. Thus, the corresponding proof is presented in Appendix B. Again, it suffices to restrict attention to one type of Fourier transformations, since our considerations carry over to the other Fourier transformations without any difficulties. Concretely, we have

\[
F_R^\dagger (F_L(g)(p^j))(x^j)
= \frac{1}{\operatorname{vol}_R} \int_{-\infty}^{+\infty} d^n_R p \int_{-\infty}^{+\infty} d^n_L y \exp(\ominus_R x^i |i^{-1}p|^k_R)_{R,L}
\]

(124)
\[
\frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d_L y \ g((\kappa x^j) \oplus_L y^m) \otimes R^\kappa(y^j) \\
= \frac{1}{\text{vol}_L} \text{vol}_R \ g((\kappa x^j) \oplus_L 0) = g((\kappa x^j)).
\]

For the first and second equality we inserted the expressions for the q-deformed Fourier transforms. For the third equality we applied the addition law for q-deformed exponentials. The fourth step uses the relation in (104) and for the sixth step we identified the expression for a q-deformed delta function. The final result is a consequence of (107), (91), and the axioms of q-translations.

In complete analogy to the identities in (92)-(95) the Fourier transformations \( \mathcal{F}_A \) fulfill

\[
\mathcal{F}_L(i \partial^j \overset{P}{\circ} f(\kappa^{-1} p^j))(\kappa^{-1} x^k) = \kappa^n x^j \overset{P}{\circ} \mathcal{F}_L^*(f)(x^k),
\]

\[
\mathcal{F}_L(i \partial^j \overset{P}{\circ} f(\kappa x^j))(\kappa x^k) = \kappa^{-n} x^j \overset{P}{\circ} \mathcal{F}_L^*(f)(x^k),
\]

\[
\mathcal{F}_R^*(f(\kappa^{-1} p^j) \overset{P}{\circ} \hat{\partial}^j)(\kappa^{-1} x^k) = \kappa^n \mathcal{F}_R^*(f)(x^k) \overset{P}{\circ} x^j,
\]

\[
\mathcal{F}_R^*(f(\kappa x^j) \overset{P}{\circ} \hat{\partial}^j)(\kappa x^k) = \kappa^{-n} \mathcal{F}_R^*(f)(x^k) \overset{P}{\circ} x^j,
\]

and

\[
\mathcal{F}_L^*(i^{-1} p^j \overset{P}{\circ} f(\kappa^{-1} p^j))(\kappa^{-1} x^k) = \kappa^n \partial^j \overset{P}{\circ} \mathcal{F}_L^*(f)(x^k),
\]

\[
\mathcal{F}_L^*(i^{-1} p^j \overset{P}{\circ} f(\kappa x^j))(\kappa x^k) = \kappa^{-n} \partial^j \overset{P}{\circ} \mathcal{F}_L^*(f)(x^k),
\]

\[
\mathcal{F}_R^*(f(\kappa^{-1} p^j) \overset{P}{\circ} (i^{-1} p^j))(\kappa^{-1} x^k) = \kappa^{-n} \mathcal{F}_R^*(f)(x^k) \overset{P}{\circ} \hat{\partial}^j,
\]

\[
\mathcal{F}_R^*(f(\kappa x^j) \overset{P}{\circ} (i^{-1} p^j))(\kappa x^k) = \kappa^n \mathcal{F}_R^*(f)(x^k) \overset{P}{\circ} \hat{\partial}^j.
\]

The above identities are a direct consequence of the relations in (122)-(125) if we take into account the relations in (109)-(113). This can be seen as follows:

\[
\mathcal{F}_L(f \overset{P}{\circ} \hat{\partial}^j)(p^k) = \mathcal{F}_L(f)(p^k) \overset{P}{\circ} (i^{-1} p^j)
\]

\[
\Rightarrow f \overset{P}{\circ} \hat{\partial}^j = \mathcal{F}_L^*(f)(p^k) \overset{P}{\circ} (i^{-1} p^j)(\kappa^{-1} x^j)
\]

\[
\Rightarrow \mathcal{F}_R^*(f)(x^j) \overset{P}{\circ} \hat{\partial}^j = \kappa^n \mathcal{F}_R^*(f(\kappa^{-1} p^m) \overset{P}{\circ} (i^{-1} p^j))(\kappa^{-1} x^j),
\]

\[
29
\]
and

\[ \mathcal{F}_L(f \otimes x^j)(p^k) = i \mathcal{F}_L(f)(p^k) \hbar \partial^j \]

\[ \Rightarrow f \otimes x^j = \mathcal{F}_R^*(i \mathcal{F}_L(f)(p^k) \hbar \partial^j)(\kappa^{-1} x^j) \]

\[ \Rightarrow \mathcal{F}_R^*(f)(x^j) \otimes x^j = \kappa^n \mathcal{F}_R^*(i f(\kappa^{-1} p^m) \hbar \partial^j)(\kappa^{-1} x^j). \] (131)

Similar calculations lead to the other relations in (126)-(129).

The relations in (94), (95), (128), and (129) describe in some sense infinitesimal translations of q-deformed Fourier transforms. We can also write down formulae for global translations of q-deformed Fourier transforms. Especially, we have

\[ \mathcal{F}_L(f)(p^k \oplus_R \bar{p}^l) \]

\[ = \int_{-\infty}^{+\infty} d_L x f(x^j) \otimes \exp(x^j | i^{-1} (p^k \oplus_R \bar{p}^l)) |_{R,L} \]

\[ = \int_{-\infty}^{+\infty} d_R^m p f(p^j) \quad \bar{p}^{|xy}_{|_{R,L}} \quad \exp((\oplus_R x^i) \oplus_L (\oplus_R y^j)| i^{-1} p^k) |_{R,L} \]

\[ = \mathcal{F}_L(f(x^j) \otimes \exp(x^j | i^{-1} \bar{p}^l) |_{R,L})(p^k), \] (132)

and

\[ \mathcal{F}_R^*(f)(x^j \oplus_R y^i) \]

\[ = \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d_R^m p f(p^j) \quad \bar{p}^{|xy}_{|_{R,L}} \quad \exp((\oplus_R x^i) \oplus_L (\oplus_R y^j)| i^{-1} p^k) |_{R,L} \]

\[ = \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d_R^m p f(p^j) \quad \bar{p}^{|y}_{|_{R,L}} \quad \exp((\oplus_R y^j)| i^{-1} p^k) |_{R,L} \]

\[ = \mathcal{F}_R^*(f(p^j) \quad \bar{p}^{|y}_{|_{R,L}} \quad \exp((\oplus_R y^j)| i^{-1} p^k) |_{R,L}), \] (133)

where again we made use of the addition law for q-exponentials and the axioms for q-translations. In a similar fashion, we get for the right versions
of q-deformed Fourier transforms:

\[ \mathcal{F}_R(f)(\tilde{p}^i \oplus_L p^k) = \int_{-\infty}^{+\infty} d^n_R x \exp(i^{-1}p^k|x^m)_R, \bar{L} \]
\[ = \mathcal{F}_R(\exp(i^{-1}\tilde{p}^i|x^j)_R, \bar{L} \oplus f(x^i)) \]
\[ = \mathcal{F}_R(\exp(i^{-1}\tilde{p}^i|x^j)_R, \bar{L} \oplus f(x^i))(p^k), \quad (134) \]

\[ \mathcal{F}_L^*(f)(y^j \oplus_L x^i) = \frac{1}{\text{vol}_L} \int_{-\infty}^{+\infty} d^n_L p \exp(i^{-1}p^m|x^i)_L \]
\[ = \mathcal{F}_L^*(\exp(i^{-1}p^k|\ominus_L y^j)_R, \bar{L} \ominus_L f(p^i))(x^i). \quad (135) \]

The corresponding relations for the other versions of q-deformed Fourier transforms are obtained most easily from the formulae in (132)-(135) by changing the labels of q-deformed objects according to

\[ L \rightarrow \bar{L}, \quad \bar{L} \rightarrow L, \quad R \rightarrow \bar{R}, \quad \bar{R} \rightarrow R. \quad (136) \]

### 3.4 Fourier transforms of q-exponentials and q-deformed delta functions

It is rather instructive to calculate q-deformed Fourier transforms of q-exponentials and q-deformed delta functions. We start our considerations with q-exponentials. For their q-deformed Fourier transforms, we find

\[ \mathcal{F}_L(\exp(i^{-1}p^k|\ominus_L y^j)_R, \bar{L}))(x^i) \]
\[ = \int_{-\infty}^{+\infty} d^n_L p \exp(i^{-1}p^k|\ominus_L y^j)_R, \bar{L} \exp(i^{-1}p^i|x^i)_R, \bar{L} \]
\[ = \delta_L^p((\ominus_L y^j) \oplus_L x^i), \quad (137) \]

\[ \mathcal{F}_R(\exp(\ominus_R y^j|i^{-1}p^k)_R, \bar{L}))(x^i) \]
\[ = \int_{-\infty}^{+\infty} d^n_R p \exp(x^i|i^{-1}p^i)_R, \bar{L} \exp(\ominus_R y^j|i^{-1}p^k)_R, \bar{L} \]
\[ = \delta_R^p(x^i \ominus_R (\ominus_R y^j)). \quad (138) \]
Concretely, we have

\[ \mathcal{F}_R^+(\exp(i y^j | p^k)_{R,L})(x^i) \]
\[ = \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_R p \exp(i \gamma | p^k)_{R,L} \otimes_R \exp(\gamma x^i | -1 p^k)_{R,L} \]
\[ = \frac{1}{\text{vol}_R} \delta^n_R (y^j \oplus_R (\gamma x^i)) \]  
(139)

\[ \mathcal{F}_L^+(\exp(p^k | y^j)_{R,L})(x^i) \]
\[ = \frac{1}{\text{vol}_L} \int_{-\infty}^{+\infty} d^n_L p \exp(i \gamma | p^k)_{R,L} \otimes_R \exp(i \gamma y^j)_{R,L} \]
\[ = \frac{1}{\text{vol}_L} \delta^n_L ((\gamma x^i) \oplus_L y^j) \]  
(140)

We see that Fourier transformations of q-exponentials lead to q-deformed delta functions.

With the identities in (116), (117), (120), and (121) we can show that the Fourier transform of a q-deformed delta function is given by a q-exponential. Concretely, we have

\[ \mathcal{F}_L(\delta^n_R(y^j \oplus_R (\gamma x^i)))(p^k) \]
\[ = \int_{-\infty}^{+\infty} d^n_L x \delta^n_R(y^j \oplus_R (\gamma x^i)) \otimes_R \exp(x^i | i p^k)_{R,L} \]
\[ = \int_{-\infty}^{+\infty} d^n_L x \delta^n_R((\gamma \kappa^{-1} y^j) \oplus_R (\kappa x^i)) \otimes_R \exp(x^i | i p^k)_{R,L} \]
\[ = \kappa^{-n} \int_{-\infty}^{+\infty} d^n_L x \delta^n_R((\gamma \kappa^{-1} y^j) \oplus_R (\kappa x^i)) \otimes_R \exp((\gamma \kappa^{-1} y^j) | i p^k)_{R,L} \]
\[ = \kappa^{-n} \text{vol}_R \exp((\gamma \kappa^{-1} y^j) | i p^k)_{R,L} \]  
(141)

\[ \mathcal{F}_R(\delta^n_L((\gamma x^i) \oplus_L y^j))(p^k) \]
\[ = \int_{-\infty}^{+\infty} d^n_R x \exp(i \gamma p^k | x^i)_{R,L} \otimes_L \delta^n_L((\gamma x^i) \oplus_L y^j) \]
\[ = \int_{-\infty}^{+\infty} d^n_R x \exp(i \gamma p^k | x^i)_{R,L} \otimes_L \delta^n_L((\kappa^{-1} x^i) \oplus_L (\gamma \kappa y^j)) \]
\[ = \kappa^n \int_{-\infty}^{+\infty} d^n_R x \exp(i \gamma p^k | \kappa x^i)_{R,L} \otimes_L \delta^n_L((\gamma x^i) \oplus_L (\gamma \kappa y^j)) \]
\[
\kappa^n \nuol \exp(i^{−1}p|\kappa y^n)_{R,L}, \quad (142)
\]
and
\[
\mathcal{F}_R^\ast (\delta^n_R((\oplus_L y^n) \oplus_L x^n))(p) \\
= \frac{1}{\nuol} \int_{−\infty}^{+\infty} d^n_R x \delta^n_R((\oplus_L y^n) \oplus_L x^n) \oplus_R \exp(i^{−1}p|\oplus_L x^n)_{R,L} \\
= \exp(i^{−1}\kappa p|\oplus_L y^n)_{R,L}, \quad (143)
\]
\[
\mathcal{F}_L^\ast (\delta^n_R(x^n \oplus_R (\oplus_R y^n)))(p) \\
= \frac{1}{\nuol} \int_{−\infty}^{+\infty} d^n_L x \exp(\oplus_R x^n|i^{−1}p^n)_{R,L} \oplus_L \delta^n_R(x^n \oplus_R (\oplus_R y^n)) \\
= \exp(\oplus_R y^n|i^{−1}\kappa^{−1}p^n)_{R,L}. \quad (144)
\]

The above relations can be derived from the results in (137)-(140) with the help of the identities in (100)-(103). Notice that the calculations in (141) and (142) make use of the identities

\[
\frac{1}{\nuol} \int_{−\infty}^{+\infty} d^n_R x \delta^n_R((\oplus_R y^n) \oplus_R x^n) {\null \overset{x}{\oplus}} \delta^n_R(x^n \oplus_R (\oplus_R y^n)) \\
= \delta^n_R((\oplus_R y^n) \oplus_R (\kappa y^n)) = \delta^n_R((\kappa y^n) \oplus_R (\oplus_R y^n)), \quad (145)
\]
\[
\frac{1}{\nuol} \int_{−\infty}^{+\infty} d^n_L x \delta^n_L((\oplus_L y^n) \oplus_L x^n) {\null \overset{x}{\oplus}} \delta^n_L(x^n \oplus_L (\oplus_L y^n)) \\
= \delta^n_L((\oplus_L y^n) \oplus_L (\kappa^{−1}y^n)) = \delta^n_L((\kappa^{−1}y^n) \oplus_L (\oplus_L y^n)), \quad (146)
\]

Lastly, it should be mentioned that we obtain further relations from the above ones by applying the substitutions in (136) together with the replacement \(\kappa \rightarrow \kappa^{−1}\).

### 3.5 Convolution products

In this subsection we turn our attention to q-analogs of the *convolution product*. They are defined by

\[
(f \ast_{A,B} g)(y^n) \equiv \int_{−\infty}^{+\infty} d^n_A x f(x^n) {\null \overset{x}{\oplus}} g((\oplus_B x^n) \oplus_B y^n),
\]
\[
(f \ast_{A,B} g)(y^n) \equiv \int_{−\infty}^{+\infty} d^n_A x g(y^n \oplus_B (\oplus_B x^n)) {\null \overset{x}{\oplus}} f(x^n), \quad (147)
\]
with $A, B \in \{L, \bar{L}, R, \bar{R}\}$.

In Ref. [45] it was shown that q-deformed convolution products are associative. In our formalism this feature of convolution products is modified as follows:

$$f *_{A,B} (g *_{A,B} h) = (\kappa_B)^{-n}(f((\kappa_B)^{-1}x^i) *_{A,B} g) *_{A,B} h,$$

$$f *_{A,B} (g *_{A,B} h) = (\kappa_B)^{-n}f *_{A,B} (g *_{A,B} h((\kappa_B)^{-1}x^i)).$$

The following calculation shall illustrate how to prove this property:

$$f *_{L,L} (g *_{L,L} h)$$

$$= \int_{-\infty}^{+\infty} d^nL_x f(x^i) \otimes \int_{-\infty}^{+\infty} d^nL_y g(y^j) \circ_L h((\oplus_L x^i \oplus_L z^m))$$

$$= \int_{-\infty}^{+\infty} d^nL_x f(x^i) \otimes \int_{-\infty}^{+\infty} d^nL_y g(y^j) \circ_L h((\oplus_L y^k \oplus_L (\oplus_L x^i)) \oplus_L z^m)$$

$$= \int_{-\infty}^{+\infty} d^nL_x f(x^i) \otimes \int_{-\infty}^{+\infty} d^nL_y g(y^j) \circ_L (\oplus_L (\kappa x^i) \oplus_L y^j)) \oplus_L z^m)$$

$$= \int_{-\infty}^{+\infty} d^nL_y \int_{-\infty}^{+\infty} d^nL_x f(x^i) \otimes g((\oplus_L \kappa x^i) \oplus_L y^j) \circ_L h((\oplus_L y^k) \oplus_L z^m)$$

$$= \kappa^{-n}(f((\kappa^{-1}x^i) *_{L,L} g) *_{L,L} h).$$

The first equality is clear, since it is a direct consequence of the definition of q-deformed convolution products. The second equality is the law of associativity for q-translations, and for the third step we applied a kind of distributivity law for q-translations. For the fourth step we made use of the fact that the q-translations $\oplus_L$ and $\oplus_R$ are related to each other by a twist. The sixth step is an application of (144) and leads to a result that can be recognized as the right-hand side of the first relation in (148).

Next, we consider opposite convolution products, which are given by

$$(f *_{A,B} g)(y^i) \equiv ((\mathcal{R}_{[2]} g) *_{A,B} (\mathcal{R}_{[1]} f))(y^i)$$

$$= \int_{-\infty}^{+\infty} d^nA_x \left[f(z^j) \circ_L g(x^i) \right]_{z^j \rightarrow ((\oplus_B x^k) \oplus_B y^j)}.$$
The first relation can be verified in the following manner:

\[(f \ast_{A,B}^L g)(y^i) = (\langle \mathcal{R}_{[1]}^{-1} \triangleright g \rangle \ast_{A,B} (\mathcal{R}_{[2]}^{-1} \triangleright f))(y^i)\]

\[= \int_{-\infty}^{+\infty} d^n_A x \left[ f(x^j) \circ_L g(x^j) \right] z^j \rightarrow (\ominus_B x^h) \oplus_B y^i, \quad (150)\]

\[(f \ast_{A,B}^R g)(y^i) = ((g \triangleleft \mathcal{R}_{[1]}^{-1}) \ast_{A,B} (f \triangleleft \mathcal{R}_{[2]}^{-1}))(y^i)\]

\[= \int_{-\infty}^{+\infty} d^n_A x \left[ f(x^j) \circ_R g(x^j) \right] z^j \rightarrow (\ominus_B x^h) \oplus_B y^i, \quad (151)\]

and

\[(f \ast_{A,B}^L g)(y^i) = ((\mathcal{R}_{[2]} \triangleright g) \ast_{A,B} (\mathcal{R}_{[1]} \triangleright f))(y^i)\]

\[= \int_{-\infty}^{+\infty} d^n_A x \left[ f(x^j) \circ_L g(x^j) \right] z^j \rightarrow y^i \ominus_B (\ominus_B x^h), \quad (152)\]

\[(f \ast_{A,B}^R g)(y^i) = ((\mathcal{R}_{[1]}^{-1} \triangleright g) \ast_{A,B} (\mathcal{R}_{[2]}^{-1} \triangleright f))(y^i)\]

\[= \int_{-\infty}^{+\infty} d^n_A x \left[ f(x^j) \circ_R g(x^j) \right] z^j \rightarrow y^i \ominus_B (\ominus_B x^h), \quad (153)\]

One can show that for opposite convolution products we have the relations

\[(f \ast_{A,B}^C g) \ast_{A,B}^C h = (\kappa_B)^n f((\kappa_C)^{-1} x^j) \ast_{A,B}^C (g \ast_{A,B}^C h(\kappa_B \kappa_C x^j)), \quad (154)\]

The first relation can be verified in the following manner:

\[(f \ast_{A,B}^L g) \ast_{A,B}^L h\]
\[(\mathcal{R}_2 \triangleright h) *_{A,B} (\mathcal{R}_1 \triangleright [(\mathcal{R}_1' \triangleright g) *_{A,B} (\mathcal{R}_1' \triangleright f)]) = (\mathcal{R}_2 \triangleright h) *_{A,B} ((\mathcal{R}_1(1) \mathcal{R}_2' \triangleright g) *_{A,B} (\mathcal{R}_1(2) \mathcal{R}_1' \triangleright f))
= (\mathcal{R}_2 \mathcal{R}_2'' \triangleright h(\kappa_L \tilde{x}^j)) *_{A,B} ((\mathcal{R}_1 \mathcal{R}_2' \triangleright g) *_{A,B} (\mathcal{R}_1' \mathcal{R}_1' \triangleright f))
= ((\mathcal{R}_2 \mathcal{R}_2' \triangleright h(\kappa_L \kappa_B \tilde{x}^j)) *_{A,B} (\mathcal{R}_1 \mathcal{R}_2' \triangleright g) *_{A,B} (\mathcal{R}_1' \mathcal{R}_1' \triangleright f))
= (\mathcal{R}_1' \mathcal{R}_2 \triangleright h(\kappa_L \kappa_B \tilde{x}^j)) *_{A,B} (\mathcal{R}_1(1) \mathcal{R}_2(2) \mathcal{R}_1(1) \triangleright g) *_{A,B} (\mathcal{R}_1' \mathcal{R}_1' \triangleright f)
= ((\mathcal{R}_2' \triangleright h(\kappa_L \kappa_B \tilde{x}^j)) *_{A,B} (\mathcal{R}_1(2) \mathcal{R}_1(1) \triangleright g)) *_{A,B} (\mathcal{R}_1' \mathcal{R}_1' \triangleright f)
= (\mathcal{R}_1' \triangleright h(\kappa_L \kappa_B \tilde{x}^j)) *_{A,B} (\mathcal{R}_1(2) \mathcal{R}_1(1) \triangleright g) *_{A,B} (\mathcal{R}_1' \mathcal{R}_1' \triangleright f)
\]

\[f((\kappa_L)^{-1} x^i) *_{A,B} (g *_{A,B} h(\kappa_L \kappa_B \tilde{x}^j)). \quad (155)\]

The first and the last step uses the definition of the opposite convolution product, while the second and seventh step result from covariance of the convolution products, i.e. convolution products can be passed through a braid-crossing up to additional scalings. (Notice that the scaling of \(h\) with \(\kappa_L\) is a consequence of the fact that the definition of a convolution product contains an integral over the whole space.) For the third and sixth equality we applied the axioms of quasi-triangularity [52–54], i.e.

\[(\text{id} \otimes \Delta) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12}, \quad (\Delta \otimes \text{id}) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23}. \quad (156)\]

The fourth step is associativity of the convolution product and for the fifth equality we made use of the Yang-Baxter equation

\[\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}. \quad (157)\]

There is a property of opposite convolution products worth recording here. It concerns the observation that q-deformed Fourier transformations map star products to opposite convolution products and vice versa. To be more specific, we have

\[\mathcal{F}_L(f)(p^i) *_{p}^{L} \mathcal{F}_L(g)(p^k) = \mathcal{F}_L(f(\kappa x^i) *_{\tilde{L},L}^{L} g)(p^k),\]
\[\mathcal{F}_L(f)(p^i) *_{p}^{L} \mathcal{F}_L(g)(p^k) = \mathcal{F}_L(f(\kappa^{-1} x^i) *_{\tilde{L},L}^{L} g)(p^k), \quad (158)\]

\[\mathcal{F}_R(f)(p^i) *_{p}^{R} \mathcal{F}_R(g)(p^k) = \mathcal{F}_R(f *_{R,R}^{R} g(\kappa^{-1} x^i))(p^k),\]
\[\mathcal{F}_R(f)(p^i) *_{p}^{R} \mathcal{F}_R(g)(p^k) = \mathcal{F}_R(f *_{R,R}^{R} g(\kappa x^i))(p^k). \quad (159)\]
The above identities can be checked in the following manner:

\[
\mathcal{F}_L(f)(p^j) \circ \mathcal{F}_L(f)(p^k)
= \int_{-\infty}^{+\infty} d_L^n x f(x^j) \oplus \exp(x^l|^{-1} p^j)_{R,L}
\circ \int_{-\infty}^{+\infty} d_L^n y g(y^m) \oplus \exp(y^r|^{-1} p^k)_{R,L}
\]

\[
= \int_{-\infty}^{+\infty} d_L^n x f(x^j) \int_{-\infty}^{+\infty} d_L^n y g(y^m) \oplus \exp(y^r \oplus_L x^l|^{-1} p^k)_{R,L}
\]

\[
= \int_{-\infty}^{+\infty} d_L^n x \int_{-\infty}^{+\infty} d_L^n y f(\kappa x^j) \oplus_L (g(y^m) \oplus \exp(y^r \oplus_L x^l|^{-1} p^k)_{R,L})
\]

\[
= \int_{-\infty}^{+\infty} d_L^n x (R_{[2]} g(y^m)) \oplus \int_{-\infty}^{+\infty} d_L^n x (R_{[1]} f(\kappa x^j))
\]

\[
\oplus_L \exp(x^l|^{-1} p^k)_{R,L}
\]

\[
= \int_{-\infty}^{+\infty} d_L^n y (R_{[2]} g(y^m)) \oplus \int_{-\infty}^{+\infty} d_L^n x (R_{[1]} f(\kappa x^j))
\]

\[
\oplus_L \exp(x^l|^{-1} p^k)_{R,L}
\]

\[
= \int_{-\infty}^{+\infty} d_L^n x \int_{-\infty}^{+\infty} d_L^n y [f(\kappa x^j) \oplus_L g(y^m)]_{x^j \rightarrow (\oplus_L y^r) \oplus_L x^j}
\]

\[
\oplus_L \exp(x^l|^{-1} p^k)_{R,L}
\]

\[
= \mathcal{F}_L(f(\kappa x^i) \ast_{L,L} g)(p^k).
\] (160)

The first step uses the definition of q-deformed Fourier transformations and for the second step we apply the addition law for q-deformed exponentials. For the third and fourth step we rearrange tensor factors, and the fifth equality holds due to (104).

Last but not least, we would like to show that q-deformed delta functions play the role of an identity element in the convolution product algebra. Concretely, it holds

\[
(\delta_{A,C}^B *_{A,C} f)(y^k) = (f(\kappa_D x^j) *_{A,C} \delta_{B}^C)(y^k)
\]

\[
= \int_{-\infty}^{+\infty} d_A^n x \delta_{B}^C(x^i) \oplus f((\oplus_C x^j) \oplus_C y^k)
\]

\[
= \text{vol}_{A,B} f(y^k),
\] (161)
\[(f *_{A,C} \delta^n B)(y^k) = (\delta^n B *_{A,C} f(\kappa x^j))(y^k)\]
\[= \int_{-\infty}^{+\infty} d^n_A x f(y^k \oplus_C (\ominus_C x^j)) \otimes \delta^n_B(x^i)\]
\[= \text{vol}_{A,B} f(y^k), \quad (162)\]

and

\[(f *_{A,C} \delta^n B)(y^k) = (\delta^n B *_{A,C} f(\kappa x^j))(y^k)\]
\[= \int_{-\infty}^{+\infty} d^n_A x f(x^i) \otimes \delta^n_B((\ominus_C x^j) \oplus_C y^k),\]
\[= (\kappa_C)^n \text{vol}_{A,B} f(\kappa_C y^k), \quad (163)\]

\[(\delta^n B *_{A,C} f)(y^k) = (f(\kappa x^j) *_{A,C} \delta^n B)(y^k)\]
\[= \int_{-\infty}^{+\infty} d^n_A x \delta^n_B(y^k \oplus_C (\ominus_C x^j)) \otimes f(x^i)\]
\[= (\kappa_C)^n \text{vol}_{A,B} f(\kappa_C y^k), \quad (164)\]

where \(A, B, C, D \in \{L, \bar{L}, R, \bar{R}\}\). Notice that the equalities concerning opposite convolution products arise from the braiding properties of q-deformed delta functions [cf. the identities in (122)]. The last equality in (161) as well as in (162) is clear from the relations in (114). The derivations of (163) and (164) need the identities (145) and (146). The following calculation referring to the case \(C = \bar{L}\) shows the line of reasonings:

\[(f *_{A,L} \delta^n B)(y^k) = \int_{-\infty}^{+\infty} d^n_A x f(x^i) \otimes \delta^n_B((\ominus_L x^j) \oplus_L y^k)\]
\[= \int_{-\infty}^{+\infty} d^n_A x f(x^i) \otimes \delta^n_B((\kappa x^j) \oplus_L (\ominus_L \kappa^{-1} y^k))\]
\[= \kappa^{-n} \int_{-\infty}^{+\infty} d^n_A x f(\kappa^{-1} x^i) \otimes \delta^n_B(x^j \oplus_L (\ominus_L \kappa^{-1} y^k))\]
\[= \kappa^{-n} \text{vol}_{A,B} f(\kappa^{-1} x^i). \quad (165)\]

Notice that for the second step we have to apply a variant of the relation in (146) while the last equality holds due to (120).
3.6 Conjugation properties of Fourier transformations

We conclude our considerations about q-deformed Fourier transformations by discussing their conjugation properties. Recalling the conjugation properties of q-integrals, q-exponentials, and star products (see Ref. [38]) we get

\[ F_L(f)(p^k) = \int_{-\infty}^{+\infty} d^n_l x f(x^i) \delta_R \exp(x^j | i^{-1} p^k)_{R,L} \]
\[ = (-1)^n \int_{-\infty}^{+\infty} d^n_l x \exp(i^{-1} p^k x^j)_{R,L} \delta_R f(x^i) \]
\[ = (-1)^n F_R(\bar{f})(p^k). \] (166)

For q-deformed exponentials and q-deformed volume elements the above relations respectively imply

\[ \delta^n_L(p^k) = F_L(1)(p^k) = (-1)^n F_R(1)(p^k) = (-1)^n \delta^n_R(p^k), \] (167)

and

\[ \text{vol}_L = \int_{-\infty}^{+\infty} d^n_{R,p} \delta^n_L(p^k) = (-1)^n \int_{-\infty}^{+\infty} d^n_{L,p} (-1)^n \delta^n_R(p^k) \]
\[ = \int_{-\infty}^{+\infty} d^n_{R,p} \delta^n_R(p^k) = \text{vol}_R. \] (168)

With the same reasonings we find the conjugation properties of the Fourier transformations in (98) and (99):

\[ F^*_R(\bar{f})(x^j) = \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_{R,p} f(p^l) \delta_R \exp(\delta_R x^j | i^{-1} p^k)_{R,L} \]
\[ = (-1)^n \frac{1}{\text{vol}_L} \int_{-\infty}^{+\infty} d^n_{L,p} \exp(i^{-1} p^k \delta_R x^j)_{R,L} \delta_R f(p^l) \]
\[ = (-1)^n F_L(\bar{f})(x^k). \] (169)

Changing the labels in the above formulae according to the substitutions of (136) yields the relations for the other q-geometries.

A short glance at our results makes it clear that up to an additional minus sign q-deformed Fourier transforms are transformed into each other by the operation of conjugation. To avoid the additional minus signs in the above formulae, we modify the definitions of Fourier transformations, delta
functions, and volume elements by carrying out the following substitutions in the defining expressions:

\[ \int d^n_L x, \int d^n_R x \rightarrow \int d^n_1 x \equiv \frac{i^n}{2} \left( \int d^n_L x + \int d^n_R x \right), \]

\[ \int d^n_L x, \int d^n_R x \rightarrow \int d^n_2 x \equiv \frac{i^n}{2} \left( \int d^n_L x + \int d^n_R x \right). \] (170)

In other words, we deal with real integrals, only, and the new objects obtained this way are distinguished from the original ones by a tilde:

\[ F_A \rightarrow \tilde{F}_A, \quad F^*_A \rightarrow \tilde{F}^*_A, \]
\[ \delta^n_A \rightarrow \tilde{\delta}^n_A, \quad \text{vol}_A \rightarrow \tilde{\text{vol}}_A. \] (171)

It should be mentioned that all considerations and identities so far remain valid under these substitutions.

4 Sesquilinear forms on quantum spaces

In this section we first introduce sesquilinear forms on q-deformed quantum spaces. In Paper II these sesquilinear forms will lead us to q-deformed analogs of the Hilbert space of square integrable functions. We discuss such properties of q-deformed sesquilinear forms that will prove useful in formulating a q-deformed version of quantum kinematics. (This task is also be done in Part II.) This way we focus attention on adjoint operators of symmetry generators and invariance properties of q-deformed sesquilinear forms. Using the results of the previous section we finally show that our sesquilinear forms fulfill q-analogs of Fourier-Plancherel identities.

4.1 Definition of sesquilinear forms

Let us briefly recall what is meant by a sesquilinear form on a vector space \( V \). It is a mapping

\[ \langle \cdot, \cdot \rangle : V \otimes V \rightarrow \mathbb{C}, \] (172)

being subject to

1. \( \langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle , \)
2. \( \langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle , \)
3. \( \langle x, \alpha y \rangle = \alpha \langle x, y \rangle , \)
4. $\langle \alpha x, y \rangle = \overline{\alpha} \langle x, y \rangle$, 

for all $x, x_i, y, y_i \in V, \alpha \in \mathbb{C}$. Likewise, we can consider the mapping 

$\langle \ldots \rangle' : V \otimes V \rightarrow \mathbb{C},$ 

with 

1. $\langle x, y_1 + y_2 \rangle' = \langle x, y_1 \rangle' + \langle x, y_2 \rangle'$, 
2. $\langle x_1 + x_2, y \rangle' = \langle x_1, y \rangle' + \langle x_2, y \rangle'$, 
3. $\langle \alpha x, y \rangle' = \alpha \langle x, y \rangle'$, 
4. $\langle x, \alpha y \rangle' = \overline{\alpha} \langle x, y \rangle'$,

for all $x, x_i, y, y_i \in V, \alpha \in \mathbb{C}$. A sesquilinear form is called *symmetrical* when it additionally fulfills 

$$\overline{\langle x, y \rangle} = \langle y, x \rangle \quad \text{for all } x, y \in V.$$  \hspace{1cm} (173)

It is not very difficult to convince yourself that the following mappings give sesquilinear forms:

$$\langle f, g \rangle_A \equiv \int_{-\infty}^{+\infty} d\alpha^A x f(x^\alpha) \overline{g(x^\alpha)},$$

$$\langle f, g \rangle'_A \equiv \int_{-\infty}^{+\infty} d\alpha^A x f(x^\alpha) \overline{g(x^\alpha)},$$  \hspace{1cm} (174)

where $A \in \{L, \bar{L}, R, \bar{R}\}$.

Due to the conjugation properties of q-deformed integrals the mappings in (174) behave under conjugation as follows:

$$\overline{\langle f, g \rangle}_L = (-1)^n \langle g, f \rangle_{\bar{R}},$$

$$\overline{\langle f, g \rangle}_R = (-1)^n \langle g, f \rangle_{\bar{R}},$$  \hspace{1cm} (175)

$$\overline{\langle f, g \rangle}'_L = (-1)^n \langle g, f \rangle'_{\bar{R}},$$

$$\overline{\langle f, g \rangle}'_R = (-1)^n \langle g, f \rangle'_{\bar{R}}.$$  \hspace{1cm} (176)
Thus, we conclude that symmetrical sesquilinear forms are given by the expressions

\[
\begin{align*}
\langle f, g \rangle_1 & \equiv \frac{in}{2} \left( \langle f, g \rangle_L + \langle f, g \rangle_R \right), \\
\langle f, g \rangle_2 & \equiv \frac{in}{2} \left( \langle f, g \rangle_L + \langle f, g \rangle_R \right), \\
\langle f, g \rangle_1' & \equiv \frac{in}{2} \left( \langle f, g \rangle'_L + \langle f, g \rangle'_R \right), \\
\langle f, g \rangle_2' & \equiv \frac{in}{2} \left( \langle f, g \rangle'_L + \langle f, g \rangle'_R \right).
\end{align*}
\] (177)

4.2 Adjoint operators and invariance properties of sesquilinear forms

The sesquilinear forms introduced in the last subsection enable us to assign linear functionals to a function. This can be achieved by the mappings \((A \in \{L, \bar{L}, R, \bar{R}\})\)

\[
\begin{align*}
f^*_{A} : A_f & \to \mathbb{C}, \quad g \to \langle g, f \rangle_A, \\
f'^*_{A} : A'_f & \to \mathbb{C}, \quad g \to \langle f, g \rangle'_A,
\end{align*}
\] (179)

and \((i \in \{1, 2\})\)

\[
\begin{align*}
f^*_{i} : A_f & \to \mathbb{C}, \quad g \to \langle g, f \rangle_i, \\
f'^*_{i} : A'_f & \to \mathbb{C}, \quad g \to \langle f, g \rangle'_i,
\end{align*}
\] (180)

where \(A_f\) and \(A'_f\) denote the subspaces of the space of formal power series on which the corresponding functionals take on finite values. This way, our sesquilinear forms should provide a natural link between suitable subspaces of the space of formal power series and their dual spaces.

It is well-known that this natural correspondence extends to the operators defined on these spaces. Each operator on the space of formal power series induces operators on the above mentioned dual spaces. They are called adjoint operators and their defining equations become

\[
\begin{align*}
\langle O^\dagger_A \triangleright f, g \rangle_A & = \langle f, O \triangleright f \rangle_A, \\
\langle O'^\dagger_A \triangleright f, g \rangle'_A & = \langle f, O \triangleright f \rangle'_A.
\end{align*}
\] (181)
and

\[ \langle O^i \triangleright f, g \rangle_i = \langle f, O \triangleright f \rangle_i, \]
\[ \langle O^i_\prime \triangleright f, g \rangle_i = \langle f, O \triangleright f \rangle_i. \] (182)

In the following it is our aim to find the adjoints of the operators generating symmetry transformations on quantum spaces.

Let \( h \) be an element of a Hopf algebra \( H \) that acts upon our quantum spaces. Furthermore, we assume that \( h \) generates infinitesimal symmetry transformations. From the definitions in (174) it follows that

\[ \langle f, h \triangleright g \rangle_A = \int_{-\infty}^{+\infty} d^nx \, f(x^i) \otimes (h \triangleright g(x^j)) \]
\[ = \int_{-\infty}^{+\infty} d^nx \, h^{(2)} \triangleright \left[ (f(x^i) \triangleright h^{(1)}) \otimes g(x^j) \right] \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(h^{(2)}) \left[ (f(x^i) \triangleright h^{(1)}) \otimes g(x^j) \right] \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(h^{(2)}) \left[ (f(x^i) \otimes h) \otimes g(x^j) \right] \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(S^{-1}(h^{(2)})) \left[ f(x^i) \otimes (h^{(1)} \triangleright g(x^j)) \right] \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(S^{-1}(h^{(2)})) \left[ f(x^i) \otimes (h^{(1)} \triangleright g(x^j)) \right] \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(S^{-1}(h^{(2)})) \left[ f(x^i) \otimes (h \triangleright g(x^j)) \right] \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(S^{-1}(h^{(2)})) \left[ f(x^i) \otimes (h \triangleright g(x^j)) \right] \] (183)

Notice that for the third equality we used the fact that q-integrals over the whole space behave like scalars (see the discussion in Ref. [38]). In very much the same way we have

\[ \langle f \triangleleft h, g \rangle_A' = \int_{-\infty}^{+\infty} d^nx \, (f(x^i) \triangleleft h) \otimes g(x^j), \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \left[ f(x^i) \otimes (h^{(1)} \triangleright g(x^j)) \right] \otimes h^{(2)} \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(S^{-1}(h^{(2)})) \left[ f(x^i) \otimes (h^{(1)} \triangleright g(x^j)) \right] \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(S^{-1}(h^{(2)})) \left[ f(x^i) \otimes (h^{(1)} \triangleright g(x^j)) \right] \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(S^{-1}(h^{(2)})) \left[ f(x^i) \otimes (h \triangleright g(x^j)) \right] \]
\[ = \int_{-\infty}^{+\infty} d^nx \, \epsilon(S^{-1}(h^{(2)})) \left[ f(x^i) \otimes (h \triangleright g(x^j)) \right] \]}
\begin{equation}
= \int_{-\infty}^{+\infty} d_A x \left[ f(x^i) \overset{x}{\oplus} (g(x^j) \triangleleft h) \right] \\
= \langle f, g \triangleleft h \rangle'_A.
\tag{184}
\end{equation}

Recalling that integrals over the whole space are invariant under q-deformed translations, we are able to modify the above considerations in a way that they carry over to the algebra of partial derivatives. In doing so, we have to take attention of the fact that for actions of partial derivatives it holds \cite{38}
\begin{align}
\hat{\partial}^i \triangleright f(x^j) &= f(x^j) \triangleleft \hat{\partial}^i, \\
\hat{\partial}^i \triangleright f(x^j) &= f(x^j) \triangleleft \hat{\partial}^i, \\
\hat{\partial}^i \triangleright f(x^j) &= f(x^j) \triangleleft \hat{\partial}^i.
\end{align}

Applying similar arguments as in \cite{183} and \cite{184} we should now end up with the equations
\begin{align}
\langle f, \hat{\partial}^i \triangleright g \rangle_A &= \langle \hat{\partial}^i \triangleright f, g \rangle_A, \\
\langle f, \hat{\partial}^i \triangleright g \rangle_A &= \langle \hat{\partial}^i \triangleright f, g \rangle_A, \\
\langle f \triangleleft \hat{\partial}^i, g \rangle_A &= \langle f, g \triangleleft \hat{\partial}^i \rangle'_A, \\
\langle f \triangleleft \hat{\partial}^i, g \rangle_A &= \langle f, g \triangleleft \hat{\partial}^i \rangle'_A.
\end{align}

Without any difficulties our reasonings also apply for the symmetrical sesquilinear forms defined in \cite{177} and \cite{178}. In this manner, we obtain
\begin{align}
\langle f, h \triangleright g \rangle_i &= \langle h \triangleright f, g \rangle_i, \\
\langle f \triangleleft h, g \rangle_i &= \langle f, g \triangleleft h \rangle_i,
\end{align}

and
\begin{align}
\langle f, \hat{\partial}^i \triangleright g \rangle_i &= \langle \hat{\partial}^i \triangleright f, g \rangle_i, \\
\langle f, \hat{\partial}^i \triangleright g \rangle_i &= \langle \hat{\partial}^i \triangleright f, g \rangle_i, \\
\langle f \triangleleft \hat{\partial}^i, g \rangle_i &= \langle f, g \triangleleft \hat{\partial}^i \rangle_i, \\
\langle f \triangleleft \hat{\partial}^i, g \rangle_i &= \langle f, g \triangleleft \hat{\partial}^i \rangle_i.
\end{align}

From what we have done so far we can read off the adjoints of symmetry
generators acting on a quantum space algebra. Our results tell us that they
are given by the hermitian conjugates of the corresponding operators:

\[(h\triangleright)_{\lambda} = \overline{h\triangleright}, \quad (h\triangleright)_{\iota} = \overline{h\triangleright}, \quad (\overline{h\triangleright})_{\lambda} = \overline{h}, \quad (\overline{h\triangleright})_{\iota} = \overline{h},\]
\[(\partial^{k}\triangleright)_{\lambda} = \overline{\partial^{k}\triangleright}, \quad (\partial^{k}\triangleright)_{\iota} = \overline{\partial^{k}\triangleright}, \quad (\overline{\partial^{k}\triangleright})_{\lambda} = \overline{\partial^{k}}, \quad (\overline{\partial^{k}\triangleright})_{\iota} = \overline{\partial^{k}},\]
\[(\overline{\partial^{k}\triangleright})_{\lambda} = \overline{\partial^{k}\triangleright}, \quad (\overline{\partial^{k}\triangleright})_{\iota} = \overline{\partial^{k}\triangleright}, \quad (\overline{\partial^{k}\triangleright})_{\lambda} = \overline{\partial^{k}}, \quad (\overline{\partial^{k}\triangleright})_{\iota} = \overline{\partial^{k}}. \quad (191)\]

In the case of partial derivatives we have to be aware of the fact that
their adjoints come along with representations being different from those
of the original operator. This observation plays an important role if we are
looking for self-adjoint operators in the algebra of partial derivatives. In this
respect, let us recall that in our approach the operation of conjugation maps
the algebra of partial derivatives onto itself. (For a discussion of this subject
and alternative approaches we refer the reader to Refs. [38, 62–65].) However,
if we consider actions of partial derivatives things become more involved,
since each partial derivative can act in different ways upon a quantum space
algebra and these actions are intertwined by the operation of conjugation.
On these grounds we have to combine different types of actions if we want
to deal with momentum operators being self-adjoint in the usual sense.

For these ideas to become more clear, we introduce as actions of momen-
tum operators

\[P^{j} \triangleright f = \frac{i}{2}(\partial^{j} \triangleright f + \partial^{j} \triangleright f),\]
\[f \triangleleft P^{j} = \frac{i}{2}(f \triangleleft \partial^{j} + f \triangleleft \partial^{j}). \quad (192)\]

It is not very difficult to show that we have

\[\overline{P^{j} \triangleright f} = \overline{f \triangleleft P^{j}} = \overline{f \triangleleft P^{j}}, \quad \overline{f \triangleleft P^{j} = P^{j} \triangleright f} = \overline{f \triangleleft P^{j}}. \quad (193)\]

These identities, in turn, imply

\[\langle f, P^{j} \triangleright f \rangle \triangleleft \bar{i} = \langle P^{j} \triangleright f, f \rangle \triangleleft \bar{i} = \langle f, P^{j} \triangleright f \rangle \triangleleft \bar{i} = \langle f, P^{j} \triangleright f \rangle \triangleleft \bar{i}, \quad (194)\]
and, likewise,
\[
\langle f \triangleleft P^j, f \rangle_i' = \langle f, f \triangleleft P^j \rangle_i',
\] (195)
As a next step we are looking for a change of basis
\[
\tilde{P}^j = \sum_k a_k P^k,
\] (196)
leading to the real components of momentum, i.e.
\[
\tilde{P}^j = \tilde{P}^j.
\] (197)

For the quantum spaces we are dealing with the explicit form of this change of bases can be found in Appendix A. Finally, a little thought shows us that the expectation values of the new components of momentum become real (for a systematic treatment of expectation values see Part II of this paper):
\[
\langle f \triangleleft \tilde{P}^j \triangleright f \rangle_i = \langle f, \tilde{P}^j \triangleright f \rangle_i,
\]
\[
\langle f \triangleright \tilde{P}^j, f \rangle_i' = \langle f \triangleleft \tilde{P}^j, f \rangle_i'.
\] (198)

Up to now we considered operators generating infinitesimal symmetry transformations on quantum spaces and derived the corresponding adjoint operators. Now, we would like to do the same for operators generating finite symmetry transformations. To reach this goal let us first recall that finite symmetry transformations like q-deformed rotations or q-deformed Lorentz transformations are described by the so-called canonical elements [66], which are given by
\[
C = \exp(\alpha | h) \equiv \sum_a e^a_\alpha \otimes e^a_h \in \mathcal{H}^* \otimes \mathcal{H},
\]
\[
C' = \exp(h | \alpha) \equiv \sum_a e^a_h \otimes e^a_\alpha \in \mathcal{H} \otimes \mathcal{H}^*.\] (199)
Notice that $\mathcal{H}^*$ denotes the dual Hopf algebra of $\mathcal{H}$. Moreover, \{$e^a_h$\} is a basis of $\mathcal{H}$ and \{$e^a_\alpha$\} is the corresponding dual basis. With the canonical elements at hand finite symmetry transformations of functions on position space can be written in the form [66]
\[
\exp(\alpha | h) \triangleright f(x^i) = \sum_a e^a_\alpha \otimes (e^a_h \triangleright f(x^i)),
\]
\[
f(x^i) \triangleleft \exp(h | \alpha) = \sum_a (f(x^i) \triangleleft e^a_h) \otimes e^a_\alpha.
\] (200)
Now, we have everything together to derive the adjoints of the operators describing finite symmetry transformations. Towards this end, we proceed as follows:

\[
\langle f(x^i), \exp(\alpha|\mathcal{h}) \triangleright g(x^j) \rangle_A = \int_{-\infty}^{+\infty} d_A x e^a_\alpha \otimes \overline{f(x^i)} \otimes (e^a_h \triangleright g(x^j))
\]

\[
= \int_{-\infty}^{+\infty} d_A x e^a_\alpha \otimes (\overline{f(x^i)} \triangleleft e^a_h) \otimes g(x^j)
\]

\[
= \int_{-\infty}^{+\infty} d_A x \left( e^a_\alpha \otimes e^a_h \triangleright f(x^i) \right) \otimes g(x^j)
\]

\[
= \int_{-\infty}^{+\infty} d_A x \left( e^a_\alpha \otimes S(e^a_h) \triangleright f(x^i) \right) \otimes g(x^j)
\]

\[
= \langle \exp(\alpha|\mathcal{h}) \triangleright f(x^i), g(x^j) \rangle_A. \tag{201}
\]

It should be mentioned that in the above calculation we used the property of the canonical element to be unitary:

\[
\mathcal{C} = \exp(\alpha |\mathcal{h}|) = \sum_a e^a_\alpha \otimes e^a_h
\]

\[
= \sum_a e^a_\alpha \otimes S(e^a_h) = \exp(\alpha|\mathcal{h}) = \mathcal{C}^{-1}. \tag{202}
\]

Notice that this property is equivalent to the requirement that

\[
\mathcal{C} \triangleright f(x^i) = \exp(\alpha|\mathcal{h}) \triangleright f(x^i) = \sum_a e^a_\alpha \otimes e^a_h \triangleright f(x^i)
\]

\[
= \sum_a e^a_\alpha \otimes S^{-1}(e^a_h) \triangleright f(x^i) = \sum_a e^a_\alpha \otimes S e^a_h \triangleleft f(x^i)
\]

\[
= \exp(\alpha|\mathcal{h}) \triangleright f(x^i) = \mathcal{C} \triangleright f(x^i). \tag{203}
\]

The explicit form of \( \mathcal{C}^{-1} \) follows from

\[
\mathcal{C} \cdot \mathcal{C}^{-1} = \exp(\alpha|\mathcal{h}) \cdot \exp(\alpha|\mathcal{h})^{-1}
\]

\[
= \sum_{a,b} e^a_\alpha \otimes e^b_\alpha \otimes e^b_h \cdot S(e^b_h)
\]

\[
= \sum_a e^a_\alpha \otimes (e^a_h)_{(1)} \cdot S((e^a_h)_{(2)})
\]

\[
= \sum_a e^a_\alpha \otimes e^a_h = 1 \otimes 1, \tag{204}
\]

47
\[ C^{-1} \cdot C = \exp(\alpha \circ h) \cdot \exp(\alpha | h) \]
\[ = \sum_{a,b} e_a^a \cdot e_b^b \otimes S(e_h^a) \cdot e_h^b \]
\[ = \sum_{a} e_a^a \otimes S((e_h^a)_1) \cdot (e_h^a)_2 \]
\[ = \sum_{a} e_a^a \otimes \epsilon(e_h^a) = 1 \otimes 1, \quad (205) \]

where we applied as characteristic properties of the canonical element [66]

\[ (\text{id}_{\mathcal{H}^*} \otimes \epsilon_{\mathcal{H}}) \circ C = \text{id}_{\mathcal{H}^*} \otimes \text{id}_{\mathcal{H}}, \quad C_{12}C_{13} = (\text{id}_{\mathcal{H}^*} \otimes \Delta_{\mathcal{H}}) \circ C. \quad (206) \]

Repeating the same steps as in (201) for the sesquilinear forms with an apostrophe we get

\[ \langle f(x^i) \triangleleft \exp(h| \alpha), g(x^j) \rangle^{'}_{A} = \int_{-\infty}^{+\infty} d_{A}^{n}x \, (f(x^i) \triangleleft e_h^a) \otimes g(x^j) \otimes e_a^a \]
\[ = \int_{-\infty}^{+\infty} d_{A}^{n}x \, f(x^i) \otimes (e_h^a \triangleright g(x^j)) \otimes e_a^a \]
\[ = \int_{-\infty}^{+\infty} d_{A}^{n}x \, f(x^i) \otimes g(x^j) \otimes e_h^a \otimes e_a^a \]
\[ = \int_{-\infty}^{+\infty} d_{A}^{n}x \, f(x^i) \otimes g(x^j) \triangleleft S(e_h^a) \otimes e_a^a \]
\[ = \langle f(x^i), g(x^j) \triangleleft \exp(h| \alpha) \rangle^{'}_{A}. \quad (207) \]

The fourth equality in the above calculation is a consequence of the identities

\[ \overline{C'} = \exp(h| \alpha) = \sum_{a} e_h^a \otimes e_a^a \]
\[ = \sum_{a} S(e_h^a) \otimes e_a^a \]
\[ = \exp(\odot h| \alpha) = (C')^{-1}. \quad (208) \]

Finally, it should be mentioned that the results in (201) and (207) remain unchanged if we use symmetrical sesquilinear forms instead. In this manner we conclude that

\[ \langle \exp(\alpha| h) \rangle^{t_1}_{\alpha/t_1} = \overline{\langle \exp(\alpha| h) \rangle} = \exp(\alpha| \odot h), \]
\[ \langle \triangleleft \exp(h| \alpha) \rangle^{t_1}_{\alpha/t_1} = \overline{\langle \triangleleft \exp(h| \alpha) \rangle} = \exp(\odot h| \alpha). \quad (209) \]
From Ref. [38] we know that q-deformed exponentials generate finite translations on q-deformed quantum spaces. It is now our aim to find the corresponding adjoint operators. To achieve this we apply a method similar to that for quantum group transformations:

\[ \langle f(x^i), \exp(y^k|\partial^j)_{\bar{R},L} \partial^{|x^i} g(x^j) \rangle_A \]

\[ = \int_{-\infty}^{+\infty} d^n_A x \left( f(x^i) \overset{\partial^{|x^i}}{\odot}_R \left( \exp(y^k|\partial^j)_{\bar{R},L} \partial^{|x^i} g(x^j) \right) \right) \]

\[ = \int_{-\infty}^{+\infty} d^n_A x \left( f(x^i) \overset{\partial^{|x^i}}{\odot} \exp(y^k|\bar{\partial}^j)_{\bar{R},L} \partial^{|x^i} g(x^j) \right) \]

\[ = \int_{-\infty}^{+\infty} d^n_A x \left( f(x^i) \overset{\partial^{|x^i}}{\odot} \exp(\odot_L \partial^{|y^k})_{\bar{R},L} \odot g(x^j) \right) \]

\[ = \langle f(x^i) \odot \exp(\odot_L \partial^{|y^k})_{\bar{R},L}, g(x^j) \rangle_A \]  \hspace{1cm} (210)

Notice that the second equality follows from the rule for integration by parts and trivial braiding of q-exponentials. The third equality makes use of the conjugation properties of q-exponentials and those of partial derivatives. In very much the same way we obtain

\[ \langle f(x^i) \odot \exp(\partial^{|y^k})_{\bar{R},L}, g(x^j) \rangle_A \]

\[ = \int_{-\infty}^{+\infty} d^n_A x \left( f(x^i) \overset{\partial^{|x^i}}{\odot} \exp(\partial^{|y^k})_{\bar{R},L} \odot g(x^j) \right) \]

\[ = \int_{-\infty}^{+\infty} d^n_A x \left( f(x^i) \odot \exp(\odot_L \partial^{|y^k})_{\bar{R},L} \odot g(x^j) \right) \]

\[ = \langle f(x^i), \exp(\odot_L \partial^{|y^k})_{\bar{R},L} \odot g(x^j) \rangle_A \]  \hspace{1cm} (211)

Applying the replacements

\[ L \leftrightarrow \bar{L}, \hspace{0.5cm} R \leftrightarrow \bar{R}, \hspace{0.5cm} \odot \leftrightarrow \odot, \hspace{0.5cm} \partial \leftrightarrow \bar{\partial} \]  \hspace{1cm} (212)

to the expressions in (210) and (211) yields further relations. Again, our results carry over to symmetrical sesquilinear forms. Finally, we can read off from what we have done so far that

\[ (\exp(x^k|\partial^j)_{\bar{R},L} \partial^{|i})^\dagger_{\lambda^i} = \overset{|\partial^{|x^k}}{\odot} \exp(\odot_R x^k)_{\bar{R},L} \]
\[
\left< \partial | \exp(\partial^j | x^k \rangle_{R,L}) \right|'_{A/L'} = \exp(\partial_L x^k | \partial^j)_{R,L} \left< \partial \right|
\]
(213)

\[
\left< \exp(x^k | \partial^j)_{R,L} \right|'_{A/L'} = \left< \partial \right| \exp(\partial | \partial_L x^k)_{R,L},
\]
(214)

For the sake of completeness let us mention that we can also assign each position operator an adjoint operator. To see this, we perform the following calculation:

\[
\langle f, X^k \triangleright g \rangle_A = \int_{-\infty}^{+\infty} d^n_A \ x f(x) \odot (X^k \triangleright g(x))
\]
(215)

where we used \(X^k = X_k\). Similar considerations show us that

\[
\langle f \triangleright X^k, g \rangle'_{A/L} = \langle f, g \triangleright X^k \rangle'_{A/L}.
\]
(216)

In this manner we find

\[
(X^k \triangleright)'_{A/L} = (X^k \triangleright)'_{A/L} = X_k \triangleright, \quad (\triangleright X^k)'_{A/L} = (\triangleright X^k)'_{A/L} = \triangleright X_k.
\]
(217)

Next, we would like to discuss how our sesquilinear forms behave under symmetry transformations. In Ref. [38] it was shown that q-deformed integrals over the whole quantum space are invariant under quantum group symmetries. For this reason, we have

\[
h \triangleright \langle f, g \rangle_A = \epsilon(h) \langle f, g \rangle_A
\]

where

\[
\langle f \triangleright X^k, g \rangle'_{A/L} = \langle f, g \triangleright X^k \rangle'_{A/L}.
\]
\[ \begin{aligned}
&= \int_{-\infty}^{+\infty} d^3x \left( S^{-1}(h_{(1)}) \triangleright f(x^i) \right) \otimes \left( h_{(2)} \triangleright g(x^j) \right) \\
&= \langle S^{-1}(h_{(1)}) \triangleright f, h_{(2)} \triangleright g \rangle_A, \quad (218)
\end{aligned} \]

where \( h \) again denotes an element of a Hopf algebra \( H \) describing the symmetry of the quantum space under consideration. Similar arguments give us

\[ \langle f, g \rangle_A \triangleright h = \epsilon(h) \langle f, g \rangle_A = \langle f \triangleright S(h_{(2)}), g \triangleright h_{(1)} \rangle_A, \quad (219) \]

and

\[ \begin{aligned}
&h \triangleright \langle f, g \rangle_A' = \epsilon(h) \langle f, g \rangle_A' = \langle h_{(1)} \triangleright f, S^{-1}(h_{(2)}) \triangleright g \rangle_A', \\
&\langle f, g \rangle_A' \triangleright h = \epsilon(h) \langle f, g \rangle_A' = \langle f \triangleright h_{(2)}, g \triangleright S(h_{(1)}) \rangle_A'. 
\end{aligned} \quad (220) \]

In the same way translation invariance of integrals over the whole space leads to

\[ \begin{aligned}
&\partial^k \triangleright \langle f, g \rangle_A = \epsilon_L(\partial^k) \langle f, g \rangle_A = \langle f \triangleright (\partial^k)_{(1)}, (\partial^k)_{(2)} \triangleright g \rangle_A, \\
&\hat{\partial}^k \triangleright \langle f, g \rangle_A = \epsilon_L(\hat{\partial}^k) \langle f, g \rangle_A = \langle f \triangleright (\hat{\partial}^k)_{(1)}, (\hat{\partial}^k)_{(2)} \triangleright g \rangle_A, \quad (221) \\
&\langle f, g \rangle_A \triangleright \hat{\partial}^k = \epsilon_R(\hat{\partial}^k) \langle f, g \rangle_A = \langle (\hat{\partial}^k)_{(1)} \triangleright f, g \triangleright (\hat{\partial}^k)_{(2)} \rangle_A, \\
&\langle f, g \rangle_A \bar{\triangleright} \partial^k = \epsilon_R(\partial^k) \langle f, g \rangle_A = \langle (\partial^k)_{(1)} \triangleright f, g \triangleright (\partial^k)_{(2)} \rangle_A, \quad (222) \\
&\langle f, g \rangle_A' \bar{\triangleright} \hat{\partial}^k = \epsilon(\hat{\partial}^k) \langle f, g \rangle_A' = \langle f \triangleright (\hat{\partial}^k)_{(1)}, (\hat{\partial}^k)_{(2)} \triangleright g \rangle_A', \\
&\langle f, g \rangle_A' \bar{\triangleright} \hat{\partial}^k = \epsilon(\hat{\partial}^k) \langle f, g \rangle_A' = \langle f \triangleright (\hat{\partial}^k)_{(1)}, (\hat{\partial}^k)_{(2)} \triangleright g \rangle_A', \quad (223) \\
&\hat{\partial}^k \triangleright \langle f, g \rangle_A' = \epsilon(\partial^k) \langle f, g \rangle_A' = \langle ((\partial^k)_{(1)} \triangleright f, g \triangleright (\partial^k)_{(2)}) \rangle_A', \\
&\partial^k \triangleright \langle f, g \rangle_A' = \epsilon(\partial^k) \langle f, g \rangle_A' = \langle ((\partial^k)_{(1)} \triangleright f, g \triangleright (\partial^k)_{(2)}) \rangle_A', \quad (224)
\end{aligned} \]

where

\[ \Delta_L(\partial) = \partial_{(1)} \otimes \partial_{(2)}, \quad \Delta_L(\hat{\partial}) = \partial_{(1)} \otimes \partial_{(2)}. \quad (225) \]

Notice that all of the above expressions in (221)–(224) vanish, since \( \epsilon_A(\partial) = 0 \) [cf. Eq. (29)].

Next we try to extend these considerations to finite symmetry transformations. For the action of the canonical element on a sesquilinear form we
have
\[
\exp(\alpha|\hbar) \triangleright \langle f,g \rangle_A = \sum_a e^a_\alpha \otimes e^a_\hbar \triangleright \langle f,g \rangle_A \\
= \sum_a e^a_\alpha \otimes \epsilon(e^a_\hbar) \langle f,g \rangle_A \\
= \langle f,g \rangle_A,
\] (226)
and
\[
\langle f,g \rangle_A \diamond \exp(h|\alpha) = \sum_a \langle f,g \rangle_A \triangleleft e^a_\hbar \otimes e^a_\alpha \\
= \sum_a \langle f,g \rangle_A e(e^a_\hbar) \otimes e^a_\alpha \\
= \langle f,g \rangle_A.
\] (227)

These equalities also hold for all other types of sesquilinear forms introduced in the previous subsection. They tell us once more that our sesquilinear forms behave like scalars.

The results in (226) and (227) can be used to show once more that finite quantum group transformations described by canonical elements are unitary:

\[
\langle f,g \rangle_A = \exp(\alpha|h) \triangleright \langle f,g \rangle_A = \sum_a e^a_\alpha \otimes e^a_\hbar \triangleright \langle f,g \rangle_A \\
= \int_{-\infty}^{+\infty} d_A^nx \sum_a e^a_\alpha \otimes f(x^i) \triangleleft (e^a_\hbar)_{(1)} \otimes (e^a_\hbar)_{(2)} \triangleright g(x^j)) \\
= \int_{-\infty}^{+\infty} d_A^nx \sum_a b_a e^a_\alpha \otimes f(x^i) \triangleleft e^b_\hbar \otimes (e^b_\hbar) \triangleright g(x^j)) \\
= \int_{-\infty}^{+\infty} d_A^nx \sum_a \overline{e^a_\alpha} \otimes S^{-1}(e^a_\hbar) \triangleright f(x^i) \cdot \sum b_b e^b_\alpha \otimes e^b_\hbar \triangleright g(x^j) \\
= \int_{-\infty}^{+\infty} d_A^nx \sum_a e^a_\alpha \otimes e^a_\hbar \triangleright f(x^i) \cdot \exp(\alpha|h) \triangleright g(x^j) \\
= \int_{-\infty}^{+\infty} d_A^nx \exp(\alpha|h) \triangleright f(x^i) \cdot \exp(\alpha|h) \triangleright g(x^j) \\
= \langle \exp(\alpha|h) \triangleright f(x^i), \exp(\alpha|h) \triangleright g(x^j) \rangle_A.
\] (228)

Likewise, we have
\[
\langle f,g \rangle_A = \langle f,g \rangle_A \diamond \exp(h \otimes \alpha)
\]
\[
\begin{align*}
&= \int_{-\infty}^{+\infty} d_A x \sum_a f(x^i) \triangleleft S(e^a_h) \otimes (g(x^j) \triangleleft e^a_h) \otimes S(e^a_h) \\
&= \int_{-\infty}^{+\infty} d_A x \sum_{a,b} f(x^i) \triangleleft \overline{S(e^a_h)} \otimes (g(x^j) \triangleleft e^b_h) \otimes S(e^b_h) \\
&= \int_{-\infty}^{+\infty} d_A x \sum_a f(x^i) \triangleleft \overline{S(e^a_h)} \otimes S^{-1}(\overline{e^a_h}) \cdot \sum_b g(x^j) \triangleleft e^b_h \otimes S(e^b_h) \\
&= \int_{-\infty}^{+\infty} d_A x \sum_a f(x^i) \triangleleft \exp(h|\ominus\alpha) \cdot (g(x^j) \triangleleft \exp(h|\ominus\alpha)) \\
&= \langle f(x^i) \triangleleft \exp(h|\ominus\alpha), g(x^j) \triangleleft \exp(h|\ominus\alpha) \rangle_A, \quad (229)
\end{align*}
\]

where
\[
\exp(h|\ominus\alpha) = \sum_a e^a_h \otimes S(e^a_h) \\
= \sum_a S(e^a_h) \otimes e^a_h = \exp(\ominus h|\alpha). \quad (230)
\]

For the fourth step in (228) as well as the third step in (229) we used the characteristic properties of the canonical element and the antihomomorphism property of the antipode. For the fourth step in (229) one has to realize that
\[
S(h) = S^{-1}(\overline{h}).
\]

The remaining steps in (229) make use of the identities
\[
\sum_a \overline{e^a_h} \otimes e^a_h = \sum_a S(e^a_h) \otimes e^a_h = \sum_a e^a_h \otimes S(e^a_h). \quad (231)
\]

The following relations can be derived along the same line of reasonings:
\[
\langle f, g \rangle_A' = \exp(\alpha|h) \triangleright \langle f, g \rangle_A \\
= \langle \exp(\alpha|h) \triangleright f, \exp(\alpha|h) \triangleright g \rangle_A', \quad (232)
\]
\[
\langle f, g \rangle_A' = \langle f, g \rangle_A \triangleleft \exp(h|\ominus\alpha) \\
= \langle f \triangleleft \exp(h|\ominus\alpha), g \triangleleft \exp(h|\ominus\alpha) \rangle_A'. \quad (233)
\]

Last but not least, it should be mentioned that these reasonings about unitarity of finite quantum group transformations also apply to symmetrical sesquilinear forms.

It arises the question whether the operators generating finite translations on quantum spaces are also subject to unitary conditions like those in (228-
That this is indeed the case can be shown by the following calculation:

\[
\langle f, g \rangle_A = \exp(y^k|\partial^l)_{R,L} \triangleright \langle f, g \rangle_A
\]

\[
= \int_{-\infty}^{+\infty} d_A x \ \exp(y^k|\partial^l)_{R,L} \triangleright (\overline{f(x^i)} \oplus g(x^i))
\]

\[
= \int_{-\infty}^{+\infty} d_A x \ \sum_a e^a_{(y,R)} \bigotimes ((e^a_{(\partial,L)})_{(1)} \triangleright \overline{f(x^i)}) \oplus ((e^a_{(\partial,L)})_{(2)} \triangleright g(x^i))
\]

\[
= \int_{-\infty}^{+\infty} d_A x \ \sum_{a,b} e^a_{(y,R)} \overline{\frac{y}{x}} e^b_{(y,R)} \bigotimes (e^b_{(\partial,L)} \triangleright \overline{f(x^i)}) \oplus (e^a_{(\partial,L)} \triangleright g(x^i))
\]

\[
= \langle f(x^i) \bigotimes \exp(\partial^r|y^m)_{R,L}, \exp(y^k|\partial^l)_{R,L} \triangleright g(x^j) \rangle_A
\]

The second equality is due to translation invariance of q-deformed integrals over the whole space. The fourth equality is an application of the addition law for q-deformed exponentials. The fifth equality results from the conjugation properties of q-deformed exponentials and the last two expressions can be viewed as a kind of shorthand notation. With the same reasonings we get

\[
\langle f, g \rangle'_A = \langle f, g \rangle_A \bigotimes \exp(\partial^r|y^m)_{R,L}
\]

\[
= \int_{-\infty}^{+\infty} d_A x \ \sum_{a,b} (f(x^i) \bigotimes e^a_{(\partial,R)}) \bigotimes \overline{e^b_{(y,R)}} \oplus (e^b_{(\partial,L)} \triangleright \overline{g(x^j)}) \bigotimes e^a_{(y,L)}
\]

\[
= \langle f(x^i) \bigotimes \exp(\partial^r|y^m)_{R,L}, \exp(y^k|\partial^l)_{R,L} \triangleright g(x^j) \rangle'_A
\]

Further relations are obtained most easily from the above ones most easily via the substitutions in (212). Again, we are allowed to replace the sesquilinear forms by their symmetrical versions.

### 4.3 Fourier-Plancherel identities

It is natural to ask about the behavior of our sesquilinear forms under q-deformed Fourier transformations of their arguments. This question leads
us to q-analogs of Fourier-Plancherel identities, for which we concretely have

\[ \langle f, g \rangle'_{L,x} = (-1)^n \langle F_L(f), F_R^*(g)(\kappa^{-1}p^i) \rangle'_{R,p}, \]
\[ \langle f, g \rangle'_{L,x} = (-1)^n \langle F_L(f), F_R^*(g)(\kappa p^i) \rangle'_{R,p}, \quad (235) \]
\[ \langle f, g \rangle'_{R,x} = (-1)^n \langle F_R^*(f)(\kappa^{-1}p^i), F_L(g) \rangle'_{L,p}, \]
\[ \langle f, g \rangle'_{R,x} = (-1)^n \langle F_R^*(f)(\kappa p^i), F_L(g) \rangle'_{L,p}, \quad (236) \]

and

\[ \langle f, g \rangle_{L,x} = (-1)^n \langle F_R(f), F_L^*(g)(\kappa^{-1}p^i) \rangle_{R,p}, \]
\[ \langle f, g \rangle_{L,x} = (-1)^n \langle F_R(f), F_L^*(g)(\kappa p^i) \rangle_{R,p}, \quad (237) \]
\[ \langle f, g \rangle_{R,x} = (-1)^n \langle F_L^*(f)(\kappa^{-1}p^i), F_R(g) \rangle_{L,p}, \]
\[ \langle f, g \rangle_{R,x} = (-1)^n \langle F_L^*(f)(\kappa p^i), F_R(g) \rangle_{L,p}. \quad (238) \]

Now, we come to the proofs of the above relations. First of all, we consider the first identity in (235), which can be proven by the following calculation:

\[
\text{vol}_L \langle f, g \rangle'_{L,x} = \text{vol}_L \int_{-\infty}^{+\infty} d_L^x f(x^i) \otimes g(x^j) \\
= \int_{-\infty}^{+\infty} d_L^x \left( \int_{-\infty}^{+\infty} d_L^y f(y^i) \otimes_L \delta_R((\otimes_R x^j) \oplus_L y^k) \right) \otimes g(x^j) \\
= \int_{-\infty}^{+\infty} d_L^x \left( \int_{-\infty}^{+\infty} d_L^y f(y^i) \right. \\
\left. \otimes_L \int_{-\infty}^{+\infty} d_R^m \exp((\otimes_R x^j) \oplus_L y^k |^{1-p^m}_{R,L}) \right) \otimes g(x^j) \\
= \int_{-\infty}^{+\infty} d_L^x \int_{-\infty}^{+\infty} d_L^y f(y^i) \otimes \left( \int_{-\infty}^{+\infty} d_R^m \exp((\otimes_R x^j) |^{1-p^m}_{R,L}) \right) \otimes g(x^j) \\
\otimes \int_{-\infty}^{+\infty} d_R^m \left( \mathcal{R}_{[1]} \otimes e_{(L,p)} \right) \otimes \mathcal{S}^{-1} \left( \mathcal{R}_{[2]} \otimes e_{(\kappa^{-1} x)} \right) \otimes g(x^j)
\]
\[
\int_{-\infty}^{+\infty} d_L y f(y^i) \otimes e_{(R,y)}^q \otimes \int_{-\infty}^{+\infty} d_L^p e_{(L,p)}^q \otimes \left( \mathcal{R}_{[1]} \triangleright e_{(L,p)}^b \right) \\
\otimes \int_{-\infty}^{+\infty} d_L^m S^{-1}(\mathcal{R}_{[2]} \triangleright e_{(R,x)}^b) \otimes g(x^i)
\]

\[
\int_{-\infty}^{+\infty} d_L^p y f(y^j) \otimes e_{(R,y)}^q \otimes \int_{-\infty}^{+\infty} d_L^p e_{(L,p)}^q \\
\otimes (-1)^n \int_{-\infty}^{+\infty} d_R^m g(x^i) \otimes S(\mathcal{R}_{[1]}^{-1} \triangleright e_{(L,x)}^b) \otimes \mathcal{R}_{[2]}^{-1} \triangleright e_{(R,p)}^b
\]

\[
\int_{-\infty}^{+\infty} d_R^m y f(y^j) \otimes e_{(R,y)}^q \otimes \int_{-\infty}^{+\infty} d_R^m e_{(L,p)}^q \\
\otimes \mathcal{R}_{[2]} \triangleright e_{(R,k^{-1}p)}^b \otimes \int_{-\infty}^{+\infty} d_R^m x \mathcal{R}_{[1]} \triangleright g(x^i) \otimes S(e_{(L,x)}^b)
\]

\[
\int_{-\infty}^{+\infty} d_R^m p \int_{-\infty}^{+\infty} d_L^m y f(y^j) \otimes e_{(R,y)}^q \otimes \exp(y^j|^{-1}p^k)_{R,L} \\
\otimes \int d_R^m x g(x^i) \mathcal{X}_{L} \exp(i^{-1}(k^{-1}p^m)(\otimes_L x'))_{R,L}
\]

\[
\int_{-\infty}^{+\infty} d_R^m p \mathcal{F}_L(f)(p^k) \otimes \mathcal{F}_R^*(g)(k^{-1}p^m) \\
\otimes \int d_R^m x g(x^i) \mathcal{X}_{L} \exp(i^{-1}(k^{-1}p^m)(\otimes_L x'))_{R,L}
\]

The first equality is the definition of the sesquilinear form. For the second equality we apply the first relation in (116). Then we rewrite the q-deformed delta function by using its definition in (88). After that we use the addition law for q-deformed exponentials. For the fifth step we rewrite our expression in a way that displays in which tensor factors the various objects live. Notice that the braided antipodes $S$ and $\tilde{S}^{-1}$ represent the operations $\otimes_L$ and $\otimes_R$, respectively. In the sixth step we rearrange these tensor factors by taking into account their braiding. This step follows most easily from diagrammatic considerations (see for example Ref. [42]). Then we extend the operation of conjugation in a way that it affects the second factor of the star product on momentum space. To reach this goal we apply the conjugation properties of q-integrals, q-exponentials, antipodes, and braiding mappings (for their discussion see Ref. [38]). The eighth equality is again rearranging of tensor factors in a way that allows us to identify the defining expressions for q-deformed Fourier transformations.

Next, we turn our attention to the other versions of Fourier-Plancherel identities. The second identity in (235) follows from the very same reason-
transformations in space which we assign functions on momentum space via the Fourier transform. To become more concrete we consider two functions \( f(x), g(x) \) in the q-deformed case the situation is a little bit more involved, since the two functions can be viewed as a change of basis or a change of representation. The last identity in (238).

Through a slight modification of the considerations leading to the identities in (235) and (236), we are able to check the Fourier-Plancherel identities in (237) and (238). To illustrate this we give in Appendix [13] the proof of the last identity in (238).

If we want to adapt the above considerations to symmetrical sesquilinear forms we need the Fourier transformations \( \mathcal{F}_A \) and \( \mathcal{F}_A^* \) [see the discussion of (170) and (171)]. Repeating the same steps as in (239) we can verify that

\[
\langle f, g \rangle'_{1,x} = \langle \mathcal{F}_L(f), \mathcal{F}_R^*(g)(\kappa^{-1}p^i) \rangle'_{1,p} = \langle \mathcal{F}_R^*(f)(\kappa^{-1}p^i), \mathcal{F}_L(g) \rangle'_{1,p},
\]

\[
\langle f, g \rangle'_{2,x} = \langle \mathcal{F}_L(f), \mathcal{F}_R^*(g)(\kappa p^i) \rangle'_{2,p} = \langle \mathcal{F}_R^*(f)(\kappa p^i), \mathcal{F}_L(g) \rangle'_{2,p},
\]

and

\[
\langle f, g \rangle_{1,x} = \langle \mathcal{F}_R(f), \mathcal{F}_L^*(g)(\kappa^{-1}p^i) \rangle_{1,p} = \langle \mathcal{F}_L^*(f)(\kappa^{-1}p^i), \mathcal{F}_R(g) \rangle_{1,p},
\]

\[
\langle f, g \rangle_{2,x} = \langle \mathcal{F}_R(f), \mathcal{F}_L^*(g)(\kappa p^i) \rangle_{2,p} = \langle \mathcal{F}_L^*(f)(\kappa p^i), \mathcal{F}_R(g) \rangle_{2,p}.
\]

We would like to make some comments about the meaning of q-deformed Fourier-Plancherel identities. Let us recall that Fourier transformations can be viewed as a change of basis or a change of representation. The Fourier-Plancherel identities tell us that sesquilinear forms are invariant under Fourier transformations of their arguments. For this reason we conclude that Fourier transformations correspond to unitary operators. However, in the q-deformed case the situation is a little bit more involved, since the two arguments of sesquilinear forms have to transform differently. For this to become more concrete we consider two functions \( \psi(x^i) \) and \( \phi(x^i) \) in position space which we assign functions on momentum space via the Fourier transformations \( \mathcal{F}_A \) and \( \mathcal{F}_A^* \):

\[
f_A(p^i) \equiv \mathcal{F}_A(\psi)(p^i), \quad g_A^*(p^i) \equiv \mathcal{F}_A^*(\phi)(p^i).
\]
With the help of the deformed Fourier-Plancherel identities we obtain
\[
\langle \psi, \phi \rangle'_{L,x} = (-1)^n \langle \mathcal{F}_L(\psi), \mathcal{F}_R^*(\phi)(\kappa^{-1}p^i) \rangle'_{R,p},
\]
\[= (-1)^n \langle f_L, g_R^*(\kappa^{-1}p^i) \rangle'_{R,p}, \tag{244}
\]
and likewise
\[
\langle \psi, \phi \rangle'_{\bar{L},x} = (-1)^n \langle f_L, g_R^*(\kappa^p) \rangle'_{R,p},
\]
\[
\langle \psi, \phi \rangle_{L,x} = (-1)^n \langle f_R, g_L^*(\kappa^{-1}p^i) \rangle_{R,p},
\]
\[= (-1)^n \langle f_R, g_L^*(\kappa^p) \rangle_{R,p}, \tag{245}
\]
\[
\langle \psi, \phi \rangle_{\bar{L},x} = (-1)^n \langle f_R, g_L^*(\kappa^p) \rangle_{R,p},
\]
\[= (-1)^n \langle f_R, g_L^*(\kappa^{-1}p^i) \rangle_{R,p}, \tag{246}
\]
In very much the same way we get
\[
\langle \phi, \psi \rangle'_{R,x} = (-1)^n \langle g_R^*(\kappa^p), f_L \rangle'_{L,p},
\]
\[
\langle \phi, \psi \rangle'_{\bar{R},x} = (-1)^n \langle g_R^*(\kappa^{-1}p^i), f_L \rangle'_{L,p}, \tag{247}
\]
\[
\langle \phi, \psi \rangle_{R,x} = (-1)^n \langle g_L^*(\kappa^p), f_R \rangle_{L,p},
\]
\[= (-1)^n \langle g_L^*(\kappa^{-1}p^i), f_R \rangle_{L,p}, \tag{248}
\]
We would like to close our considerations in this subsection by presenting examples that confirm the validity of our Fourier-Plancherel identities. To this end, we first consider the sesquilinear forms
\[
\langle \exp(y^i|i^{-1}p^k)_{\bar{R},L}, \exp(i^{-1}p^m|\ominus_L x^i)_{\bar{R},L} \rangle'_{R,L},
\]
\[= \langle \exp(i^{-1}p^k|y^j)_{\bar{R},L}, \exp(\ominus_R x^i|i^{-1}p^m)_{\bar{R},L} \rangle_{R,L},
\]
\[= \int_{-\infty}^{+\infty} d^n p \exp(i^{-1}p^m|\ominus_R x^i)_{\bar{R},L} \exp(\ominus_R x^i|i^{-1}p^m)_{\bar{R},L}
\]
\[= \delta_{\bar{R}}^2(y^j \ominus_R (\ominus_L x^i)), \tag{249}
\]
and
\[
\langle \exp(\ominus_R x^i|i^{-1}p^m)_{\bar{R},L}, \exp(i^{-1}p^k|y^j)_{\bar{R},L} \rangle'_{L,p},
\]
\[= \langle \exp(i^{-1}p^m|\ominus_L x^i)_{\bar{R},L}, \exp(y^j|i^{-1}p^k)_{\bar{R},L} \rangle_{L,p},
\]
\[= \int_{-\infty}^{+\infty} d^n p \exp(i^{-1}p^m|\ominus_L x^i)_{\bar{R},L} \exp(y^j|i^{-1}p^k)_{\bar{R},L}
\]
\[= \delta_L^2((\ominus_L x^i) \ominus_L y^j), \tag{250}
\]

58
where we used the addition law for q-exponentials and the defining expressions of q-deformed delta functions. We compare the above results with

\[
\langle \mathcal{F}_R^*(\exp(y^j|\imath^{-1}p^k)_R,L)(\kappa^{-1}\tilde{x}^l), \mathcal{F}_L(\exp(i^{-1}p^m|\odot_L x^i)_R,L)\rangle_{\tilde{L},\tilde{x}}' = \frac{1}{\text{vol}_R} (\delta_R^n(y^j \oplus_R (\odot_R \kappa^{-1} \tilde{x}^l)), \delta_L^n((\odot_L x^i) \oplus_L \tilde{x}^l))_{\tilde{L},\tilde{x}}'
\]

\[
= \langle \mathcal{F}_R^*(\exp(i^{-1}p^k|y^j)_R,L)(\kappa^{-1}\tilde{x}^l), \mathcal{F}_R(\exp(\odot_R x^i|\imath^{-1}p^m)_R,L)\rangle_{\tilde{L},\tilde{x}}
\]

\[
= \frac{1}{\text{vol}_R} (\delta_R^n(\odot_L \kappa^{-1} \tilde{x}^l) \oplus_R y^j), \delta_R^n(\tilde{x}^l \oplus_R (\odot_R x^i))_{\tilde{L},\tilde{x}}
\]

\[
= (-1)^n \delta_R^n(y^j \oplus_R (\odot_R x^i)),
\]

and

\[
\langle \mathcal{F}_L(\exp(i^{-1}p^m|\odot_L x^i)_R,L)(\tilde{x}^r), \mathcal{F}_R^*(\exp(y^j|\imath^{-1}p^k)_R,L)(\kappa^{-1}\tilde{x}^l)\rangle_{\tilde{R},\tilde{x}}' = \frac{1}{\text{vol}_L} (\delta_L^n((\odot_L x^i) \oplus_L \tilde{x}^r), \delta_R^n(y^j \oplus_R (\odot_R \kappa^{-1} \tilde{x}^l))_{\tilde{R},\tilde{x}}'
\]

\[
= \langle \mathcal{F}_R(\exp(\odot_R x^i|\imath^{-1}p^m)_R,L)(\tilde{x}^r), \mathcal{F}_L^*(\exp(i^{-1}p^k|y^j)_R,L)\rangle_{\tilde{R},\tilde{x}}
\]

\[
= \frac{1}{\text{vol}_L} (\delta_R^n(\tilde{x}^r \oplus_R (\odot_R x^i)), \delta_L^n((\odot_L \kappa^{-1} \tilde{x}^l) \oplus_L y^j))_{\tilde{R},\tilde{x}}
\]

\[
= (-1)^n \delta_L^n((\odot_L x^i) \oplus_L y^j).
\]

For the first and third step in (251) as well as in (252) we inserted the expressions we obtained in subsection 3.4 for Fourier transforms of q-exponentials. The last step in both calculations is an application of the equalities in (120) and (121).

It is not very difficult to realize that our results are in complete agreement with the Fourier-Plancherel identities in (235)-(238). For the sake of completeness it should be mentioned that we can repeat the same steps for the other geometries as well. In doing so, we are led to expressions that are obtained most easily from the above ones by changing the labels according to the interchanges

\[
L \leftrightarrow \tilde{L}, \quad R \leftrightarrow \tilde{R}.
\]

\[59\]
5 Conclusion

Let us end with some comments on what we have done so far. In this article we dealt with Fourier transformations and sesquilinear forms on q-deformed quantum spaces. This way, we laid the foundations for a q-deformed version of quantum kinematics suitable to describe free particles on q-deformed quantum spaces. In Part II these results will be applied to wave functions on position and momentum space.

At this point let us also mention that our reasonings are rather similar to the classical situation. However, we have to be aware of one remarkable difference between the deformed and the undeformed case. It has to do with the observation that for each of our q-deformed objects we can find different realizations, which in the undeformed case become identical. The reason for this lies in the fact that our constructions apply to a tensor category with braiding $\Psi$ and one with braiding $\Psi^{-1}$ [39, 40, 42]. In general, the two braidings are different and we have to deal with both categories, since they are linked via the operation of conjugation [63, 65]. In the undeformed limit, however, the braidings as well as the corresponding categories become identical and we regain the classical situation.

Acknowledgements

First of all I am very grateful to Eberhard Zeidler for very interesting and useful discussions, special interest in my work, and financial support. Furthermore I would like to thank Alexander Schmidt for useful discussions and his steady support. Finally, I thank Dieter Lüst for kind hospitality.

A Quantum spaces

In this appendix we provide some key notation for the quantum spaces we are interested in for physical reasons, i.e. Manin plane, q-deformed Euclidean space in three and four dimensions, and q-deformed Minkowski space. For each case we give the defining relations, the quantum metric, and the conjugation properties.

The coordinates of the two-dimensional q-deformed quantum plane fulfill the relation [47, 67]

$$X^1 X^2 = q X^2 X^1,$$

(254)

whereas the quantum metric is given by a matrix $\varepsilon^{ij}$ with non-vanishing entries

$$\varepsilon^{12} = q^{-1/2}, \quad \varepsilon^{21} = -q^{1/2}.$$  

(255)
Relation (254) is compatible with the conjugation assignment

\[ \overline{X}^i = -\varepsilon_{ij} X^j, \]  

(256)

where \( \varepsilon_{ij} \) denotes the inverse of \( \varepsilon^{ij} \).

The commutation relations for the q-deformed Euclidean space in three dimensions read [23]

\[
\begin{align*}
X^3 X^+ &= q^2 X^+ X^3, \\
X^- X^3 &= q^2 X^3 X^-, \\
X^- X^+ &= X^+ X^- + \lambda X^3 X^3.
\end{align*}
\]  

(257)

The non-vanishing elements of the quantum metric are

\[
\begin{align*}
g^{+-} &= -q, \quad g^{33} = 1, \quad g^{-+} = -q^{-1}.
\end{align*}
\]  

(258)

The conjugation properties of coordinates are given by

\[ \overline{X}^A = g_{AB} X^B, \]  

(259)

with \( g_{AB} \) denoting the inverse of \( g^{AB} \). If we are looking for coordinates subject to \( Y^i = Y^i \) we can choose

\[
\begin{align*}
Y^1 &= \frac{i}{q^{1/2} + q^{-1/2}} (q^{-1/2} X^+ + q^{1/2} X^-), \\
Y^2 &= \frac{1}{q^{1/2} + q^{-1/2}} (q^{-1/2} X^+ - q^{1/2} X^-), \\
Y^3 &= X^3.
\end{align*}
\]  

(260)

For the four-dimensional Euclidean space we have the relations [53]

\[
\begin{align*}
X^1 X^2 &= q X^2 X^1, \\
X^1 X^3 &= q X^3 X^1, \\
X^3 X^4 &= q X^4 X^3, \\
X^2 X^4 &= q X^4 X^2, \\
X^2 X^3 &= X^3 X^2, \\
X^4 X^1 &= X^1 X^4 + \lambda X^2 X^3.
\end{align*}
\]  

(261)
The non-vanishing components of the corresponding quantum metric read
\[ g^{14} = q^{-1}, \quad g^{23} = g^{32} = 1, \quad g^{41} = q. \quad (262) \]
If \( g_{ij} \) again denotes the inverse of \( g^{ij} \) it holds
\[ \overline{X^i} = g_{ij} X^j. \quad (263) \]
Using this relation it is easy to check that the following independent coordinates are invariant under conjugation [68]:
\[ Y^1 = \frac{1}{q^{1/2} + q^{-1/2}}(q^{1/2} X^1 + q^{-1/2} X^4), \]
\[ Y^2 = \frac{1}{2}(X^2 + X^3), \]
\[ Y^3 = \frac{i}{2}(X^2 - X^3), \]
\[ Y^4 = \frac{i}{q^{1/2} + q^{-1/2}}(q^{1/2} X^1 - q^{-1/2} X^4). \quad (264) \]
The coordinates of q-deformed Minkowski space obey the relations [23]
\[ X^\mu X^0 = X^0 X^\mu, \quad \mu \in \{0, +, -, 3\}, \]
\[ X^- X^3 - q^2 X^3 X^- = -q\lambda X^0 X^-, \]
\[ X^3 X^+ - q^2 X^+ X^3 = -q\lambda X^0 X^+, \]
\[ X^- X^+ - X^+ X^- = \lambda(X^3 X^3 - X^0 X^3). \quad (265) \]
As non-vanishing components of the corresponding metric we have
\[ \eta^{00} = -1, \quad \eta^{33} = 1, \quad \eta^{+-} = -q, \quad \eta^{-+} = -q^{-1}. \quad (266) \]
(For other deformations of Minkowski spacetime we refer to Refs. [69–75].)
The conjugation on q-deformed Minkowski space is determined by
\[ \overline{X^0} = X^0, \quad \overline{X^3} = X^3, \quad \overline{X^\pm} = -q^{\mp 1} X^\mp. \quad (267) \]
A set of independent coordinates being invariant under conjugation is now given by \( Y^0 = X^0 \) and the coordinates introduced in [260].
B Proofs

In this appendix we present some calculations that are rather lengthy or not so important for the understanding of the subject. These calculations are partly based on the relations we derived for objects of q-analysis in Ref. [38]. At some places we make use of the fact that we are dealing with objects that live in braided tensor categories. This observation enables us to apply diagrammatic considerations for certain rearrangements. For an introduction into this subject we refer the reader to Ref. [42].

B.1 Proof of the relations in (123)

We concentrate attention on the first relation in (123), since the second one follows from similar arguments. We illustrate the reasonings leading to (123) by the following calculation:

\[
\int_{-\infty}^{+\infty} d_{\tilde{L}}^{a} x g(x^{i}) \delta_{R}^{b} (\otimes_{R} x^{j}) = \int_{-\infty}^{+\infty} d_{\tilde{L}}^{a} x g(x^{i}) \delta_{R}^{b} \exp(\otimes_{R} x^{j}|\cdot^{-1})_{R,\tilde{L}} \\
= \int_{-\infty}^{+\infty} d_{\tilde{L}}^{a} x g(x^{i}) \delta_{R}^{b} S^{-1}(e_{(x,R)}^{a}) \otimes \left( \int_{-\infty}^{+\infty} d_{R}^{b} p e_{(p,\tilde{L})}^{a} \right) \\
= \int_{-\infty}^{+\infty} d_{\tilde{L}}^{a} x x^{i} \langle e^{b}_{(p,L)}(g(x^{i}))_{L,R} e^{b}_{(x,R)}(e_{(x,R)}^{a}) \otimes S^{-1}(e_{(x,R)}^{a}) \right) \left( \int_{-\infty}^{+\infty} d_{R}^{b} p e_{(p,\tilde{L})}^{a} \right) \\
= \int_{-\infty}^{+\infty} d_{\tilde{L}}^{a} x \left( S^{-1}(e^{b}_{(x,R)}), g(x^{i}) \right) L,R S^{-1}(e^{b}_{(x,R)}) S^{-1}(e_{(x,R)}^{a}) \otimes \left( \int_{-\infty}^{+\infty} d_{R}^{b} p e_{(p,\tilde{L})}^{a} \right) \\
= \int_{-\infty}^{+\infty} d_{\tilde{L}}^{a} x S^{-1}((e^{a}_{(x,R)} \rangle \left( \int_{-\infty}^{+\infty} d_{R}^{b} p e_{(p,\tilde{L})}^{a} \right) \\
= \int_{-\infty}^{+\infty} d_{\tilde{L}}^{a} x \left( S^{-1}(e^{a}_{(x,R)} \otimes e^{b}_{(x,R)}), g(x^{i}) \right) L,R \otimes \left( \int_{-\infty}^{+\infty} d_{R}^{b} p e_{(p,\tilde{L})}^{a} \right) \\
= \int_{-\infty}^{+\infty} d_{\tilde{L}}^{a} x S^{-1}(e^{a}_{(x,R)} \otimes e^{b}_{(x,R)}), g(x^{i}) \rangle L,R \otimes \left( \int_{-\infty}^{+\infty} d_{R}^{b} p e_{(p,\tilde{L})}^{a} \right)
\]
\[
\begin{align*}
&= \int_{-\infty}^{+\infty} d^n_L x S^{-1}(e^a_{(p,R)}) \left( S^{-1}(\langle e^a_{(p,L)}(R,1), g(\kappa x^i) \rangle) \right)_{L,R} \\
&\quad \otimes \int_{-\infty}^{+\infty} d^n_R p \left( e^a_{(p,L)}(\bar{R},2) \right)
&= \int_{-\infty}^{+\infty} d^n_L x \int_{-\infty}^{+\infty} d^n_R p \exp(\ominus R x^j [i^{-1} \bar{p}^k])_{R,L} \\
&= g(0) \int_{-\infty}^{+\infty} d^n_L x \int_{-\infty}^{+\infty} d^n_R p \exp(\ominus R x^j [i^{-1} \bar{p}^k])_{R,L} \\
&= \kappa^n \text{vol}_R g(0).
\end{align*}
\] (268)

For the first step we insert the definition of the q-deformed delta function [cf. the definitions in (88)]. Then we reformulate the expression in order to display the tensor product it lives in. Notice that \( S^{-1} \) corresponds to the operation \( \ominus_R \) and can be recognized as antipode on a braided space [38,42]. Next, we rewrite the function \( g \) by means of a variant of the completeness relation in (109). This step enables us to apply the identity

\[
\sum_a e^a_{(p,L)} \otimes e^a_{(x,R)} = \sum_a S^{-1}(e^a_{(p,L)}) \otimes S^{-1}(e^a_{(x,R)})
\] (269)

together with the axiom

\[
m \circ (S^{-1} \otimes S^{-1}) = S^{-1} \circ m \circ \Psi^{-1},
\] (270)

where \( m \) denotes multiplication on coordinate space and \( \psi^{-1} \) the braiding mapping induced by \( \tau \circ R^{-1} \). For the sixth equality we have to realize that the braiding involves a delta function. However, the trivial braiding of delta functions allows us to neglect the braiding. The seventh equality is the addition law for q-deformed exponentials and the eighth equality uses translation invariance of the integral over momentum space [cf. (84)]. The ninth step follows from the same arguments already applied in (107). The last equality is a consequence of the identities in (291).

B.2 Proof of the relations in (100) and (102)

\[
\begin{align*}
\mathcal{F}_L(\mathcal{F}_R^*(g)(x^i))(p^k) &= \int_{-\infty}^{+\infty} d^n_L x \mathcal{F}_R^*(g)(x^i) \otimes \exp(x^j [i^{-1} p^k])_{R,L} \\
&= \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_L x \left( \int_{-\infty}^{+\infty} d^n_R p \Gamma_R g(p^m) \otimes R \exp(\ominus R x^j [i^{-1} \bar{p}^l])_{R,L} \right).
\end{align*}
\] 64
\begin{align*}
\frac{x}{\nabla} \exp(x^j|^{-1}p^k)_{R,L} = \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_{p} g(\beta^m) \left( \frac{\tilde{\beta}}{\nabla} \int_{-\infty}^{+\infty} d^n_{p} x \exp(x^j|^{-1}p^k)_{R,L} \right) \\
= \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_{p} g(\kappa \beta^m) \left( \frac{\hat{\beta}}{\nabla} \int_{-\infty}^{+\infty} d^n_{L} x \exp(x^j|(i^{-1}k^l) \oplus_R (\bigoplus_L (i^{-1}k^l)))_{R,L} \right) \tag{271}
\end{align*}

Let us make a few comments on what we have done so far. We inserted the defining expressions for q-deformed Fourier transformations. In the left exponential we switched the antipode from space coordinates to momentum coordinates by applying

\[ \exp(\bigodot_R x^j|^{-1}p^k)_{R,L} = \exp(x^j|\bigodot_L (i^{-1}p^k))_{R,L}, \tag{272} \]

Finally, we used the addition law for q-exponentials.

In contrast to the calculation in (125), we are not yet in a position to apply the identities for q-deformed delta functions in (116) and (117). Thus, we have to work a little bit harder. In what follows, it is helpful to rewrite the last expression in (271) in such a way that tensor factors are explicitly displayed:

\begin{align*}
\frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_{R} \tilde{p} g(\kappa \beta^m) \left( \frac{\hat{\beta}}{\nabla} \int_{-\infty}^{+\infty} d^n_{L} x \exp(x^j|(i^{-1}k^l) \oplus_R (\bigoplus_L (i^{-1}k^l)))_{R,L} \right) \\
= \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_{L} x e^a(x,\tilde{R}) \otimes (e^a_{(R,L)})(R,1) \\
\otimes \int_{-\infty}^{+\infty} d^n_{R} \tilde{p} S((\bigodot_R (e^a_{(R,L)})(R,2)) \\
\otimes S^{-1}(R_{[1]} \otimes (e^a_{(\beta,L)})) g(\kappa \beta^m)) \tag{273}
\end{align*}
\[ \int_{-\infty}^{+\infty} d^n x \ e^a (x, \bar{R}) \otimes S(R|2) \triangleright (S^{-1} g(p^k))_{(R,1)} \]
\[ \otimes \int_{-\infty}^{+\infty} d^n \tilde{p} \ S(R|1) \triangleright e^a (\tilde{p}, L) \oplus (S^{-1} g(\tilde{p}^m))_{(R,2)} \]
\[ = \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n x \ e^a (x, R) \otimes S^{-1} g(\kappa^{-1} p^k)_{(R,1)} \]
\[ \otimes \int_{-\infty}^{+\infty} d^n \tilde{p} \ S(e^a (\tilde{p}, L) \otimes (S^{-1} g(\tilde{p}^m))_{(R,2)}). \] (274)

For the first step we applied the axiom
\[ S^{-1} \circ m = m \circ (S \otimes S) \circ \psi. \] (275)

The braiding mappings in the second expression determine how the function \( g \) commutes with a delta function. Due to the trivial braiding of delta functions, we are able to neglect the braiding. To understand the third step one has to realize that the expressions
\[ \int_{-\infty}^{+\infty} d^n A_{pf} (\oplus_B p^i), \] (276)
fulfill
\[ \int_{-\infty}^{+\infty} d^n A_{pf} (\oplus_B p^i) = \int_{-\infty}^{+\infty} d^n A_{pf} ((\oplus_B p^i) \oplus_C \bar{p}^j) \]
\[ = \int_{-\infty}^{+\infty} d^n A_{pf} (\bar{p}^j \oplus_C (\oplus_B p^i)), \] (277)
where \( A, B, C \in \{ L, \bar{L}, R, \bar{R} \} \). This can easily be seen by calculations of the following form:
\[ \int_{-\infty}^{+\infty} d^n A_{pf} (((\oplus_L p^i) \oplus_R \bar{p}^j) \]
\[ = \int_{-\infty}^{+\infty} d^n A_{pf} ((\oplus_L (p^i \oplus_L (\oplus_R \bar{p}^j))) = \int_{-\infty}^{+\infty} d^n A_{pf} ((\oplus_L p^i). \] (278)

The point now is that due to (277) relation (104) remains valid if we replace the integrals over the whole space by the expressions in (276) and this observation finally leads to the third equality in (274). In the last step we exploit the trivial braiding of q-integral and q-exponential to rewrite the expression
in (274) without any braiding.

To proceed any further, we need another identity, which we now derive:

\[
\begin{align*}
\int_{-\infty}^{+\infty} d^n_L x e^a_{(x,R)} \otimes \int_{-\infty}^{+\infty} d^n_R \tilde{p} S((e^a_{(\tilde{p},L)} \otimes f(\tilde{p})) &= \\
= \int_{-\infty}^{+\infty} d^n_L x e^a_{(x,R)} \otimes \int_{-\infty}^{+\infty} d^n_R \tilde{p} S(e^a_{(\tilde{p},L)} \otimes e^b_{(\tilde{p},L)}) \\
& \quad \times \langle f(p^i), e^b_{(x,R)} \rangle_{L,R} \\
= \int_{-\infty}^{+\infty} d^n_L x e^a_{(x,R)}(R,1) \otimes \int_{-\infty}^{+\infty} d^n_R \tilde{p} S(e^a_{(\tilde{p},L)}) \\
& \quad \times \langle f(p^i), (e^a_{(x,R)}(R,2))_{L,R} \\
= \int_{-\infty}^{+\infty} d^n_L x e^a_{(x,R)} \otimes \int_{-\infty}^{+\infty} d^n_R \tilde{p} S(e^a_{(\tilde{p},L)}) \langle f(p^i), 1 \rangle_{L,R}.
\end{align*}
\]

The first equality results from the completeness relation (109). The second equality is the addition law for q-exponentials. Finally, we make use of translation invariance of the integral over position space.

By virtue of the above identity the last expression in (274) becomes

\[
\begin{align*}
\frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_L x e^a_{(x,R)} \otimes S(S^{-1}g(\kappa^{-1}p^k))_{(R,1)} \\
& \quad \otimes \int_{-\infty}^{+\infty} d^n_R \tilde{p} S(e^a_{(\tilde{p},L)} \otimes (S^{-1}g(p^m))_{(R,2)}) \\
= \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_L x e^a_{(x,R)} \otimes S(S^{-1}g(\kappa^{-1}p^k))_{(R,1)} \langle (S^{-1}g(p^k))_{(R,2)}, 1 \rangle_{L,R} \\
& \quad \otimes \int_{-\infty}^{+\infty} d^n_R \tilde{p} S(e^a_{(\tilde{p},L)} \\
= \frac{1}{\text{vol}_R} \int_{-\infty}^{+\infty} d^n_L x e^a_{(x,R)} \otimes S(S^{-1}g(\kappa^{-1}p^k)) \otimes \int_{-\infty}^{+\infty} d^n_R \tilde{p} S(e^a_{(\tilde{p},L)} \\
= \kappa^{-n} g(\kappa^{-1}p^k).
\end{align*}
\]

Notice that for the second step we applied

\[
\hat{f}_{(R,1)} \otimes \langle \hat{f}_{(R,2)}, 1 \rangle_{L,R} = \hat{f}_{(R,1)} \otimes e_R(\hat{f}_{(R,2)}) = f;
\]

(281)
while the last step needs the relation
\[
\text{vol}_R = \kappa^{-n} \int_{-\infty}^{+\infty} d^n_L x \int_{-\infty}^{+\infty} d^n_R p \exp(x^i | \ominus_L (i^{-1}p^k)) \tilde{R}_{L,R}. \tag{282}
\]

It remains to prove identity (282). To this end we show that the following equalities hold:
\[
\kappa^{-n}(\text{vol}_R)^2 = \int_{-\infty}^{+\infty} d^n_L \bar{x} \int_{-\infty}^{+\infty} d^n_L x \delta^n_R((\ominus_R x^i) \oplus_R \bar{x}^j) \odot \delta^n_R(\bar{x}^j) = \text{vol}_R \int_{-\infty}^{+\infty} d^n_L x \int_{-\infty}^{+\infty} d^n_R p \exp(x^i | \ominus_L (i^{-1}p^k)) \tilde{R}_{L,R}. \tag{283}
\]

The first equality can be verified in a straightforward manner:
\[
\int_{-\infty}^{+\infty} d^n_L \bar{x} \int_{-\infty}^{+\infty} d^n_L x \delta^n_R((\ominus_R x^i) \oplus_R \bar{x}^j) \odot \delta^n_R(\bar{x}^j)
= \int_{-\infty}^{+\infty} d^n_L \bar{x} \int_{-\infty}^{+\infty} d^n_L x \left( \int_{-\infty}^{+\infty} d^n_R p \exp((\ominus_R x^i) \oplus_R \bar{x}^j | i^{-1}p^k) \tilde{R}_{L,R} \right)
\bar{x} \odot \int_{-\infty}^{+\infty} d^n_R \tilde{p} \exp(\bar{x}^j | i^{-1}p^k) \tilde{R}_{L,R}
= \int_{-\infty}^{+\infty} d^n_L \bar{x} \int_{-\infty}^{+\infty} d^n_L x \left( \int_{-\infty}^{+\infty} d^n_R \tilde{p} \exp(x^i | i^{-1}p^k) \tilde{R}_{L,R} \right)
\bar{x} \odot \int_{-\infty}^{+\infty} d^n_L \tilde{x} \int_{-\infty}^{+\infty} d^n_R \tilde{p} \exp((\kappa \bar{x}^j) \oplus_R \bar{x}^j | i^{-1}p^m) \tilde{R}_{L,R}
= \int_{-\infty}^{+\infty} d^n_L x \left( \int_{-\infty}^{+\infty} d^n_R \tilde{p} \exp(x^i | i^{-1}p^k) \tilde{R}_{L,R} \right)
\bar{x} \odot \int_{-\infty}^{+\infty} d^n_L \tilde{x} \int_{-\infty}^{+\infty} d^n_R \tilde{p} \exp((\kappa \bar{x}^j) | i^{-1}p^m) \tilde{R}_{L,R}
= \kappa^{-n}(\text{vol}_R)^2. \tag{284}
\]

For the first step we insert the defining expressions for q-deformed delta functions. Then we are ready to apply (104). The third equality is a consequence of the braiding properties of integrals over the whole space. For the
fourth step we use translation invariance of integrals over the whole space.

Now, we come to the second equality in (283), for which we likewise have

\[
\int_{-\infty}^{+\infty} d^p_L x \int_{-\infty}^{+\infty} d^p_R x \, \delta^p_R((\ominus_R x^i) \oplus_R \tilde{x}^j) \oplus \delta^p_R(\tilde{x}^j)
\]

\[
= \int_{-\infty}^{+\infty} d^p_L \tilde{x} \int_{-\infty}^{+\infty} d^p_R x \left( \int_{-\infty}^{+\infty} d^n \tilde{p} \, \exp((\kappa \tilde{x}^j) \oplus_L (\ominus_R x^i)|i^{-1}p^k)_{R,L} \right)
\]

\[
\times \int_{-\infty}^{+\infty} d^n \tilde{p} \, \exp(\tilde{x}^l|i^{-1}p^m)_{R,L}
\]

\[
= \int_{-\infty}^{+\infty} d^p_L x \left( \int_{-\infty}^{+\infty} d^n \tilde{p} \, \exp((\ominus_R x^i) \oplus_L (i^{-1}p^k)_{R,L} \right)
\]

\[
\times \int_{-\infty}^{+\infty} d^n \tilde{p} \, \exp(\tilde{x}^l|i^{-1}p^m)_{R,L}
\]

\[
= \text{vol}_R \int_{-\infty}^{+\infty} d^n \tilde{p} \, \exp(x^i|\ominus_L (i^{-1}p^k))_{R,L}.
\]

Again let us make some comments on the above calculation. First, we plug in the expressions for q-deformed delta functions. Simultaneously, we change the order of the tensor factors corresponding to the coordinates \( \tilde{x}^j \) and \( x^i \) by making use of

\[
f(x^i \oplus_L y^j) = f(y^j \ominus_R x^i).
\]

In doing so, there appears a scaling, since we integrate over the coordinate \( x^i \). Then we make use of the distributive law for q-deformed translations.

\[
f(\ominus_L (x^i \oplus_L y^j)) = f((\ominus_L x^i) \ominus_R (\ominus_L y^j)).
\]

(Notice that \( f(\ominus_L (\ominus_R x^i)) = f(x^i) \). The third equality is the property (272) and the fourth equality again results from translation invariance of integrals over the whole space.

69
For the sake of completeness it should be mentioned that in addition to (283) we also have
\[ \kappa^n (\text{vol} \bar{\mathcal{R}})^2 = \int_{-\infty}^{+\infty} d^m L \int_{-\infty}^{+\infty} d^m x \delta^n_R ((\ominus L x^i) \oplus \bar{\mathcal{R}} x^j)) \hat{\delta}^n_{\bar{\mathcal{R}}} (x^l) \]
\[ = \text{vol}_R \int_{-\infty}^{+\infty} d^m L \int_{-\infty}^{+\infty} d^m R \exp(x^i \ominus_R (i^{-1} p^k))_{R,L}, \quad (288) \]
which, in turn, leads to
\[ \text{vol}_R = \kappa^{-n} \int_{-\infty}^{+\infty} d^m L \int_{-\infty}^{+\infty} d^m R \exp(x^i \ominus_R (i^{-1} p^k))_{R,L}. \quad (289) \]
These relations can be proved in very much the same way as those in (283).

To sum up, the volume elements defined in (89) and (90) can alternatively be represented as
\[ \text{vol}_L = \text{vol}_R = \kappa^n \int_{-\infty}^{+\infty} d^m L \int_{-\infty}^{+\infty} d^m R \exp(x^i \ominus_L (i^{-1} p^k))_{R,L} \]
\[ = \kappa^{-n} \int_{-\infty}^{+\infty} d^m L \int_{-\infty}^{+\infty} d^m R \exp(x^i \ominus_R (i^{-1} p^k))_{R,L}, \quad (290) \]
\[ \text{vol}_L = \text{vol}_R = \kappa^{-n} \int_{-\infty}^{+\infty} d^m L \int_{-\infty}^{+\infty} d^m R \exp(x^i \ominus_R (i^{-1} p^k))_{R,L} \]
\[ = \kappa^n \int_{-\infty}^{+\infty} d^m L \int_{-\infty}^{+\infty} d^m R \exp(x^i \ominus_R (i^{-1} p^k))_{R,L}. \quad (291) \]

Let us note that there are similar identities for the more general volume elements \( \text{vol}_{A,B} \).

**B.3 Proof of the last relation in (238)**

To prove the last identity in (238) we adapt the calculation of (239) as follows:
\[ \text{vol}_R(\langle f, g \rangle)_{R,x} = \text{vol}_R \int_{-\infty}^{+\infty} d^m R f(x^i) \hat{\oplus} g(x^j) \]
\[ = \int_{-\infty}^{+\infty} d^m R \int_{-\infty}^{+\infty} d^m L \exp(p^m y^k \ominus_R (\ominus L x^j)) \hat{\oplus} R g(y^i) \]
\[ = \int_{-\infty}^{+\infty} d^m R \int_{-\infty}^{+\infty} d^m L \exp(p^m y^k \ominus_R (\ominus L x^j))_{R,L} \]
70
$$\begin{align*}
&= \int_{-\infty}^{+\infty} d^3_R x \int_{-\infty}^{+\infty} d^3_R y \bar{f}(x^i) \otimes \int_{-\infty}^{+\infty} d^n_R p \exp(p^\mu | \otimes \kappa x^j)_{R,L} \times \int_{-\infty}^{+\infty} d^n_R p \exp(p^m | y^k)_{R,L} \otimes g(y^l) \\
&= \int_{-\infty}^{+\infty} d^n_L p \epsilon^a_{(R,p)} \otimes \left(-1\right)^n \int_{-\infty}^{+\infty} d^n_R x \bar{S}^{-1}(\kappa^{-1} f(x^i)) \otimes \int_{-\infty}^{+\infty} d^n_R y \epsilon^b_{(L,y)} \otimes g(y^l) \\
&= \left(-1\right)^n \int_{-\infty}^{+\infty} d^n_L p \int_{-\infty}^{+\infty} d^n_R x \exp(\otimes_R x^r | \kappa^{-1} p^m)_{R,L} \otimes_R f(x^i) \times \int_{-\infty}^{+\infty} d^n_R y \exp(p^k | y^j)_{R,L} \otimes_R g(y^l) \\
&= \left(-1\right)^n \text{vol}_R \int_{-\infty}^{+\infty} d^n_L p \mathcal{F}^a_R(f)(k^{-1} p^m) \otimes \mathcal{F}_R(p^k) \\
&= \left(-1\right)^n \text{vol}_R \mathcal{F}^a_R(f)(k^{-1} p^m), \mathcal{F}_R(p^k) \bigg|_{L,p}.
\end{align*}$$

The line of reasonings is the same as for the calculation in [239].

References

[1] M. Fichtenmüller, A. Lorek and J. Wess, *q-deformed Phase Space and its Lattice Structure*, Z. Phys. C 71 (1996) 533 [hep-th/9511106].
[2] J. Wess, *q-Deformed phase space and its lattice structure*, Int. J. Mod. Phys. A **12** (1997) 4997.

[3] B. L. Cerchiai and J. Wess, *q-Deformed Minkowski Space based on a q-Lorentz Algebra*, Eur. Phys. J. C **5** (1998) 553, [math.qa/9801104](http://arxiv.org/abs/math.qa/9801104).

[4] J. Schwinger, *Quantum Electrodynamics*, Dover, New York, 1958.

[5] W. Heisenberg, *Über die in der Theorie der Elementarteilchen auftretende universelle Länge*, Ann. Phys. **32** (1938) 20.

[6] H. Grosse, C. Klimčík and P. Prešnajder, *Towards finite quantum field theory in non-commutative geometry*, Int. J. Theor. Phys. **35** (1996) 231, [hep-th/9505175](http://arxiv.org/abs/hep-th/9505175).

[7] S. Majid, *On the q-regularisation*, Int. J. Mod. Phys. A **5** (1990) 4689.

[8] R. Oeckl, *Braided Quantum Field Theory*, Commun. Math. Phys. **217** (2001) 451.

[9] C. Blohmann, *Free q-deformed relativistic wave equations by representation theory*, Eur. Phys. J. C **30** (2003) 435, [hep-th/0111172](http://arxiv.org/abs/hep-th/0111172).

[10] I. M. Gelfand and M. A. Naimark, *On the embedding of normed linear rings into the ring of operators in Hilbert space*, Math. Sbornik **12** (1943) 197.

[11] H. Hopf, *Über die Topologie der Gruppenmannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. Math. **42** (1941) 22.

[12] P. P. Kulish and N. Y. Reshetikhin, *Quantum linear problem for the Sine-Gordon equation and higher representations*, J. Sov. Math. **23** (1983) 2345.

[13] S. L. Woronowicz, *Compact matrix pseudogroups*, Commun. Math. Phys. **111** (1987) 163.

[14] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Sov. Math. Dokl. **32** (1985) 254.

[15] M. Jimbo, *A q-analogue of U(g) and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985) 63.

[16] V. G. Drinfeld, *Quantum groups*, in A. M. Gleason, ed., Proceedings of the International Congress of Mathematicians, Amer. Math. Soc., 798 (1986).
[17] N. Yu. Reshetikhin, L. A. Takhtadzhyan and L. D. Faddeev, *Quantization of Lie Groups and Lie Algebras*, Leningrad Math. J. 1 (1990) 193.

[18] M. Takeuchi, *Matrix Bialgebras and Quantum Groups*, Israel J. Math. 72 (1990) 232.

[19] U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura, *Tensor Representations of the Quantum Group SL_q(2) and Quantum Minkowski Space*, Z. Phys. C 48 (1990) 159.

[20] P. Podleś and S. L. Woronowicz, *Quantum Deformation of Lorentz Group*, Commun. Math. Phys. 130 (1990) 381.

[21] W. B. Schmidke, J. Wess and B. Zumino, *A q-deformed Lorentz Algebra in Minkowski phase space*, Z. Phys. C 52 (1991) 471.

[22] S. Majid, *Examples of braided groups and braided matrices*, J. Math. Phys. 32 (1991) 3246.

[23] A. Lorek, W. Weich and J. Wess, *Non-commutative Euclidean and Minkowski Structures*, Z. Phys. C 76 (1997) 375, [q-alg/9702025].

[24] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplane*, Nucl. Phys. B Suppl. 18 (1991) 302.

[25] U. Carow-Watamura, M. Schlieker and S. Watamura, *SO_q(N)-covariant differential calculus on quantum space and deformation of Schrödinger equation*, Z. Phys. C 49 (1991) 439.

[26] X. C. Song, *Covariant differential calculus on quantum minkowski space and q-analog of Dirac equation*, Z. Phys. C 55 (1992) 417.

[27] O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, *q-Deformed Poincaré Algebra*, Commun. Math. Phys. 150 (1992) 495.

[28] S. Majid, *Braided momentum in the q-Poincaré group*, J. Math. Phys. 34 (1993) 2045.

[29] H. Wachter and M. Wohlgenannt, *-*Products on quantum spaces*, Eur. Phys. J. C 23 (2002) 761, [hep-th/0103120].

[30] C. Bauer and H. Wachter, *Operator representations on quantum spaces*, Eur. Phys. J. C 31 (2003) 261, [math-ph/0201023].
[31] H. Wachter, *q-Integration on quantum spaces*, Eur. Phys. J. C 32 (2004) 281, [hep-th/0206083].

[32] H. Wachter, *q-Exponentials on quantum spaces*, Eur. Phys. J. C 37 (2004) 379, [hep-th/0401113].

[33] H. Wachter, *q-Translations on quantum spaces*, preprint, [hep-th/0410205].

[34] H. Wachter, *Braided products for quantum spaces*, preprint, [math-ph/0509018].

[35] D. Mikulovic, A. Schmidt and H. Wachter, *Grassmann variables on quantum spaces*, Eur. Phys. J. C 45 (2006) 529, [hep-th/0407273].

[36] A. Schmidt, H. Wachter, *Superanalysis on quantum spaces*, JHEP 0601 (2006) 84, [hep-th/0411180].

[37] V. Kac and P. Cheung, *Quantum Calculus*, Springer Verlag, Berlin (2000).

[38] H. Wachter, *Analysis on q-deformed quantum spaces*, to appear in Int. J. Mod. Phys. A, [math-ph/0604028].

[39] S. Majid, *Representations, duals and quantum doubles of monoidal categories*, Suppl. Rend. Circ. Mat. Palermo, Ser. II, 26 (1991) 197.

[40] S. Majid, *Algebras and Hopf Algebras in Braided Categories*, Lec. Notes Pure Appl. Math. 158 (1994) 55.

[41] S. Majid, *Introduction to Braided Geometry and q-Minkowski Space*, preprint (1994), [hep-th/9410241].

[42] S. Majid, *Beyond Supersymmetry and Quantum Symmetry (an introduction to braided groups and braided matrices)*, in M. L. Ge and H. J. de Vega, eds., Quantum Groups, Integrable Statistical Models and Knot Theory, World Scientific, 231 (1993).

[43] C. Chryssomalakos, P. Schupp, and P. Watts, *Translations, Integrals And Fourier Transforms In The Quantum Plane*, in A. Ali, J. Ellis and S. Randjbar-Daemi, eds., Salamfestschrift, Proceedings of the Conference on Highlights of Particle and Condensed Matter Physics, ICTP, Trieste (1993).
[44] A. Schirrmacher, *Generalized q-exponentials related to orthogonal quantum groups and Fourier transformations of noncommutative spaces*, J. Math. Phys. 36 (3) (1995) 1531.

[45] A. Kempf and S. Majid, *Algebraic q-integration and Fourier theory on quantum and braided spaces*, J. Math. Phys. 35 (1994) 6802.

[46] J. Wess, *q-deformed Heisenberg Algebras*, in H. Gausterer, H. Grosse and L. Pittner, eds., Proceedings of the 38. Internationale Universitätswochen für Kern- und Teilchenphysik, no. 543 in Lect. Notes in Phys., Springer-Verlag, Schladming (2000), [math-ph/9910013].

[47] Y. I. Manin, *Quantum Groups and Non-Commutative Geometry*, Centre de Recherche Mathématiques, Montreal (1988).

[48] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures*, Ann. Phys. 111 (1978) 61.

[49] J. E. Moyal, *Quantum mechanics as a statistical theory*, Proc. Camb. Phil. Soc. 45 (1949) 99.

[50] J. Madore, S. Schraml and P. Schupp, J. Wess, *Gauge Theory on Noncommutative Spaces*, Eur. Phys. J. C 16 (2000) 161, [hep-th/0001203].

[51] F. H. Jackson, *On q-functions and a certain difference operator*, trans. Roy. Edin. 46 (1908) 253.

[52] S. Majid, *Foundations of Quantum Group Theory*, University Press, Cambridge (1995).

[53] A. Klimyk and K. Schmüdgen, *Quantum Groups and their Representations*, Springer Verlag, Berlin (1997).

[54] M. Chaichian and A. P. Demichev, *Introduction to Quantum Groups*, World Scientific, Singapore, 1996.

[55] S. Majid, *Free braided differential calculus, braided binomial theorem and the braided exponential map*, J. Math. Phys. 34 (1993) 4843.

[56] S. Majid, *Quantum and braided linear algebra*, J. Mat. Phys. 34 (1993) 1176.

[57] U. Meyer, *q-Lorentz group and braided coaddition on q-Minkowski space*, Commun. Math. Phys. 168 (1995) 249.
[58] G. Fiore, The \(SO_q(N)\)-symmetric harmonic oscillator on the quantum Euclidean space \(R_q^N\) and its Hilbert space structure, Int. J. Mod. Phys. A 8 (1993) 4679.

[59] H. Steinacker, Integration on quantum Euclidean Space and sphere, J. Math. Phys. 37 (1996) 4738.

[60] C. Chryssomalakos, Remarks On Quantum Integration, preprint, ENSLAPP-A-562/95, [q-alg/9601014].

[61] F. H. Jackson, \(q\)-Integration, Proc. Durham Phil. Soc. 7 (1927) 182.

[62] O. Ogievetsky and B. Zumino, Reality in the differential calculus on \(q\)-Euclidean spaces, Lett. Math. Phys. 25 (1992) 121.

[63] S. Majid, \(*\)-structures on braided spaces, J. Math. Phys. 36 (1995) 4436.

[64] G. Fiore, On the hermiticity of \(q\)-differential operators and forms on the quantum Euclidean spaces \(R_q^N\), [math.QA/0403463]. New approach to Hermitian \(q\)-differential operators on \(R_q^N\), [math.QA/0410322].

[65] S. Majid, Quasi-\(*\)-structure on \(q\)-Poincaré algebras, J. Geom. Phys. 22 (1997) 14.

[66] C. Chryssomalakos, P. Schupp and P. Watts, The role of the Canonical element in the Quantized Algebra of Differential Operators \(A \rtimes U\), preprint, LBL-33274, [hep-th/9310100].

[67] M. Schlieker and W. Scholl, Spinor calculus for quantum groups, Z. Phys. C 47 (1990) 625.

[68] H. Ocampo, \(SO_q(4)\) quantum mechanics, Z. Phys. C 70 (1996) 525.

[69] J. Lukierski, A. Nowicki and H. Ruegg, New Quantum Poincaré Algebra and \(\kappa\)-deformed Field Theory, Phys. Lett. B 293 (1992) 344.

[70] L. Castellani, Differential Calculus on \(ISO_q(N)\), Quantum Poincaré Algebra and \(q\)-Gravity, preprint, [hep-th/9312179].

[71] V. K. Dobrev, New \(q\)-Minkowski space-time and \(q\)-Maxwell equations hierarchy from \(q\)-conformal invariance, Phys. Lett. B 341 (1994) 133.

[72] S. Doplicher, K. Fredenhagen and J. E. Roberts, The Quantum Structure of Space-Time at the Planck Scale and Quantum fields, Commun. Math. Phys. 172 (1995) 187.
[73] M. Chaichian and A. P. Demichev, *Quantum Poincaré group without dilatation and twisted classical algebra*, J. Math. Phys. **36** (1995) 398.

[74] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, *On a Lorentz-Invariant Interpretation of Noncommutative Space-Time and its Implications on Noncommutative QFT*, Phys. Lett. B **604** (2004) 98, [hep-th/0408062](https://arxiv.org/abs/hep-th/0408062).

[75] F. Koch and E. Tsouchnika, *Construction of θ-Poincaré Algebras and their invariants on Mθ*, Nucl. Phys. B **717** (2005) 387, [hep-th/0409012](https://arxiv.org/abs/hep-th/0409012).