Wilson Lines and Classical Solutions in Cubic Open String Field Theory

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Abstract

We construct exact classical solutions in cubic open string field theory. Through the redefinition of the string field, we find that the solutions correspond to finite deformations of the Wilson lines. The solutions have well-defined Fock space expressions, and they have no branch cut singularity of marginal parameters, which was found in the analysis using a level truncation approximation in the Feynman-Siegel gauge. We also discuss marginal tachyon lump solutions at critical radius.

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§1. Introduction

Cubic open string field theory has been a useful tool to understand various conjectures about tachyon condensation in string theory. Most works on this subject have been carried out using a level truncation scheme in Feynman-Siegel gauge. It has been shown that the tachyon vacuum solution in the level truncation scheme is BRs-invariant and that it satisfies the equation of motion outside the Feynman-Siegel gauge. The validity of the Feynman-Siegel gauge has been checked also for the tachyon lump solution. However, we are unable to understand global structure of the vacuum in the string field theory because of branch cut singularities due to the Feynman-Siegel gauge. To gain further insight into the vacuum structure, it is necessary to construct exact solutions without fixing the Feynman-Siegel gauge.

In order to obtain exact results in cubic open string field theory, we consider finite deformations of the Wilson lines background, as situation somewhat simpler than that of the tachyon vacuum. The classical solutions corresponding to marginal condensation have been constructed formally in string field theories with joining-splitting type vertices. We expect that this previously obtained knowledge will be helpful in constructing exact solutions of the condensation of marginal operators in cubic open string field theory.

There is a problem involved in the study of the Wilson lines deformations in cubic open string field theory. According to the calculation employing the level truncation scheme, it is impossible to carry out an analysis outside a critical value of marginal parameters, though, to some extent, we can describe string field configurations representing the condensations of gauge fields. Interestingly, however, from level truncated analysis for tachyon condensation, it has been shown that singularities in the tachyon potential originate in the Feynman-Siegel gauge. This gives us reason to believe that we may be able to obtain solutions beyond the critical value if we choose something other than the Feynman-Siegel gauge. It also seems possible that we might need different coordinate patches to describe the full moduli space with the string field theory.

In this paper, we construct exact solutions in cubic open string field theory, without fixing any gauge. We show that the solutions correspond to deformations of the Wilson lines background through a homogeneous redefinition of the string field. We find that there are parameters in the solutions corresponding to the vacuum expectation value of the gauge field, but the solutions have no singularity of the parameters. We conclude that the singularity in the level truncation scheme is a gauge artifact.

We also construct solutions of a marginal tachyon lump. We find that at the critical compactified radius, a tachyon lump is generated by marginal deformations of $U(1)$ currents,
and it corresponds to the Wilson line operator. In contrast to the situation for Wilson line deformations, we are able to obtain tachyon lump solutions.

The rest of this paper is organized as follows. In §2 we construct the exact solutions and redefine the string field. Then, we evaluate a solution using the oscillator expression to find that the solutions have well-defined Fock space expressions. In §3, we discuss the solutions corresponding to the marginal tachyon lump. We give a summary in §4. In the Appendix, we prove formulas related to the delta function.

§2. Exact solutions in cubic open string field theory

For simplicity, we single out a direction that is tangential to D-branes. The canonical commutation relation of the string coordinate and momentum is given by

\[ [X(\sigma), P(\sigma')] = i\delta(\sigma, \sigma'), \tag{2.1} \]

and others are zero. The delta function is defined by

\[ \delta(\sigma, \sigma') = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n \geq 1} \cos(\sigma) \cos(\sigma'). \tag{2.2} \]

We expand \( X(\sigma) \) and \( P(\sigma) \) in the oscillator modes as

\[ X(\sigma) = x + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n \cos(n\sigma), \]

\[ P(\sigma) = \frac{p}{\pi} + \frac{1}{\sqrt{2\alpha'}} \sum_{n \neq 0} \alpha_n \cos(n\sigma). \tag{2.3} \]

The mode expansions of ghost variables are given by

\[ c(\sigma) = \sum_n c_n e^{-in\sigma}, \quad b(\sigma) = \sum_n b_n e^{-in\sigma}. \tag{2.4} \]

In order to construct solutions in the string field theory, we define the operators

\[ V_L(f) = \frac{2}{\pi} \int_0^{\pi} d\sigma f(\sigma)V(\sigma), \quad V_R(f) = \frac{2}{\pi} \int_{\pi}^{2\pi} d\sigma f(\sigma)V(\sigma), \]

\[ C_L(f) = \frac{2}{\pi} \int_0^{\pi} d\sigma f(\sigma)C(\sigma), \quad C_R(f) = \frac{2}{\pi} \int_{\pi}^{2\pi} d\sigma f(\sigma)C(\sigma), \tag{2.5} \]

where \( f(\sigma) \) is an arbitrary function. The quantities \( V(\sigma) \) and \( C(\sigma) \) are defined by

\[ V(\sigma) = \frac{1}{2\sqrt{2\alpha'}} [c(\sigma) (2\pi\alpha'P(\sigma) + X'(\sigma)) + c(-\sigma) (2\pi\alpha'P(\sigma) - X'(\sigma))], \]

\[ C(\sigma) = \frac{1}{2} (c(\sigma) + c(-\sigma)). \tag{2.6} \]
The operator $V$ corresponds to the vector vertex operator with zero momentum. Substituting Eqs. (2.3) and (2.4) into Eq. (2.6), the oscillator expressions of the operators $V$ and $C$ are found to be

$$V(\sigma) = \sum_m \left( \sum_n c_{m-n} \alpha_n \right) \cos(m\sigma), \quad C(\sigma) = \sum_n c_n \cos(n\sigma). \quad (2.7)$$

The operator as $V$ can be written in terms of the commutator of the BRS charge and the string coordinate

$$V(\sigma) = \frac{i}{\sqrt{2\alpha'}} [Q_B, X(\sigma)]. \quad (2.8)$$

We can find that the commutation relations of $V_L(R)$ and $C_L(R)$ are given by

$$\{V_L(f), V_L(g)\} = -2 \{Q_B, C_L(fg)\},$$
$$\{V_R(f), V_R(g)\} = -2 \{Q_B, C_R(fg)\}, \quad (2.9)$$

while other commutation relations vanish. Here, Neumann boundary conditions are imposed on the functions $f(\sigma)$ and $g(\sigma)$. Using the formulas involving the delta function (see in Appendix A)

$$\int_0^{\frac{\pi}{2}} d\sigma \int_0^{\frac{\pi}{2}} d\sigma' f(\sigma) g(\sigma') \delta(\sigma, \sigma') = \int_0^{\frac{\pi}{2}} d\sigma f(\sigma) g(\sigma),$$
$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\sigma \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\sigma' f(\sigma) g(\sigma') \delta(\sigma, \sigma') = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\sigma f(\sigma) g(\sigma),$$
$$\int_0^{\frac{\pi}{2}} d\sigma \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} d\sigma' f(\sigma) g(\sigma') \delta(\sigma, \sigma') = 0,$$  \quad (2.10)

we can calculate the above commutation relations. For example, we have

$$\{V_L(f), V_L(g)\} = \frac{2i}{\sqrt{2\alpha'}} \int_0^{\frac{\pi}{2}} d\sigma g(\sigma') \{V_L(f), [Q_B, X(\sigma')]\}$$
$$= - \frac{4}{\pi} \left\{ Q_B, \int_0^{\frac{\pi}{2}} d\sigma \int_0^{\frac{\pi}{2}} d\sigma' f(\sigma) g(\sigma') C(\sigma) \delta(\sigma, \sigma') \right\}$$
$$= - \frac{4}{\pi} \left\{ Q_B, \int_0^{\frac{\pi}{2}} d\sigma f(\sigma) g(\sigma) C(\sigma) \right\}$$
$$= -2 \{Q_B, C_L(fg)\}. \quad (2.11)$$

From the connection conditions on the identity string field and the vertex, we find the following equations. For any pair of string fields $A$ and $B$,

$$V_L(h)I = V_R(h)I, \quad (2.12)$$
$$\ (V_R(h)A) \ast B = (-)^{|A|} A \ast (V_L(h)B), \quad (2.13)$$
$$C_L(h)I = -C_R(h)I, \quad (2.14)$$
$$\ (C_R(h)A) \ast B = (-)^{|A|} A \ast (C_L(h)B), \quad (2.15)$$
where \( I \) denotes the identity string field and \( h(\sigma) \) satisfies \( h(\pi - \sigma) = h(\sigma) \). In Eq. (2.14), \( h(\sigma) \) is more strongly constrained, so that \( C_{L(R)} \) is a regular operator on the identity string field, because the ghost operator has a singularity at the midpoint. We can find such a function \( h \) using the oscillator expression of the identity string field given below, though it can also be obtained with a more elegant procedure.\(^{17}\)

Now, we can show that exact solutions in the cubic open string field theory are given by
\[
\Psi_0(\lambda) = \sqrt{2\alpha'} a_i V_L(\lambda) I + 2\alpha' a_i^2 C_L(\lambda^2) I, \tag{2.16}
\]
where the \( a_i \) are real parameters, where \( i \) is the Chan-Paton index, *) and the function \( \lambda \) satisfies Neumann boundary conditions and \( \lambda(\pi - \sigma) = \lambda(\sigma) \), explicitly \( \lambda(\sigma) = \sum_n \lambda_n \cos(2n\sigma) \). Furthermore, the function \( \lambda(\sigma) \) must be restricted to make \( C_L(\lambda^2) \) a well-defined operator at the midpoint of the identity string field. From Eq. (2.8), we find
\[
Q_B \Psi_0 = 2\alpha' a_i^2 Q_B C_L(\lambda^2) I. \tag{2.17}
\]
Using the properties of \( V_{L(R)} \) and \( C_{L(R)} \), Eqs. (2.9) and (2.12)–(2.15), we find
\[
\Psi_0 \ast \Psi_0 = 2\alpha' a_i^2 V_L(\lambda)^2 I = -2\alpha' a_i^2 Q_B C_L(\lambda^2) I. \tag{2.18}
\]
As a result, \( \Psi_0 \) obeys the equation of motion, \( Q_B \Psi_0 + \Psi_0 \ast \Psi_0 = 0 \).

Since \( \Psi_0 \) is a solution for any values of the \( a_i \), the potential energy \( V(\Psi_0) = -S(\Psi_0) \) is equal to that for \( a_i = 0 \). Therefore, the potential energy is always 0.\(^{10}\) This implies that the parameters \( a_i \) correspond to marginal parameters.

If we expand the string field around the solution as \( \psi = \psi_0 + \psi' \), the BRS charge of the resultant theory is given by \( Q'_B = Q_B + D_{\psi_0} A \), where \( D_{\psi_0} A = \psi_0 \ast A - (-)^{|A|} A \ast \psi_0 \) for any string field \( A \). From the properties of \( V_{L(R)} \) and \( C_{L(R)} \), we find that the BRS charge in the shifted theory is given by
\[
Q'_B = Q_B + \sqrt{2\alpha'} (a_i V_L(\lambda) - a_j V_R(\lambda)) + 2\alpha' \left( a_i^2 C_L(\lambda^2) + a_j^2 C_R(\lambda^2) \right). \tag{2.19}
\]
We can easily check the nilpotency of the shifted BRS charge by using Eq. (2.9).

Let us consider the redefinition of the shifted string field. To do so, we introduce the operators
\[
X_L(f) = \frac{2}{\pi} \int_0^\pi d\sigma f(\sigma) X(\sigma), \quad X_R(f) = \frac{2}{\pi} \int_\pi^\pi d\sigma f(\sigma) X(\sigma). \tag{2.20}
\]
We find the commutators
\[
[X_L(f), Q_B] = i\sqrt{2\alpha'} V_L(f), \quad [X_L(f), V_L(g)] = i2\sqrt{2\alpha'} C_L(fg). \tag{2.21}
\]
* We omit \( \delta_{ij} \) in the identity string field.
From Eq. (2.21), it follows that the shifted BRS charge can be transformed into the original one as 

\[ e^{iB(\lambda)}Q_B' e^{-iB(\lambda)} = Q_B, \]  

(2.22)

where the operator \( B \) is defined by

\[ B(f) = a_i X_L(f) - a_j X_R(f). \]  

(2.23)

By the connection conditions of the string coordinate on the vertex, the cubic term of the action is invariant under the transformation of the string field generated by the operator \( B \)

\[ \int (e^{-iB(\lambda)}\Phi) * (e^{-iB(\lambda)}\Psi) * (e^{-iB(\lambda)}\Xi) = \int \Phi * \Psi * \Xi. \]  

(2.24)

The invariance is generalized to the \( n \)-string vertices of the midpoint interaction.

Consequently, if we redefine the string field as \( \tilde{\Psi} = e^{iB}\Psi' \), the form of the shifted action apparently becomes the original one. However, since if \( \lambda(\sigma) \) has a zero mode, the operator \( e^{iB} \) contains \( \sim e^{i(a_i - a_j) x} \) as the zero mode part, the component fields in the redefined string field are no longer periodic if the relevant direction is compactified. The effect of the \( a_i \) on the periodicity implies that the redefined theory represents the strings in the Wilson lines background. Hence, the parameters \( a_i \) correspond to the finite vacuum expectation values of the gauge field.

we now consider the oscillator expansions of the solution and show that the solutions have well-defined Fock space expressions. The operator expression of the identity string field is given by\(^{18},^{19}\)

\[ |I\rangle = \frac{1}{4i} b \left( \frac{\pi}{2} \right) b \left( -\frac{\pi}{2} \right) e^{E'} c_0 c_1 |0\rangle, \]

\[ E' = \sum_{n\geq 1} (-)^n \left[ -\frac{1}{2n} \alpha_{-n} \cdot \alpha_{-n} + c_{-n} b_{-n} \right], \]  

(2.25)

where \( |0\rangle \) is \( SL(2,R) \) invariant vacuum. For convenience, we rewrite it \(^*\)

\[ |I\rangle = e^E |0\rangle, \]

\[ E = -\sum_{n\geq 1} \frac{(-)^n}{2n} \alpha_{-n} \cdot \alpha_{-n} + \sum_{n\geq 2} (-)^n c_{-n} b_{-n} \]

\[ -2c_0 \sum_{n\geq 1} (-)^n b_{-2n} - (c_1 - c_{-1}) \sum_{n\geq 1} (-)^n b_{-2n-1}. \]  

(2.26)

\(^*\) Expanding this expression, we can check that the first few levels of terms agree with the expression obtained using the Virasoro generators in Ref. 17).
From Eq. (2.26), we find directly
\[
\left( c_0 - \frac{(-)^n}{2}(c_{2n} + c_{-2n}) \right) |I\rangle = 0,
\]
\[
(c_1 - c_{-1} - (-)^n(c_{2n+1} - c_{-2n-1})) |I\rangle = 0,
\]  
\begin{equation}
(2.27)
\end{equation}
where \( n = 1, 2, \cdots \). We can easily check that the first few equations agree with the result of Ref. 17). Applying the ghost operator to the expression in Eq. (2.26), we can evaluate the midpoint singularity on the identity string field. We obtain
\[
c(\sigma) |I\rangle = \left[ i c_0 \tan \sigma + c_1 \frac{1}{2 \cos \sigma} + c_{-1} \frac{1 + 2 \cos(2\sigma)}{\cos \sigma} 
+ 2 \sum_{n \geq 1} \left( i c_{-2n} \sin(2n\sigma) + c_{-2n-1} \cos((2n + 1)\sigma) \right) \right] |I\rangle.
\]  
\begin{equation}
(2.28)
\end{equation}
As the simplest example, we consider a solution for \( \lambda(\sigma) = 1 + \cos(2\sigma) \). In this case, we can expand the solution up to level two as follows,
\[
|\Psi_0\rangle = \left[ \sqrt{2\alpha'} a_i \left( c_1 \alpha_{-1} + 2c_{-1} \alpha_{-1} + \frac{1}{2} c_1 \alpha_{-1} (\alpha_{-1} \cdot \alpha_{-1}) \right) 
+ 2\alpha' a_i^2 \left( \frac{8}{3\pi} c_1 + \frac{88}{15\pi} c_{-1} + \frac{4}{3\pi} c_1 (\alpha_{-1} \cdot \alpha_{-1}) \right) \right] |0\rangle
+ c_0 \left[ -\sqrt{2\alpha'} a_i (\alpha_{-2} + 2c_1 b_{-2} \alpha_{-1}) - 2\alpha' a_i^2 \frac{16}{3\pi} c_1 b_{-2} \right] |0\rangle + \cdots.
\]  
\begin{equation}
(2.29)
\end{equation}
It turns out that the solution of Eq. (2.16) is well-defined in the level truncation scheme, and it is outside the Feynman-Siegel gauge because of the term proportional to \( c_0 \).

§3. Marginal tachyon lump solutions

If the compactified radius has the critical value, \( R = \sqrt{\alpha'} \), a tachyon lump is generated by a marginal operator. \(^4\), \(^5\) Let us consider the construction of tachyon lump solutions at the critical radius. In the solutions of Eq. (2.16), \( V_L \) is constructed by the marginal operator \( \partial X \) as \( V_L \sim \int c \partial X \). Therefore, we have only to replace \( \partial X \) with a \( U(1) \) current in order to obtain a solution corresponding to a marginal deformation of the \( U(1) \) current.

Here we give tachyon lump solutions. We give the mode expansion of the marginal operator corresponding to tachyon lumps as
\[
t(z) = \sqrt{2} \cos \left( \frac{X(z)}{\sqrt{\alpha'}} \right) = \sum_n t_n z^{-n-1},
\]  
\begin{equation}
(3.1)
\end{equation}
where \( X(z) \) is defined by
\[
X(z) = x - 2i\alpha' p \ln z + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n z^{-n}.
\]  
\begin{equation}
(3.2)
\end{equation}
Then, we find that $t_n$ satisfies the same commutation relations as $\alpha_n$:

$$[t_m, t_n] = m\delta_{m+n}. \quad (3.3)$$

Since the operator $t(z)$ is a $U(1)$ current, we can transform $t(z)$ into an operator in the $\rho$ plane by the mapping $\rho = \tau + i\sigma = \ln z$. Using the operator at $\tau = 0$

$$t(\sigma) = \sum_n t_n e^{-in\sigma}, \quad (3.4)$$

we define operators similar to $V_L(R)$ by

$$T_L(f) = \frac{2}{\pi} \int_0^{2\pi} d\sigma f(\sigma) T(\sigma), \quad T_R(f) = \frac{2}{\pi} \int_{2\pi}^{\pi} d\sigma f(\sigma) T(\sigma), \quad (3.5)$$

where $T(\sigma)$ is given by

$$T(\sigma) = \frac{1}{2} (c(\sigma)t(\sigma) + c(-\sigma)t(-\sigma))$$

$$= \sum_m \left( \sum_n c_{m-n} t_n \right) \cos(m\sigma). \quad (3.6)$$

Since $t_m$ satisfies the same commutation relations as $\alpha_m$ given in Eq. (3.3), algebras similar to those of $V_L(R)$ and $C_L(R)$ follow. Indeed, we have

$$\{T_L(f), T_L(g)\} = -2 \{Q_B, C_L(fg)\},$$

$$\{T_R(f), T_R(g)\} = -2 \{Q_B, C_R(fg)\}. \quad (3.7)$$

Since $t(\sigma)$ is a $U(1)$ current, it satisfies the same connection conditions on the vertex and on the identity string field as $P(\sigma)$ and $X'(\sigma)$. Therefore, we also obtain the similar identities related to $I$ and the $\ast$ product:

$$T_L(h)I = T_R(h)I, \quad (3.8)$$

$$(T_R(h)A) \ast B = (-)^{|A|} A \ast (T_L(h)B). \quad (3.9)$$

From Eqs. (3.7), (3.8) and (3.9), we find that the solutions corresponding to marginal tachyon lumps are given by

$$T_0(\lambda) = s_i T_L(\lambda)I + s_i^2 C_L(\lambda^2)I, \quad (3.10)$$

where the $s_i$ correspond to marginal parameters. If we choose $\lambda(\sigma) = 1 + \cos(2\sigma)$, the expanded form of the solution up to level two is given by

$$|T_0 \rangle = s_i \left( c_1 t_{-1} + 2 c_{-1} t_{-1} + \frac{1}{2} c_1 t_{-1}(\alpha_{-1} \cdot \alpha_{-1}) \right)$$

$$+ s_i^2 \left( \frac{8}{3\pi} c_1 + \frac{88}{15\pi} c_{-1} + \frac{4}{3\pi} c_1(\alpha_{-1} \cdot \alpha_{-1}) \right) |0 \rangle$$

$$+ c_0 \left[ -s_i \left( t_{-2} + 2 c_1 b_{-2} t_{-1} \right) - s_i^2 \frac{16}{3\pi} c_1 b_{-2} \right] |0 \rangle + \cdots. \quad (3.11)$$
§4. Summary

We constructed exact solutions in cubic open string field theory without choosing the Feynman-Siegel gauge. There are many solutions associated with the functions $\lambda(\sigma)$ that make the operator $C_{L(R)}$ well-defined on the identity string field. We showed that through the redefinition of the string field, the shifted string field theory becomes a theory with finite Wilson lines deformations for any function $\lambda(\sigma)$ with a zero mode. The solutions we constructed have no branch cut singularity. The branch singularity previously found is revealed to be a gauge artifact. Our solutions have well-defined Fock space expressions.

It is difficult to construct a solution of the tachyon vacuum and to redefine a string field to obtain a theory without physical open string excitations, which is expected to be the case for the vacuum string field theory.\(^{20) - 24\)} Though we can solve the equation of motion exactly for the Wilson lines condensation, it seems that we are unable to apply the method of the marginal case directly to the tachyon condensation.

Our solutions are well-defined in the Fock space. The finite marginal solutions in the light-cone type string field theories\(^{10}\) are rather formal, and it is very difficult to represent them in terms of the oscillator expressions. However, the solution of the dilaton condensation provides the correct result for the shift of the string coupling constant in light-cone type string field theories.\(^{11) - 13}\) We have been still faced with the problem of how to treat finite solutions in the light-cone type string field theories.\(^{10}\)

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Appendix A

--- Proof of Eq. (2.10) ---

We consider the integral

$$I_{mn} = \int_0^{\frac{\pi}{2}} d\sigma \int_0^{\frac{\pi}{2}} d\sigma' \cos (m\sigma) \cos (n\sigma') \delta(\sigma, \sigma')$$

$$= \frac{1}{\pi} \sum_l \int_0^{\frac{\pi}{2}} d\sigma \cos (m\sigma) \cos (l\sigma) \int_0^{\frac{\pi}{2}} d\sigma' \cos (n\sigma') \cos (l\sigma').$$

(A-1)

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In the case \( m \pm n \neq 0 \), this becomes

\[
I_{mn} = \frac{1}{2} \left[ \frac{1}{m + n} \sin \left( \frac{(m + n)\pi}{2} \right) + \frac{1}{m - n} \sin \left( \frac{(m - n)\pi}{2} \right) \right]
\]

\[
+ \frac{1}{2\pi} \sum_{l \neq m \pm n} \left[ \frac{1}{l(l - m + n)} \sin \left( \frac{l\pi}{2} \right) \sin \left( \frac{(l - m + n)\pi}{2} \right) \right]
\]

\[
+ \frac{1}{l(l - m - n)} \sin \left( \frac{l\pi}{2} \right) \sin \left( \frac{(l - m - n)\pi}{2} \right) \right].
\]  
(A.2)

Summing the above series by using

\[
\sum_{n \neq 0, m} \frac{1}{n(n-m)} \sin \left( \frac{n\pi}{2} \right) \sin \left( \frac{(n-m)\pi}{2} \right) = \frac{\pi^2}{4} \delta_{m,0},
\]  
(A.3)

we evaluate the integral as

\[
I_{mn} = \frac{1}{2(m + n)} \sin \left( \frac{(m + n)\pi}{2} \right) + \frac{1}{2(m - n)} \sin \left( \frac{(m - n)\pi}{2} \right), \quad (m \pm n \neq 0). \]  
(A.4)

If \( m = n = 0 \) or \( m = \pm n \neq 0 \), we find, after similar calculations,

\[
I_{00} = \frac{\pi}{2},
\]

\[
I_{nn} = I_{n, -n} = \frac{\pi}{4}, \quad (n \neq 0).
\]  
(A.5)

If \( f(\sigma) \) and \( g(\sigma) \) satisfy Neumann boundary conditions, we can expand the functions as \( f(\sigma) = \sum_n f_n \cos(n\sigma) \) and \( g(\sigma) = \sum_n g_n \cos(n\sigma) \). Then, we can easily prove one of the expressions in Eq. (2.10):

\[
\int_0^{\pi} d\sigma \int_0^{\pi} d\sigma' f(\sigma)g(\sigma') \delta(\sigma, \sigma') = \sum_{mn} f_m g_n I_{mn} = \int_0^{\pi} d\sigma f(\sigma)g(\sigma).
\]  
(A.6)

By the definition of the delta function, it follows that

\[
\int_0^{\pi} d\sigma \int_0^{\pi} d\sigma' f(\sigma)g(\sigma') \delta(\sigma, \sigma') = \int_0^{\pi} d\sigma f(\sigma)g(\sigma).
\]  
(A.7)

Considering Eqs. (A.6) and (A.7), we can obtain the other two expressions of Eq. (2.10).
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The solutions of Eq. (2.16) are locally pure gauge. If we choose as a gauge parameter functional
\[ A = -ia_iX_L(\lambda)I, \]
the solutions are generated by the gauge transformation of the zero string field
\[
e^A \ast Q_B e^{-A} = e^{-ia_iX_L(\lambda)I} \ast Q_B e^{ia_iX_L(\lambda)I}
= e^{-ia_iX_L(\lambda)I} Q_B e^{ia_iX_L(\lambda)I}
= \sqrt{2\alpha'}a_iV_L(\lambda)I + 2\alpha'a_i^2C_L(\lambda^2)I.
\]

Note added:*)

This information was provided to us by H. Hata in private communication.