DEFINABLE GROUPS IN TOPOLOGICAL FIELDS WITH A GENERIC DERIVATION

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ABSTRACT. We continue the study of a class of topological \(\mathcal{L}\)-fields endowed with a generic derivation \(\delta\), focussing on describing definable groups. We show that one can associate to an \(\mathcal{L} \cup \{\delta\}\) definable group a type \(\mathcal{L}\)-definable topological group. We use the group configuration tool in o-minimal structures as developed by K. Peterzil.

1. Introduction

Let \(\mathcal{K}\) be an \(\mathcal{L}\)-structure expanding a field of characteristic 0, endowed with a non-discrete definable field topology, as introduced by A. Pillay in [12]. We assume that \(\mathcal{K}\) is a model of an \(\mathcal{L}\)-open theory \(T\) of topological fields (see section 2.1 for a precise definition). The language \(\mathcal{L}\) is a (multisorted) language which is, on the field sort, a relational expansion of the ring language possibly with additional constants (with further assumptions on relation symbols, as defined in [7], [2]). In particular the theory \(T\) is a complete \(\mathcal{L}\)-theory admitting quantifier elimination on the field sort.

Examples of such theories \(T\) are the theories of algebraically closed valued fields, real-closed fields, real-closed valued fields, \(p\)-adically closed fields or henselian valued fields of characteristic 0. Note that the first four theories are not only dp-minimal but respectively \(C\)-minimal, \(o\)-minimal, weakly \(o\)-minimal, \(p\)-minimal.

W. Johnson established a link between topological fields and dp-minimal fields. He showed that if \(\mathcal{K}\) is an expansion of a field, \(\mathcal{K}\) infinite, and dp-minimal but not strongly minimal, then \(\mathcal{K}\) can be endowed with a non-discrete definable field topology, namely \(\mathcal{K}\) has a uniformly definable basis of neighbourhoods of zero compatible with the field operations [3, Theorem 9.1.3]. Moreover, in models of the theory of \(\mathcal{K}\), the topological dimension coincides with the dp-rank. This definable field topology is a V-topology and so it is induced either by a non-trivial valuation or an absolute value.

Here given an \(\mathcal{L}\)-open theory \(T\) of topological fields, we consider the \textit{generic} expansion of a model of \(T\) with a derivation \(\delta\). Namely, letting \(\mathcal{L}_\delta := \mathcal{L} \cup \{^{-1}\} \cup \{\delta\}\) and \(T_\delta\) the \(\mathcal{L}_\delta\)-theory consisting of \(T\) together with the axioms expressing that \(\delta\) is a derivation, we consider the class of existentially closed models of \(T_\delta\). When \(T\) is a theory of henselian fields of characteristic 0, we identify the class of existentially closed models of \(T_\delta\) (see [2, Corollary 3.3.4] for a precise statement): we give an explicit axiomatisation \(T_\delta^*\) and we show various transfer of model-theoretic properties from the theory \(T\) to
the theory $T^*_\delta$. An easy but quite useful result is that $T^*_\delta$ admits quantifier elimination as well (on the field sort).

The main aim of the present paper is to show that given an $L_\delta$-definable group (on the field sort) in a model of $T^*_\delta$, one can associate a type $L$-definable group. There are two steps in the proof. First we give a more direct and slightly more informative and more general proof that $T^*_\delta$ has the $L$-open core property [2, Theorem 6.0.8]. Second we perform two constructions. Both were first done for o-minimal expansions of a group. The first construction is due to A. Pillay [13] who, on a definable group, put a definable topology for which the group operations become continuous and the other one is due to K. Peterzil who, starting from a group configuration, constructed a type definable group. In order to perform those constructions, besides the $L$-open core property, one uses that the topological dimension function is a well-behaved dimension function in models of $T$ [2, Proposition 2.4.1, Corollary 2.4.5] (the topological dimension coincides with the algebraic dimension, has the exchange property and is a fibered dimension function [11]. Finally one uses the transfer of the elimination of imaginaries from $T$ to $T^*_\delta$ [2, Theorem 4.0.5] under the $L$-open core property of $T^*_\delta$. In [2], in the case the topology on the models of $T$ is given by a valuation, we need an intermediate result on continuity almost everywhere of definable functions to the value group and we only show it holds under certain conditions on the value group [2, Proposition 2.6.11]. Here we proceed more directly constructing, what we call an $L$-definable envelop of an $L_\delta$-definable subset. It will help us to associate with an $L_\delta$-definable group a large definable $L$-definable subset of the so-called envelop on which we can recover the group operations on the differential points.

The contents of the paper are as follows. In section 2, we review the properties that we need on the topological dimension in models of an $L$-open theory; we recall the definition of the theory $T^*_\delta$ and various transfer properties between $T$ and $T^*_\delta$. In section 3, we give a direct proof of the $L$-open core property of $T^*_\delta$ (Corollary 3.9): given a $L_\delta$-definable set $X$, one constructs a $L$-definable set that we call an $L$-open envelop (Proposition 3.8). In section 4, we prove our main result, namely we show how to associate with an $L_\delta$-definable group a type $L$-definable topological group (Theorem 4.15). In section 5 (the annex), we revisit the question of when a topological differential field embeds in a model of the scheme (DL).

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2. Preliminaries

2.1. Model theory and topological fields. Let $L_{\text{ring}} := \{\cdot, +, -, 0, 1\}$ and $L_{\text{field}} := L_{\text{ring}} \cup \{-1\}$ denote the respective language of rings and of fields. Any field is an $L_{\text{field}}$-structure by extending the multiplicative inverse to 0 by $0^{-1} = 0$.

We will follow standard model theoretic notation and terminology. Lower-case letters like $a, b, c$ and $x, y, z$ will usually denote finite tuples and we let $|x|$ denote the
length of $x$. We will sometimes use $\bar{x}$ to denote a tuple of the form $(x_1, \ldots, x_n)$ where each $x_j$ is itself a tuple. Given an $\mathcal{L}$-structure $K$ and an $\mathcal{L}$-formula $\varphi(x)$ with $|x| = n$, we let $\varphi(K)$ denote the set $\{a \in K^n \mid K \models \varphi(a)\}$. Given a subset $X \subseteq K^n$ and $a \in K^\ell$, $0 < \ell < n$, the fiber of $X$ over $a$ is denoted by $X_a := \{b \in K^{n-\ell} \mid (a, b) \in X\}$. By an $\mathcal{L}$-definable set, we mean definable with parameters. If we want to restrict the subset where the parameters vary we will use $\mathcal{L}(A)$-definable, $A \subseteq K$. If we wish to specify that it is definable without parameters, we use $\mathcal{L}(\emptyset)$-definable.

Let $\mathcal{L}_r$ be a relational extension of $\mathcal{L}_{\text{field}}$ and assume that the language $\mathcal{L}$ (possibly multi-sorted) extends $\mathcal{L}_r$ in which every sort is an imaginary $\mathcal{L}_r$-sort. In particular, the field-sort will always be the home sort of an $\mathcal{L}$-structure. Let us recall the definition of $\mathcal{L}$-open theories $T$. Let $K$ be a field of characteristic 0 in the language $\mathcal{L}$ endowed with a definable field topology, namely there is an $\mathcal{L}$-formula $\chi(x, z)$ be providing a basis of neighbourhoods of 0 [12]. Throughout the text, we will assume that the topology on $K$ is given by such formula $\chi$. Let $T$ be the $\mathcal{L}$-theory of $K$.

Then such theories $T$ with the following further assumptions are called $\mathcal{L}$-open [2]:

(A) (i) $T$ has relative quantifier elimination with respect to the field sort,
(ii) every quantifier-free $\mathcal{L}$-definable subset of the field sort is a finite union of sets of the form: an intersection of an $\mathcal{L}$-definable open set with a Zariski closed set.

One can choose the language $\mathcal{L}$ in such a way that the following theories are examples of $\mathcal{L}$-open theories: theories of real closed fields, $p$-adically closed valued fields, real closed valued fields, algebraically closed valued field of characteristic 0 and the theories of any Hahn power series field $k((t^\Gamma))$, where $k$ is a field of characteristic 0 and $\Gamma$ is an ordered abelian group [2, Examples 2.2.1], [6, Proposition 4.3].

2.2. Definable sets in models of $\mathcal{L}$-open theories. Let us recall a simple lemma in [2] on the form of $\mathcal{L}$-definable subsets in models of $\mathcal{L}$-open theories $T$. First we need to some notations.

Let $\mathcal{A}$ be a finite subset of $K[x, y]$. We let $\mathcal{A}^\circ := \{P \in \mathcal{A} \mid \deg_y(P) > 0\}$. We let the $\mathcal{L}_{\text{ring}}(K)$-formula $Z_A(x, y)$ be $Z_A(x, y) := \wedge_{P \in \mathcal{A}} P(x, y) = 0$. Thus the algebraic subset of $K^{n+1}$ (Zariski closed set) defined by $\mathcal{A}$ corresponds to $Z_A(K)$. For an element $R \in K[x, y]$ we let

$$Z_A^R(x, y) := Z_A(x, y) \land R(x, y) \neq 0. \quad (2.2.1)$$

Lemma 2.1 [2, Corollary]. Let $\mathcal{K}$ be a model of an $\mathcal{L}$-open theory $T$. Then every $\mathcal{L}$-definable set $X \subseteq K^{n+1}$ (in the field sort) is defined by an $\mathcal{L}$-formula $\varphi(x, y)$ with $x = (x_0, \ldots, x_{n-1})$ and $y$ a single variable such that

$$\varphi(x, y) \leftrightarrow \bigvee_{j \in J} Z_{A_j}^S(x, y) \land \theta_j(x, y)$$

with $J$ a finite set such that for each $j \in J$, $\theta_j$ is an $\mathcal{L}$-formula that defines an open subset of $K^n$, $Z_{A_j}^S(x, y)$ is as defined in [2.2.1] with either

(1) $A_j \subseteq K[x]$ and $S_j \in K[x, y]$ or

...
In the topological sense, let \( \text{Fr}(X) := \overline{X} \setminus X \) of elements of \( Y \) and one can relax the condition (4) by asking it only for \( m = 1 \) (see [4] Proposition 1.4).

Given a (fibered) dimension function \( d \) on \( U \), one may extend it to the space of types \( S_n^U(C) \) over a subset \( C \subseteq U \) by

\[
d(p) := \inf \{ d(\varphi(U, c) : \varphi(x, y) \text{ an } \mathcal{L}(\emptyset)-\text{formula and } \varphi(x, c) \in p) \}.
\]

Then we extend \( d \) on a tuple \( a \) of elements of \( U \) by defining \( d(a/C) \) as the dimension of the type \( tp(a/C) \) of \( a \) over \( C \). One then can show [4] Lemma 1.6]: Let \( a, b \) be two tuples of elements of \( U \), then:

\[
(2.3.1) \quad d(ab/C) = d(a/C \cup b) + d(b/C).
\]

Let \( B \subseteq U \) and let \( p(x) \) be a partial \( n \)-type. Let \( X \subseteq U^n \) with \( X = p(U) \). Then \( X \) is finitely satisfiable in \( B \) if for any formula \( \varphi(x) \in p(x) \), we have \( \varphi(M) \neq \emptyset \).

A tuple \( a \) in a definable set \( X \) is called generic over \( C \) (w.r. to the dimension \( d \)) if \( d(X) = d(a/C) \).

**Definition 2.3.** Let \( B \subseteq A \subseteq U^n \) be two definable subsets of \( U \), then \( B \) is almost equal to \( A \) (or large in \( A \)) if \( d(A \setminus B) < d(A) \) [15] section 2.

Let us recall the following result which can be found in [13], proven in the setting of o-minimal theories but it only uses the notion of a fibered dimension.
2.4. A cell decomposition theorem. Throughout this section, let $\mathcal{K} \models T$, where $T$ is $\mathcal{L}$-open. Let us first observe a few properties of the topological dimension $\dim$ on definable subsets of $\mathcal{K}$. First $\dim$ is a fibered dimension function. The proof follows the same strategy as in [4] by noting that the dimension acl-$\dim$ is induced by the field algebraic closure and has the exchange property (on the field sort) (namely $T$ is a geometric theory on the field sort) and that it coincides with the topological dimension (see [2] Proposition 2.4.1).

Note that we may also define the dimension of an $n$-tuple $\bar{a}$ of elements of $K$ as the cardinality of a maximal subtuple of acl-independent elements and it coincides with the previous definition given above (see for instance [1] section 4).

Moreover, the dimension of the frontier of a definable set has the following property. (2.4.1) $\dim(\text{Fr}(X)) < \dim(X) = \dim(X)$.  

As noted in [5] Chapter 4, Corollary 1.9], it implies for any definable set $X \subset Y$ that (2.4.2) $\dim(X \setminus \text{Int}_Y(X)) < \dim(Y)$.

Indeed, note that $X \setminus \text{Int}_Y(X) = X \cap \text{cl}_Y(Y \setminus X) = \text{cl}_Y(Y \setminus X) \setminus (Y \setminus X) \subset \text{Fr}(Y \setminus X)$ and so $\dim(X \setminus \text{Int}_Y(X)) < \dim(Y)$.

**Lemma 2.4.** [13] Lemma 2.4] Let $\bar{K}$ be an elementary saturated extension of $\mathcal{K}$ and $X$ be definable subset of $\bar{K}$ with parameters in $K$. Let $a \in X$ be generic over $K$ and $c \in \bar{K}$. Assume that $\text{tp}(a/Kc)$ is finitely satisfiable in $\mathcal{K}$, then $a$ is generic over $Kc$.

**Proof:** It amounts to show that $\dim(a/K) = \dim(a/Kc)$. Since $\dim(a/c|K) = \dim(a/Kc) + \dim(c/K)\) (see equation 2.3.1), we will show that $\dim(a/c|K) = \dim(a/Kc) + \dim(c/K)$. Since $\dim = acl$-$dim$, let $a_1$, respectively $c_1$, maximal acl-independent sub-tuples of $a$, respectively $c$ over $K$. By the way of contradiction, suppose that $a_1c_1$ is not acl-independent over $K$, then w.l.o.g. we may assume that $c_1 = c_{11}c_{12}$ with $|c_{11}| = 1$ such that $c_{11} \in acl(c_{12}a_1K)$. Let $\varphi(x, z, y; u)$ with $u \in K$ such that $\varphi(c_{11}, c_{12}, a_1; u)$ holds and $\exists^n x \varphi(x, c_{12}, a_1; u)$ for some natural number $n$. Since $\text{tp}(a/Kc)$ is finitely satisfiable in $\mathcal{K}$, there is $b \subset K$ such that $\varphi(c_{11}, c_{12}, b; u)$ holds. This contradicts that $c_1$ was chosen to be acl-independent over $K$. $\square$

Finally, one has the following description of definable subsets of topological fields models of an $\mathcal{L}$-open theory [2]; it is the analogue of the cell decomposition proven for dp-minimal fields (see [15] Proposition 4.1]). Before stating the result, we need to recall the notion of correspondences.

**Definition 2.5.** [15] section 3.1] A correspondence $f : E \rightrightarrows F$ consists of two definable subsets $E, F$ together with a definable subset $\text{graph}(f)$ of $E \times F$ such that $0 < |\{y \in F : (x, y) \in \text{graph}(f)\}| < \infty$, forall $x \in E$. 
The correspondence $f$ is continuous at $x \in E$ if for every $V \in \mathcal{B}$ there is $U \in \mathcal{B}$ such that $(f(x), f(x')) \in V$ whenever $(x, x') \in U$.

A correspondence $f$ is an $m$-correspondence if for all $x \in E$, $|\{y \in F : (x, y) \in graph(f)\}| = m$. We denote by $f(x)$ the set $\{y \in F : (x, y) \in graph(f)\}$. Note that a 1-correspondence is a function (together with its domain and image). Moreover a continuous $m$-correspondence is locally given by $m$ continuous functions [15] Lemma 3.1 (see also [2] Lemma 2.6.2). A $m$-correspondence $f$ on an open definable set $U$ is continuous on an open subset of $U$ almost equal to $U$ [2] Proposition 2.6.10.

We follow the following convention: if $f : K^n \rightarrow \mathbb{K}^n$, then $graph(f)$ is identified with a finite set and if $U$ is an open subset of $K^n$ and $f : U \rightarrow K^n$, $graph(f)$ is identified with $U$.

**Proposition 2.6.** [2] Theorem 2.7.1 Let $T$ be an $\mathcal{L}$-open theory of topological fields and $\mathcal{K}$ be a model of $T$. Let $X$ be a definable subset of $\mathcal{K}^n$. There are finitely many definable subsets $X_i$ with $X = \bigcup_i X_i$ such that $X_i$ is, up to permutation of coordinates, the graph of a definable continuous $m$-correspondence $f_i : U_i \rightarrow \mathcal{K}^{n-d_i}$, where $U_i$ is a definable open subset of $\mathcal{K}^d_i$, for some $0 \leq d_i \leq n$.

### 2.5. Generic differential expansions of topological fields.

Let $T$ be an $\mathcal{L}$-open theory of topological fields and $\mathcal{K}$ be a model of $T$. Let $\mathcal{L}_\delta$ be the language $\mathcal{L}$ extended by a unary function symbol $\delta$. Denote by $\mathcal{K}_\delta$ the expansion of $\mathcal{K}$ to an $\mathcal{L}_\delta$-structure. Let $T_\delta$ be the $\mathcal{L}_\delta$-theory $T$ together with the usual axioms of a derivation, namely,

\[
\begin{align*}
\forall x & \forall y \forall z \forall \delta (x + \delta(y)) = \delta(x) + \delta(y) \\
\forall x & \forall y \forall z \forall \delta (x y) = \delta(x) \delta(y) \\
\forall x & \forall y \forall z \forall \delta (x z) = \delta(x) \delta(z) \\
\end{align*}
\]

**Notation 2.7.** Let $\mathcal{K}_\delta \models T_\delta$. For $m \geq 0$ and $a \in \mathcal{K}$, we define

\[
\delta^m(a) := \delta \circ \cdots \circ \delta(a), \text{ with } \delta^0(a) := a,
\]

and $\delta^m(a)$ as the finite sequence $(\delta^0(a), \delta(a), \ldots, \delta^m(a)) \in \mathcal{K}^{m+1}$.

Similarly, given an element $a = (a_1, \ldots, a_n) \in \mathcal{K}^n$, we will write $\vec{\delta}^m(a)$ to denote the element $(\delta^m(a_1), \ldots, \delta^m(a_n)) \in \mathcal{K}^{(m+1)n}$. Let $\vec{m} := (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $|\vec{m}| := \sum_{i=1}^n m_i$. We will write $\vec{\delta}^m(a)$ to denote the element $(\delta^{m_1}(a_1), \ldots, \delta^{m_n}(a_n)) \in \mathcal{K}^{(m+1)n}$. For notational clarity, we will sometimes use $\nabla_{\vec{m}}$ instead of $\vec{\delta}$, especially concerning the image of subsets of $\mathcal{K}^n$. For example, when $A \subseteq \mathcal{K}$, we will use the notation $\nabla_{\vec{m}}(A)$ for $\{\delta^m(a) : a \in A\}$ instead of $\delta^m(A)$. Likewise for $A \subseteq \mathcal{K}^n$, $\nabla_{\vec{m}}(A) := \{\delta^{m_1}(a_1), \ldots, \delta^{m_n}(a_n) : a \in A\} \subseteq \mathcal{K}^{(m+1)n}$.

We will call $\delta^m(a)$ a differential tuple and we will sometimes use the notation $a \nabla$, suppressing the index $\vec{m}$.

Given $x = (x_0, \ldots, x_n)$, we let $K\{x\}$ be the ring of differential polynomials in $n + 1$ differential indeterminates $x_0, \ldots, x_n$ over $K$, namely it is the ordinary polynomial ring in formal indeterminates $\delta^i(x_j)$, $0 \leq i \leq n$, $j \in \omega$, with the convention $\delta^0(x_i) := x_i$. We extend the derivation $\delta$ to $K\{x\}$ by setting $\delta(\delta^i(x_j)) = \delta^{i+1}(x_j)$. By a rational differential function we simply mean a quotient of differential polynomials.

For $P(x) \in K\{x\}$ and $0 \leq i \leq n$, we let $\text{ord}_{x_i}(P)$ denote the order of $P$ with respect to the variable $x_i$, that is, the maximal integer $k$ such that $\delta^k(x_i)$ occurs in a non-trivial monomial of $P$ and $-1$ if no such $k$ exists. We let $\text{ord}(P)$, the order of $P$, be
max, ord\(_x\)(P). Suppose ord(P) = m. For \(\tilde{x} = (\tilde{x}_0, \ldots, \tilde{x}_n)\) a tuple of variables with \(|\tilde{x}_i| = m + 1\), we let \(P^* \in K[\tilde{x}]\) denote the corresponding ordinary polynomial such that \(P(x) = P^*(\delta^m(x))\).

Suppose ord\(_x\)(P) = m ≥ 0. Then, there are (unique) differential polynomials \(c_i \in K\{x\}\) such that ord\(_x\)(c\(_i\)) < m and

(2.5.1) \[P(x) = \sum_{i=0}^{d} c_i(x)(\delta^m(x_n))^i.\]

The separant \(s_P\) of \(P\) is defined as

\[s_P := \frac{\partial}{\partial \delta^m(x_n)} P \in K\{x\} .\]

We extend the notion of separant to arbitrary polynomials with an ordering on their variables in the natural way, namely, if \(P \in K[x]\), the separant of \(P\) corresponds to \(s_P := \frac{\partial}{\partial x_n} P \in K[x]\). By convention, we induce a total order on the variables \(\delta^j(x_i)\) by declaring that

\[\delta^k(x_i) < \delta^{k'}(x_j) \iff \begin{cases} i < j & \text{if } k = k' \\ i = j & \text{and } k < k'. \end{cases}\]

This order makes the notion of separant for differential polynomials compatible with the extended version for ordinary polynomials, i.e., \(s_P^* = s_P\).

We define an operation on \(K\{x\}\) sending \(P \mapsto P^\delta\) as follows: for \(P\) written as in (2.5.1)

\[P(x) \mapsto P^\delta(x) = \sum_{i=0}^{d} \delta(c_i(x))(\delta^m(x_n))^i.\]

A simple calculation shows that

(2.5.2) \[\delta(P(x)) = P^\delta(x) + s_P(x)\delta^{m+1}(x_n).\]

Now we will describe a scheme of \(\mathcal{L}_\delta\)-axioms generalizing the axiomatization of closed ordered differential fields (CODF) given by M. Singer in [?]. Let \(\chi(x, z)\) be an \(\mathcal{L}\)-formula providing a basis of neighbourhoods of 0. For \(a = (a_1, \ldots, a_n)\) with \(|a_i| = |z|\), we let

\[W_a := \chi(K, a_1) \times \cdots \times \chi(K, a_n).\]

**Definition 2.8.** Set \(T_\delta^* := T_\delta \cup (DL)\), where (DL) is the following list of axioms: for every differential polynomial \(P(x) \in K\{x\}\) with \(|x| = 1\) and ord\(_x\)(P) = m, for variables \(u = (u_0, \ldots, u_m)\) with \(|u_i| = |z|\) and \(y = (y_0, \ldots, y_m)\)

\[\forall u \left( \exists y (P^*(y) = 0 \land s^*_P(y) \neq 0) \rightarrow \exists x (P(x) = 0 \land s_P(x) \neq 0 \land (\delta^m(x) - y) \in W_u) \right).\]

As usual, by quantifying over coefficients, the axiom scheme (DL) can be expressed in the language \(\mathcal{L}_\delta\).

When \(T = \text{RCF}\), the theory \(\text{RCF}_\delta^*\) corresponds to CODF (which is consistent). When \(T\) is either ACVF\(_{0,p}\), RCVF, \(p\text{CF}_d\) or the RV-theory of \(\mathbb{C}(t)\) or \(\mathbb{R}(t)\), the
consistency of the theory $T^*_{\delta}$ follows by results in [7, Corollary 3.8, Proposition 3.9] (and see also [2, Theorem 3.3.2]). In [7], we showed how to embed a differential topological field $(K, \delta)$ which satisfied a hypothesis called (Hypothesis (I)) (see [7, Definition 2.21]), similar to largeness in this topological context into a differential extension model of the scheme (DL). Largeness is a notion introduced by Pop [14]). In [2], instead of assuming a largeness hypothesis, we worked with henselian topological fields. (Note that henselian fields are large [14].) In the annex, we will indicate the hypothesis we need to make these embedding proofs work (see section 5).

An immediate consequence of the axiomatisation is the density property of differential points in open subsets of models of $T^*_{\delta}$.

**Lemma 2.9** ([7, Lemma 3.17]). Let $\mathcal{K}_{\delta} \models T^*_{\delta}$, Let $O$ be an open subset of $K^n$. Then there is $a \in K$ such that $\delta^{n-1}(a) \in O$.

**Proof:** Let $\bar{u} := (u_0, \ldots, u_{n-1}) \in O$. We consider the differential polynomial $\delta^{n-1}(x)$. The corresponding algebraic polynomial is $x_{n-1}$. Then we consider the differential equation $\delta^{n-1}(x) = u_{n-1}$; its separant is 1 and so we can apply the scheme (DL) and get a differential solution close to the algebraic solution $\bar{u}$. So there is $a \in K$ such that $\delta^{n-1}(a) = u_{n-1}$ and $\delta^{n-1}(a) \in O$. \hfill $\blacksquare$

Under assumption (A) on the $\mathcal{L}$-theory $T$, different model-theoretic properties transfer from $T$ to $T^*_{\delta}$, as shown by the following results.

**Theorem 2.10.** [7, Theorem 4.1] [2] The theory $T^*_{\delta}$ admits quantifier elimination relative to the field sort in $\mathcal{L}_{\delta}$.

Using this quantifier elimination result, one can easily show the following two transfer results. (A proof of the second result may be found in [2, Appendix A.0.5].)

**Proposition 2.11** ([7, Corollary 4.3]). The theory $T^*_{\delta}$ is NIP, whenever $T$ is NIP.

**Proposition 2.12** (Chernikov). The theory $T^*_{\delta}$ is distal, whenever $T$ is distal.

Let us recall the definition of open core (in a general setting) (see also [3]).

**Definition 2.13.** Let $\mathcal{K} \models T$, let $\hat{\mathcal{L}}$ be an expansion of $\mathcal{L}$ and let $\hat{T}$ be the corresponding expansion of $T$. Let $\hat{\mathcal{K}}$ be an $\hat{\mathcal{L}}$-expansion of $\mathcal{K}$. Then $\hat{\mathcal{K}}$ has $\mathcal{L}$-open core if every $\hat{\mathcal{L}}$-definable open subset is $\mathcal{L}$-definable. An extension of $\hat{T}$ has $\mathcal{L}$-open core if every model of that extension has $\mathcal{L}$-open core.

Let $\mathcal{S}$ be a collection of sorts of $\mathcal{L}_{\text{eq}}$. We let $\mathcal{L}^\mathcal{S}$ denote the restriction of $\mathcal{L}_{\text{eq}}$ to the home sort together with the new sorts in $\mathcal{S}$.

**Theorem 2.14.** [2, Theorem 4.0.5] Suppose that $T$ admits elimination of imaginaries in $\mathcal{L}^\mathcal{S}$ and that the theory $T^*_{\delta}$ has $\mathcal{L}$-open core. Then the theory $T^*_{\delta}$ admits elimination of imaginaries in $\mathcal{L}^\mathcal{S}_{\delta}$.

Finally let us recall a result on the existence of a fibered dimension function on definable subsets (on the home sort) in models of $T^*_{\delta}$ (which uses that $T^*_{\delta}$ admits quantifier elimination).
Definition 2.15. An $\mathcal{L}$-structure $\mathcal{M}$ is called \textit{equationally bounded} if for each definable set $S \subseteq M^{m+1}$ such that for every $\bar{a} \in M^m$, $S_{\bar{a}}$ is small, there exist finitely many $\mathcal{L}(M)$-terms $f_1(x_1, \ldots, x_m, y), \ldots, f_r(x_1, \ldots, x_m, y)$ such that for every $\bar{a} \in M^m$, there exists $1 \leq i \leq r$ with $f_i(\bar{a}, y) \neq 0$ and $S_{\bar{a}} \subseteq \{ b \in M : f_i(\bar{a}, b) = 0 \}$.

Using a notion of independence built in from the algebra of all terms (t-independence), following [4], [10], one can define on $\text{Def}(\mathcal{K}_\delta)$, when $\mathcal{K}_\delta \models T^*_\delta$ a dimension function ($\text{dim}_\delta$) [8, Definition 2.3]. In order to show that this dimension is fibered, one uses the closure operator $c$ over $\mathcal{K}$ which is defined by: $a \in c(A)$ if and only if there is a differential polynomial $Q \in \mathcal{K}(A)\{X\} \setminus \{0\}$ such that $Q(a) = 0$. Then associated with this closure operator, one has a notion of independence and dimension for tuples of elements. Let $\mathcal{K}_\delta$ be a $|K|^+$-saturated extension of $\mathcal{K}_\delta$. Define for $\bar{a} = (a_1, \ldots, a_n)$ a tuple in $\hat{K}$, $\text{cl-dim}(\bar{a}) := \max\{|B| : B \subseteq \mathcal{K}(\bar{a}), B \text{ is cl-independent}\}$.

Then one shows that, for $X \in \text{Def}(\mathcal{K}_\delta)$ (see [8, Lemma 2.11]):

$$\text{dim}_\delta(X) = \max\{\text{cl-dim}(\bar{c}) : \bar{c} \in X(\hat{K})\}.$$ 

One then proves that any model of $T^*_\delta$ is equationaly bounded, which entails that $\text{dim}_\delta$ is a fibered dimension function. Note that there are infinite definable subsets $X$ with $\text{dim}_\delta(X) = 0$.

Proposition 2.16. [8, Corollary 3.10] Let $\mathcal{K}_\delta \models T^*_\delta$, then there is a dimension function $\text{dim}_\delta$ that defines a fibered dimension function on $\text{Def}(\mathcal{K}_\delta)$.

This dimension function has been further investigated in [11] and one can check that for $\bar{a}$ a tuple of elements in $\hat{K}$ that $\text{cl-dim}(\bar{a}) = \inf\{|\mathcal{K}(\varphi(K)) : \varphi(\bar{x}) \in \text{tp}(\bar{a}/K)\}$.

3. THE OPEN CORE PROPPRTY IN MODELS OF $T^*_\delta$

From now on, let $\mathcal{K}$ be a model of an $\mathcal{L}$-open theory $T$ and let $\mathcal{K}_\delta$ its expansion by a derivation $\delta$. In this section to a $\mathcal{L}_\delta$-definable set $X$, we will associate an $\mathcal{L}$-definable set where the differential prolongation of $X$ is dense. Such result was already shown in [2] using a characterization of continuous $\mathcal{L}_\delta$-correspondences (with $\mathcal{L}$-definable domain).

Notation 3.1. Under assumption (A(ii)), any quantifier-free (relative to the field sort) $\mathcal{L}_\delta$-definable set $X \subseteq K^n$ is of the form $\nabla^{-1}_m(Y)$ for a quantifier-free $\mathcal{L}$-definable set $Y \subseteq K^{n(m+1)}$ (quantifier-free relative to the field sort). Indeed, let $x = (x_1, \ldots, x_n)$ be a tuple of field sort variables. By assumption on the language $\mathcal{L}$, any $\mathcal{L}_\delta$-term $t(x)$ is equivalent, modulo the theory of differential fields, to an $\mathcal{L}$-term $t^*(\delta^{m_1}(x_1), \ldots, \delta^{m_n}(x_n))$ where $t^*$ is an $\mathcal{L}$-term, for some $(m_1, \ldots, m_n) \in \mathbb{N}^n$. Therefore, by possibly adding tautological conjunctions like $\delta^k(x_i) = \delta^k(x_i)$ if needed, we may associate with any $\mathcal{L}_\delta$-formula $\varphi(x)$ without field sort quantifiers, an equivalent $\mathcal{L}_\delta$-formula (modulo the theory of differential fields) of the form $\varphi^*(\delta^m(x))$ where $m \in \mathbb{N}$ and $\varphi^*$ is an $\mathcal{L}$-formula without field sort quantifiers. The formula $\varphi^*$ arises by uniformly replacing every occurrence of $\delta^m(x_i)$ by a new variable $y_i^m$ in $\varphi$ with the natural choice for the order of variables $\varphi^*(y_1^0, \ldots, y^m_1, \ldots, y_n^0, \ldots, y^m_n)$. Therefore, if $X$ is defined by $\varphi$, letting $Y$ be the set defined by $\varphi^*$ gives that $X = \nabla^{-1}_m(Y)$.

Definition 3.2 (Order). Let $X \subseteq K^n$ be an $\mathcal{L}_\delta$-definable set. Let $\bar{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$ and let $Z \subseteq K^{d_1+\ldots+d_n}$ such that $X = \nabla^{-1}_d(Z)$ The order of $X$, denoted by $o(X)$,
is the smallest integer $m$ such that $m = \max_{1 \leq i \leq n} d_i$ and $X = \nabla^{-1}_d(Z)$. Let $\varphi$ be a field-sort quantifier-free $\mathcal{L}_d$-formula such that $X = \{(a_1, \ldots, a_n) \in K^n : K \models \varphi^*(\delta^{d_1}(a_1), \ldots, \delta^{d_n}(a_n))\}$; we say that $d_i$ (also denoted by $d_{x_i}$) is the order of $x_i$ in $\varphi$, $1 \leq i \leq n$.

Note that $o(X) = 0$ if and only if $X$ is $\mathcal{L}$-definable.

**Definition 3.3.** [2, Definition 4.0.2] Let $X \subseteq K^n$ be a non-empty quantifier-free $\mathcal{L}_d$-definable set (relative to the field sort). Let $m$ be a positive integer and let $Z \subseteq K^{(m+1)n}$ be an $\mathcal{L}$-definable set such that:

1. $x \in X$ if and only if $\nabla_m(x) \in Z$ and
2. $Z = \nabla_m(X)$.

Then we call $(X, Z, m)$ as above a linked triple.

**Proposition 3.4.** [2, Proposition 4.0.3] The theory $T^*_d$ has $\mathcal{L}$-open core if and only if for every model $K_\delta$, for every $\mathcal{L}_d$-definable set $X \subseteq K^n$, there is an integer $m$ and an $\mathcal{L}$-definable set $Z \subseteq K^{(m+1)n}$, such that $(X, Z, m)$ is a linked triple. In addition, if $T^*_d$ has $\mathcal{L}$-open core, then there is a linked triple of the form $(X, Z, o(X))$.

Therefore, when $T^*_d$ has $\mathcal{L}$-open core, we will associate two dimension functions to $X$, the first one $\dim_d(X)$ (see Proposition 2.16) and the second one $\dim^*(X) := \min\{\frac{\dim(Z)}{m+1} : (X, Z, m)$ is a linked triple\}.

In [2, Theorem 6.0.7], we showed that $T^*_d$ has $\mathcal{L}$-open core under following hypotheses on $T$. In the ordered case, the field sort is $\mathcal{L}$-definably complete and in the valued case, the value group sort $\Gamma_\infty$ is $\mathcal{L}$-definably complete. Furthermore, in the valued case, we assume the following on the value group:

1. either there is a model $K'$ of $T$ for which $\Gamma(K')$ is a divisible ordered abelian group in which every infinite $\mathcal{L}$-definable set has an accumulation point;
2. or there is a model $K'$ of $T$ with $\Gamma(K') = \mathbb{Z}$.

This last hypotheses enabled us to show that for every $\mathcal{L}$-definable open set $V \subseteq K^n$, any $\mathcal{L}$-definable function $f : V \to \Gamma_\infty$ is continuous almost everywhere [2, Proposition 2.6.11].

### 3.1. Prolongations

At the beginning of this subsection, we simply assume that $K_\delta$ is a differential field of characteristic 0. We introduce some notations on prolongations which will be used later on for the $\mathcal{L}$-open core property in models of $T^*_d$.

**Lemma 3.5.** [2, Lemma 3.1.4] Let $x = (x_0, \ldots, x_n)$ be a tuple of variables and $y$ be a single variable. Let $P \in K\{x, y\}$ be a differential polynomial such that $k = \ord_y(P) \geq 0$. There is a sequence of rational differential functions $(f^P_i)_{i \geq 1}$ such that for every $a \in K^{n+1}$ and $b \in K$

$$K \models [P(a, b) = 0 \wedge s_P(a, b) \neq 0] \to \delta^{k+i}(b) = f^P_i(a, b).$$

In addition, each $f^P_i$ is of the form

$$f^P_i(x, y) = \frac{Q_i(x, y)}{s_P(x, y)^{\ell_i}},$$
where \( \ell_i \in \mathbb{N} \), \( \text{ord}_y(Q_i) = \text{ord}_y(P) \) and
\[
\text{ord}_{x_j}(Q_i) = \begin{cases} 
\text{ord}_{x_j}(P) + i & \text{if } \text{ord}_{x_j}(P) \geq 0 \\
-1 & \text{otherwise} 
\end{cases}
\]

We call the sequence \((f_i^P)_{i \geq 1}\) the rational prolongation along \(P\).

**Notation 3.6.** Let \(x = (x_0, \ldots, x_m)\) be a tuple of variables. For an integer \(d \geq 0\), we define a new tuple of variables \(x(d)\) which extends \(x\) by \(d\) new variables that is,
\[
x(d) := (x_0, \ldots, x_m, x_{m+1}, \ldots, x_{m+d}),
\]
with the convention that if \(d = 0\), then \(x(0) = x\). When \(\bar{x} = (\bar{x}_0, \ldots, \bar{x}_\ell)\) is a tuple of tuples of variables, we let \(\bar{x}[d] := (\bar{x}_0(d), \ldots, \bar{x}_\ell(d))\). Note that if \(d \neq 0\) and \(\bar{x}\) is not a singleton, then \(\bar{x}[d] \neq \bar{x}\) are different.

Let \(x = (x_0, \ldots, x_m)\) and \(y\) be a variable with \(|y| = 1\). Let \(P \in K\{x, y\}\) be a differential polynomial of order \(m\) and let \((f_i^P)_{i \geq 1}\) be its rational prolongation along \(P\).

Let \(\bar{y}\) be such that \(|\bar{y}| = m + 1\). We denote by \(\lambda_P^d(\bar{x}[d], \bar{y}, y_{m+1}, \ldots, y_{m+d})\) the \(L\)-formula:
\[
P^s(\bar{x}, \bar{y}) = 0 \land s_P^* (\bar{x}, \bar{y}) \neq 0 \land \bigwedge_{i \geq 1} y_{m+i} = (f_i^P)^* (\bar{x}[d], \bar{y}).
\]

Given a tuple \(a := (a_1, \ldots, a_n)\) and an element \(b\), we will call the tuple \((\bar{a}[d], b, b_{m+1}, \ldots, b_{m+d})\) satisfying the formula \(\lambda_P^d(\bar{a}[d], b, b_{m+1}, \ldots, b_{m+d})\) the rational prolongation of \((a, b)\) (along \(P\)). When \(d = 0\), we set \(\lambda_P^0(\bar{x}[0], \bar{y})\) to be the formula \(P^s(\bar{x}, \bar{y}) = 0 \land s_P^* (\bar{x}, \bar{y}) \neq 0\).

We will make the following abuse of notation. When \(P \in K\{x, y\}\) with \(x = (x_0, \ldots, x_m)\) and \(|y| = 1\), we can view \(P\) as \(P^*\) where \(\bar{P} \in K\{x, y\}\) with \(\text{ord}_x(\bar{P}) = m\). We will still denote by \(\lambda_P^d(\bar{x}(d), y(d))\) the formula \(\lambda_P^d\).

We will use the following notation concerning projections.

**Notation 3.7.** For non-zero natural numbers \(n, 1 \leq k \leq n\), we let \(\pi_k : K^n \to K^k\) denote the projection onto the first \(k\) coordinates and \(\pi_{(k)} : K^n \to K\) denote the projection onto the \(k\)th coordinate. For natural numbers \(n, m, \ell, k_i\) with \(1 \leq \ell \leq n\) and \(1 \leq k_i \leq m + 1\), we let \(\pi_{[k_1, \ldots, k_\ell]} : K^{n(m+1)} \to K^{\ell(m+1)}\) denote the projection sending the \(i\)th-block of \(m + 1\) coordinates to the first \(k_i\) coordinates of such block, that is,
\[
\pi_{[k_1, \ldots, k_\ell]}((x_1, \ldots, x_{m+1}, \ldots, x_n, \ldots, x_{n+1})) = (x_{1,1}, \ldots, x_{1,k_1}, \ldots, x_{\ell,1}, \ldots, x_{\ell,k_\ell}).
\]
For \(1 \leq i \leq n\), we will allow in \(\pi_{[k_1, \ldots, k_\ell]}\) the possibility for \(k_i\) to be equal to \(0\), in which case the corresponding subtuple \((x_{1,1}, \ldots, x_{1,k_1})\) is simply empty. Therefore the notation \(\pi_{[0, \ldots, 0, k_1]} : K^{n(m+1)} \to K^{m+1}\) denotes the projection sending the \(\ell\)th-block of \(m + 1\) coordinates to the first \(k_1\) coordinates of such block, that is,
\[
\pi_{[0, \ldots, 0, k_1]}((x_1, \ldots, x_{m+1}, \ldots, x_n, \ldots, x_{n,m+1})) = (x_{\ell,1}, \ldots, x_{\ell,k_1}).
\]
Note that \(\pi_{[k_1]} : K^{n(m+1)} \to K^{k_1}\) coincides with \(\pi_{[k_1]} : K^{n(m+1)} \to K^{k_1}\), \(k_1 \leq m = 1\).
The main result of this section is the following proposition, where we follow the same convention as in Proposition \ref{prop:finiteindex}, namely an open subset of $K^0$ is a finite set of points.

**Proposition 3.8.** Let $K \models T^*_\delta$ and let $X \subseteq K^n$ be an $\mathcal{L}_\delta$-definable set. Then there is $d \in \mathbb{N}$ and a finite family $\{Y_i \mid i \in I\}$ of $\mathcal{L}$-definable subsets of $K^{n(d+1)}$ such that

$$X = \bigcup_{i \in I} \nabla_d^{-1}(Y_i),$$

and for each $i \in I$, $Y_i$ is either open, or equal to $\{\delta^d(a)\}$ for some $a \in X$, or the graph of a continuous $\mathcal{L}$-definable correspondence $h_i$ such that for some $n$-tuples $(d_1, \ldots, d_n)$ where $0 \leq d_i \leq d + 1$, $1 \leq i \leq n$, $(d_1, \ldots, d_n)$, where $d_i := \max\{d_i, 1\}$,

1. $\pi_{(d_1, \ldots, d_n)}(Y_i)$ is open in $K^{d_1} \times \cdots \times K^{d_n}$,
2. $h_i: \pi_{(d_1, \ldots, d_n)}(Y_i) \rightarrow K^{d+1-d_1} \times \cdots \times K^{d+1-d_n}$, and
3. for every open subset $U \subseteq K^n$ such that $U \cap \pi_{(d_1, \ldots, d_n)}(Y_i) \neq \emptyset$, there is $a \in \pi_{[1, \ldots, 1]}(U \cap \pi_{(d_1, \ldots, d_n)}(Y_i))$ such that $\delta^d(a) \in Y_i$.

We will denote $\bigcup_{i \in I} Y_i$ by $X^{**}$ and call it an $\mathcal{L}$-open envelop of $X$ (in $K^{n(d+1)}$).

**Proof.** Let $m := o(X)$; since $\nabla_m^{-1}$ commutes with finite union, by Theorem \ref{thm:finitebranchingtree} and Lemma \ref{lem:finiteindex}, we may assume that $X$ is a finite union of subsets of the form $\nabla_m^{-1}(Z_i)$, where $Z_i$ is an $\mathcal{L}$-definable subset of $K^{n(m+1)}$ which is either open, finite or, up to a permutation of coordinates, the graph of a continuous $\mathcal{L}$-definable correspondence $f_i: U_i \rightarrow K^{n(m+1)-e}$, where $U_i \subset K^e$ is a non-empty open $\mathcal{L}$-definable set, $1 \leq e < n(m+1)$. We will consider each $Z_i$ separately. From now on, we drop the index $i$. The result is immediate for $Z$ in the following cases: either $Z$ is open and we apply Lemma \ref{lem:finiteindex} or $\dim(\pi_{[1, \ldots, 1]}(Z)) = 0$ and so $\nabla_m^{-1}(Z)$ is finite.

In the other cases, we proceed as follows. We associate with $Z$ a finite branching tree $(T, E)$ with root labelled by $Z = Z(0)$ and we denote the subset of the nodes at level $t$ by $T(t)$, $t \in \mathbb{N}$. If $v \in T(t)$, then we denote by $Z(v)$ the correspondence labelling the node $v$ with the property that $\nabla^{-1}Z(v) = \bigcup_{w \in T(t+1)} \nabla^{-1}Z(w)$ and $Z(v) \subset K^{n(m(v)+1)}$, where $m(v)$ has been chosen minimal such. We will show that each branch is finite and at the end of each branch the correspondence which labels the vertex is one of the $Y_i$, $i \in I$ as in the statement of the proposition. Note that by Koenig’s lemma, we get a finite tree.

To simplify notation instead of denoting a correspondence obtained at level $t$ by $Z(v)$ with $v \in T(t)$, we will simply denote it by $Z(t)$ and we will denote $m(v)$ by $m(t)$.

Given a correspondence $Z(t)$, we associate a tuple $\tilde{d}(t) := (d_1(t), \ldots, d_n(t))$ minimal in the lexicographic ordering, such that for each $1 \leq i \leq n$,

$$\dim(\pi_{[0, \ldots, 0, d_i(t)+1]}(Z(t))) = d_i(t),$$

placing ourselves in $K^{n(m(t)+2)}$ (see Notation \ref{not:finiteindex}).

For $d \in \mathbb{N}$, we use the notation $\underline{d}$ to mean $\max\{d, 1\}$. On $\mathbb{N}^n$ we also put a partial ordering $<$ defined component-wise: $(d_1, \ldots, d_n) < (\tilde{d}_1, \ldots, \tilde{d}_n)$, if for all $1 \leq i \leq n$, $d_i \leq \tilde{d}_i$ and for some $1 \leq i \leq n$, $d_i < \tilde{d}_i$. 
Define
\[ g(Z(t)) := \bar{d}(t) \quad \text{and} \quad k(Z(t)) := |\{i: 1 \leq i \leq n, d_i(t) = m(t) + 1\}|. \]

By assumption on \( Z(0) \), we have \( 0 \leq k(Z(0)) < n \) and we will show that \( k(Z(t)) \leq k(Z(t - 1)) \), \( t \geq 1 \). We decompose the tuple \( \bar{d}(t) \) into two subtuples: \( d_{[k(Z(t))]}(t) \) of length \( k(Z(t)) \) which collects (respecting the order) all the components of \( \bar{d}(t) \) equal to \( m(t) + 1 \) and \( \bar{d}_{[n - k(Z(t))]}(t) \) such that \( \bar{d}(t) = (d_{[k(Z(t))]}(t), \bar{d}_{[n - k(Z(t))]}(t)) \). We proceed by induction on \((k(Z(t)), d_{[n - k(Z(t))]}(t))\). Note that when \( k(Z(t)) = 0 \), then \( \bar{d}_{[n - k(Z(t))]}(t) = \bar{d}(t) = g(Z(t)) \) and in that case we will show that when \( t \) increases then \( \bar{d}(t) \) decreases.

To the tuple \( \bar{d}(t) \), we associate the following \( n \)-tuple of projections, with the convention that \( d_i(t)^* + 1 := \min\{d_i(t) + 1, m(t) + 1\}, 1 \leq i \leq n \),
\[ \pi_{[d_1(t)^* + 1]}, \pi_{[d_1(t)^* + 1, d_2(t)^* + 1]}, \ldots, \pi_{[d_1(t)^* + 1, \ldots, d_n(t)^* + 1]}. \]

Now let us describe how to get from \( Z(t) \) to \( Z(t + 1) \). Since this step is similar to the step getting from \( Z(0) \subset K^{m(m + 1)} \) to \( Z(1) \), in the discussion below, we will set \( t = 0 \) and \( d_i(0) = d_i, 1 \leq i \leq n \).

If \( d_1 = m + 1, \pi_{[m + 1]}(Z) \) is an open subset of \( K^{m + 1} \), then we consider the least index \( i \) such that \( d_i \leq m, 1 < i \leq n \), and the set \( \pi_{[m + 1, m + 1, d_i + 1]}(Z) \). (Note that by assumption on \( Z \), there is such an index \( i \)).

Or \( 0 \leq d_1 < m + 1 \), so \( \pi_{[d_1 + 1]}(Z) \) is a finite union of subsets \( W_{1,j}, j \in J_1 \) with \( J_1 \) finite, definable by \( \mathcal{L} \)-formulas of the form
\[ \varphi_{1,j}(x_1, y) : Z_{A_{1,j}}^{S_j}(x_1, y) \land \theta_{1,j}(x_1, y), (\dagger)_1 \]
with \( |x_1| = d_1, |y| = 1, \theta_{1,j}(x_1, y) \) an \( \mathcal{L} \)-formula defining an open subset of \( K^{d_1 + 1} \) and \( A_{1,j} \subseteq K[x_1, y] \) a finite set of non-zero polynomials such that either \( A_{1,j} \subseteq K[x_1] \) or \( A_{1,j} = \{P_1, j\} \) and \( S_j = (\partial_p P_1, j)R_j \) for some \( R_j \in K[x_1] \setminus \{0\} \) (see Lemma 2.1). Note that every \( A_{1,j} \neq \emptyset \) as otherwise we contradict the minimality of \( d_1 \). Moreover we may assume that for some formula \( \varphi_{1,j}(x_1, y) \), we have \( A_{1,j} \cap K[x_1] = \emptyset \) (by the assumption \( \pi_{[d_1]}(Z) \) is a non-empty open subset unless \( d_1 = 0 \) and we use that \( \dim \) coincides with the acl-dimension (see section 2.4)). For each \( j \in J \) such that \( A_{1,j} \cap K[x_1] = \emptyset \), we may define the rational prolongation of \((x_1, y)\) along \( P_{1,j} \) by the formula \( \lambda_{P_{1,j}}^\ell(x_1, y(\ell)), \ell \geq 0 \), where \( y_k = f_k^{P_{1,j}}(x_1, y), 1 \leq k \leq \ell \) and \( y_0 = y \) (see Notation 3.6).

In order to unify the notations, in case \( d_1 = m + 1 \), we still define \( \lambda_{P_{1,j}}^\ell(x_1(\ell)), \ell \geq 0 \) as the formula \( x_1(\ell) = x_1(\ell) \) (namely putting no conditions on the prolongation of \( x_1 \)).

We have \( \lambda_{P_{1,j}}^\ell(K) = \pi_{[m + 1]}(Z) \times K^\ell \).

In case \( A_{1,j} \cap K[x_1] \neq \emptyset \), we consider the subset \( \{\bar{z} \in Z : \pi_{[d_1]}(\bar{z}) \in W_j\} \) and we express it as a finite union of continuous correspondences \( \bar{Z}_j \), which will have the property that \( \dim(\pi_{[d_1]}(\bar{Z}_j)) < d_1 \). So for each such correspondence \( \bar{Z}_j \), we have \( g(\bar{Z}_j) < g(Z) \) and \( d_{[n - k(Z_0)]}(< d_{[n - k(Z)]}) \). So we apply the induction hypothesis. So for each \( \bar{Z}_j \) we obtained a finite number of correspondences as described in the statement of the proposition and we label the corresponding number of nodes at level 1 (that we connect to the root) by these correspondences.
Either \( d_2 = m + 1 \) and so \( \pi_{[0,d_2]}(Z) \) is an open subset of \( K^{m+1} \). In this case we proceed, namely we consider

\[
\pi_{[d_1^*+1,m+1,d_2+1]}(Z), \quad 0 \leq d_3 \leq m + 1.
\]

Assume that \( d_2 < m + 1 \) and consider \( \pi_{[d_1^*+1,d_2+1]}(Z) \). Again by Lemma 2.1, we obtain that \( \pi_{[d_1^*+1,d_2+1]}(Z) \) is a finite union of \( \mathcal{L} \)-definable sets \( W_{2j}, j \in J_2 \), each of which defined by a formula of the form:

\[
\varphi_j^2(x_1, x_2, y) := Z_{A_{2,j}}^{S_j} (x_1, x_2, y) \wedge \theta_{2,j}(x_1, x_2, y), \quad (\dagger)_2
\]

with \( |x_1| \leq d_1^* + 1, |x_2| = d_2, |y| = 1, \theta_{2,j}(x_1, x_2, y) \) an \( \mathcal{L} \)-formula defining an open subset of \( K^{d_1^*+1} \times K^{d_2+1} \) and \( A_{2,j} \subseteq K[x_1, x_2, y] \) a finite non-empty set of non-zero polynomials such that either \( A_{2,j} \subseteq K[x_1, x_2] \) or \( A_{2,j}^\mathcal{U} = \{ P_{2,j} \} \) and \( S_2 = (\partial \overline{\partial} P_2) R_2 \) for some \( R_2 \in K[x_1, x_2] \setminus \{ 0 \} \). (The fact that \( A_{2,j} \neq \emptyset \) follows from the assumption on the dimension of \( \pi_{[d_1^*+1,d_2+1]}(Z) \).) Moreover we may assume that for some formula \( \varphi_j^2 \) we have that \( A_{2,j} \cap K[x_1, x_2] = \emptyset \). For each such \( j \in J_2 \), we define the rational prolongation of \( (x_1, x_2, y) \) along \( P_{2,j} \). Namely let \( \lambda_{P_{2,j}}^\ell(x_1(\ell), x_2, y(\ell)), \ell \geq 0, \) where

\[
y_k = f_k^{P_{2,j}}(x_1(\ell), x_2, y), \quad 1 \leq k \leq \ell \quad \text{and} \quad y_0 = y \] (see Notation 3.6). Note that letting \( x_1 = (x_{10}, \ldots, x_{1m}) \) and if \( 0 \leq d \leq m \) is maximum \( \deg_{x_1d} P_{2,j} > 0 \), then \( x_1(\ell) = (x_{10}, \ldots, x_{1(\ell+d)}) \).

Again if \( d_2 = m + 1 \), we define \( \lambda_{P_{2,j}}^\ell(x_1(x\ell)), \ell \geq 0, \) as the formula \( x_2(\ell) = x_2(\ell) \), putting no conditions on the prolongation of \( (x_1, x_2, y) \).

In case \( A_{2,j} \cap K[x_1, x_2] \neq \emptyset \), we consider the subset \( \{ \tilde{z} \in Z : \pi_{[d_1^*+1,d_2]}(\tilde{z}) \in W_{2j} \} \) and express it as a finite union of continuous correspondences \( \tilde{Z}_{2,j} \), which will have the property that \( \dim(\pi_{[d_1^*+1,d_2]}(\tilde{Z}_{2,j})) < d_1 + d_2 \). So for each such correspondence \( \tilde{Z}_{2,j} \), we have \( g(\tilde{Z}_{2,j}) < g(Z), d_{n-k(\tilde{Z}_{2,j})} < d_{n-k(Z)} \) and apply the induction hypothesis.

Then we proceed with \( \pi_{[d_1^*+1,d_2+1,m+1,d_2+1]}(Z) \) in case \( d_3 \leq m \), otherwise we proceed with \( \pi_{[d_1^*+1,m+1,d_2+1]}(Z) \). At step \( 1 \leq k \leq n \) when \( d_k \leq m \), we denote the formulas we obtained by \( \varphi_k^j, j \in J_k \) finite, and if \( d_k = m + 1 \) we don’t make any modifications but in order to uniformize notations we introduce \( \lambda_k^\ell(x_1, \ldots, x_{k-1}, x_k(\ell)), \ell \geq 0, \) as the formula \( x_k(\ell) = x_k(\ell) \), putting no conditions on the prolongation of \( (x_1, \ldots, x_{k-1}, x_k) \). Then we consider the projection on the next coordinate.

Set for \( 1 \leq i \leq n: \)

1. in case \( d_i = m, \ell_i := m - d_i, \)
2. in case \( d_i = m + 1, \ell_i := 0. \)

Then define \( m(1) = m \) in case for all \( i, d_i \leq m \), otherwise let

\[
m(1) = \max_i \{ m + \sum_{j>i} \ell_j : 1 \leq i < n, d_i = m + 1 \}.
\]

Then, in the construction we replace \( Z \) by each of the following subsets \( \tilde{Z} \) as follows.

Set \( \tilde{z} := (z_1, \ldots, z_n) \) with \( z_i := (z_{i0}, \ldots, z_{im}) \), and for \( 1 \leq i \leq n, \)

1. in case \( d_i \leq m \), let \( u_i := (z_{i0}, \ldots, z_{id_i}) \),
2. in case \( d_i = m + 1 \), let \( u_i = z_i \).
For each $1 \leq i \leq n$, we pick one of the formulas $\phi_j^i(u_1, \ldots, u_i)$, $j \in J_i$, with $|u_i| = d_i^s + 1$ and $|u_s| \leq d_i^s + 1$, $1 \leq s \leq i$, as occurring above and we define

$$
\tilde{Z} := \{ \tilde{z}[m(1)-m] : \tilde{z} \in Z \land \bigwedge_{1 \leq i \leq n} \lambda_{P_{i,j}}^{\ell_i + m(1)-m}(u_1(\ell_i + m(1) - m), \ldots, u_i(\ell_i + m(1) - m)) \},
$$

making the convention that when $d_i = m + 1$, the formula $\lambda_{P_{i,j}}^{\ell_i + m(1)-m}(u_1(\ell_i + m(1) - m), \ldots, u_i(\ell_i + m(1) - m))$ is replaced by $\lambda_j^{\ell_i + m(1)-m}(u_1, \ldots, u_{i-1}, u_i(\ell_i + m(1) - m))$ if $i > 1$ and if $i = 1$ by the formula $\lambda_j^{\ell_1 + m(1)-m}(u_1(\ell_1 + m(1) - m))$. Also note that if $d \leq d_i^s + 1$ is maximal such that $\text{deg}_{zd} P_{i,j} > 0$, then in the formula $\lambda_{P_{i,j}}^{\ell_i + m(1)-m}(u_1(\ell_i + m(1) - m), u_{i+1}(\ell_{i+1})) = (z_0, \ldots, z_{sd}, z_{sd+1}, \ldots, z_{sd+\ell_i+m(1)-m})$.

Recall that along the way we took off $Z$ subsets of smaller dimension, so let us denote $Z_0$ the remaining subset. Then we get $\bigcup \nabla^{-1}(\tilde{Z}) = \nabla^{-1}(Z_0)$.

Then we apply Proposition \ref{prop:decomposition} to decompose each $\tilde{Z}$ into a finite union of correspondences to which we apply the preceding procedure. Let us denote one of these correspondences by $\bar{Z}(1)$.

Since there was nothing special going from $Z$ to $Z(1)$, replacing 0 by $t$ (and 1 by $t + 1$), $m$ by $m(t)$, $d_i$ by $d_i(t)$, $\ell_i$ by $\ell_i(t)$, we have from the construction above, that the tuple $\bar{d}(t) \in \mathbb{N}^n$ associated with $Z(t)$, $t \geq 1$, has the following properties:

1. if $d_i(0) < m + 1$, then $d_i(t) \leq d_i(0)$, $1 \leq i \leq n$,
2. if $d_i(t) < d_i(t - 1)$, then $d_i(t + 1) \leq d_i(t)$, $1 \leq i \leq n$,
3. if $d_i(t) > d_i(t - 1)$, then $d_i(t - 1) = m(t - 1)$ and for some $i < j \leq n$, $d_j(t) < d_j(t - 1)$, $1 \leq i \leq n$,
4. $d_n(t) \leq d_n(t - 1) \leq d_n(0) = d_n \leq m + 1$.

We make a similar abuse of notation by denoting $k(Z(t))$ by $k(t)$, and write the tuple $\bar{d}(t)$ as two subtuples one: $\bar{d}_{[k(t)]}(t)$ of length $k(t)$ which collects (in increasing order) all the components of $\bar{d}(t)$ equal to $m(t) + 1$ and the other one $\bar{d}_{[n-k(t)]}$ such that $\bar{d}(t) = (\bar{d}_{[k(t)]}, \bar{d}_{[n-k(t)]})$. (Recall that $k(t) + 1 \leq k(t) < n$.)

So by (4), for all $t$, $d_n(t) \leq d_n$, and by (1), (2) that the only indices $i$, $1 \leq i \leq n$, for which $d_i(t + 1) > d_i(t)$ are those where the projection onto the $i$th-block of variables has the same dimension as the ambient space and by (3) in order that this dimension increases from step $t$ to step $t + 1$, the dimension of the projection onto some $j$th-block of variables, $i < j \leq n$ has dimension strictly smaller than the dimension of the ambient space. Observe that if $k(t) = 0$, then for all $i$, $d_i(t + 1) \leq d_i(t)$ and $(d_i(t + 1), \ldots, d_n(t + 1)) \prec (d_i(t), \ldots, d_n(t)) \prec (d_1(t), \ldots, d_n(t))$, $t \geq 1$.

Furthermore if we go along a branch, then at some point $k(t) = k(t + 1)$ and we cannot have $\bar{d}_{[n-k(t+1)]} \prec \bar{d}_{[n-k(t)]}$ for infinitely many $t' > t$. So now suppose that $k(t) = k(t + 1)$ and $\bar{d}_{[n-k(t+1)]} \prec \bar{d}_{[n-k(t)]}$. By (3), we get that $\bar{d}_{[k(t+1)]} = \bar{d}_{[k(t)]}$. So $\bar{d}(t + 1) = \bar{d}(t)$.

Let us show that in this case, the differential points are dense in $Z(t+1)$ and that we can decompose $Z(t+1)$ into a finite union of correspondences ones for which we do have the required description and the other ones $Z(t+1)$ for which $g(Z(t+1)) \prec g(Z(t))$.

Let $\tilde{z} := (z_1, \ldots, z_n) \in Z(t + 1)$ with $z_i := (z_1^{i}, \ldots, z_i^{m(t)})$, $1 \leq i \leq n$. When $d_i(t) \leq$
By assumption $\pi$ 

We have $\pi_{[\tilde{d}(t)]}(Z(t)) = U_1 \times \ldots \times U_n$ where $U_i \subset K^{d_i(t)}$ are open $\mathcal{L}$-definable subsets of dimension $d_i(t)$ (with the convention that if $d_i(t) = 0$, then $U_i$ is a point), $1 \leq i \leq n$. By assumption $\pi_{[\tilde{d}(t+1)]}(Z(t+1)) = V_1 \times \ldots \times V_n$ where $\emptyset \neq V_i \subset U_i$. Let us describe a correspondence $f$ sending $(v_1, \ldots, v_n) \in V_1 \times \ldots \times V_n$ to $(u_1(\ell_1(t)), \ldots, u_n(\ell_n(t)))$, where for each $1 \leq i \leq n$, there is $j \in J_i$ such that $\lambda^i_{P_{i,j}}(u_1(\ell), \ldots, u_i(\ell))$ holds with the following convention: when $d_i(t) = m(t) + 1$, then $\lambda^i_{P_{i,j}}(u_1(\ell), \ldots, u_i(\ell))$ is replaced by $\lambda^i_{(u_1(\ell), \ldots, u_{i-1}(\ell), u_i(\ell))}$. Moreover we only keep those $(u_1(\ell_1(t)), \ldots, u_n(\ell_n(t)))$ which belongs to $Z(0)$.

The differential points are dense in $\pi_{[\tilde{d}(t+1)]}(Z(t+1))$ (Lemma 2.9), so we can pick a $n$-tuple of the form $(\delta^{d_i(t)}(a_1), \ldots, \delta^{d_i(t)-1}(a_n))$ close to $(v_1, \ldots, v_n)$. Then we use the scheme $(DL)$ and the continuity of the rational functions $f_k^{P_{i,j}}$, $1 \leq k \leq \ell$, associated with $P_{i,j}$, with $i$ such that $d_i(t) \leq m(t)$ and $j \in J_i$. We proceed by induction on $1 \leq i \leq n$, namely we assume that the differential points are dense in $\pi_{[(i-1)(m(t)+1)]}(Z(t+1))$ with the condition that if $i = 1$, this is the empty projection. Now suppose that $d_i(t) \leq m(t)$, then for $(u_1, \ldots, u_{i-1}) \in \pi_{[d_i(t)+1, \ldots, d_{i-1}(t)+1]}(Z(t+1))$, $u_i \in K^{d_i(t)+1}$, if

$$K \models P_{i,j}(u_1, \ldots, u_{i-1}, u_i) = 0 \land s_{P_{i,j}}(u_1, \ldots, u_{i-1}, u_i) \neq 0,$$

then there is a differential point $\tilde{d}^{d_i(t)+1}(u)$ close to $u_i$, for some $u \in K$, such that

$$K \models P_{i,j}(u_1, \ldots, u_{i-1}, \tilde{d}^{d_i(t)+1}(u)) = 0 \land s_{P_{i,j}}(u_1, \ldots, u_{i-1}, \tilde{d}^{d_i(t)+1}(u)) \neq 0.$$

Let $U$ be a basic open neighbourhood of $(z_1, \ldots, z_i) \in \pi_{[(m(t)+1)]}(Z(t+1))$. By induction hypothesis, there is a differential tuple close to $(z_1, \ldots, z_{i-1})$ and by scheme $(DL)$, there is a differential tuple close to $(z_1, \ldots, z_{i-1}, u_i)$. By the continuity of the functions $f_k^{P_{i,j}}$ (see Lemma 3.5), there is $V$ be a basic open neighbourhood of $(z_1, \ldots, z_{i-1}, u_i)$ such that

$$V' := V \times f_1^{P_{i,j}}(V) \times \ldots \times f_{m(t)-d_i}(V) \subseteq U.$$

The rational functions $f_k^{P_{i,j}}$, $1 \leq k \leq m(t) - d_i$, applied to $\tilde{d}^{d_i(t)+1}(u)$ give its successive derivatives.

Then by [2] Proposition 2.6.10], the correspondence $f$ is continuous on an open subset $O$ almost equal to $V_1 \times \ldots \times V_n$. It remains to apply induction to $f \restriction ((V_1 \times \ldots \times V_n) \setminus O)$. \hfill \Box

We apply the proposition above to obtain the open core property for $T^*_\delta$.

**Corollary 3.9.** Let $K \models T^*_\delta$ and let $X \subseteq K^n$ be an $\mathcal{L}_\delta$-definable set. Then there is an $\mathcal{L}$-definable subset $Z$ such that $(X, Z, o(X))$ is a linked triple. In particular $T^*_\delta$ has $\mathcal{L}$-open core.

**Proof:** By the above proposition, given an $\mathcal{L}_\delta$-definable set $X$, there is an associated linked triple $(X, X^{**}, d)$ with $X^{**}$ an open $\mathcal{L}$-envelop for $X$ and $d \in \mathbb{N}$. In general this integer $d$ will be larger than $o(X)$ but, by Proposition 3.4.1 one can construct another linked triple with $d = o(X)$. \hfill \Box


The following corollary slightly improves \[2, \text{Proposition 6.1.1}\] where we showed that one can associate with an $L_\delta$-definable $\ell$-correspondence, an $L$-definable $\ell$-correspondence. There we assumed the domain of the correspondence to be $L$-definable (but the same proof works when it is only $L_\delta$-definable). Furthermore, here we want the domain of the correspondence to have the special form described in Proposition 3.8. One can check that indeed the previous proof easily adapts.

**Corollary 3.10.** Let $K \models T_\delta^*$. Let $X \subset K^n$ and assume it is $L_\delta$-definable. Let $f : X \Rightarrow K^d$ be an $L_\delta$-definable $\ell$-correspondence with $d, \ell \geq 1$. Let $X^{**}$ be an $L$-open envelop of $X$. There is $m \in \mathbb{N}$, an $L$-definable $\ell$-correspondence $F : X^{**} \Rightarrow K^d$ such that for every $x \in X$

$$f(x) = F(\delta^m(x)).$$

\[\square\]

4. **Definable groups in models of $T_\delta$**

First we will recall a few facts about definable groups and generics in the setting of o-minimal theories \[13, 11\], since most of these notions only uses properties of the dimension function that hold in our present setting.

Then given an $L_\delta$-definable group $G$ in a model of $T_\delta^*$, where $T$ is an $L$-open theory of topological fields, we will show that on large $L$-definable subset of an $L$-open envelop of $G$ as defined in Proposition 3.8, one can recover an $L$-definable operation induced by the $L_\delta$-definable group law of $G$.

Finally we will use the work of K. Peterzil on the group configuration in o-minimal structures \[11\]. In a complete o-minimal theory admitting elimination of imaginaries, he showed that a group configuration gives rise to a transitive action of a type-definable group on infinitesimal neighbourhoods. Without appealing directly to a group configuration but using the same strategy, we will associate to an $L_\delta$-definable group a type $L$-definable group (possibly in a higher dimensional space).

In the last part of this section we will add sorts $S$ of $L^{eq}$ to the language $L$ in order that $T$ admits elimination of imaginaries in $L^S$, the restriction of $L^{eq}$ to $L$ together with $S$. Note that on the one hand, the expansion $T^S$ of the theory $T$ in $L^S$ is again an $L^S$-open theory of topological fields (see \[2, \text{Remark 2.2.1}\]) and on the other hand, by Corollary 3.9 and \[2, \text{Theorem 4.0.5}\], $T^S_\delta$ admits elimination of imaginaries.

We will first place ourselves in any $L$-structure $M$ (without our previous assumptions on the language $L$) but we will assume that $M$ is endowed with a fibered dimension which has the exchange property and that this dimension is preserved under definable bijection.

By definable group $G := (G, \cdot, 1)$ in $M$, we mean that the domain of $G$ is an $L$-definable subset of some cartesian product $M^n$ of $M$ and the graph of the group operation $\cdot$ is an $L$-definable subset of $M^{3n}$.

**Fact 4.1.** \[13\] Let $G$ be a definable group in $M$. Any element of $G$ is the product of two generic elements and given a generic element $a$ of $G$ over $b$, then $b \cdot a$ is generic in $G$. 

---

DEFINABLE GROUPS IN TOPOLOGICAL FIELDS WITH A GENERIC DERIVATION 17
**Definition 4.1.** Let $\mathcal{G}$ be a definable group in $\mathcal{M}$. A generic definable subset of $G$ is a subset of $G$ such that finitely many translates cover $G$.

Note that it entails that a generic set has the same dimension than $G$ and so it contains a generic point of $G$.

**Fact 4.2.** Let $\mathcal{G}$ be a definable group of $\mathcal{M}$. Let $X$ be a definable subset of $G$, almost equal to $G$, then $X$ is generic.

For the rest of this section, we will work in models of an $\mathcal{L}$-open theory $T$ of topological fields. In particular, we assume that $T$ satisfies hypothesis (A).

**Proposition 4.2.** Let $\mathcal{K} \models T^*_0$ be sufficiently saturated, let $\mathcal{G} := (G, f_0, f_{-1}, e)$ be an $\mathcal{L}_3$-definable group in $\mathcal{K}$ (possibly with parameters) and let $G^*$ be an $L$-open envelope of $G$. Then there exist $\mathcal{L}$-definable maps $F_{-1} : G^* \to G^*$ and $F_0 : G^* \times G^* \to G^*$, a large open subset $V$ of $G^*$ and $Y$ a definable large open subset of $G^* \times G^*$ such that

1. the map $F_{-1} : G^* \to G^*$ (respectively $F_0 : G^* \times G^* \to G^*$) coincides on differential tuples with $f_{-1}$ (respectively $f_0$),
2. the map $F_{-1} : V \to V$ is a continuous idempotent map,
3. the map $F_0 : Y \to V$ is continuous,
4. for any $a \in V$, if $b$ is generic of $V$ over $a$, then $(b, a) \in Y$ and $(F_{-1}(b), F_0(b, a)) \in Y$.

**Proof:** The proof follows the same pattern as in [13, Proposition 2.5] and we will construct a large subset $V$ of $G^*$ with the required properties by steps.

Assume that the domain of $\mathcal{G}$ is included in some $K^n$ and is an $\mathcal{L}_3$-definable set. Then by Proposition [13, 3.8] there is an $\mathcal{L}$-open envelop $G^*$ of $G$, described as follows. For some $d \in \mathbb{N}$, there is a finite family $\{Y_i \mid i \in I\}$ of $\mathcal{L}$-definable subsets of $K^{n,d}$ such that $G^* = \bigcup_{i \in I} Y_i$,

$$G = \bigcup_{i \in I} \nabla^{-1}_d(Y_i),$$

and for each $i \in I$, $Y_i$ is either open, or equal to $\{\tilde{\delta}^d(a)\}$ for some $a \in G$, or the graph of a continuous $\mathcal{L}$-definable correspondence $h_i$ such that for some n-tuple $(d_1, \ldots, d_n)$:

1. $1 \leq d_j < d + 1$, $1 \leq j \leq n$,
2. $\pi_{[d_1, \ldots, d_n]}(Y_i)$ is open in $K^{d_1} \times \cdots \times K^{d_n}$,
3. $h_i : \pi_{[d_1, \ldots, d_n]}(Y_i) \rightarrow K^{d+1-d_1} \times \cdots \times K^{d+1-d_n}$, and
4. for every open subset $U \subseteq K^{nd}$ such that $U \cap \pi_{[d_1, \ldots, d_n]}(Y_i) \neq \emptyset$, there is $a \in \pi_{[1, \ldots, 1]}(U \cap \pi_{[d_1, \ldots, d_n]}(Y_i))$ such that $\tilde{\delta}^d(a) \in Y_i$.

Moreover letting $f_{-1}$ the inverse function on $G$, by Corollary [13, 3.10] there is a definable $\mathcal{L}$-function $F_{-1}$ on $G^*$ which coincides with $f_{-1}$ on differential points. Similarly given $f_0$ the group law on $G \times G$ there is a definable $\mathcal{L}$-function $F_0$ on $G^* \times G^*$ which coincides with $f_0$ on differential points. (We also use here that a 1-correspondence is a function).

Let $I_0 := \{i \in I : \dim(Y_i) = \dim(G^*)\}$. By [2, Proposition 2.6.10], $F_{-1} \circ h_i$ is continuous on a large open subset $U_i$ of $\pi_{[d_1, \ldots, d_n]}(Y_i)$.
Let $\tilde{Y}_i := h_i(U_i)$ for $i \in I_0$ and

$$V_1 := \bigcup_{i \in I_0} \tilde{Y}_i.$$

**Claim 4.3.** Let $V'_1 := \{ \tilde{x} \in V_1 : F_{-1}(F_{-1}(\tilde{x})) = \tilde{x} \}$. Then $V'_1$ is a large definable subset of $V_1$.

**Proof of Claim:** Suppose on the contrary that we can find an open subset $U \subset V_1$ where $F_{-1}(F_{-1}(\tilde{x})) \neq \tilde{x}$. Then choose in $U$ an element $\tilde{u}^\triangledown$ for some $\tilde{u} \in G$. Since $F_{-1}$ and $f_{-1}$ coincide on differential points, we get a contradiction.  

From now on, we allow ourselves to replace $V_1$ by $V'_1$, namely we will assume that $F_{-1} \circ F_{-1}$ is the identity on $V_1$.

Using again [2] Proposition 2.6.10], $F_0 \circ (h_i, h_j)$ is continuous on a large open subset $O_{i,j}$ of $\pi_{[d_1, \ldots, d_n]}(Y_i) \times \pi_{[d_1, \ldots, d_n]}(Y_j)$. Let $\tilde{Y}_{i,j} := (h_i, h_j)(O_{i,j})$ for $i, j \in I_0$ and

$$Y_0 := \bigcup_{i,j \in I_0} \tilde{Y}_{i,j}.$$

**Claim 4.4.** Let $V_0 := \{ \tilde{y} \in V_1 : \forall \tilde{a} \in G^\ast \text{-generic of } G^\ast \text{ over } \tilde{y}, (\tilde{a}, \tilde{y}) \in Y_0 \}$. Then $V_0$ is definable and almost equal to $V_1$.

**Proof of Claim:** The set $V_0$ is $L$-definable by Fact 2.1. By the way of contradiction, assume there is a relatively open subset $U_1$ of $V_1 \subset G^\ast$ such that if $\tilde{y} \in U_1$, we have that $\{ \tilde{a} \in G^\ast : (\tilde{a}, \tilde{y}) \in Y_0 \}$ is not almost equal to $G^\ast$. In particular, we may choose in $U_1$ a generic element $\tilde{y}_0 \in V_1$ of $G^\ast$ such that the set $\{ \tilde{a} \in G^\ast : (\tilde{a}, \tilde{y}_0) \in Y_0 \}$ is not almost equal to $G^\ast$. So its complement would contain an open subset $U_2(\tilde{y}_0)$. Choose $b_0 \in G^\ast$ generic over $\tilde{y}_0$ in that open set. Then $(b_0, \tilde{y}_0)$ is generic in $G^\ast \times G^\ast$ by equation 2.3.1 but would not belong to $Y_0$, a contradiction.

**Claim 4.5.** Let $a, b \in G$ and choose $a^\triangledown$ $L$-generic over $b^\triangledown$. Then $F_0(a^\triangledown, b^\triangledown)$ is $L$-generic over $b^\triangledown$ and so it belongs to $V_0$.

**Proof of Claim:** By hypothesis, $\dim(G^\ast) = \dim(a^\triangledown / b^\triangledown)$. Let us show that $\dim(F_0(a^\triangledown, b^\triangledown) / b^\triangledown) = \dim(a^\triangledown / b^\triangledown)$. By construction, we have that $F_0$ and $F_{-1}$ coincide respectively on $G^\triangledown$ with $f_0$ and $f_{-1}$ respectively. So $F_0(a^\triangledown, b^\triangledown) = f_0(a, b)^\triangledown$, $F_{-1}(b^\triangledown) = f_{-1}(b)^\triangledown$ and $F_0(F_0(a^\triangledown, b^\triangledown), F_{-1}(b^\triangledown)) = a^\triangledown$.

So, $acl(b^\triangledown, F_0(a^\triangledown, b^\triangledown)) \subset acl(b^\triangledown, a^\triangledown) \subset acl(F_0(a^\triangledown, b^\triangledown), F_{-1}(b^\triangledown), b^\triangledown) \subset acl(F_0(a^\triangledown, b^\triangledown), b^\triangledown)$.  

**Claim 4.6.** Let $a, b \in G$ and choose $a^\triangledown$ $L$-generic over $b^\triangledown$. Then $F_{-1}(a^\triangledown)$ is $L$-generic over $b^\triangledown$ and so it belongs to $V_1$.

**Proof of Claim:** By hypothesis, $\dim(G^\ast) = \dim(a^\triangledown / b^\triangledown)$. Let us show that $\dim(F_{-1}(a^\triangledown) / b^\triangledown) = \dim(a^\triangledown / b^\triangledown)$. By construction, we have that $F_{-1}$ coincide on $G^\triangledown$ with $f_{-1}$. So $F_{-1}(a^\triangledown) = f_{-1}(a)^\triangledown$ and $F_{-1}(F_{-1}(a^\triangledown)) = F_{-1}(f_{-1}(a)^\triangledown) = f_{-1}(f_{-1}(a)^\triangledown) = a^\triangledown$. So, $acl(b^\triangledown, a^\triangledown) \subset acl(b^\triangledown, f_{-1}(a)^\triangledown) \subset acl(b^\triangledown, a^\triangledown)$.  

\[ \square \]
Let \( \bar{Y}_0(\bar{z}) := \{ (\bar{x}, \bar{y}) \in Y_0 : (\bar{y}, \bar{z}) \in Y_0 \} \)
\[
F_0(\bar{y}, \bar{z}) \in V_0 \&
(\bar{x}, F_0(\bar{y}, \bar{z})) \in Y_0 \&
F_0(\bar{x}, F_0(\bar{y}, \bar{z})) = F_0(F_0(\bar{x}, \bar{y}), \bar{z})
\]

Let \( V_0'' := \{ z \in V_0 : \bar{Y}_0(\bar{z}) \) is almost equal to \( Y_0 \} \).

The set \( V_0'' \) is definable since being almost equal is an \( L \)-definable property and all the other data is \( L \)-definable.

**Claim 4.7.** \( V_0'' \) is almost equal to \( V_0 \).

**Proof of Claim:** We proceed by contradiction. If not there would exist \( z \in V_0 \) and an open neighbourhood \( U \) of \( z \) such that for any element \( \bar{u} \in U \), the set \( Y_0(\bar{u}) \) is not almost equal to \( Y_0 \), which means that there exists an open subset \( W \) of \( Y_0 \) containing \( (\bar{x}, \bar{y}) \) where one of the following statement fails: \( (\bar{y}, \bar{u}) \in Y_0 \), \( F_0(\bar{y}, \bar{u}) \in V_0 \), \( (\bar{x}, F_0(\bar{y}, \bar{u})) \in Y_0 \), or if everything else hold, that \( F_0(\bar{x}, F_0(\bar{y}, \bar{u})) = F_0(F_0(\bar{x}, \bar{y}), \bar{u}) \) fails.

We may choose such \( \bar{u} \in V_0 \cap U \) of the form \( \bar{u} = u \) for some \( u \in G \) (\( G \) is dense in \( G^* \)). We choose \( (\bar{a}, \bar{b}) \in W \) as follows. First we choose \( \bar{b} \) \( L \)-generic over \( \bar{u} \) (in particular \( \bar{b} \in V_1 \)). By Claim 4.3 \( (\bar{b}, u) \in Y_0 \) and by Claim 4.5 \( F_0(\bar{b}, \bar{u}) \in V_0 \).
(Note that \( f_0(\bar{b}, u) \) is dense in \( G^* \).) Then we choose \( a \) \( L \)-generic over \( \bar{b} \) and over \( f_0(\bar{b}, u) \). So \( (a, b) \in Y_0 \) and \( (a, f_0(\bar{b}, u)) \in Y_0 \). Since \( G \) is a group and \( F_0 \) and \( f_0 \) coincide on differential points, we get that \( F_0(\bar{a}, F_0(\bar{b}, \bar{u})) = F_0(F_0(\bar{a}, \bar{b}), \bar{u}) \).

Let \( V_0'' := \{ z \in V_0 : \bar{X}_0(\bar{z}) \) is almost equal to \( Y_0 \} \).

The set \( V_0'' \) is definable since being almost equal is an \( L \)-definable property.

**Claim 4.8.** \( V_1'' \) is almost equal to \( V_0 \).

**Proof of Claim:** Suppose not then there would exist an open subset \( U \) of \( V_0 \) over which \( \{ \bar{x} \in V_1 : (F_0(\bar{x}), \bar{z}) \in Y_0 \& (\bar{x}, F_0(F_0(\bar{x}), \bar{z})) \in Y_0 \& F_0(\bar{x}, F_0(F_0(\bar{x}), \bar{z})) = \bar{z} \) is not almost equal to \( V_0 \).

So we can find an element of the form \( \bar{z} \in U, \bar{z} \in G \) and \( \bar{x} \in V_0 \) generic over \( \bar{z} \). By Claim 4.6 \( F_0(\bar{x}) \) is generic over \( \bar{z} \) and so it belongs to \( V_0 \) and \( (F_0(\bar{x}), \bar{z}) \in Y_0 \). By Claim 4.5 \( F_0(\bar{x}), \bar{z}) \) is generic over \( \bar{z} \). Now let us show that \( \bar{x} \) is generic over \( F_0(F_0(\bar{x}), \bar{z}) \).

We first note that \( acl(\bar{z}, \bar{x}) = acl(\bar{z}, F_0(F_0(\bar{x}), \bar{z})) \).

So \( \dim(\bar{x}, \bar{z}) = \dim(\bar{x}, F_0(F_0(\bar{x}), \bar{z})) \).

By equation 2.3.1
\[
\dim(\bar{x}, \bar{z}) = \dim(\bar{x}/\bar{z}) + \dim(\bar{z}) \quad \text{and}
\]
\[
\dim(\bar{x}, F_0(F_0(\bar{x}), \bar{z})) = \dim(\bar{x}/F_0(F_0(\bar{x}), \bar{z})) + \dim(F_0(F_0(\bar{x}), \bar{z})).
\]
So, \( \dim(\bar{x}^\nabla/F_0(F_{-1}(\bar{x}^\nabla), \bar{z}^\nabla)) = \dim(\bar{x}^\nabla/\bar{z}^\nabla) \), so \( \bar{x}^\nabla \) is generic over \( F_0(F_{-1}(\bar{x}^\nabla), \bar{z}^\nabla) \) and therefore \( (\bar{x}^\nabla, F_0(F_{-1}(\bar{x}^\nabla), \bar{z})) \in Y_0 \). Since \( F_0 \) and \( F_{-1} \) coincide with \( f_0 \) and \( f_{-1} \) on differential points, we get \( F_0(\bar{x}^\nabla, F_0(F_{-1}(\bar{x}^\nabla), \bar{z}^\nabla)) = f_0(\bar{x}^\nabla, f_0(f_{-1}(\bar{x}^\nabla), \bar{z}^\nabla)) = \bar{z}^\nabla \). 
\[ \square \]

Since both \( V_0'' \), \( V_1'' \) are \( L \)-definable and almost equal to \( G'' \) (see Claims 4.7, 4.8), \( V_0'' \cap V_1'' \) is \( L \)-definable and almost equal to \( G'' \). So it can be expressed as a finite union of continuous correspondences by Proposition 2.6. Let \( V_2 \) be the union of those with the same \( L \)-dimension as \( G'' \).

**Claim 4.9.** Let \( V := V_2 \cap F_{-1}(V_2) \). Then \( V \) is \( L \)-definable, relatively open in \( G'' \) and almost equal to \( G'' \). Moreover \( F_{-1} \) maps \( V \) to \( V \) and is continuous.

**Proof of Claim:** It remains to show that \( F_{-1} \) maps \( V \) to \( V \). An element of \( V \) can be written as \( \bar{a} \in V_2 \) and \( F_{-1}(\bar{b}) \) for \( \bar{b} \in V_2 \). Since \( V_2 \subset V_1 \), we get that \( F_{-1}(F_{-1}(\bar{b})) = \bar{b} \). 
\[ \square \]

By construction, \( V_2 \) is a relatively open definable large subset of \( G'' \). We define \( Y := \{(\bar{x}, \bar{y}) \in (V \times V) \cap Y_0 : F_0(\bar{x}, \bar{y}) \in V \} \). Note that \( Y \) is large in \( V \times V \). If we choose \( \bar{x}^\nabla \in V \) generic, then \( \bar{y}^\nabla \in V \) generic over \( \bar{x}^\nabla \). Then \( F_0(\bar{x}^\nabla, \bar{y}^\nabla) \) is generic over \( \bar{x}^\nabla \) by Claim 4.5 and so belongs to \( Y \). Note that on \( Y \), the map \( F_0 \) is continuous.

By Claim 4.9, \( V \) is a large definable relatively open subset of \( G'' \). By the paragraph above \( Y \) is a large definable relatively open subset of \( G'' \times G'' \) and \( F_0 \) is continuous on \( Y \) (statement (3) of the proposition). Statement (1) of the proposition follows by construction, (2) is Claim 4.9 and (4) is proven in the same way as we did with \( V_0 \) and \( Y_0 \) (see Claim 4.4). 
\[ \square \]

Now let us recall the notion of infinitesimal neighbourhoods in the context of \( L \)-open theories \( T \) of topological fields.

**Definition 4.10.** [11] Definition 2.3] Let \( \mathcal{M} \models T \) and let \( a \in M^n \). Then the \( M \)-infinitesimal neighbourhood of \( a \in M^n \) is the partial type consisting of all formulas with parameters in \( M \) that define an open subset of \( M^n \) containing \( a \). Given \( \mathcal{N} \) an \( |M|^+ \)-saturated extension of \( \mathcal{M} \), we denote by \( \mu_a(\mathcal{N}) \) its realization in \( \mathcal{N} \). Let \( X \) be a \( M \)-definable subset of \( M^n \) containing \( a \), then \( \mu_a(X) = \mu_a(\mathcal{N}) \cap X(\mathcal{N}) \).

We will also use the notation \( u \sim_M 0 \) to mean that \( u \in \mu_0(\mathcal{N}) \) and use the term for such elements \( u \), \( M \)-infinitesimals.

If \( a \) is generic in \( X \), then \( \mu_a(X) \) is independent of the choice of \( X \), namely we have the analog of [11] Fact 2.4, using property 2.3.1 of the dimension function in models of \( T \). For convenience of the reader, we prove it below.

**Lemma 4.11.** [11] Fact 2.4] Let \( X \) be an \( A \)-definable subset of \( M^n \) and \( a \) a generic element of \( X \) over \( A \). Then for any \( A \)-definable set \( Y \), if \( a \in Y \), then \( \mu_a(X) \subset \mu_a(Y) \). In particular if \( \dim(X) = \dim(Y) \), then \( \mu_a(X) = \mu_a(Y) \).

**Proof:** Consider \( X \cap Y \). Then since \( a \) is generic in \( X \), \( a \in \text{Int}_X(X \cap Y) \). Since the topology is definable, there exists an open \( A \)-definable set \( U \) containing \( a \) such that \( U \cap X \subset X \cap Y \). So, \( U \cap X = U \cap X \cap Y \). It follows that \( \mu_a(X) = \mu_a(X \cap Y) \) and
so $\mu_a(X) \subset \mu_a(Y)$. If $\dim(X) = \dim(Y)$, then $a$ is also generic in $Y$ and the reverse inclusion holds, namely $\mu_a(Y) \subseteq \mu_a(X)$. 

The above lemma allows us to introduce the following notation.

**Notation 4.12.** [II] Definition 2.5] Given $a \in M^n$, $A \subseteq M$ and $X$ an $A$-definable subset of $M^n$ with $a$ generic in $X$ over $A$, we denote $\mu^A(a/A)$ (or $\mu(a/A)$) the set $\mu_a(X)$.

**Notation 4.13.** [II] Definition 2.10] Let $M \models T$ and let $O$ be an open subset of $M^n$. Let $p_1, p_2 : O \to M$ be two maps and let $y \in O$. Then $p_1 \sim_Y p_2$ means that for some open neighbourhood $U \subseteq O$ of $y$, $p_1 \mid U = p_2 \mid U$.

**Lemma 4.14.** [II] Lemma 2.11] Let $M$ be a sufficiently saturated model of $T$. Let $V \subseteq M^n$ be a $\mathcal{L}$-definable open subset in $M$; let $F := \{p_b : V \to V : b \in V\}$ be a family of $\mathcal{L}$-definable bijections of $V$. Let $x$ be a generic element of $V$ and let $a_1 \in V$ generic over $\{x\}$. Then there exist definable open subsets $W$ containing $x$ and $U$ containing $a_1$ such that for every $a'_1, a''_1 \in U$ and $y \in W$, if $p_{a'_1} \sim_y p_{a''_1}$, then $p_{a'_1} \mid W = p_{a''_1} \mid W$.

**Proof:** Denote by $[a_1]_x := \{u \in V : p_u \sim_x p_{a_1}, \forall p_u \in F\}$. Let $a \in [a_1]_x$ be generic over $\{a_1, x\}$. Let $W_0$ be an open neighbourhood of $x$ such that $p_a \mid W_0 = p_a \mid W_0$. Recall that the topology on $K$ is definable and let $d_0$ be parameters such that $W_0 = \tilde{x}(M, d_0)$. We may assume by shrinking $W_0$ if necessary that $d_0$ is independent from $a_1, a, x$. We can express by an $\mathcal{L}$-formula in $a_1, a, x, d_0$ the following property:

$p_{a_1} \sim_x p_a \to p_{a_2} \mid W_0 = p_a \mid W_0$.

So it continues to hold for all $a'$ in an open neighbourhood $U_0$ of $a_1$ and this last property can be expressed by an $\mathcal{L}$-formula in $a_1, x$ and $d_0$:

$$\forall a' \in U_0 (p_{a_1} \sim_x p_{a'} \to p_{a_1} \mid W_0 = p_{a'} \mid W_0).$$

Since $a_1$ and $x$ are independent, this formula continues to hold in an open neighbourhood $U_1$ of $a_1$ and an open neighbourhood $W_1$ of $x$. Then $U := U_1 \cap U_0$ and $W := W_0 \cap W_1$ are the sought neighbourhoods. 

**Theorem 4.15.** Let $S$ be sorts in $\mathcal{L}^{eq}$ such that $T$ admits elimination of imaginaries in $\mathcal{L}^S$. Let $K \models T^*_S$ and assume that $K$ is sufficiently saturated. Let $G := (G, f_0, f_{-1}, e)$ be an $\mathcal{L}$-definable group in (the field sort of) $K$ (over a subset of parameters). Let $G^{**}$ be an $\mathcal{L}$-open envelop of $G$. Then there exists a type $\mathcal{L}$-definable topological group $H$ (over some parameters) with $\dim(H) = \dim(G^{**})$.

**Proof:** We keep the same notations as in Proposition 4.2, in particular we will use $F_{-1}$, $F_0$, $V$ and $Y$. In order to simplify notations we will not indicate the parameters we are working with and simply use $\mathcal{L}$; however we will indicate all the additional parameters.

Let $a_1 \in G^{**}$ be $\mathcal{L}$-generic and let $a_2$ be $\mathcal{L}$-generic over $a_1$. Then by Proposition 4.2 (4), we have: $(a_2, a_1) \in Y$ and $(F_{-1}(a_2), F_0(a_2, a_1)) \in Y$. By Proposition 4.2 (3), $a_3 \in V$.

**Claim 4.16.** Let us show that $a_3 = F_0(a_2, a_1)$ is $\mathcal{L}$-generic in $V$.

**Proof of Claim:** This is similar to the proof of Claim 4.5. We have that
Note that \( \forall \) since \( h \rightarrow \).

Claim 4.18.

Proof of Claim:

(a) Denote for \( a'_i \in \mu(a_i) \), \( p_{a'_i} : V \rightarrow V : u \mapsto F_0(a'_i, u) \) and by 

\[ \mathcal{P} := \{ p_{a'_i} : a'_i \in \mu(a_1) \} \] 

(b) Denote for \( a'_2 \in \mu(a_2) \), \( q_{a'_2} : V \rightarrow V : u \mapsto F_0(a'_2, u) \) and by 

\[ \mathcal{Q} := \{ q_{a''_2} : a''_2 \in \mu(a_2) \} \] 

(c) Denote for \( a'_3 \in \mu(a_3) \), \( h_{a'_3} : V \rightarrow V : u \mapsto F_0(a'_3, u) \) and by 

\[ \mathcal{H} := \{ h_{a'_3} : a'_3 \in \mu(a_3) \} \] 

Let \( x \in V \) be \( L \)-generic over \( \{ a_1, a_2 \} \). In particular \( \mu(x) = \mu(x/\{ a_1, a_2 \}) \). As in the group configuration theorem [11] Theorem 3.4], we consider the following subgroup \( H \) of the group \( Sym(\mu(x)) \) of permutations on \( \mu(x) \) generated by: \( \{ p_{a'_i}^{-1} \circ p_{a''_i} : a'_i, a''_i \in \mu(a_1) \} \).

Claim 4.17. \( \forall p \in \mathcal{P} \forall q \in \mathcal{Q} \exists h \in \mathcal{H} \) \( q \circ p = h \) on \( \mu(x) \).

Proof of Claim: Let \( a'_1 \in \mu(a_1) \) be such that \( p = p_{a'_1} \), let \( a'_2 \in \mu(a_2) \) be such that \( q = q_{a''_2} \). The map \( F_0 \) is continuous on \( Y \), so for every neighbourhood \( W \) of \( \mu(a_2) \) there exists \( W_1 \) a neighbourhood of \( a_1 \) and \( W_2 \) a neighbourhood of \( a_2 \) such that \( F_0(W_2, W_1) \subset W \).

First let us show that \( F_0(a_2, F_0(a_1, x)) = F_0(F_0(a_2, a_1), x) \). By Claim 4.7, we know that \( \tilde{Y}_0(x) \) is almost equal to \( Y_0 \). Since \( x \) is \( L \)-generic over \( \{ a_2, a_1 \} \), we get that \( (a_2, a_1) \in \tilde{Y}_0(x) \). So, \( (a_2, F_0(a_1, x)) \in Y_0 \) and \( F_0(a_2, F_0(a_1, x)) = F_0(F_0(a_2, a_1), x) \). Furthermore, \( (a_2, a_1) \in \text{Int}_{Y_0}(\tilde{Y}_0(x)) \). So for \( x' \in \mu(x) \) and \( a'_2 \in \mu(a_2) \), we get that \( (a'_2, a'_3) \in \tilde{Y}_0(x) \) and so \( F_0(a'_3, F_0(a'_2, x)) = F_0(F_0(a'_2, a'_3), x) \).

Now we can express by a \( L(\{ a_1, a_2 \}) \)-formula (in \( x \)) the property that \( (a_2, a_1) \in \text{Int}_{Y_0}(\tilde{Y}_0(x)) \). So for any \( x' \in \mu(x) = \mu(x/\{ a_1, a_2 \}) \), since \( x \) is generic in \( V \), we get that \( (a_2, a_1) \in \text{Int}_{Y_0}(\tilde{Y}_0(x')) \). Therefore, \( F_0(a_2, F_0(a_1, x')) = F_0(F_0(a_2, a_1), x') \). By the same reasoning as above, it also holds for \( a'_1, a'_2 \) in place of \( a_1, a_2 \).

Claim 4.18. \( \forall h \in \mathcal{H} \forall q \in \mathcal{Q} \exists p \in \mathcal{P} \) \( q \circ p = h \) on \( \mu(x) \).

Proof of Claim: Let \( a'_2 \in \mu(a_2) \) be such that \( q = q_{a''_2} \), let \( a'_3 \in \mu(a_3) \) be such that \( h = h_{a'_3} \). Then define a map \( p \) on \( u \in \mu(a_2) \) as follows \( p(u) := F_0(F_0(F_{-1}(a'_2), a'_2), u) \). Note that \( F_{-1}(V) = V \) and \( F_0(a_2, F_0(F_{-1}(a_2), a_3)) = a_3 \) by Claim [4.8] \( (a_2) \) is \( L \)-generic.

Since \( F_0 \) and \( F_{-1} \) are continuous, it holds in a neighbourhood of \( a_2 \), respectively \( a_3 \).
Claim 4.19. \( \forall h \in \mathcal{H} \forall p \in \mathcal{P} \exists q \in \mathcal{Q} \ q \circ p = h \) on \( \mu(x) \).

Proof of Claim: Let \( a_1' \in \mu(a_1) \) be such that \( p = p_{a_1'} \), let \( a_3' \in \mu(a_3) \) be such that \( h = h_{a_3'} \). Then define a map \( q \) on \( u \in \mu(F_3(a_1, x)) \) by \( q(u) := F_3(F_0(a_3', F_{-1}(a_1')), u) \). Note that \( F_{-1}(V) = V \) and \( F_3(F_0(a_3, F_{-1}(a_1)), a_1) = a_3 \) by Claim 4.22 (\( a_1 \) is \( \mathcal{L} \)-generic and \( a_3 \) is \( \mathcal{L} \)-generic over \( a_1 \)). Since \( F_0 \) and \( F_{-1} \) are continuous, it holds in a neighbourhood of \( a_1 \), respectively \( a_3 \).

\( \square \)

Claim 4.20. \( \forall p_1 \in \mathcal{P} \forall p_2 \in \mathcal{P} \exists p_3 \in \mathcal{P} \exists p_4 \in \mathcal{P} \ p_1^{-1}p_2 = p_3^{-1}p_4 \) on \( \mu(x) \).

Proof of Claim: We apply the previous claims in order to write first \( p_3 \) as \( q_3^{-1} \circ h_3 \) with \( q_3 \in \mathcal{Q} \) and \( h_3 \in \mathcal{H} \) by Claim 3.8. Then write \( p_1 \) as \( q_1^{-1} \circ h_1 \) with \( q_1 \in \mathcal{Q} \) by Claim 3.10 and finally \( p_2 = q_1^{-1} \circ h_2 \) with \( h_2 \in \mathcal{H} \) by Claim 3.8. Composing these maps, we get \( p_3p_1^{-1}p_2 = q_3^{-1}h_2 \in \mathcal{P} \) by Claim 3.9.

\( \square \)

Claim 4.21. Let \( a \) be a fixed element of \( \mu(a_1) \) \( \mathcal{L} \)-generic in \( V \) over \( a_1 \). Then group \( H \) is equal to \( \{ p_{a_1}^{-1} \circ p_{a_1'} : a_1' \in \mu(a_1) \} \).

Proof of Claim: We apply the previous claim with \( p_3 = p_a \).

\( \square \)

Claim 4.22. The action of the group \( H \) is transitive on \( \mu(x) \).

Proof of Claim: Since \( x \in V \), \( F_{-1}(x) \in V \) and since \( x \) is generic over \( a_1 \), \( (a_1, x) \in Y \).

By Claim 4.7 \( F_0(F_0(a_1, x), F_{-1}(x)) = a_1 \). So given \( x_2' \in \mu(x) \) and \( x_3' \in \mu(F_0(a_1, x)) \), we may define \( a_1' := F_0(x_3', F_{-1}(x_2')) \) and this element belongs to \( \mu(a_1) \).

Then we re-apply the same reasoning to \( x_2'' \in \mu(x) \) and \( x_3'' \), and define \( a_1'' := F_0(x_3'', F_{-1}(x_2'')) \). Then \( a_1'' \in \mu(a_1) \). Finally consider \( p_{a_1'}^{-1} \circ p_{a_1} \); by Claim 4.21 it belongs to \( H \) and \( p_{a_1'}^{-1} \circ p_{a_1}(x_2') = x_2'' \).

Finally by Lemma 4.14 there is a \( \mathcal{L} \)-definable open neighbourhoods \( W \subset V \) of \( x \) and \( U \subset V \) of \( a_1 \) such that if \( p_{a_1} \sim_x p_{a_1'} \), then \( p_{a_1} \downarrow W = p_{a_1'} \downarrow W \). For \( a_1' \in U \), let \( [a_1']_x := \{ a_1' \in U : p_{a_1'} \sim_x p_{a_1'} \} = \{ a_1 \in U : p_{a_1'} \downarrow W = p_{a_1'} \downarrow W \} \). Now suppose that \( p_{a_1}^{-1}p_{a_1}(u) = p_{a_1'}^{-1}p_{a_1'}(u) \), namely \( F_0(F_{-1}(a), F_0(a_1', u)) = F_0(F_0(F_{-1}(a), a_1', u), u) \). So \( F_{arc}(a, F_0(F_{-1}(a), F_0(a_1', u))) = F_0(a, F_0(F_{-1}(a), a_1', u)) \) which implies that \( F_0(a_1', u) = F_0(a, F_0(a_1', u)) \). So, two elements of \( H \) are equal on a neighbourhood of \( x \), then they coincide on \( U \) by Lemma 4.14. This will allow us to identify \( H := \{ p_{a_1}^{-1} \circ p_{a_1'} : a_1' \in \mu(a_1) \} \) with \( \mu(a_1)/E \) where \( \mu(a_1) \) is type-definable over \( M \) and \( E \) is a definable equivalence relation on \( V \), defined by \( E(a_1', a_1'') \) if and only if \( p_{a_1'} \downarrow W = p_{a_1''} \downarrow W \).

The group law is definable since the action of the elements of \( H \) on \( \mu(x) \) is defined using the \( \mathcal{L} \)-definable functions \( F_0 \) and \( F_{-1} \). Finally since \( H \) acts transitively on \( \mu(x) \) (Claim 4.22) its dimension is equal to \( \dim(x) = \dim(V) = \dim(G^{**}) \). Furthermore \( H \) a topological group, namely the group laws are continuous. (It follows from the fact that \( F^{-1} \) is continuous on \( V \) and \( F_0 \) is continuous on \( Y \).)

\( \square \)

Remark 4.23. Note that we may associate with \( H \) a \( \dim(G^{**}) \)-group configuration [11] Definition 3.1].

5. ANNEX: LARGENESS

In the following proposition, we put an extra-hypothesis besides largeness on a differential topological field that we will call c-largeness, in order to embed a differential
topological field \((K, \delta)\) endowed with a definable topology into a differential extension model of the scheme (DL).

A topological field \(K\) is \(c\)-large if it is large and the following holds. We consider an embedding \(K \hookrightarrow K((s_0, \ldots, s_{n-1})) \hookrightarrow K^s\) with \(K^s\) a non principal ultrapower of \(K\), \(s_0, \ldots, s_{n-1}\) algebraically independent over \(M\) and \(s_i \sim_K 0\), \(0 \leq i \leq n-1\) (see Definition 4.10). We endow \(K((s_0, \ldots, s_{n-1}))\) with a valuation \(v\), trivial on \(K\) and with \(0 < v(s_0) < v(s_1) < \cdots < v(s_{n-1})\). Let \(\mathcal{M}_v\) be the maximal ideal of \(K[[s_0, \ldots, s_{n-1}]]\), consisting of elements of strictly positive valuation. Then the additional condition we ask is the following: we are in the conditions of Hensel's Lemma since \(v(\bar{n}) \neq 0\).

### Proposition 5.1

Let \(K_\delta\) be a \(c\)-large topological differential field. Then we can embed \(K_\delta\) into a model of the scheme (DL) and with the same \(L\)-theory as \(K\).

**Proof:** We only show the main step. As classically done in the construction of existentially closed models, one first enumerate the existential formulas (with parameters in the ground field \(K\), belonging to a certain family) one wants to satisfy in an extension and then one redo the construction \(\omega\) times in order to get an existentially closed model (with respect to the family of formulas) containing \(K\) as the union of elementary extensions of \(K\).

Consider a differential polynomial \(p(x)\) in one variable in \(K\{x\}\) of order \(n > 0\) such that \(p^*(\bar{a}) = 0 \& s_0^*(\bar{a}) \neq 0\) for some \(\bar{a} := (a_0, \ldots, a_n) \in K^{n+1}\). Then we want to find an \(L\)-elementary extension of \(K\) and an extension of \(\delta\) such that in that elementary extension there is a differential solution of \(p(x) = 0\) with \(\delta^0(b) \sim_K \bar{a}\). We first consider an ultrapower \(K^\delta\) of \(K\) which is \(|K|^\omega\)-saturated and in that ultrapower \(n\) elements \(s_0, \ldots, s_{n-1}\) with \(s_0 \sim_K 0\) and for \(i \geq 1\), \(s_i \sim_K (s_0, \ldots, s_{i-1}) 0\). Then \(s_0, \ldots, s_{n-1}\) are algebraically independent over \(K\). Since \(K\) is large, we have that \(K \subset K((s_0, \ldots, s_{n-1})) \subset K^{**}\) for some ultrapower of \(K\). Furthermore, we may assume that \(s_i, 0 \leq i \leq n-1\), are still \(K\)-infinitesimals in \(K^{**}\).

Set \(\bar{s}_n := (s_0, \ldots, s_{n-1})\). We rewrite \(\bar{a}\) as \((\bar{a}_n, a_n)\) with \(|\bar{a}_n| = n\), \(|a_n| = 1\) and \(p^*(\bar{x}) = p^*(\bar{a}_n + \bar{s}_n, x_n)\) with \(|\bar{x}_n| = n\), \(|x_n| = 1\). We consider the polynomial in \(x_n\):

\[
p^*(\bar{a}_n + \bar{s}_n, x_n).
\]

We endow the field \(K((s_0, \ldots, s_{n-1}))\) with a valuation \(v\), trivial on \(K\) and with \(0 < v(s_0) < v(s_1) < \cdots < v(s_{n-1})\). (So the value group is isomorphic to \(Z^n\) with the lexicographic order.) We are in the conditions of Hensel’s Lemma since \(v(p^*(\bar{a}_n + \bar{s}_n, a_n)) > 0\) and \(v(s_0^*(\bar{a}_n + \bar{s}_n, a_n)) = 0\). So there is \(b \in K((s_0, \ldots, s_{n-1}))\) such that \(v(b - a_n) > 0\) and

\[
(5.0.1) \quad p^*(\bar{a}_n + \bar{s}_n, b) = 0 \& s_0^*(\bar{a}_n + \bar{s}_n, b) \neq 0.
\]

By \(c\)-largeness, we have \(b \sim_K a_n\). We extend \(\delta\) on \(K((s_0, \ldots, s_{n-1}))\) by setting \(\delta_1(a_i + s_i) = a_{i+1} + s_{i+1}\), \(0 \leq i < n - 1\) and \(\delta_1(a_n + s_{n-1}) = b \in K^*\). Thanks to equation (5.0.1), \(\delta_1(b)\) is completely determined and belongs to the subfield of \(K((s_0, \ldots, s_{n-1}))\) of \(K^*\). We set \(K_1 = K((s_0, \ldots, s_{n-1}))\) and extend \(\delta_1\) on the relative algebraic closure of \(K_1\) in \(K^*\). We apply Lowenheim-Skolem theorem to embed \(K_1\) in an \(L\)-elementary extension \(\bar{K}\) of \(K\) of the same cardinality and extend \(\delta_1\) on \(\bar{K}\).

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