On a class of optimal constant weight ternary codes

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Abstract

A weighing matrix $W$ of order $n = \frac{p^{m+1} - 1}{p-1}$ and weight $p^m$ is constructed and shown that the rows of $W$ and $-W$ together form optimal constant weight ternary codes of length $n$, weight $p^m$ and minimum distance $p^{m-1}(\frac{p+3}{2})$ for each odd prime power $p$ and integer $m \geq 1$ and thus

$$A_3\left(\frac{p^{m+1} - 1}{p-1}, p^{m-1}(\frac{p+3}{2}), p^m\right) = 2\left(\frac{p^{m+1} - 1}{p-1}\right).$$

Keywords Weighing matrix · Ternary code · Constant weight code · Optimal code · Johnson bound

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1 Introduction

A weighing matrix of order $n$ and weight $p$, shown as $W(n, p)$, is a $(0, \pm1)$-matrix $W$ of order $n$ such that $WW^t = pI_n$. The case where $n = p + 1$ is called a conference matrix and $n = p$ is a Hadamard matrix. We refer the reader to [9] for conference matrices. Optimal binary codes obtained from Hadamard matrices constitute an important class of codes due to their error-correcting capability. One expects that weighing matrices also provide sets of
useful codes. The rows of a weighing matrix \( W(n, p) \), \( n \neq p \), form a set of constant weight ternary codes.

Optimal constant weight (CW) ternary codes have been the subject of study in several papers. Fu et al. constructed several optimal CW binary and ternary codes in [3]. Ge [4], Zhang et al. [12, 13] worked on optimal CW ternary codes with small weight and distance. Bogdanova et al. [1] studied equidistant CW codes with small parameters.

The main references for ternary codes in this work are [2, 3, 8, 11]. Theorem 16 of [8] and Propositions 5.3 and 5.4 in [3] relate, though in disguise, to optimal classes of constant ternary codes from weighing matrices. There seem to be no more optimal constant weight codes explicitly related to the weighing matrices in the literature. A large class of optimal constant weight ternary codes are shown to arise from weighing matrices in Sects. 3 and 4 of this paper.

2 Preliminaries

Let \( S_3 = \{0, 1, -1\} \). A ternary code of length \( n \) is any subset \( C \) of \( S_3^n \). Elements of \( C \) are called codewords. The Hamming distance between two ternary codewords of length \( n \) is the number of coordinates in which they differ. The number of nonzero entries of a codeword is the weight of the code. A ternary code of length \( n \) containing \( M \) codewords and having minimum Hamming distance \( d \) is denoted \( (n, M, d) \)-code. If all the codewords have the same number of nonzero entries the code is said to be of constant weight. The largest value of \( M \) for which there is a ternary code of length \( n \), minimum distance \( d \) and weight \( w \) is denoted by \( A_3(n, d, w) \) and the code is said to be optimal. Östergård et al in [8] among other interesting results have shown that if \( p \geq 3 \) is a prime power and \( m \geq 1 \), then 

\[
A_3(p^m + 1, (p^m + 3)/2, p^m) = 2(p^m + 1).
\]

To show an extension of this result the restricted Johnson bound for \( A_3(n, d, w) \) is essential, see Theorem 2.3.4 of [2].

**Theorem 1** If \( 3w^2 - 4nw + 2nd > 0 \), then

\[
A_3(n, d, w) \leq \left\lfloor \frac{2nd}{3w^2 - 4nw + 2nd} \right\rfloor.
\] (1)

A weighing matrix \( W(n, p) \) is said to be in normal form if it has the block configuration

\[
\begin{bmatrix}
0_{n-p} & R \\
1_p & D
\end{bmatrix}
\]

for some \((0, \pm 1)\)-matrices \( R \) and \( D \), where \( 0_{n-p} \) is the \((n - p) \times 1\) column vector with all entries 0 and \( 1_p \) is the \( p \times 1 \) column vector of all entries 1. We call \( R \) the residual and \( D \) the derived parts of the weighing matrix. It follows that \( RR^t = pI_{n-p} \), and \( DD^t = pI_p - J_p \), and \( RD^t = DR^t = 0 \). By permuting and negating some rows, if necessary, every weighing matrix can be assumed to be in normal form. The Jacobsthal matrix, as described below, is used extensively in this paper, see [10].

**Theorem 2** There is a \((0, \pm 1)\)-matrix \( Q \) of an odd prime power order \( p \) having zero on the diagonal and \( \pm 1 \) off-diagonal with row and column sum zero and \( QQ^t = pI_p - J_p \).

**Theorem 3** Let \( C \) be a conference matrix \( W(n + 1, n) \) with the \( n \times n \) matrix \( D \) being the derived part of \( C \). The rows of \( D \) form an optimal constant weight ternary code with minimum distance \( \frac{n+3}{2} \) and \( A_3(n, \frac{n+3}{2}, n - 1) = n \).
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Proof The inner product of two distinct rows of $D$ is $-1$. There are thus $(n-1)/2$ minus ones and $(n-3)/2$ plus ones in the inner product. It follows that the distance between any two rows is $\frac{n-1}{2} + 2 = \frac{n+1}{2}$. Considering that $3w^2 - 4nw + 2nd = 3(n-1)^2 - 4n(n-1) + 2n(\frac{n+3}{2}) = n + 3$, it follows from Johnson bound (1) above that $A_3(n, \frac{n+3}{2}, n-1) \leq n$. This completes the proof.

A second Johnson bound which will be used is as follows.

Theorem 4

$$A_3(n, d, w) \leq \left\lfloor \frac{2n}{w} A_3(n-1, d, w-1) \right\rfloor$$

(2)

As an application of Johnson bound (2) and Theorem 3 above a large set of optimal ternary codes are obtained in the next theorem.

Theorem 5 Let $C$ be a conference matrix of order $n+1$. Then the rows of $C$ and $-C$ together form an optimal constant weight ternary code and so

$$A_3(n+1, \frac{n+3}{2}, n) = 2(n+1).$$

Proof From Theorem 3 we know that $A_3(n, \frac{n+3}{2}, n-1) = n$. By Johnson bound (2):

$$A_3(n+1, \frac{n+3}{2}, n) \leq \frac{2(n+1)}{n} A_3(n, \frac{n+3}{2}, n-1) = \frac{2(n+1)}{n} n = 2(n+1).$$

For convenience we may assume that

$$[C] = \begin{bmatrix} 0 & 1^t_n \\ 1_n & D \\ 0 & -1^t_n \\ -1_n & -D \end{bmatrix}.$$ 

The codewords are of length $n+1$ and weight $n$. The minimum distance in $C$ and $-C$ stays as $\frac{n+3}{2}$. The only part requiring attention is taking a row $i$ of $D$ and $-j$ in $-D$. If $i = | - j |$, then the distance between row $i$ of $D$ and $-j$ of $-D$ is $n-1$ and thus the distance between the two longer rows is $n$. For $i \neq | - j |$ the inner product of row $i$ of $D$ and $-j$ of $-D$ is one and the distance between the two rows is $\frac{n-3}{2} + 2 = \frac{n+1}{2}$. The entries in the first column now make the difference and the minimum distance between the longer rows stays as $\frac{n+3}{2}$. This completes the proof.

Theorem 5 extends the following Theorem 16 of [8].

Corollary 6 Let $p$ be an odd prime power, then

$$A_3(p^m + 1, \frac{p^m + 3}{2}, p^m) = 2(p^m + 1)$$

for every positive integer $m$.

Proof There is a conference matrix $W(p^m + 1, p^m)$ for every odd prime power $p$ and positive integer $m$ (see [9]), and the result follows.
Remark  It is conjectured that conference matrices $W(n, n - 1)$ exist for all values of $n \equiv 0 \pmod{4}$ and all $n \equiv 2 \pmod{4}$ provided $n - 1$ is a sum of two squares, see [9]. There are many orders not covered by the Theorem 16 of [8]. For example, there is a conference matrix $W(16, 15)$, see [9], and 15 is not a prime number.

The main result of the paper is in part an application of orthogonal arrays.

Definition 7  Let $S = \{1, 2, \ldots, q\}$ be some finite alphabet. An orthogonal array of strength $t$ and index $\lambda$ is an $N \times k$ matrix over $S$ such that in every $N \times t$ subarray, each $t$-tuple in $S^t$ appears $\lambda$ times. We denote this property as $OA(\lambda(N, k, q, t))$.

Theorem 8  For the prime power $p$ and the positive integer $m$ there is an $\lambda$-array $O$ in $p$ symbols such that any two distinct rows share a common symbol in exactly

$$\frac{pm - 1}{p - 1}$$

columns.

The main construction is recursive and consists of two parts. Before proceeding to the main construction an example is helpful.

Example 1  For the prime $p = 3$

$$A_3(13, 9, 9) = 26. \quad \text{(3)}$$

In order to show equation (3) the construction is broken into seven steps.

1. Starting with a normalized $W(4, 3)$ ($-1$ is denoted by $-$):

$$W = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & - \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & - \\
\end{bmatrix}.$$
2. The derived part of $W$ is

$$D = \begin{bmatrix} 0 & 1 & - \\ - & 0 & 1 \\ 1 & - & 0 \end{bmatrix}.$$

3. A corresponding orthogonal array from Theorem 8 on symbols $\{1, 2, 3\}$ is

$$\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 3 & 3 & 3 \\
2 & 1 & 2 & 3 \\
2 & 3 & 1 & 2. \\
2 & 2 & 3 & 1 \\
3 & 1 & 3 & 2 \\
3 & 2 & 1 & 3 \\
3 & 3 & 2 & 1 \\
\end{array}$$

4. Replacing the symbol $i$ with row $i$ of $D$ to find a matrix $T_{32}$:

$$T_{32} = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Applying the restricted Johnson bound from Theorem 1, we can see that the rows of this matrix form optimal constant weight ternary codes of length $n = 12$, weight $w = 8$, minimum distance $d = 9$ and $A_3(12, 9, 8) = 9$. Applying the second Johnson bound, Theorem 4, we note that $A_3(13, 9, 9) \leq 26$.

5. Forming a matrix $R = W \otimes [111]$:

$$R = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

6. Adding the two matrices $T_{32}$ and $R$ together with an additional column of 1’s and 0’s the matrix

$$C = \begin{bmatrix} 0 & R \\ 1 & T_{32} \end{bmatrix}$$

is obtained. $C$ is a weighing matrix $W(13, 9)$ and the rows of $C$ consist of 13 constant weight ternary codewords of length $n = 13$, weight $w = 9$ and constant distance $d = 9$.

7. The final step is to form a $26 \times 13$ matrix

$$T_{13,9,9} = \begin{bmatrix} C \\ -C \end{bmatrix}.$$
whose rows form the codewords of optimal constant weight ternary codes of length $n = 13$, $w = 9$, and $d = 9$, demonstrating that $A_3(13, 9, 9) = 26$.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & -
0 & 1 & 1 & 1 & - & - & - & 0 & 0 & 1 & 1 & - & -
0 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 0 & 0 & 0
1 & 0 & 1 & - & 0 & 1 & - & 0 & 1 & - & 0 & 1
1 & 0 & 1 & - & 0 & 1 & - & 0 & 1 & - & 0 & 1
1 & 0 & 1 & - & 0 & 1 & - & 0 & 1 & - & 0 & 1
1 & 0 & 1 & - & 0 & 1 & - & 0 & 1 & - & 0 & 1
1 & 0 & 1 & - & 0 & 1 & - & 0 & 1 & - & 0 & 1
1 & 0 & 1 & - & 0 & 1 & - & 0 & 1 & - & 0 & 1
0 & 0 & 0 & - & - & - & - & - & - & - & - & - & -
0 & - & - & - & 0 & 0 & 0 & - & - & - & - & - & -
0 & - & - & - & 1 & 1 & 1 & 0 & 0 & 0 & - & -
- & 0 & - & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & - & -
- & 0 & - & 1 & 0 & - & 1 & 0 & 1 & 0 & - & - & -
- & 0 & - & 1 & 0 & - & 1 & 0 & 1 & 0 & - & - & -
- & 0 & - & 1 & 0 & - & 1 & 0 & 1 & 0 & - & - & -
- & 1 & 0 & - & 1 & 0 & 1 & 0 & - & - & - & - & -
- & 1 & 0 & - & 1 & 0 & 1 & 0 & - & - & - & - & -
- & 1 & 0 & - & 1 & 0 & 1 & 0 & - & - & - & - & -
- & 1 & 0 & - & 1 & 0 & 1 & 0 & - & - & - & - & -
- & 1 & 0 & - & 1 & 0 & 1 & 0 & - & - & - & - & -
\end{bmatrix}$$

**Remark** The construction of ternary codes is recursive. The matrix $C$ obtained in the preceding construction is used next to show that $A_3(40, 27, 27) = 80$, etc.

The first class of optimal ternary codes which will be used in the proof of the second class is introduced next.

**Theorem 10** For any odd prime power $p$ and positive integer $m$

$$A_3\left(p \left(\frac{p^{m+1} - 1}{p - 1}\right), p^m \left(\frac{p + 3}{2}\right), p^{m+1} - 1\right) = p^{m+1}.$$  

**Proof** Let $D$ be the Jacobsthal matrix of order $p$ described in Theorem 2 and note that the distance between any two rows is $\frac{p^{m+2} - 1}{2}$. Consider the orthogonal array $O$ in $p$ symbols of Theorem 8 corresponding to the positive integer $m$. By replacing the $p$ symbols with the rows of $D$, a $p^{m+1} \times p\left(\frac{p^{m+1} - 1}{p - 1}\right)$ array, say $T_{p^{m+1}}$, is obtained in which any two distinct rows have exactly $\frac{p^{m+1} - 1}{p - 1}$ rows of $D$ in the same columns. Noting this, a careful calculation shows that the distance between any two rows, considered as ternary codes, is

$$\left(p^{m+1} - 1\right) \left(p^{m} - 1\right) \left(\frac{1}{p - 1}\right) = p^m \left(\frac{p + 3}{2}\right).$$

The weight of each codeword is $w = \left(\frac{p^{m+1} - 1}{p - 1}\right)(p - 1) = p^{m+1} - 1$. The rows of $T_{p^{m+1}}$ form the codewords of constant weight ternary code of length $n = p\left(\frac{p^{m+1} - 1}{p - 1}\right)$, minimum distance $d = p^m \left(\frac{p + 3}{2}\right)$ and weight $w = p^{m+1} - 1$. The condition for the Johnson bound (1)
is obtained to be $3w^2 - 4nw + 2nd = \binom{p+3}{p-1}(p^{m+1} - 1)$. This and the Johnson bound show that

$$A_3 \left( p \left( \frac{p^{m+1} - 1}{p - 1} \right), p^m \left( \frac{p + 3}{2} \right), p^{m+1} - 1 \right) \leq p^{m+1}.$$ 

This completes the proof. \hfill \square

**Remark** The authors were unaware of a proof of Theorem 10 in Proposition 5.4 of [3], which was disclosed by an anonymous referee.

The second class of optimal ternary codes is introduced next.

**Theorem 11** There is a weighing matrix $W$ of order $n = \frac{p^{m+1} - 1}{p - 1}$ and weight $p^m$ for which the rows of $W$ and $-W$ together form the codewords of an optimal constant weight ternary code of length $n$, weight $w = p^m$, and minimum distance $d = p^{m-1} \left( \frac{p+3}{2} \right)$ for each odd prime power $p$ and integer $m \geq 1$ demonstrating that

$$A_3 \left( \frac{p^{m+1} - 1}{p - 1}, p^{m-1} \left( \frac{p + 3}{2} \right), p^m \right) = 2 \left( \frac{p^{m+1} - 1}{p - 1} \right).$$ 

**Proof** The proof is by induction on $m$. For $m = 1$ the statement is to show that $A_3(p + 1, \frac{p+3}{2}, p) = 2(p + 1)$. By Theorems 2, 3 and Corollary 6, there is a conference matrix $C = W(p + 1, p)$ for which $A_3(p + 1, \frac{p+3}{2}, p) = 2(p + 1)$.

Assuming the existence of a weighing matrix $C_m$ of order $n = \frac{p^{m+1} - 1}{p - 1}$, weight $w = p^m$ and minimum distance $d = p^{m-1} \left( \frac{p+3}{2} \right)$ we proceed in three steps.

1. Considering the $p^{m+1} \times p\left( \frac{p^{m+1} - 1}{p - 1} \right)$ matrix $T_{p^{m+1}}$ as noted in Theorem 10, the rows of $T_{p^{m+1}}$ form $p^{m+1}$ codewords of length $p\left( \frac{p^{m+1} - 1}{p - 1} \right)$, weight $w = p^{m+1} - 1$, and minimum distance $p^m \left( \frac{p+3}{2} \right)$.

2. Let $R = C_m \otimes \left[ I_p \right]$. $R$ is a $p^{m+1} - 1 \times p\left( \frac{p^{m+1} - 1}{p - 1} \right)$ $(0, \pm 1)$-matrix. The weight of each row is $w = p(p^m) = p^{m+1}$ and the minimum distance is $d = p^m \left( \frac{p+3}{2} \right)$.

3. The rows of the matrices

$$W = \begin{bmatrix} 0_{p^{m+1} - 1} & R & 1_{p^{m+1}} \end{bmatrix},$$

and $-W$ provide all the codewords.

There remains to show that the distance between one codeword from $W$ and one from $-W$ is not smaller than $p^{m-1} \left( \frac{p+3}{2} \right)$. The distance of a codeword $k$ in $W$ from a codeword $-j$ in $-W$ is either $p^{m+1}$ or $p^m \left( \frac{p+3}{2} \right)$ depending on whether $k = j$ or $k \neq j$, respectively. Since $\frac{p+3}{2} \leq p$ for all $p \geq 3$ it follows that $\left( \frac{p+3}{2} \right)p^m \leq p^{m+1}$. This completes the proof. \hfill \square

**Remark** The ternary codes from Theorem 11 corresponding to $p = 3$ are in addition equidistant. So, there is an optimal set of equidistant constant weight ternary codes consisting of $3^{m+1} - 1$ codewords of length $n = \frac{3^{m+1} - 1}{2}$, weight $w = 3^m$ and minimum distance $d = 3^m$ for each positive integer $m$. All other ternary codes obtained from Theorem 11 are bidistance codes.

The method used in the construction of optimal constant weight ternary codes in Theorems 10 and 11 seems extendable to the binary and $q$-ary codes.
4 Ternary codes from balanced weighing matrices

A weighing matrix \( W(v, k) = [w_{ij}] \) is said to be balanced if the matrix of absolute values \( \| w_{ij} \| \) is the incidence matrix of a symmetric balanced incomplete design with parameters \((v, k, \lambda)\), see [7] for details. To emphasize that a weighing matrix is balanced it is denoted by \( BW(v, k, \lambda) \), where \( \lambda = \frac{k(k-1)}{v-1} \). Examples of balanced weighing matrices include conference matrices, those with classical parameters \( W \left( \frac{p^{n+1}-1}{p-1}, p^m \right) \) and a few others. The balanced structure of weighing matrices used in previous sections is instrumental in the generation of optimal codes. A natural question is if all balanced weighing matrices lead to optimal codes. The answer in general depends on the parameters \((v, k, \lambda)\) and it seems to be difficult.

**Lemma 12** Let \( C_v \) be the set of ternary codes consisting of the rows of a \( BW(v, k, \lambda) = [w_{ij}] \). Then the distance between the codewords is the constant

\[
d_3 = \frac{4k(v-1) - 3k(k-1)}{2(v-1)},
\]

and the distance between the binary codewords consisting of the rows of \( |W| = \| w_{ij} \| \) is the constant

\[
d_2 = 2 \left( k - \frac{k(k-1)}{v-1} \right).
\]

**Proof** The matrix \( |W| = \| w_{ij} \| \) is the incidence matrix of a symmetric \((v, k, \lambda)\) design, where \( \lambda = \frac{k(k-1)}{v-1} \). Any two distinct binary codewords have \( \lambda \) ones in the same columns and the remaining \( k-\lambda \) ones have zero in the same columns. This shows that \( d_2 = 2 \left( k - \frac{k(k-1)}{v-1} \right) \). The same arrangement happens for the codewords in \( C_v \). Since the two rows are orthogonal, there are \( \lambda/2 \) (note that this forces \( \lambda \) to be even) \(-1\)’s in the same column contributing the same number to the distance in addition to the \( d_2 \). Therefore,

\[
d_3 = 2 \left( k - \frac{k(k-1)}{v-1} \right) + \frac{k(k-1)}{2(v-1)} = \frac{4k(v-1) - 3k(k-1)}{2(v-1)}.
\]

This completes the proof. \( \square \)

In the remaining part of the paper, the constants \( d_2 \) and \( d_3 \) are used in reference to Lemma 12.

**Theorem 3** is extended to the following. The rows of the derived part of a balanced weighing matrix form an optimal constant weight ternary code.

**Theorem 13** Let \( (v, k, \frac{k(k-1)}{v-1}) \) be the parameters of a balanced weighing matrix.

Then

\[
A_3(v-1, d_3, k-1) = k.
\]

**Proof** Let \( W \) be a balanced weighing matrix \( BW(v, k, \lambda) \). The rows of the derived part of \( W \) form an optimal constant weight ternary code of length \( n = v-1 \), constant weight \( w = k-1 \) and the constant distance \( d_3 \).

The condition \( 3w^2 - 4nw + 2nd_3 = 4(v-1) - 3(k-1) > 0 \) by the assumption for the given parameters. By inequality (1) of Theorem 1

\[
A_3(v-1, d_3, k-1) \leq \frac{4k(v-1) - 3k(k-1)}{4(v-1) - 3(k-1)} = k.
\]

This completes the proof. \( \square \)
An infinite family \(BW(1 + 18 \cdot \frac{9^m + 1}{8}, 9^m + 1, 4 \cdot 9^m)\) including \(BW(19, 9, 4)\) was constructed in [5]. As a corollary, we obtain:

**Corollary 14** \(A_3(18 \cdot \frac{9^m + 1}{8}, 12 \cdot 9^m, 9^m + 1 - 1) = 9^m + 1\) for every non-negative integer \(m\).

**Theorem 15** Let \((v, k, \frac{k(k-1)}{v-1})\) be the parameters of a balanced weighing matrix for which \(v > \frac{3k-1}{2}\). Then

\[v \leq A_3(v, d_3, k) < 2v.\]

**Proof** Working out the condition in the restricted Johnson bound Theorem 1

\[3k^2 - 4vk + 2vd_3 = 3k \left( k - \frac{v(k-1)}{v-1} \right),\]

which is positive. The upper bound from Theorem 1 is \(\frac{v(4v-3k-1)}{3(v-k)}\). It follows that

\[v \leq A_3(v, d_3, k) < 2v\]

if and only if \(v > \frac{3k-1}{2}\). \(\square\)

**Remark** There is a \(BW(19, 9, 4)\), see [5]. The rows of the derived part of this weighing matrix form an optimal constant weight 8 and equidistant \(d = 12\) ternary codes of length 18 by Theorem 13. The condition in Theorem 15 holds and

\[19 \leq A_3(19, 12, 9) < 38.\]

The Johnson upper bound in Theorem 1 provides a smaller number of 30 for the possible number of codewords.

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