EXTREMES OF LOCALLY-STATIONARY CHI-SQUARE PROCESSES ON DISCRETE GRIDS

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Abstract: For \( X_i(t), i = 1, \ldots, n, t \in [0, T] \) centered Gaussian processes, the chi-square process \( \sum_{i=1}^{n} X_i^2(t) \) appears naturally as limiting processes in various statistical models. In this paper, we are concerned with the exact tail asymptotics of the supremum taken over discrete grids of a class of locally stationary chi-square processes where \( X_i(t), 1 \leq i \leq n \) are not identical. An important tool for establishing our results is a generalisation of Pickands lemma under the discrete scenario. An application related to the change-point problem is discussed.

Key Words: Chi-square processes; asymptotic methods; Pickands constant; change-point problem.

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction and Main Result

Numerous applications, especially from statistics are concerned with the supremum of chi-square processes over a discrete grid, which is threshold dependent.

The investigation of the extremes of chi-square process \( \sum_{i=1}^{n} X_i^2(t) \) is initiated by the studies of high excursions of envelope of a Gaussian process, see e.g., [1] and generalized in [2–4]. When \( X_i \)'s are stationary, [5, 6] develop the Berman’s approach in [7] to obtain an asymptotic behavior of large deviation probabilities of the stationary chi-square processes. When \( X_i, i = 1, \ldots, n \) are locally-stationary Gaussian processes, [8] obtains the extreme of the supremum of the locally-stationary Gaussian process. See [9, 10] for more literature about locally stationary Gaussian processes.

In the case that the grid is very dense, then the tail asymptotics of supremum over \([0, T]\) or the dense grid is the same. The deference up to the constant appears for so-called Pickands grids, and in the case of rare grids, the asymptotics of supremum is completely different and usually very simple, see [11].

Before giving the locally-stationary chi-square processes, we introduce a class of locally stationary Gaussian processes, considered by Berman in [12], see also [13–18]. Specifically, let \( X_i(t), i = 1, \ldots, n, t \in [0, T] \) be centered Gaussian processes with unit variance and correlation function \( r \) satisfying

\[
\lim_{\varepsilon \to 0} \sup_{t, t+s \in [0, T], |s| < \varepsilon} \left| \frac{1 - r_i(t, t+s)}{|s|^{\alpha}} - a_i(t) \right| = 0,
\]

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where \( a_i(\cdot) \) are a continuous positive function on \([0, T]\) and \( \alpha \in (0, 2] \).
Then the locally-stationary chi-square process can be defined as

\[
\chi^2(t) = \sum_{i=1}^{n} X_i^2(t), \quad t \in [0, T].
\]

Let \( \mathcal{H}_\alpha^\eta(a) \) be the Pickands constant defined for \( \eta \geq 0 \) and \( a > 0 \) by

\[
\mathcal{H}_\alpha^\eta(a) = \lim_{S \to \infty} \frac{1}{S} \mathbb{E} \left\{ \sup_{t \in [0, S] \cap \eta \mathbb{Z}} e^{\sqrt{n} B_\alpha(t) - a |t|^\alpha} \right\},
\]

where \( \eta \mathbb{Z} = \mathbb{R} \) if \( \eta = 0 \). See [19–30] for various properties of \( \mathcal{H}_\alpha^\eta(a) \).

**Theorem 1.1.** Let \( X_i(t), \ 1 \leq i \leq n, \ t \in [0, T] \) be centered, sample path continuous Gaussian processes with unit variance and correlation functions satisfying assumption (1.1). Suppose that \( \eta_u, u > 0 \) are such that

\[
\lim_{u \to \infty} \eta_u u^{1/\alpha} = \eta \in [0, \infty).
\]

If further \( r(s, t) < 1 \) for all \( s, t \in [0, T], s \neq t \), we have as \( u \to \infty \)

\[
P \left\{ \sup_{t \in [T_1, T_2] \cap u^{-1} \mathbb{Z}} \chi^2(t) > u \right\} \sim \int_{T_1}^{T} \int_{v \in \mathbb{S}_{n-1}} \mathcal{H}_\alpha^\eta \left( \sum_{i=1}^{n} v_i^2 a_i(t) \right) \, dv \, dt u^{1/\alpha} P \left\{ \chi^2(0) > u \right\},
\]

where \( \mathbb{S}_{n-1} = \{ v \in \mathbb{R}^n : \sum_{i=1}^{n} v_i^2 = 1 \} \).

Next in Section 2, we show an application related to change-point problem. The proof of Theorem 1.1 and some lemmas are relegated to Section 3 and Section 4.

**2. Applications**

In [31], they detect the change-point problem and give the generalized edge-count scan statistic expressed as

\[
S(t) = X_1^2(t) + X_2^2(t),
\]

where \( X_i, i = 1, 2 \) are two independent Gaussian processes which, respectively, have covariance functions

\[
\text{Cov} (X_1(s), X_1(t)) = \frac{(s \land t)(1 - (s \lor t))}{(s \lor t)(1 - (s \land t))}, \quad \text{Cov} (X_2(s), X_2(t)) = \frac{(s \land t)(1 - (s \lor t))}{\sqrt{(s \land t)(1 - (s \land t))(s \lor t)(1 - (s \lor t))}}.
\]

Then in order to give the asymptotic \( p \)-value approximations, we need to investigate when \( u \) is large enough for \( 0 < T_1 < T_2 < \infty \)

\[
P \left\{ \sup_{t \in [T_1, T_2] \cap u^{-1} \mathbb{Z}} S(t) > u \right\}.
\]
Theorem 2.1. we have as \( u \to \infty \)

\[
P \left\{ \sup_{t \in [T_1,T_2] \cap \mathbb{Z}^{-1}} S(t) > u \right\} \sim \frac{ue^{-u/2}}{2\pi} \int_{v_1^2 + v_2^2 = 1} \int_{T_1}^{T_2} H_1^l \left( \frac{v_1^2 + \frac{1}{2}v_2^2}{t(1-t)} \right) dv_1dv_2dt.
\]

3. Proofs

Below \( \lfloor x \rfloor \) stands for the integer part of \( x \) and \( \lceil x \rceil \) is the smallest integer not less than \( x \). Further \( \Psi \) is the survival function of an \( N(0,1) \) random variable. During the following proofs, \( Q_i, i \in \mathbb{N} \) are some positive constants which can be different from line by line and for interval \( \Delta_1, \Delta_2 \subseteq [0,\infty) \) we denote

\[
K_u(\Delta_1) := P \left\{ \sup_{t \in \Delta_1} \chi^2(t) > u \right\}, \quad K_u(\Delta_1, \Delta_2) := P \left\{ \sup_{t \in \Delta_1} \chi^2(t) > u, \sup_{t \in \Delta_2} \chi^2(t) > u \right\}.
\]

Proof of Theorem 1.1 For any \( \theta > 0 \) and \( \lambda > 0 \), set

\[
I_k(\theta) = [k\theta, (k+1)\theta] \cap \eta_u \mathbb{Z}, \quad k \in \mathbb{N}, \quad N(\theta) = \left\lceil \frac{T}{\theta} \right\rceil, \quad J^k_i(u) = [k\theta + lu^{-1/\alpha} \lambda, k\theta + (l+1)u^{-1/\alpha} \lambda] \cap \eta_u \mathbb{Z}, \quad M(u) = \left\lceil \frac{\theta u^{1/\alpha}}{\lambda} \right\rceil, \quad K^k_i(u) = [k\theta + lu^{-1/\alpha} \lambda, k\theta + (l+1)u^{-1/\alpha} \lambda].
\]

We have

\[
\sum_{k=0}^{N(\theta)-1} \left( \sum_{l=0}^{M(u)-1} K_u(J^k_i(u)) \right) - \sum_{i=1}^{4} A_i(u) \leq K_u([0, T] \cap \eta_u \mathbb{Z}) \leq \sum_{k=0}^{N(\theta)} K_u(I_k(\theta)) \leq \sum_{k=0}^{N(\theta)} \left( \sum_{l=0}^{M(u)} K_u(J^k_i(u)) \right),
\]

where

\[
A_i(u) = \sum_{(k_1,l_1,k_2,l_2) \in \mathcal{L}_i} K_u(K^{k_1}_{l_1}(u), K^{k_2}_{l_2}(u)), \quad i = 1, 2, 3, 4,
\]

with

\[
\mathcal{L}_1 = \{0 \leq k_1 = k_2 \leq N(\theta) - 1, 0 \leq l_1 + 1 = l_2 \leq M(u) - 1\},
\]

\[
\mathcal{L}_2 = \{0 \leq k_1 + 1 = k_2 \leq N(\theta) - 1, l_1 = M(u), l_2 = 0\},
\]

\[
\mathcal{L}_3 = \{0 \leq k_1 + 1 < k_2 \leq N(\theta) - 1, 0 \leq l_1, l_2 \leq M(u) - 1\},
\]

\[
\mathcal{L}_4 = \{0 \leq k_1 \leq k_2 \leq N(\theta) - 1, k_2 - k_1 \leq 1, 0 \leq l_1, l_2 \leq M(u) - 1\} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2).
\]

By Lemma 4.1, we have as \( u \to \infty, \lambda \to \infty, \theta \to 0 \)

\[
\sum_{k=0}^{N(\theta)} \left( \sum_{l=0}^{M(u)} K_u(J^k_i(u)) \right) \leq \sum_{k=0}^{N(\theta)} \left( \sum_{l=0}^{M(u)} P \left\{ \sup_{t \in J^k_i(u)} \chi^2(t) > u \right\} \right)
\]

\[
\leq \sum_{k=0}^{N(\theta)} \left( \sum_{l=0}^{M(u)} \int_{v \in S_{n-1}} H_{\alpha}^n \left( \sum_{i=1}^{n} v_i^2(a_i(k\theta) + \varepsilon_\theta) \right) \lambda dv \mathbb{P} \left\{ \chi^2(0) > u \right\} \right).
\]
Further, by Lemma 4.1

\[ \sum_{k=0}^{N(\theta)} \theta \int_{\nu \in S_{n-1}} \mathcal{H}_{\alpha}^n \left( \sum_{i=1}^{n} v_i^2 (a_i(k\theta) + \varepsilon_\theta) \right) d\nu u^{1/\alpha} \mathbb{P} \{ \chi^2(0) > u \} \]

(3.1)

\[ \sim \int_{0}^{T} \int_{\nu \in S_{n-1}} \mathcal{H}_{\alpha}^n \left( \sum_{i=1}^{n} v_i^2 a_i(t) \right) d\nu dt u^{1/\alpha} \mathbb{P} \{ \chi^2(0) > u \} . \]

Similarly, we have as \( u \to \infty, \lambda \to \infty, \theta \to 0 \)

\[ \sum_{k=0}^{N(\theta)-1} \sum_{l=0}^{M(u)-1} \mathcal{K}_u \left( J_t^k(u) \right) \geq \int_{0}^{T} \int_{\nu \in S_{n-1}} \mathcal{H}_{\alpha}^n \left( \sum_{i=1}^{n} v_i^2 a_i(t) \right) d\nu dt u^{1/\alpha} \mathbb{P} \{ \chi^2(0) > u \} . \]

Next we focus on the analysis of \( \mathcal{A}_i(u), i = 1, 2. \) For \((k_1, l_1, k_2, l_2) \in \mathcal{L}_1, \) without loss of generality, we assume \( k_1 = k_2 \) and \( l_1 + 1 = l_1. \) Then set

\[ (K_{l_1}^{k_1}(u))^1 = \left[ k_1 \theta + l_1 u^{-1/\alpha} \lambda, k_1 \theta + (l_1 + 1) u^{-1/\alpha} (\lambda - \sqrt{\lambda}) \right], \]

\[ (K_{l_1}^{k_1}(u))^2 = \left[ k_1 \theta + (l_1 + 1) u^{-1/\alpha} (\lambda - \sqrt{\lambda}), k_\theta + (l_1 + 1) u^{-1/\alpha} \lambda \right] . \]

Then we have

\[ \mathcal{A}_1(u) \leq \sum_{(k_1, l_1, k_2, l_2) \in \mathcal{L}_1} \left( \mathcal{K}_u \left( (K_{l_1}^{k_1}(u))^1, (K_{l_2}^{k_2}(u) \right) + \mathcal{K}_u \left( (K_{l_1}^{k_2}(u))^2 \right) \right) . \]

Analogously as in (3.1), we have as \( u \to \infty, \lambda \to \infty, \theta \to 0 \)

\[ \sum_{(k_1, l_1, k_2, l_2) \in \mathcal{L}_1} \mathcal{K}_u \left( (K_{l_1}^{k_1}(u))^2 \right) \leq \sum_{k_1=0}^{N(\theta)-1} \sum_{l_1=0}^{M(u)-1} \mathcal{K}_u \left( (K_{l_1}^{k_1}(u))^2 \right) \]

\[ \leq \sum_{k=0}^{N(\theta)-1} \sum_{l=0}^{M(u)-1} \int_{\nu \in S_{n-1}} \mathcal{H}_{\alpha}^n \left( \sum_{i=1}^{n} v_i^2 (a_i(k\theta) + \varepsilon_\theta) \right) \sqrt{\lambda} d\nu \mathbb{P} \{ \chi^2(0) > u \} \]

\[ \sim \frac{1}{\sqrt{\lambda}} \int_{0}^{T} \int_{\nu \in S_{n-1}} \mathcal{H}_{\alpha}^n \left( \sum_{i=1}^{n} v_i^2 a_i(t) \right) d\nu dt u^{1/\alpha} \mathbb{P} \{ \chi^2(0) > u \} , \]

\[ = o \left( u^{1/\alpha} \mathbb{P} \{ \chi^2(0) > u \} \right) . \]

Further, by Lemma 4.1

\[ \mathcal{A}_1(u) \leq \sum_{k=0}^{N(\theta)-1} \left( \sum_{l=0}^{M(u)-1} \left( \mathcal{K}_u \left( K_{l_1}^{k_1}(u) + \mathcal{K}_u \left( K_{l_1}^{k_1}(u) \right) \right) - \mathcal{K}_u \left( K_{l_1+1}^{k_1}(u) \right) \right) \right) \]

\[ \sim \sum_{k=0}^{N(\theta)-1} \left( \left( \mathcal{H}_{\alpha}[0, (a_k + \varepsilon_\theta)^{1/\alpha} d^{-1/\alpha} \lambda] + \mathcal{H}_{\alpha}[0, (a_k + \varepsilon_\theta)^{1/\alpha} d^{-1/\alpha} \lambda] - \mathcal{H}_{\alpha}[0, 2(a_k - \varepsilon_\theta)^{1/\alpha} d^{-1/\alpha} \lambda] \right) \right) \]

\[ \times \sum_{l=0}^{M(u)-1} \mathbb{P} \{ \chi^2(0) > u \} \right) \)
\[ \leq Q_1 \left( \sum_{k=0}^{N(\theta) - 1} \left( (a_k + \varepsilon_\theta)^{\frac{\alpha}{\lambda}} - (a_k - \varepsilon_\theta)^{\frac{\alpha}{\lambda}} \right) \right) u^{2/\alpha \ast} P \{ \chi^2(0) > u \} \]

\[ = o \left( u^{2/\alpha \ast} P \{ \chi^2(0) > u \} \right), \quad u \to \infty, \lambda \to \infty, \theta \to 0. \]

Similarly, by Lemma 4.1

\[ \mathcal{A}_2(u) = \sum_{k=0}^{N(\theta) - 1} K_u(J_{M(u) - 1}^k(u), J_{0}^{k+1}(u)) \]

\[ \leq \sum_{k=0}^{N(\theta) - 1} P \left\{ \sup_{t \in [0,2\lambda]} \chi^2((k + 1)\theta - u^{-2/\alpha}t) > u, \sup_{t \in [0,2\lambda]} \chi^2((k + 1)\theta + u^{-2/\alpha}t) > u \right\} \]

\[ = \sum_{k=0}^{N(\theta) - 1} \left( P \left\{ \sup_{t \in [0,2\lambda]} \chi^2((k + 1)\theta - u^{-2/\alpha}t) > u \right\} + P \left\{ \sup_{t \in [0,2\lambda]} \chi^2((k + 1)\theta + u^{-2/\alpha}t) > u \right\} \right) \]

\[ - P \left\{ \sup_{t \in [-2\lambda,2\lambda]} \chi^2((k + 1)\theta - u^{-2/\alpha}t) > u \right\} \]

\[ \sim \sum_{k=0}^{N(\theta) - 1} \left( \left( 2\mathcal{H}_0[0, 2(a_{k+1} + \varepsilon_\theta)^{\frac{\alpha}{\lambda}} d^{-1/\alpha} \lambda] - \mathcal{H}_0[-2(a_k - \varepsilon_\theta)^{\frac{\alpha}{\lambda}} d^{-1/\alpha} \lambda, 2(a_k - \varepsilon_\theta)^{\frac{\alpha}{\lambda}} d^{-1/\alpha} \lambda] \right) \times \sum_{l=0}^{M(u) - 1} P \{ \chi^2(0) > u \} \right) \]

\[ \leq Q_2 \left( \sum_{k=0}^{N(\theta) - 1} \left( (a_k + \varepsilon_\theta)^{\frac{\alpha}{\lambda}} - (a_k - \varepsilon_\theta)^{\frac{\alpha}{\lambda}} \right) \right) u^{2/\alpha \ast} P \{ \chi^2(0) > u \} \]

\[ = o \left( u^{2/\alpha \ast} P \{ \chi^2(0) > u \} \right), \quad u \to \infty, \lambda \to \infty, \theta \to 0. \]

For any \( \theta > 0 \)

\[ \mathbb{E} \{ X_i(t)X_i(s) \} = r(s,t) \leq 1 - \delta(\theta) \]

for \( (s,t) \in J_{i_1}^{k_1}(u) \times J_{i_2}^{k_2}(u), (j_1, k_1, j_2, k_2) \in L_3 \) where \( \delta(\theta) > 0 \) is related to \( \theta \). Then we have

\[ \mathcal{A}_3(u) \leq N(\theta)M(u)2\Psi \left( \frac{2u - Q_3}{d \sqrt{4 - \delta(\theta)}} \right) \]

\[ \leq \frac{T}{\lambda} u^{2/\alpha \ast} 2\Psi \left( \frac{2u - Q_3}{d \sqrt{4 - \delta(\theta)}} \right) \]

\[ = o \left( u^{2/\alpha \ast} P \{ \chi^2(0) > u \} \right), \quad u \to \infty, \lambda \to \infty, \theta \to 0. \]

where \( Q_3 \) is a large constant. Finally by Lemma 4.2 for \( u \) large enough and \( \theta \) small enough

\[ \mathcal{A}_4(u) \leq \sum_{k=0}^{N(\theta) - 1} \sum_{l=0}^{2M(u)} \sum_{i=2}^{2M(u)} K_u(J_{i}^{k}(u), J_{l+i}^{k}(u)) \]
2.1 For the correlation function shows that
Then by Theorem □

Let Lemma 4.1.

\[ \sum_{i=0}^{n} Q_{4} \exp \left( -\frac{Q_{5}}{8} |i\lambda|^\alpha \right) \]

Thus the claim follows. □

4. Appendix

**Proof of Theorem 2.1** For the correlation function \( r_i(t) \) of \( X_i(t) \), \( i = 1, 2 \), a simple calculation shows that

\[
\lim_{\epsilon \to 0} \sup_{t, t+s \in [T_1, T_2], |s| < \epsilon} \left| \frac{1 - r_i(t, t+s)}{|s|} - a_i(t) \right| = 0, \quad i = 1, 2,
\]

where \( a_1(t) = \frac{1}{\theta(1-t)} \) and \( a_2(t) = \frac{1}{2\theta(1-t)} \).

Then by Theorem 1.1, the result follows. □

**Lemma 4.1.** Let \( X_i(t), 1 \leq i \leq n, \ t \in [0, T] \) be centered, sample path continuous Gaussian processes with unit variance and correlation functions satisfying assumption (1.1). Suppose that \( \eta, u > 0 \) are such that

\[
(4.1) \quad \lim_{u \to \infty} \eta u^{1/\alpha} = \eta \in [0, \infty).
\]

Set \( a := a(t_0), \ t_0 \in \mathbb{R}, \) and \( K_u \) a family of index sets. If \( \lim_{u \to \infty} \sup_{k \in K_u} |ku^{-2/\alpha}| \leq \theta \) for some small enough \( \theta \geq 0 \), we have for some constant \( S > 0, S_1, S_2 \in \mathbb{R} \) with \( S_1 < S_2 \) when \( u \) large enough

\[
\int_{v \in S_{n-1}} \mathcal{H}_\alpha^n \left( \sum_{i=1}^{n} v_i^2 (a_i - \varepsilon \theta) \right) dv \leq \lim_{u \to \infty} \sup_{k \in K_u} \mathbb{P} \left\{ \sup_{t \in S_1, S_2, k \in K_u} \chi^2 \left( t_0 + u^{-1/\alpha} k S + t \right) > u \right\}
\]

\[
= \mathbb{P} \left\{ \sup_{(t, v) \in (S_1, S_2) \times (S_{d-1})} Y_u(t, v) > u^{1/2} \right\},
\]

where \( Y_u(t, v) = \sum_{i=1}^{n} v_i X_i(t_0 + u^{-1/\alpha} k S + u^{-1/\alpha} t) \). We know that the variance function of \( Y_u(t, v) \) always equal to 1 over \( ([S_1, S_2] \cap \eta_2) \times (S_{d-1}) \). The fields \( Y_u(t, v) \) can be represented as

\[
Y_u(t, \tilde{v}) = \sum_{i=2}^{n} v_i X_i(t_0 + u^{-1/\alpha} k S + u^{-1/\alpha} t) + \left( 1 - \sum_{i=2}^{n} v_i^2 \right)^{1/2} X_1(t_0 + u^{-1/\alpha} k S + u^{-1/\alpha} t), \ 
\tilde{v} = (v_2, \cdots, v_n),
\]

Thus the claim follows. □
which is defined in $[-S_1, S_2] \times \mathbb{S}^{d-1}$ where

$$
\widetilde{\mathbb{S}}^{d-1} = \left\{ \tilde{v} : \left( 1 - \sum_{i=2}^{n} v_i^2 \right)^{1/2}, v_2, \ldots, v_n, \right\} \in \mathbb{S}^{d-1}. 
$$

Furthermore, following the arguments as in [32] we conclude that the correlation function $r_u(t, \tilde{v}, s, \tilde{w})$ of $Y_u(t, \tilde{v})$ satisfies for $u$ large enough

$$
r_u(t, \tilde{v}, s, \tilde{w}) \geq 1 - u^{-2} \left( \sum_{i=1}^{n} v_i^2(a_i + \varepsilon_\theta) \right) t - s)^{\alpha} - \frac{1 + \varepsilon_\theta}{2} \sum_{i=2}^{n} (v_i - w_i)^2,
$$

$$
r_u(t, \tilde{v}, s, \tilde{w}) \leq 1 - u^{-2} \left( \sum_{i=1}^{n} v_i^2(a_i - \varepsilon_\theta) \right) t - s)^{\alpha} - \frac{1 - \varepsilon_\theta}{2} \sum_{i=2}^{n} (v_i - w_i)^2.
$$

Then the proof follows by similar arguments as in the proof of [33] [Theorem 6.1] with the case $\mu = \nu$. Consequently, we get

$$
P \left\{ \sup_{(t,v) \in ([S_1,S_2] \cap \eta_0 Z) \times (\mathbb{S}^{d-1})} Y_u(t,v) > u^{1/2} \right\} = P \left\{ \sup_{(t,v) \in ([S_1,S_2] \cap \eta_0 Z) \times (\mathbb{S}^{d-1})} Y_u(t,v) > u^{1/2} \right\}
$$

$$
\leq \int_{v \in \mathbb{S}_{n-1}} \mathcal{H}_\alpha^n \left( \sum_{i=1}^{n} v_i^2(a_i + \varepsilon_\theta) \right) d\mathbb{P} \left\{ \chi^2(0) > u \right\},
$$

and

$$
P \left\{ \sup_{(t,v) \in ([S_1,S_2] \cap \eta_0 Z) \times (\mathbb{S}^{d-1})} Y_u(t,v) > u^{1/2} \right\} \geq \int_{v \in \mathbb{S}_{n-1}} \mathcal{H}_\alpha^n \left( \sum_{i=1}^{n} v_i^2(a_i - \varepsilon_\theta) \right) d\mathbb{P} \left\{ \chi^2(0) > u \right\}.
$$

\[ \square \]

**Lemma 4.2.** Assume the same assumptions as in Lemma 4.1. Further, let $\varepsilon_0$ be such that for all $s, t \in [t_0 - \varepsilon_0, t_0 + \varepsilon_0]$ and $1 \leq i \leq n$

$$
\frac{a}{2} |t - s|^\alpha \leq 1 - r_i(s,t) \leq 2a |t - s|^\alpha.
$$

Then we can find a constant $C$ such that for all $S > 0$ and $T_2 - T_1 > S$,

$$
\limsup_{u \rightarrow \infty} \sup_{k \in K_u} \frac{P \{ A_1(u_k), A_2(u_k) \}}{P \{ \chi^2(t_0) > u \}} \leq C \exp \left( -\frac{a}{8} (T_2 - T_1 - S)^\alpha \right),
$$

where $A_i(u_k) = \{ \sup_{t \in [T_i, T_i + S]} \chi^2(u^{-2/\alpha} (t + kS) + t_0) > u \}, i = 1, 2,$ and

$$
\lim_{u \rightarrow \infty} \sup_{k \in K_u} \left| u^{-2/\alpha} kS \right| \leq \varepsilon_0.
$$

**Proof of Lemma 4.2** Through this proof, $C_i, i \in \mathbb{N}$ are some positive constant.

Set $Y_u(t,v) = \sum_{i=1}^{n} v_i X_i(u^{-2/\alpha} (t + kS) + t_0), (t,v) \in \mathbb{R} \times \mathcal{S}_q$ which is a centered Gaussian field and
\[ S_q^\delta = \{ v \in S_q : 1 - \sum_{i=1}^{n} v_i^2 \leq \delta \}, \delta > 0. \]

Below for \( \Delta_1, \Delta_2 \subseteq \mathbb{R}^{n+1} \), denote

\[
\mathcal{Y}_u(\Delta_1, \Delta_2) = \mathbb{P} \left\{ \sup_{(t,v) \in \Delta_1} Y_u(t,v) > u_k, \sup_{(t,v) \in \Delta_2} Y_u(t,v) > u_k \right\}.
\]

We have

\[
\mathbb{P} \{ A_1(u_k), A_2(u_k) \} \geq \mathcal{Y}_u([T_1, T_1+S] \times S_q^\delta, [T_2, T_2+S] \times S_q^\delta),
\]

\[
\mathbb{P} \{ A_1(u_k), A_2(u_k) \} \leq \mathcal{Y}_u([T_1, T_1+S] \times S_q^\delta, [T_2, T_2+S] \times S_q^\delta) + \mathcal{Y}_u([T_1, T_1+S] \times S_q^\delta, [T_2, T_2+S] \times (S_q \setminus S_q^\delta))
\]

\[
+ \mathcal{Y}_u([T_1, T_1+S] \times (S_q \setminus S_q^\delta), [T_2, T_2+S] \times S_q^\delta),
\]

and

\[
\mathcal{Y}_u([T_1, T_1+S] \times S_q^\delta, [T_2, T_2+S] \times (S_q \setminus S_q^\delta)) \leq \mathbb{P} \left\{ \sup_{(t,v) \in [T_2, T_2+S] \times (S_q \setminus S_q^\delta)} Y_u(t,v) > u_k \right\}
\]

\[
\leq \exp \left( \frac{-(u_k - C_1)^2}{2(d^2 - \delta)} \right)
\]

\[
= o(\mathbb{P} \{ Z(t_0) > u_k \}),
\]

as \( u \to \infty \) where the last second inequality follows from Borell inequality and the fact that

\[
\sup_{(t,v) \in [T_2, T_2+S] \times (S_q \setminus S_q^\delta)} \text{Var}(Y_u(t,v)) = \sup_{v \in (S_q \setminus S_q^\delta)} \left( \sum_{i=1}^{n} d_i v_i^2 \right) \leq d^2 - \delta.
\]

Similarly, we have

\[
\mathcal{Y}_u([T_1, T_1+S] \times (S_q \setminus S_q^\delta), [T_2, T_2+S] \times S_q^\delta) = o(\mathbb{P} \{ Z(t_0) > u_k \}), \ u \to \infty.
\]

Then we just need to focus on

\[
\Pi(u) := \mathcal{Y}_u([T_1, T_1+S] \times S_q^\delta, [T_2, T_2+S] \times S_q^\delta).
\]

We split \( S_q^\delta \) into sets of small diameters \( \{ \partial S_i, 0 \leq i \leq N^* \} \), where

\[
N^* = \# \{ \partial S_i \} < \infty.
\]

Further, we see that \( \Pi(u) \leq \Pi_1(u) + \Pi_2(u) \) with

\[
\Pi_1(u) = \sum_{0 \leq i,j \leq N^*, \partial S_i \cap \partial S_i = \emptyset} \mathcal{Y}_u([T_1, T_1+S] \times \partial S_i, [T_2, T_2+S] \times \partial S_i),
\]

\[
\Pi_2(u) = \sum_{0 \leq i,j \leq N^*, \partial S_i \cap \partial S_i \neq \emptyset} \mathcal{Y}_u([T_1, T_1+S] \times \partial S_i, [T_2, T_2+S] \times \partial S_i),
\]
where \( \partial S_i \cap \partial S_l \neq \emptyset \) means \( \partial S_i, \partial S_l \) are identical or adjacent, and \( \partial S_i \cap \partial S_l = \emptyset \) means \( \partial S_i, \partial S_l \) are neither identical nor adjacent. Denote the distance of two set \( \partial S_i \cap \partial S_l \) as

\[
\rho(A, B) = \inf_{x \in A, y \in B} \|x - y\|_2.
\]

if \( \partial S_i \cap \partial S_l = \emptyset \), then there exists some small positive constant \( \rho_0 \) (independent of \( i, l \)) such that \( \rho(\partial S_i, \partial S_l) > \rho_0 \). Next we estimate \( \Pi_1(u) \). For any \( u \geq 0 \)

\[
\Pi_1(u) \leq \mathbb{P} \left\{ \sup_{(t,s) \in [T_1, T_1 + S] \times [T_2, T_2 + S], w \in \partial S_i, \bar{w} \in \partial S_l} Z_u(t, s, w) > 2u_k \right\},
\]

where \( Z_u(t, s, w) = Y_u(t, v) + Y_u(s, v) \), \( v, \bar{v}, w, \bar{w} \in \mathbb{R}^n \).

When \( u \) is sufficiently large for \( (t, s) \in [T_1, T_1 + S] \times [T_2, T_2 + S], v \in \partial S_i \subset [-2, 2]^n, w \in \partial S_l \subset [-2, 2]^n \), with \( \rho(\partial S_i, \partial S_l) > \rho_0 \) we have

\[
Var(Z_u(t, s, w)) \leq \sum_{i=1}^{n} (v_i^2 + w_i^2 + 2v_i w_i) d_i^2 \\
\leq 4d^2 - 2 \sum_{i=1}^{n} (v_i - w_i)^2 d_i^2 \\
= 4d^2 - 2d^2 \rho_0 \\
\leq d^2 (4 - \delta_0),
\]

for some \( \delta_0 > 0 \). Therefore, it follows from the Borell inequality that

\[
\Pi_1(u) \leq C_2 N^* \exp \left( -\frac{(2u_k - C_3)^2}{2d^2 (4 - \delta_0)} \right) = o(\mathbb{P} \{ Z(t_0) > u_k \}) , u \to \infty,
\]

with

\[
C_3 = \mathbb{E} \left\{ \sup_{(t,s) \in [T_1, T_1 + S] \times [T_2, T_2 + S]} Z_u(t, s, w) \right\} < \infty.
\]

Now we consider \( \Pi_2(u) \). Similar to the argumentation as in Step 1 of the proof of Lemma 4.1, we set \( \tilde{Y}_u(t, \tilde{v}) = Y_u(t, Q \tilde{v}) \) and \( \tilde{Z}_u(t, \tilde{v}, s, \tilde{w}) = \tilde{Y}_u(t, \tilde{v}) + \tilde{Y}_u(s, \tilde{w}) \) with \( \tilde{v}, \tilde{w} \in \mathbb{R}^{n-1} \). Since for \( (t, s) \in [T_1, T_1 + S] \times [T_2, T_2 + S], \tilde{v} \in [-2, 2]^{n-1}, \tilde{w} \in [-2, 2]^{n-1} \), we have

\[
2d^2 \leq Var(\tilde{Z}_u(t, \tilde{v}, s, \tilde{w})) \leq \sum_{i=1}^{n} (v_i^2 + w_i^2 + 2r(u^{-2/\alpha}(t + kS) + t_0, u^{-2/\alpha}(s + kS) + t_0)v_i w_i) d_i^2 \\
\leq 2d^2 + 2 \left( 1 - \frac{a}{2} u^{-2} |t - s| \right) \sum_{i=1}^{n} v_i w_i d_i^2 \\
\leq 4d^2 - d^2 au^{-2} |t - s| \alpha \\
\leq 4d^2 - d^2 au^{-2} |T_2 - T_1 - S| \alpha.
\]
Set
\[ Z_u(t, \tilde{v}, s, \tilde{w}) = \frac{\tilde{Z}_u(t, \tilde{v}, s, \tilde{w})}{\text{Var}(Z_u(t, \tilde{v}, s, \tilde{w}))}. \]

Borrowing the arguments of the proof in [34] [Lemma 6.3] we show that
\[ \mathbb{E} \left\{ \left( Z_u(t, \tilde{v}, s, \tilde{w}) - Z_u(t', \tilde{v}', s', \tilde{w}') \right) \right\} \leq 4 \left( \mathbb{E} \left\{ (\tilde{Y}_u(t, \tilde{v}) - Y_u(t', \tilde{v}'))^2 \right\} + \mathbb{E} \left\{ (Y_u(s, \tilde{w}) - Y_u(s', \tilde{w}'))^2 \right\} \right). \]

Moreover,
\[ r_1(t, \tilde{v}, s, \tilde{v}') = 1 - u^{-2}a(t - s)^\alpha - \frac{1}{2} \sum_{i=2}^{n} d_i^2 (v_i - v_i')^2 + o \left( \sum_{i=2}^{n} d_i^2 (v_i - v_i')^2 + u^{-2} \right), \tilde{v}, \tilde{v}' \to \tilde{0}, u \to \infty. \]

Then we have
\[ \mathbb{E} \left\{ (Y_u(t, \tilde{v}) - Y_u(t', \tilde{v}'))^2 \right\} \leq 4d^2 au^{-2} |t - t'|^\alpha + 2 \sum_{i=2}^{n} (v_i - v_i')^2. \]

Therefore
\[ \mathbb{E} \left\{ \left( Z_u(t, \tilde{v}, s, \tilde{w}) - Z_u(t', \tilde{v}', s', \tilde{w}') \right) \right\} \leq 16d^2 au^{-2} |t - t'|^\alpha + 16d^2 au^{-2} |s - s'|^\alpha + 8 \sum_{i=2}^{n} (v_i - v_i')^2 + 8 \sum_{i=2}^{n} (w_i - w_i')^2. \]

Set \( \zeta(t, s, \tilde{v}, \tilde{w}), t, s \geq 0, \tilde{v}, \tilde{w} \in \mathbb{R}^{n-1} \) is a stationary Gaussian field with unit variance and correlation function
\[ r_\zeta(t, s, \tilde{v}, \tilde{w}) = \exp \left( -9d^2 at^\alpha - 9d^2 as^\alpha - 5 \sum_{i=2}^{n} v_i^2 - 5 \sum_{i=2}^{n} w_i^2 \right). \]

Then
\[ \Pi_2(u) \leq \mathbb{P} \left\{ \sup_{(t,s) \in [T_1,T_1+S] \times [T_2,T_2+S]} \tilde{Z}_u(t, \tilde{v}, s, \tilde{w}) > 2u_k \right\} \leq \mathbb{P} \left\{ \sup_{(t,s) \in [T_1,T_1+S] \times [T_2,T_2+S]} \zeta(u^{-2/\alpha}t, u^{-2/\alpha}s, \tilde{v}, \tilde{w}) > 2u_k \sqrt{4d^2 - d^2 au^{-2} |T_2 - T_1 - S|^\alpha} \right\}. \]

Then following the similar argumentation as in [35], we have
\[ \Pi_2(u) \leq C_4 u_k^{M-2} \exp \left( -\frac{u_k^2}{2d^2} - \frac{a}{8} |T_2 - T_1 - S|^\alpha \right) \]
where \( M = 0 \) when \( p \in (1, 2) \cup (2, \infty) \) and \( M = m \) when \( p = 2 \). Thus we have
\[ \limsup_{u \to \infty} \frac{\Pi_2(u)}{\mathbb{P} \{ Z(t_0) > u_k \}} \leq C_5 \exp \left( -\frac{a}{8} |T_2 - T_1 - S|^\alpha \right). \]

Thus we complete the proof.
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