EXTENDED GENERALISED FLETT'S MEAN VALUE THEOREM

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Various forms of Mean Value Theorems are available in the literature. If we use Flett’s Mean Value Theorem in Extended Generalized Mean Value Theorem then what would the new theorem look like. A sincere effort is done to develop this theorem. This theorem is named as Extended Generalised Flett’s Mean Value Theorem.

1. INTRODUCTION

Lagrange’s Mean Value Theorem (LMVT) often called Mean value Theorem (MVT) is one of the most important result in mathematical analysis. It is important tool used in differential and integral calculus. For example, it is useful in proving Fundamental Theorem of Calculus. Every mathematics student knows the Lagranges mean value theorem which has appeared in Lagranges book Théorie des fonctions analytiques in 1797 as an extension of Rolles result from 1691. Mean Value Theorem says something about the slope of a function on closed interval based on the values of the function at the two endpoints of the interval. It relates local behavior of the function to it’s global behavior. This theorem turns out to be the key to many other theorems about the graph of functions and their behavior. The MVT can be stated as below.

**Lagrange’s Mean Value Theorem :** Suppose \( f : [a, b] \to \mathbb{R} \) is a function satisfying two conditions:
1. \( f \) is continuous on closed interval \([a, b]\)
2. \( f \) is differentiable on open interval \((a, b)\)
Then there \( \exists \) a number \( c \) in \((a, b)\) such that
\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Many extension, generalization and forms of MVT were proposed by many mathematicians from all over the worlds. In this article we will discuss Flett’s Mean Value Theorem (FMVT) \[\] and Extended Generalized Mean Value Theorem (EGMVT) \[\]. Then we will state a new form of MVT based on FMVT and EGMVT and we called it Extended Generalized Fleet’s Mean Value Theorem (EGFMVT). We also provide an example for the support of our proposed theorem.

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2. Flett’s Mean Value Theorem

In 1958, Thomas Muirhead Flett [1] gave the idea of a Mean Value Theorem using different conditions as used in Mean Value Theorem, we called it Flett’s mean value theorem (FMVT). The FMVT was based on some observations: 1. To notice the changes if in Rolle’s Theorem the hypothesis $f(a) = f(b)$ refers to higher order derivatives? 2. To see if there is any analogy with LMVT? 3. To see which geometrical consequences do such result have? [2].

**Notations:** Throughout in this article we will use following notations.
1. $C(M) \rightarrow$ space of continuous functions.
2. $D^n(M) \rightarrow n$ -times differentiable real-valued function on set $M \subseteq \mathbb{R}$.
3. Under continuity of a function on $[a,b]$ we understand its continuity on $(a,b)$ and one- sided continuity at end points of interval.
4. Similarly differentiability holds under open interval.
5. For a function $f, g$ on an interval $[a,b]$ the expression of the form.

$$\frac{f^n(b) - f^n(a)}{g^n(b) - g^n(a)}, n \in \mathbb{N} \cup \{0\}$$

will be denoted by symbol $\frac{b-a}{a}K(f^n, g^n)$. If denominator is equal to $b-a$ we will only write $\frac{b-a}{a}K(f^n)$.

So the Lagrange Theorem in the introduced notation has the form: if $f \in C[a,b] \cap D(a,b)$ then $\exists \eta \in (a,b)$ s.t. $f' (\eta) = \frac{b-a}{a}K(f)$, where we use usual convection $f^{(0)} = f$.

Basically, Flett focus on the theme of Rolle’s Theorem where condition $f(a) = f(b)$ is replaced by $f'(a) = f'(b)$. It is a Lagrange’s Type Mean Value Theorem with Rolle’s Type Condition.

**Theorem 2.1. Fletts Mean Value Theorem-I** [1] If $g \in C[a,b]$, then from the integral mean value theorem $\exists \eta \in (a,b)$ s.t.

$$g(\eta) = \frac{1}{b-a} \int_a^b g(t) dt$$

(1)

**Theorem 2.2. Fletts Mean Value Theorem-II** [1] If $f \in C[A,b] \cap D(a,b)$ and $f'(a) = f'(b)$, then $\exists \eta \in (a,b)$ s.t. $f'(\eta) = \frac{b-a}{a}K(f)$.

We refer the reader [1, 2] for the proof of the above two theorems.

**Example 2.3.** [2] In which point of curve $y = x^3$ the tangent passes through the point $X = [-2, 2]$?

**Solution** - To verify $X$ lies on curve and $y = x^2$ differentiable on $\mathbb{R}$. Since $y' = 3x^2$ is even function on $\mathbb{R}$.

$y'(-2) = 12 = y'(2)$
By Flett’s Theorem, \( \exists \eta \in (-2, 2) \) s.t.
\[
f'(\eta) = \frac{\eta}{2} K(f)
\]
\[
\Rightarrow 3\eta^2 = \frac{n^3 + 8}{\eta + 2}
\]
\[
\Leftrightarrow \eta^2 + 3\eta - 4 = 0
\]
\[
\Leftrightarrow (\eta + 4)(\eta - 1) = 0
\]
Because \(-4 \notin (-2, 2)\), we consider only \(\eta = 1\). Then \(y(\eta) = 1\)
The desired point is \(T = [-1, 1]\).

3. Extended Generalized Mean Value Theorem

In 2014, Phillip Mafuta [3], stated and proved a theorem of a similar flavor to the Generalized Mean Value Theorem for functions of one variable. For lack of a better term, he called the theorem “Extended Generalized Mean Value Theorem (EGMVT)”. In addition, he applied Rolle’s Theorem to prove the EGMVT. Also, he deduced some corollaries for Mean Value Theorems. In addition, the EGMVT is verified by an example.

**Theorem 3.1.** [3] Let \(n \in \mathbb{N}\) and let \(f(x), g_1(x), g_2(x), g_3(x), \ldots, g_n(x)\) be \(n + 1\) continuous functions on a closed bounded interval \([a, b]\) and differentiable in an open interval \((a, b)\) with \(g'_i(x) \neq 0\), \(\forall x \in (a, b)\) for \(i = 1, 2, 3, \ldots, n\). Then \(\exists \xi \in (a, b) : \)

\[
f'(')(\xi) = \frac{f(b) - f(a)}{b - a} \left( \sum_{i=1}^{n} \frac{g'_i(\xi)}{g_i(b) - g_i(a)} \right)
\]

*Proof.* Define a function \(F(x)\) by:

\[
(2) \quad F(x) = nf(x) - f(a) - \sum_{i=1}^{n} \frac{f(b) - f(a)}{g_i(b) - g_i(a)} (g_i(x) - g_i(a))
\]

Since \(g'(x) \neq 0 \Rightarrow g_i(b) - g_i(a) \neq 0\), for \(i = 1, 2, 3, \ldots, n\)
Moreover, \(n\) is finite. So \(F(x)\) is well defined on closed interval \([a, b]\), by algebra of continuous functions.
In addition, it follows by algebra of differentiable functions that \(F(x)\) is differentiable function in \((a, b)\).
Hence the condition for Rolle’s Theorem are satisfied, so \(\exists \xi \in (a, b) \) s.t. \(F'(\xi) = 0\)

\[
(3) \quad \Rightarrow nf'(\xi) - \sum_{i=1}^{n} \frac{f(b) - f(a)}{g_i(b) - g_i(a)} g'_i(\xi) = 0
\]

Thus, \(\exists \xi \in (a, b) : f'(\xi) = \frac{n}{\sum_{i=1}^{n} \frac{g'_i(\xi)}{g_i(b) - g_i(a)}}\).

*Example 3.2.* [3] Consider \(f(x) = x + 1, g_1(x) = x^2 + 4x - 4\) and \(g_2(x) = x^2 + 3x\) on an interval \([0, 3]\).
Solution : 1. All functions are continuous on $[0, 3]$.  
2. All functions are differentiable on $(0, 3)$.

with $g_i'(x) \neq 0, \forall x \in (0, 3)$ for $i = 1, 2$

So condition of EGMVT are satisfied hence, $\exists \xi \in (0, 3)$ :

\[ f'(\xi) = \frac{f(3) - f(0)}{2} - \sum_{i=1}^{2} \frac{g_i'(\xi)}{g_i(3) - g_i(0)} \]

$\Rightarrow 78\xi = 117$

so, $\xi = 1.5 \in (0, 3)$ as required.

Corollary 3.3. Let $n \in \mathbb{N}$ and let $f(x), g_1(x), g_2(x), g_3(x), \ldots, g_n(x)$ be $n+1$ continuous functions on closed bounded interval $[a, b]$ and are $k$ times differentiable in an open interval $(a, b)$ with $g_i^{(k)}(x) \neq 0, \forall x \in (a, b)$ for $i = 1, 2, 3, \ldots, n$. Then $\exists \xi \in (a, b)$ s.t.

\[ f^k(\xi) = \left( \frac{f^{k-1}(b) - f^{k-1}(a)}{n} \right) \sum_{i=1}^{n} \frac{g_i^{(k)}(\xi)}{g_i^{(k)}(b) - g_i^{(k)}(a)} \]

Proof. Set $\phi(x) = f^{k-1}(x)$

$\psi_i(x) = g_i^{(k-1)}(x)$ for $i = 1, 2, 3, \ldots, n$

$\Rightarrow \psi_i'(x) = g_i^{(k)}(x) \neq 0$

Moreover both $\phi(x)$ and $\psi_i(x)$ are continuous on $[a, b]$ and differentiable on $(a, b)$. Hence by EGMVT, $\exists \xi \in (a, b)$ s.t.

\[ \phi'(\xi) = \frac{\phi(b) - \phi(a)}{n} \sum_{i=1}^{n} \frac{\psi_i'(\xi)}{\psi_i(b) - \psi_i(a)} \]

4. **Extended Generalized Flett’s Mean Value Theorem**

If we use Flett’s Theorem in Extended Generalized Mean Value Theorem then what would the new theorem look like. A sincere effort is done to develop this theorem. This theorem is named as Extended Generalised Flett’s Mean Value Theorem (EGMVT).

Theorem 4.1. Let $n \in \mathbb{N}$ and let $f, g_1, g_2, g_3, \ldots, g_n \in C[a, b] \cap D(a, b)$ also $f, g_1, g_2, g_3, \ldots, g_n$ are increasing functions and $f^{n+1}(a) = f^{n+1}(b) = 0$ then $\exists \alpha \in (a, b)$ s.t.

\[ f^{n+1}(\xi) = \frac{b}{a}K(f) \sum_{i=1}^{n} \frac{g_i'(\xi)}{n!} \frac{b}{a}K(g_i) \]

Proof. Without loss of generality assume $f^{n+1}(a) = f^{n+1}(b) = 0$. Define a function

\[ G(x) = \begin{cases} 
  n(x - a)\frac{b}{a}K(f) - \sum_{i=1}^{n} \frac{b}{a}K(g_i)(g_i(x) - g_i(a)), & \forall x \in [a, b] \\
  \frac{1}{n+1}f^{n+1}(a) & x = a
\end{cases} \]
Now \( G \in C[a,b] \cap D^{n+1}(a,b) \)

\[
(7) \quad G(x) = n(x-a)\left(\frac{f^n(x) - f^n(a)}{x-a}\right) - \sum_{i=1}^{n} \frac{f(b) - f(a)}{g_i(b) - g_i(a)} (g_i(x) - g_i(a))
\]

\[
(8) \quad G'(x) = nf^{n+1}(x) - (f(b) - f(a)) \sum_{i=1}^{n} \frac{g'_i(x)}{g_i(b) - g_i(a)}, x \in [a,b]
\]

So we can say \( \exists \xi \in (a,b) \) s.t. \( G'\xi = 0 \)

From definition of \( G \) we can say \( G(a) = 0 \).
If \( G(b) = 0 \) then Rolle's Theorem guarantees existence of a point \( \xi \in (a,b) \) s.t. \( G'\xi = 0 \).
Let \( G(b) \neq 0 \) and \( G(b) > 0 \) (or \( G(b) < 0 \)). Then

\[
G'(b) = nf^{n+1}(b) - (f(b) - f(a)) \sum_{i=1}^{n} \frac{g'_i(b)}{g_i(b) - g_i(a)}
\]

\[
= -(f(b) - f(a)) \sum_{i=1}^{n} \frac{g'_i(b)}{g_i(b) - g_i(a)} \quad (since f^{n+1}(b) = 0)
\]

\[
< 0
\]

Now \( G \in C[a,b] \) and \( G'(b) < 0 \)
\( \Rightarrow \) \( G \) is strictly decreasing in \( b \)
\( \Rightarrow \exists x_1 \in (a,b) \) s.t. \( G(x_1) > G(b) \)

From continuity of \( G \) on \( [a,x_1] \) and from relation \( 0 = G(a) < G(b) < G(x_1) \). Therefore we deduce from Darboux Intermediate Value Theorem, \( \exists x_2 \in (a,x_1) \) s.t. \( G(x_2) = G(b) \)
since \( G \in C[x_2,b] \cap D^{n+1}(x_2,b) \) therefore, from Rolle's Theorem we have \( G'\xi = 0 \) for some \( \xi \in (x_2,b) \subset (a,b) \)

\[
(9) \quad \Rightarrow nf^{n+1}(\xi) - (f(b) - f(a)) \sum_{i=1}^{n} \frac{g'_i(\xi)}{g_i(b) - g_i(a)} = 0
\]

\[
(10) \quad \Rightarrow nf^{n+1}(\xi) = (f(b) - f(a)) \sum_{i=1}^{n} \frac{g'_i(\xi)}{g_i(b) - g_i(a)}
\]

\[
\Rightarrow f^{n+1}(\xi) = \frac{f(b) - f(a)}{n} \sum_{i=1}^{n} \frac{g'_i(\xi)}{g_i(b) - g_i(a)}
\]

\[
= \frac{b}{n}K(f) \sum_{i=1}^{n} \frac{g'_i(\xi)}{nK(g_i)}
\]
Example 4.2. Consider $f(x) = x^4 + 1, g_1(x) = x^2 - 2x + 12, g_2(x) = x^2 + 4x$ on an interval $[0, 4]$

Solution- We are given 3 functions $f, g_1, g_2$. We will check the conditions of EGFMVT one by one as follows:

1. All $f, g_1, g_2$ are continuous on $[0, 4]$
2. All functions are differentiable on $(0, 4)$ with $g_i'(x) \neq 0, \forall x \in (0, 4)$ for $i = 1, 2$
3. Also $f, g_1, g_2$ are increasing functions at the endpoint 4.

All the conditions of theorem are satisfied, hence $\exists \xi \in (0, 4)$ s.t.

$$f^{(3)}(\xi) = \frac{4}{3} K(f) \left[ \sum_{i=1}^{2} \frac{g_i'(\xi)}{2^i K(g_i)} \right]$$

$\Rightarrow 4\xi = 4$
$\Rightarrow \xi = 1$
$\Rightarrow \xi \in (0, 4)$

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