Motion on Constant Curvature Spaces
and Quantization Using Noether Symmetries

Paul Bracken
Department of Mathematics,
University of Texas,
Edinburg, TX
78540

Abstract
A general approach is presented for quantizing a metric nonlinear system on a manifold of constant curvature. It makes use of a curvature dependent procedure which relies on determining Noether symmetries from the metric. The curvature of the space functions as a constant parameter. For a specific metric which defines the manifold, Lie differentiation of the metric gives these symmetries. A metric is used such that the resulting Schrödinger equation can be solved in terms of hypergeometric functions. This permits the investigation of both the energy spectrum and wave functions exactly for this system.

PACs: 03.65.Ge, 03.65.Ta, 03.65.Aa

Keywords: curvature, vector field, Hamiltonian, quantization


1 Introduction

The standard approach to quantization in quantum mechanics usually takes place on a Euclidean space which is characterized by a zero curvature scalar [1-2]. The quantization of physical models on a curved space, even a space of constant curvature is a problem which impacts many different areas of physics such as gravitation [3-4]. A specific example of physical importance is the existence of Landau levels for the motion of a charged particle under perpendicular fields, which has been investigated in the case of non-Euclidean geometries [5]. The quantum dot has also given rise to the use of models which are based in the area of quantum mechanics on spaces of constant curvature. In fact, the entire area of gravitation and cosmology is approached at the present time on a geometric foundation. This relies on specifying a space-time characterized by a metric, whose components are used to calculate the curvature of the spacetime manifold. The result need not always be constant, but this case is easier to study mathematically. Schrödinger first made use of a factorization method for the study of the hydrogen atom in a spherical geometry [6], and dynamical symmetries in a spherical geometry have been worked out by Higgs [7]. A more esoteric problem is the study of polygonal billiards, or systems which are enclosed by geodesic arcs on surfaces with curvature. Some motions that are integrable in the Euclidean case can become ergodic when the curvature is negative, so this subject overlaps with the study of chaos in quantum systems [8].

It is the objective here to look at the quantization of a geometric model on a curved space which can be thought of as a two-dimensional oscillator under a particular choice of potential. An approach is proposed here which ought to be applicable to many types of models which are specified by a metric in the sense that a metric is defined and the components of the metric are used to construct the Lagrangian [9-11]. The two-dimensional problem, originally introduced as a nonlinear deformation of a linear system, can in fact be interpreted as a potential model on a space with constant curvature. The three spaces with constant curvature $\kappa$, the sphere $S^2_\kappa(\kappa > 0)$, Euclidean plane $E^2$, and hyperbolic plane $H^2_\kappa(\kappa < 0)$ can be thought of as three different cases arising from a family of Riemannian manifolds $M^2_\kappa = (S^2, E^2, H^2)_\kappa$ with the curvature $\kappa \in \mathbb{R}$.
appearing as a parameter. The components of the metric will be selected according to this geometry, but everything is done in such a way that applications to other types of systems whose Lagrangian is specified by a metric should be straightforward. Other spaces to which the procedure applies would be classically diffeomorphic to either a sphere of constant curvature, or to the hyperbolic plane depending on the sign of the curvature. The curvature is thus considered as a parameter and all mathematical expressions are presented in a curvature dependent way in terms of this parameter. The ideas of the procedure can be enunciated in a very general, mathematical framework than has been done. It is hoped that these ideas can be applied to other types of metrics to achieve similar results [12-14]. With the metric adopted here, it will be seen that eigenfunctions can be obtained for the Schrödinger equation.

The model can be formulated in Cartesian and cylindrical coordinates and the transformation properties of dynamical variables such as the Lagrangian under coordinate changes is studied. The Lagrangian is determined once the components of the metric have been defined, and depends on the coordinates of the underlying manifold. It is shown how the Killing vector fields can be calculated by Lie differentiation of the metric. These Killing vectors can be shown to be specified by a coupled system of partial differential equations which can be solved in closed form for the component functions in these vector fields. The Poisson brackets of the classical variables can be calculated as well as the commutator brackets of the Killing vector fields. This work can be done quickly by using symbolic manipulation [15]. The Hamiltonian is calculated by means of the usual canonical transformation. It is necessary to know the Hamiltonian in order to quantize the system. The Killing vector fields will provide the Noether momenta for the system. The canonical momenta do not coincide with the Noether momenta when $\kappa \neq 0$, and so the quantization procedure is more complicated in a constant curvature space. The usual quantization prescription can be applied to the components of the Noether momenta which appear in the Hamiltonian. It may be noted that the Poisson momenta do not Poisson commute, and the corresponding self-adjoint quantum counterparts do not commute as operators. Verifying these statements is possible by using symbolic manipulation again. To summarize, it is explicitly indicated how the transition from the classical curvature dependent picture to a quantum system in terms of operators can
be done using this approach based on the quantization of the Noether momenta. Finally, it is explained how the exact resolution of the curvature dependent Schrödinger equation can be accomplished. The eigenfunctions and energies can be calculated and studied in detail for the metric of this problem, and will lead to the concept of curvature dependent plane waves.

2 Metric and Lagrangian

A model will be constructed and examined which is defined by a specific metric and relates to geodesic motion. The dynamics is obtained from a Lagrangian whose kinetic energy term depends on the curvature parameter of the underlying space and is related to the metric. Let $M$ be a Riemannian or pseudo-Riemannian manifold whose metric evaluated at a point $p \in M$ is $g(p)$ [16]. On the tangent space $TM$, consider the Lagrangian given by the kinetic energy of the metric

$$T(v) = \frac{1}{2} g_{ij} v^i v^j.$$  \hfill (2.1)

The general Lagrangian is obtained from (2.1) by adding a potential term. The exact form of the metric $g$ in the case studied here is defined in cylindrical coordinates to be

$$g = \frac{1}{2} \frac{1}{1 - \kappa r^2} dr \otimes dr + \frac{1}{2} r^2 d\varphi \otimes d\varphi.$$  \hfill (2.2)

In (2.2), $\kappa$ is the curvature scalar and this metric can be put into an equivalent form in which the geometry of the manifold is clearer. The three spaces of constant curvature which occur here, the sphere $S^2_\kappa (\kappa > 0)$, the Euclidean plane $E^2(\kappa = 0)$, and hyperbolic plane $H^2_\kappa (\kappa < 0)$, can be considered different situations inside a family of Riemannian manifolds $M^2_\kappa = (S^2_\kappa, E^2, H^2_\kappa)$ with the curvature $\kappa \in \mathbb{R}$ as a parameter. Taking the components from (2.2) and putting them in (2.1), the general Lagrangian is given in terms of cylindrical variables with $v_r = \dot{r}$ and $v_\varphi = \dot{\varphi}$ as

$$L(\kappa) = \frac{1}{2} \left( \frac{v_r^2}{1 - \kappa r^2} + r^2 v_\varphi^2 \right) + V(r).$$  \hfill (2.3)

It is useful to study this system at the classical level before proceeding to look at its quantization. The standard transformations of Lagrangian mechanics can be established as well as the transformations between Cartesian and cylindrical forms. To this end, begin with the transformation
given by
\[ x = r \cos \varphi, \quad y = r \sin \varphi. \]  
(2.4)

All the variables appearing in (2.4) depend on an evolution parameter \( \kappa \). Differentiating the variables in the transformation (2.4) with respect to the time parameter \( t \), setting \( v_x = \dot{x} \) and \( v_y = \dot{y} \), these additional relationships are obtained

\[ v_x = v_r \cos \varphi - r \sin \varphi \frac{\partial\varphi}{\partial t}, \quad v_y = v_r \sin \varphi + r \cos \varphi \frac{\partial\varphi}{\partial t}. \]  
(2.5)

From (2.5), it follows that

\[ v_x^2 + v_y^2 - \kappa(\kappa x v_y - y v_x)^2 = v_r^2 + r^2(1 - \kappa r^2)v_\varphi^2. \]  
(2.6)

The Lagrangian (2.3) in Cartesian coordinates is given by

\[ L(\kappa) = \frac{1}{2} \left[ \frac{\partial}{\partial \dot{x}} \left( \frac{\partial L}{\partial \dot{x}} \right) + \frac{\partial}{\partial \dot{y}} \left( \frac{\partial L}{\partial \dot{y}} \right) \right] - \kappa(x v_y - y v_x)^2 + V(x, y). \]  
(2.7)

Using formulas such as (2.6), the Lagrangian can be transformed from Cartesian to cylindrical form or from cylindrical to Cartesian by means of (2.6).

To obtain a quantum formulation for the system, it is important to study the Hamiltonian. The Hamiltonian is determined from the Lagrangian (2.7) by first obtaining the momenta by differentiating \( L \),

\[ p_x = \frac{\partial L}{\partial \dot{x}} = \frac{v_x + \kappa(x v_y - y v_x)y}{1 - \kappa r^2}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = \frac{v_y - \kappa(x v_y - y v_x)x}{1 - \kappa r^2}. \]  
(2.8)

Solving (2.8) for \( v_x \) and \( v_y \), we obtain,

\[ v_x = (1 - \kappa x^2)p_x - \kappa x y p_y, \quad v_y = (1 - \kappa y^2)p_y - \kappa x y p_x. \]  
(2.9)

The Hamiltonian is calculated from (2.9) and the Lagrangian (2.7) by means of the usual transformation,

\[ H = p_x v_x + p_y v_y - L(\kappa) = \frac{1}{2}(p_x^2 + p_y^2 - \kappa(x p_x + y p_y)^2) - V(x, y). \]  
(2.10)

If (2.8) is taken and \( p_x, p_y \) are substituted back into \( L(\kappa) \), the Lagrangian (2.7) is recovered. Finally, to obtain the Hamiltonian in terms of the cylindrical \( r, \varphi \) coordinates, the momenta
which correspond to (2.3) are determined by differentiating the Lagrangian in (2.3),

\[ p_r = \frac{\partial L}{\partial \dot{r}} = \frac{v_r}{1 - \kappa r^2}, \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = r^2 \dot{\varphi}. \]  

(2.11)

Therefore, the Hamiltonian in these coordinates is given by

\[ H(\kappa) = p_r v_r + p_\varphi v_\varphi - L(\kappa) = \frac{1}{2} \left[ (1 - \kappa r^2) p_r^2 + \frac{1}{r^2} p_\varphi^2 \right] - V(r). \]  

(2.12)

Finally, the equations of motion can be written down in the \( x, y \) coordinate system. With \( q = x, y \), the Euler-Lagrange equation will produce these,

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0. \]  

(2.13)

Calculating (2.13) for both variables, two coupled equations are obtained each of which contain both derivatives \( \ddot{x} \) and \( \ddot{y} \). Solving this pair as a system in the two variables \( \{ \dot{x}, \dot{y} \} \) with the potential function unspecified for the moment, the equations of motion are found to be

\[ (1 - \kappa r^2) \ddot{x} + \kappa (\dot{x}^2 + \dot{y}^2 - \kappa (x \dot{y} - y \dot{x})^2) x = (1 - \kappa r^2)((1 - \kappa x^2) \frac{\partial V}{\partial x} - \kappa xy \frac{\partial V}{\partial y}), \]

\[ (1 - \kappa r^2) \ddot{y} + \kappa (\dot{x}^2 + \dot{y}^2 - \kappa (x \dot{y} - y \dot{x})^2) y = (1 - \kappa r^2)(-\kappa xy \frac{\partial V}{\partial x} + (1 - \kappa y^2) \frac{\partial V}{\partial y}). \]

The potential that will be used in the subsequent analysis will have the form,

\[ V(r) = -\frac{\alpha^2}{2} \left( \frac{r^2}{1 - \kappa r^2} \right). \]

In fact, many other forms for the potential are admitted by the method proposed here. This potential has the advantage that eigenfunctions can be obtained for its Schrödinger equation. Substituting this potential into the equation of motion, they become the following

\[ (1 - \kappa r^2) \ddot{x} + \kappa [\dot{x}^2 + \dot{y}^2 - \kappa (x \dot{y} - y \dot{x})^2] x + \alpha^2 x = 0, \]

\[ (1 - \kappa r^2) \ddot{y} + \kappa [\dot{x}^2 + \dot{y}^2 - \kappa (x \dot{y} - y \dot{x})^2] y + \alpha^2 y = 0. \]  

(2.14)

Since the Schrödinger equation can be studied, it is worth remarking about classical solutions. The general solution to the Euler-Lagrange equations (2.14) will have the structure given as (1) if \( \kappa > 0 \) the dynamics is restricted to the region \( r^2 < 1/|\kappa| \) where the kinetic energy is positive
definite and the general solution would be  \( x = A \sin(\omega t + \phi_1), \ y = B \sin(\omega t + \phi_2) \). (ii) If \( \kappa < 0 \) then the most general solution is  \( x = A \sin(\omega t + \phi_1), \ y = B \sin(\omega t + \phi_2) \) when the energy \( E \) is smaller than a certain value \( E_0 \) and by  \( x = A \sinh(\Omega t + \phi_1), \ y = B \sinh(\Omega t + \phi_2) \) when the energy \( E \) is greater than this value. The coefficients \( A \) and \( B \) are related to \( \alpha \) and the frequency \( \omega \), which is oscillatory motion, or with \( \alpha \) and \( \Omega \), unbounded motion.

3 Calculation of Killing Vector Fields

It is required to determine three linearly independent Killing vector fields for the metric. For a vector field to be such a vector field, the Lie derivative of the metric (2.2) with respect to this vector field must vanish. A procedure will be presented which allows the determination of these vector fields from the metric in closed form. Define a general vector field  \( X \) with unknown coefficients which depend on the cylindrical coordinates of the manifold as follows

\[
X = f(r, \varphi) \frac{\partial}{\partial r} + h(r, \varphi) \frac{\partial}{\partial \varphi}.
\] (3.1)

The functions \( f \) and \( h \) are determined in such a way that the Lie derivative of metric (2.2) with respect to \( X \) in (3.1) vanishes,

\[
\mathcal{L}_X g = 0.
\] (3.2)

This is the condition that must be satisfied for \( X \) to be a Killing vector field. Using (3.1) in (3.2), the Lie derivative of \( g \) is found to be

\[
\mathcal{L}_X g = X \left( \frac{1}{1 - \kappa r^2} \right) dr \otimes dr + \frac{1}{1 - \kappa r^2} \left\{ \frac{\partial f}{\partial r} dr \otimes dr + \frac{\partial f}{\partial \varphi} d\varphi \otimes dr + \frac{\partial f}{\partial r} dr \otimes d\varphi + \frac{\partial f}{\partial \varphi} d\varphi \otimes d\varphi \right\} + X (r^2) d\varphi \otimes d\varphi
\]

\[
+ r^2 \left\{ \frac{\partial h}{\partial r} dr \otimes d\varphi + \frac{\partial h}{\partial \varphi} d\varphi \otimes d\varphi + \frac{\partial h}{\partial r} d\varphi \otimes dr + \frac{\partial h}{\partial \varphi} d\varphi \otimes d\varphi \right\}.
\] (3.3)

In order for (3.2) to hold, the coefficient of each tensor product in (3.3) must vanish. Collecting like products, condition (3.2) is satisfied when the following system of partial differential equations is satisfied

\[
\frac{\partial f}{\partial r} + \kappa r \frac{1}{1 - \kappa r^2} f = 0, \quad r^2 \frac{\partial h}{\partial r} + \frac{1}{1 - \kappa r^2} \frac{\partial f}{\partial \varphi} = 0, \quad r \frac{\partial h}{\partial \varphi} = -f.
\] (3.4)
The general solution to system (3.4) is given as
\[ f(r, \varphi) = \sqrt{1 - \kappa r^2} (c_1 \sin \varphi + c_2 \cos \varphi), \quad h(r, \varphi) = \frac{1}{r} \sqrt{1 - \kappa r^2} (c_1 \cos \varphi - c_2 \sin \varphi) + c_3. \] (3.5)

A set of three independent vector fields which satisfies (3.2) will suffice. By picking three independent sets of constants \( \{c_i\}_{i=1}^3 \) appropriately, these vector fields take the form
\[ X_1 = \sqrt{1 - \kappa r^2} (\cos \varphi \frac{\partial}{\partial r} - \frac{1}{r} \sin \varphi \frac{\partial}{\partial \varphi}), \quad X_2 = \sqrt{1 - \kappa r^2} (\sin \varphi \frac{\partial}{\partial r} + \frac{1}{r} \cos \varphi \frac{\partial}{\partial \varphi}), \quad X_J = \frac{\partial}{\partial \varphi}. \] (3.6)

The set of vector fields (3.6) are the required Noether symmetries, and the coefficient functions satisfy system (3.4). The associated constants of the motion are given by
\[ P_1 = \sqrt{1 - \kappa r^2} (\cos \varphi p_r - \frac{1}{r} \sin \varphi p_\varphi), \quad P_2 = \sqrt{1 - \kappa r^2} (\sin \varphi p_r + \frac{1}{r} \cos \varphi p_\varphi), \quad J = p_\varphi, \] (3.7)
in the Hamiltonian formalism. The classical Poisson bracket of any of the quantities \( F, G \) from (3.7) is defined by
\[ \{F, G\} = \frac{\partial F}{\partial r} \frac{\partial G}{\partial p_r} + \frac{\partial F}{\partial \varphi} \frac{\partial G}{\partial p_\varphi} - \frac{\partial F}{\partial p_r} \frac{\partial G}{\partial r} - \frac{\partial F}{\partial p_\varphi} \frac{\partial G}{\partial \varphi}. \] (3.8)

Substitute the variables in (3.7) for \( F \) and \( G \) in (3.8), and the following brackets are obtained
\[ \{P_1, P_2\} = \kappa J, \quad \{P_1, J\} = -P_2, \quad \{P_2, J\} = P_1. \] (3.9)

Moreover, using the Hamiltonian (2.12) and (3.7), the following brackets can be calculated
\[ \{P_1, H\} = 0, \quad \{P_2, H\} = 0, \quad \{J, H\} = 0. \] (3.10)

The Lie brackets of the vector fields in (3.6) can also be worked out. It is found that they close in the following way,
\[ [X_1, X_2] = -\kappa X_J, \quad [X_1, X_J] = X_2, \quad [X_2, X_J] = -X_1. \] (3.11)

Depending on the sign of \( \kappa \), the Lie algebra of the group of isometries of the spherical, Euclidean and hyperbolic spaces is obtained. Only in the Euclidean case \( \kappa = 0 \) do \( X_1 \) and \( X_2 \) commute.

It should be mentioned that although some of these calculations appear tedious, they can be carried out very efficiently by means of symbolic manipulation [15]. Now (3.7) can be solved
as a system for the variables $p_r, p_\phi$, which are then placed into (2.12) for the Hamiltonian. The Hamiltonian then assumes the form,

$$H(\kappa) = \frac{1}{2m}[P_1^2 + P_2^2 + \kappa J^2] - V(r).$$

(3.12)

This Hamiltonian will be used in the quantization procedure. A factor of mass $m$ has been included to that end. The only measure on the space $\mathbb{R}^2$ that is invariant under the action of the vector fields (3.6) in the sense that the Lie derivative vanishes,

$$\mathcal{L}_{X_i} d\mu_\kappa = 0, \quad i = 1, 2, J$$

is given up to a constant factor by

$$d\mu_\kappa = \frac{r}{\sqrt{1 - \kappa r^2}} dr \wedge d\varphi.$$  

(3.13)

To verify (3.13), the Lie derivative of the right-hand side is evaluated with respect to each vector field in (3.6) using the usual rules for Lie differentiation.

## 4 Quantization and Schrödinger Equation

This property of the measure suggests a way to quantize the Hamiltonian for the model [12-13]. The idea is to consider functions and linear operators which are defined on a related space. This is obtained by taking the two-dimensional real plane $\mathbb{R}^2$ and using the measure (3.13) on it. To put it another way, the operators $\hat{P}_1$ and $\hat{P}_2$, which represent the quantum version of the Noether momenta in (3.7), must be self-adjoint not in the space $L^2(\mathbb{R})$, but in the space $L^2(\mathbb{R}, d\mu_\kappa)$ which is endowed with the measure (3.13).

Assuming the usual correspondence for the momenta in (3.7), the quantum operators which will form the quantum Hamiltonian are given by

$$P_1 \rightarrow \hat{P}_1 = -i\hbar \sqrt{1 - \kappa r^2} \left[ \cos \frac{\varphi}{r} \frac{\partial}{\partial r} - \frac{1}{r} \sin \frac{\varphi}{r} \frac{\partial}{\partial \varphi} \right],$$

$$P_2 \rightarrow \hat{P}_2 = -i\hbar \sqrt{1 - \kappa r^2} \left[ \sin \frac{\varphi}{r} \frac{\partial}{\partial r} + \frac{1}{r} \cos \frac{\varphi}{r} \frac{\partial}{\partial \varphi} \right].$$

(4.1)
\[ J \rightarrow \hat{J} = -i\hbar \frac{\partial}{\partial \varphi}. \]

Transforming (3.12) by means of (4.1), the quantum Hamiltonian is found to be

\[ \hat{H} = -\frac{\hbar^2}{2m} \left\{ (1 - \kappa r^2) \frac{\partial^2}{\partial r^2} + (1 - 2\kappa r^2) \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right\} + \frac{1}{2} \alpha^2 \frac{r^2}{1 - \kappa r^2}. \] (4.2)

Using the operator (4.2), it is straightforward to form the Schrödinger equation, \( \hat{H} \Psi = E \Psi \), as follows

\[ -\frac{\hbar}{2m} \left\{ (1 - \kappa r^2) \frac{\partial^2}{\partial r^2} + (1 - 2\kappa r^2) \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right\} \Psi + \frac{1}{2} \alpha^2 \frac{r^2}{1 - \kappa r^2} \Psi = E \Psi. \] (4.3)

Before studying a class of solutions to (4.3), it is helpful to scale out the physical constants from equation (4.3). To do this, let us substitute the new variables \( \bar{r}, \bar{\kappa} \) and \( \mathcal{E} \) into the equation. To this end, let us set \( \alpha = \sqrt{m\beta} \) and then make the replacement \( \beta^2 \rightarrow \beta^2 - (\kappa \hbar \beta)/m \) in the equation. The reason for the latter transformation is simply that an \( r \)-dependent term can be factored from the equation thereby simplifying it. Define

\[ r = \sqrt{\frac{\hbar}{m\beta}} \bar{r}, \quad \kappa = \frac{m\beta}{\hbar} \bar{\kappa}, \quad E = (\hbar \beta) \mathcal{E}. \] (4.4)

All of the physical constants in (4.3) simplify and factor out. The Schrödinger equation takes the equivalent form,

\[ \left\{ (1 - \kappa r^2) \frac{\partial^2}{\partial r^2} + (1 - 2\kappa r^2) \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} - (1 - \kappa) \frac{r^2}{1 - \kappa r^2} \right\} \Psi = -2\mathcal{E} \Psi. \] (4.5)

In writing (4.5), all the bars from (4.4) have subsequently been dropped for ease of writing.

5 Spectrum and Wavefunctions

It is well worth studying Schrödinger equation (4.5) in this case since many things can be learned with regard to the nature of the spectrum and eigenfunctions of the equation. First of all, (4.5) is a separable equation, and there exist solutions to it of the form

\[ \Psi(r, \varphi) = R(r)\Phi(\varphi), \] (5.1)
where $R$ and $\Phi$ are functions of the variables $r$ and $\varphi$. Substituting $\Psi$ from (5.1) into (4.5), the equation becomes

$$\Phi\{(1 - \kappa r^2)R'' + (1 - 2\kappa r^2)\frac{R'}{r}\} + \frac{1}{r^2}R\dot{\Phi} - (1 - \kappa)(\frac{r^2}{1 - \kappa^2})R\Phi = -2\mathcal{E}R\Phi.$$ (5.2)

Introducing a separation constant $\mu$, this can be written in the separated form

$$\frac{r^2}{R}\{(1 - \kappa r^2)R'' + (1 - 2\kappa r^2)\frac{R'}{r}\} - (1 - \kappa)(\frac{r^4}{1 - \kappa^2}) + 2\mathcal{E}r^2 = -\frac{\ddot{\Phi}}{\Phi} = \mu^2.$$ (5.3)

This is equivalent to the following pair of ordinary equations

$$\ddot{\Phi} + \mu^2\Phi = 0,$$

$$r^2(1 - \kappa r^2)R'' + ((1 - 2\kappa r^2)rR' - (1 - \kappa)\frac{r^4}{1 - \kappa^2}R + 2\mathcal{E}r^2R - \mu^2R = 0.$$ (5.4)

The equation in $\Phi$ is easy to solve and has exponential solutions $e^{\pm i\mu\varphi}$. The parameter $\kappa$ appears only in the radial equation in the end.

The radial equation is the most work to solve, so consider it. This equation has a factorized solution of the form

$$R(r, \kappa) = F(r, \kappa)(1 - \kappa r^2)^s.$$ (5.5)

Substituting (5.5) into (5.4), the resulting equation in terms of the parameter $s$ is given by

$$r^2(1 - \kappa r^2)^2F'' - (1 - \kappa r^2)(2(2s + 1)\kappa r^2 - 1)rF' + ((4\kappa^2 s^2 + 2\kappa^2 s + \kappa - 1 - 2\kappa\mathcal{E})r^4
+ (\mu^2\kappa - 4\kappa s + 2\mathcal{E}r^2 - \mu^2)F = 0.$$ (5.6)

If $s$ is taken to be

$$s = \frac{1}{2} - \frac{1}{2\kappa},$$

the $r^4$ term disappears from the coefficient of the last term. Moreover, evaluating the limit $\kappa \to \infty$ with this $s$ at fixed $r$ in (5.5), it is found to exist and is given by

$$\lim_{\kappa \to 0} R(r, \kappa) = F(r)e^{r^2/2}.$$ (5.8)

Finally, equation (5.6) simplifies to the form,

$$r^2(1 - \kappa r^2)F'' + (2(1 - 2\kappa)r^2 + 1)rF' + (2(1 - \kappa + \mathcal{E})r^2 - \mu^2)F = 0.$$ (5.7)
This equation can be studied by means of the method of Frobenius and the indicial equation for (5.7) implies that it has a regular solution of the form

$$ F(r) = r^\mu \cdot f(r). \quad (5.8) $$

The function $f(r)$ is regular at $r = 0$. Substituting (5.8) into (5.7), it is found that $f(r)$ satisfies the equation

$$ r(1 - \kappa r^2)f'' + (2(1 - \kappa \mu - 2\kappa)r^2 + 2\mu + 1)f' + (2E + 2\mu - (\mu + 1)(\mu + 2)\kappa + 2)rf = 0. \quad (5.9) $$

Assume now a $\kappa$-dependent power series for the function $f$ in (5.9) of the form

$$ f(r, \kappa) = \sum_{n=0}^{\infty} a_n(\kappa) r^n \quad (5.10) $$

and substitute (5.10) into (5.9). The following $\kappa$-dependent recursion relation is obtained for the coefficients $a_n(\kappa)$,

$$ a_{n+1}(\kappa) = \frac{(n + \mu)(n + \mu + 1)\kappa - 2(E + \mu + m)}{(n + 1)(n + 1 + 2\mu)} a_{n-1}(\kappa), \quad n = 1, 2, \cdots $$

with $a_1(\kappa) = 0$ and $a_0(\kappa)$ is an arbitrary constant. The radius of convergence of the series (5.10) is given by $r_c = 1/\sqrt{\kappa}$. The even powers dependence implied by the recursion relation suggests the introduction of the new variable $t = r^2$

$$ t(1 - \kappa t)f''(t) + [(1 + \mu + (1 - \kappa \mu - \frac{5}{2}\kappa)t)f'_t + \frac{1}{4}(2E + 2(\mu + 1) - \kappa(\mu + 1)(\mu + 2))f] = 0. \quad (5.11) $$

When $\kappa = 0$, which is the Euclidean case, the equation reduces to

$$ tf''(t) + [\mu + 1 + t]f'(t) + \frac{1}{2}(E + \mu + 1)f(t) = 0. \quad (5.12) $$

There exists a solution which is regular at $r = 0$ given by

$$ f(r) = c_0 e^{-r^2} K_M(a, \mu + 1; r^2), \quad a = \frac{1}{2}(1 + \mu - E). \quad (5.13) $$

In (5.13), the function $K_M$ is the Kummer M-function. The physically acceptable solutions are the polynomial solutions which appear when the parameter $a$ takes the values $a = -n_r, n_r = 0, 1, 2, \cdots$. 
When $\kappa \neq 0$, introduce the variable $s = \kappa t$ so the equation is transformed into

$$s(1 - s)f_{ss} + (1 + \mu + \frac{2 - 2\kappa \mu - 5\mu}{2\kappa} s)f_s + \frac{1}{4\kappa}(2\mathcal{E} + 2\mu + 2 - (\mu + 1)(\mu + 2)\kappa)f = 0. \quad (5.14)$$

This is the Gauss hypergeometric equation in the standard form

$$s(1 - s)f_{ss} + [c - (1 + a_\kappa + b_\kappa)s]f_s - a_\kappa b_\kappa f = 0. \quad (5.15)$$

The constants in (5.15) are given by comparing with (5.14),

$$c = \mu + 1, \quad a_\kappa + b_\kappa = \frac{(2\mu + 3)\kappa - 2}{2\kappa}, \quad a_\kappa b_\kappa = -\frac{1}{2\kappa}(\mathcal{E} + \mu + 1) + \frac{1}{4}(\mu + 1)(\mu + 2). \quad (5.16)$$

The solution to (5.14) is given by the hypergeometric function

$$f(t) = \, _2F_1(a_\kappa, b_\kappa; c; t), \quad (5.17)$$

where $a_\kappa$ and $b_\kappa$ in (5.17) are found to be given by

$$a_\kappa = \frac{3\kappa + 2\mu \kappa - 2 - \Delta}{4\kappa}, \quad b_\kappa = \frac{3\kappa + 2\mu \kappa - 2 + \Delta}{4\kappa}, \quad \Delta = \sqrt{(\kappa - 2)^2 + 8\mathcal{E}\kappa}. \quad (5.18)$$

The physically acceptable solutions which are determined as eigenfunctions of the singular $\kappa$-dependent Sturm-Liouville problem appear when one of the two $\kappa$-dependent coefficients $a_\kappa$, $b_\kappa$ coincides with zero or a negative integer number

$$a_\kappa = -N_r, \quad b_\kappa = -N_r, \quad N_r = 0, 1, 2, \cdots. \quad (5.19)$$

This restricts the energy to one of the following values

$$\mathcal{E} = (2N_r + \mu + 1)(\frac{1}{2}(2N_r + \mu + 2)\kappa - 1). \quad (5.20)$$

The hypergeometric series then reduces to a polynomial of degree $N_r$. Introducing the new quantum number $n = 2N_r + \mu$, the energy levels are given by

$$\mathcal{E} = (n + 1)(\frac{1}{2}(n + 2)\kappa - 1).$$

Summarizing what has been obtained for this case, the wavefunctions for the system on a space with constant curvature are given by

$$\Psi_{N_r,\mu}(r, \varphi; \kappa) = C_\kappa r^\mu(1 - \kappa r^2)^{\frac{1}{2}} \, _2F_1(-N_r, b_r; \mu + 1; \kappa r^2) e^{\pm i\mu \varphi}, \quad (5.21)$$
where $C_\kappa$ is a normalization constant. Inverting transformation (4.4), the energies for the model are then given by

$$E_n(\kappa) = \frac{\hbar \alpha}{\sqrt{m}}(n + 1)(\frac{1}{2}(n + 2)\kappa - 1).$$

The energy is a linear function of the curvature parameter $\kappa$ and it depends on the combination of the two quantum numbers $2N_r + \mu$. Finally, $\Psi_{N_r, \mu}$ is well defined for both $\kappa > 0$ and $\kappa < 0$, and the degeneracy of the energy levels is the same as in the Euclidean case.

## 6 Concluding Observations

It is worth summarizing some of the physical consequences of what has been found here. The consequence of taking the metric and potential in the form chosen is that the spectrum and wavefunctions can be obtained for the system. The spherical case corresponds to the parameter $\kappa$ in the interval $\kappa > 0$. The quantum Hamiltonian describes a quantum motion of a mass $m$ on a sphere $S^2_\kappa$. The particle possesses a countable infinite set of bound states $\Psi_{N_r, \mu}$ and the energy spectrum is unbounded, not equidistant and possesses a gap between every consecutive pair that increases with $n$. The values are higher than in the Euclidean case.

The other case which has to be mentioned is the hyperbolic case $\kappa < 0$. The Hamiltonian describes quantum dynamics on the hyperbolic plane $H^2_\kappa$. In order for the wave function to be normalizable with respect to the measure $d\mu_\kappa$, the limiting behavior of the square of the wave function times the $r$-dependent factor in the measure enforces a limit on the number of bound states in this case. The energy spectrum then becomes bounded not equidistant, with a gap between every two levels that decreases. In the hyperbolic case, the energies are lower than in the Euclidean case as well.

## 7 References

[1] L. D. Landau and E. M. Lifshitz, Quantum Mechanics, (Pergamon Press Ltd, Oxford, 1977).
[2] E. Prugovecki, Quantum Mechanics in Hilbert Space, (Academic Press, New York, 1971).
[3] N. D. Birrell and P. C. Davies, Quantum Fields in Curved Space, (Cambridge University Press,
Cambridge, England, 1994).
[4] L. E. Parker and D. J. Toms, Quantum Field Theory in Curved Spacetime, (Cambridge University Press, Cambridge, England, 2009).
[5] P. Bracken, Hamiltonians for the Quantum Hall Effect on Spaces with Non-Constant Metric, Int. J. Theoretical Phys., 46, 119-132, (2007).
[6] E. Schrödinger, Proc. Roy. Irish Acad. Sect. A 46, 9 (1940).
[7] P. W. Higgs, Dynamical Symmetries in a Spherical Geometry, J. Phys. A 12, 309-323, (1979).
[8] M. Lekshmanan and S. Rajackar, Nonlinear Dynamics, Integrability, Chaos and Patterns (Springer-Verlag, Berlin, 2003).
[9] N. Woodhouse, Geometric Quantization, (Oxford University Press, Oxford, 1980).
[10] J. Sniatycki, Geometric Quantization and Quantum Mechanics (Springer, New York, 1980).
[11] I. H. McKenna and K. K. Wan, The role of the connection in geometric quantization, J. Math. Phys. 25, 1798-1803, (1984).
[12] J. F. Cariñena, M. F. Rañada and M. Santander, The quantum free particle on spherical and hyperbolic spaces: A curvature dependent approach, J. Math. Phys. 52, 072104, (2011).
[13] J. F. Cariñena, M. F. Rañada and M. Santander, The quantum Harmonic oscillator on the sphere and the hyperbolic plane: \( \kappa \)-dependent formalism, polar coordinates and hypergeomrtic functions, J. Math. Phys. 48, 102106, (2007).
[14] J. F. Cariñena, M. F. Rañada and M. Santander, A quantum exactly solvable non-linear oscillator with quasi-harmonic behavior, Ann. Phys., 322, 434-459, (2007).
[15] B. W. Char, K. O. Geddes, G. H. Gonnet, B. L. Leong, M. B. Monagen and S. M. Watt, Maple V Library Reference Manual (Springer, New York, 1991).
[16] J. E. Marsden and T. S. Ratiu, Introduction to Mechanics and Symmetry (Springer-Verlag, New York, 1994).