Greed is Still Good: Maximizing Monotone Submodular+Supermodular Functions

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Abstract

We analyze the performance of the greedy algorithm, and also a discrete semi-gradient based algorithm, for maximizing the sum of a submodular and supermodular (BP) function (both of which are non-negative monotone non-decreasing) under two types of constraints, either a cardinality constraint or $p \geq 1$ matroid independence constraints. These problems occur naturally in several real-world applications in data science, machine learning, and artificial intelligence. The problems are ordinarily inapproximable to any factor (as we show). Using the curvature $\kappa_f$ of the submodular term, and introducing $\kappa_g$ for the supermodular term (a natural dual curvature for supermodular functions), however, both of which are computable in linear time, we show that BP maximization can be efficiently approximated by both the greedy and the semi-gradient based algorithm. The algorithms yield multiplicative guarantees of $\frac{1}{\kappa_f} \left[ 1 - e^{-(1-\kappa_g)\kappa_f} \right]$ and $\frac{1-\kappa_g}{(1-\kappa_g)\kappa_f + p}$ for the two types of constraints respectively. For pure monotone supermodular constrained maximization, these yield $1 - \kappa_g$ and $(1 - \kappa_g)/p$ for the two types of constraints respectively. We also analyze the hardness of BP maximization and show that our guarantees match hardness by a constant factor and by $O(\ln(p))$ respectively. Computational experiments are also provided supporting our analysis.
1 Introduction

The Greedy algorithm [3, 8] is a technique in combinatorial optimization that makes a locally optimal choice at each stage in the hope of finding a good global solution. It is one of the simplest, most widely applied, and most successful algorithms in practice [32, 58, 31, 47, 57]. Due to its simplicity, and low time and memory complexities, it is used empirically even when no guarantees are known to exist although, being inherently myopic, the greedy algorithm’s final solution can be arbitrarily far from the optimum solution [2].

On the other hand, there are results going back many years showing where the greedy algorithm is, or almost is, optimal, including Huffman coding [25], linear programming [13, 11], minimum spanning trees [36, 46], partially ordered sets [16, 11], matroids [15, 12], greedoids [34], and so on, perhaps culminating in the association between the greedy algorithm and submodular functions [14, 45, 7, 21].

Submodular functions have recently shown utility for a number of machine learning and data science applications such as information gathering [35], document summarization [39], image segmentation [33], and string alignment [40], since such functions are natural for modeling concepts such as diversity, information, and dispersion. Defined over an underlying ground set $V$, a set function $f : 2^V \rightarrow \mathbb{R}$ is said to be submodular when for all subsets $X, Y \subseteq V$, $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$. Defining $f\{v\}|X = f\{v\} \cup X) - f(X)$ as the gain of adding the item $v$ in the context of $X \subseteq V$, an equivalent characterization of submodularity is via diminishing returns: $f\{v\}|X) \geq f\{v\}|Y)$, for any $X \subseteq Y \subseteq V$ and $v \in V \setminus Y$. A set function $f$ is monotonically non-decreasing if $f\{v\}|S) \geq 0$ for all $v \in V \setminus S$ and it is normalized if $f(\emptyset) = 0$. In addition to being useful utility models, submodular functions also have amiable optimization properties — many submodular optimization problems (both maximization [57] and minimization [9]) admit polynomial time approximation or exact algorithms. Most relevant presently, the greedy algorithm has a good constant-factor approximation guarantee, e.g., the classic $1 - 1/e$ and $1/(p + 1)$ guarantees for submodular maximization under a cardinality constraint or $p$ matroid constraints [44, 20].

Certain subset selection problems in data science are not purely submodular, however. For example, when choosing a subset of training data in a machine learning system [56], there might be not only redundancies but also complementarities amongst certain subsets of elements, where the full collective utility of these elements are seen only when utilized together. Submodular functions can only diminish, rather than enhance, the utility of a data item in the presence other data items. Supermodular set functions can model such phenomena, and are widely utilized in economics and social sciences, where the notion of complementary [53] is naturally needed, but are studied and utilized less frequently in machine learning. A set function $g(X)$ is said to be supermodular if $-g(X)$ is submodular.

In this paper, we advance the state of the art in understanding when the greedy (and the semigradient) algorithm offers a guarantee, in particular for approximating the constrained maximization of an objective that may be decomposed into the sum of a submodular and a supermodular function (applications are given in Section 1.1). That is, we consider the following problem

$$\text{Problem 1. } \max_{X \in \mathcal{C}} h(X) := f(X) + g(X),$$

where $\mathcal{C} \subseteq 2^V$ is a family of feasible sets, $f$ and $g$ are normalized ($f(\emptyset) = 0$), monotonic non-decreasing ($f\{s\}|S) \geq 0$ for any $s \in V$ and $S \subseteq V$) submodular and supermodular functions respectively and hence are non-negative. We call this problem submodular-supermodular (BP) maximization, and $f + g$ a BP function, and we say $h$ admits a BP decomposition if $\exists f, g$ such that $h = f + g$ where $f$ and $g$ are defined as above. In the paper, the set $\mathcal{C}$ may correspond

\[\text{Throughout, } f & g \text{ are assumed monotonic non-decreasing submodular/supermodular functions respectively.}\]
either to a cardinality constraint (i.e., \( \mathcal{C} = \{ A \subseteq V \mid |A| \leq k \} \) for some \( k \geq 0 \)), or alternatively, a more general case where \( \mathcal{C} \) is defined as the intersection of \( p \) matroids. Hence, we may have \( \mathcal{C} = \{ X \subseteq V \mid X \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \cdots \cap \mathcal{I}_p \} \), where \( \mathcal{I}_i \) is the set of independent sets for the \( i \)-th matroid \( \mathcal{M}_i = (V, \mathcal{I}_i) \). A matroid generalizes the concept of independence in vector spaces, and is a pair \((V, \mathcal{I})\) where \( V \) is the ground set and \( \mathcal{I} \) is a family of subsets of \( V \) that are independent with the following three properties: (1) \( \emptyset \in \mathcal{I} \); (2) \( Y \in \mathcal{I} \) implies \( X \in \mathcal{I} \) for all \( X \subseteq Y \subseteq V \); and (3) if \( X, Y \in \mathcal{I} \) and \( |X| > |Y| \), then there exists \( v \in X \setminus Y \) such that \( Y \cup \{v\} \in \mathcal{I} \). Matroids are often used as combinatorial constraints, where a feasible set of an optimization problem must be independent in all \( p \) matroids.

The performance of the greedy algorithm for some special cases of BP maximization has been studied before. For example, when \( g(X) \) is modular, the problem reduces to submodular maximization where, if \( f \) and \( g \) are also monotone, the greedy algorithm is guaranteed to obtain an \( 1 - 1/e \) approximate solution under a cardinality constraint [44] and \( 1/p+1 \) for \( p \) matroids [20, 7]. The greedy algorithm often does much better than this in practice. Correspondingly, the bounds can be significantly improved if we also make further assumptions on the submodular function. One such assumption is the (total) curvature, defined as \( \kappa_f = 1 - \min_{v \in V} \frac{f(v \setminus \{v\})}{f(v)} \) — the greedy algorithm has a \( \frac{1}{\kappa_f} (1 - e^{-\kappa_f}) \) and a \( \frac{1}{\kappa_f + p} \) guarantee [7] for a cardinality and for \( p \) matroid constraints, respectively. Curvature is also attractive since it is linear time computable with only oracle function access. Liu et al. [41] shows that \( \kappa_f \) can be replaced by a similar quantity, i.e., \( b = 1 - \min_{v \in A \subseteq \mathcal{I}} \frac{f(|A| \setminus \{v\})}{f(A)} \) for a single matroid \( \mathcal{M} = (V, \mathcal{I}) \), a quantity defined only on the independent sets of the matroid, thereby improving the bounds further. In the present paper, however, we utilize the traditional definition of curvature. The current best guarantee is \( 1 - \kappa_f/e \) for a cardinality constraint using modifications of the continuous greedy algorithm [51] and \( \frac{1}{1 + p} \) for multiple matroid constraints based on a local search algorithm [37]. In another relevant result, Sarpatwar et al. [48] gives a bound of \( (1-e^{-\frac{1}{p+1}})/(p+1) \) for submodular maximization with a single knapsack and the intersection of \( p \) matroid constraints.

When \( g(X) \) is not modular, the problem is much harder and is NP-hard to approximate to any factor (Lemma 3.1). In our paper, we show that bounds are obtainable if we make analogous further assumptions on the supermodular function \( g \). That is, we introduce a natural curvature notion to monotone non-decreasing nonnegative supermodular functions, defining the supermodular curvature as \( \kappa^g = \kappa_g(V) - g(V \setminus X) = 1 - \min_{v \in V} \frac{g(v)}{g(v \setminus \{v\})} \). We note that \( \kappa^g \) is distinct from the steepness [26, 51] of a nonincreasing supermodular function (see Section 3.1). The function \( g(V) - g(V \setminus X) \) is a normalized monotone non-decreasing submodular function, known as the submodular function dual to the supermodular function \( g \) [21]. Supermodular curvature is a natural dual to submodular curvature and, like submodular curvature, is computationally feasible to compute, requiring only linear time in the oracle model, unlike other measures of non-submodularity (Section 1.2). Hence, given a BP decomposition of \( h = f + g \), it is possible, as we show below, to derive practical and useful quality assurances based on the curvature of each component of the decomposition.

We examine two algorithms, GREED MAX (Alg. 1) and SEMI GRAD (Alg. 2) and show that, despite the two algorithms being different, both of them have a worst case guarantee of \( \frac{1}{\kappa_f} \left[ 1 - e^{-(1-\kappa^g)\kappa_f} \right] \) for a cardinality constraint (Theorem 3.7) and \( \frac{1 - \kappa^g}{(1 - \kappa^g)\kappa_f + p} \) for \( p \) matroid constraints (Theorem 3.10). If \( \kappa^g = 0 \) (i.e., \( g \) is modular), the bounds reduce to \( \frac{1}{\kappa_f} (1 - e^{-\kappa_f}) \) and \( \frac{1}{\kappa_f + p} \), which recover the aforementioned bounds. If \( \kappa^g = 1 \) (i.e., \( g \) is fully curved) the bounds are 0 since, in general, the problem is NP-hard to approximate (Lemma 3.1). For pure monotone supermodular function maximization, the bounds yield \( 1 - \kappa^g \) and \( (1 - \kappa^g)/p \) respectively. We also show that no polynomial algorithm can do better than \( 1 - \kappa^g + \epsilon \) or \( (1 - \kappa^g)O\left(\frac{\log p}{p}\right) \) for cardinality or multiple matroid
| cardinality constraint | bound | hardness |
|------------------------|-------|----------|
|                        | $\frac{1}{\kappa_f} \left[ 1 - e^{-(1-\kappa^g)\kappa_f} \right]$ | $1 - \kappa^g + \epsilon$ |
| $p$ matroid constraints| $\frac{1-\kappa^g}{(1-\kappa^g)\mu_f + p}$ | $(1 - \kappa^g)O\left(\frac{\ln(p)}{p}\right)$ |

Table 1: Lower bounds for GreedMax (Alg. 1)/SemiGrad (Alg. 2) and BP maximization hardness.

Constraints respectively unless $P=NP$. Therefore, no polynomial algorithm can beat GreedMax by a factor of $\frac{1+\epsilon}{1-e^{-\epsilon}}$ or $O(\ln(p))$ for the two constraints unless $P=NP$.

### 1.1 Applications

Problem 1 naturally applies to a number of machine learning and data science applications.

**Summarization with Complementarity** Submodular functions are an expressive set of models for summarization tasks where they capture how data elements are mutually redundant. In some cases, however, certain subsets might be usefully chosen together, i.e., when their elements have a complementary relationship. For example, when choosing a subset of training data samples for supervised machine learning system [56], nearby points on opposite sides of a decision boundary would be more useful to characterize this boundary if chosen together. Also, for the problem of document summarization [39, 38], where a subset of sentences is chosen to represent a document, there are some cases where a single sentence makes sense only in the context of other sentences, an instance of complementarity. In such cases, it is reasonable to allow these relationships to be expressed via a monotone supermodular function. One such complementarity family takes $g$ to be a weighted sum of monotone convex functions composed with non-negative modular functions, as in $g(A) = \sum_i w_i \psi_i(m_i(A))$. A still more expressive family includes the “deep supermodular functions” [5] which consist of multiple nested layers of such transformations. A natural formulation of the summarization with complementary problem is to maximize an objective that is the weighted sum of a monotone submodular utility function and one of the above complementarity functions. Hence, such a formulation is an instance of Problem 1. In either case, the supermodular curvature is easy to compute, and for many instances is less than unity leading to a quality assurance based on the results of this paper.

**Generalized Bipartite Matching** Submodularity has been used to generalize bipartite matching. For example, a generalized bipartite matching [40] procedure starts with a non-negative weighted bipartite graph $(V,U,E)$, where $V$ is a set of left vertices, $U$ is a set of right vertices, $E \subseteq V \times U$ is a set of edges, and $h : 2^E \rightarrow \mathbb{R}_+$ is a score function on the edges. Note that a matching constraint is an intersection of two partition matroid constraints, so a matching can be generalized to the intersection of multiple matroid constraints. Word alignment between two sentences of different languages [42] can be viewed as a matching problem, where each word pair is associated with a score reflecting the desirability of aligning that pair, and an alignment is formed as the highest scored matching under some constraints. Lin and Bilmes [40] use a submodular objective functions that can represent complex interactions among alignment decisions. Also in [1], similar bipartite matching generalizations are used for the task of peptide identification in tandem mass spectrometry. By utilizing a BP function in Problem 1, our approach can extend this to allow also for complementarity to be represented amongst sets of matched vertices.
1.2 Approach, and Related Studies

An arbitrary set function can always be expressed as a difference of submodular (DS) functions [43, 27]. Although finding such a decomposition itself can be hard [27], the decomposition allows for additional optimization strategies based on discrete semi-gradients (Equation (2)) that do not offer guarantees, even in the unconstrained case [27]. Our problem is a special case of constrained DS optimization since a negative submodular function is supermodular. Our problem also asks for a BP decomposition of $h$ which is not always possible even for monotone functions (Lemma 3.2). Constrainedly optimizing an arbitrary monotonic non-decreasing set function is impossible in polynomial time and not even approximable to any positive factor (Lemma 3.1). In general, there are two ways to approach such a problem: one is to offer polynomial time heuristics without any theoretical guarantee (and hence possibly performing arbitrarily poorly in worst case); another is to analyze (using possibly exponential time itself, e.g., see below starting with the submodularity ratio) the set function in order to provide theoretical guarantees. In our framework, as we will see, the BP decomposition not only allows for additional optimization strategies as does a DS decomposition, but also, given additional information about the curvature of the two components (computable easily in linear time), allows us to show how the set function can be approximately maximized in polynomial time with guarantees. With a curvature analysis, not only the greedy algorithm but also a semi-gradient optimization strategy (Alg. 2) attains a guarantee even in the constrained setting. We also argued, in Section 1.1, that BP functions, even considering their loss of expressivity relative to DS functions, are still quite natural in applications.

Submodularity ratio and curvature Bian et al. [4] introduced a form of bound based on both the submodularity ratio and introduced a generalized curvature. The submodularity ratio [10] of a non-negative set function $h$ is defined as the largest scalar $\gamma$ s.t. $\sum_{\omega \subseteq \Omega \setminus S} h(\Omega \setminus S) \geq \gamma h(\omega | S), \forall \Omega, S \subseteq V$ and is equal to one if and only if $h$ is submodular. It is often defined as $\gamma_{U,k}(h) = \min_{L \subseteq U, |S| \leq k, S \cap L = \emptyset} \sum_{x \in S} h(x | L)$ for $U \subseteq V$ and $1 \leq k \leq |V|$, and then $\gamma = \gamma_{V,|V|}(h)$. The generalized curvature [4] of a non-negative set function $h$ is defined as the smallest scalar $\alpha$ s.t. $h(i | S \setminus \{i\} \cup \Omega) \geq (1 - \alpha) h(i | S \setminus \{i\}), \forall \Omega, S \subseteq V, i \in S \setminus \Omega$. [4] offers a lower bound of $\frac{1}{\alpha}(1 - e^{-\alpha \gamma})$ for the greedy algorithm. Computing this bound is not computationally feasible in general because both the submodularity ratio and the generalized curvature are information theoretically hard to compute under the oracle model, as we show in Section J.2. This is unlike curvatures $\kappa_f, \kappa^g$ which are both computable in linear time given only oracle access to both $f$ and $g$. We make further comparisons between the pair $\kappa_f, \kappa^g$ with the submodularity ratio in Section J.

Approximately submodular functions A function $h$ is said to be $\epsilon$-approximately submodular if there exists a submodular function $f$ such that $(1 - \epsilon)f(S) \leq h(S) \leq (1 + \epsilon)f(S)$ for all subsets $S$. Horel and Singer [24] show that the greedy algorithm achieves a $(1 - 1/e - O(\delta))$ approximation ratio when $\epsilon = \frac{\delta}{e}$. Furthermore, this bound is tight: given a $1/k^{1-\delta}$-approximately submodular function, the greedy algorithm no longer provides a constant factor approximation guarantee.

Elemental Curvature and Total Primal Curvature Wang et al. [55] analyze the approximation ratio of the greedy algorithm on maximizing non-submodular functions under cardinality constraints. Their bound is $1 - \left(1 - \left(\sum_{i=1}^{k} \alpha^i\right)^{-1}\right)^k$ based on the elemental curvature with $\alpha = \max_{S \subseteq X, i \in X} \frac{f(i | S \setminus \{i\})}{f(S)}$, and $\alpha^i$ the $i$th power of $\alpha$. Smith and Thai [49] generalize this definition to total primal curvature, $\Gamma(x | B, A) = \frac{f(x | A \cup B)}{f(x | A)}$ and define an estimator $\hat{\Gamma}(i, S)$ satisfying
\[ \forall |T| \leq k, S \subseteq T, i = |T \setminus S|, x \notin T \cup S : \Gamma(x|T, S) \leq \hat{\Gamma}(i, S) + \epsilon_i. \]

They claim a bound of \[ 1 + \left( \frac{f(S^*)}{f(S)} - 1 \right) \sum_{i=0}^{k-1} (\hat{\Gamma}(i, S) + \epsilon_i) \] \[ f(S^*) \leq f(S) \] where \( S \) is the greedy solution, and \( S^+ \) is the greedy solution for an identical problem for \( k + 1 \) cardinality constraints. They also claim that finding a deterministic strict estimator \( \hat{\Gamma} \) is not feasible and therefore, they provide an algorithm for finding a probabilistic estimator based on Monte-Carlo simulation.

**Supermodular Degree** Feige and et al. [17] introduce a parameter, the supermodular degree, for solving the welfare maximization problem. Feldman and et al. [19, 18] use this concept to analyze monotone set function maximization under a \( p \)-extendable system constraint with guarantees. A supermodular degree of one element \( u \in V \) by a set function \( h \) is defined as the cardinality of the set \( D_h^+(u) = \{ v \in V | \exists S \subseteq V, h(u|S + v) > h(u|S) \} \), containing all elements whose existence in a set might increase the marginal contribution of \( u \). The supermodular degree of \( h \) is \( D_h^+ = \max_{u \in V} |D_h^+(u)| \). A set system \( (V, \mathcal{I}) \) is called \( p \)-extendable [19, 18] if for every two subsets \( T \subseteq S \subseteq I \) and element \( u \notin T \) for which \( T \cup u \in \mathcal{I} \), there exists a subset \( Y \subseteq S \setminus T \) of cardinality at most \( p \) for which \( S \setminus Y + u \in \mathcal{I} \), which is a generalization of the intersection of \( p \) matroids. They offer a greedy algorithm for maximizing a monotonic non-decreasing set function \( h \) subject to a \( p \)-extendable system with an guarantee of \( \frac{1}{p(D_h^++1)+1} \) and time complexity polynomial in \( n \) and \( 2^{D_h^+} \) [19, 18], where \( n = |V| \). But again, \( D_h^+ \) can not be calculated in polynomial time in general unlike our curvatures. Moreover, if we consider a simple supermodular function \( g(X) = |X|^{\alpha} \) where \( \alpha \) is a small positive number. Then \( D_h^+ = n - 1 \) since all elements have supermodular interactions. Therefore, the time complexity of their algorithm is polynomial in \( 2^{n-1} \) and their bound is \( \frac{1}{pn+1} \), while our algorithm requires at most \( n^2 \) quires with a performance guarantee of \( \frac{1-\log(n)\kappa^\alpha}{p} \) where \( \kappa^\alpha = 1 - \frac{1}{n^{\alpha} - (n-1)^{\alpha}} \). When \( \alpha \) is small, our bound is around \( n \) times better than theirs; e.g., \( n = 10, \ p = 5, \ \alpha = 0.05 \), ours is around \( \frac{1}{7.61} \) while theirs is \( \frac{1}{5.5} \).

**Proportional Submodularity** Borodin et al. [6] define the notion of proportionally submodular functions defined as those set functions \( h \) satisfying \( |X|h(Y) + |Y|h(X) \geq |X \cap Y|h(X \cup Y) + |X \cup Y|h(X \cap Y) \) for all \( X, Y \subseteq V \). The class of proportionally submodular functions includes both submodular functions and also some supermodular functions, although there are instances of BP functions, e.g., \( h(X) = |X|^4 \), that are not proportionally submodular ([6] proposition 3.12).

**Discussion** The above results are both useful and complementary with our analyses below for BP-decomposable functions, thus broadening our understanding of settings where the greedy and semi-gradient algorithms offer a guarantee. We say our analysis is complementary in a sense the following example demonstrates. Should a given function \( h \) have a BP decomposition \( h = f + g \), then it is easy, given oracle access to both \( f \) and \( g \), to compute curvatures and establish bounds. On the other hand, if we do not know \( h \)'s BP decomposition, or if \( h \) does not admit a BP decomposition (Lemma 3.2), then we would need to resort, for example, to the submodularity ratio and generalized curvature bounds of Bian et al. [4].
2 Approximation Algorithms for BP Maximization

Algorithm 1: GreedMax for BP maximization

1: Input: $f$, $g$ and constraint set $C$.
2: Output: An approximation solution $\hat{X}$.
3: Initialize: $X_0 \leftarrow \emptyset$, $i \leftarrow 0$ and $R \leftarrow V$
4: while $\exists v \in R$ s.t. $X_i \cup v \in C$ do
5: $v \leftarrow \arg\max_{v \in R \cup \cup C} f(v|X_i) + g(v|X_i)$.
6: $X_{i+1} \leftarrow X_i \cup v$.
7: $R \leftarrow R \setminus v$.
8: $i \leftarrow i + 1$.
9: end while
10: Return $\hat{X} \leftarrow X_i$.

Algorithm 2: SemiGrad for BP maximization

1: Input: $f$, $g$, constraint set $C$ and an initial set $X_0$
2: Output: An approximation solution $\hat{X}$.
3: Initialize: $i \leftarrow 0$.
4: repeat
5: pick a semigradient $g_i$ at $X_i$ of $g$
6: $X_{i+1} \leftarrow \arg\max_{X \in C} f(X) + g_i(X) \setminus \frac{1}{\kappa_f}(1 - e^{-\kappa_f})$—Approximately solved by Alg. 1
7: $i \leftarrow i + 1$
8: until we have converged ($X_i = X_{i-1}$)
9: Return $\hat{X} \leftarrow X_i$

GreedMax (Alg. 1) The simplest and most well known algorithm for approximate constrained non-monotone submodular maximization is the greedy algorithm [44]. We show that this also works boundedly well for BP maximization when the functions are not both fully curved ($\kappa_f \leq 1, \kappa_g < 1$). At each step, a feasible element with highest gain with respect to the current set is chosen and added to the set. Finally, if no more elements are feasible, the algorithm returns the greedy set.

SemiGrad (Alg. 2) Akin to convex functions, supermodular functions have tight modular lower bounds. These bounds are related to the subdifferential $∂_g(Y)$ of the supermodular set function $g$ at a set $Y \subseteq V$, which is defined [21]

$$∂_g(Y) = \{ y \in \mathbb{R}^n : g(X) - y(X) \geq g(Y) - y(Y) \text{ for all } X \subseteq V \}$$

(2)

It is possible, moreover, to provide specific semigradients [29, 30] that define the following two modular lower bounds:

$$m_{g,X,1}(Y) \triangleq g(X) - \sum_{j \in X \setminus Y} g(j|X \setminus j) + \sum_{j \in Y \setminus X} g(j|\emptyset),$$

(3)

$$m_{g,X,2}(Y) \triangleq g(X) - \sum_{j \in X \setminus Y} g(j|V \setminus j) + \sum_{j \in Y \setminus X} g(j|X).$$

(4)

Then $m_{g,X,1}(Y), m_{g,X,2}(Y) \leq g(Y), \forall Y \subseteq V$ and $m_{g,X,1}(X) = m_{g,X,2}(X) = g(X)$. Removing constants yields normalized non-negative (since $g$ is monotone) modular functions for $g_i$ in Alg. 2.

Having formally defined the modular lower bound of $g$, we are ready to discuss how to apply this machinery to BP maximization. SemiGrad consists of two stages. In the first stage, it is initialized by an arbitrary set (e.g., $\emptyset$, $V$, or the solution of GreedMax). In the second stage, SemiGrad replaces $g$ by its modular lower bound, and solves the resulting problem using GreedyMax. The algorithm repeatedly updates the set and calculates an updated modular lower bound until convergence.

Since SemiGrad does no worse than the arbitrary initial set, we may start with the solution of GreedMax and show that SemiGrad is always no worse than GreedMax. Interestingly, we

\[\text{(21)}\] defines the subdifferential of a submodular set function. The subdifferential definition for a supermodular set function takes the same form, although instances of supermodular subdifferentials (e.g., Eq. (3)-(4)) take a form different than instances of submodular subdifferentials.
obtain the same bounds for SEMIGRAD even if we start with the empty set (Theorems 3.11 and 3.12) despite that they may behave quite differently empirically and yield different solutions (Section 5).

3 Analysis of Approximation Algorithms for BP Maximization

We next analyze the performance of two algorithms GREEDMAX (Alg. 1) and SEMIGRAD (Alg. 2) under a cardinality constraint and under \( p \) matroid constraints. First, we claim that BP maximization is hard and cannot be approximately solved to any factor in polynomial time in general.

**Lemma 3.1.** [54] There exists an instance of a BP maximization problem that cannot be approximately solved to any positive factor in polynomial time.

**Proof.** For completeness, Appendix A offers a detailed proof based on [54].

It is also important to realize that not all monotone functions are BP-decomposable, as the following demonstrates.

**Lemma 3.2.** There exists a monotonic non-decreasing set function \( h \) that is not BP decomposable.

**Proof.** See Appendix B.

3.1 Supermodular Curvature

Although BP maximization is therefore not possible in general, we show next that we can get worst-case lower bounds using curvature whenever the functions in question indeed have limited curvature.

The (total) curvature of a submodular function \( f \) is defined as \( \kappa_f = 1 - \min_{v \in V} \frac{f(v|V \setminus \{v\})}{f(v)} \) [7]. Note that \( 0 \leq \kappa_f \leq 1 \) since \( 0 \leq f(v|V \setminus \{v\}) \leq f(v) \) and if \( \kappa_f = 0 \) then \( f \) is modular. We observed that for any monotonically non-decreasing supermodular function \( g(X) \), the dual submodular function [21] \( g(V) - g(V \setminus X) \) is always monotonically non-decreasing and submodular. Hence, the definition of submodular curvature can be naturally extended to supermodular functions \( g \):

**Definition 3.3.** The supermodular curvature of a non-negative monotone nondecreasing supermodular function is defined as \( \kappa^g = \kappa_{g(V)} - g(V \setminus X) = 1 - \min_{v \in V} \frac{g(v)}{g(v|V \setminus \{v\})} \).

For clarity of notation, we use a superscript for supermodular curvature and a subscript for submodular curvature, which also indicates the duality between the two. In fact, for supermodular curvature, we can recover the submodular curvature.

**Corollary 3.3.1.** \( \kappa_f = \kappa^f(V) - f(V \setminus X) \).

The dual form also implies similar properties, e.g., we have that \( 0 \leq \kappa^g \leq 1 \) and if \( \kappa^g = 0 \) then \( g \) is modular. In both cases, a form of curvature indicates the degree of submodularity or supermodularity. If \( \kappa_f = 1 \) (or \( \kappa^g = 1 \)), we say that \( f \) (or \( g \)) is fully curved. Intuitively, a submodular function is very (or fully) curved if there is a context \( B \) and element \( v \) at which the gain is close to (or equal to) zero \( (f(v|B) \approx 0) \), whereas a supermodular function is very (or fully) curved if there is an element \( v \) whose valuation is close to (or equal to) zero \( (g(v) \approx 0) \). We can calculate both submodular and supermodular curvature easily in linear time. Hence, given a BP decomposition of \( h = f + g \), we can easily calculate both curvatures, and the corresponding bounds, with only oracle access to \( f \) and \( g \).

**Proposition 3.4.** Calculating \( \kappa_f \) or \( \kappa^g \) requires at most \( 2|V| + 1 \) oracle queries of \( f \) or \( g \).
The steepness \cite{26, 51} of a monotone nonincreasing supermodular function \( g' \) is defined as 
\[ s = 1 - \min_{v \in V} \frac{g'(v) \mathbb{1}_{V \setminus \{v\}}}{g'(v) \mathbb{1}_{V \setminus \{v\}}} \]. Here, the numerator and denominator are both negative and \( g \) need not be normalized. Steepness has a similar mathematical form to the submodular curvature of a nondecreasing submodular function \( f \), i.e., \( \kappa_f = 1 - \min_{v \in V} \frac{f(v) \mathbb{1}_{V \setminus \{v\}}}{f(v) \mathbb{1}_{V \setminus \{v\}}} \), but is distinct from the supermodular curvature. Steepness may be used to offer a bound for the minimization of such nonincreasing supermodular functions \cite{51}, whereas we in the present work are interested in maximizing nondecreasing BP (and hence also supermodular) functions.

3.2 Theoretical Guarantees for GreedMax

Before analyzing specific constraints, we first analyze each step of GreedMax based on submodular and supermodular curvature.

The following holds for any chain of sets, not just those produced by the greedy algorithm.

**Lemma 3.5.** For any chain of solutions \( \emptyset = S_0 \subset S_1 \subset \ldots \subset S_k \), where \( |S_i| = i \), the following holds for all \( i = 0 \ldots k - 1 \),
\[ h(X^*) \leq \kappa_f \sum_{j : s_j \in S_i \setminus X^*} a_j + \sum_{j : s_j \in S_i \cap X^*} a_j + h(X^* \setminus S_i \setminus S_i) \tag{5} \]
where \( \{s_i\} = S_i \setminus S_{i-1}, a_i = h(s_i | S_{i-1}) \) and \( X^* \) is the optimal set.
Proof. See Appendix C.

3.2.1 Cardinality constraints

In this section, we provide a lower bound for Greedy maximization of a BP function under a cardinality constraint, inspired by the proof in \cite{7} where they focus only on submodular functions.

**Lemma 3.6.** GreedMax is guaranteed to obtain a solution \( \hat{X} \) such that
\[ h(\hat{X}) \geq \frac{1}{\kappa_f} \left[ 1 - \left( 1 - \frac{(1 - \kappa_g)\kappa_f}{k} \right)^k \right] h(X^*) \] \tag{6}
where \( X^* \in \arg\max_{|X| \leq k} h(X) \), \( h(X) = f(X) + g(X) \), \( \kappa_f \) is the curvature of submodular \( f \) and \( \kappa_g \) is the curvature of supermodular \( g \).
Proof. See Appendix D.

**Theorem 3.7.** Theoretical guarantee in the cardinality constrained case. GreedMax is guaranteed to obtain a solution \( \hat{X} \) such that
\[ h(\hat{X}) \geq \frac{1}{\kappa_f} \left[ 1 - e^{-(1-\kappa_g)\kappa_f} \right] h(X^*) \] \tag{7}
where \( X^* \in \arg\max_{|X| \leq k} h(X) \), \( h(X) = f(X) + g(X) \), \( \kappa_f \) is the curvature of submodular \( f \) and \( \kappa_g \) is the curvature of supermodular \( g \).
Proof. This follows Lemma 3.6 and uses the inequality \( (1 - \frac{a}{k})^k \leq e^{-a} \) for all \( a \geq 0 \) and \( k \geq 1 \).

Theorem 3.7 gives a lower bound of GreedMax in terms of the submodular curvature \( \kappa_f \) and the supermodular curvature \( \kappa_g \). We notice that this bound immediately generalizes known results and provides one new one.
1. $\kappa_f = 0, \kappa^g = 0$, $h(\hat{X}) = h(X^*)$. In this case, the BP problem reduces to modular maximization under a cardinality constraint, which is solved exactly by the greedy algorithm.

2. $\kappa_f > 0, \kappa^g = 0$, $h(\hat{X}) \geq \frac{1}{\kappa_f} [1 - e^{-\kappa_f}] h(X^*)$. In this case, BP problem reduces to submodular maximization under a cardinality constraint, and with the same $\frac{1}{\kappa_f} [1 - e^{-\kappa_f}]$ guarantee for the greedy algorithm [7].

3. If we take $\kappa_f \to 0$, we get $1 - \kappa^g$, which is a new curvature-based bound for monotone supermodular maximization subject to a cardinality constraint.

4. $\kappa^g = 1$, $h(\hat{X}) \geq 0$ which means, in the general fully curved case for $g$, this offers no theoretical guarantee for constrained BP or supermodular maximization, consistent with [54] and Lemma 3.1.

### 3.2.2 Weaker bound in the cardinality constrained case

The bound in Equation (7) is one of the major contributions of this paper. Another bound can be achieved using a surrogate objective $h'(X) = f(X) + \sum_{v \in X} g(v)$, similar to an approach used in [28]. We have that $h'(X) \leq h(X)$ thanks to the supermodularity of $g$, and we can apply GreedMax directly to $h'$, the solution of which has a guarantee w.r.t. the original objective $h$. The proof of this bound is quite a bit simpler, so we first offer it here immediately. On the other hand, we also show that the bound obtained by this method is worse than Equation (7) for all $0 < \kappa_f, \kappa^g < 1$, sometimes appreciably.

**Lemma 3.8. Weak bound in cardinality constrained case.** GreedMax maximizing $h'(X) = f(X) + \sum_{v \in X} g(v)$ is guaranteed to obtain a solution $\hat{X}$ such that

\[
h(\hat{X}) \geq \frac{1 - \kappa^g}{\kappa_f} [1 - e^{-\kappa_f}] h(X^*)
\]

where $X^* \in \text{argmax}_{|X| \leq k} h(X)$, $h(X) = f(X) + g(X)$, $\kappa_f$ is the curvature of submodular $f$ and $\kappa^g$ is the curvature of supermodular $g$.

**Proof.** According to lemma C.1 (iv), $(1 - \kappa^g) h(X) \leq h'(X)$ for all $X \subseteq V$. Also we have $h'(X) \leq h(X)$. And $h'$ is a monotone submodular function with $\kappa_{h'} = 1 - \min_{v \in V} \frac{g(v)}{f(v)} = 1 - \min_{v \in V} \frac{f(v \setminus \{v\}) + g(v)}{f(v)} \leq 1 - \min_{v \in V} \frac{f(v \setminus \{v\})}{f(v)} = \kappa_f$ since $0 \leq f(v \setminus \{v\}) \leq f(v)$.

Using the traditional curvature bound for submodular maximization [7], the greedy algorithm to maximize $h'$ provides a solution $\hat{X}$ s.t. $h'(\hat{X}) \geq \frac{1}{\kappa_{h'}} [1 - e^{-\kappa_{h'}}] h'(X^*)$ where $X^* \in \text{argmax}_{|X| \leq k} h(X)$. Thus, we have

\[
h(\hat{X}) \geq h'(\hat{X}) \geq \frac{1}{\kappa_{h'}} [1 - e^{-\kappa_{h'}}] h'(X^*) \geq \frac{1}{\kappa_f} [1 - e^{-\kappa_f}] h'(X^*) \geq \frac{1 - \kappa^g}{\kappa_f} [1 - e^{-\kappa_f}] h(X^*)
\]

Next, we show that this bound is almost everywhere worse than Equation (7).

**Lemma 3.9.** $\frac{1}{\kappa_f} [1 - e^{-(1-\kappa^g)\kappa_f}] \geq \frac{1 - \kappa^g}{\kappa_f} [1 - e^{-\kappa_f}]$ for all $0 \leq \kappa_f, \kappa^g \leq 1$ where equality holds if and only if $\kappa_f = 0$ or $\kappa^g = 0$ or $\kappa_f = 1$. For simplicity, dividing by 0 is defined using limits, e.g.,

\[
\frac{1}{\kappa_f} [1 - e^{-(1-\kappa^g)\kappa_f}] = \lim_{\kappa_f \to 0^+} \frac{1}{\kappa_f} [1 - e^{-(1-\kappa^g)\kappa_f}] = 1 - \kappa^g \text{ when } \kappa_f = 0.
\]
Proof. Let \( \phi(\kappa_f, \kappa^g) = \frac{1}{\kappa_f} \left[ 1 - e^{-(1-\kappa^g)\kappa_f} \right] \) and \( \psi(\kappa_f, \kappa^g) = \frac{1-\kappa^g}{\kappa_f} \left[ 1 - e^{-\kappa_f} \right] \). Specifically, \( \phi(0, \kappa^g) = \lim_{\kappa_f \to 0^+} \phi(\kappa_f, \kappa^g) = 1 - \kappa^g \) and \( \psi(0, \kappa^g) = \lim_{\kappa_f \to 0^+} \psi(\kappa_f, \kappa^g) = 1 - \kappa^g \). So if \( \kappa_f = 0 \), \( \phi(\kappa_f, \kappa^g) = \psi(\kappa_f, \kappa^g) \).

When \( 0 < \kappa_f \leq 1 \), we notice that \( \phi(\kappa_f, \kappa^g) = \psi(\kappa_f, \kappa^g) \) when \( \kappa^g = 0 \) or \( \kappa^g = 1 \). When \( 0 < \kappa^g < 1 \), we have \( \phi(\kappa_f, \kappa^g) > \psi(\kappa_f, \kappa^g) \) since \( \phi(\kappa_f, \kappa^g) \) is a strictly concave function in \( \kappa^g \) and \( \psi(\kappa_f, \kappa^g) \) is linear in \( \kappa^g \).

A simple computation shows the maximum ratio of these two bounds is \( 1/(1 - e^{-1}) \approx 1.5820 \) when \( \kappa_f = 1 \) and \( \kappa^g \to 1 \). As another example, with \( \kappa_f = 1 \) and \( \kappa^g = \ln(e - 1) \approx 0.541, \) the ratio is \( \approx 1.2688 \).

3.2.3 Multiple matroid constraints

Matroids are useful combinatorial objects for expressing constraints in discrete problems, and which are made more useful when taking the intersection of the independent sets of \( p > 1 \) matroids defined on the same ground set [44]. In this section, we show that the greedy algorithm on a BP function subject to \( p \) matroid independent constraints has a guarantee if \( g \) is not fully curved.

Theorem 3.10. Theoretical guarantee in the \( p \) matroids case. GREEDMAX is guaranteed to obtain a solution \( \hat{X} \) such that

\[
h(\hat{X}) \geq \frac{1 - \kappa^g}{(1 - \kappa^g)\kappa_f + p} h(X^*)
\]

where \( X^* \in \arg\max_{X \in M_1 \cap \cdots \cap M_p} h(X) \), \( h(X) = f(X) + g(X) \), \( \kappa_f \) is the curvature of submodular \( f \) and \( \kappa^g \) is the curvature of supermodular \( g \).

Proof. See Appendix E.

Theorem 3.10 gives a theoretical lower bound of GREEDMAX in terms of submodular curvature \( \kappa_f \) and supermodular curvature \( \kappa^g \) for the \( p \) matroid constraints case. Like in the cardinality case, this bound also generalizes known results and yields a new one.

1. \( \kappa_f = 0, \kappa^g = 0 \), \( h(\hat{X}) \geq \frac{1}{p} h(X^*) \). In this case, the BP problem reduces to modular maximization under \( p \) matroid constraints [7].

2. \( \kappa_f > 0, \kappa^g = 0 \), \( h(\hat{X}) \geq \frac{1}{p + \kappa_f} h(X^*) \). In this case, the BP problem reduces to submodular maximization under \( p \) matroid constraints [7].

3. If we take \( \kappa_f \to 0 \), we get \( (1 - \kappa^g)/p \), which is a new curvature-based bound for monotone supermodular maximization subject to a \( p \) matroid constraints.

4. \( \kappa^g = 1 \), \( h(\hat{X}) \geq 0 \) which means that, in general, there is no theoretical guarantee for constrained BP or supermodular maximization.

3.3 Theoretical guarantee of SemiGrad

In this section, we show a perhaps interesting result that SEMIGRAD achieves the same bounds as GREEDMAX even if we initialize SEMIGRAD with \( \emptyset \) and even though the two algorithms can produce quite different solutions (as demonstrated in Section 5).

Theorem 3.11. SEMIGRAD initialized with the empty set is guaranteed to obtain a solution \( \hat{X} \) for the cardinality constrained case such that

\[
h(\hat{X}) \geq \frac{1}{\kappa_f} \left[ 1 - e^{-(1-\kappa^g)\kappa_f} \right] h(X^*)
\]

where \( X^* \in \arg\max_{|X| \leq k} h(X) \), \( h(X) = f(X) + g(X) \), \( \emptyset \) \( \kappa_f \) (resp. \( \kappa^g \)) is the curvature of \( f \) (resp. \( g \)).
Figure 1: Guarantees of GreedMax for two constraint types. The x and y axes are \( \kappa_f \) and \( \kappa_g \), respectively, and the z axis is the guarantee. In (b), from top to bottom, the surfaces represent \( p = 2, 5, 10 \).

**Proof.** See Appendix F.

**Theorem 3.12.** SEMIGRAD initialized with the empty set is guaranteed to obtain a solution \( \hat{X} \), feasible for the \( p \) matroid constraints, such that

\[
h(\hat{X}) \geq \frac{1 - \kappa^g}{(1 - \kappa^g)\kappa_f + p} h(X^*)
\]

(13)

where \( X^* \in \text{argmax}_{X \in \mathcal{M}_1 \cap \ldots \cap \mathcal{M}_p} h(X) \), \( h = f + g \) with supermodular curvature \( \kappa^g = \beta \) (resp. \( \kappa^g \)) is the curvature of \( f \) (resp. \( g \)).

**Proof.** See Appendix G.

All the above guarantees are plotted in Figure 1 (in the matroid case for \( p = 2, 5, \) or 10 matroids).

### 4 Hardness

We next show that the curvature \( \kappa^g \) limits the polynomial time approximability of BP maximization.

**Theorem 4.1.** Hardness for cardinality constrained case. For all \( 0 \leq \beta \leq 1 \), there exists an instance of a BP function \( h = f + g \) with supermodular curvature \( \kappa^g = \beta \) such that no poly-time algorithm solving Problem 1 with a cardinality constraint can achieve an approximation factor better than \( 1 - \kappa^g + \epsilon \), for any \( \epsilon > 0 \).

**Proof.** See Appendix H.

For the \( p \) matroid constraints case, Hazan et al. [23] studied the complexity of approximating \( p \)-set packing which is defined as follows: given a family of sets over a certain domain, find the maximum number of disjoint sets, which is actually a special case of finding the maximum intersection of \( p \) matroids. They claim that this problem cannot be efficiently approximated to a factor better than \( O(\ln p/p) \) unless \( P = NP \). We generalize their result to BP maximization.
Figure 2: Empirical test of our guarantee. The upper and middle surface indicate the performance of SemiGrad and GreedMax respectively, and the lower surface is the theoretical worst case guarantee. (a) and (b) are two sets of experiments.

**Theorem 4.2.** Hardness for $p$ matroids constraint case. For all $0 \leq \beta \leq 1$, there exists an instance of a BP function $h = f + g$ with supermodular curvature $\kappa^g = \beta$ such that no poly-time algorithm can achieve an approximation factor better than $(1 - \kappa^g)O(\frac{\ln p}{p})$ unless P=NP.

*Proof.* See Appendix I.

**Corollary 4.2.1.** No polynomial algorithm can beat GreedMax or SemiGrad by a factor of $\frac{1+\epsilon}{1-e^{-\epsilon}}$ for cardinality, or $O(\ln(p))$ for $p$ matroid constraints, unless P=NP.

5 Computational Experiments

We empirically test our guarantees for BP maximization subject to a cardinality constraint on contrived functions using GreedMax and SemiGrad. For the first experiment, we let $|V| = 20$ set the cardinality constraint to $k = 10$, and partition the ground set into $|V_1| = |V_2| = k$, $V_1 \cup V_2 = V$ where $V_1 = \{v_1, v_2, \ldots, v_k\}$. Let $w_i = \frac{1}{\alpha} \left((1 - \frac{\alpha}{k})^i - (1 - \frac{\alpha}{k})^{i+1}\right)$ for $i = 1, 2, \ldots, k$. Then we define the submodular and supermodular functions as follows, $f(X) = \left[k - \alpha|X \cap V_2|\right] \sum_{i \in V_1 \cap X} w_i \cdot \frac{|X \cap V_2|}{k}$, $g(X) = |X| - \beta \min(1 + |X \cap V_1|, |X|, k) + \epsilon \max(|X|, |X| + \frac{\beta}{1-\beta}(|X \cap V_2| - k + 1))$ and $h(X) = \lambda f(X) + (1 - \lambda) g(X)$ for $0 \leq \alpha, \beta, \lambda \leq 1$ and $\epsilon = 1 \times 10^{-5}$. Immediately, we notice that $\kappa_f = \alpha$ and $\kappa^g = \beta$. In particular, we choose $\alpha, \beta, \lambda = 0, 0.01, 0.02, \ldots, 1$ and for all cases, we normalize $h(X)$ using either exhaustive search so that $\text{OPT} = h(X^*) = 1$. Since we are doing a proof-of-concept experiment to verify the guarantee, we are interested in the worst case performance at curvatures $\kappa_f$ and $\kappa^g$. In Figure 2(a), we see that both methods are always above the theoretical worst case guarantee, as expected. Interestingly, SemiGrad is doing significantly better than GreedMax demonstrating the different behavior of the algorithms, despite their identical guarantee. Moreover, the gap between GreedMax and the bound layer is small (the maximum difference is 0.1852), which suggests the guarantee for greedy may be almost tight in this case.
The above example is designed to show the tightness of GreedMax and the better potential performance of SemiGrad. For a next experiment, we again let $|V| = 20$ and $k = 10$, partition the ground set into $|V_1| = |V_2| = k$, $V_1 \cup V_2 = V$. Let $f(X) = |X \cap V_1|^\alpha$ and $g(X) = \max(0, \frac{|X \cap V_2| - \beta}{1-\beta})$ $0 \leq \alpha, \beta \leq 1$, and normalize $h$ (by exhaustive search) to ensure $\text{OPT} = h(X^*) = 1$. Immediately, we notice that the curvature of $f$ is $\kappa_f = 1 - k\alpha + (k - 1)\alpha$ and the curvature of $g$ is $\kappa_g = \beta$. The objective BP function is $h(X) = f(X) + g(X)$. We see that SemiGrad is again doing better than GreedMax in most but not all cases (Figure 2(b)) and both are above their bounds, as they should be.
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A Proof of Lemma 3.1

Lemma A.1. [54] There exists an instance of a BP maximization problem that cannot be approximately solved to any positive factor in polynomial time.

Proof. We consider the BP problem with ground set $n$ and a cardinality constraint $|X| \leq k = n/2$. Let $R \subseteq V$ be an arbitrary set with $|R| = k$. Let $f = 0$ and $g'(X) = \max(|X| - k, 0)$ so that $g'(X) = 0$ for all $|X| = k$. $g'(X)$ is clearly supermodular.

Let $g(X) = g'(X)$ for all $X \neq R$ but $g(R) = 0.5$. We notice that for $X \subset V$ and $v \notin X$, $g(v|X) = 0$ if $|X| \leq k - 2$, $g(v|X) = 0$ or $0.5$ if $|X| = k - 1$, $g(v|X) = 0.5$ or $1$ if $|X| = k$, and $g(v|X) = 1$ if $|X| \geq k + 1$. Immediately, we have for all $X \subset Y \subset V$ and $v \notin Y$, $g(v|X) \leq g(v|Y)$. Therefore, $g(X)$ is also supermodular.

Next, we use a proof technique similar to [52]. Note that $g'(X) = g(X)$ if and only if $X \neq R$. So for any algorithm maximizing $g(X)$, before it evaluates $g(R)$, all function evaluations are the same with maximizing $g'(X)$. Additionally, since $g'(X) = \max(|X| - k, 0)$, it is permutation symmetric. Therefore, the algorithm can only do random search to find $R$. If the algorithm acquires a polynomial number $O(n^m)$ of sets of size $k$, the probability of finding $R$ is

$$O(n^m) \leq O(n^m) \leq O(2^{-n/2+\epsilon n})$$

for all $\epsilon > 0$. Therefore, no polynomial time algorithm can distinguish $g$ and $g'$ with probability greater than $1 - O(2^{-n/2+\epsilon n})$ and will return 0 in almost all cases.

Hence, we have $\max_{|X| \leq k} f(X) + g(X) = 0.5 > 0$ so no polynomial algorithm can do better than $\max_{|X| \leq k} f(X) + g'(X) = 0$ with high probability, or has any positive guarantee.

B Proof of Lemma 3.2

Lemma B.1. There exists a monotonic non-decreasing set function $h$ that is not BP decomposable.

Proof. Let $h(X) = \min(\max(|X|, 1), 3) - 1$. This function is monotonic, and we wish to show it is not BP decomposable. Let $A \subset B$ be subsets of $V$ with $|A| = 1$ and $|B| = 3$. Let $v \in V \setminus B$. We calculate that $h(v|\emptyset) = 0$, $h(v|A) = 1$, $h(v|B) = 0$. So $h(v|\emptyset) + h(v|B) < h(v|A)$.

Assume $h(X) = f(X) + g(X)$ where $f$ is submodular, $g$ is supermodular and both are monotonic non-decreasing. We have $f(v|\emptyset) + f(v|B) \geq f(v|\emptyset) \geq f(v|A)$ and $g(v|\emptyset) + g(v|B) \geq g(v|B) \geq g(v|A)$. Therefore $h(v|\emptyset) + h(v|B) \geq h(v|A)$ by summing the two inequalities, which is a contradiction. We thus have that $h$ is not BP decomposable.

C Proof of Lemma 3.5

We begin with the following four-part lemma,

Lemma C.1. For a BP function $h(X) = f(X) + g(X)$, we have

(i) $h(v|Y) \geq (1 - \kappa_f)h(v|X)$ for all $X \subseteq Y \subset V$ and $v \notin Y$

(ii) $h(v|Y) \leq \frac{1}{1 - \kappa_f} h(v|X)$ for all $X \subseteq Y \subset V$ and $v \notin Y$

(iii) $h(X|Y) \geq (1 - \kappa_f) \sum_{v \in X \setminus Y} h(v|Y)$ for all $X, Y \subseteq V$

(iv) $h(X|Y) \leq \frac{1}{1 - \kappa_f} \sum_{v \in X \setminus Y} h(v|Y)$ for all $X, Y \subseteq V$

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Proof. (i) $\kappa_f = 1 - \min_v f(v|Y \setminus \{v\})$, therefore, $f(v|Y \setminus \{v\}) \geq (1 - \kappa_f) f(v)$ for all $v$.

So we have $f(v|Y) \geq f(v|Y \setminus \{v\}) \geq (1 - \kappa_f) f(v) \geq (1 - \kappa_f) f(v|X)$ and $g(v|Y) \geq g(v|X)$ for all $X \subseteq Y \subseteq V$ and $v \notin Y$. Therefore, $h(v|Y) \geq (1 - \kappa_f) h(v|X)$ for all $X \subseteq Y \subseteq V$ and $v \notin Y$.

(ii) $\kappa^g = 1 - \min_v g(v|Y \setminus \{v\})$, therefore, $g(v|Y \setminus \{v\}) \leq \frac{1}{1 - \kappa^g} g(v)$ for all $v$.

So we have $g(v|Y) \leq g(v|Y \setminus \{v\}) \leq \frac{1}{1 - \kappa^g} g(v) \leq \frac{1}{1 - \kappa^g} f(v|X)$ for all $X \subseteq Y \subseteq V$ and $v \notin Y$. Therefore, $h(v|Y) \leq \frac{1}{1 - \kappa^g} h(v|X)$ for all $X \subseteq Y \subseteq V$ and $v \notin Y$.

(iii) Let $X \setminus Y = \{v_1, \ldots, v_m\}$, $h(X|Y) = \sum_{i=1,2,\ldots,m} h(v_i|Y \cup \{v_i\} \cup \{v_1\} \cup \{v_2\} \cup \ldots \cup \{v_{i-1}\}) \geq (1 - \kappa_f) \sum_{i=1,2,\ldots,m} h(v_i|Y) = (1 - \kappa_f) \sum_{v \in X \setminus Y} h(v|Y)$, according to (i).

(iv) Let $X \setminus Y = \{v_1, \ldots, v_m\}$, $h(X|Y) = \sum_{i=1,2,\ldots,m} h(v_i|Y \cup \{v_i\} \cup \{v_1\} \cup \{v_2\} \cup \ldots \cup \{v_{i-1}\}) \leq \frac{1}{1 - \kappa^g} \sum_{i=1,2,\ldots,m} h(v_i|Y) = \frac{1}{1 - \kappa^g} \sum_{v \in X \setminus Y} h(v|Y)$, according to (ii).

\[\square\]

Lemma C.2. For any chain of solutions $\emptyset = S_0 \subset S_1 \subset \ldots \subset S_k$, where $|S_i| = i$, the following holds for all $i = 0 \ldots k - 1$,

\[h(X^*) \leq \kappa_f \sum_{j:s_j \in S_i \setminus X^*} a_j + \sum_{j:s_j \in S_i \cap X^*} a_j + h(X^* \setminus S_i|S_i) \quad (5)\]

where $\{S_i\} = S_i \setminus S_{i-1}$, $a_i = h(s_i|S_{i-1})$ and $X^*$ is the optimal set.

Proof. For any $i = 0, \ldots, k - 1$, we focus on the term $h(X^* \cup S_i)$.

According to basic set operations,

\[h(X^* \cup S_i) = h(S_i) + h(X^*|S_i) = \sum_{j:s_j \in S_i \setminus X^*} a_j + \sum_{j:s_j \in S_i \cap X^*} a_j + h(X^* \setminus S_i|S_i). \quad (14)\]

We can also express $h(X^* \cup S_i)$ the other way around, $h(X^* \cup S_i) = h(X^*) + h(S_i \setminus X^*|X^*)$. Since we already have an order of element in $S_i$, we can expand $h(S_i \setminus X^*|X^*)$. When adding $s_j$ to the context $S_{j-1} \cup X^*$ we do not need add elements that are not in $S_i \setminus X^*$ since $h(s_j|X^* \cup S_{j-1}) = 0$ if $s_j \in X^*$. Thus, using Lemma C.1 (i), we get $h(X^* \cup S_i) = h(X^*) + \sum_{j:s_j \in S_i \setminus X^*} h(s_j|X^* \cup S_{j-1}) \geq h(X^*) + (1 - \kappa_f) \sum_{j:s_j \in S_i \setminus X^*} h(s_j|S_{j-1})$.

Therefore, we have inequalities on both sides of $h(X^* \cup S_i)$ and we can join them together to get:

\[h(X^*) + (1 - \kappa_f) \sum_{j:s_j \in S_i \setminus X^*} a_j \leq \kappa_f (1 - \kappa_f) \sum_{j:s_j \in S_i \setminus X^*} a_j + \sum_{j:s_j \in S_i \cap X^*} a_j + h(X^* \setminus S_i|S_i), \quad (16)\]

or

\[h(X^*) \leq \kappa_f \sum_{j:s_j \in S_i \setminus X^*} a_j + \sum_{j:s_j \in S_i \cap X^*} a_j + h(X^* \setminus S_i|S_i). \quad (17)\]

\[\square\]
D Proof of Lemma 3.6

Lemma D.1. GreedMax is guaranteed to obtain a solution $\hat{X}$ such that

$$h(\hat{X}) \geq \frac{1}{\kappa_f} \left[ 1 - \left( 1 - \frac{(1-\kappa^g)\kappa_f}{k} \right)^k \right] h(X^*)$$

where $X^* \in \text{argmax}_{|X| \leq k} h(X)$, $h(X) = f(X) + g(X)$, $\kappa_f$ is the curvature of submodular $f$ and $\kappa^g$ is the curvature of supermodular $g$.

Proof. According to Lemma 3.5, for all $i = 0, \ldots, k - 1$,

$$h(X^*) \leq \kappa_f \sum_{j: s_j \in S_i \setminus X^*} a_j + \sum_{j: s_j \in S_i \cap X^*} a_j + h(X^* \setminus S_i | S_i) \tag{18}$$

Since GreedMax is choosing the feasible element with the largest gain, we have $h(v|S_i) \leq h(s_{i+1}|S_i)$ for all feasible $v \in X^*$. In fact, all elements in $X^* \setminus S_j$ are feasible since we are considering a cardinality constraint and $|S_j| \leq k - 1$. Also, $|X^* \setminus S_j| = |X^*| - |X^* \cap S_j| = k - |X^* \cap S_j|$, and therefore from Lemma 3.5 and Lemma C.1(iv), we have that:

$$h(X^*) \leq \kappa_f \sum_{j: s_j \in S_i \setminus X^*} a_j + \sum_{j: s_j \in S_i \cap X^*} a_j + \frac{k - |X^* \cap S_i|}{1 - \kappa^g} a_{i+1} \tag{19}$$

Next, we use a nested lemma, Lemma D.2, to get Equation (6).

Lemma D.2. Given any chain of solutions $\emptyset = S_0 \subset S_1 \subset \ldots \subset S_k$ such that $|S_i| = i$, if the following holds for all $i = 0 \ldots k - 1$:

$$h(X^*) \leq \alpha \sum_{j: s_j \in S_i \setminus X^*} a_j + \sum_{j: s_j \in S_i \cap X^*} a_j + \frac{k - |X^* \cap S_i|}{1 - \beta} a_{i+1} \tag{20}$$

where $0 \leq \alpha, \beta \leq 1$ and $s_i = S_i \setminus S_{i-1}$, and $a_i = h(s_i|S_{i-1})$, then we have

$$h(S_k) \geq \frac{1}{\alpha} \left[ 1 - \left( 1 - \frac{(1-\beta)\alpha}{k} \right)^k \right] h(X^*). \tag{21}$$

Proof. Assume $\beta < 1$ as otherwise the bound is immediate. This lemma aims to show one inequality (Equation (21)) based on $k$ other inequalities (Equation (20)) with $k$ variables $a_1, \ldots, a_k$. In the inequalities, $s_j \in S_k \cap X^*$ and $s_j \in S_k \setminus X^*$ are not treated identically. We will, in fact, correspondingly treat the indices of the elements in $S_k \cap X^*$ as parameters. Recall, $S_k = \{s_1, s_2, \ldots, s_k\}$ is an ordered set and $S_k$ has index set $\{1, 2, \ldots, k\} = [k]$. Let $B = \{b_1, \ldots, b_p\} \subseteq [k]$ be the set of indices of $S_k \cap X^*$ where $b_i$’s are in increasing order (so $b_i < b_{i+1}$) and $p = |S_k \cap X^*|$. Thus, $i \in B$ means $s_i \in S_k \cap X^*$, and $i \in [k] \setminus B$ means $s_i \in S_k \setminus X^*$.

Our next step is to view this problem as a set of parameterized (by $B$) linear programming problems. Each linear programming problem is characterized as finding:

$$T(B) = T(b_1, b_2, \ldots, b_p) = \min_{a_1, a_2, \ldots, a_k} \sum_{i=1}^{k} a_i \tag{22}$$

subject to

$$h(X^*) \leq \alpha \sum_{j \in [i-1]\setminus B_{i-1}} a_j + \sum_{j \in B_{i-1}} a_j + \frac{k - |B_{i-1}|}{1 - \beta} a_i, \text{ for } i = 1, \ldots, k. \tag{23}$$
where $B_i = \{b \in B | b \leq i\}$. In this LP problem, $a_1, \ldots, a_k$ are non-negative variables, and $k, \alpha, \beta$ and $h(X^*)$ are fixed values. Different indices $B = \{b_1, b_2, \ldots, b_p\}$ define different LP problems, and our immediate goal is to show that $T(\emptyset) \leq T(b_1, b_2, \ldots, b_p)$ for all $b_1, b_2, \ldots, b_p$ and $p \geq 0$. In the below, we will use $\Upsilon(B, a, i)$ to refer to the right hand side of Equation (23) for a given set $B$, vector $a$, and index $i = 1, \ldots, k$, and hence Equation (23) becomes $h(X^*) \leq \Upsilon(B, a, i)$ for $i = 1, \ldots, k$.

Note that $\Upsilon(B, a, i)$ is linear in $a$ with non-negative coefficients.

First, we show that there exists an optimal solution\(^3\) $a_1, a_2, \ldots, a_k$ s.t. for all $r \leq k - 1$ with $r \in B$, $a_r \leq a_{r+1}$. Let $r_a$ be the largest $r$ s.t. $r \leq k - 1$, $r \in B$ and $a_r > a_{r+1}$; if such an $r$ does not exist, let $r_a = 0$. Our goal here is equivalent to showing, for any feasible solution $\{a_i\}_{i=1}^k$ with $a_r > 0$, we can create another feasible solution $\{a'_i\}_{i=1}^k$ with $r_{a'} = 0$ and the objective $\sum_{i=1}^k a'_i \leq \sum_{i=1}^k a_i$. We do this iteratively, by in each step showing that for any feasible solution $\{a_i\}_{i=1}^k$ with $r_a > 0$, we can create another feasible solution $\{a'_i\}_{i=1}^k$ with $r_{a'} \leq r_a - 1$ and with objective having $\sum_{i=1}^k a'_i \leq \sum_{i=1}^k a_i$. Repeating this argument leads ultimately to $r_{a'} = 0$.

Let $r = r_a$ for notational simplicity. Consider the $r^{th}$ and $(r+1)^{th}$ inequalities:

$$h(X^*) \leq \alpha \sum_{j \leq [r-1]|B_{r-1}} a_j + \sum_{j \in B_{r-1}} a_j + \frac{k-|B_{r-1}|}{1-\beta} a_r$$

and

$$h(X^*) \leq \alpha \sum_{j \leq [r-1]|B_{r-1}} a_j + \sum_{j \in B_{r-1}} a_j + a_r + \frac{k-|B_{r-1}|-1}{1-\beta} a_{r+1}.$$  

Since $a_r > a_{r+1}$ and $\beta < 1$, $\frac{k-|B_{r-1}|}{1-\beta} a_r > \frac{k-|B_{r-1}|-1}{1-\beta} a_{r+1} + a_r$ and thus the r.h.s. of Eq. (24) is always strictly larger than the r.h.s. of Eq. (25).

Therefore, Eq. (24) is not tight and it is possible to decrease $a_r$ a little bit. Let $\{a'_i\}$ be another set of solutions with $a'_i = a_i$ for all $i = 1, 2, \ldots, r - 1$; $a'_r = a_r - \epsilon$; $a'_i = a_i + \epsilon/(k - |B_r|)$ for $i = r + 1, r + 2, \ldots, k$ and $\epsilon = \left[1 - \frac{1-\beta}{k-|B_{r-1}|}\right] a_r$. It is easy to see that $\epsilon > 0$ since $|B_{r-1}| \leq r - 1 \leq k - 2$.

Below, we show that $a'_r \leq a'_{r+1}$. First, we notice $\sum_{i=1}^k a'_i \leq \sum_{i=1}^k a_i$ since $|B_r| \leq r$ and $-\epsilon + \frac{k-|B_{r-1}|}{k-|B_r|} \epsilon \leq 0$. Next, we want to show that $a'_1, a'_2, \ldots, a'_k$ is still feasible. As mentioned above, define $\Upsilon(B, a, i) = \alpha \sum_{j \in [i-1]|B_{i-1}} a_j + \sum_{j \in B_{i-1}} a_j + \frac{k-|B_{i-1}|}{1-\beta} a_i$.

We examine if $h(X^*) \leq \Upsilon(B, a', i)$ or not for each $i$.

1. For $i = 1, 2, \ldots, r - 1$, $\Upsilon(B, a', i) = \Upsilon(B, a, i) \geq h(X^*)$.

2. For $i = r$, $\Upsilon(B, a', r) - \Upsilon(B, a, r+1) = \frac{k-|B_{r-1}|}{1-\beta} [a_r - \epsilon] - a_r - \frac{k-|B_{r-1}|-1}{1-\beta} a_{r+1} \geq \frac{k-|B_{r-1}|}{1-\beta} [a_r - a_{r+1}] + a_{r+1} - a_r - \frac{k-|B_{r-1}|}{1-\beta} \epsilon = \left[\frac{k-|B_{r-1}|}{1-\beta} - 1\right] [a_r - a_{r+1}] - \frac{k-|B_{r-1}|}{1-\beta} \left[1 - \frac{1-\beta}{k-|B_{r-1}|}\right] [a_r - a_{r+1}] = 0$. So $\Upsilon(B, a', r) \geq \Upsilon(B, a, r+1) \geq h(X^*)$.

3. For $i = r + 1, r + 2, \ldots, k$, we compare $\Upsilon(B, a', i)$ with $\Upsilon(B, a, i)$. Note that $\Upsilon(B, a, i) = \alpha \sum_{j \in [i-1]|B_{i-1}} a_j + \sum_{j \in B_{i-1}} a_j + \frac{k-|B_{i-1}|}{1-\beta} a_i$ and it has three terms, that we consider individually.

(a) The first term is not decreasing since $a'_i < a_i$ only if $i = r$, but $r \notin [i-1] \setminus B_{i-1}$. The increment therefore is at least 0.

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\(^3\)Optimal in this case means for the LP, distinct from the optimal BP maximization solution $X^*$.  

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(b) $a_r$ appears in the second term once, and when changing to $a'_r$, will decreases the value by $\epsilon$. However, $a'_j = a_j + \epsilon/(k - |B_j|)$ for all $j = r + 1, r + 2, \ldots, k$. Immediately, we notice the number of such $a_j$ in the second term is $\sum_{j \in B_{i-1}, j \geq r+1} 1 = \sum_{j \in B_{i-1}, j \notin B_r} 1 = |B_{i-1}|-|B_r|$. So the increment of the second term is $|B_{i-1}|-|B_r| \cdot \epsilon - \epsilon$.

\[ \begin{align*} 
&\text{(c) The third term is increased by } \frac{k - |B_{i-1}|}{1 - \beta} \frac{k - |B_r|}{k - |B_r|} \epsilon \geq k - |B_{i-1}| \epsilon. 
&\text{So overall, the increment is greater than or equal to } \frac{|B_{i-1}|-|B_r|}{k - |B_r|} \epsilon - \epsilon + \frac{k - |B_{i-1}|}{k - |B_r|} \epsilon \geq 0, \text{ which means } \Upsilon(B, a', i) \geq \Upsilon(B, a, i) = h(X^*). 
\end{align*} \]

Therefore, $\{a'_i\}_{i=1}^k$ still satisfies all the constraints but $\sum_{i=1}^k a'_k \leq \sum_{i=1}^k a_k$. Note that $r = r_a = \max(\{r' \in B | r' \leq k - 1, a_r > a_{r+1}\})$ by definition. And we have $a'_i = a_i + \epsilon \frac{k - |B_r|}{k - |B_r|}$ for $i = r + 1, r + 2, \ldots, k$. Therefore, $a'_r \leq a'_{r+1}$ for any $r' \in B \cap [r + 1, k - 1]$. Next we calculate $a'_r - a'_{r+1} = a_r - a_{r+1} - \epsilon - \epsilon \frac{k - |B_r|}{k - |B_r|} = \left(1 - \frac{1}{k - |B_r|}\right) \left(1 - \frac{1 - \beta}{k - |B_r|}\right) (a_r - a_{r+1}) \leq 0$. Therefore, $a'_r \leq a'_{r+1}$ for all $r' \in B \cap [r, k - 1]$ which implies $r_{a'} \leq r_a - 1$.

By repeating the above steps, we can get a feasible solution $\{a''_i\}$ s.t. $r_{a''} = 0$ and $\sum_{i=1}^k a''_k \leq \sum_{i=1}^k a_k$. Therefore, from any optimal solution $\{a_i\}_{i=1}^k$, we can also create another optimal solution $\{a'_i\}_{i=1}^k$ s.t. for all $r \in B$ and $r \leq k - 1$, we have $a''_r \leq a'_{r+1}$. W.l.o.g., we henceforth consider only the optimal solutions $\{a_i\}_{i=1}^k$ with $r_a = 0$.

**Second,** we assume $r \in B$ but $r + 1 \notin B$ for some $r \leq k - 1$. We can create $B' = B \cup \{r + 1\} \setminus \{r\}$ and show for all $\{a_i\}_{i=1}^k$ that satisfies the constraints of $B$, $\{a_i\}_{i=1}^k$ will also satisfy the constraints of $B'$ by showing that $\Upsilon(B', a, i) \geq \Upsilon(B, a, i)$ for $i = 1, \ldots, k$. We consider each $i$ in turn.

1. For $i = 1, 2, \ldots, r$, $\Upsilon(B, a, i) = \Upsilon(B', a, i)$.

2. If $i = r + 1$, we notice $a_r$ moves from the second term to the first, and the third term is changed from $\frac{k - |B_r|}{1 - \beta} a_{r+1}$ to $\frac{k - |B_r|}{1 - \beta} a_{r+1}$ and $|B'_r| = |B_r| - 1$. So the overall value is increased by $\Upsilon(B', a, i) - \Upsilon(B, a, i) = \frac{1}{1 - \beta} a_{r+1} - (1 - \alpha) r_a \geq 0$ since $a_r \leq a_{r+1}$.

3. For $i = r + 2, r + 3, \ldots, k$, we notice that the third term does not change but $a_r$ moves from the second term to the first and $a_{r+1}$ moves from the first term to the second. Thus, the value is increased by $\Upsilon(B', a, i) - \Upsilon(B, a, i) = (1 - \alpha) (a_{r+1} - a_r) \geq 0$ since $a_r \leq a_{r+1}$.

Since $\Upsilon(B', a, i) \geq \Upsilon(B, a, i)$ for $i = 1, \ldots, k$, we have that $T(B') \leq T(B)$. Therefore, if we see two indexes in $B$ differ by at least 2, we can increase the first index by 1. Repeating this process, we get

\[
T(B) \geq T(k - p + 1, k - p + 2, \ldots, k). \tag{26}
\]

**Third,** if $\{a_i\}_{i=1}^k$ satisfies the constraints for $B = \{k-p+1, k-p+2, \ldots, k\}$ and $a_{k-p+1} \leq \ldots \leq a_k$, then $\{a'_i\}_{i=1}^k$ also must satisfy the constraints for $B' = \{k - p + 2, k - p + 3, \ldots, k\}$. We show that $\Upsilon(B', a, i) \geq \Upsilon(B, a, i)$ for $i = 1, \ldots, k$ and again consider each $i$ in turn.

1. For $i = 1, 2, \ldots, k - p + 1$, $\Upsilon(B', a, i) = \Upsilon(B, a, i)$.

2. For $i = k - p + 2, k - p + 3, \ldots, k$, the change of the value is $\Upsilon(B', a, i) - \Upsilon(B, a, i) = (\alpha - 1) a_{k-p+1} + \frac{1}{\beta} a_i$. We notice that $a_i \geq a_{k-p+1}$ since $k - p + 1, k - p + 2, \ldots, i - 1 \in B$. Thus, we have $\Upsilon(B', a, i) - \Upsilon(B, a, i) \geq 0$ and correspondingly $T(B) \geq T(B')$. 
Repeating this process, therefore, we have that

\[ T(B) \geq T(\emptyset) \]  

(27)

Next, we calculate \( T(\emptyset) \). For \( B = \emptyset \) and any feasible (for Equation \((23)\)) \( a_1, a_2, \ldots, a_k \), let \( T_i \) be the partial sum \( T_i = \sum_{j=1}^{i} a_j \) for \( i = 0, \ldots, k \) with \( T_0 = 0 \). We get, for \( i = 1, \ldots, k \) that \( h(X^*) \leq \Upsilon(\emptyset, a, i) \) which takes the form

\[ h(X^*) \leq \alpha \sum_{j \in [i-1]} a_j + \frac{k}{1 - \beta} a_i, \]  

(28)

which is the same as

\[ h(X^*) \leq \alpha T_{i-1} + \frac{k}{1 - \beta} (T_i - T_{i-1}), \]  

(29)

and also, after multiplying both sides by \((1 - \beta)/k\) and then adding \((1/\alpha)h(X^*)\) to both sides, the same as

\[ \frac{1}{\alpha} h(X^*) - T_i \leq \left( 1 - \frac{(1 - \beta)\alpha}{k} \right) \left( \frac{1}{\alpha} h(X^*) - T_{i-1} \right). \]  

(30)

We then repeatedly apply all \( k \) inequalities from \( i = k, \ldots, 1 \), to get

\[ \frac{1}{\alpha} h(X^*) - T_k \leq \left( 1 - \frac{(1 - \beta)\alpha}{k} \right)^k \left( \frac{1}{\alpha} h(X^*) - T_0 \right) \]  

(32)

yielding

\[ T_k \geq \frac{1}{\alpha} \left[ 1 - \left( 1 - \frac{(1 - \beta)\alpha}{k} \right)^k \right] h(X^*). \]  

(33)

Let \( \gamma = \frac{1}{\alpha} \left[ 1 - \left( 1 - \frac{(1 - \beta)\alpha}{k} \right)^k \right] \). So, for \( B = \emptyset \) and any feasible \( a_1, a_2, \ldots, a_k \), we have \( \sum_{j=1}^{k} a_j = T_k \geq \gamma h(X^*) \). Therefore \( T(\emptyset) = \min_{a_1, a_2, \ldots, a_k} \sum_{i=1}^{k} a_i \geq \gamma h(X^*) \).

Recall that \( T(B) \geq T(\emptyset) \) for all \( B \). We thus have, with \( a_i = h(s_i \{ s_1, \ldots, s_{i-1} \}) \) (which are also feasible for Equation \((23)\) with \( B \) again the indices of \( S_k \cap X^* \), which follows from Equation \(19)\), \( h(S_k) = \sum_{i} a_i \geq T(B) \geq T(\emptyset) \geq \gamma h(X^*) \).

\[ \square \]

Lemma D.2 yields Equation \((6)\) which shows the result for Lemma 3.6.

\[ \square \]

E  Proof of Theorem 3.10

**Theorem E.1. Theoretical guarantee in the \( p \) matroids case.** \textsc{GreedMax} is guaranteed to obtain a solution \( \hat{X} \) such that

\[ h(\hat{X}) \geq \frac{1 - \kappa^g}{(1 - \kappa^g)\kappa_f + p} h(X^*) \]  

(11)

where \( X^* \in \arg\max_{X \in M_1 \cap \ldots \cap M_p} h(X) \), \( h(X) = f(X) + g(X) \), \( \kappa_f \) is the curvature of submodular \( f \) and \( \kappa^g \) is the curvature of supermodular \( g \).
Proof. The greedy procedure produces a chain of solutions $S_0, S_1, \ldots, S_k$ such that $|S_i| = i, S_i \subseteq S_{i+1}$, where $k$ is the iteration after which any addition to $S_k$ is infeasible in at least one matroid, and hence $|\hat{X}| = k$. Immediately, we notice all $S_i$ and $X^*$ are independent sets for all $p$ matroids.

For $j = 0, \ldots, k$ and $l = 1, \ldots, p$, there exist at least max$(|X^*| - j, 0)$ elements $v \in X^* \setminus S_j$ s.t. $v \notin S_j$ and $S_j + v \in I(M_l)$, which follows from the third property in the matroid definition. Therefore, for $j = 0, \ldots, k-1, l = 1, \ldots, p$, there are at most $j$ elements of $X^*$ that can not be added to $S_j$.

We next consider the intersection of all $p$ matroids. For $j = 0, \ldots, k$, since in each matroid, there are at most $j$ elements of $X^*$ that cannot be added to $S_j$, the total possible number of elements for which there exists at least one matroid preventing us from adding to $S_j$ is $j p$ (the case that the $p$ sets of at most $j$ elements are disjoint). In other words, there are at least max$(|X^*| - p j, 0)$ different $v \in |X^*|$ s.t. $v \notin S_j, S_j \cup \{v\} \in M_1 \cap \ldots \cap M_p$.

We claim $|X^*| \leq pk$ as otherwise, by setting $j = k$ above, there are still feasible elements in $X^* \setminus S_k$ in the context of $S_k$, which indicates that GREEDYMAX has not ended at iteration $k$. Therefore, we are at liberty to create $pk - |X^*|$ dummy elements, that are always feasible (i.e., independent in all matroids) and that have $h(v|X) = 0$ for all $X \subseteq V$ for each dummy $v$. We add these dummy elements to $X^*$ and henceforth assume, w.l.o.g., that $|X^*| = pk$.

We next form an ordered $k$-partition of $X^* = X_0 \cup X_1 \cup \ldots \cup X_{k-1}$. We show below that it is possible to form this partition so that it has the following properties for $j = 0, \ldots, k-1$:

1. $|X_j| = p$;
2. for all $v \in X_j$, we have $v \notin S_j$ and $S_j \cup \{v\} \in M_1 \cap \ldots \cap M_p$ (i.e., $v$ can be added to $S_j$);
3. and for all $j$ s.t. $s_{j+1} \in X^* \cap S_k$, we have $s_{j+1} \in X_j$.

Immediately, we notice that property 3 is compatible with property 2.

We construct this partition in an order reverse from that of the greedy procedure, that is we create $X_j$ from $j = k - 1$ to 0. Recall that, at each step with index $j = k - 1, k - 2, \ldots, 0$, there are at least $|X^*| - p j = p(k - j)$ elements in $X^*$ can be added to $S_j$.

When $j = k - 1$, there are at least $p$ candidate elements in $X^*$ and we choose $p$ of them to form $X_{k-1}$. The element $s_k$ can be added to $S_{k-1}$ because the greedy algorithm only adds feasible elements and hence, if also $s_k \in X^*$, then $s_k$ can be one of the elements in $X_{k-1}$. Thus, abiding property 3 above, we place $s_k \in X_{k-1}$.

Continuing, for $j = k - 2, k - 3, \ldots, 0$, there are at least $p$ candidate elements in $X^* \setminus [X_{k-1} \cup X_{k-2} \cup \ldots \cup X_{j+1}]$ since $|X_{k-1} \cup X_{k-2} \cup \ldots \cup X_{j+1}| = p(k - j - 1)$ and we choose $p$ of them for $X_j$. Moreover, if $s_{j+1} \in X^*$, we notice $s_{j+1}$ may be one of those candidate elements because of the greedy properties and since $s_{j+1} \notin [X_{k-1} \cup X_{k-2} \cup \ldots \cup X_{j+1}]$ (this follows because $s_{j+1} \in S_{j'}$ for any $j' \geq j + 1$, so $s_{j+1}$ is not a candidate element at step $j' = k - 2, \ldots, j + 1$). Similar to what was done in step $k - 1$, we again choose $p$ candidate elements to form $X_j$, and, if $s_{j+1} \in X^*$, we place $s_{j+1} \in X_j$.

We then arrive at partition $X^* = X_0 \cup X_1 \cup \ldots \cup X_{k-1}$ with the aforementioned three properties.

Next, we order the elements in $X^* = \{x_1, \ldots, x_{pk}\}$ where $\{x_{jp+1}, x_{jp+2}, \ldots, x_{(j+1)p}\} = X_j$ for $j = 0, 1, \ldots, k - 1$. According to greedy, we have $h(x_{jp+1}|S_j) \leq h(s_{j+1}|S_j) = a_{j+1}$ for $t = 1, \ldots, p$. Recall that $a_i$ is defined to be $h(s_i|S_{i-1})$. Moreover, if $x_{jp+t} \in X^* \cap S_k$, we have $x_{jp+t} = s_{j+1}$.

\footnote{There should be no confusion here that the $k$ we refer to in this section is not any cardinality constraint, but rather the size of the greedy solution.}
According to Lemma 3.5 above,
\[
\begin{align*}
    h(X^*) &\leq \kappa_f \sum_{j:s_j \in S_k \setminus X^*} a_j + \sum_{j:s_j \in S_k \cap X^*} a_j + h(X^* \setminus S_k | S_k) \\
    &= \kappa_f \sum_{j:s_j \in S_k \setminus X^*} a_j + \sum_{j:s_j \in S_k \cap X^*} h(s_j | S_{j-1}) + \sum_{i=1}^{p_k} h(x_i | S_k \cup \{x_1\} \ldots \cup \{x_{i-1}\}) 1_{\{x_i \in X^* \setminus S_k\}} \\
    &\leq \kappa_f \sum_{j:s_j \in S_k \setminus X^*} a_j + \frac{1}{1 - \kappa^g} \sum_{j:s_j \in S_k \cap X^*} h(s_j | S_{j-1}) + \frac{1}{1 - \kappa^g} \sum_{j=0}^{k-1} \sum_{t=1}^{p} h(x_{jp+t} | S_j) 1_{\{x_{jp+t} \in X^* \setminus S_k\}} \\
    &= \kappa_f \sum_{j:s_j \in S_k \setminus X^*} a_j + \frac{1}{1 - \kappa^g} \left[ \sum_{j:s_j \in S_k \cap X^*} h(s_j | S_{j-1}) + \sum_{j=0}^{k-1} \sum_{t=1}^{p} h(x_{jp+t} | S_j) - \sum_{j:s_j \in S_k \cap X^*} h(s_j | S_{j-1}) \right] \\
    &\leq \kappa_f \sum_{j:s_j \in S_k \setminus X^*} a_j + \frac{1}{1 - \kappa^g} \sum_{j=0}^{k-1} \sum_{t=1}^{p} a_{j+1} \\
    &\leq \left[ \kappa_f + \frac{p}{1 - \kappa^g} \right] \sum_{j=0}^{k-1} a_{j+1} = \left[ \kappa_f + \frac{p}{1 - \kappa^g} \right] h(\hat{X}) 
\end{align*}
\]

where \(1_{\text{condition}}\) equals 1 if the condition is met and is 0 otherwise. Line 35 to 36 hold because of Lemma C.1 (ii). As for Line 37 to 38, we notice \(x_{jp+t} = s_{j+1}\) if \(x_{jp+t} \in X^* \cap S_k\). Line 38 to line 39 follows via the greedy procedure.

Therefore, we have our result which is
\[
h(\hat{X}) \geq \frac{1 - \kappa^g}{(1 - \kappa^g)\kappa_f + p} h(X^*). \tag{41}
\]

\[\square\]

F  Proof of Theorem 3.11

**Theorem F.1.** **SEMI**\(\text{GRAD}\) initialized with the empty set is guaranteed to obtain a solution \(\hat{X}\) for the cardinality constrained case such that
\[
h(\hat{X}) \geq \frac{1}{\kappa_f} \left[ 1 - e^{-(1 - \kappa^g)\kappa_f} \right] h(X^*) \tag{12}
\]
where \(X^* \in \arg\max_{|X| \leq k} h(X), h(X) = f(X) + g(X), \exists \kappa_f \text{ (resp. } \kappa^g)\text{ is the curvature of } f \text{ (resp. } g).\)
Proof. If \textsc{SemiGrad} is initialized by empty set, we need to calculate the semigradient of \( g \) at \( \emptyset \). By definition, we have

\[
m_{g,\emptyset,1}(Y) = m_{g,\emptyset,2}(Y) = \sum_{v \in Y} g(j)
\]  

(42)

So in the first step of \textsc{SemiGrad}, we are optimizing \( h'(X) = f(X) + m_g(X) = f(X) + \sum_{v \in X} g(v) \) by \textsc{GreedMax}. We will focus elusively on this step as later iterations can only improve the objective value.

According to Lemma 3.5, we have

\[
h(X^*) \leq \kappa_f \sum_{j : s_j \in S_i \setminus X^*} h(s_j|S_{j-1}) + \sum_{j : s_j \in S_i \cap X^*} h(s_j|S_{j-1}) + h(X^* \setminus S_j|S_j)
\]

(43)

Since \textsc{GreedMax} is choosing the feasible element with the largest gain, in the semigradient approximation we have \( h'(v|S_i) \leq h'(s_{i+1}|S_i) \) instead of \( h(v|S_i) \leq h(s_{i+1}|S_i) \). We get:

\[
h(X^* \setminus S_j|S_j) = f(X^* \setminus S_j|S_j) + g(X^* \setminus S_j|S_j)
\]

(44)

\[
\leq \sum_{v \in X^* \setminus S_j} f(v|S_j) + \frac{1}{1 - \kappa^g} \sum_{v \in X^* \setminus S_j} g(v)
\]

(45)

\[
\leq \frac{1}{1 - \kappa^g} \sum_{v \in X^* \setminus S_j} h'(v|S_j)
\]

(46)

\[
\leq \frac{1}{1 - \kappa^g} \sum_{v \in X^* \setminus S_j} h'(s_{j+1}|S_j)
\]

(47)

\[
= \frac{1}{1 - \kappa^g} \sum_{v \in X^* \setminus S_j} f(s_{j+1}|S_j) + g(s_{j+1})
\]

(48)

\[
\leq \frac{1}{1 - \kappa^g} \sum_{v \in X^* \setminus S_j} f(s_{j+1}|S_j) + g(s_{j+1}|S_j)
\]

(49)

\[
= \frac{|X^* \setminus S_j|}{1 - \kappa^g} h(s_{j+1}|S_j)
\]

(50)

And hence,

\[
h(X^*) \leq \kappa_f \sum_{j : s_j \in S_i \cap X^*} a_i + \sum_{j : s_j \in S_i \cap X^*} a_i + \frac{k - |X^* \cap S_i|}{1 - \kappa^g} s_{i+1}.
\]

(51)

We can then use Lemma D.2 to \( h \) to finish the proof. \( \square \)

G Proof of Theorem 3.12

Theorem G.1. \textsc{SemiGrad} initialized with the empty set is guaranteed to obtain a solution \( \hat{X} \), feasible for the \( p \) matroid constraints, such that

\[
h(\hat{X}) \geq \frac{1 - \kappa^g}{(1 - \kappa^g)\kappa_f + p} h(X^*)
\]

(13)

where \( X^* \in \arg\max_{X \in \mathcal{M}_1 \cap \ldots \cap \mathcal{M}_p} h(X) \), \( h = f + g \), \( \kappa_f \) (resp. \( \kappa^g \)) is the curvature of \( f \) (resp. \( g \)).
Proof. If SemiGrad is initialized by empty set, we need to calculate the semigradient of $g$ at $\emptyset$. By definition, we have

$$m_{g,\emptyset,1}(Y) = m_{g,\emptyset,2}(Y) = \sum_{v \in Y} g(j)$$  \hspace{1cm} (52)$$

So in the first step of SemiGrad, we are optimizing $h'(X) = f(X) + m_g(X) = f(X) + \sum_{v \in X} g(v)$ by GreedMax. We will focus on this step.

According to Lemma 3.5, we have

$$h(X^*) \leq \kappa_f \sum_{j: s_j \in S_i \setminus X^*} h(s_j | S_{j-1}) + \sum_{j: s_j \in S_i \cap X^*} h(s_j | S_{j-1}) + h(X^* \setminus S_j | S_j)$$  \hspace{1cm} (53)$$

We then follow the proofs of Theorems 3.10 and 3.11. The only difference is that in Theorem 3.10 we have $h(v | S_i) \leq h(s_{i+1} | S_i)$ for all feasible $v$, but in this proof, we have $h'(v | S_i) \leq h'(s_{i+1} | S_i)$, which does not affect the proof as shown in the proof of Theorem 3.11. \qed

H Proof of Theorem 4.1

Lemma H.1. (Lemma 4.1 from [52]) Let $R$ be a random subset of $V$ of size $\alpha = \frac{x\sqrt{n}}{\omega}$, let $\beta = \frac{x^2}{n}$, and let $x$ be any parameter satisfying $x^2 = \omega(\ln n)$ and such that $\alpha$ and $\beta$ are integer. Let $f_1(X) = \min(|X|, \alpha)$ and $f_2(X) = \min(\beta + |X \cap \hat{R}|, |X|, \alpha)$. Any algorithm that makes a polynomial number of oracle queries has probability $n^{-\omega(1)}$ of distinguishing the functions $f_1$ and $f_2$.

Theorem H.2. Hardness for cardinality constrained case. For all $0 \leq \beta \leq 1$, there exists an instance of a BP function $h = f + g$ with supermodular curvature $\kappa^g = \beta$ such that no poly-time algorithm solving Problem 1 with a cardinality constraint can achieve an approximation factor better than $1 - \kappa^g + \epsilon$, for any $\epsilon > 0$.

Proof. $\kappa^g = \alpha = 0$ is trivial since no algorithm can do better than $1$.

The case when $\kappa^g = 1$ can be proven using the example in Lemma 3.1. $g(X) = \max\{|X| - k, 0\}$, except for a special set $R$ where $g(R) = 0.5$ and $|R| = k$.

For the other case, we prove this result using the hardness construction from [22, 52]. The intuition is to construct two supermodular functions, $g$ and $g'$ both with curvature $\kappa^g$ which are indistinguishable\footnote{Indistinguishable means for all sets $X$ that the algorithm evaluates, $g(X) = g'(X)$.} with high probability in polynomially many function queries. Therefore, any polynomial time algorithm to maximize $g(X)$ can not find $\hat{X} \subseteq V$ with $|\hat{X}| \leq k$ s.t. $g(\hat{X}) > \max_{X \leq k} g'(X)$; otherwise we will have $g(\hat{X}) > \max_{X \leq k} g'(X) \geq g'(\hat{X})$ which contradicts the indistinguishability. In this case, the approximate ratio $\frac{g(X)}{OPT} \leq \frac{OPT'}{OPT}$ where $OPT = \max_{X \leq k} g(X)$ and $OPT' = \max_{X \leq k} g'(X)$. The guarantee, by definition, is the best case approximate ratio and, thus no greater than $\frac{OPT'}{OPT}$. If any polynomial algorithm has a guarantee greater than $\frac{OPT'}{OPT}$, then it contradicts the information theoretic hardness. This is meaningful if $OPT' < OPT$.

$$g(X) = |X| - \beta \min\{|X \cap \hat{R}|, |X|, \alpha\}$$ and $g'(X) = |X| - \beta \min\{|X|, \alpha\}$, where $R \subseteq V$ is a random set of cardinality $\alpha$. Let $\alpha = x\sqrt{n}/5$ and $\gamma = x^2/5$ and let $x$ be any parameter satisfying $x^2 = \omega(\ln n)$ s.t. $\gamma < \alpha$ are positive integers and $\alpha \leq \frac{n}{2} - 1$.\footnote{These examples and the specific parameters like $5$ are adopted from [52].} $g$ and $g'$ are modular minus submodular functions, which implies supermodularity. Monotonicity follows from $g(v | X), g'(v | X) \geq 0$. Also, $OPT = \alpha - \beta \gamma > OPT' = \alpha(1 - \beta)$.  


Next, we calculate the supermodular curvature. \( g(0) = g'(0) = 0 \). \( g(v) = g'(v) = 1 - \beta \) for all \( v \in V \) since \( \alpha, \gamma \geq 1 \). \( g(V \setminus \{v\}) = g'(V \setminus \{v\}) = n - 1 - \beta \alpha \) and \( g(V) = g'(V) = n - \beta \alpha \) for all \( v \in V \) since \( \alpha \leq \frac{n}{2} - 1 \). Therefore, \( \kappa^3 = 1 - \min_{v \in V} \frac{g(v)}{g(v) - \bar{g}(v)} = \beta \). \( \kappa^9 = 1 - \min_{v \in V} \frac{g'(v)}{g'(v) - \bar{g}'(v)} = \beta \). So \( g \) and \( g' \) are monotone non-decreasing supermodular functions with curvature \( \beta \). Let \( f(X) = 0 \) for all \( X \) and \( h(X) = f(X) + g(X) = g(X) \) is the objective BP function.

Any algorithm that uses a polynomial number of queries can distinguish \( g \) and \( g' \) with probability only \( n^{-\omega(1)} \) according to lemma H.1 [52]. More precisely, \( g(X) > g'(X)^8 \) if and only if \( \gamma + |X \cap \tilde{R}| < |X| \) and \( \gamma + |X \cap \tilde{R}| < \alpha \). It is equivalent with asking \( |X \cap \tilde{R}| > \gamma \) and \( |X \cap \tilde{R}| < \alpha - \gamma \). Moreover, \( \Pr(g(X) \neq g'(X)) \), where randomness is over random subsets \( R \subseteq V \) of size \( \alpha \), is maximized when \( |X| = \alpha \) [52]. In this case, the two conditions become identical, and since \( |X| = |X \cap \tilde{R}| + |X \cap \tilde{R}| \), the condition \( g(X) > g'(X) \) happens when only \( |X \cap \tilde{R}| > \gamma \). Intuitively, \( E|X \cap \tilde{R}| = \frac{n^2}{n} = \frac{2}{5} \) where \( R \) is a random set (of arbitrary size) and \( X \) is an arbitrary but fixed set of size \( \alpha \). So \( |X \cap \tilde{R}| \) is located in small interval around \( \frac{2}{5} \) and is hardly ever be larger than \( \gamma \) for large \( n \) according to the law of large numbers. While this is only the intuition, a similar reasoning in [52] offers more details.

Therefore, the output \( \hat{X} \) of any polynomial algorithm must satisfies \( g(\hat{X}) \leq \max_{X \subseteq k} g'(X) \) since, otherwise the algorithm actually distinguishes the two function at \( \hat{X} \), \( g(\hat{X}) > \max_{X \subseteq k} g'(X) \geq g'(\hat{X}) \). The approximate ratio \( \frac{g(\hat{X})}{\text{OPT}} \leq \frac{\text{OPT}}{\text{OPT}} = \frac{g(\hat{X})}{\text{OPT}} = (1 - \kappa^9) \frac{1}{\omega(\ln \kappa \ln n)} \leq 1 - \kappa^9 + \epsilon \). Therefore, the guarantee of any polynomial algorithm, that, by definition, the best case approximate ratio, is no greater than \( 1 - \kappa^9 + \epsilon \) for any \( \epsilon > 0 \) since, otherwise contradicts the information theoretic hardness.

\[ \square \]

### I Proof of Theorem 4.2

**Theorem I.1. Hardness for \( p \) matroids constraint case.** For all \( 0 \leq \beta \leq 1 \), there exists an instance of a BP function \( h = f + g \) with supermodular curvature \( \kappa^3 = \beta \) such that no poly-time algorithm can achieve an approximation factor better than \((1 - \kappa^9)O\left(\frac{\ln p}{p}\right)\) unless \( P=NP \).

**Proof.** Consider the \( p \)-set problem [23], let \( R \) be the maximum disjoint sets of these \( p \)-sets. No polynomial algorithm can find a larger number of disjoint sets than \( O\left(\frac{\ln p}{p}\right)R \) [23]. Let \( k = O\left(\frac{\ln p}{p}\right)R \). So no polynomial algorithm can find a feasible set with size larger than \( k \) unless \( P=NP \).

Let \( h(X) = (1 - \beta)|X| + \beta \max\{|X| - k, 0\} \). It is easy to check that \( h \) is a BP function with \( f = 0 \) and \( g = h \) with \( \kappa^9 = \beta \).

Therefore, the output \( \hat{X} \) of any polynomial algorithm that maximizes \( h \) under the \( p \)-set constraint (expressible via the intersection of \( p \) matroids) must satisfy that \( |X| \leq k \) and, therefore, \( h(\hat{X}) \leq (1 - \beta)k \) unless \( P=NP \). But \( h(X^*) \geq h(R) = (1 - \beta)|R| + \beta(|R| - k) = |R| - \beta k \).

Since \( \kappa^9 \geq \frac{1}{2} \), the approximate ratio

\[
\frac{h(\hat{X})}{h(X^*)} \leq \frac{(1 - \beta)k}{|R| - \beta k} \leq \frac{(1 - \beta)O\left(\frac{\ln p}{p}\right)}{1 - \beta O\left(\frac{\ln p}{p}\right)} \leq \frac{(1 - \beta)O\left(\frac{\ln p}{p}\right)}{\frac{1}{2}} = (1 - \kappa^9)O\left(\frac{\ln p}{p}\right).
\]

since the denominator \( 1 - \beta O\left(\frac{\ln p}{p}\right) \geq \frac{1}{2} \) asymptotically and \( 2O\left(\frac{\ln p}{p}\right) = O\left(\frac{\ln p}{p}\right) \).

\[ \square \]

### J Submodularity Ratio and Generalized Curvature

In this section, we compare the pair \( \kappa_f, \kappa^9 \) of curvatures with the submodularity ratio [10, 4]. We also show that both the generalized curvature introduced in [4] and the submodularity ratio [10].

---

\( ^8 \)Note that \( g(X) \geq g'(X) \) for all \( X \subseteq V \) for any \( \alpha \) and \( \gamma \).
also note, if \( \epsilon > 0 \) where \( \kappa^g \) with another notion of curvature introduced in [50], showing a simple inequality relationship in general and a correspondence when \( h = g \).

\section*{J.1 Submodularity ratio}

The submodularity ratio is defined as

\[
\gamma_{U,k}(h) = \min_{L \subseteq U, S : |S| \leq k, S \cap L = \emptyset} \frac{\sum_{x \in S} h(x) |L|}{h(S|L)}
\]

with \( U \subseteq V \) and \( 1 \leq k \leq |V| = n \), and typically we consider \( \gamma_{V,n} \). We can establish a simple lower bound of the submodularity ratio based on the supermodular curvature as follows.

\textbf{Lemma J.1.} \( \gamma_{V,n}(h) \geq 1 - \kappa^g \) when \( h = f + g \).

\textit{Proof.} For all \( L \subseteq V \) and \( S \cap L = \emptyset \), we have \( \sum_{x \in S} \frac{h(x) |L|}{h(S|L)} \geq 1 - \kappa^g \) which follows from Lemma C.1(iv) Thus, \( \gamma_{V,n}(h) \geq 1 - \kappa^g \).

The function \( h \) is submodular if and only if \( \gamma_{V,n} = 1 \) so one might hope that given a BP function \( h = f + g \), that as \( \gamma_{V,n}(h) \to 1 \), correspondingly \( \kappa^g \to 0 \). This is not the case, however, as can be seen by considering the following example.

Let \( a \) be an element of \( V \) and define the function \( g(A) = |A \cap (V \setminus \{a\})| + \epsilon|A \cap (V \setminus \{a\})| |A \cap \{a\}| \), where \( \epsilon > 0 \) is a very small number. Immediately, we have that \( g \) being supermodular and monotone. Also note, if \( a \notin A \) then \( g(A) = |A| \); if \( a \in A \) then \( g(A) = (|A| - 1)(1 + \epsilon) \).

First, we calculate the supermodular curvature \( \kappa^g \). We have that \( g(a) = 0 \) and also \( g(a|V \setminus \{a\}) = \epsilon(n - 1) \). Therefore, the function is fully curved, \( \kappa^g = 1 \).

Next, we calculate the submodularity ratio \( \gamma_{V,n} = \min_{L, S \subseteq V, S \cap L = \emptyset} \frac{\sum_{v \in S} g(v|L)}{g(S|L)} \). When \( |S| = 1 \), \\
\( \frac{\sum_{v \in S} g(v|L)}{g(S|L)} = 1 \). When \( |S| \geq 2 \), we have the following 3 cases (recall that \( S \cap L = \emptyset \) so there is no forth case):

\begin{itemize}
  \item \( a \in S \). \( g(S|L) = g(S \cup L) - g(L) = (|S| + |L| - 1)(1 + \epsilon) - |L| \) is very close to \( |S| - 1 \) for very small \( \epsilon \). \( \sum_{v \in S} g(v|L) = \epsilon |L| + |S| - 1 \), which is also very close to \( |S| - 1 \) for small \( \epsilon \). So \( \frac{\sum_{v \in S} g(v|L)}{g(S|L)} \approx 1 \) for small \( \epsilon \).
  
  \item \( a \in L \). \( g(S|L) = g(S \cup L) - g(L) = |S|(1 + \epsilon) \). \( \sum_{v \in S} g(v|L) = |S|(1 + \epsilon) \). So \( \frac{\sum_{v \in S} g(v|L)}{g(S|L)} = 1 \)
  
  \item \( a \notin S \cup L \). \( g(S|L) = |S| \) and \( \sum_{v \in S} g(v|L) = |S| \). Therefore, \( \frac{\sum_{v \in S} g(v|L)}{g(S|L)} = 1 \).
\end{itemize}

In all cases, \( \frac{\sum_{v \in S} g(v|L)}{g(S|L)} \) is either 1 or very close to 1 for small \( \epsilon \), so \( \gamma_{V,n} \) has only 1 as an upper bound. That is, we have an example function that is purely supermodular and fully curved (\( \kappa^g = 1 \)) for all non-zero values of \( \epsilon \), but the submodularity ratio can be arbitrarily close to 1. If we consider a weighted sum of a submodular function and this supermodular function, the submodularity ratio is again arbitrarily close to 1. Therefore, there does not seem to be an immediately accessible strong relationship between the supermodular curvature and the submodularity ratio.
J.2 Hardness of Generalized Curvature and Submodularity Ratio

The generalized curvature Bian et al. [4] of a non-negative function $h$ is the smallest scalar $\alpha$ s.t.

$$h(v) \leq (1-\alpha)h(v)$$

for all $S, \Omega \subseteq V$ and $v \in V \setminus \Omega$ and this is used, in concert with the submodularity ratio, to produce bounds such as $\frac{1}{\alpha}(1-e^{-\alpha \gamma})$ for the greedy algorithm. Unfortunately, the generalized curvature is hard to compute under the oracle model. We have the following.

**Lemma J.2.** There exists an instance of a non-negative function $h$ whose generalized curvature can not be calculated in polynomial time, when we have only oracle access to the function.

**Proof.** We consider a non-negative function $h': 2^V \rightarrow R$ with ground set size equals $n$ ($n$ is even number). Let $h'(X) = |X|$ for all $X \subseteq V$. Let $R \subseteq V$ be an arbitrary set with $|R| = \frac{n}{2}$. Define another set function $h: 2^V \rightarrow R$, $h(X) = h'(X)$ for all $X \subseteq V$ and $X \neq R$. If $h(R) = \frac{n}{2} - 1$.

First, we can easily calculate the generalized curvature of $h'$ and $h$. We have that $\alpha_{h'} = 0$ since $h'$ is a non-decreasing modular function. For $h$, let $S \cup \Omega = R$, $S \cap \Omega = \emptyset$, $|S|, |\Omega| \geq 1$ and $v \in S$, we have $h(v) = 0$ and $h(v) = 1$. Therefore $\alpha = 1$ is the smallest scalar s.t. $h(v) \leq (1-\alpha)h(v)$. So, as a conclusion of this part, the generalized curvature of the two functions are not the same.

Next we use a proof technique similar to [52]. Note that $h'(X) = h(X)$ if and only if $X \neq R$. So for any algorithm trying to calculate $\alpha_{h'}$, before it evaluates $h(R)$, all function evaluations are the same with calculating $\alpha_{h'}$. Additionally, since $h(X) = |X|$, it is permutation symmetric. Therefore, the algorithm can only do random search to find $R$. If the algorithm acquires a polynomial number $O(n^2)$ of sets of size $\frac{n}{2}$, the probability of finding $R$ is $\frac{O(n^2)}{(\frac{n}{2})^2} \leq \frac{O(n^2)}{2^n} \leq O(2^{-n/2+\epsilon n})$ for all $\epsilon > 0$.

Therefore, no algorithm can be guaranteed to distinguish $h$ and $h'$ in polynomial time. Since the generalized curvature of $h$ and $h'$ are different, neither of them can be calculated in polynomial time.

Likewise, the submodularity ratio is unfortunately also hard to compute exactly, in the oracle model.

**Lemma J.3.** There exists an instance of a non-negative function $h$ whose submodularity ratio (Equation (55)) can not be calculated in polynomial time under only oracle access to that function.

**Proof.** We consider a non-negative function $h': 2^V \rightarrow R$ with ground set size equals $n$ ($n$ is an even number). Let $h'(X) = |X|$ for all $X \subseteq V$. Let $R \subseteq V$ be an arbitrary set with $|R| = \frac{n}{2}$. Define another set function $h: 2^V \rightarrow R$, $h(X) = h'(X)$ for all $X \subseteq V$ and $X \neq R$.

We can easily calculate the submodularity ratio of both $h'$ and $h$ as follows. We have that $\gamma_{V,n}(h') = 1$ since $h'$ is a non-decreasing modular (and thus submodular) function. For $h$, choose an element $v_1 \in R$ and another element $v_2 \in V \setminus R$, and let $L = R \setminus \{v_1\}$ and $S = \{v_1, v_2\}$. We have

$$\frac{\sum_{h \in S} h(v[L])}{h(S[L])} = \frac{h(R) + h(R \setminus \{v_1\}) - 2h(R \setminus \{v_1\})}{h(R \setminus \{v_1\})} = \frac{1}{2}$$

and thus $\gamma_{V,n}(h) = \min_{L, S \subseteq V, S \cap L = \emptyset} \frac{\sum_{h \in S} h(v[L])}{h(S[L])} \leq \frac{1}{2}$. Therefore, the submodularity ratio of the two functions are not the same. Given the submodularity ratio of the two functions, we would be able to tell them apart.

Next we use a proof technique similar to [52]. We have that $h'(X) = h(X)$ if and only if $X \neq R$. So for any algorithm trying to calculate $\gamma_{V,n}(h)$, before it evaluates $h(R)$, all function evaluations are the same with calculating $\gamma_{V,n}(h')$. Additionally, since $h(X) = |X|$ is permutation symmetric,
the algorithm can only do a random search to find \( R \). If the algorithm queries a polynomial number \( O(n^m) \) of sets of size \( \frac{n}{2} \), the probability of finding \( R \) is
\[
\frac{O(n^m)}{(\frac{n}{2})^2} \leq \frac{O(n^m)}{(n/2)^2} = O(2^{-n/2+\epsilon n})
\]
for all \( \epsilon > 0 \).

Therefore, no algorithm can guarantee to distinguish \( h \) and \( h' \) in polynomial time. Since the submodularity ratio of \( h \) and \( h' \) are different, this means that neither of them can be calculated in polynomial time.

\[ \square \]

J.3 Comparison to Sviridenko et al. [50]'s curvature

Sviridenko et al. [50] (in their Section 8) define a notion of curvature as follows:

\[
1 - c = \min_j \min_{A,B \subseteq V \setminus j} \frac{h(j|A)}{h(j|B)} \tag{57}
\]

We can establish a simple upper bound on \( c \) based on submodular and supermodular curvature as follows. We calculate \( \frac{h(j|A)}{h(j|B)} \) given \( h = f + g \) and \( \kappa_f \) and \( \kappa_g \) as follows. First, \( f(j|B) \leq f(j) \leq \frac{1}{1-\kappa_f} f(j|A) \) which follows from Lemma C.1 (i). Thus \( \frac{f(j|A)}{f(j|B)} \geq 1 - \kappa_f \). Next, \( g(j|A) \geq g(j) \geq (1 - \kappa^g) g(j|B) \) which follows from Lemma C.1 (ii). Thus, \( \frac{g(j|A)}{g(j|B)} \geq 1 - \kappa^g \). Therefore,

\[
\frac{h(j|A)}{h(j|B)} = \frac{f(j|A) + g(j|A)}{f(j|B) + g(j|B)} \geq \frac{(1 - \kappa_f) f(j|B) + (1 - \kappa^g) g(j|B)}{f(j|B) + g(j|B)} \geq \frac{\min(1 - \kappa_f, 1 - \kappa^g)(f(j|B) + g(j|B))}{f(j|B) + g(j|B)} \geq \min(1 - \kappa_f, 1 - \kappa^g) \tag{58}
\]

Thus we have \( 1 - c \geq \min(1 - \kappa_f, 1 - \kappa^g) \), or \( c \leq \max(\kappa_f, \kappa^g) \).

Note that for purely supermodular functions, \( \kappa_f = 0 \) and, considering Equation (57), we have \( c = \kappa^g \). This coincides with the \( 1 - \kappa^g \) bound and hardness for monotone supermodular functions — compare Theorem 8.1 of Sviridenko et al. [50] with the present paper’s item 3 in Section 3.2.1 and Theorem 4.1.