Dynamic stability of viscoelastic orthotropic shells with concentrated mass

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Abstract. Viscoelastic thin-walled structures such as plates, panels and shells, with mounted objects in the form of additional masses are widely used in modern technology. The role of such additional masses is often played by longitudinal and transverse ribs, tie-plates and fixtures. When designing such structures, it is relevant to study their dynamic behavior depending on the mass distribution, viscoelastic and inhomogeneous properties of the material, etc. In this paper, the dynamic stability of a viscoelastic shell carrying concentrated masses is considered, taking into account the nonlinear and inhomogeneous properties of the material. A mathematical model of the problem is described by a system of integro-differential equations in partial derivatives. With the Bubnov-Galerkin method, the problem is reduced to solving a system of ordinary nonlinear integro-differential equations. To solve the resulting system with the Koltunov-Rzhanitsyn singular kernel, a numerical method based on the use of quadrature formulas is applied. The effect of the viscoelastic and inhomogeneous properties of the shell material, location, and the amount of concentrated masses on stability is studied.

Keywords: viscoelastic orthotropic shell, concentrated mass, geometrical nonlinearity, dynamic stability, relaxation kernel, numerical method.

1 Introduction

Thin-walled structural elements such as plates and shells are widely used in engineering and construction. Various types of tie-plates, fasteners or components of devices are mounted on them. In calculations, such additional masses are considered as concentrated at certain points and rigidly fixed. The optimal design of such structures is impossible without the creation of mathematical models that allow taking into account the maximum possible number of factors affecting their performance. One of these factors is the viscoelastic and inhomogeneous property of the material.

There is a number of publications which investigate the problems of strength, vibrations and stability of elastic and viscoelastic plates and shells with concentrated masses.

The study in [1] was devoted to the nonlinear forced vibrations of fixed plates with a mass concentrated in the center.

In [2], various methods for calculating the eigenfrequencies of beams and plates with concentrated masses were presented and compared. A new effective method for solving the problems under consideration was proposed there.

Experimental and numerical studies of forced oscillations of rigidly fixed rectangular plates with concentrated masses were presented in [3].

Based on the theory of shallow shells, the free vibrations of thin shells with added small mass were studied in [4]. The solution of mathematical model was obtained using the Bubnov-Galerkin method.

The study in [5] was devoted to free vibrations of cylindrical, spherical, elliptical and other forms of shells with cutouts and concentrated mass. To solve the problem, the finite element method was used.
In [6], analytical and experimental studies of the dynamic instability of a hinge-supported rectangular plate with arbitrarily concentrated masses were presented. The equation of motion was obtained using the theory of large strains, solved by the Bubnov-Galerkin method. The study of forced vibrations of shells with a small added mass was considered in [7]. New mathematical models based on the results of experimental studies were presented.

As the analysis of published works shows, the problems are mainly considered in elastic and linear statements. But, the composite materials with pronounced viscoelastic properties have recently become widely used [8].

There is a number of publications in literature devoted to the dynamics problems of viscoelastic thin-walled structures with concentrated masses. In these works, to solve the problems, either the Voigt differential model or the Boltzmann-Volterra integral model with a relaxation kernel in the form of an exponential kernel were used [8]; however these models cannot describe the real properties of the structures.

As studies show, the Koltunov-Rzhanitsyn relaxation kernel more accurately describes the real processes occurring in viscoelastic structures [9]. Moreover, the resulting systems of integro-differential equations have a high order and variable coefficients; this narrows the range of considered problems of the hereditary theory of viscoelasticity. Thanks to the numerical methods developed in recent years, it became possible to significantly expand the class of problems to be solved.

Hereditary viscoelastic models that describe the soil properties under various static and dynamic effects were considered in [10–12]. The parameters of the relaxation kernel were determined for various types of soil.

A method for calculating dynamic characteristics of thin-walled multilayer viscoelastic composite pipes taking into account internal hydrodynamic pressure was considered in [13]. The equation of motion was described by a coupled system of ordinary integro-differential equations with variable coefficients.

A mathematical model of nonlinear supersonic flutter of viscoelastic shells was given in [14]. There, the Boltzmann-Volterra integral model was used to describe the strain processes in shallow shells.

Dynamic stability of viscoelastic plates of variable stiffness was studied in [15]. Using the Bubnov-Galerkin method, the problem was reduced to solving a system of ordinary integro-differential equations.

In [16–19], oscillations and dynamic stability of viscoelastic isotropic plates and panels were considered accounting geometrical and physical nonlinearity.

The paper considers the problem of dynamic stability of viscoelastic orthotropic shells with concentrated masses.

**2 Materials and methods**
A rectangular viscoelastic orthotropic shell is considered of constant thickness $h$ with radii of curvature of the middle surface $R_1$ and $R_2$, rectangular in plan with dimensions $a \times b$ and carrying concentrated masses $M_i$ at points with coordinates $(x_i, y_i), i=1,2,\ldots,I$. Along one of the generatrices the shell is subjected to dynamic compression force $P(t)=\nu t$ ($\nu$ - is the loading velocity). Assume that the shell has initial deflections $w_0$.

Considering the rapidly increasing compressive force $P(t)\frac{\partial^2 w}{\partial x^2}$ and initial deflections obtained from the results given in [18], we construct the following mathematical model of the problem with respect to the displacements $u$, $v$ and deflection $w$ under the corresponding initial and boundary conditions.
\[ B_{11}(l - \Gamma_{11}) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) - \left[ k_y B_{11}(l - \Gamma_{11}^*) + k_y B_{12}(l - \Gamma_{12}^*) \right] \frac{\partial w}{\partial x} + \]
\[ B_{12}(l - \Gamma_{12}^*) + 2B(l - \Gamma^*) \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) + \]
\[ + 2B(l - \Gamma^*) \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} \right) - \left[ \rho + \frac{1}{h} \sum_{i=1}^{l} M_i \delta(x - x_i) \delta(y - y_i) \right] \frac{\partial^2 u}{\partial t^2} = 0, \]
\[ B_{22}(l - \Gamma_{22}^*) \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} \right) - \left[ k_y B_{22}(l - \Gamma_{22}^*) + k_y B_{21}(l - \Gamma_{21}^*) \right] \frac{\partial w}{\partial y} + \]
\[ + B_{21}(l - \Gamma_{21}^*) + 2B(l - \Gamma^*) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) + \]
\[ + 2B(l - \Gamma^*) \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) - \left[ \rho + \frac{1}{h} \sum_{i=1}^{l} M_i \delta(x - x_i) \delta(y - y_i) \right] \frac{\partial^2 v}{\partial x^2} = 0, \]
\[ \frac{\hbar^2}{12} B_{11}(l - \Gamma_{11}^*) \frac{\partial^2 (w - w_0)}{\partial x^4} + 8B(l - \Gamma^*) B_{12}(l - \Gamma_{12}^*) B_{21}(l - \Gamma_{21}^*) \frac{\partial^2 (w - w_0)}{\partial x^2 \partial y^2} + \]
\[ + B_{22}(l - \Gamma_{22}^*) \frac{\partial^2 (w - w_0)}{\partial y^4} \]
\[ - \left[ \left( k_y B_{11}(l - \Gamma_{11}^*) + k_y B_{21}(l - \Gamma_{21}^*) \right) \frac{\partial u}{\partial x} + \left( k_y B_{22}(l - \Gamma_{22}^*) + k_y B_{21}(l - \Gamma_{21}^*) \right) \frac{\partial v}{\partial y} \right] \]
\[ - \left[ (k_y B_{11}(l - \Gamma_{11}^*) + k_y B_{12}(l - \Gamma_{12}^*) + k_y B_{22}(l - \Gamma_{22}^*) + k_y B_{21}(l - \Gamma_{21}^*) \right] w + \]
\[ + \frac{1}{2} \left( k_y B_{11}(l - \Gamma_{11}^*) + k_y B_{21}(l - \Gamma_{21}^*) \right) \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left( k_y B_{22}(l - \Gamma_{22}^*) + k_y B_{21}(l - \Gamma_{21}^*) \right) \left( \frac{\partial w}{\partial y} \right)^2 \right] - \]
\[ - \frac{\partial w}{\partial x} B_{11}(l - \Gamma_{11}^*) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right) - \left( k_y B_{11}(l - \Gamma_{11}^*) + k_y B_{21}(l - \Gamma_{21}^*) \right) \frac{\partial w}{\partial x} + \]
\[ + B_{12}(l - \Gamma_{12}^*) + 2B(l - \Gamma^*) \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) + 2B(l - \Gamma^*) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) + \]
\[ - \frac{\partial^2 w}{\partial x^2} B_{11}(l - \Gamma_{11}^*) \left( \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial w}{\partial x} \right)^2 + B_{12}(l - \Gamma_{12}^*) \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \frac{\partial w}{\partial y} \right]^2 \right] - \]
\[ - \left( k_y B_{11}(l - \Gamma_{11}^*) + k_y B_{12}(l - \Gamma_{12}^*) \right) w - \frac{\partial w}{\partial y} B_{22}(l - \Gamma_{22}^*) \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} \right) - \]
\[ - \left( k_y B_{21}(l - \Gamma_{21}^*) + k_y B_{22}(l - \Gamma_{22}^*) \right) w + \left( B_{21}(l - \Gamma_{21}^*) + 2B(l - \Gamma^*) \right) \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) + \]
\[ + B(l - \Gamma^*) \left( \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} \right) - \frac{\partial^2 w}{\partial x^2} B_{21}(l - \Gamma_{21}^*) \left( \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial w}{\partial x} \right)^2 + \]
are the coordinate functions satisfying the given boundary conditions of the problem.

\[ \varphi(t) = \int_0^t \varphi(t) \, dt, \quad \varphi(t) = \int_0^t \varphi(t) \, dt, \quad \varphi(t) = \int_0^t \varphi(t) \, dt, \quad \varphi(t) = \int_0^t \varphi(t) \, dt \]

\[ \Gamma_1(t) - \Gamma_2(t), \quad \Gamma_3(t), \quad \Gamma_4(t), \quad \Gamma_5(t) - \text{are the kernels of relaxation}; \]

\[ B_1 = \frac{E_1}{1 - \mu_1 \mu_2}, \quad B_2 = -\frac{\mu_2 E_1}{1 - \mu_1 \mu_2} = -\frac{\mu_2 E_2}{1 - \mu_1 \mu_2}, \quad B_{22} = \frac{E_2}{1 - \mu_1 \mu_2}, \quad 2B = G. \]

\[ E_1, E_2 - \text{are the linear elastic moduli}, \quad G - \text{is the shear modulus}, \quad \mu_1, \mu_2 - \text{are the Poisson's ratios}, \quad \rho \]

is the density of the shell material; \( q \) is the external static load.

Further, in calculations, the Koltunov-Rzhanitsyn weakly singular kernel of the form [9] is used as a relaxation kernel.

\[ \Gamma(t) = Ae^{-\beta t} \sin \alpha t, \quad A > 0, \quad \beta > 0, \quad 0 < \alpha < 1. \]

The solution of the obtained system (1) is sought in the form

\[ u(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{M} u_{nm}(t) \phi_{nm}(x,y), \quad v(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{M} v_{nm}(t) \psi_{nm}(x,y), \]

\[ w(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{nm}(t) \psi_{nm}(x,y), \quad w_0(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{0nm}(x,y), \quad (2) \]

where \( u_{nm} = u_{nm}(t), \quad v_{nm} = v_{nm}(t), \quad w_{nm} = w_{nm}(t) - \text{are the unknown time functions}; \quad \phi_{nm}(x,y), \quad \psi_{nm}(x,y), \quad \psi_{nm}(x,y), \quad n=1,2,...,N; \quad m=1,2,...,M \]

are the coordinate functions satisfying the given boundary conditions of the problem.

We substitute (2) into system (1) and apply the Bubnov-Galerkin method. Introducing the following dimensionless quantities

\[ \frac{u}{h}, \quad \frac{v}{h}, \quad \frac{w}{h}, \quad \frac{w_0}{h}, \quad \frac{x}{a}, \quad \frac{y}{b}, \quad \lambda = \frac{a}{h}, \quad \delta = \frac{b}{h}, \quad t^* = \frac{P^*}{P_{cr}} = \frac{ut}{P_{cr}} = \frac{ot}{P_{cr}} = \frac{P^*}{P_{cr}}, \]

\[ P^* = \frac{P}{\sqrt{E_1 E_2}} \left( \frac{h}{h} \right)^2, \quad q = \frac{q}{\sqrt{E_1 E_2}} \left( \frac{h}{h} \right)^4, \quad S = P_{cr} \left( \frac{\pi \sqrt{E_1 E_2}}{P_{cr}} \right)^2, \]

\[ P_{cr}^* = \frac{P_{cr}}{\sqrt{E_1 E_2}} \left( \frac{h}{h} \right)^2, \quad P_{cr}^* = \frac{P_{cr}}{\sqrt{E_1 E_2}} \left( \frac{h}{h} \right)^2, \quad \Gamma(t) \sqrt{S}, \quad \Gamma(t) \sqrt{S}, \quad i,j=1,2 \]

and maintaining the previous notation to determine the unknowns \( w_{nm} = w_{nm}(t), \quad u_{nm} = u_{nm}(t), \quad v_{nm} = v_{nm}(t), \quad \psi_{nm} = \psi_{nm}(t) \) we obtain a system of nonlinear integro-differential equations.

Here \( \omega = \sqrt{\pi^2 \left( \frac{E_1 E_2}{P_{cr}} \right)^2} \) is the frequency of the eigenmode of vibrations; \( c = \sqrt{\frac{E_1 E_2}{\rho}} \) - is the sound velocity in the shell material.

To integrate the obtained system, we apply the numerical method [20], based on the elimination of a singularity in the relaxation kernel.
To study the dynamic behavior of a shell, we use a dynamic coefficient $C_D$ equal to the ratio of the dynamic critical load to the upper static one [21]. The results of calculations at various values of physico-mechanical and geometrical parameters are shown in the graphs (Figs. 1-4).

3 Results and discussion

Figure 1 shows the relation between the bending deflection and time at various values of the loading velocity parameter $S$. At these values of $S$, the $C_D$ factors are 5.75; 5.03; 4.46, respectively. A decrease in the dimensionless parameter of the loading velocity $S$ leads to an increase in the critical load coefficient and time.

![Figure 1. Deflection dependence on time at $S$=0.5 (1); 1(2); 2 (3).](image)

The effect of geometrical parameter $k_y$ was also studied (Figure 2). At values of $k_y = 10; 15; 20$, the $C_D$ coefficients are 5.03; 5.43; 5.92, respectively. It is seen that an increase in dimensionless geometric parameter $k_y$ leads to an increase in critical load and time.

![Figure 2. Deflection dependence on time at $k_y$=10(1); 15(2); 20(3).](image)

The effect of inhomogeneous material properties on the process of panel stability was studied (Figure 3). As can be seen from the figure, an increase in the parameter $\Delta = \sqrt{\frac{E_1}{E_2}}$ determining the
degree of anisotropy (curve 1 - Δ= 1; curve 2 - Δ= 1.5 and curve 3 - Δ= 2) leads to a later intense increase in deflections, and accordingly, to an increase in critical value of $C_D$.

Figure 3. Deflection dependence on time at Δ=1 (1); 1.5 (2); 2 (3).

Figure 4 shows the results of the effect of concentrated masses on the behavior of a viscoelastic shell.

Figure 4. Deflection dependence on time at $M_1=0$ (10; 0.1 (2); 0.3 (3); 0.5 (4).

We believe that the concentrated mass is applied to the center of the shell and the loading velocity parameter $S$ is 0.2. With the concentrated mass parameter $M_1 = 0.0; 0.1; 0.3$ and 0.5, the $C_D$ values are 6.61, 6.96; 7.20 and 7.29, respectively. It can be seen that as the parameters of concentrated masses increase, the entire curve moves to the right toward large values of $t$. It should be noted here that the effect of concentrated masses manifests itself significantly in cases of small values of the loading velocity parameter $S$.

4 Conclusion
1. The dynamic stability of viscoelastic orthotropic shells with concentrated masses was investigated.
2. The effect of the change in shell physico-mechanical and geometrical parameters on dynamic stability of a shell was shown.
3. It was shown that an increase in loading velocity $v$, in the viscoelastic case led to an increase in critical load coefficient and time.
4. It was determined that an account for inhomogeneous properties of the shell material leads to a later intense increase in deflections, and accordingly, to an increase in critical value of $C_D$.

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