The commutativity of prime rings with homoderivations

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ABSTRACT

Let R be a ring with center Z(R), and I be a nonzero left ideal. An additive mapping h: R → R is called a homoderivation on R if h(xy) = h(x)h(y) + h(x)y + xyh(y) for all x, y ∈ R. In this paper, we prove the commutativity of R if any of the following conditions is satisfied for all x, y ∈ R: (i) xh(y) ± xy ∈ Z(R). (ii) xh(y) ± yx ∈ Z(R). (iii) xh(y) ± [x, y] ∈ Z(R) (iv) xh(y) ± yx ∈ Z(R). (v) h(x, y) ± xy ∈ Z(R).

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1. Introduction

Throughout, R denotes a ring with a center Z(R). We write [x, y] for xy − yx and is called the commutator. A ring R is called prime if aRb = 0 implies a = 0 or b = 0 and is called semiprime if aRa = 0 then a = 0. A derivation on R is an additive mapping d: R → R satisfying d(xy) = d(x)y + xd(y) for all x, y ∈ R. The only additive map which is both derivation and homoderivation on prime ring is the zero map. If S ⊆ R, then a mapping f: R → R preserves S if f(S) ⊆ S. A mapping f: R → R is said to be zero-power valued on S if f preserves S and if for each x ∈ S, there exists a positive integer n(x) > 1 such that f n(x) = 0 (El Sofy, 2000). Ashraf and Rehman (2001) had shown that if R is a prime ring, I an ideal of R and d: R → R is a derivation of R, then R is a commutative ring if and only if R satisfies any one of the properties:

d(xy) ± xy ∈ Z(R), d(xy) + xy ∈ Z(R), d(xy) ± yx ∈ Z(R), d(xy) + yx ∈ Z(R), d(xy) ± xy ∈ Z(R) and d(xy) ± xy ∈ Z(R) for all x, y ∈ I.

Motivated by these results, we prove a similar result regarding homoderivations. To achieve our aim, we will use the following lemma.

Lemma 1.1 (Lemma 4): Let b and ab be in the center of a prime ring R. If b ≠ 0, then a is in Z(R) (Mayne, 1984).

Remark 1.2 (Remark 3): Let R be a prime ring. If R contains a nonzero commutative left ideal, then R is a commutative ring (Bresar, 1993).

Lemma 1.3 (Lemma 1.1): Let R be a ring and 0 ≠ l a right ideal of R. Suppose that ael such that a^n = 0 for a fixed integer n. Then R has a nonzero nilpotent ideal (Herstein, 1969).

Lemma 1.4 (Corollary 2.5): Let R be a prime ring of characteristic not 2 and I a nonzero left ideal. If R admits a nonzero homiderivation h which is centralizing on I, then R is commutative.

2. On the commutative conditions

Theorem 2.1: Let R be a prime ring of characteristic not 2, and I be a nonzero left ideal in R. If h is a nonzero homiderivation which is zero-power valued on I. Then, for all x, y ∈ I, the following conditions are equivalent:

i. xh(y) ± xy ∈ Z(R)
ii. xh(y) ± yx ∈ Z(R)
iii. xh(y) ± [x, y] ∈ Z(R)
iv. h(y)x ± [x, y] ∈ Z(R)
v. [h(x), y] ± xy ∈ Z(R)
vi. [h(x), y] ± yx ∈ Z(R)
vii. R is commutative.

Proof: If (vii) holds then all other conditions are true. To prove (i) ⇒ (vii). By hypothesis, we have

\[ xh(y) ± xy ∈ Z(R) \text{ for all } x, y ∈ I. \] (1)
Replacing $x$ by $yx$ in (1), we get $y(xh(y) \pm xy) \in Z(R)$.

By Lemma 1.1, $y \in Z(R)$ or $xh(y) \pm xy = 0$. If $y \in Z(R)$ for all $y \in I$, hence $I$ is commutative. By Remark 1.2, $R$ is commutative. If

$$xh(y) \pm xy = 0 \text{ for all } x, y \in I.$$  \hspace{1cm} (2)

Replace $y$ by $yx$ in (2), we have $x(h(y) \pm y)h(x) = 0$. Since $h$ is zero-power valued on $I$, there exists an integer $n(y) > 1$ such that $h^{n(y)}(y) = 0$ for all $y \in I$. Replacing $y$ by

$$y = h(y) + h^2(y) + \cdots + (-1)^{n(y)-1}h^{n(y)-1}(y)$$

in the last relation. We have $xh(y) = 0$ for all $x, y \in I$. Hence, $xRh(x) = 0$ for all $x \in I$. But $I \neq 0$. So $lh(x) = 0$ for all $x \in I$. Hence the Eq. 2 implies $x^2 = 0$ for all $x \in I$. By Lemma 1.3, $R$ has a nonzero nilpotent ideal which contradicts that $R$ is prime ring.

To prove (ii) $\rightarrow$ (vii). By hypothesis, we have

$$xh(y) \pm xy \in Z(R) \text{ for all } x, y \in I.$$  \hspace{1cm} (3)

Replace $x$ by $yx$ in (3), we have $y(xh(y) \pm xy) \in Z(R)$. By Lemma 1.1, $y \in Z(R)$ or $xh(y) \pm xy = 0$. If $y \in Z(R)$ for all $y \in I$, hence $I \subseteq Z(R)$. Therefore $I$ is commutative. By Remark 1.2, $R$ is commutative. If

$$xh(y) \pm xy = 0 \text{ for all } x, y \in I.$$  \hspace{1cm} (4)

Replace $y$ by $xy$ in (4) we have $xh(y)(y) \pm y) = 0$ for all $x, y \in I$. Since $h$ is zero-power valued on $I$, so $xh(x)y = 0$ for all $x, y \in I$. Hence, $xh(x)RI = 0$ for all $x \in I$. By primeness of $R$, we get $xh(x) = 0$ for all $x \in I$. So, by (3) we get $x^2 = 0$ for all $x \in I$. By Lemma 1.3, this is contradiction. To prove (iii) $\rightarrow$ (vii). By hypothesis, we have

$$xh(y) \pm [x, y] \in Z(R) \text{ for all } x, y \in I.$$  \hspace{1cm} (5)

Replace $x$ by $yx$ in (5), $y(xh(y) \pm [x, y]) \in Z(R)$ either $y \in Z(R)$ or $xh(y) \pm [x, y] = 0$. If $y \in Z(R)$ and $I \subseteq Z(R)$ then $I$ is commutative ideal. By Remark 1.2, $R$ is commutative. If

$$xh(y) \pm [x, y] = 0 \text{ for all } x, y \in I.$$  \hspace{1cm} (6)

Replace $y$ by $yx$ in (6) we get:

$$x(h(y) \pm y)h(x) = 0 \text{ for all } x, y \in I.$$  

Since $h$ is zero-power valued on $I$, so we get $xh(x) = 0$ for all $x \in I$ which implies $xRh(x) = 0$ for all $x \in I$. By primeness of $R$ either $x = 0$ or $lh(x) = 0$. But $I \neq 0$. So $lh(x) = 0$ for all $x \in I$. Form (6) we have $[x, y] = 0$ for all $x, y \in I$. Then $I$ is commutative ideal. By Remark 1.2, we have $R$ is commutative. To prove (iv) $\rightarrow$ (vii) by hypothesis we get:

$$h(y)x \pm [x, y] \in Z(R) \text{ for all } x, y \in I.$$  \hspace{1cm} (7)

Replace $x$ by $xy$ in (7)

$$h(y)x \pm [x, y] \in Z(R).$$

Since $h(y)x \pm [x, y] \in Z(R)$, we get $(h(y)x \pm [x, y])y = y(h(y)x \pm [x, y])$. So $h(y)x \pm [x, y] \in Z(R)$. By Lemma 1.1, we have $y \in Z(R)$ or $h(y)x \pm [x, y] = 0$. If $y \in Z(R)$ for all $y \in I$, if $I \subseteq Z(R)$. Then $I$ is commutative, by Remark 1.2, $R$ is commutative. If

$$h(y)x \pm [x, y] = 0 \text{ for all } x, y \in I.$$  \hspace{1cm} (8)

Replace $y$ by $xy$ in (8), $h(x)(h(y) \pm y)x = 0$. Since $h$ is zero-power valued on $I$, so $h(x)yx = 0$ for all $x, y \in I$. Then we get $h(x)RI = 0$.

By primeness of $R$ we have $h(x) = 0$ since $Ix \neq 0$ for all $x \in I$.

If $h(x) = 0$ for all $x \in I$, we have $[x, y] = 0$ by (8), then $I$ is commutative. By Remark 1.2, $R$ is commutative.

To prove (v) $\rightarrow$ (vii). By hypothesis, we have

$$[h(x), y] \pm xy \in Z(R) \text{ for all } x, y \in I.$$  \hspace{1cm} (9)

Replace $y$ by $yx$ in (8) for all $x, y \in I$, we have,

$$([h(x), y] \pm xy)h(x) \in Z(R) \text{ for all } x, y \in I.$$  \hspace{1cm} (10)

By Lemma 1.1 either $h(x) \in Z(R)$ or $[h(x), y] + xy = 0$ for all $x, y \in I$.

If $h(x) \in Z(R)$ for all $x \in I$, then $[h(x), x] = 0$. By Lemma 1.4, $R$ is commutative.

By primeness of $R$ and $I \neq 0$, we have $[h(x), x] = 0$ for all $x \in I$. By Lemma 1.4, $R$ is commutative.

To prove (vi) $\rightarrow$ (vii). By hypothesis, we have

$$[h(x), y] \pm xy \in Z(R) \text{ for all } x, y \in I.$$  \hspace{1cm} (11)

Replace $y$ by $h(y)$ in (11)

$$h((h(x), y) + xy) \in Z(R) \text{ for all } x, y \in I.$$  

By Lemma 1.1 either $h(x) \in Z(R)$ or $[h(x), y] + yx = 0$.

If $h(x) \in Z(R)$ for all $x \in I$, then $[h(x), x] = 0$. By Lemma 1.4, $R$ is commutative.

$$[h(x), y] + xy = 0 \text{ for all } x, y \in I.$$  \hspace{1cm} (12)
3. On condition $h(x, y) = -[x, y]$

Daif and Bell (1992) proved that a prime ring $R$ with a nonzero ideal $I$ must be commutative if it admits a derivation $d$ such that $d([x, y]) = -[x, y]$. Motivated by their results, we investigate the commutativity of rings admitting a homoderivation $h$ such that $h([x, y]) = -[x, y]$. We begin with the following useful lemma.

**Lemma 3.1 (Corollary 3.4.2):** Let $R$ be a prime ring of char($R$) $\neq 2$, and $I \neq (0)$, a two sides ideal of $R$. If $R$ admits a a nonzero homoderivation $h$ on $I$ such that $h([x, y]) = [x, y]$ for all $x, y \in I$. Then $R$ is commutative (El Sofy, 2000).

**Theorem 3.2:** Let $I$ be nonzero left ideal in a prime ring $R$ that admits a homoderivation $h$ which is zero-power valued on $I$ satisfying $xy + h(xy) = yx + h(xy)$ for all $x, y \in I$. Then $R$ is commutative.

**Proof:** By hypothesis,

$$xy + h(xy) = yx + h(xy) \text{ for all } x, y \in I.$$  

i.e.,

$$h([x, y]) = -[x, y] \text{ for all } x, y \in I.$$  

Therefore

$$[h(x) + h(y)] + [h(x), y] = -[x, y] \text{ for all } x, y \in I$$

Since $h$ is zero-power valued on $I$, so there exists an integer $n(y) > 1$ such that $h^n(y)(y) = 0$ for all $y \in I$. Replacing $y$ by

$$y - h(y) + h^2(y) + \cdots + (-1)^{n(y)-1}h^{n(y)-1}(y)$$

in the last relation. Also, there exists an integer $n(x) > 1$ such that $h^n(x)(x) = 0$ for all $x \in I$ replacing $x$ by $x-h(x)+h^2(x)+\cdots+(-1)^{n(x)-1}h^{n(x)-1}(x)$ in the last relation, we get

$$[x, y] = 0 \text{ for all } x, y \in I$$

Then $I$ is a commutative ideal in prime ring. By Remark 1.2, $R$ is commutative.

**Theorem 3:** Let $R$ be a prime ring with char $(R) \neq 2$ and $I$ be a nonzero ideal of $R$. Suppose $h$ is a nonzero homoderivation which is zero-power valued on $I$. If one of the following conditions are satisfied for all $x, y \in I$:

i. $h(xy) = xy$.

ii. $h(xy) = yx$.

Then $R$ is commutative.

**Proof:** Suppose (i) is satisfies for all $x, y \in I$ we get

$$h(xy - yx) = xy - yx \text{ for all } x, y \in I$$

$$h(xy) = h(yx) = xy \text{ for all } x, y \in I$$

$$h([x, y]) = [x, y] \text{ for all } x, y \in I$$

By Lemma (3.1), $R$ is commutative. Suppose (ii) is satisfies for all $x, y \in I$. We get

$$h(xy) = h(yx) = yx - yx \text{ for all } x, y \in I$$

$$h([x, y]) = -[x, y] \text{ for all } x, y \in I.$$

By Theorem (3.2), we obtain $R$ is commutative.

4. Conclusion

The goal of this paper is to prove the commutativity of prime rings with homoderivation which satisfying some algebraic conditions. This article is divided into two sections; in the first section, the commutativity of prime rings $R$ was proved of the homoderivation on $R$ satisfies following conditions for all

$$x, y \in R: (i) xh(y) \pm xy \in Z(R), (ii) xh(y) \pm xy \in Z(R), (iii) xh(y) \in [x, y] \in Z(R), (iv) h(xy) \pm x, y \in Z(R), (v) h([x, y]) \pm xy \in Z(R) \text{and (vi) } h(x, y) \pm xy \in Z(R).$$

In the second section, we investigate the commutativity of prime ring, if $R$ admits a nonzero homoderivation $h$ such that $h([x, y]) = \pm [x, y]$ for all $x, y$ in a nonzero left ideal.

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