Marshall’s lemma for convex density estimation

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Abstract: Marshall’s [Nonparametric Techniques in Statistical Inference (1970) 174–176] lemma is an analytical result which implies \sqrt{n}-consistency of the distribution function corresponding to the Grenander [Skand. Aktuarietidskr. 39 (1956) 125–153] estimator of a non-decreasing probability density. The present paper derives analogous results for the setting of convex densities on \([0, \infty)\).

1. Introduction

Let \(F\) be the empirical distribution function of independent random variables \(X_1, X_2, \ldots, X_n\) with distribution function \(F\) and density \(f\) on the halfline \([0, \infty)\). Various shape restrictions on \(f\) enable consistent nonparametric estimation of it without any tuning parameters (e.g. bandwidths for kernel estimators).

The oldest and most famous example is the Grenander estimator \(\hat{f}\) of \(f\) under the assumption that \(f\) is non-increasing. Denoting the family of all such densities by \(\mathcal{F}\), the Grenander estimator may be viewed as the maximum likelihood estimator,

\[ \hat{f} = \arg\max \left\{ \int \log h \, dF : h \in \mathcal{F} \right\}, \]

or as a least squares estimator,

\[ \hat{f} = \arg\min \left\{ \int_0^\infty h(x)^2 \, dx - 2 \int h \, dF : h \in \mathcal{F} \right\}; \]

cf. Robertson et al. [5]. Note that if \(F\) had a square-integrable density \(F'\), then the preceding argmin would be identical with the minimizer of \(\int_0^\infty (h - F')^2 \, dx\) over all non-increasing probability densities \(h\) on \([0, \infty)\).

A nice property of \(\hat{f}\) is that the corresponding distribution function \(\hat{F}\),

\[ \hat{F}(r) := \int_0^r \hat{f}(x) \, dx, \]

is automatically \(\sqrt{n}\)-consistent. More precisely, since \(\hat{F}\) is the least concave majorant of \(F\), it follows from Marshall’s [4] lemma that

\[ \|\hat{F} - F\|_\infty \leq \|F - F\|_\infty. \]

A more refined asymptotic analysis of \(\hat{F} - F\) has been provided by Kiefer and Wolfowitz [2].
2. Convex densities

Now we switch to the estimation of a convex probability density \( f \) on \([0, \infty)\). As pointed out by Groeneboom et al. [2], the nonparametric maximum likelihood estimator \( \hat{f}_{ml} \) and the least squares estimator \( \hat{f}_{ls} \) are both well-defined and unique, but they are not identical in general. Let \( \mathcal{K} \) denote the convex cone of all convex and integrable functions \( g \) on \([0, \infty)\). (All functions within \( \mathcal{K} \) are necessarily nonnegative and non-increasing.) Then

\[
\hat{f}_{ml} = \text{argmax}_{h \in \mathcal{K}} \left( \int \log h \, dF - \int_0^\infty h(x) \, dx \right),
\]

\[
\hat{f}_{ls} = \text{argmin}_{h \in \mathcal{K}} \left( \int_0^\infty h(x)^2 \, dx - 2 \int h \, dF \right).
\]

Both estimators have the following property:

**Proposition 1.** Let \( \hat{f} \) be either \( \hat{f}_{ml} \) or \( \hat{f}_{ls} \). Then \( \hat{f} \) is piecewise linear with

- at most one knot in each of the intervals \((X(i), X(i+1))\), \(1 \leq i < n\),
- no knot at any observation \(X_i\), and
- precisely one knot within \((X(n), \infty)\).

The estimators \( \hat{f}_{ml} \), \( \hat{f}_{ls} \) and their distribution functions \( \hat{F}_{ml} \), \( \hat{F}_{ls} \) are completely characterized by Proposition 1 and the next proposition.

**Proposition 2.** Let \( \Delta \) be any function on \([0, \infty)\) such that \( \hat{f}_{ml} + t \Delta \in \mathcal{K} \) for some \( t > 0 \). Then

\[
\int \Delta \, d\hat{F}_{ml} \leq \int \Delta(x) \, dx.
\]

Similarly, let \( \Delta \) be any function on \([0, \infty)\) such that \( \hat{f}_{ls} + t \Delta \in \mathcal{K} \) for some \( t > 0 \). Then

\[
\int \Delta \, d\hat{F}_{ls} \leq \int \Delta \, d\hat{F}_{ls}.
\]

In what follows we derive two inequalities relating \( \hat{F} - F \) and \( F - \hat{F} \), where \( \hat{F} \) stands for \( \hat{F}_{ml} \) or \( \hat{F}_{ls} \):

**Theorem 1.**

\[
(1) \quad \inf_{[0, \infty)} (\hat{F}_{ml} - F) \geq \frac{3}{2} \inf_{[0, \infty)} (F - F) - \frac{1}{2} \sup_{[0, \infty)} (F - F),
\]

\[
(2) \quad \| \hat{F}_{ls} - F \|_\infty \leq 2 \| F - F \|_\infty.
\]

Both results rely on the following lemma:

**Lemma 1.** Let \( F, \hat{F} \) be continuous functions on a compact interval \([a, b]\), and let \( \hat{F} \) be a bounded, measurable function on \([a, b]\). Suppose that the following additional assumptions are satisfied:

\[
(3) \quad \hat{F}(a) = F(a) \text{ and } \hat{F}(b) = F(b),
\]

\[
(4) \quad \hat{F} \text{ has a linear derivative on } (a, b),
\]

\[
(5) \quad F \text{ has a convex derivative on } (a, b),
\]

\[
(6) \quad \int_r^b \hat{F}(y) \, dy \leq \int_r^b F(y) \, dy \text{ for all } r \in [a, b].
\]
Then
\[ \sup_{[a,b]} (\hat{F} - F) \leq \frac{3}{2} \sup_{[a,b]} (\overline{F} - F) - \frac{1}{2}(\overline{F} - F)(b). \]

If condition \((6)\) is replaced with
\[ \int_r^a \hat{F}(x) \, dx \geq \int_a^r F(x) \, dx \quad \text{for all } r \in [a, b], \]
then
\[ \inf_{[a,b]} (\hat{F} - F) \geq \frac{3}{2} \inf_{[a,b]} (\overline{F} - F) - \frac{1}{2}(\overline{F} - F)(a). \]

The constants \(3/2\) and \(1/2\) are sharp. For let \([a, b] = [0, 1]\) and define
\[
F(x) := \begin{cases} 
  x^2 - c & \text{for } x \geq \epsilon, \\
  (x/\epsilon)(\epsilon^2 - c) & \text{for } x \leq \epsilon,
\end{cases}
\]
\[
\hat{F}(x) := 0, 
\]
\[
\overline{F}(x) := 1 \{0 < x < 1\}(x^2 - 1/3)
\]
for some constant \(c \geq 1\) and some small number \(\epsilon \in (0, 1/2]\). One easily verifies conditions \((3)\)–\((6)\). Moreover,
\[
\sup_{[0,1]} (\hat{F} - F) = c - \epsilon^2, \quad \sup_{[0,1]} (\overline{F} - F) = c - 1/3 \quad \text{and} \quad (\overline{F} - F)(1) = c - 1.
\]

Hence the upper bound \((3/2)\sup(\overline{F} - F) - (1/2)(\overline{F} - F)(1)\) equals \(\sup(\hat{F} - F) + \epsilon^2\) for any \(c \geq 1\). Note the discontinuity of \(F\) at 0 and 1. However, by suitable approximation of \(F\) with continuous functions one can easily show that the constants remain optimal even under the additional constraint of \(F\) being continuous.

**Proof of Lemma.** We define \(G := \hat{F} - F\) with derivative \(g := G'\) on \((a, b)\). It follows from \((5)\) that
\[ \max_{(a,b)} G = \max_{(a,b)} (\overline{F} - F) \leq \frac{3}{2} \sup_{[a,b]} (\overline{F} - F) - \frac{1}{2}(\overline{F} - F)(b). \]
Therefore it suffices to consider the case that \(G\) attains its maximum at some point \(r \in (a, b)\). In particular, \(g(r) = 0\). We introduce an auxiliary linear function \(\tilde{g}\) on \([r, b]\) such that \(\tilde{g}(r) = 0\) and
\[
\int_r^b \tilde{g}(y) \, dy = \int_r^b g(y) \, dy = G(b) - G(r).
\]
Note that \(g\) is concave on \((a, b)\) by \((4)\)–\((5)\). Hence there exists a number \(y_o \in (r, b]\) such that
\[ g - \tilde{g} \begin{cases} 
  \geq 0 & \text{on } [r, y_o], \\
  \leq 0 & \text{on } [y_o, b].
\end{cases} \]
This entails that
\[ \int_r^y (g - \tilde{g})(u) \, du = -\int_y^b (g - \tilde{g})(u) \, du \geq 0 \quad \text{for any } y \in [r, b]. \]
Consequently,

\[ G(y) = G(r) + \int_r^y g(u) \, du \]

\[ \geq G(r) + \int_r^y \tilde{g}(u) \, du \]

\[ = G(r) + \frac{(y - r)^2}{(b - r)^2} [G(b) - G(r)], \]

so that

\[ \int_r^b G(y) \, dy \geq (b - r)G(r) + \frac{G(b) - G(r)}{(b - r)^2} \int_r^b (y - r)^2 \, dy \]

\[ = (b - r) \left[ \frac{2}{3} G(r) + \frac{1}{3} G(b) \right] \]

\[ = (b - r) \left[ \frac{2}{3} G(r) + \frac{1}{3} (F - F)(b) \right]. \]

On the other hand, by assumption (6),

\[ \int_r^b G(y) \, dy \leq \int_r^b (F - F)(y) \, dy \leq (b - r) \sup_{[a, b]} (F - F). \]

This entails that

\[ G(r) \leq \frac{3}{2} \sup_{[a, b]} (F - F) - \frac{1}{2} (F - F)(b). \]

If (6) is replaced with (7), then note first that

\[ \min_{\{a, b\}} G = \min_{\{a, b\}} (F - F) \geq \frac{3}{2} \min_{\{a, b\}} (F - F) - \frac{1}{2} (F - F)(a). \]

Therefore it suffices to consider the case that \( G \) attains its minimum at some point \( r \in (a, b) \). Now we consider a linear function \( \tilde{g} \) on \([a, r]\) such that \( \tilde{g}(r) = 0 \) and

\[ \int_a^r \tilde{g}(x) \, dx = \int_a^r \tilde{g}(x) \, dx = G(r) - G(a). \]

Here concavity of \( g \) on \((a, b)\) entails that

\[ \int_a^x (g - \tilde{g})(u) \, du = - \int_x^r (g - \tilde{g})(u) \, du \leq 0 \quad \text{for any } x \in [a, r], \]

so that

\[ G(x) = G(r) - \int_x^r \tilde{g}(u) \, du \]

\[ \leq G(r) - \int_x^r g(u) \, du \]

\[ = G(r) - \frac{(r - x)^2}{(r - a)^2} [G(r) - G(a)]. \]

Consequently,

\[ \int_a^r G(x) \, dx \leq (r - a)G(r) - \frac{G(r) - G(a)}{(r - a)^2} \int_a^r (r - x)^2 \, dx \]

\[ = (r - a) \left[ \frac{2}{3} G(r) + \frac{1}{3} (F - F)(a) \right]. \]
and it follows from Proposition 2 that

\[ F_k \]

Similarly let \( \Delta_2 \) be continuous and piecewise linear with knots at \( t_k-1, t_k-1+\epsilon, t_k-\epsilon \) and \( t_k \). Precisely, let \( \Delta_2(x) := 0 \) for \( x \notin (t_k-1, t_k) \) and

\[
\Delta_2(x) := \begin{cases} 
-\hat{f}_{ml}(x) & \text{if } \hat{f} = \hat{f}_{ml}, \\
-1 & \text{if } \hat{f} = \hat{f}_{ls} 
\end{cases}
\]

for \( x \in [t_k-1+\epsilon, t_k-\epsilon] \).

The limit of \( \Delta_2(x) \) as \( \epsilon \to 0 \) equals

\[
\begin{cases} 
-1 \{t_k-1 < x < t_k\} & \text{if } \hat{f} = \hat{f}_{ml}, \\
-1 \{t_k-1 < x < t_k\} & \text{if } \hat{f} = \hat{f}_{ls} 
\end{cases}
\]

and it follows from Proposition 2 that

\[
F(t_k) - F(t_{k-1}) \geq \hat{F}(t_k) - \hat{F}(t_{k-1}).
\]

This shows that \( F(t_k) - F(t_{k-1}) = \hat{F}(t_k) - \hat{F}(t_{k-1}) \) for \( k = 1, \ldots, m \). Since \( \hat{F}(0) = 0 \), one can rewrite this as

\[ F(t_k) = \hat{F}(t_k) \quad \text{for } k = 0, 1, \ldots, m. \]  

Now we consider first the maximum likelihood estimator \( \hat{f}_{ml} \). For \( 0 \leq k < m \) and \( r \in (t_k, t_{k+1}] \) let \( \Delta(x) := 0 \) for \( x \notin (t_k-\epsilon, r) \), let \( \Delta \) be linear on \([t_k-\epsilon, t_k]\),

whereas

\[
\int_a^r G(x) \, dx \geq \int_a^r (F - F)(x) \, dx \geq (r - a) \inf_{[a,b]} (F - F),
\]

by assumption (7). This leads to

\[
G(r) \geq \frac{3}{2} \inf_{[a,b]} (F - F) - \frac{1}{2}(F - F)(a).
\]

\( \square \)

\( \square \)

**Proof of Theorem 2** Let 0 =: \( t_0 < t_1 < \cdots < t_m \) be the knots of \( \hat{f} \), including the origin. In what follows we derive conditions (3)–(5) and (6/7) of Lemma 1 for any interval \([a,b] = [t_k, t_{k+1}] \) with \( 0 \leq k < m \). For the reader’s convenience we rely entirely on Proposition 2 in case of the least squares estimator, similar inequalities and arguments may be found in Groeneboom et al. [2].

Let \( 0 < \epsilon < \min_{1 \leq i \leq m} (t_i - t_{i-1})/2 \). For a fixed \( k \in \{1, \ldots, m\} \) we define \( \Delta_1 \) to be continuous and piecewise linear with knots \( t_{k-1} - \epsilon \) (if \( k > 1 \)), \( t_{k-1}, t_k \) and \( t_k + \epsilon \). Namely, let \( \Delta_1(x) = 0 \) for \( x \notin (t_{k-1} - \epsilon, t_k + \epsilon) \) and

\[
\Delta_1(x) := \begin{cases} \hat{f}_{ml}(x) & \text{if } \hat{f} = \hat{f}_{ml}, \\
1 & \text{if } \hat{f} = \hat{f}_{ls} \end{cases}
\]

for \( x \in [t_{k-1}, t_k] \).

This function \( \Delta_1 \) satisfies the requirements of Proposition 2. Letting \( \epsilon \downarrow 0 \), the function \( \Delta_1(x) \) converges pointwise to

\[
\{ 1 \{t_{k-1} \leq x \leq t_k\} \hat{f}_{ml}(x) \text{ if } \hat{f} = \hat{f}_{ml}, \\
1 \{t_{k-1} \leq x \leq t_k\} \text{ if } \hat{f} = \hat{f}_{ls},
\]

and the latter proposition yields the inequality

\[
F(t_k) - F(t_{k-1}) \leq \hat{F}(t_k) - \hat{F}(t_{k-1}).
\]

Similarly let \( \Delta_2 \) be continuous and piecewise linear with knots at \( t_{k-1}, t_{k-1} + \epsilon, t_k - \epsilon \) and \( t_k \). Precisely, let \( \Delta_2(x) := 0 \) for \( x \notin (t_{k-1}, t_k) \) and

\[
\Delta_2(x) := \begin{cases} -\hat{f}_{ml}(x) & \text{if } \hat{f} = \hat{f}_{ml}, \\
-1 & \text{if } \hat{f} = \hat{f}_{ls} \end{cases}
\]

for \( x \in [t_{k-1} + \epsilon, t_k - \epsilon] \).

The limit of \( \Delta_2(x) \) as \( \epsilon \to 0 \) equals

\[
\begin{cases} -1 \{t_{k-1} < x < t_k\} & \text{if } \hat{f} = \hat{f}_{ml}, \\
-1 \{t_{k-1} < x < t_k\} & \text{if } \hat{f} = \hat{f}_{ls} \end{cases}
\]
and let $\Delta(x) := (r - x)\hat{f}_{ml}(x)$ for $x \in [t_k, r]$. One easily verifies, that this function $\Delta$ satisfies the conditions of Proposition 2 too, and with $\epsilon \searrow 0$ we obtain the inequality

$$
\int_{t_k}^{r} (r - x) F(dx) \leq \int_{t_k}^{r} (r - x) \hat{F}(dx).
$$

Integration by parts (or Fubini’s theorem) shows that the latter inequality is equivalent to

$$
\int_{t_k}^{r} (F(x) - F(t_k)) dx \leq \int_{t_k}^{r} (\hat{F}(x) - \hat{F}(t_k)) dx.
$$

Since $F(t_k) = \hat{F}(t_k)$, we end up with

$$
\int_{t_k}^{r} F(x) dx \leq \int_{t_k}^{r} \hat{F}(x) dx \quad \text{for } k = 0, 1, \ldots, m - 1 \text{ and } r \in (t_k, t_{k+1}].
$$

Hence we may apply Lemma 1 and obtain (1).

Finally, let us consider the least squares estimator $\hat{f}_{ts}$. For $0 \leq k < m$ and $r \in (t_k, t_{k+1}]$ let $\Delta(x) := 0$ for $x \notin (t_k - \epsilon, t_k)$, let $\Delta$ be linear on $[t_k - \epsilon, t_k]$ as well as on $[t_k, r]$ with $\Delta(t_k) := r - t_k$. Then applying Proposition 2 and letting $\epsilon \searrow 0$ yields

$$
\int_{t_k}^{r} (r - x) F(dx) \leq \int_{t_k}^{r} (r - x) \hat{F}(dx),
$$

so that

$$
\int_{t_k}^{r} F(x) dx \leq \int_{t_k}^{r} \hat{F}(x) dx \quad \text{for } k = 0, 1, \ldots, m - 1 \text{ and } r \in (t_k, t_{k+1}].
$$

Thus it follows from Lemma 1 that

$$
\inf_{[0, \infty)} (\hat{F} - F) \geq \frac{3}{2} \inf_{[0, \infty)} (F - F) - \frac{1}{2} \sup_{[0, \infty)} (F - F) \geq -2 \|F - F\|_{\infty}.
$$

Alternatively, for $1 \leq k \leq m$ and $r \in [t_{k-1}, t_k]$ let $\Delta(x) := 0$ for $x \notin (r, t_k + \epsilon)$, let $\Delta$ be linear on $[r, t_k]$ as well as on $[t_k, t_k + \epsilon]$ with $\Delta(t_k) := -(t_k - r)$. Then applying Proposition 2 and letting $\epsilon \searrow 0$ yields

$$
\int_{r}^{t_k} (t_k - x) F(dx) \geq \int_{r}^{t_k} (t_k - x) \hat{F}(dx),
$$

so that

$$
\int_{r}^{t_k} F(x) dx \geq \int_{t_k}^{r} \hat{F}(x) dx \quad \text{for } k = 1, 2, \ldots, m \text{ and } r \in [t_{k-1}, t_k].
$$

Hence it follows from Lemma 1 that

$$
\sup_{[0, \infty)} (\hat{F} - F) \leq \frac{3}{2} \sup_{[0, \infty)} (F - F) - \frac{1}{2} \inf_{[0, \infty)} (F - F) \leq 2 \|F - F\|_{\infty}.
$$

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