Two-Loop Four-Gluon Amplitudes in N=4 Super-Yang-Mills

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Abstract

Using cutting techniques we obtain the two-loop $N = 4$ super-Yang-Mills helicity amplitudes for four-gluon scattering in terms of scalar integral functions. The $N = 4$ amplitudes are considerably simpler than corresponding QCD amplitudes and therefore provide a testing ground for exploring two-loop amplitudes. The amplitudes are constructed directly in terms of gauge invariant quantities and therefore remain relatively compact throughout the calculation. We also present a conjecture for the leading color four-gluon amplitudes to all orders in the perturbative expansion.
1 Introduction

Recent years have seen improvements in the calculational techniques for one-loop amplitudes \cite{1}. Besides calculations of new amplitudes for phenomenological \cite{2} purposes, a number of infinite sequences of amplitudes have been obtained \cite{3, 4, 5, 6}. At two- or higher-loops there has not been analogous progress as yet, although there is a need for such calculations in the analysis of experiments. For example, experiments at LEP are currently sensitive to next-to-next-to leading order corrections to $Z \rightarrow 3$ jets \cite{7}; this requires two-loop amplitudes as an input.

Before this and other two-loop calculations can be undertaken, a number of technical issues must be addressed. It would be useful to have an efficient technique for obtaining compact analytic expressions for two-loop amplitudes. Some first steps in developing such a technique have recently been reported using ideas motivated by string theory \cite{8, 9}. In particular, Reuter, Schmidt and Schubert \cite{10} have performed an elegant evaluation of two-loop quantities using a world-line approach. Kim and Nair \cite{11} have also set up a multi-loop formulation of the recursive approach \cite{12}.

Another technical hurdle is that many of the required integrals are uncalculated \cite{13}. The construction of a numerical program to extract jet cross-sections would also require significant work; in particular, one would need to extend the known techniques for handling the infrared divergent corners of phase space \cite{14}.

In this letter we explore the use of cutting rules \cite{15, 1} to obtain compact analytical expressions for two-loop amplitudes. In particular, we compute the $\mathcal{N}=4$ supersymmetric four-gluon amplitudes in terms of a basic set of scalar integral functions. Amplitudes in $\mathcal{N}=4$ super-Yang-Mills are particularly simple and are therefore a good starting point.

At tree-level \cite{16} and one-loop, infinite sequences of gauge theory amplitudes have been explicitly constructed. Can one find such sequences for higher-loop amplitudes? In this letter we also present a pattern for the leading-color four-gluon amplitudes to all orders of perturbation theory consistent with two-particle cuts. We do not, however, have a proof that the pattern is the complete answer and it is possible that a more complete analysis including three- or higher-particle cuts might reveal additional terms.

In the past, cutting rules \cite{15} have been widely used in field theory and provide powerful constraints on the form of amplitudes. In order to construct complete amplitudes one must have control of the ambiguities and subtractions. At one-loop it has been shown that for amplitudes satisfying a power counting criterion, such as supersymmetric ones, one can obtain the complete answer from four-dimensional cuts and knowledge of the set of integral functions that can appear in the result \cite{1}. For amplitudes not satisfying the power counting criterion one may unambiguously obtain the amplitudes by computing the cuts to all orders in the dimensional regularization parameter \cite{17, 18, 1}, since all terms develop cuts. At two loops much less is known about the integral functions. Nevertheless, for the supersymmetric two-loop amplitudes considered in this letter one can perform a complete reconstruction.
Two-loop color decomposition

At tree and one-loop levels it has proven useful to separate the color factors from the kinematics [19]. (The reader may consult various review articles [20, 1] for notation and further details.) The same type of color decomposition is also useful at the two-loop level. The color decomposed expression for a four-point amplitude with all particles in the adjoint representation is

\[
A_{4;1,1}^{2\text{-loop}}(1, 2, 3, 4) = g^6 \sum_{\sigma \in S_4/Z_4} N_c^2 \text{Tr}[T^{a_\sigma(1)} T^{a_\sigma(2)} T^{a_\sigma(3)} T^{a_\sigma(4)}] \left( A_{4;1,1}^{\text{LC}}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) + \frac{1}{N_c^2} A_{4;1,1}^{\text{SC}}(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \right) + g^6 N_c \text{Tr}[T^{a_\sigma(1)} T^{a_\sigma(2)}] \text{Tr}[T^{a_\sigma(3)} T^{a_\sigma(4)}] A_{4;1,3}(\sigma(1), \sigma(2); \sigma(3), \sigma(4)),
\]

where \( A_{4;1,1} \) and \( A_{4;1,3} \) are ‘partial amplitudes’. Our notation for \( A_{n;j,k} \) is that \( n \) is the number of external legs, and \( j \) and \( k \) label the position at which the color traces are split; for \( n \geq 6 \) there can be up to three color traces so for a generally consistent notation two indices are required. We have explicitly broken up the single color trace partial amplitude into leading color \( A_{4;1,1}^{\text{LC}} \) and subleading color \( A_{4;1,1}^{\text{SC}} \) pieces. (By subleading color we mean expressions suppressed in powers of \( N_c \).) For each external leg we have abbreviated the dependence on the outgoing external momenta, \( k_i \), and polarizations, \( \epsilon_i \), by the label \( i \). The notation ‘\( S_4/Z_4 \)’ denotes the set of all permutations of four objects \( S_4 \), omitting the cyclic transformations. The notation ‘\( S_4/Z_2^3 \)’ refers again to the set of permutations of four objects omitting those permutations which exchange labels within a single trace or exchange the two traces. That is, \( S_4/Z_2^3 = \{(1 2 3 4), (1 3 2 4), (1 4 2 3)\} \).

This decomposition has a straightforward generalization to an arbitrary number of external legs. A similar decomposition exists when some of the particles are in the fundamental representation. The partial amplitudes are independently gauge invariant and may therefore be calculated separately.

3 Properties of \( N = 4 \) amplitudes

The high degree of supersymmetry present in \( N = 4 \) amplitudes considerably simplifies their analytic structure. Supersymmetic amplitudes with external legs that have either all helicities identical or all but one identical vanish by a supersymmetry identity [21]. The non-vanishing maximally helicity violating (MHV) amplitudes, where two of the external helicities are of one type and the rest of the other type, are especially simple. At four and five points all the non-vanishing amplitudes are MHV.

For these reasons four-point \( N = 4 \) supersymmetric amplitudes are a natural starting point for evaluating higher-loop amplitudes. At one-loop \( N = 4 \) amplitudes may be considered as pieces of QCD amplitudes; similarly at two and higher loops we may expect a study of \( N = 4 \) amplitudes to be of some use for obtaining QCD amplitudes.
The tree-level four-gluon partial amplitudes are
\[ A_4^{\text{tree}}(1^\pm, 2^+, 3^+, 4^+) = 0, \]
\[ A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) = i \frac{(12)(34)}{(12)(34)(41)}, \]
\[ A_4^{\text{tree}}(1^-, 2^+, 3^-, 4^+) = i \frac{(12)(23)(34)}{(12)(23)(34)(41)}. \]

These are identical to the ones of standard QCD, since intermediate scalars or fermions do not propagate at tree level. The ± superscripts on the leg labels represent the helicities of the external gluons. We have chosen to represent the amplitudes in terms of the spinor helicity formalism [22], where gluon polarizations are replaced by spinor inner-products. (The reader may consult review articles for details [20].) We use the compact notation \[ \langle \cdots \rangle \] where gluons. We have chosen to represent the amplitudes in terms of the spinor helicity formalism [22], where gluon polarizations are replaced by spinor inner-products. (The reader may consult review articles for details [20].) We use the compact notation \[ \langle k_i | k_j^\pm \rangle \equiv \langle ij \rangle, \langle k_i^\pm | k_j^- \rangle \equiv [ij], \] where \[ |k^\pm\rangle \] are massless Weyl spinors labeled by the sign of their helicities and with normalization \[ \langle i,j | [j,i] = 2k_i \cdot k_j. \]

At one-loop the leading color \( N = 4 \) partial amplitudes are rather simple and given by
\[ A_{4,1}^{\text{1-loop}}(1, 2, 3, 4) = ist A_4^{\text{tree}}(1, 2, 3, 4) \mathcal{I}_{4}^{\text{1-loop}}(s, t), \]
where
\[ \mathcal{I}_{4}^{\text{1-loop}}(s, t) = \int \frac{d^{1-2\epsilon} p}{(2\pi)^{1-2\epsilon}} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_1)^2}, \]
is the one-loop scalar integral. The Mandelstam variables are defined as \[ s \equiv (k_1 + k_2)^2 \] and \[ t \equiv (k_2 + k_3)^2. \] The massless \( N = 4 \) super-multiplet consists of one gluon, four Weyl fermions and six real scalars, whose contributions are summed over to obtain the result (3). The integral \( (4) \) has a simple representation in terms of logarithms. (See for example ref. [8].) The remaining subleading color partial amplitudes may easily be computed in terms of sums of permutations of the expression (3).

The one-loop \( N = 4 \) four-gluon amplitude, in \( D \) dimensions, was first obtained by Green, Schwarz and Brink from the low energy limit of superstring theory [23].

Amplitudes, of course, depend on the particular form of dimensional regularization. The amplitudes (3) are quoted in the dimensional reduction scheme [24] or equivalently the four-dimensional helicity scheme [8], which preserve supersymmetry. The above expression turns out to be valid to all orders of of the dimensional regularization parameter \( \epsilon [24, 25] \). Furthermore, when \( A_4^{\text{tree}} \) is expressed in terms of formal polarization vectors and spinors instead of helicities, it is valid in any dimension. This will be useful later when we construct two-loop amplitudes.

When using cuts to obtain the \( N = 4 \) gluon amplitudes we also need tree and one-loop amplitudes with either two external scalars or fermions, since these particles can also cross the cuts. The maximally helicity violating amplitudes with two external fermions or scalars may be obtained directly from the gluon amplitudes using the supersymmetry identity [21, 20]
\[ \mathcal{A}_n^{\text{SUSY}}(1^-, 2^+, \cdots, j^-, \cdots, (n-1)^+, n_P^+) = \left( \frac{|j|}{|j|} \right)^{2-2|h_P|} \mathcal{A}_n^{\text{SUSY}}(1^-, 2^+, \cdots, j^-, \cdots, (n-1)^+, n^+), \]
where leg \( j \) is the only negative helicity gluon on the left-hand-side, \( P \) is either a scalar or fermion and \( |h_P| \) is the absolute value of its helicity (0 or 1/2). The legs without subscripts are taken to be
The MHV $N = 4$ gluon amplitudes also satisfy the identity \[ \mathcal{A}_n(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots n^+) = \frac{\langle ij \rangle^4}{\langle ab \rangle^4} \mathcal{A}_n(1^+, 2^+, \ldots, a^-, \ldots, b^-, \ldots n^+) , \] where $i$ and $j$ are the only negative helicity legs on the left-hand-side and $a$ and $b$ are the only negative helicities on the right-hand-side. Furthermore, it implies that different cuts are related by simple relabelings, up to an overall prefactor of $\langle ij \rangle^4$, where $i$ and $j$ label the negative helicities \[ 25. \]

In four dimensions the $n$-loop $N = 4$ super-Yang-Mills amplitudes are ultraviolet finite \[ 26. \]

An important ingredient in Mandelstam’s demonstration of finiteness is that for each external leg a power of external momentum may be extracted from each superspace Feynman diagram. More generally, in $D$ dimensions with $N$ supersymmetries (where $N$ is counted in four dimensions), power counting constraints on the form of the $L > 1$ loop effective action lead to the ultraviolet finiteness condition \[ 27. \]

\[ L < 2 \frac{N - 1}{D - 4} . \] In particular, at two-loops, $N = 4$ amplitudes are ultraviolet finite for $D < 7$.

### 4 Two-loop cut construction

![Diagram](image)

Figure 1: The two-particle $s$-channel cut has two contributions: one with the four-point one-loop amplitude ‘1’ to the left and the tree amplitude ‘T’ to the right (a) and the other being the reverse (b).

First we briefly review the cut construction method \[ 5, 6, 1, \] extending it to the case of two loops. Following the discussion in ref. \[ 1, \] a convenient way to obtain amplitudes is by considering cuts of unrestricted loop momentum integrals. In this way, one may simultaneously construct the imaginary and associated real parts of the cuts. Consider for example, the cuts of the two-loop amplitude $\mathcal{A}_4(1, 2, 3, 4)$. At two loops one must consider both two- and three-particle cuts. In each channel there may be multiple contributing cuts. For example, in the $s$ channel there are two two-particle cuts, as depicted in fig. \[ 1. \] The first of these has the explicit representation

\[ \mathcal{A}_{4}^{2\text{-loop}}(1, 2, 3, 4) \bigg|_{\text{cut(a)}} = \int \sum_{P_1, P_2} \frac{d^4 p}{(2\pi)^4 2 \pi^2} \frac{i}{\ell_2} \mathcal{A}_{4}^{1\text{-loop}}(-\ell_2, 3, 4, \ell_1) \frac{i}{\ell_1} \mathcal{A}_{4}^{\text{tree}}(-\ell_1, 1, 2, \ell_2) \bigg|_{\ell_1^2 = \ell_2^2 = 0} , \]}

where $\ell_1$ and $\ell_2$ are the momenta of the cut legs and the sum runs over all particle types (including helicity) $P_1$ and $P_2$ which may propagate across the two cut lines. We may use the on-shell conditions
\( \ell_1^2 = 0 \) and \( \ell_2^2 = 0 \) in the integrand (but not on the cut propagators) since this equation is valid only for those terms which have explicit \( \ell_1 \) and \( \ell_2 \) propagators. It is convenient to apply the color decomposition \(^{[1]}\) before computing the various cut contributions.

The three-particle s-channel cut is depicted in fig. 2. The t-channel cuts, of course, have a similar structure to the s-channel cuts. By combining all cuts into a single function, one obtains the full amplitude. When combining the cuts, care must be exercised not to over-count a particular term.

![Figure 2: The three-particle s-channel cut.](image)

As discussed in refs. \(^{[17, 18, 1]}\), by computing cuts to all orders in the dimensional regularization parameter, one may perform a complete reconstruction of a massless loop amplitude. This follows from dimensional analysis since every term in an amplitude must have a prefactor of powers of \((-s_{ij})^{-\epsilon}\), which necessarily have cuts. For reasons of technical simplicity, it is convenient to use helicity amplitudes which implicitly replace the \((4-2\epsilon)\)-dimensional momenta of the cut lines by four-dimensional momenta. However, to do so we must ensure that errors, especially through \(O(\epsilon^0)\), are not introduced. In section 6, we shall argue that at least for the \(N = 4\) four-point amplitudes there are no such errors.

We also find it convenient to perform the cut construction in components instead of using superfields. The potential advantage of a superfield formalism would be that one would simultaneously include contributions from all particles in a supersymmetry multiplet. However, for maximally helicity violating amplitudes, the supersymmetry identities are sufficiently powerful that once the contribution from one component is known the others immediately follow. A component formulation is also more natural for extensions to QCD.

## 5 Computation of amplitudes

We now proceed with the explicit construction of the \(N = 4\) four gluon partial amplitudes. First we consider the leading color amplitude \(A_{^{4}\text{c},1,1}^{\text{LC}}(1^-, 2^-, 3^+, 4^+)\). In the cut construction we use the amplitudes \(^{[2]}\) and \(^{[3]}\) as inputs. For the s-channel cuts depicted in fig. 1, with the helicity configuration \((1^-, 2^-, 3^+, 4^+)\), only the internal gluon loop contributes in the configuration where the cut gluons on each side of the cut have helicity opposite to that of the external legs. For fermion, scalars or gluons with a different helicity configuration, propagating across the cut, the amplitudes
on both sides of the cut vanish by a supersymmetry identity. This then gives for eq. (8),

\[ A_{4;1:1}^{\text{LC}}(1^-, 2^-, 3^+, 4^+) \big|_{\text{cut(a)}} = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2} A_4^{\text{tree}}(-\ell_2^-, 3^+, 4^+, \ell_1^+) \bigg|_{\ell_1^2 = \ell_2^2 = 0} \]

\[ = A_4^{\text{tree}} \left[ \int \frac{d^4 p}{(2\pi)^4} p^2 (p - k_1)^2 (p - k_1 - k_2)^2 (p + k_4)^2 \right] N \bigg|_{\ell_1^2 = \ell_2^2 = 0} \]

where \( \ell_1 = p \) and \( \ell_2 = p - k_1 - k_2 \) and we have used the amplitudes in eqs. (2) and (3). We have also used, for example, \( 1/\ell_2 = -[2 \ell_2]/(p - k_1)^2 \) to rationalize the denominators. The numerator of the integrand is

\[ N = \left[ \ell_1 \right] \left[ 14 \right] \left[ 4 \ell_1 \right] \left[ \ell_1 \ell_2 \right] \left[ 2 \ell_2 \right] \left[ \ell_2 \ell_1 \right] \]

\[ = \text{tr}_+ \left[ \ell_1 \ell_2 \ell_2 \ell_1 \right] \]

\[ = -st (p - k_1)^2 (p + k_4)^2, \]

where \( \text{tr}_+ \left[ \cdots \right] = \frac{1}{2} \text{tr} \left[ (1 + \gamma_5) \cdots \right] \). The \( \gamma_5 \) term in the trace does not contribute because a four-point amplitude has only three independent momenta to contract into the totally anti-symmetric Levi-Civita tensor. The above sewing algebra is particularly simple because it is identical to that of the one-loop case, discussed in ref. [1].

Thus, after canceling numerator factors against propagators and identifying the remaining integral as a two-loop scalar integral we obtain

\[ A_{4;1:1}^{\text{LC}}(1^-, 2^-, 3^+, 4^+) \big|_{\text{cut(a)}} = -s^2 t A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) T_4^\text{P}(s, t) \big|_{\text{cut(a)}}, \]

where the two-loop planar scalar double-box integral is

\[ T_4^\text{P}(s, t) = \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{1}{p^2 (p - k_1)^2 (p - k_1 - k_2)^2 (p + q)^2 (q - k_4)^2 (q - k_3 - k_4)^2}. \]

The evaluation of the two-particle \( s \)-channel cut in fig. [1]b is similar and gives the same result. The evaluation of the \( t \)-channel two-particle cuts are again similar, but a bit more involved since all particles in the super-multiplet contribute. However, after summing over the contribution of all particles, with the help of the supersymmetry identities and spinor identities, the integral appearing in the \( t \)-channel cut coincides with the one appearing in the \( s \)-channel cut in eq. (8), but with \( s \) and \( t \) interchanged.

\[ \text{- st A_4^{\text{tree}}} \left\{ s \begin{array}{c} 3 \hline \hline 1 \hline \hline 2 \end{array} \frac{1}{2} + s \begin{array}{c} 3 \hline \hline \hline 2 \end{array} \frac{1}{2} + t \begin{array}{c} 3 \hline \hline \hline 1 \end{array} \frac{1}{2} \right\} \]

Figure 3: The result of evaluating the three-particle \( s \)-channel cuts in terms of double-box scalar integrals. The dashed lines indicate the cuts.

The three-particle cuts reproduce the results obtained from the two-particle cuts without adding any new integral functions. The result of evaluating the three-particle \( s \)-channel cut in fig. [2] is
depicted in fig. 3. The scalar box integral \( I_4(s, t) \) appears twice because it has two distinct three-particle cuts. This calculation is more complicated than the two-particle cut calculation since one must sum over a total of sixteen intermediate particle and helicity configurations. Nevertheless the \( N = 4 \) supersymmetry ensures that the various terms combine neatly.

Combining all cuts into a single function that has the correct cuts in all channels yields the final result for the leading color \( N = 4 \) four-gluon partial amplitude,

\[
A_{4;1;1}^{LC}(1^-, 2^-, 3^+, 4^+) = -st A_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+) \left( s I_4^P(s, t) + t I_4^P(t, s) \right) .
\]

This is depicted in fig. 4. We comment that although substantial progress has been made in the calculation of such integrals \([13]\), the scalar integral appearing in this amplitude has not yet been evaluated in terms of known functions. This amplitude exhibits the cyclic symmetry for the leading color partial amplitude with non-adjacent negative helicities.

\[
- st A_4^{\text{tree}} \left\{ s \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 1 \end{array} + t \begin{array}{c} 4 \\ 3 \end{array} \begin{array}{c} 1 \\ 1 \end{array} \right\}
\]

Figure 4: The result for the leading color two-loop amplitude, corresponding to eq. (13)

The computation of the subleading color pieces is similar to the leading color computation, so we only quote the results. There are two types of subleading color contributions. Firstly, there are contributions arising from planar diagrams. These can be determined by evaluating the color algebra for the Feynman diagrams and are given by nothing more than sums of permutations of arguments of the same functions that appear at leading color. Secondly, there is a new contribution characterized by diagrams with non-planar topology. The cut construction of the non-planar contributions is quite similar to the planar one, with the main difference being the appearance of non-planar scalar integrals in the final results.

The results for the subleading in \( N_c \) partial amplitudes are

\[
A_{4;1;1}^{SC}(1, 2, 3, 4) = 2A_4^P(1, 2; 3, 4) + 2A_4^P(3, 4; 2, 1) + 2A_4^P(1, 4; 2, 3) + 2A_4^P(2, 3; 4, 1) - 4A_4^P(1, 3; 2, 4) - 4A_4^P(2, 4; 3, 1) + 2A_4^{NP}(1; 2; 3, 4) + 2A_4^{NP}(3; 4; 2, 1) + 2A_4^{NP}(1; 4; 2, 3) + 2A_4^{NP}(2; 3; 4, 1) - 4A_4^{NP}(1; 3; 2, 4) - 4A_4^{NP}(2; 4; 3, 1) ,
\]

\[
A_{4;1;3}(1; 2; 3, 4) = 6A_4^P(1; 2, 3, 4) + 6A_4^P(1; 2; 4, 3) + 4A_4^{NP}(1; 2; 3, 4) + 4A_4^{NP}(3; 4; 2, 1) - 2A_4^{NP}(1; 4; 2, 3) - 2A_4^{NP}(2; 3; 4, 1) - 2A_4^{NP}(1; 3; 2, 4) - 2A_4^{NP}(2; 4; 3, 1) ,
\]

where

\[
A_4^P(1; 2, 3; 4) \equiv -s_{12}^2 s_{23} A_4^{\text{tree}}(1, 2, 3, 4) I_4^P(s_{12}, s_{23}) ,
\]

\[
A_4^{NP}(1; 2; 3, 4) \equiv -s_{12}^2 s_{23} A_4^{\text{tree}}(1, 2, 3, 4) I_4^{NP}(s_{12}, s_{23}) ,
\]

and \( s_{ij} = (k_i + k_j)^2 \). The planar scalar integral is defined in eq. (12), while the non-planar one,
depicted in fig. 5, is

\[ I_{NP}^{4} (s, t) = \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{d^{4-2\epsilon} q}{(2\pi)^{4-2\epsilon}} \frac{1}{p^{2} (p - k_{2})^{2} (p + q)^{2} (p + q + k_{1})^{2} q^{2} (q - k_{3})^{2} (q - k_{3} - k_{4})^{2}}. \]  

(16)

Figure 5: The non-planar integral function, \( I_{NP}^{4} (s_{12}, s_{23}) \), appearing in the amplitude.

A simple consistency check on the amplitudes (13) and (14) is that they satisfy the expected finiteness condition (7).

6 Validity of cut construction

We now discuss potential errors arising from our use of four-dimensional helicity amplitudes in the cuts. Such errors would not occur if the sewing were done using formal spinors and polarization vectors. When using the four-dimensional spinor helicity formalism potential errors will be of the form,

\[ \int \frac{d^{4} p}{(2\pi)^{4}} \frac{d^{4-2\epsilon} \mu_{p}}{(2\pi)^{4-2\epsilon}} \frac{d^{4} q}{(2\pi)^{4}} \frac{d^{4-2\epsilon} \mu_{q}}{(2\pi)^{4-2\epsilon}} f(p, q, k_{i}) \times \left\{ \mu_{p}^{2}, \mu_{q}^{2}, \mu_{p} \cdot \mu_{q}, \cdots \right\}, \]

(17)

where we have explicitly separated the loop momenta into four- and \((-2\epsilon)\)-dimensional parts. (A discussion of the \((-2\epsilon)\)-dimensional parts of loop momenta can be found, for example, in refs. [4, 18].) Terms in the integrand proportional to \((-2\epsilon)\)-dimensional momenta \( \mu_{p} \) or \( \mu_{q} \), may in principle have been dropped in the above cut construction. Note that terms containing such factors are necessarily suppressed by a power of \( \epsilon \). However, this suppression may be cancelled if the integrals contain poles in \( \epsilon \).

The \( N = 4 \) amplitudes are, however, special. We may use the fact that the algebra associated with the two-particle cuts is identical at one and two loops. As discussed in section 4, the one-loop amplitudes (3) are valid in any dimension when expressed in terms of formal polarizations and spinors. Since the one-loop amplitudes are proportional to \( A_{4}^{\text{tree}} \), the two-loop sewing procedure for the two-particle cuts using formal spinors and polarizations reproduces the one-loop result, except for the prefactors and the two-loop scalar integral function. This implies that all Feynman diagrams with two-particle cuts have been accounted for exactly.

This, of course, does not rule out errors of the type in eq. (17) from diagrams with no two-particle cuts. Two examples of such diagrams are given in fig. 6. Here we may appeal to the superspace power counting rule that requires the extraction of a power of external momentum for each external leg of a superspace Feynman diagram [26]. After extracting four powers of external momenta, four-point diagrams with no two-particle cuts cannot contain any powers of \( \mu \) in the numerators of their integrands since there are no remaining powers of loop momenta in the non-vanishing diagrams. This means that there are no potential errors of the type in eq. (17) arising from three-particle cuts.
A more complete discussion of the $\mu$ terms, especially for theories with less supersymmetries, will be presented in the future.

![Diagrams](image)

Figure 6: Examples of diagrams with no two-particle cuts. The external lines are gluons, but the internal lines are summed over all states in the supermultiplet.

### 7 Structure of higher loop amplitudes

Following the same cut construction procedure used for the two-loop amplitudes, we have found a pattern for the $n$-loop $N=4$ four-gluon leading color partial amplitudes.

The three-loop leading color partial amplitude is given in fig. 7. Note that there are one-loop pentagon sub-diagrams. This complicates the analysis of the three-particle cuts since one-loop pentagons can be reduced to sums over box integrals. In some cuts it is the box integrals that appear and in some it is the pentagon; this must be disentangled in order to identify the form appearing in fig. 7.

$$-i st A_4^{1\text{tree}} \left\{ s^2 \left( t + s \right)^2 + s(t + k_2)^2 + s(t + k_3)^2 \right\}$$

Figure 7: A pictorial representation of the three-loop four-point $N = 4$ leading color amplitude. Note the prefactors that involve $\ell$ (where $\ell$ is the internal loop momentum indicated by the arrow in each term) are part of the integrand.

Observing the results for the leading color one-, two- and three-loop $N = 4$ amplitudes one can recognize a pattern which can be used to construct the $(n + 1)$-loop amplitude from the $n$-loop result. The pattern is that one takes each $n$-loop graph in the $n$-loop amplitude and generates all the possible $(n + 1)$-loop graphs by inserting a new leg between each possible pair of internal legs. Diagrams where triangle or bubble subgraphs are created should not be included. The new loop momentum including an additional factor of $i(\ell_1 + \ell_2)^2$ in the numerator is integrated over, where $\ell_1$ and $\ell_2$ are the momenta flowing through each of the legs to which the new line is joined. (This is depicted in fig. 8). Momentum conservation ensures that it does not matter on which side of the new line the momentum pair $\ell_1$ and $\ell_2$ are taken. Note that no four-point vertices are created by this procedure. Each distinct $(n + 1)$-loop graph should be counted once, even though they can be generated in multiple ways. The $(n + 1)$-loop amplitude is then the sum of all distinct $(n + 1)$-loop graphs.
8 Conclusions

In this letter we have made an initial step in applying the one-loop cut construction technique \cite{5,6} reviewed in ref. \cite{1} to two-loop amplitudes. As an explicit example we computed the two-loop $N = 4$ supersymmetric four-gluon amplitudes in terms of a basic set of scalar integrals. (The amplitudes with two fermions or scalars and two gluons trivially follow from a supersymmetry identity \cite{21}.) We also presented a pattern for the leading-color $N = 4$ four-gluon amplitudes to all loop orders consistent with two-particle cuts.

For the two-loop $N = 4$ amplitudes presented in this letter, we showed that the cut construction yields the complete amplitude without any rational function or subtraction ambiguities. This argument was based on the fact that the one-loop four-point amplitudes are known in any dimension and that the amplitudes satisfy a power counting criterion related to the finiteness of the theory.

A next step, along the lines of this letter, would be to obtain amplitudes in $N = 1, 2$ supersymmetric theories as a precursor to computing QCD amplitudes. However, before two-loop amplitudes can be used in phenomenological studies, one would need a numerically stable method for evaluating the basic integral functions that occur \cite{13}.

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Figure 8: Starting from an $n$-loop integral function we may add an extra line. The two-lines on the left represent two lines buried in some $n$-loop integral.

We have verified that this pattern is consistent with two-particle cuts to all loop orders and with four-dimensional three particle cuts, up to five loops. The two-particle cuts are not difficult to check because the same algebra appears at any loop order. It is, however, possible that there are further terms in the amplitudes. In order to show that nothing has been missed, one would need to demonstrate that higher order in $\epsilon$ terms in lower-loop amplitudes do not cause errors through $O(\epsilon^0)$. One would also need to verify the consistency of higher-particle cuts. Presumably, a similar pattern can be found for subleading in color contributions.
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