Phase Transition and Thermal Fluctuations of Quintessential Kerr-Newman-AdS Black Hole

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Abstract

This paper is devoted to analyzing the critical phenomenon and phase transition of quintessential Kerr-Newman-anti-de Sitter black hole in the framework of Maxwell equal-area law. For this purpose, we first derive thermodynamic quantities such as Hawking temperature, entropy and angular momentum in the context of extended phase space. These quantities satisfy Smarr-Gibbs-Dehum relation in the presence of quintessence matter. We then discuss the critical behavior of thermodynamic quantities through two approaches, i.e., van der Waals-like equation of state and Maxwell equal-area law. It is found that the latter approach is more effective to analyze the critical behavior of the complicated black holes. Using equal-area law, we also study phase diagram in $T - S$ plane and find an isobar which shows the coexistence region of two phases. We conclude that below the critical temperature, black holes show a similar phase transition as that of van der Waals fluid. Finally, we study the effects of thermal fluctuations on the stability of this black hole.

Keywords: Black hole; Thermodynamics; Equal-area law; Phase transition; Thermal fluctuations.

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1 Introduction

Cosmological evidences indicate that our universe is undergoing an accelerated expansion due to some large negative pressure. The most prevailed conjecture to discuss the evolution of the cosmos is an anti-gravitational force named as dark energy (DE). Despite of enormous astronomical observations, the essential features as well as the origin of DE is still unclear and has become a source of vivid debate. This can be expressed by different models such as the cosmological constant ($\Lambda$), quintessence energy, etc. The homogeneous cosmological constant or vacuum energy has fixed value in space i.e., $\Lambda \approx 1.3 \times 10^{-56}$ cm$^{-2}$ $[1]$ whereas the quintessence energy is inhomogeneous as well as dynamical scalar field which can be characterized by the equation of state (EoS) $w = \frac{P}{\rho}$, where $P$ and $\rho$ are the pressure and energy density, respectively. It is believed that the late-time evolution may be the consequence of cosmological constant or quintessence matter which permeates throughout the universe. Thus, if quintessence matter is spread out all over the spacetime, it must cover the black hole (BH) surrounding which alters its spacetime structure as well as asymptotic features of the cosmological horizon.

Wei and Liu $[2]$ studied the relation between pressure and cosmological constant for charged spherically symmetric anti-de Sitter (AdS) BH in higher-dimensional spacetime. They found an analogy between BH and the van der Waals (vdW) liquid-gas system which kept the foundation of BH thermodynamics in extended phase space. The extended phase space is based on the fact that thermodynamic pressure is identified with the cosmological constant and its conjugate quantity to the volume of BH. This implies that the conventional phase space is quite different from the extended phase space where the extra PdV term is present which modifies the Smarr relation as well as the first law of BH. In this phase space, the whole BH system is mapped to the vdW fluid system consolidating the analogy between small/large BH with the vdW liquid/gas phase transitions. It is known that above the critical temperature, the isothermal curves in the vdW system depict similar behavior to the experimental results. However, below the critical temperature, there exists an oscillating region which violates the condition of stable equilibrium. Using Maxwell equal-area law, the oscillating part can be replaced with an isobar which yields the correspondence with the experimental results. After this breakthrough, several aspects of charged AdS BH have been discussed, such as $P - V$ or $T - S$ criticality, first-order phase transition, etc $[3]$-$[6]$. 
Gunasekaran et al. [7] derived the critical thermodynamic quantities of charged and rotating AdS BH in extended phase space and found its analogy with the vdW liquid-gas system. Cheng et al. [8] investigated the critical behavior of Kerr-Newman-AdS BH in extended phase space and numerically solved the critical points for the vdW-like phase transition. Wei and Liu [9] examined the phase transition of charged AdS black hole in Gauss-Bonnet gravity. It is found that the charged AdS BH, in the presence of quintessence, shows a small-large BH phase transition similar to the liquid-gas phase transition of the vdW system. Li [10] examined the effect of DE on $P-V$ criticality of Reissner-Nordstrom (RN) AdS BH and showed that quintessence matter does not affect the existence of small/large BH phase transition. Guo [11] used Maxwell equal-area law to determine the phase transition of charged AdS BHs in the presence of quintessence matter and concluded that BHs have the same phase transition as that of the vdW system.

In quantum gravity, one of the important issues is the consideration of statistical perturbations which modify the BH geometry. These perturbations lead to thermal fluctuations that do not affect thermodynamics of large BHs but have great implications on the BHs whose size and temperature decrease and increase, respectively, due to Hawking radiation. [12]. Thus, thermal fluctuations play a critical role in the small BHs thermodynamics due to sufficient increase in temperature. Pourhassan et al. [13] derived logarithmic corrections to entropy around the equilibrium state and investigated their influence on the thermodynamics of higher-dimensional charged BHs. They also investigated the vdW fluid duality as well as the validity of the first law of thermodynamics. Upadhyay [14] discussed the effects of leading order corrections on the stability of charged rotating AdS BHs. They found that thermodynamic potentials satisfy the first law of BH thermodynamics. Moreover, the effect of first-order corrections on the phase transition of Kerr-Newman-AdS BH has been analyzed [15]. It is found that these corrections modify the thermodynamic potentials and its phase transition. In the same perspective, Zhang [16] analyzed the physical behavior of thermodynamic potentials of RN AdS and Kerr-Newman BHs in the presence of first-order corrections. Recently, Sharif and Akhtar [17] studied quasi-normal modes and thermal fluctuations of charged black hole with Weyl corrections.

This work aims to explore the effects of DE on critical behavior and phase transition of charged rotating BH surrounded by the quintessential field. We discuss thermodynamic properties including angular velocity and Smarr relation. We also derive the exact expression of critical quantities through two
approaches and discuss the phase transition of BH. The paper is assembled as follows. In the next section, we elaborate the spacetime structure and calculate its thermodynamics quantities. Further, the $P - V$ criticality of quintessential Kerr-Newman BH is investigated in extended phase space. In section 3, we use Maxwell equal-area law in $T - S$ conjugate variables to study the conditions satisfied by phase transition. Section 4 provides the general expression of corrected entropy as well as explores the effects of thermal fluctuations on the stability of considered BH. Finally, we compile our results in the last section.

2 Quintessential Kerr-Newman-AdS Black Hole

Using Newman-Penrose formalism \cite{[18]}, Xu and Wang \cite{[19]} derived the Kerr-Newman-AdS BH in the presence of quintessence field whose line-element, in Boyer-Lindquist coordinates, reads

$$ds^2 = -\frac{\chi}{\Omega}\left[dt - \frac{a \sin^2 \theta}{k} d\phi\right]^2 + \frac{\Omega}{\chi} dr^2 - \frac{\Omega}{\tilde{P}} d\theta^2 - \frac{\tilde{P} \sin^2 \theta}{\Omega}\left[adt - \frac{(r^2 + a^2)}{k} d\phi\right]^2,$$

with

$$\chi = (r^2 + a^2)(1 + \frac{r^2}{l^2}) - 2mr + q^2 - \alpha r^{1-3w},$$

$$\Omega = r^2 + a^2 \cos^2 \theta, \quad k = 1 - \frac{a^2}{l^2}, \quad \tilde{P} = 1 - \frac{a^2}{l^2} \cos^2 \theta. \quad (3)$$

Here $a$ corresponds to rotation parameter and $q$ is defined as $q^2 = q_e^2 + q_m^2$, where $q_e$ and $q_m$ represent the electric and magnetic charges, respectively. Moreover, $m$ is the mass of BH, $l^2 = -\frac{2}{\Lambda}$ determines the radius of AdS BH, $w$ is the dimensionless state parameter with $-1 < w < -\frac{1}{3}$ and $\alpha$ is the quintessence parameter which measures the intensity of quintessential field around a BH, satisfying the following inequality \cite{[19]}

$$\alpha \leq \frac{2}{1 - 3w} 8^w. \quad (4)$$

It is analyzed that the above relation holds until the cosmological horizon determined by quintessential DE exists. The line element (1) can be rewritten as

$$ds^2 = -\mathcal{F}(r, \theta) dt^2 + \frac{dr^2}{\mathcal{G}(r, \theta)} + \Sigma(r, \theta)d\theta^2 + K(r, \theta)d\phi^2 - 2H(r, \theta)dt d\phi, \quad (5)$$

4
where

\[
F(r, \theta) = \frac{\chi - a^2 \tilde{P} \sin^2 \theta}{\Omega}, \quad G(r, \theta) = \frac{\chi}{\Omega}, \quad \Sigma(r, \theta) = \frac{\Omega}{\tilde{P}},
\]

\[
K(r, \theta) = \frac{\tilde{P} \sin^2 \theta (a^2 + r^2) - \chi (a \sin^2 \theta)^2}{k^2 \Omega},
\]

\[
H(r, \theta) = \frac{a \tilde{P} \sin^2 \theta (a^2 + r^2) - \chi a \sin^2 \theta}{k \Omega}.
\]

The associated electromagnetic potential is given as

\[
B = \frac{1}{\Omega} \left[-(-q_e r + q_m a \cos \theta) d t + \frac{1}{k} (a \sin^2 \theta q_e r + q_m (r^2 + a^2) \cos \theta) d \phi \right].
\]

For \(\alpha = 0\), the line element \(\text{(1)}\) reduces to Kerr-Newman-AdS BH while the Kerr-AdS BH solution can be obtained for \(\alpha = 0 = q\).

### 2.1 Critical Phenomenon and Thermodynamical Structure in Extended Phase Space

Black hole as an interdisciplinary area provides a possible bridge between classical general relativity and quantum theory of gravity. Based on the pioneering work of Hawking and Bekenstein \[20\], BH thermodynamic quantities such as temperature and entropy can be mapped on the the laws of ordinary thermodynamics which have opened many interesting aspects of unification of gravity, quantum mechanics and thermodynamics \[21\]-\[23\]. In this section, we will discuss thermal properties as well as \(P - V\) criticality of the quintessential Kerr-Newman-AdS BH in an extended phase space. Using the horizon condition \(\chi(r_+) = 0\), the mass of the BH in terms of horizon radius \(r_+\) reads

\[
m = \frac{1}{2r_+} \left(r_+ \left(\frac{a^2 + l^2 + r_+^2}{l^2} - \alpha r_+^{-3w}\right) + a^2 + q^2\right).
\]

The entropy in terms of horizon area is defined as

\[
S = \frac{A}{4} = \frac{\pi (a^2 + r_+^2)}{k}, \quad \text{with} \quad A = \int_0^{2\pi} \int_0^{\pi} \sqrt{g_{\theta \theta} g_{\phi \phi}} |_{r=r_+} d \theta d \phi,
\]

which plays a significant role to study the thermodynamic evolution of BH.

It is observed that the entropy depends upon rotation and horizon radius.
Figure 1: Hawking temperature vs $r_+$ for $a = 0.1 = q$. Left plot: $w = \frac{1}{3}$ and $\alpha = 0$ (red), 0.2 (green), 0.4 (blue). Right plot: $\alpha = 0.2$ and $w = \frac{1}{2}$ (red), $w = \frac{3}{2}$ (green), $w = \frac{2}{3}$ (blue).

whereas the cosmological constant, electric charge and quintessence parameter have no explicit effect on it.

The Hawking temperature \( T = \frac{\chi'(r)}{4\pi(r^2+a^2)} \biggr|_{r=r_+} \) for the considered BH is evaluated as

\[
T = \frac{1}{4\pi(a^2+r_+^2)} \left(2\left(\frac{r_+\left(a^2+l^2+2r_+^2\right)}{l^2} - m\right) + \alpha(3w-1)r_+^{3w-1}\right). \tag{11}
\]

Figure 1 shows the graphical behavior of Hawking temperature with respect to horizon radius. It is observed that the temperature of BH decreases for larger values of quintessence parameter (left plot). For smaller values of state parameter, the temperature increases and decreases, respectively, before and after the critical radius (right plot). The corresponding angular velocity is computed as

\[
\Pi_r = \frac{g_{t\phi}}{g_{\phi\phi}} = \frac{aP \sin^2(\theta) \left(a^2 + r^2\right) - \chi a \sin^2(\theta)}{P \sin^2(\theta) \left(a^2 + r^2\right)^2 - \chi \left(a \sin^2(\theta)\right)^2}. \tag{12}
\]

The radial function $\chi$ vanishes at the event horizon which reduces the above expression to the following form

\[
\Pi_H = \frac{ak}{\left(a^2 + r_+^2\right)}, \tag{13}
\]

where $\Pi_H$ is the angular velocity of BH horizon. It is known that the angular velocity of rotating BHs in AdS space does not vanish at infinity, i.e., $\Pi_\infty \neq 0$. 

0 for \( r \to \infty \) \[24, 25\]. This salient feature differentiates them from the asymptotically flat spacetime where \( \Pi_\infty = 0 \). For the rotating BHs, the angular velocity is defined as

\[
\Pi = \Pi_H - \Pi_\infty = \frac{a (l^2 + r_+^2)}{l^2 (a^2 + r_+^2)},
\]

where the angular potential at infinity reads

\[
\Pi_\infty = \frac{a}{l^2}.
\]

In extended phase space, the cosmological constant is treated as pressure and its conjugate quantity to the volume of BH \[7, 26\], given by

\[
P = \frac{3}{8\pi l^2},
\]

\[
V = \left( \frac{\partial M}{\partial P} \right)_{S,Q,J,\alpha} = 2\pi \left( a^2 l^2 \left( q^2 - \alpha r_+^{1-3w} \right) + \left( a^2 + r_+^2 \right) \left( a^2 l^2 - a^2 r_+^2 + 2l^2 r_+^2 \right) \right) \frac{1}{3k^2 l^2 r_+};
\]

Here mass \( M \), angular momentum \( J \) and charge \( Q \), are related to parameters \( m, a \) and \( q \) as follows

\[
M = \frac{m}{k^2}, \quad J = \frac{ma}{k^2}, \quad Q = \frac{q}{k};
\]

Through Eqs.\((10)\) and \((11)\), we obtain

\[
TS = \frac{1}{4k} \left( -r_+^2 (8\pi a^2 P + 3) + 3 (a^2 + q^2) + 8\pi P r_+^4 \right) \frac{1}{3r_+} + \frac{2}{3} r_+ \left( 8\pi a^2 P_+ + 16\pi P r_+^2 + 3 \right) + \alpha (3w - 1) r_+^{-3w} + \alpha r_+^{-3w}\right),
\]

which in terms of physical parameters \( M, J \) and \( Q \) satisfies Smarr-Gibbs-Dehum relation as

\[
M = 2(T_k S - PV + \Pi J) + Q \Phi + \alpha \Psi \left( -\frac{2a^2}{a^2 + r_+^2} + 3w + 1 \right),
\]

where the electric potential \( \Phi \) and physical quantity \( \Psi \) (conjugate to the parameter \( \alpha \)) \[27\] read

\[
\Phi = \frac{q r_+}{a^2 + r_+^2}, \quad \Psi = -\frac{r_+^{-3w}}{2k}.
\]
It is noted that for $\alpha = 0$, all the derived quantities reduce to charged rotating AdS BH \cite{7}.

Now we discuss the effects of quintessence matter on $P - V$ criticality of Kerr-Newman-AdS BH. Using Eqs. (9) and (16), the Hawking temperature takes the form

$$T = \frac{2r_+ (8\pi P (a^2 + 2r_+^2) + 3) - r_+ (8\pi a^2 P + 3) - \frac{3(a^2+q^2)}{r_+} - 8\pi P r_+^3 + 9\alpha wr_+^{-3w}}{12\pi (a^2 + r_+^2)}.$$  

(21)

In terms of charge and angular momentum, the EoS can be written as

$$P = \frac{T}{2r_+} + \frac{Q^2}{8\pi r_+^4} - \frac{1}{8\pi r_+^2} - \frac{3\alpha wr_+^{-3w-3}}{8\pi} + \frac{3J^2 (Q^2r_+^2 + 2Q^4 - 8\pi Q^2 r_+^3 T + 6\alpha Q^2 w r_+^{1-3w} + 8\pi r_+^5 T + 3\alpha w r_+^{3-3w} + 4r_+^4)}{8\pi r_+^6 (2Q^2 + 2r_+^3 T - \frac{3}{2}\alpha (w + 1) r_+^{1-3w} + r_+^2)^2} + O(J^4),$$

(22)

which is an expansion in powers of $J$. To avoid the complex computation, we expand all quantities to $O(J^2)$ by neglecting all higher order terms in $J^2$. Introducing a new variable (which corresponds to the specific volume of the associated vDW fluid) as

$$\nu = 2 \left( \frac{3V}{4\pi} \right)^{\frac{1}{3}} = 2r_+ + \frac{12J^2}{r_+} \left( \frac{8\pi P r_+^4 + Q^2 - \alpha r_+^{1-3w} + 3r_+^2}{8\pi r_+^4 + 3Q^2 - 3\alpha r_+^{1-3w} + 3r_+^2} \right)^2,$$

(23)

which leads Eq. (22) to the form

$$P = \frac{T}{\nu} - \frac{1}{2\pi \nu^2} + \frac{2Q^2}{\pi \nu^4} - \frac{3\alpha 2^{3w} w \nu^{-3(w+1)}}{\pi} + \frac{48J^2}{\pi \nu^6} + \frac{48J^2}{\pi \nu^6 (\nu^2 + 8Q^2 + \pi^3 \nu T - 3\alpha 8^w (w + 1) \nu^{1-3w})^2} \left( -48q^4 - 12\pi^3 q^2 T \right) - 9\alpha q^2 2^{2w+2} \nu^{1-3w} - 10\nu^2 Q^2 + 5\alpha Q^2 8^{w+1} \nu^{1-3w} + 3\alpha Q^2 2^{3w+2} w \nu^{1-3w} - 9\pi \alpha T 8^w \nu^{4-3w} + 5\pi \alpha T 8^w \nu^{4-3w} - 3\pi \alpha T 2^{3w+1} w \nu^{4-3w} + 27\alpha^2 6^4 w \nu^{2-6w} + 27\alpha^2 6^4 w \nu^{2-6w} - 9\alpha 8^w w \nu^{3-3w} - 9\alpha^2 64^w \nu^{2-6w} - 9\alpha^2 64^w w \nu^{2-6w} + 7\alpha 8^w \nu^{3-3w} + 3\alpha 8^w w \nu^{3-3w} \right).$$

(24)
The critical points can be obtained through the constraints

$$\frac{\partial P}{\partial \nu} |_{T=T_c} = 0, \quad (25)$$

$$\frac{\partial^2 P}{\partial \nu^2} |_{T=T_c} = 0. \quad (26)$$

Setting \((\nu^2 + 8Q^2 + \pi \nu^3 T - 3\alpha \delta w (w + 1) \nu^{1-3w}) = 1\) and utilizing Eqs.(24) and (25), one can get the critical temperature \(T_c\) as

$$T_c = r^{-3w-3} \left( 8q^2 \nu_c^{6w+2} - \nu_c^{6w+4} - 9\alpha 2^{3w} w^2 \nu_c^{3w+3} - 9\alpha 2^{3w} w^2 \nu_c^{3w+3} + 288 J^2 \nu_c^{6w} ight)$$

$$- 13824 J^2 q^2 \nu_c^{6w} - 81 \alpha J^2 q^2 2^{3w+6} w^3 \nu_c^{3w+1} - 1920 \alpha J^2 q^2 2^{3w+6} w^3 \nu_c^{3w+1}$$

$$+ 243 \alpha J^2 2^{6w+5} w^4 \nu_c^2 + 405 \alpha J^2 2^{6w+5} w^3 \nu_c^2 - 81 \alpha J^2 2^{3w+4} w^3 \nu_c^2$$

$$+ 10 \alpha J^2 2^{6w+5} w^2 \nu_c^2 - 27 \alpha J^2 2^{3w+5} w^2 \nu_c^2 + 9 \alpha J^2 2^{3w+5} w^6 \nu_c^2$$

$$- 135 \alpha J^2 2^{6w+5} w^3 \nu_c^2 + 63 \alpha J^2 2^{3w+4} \nu_c^{3w+3} + 45 \alpha J^2 2^{3w+5} w^2 \nu_c^{3w+3} \right) \times \left( \pi (1728 J^2 q^2 \nu_c^{3w} + 81 \alpha J^2 2^{3w+4} w^3 \nu_c + 27 \alpha J^2 2^{3w+4} w^2 \nu_c \right.

$$- 15 \alpha J^2 2^{3w+5} \nu_c - 9 \alpha J^2 2^{3w+4} w \nu_c - \nu_c^{3w+2}) \right)^{-1}, \quad (27)$$

where \(\nu_c\) corresponds to critical specific volume whose expression can easily be evaluated through Eqs.(24), (25) and (27). Using Eqs. (24) and (27), the critical pressure can be put in the form

$$P_c = \left( r^{-3(w+1)} \nu_c^{-6(w+1)} \left( 96 J^2 \left( 120 \left( 144 J^2 - 1 \right) Q^4 + 1 \right) \nu_c^{2w3w+3} + 165888 \right) \times J^4 Q^2 \left( 48 Q^4 - 1 \right) \nu_c^{6w+3} + 3 \nu_c^{3w+1} \left( 16 \left( 9 w (3 w^2 + w - 4) - 16 \right) J^2 \right)$$

$$3 w (w + 1) \alpha \nu_c^8 + (-9 \nu_c^{3w+3} - 9 \nu_c^{3w+7} J^2 Q^2 (16 (3 w (3 w (w + 1) - 10)))$$

$$- 113 J^2 + (-w - 1) (9 w^2 - 20) \alpha \nu_c^6 + 4 (192 J^2 + 1) Q^2 \nu_c^{3w+3} - 27$$

$$2^{3w+8} J^4 \left( 8 (9 w (w (3 w + 1) - 1) - 80) Q^4 - 3 w (3 w + 2) + 5 \right) \alpha \nu_c^9 \nu_c^{9w} + 92^{6w+5} J^2 \alpha^2 (64 J^2 (3 w (3 w (w + 1) - 8) - 76) + 47) + 181) Q^2$$

$$+ (16 (3 w + 2) (9 w^2 - 3 w - 7) (9 w^2 + 6 w - 5) J^2 + w (3 w (3 w + 5) + 2) - 13) - 3 \nu_c^2 \right) r^{3w+3} + 9 2^{3w+5} J^2 (w + 1) (3 w + 2) (3 w^2 - 1) (9 w^2$$

9
\begin{align*}
+ & \ 6w - 5) \alpha \nu_c^4 \nu_c^{3w+2} - 3 \ 2^{3w+1} \alpha (-96J^2 Q^2 (16(3w(3w(15w + 38) - 23)
- & \ 176)J^2 - 3w(3w(w + 2) - 7) + 10) \nu_c^2 r^{3w+3} - 768J^4 (48(3w(3w(w + 7) - 4) - 40)Q^4 + 3w - 9w^2(3w + 4) + 10)r^{3w+3} + 9 \ 2^{3w+4} J^2 (w + 1) \\
\times & (16(9w^2 - 3w - 7)(9w^2 + 6w - 5)J^2 + w(9w(3w + 2) - 11) - 4) \\
\times & \alpha \nu_c^6 + ((w - 8J^2 (3w(3w + 1)^2 + 4)) r^{3w+3} + 3 \ 2^{3w+10} J^4 Q^2 (3w \\
\times & (81w^4 - 378w^2 - 354w + 65) + 358) \alpha \nu_c^4) \nu_c^{6w+1} - 2 (24J^2 Q^2 - \nu_c^2) \\
\times & (\nu_c^4 + 8 (240J^2 - 1) Q^2 \nu_c^2 + 288J^2 (48Q^4 - 1)) \nu_c^{12w+3} - 81 \ 512^{w+1} \\
\times & J^4 r^{3w+3} (w + 1)(3w + 2) (3w^2 - 1) (9w^2 + 6w - 5) \alpha^3 \nu_c^3 \right) \left(2(\pi \nu_c^{3w} \\
\times & (\nu_c^2 - 1728J^2 Q^2) - 3 \ 2^{3w+4} J^2 \pi (3w + 2) (9w^2 + 6w - 5) \alpha \nu_c\right)^{-1}. \tag{28}
\end{align*}

For \( w = \frac{-2}{3} \) and \( Q = 0 \), the analytical critical points can be computed as

\begin{align*}
\nu_c &= 2\sqrt{3} \sqrt[3]{10} J, \quad T_c = -\frac{\alpha}{2\pi} - \frac{12\alpha J^2}{\pi} + \frac{2^{3/4}}{5\sqrt{3} \sqrt[5]{5\pi} \sqrt{J}}, \\
P_c &= \frac{4 \ 2^{3/4} \sqrt[3]{3} \alpha J^{3/2}}{\sqrt{5\pi}} - \frac{720\alpha^2 J^4}{\pi} - \frac{27\alpha^2 J^2}{\pi} + \frac{1}{36 \sqrt{10\pi} J}. \tag{29}
\end{align*}

It is observed that for \( \alpha = 0 \), the usual critical behavior of Kerr BH can be recovered [7]. Similarly, for \( w = \frac{-2}{3} \) and \( J = 0 \), the critical values of thermodynamic quantities for charged AdS BH can be obtained as [10]

\begin{align*}
\nu_c &= 2\sqrt{6} Q, \quad T_c = \frac{1}{3 \sqrt[3]{6\pi} Q} - \frac{\alpha}{2\pi}, \quad P_c = \frac{1}{96\pi Q^2}, \tag{30}
\end{align*}

which shows that the critical temperature as well as pressure have the same critical behavior as that of the vdW fluid. Notice that for \( w \neq \frac{-2}{3} \), the exact critical points of charged as well as rotating BHs cannot be computed due to lengthy expressions of the critical specific volume. However, to observe the behavior for other values of state parameter, one can numerically solve \( \nu_c \) through Mathematica programming. Then the critical temperature and pressure can be obtained through Eqs. (27) and (28), respectively.
3 Construction of Equal-Area Law in $T - S$ Diagram

In this section, we will study the phase diagram of quintessential Kerr-Newman-AdS BH with the help of Maxwell equal-area law in $T - S$ differential conjugate variables. For this purpose, we consider $Q, J, l, w$ and $\alpha$ as constants. The vertical axis is $T_0$ ($T_0 \leq T_c$) depending upon the horizon radius while the horizontal axes of the two-phase coexistence region are $S_2$ and $S_1$, respectively. In this scenario, the Maxwell equal-area law takes the form

$$T_0(S_2 - S_1) = \int_{S_1}^{S_2} T dS,$$

$$= \int_{r_1}^{r_2} \left(\frac{4\pi Pr^4 - \frac{q^2}{2} + \frac{3}{2} \alpha wr_1 - 3w}{r^2} + \frac{a^2 \left(-\frac{8}{3} \pi Pr^4 + \frac{q^2}{2} - 2\alpha wr_1 - r^2\right)}{r^4}\right)dr,$$

where points $r_2$ and $r_1$ should satisfy

$$T_0 = -\frac{a^2 \left(\frac{3}{r_2} - 8\pi Pr_2\right) - 24\pi Pr_2^3 + \frac{3q^2}{r_2} - 9\alpha wr_2 - 3r_2}{12\pi (a^2 + r_2^2)},$$

$$T_0 = -\frac{a^2 \left(\frac{3}{r_1} - 8\pi Pr_1\right) - 24\pi Pr_1^3 + \frac{3q^2}{r_1} - 9\alpha wr_1 - 3r_1}{12\pi (a^2 + r_1^2)}.$$

Using the above equations, one can derive

$$0 = 8\pi P(r_2 - r_1) - q^2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right) + 3w\alpha \left(\frac{1}{r_2 + 3w} - \frac{1}{r_1 + 3w}\right) + \left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$+ a^2 \left(-\frac{16}{3} \pi P \left(\frac{1}{r_2} - \frac{1}{r_1}\right) + q^2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right) - 3\alpha w \left(\frac{1}{r_2 + 3w} - \frac{1}{r_1 + 3w}\right) - 2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right)\right),$$

and

$$8\pi T_0 = +8\pi P(r_2 + r_1) - q^2 \left(\frac{1}{r_2} + \frac{1}{r_1}\right) + 3w\alpha \left(\frac{1}{r_2 + 3w} + \frac{1}{r_1 + 3w}\right) + \left(\frac{1}{r_2} + \frac{1}{r_1}\right)$$

$$+ a^2 \left(-\frac{16}{3} \pi P \left(\frac{1}{r_2} + \frac{1}{r_1}\right) + q^2 \left(\frac{1}{r_2} + \frac{1}{r_1}\right) - 3\alpha w \left(\frac{1}{r_2 + 3w} + \frac{1}{r_1 + 3w}\right) - 2 \left(\frac{1}{r_2} + \frac{1}{r_1}\right)\right).$$
After integrating Eq. (31), we obtain
\[
\pi T_0 \left( r_2^2 - r_1^2 \right) = \frac{r_2 - r_1}{2} + \frac{4}{3} \pi P \left( \frac{1}{r_2} - \frac{1}{r_1} \right) + \frac{q^2}{2} \left( \frac{r_1 - r_2}{r_1 r_2} \right) - \frac{\alpha}{2} \left( \frac{1}{r_2^w} - \frac{1}{r_1^w} \right) + a^2 \left( \frac{r_1 - r_2}{r_1 r_2} - \frac{8}{3} \pi P (r_2 - r_1) - \frac{q^2}{6} \left( \frac{1}{r_2} - \frac{1}{r_1} \right) + \frac{3\alpha w}{2(3w + 2)} \left( \frac{1}{r_2^w} - \frac{1}{r_1^w} \right) \right).
\]
(36)

Setting \( x = \frac{r_2}{r_1} \) with \( 0 \leq x \leq 1 \), Eqs. (34)-(36) turn into
\[
8\pi T_0 = +8\pi P r_2(1 + x) - q^2 \left( \frac{1 + x^3}{x^3 r_2^3} \right) + 3w\alpha \left( \frac{1 + x^2+3w}{x^2+3w r_2^2+3w} \right) + \left( \frac{1 + x}{x r_2} \right) + a^2 \left( \frac{16}{3} \pi P \left( \frac{1 + x}{x r_2} \right) + q^2 \left( \frac{1 + x^5}{x^5 r_2^5} \right) - 3\alpha w \left( \frac{1 + x^4+3w}{x^4+3w r_2^4+3w} \right) - 2 \left( \frac{1 + x^3}{x^3 r_2^3} \right) \right),
\]
(37)

and
\[
8\pi T_0 = +8\pi P r_2(1 + x) - q^2 \left( \frac{1 + x^3}{x^3 r_2^3} \right) + 3w\alpha \left( \frac{1 + x^2+3w}{x^2+3w r_2^2+3w} \right) + \left( \frac{1 + x}{x r_2} \right) + a^2 \left( \frac{16}{3} \pi P \left( \frac{1 - x}{x r_2} \right) - q^2 \left( \frac{1 - x^5}{x^5 r_2^5} \right) + 3\alpha w \left( \frac{1 - x^4+3w}{x^4+3w r_2^4+3w} \right) + 2 \left( \frac{1 - x^3}{x^3 r_2^3} \right) \right).
\]
(38)

Through Eqs. (37) and (39), we get \( T_0 \)-free relation as
\[
-8\pi P r_2(1 - x)^2 = \frac{4}{3} \pi P r_2^3 (1 - x^3) - \frac{q^2}{2} \left( \frac{1 - x}{x r_2} \right) + \alpha \left( \frac{1 - x^3}{2x^3 r_2^3} \right) + \left( \frac{r_2(1 - x)}{2} \right) + a^2 \left( \frac{q^2}{6x^3 r_2^3} - \frac{(1 - x)}{r_2 x} \right) - 8 \pi P r_2(1 - x) - 3\alpha w \left( \frac{1 - x^2+3w}{2(2 + 3w)x^2+3w r_2^2+3w} \right).
\]
(39)

Through Eqs. (37) and (39), we get \( T_0 \)-free relation as
\[
-8\pi P r_2(1 - x)^2 = \frac{4}{3} \pi P r_2^3 (1 - x^3) - \frac{q^2}{2} \left( \frac{1 - x}{x r_2} \right) + \alpha \left( \frac{1 - x^3}{2x^3 r_2^3} \right) + \left( \frac{r_2(1 - x)}{2} \right) + a^2 \left( \frac{q^2}{6x^3 r_2^3} - \frac{(1 - x)}{r_2 x} \right) - 8 \pi P r_2(1 - x) - 3\alpha w \left( \frac{1 - x^2+3w}{2(2 + 3w)x^2+3w r_2^2+3w} \right).
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\]
(39)
Utilizing Eq. (38), the explicit expression of pressure reads

\[
8\pi (1-x)P = \frac{-1}{3r_2^6x^5} \left( q^2 (-q^2 (-3x^5 - 2x^4 + 2x + 3) + 3\alpha wr_2^{-3w-3}x^{1-3w} \right.
\]

\[
\times \left[ ((3x + 2)x^{3w+3} + 2x + 3) + 2r_2^3x^2 (-3x^3 - x^2 + x + 3) \right]
\]

\[
+ \frac{q^2 (1-x^3)}{r_2^4x^3} - \frac{1-x}{r_2^2x} - 3\alpha wr_2^{-3w-3}x^{1-3w} \right) (1-x^{3w+2}),
\]

Substituting the above expression in Eq. (40) yields

\[
\frac{1}{3}q^2 \left( -q^2 (x^2 + x + 1) (3x^2 + 11x + 3) r_2^{-4x-4} + 2 (3x^2 + 13x + 3) r_2^{-2x-2} \right.
\]

\[
+ \frac{3\alpha wr_2^{-3w-3}x^{1-3w}}{(1-x)^3} (-2x^{3w+3} + 28x^{3w+4} - 5x^{3w+5} - 3x^{3w+6} + 2x^3 - 28x^2
\]

\[
+ 5x + 3) \left) + \frac{q^2 (x^2 + 4x + 1)}{r_2^2x^2} - \frac{3\alpha r_2^{-3w-1}x^{1-3w}}{(1-x)^3} \right)
\]

\[
- 2x^2 (1-x^{3w}) - 2x^2 (1-x^{3w}) = 1,
\]

which further can be written as

\[
r_2^2 = q^2 f_1(x) - \frac{q^2\alpha^2 f_2(x)}{r_2} + \sigma (-f_3(x,w)r^2 + \alpha^2 f_4(x,w)) + \alpha^2 f_5(x),
\]

with

\[
f_1(x) = \frac{x^2 + 4x + 1}{x^2}, \quad f_2(x) = \frac{(x^2 + x + 1) (3x^2 + 11x + 3)}{3x^4},
\]

\[
f_3(x,w) = \frac{x^{-3w} (w ((x + 1) (1 - x^{3w+3}) - 2x^2 (1 - x^{3w})) - 2x^2 (1 - x^{3w}))}{(1-x)^3},
\]

\[
f_4(x,w) = \frac{wx^{-3w-3}}{3(1-x)^3} (-2x^{3w+3} + 28x^{3w+4} - 5x^{3w+5} - 3x^{3w+6} - 2x^3
\]

\[
- 28x^2 + 5x + 3), \quad f_5(x) = \frac{2 (3x^2 + 13x + 3)}{3x^2},
\]

\[
\sigma = - \frac{B_c r_c^{-3w-1}r+c^{3w+1}}{2\pi w}, \quad B_c = - \frac{6w\pi \alpha}{r_c^{3w+1}},
\]

where \(B_c\) is the quintessence of unit thickness of BH horizon.
It is observed that for \( w = -\frac{2}{3} \), \( f_3(x, w) \) vanishes and when \( a = 0 \), the results are identical with the one obtained in charged AdS BH \([11]\). For \( x \to 1 \), there must exist \( r_1 = r_2 = r_c \) (\( r_c \) is the horizon of critical radius) which leads Eq.(43) to

\[
\frac{1}{6} = \phi_c^2 - \frac{B_c}{8\pi} (w + 1)(3w + 2) + \frac{4\pi a^2}{A_c} \left(-\frac{17\phi_c^2}{6} + \frac{19}{9} + \frac{B_c}{54\pi}\right) \times (-50 + w(1 + 6w)(41 + 33w)), \tag{44}
\]

where \( \phi_c^2 = q^2 \) denotes the charged potential and \( A_c = 4\pi r_c^2 \) corresponds to the area of critical horizon. When the values of \( B_c, A_c \) and \( w \) are given, \( \phi_c \) can be evaluated in terms of rotation parameter. Further, for given values of \( q \) and \( a \), we can obtain the critical radius. Thus from Eq.(44), it is noted that the position of phase transition point not only depend on the size of BH but also depends on \( a, B_c, \phi_c \) and \( w \). Using \( x \to 1 \) in Eq.(40), one can obtain the critical pressure as

\[
P_c = \frac{3}{8\pi} \left( \frac{1}{r_c^2} - \frac{5q^2}{r_c^4} - \frac{B(3w + 2)(3w + 4)R^{3w + 1}r_c^{-3w - 3}}{6\pi} + 16J^2 \left(-\frac{11}{r_c^4}\right) \right)
+ \frac{5q^2}{3r_c^6} - \frac{B(3w + 4)(18w^2 + 18w - 2)R^{3w + 1}r_c^{-3w - 5}}{6\pi}, \tag{45}
\]

and critical temperature takes the form

\[
T_c = \frac{4q^2}{\pi r_c^3} + \frac{27\alpha w(w + 1)^2 r_c^{-3w - 2}}{4\pi} + \frac{1}{\pi r_c^2} + 16J^2 \left(-\frac{37}{4\pi r_c^3} + \frac{4q^2}{\pi r_c^5}\right)
+ \frac{\alpha w \left(81w^3 + 90w^2 + \frac{81w^2}{2} - \frac{43}{3} \right)}{\pi} r_c^{-3w - 4}. \tag{46}
\]

Using the above equations, we can derive the critical behavior of thermodynamic quantities. It can easily be observed that this method is more efficient than the usual approach to determine the critical points of complicated BHs.

In order to find the explicit expression of \( r_2 \), Eq.(43) can be written as

\[
r_2^2 = \frac{f_3^2(x, w)\sigma + 1}{f_3(x, w)\sigma + 1} + \frac{f_4(x, w)\sigma + 1}{f_3(x, w)\sigma + 1} + \frac{f_1(x)q^2}{f_3(x, w)\sigma + 1}. \tag{47}
\]
Table 1: Numerical solutions for $x$, $r_2$ and $T_0$ at constant pressure with $J = q = 0.1$ and $w = \frac{-1}{3}$.

| $B_c$ | $\chi$ | $x$    | $r_2$  | $T_0$  |
|-------|--------|--------|--------|--------|
| 0.05  | 0.7    | 0.8658 | 1.5853 | 0.059999 |
|       | 0.8    | 0.8682 | 1.5949 | 0.06175  |
|       | 0.9    | 0.87068| 1.60566| 0.06353  |
| 0.15  | 0.7    | 0.9442 | 5.488  | 0.052899 |
|       | 0.8    | 0.941  | 5.1549 | 0.054042 |
|       | 0.9    | 0.93934| 4.8704 | 0.056097 |
| 0.25  | 0.7    | 0.9431 | 7.0782 | 0.0428893|
|       | 0.8    | 0.9421 | 6.66225| 0.044072 |
|       | 0.9    | 0.9397 | 6.30903| 0.04678  |

Substituting the above expression in Eq. (41) and setting $P_0 = \chi P_c$ with $0 \leq \chi \leq 1$, we obtain $r_2$-free relation. For given values of $q$, $J$, $w$, $B_c$ and $\chi$, the numerical value of $x$ can be evaluated. Inserting the obtained value of $x$ in Eq. (41), we get $r_2$ and from $r_1 = xr_2$, we can have $r_1$. Finally, the value of $T_0$ (from Eq. (37)) can be obtained. Table 1 provides the numerical values of $x$, $r_2$ and $T_0$ for $J = 0.1$, $q = 0.1$ and $w = \frac{-1}{3}$. From

$$T = \frac{1}{8\pi}(16J^2 \left( \frac{B_c r_+^{-3w-4} r_c^{3w+1}}{\pi} - \frac{32\pi P \chi}{3r_+} + \frac{2q^2}{r_+^3} - \frac{4}{r_+^3} \right) - \frac{B_c r_+^{-3w-2} r_c^{3w+1}}{\pi})$$

$$+ 16\pi Pr_+\chi - \frac{2q^2}{r_+^3} + \frac{2}{r_+},$$ (48)

we plot $T - S$ diagram for pressure $P = \chi P_c$. To analyze the impact of parameters $w$ and $B_c$ on the phase transition point, we plot the curves at the same pressure as shown in Figures 2 and 3. The horizontal black line in Figure 2 denotes the coexistence region of two phases and its intersection with the curve gives the position of the first-order phase transition point. At a given pressure $P(< P_c)$, when the horizon radius $r_+ < r_1$, BH corresponds to the liquid phase of the vdW system. For $r_+ > r_2$, the BH corresponds to the vapor phase of the vdW system. When the event horizon of BH lies in the range $r_1 \leq r_+ \leq r_2$, the BH corresponds to the vapor-liquid coexistence phase of the vdW system.
It is observed that phase transition point and the coexistence region of the two phases increase with increasing $B_c$ at the same pressure. Thus, the phase transition in a charged as well as rotating BH does not merely depend on the size of BH but electric potential, angular momentum and state parameter also affect its position. Moreover, the temperature of BH decreases for larger choices of $B_c$ and $w$. It is found that isobar in the isobaric decreases with increasing pressure. In the coexisting regions, the evolution of BH is different from Hawking behavior. As the temperature of BH should be higher due to Hawking radiation but in coexisting regions, the radiation does not affect the temperature as well as pressure of BH.

4 Thermal Fluctuations

In this section, we investigate the effects of thermal fluctuations on stability of Kerr-Newman-AdS BH surrounded by quintessence. We firstly derive the exact expression of entropy in the presence of statistical perturbations
around the equilibrium state which further yields modification in other thermodynamic potentials. To compute the corrected entropy against thermal fluctuations, the canonical partition function is taken to be
\[
Z(\beta) = \int_0^\infty \exp^{-\beta E} \sigma(E) dE,
\]
where \(T = \frac{1}{\beta}\), \(E\) and \(\sigma(E)\) are the average energy and quantum density of the system, respectively. Using inverse Laplace transformation, the density of states is calculated as
\[
\sigma(E) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \exp^{\beta E} Z(\beta) d\beta = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \exp^{S_0(\beta)} d\beta,
\]
where \(b > 0\) and \(S_0 = \ln Z + \beta E\) is the corrected entropy. Through the method of steepest descent around saddle point \(\beta\), the above integral can be put in the form
\[
S_0(\beta) = S + \frac{1}{2} (\beta - b)^2 \frac{d^2 S}{d\beta^2} |_{\beta = b} + \text{higher-order terms},
\]
where \(S = S_0(\beta)\) is the zero-order entropy with \(\frac{dS}{d\beta} = 0\) and \(\frac{d^2 S}{d\beta^2} > 0\) at \(\beta = b\). Through Eqs. (50) and (51), we get
\[
\sigma(E) = \frac{e^S}{2\pi i} \int_{b-i\infty}^{b+i\infty} \exp^{\frac{1}{2}(\beta - b)^2 \frac{d^2 S}{d\beta^2}} d\beta,
\]
which can further be simplified as

$$\sigma(E) = \frac{e^{S}}{\sqrt{2\pi \frac{d^2 S_0}{d\beta^2}}}.$$  \hspace{1cm} (53)

Eventually, this leads to

$$S_0 = S - \beta \ln(ST^2) + \frac{\beta_1}{S},$$

where $\beta$ and $\beta_1$ are correction parameters.

- For $\beta, \beta_1 \to 0$, the original BH entropy (entropy without any correction parameters) can be recovered.
- For $\beta \to 1, \beta_1 \to 0$, the usual logarithmic corrections can be obtained.
- For $\beta \to 0, \beta_1 \to 1$, the second order correction term can be obtained which is inversely proportional to original BH entropy.
- Finally, for $\beta, \beta_1 \to 1$, the effects of higher order correction terms can be observed.

Here, we consider the second case ($\beta \to 1, \beta_1 \to 0$). It is noted that second term in the above expression is logarithmic in nature which appears due to small fluctuations around the equilibrium state. Since BH is regarded as...
a macroscopic object while thermal fluctuations become effective on Planck scale level, therefore the logarithmic correction terms have a small contribution in the equilibrium entropy. Inserting Eqs. (10) and (11) in the above expression, the corrected entropy is evaluated as

\[
S_0 = \frac{\pi}{k} \left( a^2 + r_+^2 \right) - \beta \ln \left( \frac{-\frac{8}{3} \pi P r_+^2 (a^2 + r_+^2) + a^2 + q^2 - \alpha r_+^{1-3w} + r_+^2}{r_+} \right) + \frac{16}{3} \pi a^2 P r_+ + \frac{32}{3} \pi P r_+^3 + r_+^{3w} \left( - (\alpha - 3 \alpha w) \right) + 2r_+^2 \right) 16\pi k (a^2 + r_+^2) \right)^{-1}.
\]  

(54)

To analyze the impact of correction terms, we plot corrected as well as uncorrected entropy for different values of \( w \) and \( \alpha \). Figures 4 and 5 show that equilibrium entropy (\( \beta = 0 \)) is positive valued as well as monotonically increasing function which satisfies the second law of BH thermodynamics (i.e., entropy of BH always increases). However, in the presence of thermal fluctuations, the entropy of small BHs becomes negative for larger choices of correction parameter. We observe that BH gains more entropy for larger (smaller) choices of state parameter (quintessence parameter) which implies an increase in the area of BH geometry. It is noted that for larger horizon radius, the corrected entropy shows the same behavior as that of uncorrected one which yields an important fact that thermal fluctuations do not affect the thermodynamics of large BH.

The Helmholtz free energy \( F = M - TS_0 - PV \) is the direct measure of work that one can get out of a system. It becomes constant once a reversible
equilibrium is achieved. The first-order corrected Helmholtz free energy as the Legendre transformation of the internal energy is calculated by

\[
F = \frac{1}{4} r_+^{-3w-1} \left( \left( \beta \left( r_+^{3w} \left( -l^2 (a^2 + q^2) + r_+^2 \left( a^2 + l^2 \right) + 3r_+^4 \right) + 3\alpha l^2 r_+ w \right) \right) \times \ln \left( \frac{r_+^{3w-2} \left( r_+^{2w} (l^2 (a^2 + q^2) - r_+^2 \left( a^2 + l^2 \right) - 3r_+^4) - 3\alpha l^2 r_+ w \right)^2}{16\pi l^2 (l^2 - a^2) (a^2 + r_+^2)} \right) \right) \\
\times \left( \pi l^2 (a^2 + r_+^2) \right)^{-1} + \left( \alpha l^2 r_+ \left( a^2 (3w + 1) - l^2 (3w + 2) \right) + r_+^{3w} (l^2 \right) \\
\times (3l^2 - 2a^2) \left( a^2 + q^2 \right) + r_+^4 \left( 4a^2 - 3l^2 \right) + r_+^2 \left( 2a^4 - a^2 l^2 + l^4 \right)) \\
\times \left( (a^2 - l^2)^2 \right)^{-1}.
\]

Figure 6 shows that when \(-\frac{2}{3} < w \leq -\frac{1}{3}\), the Helmholtz free energy attains positive values for small BH whereas, for large BH, it shows negative behavior (left plot). However, for \(-1 < w \leq -\frac{2}{3}\), the Helmholtz free energy remains positive throughout the considered domain as shown in the right plot of Figure 6. It is noted that the correction parameter increases and decreases the Helmholtz free energy before and after the critical horizon radius, respectively.

In the extended phase space, the mass of BH is interpreted as enthalpy while Gibbs free energy is used to measure the maximum amount of reversible work that may be performed by a thermodynamic system. The corresponding Gibbs free energy \((G = M - TS_0)\) is derived to be

\[
G = \frac{1}{4r_+} \left( r_+^{3w} \left( l^2 (a^2 + q^2) - r_+^2 \left( a^2 + l^2 \right) - 3r_+^4 \right) - 3\alpha l^2 r_+ w \right)
\]
Figure 7: Gibbs free energy vs $r_+$ for $a = 1 = q$, $\alpha = 2$ with $w = -\frac{1}{3}$ (left plot) and $w = -\frac{2}{3}$ (right plot). Here $\beta = 0, 0.5$ and $1$ are denoted by red, green and blue lines, respectively.

\[
\begin{align*}
&\times \left( \ln \left( \frac{r_+^{6w-2} (r_+^{3w} (l^2 (a^2 + q^2) - r_+^2 (a^2 + l^2) - 3r_+^4) - 3\alpha l^2 r_+ w)^2)}{16\pi l^2 (l^2 - a^2) (a^2 + r_+^2)} \right) \\
&\times \beta (a^2 - l^2) + \pi l^2 (a^2 + r_+^2)) \left( \pi l^2 (l^2 - a^2) (a^2 + r_+^2) \right)^{-1} \\
&2 \left( \frac{r_+^2 (a^2 + r_+^2)}{l^2} + a^2 + q^2 - \alpha r_+^{1-3w} + r_+^2 \right) \\
&+ \frac{\left( \frac{a^2}{l^2} - 1 \right)^2}{\left( \frac{a^2}{l^2} - 1 \right)^2}. \quad (56)
\end{align*}
\]

Figure 7 indicates that Gibbs free energy depicts the same trend as that of Helmholtz free energy. It is known that positive values of Gibbs energy correspond to non-spontaneous reactions that requires an external source of energy whereas its negative values correspond to spontaneous reactions which can be driven without any external source. Black holes with negative Gibbs energy are thermodynamically stable as they release their energy in the surroundings to acquire the low energy state. It can be seen that large BHs with $-\frac{2}{3} < w \leq -\frac{1}{3}$ (left plot) are thermodynamically stable as $G < 0$. For $-1 < w \leq -\frac{2}{3}$, the system is unstable due to positive range of Gibbs free energy. This indicates that larger values of state parameters yield a stable model.

To analyze the local stability as well as phase transition, we calculate specific heat by incorporating thermal fluctuation effects. The divergence points of heat capacity are known as phase transition points whereas the signature of specific heat determines thermal stability of BH. The positive values of specific heat ensure thermodynamical stable phase while its negative values lead the system towards instability. The specific heat ($C = T \frac{\partial S}{\partial T}$) is
computed as

\[
C = \left(18\alpha r_+ w(\beta k (3w (a^2 + r_+^2) + r_+^2) + \pi r_+^2 (a^2 + r_+^2)) + 2r_+^3w(\pi r_+^2 \\
\times (a^2 + r_+^2) (r_+^2 (8\pi a^2 P + 3) - 3 (a^2 + q^2) + 24\pi P r_+^4) - \beta k(a^4 \\
\times (8\pi P r_+^2 + 3) + 3a^2(3 (8\pi P r_+^4 + r_+^2) + q^2) + 6r_+^2 (8\pi r_+^2 + q^2)) \right)
\times \left(kr_+^3w (r_+^4 (64\pi a^2 P - 3) + 3a^2 (a^2 + q^2) + r_+^2 (8\pi a^4 P + 12a^2 + 9q^2) \\
+ 24\pi P r_+^6) - 9\alpha kr_+ w (3a^2 w + r_+^2 (3w + 2)) \right)^{-1}.
\]

From graphical analysis of Figure 8, one can observe that the specific heat diverges at critical radius \(r_+ = 1\) which indicates that the system undergoes the first-order phase transition (left plot). Moreover, it is noted that the thermal fluctuations do not affect the position of phase transition. For \(\frac{-2}{3} < w \leq \frac{-1}{3}\), BHs with large horizon radius are thermally stable as the specific heat lies in the positive range whereas small BHs attain its negative values which leads the system towards instability. However, the right plot displays the negative values of specific heat for all values of \(r_+\) which indicates thermally unstable phase of BH. It is found that larger values of correction parameter yield more negative range of specific heat for small horizon radius without affecting the large BH geometries. This indicates that thermal fluctuations affect the stability of small BHs while the large BHs mostly remain unaffected.

5 Conclusions

The study of critical phenomenon as well as thermal properties of BHs has always been an interesting topic in theoretical physics. In this paper, we have studied the effects of DE on critical behavior and phase transition of quintessential Kerr-Newman-AdS BH. In this regard, we have derived the exact expression of thermodynamic quantities that satisfy Smarr-Gibbs-Dehum relation in extended phase space. The graphical analysis of Hawking temperature shows that the BH temperature decreases against the quintessence parameter. The critical behavior of thermodynamic quantities are investigated through two approaches, i.e., vdW-like equation of state and Maxwell
equal-area law. It is found that the latter technique can efficiently discuss the critical behavior of the complicated BH. Using equal-area law, we have also constructed the phase diagram in $T - S$ conjugate variables and found an isobar which shows the coexistence region of two phases. Moreover, we have computed leading order thermal corrections to entropy to investigate the effects of thermal fluctuations.

It is known that BH behaves like vdW liquid-gas system if isobaric contains a region where the condition of stable equilibrium is not satisfied. Similar to the vdW system, we have replaced un-physical oscillating region with an isobar. Here, isobar is represented by a black line that shows the coexistence region of two phases. Using Maxwell equal-area law, we have found the position of isobar in $T - S$ plane at different pressures. It is analyzed that the higher the pressure is, the shorten of isobar will be. The temperature of BH increases and decreases, respectively, for larger values of $B_c$ and $w$. It is noted that the coexistence region also increases for larger choices of $B_c$. We have also observed that the considered BH has first-order phase transition similar to that of the vdW system which does not merely depend on the size of BH but electric potential, angular momentum and state parameter also affect its position.

Finally, we have analyzed the effects of thermal fluctuations by plotting the corrected as well as uncorrected entropy for different choices of $w$ and $\alpha$. It is interesting to mention here that the existence of quintessence matter does not affect the uncorrected entropy. However, the corrected entropy increases and decreases, respectively, for larger values of $w$ and $\alpha$. The increase in entropy BH implies an increase in the area of BH geometry. The
logarithmic corrections disturb the entropy of small BH while for large BH, the corrected entropy coincides with the equilibrium state indicating that statistical perturbations do not affect thermodynamics of large BH. We have found that for $\frac{-2}{3} < w \leq \frac{-1}{3}$, the Gibbs free energy attains positive values for small BH whereas, for large BH, it shows negative behavior. However, for $-1 < w \leq \frac{-2}{3}$, it remains positive throughout the considered domain which indicates that smaller values of state parameters yield an unstable model.

To analyze the local stability, we have studied the physical behavior of specific heat with respect to horizon radius and found the same cut off values of $w$ as found in Gibbs energy. Thus the model is located in thermally stable region for $\frac{-2}{3} < w \leq \frac{-1}{3}$. It is observed that BH undergoes first-order phase transition and its position remains unchanged under logarithmic corrections. We conclude that thermal fluctuations yield more unrealistic regions in small BH geometry while large BHs mostly remain unaffected. It is worthwhile to mention here that for $\alpha = 0$, all derived quantities reduce to charged rotating AdS BH [7] and in the absence of rotation parameter, it leads to quintessential charged AdS BH solution [10][11].

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