Magnetically Charged Black Holes with Hair

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Abstract

In these lectures the properties of magnetically charged black holes are described. In addition to the standard Reissner-Nordstrom solution, there are new types of static black holes that arise in theories containing electrically charged massive vector mesons. These latter solutions have nontrivial matter fields outside the horizon; i.e., they are black holes with hair. While the solutions carrying unit magnetic charge are spherically symmetric, those with more than two units of magnetic charge are not even axially symmetric. These thus provide the first example of time-independent black hole solutions that have no rotational symmetry.

1. Introduction

Almost two centuries ago Laplace published his *Exposition du system du monde* [1]. Its final chapter contained a number of speculations on the future of astronomy. Among these was the observation that “A luminous star of the same density as the Earth, and whose diameter was two hundred and fifty times larger than that of the Sun, would not, because of its attraction, allow any of its rays to reach us; it is thus possible that the largest luminous bodies in the universe may, by this means, be invisible.” Two centuries later, black holes have indeed become objects of great astrophysical interest. There is strong evidence for their production by the collapse of dying stars and for the existence of black holes of great size in the cores of some galaxies. It has also been speculated that microscopic black holes produced shortly after the big bang may account for a portion of the dark matter in the universe.
However, there is a second aspect of these objects that could not have been anticipated by Laplace. With the advent of special relativity, an escape velocity equal to the speed of light indicates not merely an astronomically fascinating object but also suggests a sharp departure from the normal structure of space-time. This takes black holes from the realm of astrophysical phenomena into the arena of fundamental physics.

Closer theoretical consideration of these objects reveals several remarkable features. One is the occurrence of space-time singularities. These are first encountered in the simplest black hole solution, that of Schwarzschild. In that context, one might plausibly guess that the singularity was an artifact of the great symmetry of the solution and that it would disappear with even the slightest departure from spherical symmetry. To the contrary, it turns out that, under rather general conditions, the classical equations describing gravitational collapse from an initially nonsingular configuration inevitably lead to a space-time singularity. The precise nature of the singularity, and the question of whether or not it is hidden by a horizon from the view of distant observers, remain topics of active research. A second line of investigation follows from the discovery \[2\] that quantum mechanics implies that a black hole loses mass by the emission of thermal radiation, with the strong possibility that this could lead to complete evaporation of the black hole. This process leads to a number of puzzles concerning the fate of the information associated with matter that falls into the black hole, and has raised the question of whether the fundamental laws of quantum mechanics must be modified.

I will be concerned in these lectures primarily with a third aspect of black holes, namely the symmetry and simplicity of the classical black hole solutions. It is not particularly surprising that the Schwarzschild and other known closed-form solutions have these properties; after all, the solutions most likely to be found analytically are those that are symmetric and algebraically simple. What is unusual is the degree to which these properties are generic. Consider, for example, the case of gravity coupled to electromagnetism with no other types of matter present. All black hole solutions that are both time-independent and invariant under time reversal are spherically symmetric. If the latter condition is relaxed, there are also solutions with nonzero angular momentum; the direction of this angular momentum removes the isotropy, but an axial symmetry remains. Furthermore, these solutions are completely specified by the associated conserved quantities — their total energy, angular momentum, and electromagnetic charges. It was also shown for a number of cases that even with the inclusion of additional matter fields in the theory one could not produce black holes with more structure: the failure of such attempts led to the statement that “black holes have no hair.” Although these results were all obtained for specific cases, one might have guessed that there was in fact a more general result. This turns out not to be the case — theories containing electrically charged vector mesons admit magnetically charged black hole solutions with rather nontrivial structure. In these lectures I will describe these fascinating objects and the methods by which they were found.
I begin with two sections reviewing some standard results. The first, Sec. 2, contains a brief survey of some topics in black hole physics, with an emphasis on the properties of the classical solutions. The Reissner-Nordström charged black hole solution is described in this section. I then recall, in Sec. 3, some of the properties of magnetic monopoles in flat space-time, including in particular those of the nonsingular topological monopoles [3] that arise in spontaneously broken gauge theories. Next, in Sec. 4, I describe the effects of gravity on these monopoles and show that the equations describing these curved space monopoles also admit a new class of black hole solutions, carrying a single unit of magnetic charge, that have a cloud of charged vector mesons (i.e., “hair”) outside the horizon. I also show that in the context of this Higgs theory a Reissner-Nordström black hole carrying unit magnetic charge has a classical instability leading to the formation of this new type of black hole if its horizon radius is sufficiently small. In Sec. 5 these results are extended to a more general class of theories containing electrically charged massive vector mesons that are not necessarily generated via the Higgs mechanism from a non-Abelian gauge theory. It is shown that for certain choices of parameters such theories have finite energy monopole solutions in flat space and that, even if these conditions are relaxed, they give rise to magnetically charged black holes with hair. The magnetically charged solutions considered up to this point are all spherically symmetric and all carry unit magnetic charge. Black hole solutions with higher magnetic charge are also possible. However, essentially for reasons arising from the anomalous angular momentum of a charge-monopole pair, these cannot (except for the Reissner-Nordström case) cannot be spherically symmetric. A useful tool for investigating these nonsymmetric solutions is the formalism of monopole scalar and vector spherical harmonics, which is developed in Sec. 6. In Sec. 7 this formalism is employed to extend the stability analysis of Sec. 4 to Reissner-Nordström black holes in an arbitrary theory with charged vector mesons, with no restriction on the magnetic charge. The results of this analysis imply the existence of new black holes with hair that, because they carry other than unit magnetic charge, are not spherically symmetric. While an exact closed form solution for these objects is probably unattainable, a perturbative scheme for constructing them can be developed under certain conditions. This scheme is described in Sec. 8 and is used to show that, at least for certain choices of parameters, these new solutions are not even axially symmetric.

2. Black Holes

2.1 The Schwarzschild Solution

The most elementary problem to be solved in Newtonian gravity is the calculation of the gravitational field of a point mass at rest. Let us consider the general relativistic analogue of this problem [4]. We expect the metric to be static (i.e., both independent
of time and invariant under time reversal) and spherically symmetric. Any such metric can be written in the form

\[ ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(1)

The energy-momentum tensor should vanish everywhere except at the position of the point mass. Hence, one should insert this form for the metric into the source-free Einstein equations and solve, subject to the boundary condition that the spacetime be asymptotically flat. This leads, by well-known steps, to the Schwarzschild [5] metric

\[ ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(2)

where \( M \) is an arbitrary constant that arises in the course of integrating the field equations. The meaning of \( M \) is obtained by examining the behavior of the metric at large distance. If we recall that for weak gravitational fields the Newtonian gravitational potential is related to the spacetime metric by

\[ g_{tt} = -1 - 2\phi_{\text{Newton}} \]  

(3)

we see that \( M \) is just the mass that an observer at large distances would measure by, for example, observations of Keplerian orbits.

This metric has singularities at \( r = 2M \), where \( g_{tt} = 0 \) and \( g_{rr} = \infty \), and at \( r = 0 \), where \( g_{tt} = \infty \) and \( g_{rr} = 0 \). One possibility is that these are coordinate singularities that reflect singularities in the definition of the coordinates rather than any singularity in spacetime. An two-dimensional example of this is provided by the metric

\[ ds^2 = dr^2 + r^2 d\theta^2. \]  

(4)

Transforming from polar to Cartesian coordinates removes the singularity at \( r = 0 \) and shows that the space is simply the flat two-dimensional plane. By contrast, for \( \alpha < 1 \) the metric

\[ ds^2 = dr^2 + \alpha r^2 d\theta^2 \]  

(5)

describes a cone and has a true singularity at \( r = 0 \), as can be verified by a calculation of the curvature.

The curvature scalar \( R \) and the Ricci tensor \( R_{\mu\nu} \) vanish everywhere in this spacetime. However, the scalar \( R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} \) is nonzero and equal to \( 48M^2G^2/r^6 \), thus showing that the metric singularity at \( r = 0 \) does indeed reflect a true singularity of the spacetime.
The singularity at $r = 2M$, on the other hand, is only a coordinate singularity. This can be demonstrated [6] by introducing Kruskal coordinates $X$ and $T$ that are related to $r$ and $t$ by

$$X^2 - T^2 = \left(\frac{r}{2MG} - 1\right)e^{r/2MG}$$

$$\frac{T}{X} = \tanh(t/4MG).$$

(6)

The metric can then be written in the form

$$ds^2 = \frac{32M^3G^3e^{-r/2MG}}{r} (-dT^2 + dX^2) + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

(7)

whose only singularity is at $r = 0$.

The full range, $0 \leq r < \infty$, $-\infty < t < \infty$, of the original Schwarzschild coordinates corresponds to the region in which $T^2 - X^2 \geq 1$ and $X + T \geq 0$. However, since the metric is perfectly regular along the surface $X + T = 0$, there is no bar to extending the range of the Kruskal coordinates beyond this boundary to obtain the extended Schwarzschild spacetime shown in Fig. 1. In this figure the coordinates have been drawn in such a manner that light cones are $45^\circ$ lines. The lines $X = \pm T$ divide the extended spacetime into four quadrants. The first, $X > |T|$, is the “exterior” region $r > 2M$ of the original spacetime. The second quadrant, $|X| < T < (X^2 + 1)^{1/2}$, corresponds to the “interior” region $0 < r < 2M$. Note the causal relation between these two quadrants: a particle can go from the first to the second quadrant by crossing the line $X = T$ ($r = 2M$), but once inside this quadrant it can never leave. This region is called a black hole, and its boundary the horizon. The third quadrant, $(X^2 + 1)^{1/2} < T < -|X|$, is in a sense the opposite of the second. Any particle inside it must eventually leave and can never return. Finally, the fourth quadrant, $X < -|T|$, is a second exterior region quite similar to the first quadrant but completely inaccessible to it.

We can see from this that the interpretation of the Schwarzschild coordinates $r$ and $t$ is not as simple as one might have thought when first writing down the metric (2). Inside the horizon, $t$ is a spatial coordinate rather than a timelike one. Thus, although our metric is indeed independent of $t$, the spacetime is not truly static for $r < 2M$. Similarly, $r$ is timelike inside the horizon, although it does continue to specify the circumferences of the two-spheres generated by the $SO(3)$ isometries of the metric. The “point” $r = 0$ is in fact a time, not a position. Examination of Fig. 1 shows that there are in fact many spacelike surfaces on which $r$ is never zero. Although we began by seeking the metric generated by a point mass at the origin, and have indeed found a solution that at large distances is consistent with the Newtonian potential from such a source, our spacetime has no static point mass inside it.
Fig. 1. The extended Schwarzschild spacetime. Each point in the plane represents a two-sphere of constant $X$ and $T$ (and thus of constant $r$ and $t$). The $X$ and $T$ axes, which are not shown here, would be the usual Cartesian axes, so lightcones correspond to $45^\circ$ lines. Lines of constant $r$ and $t$ are indicated by long and short dashes, respectively. The singularities at $r = 0$ are indicated by the heavy solid lines, while the lighter solid lines indicate the horizon, $r = 2MG$.

Because the horizon at $r = 2M$ is merely a coordinate singularity, from a purely local viewpoint one would not perceive any distinction between points just outside the horizon and those just inside it. Viewed globally, of course, the causal structure of the spacetime makes a sharp distinction between these. To illustrate this difference between the local and the global viewpoints, consider two classes of observers near the horizon. The first falls freely toward, and then through, the horizon. Although they suffer strong gravitational tidal forces as they approach the neighborhood of the horizon, and are crushed to death in the $r = 0$ singularity soon after entering the black hole, they feel no distinct effect at the moment of crossing the horizon. The second are a set of static observers who remain at fixed values of the Schwarzschild coordinate $r$. To keep from falling into the black hole, some external force must be applied; one might imagine them to be hanging from long ropes whose other ends were held by stationary assistants far from the black hole. For these observers the horizon is quite special; for example, the value they measure for the acceleration of gravity, $g$, diverges as their position approaches the horizon.
To the freely falling observers, the static observers at the horizon appear to be undergoing an infinite acceleration. From this local point of view, the divergence in the measurements of \( g \) reflects the unusual behavior of the observer rather than any special property of spacetime. For the behavior of the static observers to seem anything but bizarre, one must step back and view the spacetime as a whole. The static nature of the metric in the exterior regions is then apparent, and it becomes quite reasonable to consider the measurements of an observer with fixed Schwarzschild position.

Finally, one must keep in mind that the full extended Schwarzschild spacetime, although a solution of Einstein’s equations, is unlikely to occur in the real world, where black holes are expected to be formed by gravitational collapse from nonsingular initial conditions. After the collapse, such black holes will settle down to almost static configurations that are well described by the Schwarzschild metric. These asymptotic spacetimes have exterior and interior regions that approximate portions of the first two quadrants of Fig. 1. The third and fourth quadrants, however, are completely absent.

### 2.2 Other Black Holes

By endowing the black hole with electric or magnetic charge, one obtains the spherically symmetric Reissner-Nordström solution [7]. This has radial electric and magnetic fields

\[
E_r = \frac{Q_E}{r^2} \\
B_r = \frac{Q_M}{r^2}
\]  

and a metric

\[
ds^2 = -B_{RN}(r) dt^2 + B_{RN}^{-1}(r) dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)
\]

where

\[
B_{RN}(r) = 1 - \frac{2MG}{r} + \frac{4\pi G \sqrt{Q_E^2 + Q_M^2}}{r^2}.
\]

The nature of this spacetime depends on the value of the mass \( M \). If \( M \) is greater than the extremal mass

\[
M_{\text{ext}} = \sqrt{4\pi (Q_E^2 + Q_M^2)} M_{\text{Pl}}
\]

(where \( M_{\text{Pl}} = G^{-1/2} \) is the Planck mass), the metric has coordinate singularities at
the zeroes of $B(r)$, which occur at

$$r_{\pm} = MG \left[ 1 \pm \sqrt{1 - \left( \frac{M_{\text{ext}}}{M} \right)^2} \right].$$  \hspace{1cm} (12)$$

In addition, there is a true physical singularity at $r = 0$. As with the Schwarzschild solution, this spacetime can be extended beyond the region covered by the original coordinates $r$ and $t$. The causal structure of this extended spacetime is more complex than in the Schwarzschild case. However, as in that case, there is a horizon at $r = r_+$ with the property that objects that cross inside it can never return to the exterior region $r_+ < r < \infty$.

If $M = M_{\text{ext}}$, so that the two zeroes of $B_{RN}(r)$ coincide, one obtains the extremal Reissner-Nordström black hole. There is still a horizon, but the double zero in $g_{rr}^{-1}$ implies that it lies at an infinite proper distance from any point in the exterior region $r > r_H$. One also has the curious fact that there is no long-range static force between two extremal Reissner-Nordström black holes with charges of the same sign, because their mutual gravitational attraction is exactly cancelled by their electrostatic and magnetostatic repulsion. (In fact, static multi-black hole solutions for the extremal case are known [8].)

The final possibility is $M < M_{\text{ext}}$. In this case one has a “naked singularity” in which there is no horizon shielding the asymptotic region from the effects of the physical singularity at $r = 0$. (The Schwarzschild solution with negative $M$ also has a naked singularity.) For the remainder of these lectures I will exclude naked singularities from the discussion. Indeed, it has been conjectured that there is a “cosmic censorship” that prevents the evolution of a naked singularity from generic nonsingular initial conditions.

There is a simple physical explanation for the existence of a minimum horizon radius about a charged black hole. Roughly speaking, one expects a horizon to form whenever a region of radius $r$ contains a total energy much greater than $r/G$. If a charged black hole were to have a horizon at $r_H \ll Q G^{1/2}$, the energy in the Coulomb field just outside the horizon (say, in the region $r_H < r < 2r_H$) would be much greater than $r_H/G$, thus implying the existence of a horizon at a radius much greater than the hypothesized $r_H$.

The Schwarzschild and Reissner-Nordström solutions are both static. It is also possible to have a metric that is stationary (i.e., time-independent) but, because of the presence of nonzero time-space components, not invariant under time reversal. An example of this is the axisymmetric Kerr metric [9], which describes a black hole with a mass $M$ and an angular momentum $J$ along the axis of symmetry; it can be generalized [10] to include the possibility of nonzero electric and magnetic charges. As with the previous black hole solutions, there is a region, bounded by a horizon,
from which one cannot escape to the asymptotically flat external region. However, in contrast with these solutions, the surface on which $g_{tt}$ vanishes lies outside the horizon, meeting it only on the axis of symmetry. The region lying between these two surfaces is known as the ergosphere (the significance of this name will be explained below). Within the ergosphere a particle cannot be stationary, but must have an angular velocity greater than an ($r$-dependent) minimum.

One might well expect there to be additional types of black holes. For example, one might seek solutions that corresponded at large distances to the Newtonian gravitational field due to a mass distribution with higher multipole moments. Remarkably, if we restrict ourselves to gravity plus electromagnetism, there are no more black holes; the only time-independent solutions are those described above. These can be characterized completely by their mass, angular momentum, and electromagnetic charges. They are all at least axially symmetric, and the static solutions are all spherically symmetric.

The next step, of course, is to see if more structure can be obtained by including other types of matter fields in the theory. This possibility has been ruled out for a number of cases, giving rise to the statement that “a black hole has no hair”. However, it should be realized that there is no general no-hair theorem but only a collection of theorems applicable to very specific cases. As an example of these, consider the case [11] of a massive scalar field $\phi$ with the matter portion of the action taking the form

$$S_{\text{matter}} = \int d^4x \sqrt{g} \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right].$$

(13)

For the sake of simplicity, let us assume spherical symmetry, which implies that the metric can be written in the form (1). For a static solution, the field equations take the form

$$\frac{1}{r^2 \sqrt{AB}} \frac{d}{dr} \left[ \frac{r^2 \sqrt{AB}}{A} \phi' \right] = \frac{dV}{d\phi}.$$

(14)

where a prime denotes a derivative with respect to $r$. Let us now multiply both sides of this equation by $\phi - \phi_0$, where $\phi_0$ is the value which minimizes $V(\phi)$. After some rearranging, we obtain

$$(\phi - \phi_0) \frac{d}{dr} \left[ \frac{r^2 \sqrt{AB}}{A} \phi' \right] = (\phi - \phi_0)r^2 \sqrt{AB} \frac{dV}{d\phi}.$$

(15)

Integrating this over $r$ from the horizon to infinity leads to

$$\int_{r_H}^{\infty} dr \frac{d}{dr} \left[ \frac{r^2 \sqrt{AB}}{A} (\phi - \phi_0) \phi' \right] = \int_{r_H}^{\infty} dr r^2 \sqrt{AB} \left[ \frac{(\phi')^2}{A} + (\phi - \phi_0) \frac{dV}{d\phi} \right].$$

(16)
The left-hand side of this equation can be immediately integrated to give two surface terms. The one from \( r = \infty \) vanishes because \( \phi \) tends asymptotically to its vacuum value, \( \phi_0 \), with the deviation from \( \phi_0 \) falling exponentially fast because we are dealing with a massive field. The surface term from \( r = r_H \) is also zero, since \( g^{rr} = A^{-1}(r) \) vanishes on the horizon. On the right-hand side, the contribution from the first term in brackets is clearly positive, since \( A(r) > 0 \) for all \( r > r_H \). The contribution due to the second term is also positive if we assume that \( \phi_0 \) is the only minimum of \( V \). When this is the case, we have two positive terms whose sum must vanish. This implies that the two vanish separately, and allows only the trivial solution in which \( \phi \) is identically equal to its vacuum value everywhere outside the horizon. Note that this argument fails if \( V(\phi) \) has minima other than \( \phi_0 \), even if these are only local minima.

2.3 Energy Extraction and Black Hole Thermodynamics

It is clear that the mass of a black hole can increase by absorption of infalling matter. Given the fact that particles cannot escape from inside the horizon, it might seem equally clear that the mass can never decrease. In fact, this is not so. One example of such a process is associated with the Kerr black hole. Recall that there is a region just outside the horizon, known as the ergosphere, in which \( g_{tt} \) is positive (i.e., of the nonstandard sign). Using this fact, one can show that it is possible for a particle in the ergosphere to have a negative total energy; very roughly speaking, one might say that it has a negative gravitational potential energy that is larger in magnitude than its rest energy. Several mechanisms for extracting energy from the black hole then suggest themselves. For example, one might send a positive-energy object into the ergosphere, have it produce some negative-energy particles that are then sent inside the horizon, and then have the object emerge from the ergosphere with more energy than it started with. The net effect of this would be to reduce the mass of the black hole.

This process cannot be repeated indefinitely. Particles inside the ergosphere cannot be static, but must instead have a minimum angular velocity whose sign is such that any negative-energy particles falling inside the horizon decrease the total angular momentum of the hole. This in turn reduces the size of the ergosphere. Eventually, the hole loses all of its angular momentum, thus becoming a Schwarzschild black hole, and the possibility of energy extraction ceases.

Energy can also be extracted from a Reissner-Nordström black hole. To see this, consider a particle near the horizon that carries a charge of the same type (i.e., electric or magnetic) but opposite sign as the black hole. If the charge to mass ratio of the particle is large enough, its negative potential energy can be of sufficient magnitude to cancel the rest mass and give the particle a negative total energy. As with the Kerr example, one can use such negative-energy particles to reduce the mass of the
black hole. One simple mechanism is to produce a pair of particles with opposite sign near the horizon, letting the one with negative energy fall through the horizon while the one with positive energy goes off to spatial infinity. Again, this process is self-limiting, in that the negative-energy particles reduce the charge of the black hole, with energy extraction no longer possible once the hole is completely neutralized.

Further analysis leads to the formulation of several laws of black hole dynamics that govern these and other processes:

0. One can define a quantity $\kappa$, known as the surface gravity, that is constant over the horizon of a stationary black hole.

1. The change in black hole mass can be written as a sum of a term proportional to the change in the area $A$ of the horizon and “work terms”. In the case of a Kerr black hole, for example, one has

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ,$$  \hspace{1cm} (17)

where $\Omega_H$ may be interpreted as the angular velocity of the horizon.

2. The area of the black hole horizon can never decrease; i.e.,

$$dA \geq 0.$$  \hspace{1cm} (18)

3. One cannot reach the limit $\kappa = 0$ by any physical process.

These are remarkably similar in form to the corresponding laws of thermodynamics. To convert one set to the other, we need only replace the work term in the first law by $PdV$ and make the substitutions $\kappa \rightarrow T/a$, $M \rightarrow E$, and $A \rightarrow 8\pi a S$, where $T$, $E$, and $S$ are the temperature, energy, and entropy, respectively, and $a$ is an arbitrary constant. Hawking's discovery \[2\] that quantum mechanically black holes radiate with a temperature $T_H = \hbar \kappa/2\pi$ fixes the constant $a$, and indicates a deeper meaning to this analogy.

There are a several ways to determine the Hawking temperature of a black hole. A particularly convenient method \[12\] is based on the formulation of thermodynamics in terms of a Euclidean path integral over configurations with periodicity $\hbar/T$. To apply this to a spherically symmetric spacetime with a metric of the form (1), we first make the replacement $t \rightarrow i\tau$, thus obtaining the Euclidean metric

$$ds_E^2 = B(r) d\tau^2 + A(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$  \hspace{1cm} (19)

We want $\tau$ to be a periodic variable with period $\hbar/T$; this suggests that we define an
angle
\[ \alpha = \frac{2\pi T}{\hbar} \tau \]  \hspace{1cm} (20)

that is understood to have period 2\pi. If we also change variables from \( r \) to a quantity
\[ R(r) = \frac{\hbar}{2\pi T} \sqrt{B(r)}, \]  \hspace{1cm} (21)

the Euclidean metric takes the form
\[ ds_E^2 = \frac{4AB}{(B')^2} \left( \frac{4\pi T}{\hbar} \right)^2 dR^2 + R^2 d\alpha^2 + r^2 \sin^2 \theta d\phi^2. \]  \hspace{1cm} (22)

The horizon of the Lorentzian signature spacetime lies at the zero of \( B(r) \). In the Euclidean metric (22) this corresponds to the zero of \( R(r) \), which may be viewed as the origin of a two-dimensional subspace of constant \( \theta \) and \( \phi \) that is spanned by polar coordinates \( R \) and \( \alpha \). The temperature is fixed by requiring that the singularity at this point be merely a coordinate singularity rather than a true conical singularity. This is achieved by arranging that the coefficient of \( dR^2 \) be equal to unity at \( r = r_H \), which gives
\[ T = \frac{\hbar}{4\pi} \left( \frac{B'}{\sqrt{AB}} \right)_{r=r_H}. \]  \hspace{1cm} (23)

For the Schwarzschild metric, we obtain a temperature
\[ T_H = \frac{\hbar}{8\pi MG} \]  \hspace{1cm} (24)

that increases monotonically as the mass decreases. For the Reissner-Nordström black hole the temperature is
\[ T_H = \frac{\hbar}{2\pi G} \frac{\sqrt{M^2 - M_{\text{ext}}^2}}{\left[ M + \sqrt{M^2 - M_{\text{ext}}^2} \right]^2}. \]  \hspace{1cm} (25)

For large \( M \) this also increases with decreasing mass. However, as the mass approaches its extremal value the temperature turns over and begins to decrease, finally tending to zero as \( M \to M_{\text{ext}} \).
The Hawking radiation removes mass from the black hole at a rate proportional to the area of the horizon times the fourth power of the temperature. In the Schwarzschild case, this is proportional to $1/M^2$, suggesting that the black hole evaporates completely in a time

$$t_{\text{evap}} \sim \left( \frac{M}{M_{\text{Sun}}} \right)^3 (10^{71} \text{ seconds}) .$$

(26)

For the Reissner-Nordström case, on the other hand, the evaporation turns off as the black hole approaches the extremal limit, assuming that the hole has not in the meantime been discharged by some process.

It must be kept in mind, of course, that the semiclassical approximation used to derive the Hawking process breaks down as the black hole approaches Planck size. Whether complete evaporation actually occurs and whether this leads to information loss and a breakdown of the unitary evolution of quantum mechanics are important issues that lie beyond the scope of these lectures.

3. Magnetic Monopoles in Flat Spacetime

Let us now recall some facts about magnetic monopoles in flat space-time. A monopole with magnetic charge $Q$ gives rise to a radial magnetic field

$$B = \hat{r} \frac{Q}{r^2}$$

(27)

that can be derived from the Dirac vector potential with components

$$A_{r}^{\text{Dirac}} = 0$$

$$A_{\theta}^{\text{Dirac}} = 0$$

$$A_{\phi}^{\text{Dirac}} = Q(1 - \cos \theta) .$$

(28)

This potential has a singularity (“the Dirac string”) along the negative $z$-axis. Electromagnetic gauge-transformations can change the orientation of this singularity, but cannot remove it. By requiring that the string be unobservable (for example, by Aharonov-Bohm experiments) one obtains the Dirac quantization condition [13]

$$Q = \frac{q}{e}$$

(29)

where $e$ is the smallest electric charge in the theory and $q$ is either an integer or an integer plus one half.
Even when the Dirac quantization condition is satisfied, a singularity at the origin remains; this is, of course, no more remarkable than the singularity of a point electric charge. However, it was shown by ’t Hooft and Polyakov [3] that certain spontaneously broken gauge theories have completely nonsingular classical solutions that carry magnetic charge. The simplest case is an $SU(2)$ gauge theory that is spontaneously broken to $U(1)$ when a triplet Higgs field $\phi$ acquires a nonzero vacuum expectation value of magnitude $v$; I will use the language of electromagnetism to describe this unbroken $U(1)$. This theory is governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} + \frac{1}{2} (D_{\mu} \phi)^a (D^\mu \phi)^a - \frac{\lambda}{2} (\phi^a \phi^a - v^2)^2$$

where the field strength

$$F^{a\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} - e \epsilon^{abc} A^{b}_{\mu} A^{c}_{\nu}$$

and the covariant derivative

$$D_{\mu} \phi^a = \partial_{\mu} \phi^a - e \epsilon^{abc} A^{b}_{\mu} \phi^c.$$ 

In the vacuum, $\phi$ may be chosen to have a constant direction in internal space; to be definite, let $\phi^a = v \delta^a_3$. The particle content of the theory then includes a massless photon (corresponding to the 3-component of the gauge field), two charged vector particles with mass $ev$ and charges $\pm e$ (corresponding to linear combinations of the 1- and 2-components of the gauge field), and a massive neutral scalar.

The monopole solution is obtained by allowing the orientation of $\phi$ in internal space to vary from point to point. Specifically, let us suppose that the direction of $\phi$ in internal space is aligned with the direction from the origin in physical space, so that $\phi^a \to v \hat{r}^a$ as $r \to \infty$. Configurations with this asymptotic behavior are topologically distinct from the vacuum; i.e., they cannot be smoothly deformed into a configuration where $\phi$ has a uniform direction. Minimizing the energy among configurations of this topological class gives a new static solution. To see that this solution is a magnetic monopole, we must examine the long-range behavior of the gauge fields. First, note that the total energy is finite only if $D_{\mu} \phi$ vanishes asymptotically. Given the asymptotic form of $\phi$, this means that, up to a $U(1)$ gauge transformation, the spatial components of the asymptotic gauge field must be

$$A^a_i = \epsilon_{iak} \hat{r}_k \frac{1}{er}.$$  

From this we find that at large distance the magnetic components of the field strength are

$$F^{a}_{ij} = \epsilon_{ijk} \hat{r}_k \frac{1}{er^2},$$

which is just the Coulomb magnetic field of a monopole of magnetic charge $1/e$. 

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This charge is twice the minimum value allowed by the Dirac quantization condition. Indeed, analysis of the possible topologies of the Higgs field at spatial infinity shows that nonsingular configurations can only have integral magnetic charges. Although this would seem to allow nonsingular magnetic monopoles with multiple charge, further analysis of the field equations shows that such solutions are absent, except in the Prasad-Sommerfield limit, $\lambda \to 0$ [14].

The monopole solution can be studied in further detail by introducing the spherically symmetric ansatz

$$
\phi^a = \hat{r}^a vh(r)
$$

$$
A^a_i = \epsilon_{iak} \hat{r}^k \left[ \frac{1 - u(r)}{er} \right]. \tag{35}
$$

The requirement that the fields be nonsingular at the origin gives the boundary conditions

$$
h(0) = 0, \quad u(0) = 1, \tag{36}
$$

while the asymptotic behavior described above implies that

$$
h(\infty) = 1, \quad u(\infty) = 0. \tag{37}
$$

It is instructive to apply an $SU(2)$ gauge transformation that makes the orientation of $\phi$ spatially uniform. This reintroduces the Dirac string, but has the advantage of making clearer the correspondence between the physical fields and the radial functions in the ansatz of Eq. (35). The configuration that results takes the form

$$
\phi^a = \delta^{a3} vh(r)
$$

$$
A^{EM}_i \equiv A^3_i = A^\text{Dirac}_i
$$

$$
W_i \equiv \frac{1}{\sqrt{2}} (A^1_i + i A^2_i) = f_i(\theta, \phi) \frac{u(r)}{er} \tag{38}
$$

where the $f_i(\theta, \phi)$ are functions whose detailed form will not concern us at present. We see that $u(r)$ determines the magnitude of the charged vector field $W_i$, while, as could already be seen from Eq. (35), $h(r)$ gives the magnitude of the Higgs field.

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1 One way of understanding this is to realize that one could easily introduce an $SU(2)$ doublet of scalar fields into the theory, with the interactions arranged so that the unbroken gauge group remained $U(1)$ and so that the ‘t Hooft-Polyakov monopole remained a classical solution. After symmetry breaking, this components of the doublet would have electric charges $\pm e/2$, and so the Dirac condition would become $Q = n/e$, with $n$ an integer.
Inserting the ansatz of Eq. (35) into the fields equations leads to a pair of coupled differential equations. In general, these can only be solved numerically. Even without doing so, we can obtain a rough picture of the solution. Near the origin there is a monopole core of radius $R_{\text{mon}}$ within which $u$ is nonzero and $h$ is close to unity. Outside this core the two radial functions, each of which corresponds to a massive field, tend exponentially fast to their asymptotic values, leaving only the Coulomb magnetic field. If we approximate the energy density within the core as a constant, $\rho_0$, then the total energy of the solution is

$$M_{\text{mon}} \approx \frac{4\pi}{3} \rho_0 R_{\text{mon}}^3 + \frac{2\pi}{3} \frac{Q_M^2}{R_{\text{mon}}}$$ \hspace{1cm} (39)$$

where the first term is the core contribution and the second is the energy in the Coulomb field outside the core. Adjusting the core radius to minimize the energy, we obtain

$$R_{\text{mon}} \sim \sqrt[4]{\frac{Q_M}{\rho_0}}$$ \hspace{1cm} (40)$$

which gives

$$M_{\text{mon}} \sim Q_M^{3/2} \rho_0^{1/4}.$$ \hspace{1cm} (41)$$

If the vector and scalar masses are comparable (i.e., if $\lambda \sim e^2$), then a reasonable estimate of the core energy density is $\rho_0 \sim \lambda^4 \sim e^2 v^4$. Substituting this into the above equations and using the fact that $Q_M = 1/e$, we obtain

$$R_{\text{mon}} \sim \frac{1}{e v}$$ \hspace{1cm} (42)$$

and

$$M_{\text{mon}} \sim \frac{v}{e}.$$ \hspace{1cm} (43)$$

More detailed calculations show that these estimates are in fact valid even if $\lambda$ is not comparable to $e^2$. In particular,

$$M_{\text{mon}} = \frac{4\pi v}{e} f \left( \frac{\lambda}{e^2} \right)$$ \hspace{1cm} (44)$$

where $f(0) = 1$ and $f(\infty) = 1.787$ [15].
4. Magnetic Monopoles in Curved Spacetime

4.1 Nonsingular Magnetic Monopoles

Let us now examine the effects of gravity on the monopole solution[16-19]. In particular, consider what happens if the monopole mass is increased. The Schwarzschild radius of the monopole is

\[ R_S = 2M_{\text{mon}} G \sim \frac{v}{eM_{\text{Pl}}^2}. \]  

(45)

If the radius of the monopole core were to be much less than this, we would expect the monopole to become a black hole. Comparison of Eqs. (42) and (45) suggests that this would happen if \( v \) exceeded a critical value of order \( M_{\text{Pl}} \). For such values the monopole mass exceeds the Planck mass by a factor of roughly \( 1/e \). However, because the energy density within the monopole is of order \( e^2 M_{\text{Pl}}^4 \), we should be able to safely neglect quantum gravity effects as long as \( e^2 \) is sufficiently small.

To see how this works, let us examine the curved space solutions more closely. If we assume that the solution is static and spherically symmetric, then the metric can be written in the form of Eq. (1), while the matter fields can be described by the ansatz of Eq. (35), suitably modified for a curved space. The action then reduces to

\[
S_{\text{matter}} = -4\pi \int dt \, dr \, r^2 \sqrt{AB} \left[ \frac{K}{A} + U \right]
\]

(46)

where

\[
K = \frac{(u')^2}{e^2 r^2} + \frac{1}{2} v^2 (h')^2
\]

(47)

and

\[
U = \frac{(1 - u^2)^2}{2e^2 r^4} + \frac{u^2 h^2 v^2}{r^2} + \frac{\lambda}{2} v^4 (1 - h^2)^2.
\]

(48)

Thus, the theory has been effectively reduced to a theory of two scalar fields in one space and one time dimension, but with a slightly unconventional gradient term and with a scalar field potential \( U(u, h) \) that is position dependent. At large \( r \) the potential is minimized by setting \( u = 0 \) and \( h = 1 \), while at small \( r \) its minimum is at \( u = 1, h = 0 \). More precisely, the former configuration is at least a local minimum for \( 1/(ev) \leq r < \infty \), while the latter is a local minimum for \( 0 \leq r \leq 1/(\sqrt{\lambda}v) \). If \( \lambda < e^2 \) these two intervals overlap; the large distance minimum becomes the global minimum.
at $r = 1/(\lambda e^2)^{-1/4}v$. If $\lambda > e^2$, the two intervals are disjoint; in the region between them there is an $r$-dependent minimum that interpolates between the short-distance and large-distance minima.

The field equations that follow from variation of the matter action are

\[
\frac{1}{\sqrt{AB}} \frac{d}{dr} \left[ \sqrt{AB} u' \right] = \frac{e^2 r^2}{2} \frac{\partial U}{\partial u}
\]

\[
= \frac{u(u^2 - 1)}{r^2} + e^2 u h^2 v^2
\]  

(49)

and

\[
\frac{1}{r^2 \sqrt{AB}} \frac{d}{dr} \left[ \frac{r^2 \sqrt{AB} h'}{A} \right] = \frac{1}{v^2} \frac{\partial U}{\partial h}
\]

\[
= \frac{2u^2 v^2 h}{r^2} + \lambda v^4 h(h^2 - 1).
\]  

(50)

These must be supplemented by Einstein’s equations. If we define a quantity $\mathcal{M}(r)$ by

\[
A^{-1}(r) = 1 - \frac{2\mathcal{M}(r)G}{r},
\]

(51)

these may be reduced to the two equations

\[
\mathcal{M}' = 4\pi r^2 \left( \frac{K(u, h)}{A} + U(u, h) \right)
\]

(52)

and

\[
\frac{(AB)'}{AB} = 16\pi GrK(u, h).
\]

(53)

The last of these equations can be used to eliminate $AB$ from the first three. This leaves two second-order and one first-order equation for the functions $h$, $u$, and $\mathcal{M}$. These require five boundary conditions. Four are provided by the conditions obtained above on the values of $h$ and $u$ at the origin and at spatial infinity. The fifth follows from noting that the metric is nonsingular at $r = 0$ only if $\mathcal{M}(0) = 0$. Note that $\mathcal{M}(\infty)$ is unconstrained; it gives the total mass of the monopole.

This set of differential equations can be solved numerically. It is interesting to compare the behavior of the solutions as the quantity $v/M_{\text{Pl}}$ is increased while the other parameters are held fixed [17-19]; this corresponds to increasing the effects of gravity. One finds that the monopole core (as measured by the behavior of $h(r)$ and
$u(r)$ is pulled in to smaller values of $r$, until a limiting case is reached for a critical value of $v$ of order $M_{\text{Pl}}$. For this limiting case, the variation of $h(r)$ and $u(r)$ is confined entirely to the range $0 < r < r_{\text{ext}}$, where $r_{\text{ext}} = (\sqrt{4\pi/\epsilon})M_{\text{Pl}}^{-1}$ is the horizon radius of the extremal Reissner-Nordström black hole with unit magnetic charge; for all $r > r_{\text{ext}}$, $h(r) = 1$ and $u(r) = 0$. Similarly, the metric is precisely that of the extremal Reissner-Nordström solution for $r > r_{\text{ext}}$, but is smooth and nonsingular for $0 < r < r_{\text{ext}}$. For values of $v/M_{\text{Pl}}$ greater than this critical value, we find no nonsingular monopole solutions.

This description suggests a monopole core being pulled inward by an increasingly strong gravitational force. A somewhat different picture is obtained if, instead of plotting the solutions as functions of the coordinate $r$, one plots them as functions of the invariant proper distance from the origin, $\ell(r) = \int_0^r dr' g_{rr}^{1/2}$. Plotted this way, the radial functions $h$ and $u$ show very little variation as the strength of gravity is increased. This remains true even when the limiting case is reached. This is possible because in the extremal case the double zero of $g_{rr}^{-1}$ implies that the position of the zero lies an infinite proper distance from the origin.

4.2 Magnetically Charged Black Holes

Let us now turn to the magnetically charged black holes of this theory. First of all, we can trivially extend the Reissner-Nordström black hole to this theory. From examining the field equations, it is clear that a solution can be obtained by taking the metric to be exactly the same as in the electromagnetic case, and setting $u(r) = 0$ and $h(r) = 1$ everywhere. As before, to avoid a naked singularity we must require that the black hole mass $M \geq M_{\text{ext}}$.

In addition, we can obtain an entirely new type of magnetic black hole [17-18] by making only a slight variation on the procedure used to obtain the nonsingular curved space monopoles. To be specific, suppose that we allowed $\mathcal{M}(0)$ to be nonzero and positive. This would give a singularity at $r = 0$ that we would expect to be hidden behind a horizon at some $r_H \sim \mathcal{M}(0)G$. If $\mathcal{M}(0)$ were small enough that $\mathcal{M}(0)G \ll 1/(\epsilon v)$, we might thus obtain an object that could be viewed as a tiny Schwarzschild black hole in the center of an ’t Hooft-Polyakov monopole. This would be a black hole surrounded by nontrivial massive charged vector and Higgs fields; in other words, a black hole with hair.

To understand heuristically how the no-hair theorems are evaded, recall that the proof given above depended on the fact that the scalar potential had only a single minimum. In the context of our spherically symmetric ansatz, the Higgs theory reduces to one involving two scalar fields governed by a potential $U(u, h)$ that has different minima at small and large $r$. This suggests that a black hole with hair might exist provided that the horizon lies in the region where $U$ is minimized by the
short-distance minimum. Indeed, one can show that a necessary condition for such a solution is that \( r_H \) be less than the maximum of \( 1/(ev) \) and \( 1/(\sqrt{\lambda}v) \).

We can proceed further by returning to Eqs. (49)-(53). Since we have a curvature singularity at \( r = 0 \), there is no longer any reason to fix the values of \( h(0) \) and \( u(0) \). These boundary conditions at the origin are replaced by new boundary conditions at the horizon. These arise because the coefficients of \( u'' \) and \( h'' \) in Eqs. (49) and (50) vanish at the zeroes of \( 1/A \). Requiring that the matter fields be nonsingular at \( r_H \) then gives two constraints (one for each equation) involving the values of the fields and their derivatives at the horizon. If there were more than one horizon, there would be two such conditions at each horizon. Together with the conditions at spatial infinity this would give at least six boundary conditions, which would be too many for our set of differential equations. Hence, apart from exceptional cases (e.g., the Reissner-Nordström solution) we should not expect to find solutions with multiple horizons. (The existence of boundary conditions at the horizon, even though there is no local physical singularity there, reflects the global nature of the requirement we have imposed on our solutions, namely, that they be static.)

It is straightforward now to integrate the differential equations numerically. However, a somewhat different approach may be more instructive. By integrating Eq. (52) we obtain

\[
\mathcal{M}(r) = \mathcal{M}(0) + 4\pi \int_0^r dr r^2 \rho(r)
\]

(54)

where \( \rho = K/A + U \). We would expect \( \rho \) to be roughly constant out to \( r \sim R_{\text{mon}} \sim 1/(ev) \), and then to fall as \( 1/r^4 \). The behavior that this implies for \( \mathcal{M}(r) \) is indicated by the solid lines in Fig. 2 for the three cases \( \mathcal{M}(0) = 0, 0 < \mathcal{M}(0) \ll M_{\text{mon}}, \) and \( \mathcal{M}(0) \gg M_{\text{mon}} \). In all three cases the graph begins to level off at \( r \sim 1/(ev) \). The
dashed lines in these plots show the corresponding behavior for $\mathcal{M}(r)/r$; Eq. (51) shows that there will be a horizon whenever this quantity is equal to $1/(2G)$. For the first case ($\mathcal{M}(0) = 0$), the plot of $\mathcal{M}/r$ has a single peak. For small values of $v$ the height of this peak is less than $1/(2G)$, and there is no horizon. As $v$ is increased, this peak eventually becomes high enough that a horizon forms; this is just the critical case described previously, for which we found that the total mass of the system was precisely that of the extremal Reissner-Nordström solution. In the second case, the divergence of $\mathcal{M}/r$ at $r = 0$ ensures that there is always some small value of $r$ at which there is a horizon. However, the discussion above suggests that we should not expect to find a solution with two horizons. Hence, the solutions of this type should disappear when $v$ becomes large enough that the peak at $r \sim 1/(ev)$ reaches $1/(2G)$. But, at least for small values of $\mathcal{M}(0)$, the behavior near this peak should be just as for the first case, and so the maximum mass here also should be simply the extremal Reissner-Nordström mass. Finally, for the third case the plot would suggest that there is always a horizon. However, as was argued above, we should not expect to find a solution with nontrivial fields outside the horizon if $r_H \gtrsim R_{\text{mon}}$. If $r_H \approx \mathcal{M}(0) G$, this means that new black hole solutions should exist for this case only if $M = \mathcal{M}(\infty) \approx \mathcal{M}(0) \lesssim M_{\text{ext}}^2/M_{\text{mon}}$.

![Fig. 3. Schematic phase diagram of solutions. “R-N” refers to a Reissner-Nordstrom solution with a horizon, while “Mon” refers to the solutions with a black hole inside a nontrivial monopole configuration.](image)
These results are summarized by the phase diagram in Fig. 3, which indicates the types of magnetic black holes that exist for various values of the total mass $M$ and the flat space monopole mass $M_{\text{mon}}$. These new solutions, which may be described as black holes inside magnetic monopoles, occur in the region on the left side of the plot; the diagonal line which bounds this region from below corresponds to the nonsingular magnetic monopoles. In addition to these solutions, there are also the purely Reissner-Nordström black holes for all $M \geq M_{\text{ext}}$; these are of course independent of the value of $M_{\text{mon}}$. Examining this plot, one sees that there is a region where both types of black holes are allowed. Thus, there can be two different black hole solutions with the same mass and the same magnetic charge. Can one of these be transformed into the other?

The second law of black hole dynamics prevents the black hole with hair from going over into the Reissner-Nordström solution (at least classically), since for a given mass the latter has the smaller area. On the other hand, there is no general principle forbidding the reverse transition. We can pursue further the question of whether it actually occurs by studying the stability of the Reissner-Nordström solution under small fluctuations.

4.3 Instabilities of the Reissner-Nordström Black Hole — A First Examination

For the moment, let us consider only spherically symmetric fluctuations about the Reissner-Nordström solution with unit magnetic charge [20]. If growing modes can be found among these, then the Reissner-Nordström solution is certainly unstable, although the converse need not be the case. After having developed the necessary formalism, I will return later to the general stability analysis.

The spherically symmetric fluctuations are completely described by four functions: the deviations of the metric functions $A(r)$ and $B(r)$ from their Reissner-Nordström form, the deviation of the Higgs field function $h(r)$ from unity, and the function $u(r)$ that measures the magnitude of the massive vector field and that vanishes for the Reissner-Nordström black hole. When the field equations are linearized about the Reissner-Nordström solution, they separate into three decoupled sets of equations. One, involving $\delta A(r, t)$ and $\delta B(r, t)$, is the same as for the purely electromagnetic case, where it was shown some time ago [21] that there is no instability. One can easily show that the second, involving only $\delta h(r, t)$, also does not lead to any instability. Finally, there is the equation for $u(r, t)$. The generalization of Eq. (49) to the time-dependent case is

$$
1 \frac{\partial}{\sqrt{AB} \partial t} \left( \frac{\sqrt{AB}}{B} \dot{u} \right) - \frac{1}{\sqrt{AB}} \frac{\partial}{\partial r} \left( \frac{\sqrt{AB}}{A} u' \right) + \left( \frac{(u^2 - 1)}{r^2} - e^2 v^2 h^2 \right) u = 0. \quad (55)
$$

By setting $A$ and $B$ equal to their Reissner-Nordström values and $h = 1$, and keeping
terms linear in $u$, we obtain

$$
-\frac{1}{B_{RN}} \ddot{u} - \frac{\partial}{\partial r} (B_{RN} u') + \left( \frac{1}{r^2} - e^2 v^2 \right) u = 0 .
$$

(56)

Any instability will appear as an exponentially growing solution of this equation; i.e., one of the form

$$
u(r,t) = u(r) e^{\omega t} .
$$

(57)

Let us define a tortoise coordinate $x$ by

$$
\frac{dr}{dx} = B_{RN}(r).
$$

(58)

(Note that the range $r_H \leq r < \infty$ corresponds to $-\infty < x < \infty$.) Making this change of variables in Eq. (56) and requiring $u$ to behave as in Eq. (57), we obtain

$$
-\frac{d^2 u}{dx^2} + \frac{B_{RN}(r)}{r^2} (e^2 v^2 r^2 - 1) u = -\omega^2 u .
$$

(59)

This is of the form of the Schrödinger equation for a particle in the potential

$$
V(x) = \frac{B_{RN}(r)}{r^2} (e^2 v^2 r^2 - 1) .
$$

(60)

The existence of an unstable mode about the Reissner-Nordström solution is equivalent to the existence of a negative energy bound state in the potential $V(x)$. If $r_H > 1/(ev)$, this potential is positive over the whole range of $x$, and so there are clearly no bound states. On the other hand, as $r_H$ falls below $1/(ev)$ the potential becomes negative near the horizon, raising the possibility of a bound state. Numerical calculations [22] reveal that a bound state is present if

$$
r_H < r_{cr} \approx \frac{0.557}{ev} .
$$

(61)

Thus, the Reissner-Nordström black hole becomes unstable when its horizon is of the order of the monopole radius. In terms of mass, the instability occurs whenever the black hole mass $M < M_{\text{unstable}} \approx M_{Pl}^2/(ev)$. This is greater than the mass of the extremal Reissner-Nordström solution as long as $v < M_{Pl}$, i.e., for essentially any value of $v$ consistent with the existence of nonsingular monopoles.
We can now use these results to follow the evolution in such a theory of an initially large black hole carrying a unit magnetic charge, such as might have formed from the collapse of a star containing a single magnetic monopole. (The multiply-charged case will be described later.) After the collapse, the black hole eventually settles down to the Reissner-Nordström form, and slowly evaporates via the Hawking process. If the black hole were to remain Reissner-Nordström, this evaporation would cease as it reached the extremal solution with vanishing Hawking temperature. Instead, as the evaporation causes the horizon to fall below the monopole radius, the core of an 't Hooft-Polyakov monopole begins to form outside the horizon. As the evaporation continues, the receding horizon reveals more of the monopole core, while the Hawking temperature (which can be obtained from Eq. (23)) rises monotonically. This continues until the horizon reaches Planck size, at which point the black hole presumably evaporates completely, leaving behind a nonsingular monopole.

5. Nontopological Magnetic Monopoles and New Magnetic Black Holes

The discussion of magnetic monopoles in the previous two sections was framed largely in context of a spontaneously broken gauge theory in which monopoles appear as classical topological solitons. However, one can construct a wider class of theories which have classical solutions carrying magnetic charge even though there is no nontrivial topology [23]. Imposing a certain constraint on the parameters ensures that the energy density is nonsingular at the origin. This condition can be relaxed if one is interested in black hole solutions, where the singularity can be hidden within the black hole horizon.

Thus, consider a theory with electromagnetism, a massive charged spin-1 particle represented by a complex vector field $W_\mu$, and a neutral massive scalar $\phi$. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2} |D_\mu W_\nu - D_\nu W_\mu|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{g}{4} d_{\mu\nu} d^{\mu\nu} - \frac{\lambda}{4} d_{\mu\nu} d^{\mu\nu}$$

$$+ m^2(\phi) |W_\mu|^2 + \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi)$$

(62)

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(63)

and

$$D_\mu = \partial_\mu - i e A_\mu$$

(64)

are the ordinary electromagnetic field strength and covariant derivative, respectively,
and

\[ d_{\mu\nu} = ie(W_{\mu}^*W_{\nu} - W_{\nu}^*W_{\mu}). \]  

(65)

The term involving \( d_{\mu\nu}F_{\mu\nu} \) gives an anomalous magnetic moment to the vector particles that combines with the contribution from the covariant derivative terms to give a total gyromagnetic ratio \((g + 2)/2\). The quantity \( m(\phi) \) represents a \( \phi \)-dependent mass for the \( W \); let us denote by \( m_W \) the value that this takes when \( \phi \) is chosen to minimize the potential \( V(\phi) \).

If \( g = 2, \lambda = 1, \) and \( m(\phi) = e\phi \), this theory is simply the \( SU(2) \) gauge theory of Eq. (30) written in the unitary gauge where \( \phi^a \) has a single nonzero component, \( \phi^3 \equiv \phi. \) The vectors fields in Eqs. (30) and (62) are related by \( A_3^\mu = A_\mu \) and \((A_\mu^1 + i A_\mu^2)/\sqrt{2} = W_\mu \). Similarly, for \( g = 2, \lambda = (\sin \theta_W)^{-2}, \) and \( m(\phi) = e\phi/2, \) Eq. (62) gives the unitary gauge form of the standard electroweak theory, but with all terms involving the \( Z \) or fermions omitted. For generic values of the parameters, however, the extension to the standard non-Abelian theory is not possible. In such cases \( \phi \) cannot be viewed as a component of a Higgs multiplet and the usual topological arguments for the existence of a solution cannot be applied. Nevertheless, energetic considerations show that magnetically charged solutions exist. (An extension to a somewhat nonstandard model where a topological charge can be defined is given in [24].)

To see this, let us begin with a configuration where \( W_\mu \) is identically zero, \( \phi \) is everywhere equal to its vacuum value, and the electromagnetic vector potential \( A_i(x) \) is given by the Dirac potential of Eq. (28) with \( Q = 1/e. \) This carries unit magnetic charge, but has infinite energy because of the divergent Coulomb magnetic field near the origin. We can reduce, and possibly even remove, this divergence by surrounding this singularity with a cloud of \( W \)-particles with their magnetic moments oriented so as to cancel the original magnetic field. To examine this in further detail, we need the explicit expression for the energy density of a static configuration. For the sake of simplicity, let us assume that the configuration carries no electric charge, and that \( A_0 = W_0 = 0. \) The energy density is then

\[
\mathcal{E} = \frac{1}{4} \left( 1 - \frac{g^2}{4\lambda} \right) F_{ij}^2 + \frac{g^2}{16\lambda} \left( F_{ij} - \frac{2\lambda}{g} d_{ij} \right)^2 + \frac{1}{2} \left| D_i W_j - D_j W_i \right|^2 + m^2(\phi) |W_i|^2 + \frac{1}{2} (\partial_i \phi)^2 + V(\phi).
\]

(66)

The first term on the right hand side behaves as \( 1/r^4 \) near the origin; it represents that part of the Coulomb energy that cannot be cancelled by the \( W \) magnetic moments. Let us assume for the moment that \( g^2 = 4\lambda, \) so that this term is absent. The next term can be made finite by choosing \( W_i \) in such a fashion that \( d_{ij} \) has a \( 1/r^2 \) singularity at the origin that cancels the singularity in \( F_{ij} \). This requires that
The magnitude of $W_i$ grow as $1/r$ near the origin, which would lead one to expect $1/r^4$ contribution to the energy density from the angular derivatives in the next term. However, it turns out to be possible to arrange the vector field so that these angular derivatives cancel completely, leaving only radial derivative terms that are easily made finite. (The discussion in the next section will clarify this.) Next is the mass term for the $W$. With $W_i \sim 1/r$, this term would have an integrable $1/r^2$ singularity; this singularity can be eliminated completely by choosing $\phi(x)$ so that $m(\phi)$ vanishes at the origin. Finally, the last two terms, involving only the scalar field, cause no difficulty.

We can thus obtain a finite energy configuration carrying unit magnetic charge. There is a Dirac string singularity, but we know that this is not physically observable. There may also be singularities in the vector fields at the origin, but these do not lead to any singularity in the energy density. With parameters chosen so the theory is actually the unitary gauge form of the $SU(2)$ Higgs theory, this configuration is simply the unitary gauge form of the 't Hooft-Polyakov monopole given in Eq. (38). For other choices of parameters, but with $g^2 = 4\lambda$, it describes a nontopological, but finite energy, magnetic monopole.

Now consider relaxing the condition $g^2 = 4\lambda$. If $g^2 > 4\lambda$, the first term in Eq. (66) leads to an infinite negative energy at the center of any monopole configuration, implying that the vacuum is unstable against formation of monopole-antimonopole pairs. If instead $g^2 < 4\lambda$, any monopoles will have infinite energy. Although this rules out the existence of such objects in flat spacetime, it leaves the possibility of magnetic black holes in which this singularity is hidden behind the horizon. We can pursue this possibility using methods similar to those of the previous section. Thus, let us assume spherical symmetry, with the metric being of the form of Eq. (1) and the vector field determined by a single function $u(r)$, as before, while the scalar field is a function only of $r$. The matter action is again of the form of Eq. (46), with

$$K = \frac{u'^2}{e^2r^2} + \frac{1}{2}\phi'^2 \quad (67)$$

and

$$U = \frac{\lambda}{2e^2r^4} \left(u^2 - \frac{g}{2\lambda}\right)^2 + \frac{u^2m^2(\phi)}{r^2} + V(\phi) + \frac{1}{2e^2r^4} \left(1 - \frac{g^2}{4\lambda}\right) \quad (68)$$

If we define a function $F(r)$ by

$$A(r) = \left[1 - \frac{2G F(r)}{r} + \frac{4\pi G}{r^2} \left(1 - \frac{g^2}{4\lambda}\right)\right]^{-1}, \quad (69)$$

the gravitational field equations imply that

$$F' = 4\pi r^2 \left(\frac{K}{A} + U_1\right) \quad (70)$$
The horizon, \( r_H \), is a zero of \( A^{-1}(r) \). The existence of such a zero places a lower bound

\[
F(r_H) \geq \frac{\sqrt{4\pi}}{e} m_{\text{Pl}} \left( 1 - \frac{g^2}{4\lambda} \right)^{1/2} = M_{\text{ext}}^{\text{RN}} \left( 1 - \frac{g^2}{4\lambda} \right)^{1/2}
\]

(71)
on the value of \( F(r) \) at the horizon, and implies a lower bound on the total mass \( M = F(\infty) \approx F(r_H) + m_W/e^2 \), where the second term is an estimate of the sum of the energy in the portion of the monopole core outside the horizon and that contained in the long-range Coulomb field. A rough upper bound on the mass can be obtained by requiring that the horizon radius be less than that of the monopole core. Together, these considerations lead us to expect new magnetic black hole solutions with masses in the range

\[
\left( 1 - \frac{g^2}{4\lambda} \right)^{1/2} M_{\text{ext}}^{\text{RN}} \lesssim M \lesssim \frac{\sqrt{g} m_{\text{Pl}}^2}{m_W},
\]

(72)

where it has been assumed that \( m_W \ll M_{\text{Pl}} \). The solutions saturating the lower bound will be new types of extremal black holes. Since they have a lower mass, but the same magnetic charge, as the extremal Reissner-Nordström solutions, there will be a repulsive force between any two extremal holes with magnetic charges of the same sign.

We can also envision solutions with other values of the magnetic charge, subject to the constraint that \( q = eQ_M \) be either an integer or an integer plus one-half. (This quantization condition can be obtained classically by requiring that the Dirac string not affect the scattering of classical \( W \)-field waves.) These solutions cannot be spherically symmetric and so require a more complicated analysis than those with \( q = 1 \). The first step toward dealing with these is to develop the formalism of monopole spherical harmonics; this is done in the next section.

6. Angular Momentum and Monopole Spherical Harmonics

It is well known that in the presence of a magnetic monopole an electrically charged particle acquires an additional angular momentum oriented toward the position of the monopole and with a magnitude equal to the product of the electric and magnetic charges. This extra term changes the spectrum of eigenvalues of the orbital angular momentum in the quantum theory, and leads to a modification of the corresponding eigenfunctions, the spherical harmonics.

Thus, a spinless particle with electric charge \( e \) moving in the field of a monopole
with magnetic charge $q/e$ has a conserved orbital angular momentum

$$L = r \times (m \mathbf{v}) - q \hat{r}$$
$$= r \times (p - eA) - q \hat{r}. \tag{73}$$

This can be represented in the usual fashion by the differential operator

$$L = -ir \times D - q \hat{r} \tag{74}$$

where

$$D = \nabla - ieA. \tag{75}$$

The components of $L$ obey the standard angular momentum commutation relations. The monopole spherical harmonics[25,26] $Y_{qlm}(\theta, \phi)$ are eigenfunctions of $L^2$ and $L_z$ obeying

$$L^2 Y_{qlm}(\theta, \phi) = l(l + 1)Y_{qlm}(\theta, \phi)$$

$$L_z Y_{qlm}(\theta, \phi) = mY_{qlm}(\theta, \phi). \tag{76}$$

The eigenvalues $m$ run in integer steps from $l$ down to $-l$, as usual. The allowed values of $l$, however, are not the usual ones. Classically, the fact that the two terms on the right hand side of Eq. (73) are perpendicular to each other implies that $L^2 \geq q^2$. Correspondingly, the minimum value of $l$ is not 0, but rather $q$, with all integer increments above this also allowed.

As with the usual spherical harmonics, the monopole harmonics for a given value of $q$ form a complete and orthonormal set, with

$$\int d\Omega Y_{qlm}^*(\theta, \phi)Y_{q'l'm'}(\theta, \phi) = \delta_{ll'}\delta_{mm'}. \tag{77}$$

The precise form of the monopole harmonics is gauge-dependent, reflecting the gauge dependence of the vector potential. In addition, they have singularities corresponding to the Dirac string singularity of the potential. In a gauge where

$$A_\phi = \frac{q}{e}(\pm 1 - \cos \theta) \tag{78}$$

the monopole harmonics are of the form

$$Y_{qlm}(\theta, \phi) = e^{i(m\pm q)\phi}F_{qlm}(\theta). \tag{79}$$
For example, with $q = 1/2$,

$$F_{1/2, 1/2, 1/2} = -\frac{1}{\sqrt{4\pi}} \sqrt{1 - \cos \theta}$$

$$F_{1/2, -1/2, 1/2} = -\frac{1}{\sqrt{4\pi}} \sqrt{1 + \cos \theta}.$$  \hfill (80)

With the upper choice of sign, $Y_{1/2, 1/2, 1/2}$ is singular along the negative $z$-axis, where the Dirac string is located. If one chooses instead the lower sign, then these singularities are replaced by a Dirac string along the positive $z$-axis, with a corresponding singularity in $Y_{1/2, 1/2, -1/2}$. As emphasized by Wu and Yang [26], one can combine these, using one gauge for the region $0 \leq \theta < (\pi/2) + \delta$ and the other for $(\pi/2) - \delta < \theta \leq \pi$, to obtain an object that is nonsingular everywhere but at the origin.

For dealing with charged vector fields we will need monopole vector spherical harmonics, which are eigenfunctions of the total angular momentum $J = L + S$. By the usual rules for adding angular momenta, one sees that the minimum value for the total angular momentum quantum number $J$ is $q - 1$, except in the two cases $q = 0$ and $q = 1/2$, where $J_{\text{min}} = q$. Note that $J = 0$ is possible only if $q = 0$ of $q = 1$; this is why nonsingular spherically symmetric monopoles can only have unit magnetic charge in the spontaneously broken $SU(2)$ theory [27].

In general, there are several multiplets of monopole harmonics with a given value of $J$. These can be chosen to be eigenfunctions of $L^2$ [28]. However, for our purposes it will be more convenient to choose harmonics [29] that are eigenfunctions of

$$\lambda \equiv \hat{r} \cdot S.$$  \hfill (81)

In general, the eigenvalues of $\lambda$ are $-1$, $0$, and $1$. However, the identity

$$\hat{r} \cdot J = \hat{r} \cdot L + \hat{r} \cdot S = -q + \lambda$$  \hfill (82)

leads to the restriction

$$-J \leq \lambda - q \leq J.$$  \hfill (83)

From this, one sees that the multiplet structure is as follows:

For $J = q - 1 \geq 0$, one multiplet, with $\lambda = 1$.

For $J = q > 0$, two multiplets, with $\lambda = 1$ and $0$.

For $J = q = 0$, one multiplet, with $\lambda = 0$.

For $J > q$, three multiplets, with $\lambda = 1$, $0$, and $-1$. 
Thus, let us define monopole vector harmonics \( C^{(\lambda)}_{qJM} \) that obey

\[
J^2 C^{(\lambda)}_{qJM} = J(J + 1)C^{(\lambda)}_{qJM}
\]
\[
J_z C^{(\lambda)}_{qJM} = MC^{(\lambda)}_{qJM}
\]
\[
\hat{\mathbf{r}} \cdot \mathbf{S} C^{(\lambda)}_{qJM} = \lambda C^{(\lambda)}_{qJM}.
\]

Because the spin matrices \((S^k)_{ij} = -i\epsilon_{ijk}\), the last of these can be rewritten as

\[
\hat{\mathbf{r}} \times C^{(\lambda)}_{qJM} = -i\lambda C^{(\lambda)}_{qJM}.
\]

From this we see that the harmonics with \( \lambda = 0 \) are purely radial, while those with \( \lambda = \pm 1 \) have only angular components. Further, we find that harmonics with different values of \( \lambda \) are orthogonal vectors at every point, in the sense that

\[
C^{(\lambda)*}_{qJM} \cdot C^{(\lambda')}_{qJM'} = 0, \quad \lambda \neq \lambda'.
\]

Harmonics with different values of \( j, M, \) or \( \lambda \) are also orthogonal in the functional sense. Using a convenient choice of normalization, we have

\[
\int d\Omega C^{(\lambda)*}_{qJM} \cdot C^{(\lambda')}_{qJM'} = \frac{\delta_{JJ'}\delta_{MM'}\delta_{\lambda\lambda'}}{r^2}.
\]

Experience with the standard vector harmonics would suggest that the monopole spherical harmonics could be constructed by applying vector differential operators to the scalar monopole harmonics. This construction does in fact work if \( J \geq q \), although not for the case \( J = q - 1 \), which will be treated separately below. Let \( \mathbf{v} \) be any vector constructed from \( \mathbf{r} \) and \( \mathbf{D} \). It follows from the commutation relations

\[
[L^2, v_k] = -2i\epsilon_{ijk}L_i v_j - 2v_k
\]
\[
= -2[(L \cdot S + 1)v]_k
\]

that

\[
(L + S)^2 v Y_{qKM} = v L^2 Y_{qKM} = K(K + 1)v Y_{qKM}.
\]

Hence, if \( \mathbf{v}_\lambda \) is a vector satisfying \( \hat{\mathbf{r}} \times \mathbf{v}_\lambda = -i\lambda \mathbf{v}_\lambda \), the desired harmonics are, up to
a normalization constant, equal to $v_Y$. Explicitly,

$$C^{(1)}_{qJM} = \frac{1}{\sqrt{2(J^2 + q)}} [D + i \hat{r} \times D] Y_{qJM}, \quad J \geq q > 0$$

$$C^{(0)}_{qJM} = \frac{\hat{r}}{r} Y_{qJM}, \quad J \geq q \geq 0 \quad (90)$$

$$C^{(-1)}_{qJM} = \frac{1}{\sqrt{2(J^2 - q)}} [D - i \hat{r} \times D] Y_{qJM}, \quad J > q \geq 0$$

where

$$J^2 \equiv J(J + 1) - q^2. \quad (91)$$

Formulas for the covariant divergence and covariant curl of these vector harmonics are easily derived. Consider first the $\lambda = 0$ harmonics. For their curl, we have

$$D \times C^{(0)}_{qJM} = D \times \frac{\hat{r}}{r} Y_{qJM} = -\frac{\hat{r}}{r} \times D Y_{qJM}$$

$$= \frac{i}{r} \left[ \sqrt{\frac{J^2 + q}{2}} C^{(1)}_{qJM} - \sqrt{\frac{J^2 - q}{2}} C^{(-1)}_{qJM} \right], \quad (92)$$

while for their divergence,

$$D \cdot C^{(0)}_{qJM} = D \cdot \left( \frac{\hat{r}}{r} Y_{qJM} \right) = \frac{1}{r^2} Y_{qJM} \quad (93)$$

For the $\lambda = \pm 1$ harmonics, the first step is to note the identity

$$r \times \left( D \times C^{(\pm 1)}_{qJM} \right) = D \left( r \cdot C^{(\pm 1)}_{qJM} \right) - C^{(\pm 1)}_{qJM} - (r \cdot D) C^{(\pm 1)}_{qJM} \quad (94)$$

The first term on the right vanishes because the $\lambda = \pm 1$ harmonics have only angular components, while the last two cancel because the harmonics are homogeneous of degree $-1$ in the Cartesian coordinates. Hence the entire right hand side vanishes, implying that $D \times C^{(\pm 1)}_{qJM}$ is purely radial and thus a linear combination of the $\lambda = 0$
harmonics. The coefficients can be obtained from

\[ \int d\Omega \mathbf{C}_{qJ' M'}^* \cdot \mathbf{D} \times \mathbf{C}_{qJM}^{(\pm 1)} = - \int d\Omega \mathbf{D} \times \mathbf{C}_{qJ' M'}^* \cdot \mathbf{C}_{qJM}^{(\pm 1)}. \]  

(95)

Using the result already obtained for the curl of \( \mathbf{C}_{qJM}^{(0)} \), together with the normalization condition (87), one then obtains

\[ \mathbf{D} \times \mathbf{C}_{qJM}^{(\pm 1)} = \pm i \frac{\mathcal{J}^2 \pm q}{2} \mathbf{C}_{qJM}^{(0)}. \]  

(96)

Finally, for the covariant divergence we have

\[ \mathbf{D} \cdot \mathbf{C}_{qJM}^{(\pm 1)} = \pm i \mathbf{D} \cdot \left( \hat{\mathbf{r}} \times \mathbf{C}_{qJM}^{(\pm 1)} \right) = \pm i \mathbf{D} \times \mathbf{C}_{qJM}^{(\pm 1)} = \frac{1}{r^2} \sqrt{\mathcal{J}^2 \pm q} \mathbf{Y}_{qJM}. \]  

(97)

The vector harmonics with \( J = q - 1 \) form a single multiplet, with \( \lambda = 1 \). They clearly cannot be obtained by a construction of the above type, since there are no scalar harmonics with \( J = q - 1 \). To begin, I will show that both their covariant curls and their covariant divergences vanish. (This gives the cancellation of angular derivatives that was asserted in the discussion following Eq. (66).) To simplify the notation, let

\[ \mathbf{U}_M = \mathbf{C}_{q,q-1,M}^{(1)}. \]  

(98)

First consider the divergence \( \mathbf{D} \cdot \mathbf{U}_M \). Since this is a scalar, it can be expanded as a linear combination of the scalar monopole harmonics, with the coefficient of \( \mathbf{Y}_{qJ'M'} \) being

\[ I_{J'M'} = \int d\Omega Y_{qJ'M'}^* \mathbf{D} \cdot \mathbf{U}_M \]

\[ = - \int d\Omega (\mathbf{D} Y_{qJ'M'}^*) \cdot \mathbf{U}_M \]

\[ = - \int d\Omega \left[ \sqrt{\frac{\mathcal{J}^2 + q}{2}} \mathbf{C}_{qJ'M'}^{(1)*} + \sqrt{\frac{\mathcal{J}^2 - q}{2}} \mathbf{C}_{qJ'M'}^{(-1)*} \right] \cdot \mathbf{U}_M. \]  

(99)

The last integral is zero, since the harmonics with \( J' \geq q \) are orthogonal to those with \( J = q - 1 \). Since all of the \( I_{J'M'} \) vanish, so must \( \mathbf{D} \cdot \mathbf{U}_M \). Next, manipulations
paralleling those in Eq. (94) show that the angular components of $D \times U_M$ vanish. Finally, the analog of Eq. (97) shows that the radial component of the curl is proportional to the divergence, which has just been shown to be zero. Hence, $D \times U_M = 0$. The $J = q - 1$ vector harmonics can therefore be viewed as a set of curl-free and divergenceless vector fields on the unit two-sphere. One can derive an index theorem to the effect that there are precisely $2q - 1 = 2(q - 1) + 1$ linearly independent fields of this sort on the two-sphere, giving just the number needed for a $J = q - 1$ multiplet.

Let us construct these explicitly, working in a gauge where the vector potential is given by Eq. (78). When acting on a scalar quantity, $J_z = L_z = -i \partial \phi \mp q$. Let us apply this to the scalar $\hat{z} \cdot U_M$, where $\hat{z}$ denotes the unit vector along the $z$-axis. Using the fact that $\hat{z}$ is invariant under rotations about the $z$-axis and therefore commutes with $J_z$, we have

$$J_z(\hat{z} \cdot U_M) = M(\hat{z} \cdot U_M) = (-i \partial \phi \mp q)(\hat{z} \cdot U_M)$$

(100)

from which it follows that

$$(-i \partial \phi \mp q)(U_M)_\theta = M(U_M)_\theta.$$  

(101)

Hence, the $\theta$-components of these harmonics must be of the form

$$(U_M)_\theta = e^{i(M \pm q)\phi} f_{qM}(\theta).$$

(102)

Next, applying Eq. (85) to the the $J = q - 1$ harmonics gives

$$(U_M)_\phi = i \sin(\theta)(U_M)_\theta.$$  

(103)

This identity, together with the vanishing of the curl, leads to the differential equation

$$\partial_\theta(\sin \theta f_{qM}) - (M + \cos \theta)f_{qM} = 0.$$  

(104)

Solving this for the $f_{qM}$, we obtain

$$(U_M)_\theta = a_{qM}e^{i(M \pm q)\phi}(1 - \cos \theta)^M(\sin \theta)^{q - M + 1}$$

$$= a_{qM}e^{i(M \pm q)\phi}(1 + \cos \theta)^M(\sin \theta)^{q + M + 1}.$$  

(105)

From the form given on the first line it is clear that the harmonic is singular along the positive $z$-axis unless $q - M + 1 \geq 0$. Similarly, the form on the second line is manifestly singular along the negative $z$-axis unless $q + M - 1 \geq 0$. To satisfy both constraints, $M$ must lie in the interval $-(q - 1) \leq M \leq q - 1$. 

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7. Stability Analysis of the Reissner-Nordström Solution

Armed with the results of the previous section, we are now prepared to carry out the full stability analysis [22] for a Reissner-Nordström black hole with arbitrary magnetic charge in the more general theory described by the matter Lagrangian (62). This will include as a special case the analysis in Sec. 4.3 of the spherically symmetric perturbations about a singly-charged black hole in the context of the Higgs theory. Just as in that case, the fluctuations of the massive vector fields separate from those of the metric and of the electromagnetic field; since it is known that the latter two do not give rise to any instability, we can restrict our attention to the fluctuations in $W_\mu$. Keeping only terms in the matter action that are quadratic in $W_\mu$, we obtain

$$S_{\text{quad}} = \int d^4x \sqrt{-\det(g_{\mu\nu})} \left[ -\frac{1}{2} |D_\mu W_\nu - D_\nu W_\mu|^2 - m^2 |W_\mu|^2 - \frac{ieq}{4} F^{\mu\nu} (W_\mu^* W_\nu - W_\nu^* W_\mu) \right].$$

(106)

Here, and for the remainder of this section, $m(\phi)$ should be understood to be at its vacuum value $m_W$.

We can exploit the spherical symmetry of the unperturbed solution by expanding $W_\mu$ in terms of scalar and vector spherical harmonics:

$$W_t = \sum_{J=q}^{\infty} \sum_{M=-J}^{J} a^{JM}(r, t) Y_{qJM},$$

$$W_r = \frac{1}{r} \sum_{J=q}^{\infty} \sum_{M=-J}^{J} b^{JM}(r, t) Y_{qJM},$$

$$W_a = \sum_{J=q-1}^{\infty} \sum_{M=-J}^{J} f_{+JM}(r, t) \left[ C^{(1)}_{qJM} \right]_a + \sum_{J=q+1}^{\infty} \sum_{M=-J}^{J} f_{-JM}(r, t) \left[ C^{(-1)}_{qJM} \right]_a,$$

where a Roman subscript from the beginning of the alphabet stands for $\theta$ or $\phi$. When these expansions are substituted into the quadratic action, it separates into a sum of terms, each of which contains only contributions from modes with fixed values of $J$ and $M$; i.e.,

$$S_{\text{quad}} = \sum_{J=q-1}^{\infty} \sum_{M=-J}^{J} S_{\text{quad}}^{JM}.$$  

(108)

For $J = q - 1$ the action is particularly simple, both because there is only a single radial function, $f_+(r)$, and because the vanishing of the covariant curl of the $J = q - 1$
vector harmonics eliminates some terms. One finds that

$$S_{\text{quad}}^{(q-1)M} = \int dt \int dr \left\{ \frac{1}{B} |f_+|^2 - B |f_+^2 - m^2 f_+|^2 + \frac{gg}{2r^2} |f_+|^2 \right\}.$$  \hspace{1cm} (109)

For $J \geq q$ the action is

$$S_{\text{quad}}^{JM} = \int dt \int dr \left\{ |b - r a'|^2 + \frac{1}{B} \left[ |f_+ - k_+ a|^2 + |f_- - k_- a|^2 \right] - B \left[ |f_+^2 - \frac{1}{r} k_+ b|^2 + |f_-^2 - \frac{1}{r} k_- b|^2 \right] - \frac{1}{r^2} |k_+ f_+ - k_- f_-|^2 \right\}$$

$$- m^2 \left[ |f_+|^2 + |f_-|^2 + B |b|^2 - r^2 \frac{1}{B} |a|^2 \right] + \frac{gg}{2r^2} \left[ |f_+|^2 - |f_-|^2 \right] \right\}.$$  \hspace{1cm} (110)

where

$$k_\pm = \sqrt{\frac{J^2 \pm q}{2}} = \sqrt{\frac{J(J+1) - q^2 \pm q}{2}}.$$  \hspace{1cm} (111)

(There is some simplification in the case $J = q$, because the $f_-$ terms are absent.)

Apart for the $J = q - 1$ case, these are daunting expressions. However, some simplifications are possible. Because its time derivative does not enter the action, $W_t$ (or $a^{JM}$ in the reduced actions) is a nondynamical field that can be eliminated. To carry this out in detail, it is it is convenient to adopt a more compact notation. For a given $J$ and $M$, let us combine the radial functions entering the spatial components of $W_\mu$ into a vector $z = \left( f_+ / \sqrt{B}, f_- / \sqrt{B}, b \right)^T$. The action can then be written as

$$S_{\text{quad}}^{JM} = \int dt \int dr \left[ \dot{z} \dot{z}^\dagger + \left( \dot{z}^\dagger Fa + a^* F^\dagger \dot{z} + a^* Ga - z^\dagger H z \right) \right],$$  \hspace{1cm} (112)

where

$$F = \left( - \frac{k_+}{\sqrt{B}}, - \frac{k_-}{\sqrt{B}}, r \frac{\partial}{\partial r} \right)^T.$$  \hspace{1cm} (113)

---

2 To simplify the notation, the superscripts $JM$ on the radial functions have been omitted in these and later equations.
\[ G = -\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{B} \left( k_+^2 + k_-^2 + r^2 m^2 \right) \]
\[ = F^\dagger F + B^{-1} r^2 m^2, \]  
(114)

and \( H \) is a \( 3 \times 3 \) matrix whose form we do not yet need. Variation with respect to \( a^* \) and \( z^\dagger \) yields the equations

\[ 0 = F^\dagger \dot{z} + Ga \]  
(115)

and

\[ 0 = \ddot{z} + F \dot{a} + Hz, \]  
(116)

respectively. Using the first of these to solve for \( a \) and then substituting the result into the second equation, we obtain

\[ 0 = \left( I - FG^{-1} F^\dagger \right) \ddot{z} + Hz. \]  
(117)

The coefficient of \( \ddot{z} \),

\[ I - FG^{-1} F^\dagger = I - F \left. \frac{1}{F^\dagger F + B^{-1} r^2 m^2 F} \right| \]  
(118)

is a positive operator. Using this fact, we see that unstable modes growing as \( e^{\alpha t} \) are possible if and only if the matrix \( H \) has negative eigenvalues. (However, because of the nontrivial matrix coefficient of \( \ddot{z} \) in Eq. (117), the actual values of \( \alpha \) cannot be determined directly from the negative eigenvalues.) Thus, the stability of the Reissner-Nordström solution is equivalent to the positivity of \( H \).

A second simplification follows from the absence of the radial derivatives of \( W_r \) from the action. This allows us to write the potential energy contribution associated with \( H \) as the sum of the integral of a perfect square involving \( b(r) \), but not its derivative, and a quantity that is independent of \( b(r) \); i.e.,

\[ \int_{r_H}^{\infty} dr \ z^\dagger H z = E_{JM}(f_+, f_-) + \int_{r_H}^{\infty} dr \ B \left| k_+ f_+^\prime + k_- f_-^\prime - \frac{(r^2 m^2 + J^2)}{r} b \right|^2. \]  
(119)

By choosing \( b(r) \) appropriately, the second term on the right hand side can always be made to vanish. Hence, the positivity of of \( H \) is equivalent to the positivity of
\[ E_{JM}(f_+, f_-), \] which may be written in the form

\[ E_{JM}(f_+, f_-) = \int_{r_H}^{\infty} dr \left[ B f'^\dagger K_J f' + f'^\dagger \left( m^2 I - \frac{V_J}{r^2} \right) f \right]. \] (120)

where \( f \equiv f_+ \) if \( J = q - 1 \) or \( q \), and otherwise \( f \equiv (f_+, f_-)^T \). For the two lowest values of \( J \), the matrices \( K_J \) and \( V_J \) are simply

\[ K_{q-1} = 1, \quad V_{q-1} = \frac{gq}{2} \] (121)

and

\[ K_q = \frac{r^2 m^2}{r^2 m^2 + q}, \quad V_q = \frac{(g-2)q}{2}, \] (122)

while for \( J > q \) they are given by

\[ K_J = I - \frac{1}{r^2 m^2 + J^2} \begin{pmatrix} k_+^2 & k_+k_- \\ k_+k_- & k_-^2 \end{pmatrix}, \quad J > q \] (123)

\[ V_J = \begin{pmatrix} -k_+^2 + \frac{gq}{2} & k_+k_- \\ k_+k_- & -k_-^2 - \frac{gq}{2} \end{pmatrix}, \quad J > q. \] (124)

The eigenvalues of \( K_J \) are always positive. It follows that a necessary condition for the existence of an instability in a mode with angular momentum \( J \) is that the potential energy terms in Eq. (120) be negative somewhere outside the horizon. This in turn requires that

\[ m^2 - \frac{V_J}{r_H^2} < 0 \] (125)

where, for \( J > q \), \( V_J \) should be understood to signify the larger of its two eigenvalues. This condition can be phrased as an upper bound \( r_H < r_0(J) \) on the horizon radius.
For $J = q - 1$, Eq. (125) can be satisfied only if $g > 0$, in which case

$$mr_0(q - 1) = \sqrt{gq/2}. \quad (126)$$

For $J = q$, instability is possible only if $g > 2$, and

$$mr_0(q) = \sqrt{(g - 2)q/2}. \quad (127)$$

For $J > q$, the two eigenvalues of $V_J$ are $[-J^2 \pm \sqrt{J^4 + g(q - 2)q^2}] / 2$. If $0 \leq g \leq 2$, both of these are negative, thus ruling out any instability. Outside of this range there is one positive eigenvalue, and

$$mr_0(J) = \frac{1}{\sqrt{2}} \left[-J^2 \pm \sqrt{J^4 + g(q - 2)q^2}\right]^{1/2}. \quad (128)$$

Note that, for fixed $q$ and $g$, $r_0$ is a decreasing function of $J$.

Because of the gradient terms in the energy, Eq. (125) is not sufficient for the existence of an unstable mode. Instead, there is a somewhat stronger condition that may be written as

$$r_H < r_{cr}(J) < r_0(J), \quad (129)$$

where $r_{cr}(J)$ is the largest value of $r_H$ for which

$$-\frac{d}{dr} \left[ \frac{1}{A} K_J f' \right] + \left( m^2 - \frac{V_J}{r^2} \right) f = -\omega^2 f \quad (130)$$

has solutions with real $\omega$ that satisfy $f(r_H) = f(\infty) = 0$. (For $g = 2$, $q = 1$, and $J = 0$, this is equivalent to Eq. (59).) The values of $r_{cr}(J)$ must be determined numerically. To do this for $J = q - 1$ or $q$, we first by set $\omega = 0$ in Eq. (130). We then choose a trial value of $r_H$ and integrate in from large $r$. By appropriately adjusting the trial value, one can fairly quickly zero in on the value which gives $f(r_H) = 0$; this is the desired $r_{cr}(J)$. For $J > q$, where Eq. (130) is actually a pair of coupled equations, this procedure cannot be applied directly. One approach is to replace these coupled equations by a single equation of the same order. One such equation is obtained by replacing the matrix $K_J$ by its smallest eigenvalue and the matrix $V_J$ by its positive eigenvalue. This gives an underestimate of $E_{JM}$, and thus an overestimate of $r_{cr}(J)$. A second equation is obtained by going to a basis where $V_J$ is diagonal and then restricting $f$ to the subspace corresponding to the positive eigenvalue of $V_J$. This gives an overestimate of $E_{JM}$, and thus an underestimate of $r_{cr}(J)$. Numerically, these two bounds on $r_{cr}(J)$ turn out to be rather close, and thus give a rather good estimate of the true value.
Numerical calculations of $r_{\text{ct}}(J)$ for a variety of values of $q$ and $g$ indicate that $r_{\text{ct}}(J)$, like $r_0(J)$, is a decreasing function of $J$. They also show that $r_{\text{ct}}(J)$ and $r_0(J)$ are roughly proportional, with their ratio lying between $1/2$ and 1, except for the case $J = q - 1$ with very small values of $g$, where $r_{\text{ct}}(J) \sim r_0^2(J)$. These results indicate that as the horizon of the Reissner-Nordström solution is decreased, instabilities occur first in the mode with the lowest possible $J$. Thus, for $q \geq 1$, the first unstable mode has $J = q - 1$ if $g > 0$ and $J = q + 1$ if $g < 0$. For $q = 1/2$, the first unstable mode has $J = 1/2$ if $g > 2$ and $J = 3/2$ if $g < 0$. Finally, if $q = 1/2$ and $0 \leq g \leq 2$, the Reissner-Nordström solution is always stable.

8. Construction of New, Nonsymmetric, Black Holes

We have seen that if the horizon radius, and thus the mass, is sufficiently small, the Reissner-Nordström black hole solution becomes unstable. What happens when such a black hole is perturbed infinitesimally? For a black hole with unit magnetic charge, the answer is clear from the previous discussion. The instability leads to the formation of a cloud of $W$ field outside the horizon and the black hole presumably settles down in a new static configuration with hair. If the matter is described by the Higgs theory, this will be one of the new black hole solutions described in Sec. 4.2; the extension of these to the more general class of theories described by Eq. (62) should be straightforward. Now consider the case of black holes with higher charge. If the magnetic charge of the hole does not change, we would expect the black hole to evolve into a new static black hole with hair, much as happens in the singly-charged case. However, in contrast with the previous situation, this new solution could not be spherically symmetric, because there would be no $J = 0$ vector spherical harmonic available for the $W$-field configuration. Matters might be different if the black hole could reduce its magnetic charge, either by emission of a finite energy monopole or by fissioning into two or more holes of lower charge, since the hole might then evolve toward one of the singly-charged solutions we have already found. The former possibility seems plausible, provided that $\lambda = 4g^2$ so that finite energy monopoles exist, but energy conservation and the requirement that the horizon area not decrease would allow this only if the initial Reissner-Nordström black hole were well below the critical radius for instability. The latter possibility could occur, if at all, only through quantum mechanical tunneling, since such splitting of a horizon is classically forbidden.

Thus, the instability of the Reissner-Nordström solutions leads us to expect new black hole solutions that are not spherically symmetric. Without the simplifications that follow from spherical symmetry, any attempt to obtain exact closed form expressions for these solutions is probably hopeless. However, if the Reissner-Nordström is just barely unstable (i.e., if its horizon is just slightly less than the critical radius) a perturbative construction of these solutions is possible [30].
The intuition behind this construction arises from the observation that the problem of finding a stable static solution near an unstable one is essentially that of finding a local minimum of a function for which one has been given a saddle point. If the matrix of second derivatives at a saddle point has only a single negative eigenvalue, of sufficiently small magnitude, then we may expect to find a local minimum near the saddle point, with the eigenvector corresponding to the negative eigenvalue indicating the direction in which that minimum is to be found.

To illustrate this, consider the problem of minimizing a function of $N$ real variables $x_i$ of the form

$$F = \frac{1}{2} x_i M_{ij} x_j + \frac{1}{4} \lambda_{ijkl} x_i x_j x_k x_l$$  \hspace{1cm} (131)

where the $M_{ij}$ and $\lambda_{ijkl}$ are real, and the latter are such that the quartic term in $F$ is positive definite. We may assume that $M_{ij}$ is symmetric and hence can be diagonalized by an orthogonal transformation. Let us assume that it is diagonal in the basis in which we are working, with diagonal matrix elements $M_{ii} = b_i$. The stationary points of this function are determined by the equations

$$0 = b_i x_i + \sum_{jkl} \lambda_{ijkl} x_j x_k x_l.$$  \hspace{1cm} (132)

If the eigenvalues of $M$ are all positive, the only stationary point is a minimum at the origin, $x_j = 0$. Now suppose instead that $M$ has a single negative eigenvalue, which we may take to be $b_1 \equiv -\beta^2$. The minimum of $F$ no longer lies at the origin, which is now a saddle point. As a first step toward the determination of this minimum, note that in the subspace defined by $x_1 = 0$ Eq. (132) has only the trivial solution $x_j = 0$. This suggests an iterative approach to the solution in which one first determines a nonzero approximation to $x_1$ and then uses this to determine the remaining $x_j$. Thus, at the first step we set $x_j = 0$ for all $j \geq 2$. Eq. (132) then gives the zeroth order approximation

$$x_1^{(0)} = \sqrt{-\frac{b_1}{\lambda_{1111}}} = \frac{\beta}{\sqrt{\lambda_{1111}}}.$$  \hspace{1cm} (133)

Substituting this back into Eq. (132) leads to

$$x_j^{(0)} = -\frac{\lambda_{j111} \left[ x_1^{(0)} \right]^3}{b_j} = \frac{\beta^3 \lambda_{j111}}{b_j \lambda_{1111}^{3/2}}, \hspace{1cm} j \geq 2.$$  \hspace{1cm} (134)

The next step is to substitute these lowest order approximations back into Eq. (132) to find higher order corrections. For sufficiently small $\beta$, these higher order are small and can be calculated as power series in $\beta$. 
Let us now generalize this method to a field theory and use it to obtain new black hole solutions. To be specific, let us take the matter Lagrangian to be essentially that of Eq. (62), but with the $\phi$-dependent $W$ mass replaced by a constant $m$ and the remaining scalar field terms omitted. (This elimination of the scalar field simplifies the calculations somewhat, but is not essential.) The first step is to examine the normal modes about the Reissner-Nordström solution and identify those with negative eigenvalues. From previous remarks, it should be clear that the modes involving electromagnetic or metric perturbations all have positive eigenvalues. Thus, the negative eigenmodes involve only $W_{\mu}$ and can be found by diagonalizing the action of Eq. (106). Specifically, let us define $M_{\mu\nu}$ by

$$M_{\mu\nu}W_{\nu} = -\frac{1}{\sqrt{g}}D_{\alpha}(\sqrt{g}W^{\alpha\mu}) + m^2W_{\mu} - \frac{ieg}{2}F^{\alpha\mu}W_{\alpha}$$

(135)

where overbars denote the use of the unperturbed metric and electromagnetic potential. The desired modes satisfy

$$M_{\mu\nu}\psi_{\nu} = -\beta^2m^2\psi_{\mu}$$

(136)

with real $\beta$; for our perturbative method to succeed, we will need that $\beta \ll 1$. (A factor of $m^2$ has been extracted to make $\beta$ dimensionless.)

As in the stability analysis, the modes can be chosen to have definite angular momentum. The requirements for our perturbative method are met by choosing the horizon radius $r_H$ so that unstable modes occur for only one value $\bar{J}$ of the total angular momentum, and by furthermore requiring that $r_H$ be just less than $r_{cr}(\bar{J})$; one can then show that $\beta \sim [r_{cr}(\bar{J}) - r_H]/r_{cr}(\bar{J})$. There will then be $2\bar{J} + 1$ degenerate negative eigenvalues modes $\psi^M_{\mu}$ distinguished by the eigenvalue of $J_z$; it is convenient to normalize these so that

$$\int d^3x\sqrt{g}\psi^M_{\mu}^*\psi^M_{\mu} = 1,$$

(137)

where the integration is over all space outside the horizon.

We now write $W_{\mu}$ as a linear combination of these modes plus a remainder, orthogonal to these, which we expect to be subdominant:

$$W_{\mu} = W_{\mu}^{(0)} + \tilde{W}_{\mu} = m^{-1/2}\sum_{M=-\bar{J}}^{\bar{J}} k_M\psi^M_{\mu} + \tilde{W}_{\mu}.$$  

(138)

The actual values of the $k_M$ reflect a balancing of the effects of the quadratic terms in the action against those of the dominant higher order terms. For identifying which
terms are dominant, it is useful to define a dimensionless quantity \( a \) by

\[
\sum_{M = -J}^{j} |k_M|^2 = a^2.
\] (139)

Thus, \( W_\mu^{(0)} \) is of order \( a \). The source for the electromagnetic perturbations is quadratic in \( W_\mu \) and contains an explicit factor of \( e \), implying that \( \delta A_\mu = O(ea^2) \). The leading perturbation to the energy-momentum tensor is of order \( a \), arising both from terms quadratic in \( W_\mu \) and terms linear in \( \delta A_\mu \). These enter Einstein's equations multiplied by a factor of \( G \), implying that the metric perturbations are of order \( Gm^2a^2 \), where the \( m^2 \) follows from dimensional considerations. Substituting these back into the \( W_\mu \) field equations, one finds that \( \tilde{W} \) is of order \( e^2a^3 \) and thus subdominant, as expected, provided that \( a \) is small.

We can now use these estimates to identify the relevant terms in the action. If we assume that \( Gm^2 \) is small, and in particular that \( Gm^2 \ll a^2 \), it is sufficient to consider only those terms that are of up to order \( e^2a^4 \). These involve only \( W_\mu^{(0)} \) and \( \delta A_\mu \). To leading order, the latter is determined in terms of \( W_\mu^{(0)} \) by the linearized version of the electromagnetic field equation,

\[
\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} \delta F^{\mu\nu}) = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} p^{\mu\nu}) + j^\nu
\] (140)

where

\[
p_{\mu\nu} = -\frac{ieg}{2} \left( W_\mu^{(0)*} W_\nu^{(0)} - W_\nu^{(0)*} W_\mu^{(0)} \right)
\] (141)

and

\[
j^\nu = ie \left[ W_\mu^{(0)*} \left( \bar{D}^\mu W^{(0)\nu} - \bar{D}^\nu W^{(0)\mu} \right) - W_\mu^{(0)} \left( \bar{D}^\mu W^{(0)\nu*} - \bar{D}^\nu W^{(0)\mu*} \right) \right].
\] (142)

With the aid of these equations, as well as Eqs. (136) and (137), the relevant terms in the action may be written as

\[
I = -\beta^2 ma^2 + \int d^3x \sqrt{g} \left[ \frac{\lambda e^2}{4} \left| W_\mu^{(0)*} W_\nu^{(0)} - W_\nu^{(0)*} W_\mu^{(0)} \right|^2 - \frac{1}{4} \delta F_{\mu\nu} p^{\mu\nu} + \frac{1}{2} \delta A_\nu j^\nu \right].
\] (143)

Since \( \delta A_\mu \) should be understood here to be given in terms of the \( k_M \) through Eq. (140), \( I \) is simply a polynomial in the \( k_M \). The leading approximation to \( W_\mu \) corresponds to the choice for the \( k_M \) that minimizes this polynomial. Using the order of magnitude
estimates given above, together with the fact that the typical spatial scale of the $\psi^M_\mu$ is $m^{-1}$, one finds that the contribution from the first term in the integral in Eq. (143) is of order $\lambda \epsilon^2 m a^4$, while that from the remaining terms in the integral is of order $\epsilon^2 m a^4$. It follows that the $k_M$ that minimize $I$ will give a value for $a$ that is of order $\beta/\epsilon \sqrt{\lambda + 1}$, and thus can be made arbitrarily small by choosing $r_H$ sufficiently close to $r_{cr}$.

Having thus determined $W_\mu^{(0)}$ and the leading approximation to $\delta A_\mu$, one can use the linearized Einstein equations to obtain the metric perturbations to first order, and then iterate the equations to obtain the higher order corrections to the various fields. These results are described in detail elsewhere [30]. I will not go into them any further here, but will instead spend the remainder of this lecture discussing the symmetry of these solutions.

We have seen that if $q \neq 1$ there are no $J = 0$ vector spherical harmonics, and hence that the solution cannot be spherically symmetric. This still leaves the possibility that the black hole might possess an axial symmetry. We can decide whether this is the case by examining the relative magnitudes of the various $k_M$. If the solution is axially symmetric we can, without any loss of generality, take the $z$-axis to be the axis of symmetry. The only possibilities then are either that only $k_0$ is nonvanishing, in which case the solution is manifestly invariant under rotations about the $z$-axis, or else that a single one of the other $k_M$ is nonvanishing, in which case the solution is unchanged if such a rotation is supplemented by a gauge transformation. Thus, the new black hole solution is axially symmetric if and only if Eq. (143) can be minimized with only a single nonzero $k_M$.

Because the first term on the right hand side depends only on the overall scale $a$, but not on the relative magnitudes of the various $k_M$, we need only examine the integral on the right hand side of Eq. (143). This is particularly simple in the case where $\lambda$ is large, so that the integral is dominated by its first term, and $g \geq 0$ and $q \neq 1/2$, so that the unstable mode has $J = q - 1$. Because there is only a single vector harmonic with this total angular momentum, $W^{(0)}_\mu$ then involves only a single radial function and is of the form

$$W^{(0)}_\mu = f(r) \sum_{M=-\hat{J}}^{-\hat{J}} k_M C_{q,q-1,M}^{\mu}. \tag{144}$$

The integral in Eq. (143) can then approximated by

$$I' = \frac{\lambda \epsilon^2}{2} P(k_M) \int_0^\infty dr \frac{f(r)^4}{r^2}. \tag{145}$$

where $P(k_M)$ is a quartic polynomial whose coefficients are obtained from the angular integration of products of four vector harmonics. The problem of determining the
symmetry of the solution is thus reduced to the minimization of a polynomial in $2J + 1 = 2q - 1$ complex variables, subject to the constraint that the sums of their squares be held fixed. For $q = 2$, one finds that the minimum can be achieved with a single nonzero $k_M$, showing that the solution is indeed axially symmetric. However, when $q > 2$ this axial symmetry disappears; the static solutions with these higher charges have no continuous rotational symmetry.

Thus, to summarize, we have found that theories with charged vector mesons admit a rich variety of magnetically charged black holes that provide counterexamples to generalizations suggested by the properties of black holes in simpler theories. These new solutions include black holes with hair, new types of extremal black holes with repulsive mutual forces, and, finally, the first examples of static black hole solutions that have no rotational symmetry.

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References

1. P.-S. Laplace, *Exposition du system du monde*, vol II, (Paris, 1796)
2. S.W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
3. G. ’t Hooft, *Nucl. Phys.* **B79**, 276 (1974); A.M. Polyakov, *Pisma v. Zh. E.T.F.*, **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].
4. For a fuller treatment of the material in this section, see C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, (Freeman, San Francisco, 1973); R.M. Wald, *General Relativity*, (University of Chicago, Chicago, 1984).
5. K. Schwarzschild, *Sitzber. Deut. Akad. Wiss. Berlin*, Kl. Math.-Phys. Tech. 189 (1916).
6. M.D. Kruskal, *Phys. Rev.* **119**, 1743 (1960).
7. H. Reissner, *Ann. Physik* **50**, 106 (1916); G. Nordström, *Proc. Kon. Ned. Acad. Wet.* **20**, 1238 (1918).
8. S.D. Majmudar, *Phys. Rev.* **72**, 390 (1947); A. Papapetrou, *Proc. Roy. Irish Acad.* **51**, 191 (1947).
9. R.P. Kerr, *Phys. Rev. Lett.* **11**, 237 (1963).
10. E.T. Newman et al, *J. Math. Phys.* **6**, 918 (1965).
11. J.D. Bekenstein, *Phys. Rev. D* **5**, 1239 (1972).
12. G.W. Gibbons and S.W. Hawking, *Phys. Rev. D* **15**, 2752 (1977).
13. P.A.M. Dirac, *Proc. Roy. Soc. London* A**133**, 60 (1931).
14. M.K. Prasad and C.M. Sommerfield, *Phys. Rev. Lett.* **35**, 760 (1975).
15. T. Kirkman and C.K. Zachos, *Phys. Rev. D* **24**, 999 (1981).
16. P. van Nieuwenhuizen, D. Wilkinson, and M.J. Perry, *Phys. Rev. D* **13**, 778 (1976).
17. K. Lee, V.P. Nair and E.J. Weinberg, *Phys. Rev. D* **45**, 2751 (1992).
18. P. Breitenlohner, P. Forgács, and D. Maison, *Nucl. Phys.* B**383**, 357 (1992); Max-Planck-Institut preprint MPI-PhT/94-87 (1994).
19. M.E. Ortiz, *Phys. Rev. D* **45**, 2586 (1992).
20. K. Lee, V.P. Nair and E.J. Weinberg, *Phys. Rev. Lett.* **68**, 1100 (1992).
21. V. Moncrief, *Phys. Rev. D* **9**, 2707 (1974); *Phys. Rev. D* **10**, 1057 (1974); *Phys. Rev. D* **12**, 1526 (1975).
22. S.A. Ridgway and E.J. Weinberg, *Phys. Rev. D* **51**, 638 (1995).
23. K. Lee and E.J. Weinberg, *Phys. Rev. Lett.* **73**, 1203 (1994).
24. C. Lee and P. Yi, Seoul National University/Caltech preprint SNUTP-94-73/CALT-68-1944 (1994).
25. I. Tamm, *Z. Phys.* **71**, 141 (1931).
26. T.T. Wu and C.N. Yang, *Nucl. Phys.* B**107**, 365 (1976).
27. A.H. Guth and E.J. Weinberg, *Phys. Rev. D* **14**, 1660 (1976).
28. H.A. Olsen, P. Osland, and T.T. Wu, *Phys. Rev. D* **42**, 665 (1990).
29. E.J. Weinberg, *Phys. Rev. D* **49**, 1086 (1994).
30. S.A. Ridgway and E.J. Weinberg, Columbia preprint CU-TP-673.