Self-similarly corrected Padé approximants for nonlinear equations

S. Gluzman\textsuperscript{1} and V.I. Yukalov\textsuperscript{2,3}

\textsuperscript{1}Bathurst St. 3000, 606, Ontario M6B 3B4, Toronto, Canada
E-mail: simongluzmannew@gmail.com

\textsuperscript{2}Bogolubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, Dubna 141980, Russia

\textsuperscript{3}Instituto de Fisica de S\~{a}o Carlos, Universidade de S\~{a}o Paulo,
CP 369, S\~{a}o Carlos 13560-970, S\~{a}o Paulo, Brazil
E-mail: yukalov@theor.jinr.ru

Abstract

We consider the problem of finding approximate analytical solutions for nonlinear equations typical of physics applications. The emphasis is on the modification of the method of Padé approximants that are known to provide the best approximation for the class of rational functions, but do not provide sufficient accuracy or cannot be applied at all for those nonlinear problems, whose solutions exhibit behaviour characterized by irrational functions. In order to improve the accuracy, we suggest a method of self-similarly corrected Padé approximants, taking into account irrational functional behaviour. The idea of the method is in representing the sought solution as a product of two factors, one of which is given by a self-similar root approximant, responsible for irrational functional behaviour, and the other being a Padé approximant corresponding to a rational function. The efficiency of the method is illustrated by constructing very accurate solutions for nonlinear differential equations. A thorough investigation is given proving that the suggested method is more accurate than the method of standard Padé approximants.

Keywords: Nonlinear differential equations, Approximation theory, Padé approximants, Self-similar root approximants, Thomas-Fermi equation, Nonlinear Schrödinger equation, Ruina-Dieterich equation

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1 Introduction

Nonlinear problems are widespread in different branches of physics. As a rule, such problems can very rarely be solved exactly. But it is often useful to have in hands an approximate solution in analytic form. Various ways of constructing asymptotic approximate solutions for nonlinear equations have recently been reviewed by He [1]. Another way of constructing approximate solutions is by using the method of Padé approximants [2]. It is well known that the latter provide the best approximation for the class of rational functions. However, approximating irrational functions by Padé approximants can result in poor accuracy and sometimes is not possible at all (see, e.g., Refs. [3,4]). When solutions to nonlinear equations exhibit irrational functional behavior, to reach a good accuracy, it is necessary to go to high-order Padé approximants. In these cases Padé approximants may display unphysical poles and zeroes [2,4,5].

In the present paper we suggest a method for overcoming the above mentioned problems, when approximating irrational functions. The idea of the method is to correct Padé approximants by combining them with the other type of approximants that would take into account the irrational behavior of the sought functions. We show that the method is applicable not only to rather simple problems, but can be used for solving nonlinear differential equations. By a careful comparison, we prove that the corrected Padé approximants are essentially more accurate than the standard Padé approximants.

The outline of the paper is as follows. In sec. 2, we present the main ideas of the suggested method and explain why self-similar root approximants can efficiently describe the behaviour of irrational functions. In sec. 3, we solve the Thomas-Fermi equation, then in sec. 4, the nonlinear Schrödinger equation, and in sec. 5, the Ruina-Dieterich equation. In all the cases, we thoroughly compare our solutions with the standard Padé approximants and show that the corrected Padé approximants are more accurate. In Sec. 6, we discuss possible extensions of the method. One such an extension is the procedure of finding the large-variable behavior of a function from its small-variable asymptotic form. We also discuss the possible generalization of the method for partial differential equations. Section 7 concludes.

2 Self-similarly corrected Padé approximants

The straightforward way of how it would be possible to improve the accuracy of approximants for irrational solutions is to take into account this irrational behavior by an appropriate approximation procedure. Approximate solutions in the class of irrational functions can be constructed by invoking self-similar approximation theory that was advanced in Refs. [6–11]. This approach combines the ideas of dynamical theory, optimal control theory, and renormalization group [12–16]. A convenient type of approximants, resulting from the self-similar approximation theory and representing well irrational functional forms, is given by self-similar root approximants [17,22]. In principle, it could be possible to employ the self-similar approximants as such. But it would, probably, be a pity to forget the well developed techniques of Padé approximants. So, the important question is whether it would be possible to modify Padé approximants in such a way that to improve their accuracy for the class of irrational functions.
The desired modification could be done by splitting the sought solution into two factors, one, represented by root approximants or factor approximants, taking care of the irrational part of the solution, and the other being a Padé approximant, characterizing the rational part of the solution. The so corrected Padé approximants would be applicable to a larger class of problems and be well defined even for those cases, where the standard Padé approximants could not be used. Below, we present the main ideas of the method.

Suppose we wish to find a real function \( f(x) \) of a real variable \( x \), which is defined by a complicated nonlinear equation, whose exact solution is not available. But it is often possible to derive an asymptotic form of the sought solution, say, at small values of the variable,

\[
\begin{align*}
  f(x) &\simeq f_k(x) \quad (x \to 0), \\
  f_k(x) &= f_0(x) \left(1 + \sum_{n=1}^{k} a_n x^n\right),
\end{align*}
\]

where it is presented by an expansion in powers of \( x \):

\[
f_k(x) = f_0(x) \left(1 + \sum_{n=1}^{k} a_n x^n\right),
\]

with \( f_0(x) \) being a known function.

And let the asymptotic form of the sought function at large values of the variable,

\[
\begin{align*}
  f(x) &\simeq f^{(p)}(x) \quad (x \to \infty), \\
  f^{(p)}(x) &= \sum_{n=1}^{p} b_n x^{\beta_n},
\end{align*}
\]

be given as an expansion

\[
f^{(p)}(x) = \sum_{n=1}^{p} b_n x^{\beta_n},
\]

with the descending order of the powers,

\[\beta_{n+1} < \beta_n \quad (n = 1, 2, \ldots, p - 1).\]

Assume that we can construct an irrational function \( f_{irr}^*(x) \) satisfying the boundary conditions represented by the above asymptotic expansions,

\[
\begin{align*}
  f_{irr}^*(x) &\simeq f_k(x) \quad (x \to 0), \\
  f_{irr}^*(x) &\simeq f^{(p)}(x) \quad (x \to \infty).
\end{align*}
\]

Then we introduce a correcting function

\[
C_k(x) \equiv \frac{f_k(x)}{f_{irr}^*(x)}
\]

that, being expanded in powers of \( x \), acquires the form

\[
C_k(x) \simeq 1 + \sum_{n=1}^{k} d_n x^n \quad (x \to 0).
\]

The first term here is one, since the function \( f_{irr}^*(x) \) tends to \( f_k(x) \) at small \( x \).
On the basis of expansion (7), it is straightforward to generate a diagonal Padé approximant $P_{N/N}(x)$, such that

$$P_{N/N}(x) \simeq C_k(x) \quad (x \to 0).$$

Then the approximate solution is represented as the product

$$f^*_k(x) = f^*_{irr}(x) P_{N/N}(x). \quad (8)$$

If the irrational factor $f^*_{irr}(x)$ is assumed to satisfy the boundary conditions (5), then the Padé approximant has to obey the boundary conditions

$$P_{N/N}(0) = P_{N/N}(\infty) = 1. \quad (9)$$

However, in general, it is not compulsory that conditions (5) and (9) be valid separately, but is sufficient that the product solution (8) as a whole would satisfy the asymptotic conditions

$$f^*_k(x) \simeq f_k(x) \quad (x \to 0),$$
$$f^*_k(x) \simeq f^{(p)}(x) \quad (x \to \infty). \quad (10)$$

The so constructed approximate solution (8) takes into account the irrational functional behaviour of the sought solution and allows one to invoke the techniques of Padé approximants for reaching good accuracy of this solution.

The irrational factor can be presented in different forms. Here we show that a very convenient form is provided by self-similar root approximants [17–22] resulting from the self-similar approximation theory [6–11].

Suppose we have the small-variable expansion (2) that can be divergent for finite values of the variable $x$. To make sense of divergent series, it is necessary to introduce control parameters $s_k$ generating control functions $s_k(x)$ such that the series of the terms $f_k(x)$ is renormalized into the series of the expressions $F_k(x, s_k(x))$ that become convergent. This renormalization can be formalized as

$$F_k(x, s_k) = \hat{R}[s_k]f_k(x). \quad (11)$$

The control functions should be defined from asymptotic and optimization conditions.

The second idea is the reformulation of the study of the sequence $\{F_k\}$ into the language of dynamical theory in order to resort to the powerful techniques of the latter. For this purpose, it is necessary to introduce a sequence of endomorphisms $y_k$ in the space of approximations, which is bijective to the approximation sequence $\{F_k\}$. These endomorphisms are constructed in the following way. We define the expansion function $x = x_k(f)$ by the reonomic constraint

$$F_0(x, s_k(x)) = f, \quad x = x_k(f). \quad (12)$$

This makes it possible to introduce the endomorphism

$$y_k(f) \equiv F_k(x_k(f), s_k(x_k(f))) \quad (13)$$

acting in the space of approximations. Treating the passage from an approximation $y_k$ to another approximation $y_{k+p}$ as the motion with respect to the approximation order $k$, we
can consider the sequence of endomorphisms \( \{y_k\} \) in the space of approximations as the trajectory of a dynamical system, with \( k \) playing the role of discrete time.

Since by construction the sequences \( \{F_k\} \) and \( \{y_k\} \) are bijective, the convergence of \( \{F_k\} \) implies the existence of an attractive fixed point \( y^* \) for the sequence \( \{y_k\} \), where \( y_k(y^*) = y^* \). In the vicinity of a fixed point, the dynamical system enjoys the property of self-similarity, such that the relation is valid:

\[
y_{k+p}(f) = y_k(y_p(f)) .
\]

This relation defines a cascade, that is a discrete dynamical system.

The next step is the embedding of the cascade into a flow, whose trajectory \( \{y(t, f)\} \) passes through all points of the cascade trajectory,

\[
y(t, f) = y_k(f) \quad (k = t) ,
\]

so that the self-similar relation (14) be preserved,

\[
y(t + t', f) = y(t, y(t', f)) .
\]

The latter relation can be rewritten in the differential form of the Lie equation

\[
\frac{\partial}{\partial t} y(t, f) = v(y(t, f)) ,
\]

where \( v \) is the velocity of the dynamical system. Integrating the Lie equation between \( y_k \) and \( y_k^* \), with the cascade velocity

\[
v_k(f) = y_k(f) - y_{k-1}(f) ,
\]

defines an approximate fixed point \( y_k^* \). Performing the transformation inverse to (11), we obtain the self-similar approximant

\[
f_k^*(x) = \hat{R}^{-1}[s_k] y_k^*(F_0(x, s_k(x))) .
\]

For deriving self-similar root approximants, we introduce control functions through the fractal transform

\[
F_k(x, s_k) = x^{s_k} f_k(x) .
\]

This transform allows us to extract the scaling properties of the resulting sequence of approximants.

Then we integrate the Lie equation (17) between the quasi-fixed points \( y_{k-1}^* \) and \( y_k^* \), which gives the recurrent relation

\[
y_k^* = y_k^*(y_{k-1}^*) .
\]

Iterating this recurrent relation \( k - 1 \) time yields

\[
y_k = y_k^*(y_{k-1}^*(y_{k-2}^* \ldots (y_0^*)) ) .
\]

With the inverse fractal transform

\[
f_k^*(x) = x^{-s_k} y_k^*(x^{s_k}) ,
\]
we come to the recurrent relation

$$f_k^* = f_k^* \left( f_{k-1}^* \left( f_{k-2}^* \cdots (f_0^*) \right) \right). \quad (24)$$

Accomplishing the described steps for the asymptotic expansion of the sought function in equation (2), we obtain the self-similar root approximant

$$f_k^*(x) = f_0(x) \left( (1 + A_1 x^{\alpha})^{n_1} + A_2 x^{2\alpha} + \ldots + A_k x^{k\alpha} \right)^{n_k}. \quad (25)$$

The parameters $A_i$ and powers $n_i$ are defined so that to satisfy the boundary asymptotic conditions (10). It has been shown [23] that this procedure can be uniquely defined. Thus, if $f_0(x) = A x^{\alpha}$ and $p < k$, then the boundary conditions uniquely define all parameters of the root approximant (25), prescribing the powers by the equalities

$$n_j = \frac{j + 1}{j} \quad (j = 1, 2, \ldots, k-p),$$

$$jn_j = j + 1 + \beta_{k-j+1} - \beta_{k-j} \quad (j = k-p+1, k-p+2, \ldots, k-1),$$

$$kn_k = \beta_1 - \alpha.$$

Of course, the order of the root approximant is to be lower than the order of expansion (2), since we need the remaining terms for constructing Padé approximants. As is evident, the root approximant, having irrational form, can efficiently take into account the corresponding features of the sought function.

One can remember that some irrational functions that yield the asymptotic expansions in noninteger powers,

$$f_k(x) = f_0(x) \left( 1 + \sum_{n=1}^{k} c_n x^{n\gamma} \right),$$

where $\gamma$ is not integer, can easily be approximated by the Padé approximants of the form

$$P_{M/N}(x) = \frac{\sum_{m=0}^{M} a_m x^{m\gamma}}{\sum_{n=0}^{N} b_n x^{n\gamma}}.$$  

As is clear, this kind of approximants is nothing but the standard Padé approximants with a straightforward change of the variable $z = x^{\gamma}$. Hence the above approximant becomes rational with respect to the variable $z$. Respectively, this approximant represents a function that is rational with respect to $z$. This type of irrationality that can be easily reduced to a rational form can be called the reducible irrationality. This fact is well known and we use it without special comments.

Contrary to this, the root approximant (25) cannot be reduced to a rational form by a simple change of the variable. In that sense, it describes a more general, nonreducible irrationality. While the correcting function (7) can have an expansion in noninteger powers, which will lead to the related Padé approximant composed of polynomials of noninteger powers.

Taking into account the irrational behavior by means of root approximants accelerates the convergence of the approximation procedure. As is mentioned above, in general, Padé
approximants can also approximate irrational functions. At lower orders, such Padé approximants oscillate, or give insufficiently accurate results, or display unphysical poles and zeros. Sometimes (although not always), as stated earlier, these drawbacks can simultaneously disappear by resorting to higher-order Padé approximants. In fact, the reader should be reminded that the only acceptable results from the table of Padé approximants are those that have converged/stabilized, and these generally necessitate resorting to high-orders. However, taking into account the irrational behavior from the very beginning allows a quicker attainment of the desired accuracy (i.e. with lower orders).

3 Thomas-Fermi equation

The Thomas-Fermi equation describes the screened Coulomb potential caused by a heavy charged nucleus surrounded by a cloud of electrons [24]. The equation reads as

\[
\frac{d^2 f(x)}{dx^2} = \frac{f^{3/2}(x)}{\sqrt{x}}. \tag{26}
\]

For neutral atoms, the boundary conditions are

\[
f(0) = 1 , \quad f(\infty) = 0 . \tag{27}
\]

At small \( x \to 0 \), the solution to the equation can be written [25, 26] in the form of the expansion

\[
f(x) \simeq 1 - B x + \frac{1}{3} x^3 - \frac{2B}{15} x^4 + \frac{4}{3} - \frac{2B}{5} x + \frac{3B^2}{70} x^2 + \left( \frac{2}{27} + \frac{B^3}{252} \right) x^3 ,
\]

where \( B = 1.588071 \) is found numerically [27]. This gives

\[
f(x) \simeq 1 + \sum_{n=1}^{9} a_n x^{n/2} \quad (x \to 0) , \tag{28}
\]

with the coefficients

\[
a_1 = 0 , \quad a_2 = -1.588071 , \quad a_3 = \frac{4}{3} , \quad a_4 = 0 , \quad a_5 = -0.635228 , \quad a_6 = \frac{1}{3} , \quad a_7 = 0.108084 , \quad a_8 = -0.211743 , \quad a_9 = 0.0899672 .
\]

At large \( x \), the asymptotic behaviour is [28]

\[
f(x) \simeq b_1 x^\beta_1 + b_2 x^\beta_2 \quad (x \to \infty) , \tag{29}
\]

where

\[
b_1 = 144 , \quad \beta_1 = -3 , \quad b_2 = 1911.02 , \quad \beta_2 = -3.772 .
\]
Following the general method of constructing self-similar root approximants, using expansion (28), to second order, we have

$$f^*_2(x) = \left( (1 + A_1 x)^{n_1} + A_2 x^{3/2} \right)^{-2},$$

with the parameters defined by the asymptotic boundary conditions (10),

$$A_1 = 0.443153, \quad A_2 = 0.0833333, \quad n_1 = 0.727998.$$

To third order, we find

$$f^*_3(x) = \left( (1 + B_1 x)^{n_1} + B_2 x^{3/2} \right)^{-3/2},$$

where the parameters are

$$B_1 = 1.7764, \quad n_1 = 0.727998,$$

$$B_2 = 0.250555, \quad n_2 = 0.818665, \quad B_3 = 0.0363992.$$

As the irrational factor, we can take

$$f^*_{irr}(x) = \frac{1}{2} [f^*_2(x) + f^*_3(x)].$$

Introducing the correcting function

$$C_8(x) = \frac{f_8(x)}{f^*_{irr}(x)},$$

which is defined by the eight order of expansion (28), we obtain the small-variable expansion

$$C_8(x) \simeq 1 + \sum_{n=1}^{8} d_n x^{n/2} \quad (x \to 0),$$

with the coefficients

$$d_1 = 0, \quad d_2 = -0.471421, \quad d_3 = 1.57051, \quad d_4 = -2.01043,$$

$$d_5 = 0.482756, \quad d_6 = 1.41347, \quad d_7 = 0.838164, \quad d_8 = -1.07168.$$

On the basis of expansion (34), we define the Padé approximant

$$P_{4/4}(x) = \frac{1 + \bar{a}_1 x^{1/2} + \bar{a}_2 x + \bar{a}_3 x^{3/2} + \bar{a}_4 x^2}{1 + \bar{b}_1 x^{1/2} + \bar{b}_2 x + \bar{b}_3 x^{3/2} + \bar{b}_4 x^2},$$

satisfying the boundary conditions

$$P_{4/4}(0) = P_{4/4}(\infty) = 1.$$

The parameters of the approximant (35) are

$$\bar{a}_1 = 2.79159, \quad \bar{a}_2 = 4.56393, \quad \bar{a}_3 = 6.14842, \quad \bar{a}_4 = 4.01834,$$
The sought corrected Padé approximant becomes
\[ f^*_8(x) = f^*_{irr}(x) P_{4/4}(x). \] (37)

The accuracy of the approximant can be characterized by the relative errors
\[ \varepsilon^*_8(x) \equiv \frac{f^*_8(x) - f(x)}{f(x)} \times 100\%, \] (38)

where \( f(x) \) is the exact numerical solution of the Thomas-Fermi equation \([26]\).

For completeness, we also present the lower-order corrected approximants, \( f^*_4(x), \) \( f^*_6(x), \)
\[ f^*_4(x) = f^*_{irr}(x) P_{2/2}(x). \] (39)

with
\[ P_{2/2}(x) = \frac{1 + 3.33143\sqrt{x} + 6.36239x}{1 + 3.33143\sqrt{x} + 6.83382x}, \]
and \( f^*_6(x), \)
\[ f^*_6(x) = f^*_{irr}(x) P_{3/3}(x), \] (40)

with
\[ P_{3/3}(x) = \frac{1 + 2.37227\sqrt{x} + 3.16702x + 3.48058x^{3/2}}{1 + 2.37227\sqrt{x} + 3.63844x + 3.02841x^{3/2}}. \]

The related values of the approximants and their errors are shown in Table 1.

Among empirical analytic forms, it is worth mentioning the empirical Sommerfeld solution \([29]\)
\[ f_S(x) = \frac{1}{(1 + 0.278343x^{0.772002})^{3.886}} \] (41)
and the Andrianov-Awrejcewicz empirical solution \([30]\),
\[ f_A(x) = \left[ 1 + 0.1336x^{1/2} - 1.3038x + 0.9598x^{3/2} - 0.2523x^2 + x^{5/2} \right] \times \]
\[ \times \left[ 1 + 0.1336x^{1/2} + 0.2842x - 0.1614x^{3/2} + 0.0209x^2 + F(x)x^{5/2} \right]^{-1}, \] (42)
in which
\[ F = \left[ 1 + \frac{0.2783x}{(1 + x)^{0.228}} \right]^{3.886}. \]

The relative errors \( \varepsilon_S(x) \) and \( \varepsilon_A(x) \) can be defined similarly to the relative errors \([38]\).

In order to compare our results with the standard Padé approximants, defined so that to satisfy the boundary conditions \([10]\), we calculate the Padé approximants having no unphysical poles. To construct \( P_{1/4}(x) \) and \( P_{2/5}(x), \) we add to the series an additional zero term. Thus we obtain
\[ P_{0/3}(x) = \frac{Q_0(x)}{Q_3(x)}, \]

where
\[ Q_0(x) = 1, \]
\[ Q_3(x) = 1 + x(5.44951x^2 - 3.59963x^{3/2} + 2.52197x - 1.33333\sqrt{x} + 1.58807) , \]

\[ P_{1/4}(x) = \frac{Q_1(x)}{Q_4(x)} , \]

where

\[ Q_1(x) = 1 + 7.29513\sqrt{x} + 8.70365x , \]
\[ Q_4(x) = 1 + 7.29513\sqrt{x} + 10.2917x + 10.2519x^{3/2} + 6.61714x^2 + 
\quad + 3.19361x^{5/2} + 1.14009x^3 + 0.246624x^{7/2} + 0.0573431x^4 , \]

\[ P_{2/5}(x) = \frac{Q_2(x)}{Q_5(x)} , \]

where

\[ Q_2(x) = 1 + 0.0611225\sqrt{x} - 0.75871x + 2.75597x^{3/2} + 0.957022x^2 , \]
\[ Q_5(x) = 1 + 0.0611225\sqrt{x} + 0.829361x + 1.51971x^{3/2} + 2.19261x^2 + 
\quad + 1.94282x^{5/2} + 1.16124x^3 + 0.560225x^{7/2} + 
\quad + 0.147749x^4 + 0.0609348x^{9/2} - 0.00321342x^5 . \]

We have also constructed \( P_{3/6}(x) \), but its accuracy is rather bad, with the maximal error around 80\% at \( x = 1000 \).

The other admissible sequence of approximants is given by

\[ P_{1/7}(x) = \frac{1 + a_1x^{1/2}}{1 + \sum_{n=1}^7 b_nx^{n/2}} , \]

with the parameters

\[ a_1 = 1.502670 = b_1 , \quad b_2 = 1.588071 , \quad b_3 = 1.053015 , \quad b_4 = 0.518408 , \]
\[ b_5 = 0.190063 , \quad b_6 = 0.040455 , \quad b_7 = 0.010435 , \]

and the Padé approximant

\[ P_{2/8}(x) = \frac{1 + a_1x^{1/2} + a_2x}{1 + \sum_{n=1}^8 b_nx^{n/2}} , \]

with the parameters

\[ a_1 = -8.448419 = b_1 , \quad a_2 = -14.953104 , \quad b_2 = -13.365089 , \]
\[ b_3 = -14.750023 , \quad b_4 = -9.960151 , \quad b_5 = -4.968737 , \]
\[ b_6 = -1.850739 , \quad b_7 = -0.392334 , \quad b_8 = -0.103841 . \]

The results for the Sommerfeld solution \( f_S(x) \), the Andrianov-Awrejscewicz solution \( f_A(x) \), and for the Padé approximants \( P_{0/3}(x) \) and \( P_{1/7}(x) \), with their errors, are shown in
There have been suggested approximate solutions derived by employing the Lagrange variational techniques \[31–33\]. However, these solutions are not sufficiently accurate, yielding relative errors of order 100\%. The empirical Andrianov-Awrejcewicz solution \[30\] has been the most accurate of analytical solutions known till now. But our solution \[37\] is an order more accurate.

The Padé approximants \( P_{0/3}(x) \), \( P_{1/4}(x) \), and \( P_{2/5}(x) \) are shown in Table 3. The approximant \( P_{2/8}(x) \) practically coincides with \( P_{1/7}(x) \), because of which it is not shown. The percentage errors of the Padé approximants \( P_{M/N}(x) \) are denoted as

\[
\varepsilon_{M/N}(x) \equiv \frac{P_{M/N}(x) - f(x)}{f(x)} \times 100\% .
\]

Table 2. The approximant \( P_{2/8}(x) \) practically coincides with \( P_{1/7}(x) \), because of which it is not shown. The percentage errors of the Padé approximants \( P_{M/N}(x) \) are denoted as

\[
\varepsilon_{M/N}(x) \equiv \frac{P_{M/N}(x) - f(x)}{f(x)} \times 100\% .
\]

### 4 Nonlinear Schrödinger equation

The nonlinear Schrödinger equation can be met in many chemical and physical problems. Here we keep in mind the variant describing a spherically symmetric function \( f = f(r) \), with \( r \geq 0 \), satisfying the equation \( E[f] = 0 \) in the form

\[
\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{f}{r^2} + f - f^3 = 0 .
\]
Table 3: The Padé approximants $P_{0/3}(x)$, $P_{1/4}(x)$, $P_{2/5}(x)$. Numerical solution from [28], named as “exact”, is shown as well.

| $x$    | 0.1 | 1   | 40  | 100 | 1000 |
|--------|-----|-----|-----|-----|------|
| $P_{0/3}(x)$ | 0.880 | 0.178 | 3.2×10⁻⁶ | 2×10⁻⁷ | 1.9×10⁻¹⁰ |
| $\varepsilon_{0/3}(x)$ | -0.15 | -58.083 | -99.72 | -99.80 | -99.86 |
| $P_{1/4}(x)$ | 0.882 | 0.424 | 0.00108 | 0.000097 | 1.346×10⁻⁷ |
| $\varepsilon_{1/4}(x)$ | -0.00033 | -0.005 | -2.93 | -3.22 | -0.4 |
| $P_{2/5}(x)$ | 0.882 | 0.424 | 0.00162 | 0.00024 | -9.318×10⁻⁷ |
| $\varepsilon_{2/5}(x)$ | -0.00034 | -0.024 | 45.1 | 141.8 | -780 |
| exact [28] | 0.8817 | 0.4240 | 0.001114 | 0.0001002 | 1.351275×10⁻⁷ |

Under the boundary conditions

$$f(0) = 0 \ , \quad f(\infty) = 1 \ , \quad (44)$$

the solution to this equation represents a vortex line.

At small values of the dimensionless variable $r$, the solution is written [34] [35] as the expansion in powers of $r^2$,

$$f(r) \simeq cr \left( 1 + a_1 r^2 + a_2 r^4 + a_3 r^6 + a_4 r^8 \right) \quad (r \to 0) \ , \quad (45)$$

where

$$a_1 = \frac{-1}{8}, \quad a_2 = \frac{1 + 8c^2}{192}, \quad a_3 = \frac{-1 + 80c^2}{9216}, \quad a_4 = \frac{1 + 656c^2 + 1152c^4}{737280},$$

and the value of $c$ is prescribed by the second of the boundary conditions (44). Here we take $c = 0.58319$, as defined by Ginzburg and Sobyanin [36].

At large $r$, one has [34] the asymptotic expansion

$$f(r) \simeq 1 + \frac{b_1}{r^2} + \frac{b_2}{r^4} + \frac{b_3}{r^6} \quad (r \to \infty) \ , \quad (46)$$

with

$$b_1 = -\frac{1}{2}, \quad b_2 = -\frac{9}{8}, \quad b_3 = -\frac{161}{16} .$$

As an irrational factor, we can take the first-order root approximant

$$f_{irr}^*(r) = \frac{cr}{\sqrt{1 + Ar^2}} \ , \quad (47)$$

with $A = 0.0163972$. Then, following the scheme of Sec. 2, we define the correcting function $f(r)/f_{irr}^*(r)$, expand it in powers of $r$, and construct the related Padé approximant

$$P_{2/2}(r) = \frac{1 + A_1 r^2 + A_2 r^4}{1 + B_1 r^2 + B_2 r^4} \ , \quad (48)$$

in which

$$A_1 = 0.0674195 \ , \quad A_2 = 0.000899209 \ .$$
Thus the corrected Padé approximant is

\[ f_4^*(r) = f_{\text{irr}}^*(r) P_{2/3}(r) . \]  

The accuracy of an approximate solution \( f_{\text{app}}(r) \) to the considered equation, that is denoted as

\[ E[f(r)] = 0 , \]  

can be conveniently characterized by the solution defect

\[ D[f_{\text{app}}(r)] \equiv | E[f_{\text{app}}(r)] | , \]  

which defines the maximal solution defect

\[ D[f_{\text{app}}] \equiv \sup_r D[f_{\text{app}}(r)] . \]

The found solution (49) turns out to be very accurate, having the maximal defect \( D[f_4^*] = 0.0002 \). The standard Padé approximants are not applicable, since they cannot satisfy the correct asymptotic behavior as \( r \to \infty \). Approximate solutions, represented in the form of modified Padé approximants [37], are less accurate than the corrected approximant (49). For example, the modified Padé approximant

\[ P_{3/3}(r) = \left[ \frac{c_1 r^2 + c_2 r^4 + c_3 r^6}{1 + d_1 r^2 + d_2 r^4 + d_3 r^6} \right]^{1/2} , \]

with

\[ c_1 = 0.340111 , \quad c_2 = 0.0745487 , \quad c_3 = 0.0181768 , \]
\[ d_1 = 0.469190 , \quad d_2 = 0.0927255 , \quad d_3 = 0.0181768 , \]

has the maximal defect \( D[P_{3/3}] = 0.04 \), which is two orders larger than the maximal solution defect of the approximant (49).

It is possible to try Padé approximants considered as fractional with respect to the powers of expansions (45) and (46). For such fractional approximants we have

\[ p_{2/2}(r) = \frac{0.58319r (1 + 0.758279r + 0.600453r^2 + 0.971322r^3)}{1 + 0.758279r + 0.725453r^2 + 0.350178r^3 + 0.566465r^4} \]  

and

\[ p_{2/3}(r) = \frac{Q_2(r)}{Q_3(r)} , \]

where

\[ Q_2(r) = 0.58319r (1 + 0.691638r + 0.307419r^2 + 0.0996674r^3 + 0.0234523r^4) \]
\[ Q_3(r) = 1 + 0.691638r + 0.432419r^2 + 0.186122r^3 + 0.058125r^4 + 0.0136771r^5 . \]

The latter approximants can provide a reasonable accuracy for not too large \( r \), although they fail for \( r \to \infty \), not satisfying the boundary conditions (44).
Table 4: Different types of Padé approximants and the corrected approximant $f^*_4(r)$ for the nonlinear Schrödinger equation, with the corresponding solution defects

| $r$ | $p_{2/3}(r)$ | $D[p_{2/3}(r)]$ | $p_{3/3}(r)$ | $D[p_{3/3}(r)]$ | $P_{3/3}(r)$ | $D[P_{3/3}(r)]$ | $f^*_4(r)$ | $D[f^*_4(r)]$ |
|-----|-------------|----------------|-------------|----------------|-------------|----------------|-------------|-------------|
| 0.1 | 0.0583      | -0.12          | 0.856       | 0.097          | -0.021      | -0.00174       | 0.079       | 0.989       |
| 1   | 0.049       | -0.097         | 0.937       | -0.021         | -0.00174    | -0.000303      | 0.973       | 0.988       |
| 2   | 0.0582      | 0.801          | 0.912       | 0.973          | 0.988       | -0.000078      | 0.0002      | 0.989       |
| 3   | -6.6×10^{-6} | 0.0057         | 0.013       | 0.0058         | 0.000078    | 0.00002       | 0.978       | 0.989       |
| 5   | 0.0582      | 0.523          | 0.82        | 0.978          | 0.989       | -0.000071      | 0.00002     | 0.989       |
| 7   | 0.0582      | 0.52           | 0.805       | 0.977          | 0.989       | 0.000032       | 0.000011    | 0.991       |

The results for different approximants are presented in Tables 4 and 5, from where it is seen that the corrected approximant $f^*_4(r)$ is the most accurate.

Several approximants, that are close to the numerical solution [37], are shown in Fig. 4. The Padé approximant $p_{3/3}(r)$ (dot-dashed line), with the maximal error 0.8%; the Padé approximant $P'_{3/3}(r)$ (dotted line), with the maximal error of 1.6%. The corrected Padé approximant is the most accurate, with the maximal error less than 0.4%.

Table 5: Different types of Padé approximants and the corrected approximant $f^*_4(r)$ for the nonlinear Schrödinger equation, with the corresponding errors.

| $r$ | $p_{3/3}(r)$ | $\varepsilon_{p_{3/3}}$ | $P_{3/3}(r)$ | $\varepsilon_{P_{3/3}}$ | $f^*_4(r)$ | $\varepsilon_{f^*_4(r)}$ | numerical |
|-----|-------------|----------------|-------------|----------------|-------------|----------------|-----------|
| 0.1 | 0.0582      | -0.0065       | 0.0582      | -0.0064       | 0.0582      | -0.065         | 0.05825    |
| 1   | 0.52        | -0.46         | 0.523       | 0.26          | 0.52        | -0.37          | 0.522      |
| 2   | 0.801       | -0.75         | 0.82        | 1.64          | 0.805       | -0.25          | 0.807      |
| 3   | 0.912       | -0.81         | 0.927       | 0.92          | 0.918       | -0.05          | 0.919      |
| 5   | 0.973       | -0.47         | 0.978       | -0.022        | 0.977       | -0.13          | 0.978      |
| 7   | 0.988       | -0.3          | 0.989       | -0.18         | 0.989       | -0.18          | 0.991      |

5 Ruina-Dieterich equation

The Ruina-Dieterich equation describes the law of friction between two solid surfaces sliding against each other [38]. In dimensionless form the equation reads as

$$\frac{df}{dt} = b - f^{1-m},$$

(55)

defining a semi-positive function $f = f(t) \geq 0$ of dimensionless time $t \geq 0$. Here we consider the values $b = 0.526$ and $m = 3/2$. The initial condition is $f(0) = 0.5$.

In the asymptotic limit of short time,

$$f(t) \simeq f_{10}(t) \quad (t \to 0),$$

(56)
Figure 1: Different approximants are compared: The corrected Padé approximant $f_4^\ast(r)$ (solid line); the Padé approximant $P'_{3/3}(r)$ (dotted line); the Padé approximant $p_{3/3}(r)$ (dot-dashed line). Naive Berloff’s formula $R(r) = \sqrt{r^2(0.0286^2 + 0.3437\, r^2) + 0.3333}$ from [37] is shown with dashed line. The numerical data from [37] are marked with dots: $R(0.1) = 0.05825$, $R(1) = 0.522$, $R(2) = 0.807$, $R(3) = 0.919$, $R(4) = 0.962$, $R(5) = 0.978$, $R(6) = 0.987$, $R(7) = 0.991$.

The solution can be written as the expansion

$$f_{10}(t) = \sum_{n=0}^{10} a_n t^n, \quad (57)$$

with the coefficients

$$a_0 = \frac{1}{2}, \quad a_1 = -0.888214, \quad a_2 = -0.628062,$$

$$a_3 = -0.853924, \quad a_4 = -1.51297, \quad a_5 = -3.06015, \quad a_6 = -6.70249,$$

$$a_7 = -15.4836, \quad a_8 = -37.1618, \quad a_9 = -91.7923, \quad a_{10} = -231.875.$$

As is evident, the above series diverge.

Contrary to the previous two cases, the solution to this equation does not extend to infinite times, but is limited from the right by a critical point $t_c = 0.329956$, where

$$f(t) \simeq f_{\text{irr}}(t) \quad (t \to t_c - 0), \quad (58)$$

and in the vicinity of this point the function exhibits the irrational behavior in the form of the root approximant

$$f_{\text{irr}}(t) = \frac{1}{2} \left(1 - \frac{t}{t_c}\right)^{2/3}. \quad (59)$$

Introducing the correcting function

$$C_{10}(t) = \frac{f_{10}(t)}{f_{\text{irr}}(t)}, \quad (60)$$
we expand the latter in powers of time, getting
\[ C_{10}(t) \simeq \sum_{n=0}^{10} b_n t^n \quad (t \to 0) , \] (61)
with the coefficients
\[ b_0 = 1 , \quad b_1 = 0.244047 , \quad b_2 = 0.257547 , \]
\[ b_3 = 0.436291 , \quad b_4 = 0.884281 , \quad b_5 = 1.96929 , \quad b_6 = 4.64878 , \]
\[ b_7 = 11.4182 , \quad b_8 = 28.8636 , \quad b_9 = 74.5732 , \quad b_{10} = 196.0 . \]

On the basis of expansion (61), we define the Padé approximant
\[ P_{5/5}(t) = \sum_{n=0}^{5} c_n t^n / \sum_{n=0}^{5} d_n t^n , \] (62)
whose parameters are
\[ c_0 = 1 , \quad c_1 = -6.89716 , \quad c_2 = 16.4086 , \]
\[ c_3 = -15.5576 , \quad c_4 = 4.88054 , \quad c_5 = -0.15572 , \]
\[ d_0 = 1 , \quad d_1 = -7.14121 , \quad d_2 = 17.8938 , \]
\[ d_3 = -18.5216 , \quad d_4 = 7.02358 , \quad d_5 = -0.560998 . \]

Thus the sought approximate solution is
\[ f_{10}^*(t) = f_{\text{irr}}(t) P_{5/5}(t) . \] (63)
This solution very well approximates the sought function \( f(t) \). The maximal deviation of the corrected Padé approximant (63) from the exact numerical solution \( f(t) \), in the whole range of its definition, is
\[ \sup_{t \in [0,t_\text{c}]} |f(t) - f_{10}^*(t)| = 0.0002 . \] (64)

To study convergence, we consider the lower-order corrected approximants
\[ f_8^*(t) = f_{\text{irr}}(t) P_{4/4}(t) , \] (65)
with
\[ P_{4/4}(t) = \frac{1 - 5.37518 t + 8.8021 t^2 - 4.37466 t^3 + 0.243023 t^4}{1 - 5.61923 t + 9.91591 t^2 - 5.78369 t^3 + 0.668044 t^4} , \]
\[ f_6^*(t) = f_{\text{irr}}(t) P_{3/3}(t) , \] (66)
with
\[ P_{3/3}(t) = \frac{1 - 3.85376 t + 3.51763 t^2 - 0.372822 t^3}{1 - 4.09781 t + 4.26014 t^2 - 0.793411 t^3} . \]
and
\[ f_1^*(t) = f_{irr}(t)P_{2/2}(t), \quad (67) \]
with
\[ P_{2/2}(t) = \frac{1 - 2.33262t + 0.560183t^2}{1 - 2.57667t + 0.931464t^2}. \]

The related results are presented in Table 6. These results demonstrate good convergence of the corrected approximants.

For comparison, we also study the Padé approximants
\[ p_{2/2}(t) = \frac{0.5 - 1.99743t + 1.64601t^2}{1 - 2.21843t + 0.60727t^2}, \]
\[ p_{3/3}(t) = \frac{0.5 - 2.76077t + 4.42424t^2 - 1.83935t^3}{1 - 3.74512t + 3.45167t^2 - 0.543545t^3}, \]
\[ p_{4/4}(t) = \frac{0.5 - 3.52899t + 8.39113t^2 - 7.53882t^3 + 1.92114t^4}{1 - 5.28156t + 8.65607t^2 - 4.62722t^3 + 0.501267t^4}, \]
and
\[ p_{5/5}(t) = \frac{0.5 - 4.30291t + 13.5796t^2 - 19.0149t^3 + 11.167t^4 - 1.96629t^5}{1 - 6.82939t + 16.2834t^2 - 15.9743t^3 + 5.77314t^4 - 0.478209t^5}. \]

The corresponding results are shown in Table 7. As is seen, the corrected approximants are more accurate than the Padé approximants.

For further comparison, we also study the non-diagonal Padé approximants
\[ p_{1/2}(t) = \frac{1/2 - 1.33466t}{1 - 0.892888t - 0.330028t^2}, \]
\[ p_{2/3}(t) = \frac{0.5 - 2.20204t + 2.19219t^2}{1 - 2.62765t + 0.972664t^2 + 0.135056t^3}, \]
\[ p_{3/4}(t) = \frac{0.5 - 3.00676t + 5.5076t^2 - 2.91787t^3}{1 - 4.2371t + 4.74443t^2 - 1.02208t^3 - 0.0664454t^4}, \]
and
\[ p_{4/5}(t) = \frac{0.5 - 3.79116t + 9.96764t^2 - 10.4266t^3 + 3.45104t^4}{1 - 5.80588t + 10.8777t^2 - 7.11483t^3 + 1.03717t^4 + 0.0348388t^5}. \]

The results are shown in Table 8. Again, we see that the corrected Padé approximants are essentially more accurate than the standard Padé approximants.

6 Possible further extensions

The described method of self-similarly corrected Padé approximants can be extended to other problems. Here we mention some of possible applications.
Table 6: Self-similarly corrected Padé approximants with their percentage errors

| $t$ | 0.05 | 0.1 | 0.2 | 0.25 | 0.3 | 0.32 |
|-----|------|-----|-----|------|-----|------|
| $f_4^*(t)$ | 0.454 | 0.404 | 0.286 | 0.214 | 0.126 | 0.0834 |
| $\varepsilon_4^*(t)$ | 0.00080016 | -0.00015 | -0.02 | -0.10 | -0.71 | -1.98 |
| $f_6^*(t)$ | 0.454 | 0.404 | 0.286 | 0.212 | 0.115 | 0.0562 |
| $\varepsilon_6^*(t)$ | 0.0000020 | 0.000047 | -0.00072 | -0.012 | -0.2 | -0.94 |
| $f_8^*(t)$ | 0.454 | 0.404 | 0.286 | 0.212 | 0.115 | 0.0566 |
| $\varepsilon_8^*(t)$ | 0.0000020 | 0.000049 | 0.00014 | -0.0010 | -0.059 | -0.46 |
| $f_{10}^*(t)$ | 0.454 | 0.404 | 0.286 | 0.212 | 0.115 | 0.0566 |
| $\varepsilon_{10}^*(t)$ | 0.0000020 | 0.000049 | 0.00018 | -0.0016 | -0.23 | 0.00024 |
| $exact$ | 0.453902 | 0.403853 | 0.286184 | 0.212307 | 0.114745 | 0.0567535 |

Table 7: Diagonal Padé approximants with their percentage errors.

| $t$ | 0.05 | 0.1 | 0.2 | 0.25 | 0.3 | 0.32 |
|-----|------|-----|-----|------|-----|------|
| $p_{2/2}(t)$ | 0.454 | 0.404 | 0.287 | 0.214 | 0.126 | 0.0834 |
| $\varepsilon_{2/2}(t)$ | 0.000023 | 0.00108 | 0.12 | 0.88 | 9.55 | 46.9 |
| $p_{3/3}(t)$ | 0.454 | 0.404 | 0.286 | 0.213 | 0.118 | 0.068 |
| $\varepsilon_{3/3}(t)$ | 2.4x10^-6 | 0.000013 | 0.0059 | 0.0972 | 2.53 | 19.73 |
| $p_{4/4}(t)$ | 0.454 | 0.404 | 0.286 | 0.212 | 0.116 | 0.0618 |
| $\varepsilon_{4/4}(t)$ | 2.4x10^-6 | 4.9x10^-6 | 0.00033 | 0.011 | 0.70 | 8.92 |
| $p_{5/5}(t)$ | 0.454 | 0.404 | 0.286 | 0.212 | 0.115 | 0.0591 |
| $\varepsilon_{5/5}(t)$ | 2.4x10^-6 | 4.8x10^-6 | 0.000046 | 0.0013 | 0.19 | 4.17 |

Table 8: Non-diagonal Padé approximants with their percentage errors.

| $t$ | 0.05 | 0.1 | 0.2 | 0.25 | 0.3 | 0.32 |
|-----|------|-----|-----|------|-----|------|
| $p_{1/2}(t)$ | 0.454 | 0.404 | 0.288 | 0.22 | 0.142 | 0.107 |
| $\varepsilon_{1/2}(t)$ | 0.0009 | 0.02 | 0.76 | 3.61 | 23.6 | 88.8 |
| $p_{3/3}(t)$ | 0.454 | 0.404 | 0.286 | 0.213 | 0.121 | 0.0753 |
| $\varepsilon_{3/3}(t)$ | 3.61x10^-6 | 0.000138 | 0.034 | 0.357 | 5.5 | 32.74 |
| $p_{4/4}(t)$ | 0.454 | 0.404 | 0.286 | 0.212 | 0.116 | 0.0648 |
| $\varepsilon_{4/4}(t)$ | 2.382x10^-6 | 5.818x10^-6 | 0.00168 | 0.0386 | 1.472 | 14.1 |
| $p_{5/5}(t)$ | 0.454 | 0.404 | 0.286 | 0.212 | 0.115 | 0.0604 |
| $\varepsilon_{5/5}(t)$ | 2.381x10^-6 | 4.844x10^-6 | 0.000114 | 0.00436 | 0.407 | 6.47 |
6.1 Finding large-variable behavior

The method can be used for finding the large-variable behavior of functions from their small-variable asymptotic expansions. For instance, suppose we have a small-variable approximation \( f_k(x) \) at \( x \to 0 \). And we need to find out the behaviour of the function at large variables, when \( x \to \infty \). As is evident, the small-variable series \( f_k(x) \) has no sense for \( x \to \infty \). But to find the large-variable exponent, it is possible to proceed as follows. We define the function

\[
\beta_k(x) = \frac{d \ln f(x)}{d \ln x}
\]  

(68)

expanding it in powers of \( x \). Then, according to sec. 2, we derive the self-similarly corrected Padé approximant

\[
\beta_k^*(x) = \beta_{irr}^*(x) P_{N/N}(x)
\]  

(69)

Taking the limit

\[
\beta_k = \lim_{x \to \infty} \beta_k^*(x),
\]  

(70)

we obtain the large-variable exponent defining the behaviour of the sought function at large variables as

\[
f_k^*(x) \simeq B_k x^{\beta_k}.
\]  

(71)

We have checked this way of defining the large-variable exponents for several physical problems and compared the accuracy of the corrected approximants with the standard Padé approximants. As expected, the corrected approximants are always more accurate than the standard Padé approximants and, moreover, exist in the cases where the standard Padé approximants cannot be defined at all. Root approximants can also be used as an initial approximation for calculating critical exponents at phase transitions [39]. In order not to overload the present paper, we do not go here into the details of calculating the large-variable and critical exponents, since our main aim has been to demonstrate the applicability of the method for an accurate solution of nonlinear differential equations.

6.2 Partial differential equations

As we have demonstrated, the method can be applied for solving nonlinear ordinary differential equations. Its generalization to partial differential equations can be done in the following cases.

(i) Equation allows for the standard separation of variables. Suppose we consider a partial differential equation for which the variables can be separated by the standard procedure [40]. Then the problem can be reduced to the set of equations in ordinary derivatives. For each of the separated equations, containing a single variable, it is straightforward to use the developed method.

(ii) Functional separation of variables is allowed. The reduction of a partial differential equation to several ordinary differential equations is also admissible under the functional separation of variables [41]. For example, we consider an equation for a function \( f(x, t) \) of two variables. One says that the equation allows for the functional separation of variables if
there exist functions $F(f)$, $\psi(x)$, and $\varphi(t)$, for which

$$F(f) = \psi(x) + \varphi(t).$$

The separation is called additive for $F(f) = f$ and it is called product separation, if $F(f) = \ln f$. Then, instead of one equation in partial derivatives for $f$, one gets two equations in ordinary derivatives [41].

More generally, when the sought function depends on several variables, say $f = f(x_1, x_2, \ldots)$, they can be functionally separated if there exists a function $F(f)$ such that

$$F(f) = f_1(x_1) + f_2(x_2) + \ldots.$$ 

Then one equation in partial derivatives separates in a set of equations in ordinary derivatives.

(iii) Generalized separation of variables is admissible. For a function of two variables $f(x, t)$, this means the following. A partial differential equation can be reduced to two ordinary differential equations, when there exist functions $g(x/\varphi)$, $\varphi(t)$, and $y(t)$, such that

$$f(x, t) = g(x/\varphi(t))y(t).$$

Again, instead of one equation in partial derivatives, one obtains two equations in ordinary derivatives [42, 43].

(iv) Perturbation theory is used, starting from an approximation with separated variables. Then the solution for the initial approximation can be constructed by employing the method of corrected Padé approximants for each of the separated equations.

(v) Solving an eigenvalue problem, one is interested in eigenvalues. Then it is possible to use perturbation theory in powers of some parameter, considering the eigenvalue as a function of this parameter. For instance, we can be interested in the energy levels of a three-dimensional Schrödinger equation as a function $E(g)$ of a coupling parameter $g$. In that case, the problem is reduced to studying the perturbative series for the eigenvalues $E(g)$ depending on one variable $g$. Such perturbative series can be derived for linear as well as for nonlinear equations [44–46].

(vi) The main aim is the construction of an effective equation interpolating between small and large parameters. The typical example is the construction of an effective Schrödinger equation, with the energy term interpolating between weak and strong coupling [47]. In the latter case, one is looking for the expression of an effective energy interpolating between the weak-coupling Lee-Huang-Yang form [48–50] and the strong-coupling limit corresponding to unitarity [51, 52]. The interpolation can be done by using a kind of a two-point Padé approximation [47]. Because of the small number of the available interpolation terms, the used two-point Padé approximation omits some of the coefficients of the two-point approximants. Also, the number of available terms is not sufficient for using corrected Padé approximants. Nevertheless, despite the simplicity, the constructed interpolation for the energy describes reasonably well the crossover between weak-coupling and strong-coupling limits. Then this energy term is substituted into the equation that is solved numerically [47]. Although in this case, the method of self-similar approximants, because of the small number of the known asymptotic terms, cannot been used, this example illustrates how the method could be applied, provided a sufficient number of expansion terms would be available.
7 Conclusion

When a problem is expected to possess a solution with rational functional behavior, this solution can be described well by Padé approximants that provide the best representation for rational functions [2]. Although a solution exhibiting irrational functional behavior can also be approximated by the higher-order Padé approximants, the results in reasonably low orders are not sufficiently accurate. We are presently focused upon improving the convergence rate of the standard Padé approximants for lower-orders as well as for higher orders. This problem is of high importance in describing finite quantum systems [53] and structured media [54].

An approach is presented allowing for the extension of the method of Padé approximants to the problems with irrational functional behaviour of solutions. The main idea of the approach is in splitting the sought solution into two factors. One of them is defined through self-similar root approximants taking into account irrational functional behaviour. While the second factor is constructed as a Padé approximant corresponding to a rational function, or a function with reducible irrationality. This method of corrected Padé approximants is illustrated by finding approximate analytical solutions to nonlinear differential equations typical of physics, and several branches of applied sciences. We present accurate solutions to Thomas-Fermi equation, nonlinear Schrödinger equation, and Ruina-Dieterich equation. The method is shown to provide a very high accuracy of solutions, essentially better than that of the standard Padé approximants. Since the formulation of the method is general, the described approach can be applied to any nonlinear ordinary differential equation. Possible use of the method for partial differential equations is discussed.

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