STATED SKEIN ALGEBRAS OF SURFACES

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Abstract. We study the algebraic and geometric properties of stated skein algebras of surfaces with punctured boundary. We prove that the skein algebra of the bigon is isomorphic to the quantum group \( \mathcal{O}_{q^2}(\text{SL}(2)) \) providing a topological interpretation for its structure morphisms. We also show that its stated skein algebra lifts in a suitable sense the Reshetikhin-Turaev functor and in particular we recover the dual \( R \)-matrix for \( \mathcal{O}_{q^2}(\text{SL}(2)) \) in a topological way. We deduce that the skein algebra of a surface with \( n \) boundary components is an algebra-comodule over \( \mathcal{O}_{q^2}(\text{SL}(2))^{\otimes n} \) and prove that cutting along an ideal arc corresponds to Hochshild cohomology of bicomodules. We give a topological interpretation of braided tensor product of stated skein algebras of surfaces as “glueing on a triangle”; then we recover topologically some braided bialgebras in the category of \( \mathcal{O}_{q^2}(\text{SL}(2)) \)-comodules, among which the “transmutation” of \( \mathcal{O}_{q^2}(\text{SL}(2)) \). We also provide an operadic interpretation of stated skein algebras as an example of a “geometric non symmetric modular operad”. In the last part of the paper we define a reduced version of stated skein algebras and prove that it allows to recover Bonahon-Wong’s quantum trace map and interpret skein algebras in the classical limit when \( q \to 1 \) as regular functions over a suitable version of moduli spaces of twisted bundles.

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1. Introduction

This paper is devoted to study the notion of stated skein algebra of surfaces introduced by
the second author in [Le2] in order to reinterpret in skein theoretical terms the construction
of the quantum trace by Bonahon and Wong [BW] as well as incorporating Muller’s version
of skein algebra [Mu]. Although the definition of the stated skein module applies to 3-
manifolds, this paper is entirely devoted to the case of surfaces: a forthcoming paper will
describe how this fits in the framework of an extended topological field theory in dimensions
1, 2, 3. Indeed the case of surfaces is sufficiently rich in algebraic and geometrical terms to
deserve a separate treatment and we will now outline the results of this paper.

1.1. Skein algebras. Let \( \mathcal{R} = \mathbb{Z}[q^{\pm 1/2}] \) be the ring of Laurent polynomials in a variable
\( q^{1/2} \). Suppose \( \mathcal{S} \) is the result of removing a finite number of points, called punctures, from a
compact oriented 2-dimensional manifold with possibly non-empty boundary. The ordinary
skein algebra \( \mathcal{S}(\mathcal{S}) \), introduced by Przytycki [Pr] and Turaev [Tu2], is defined to be the
\( \mathcal{R} \)-module generated by isotopy classes of framed unoriented links in \( \mathcal{S} \times (0, 1) \) modulo the
Kauffman relations [Kau]

\[
\begin{align*}
\left( \frac{1}{q} \right) & = q \left( \frac{1}{q} \right) + q^{-1} \\
\circ & = (-q^2 - q^{-2})
\end{align*}
\]

The product of two links \( \alpha_1 \) and \( \alpha_2 \) is the result of stacking \( \alpha_1 \) above \( \alpha_2 \). The skein algebra
has played an important role in low-dimensional topology and quantum topology and it
serves as a bridge between classical topology and quantum topology. The skein module has
connections to the \( \text{SL}_2(\mathbb{C}) \)-character variety [Bul, PS1], the quantum group of \( \text{SL}_2(\mathbb{C}) \), the
Witten-Reshetikhin-Turaev topological quantum field theory [BHMV], the quantum Teich-
müller spaces [CF1, Kas, BW, Le1], and the quantum cluster algebra theory [Mu].

In the definition of the skein algebra \( \mathcal{S}(\mathcal{S}) \) the boundary \( \partial \mathcal{S} \) does not play any role, and
we have \( \mathcal{S}(\mathcal{S}) = \mathcal{S}(\mathcal{S}) \), where \( \mathcal{S} \) is the interior of \( \mathcal{S} \). In an attempt to introduce excision
into the study the skein algebra, the second author [Le2] introduce the notion of stated skein
algebra, denoted in this paper by \( \hat{\mathcal{S}}(\mathcal{S}) \), whose definition involves tangles properly embedded
into \( \hat{\mathcal{S}} \times (0, 1) \). These tangles can have end-points only on boundary edges of \( \mathcal{S} \), which are
open interval connected components of the boundary. For details see Section 2.
A key result about stated skein algebras is that they behave well under cutting along an ideal arc. Here an ideal arc is a proper embedding \( c : (0, 1) \hookrightarrow \mathcal{G} \) (so that its end points are the punctures). Cutting \( \mathcal{G} \) along \( c \) one gets a 2-manifold \( \mathcal{G}' \) whose boundary contains two open intervals \( a \) and \( b \) so that one can recover \( \mathcal{G} \) from \( \mathcal{G}' \) by gluing \( a \) and \( b \) together, see Figure 1.

**Figure 1.** Cutting \( \mathcal{G} \) along ideal arc \( c \) to get \( \mathcal{G}' \), which might be disconnected

Then [Le2, Theorem 1] (see the splitting Theorem 2.14 below) says that there is an natural injection of algebras

\[
\theta_c : \mathcal{S}(\mathcal{G}) \hookrightarrow \mathcal{S}(\mathcal{G}'),
\]

given by a simple state sum. The extension from \( \mathcal{S}(\mathcal{G}) \) to \( \mathcal{S}(\mathcal{G}) \) is unique (or canonical) if one wants the splitting theorem and a consistency requirement to hold.

The paper is a systematic study of the stated skein algebra \( \mathcal{S}(\mathcal{G}) \). Let us now list the main results of the paper.

1.2. **Bigon and quantum \( SL_2(\mathbb{C}) \) coordinate ring.** The quantized enveloping algebra \( U_q(sl_2) \) and its Hopf dual \( \mathcal{O}_q(SL(2)) \), known as the quantum coordinate ring of the Lie group \( SL_2(\mathbb{C}) \), play an important role in many branches of mathematics, see [Kass, Maj]. These algebras are usually defined by rather complicated presentations which are hard to comprehend.

A first consequence of the splitting theorem is that the quantum coordinate ring \( \mathcal{O}_q(SL(2)) \) can be described by simple geometric terms, namely, it is naturally isomorphic to the stated skein algebra of the bigon \( \mathcal{B} \), which is the standard disk without two points on its boundary, see Figure 2.

**Figure 2.** Left: bigon. Right: splitting the bigon along the dashed ideal arc

By splitting the bigon along an ideal arc \( c \) (which is the dashed arc in Figure 2) we get a homomorphism \( \Delta := \theta_c \),

\[
\Delta : \mathcal{S}(\mathcal{B}) \to \mathcal{S}(\mathcal{B}) \otimes \mathcal{S}(\mathcal{B}),
\]
which turns out to be compatible with the product and makes $\mathcal{S}(B)$ a bialgebra. Moreover, we will define using topological terms the counit, antipode and co-$R$-matrix which turns $\mathcal{S}(B)$ into a cobrained Hopf algebra, and will prove the following.

**Theorem 1** (Theorems 3.4 and 3.5). The cobrained Hopf algebra $\mathcal{S}(B)$ is isomorphic in a natural way to the quantum coordinate ring $O_{q^2}(SL(2))$.

This result allows to use skein theoretical techniques to study $O_{q^2}(SL(2))$. We will show that many complicated algebraic objects and facts concerning the quantum groups $O_{q^2}(SL(2))$ and $U_{q^2}(sl_2)$ have simple transparent picture interpretations. For example, the above mentioned co-$R$-matrix has a very simple geometric picture description, see Theorem 3.5. Another example is given by the reconstruction of Kashiwara’s crystal basis, see Proposition 3.9. One can even “import” in $O_{q^2}(SL(2))$ natural skein theoretical objects: in Subsection 3.8 we define and provide some properties of the Jones-Wenzl idempotents in $O_{q^2}(SL(2))$.

1.3. **Lift of the Reshetikhin-Turaev invariant.** Suppose $T$ is a tangle diagram in the bigon whose boundary $\partial T$ is in $\partial B$ and the boundary points are labeled by signs $\pm$. The Reshetikhin-Turaev operator invariant theory $[\text{RT}]$ assigns to $T$ a scalar $Z(T) \in \mathbb{Q}(q^{1/2})$, see Section 5. On the other hand, such a labeled tangle $T$ defines an element in our skein algebra $\mathcal{S}(B)$. We have the following result which shows that our “invariant”, which is $T$ considered as an element of $\mathcal{S}(B)$, is a lift of the Reshetikhin-Turaev invariant.

**Theorem 2** (Theorem 5.2). One has $\epsilon(T) = Z(T)$, where $\epsilon : \mathcal{S}(B) \to \mathbb{Q}[q^{\pm 1/2}]$ is the counit.

It would be interesting to understand this lift of the Reshetikhin-Turaev invariant in terms of categorification.

1.4. **Skein algebras as comodule over $O_{q^2}(SL(2))$. Hochshild cohomology.** One important consequence of the identification of the bigon algebra with $O_{q^2}(SL(2))$ is that for every boundary edge $e$ of a surface $\mathcal{S}$, the skein algebra $\mathcal{S}(\mathcal{S})$ has a right $O_{q^2}(SL(2))$-comodule structure

$$\Delta_e : \mathcal{S}(\mathcal{S}) \to \mathcal{S}(\mathcal{S}) \otimes \mathcal{S}(B).$$

This map $\Delta_e$ is the splitting homomorphism (3) applied to the an ideal arc parallel to $e$ which cuts off an ideal bigon from $\mathcal{S}$ whose right edge is $e$, see Figure 3. Similarly identifying the left edge of $B$ to $e$ we get a left $O_{q^2}(SL(2))$-comodule structure on $\mathcal{S}(\mathcal{S})$.

![Figure 3. Geometric definition of the coaction: splitting the bigon along the dashed ideal arc](image)

Figure 3. Geometric definition of the coaction: splitting the bigon along the dashed ideal arc.
Using the comodule structure one can refine the splitting theorem by identifying the image of the splitting homomorphism, as follows. Let us cut \( S \) along an ideal arc \( c \) to get \( S' \) as in Figure 1. Then \( \mathcal{I}(S') \) has a right \( O_{q^2}(SL(2)) \)-module structure coming from edge \( a \) and a left \( O_{q^2}(SL(2)) \)-module structure coming from edge \( b \). Thus \( \mathcal{I}(S') \) is a \( O_{q^2}(SL(2)) \)-bicomodule, and hence there is defined the Hochshild cohomology \( HH^0(\mathcal{I}(S')) \), for details see Section 4.

**Theorem 3** (Theorem 4.7). Under the splitting homomorphism the skein algebra \( \mathcal{I}(S) \) embeds isomorphically into the Hochshild cohomology \( HH^0(\mathcal{I}(S')) \). In particular, when \( c \) cuts \( S \) into two surfaces \( S_1 \) and \( S_2 \), the splitting homomorphism maps \( \mathcal{I}(S) \) isomorphically onto the cotensor product of \( \mathcal{I}(S_1) \) and \( \mathcal{I}(S_2) \).

### 1.5. Skein algebra \( \mathcal{I}(S) \) as module over \( U_{q^2}(sl_2) \).

Since the Hopf algebra \( U_{q^2}(sl_2) \) is the Hopf dual of \( O_{q^2}(SL(2)) \), then after tensoring with \( \mathbb{Q}(q^{1/2}) \) each right \( O_{q^2}(SL(2)) \)-comodule is automatically a left \( U_{q^2}(sl_2) \)-module. Thus each boundary edge \( e \) of \( S \) gives \( \mathcal{I}(S) \) a left \( U_{q^2}(sl_2) \)-module structure. Note that finite-dimensional \( U_{q^2}(sl_2) \)-modules are well-understood as they are quantum deformations of modules over the Lie algebra \( sl_2(\mathbb{C}) \).

**Theorem 4** (Part of Theorem 4.5). Over the field \( \mathbb{Q}(q^{1/2}) \) the \( U_{q^2}(sl_2) \)-module \( \mathcal{I}(S) \) is integrable, i.e. it is the direct sum of finite-dimensional irreducible \( U_{q^2}(sl_2) \)-modules.

Actually Theorem 4.5 is much stronger: it provides an explicit decomposition and contains much more information about the decompositions as it deals also with the decomposition over Lusztig’s integral version of \( U_{q^2}(sl_2) \).

Using this result we also prove a dual version of Theorem 3 which, with the notation of the theorem, shows that \( HH_0(\mathbb{Q}(q^{1/2}) \otimes \mathcal{R} \mathcal{I}(S')) = \mathbb{Q}(q^{1/2}) \otimes \mathcal{R} \mathcal{I}(S) \) (see Theorem 4.9).

### 1.6. Braided tensor product.

The co-\( R \)-matrix makes the category of \( O_{q^2}(SL(2)) \)-comodules a braided category and in general given two algebras in that category (which are then \( O_{q^2}(SL(2)) \)-comodule algebras) their tensor product can be endowed with the structure of an algebra by using appropriately the braiding: this is the braided tensor product of the algebras, see [Maj].

Suppose \( S \) is obtained by gluing two surfaces \( S_1, S_2 \) to two distinct edges of an ideal triangle as in Figure 4. Then both \( \mathcal{I}(S_1) \) and \( \mathcal{I}(S_2) \) have a natural structure of \( O_{q^2}(SL(2)) \)-

![Figure 4](image-url)
**Theorem 5** (Theorem 4.13). As a $O_q^2(SL(2))$-comodule algebra $\mathcal{I}(\mathcal{S})$ is canonically isomorphic to the braided tensor product of the $O_q^2(SL(2))$-comodule algebras $\mathcal{I}(\mathcal{S}_1)$ and $\mathcal{I}(\mathcal{S}_1)$.

Through this theorem we easily compute the skein algebra of all “polygons” and “punctured monogons” in Subsection 4.8. It is remarkable that the skein algebras of the latter turn out to be braided bialgebra objects in the category of $O_q^2(SL(2))$-comodules and that their structure morphism have natural topological interpretation. In particular the punctured monogon yields the “transmutation” of $O_q^2(SL(2))$.

1.7. **Modular operad.** The splitting homomorphism allows to put the theory of stated skein algebras of surfaces in the framework of operad theory. We define the notion of geometric non-symmetric modular operad in Section 6 and prove the following.

**Theorem 6** (Precise statement given by Theorem 6.1). The stated skein algebra of surfaces gives rise to a non-symmetric modular operad in a category of bimodules over $U_q^2(sl_2)$.

To be more specific while leaving the details for Section 6, let us recall that, according to Markl ([Mark]) a “Non-symmetric modular operad in a monoidal category $Cat$” is a monoidal functor $NSO : MultiCyc \to Cat$, where MultiCyc is a suitable category of “multicyclic sets”. In Section 6 we re-cast Markl’s definition, by defining a category $\text{TopMultiCyc}$ whose objects are punctured surfaces $\mathcal{S}$ and whose morphisms are finite sets of ideal arcs (describing a way of cutting the surfaces). From this point of view, we then show in Theorem 6.1 that stated skein algebras provide a symmetric monoidal functor from this category into a suitable category of modules and bimodules over copies of $U_q^2(sl_2)$, thus providing a topological example of a NS modular operad.

1.8. **Reduced stated skein algebra, quantum torus, and quantum trace map.** The stated skein algebra $\mathcal{I}(\mathcal{S})$ has a quotient $\overline{\mathcal{I}}(\mathcal{S}) = \mathcal{I}(\mathcal{S})/\mathcal{I}^{\text{bad}}$, called the reduced stated skein algebra, whose algebraic structure is much simpler as it can be embedded into the so called quantum tori. Here $\mathcal{I}^{\text{bad}}$ is the ideal generated by elements, called bad arcs, described in Figure 5 and is explained in Section 7.

![Figure 5. A bad arc.](image)

We will show that the ordinary skein algebra $\hat{\mathcal{I}}(\mathcal{S})$ still embeds into $\overline{\mathcal{I}}(\mathcal{S})$ and hence we can use $\overline{\mathcal{I}}(\mathcal{S})$ to study $\hat{\mathcal{I}}(\mathcal{S})$. Most importantly, the splitting theorem still holds for $\overline{\mathcal{I}}(\mathcal{S})$.

**Theorem 7** (Theorem 7.6). If $\mathcal{S}'$ is the result of cutting $\mathcal{S}$ along an ideal arc $c$, then the splitting homomorphism $\theta_c$ descends to an algebra embedding $\overline{\theta}_c : \overline{\mathcal{I}}(\mathcal{S}) \hookrightarrow \overline{\mathcal{I}}(\mathcal{S}')$. 
The non-trivial fact here is that $\bar{\theta}$ is injective.

Except for a few simple surfaces, we can always cut $\mathcal{S}$ along ideal arcs so that the result is a collection of ideal triangles $T_1, \ldots, T_k$. It follows that there is an embedding

$$\Theta : \mathcal{F}(\mathcal{S}) \hookrightarrow \bigotimes_{i=1}^{k} \mathcal{F}(T_i).$$

The important thing with the reduced version is that for an ideal triangle $T$, unlike the full fledged $\mathcal{S}(T)$, the reduced stated skein algebra $\mathcal{F}(T)$ is a quantum torus in three variables:

**Theorem 8 (Theorem 7.11).** The reduced stated skein algebra $\mathcal{F}(T)$ of an ideal triangle has presentation

$$\mathcal{F}(T) = \mathcal{R}(\alpha^\pm 1, \beta^\pm 1, \gamma^\pm 1)/(\beta \alpha = q \alpha \beta, \gamma \beta = q \beta \gamma, \alpha \gamma = q \gamma \alpha).$$

Moreover, the reduced stated skein algebra of the bigon is naturally isomorphic to the algebra $\mathcal{R}[x^\pm 1]$ of Laurent polynomial in one variables, see Proposition 7.10.

Consequently, the map $\Theta$ of (4) embeds the reduced stated skein algebra $\mathcal{F}(\mathcal{S})$ into a quantum torus in $3k$ variables.

Geometrically the variables $\alpha, \beta, \gamma$ in Theorem 8 come from the corner arcs of the ideal triangle. There is a similar quantum torus $\mathbb{T}'(T)$ in 3 variables corresponding to the edges of $T$, and a simple change of variables gives us an embedding $\mathcal{F}(T) \hookrightarrow \mathbb{T}'(T)$. Combining with $\Theta$ of (4) we get an algebra embedding

$$\text{tr}_q : \mathcal{F}(\mathcal{S}) \hookrightarrow \bigotimes_{i=1}^{k} \mathcal{F}(T_i) \hookrightarrow \bigotimes_{i=1}^{k} \mathbb{T}(T_i).$$

There is a subalgebra $\mathcal{Y}$ of $\bigotimes_{i=1}^{k} \mathbb{T}(T_i)$, known as the Chekhov-Fock algebra associated to the triangulation. The famous quantum trace map of Bonahon and Wong [BW] is an injective algebra homomorphism from the ordinary skein algebra $\mathcal{F}(\mathcal{S})$ to $\mathcal{Y}$.

**Theorem 9 (See Theorem 7.12).** The image of $\text{tr}_q$ is in $\mathcal{Y}$. Thus $\text{tr}_q$ restricts to an algebra embedding $\text{tr}_q : \mathcal{F}(\mathcal{S}) \hookrightarrow \mathcal{Y}$ which on $\mathcal{F}(\mathcal{S})$ is equal to the quantum trace map of Bonahon and Wong.

The existence of the quantum trace map (for $\mathcal{F}(\mathcal{S})$) was conjectured by Chekhov and Fock [CF2], and was established by Bonahon and Wong [BW] using difficult calculations. It is called the quantum trace map since when $q = 1$ it becomes a formula expressing the trace of a curve under the holonomy representation of the hyperbolic metric in terms of the shear coordinates of the Teichmüller space. The second author [Le1] gave another proof of the existence of the quantum trace map based on the Muller skein algebra, which is actually a subspace of the state skein algebra $\mathcal{F}(\mathcal{S})$. The above approach using the reduced stated skein algebra is probably the most conceptual one.

1.9. **Classical limit.** The last section explores the natural question of "what is the classical limit of $\mathcal{F}(\mathcal{S})$?" In the case of the standard skein algebra $\mathcal{F}(\mathcal{S})$ it is known [Bul, BP] (see also [CM]) that when the quantum parameter $q$ is $-1$ and the ground ring is $\mathbb{C}$ then $\mathcal{F}(\mathcal{S})$ is isomorphic as an algebra to the coordinate ring the $\text{SL}_2(\mathbb{C})$-character variety of $\mathcal{S}$ and
that in general the algebras at $q$ and $-q$ are isomorphic via the choice of a spin structure on $\mathcal{S}$ ([Ba]). Note that our stated skein algebra is not commutative when $q = -1$ though it is commutative when $q = 1$.

We introduce the variety $\text{tw}(\mathcal{S})$ of “twisted $\text{SL}_2(\mathbb{C})$-bundles” over $\mathcal{S}$, which, roughly speaking, are flat $\text{SL}_2(\mathbb{C})$-bundles over the unit tangent bundle $U\mathcal{S}$ of $\mathcal{S}$ with holonomy $-\text{Id}$ around the fibers of $\pi: U\mathcal{S} \to \mathcal{S}$ and trivializations along the edges of $\mathcal{S}$, but which we re-formulate in terms of groupoid representations. To deal with the non-oriented nature of the arcs of the stated skein algebra we have to use a trick smoothing the arcs at their end-points so that one can compose arcs.

**Theorem 10** (Theorem 8.12). When $q = 1$ and the ground ring is $\mathbb{C}$ the stated skein algebra $\mathcal{H}(\mathcal{S})$ is naturally isomorphic to the coordinate ring of $\text{tw}(\mathcal{S})$.

In classical terms, the splitting theorem becomes an instance of a van-Kampen like theorem for groupoid representations.

Theorem 8.12 highlights a relation between $\mathcal{H}(\mathcal{S})$ and the coordinate ring of the character variety of $\mathcal{S}$. The study of quantizations of such rings has been performed with different techniques (based on Hopf algebras and lattice gauge theory) by Alekseev-Grosse-Schomerus ([AGS]), Buffenoir-Roche [BR], Fock-Rosly ([FR]) and, later, via skein theoretical approaches by Bullock-Frohman-Kania-Bartoszynska ([BFK]). The relation of our work with these previous ones is still to be clarified, although it seems that one of the main differences between our approach and some of the above cited ones, is that we allow for “observables with boundary” and, as explained in the preceding paragraph, this endows the algebras we work with rich algebraic structures which in particular make computations much easier.

1.10. **Related results.** While the authors were completing the present work, D. Ben-Zvi, A. Brochier and D. Jordan [BBJ] constructed a theory of quantum character variety for general Hopf algebras, based on completely different techniques, and part of the results of this paper could probably recast in that theory, though we don’t know the precise relation between the two theories. The substantial difference of the techniques used makes these works complementary. K. Habiro informed us that his “quantum fundamental group theory” also gives an alternative approach to the theory of quantum character variety.

When the authors presented their works at conferences, Korkinman informed us that he in joint work A. Quesney obtained results similar to Theorem 3 and Theorem 10, see their recent preprint [KQ].

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developments in quantum topology” (Berkeley June 2019), “Expansions, Lie algebras and invariants” (Montreal July 2019) and would like to thank the organizers for the opportunities to present their work.

2. Stated skein algebras

We will present the basics of the theory of stated skein algebras: definitions, bases of skein algebras, the splitting homomorphism, filtrations and gradings. New results involve Proposition 2.7 describing the inversion homomorphism and Proposition 2.16 giving the exact value of the splitting homomorphism in the associated graded algebra.

2.1. Notations. Throughout the paper let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ be the set of non-negative integers, $\mathbb{C}$ be the set of complex numbers. The ground ring $\mathcal{R}$ is a commutative ring with unit 1, containing a distinguished invertible element $q^{1/2}$. For a finite set $X$ we denote by $|X|$ the number of elements of $X$.

We will write $x \cdot y$ if there is $k \in \mathbb{Z}$ such that $x = q^k y$.

2.2. Punctured bordered surface. By a punctured bordered surface $\mathcal{S}$ we mean a surface of the form $\mathcal{S} = \mathcal{S} \setminus P$, where $\mathcal{S}$ is a compact oriented surface with (possibly empty) boundary $\partial \mathcal{S}$, and $P$ is a finite non empty set such that every connected component of the boundary $\partial \mathcal{S}$ has at least one point in $P$. We don’t require $\mathcal{S}$ to be connected. It is easy to see that $\mathcal{S}$ is uniquely determined by $\mathcal{G}$. Throughout this section we fix a punctured bordered surface $\mathcal{S}$.

An ideal arc on $\mathcal{S}$ is an immersion $a : [0, 1] \rightarrow \mathcal{S}$ such that $a(0), a(1) \in P$ and the restriction of $a$ onto $(0, 1)$ is an embedding into $\mathcal{S}$. Isotopy of ideal arcs are considered in the class of ideal arcs.

A connected component of $\partial \mathcal{S}$ is called a boundary edge of $\mathcal{S}$ (or simply edge), which is an ideal arc.

Remark 2.1. The fact that each connected component of $\partial \mathcal{S}$ is an open interval is not a serious restriction as for the purpose of the constructions of this paper a point-less boundary component is treated as a puncture; so that in the end the only excluded surfaces are closed ones without punctures.

2.3. Ordinary skein algebra. Let $M = \mathcal{S} \times (0, 1)$. For a point $(z, t) \in \mathcal{S} \times (0, 1)$ its height is $t$. A vector at $(z, t)$ is called vertical if it is a positive vector of $z \times (0, 1)$. A framing of a 1-dimensional submanifold $\alpha$ of $M = \mathcal{S} \times (0, 1)$ is a continuous choice of a vector transverse to $\alpha$ at each point of $\alpha$.

A framed link in $\mathcal{S} \times (0, 1)$ is a closed 1-dimensional unoriented submanifold $\alpha$ equipped with a framing. The empty set, by convention, is considered a framed link.

A link diagram on $\mathcal{S}$ determines an isotopy class of framed links in $\mathcal{S} \times (0, 1)$, where the framing is vertical everywhere. Every isotopy class of framed links in $\mathcal{S} \times (0, 1)$ is presented by a link diagram.

The skein module $\mathcal{J}(\mathcal{S})$, first introduced by Przytycki [Pr] and Turaev [Tu1], is defined to be the $\mathcal{R}$-module generated by the isotopy classes of framed unoriented links in $\mathcal{S} \times (0, 1)$.
modular the Kauffman relations

\begin{align}
\begin{array}{crr}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1} \\
\includegraphics[width=0.2\textwidth]{diagram2}
\end{array} & = & q \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3} \\
\includegraphics[width=0.2\textwidth]{diagram4}
\end{array} + q^{-1} \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram5} \\
\includegraphics[width=0.2\textwidth]{diagram6}
\end{array} \\
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram7} \\
\includegraphics[width=0.2\textwidth]{diagram8}
\end{array} & = & (-q^2 - q^{-2}) \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram9} \\
\includegraphics[width=0.2\textwidth]{diagram10}
\end{array}
\end{array}
\end{align}

Here is the convention about pictures in these identities, as well as in other identities in this paper. Each shaded part is a part of $\mathcal{S}$, with a link diagram on it. Relation (5) says that if link diagrams $D_1, D_2,$ and $D_3$ are identical everywhere except for a small disk in which $D_1, D_2, D_3$ are like in respectively the first, the second, and the third shaded areas, then $[D_1] = q[D_2] + q^{-1}[D_3]$ in the skein module $\mathcal{S}(\mathcal{S})$. Here $[D_i]$ is the isotopy class of links determined by $D_i$. The other relation is interpreted similarly.

For two framed links $\alpha_1$ and $\alpha_2$ the product $\alpha_1 \alpha_2$ is defined as the result of stacking $\alpha_1$ above $\alpha_2$. That is, first isotope $\alpha_1$ and $\alpha_2$ so that $\alpha_1 \subset \mathcal{S} \times (1/2, 1)$ and $\alpha_2 \subset \mathcal{S} \times (0, 1/2)$. Then $\alpha_1 \alpha_2 = \alpha_1 \cup \alpha_2$. It is easy to see that this gives rise to a well-defined product and hence an $\mathcal{R}$-algebra structure on $\mathcal{S}(\mathcal{S})$.

2.4. Tangles and order. In order to include the boundary of $\mathcal{S}$ into the picture, we will replace framed links by more general objects called $\partial M$-tangles. Recall that $M = \mathcal{S} \times (0, 1)$ and its boundary is $\partial M = \partial \mathcal{S} \times (0, 1)$.

In this paper, a $\partial M$-tangle is an unoriented, framed, compact, properly embedded 1-dimensional submanifold $\alpha \subset M = \mathcal{S} \times (0, 1)$ such that:

- at every point of $\partial \alpha = \alpha \cap \partial M$ the framing is vertical, and
- for any boundary edge $b$, the points of $\partial_b(\alpha) := \partial \alpha \cap (b \times (0, 1))$ have distinct heights.

Isotopy of $\partial M$-tangles are considered in the class of $\partial M$-tangles. The empty set, by convention, is a $\partial \mathcal{S}$-tangle which is isotopic only to itself.

For a $\partial M$-tangle $\alpha$ define a partial order on $\partial \alpha$ by: $x > y$ if there is a boundary edge $b$ such that $x, y \in b \times (0, 1)$ and $x$ has greater height. If $x > y$ and there is no $z$ such that $x > z > y$, then we say $x$ and $y$ are consecutive.

As usual, $\partial M$-tangles are depicted by their diagrams on $\mathcal{S}$, as follows. Every $\partial \mathcal{S}$-tangle is isotopic to one with vertical framing. Suppose a vertically framed $\partial M$-tangle $\alpha$ is in general position with respect to the standard projection $\pi : \mathcal{S} \times (0, 1) \to \mathcal{S}$, i.e. the restriction $\pi|_{\alpha} : \alpha \to \mathcal{S}$ is an immersion with transverse double points as the only possible singularities and there are no double points on the boundary of $\mathcal{S}$. Then $D = \pi(\alpha)$, together with

- the over/underpassing information at every double point, and
- the linear order on $\pi(\alpha) \cap b$ for each boundary edge $b$ induced from the height order

is called a $\partial M$-tangle diagram, or simply a tangle diagram on $\mathcal{S}$. Isotopies of $\partial M$-tangle diagrams are ambient isotopies in $\mathcal{S}$.

Clearly the $\partial M$-tangle diagram of a $\partial M$-tangle $\alpha$ determines the isotopy class of $\alpha$. When there is no confusion, we identify a $\partial M$-tangle diagram with its isotopy class of $\partial M$-tangles.

Let $\sigma$ be an orientation of $\partial \mathcal{S}$, which on a boundary edge may or may not be equal to the orientation inherited from $\mathcal{S}$. A $\partial M$-tangle diagram $D$ is $\sigma$-ordered if for each boundary edge $b$ the order of $\partial D$ on $b$ is increasing when one goes along $b$ in the direction of $\sigma$. It is
clear that every \( \partial M \)-tangle, after an isotopy, can be presented by an \( \sigma \)-ordered \( \partial M \)-tangle diagram. If \( \sigma \) is the orientation coming from \( \mathcal{G} \), the \( \sigma \)-order is called the positive order.

2.5. Stated skein algebra. A state on a finite set \( X \) is a map \( s : X \to \{\pm\} \). We write \( \#s = |X| \). A stated \( \partial M \)-tangle (respectively a stated \( \partial M \)-tangle diagram) is a \( \partial M \)-tangle (respectively a \( \partial M \)-tangle diagram) equipped with a state on its set of boundary points.

The stated skein algebra \( \mathcal{I}(\mathcal{G}) \) is the \( \mathcal{R} \)-module freely spanned by isotopy classes of stated \( \partial M \)-tangles modulo the defining relations, which are the old skein relation (7) and the trivial loop relation (8), and the new boundary relations (9) and (10):

\[
\begin{align*}
(7) & \quad \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array} = q \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array} + q^{-1} \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array} \\
(8) & \quad \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array} = (-q^2 - q^{-2}) \\
(9) & \quad \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array} = q^{-1/2}, \quad \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array} = 0, \quad \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array} = 0 \\
(10) & \quad \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array} = q^2 \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array} + q^{-1/2} \begin{array}{c}
\includegraphics[width=0.5cm]{circle.png}
\end{array}
\end{align*}
\]

Here each shaded part is a part of \( \mathcal{G} \), with a stated \( \partial M \)-tangle diagram on it. Each arrowed line is part of a boundary edge, and the order on that part is indicated by the arrow and the points on that part are consecutive in the height order. The order of other end points away from the picture can be arbitrary and are not determined by the arrows of the picture. On the right hand side of the first identity of (9), the arrow does not play any role; it is there only because the left hand side has an arrow.

Again for two \( \partial M \)-tangles \( \alpha_1 \) and \( \alpha_2 \) the product \( \alpha_1 \alpha_2 \) is defined as the result of stacking \( \alpha_1 \) above \( \alpha_2 \). The product makes \( \mathcal{I}(\mathcal{G}) \) an \( \mathcal{R} \)-algebra. In [Le2] it is proved that if \( \mathcal{R} \) is a domain then \( \mathcal{I}(\mathcal{G}) \) is does not have non-trivial zero-divisors.

If \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are two punctured bordered surfaces, then there is a natural isomorphism

\[
\mathcal{I}(\mathcal{G}_1 \sqcup \mathcal{G}_2) \cong \mathcal{I}(\mathcal{G}_1) \otimes_{\mathcal{R}} \mathcal{I}(\mathcal{G}_2).
\]

Remark 2.2. Stated skein algebra was first introduced in [Le2] in order to provide skein theoretical proofs of previous results in constructions of Bonahon and Wong.

Since the interior \( \mathcal{G} \) of \( \mathcal{G} \) does not have boundary, we have \( \mathcal{I}(\mathcal{G}) = \mathcal{I}(\mathcal{G}) \).

The subalgebra \( \mathcal{I}^+(\mathcal{G}) \) spanned by \( \partial M \)-tangles whose states are all \( + \) is naturally isomorphic to the skein algebra defined by Muller [Mu], see [Le2, LP].

2.6. Consequences of defining relations. Define \( C_{\nu\nu'}^\prime \) for \( \nu, \nu' \in \{\pm\} \) by

\[
(11) \quad C_+^+ = C_-^- = 0, \quad C_+^- = q^{-1/2}, \quad C_-^+ = -q^{-5/2}.
\]

In the next lemma we have the values of all the trivial arcs.
Lemma 2.3 (Lemma 2.3 of [Le2]). In $\mathcal{S}(\mathcal{G})$ one has
\begin{align*}
(-q^{-3}) & = -q^3 \nu \quad (12) \\
C_{\nu} & = C_{\nu}^\prime \quad (13) \\
(-q^3)C_{\nu}^\prime & = -q^3C_{\nu}^\prime \quad (14)
\end{align*}

The next lemma describes how a skein behaves when the order of two consecutive boundary points is switched.

Lemma 2.4 (Height exchange move, Lemma 2.4 of [Le2]). (a) One has
\begin{align*}
q^{-1} & = q^{-1} \quad (15) \\
q^2 - q^{-2} & = (q^2 - q^{-2}) \quad (16)
\end{align*}

(b) Consequently, if $q = 1$ or $q = -1$, then for all $\nu, \nu' \in \{\pm\}$,
\begin{align*}
\nu \nu' & = q \nu \nu' \quad (17)
\end{align*}

Remark 2.5. Because of relation (17), in general $\mathcal{S}(\mathcal{G})$ is not commutative when $q = -1$. This should be contrasted with the case of the usual skein algebra $\mathcal{S}(\mathcal{G})$, which is commutative and is canonically equal to the SL$_2(\mathbb{C})$ character variety of $\pi_1(\mathcal{G})$ if $R = \mathbb{C}$ and $q = -1$ (assuming $\mathcal{G}$ is connected), see [Bul, PS1].

2.7. Reflection anti-involution.

Proposition 2.6 (Reflection anti-involution, Proposition 2.7 in [Le2]). Suppose $R = \mathbb{Z}[q^{\pm 1/2}]$. There exists a unique $\mathbb{Z}$-linear anti-automorphism $\chi : \mathcal{S}(\mathcal{G}) \to \mathcal{S}(\mathcal{G})$, such that $\chi(q^{1/2}) = q^{-1/2}$ if $\alpha$ is a stated $\partial M$-tangle then $\chi(\alpha) = \bar{\alpha}$, where $\bar{\alpha}$ is the image of $\alpha$ under the reflection of $\mathcal{G} \times (0,1)$, defined by $(z,t) \to (z,1-t)$. Here $\chi$ is an anti-automorphism means for any $x, y \in \mathcal{S}(\mathcal{G})$
\begin{align*}
\chi(x+y) & = \chi(x) + \chi(y), \quad \chi(xy) = \chi(y)\chi(x). \\
\chi^2 & = \text{id}. \quad \text{We call $\chi$ the reflection anti-involution.}
\end{align*}

2.8. Inversion along an edge. Suppose $e$ is a boundary edge of $\mathcal{G}$. For $\nu \in \{\pm\}$ define $C(\nu) = C_{\nu}^{-\nu}$. Explicitly,
\begin{align*}
C(+) & = -q^{-5/2}, \quad C(-) = q^{-1/2} \quad (18)
\end{align*}

For a stated tangle diagram $\alpha$ with state $s$ on the boundary edge $e$ define
\begin{align*}
C_{e}(\alpha) & = \prod_{x \in \alpha \cap e} C(s(x)) = \prod_{x \in \alpha \cap e} C_{s(x)}^{-s(x)} \quad (19)
\end{align*}

Proposition 2.7. There exists a unique $R$-linear homomorphism $\text{inv}_e : \mathcal{S}(\mathcal{G}) \to \mathcal{S}(\mathcal{G})$ such that if $\alpha$ is a stated $\partial \mathcal{G}$-tangle diagram whose height order on $e$ is given by the positive direction of $e$, then
\begin{align*}
\text{inv}_e([\alpha]) & = C_{e}(\alpha)[\alpha'] \quad (20)
\end{align*}
where $\alpha'$ is the same $\alpha$ except that the height order of $\alpha'$ on $e$ is given by the negative direction of $e$ and the state of $\alpha'$ on $e$ is obtained from that of $\alpha$ by switching $\nu \in \{\pm\}$ to $-\nu$ at every boundary points in $\alpha \cap e$.

If $e$ and $e'$ are two boundary edges, then

\[ \text{inv}_e \circ \text{inv}_{e'} = \text{inv}_{e'} \circ \text{inv}_e. \]

**Proof.** It is sufficient to check that relations (9) and (10) are preserved under $\text{inv}_e$. By (14) we have:

\[ \text{inv}_e \left( \begin{array}{c} \pi \\ \uparrow \end{array} \right) = C(+)C(-) \begin{array}{c} \pi \\ \uparrow \end{array} = -q^{-3} \begin{array}{c} \pi \\ \uparrow \end{array} = -q^{-3}(-q^3)C^+ = q^{-1/2}, \]

which proves the first identity of (9). The other two identities of (9) are trivial.

We now verify (10) using Lemma 2.4:

\[ \text{inv}_e \left( \begin{array}{c} \pi \\ \uparrow \end{array} \right) = C(+)C(-) \begin{array}{c} \pi \\ \uparrow \end{array} = -q^{-2} \begin{array}{c} \pi \\ \uparrow \end{array} \]

\[ \text{inv}_e \left( q^2 \begin{array}{c} \pi \\ \uparrow \end{array} \right) = q^2(-q^{-3}) \begin{array}{c} \pi \\ \uparrow \end{array} + q^{-\frac{1}{2}} \begin{array}{c} \pi \\ \uparrow \end{array} \]

\[ = -q^{-1} \left( q^{-3} \begin{array}{c} \pi \\ \uparrow \end{array} + q^{-\frac{3}{2}}(q^2 - q^{-2}) \begin{array}{c} \pi \\ \uparrow \end{array} + q^{-\frac{3}{2}} \begin{array}{c} \pi \\ \uparrow \end{array} \right) \]

\[ = -q^{-4} \begin{array}{c} \pi \\ \uparrow \end{array} + q^{-\frac{3}{2}} \begin{array}{c} \pi \\ \uparrow \end{array}, \]

and the right hand sides of (22) and (23) are equal due to (10).

Identity (21) follows immediately from the definitions. \(\square\)

Note that $\text{inv}_e$ is not an algebra homomorphism.

2.9. **Basis of stated skein module.** A $\partial M$-tangle diagram $D$ is *simple* if it has neither double point nor trivial component. Here a closed component of $D$ is *trivial* if it bounds a disk in $\mathcal{G}$, and an arc component of $\alpha$ is *trivial* if it can be homotoped relative to its boundary to a subset of a boundary edge. By convention, the empty set is considered as a simple stated $\partial M$-tangle diagram with 0 components.

Define an order on $\{\pm\}$ so that the sign $-$ is less than the sign $+$. If $X$ is a partially ordered set, then a state $s : X \rightarrow \{\pm\}$ is *increasing* if $s$ is an increasing function, i.e. $f(x) \leq f(y)$ whenever $x \leq y$.

Choose an orientation $\mathfrak{o}$ of $\partial \mathcal{G}$. Let $B(\mathcal{G}; \mathfrak{o})$ be the set of of all isotopy classes of increasingly stated, $\mathfrak{o}$-ordered simple $\partial M$-tangle diagrams. From the defining relations it is easy to show that the set $B(\mathcal{G}; \mathfrak{o})$ spans $\mathcal{H}(\mathcal{G})$ over $\mathcal{R}$.

**Theorem 2.8** (Theorem 2.8 in [Le2]). Suppose $\mathcal{G}$ is a punctured bordered surface and $\mathfrak{o}$ is an orientation of $\partial \mathcal{G}$. Then $B(\mathcal{G}; \mathfrak{o})$ is an $\mathcal{R}$-basis of $\mathcal{H}(\mathcal{G})$.

**Remark 2.9.** Theorem 2.8 means that the coefficients given in the defining relations (9) and (10) are consistent in the sense that they do not lead to any more relations among the set $B(\mathcal{G}; \mathfrak{o})$. 

The subset $\hat{B}(\mathcal{S};o) \subset B(\mathcal{S};o)$ consisting of $\alpha \in B(\mathcal{S};o)$ having no arcs is a basis of the ordinary skein algebra $\mathcal{J}(\mathcal{S})$. Similarly, the subset $B^+(\mathcal{S};o) \subset B(\mathcal{S};o)$ consisting of $\alpha \in B(\mathcal{S};o)$ having only positive states is a basis of the Muller skein algebra $\mathcal{J}^+(\mathcal{S})$, see [Mu, Le2, LP]. Hence we have the following.

**Corollary 2.10.** Both the ordinary skein algebra $\mathcal{J}(\mathcal{S})$ and the Muller skein algebra $\mathcal{J}^+(\mathcal{S})$ are subalgebras of the stated skein algebra $\mathcal{J}(\mathcal{S})$.

### 2.10. Filtration and grading.

Suppose $a$ is either an ideal arc or a simple closed curve on $\mathcal{S}$ and $\alpha$ is a simple $\partial M$-tangle diagram on $\mathcal{S}$. The geometric intersection index $I(a, \alpha)$ is

$$I(a, \alpha) = \min |a \cap \alpha|,$$

where the minimum is over all the simple $\partial M$-tangle diagrams $\alpha'$ isotopic to $\alpha$.

For a collection $\mathfrak{A} = \{a_1, \ldots, a_k\}$, where each $a_i$ is either an ideal arc or a simple closed curve, and $n \in \mathbb{N}$ let $F_n^\mathfrak{A}(\mathcal{J}(\mathcal{S}))$ be the $\mathcal{R}$-submodule of $\mathcal{J}(\mathcal{S})$ spanned by all stated simple $\partial M$-diagrams $\alpha$ such that $\sum_{i=1}^{k} I(a_i, \alpha) \leq n$. It is easy to see that the collection $\{F_n^\mathfrak{A}(\mathcal{J}(\mathcal{S}))\}_{n \in \mathbb{N}}$ forms a filtration of $\mathcal{J}(\mathcal{S})$ compatible with the algebra structure, i.e. with $F_n = F_n^\mathfrak{A}(\mathcal{J}(\mathcal{S}))$ one has

$$F_n \subset F_{n+1}, \bigcup_{n \in \mathbb{N}} F_n = \mathcal{J}(\mathcal{S}), F_n F_{n'} \subset F_{n+n'}.$$

One can define the associated graded algebra $\text{Gr}^\mathfrak{A}(\mathcal{J}(\mathcal{S}))$:

$$\text{Gr}^\mathfrak{A}(\mathcal{J}(\mathcal{S})) = \bigoplus_{n=0}^{\infty} \text{Gr}^\mathfrak{A}_n(\mathcal{J}(\mathcal{S})) \text{ with } \text{Gr}^\mathfrak{A}_n(\mathcal{J}(\mathcal{S})) = F_n/F_{n-1} \forall n \geq 1 \text{ and } \text{Gr}_0 = F_0.$$

This type of filtration has been used extensively in the theory of the ordinary skein algebra, see e.g. [Le1, FKL, LP, Marc].

The following is a consequence of Theorem 2.8:

**Proposition 2.11** (Proposition 2.12 in [Le2]). Let $o$ be an orientation of the boundary of a punctured bordered surface $\mathcal{S}$, and $\mathfrak{A} = \{a_1, \ldots, a_k\}$ be a collection of boundary edges of $\mathcal{S}$.

(a) The set $\{\alpha \in B(\mathcal{S};o) \mid \sum_{i=1}^{k} I(\alpha, a_i) \leq n\}$ is an $\mathcal{R}$-basis of $F_n^\mathfrak{A}(\mathcal{J}(\mathcal{S}))$.

(b) The set $\{\alpha \in B(\mathcal{S};o) \mid \sum_{i=1}^{k} I(\alpha, a_i) = n\}$ is an $\mathcal{R}$-basis of $\text{Gr}_n^\mathfrak{A}(\mathcal{J}(\mathcal{S}))$.

For what concerns the grading, for each non-negative integer $m$ and a boundary edge $e$ let $G^e_m$ be the $\mathcal{R}$-subspace of $\mathcal{J}(\mathcal{S})$ spanned by stated $\partial M$-tangle diagrams $\alpha$ with $\delta_e(\alpha) := \sum_{u \in (\partial M \cap e)} s(u) = m$, where $s$ is the state and we identify $+$ with $+1$ and $-$ with $-1$.

From the defining relations it is clear that $\mathcal{J}(\mathcal{S}) = \bigoplus_{m \in \mathbb{Z}} G^e_m$ and $G^e_m G^e_{m'} \subset G^e_{m+m'}$. In other words, $\mathcal{J}(\mathcal{S})$ is a graded algebra with the grading $\{G^e_m\}_{m \in \mathbb{Z}}$.

Also the following is a consequence of Theorem 2.8:

**Proposition 2.12.** Let $\mathcal{S}$ be a punctured bordered surface and $o$ be an orientation of $\partial \mathcal{S}$. The set $\{\alpha \in B(\mathcal{S};o) \mid \delta_e(\alpha) = m\}$ is an $\mathcal{R}$-basis of $G^e_m(\mathcal{J}(\mathcal{S}))$. 
If \( o' \) is another orientation of the boundary \( \partial \mathcal{S} \), the change from basis \( B(\mathcal{S}; o) \) to \( B(\mathcal{S}; o') \) might be complicated. For the associated space \( \text{Gr}^\mathfrak{a}_k(\mathcal{H}(\mathcal{S})) \), the change of bases is simpler.

Recall that \( \alpha \overset{\bullet}{=} \alpha' \) means \( \alpha = q^m \alpha' \) for some \( m \in \mathbb{Z} \).

**Proposition 2.13.** Suppose \( \alpha \) is stated tangle diagram on \( \mathcal{S} \) and \( I(\alpha, e) = k \) where \( e \) is a boundary edge. Let’s alter \( \alpha \) to get \( \alpha' \) by changing the height order on \( e \) and the states on \( e \) such that \( \delta_e(\alpha) = \delta_e(\alpha') \). Then one has

\[
(24) \quad \alpha \overset{\bullet}{=} \alpha' \quad \text{in} \quad \text{Gr}^\mathfrak{a}_k(\mathcal{H}(\mathcal{S})).
\]

**Proof.** One can get \( \alpha' \) from \( \alpha \) by a sequence of moves, each is either (i) an exchange of the heights of two consecutive vertices on \( e \), or (ii) an exchange of states of two consecutive vertices on \( b \). We can assume that \( \alpha' \) is the result of doing a move of type (i) or type (ii).

In case of move (i), the identities \((15) \) and \((16) \) prove \((24) \).

In case of move (ii), the identity \((10) \) prove \((24) \). \( \square \)

### 2.11. Splitting/Gluing punctured bordered surfaces.

Suppose \( a \) and \( b \) are distinct boundary edges of a punctured bordered surface \( \mathcal{S}' \) which may not be connected. Let \( \mathcal{S} = \mathcal{S}'/(a = b) \) be the result of gluing \( a \) and \( b \) together in such a way that the orientation is compatible. The canonical projection \( \text{pr} : \mathcal{S}' \to \mathcal{S} \) induces a projection \( \tilde{\text{pr}} : M = \mathcal{S}' \times (0, 1) \to M = \mathcal{S} \times (0, 1) \). Let \( c = \text{pr}(a) = \text{pr}(b) \), which is an interior ideal arc of \( \mathcal{S} \).

Conversely if \( c \) is an ideal arc in the interior of \( \mathcal{S} \), then there exists \( \mathcal{S}', a, b \) as above such that \( \mathcal{S} = \mathcal{S}'/(a = b) \), with \( c \) being the common image of \( a \) and \( b \). We say that \( \mathcal{S}' \) is the result of splitting \( \mathcal{S} \) along \( c \).

A \( \partial M \)-tangle \( \alpha \subset M = \mathcal{S} \times (0, 1) \), is said to be **vertically transverse to** \( c \) if

- \( \alpha \) is transverse to \( c \times (0, 1) \),
- the points in \( \partial_c \alpha := \alpha \cap (c \times (0, 1)) \) have distinct heights, and have vertical framing.

Suppose \( \alpha \) is a stated \( \partial M \)-tangle vertically transverse to \( c \). Then \( \tilde{\alpha} := \tilde{\text{pr}}^{-1}(\alpha) \) is a \( \partial M' \)-tangle which is stated at every boundary point except for newly created boundary points, which are points in \( \tilde{\text{pr}}^{-1}(\partial_c \alpha) \). A lift of \( \alpha \) is a stated \( \partial M' \)-tangle \( \beta \) which is \( \tilde{\alpha} \) equipped with states on \( \tilde{\text{pr}}^{-1}(\partial_c \alpha) \) such that if \( x, y \in \tilde{\text{pr}}^{-1}(\partial_c \alpha) \) with \( \tilde{\text{pr}}(x) = \tilde{\text{pr}}(y) \) then \( x \) and \( y \) have the same state. If \( |\partial_c \alpha| = k \), then \( \alpha \) has \( 2^k \) lifts.

**Theorem 2.14** (Splitting Theorem, Theorem 3.1 in \([Le2]\)). Suppose \( c \) is an ideal arc in the interior of a punctured bordered surface \( \mathcal{S} \) and \( \mathcal{S}' \) is the result of splitting \( \mathcal{S} \) along \( c \).

(a) There is a unique \( \mathcal{R} \)-algebra homomorphism \( \theta_c : \mathcal{H}(\mathcal{S}) \to \mathcal{H}(\mathcal{S}') \), called the splitting homomorphism along \( c \), such that if \( \alpha \) is a stated \( \partial M \)-tangle vertically transverse to \( c \), then

\[
(25) \quad \theta(\alpha) = \sum \beta,
\]

where the sum is over all lifts \( \beta \) of \( \alpha \).

(b) In addition, \( \theta_c \) is injective.

(c) If \( c_1 \) and \( c_2 \) are two non-intersecting ideal arcs in the interior of \( \mathcal{S} \), then

\[
\theta_{c_1} \circ \theta_{c_2} = \theta_{c_2} \circ \theta_{c_1}.
\]

**Remark 2.15.** The coefficients of the right hand sides of the defining relations \((9) \) and \((10) \) were chosen so that one has the consistency (see Remark 2.9) and the splitting theorem. It
can be shown if one requires the consistency and the splitting theorem, then the coefficients
are unique, up symmetries of a group isomorphic to \( \mathbb{Z}/2 \times \mathbb{Z}/2 \).

2.12. **Splitting homomorphism and filtration.** Fix an orientation \( o \) of the boundary
edges of \( \partial \mathcal{S} \). Let \( \mathcal{S}' \) be the result of splitting \( \mathcal{S} \) along an ideal arc \( c \), with \( c \) being lifted
to boundary edges \( a \) and \( b \) of \( \mathcal{S}' \). Choose an orientation of \( c \) and lift this orientation to \( a \) and \( b \) which, together with \( o \), gives an orientation \( o' \) for \( \mathcal{S}' \). Assume \( D \) is a stated simple \( o \)-ordered \( \partial M \)-tangle diagram which is
taut with respect to \( c \), i.e. \( |D \cap c| = I(\alpha, c) \). For each each \( i = 0, 1, \ldots, m := |D \cap c| \) let \( (\tilde{D}, s_i) \) be the \( \partial M' \)-tangle diagram where \( \tilde{D} = \text{pr}^{-1}(D) \),
and the states on both \( a \) and \( b \) are \( o' \)-increasing and having exactly \( i \) minus signs. Then each
\( (\tilde{D}, s_i) \) is in the basis of the free \( \mathcal{R} \)-module \( \text{Gr}_{2m}^{(a,b)}(\mathcal{S}'(\mathcal{S}')) \) described in Proposition 2.11.

For non-negative integers \( n, i \) the quantum binomial coefficient is defined by

\[
\binom{n}{i}_q = \frac{\prod_{j=n-i+1}^{n}(1 - q^j)}{\prod_{j=1}^{i}(1 - q^j)}.
\]

**Proposition 2.16.** In \( \text{Gr}_{2m}^{(a,b)}(\mathcal{S}'(\mathcal{S}')) \) one has

\[
\theta_c(D) = \sum_{i=0}^{m} \binom{m}{i}_q (\tilde{D}, s_i).
\]

**Proof.** For \( s : D \cap c \rightarrow \{ \pm \} \) let \( (\tilde{D}, s) \) be the stated \( o' \)-ordered \( \mathcal{S}' \)-tangle diagram with state
\( s \) on \( a \) and \( b \). By definition,

\[
\theta(D) = \sum_{s : D \cap c \rightarrow \{ \pm \}} (\tilde{D}, s) \in \mathcal{J}(\mathcal{S}').
\]

Taking into account the filtration, from relation (10), we see that in \( \text{Gr}_{2m}^{(a,b)}(\mathcal{S}'(\mathcal{S}')) \) we have

\[
\begin{array}{c}
\begin{array}{c}
|a| \\

\end{array}
\end{array}
= q^2 \begin{array}{c}
\begin{array}{c}
|a| \\

\end{array}
\end{array},
\begin{array}{c}
\begin{array}{c}
|b| \\

\end{array}
\end{array}
= q^2 \begin{array}{c}
\begin{array}{c}
|b| \\

\end{array}
\end{array}.
\]

It follows that \( \text{Gr}_{2m}^{(a,b)}(\mathcal{S}'(\mathcal{S}')) \) we have

\[
\begin{array}{c}
\begin{array}{c}
|a| \\

\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
|b| \\

\end{array}
\end{array}
= q^4 \begin{array}{c}
\begin{array}{c}
|a| \\

\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
|b| \\

\end{array}
\end{array}.
\]

Suppose \( s : D \cap c \rightarrow \{ \pm \} \) has \( i \) minus values. For \( k = 1, \ldots, i \) let \( x_k \) be the number of
plus state (of \( s \)) lying below the \( k \)-th minus state. By doing many switches, each changing
a pair of consecutive \((-+, +)\) to \((+,-)\), we can transform \( s \) into \( s_i \). The number of switches
is \( x_1 + \cdots + x_i \). Hence from (29) we see that

\[
(\tilde{D}, s) = q^{4(x_1+\cdots+x_i)}(\tilde{D}, s_i).
\]

Taking the sum over all \( s : D \cap c \rightarrow \{ \pm \} \) with \( i \) minus values, we get

\[
\theta(D) = \sum_{i=0}^{m} \left( \sum_{0 \leq x_1 \leq \cdots \leq x_i \leq m-i} q^{x_1+\cdots+x_i} \right) (\tilde{D}, s_i) \text{ in } \text{Gr}_{2m}^{(a,b)}(\mathcal{S}'(\mathcal{S}')).
\]
By induction on $i$ one can easily prove that
\begin{equation}
\sum_{0 \leq x_1 \leq \cdots \leq x_i \leq n} q^{x_1 + \cdots + x_i} = \binom{n+i}{i} q^i,
\end{equation}
from which and (30) we get (26). \qed

2.13. **The category of punctured bordered surfaces and the functor $\mathcal{S}$.** A morphism from one bordered punctured surface $\mathcal{S}$ to another one $\mathcal{S}'$ is an isotopy class of orientation-preserving embeddings from $\mathcal{S}$ to $\mathcal{S}'$. Here we assume that the embeddings map a boundary edge of $\mathcal{S}$ into (but not necessarily onto) a boundary edge of $\mathcal{S}'$.

Very often we confuse an embedding $f : \mathcal{S} \hookrightarrow \mathcal{S}'$ with its isotopy class.

Suppose $f : \mathcal{S} \to \mathcal{S}'$ is an embedding representing a morphism from $\mathcal{S}$ to $\mathcal{S}'$. Define an $\mathcal{R}$-linear homomorphism $f_* : \mathcal{S}(\mathcal{S}) \to \mathcal{S}(\mathcal{S}')$ such that if $\alpha$ is a stated tangle diagram on $\mathcal{S}$ with positive order then $f_*(\alpha)$ is given by the stated tangle diagram $f(\alpha)$, also with positive order. It is clear that $f_*$ is an $\mathcal{R}$-linear homomorphism, and it does not change under isotopy of $f$.

In general $f_*$ is not an $\mathcal{R}$-algebra homomorphism. However, if every edge of $\mathcal{S}'$ contains the image of at most one edge of $\mathcal{S}$, then $f_*$ is an $\mathcal{R}$-algebra homomorphism.

**Example 2.17.** Let $e$ be a boundary edge of $\mathcal{S}$ and $\mathcal{S}' = \mathcal{S} \setminus \{v\}$, where $v \in e$. The embedding $\iota : \mathcal{S}' \hookrightarrow \mathcal{S}$ induces an $\mathcal{R}$-linear homomorphism $\iota_* : \mathcal{S}(\mathcal{S}') \to \mathcal{S}(\mathcal{S})$ which is surjective but not injective in general.

Suppose $e' \subset e$ is one of the two boundary edges of $\mathcal{S}'$ which is part of $e$. There is a diffeomorphism $g : \mathcal{S} \to \mathcal{S}' \setminus \{e'\}$ which is unique up to isotopy. Thus, we have a morphism $f : \mathcal{S} \to \mathcal{S}'$, which is the composition
\[ \mathcal{S} \xrightarrow{g} \mathcal{S}' \setminus \{e'\} \hookrightarrow \mathcal{S}'. \]

The morphism $f$ induces an injective (but not surjective) algebra morphism $f_* : \mathcal{S}(\mathcal{S}) \hookrightarrow \mathcal{S}(\mathcal{S}')$.

3. **Hopf algebra structure of the bigon and $\mathcal{O}_{q^2}(\text{SL}(2))$**

We will define using geometric terms a cobraided Hopf algebra structure on the stated skein algebra $\mathcal{S}(\mathcal{B})$ of the bigon $\mathcal{B}$ and then show that it is naturally isomorphic to the cobraided Hopf algebra $\mathcal{O}_{q^2}(\text{SL}(2))$. We also show simple pictures of the canonical basis of $\mathcal{O}_{q^2}(\text{SL}(2))$, and discuss the Jones-Wenzl idempotents in $\mathcal{S}(\mathcal{B})$. In this section $\mathcal{R} = \mathbb{Z}[q \pm 1/2]$ unless otherwise stated.

3.1. **Monogon and Bigon.** Let $D$ be the standard disk
\[ D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \]
and $v_1 = (0, -1)$ and $v_2 = (0, 1)$ are two points on the circle $\partial D$. The punctured bordered surface $\mathcal{M} = D \setminus \{v_1\}$ is called the *monogon*, and $\mathcal{B} = D \setminus \{v_1, v_2\}$ is called the *bigon*. Let $e_l, e_r$ be the two boundary edges of $\mathcal{B}$ as depicted in Figure 6. For $\vec{\mu} = (\mu_1, \ldots, \mu_k)$ and $\vec{\eta} = (\eta_1, \ldots, \eta_k)$ in $\{\pm\}^k$ let $\alpha_{\vec{\eta}\vec{\mu}} \in \mathcal{S}(\mathcal{B})$ be the element presented by $k$ parallel arcs as in Figure 6, with $(\eta_1, \ldots, \eta_k)$ being the states on $e_l$ in increasing order and $(\mu_1, \ldots, \mu_k)$ being the states on $e_r$ in increasing order.
We have $\mathcal{S}(\mathcal{M}) = \mathcal{R}$. We study the algebra $\mathcal{S}(\mathcal{B})$ in this section.

Let $\text{rot} : \mathcal{B} \to \mathcal{B}$ be the rotation (of the plane containing $\mathcal{B}$) by $180^\circ$ about the center of $\mathcal{B}$, which is an isomorphism of $\mathcal{B}$ and induces an $\mathcal{R}$-algebra isomorphism $\text{rot}^* : \mathcal{S}(\mathcal{B}) \to \mathcal{S}(\mathcal{B})$.

3.2. Coproduct. Suppose $e$ is a boundary edge of a punctured bordered surface $\mathcal{S}$. Let $\mathcal{S}'$ be the result of cutting out of $\mathcal{S}$ a bigon $\mathcal{B}$ whose right edge $e_r$ is identified with $e$. Note that $\mathcal{S}'$ is canonically isomorphic to $\mathcal{S}$ in the category of punctured bordered surfaces. Hence we will identify $\mathcal{S}(\mathcal{S})$ with $\mathcal{S}(\mathcal{S}')$. The splitting homomorphism gives us an injective algebra homomorphism

$$\mathcal{S}(\mathcal{S}) \hookrightarrow \mathcal{S}(\mathcal{S} \sqcup \mathcal{B}) \equiv \mathcal{S}(\mathcal{S}) \otimes_\mathcal{R} \mathcal{S}(\mathcal{B}).$$

Since we identify $\mathcal{S}(\mathcal{S})$ with $\mathcal{S}(\mathcal{S}')$, this map becomes an $\mathcal{R}$-algebra homomorphism

$$\Delta_e : \mathcal{S}(\mathcal{S}) \hookrightarrow \mathcal{S}(\mathcal{S}) \otimes_\mathcal{R} \mathcal{S}(\mathcal{B}).$$

![Figure 7. The coaction $\Delta_e$.](image)

Suppose $x \in B(\mathcal{S}, o)$ is a basis element, where $o$ is a given orientation of $\partial \mathcal{S}$. Assume the state of $x$ on $e$ is $\bar{\mu}$, then we have (see Figure 7):

$$\Delta_e(x) = \sum_{\bar{\eta} \in S_{x \cap e}} x_{\bar{\eta}} \otimes \alpha_{\bar{\eta}\bar{\mu}},$$

where $S_{x \cap e}$ is the set of all states of $x \cap e$ and $x_{\bar{\eta}}$ is $x$ with the state on $e$ switched to $\bar{\eta}$.

In particular, when $\mathcal{S} = \mathcal{B}$ and $e = e_r$, we get an $\mathcal{R}$-algebra homomorphism $\Delta = \Delta_{e_r}$,

$$\Delta : \mathcal{S}(\mathcal{B}) \to \mathcal{S}(\mathcal{B}) \otimes_\mathcal{R} \mathcal{S}(\mathcal{B}).$$

Theorem 2.14(c) implies that $\Delta$ is coassociative, i.e.

$$\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta.$$
Applying (33) to $x = \alpha_{\nu\mu}$ with $\nu, \mu \in \{\pm\}$, we get
\begin{equation}
\Delta(\alpha_{\nu\mu}) = \sum_{\eta \in \{\pm\}} \alpha_{\eta\nu} \otimes \alpha_{\eta\mu}.
\end{equation}

3.3. Presentation of $\mathcal{S}(B)$. A presentation of the algebra $\mathcal{S}(B)$ was given in [Le2]. We give here a presentation of $\mathcal{S}(B)$ in a form which is suitable for us. Recall that $C(\eta) = C^\eta_\eta$ for $\eta \in \{\pm\}$ were defined by (11). We form the following matrix
\begin{equation}
C := \begin{pmatrix} C^+_+ & C^+_\pm \\ C^+_\pm & C^-_- \end{pmatrix} = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{-5/2} & 0 \end{pmatrix}
\end{equation}

**Lemma 3.1.** The $R$-algebra $\mathcal{S}(B)$ is generated by $\{\alpha_{\nu,\mu} \mid \nu, \mu \in \{\pm\}\}$ with the following relations:
\begin{align}
C^\nu_\mu \cdot 1 &= \sum_{\eta \in \{\pm\}} C(\bar{\eta}) \alpha_{\eta\nu} \alpha_{\eta\mu} = C^+_+ \alpha_{\nu+} \alpha_{\mu+} + C^+_\pm \alpha_{\nu+} \alpha_{\mu-} + C^-_- \alpha_{\nu-} \alpha_{\mu+} + C^-_+ \alpha_{\nu-} \alpha_{\mu-} \quad \forall \nu, \mu \in \{\pm\} \\
C^\mu_\nu \cdot 1 &= \sum_{\eta \in \{\pm\}} C(\bar{\eta}) \alpha_{\nu\eta} \alpha_{\mu\eta} = C^+_+ \alpha_{\nu+} \alpha_{\mu-} + C^+_\pm \alpha_{\nu+} \alpha_{\mu-} + C^-_- \alpha_{\nu-} \alpha_{\mu+} + C^-_+ \alpha_{\nu-} \alpha_{\mu-} \quad \forall \nu, \mu \in \{\pm\}.
\end{align}

Proof. Explicitly, the relations (38) and (39) are respectively
\begin{align}
C^\nu_\mu \cdot 1 &= \sum_{\eta \in \{\pm\}} C(\bar{\eta}) \alpha_{\eta\nu} \alpha_{\eta\mu} = C^+_+ \alpha_{\nu+} \alpha_{\mu+} + C^+_\pm \alpha_{\nu+} \alpha_{\mu-} + C^-_- \alpha_{\nu-} \alpha_{\mu+} + C^-_+ \alpha_{\nu-} \alpha_{\mu-} \quad \forall \nu, \mu \in \{\pm\} \\
C^\mu_\nu \cdot 1 &= \sum_{\eta \in \{\pm\}} C(\bar{\eta}) \alpha_{\nu\eta} \alpha_{\mu\eta} = C^+_+ \alpha_{\nu+} \alpha_{\mu-} + C^+_\pm \alpha_{\nu+} \alpha_{\mu-} + C^-_- \alpha_{\nu-} \alpha_{\mu+} + C^-_+ \alpha_{\nu-} \alpha_{\mu-} \quad \forall \nu, \mu \in \{\pm\}.
\end{align}

Let $e$ be the only boundary edge of the monogon $M$. Because $\mathcal{S}(M) = R$ and $R \otimes \mathcal{S}(B) = \mathcal{S}(B)$, the $R$-algebra map $\Delta_e : \mathcal{S}(M) \to \mathcal{S}(M) \otimes_R \mathcal{S}(B)$ is an $R$-algebra map $\Delta_e : R \to \mathcal{S}(B)$. As any $R$-linear map, we must have $\Delta_e(c) = c \cdot 1$, where 1 is the unit of $\mathcal{S}(B)$. Apply $\Delta_e$ to the simple arc in the monogon whose endpoints are stated by $\nu$ and $\mu$ and we get a proof of (40) as follows:

\begin{equation}
C^\nu_\mu = \sum_{\eta, \eta' \in \{\pm\}} C(\bar{\eta}) \alpha_{\eta\nu} \alpha_{\eta'\mu} = \sum_{\eta \in \{\pm\}} C(\bar{\eta}) \alpha_{\eta\nu} \alpha_{\eta\mu}.
\end{equation}

Equation (41) is obtained from Equation (40) by applying $\text{rot}_+$. Using Theorem 2.8 one sees that the set
\begin{equation}
B = \{\alpha_{++}^h + \alpha_{+-}^k \mid h, k \in \mathbb{N}\} \cup \{\alpha_{++}^h + \alpha_{--}^k \mid h, k \in \mathbb{N}, k \geq 1\}
\end{equation}
is an $R$-basis of $\mathcal{S}(B)$. In particular, $\mathcal{S}(B)$ is generated by $\alpha_{\nu\mu}$ with $\nu, \mu \in \{\pm\}$. Using these relations it is easy to check that any monomial in the $\alpha_{\nu\mu}$ can be expressed as an $R$-linear combinations of $B$. The proposition follows. \qed
3.4. **Counit.** The embedding \( \imath : \mathcal{B} \hookrightarrow \mathcal{M} \) gives rise to an \( \mathcal{R} \)-linear map \( \imath_* : \mathcal{S}(\mathcal{B}) \to \mathcal{S}(\mathcal{M}) = \mathcal{R} \). Define \( \epsilon : \mathcal{S}(\mathcal{B}) \to \mathcal{R} \) as the composition \( \epsilon = \imath_* \circ \text{Inv} \),

where \( \text{Inv} \) is defined in Section 2.8. Explicitly, if \( \alpha \) is a stated \( \partial \mathcal{B} \)-tangle diagram as in Figure 8, then

\[
\epsilon(\alpha) = C_{e_r}(\alpha) \alpha',
\]

where \( \alpha' \) is described in Figure 8 and \( C_{e_r}(\alpha) \) is defined by (19).

**Figure 8.** How to obtain \( \alpha' \) and \( \alpha'' \) from \( \alpha \) in the definition of counit and antipode. Height order is indicated by the arrows on the boundary edges. Then \( \alpha' \) is the same \( \alpha \), but considered as a tangle diagram in \( \mathcal{M} \) with its states on the edge \( e_r \) switched from \( \nu \) to \( \bar{\nu} = -\nu \). And \( \alpha'' \) is obtained from \( \alpha \) by a rotation of \( 180^\circ \), and switching all the states \( \nu \) to \( \bar{\nu} \).

Using (43) and the values of \( C(\eta) \), one can check that

\[
\epsilon(\alpha_{\nu \mu}) = \delta_{\nu \mu} := \begin{cases} 1 & \text{if } \nu = \mu \\ 0 & \text{if } \nu \neq \mu. \end{cases}
\]

**Proposition 3.2.** The algebra \( \mathcal{S}(\mathcal{B}) \) is a bialgebra with counit \( \epsilon \) and coproduct \( \Delta \).

**Proof.** We already saw that \( \Delta \) is an algebra homomorphism and is associative. It remains to show that \( \epsilon \) is an algebra homomorphism, and

\[
(\epsilon \otimes \text{id}) \circ \Delta(x) = x = (\text{id} \otimes \epsilon) \circ \Delta(x).
\]

Let \( x, y \in \mathcal{S}(\mathcal{B}) \) be presented by tangle diagrams schematically depicted as in Figure 9.

**Figure 9.** Elements \( x, y \in \mathcal{S}(\mathcal{B}) \). Each horizontal strand stands for several horizontal lines which are tangled in the two small disks.
The following shows that $\epsilon(xy) = \epsilon(x)\epsilon(y)$, i.e. $\epsilon$ is an algebra homomorphism:

$$
\epsilon(xy) = \epsilon \left( \begin{array}{c}
C_{e_r}(x) C_{e_r}(y)
\end{array} \right) = \epsilon(y) C_{e_r}(x) = \epsilon(y)\epsilon(x).
$$

As both $\Delta$ and $\epsilon$ are algebra homomorphisms, one only needs to check (45) for the generators $x = \alpha_{\nu\mu}$ with $\nu, \mu \in \{\pm\}$. Using (36) and (44), we have

$$(\epsilon \otimes \text{id}) \circ \Delta(\alpha_{\nu\mu}) = (\epsilon \otimes \text{id}) \sum_\eta \alpha_{\nu\eta} \otimes \alpha_{\eta\mu} = \alpha_{\nu\mu},$$

which proves the first identity of (45). The other identity is proved similarly. \hfill \Box

3.5. **Antipode.** Define $S : \mathcal{H}(B) \to \mathcal{H}(B)$, by $S := \text{rot}_r \circ (\text{inv}_{e_r} \circ (\text{inv}_{e_l})^{-1})$. Explicitly, if $\alpha$ is a stated $\partial B$-tangle diagram as in Figure 8, then

$$S(\alpha) = \frac{C_{e_r}(\alpha)}{C_{e_l}(\alpha)} \alpha'',$$

where $\alpha''$ is described in Figure 8. In particular, we have

$$S(\alpha_{\nu\mu}) = \frac{C(\mu)}{C(\nu)} \alpha_{\overline{\nu}\overline{\mu}}.$$

Explicitly,

$$S(\alpha_{++}) = \alpha_{--}, \ S(\alpha_{-+}) = \alpha_{++}, \ S(\alpha_{+--}) = -q^2 \alpha_{++}, \ S(\alpha_{--}) = -q^{-2} \alpha_{--}.$$

**Proposition 3.3.** The map $S$ is an antipode of the bialgebra $\mathcal{H}(B)$, making $\mathcal{H}(B)$ a Hopf algebra.

**Proof.** From the definition (46) one sees that $S$ is an anti-homomorphism, i.e.

$$S(xy) = S(y)S(x).$$

It remains to check the following property of an antipode:

$$\sum S(x')x'' = \epsilon(x)1 = \sum x'S(x''),$$

where we use the Sweedler’s notation for the coproduct $\Delta x = \sum x' \otimes x''$. Since $S$ is an anti-homomorphism and $\epsilon$ is an algebra homomorphism, it is enough to check (49) for generators $x = \alpha_{\nu\mu}$. In that case, using (36) we have $\Delta(x) = \Delta(\alpha_{\nu\mu}) = \sum_\eta \alpha_{\nu\eta} \otimes \alpha_{\eta\mu}$, and

$$\sum S(x')x'' = \sum_{\eta \in \{\pm\}} S(\alpha_{\nu\eta}) \alpha_{\eta\mu}
= \sum_{\eta \in \{\pm\}} \frac{C(\eta)}{C(\nu)} \alpha_{\overline{\eta}\overline{\mu}} \alpha_{\eta\mu} \quad \text{by (47)}
= \frac{C(\nu)}{C(\nu)} \cdot 1 \quad \text{by (40)}
= \delta_{\nu\mu} \cdot 1 = \epsilon(\alpha_{\nu\mu}) \cdot 1 \quad \text{by definition of } C(\nu) \text{ and (44)},$$

which proves the first identity of (47). The second identity of (47) is proved similarly. \hfill \Box
3.6. **Quantum algebra** $\mathcal{O}_{q^2}(\text{SL}(2))$. Let us recall the definition of the quantum coordinate ring $\mathcal{O}_{q^2}(\text{SL}(2))$ of $\text{SL}_2(\mathbb{C})$, which is the Hopf dual of the quantum group $U_{q^2}(\mathfrak{sl}_2)$. See [Maj].

**Definition 1** ($\mathcal{O}_{q^2}(\text{SL}(2))$). The Hopf algebra $\mathcal{O}_{q^2}(\text{SL}(2))$ is the $\mathcal{R}$-algebra generated by $a, b, c, d$ with relations

\[(50) \quad ca = q^2ac, \quad db = q^2bd, \quad ba = q^2ab, \quad dc = q^2cd,\]
\[(51) \quad bc = cb, \quad ad - q^{-2}bc = 1 \quad \text{and} \quad da - q^2cb = 1.\]

Its coproduct structure is given by

\[\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \quad \Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d.\]

Its counit is defined as $\epsilon(a) = \epsilon(d) = 1, \epsilon(b) = \epsilon(c) = 0$ and its antipode is defined by $S(a) = d, S(d) = a, S(b) = -q^2b, S(c) = -q^{-2}c$.

**Theorem 3.4.** There exists a Hopf algebra isomorphism $\phi : \mathcal{I}(\mathcal{B}) \to \mathcal{O}_{q^2}(\text{SL}(2))$ given on the generators by

\[(52) \quad \phi(\alpha_{+,+}) = a, \quad \phi(\alpha_{+,-}) = b, \quad \phi(\alpha_{-,+}) = c, \quad \phi(\alpha_{-,-}) = d.\]

Furthermore, let $\text{rot} : \mathcal{B} \to \mathcal{B}$ be the rotation of $180^\circ$ around the center of the bigon w.r.t. the axis orthogonal to the bigon, and $r : \mathcal{O}_{q^2}(\text{SL}(2)) \to \mathcal{O}_{q^2}(\text{SL}(2))$ be $r = \phi \circ \text{rot} \circ \phi^{-1}$. Then $r$ is the algebra automorphism such that $r(a) = a, r(b) = c, r(c) = b, r(d) = d$ and $(r \otimes r) \circ \Delta^{\text{cot}} = \Delta \circ r$.

**Proof.** Let $\phi$ be the map given by (52). We need to show that $\phi$ respects the relations (50) and (51).

Identity (40) with $(\nu, \mu) = (-,-)$ and $(\nu, \mu) = (+,+)$, and Identity (41) with $(\nu, \mu) = (-, -)$ and $(\nu, \mu) = (+,+)$, are respectively the 4 identities of Relation (50) (under $\phi$).

By subtracting the identity (41) from (40) with $(\nu, \mu) = (-,+) \quad \text{in both, we get the first identity of (51) (under $\phi$).}$

Identity (40) with $(\nu, \mu) = (+,-)$ and Identity (41) with $(\nu, \mu) = (-,+)$ are respectively the second and the third identities of (51).

Thus $\phi$ respects all the defining relations of $\mathcal{O}_{q^2}(\text{SL}(2))$ and gives rise to an $\mathcal{R}$-algebra homomorphism from $\mathcal{I}(\mathcal{B})$ to $\mathcal{O}_{q^2}(\text{SL}(2))$.

Recall that the set $B$ given in (42) is an $\mathcal{R}$-basis of $\mathcal{I}(\mathcal{B})$. It is well known that the image of $B$ under $\phi$ is an $\mathcal{R}$-basis of $\mathcal{O}_{q^2}(\text{SL}(2))$. Hence $\phi$ is an isomorphism.

To check that $\phi$ is a Hopf algebra isomorphism it is sufficient to check this on the level of generators where it is straightforward.

The last statement is a direct verification. \[\square\]

3.7. **Geometric depiction of co-$R$-matrix, a lift of the co-$R$-matrix.** The Hopf algebra $\mathcal{O}_{q^2}(\text{SL}(2))$ is cobraided, i.e. it has a co-$R$-matrix with the help of which one can make the category of $\mathcal{O}_{q^2}(\text{SL}(2))$-modules a braided category. Formally, a co-$R$-matrix is a bilinear form

\[\rho : \mathcal{O}_{q^2}(\text{SL}(2)) \otimes \mathcal{O}_{q^2}(\text{SL}(2)) \to \mathcal{R}\]
such that there exists another bilinear form $\bar{\rho} : O_q^2(\text{SL}(2)) \otimes O_q^2(\text{SL}(2)) \to \mathcal{R}$ (the “inverse” of $\rho$) satisfying for any $x, y, z \in U$,

\begin{align}
(53) & \sum \rho(x' \otimes y') \bar{\rho}(x'' \otimes y'') = \sum \bar{\rho}(x' \otimes y') \rho(x'' \otimes y'') = \epsilon(x)\epsilon(y) \\
(54) & yx = \sum \rho(x' \otimes y') x'' y'' \bar{\rho}(x''' \otimes y''') \\
(55) & \rho(xy \otimes z) = \sum \rho(x' \otimes z') \rho(y'' \otimes z'') \epsilon(x'') \epsilon(y') \\
(56) & \rho(x \otimes yz) = \sum \rho(x' \otimes z') \rho(x'' \otimes y'') \epsilon(z'') \epsilon(y')
\end{align}

Here we use Sweedler’s notation for the coproduct. Relations (55) and (56) show that $\rho$ is totally determined by its values at a set of generators of the algebra $O_q^2(\text{SL}(2))$, and the values $r$ at a set of generators are given by

\begin{equation}
(57) \quad \rho \begin{pmatrix} a \otimes a & b \otimes b & a \otimes b & b \otimes a \\ c \otimes c & d \otimes d & c \otimes d & d \otimes c \\ a \otimes c & b \otimes d & a \otimes b & b \otimes c \\ c \otimes a & d \otimes b & c \otimes b & d \otimes a \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^{-1} & q - q^{-3} \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}
\end{equation}

**Theorem 3.5.** Under the identification of $\mathcal{S}(B)$ and $O_q^2(\text{SL}(2))$ via the isomorphism $\phi$, the co-$R$-matrix $\rho$ of $O_q^2(\text{SL}(2))$ has the following geometric description

\begin{equation}
(58) \quad \rho \begin{pmatrix} \hline a & b \\ \hline c & d \\ \hline a & b \end{pmatrix} = \epsilon \begin{pmatrix} \hline x & y \\ \hline x' & y' \\ \hline x'' & y'' \end{pmatrix}
\end{equation}

**Proof.** Let $\rho'$ be the map defined by the r.h.s. of (58): we will show that $\rho' = \rho$. For this it is enough to show that $\rho'$ satisfy (55), (56), and the initial values identity (57), all with $\rho$ replaced by $\rho'$. We have

\begin{align}
\rho'(xy \otimes z) &= \epsilon \begin{pmatrix} \hline z \\ \hline x \end{pmatrix} = \sum \epsilon \begin{pmatrix} \hline z \\ \hline x \end{pmatrix} \epsilon \begin{pmatrix} \hline y' \\
\hline y'' \end{pmatrix} \text{ because } \epsilon(u) = \sum \epsilon(u')\epsilon(u'') \\
&= \sum r(x' \otimes z') r(y'' \otimes z'') \epsilon(x'') \epsilon(y').
\end{align}

This proves (55) for $\rho'$. The proof of (56) is similar.

To check (57) we have to check 16 identities, all of which are easy. For example, the most difficult one is the identity of the $(3, 4)$ entries:

\begin{align}
\rho'(b \otimes c) &= \epsilon \begin{pmatrix} \hline b \\ \hline c \\ \hline a \end{pmatrix} = q \epsilon \begin{pmatrix} \hline b \\ \hline c \\ \hline a \end{pmatrix} + q^{-1} \epsilon \begin{pmatrix} \hline b \\ \hline c \\ \hline a \end{pmatrix} \\
&= q - q^{-3} \text{ by (44) and (14)}.
\end{align}

This proves (57) for the $(3, 4)$ entries. Identity (57) for other entries are similar. \qed
3.8. The Jones-Wenzl idempotents as elements of the bigon algebra. In this subsection we will work over the ring $\mathcal{R}^{\text{loc}}$ obtained by localizing $\mathcal{R}$ over the multiplicative set generated by $\{[n] = \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}, n \geq 1\}$ (or if the reader prefers, he can work over the field of fractions of $\mathcal{R}$). Recall that the $n^{\text{th}}$ Temperley-Lieb algebra $T_n$ is the finite dimensional $\mathcal{R}^{\text{loc}}$-algebra generated by non-stated $(n,n)$-planar diagrams in $\mathcal{B}$ modulo isotopy (rel to the boundary) and relation (8). The product structure is induced by gluing two copies of the bigon $\mathcal{B}$ and $\mathcal{B}'$ by identifying the edges $e_l$ and $e_r'$ (see Figure 6 for the notation). A set of generators as an algebra is obtained by considering the $(n,n)$-diagrams $c_i$ containing $i - 1$ horizontal strands, two $C$-shaped arcs connecting the $i^{\text{th}}$ and the $(i + 1)^{\text{th}}$ boundary components in $e_l$ and in $e_r$, and $n - i - 1$ horizontal strands (see Figure 10).

![Figure 10](image-url)

**Figure 10.** On the left the unit of $T_3$. On the right the element $c_2$ of $T_3$

A $\mathcal{R}^{\text{loc}}$-basis of $T_n$ is given by planar diagrams without closed components; define $\epsilon : T_n \to \mathcal{R}^{\text{loc}}$ be the dual of the element 1 with respect to this basis. The $n^{\text{th}}$ Jones-Wenzl idempotent is an element $JW_n \in T_n$ defined by recursion as explained in Figure 11.

![Figure 11](image-url)

**Figure 11.** The recursion relation for $JW_n \in T_n$. By definition $JW_1 = 1 \in T_1$.

The following is the key property of the $JW_n$:

**Proposition 3.6.** It holds $JW_n x = xJW_n = \epsilon(x)JW_n \forall x \in T_n$. In particular $JW_n^2 = JW_n$.

The following definition is well-posed because of relation (8):

**Definition 2.** Let $\vec{\mu}_l, \vec{\mu}_r$ be two non-necessarily increasing states on $\mathcal{B}$ each having $n$ signs and let $T_n(\vec{\mu}_l, \vec{\mu}_r) : T_n \to \mathcal{J}(\mathcal{B})$ be the $\mathcal{R}^{\text{loc}}$-linear map obtained by endowing each element of $T_n$ with states $\vec{\mu}_l$ and $\vec{\mu}_r$ along $e_l$ and $e_r$ respectively. In particular we let $JW(\vec{\mu}_l, \vec{\mu}_r) := T_n(\vec{\mu}_l, \vec{\mu}_r)(JW_n)$.

**Example 3.7.** Let $\vec{\epsilon}$ or $\vec{\eta}$ be of the form $++ + + + +$ or $-- -- -- --$ (i.e. they contain all equal signs). Then the skein $JW_n(\vec{\epsilon}, \vec{\eta})$ is represented by $n$ parallel strands with states $\vec{\epsilon}, \vec{\eta}$. Indeed any other term in the linear combination of diagrams representing $JW_n$ contains an arc whose endpoints are both in $e_l$ or in $e_r$ and thus are stated equally, thus it is 0 in $\mathcal{J}(\mathcal{B})$. 

Proposition 3.8. The following hold in $S(B)$:

1. \( JW(\bar{\mu}_l, \bar{\mu}_r) = q^2 n_0(\bar{\mu}_l) JW(o(\bar{\mu}_l), o(\bar{\mu}_r)) \) where \( o(\bar{\mu}) \) is obtained by reordering increasingly the states of \( \bar{\mu} \) and \( n_0(\bar{\mu}) \) is the minimal number of exchanges needed to do so.

2. \( \Delta(JW(\bar{\mu}_l, \bar{\mu}_r)) = \sum_{j=0}^{n} \binom{n}{j} q^{4j} JW(\bar{\eta}_j, \bar{\mu}_r) \otimes JW(\bar{\eta}_j, \bar{\mu}_l) \)

where \( \bar{\eta}_j \) is the increasing state containing \( j \) signs + and \( n-j \) signs −.

Proof. Observe that if one exchanges a sign − and a + which are not in the increasing order along \( e_r \) then by (10) one gets \( q^2 \) times the reordered term and \( q^2 \) times a term killed by \( JW_n \). A similar argument (using Lemma 2.4) shows that each reordering along \( e_l \) multiplies \( JW \) by \( q^2 \). The first statement follows. The second statement is a consequence of the fact that in \( T_n \) it holds \( JW^2 = JW \) and of Proposition 2.16. \( \square \)

3.9. Kashiwara’s basis for \( O_{q^2}(SL(2)) \). Recall that for \( \bar{\eta}, \bar{\mu} \in \{\pm\}^k \) the elements \( \alpha_{\bar{\eta}\bar{\mu}} \in S(B) \) is described in Figure 12. Let \( B_{\text{can}} \) be the set of all elements \( \alpha_{\bar{\eta}\bar{\mu}} \) where \( \bar{\eta} \) is decreasing and \( \bar{\mu} \) is increasing. Then \( B_{\text{can}} \) is an \( R \)-basis of \( S(B) \).

![Figure 12. Element \( \alpha_{\bar{\eta}\bar{\mu}} \)](image-url)

Proposition 3.9. Via the isomorphism of Theorem 3.4, the basis \( B_{\text{can}} \) coincides with the canonical basis defined by Kashiwara in Proposition 9.1.1 of [Ka]. In particular it is positive with respect to the product and to the coproduct, i.e.:

\[ \forall \alpha, \beta \in B_{\text{can}}, \alpha \cdot \beta \in \mathbb{N}[q^{\pm 1}] \cdot B_{\text{can}} \]

\[ \forall \alpha \in B_{\text{can}}, \Delta(\alpha) \in \mathbb{N}[q^{\pm 1}] \cdot B_{\text{can}} \otimes B_{\text{can}}. \]

Proof. The first statement is an observation directly following Theorem 3.4: in Proposition 9.1.1 of [Ka] the basis is \( \{c^l a^m b^n, l, m, n \geq 0\} \sqcup \{c^l d^m b^n, l, n \geq 0, m > 0\} \). For positivity of multiplication, it is sufficient to check it on pairs of generators: there are then 16 cases. All of them are straightforward; we provide here some instances among the most complicated cases here below where the r.h.s. are elements of \( B_{\text{can}} \):

\[ \alpha_{+-} \cdot \alpha_{-+} = \alpha_{-+} \cdot \alpha_{+-}, \alpha_{++} \cdot \alpha_{--} = q^{-2} \alpha_{--} \cdot \alpha_{++}, \alpha_{--} \cdot \alpha_{++} = q^2 \alpha_{+-} \cdot \alpha_{-+} \]

\[ \alpha_{++} \cdot \alpha_{++} = q^2 \alpha_{++} \cdot \alpha_{++}, \alpha_{++} \cdot \alpha_{-+} = q^{-2} \alpha_{--} \cdot \alpha_{++}, \alpha_{++} \cdot \alpha_{--} = q^{-2} \alpha_{++} \cdot \alpha_{--}. \]

Once positivity is known for multiplication, the statement for comultiplication can be checked on generators where it is straightforward. \( \square \)

Remark 3.10. A direct proof of positivity using pictures is also easy and left to the reader.
4. Comodule structures, co-tensor products and braided tensor products

In this section we show that given any edge of $\mathcal{S}$ the skein algebra $\mathcal{I}(\mathcal{S})$ has a natural structure of $\mathcal{O}_q(SL(2))$-comodule algebra. We show how to decompose this comodule into finite dimensional comodules. We then identify the image of the splitting homomorphism using the Hochshild cohomology, and give a dual result using Hochshild homology. When $\mathcal{S}$ is the result of gluing two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ to two edges of an ideal triangle, we show that the skein algebra $\mathcal{I}(\mathcal{S})$ is canonically isomorphic to the braided tensor product of $\mathcal{I}(\mathcal{S}_1)$ and $\mathcal{I}(\mathcal{S}_2)$. In this section $\mathcal{R} = \mathbb{Z}[q^{\pm 1/2}]$.

4.1. Comodule. Suppose $e$ is a boundary edge of a punctured bordered surface $\mathcal{S}$. Recall that by cutting out of $\mathcal{S}$ a bigon $\mathcal{B}$ whose right edge is $e$ and canonically identifying $\mathcal{S} \setminus \text{int}(\mathcal{B})$ with $\mathcal{S}$, we get an $\mathcal{R}$-linear map

$$\Delta_e : \mathcal{I}(\mathcal{S}) \to \mathcal{I}(\mathcal{S}) \otimes \mathcal{I}(\mathcal{B}),$$

see Figure 7. Similarly cutting out of $\mathcal{S}$ a bigon $\mathcal{B}$ whose left edge is $e$ and canonically identifying $\mathcal{S} \setminus \text{int}(\mathcal{B})$ with $\mathcal{S}$, we get an $\mathcal{R}$-linear algebra homomorphism

$$\epsilon \Delta : \mathcal{I}(\mathcal{S}) \to \mathcal{I}(\mathcal{B} \sqcup \mathcal{S}) \equiv \mathcal{I}(\mathcal{B}) \otimes \mathcal{I}(\mathcal{S}).$$

**Proposition 4.1.** (a) The map $\Delta_e : \mathcal{I}(\mathcal{S}) \to \mathcal{I}(\mathcal{S}) \otimes \mathcal{I}(\mathcal{B})$ gives $\mathcal{I}(\mathcal{S})$ a right comodule-algebra structure over the Hopf algebra $\mathcal{I}(\mathcal{B})$. Similarly $\epsilon \Delta$ gives $\mathcal{I}(\mathcal{S})$ a left comodule-algebra structure over the Hopf algebra $\mathcal{I}(\mathcal{B})$.

(b) It holds $\Delta = \text{fl} \circ (\text{id}_{\mathcal{I}(\mathcal{S})} \otimes \text{rot}_*) \circ \Delta_e$ where $\text{fl}(x \otimes y) = y \otimes x$ and $\text{rot}_* : \mathcal{I}(\mathcal{B}) \to \mathcal{I}(\mathcal{B})$ is the algebra involution induced by the rotation of $\pi$ around the center of the bigon (as in Theorem 3.4).

(c) If $e_1, e_2$ are two distinct boundary edges, the coactions on the two edges commute, i.e. for instance

$$(\Delta_{e_2} \otimes \text{id}) \circ \Delta_{e_1} = (\Delta_{e_1} \otimes \text{id}) \circ \Delta_{e_2}.$$  

**Proof.** (a) The associativity of $\Delta_e$ follows from the the commutativity of the splitting maps. Applying $(\text{id} \otimes \epsilon)$ to Equation (33) and using the value of $\epsilon(a_{1m})$ from (44), we get that

$$(\text{id} \otimes \epsilon)\Delta_e(x) = x_{12} = x, \quad \forall x \in B(\mathcal{S}, o).$$

Hence $\Delta_e$ gives $\mathcal{I}(\mathcal{S})$ the structure of a right $\mathcal{I}(\mathcal{B})$-comodule.

Recall that $\mathcal{I}(\mathcal{S})$ is a comodule-algebra over the bialgebra $\mathcal{I}(\mathcal{B})$, see e.g. [Kass, Proposition III.7.2], if and only if the map $\Delta_e : \mathcal{I}(\mathcal{S}) \to \mathcal{I}(\mathcal{S}) \otimes \mathcal{I}(\mathcal{B})$ is an algebra homomorphism. The last fact follows easily from the definition of $\Delta_e$.

(b) Observe that $(\text{rot}_* \otimes \text{rot}_*) \circ \Delta_{mp} = \Delta \circ \text{rot}_*$.

(c) is clear from the definition. \hfill \square

By identifying $\mathcal{I}(\mathcal{B})$ with $\mathcal{O}_q^*(SL(2))$ using Theorem 3.4, the above proposition also provides $\mathcal{I}(\mathcal{S})$ with the structure of a $\mathcal{O}_q^*(SL(2))$-comodule. More in general, we will use the following terminology:

**Definition 3** (Surfaces with indexed boundary). A punctured bordered surface $\mathcal{S}$ has indexed boundary if its boundary edges are partitioned into two ordered sets (the left and right ones, with indices $L$ and $R$ respectively): $e_1^L, \ldots, e_n^L, e_1^R, \ldots, e_m^R$. 

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If \( \mathcal{S} \) has indexed boundary then \( \mathcal{S}(\mathcal{S}) \) is naturally endowed with a structure of
\[
(\mathcal{O}_{q^2}(SL(2))^{\otimes n}, \mathcal{O}_{q^2}(SL(2))^{\otimes m}) - \text{bicomodule}
\]
by the left coaction \( \Delta^L : \mathcal{S}(\mathcal{S}) \rightarrow \mathcal{O}_{q^2}(SL(2))^{\otimes n} \otimes \mathcal{S}(\mathcal{S}) \) defined by
\[
\Delta^L := (\text{Id}_{\mathcal{O}_{q^2}(SL(2))}^{\otimes n-1} \otimes e_n^L \Delta) \circ (\text{Id}_{\mathcal{O}_{q^2}(SL(2))}^{\otimes n-2} \otimes e_{n-1}^L \Delta) \circ \cdots \circ (\text{Id}_{\mathcal{O}_{q^2}(SL(2))} \otimes e_1^L \Delta) \circ e_1^L
\]
and the right coaction
\[
\Delta^R := (\Delta e_1^R \otimes \text{Id}_{\mathcal{O}_{q^2}(SL(2))}^{\otimes m-1}) \circ (\Delta e_2^R \otimes \text{Id}_{\mathcal{O}_{q^2}(SL(2))}^{\otimes m-2}) \circ \cdots \circ (\Delta e_{m-1}^R \otimes \text{Id}_{\mathcal{O}_{q^2}(SL(2))}) \circ \Delta e_1^R.
\]
Furthermore notice that \( \mathcal{S}(\mathcal{S}) \) is not only a bicomodule but a bicomodule-algebra as each of the above maps \( \Delta e^R \) or \( e^L \Delta \) are also morphisms of algebras.

### 4.2. Quantum group \( U_{q^2}(\mathfrak{sl}_2) \)

Recall that the quantized enveloping algebra \( U_{q^2}(\mathfrak{sl}_2) \) is the Hopf algebra generated over the field \( \mathbb{Q}(q^{1/2}) \) by \( K, E, F \) with relations
\[
KE = q^4 EK, \quad KF = q^{-4} FK, \quad [E, F] = \frac{K - K^{-1}}{q^2 - q^{-2}}.
\]
The coproduct and the antipode are given by
\[
\Delta(K) = K \otimes K, \quad \Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1
\]
\[
S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.
\]

We emphasize that \( U_{q^2}(\mathfrak{sl}_2) \) is defined over the field \( \mathbb{Q}(q^{1/2}) \). There is an integral version \( U_{q^2}^I(\mathfrak{sl}_2) \), defined by Lusztig [Lus], which is the \( \mathcal{R} \)-subalgebra of \( U_{q^2}(\mathfrak{sl}_2) \) generated by \( K^{\pm 1} \) and the divided powers \( E^{(r)} := E^{r \leftarrow [r]}, F^{(r)} := F^{r \leftarrow [r]} \). Here \( [r] = \prod_{i=1}^{r} (q^{2i} - q^{-2i})/(q^2 - q^{-2}) \).

One has a non degenerate Hopf pairing
\[
\langle \cdot, \cdot \rangle : U_{q^2}(\mathfrak{sl}_2) \otimes_{\mathcal{R}} \mathcal{O}_{q^2}(SL(2)) \rightarrow \mathbb{Q}(q^{1/2}).
\]
This is a Hopf duality since it satisfies (with Sweedler’s coproduct notation)
\[
\langle x, y_1 y_2 \rangle = \sum \langle x', y_1 \rangle \langle x'', y_2 \rangle, \quad \langle x_1 x_2, y \rangle = \sum \langle x_1, y' \rangle \langle x_2, y'' \rangle
\]
The values of the form on generators are given by
\[
\langle K, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle = \begin{pmatrix} q^2 & 0 \\ 0 & q^{-2} \end{pmatrix}, \quad \langle E, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \langle F, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

**Lemma 4.2** (Integrality of \( \langle \cdot, \cdot \rangle \)). The form (62) on \( U_{q^2}^I(\mathfrak{sl}_2) \) is integral, i.e. its restricts to a map
\[
U_{q^2}^I(\mathfrak{sl}_2) \otimes_{\mathcal{R}} \mathcal{O}_{q^2}(SL(2)) \rightarrow \mathcal{R} = \mathbb{Z}[q^{\pm 1/2}].
\]

**Proof.** It is enough to check that \( \forall r \geq 1, \forall x \in \mathcal{O}_{q^2}(SL(2)) \) it holds \( \langle E^{(r)}, x \rangle, \langle F^{(r)}, x \rangle \in \mathcal{R} \). Since \( \Delta(E^{(r)}) = \sum_{i=0}^{r} q^{2(r-i)} E^{(i)} \otimes E^{(r-i)} K^i \) it is sufficient to check the statement for the evaluations of \( E^{(i)} \) and \( K^j \) on \( a, b, c, d \) where this is a straightforward computation. Similarly for \( F^{(r)} \).

**Lemma 4.3** (Left and right modules). The following hold:
(1) Let \( r^* : U_q(\mathfrak{sl}_2) \to U_q^1(\mathfrak{sl}_2) \) be the adjoint of the map \( r : \mathcal{O}_q(\text{SL}(2)) \to \mathcal{O}_q^2(\text{SL}(2)) \) defined in Theorem 3.4. Then \( r^* \) is an algebra antihomorphism involution and a coalgebra morphism, i.e. \( \forall x, y \in U_q(\mathfrak{sl}_2) \) it holds:

\[
(r^*)^2(x) = x \quad r^*(xy) = r^*(y)r^*(x) \quad \Delta(r^*(x)) = (r^* \otimes r^*) \circ \Delta(x).
\]

Explicitly, the value of \( r^* \) on the generators is

\[
r^*(E) = q^2K, \quad r^*(K) = K, \quad r^*(F) = q^{-2}EK^{-1}.
\]

(2) \( r^* \) induces monoidal functors

\[ LR : U_q^2(\mathfrak{sl}_2) - \text{Mod} \to \text{Mod} - U_q^2(\mathfrak{sl}_2), \quad \text{and} \quad RL : \text{Mod} - U_q^2(\mathfrak{sl}_2) \to U_q^2(\mathfrak{sl}_2) - \text{Mod} \]

which are inverse to each other as follows: for each left (resp. right) module \( M \) then \( LR(M) \) (resp. \( RL(M) \)) is the right (resp. left) module whose underlying vector space is \( M \) and on which the action of \( x \in U_q^2(\mathfrak{sl}_2) \) is given by

\[
\forall \alpha \in M, \quad \alpha \cdot x := r^*(x) \cdot \alpha \quad (\text{resp.} \quad x \cdot \alpha := \alpha \cdot r^*(x)).
\]

\[ \text{Proof. } (1) \text{ We know that } r : \mathcal{O}_q(\text{SL}(2)) \to \mathcal{O}_q^2(\text{SL}(2)) \text{ satisfies the following equalities:}
\]

\[
r^2(\alpha) = \alpha, \quad r(\alpha \beta) = r(\alpha)r(\beta), \quad \Delta(r(\alpha)) = (r \otimes r)\Delta^q(\alpha) \quad \forall \alpha \in \mathcal{O}_q^2(\text{SL}(2))
\]

then since the Hopf pairing is non-degenerate, \( \forall x, y \in U_q^2(\mathfrak{sl}_2) \) it holds:

\[
(r^*)^2(x) = x \quad r^*(xy) = r^*(y)r^*(x) \quad \Delta(r^*(x)) = (r^* \otimes r^*) \circ \Delta(x).
\]

Thus it is sufficient to compute it on the generators where one can verify the values provided in the statement. For instance:

\[
\langle r^*(E), a \rangle = \langle E, a \rangle = 0 = \langle q^2KF, a \rangle = \langle q^2K \otimes F, a \otimes a + c \otimes b \rangle
\]

\[
\langle r^*(E), b \rangle = \langle E, c \rangle = 0 = \langle q^2KF, b \rangle = \langle q^2K \otimes F, a \otimes b + b \otimes d \rangle
\]

\[
\langle r^*(E), c \rangle = \langle E, b \rangle = 1 = \langle q^2KF, c \rangle = \langle q^2K \otimes F, c \otimes a + d \otimes c \rangle
\]

\[
\langle r^*(E), d \rangle = \langle E, d \rangle = 0 = \langle q^2KF, d \rangle = \langle q^2K \otimes F, c \otimes b + d \otimes d \rangle
\]

The verification for the pairings with \( a, b, c, d \) for \( r^*(F) \) and \( r^*(K) \) are similar.

(2) \( LR \) and \( RL \) are functors as \( r^* : U_q^2(\mathfrak{sl}_2) \to U_q^2(\mathfrak{sl}_2) \) is an algebra antihomorphism; they are inverse to each other as \( r^* \) is an involution. Monoidality is a consequence of the fact that \( r^* \) is a coalgebra morphism.

\[ \square \]

4.3. Module structure of \( \mathcal{S}(\mathcal{G}) \). As usual, the Hopf duality implies that every right (resp. left) \( \mathcal{O}_q^2(\text{SL}(2)) \)-comodule \( V \) has a natural structure of a left (resp. right) \( U_q^2(\mathfrak{sl}_2) \)-module, via the following construction. For \( a \in U_q^2(\mathfrak{sl}_2) \) and \( v \in V \), one has

\[
a \cdot v := \sum \langle a, b' \rangle v', \quad \text{where} \quad \Delta_r(v) = \sum v' \otimes b'.
\]

To be precise, we have to replace \( V \) by \( V \otimes_R \mathbb{Q}(q^{1/2}) \), since \( U_q^2(\mathfrak{sl}_2) \) is defined over \( \mathbb{Q}(q^{1/2}) \).

In particular, for an edge \( e \) of \( \mathcal{G} \) the right comodule structure \( \Delta : \mathcal{S}(\mathcal{G}) \to \mathcal{S}(\mathcal{G}) \otimes \mathcal{S}(\mathcal{B}) \) gives \( \mathcal{S}(\mathcal{G}) \otimes_R \mathbb{Q}(q^{1/2}) \) a left module structure over \( U_q^2(\mathfrak{sl}_2) \), and we want to understand this module structure.
Fix an orientation $\sigma$ of the boundary $\partial \mathcal{G}$. Recall that $B(\mathcal{G}; \sigma)$ is an $\mathcal{R}$-basis of $\mathcal{S}(\mathcal{G})$. For each edge $e$ let $B_{e,d}(\mathcal{G}; \sigma) \subset B(\mathcal{G}; \sigma)$ be the set of all $\alpha \in B(\mathcal{G}; \sigma)$ such that $|\alpha \cap e| = d$ and all the states on $\alpha \cap e$ are signs $\pm$. Let $B_e(\mathcal{G}; \sigma) = \bigcup_{d=0}^{\infty} B_{e,d}(\mathcal{G}; \sigma)$. For $\alpha \in B_{e,d}(\mathcal{G}; \sigma)$ and $\bar{\eta} \in \{\pm\}^d$, let $\alpha(\bar{\eta})$ be the same $\alpha$ except for the states of $e \cap \alpha$ which are given by $s(x_i) = \eta_i$, where $x_1, \ldots, x_d$ are the points of $\alpha \cap e$ listed in decreasing order. In particular let $\alpha_j := \alpha(\pm, \pm, \cdots, \pm, \cdots, \pm)$ where the number of $-$ is $j$. For example, $\alpha = \alpha_0$.

**Lemma 4.4** (Module structures of $\mathcal{S}(\mathcal{G})$ along an edge $e$). The left action of $U_{q^2}(\mathfrak{sl}_2)$ on $\mathcal{S}(\mathcal{G}) \otimes_{\mathcal{R}} \mathbb{Q}(q^{1/2})$, dual to $\epsilon \Delta$, is:

$$K_{left}(\alpha_j) = q^{2(d-2j)}\alpha_j$$

$$E_{left}(\alpha_0) = 0, \text{ and } E_{left}(\alpha_j) = [j] q^2\alpha_{j-1} + F_{d-1}^e$$

$$F_{left}(\alpha_0) = 0, \text{ and } F_{left}(\alpha_j) = [d-j] q^2\alpha_{j+1} + F_{d-1}^e$$

where $F_{d-1}^e = F_{d-1}(\mathcal{S}(\mathcal{G}) \otimes_{\mathcal{R}} \mathbb{Q}(q^{1/2}))$ is the $\mathbb{Q}(q^{1/2})$-span of elements $\beta \in B(\mathcal{G}; \sigma)$ with $|\beta \cap e| < d$. The right action, dual to $\Delta_e$, is given by the left action of $r^*(K), r^*(E), r^*(F)$ (see Lemma 4.2).

**Proof.** Recall that $\Delta(\alpha(\bar{\eta})) = \sum_{\bar{\epsilon}} \alpha(\bar{\epsilon}) \otimes \alpha_{\bar{\epsilon}, \bar{\eta}}$ where the sum is over all the $\bar{\epsilon} \in \{\pm\}^m$ and $\alpha_{\bar{\epsilon}, \bar{\eta}} = \alpha_{\epsilon_1, \eta_1} \cdots \alpha_{\epsilon_d, \eta_d} \in \mathcal{S}(\mathcal{B})$. The computation of the action of $K, E, F$ comes from the following formulas for their coproducts, the Hopf duality and the above description of the coproduct of $\mathcal{S}(\mathcal{B})$:

(66) $\Delta^{[d]}(K) = K^{\otimes d}$

(67) $\Delta^{[d]}(E) = \sum_{j=1}^{d} 1^{\otimes (j-1)} \otimes E \otimes K^{\otimes (d-j)}$

(68) $\Delta^{[d]}(F) = \sum_{j=1}^{d} (K^{-1})^{\otimes j} \otimes F \otimes 1^{\otimes (d-j-1)}$

Applying it to compute the Hopf pairing of $K, E, F$ with $\alpha_{\bar{\epsilon}, \bar{\eta}}$ we get:

$$K_{left}(\alpha(\bar{\eta})) = q^{2\sum \eta_k} \alpha(\bar{\eta}),$$

$$E_{left}(\alpha(\bar{\eta})) := \sum_{j=1}^{d} (\delta_{\eta_j, -}) q^{2\sum_{k=j+1}^{d} \eta_k} \alpha(\eta_1, \cdots, \eta_{j-1}, +, \eta_{j+1}, \cdots, \eta_d),$$

$$F_{left}(\alpha(\bar{\eta})) := \sum_{j=1}^{d} (\delta_{\eta_j, +}) q^{-2\sum_{k=1}^{j-1} \eta_k} \alpha(\eta_1, \cdots, \eta_{j-1}, -, \eta_{j+1}, \cdots, \eta_d).$$

Now the main claim is a direct computation using relation (10). □

Let $\mathcal{G}$ have indexed boundary $\partial \mathcal{G} = \{e_1^L, \ldots, e_m^L, e_1^R, \ldots, e_n^R\}$ as explained in Subsection 4.1. The Hopf duality gives $\mathcal{S}(\mathcal{G}) \otimes_{\mathcal{R}} \mathbb{Q}(q^{1/2})$ an algebra bimodule structure over $(U_{q^2}(\mathfrak{sl}_2)^{\otimes m}, U_{q^2}(\mathfrak{sl}_2)^{\otimes m})$ (notice the inversion between left and right when passing to modules).
For each $\vec{m} \in \mathbb{N}^m$ and $\vec{n} \in \mathbb{N}^n$, let $B_{\vec{m}, \vec{n}}(\mathcal{S}; \mathfrak{o})$ be defined as:

$$B_{\vec{m}, \vec{n}}(\mathcal{S}; \mathfrak{o}) = \left( \bigcap_{i=1}^{m} B_{e_i^L, m_i}(\mathcal{S}; \mathfrak{o}) \right) \cap \left( \bigcap_{j=1}^{n} B_{e_j^R, n_j}(\mathcal{S}; \mathfrak{o}) \right).$$

Let also, for each $\vec{j} \leq \vec{m}$ and $\vec{h} \leq \vec{n}$ (component-wise) and each $\alpha \in B_{\vec{m}, \vec{n}}(\mathcal{S}; \mathfrak{o})$ let $\alpha_{\vec{j}, \vec{h}} \in B(\mathcal{S}; \mathfrak{o})$ be the skein identical to $\alpha$ but for its state which is increasing and contains $\vec{j}$ (resp. $\vec{h}$) signs — on the left (resp. right) edges.

**Theorem 4.5.** Suppose that $\mathcal{S}$ has indexed boundary $\partial \mathcal{S} = \{e_1^L, \ldots, e_n^L, e_1^R, \ldots, e_m^R\}$.

(a) For each $\vec{m} \in \mathbb{N}^m$ and $\vec{n} \in \mathbb{N}^n$ and each $\alpha \in B_{\vec{m}, \vec{n}}(\mathcal{S}; \mathfrak{o})$, the $(U_q^2(\mathfrak{sl}_2)^{\otimes n}, U_q^2(\mathfrak{sl}_2)^{\otimes m})$-bimodule generated by $\alpha$ (namely $U_q^2(\mathfrak{sl}_2)^{\otimes n} \cdot \alpha \cdot U_q^2(\mathfrak{sl}_2)^{\otimes m}$) is irreducible and isomorphic to $V_{n_1}^L \otimes \cdots \otimes V_{n_n}^L \otimes V_{m_1}^R \otimes \cdots \otimes V_{m_m}^R$, where $V_k^L$ (resp. $V_k^R$) is the irreducible left (resp. right) module on $U_q^2(\mathfrak{sl}_2)$ with highest weight $k$.

(b) As $(U_q^2(\mathfrak{sl}_2)^{\otimes n}, U_q^2(\mathfrak{sl}_2)^{\otimes m})$-bimodules, we have

$$\mathcal{S}(\mathcal{S}) \otimes \mathcal{R}(q^{1/2}) = \bigoplus U_q^2(\mathfrak{sl}_2)^{\otimes n} \cdot \alpha \cdot U_q^2(\mathfrak{sl}_2)^{\otimes m}. $$

where the sum is taken over all $\vec{m} \in \mathbb{N}^m, \vec{n} \in \mathbb{N}^n$, and all $\alpha \in B_{\vec{m}, \vec{n}}(\mathcal{S}; \mathfrak{o})$. In particular, the bimodule $\mathcal{S}(\mathcal{S})$ is a direct sum of finite dimensional bimodules over $(U_q^2(\mathfrak{sl}_2)^{\otimes n}, U_q^2(\mathfrak{sl}_2)^{\otimes m})$.

(c) Furthermore the bimodule structure restricts to that of a $(U_q^2(\mathfrak{sl}_2)^{\otimes n}, U_q^2(\mathfrak{sl}_2)^{\otimes m})$-bimodule (where $U_q^2(\mathfrak{sl}_2)$ is the integral version of $U_q^2(\mathfrak{sl}_2)$) and a decomposition similar to the above one holds:

$$\mathcal{S}(\mathcal{S}) = \bigoplus \bigoplus \left( U_q^2(\mathfrak{sl}_2)^{\otimes n} \cdot \alpha_{\vec{j}, \vec{h}} \cdot U_q^2(\mathfrak{sl}_2)^{\otimes m} \right)$$

where the direct sum is taken over all $\vec{m} \in \mathbb{N}^m, \vec{n} \in \mathbb{N}^n$ and all $\alpha \in B_{\vec{m}, \vec{n}}(\mathcal{S}; \mathfrak{o})$, and the $\bigoplus_{\vec{j}, \vec{h}}$ symbol stands for the non direct sum.

**Proof.** (a) Fix $m_i, n_j$ and $\alpha$ as in the statement and let $JW(\alpha)$ be the skein obtained by inserting a $JW_{m_i}$ near $e_i^L$ and a $JW_{n_j}$ near $e_j^R$ for all $i, j$. By Example 3.7 it is clear that $\alpha = JW(\alpha)$ and by Lemma 4.4 that it is a highest weight vector of weight $q^{2m_i}$ for the action of the $i$th-copy of $U_q^2(\mathfrak{sl}_2)$ for each $i \leq m$; similarly it is a highest weight vector of weight $q^{2n_j}$ for the right action of the $j$th-copy of $U_q^2(\mathfrak{sl}_2)$. Furthermore, by Lemma 4.4 and the fact that the $m^th$-Jones Wenzl projector kills the self-returns, the orbit of $\alpha$ is exactly the span of the vectors $JW(\alpha_{\vec{j}, \vec{h}})$ with $\vec{j} \leq \vec{m}$ and $\vec{h} \leq \vec{n}$.

(b) It is straightforward from (a) and from Theorem 2.8.

(c) If a left (resp. right) $U_q^2(\mathfrak{sl}_2)$-module weight $M$ (over $\mathcal{R}(q^{1/2})$) has a basis formed by weight vectors over which the action of $E^{(r)}, F^{(r)}, r \geq 1$ has coefficients in $\mathcal{R}$, then $M$ restricts to a $U_q^2(\mathfrak{sl}_2)$-module; we claim that the basis $B(\mathcal{S}; \mathfrak{o})$ of $\mathcal{S}(\mathcal{S})$ has this property. Indeed since the structure of module is induced by the right (resp. left) comodule structure on each edge and the Hopf pairing between $U_q^2(\mathfrak{sl}_2)$ and $\mathcal{O}_q^2(\text{SL}(2))$, and since the comodule structure is integral in the basis $B(\mathcal{S}; \mathfrak{o})$, it is sufficient to observe that the Hopf pairing between $U_q^2(\mathfrak{sl}_2)$ and $\mathcal{O}_q^2(\text{SL}(2)) = \mathcal{S}(\mathcal{B})$ extends to a $\mathcal{R}$-bilinear Hopf pairing between $U_q^2(\mathfrak{sl}_2)$ in the basis $B(\mathcal{B}, \mathfrak{o})$: this is the content of point (1) of Lemma 4.2.
To prove that the direct sum decomposition still holds, let
\[ B_{\bar{m}, \bar{n}}(\mathcal{G}; \mathfrak{o}) := \{ \alpha \in B(\mathcal{G}; \mathfrak{o}) | \#(\alpha \cap e_i^L) \leq m_i, \#(\alpha \cap e_j^R) \leq n_j, \forall i \leq m, \forall j \leq n \} \setminus B_{\bar{m}, \bar{n}}(\mathcal{G}; \mathfrak{o}). \]

To prove the claim, we will show that for each \( \alpha \in B_{\bar{m}, \bar{n}}(\mathcal{G}; \mathfrak{o}) \), the following holds:
\[
(\mathcal{R} \cdot B_{\bar{m}, \bar{n}}^{\leq}(\mathcal{G}; \mathfrak{o})) \bigcap \left( \bigoplus_{\bar{j}\leq \bar{m}, \bar{h}\leq \bar{n}} \left( \bigwedge_{q}^L(\mathfrak{sl}_2)^{\otimes n} \cdot \alpha_{\bar{j}, \bar{h}} \cdot \bigwedge_{q}^L(\mathfrak{sl}_2)^{\otimes m} \right) \right) = \{0\}.
\]

We start by remarking that if \( \alpha \in B_{\bar{m}, \bar{n}}(\mathcal{G}; \mathfrak{o}) \), then \( \alpha = JW(\alpha) \) and so, over \( \mathbb{Q}(q^{1/2}) \), its orbit is a direct summand of \( \mathcal{H}(\mathcal{G}) \) and thus it has trivial intersection with the \( \mathcal{R} \)-span of \( B_{\bar{m}, \bar{n}}^{\leq}(\mathcal{G}; \mathfrak{o}) \):
\[
(\mathcal{R} \cdot B_{\bar{m}, \bar{n}}^{\leq}(\mathcal{G}; \mathfrak{o})) \bigcap \left( \bigwedge_{q}^L(\mathfrak{sl}_2)^{\otimes n} \cdot \alpha \cdot \bigwedge_{q}^L(\mathfrak{sl}_2)^{\otimes m} \right) = \{0\}.
\]

Furthermore, by the point (b), given \( \bar{j} \leq \bar{m}, \bar{h} \leq \bar{n} \), there exist \( c(\bar{j}, \bar{h}) \in \mathcal{R} \setminus 0 \) and \( \ell(\bar{j}, \bar{h}) \in \mathcal{R} \cdot B_{\bar{m}, \bar{n}}^{\leq}(\mathcal{G}; \mathfrak{o}) \) such that \( c(\bar{j}, \bar{h}) \alpha_{\bar{j}, \bar{h}} + \ell(\bar{j}, \bar{h}) \) is in the orbit of \( \alpha \):
\[
c(\bar{j}, \bar{h}) \alpha_{\bar{j}, \bar{h}} + \ell(\bar{j}, \bar{h}) \in \bigwedge_{q}^L(\mathfrak{sl}_2)^{\otimes n} \cdot \alpha \cdot \bigwedge_{q}^L(\mathfrak{sl}_2)^{\otimes m}.
\]

Now suppose that for some \( l_i \in \bigwedge_{q}^L(\mathfrak{sl}_2)^{\otimes m} \) and \( r_i \in \bigwedge_{q}^L(\mathfrak{sl}_2)^{\otimes n} \) and some \( \bar{j}_i, \bar{h}_i \) it holds
\[
\sum_i l_i \cdot \alpha_{\bar{j}_i, \bar{h}_i} \cdot r_i \in \mathcal{R} \cdot B_{\bar{m}, \bar{n}}^{\leq}(\mathcal{G}; \mathfrak{o}) \setminus \{0\}.
\]

Then, multiplying by \( \prod_i c(\bar{j}_i, \bar{h}_i) \) (which gives a non-zero vector as \( \mathcal{H}(\mathcal{G}) \) is free as a \( \mathcal{R} \)-module) we also get that \( \sum_i l_i \cdot \alpha \cdot r_i \in \mathcal{R} \cdot B_{\bar{m}, \bar{n}}^{\leq}(\mathcal{G}; \mathfrak{o}) \setminus \{0\} \), which as already argued is impossible.

\[\square\]

**Example 4.6.** Let \( \mathcal{B} \) be the bigon whose edges \( e_l \) and \( e_r \) are declared to be respectively of type \( L \) and \( R \). Then by Theorem 3.4, \( \mathcal{H}(\mathcal{B}) \) is the right and left module \( U_q^L(\mathfrak{sl}_2) \)-module \( \mathcal{O}_q^R(SL(2)) \): the left action is induced by the right comodule structure coming from \( e^R \) and the right action from \( e^L \). If we let \( \mathcal{B}^R \) be the bigon where both \( e_l \) and \( e_r \) are declared to be of type \( R \) (right), then \( \mathcal{H}(\mathcal{B}^R) \) is a left \( (U_q^L(\mathfrak{sl}_2))^\otimes 2 \)-module; the action of \( x \otimes y \) on a skein \( b \in \mathcal{H}(\mathcal{B}^R) \) is given by
\[
(x \otimes y) \cdot b = x \cdot b \cdot r^*(y)
\]
where \( r^*(y) \) is the algebra antomorphism provided in Lemma 4.3 and the left and right actions are those on \( \mathcal{H}(\mathcal{B}) \) described above.

4.4. **Recalls on co-tensor product.** Suppose \( U \) is a coalgebra over a ground ring \( \mathcal{R} \). Assume \( M \) is a left \( U \)-comodule with coaction \( \Delta_M : M \to U \otimes \mathcal{R} M \), and \( N \) a right \( U \)-comodule with coaction \( \Delta_N : N \to N \otimes \mathcal{R} U \). Then the cotensor product \( N \square_U M \) is
\[
N \square_U M := \{ v \in N \otimes M | (\Delta_N \otimes \text{id}_M)(v) = (\text{id}_N \otimes \Delta_M)(v) \}.
\]

Cotensor product is a special case of the following notion of Hochschild cohomology. Assume \( V \) is a \( \mathcal{R} \)-module with a left \( U \)-coaction and a right \( U \)-coaction:
\[
\Delta_r : V \to V \otimes U, \quad \Delta_l : V \to U \otimes V.
\]
The 0-th Hochshild cohomology of $V$ is defined by

$$HH^0(V) = \{ x \in V \mid \Delta_r(x) = \text{fl}(i\Delta(x)) \},$$

where $\text{fl} : V \otimes U \to U \otimes V$ if the flip $\text{fl}(x \otimes y) = y \otimes x$.

With $M$ and $N$ as above, define a left $U$-coaction and a right $U$ coaction on $N \otimes_R M$ by

$$\Delta_r : N \otimes_R M \to N \otimes_R M \otimes_R U, \quad \Delta_r(n \otimes m) = \sum n' \otimes m \otimes u' \text{ if } \Delta_N(n) = \sum n' \otimes u'$$

$$i\Delta : N \otimes_R M \to U \otimes_R N \otimes_R M, \quad i\Delta(n \otimes m) = \sum u'' \otimes n \otimes m'' \text{ if } \Delta_M(m) = \sum u'' \otimes m''.$$

Then the cotensor product $N \square_U M = HH^0(N \otimes_R M)$.

4.5. **Splitting as co-tensor product and Hochshild cohomology.** Suppose $c_1, c_2$ are distinct boundary edges of a punctured bordered surface $\mathcal{S}'$ and $\mathcal{S} = \mathcal{S}'/(c_1 = c_2)$, with $c \subset \mathcal{S}$ being the common image of $c_1$ and $c_2$. The splitting homomorphism gives an embedding $\theta_c : \mathcal{I}(\mathcal{S}) \hookrightarrow \mathcal{I}(\mathcal{S}')$ and we will make precise the image of $\theta_c$.

![Figure 13](image.png)

**Figure 13.** (a) The middle shaded part is the bigon, while the left and the right shaded parts are part of $\mathcal{S}'$. Gluing $c_1 = e_l$ gives the right coaction $\Delta_r$ and gluing $e_r = c_2$ gives the left coaction $i\Delta$. (b) Element $x\bar{\mu}$ $\in$ $\mathcal{I}(\mathcal{S}')$. The horizontal lines are part of $x$. Note the order of indices.

**Theorem 4.7.** Suppose $c_1, c_2$ are distinct boundary edges of a punctured bordered surface $\mathcal{S}'$ and $\mathcal{S} = \mathcal{S}'/(c_1 = c_2)$. The splitting homomorphism

$$\theta_c : \mathcal{I}(\mathcal{S}) \hookrightarrow \mathcal{I}(\mathcal{S}').$$

maps $\mathcal{I}(\mathcal{S})$ isomorphically onto the Hochshild cohomology $HH^0(\mathcal{I}(\mathcal{S}'))$, which is a $\mathcal{I}(\mathcal{B})$-bimodule via the left coaction $i\Delta := e_2\Delta$ and the right coaction $\Delta_r := \Delta_{c_1}$ (see Figure 13(a)).

In particular, if $c_1$ is a boundary edge of $\mathcal{S}'_1$ and $c_2$ is a boundary edge of $\mathcal{S}'_2$ which is disjoint from $\mathcal{S}'_1$, and $\mathcal{S} = (\mathcal{S}'_1 \cup \mathcal{S}'_2)/(c_1 = c_2)$, then $\theta_c$ maps $\mathcal{I}(\mathcal{S})$ isomorphically onto the cotensor product of $\mathcal{I}(\mathcal{S}'_1)$ and $\mathcal{I}(\mathcal{S}'_2)$ over $\mathcal{I}(\mathcal{B})$.

**Proof.** Let us identify $\mathcal{I}(\mathcal{S})$ with its image under $\theta_c$. From the splitting formula (25) it is easy to see that $\mathcal{I}(\mathcal{S}) \subset HH^0(\mathcal{I}(\mathcal{S}'))$. Let us prove the converse inclusion. Assume $0 \neq v \in HH^0(\mathcal{I}(\mathcal{S}'))$. By definition, this means

$$\Delta_r(v) = \text{fl}(i\Delta(v)) = 0.$$

Choose an orientation $\sigma$ of $\partial \mathcal{S}$ and an orientation of $c$. Then $\sigma$ and the orientation of $c_1$ and $c_2$ induced from $c$ give an orientation $\sigma'$ of $\partial \mathcal{S}'$. Recall that $B(\mathcal{S}' ; \sigma')$ is a free $\mathcal{R}$-basis of $\mathcal{I}(\mathcal{S}')$. Let $\tilde{B}(\mathcal{S}' ; \sigma')$ be the set of all isotopy classes $x$ of $\sigma'$-ordered $\partial M'$-tangle diagrams which are increasingly stated on every boundary edge except for $c_1$ and $c_2$. If $\bar{\mu}$ is a state
of $x \cap c_1$ and $\tilde{\nu}$ is a state of $x \cap c_2$, let $x_{\tilde{\nu}\tilde{\mu}}$ be the stated $\sigma'$-ordered $\partial M'$-tangle diagram whose states on $x \cap c_1$ and $x \cap c_2$ are respectively $\tilde{\mu}$ and $\tilde{\nu}$. See Figure 13(b). If $\tilde{\mu}$ and $\tilde{\nu}$ are increasing, then $x_{\tilde{\nu}\tilde{\mu}} \in B(\mathcal{G}', \sigma')$ is a basis element. For each $i = 1, 2$ let $S_{x\cap c_i}$ and $S'_{x\cap c_i}$ be respectively the set of all states and the set of all increasing states of $(77)$.

Using the above $\mathcal{R}$-basis $B(\mathcal{G}', \sigma')$ of $\mathcal{R}(\mathcal{G}')$, we can present $v \in \mathcal{R}(\mathcal{G}')$ in the form

$$v = \sum_{x \in X(v)} \sum_{\tilde{\mu} \in S'_{x\cap c_1}} \sum_{\tilde{\nu} \in S'_{x\cap c_2}} \text{coef}(v, x_{\tilde{\nu}\tilde{\mu}}) x_{\tilde{\nu}\tilde{\mu}},$$

where $X \subset \tilde{B}(\mathcal{G}', \sigma')$ is a minimal finite set, so that for each $x \in X$, there are $\tilde{\mu}, \tilde{\nu}$ such that the coefficient $\text{coef}(v, x_{\tilde{\nu}\tilde{\mu}})$ is non-zero.

Let $m(v) = \max\{|x \cap c_1|, |x \cap c_2|, x \in X(v)\}$. We show by induction on $m(v)$ that $v \in \mathcal{R}(\mathcal{G})$.

For $i = 1, 2$ let $X_i(v) = \{x \in X(v), |x \cap c_i| = m\}$. If $x \in X_1(v) \cap X_2(v)$, then $|x \cap c_1| = |x \cap c_2| = m(v)$, and there is an element $\bar{x} \in \mathcal{B}(\mathcal{G}, \sigma)$ such that $x$ has coefficient non-zero in the result of splitting $\bar{x}$ along $\sigma$. From the definition of the splitting map we have

$$\text{coef}(\theta(\bar{x}), x_{\bar{x}}) = 1,$$

where $\bar{x} = (+)^m$ is the state consisting of $m$ plus signs. Let

$$v' = v - \sum_{x \in X_1(v) \cap X_2(v)} \text{coef}(v, x_{\bar{x}}) \theta(\bar{x}).$$

If $m(v') < m(v)$ then we are done by induction. Assume that $m(v') = m(v)$. One of $X_1(v')$, $X_2(v')$ is not empty, and without loss of generality assume $X_2(v') \neq \emptyset$. Formula (73) for $v'$ has the form

$$v' = \sum_{x \in X(v')} \sum_{\tilde{\mu} \in S'_{x\cap c_1}} \sum_{\tilde{\nu} \in S'_{x\cap c_2}} \text{coef}(v', x_{\tilde{\nu}\tilde{\mu}}) x_{\tilde{\nu}\tilde{\mu}},$$

and because of (74) we can assume that there is no $x_{\bar{x}}$ on the right hand side of (75).

Let $p_m^{c_2} : \mathcal{R}(\mathcal{G}') \to \mathcal{R}(\mathcal{G}')$ be the projection onto the homogeneous part $G_m^{c_2}(\mathcal{R}(\mathcal{G}'))$, and $p_m^{c_1} : \mathcal{R}(\mathcal{B}) \to \mathcal{R}(\mathcal{B})$ be the projection onto the homogeneous part $G_m^{c_1}(\mathcal{R}(\mathcal{B}))$, see the end of Section 2.10. Explicitly, for $x \in X(v')$ we have

$$p_m^{c_2}(x_{\tilde{\nu}\tilde{\mu}}) = \begin{cases} 0 & \text{if } \tilde{\nu} \neq \tilde{\tau} \\ x_{\bar{x}_{\tilde{\mu}}} & \text{if } \tilde{\nu} = \tilde{\tau} \end{cases}, \quad p_m^{c_1}(\alpha_{\tilde{\nu}\tilde{\mu}}) = \begin{cases} 0 & \text{if } \tilde{\mu} \neq \tilde{\tau} \\ \alpha_{\bar{x}_{\tilde{\mu}}} & \text{if } \tilde{\mu} = \tilde{\tau}. \end{cases}$$

From Formula (33) for the coaction, we have, for $x \in X(v')$ and $(\tilde{\nu}, \tilde{\mu}) \neq (\tilde{\tau}, \tilde{\tau})$,

$$x_{\tilde{\nu}\tilde{\mu}} \xrightarrow{\Delta} \sum_{\bar{\eta} \in S_{x\cap c_1}} x_{\bar{\eta}\bar{\eta}} \otimes \alpha_{\bar{\eta}\bar{\eta}} \xrightarrow{p_m^{c_2} \otimes p_m^{c_1}} 0,$$

$$x_{\tilde{\nu}\tilde{\mu}} \xrightarrow{i \Delta} \sum_{\bar{\eta} \in S_{x\cap c_2}} \alpha_{\bar{\eta}\bar{\eta}} \otimes x_{\bar{\eta}\bar{\eta}} \xrightarrow{\bar{\eta}} \sum_{\bar{\eta}} x_{\bar{\eta}\bar{\eta}} \otimes \alpha_{\bar{\eta}\bar{\eta}} \xrightarrow{p_m^{c_2} \otimes p_m^{c_1}} \begin{cases} 0 & \text{if } x \notin X_2(v') \\ x_{\bar{x}_{\tilde{\mu}}} \otimes \alpha_{\bar{x}_{\tilde{\mu}}} & \text{if } x \in X_2(v'). \end{cases}$$
It follows that
\[ 0 = (p_m^c \otimes p_m^c)(f(\Delta(v'))) - \Delta_r(v') = \sum_{x \in X_2(v')} \sum_{\bar{\mu} \in S_{n+c}^2} \sum_{\bar{\nu} \in S_{n+c}^2} \text{coef}(v', x_{\bar{\mu} \bar{\nu}}) \cdot x_{\bar{\mu} \bar{\nu}} \otimes \alpha_{\bar{\nu} \bar{\tau}}. \]

As the right hand side is a linear combination of elements of a basis, all the coefficients \(v\) there are 0. This means \(X_2(v') = \emptyset\), a contradiction. Thus \(m(v') < m(v)\) and we are done. \(\Box\)

**Remark 4.8.** Theorem 4.7 holds also if we change the base ring to \(\mathbb{C}\) by evaluating \(q\) to a non-zero complex number.

Using the above result together with Theorem 4.5 we can deduce a similar result for \(U_q(\text{sl}_2)\)-modules. Recall that for a bi-module \(V\) over a \(\mathbb{Q}(q^{1/2})\)-algebra \(U\) the 0-homology group is
\[ HH_0(V) = V/\mathbb{Q}(q^{1/2}) \cdot \text{span}\langle a \cdot v - v \cdot a | a \in U, v \in V \rangle. \]

**Theorem 4.9.** Suppose \(c_1, c_2\) are distinct boundary edges of a punctured bordered surface \(\mathcal{S}'\) and \(\mathcal{S} = \mathcal{S}'/(c_1 = c_2)\), with \(c\) being the common image of \(c_1\) and \(c_2\). Then the composition
\[ \mathcal{J}(\mathcal{S}) \otimes_R \mathbb{Q}(q^{1/2}) \xrightarrow{\delta_c} \mathcal{J}(\mathcal{S}') \otimes_R \mathbb{Q}(q^{1/2}) \to HH_0(\mathcal{J}(\mathcal{S}') \otimes_R \mathbb{Q}(q^{1/2})) \]

is an isomorphism of \(\mathbb{Q}(q^{1/2})\)-vector spaces. Here \(\mathcal{J}(\mathcal{S}') \otimes_R \mathbb{Q}(q^{1/2})\) is a \(U_q(\text{sl}_2)\)-bimodule via the dual actions of \(\Delta_{c_1}\) and \(\Delta_{c_2}\).

In particular, if \(\mathcal{S}' = \mathcal{S}_1 \sqcup \mathcal{S}_2\) with \(c_1 \subset \mathcal{S}_1\) and \(c_2 \subset \mathcal{S}_2\), then the map in (79) is an isomorphism between \(\mathcal{J}(\mathcal{S}) \otimes_R \mathbb{Q}(q^{1/2})\) and \((\mathcal{J}(\mathcal{S}_1) \otimes_R \mathbb{Q}(q^{1/2})) \otimes_{U_q(\text{sl}_2)} (\mathcal{J}(\mathcal{S}_2) \otimes_R \mathbb{Q}(q^{1/2}))\).

**Proof.** The decomposition of \(\mathcal{J}(\mathcal{S}') \otimes_R \mathbb{Q}(q^{1/2})\) given by part (b) of Theorem 4.5 shows that it is sufficient to prove that if \(V_{m_2}^R\) (resp. \(V_{m_1}^L\)) is the irreducible \(m_2 + 1\)-dimensional (resp. \(m_1 + 1\)-dimensional) right (resp. left) \(U_q(\text{sl}_2)\)-module, then the composition of natural map
\[ HH_0(V_{m_1}^L \otimes V_{m_2}^R) \hookrightarrow V_{m_1}^L \otimes V_{m_2}^R \to HH_0(V_{m_1}^L \otimes V_{m_2}^R) \]
is an isomorphism of vector spaces. We will see that this follows from the fact that every finite-dimensional \(U_q(\text{sl}_2)\)-module \(V\) is equivalent to its dual \(V^*\).

First observe that since the pairing between \(O_{SU}(\text{SL}(2))\) and \(U_q(\text{sl}_2)\) is non-degenerate, \(HH_0(V_{m_1}^L \otimes V_{m_2}^R)\) can be equivalently defined as
\[ HH_0(V_{m_1}^L \otimes V_{m_2}^R) = \{ v \otimes w \in V_{m_1}^L \otimes V_{m_2}^R | v \cdot v \otimes w = v \otimes w \cdot x, \forall x \in U_q(\text{sl}_2) \}. \]

Then using the isomorphism between \(V_{m_2}^R\) and \((V_{m_2}^L)^*\) we have:
\[ HH_0(V_{m_1}^L \otimes V_{m_2}^R) = \text{Hom}_{U_q(\text{sl}_2)}(V_{m_2}^R, V_{m_1}^L) = \delta_{m_1, m_2} \mathbb{Q}(q^{1/2}) \]
by Schur’s lemma.

Now let’s take the dual of the above equation and get:
\[ (HH_0(V_{m_1}^L \otimes V_{m_2}^R))^* \leftrightarrow (V_{m_2}^R)^* \otimes (V_{m_1}^L)^* \to (HH_0(V_{m_1}^L \otimes V_{m_2}^R))^* \]
where the first arrow maps an element of \((HH_0(V_{m_1}^L \otimes V_{m_2}^R))^*\) to some \(f \in (V_{m_2}^L \otimes V_{m_2})^*\) such that \(f(x \cdot v \otimes w) = f(v \otimes w \cdot x)\) for all \(x \in U_q(\text{sl}_2)\) and \(v \otimes w \in V_{m_1}^L \otimes V_{m_2}^R\). Using again
the isomorphism between \( (V_{m_2}^R)^* \) and \( V_{m_2}^L \) we have that the image of \( (HH_0(V_{m_1}^L \otimes V_{m_2}^R))^* \) in \( V_{m_2}^L \otimes (V_{m_1}^L)^* \) is \( \text{Hom}_{U_q^r(sl_2)}(V_{m_1}^L, V_{m_2}^L) = \delta_{m_1, m_2} \mathbb{Q}(q^{1/2}) \) by Schur’s lemma.

To conclude, observe that if \( m_1 = m_2 \) then the image of the inclusion \( HH^0(V_{m_1}^L \otimes V_{m_1}^R) \hookrightarrow V_{m_1}^L \otimes V_{m_1}^R \simeq V_{m_1}^L \otimes (V_{m_1}^L)^* = \text{Hom}(V_{m_1}^L, V_{m_1}^L) \) is given by the multiples of the identity map. But the kernel of the projection \( V_{m_1}^L \otimes V_{m_1}^R \rightarrow HH_0(V_{m_1}^L \otimes V_{m_1}^R) \) is the sub vector space of \( \text{Hom}(V_{m_1}^L, V_{m_1}^L) \) spanned by the matrices of the form \( xM - Mx \) where \( x \) represents the action of an element of \( U_q^r(sl_2) \) and \( M \in \text{Hom}(V_{m_1}^L, V_{m_1}^L) \); thus it is contained the set of matrices with zero trace and so the projection of \( HH^0(V_{m_1}^L \otimes V_{m_1}^R) \) in \( HH_0(V_{m_1}^L \otimes V_{m_1}^R) \) is nonzero.

\[ \square \]

**Remark 4.10.** By the splitting theorem and Proposition 4.9, \( \mathcal{S}(\mathcal{G}) \) is both a submodule and a quotient module of \( \mathcal{S}(\mathcal{G}') \).

**Example 4.11.** Clearly, if in Theorem 4.9 \( c_1 = e_i^R \) and \( c_2 = e_j^L \) belong to two distinct connected components of \( \mathcal{G}' \), then one can restate the \( HH_0 \) simply as a tensor product over a copy of \( U_q^r(sl_2) \) acting on the left on the skein algebra of one component and on the right on the other.

In particular, if \( \mathcal{G} \) is obtained by gluing a bigon \( \mathcal{B} \) along its right edge to a left edge of \( \mathcal{G}' \) then \( \mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{G}') \otimes U_q^r(sl_2) \mathcal{S}(\mathcal{B}) \) is isomorphic to \( \mathcal{S}(\mathcal{G}') \) as it can be seen directly by Theorem 2.14.

If \( \mathcal{B}^R \) is the bigon whose edges are both declared to be of type \( R \) (right), then \( \mathcal{S}(\mathcal{B}^R) \) is a left module over \( U_q^r(sl_2)^{\otimes 2} \) (see Example 4.6). Then gluing \( \mathcal{B}^R \) to \( \mathcal{G}' \) along one edge of type \( L \), shows that

\[ \mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{G}') \otimes U_q^r(sl_2) \mathcal{S}(\mathcal{B}^R). \]

The resulting surface \( \mathcal{G} \) is still homeomorphic to \( \mathcal{G}' \) but the edge on which the gluing has been performed has been transformed from an edge of type \( L \) to one of type \( R \). This corresponds to applying Lemma 4.3 to the module structure coming from that edge.

**Remark 4.12.** If \( c_1', c_2' \) are two other edges of \( \partial \mathcal{G}' \) (and then of \( \partial \mathcal{G} \)), (79) is an isomorphism of \( U_q^r(sl_2) \)-modules for the structure associated to \( c_1' \) and \( c_2' \). Furthermore the theorem can be applied independently to glue also \( c_1' \) and \( c_2' \) and the final isomorphism between \( \mathcal{S}(\mathcal{G}')/(c_1 = c_2, c_1' = c_2') \otimes \mathbb{Q}(q^{1/2}) \) and \( HH^0(\mathcal{S}(\mathcal{G}') \otimes \mathbb{Q}(q^{1/2})) \) (with respect to the \( U_q^r(sl_2)^{\otimes 2} \)-bimodule structure) does not depend on the order in which the gluing was performed.

### 4.6. Recalls on the braided tensor product.

Suppose \( M, N \) are both right comodule-algebras over a cobranded Hopf algebra \( U \). The tensor product \( M \otimes_R N \) has a natural structure of a right \( U \)-comodule. On \( M \otimes N \) the usual product is

\[ (x \otimes y)(z \otimes t) = xy \otimes zt \]

but it is not compatible with the \( U \)-comodule structure, i.e. the map

\[ (M \otimes_R N) \otimes_R (M \otimes N) \rightarrow M \otimes_R N, \quad (x \otimes y) \otimes (z \otimes t) \rightarrow xy \otimes zt \]

is not a \( U \)-morphism. In the presence of the braiding, there is a new algebra structure on \( M \otimes_R N \) for which the product map is a \( U \)-morphism. This product is defined by

\[ (x \otimes y) \ast (z \otimes t) = (x \otimes 1)c_{M, N}(y \otimes z)(1 \otimes t) = (x \otimes 1)\left( \sum \rho(z'' \otimes y'')(z' \otimes y') \right)(1 \otimes t). \]
Here $c_{M,N}(y \otimes z) = \sum \rho(z'' \otimes y'')(z' \otimes y')$ is the braiding. In other words, if we identify $M$ with $M \otimes \{1\}$ and $N$ with $\{1\} \otimes N$ (as subsets of $M \otimes_R N$), then the new product is defined by

\begin{align*}
xy &= \begin{cases} 
xy & \text{if } x, y \in M \text{ or } x, y \in N \\
x \otimes y & \text{if } x \in M, y \in N \\
\sum \rho(y'' \otimes x'')(y' \otimes x') & \text{if } x \in N, y \in M.
\end{cases}
\end{align*}

(83)

Then $M \otimes_R N$ with this new product is called the braided tensor product of $M$ and $N$, and is denoted by $M \underline{\otimes}_U N$. For details see [Maj].

4.7. **Glueing along a triangle is a braided tensor product.** An ideal triangle is the disk without 3 points on its boundary. Suppose $e, a', a''$ are edges of an ideal triangle $P_3$, as depicted in Figure 14.

**Figure 14.** Left: Ideal triangle $P_3$. Middle: Glueing $S', P_3, S''$ by $e' = a'$ and $e'' = a''$. Right: Elements $x \in \mathcal{I}(S'')$ and $y \in \mathcal{I}(S')$

Suppose $S', S''$ are disjoint punctured bordered surfaces, with distinguished edges $e' \subset \partial S'$ and $e'' \subset \partial S''$. Let $\mathcal{G} = (S' \sqcup P_3 \sqcup S'')/(a' = e', a'' = e'')$, see Figure 14. Choose an orientation $\sigma$ of $\partial \mathcal{G}$ such that the orientation of $e$ is as in Figure 14. Let $\sigma'$ (resp. $\sigma''$) be the orientation of $S'$ (resp. $S''$) inherited from $\sigma$ with the orientation on $e'$ (resp. $e''$) as in Figure 14. Let $\bar{\sigma} : \mathcal{I}(S') \hookrightarrow \mathcal{I}(\mathcal{G})$ (resp. $\bar{\sigma} : \mathcal{I}(S'') \hookrightarrow \mathcal{I}(\mathcal{G})$) be the $\mathcal{R}$-linear map defined such that if $x \in B(\mathcal{G}; \sigma')$ (resp. $x \in B(\mathcal{G}; \sigma'')$) is a stated diagram then $\tilde{x}$ is the diagram obtained from $x$ by extending the strands ending on $e'$ until they end on $e$ (resp. the one obtained symmetrically), see Figure 15. More precisely if $|x \cap e'| = k$ then $\tilde{x} = x \cup y$ where $y \subset P_3$ consists of $k$ straight lines, each has endpoints in $e'$ and $e$, with $y \cap e' = x \cap e'$, and the states of $\tilde{x} \cap e$ come from the corresponding ones of $x \cap e$. Since this map defined an embedding $B(\mathcal{G}; \sigma') \hookrightarrow B(\mathcal{G}; \sigma)$ it is an injective algebra morphism. Define $\bar{\sigma} : \mathcal{I}(S'') \hookrightarrow \mathcal{I}(\mathcal{G})$ similarly.

**Figure 15.** From $x$ to $\tilde{x}$

Let also $S''' = (S' \sqcup S'')/(a' = a'')$ i.e. the surface obtained by gluing $S'$ and $S''$ along $a'$ and $a''$ via orientation reversing diffeomorphisms.
Let $\Delta := \Delta_e : \mathcal{S}(\mathcal{G}) \to \mathcal{S}(\mathcal{G}) \otimes \mathcal{O}_q(\text{SL}(2))$ be the right $\mathcal{O}_q(\text{SL}(2))$-comodule algebra structure given by the edge $e$. It is easy to see that under the map $x \to \tilde{x}$, each of $\mathcal{S}(\mathcal{G}')$ and $\mathcal{S}(\mathcal{G}'')$ become a right $\mathcal{O}_q(\text{SL}(2))$-sub-comodule algebra of $\mathcal{S}(\mathcal{G})$.

**Theorem 4.13.** (a) The right $\mathcal{O}_q(\text{SL}(2))$-comodule algebra $\mathcal{S}(\mathcal{G})$ is canonically isomorphic to the braided tensor product $\mathcal{S}(\mathcal{G}') \otimes \mathcal{S}(\mathcal{G}'')$. More precisely, the $\mathcal{R}$-linear map $f : \mathcal{S}(\mathcal{G}') \otimes \mathcal{S}(\mathcal{G}'') \to \mathcal{S}(\mathcal{G})$, defined by $f(x \otimes y) = \tilde{x}\tilde{y}$, is an isomorphism of right $\mathcal{O}_q(\text{SL}(2))$-comodule algebras.

(b) The subalgebra of $\mathcal{S}(\mathcal{G})$ consisting of $\mathcal{O}_q(\text{SL}(2))$-invariant elements is isomorphic to $\mathcal{S}(\mathcal{G}'')$.

**Proof.** (a) It is enough to show that $f$ is an $\mathcal{R}$-algebra isomorphism. Recall that as $\mathcal{R}$-modules, $\mathcal{S}(\mathcal{G}') \otimes_{\mathcal{R}} \mathcal{S}(\mathcal{G}'')$ and $\mathcal{S}(\mathcal{G}') \otimes \mathcal{S}(\mathcal{G}'')$ are the same.

**Lemma 4.14.** $f$ is an algebra homomorphism.

**Proof.** For $x, y \in \mathcal{S}(\mathcal{G}') \otimes \mathcal{S}(\mathcal{G}'')$ we have to show that $f(xy) = (f(x)f(y))$. This is clearly true if both $x, y$ are in one of the two subsets $\mathcal{S}(\mathcal{G}')$ or $\mathcal{S}(\mathcal{G}'')$. It is also true if $x \in \mathcal{S}(\mathcal{G}')$ and $y \in \mathcal{S}(\mathcal{G}'')$. It remains to consider the case $x \in \mathcal{S}(\mathcal{G}')$ and $y \in \mathcal{S}(\mathcal{G}'')$. We present $x$ and $y$ schematically as in Figure 14. Then

\[
f(x)f(y) = \tilde{x}\tilde{y} = \sum (\tilde{y}' \tilde{x}') \otimes \varepsilon(\tilde{y}'\tilde{x}') \quad \text{because } u = \sum u' \varepsilon(u'')
\]

\[
= \sum (\tilde{y}' \tilde{x}') \rho(\tilde{y}'' \tilde{x}'') \quad \text{by (58)}
\]

\[
= f(xy) \quad \text{by (83)}.
\]

Thus $f$ is an algebra homomorphism. □

**Lemma 4.15.** $f$ is an $\mathcal{R}$-linear isomorphism.

**Proof.** For $l \in \mathbb{N}$ let $F_l(\mathcal{S}(\mathcal{G}') \otimes_{\mathcal{R}} \mathcal{S}(\mathcal{G}'')) \subset \mathcal{S}(\mathcal{G}') \otimes_{\mathcal{R}} \mathcal{S}(\mathcal{G}'')$ be the $\mathcal{R}$-submodule spanned by $u \otimes v$, where $u, v$ are simple diagrams such that $I(u, e') + I(v, e'') \leq l$. Let $F_l(\mathcal{S}(\mathcal{G})) \subset \mathcal{S}(\mathcal{G})$ be the $\mathcal{R}$-submodule spanned by $u$ such that $I(u, e') + I(u, e'') \leq l$. Denote by $\text{Gr}_*$ the corresponding graded $\mathcal{R}$-modules.

It is clear that $f$ preserves the filtrations $F_l$. It is enough to show that $\text{Gr}(f)$ is a bijection.

Let $B_{\alpha, m}(\mathcal{G}; \sigma')$ (resp. $B_{\alpha, m}(\mathcal{G}; \sigma'')$) be the set of $\alpha \in B(\mathcal{G}; \sigma')$ (resp. $\alpha \in B(\mathcal{G}; \sigma'')$) such that $I(\alpha, e') = m$ (resp. $I(\alpha, e'') = n$). Then

\[
\text{Gr}_l(\mathcal{S}(\mathcal{G}') \otimes_{\mathcal{R}} \mathcal{S}(\mathcal{G}'')) = \bigoplus_{m+n=l, \; x \in B_{\alpha, m}(\mathcal{G}; \sigma'), \; y \in B_{\alpha, n}(\mathcal{G}; \sigma'')} V(x, y)
\]

\[
\text{Gr}_l(\mathcal{S}(\mathcal{G})) = \bigoplus_{m+n=l, \; x \in B_{\alpha, m}(\mathcal{G}; \sigma'), \; y \in B_{\alpha, n}(\mathcal{G}; \sigma'')} W(x, y).
\]
Where \( V(x, y) \) is the \( \mathcal{R} \)-submodule of \( \text{Gr}_l(\mathcal{S}'(\mathcal{G})) \otimes_R \mathcal{S}(\mathcal{G}'') \) spanned by \( x(\vec{\nu}) \otimes y(\vec{\mu}) \), where \( \vec{\nu} \) and \( \vec{\mu} \) range over all the increasing states of \( x \) and \( y \) respectively, and \( W(x, y) \) is the \( \mathcal{R} \)-submodule of \( \text{Gr}_l(\mathcal{S}(\mathcal{G})) \) spanned by \( z \in B(\mathcal{G}; \partial) \) such that \( z \cap \mathcal{G}' = x \) and \( z \cap \mathcal{G}'' = y \). It is enough to show that \( \text{Gr}(f) \) is an isomorphism from \( V(x, y) \) to \( W(x, y) \) for \( x \in B_{e', m}(\mathcal{G}'; \partial') \), \( y \in B_{e'' m}(\mathcal{G}''; \partial'') \). Note that both \( V(x, y) \) and \( W(x, y) \) are free \( \mathcal{R} \)-modules, and

\[
\text{rk}_\mathcal{R}(V(x, y)) = \text{rk}_\mathcal{R}(W(x, y)) = (m + 1)(n + 1).
\]

Indeed there are \( m + 1 \) increasing states on \( x \) and \( n + 1 \) increasing states on \( y \) and these can be chosen independently, thus \( \text{rk}_\mathcal{R}(V(x, y)) = (m + 1)(n + 1). \) For what concerns \( W(x, y) \), observe that an element \( z \in B(\mathcal{G}; \partial) \) such that \( z \cap \mathcal{G}' = x \) and \( z \cap \mathcal{G}'' = y \) is composed by \( k \) arcs (for some \( k \in [0, \min(m, n)] \)) connecting \( \partial x \) and \( \partial y \), \( m - k \) arcs extending some component \( a \) of \( x \) to \( \tilde{a} \) and finally \( n - k \) arcs extending some component \( b \) of \( y \) to \( \tilde{b} \); furthermore \( z \cap e \) is increasingly stated so that there are exactly \( m + n - 2k + 1 \) such \( z \) and we denote \( W^k(x, y) \) the free \( \mathcal{R} \)-modules they span in \( W(x, y) \). Thus we have:

\[
\text{rk}_\mathcal{R}(W(x, y)) = \sum_{k=0}^{\min(m, n)} \text{rk}_\mathcal{R}(W^k(x, y)) = \sum_{k=0}^{\min(m, n)} (m + n - 2k + 1) = (m + 1)(n + 1).
\]

Hence to prove the isomorphism it is sufficient to show that \( \text{Gr}(f) : V(x, y) \to W(x, y) \) is surjective because both modules are \( \mathcal{R} \)-free and have the same rank. To prove this observe first that the reordering relation (10) implies the relation in Figure 16. Then if \( z \in W^k(x, y) \), applying \( k \) times the relation in Figure 16, we can express \( z \) as a linear combination of stated diagrams of the form \( \widehat{x(\mu)y(\nu)}(\vec{\nu}) \) where \( \mu \) and \( \nu \) are non-necessarily increasing states on \( x \cap e' \) and \( y \cap e'' \) respectively. But by the reordering relation (10) \( x(\mu) \) is equivalent in \( \text{Gr}_l(\mathcal{S}'(\mathcal{G})) \otimes_R \mathcal{S}(\mathcal{G}'') \) to a multiple of an increasing state \( x(\vec{\mu}) \) and similarly for \( y(\nu) \). Hence \( z \) is a linear combination of \( x(\vec{\mu})y(\vec{\nu}) \) as claimed.

(b) Observe that \( \mathcal{G} \) is obtained from \( \mathcal{G}''' \) by doubling the boundary vertex corresponding to the endpoint of \( e' \) (initial point of \( e'' \)) in Figure 14. Hence there is an injective algebra morphism \( i : \mathcal{S}(\mathcal{G}''') \hookrightarrow \mathcal{S}(\mathcal{G}'') \) whose image is exactly the span of the subset of \( B(\mathcal{G}'') \) consisting of elements not intersecting \( e \). Stated differently, \( i(\mathcal{S}(\mathcal{G}'')) \) is the subalgebra of invariant vectors by the action of \( U_{q^2}(\mathfrak{sl}_2) \) corresponding to \( e \).

4.8. Some examples: algebra of polygons and braided bialgebras of punctured monogons. We apply now Theorem 4.13 to provide some examples of computations of stated skein algebras.

\[
\includegraphics[width=0.5\textwidth]{figure16}
\]

Figure 16.
**Figure 17.** On the left the inclusion $i : M_4 \hookrightarrow M_8$ (the segments used in the construction are dotted). On the right the decomposition of $M_8$ in a triangle and two copies of $M_4$ used to fix the isomorphism $\mathcal{I}(M_8) = \mathcal{I}(M_4) \otimes \mathcal{I}(M_4)$.

**Definition 4 (Polygons).** Let $P_n, n \geq 1$ be the $n$-polygon, namely the oriented disc with $n$ punctures $v_1, \ldots, v_n$ in its boundary numbered in the clockwise order and let $e = v_i v_{i+1}$ and $e_i = v_i v_{i+1}, i \in \{1, \ldots, n-1\}$.

Arguing by induction using Theorem 4.13 one has the following:

**Corollary 4.16.** For $n \geq 3$, in the category of comodule-algebras over $O_q^e(\text{SL}(2)) = \mathcal{I}(B_e)$ it holds $\mathcal{I}(P_n) = O_q^e(\text{SL}(2)) \otimes \cdots \otimes O_q^e(\text{SL}(2))$ where there are $n-1$-copies of $O_q^e(\text{SL}(2)) = \mathcal{I}(B_{e_i})$ corresponding to the bigons over the edges $e_1, \ldots, e_{n-1}$. Furthermore $\mathcal{I}(P_n)$ is isomorphic to the sub-algebra of invariant vectors in $\mathcal{I}(P_{n+1})$ with respect to action of $U_q^e(\mathfrak{sl}_2)$ on $e = v_1 v_{n+1}$.

For the following constructions we refer to Figure 17.

**Definition 5 (Punctured monogons).** Let $M_n$ be the monogon with one vertex in its boundary and $n \geq 0$ punctures in its interior (e.g. $M_0$ is the monogon $M$ already introduced in Subsection 3.1).

Let $i : M_n \hookrightarrow M_{2n}$ be the inclusion identifying $M_n$ with the complement of $n$ closed disjoint arcs connecting the $2n$-punctures of $M_{2n}$. Consider two disjoint ideal arcs attached to the boundary vertex of $M_{2n}$ each bounding a disjoint copy of $M_n$: since the complement of these two copies of $M_n$ is a triangle, by Theorem 4.13 we have that $\mathcal{I}(M_{2n}) = \mathcal{I}(M_n) \otimes \mathcal{I}(M_n)$ where the comodule structure is induced by the only boundary edge. Let then $\Delta : \mathcal{I}(M_n) \hookrightarrow \mathcal{I}(M_{2n}) = \mathcal{I}(M_n) \otimes \mathcal{I}(M_n)$ be the map induced by $i_*$ and this isomorphism.

**Proposition 4.17.** For each $n \geq 0$ the algebra $\mathcal{I}(M_n)$ endowed with the map $\Delta$ and the map $\epsilon : \mathcal{I}(M_n) \rightarrow \mathcal{I}(M_0)$ induced by inclusion is a braided bialgebra object in the category of $O_q^e(\text{SL}(2))$-comodules. In particular for $n = 1$ it is isomorphic as a braided Hopf algebra to $BSL_q(2)$ the “transmutation” of $O_q^e(\text{SL}(2))$, or “braided version” or “covariant version” of $O_q^e(\text{SL}(2))$ (see [Maj] Examples 4.3.4 and 10.3.3).

**Proof.** It is clear that the inclusion $i$ induces an injective (but not surjective) morphism of $O_q^e(\text{SL}(2))$-algebra comodules $i_* : \mathcal{I}(M_n) \hookrightarrow \mathcal{I}(M_{2n}) = \mathcal{I}(M_n) \otimes \mathcal{I}(M_n)$. Coassociativity is a direct consequence of the fact that applying twice $i$ induces the same morphism as the identification of $M_n$ with the complement of $n$ disjoint arcs in $M_{3n}$ connecting $2n$
distinct punctures and each containing a single puncture in its interior as depicted here:

Furthermore it is straightforward to see that the map $\epsilon$ is a counit for $\Delta$ and that is a morphism. The last statement is a direct computation which we outline in what follows. Fix an orientation for $\partial M_0$, let $\alpha(x, y)$ be the stated simple arc depicted in the left part of the following figure:

In the right part we represent the skein $\alpha(x, y) \cdot \alpha(z, t)$ where $x, y, z, t \in \{\pm\}$ are the boundary states. A long but straightforward computation shows that if we let

$$a = q^2 \alpha(-, +), \quad b = -q^3 \alpha(+, +), \quad c = q^2 \alpha(-, -), \quad d = -q^2 \alpha(+, -)$$

then the following relations are satisfied:

$$ba = q^{-4} ab, \quad ca = q^4 ac, \quad da = ad, \quad bc = cb + (1 - q^4)a(d-a)$$

$$db = bd + (1 - q^4)ab, \quad cd = dc + (1 - q^4)ca, \quad ad - q^{-4} cb = 1$$

which are those provided in [Maj] Example 4.3.4, up to exchanging our $q^2$ with $q^{-1}$. Remark for instance that the element $q^2 a + q^{-2} d$ is central as it equals the skein represented by the meridian of the puncture. It is clear by the definition of $\epsilon$ that $\epsilon(a) = \epsilon(d) = 1$ and $\epsilon(b) = \epsilon(c) = 0$. Furthermore to compute explicitly the coproduct on the generators, we use (10) as follows:

Then it is straightforward to verify that the following holds:

$$\Delta \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \otimes \left( \begin{array}{cc} a & b \\ c & d \end{array} \right).$$
5. A lift of the Reshetikhin-Turaev operator invariant

In this section we show that a Reshetikhin-Turaev operator invariant of tangles can be lifted to an invariant with values in $O_{q^2}(SL(2))$. In this section $R = \mathbb{Z}[q^{\pm 1/2}]$.

5.1. Category of non-directed ribbon graphs. We will present the category of non-directed ribbon graphs [Tu3], also known as framed tangles [Oh], in the form convenient for us.

The bigon is canonically isomorphic (in the category of punctured bordered surface) to the square $S = [0,1] \times (0,1)$. Under the isomorphism $e_l$ and $e_r$ are mapped respectively to $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$, and abusing notation we also denote $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$ respectively $e_l$ and $e_r$. We identify $S$ with $S \times \{0\}$ in $M := S \times (-1,1)$. We have $\partial M = \partial S \times (-1,1) = (e_l \cup e_r) \times (-1,1)$.

\[ \text{Figure 18. Left: Square } S = [0,1] \times (0,1), \text{ with edges } e_l \text{ and } e_r. \text{ Middle: tensor product } \beta \otimes \beta', \text{ Right: composition } \beta \circ \beta', \text{ which can be defined only when } |\partial_r \beta| = |\partial_l \beta'|. \]

Recall that in the definition of a $\partial M$-tangle we require the boundary points over one boundary edge have distinct heights (see Subsection 2.4). If we change this requirement to: all boundary points are in $\partial S$ (in particular they all have the same height) we get the notion of a $\partial S$-tangle. Formally, a $\partial S$-tangle is a framed compact 1-dimensional unoriented manifold $\beta$ properly embedded in $M = S \times (-1,1)$ such that $\partial \beta$ has height 0, i.e. $\partial \beta \subset \partial S = e_l \cup e_r$, and the framing at every boundary point of $\beta$ is vertical. Let $\partial_r \beta = \beta \cap e_r$ and $\partial_l \beta = \beta \cap e_l$. Two $\partial S$-tangles are $\partial S$-isotopic if they are isotopic in the class of $\partial S$-tangles. If $|\partial_r \beta| = k$ and $|\partial_l \beta| = l$, then our notion of a $\partial S$-tangle is the notion of a non-directed ribbon $(k,l)$-graph without coupons in [Tu3].

After an isotopy we can bring $\beta$ to a generic position (with respect to the projection from $S \times (-1,1)$ onto $S$) and make the framing vertical everywhere, and the projection of $\beta$ together with the over/under information at every crossing, is called a $\partial S$-tangle diagram of $\beta$. The isotopy class of $\beta$ is totally determined by any of its diagrams.

The non-directed ribbon graph category is the category whose set of objects is $\mathbb{N}$ and a morphism from $k$ to $l$ is an isotopy class of $\partial S$-tangle $\beta$ such that $|\partial_r \beta| = k$ and $|\partial_l \beta| = l$, with the usual composition (see Figure 18). If the tangles are oriented, then one would get the usual ribbon tangle category.

If $\beta, \beta'$ are two $\partial S$-tangles, define their tensor product $\beta \otimes \beta'$ as the result of putting $\beta$ above $\beta'$ as in Figure 18. Under the tensor product and the composition, morphisms of the
non-directed ribbon tangle category are generated by the five elementary $\partial S$-tangles depicted in Figure 19.

![Diagram of five elementary tangles](image)

**Figure 19.** Five elementary tangles

From the ribbon category of finite-dimensional modules over the quantum group $U_q(\mathfrak{sl}_2)$ we get the Reshetikhin-Turaev operator invariant of $\partial S$-tangles, see [Tu3]. Let us describe this operator invariant in a special case. Let $V$ be the free $\mathcal{R}$-module with basis $g_+, g_-$. The above mentioned operator invariant is the unique functor $Z$ from the non-directed ribbon tangle category to the category of $\mathcal{R}$-modules preserving the tensor product such that $Z(n) = V^\otimes n$ and the values of the elementary tangles are given by

\begin{align*}
(87) & \quad Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} \right) : V^\otimes 2 \to R, \quad g_+ \otimes g_- \to q^{-\frac{1}{2}}, \quad g_- \otimes g_+ \to -q^{-\frac{3}{2}}, \quad g_+ \otimes g_+ \to 0, \quad g_- \otimes g_- \to 0
\end{align*}

\begin{align*}
(88) & \quad Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \end{array} \right) : R \to V^\otimes 2, \quad 1 \to -q^2 (g_+ \otimes g_-) + q^3 (g_- \otimes g_+)
\end{align*}

\begin{align*}
(89) & \quad Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \end{array} \right) : V^\otimes 2 \to V^\otimes 2, \quad Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \end{array} \right) = q \text{id} + q^{-1} \left( Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \end{array} \right) \circ Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \end{array} \right) \right),
\end{align*}

\begin{align*}
(90) & \quad Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \end{array} \right) : V^\otimes 2 \to V^\otimes 2, \quad Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \end{array} \right) = q^{-1} \text{id} + q \left( Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \end{array} \right) \circ Z \left( \begin{array}{c}
\begin{array}{c}
\vdots
\end{array} \end{array} \right) \right),
\end{align*}

see [Co]. Here our $g_{\pm}$ are related to the basis vectors $g_{\frac{1}{2}}$ in [Co] by

\[ g_+ = -\sqrt{-1} q^{-3/2} g_{\frac{1}{2}}, \quad g_- = g_{-\frac{1}{2}}. \]

Thus if $\beta$ is a $\partial S$-tangle with $|\partial_l \beta| = l$ and $|\partial_r \beta| = k$ then $Z(\beta)$ is an $\mathcal{R}$-linear map $V^\otimes k \to V^\otimes l$ which depends only on the isotopy class of $\beta$.

For $\bar{\nu} = (\nu_1, \ldots, \nu_l) \in \{\pm\}^l$ and $\bar{\mu} = (\mu_1, \ldots, \mu_k) \in \{\pm\}^k$ we can define the matrix entry $\nu Z(\beta) \bar{\mu} \in R$ such that

\[ Z(\beta)(g_{\nu_1} \otimes \ldots \otimes g_{\nu_l}) = \sum_{\bar{\nu} \in \{\pm\}^l} (\nu Z(\beta) \bar{\mu}) g_{\nu_1} \otimes \ldots \otimes g_{\nu_l}. \]

**Remark 5.1.** In fact $V$ and all its tensor powers are modules over the quantum group $U_q(\mathfrak{sl}_2)$, and all the operators $Z(\beta)$ are $U_q(\mathfrak{sl}_2)$-morphisms. But we don’t need the structure of $U_q(\mathfrak{sl}_2)$-modules here. When $k = l = 0$, we have $Z(\beta) \in R$, which is equal to the Kauffman bracket polynomial of $\beta$.

5.2. **From $\partial M$-tangles to $\partial S$-tangles.** Suppose $\gamma$ is a $\partial M$-tangle. We can $\partial M$-isotope $\gamma$ so that its diagram $D$ has the height order on $e_l$ and $e_r$ determined by the arrows in Figure 20. This diagram determines a unique class of $\partial S$-tangle, denoted by $\tilde{\gamma}$. Note that the arrows of $e_r, e_l$ are irrelevant for $\tilde{\gamma}$. It is easy to see that the map $\gamma \to \tilde{\gamma}$ is a bijection from the set of $\partial M$-isotopy classes of $\partial M$-tangles to the set of $\partial S$-isotopy classes of $\partial S$-tangles.

Suppose $|\gamma \cap e_l| = l$ and $|\gamma \cap e_r| = k$, and $\bar{\nu} = (\nu_1, \ldots, \nu_l) \in \{\pm\}^l$ and $\bar{\mu} = (\mu_1, \ldots, \mu_k) \in \{\pm\}^k$. Let $\nu \gamma \bar{\mu}$ be the stated $\partial M$-tangle $\gamma$ whose underlying tangle is $\gamma$ and whose states
on $\gamma \cap e_l$ (respectively on $\gamma \cap e_r$) from top to bottom by the height order are $\nu_1, \ldots, \nu_l$ (respectively $\mu_1, \ldots, \mu_k$).

**Theorem 5.2.** Assume the above notation. Consider $\vec{\nu} \gamma \vec{\mu}$ as an element of $\mathcal{S}(B)$. Then

$$\epsilon(\vec{\nu} \gamma \vec{\mu}) = \vec{\nu} Z(\bar{\gamma}) \vec{\mu}. \quad (92)$$

Thus we see that the tangle invariant of $\vec{\nu} \gamma \vec{\mu}$ with values in $\mathcal{S}(B)$ is stronger than the Reshetikhin-Turaev operator invariant.

**Proof.** Suppose $\gamma_1, \gamma_2$ are $\partial M$-tangles. Since $\bar{\gamma_1 \gamma_2} = \bar{\gamma_1} \otimes \bar{\gamma_2}$, if (92) is true for $\gamma = \gamma_1$ and $\gamma = \gamma_2$, it is true for $\gamma = \gamma_1 \gamma_2$.

Now suppose $\gamma_1, \gamma_2$ are obtained by splitting a $\partial M$-tangle $\beta$ along an ideal edge. By the splitting formula (36) and the definition of $\Delta,$

$$\Delta(\vec{\nu} \beta \vec{\mu}) = \sum_{\bar{\eta}} \vec{\nu}(\bar{\gamma_1}) \vec{\eta}(\bar{\gamma_2}) \vec{\mu}. \quad (\text{36})$$

Applying $\epsilon \otimes \text{id}$ to the above, we get

$$\vec{\nu} \beta \vec{\mu} = \sum_{\bar{\eta}} \epsilon(\nu(\bar{\gamma_1}) \eta(\bar{\gamma_2}) \mu). \quad (\text{87})$$

Applying $\epsilon$ to the above, we get

$$\epsilon(\vec{\nu} \beta \vec{\mu}) = \sum_{\bar{\eta}} \epsilon(\nu(\bar{\gamma_1}) \eta(\bar{\gamma_2}) \mu), \quad (\text{90})$$

which shows that if (92) holds for $\gamma = \gamma_1$ and $\gamma = \gamma_2$ then it holds for $\beta = \gamma_1 \circ \gamma_2$.

Thus it is enough to check (92) for the elementary tangles, for which (92) follows from the explicit formulas (87)–(90). \qed

5.3. **A 1 + 1-TQFT.** Let $\text{Cob}_{1,1}$ be the symmetric monoidal category whose:

- **Objects** are numbered disjoint unions of open unoriented segments.
- **Morphisms** are diffeomorphism classes of punctured bordered surfaces $\mathcal{S}$ with indexed boundary. Explicitly if $\partial \mathcal{S} = e_1^L, \ldots, e_m^L, e_1^R, \ldots, e_n^R$ then $\mathcal{S} \in \text{Mor}(e_1^L \sqcup \cdots \sqcup e_m^L, e_1^R \sqcup \cdots \sqcup e_n^R)$ and the composition of morphisms is given by the gluing of marked surfaces explained above (associativity of compositions is ensured by the fact that we consider diffeomorphism classes of surfaces). In particular the identity morphism of $e_1 \sqcup \cdots \sqcup e_n$ is a disjoint union of $n$ copies of $\mathcal{S}$.
- **Tensor product** is the disjoint union, where the components of $(e_1 \sqcup \cdots \sqcup e_n) \sqcup (e_1' \sqcup \cdots \sqcup e_m')$ are ordered as $e_1 \sqcup \cdots \sqcup e_n \sqcup e_1' = e_{n+1} \sqcup \cdots \sqcup e_m' = e_{m+n}$. 

![Figure 20. Direction of boundary edges, used to determine the height order](image)
In order to define the target category of our TQFT functor, let us fix some notation. Given a finite set $C$, we will then denote by $U_q(\mathfrak{sl}_2)^\otimes C$ the algebra obtained as the tensor product $\bigotimes_{c \in C} U_q(\mathfrak{sl}_2)$ where each copy of $U_q(\mathfrak{sl}_2)$ in the tensor product is indexed by a distinct element of $C$.

**Definition 6** ($U_q(\mathfrak{sl}_2) - \text{finBim}$). Let $U_q(\mathfrak{sl}_2) - \text{finBim}$ be the category whose objects are pairs $(C, [M])$ where $C$ is a finite set, $M$ is a right module over $U_q(\mathfrak{sl}_2)^\otimes C$ which is a direct sum of finite dimensional modules and $[M]$ is its isomorphism class. A morphism from $(C, [M])$ to $(C', [M'])$ in $U_q(\mathfrak{sl}_2) - \text{finBim}$ is the isomorphism class of a bimodule $B$ over $(U_q(\mathfrak{sl}_2)^\otimes C, U_q(\mathfrak{sl}_2)^\otimes C')$ which is a direct sum of finite dimensional bimodules and such that $[M \otimes_{U_q(\mathfrak{sl}_2)^\otimes C} B] = [M']$. The composition of $[B] : (C, [M]) \to (C', [M'])$ and $[B'] : (C', [M']) \to (C'', [M''])$ is $[B \otimes_{U_q(\mathfrak{sl}_2)^\otimes C} B']$ (the composition is associative as we consider bimodules up to isomorphisms). The monoidal structure on $U_q(\mathfrak{sl}_2) - \text{finBim}$ is given by $(C, [M]) \otimes (C', [M']) := (C \cup C', [M \otimes_R M'])$ and its symmetry is given by exchanging $(C, [M])$ and $(C', [M'])$.

Then let $\mathcal{I} : \text{Cob}_{1,1} \to U_q(\mathfrak{sl}_2) - \text{finBim}$ be defined as

$$\mathcal{I}(e_1 \sqcup \cdots \sqcup e_n) = (C = \{e_1, \ldots, e_n\}, [\mathbb{Q}(q^{1/2}) \otimes_R \mathcal{I}(\mathcal{B})^\otimes C])$$

and for a punctured bordered surface $\mathcal{G}$ whose boundary is $C = \{e_1^L, \ldots, e_n^L\} \cup C' = \{e_1^R, \ldots, e_m^R\}$ let $\mathcal{I}(\mathcal{G})$ be the isomorphism class of the $(U_q(\mathfrak{sl}_2)^\otimes C, U_q(\mathfrak{sl}_2)^\otimes C')$-bimodule $\mathbb{Q}(q^{1/2}) \otimes_R \mathcal{I}(\mathcal{G})$.

**Theorem 5.3** (Skein algebra as a TQFT). The functor $\mathcal{I}$ is a symmetric monoidal functor into $U_q(\mathfrak{sl}_2) - \text{finBim}$.

**Proof.** By point b) of Theorem 4.5 it holds $\mathbb{Q}(q^{1/2}) \otimes_R \mathcal{I}(\mathcal{B}) = \bigoplus_{i \geq 0} V_i^L \otimes V_i^R$ where $V_i^L$ (resp. $V_i^R$) is the irreducible $i + 1$-dimensional left (resp. right) module over $U_q(\mathfrak{sl}_2)$. Then, arguing exactly as in the proof of Theorem 4.9 one sees that for each $j \geq 0$ it holds

$$[V_j^R \otimes_{U_q(\mathfrak{sl}_2)} (\mathbb{Q}(q^{1/2}) \otimes_R \mathcal{I}(\mathcal{B}))] = [V_j^R].$$

Then $\mathbb{Q}(q^{1/2}) \otimes_R \mathcal{I}(\mathcal{B})$ represents the identity morphism $((e), [M]) \to ((e), [M])$ (for any edge $e$) if restricted to finite dimensional right $U_q(\mathfrak{sl}_2)$-modules (which are all direct sums of $V_j^R$). Let $\mathcal{G}'$ and $\mathcal{G}''$ be two bordered punctured surfaces with boundaries indexed so that $\partial^L \mathcal{G}' = \{e_1, \ldots, e_n\} = \partial^R \mathcal{G}''$ and let $\mathcal{G}$ be the surface obtained by glueing $\mathcal{G}'$ and $\mathcal{G}''$ by identifying the corresponding edges of $\partial^L \mathcal{G}'$ and $\partial^R \mathcal{G}''$ via an orientation reversing diffeomorphism. Then $\mathcal{I}(\mathcal{G}')$ (resp. $\mathcal{I}(\mathcal{G}'')$) is a right (resp. left) module over $U_q(\mathfrak{sl}_2)^{\partial^L \mathcal{G}'}$ (resp. over $U_q(\mathfrak{sl}_2)^{\partial^R \mathcal{G}'' = \partial^L \mathcal{G}'}$). To conclude, a repeated application of Theorem 4.9 shows that the following holds up to isomorphism:

$$\mathcal{I}(\mathcal{G}) = \mathcal{I}(\mathcal{G}') \otimes_{U_q(\mathfrak{sl}_2)^{\partial^L \mathcal{G}'}} \mathcal{I}(\mathcal{G}).$$

**Remark 5.4.** The previous construction can be improved by passing to the setting of 2-categories in order to consider objects no longer up to isomorphisms. This requires to consider marked surfaces and bimodules and will be dealt with in an another work.
6. A non-symmetric modular operad

In this section we show that stated skein algebras provide an example of "non symmetric geometric modular operad". Such objects were defined by Markl ([Mark]) as a generalisation of "modular operads" initially defined by Gezelter and Kapranov ([GK]). Given a monoidal category $C$, Markl defined a NS modular operad in $C$ as a monoidal functor $NSO : \text{MultiCyc} \rightarrow C$ where MultiCyc is a suitable category of MultiCyc "multicyclic sets". In this section we rephrase Markl’s definition in the case of a suitable category of punctured bordered surfaces $\text{TopMultiCyc}$; then we define a NS geometric modular operad as a monoidal functor $NSO : \text{TopMultiCyc} \rightarrow C$. Finally we re-interpret skein algebras as an example of an NS geometric modular operad with values in $U_q(sl_2) - \text{finBim}$ (see Definition 6).

6.1. The category of topological multicyclic sets $\text{TopMultiCyc}$. In this section all surfaces will be oriented and all homeomorphisms will preserve the orientation.

A cutting system in a bordered punctured surface $\mathcal{S}$ is a finite linearly ordered set $\alpha$ of pairwise disjoint ideal oriented arcs $\alpha_1, \cdots, \alpha_k \subset \mathcal{S}$ (see Subsection 2.2); a homeomorphism of cutting systems $\alpha$ and $\beta$ in $\mathcal{S}$ is a homeomorphism $\phi : \mathcal{S} \rightarrow \mathcal{S}$ such that $\phi(\alpha) = \beta$ so that it preserves the ordering and the orientations of the arcs. Cutting along all the arcs of a cutting system $\alpha$ produces a bordered punctured surface $\text{cut}_\alpha(\mathcal{S})$ whose homeomorphism class depends only on the homeomorphism class of $\alpha$. We will say that a cutting system $\alpha$ is disconnecting if each arc in $\alpha$ disconnects $\mathcal{S}$.

If the connected components of $\mathcal{S}$ are linearly ordered then one can order the connected components of $\text{cut}_\alpha(\mathcal{S})$ as follows. Since $\text{cut}_\alpha(\mathcal{S}) = \text{cut}_{\alpha_k} \left( \cdots \text{cut}_{\alpha_1}(\mathcal{S}) \right)$, it is sufficient to define how to do it for the cut along a single ideal arc $\alpha$. If $\alpha$ does not disconnect, then there is a natural bijection between the components of $\mathcal{S}$ and $\text{cut}_\alpha(\mathcal{S})$ which induces the ordering on those of the latter surface. If $\alpha$ disconnects $\mathcal{S}$, since both $\alpha$ and $\mathcal{S}$ are oriented there is a well defined notion of the connected component of $\text{cut}_\alpha(\mathcal{S})$ “lying at the left” and “at the right of $\alpha$” we then order them so that left precedes right and they are in the same position in the global ordering of the components of $\mathcal{S}$ as the component they come from.

**Definition 7** (TopMultiCyc, TopForest). Let $\text{TopMultiCyc}$ be the category whose objects are homeomorphism classes of punctured bordered surface whose connected components are linearly ordered, and where a morphism $\mathcal{S}' \rightarrow \mathcal{S}$ is a homeomorphism class of a cutting system $\alpha$ in $\mathcal{S}$ such that $\text{cut}_\alpha(\mathcal{S})$ is homeomorphic to $\mathcal{S}'$. The category $\text{TopForest}$ is the subcategory whose objects are disjoint unions of polygons (see Definition 4) and whose morphisms are those represented by disconnecting cutting systems.

If $\phi : \mathcal{S}' \rightarrow \text{cut}_\alpha(\mathcal{S})$ and $\psi : \mathcal{S} \rightarrow \text{cut}_\beta(\mathcal{S}'')$ are homeomorphisms, then the composition of the morphisms associated to $\alpha$ and $\beta$ is the homeomorphism class of $\psi(\alpha) \sqcup \beta \subset \mathcal{S}''$ where the numbering of the arcs of $\psi(\alpha)$ is lower than those of $\beta$. The identity morphism is represented by the class of the empty cutting system and it is straightforward to check that the composition is associative, so that the above are indeed categories.

Both $\text{TopMultiCyc}$ and $\text{TopForest}$ are symmetric monoidal categories. Indeed the tensor product of $\mathcal{S}' = \mathcal{S}'_1 \sqcup \cdots \sqcup \mathcal{S}'_k$ and $\mathcal{S} = \mathcal{S}_1 \sqcup \cdots \sqcup \mathcal{S}_h$ (where $\mathcal{S}'_i, \mathcal{S}_j$ are connected for all
Determine the linear order of the components is increasing from left to right) is defined as
\[ G' \otimes G := G'_1 \sqcup \cdots \sqcup G'_n \sqcup G_1 \sqcup \cdots \sqcup G_h. \]

On the level of morphisms, if \( \alpha \subset G_1 \) and \( \beta \subset G_2 \) are two cutting systems then \( \alpha \otimes \beta = \alpha \sqcup \beta \) where the linear order of the arcs of \( \alpha \) is lower than that of the arcs in \( \beta \). The symmetry is given by exchanging the components, so with the above notations \( s(G' \otimes G) = G \otimes G' \) and \( s(\alpha \otimes \beta) = \beta \otimes \alpha \).

The following definition is a reformulation of Markl’s [Mark] (Definition 4.1) in the context of punctured bordered surfaces:

**Definition 8 (NS Modular Operads).** Let \( C \) be a symmetric monoidal category. A NS (non symmetric) geometric modular operad in \( C \) is a symmetric monoidal functor

\[ O : \text{TopMultiCyc} \to C. \]

A NS cyclic operad in \( C \) is a symmetric monoidal functor \( O : \text{TopForest} \to C. \)

6.2. **NS geometric modular operads from skein algebras.** Recall that if \( B \) is the bigon with one edge of type “left” and one of type “right”, then \( \mathcal{S}(B) = O_{SL(2)}(q) \) as a \((U_q(\mathfrak{sl}_2), U_q(\mathfrak{sl}_2))-bimodule\). Let also \( B^R \) be the bigon whose edges are declared to be both of type \( R \) (right edges) then \( \mathcal{S}(B^R) \) is the left module over \( U_q(\mathfrak{sl}_2)^{\otimes 2} \) whose underlying space is \( O_{SL(2)}(q) \) and on which the action of \( x \otimes y \in U_q(\mathfrak{sl}_2)^{\otimes 2} \) is given by \( x \otimes y \cdot b = x \cdot b \cdot r^r(y) \) (see Example 4.6).

**Theorem 6.1 (Skein algebras as non symmetric operads).** There is a geometric NS-modular operad \( NSO \) in \( U_q(\mathfrak{sl}_2) - \text{finBim} \) defined on an object \( G \) of \( \text{TopMultiCyc} \) as

\[ NSO(G) = (\text{Edges}(G), [Q(q^{1/2}) \otimes R \mathcal{S}(G)]) \]

where \( G \) is the surface whose edges are all indexed to be of type \( L \) (left) and where we see \( Q(q^{1/2}) \otimes R \mathcal{S}(G) \) as a right module over \( U_q(\mathfrak{sl}_2)^{\otimes 2} \text{Edges}(G) \) as explained in Subsection 4.3.

If \( \phi : G' \to G \) is a morphism associated to a cutting system \( \alpha \), then let

\[ NSO(\phi) = [Q(q^{1/2}) \otimes R \mathcal{S}(B)^{\otimes \text{Edges}(G)} \otimes \mathcal{S}(B^R)^{\otimes \alpha}] \]

where \( \mathcal{S}(B^R)^{\otimes \alpha} \) is the skein algebra of a disjoint union of one copy of \( B' \) per arc \( \alpha_i \in \alpha \) whose boundary edges correspond to the edges of \( \partial G' \) lying respectively at the left and at the right of \( \alpha_i \).

**Proof.** First of all observe that the functor is well defined as all surfaces are seen up to orientation preserving diffeomorphism and all modules and bimodules in \( U_q(\mathfrak{sl}_2) - \text{finBim} \) are seen up to isomorphism. Then we observe that the skein algebra of a disjoint union of \( n \) bigons

\[ Q(q^{1/2}) \otimes R \mathcal{S}(\sqcup_{j=1}^n B_j) = Q(q^{1/2}) \otimes R \mathcal{S}(B)^{\otimes n} = Q(q^{1/2}) \otimes R O_{SL(2)}^{\otimes n} \]

is the identity of \( (\{1, 2, \cdots, n\}, [M]) \) (where \( i \) is the left edge of the \( i^{th} \) bigon) for any right module \( M \) which is a direct sum of finite dimensional modules over \( U_q(\mathfrak{sl}_2)^{\otimes n} \). Indeed \( M \) is a direct sum of modules of the form \( W \otimes V_j^R \) where \( V_j^R \) is the \( j + 1 \)-dimensional irreps of \( U_q(\mathfrak{sl}_2) \) and \( W \) is a right module over \( U_q(\mathfrak{sl}_2)^{\otimes n-1} \) which is itself a tensor product of finite
dimensional modules. As proved in Theorem 4.5 \( \mathcal{S}(\mathcal{B}) = \mathcal{O}_{q^2}(\mathrm{SL}(2)) = \bigoplus_i V^L_i \otimes V^R_i \) so that by the same arguments as in the proof of Theorem 4.9 it holds:

\[
[W \otimes V^R_j \otimes U_{q^2(\mathfrak{sl}_2)}(s_i)] = \bigoplus_i [W \otimes \left( V^R_j \otimes U_{q^2(\mathfrak{sl}_2)}(s_i) V^L_i \right) \otimes V^R_i] = [W \otimes V^R_j].
\]

This shows that tensoring over \( U_{q^2(\mathfrak{sl}_2)} \) with a single copy of \( \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{R}} \mathcal{S}(\mathcal{B}) \) provides the identity morphism; by Remark 4.12 repeating this along all the boundary edges one gets that tensoring with \( \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{R}} \mathcal{O}_{q^2}(\mathrm{SL}(2))^{\otimes n} \) is the identity of \( \{1, 2, \ldots, n\}, [M] \) for any \( M \) decomposing into a direct sum of finite dimensional modules.

Now we prove that if \( i, j \) are two distinct boundary edges of a (possibly disconnected) surface \( \mathcal{S}' \), then

\[
[\mathcal{S}(\mathcal{S}') \otimes U_{q^2(\mathfrak{sl}_2)}^{\{i,j\}}, \mathcal{S}(\mathcal{B}^R)] = [\mathcal{S}(\mathcal{B})]
\]

where \( \mathcal{S}(\mathcal{B}^R) \) is seen as left module over \( U_{q^2(\mathfrak{sl}_2)}^{\{i,j\}} \) and \( \mathcal{S} \) is the surface obtained by glueing the edges \( i, j \) by an orientation reversing homeomorphism. Indeed by Remark 4.12 and Example 4.11 to glue \( \mathcal{B}^R \) along \( i \) and \( j \), one can first glue \( \mathcal{S}' \) and \( \mathcal{B}^R \) along \( i \) thus obtaining the surface \( \mathcal{S}' \) whose edge \( i \) has been changed to type \( R \) (see Example 4.11) and then operating a self-glueing along \( i \) and \( j \) on this surface. By Theorem 4.9 the overall result is \( \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{R}} \mathcal{S}(\mathcal{S}) \). Then if \( \alpha \) is a cutting system given by \( c \) arcs, by Remark 4.12 applying \( c \) times (93) we get that tensoring with \( \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{R}} \mathcal{S}(\mathcal{B}^R)^{\otimes \alpha} \) is performing the glueing inverting the cut associated to the cutting system \( \alpha \).

7. Reduced skein algebra

We show that the stated skein algebra \( \mathcal{S}(\mathcal{S}) \) has a nice quotient \( \overline{\mathcal{S}}(\mathcal{S}) \), called the reduced stated skein algebra, which can be embedded in a quantum torus. This quotient is still big enough to contain the ordinary skein algebra and the Muller skein algebra. Unlike the case of the full fledged version \( \mathcal{S}(\mathcal{S}) \), when \( \mathcal{S} \) is an ideal triangle, the reduced version \( \overline{\mathcal{S}}(\mathcal{S}) \) is a quantum torus. The construction of the quantum trace map follows immediately from the splitting theorem for the reduced stated skein algebra.

Throughout we fix a punctured bordered surface \( \mathcal{S} = \overline{\mathcal{S}} \setminus \mathcal{P} \) and we will denote \( \mathcal{S} = \mathcal{S}(\mathcal{S}) \).

7.1. Definition. A non-trivial arc \( \alpha \subset \mathcal{S} \), which is the closed interval \([0, 1]\) properly embedded in \( \mathcal{S} \) not homotopic relative its endpoints to a subset of the boundary \( \partial \mathcal{S} \), is called a corner arc if it is as that depicted in Figure 21 (a), i.e. it cuts off from \( \mathcal{S} \) a triangle with one ideal vertex. Such an ideal vertex is said to be surrounded by the corner arc \( \alpha \).

A bad arc is a stated corner arc whose states are as in the figure Figure 21, i.e. they are followed by \( + \) if we go along the arc counterclockwise around a surrounded vertex. The reduced stated skein algebra \( \overline{\mathcal{S}}(\mathcal{S}) \) is defined to be the quotient of \( \mathcal{S} \) by the 2-sided ideal \( \mathcal{I}^{\text{bad}} \) generated by bad arcs.

7.2. Basis. Let \( o_+ \) be the orientation of \( \partial \mathcal{S} \) induced by that of \( \mathcal{S} \), i.e. every boundary edge has positive orientation. Then \( B := B(\mathcal{S}; o_+) \) is an \( \mathcal{R} \)-basis of \( \mathcal{S}(\mathcal{S}) \). Let \( \overline{B} = \overline{B}(\mathcal{S}) \subset B \) be the subset consisting of all elements in \( B \) which contain no bad arc.
Figure 21. (a) a bad arc (b) the splitting of a bad arc

Theorem 7.1. The set $\overline{B}$ is a free $\mathcal{R}$-basis of the $\mathcal{R}$-module $\mathcal{F}(\mathcal{S})$.

Proof. Let $A \subset \mathcal{S}$ be the $\mathcal{R}$-span of $\overline{B}$ and $A' \subset \mathcal{S}$ be the $\mathcal{R}$-span of $B \setminus \overline{B}$. One has $\mathcal{S} = A \oplus A'$. Let us prove that the ideal $\mathcal{I}^{bad}$ is equal to $A'$.

Proof that $A' \subset \mathcal{I}^{bad}$. Let $\gamma \in (B \setminus \overline{B})$, i.e. $\gamma$ contains a bad arc. We have to show that $\gamma \in \mathcal{I}^{bad}$. If an arc in $\gamma$ (at some corner) is bad, then the positive orientation and increasing states imply that all the arcs closer to the vertex of that corner are bad, see Figure 22.

Figure 22. If the outer arc is bad, then all inner arcs are bad, too.

Thus we assume that $\gamma$ has a bad arc which is an innermost arc, see Figure 23(b).

Figure 23. (a) The product $\alpha \beta$, here $\alpha$ (in red) is a bad arc, (b) an element $\gamma \in B$ which has a bad innermost arc (in red).

We have the relations in Figure 24, which are part of Lemma 2.4. The first relation

Figure 24. Moving endpoint with negative state (left) and positive state (right) allows us to move the end of the red arc with state $-$ (in $\gamma$) up until we get the diagram
in Figure 23(a), which is of the form \( \alpha \beta \), where \( \alpha \) is a bad arc. The result is that \( \gamma = \alpha \beta \).

Thus, \( \gamma \in \mathcal{I}^{\text{bad}} \).

Proof that \( \mathcal{I}^{\text{bad}} \subset A' \). We have to show that \( \alpha \beta, \beta \alpha \in A' \) for any bad arc \( \alpha \) and any \( \beta \in B \).

The product \( \alpha \beta \): In this case, \( \alpha \beta \) is presented as in Figure 23(a). We already saw that \( \alpha \beta = \gamma \), where \( \gamma \) is as in Figure 23(b). Since \( \gamma \in A' \), we see that \( \alpha \beta \in A' \).

![Figure 25. (a) Product \( \beta \alpha \), where \( \alpha \) is a bad arc (in red), (b) the diagram \( \gamma \)](image)

The product \( \beta \alpha \): In this case, \( \beta \alpha \) is presented as in Figure 25(a). Using the 2nd relation in Figure 24, we get that \( \beta \alpha = \gamma \), where \( \gamma \) is as in Figure 25(b). Since \( \gamma \in A' \), we see that \( \alpha \beta \in A' \).

Thus, \( \mathcal{I}^{\text{bad}} = A' \). Hence as \( \mathcal{R} \)-modules, \( \mathcal{F}(\mathcal{G}) = \mathcal{I}/J \cong A \), which has \( \mathcal{B} \) as an \( \mathcal{R} \)-basis. \( \square \)

Remark 7.2. Positive order is used substantially in the proof. For other orientation of \( \partial \mathcal{G} \), the set similar to \( \mathcal{B} \) might not be the basis of \( \mathcal{F}(\mathcal{G}) \).

Corollary 7.3. The ordinary skein algebra \( \mathcal{F}(\mathcal{G}) \) and the Muller skein algebra \( \mathcal{F}^+(\mathcal{G}) \) embed naturally into the reduced skein algebra \( \mathcal{F}(\mathcal{G}) \).

Proof. Clearly the standard basis of the ordinary skein algebra and the standard basis of the Muller skein algebra (where all the states are +) are subsets of the basis \( \mathcal{B} \) of \( \mathcal{F}(\mathcal{G}) \). \( \square \)

7.3. Corner elements.

Proposition 7.4. Let \( u \) be a stated corner arc with both states positive and \( v \) be the same arc with both states negative. Then \( uv = vu = 1 \) in \( \mathcal{F}(\mathcal{G}) \).

Proof. In \( \mathcal{F}(\mathcal{G}) \) we have

\[
\begin{align*}
u u &= \begin{array}{c}
  \text{+} \\
  \text{+}
\end{array} = q^2 + q^{-1/2} = q^{-1/2} = 1, \\
u v &= \begin{array}{c}
  \text{+} \\
  \text{+}
\end{array} = q^{-2} - q^{-5/2} = q^{-5/2} = 1,
\end{align*}
\]

where the second identity follows from (10) and the last follows from (14). Similarly,

\[
\begin{align*}
u v &= \begin{array}{c}
  \text{+} \\
  \text{+}
\end{array} = q^{-2} + q^{-5/2} = q^{-5/2} = 1,
\end{align*}
\]

where the second identity follows from (10) and the last follows from (14). \( \square \)
7.4. Filtration. For a finite collection \( \mathcal{A} \) of ideal arcs or simple closed loops let \( F_n^\mathcal{A}(\mathcal{T}(\mathcal{S})) \) be the \( \mathcal{R} \)-submodule of \( \mathcal{T}(\mathcal{S}) \) spanned by stated tangle diagrams \( \alpha \) such that \( \sum_{\alpha \in \mathcal{A}} I(\alpha, \alpha) \leq n \). Then \( (F_n^\mathcal{A}(\mathcal{T}(\mathcal{S})))_{n=0}^\infty \) is a filtration of \( \mathcal{T}(\mathcal{S}) \) compatible with the algebra structure. Denote by \( \text{Gr}^\mathcal{A}(\mathcal{T}(\mathcal{S})) \) the associated graded algebra:

\[
\text{Gr}^\mathcal{A}(\mathcal{T}(\mathcal{S})) = \bigoplus_{n=0}^\infty \text{Gr}^\mathcal{A}_n(\mathcal{T}(\mathcal{S})), \quad \text{where} \quad \text{Gr}^\mathcal{A}_n(\mathcal{T}(\mathcal{S})) = F_n^\mathcal{A}(\mathcal{T}(\mathcal{S}))/F_{n-1}^\mathcal{A}(\mathcal{T}(\mathcal{S})).
\]

From Theorem 7.1 we have the following analog of Proposition 2.11.

**Proposition 7.5.** Suppose \( \mathcal{A} \) is a collection of boundary edges of \( \mathcal{S} \).

(a) The set \( \{ \alpha \in B \mid \sum_{\alpha \in \mathcal{A}} I(\alpha, \alpha) \leq n \} \) is an \( \mathcal{R} \)-basis of \( F_n^\mathcal{A}(\mathcal{T}(\mathcal{S})) \).

(b) The set \( \{ \alpha \in B \mid \sum_{\alpha \in \mathcal{A}} I(\alpha, \alpha) = n \} \) is an \( \mathcal{R} \)-basis of \( \text{Gr}_n^\mathcal{A}(\mathcal{T}(\mathcal{S})) \).

7.5. Splitting theorem.

**Theorem 7.6.** Suppose \( \mathcal{S}' \) is the result of splitting \( \mathcal{S} \) along an interior ideal arc \( a \). The splitting algebra embedding \( \tilde{\theta}_a : \mathcal{T}(\mathcal{S}) \hookrightarrow \mathcal{T}(\mathcal{S}') \) descends to an algebra embedding

\[
(94) \quad \tilde{\theta}_a : \mathcal{T}(\mathcal{S}) \rightarrow \mathcal{T}(\mathcal{S}').
\]

Besides, if \( a \) and \( b \) are two disjoint ideal arcs in the interior of \( \mathcal{S} \), then

\[
(95) \quad \tilde{\theta}_a \circ \tilde{\theta}_b = \tilde{\theta}_b \circ \tilde{\theta}_a
\]

**Proof.** Suppose \( \alpha \subset \mathcal{S} \) is a bad arc. The geometric intersection \( I(\alpha, a) \) is 0 or 1. In the first case \( \theta_a(\alpha) = \alpha \) is also a bad arc in \( \mathcal{S}' \). In the second case the splitting of \( \alpha \), given in Figure 21(b), has a bad arc for both values of \( \nu \in \{ \pm \} \). It follows that \( \theta_a(T_{\text{bad}}) \subset T_{\text{bad}} \). Hence \( \theta_a \) descends to an algebra homomorphism \( \tilde{\theta}_a : \mathcal{T}(\mathcal{S}) \rightarrow \mathcal{T}(\mathcal{S}') \) and we also have (95).

It remains to show that \( \tilde{\theta}_a \) is injective. Let \( 0 \neq x \in \mathcal{T}(\mathcal{S}). \) We have to show that \( \tilde{\theta}_a(x) \neq 0 \). Since \( B(\mathcal{S}) \) is an \( \mathcal{R} \)-basis, there is a non-empty finite set \( S \subset B(\mathcal{S}) \) such that

\[
(96) \quad x = \sum_{\alpha \in S} c_\alpha \alpha, \quad 0 \neq c_\alpha \in \mathcal{R}.
\]

Let \( k = \max_{\alpha \in S} I(\alpha, a) \). Then \( S' := \{ \alpha \in S \mid I(\alpha, a) = k \} \) is non-empty.

Let \( \text{pr} : \mathcal{S}' \rightarrow \mathcal{S} \) be the projection and \( a', a'' \subset \mathcal{S}' \) be the boundary edges which are \( \text{pr}^{-1}(a) \). To simplify the notations we write \( F_n^{a'} \) and \( \text{Gr}^a_n \) for respectively \( F_n^{a'}(\mathcal{T}(\mathcal{S}')) \) and \( \text{Gr}^a_n(\mathcal{T}(\mathcal{S}')) \). From the formula of the splitting homomorphism, for every \( \alpha \in S \),

\[
\tilde{\theta}_a(\alpha) \in F_{k}^{a'} \cap F_{k}^{a''} \subset F_{2k}^{a',a''}.
\]

Let \( P : F_{2k}^{a',a''} \rightarrow \text{Gr}_{2k}^{a',a''} \) be the canonical projection. Clearly if \( \alpha \in S' \setminus S' \) then \( P(\alpha) = 0 \). We consider Case 1 and Case 2 below.

**Case 1:** There exists \( \beta \in S \) such that \( P(\tilde{\theta}_a(\beta)) \neq 0 \).

Choose an orientation of \( a \) such that the induced orientation on \( a'' \) is positive. Then the induced orientation on \( a' \) is negative, see Figure 26. Let \( \delta' \) be the orientation of \( \partial \mathcal{S}' \) which is positive everywhere except for the edge \( a' \) where it is negative. For \( \alpha \in S' \) its lift \( \tilde{\alpha} = \text{pr}^{-1}(\alpha) \) is a *partially stated tangle diagram*; it is stated everywhere except for endpoints on \( a' \cap a'' \), and the endpoints on each of \( a' \) and \( a'' \) are ordered by \( \delta' \). Let \( \tilde{\alpha}^+ \) be the same \( \tilde{\alpha} \) except that the order on \( a' \) (and hence on all edges) is given by the positive orientation.
For $0 \leq j \leq k$ let $s_j(\tilde{\alpha})$ (respectively $s_j(\tilde{\alpha}^+)$) be the stated tangle diagram which is $\tilde{\alpha}$ (respectively $\tilde{\alpha}^+$) where the states on each of $a'$ and $a''$ are increasing and having exactly $j$ minus signs. Then $s_j(\tilde{\alpha}^+)$ is either equal to $0$ in $\mathcal{F}(\mathcal{G}')$ or belongs to the basis set $\mathcal{B}(\mathcal{G}')$.

By Proposition 2.16 and then Proposition 2.13 we have, for some $f(\alpha, j) \in \mathbb{Z}$,

$$P(\tilde{\theta}_a(\alpha)) = \sum_{j=0}^{k} \binom{k}{j} q^j s_j(\tilde{\alpha}) = \sum_{j=0}^{k} \binom{k}{j} q^j q^{f(\alpha,j)} s_j(\tilde{\alpha}^+).$$

Since $P(\tilde{\theta}_a(\beta)) \neq 0$, there is $l$ such that $s_l(\tilde{\beta}^+) \neq 0$ in $\mathcal{F}(\mathcal{G}')$ and hence $s_l(\tilde{\beta}^+) \in \mathcal{B}(\mathcal{G}')$.

Using (97) we have

$$P(\tilde{\theta}_a((x)) = \sum_{\alpha \in S'} \sum_{j=0}^{k} \binom{k}{j} q^j c_\alpha s_j(\tilde{\alpha}^+).$$

As $\alpha \in S'$ can be recovered from $\tilde{\alpha}$, if $\alpha \neq \beta$ then the two partially stated diagrams $\tilde{\alpha}$ and $\tilde{\beta}$ are not isotopic. It follows that $s_l(\tilde{\beta}^+) \neq s_j(\tilde{\alpha}^+)$ for all $j$ and all $\alpha \neq \beta$. It is also clear that $s_j(\tilde{\theta}^+) \neq s_l(\tilde{\theta}^+)$ for $j \neq l$. Hence the right hand side of (98) is not $0$, since the basis element $s_l(\tilde{\beta}^+)$ has non-zero coefficient, and all other elements $s_j(\tilde{\alpha}^+)$ is either $0$ or a basis element different from $s_l(\tilde{\beta}^+)$. Thus $P(\tilde{\theta}_a((x)) \neq 0$ and consequently $\tilde{\theta}_a(x) \neq 0$. This completes the proof in Case 1.

**Case 2:** For all $\alpha \in S$ we have $P(\tilde{\theta}_a(\alpha)) = 0$. Identity (97) shows that $s_j(\tilde{\alpha}^+) = 0$ for all $0 \leq j \leq k$ and all $\alpha \in S'$.

Incident with $a'$ there are two corners, the top left corner and the bottom left corner. Similarly, incident with $a''$ there are the top right corner and the bottom right corner, see Figure 26. A corner arc of $\tilde{\alpha}$ at one of these four corners has one end stated and one end not stated, and it is called a negative (respectively positive) corner arc if this only state is negative (respectively positive).

For $\alpha \in S'$ and $\nu \in \{\pm\}$ let $TL_\nu(\alpha)$ be the number of top left corner arcs whose only state is $\nu$. Define $TR_\pm(\alpha), BL_\pm(\alpha), BR_\pm(\alpha)$ similarly.

**Lemma 7.7.** Suppose $\alpha \in S'$. One of the following two mutually exclusive cases happens:

(i) $TL_-(\alpha) > 0$ and $BL_+(\alpha) > 0$, or
(ii) $BR_- (\alpha) > 0$ and $BR_+ (\alpha) > 0$.

**Proof.** Since $s_0(\tilde{\alpha}^+) = 0$ in $\mathcal{F}(\mathcal{G}')$, it has a bad arc. This implies either $TL_-(\alpha) > 0$ or $BR_-(\alpha) > 0$. 

![Figure 26. The split surface $\mathcal{G}'$, with orientations $a'$ on $\partial \mathcal{G}'$. The top left, top right, bottom left, and bottom right corners are marked respectively $TL, TR, BL, BR$.](image)
Assume $\TL_-(\alpha) > 0$. Since $\alpha$ does not have a bad arc, we conclude that $\TR_+(\alpha) = 0$. Then from $s_k(\tilde{\alpha}^+) = 0$ we see that $\BR_+(\alpha) > 0$. Again since $\alpha$ does not have a bad arc, we conclude that $\BR_-(\alpha) = 0$. Thus we have case (i) but not case (ii).

Assume $\BR_-(\alpha) > 0$. Since $\alpha$ does not have a bad arc, we conclude that $\BL_+(\alpha) = 0$. Then $s_0(\tilde{\alpha}^+) = 0$ we see that $\TR_+(\alpha) > 0$. Again since $\alpha$ does not have a bad arc, we conclude that $\BL_-(\alpha) = 0$. Thus we have case (ii) but not case (i).

The cases (i) and (ii) of Lemma 7.7 partition $S' = S'_L \sqcup S'_R$, where

$$S'_L = \{\alpha \in S' \mid TL_-(\alpha)BL_+(\alpha) > 0\}, \quad S'_R = \{\alpha \in S' \mid BR_-(\alpha)BR_+(\alpha) > 0\}.$$ 

**Lemma 7.8.** If $\alpha \in S'_L$ then $\Bar{\alpha}(\alpha) = 0$ in $\Gr_\alpha'$. Similarly, if $\alpha \in S'_R$ then $\Bar{\alpha}(\alpha) = 0$ in $\Gr_\alpha''$.

**Proof.** Suppose $\alpha \in S'_L$. For $\Bar{\nu} = (\nu_1, \ldots, \nu_k)$ the stated tangle diagram $(\tilde{\alpha}, \Bar{\nu})$ is defined to be $\tilde{\alpha}$ with states on both $a'$ and $a''$ are sequence $\Bar{\nu}$ listed from top to bottom. By definition,

$$\Bar{\alpha}(\alpha) = \sum_{\Bar{\nu} \in \{\pm\}^k} (\tilde{\alpha}, \Bar{\nu}).$$

Let $(\tilde{\alpha}^+, \Bar{\nu})_L$ be $\tilde{\alpha}^+$ whose states on $a''$ is given by $\Bar{\nu}$ but whose state on $a'$ is given by a permutation of $\Bar{\nu}$ such that the states are increasing on $a'$. By Proposition 2.13

$$(\tilde{\alpha}, \Bar{\nu}) = (\tilde{\alpha}^+, \Bar{\nu})_L \text{ in } \Gr_\alpha'.$$

If $\Bar{\nu}$ has at least one negative sign then $(\tilde{\alpha}^+, \Bar{\nu})_L$ has a bad arc in the bottom left corner (because $BL_+(\alpha) > 0$) and hence is equal to 0 in $\F(S')$. If $\Bar{\nu}$ has at least one positive sign then $(\tilde{\alpha}^+, \Bar{\nu})_L$ has a bad arc in the top left corner (because $TL_-(\alpha) > 0$) and hence is equal to 0 in $\F(S')$. Thus we always have $(\tilde{\alpha}^+, \Bar{\nu})_L = 0$ in $\F(S')$. From (100) and (99) we conclude that $\Bar{\alpha}(\alpha) = 0$ in $\Gr_\alpha''$.

The other case follows from the above case by noticing that if one rotates the Figure 26 by $180^\circ$, then the top left corner becomes the bottom right corner. \qed

As $\emptyset \neq S' = S'_L \sqcup S'_R$, one of $S'_L$ and $S'_R$ is non-empty. Without loss of generality we can assume that $S'_L$ is not empty.

Let $d = \min\{\TL_-(\alpha) \mid \alpha \in S'_L\}$ and $S'' = \{\alpha \in S'_L \mid TL_-(\alpha) = d\}$. Then $S'' \neq \emptyset$.

Let $P_+ : \F(S') \rightarrow \F(S')$ be the $\R$-linear map defined on basis elements $\gamma \in \Bar{B}(S')$ by

$$P_+(\gamma) = \begin{cases} 
\gamma & \text{if } I(\gamma, a') = d, I(\gamma, a'') = k, \text{ all states on } a' \text{ are } + \\
0 & \text{otherwise}. 
\end{cases}$$

For $\alpha \in S''$ let $\tilde{\alpha}$ be the stated tangle diagram obtained from $\tilde{\alpha}^+$ by first removing the $d$ negative top left corner arcs then providing states on $a'$ and $a''$ so that all states on $a'$ are + and the states on $a''$ are increasing and having exactly $d$ negative signs. Since $BR_-(\alpha) = 0$, we see that $\tilde{\alpha}$ is an element of the basis $\Bar{B}(S')$. As $\alpha$ can be recovered from $\tilde{\alpha}$, the map $\alpha \rightarrow \tilde{\alpha}$ from $S''$ to $\Bar{B}(S')$ is injective.

Let $u$ be a top left corner arc whose both states are +.
Lemma 7.9. For $\alpha \in S$ one has
\[
P_+(\tilde{\theta}_a(\alpha) u^d) = \begin{cases} q^{-(k-d)(k-d-1)/2} \tilde{\alpha} & \text{if } \alpha \in S'' \\ 0 & \text{if } \alpha \notin S''. \end{cases}
\]

Proof. One has $S = (S'_R \sqcup S'_L) \sqcup (S \setminus S')$. If $\alpha \in (S \setminus S')$ then $I(\alpha, a) < k$ and hence $P^+(\tilde{\theta}_a(\alpha)) = 0$.

If $\alpha \in S'_R$ then by Lemma 7.8 one has $\tilde{\theta}_a(\alpha) = 0$ in $Gr''$ which means $\tilde{\theta}_a(\alpha)$ is a linear combination of elements $\gamma \in B$ with $I(\gamma, a'') < k$. It follows that $P_+(\tilde{\theta}_a(\alpha)) = 0$.

It remains to consider the case $\alpha \in S'_L = S'' \sqcup (S'_L \setminus S'')$. From (99),
\[
P_+(\theta_a(\alpha) u^d) = \sum_{\nu \in \{\pm\}^k} P_+((\tilde{\alpha}, \nu) u^d).
\]
Recall that for $\beta \in \bar{B}(\mathcal{S}')$ one defines $\delta_{\beta'}(\beta)$ as the sum of all the states of $\beta \cap a'$. From the definition, if $\delta_{\beta'}(\beta) \neq k - d$ then $P^+(\beta) = 0$. If $\nu$ has $m$ negative signs where $m > d$ then
\[
\delta_{\beta'}((\tilde{\alpha}, \nu) u^d) = k - 2m + d < k - d,
\]
and hence $P_+((\tilde{\alpha}, \nu) u^d) = 0$. Thus we can assume that in the sum in (102), the number of negative signs in $\nu$ is $\leq d$.

Assume that $TL_-(\alpha) = m$. Note that the $m$ negative top left corner arcs of $\tilde{\alpha}$ are below any other components of $\tilde{\alpha}$. Hence the number of the first $m$ components of $\nu$ must be negative since otherwise one of the $m$ top left corner arcs is bad and $(\tilde{\alpha}, \nu) = 0$ in $\mathcal{F}(\mathcal{S}')$.

We conclude that if $TL_-(\alpha) > d$ (that is, if $\alpha \in S'_L \setminus S''$), then
\[
P_+(\theta_a(\alpha) u^d) = 0.
\]
Moreover, if $TL_-(\alpha) = d$ (that is, $\alpha \in S''$), then
\[
P_+(\theta_a(\alpha) u^d) = P_+((\tilde{\alpha}, \nu_d) u^d),
\]
where $\nu_d \in \{\pm\}^k$ is the sequence whose first $d$ components are $-$ and all other components are $+$. The $d$ negative top left corner arcs of $(\tilde{\alpha}, \nu_d)$ are all $\nu$, the corner edge with negative states on both ends. Hence we have $(\tilde{\alpha}, \nu) = \tilde{\alpha}' \nu^d$, where $\tilde{\alpha}'$ is the same as $\tilde{\alpha}$ except that the order on $a'$ is negative. Using the height exchange move between positive states, see Equation (15), and relation $\nu u = 1$ we get that
\[
P_+((\tilde{\alpha}, \nu_d) u^d) = q^{-(k-d)(k-d-1)/2} \tilde{\alpha},
\]
which proves (101) and completes the proof of the lemma. \hfill \Box

Let us continue the proof of the theorem for Case 2. From Lemma 7.9 we have
\[
P_+(\tilde{\theta}_a(x) u^d) = \sum_{\alpha \in S''} c_{\alpha} q^{-(k-d)(k-d-1)/2} \tilde{\alpha},
\]
which is non-zero since $\{\tilde{\alpha}\}$ are distinct elements of the basis $\bar{B}(\mathcal{S}')$. The theorem is proved. \hfill \Box
7.6. The bigon. The elements $\alpha_{\mu\nu} \in \mathcal{H}(\mathcal{B})$ are defined in Section 3.

**Proposition 7.10.** Let $\mathcal{B}$ be the bigon. There is an algebra isomorphism $\mathcal{T}(\mathcal{B}) \cong \mathcal{R}[x^{\pm 1}]$ given by $\alpha_{++} \rightarrow x$, $\alpha_{--} \rightarrow x^{-1}$, $\alpha_{+-} \rightarrow 0$, $\alpha_{-+} \rightarrow 0$.

**Proof.** A presentation of the algebra $\mathcal{T}(\mathcal{B}) \cong \mathcal{O}_q^2(\text{SL}(2))$ is given by Theorem 3.4, with generators $a = \alpha_{++}$, $b = \alpha_{--}$, $c = \alpha_{+-}$, $d = \alpha_{-+}$ and relations (50) and (51). The only bad arcs in $\mathcal{B}$ are $\alpha_{--} = c$ and $\alpha_{+-} = b$. Thus $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B})/\mathcal{I}_{\text{bad}}$ has a presentation like that of $\mathcal{O}_q^2(\text{SL}(2))$, with additional relations $b = c = 0$. From the quantum determinant relation in (51) we get $ad = 1$ in $\mathcal{T}(\mathcal{B})$.

On the other hand it is easy to check that the relations $b = c = 0$ and $ad = 1$ imply all other relations in (50) and (51). Hence $\mathcal{T}(\mathcal{B}) \cong \mathcal{R}\langle a, b, c, d \rangle/(ad = 1, b = c = 0) \cong \mathcal{R}[a^{\pm 1}]$. □

7.7. The triangle. Let $\mathcal{P}_3$ be the ideal triangle, with boundary edges $a, b, c$ as in Figure 27. Let $\alpha, \beta, \gamma$ be the corner arcs which are opposite respectively to $a, b$ and $c$. For $\mu, \nu \in \{\pm\}$ and $\xi \in \{\alpha, \beta, \gamma\}$ let $\xi(\mu\nu)$ be the arc $\xi$ with states $\mu$ and $\nu$ on the end points such that $\nu$ follows $\mu$ along $\xi$ counter-clockwise (with respect to the vertex surrounded by $\xi$).

![Figure 27. Edges $a, b, c$ opposite to corner arcs $\alpha, \beta, \gamma$](image)

For an anti-symmetric $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ the quantum torus associated to $A$ is the algebra with presentation

$$\mathcal{R}\langle x_i^{\pm 1}, i = 1, \ldots, n \rangle/(x_i x_j = q^{a_{ij}} x_j x_i).$$

For basic properties of quantum tori see for example [Le1, Section 2].

To the triangle $\mathcal{P}_3$ we associate the quantum torus $\mathbb{T}$ with presentation

$$\mathbb{T} := \mathcal{R}\langle \alpha^{\pm 1}, \beta^{\pm 1}, \gamma^{\pm 1} \rangle/(q^{\alpha\beta} = \beta\alpha, q^{\beta\gamma} = \gamma\beta, q^{\gamma\alpha} = \gamma\alpha).$$

The cyclic group $\mathbb{Z}/3 = \langle \tau \mid \tau^3 = 1 \rangle$ acts by algebra automorphisms on each of the algebras $\mathcal{H}(\mathcal{P}_3)$, $\mathcal{T}(\mathcal{P}_3)$, and $\mathbb{T}$ as follows. In short $\tau$ is rotation $\tau$ by $2\pi/3$ counterclockwise about the center of the triangle. This rotation induces the algebra automorphism $\tau$ of $\mathcal{H}(\mathcal{P}_3)$; it also induces the algebra automorphism $\tau$ of $\mathcal{T}(\mathcal{P}_3)$. On $\mathbb{T}$ and $\tau$ is given by

$$\tau(\alpha) = \beta, \ \tau(\beta) = \gamma, \ \tau(\gamma) = \alpha.$$

**Theorem 7.11.** The reduced skein algebra $\overline{\mathcal{T}}(\mathcal{P}_3)$ of the ideal triangle is isomorphic to the quantum torus $\mathbb{T}$. The isomorphism is $\mathbb{Z}/3$-equivariant and given by

$$\alpha(++) \rightarrow \alpha, \ \alpha(+-) \rightarrow q^{-\frac{2}{3}}\gamma\beta, \ \alpha(-+) \rightarrow 0, \ \alpha(--) \rightarrow \alpha^{-1}.$$
Proof. By [Le2, Theorem 4.6] the algebra $\mathcal{S}(P_3)$ is generated by

$$X = \{\alpha(\nu, \nu'), \beta(\nu, \nu'), \gamma(\nu, \nu') \mid \nu, \nu' \in \{\pm\}\}$$

subject to the following relations and their images under $\tau$ and $\tau^2$:

\begin{align}
\beta(\mu, \nu) \alpha(\mu', \nu') &= q \alpha(\nu, \nu') \beta(\mu, \mu') - q^2 \nu C_{\nu'}(\gamma(\nu', \mu)) \\
\alpha(-, \nu) \alpha(+, \nu') &= q^2 \alpha(+, \nu) \alpha(-, \nu') - q^{5/2} C_{\nu'} \\
\alpha(\nu, -) \alpha(\nu', +) &= q^2 \alpha(\nu, +) \alpha(\nu', -) - q^{5/2} C_{\nu'} \\
\alpha(-, \nu) \beta(\nu', +) &= q^2 \alpha(\nu, +) \beta(\nu', -) - q^{5/2} \nu(\nu, \nu') \\
\alpha(\nu, -) \gamma(+, \nu') &= q^2 \alpha(\nu, +) \gamma(-, \nu') + q^{-1/2} \beta(\nu', \nu).
\end{align}

As the only bad arcs are $\alpha(-, +), \beta(-, +), \gamma(-, +)$, the quotient $\mathcal{S}(P_3)$ is obtained by adding the relations $\alpha(-, +) = \beta(-, +) = \gamma(-, +) = 0$, and from this presentation one can check that the map given by (105) and its images under the action of $\mathbb{Z}/3$ is an isomorphism.

Here is an alternative, more geometric proof. First in $\mathcal{S}(P_3)$ we have

$$\alpha(++) \alpha(--) = 1, \quad \beta(++) \alpha(++) = q \alpha(++) \beta(++),$$

and all its images under $\mathbb{Z}/3$. In fact the first identity follows from Proposition 7.4 and the second follows from the height exchange identity (15). It follows that the $\mathbb{Z}/3$-equivariant map $f : \Gamma \to \mathcal{S}(P_3)$ given by

$$f(\alpha) = \alpha(++) , \quad f(\alpha^{-1}) = \alpha(--),$$

gives a well-defined algebra homomorphism, as all the defining relations of $\mathbb{Z}/3$ are preserved under $f$. In $\mathcal{S}(P_3)$ we have

$$\gamma(+-) = q^{-1/2} \beta(++) \gamma(--),$$

which follows from the identity in Figure 16 (where the left hand arc is stated to become $\alpha(+-)$). Thus all elements in the generator set $X$ are in the image of $f$. This shows that $f$ is surjective.

Let us show $f$ is injective. The set $\{\alpha^k \beta^m \gamma^n \mid k, m, n \in \mathbb{Z}\}$ is an $\mathcal{S}$-basis of $\Gamma$. Assume that there is a finite set $S \subset \mathbb{Z}^3$ such that

\begin{equation}
\sum_{(k,m,n) \in S} c_{k,m,n} \alpha^k \beta^m \gamma^n = 0, \quad c_{k,m,n} \in \mathcal{S}.
\end{equation}

Multiplying the identity (113) on the left by $f(\alpha^{k'} \beta^{m'} \gamma^{n'})$ with large $k', m', n'$ and using the $q$-commutations between $\alpha, \beta, \gamma$ we can assume that $k, m, n > 0$ in (113). For each $(k, m, n) \in \mathbb{N}^3$ let $z(k, m, n)$ be the stated simple tangle diagram consisting of $k$ arcs parallel to $\alpha$, $m$ arcs parallel to $\beta$, and $n$ arcs parallel to $\gamma$, with all state positive. Note that $z(k, m, n) \in \mathbb{S}(P_3)$. Clearly the map $z : \mathbb{N}^3 \to \mathbb{S}(P_3)$ is injective. As the diagram of $f(\alpha^k \beta^m \gamma^n)$ can be obtained from $z(k, m, n)$ by a sequence of height change moves of positively stated endpoints, the first identity of (15) shows that

$$f(\alpha^k \beta^m \gamma^n) = q^{g(k,m,n)} z(k, m, n)$$

by checking that $f(\alpha^k \beta^m \gamma^n)$ has the stated simple tangle diagram consisting of $k$ arcs parallel to $\alpha$, $m$ arcs parallel to $\beta$, and $n$ arcs parallel to $\gamma$, with all state positive. Note that $z(k, m, n) \in \mathbb{S}(P_3)$. Clearly the map $z : \mathbb{N}^3 \to \mathbb{S}(P_3)$ is injective. As the diagram of $f(\alpha^k \beta^m \gamma^n)$ can be obtained from $z(k, m, n)$ by a sequence of height change moves of positively stated endpoints, the first identity of (15) shows that
for some \( g(k, m, n) \in \mathbb{Z} \). From (98) we get
\[
\sum_{(k, m, n) \in S} c_{k, m, n} q^{g(k, m, n)} z(k, m, n) = 0.
\]
As \( z(k, m, n) \) are distinct elements of the basis \( \bar{B}(\mathcal{P}_3) \), this forces all \( c_{k, m, n} = 0 \). Hence \( f \) is injective.

\[ \square \]

7.8. **The quantum trace map.** Assume that \( \mathcal{S} \) is triangulable, i.e. \( \mathcal{S} \) is not one of the following: a monogon, a bigon, a sphere with one or two punctures. A triangulation \( \mathcal{E} \) of \( \mathcal{S} \) is a collection consisting of all boundary edges and several ideal arcs in the interior of \( \mathcal{S} \) such that

(i) no two arcs in \( \mathcal{E} \) intersect and no two are isotopic, and

(ii) if \( a \) is an ideal arc not intersecting any ideal arc in \( \mathcal{E} \) then \( a \) is isotopic to one in \( \mathcal{E} \).

It is known that if \( \mathcal{S} \) is triangulable, then by splitting \( \mathcal{S} \) along all interior ideal arcs in \( \mathcal{E} \) we get a collection \( \mathcal{F}(\mathcal{E}) \) of ideal triangles. By the splitting theorem, we get an algebra embedding of \( \mathcal{T}(\mathcal{S}) \) into a quantum torus
\[
\Theta : \mathcal{T}(\mathcal{S}) \to \bigotimes_{\mathcal{F}(\mathcal{E})} \mathbb{T}.
\]

In addition to the quantum torus \( \mathbb{T} \) we associate the quantum torus \( \mathbb{T}' \) to the standard ideal triangle \( \mathcal{P}_3 \):

\[
\mathbb{T}' := \mathcal{R}\langle a^{\pm1}, b^{\pm1}, c^{\pm1} \rangle / (qab = ba, qbc = cb, qca = ac).
\]

One should think of \( a, b, c \) as the edges opposite to \( \alpha, \beta, \gamma \), see Figure 27.

The cyclic group \( \mathbb{Z}/3 = \langle \tau \mid \tau^3 = 1 \rangle \) acts by algebra automorphisms on \( \mathbb{T}' \) by
\[
\tau(a) = b, \quad \tau(b) = c, \quad \tau(c) = a.
\]

There is a \( \mathbb{Z}/3 \)-equivariant algebra embedding \( \mathbb{T} \hookrightarrow \mathbb{T}' \), defined by
\[
\alpha \to q^{1/2}bc, \quad \beta \to q^{1/2}ca, \quad \gamma \to q^{1/2}ab.
\]

Consider the composition \( \text{tr}_q \) given by
\[
\text{tr}_q : \mathcal{T}(\mathcal{S}) \xrightarrow{\Theta} \bigotimes_{\mathcal{F}(\mathcal{E})} \mathbb{T} \hookrightarrow \bigotimes_{\mathcal{F}(\mathcal{E})} \mathbb{T}'.
\]

On the collection of all edges of all the triangles in \( \mathcal{F}(\mathcal{E}) \) define the equivalence relation generated by \( a' \cong a'' \) if they are glued together in the triangulation. Then the set of equivalence classes is canonically isomorphic to \( \mathcal{E} \). Let \( \mathcal{Y}(\mathcal{E}) \) be the subalgebra of \( \bigotimes_{\mathcal{F}(\mathcal{E})} \mathbb{T}' \) generated by all \( a' \otimes a'' \) with \( a \cong a' \) and all boundary edges (each boundary edge is equivalent only to itself). It is easy to see that the image of \( \text{tr}_q \) is in \( \mathcal{Y}(\mathcal{E}) \). Thus, \( \text{tr}_q \) restricts to
\[
\text{tr}_q : \mathcal{T}(\mathcal{S}) \to \mathcal{Y}(\mathcal{E}).
\]

The algebra \( \mathcal{Y}(\mathcal{E}) \) is a quantum torus, known as the Chekhov-Fock algebra associated to a triangulation \( \mathcal{E} \) of \( \mathcal{S} \), see [BW, CF1, Le2]. The restriction of \( \text{tr}_q \) to \( \mathcal{T}(\mathcal{S}) \) is the quantum trace map of Bonahon and Wong, see [Le2]. Thus we have proved the following.
Theorem 7.12. If $\mathcal{E}$ is a triangulation of $\mathcal{S}$ then there is an algebra embedding

$$\text{tr}_q : \mathcal{F}(\mathcal{S}) \to \mathcal{Y}(\mathcal{E})$$

extending the quantum trace map of Bonahon and Wong.

Remark 7.13. Actually in [Le2] the reduced stated skein algebra was not introduced, instead a direct algebra homomorphism from $\mathcal{F}(\mathcal{P}_3)$ to $\mathbb{T}$ is constructed.

7.9. Co/module structure for $\mathcal{F}(\mathcal{S})$. The Hopf algebra structure of $\mathcal{F}(\mathcal{B})$ descends to a Hopf algebra of $\mathcal{F}(\mathcal{B})$. We identify $\mathcal{F}(\mathcal{B}) \equiv \mathcal{R}[x^{\pm 1}]$ using the isomorphism of Proposition 7.10. Then $\Delta(x) = x \otimes x$ and $e(x) = 1$.

Arguing exactly as in Subsection 4.1, one sees that for each surface $\mathcal{S}$ and each edge $e$ of $\mathcal{S}$, the algebra $\mathcal{F}(\mathcal{S})$ has both a left and a right $\mathcal{R}[x^{\pm 1}]$-comodule algebra structure (which is equivalent to a $\mathbb{Z}$-valued grading counting the number of $+$ and $-$ states of each skein along $e$):

Proposition 7.14. (a) The map $\Delta_e : \mathcal{F}(\mathcal{S}) \to \mathcal{F}(\mathcal{S}) \otimes \mathcal{F}(\mathcal{B})$ gives $\mathcal{F}(\mathcal{S})$ a right comodule-algebra structure over the Hopf algebra $\mathcal{R}[x^{\pm 1}]$. Similarly $\Delta$ gives $\mathcal{F}(\mathcal{S})$ a left comodule-algebra structure over the Hopf algebra $\mathcal{R}[x^{\pm 1}]$.

(b) If $e_1, e_2$ are two distinct boundary edges, the coactions on the two edges commute, i.e. for instance

$$(\Delta_{e_2} \otimes \text{id}) \circ \Delta_{e_1} = (\Delta_{e_1} \otimes \text{id}) \circ \Delta_{e_2}.$$ 

In the reduced setting, though, Theorem 4.7 does no longer hold: indeed, with the notation in there, if $\beta \in \overline{B}(\mathcal{S})$ is a basis element intersecting a cutting edge exactly once, then its image $\theta(\beta)$ under the cutting morphism is $\theta(\beta) = \beta'_{++} + \beta'_{--}$ where $\beta'_{++}, \beta'_{--} \in \overline{B}(\mathcal{S}')$ are identical except for their states on $\beta' \cap (c_1 \cup c_2)$. But it is not difficult to check that $\beta'_{++}$ is balanced:

$$\Delta_{c_1}(\beta'_{++}) = \beta'_{++} \otimes \alpha_{++} + \beta'_{++} \otimes \alpha_{--} = \beta'_{++} \otimes \alpha_{++} = c_2 \Delta(\beta'_{++})$$

because the class of $\alpha_{--} = \alpha_{++} = 0 \in \mathcal{F}(\mathcal{S})(\mathcal{B})$, still $\beta'_{++}$ it is not in the image of $\theta$.

8. The classical case: twisted bundles

In this section we will suppose that $\mathcal{S}$ is a connected, oriented surface with a non-empty set of boundary edges and let $\mathcal{S}$ be the positive orientation of $\partial \mathcal{S}$ i.e. that induced by the orientation of $\mathcal{S}$. We will prove that if $q^{\frac{1}{2}} = 1$ then $\mathcal{F}(\mathcal{S})$ is isomorphic to the algebra of regular functions on the affine variety of “twisted bundles” on $\mathcal{S}$. The similar result for the case when $\partial \mathcal{S} = \emptyset$ is well known (see for instance [Thu]).

Fix an arbitrary Riemannian metric and let $U\mathcal{S}$ be the unit tangent bundle over $\mathcal{S}$, with the canonical projection $\pi : U\mathcal{S} \to \mathcal{S}$. A point in $U\mathcal{S}$ is a pair $(p, v)$, where $p \in \mathcal{S}$, $v \in T_p \mathcal{S}$, $\|v\| = 1$. For each immersion $\alpha : [0, 1] \to \mathcal{S}$ its canonical lift is path $(\alpha(t), \frac{\alpha'(t)}{\|\alpha'(t)\|})$ in $U\mathcal{S}$. In particular, since each edge $e$ of $\partial \mathcal{S}$ is oriented by $\mathcal{S}$, it has a canonical lift $\tilde{\mathcal{S}} \subset \partial U\mathcal{S}$; we will denote $\partial \tilde{\mathcal{S}} := \cup_{\mathcal{E}} \partial \mathcal{S} \tilde{\mathcal{S}}$. If we let $-e$ be the edge oriented in the opposite way, then we get a different lift which we will denote $(-e)^\sim$. Let $-\partial \tilde{\mathcal{S}} = \cup_{\mathcal{E}} \partial \mathcal{S}(-e)^\sim$ and $\pm \partial \mathcal{S} = \partial \tilde{\mathcal{S}} \cup -\partial \tilde{\mathcal{S}}$. 
For a point $x \in \mathcal{G}$ the fiber $O = \pi^{-1}(x)$ is a circle, and we will orient it according to the orientation of $\mathcal{G}$. It is clear that the homotopy class of $O$ does not depend on $x$.

For each boundary edge $e$ choose a point $x \in e$. Let $v \in T_x(e)$ be the unit tangent vector with orientation $o$. Then both $(x, v)$ and $(x, -v)$ are in $\pi^{-1}(x)$, and the half circle of $\pi^{-1}(x)$ going from $(x, v)$ to $(x, -v)$ in the positive direction is denoted by $\sqrt{O}_e$. The exact position of $x$ on $e$ will not be important in what follows.

**Definition 9** (Fundamental Groupoids). Let $X$ be a path connected topological space and $\{E_i\}_{i \in I}$ disjoint contractible subspaces of $X$. The fundamental groupoid $\pi_1(X, \{E_i\}_{i \in I})$ is the groupoid (i.e. a category with invertible morphisms) whose objects are $\{E_i, i \in I\}$ and whose morphisms are the homotopy classes of oriented paths in $X$ with endpoints in $\bigcup_{i \in I} E_i$. A morphism of groupoids if a functor of the corresponding categories.

Recall that a group is a groupoid with only one object. The proof of the following is standard and left to the reader:

**Lemma 8.1** (Extension of morphisms). With the above notation, let $E \subset X$ be a contractible subspace disjoint from $\bigcup_{i \in I} E_i$. Then given a morphism $\rho : \pi_1(X, \{E_i\}_{i \in I}) \to G$ for some group $G$, an oriented path $\gamma$ connecting some $E_i$ to $E$, and an arbitrary $g \in G$ there is a unique extension $\rho' : \pi_1(X, \{E_i\}_{i \in I} \cup \{E\}) \to G$ of $\rho$ such that $\rho'(\gamma) = g$ and $\rho'(\alpha) = \rho(\alpha)$ for all $\alpha \in \text{Mor}(E_i, E_j)$ for some $i, j$.

We shall be interested in two particular groupoids: $\pi_1(\mathcal{G}, \partial \mathcal{G})$ and $\pi_1(U\mathcal{G}, \tilde{\partial} \mathcal{G})$. Note that $\pi : (U\mathcal{G}, \tilde{\partial} \mathcal{G}) \to (\mathcal{G}, \partial \mathcal{G})$ induces a surjective morphism $\pi_*$ of groupoids.

**Definition 10** (Twisted bundle). A twisted bundle on $\mathcal{G}$ is a morphism $\rho : \pi_1(U\mathcal{G}; \tilde{\partial} \mathcal{G}) \to \text{SL}_2(\mathbb{C})$ such that $\rho(O) = -Id$.

By Lemma 8.1 we extend $\rho$ to a morphism (with the same notation) $\rho : \pi_1(U\mathcal{G}; \pm \tilde{\partial} \mathcal{G}) \to \text{SL}_2(\mathbb{C})$ such that for every boundary edge $e$,

\begin{equation}
(115) \quad \rho(\sqrt{O}_e) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{equation}

Since $\mathcal{G}$ is not a closed surface, its fundamental group $\pi_1(\mathcal{G})$ is a free group.

**Lemma 8.2.** Suppose that $\partial \mathcal{G} \neq \emptyset$. Then the set $\text{tw}(\mathcal{G})$ of twisted bundles on $\mathcal{G}$ is the affine algebraic variety $\text{SL}_2(\mathbb{C})^{n+k}$ where

\[ n = -1 + \#\{e \subset \partial \mathcal{G}\} \text{ and } k = \text{rank}(\pi_1(\mathcal{G})). \]

In particular the algebra $\chi(\mathcal{G})$ of its regular functions is generated by the matrix entries of each of the copies of $\text{SL}_2(\mathbb{C})$.

**Proof.** We claim that there are no canonical injective morphisms of fundamental groupoids $s_* : \pi_1(\mathcal{G}; \partial \mathcal{G}) \to \pi_1(U\mathcal{G}, \tilde{\partial} \mathcal{G})$. To build one, pick any non zero vector field on $\mathcal{G}$ which is positively tangent to the edges of $\partial \mathcal{G}$: it exists because we are not prescribing its behavior near the (non compact) cusps. This trivializes $U\mathcal{G}$ as $\mathcal{G} \times S^1$; let $s : \mathcal{G} \to \mathcal{G} \times \{1\}$ be a section of $\pi : U\mathcal{G} \to \mathcal{G}$. To each twisted bundle $\rho : \pi_1(U\mathcal{G}; \tilde{\partial} \mathcal{G}) \to \text{SL}_2(\mathbb{C})$ we associate
\( \rho' : \pi_1(\mathcal{G}; \partial \mathcal{G}) \to \text{SL}_2(\mathbb{C}) \) defined as \( \rho' = \rho \circ s_* \). Reciprocally given \( \rho' : \pi_1(\mathcal{G}; \partial \mathcal{G}) \to \text{SL}_2(\mathbb{C}) \) we extend it to \( \rho : \pi_1(U \mathcal{G}; \partial \mathcal{G}) \to \text{SL}_2(\mathbb{C}) \) by setting \( \rho(\emptyset) = -\text{Id} \) and \( \rho|_{\pi_1(\mathcal{G} \times \{1\})} = \rho' \).

Now fix a set of immersed smooth paths \( \alpha_1, \ldots, \alpha_n \subset \mathcal{G} \) connecting a fixed edge \( e_0 \subset \partial \mathcal{G} \) to each other edge of \( \partial \mathcal{G} \) as well as a set of paths whose endpoints are in \( e_0 \) representing generators \( g_1, \ldots, g_k \) of \( \pi_1(\mathcal{G}; e_0) \) (which is free because \( \partial \mathcal{G} \neq \emptyset \)). The image of each \( \alpha_i \) and \( g_j \) in \( \text{SL}_2(\mathbb{C}) \) provides the sought non canonical isomorphism.

**Example 8.3.** Let \( \mathcal{P}_n \) be the \( n \)-polygon with vertices numbered in the orientation sense from 0 to \( n-1 \); then \( tw(\mathcal{P}_n) = \text{SL}_2(\mathbb{C})^{n-1} \) where the \( n-1 \) matrices are given by the holonomies of the diagonals connecting the edge \( v_0v_1 \) to each other edge. Then \( \chi(\mathcal{P}_n) = O(\text{SL}_2)^{\otimes n-1} \) and in particular \( \chi(\mathcal{B}) = O(\text{SL}_2) \) and \( \chi(\mathcal{P}_3) = O(\text{SL}_2) \otimes O(\text{SL}_2) \) (where by “equal” we mean “non-canonically isomorphic to”).

**Remark 8.4.** The notion of twisted bundle is closely related to the one considered in [Thu].

### 8.1. Trace functions for non oriented curves

We will identify the states of a stated tangles with vector in \( \mathbb{C}^2 \) as follows:

\[
+ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, - := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

If \( \vec{x}, \vec{y} \in \mathbb{C}^2 \) let \( \det(\vec{x}|\vec{y}) \) denote the determinant of the matrix whose first column is \( \vec{x} \) and second is \( \vec{y} \).

We will say an immersion \( a : [0, 1] \to \mathcal{G} \) is in good position if \( a(0), a(1) \in \partial \mathcal{G} \) and the tangent vectors \( \dot{a}(0), \dot{a}(1) \) have orientation \( \alpha \).

An immersion \( \alpha : [0, 1] \to \mathcal{G} \) is transversal if \( \alpha(0), \alpha(1) \in \partial \mathcal{G} \) and \( \alpha \) is transversal to \( \partial \mathcal{G} \) at 0 and 1. One can bring such a transversal \( \alpha \) to an arc \( a \) in good position by an isotopy (relative 0 and 1) in a small neighborhood of \( \alpha(0) \) and \( \alpha(1) \). The canonical lift of \( a \) will be denoted by \( \hat{a} \) and is called the good lift of \( \alpha \). Note that the homotopy class of \( \hat{a} \) is uniquely determined by \( \alpha \), and we will consider \( \hat{a} \) as an element of \( \pi_1(U \mathcal{G}; \partial \mathcal{G}) \). Note that the good lift of the inverse path \( \alpha^{-1} \), defined by \( \alpha^{-1}(t) = \alpha(1-t) \), is not the inverse of \( \hat{a} \), since before lifting one has to isotope \( \alpha^{-1} \) to good position.

A stated transversal immersion is a transversal immersion whose end points are stated \( \{ \pm \} \).

**Definition 11** (Trace). Let \( \rho \) be a twisted bundle on \( \mathcal{G} \).

Assume \( \alpha : [0, 1] \to \mathcal{G} \) is a stated transversal immersion with state \( \varepsilon \) at \( \alpha(0) \) and \( \eta \) at \( \alpha(1) \). Define the trace of \( \alpha \) by

\[
\text{tr}(\alpha) := \det(\eta|\rho(\hat{\alpha}) \cdot \varepsilon).
\]

Assume \( \beta : [0, 1] \to \mathcal{G} \) is an immersed closed curve (i.e. \( \beta(0) = \beta(1) \) and \( \beta \) has the same tangent at 0 and 1). Define the trace of \( \beta \) by

\[
\text{tr}(\beta) = \text{tr}(\rho(\hat{\beta})),
\]

where \( \beta' \) is any smooth closed curve isotopic to \( \beta \) such that \( \beta'(0) \in \partial \mathcal{G} \).

In the first case if \( \alpha' \) is homotopic to \( \alpha \) through stated transversal immersions and has the same states as \( \alpha \) then \( \text{tr}(\alpha') = \text{tr}(\alpha) \) (indeed the homotopy lifts to a homotopy of \( \hat{\alpha} \) and \( \hat{\alpha}' \)).

In the second case it is easy to see that \( \text{tr}(\beta) \) does not depend on the choice of \( \beta' \), as the images under \( \rho \) of any two such \( \beta' \) are conjugate in \( \text{SL}_2(\mathbb{C}) \) and hence have the same trace.
Example 8.5. Let $\alpha : [0, 1] \to S$ be a stated transversal immersion with state $\varepsilon$ at $\alpha(0)$ and $\eta$ at $\alpha(1)$.

If $\rho(\hat{\alpha}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\text{tr}(\alpha) = \det(\eta, \rho(\alpha) \cdot \varepsilon) = \frac{\eta \varepsilon_{1} | + - c}{+} + \frac{-a -b}{-d}$. 

Then remark that the matrix on the right, expressing the values of the traces for an immersed transverse stated arc $\alpha$ is $\rho(\sqrt{O}^{-1})\rho(\hat{\alpha})$ (see Equation (115)).

Remark 8.6. The notion of trace here is similar to the one introduced in [Mu], where trace is defined only for oriented arcs. The novelty here is the good lift, which is used to define traces for unoriented arcs and the use of twisted bundles as representations of fundamental groupoids.

When $\alpha$ is stated, we provide $\alpha^{-1}$ with states so that the state of $\alpha^{-1}(t)$ is equal to the state of $\alpha(1-t)$ for $t = 0, 1$.

Lemma 8.7. Suppose $\rho$ is a twisted bundle on $S$. (a) Let $\alpha$ be a stated transversal immersion. One has

$$\rho(\alpha^{-1}) = -\rho(\hat{\alpha})^{-1}.$$ 

As a consequence, $\text{tr}(\alpha) = \text{tr}(\alpha^{-1})$.

(b) Let $\beta : [0, 1] \to S$ be an immersed closed curve such that $\beta(0) \in \partial S$. Then

$$\rho(\tilde{\beta})^{-1} = \rho(\tilde{\beta}^{-1}).$$ 

As a consequence, if $\gamma$ is any immersed closed curve then $\text{tr}(\gamma) = \text{tr}(\gamma^{-1})$.

Proof. (a). A direct inspection shows that the homotopy class of the closed simple loop in $U S$ given by the concatenation $\alpha^{-1} \circ \hat{\alpha}$ is $O$ (see the left hand side of Figure 28). The first equality follows as by definition $\rho$ is a functor such that $\rho(O) = -Id$.

To prove that $\text{tr}(\alpha) = \text{tr}(\alpha^{-1})$, we compute the traces using the notation and content of Example 8.5:

If $\rho(\alpha^{-1}) = -\rho(\hat{\alpha})^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$, then $\text{tr}(\alpha) = \det(\eta', \rho(\alpha) \cdot \varepsilon') = \frac{\eta' \varepsilon'_{1} | + - c}{+} + \frac{-a -b}{-d}$. 

But since the state at $\alpha^{-1}(0)$ is $\eta$ and that at $\alpha^{-1}(1)$ is $\varepsilon$, we get the claim in this case by directly comparing with the transpose of the matrix of values provided in Example 8.5.

(b). Observe that if $\tilde{\beta}$ is the black curve depicted in the r.h.s. of Figure 28, then $\tilde{\beta}^{-1}$ is regularly homotopic to the dotted curve $\beta'$ in the same picture. By construction $\hat{\beta}'(0) = \hat{\beta}(0)$ and $\beta' \circ \beta$ is regularly homotopic to an eight-shaped immersed curve in a disc. Then $\tilde{\beta}^{-1}$ is homotopic to $\tilde{\beta}'$ in $U S$ and it holds $\rho(\tilde{\beta}' \circ \tilde{\beta}) = Id$, thus $\rho(\tilde{\beta}^{-1}) = \rho(\tilde{\beta})^{-1}$. The last statement now follows because $\rho(\tilde{\beta}) \in \text{SL}_2(\mathbb{C})$ so that $\text{tr}(\rho(\tilde{\beta})) = \text{tr}(\rho(\tilde{\beta})^{-1})$. 

□
Lemma 8.8. If non-oriented immersed closed curved. Define an isomorphism of algebras. The same holds for \( \text{tr} : \).

Corollary 4.16 at \( q \).

Applying Theorem 3.4 at \( \text{proff} \).

the map \( \text{tr} \) compositions and arc oriented arbitrarily, let \( S \).

EXAMPLE 8.3 is isomorphic to \( e \) the arcs connecting the edge \( S \).

by the proof of Corollary 4.16 a system of algebra generators of \( \mathcal{S}(\mathcal{P}_n) \) is easily seen to be the arcs connecting the edge \( e_0 \) to each other edge and stated arbitrarily. By Example 8.5, the map \( \text{tr} \) on these generators provides a system of generators of \( O(SL_2)^{\otimes n-1} \) which by Example 8.3 is isomorphic to \( \chi(\mathcal{P}_n) \).

Lemma 8.8. If \( q^\frac{1}{2} = 1 \) the map \( \text{tr} : \mathcal{S}(\mathcal{B}) \to \chi(\mathcal{B}) \) sending a stated skein to its trace is an isomorphism of algebras. The same holds for \( \text{tr} : \mathcal{S}(\mathcal{P}_n) \to \chi(\mathcal{P}_n) \) for every \( n \geq 2 \).

Proof. Applying Theorem 3.4 at \( q^\frac{1}{2} = 1 \) we get an explicit algebra isomorphism \( \phi : \mathcal{S}(\mathcal{B}) \to O(SL_2) \); by Example 8.3, we know that \( \chi(\mathcal{B}) \) is isomorphic to \( O(SL_2) \). Furthermore, by Example 8.5, the map \( \text{tr} \) is an algebra isomorphism. One argues similarly for \( \mathcal{P}_n \): applying Corollary 4.16 at \( q^\frac{1}{2} = 1 \) we get an explicit algebra isomorphism \( \phi : \mathcal{S}(\mathcal{P}_n) \to O(SL_2)^{\otimes n-1} \); by the proof of Corollary 4.16 a system of algebra generators of \( \mathcal{S}(\mathcal{P}_n) \) is easily seen to be the arcs connecting the edge \( e_0 \) to each other edge and stated arbitrarily. By Example 8.5, the map \( \text{tr} \) on these generators provides a system of generators of \( O(SL_2)^{\otimes n-1} \) which by Example 8.3 is isomorphic to \( \chi(\mathcal{P}_n) \).

8.2. Splitting theorem for trace functions. In all this subsection, let \( c \subset \mathcal{S} \) be an ideal arc oriented arbitrarily, let \( \mathcal{S}' \) be the result of cutting \( \mathcal{S} \) along \( c \) and let \( \text{pr} : \mathcal{S}' \to \mathcal{S} \) be the projection and \( \tilde{\text{pr}} : U\mathcal{S}' \to U\mathcal{S} \) the projection induced on the unit tangent bundles. Let \( \text{pr}^{-1}(c) = c_1 \cup c_2 \subset \partial \mathcal{S}' \) so that \( c_1 \) has the positive orientation and \( c_2 \) the negative one with respect to the orientation induced by that of \( \mathcal{S}' \) on the boundary. For each \( c_i \) let \( \tilde{c}_i \subset U\mathcal{S}' \) be its canonical lift and \( (-c_i)^{\sim} \) the canonical lift of \( -c_i \). Similarly let \( \tilde{c} \) be the canonical lift of \( c \) in \( U\mathcal{S} \) and \( (-c)^{\sim} \) be the canonical lift of \( -c \).

Lemma 8.9. Each \( \alpha \in \pi_1(U\mathcal{S}; \tilde{\mathcal{S}}) \) can be written as a composition \( [\alpha_k] \circ [\alpha_{k-1}] \circ \cdots \circ [\alpha_1] \) of homotopy classes of immersed paths \( \alpha_i : [0,1] \to U\mathcal{S} \) such that \( \alpha_i([0,1]) \subset \tilde{\mathcal{S}} \cup \tilde{c} \) and \( \alpha_i \cap \pi^{-1}(c) = \partial \alpha_i \cap \tilde{c} \). Such a decomposition is unique up to insertion/deletions of compositions \( \alpha' \circ [\alpha^{-1}] \) for \( \alpha' \in \pi_1(U\mathcal{S}; \tilde{c}) \) and replacement of \( [\alpha_j] \) by \( [\alpha_j''] \circ [\alpha_j'] \circ [\alpha_j']^\prime \).
some $[\alpha'_j]$, $[\alpha''_j] \in \pi_1(U \tilde{\mathcal{S}}; \partial \tilde{\mathcal{S}} \cup \tilde{c})$ and $[\alpha''_j] \in \pi_1(U \tilde{\mathcal{S}}; \tilde{c})$ such that $[\alpha_j] = [\alpha''_j] \circ [\alpha'_j]$ (or reciprocally).

Proof. Observe that $\pi^{-1}(c) \subset U \mathcal{S}$ is homeomorphic to an annulus $A = \mathbb{R} \times S^1$ so that $\tilde{c} = \mathbb{R} \times \{1\}$. Represent the class $[\alpha]$ by a smooth curve $\alpha : [0, 1] \to U \mathcal{S}$ so that it is transverse to $A$; then homotope it so that it intersects $A$ exactly along $\tilde{c}$: this provides an instance of the claimed splitting. If $\alpha' : ([0, 1], \{0, 1\}) \to (U \mathcal{S}, \partial \tilde{\mathcal{S}})$ is another smooth representative of the same class intersecting $A$ exactly along $\tilde{c}$, let $h(t, s) : [0, 1] \times [0, 1] \to U \mathcal{S}$ be a smooth homotopy between $\alpha$ and $\alpha'$ which is transverse to $A$. Then $h^{-1}(A)$ is a disjoint union of arcs and circles embedded in $[0, 1] \times [0, 1]$ with boundary in $[0, 1] \times \{0, 1\}$ containing a finite number of maxima and minima with respect to the height function given by the second coordinate $s$. Pick a finite number of heights $s_0 = 0 < s_1 < \cdots < s_n = 1$ so that each strip $[0, 1] \times [s_i, s_{i+1}]$ contains at most one maximum or minimum of the diagram of $h^{-1}(A)$. Each immersed path $\alpha_{s_i}(t) := h(t, s_i) : [0, 1] \to U \mathcal{S}$ intersects $A$ transversally a finite number of times and we can then modify $h$ locally around $h^{-1}(A) \cap ([0, 1] \times \{s_i\})$ without inserting new maxima and minima so that $\alpha_{s_i}(t)$ intersects $A$ only along $\tilde{c}$. Then the homotopies $h|_{[s_i, s_{i+1}]}$ transform the immersed path $\alpha_{s_i}$ into $\alpha_{s_{i+1}}$ by the moves described in the thesis: passing through a minimum replaces a smooth curve $\alpha$ with a composition $\alpha' \circ \alpha'' \circ \alpha''$ all were of $\alpha, \alpha', \alpha''$ intersect $A$ only along $\tilde{c}$ and in their boundary; passing through a maximum has the converse effect. Finally a strip containing no maxima and minima corresponds to a finite number of moves consisting in rewriting $\alpha \circ \alpha'$ with $\alpha \circ \beta \circ \beta^{-1} \circ \alpha'$ where $\beta \in \pi_1(U \mathcal{S}; \tilde{c})$ is the homotopy class represented by the restriction of $h$ to a "vertical arc" of $h^{-1}(A) \cap [0, 1] \times [s_i, s_{i+1}]$ (i.e. an arc joining $[0, 1] \times \{s_i\}$ and $[0, 1] \times \{s_{i+1}\}$).

If $\rho' : \pi_1(U \tilde{\mathcal{S}}; \partial \tilde{\mathcal{S}}') \to \text{SL}_2(\mathbb{C})$ is a twisted bundle, then by Lemma 8.1 we can extend it to a twisted bundle $\rho'' : \pi_1(U \tilde{\mathcal{S}}'; \partial \tilde{\mathcal{S}}' \cup (-c_2)\sim) \to \text{SL}_2(\mathbb{C})$ by setting $\rho''(\sqrt{D_{c_2}}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where $\sqrt{D_{c_2}}$ is the path connecting $\tilde{c}_2$ and $(-c_2)\sim$ by following in the positive direction the fiber $\pi^{-1}(x)$ for some $x \in c_2$.

**Proposition 8.10.** There is a surjective map $i^* : \text{tw}(\tilde{\mathcal{S}}') \to \text{tw}(\tilde{\mathcal{S}})$ defined as follows. Given $\alpha \in \pi_1(U \tilde{\mathcal{S}}; \partial \tilde{\mathcal{S}})$, decompose it as $\alpha = \alpha_k \circ \alpha_{k-1} \cdots \circ \alpha_1$ where each $\alpha_i \in \pi_1(U \tilde{\mathcal{S}}; \partial \tilde{\mathcal{S}} \cup \tilde{c})$ intersects $\pi^{-1}(c)$ at most in its endpoints and exactly along $\tilde{c}$ (such a decomposition exists by Lemma 8.9). Then for each $\rho' \in \text{tw}(\tilde{\mathcal{S}}')$ let

$$i^*(\rho')(\alpha) = \rho''(\alpha'_k) \rho''(\alpha'_{k-1}) \cdots \rho''(\alpha'_1)$$

where $\alpha'_i = \overline{\rho}^{-1}(\alpha_i)$ is the lift of $\alpha_i$ to $\pi_1(U \tilde{\mathcal{S}}; \partial \tilde{\mathcal{S}} \cup (-c_2)\sim)$. Passing to the algebras $\chi(\tilde{\mathcal{S}})$ and $\chi(\tilde{\mathcal{S}}')$ of regular functions on the algebraic varieties $\text{tw}(\tilde{\mathcal{S}})$ and $\text{tw}(\tilde{\mathcal{S}}')$, $i^*$ induces an injective algebra morphism $i : \chi(\tilde{\mathcal{S}}) \hookrightarrow \chi(\tilde{\mathcal{S}}')$ which we will call the "cutting morphism" associated to $c$.

Proof. By Lemma 8.9 to check that $i^*$ is well defined it is sufficient to check that for each $\alpha$ the choice of the decomposition does not affect the result of $i^*(\rho')(\alpha)$. But this is evident if we make an exchange $\alpha \circ \alpha_1 \leftrightarrow \alpha_2 \circ \alpha' \circ \alpha'^{-1} \circ \alpha_1$ or $\alpha \leftrightarrow \alpha' \circ \alpha'' \circ \alpha'''$ as in the statement of Lemma 8.9 because $\rho'$ is a functor.
To prove surjectivity observe that by Lemma 8.1 we can extend any morphism \( \rho : \pi_1(U\mathcal{G}; \partial \mathcal{G}) \to \text{SL}_2(\mathbb{C}) \) to \( \rho : \pi_1(U\mathcal{G}; \partial \mathcal{G} \cup \mathcal{C} \cup (-c)^{-}) \to \text{SL}_2(\mathbb{C}) \) setting in particular

\[
\rho(\sqrt{\mathcal{C}}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Then if \( \alpha' \in \pi_1(U\mathcal{G}'; \partial \mathcal{G}' \cup (-c_2)^{-}) \) define \( \rho' : \pi_1(U\mathcal{G}' ; \partial \mathcal{G}' \cup (-c_2)^{-}) \to \text{SL}_2(\mathbb{C}) \) by \( \rho'(\alpha') = \rho(\tilde{\rho}_*(\alpha')) \) where \( \tilde{\rho}_* : \pi_1(U\mathcal{G}' ; \partial \mathcal{G}' \cup (-c_2)^{-}) \to \pi_1(U\mathcal{G}; \partial \mathcal{G} \cup (-c)^{-}) \) is the morphism induced by the continuous map \( \tilde{\rho} : U\mathcal{G}' \to U\mathcal{G} \). Then letting \( \rho' \) be the restriction of \( \rho' \) to \( \pi_1(U\mathcal{G}'; \partial \mathcal{G}') \) we have that by construction \( \rho = i^*(\tilde{\rho}') \). Surjectivity of \( i^* \) implies the injectivity of \( i \).

The following proposition tells us that the trace functions behave exactly as the skeins under cutting along an ideal arc (see Theorem 2.14). Suppose \( \alpha \) is a stated transverse smooth simple curve intersecting transversally \( c \). Then \( \alpha' := \text{pr}^{-1}(\alpha) \) is a transverse smooth simple curve which is stated at every boundary point except for newly created boundary points, which are points in \( \text{pr}^{-1}(c) \cap \alpha' = (c_1 \cup c_2) \cap \alpha' \). A lift in \( \mathcal{G}' \) of \( \alpha \) is a stated transverse smooth simple curve \( \beta \) in \( \mathcal{G}' \) which is \( \alpha' \) equipped with states on \( \text{pr}^{-1}(c \cap \alpha) \) such that if \( x, y \in \text{pr}^{-1}(c \cap \alpha) \) with \( \text{pr}(x) = \text{pr}(y) \) then \( x \) and \( y \) have the same state. If \( |c \cap \alpha| = k \), then \( \alpha \) has \( 2^k \) lifts in \( \mathcal{G}' \).

**Proposition 8.11** (Cutting trace functions). Let \( \alpha \) be a stated transverse smooth simple curve intersecting transversally \( c \). Then

\[
(116) \quad i(\text{tr}(\alpha)) = \sum \text{tr}(\beta)
\]

where the sum is taken on all the lifts in \( \mathcal{G}' \) of \( \alpha \) (i.e. as in Theorem 2.14). Furthermore if \( c' \subset \mathcal{G} \) is another ideal arc disjoint from \( c \) and \( i' \) is the associated cutting morphism, it holds \( i' \circ i = i \circ i' \).

**Proof.** Since by Proposition 8.10 we already know that \( i \) is a well defined injective algebra morphism, it is sufficient to check the statement for a system of stated transverse smooth curves \( \{\gamma_i \in I\} \) which generate \( \chi(\mathcal{G}) \) as an algebra. By the proof of Lemma 8.2, we can choose a finite system of such \( \gamma_i \) such that \( |\gamma_i \cap c| \leq 2, \forall i \). Let \( \alpha \in \{\gamma_i, i \in I\} \) be represented by a smooth immersion \( \alpha : [0,1] \to \mathcal{G} \) intersecting transversally \( c \) with states \( st(\alpha(0)) = \epsilon, st(\alpha(1)) = \eta \); if \( |\alpha \cap c| = 0 \) the statement is true. If \( |\alpha \cap c| = 1 \) then \( \alpha = \alpha_2 \circ \alpha_1 \) where \( \alpha_1 \) are transverse smooth simple curves with \( \alpha_1(1) = \alpha_2(0) \in c \) and are partially stated by \( st(\alpha_1(0)) = \epsilon, st(\alpha_2(1)) = \eta \). Furthermore, up to switching \( \alpha \) to \( \alpha^{-1} \) we can suppose \( (\alpha_1(1), \hat{c}) \) form a positive basis of \( \mathcal{G} \).

Let then \( A_i \) (resp. \( A \)) be the \( 2 \times 2 \) matrix expressing the values of \( \text{tr}(\alpha_i) \) (resp. \( \text{tr}(\alpha) \)) with states in \( \{\pm\} \) as in Example 8.5; then, as remarked in the example \( A_i = \rho(\sqrt{\mathcal{C}})^{-1} \rho(\hat{\alpha}_i) \) (resp. \( A = \rho(\sqrt{\mathcal{C}})^{-1} \rho(\hat{\alpha}) \)) so that Equation (116) rewrites in this case as \( A = A_2 \cdot A_1 \).

Now since the orientation induced by \( \text{pr}^{-1}(c) \) is negative on \( c_2 \) then in \( U\mathcal{G} \) the good lift \( \hat{\alpha} \) of \( \alpha \) is homotopic to \( \tilde{\rho}(\hat{\alpha}_2) \circ \sqrt{\mathcal{C}}^{-1} \circ \tilde{\rho}(\hat{\alpha}_1) \) where \( \hat{\alpha}_i \) are depicted in the left hand side of Figure 29, therefore

\[
A = \rho(\sqrt{\mathcal{C}})^{-1} \rho(\hat{\alpha}) = \rho(\sqrt{\mathcal{C}})^{-1} \rho(\hat{\alpha}_2) \circ \rho(\sqrt{\mathcal{C}})^{-1} \circ \rho(\hat{\alpha}_1) = A_2 \cdot A_1
\]
and the claim is proved.

Suppose now that $|\alpha \cap c| = 2$ where $\alpha$ is a stated smooth immersion transverse to $c$; by the proof of Lemma 8.2 we can suppose that the sign of the intersections of $\alpha$ and $c$ is opposite and we can split $\alpha$ as $\alpha_3 \circ \alpha_2 \circ \alpha_1$ where $\alpha_i$ are transverse smooth immersions with $\alpha_1(1) = \alpha_2(0), \alpha_2(1) = \alpha_3(0)$ and partially stated so that $st(\alpha_1(0)) = \epsilon$ and $st(\alpha_3(1)) = \eta$. Furthermore, up to switching $\alpha$ and $\alpha^{-1}$ we can suppose that $(\dot{\alpha}_1(1), \dot{\alpha}^*)$ form a positive basis of $\mathfrak{g}$.

As above let $A_i = \rho(\mathbb{O}_c)^{-1} \cdot \rho(\hat{\alpha}_i)$ and $A = \rho(\mathbb{O}_c)^{-1} \cdot \rho(\hat{\alpha})$ and Equation (116) is equivalent to $A = A_3 \cdot A_2 \cdot A_1$. Then again, as shown in the right hand side of Figure 29, $\alpha$ is regularly homotopic in $U\mathfrak{g}$ to $\text{pr}(\hat{\alpha}_3) \circ \mathbb{O}_c^{-1} \circ \text{pr}(\hat{\alpha}_2) \circ \mathbb{O}_c^{-1} \circ \text{pr}(\hat{\alpha}_1)$. Therefore we have:

$$A = \rho(\mathbb{O}_c)^{-1} \cdot \rho(\hat{\alpha}) = \rho(\mathbb{O}_c)^{-1} \cdot \rho(\hat{\alpha}_3) \cdot \rho(\mathbb{O}_c)^{-1} \cdot \rho(\hat{\alpha}_2) \cdot \rho(\mathbb{O}_c)^{-1} \cdot \rho(\hat{\alpha}_1) = A_3 \cdot A_2 \cdot A_1$$

and the thesis follows.

\[
\square
\]

### 8.3. The classical limit of stated skein algebras.

**Theorem 8.12.** Suppose $q^{1/2} = 1$. The map sending a skein to its trace induces an algebra isomorphism

$$\text{tr} : \mathcal{S}(\mathfrak{g}) \rightarrow \chi(\mathfrak{g}).$$

**Proof.** We first claim that the relations (7), (8), (9), (10) with $q^{1/2} = 1$ are satisfied by the trace functions. By Lemma 8.8 the claim is true for bigons. But by Proposition 8.10 cutting induces an injective algebra map, thus to verify local relations we can verify them in a bigon containing the disc where the relations are depicted: this proves the claim in general.

The algebra isomorphism is proved as follows: pick an ideal triangulation of $\mathfrak{g}$ and apply to each edge of the triangulation Proposition 8.11 on the side of $\chi(\mathfrak{g})$ and Theorem 2.14 on $\mathcal{S}(\mathfrak{g})$. We get the following diagram of algebra morphisms of which the horizontal lines are...
injective and which is commutative by Proposition 8.11 and Theorem 2.14:

\[
\mathcal{J}(S) \xrightarrow{\hookrightarrow} \bigotimes_i \mathcal{J}(T_i) \\
\text{tr} \downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow \text{tr} \\
\chi(S) \xrightarrow{\hookrightarrow} \bigotimes_i \chi(T_i).
\]

(117)

Since by Lemma 8.8 the right vertical arrow is an isomorphism we conclude. □

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