Analyzing the features of material nonlinearity evaluation in a rectangular aluminum beam using Rayleigh waves: theoretical and experimental study

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Keywords: rectangular beam, lattice elasticity, material nonlinearity, nonlinearity parameter, Rayleigh waves, experimentation

Abstract

This study proposes a new parameter to evaluate the material nonlinearity in a thick Aluminum (Al) beam having rectangular cross section using Rayleigh waves. This parameter yields a true value of material nonlinearity using the amplitudes of Rayleigh wave harmonics, in contrast to the relative value yielded by the conventional nonlinearity parameter $\beta'$. The Rayleigh wave harmonics are generated in a thick Al 1100 specimen through experiments to estimate its inherent material nonlinearity. This inherent nonlinearity is embedded in the material via lattice elasticity and reckoned using the higher order elastic coefficients. With this experimental investigation, it is found that the accurate evaluation of material nonlinearity is highly dependent on the tone burst cycles in the excitation signal. It is also found that there is a small amount of contribution to the material nonlinearity parameter from the imaginary part of the shear wave component. Furthermore, the relationship between material nonlinearity evaluated using the proposed parameter, excitation frequency, propagation distance, and tone burst cycles in the excitation signal have been unveiled. After knowing these relationships, the material nonlinearity evaluated using the proposed parameter is compared with that obtained from a physics-based nonlinearity parameter containing higher order elastic coefficients. The deviation between the results is minimal. Thus, with the use of amplitudes of harmonics of the Rayleigh wave generated through the experiments, the proposed parameter can evaluate the true material nonlinearity of thick Al beams with fair accuracy.

1. Introduction

In the manufacturing of metallic plates or beams, impurities and micro-damages can be introduced. Ignoring such impurities and micro-damages may result in the formation of numerous micro-cracks due to fatigue. These cracks can ultimately make the plate materials break into pieces. Up to 90% of mechanical failures are caused by fatigue due to weak durability, which results from impurities in the material that ultimately become micro-cracks under loads. Therefore, appraisal of their caliber is essential prior to using them for an uninterrupted service as well as for safety [1–8].

Damage identification in these metallic structures is a usual purpose of such appraisals. Nonetheless, it is indispensable to scrutinize the impurities present in the material and to distinguish the material at any moment of time with reference to its intact state. This can be accomplished using the ultrasonic waves. When the ultrasonic wave at a single excitation frequency is launched into a nonlinear material, higher harmonic waves will be generated. These higher harmonic waves are generated at the intact state because of the lattice elasticity of the material. The nonlinearity induced in the material through the lattice elasticity should also be assessed since it is crucial to unveil the health of the ‘intact’ state materials prior to using them for service. Moreover, the
nonlinearity is intensified further by the evolution of dislocation substructures comprising the dipoles and monopoles for a fatigued or elasto-plastically loaded material [6–9].

The fracturing in a material is usually instigated from the surface, which suggests evaluating the degradation of the surface at an early stage for the prevention of fracture [10–16]. As the fracturing is very small and hardly detected by the linear approach such as measurement of attenuation, time of flight [3, 4, 10], etc, most of the researchers employ the acoustic nonlinearity parameter to quantify the degradation on the surface of the material [11–16].

The higher harmonics generation and measurement setups determine the material nonlinearity using longitudinal waves, Rayleigh waves or Lamb waves. Note that, most published works in nonlinear ultrasonics (NLU) is carried out using bulk longitudinal waves. However, longitudinal waves demand access to both sides of the specimen, making it difficult for a field test. For Lamb waves, locating the mode pairs for cumulative propagation may consume time, and these mode pairs are finite in number depending on the thickness and properties of the material. Rayleigh waves have the advantage of requiring access to one side, making them a good candidate for field applications [11–16].

The elastic displacements and stresses produced by the Rayleigh waves are limited to the surface depth of nearly one wavelength where material damage is usually instigated. This concentration of energy close to specimen surface makes Rayleigh waves a good candidate for evaluating surface conditions of the specimen. Moreover, in comparison to bulk waves, they can propagate with less beam spreading for longer distances. Furthermore, Rayleigh waves are attractive for the detection of the various type of defects, as these waves are nondispersive. With that said, this nondispersive feature of Rayleigh wave provides a greater relaxation in choosing the excitation frequency in contrast to Lamb waves. That is, at any frequency, the Rayleigh wave harmonics will travel at the same phase velocity, which is a prerequisite condition for cumulative effect in the nonlinear analysis [1–16].

The characterization of the material state is explored in the literature using longitudinal waves [17–19], Lamb waves [6–9, 20, 21], and Rayleigh waves [1, 5, 11–16]. The estimation of the true value of material nonlinearity using longitudinal waves can be obtained using the following equation [5, 15, 18],

$$\beta = \frac{8}{k^2x} \frac{A_2}{A_1^2}$$

where $A_1$ and $A_2$ denote the amplitudes of the first and second harmonics of the propagating waves respectively, $k$ being wave number, and $x$ as the distance. However, for interrogation using Lamb or Rayleigh waves, a relative nonlinearity parameter $\beta^\prime$ is used to quantify the material nonlinearity. The relation for $\beta^\prime$ is given as [6–9],

$$\beta^\prime = \frac{A_2}{A_1}$$

The parameter $\beta^\prime$ is a dimensional index and therefore gives only an indication of the increase of the nonlinearity with the propagation distance or increase in fatigue. Therefore, authors proposed [6–8] two amplitude-based nonlinearity parameters $\gamma^\prime_{amp}$ and $\gamma^\prime_{amp}$ separately for the ‘symmetric’ and ‘anti-symmetric’ particle motion of Lamb modes respectively that can accurately determine the nonlinearity at the intact state as well as at the fatigued state. Thus, the experimental [6, 8] and numerical studies [7] revealed that $\gamma^\prime_{amp}$ yields a true nonlinearity value for the plate rather than just a relative value, as yielded by the conventional $\beta^\prime$. These studies were carried out on different states of material, propagation distances, excitation frequencies etc which ensures reproducibility and repeatability of the results.

While quantifying nonlinearity using Rayleigh waves, either equation (2) or the following equations are commonly used [12, 15],

$$\beta^R = \frac{8A_2}{\omega^2xA_1^2} \frac{V_L \sqrt{V_L^2 - V_R^2}}{2 \left(\frac{V_L}{V_R}\right)}$$

$$\beta^R = \frac{A_2}{xA_1^2}$$

where $\omega$ is the angular frequency; $V_L$, $V_T$, and $V_R$ are the longitudinal, transverse, and Rayleigh wave velocity respectively. Moreover, $\beta^R$ is used with an intention to segregate the material nonlinearity of the tested specimen from instrumentation nonlinearity. However, it is found in the later part of the present study that even these parameters in equations (3) and (4) yields only a relative value of the material nonlinearity.

Thus, it is noted that, a nonlinearity parameter for evaluating the true value of material nonlinearity for interrogation using Rayleigh waves is not available. This immediately defines the objective of the current research. A new parameter $\delta^R$ is therefore proposed in this study to evaluate the true value of material nonlinearity using the Rayleigh wave harmonics. Moreover, it was found in the authors’ previous study [7] using
lamb waves, that there exists a relationship between the material nonlinearity evaluated using the amplitude-based parameter, excitation frequency, tone burst cycles in the excitation signal, and propagation distance etc. Therefore, extensive experiments were carried out to study if there exist such relationships for quantifying material nonlinearity using Rayleigh waves.

In practice, it is very important to estimate the true value of material nonlinearity as opposed to the relative value of material nonlinearity because of the following reasons. First, the true value of the material nonlinearity can help to distinguish the in-service materials with reference to its intact state. Because it is convenient to compare the material nonlinearity evaluated using $\delta_R$ with that obtained from the physics-based nonlinearity parameter $\gamma_{phy}$ containing higher order elastic coefficients $[6–8]$. Second, it is also possible to verify the ultrasonic nonlinearity measurements by comparing the nonlinearity obtained using $\delta_R$ with that of $\gamma_{phy}$. Third, the evaluation of true nonlinearity induced from the evolution of dislocations due to fatigue or elastoplastic loading is extremely essential in predicting the remaining useful life (RUL). The relative estimate of the material nonlinearity cannot be used for predicting the RUL of the specimens. Fourth, evaluation of the elastic material nonlinearity using $\delta_R$ will help to understand the level of impurities present in the material when compared to its intact state. Fifth, the proposed parameter $\delta_R$ can help to distinguish the portion of nonlinearity that could have been induced by lattice elasticity or fatigue. This cannot be done using the relative nonlinearity parameters as shown in equations (2)–(4).

This study proposes a new parameter $\delta_R$ to estimate the true nonlinearity of a thick Al beam specimen using the amplitudes of harmonics of the Rayleigh wave. These amplitudes are obtained through the experiments carried out on an intact Al beam specimen. Thus, the nonlinearity in such a specimen is embedded by the lattice elasticity and reckoned using the higher order elastic coefficients. With this experimental investigation, it is confirmed that the accurate evaluation of material nonlinearity is highly dependent on the tone burst cycles in the excitation signal. It is also found that there is a small amount of contribution to the material nonlinearity parameter from the imaginary part of the shear wave component. After knowing this, several other possibilities were explored by carrying out extensive experiments that could impact the evaluation of $\delta_R$. There exists a relationship between the material nonlinearity evaluated using $\delta_R$, excitation frequency, tone burst cycles in the excitation signal, propagation distance etc for accurately evaluating the material nonlinearity. Finally, the material nonlinearity of the tested specimen was obtained using $\delta_R$ and experimental results and then compared with $\gamma_{phy}$. The deviation between the results was found to be minimal. The following section presents the formulation of amplitude-based parameter $\delta_R$ proposed to evaluate the material nonlinearity.

2. Amplitude based nonlinearity parameter for Rayleigh waves

The elastic strain energy function in terms of second and third order elastic constants of the material can be written as $[20, 22]$,

$$P(E) = \frac{1}{2} \lambda (tr(E))^2 + \mu tr(E^2) + \frac{1}{3} C (tr(E))^3 + B tr(E) tr(E^3) + \frac{1}{3} A (tr(E^3))$$  \hspace{1cm} (5)

Where $\lambda, \mu$ are the Lame’s constants, $E$ is the Lagrangian strain tensor, $tr()$ denotes the trace and $A, B, C$ are the third order elastic constants of the material. The first Piola–Kirchoff stress ($\sigma$) can be obtained as,

$$\sigma = F \frac{\partial P(E)}{\partial E},$$  \hspace{1cm} (6)

Where $F$ is the deformation gradient tensor. With a plain strain assumption, it is assumed that any displacement $u = (u_x, 0, u_z)$ that occurs is in the plane formed by the $x$ and $z$ unit vectors as shown in figure 1. The nonlinear stress-strain relationship can be written as follows,
The two coupled hyperbolic partial differential equations describing the wave propagation along $x$ and $z$ direction can be written as,

\[ \sigma_{xx} = (\lambda + 2\mu) \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) - 2\mu \frac{\partial u_x}{\partial x} + \left[ \left( \frac{3}{2} \lambda + 3\mu + A + 3B + C \right) \left( \frac{\partial u_x}{\partial x} \right)^2 \right] \\
+ \left( \frac{1}{2} \lambda + B + C \right) \left( \frac{\partial u_x}{\partial z} \right)^2 + \left( \frac{1}{2} \lambda + \mu + \frac{1}{4} A + \frac{1}{2} B + \left( \frac{\partial u_z}{\partial x} \right)^2 \right) \]

\[ \sigma_{zz} = \mu \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) + \left( \frac{1}{2} \lambda + A + B \right) \left( \frac{\partial u_x}{\partial x} \right)^2 + \frac{1}{2} B \left( \frac{\partial u_z}{\partial x} \right)^2 + \left( \lambda + 2B + 2C \right) \frac{\partial u_x}{\partial z} \frac{\partial u_z}{\partial z} \]

\[ + \left( \frac{2}{2} \lambda + 3\mu + A + 3B + C \right) \left( \frac{\partial u_z}{\partial z} \right)^2 \]  

where the Murnaghan and Landau & Lifshitz constants are related as $l = B + C$, $m = \frac{1}{2} A + B$, and $n = A$. In the classical theory of elasticity, a deformation may be termed as infinitesimal when the space derivatives of $u$ of a random particle in the medium are so small that their squares and products may be neglected. Our immediate interest is therefore the identification of the terms which do not create a significant impact on the analysis and can be neglected on a general basis for small deformation studies. In order to quantitatively evaluate the impact of the higher order terms present in the brackets $[I_s, ]_{s5s}$ and $[I_{s5s}]$ in equations (12)–(14) respectively, the normal strains $\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$ and $\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$ were evaluated using experiments with a further numerical validation. The arrangement for the measurement of strains and the obtained results are presented in Appendix A. The maximum value of normal strain recorded for $\varepsilon_{xx}$ and $\varepsilon_{zz}$ are $1.203 \times 10^{-6}$ and $1.192 \times 10^{-7}$ respectively as shown in figure A.2.

With these values of strains and the third order elastic constants as mentioned in table 1, the summation of the terms in brackets $[I_5]$ and $[I_{s5s}]$ in equations (12) and (14) are calculated to be $-1.38158 \text{ Pa}^3$ higher order terms and $-0.45015 \text{ Pa}^3$ higher order terms respectively. Based on these results, it can be said that the effect of these higher order terms on the stress components $\sigma_{xx}$ and $\sigma_{zz}$ is significantly small as compared to the linear terms. Note that, the shear strains will be even less than the normal strains making the terms in brackets $[I_5, ]_{s5s}$,
The arguments presented above are further validated using a numerical solution of linear and nonlinear stress-strain equations. The stress-strain curves for a linear material (Hooke’s Law) and non-linear material (Neo-Hooke’s and Murnaghan material model) are compared to further verify the effect of these higher order terms for the strain range under consideration. The results obtained are presented in Appendix B. It can be seen from figures B(a)–(c) that the difference between the linear and nonlinear stresses (represented by Neo-Hookean and Murnaghan material model) is extremely small for the strain range under consideration. For the maximum strains of $1.203 \times 10^{-6}$ and $1.192 \times 10^{-7}$ generated during the experiments, the stress level difference is less than 0.2% (between Neo-Hookean and Murnaghan model) and 0.08% (between Neo-Hookean and Murnaghan model) respectively for the x and z direction. This shows that the stress level difference is almost negligible and the nonlinear effect of Neo-Hookean material model is comparable to the Murnaghan (5 constants) material model. Therefore, the nonlinearity coming from the higher order terms can be even approximated by the Neo-Hookean parameters which depend on the second order elastic constants of the material. The presented arguments are in agreement with previous published result [20]. It can be concluded that the higher order terms from equations (12)–(14) are not bringing in significant effect for the strains induced due to the propagation of a Rayleigh surface wave in the experiments carried out in present study.

Furthermore, it may be noted that for strains as high as $10^{-3}$, the difference between the linear and nonlinear stresses is still very small. The deviation between the linear and nonlinear stresses may be more apparent for large strains of the order of $10^{-2}$. However, such strains are impossible to generate in the specimens during the propagation of ultrasonic waves. Thus, the higher order terms can be excluded from the analysis on a general basis, as even for strains as high as $10^{-3}$, the stress level difference is very small (figures B(a)–(c)). From a practical standpoint, it is therefore sensible, to drop these higher order terms with a sole objective of reducing the mathematical complexity without losing much of the physics, and to derive an expression which can be used practically for the evaluation of the material nonlinearity using Rayleigh waves.

The physics-basis for the generation of the nonlinearity in Rayleigh wave surface comes from the nonlinear self (L-L, S-S) and mixing (L-S) interactions of these propagating longitudinal (L) and shear (S) waves at the stress-free surface which results in a second harmonic wave. With the support of experiments and numerical simulations carried out in present study as well as in [20], the stress components acting on the surface of the plate are further simplified and written as follows [6, 8],

$$\sigma_{xz} = \mu \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right),$$

(15)

$$\sigma_{zz} = (\lambda + 2\mu) \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) + \delta^{R} \left[ \frac{\partial u_x}{\partial x} \frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial x} \frac{\partial u_x}{\partial z} + 2 \frac{\partial u_x}{\partial x} \frac{\partial u_z}{\partial z} \right] - 2\mu \frac{\partial u_x}{\partial x},$$

(16)

It is known from previous studies that, in case of an isotropic material, the second and third order elastic constants along the longitudinal and transverse wave directions are roughly the same [23]. Thus, basically there is no need of defining two different nonlinearity parameters for the two stress components acting on the surface of the plate as shown in equations (15) and (16). Furthermore, as analyzed before, the nonlinearity effect from the higher order terms of $\sigma_{xz}$ will be infinitesimal due to significantly small shear strains (figure B(c)). Therefore, equation (16) is employed as a legitimate nonlinear stress strain relationship ($\sigma_{zz}$), wherein, the nonlinearity parameter $\delta^{R}$ is introduced as a coefficient of the nonlinear term. The second order term of $(\epsilon_{xx} + \epsilon_{zz})$ is appended along with the parameter $\delta^{R}$ as its coefficient which makes the normal stress-strain equation nonlinear as shown in equation (16). The second order term of only $(\epsilon_{xx} + \epsilon_{zz})$ is considered as it consists of the normal strain $\epsilon_{zz}$ [6, 8]. Equation (16), therefore, accommodates the underlying physics responsible for the generation of a second harmonic wave propagating along a stress-free surface.
If the nonlinear stress-strain relationships from equations (12)–(14) are substituted into the equations (10) and (11), nonlinear version of the wave propagation along $x$ and $z$ direction can be obtained as follows,

$$
\rho \frac{\partial^2 u_x}{\partial t^2} - (\lambda + 2\mu) \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_z}{\partial x\partial z} \right) + 2\mu \left( \frac{\partial^2 u_z}{\partial z^2} \right) = T_1(u)
$$

$$
\rho \frac{\partial^2 u_z}{\partial t^2} - (\lambda + 2\mu) \left( \frac{\partial^2 u_x}{\partial z\partial x} + \frac{\partial^2 u_z}{\partial z^2} \right) + 2\mu \left( \frac{\partial^2 u_x}{\partial x\partial z} \right) = T_2(u)
$$

The left side of equation (17) is linear whereas the right side is nonlinear. In order to solve the displacement components $u_x$ and $u_z$ of such nonlinear equations, a perturbation approach, is employed. This approach involves finding an approximate solution, by firstly starting from a precise solution of an associated simpler problem. The displacement vector can be written as follows,

$$
u = u^0 + u^{nl}.
$$

where $u^0$ is the solution of the linear equation when $T_1(u) = T_2(u) = 0$ and $u^{nl}$ is a small correction arising as a result of material nonlinearity. Then to solve equation (17), $u = u^0 + u^{nl}$ should be substituted to the left side and $u = u^0$ should be substituted to the right side of equation (17). Here, $u^0$ being the solution of linear equation will disappear from the left side. Note that, in practical applications, the displacement vector $u$ is small and therefore $u^{nl}$ will be very small in comparison to $u^0$. In order to evaluate $u^{nl}$, firstly $u^0$ is obtained considering the equation of motion for an isotropic and linear elastic solid ($T_1(u) = T_2(u)= 0$ in equation (17)). The solution of the displacement vector $u^0$ can be obtained using the Helmholtz decomposition, which results into two wave equations, one for longitudinal wave and one for the shear wave as below [2],

$$
\frac{\partial^2 \phi}{\partial t^2} = V_0^2 \nabla^2 \phi,
$$

$$
\frac{\partial^2 \psi}{\partial t^2} = V_0^2 \nabla^2 \psi,
$$

Where, $u_{0x} = \phi_x + \psi_x$, $u_{0z} = \phi_z - \psi_z$, $V_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ and $V_T = \sqrt{\frac{\mu}{\rho}}$ being the longitudinal and shear wave velocity respectively. Assuming the harmonic wave motion of the form $\phi = G(z)e^{i(\omega t - kx)}$ and $\psi = H(z)e^{i(\omega t - kx)}$, where $k = \omega/\nu$, $\nu$ being the Rayleigh velocity, and substituting the above-assumed solution into equations (19) and (20), one may arrive,

$$
G(z)_{zz} = q^2 \ G(z) = 0,
$$

$$
H(z)_{zz} = s^2 \ H(z) = 0.
$$

where $q^2 = \frac{\omega^2}{c_L^2} - k^2$, and $s^2 = \frac{\omega^2}{c_T^2} - k^2$. The standard form of solution to the above equation (21) may then be written as,

$$
G(z) = Pe^{-qz} + Qe^{qz}, \quad \text{and} \quad H(z) = Re^{-sz} + Se^{sz},
$$

and $P, Q, R,$ and $S$ are the constants. Practically, $Q$ and $S$ should be equated to zero, since we cannot have the amplitudes increasing without bounds, therefore,

$$
\phi = Pe^{-qz}i(\omega t - kx), \quad \text{and} \quad \psi = Re^{-sz}i(\omega t - kx).
$$

Using the scalar ($\phi$) and vector ($\psi$) potential functions, the displacements can be written as below,

$$
u^0 = -[i(kP e^{-qz} + sR e^{sz}],
$$

$$
u^{nl} = [qP e^{-qz} + ikS e^{sz}].
$$

Note that for $u_{nl}$, the constants $P$ and $R$ are related (so that the boundary condition is satisfied) as follows,

$$
P = \frac{2\mu sk}{[(\lambda + 2\mu)(q^2 - k^2) + 2\mu k^2]} R = \frac{2\mu sk}{2\mu k} R.
$$

In order to obtain $u^{nl}$, these displacement components $u_{nl}^0$ and $u_{nl}^z$ from equations (24) and (25) should be substituted into the right side of equation (17). Note that, the terms in $T_1(u)$ and $T_2(u)$ include either the squares of the normal strains or the product of normal and shear strains with the Murnaghan ($l, m,$ and $n$) or Landau & Lifshitz constants ($A, B,$ and $C$) as their coefficients. For the strain range under consideration (from experiments) and the numerical results presented in Appendix B, it is impractical to evaluate the exact expressions for $u^{nl}$ unless the terms bring significant impact to the analysis. Furthermore, the mathematical equations will be cumbersome to solve for the constants $P$ and $R$ using the boundary conditions as the terms will also include the product of these two constants. Previous studies [7, 20] have also demonstrated that, when a linear and nonlinear elastic solids are sonified with an ultrasonic wave, the displacement of the particles in time domain will be roughly same for a linear and nonlinear material due to the arguments presented before. For the strains, that
have been estimated using experiments carried out in present study, it is therefore sensible, to use the
displacements in equations (24), (25) to evaluate the secondary displacements \(u^u\) and \(u^u\), but the constants \(P\) and \(R\) in equations (24), (25), evaluated for the corresponding higher order terms of the nonlinear stress
components (second order in this case). This will be a good approximation since \(u^u \ll u\). The components
\(E_{xx_1}, E_{xx} \) and \(E_{zz} \) of \(E\) as in equation (5), can be written as follows,

\[
E_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u_x}{\partial x} \right)^2 + \left( \frac{\partial u_x}{\partial z} \right)^2 \right]
\]

(26)

\[
E_{xx} = \frac{1}{2} \left[ \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right)^2 + \left( \frac{\partial u_x}{\partial x} \right)^2 + \left( \frac{\partial u_z}{\partial z} \right)^2 \right]
\]

(27)

\[
E_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2} \left[ \left( \frac{\partial u_z}{\partial z} \right)^2 + \left( \frac{\partial u_z}{\partial z} \right)^2 \right]
\]

(28)

The displacements \(u^u\) and \(u^u\) from equations (24)–(25) can be substituted into equations (26)–(28) to obtain the
nonlinear version of the strains, which can be eventually used to evaluate the nonlinear stresses (equations (15)
and (16)). However, substituting the equations (24), (25) directly into equations (26)–(28), to evaluate the
nonlinear stresses may not be feasible from a mathematical perspective. Furthermore, the quadratic terms in
equations (26)–(28) is either a product of normal and shear strain or the squares of normal strain, and for the
strain range under consideration, these terms will be significantly smaller, and can be therefore dropped from
the analysis to reduce the mathematical complexity. Thus, it is important to note that, even if these higher order
terms are included, the nonlinear effect contributed from these terms will be significantly smaller. Thus, the
nonlinear stress components previously assumed as in equations (15), (16) are obtained using equations (24),
(25) as follows,

\[
\sigma_{xx} = \mu \left[ 2iPqk e^{-i(\omega t - kx)} \right] + R(s^2 + k^2) e^{-i(\omega t - kx)}
\]

(29)

and,

\[
\sigma_{zz} = [\lambda + 2\mu](q^2 - k^2) Pe^{-i(\omega t - kx)} - 2iPsk RE^{-i(\omega t - kx)}
\]

(30)

In an ideal case, there would have been higher order terms in equations (29) and (30) whose physical existence
would be infinitesimal. Since the Rayleigh wave and the generated second harmonic will propagate along a
stress-free surface, which implies \(\sigma_{xx} = \sigma_{zz} = 0\) at \(z = 0\), gives the constants \(P\) and \(R\) as,

\[
P = \frac{(k^2 + s^2)(\lambda + 2\mu)\delta e^{-i(\omega t - kx)}}{4i\mu q k^2 - (k^2 + s^2)[(\lambda + 2\mu)(q^2 - k^2) + 2iPsk]}
\]

(31)

\[
R = \frac{2i\mu q k^2 - (k^2 + s^2)[(\lambda + 2\mu)(q^2 - k^2) + 2iPsk]}{4i\mu q k^2 - (k^2 + s^2)[(\lambda + 2\mu)(q^2 - k^2) + 2iPsk]}
\]

(32)

Based on the arguments presented before, substituting these constants equations (31) and (32) into
equations (24) and (25), and taking into account only the real part, the displacement components indicating the
longitudinal and shear wave contributions for the secondary Rayleigh wave can be written as,

\[
u^u_{xx} = i\delta \left( \begin{array}{cc}
\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \\
\frac{\partial u_z}{\partial z}
\end{array} \right) \left( \lambda + 2\mu \right) \xi^2 \left[ 2kq - k(k^2 + s^2) \right] \times \left[ \cos(\omega t - kx) \right]^2,
\]

(33)

\[
u^u_{zz} = \delta \left( \begin{array}{cc}
\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \\
\frac{\partial u_z}{\partial z}
\end{array} \right) \left( \lambda + 2\mu \right) \xi^2 \left( 2kq - k(k^2 + s^2) \right) \left[ \cos(\omega t - kx) \right]^2,
\]

(34)

Where,

\[
\xi^2 = 4\mu q k^2 - \frac{(k^2 + s^2)}{s}[\lambda + 2\mu] q^2 - \lambda k^2,
\]

(35)

and

\[
\xi^2 = 4\mu q k^2 - \frac{(k^2 + s^2)}{s}[\lambda + 2\mu] q^2 - \lambda k^2.
\]

(36)

Thus, for the strain range under consideration, the solution for the displacement components \(u_x\) and \(u_z\) in
equation (17) can be obtained as \(u_x = u^u_{xx} + u^u_{zz}\) and \(u_z = u^u_{zz} + u^u_{zz}\). Finally, the effective displacement of
the second harmonic wave in the form of resultant \(u^u\) is obtained from \(u^u_{xx}\) and \(u^u_{zz}\) in equations (33) and (34) as
[6, 8].
In previous studies, the nonlinear term 

\[ \delta R = \delta \left( \frac{\partial u_x}{\partial x} \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \frac{\partial u_z}{\partial z} + \frac{\partial u_y}{\partial y} \frac{\partial u_y}{\partial y} \right) \sqrt{\left( \zeta^{R}_{2} \right)^2 - \left( \zeta^{R}_{1} \right)^2} \left[ \cos (\omega t - kx) \right]^2, \]  

(37)

Where,

\[ \zeta^{R}_{1} = \left\{ \frac{\lambda + 2\mu}{\xi^4} \right\} 2\eta k - k(k^2 + s^2) \]  

(38)

\[ \zeta^{R}_{2} = \left\{ \frac{\lambda + 2\mu}{\xi^4} \right\} [2k^2 - (k^2 + s^2)] \]  

(39)

It is evident from equations (38) and (39), that the contribution to \( \zeta^{R}_{1} \) is dominant from the longitudinal wave, whereas, for \( \zeta^{R}_{2} \) is mainly from the shear wave. The nonharmonic term is neglected after the expansion of equation (37), which yields a secondary solution as,

\[ u^R = \delta R \left( \frac{\partial u_x}{\partial x} \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \frac{\partial u_z}{\partial z} + \frac{\partial u_y}{\partial y} \frac{\partial u_y}{\partial y} \right) \sqrt{\left( \zeta^{R}_{2} \right)^2 - \left( \zeta^{R}_{1} \right)^2} \cos (\omega t - kx). \]  

(40)

Finally, \( u \) can be written with its linear and nonlinear part as,

\[ u = \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) \cos (\omega t - kx) + \frac{\delta R}{2} \left( \frac{\partial u_x}{\partial x} \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \frac{\partial u_z}{\partial z} + \frac{\partial u_y}{\partial y} \frac{\partial u_y}{\partial y} \right) \sqrt{\left( \zeta^{R}_{2} \right)^2 - \left( \zeta^{R}_{1} \right)^2} \times \cos (\omega t - kx). \]  

(41)

The above equation (41) is then compared with a standard solution [6, 8],

\[ u = A_1 \cos(\omega t - kx) + A_2 \cos(2\omega t - kx) + ... \]  

(42)

yields, \( A_1 = \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) \) and \( A_2 = \frac{\delta R}{2} \left( \frac{\partial u_x}{\partial x} \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \frac{\partial u_z}{\partial z} + \frac{\partial u_y}{\partial y} \frac{\partial u_y}{\partial y} \right) \sqrt{\left( \zeta^{R}_{2} \right)^2 - \left( \zeta^{R}_{1} \right)^2} \). With this, the parameter \( \delta R \) related to the interaction between material nonlinearity and Rayleigh waves is,

\[ \delta R = \left[ \frac{A_2}{A_1} \right] \frac{2}{\sqrt{\left( \zeta^{R}_{2} \right)^2 - \left( \zeta^{R}_{1} \right)^2}}. \]  

(43)

Note that, the above developed equation for \( \delta R \) is based on an asymptotic solution and can be used for wave propagation problems with small deformations. For simplicity, we denote the feature function as \( F(\zeta^{R}) = \frac{2}{\sqrt{\left( \zeta^{R}_{2} \right)^2 - \left( \zeta^{R}_{1} \right)^2}} \) in equation (43). Thus, equation (43) takes the following form,

\[ \delta R = \left[ \frac{A_2}{A_1} \right] F(\zeta^{R}). \]  

(44)

In previous studies, the nonlinear term \( A_2/A_1^2 \) (equation (2)) arising from the classical development of the nonlinearity parameter \( \beta \) is shown to account for the nonlinearity induced in the material as a function of propagation distance, either from the lattice anharmonicity (at the pristine state) or from the formation of dislocation monopoles and dipoles due to fatigue or elastoplastic loading of the material. Therefore, the feature function arising as a result of Rayleigh wave propagation \( F(\zeta^{R}) \) should be independent of these effect. Even though the feature function of longitudinal waves \( \left( 8/k^3 \right) \) as in equation (1) is independent of any second and third order elastic constants of the material, it can evaluate the nonlinearity at the pristine state as well as the in-service state. With that said, the feature function remains same for any state of material (at the desired propagation distance), only the term \( A_2/A_1^2 \) changes which accounts for the nonlinearity effects. That is, it is not required to evaluate the feature function as a function of the second and third order elastic constants. This will be a case when the strain energy function (equation (5)) is dependent on the microstructure. Note that, this will be a tedious process to evaluate the third order elastic constants for every stage of the material.

Above all, previous studies have explicitly demonstrated that, with this change in third order elastic constants the term \( A_2/A_1^2 \) changes. This simply means that the effect of lattice anharmonicity at the pristine state and formation of dislocations at the fatigued state will already be accounted by \( A_1 \) and \( A_2 \) which can be obtained experimentally on the current state of the material. Although in nonlinear ultrasonics, the change in linear elastic properties (\( \lambda \) and \( \mu \)) is smaller when compared to the change in third order elastic constants, \( A, B, C \), however, this should be already accounted by \( A_1 \) and \( A_2 \) and not the feature function. This is also verified in authors’ previous study [5] wherein \( A_1 \) and \( A_2 \) are shown to account for the increase in material plasticity and the Lamb wave-based feature function remains constant for different loading. Here, \( F(\zeta^{R}) \) is a feature function defined for Rayleigh waves, and is a constant value at a specific frequency. Its magnitude depends on the Lamé’s constants of the material, frequency, wavenumber, and Rayleigh wave velocity. Equation (44) gives the true value of material nonlinearity for any state of material, using the amplitudes of harmonics of the Rayleigh waves obtained either through the experiments or simulations. The following section discusses the experimental verification of the proposed amplitude-based parameter.
3. Experimental study

3.1. The specimen
The specimen used in this study is a $1000 \times 70 \times 20$ cu.mm (length $\times$ width $\times$ thickness) in size and made of Al 1100. Firstly, the specimen was gently hand polished to remove any dust particles present on the specimen surface and to reduce the impact of surface conditions on the Rayleigh surface waves. There were no scratches obtained during the polishing process. If there were some scratches not visible, they will be surely very small in comparison to the wavelength of the Rayleigh wave for the inspection frequencies considered here, and will neither affect the nonlinearity nor the attenuation of the Rayleigh waves. A 20 mm diameter with 2 mm thickness Piezoelectric-Wafer (PW) transducer was bonded on the specimen surface at a location as shown schematically in figure 2. Retro-reflective tapes were applied on the specimen surface to enhance the optical sensitivity and receive the waveform with a high signal-to-noise ratio (SNR).

3.2. Material properties
The material under consideration is an Al 1100 beam specimen having a rectangular cross-section. The Modulus of Elasticity, poison’s ratio, density, and Rayleigh wave velocity are shown in table 2. The properties of the material are similar to that used in authors’ previous studies while investigating the material nonlinearity of a thin Al 1100 plate using Lamb waves [8]. However, the specimens differ in their dimensions and shape. That is, the specimens were fabricated with a dog-bone shape and thickness was 1.6 mm which ensures Lamb wave propagation at the considered frequency. Here, the specimen thickness is roughly 12 times more than that used in [8], and therefore Rayleigh waves will be dominantly generated in such a thick specimen.

The true material nonlinearity at the intact state of the test specimen induced from the lattice elasticity for Al 1100 is evaluated as $17\%$ [8]. The phase velocity dispersion curves are shown in figure 3 and the inspection frequencies used in the experimental study are highlighted. Here, 0.35 MHz excitation frequency is an onset frequency used in previous studies while investigating the material nonlinearity of a thin Al 1100 plate using Lamb waves [8]. However, the specimens differ in their dimensions and shape. That is, the specimens were fabricated with a dog-bone shape and thickness was 1.6 mm which ensures Lamb wave propagation at the considered frequency. Here, the specimen thickness is roughly 12 times more than that used in [8], and therefore Rayleigh waves will be dominantly generated in such a thick specimen.

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3.3. Variation of $\zeta^R$ and $F(\zeta^R)$ with frequency for Al 1100
In this subsection, the dependency of the feature function $F(\zeta^R)$ on the frequency is studied and several observations are noted. Figures 4 and 5 are plotted using the properties of Al 1100 as specified in table 2. The feature function $F(\zeta^R)$ is dependent on $\zeta^R_1$ and $\zeta^R_2$ emerging because of longitudinal and shear wave contribution respectively as shown in equation (44).

In figure 4, it can be seen that $\zeta^R_1$ is a real number for all the frequencies up to 5 MHz, whereas $\zeta^R_2$ has a real as well as positive imaginary part. In case of $\zeta^R_2$ which is from the shear wave contribution, the real part vanishes, whereas, the imaginary part follows the same pattern for different frequencies like the longitudinal wave component from the positive side as shown in figure 4. Both the components i.e. $\zeta^R_1$ and Imag($\zeta^R_1$) change dramatically at lower frequencies, whereas, the change is seen to be minimal after around 2 MHz. Moreover, it can be further seen from figure 5 that the imaginary part of $\zeta^R_1$ resulting from the shear component does contribute to lower the factor $(2/\sqrt{1(\zeta^R_1)^2 - (\zeta^R_2)^2})$ in equation (44), by several hundred in magnitude. The reduction in magnitude is 19.56% at 0.35 MHz and 19.59% at 1.0 MHz due to contribution from the imaginary part of the shear wave component. Thus, the reduction is almost constant with increasing frequency. Therefore, it is clear that, if only $(2/\sqrt{1(\zeta^R_1)^2})$ (from the longitudinal wave) is considered, this may result in an incorrect
value of the nonlinearity parameter. It is also seen that the feature function $F(\zeta^R)$ increases linearly with frequency. The variation between $(2 / \sqrt{(\zeta_1^R)^2})$ and $(2 / \sqrt{(\zeta_2^R)^2})$ is less at smaller frequencies, whereas, it increases linearly with increasing frequency. It is known from the previous study [24] that the longitudinal wave contributes dominantly to the nonlinearity parameter in an isotropic material. Thus, this study also corroborates that the longitudinal wave component contributes dominantly to the nonlinearity parameter as compared to the shear wave component which is in accordance with the theoretical analysis of [24]. The following section discusses the description of the experimental setup used in the present study.

3.4. Description of the experimental setup
A tone burst waveform is produced by the function generator (Keysight 33600A) and amplified by the RITEC RPR-4000 system. This amplified signal having high-voltage is allowed to pass through a RITEC RT-150 ohm load to compensate for the electrical impedance mismatch between the PW transducer and the RITEC amplifier.
The propagating Rayleigh waves and its reflections are sensed using a 1D Laser vibrometer (Class 2 laser with wavelength of 633 nm) and recorded by an oscilloscope (Tektronix MD04054C) having the maximum sampling rate of 2.5 GS \( s^{-1} \) and averaged 512 times to increase the SNR before transferring to a personal computer for the data analysis and signal processing. Measurements are taken with a uniform increment of 10 mm from the exciter location. The complete view of the components used in the experimental setup is displayed in figure 6.

The results and discussions are presented in the following section.

4. Results and discussions

4.1. Influence of the propagation distance and frequency bandwidth on \( \delta^R \)

In this subsection, the variation of \( \delta^R \) with the wave propagation distance and frequency bandwidth are studied simultaneously. For this purpose, experiments were carried out for the different tone burst cycles (N) in the excitation signal at 0.65 MHz and the data was recorded at different propagation distances. One of the signals
recorded at a distance of 60 mm from the exciter and for $N = 20$ is displayed in figure 7. It can be seen that the time of arrival ($t_R$) of the first wave packet in figure 7 corresponds to the incident Rayleigh wave (direct wave) as confirmed according to $0.06/2876.7 \approx 2.1 \times 10^{-5}$ s. Note that, reflections of the incident Rayleigh wave will appear in the time domain waveform after the arrival of the direct wave. Because the PZT is an Omni-directional wave exciter, some reflections are expected in the time domain signals.

However, as seen from figure 7, they all will arrive later than the direct wave. The second wave packet in figure 7 corresponds to the reflection of the direct wave from the left end referring to figure 2. The Fast Fourier Transform (FFT) of the originally received waveform is displayed in figure 8. However, these amplitudes of the harmonics depicted in figure 8 cannot be used in equation (44) to evaluate the material nonlinearity. This is because the reflected wave packets also contribute to amplitudes of the harmonics. Therefore, the incident wave packet with 20 cycles is selected and highlighted with a rectangle in figure 7. This selected wave packet is the
incident Rayleigh wave and is shown in figure 9. The corresponding FFT of this direct wave is shown in figure 10. These amplitudes correspond to the fundamental and second harmonics and are used in equation (44) to evaluate the true material nonlinearity.

The amplitudes $A_1$ and $A_2$ are similarly obtained for all the propagation distances and used in equation (44). Figures 11–14 show the results of $\delta^R$ when the tone burst cycles are varied from 5 to 20. The true material nonlinearity at the intact state induced by lattice elasticity is reckoned using the higher order elastic coefficients and is denoted by $\gamma_{phy}$. This value of $\gamma_{phy}$ for Al 1100 is evaluated in the author’s previous study as 17 [8]. A horizontal line intersecting the ordinate at $\gamma_{phy}$ is plotted in figures 11–14, to examine the relationship between $\delta^R$ and $\gamma_{phy}$. It is found that $\delta^R$ consists of several peaks (crests) and nadir (troughs) between a wave propagation distances of 50 mm and 210 mm. However, only the first peak in $\delta^R$ reaches close to $\gamma_{phy}$ which is shown by a ‘dashed’ line in figures 11–14.
Moreover, in the case of $N = 20$, $\delta^R \approx \gamma_{\text{phy}}$ at first peak. This signifies that $\delta^R$ estimated over different propagation distance reaches $\gamma_{\text{phy}}$ at its first peak when the tone burst cycles in excitation signal are 20. Thus, the difference between the two horizontal lines evaluated using $\delta^R$ at its first peak and $\gamma_{\text{phy}}$ is more for $N = 5$ and significantly less for $N = 20$. For convenience, the term ‘peak distance’ will be used from now to refer the distance at which the amplitude based nonlinearity parameter $\delta^R$ attains its first peak.

Recently, Masurkar et al [7] found that the number of tone burst cycles in excitation signal is critical in evaluating the true value of material nonlinearity using Lamb waves. This is because the possibility of phase mismatch between the Lamb wave harmonics is more pronounced for a small value of $N$. It was shown that [7] in case of low-frequency and approximate matching of phase velocity of $S_0 - S_0$ Lamb mode pairs, $N \geq 40$ is required whereas, for a stringent matching condition of the phase and group velocity of $S_1 - S_1$ Lamb modes, $N \geq 10$ is required [7]. Thus, for Rayleigh waves, it has been found in the present study that $N \geq 20$ is required.

Figure 11. Variation of $\delta^R$ with propagation distance for 5 tone burst cycles in excitation signal.

Figure 12. Variation of $\delta^R$ with propagation distance for 10 tone burst cycles in excitation signal.
Despite the Rayleigh wave harmonics travel at the same phase velocity, a minimum \( N \) is still required for evaluating \( \delta^R \) accurately. Moreover, figures 11–14 show that the influence of tone burst cycles on \( \delta^R \) is not as significant as that was observed for a low-frequency and approximate matching of phase velocity of \( S_0 - S_0 \) Lamb mode pairs on \( \gamma_{amp} \) [7]. Nevertheless, it is extremely essential to evaluate an accurate value of material nonlinearity for the prediction of Remaining Useful Life (RUL) of fatigued specimens. It is also observed that the remaining peaks do not reach \( \gamma_{phy} \) and the relationship between these two parameters at the second peak is illustrated in figures 11–14. This is because the fundamental wave attenuates and loses its energy in continuous energy transfer to the second harmonic wave during its propagation to fulfill the cumulative effect [6–8]. In an ideal case, it may be expected that all the peaks in \( \delta^R \) reach \( \gamma_{phy} \). Furthermore, no obvious relationship can be derived between the consecutive peaks of \( \delta^R \) in terms of the wave propagation distance. However, it is interesting to see that the distance at which these peaks occur do not change with the increase in \( N \). This observation is in accordance with the evaluation of the material nonlinearity using the nonlinearity parameter \( \gamma_{amp}^s \) proposed for

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**Figure 13.** Variation of \( \delta^R \) with propagation distance for 15 tone burst cycles in excitation signal.

**Figure 14.** Variation of \( \delta^R \) with propagation distance for 20 tone burst cycles in excitation signal.
Lamb waves [7]. That is, for example, the first peak in $\delta^R$ is reached at 60 mm from the exciter location irrespective of change in N. Furthermore, the number of peaks obtained over the wave propagation distance is highly dependent on the composition of the material and the excitation frequency.

It may also be noted that, in previous studies, only the linear increase of the relative nonlinearity parameter $\beta'$ was studied. However, in this study, it has been found that the nonlinearity parameter $\delta^R$ increases to its first maximum until it covers a certain distance (crest), decreases after this distance until it reaches its first minimum (trough), and again increases. This process persists until the full attenuation of the fundamental wave. This phenomenon for Rayleigh waves is not yet reported in the literature. This shows that the behavior of the nonlinearity parameter $\delta^R$ of Rayleigh waves as shown in figures 11–14 is similar to the parameter $\gamma_{amp}$ of Lamb waves [6, 7].

It was found that the first peak in $\delta^R$ is reached at 60 mm for an excitation frequency of 0.65 MHz. After knowing this distance, a qualitative and quantitative comparison of the nonlinearity parameters given by equations (2)–(4) and the proposed equation (44) is carried out and presented in figure 15. It is seen that the value of material nonlinearity given by the relative nonlinearity parameters is quite far away from $\gamma_{phy}$ as shown in figure 15(a). The deviation between the material nonlinearity evaluated using $\delta^R$ and $\gamma_{phy}$ for 20 cycles is close to zero as shown in figure 15(b).

Thus, with the use of amplitudes of harmonics of the Rayleigh waves from the experiments, it was confirmed that $\delta^R$ is more accurate than the relative nonlinearity parameters for studying material nonlinearity in beam structures. Moreover, $\delta^R$ can help characterize the in-service beam structure with respect to its intact state. For example, when the elastic nonlinearity reflecting the original inherent nonlinearity is known, any further increase in this nonlinearity would have been caused by the action of repetitive fatigue or elastoplastic loading. As it is known from previous studies [6] that, with an increase in external loading, the population of dislocations increases, which will eventually increase the nonlinearity value. Hence, knowing the initial inherent value will help in determining whether the later measured nonlinearity is solely due to lattice elasticity or is a combined effect of lattice elasticity and plasticity. That is, any measured nonlinearity value that is higher than this initial inherent value should be caused by the increase of cyclic loads. This is only possible if one is able to estimate the true nonlinearity of the material. Furthermore, evaluation of the true nonlinearity value is extremely essential as far as predicting the remaining useful life (RUL) of an in-service specimen is concerned. In contrast, evaluation of the relative value of material nonlinearity cannot be used for obtaining the RUL. Through this proposed parameter $\delta^R$, it may also be possible to study the effect of fatigue or elastoplastic loading separately. The following subsection discusses the reproducibility and repeatability of the experimental results.

### 4.2. Verifying the reproducibility and repeatability of experimental results

In this subsection, extensive experiments were conducted to verify the reproducibility and repeatability of the experimental results. Firstly, to verify whether the results are reproducible, the experiments were carried out with 20 cycles in tone burst for an excitation frequency of 0.35 MHz, 0.6 MHz, 0.65 MHz, and 1 MHz. The time

![Figure 15. Qualitative and Quantitative comparison of relative nonlinearity parameters ($\beta'$, $\beta^R$, $\beta_{amp}$) and $\delta^R$.](image-url)
domain responses recorded at 550 mm and 350 mm for an excitation frequency of 0.350 MHz and 0.6 MHz respectively are shown in figures 16(a) and (b). The corresponding FFT of the selected part of the time domain waveform representing the incident Rayleigh wave is shown in figures 16(c) and (d). Note that, these are the distances at which $\delta_R$ attains its first peak for 0.35 MHz and 0.6 MHz excitation frequencies respectively. For brevity, the time domain waveform and its FFT for 1 MHz excitation frequency is not shown here.

Figure 17 shows the evaluated value of $\delta_R$ for different excitation frequencies and data recorded at their respective peak distances. It can be seen that the error between the material nonlinearity evaluated using $\delta_R$ and $\gamma_{phy}$ keeps on decreasing with increase in excitation frequency. From the phase velocity dispersion curve as shown in figure 3, it can be said that 0.35 MHz excitation frequency is an onset, for the two fundamental Lamb wave modes to merge into a Rayleigh wave. Therefore, a frequency of 0.35 MHz was intentionally selected to study the deviation between $\delta_R$ and $\gamma_{phy}$. The deviation between $\delta_R$ and $\gamma_{phy}$ at the inspection frequencies of 0.65 and 1 MHz is seen to be minimal. The wavelength of the incident Rayleigh wave is about 22.13% and 14.38% of
the beam thickness at 0.65 MHz and 1 MHz respectively which ensures efficient Rayleigh wave generation and propagation.

It is also evident from figure 17, that the distance at which the first peak is reached in $\delta^R$ decreases with an increase in excitation frequency. The peak distance is 550 mm at 0.35 MHz, whereas, it is just 20 mm for 1 MHz. This shows that, with higher excitation frequency, the second harmonic wave is capable to build its full cumulative effect within shorter propagation distances. Here, by full cumulative effect, it is meant that the distance up to which, the harmonics of Rayleigh wave travels exactly in the same phase. Thus after this distance, due to the initiation of phase mismatch, the secondary wave amplitude and eventually $\delta^R$ will show a decreasing trend. Again, after some distance, the Rayleigh wave harmonics will travel exactly in the same phase, leading to the increasing amplitude of the second harmonic, and eventually, $\delta^R$ will have an increasing trend. This process persists until the full attenuation of the fundamental wave. However, this conjecture needs further investigation. Thus, it is clear that, as $\gamma_{phy}$ is independent of the wave propagation distance [6–8], it yields a global value of material nonlinearity. Therefore, $\delta^R$ reaches the value of $\gamma_{phy}$ when the second harmonic wave builds its full cumulative effect at the peak distance. This phenomenon was seen in figures 11–14. Another observation that could be noticed in figure 17 is the decreasing trend of $\beta'$ with an increase in excitation frequency which is similar for the relationship between excitation frequency and the propagation distance. This makes sense, as the feature function $F(\zeta^R)$ increases linearly with increase in frequency as shown in figure 5, and therefore the ratio $\beta' = A_2/A_1^2$ will decrease, to yield a constant value of $\delta^R$. Since as per the proposed equation (44), the product of the feature function $F(\zeta^R)$ and $\beta'$ yields $\delta^R$.

Next, experiments were conducted to verify whether the measurements are repeatable. For this purpose, three measurements were carried out at the same peak distances of 550 mm, 350 mm, 60 mm, and 20 mm at the respective excitation frequencies of 0.35 MHz, 0.6 MHz, 0.65 MHz, and 1 MHz. Note that, between each measurement, all the components of the experimental setup shown in figure 6 were switched off and connections were removed. For taking the measurements, connections were done again, all components were switched on, and data was recorded. The results with these new measurements are presented in figure 18. It can be seen that the variation between the values of $\delta^R$ with the 3 measurement sets is marginal confirming that the measurements are repeatable. This repeatability is observed in the case of all the inspection frequencies.

Furthermore, a possibility of variation in the value of $\delta^R$ with a different PZT (as exciter) was also exploited. This will also confirm, if the bonding agent by itself, has any significant influence on the amplitudes of harmonics of Rayleigh wave, and eventually on $\delta^R$. For this purpose, the existing PZT was carefully removed, and the residue of the glue was removed by hand polishing the surface. In this process, no obvious cracks were seen on the surface, where the PZT was bonded. The evaluation of $\delta^R$ with the measurement from this new PZT at 1 MHz is shown in figure 18. Note that, a single measurement was carried out at the peak distance of 20 mm from the exciter location for 1 MHz excitation frequency. It can be seen that the value of material nonlinearity evaluated using $\delta^R$ is close to $\gamma_{phy}$. The following subsection deals with the examination of the instrumentation nonlinearity.
4.3. Influence of increasing transducer input voltage on $\delta_R$

It is well known that for a nonlinear ultrasonic measurement setup, the reliability of measurement system should also be verified, since the experimental components, the coupling medium, and the transducer itself would bring in the nonlinearity. The nonlinear effects in the material are generally very small and difficult to detect. Due to this, the nonlinear effects which are introduced by the experimental setup itself can overwhelm the nonlinearity induced by the material. In order to examine the instrumentation nonlinearity, measurements were performed with an increase in the input voltage fed to the transducer for a total of four propagation distances. A significant increase of the nonlinearity parameter $\delta_R$ with increasing voltage would imply a strong influence of the instrumentation nonlinearity on the measured results. This is because, with an increase in the input voltage fed to the transducer, the nonlinearity will further intensify, consequently causing an increase in the non-linearity parameter [8]. The experimental results along with $\delta_R$ presented in figure 19 show that the nonlinearity parameter $\delta_R$ remain almost constant with increasing input voltage. The transducer input voltage was varied from 175 to 400 volts. These voltage values were measured from the front panel of the RITEC system as shown in figure 6.

The results confirm that the material nonlinearity evaluated with experimental results and $\delta_R$ is solely because of the nonlinearity of the tested beam structure. The curve overlapped onto the figure 19 shows the change of nonlinearity parameter $\delta_R$ with the measurements taken at a uniform increment of 5 mm for 0.65 MHz excitation frequency and spline interpolated. The conclusions emerging from the present study are outlined in the following section.

5. Conclusions

In the present study, a new parameter $\delta_R$ is proposed for interrogating the material nonlinearity of Al 1100 beam having a rectangular cross-section using Rayleigh waves. This parameter $\delta_R$ can be employed to distinguish the tested specimens at their ‘intact state’ and can also be useful for predicting the Remaining Useful Life (RUL) at their ‘fatigued state’. This is because $\delta_R$ can evaluate the true material nonlinearity, in contrast to the relative value of material nonlinearity yielded by the conventional nonlinearity parameter $\beta'$. To verify the formulation of $\delta_R$, extensive experiments were conducted on the specimen. The experimental results along with $\delta_R$ unveil some of the critical aspects and relationships in evaluating the true material nonlinearity. They are summarized as follows:

1. There is a small amount of contribution to the amplitude-based nonlinearity parameter from the imaginary part of the shear wave component.
2. The Qualitative and Quantitative comparison of the parameters $\beta'$, $\beta^R$, $\delta^R$, and $\delta^R$ show that the trend of all the nonlinearity parameters is similar with respect to the peak distance, however, $\delta^R$ differs from the relative nonlinearity parameters in two perspectives: $\delta^R$ is a dimensionless index, and it gives the true nonlinearity induced in the material.

3. The evaluation of $\delta^R$ with 20 cycles in tone burst gave a better prediction of the material nonlinearity.

4. The first peak in $\delta^R$ is not a unique one and changes with excitation frequency. With that said, the increase in excitation frequency will decrease the peak distance. Thus, the relationship between excitation frequency and peak distance is inversely proportional. Nevertheless, the condition of $\delta^R$ attaining the value of $\gamma_{phy}$ at its first peak still holds true similar to Lamb waves.

5. There is no relationship observed between the peak distance and the tone burst cycles in the excitation signal. Furthermore, all the peaks in $\delta^R$ bear the same distance even though the tone burst cycle changes.

6. The frequency selection is critical, as it is found that with an increase in excitation frequency from 0.35 MHz to 1 MHz, the accuracy in evaluating $\delta^R$ increases as confirmed with $\gamma_{phy}$. However, this is completely related to the efficient Rayleigh wave generation and propagation.

7. There is no obvious relationship found between the consecutive peaks of $\delta^R$ in terms of wave propagation distance since this highly depends on the attenuation of the wave in the inspected specimen and its composition.

Acknowledgments

The work described in this paper was fully supported by a grant from City University of Hong Kong (Project No. 7005120) and a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. [T32-101/15-R]).

Declaration of conflicting interests

The author(s) declare no conflicts of interest with respect to the publication, authorship, and/or research reported in this article.

Appendix A. Measurement of strain field induced on the surface of a Al 1100 beam due to propagation of Rayleigh surface wave using a three dimensional (3D) scanning Laser Doppler vibrometer (SLDV)

The strain field generated on the surface of the specimen is evaluated using a 3D-SLDV as shown in the figure A.1. Estimation of strain field using a 3D-SLDV is explained in [25] and will not be repeated here. The measurement location and scan points defined for obtaining the strain field is shown in the figure A.1.
The figures A.2(a) and (b) show the normal strains $\varepsilon_{xx}$ and $\varepsilon_{zz}$ generated on the surface of the material due to propagation of a Rayleigh wave having 20 cycles in a tone burst with an excitation frequency of 0.65 MHz. The strain field is almost uniform across the user-defined measurement area which is intentionally selected close to the exciter, so as to capture the maximum amount of induced strain.

**Appendix B. Numerical evaluation of the linear and nonlinear stresses for a thick Al beam specimen with rectangular cross section**

In this section, the linear and nonlinear stresses are evaluated using the numerical simulations. The simulations are carried out in a software COMSOL Multiphysics for an excitation frequency of 0.65 MHz with 20 cycles in the tone burst excitation signal. The excitation amplitude is set to 1 MPa. The mesh size and time steps are chosen according to [7] as $(1/20) \lambda_{\text{min}}$ and $(1/20) f_{\text{max}}$, where $\lambda_{\text{min}}$ is the smallest wavelength to be analyzed and $f_{\text{max}}$ is the maximum frequency of interest. The results are presented in figures B(a)–(c).
Figure B. Stress-Strain curves (a) normal stress along x direction (b) normal stress along z direction (c) Shear stress along the x-z plane.

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