Abstract

In this paper we construct some multi-time geometrical extensions of the KCC-invariants, which characterize a given second-order system of PDEs on the 1-jet space $J^1(T, M)$. A theorem of characterization of these multi-time geometrical KCC-invariants is given.

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1 Geometrical objects on multi-time 1-jet spaces

We remind first few differential geometrical properties of the multi-time 1-jet spaces. The multi-time 1-jet bundle

$$\xi_1 = (J^1(T, M), \pi_1, T \times M)$$

is a vector bundle over the product manifold $T \times M$, having the fibre type $\mathbb{R}^{m n}$, where $m$ (resp. $n$) is the dimension of the temporal (resp. spatial) manifold $T$ (resp. $M$). If the temporal manifold $T$ has the local coordinates $(t^\alpha)_{\alpha = 1, m}$ and the spatial manifold $M$ has the local coordinates $(x^i)_{i = 1, n}$, then we denote the local coordinates of the multi-time 1-jet space $J^1(T, M)$ by $(t^\alpha, x^i, x^i_\alpha)$. These transform by the rules [8]

$$\begin{align*}
\tilde{t}^\alpha &= \tilde{t}^\alpha(t^\beta) \\
\tilde{x}^i &= \tilde{x}^i(x^j) \\
\tilde{x}^i_\alpha &= \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial t^\beta}{\partial t^\alpha} x^j,
\end{align*}$$

(1.1)

where $\det(\partial \tilde{t}^\alpha/\partial t^\beta) \neq 0$ and $\det(\partial \tilde{x}^i/\partial x^j) \neq 0$.

Remark 1.1 In this work the greek indices $\alpha, \beta, \gamma, \delta, \varepsilon, \mu, \nu, \rho...$ run over the set $\{1, 2, ..., m\}$ and the latin indices $i, j, k, l, p, q, r, s...$ run over the set $\{1, 2, ..., n\}$. The Einstein convention of summation is also adopted all over this paper.
In the geometrical study of the multi-time 1-jet vector bundle, a central rôle is played by the distinguished tensors \((d-\text{tensors})\).

**Definition 1.2** A geometrical object \(D = (D_{\gamma k(\beta)(l)}^{(j)(\nu)})\) on the 1-jet vector bundle \(J^1(T, M)\), whose local components transform by the rules

\[
D_{\gamma k(\beta)(l)}^{(j)(\nu)} = \tilde{D}_{\delta p(\eta)}^{(\mu)} \frac{\partial}{\partial \tilde{x}^p} \frac{\partial}{\partial \tilde{t}^\gamma} \frac{\partial}{\partial x^k} \frac{\partial}{\partial \tilde{t}^\beta} \frac{\partial}{\partial x^l} \frac{\partial}{\partial \tilde{t}^\nu} \ldots,
\]

is called a \(d-\text{tensor field}\).

**Remark 1.3** The use of parentheses for certain indices of the local components of the distinguished tensor field \(D\) on the 1-jet space is motivated by the fact that the pair of indices \(^{(j)}_{(\beta)}\) or \(^{(\nu)}_{(l)}\) behaves like a single index.

**Example 1.4** The geometrical object

\[
C = C^{(i)}_{(\alpha)} \frac{\partial}{\partial x_\alpha} \cdot
\]

where \(C^{(i)}_{(\alpha)} = x^i_\alpha\), represents a \(d-\text{tensor field}\) on the 1-jet space; this is called the canonical Liouville \(d-\text{tensor field}\) of the 1-jet vector bundle \(J^1(T, M)\) and it is a global geometrical object.

**Example 1.5** Let \(h = (h_{\alpha\beta}(t))\) be a Riemannian metric on the temporal manifold \(T\). The geometrical object

\[
J_h = J^{(i)}_{(\alpha)j} \frac{\partial}{\partial x_\alpha} \odot dt^\beta \odot dx^j,
\]

where \(J^{(i)}_{(\alpha)j} = h_{\alpha\beta} \delta^i_j\) is a \(d-\text{tensor field}\) on \(J^1(T, M)\), which is called the h-normalization \(d-\text{tensor field}\) of the 1-jet space \(J^1(T, M)\). Obviously, it is also a global geometrical object.

In the Riemann-Lagrange differential geometry of the 1-jet spaces developed in [8], [9] important rôles are also played by geometrical objects as the temporal or spatial semisprays, together with the multi-time jet nonlinear connections.

**Definition 1.6** A set of local functions \(H = \left( H^{(i)}_{(\alpha)j} \right)\) on \(J^1(T, M)\), which transform by the rules

\[
2 \tilde{H}^{(i)}_{(\alpha)j} = 2H^{(k)}_{(\gamma)l} \frac{\partial^2}{\partial x^k} \frac{\partial}{\partial \tilde{t}^\gamma} \frac{\partial}{\partial t^\nu} - \frac{\partial}{\partial \tilde{t}^\beta} \frac{\partial^2}{\partial x^k} \frac{\partial}{\partial \tilde{t}^\gamma} \frac{\partial}{\partial t^\nu},
\]

is called a temporal semispray on \(J^1(T, M)\).
Example 1.7 Let us consider a Riemannian metric \( h = (h_{\alpha\beta}(t)) \) on the temporal manifold \( T \) and let

\[
H^\alpha_{\beta\gamma} = \frac{h^{\alpha\mu}}{2} \left( \frac{\partial h_{\beta\mu}}{\partial t^\gamma} + \frac{\partial h_{\gamma\mu}}{\partial t^\beta} - \frac{\partial h_{\beta\gamma}}{\partial t^\mu} \right)
\]
be its Christoffel symbols. Taking into account that we have the transformation rules

\[
H^\delta_{\nu\rho} = H^\alpha_{\beta\gamma} \frac{\partial \tilde{x}^\delta}{\partial x^\alpha} \frac{\partial \tilde{t}^\beta}{\partial t^\nu} \frac{\partial \tilde{t}^\gamma}{\partial t^\rho},
\]
we deduce that the local components

\[
\hat{H}^{(i)}_{\alpha\beta} = -\frac{1}{2} h^{\alpha\mu} \tilde{x}_i^\mu
\]
define a temporal semispray \( \hat{H} = \left( \hat{H}^{(i)}_{\alpha\beta} \right) \) on \( J^1(T, M) \). This is called the canonical temporal semispray associated to the temporal metric \( h_{\alpha\beta}(t) \).

Definition 1.8 A set of local functions \( G = \left( G^{(i)}_{\alpha\beta} \right) \), which transform by the rules

\[
2 \tilde{G}^{(i)}_{\gamma\nu} = 2 G^{(k)}_{\gamma\nu} \frac{\partial x^k}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^\gamma} - \frac{\partial x^\gamma}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^\nu},
\]
is called a spatial semispray on \( J^1(T, M) \).

Example 1.9 Let \( \phi = (\varphi_{ij}(x)) \) be a Riemannian metric on the spatial manifold \( M \) and let us consider

\[
\gamma^i_{jk} = \frac{\varphi^{im}}{2} \left( \frac{\partial \varphi_{jm}}{\partial x^k} + \frac{\partial \varphi_{km}}{\partial x^j} - \frac{\partial \varphi_{jk}}{\partial x^m} \right)
\]
its Christoffel symbols. Taking into account that we have the transformation rules

\[
\tilde{\gamma}^p_{\gamma q r} = \gamma^i_{jk} \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^j}{\partial x^q} \frac{\partial \tilde{x}^k}{\partial x^r} + \frac{\partial \tilde{x}^p}{\partial x^i} \frac{\partial \tilde{x}^q}{\partial x^j} \frac{\partial \tilde{x}^r}{\partial x^k},
\]
we deduce that the local components

\[
\hat{G}^{(i)}_{\alpha\beta} = \frac{1}{2} \gamma^i_{pq} \tilde{x}^p \tilde{x}^q
\]
define a spatial semispray \( \hat{G} = \left( \hat{G}^{(i)}_{\alpha\beta} \right) \) on \( J^1(T, M) \). This is called the canonical spatial semispray associated to the spatial metric \( \varphi_{ij}(x) \).

Definition 1.10 A set of local functions \( \Gamma = \left( M^{(i)}_{\alpha\beta}, N^{(i)}_{\alpha\beta} \right) \) on \( J^1(T, M) \), which transform by the rules

\[
\tilde{M}^{(i)}_{\alpha\beta} = M^{(k)}_{\gamma\nu} \frac{\partial \tilde{x}^k}{\partial x^\alpha} \frac{\partial \tilde{x}^\gamma}{\partial x^\beta} - \frac{\partial x^\gamma}{\partial t^\alpha} \frac{\partial \tilde{x}^k}{\partial t^\nu},
\]
is called the canonical spatial semispray associated to the spatial metric \( \varphi_{ij}(x) \).
and
\[ \tilde{N}^{(i)}_{(\alpha)j} = N^{(i)}_{(\gamma)k} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{U}}{\partial \bar{x}^j} + \frac{\partial x^r}{\partial \bar{x}^i} \frac{\partial \bar{U}^{(i)}}{\partial \bar{x}^r}, \]
\[ (1.8) \]

is called a nonlinear connection on the 1-jet space \( J^1(T, M) \).

**Example 1.11** Let us consider that \((T, h_{\alpha\beta}(t))\) and \((M, \phi_{ij}(x))\) are Riemannian manifolds having the Christoffel symbols \( H^\alpha_{\beta\gamma}(t) \) and \( \gamma^i_{jk}(x) \). Then, using the transformation rules (1.1), (1.4) and (1.6), we deduce that the set of local functions
\[ \tilde{\Gamma} = \left( \tilde{M}^{(i)}_{(\alpha)\beta}, \tilde{N}^{(i)}_{(\alpha)j} \right), \]
where
\[ \tilde{M}^{(i)}_{(\alpha)\beta} = -H^\mu_{\alpha\beta} x^i_\mu \quad \text{and} \quad \tilde{N}^{(i)}_{(\alpha)j} = \gamma^i_{jr} x^r_\alpha, \]
represents a nonlinear connection on the 1-jet space \( J^1(T, M) \). This multi-time jet nonlinear connection is called the canonical nonlinear connection attached to the pair of Riemannian metrics \((h_{\alpha\beta}(t), \phi_{ij}(x))\).

In the sequel, let us study the geometrical relations between temporal or spatial semisprays and multi-time nonlinear connections on the 1-jet vector bundle \( J^1(T, M) \). In this direction, using the local transformation laws (1.3), (1.7) and (1.1), respectively the transformation laws (1.5), (1.8) and (1.1), by direct local computations, we find the following geometrical results:

**Theorem 1.12** The temporal semisprays \( H = \left( H^{(i)}_{(\alpha)\beta} \right) \) and the sets of temporal components of nonlinear connections \( \Gamma_{\text{temporal}} = \left( M^{(i)}_{(\alpha)\beta} \right) \) are in one-to-one correspondence on the 1-jet space \( J^1(T, M) \), via:
\[ M^{(i)}_{(\alpha)\beta} = 2H^{(i)}_{(\alpha)\beta}, \quad H^{(i)}_{(\alpha)\beta} = \frac{1}{2} M^{(i)}_{(\alpha)\beta}; \]

**Theorem 1.13** (i) If \( G^{(i)}_{(\alpha)\beta} \) are the components of a spatial semispray on \( J^1(T, M) \), where \((T, h_{\alpha\beta}(t))\) is a Riemannian manifold, then the components
\[ N^{(i)}_{(\alpha)j} = \frac{\partial G^i}{\partial x^j_\mu} h_{\alpha\mu}, \]
where \( G^i = h^{\delta\epsilon} G^{(i)}_{(\delta)\epsilon} \), represent a spatial nonlinear connection on \( J^1(T, M) \).

(ii) Conversely, the spatial nonlinear connection \( \Gamma_{\text{spatial}} = \left( N^{(i)}_{(\alpha)j} \right) \) produces the spatial semispray components
\[ G^{(i)}_{(\alpha)\beta} = \frac{1}{2} N^{(i)}_{(\alpha)j} x^r_\beta. \]
In this Section we construct some multi-time generalizations on the 1-jet space $J^1(T, M)$ for the basic objects of the KCC-theory ([1], [2], [3], [10]). In this respect, let us consider on $J^1(T, M)$ a second-order system of partial differential equations of local form

$$\frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} + F^{(i)}_{(\alpha)\beta}(t^\gamma, x^k, x^\gamma_k) = 0, \quad \alpha, \beta = 1, m, \quad i = 1, n, \quad (2.1)$$

where $x^k = \partial x^k / \partial t^\gamma, F^{(i)}_{(\alpha)\beta} = F^{(i)}_{(\beta)\alpha}$ and the local components $F^{(i)}_{(\alpha)\beta}(t^\gamma, x^k, x^\gamma_k)$ transform under a change of coordinates (1.1) by the rules

$$\tilde{F}^{(i)}_{(\alpha)\beta} = 2\tilde{F}^{(k)}_{(\gamma)\nu} \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial t^\gamma}{\partial t^\nu} - \frac{\partial \tilde{t}^\alpha}{\partial t^\nu} \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial t^\beta}{\partial t^\nu} - \frac{\partial x^r}{\partial \tilde{t}^\mu} \frac{\partial \tilde{x}^s}{\partial \tilde{t}^\mu} \frac{\partial \tilde{x}^i}{\partial \tilde{x}^s} \frac{\partial \tilde{x}^i}{\partial \tilde{x}^s}, \quad (2.2)$$

Remark 2.1 The second-order system of partial differential equations (2.1) is invariant under a change of coordinates (1.1).

Example 2.2 Let us consider that $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$ are Riemannian manifolds having the Christoffel symbols $H^{\alpha}_{\beta\gamma}(t)$ and $\gamma_{ij}^k(x)$. Then, the local components

$$\tilde{F}^{(i)}_{(\alpha)\beta} = -H^{\mu}_{\alpha\beta} x^i_\mu + \gamma^q_{pq} x^p_{\alpha} x^q_{\beta}$$

transform under a change of coordinates (1.1) by the rules (2.2). In this particular case, the PDE system (2.1) becomes

$$\frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - H^{\mu}_{\alpha\beta} x^i_\mu + \gamma^q_{pq} x^p_{\alpha} x^q_{\beta} = 0, \quad \alpha, \beta = 1, m, \quad i = 1, n, \quad (2.3)$$

that is the PDE system of the affine maps between the Riemannian manifolds $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$. We recall that these affine maps carry the geodesics of the temporal Riemannian manifold $(T, h_{\alpha\beta}(t))$ into the geodesics of the spatial Riemannian manifold $(M, \varphi_{ij}(x))$. Moreover, the $h$–trace of the equations (2.3) produces the equations of the harmonic maps between the Riemannian manifolds $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$. For more details, please see [6].

Using a temporal Riemannian metric $h_{\alpha\beta}(t)$ on $T$ and taking into account the transformation rules (1.3), (1.5) and (2.2), we can rewrite the PDE system (2.1) in the following form:

$$\frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - H^{\mu}_{\alpha\beta} x^i_\mu + 2G^{(i)}_{(\alpha)\beta}(t^\gamma, x^k, x^\gamma_k) = 0, \quad \alpha, \beta = 1, m, \quad i = 1, n,$$

where

$$G^{(i)}_{(\alpha)\beta} = \frac{1}{2} F^{(i)}_{(\alpha)\beta} + \frac{1}{2} H^{\mu}_{\alpha\beta} x^i_\mu$$

2 Multi-time geometrical KCC-theory
are the components of a spatial semispray on $J^1(T, M)$. The coefficients of the spatial semispray $G^{(i)}_{(\alpha)\beta}$ produce the spatial components $N^{(i)}_{(\alpha)j}$ of a nonlinear connection $\Gamma$ on the 1-jet space $J^1(T, M)$, by putting

$$N^{(i)}_{(\alpha)j} = \frac{h^{\mu\nu} \partial G^{(i)}_{(\mu)\nu}}{\partial x^j} h_{\gamma\alpha} = \frac{h^{\mu\nu} \partial F^{(i)}_{(\mu)\nu}}{2 \partial x^j} h_{\gamma\alpha} + \frac{h^{\mu\nu} h_{\gamma\alpha} \delta^i_j}{2}.$$

In order to find the basic jet multi-time geometrical invariants of the PDE system (2.1), we define the $h-$KCC-covariant derivative of a $d-$tensor of type $T^{(i)}_{(\alpha)\beta}$ on the 1-jet space $J^1(T, M)$, via

$$\frac{h}{\partial t^\beta} T^{(i)}_{(\alpha)\beta} = \frac{\partial T^{(i)}_{(\alpha)\beta}}{\partial t^\beta} + N^{(i)}_{(\alpha)\beta} T^{(r)}_{(\beta)} - H^{\mu}_{\alpha\beta} T^{(i)}_{(\mu)} =$$

$$= \frac{\partial T^{(i)}_{(\alpha)\beta}}{\partial t^\beta} + \frac{h^{\mu\nu} \partial F^{(i)}_{(\mu)\nu}}{2 \partial x^\gamma} h_{\gamma\alpha} T^{(r)}_{(\beta)} + \frac{h^{\mu\nu} h_{\gamma\alpha} T^{(i)}_{(\beta)}}{2} - H^{\mu}_{\alpha\beta} T^{(i)}_{(\mu)}.$$

**Remark 2.3** The $h-$KCC-covariant derivative components $\frac{h}{\partial t^\beta} T^{(i)}_{(\alpha)\beta}$ transform under a change of coordinates (1.1) as a $d-$tensor of type $T^{(i)}_{(\alpha)\beta}$.

In such a geometrical context, if we use the notation $x^i_\alpha = \partial x^i / \partial t^\alpha$, then the PDE system (2.1) can be rewritten in the following distinguished tensorial form:

$$\frac{h}{\partial t^\beta} x^i_\alpha = -F^{(i)}_{(\alpha)\beta} (T, x^k, x^\gamma) + N^{(i)}_{(\alpha)\beta} x^r_\beta - H^{\mu}_{\alpha\beta} x^i_\mu =$$

$$= -F^{(i)}_{(\alpha)\beta} + \frac{h^{\mu\nu} \partial F^{(i)}_{(\mu)\nu}}{2 \partial x^\gamma} h_{\gamma\alpha} x^r_\beta + \frac{h^{\mu\nu} h_{\gamma\alpha} x^i_\mu}{2} - H^{\mu}_{\alpha\beta} x^i_\mu.$$

**Definition 2.4** The distinguished tensor

$$\frac{h}{\partial t^\beta} = -F_{(\alpha)\beta} + \frac{h^{\mu\nu} \partial F^{(i)}_{(\mu)\nu}}{2 \partial x^\gamma} h_{\gamma\alpha} x^r_\beta + \frac{h^{\mu\nu} h_{\gamma\alpha} x^i_\mu}{2} - H^{\mu}_{\alpha\beta} x^i_\mu$$

is called the **first multi-time $h-$KCC-invariant** on the 1-jet space $J^1(T, M)$ of the PDEs (2.1), which can be interpreted as an **external force** [1], [3].

**Example 2.5** For the second-order PDE system (2.3), which gives the affine maps between the Riemannian manifolds $(T, h_{\alpha\beta}(t))$ and $(M, \varphi_{ij}(x))$, the first multi-time $h-$KCC-invariant is zero.
Example 2.6 It can be easily seen that for the particular first order PDE system
\[
\frac{\partial x^i}{\partial t^\alpha} = X^{(i)}(t^\gamma, x^k) = \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} = \frac{\partial X^{(i)}(t^\gamma)}{\partial x^r} x^r_{\beta},
\]
where \(X^{(i)}(t, x)\) is a given \(d\)-tensor on \(J^1(T, M)\), the first multi-time \(h\)-KCC-invariant has the form
\[
h^{(i)}_{\alpha \beta}(x^i) = \frac{\partial X^{(i)}(t^\gamma)}{\partial t^\beta} + \frac{1}{2} \frac{\partial X^{(i)}(t^\gamma)}{\partial x^r} x^r_{\beta} + \frac{h^\mu_\alpha}{2} h^\gamma_\mu h^{\gamma \alpha} x^i_{\beta} - H^\gamma_{\alpha \beta} x^i_{\mu}.
\]

In the sequel, let us vary the solutions \(x^i(t^\gamma)\) of the PDE system (2.1) by the nearby smooth maps \((x^i(t^\gamma, s))_{s \in (-\varepsilon, \varepsilon)}\), where \(x^i(t^\gamma, 0) = x^i(t^\gamma)\). Then, if we consider the variation \(d\)-tensor field \(\xi^i(t^\gamma) = \frac{\partial x^i}{\partial s} \big|_{s=0}\), we get the variational equations
\[
\frac{\partial^2 \xi^i}{\partial s^\alpha \partial s^\beta} + \frac{\partial F^{(i)}_{(\alpha \beta \gamma)}}{\partial x^k} \xi^k + \frac{\partial F^{(i)}_{(\alpha \beta \gamma)}}{\partial x^r} \xi^r = 0, \quad \alpha, \beta = 1, m, \quad i = 1, n.
\]
(2.5)
It is obvious that the equations (2.5) imply the \(h\)-trace variational equations
\[
h^{\alpha \beta} \frac{\partial^2 \xi^i}{\partial s^\alpha \partial s^\beta} + \frac{\partial F^i}{\partial x^k} \xi^k + \frac{\partial F^i}{\partial x^r} \xi^r = 0, \quad i = 1, n.
\]
(2.6)
where \(F^i = h^{\alpha \beta} F^{(i)}_{(\alpha \beta \gamma)}\).

To find other multi-time geometrical invariants for the PDE system (2.1), we also introduce the \(h\)-KCC-covariant derivative of a \(d\)-tensor of type \(\xi^i(t^\gamma)\) on the 1-jet space \(J^1(T, M)\), via
\[
\nabla \xi^i = \frac{\partial \xi^i}{\partial t^\alpha} + N^{(i)}_{(\alpha \mu \gamma)} \xi^\gamma = \frac{\partial \xi^i}{\partial t^\alpha} + \frac{1}{2} \frac{\partial F^i}{\partial x^r} h^\gamma_{\alpha \beta} \xi^r + \frac{1}{2} H^\gamma h^{\gamma \alpha} \xi^i,
\]
where \(H^\gamma = h^\mu_\nu H^\gamma_{\mu \nu}\).

Remark 2.7 The \(h\)-KCC-covariant derivative components \(\nabla \xi^i = \frac{\partial \xi^i}{\partial t^\alpha}\) transform under a change of coordinates (1.1) as a \(d\)-tensor of type \(T^{(i)}_{(\alpha \mu \gamma)}\).

In this geometrical context, the \(h\)-trace variational equations (2.6) can be rewritten in the following distinguished tensorial form:
\[
h^{\alpha \beta} \nabla \xi^i = \frac{h^{\nabla \xi^i}}{\partial t^\alpha} \left[ \frac{h^{\nabla \xi^i}}{\partial t^\alpha} \right] = F^i \xi^r.
\]
where
\[
\frac{h}{P^i_j} = -\frac{\partial F^i}{\partial x^j} + \frac{1}{2} \frac{\partial^2 F^i}{\partial x^j \partial x^l} \gamma_{ij}^l + \frac{1}{2} \frac{\partial^2 F^i}{\partial x^j \partial x^r} x^r_{ij} - \frac{1}{2} \frac{\partial^2 F^i}{\partial x^j \partial x^r} F^{(l)}_{(\gamma)\mu^l} + \\
+ \frac{h_{\gamma \mu}}{4} \frac{\partial F^i}{\partial x^r} + \frac{h_{\gamma \mu}}{2} \frac{\partial h_{\mu \gamma}}{\partial x^r} + \frac{h_{\gamma \mu}}{4} \frac{\partial H_{\mu}}{\partial x^r} + \frac{h_{\gamma \mu}}{4} H_{\gamma} H_{\mu} \right] \delta^i_j.
\]

**Definition 2.8** The \(d\)-tensor \(\frac{h}{P^i_j}\) is called the **second multi-time \(h\)-KCC-invariant** on the 1-jet space \(J^1(T, M)\) of the PDE system \([2.7]\), or the **multi-time \(h\)-deviation curvature \(d\)-tensor**.

**Example 2.9** If we consider the second-order PDE system of the affine maps between the Riemannian manifolds \((T, h_{\alpha \beta}(t))\) and \((M, \varphi_{ij}(x))\), system which is given by \([2.3]\), then the second multi-time \(h\)-KCC-invariant has the form
\[
\frac{h}{P^i_j} = -h_{\alpha \beta} R^i_{pq} x^p_{\alpha} x^q_{\beta},
\]
where
\[
R^i_{pq} = \frac{\partial^i p}{\partial x^p} - \frac{\partial^i p}{\partial x^q} + \gamma_{pq}^r \gamma_{ij}^r - \gamma_{pq}^r \gamma_{riq}^r
\]
are the components of the curvature of the spatial Riemannian metric \(\varphi_{ij}(x)\).

Consequently, the \(h\)-trace variational equations \([2.6]\) become the following **multi-time \(h\)-Jacobi field equations**:
\[
h^{\alpha \beta} \left\{ \frac{h}{\partial t^\beta} \left[ \frac{h}{\partial t^\alpha} \left( \frac{h}{\partial x^i} \frac{h}{\partial x^j} \right) \right] + R^i_{pq} x^p_{\alpha} x^q_{\beta} \right\} = 0,
\]
where
\[
\frac{h}{\partial t^\alpha} = \frac{\partial x^\alpha}{\partial t^\alpha} + \gamma_{\alpha \beta} x^p_{\beta} \xi^p.
\]

**Example 2.10** For the particular first order PDE system \([2.3]\) the multi-time \(h\)-deviation curvature \(d\)-tensor is given by
\[
\frac{h}{P^i_j} = \frac{h_{\alpha \beta}}{2} \left[ \frac{\partial^2 X^{(i)}_{(\alpha)}}{\partial t^\beta \partial x^j} + \frac{\partial^2 X^{(i)}_{(\alpha)}}{\partial x^j \partial x^r} x^r_{ij} + \frac{1}{2} \frac{\partial X^{(i)}_{(\alpha)}}{\partial x^j} \frac{\partial X^{(r)}_{(\beta)}}{\partial x^j} \right] + \\
+ \frac{1}{2} \frac{\partial H_{\gamma}}{\partial t^\gamma} + \frac{h_{\gamma \mu}}{2} \frac{\partial h_{\mu \gamma}}{\partial x^r} H_{\mu} - \frac{h_{\gamma \mu}}{4} H_{\gamma} H_{\mu} \right] \delta^i_j.
\]
Definition 2.11  The distinguished tensors

\[
\begin{align*}
R_{j_k}^{i} &= \frac{1}{3} \left[ \frac{\partial P^i_j}{\partial x^k} - \frac{\partial P^i_k}{\partial x^j} \right], \\
B_{j_k}^{i} &= \frac{\partial R_{j_k}^{i}}{\partial x^i}
\end{align*}
\]

and

\[
D^{(i)}_{(\alpha)\beta(j)(k)} = \frac{\partial^3 F^{(i)}_{(\alpha)\beta}}{\partial x^i \partial x^j \partial x^k}
\]

are called the third, fourth and fifth multi-time \( h - \text{KCC-invariant} \) on the 1-jet vector bundle \( J^1(T, M) \) of the PDE system (2.1).

Remark 2.12  Taking into account the transformation rules (2.2) of the components \( F^{(i)}_{(\alpha)\beta} \), we immediately deduce that the components \( D^{(i)}_{(\alpha)\beta(j)(k)} \) behave like a \( d - \)tensor on the 1-jet space \( J^1(T, M) \).

Example 2.13  For the second-order PDE system (2.3) of the affine maps between the Riemannian manifolds \( (T, h_{\alpha\beta}(t)) \) and \( (M, \varphi_{ij}(x)) \), the third, fourth and fifth multi-time \( h - \text{KCC-invariants} \) are given by

\[
\begin{align*}
R_{j_k}^{i} &= h^{\alpha\mu}R_{\mu jk}^i x^p_{\alpha}, \\
B_{j_k}^{i} &= h^{\alpha\beta}R_{\alpha jk}^i, \\
D^{(i)}_{(\alpha)\beta(j)(k)} &= 0.
\end{align*}
\]

Example 2.14  For the first order PDE system (2.4) the third, fourth and fifth multi-time \( h - \text{KCC-invariants} \) are zero.

Theorem 2.15  (of characterization of the multi-time KCC-invariants)  Let \( (T, h) \) be a Riemannian manifold, where \( m = \dim T \geq 3 \). If the first and the fifth KCC-invariants of the PDE system (2.1) cancel on \( J^1(T, M) \), then there exist on \( J^1(T, M) \) some local functions \( \Gamma_{pq}^i(t, x) \), where \( i, p, q = 1, n, n = \dim M \), and \( S_{\alpha p q}^{\nu} \alpha (t, x), \alpha \neq \nu \in \{1, 2, \ldots, m\}, i = 1, n, p \neq q \in \{1, 2, \ldots, n\} \), which have the properties

\[
\Gamma_{pq}^i = \Gamma_{qp}^i, \quad S_{\alpha p q}^{\nu} + S_{\alpha q p}^{\nu} = 0
\]

and (no sum by \( \alpha \) or \( \nu \))

\[
2S_{\alpha p q}^{\nu} = \sum_{\nu \neq \nu}^{m} \left[ h_{\nu \nu} S_{\nu p q}^{\nu} - h_{\nu \nu} S_{\nu p q}^{\nu} \right] h_{\nu \nu}, \quad (2.7)
\]

such that (no sum by \( \alpha \) or \( \beta \))

\[
F^{(i)}_{(\alpha)\beta} = \Gamma_{pq}^i x^p_{\alpha} x^q_{\beta} - H^{\mu}_{\alpha\beta} x^\mu + 2\delta_{\alpha\beta} \sum_{\nu \neq \nu}^{m} \sum_{p \neq q \in \{1, 2, \ldots, n\}} S_{\alpha p q}^{\nu} x^p_{\alpha} x^q_{\nu}, \quad (2.8)
\]

where \( \delta_{\alpha\beta} \) is the Kronecker symbol and \( H^{(\gamma)}_{\alpha\beta} \) are the Christoffel symbols of the Riemannian metric \( h_{\alpha\beta}(t) \).
Proof. By integration, the relations

$$D_{\alpha\beta;\eta\rho}(\gamma(\epsilon)) = \frac{\partial^3 F_{\alpha\beta}}{\partial x^\eta \partial x^\rho \partial x^\gamma} = 0,$$

where $F_{\alpha\beta} = F_{\beta\alpha}$, subsequently lead to

$$\frac{\partial^2 F_{\alpha\beta}}{\partial x^\eta \partial x^\gamma} = 2\Gamma_{\alpha\beta}(\gamma(\epsilon))(t, x) \Rightarrow \frac{\partial F_{\alpha\beta}}{\partial x^\eta} = 2\Gamma_{\alpha\beta}(\gamma(\epsilon))(\nu) \Rightarrow F_{\alpha\beta} = \Gamma_{\alpha\beta}(\gamma(\epsilon))(\nu) x^\nu + U_{\alpha\beta}(\gamma(\epsilon))(t, x),$$

where

$$\Gamma_{\alpha\beta}^{(i)}(\gamma(\epsilon)) = \Gamma_{\alpha\beta}^{(i)}(\gamma(\epsilon)) = \Gamma_{\alpha\beta}(\gamma(\epsilon)), \quad U_{\alpha\beta}^{(i)}(\gamma(\epsilon)) = U_{\alpha\beta}(\gamma(\epsilon)), \quad V_{\alpha\beta}^{(i)} = V_{\alpha\beta}(\gamma(\epsilon)). \quad (2.9)$$

The equalities $\epsilon_{\alpha\beta} = 0$ on $J^i(T, M)$ lead us to

$$\Gamma_{\alpha\beta}(\gamma(\epsilon))(\mu(\nu)) = \frac{1}{2} \left[ \Gamma_{\alpha\beta}(\gamma(\epsilon))(\nu) \delta_{\beta}^\mu + \Gamma_{\alpha\beta}(\gamma(\epsilon))(\mu) \delta_{\beta}^\nu \right] h_{\alpha\rho},$$

$$U_{\alpha\beta}^{(i)}(\nu) = \frac{1}{2} U_{\alpha\beta}^{(i)}(\nu) h_{\alpha\beta} + \frac{1}{2} H_{\alpha\beta}^{\nu\rho} \delta^\nu_{\beta} \delta^\rho_{\alpha} - H_{\alpha\beta}^{\nu\nu}, \quad (2.10)$$

where

$$\Gamma_{\alpha\beta}(\gamma(\epsilon))(\mu(\nu)) = h_{\alpha\beta} \Gamma_{\alpha\beta}(\gamma(\epsilon))(\nu) \quad \text{and} \quad U_{\alpha\beta}^{(i)}(\nu) = h_{\alpha\beta} U_{\alpha\beta}(\gamma(\epsilon))(\nu).$$

Applying an $h-$trace in the second relation of (2.10), we deduce that

$$U_{\alpha\beta}(\gamma(\epsilon))(\nu) = -H_{\alpha\beta}^{\nu\nu} \delta_q^i.$$  

The first relation of (2.10) and the first symmetry properties of (2.9) imply the following equalities:

1. for every $\alpha \neq \beta$ we have (no sum by $\alpha$ or $\beta$):
   (a) $\mu, \nu \notin \{\alpha, \beta\} \Rightarrow \Gamma_{\alpha\beta}(\gamma(\epsilon))(\mu(\nu)) = 0$;
   (b) $\mu = \alpha, \nu = \alpha \Rightarrow \Gamma_{\alpha\beta}(\gamma(\epsilon))(\alpha(\alpha)) = 0$;
   (c) $\mu = \alpha, \nu = \beta \Rightarrow \Gamma_{\alpha\beta}(\gamma(\epsilon))(\alpha(\beta)) = \frac{1}{2} \Gamma_{\alpha\beta}(\gamma(\epsilon))(\alpha(\alpha)) h_{\alpha\beta}^{\alpha(\alpha)} n_{\alpha(\alpha)} = T_{\alpha(\alpha)}^{\alpha(\alpha)} = T_{\alpha(\alpha)}^{\alpha(\alpha)}$;
   (d) $\mu = \beta, \nu = \alpha \Rightarrow \Gamma_{\alpha\beta}(\gamma(\epsilon))(\beta(\alpha)) = \frac{1}{2} \Gamma_{\alpha\beta}(\gamma(\epsilon))(\beta(\beta)) h_{\alpha\beta}^{\beta(\beta)} n_{\beta(\beta)} = T_{\beta(\beta)}^{\beta(\beta)} = T_{\beta(\beta)}^{\beta(\beta)}$.

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(c) $\mu = \beta, \nu = \beta \Rightarrow \Gamma^{(i)}_{(\alpha)(\beta)(p)(q)}(\beta(\beta) = \Gamma^{(i)}_{(\beta)(p)(q)} = \\
= \frac{1}{2} \left[ I^{(i)(\beta)(\beta)}_{(p)(q)} h_{\beta\alpha} + \Gamma^{(i)(\beta)(\beta)}_{(p)(q)} h_{\beta\alpha} \right] = S_{\alpha pq} + T_{\alpha pq} = 0;
2. for every $\alpha = \beta \in \{1, 2, ..., m\}$ we obtain (no sum by $\alpha$):

(a) $\mu \neq \alpha, \nu \neq \alpha \Rightarrow \Gamma^{(i)}_{(\alpha)(\alpha)(p)(q)} = 0$;
(b) $\mu = \alpha, \nu \neq \alpha \Rightarrow \Gamma^{(i)}_{(\alpha)(\alpha)(p)(q)}(\alpha(\alpha) = 1/2 I^{(i)(\alpha)(\alpha)}_{(p)(q)} h_{\alpha\alpha} = S_{\alpha pq} + T_{\alpha pq};
(c) $\mu \neq \alpha, \nu = \alpha \Rightarrow \Gamma^{(i)}_{(\mu)(\alpha)(p)(q)} = 1/2 I^{(i)(\mu)(\alpha)}_{(p)(q)} h_{\alpha\alpha} = I_{\alpha pq} + S_{\alpha pq} =
\varepsilon = \epsilon\epsilon = \epsilon\eta \alpha_{1}, \eta_{2} = 1/2 \left[ I^{(i)(\alpha)(\alpha)}_{(p)(q)} h_{\alpha\alpha} + \Gamma^{(i)(\alpha)(\alpha)}_{(p)(q)} h_{\alpha\alpha} \right] = T_{\alpha pq} + S_{\alpha pq};

The first symmetry condition from (2.9), together with 1.(c) and 1.(d), give
us $(m = \dim T = 3)$

\begin{align*}
S_{1 pq} & = T_{2 pq} = T_{3 pq} = T_{4 pq} = \cdots = T_{mpq} = \not\Gamma \frac{1}{2} T_{pq} \\
S_{2 pq} & = T_{1 pq} = T_{3 pq} = T_{4 pq} = \cdots = T_{mpq} = \not\Gamma \frac{1}{2} T_{pq} \\
\cdots & = \cdots = \cdots = \cdots \cdots \cdots \cdots \cdots \cdots \\
S_{mpq} & = T_{1 pq} = T_{2 pq} = T_{3 pq} = \cdots = T_{(m-1) pq} = \not\Gamma \frac{1}{2} T_{pq}.
\end{align*}

Consequently, for every $\alpha \neq \beta \in \{1, 2, ..., m\}$ we have

$$\Gamma^{(i)}_{(\alpha)(\beta)(p)(q)} = \frac{1}{2} T_{pq} \left[ \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} + \delta^{\nu}_{\alpha} \delta^{\mu}_{\beta} \right]$$

and for every $\alpha \in \{1, 2, ..., m\}$ we have

$$\Gamma^{(i)}_{(\alpha)(\alpha)(p)(q)} = \Gamma_{pq}^{i} = \Gamma^{(i)}_{(\alpha)(\alpha)(p)(q)} = \Gamma_{qp}^{i}.$$

Using now all the preceding properties, together with the equality 2.(b), we find the equations (2.7). Moreover, for every $\alpha \neq \nu \in \{1, 2, ..., m\}$, it is obvious that we have

$$S_{\alpha pq} + S_{\alpha qp} = 0 \Rightarrow S_{\alpha pq} = 0.$$

All the preceding situations can be briefly written in the general formula

$$\Gamma^{(i)}_{(\alpha)(\beta)(p)(q)} = \frac{1}{2} T_{pq} \left[ \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} + \delta^{\nu}_{\alpha} \delta^{\mu}_{\beta} \right] + S^{\mu}_{\alpha pq} \delta_{\alpha \beta} \delta^{\mu}_{\alpha} \left[ 1 - \delta^{\mu}_{\alpha} \right] + S^{\mu}_{\alpha qp} \delta_{\alpha \beta} \delta^{\mu}_{\alpha} \left[ 1 - \delta^{\mu}_{\alpha} \right].$$

In conclusion, we obtain the equalities (2.8) on the 1-jet space $J^{1}(T, M)$. n
Open problem. If we fix the indices $i$ and $p \neq q$ in the set $\{1, 2, ..., n\}$, then we deduce that the system of equations (2.7) is an homogenous linear system of order $m(m - 1)$. Consequently, it has at least the zero solution. Because the coefficients of the system depend only by the metric $h_{\alpha\beta}(t)$, there exist a temporal Riemannian metric $h_{\alpha\beta}(t)$ such that the system of equations (2.7) to admit only the banal solution?

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Authors’ address:

Mircea NEAGU
University Transilvania of Brașov, Faculty of Mathematics and Informatics,
Department of Algebra, Geometry and Differential Equations,
B-dul Iuliu Maniu, Nr. 50, BV 500091, Brașov, Romania.
E-mail: mircea.neagu@unitbv.ro
Website: [http://www.2collab.com/user:mirceaneagu](http://www.2collab.com/user:mirceaneagu)