STABILITY OF PULLBACK OF ORBIFOLD BUNDLES

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Abstract. In this article, we study the behavior of the stability of pullback of a vector bundle under a finite morphism from a (not necessarily smooth) stacky curve to an orbifold curve. We establish a categorical equivalence between proper formal orbifold curves and proper orbifold curves in the sense of Deligne-Mumford stacks. Using this identification, we define the notion of slope $P$-(semi)stability of vector bundles on proper formal orbifold curves $(X, P)$. We establish some equivalent conditions for a stacky genuinely ramified morphism, analogous to the case of curves. Finally, we show that for a cover of an orbifold curve arising as a cartesian pullback via a genuinely ramified morphism of smooth projective connected curves, the orbifold slope stability is preserved under the pullback.

1. Introduction

We work over a fixed algebraically closed field $k$ of arbitrary characteristic. The stability of vector bundles on a smooth projective curve was introduced by Mumford in [15]. A natural question is to understand how do this stability of vector bundles behave under the pullback via a reasonable morphism of curves.

Let $f : Y \to X$ be a finite, generically separable map of smooth projective connected $k$-curves (henceforth referred to as a finite cover), and let $E$ be a vector bundle on $X$. It is well known that $E$ is semistable if and only if the pullback bundle $f^* E$ on $Y$ is also semistable. Moreover, if the bundle $f^* E$ is stable, then $E$ is a stable bundle. But the converse need not be true; [2, Proposition 5.2] constructs such an example. [2] establishes a classification of all finite covers $f$ of $X$ such that $f^* E$ is stable for any stable vector bundle $E$ on $X$. These covers, called the genuinely ramified covers, are precisely the ones for which the induced map

$$f_* : \pi_1(Y) \to \pi_1(X)$$

of the étale fundamental groups is a surjection. The aforementioned paper also establishes other equivalent conditions for a cover to be genuinely ramified, in terms of the Harder-Narasimhan filtration of the vector bundle $f^* f_* O_Y$, of its global sections and also in terms of the topology of the fiber product $Y \times_X Y$. For a similar classification for higher dimensional normal varieties, see [3]. In this article, we study the stability for bundles over formal orbifold curves and their behavior under finite covers.

Formal orbifold curves were defined in [17], [2]. Roughly, a formal orbifold curve is a pair $(X, P)$ consisting of a smooth $k$-curve $X$ together with the data $P$ given by certain local Galois extensions attached to finitely many points of the curve. Vector bundles on a formal orbifold curve were introduced and studied in [12]. It can be seen that any such formal orbifold curve can be dominated by a smooth connected Galois cover $Z \to X$, and the vector bundles on $(X, P)$ can be realized as equivariant vector bundles on $Z$. This enables us to define the slope stability for vector bundles on $(X, P)$, and we can study the properties of...
the pullback bundles under a finite cover of formal orbifold curves. We succumb to another view for this formal orbifold curves, namely, we can see them as smooth connected proper Deligne-Mumford stacks of dimension one that is generically a \(k\)-curve. We call these corresponding stacky objects as simply ‘orbifold curves’ (see Section § 3.1 for details).

Our first objective is to establish a categorical equivalence between formal orbifold curves and orbifold curves (see Section § 4). Moreover, we show that the the categories of vector bundles on them coincide (Theorem 4.7). This identification further allows us to borrow the definition and theories developed for algebraic stacks in the formal orbifold curve set up. We can define the ‘degree’ (\(\deg_P\)) of a vector bundle on \((X, P)\), its ‘slope’ (\(\mu_P\)), and (semi)stability (\(P\)-(semi)stability; Definition 5.4) using the data \(P\) itself, without using a cover or an equivariant set up (Section 5.2). Another generality we consider is that instead of considering finite covers between two (formal) orbifold curves, we consider finite covers of an orbifold curve from a stacky curve, which need not be smooth. On the non-smooth stacky curve \(\mathcal{Y}\), we lose this intrinsic notion of slope coming from field extensions, but nevertheless, we can use equivariant set up to consider the stability of a vector bundle on \(\mathcal{Y}\) with respect to this slope \(\mu_\mathcal{Y}\) (Section 3.3). We check that these stability criteria satisfy the expected properties, and they coincide with the already existing definitions under suitable categorical equivalences (in particular, when \(k = \mathbb{C}\), the \(P\)-slope coincides with the parabolic slope of the parabolic bundle corresponding to an orbifold curve; see Remark 5.9).

Next, we study how this orbifold stability criterion behave with respect to the pullback under a finite morphism from a (not necessarily smooth) stacky curve. The equivalent conditions from [2] for a genuinely ramified map is shown to hold in our context.

**Proposition 1.1** (Proposition 5.11). Let \(\mathcal{X} = (X, P)\) be a connected proper orbifold \(k\)-curve. Let \(f : \mathcal{Y} \longrightarrow (X, P)\) be a finite cover of proper stacky curves. The maximal destabilizing sub-bundle \(HN(f_!\mathcal{O}_\mathcal{Y})_1 \subset f_!\mathcal{O}_\mathcal{Y}\) is a sheaf of \(\mathcal{O}_{X,P}\)-algebras, and is a \(P\)-semistable vector bundle of \(P\)-degree \(0\). Moreover, the following are equivalent for the finite cover \(f : \mathcal{Y} \longrightarrow (X, P)\).

1. \(HN(f_!\mathcal{O}_\mathcal{Y})_1 = \mathcal{O}_{X,P}\).
2. The map \(f\) does not factor through any non-trivial étale sub-cover.
3. The homomorphism between étale fundamental groups \(f_! : \pi_1(\mathcal{Y}) \longrightarrow \pi_1(X)\) induced by \(f\) is a surjection.
4. The fiber product Deligne-Mumford stack \(\mathcal{Y} \times_X \mathcal{Y}\) is connected.
5. \(\dim H^0(\mathcal{Y}, f^* f_! \mathcal{O}_\mathcal{Y}) = 1\).

Finally, the above conditions imply that the finite cover \(f_0\) induced on the Coarse moduli curves is a genuinely ramified morphism.

The rest of the article is devoted to study the orbifold stability of the pullback of an orbifold stable bundle. We show that the pullback of an orbifold stable bundle need not be orbifold stable when the finite cover induced on the Coarse moduli curves is not genuinely ramified (Theorem 6.10). Moreover, only assuming that the induced cover of Coarse moduli curves is genuinely ramified is not enough. We expect that the sufficient condition for a cover to preserve stability is the equivalent conditions of the above proposition. We show this by restricting ourselves to the following set up where the finite cover of the formal orbifold curve arises as a fiber product under a genuinely ramified morphism of the Coarse moduli curves (this implies the above mentioned equivalent conditions; see Remark 6.5).

Let \(\mathcal{X} = (X, P)\) be a proper orbifold curve, and \(f_0 : Y \longrightarrow X\) be a genuinely ramified morphism of smooth projective connected \(k\)-curves. Consider the fiber product stack \(\mathcal{Y} := \mathcal{X} \times_X \mathcal{Y}\).
Theorem 1.2 (Theorem 6.8). Under the above set up, for every $P$-stable vector bundle $E_\ast$ on $(X, P)$, the pullback bundle $f^\ast E_\ast$ on $\mathcal{Y}$ is $\mu_0$-stable.

For the proof, we closely follow the strategy from [4], generalized to the above set up involving algebraic stacks. One key ingredient is the following technical result.

Lemma 1.3 (Lemma 6.4, Lemma 6.7). Suppose that the above set up hold. Additionally, if $f_0: Y \to X$ is a $G$-Galois cover, we have

$$f^\ast \left( (f_0^\ast \mathcal{O}_Y) / \mathcal{O}_{(X, P)} \right) \subset \bigoplus_{1 \leq i \leq [G : 1]} M_i$$

in $\text{Vect}(\mathcal{Y})$ where each $M_i \in \text{Vect}(\mathcal{Y})$ is a line bundle with $\mu_0(M_i) < 0$.

Moreover, if $E_\ast, F_\ast$ are $P$-semistable with $\mu_P(E_\ast) = \mu_P(F_\ast)$, we have

$$\text{Hom}_{\text{Vect}(\mathcal{Y})}(f^\ast E_\ast, f^\ast F_\ast) = \text{Hom}_{\text{Vect}(X, P)}(E_\ast, F_\ast).$$

This allows us to construct certain bundles on formal orbifold curves, and finally a descent argument proves the theorem.

While preparing this manuscript, Prof. Indranil Biswas has pointed out to us the article [4] where the authors study the above question about stability of pullback of vector bundles on formal orbifold curves. Under their hypothesis, the stacky curve $\mathcal{Y}$ in our set up is indeed the formal orbifold curve $(Y, f^\ast P)$, and we recover their result in this case.

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2. Notation and Convention

Throughout this article, we work over an algebraically closed field $k$ of arbitrary characteristics. In this paper, the curves we consider are reduced $k$-curves. For any $k$-scheme $W$ and a closed point $w \in W$, we denote the completion of the local ring $\mathcal{O}_{W, w}$ of $W$ at $w$ by $\hat{\mathcal{O}}_{W, w}$. When this complete local ring is a domain, we set $K_{W, w}$ as the quotient field $QF(\hat{\mathcal{O}}_{W, w})$. For a smooth $k$-curve $X$, any finite extension $L/K_{X, x}$ is understood as an extension in a fixed separable closure $K_{X, x}^{\text{sep}}$ of $K_{X, x}$.

A cover or a finite cover $f: Y \to X$ of curves refers to a finite surjective morphism $f$ that is generically separable. For a finite group $G$, a $G$-Galois cover $Y \to X$ is a finite cover together with a $G$-action on $Y$ such that $G$ acts simply transitively on each generic geometric fiber. Any finite cover (in particular, a Galois cover) is étale away from finitely many points on the base curve, which may be empty. For a $G$-Galois cover $f: Y \to X$, the group $G$ acts transitively on the fiber $f^{-1}(x) \subset Y$ for each point $x \in X$; the stabilizer groups at points in $f^{-1}(x)$ are conjugate to each other in $G$. (Up to conjugacy) we define an inertia group above a point $x \in X$ to be a stabilizer group $\text{Stab}_G(y)$ for some $y \in f^{-1}(x)$. In particular, the cover $f$ is étale above $x \in X$ if and
only if the inertia group above $x$ is the trivial group. When the order of the inertia group at $x$ is invertible in $k$, we say that the cover $f$ is *tamely ramified* over the point $x$.

### 3. Preliminaries

#### 3.1. Stacky curves, Orbifold Curves

We fix an algebraically closed base field $k$. For the definition and properties of a Deligne-Mumford stack (DM stack), we refer to [10, 6] and [18, Appendix A]. We mention some important defining properties and convention.

We always consider DM stacks that are separated and of finite type over $k$. A *representable morphism* $Y \to X$ in this article is a morphism representable by a scheme [1], i.e., for any scheme $Z$ and a morphism $Z \to X$ of stacks, the fibre product $Y \times_X Z$ is a scheme. A representable morphism $Y \to X$ is said to be *unramified* if for any scheme $Z$ and a morphism $Z \to X$ of stacks, the morphism $Y \times_X Z \to Z$ is a formally unramified morphism of schemes. For a DM stack $X$ over $k$, the following hold.

1. The diagonal morphism
   $$\Delta_X : X \to X \times_{\text{Spec}(k)} X$$
   is a representable, unramified morphism (see [18 Proposition 7.15], [16 Theorem 8.3.3]). The ‘separated’ assumption on $X$ means that for any morphism $Y \to X \times_{\text{Spec}(k)} X$ of stacks where $Y$ is a scheme, the morphism
   $$Y \to X \times_{\text{Spec}(k)} X$$
   of schemes is a proper (equivalently, finite) morphism.

2. There exists an étale surjective morphism $Z \to X$ from a scheme $Z$ (the morphism $Z \to X$ is called an *atlas* of $X$). We say that $\tilde{X}$ is smooth if there exists an atlas $Z \to \tilde{X}$ where $Z$ is a smooth scheme (equivalently, for every atlas $Z' \to \tilde{X}$, $Z'$ is a smooth scheme).

3. Since a DM stack is assumed to be separated, of finite type over $k$, it admits a coarse moduli scheme $\pi : X \to X$ ([16 Theorem 11.1.2]) satisfying the following properties.
   
   a. The morphism $\pi$ is initial among all morphisms from $X$ to $k$-schemes.
   
   b. $\pi$ induces a bijective correspondence between the $k$-points of $X$ and the isomorphism classes of $k$-points of $X$.
   
   c. $X$ is separated and of finite type over $k$.
   
   d. $\pi$ is a proper morphism of stacks and $\pi_* O_X = O_X$.

   e. ([16 Theorem 11.3.6]) For any morphism of schemes $h : X' \to X$, the coarse moduli scheme of the fiber product DM stack $X' \times_X \tilde{X}$ is universally homeomorphic to $X'$, and it is an isomorphic if either $h$ is flat or $\tilde{X}$ is a tame stack.

   If the stack $\tilde{X}$ is one-dimensional (admits an atlas from a $k$-curve), the coarse moduli space $X$ is also a $k$-curve.

   We refer to [6 Section 4] for the defining properties of a DM stack and for the morphisms. In particular, a noetherian DM stack is a disjoint union of its connected components; each connected component is a union of its irreducible components ([6 Proposition 4.13, Proposition 4.15]) in a unique way. It can be seen that for a DM stack $\tilde{X}$ (which we are assuming to be separated and of finite type over $k$) with its coarse moduli scheme $X$, the stack is connected if and only if $X$ is connected. One direction is clear: if $\tilde{X}$ is a disjoint

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1This is done to avoid the language of algebraic spaces
union of non-void sub-stacks, then \( X \) is a disjoint union of non-empty connected components; conversely, if \( X_1 \) is a connected component of \( X \), via base change under the open and closed imbedding \( X_1 \hookrightarrow X \), we obtain an open and closed imbedding \( X_1 \times_X \bar{x} \hookrightarrow \bar{x} \).

**Example 3.1.** One important example of a DM stack is a quotient stack. Let \( Y \) be a quasi-projective \( k \)-variety. Let \( G \) be a finite group acting on \( Y \) such that there is a quotient scheme \( X \coloneqq Y/G \). We can assign a DM stack \( [Y/G] \) to this data (see [16 Example 8.1.12]). The \( k \)-points of \( [Y/G] \) (i.e., the isomorphism classes of objects over \( k \)) and the closed points of \( X \) are both canonically identified with the \( G \)-orbits of the closed points of \( Y \). For each such point \( x \in [Y/G](k) \), we obtain a stabilizer group \( G_x \), which is the group of automorphisms lying over \( \text{Id}_{\text{Spec}(k)} \). More precisely, the fiber product \([Y/G] \times_{(Y/G)(k)} \text{Spec}(k)\) is a constant \( k \)-group scheme associated to the finite group \( G_x \). A point \( x \in [Y/G](k) \) is called a **stacky point** if the stabilizer group \( G_x \) is non-trivial. The canonical morphism \( Y \rightarrow [Y/G] \) is an atlas, and so \( [Y/G] \) is a smooth (respectively, proper) DM stack if and only if \( Y \) is smooth (respectively, proper). Moreover, the stack \( [Y/G] \) admits a Coarse moduli morphism \([Y/G] \rightarrow X\).

Assume that \( Y \) is a projective curve (not necessarily connected or smooth). Since \( G \) is a finite group, the branch locus \( B \) of the \( G \)-Galois cover \( Y \rightarrow X \) consists of finitely many closed points of \( X \). Set \( U = X - B \). Then a point \( x \in [Y/G](k) \) is a stacky point if and only if the image of \( x \) in \( X \) is in \( B \). It follows that \([Y/G] \times_X U \equiv U\).

We come to the definition of a stacky curve and an orbifold curve. In some literature (see [19]), these objects have the same definition; but for our context, we distinguish them: a stacky curve is an orbifold curve if it is smooth.

**Definition 3.2** (Stacky Curve). A reduced separated DM stack \( \bar{x} \) of finite type over \( k \) is said to be a **stacky curve** if it satisfies the following properties.

1. every irreducible component of \( \bar{x} \) is one-dimensional and is generically an integral \( k \)-curve;
2. \( \bar{x} \) admits an irreducible smooth \( k \)-curve \( X \) as its Coarse moduli space.

A **finite cover** of stacky curves is defined to be a finite surjective morphism that is generically separable.

Before proceeding, we make the following notes.

**Remark 3.3.** By our definition, the Coarse moduli curve is connected, hence any stacky curve is connected. Further, there is a smooth \( k \)-curve \( Z \) together with a Galois cover \( Z \rightarrow X \) dominating the Coarse moduli morphism \( \bar{x} \rightarrow X \). To see this, we can follow the proof of [16 Theorem 11.4.1, Chow’s Lemma, pg. 233] to find an étale surjection \( W \rightarrow \bar{x} \) with \( W \) a reduced \( k \)-curve such that \( W \rightarrow X \) is a finite cover of \( k \)-curves. Further, taking the normalization of the function field \( k(\bar{x}) \) in suitable Galois extensions containing all the function fields of the generic points of \( W \), we conclude the statement.

We will frequently consider stacky curves which are proper; in this case, the above Galois cover \( Z \rightarrow \bar{x} \) is naturally a cover of smooth projective \( k \)-curves.

**Remark 3.4.** The definition of a finite morphism of stacky curves \( f : \bar{\mathcal{Y}} \rightarrow \bar{x} \) is taken analogous to the definition of a proper morphism in [6 Definition 4.11]. So a finite morphism is not necessarily representable; the morphism is finite if it is dominated by a morphism \( \mathcal{Z} \rightarrow \bar{x} \) of stacky curves that is representable and finite, and \( \mathcal{Z} \rightarrow \bar{\mathcal{Y}} \) is a surjection. The generically separable condition can be checked for the induced morphism of the moduli curves since a stacky curve is generically isomorphic to its Coarse moduli curve. So a
finite cover \( f: \mathcal{Y} \rightarrow \mathfrak{X} \) induces a finite cover \( f_0: Y \rightarrow X \) of the Coarse moduli curves. We set

\[
\deg(f) \equiv \deg(f_0).
\]

Moreover, when \( \mathcal{Y} = Y \times_X \mathfrak{X} \), the projection \( \mathcal{Y} \rightarrow \mathfrak{X} \) is a finite cover if and only if \( f_0 \) is a finite cover. This is also equivalent to \( Y \times_X Z \rightarrow Z \) being a finite cover where \( Z \rightarrow X \) is any finite cover dominating the Coarse moduli morphism \( \mathfrak{X} \rightarrow X \).

Remark 3.5. We also note that a stacky curve in our definition need not be smooth although its Coarse moduli curve is smooth. Recall that ([6, Section 4, pg. 100]) a stacky curve \( \mathfrak{X} \) is smooth if for one (and hence for every) surjective étale morphism \( f: Y \rightarrow \mathfrak{X} \) the Coarse moduli curve is smooth.

\[\text{Definition 3.6 (Orbifold Curve).} \quad \text{A stacky curve } \mathfrak{X} \text{ is said to be an orbifold curve if it is smooth (i.e. any atlas is a smooth } k\text{-curve).}\]

When \( Y \rightarrow X \) is a \( G \)-Galois cover of smooth \( k\)-curves for some finite group \( G \) and \( X \) is connected, the quotient stack \([Y/G]\) in Example 3.1 is an orbifold curve.

3.2. Formal Orbifold Curves. The notion of a formal orbifold curve was introduced in [17] when \( k = \mathbb{C} \), and was later generalized over fields of arbitrary characteristic in [11]. In this section, we recall the definitions of the formal orbifold curves, the morphisms among them and some properties.

\[\text{Definition 3.7 (Formal Orbifold Curve).} \quad \text{Let } k \text{ be an algebraically closed field and } X \text{ be a smooth } k\text{-curve.}\]

(a) A quasi-branch data on \( X \) is a function \( P \) that to every closed point \( x \in X \) associates a finite Galois extension \( P(x) \) of \( K_{X,x} \) (in some fixed separable algebraic closure of \( K_{X,x} \)). For two quasi-branch data \( P' \) and \( P \) on \( X \), we define \( P' \geq P \) if \( P'(x) \supset P(x) \) as finite Galois extensions of \( K_{X,x} \) for all closed points \( x \in X \).

The support \( \text{Supp}(P) \) of a quasi-branch data \( P \) on \( X \) is defined to be

\[
\text{Supp}(P) := \{ x \in X \mid P(x) \text{ is a nontrivial extension of } K_{X,x} \}.\]

A quasi-branch data \( P \) on \( X \) is said to be a branch data if \( \text{Supp}(P) \subset X \) is a finite set.

(b) A formal orbifold curve is an ordered pair \((X, P)\) where \( X \) is a smooth \( k\)-curve \( X \) and \( P \) is a branch data \( P \) on \( X \).

A formal orbifold curve \((X, P)\) is said to be connected (respectively, projective) if the \( k\)-curve \( X \) is connected (respectively, projective).

Thus a formal orbifold curve is a smooth curve \( X \) together with the data of a finite set \( B \) (which may be empty) of closed points in \( X \), and for each \( x \in B \), a finite Galois extension \( P(x)/K_{X,x} \). When \( \text{char}(k) = 0 \) or \( \text{Gal} \{ P(x)/K_{X,x} \} \) is coprime to \( p = \text{char}(k) \) for each closed point \( x \in X \), the Galois extensions \( P(x)/K_{X,x} \) are uniquely determined by their order. In these situations, \((X, P)\) is determined by \( X \) together with finitely many points and a positive integer (that is invertible in \( k \)) attached to each of these points. We call \( P \) a tame branch data.

The branch data with empty support is denoted by \( O \) and is called the trivial branch data. We identify the formal orbifold curve \((X, 0)\) with the \( k\)-curve \( X \).
Example 3.8. For any finite surjective morphism \( f: Y \to X \) of smooth \( k \)-curves, we can associate a branch data \( B_f \) on \( X \) as follows. For any closed point \( x \in X \), define \( B_f(x) \) to be the compositum of the Galois closures of \( K_{Y_f}/K_{X,x} \) for all \( y \in f^{-1}(x) \subset Y \) (where the compositum is taken in some fixed algebraic separable closure of \( K_{X,x} \)). Since \( f \) is a finite cover of smooth \( k \)-curves, it is branched only at finitely many points (possibly empty) of \( X \). So \( B_f \) is a branch data on \( X \) and \( \text{Supp}(B_f) \) is the branch locus \( \text{BL}(f) \) of \( f \).

Let \( \tilde{f}: \tilde{Y} \to X \) be the Galois closure of \( f \). Then it follows that \( B_{\tilde{f}} = B_f \).

Example 3.9. For any finite morphism \( f: Y \to X \) of smooth \( k \)-curves and a branch data \( P \) on \( X \), we can define the pullback of the branch data \( P \) under \( f \) on \( Y \), denoted by \( f^*P \). For any closed point \( y \in Y \), set
\[
f^*P(y) = P(f(y)) \cdot K_{Y_f}
\]
as the compositum of extensions over \( K_{X,f(y)} \). Then \( f^*P(y) \) is a Galois extension of \( K_{Y_f} \), and the Galois group \( \text{Gal}\left(f^*P(y)/K_{Y_f}\right) \) is a subgroup of \( \text{Gal}\left(P(f(y))/K_{X,f(y)}\right) \). So \( \text{Supp}(f^*P) \subseteq f^{-1}(\text{Supp}P) \) is a finite set, and \( f^*P \) is a branch data on \( Y \).

Definition 3.10. Let \((Y,Q)\) and \((X,P)\) be two formal orbifold curves. A morphism
\[
f: (Y,Q) \to (X,P)
\]
is defined to be a finite morphism \( f: Y \to X \) such that for all closed point \( y \in Y \), we have \( P(f(y)) \subset Q(y) \) as extensions of \( K_{X,f(y)} \).

The morphism \( f: (Y,Q) \to (X,P) \) is called étale at \( y \) if \( Q(y) = P(f(y)) \). It is called étale if it is étale for all closed point \( y \in Y \).

Let \((X,P)\) be a formal orbifold curve. A finite morphism \( f: Y \to X \) of smooth \( k \)-curves is said to be an essentially étale cover of \((X,P)\) if for any closed point \( y \in Y \),
\[
K_{Y_f} \subset P(f(y))
\]
as extensions of \( K_{X,f(y)} \).

Note that if a finite cover \( f: Y \to X \) of a smooth projective \( k \)-curves is an essentially étale cover of the orbifold \((X,P)\), then its Galois closure \( \tilde{f}: \tilde{Y} \to Y \) is also an essentially étale cover of \((X,P)\).

Example 3.11. Let \( P \) and \( P' \) be two branch data on \( X \). The identity map \( \text{Id}_X \) defines a morphism of formal orbifolds \( \iota: (X,P') \to (X,P) \) if and only if \( P' \geq P \). The morphism \( \iota \) is étale if and only if \( P' = P \).

In Example 3.9, we saw that to any finite morphism \( f: Y \to X \) and a branch data \( P \) on \( X \), we obtain a branch data \( f^*P \) on \( Y \). For any closed point \( y \in Y \), we have \( f^*P(y) = P(f(y)) \cdot K_{Y_f} \supset P(f(y)) \) as field extensions of \( K_{X,f(y)} \). So \( f \) induces a morphism \( f: (Y,f^*P) \to (X,P) \) of formal orbifold curves. By [11] Lemma 2.12, the following hold.

1. There is a morphism \( f: (Y,Q) \to (X,P) \) of formal orbifold curves if and only if \( Q \geq f^*P \).

2. For any finite cover \( f: Y \to X \) of smooth \( k \)-curves, the induced morphism \( f: (Y,f^*P) \to (X,P) \) of formal orbifold curves is étale if and only if \( f \) is an essentially étale cover of \((X,P)\).

When \( f: Y \to X \) is a finite Galois cover of smooth connected \( k \)-curves, the induced morphism \( f: (Y,O) \to (X,B_f) \) is étale ([11] Corollary 2.13]). In view of this, we have the following definition.

Definition 3.12. A connected formal orbifold curve \((X,P)\) is said to be geometric if there exists a connected étale cover \((Y,O) \to (X,P)\) of formal orbifold curves where \( O \) is the trivial branch data on \( Y \). In this case, \( P \) is called a geometric branch data on \( X \).
By [11, Proposition 2.30], given any branch data $P$ on $X$, we can always find a branch data $Q$ such that $Q \geq P$ and $Q$ is geometric.

3.3. **Slope stability for stacky curves.** In this section, we consider the slope stability conditions for vector bundles on a proper stacky curve $\mathfrak{x}$ over $k$. We adapt notions from [18, Definition 7.18]. So a vector bundle (respectively, a quasi-coherent or a coherent sheaf) on $\mathfrak{x}$ is the data given by a vector bundle (respectively, a quasi-coherent or a coherent sheaf) on each atlas that satisfy certain co-cycle conditions. The structure sheaf $O_{\mathfrak{x}}$ on an orbifold curve $\mathfrak{x}$ is the quasi-coherent sheaf defined by associating the structure sheaf $O_{Z}$ for every atlas $Z$ of $\mathfrak{x}$. It should be noted that a quasi-coherent sheaf in the above sense is actually a quasi-coherent sheaf of $O_{X}$-modules as in [16, Definition 9.1.14, Proposition 9.1.15]. Under the hypothesis on $\mathfrak{x}$, we can always find a Galois cover $Z \rightarrow X$ of projective curves dominating the Coarse moduli morphism (see Remark 3.3). For any morphism $f : \mathfrak{y} \rightarrow \mathfrak{x}$ of stacky curve, we have the following functors

$$\text{Vect}(\mathfrak{y}) \xrightarrow{f^*} \text{Vect}(\mathfrak{x}) \quad \text{and} \quad \text{Vect}(\mathfrak{x}) \xrightarrow{f^*} \text{Vect}(\mathfrak{y})$$

of the categories of vector bundles ([16, Section 9.2.5, pg. 198 and Section 9.3.]; defined up to a canonical natural isomorphism for the choice of charts). We start with the following observations.

First, suppose that $Z$ is a projective (not necessarily smooth or connected) $k$-curve equipped with an action of a finite group $G$ such that $X := Z/G$ is a smooth projective connected $k$-curve. Consider the quotient stacky curve $\mathfrak{x} = [Z/G]$ (see Example 3.1). Then the $G$-Galois cover $f : Z \rightarrow X$ factors as a composition of a $G$-Galois étale cover $g : Z \rightarrow \mathfrak{x}$ followed by the Coarse moduli morphism $\iota : \mathfrak{x} \rightarrow X$. The functor $g^*$ defines an equivalence of categories

$$(3.2)\quad g^* : \text{Vect}(\mathfrak{x}) \xrightarrow{\sim} \text{Vect}^G(Z)$$

of vector bundles on $\mathfrak{x}$ with the $G$-equivariant vector bundles on $Z$, with a quasi-inverse defined by $g^*_G$ (to see that this defines a quasi-inverse, one can work over charts and use the Galois étale descent for schemes).

Now consider a stacky curve $\mathfrak{x}'$ with its Coarse moduli curve $X$. By Remark 3.3 for some finite group $G$, there exists a $G$-Galois cover $f : Z \rightarrow X$ that factors as the composition

$$f : Z \xrightarrow{g} \mathfrak{x} := [Z/G] \xrightarrow{\iota} \mathfrak{x}' \xrightarrow{\iota'} X$$

where $\iota'$ and $\iota' \circ \iota$ are the Coarse moduli morphisms. By definition, any vector bundle $E$ on $\mathfrak{x}'$ can be seen as a $G$-equivariant vector bundle on $Z$, and under the equivalence $(3.2)$, as a vector bundle on $\mathfrak{x}$. Further, for any two vector bundles $E, F \in \text{Vect}(\mathfrak{x}')$, by [16, Proposition 9.3.6, pg. 205] and [8, Proposition 1.12], we have the following.

$$\text{Hom}_{\text{Vect}(\mathfrak{x})}(\iota^*E, \iota^*F) = \text{Hom}_{\text{Vect}(\mathfrak{x}')}\left(\iota_*E, \iota_*F\right)$$

Note that $\iota_*O_{\mathfrak{x}} = O_X$. This shows that we have an inclusion of categories

$$(3.3)\quad \iota^* : \text{Vect}(\mathfrak{x}') \hookrightarrow \text{Vect}(\mathfrak{x})$$

In view of the above discussion, we make the following definition.

**Definition 3.13.** Let $\mathfrak{x}$ be a proper stacky curve with Coarse moduli space $X$. Let $Z \rightarrow X$ be a Galois cover of curves with group $G$, dominating the Coarse moduli map $\mathfrak{x} \rightarrow X$. For a vector bundle $E \in \text{Vect}(\mathfrak{x})$, define the **degree** and the **slope** of $E$ as follows.
Let the $G$-equivariant vector bundle $\mathcal{E}$ be the image of $E$ under the inclusion functor $\text{Vect}(\mathfrak{X}) \hookrightarrow \text{Vect}^G(\mathfrak{Z})$ (as the composition of the functors in Equations (3.2), (3.3)). Define

$$\deg_\mathfrak{X}(E) := \frac{1}{|G|} \deg(\mathcal{E}),$$

and

$$\mu_\mathfrak{X}(E) := \frac{1}{|G|} \mu(\mathcal{E}).$$

**Remark 3.14.** To see that the above notion are well defined, it is enough to consider the case $\mathfrak{X} = [\mathfrak{Z}/G]$. Suppose that $[\mathfrak{Z}/G] = [\mathfrak{Z}'/G']$ and $\mathcal{E}'$ be the $G'$-equivariant vector bundle on $\mathfrak{Z}'$ corresponding to $E$. As $\mathcal{E}$ and $\mathcal{E}'$ pullback to the same equivariant bundle on $\mathfrak{Z} \times X \mathfrak{Z}'$ of degree $|G'| \deg(\mathcal{E}) = |G| \deg(\mathcal{E}')$, we see that $\deg_\mathfrak{X}$ and $\mu_\mathfrak{X}$ do not depend on the choice of the cover $\mathfrak{Z} \to X$.

In view of the above definition, we can define $\mu_\mathfrak{X}$-(semi)stable or $\mu_\mathfrak{X}$-polystable vector bundles using the slope $\mu_\mathfrak{X}$.

**Definition 3.15.** Let $\mathfrak{X}$ be a proper stacky curve with Coarse moduli curve $X$. A vector bundle $E \in \text{Vect}(\mathfrak{X})$ is called $\mu_\mathfrak{X}$-(semi)stable if for all its sub-bundle $0 \neq F \subset E$ in $\text{Vect}(\mathfrak{X})$, we have

$$\mu_\mathfrak{X}(F) \leq \mu_\mathfrak{X}(E).$$

A $\mu_\mathfrak{X}$-polystable bundle on $\mathfrak{X}$ bundle that is a finite sum of $\mu_\mathfrak{X}$-stable vector bundles in $\text{Vect}(\mathfrak{X})$ having the same $\mu_\mathfrak{X}$.

We have the following relation between a vector bundle on $\mathfrak{X}$ and the equivariant vector bundle on $\mathfrak{Z}$.

**Proposition 3.16.** Let $\mathfrak{X}$ be a proper stacky curve with Coarse moduli curve $X$. Let $\mathfrak{Z} \to X$ be a Galois cover of curves with group $G$, dominating the Coarse moduli map $\mathfrak{X} \to X$. Let $E \in \text{Vect}(\mathfrak{X})$. Let the $G$-equivariant vector bundle $\mathcal{E}$ be the image of $E$ under the inclusion functor $\text{Vect}(\mathfrak{X}) \hookrightarrow \text{Vect}^G(\mathfrak{Z})$ (as the composition of the functors in Equations (3.2), (3.3)). Then $E$ is $\mu_\mathfrak{X}$-(semi)stable if and only if $\mathcal{E}$ is $G$-(semi)stable. Moreover, $E$ is $\mu_\mathfrak{X}$-polystable if and only if $\mathcal{E}$ is $G$-polystable.

**Proof.** The first conclusion is immediate from the slope relation $\mu_\mathfrak{X} = \frac{1}{|G|} \mu$. This relation together with the fact that the equivalence and the inclusion functors in Equation (3.2), (3.3) preserve finite direct sum imply the second statement. □

It should be noted that a $G$-equivariant vector bundle $\mathcal{E}$ on $\mathfrak{Z}$ is $G$-semistable (respectively, $G$-polystable) if and only if $\mathcal{E}$ is a semistable (respectively, polystable) in the usual sense (for example, see [1] Lemma 2.7]; these follow from the uniqueness of the Harder-Narasimhan filtration and the socle of a semistable bundle). Whereas, $G$-stability need not be same as the usual stability – consider any irreducible $k[G]$-module $V$ of dimension $\geq 2$ and equip the trivial bundle $O_Z \otimes k V$ with the diagonal $G$-action. This the $G$-equivariant bundle is $G$-stable, but non stable in the usual sense.

We list some properties of the stability condition.

**Proposition 3.17.**

1. **Under the hypothesis of Proposition 3.16** we have the following.

---

2. As in the case of schemes, the notation $(\leq)$ means that $E$ is $\mu_\mathfrak{X}$-semistable if we have $\leq$, and it is $\mu_\mathfrak{X}$-stable if we have the strict inequality $<$.  
3. This example was pointed out to us by the referee
and the usual results for curves.

All of the above are easy consequences of Proposition 3.16, the previous definitions (Definition 4.4, Definition 4.5), and the previous definitions (Definition 3.18) of vector bundles and their morphisms on an orbifold curve following [12].

3.4. Vector Bundles on Formal Orbifold Curves. In this section, we briefly recall the concepts coincide.

In (i) above, we needed the assumption that \( \mathcal{X} = [Z/G] \). For an arbitrary proper stacky curve, one can still construct a 'maximal destabilizing sub-bundle' \( \mathcal{E}_1 \subseteq \mathcal{E} \) or a 'socle' \( \mathcal{E}(\mathcal{E}_1) \) of \( \mathcal{E} \) using the slope \( \mu_\mathcal{E} \). But, we do not know whether the corresponding \( G \)-equivariant bundles on \( Z \) coincide with the destabilizing sub-bundle \( \mathcal{E}_1 \subseteq \mathcal{E} \) or the socle \( \mathcal{E}(\mathcal{E}_1) \) of \( \mathcal{E} \). When \( \mathcal{X} \) is an orbifold curve, we will see that these concepts coincide.

(a) Suppose that \( \mathcal{X} = [Z/G] \). There is a unique Harder-Narasimhan filtration for \( E \), and \( \mu_{\mathcal{X},\max} \) coincides with the slope \( \mu_\mathcal{X} \) of the maximal destabilizing sub-bundle. If \( E \) is also \( \mu_{[Z/G]} \)-semistable, there is a unique socle for \( E \).

(b) If \( L \) is a line bundle on \( \mathcal{X} \), the tensor product \( E \otimes L \) is \( \mu_\mathcal{X} \)-semistable if and only if \( E \) is \( \mu_\mathcal{X} \)-semistable.

(2) Let \( f : \mathcal{Y} \longrightarrow \mathcal{X} \) be a finite cover of proper stacky curves. This necessarily induce a finite cover \( f_0 : Y \longrightarrow X \) of the Coarse moduli curves. We have the following.

(a) \[
S\deg(f^*E) = \deg(f_0)\deg_\mathcal{X}(E).
\]

The same holds for \( \mu_- \) and \( \mu_{-\max} \).

(b) \( E \) is \( \mu_\mathcal{X} \)-semistable (respectively, \( \mu_\mathcal{X} \)-polystable) if and only if \( f^*E \in \text{Vect}(\mathcal{Y}) \) is \( \mu_\mathcal{Y} \)-semistable (respectively, \( \mu_\mathcal{Y} \)-polystable).

(c) If \( f^*E \in \text{Vect}(\mathcal{Y}) \) is \( \mu_\mathcal{Y} \)-stable, then \( E \) is \( \mu_\mathcal{X} \)-stable.

\textbf{Proof.} All of the above are easy consequences of Proposition 3.16 and the usual results for curves.

In (i) above, we needed the assumption that \( \mathcal{X} \) is the quotient stacky curve \([Z/G]\). For an arbitrary proper stacky curve, one can still construct a 'maximal destabilizing sub-bundle' \( \mathcal{E}_1 \subseteq \mathcal{E} \) or a 'socle' \( \mathcal{E}(\mathcal{E}_1) \) of \( \mathcal{E} \) using the slope \( \mu_\mathcal{X} \). But, we do not know whether the corresponding \( \mathcal{G} \)-equivariant bundles on \( Z \) coincide with the destabilizing sub-bundle \( \mathcal{E}_1 \subseteq \mathcal{E} \) or the socle \( \mathcal{E}(\mathcal{E}_1) \) of \( \mathcal{E} \). When \( \mathcal{X} \) is an orbifold curve, we will see that these concepts coincide.

A vector bundle on \((X, P)\) is a triple \( E_\ast = (E, \{\Phi_x\}_{x \in B}, \{\eta_x\}_{x \in B}) \) where \( E \) is a vector bundle on \( X \) together with the following data for each \( x \in B \).

(i) A group homomorphism \( \Phi_x : G_x \longrightarrow \text{Aut}_{\text{Ab}}(E_x \otimes O_{x_x}, R_x) \) satisfying the following condition.

\[ \Phi_x(g)(r \cdot m) = \Phi_x(g)(r) \cdot \Phi_x(g)(m) \text{ for any } g \in G_x, r \in R_x, m \in E_x \otimes O_{x_x}, R_x. \]

(ii) For the induced generic actions

\[ \Phi^0_x : G_x \xrightarrow{\Phi_x} \text{Aut}_{\text{Ab}}(E_x \otimes O_{x_x}, R_x) \longrightarrow \text{Aut}_{\text{Ab}}(E_x \otimes O_{x_x}, P(x)) \]

and \( \phi^0_x : G_x \xrightarrow{\phi_x} \text{Aut}_{\text{Ring}}(R_x) \longrightarrow \text{Aut}_{\text{Ring}}(P(x)) \).

In (i) above, we needed the assumption that \( \mathcal{X} \) is the quotient stacky curve \([Z/G]\). For an arbitrary proper stacky curve, one can still construct a 'maximal destabilizing sub-bundle' \( \mathcal{E}_1 \subseteq \mathcal{E} \) or a 'socle' \( \mathcal{E}(\mathcal{E}_1) \) of \( \mathcal{E} \) using the slope \( \mu_\mathcal{X} \). But, we do not know whether the corresponding \( \mathcal{G} \)-equivariant bundles on \( Z \) coincide with the destabilizing sub-bundle \( \mathcal{E}_1 \subseteq \mathcal{E} \) or the socle \( \mathcal{E}(\mathcal{E}_1) \) of \( \mathcal{E} \). When \( \mathcal{X} \) is an orbifold curve, we will see that these concepts coincide.
$\eta_s$ is a $G_s$-equivariant isomorphism
\[ \eta_s: E_x \otimes_{O_{X_s}} P(x) \xrightarrow{\sim} E_x \otimes_{O_{X_s}} P(x), \]
where $G_s$ acts on the left via $\text{Id}_{E_x} \otimes \phi_s^0$, and on the right via $\Phi_s^0$.

For vector bundles $E_\ast = (E, \{ \Phi_{x} \}_{x \in B}, \{ \eta_s \}_{s \in B})$ and $F_\ast = (F, \{ \Psi_{x} \}_{x \in B}, \{ \vartheta_s \}_{s \in B})$, a morphism $E_\ast \rightarrow F_\ast$ is defined to be a pair $(g, \{ \sigma_s \}_{s \in B})$ where $g: E \rightarrow F$ is a homomorphism of vector bundles on $X$, and for each $x \in B$, $\sigma_s: E_x \otimes_{O_{X_s}} R_x \rightarrow F_x \otimes_{O_{X_s}} R_x$ is a $G_s$-equivariant (with respect to the $\Phi_s$ and $\Psi_s$-actions) homomorphism of $R_x$-modules such that the following diagram is commutative.

\[
\begin{array}{ccc}
E_x \otimes_{O_{X_s}} P(x) & \xrightarrow{g \otimes \text{Id}_{P(x)}} & F_x \otimes_{O_{X_s}} P(x) \\
\downarrow \eta_s & & \downarrow \theta_s \\
E_x \otimes_{O_{X_s}} P(x) & \xrightarrow{\sigma_s^0} & F_x \otimes_{O_{X_s}} P(x)
\end{array}
\]

Here $\sigma_s^0$ is the homomorphism induced by $\sigma_s$.

We denote the category of vector bundles on a formal orbifold $(X, P)$ by $\text{Vect}(X, P)$.

To simplify the notation, we write $E_\ast = (E, \Phi, \eta)$ when the context is clear.

**Example 3.19.** For any formal orbifold curve $(X, P)$, we have the trivial vector bundle $O_{(X, P)} \in \text{Vect}(X, P)$. The underlying vector bundle on $X$ is the structure sheaf $O_X$, and for any $x \in B$, the action maps are given by
\[ \Phi_x = \phi_x: G_x \rightarrow \text{Aut}_{\text{Ab}}(O_{X_s} \otimes_{O_{X_s}} R_x) = \text{Aut}_{\text{Ab}}(R_x), \]
and so $\eta_s$ is the identity map.

**Example 3.20.** Suppose that $\text{char}(k) = 0$. Let $(X, P)$ be a formal orbifold curve. So $(X, P)$ is determined by a finite set $B$ of closed points and for each $x \in B$, a positive integer $n_s$. By [12] Proposition 5.15, $\text{Vect}(X, P)$ is equivalent to the category of parabolic vector bundles on $X$ with respect to the divisor $\sum_{x \in B} x$ and over each $x \in B$, the weights are of the form $a/n_s$, $0 \leq a < n_s$.

**Definition 3.21.** Let $E_\ast = (E, \Phi, \eta) \in \text{Vect}(X, P)$. A vector bundle $F_\ast = (F, \Psi, \vartheta) \in \text{Vect}(X, P)$ is called a sub-bundle of $E_\ast$ if there is a morphism $(g, \sigma): F_\ast \rightarrow E_\ast$ such that $g: F \rightarrow E$ is an injective homomorphism making $F$ into a sub-bundle of $E$ (so the quotient $E/g(F)$ is a vector bundle) and for each $x \in \text{Supp}(P)$, the $R_x$-module homomorphism $\sigma_x: F_x \otimes_{O_{X_s}} R_x \rightarrow E_x \otimes_{O_{X_s}} R_x$ is a $G_s$-equivariant monomorphism.

**Remark 3.22.** For a sub-bundle $F_\ast \subset E_\ast$ in $\text{Vect}(X, P)$ as above, for any $g \in G_x$ and $f \in F_x \otimes_{O_{X_s}} R_x$, we have $\sigma_x(\Psi_x(g)(f)) = \Phi_x(g)(\sigma_x(f))$. Thus for any $g, f$, and $r \in R_x$, we have the following.

\[
\Phi_x(g)(r \cdot \sigma_x(f)) = \phi_x(g)(r) \cdot \Phi_x(g)(\sigma_x(f)) = \phi_x(g)(r) \cdot \sigma_x(\Psi_x(g)(f)) = \sigma_x(\Psi_x(g)(rf)).
\]

Thus the action morphisms $\Psi_s$, and hence the $G_s$-equivariant isomorphisms $\theta_s$ are restriction of $\Phi_s$ and $\eta_s$, respectively.

We also recall that [12 Theorem 4.12)](12) for a $(X, P)$ geometric formal orbifold curve with $g: (Z, O) \rightarrow (X, P)$ a $G$-Galois étale cover for a finite group $G$, there is an equivalence of categories
\[
(3.4) \quad T^G_{(X, P)}: \text{Vect}(X, P) \xrightarrow{\sim} \text{Vect}^G(Z),
\]
of vector bundles on \((X, P)\) and the \(G\)-equivariant vector bundles on \(Z\). The quasi-inverse to \(T^Z_{X, h}P\) is denoted by \(S^Z_{X, h}P\), which is constructed using the invariant pushforward functor.

For other important properties and relations, we refer to [12].

4. Formal orbifold curves: a stacky view

This section is devoted to showing that a formal orbifold curve is the same as an orbifold curve (see Definition 3.6). Moreover, the categories of vector bundles on them coincide. This view lets us define concepts like the degree of a vector bundle on a formal orbifold curve, and a stability condition in a more intrinsic way.

Let \(\mathfrak{X}\) be an orbifold curve. For any point \(x \in \mathfrak{X}(k)\), the fiber product \(\mathfrak{X} \times_{\mathfrak{X}} \mathfrak{X}_x \mathfrak{X}(x, x)\) \(\text{Spec}(k) = \text{Isom}(x, x)\) is a constant group \(k\)-scheme associated to a finite group \(G_x\) (up to a canonical isomorphism). We say that \(G_x\) is the stabilizer group at \(x\). So a closed geometric point in \(\mathfrak{X}\) is identified with a closed point \(x\) of \(X\) together with the group \(G_x\) that acts as the group of automorphisms on \(x\). It follows that \(G_x\) is the trivial group if and only if \(x\) lies in the open sub-scheme of \(\mathfrak{X}\).

**Proposition 4.1.** Let \(\mathfrak{X}\) be an orbifold curve and \(\pi : \mathfrak{X} \longrightarrow X\) be a Coarse moduli \(k\)-curve. Let \(U \subset X\) be the maximal open subset such that \(\mathfrak{X} \times_X U \cong U\). Set \(B = X - U\). For each point \(x \in B\), let \(G_x\) be the stabilizer group at \(x\). Then for each point \(x \in B\), the following hold.

There is an open subset \(U_x \subset X\) together with a \(\Gamma_x\)-Galois cover \(V_x \longrightarrow U_x\) (for some finite group \(\Gamma_x\) that is étale away from \(x\), \(U_x \cap B = \{x\}\) the inertia groups at points in \(V_x\) lying above \(x\) are conjugates of \(G_x\) in \(\Gamma_x\) such that

\[\mathfrak{X} \times_X U_x \cong [V_x/\Gamma_x].\]

**Proof.** Let \(V \longrightarrow \mathfrak{X}\) be an atlas where \(V\) is a smooth \(k\)-curve. The induced morphism \(g : V \longrightarrow X\) is a finite surjective morphism of \(k\)-curves, étale over \(U\). In particular, \(g\) is a finite cover. By the maximality of \(U\), the branched locus of \(g\) is \(B\). Let \(x \in B\). Then

\[V \times_X \text{Spec} (\mathcal{O}_{\mathfrak{X}, x}) = \sqcup_{1 \leq i \leq n} \text{Spec} (S_i),\]

a finite disjoint union, where each \(S_i\) is a complete discrete valuation domain and the maps \(\text{Spec} (S) \longrightarrow \text{Spec} (\mathcal{O}_{\mathfrak{X}, x})\) are finite morphisms of regular schemes. Set \(\mathfrak{X}_i = \mathfrak{X} \times_X \text{Spec} (\mathcal{O}_{\mathfrak{X}, x})\). Let \(W_i = \text{Spec} (S_i)\).

We first show that \(W_1\) is equipped with a \(G_x\)-action such that \(\mathfrak{X}_i \cong [W_i/G_x]\) (the same holds for each \(i, 1 \leq i \leq n\)).

Both the projection maps \(W_1 \times X, W_1 \underset{p_1}{\longrightarrow} W_1, W_1 \underset{p_2}{\longrightarrow} W_1\) are étale. So via the first projection map \(p_1\), each connected component of \(W_1 \times X, W_1\) is identified with \(W_1\). Since

\[(W_1 \times X, W_1) \times_{\text{Spec} (\mathcal{O}_{\mathfrak{X}, x})} \text{Spec} (k) \cong \text{Isom}(x, x) \cong G_x,\]

we have an isomorphism \(W_1 \times X, W_1 \cong W_1 \times G_x\), where the composition \(W_1 \times X, W_1 \overset{\sim}{\longrightarrow} W_1 \times G_x \longrightarrow W_1\) is \(p_1\). We also obtain a morphism \(\rho_{x, 1} : W_1 \times G_x \overset{\sim}{\longrightarrow} W_1 \times X, W_1 \overset{p_1}{\longrightarrow} W_1\).

As in the proof of [16] Theorem 11.3.1, it follows that \(\rho_{x, 1}\) equips \(W_1\) with a \(G_x\)-action such that \(\mathfrak{X}_i \cong [W_i/G_x]\).

Since \(g\) is a finite morphism of smooth curves, there is an open irreducible smooth subset \(V_x \subset V\) containing \(W_1\) such that the map \(g\) restricts to a finite surjective cover

\[\mathfrak{X}_i \cong [V_x/G_x].\]
Let \( g : V_1 \to U_s = g(V_1) \) of smooth connected \( k \)-curves, étale away from \( x \), \( U_s \cap B = \{ x \} \), and there a point \( v_1 \in V_1 \) in the fiber over \( x \) with \( \overline{\partial}_{V_1,v_1} = S_1 \).

Let \( g_\lambda : V_\lambda \to U_\lambda \) be the Galois closure of the cover \( V_1 \to U_s \), with Galois group \( \Gamma_\lambda \), say. For each point \( v_1 \in V_1 \) above \( x \), we have \( \overline{\partial}_{V_1,v_1} \cong S_1 \) for some \( i \), and so \( K_{V_1,v} \cong K_{V_1,v_1} \) as \( G_x \)-Galois extensions of \( K_{X,x} \). Also since \( V_1 \to U_\lambda \) is étale away from \( x \), the induced map \( V_\lambda \to V_1 \) is a Galois étale cover. So the \( \Gamma_\lambda \)-Galois cover \( g_\lambda \) is a finite surjective cover of smooth connected \( k \)-curves, étale away from \( x \), and the inertia groups at the points in the fiber \( g_\lambda^{-1}(x) \) are conjugate to \( G_x \) in \( \Gamma_\lambda \). Then the induced map \( V_\lambda \to \mathfrak{X} \times_X U_\lambda \) is an étale surjective morphism,

\[
\left[ (V_\lambda - g_\lambda^{-1}(x))/\Gamma_\lambda \right] \cong U_\lambda - \{ x \} = U \cap U_\lambda \cong \mathfrak{X} \times_X (U_\lambda \cap U), \text{(1)}
\]

\[
\left[ (\{ x \} \times \text{Spec}(\overline{\partial}_{V_\lambda,v})) / \Gamma_\lambda \right] \cong \left[ \text{Spec} \left( \overline{\partial}_{V_\lambda,v_1} \right) / G_x \right] \cong \mathfrak{X}_s.
\]

This implies

\[
\mathfrak{X} \times_X U_\lambda \cong [V_\lambda / \Gamma_\lambda].
\]

\[\square\]

**Notation 4.2.** Let \( \mathfrak{X} \) be any orbifold curve. For a point \( x \in B \) as in Proposition 4.1 and the \( \Gamma_x \)-Galois cover \( V_x \to U_x \), we have

\[
\overline{\partial}_{V_x,v} \cong \overline{\partial}_{V_x,v'}
\]

for any two points \( v, v' \in V_x \) over \( x \), as \( G_x \)-Galois extensions of \( \overline{\partial}_{X,x} \)-algebras. Moreover,

\[
\mathfrak{X} \times_X \text{Spec}(\overline{\partial}_{X,x}) \cong \left[ \text{Spec} \left( \overline{\partial}_{V_x,v} \right) / G_x \right] \cong \left[ \text{Spec} \left( \overline{\partial}_{V_x,v'} \right) / G_x \right].
\]

We fix a point \( v \in V_x \) above \( x \) and set

\[
R_x := \overline{\partial}_{V_x,v}.
\]

To simplify the notation, to any orbifold curve \( \mathfrak{X} \) with \( B = X - U \) as before, to each point \( x \in B \) we associate the tuple

\[
(U_x, V_x, R_x).
\]

**Remark 4.3.** Let \( \mathfrak{X} \) be an orbifold curve. For \( x \in B \) as in Proposition 4.1 we have a tuple \((U_x, V_x, R_x)\). For a different choice of \((U'_x, V'_x, R'_x)\), by shrinking \( U_x \) and \( U'_x \), and taking a \( \Gamma'_x \)-Galois cover \( V'_x \to U'_x \) dominating the \( \Gamma_x \) and \( \Gamma'_x \)-Galois covers, we can find another tuple \((U''_x, V''_x, R''_x)\) such that

\[
R_x \cong R'_x \cong R''_x
\]

as \( G_x \)-Galois extensions of \( \overline{\partial}_{X,x} \). So for different choices of tuples which arise from the choices made in the proof of Proposition 4.1 the complete discrete valuation domains are all \( G_x \)-equivariantly isomorphic.

Let \( \mathfrak{X} \) be a proper orbifold curve and \( X \) be its Coarse moduli scheme. In particular, \( X \) is a smooth ([16] Exercise 11.H]) connected projective \( k \)-curve. We construct a connected formal orbifold curve \( \lambda(\mathfrak{X}) \) as follows.

Let \( U \subset X \) be the maximal open subset such that \( \mathfrak{X} \times_X U = U \). Set \( B := X - U \). Then \( B \) is a finite set of closed points of \( X \). The set \( B \) is empty if and only if \( \mathfrak{X} \cong X \), in which case we set \( \lambda(\mathfrak{X}) = (X, O) \) where \( O \) is the trivial branch data on \( X \). By Notation 4.2 we obtain a tuple \((U_x, V_x, R_x)\) associated to \( \mathfrak{X} \). More precisely, for each point \( x \in B \), we have an open connected subset \( U_x \subset X \) and a \( \Gamma_x \)-Galois cover \( g_x : V_x \to U_x \) of smooth connected
$k$-curves, étale away from $x$, and for a point $v_x \in V_x$ over $x$, $R_x = \widehat{O}_{V_x,v_x}$ is a $G_x$-Galois $\hat{O}_{X,x}$-algebra which induces an isomorphism

$$\mathfrak{X} \times_X \text{Spec}(\hat{O}_{X,x}) \cong [\text{Spec}(R_x) / G_x].$$

For any closed point $x \in X$, consider the $G_x$-Galois extensions of $K_{X,x}$ given by

$$P(x) = \begin{cases} QF(R_x) = K_{V_x,v_x}, & \text{if } x \in B \\ K_{X,x}, & \text{if } x \in U. \end{cases}$$

By Remark 4.3, the $G_x$-Galois extension over each $K_{X,x}$ is independent (up to isomorphism) of the choices of $(U_x, V_x, R_x)$. We define $\lambda(x)$ to be the connected formal orbifold curve $(X, P)$. Note that $\text{Supp}(P) = B$.

Now we construct $\lambda(f)$ for any finite covering map $f : Y \rightarrow X$ of proper orbifold curve. Suppose that $\lambda(Y) = (Y, Q)$ and $\lambda(X) = (X, P)$. For a point $y \in Y(k)$ with image $x \in X(k)$, let $G_y'$ and $G_x$ denote the stabilizer groups at $y$ and at $x$, respectively. Then $f$ induces a finite cover $f : Y \rightarrow X$ of smooth projective connected $k$-curves. Moreover, for each closed point $y \in Y$, we obtain tuples $(U'_y, V'_y, R'_y)$ and $(U_x, V_x, R_x)$ with $x = f(y)$ that are compatible with the given morphism $f$ of orbifold curves. More precisely, $f$ restricts to a finite surjective cover $f : U'_y \rightarrow U_x$ and there is a finite surjective cover $h : V'_y \rightarrow V_x$ making the following diagram commute.

\[
\begin{array}{ccc}
V'_y & \xrightarrow{h} & V_x \\
\downarrow{g'_y} & & \downarrow{g_x} \\
U'_y & \xrightarrow{f} & U_x \\
\downarrow{g'_y} & & \downarrow{g_x} \\
Y & \xrightarrow{f} & X
\end{array}
\]

(4.1)

The above diagram induces the following commutative diagram.

$$\begin{array}{ccc}
\text{Spec}(R'_y) & \xrightarrow{h} & \text{Spec}(R_x) \\
\downarrow & & \downarrow \\
\text{Spec}(\hat{O}_{Y,y}) & \xrightarrow{f} & \text{Spec}(\hat{O}_{X,x})
\end{array}$$

So for every closed point $y \in Y$, we have

$$Q(y) \supset P(x) \cdot K_{Y,y} \supset P(x),$$

and $f : Y \rightarrow X$ defines a morphism $\lambda(f) : (Y, Q) \rightarrow (X, P)$ of connected formal orbifold curves (up to isomorphism).

The above construction produces a faithful functor $\lambda$ of categories from the isomorphism classes of proper orbifold curves together with finite morphisms to the category of isomorphism classes of connected formal orbifold curves.

**Theorem 4.4.** The functor $\lambda$ defines an equivalence of categories

$$\left(\text{Isomorphism classes of proper orbifold curves}\right) \overset{\sim}{\longrightarrow} \left(\text{Isomorphism classes of connected projective formal orbifold curves}\right).$$

We will denote the quasi-inverse of $\lambda$ by $\alpha$. 
Proof. Suppose that \( Y \) and \( X \) be two proper orbifold curves. Let \( \lambda(Y) = (Y, Q) \) and \( \lambda(X) = (X, P) \). We first show that

\[
\text{Hom}_{k}\text{-stacks} \ (Y, X) = \text{Mor} \ ((Y, Q), (X, P)) .
\]

Since \( \lambda \) is faithful, we need to show that for any morphism \( f : (Y, Q) \to (X, P) \), there is a finite covering morphism \( h : \mathcal{Y} \to \mathcal{X} \) such that \( \lambda(h) = f \). We have an induced finite cover \( f : Y \to X \) of smooth projective connected \( k \)-curves. Also \( f \ (Y - \text{Supp}(Q)) \subset X - \text{Supp}(P) \). We take \( h = f \) on \( Y - \text{Supp}(Q) \equiv \mathcal{Y} \times_Y (Y - \text{Supp}(Q)) \). So it is enough to construct \( h \) étale locally on \( \mathcal{Y} \) and \( \mathcal{X} \) for each point \( y \in \text{Supp}(Q) \).

Let \( y \in \text{Supp}(Q) \), \( x = f(y) \) with respective stabilizer groups \( G_y \) at \( y \) and \( G_x \) at \( x \). Consider tuples \((U'_y, V'_y, R'_y)\) and \((U_x, V_x, R_x)\) as in Notation 4.2. We have the inclusion \( Q(y) \supset P(x) \cdot K_{y,x} \supset K_{x,x} \). Since \( R'_y \) is the integral closure of \( \tilde{O}_{Y,y} \) in \( Q(y) \) and \( R_x \) is the integral closure of \( \tilde{O}_{X,x} \) in \( P(x) \), we obtain

\[
R'_y \supset R_x
\]
as \( \tilde{O}_{X,x} \)-algebras, and a morphism

\[
\mathcal{Y} \times_Y \tilde{O}_{Y,y} \equiv \left[ R'_y/G'_y \right] \xrightarrow{h} \left[ R_x/G_x \right] \equiv \mathcal{X} \times_X \tilde{O}_{X,x}.
\]

By shrinking around \( y \) and \( x \), we assume that \( U'_y \to Y \) and \( U_x \to X \) are étale neighborhoods of \( y \) and \( x \), respectively, and the following hold. \( \Gamma'_y = G'_y \), \( \Gamma_x = G_x \), the morphism \( f \) induces maps \( f : U'_y \to U_x \) and \( h : V'_y \to V_x \) so that the diagram (4.1) commutes, and that the inclusion \( R'_y \supset R_x \) of \( \tilde{O}_{X,x} \)-algebras is obtained via the pullback of \( h \) to \( \text{Spec} \ (\tilde{O}_{X,x}) \). This produces our required étale local morphism

\[
\mathcal{Y} \times_Y U'_y \equiv \left[ V'_y/G'_y \right] \xrightarrow{\alpha(f)} \left[ V_x/G_x \right] \equiv \mathcal{X} \times_X U_x.
\]

Now we show that \( \lambda \) is essentially surjective. Let \((X, P)\) be a connected formal orbifold curve. For any closed point \( x \in X \), set \( G_x := \text{Gal} \ (P(x)/K_{x,x}) \). Let \( B \) be the support of the branch data \( P \), and set \( U = X - B \). Let \( x \in B \). Identifying the complete local ring \( \tilde{O}_{X,x} \) at \( x \) with \( \mathcal{O}_{P^{1},x} \), we have a \( G_x \)-Galois field extension \( P(x) \supset K_{x,x} \). Let \( \phi_x : W_x \to \mathbb{P}^1 \) be the Harbater-Katz-Gabber cover (the Kummer cover if \( \text{char}(k) = 0 \); [10] Theorem 1.4.1) corresponding to the extension \( P(x)/K_{x,x} \). So \( \phi_x \) is a \( G_x \)-Galois cover of smooth projective connected \( k \)-curves, étale away from \( \infty \), namely ramified over \( 0 \), and over the unique point \( w_x \in W_x \) over \( \infty \), we have

\[
K_{W_{x,w_x}} \equiv P(x)
\]
as \( G_x \)-Galois extensions of \( K_{x,x} \). By the Noether Normalization Lemma ([2], Corollary 16.18)], there exists a finite cover \( g_x : X \to \mathbb{P}^1 \) of smooth connected projective \( k \)-curves that is étale at \( x \). Consider the component \( V_x \) in the normalization of \( X \times_{g_x,\mathbb{P}^1} W_x \). So we obtain a Galois cover \( \theta_x : V_x \to X \) with the Galois group contained in \( G_x \). Since \( g_x \) is étale at the point \( x \), the cover \( \theta_x \) is \( G_x \)-Galois, and at the unique point \( v_x \in V_x \) over \( x \), we have \( K_{V_{x,v_x}} \equiv P(x) \) as \( G_x \)-Galois extensions of \( K_{x,x} \). Consider the open subset \( U_x \subset X \) by removing the branched points of \( \theta_x \), except for the point \( x \), i.e. \( U_x = X - \text{BL} (\theta_x) \cup \{ x \} \). Set \( Z_x := \theta_x^{-1}(U_x) \).

So for each \( x \in B \), we have constructed an open subset \( U_x \subset X \) such that \( x' \notin U_x \) for \( x' \in B - \{ x \} \), together with a \( G_x \)-Galois cover \( V_x \to U_x \) of smooth connected \( k \)-curves, étale away from \( x \), and \( P(x)/K_{x,x} \) occurs as the local Galois extension near \( x \). We obtain a stack \( \mathcal{X} \) by gluing the quotients stacks \([Z_x/G_x]\), \( x \in B \), with the curve \( U \). From the above construction, \( \lambda(\mathcal{X}) \) is isomorphic to the formal orbifold curve \((X, P)\). \( \square \)
Remark 4.5. The above theorem seems to be a folklore result, known to the experts (for example, see [Appendix B, Theorem B.1, [14]]. As we could not find a proof, we provide a rigorous one.

Remark 4.6. We remark some important properties of the functor \( \lambda \) which will be used frequently for the rest of this section. For any proper orbifold curve \( \mathcal{X} \) with Coarse moduli \( k \)-curve \( X \) and \( \lambda(X) = (X, P) \), we have the following.

1. The support \( \text{Supp}(P) \) of the branch data is given by \( X - U \), where \( U \) is the maximal open subset of \( X \) satisfying \( \mathcal{X} \times_X U \).

2. For any closed point \( x \in X \), we have
   \[
   G_x = \text{Gal}(P(x)/K_{X,x})
   \]
   where \( G_x \) is the stabilizer group at \( x \). By (1), \( x \in X(k) \) corresponds to a stacky point in \( \mathcal{X}(k) \) if and only if \( G_x \) is a non-trivial group if and only if \( x \in \text{Supp}(P) \).

3. For any closed point \( x \in X \), let \( R_x \) denote the integral closure of \( \hat{O}_{X,x} \) in \( P(x) \). Then we have the isomorphism
   \[
   \mathcal{X} \times_X \text{Spec}(\hat{O}_{X,x}) \cong \text{Spec}(R_x) / G_x.
   \]

Under the above equivalence, we can view a connected formal orbifold curve as a proper orbifold curve. We have a well developed theory of quasi-coherent sheaves on an orbifold curve (see [18] or [16]). We will show that the vector bundles on a proper orbifold curve \( X \) are the same as the vector bundles on \( (X, P) = \lambda(X) \).

Theorem 4.7. Let \( \mathcal{X} \) be a proper orbifold curve. Consider the formal orbifold curve \( (X, P) = \lambda(X) \), where \( \lambda \) is the categorical equivalence defined in Theorem 4.4. We have an equivalence of categories

\[
\lambda_{\text{Vect}, X} : \text{Vect}(\mathcal{X}) \xrightarrow{\sim} \text{Vect}(X, P)
\]

with a quasi-inverse \( \alpha_{\text{Vect}, X} \).

Proof. The smooth projective connected \( k \)-curve \( X \) is the Coarse moduli space for \( \mathcal{X} \). Let \( U \subset X \) be maximal open such that \( \mathcal{X} \times_X U \cong U \). Set \( B = X - U \). By Remark 4.6, \( B = \text{Supp}(P) \), and for \( x \in B \), the group \( \text{Gal}(P(x)/K_{X,x}) \) is the stabilizer group \( G_x \) at \( x \), and we have

\[
\mathcal{X} \times_X \hat{O}_{X,x} \cong \text{Spec}(R_x) / G_x
\]

where \( R_x \) is the integral closure of \( \hat{O}_{X,x} \) in \( P(x) \).

Let \( E \in \text{Vect}(\mathcal{X}) \). Then \( E \) restricts to a vector bundle \( E_{|U} \) on \( U \). Let \( x \in B \). By Notation 4.2 we have the tuple \( (U_x, V_x, R_x) \) where \( U_x \to X \) is an open connected subset, \( U \cap U_x = \{ x \} \), \( V_x \to U_x \) is a finite surjective \( \Gamma_x \)-Galois cover that is étale away from \( x \), and \( \mathcal{X} \times_X U_x \cong [V_x/\Gamma_x] \) which in turn induces the isomorphism in Equation (4.2) above. So \( E \) restricts to a vector bundle \( E^{(x)} \) on the quotient sub-stack \( \mathcal{X} \times_X U_x \).

By our construction of \( \lambda \), we have

\[
\lambda([V_x/\Gamma_x]) = (U_x, P_{|U_x})
\]

where \( P_{|U_x} \) is the restriction of branch data on the connected curve \( U_x \). We have the following equivalences of categories

\[
\text{Vect}([V_x/\Gamma_x]) \xrightarrow{\sim} \text{Vect}^{\Gamma_x}(V_x) \xrightarrow{\sim} \text{Vect}(U_x, P_{|U_x}),
\]
where the first equivalence is \([18] \text{ Example 7.21}\) and the second one is \([12] \text{ Theorem 4.12}\). So the vector bundle \(E^{(3)}\) on \(\mathfrak{X} \times_X U_x\) under the above equivalence corresponds to a bundle \(E^{(3)}_x \in \text{Vect}(U_x, P(U_x))\). Let \(E^{(3)}_0\) be the underlying vector bundle on \(U_x\). Also notice that since \(g_x\) is étale over \(U_x - \{x\}\), the restriction of \(E^{(3)}_0\) to \(U_x - \{x\} = U_x \cap U\) is isomorphic to the restriction of \(E\mid_U\) to \(U_x \cap U\). We obtain a bundle \(E_0\) on \(X\) via gluing. Set \(\lambda_{\text{Vect}}(E)\) to be the bundle \(E_0\) together with the action and equivariant isomorphism maps obtained from \(E^{(3)}_x\).

We note that \(\lambda_{\text{Vect}}\) is functorial and up to an isomorphism of the vector bundles in \(\text{Vect}(X, P)\), it is independent of the choice of atlases for the open sub-stacks \(\mathfrak{X} \times_X U\).

To construct the quasi-inverse functor \(\alpha_{\text{Vect}}\), start with a bundle \(E_e \in \text{Vect}(X, P)\). Let \(E_0\) be the underlying vector bundle on \(X\). Via restriction, for each \(x \in B\), we obtain the vector bundle \(E^{(3)}_x \in \text{Vect}(U_x, P(U_x))\). Again using the previous equivalences, we obtain a vector bundle \(E_{U_x}\) on the open sub-stack \(\mathfrak{X} \times_X U_x\). Then the restriction of \(E_{U_x}\) to the curve \(U \cap U_x\) is isomorphic to \(E_{0|U \cap U_x}\). Now we glue together the bundles \(E_{U_x}\) to obtain \(\alpha_{\text{Vect}}(E_e)\). The construction is again functorial and independent of the choices on local atlases, up to an isomorphism.

\(\Box\)

**Remark 4.8.**

1. When \(\mathfrak{X} = [Z/G]\) corresponding to a \(G\)-Galois cover \(g : Z \to X\) of smooth projective connected \(k\)-curves, we have \(\lambda([Z/G]) = (X, B_g)\) and the equivalence \(\lambda_{\text{Vect},X}\) is the equivalence

\[
\text{Vect}([Z/G]) \sim_{g^*} \text{Vect}^G(Z) \sim_{S^Z_{(X,B_g)}} \text{Vect}(X, B_g),
\]

where \(S^Z_{(X,B_g)}\) is the equivalence of categories in \([12] \text{ Theorem 4.12}\) (see Equation \([3.4]\)). While working with formal orbifold curves, we will adapt the above viewpoint. Then in the above situation, we identify \(T^Z_{(X,B_g)}\) with \(g^*\) and \(S^Z_{(X,B_g)}\) with \(g^G\).

2. Under \(\lambda_{\text{Vect},X}\), the trivial vector bundle \(O_{\lambda(\mathfrak{X})}\) on \(\lambda(\mathfrak{X})\) corresponds to the structure sheaf \(O_X\).

**Remark 4.9.** Under a morphism of algebraic stacks, one can define the pushforward and the pullback of quasi-coherent sheaves or vector bundles and can consider the tensor product of the bundles (see \([16]\)). The same operations are defined and studied in \([12]\) for formal orbifold curves. Under the equivalence in Theorem \([4.7]\) it is not hard to check that these definitions and relations coincide. We mention that some of the definitions in \([12]\) assume special situations, e.g., étale morphism, which can be generalized to more wider context. Nevertheless, we follow \([16]\) for them.

**Remark 4.10.** For any smooth curves \(X\) (not necessarily connected or projective), we can define a quasi-coherent sheaf on a formal orbifold curve \((X, P)\) and morphisms of quasi-coherent sheaves similar to Definition \([3.13]\) by replacing the underlying vector bundle by a quasi-coherent sheaf of \(O_X\)-modules. One can see that the equivalence \([12] \text{ Theorem 4.12}\) holds. As in Theorem \([4.7]\) we can then establish that there is an equivalence of categories

\[
\text{Qcoh}(\mathfrak{X}) \sim \text{Qcoh}(\lambda(X, P)),
\]

for any proper orbifold curve \(\mathfrak{X}\), where \(\text{Qcoh}(X, P)\) denote the category of quasi-coherent sheaves on an orbifold curve \((X, P)\), and \(\text{Qcoh}(\mathfrak{X})\) is the category of the quasi-coherent sheaves on \(\mathfrak{X}\). Moreover, the equivalence restricts to the equivalence of Theorem \([4.7]\).  

\(^4\)Although the statement is for smooth projective curves, projectivity is not used in the proof of the result.
5. Stability conditions for vector bundles on formal orbifold curves

5.1. Degree of a line bundle on an orbifold curve. Let $\mathfrak{X}$ be a proper orbifold curve with coarse moduli space $X$. We make the following definitions with an analogy as in scheme theory.

- The (Weil) divisor group of $\mathfrak{X}$, denoted by $\text{Div}(\mathfrak{X})$, is the free abelian group generated by the closed points of $\mathfrak{X}$.
- A principal (Weil) divisor on $\mathfrak{X}$ is a divisor $\text{div}(f)$ associated to a rational section $f$ of $O_{\mathfrak{X}}$ given by
  \[\text{div}(f) := \sum_x n_x f(x),\]
  where $n_x(f)$ is the valuation of $f$ in the local ring $O_{\mathfrak{X},x}$. Two divisors $D, D' \in \text{Div}(\mathfrak{X})$ are said to be linearly equivalent, denoted by $D \sim D'$ if $D = D' + \text{div}(f)$ for some rational section $f : \mathfrak{X} \to \mathbb{P}_k^1$ of $\mathfrak{X}$.
- The degree of a divisor $D = \sum_x n_x x \in \text{Div}(\mathfrak{X})$, denoted by $\deg(D)$ is the sum
  \[\deg(D) = \sum_x n_x \deg(x) = \sum_x n_x \frac{1}{|G_x|},\]
  where $G_x$ is the stabilizer at $x$.
- For a closed point $x \in \mathfrak{X}$, we denote the ideal sheaf at $x$ by $O_{\mathfrak{X}}(x)$. Defining as usual
  \[O_{\mathfrak{X}}(x) = O_{\mathfrak{X}}(-x)^\vee = \text{Hom}(O_{\mathfrak{X}}(-x), O_{\mathfrak{X}}),\]
  and then extending this definition linearly to any (Weil) divisor on $\mathfrak{X}$, we obtain a line bundle $O_{\mathfrak{X}}(D)$ associated to a (Weil) divisor $D$.

**Lemma 5.1.** [19, Remark 5.4.4] Let $\mathfrak{X}$ be a proper orbifold curve over $k$ with coarse morphism $\pi : \mathfrak{X} \to X$. Then for any non-constant map $f : \mathfrak{X} \to \mathbb{P}_k^1$, we have
  \[\text{div}(f) = \pi^* \text{div}(f'),\]
  where $f' : X \to \mathbb{P}_k^1$ is the unique map making the following diagram commutative:

  $\mathfrak{X} \xrightarrow{f} X \xrightarrow{\pi} \mathbb{P}_k^1$

**Lemma 5.2.** [19, Lemma 5.4.5] Every line bundle $L$ on $\mathfrak{X}$ is of the form $O_{\mathfrak{X}}(D)$ for some divisor $D \in \text{Div}(\mathfrak{X})$. Moreover, $O_{\mathfrak{X}}(D) \cong O_{\mathfrak{X}}(D')$ if and only if $D \sim D'$ in $\text{Div}(\mathfrak{X})$.

**Definition 5.3.** Let $L$ be a line bundle on a proper orbifold curve $\mathfrak{X}$. Then $L \cong O_{\mathfrak{X}}(D)$ where $D$ is a divisor on $\mathfrak{X}$, up to a numerical equivalence (Lemma 5.2). Define the degree of $L$ to be
  \[\deg(L) := \deg(D)\]

5.2. Degree, slope and (semi)stability. In this section, we introduce the notion of stability and semistability for any vector bundle on a formal orbifold curve. Throughout the section, any formal orbifold curve will be assumed to be connected and proper. In view of Section 4 we will not distinguish between an orbifold curve and the corresponding formal orbifold curve. For simplicity of the notation, we will write $\mathfrak{X} = (X, P)$, instead of $\lambda(\mathfrak{X}) = (X, P)$.
Let \( \mathfrak{x} = (X, P) \) be a proper orbifold curve. Let \( E_\ast \in \text{Vect}(X, P) \) be a vector bundle on \( (X, P) \). Then the \textit{rank} of the bundle \( E_\ast \) is defined to be the rank of the underlying vector bundle \( E \) on \( X \). Let \( n \) be the rank of \( E_\ast \). Then we define its \textit{determinant bundle}

\[
\det(E_\ast) := (\wedge^n E, \wedge^n \Phi, \wedge^n \eta) \in \text{Vect}(X, P).
\]

Since \( \wedge \) commutes with \( \otimes \), we have a line bundle \( \det(E_\ast) \in \text{Vect}(X, P) \) whose underlying vector bundle is the usual determinant line bundle \( \det(E) \) on \( X \).

We define the \textit{degree} of \( E_\ast \) with respect to \( P \) to be

\[
\deg_P(E_\ast) := \deg_P(\det(E_\ast)) := \deg(\alpha_{\text{Vect}, X}(\det(E_\ast)));
\]

where \( \alpha_{\text{Vect}, X} \) is the functor defined in Theorem 4.7 and \( \deg(\alpha_{\text{Vect}, X}(\det(E_\ast))) \) is defined in Definition 5.4.

The \textit{P-slope} of \( E_\ast \), denoted by \( \mu_P(E_\ast) \), is the fraction

\[
\mu_P(E_\ast) = \frac{\deg_P(E_\ast)}{\text{rank}(E_\ast)}.
\]

\textbf{Definition 5.4.} A vector bundle \( E_\ast \in \text{Vect}(X, P) \) is called \( P \)-\( (\text{semi}) \)-stable if for all its subbundle \( 0 \neq F_\ast \subset E_\ast \) in \( \text{Vect}(X, P) \), we have

\[
\mu_P(F_\ast) \leq \mu_P(E_\ast). \tag{5.5}
\]

A \( P \)-\( \text{polystable} \) bundle on \( (X, P) \) bundle that is a finite sum of \( P \)-\( \text{stable} \) vector bundles in \( \text{Vect}(X, P) \) having the same \( P \)-slope.

We will show that \( \mu_P = \mu_{\mathfrak{x}} \). First, we see how the degrees with respect to branch data behave for the pullback under a morphism of formal orbifold curves.

\textbf{Lemma 5.5.} Let \( P' \geq P \) be two branch data on a smooth projective connected \( k \)-curve \( X \). The identity map on \( X \) induces \( \iota : (X, P') \longrightarrow (X, P) \). Consider the embedding \( \iota^* : \text{Vect}(X, P) \longrightarrow \text{Vect}(X, P') \). Then \( \deg_P(\iota^* E_\ast) = \deg_P(E_\ast) \) for any \( E_\ast \in \text{Vect}(X, P) \).

\textit{Proof.} Let \( \mathfrak{x} \) and \( \mathfrak{x}' \) be the orbifold curves such that \( \lambda(\mathfrak{x}) = (X, P) \) and \( \lambda(\mathfrak{x}') = (X, P') \). From the definition, it follows that for any \( F_\ast \in \text{Vect}(X, P) \), we have

\[
\iota^* \det(F_\ast) = \det(\iota^* E_\ast).
\]

We defined the \( P \)-degree of any bundle \( F_\ast \in \text{Vect}(X, P) \) as the degree of the line bundle \( \alpha_{\text{Vect}}(\det(F_\ast)) \) on \( \mathfrak{x} \), which in turn is defined to be the degree of a (Weil) divisor associated to it. So it is enough to show that for any closed point \( x \in \mathfrak{x} \), the degree of the point \( x \) considered as a divisor on \( \mathfrak{x} \) and the divisor \( \alpha(\iota)^* x \) on \( \mathfrak{x}' \) have the same degree. Note that

\[
\alpha(\iota)^* x = [P'(x) : P(x)] x.
\]

Now note that \( \deg_{\mathfrak{x}}(\alpha(\iota)^* x) = \frac{[P'(x) : P(x)]}{[P'(x) : P(x)]} = \deg_{\mathfrak{x}}(x). \quad \square
\]

\textbf{Lemma 5.6.} Let \( f : (Y, Q) \longrightarrow (X, P) \) be a morphism of formal orbifold curves. Then for any \( E_\ast \in \text{Vect}(X, P) \), we have the following.

\[
\deg_Q(f^* E_\ast) = \deg(f) \deg_P(E_\ast).
\]

\textsuperscript{5} As in the case of schemes, the notation \( (\leq) \) means that \( E_\ast \) is \( P \)-semistable if we have \( \leq \), and it is \( P \)-stable if we have the strict inequality \( < \).
Proof. The map $f$ factors as $(Y, Q) \xrightarrow{g} (Y, f^*P) \xrightarrow{f} (X, P)$. By Lemma 5.5, we have $\text{deg}_Q(f^*E_s) = \text{deg}_{f^*P}(f^*E_s)$. So it is enough to show that

$$\text{deg}_{f^*P}(f^*E_s) = \text{deg}(f) \text{deg}_{P}(E_s).$$

Let $\alpha(X, P) = X$ and $\alpha(Y, f^*P) = X'$ be the orbifold curves as in Theorem 4.4. Using the definition, it is again enough to show that for any closed point $x \in X$, the degree of the divisor $\alpha(f)^*x$ in $X'$ is equal to the degree of the point $x \in X$ considered as a divisor, multiplied by $\text{deg}(f)$. We have

$$\alpha(f)^*(x) = \sum_{f(y) = x} [f^*P(y) : P(x)] y.$$

So

$$\text{deg}_X(\alpha(f)^*x) = \sum_{f(y) = x} \frac{[f^*P(y) : P(x)]}{[P(x)_y : K_{x_y}]} = \sum_{f(y) = x} \frac{[P(x)_y : K_{x_y}] [P(x)_y : K_{x_y}]}{[P(x)_y : K_{x_y}]} = \sum_{f(y) = x} \frac{[K_{x_y} : K_{x_y}]}{[K_{x_y} : K_{x_y}]} = \text{deg}(f) \text{deg}_X(x).$$

This completes the proof. \hfill \Box

We have the following immediate consequences.

**Proposition 5.7.** Let $\tilde{X} = (X, P)$ be a connected projective orbifold curve. For any vector bundle $E_s \in \text{Vect}(X, P)$, we have

$$\mu_\tilde{X}(E_s) = \mu_P(E_s).$$

**Proof.** Let $Q \geq P$ be a geometric branch data on $X$ such that there is a finite $G$-Galois étale cover $g: Z \rightarrow (X, Q)$ where $Z$ is a smooth projective connected $k$-curve. The $G$-Galois cover $f: Z \rightarrow X$ of smooth projective connected $k$-curves factors as follows.

$$f: Z \xrightarrow{g} \tilde{X} := (X, Q) \xrightarrow{f} \tilde{X} = (X, P) \xrightarrow{\iota} X.$$

Now the result follows from Lemma 5.5, Lemma 5.6, and Definition 3.13. \hfill \Box

Thus we can interchangeably talk about the stability conditions with respect to $\mu_P$ or $\mu_\tilde{X}$, and Proposition 3.17 remains valid with respect to $\mu_P$. One extra advantage of working with formal orbifold curves is that we can define the maximal destabilizing sub-bundle and socle even when the orbifold curve is not a quotient stack (see the discussion after Proposition 3.17).

**Proposition 5.8.** Let $(X, P)$ be a connected projective formal orbifold curve and $E_s \in \text{Vect}(X, P)$.

1. (Harder-Narasimhan Filtration) There is a unique filtration

$$0 = \text{HN}(E_s)_0 \subset \text{HN}(E_s)_1 \subset \cdots \subset \text{HN}(E_s)_l = E_s$$

such that $\text{HN}(E_s)/\text{HN}(E_s)_{i-1}$ are $P$-semistable and their slopes satisfy

$$\mu_{P, \text{max}}(E_s) := \mu_P(\text{HN}(E_s)_1) > \cdots > \mu_P(\text{HN}(E_s)/\text{HN}(E_s)_{l-1}).$$

2. The ‘maximal destabilizing sub-bundle’ $\text{HN}(E_s)_1$ has the following property: for any sub-bundle $F_s \subseteq E_s$, we have $\mu_P(\text{HN}(E_s)_1) \geq \mu_P(F_s)$; when $\mu_P(\text{HN}(E_s)_1) = \mu_P(F_s)$, we have $F_s \subset \text{HN}(E_s)_1$.

3. If $E_s$ is $P$-semistable, there exists a filtration

$$0 = E_s^{(0)} \subset E_s^{(1)} \subset \cdots \subset E_s^{(l-1)} \subset E_s^{(l)}$$

such that $E_s^{(i)}/E_s^{(i-1)}$ are $P$-stable, having the same $P$-slope as $\mu_P(E_s)$. 
(4) If there is a $G$-Galois cover $g: Z \rightarrow (X, P)$, the pullback of the filtration in (1) is the unique Harder-Narasimhan filtration of $g^*(E_\ast)$, and $g^*(\text{HN}(E_\ast))$ is the maximal destabilizing sub-bundle of $g^*(E_\ast)$. When $E_\ast$ is also $P$-semistable, the pullback of the filtration in (3) is a Jordan-Hölder filtration. In particular, $g^*(\oplus_{i}E_i^{(i)}/E_i^{(i-1)})$ is the socle $\Xi(g^*E_\ast)$ of the semistable $G$-bundle $g^*E_\ast$.

Proof. (1)–(3) are obtained as in the proof of [9, Lemma 1.3.5, pg. 17 and Proposition 1.5.2, pg. 23] using the slope $\mu_P$. By Proposition 5.7 and Proposition 5.17 (1c), the statement (4) follows when the map $g$ is also étale. So to prove (4), it is enough to show that for branch data $P' \geq P$ on $X$ with induced morphism $\iota: (X, P') \rightarrow (X, P)$ and $E_\ast \in \text{Vect}(X, P)$, we have

$$\text{HN}(\iota^*(E_\ast))_1 = \iota^*(\text{HN}(E_\ast))_1,$$

and for $P'$-semistable $E_\ast$, $\Xi(\iota^*E_\ast) = \iota^*(\Xi(E_\ast))$.

We have a vector bundle inclusion

$$\iota^*(\text{HN}(E_\ast))_1 \subseteq \text{HN}(\iota^*(E_\ast))_1 \subseteq \iota^*(E_\ast)$$

on $(X, P')$. In particular, $\text{HN}(\iota^*(E_\ast))_1 = \iota^*F_\ast$ for some sub-bundle $F_\ast \subseteq E_\ast$ (see Remark 3.22). Then $\mu_P(\text{HN}(\iota^*(E_\ast))_1) = \mu_P(F_\ast) \leq \mu_P(\text{HN}(E_\ast)_1)$. By the maximality of $\mu_P(\text{HN}(E_\ast)_1)$, we have $\text{HN}(E_\ast)_1 = F_\ast$. Thus $\iota^*(\text{HN}(E_\ast))_1$ is the maximal destabilizing sub-bundle of $\iota^*E_\ast$. Since the Harder-Narasimhan filtration is constructed inductively, we obtain the first equality.

By Remark 3.22, every vector sub-bundle of $\iota^*E_\ast$ is of the form $\iota^*F_\ast$ for some sub-bundle $F_\ast \subseteq E_\ast$. So each direct summand of $\Xi(\iota^*E_\ast)$ is also a direct summand of $\iota^*\Xi(E_\ast)$. Also since $\iota^*$ preserves polystability (see Proposition 3.16), we obtain the second equality.

Remark 5.9. Suppose that $\text{char}(k) = 0$. Let $(X, P)$ be a connected projective formal orbifold curve, determined by a finite set $B \subset X$ of closed points and a positive integers $n_x$ for each $x \in B$. Let $D = \sum_{x \in B} x \in \text{Div}(X)$. As noted in Example 5.21, there is an equivalence of categories

$$(5.1) \quad \text{Vect}(X, P) \sim \text{Vect}_{\text{par, rat}}(X, D)$$

where $\text{Vect}_{\text{par, rat}}(X, D)$ is the category of parabolic vector bundles on $X$ with respect to the divisor $D$ and over each $x \in B$, the weights are of the form $a/n_x$, $0 \leq a < n_x$.

There exists a connected $G$-Galois cover $g: Z \rightarrow X$ of smooth projective connected $k$-curves that is branched over the set $B$, and for each point $x \in B$, the integer $n_x$ divides the ramification index at any point $z \in g^{-1}(x)$. In other words, $B_g \supseteq P$ and $g$ induces $Z \rightarrow (X, B_g) \rightarrow (X, P)$, where $g^*$ is a $G$-Galois étale map and $\iota$ is induced from $\text{Id}_Z$. By [11, 17], for each parabolic vector bundle $V_\ast \in \text{Vect}_{\text{par, rat}}(X, D)$, there is a unique $G$-bundle $\hat{V} \in \text{Vect}^g(Z)$, and

$$\mu(\hat{V}) = |G|\mu_{\text{para}}(V_\ast).$$

Moreover, $\hat{V}$ is $G$-(semi)stable (respectively, $G$-polystable) if and only if $V_\ast$ is parabolic (semi)stable (respectively, parabolic polystable).

Let $E_\ast \in \text{Vect}(X, P)$ be the vector bundle corresponding to $V_\ast$. By Lemma 5.6 and Lemma 5.8, we have

$$|G|\mu_{\text{para}}(E_\ast) = |G|\mu_{\text{para}}(\iota^*E_\ast) = \mu\text{polystable}(E_\ast).$$

Moreover, using Proposition 3.16 $E_\ast$ is $P$-(semi)stable if and only if $\iota^*E_\ast$ is $B_x$-(semi)stable. So by Proposition 3.16 and Proposition 5.7, $E_\ast$ is $P$-(semi)stable or $P$-polystable if and only if $g^*E_\ast$ is $G$-(semi)stable or $G$-polystable.
From the above discussion, we see that under the equivalence \((\ref{5.1})\), the parabolic slope is the same as \(P\)-slope, and parabolic (semi)stability or parabolic polystability are the same as \(P\)-(semi)stability.

5.3. Genuinely Ramified Morphisms. In this section, we extend the definition of a genuinely ramified morphism for stacky curves.

Consider any finite cover \(f : Y \to X\) of smooth projective connected \(k\)-curves. The maximal destabilizing sub-bundle \(\text{HN}(f_*O_Y)_1 \subset f_*O_Y\) is a sheaf of algebras that is a semistable sub-bundle of degree 0 containing \(O_X\) (\([2]\) Equation (2.7), Lemma 2.4). Moreover, \(f\) factors as a composition

\[
f : Y \to \hat{X} := \text{Spec}(\text{HN}(f_*O_Y)_1) \to X
\]

where \(\hat{X} \to X\) the maximal étale cover of \(X\) via which the map \(f\) factors. Moreover, the induced finite cover \(Y \to \hat{X}\) is genuinely ramified. The cover \(f\) is said to be genuinely ramified if \(f_*O_Y\) is \(O_X\). This is equivalent to: the homomorphism between étale fundamental groups \(f_* : \pi_1(Y) \to \pi_1(X)\) induced by \(f\) is a surjection.

Other equivalent conditions are given in \([2]\) Proposition 2.6, Lemma 3.1]. The main result of \([2]\) gives another important criterion of a map to be genuinely ramified in terms of the stability of vector bundles under pullback maps.

**Theorem 5.10** (\([2]\) Theorem 5.3]). Let \(f : Y \to X\) be a finite cover of smooth projective connected \(k\)-curves. The map \(f\) is genuinely ramified if and only if the vector bundle \(f^*E\) is stable on \(Y\) for every stable vector bundle \(E\) on \(X\).

In the following, we see that there is also an analogue equivalence among the conditions for a finite cover of stacky curves.

**Proposition 5.11.** Let \(\mathfrak{X} = (X, P)\) be a connected proper orbifold \(k\)-curve. Let \(f : \mathfrak{Y} \to (X, P)\) be a finite cover of connected proper stacky curves. The maximal destabilizing sub-bundle \(\text{HN}(f_*O_\mathfrak{Y})_1 \subset f_*O_\mathfrak{Y}\) is a sheaf of \(O_{(X,P)}\)-algebras, and is a \(P\)-semistable vector bundle of \(P\)-degree 0. Moreover, the following are equivalent for the finite cover \(f : \mathfrak{Y} \to (X, P)\),

1. \(\text{HN}(f_*O_\mathfrak{Y})_1 = O_{(X,P)}\).
2. The map \(f\) does not factor through any non-trivial étale sub-cover.
3. The homomorphism between étale fundamental groups \(f_* : \pi_1(\mathfrak{Y}) \to \pi_1(\mathfrak{X})\) induced by \(f\) is a surjection.
4. The fiber product DM stack \(\mathfrak{Y} \times_X \mathfrak{Y}\) is connected.
5. \(\dim H^0(\mathfrak{Y}, f^*f_*O_\mathfrak{Y}) = 1\).

Finally, the above conditions imply that the finite cover \(f_0\) induced on the Coarse moduli curves is a genuinely ramified morphism.

**Proof.** Let \(Z \to (X, P)\) be a \(G\)-Galois cover where \(Z\) is a smooth projective connected \(k\)-curve. Choose a finite cover \(Z' \to \mathfrak{Y} \times_{(X,P)} Z\) where \(Z'\) is a projective \(k\)-curve (since \(\mathfrak{Y} \times_{(X,P)} Z\) is a stacky curve, such a cover always exists; see Remark \(\ref{5.3}\)). Note that \(\[2\] Lemma 2.2 and Lemma 2.4\) are valid even when the source is a singular curve. Thus, under the finite cover \(g : Z' \to Z\), we conclude the following. The maximal destabilizing sub-sheaf of \(g_*O_{Z'}\) is a semistable \(G\)-equivariant bundle of degree 0, and is a sheaf of \(O_{Z'}\)-algebras containing \(O_Z\). By definition, \(f_*O_{\mathfrak{Y}}\) corresponds to the \(G\)-equivariant vector bundle \(g^*O_{Z'}\). By Proposition \(\ref{5.8}\) and Proposition \(\ref{5.11}\) \(\ref{13}\), we conclude that \(\text{HN}(f_*O_{\mathfrak{Y}})_1 \subset f_*O_{\mathfrak{Y}}\) is a sheaf of \(O_{(X,P)}\)-algebras, and is a \(P\)-semistable vector bundle of \(P\)-degree 0.
The equivalence between (2) and (3) is a tautology from the formalism of Galois categories (for a detail argument for varieties, see [3] Theorem 2.4]). The same argument from [2] Proposition 2.6] used in our context establishes the equivalence of (1) and (2).

We show the equivalence (4)⇔(5). By [13] Proposition 13.1.9, pg. 122] or [5] Proposition A.1.7.4, we have an isomorphism

\[ f^* f_! O_Y \cong (p_1)_! \circ (p_2)_! O_Y \]

where \( p_1 \) and \( p_2 \) are the projection morphisms. As we have \( p_2^* O_Y = O_{Y \times X Y} \), we conclude

\[ H^0(\mathcal{Y}, f^* f_! O_Y) = H^0(\mathcal{Y}, (p_1)_! O_{Y \times X Y}) = H^0(\mathcal{Y} \times_X \mathcal{Y}, O_{Y \times X Y}). \]

We note that \( \mathcal{Y} \times_X \mathcal{Y} \) is a reduced DM stack each of whose irreducible components is one-dimensional and generically an integral curve. The later cohomology can be calculated using the Leray spectral sequence (see [16, (11.6.2.2), pg. 237])

\[ E_2^{pq} = H^0(S, R^p \pi_* O_{Y \times X Y}) \Rightarrow H^{p+q}(\mathcal{Y} \times_X \mathcal{Y}, O_{Y \times X Y}) \]

where \( \pi: \mathcal{Y} \times_X \mathcal{Y} \to S \) denote the Coarse moduli map. In particular, \( H^0(\mathcal{Y} \times_X \mathcal{Y}, O_{Y \times X Y}) = H^0(S, \pi_* O_{Y \times X Y}) = H^0(S, O_S) \) using the property of the Coarse moduli space. Since \( \mathcal{Y} \times_X \mathcal{Y} \) is connected if and only if \( S \) is connected, the desired equivalence follows.

Now we show that (4)⇒(2). Suppose that \( f \) factors as a composition \( \mathcal{Y} \overset{\xi}{\to} \mathcal{X}' \overset{h}{\to} \mathcal{X} \) where \( h \) is a non-trivial finite étale cover. Then the fiber product \( \mathcal{X}' \times_X \mathcal{X} \) is disconnected as it contains \( \mathcal{X}' \) as a connected component (the diagonal morphism is an open imbedding) and \( \mathcal{X}' \times_X \mathcal{X} \to \mathcal{X} \) is an étale cover of degree \( >\deg(h) \).

We will show (1)⇒(5). We have

\[ H^0(\mathcal{Y}, f^* f_! O_Y) = H^0(\mathcal{X}, f^* f_! O_Y) = O_{\mathcal{X}}. \]

By Proposition 3.17, \( \mu_{\mathcal{Y}, max}(f^* f_! O_Y) = H^0(\mathcal{Y}, f_! O_Y) < 0. \) So there is no vector bundle inclusion \( O_{\mathcal{X}} \hookrightarrow f^* f_! O_Y / f^* H^0(\mathcal{Y}, f_! O_Y) \), and consequently,

\[ H^0(\mathcal{Y}, f^* f_! O_Y / f^* H^0(\mathcal{Y}, f_! O_Y)) = 0. \]

Now the implication follows from the long exact sequence of cohomologies associated to the following exact sequence of vector bundles on \( \mathcal{Y} \):

\[ 0 \to f^* H^0(\mathcal{Y}, f_! O_Y) \to f^* f_! O_Y \to f^* f_! O_Y / f^* H^0(\mathcal{Y}, f_! O_Y) \to 0. \]

To see the last statement, assume that \( f_0 \) is not genuinely ramified. Then there is a non-trivial étale sub-cover \( X' \to X \) via which \( f_0 \) factors. Then \( X' \times_X (X, P) \to (X, P) \) is a non-trivial étale sub-cover via which \( f \) factors. \( \Box \)

6. Stability of pullback under genuinely ramified map

Throughout this section, all formal orbifold curves considered are connected and projective, unless otherwise specified. In view of the equivalence \( \lambda \) from Theorem 4.2, we also do not distinguish between an orbifold curve and a formal orbifold curve. We first show that ‘being a genuinely ramified morphism is the property of cartesian diagrams’. More precisely, under a base change morphism of genuinely ramified morphism of coarse moduli curves, the pullback of each orbifold stable bundle remains orbifold stable. We fix the following notation.

**Notation 6.1.** Let \( \mathcal{X} = (X, P) \) be a formal orbifold curve where \( X \) is a smooth projective \( k \)-curve and \( P \) is a branch data on \( X \). Let \( f_0: Y \to X \) be a genuinely ramified morphism of smooth projective connected \( k \)-curves. Consider the fiber product stack \( \mathcal{Y} := \mathcal{X} \times_X Y \) which admits the projection morphism \( \mathcal{Y} \to Y \) as the coarse moduli space ([16] Theorem 11.3.6,
Let \( f : \mathcal{Y} \to (X, P) \) be the projection morphism which is necessarily a finite covering morphism.

**Remark 6.2.** Under Notation 6.1 the stack \( \mathcal{Y} \) is a connected proper DM stack, but not necessarily smooth. Let \( Z \to (X, P) \) be a \( G \)-Galois covering morphism of connected stacks for a finite group \( G \). Let \( W \) be the smooth connected curve with function field \( k(Z) \cap k(Y) \). Then the covers \( Z \to X \) and \( Y \to X \) factor as the composition \( Z \to W \to X \) and as \( Y \to W \to X \), respectively. Here \( Z \to W \) is a Galois cover for some subgroup \( H \leq G \), and the cover \( W \to X \) is either the trivial cover or it is genuinely ramified as \( f_0 \) is genuinely ramified. If \( W \to X \) is a non-trivial cover (or equivalently, \( H \neq G \)), the fiber product \( W \times_X W \) is a singular curve with each irreducible component isomorphic to \( W \) (when \( W = X \) i.e., \( Z \) and \( Y \) are linearly disjoint over \( X \), the fiber product is equal to \( X \)). Since \( Z \) and \( Y \) are linearly disjoint over \( W \), the \( k \)-algebra \( k(Z) \otimes_{k(W)} k(Y) \) is the function field of some smooth projective connected \( k \)-curve \( W' \). These lead to the following description for the cover \( Z \times_X Y \) and the stack \( \mathfrak{X} \times_X Y \).

1. If \( Z \) and \( Y \) are linearly disjoint over \( X \), the fiber product \( Z \times_X Y \) is a smooth projective connected \( k \)-curve, and the base change stack \( (X, P) \times_X Y \) corresponds to the formal orbifold curve \( (Y, f^* P) \).
2. Otherwise, the fiber product \( Z \times_X Y \) is a connected \( k \)-curve which is singular and each of its irreducible components is isomorphic to \( W' \). The stack \( (X, P) \times_X Y \) is a connected proper DM stack which is not smooth and with coarse moduli space \( Y \).

**Remark 6.3.** We also remark that if \( f_0 \) in Notation 6.1 is an étale cover of smooth projective connected curves and one can consider a curve \( W \) as in Remark 6.2. When \( W \to X \) is non-trivial, the fibre product \( W \times_X W \) is a smooth disconnected curve, and consequently, so is the fibre product \( Z \times_X Y \). So, in this case, we see that \( \mathcal{Y} = (X, P) \times_X Y \) is indeed a connected formal orbifold curve \( (X, f^* P) \).

**Lemma 6.4.** Suppose that Notation 6.1 hold. Additionally, if \( f_0 : Y \to X \) is a \( G \)-Galois cover, we have

\[
\psi : f_0^*((f_0, O_Y)/(O_X)) \to O_Y^{G-1}
\]

in \( \text{Vect}(\mathcal{Y}) \) where \( \psi \in \text{Vect}(\mathcal{Y}) \) is a line bundle with \( \mu_0(M_i) < 0 \).

**Proof.** As in the proof of [2, Proposition 3.5], there is an injective homomorphism ([2, Equation 3.17])

\[
\Psi : f_0^*((f_0, O_Y)/(O_X)) \to O_Y^{G-1}
\]

which is an isomorphism on the étale locus of \( f_0 \), and there are points (2 Equation 3.18)]

\( \psi_i \in \mathcal{Y}, 1 \leq i \leq |G| - 1 \), such that for each projection map \( \pi_i : O_Y^{G-1} \to O_Y \) to the \( i \)th factor \((1 \leq i \leq |G| - 1)\), the composition morphism \( \pi_i \circ \psi \) vanishes at the point \( \psi_i \). This implies

\[
\pi_i \circ \psi \bigg( f_0^*((f_0, O_Y)/(O_X)) \bigg) \subset L_i \subseteq O_Y(\psi_i) \subseteq O_Y,
\]

and \( f_0^*((f_0, O_Y)/(O_X)) \subset \bigoplus_{1 \leq i \leq |G|} L_i \). We have the coarse moduli curves \( j : \mathcal{Y} \to Y \) and \( \iota : \mathcal{X} = (X, P) \to X \). By [13, Proposition 13.1.9, pg. 122] or [5, Proposition A.1.7.4], we have an isomorphism

\[
\iota^*(f_0, O_Y) \cong f_0^* j^*(f_0, O_Y) = f_0^* f_0^* O_Y.
\]

This implies

\[
j^* f_0^*(f_0, O_Y) \cong f^* \iota^*(f_0, O_Y) = f^* f_0^* O_Y.
\]
Thus we have $f^* \left( \left( f_0^* (f_0), O_T \right) / O_X \right) \cong f^* \left( (f, O_Y) / O_{(X,P)} \right)$. Then applying $f^*$ to the composite morphism $p_i \circ \Psi$ for each $1 \leq i \leq |G| - 1$, we obtain the following morphism

\[
f^* \left( (f, O_Y) / O_{(X,P)} \right) \cong f^* \left( \left( f_0^* (f_0), O_T \right) / O_X \right) \xrightarrow{f^* \Psi} O_0^{|G|} \cong f^*_P O_0.
\]

Since for each $i$, the morphism $p_i \circ \Psi$ vanishes at the point $z_i$, the morphism $f^* p_i \circ f^* \Psi$ vanishes along the effective Cartier divisor $f^* z_i$. So the image of $f^* \left( (f, O_Y) / O_{(X,P)} \right)$ is contained in the line bundle $M_i := O_y(-f^* z_i) \subset O_y$. This shows that

\[
f^* \left( (f, O_Y) / O_{(X,P)} \right) \subset \bigoplus_{1 \leq i \leq |G| - 1} M_i.
\]

\[\square\]

**Remark 6.5.** The above proof also establishes that under the given hypothesis, the maximal destabilizing subsheaf $HN(f, O_Y)_1$ on $(X,P)$ is $\iota^* HN((f_0), O_T)_1$. Since $f_0$ is genuinely ramified by assumption, $HN((f_0), O_T)_1 = O_X$. By Proposition [5.11], this is equivalent to the induced map $f : \pi_1(\mathcal{Y}) \to \pi_1(X,P)$ being a surjection.

**Lemma 6.6.** Suppose that Notation [6.7] hold. Additionally, suppose that $f_0 : Y \to X$ is a $G$-Galois cover. Then for any $P$-semistable vector bundle $F_\ast \in \text{Vect}(X,P)$, we have

\[
\mu_{P,\max} \left( F_\ast \otimes_{O_{(X,P)}} \left( f_0, O_Y / O_{(X,P)} \right) \right) < \mu_{P}(F_\ast).
\]

**Proof.** Since $F_\ast$ is $P$-semistable, the vector bundle $f^* F_\ast \in \text{Vect}(\mathcal{Y})$ is $\mu_{\mathcal{Y}}$-semistable (see Proposition [3.17](2b)). We have

\[
\deg(f_0) \times \mu_{P,\max} \left( F_\ast \otimes_{O_{(X,P)}} \left( f_0, O_Y / O_{(X,P)} \right) \right) = \mu_{\mathcal{Y},\max} \left( f^* \left( F_\ast \otimes_{O_{(X,P)}} \left( f_0, O_Y / O_{(X,P)} \right) \right) \right)
\]

\[
= \mu_{\mathcal{Y},\max} \left( f^* F_\ast \otimes_{O_Y} f^* \left( (f, O_Y) / O_{(X,P)} \right) \right)
\]

By Lemma [6.4] we have

\[
f^* \left( (f, O_Y) / O_{(X,P)} \right) \subset \bigoplus_{1 \leq i \leq |G| - 1} M_i
\]

where each $M_i \in \text{Vect}(\mathcal{Y})$ is a line bundle with $\mu_{\mathcal{Y}}(M_i) < 0$. So the last slope in Equation (6.2) is

\[
\leq \mu_{\mathcal{Y},\max} \left( f^* F_\ast \otimes_{O_Y} \bigoplus_{1 \leq i \leq |G| - 1} M_i \right) \leq \max_{1 \leq i \leq |G| - 1} \left\{ \mu_{\mathcal{Y}} \left( f^* F_\ast \otimes_{O_Y} M_i \right) \right\}.
\]

Here we have used that each $f^* F_\ast \otimes_{O_Y} M_i$ is semistable (Proposition [X.17](1d)) since $f^* F_\ast$ is semistable. Since $M_i$ is a line bundle on $\mathcal{Y}$ of negative degree, we have

\[
\deg(f_0) \times \mu_{P,\max} \left( F_\ast \otimes_{O_{(X,P)}} \left( f_0, O_Y / O_{(X,P)} \right) \right) = \deg(f_0) \times \mu_{P}(F_\ast).
\]

We conclude that

\[
\mu_{P,\max} \left( F_\ast \otimes_{O_{(X,P)}} \left( f_0, O_Y / O_{(X,P)} \right) \right) < \mu_{P}(F_\ast).
\]

\[\square\]

**Lemma 6.7.** Under Notation [6.7] for any two $E_\ast, F_\ast \in \text{Vect}(X,P)$, we have

\[
\text{Hom}_{\text{Vect}(X,P)}(E_\ast, F_\ast) \subset \text{Hom}_{\text{Vect}(\mathcal{Y})}(f^* E_\ast, f^* F_\ast).
\]

Moreover, if $f_0$ is a Galois cover and $E_\ast, F_\ast$ are $P$-semistable with $\mu_P(E_\ast) = \mu_P(F_\ast)$, we have

\[
\text{Hom}_{\text{Vect}(\mathcal{Y})}(f^* E_\ast, f^* F_\ast) = \text{Hom}_{\text{Vect}(X,P)}(E_\ast, F_\ast).
\]
Theorem 6.8. Let $X = (X, P)$ be a formal orbifold curve where $X$ is a smooth projective $k$-curve and $P$ is a branch data on $X$. Let $f_0 : Y \to X$ be a genuinely ramified morphism of smooth projective connected $k$-curves. Consider the fiber product stack $\mathbb{Y} := X \times_X Y$. Let $f : \mathbb{Y} \to (X, P)$ be the projection morphism (necessarily, it is a finite covering morphism). Then for every $P$-stable vector bundle $E_\ast$ on $(X, P)$, the pullback bundle $f^\ast E_\ast$ on $\mathbb{Y}$ is $\mu_{0\ast}$-stable.

Proof. Suppose that $E_\ast \in \text{Vect}(X, P)$ is $P$-stable. By Proposition [3, 11] [2b], the vector bundle $f^\ast E_\ast$ on $\mathbb{Y}$ is $\mu_{0\ast}$-polystable.

Let $\tilde{f}_0 : \tilde{Y} \to Y$ be the Galois closure of $f_0$ with Galois group $\tilde{G}$. Then $\tilde{f}_0 : \tilde{Y} \to Y$ is also Galois. The map $\tilde{f}_0$ factors through the maximal finite étale sub-cover

$$g_0 : \tilde{X} := \text{Spec} \left( \text{HN}(\tilde{f}_0)_!(O_Y) \right) \to X$$

where $\text{HN}(\tilde{f}_0)_!(O_Y)$ is the maximal destabilizing sub-bundle of $(\tilde{f}_0)_!(O_Y)$. We see that $\tilde{f}_0$ is genuinely ramified if and only if $\tilde{X} = X$ and $g_0 = 1_{\text{Id}_X}$ [3]. By the maximality, $g_0$ is a Galois cover with group $G$, say. By [2] Corollary 2.7, the induced map $\hat{g}_0 : \hat{Y} \to \hat{X}$ is a Galois genuinely ramified morphism. We have the following commutative diagram.

\[
\begin{array}{c}
\tilde{Y} \\
\downarrow \tilde{f}_0 \\
\hat{Y} := Y \times_X \hat{X} \\
\downarrow p_{1\ast} \\
Y \\
\end{array}
\]

\[
\begin{array}{c}
\tilde{f}_0 \\
\downarrow \hat{f}_0 \\
\hat{X} \\
\end{array}
\]

\[
\begin{array}{c}
\hat{X} \\
\downarrow \hat{p}_0 \\
X \\
\end{array}
\]

The Galois closure of a genuinely ramified map need not be genuinely ramified; one can construct easy examples over any characteristic base field.
Since \( g_0 \) is a \( G \)-Galois étale map and \( f_0 \) is genuinely ramified, from the above cartesian square we see that \( \hat{Y} \) is a smooth connected curve, the first projection \( p_{1,0}: \hat{Y} \rightarrow Y \) is a \( G \)-Galois étale cover, and the second projection \( p_{2,0}: \hat{Y} \rightarrow \hat{X} \) is a finite genuinely ramified morphism. As all these maps are flat morphisms, we can realize the base changes of \( \hat{X} \) under these maps as connected proper one dimensional DM stacks admitting the corresponding base as its coarse moduli space and having a description as in Remark 6.2. Set

\[
(X, P) \times_X \hat{Y} = \hat{Y}, \quad \text{and} \quad (X, P) \times_X Y = \hat{Y}.
\]

By Remark 6.3 we have \((X, P) \times_X \hat{X} = (\hat{X}, \hat{P})\) which we denote by \((\hat{X}, \hat{P})\). We also obtain a commutative diagram as above involving the above formal orbifold curves, where we denote the corresponding morphisms by removing the subscript ‘0’.

Now let \( 0 \neq S , \subseteq f^* E \) be a \( \mu_\theta \)-stable sub-bundle of \( f^* E \), such that \( \mu_\theta(S, \cdot) = \mu_\theta(f^* E, \cdot) \). We will show that \( f^* E = S \). For this, \( 0 \) will first construct a sub-bundle \( V \), \( \subseteq g^* E \), having the same \( \mu_\hat{p} \) such that \( p^*_1 V = p^*_1 S \). Using the fact that \( p^*_1 S \subseteq (f \circ p_1)^* E \) is \( G \)-invariant, we will descend the bundle \( V \) to a sub-bundle of \( E \), on \( (X, P) \) having the same \( \mu_\theta \), and this will conclude the proof.

First, taking pullback under the Galois morphism \( \hat{f}: \hat{Y} \rightarrow Y \), we obtain a sub-bundle \( \hat{f}^* S_\gamma \subseteq \hat{f}^* f^* E = \hat{f}^* E \). By Proposition 5.17 (2a), we have \( \mu_\theta(\hat{f}^* S) = \mu_\theta(\hat{f}^* E) \). Since \( S \) is an \( \mu_\theta \)-stable bundle on \( Y \) and \( E \) is a \( P \)-stable bundle on \( (X, P) \), both \( \hat{f}^* S \) and \( \hat{f}^* E \) are \( \mu_\theta \)-polystable by (Proposition 5.17 (2b)). Define the right ideal \( \Theta_{\hat{f}} \) of the associative algebra \( \text{End}_{\text{Vect}(\hat{f}^* E)}(\hat{f}^* E) \) by

\[
\Theta_{\hat{f}} := \{ \gamma \in \text{End}_{\text{Vect}(\hat{f}^* E)}(\hat{f}^* E) \mid \gamma(\hat{f}^* E) \subset \hat{f}^* S \}.
\]

Since \( \hat{f}^* S \) is a direct summand of \( \hat{f}^* E \), the bundle \( \hat{f}^* S \) coincides with the vector sub-bundle of \( \hat{f}^* E \), generated by the images of the endomorphisms in \( \Theta_{\hat{f}} \).

Similarly, via a pullback under the \( G \)-Galois étale cover \( p_1: \hat{Y} \rightarrow \hat{Y} \), we obtain a sub-bundle \( p_1^* S = p_1^* f^* E \) on \( \hat{Y} \), which are also \( \mu_\theta \)-polystable, and \( \mu_\theta(p_1^* S, \cdot) = \mu_\theta((f \circ p_1)^* E, \cdot) \). Further, \( p_1^* S \) is generated by the images of endomorphisms in the right ideal \( \Theta_{\hat{f}} \subset \text{End}_{\text{Vect}(\hat{f}^* E)}((f \circ p_1)^* E) \) defined as

\[
\Theta_{\hat{f}} := \{ \gamma' \in \text{End}_{\text{Vect}(\hat{f}^* E)}((f \circ p_1)^* E) \mid \gamma'(f \circ p_1)^* E \subset (f \circ p_1)^* S \}.
\]

Applying Lemma 6.7 to the genuinely ramified Galois map \( \hat{g}_0: \hat{Y} \rightarrow \hat{X} \), we obtain

\[
(6.4) \quad \text{End}_{\text{Vect}(\hat{X}, \hat{P})}(g^* E) = \text{End}_{\text{Vect}(\hat{X}, \hat{P})}(\hat{f}^* E).
\]

As an element \( \gamma \in \text{End}_{\text{Vect}(\hat{X}, \hat{P})}(g^* E) \) is mapped to \( \hat{g}\gamma \in \text{End}_{\text{Vect}(\hat{f}^* E)}(\hat{f}^* E) \), the associative algebra structures are preserved. Let the right ideal \( \Theta_{\hat{X}} \subset \text{End}_{\text{Vect}(\hat{X}, \hat{P})}(g^* E) \) be the image of \( \Theta_{\hat{f}} \). Since \( g^* E \) is a \( \hat{P} \)-polystable bundle on \( (\hat{X}, \hat{P}) \), the image of any endomorphism of it is a sub-bundle. Let \( V \) be the sub-bundle of \( g^* E \) generated by the images \( \gamma(g^* E) \) for \( \gamma \in \Theta_{\hat{X}} \). Then we have \( \hat{g}^* V = \hat{f}^* S \). Moreover, \( V \) is again a \( \hat{P} \)-polystable bundle on \( (\hat{X}, \hat{P}) \), and by Lemma 5.6 we have \( \mu_\hat{p}(V) = \mu_\hat{p}(g^* E) \).

Now applying the first part of Lemma 6.7 to the finite covers \( p_{2,0} \) and \( \delta_0 \), we obtain the following.

\[
\text{End}_{\text{Vect}(\hat{X}, \hat{P})}(g^* E) \subseteq \text{End}_{\text{Vect}(\tilde{f}^* E)}((g \circ p_2)^* E) \subseteq \text{End}_{\text{Vect}(\hat{f}^* E)}(\hat{f}^* E).
\]

By (6.4), each of the above containment is an equality, and the associative algebra structures are preserved. Thus the ideal \( \Theta_{\hat{X}} \) maps onto the ideal \( \Theta_{\hat{f}} \) which maps onto the ideal
$\Theta_{\mathcal{F}}$. We have seen that $V_*$ is generated by the images of the endomorphisms in $\Theta_{\mathcal{F}}$, and $p_1^*S_*$ is generated by the images of the endomorphisms in $\Theta_{\mathcal{F}}$. Thus we have
\begin{equation}
(6.5)
p_1^*S_* = p_2^*V_*.
\end{equation}

Since $p_1$ is a $G$-Galois étale cover, the injective morphism $p_1^*S_0 \hookrightarrow p_1^*f_0^*E$ underlying the sub-bundle $p_1^*S_0 \subseteq p_1^*f_0^*E$ is $G$-equivariant. From (6.5), the injective morphism $V \hookrightarrow g_0^*E$ underlying the sub-bundle $V_0 \subseteq g_0^*E_*$ is also preserved under the $G$-action. By Galois étale descent under the map $g_0$, there is a sub-bundle $W \subseteq E$ on $X$ such that $g_0^*W = V$. Let $W_* \subseteq E_*$ be the sub-bundle on $(X, P)$ with underlying bundle $W$ on $X$ and equipped with the induced action maps from $E_*$. Thus $g^*W_* = V_*$, and $f^*W_* = S_*$. By Lemma 6.4, we have
\[ \mu_p(W_*) = \mu_p(E_*). \]
Since $E_*$ was assumed to be $P$-stable, we obtain $W_* = E_*$, and consequently, $S_* = f^*E_*$. \hfill $\Box$

Remark 6.9. We remark that, in [4], the authors have put necessary conditions on the orbifold curve $(X, P)$ such that the fiber product $Y \times_X (X, P)$ is again an orbifold curve and concluded the above result in this case. In this respect, our proof is more general and functorial.

One general question remains: whether the conclusion of Theorem 6.8 remains true under the equivalent conditions of Proposition 5.11. We note that one of the important ingredients in the proof is Lemma 6.4. It is not yet clear to us how to achieve this result for general finite cover of stacky curves.

Finally, we see that the conclusion of Theorem 6.8 fails if the cover $f_0: Y \to X$ is not genuinely ramified.

**Theorem 6.10.** Let $f: \mathcal{Y} \to (X, P)$ be a morphism of connected proper stacky curves where $(X, P)$ is a formally orbifold curve. If the induced cover $f_0: Y \to X$ on Coarse moduli curves is not genuinely ramified, there is a $P$-stable vector bundle $E_* \in \text{Vect}(X, P)$ whose pullback $f^*E_* \in \text{Vect}(\mathcal{Y})$ is not $\mu_0$-stable.

**Proof.** By [2, Proposition 5.2], there is a stable vector bundle $E \in \text{Vect}(X)$ such that $f_0^*E \in \text{Vect}(Y)$ is not stable. Let $\iota: (X, P) \to X$ and $j: \mathcal{Y} \to Y$ denote the corresponding Coarse moduli morphisms. Then using Proposition 5.16, $\iota^*E \in \text{Vect}(X, P)$ is $P$-stable, but $f^*(\iota^*E) = j^*f_0^*E$ is not $\mu_0$-stable. \hfill $\Box$

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