Right-angled Artin groups in the $C^\infty$ diffeomorphism group of the real line

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Abstract. We prove that every right-angled Artin group embeds into the $C^\infty$ diffeomorphism group of the real line. As a corollary, we show every limit group, and more generally every countable residually RAAG group, embeds into the $C^\infty$ diffeomorphism group of the real line.

1. Introduction

The right-angled Artin group on a finite simplicial graph $\Gamma$ is the following group presentation:

$$A(\Gamma) = \langle V(\Gamma) \mid [v_i, v_j] = 1 \text{ if and only if } \{v_i, v_j\} \in E(\Gamma) \rangle.$$ 

Here, $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of $\Gamma$, respectively.

For a smooth oriented manifold $X$, we let $\text{Diff}^\infty_+(X)$ denote the group of orientation preserving $C^\infty$ diffeomorphisms on $X$. A group $G$ is said to embed into another group $H$ if there is an injective group homomorphism $G \to H$. Our main result is the following.

Theorem 1. Every right-angled Artin group embeds into $\text{Diff}^\infty_+(\mathbb{R})$.

Recall that a finitely generated group $G$ is a limit group (or a fully residually free group) if for each finite set $F \subset G$, there exists a homomorphism $\phi_F$ from $G$ to a nonabelian free group such that $\phi_F$ is an injection when restricted to $F$. The class of limit groups fits into a larger class of groups, which we call residually RAAG groups. A group $G$ is in this class if for each $1 \neq g \in G$, there exists a graph $\Gamma_g$ and a homomorphism $\phi_g : G \to A(\Gamma_g)$ such that $\phi_g(g) \neq 1$. Since the class of right-angled Artin groups is closed under taking finite direct products, we could replace $g$ with an arbitrary finite subset of $G$. Theorem 1 can be strengthened as follows.

Corollary 2. Every countable residually RAAG group embeds into $\text{Diff}^\infty_+(\mathbb{R})$.

A similar argument to the proof of Theorem 1 also applies to the group $\text{PL}_+(\mathbb{R})$ of orientation preserving piecewise-linear homeomorphisms of $\mathbb{R}$.

Corollary 3. Every countable residually RAAG group embeds into $\text{PL}_+(\mathbb{R})$. 

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Our construction requires infinitely many domains of linearity in $\mathbb{R}$, so that we must take infinite subdivisions of $\mathbb{R}$ in order to get residually RAAG groups inside of $\text{PL}_+(\mathbb{R})$. The reader may compare Corollary 3 with the work of Brin and Squier [6], which shows that the group $\text{PLF}(\mathbb{R})$ of piecewise-linear homeomorphisms of $\mathbb{R}$ with finite subdivisions contains no nonabelian free subgroup.

Recall that a group $G$ is called virtually special if a finite index subgroup $H \leq G$ acts properly and cocompactly on a CAT(0) cube complex $X$ such that the quotient $H \backslash X$ avoids certain pathologies of its half–planes (see [16]). A consequence of such an action is that the subgroup $H$ embeds in a right-angled Artin group.

**Corollary 4.** Let $G$ be a group which is virtually special. Then there is a finite index subgroup $H \leq G$ which embeds into $\text{Diff}^\infty_+ (\mathbb{R})$ and also into $\text{PL}_+(\mathbb{R})$.

Examples of virtually special groups include fundamental groups of closed surfaces and finite volume hyperbolic 3–manifold groups [1, 22]. Combining the work of Bergeron–Haglund–Wise [3] and Bergeron–Wise [4], we have that there are virtually special closed hyperbolic manifolds in all dimensions.

The finitely presented subgroups of diffeomorphism groups are generally very complicated. In [5], Bridson used a virtually special version of the Rips machine to produce finitely presented subgroups of right-angled Artin groups with exotic algorithmic properties. An arbitrary class of groups which contains sufficiently complicated right-angled Artin groups thereby also has finitely presented subgroups with exotic algorithmic properties (cf. [18]):

**Corollary 5.** Suppose $G = \text{Diff}^\infty_+ (\mathbb{R})$ or $G = \text{PL}_+(\mathbb{R})$. Then there is a finitely presented subgroup $H \leq G$ such that the conjugacy problem in $H$ is unsolvable. Furthermore, the isomorphism problem for the class of finitely presented subgroups of $G$ is unsolvable.

The reader may contrast Corollary 5 with Thompson’s groups $F$, $T$, and $V$, in which the conjugacy problem is generally solvable [2]. The group $T$ can be embedded into $\text{Diff}^\infty_+ (S^1)$ [15].

The group $H$ in Corollary 5 is not conjugacy separable, since conjugacy separable groups have solvable conjugacy problems. In [13], Farb and Franks showed that the group of real analytic diffeomorphisms of $\mathbb{R}$ contains non–solvable Baumslag–Solitar groups, which are not even residually finite.

### 1.1 Notes and references.

It is well-known that right-angled Artin groups embed in $\text{Homeo}(\mathbb{R})$, as follows from the fact that right-angled Artin groups admit left–invariant orders. Similarly, right-angled Artin groups embed in $\text{Homeo}(S^1)$ because they admit left–invariant cyclic orderings. Alternatively, one can embed an arbitrary right-angled Artin group into the mapping class group $\text{Mod}(S)$ of a compact surface with one boundary, and then embed $\text{Mod}(S)$ into $\text{Homeo}(S^1)$. As noted in [12, p.47], it is generally difficult to
smoothen these embeddings. The reader may consult [20] for a general discussion of the relationship between linear and cyclic orderings of groups and embeddings into Homeo(\(\mathbb{R}\)) and Homeo(\(S^1\)).

Farb and Franks [14] proved that every residually torsion–free nilpotent group embeds in the group of \(C^1\) diffeomorphisms of the interval and of the circle. This implies that residually RAAG groups embed in the group of \(C^1\) diffeomorphisms of both the interval and of the circle. Their construction does not allow for twice differentiable diffeomorphisms. In fact, Plante and Thurston [21] showed that nilpotent groups of \(C^2\) diffeomorphisms of the interval or of the circle are abelian. In the same vein, Farb and Franks [14] show that every nilpotent subgroup of \(C^2\) diffeomorphisms of \(\mathbb{R}\) is metabelian.

For the case when the dimension is two, Calegari and Rolfsen proved that every right-angled Artin group embeds into the piecewise linear homeomorphism group of a square fixing the boundary [7]. M. Kapovich showed that every right-angled Artin group embeds into the symplectomorphism group of the sphere [17]. The second and the third named authors refined this result by embedding every right-angled Artin group into the symplectomorphism groups of the disk and of the sphere by quasi-isometric group embeddings [18].

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3. Building an injective homomorphism

Throughout this section, we let \(\Gamma\) be a finite graph. Consider an element \(g\) of \(A(\Gamma)\). A reduced word representing \(g\) means a minimal length word in the standard generating set \(V(\Gamma) \cup V(\Gamma)^{-1}\) representing \(g\). The support of \(g\) is the set of vertices \(v\) of \(\Gamma\) such that \(v\) or \(v^{-1}\) appears in a reduced word representing \(g\). We denote the support of \(g\) by \(\text{supp}(g)\). We say \(g\) is a clique word if every pair of vertices in \(\text{supp}(g)\) are adjacent in \(\Gamma\). A clique word decomposition for \(g\) is the concatenation \(w_k \cdots w_1\) of clique words \(w_1, w_2, \ldots, w_k\) such that the concatenation is still reduced and represents \(g\) in \(A(\Gamma)\). Since vertices themselves are clique words, every element in \(A(\Gamma)\) has a clique word decomposition. A clique word decomposition \(w_k \cdots w_1\) is left–greedy if for each \(i < k\) and for each \(v \in \text{supp}(w_i)\), there exists a vertex \(v' \in \text{supp}(w_{i+1})\) such that \([v, v'] \neq 1\). The left–greedy clique word decomposition can be compared to the left–greedy normal form used in [19].
Lemma 6. Every element of \( A(\Gamma) \) admits a left–greedy clique word decomposition.

Proof. Fix an element \( g \in A(\Gamma) \). Let us define the complexity of a clique word decomposition \( w_kw_{k-1} \cdots w_1 \) for \( g \) as the \( k \)-tuple consisting of the word lengths \( (|w_k|, |w_{k-1}|, \ldots, |w_1|) \). In the lexicographical order, this complexity is bounded above by \( (|g|) \). Let us assume \( w_kw_{k-1} \cdots w_1 \) is the maximal clique word decomposition for \( g \) in this order. If \( w_k \cdots w_1 \) is not left–greedy, then there exists \( i \) and \( v \in \text{supp}(w_{i-1}) \) be such that \( v \cup \text{supp}(w_i) \) spans a complete subgraph of \( \Gamma \). Then we can move an occurrence of \( v \) or \( v^{-1} \) in \( w_{i-1} \) to \( w_i \), i.e. we slide to the left. This is a contradiction to the maximality. □

Let \( X \) be a smooth oriented manifold. If \( f \in \text{Diff}_{\infty}^\infty(X) \), then we write the fixed point set of \( f \) as \( \text{Fix}(f) \). The closure of \( X \setminus \text{Fix}(f) \) will be denoted as \( \text{supp}(f) \). For \( f_1, f_2, \ldots, f_n \in \text{Diff}_{\infty}^\infty(X) \), let us define the disjointness graph \( \Lambda \) by \( V(\Lambda) = \{v_1, v_2, \ldots, v_n\} \) and

\[
E(\Lambda) = \{(v_i, v_j) : i \neq j \text{ and } \text{supp}(f_i) \cap \text{supp}(f_j) = \emptyset\}.
\]

Note that two self-maps with disjoint supports commute. Hence, we have a group homomorphism \( \phi : A(\Lambda) \to \text{Diff}_{\infty}^\infty(X) \) satisfying \( \phi(v_i) = f_i \). Often, it is not an obvious task at all to decide whether or not such a map \( \phi \) is injective; see [11, 17, 18, 7] for related work on diffeomorphism groups and [2, 10, 19, 8] on mapping class groups.

Let \( H \) be a group. Another group \( G \) is residually \( H \) if for each nontrivial element \( g \) in \( G \) there exists a group homomorphism \( \phi_g : G \to H \) such that \( \phi_g(g) \neq 1 \). For a smooth oriented manifold \( X \), we let \( \text{Diff}_{\infty}^\infty(X, \partial X) \) denote the group of orientation preserving \( C^\infty \) diffeomorphisms on \( X \) which restrict to the identity near \( \partial X \). The crucial fact we need is the following:

Lemma 7. The group \( A(\Gamma) \) is residually \( \text{Diff}_{\infty}^\infty(I, \partial I) \).

Proof. Let us fix an element \( g \) in \( A(\Gamma) \). We let \( w_k \cdots w_1 \) be a left–greedy clique decomposition representing \( g \). We will construct a group homomorphism

\[
\phi_g : A(\Gamma) \to \text{Diff}_{\infty}^\infty(I, \partial I)
\]

such that \( \phi_g(g) \neq 1 \). We can inductively choose a (possibly redundant) sequence

\[
v_1 \in \text{supp}(w_1), v_2 \in \text{supp}(w_2), \ldots, v_k \in \text{supp}(w_k)
\]

such that \([v_i, v_{i+1}] \neq 1\) for each \( i = 1, 2, \ldots, k-1 \). There exists \( \sigma_i \in \{-1, 1\} \) and \( n_i > 0 \) such that \( v_i^{\sigma_i \cdot n_i} \) is the highest power of \( v_i \) in \( w_i \). This means that \( v_i \not\in \text{supp}(w_i \cdot v_i^{-\sigma_i \cdot n_i}) \).

We choose \( \rho \in \text{Diff}_{\infty}^+((\mathbb{R})\) such that \( \rho(1/4) = 5/4 \) and \( \rho(x) = x \) for \( x \leq 0 \) or \( x \geq 3/2 \). Put \( \rho_i(x) = \rho(x-i) + i \) and \( I_i = [i, i+3/2] \), so that \( \text{supp} \rho_i \subseteq I_i \).

Note that \( I_i \cap I_j = \emptyset \) for \( |i - j| > 1 \).

For each \( v \in V(\Gamma) \), we define

\[
\psi_g(v) = \prod_{v_j = v} \rho_j^{s_j}.
\]
This means that if \( \{ j : v_j = v \} = \{ j_1 < j_2 < \cdots < j_n \} \) then \( \psi_g(v) = \rho_{j_1}^{a_1} \rho_{j_2}^{a_2} \cdots \rho_{j_n}^{a_n} \). We use the convention that the empty multiplication is trivial. Let us write \( J_v = \cup_{v_j = v} I_j \supseteq \text{supp} \psi_g(v) \). Note that if \( v_i = v = v_j \) and \( i \neq j \), then \( I_i \cap I_j = \emptyset \). For each \( \{ u, v \} \in E(\Gamma) \), the choice of \( v_1, v_2, \ldots, v_k \) implies that \( J_u \cap J_v = \emptyset \). In other words, \( \Gamma \) is a subgraph of the disjointness graph of \( \{ \psi_g(v) : v \in V(\Gamma) \} \). It follows that \( \psi_g \) defines a group homomorphism from \( A(\Gamma) \) to \( \text{Diff}_\infty^+({\mathbb R}) \).

Suppose \( \ell \in \{ 1, 2, \ldots, k \} \). For every \( u \in \text{supp} w_\ell \setminus \{ v_\ell \} \), we have \( J_u \cap J_v = \emptyset \). Since \( \ell + 1/4 \in I_\ell \subseteq J_{v_\ell} \), we see that if \( x \in [\ell+1/4, \ell+1/2] \) is an arbitrary point then

\[
\psi_g(w_\ell)(x) = \psi_g(v_\ell^{a_\ell}) \cdot \rho_\ell^{a_\ell}(x) = \rho_\ell^{a_\ell}(x) \in [\ell + 5/4, \ell + 3/2].
\]

It follows that if \( y \in [5/4, 3/2] \) is arbitrary then

\[
\psi_g(g)(y) = \psi_g(w_\ell w_{\ell-1} \cdots w_1)(y) \in [k + 5/4, k + 3/2].
\]

In particular, \( \psi_g(g) \) is not the identity. By restricting the image of \( \psi_g \) onto the interval \([0, k + 2]\) and conjugating by a diffeomorphism \([0, k + 2] \approx I \) we obtain a desired group homomorphism \( \phi_g \). \( \square \)

We have the following general fact.

**Lemma 8.** Let \( G \) and \( H \) be groups. If \( G \) is countable and residually \( H \), then \( G \) embeds into the countable direct product \( \prod_{\ell} H \).

**Proof.** Define

\[
\psi : G \to \prod_{g \in G} H
\]

by \( \psi(x)(g) = \phi_g(x) \), where \( \phi_g \) is as in the definition of residually \( H \) group. \( \square \)

Since \( \text{Diff}_\infty^+([0, \infty), \{0\}) \hookrightarrow \text{Diff}_\infty^+({\mathbb R}) \), Theorem 1 is a trivial consequence of the following:

**Theorem 9.** Every right-angled Artin group embeds into \( \text{Diff}_\infty^+([0, \infty), \{0\}) \).

**Proof.** Immediate from Lemmas 7 and 8 as well as the fact that

\[
\prod_{z} \text{Diff}_\infty^+(I, \partial I) \hookrightarrow \text{Diff}_\infty^+([0, \infty), \{0\}).
\]

**Proof of Corollary 2.** Lemma 7 implies that a residually RAAG is residually \( \text{Diff}_\infty^+(I, \partial I) \). We proceed as Theorem 9. \( \square \)

**Proof of Corollary 3.** It suffices to show that every right-angled Artin group is residually \( \text{PL}_+(I) \). For this, we follow the proof of Lemma 7 by using \( \rho_0 \in \text{PL}_+(\mathbb R) \) defined by

\[
\rho_0(x) = \begin{cases} 
   x & \text{if } x < 0 \text{ or } x \geq \frac{3}{2} \\
   5x & \text{if } 0 \leq x < \frac{1}{4} \\
   (x + 6)/5 & \text{if } \frac{1}{4} \leq x < \frac{3}{2}
\end{cases}
\]
instead of $\rho$. 

\begin{thebibliography}{99}

1. Ian Agol, \textit{The virtual Haken conjecture}, Doc. Math. \textbf{18} (2013), 1045–1087, With an appendix by Agol, Daniel Groves, and Jason Manning. MR 3104553

2. James Belk and Francesco Matucci, \textit{Conjugacy and dynamics in Thompson’s groups}, Geom. Dedicata \textbf{169} (2014), 239–261. MR 3175247

3. Nicolas Bergeron, Frédéric Haglund, and Daniel T. Wise, \textit{Hyperplane sections in arithmetic hyperbolic manifolds}, J. Lond. Math. Soc. (2) \textbf{83} (2011), no. 2, 431–448. MR 2776645 (2012f:57037)

4. Nicolas Bergeron and Daniel T. Wise, \textit{A boundary criterion for cubulation}, Amer. J. Math. \textbf{134} (2012), no. 3, 843–859. MR 2931226

5. Martin R. Bridson, \textit{On the subgroups of right angled artin groups and mapping class groups}, To appear in Math. Res. Lett.

6. Matthew G. Brin and Craig C. Squier, \textit{Groups of piecewise linear homeomorphisms of the real line}, Inventiones Mathematicae \textbf{79} (1985), no. 3, 485–498.

7. Danny Calegari and Dale Rolfsen, \textit{Groups of PL homeomorphisms of cubes}, preprint (2014).

8. Matt T. Clay, Christopher J. Leininger, and Johanna Mangahas, \textit{The geometry of right-angled Artin subgroups of mapping class groups}, Groups Geom. Dyn. \textbf{6} (2012), no. 2, 249–278. MR 2914860

9. John Crisp and Luis Paris, \textit{The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group}, Invent. Math. \textbf{145} (2001), no. 1, 19–36. MR 1839284 (2002j:20069)

10. John Crisp and Bert Wiest, \textit{Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups}, Algebr. Geom. Topol. \textbf{4} (2004), 439–472. MR 2077673 (2005c:20052)

11. Benson Farb, \textit{Some problems on mapping class groups and moduli space}, Problems on mapping class groups and related topics, Proc. Sympos. Pure Math., vol. 74, Amer. Math. Soc., Providence, RI, 2006, pp. 11–55. MR 2264130 (2007h:57035)

12. Benson Farb and John Franks, \textit{Groups of homeomorphisms of one-manifolds, I: Actions of nonlinear groups}, Preprint.

13. Benson Farb and John Franks, \textit{Groups of homeomorphisms of one-manifolds, III: Nilpotent subgroups}, Ergodic Theory Dynam. Systems \textbf{23} (2003), no. 5, 1467–1484.

14. Étienne Ghys and Vlad Sergiescu, \textit{Sur un groupe remarquable de difféomorphismes du cercle}, Comment. Math. Helv. \textbf{62} (1987), no. 2, 185–239. MR 896095 (90c:57035)

15. Frédéric Haglund and Daniel T. Wise, \textit{Special cube complexes}, Geom. Funct. Anal. \textbf{17} (2008), no. 5, 1551–1620. MR 2377497 (2009a:20061)

16. Michael Kapovich, \textit{RAAGs in Ham}, Geom. Funct. Anal. \textbf{22} (2012), no. 3, 733–755. MR 2972607

17. Sang-hyun Kim and Thomas Koberda, \textit{Anti–trees and right-angled Artin subgroups of planar braid groups}, preprint.

18. Tomás Koivisto, \textit{Right-angled Artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups}, Geom. Funct. Anal. \textbf{22} (2012), no. 6, 1541–1590. MR 3000498

19. Andrés Navas, \textit{Groups of circle diffeomorphisms}, Spanish ed., Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2011. MR 2809110

20. Joseph Plante and William P. Thurston, \textit{Polynomial growth in holonomy groups of foliations}, Comment. Math. Helvetic 

1. \textbf{51} (1976), no. 39, 567–584.

\end{thebibliography}
22. Daniel T. Wise, *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics, vol. 117, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2012. MR 2986461

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