ON A PERTURBED COMPOUND POISSON RISK MODEL UNDER A PERIODIC THRESHOLD-TYPE DIVIDEND STRATEGY

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(Communicated by Hailiang Yang)

Abstract. In this paper, we model the insurance company’s surplus flow by a perturbed compound Poisson model. Suppose that at a sequence of random time points, the insurance company observes the surplus to decide dividend payments. If the observed surplus level is larger than the maximum of a threshold $b > 0$ and the last observed level (after dividends payment if possible), then a fraction $0 < \theta < 1$ of the excess amount is paid out as a lump sum dividend. We assume that the solvency is also discretely monitored at these observation times, so that the surplus process stops when the observed value becomes negative. Integro-differential equations for the expected discounted dividend payments before ruin and the Gerber-Shiu expected discounted penalty function are derived, and solutions are also analyzed by Laplace transform method. Numerical examples are given to illustrate the applicability of our results.

1. Introduction. In this paper, the surplus process of an insurance company is described by the following perturbed compound Poisson risk model,

$$U(t) = u + ct - \sum_{n=1}^{N(t)} X_n + \sigma B(t), \quad t \geq 0,$$

where $u = U(0) \geq 0$ is the initial surplus level, and $c > 0$ is the constant premium rate per unit time. The claim number process $\{N(t)\}_{t \geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, and the individual claim sizes $\{X_n\}_{n \geq 1}$ form a sequence of independent and identically distributed (i.i.d.) positive random variables.

2010 Mathematics Subject Classification. Primary: 91B30; Secondary: 62P05.

Key words and phrases. Dividend payments, Gerber-Shiu function, integro-differential equation, threshold-type dividend strategy, periodic observation.

The research of Xuanhua Peng was supported by the Chongqing Social Science Planning Project (No. 2017YBGL151), the Chongqing Municipal Education Commission Humanities and Social Sciences Research Project (No. 18SKGH006) and the Southwest University of Political Science and Law Research Project (No. 2018XZQN-35). The research of Zhimin Zhang was supported by the National Natural Science Foundation of China (Nos. 11471058, 11871121), MOE (Ministry of Education in China) Project of Humanities and Social Sciences (No. 16YJC910005) and Fundamental Research Funds for the Central Universities (No. 2018CDQYST0016).

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with common probability density function $f_X$ and mean $\mu_X$. In addition, suppose that $c > \lambda \mu_X$ holds true, so that the process $U$ has a positive drift with probability one. Finally, $\{B(t)\}_{t \geq 0}$ is a standard Brownian motion and $\sigma > 0$ is the volatility parameter. Throughout this paper, we suppose that $\{N(t)\}_{t \geq 0}$, $\{X_n\}_{n \geq 1}$ and $\{B(t)\}_{t \geq 0}$ are mutually independent.

The perturbed compound Poisson risk model (1) was first proposed by Gerber [14] to extend the classical risk model, where the diffusion perturbation can be used to describe uncertainty of the premium income. Since then, model (1) has been studied by many researchers in the literature. See e.g. [12, 15, 24, 25, 27, 32].

In his seminal paper, de Finetti [9] first proposed that insurance company should redistribute part of its surplus to its shareholders when the surplus process attains some given levels. Since de Finetti [9], a lot of contributions have been made to dividend problems by actuarial researchers, in which there exist two widely used dividend strategies, namely the constant barrier strategy and the threshold dividend strategy. See e.g. [17, 22, 23, 28, 29, 30, 38]. In particular, some dividend problems in the perturbed compound Poisson model have been studied by Li [20] and Wan [26] under the constant barrier strategy and the threshold dividend strategy, respectively. Some of the results in Wan [26] are extended by Gao and Yin [13], where a perturbed Sparre Andersen renewal process is used to describe the surplus process.

In all the aforementioned papers on dividend problems, actuarial researchers only consider the continuous-time dividend problems, where the surplus process is observed continuously so that decisions on dividend payments are made immediately when the surplus process attains some given levels. However, in practical situations the insurer only checks the surplus level periodically at some discrete time points. To overcome this problem, Albrecher et al. [2] first proposed the discrete observation of the surplus process, where dividend decisions are made periodically according to the observed surplus levels. Since then, the discrete-time dividend problems have been studied by many researchers. For example, Avanzi et al. [5] considered the periodic dividend barrier strategy in a dual risk model where the solvency is assumed to be continuously monitored; Zhang [31] studied a perturbed compound Poisson model under the periodic dividend barrier strategy, and some results were extended to the Markov additive risk model and the Lévy risk model by Zhang and Cheung [33, 34], respectively; Zhang and Liu [37] studied the higher order moments of the discounted dividend payments before ruin; Dong et al. [11] considered the periodic barrier dividend strategy in a general spectrally negative Lévy risk model with Erlang inter-dividend-decision times; Some optimality results on periodic dividend strategy were studied by Albrecher et al. [1] and Avanzi et al. [6, 7].

Recently, Cheung and Zhang [8] proposed a new threshold-type dividend strategy in the classical compound Poisson model. In comparison with the continuous-time threshold dividend strategy considered in Lin and Pavlova [22], Cheung and Zhang [8] supposed that the surplus process is observed periodically and lump sum dividends can only be paid out at the observation times. In their paper, the expected present value of dividend payments before ruin is studied when the inter-observation times follow an Erlang distribution. In this paper, we shall use the periodic threshold-type dividend strategy in Cheung and Zhang [8] to modify the surplus process (1), which is described as follows. Let $0 < Z_1 < Z_2 < \cdots$ denote a sequence of increasing observation times, at which the insurer observes the surplus process $\{U_t\}$ to decide dividend payments. Put $Z_0 = 0$ for convenience, but it is
not a dividend-payment time. At every observation time $Z_j \ (j \geq 1)$, if the observed surplus process is larger than the maximum of a threshold $b > 0$ and the last observed (post-dividend) surplus level, then a fraction $0 < \theta < 1$ of the excess is paid as a lump sum dividend. Let $U^b = \{U^b(t)\}_{t \geq 0}$ denote the surplus process after this modification. In order to describe $U^b$ as a lump sum dividend. Let

$$U^b = \begin{cases} U(t), & 0 \leq t < Z_1, \\ C^b_{Z_j} - \theta[C^b_{Z_j} - \max(U^b(Z_{j-1}), b)]_+ + U(t) - U(Z_j), & Z_j \leq t < Z_{j+1}; \\ j = 1, 2, \ldots \end{cases}$$

and

$$C^b_{Z_j} = U^b(Z_{j-1}) + U(Z_j) - U(Z_{j-1}), \quad j = 1, 2, \ldots,$$

where $x_+ = \max(0, x)$.

Let $T_1 = Z_1$ be the time until the first observation, and for $j \geq 2$ let $T_j = Z_j - Z_{j-1}$ be the inter-observation time between the $(j-1)$-th and $j$-th observations. In practical applications, it is natural to assume that $T_j$’s are constant, since the surplus level is usually checked by the board of directors at some deterministic times. However, it is very hard to obtain desirable mathematical results under the constant inter-observation time assumption. To get over this problem, the Erlang($n$) distribution is usually used to model the inter-observation times not only because it can lead to some nice explicit expressions for the quantities of interest but also it can approximate the deterministic time horizons. See e.g. [2, 3, 4, 8]. One particular example of Erlang($n$) distributions is the exponential distribution (with $n = 1$). Under this assumption, the observation times $\{Z_j\}_{j=1}^{\infty}$ are the arrival epochs of a Poisson process, and the risk model is also called Poisson observation model. In actuarial mathematics, the Poisson observation model was first proposed by Albrecher et al. [2, 3], and it has also been adopted by other researchers, e.g. [19, 35]. As commented in Landriault et al. [19], the Poisson observation scheme may be used to model the monitoring frequency by an exogenous regulatory authority of an insurer’s surplus. In this paper, we shall pay attention to the Poisson observation model, and assume that the inter-observation times are i.i.d. with exponential density function

$$f_T(t) = \gamma e^{-\gamma t}, \quad t \geq 0, \ \gamma > 0.$$  

Furthermore, suppose that $T_j$’s are independent of other stochastic quantities. We also remark that some results obtained in this paper can be extended to the Erlang renewal observation scheme, but the arguments would be very involved. See e.g. [8, 36].

Throughout this paper, suppose that the solvency is also discretely monitored at the time points $Z_j$’s, and define the ruin time by $\tau_b = Z_{J_b}$, where $J_b = \min\{j \geq 1 : U^b(Z_j) < 0\}$ is the number of observations until ruin. We put $\tau_b = \infty$ if $U^b(Z_j) \geq 0$ for any $j \geq 0$. For a positive interest force $\delta > 0$, the expected present value of dividends paid until ruin is given by

$$V(u; b) \equiv \theta \mathbb{E} \left[ \sum_{j=1}^{J_b} e^{-\delta Z_j} [C^b_{Z_j} - \max(U^b(Z_{j-1}), b)]_+ \bigg| U^b(0) = u \right], \quad u \geq 0.$$  

We are also interested in the study of ruin related quantities. To this end, we shall consider the Gerber-Shiu expected discounted penalty function (Gerber and Shiu [16]) defined by

$$\phi(u; \delta) = \mathbb{E}[e^{-\delta T_{j,b}}w(U(T_{j,b}))1_{(T_{j,b} < \infty)} | U(0) = u], \quad u \geq 0,$$

where $w(\cdot)$ is a nonnegative measurable function of the deficit at ruin, and $1_{(\cdot)}$ is an indicator function. Whenever we study the Gerber-Shiu function, $\delta = 0$ is also allowed. For the limit case $b = \infty$, the insurance company will never pay any dividends, then our model reduces to a perturbed compound risk model where the solvency is observed at the discrete time points $Z_j$’s. In this model, the ruin time is defined by $T_\infty = Z_{j,\infty}$ with $J_{\infty} = \min\{j \geq 1 : U(Z_j) < 0\}$, where $T_\infty = \infty$ if $U(Z_j) \geq 0$ for any $j \geq 0$. The Gerber-Shiu function is defined by

$$\phi(u; \infty) = \mathbb{E}[e^{-\delta T_\infty}w(U(Z_{J_{\infty}}))1_{(Z_{J_{\infty}} < \infty)} | U(0) = u], \quad u \geq 0.$$

The remainder of this paper is organized as follows. In Section 2, we study the discounted increments of the surplus process $U$ observed at the observation times $\{Z_j\}_{j=1}^\infty$. In Section 3, we study the expected discounted dividend payments until ruin, where integro-differential equations for $V(u; b)$ are derived and solutions are also obtained by Laplace transform method. In Section 4, the Gerber-Shiu function is analyzed by the same method. In Section 5, we assume that the claim size density has a rational Laplace transform, and derive some more explicit formulae for some functions of importance. Finally, some numerical examples are given in Section 6.

2. Discounted increments between successive observations. For the i.i.d. sequence of bivariate random vectors $\{(Z_j - Z_{j-1}, U(Z_j) - U(Z_{j-1}))\}_{j=1}^\infty$, we introduce the discounted density of $U(Z_j) - U(Z_{j-1})$ discounted at rate $\delta$, say $g_\delta$, which satisfies

$$\mathbb{E}[e^{-\delta Z_1}; U(Z_1) - U(Z_0) \in dx] = g_\delta(x)dx.$$ 

Due to that $Z_1$ is exponentially distributed with rate $\gamma$, we have

$$\mathbb{E}[e^{-\delta Z_1}; U(Z_1) - U(Z_0) \in dx] = \int_0^\infty e^{-\delta t} \mathbb{P}(U(t) - U(0) \in dx)e^{-\gamma t}dt = \gamma \int_0^\infty \mathbb{P}(U(t) - U(0) \in dx)e^{-(\gamma + \delta)t}dt. \quad (6)$$

Note that the perturbed compound Poisson process (1) is a spectrally negative Lévy process having Laplace exponent

$$\psi(s) := \frac{1}{t} \ln \mathbb{E}[e^{xt(U(t) - U(0))}] = cs + \frac{\sigma^2}{2} s^2 - \lambda(1 - \hat{f}_X(s)), \quad s \geq 0, \quad (7)$$

where $\hat{f}_X(s) = \int_0^\infty e^{-sx}f_X(x)dx$ is the Laplace transform of $f_X$. For $q \geq 0$, the $q$-scale function $W^{(q)}(x)$ is a function such that $W^{(q)}(x) = 0$ for $x < 0$, and for $x \geq 0$ it is determined by the Laplace transform

$$\int_0^\infty e^{-sx}W^{(q)}(x)dx = \frac{1}{\psi(s) - q}, \quad s > \Phi(q),$$

where $\Phi(q) = \sup\{s \geq 0 : \psi(s) = q\}$ is the right inverse of $\psi$.

By Corollary 8.9 in Kyprianou [18] we can write (6) as

$$\mathbb{E}[e^{-\delta Z_1}; U(Z_1) - U(Z_0) \in dx] = \gamma \Phi'(\gamma + \delta)e^{-\Phi(\gamma + \delta)} - W^{(\gamma + \delta)}(-x)|dx. \quad (8)$$
Furthermore, since \( \psi(\Phi(q)) = q \), differentiating both sides of this equation with respect to \( q \) gives \( \Phi'(q) = \frac{1}{\psi'(\Phi(q))} \). Hence, (8) becomes

\[
\mathbb{E}[e^{-\delta Z_1}; U(Z_1) - U(Z_0) \in dx] = \frac{\gamma}{\psi'(\Phi(\gamma + \delta))} e^{-\Phi(\gamma + \delta)x} dx - \gamma W(\gamma + \delta)(-x)dx,
\]

which implies that the discounted density function \( g_\delta \) can be expressed as follows,

\[
g_\delta(x) = a_\gamma e^{-\rho_\gamma x} - \gamma W(\gamma + \delta)(-x),
\]

where we have written \( a_\gamma = \frac{\gamma}{\psi'(\Phi(\gamma + \delta))} \) and \( \rho_\gamma = \Phi(\gamma + \delta) \) for notational convenience.

When we study the expected discounted dividend payments before ruin and the Gerber-Shiu function, it is more convenient to write the two-sided density function \( g_\delta \) as

\[
g_\delta(x) = g_{\delta, -}(x)1_{(x<0)} + g_{\delta, +}(x)1_{(x>0)}.
\]

By formula (10) we have

\[
g_{\delta, +}(x) = a_\gamma e^{\rho_\gamma x}, \quad x > 0,
\]

and

\[
g_{\delta, -}(x) = a_\gamma e^{\rho_\gamma x} - \gamma W(\gamma + \delta)(x), \quad x > 0.
\]

Remark 1. We can use the discounted density \( g_{\delta, +} \) to derive an upper bound for \( V(u; b) \). It follows from the definition of \( V(u; b) \) that

\[
V(u; b) \leq \theta \mathbb{E} \left[ \sum_{j=1}^{\infty} e^{-\delta Z_j} \left[ C_{Z_j}^b - \max\{U^b(Z_j-1), b\} \right] \right]
\]

\[
\leq \theta \sum_{j=1}^{\infty} \mathbb{E} \left[ e^{-\delta Z_j} \left[ C_{Z_j}^b - U^b(Z_j-1) \right] \right]
\]

\[
= \theta \sum_{j=1}^{\infty} \mathbb{E} \left[ e^{-\delta Z_j} (U(Z_j) - U(Z_j-1)) \right]
\]

\[
= \theta \sum_{j=1}^{\infty} \left[ \mathbb{E} e^{-\delta Z_1} \right]^{j-1} \int_0^{\infty} xg_{\delta, +}(x) dx
\]

\[
= \frac{\theta}{1 - \frac{\gamma}{\gamma + \delta}} \int_0^{\infty} xe^{\rho_\gamma x} dx
\]

\[
= \frac{a_\gamma \theta(\gamma + \delta)}{\delta \rho_\gamma^2} < \infty,
\]

where the third step follows from (3). The above upper bound is useful for solving the integro-differential equations satisfied by \( V(u; b) \).

3. Analysis of \( V(u; b) \). In this section, we study the expected discounted dividend payments before ruin. By distinguishing the initial surplus level for \( 0 \leq u \leq b \) and \( u > b \), we decompose \( V(u; b) \) as follows,

\[
V(u; b) = \begin{cases} 
V_1(u), & 0 \leq u \leq b, \\
V_2(u), & u > b.
\end{cases}
\]

We shall derive some expressions for \( V_1(u) \) and \( V_2(u) \) in the sequel.
First, we consider the case $0 \leq u \leq b$. Using the same arguments leading to equation (16) in Cheung and Zhang [8], we can easily obtain the following integral equation,

$$V_1(u) = \int_{b-u}^{\infty} \left[ \theta(u + x - b) + V_2(u + x - \theta(u + x - b)) \right] g_{\delta,+}(x) \, dx$$

$$+ \int_{0}^{b-u} V_1(u + x) g_{\delta,+}(x) \, dx + \int_{0}^{u} V_1(u - x) g_{\delta,-}(x) \, dx,$$

which, together with (11) and some changes of variables, yields

$$V_1(u) = \theta a_1 \int_{0}^{\infty} xe^{-\rho_{\gamma} (x+b-u)} \, dx + \frac{a_2}{1 - \theta} \int_{b}^{\infty} V_2(x) e^{-\rho_{\gamma} \left( \frac{x-b}{\rho_{\delta}} \right)} \, dx$$

$$+ a_1 \int_{u}^{b} V_1(x) e^{-\rho_{\gamma} (x-u)} \, dx + \int_{0}^{u} V_1(u - x) g_{\delta,-}(x) \, dx.$$  

(15)

Suppose that the discounted density function $g_{\delta,-}$ is differentiable, then (15) implies that $V_1(u)$ is also differentiable with respect to $u$ in $(0, b)$. Applying the operator $\frac{d}{du} - \rho_{\gamma}$ to both sides of (15) gives

$$V'_1(u) + (a_1 - \rho_{\gamma}) V_1(u) = \left( \frac{d}{du} - \rho_{\gamma} \right) \int_{0}^{u} V_1(u - x) g_{\delta,-}(x) \, dx, \quad 0 < u < b,$$

(16)

which is a homogeneous integro-differential equation independent of $V_2(u)$.

Let $v(u)$ be a solution to the following homogeneous integro-differential equation

$$v'(u) + (a_1 - \rho_{\gamma}) v(u) = \left( \frac{d}{du} - \rho_{\gamma} \right) \int_{0}^{u} v(u - x) g_{\delta,-}(x) \, dx, \quad u \geq 0,$$

(17)

with initial condition $v(0) = 1$. Then we have

$$V_1(u) = K v(u), \quad 0 \leq u \leq b,$$

(18)

where $K$ is an unknown constant to be determined later. The solution $v$ can be determined by Laplace transform. In the sequel, we use $\hat{f}(s)$ to denote the Laplace transform of a function $f$ defined on $[0, \infty)$, and let $\mathcal{L}_s^{-1}$ be the Laplace inversion operator with respect to the argument $s$. Taking Laplace transform on both sides of (17) one easily obtains

$$\hat{v}(s) = \frac{1}{s + a_1 - \rho_{\gamma} - (s - \rho_{\gamma}) \hat{g}_{\delta,-}(s)}.$$  

(19)

On one hand, using the Laplace inversion operator we can obtain

$$v(u) = \left( \mathcal{L}_s^{-1} \frac{1}{s + a_1 - \rho_{\gamma} - (s - \rho_{\gamma}) \hat{g}_{\delta,-}(s)} \right) (u).$$

It follows from Section 5 that the above Laplace inversion can lead to an explicit formula for $v(u)$ when the claim size density function has rational Laplace transform. On the other hand, we can use the $q$-scale function to invert the Laplace transform in (19). By formula (12) we have for $s > \rho_{\gamma}$

$$\hat{g}_{\delta,-}(s) = \frac{a_1}{s - \rho_{\gamma}} - \frac{\gamma}{\psi(s) - \gamma - \delta}.$$  

(20)

Then we can rewrite (19) as follows,

$$\hat{v}(s) = \frac{1}{1 + \frac{a_1}{s - \rho_{\gamma}} \hat{g}_{\delta,-}(s)} = \frac{1}{1 + \frac{s - \rho_{\gamma}}{\psi(s) - \gamma - \delta}} = \frac{1}{s - \rho_{\gamma}} - \frac{1}{s - \rho_{\gamma} \psi(s) - \delta}.$$  

(21)
After inverting the above Laplace transform we obtain
\[ v(u) = e^{\rho s}u - \gamma \int_0^u e^{\rho s} W(s)(u - x)dx, \quad u \geq 0. \tag{22} \]

Next, we consider the case \( u > b \). In order to make the following analysis more transparent, we introduce the Dickson-Hipp operator \( T_s \) (see e.g. \([10, 21]\)), which for any integrable function \( f \) on \((0, \infty)\) and any complex number \( s \) with \( \text{Re}(s) \geq 0 \) is defined as
\[ T_s f(y) = \int_y^\infty e^{-s(x-y)}f(x)dx = \int_0^\infty e^{-sx} f(x+y)dx, \quad y \geq 0. \]

The Dickson-Hipp operator has been widely used in ruin theory to simplify the expressions of ruin related functions. For properties on this operator, we refer the interested readers to Li and Garrido \([21]\).

Now using the same arguments leading to equation (21) in Cheung and Zhang \([8]\), we obtain, for \( u > b \)
\[ V_2(u) = \int_0^\infty [\theta x + V(u + x - \theta x; b)] g_\delta,+(x) dx + \int_0^u V(u - x; b) g_\delta,-(x) dx \\
= \theta a_\gamma \int_0^\infty xe^{-\rho s} dx + a_\gamma \int_0^\infty V_2(u + (1 - \theta)x)e^{-\rho s} dx \\
+ \int_0^u V(u - x; b) g_\delta,-(x) dx \\
= Q_\delta + \frac{a_\gamma}{1 - \theta} \int_u^\infty V_2(x)e^{-\frac{\rho s}{1 - \theta} x} dx + \int_0^u V(u - x; b) g_\delta,-(x) dx \\
= Q_\delta + \frac{a_\gamma}{1 - \theta} T_{\rho s/(1 - \theta)} V_2(u) + \int_0^u V(u - x; b) g_\delta,-(x) dx, \tag{23} \]
where
\[ Q_\delta = \theta a_\gamma \int_0^\infty xe^{-\rho s} dx = \frac{a_\gamma \theta}{\rho^2}. \]

Setting \( u = b \) in (15) and taking limit \( \lim_{u \to b+} V_2(u) \) in (23), we find that \( V(u; b) \) is continuous at \( u = b \), i.e.
\[ V(b-; b) = V(b+; b), \tag{24} \]
which can be used to determine the unknown constant \( K \).

We use Laplace transform to find an expression for \( V_2(u) \). For \( \text{Re}(s) > 0 \), multiplying both sides of (23) by \( e^{-s(u-b)} \) and then performing integration from \( b \) to \( \infty \), we can obtain
\[
T_s V_2(b) = s^{-1}Q_\delta + \frac{a_\gamma}{1 - \theta} T_s T_{\rho s/(1 - \theta)} V_2(b) \\
+ \int_b^\infty e^{-s(u-b)} \int_0^u V(u - x; b) g_\delta,-(x) dx du \\
= s^{-1}Q_\delta + \frac{a_\gamma}{1 - \theta} \frac{T_s V_2(b) - T_{\rho s} V_2(b)}{\rho s/(1 - \theta) - s} + g_\delta,-(s) T_s V_2(b) \\
+ \int_0^b V_1(x) T_s g_\delta,-(b - x) dx. \tag{25} 
\]
Solving the above equation for $T_sV_2(b)$ yields

$$
T_sV_2(b) = \frac{s^{-1}Q_\delta + \int_0^b v(x)T_s g_{\delta,-}(b-x)dx - \frac{a_\gamma}{\rho_\gamma - (1-\theta)s} \mathcal{T}_{\frac{\rho_\gamma}{\rho_\gamma - (1-\theta)s}} V_2(b)}{1 - \frac{a_\gamma}{\rho_\gamma - (1-\theta)s} - \hat{g}_{\delta,-}(s)},
$$

where the second step follows from the substitution of formula (18). Note that there are still two unknown constants in the numerator of (26), namely $K$ and $\mathcal{T}_{\frac{\rho_\gamma}{\rho_\gamma - (1-\theta)s}} V_2(b)$.

In order to determine the constant $\mathcal{T}_{\frac{\rho_\gamma}{\rho_\gamma - (1-\theta)s}} V_2(b)$, we consider the root of the following equation

$$
1 - \frac{a_\gamma}{\rho_\gamma - (1-\theta)s} - \hat{g}_{\delta,-}(s) = 0. \tag{27}
$$

Let $h(s) = 1 - \frac{a_\gamma}{\rho_\gamma - (1-\theta)s} - \hat{g}_{\delta,-}(s)$. Because

$$
h(0) = 1 - \frac{a_\gamma}{\rho_\gamma} - \hat{g}_{\delta,-}(0) = 1 - \hat{g}_0 = 1 - \mathbb{E}[e^{-\delta Z_1}] > 0
$$

and

$$
\lim_{s \to \frac{\rho_\gamma}{\rho_\gamma - (1-\theta)s}} h(s) = 1 - \lim_{s \to \frac{\rho_\gamma}{\rho_\gamma - (1-\theta)s}} \frac{a_\gamma}{\rho_\gamma - (1-\theta)s} - \lim_{s \to \frac{\rho_\gamma}{\rho_\gamma - (1-\theta)s}} \hat{g}_{\delta,-}(s) = -\infty,
$$

we conclude that there exists a positive root, say $s_\gamma$, between 0 and $\rho_\gamma/(1-\theta)$. Due to the upper bound (13) we have, for $s > 0$

$$
T_sV_2(b) = \int_0^\infty e^{-su}V_2(b + u)du \leq \int_0^\infty e^{-su} \frac{a_\gamma \theta(\gamma + \delta)}{\delta \rho_\gamma^2} du \leq \frac{1}{s} \frac{a_\gamma \theta(\gamma + \delta)}{\delta \rho_\gamma^2} < \infty,
$$

which implies that $s_\gamma$ is also a zero point of the numerator of (26), and this leads to

$$
\mathcal{T}_{\frac{\rho_\gamma}{\rho_\gamma - (1-\theta)s}} V_2(b) = \frac{\rho_\gamma - (1-\theta)s}{a_\gamma} \left( s_\gamma^{-1}Q_\delta + K \int_0^b v(x)T_s g_{\delta,-}(b-x)dx \right). \tag{28}
$$

Now plugging (28) back into (26) we obtain

$$
T_sV_2(b) = \frac{s^{-1}Q_\delta - \frac{\rho_\gamma - (1-\theta)s}{\rho_\gamma - (1-\theta)s} \frac{1}{a_\gamma} s_\gamma^{-1}Q_\delta}{1 - \frac{a_\gamma}{\rho_\gamma - (1-\theta)s} - \hat{g}_{\delta,-}(s)} + K \frac{\int_0^b v(x)T_s g_{\delta,-}(b-x)dx - \frac{\rho_\gamma - (1-\theta)s}{\rho_\gamma - (1-\theta)s} \int_0^b v(x)T_s g_{\delta,-}(b-x)dx}{1 - \frac{a_\gamma}{\rho_\gamma - (1-\theta)s} - \hat{g}_{\delta,-}(s)}.
$$

Taking Laplace inversion transform on both sides of the above equation we obtain

$$
V_2(u + b) = H_V(u) + KH(u), \quad u \geq 0, \tag{29}
$$

where

$$
H_V(u) = \left( \mathcal{L}_{\frac{\rho_\gamma}{\rho_\gamma - (1-\theta)s}}^{-1} s^{-1}Q_\delta - \frac{\rho_\gamma - (1-\theta)s}{\rho_\gamma - (1-\theta)s} \frac{1}{a_\gamma} s_\gamma^{-1}Q_\delta \right)(u),
$$

$$
H(u) = \left( \mathcal{L}_{\frac{\rho_\gamma}{\rho_\gamma - (1-\theta)s}}^{-1} \int_0^b v(x)T_s g_{\delta,-}(b-x)dx - \frac{\rho_\gamma - (1-\theta)s}{\rho_\gamma - (1-\theta)s} \int_0^b v(x)T_s g_{\delta,-}(b-x)dx \right)(u).
$$
It remains to determine the unknown constant $K$. To this end, we use the continuity condition (24), which together with formulae (18) and (29), gives

$$Kv(b) = HV(0) + KH(0).$$

After solving this equation for $K$ we obtain

$$K = \frac{HV(0)}{v(b) - H(0)}.$$

Finally, we find that the expected discounted dividend payments before ruin can be expressed as follows,

$$V(u; b) = \begin{cases} \frac{HV(0)}{v(b) - H(0)}v(u), & 0 \leq u \leq b, \\ HV(u - b) + \frac{HV(0)}{v(b) - H(0)}H(u - b), & u > b. \end{cases}$$

(30)

4. Analysis of the Gerber-Shiu function. In this section, we study the Gerber-Shiu expected discounted penalty function. As in the analysis of $V(u; b)$, we decompose $\phi(u; b)$ according to $0 \leq u \leq b$ and $u > b$ as follows,

$$\phi(u; b) = \begin{cases} \phi_1(u), & 0 \leq u \leq b, \\ \phi_2(u), & u > b. \end{cases}$$

First, we consider the case $0 \leq u \leq b$. By conditioning on the increment of the surplus process $U$ from time 0 to time $Z_1$, we obtain, for $0 \leq u \leq b$

$$\phi_1(u) = \int_{b-u}^{\infty} \phi_2(u + x - \theta(u + x - b))g_{\delta_+}(x)dx + \int_{u}^{\infty} w(x - u)g_{\delta_-}(x)dx$$

$$+ \int_{0}^{b-u} \phi_1(u + x)g_{\delta_+}(x)dx + \int_{u}^{b} \phi_1(u - x)g_{\delta_-}(x)dx$$

$$= \int_{b-u}^{\infty} \phi_2(u + x - \theta(u + x - b))g_{\delta_+}(x)dx$$

$$+ \int_{0}^{b-u} \phi_1(u + x)g_{\delta_+}(x)dx + \int_{0}^{b} \phi_1(u - x)g_{\delta_-}(x)dx + \omega(u),$$

(31)

where $\omega(u) = \int_{u}^{\infty} w(x - u)g_{\delta_-}(x)dx$. Furthermore, by some changes of variables, we can rewrite (31) as

$$\phi_1(u) = \frac{a_\gamma}{1 - \theta} \int_{b-u}^{\infty} \phi_2(x)e^{-\rho_\gamma(x-b-u)}dx + a_\gamma \int_{u}^{b} \phi_1(x)e^{-\rho_\gamma(x-u)}dx$$

$$+ \int_{0}^{u} \phi_1(u - x)g_{\delta_-}(x)dx + \omega(u).$$

(32)

If the discounted density function $g_{\delta_-}$ is differentiable, we can apply the operator $\frac{d}{du} - \rho_\gamma$ to both sides of (32) to obtain, for $0 \leq u \leq b$

$$\phi_1'(u) + (a_\gamma - \rho_\gamma)\phi_1(u) = \left(\frac{d}{du} - \rho_\gamma\right)\int_{0}^{u} \phi_1(u - x)g_{\delta_-}(x)dx + \omega'(u) - \rho_\gamma\omega(u).$$

(33)

Note that the integro-differential equation (33) is independent of $\phi_2(u)$. By exactly the same arguments as above, we can prove that the Gerber-Shiu function $\phi(u; \infty)$ satisfies the same type integro-differential equation, i.e. for $u \geq 0$

$$\phi'(u; \infty) + (a_\gamma - \rho_\gamma)\phi(u; \infty) = \left(\frac{d}{du} - \rho_\gamma\right)\int_{0}^{u} \phi(u - x; \infty)g_{\delta_-}(x)dx + \omega'(u) - \rho_\gamma\omega(u).$$

(34)
Hence, by general theory of integro-differential equations we can express \( \phi_1(u) \) as
\[
\phi_1(u) = \phi(u; \infty) + Lv(u),
\]
where \( L \) is an unknown constant to be determined later.

The Gerber-Shiu function \( \phi(u; \infty) \) can be determined by the Laplace transform method. Taking Laplace transform on both sides of (34) we obtain
\[
\hat{\phi}(s; \infty) = \frac{\phi(0; \infty) - \omega(0) + (s - \rho_\gamma)\hat{\omega}(s)}{s + a_\gamma - \rho_\gamma - (s - \rho_\gamma)\hat{g}_{\delta,-}(s)},
\]
where \( \phi(0; \infty) \) is still unknown. To find an expression for \( \phi(0; \infty) \), we consider the zero point of the denominator in (36). Define
\[
\xi(s) = s + a_\gamma - \rho_\gamma - (s - \rho_\gamma)\hat{g}_{\delta,-}(s).
\]
Then we have \( \lim_{s \to \infty} \xi(s) = \infty \), and
\[
\xi(0) = a_\gamma - \rho_\gamma + \rho_\gamma\hat{g}_{\delta,-}(0) = \rho_\gamma \left( \frac{a_\gamma}{\rho_\gamma} + \hat{g}_{\delta,-}(0) - 1 \right) = \rho_\gamma \left( \int g_\delta(x)dx - 1 \right) = \rho_\gamma (E[e^{-\delta Z}] - 1) < 0.
\]
Hence, we conclude that equation \( \xi(s) = 0 \) has a positive root and we denote it by \( \beta_\delta \). Note that \( \beta_0 = 0 \) as \( \delta = 0 \). Suppose that the Gerber-Shiu function is integrable, then we have \( \hat{\phi}(s) < \infty \) for any \( s \) with positive real part, which implies that \( \beta_\delta \) is also a zero point of the numerator of (36), and this argument leads to
\[
\phi(0; \infty) = \omega(0) - (\beta_\delta - \rho_\gamma)\hat{\omega}(\beta_\delta).
\]
Substituting this formula into (36) yields
\[
\hat{\phi}(s; \infty) = \frac{(s - \rho_\gamma)\hat{\omega}(s) - (\beta_\delta - \rho_\gamma)\hat{\omega}(\beta_\delta)}{s + a_\gamma - \rho_\gamma - (s - \rho_\gamma)\hat{g}_{\delta,-}(s)},
\]
which together with Laplace inversion gives
\[
\phi(u; \infty) = \left( L^{-1}_s \frac{(s - \rho_\gamma)\hat{\omega}(s) - (\beta_\delta - \rho_\gamma)\hat{\omega}(\beta_\delta)}{s + a_\gamma - \rho_\gamma - (s - \rho_\gamma)\hat{g}_{\delta,-}(s)} \right)(u), \quad u \geq 0.
\]
Using the same arguments leading to (21), we can obtain from (37) that
\[
\hat{\phi}(s; \infty) = \left( \hat{\omega}(s) - \hat{\omega}(\beta_\delta) \right) \frac{\beta_\delta - \rho_\gamma}{s - \rho_\gamma} \left( 1 - \frac{\gamma}{\psi(s; \delta)} \right),
\]
which immediately yields
\[
\phi(u; \infty) = \omega(u) - (\beta_\delta - \rho_\gamma)\hat{\omega}(\beta_\delta)e^{\rho_\gamma u} - \gamma \int_0^u [\omega(x) - (\beta_\delta - \rho_\gamma)\hat{\omega}(\beta_\delta)e^{\rho_\gamma x}]W(\delta)(u - x)dx, \quad u \geq 0.
\]
Next, we consider the case \( u > b \). By conditioning on the increment of the surplus process until the first observation time, we obtain, for \( u > b \)
\[
\phi_2(u) = \int_0^\infty \phi_2(u + (1 - \theta)x)g_{\delta,+}(x)dx + \int_0^u \phi(u - x; b)g_{\delta,-}(x)dx
\]
\[
+ \int_u^\infty w(x - u)g_{\delta,-}(x)dx
\]
\[
= \int_0^\infty \phi_2(u + (1 - \theta)x)e^{-\rho_\gamma x}dx + \int_0^u \phi(u - x; b)g_{\delta,-}(x)dx + \omega(u)
\]
where the second step follows from the substitution of formula (35).

From which we can obtain

\[ \phi(b^-; b) = \phi(b^+; b), \]  

(42)

which can be used to determine the unknown constant \( L \).

Now we use Laplace transform to solve the integral equation (41). For \( s > 0 \), multiplying both sides of equation (41) by \( e^{-s(u-b)} \) and then performing integration from \( b \) to \( \infty \), we obtain

\[
T_s \phi_2(b) = \frac{a_\gamma}{1 - \theta} T_s \frac{\phi_2(b)}{s} + \int^b_0 e^{-s(u-b)} \int^u_0 \phi(u-x; b) g_{b,-}(x) dx du + T_s \omega(b) \\
= \frac{a_\gamma}{1 - \theta} T_s \phi_2(b) - T_s \frac{\phi_2(b)}{s} + \tilde{g}_{b,-}(s) T_s \phi_2(b) \\
+ \int^b_0 \phi_1(x) T_s g_{b,-}(b-x) dx + T_s \omega(b),
\]

(43)

from which we can obtain

\[
T_s \phi_2(b) = \frac{T_s \omega(b) + \int^b_0 \phi_1(x) T_s g_{b,-}(b-x) dx - \frac{a_\gamma}{p_{\gamma} - (1-s)\gamma} T_s \frac{\phi_2(b)}{s}}{1 - \frac{a_\gamma}{p_{\gamma} - (1-s)\gamma} - \tilde{g}_{b,-}(s)} \\
= \frac{T_s \omega(b) + \int^b_0 \phi(x; \infty) T_s g_{b,-}(b-x) dx}{1 - \frac{a_\gamma}{p_{\gamma} - (1-s)\gamma} - \tilde{g}_{b,-}(s)} \\
+ \frac{L \int^b_0 v(x) T_s g_{b,-}(b-x) dx - \frac{a_\gamma}{p_{\gamma} - (1-s)\gamma} T_s \frac{\phi_2(b)}{s}}{1 - \frac{a_\gamma}{p_{\gamma} - (1-s)\gamma} - \tilde{g}_{b,-}(s)},
\]

where the second step follows from the substitution of formula (35).

Note that we have to determine the unknown constants \( L \) and \( T_s \frac{\phi_2(b)}{s} \). To this end, we can use the same arguments as in the analysis of \( V_2 \) to find them. Because \( T_s \phi_2(b) \) is finite for \( s > 0 \) and \( s_* \) is a zero point of the denominator in (44), \( s_* \) must be also a zero point of the numerator in (44). Hence, we have

\[
T_s \frac{\phi_2(b)}{s} = \frac{\rho_{\gamma} - (1-s)s_\gamma}{a_\gamma} \left( T_s \omega(b) + \int^b_0 \phi(x; \infty) T_s g_{b,-}(b-x) dx \\
+ L \int^b_0 v(x) T_s g_{b,-}(b-x) dx \right).
\]

Plugging this formula back into (44) we obtain

\[
T_s \phi_2(b) = \frac{T_s \omega(b) + \int^b_0 \phi(x; \infty) T_s g_{b,-}(b-x) dx}{1 - \frac{a_\gamma}{p_{\gamma} - (1-s)\gamma} - \tilde{g}_{b,-}(s)} \\
- \frac{\rho_{\gamma} - (1-s)s_\gamma}{p_{\gamma} - (1-s)\gamma} \left( T_s \omega(b) + \int^b_0 \phi(x; \infty) T_s g_{b,-}(b-x) dx \right) \frac{1}{1 - \frac{a_\gamma}{p_{\gamma} - (1-s)\gamma} - \tilde{g}_{b,-}(s)}
\]
order to compute (30) and (46), we still need to study the following functions, which together with Laplace inversion transform gives

\[
\phi_2(u + b) = H_\phi(u) + LH(u), \quad u \geq 0,
\]

where

\[
H_\phi(u) = \mathcal{L}^{-1}_s \left( \frac{T_s \omega(b) + \int_0^b \phi(x; \infty) T_s g_{\delta,-}(b - x)dx}{1 - \frac{\alpha_\gamma}{\rho_\gamma - (1 - \theta)s} - \hat{g}_{\delta,-}(s)} \right)(u).
\]

As for determining the constant \( L \), using the continuity condition (42) and formulæ (35) and (45), we can obtain

\[
L = \frac{H_\phi(0) - \phi(b; \infty)}{v(b) - H(0)}.
\]

Finally, we find that the Gerber-Shiu expected discounted penalty function can be expressed as follows,

\[
\phi(u; b) = \left\{ \begin{array}{ll}
\phi(u; \infty) + \frac{H_\phi(0) - \phi(b; \infty)}{v(b) - H(0)} v(u), & 0 \leq u \leq b, \\
H_\phi(u - b) + \frac{H_\phi(0) - \phi(b; \infty)}{v(b) - H(0)} H(u - b), & u > b.
\end{array} \right.
\]

5. Some explicit results for claims with rational Laplace transform. In order to compute (30) and (46), we still need to study the following functions, when the individual claim sizes have rational Laplace transform. More precisely, we suppose that the Laplace transform \( \hat{f}_X(s) \) is given by

\[
\hat{f}_X(s) = \frac{T_{2,m-1}(s)}{T_{1,m}(s)},
\]

where \( T_{1,m}(s) \) is a polynomial in \( s \) of degree \( m \) and \( T_{2,m-1}(s) \) is also a polynomial in \( s \) but of degree at most \( m - 1 \). Furthermore, suppose that \( T_{1,m}(s) \) and \( T_{2,m-1}(s) \) have no common zeros and the leading coefficient of \( T_{1,m}(s) \) is 1. Under this assumption, equation \( \psi(s) = \gamma + \delta \) (in \( s \)) has exactly \( m + 2 \) roots, and we assume that they are distinct for simplicity. Note that one root is \( \rho_\gamma \), and all the other roots, say \( -R_1, \ldots, -R_{m+1} \), have negative real parts.

First, we identify the discounted density function \( g_{\delta,-} \) by Laplace transform. For the scale function \( W^{(\gamma + \delta)}(x) \), its Laplace transform is given by

\[
\int_0^\infty e^{-sx}W^{(\gamma + \delta)}(x)dx = \frac{1}{\psi(s) - \gamma - \delta} = \frac{\hat{z}_2 T_{1,m}(s)}{(s - \rho_\gamma) \prod_{k=1}^{m+1}(s + R_k)} = \frac{A_\gamma}{s - \rho_\gamma} - \sum_{k=1}^{m+1} \frac{B_k}{s + R_k},
\]

(47)
where by partial fraction
\[ A_\gamma = \frac{2\alpha}{T_{1,m}(\rho_\gamma)}; \quad B_k = \frac{2\alpha}{T_{1,m}(-R_k)} \frac{1}{(R_k + \rho_\gamma) \prod_{j=1, j \neq k}^{m+1}(R_j - R_k)}, \quad k = 1, \ldots, m + 1. \]

Applying Laplace inversion transform to (47) we obtain
\[ W^{(\gamma + \delta)}(x) = A_\gamma e^{\rho_\gamma x} - \sum_{k=1}^{m+1} B_k e^{-R_k x}, \quad x \geq 0. \quad (48) \]

By formula (12) we obtain, for \( x > 0 \)
\[ g_{\delta,-}(x) = a_\gamma e^{\rho_\gamma x} - \gamma \left( A_\gamma e^{\rho_\gamma x} - \sum_{k=1}^{m+1} B_k e^{-R_k x} \right) = \sum_{k=1}^{m+1} \gamma B_k e^{-R_k x}, \quad (49) \]

where the second equality follows from the fact that \( A_\gamma = \frac{1}{\psi(\rho_\gamma)} \). It follows from formula (49) that the Laplace transform \( \hat{g}_{\delta,-}(s) \) is given by
\[ \hat{g}_{\delta,-}(s) = \sum_{k=1}^{m+1} \frac{\gamma B_k}{s + R_k}. \quad (50) \]

Then by formula (22) we have
\[ \hat{v}(s) = \frac{1}{s + a_\gamma - \rho_\gamma - (s - \rho_\gamma) \sum_{k=1}^{m+1} \frac{\gamma B_k}{s + R_k}} = \frac{\prod_{j=1}^{m+1}(s + R_j)}{\prod_{k=1}^{m+2}(s + \xi_k)} = \sum_{k=1}^{m+2} \frac{\alpha_k}{s + \xi_k}, \quad (51) \]

where \(-\xi_1, \ldots, -\xi_{m+2}\) are distinct roots of equation \( s + a_\gamma - \rho_\gamma - (s - \rho_\gamma) \hat{g}_{\delta,-}(s) = 0 \), and by partial fraction
\[ \alpha_k = \frac{\prod_{j=1}^{m+1}(R_j - \xi_k)}{\prod_{j=1, j \neq k}^{m+2}(\xi_j - \xi_k)}, \quad k = 1, \ldots, m + 2. \quad (52) \]

It can be verified that one of the roots \(-\xi_k's\) is equal to \( \beta_{\delta} \) and the other roots have negative real parts. For convenience, we set \(-\xi_{m+2} = \beta_{\delta} \). Applying Laplace inversion transform to (51) we obtain
\[ v(u) = \sum_{k=1}^{m+2} \alpha_k e^{-\xi_k u}, \quad u \geq 0. \quad (53) \]

Next, we consider \( H_V(u) \) and \( H(u) \). Using formula (50) we find that the Laplace transform of \( H_V(u) \) is given by
\[ \hat{H}_V(s) = \frac{s^{-1} Q_\delta - \rho_\gamma (1 - \theta) s^{-1} Q_\delta}{\rho_\gamma (1 - \theta) s^{-1} - \hat{g}_{\delta,-}(s)} = \frac{s(\rho_\gamma - (1 - \theta) s) \prod_{k=1}^{m+1}(s + R_k)}{(s - \rho_\gamma) \prod_{k=1}^{m+1}(s + R_k)} \]
\[ = \frac{Q_\delta \rho_\gamma s^{-1} (s - s_\gamma) \prod_{k=1}^{m+1}(s + R_k)}{(1 - \theta) s (s - s_\gamma) \prod_{k=1}^{m+1}(s + \xi_k)} = \frac{\beta_{\delta}}{s} + \sum_{k=1}^{m+1} \frac{\beta_k}{s + \xi_k}, \quad (54) \]

where \(-\xi_1, \ldots, -\xi_{m+1}\) are distinct negative roots of equation
\[ 1 - \frac{a_\gamma}{\rho_\gamma (1 - \theta)} - \hat{g}_{\delta,-}(s) = 0, \]
and by partial fraction
\[ \beta_0 = \frac{Q \rho \gamma s^{-1} \prod_{k=1}^{m+1} R_k}{(1-\theta) \prod_{k=1}^{m+1} \zeta_k}; \quad \beta_k = \frac{Q \rho \gamma s^{-1} \prod_{j=1}^{m+1} (R_j - \zeta_k)}{(1-\theta)(-\zeta_k) \prod_{j=1}^{m+1,j \neq k} (\zeta_j - \zeta_k)}, \quad k = 1, \ldots, m+1. \]

Then applying Laplace inversion transform to (54) gives
\[ H_V(u) = \beta_0 + \sum_{k=1}^{m+1} \beta_k e^{-\zeta_k u}, \quad u \geq 0. \]

Now let us consider \( H(u) \). By the definition of Dickson-Hipp operator and formula (49), we have
\[ T_s g_{\delta, -}(b-x) = \int_0^\infty e^{-sy} g_{\delta, -}(y+b-x)dy = \sum_{k=1}^{m+1} \frac{\gamma B_k}{s+R_k} e^{-R_k(b-x)}, \]

which together with formula (53) gives
\[ \int_0^b v(x) T_s g_{\delta, -}(b-x)dx = \sum_{i=1}^{m+2} \sum_{j=1}^{m+1} \gamma \alpha_i B_j \int_0^b e^{-\xi_j x} e^{-R_j(b-x)} dx = \sum_{i=1}^{m+2} \sum_{j=1}^{m+1} \gamma \alpha_i B_j \frac{e^{-\xi_j b} - e^{-R_j b}}{R_j - \xi_i}. \]

Hence, we have
\[ \hat{H}(s) = \frac{\int_0^b v(x) T_s g_{\delta, -}(b-x)dx - \frac{\rho \gamma s^{-1} \prod_{k=1}^{m+1} \zeta_k}{\rho \gamma (1-\theta)s} \int_0^b v(x) T_s g_{\delta, -}(b-x)dx}{1 - \frac{\rho \gamma s^{-1} \prod_{k=1}^{m+1} \zeta_k}{\rho \gamma (1-\theta)s} g_{\delta, -}(s)} = \sum_{i=1}^{m+2} \sum_{j=1}^{m+1} \gamma \alpha_i B_j \frac{e^{-\xi_j b} - e^{-R_j b}}{R_j - \xi_i} \frac{(s+R_j)^{-1} - \frac{\rho \gamma s^{-1} \prod_{k=1}^{m+1} \zeta_k}{\rho \gamma (1-\theta)s} (s+R_j)^{-1}}{1 - \frac{\rho \gamma s^{-1} \prod_{k=1}^{m+1} \zeta_k}{\rho \gamma (1-\theta)s} g_{\delta, -}(s)} = \frac{\text{Num}_1(s)}{\prod_{k=1}^{m+1} (s+\zeta_k)} = \sum_{k=1}^{m+1} \frac{C_k}{s+\zeta_k}, \]

where
\[ \text{Num}_1(s) = \sum_{i=1}^{m+2} \sum_{j=1}^{m+1} \gamma \alpha_i B_j \frac{e^{-\xi_j b} - e^{-R_j b}}{R_j - \xi_i} \rho \gamma + (1-\theta) R_j \prod_{k=1, k \neq j}^{m+1} (s+R_k) \]
is a polynomial of degree \( m \), and by partial fraction
\[ C_k = \frac{\text{Num}_1(-\zeta_k)}{\prod_{j=1, j \neq k}^{m+1} (\zeta_j - \zeta_k)}, \quad k = 1, \ldots, m+1. \]

Applying Laplace inversion transform we get
\[ H(u) = \sum_{k=1}^{m+1} C_k e^{-\zeta_k u}, \quad u \geq 0. \]
Finally, we study the Gerber-Shiu function with \( w = 1 \). In this case, we have
\[
\hat{\omega}(s) = \sum_{i=1}^{m+1} \frac{\gamma B_i R_i^{-1}}{s + R_i},
\]
and by formula (37), we obtain
\[
\hat{\phi}(s; \infty) = \frac{(s - \rho) \hat{\omega}(s) - (\beta_\delta - \rho_\gamma) \hat{\omega}(\beta_\delta)}{s + \alpha_\gamma - \rho_\gamma - (s - \rho_\gamma) g_{\delta_-}(s)} = \frac{\sum_{i=1}^{m+1} \gamma B_i R_i \frac{R_i}{s + \beta_\delta + R_i} \prod_{j=1, j \neq i}^{m+1} (s + R_j)}{\prod_{k=1}^{m+1} (s + \xi_k)},
\]
\[
:= \frac{\text{Num}_2(s)}{\prod_{k=1}^{m+1} (s + \xi_k)}.
\]
Hence,
\[
\hat{\phi}(u; \infty) = \sum_{k=1}^{m+1} D_k e^{-\xi_k u}, \quad u \geq 0,
\]
where
\[
D_k = \frac{\text{Num}_2(-\xi_k)}{\prod_{j=1, j \neq k}^{m+1} (\xi_j - \xi_k)}, \quad k = 1, \ldots, m + 1.
\]
Using the above result we have
\[
\int_0^b \phi(x; \infty) T_s g_{\delta_-}(b - x) dx = \sum_{i=1}^{m+1} \sum_{j=1}^{m+1} \gamma D_i B_j \frac{e^{-\xi_i} - e^{-R_i b}}{s + R_j} R_j - \xi_i,
\]
which leads to
\[
\hat{H}_\phi(s) = T_s \omega(b) + \int_0^b \phi(x; \infty) T_s g_{\delta_-}(b - x) dx = \frac{1 - \frac{\rho_\gamma - \rho - (1 - \theta) s}{\rho_\gamma - \rho - (1 - \theta) s} \hat{\omega}_\gamma(s)}{1 - \frac{\rho_\gamma - \rho - (1 - \theta) s}{\rho_\gamma - \rho - (1 - \theta) s} \hat{g}_{\delta_-}(s)} \left( T_s, \omega(b) + \int_0^b \frac{\rho_\gamma - \rho - (1 - \theta) s}{\rho_\gamma - \rho - (1 - \theta) s} \hat{\omega}_\gamma(s) \right)
\]
\[
= \sum_{i=1}^{m+1} \sum_{j=1}^{m+1} \gamma D_i B_j e^{-\xi_i} e^{-R_i b} \frac{\rho_\gamma + (1 - \theta) R_i}{s + R_j} \prod_{k=1, k \neq j}^{m+1} (s + R_k)
\]
\[
= \frac{\prod_{k=1}^{m+1} (s + \xi_k)}{\prod_{k=1}^{m+1} (s + \xi_k)} \cdot \frac{\prod_{k=1}^{m+1} (s + \xi_k)}{\prod_{k=1}^{m+1} (s + \xi_k)}
\]
\[
:= \frac{\text{Num}_3(s)}{\prod_{k=1}^{m+1} (s + \xi_k)},
\]
Then applying Laplace inversion transform we have
\[
H_\phi(u) = \sum_{k=1}^{m+1} E_k e^{-\xi_k u}, \quad u \geq 0,
\]
where
\[
E_k = \frac{\text{Num}_3(-\xi_k)}{\prod_{j=1, j \neq k}^{m+1} (\xi_j - \xi_k)}, \quad k = 1, \ldots, m + 1.
\]
6. Numerical illustrations. In this section, we present some numerical examples. Let $c = 1.5, \lambda = 1, \delta = 0.01, \gamma = 1, \theta = 0.2$ and $\sigma = 1$. We consider the following size density functions:

- Erlang(2): $f(x) = 4xe^{-2x}$;
- Combination of exponentials: $f(x) = 3e^{-1.5x} - 3e^{-3x}$;
- Exponential: $f(x) = e^{-x}$;
- Mixture of exponentials: $f(x) = \frac{2}{3}e^{-2x} + \frac{8}{15}e^{-0.8x}$.

All the above densities have rational Laplace transforms, so that we can use the results in Section 5 to compute $V(u; b)$ and $\phi(u; b)$. Furthermore, it is easily seen that the above densities have a common mean 1, but they possess increasing variances.

In Figure 1, we plot $V(u; b)$ as a function of $u$ for $b = 1, 5, 10$ with different claim size densities. It is easily seen that $V(u; b)$ is increasing with respect to $u$ for each $b$ and each claim size density. It is due to that for a larger initial surplus, ruin is less likely to happen so that dividends are more likely to be paid out forever since the surplus level will ultimately always stay above $b$. In Figure 2, we display $V(u; b)$ as a function of $b$ for different $u$ and different claim size densities. By careful observation, we find that in each case $V(u; b)$ is a concave function, and this implies that the optimal threshold level, say $b_{opt}$, that maximizes the expected discounted dividend payments before ruin exists. By comparing the values of $V(u; b)$ we can obtain $b_{opt}$, which are given in Figure 2. Furthermore, it is easily observed that the values of $b_{opt}$ are independent of the initial surplus level $u$.

In Figure 3, we plot the Gerber-Shiu function as a function of $u$ for $b = 1, 5, 10$ with different claim size densities. We find that the Gerber-Shiu function is a decreasing function of $u$, which is due to that ruin is less likely to happen for a larger initial surplus. In Figure 4, we plot the Gerber-Shiu function as a function of $b$ for $u = 3, 5, 8$. In this figure we also observe that $\phi(u; b)$ is decreasing with respect to $b$, which can be explained by the obvious reason: for a larger threshold $b$, more surplus will be kept as the insurance company’s capital for its solvency. Finally, we analyze the impact of variance of the claim size distribution on $\phi(u; b)$. In Figure 5, we plot $\phi(u; b)$ with different claim size densities. We find that $\phi(u; b)$ is increasing with respect to the variance, which is consistent with the usual intuition that ruin is more likely to occur for claims with larger variance.

Acknowledgments. We would like to thank the Editor and two anonymous referees for providing very helpful comments and suggestions.

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Figure 1. $V(u; b)$ as a function of $u$: (a) Exponential; (b) Erlang(2); (c) Mixture of exponentials; (d) Compound of exponentials.

Figure 2. $V(u; b)$ as a function of $b$: (a) Exponential; (b) Erlang(2); (c) Mixture of exponentials; (d) Compound of exponentials.
Figure 3. $\phi(\mu; b)$ as a function of $\mu$: (a) Exponential; (b) Erlang(2); (c) Mixture of exponentials; (d) Combination of exponentials.

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A MODEL WITH PERIODIC THRESHOLD-TYPE DIVIDEND STRATEGY 1985

Figure 4. $\phi(u; b)$ as a function of $b$: (a) Exponential; (b) Erlang(2); (c) Mixture of exponentials; (d) Combination of exponentials.

Figure 5. The Gerber-Shiu function $\phi(u; b)$: (a) $b = 5$; (b) $u = 5$.

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Received July 2018; revised November 2018.

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