Existence and smoothness of the solution to the Navier-Stokes equation

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Abstract. A fundamental problem in analysis is to decide whether a smooth solution exists for the Navier-Stokes equations in three dimensions. In this paper we shall study this problem. The Navier-Stokes equations are given by:

\[ u_t(x, t) - \rho \Delta u_i(x, t) - u_j(x, t) \cdot \nabla u_i(x, t) + \nu \nabla u_i(x, t) = f_i(x, t) \quad \text{div } u(x, t) = 0, \quad i = 1, 2, 3 \]

with initial conditions \( u|_{t=0} \cap \partial \Omega = 0 \). We introduce the unknown vector-function: \( \left( w_i(x, t) \right)_{i=1,2,3} : u_t(x, t) - \rho \Delta u_i(x, t) - \frac{\partial p(x, t)}{\partial x_i} = w_i(x, t) \) with initial conditions: \( u_i(x, 0) = 0, u_i(x, t) |_{\partial \Omega} = 0 \). The solution \( u_i(x, t) \) of this problem is given by:

\[ u_i(x, t) = \int_0^t \int_\Omega G(x, t; \xi, \tau) \left( w_i(\xi, \tau) + \frac{\partial p(\xi, \tau)}{\partial x_i} \right) d\xi d\tau \]

where \( G(x, t; \xi, \tau) \) is the Green function. We consider the following Navier-Stokes -2 problem: find a solution \( w(x, t) \in L_2(\Omega_I), p(x, t) : p_\nu(x, t) \in L_2(\Omega_I) \) of the system of equations:

\[ w_i(x, t) - G \left( w_j(x, t) + \frac{\partial p(x, t)}{\partial x_j} \right) \cdot G_{ij} \left( w_i(x, t) + \frac{\partial p(x, t)}{\partial x_i} \right) = f_i(x, t), \ i = 1, 2, 3 \]

satisfying almost everywhere on \( \Omega_I \).

1. Introduction

The Navier-Stokes equations are given by:

\[ \frac{\partial u_i(x, t)}{\partial t} - \rho \Delta u_i(x, t) + \sum_{j=1}^{n} u_j(x, t) \frac{\partial u_i(x, t)}{\partial x_j} + \frac{\partial p(x, t)}{\partial x_i} = f_i(x, t) \]

\[ \text{div } u(x, t) = 0; \ i = 1, \ldots, n \]

with the initial condition:

\[ u(x, 0) = u_0(x) \quad (1.2) \]
where $u(x,t) = (u_i(x,t))_{i=1,2,3}$ and $p(x,t) \in R$ are the unknown velocity vector and pressure defined for position $x \in R^3$ and time $t \geq 0$. Here, $u_0(x)$ is a given divergence-free vector field on $R^3$, $f_i(x,t)$ are the components of a given externally force, $\rho > 0$ is a positive coefficient and $\Delta = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the space variables.

Starting with Leray [1], important progress has been made in understanding weak solutions of the Navier-Stokes equations. For instance, if (1.1) and (1.2) hold, then for any smooth vector field $\varphi(x,t)$ satisfying the equation (2.1) and (2.3) in three space dimensions always have a weak solution $(u(x,t), p(x,t))$. The uniqueness of weak solutions of the Navier-Stokes equation is not known. In two dimensions the existence, uniqueness and smoothness of weak solutions have been known for a long time (R. Temam [2], O. Ladyzhenskaya [3], I. Lions[4]).

In three dimensions, this questions studied for the initial velocity $u_0(x)$ satisfying smallness condition. For the initial data $u_0(x)$ not assumed to be small, it is known that the existence of smooth weak solutions holds if the time interval $[0, T]$ is replaced by a small time interval $[0, t]$ depending on the initial data.

A fundamental problem in analysis is to decide whether a smooth solution exists for the Navier-Stokes equations in three dimensions.

2. Results

Let $\Omega \subset R^3$ be a finite domain bounded by the Lipchitz surface $\partial \Omega$. $Q_t = \Omega \times [0,t]$, $x = (x_1, x_2, x_3)$ and $u(x,t) = (u_i(x,t))_{i=1,2,3}$, $f(x,t) = (f_i(x,t))_{i=1,2,3}$ are vector-functions. Here $t > 0$ is an arbitrary real number. The Navier-Stokes equations are given by

$$\frac{\partial u_i(x,t)}{\partial t} - \rho \Delta u_i(x,t) = \sum_{j=1}^3 u_j(x,t) \frac{\partial u_i(x,t)}{\partial x_j} + \frac{\partial p(x,t)}{\partial x_i} = f_i(x,t) \quad (2.1),$$

$$\text{div } u(x,t) = \sum_{i=1}^3 \frac{\partial u_i(x,t)}{\partial x_i} = 0 \quad i = 1, 2, 3$$

The Navier-Stokes problem 1. Find a vector-function $u(x,t) = (u_i(x,t))_{i=1,2,3}: \Omega \times [0,t] \rightarrow R^3$ and a scalar function $p(x,t): \Omega \times [0,t] \rightarrow R^1$ satisfying the equation (2.1) and the following initial condition

$$u(x,0) = 0 \quad u(x,t) |_{\partial \Omega \times [0,t]} = 0 \quad (2.2)$$
Let $p > 1, r > 1$ be real numbers. We shall use the following functional spaces. $L_{p,r}(Q_t)$ is the Banach space with the norm $\|u(x,t)\|_{L_{p,r}(Q_t)} = \left[ \int_0^t \left( \int_\Omega |u(x,t)|^p dx \right)^{r/p} dt \right]^{1/r}$, $L_p(Q_t) = L_p(Q_t)$. $W^{2,1}_p(Q_t)$ is the Banach space supplied by the norm $\|u(x,t)\|_{W^{2,1}_p(Q_t)} = \sqrt{\|u\|^p_{L_p(Q_t)} + \|u_t\|^p_{L_p(Q_t)} + \|ux\|^p_{L_p(Q_t)} + \|uxx\|^p_{L_p(Q_t)}}$.

$L_{p,r}(Q_t)$ is the Banach vector-space with the norm $\|u(x,t)\|_{L_{p,r}(Q_t)} = \sum_{i=1}^3 \|u_i(x,t)\|_{L_{p,r}(Q_t)}$.

$L_2(Q_t)$ is the Hilbert vector-space with the inner product $(u(x,t), v(x,t))_{L_2(Q_t)} = \sum_{i=1}^3 (u_i(x,t), v_i(x,t))_{L_2(Q_t)}$.

$V_0(Q_t)$ ($V(Q_t)$) are the vector-spaces of smooth functions $V_0(Q_t) = \{u(x,t) \in C^2(\overline{Q_t}), \text{ div } u(x,t) = 0, u \cdot n|_{\partial \Omega} = \sum_{i=1}^3 u_i(x,t)\cos(n, x_i)|_{\partial \Omega} = 0\}$,

$V(Q_t) = \{u(x,t) \in C^2(\overline{Q_t}) : \text{ div } u(x,t) = 0\}$.

$H_2(Q_t)$ is the closure of $V_0(Q_t)$ in the norm of $L_2(Q_t)$. [2 p.13] I.e.

$H_2(Q_t) = \{u(x,t) : u(x,t) \in L_2(\Omega), \text{ div } u(x,t) = 0, u \cdot n|_{\partial \Omega} = 0\}$

$E_2(Q_t)$ is the closure of $V(Q_t)$ in the norm of $L_2(Q_t)$. [2 p. 13] I.e.

$E_2(Q_t) = \{u(x,t) : u(x,t) \in L_2(Q_t), \text{ div } u(x,t) = 0\}$

It is obvious that $H_2(Q_t) \subseteq E_2(Q_t)$. Further, we shall denote the vector-functions and vector-spaces by bold type. The following is principal result.

**Theorem 2.1.** For any right-hand side $f(x,t) \in L_2(Q_t)$ in equation (2.1) and for any real numbers $\rho > 0, \tau > 0$, the Navier-Stokes problem-1 has a unique smooth solution $u(x,t) : u(x,t) \in W^{2,1}_p(Q_t) \cap H_2(Q_t)$ and a scalar function $p(x,t) : p_x(x,t) \in L_2(Q_t)$ satisfying (2.1) almost everywhere on $Q_t$, and the following estimates are valid:

$$
\|u(x,t)\|_{W^{2,1}_p(Q_t)} \leq c \|f\|_{L_2(Q_t)}, \quad \left\| \frac{\partial p(x,t)}{\partial x_i} \right\|_{L_2(Q_t)} \leq c \|f\|_{L_2(Q_t)} \quad (2.3)
$$

Here and bellow by symbol $c$, we denote a generic constant, independent on the solution and right-hand side whose value is inessential to our aims, and it may change from line to line.
Remark 2.1. The case when the right-hand side \( f(x,t) \) has a small norm or a time \( t \ll 1 \) (or \( \rho \gg 1 \)) is well-known and not interesting. But in Theorem 1 \( f(x,t) \in L_2(Q_t) \) is an arbitrary vector-function and \( t > 0 \), \( \rho > 0 \) are arbitrary real numbers. In recent paper [6] Ladyzhenskaja formulates the Navier-Stokes problem as in the formulas (2.1) - (2.2) and in Theorem 1. For simplicity, we consider the Navier-Stokes problem for the homogeneous case (i.e. \( u(x,0) = 0 \), \( u(x,t) \mid \partial \Omega = 0 \)). We consider the inhomogeneous case (i.e. \( u(x,0) = u_0(x) \), \( u(x,t) \mid \partial \Omega = 0 \)) in Section 4.

Definition 2.1. A vector-function \( u(x,t) : u(x,t) \mid_{(t=0),\partial \Omega} = 0 \) and a scalar function \( p(x,t) \) are called a smooth solution to the Navier-Stokes problem-1, if \( u(x,t) \in W_2^{1,1}(Q_t) \cap H_2(Q_t) \) and \( p_{x_i}(x,t) \in L_2(Q_t) \).

We adduce the well-known definition of the Hopf solution to the Navier-Stokes equation.

Definition 2.2 (the Hopf’s solution). Let a right-hand side \( f(x,t) \in L_2(Q_t) \). A vector-function \( u(x,t) \in L_2([0,t]; H_0^2(\Omega)) \cap L^\infty([0,t]; L_2(\Omega)) \) is called the Hopf’s solution, if the following equality [2 p.225].

\[
\frac{\partial(u(x,t),v(x))}{\partial t} + \rho (u_x(x,t),v(x)) - \sum_{i=1}^3 u_j(x,t)(u_{x_j}(x,t) , v(x)) = \int_\Omega f(x,t) \cdot v(x)dx
\]

is fulfilled for all vector-functions \( v(x) \in H_0^2(\Omega) = \{u(x) : \text{div } u(x) = 0, u(x)\mid_{\partial \Omega} = 0, u(x), u_{x_i}(x) \in L_2(\Omega)\} [2p.24] \).

Remark 2.2. By Theorem 2.1 it follows that the Hopf’s solution is the smooth solution.

For the proof of Theorem 2.1 we shall use the following known propositions.

Theorem of Weyl H. In the book [2 p.22] the following equalities are proved:

\[
L_2(Q_t) = H_2(Q_t) \oplus G_2(Q_t) \quad \text{where } H_2(Q_t) = \{u(x,t) : u(x,t) \in L_2(Q_t), \text{div } u(x,t) = 0, u \cdot n\mid_{\partial \Omega \times [0,t]} = 0\}. \quad G_2(Q_t) = \{u(x,t) : u(x,t) \in L_2(Q_t), u(x,t) = \text{grad}p(x,t) : p_{x_i}(x,t) \in L_2(Q_t)\}.
\]

I.e. for any \( f(x,t) \in L_2(Q_t) \), the following equality: \( f(x,t) = H(f(x,t)) + G(f(x,t)) \) is valid where \( H : L_2(Q_t) \Rightarrow H_2(Q_t) \), \( G : L_2(Q_t) \Rightarrow G_2(Q_t) \)- are the projection operators.

Proposition 1. (The Holder inequality). Let \( p_1 > 1, p_2 > 1, r_1 > 1, r_2 > 1 \) be real numbers. Then, the following Holder inequality is valid [5 p.75].

\[
\|u(x,t)v(x,t)\|_{L_{p_1+p_2}(\Omega)} \leq \|u(x,t)\|_{L_{p_1}(Q_t)} \|v(x,t)\|_{L_{p_2}(Q_t)} \quad (2.4)
\]

Proposition 2. (The system equations of Volterra V.). On the space of vector-functions \( u(x,t) = (u_i(x,t))_{i=1,2,3} \in L_2(Q_t) \) we shall consider the following system of nonlinear integral equations of Volterra: [7 p.59, p.62]

\[
u_i(x,t) - \sum_{s=1}^3 \int_0^t \int_\Omega K_{i,s}(x,t;\xi,\tau)u_s(\xi,\tau) \ u_s(\xi,\tau) \ d\xi \ d\tau = f_i(x,t) \quad (2.5)
\]

\[
l = 1, 2, 3. \quad \text{Or, in the vector form}
\]

\[
u(x,t) - Ku(x,t) = f(x,t) \in L_2(Q_T) \quad (2.6)
\]

This system of equations under some conditions to the nonlinear kernel \( K(x,t;\xi,\tau;u(\xi,\tau)) \) has been studied in the book [7 p.61]. We shall study this nonlinear system of equations by using the theorem of Leray J., Schauder J.
Proposition 3. (Theorem of Hardy G.H., Littlewood J.E.) Let \( \mu : 0 < \mu < 1 \) be a real number. We shall consider the following operator of the fractional integration \( J^\mu u(t) = \int_0^t \frac{u(\tau) \, d\tau}{(t-\tau)^\mu} \). Then:

a) If \( 1 < p < \frac{1}{1-\mu} \), then the operator \( J^\mu \) is bounded from the space \( L_p(0, t) \) into the space \( L_q(0, t) \) where \( q = \frac{p}{1-\mu(p-1)} \) and \( \|J^\mu u(t)\|_{L_q(0,t)} \leq c \|u(t)\|_{L_p(0,t)} \). [8 p.64].

Proposition 4. (Theorem of Sobolev S.L.) Let a function \( u(x) \) be represented as the potential of a function \( f(x) \), i.e., \( u(x) = \int_\Omega \frac{f(\xi) \, d\xi}{|x-\xi|^{\mu+1}} \). Then:

a) If \( 0 < \lambda < 3/p \) and \( f(x) \in L_p(\Omega) \), then \( u(x) \in L_q(\Omega) \) where \( q \leq \frac{3p}{3-p} \lambda \) and \( \|u(x)\|_{L_q(\Omega)} \leq c \|f(x)\|_{L_p(\Omega)} \).

b) If \( \lambda = \frac{3}{p} \) and \( f(x) \in L_p(\Omega) \), then \( u(x) \in L_\infty(\Omega) \) and \( \|u(x)\|_{L_\infty(\Omega)} \leq c \|f(x)\|_{L_p(\Omega)} \).

Proposition 5. We shall consider the following problem on the domain \( Q_t \):

\[
 u_t(x, t) - \rho \Delta u(x, t) = g(x, t), \quad u(x, 0) = 0, \quad u(x, t) |_{\partial Q_t \times [0, t]} = 0 \tag{2,7}
\]

The solution \( u(x, t) \) of this problem is given by

\[
 u(x, t) = Gg(\xi, \tau) = \int_0^t \int_\Omega G(x, t; \xi, \tau) \, g(\xi, \tau) \, d\xi \, d\tau \tag{2,8}
\]

where \( G(x, t; \xi, \tau) \) is the Green function for \( Q_t \). The construction of the Green function is resulted in book [9 p.111]. The following estimates are valid.[9 p.170]

\[
 |G(x, t; \xi, \tau)| \leq \frac{\text{const}}{(t-\tau)^\mu} \frac{1}{|x-\xi|^{3-2\mu}}, \quad 0 < \mu < 1 \tag{2,9}
\]

\[
 \left| \frac{\partial}{\partial x} G(x, t; \xi, \tau) \right| \leq \frac{\text{const}}{(t-\tau)^\mu} \frac{1}{|x-\xi|^{3-(2\mu-1)}}, \quad 1/2 < \mu < 1 \tag{2,10}
\]

From estimates in Propositions 3, 4 and the estimates (2,9),(2,10) follow that:

a) If \( g(x, t) \in L_2(Q_t) \) and \( \mu = 5/8 \), then

\[
 Gg(x, t) \in L_{12,8}(Q_t), \quad G_x g(x, t) = \frac{\partial Gg(x, t)}{\partial x} \in L_{12/5,8}(Q_t), \tag{2,11}
\]

\[
 \|Gg(x, t)\|_{L_{12,8}(Q_t)} \leq \|Gg(x, t)\|_{L_{12/5,8}(Q_t)} \|G_x g(x, t)\|_{L_{12/5,8}(Q_t)} \leq c \|g(x, t)\|_{L_2(Q_t)}^2
\]

b) If \( g_1(x, t), \ g_2(x, t) \in L_2(Q_t) \) and \( g_i(t) = \|g_i(x, t)\|_{L_2(\Omega)} \), then for any \( \mu : 5/8 < \mu < 1 \) the following estimates are valid:

\[
 \|Gg_i(x, t)\|_{L_{\frac{3}{3-4\mu}}(\Omega)} \leq c \int_0^t g_i(\tau) \, d\tau ; \quad \|G_x g_i(x, t)\|_{L_{\frac{5}{5-4\mu}}(\Omega)} \leq c \int_0^t g_i(\tau) \, d\tau \quad \text{for} \quad \mu : \frac{5}{8} \leq \mu < 1 \tag{2.12.1}
\]

and for \( \mu : \frac{5}{8} \leq \mu < 1 \) follows that \( 2 \leq \frac{3}{4(1-\mu)} \). Therefore,

\[
 \|Gg_1(x, t) G_x g_2(x, t)\|_{L_2(\Omega)} \leq
\]
Let \( \parallel \) in the second inequality of (2.12.1) satisfy all conditions of parameters (estimate (2.9) with inequality (12.1) follows the inequality operator on \( L \)). Here we used the fact that from Proposition 4 (Green function (2,9), (2,10), Proposition L_1) do not fall outside the bounds of some sphere side (Theorem of Leray J., Schauder J.) Let \( A \) be a compact nonlinear equation of Abel-Carleman.) The equation of Abel-Carlleman is (The equation of Abel-Carlleman is)

\[ \int_{0}^{t} g_1(\tau) \, d\tau = f(t); \quad u(t) = \frac{\sin \pi \mu}{\pi} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\mu}} \, d\tau \]  

(2.13)

Let \(-\infty < \mu_1 < 1; \quad -\infty < \mu_2 < 1.\) Then

\[ \int_{0}^{t} \frac{d\tau}{(t-\tau)^{\mu_1}} \int_{0}^{\tau} \frac{g(\tau_1) \, d\tau_1}{(\tau-\tau_1)^{\mu_2}} = \int_{0}^{t} g(\tau_1) \, d\tau_1 \int_{\tau_1}^{t} \frac{d\tau}{(t-\tau)^{\mu_1}} \frac{d\tau}{(\tau-\tau_1)^{\mu_2}} = \Gamma_{\mu_2}^{\mu_1} \int_{0}^{t} \frac{g(\tau) \, d\tau}{(t-\tau)^{\mu_1+\mu_2-1}} \quad ; \quad \Gamma_{\mu_1}^{\mu_2} = \frac{\Gamma(1-\mu_1) \Gamma(1-\mu_2)}{\Gamma(2-\mu_1-\mu_2)} \]  

(2.14)
Proposition 8. (The linear Navier-Stokes equation) Let \( t > 0 \) be an arbitrary real number. We consider the following linear Navier-Stokes problem on the domain \( Q_t \): \[ 3, \text{p.95} \]

\[
\begin{align*}
\frac{\partial u_i(x,t)}{\partial t} - \rho \Delta u_i(x,t) - \frac{\partial p(x,t)}{\partial x_i} &= w_i(x,t) \quad (2.15), \\
div \mathbf{u}(x,t) &= 0; \quad \mathbf{u}(x,0) = 0, \quad \mathbf{u}(x,t) |_{\partial \Omega \times [0,t]} = 0
\end{align*}
\]

For this problem in the manuscript of author [12] is received the explicit expression to the pressure function \( p(x,t) \), depending on the right-hand side \( w_i(x,t) \):

\[
p(x,t) = -T * \Delta^{-1} \ast \int_0^t \int_{\Omega} \sum_i^3 \frac{dG(x,t;\xi,\tau)}{dx_i} w_i(\xi,\tau) d\xi d\tau = (2.16)
\]

\[
= -\frac{d}{dt} T * \Delta^{-1} \ast \int_0^t \int_{\Omega} \sum_i^3 \frac{dG(x,t;\xi,\tau)}{dx_i} w_i(\xi,\tau) d\xi d\tau + \rho \cdot \int_0^t \int_{\Omega} \sum_i^3 \frac{dG(x,t;\xi,\tau)}{dx_i} w_i(\xi,\tau) d\xi d\tau
\]

where \( T * u(x,t) = \frac{\partial}{\partial t} u(x,t) - \rho \Delta u(x,t) \) is the parabolic operator, \( \Delta^{-1} \) is the inverse operator to Dirichlet problem for Laplace equation on the domain \( \Omega \). \( G(x,t;\xi,\tau) \) is the Green function of Dirichlet problem for the parabolic equation on the domain \( \Omega \) \[ 9 \text{p.106} \]. If \( \mathbf{w}(x,t) \in L_2(Q_t) \), then \( \int_0^t \int_{\Omega} G(x,t;\xi,\tau) \mathbf{w}(\xi,\tau) d\xi d\tau \in W^{2,1}_2(Q_t) \). It is obvious that:

\[
\left| \int_0^t \int_{\Omega} \sum_i^3 \frac{dG(x,t;\xi,\tau)}{dx_i} w_i(\xi,\tau) d\xi d\tau \right|_{W^{2,1}_2(Q_t)} < c \cdot |\mathbf{w}(x,t)|_{L_2(Q_t)}
\]

From this estimate follows the following estimate:

\[
\int_0^t \left( |\frac{d}{dt} T * \Delta^{-1} \ast \int_{\Omega} \sum_i^3 \frac{dG(x,t;\xi,\tau)}{dx_i} w_i(\xi,\tau) d\xi d\tau|_{W^{2,1}_2(\Omega)} \right)^2 d\tau < c |\mathbf{w}(x,t)|^2_{L_2(Q_t)}
\]

We consider the formula (2.16) in detail. It is known that the following classical problem for the parabolic equations: \( u_t - \rho \Delta u(x,t) = f(x,t) \in L_2(Q_t) \), \( u_{t=0,\partial \Omega} = 0 \) has the unique solution \( u(x,t) \in W^{3,1}_2(Q_t) \) and \( |u(x,t)|_{W^{3,1}_2(Q_t)} < c |f(x,t)|_{L_2(Q_t)} \) where \( Q_t = \Omega \times [0,t] \). Therefore, from the formula (2.16) follow the following estimates:

\[
\left| \int_0^t \int_{\Omega} \sum_i^3 \frac{dG(x,t;\xi,\tau)}{dx_i} w_i(\xi,\tau) d\xi d\tau \right|_{L_2(Q_t)} < c |\mathbf{w}(x,t)|_{L_2(Q_t)}
\]

\[
\left| \int_0^t \int_{\Omega} \sum_i^3 \frac{dG(x,t;\xi,\tau)}{dx_i} w_i(\xi,\tau) d\xi d\tau \right|_{L_2(Q_t)} < c |\mathbf{w}(x,t)|_{L_2(Q_t)}
\]

Differentiating the function of pressure \( p(x,t) \) in formula (2.16) by \( x_i \), we find \( \frac{\partial p(x,t)}{\partial x_i} \), \( i = 1, 2, 3 \), depending on the right-hand side \( w_i(x,t) \). And by these estimates, integrating on the domain \( Q_t \), we obtain the following estimate:

\[
\int_0^t \sum_i^3 \left| \frac{\partial p(x,t)}{\partial x_i} \right|_{L_2(\Omega)}^2 d\tau < c \cdot \int_0^t \sum_i^3 |w_i(x,\tau)|_{L_2(\Omega)}^2 d\tau \quad (2.17)
\]
By the formula (2.16) and estimate (2.17) on the vector space \( \mathbf{w}(x,t) = \left( w_i(x,t) \right)_{i=1,2,3} \in \mathbf{L}_2(Q_t) \) we define the following linear and bounded operator:
\[
P \left( w_i(x,t) \right)_{i=1,2,3} = \left( \frac{dp(x,t)}{dx_i} \right)_{i=1,2,3} \in \mathbf{L}_2(x,t)
\]
where the functions \( \frac{dp(x,t)}{dx_i} \) is defined by the functions \( w_i(x,t) \) from the formula (2.16).

Using the Green function \( G(x,t;\xi,\tau) \), from the equation (2.15) we find:
\[
u_i(x,t) = \int_0^t \int_\Omega G(x,t;\xi,\tau) \left( w_i(\xi,\tau) + \frac{\partial p(\xi,\tau)}{\partial x_i} \right) d\xi d\tau
\]
And present the nonlinear Navier-Stokes equations (2.1) as:
\[
w_i(x,t) = \sum_{j=1}^3 G \left( w_j(x,t) + \frac{\partial p(x,t)}{\partial x_j} \right) \cdot G_{x_j} \left( w_i(x,t) + \frac{\partial p(x,t)}{\partial x_i} \right) = f_i(x,t)
\]
where \( \nu_j w_i(x,t) = \frac{\partial G_{x_j}(x,t)}{\partial x_j} \). Differentiating the pressure function \( p(x,t) \) in the formula (2.16) by \( x_i \), we find \( \frac{dp(x,t)}{dx_i}, i=1,2,3 \), depending on the functions \( w_i(x,t) \), and substitute these functions to the equation (2.20). Then, for the definition of the three unknown functions \( (w_i(x,t))_{i=1,2,3} \) we obtain the three system of nonlinear equations of Volterra (2.20).

Remark 2.3. It is obvious that the vector function \( \left( \frac{dp(x,t)}{dx_i} \right)_{i=1,2,3} \) depends on the vector function \( (w_i(x,t))_{i=1,2,3} \) linearly. But we shall not write the expressions of these depends, we shall use the estimate (2.17).

Navier-Stokes problem 2. Find the vector-function \( (w_i(x,t))_{i=1,2,3} \in \mathbf{L}_2(Q_t) \), satisfying the equation (2.20) almost everywhere on \( Q_t \).

We will find the unknown vector-function \( \mathbf{w}(x,t) = (w_i(x,t))_{i=1,2,3} \in \mathbf{L}_2(Q_t) \).

Theorem 2.2 For any right-side \( \mathbf{f}(x,t) \in \mathbf{L}_2(Q_T) \) in the system of equations (2.20) there exists a unique vector-function \( \mathbf{w}(x,t) \in \mathbf{L}_2(Q_T) \), satisfying almost everywhere on \( Q_T \), the system equations (2.20). And for any possible solution \( \mathbf{w}(x,t) : \| \mathbf{w}(x,t) \|_{L_2(Q_T)} = \| \mathbf{w}(x,t) \|_{L_2(Q_T)} = \| w(t) \|_{L_2(0,T)} < \infty \) to the basis equation (2.20), the following a priori estimate is valid
\[
\| \mathbf{w}(x,t) \|_{L_2(Q_T)} < \sqrt{2} \cdot \| \mathbf{f}(x,t) \|_{L_2(Q_T)}
\]
where \( Q_T = \Omega \times [0,T], \quad T > 0 \) is an arbitrary real number.

3. Proof of Theorem 2.2.

The following is the key Lemma for the proof of Theorem 2.2.

Lemma 3.1. Let the vector-function \( \mathbf{g}(x,t) = (g_i(x,t))_{i=1,2,3} \in \mathbf{L}_2(Q_t) \). I.e.
\[
\| \mathbf{g}(x,t) \|_{L_2(Q_t)} = \sum_{i=1}^3 \left( \int_{Q_t} g_i^2(x,t) \ dxdt \right)^{1/2} < \infty
\]
We define the following vector-function:
\[
G g_j(x,t) G_{x_j} \mathbf{g}(x,t) \equiv \left( \sum_{j=1}^3 G g_j(x,t) G_{x_j} g_i(x,t) \right)_{i=1,2,3} = \left( \sum_{j=1}^3 G g_j(x,t) \frac{\partial G g_i(x,t)}{\partial x_j} \right)_{i=1,2,3}
\]
Then for any $t > 0$ there exists a constant $b > 0$ independent on $g(x,t)$ such that the following inequality is valid:

$$
\left\| Gg_j(x,t)G_{x,j}g(x,t) \right\|_{L^2(\Omega)} \leq b \left( \int_0^t \frac{g(\tau)d\tau}{(t-\tau)^{\mu}} \right)^2 \tag{3.1}
$$

where $\mu : \frac{5}{8} \leq \mu < 1$, $g(\tau) = \|g(x,\tau)\|_{L^2(\Omega)} = \sum_{i=1}^{3}(\int_{\Omega} g_i^2(x,\tau) \, dx)^{1/2}$.

Let $\mu : \frac{5}{8} \leq \mu < 1$. By the estimate (2.12.2) in Proposition 5 and the following inequality: $\sum_{i,j=1}^{3} c_i c_j \cdot a_i \cdot a_j \leq b \left( \sum_{i=1}^{3} a_i \right)^2$ we have

$$
\left\| Gg_j(x,t)G_{x,j}g(x,t) \right\|_{L^2(\Omega)} \leq \sum_{i,j=1}^{3} \|Gg_j(x,t)G_{x,j}g_i(x,t)\|_{L^2(\Omega)} < \sum_{i,j=1}^{3} c_i c_j \int_0^t g_i(\tau) \frac{d\tau}{(t-\tau)^{\mu}}, \int_0^t g_j(\tau) \frac{d\tau}{(t-\tau)^{\mu}} < b \left( \int_0^t \frac{g(\tau)d\tau}{(t-\tau)^{\mu}} \right)^2
$$

Lemma is proved. △

From the basis equation (2.20) and the inequality (3.1), integrating over the domain $\Omega$, and, using the Holder inequality, the estimates (2.9), (2.10) for Green function, we obtain the following estimate:

$$
w(t) < f(t) + b \left( \int_0^t \frac{w(\tau) + p(\tau)}{(t-\tau)^{\mu}} \, d\tau \right)^2 \tag{3.2}
$$

where

$$
w(\tau) = \|w(x,\tau)\|_{L^2(\Omega)} = \sum_{i=1}^{3} \left( \int_{\Omega} w_i^2(x,\tau) \, dx \right)^{1/2} \geq 0;
$$

$$
p(\tau) = \sum_{i=1}^{3} \left\| \frac{\partial p(x,\tau)}{\partial x_i} \right\|_{L^2(\Omega)} \geq 0; \quad f(t) = \|f(x,t)\|_{L^2(\Omega)} = \sum_{i=1}^{3} \left( \int_{\Omega} f_i^2(x,\tau) \, dx \right)^{1/2} \geq 0 \tag{3.2'}
$$

**Remark 3.1.** Let $T > 0$ be an arbitrary number. We have proved that for all solutions $w(x,t), p(x,t)$ of the basis equation (2.20) the functions $w(\tau) = \|w(x,\tau)\|_{L^2(\Omega)}$, $p(\tau) = \sum_{i=1}^{3} \left\| \frac{\partial p(x,\tau)}{\partial x_i} \right\|_{L^2(\Omega)}$ satisfy to the inequality (3.2). The inequality (3.2) does not exclude the functions of the type $w(t) + p(t) = \frac{t}{t-\tau} \notin L_2(0,T)$. I.e. these functions satisfy the inequality (3.2). Note that $\mu : \frac{3}{8} \leq \mu < \frac{3}{4}$ and $w(t) \in L_2(0,T)$. By Proposition 3 and estimate (2.17) in Proposition 8 it follows that:

$$
\left\| \left( \int_0^t \frac{(w(\tau) + p(\tau)) \, d\tau}{(t-\tau)^{\mu}} \right)^2 \right\|_{L^2(0,T)} < c \cdot \|w(t) + p(t)\|_{L^2(0,T)} < c \cdot \|w(t)\|_{L^2(0,T)} \tag{3.3}
$$

In the Theorem 2.2 we assumed that for all possible solutions $w(x,t)$ of the basis equation (2.20) the function $w(t) = \|w(x,t)\|_{L^2(\Omega)} \in L_2(0,T)$.

By the basis equation (2.20) we have proved the inequality (3.2). Below (see Theorem 3.1), we shall prove that for all functions $w(t), p(t) \in L_2(0,T) : \|p(t)\|_{L^2(0,T)} < c \|w(t)\|_{L^2(0,T)} < \infty$, satisfying the inequality (3.2), the following a priori estimate: $\|w(t)\|_{L^2(0,T)} < \sqrt{2} \|f(t)\|_{L^2(0,T)}$ holds. △
Lemma 3.2. From the estimates (2.17), (3.2) follows that:

\[ w(t) < f(t) + b_1 \cdot \int_0^t \frac{w^2(\tau)}{(t-\tau)^\mu} \, d\tau \]

(3.4)

where \( b_1 = b + c \) is constant independent on function \( w(t) \).

By H"older inequality and the inequality \((a + b)^2 < 2a^2 + 2b^2\) from the basic inequality (3.2) follows that:

\[ w(t) < f(t) + b \left( \int_0^t \frac{w(\tau) + p(\tau)}{(t-\tau)^{\mu/2}} \cdot \frac{1}{(t-\tau)^{\mu/2}} \, d\tau \right)^2 < f(t) + \frac{2b}{1-\mu} T^{1-\mu} \int_0^t \frac{w^2(\tau) + p^2(\tau)}{(t-\tau)^\mu} d\tau \]

Using the estimate (2.17) in Proposition 8 and the inequality \((a_1 + a_2 + a_3)^2 < 3 \cdot (a_1^2 + a_2^2 + a_3^2)\), we infer:

\[ \int_0^t p^2(\tau) \, d\tau = \int_0^t \left( \sum_1^3 \|p_{x_i}(x, \tau)\|_{L_2(\Omega)}^2 \right)^2 \, d\tau < 3 \cdot \int_0^t \sum_1^3 \|p_{x_i}(x, \tau)\|_{L_2(\Omega)}^2 \, d\tau \]

\[ < 3c \cdot \int_0^t \sum_1^3 \|w_i(x, \tau)\|_{L_2(\Omega)}^2 \, d\tau < c \int_0^t \left( \sum_1^3 \|w_{x_i}(x, \tau)\|_{L_2(\Omega)}^2 \right)^2 \, d\tau = 3c \cdot \int_0^t w^2(\tau) \, d\tau \]

I.e.:

\[ w(t) < f(t) + b_1 \int_0^t \frac{w^2(\tau) + p^2(\tau)}{(t-\tau)^\mu} \, d\tau; \quad \int_0^t p^2(\tau) \, d\tau < c \cdot \int_0^t w^2(\tau) \, d\tau \quad (3.4') \]

where \( b_1 = \frac{2b}{1-\mu} \cdot T^{1-\mu} \). We shall prove by the second inequality of (3.4') that there exists a constant \( c > 0 \):

\[ \int_0^t \frac{p^2(\tau)}{(t-\tau)^\mu} \, d\tau < c \cdot \int_0^t \frac{w^2(\tau)}{(t-\tau)^\mu} \, d\tau \quad (3.4'') \]

We shall prove this estimate by contradiction method and assumed that there exists a constants \( c_n \to \infty \):

\[ \int_0^t \frac{p^2(\tau)}{(t-\tau)^\mu} \, d\tau > c_n \cdot \int_0^t \frac{w^2(\tau)}{(t-\tau)^\mu} \, d\tau \]

Let us apply to this inequality the following operator: \( J^{1-\mu} u(t) = \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\mu}} \, d\tau \). Then: \( \int_0^t p^2(\tau) \, d\tau > c \cdot \int_0^t w^2(\tau) \, d\tau \). But: \( \int_0^t p^2(\tau) \, d\tau < c \cdot \int_0^t w^2(\tau) \, d\tau \). And this contraction proves the estimate (3.4''). Lemma 3.2 is proved. □

Remark 3-2. Below, using the Riccati's replacement of the function \( w(t) \), from the estimate (3.4) we derive the following estimate: \( \|w(t)\|_{L_2(0,T)} < \sqrt{2} \cdot \|f(t)\|_{L_2(0,T)} \). For these aims we consider the following equation.

The equation of Riccati. In 1715, Riccati has studied the following nonlinear equation on the segment \([0,T]\) where \( T > 0 \) is an arbitrary real number [10 p.41]:

\[ \frac{dz(t)}{dt} = f(t) + b \cdot z^2(t); \quad z(0) = 0 \]

By the replacement of the unknown function \( z(t) = -\frac{1}{b} \cdot \frac{u(t)}{u(t)} \), this nonlinear equation is reduced to the following linear equation of the second order:

\[ \frac{d^2 u(t)}{dt^2} + bf(t) u(t) = 0; \quad \frac{du(t)}{dt} \big|_{t=0} = 0. \]
Using the Riccati’s result, we prove the following key proposition.

**Theorem 3.1.** For all functions $w(t) \in L_2(0, T)$ satisfying the inequality (3.4) the following estimate holds:

$$
\|w(t)\|_{L_2(0, T)} < \sqrt{2} \cdot \|f(t)\|_{L_2(0, T)}
$$

(3.5)

This estimate does not depend on the number $b_1$ in (3.4).

In (3.4) we make the replacement of the function:

$$
w_1(t) = \int_0^t \frac{w^2(\tau)}{(t-\tau)^{1-\mu}} d\tau
$$

(3.6)

and, using the inequality $(a + b)^2 < 2a^2 + 2b^2$, we rewrite the basis inequality (3.4) as:

$$
\int_0^t \frac{dw_1(\tau)}{(t-\tau)^{1-\mu}} d\tau < 2f^2(t) + 2b_1^2 \cdot w_1^2(t)
$$

(3.7)

Let $k : 0 < k < \infty$ and $s : 0 < s < 1$ - are an arbitrary real numbers and for $t > 1$ we present the inequality (3.7) as:

$$
\int_0^t \frac{dw_1(\tau)}{(t-\tau)^{1-\mu}} d\tau < 2f^2(t) + k \int_0^t \frac{w_1^2(\tau)d\tau}{(t-\tau)^{1-\mu}} + 2b_1^2 \cdot t^\mu w_1^2(t) - k \int_0^t \frac{w_1^2(\tau)d\tau}{(t-\tau)^{1-\mu}}
$$

(3.8)

Applying the Mellin transformation, for $s : 0 < s + \mu < 1$ we obtain:

$$
\int_0^\infty t^{s-1} \cdot \left(2b_1^2 \cdot t^\mu w_1^2(t) - k \int_0^t \frac{w_1^2(\tau)d\tau}{(t-\tau)^{1-\mu}}\right) dt =
$$

$$
= \left(2b_1^2 - k \cdot \int_0^1 \frac{d\tau}{\tau^{s+\mu} \cdot (1-\tau)^{1-\mu}}\right) \cdot \int_0^\infty \tau^{s+\mu-1} w_1^2(\tau) d\tau
$$

From this equality follows that: $\left(2b_1^2 - k \cdot \int_0^1 \frac{d\tau}{\tau^{s+\mu} \cdot (1-\tau)^{1-\mu}}\right) < 0$ for all numbers $k \gg 1$:

$$
\left(2b_1^2 - k \cdot \int_0^1 \frac{d\tau}{\tau^{s+\mu} \cdot (1-\tau)^{1-\mu}}\right) < 0
$$

(3.9)

where $s : 0 < s + \mu < 1$. Using this inequality, we rewrite the inequality (3.8) as follows:

$$
\int_0^t \frac{dw_1(\tau)}{(t-\tau)^{1-\mu}} d\tau < 2f^2(t) + k \int_0^t \frac{w_1^2(\tau)d\tau}{(t-\tau)^{1-\mu}}
$$

(3.10)

In this inequality, as Riccati, we shall make the replacement of the function $w_1(\tau)$

$$
w_1(\tau) = - \frac{1}{k} \frac{z'(\tau)}{z(\tau)} ; \quad \frac{dw_1(\tau)}{d\tau} = - \frac{1}{k} \cdot \frac{z''(\tau)}{z(\tau)} + \frac{1}{k} \left(\frac{z'(\tau)}{z(\tau)}\right)^2
$$

(3.11)

From the definition of the function $w_1(t)$ by (3.6) follows that:

$$
z(t) = z(0) \cdot e^{-k \int_0^t w_1(\tau)d\tau} = z(0) \cdot e^{-k \int_0^t \tau w_1^2(t-\tau)^{1-\mu} d\tau}
$$

(3.11')
Since $w_1(0) = 0$, then $z'(0) = \frac{dz(t)}{dt}|_{t=0} = 0$. From the inequality (3.10) we have

$$-\frac{1}{k} \int_0^t \frac{1}{z(\tau)} \frac{d^2 z(\tau)}{d\tau^2} d\tau < 2f_1^2(t) \quad (3.12)$$

Or

$$-\frac{1}{k} \int_0^t \frac{1}{z(\tau)} \frac{d^2 z(\tau)}{d\tau^2} d\tau < \frac{2}{\Gamma_{1-\mu}} \int_0^t \frac{f_1^2(\tau)}{(t-\tau)^\mu} d\tau \quad (3.13)$$

Let us denote:

$$\frac{1}{k} \int_0^t \frac{1}{z(\tau)} \frac{d^2 z(\tau)}{d\tau^2} d\tau + \frac{2}{\Gamma_{1-\mu}} \int_0^t \frac{f_1^2(\tau)}{(t-\tau)^\mu} d\tau = g(t) > 0$$

Then

$$\frac{1}{k} \cdot \frac{d^2 z(t)}{dt^2} + \frac{2}{\Gamma_{1-\mu}} \cdot \frac{z(t)}{d\tau} \int_0^t \frac{f_1^2(\tau)}{(t-\tau)^\mu} d\tau = \int_0^t \frac{d g(t)}{d\tau} \cdot g(\tau) d\tau > 0 \quad (3.13')$$

Since $g(t) > 0, g(0) = 0, -\frac{d^2 z(t)}{d\tau^2} > 0$, integrating by parts, we obtain:

$$\int_0^t z(\tau) \cdot \frac{d}{d\tau} g(\tau) = z(t)g(t) + \int_0^t \left( -\frac{d z(\tau)}{d\tau} \right) \cdot g(\tau) d\tau > 0$$

Since $\frac{dz(t)}{dt}|_{t=0} = 0$, integrating the equation (3.13') over $[0,t]$, we obtain the following important inequality:

$$\frac{dz(t)}{dt} < k \cdot \int_0^t z(\tau) \frac{d F_\mu(t)}{d\tau} d\tau \quad (3.14)$$

where

$$F_\mu(t) = \frac{2}{\Gamma_{1-\mu}} \int_0^t \frac{f_1^2(\tau)}{(t-\tau)^\mu} d\tau \quad \blacktriangle$$

Integrating by parts from the basis inequality (3.14), we have:

$$-\frac{1}{k} \cdot \frac{dz(t)}{dt} < z(t) \cdot F_\mu(t) + \int_0^t F_\mu(t) \left( -\frac{d z(\tau)}{d\tau} \right) \cdot d\tau \quad (3.16)$$

In order to proof the a priori estimate, it is necessary that the function $F_\mu(t) = \frac{2}{\Gamma_{1-\mu}} \cdot \int_0^t \frac{f_1^2(\tau)}{(t-\tau)^\mu} d\tau$ is increasing on $[0,T]$. Otherwise, the proof of this estimate is difficult.

**Remark 3.3.** Not for all positive right-hand side $f_1^2(t)$: $f_1^2(t) \in L_1(0, T)$ and real numbers $\mu : 1/2 < \mu < 1$ the function $f_\mu(t) = \frac{\sin\pi\mu}{\pi} \int_0^t \frac{f_1^2(\tau)}{(t-\tau)^\mu} d\tau$ is increasing on $[0,T]$. For example, we consider the following function $f_\mu(t) = \frac{\sin\pi\mu}{\pi} \int_0^t \frac{1-k\tau}{(t-\tau)^\mu} d\tau$ where $(1-k\tau) > 0$ on $(0,t)$, i.e. $0 < \tau < 1/k$. Then $\frac{df_\mu(t)}{dt} = \frac{\sin\pi\mu}{\pi} \cdot \frac{1}{\tau^2} \left( 1 - \frac{k\tau}{1-k} \right)$, and for $t : \frac{1-\mu}{k} < t < \frac{1}{k}$ the function $f_\mu(t)$ is decreasing on $(\frac{1-\mu}{k}, \frac{1}{k})$. $\blacktriangle$

Let $f(t) : f(t) \in L_2(0,T)$ - is an arbitrary function. To define the right-hand side $f(t)$, for which the function $F_\mu(t)$ is increasing on $[0,T]$, we introduce the following functional spaces:

$$L_2^+(0,T) = \{ f^2(t) : f(t) > 0, \int_0^T f^2(\tau) d\tau < \infty \}$$
Since \( g(t) \in L^2_{1-\mu}(0, T) \) is an arbitrary function.

**Remark 3-4.** For all functions \( \{ f_{1-\mu}(t) = \int_0^t \frac{g^2(\tau)}{(t-\tau)^{1-\mu}} \, d\tau \} \in L^+_{1-\mu}(0, T) \) the functions

\[
F_\mu(t) = \frac{2}{\Gamma_{1-\mu}} \cdot \int_0^t f_{1-\mu}(\tau) \, d\tau = \frac{2}{\Gamma_{1-\mu}} \cdot \int_0^t d\tau \int_0^\tau \frac{g^2(\tau_1) \, d\tau_1}{(\tau-\tau_1)^{1-\mu}} = 2 \int_0^t g^2(\tau) \, d\tau
\]

are increasing on \([0, T]\). This remark is **important** for the proof the a priori estimate.

**Lemma 3.3** For any right-side \( f(t) = f_{1-\mu}(t) = \int_0^t \frac{g^2(\tau)}{(t-\tau)^{1-\mu}} \, d\tau \in L^+_{1-\mu}(0, T) \) in the basis equations (3,14) and (3,15) the following a priori estimate holds:

\[
\|w(t)\|_{L^2(0,T)} < \sqrt{2} \cdot \|f_{1-\mu}(t)\|_{L^2(0,T)}
\]

Since \(- \frac{dz(t)}{dt} > 0\) and for any function \( f_{1-\mu}(t) \in L^+_{1-\mu}(0, T) \) the function \( F_\mu(t) = 2 \cdot \int_0^t g^2(\tau) \, d\tau \) is increasing (Remark 3.4), we rewrite the inequality (3,16) as:

\[
-\frac{1}{k} \cdot \frac{dz(t)}{dt} < z(t) \cdot F_\mu(t) + F_\mu(t) \cdot \int_0^t \left( - \frac{dz(\tau)}{d\tau} \right) \, d\tau
\]

Let us denote: \( \int_0^t \left( - \frac{dz(\tau)}{d\tau} \right) \, d\tau = z_1(t) \). Then this equation will accept the following kind:

\[
\frac{dz_1(t)}{dt} - k \cdot F_\mu(t) \cdot z_1(t) < k \cdot F_\mu(t) \cdot z(t)
\]

Since \( z_1(0) = 0 \) and \( k \cdot F_\mu(t) \cdot z(t) > 0 \), from this inequality and Gronwall’s Lemma we infer that:

\[
z_1(t) = \int_0^t \left( - \frac{dz(\tau)}{d\tau} \right) \, d\tau < k \cdot \left( \int_0^t F_\mu(\tau) \cdot z(\tau) \cdot e^{-k \int_0^\tau F_\mu(\tau_1) \, d\tau_1} \, d\tau \right) \cdot e^{k \int_0^t F_\mu(\tau) \, d\tau}
\]

Since \( z(t) > 0 \) and the function \( F_\mu(t) > 0 \) is increasing, we rewrite this inequality:

\[
z(0) - z(t) < k \cdot F_\mu(t) \left( \int_0^t z(\tau)e^{-k \int_0^\tau F_\mu(\tau_1) \, d\tau_1} \, d\tau \right) \cdot e^{k \int_0^t F_\mu(\tau) \, d\tau}
\]

Let us denote:

\[
z_2(t) = \int_0^t z(\tau)e^{-k \int_0^\tau F_\mu(\tau_1) \, d\tau_1} \, d\tau
\]

and we present (3.21) in the following form:

\[
\frac{dz_2(t)}{dt} + k \cdot F_\mu(t) \cdot z_2(t) > z(0) \cdot e^{-k \int_0^t F_\mu(\tau) \, d\tau}
\]

Using the Gronwall’s Lemma and \( z_2(0) = 0 \), we infer that:

\[
z_2(t) > z(0) \cdot t \cdot e^{-k \int_0^t F_\mu(\tau) \, d\tau}
\]
Since \( z_2(t) = \int_0^t z(\tau) e^{-k \int_0^\tau F_\mu(\tau')d\tau'} d\tau < \int_0^t z(\tau)d\tau \), it follows from (3,22),(3,24) that
\[
z(0) \cdot t < \left( \int_0^t z(\tau)d\tau \right) \cdot e^{k \int_0^t F_\mu(\tau)d\tau}
\] (3,25)
where the function \( z(t) \) is defined by (3,11') and (3,6):
\[
z(t) = z(0) \cdot e^{-k \int_0^t w_t(\tau)d\tau} = z(0) \cdot e^{-\frac{k}{\tau_0} \int_0^t w^2(\tau)(t-\tau)^{1-\mu}d\tau}
\] (3,25')

**Remark 3.5.** From (3,25') we have:
\[
\frac{dz(t)}{dt} = -z(0) \cdot k \cdot \left( \int_0^t \frac{w^2(\tau)}{(t-\tau)^\mu} d\tau \right) \cdot e^{\frac{k}{\tau_0} \int_0^t w^2(\tau)(t-\tau)^{1-\mu}d\tau} < 0.
\]
If for a some \( t_0 \in (0, \infty) \) \( \frac{dz(t)}{dt} \bigg|_{t=t_0} = 0 \), then the following two cases are possible:
1. \( \int_0^{t_0} \frac{w^2(\tau)}{(t_0-\tau)^\mu} d\tau = 0 \), or
2. \( \int_0^{t_0} \frac{w^2(\tau)}{(t_0-\tau)^\mu} d\tau = 0 \), then \( w(t) \equiv 0 \) on \([0,t_0] \), since \( w(t) \geq 0 \) on \([0,t_0] \). If 2. \( \int_0^{t_0} \frac{w^2(\tau)}{(t_0-\tau)^\mu} d\tau = \infty \), then below we shall prove that for all positive functions \( w^2(t) \) satisfying to the basis inequality (3,25):
\[
\int_0^{t_0} w^2(\tau)(t_0-\tau)^{1-\mu}d\tau < \infty.
\]

The following is the key Lemma for the proof of Theorem 2.2.

**Lemma 3.5.** Let \( t > 0 \) is an arbitrary real number. For all positive functions \( w^2(\tau) \) satisfying to the basis inequality (3,25) follows that:
\[
\int_0^t w^2(\tau)(t-\tau)^{1-\mu}d\tau < \infty
\] (3,36)
where \( \mu : 1/2 < \mu < 1 \). Or, passing to limit \( \mu \to 1 \), we obtain:
\[
\int_0^t w^2(\tau) d\tau < \infty
\] (3,36')

We shall prove this Lemma by contradiction method and rewrite the basis inequality (3,25):
\[
z(0) \cdot t < \left( \int_0^t z(\tau)d\tau \right) \cdot e^{k \int_0^t F_\mu(\tau)d\tau}
\] (3,25)
where the function \( z(t) \) is defined by (3,11'):
\[
z(t) = z(0) \cdot e^{-\frac{k}{\tau_0} \int_0^t w^2(\tau)(t-\tau)^{1-\mu}d\tau}
\] (3,25')
Let for a some number \( t_0 : \int_0^{t_0} w^2(\tau)(t_0-\tau)^{1-\mu} d\tau = \infty \). I.e. \( w^2(\tau) \approx \frac{e}{|t_0-\tau|^\lambda |t_0-\tau|^{1-\mu}} \)
where \( \lambda \geq 1 \) is a real number. Then for all \( t \geq t_0 \): \( \int_0^t w^2(\tau)(t-\tau)^{1-\mu} d\tau = \infty \). And \( z(t) = z(0) \cdot e^{-\frac{k}{\tau_0} \int_0^t w^2(\tau)(t-\tau)^{1-\mu}d\tau} \equiv 0 \) for \( t \geq t_0 \). Since \( e^{-\frac{k}{\tau_0} \int_0^t w^2(\tau)(t-\tau)^{1-\mu} d\tau} \bigg|_{t_0}^t = 0 \), then, integrating by part, we have:
\[
\frac{1}{z(0)} \int_0^t z(\tau)d\tau = + \int_0^t t \cdot e^{-\frac{k}{\tau_0} \int_0^t w^2(\tau)(t-\tau)^{1-\mu} d\tau} d\left( \frac{k}{1-\mu} \int_0^t w^2(\tau)(t-\tau)^{1-\mu} d\tau \right)
\]
In this equality we shall make the replacement of variable \( \frac{k}{1-\mu} \int_{t}^{t_0} w^2(\tau)(t-\tau)^{1-\mu} d\tau = t_1 \). Then:

\[
\frac{1}{z(0)} \int_{t}^{t_0} z(\tau) d\tau < t_0 \cdot \int_{t}^{\infty} e^{-t_1} d t_1 = t_0
\]

From the definition of the function \( F_\mu(t) \) by (3,15) we obtain:

\[
\int_{0}^{t} F_\mu(\tau) d\tau = \frac{2}{\Gamma_{1-\mu}} \cdot \frac{1}{1-\mu} \cdot \int_{0}^{t} f^2(\tau)(t-\tau)^{1-\mu} d\tau \tag{3,15'}
\]

Using the following limits: \( \lim_{\mu \to 1} \Gamma(1-\mu) \cdot (1-\mu) = 1 \) and \( \lim_{\mu \to 1} (t-\tau)^{1-\mu} = 1 \), we get: \( \lim_{\mu \to 1} \int_{0}^{t} F_\mu(\tau) d\tau = \int_{0}^{t} f^2(\tau) d\tau \). Using these facts, from the basis inequality (2,25) we have:

\[
t < t_0 \cdot e^{\int_{0}^{t} f^2(\tau) d\tau}
\]

But, for \( t \gg 1 \) we have received the contradiction. Therefore \( \int_{0}^{t} w^2(\tau)(t-\tau)^{1-\mu} d\tau < \infty \).

Lemma is proved. ▮

**Remark 3.35**'. Hence, for any positive function \( w^2(t) \geq 0 \) satisfying to the basis inequality (3,25) the function \( z(t) > 0 \) is continuous on \([0, \infty)\) and monotonously decreasing from \( z(0) \) to zero on \([0, \infty)\).

Let \( T > 0 \) be an arbitrary real number. We rewrite the basis inequality (3,25) for \( t = T \) as:

\[
T \cdot z(0) < \left( \int_{0}^{T} z(t) dt + \int_{T}^{2T} z(t) dt \right) \cdot e^{k \int_{0}^{T} F_\mu(\tau) d\tau} \tag{3,27}
\]

Since the function \( z(t) = z(0) \cdot e^{-\frac{k}{1-\mu} \int_{0}^{t} w^2(\tau)(t-\tau)^{1-\mu} d\tau} \) is continuous on \([0, \infty)\) and monotonously decreasing from \( z(0) \) to zero, there exists the numbers \( t_1, t_2 \):

\[
t_1: 0 < t_1 < T; \quad ; t_2: T < t_2 < 2T \tag{3,27'}
\]

such that the following equalities holds:

\[
\int_{0}^{T} e^{-\frac{k}{1-\mu} \int_{0}^{t} w^2(\tau)(t-\tau)^{1-\mu} d\tau} dt = \left( e^{-\frac{k}{1-\mu} \int_{0}^{t_1} w^2(\tau)(t-\tau)^{1-\mu} d\tau} \right) \cdot T \tag{3,28}
\]

\[
\int_{T}^{2T} e^{-\frac{k}{1-\mu} \int_{0}^{t} w^2(\tau)(t-\tau)^{1-\mu} d\tau} dt = \left( e^{-\frac{k}{1-\mu} \int_{0}^{t_2} w^2(\tau)(t-\tau)^{1-\mu} d\tau} \right) \cdot T \tag{3,28'}
\]

Using these equalities, we rewrite the inequality (3,27) as:

\[
1 < \left( e^{-w_2(t_1)} + e^{-w_2(t_2)} \right) \cdot e^{k \int_{0}^{T} F_\mu(\tau) d\tau} \tag{3,29}
\]

where

\[
w_2(t_1) = \frac{k}{1-\mu} \int_{0}^{t_1} w^2(\tau)(t-\tau)^{1-\mu} d\tau; \quad ; w_2(t_2) = \frac{k}{1-\mu} \int_{0}^{t_2} w^2(\tau)(t-\tau)^{1-\mu} d\tau \tag{3,29'}
\]

and present the inequality (3,29) as:

\[
e^{w_2(t_2)} < \left( e^{w_2(t_2)} + 1 \right) \cdot e^{k \int_{0}^{T} F_\mu(\tau) d\tau} \tag{3,30}
\]
Let in formula (3.15) the right-side

\[ f^2(t) = f_{1-\mu}^2(t) = \int_0^t \frac{g^2(\tau)}{(t-\tau)^{1-\mu}} d\tau \in L_{1-\mu}^+(0, T) \quad (3.31) \]

where the \( g(t) : g(t) \in L_2(0, T) \) is an arbitrary function. Then, from (3.15) we have:

\[ F_\mu(t) = \frac{2}{\Gamma_{1-\mu}} \cdot \int_0^t \frac{f_{1-\mu}^2(\tau)}{(t-\tau)^\mu} d\tau = \]

\[ = 2 \cdot \frac{1}{\Gamma_{1-\mu}} \cdot \int_0^t \frac{1}{(t-\tau)^\mu} \int_0^\tau \frac{g^2(\tau_1)}{(\tau-\tau_1)^{1-\mu}} d\tau_1 = 2 \cdot \int_0^t g^2(\tau) d\tau \quad (3.32') \]

From (3.31) and (3.32) follows that:

\[ g^2(t) = \frac{d}{dt} \int_0^t \frac{f_{1-\mu}^2(\tau)}{(t-\tau)^\mu} d\tau; \quad F_\mu(t) = 2 \int_0^t g^2(\tau) d\tau = 2 \cdot \int_0^t \frac{f_{1-\mu}^2(\tau)}{(t-\tau)^\mu} d\tau \quad (3.32') \]

and from (3.32') we have:

\[ \int_0^t F_\mu(\tau) d\tau = \frac{2}{1-\mu} \cdot \int_0^t (t-\tau)^{1-\mu} \cdot f_{1-\mu}^2(\tau) d\tau \quad (3.33) \]

Since the function \( e^{-w_2(t)} \) is decreasing on \((0, \infty)\) and \( t_2 > t_1 \), we have: \( e^{w_2(t_2)} / e^{w_2(t_1)} < 1 \).

And from (3.30) follows that

\[ e^{\frac{1}{1-\mu} \int_0^t w^2(\tau)(t-\tau)^{1-\mu} d\tau} < 2 \cdot e^{2 \cdot \frac{1}{1-\mu} \int_0^t f_{1-\mu}^2(\tau)(t-\tau)^{1-\mu} d\tau} \]

or

\[ k \int_0^{t_2} w^2(\tau)(t-\tau)^{1-\mu} d\tau < (1-\mu) \cdot ln2 + 2k \cdot \int_0^T f_{1-\mu}^2(\tau)(t-\tau)^{1-\mu} d\tau \quad (3.34) \]

Passing to the limit \( \mu \to 1 \), from \( t_2 > T \), \( \lim_{\mu \to 1} (t-\tau)^{1-\mu} = 1 \) and this inequality we obtain:

\[ \int_0^T w^2(\tau) d\tau < 2 \cdot \int_0^T f_{1-\mu}^2(\tau) d\tau \quad (3.34') \]

Lemma 3.3 is proved. \( \blacktriangleleft \)

**The proof of Theorem 3.1.**  **Lemma 3-4.** The space of functions \( L_{1-\mu}^+(0, T) \) is dense in the space of functions \( L_2^+(0, T) \) in the norm of the space \( L_2(0, T) \). I.e. for all functions \( f(t) \in L_2^+(0, T) \), there exists a sequence of functions: \( f_{n_{1-\mu}}(t) \in L_{1-\mu}^+(0, T) \):

\[ \lim_{n \to \infty} \left( ||f_{n_{1-\mu}}(t)||_{L_2(0,T)} - ||f(t)||_{L_2(0,T)} \right) < \lim_{n \to \infty} ||f_{n_{1-\mu}}(t) - f(t)||_{L_2(0,T)} \to 0 \quad (3.35) \]

or

\[ \lim_{n \to \infty} \left( ||f_{n_{1-\mu}}^2(t)||_{L_1(0,T)} - ||f^2(t)||_{L_1(0,T)} \right) < \lim_{n \to \infty} ||f_{n_{1-\mu}}^2(t) - f^2(t)||_{L_1(0,T)} \to 0 \quad (3.35') \]

**Remark 3.6.** Let us note that for any \( n = 1, 2, \ldots \), the functions \( f_{n_{1-\mu}}^2(t) \in L_{1-\mu}^+(0, T) \). From Remark 3.4 follows that the functions \( F_{n_{1-\mu}}(t) = \frac{2}{\Gamma_{1-\mu}} \cdot \int_0^t f_{n_{1-\mu}}^2(\tau) d\tau \) are
increasing on $[0,T]$. And from (3.34) in Lemma 3.3 follow the following estimates: 

$$
\| w(t) \|_{L_2(0,T)} < 2 \cdot \| f_{n-\mu}^\prime(t) \|_{L_2(0,T)}.
$$

We prove Lemma 3.4 by the contradiction method. Let there exists a function $f_0(t): f_0(t) \geq 0, f_0(t) \neq 0, f_0(t) \in L_2(0,T)$, such that for all functions $f(t): f(t) \geq 0, f(t) \in L_2(0,T)$ the following equality is valid

$$
\int_0^T f_0(t) \cdot \int_0^t f(\tau)d\tau dt = \int_0^T f(\tau) \cdot \left( \int_\tau^T f_0(t) dt \cdot \frac{1}{(t-\tau)^{1-\mu}} \right) d\tau = 0.
$$

Since $f(\tau) \geq 0$ is an arbitrary function, by this equality it follows that for all $\tau \in [0,T]:$

$$
f_0(t) \equiv 0.\text{ Then } f_0(t) \equiv 0. \text{ Lemma is proved. △}
$$

Let $\epsilon_n: 0 < \epsilon_n \ll 1, \lim_{n \to \infty} \epsilon_n = 0$ are an arbitraries real numbers. Then for any function $f(t) \in L_2(0,T), f(t) > 0$ by (3.35') and for $n \gg 1$ it follows that:

$$
f^2(t) - f^2_{n-\mu}(t) < | f^2(t) - f^2_{n-\mu}(t) | + f^2_{n-\mu}(t) < \epsilon_n + f^2_{n-\mu}(t)
$$

Then:

$$
F_\mu^\prime(t) = \frac{2}{\Gamma_{1-\mu}} \cdot \int_0^t f^2(\tau) d\tau < \frac{2}{\Gamma_{1-\mu}} \cdot \int_0^t \epsilon_n + f^2_{n-\mu}(\tau) d\tau = \frac{2}{\Gamma_{1-\mu}} \cdot \left( \frac{t^{1-\mu}}{1-\mu} \cdot \epsilon_n + \int_0^t f^2_{n-\mu}(\tau) \frac{d\tau}{(t-\tau)^{1-\mu}} \right)
$$

Let us denote:

$$
F_\mu^n(t) = \frac{2}{\Gamma_{1-\mu}} \cdot \left( \frac{t^{1-\mu}}{1-\mu} \cdot \epsilon_n + \int_0^t f^2_{n-\mu}(\tau) \frac{d\tau}{(t-\tau)^{1-\mu}} \right)
$$

(3.36)

Since the functions $f^2_{n-\mu} \in L_1^{1-\mu}(0,T)$ and the function $t^{1-\mu}$ is increasing, the functions $F_\mu^n(t)$ are increasing on $[0,T]$ and $F_\mu^n(t) < F_\mu^n(t)$. Therefore, from the basis inequality (3.25) we have:

$$
z(0) \cdot t < \left( \int_0^t z(\tau) d\tau \right) \cdot e^{k \int_0^t F_\mu^n(\tau) d\tau}
$$

(3.25')

and

$$
\int_0^t F_\mu^n(\tau) d\tau = \frac{2}{\Gamma_{1-\mu}} \cdot \left( \frac{t^{2-\mu}}{(1-\mu)(2-\mu)} \epsilon_n + \frac{1}{1-\mu} \cdot \int_0^t (t-\tau)^{1-\mu} f^2_{n-\mu}(\tau) d\tau \right)
$$

Further similarly to the proof of an estimate (3.34), we obtain:

$$
k \int_0^{t_2} w^2(\tau) (t-\tau)^{1-\mu} d\tau < (1-\mu) \cdot ln2 + \int_0^T F_\mu^n(\tau) d\tau
$$

(3.34')

Using the following limits: $\lim_{\mu \to 1} \Gamma(1-\mu) \cdot (1-\mu) = 1, \lim_{n \to \infty} \| f^2_{n-\mu}(t) \|_{L_1(0,T)} = \| f^2(t) \|_{L_1(0,T)}, \lim_{n \to \infty} \epsilon_n = 0$ and Remark 3.6, passing to the limits $\mu \to 1$ and $n \to \infty$, from the inequality (3.34') we get:

$$
\int_0^T w^2(\tau) d\tau < 2 \cdot \int_0^T f^2(\tau) d\tau
$$

(3.37)

Theorem 3.1 is proved. △△
Proof of Theorem 2.1.

Definition 3. If the sequence of vector-functions \( \{w_n(x, t)\} \) weakly converges to the vector-function \( w_0(x, t) \) in the space \( L_2(Q_t) \), then we denote: \( w_n(x, t) \rightharpoonup w_0(x, t) \). I.e. for an arbitrary vector-function \( u(x, t) \in L_2(Q_t) \) the following convergence is valid: \( (w_n(x, t) , u(x, t))_{L_2(Q_t)} \to (w_0(x, t) , u(x, t))_{L_2(Q_t)} \) as \( n \to \infty \).

2) If the sequence of vector-functions \( \{w_n(x, t)\} \) strongly converges to the vector-function \( w_0(x, t) \) in the space \( L_2(Q_t) \), then we denote: \( w_n(x, t) \Rightarrow w_0(x, t) \). I.e. \( \lim_{n \to \infty} ||w_n(x, t) - w_0(x, t)||_{L_2(Q_t)} \to 0 \).

Lemma 3.4 Let the vector-function \( w(x, t) = (w^i(x, t))_{i=1,2,3} \in L_2(Q_t) \). I.e. \( ||w(x, \tau)||_{L_2(Q_t)} = \sum_{i=1}^{3} G(w_i(x, \tau)) = 3/2 < \infty \). We define the following nonlinear operator \( K \) on the vector-space \( L_2(Q_t) \):

\[
K \ast \left( w(x, t) \right) = Gw^j(x, t)G_{xj}w(x, t) = \left( \sum_{j=1}^{3} Gw^j(x, t)G_{xj}w^j(x, t) \right)_{i=1,2,3}
\]

Let us prove that

\[
K \ast w_n(x, t) \Rightarrow K \ast w_0(x, t)
\]

as \( w_n(x, t) \rightharpoonup w_0(x, t) \).

It follows from this proposition that the operator \( K \) is compact on the vector-space \( L_2(Q_t) \). It is follows from book [3 p.42]

Then

\[
||K \ast w_n(x, t) - K \ast w_0(x, t)||_{L_2(Q_t)} \leq c \sum_{i,j=1}^{3} ||Gw_n^j(x, t)G_{xj}w^j_n(x, t) - Gw_0^j(x, t)G_{xj}w_0^j(x, t)||_{L_2(Q_t)}
\]

Let us estimate a each member:

\[
||Gw_n(x, t)G_{xj}w_n(x, t) - Gw_0(x, t)G_{xj}w_0(x, t)||_{L_2(Q_t)} \leq \quad (3.39)
\]

\[
\leq ||G_{xj}w_n(x, t) \left( Gw_n(x, t) - Gw_0(x, t) \right) ||_{L_2(Q_t)} + ||Gw_0(x, t) \left( G_{xj}w_n(x, t) - G_{xj}w_0(x, t) \right) ||_{L_2(Q_t)}
\]

We obtain the following estimates, using the formula (2.12.2) in Proposition 5, the inequality (2.4) in Proposition 1 and Proposition 3 for \( \mu : 5/8 < \mu < 1 \):

\[
||G_{xj}w_n(x, t) \left( Gw_n(x, t) - Gw_0(x, t) \right) ||_{L_2(Q_t)} \leq \quad (3.40)
\]

We obtain the following estimates , using the formula (2.12.1) in Proposition 5 and the inequality (2.4) in Proposition 1 for \( \mu : 5/8 < \mu < 1 \):

\[
||Gw_0(x, t) \left( G_{xj}w_n(x, t) - G_{xj}w_0(x, t) \right) ||_{L_2(Q_t)} \leq
\]
\[
\leq c \left\| Gw_0(x, t) \right\|_{L^\frac{2n}{2n - 1} \left( Q_t \right)} \cdot \left\| G_x \left( w_n(x, t) - w_0(x, t) \right) \right\|_{L^\frac{2n}{2n - 1} \left( Q_t \right)} \\
\leq c \left\| w_0(x, t) \right\|_{L^2 \left( Q_t \right)} \cdot \left\| G_x \left( w_n(x, t) - w_0(x, t) \right) \right\|_{L^\frac{2n}{2n - 1} \left( Q_t \right)} \to 0 \quad (3.41)
\]

Note that by the weakly convergence \( w_n(x, t) - w_0(x, t) \to 0 \) it follows that for any number \( n \) there is a constant \( c \) such that: \( \|w_n(x, t)\|_{L^2 \left( Q_t \right)} < c\|w_0(x, t)\|_{L^2 \left( Q_t \right)} \). Since \( G \left( w_n(x, t) - w_0(x, t) \right) \in W^2_2 \left( Q_t \right) \), the space \( W^2_2 \left( Q_t \right) \) is compactly enclosed into the space \( L^\frac{3 n}{2 n - 1} \left( Q_t \right) \) [5, p.78], and on the space \( L^\frac{3 n}{2 n - 1} \left( Q_t \right) \) the compact operator \( G \) translates the weakly convergence \( w_n(x, t) - w_0(x, t) \to 0 \) to the strongly convergence, then the strongly convergence (3.40) is valid.

Since \( G_x \left( w_n(x, t) - w_0(x, t) \right) \in W^1_2 \left( Q_t \right) \), the space \( W^1_2 \left( Q_t \right) \) is compactly enclosed into the space \( L^\frac{3}{2 n - 1} \left( Q_t \right) \) [5, p.78], and on the space \( L^\frac{3 n}{2 n - 1} \left( Q_t \right) \) the compact operator \( G_x \) translates the weakly convergence \( w_n(x, t) - w_0(x, t) \to 0 \) to the strongly convergence, then the strongly convergence (3.41) is valid. The strongly convergence (3.38) follows by (3.39), (3.40), (3.41). Lemma is proved. ▲

**The proof of Theorem 2.1.** Let \( z(x, t) = (z_i(x, t))_{i=1,2,3} \in L_2 \left( Q_t \right) \) be an arbitrary vector-function. And the sequence vector-functions \( w^n(x, t) \) weakly converges to the vector-function \( w^0(x, t) \), i.e. \( w^n(x, t) \rightharpoonup w^0(x, t) \). On the vector-space \( L_2 \left( Q_t \right) \) we define the following nonlinear operator \( K_p \):

\[
K_p \ast (w(x, t)) = \left( \sum_{j=1}^{3} G \left( w_j(x, t) + \frac{\partial p(x, t)}{\partial x_j} \right) \cdot G_{x_j} \left( w_i(x, t) + \frac{\partial p(x, t)}{\partial x_i} \right) \right)_{i=1,2,3} \quad (3.42)
\]

where the functions \( \frac{\partial p(x, t)}{\partial x_i} \) are defined by the functions \( w_i(x, t) \) from the formula (2.16) in Proposition 8. On the vector space \( w(x, t) = (w_i(x, t))_{i=1,2,3} \in L_2 \left( Q_t \right) \) by the formula (2.18) in Proposition 8 we have defined the following linear and bounded operator:

\[
P \ast (w_i(x, t))_{i=1,2,3} = \left( \frac{\partial p(x, t)}{\partial x_i} \right)_{i=1,2,3} \in L_2 \left( x, t \right) \quad (3.43)
\]

where the functions \( \frac{\partial p(x, t)}{\partial x_i} \) is defined by the functions \( w_i(x, t) \) from the formula (2.16) in Proposition 8. Since \( P \) is linear and bounded operator on \( L_2 \left( Q_t \right) \), there exists the linear and bounded connected operator \( P^* \). Let \( z(x, t) \in L_2 \left( Q_t \right) \) is an arbitrary vector-function and \( w^n(x, t) \rightharpoonup w^0(x, t) \). Then:

\[
\left( \left( w^n_i(x, t) + \frac{\partial p(x, t)}{\partial x_i} \right)_{i=1,2,3} \cdot z(x, t) \right)_{L_2 \left( Q_t \right)} = \left( \left( w^n_i(x, t) + P \ast (w^n_i(x, t)) \right)_{i=1,2,3} \cdot z(x, t) \right)_{L_2 \left( Q_t \right)} = \\
= \left( w^n(x, t) \cdot z(x, t) + P^* \ast z(x, t) \right)_{L_2 \left( Q_t \right)} = \left( w^0(x, t) \cdot z(x, t) + P^* \ast z(x, t) \right)_{L_2 \left( Q_t \right)} = \\
= \left( w^0(x, t) + P^* \ast w^0(x, t) \cdot z(x, t) \right)_{L_2 \left( Q_t \right)}
\]

as \( n \to \infty \). I.e, it is proved that: \( w^n(x, t) + P^* \ast w^n(x, t) \rightharpoonup w^0(x, t) + P^* \ast w^0(x, t) \). Then, similarly to the proof in Lemma 3.4, proves that: \( K_p \ast w^n(x, t) \Rightarrow K_p \ast w^0(x, t) \).
Hence, it follows that on the vector-space \( L_2(Q_t) \) the nonlinear operator \( K_p \) is compact. [3 p.42]. Therefore, it follows by the Leray-Schauder’s theorem in Proposition 6, that the basis equation (2.20) has at least one solution \( w(x, t) \in L_2(Q_t) \) and it follows from Theorem 3.1 that: \( \|w(x, t)\|_{L_2(Q_T)} \leq \sqrt{2} \cdot \|f(x, t)\|_{L_2(Q_T)} \). Then, it follows by Proposition 8 that there exists the smooth solution \( u(x, t) \in W^{2,1}_x(Q_T) \cap H_2(Q_t) \). But the Navier-Stokes problem has the unique smooth solution [3p.139]. Therefore, \( w(x, t) \in L_2(Q_T) \) is the unique solution to (2.20). The existence and smoothness of the solution to Navier-Stokes equation is proved.\

4. The Navier-Stokes problem for the inhomogeneous boundary condition.

Let \( \Omega \subset R^3 \) be a finite domain bounded by a Lipschitz surface \( \partial\Omega \) and \( Q_T = \Omega \times [0, T], S = \partial\Omega \times [0, T], x = (x_1, x_2, x_3) \) and \( u(x, t) = (u_i(x, t))_{i=1,2,3}, f(x, t) = (f_i(x, t))_{i=1,2,3} \) are vector-functions. Here \( T > 0 \) is an arbitrary real number. The Navier-Stokes equations are given by:

\[
\frac{\partial u_i(x, t)}{\partial t} - \rho \Delta u_i(x, t) - \sum_{j=1}^3 u_j(x, t) \frac{\partial u_i(x, t)}{\partial x_j} + \frac{\partial p(x, t)}{\partial x_i} = f_i(x, t) \quad (4.1),
\]

\[
\text{div } u(x, t) = \sum_{i=1}^3 \frac{\partial u_i(x, t)}{\partial x_i} = 0 \quad , i = 1, 2, 3
\]

The Navier-Stokes problem 1. Find a vector-function \( u(x, t) = (u_i(x, t))_{i=1,2,3} : \Omega \times [0, T] \to R^3 \), the scalar function \( p(x, t) : \Omega \times [0, T] \to R^1 \) satisfying the equation (4.1) and the following initial condition

\[
u(x, 0) = a(x) , \quad u(x, t) |_{\partial\Omega \times [0, T]} = 0 \quad (4.2)
\]

where \( \text{div } a(x) = \frac{\partial a_1(x)}{\partial x_1} + \frac{\partial a_2(x)}{\partial x_2} + \frac{\partial a_3(x)}{\partial x_3} = 0 \) and \( a(x) \in W_2^1(\Omega) \).

**Theorem 4.1.** For any right-hand side \( f(x, t) \in L_2(Q_t) \) in equation (4.1) and for any real numbers \( \rho > 0, t > 0 \), the Navier-Stokes problem 1 has a unique smooth solution \( u(x, t) : u(x, t) \in W_2^1(Q_t) \cap H_2(Q_t) \), the scalar function \( p(x, t) : p_{x_i}(x, t) \in L_2(Q_t) \) satisfying to (4.1) almost everywhere on \( Q_t \), and to the initial conditions (4.2). The following estimate holds:

\[
\|u(x, t)\|_{W_2^1(Q_t)} + \left\| \frac{\partial p(x, t)}{\partial x_i} \right\|_{L_2(Q_t)} \leq c \left( \|f\|_{L_2(Q_t)} + \|a(x)\|_{W_2^1(\Omega)} \right) \quad (4.3)
\]

\[\blacktriangleleft\]

In 1941 Hopf proved that this problem has a weak solution \( u(x, t) \):

\[
\|u(x, t)\|_{L_2(\Omega)} + 2\rho \int_0^t \|u(x, \tau)\|_{L_2(\Omega)} d\tau < \|a(x)\|_{W_2^1(\Omega)} + c \int_0^t \|f(x, \tau)\|_{L_2(\Omega)} d\tau \quad (4.4)
\]

and \( \lim_{t \to 0} \|u(x, t) - a(x)\|_{L_2(\Omega)} = 0 \). [3p.143] \[\blacktriangleleft\]

The problem 2. Find a vector-function \( u^0(x, t) = (u_i^0(x, t))_{i=1,2,3} : \Omega \times [0, T] \to R^3 \) satisfying the following equation and the initial condition:

\[
\frac{d u^0(x, t)}{dt} - \rho \Delta u^0(x, t) = 0; \quad u^0(x, 0) = a(x), \quad u^0(x, t)|_{\partial\Omega \times [0, T]} = 0 \quad (4.5)
\]

\[\blacktriangleleft\]

It follows by \( \text{div } a(x) = 0 \) and \( u^0(x, t)|_{\partial\Omega \times [0, T]} = 0 \) that: \( \text{div } u^0(x, t) = 0 \) for any \( t > 0 \). And it follows by \( a(x) \in W_2^1(\Omega) \) that: \( |u^0(x, t)| \leq c \|a(x)\|_{W_2^1(\Omega)} \cdot \left| \frac{\partial u^0(x, t)}{\partial x_i} \right| \leq c \|a(x)\|_{W_2^1(\Omega)} \).
Then the vector-function: \( \mathbf{v}(x, t) = \mathbf{u}(x, t) - \mathbf{u}^0(x, t) \) satisfies the following system of equations:

\[
\frac{\partial v_i(x, t)}{\partial t} - \rho \triangle v_i(x, t) - \sum_{j=1}^{3} v_j(x, t) \frac{\partial v_i(x, t)}{\partial x_j} - \sum_{j=1}^{3} u_j^0(x, t) \frac{\partial v_i(x, t)}{\partial x_j} - \\
- \sum_{j=1}^{3} v_j(x, t) \frac{\partial u_j^0(x, t)}{\partial x_j} - \sum_{j=1}^{3} u_j^0(x, t) \frac{\partial u_j^0(x, t)}{\partial x_j} + \frac{\partial p(x, t)}{\partial x_i} = f_i(x, t) \quad (4.6),
\]

\[
div \mathbf{v}(x, t) = \sum_{i=1}^{3} \frac{\partial v_i(x, t)}{\partial x_i} = 0 \quad , i = 1, 2, 3
\]

and the following initial conditions:

\[
\mathbf{v}(x, 0) = 0, \quad \mathbf{v}(x, t)|_{\partial \Omega \times [0, T]} = 0 \quad (4.7)
\]

Similarly, we introduce the unknown vector-function \( \left( w_i(x, t) \right)_{i=1,2,3} \in L_2(Q_t) : \)

\[
\frac{\partial v_i(x, t)}{\partial t} - \rho \triangle v_i(x, t) - \frac{\partial p(x, t)}{\partial x_i} = w_i(x, t) \quad (4.8)
\]

\[
div \mathbf{v}(x, t) = \sum_{i=1}^{3} \frac{\partial v_i(x, t)}{\partial x_i} = 0 \quad , i = 1, 2, 3
\]

and the following initial conditions:

\[
\mathbf{v}(x, 0) = 0, \quad \mathbf{v}(x, t)|_{\partial \Omega \times [0, T]} = 0 \quad (4.8')
\]

In the Proposition 8 we have proved that for any right-side \( \mathbf{w}(x, t) \in L_2(Q_t) \) this problem has a unique solution \( \left( v_i(x, t) \right)_{i=1,2,3} \in W^{2,1}_2(Q_t) \) and \( \| \mathbf{v}(x, t) \|_{W^{2,1}_2(Q_t)} < c \cdot \| \mathbf{w}(x, t) \|_{L_2(Q_t)} \).

It follows from (4.8) that.

\[
v_i(x, t) = \int_0^t \int_{\Omega} G(x, t; \xi, \tau) \left( w_i(\xi, \tau) + \frac{\partial p(\xi, \tau)}{\partial \xi_i} \right) d\xi d\tau; \quad i = 1, 2, 3 \quad (4.9)
\]

Using (4.8) and this formula, we rewrite the Navier-Stokes equation (4.6) as:

\[
w_i(x, t) - \sum_{j=1}^{3} G \left( w_i(\xi, \tau) + \frac{\partial p(\xi, \tau)}{\partial \xi_i} \right) \cdot G_{x_j} \left( w_i(\xi, \tau) + \frac{\partial p(\xi, \tau)}{\partial \xi_i} \right) - \\
- \sum_{j=1}^{3} \frac{\partial p(\xi, \tau)}{\partial \xi_i} \cdot u_j^0(x, t) - \sum_{j=1}^{3} G \left( w_j(\xi, \tau) + \frac{\partial p(\xi, \tau)}{\partial \xi_j} \right) \cdot \frac{du_j^0(x, t)}{dx_j} - (4.10)
\]

\[
- \sum_{j=1}^{3} u_j^0(x, t) \cdot \frac{du_j^0(x, t)}{dx_j} = f_i(x, t)
\]
Similarly to the inequality (3.2), using the estimates (4.4), (4.5), it follows by the equation (4.10) that:

\[
\int_{0}^{t} \frac{w(\tau) + p(\tau)}{(t-\tau)^{\mu}} \, d\tau = \int_{0}^{t} \frac{w(\tau) + p(\tau)}{(t-\tau)^{\mu/2}} \cdot \frac{1}{(t-\tau)^{\mu/2}} \, d\tau < \int_{0}^{t} \frac{(w(\tau) + p(\tau))^{2}}{(t-\tau)^{\mu}} \, d\tau + \frac{t^{1-\mu}}{1-\mu}
\]

From the inequality \(a \cdot b < a^{2} + b^{2}\) the following inequality holds:

\[
\int_{0}^{t} \frac{w(\tau)}{(t-\tau)^{\mu}} \, d\tau = \int_{0}^{t} \frac{w(\tau)}{(t-\tau)^{\mu/2}} \cdot \frac{1}{(t-\tau)^{\mu/2}} \, d\tau < \int_{0}^{t} \frac{w(\tau) + p(\tau)}{(t-\tau)^{\mu/2}} \, d\tau + \frac{t^{1-\mu}}{1-\mu}
\]

By this inequality and the inequality (3.4) in Lemma 3.2 we rewrite the inequality (4.11) as:

\[
w(t) < \left( f(t) + \frac{t^{1-\mu}}{1-\mu} \right) + \left( b_{1} + c(T) \right) \cdot (1 + c) \int_{0}^{t} \frac{w^2(\tau)}{(t-\tau)^{\mu}} \, d\tau \quad \text{(4.12)}
\]

From the Theorem 3.1, using this inequality, similarly we obtain:

\[
\|w(x,t)\|_{L_{2}(Q_{t})} \leq \sqrt{2} \cdot c(T) \cdot \|f_{u}(x,t)\|_{L_{2}(Q_{t})} \quad \text{(4.13)}
\]

where the vector-function \(f_{u}(x,t) = \left( f_{i}(x,t) \right)_{i=1,2,3} + \left( \sum_{i=1}^{3} u_{j}^{0}(x,t) \cdot \frac{du_{j}^{0}(x,t)}{dx_{j}} \right)_{i=1,2,3} \) and \(c(T)\) are defined by (4.11'). And we present the basic equation (4.10) as:

\[
w(x,t) - (K_{p} + K_{1}) \ast (E + P)w(x,t) = f(x,t) + \sum_{j=1}^{3} u_{j}^{0}(x,t) \cdot \frac{du_{j}^{0}(x,t)}{dx_{j}} \quad \text{(4.14)}
\]

where the operator \((K_{p})\) is defined by (3.42), the operator \(P\) is defined by (3.43) and \(E\) is an identify operator, i.e. \(E(w(x,t)) = w(x,t)\). The operator \((K_{1}) \ast (E + P)\) is defined as:

\[
K_{1} \ast (E + P)(w(x,t)) = \sum_{j=1}^{3} \left( \frac{du_{j}^{0}(x,t)}{dx_{j}} \cdot G + u_{j}^{0}(x,t) \cdot G_{x_{j}} \right) \ast (E + P)(w(x,t)) \quad \text{(4.15)}
\]

As the proof of Theorem 2.1 (p. 19) and definitions of the operators \(K_{p}, P, K_{1}\) by the formulas (3.42), (3.43) and (4.15) it is proves that the following operators: \((K_{p}) \ast (E + P)\); \((K_{1}) \ast (E + P)\) are compact on the vector-space \(L_{2}(Q_{t})\). Similarly, using Leray-Schauder's theorem and the estimate (4.13), the existence and smoothness of solution to the Navier-Stokes problem with \(u(x,0) = a(x) \neq 0\) is proved. Thank you for your attention.
Declarations

Competing interests
The authors declare that they have no competing interests.

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