REPRESENTATIONS OF A $p$-ADIC GROUP IN CHARACTERISTIC $p$

G. HENNIART AND M.-F. VIGNÉRAS

Abstract. Let $F$ be a locally compact non-archimedean field of residue characteristic $p$, $G$ a connected reductive group over $F$, and $R$ a field of characteristic $p$. When $R$ is algebraically closed, the irreducible admissible $R$-representations of $G = G(F)$ are classified in [AHHV] in term of supersingular $R$-representations of the Levi subgroups of $G$ and parabolic induction; there is a similar classification for the simple modules of the pro-$p$ Iwahori Hecke $R$-algebra $H(G)_R$ in [Abe]. In this paper, we show that both classifications hold true when $R$ is not algebraically closed.

Contents

I. Introduction 1
II. Some general algebra 7
III. Classification theorem for $G$ 12
IV. Classification theorem for $H(G)$ 18
V. Applications 25
VI. Appendix: Eight inductions $\text{Mod}_R(H(M)) \to \text{Mod}_R(H(G))$ 29
References 31

I. Introduction

I.1. In this paper, $p$ is a prime number, $F$ is a locally compact non-archimedean field of residual characteristic $p$, $G$ is a connected reductive group over $F$, finally $R$ is a field; in this introduction $R$ has characteristic $p$ - except in §II.

Recent applications of automorphic forms to number theory have imposed the study of smooth representations of $G = G(F)$ on $R$-vector spaces; indeed one expects a strong relation, à la Langlands, with $R$-representations of the Galois group of $F$ - the only established case, however, is that of $GL(2, \mathbb{Q}_p)$.

The first focus is on irreducible representations. When $R$ is an algebraically closed, the irreducible admissible $R$-representations of $G$ have been classified in terms of parabolic induction of supersingular $R$-representations of Levi subgroups of $G$ [AHHV]. But the restriction to algebraically closed $R$ is undesirable: for example, in the work of Breuil and Colmez on $GL(2, \mathbb{Q}_p)$, $R$ is often finite. Here we extend to any $R$ the classification of [AHHV] and its consequences.

Let $I$ be a pro-$p$ Iwahori subgroup of $G$. If $W$ is a smooth $R$-representation of $G$, the fixed point $W^I$ is a right module over the Hecke ring $H(G)$ of $I$ in $G$; it is non-zero if $W$ is, and finite dimensional if $W$ is admissible. Even though $W^I$ might not be simple over $H(G)$ when $W$ is irreducible, it is important to study simple $R \otimes H(G)$-modules. When
Before we state our main results more precisely, let us describe our principal tool developed in section II.

The idea is to introduce an algebraic closure $R^{alg}$ of $R$, and study the scalar extension $W \mapsto R^{alg} \otimes_R W$ from $R$-representations of $G$ to $R^{alg}$-representations of $G$, or from $R \otimes H(G)$-modules to $R^{alg} \otimes H(G)$-modules. The important remark is that when $W$ is an irreducible admissible $R$-representation of $G$, or a simple $R \otimes H(G)$-module, its commutant has finite dimension over $R$. The following result examines what happens for more general extensions $R'$ of $R$.

**Theorem 1.** [Decomposition theorem] Let $R$ be a field, $A$ an $R$-algebra, and $V$ a simple $A$-module with commutant $D = \text{End}_A V$ of finite dimension over $R$. Let $E$ denote the center of the skew field $D$, $\delta$ the reduced degree of $D$ over $E$ and $E_{sep}$ the maximal separable extension of $R$ contained in $E$.

Let $R'$ be a normal extension of $R$ containing a finite separable extension of $E$ splitting $D$. Then the scalar extension $V_{R'}$ of $V$ to $R'$ has length $\delta[E : R]$ and is a direct sum

$$V_{R'} \simeq \oplus \delta \otimes_{j \in \text{Hom}_R(E_{sep}, R')} V_{R', j}$$

of $\delta$ copies of a direct sum of $[E_{sep} : R]$ indecomposable $A_{R'}$-modules $V_{R', j}$ of commutant the local artinian ring $R' \otimes_{j, E_{sep}} E$. For each $j$, $V_{R', j}$ has length $[E : E_{sep}]$, simple subquotients all isomorphic to $R' \otimes_{(R' \otimes E_{sep}) E} V_{R', j}$ of commutant $R'$, and descends to a finite extension of $R$. The $V_{R', j}$ are not isomorphic to each other and form a single $\text{Aut}_R(R')$-orbit.

The map sending $V$ to the $\text{Aut}_R(R')$-orbit of $R' \otimes_{(R' \otimes E_{sep}) E} V_{R', j}$ induces a bijection

- from the set of isomorphism classes $[V]$ of simple $A$-modules $V$ with commutant of finite dimension over $R$ (resp. $V$ of finite dimension over $R$),
- to the set of $\text{Aut}_R(R')$-orbits of the isomorphism classes $[V']$ of simple $A_{R'}$-modules $V'$ with commutant of finite dimension over $R'$ descending to a finite extension of $R$ (resp. $V'$ of finite dimension over $R'$).

Thm\[I\] implies without difficulty:

**Corollary 1.** For any extension $L/R$, the length of $V_L$ is $\leq \delta[E : R]$, and the dimension over $L$ of the commutant of any subquotient of $V_L$ is finite.

The second theorem is a criterion, inspired by [AHenV1, Lemma 3.11], for a functor to preserve the lattice $\mathcal{L}_W$ of submodules of a module $W$.

**Theorem 2.** [Lattice isomorphism] Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between abelian categories of right adjoint $G$, unit $\eta : \text{id} \rightarrow G \circ F$ and counit $\epsilon : F \circ G \rightarrow \text{id}$.

Then $F$ and $G$ induce lattice isomorphisms inverse from each other between $\mathcal{L}_W$ and $\mathcal{L}_{F(W)}$, for any object $W \in \mathcal{C}$ of finite length satisfying the properties:

a) The unit morphism $\eta_W : W \rightarrow (G \circ F)(W)$ is an isomorphism.
Theorem 3. [Lattice isomorphism and tensor product] Let $R$ be a field, $A$ an $R$-algebra, $V$ a simple $A$-module with commutant $R$, and $W$ an $R$-vector space. Then,

$$W \otimes_R V$$

is an isotypical $A$-module of type $V$,

the map $Y \mapsto Y \otimes_R V : \mathcal{L}_W \to \mathcal{L}_{W \otimes_R V}$ is a lattice isomorphism,

the map $W \to \text{Hom}_A(V, W \otimes_R V)$, sending $w \in W$ to $v \mapsto w \otimes v$ is an isomorphism. Let moreover, $b_W \in \text{End}_R(W)$, $b_V \in \text{End}_R(V)$ and a subspace $Y$ of $W$. If $Y$ is stable by $b_W$, then $Y \otimes_R V$ is stable by $b_W \otimes b_V$. Conversely, if $Y \otimes_R V$ is stable by $b_W \otimes b_V$, then $Y$ is stable by $b_W$ provided that $b_V \neq 0$.

In our applications, we have an $R$-algebra $A'$ containing $A$ and an $A'$-module $V$ which is simple with commutant $R$ as an $A$-module. We also have a basis $B'$ of $A'$ containing a basis $B$ of $A$ and elements of $B' \setminus B$ act invertibly on $V$. Moreover, we deal with $A'$-modules $W$ where elements of $B$ act as identity, and such that the tensor product action of $B'$ on $W \otimes_R V$ yields an $A'$-module.

Corollary 2. In the above situation,

The map $Y \mapsto Y \otimes_R V : \mathcal{L}_W \to \mathcal{L}_{W \otimes_R V}$ yields a lattice isomorphisms between $A'$-submodules of $W$ and $A'$-submodules of $W \otimes_R V$.

The map $W \to \text{Hom}_A(V, W \otimes_R V)$ is an isomorphism of $A'$-modules, if we let $b \in B'$ act on $\varphi \in \text{Hom}_A(V, W \otimes_R V)$ by $b \varphi = b_W \otimes b_V \circ \varphi \circ b_V^{-1}$.

Note that the natural map $\text{Hom}_A(V, W \otimes_R V) \otimes_R V \to W \otimes_R V$ is also an isomorphism of $A'$-modules if we let $b \in B'$ act by the tensor product action.

L.3. In [III] for a field $R$ of characteristic $p$, we prove the classification of the irreducible admissible $R$-representations of $G$ in terms of supersingular irreducible admissible $R$-representations of Levi subgroups of $G$.

An $R$-triple $(P, \sigma, Q)$ of $G$ consists of a parabolic subgroup $P = MN$ of $G$, a smooth $R$-representation $\sigma$ of $M$, and a parabolic subgroup $Q$ of $G$ satisfying $P \subset Q \subset P(\sigma)$, where $P(\sigma) = M(\sigma)N(\sigma)$ is the maximal parabolic subgroup of $G$ where $\sigma$, extends trivially on
N; let $\epsilon_Q(\sigma)$ denote the restriction to $M_Q$ of this extension. By definition

\begin{align}
(3.1) \quad I_G(P, \sigma, Q) &= \text{Ind}^G_{P(\sigma)}(\text{St}^M_Q(M(\sigma)(\sigma))) \\
(3.2) \quad \text{St}^M_Q(\sigma) &= \text{Ind}^M_Q(\epsilon_Q(\sigma))/\sum_{Q \leq Q' \subset P(\sigma)} \text{Ind}^M_{Q'}(\epsilon_Q(\sigma')),
\end{align}

is the generalized Steinberg $R$-representation of $M(\sigma)$ and $\text{Ind}^M_Q(\sigma)$ stands for the parabolic smooth induction $\text{Ind}^M_{Q \cap M(\sigma)}$. In [III.3] Prop.3 we show that $I_G(P, - ,Q)$ and scalar extension are compatible: for any $R$-triple $(P, \sigma, Q)$ of $G$, we have $R' \otimes_R I_G(P, \sigma, Q) \simeq I_G(P, R' \otimes_R \sigma, Q)$ for any extension $R'/R$ and $I_G(P, \sigma, Q)$ descends to a subfield of $R$ if and only if $\sigma$ does.

What supersingular means for an irreducible smooth $R$-representation $\pi$ of $G$? We know what it means to be a supersingular $H(\sigma)_R = R \otimes H(G)$-module: for all $P \neq G$, a certain central element $T_P$ of the pro-$p$ Iwahori Hecke ring $H(G)$ should act locally nilpotently [Vig99]. We say that $\pi$ is supersingular if $\pi^I$ (the $I$-invariants) is supersingular as a right $H(G)_R$-module (Definition 2 in [III.3.]). When $R$ is algebraically closed, the definition given in [AHIV] is equivalent by [OV]. In [III.4] Lemma 5 we show that supersingularity is compatible with scalar extension.

**Theorem 4.** [Classification theorem for $G$]

For any $R$-triple $(P, \sigma, Q)$ of $G$ with $\sigma$ irreducible admissible supersingular, $I_G(P, \sigma, Q)$ is an irreducible admissible $R$-representation of $G$.

If $(P, \sigma, Q)$ and $(P_1, \sigma_1, Q_1)$ are two $R$-triples of $G$ with $\sigma$ and $\sigma_1$ irreducible admissible supersingular and $I_G(P, \sigma, Q) \simeq I_G(P_1, \sigma_1, Q_1)$, then $P = P_1, Q = Q_1$ and $\sigma \simeq \sigma_1$.

For any irreducible admissible $R$-representation $\pi$ of $G$, there is a $R$-triple $(P, \sigma, Q)$ of $G$ with $\sigma$ irreducible admissible supersingular, such that $\pi \simeq I_G(P, \sigma, Q)$.

When $R$ is algebraically closed, this is the classification theorem of [AHIV]. In [III.5] we descend the classification theorem from $R^{alg}$ to $R$ by a formal proof using the decompostion theorem (Thm 1) and a lattice isomorphism $\mathcal{L}_{\sigma^{alg}} \simeq \mathcal{L}_{\sigma_R^{alg}, Q}$ when $\sigma$ is irreducible admissible supersingular of scalar extension $\sigma_R^{alg}$ to $R^{alg}$ (Prop.1 in [III.3]. Remark 11 in [III.3].)

**I.4.** In [IV] we prove a similar classification for the simple right $H(G)_R$-modules when $R$ is a field of characteristic $p$. As in [AHenV2] when $R$ is algebraically closed, this classification uses the parabolic induction functor

$$\text{Ind}_P^{H(G)} : \text{Mod}_R(H(M)) \to \text{Mod}_R(H(G))$$

from right $H(M)_R$-modules to right $H(G)_R$-modules, analogue of the parabolic smooth induction: indeed (Ind$^G_P\sigma)^I$ is naturally isomorphic to Ind$^G_P(H(M))$ for a smooth $R$-representation $\sigma$ of $G$ [OV]. An $R$-triple $(P, \mathcal{V}, Q)$ of $H(G)$ consists of parabolic subgroups $P = MN \subset Q$ of $G$ (containing $B$) and of a right $H(M)_R$-module $\mathcal{V}$ with $Q \subset P(\mathcal{V})$ (Definition 3); as for the group, it defines a right $H(G)_R$-module $I_{H(G)}(P, \mathcal{V}, Q)$.

**Theorem 5.** [Classification theorem for $H(G)$]

Any simple right $H(G)_R$-module $\mathcal{X}$ is isomorphic to $I_{H(G)}(P, \mathcal{V}, Q)$ for some $R$-triple $(P, \mathcal{V}, Q)$ of $H(G)$ with $\mathcal{V}$ simple supersingular.
For any $R$-triple $(P, V, Q)$ of $H(G)$ with $V$ simple supersingular, $I_{H(G)}(P, V, Q)$ is a simple $H(G)_R$-module.

If $(P, V, Q)$ and $(P_1, V_1, Q_1)$ are $R$-triples of $H(G)$ with $V$ and $V_1$ simple supersingular, and $I_{H(G)}(P, V, Q) \simeq I_{H(G)}(P_1, V_1, Q_1)$, then $P = P_1, Q = Q_1$ and $V \simeq V_1$.

The proof follows the same pattern as for the group $G$, by a descent to $R$ of the classification over $R^{alg}$ [AhLenV2].

In Proposition 7 we prove that $I_{H(G)}(P, -, Q)$ and scalar extension are compatible, as in the group case (Prop. 3).

Assuming that $R$ contains a root of unity of order the exponent of $Z_k$ (the quotient of the parahoric subgroup of $Z$ by its pro-$p$ Sylow subgroup), the simple supersingular $H(G)_R$-modules are classified [Oss], [VigP1wss] Thm.1.6; in particular when $G$ is semisimple and simply connected, they have dimension 1. With Theorem 5 we get a complete classification of the $H(G)_R$-modules.

Note that the ring $H(M)$ does not embed in the ring $H(G)$ and different inductions from $\text{Mod}_R(H(M))$ to $\text{Mod}_R(H(G))$ are possible. We denote $CI^{H(G)}_P : \text{Mod}_R(H(M)) \to \text{Mod}_R(H(G))$ the parabolic coinduction functor and $CI_{H(G)}(P, V, Q)$ the corresponding $H(G)_R$-module associated to an $R$-triple $(P, V, Q)$ of $H(G)$, used in [Abe]. The classification theorem (Thm. 5) can be equivalently expressed with $CI_{H(G)}(P, V, Q)$ instead of $I_{H(G)}(P, V, Q)$, as in the case where $R$ is algebraically closed [AhLenV2], Cor. 4.24. In the appendix we recall results of Abe on the different inductions $\text{Mod}_R(H(M)) \to \text{Mod}_R(H(G))$ and their relations.

I.5. In [V] we give applications (Theorems 6, 7, 8, 9) of the classification for $G$ (Thm 4) and for $H(G)$ (Thm 5); they were already known when $R$ is algebraically closed, except for parts (ii), (iii) of Theorem 7 below.

**Theorem 6.** [Vanishing of the smooth dual] The smooth dual of an infinite dimensional irreducible admissible $R$-representation of $G$ is 0.

This was proved by different methods when the characteristic of $F$ is 0 in [Kohl] and when $R$ is algebraically closed in [AhLenV2] Thm. 6.4. In [V, 1] we deduce easily the theorem from the theorem over $R^{alg}$ using the scalar extension to $R^{alg}$ (Theorem 1).

[Description of $\text{Ind}_P^G \sigma$ for an irreducible admissible $R$-representation of $M$, and of $\text{Ind}_P^{H(G)} V$ for a simple $H(M)_R$-module $V$]

We write $\mathcal{L}_\pi$ for the lattice of subrepresentations of an $R$-representation $\pi$ of $G$, and $\mathcal{L}_X$ for the lattice of submodules of an $H(G)_R$-module $X$.

Recall that for a set $X$, an upper set in $\mathcal{P}(X)$ is a set $Q$ of subsets of $X$, such that if $X_1 \subset X_2 \subset X$ and $X_1 \in Q$ then $X_2 \in Q$. Write $\mathcal{L}_{\mathcal{P}(X), \geq}$ for the lattice of upper sets in $\mathcal{P}(X)$. For two subsets $X_1, X_2$ of $X$ write $X_1 \setminus X_2$ for the complementary set of $X_1 \cap X_2$ in $X_1$.

By the classification theorems, $\sigma \simeq I_M(P_1 \cap M, \sigma_1, Q \cap M)$ with $(P_1, \sigma_1, Q)$ an $R$-triple of $G$, $Q \subset P$ and $\sigma_1$ irreducible admissible supersingular and $\mathcal{V} \simeq I_{H(M)}(P_1 \cap M, \mathcal{V}_1, Q \cap M)$ with $(P_1, \mathcal{V}_1, Q)$ an $R$-triple of $H(G)$, $Q \subset P$, and $\mathcal{V}_1$ simple supersingular.

**Theorem 7.** [Lattices $\mathcal{L}_{\text{Ind}_P^G \sigma}$ and $\mathcal{L}_{\text{Ind}_P^{H(G)} \mathcal{V}}$]

(i) $\text{Ind}_P^G \sigma$ is multiplicity free of irreducible subquotients (isomorphic to) $I_G(P_1, \sigma_1, Q')$ for $R$-triples $(P_1, \sigma_1, Q')$ of $G$ with $Q' \cap P = Q$ (notations as above).
Sending $I_G(P_1, \sigma_1, Q')$ to $\Delta_{Q'} \cap (\Delta_{P(\sigma_1)} \setminus \Delta_P)$ gives a lattice isomorphism\footnote{see the discussion in [He] and [AHBr] on the lattice of submodules of a multiplicity free module}

\[ \mathcal{L}_{\text{Ind}^G_\sigma} \to \mathcal{L}_{\text{Ind}^G_{\sigma(\Delta_{P(\sigma_1)} \setminus \Delta_P)}}. \]

(ii) The $H(G)_R$-module $\text{Ind}^{H(G)}_P V$ satisfies the analogue of (i).

(iii) If $\sigma^{\text{IM}}$ is simple and the natural map $\sigma^{\text{IM}} \otimes_{H(M)} \mathbb{Z}[(I \cap M) \setminus M] \to \sigma$ is injective, then the $I$-invariant functor $(-)^I$ gives a lattice isomorphism $\mathcal{L}_{\text{Ind}^G_\sigma} \to \mathcal{L}_{\text{Ind}^{H(G)}_H(\sigma^{\text{IM}})}$ with inverse given by $- \otimes_{H(G)_R} R[I \setminus G]$.

When $R$ is algebraically closed (i) is proved in [AHenV1] §3.2. In §V.2 we prove (i) and (ii); (iii) follows from (i), (ii), Theorem 2 and the commutativity of the parabolic inductions with $(-)^I$ and $- \otimes_{H(G)_R} R[I \setminus G]$ [OV].

**Corollary 3.** 1. The socle and the cosocle of $\text{Ind}^G_\sigma$ are irreducible; $\text{Ind}^G_\sigma$ is irreducible if and only if $P$ contains $P(\sigma_1)$. The same is true for $\text{Ind}^{H(G)}_P V$.

2. Let $\pi$ be an irreducible admissible $R$-representation of $G$; write $\pi \simeq I_G(P, \sigma, Q)$ with $\sigma$ irreducible admissible supersingular.

If $\sigma^{\text{IM}}$ is simple and the natural map $\sigma \to \sigma^{\text{IM}} \otimes_{H(M)} \mathbb{Z}[(I \cap M) \setminus M]$ is bijective, then $\pi^I$ is simple and $\pi \simeq \pi^I \otimes_{H(G)} \mathbb{Z}[I \setminus G]$.

The first assertion for $\sigma$ and $R$ is algebraically closed is [AHenV1] Cor. 3.3 and 4.4. The second assertion follows from Theorem 7 (iii).

[Computation of the left and right adjoints of the parabolic induction, of $\pi^I$ for an irreducible admissible $R$-representation $\pi$ of $G$ and of $\mathcal{X} \otimes_{H(G)} \mathbb{Z}[I \setminus G]$ for a simple $H(G)_R$-module $\mathcal{X}$]

For a parabolic subgroup $P_1$ of $G$, write $L^G_{P_1}$ and $R^G_{P_1}$ for the left and right adjoints of $\text{Ind}^G_{P_1}$, and $L^{H(G)}_{P_1}$ and $R^{H(G)}_{P_1}$ for the left and right adjoints of $\text{Ind}^{H(G)}_{P_1}$ [Vigadjoint].

There is nothing new in Theorem 8 below, now that we know that $\pi \simeq I_G(P, \sigma, Q)$ with $\sigma$ irreducible admissible supersingular, and $\mathcal{X} \simeq I_{H(G)}(P, \mathcal{V}, Q)$ with $\mathcal{V}$ simple supersingular. It suffices to quote: for $R^G_{P_1}(\pi)$ [AHenV1, Corollary 6.5], for $L^G_{P_1}(\pi)$ [AHenV1, Cor. 6.2, 6.8], for $L^{H(G)}_{P_1}(\mathcal{X})$ and $R^{H(G)}_{P_1}(\mathcal{X})$ ([Abeparim] Thm. 5.20 when $R$ is algebraically closed, but this hypothesis is not used), for $\pi^I$ and $\mathcal{X} \otimes_{H(G)} \mathbb{Z}[I \setminus G]$ [AHenV2, Thm.4.17, Thm.5.11].

**Theorem 8.** [Adjoint functors and $I$-invariant]

(i) $L^G_{P_1}(\pi)$ and $R^G_{P_1}(\pi)$ are 0 or irreducible admissible.

\[ L^G_{P_1}(\pi) \neq 0 \iff P_1 \supset P, (P_1, Q) \supset P(\sigma) \iff L^G_{P_1}(\pi) \simeq I_{M_1}(P \cap M_1, \sigma, Q \cap M_1). \]

\[ R^G_{P_1}(\pi) \neq 0 \iff P_1 \supset Q \iff R^G_{P_1}(\pi) \simeq I_{M_1}(P \cap M_1, \sigma, Q \cap M_1). \]

(ii) $L^{H(G)}_{H(M)}(\mathcal{X})$ and $R^{H(G)}_{H(M)}(\mathcal{X})$ satisfy (i) with the obvious modifications.

(iii) $\pi^I \simeq I_{H(G)}(P, \sigma^{\text{IM}}, Q)$ and $\mathcal{X} \otimes_{H(G)_R} R[I \setminus G] \simeq I_{H(G)}(P, \mathcal{V} \otimes_{H(M)_R} R[I \setminus M], Q)$.

Example: $L^G_{P(\sigma)}(I_G(P, \sigma, Q)) \simeq R^G_{P(\sigma)}(I_G(P, \sigma, Q)) \simeq \text{St}_Q M(\sigma)^\sigma$ and the analogous for $I_{H(G)}(P, \mathcal{V}, Q)$. 
An irreducible admissible $R$-representation $\pi$ of $G$ is said to be
- **supercuspidal** if it is not a subquotient of $\text{Ind}^G_P \pi$ with $\pi \in \text{Mod}^R_P(M)$ irreducible admissible for all parabolic subgroups $P = MN \subset G$.
- **cuspidal** if $L^G_P(\pi) = R^G_P(\pi) = 0$ for all parabolic subgroups $P \subset G$.

**Theorem 9.** $\pi$ is supersingular if and only if $\pi$ is supercuspidal if and only if $\pi$ cuspidal.

The equivalence of supersingular with supercuspidal, resp. cuspidal, follows immediately from Theorem 7, resp. Theorem 8. When $R$ is algebraically closed, the first equivalence was proved in [AHHV, Thm VI.2] and the second one in [AHenV1, Cor.6.9].

**Remark 1.** Let $W$ be a (left or right) $A$-module, $W_{R'}$ is an $A_{R'}$-module. An $A_{R'}$-module of the form $W_{R'}$ is said to descend to $R$.

**Remark 2.** If $V, W$ are $A$-modules, the natural map
\begin{equation}
(\text{Hom}_A(V, W))_{R'} \to \text{Hom}_{A_{R'}}(V_{R'}, W_{R'})
\end{equation}
is injective, and bijective if $R'/R$ is finite or if $V$ is a finitely generated $A$-module [BkiA8 §12, n°2, Lemme 1], [BkiA2 II prop.16 p.110].

We assume from now on in 111 that $V$ is a simple $A$-module (in particular finitely generated); we write $D$ for the commutant $\text{End}_A(V)$, so that $D$ is a division algebra, and $E$ for the center of $D$. By Remark 2 the commutant of $V_{R'}$ is $D_{R'}$ and its center is $E_{R'}$ (consider $V$ as an $A \otimes R D$-module). That $V$ is simple has further consequences:

(P1) As an $A$-module $V_{R'}$ is a direct sum of $A$-modules isomorphic to $V$, i.e. $V$-isotypic of type $V$ [BkiA8 §4, n°4, Prop.1].

(P2) The map $J \mapsto JV_{R'}$ is a lattice isomorphism of the lattice of right ideals $J$ of $D_{R'}$ onto the lattice of $A_{R'}$-submodules of $V_{R'}$, with inverse $W \mapsto \{d \in D_{R'}, dV_{R'} \subset W\}$ [BkiA8 §12, n°2, Thm.2b)].
(P3) The map \( I \mapsto ID_{R'} \) is a lattice isomorphism of the lattice of ideals \( I \) of \( E_{R'} \) onto the lattice of two-sided ideals \( J \) of \( D_{R'} \), the inverse map sending \( J \mapsto J \cap E_{R'} \) [KiiAS §12, n°4, Prop.3a)].

(P4) If \( R'/R \) is finite, \( V_{R'} \) has finite length as an \( A \)-module, so also as an \( A_{R'} \)-module; then \( D_{R'} \) is left and right artinian and \( E_{R'} \) is artinian [KiiAS §12, n°5, Prop.5]. If moreover \( R'/R \) is separable, \( V_{R'} \) is semisimple [KiiAS §12, n°5, Cor.].

(P5) If \( D \) has finite dimension over \( R \), \( D_{R'} \) has the same dimension over \( R' \), and by (P2) \( V_{R'} \) has finite length \( \leq [D : R] \) over \( A' \).

In the reverse direction:

**Lemma 1.** Let \( R'/R \) be an extension and \( V' \) a simple \( A_{R'} \)-module descending to a finite extension of \( R \). Then \( \text{Hom}_{A_{R'}}(V', V_{R'}) \neq 0 \) for some simple \( A \)-module \( V \). For any such \( V \), \( \text{dim}_R V \) is finite if \( \text{dim}_{R'} V' \) is, and \( \text{dim}_R \text{End}_A V \) is finite if \( \text{dim}_{R'} \text{End}_{A_{R'}} V' \) is.

**Proof.** a) Assume first that \( R'/R \) finite. Then \( A_{R'} \) is a (free) finitely generated \( A \)-module, so \( V' \) as an \( A \)-module is finitely generated, and in particular has a simple quotient \( V \): \( \text{Hom}_A(V', V) \neq 0 \). By Remark 2 \( \text{Hom}_{A_{R'}}(V_{R'}, V_{R'}) \neq 0 \); but \( V_{R'} \) is the sum of \([R' : R]\) copies of \( V' \) so \( \text{Hom}_{A_{R'}}(V', V_{R'}) \neq 0 \).

Let \( V \) be any simple \( A \)-module with \( \text{Hom}_{A_{R'}}(V', V_{R'}) \neq 0 \). Then by the same reasoning \( \text{Hom}_A(V', V) \neq 0 \) so \( \text{dim}_R V \) is finite if \( \text{dim}_{R'} V' \) is. Put \( D = \text{End}_A(V) \) and \( D' = \text{End}_{A_{R'}}(V') \) and let \( W \) be the maximal \( V' \)-isotypic submodule of \( V_{R'} \). Then \( W \) is \( D_{R'} \)-stable and we get an endomorphism \( D_{R'} \rightarrow \text{End}_{A_{R'}} W \) which is necessarily injective on \( D \), since \( D \) is a division algebra. By (P4), \( V_{R'} \) has finite length, so is \( W \) and \( \text{End}_{A_{R'}} W \) is a matrix algebra over \( D' \); it follows that if \( \text{dim}_{R'} D' \) is finite, so is \( \text{dim}_{R'} \text{End}_{A_{R'}} W \) hence also \( \text{dim}_R(\text{End}_{A_{R'}} W) \) and \( \text{dim}_R(D) \).

b) Let us treat the general case. By assumption there is a finite subextension \( L \) of \( R \) in \( R' \) and an \( A_L \)-module \( U \) such that \( V' = R' \otimes_L U \) - then \( U \) is necessarily simple. By a) \( \text{Hom}_{A_L}(U, V_L) \neq 0 \) for some simple \( A \)-module \( A \) and by Remark 2 \( \text{Hom}_{A_{R'}}(V', V_{R'}) \neq 0 \).

Conversely if \( V \) is some simple \( A \)-module with \( \text{Hom}_{A_{R'}}(V', V_{R'}) \neq 0 \) then by Remark 2 again \( \text{Hom}_{A_L}(U, V_L) \neq 0 \), so the other assertions follow from a).

**Remark 3.** A non-zero \( A \)-module \( W \) is called absolutely simple if \( W_{R'} \) is simple for any extension \( R'/R \). A simple \( A \)-module \( V \) of commutant \( D \) is absolutely simple if and only if \( D = R \). For \( \Rightarrow \) [KiiAS §3,n°2,Cor.2,p.44]. For \( \Leftarrow \), the commutant of \( V_{R'} \) is \( R' \) and P1) implies that \( V_{R'} \) is simple.

If \( R \) is algebraically closed of cardinal \( > \text{dim}_R V \), then \( D = R \) [KiiAS §3,n°2,Thm.1, p.43].

There exists an algebraically closed extension \( R'/R \) of cardinal \( > \text{dim}_R V \) [KiiAS §3,n°2, proof of Cor.3, p.44].

### II.2. A bit of ring theory.

**Lemma 2.** Let \( L/K \) be a field extension and \( E/K \) be a finite purely inseparable extension. Then \( L \otimes_K E \) is an artinian local ring with residue field \( L \).

**Proof.** This is probably well known but I do not know a reference. Here is a proof.

When \( E = K \), this is obvious. Assume \( E \neq K \). Then, the characteristic of \( K \) is positive, say \( p \). There is a finite filtration from \( K \) to \( E \) by subfields \( K = E_0 \subset \ldots \subset \)
$E_i \subset \ldots \subset E_n = E$ such that $E_i \simeq E_{i-1}[X]/(X^p)$. By induction on $i$, we suppose that $A_{i-1} := L \otimes_K E_{i-1}$ is an artinian local ring with residue field $L$. We show that $A_i$ has the same property. Clearly $A_i$ is an artinian commutative ring, hence a finite product of local (artinian) rings \cite[Cor. 2.16]{Eis}. We have $A_i \simeq A_{i-1}[X]/(X^p)$. The only idempotents in $A_i$ are trivial (if $P(X) \in A_{i-1}[X]$ is unitary and satisfies $P(X)^2 \equiv P(X)$ modulo $(X^p)$ then $P(X) = 1$) hence $A_i$ is local. As $A_{i-1}$ is a quotient of $A_i$ and $L$ is a quotient of $A_{i-1}$, the field $L$ is a quotient of $A_i$.

\[\square\]

II.3. Proof of the decomposition theorem (Thm 11 and Cor. 11). Let $V$ a simple $A$-module of commutant $D$ of finite dimension over $R$, $\delta^2$ the dimension of $D$ over its center $E$.

We recall that a finite extension $E'/E$ splits $D$, i.e. $E' \otimes_E D \simeq M(\delta, E')$, if and only if $E'$ is isomorphic to a maximal subfield of a matrix algebra over $D$ \cite[§15, n°3, Prop.5]{BkiAS}. We recall also that $D$ contains a maximal subfield, which a separable extension $E'/E$ of degree $\delta$ \cite[Prop 7.24]{CR} or \cite[lo.cit. and §14, n°7]{BkiAS}.

Let $R'/R$ be a normal extension containing a finite Galois extension $E'/E$ splitting $D$ (for example an algebraic closure $R_{alg}$ of $R$).

For $i \in \text{Hom}_R(E, R')$, $R' \otimes_i E$ is a quotient field of $E_{R'}$ isomorphic to $R'$ so

$$\prod_{i \in \text{Hom}_R(E, R')} R' \otimes_i E$$

is a semi-simple quotient of $E_{R'}$ of dimension $[E_{sep} : R]$. It is equal to $E_{R'}$ if and only if $E/R$ is separable: $[E : R] = [E_{sep} : R]$. By the same argument,

$$R' \otimes R E_{sep} = \prod_{j \in \text{Hom}_R(E_{sep}, R')} R' \otimes_j E_{sep} E_{sep}.$$  

Recall from \cite{112} the bijective map $i \mapsto j = [i]_{E_{sep}} : \text{Hom}_R(E, R') \rightarrow \text{Hom}_R(E_{sep}, R') = J$. Tensoring by $- \otimes_{E_{sep}} E$, we get a product decomposition

$$E_{R'} \simeq \bigoplus_{j \in J} R' \otimes_j E_{sep} E.$$  

inducing decompositions

$$D_{R'} \simeq \bigoplus_{j \in J} R' \otimes_j E_{sep} D, \quad V_{R'} \simeq \bigoplus_{j \in J} R' \otimes_j E_{sep} V,$$

where $R' \otimes_j E_{sep} D$ is the commutant of the $A_{R'}$-module $R' \otimes_j E_{sep} V$. As $R'$ contains a Galois extension $E'/E$ splitting $D$,

$$R' \otimes_j E_{sep} D \simeq M(\delta, R' \otimes_j E_{sep} E), \quad R' \otimes_j E_{sep} V \simeq \oplus \delta V_{R', j},$$

for an $A_{R'}$-submodule $V_{R', j}$ of commutant isomorphic to $R' \otimes_j E_{sep} E$. The first isomorphism implies the second one \cite[§6 n°7, cor.2, p.103]{BkiAS}. To prove the first isomorphism, we choose $g \in \text{Hom}_R(E', R')$ extending $j$ and $i$ and we compute:

$$R' \otimes_j E_{sep} D \simeq R' \otimes_j E_{sep} E \otimes_E D \simeq R'^G \otimes_R R_{sep}^i(E) \otimes_E D \simeq R'^G \otimes_R R_{sep}^i(E) \otimes g.E' \otimes E D \simeq R'^G \otimes_R R_{sep}^i(E) \otimes g.E' M(\delta, E') \simeq M(\delta, R'^G \otimes_R R_{sep}^i(E) \otimes g.E' E') \simeq M(\delta, R' \otimes_j E_{sep} E).$$

For any extension $L/R'$, we still have $\text{Hom}_R(E, L) = \text{Hom}_R(E, R') \simeq J$,

$$E_L \simeq \bigoplus_{j \in J} L \otimes_j E_{sep} E, \quad D_L \simeq \bigoplus_{j \in J} L \otimes_j E_{sep} D, \quad V_L \simeq \bigoplus_{j \in J} \oplus \delta V_{L,j},$$
with $V_{L,j} = (V_{R,j})_L$ and $\text{End}_{A_L} V_{L,j} \cong L \otimes_{j,E_{sep}} E$.

By [2, Lemma 3.12] $L \otimes_{j,E_{sep}} E$ is an Artinian local ring. We deduce that the $A_L$-module $V_{L,j}$ of commutant $L \otimes_{j,E_{sep}} E$ is indecomposable [3, §2 n°3 Prop.4] of length $[E : E_{sep}]$ and that its simple subquotients are all isomorphic to

$$L \otimes (L \otimes_{j,E_{sep}} E) V_{L,j}.$$ 

The decomposition of $D_L$ shows that there are no non-zero $A_L$-homomorphism between $V_{L,j}$ and $V_{L,j'}$ if $j \neq j'$ (also between the simple subquotients of $V_{L,j}$ and of $V_{L,j'}$).

The $A_L$-module $V_{L,j} = L \otimes_{R'} V_{R',j}$ and the $A_{R'}$-module $V_{R',j}$ have the same length, the scalar extension to $L$ of the $A_{R'}$-module $R' \otimes_{R,j,E_{sep}} E V_{R',j}$ is simple. This being true for all $L$, the simple subquotients of $V_{R'}$ are absolutely simple. The same is true for their scalar extension to $L$. Taking $R' = E'$, the $A_L$-subquotients of $V_L$ are defined over $E'$.

An $R$-automorphism $g$ of $R'$ induces an $R$-isomorphism $r' \otimes_v \rightarrow g(r') \otimes_v : R' \otimes_{j,E_{sep}} V \rightarrow R' \otimes_{g_j,E_{sep}} V$ for $i \in J$. This action which corresponds to the transitive action of $G$ on $J$, induces a transitive action of $G$ on the set of isomorphism classes $[V_{R',j}]$ of $V_{R',j}$ for $j \in J$.

So the map $[V'] \rightarrow \text{Aut}_R(R'[V'])$ where $V'$ is a simple subquotient of $V_{R'}$ is well defined. It is injective because $V'$ is seen as an $A$-module is $V$-isotypic, and it is surjective by Lemma [1]. This ends the proof of Thm [1].

Remark 4. In Thm [1] we note that the subquotients of $V_{R'}$ descend to the finite Galois extension $E'/R$.

We prove now Corollary [1].

We choose algebraic closures $L^{alg} \supset R^{alg}$ of $L \supset R$ containing a finite Galois extension $E'/R$ splitting $D$.

(i) The length of $V_L$ is less or equal to the length of $V_{L^{alg}}$ and the length of $V_{L^{alg}}$ is $\delta [E : R]$.

(ii) The length of $V_{E'}$ is $\delta [E : R]$. The commutant of any subquotient $W$ of $V_{E'}$ is contained in the commutant of $W$ seen as an $A$-module, $W$ is $V$-isotypic of finite length as an $A$-module, and the dimension of the commutant $D$ of $V$ is finite. Hence the dimension of the commutant of $W$ is finite. The scalar extension from $E'$ to $L^{alg}$ induces a lattice isomorphism $L_{V_{E'}} \rightarrow L_{V_{L^{alg}}}$, Any subquotient of $V_{L^{alg}}$ is the scalar extension $W_{L^{alg}}$ from $E'$ to $L^{alg}$ of some subquotient $W$ of $V_{E'}$. The dimension of the commutant of $W_{L^{alg}}$ is equal to the dimension of the commutant of $W$. The scalar extension $W_{L^{alg}}$ from $L$ to $L^{alg}$ of a subquotient $W'$ of $V_L$ is a subquotient of $V_{L^{alg}}$. The dimension of the commutant of $W'$ is equal to the dimension of the commutant of $W_{L^{alg}}$, hence is finite.

II.4. Proof of the lattice theorems (Theorems [2], [3], Corollary 2).

Motivated by the parabolic induction and the pro-$p$ Iwahori invariant functor when $R$ is a field of characteristic $p$, we prove the lattice theorem (Theorem [2]) which generalizes [AHenV1] Lemma 3.11] to the setting of:

- an adjunction $(F,G,\eta,\epsilon)$ where $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor between abelian categories of right adjoint $G$, $\eta : \text{id} \rightarrow G \circ F$ is the unit and $\epsilon : F \circ G \rightarrow \text{id}$ is the counit.
- $W$ is a finite length object of $\mathcal{C}$ [KS, Ex. 8.20, p. 205].

Familiar notions for modules extend to abelian categories.

Lemma 3. The partially ordered set $\mathcal{L}_W$ of subobjects of $W$ is a lattice, i.e. for any pair of subobjects $X, X'$ of $W$, the common subobjects of $X$ and of $X'$ have a largest element
Clearly, a subobject of $W$ with $X$ and $X'$ as subobjects have a smallest element $X + X'$. Writing $X \oplus X'$ for the direct sum we have an exact sequence

$$0 \to (X \cap X') \to (X \oplus X') \to X + X' \to 0.$$ 

Proof. [KS, Prop. II.5.4 Axiom 3, Notations 8.3.10, II.5. Ex. 8.20] \hfill \square

We prove now Theorem 2.

Proof. Step 1. As $G$ is left exact, it defines a map of ordered sets $\mathcal{L}_F(W) \xrightarrow{G} \mathcal{L}_{(G \circ F)}(W)$. If $Y_1 \subset Y_2 \subset W$ and $X$ is the kernel of $F(Y_1) \to F(Y_2)$, then $G(X)$ is the kernel of $(G \circ F)(Y_1) \to (G \circ F)(Y_2)$. By a) $\eta_W$ is an isomorphism, so $\eta_Y$ is an isomorphism for all $Y \subset W$. So $G(X) = 0$. By b) $X = 0$ so $F$ defines a map $\mathcal{L}_W \xrightarrow{F} \mathcal{L}_F(W)$.

The composite map $\mathcal{L}_W \xrightarrow{F} \mathcal{L}_F(W) \xrightarrow{G} \mathcal{L}_{(G \circ F)}(W)$ is an isomorphism by a). So $\mathcal{L}_W \xrightarrow{F} \mathcal{L}_F(W)$ is injective.

Step 2. We prove Step 1 and c) implies b') and c'). By Step 1 the image of a Jordan-Holder sequence of $W$ by $F$ has length $\geq \ell(W)$. It has length $\ell(W)$ and is a Jordan-Holder sequence of $F(W)$ if and only if $F(Y)$ is simple for any irreducible subquotient $Y$ of $W$, i.e. c) holds. Then $G(X)$ is simple for any simple subquotient $X$ of $F(W)$ by a). So Step 1 and c) imply b') and c').

We prove in Steps 3, 4, 5 that b') and c') imply the injectivity of $\mathcal{L}_F(W) \xrightarrow{G} \mathcal{L}_{(G \circ F)}(W)$, therefore $\mathcal{L}_W \xrightarrow{F} \mathcal{L}_F(W)$ is a lattice isomorphism of inverse $\mathcal{L}_F(W) \xrightarrow{\eta^{-1}G} \mathcal{L}_W$.

Step 3. Let $X$ be a non-zero subquotient of $F(W)$ of length $\lg(X)$. We prove by induction on the length that b') implies $\lg(G(X)) \leq \lg(X)$. Let $X \to X''$ a simple quotient of kernel $X'$. Then $G(X')$ is the kernel of $G(X) \to G(X'')$ as $G$ is left exact. By c') $G(X'')$ is simple and by induction on the length, we get $\lg(G(X)) \leq \lg(G(X')) + 1 \leq \lg(X') = \lg(X)$.

Step 4. If $X$ is a non-zero subobject of $F(W)$, we prove that c') and Step 3 imply $\lg(G(X)) = \lg(X)$. Seeing $X$ as the kernel of the quotient map $F(W) \to F(W)/X$, $G(X)$ is the kernel of the quotient map $(G \circ F)(W) \to G(F(W)/X)$ by left exactness of $G$. By Step 3, $\lg(G(X)) \leq \lg(X)$ and $\lg(G(F(W)/X)) \leq \lg(F(W)/X)$. By c'), $\ell(W) = \ell(F(W))$. So $\lg(G(X)) = \lg(X)$ and $\lg(G(F(W)/X)) = \lg(F(W)/X)$.

Step 5. Let $X, X'$ be subobjects of $F(W)$ such that $G(X) = G(X')$. We show that Step 4 implies $X = X'$. A functor between abelian categories commutes with finite direct sums [KS, II.5. Axiom A3] and a right adjoint functor is left exact [KS, II.6.20 p.137]. Applying $G$ to the exact sequence of Lemma 2, $G(X \cap X')$ is the kernel of $G(X \oplus X') = G(X) \oplus G(X') \to G(X + X')$. By Step 4 and length count, the last map is surjective. But then $G(X + X') = G(X) + G(X')$. So $G(X + X') = G(X) = G(X')$ and by length preservation $X + X' = X = X'$.

Step 6. We showed that the properties a), b) and c) imply the properties a), b') and c'). Conversely, assume the properties a), b') and c'). As $G$ is left exact and $F(W)$ has finite length by c'), b') implies that $G(X) \neq 0$ for any non-zero object of $F(W)$, hence b') and c') imply b). We showed that a), b), b'), c') imply that $\mathcal{L}_W \xrightarrow{F} \mathcal{L}_F(W)$ is a lattice isomorphism, and in particular c). Therefore, the properties a), b) and c) hold true.

Step 7. Assume that $W$ satisfies a), b) and c), or equivalently a), b') and c') by Step 6. Clearly, a subobject of $W$ has finite length satisfies a), b) and c) and a quotient $W \to W'$
has finite length and satisfies a), c); it satisfies also b') as \( F(W') \) is a quotient of \( F(W) \) and has finite length. As b') implies b), \( W' \) satisfies a), b), c). 

\[\text{Proof of Theorem 3}\] The first assertions of the theorem are \([BkiA8, \S 4\) n°4 Prop. 3, n°4 Thm. 2]. If \( Y \) is stable by \( b_W \), it is clear that \( Y \otimes_R V \) is stable by \( b_W \otimes b_V \). Conversely, assume that \( Y \otimes_R V \) is stable by \( b_W \otimes b_V \); then for \( y \in Y \) and \( v \in V \), \( b_W y \otimes b_V v \) belongs to \( W \otimes_R V \). Applying an \( R \)-linear form \( \lambda \) on \( V \) we see that \( \lambda(b_V v)b_W y \) belongs to \( Y \). If \( b_V \neq 0 \) we can choose \( v \in V \) and \( \lambda \) such that \( \lambda(b_V v) \neq 0 \) and then \( b_W y \in Y \).

\[\text{Proof of Corollary 2}\] We apply Theorem 3 with the endomorphisms \( b_W \) and \( b_V \) attached to elements \( b \) of \( B' \); if \( b \in B \), \( b_W = \text{Id}_W \) so any \( Y \) is stable by \( b_W \) and if \( b \in B' - B \) then \( b_V \neq 0 \) by hypothesis. The assertions of \( A' \)-linearity are straightforward to check on the action of \( B' \).

\[\text{Remark 5.}\] Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor between abelian categories and \( W \) a finite length object of \( \mathcal{C} \) satisfying:

\[ X \mapsto F(X) : \mathcal{L}_W \to \mathcal{L}_{F(W)} \quad \text{is a lattice isomorphism.} \]

Then any subquotient of \( W \) satisfies the same property. Indeed, this is clear for a submodule \( W' \) of \( W \), and \( \ell(W') = \ell(F(W')) \). For any exact sequence \( 0 \to W_1 \to W_2 \to W_3 \to 0 \) in \( \mathcal{C} \) with \( W_2 \) a subobject of \( W \), the sequence \( 0 \to F(W_1) \to F(W_2) \to F(W_3) \to 0 \) in \( \mathcal{D} \) is exact by length count. Let \( \mathcal{L}_{W_2}(W_1) \) denote the lattice of subobjects \( Y \) of \( W_2 \) containing \( W_1 \). The map \( Y \mapsto F(Y) : \mathcal{L}_{W_2}(W_1) \to \mathcal{L}_{F(W_2)}(F(W_1)) \) is a lattice isomorphism. Taking the cokernels, it corresponds to a lattice isomorphism \( Z \mapsto F(Z) : \mathcal{L}_{W_3} \to \mathcal{L}_{F(W_3)} \).

\[\text{Remark 6.}\] \([\text{Vigadjoint, Prop. 2.4}], [\text{KS, Thm. 8.5.8}]\):

For any adjunction \( (F, G, \eta, \epsilon) \) between two categories,

- \( F \) is fully faithful if and only if the unit \( \eta \) is an isomorphism,
- \( G \) is fully faithful if and only if the counit \( \epsilon \) is an isomorphism,
- the following equivalent properties imply that \( F, G \) are quasi-inverse of each other:
  - \( F \) and \( G \) are fully faithful,
  - \( F \) is an equivalence,
  - \( G \) is an equivalence.

III. Classification theorem for \( R \)

III.1. Admissibility, \( K \)-invariant, and scalar extension.

In this section \( R \) is any field and \( G \) is a locally profinite group. An \( R[G]\)-module \( \pi \) is smooth if \( \pi = \bigcup_K \pi^K \) with \( K \) running through the open compact subgroups of \( G \), and is admissible if it is smooth and \( \dim_R \pi^K \) is finite for all \( K \). Note that \( \text{End}_{R[G]} \pi \subset \text{End}_R \pi^K \) if \( \pi^K \) generates \( \pi \) so that \( \dim_R \text{End}_{R[G]} \pi \) is finite if \( \dim_R \pi^K \) is finite.

The category \( \text{Mod}_{R}(G) \) of \( R[G]\)-modules and the subcategory \( \text{Mod}^a_{R}(G) \) of smooth \( R[G]\)-modules are abelian, but not the additive subcategory \( \text{Mod}_{R}(G)^a \) of admissible \( R[G]\)-modules. The subcategory \( \text{Mod}^K_{R}(G) \) of \( R[G]\)-modules \( \pi \) generated by \( \pi^K \) is additive with a generator \( R[K\setminus G] \) is not abelian in general \( [3] \). The commutant of \( R[K\setminus G] \) is the Hecke \( R \)-algebra

\[ \text{End}_{R[G]} R[K\setminus G] \simeq R \otimes_{\mathbb{Z}} H(G, K) = H(G, K)_R \]

\(3\)If \( \text{Mod}^K_{R}(G) \) is abelian and \( G \) second countable, \( \text{Mod}^K_{R}(G) \) is a Grothendieck category (same proof than for \( \text{Mod}_{R}(G) \) \([\text{Vigadjoint, lemma 3.2}]\))
(no index if $R = \mathbb{Z}$). We have the abelian category $\text{Mod}_R(H(G, K))$ of right $H(G, K)_R$-modules ($H(G, K)$-modules over $R$). The functor

$$T := - \otimes_{H(G, K)} \mathbb{Z}[K\backslash G] : \text{Mod}_R(H(G, K)) \to \text{Mod}_R(G)$$

of image $\text{Mod}_R^P(G)$ is left adjoint to the $K$-invariant functor $(-)^K$.

The unit $\varepsilon : \text{id}_{\text{Mod}_R(H(G, K))} \to (-)^K \circ T$ and the counit $\eta : T \circ (-)^K \to \text{id}_{\text{Mod}_R^P(G)}$ of the adjunction correspond to the natural maps $\mathcal{X} \xrightarrow{\epsilon_{\mathcal{X}}} T(\mathcal{X})^K, \epsilon_{\mathcal{X}}(x) = x \otimes 1$ for $x \in \mathcal{X} \in \text{Mod}_R(H(G, K))$ and $T(\pi^K) \xrightarrow{\eta_{\pi}} \pi, \eta_{\pi}(v \otimes K g) = gv$ for $g \in G, v \in \pi^K, \pi \in \text{Mod}_R(G)$.

**Lemma 4.** (i) If $\pi$ is admissible and simple, then $\dim_R \text{End}_R[G] \pi$ is finite.

(ii) Let $R'$ be an extension of $R$.

$\pi$ is admissible if and only if the scalar extension $\pi_{R'}$ of $\pi$ from $R$ to $R'$ is admissible.

The adjoint functors $T, (-)^K$, the unit $\eta$ and the counit $\epsilon$ commute with scalar extension from $R$ to $R'$: $T(\mathcal{X})_{R'} \simeq T(\mathcal{X}_{R'})$, $(\pi^K)_{R'} \simeq (\pi_{R'})^K$, $(\eta_{\pi})_{R'} \simeq \eta_{\pi_{R'}}, (\epsilon_{\mathcal{X}})_{R'} \simeq \epsilon_{\mathcal{X}_{R'}}$.

**Proof.** Clear.

We deduce that if the unit (resp. counit) of the adjunction is an automorphism of $\text{Mod}_R^P(H(G, K))$ (resp. $\text{Mod}_R^P(G)$), it is an automorphism for any subfield of $R$. Recalling Remark 6.

**Lemma 5.** If the functor $(-)^K : \text{Mod}_R^P(G) \to \text{Mod}_R(H(G, K))$ over $R$ is an equivalence, then it is an equivalence for any subfield of $R$.

**Remark 7.** Assume that $R$ is a field of characteristic $p$. When $K$ is a pro-$p$-Iwahori subgroup the functor $(-)^K$ of Lemma 5 is an equivalence if $G = GL(2, \mathbb{Q}_p)$ and $p \neq 2$, or if $G = SL(2, \mathbb{Q}_p)$.

Indeed, for $GL(2, \mathbb{Q}_p)$ this is proved with the extra-hypothesis that $R$ contains a $(p - 1)$-th root of 1 ([$\Omega$] plus [K]), that we can remove with Lemma 5. For $G = SL(2, \mathbb{Q}_p)$, see [OS, Prop. 3.25].

### III.2. Decomposition Theorem for $G$.

Let $G$ be a locally profinite group and let $R$ be a field. For any irreducible admissible $R$-representation $\pi$ of $G$, its commutant $D = \text{End}_R[G] \pi$ has finite dimension (Lemma 4 (ii)), and for any extension $R'/R$, $\pi_{R'}$ is admissible (loc. cit.). Let $R^{alg}$ be an algebraic closure of $R$.

**Theorem 10.** a) Let $\pi$ be an irreducible admissible $R$-representation $\pi$ of $G$, $D$ its commutant, $E$ the center of $D$ and $\delta$ the reduced degree of $D$ over $E$. Then $\pi_{R^{alg}}$ has length $\delta[E : R]$, its simple subquotients are isomorphic to submodules; they are admissible, descend to a finite extension of $R$, their commutant is $R^{alg}$ and their isomorphism classes form a single orbit under $\text{Aut}_R(R^{alg})$.

b) The map which to $\pi$ as above associates the $\text{Aut}_R(R^{alg})$-orbit of the irreducible subquotients of $\pi_{R^{alg}}$ is a bijection from the set of (isomorphism classes of) irreducible admissible $R$-representations of $G$ onto the set of $\text{Aut}_R(R^{alg})$-orbit of (isomorphism classes of) irreducible admissible $R^{alg}$-representations of $G$ descending to some finite extension of $R$.

This is immediate from Theorem 4 if we remark that a submodule of the admissible representation $\pi_{R^{alg}}$ is also admissible. Of course we could apply the more precise results of Theorem 4 for intermediate extensions $R'/R$ and Corollary 4 (use Lemma 3):
Corollary 4. Let $L$ be an extension of $R$ and $\pi$ as in Theorem 4. Then $\pi_L$ has length $\leq \delta [E:R]$, its simple subquotients are admissible.

Proof. Let $R^{alg} \subset L^{alg}$ be algebraic closures of $R \subset L$. The scalar extension of $\pi$ in $R^{alg}$ has length $\delta [E:R]$, the irreducible subquotients of $\pi_{R^{alg}}$ are all absolutely irreducible (their commutant is $R^{alg}$). Therefore $\pi_L$ has length $\leq \delta [E:R]$. Let $\tau$ be an irreducible subquotient of $\pi_L$. Then $\tau_{L^{alg}}$ is a subquotient of $\pi_{L^{alg}}$. All the subquotients of $\pi_{R^{alg}}$ are admissible, the same is true for those of $\pi_{L^{alg}}$, hence $\tau_{L^{alg}}$ is admissible. Applying Lemma 4, $\tau$ is admissible. \hfill \Box

III.3. The representations $I_G(P, \sigma, Q)$.

From now on $G$ is a $p$-adic reductive group and $R$ is a field of characteristic $p$.

As stated in the introduction, our base field $F$ is locally compact non-archimedean of residue characteristic $p$. A linear algebraic group over $F$ is written with a boldface letter like $\bf{H}$, and its group of $F$-points by the corresponding ordinary letter $H = \bf{H}(F)$. We fix an arbitrary connected reductive $F$-group $G$, a maximal $F$-split torus $T$ in $G$ and a minimal $F$-parabolic subgroup $B$ of $G$ containing $T$; we write $Z$ for the centralizer of $T$ in $G$ and $U$ for the unipotent radical of $B$, $G^{is}$ for the product of the isotropic simple components of the simply connected cover of the derived group of $G$; the image of $G^{is}$ in $G$ is the normal subgroup $G'$ of $G$ generated by $U$, and $G = ZG'$. Let $\Phi^+$ denote the set of roots of $T$ in $U \cap M$ and $\Delta \subset \Phi^+$ the set of simple roots.

We say that $P$ is a parabolic subgroup of $G$ and write $P = MN$ to mean that $P$ is an $F$-parabolic subgroup of $G$ containing $B$, $M$ the Levi subgroup containing $Z$ and $N$ the unipotent radical; so $P = MB = MN$, the parabolic subgroups $P$ of $G$ are in bijection $P \mapsto \Delta_P$ with the subsets $\Delta$. We have $G = M(GN)$ for the normal subgroup $(G)N$ of $G$ generated by $N$. For $J \subset \Delta$ we write $M_J = M_JN_J$ for the corresponding parabolic subgroup such that $J = \Delta_{P_J}$; for a singleton $J = \{\alpha\}$ we rather write $P_{\alpha} = M_{\alpha}N_{\alpha}$. Set $P^{is}$ for the parabolic subgroup of $G^{is}$ of image $P \cap G'$ in $G$.

The smooth parabolic induction $\text{Ind}^P_M : \text{Mod}^P_{\mathbb{R}}(M) \to \text{Mod}^G_{\mathbb{R}}(G)$ is fully faithful, and admits a right adjoint $R_P^G$ and a left adjoint $L_P^G$ [Vigadjoint].

For a pair of parabolic subgroups $Q \subset P$, write $\text{Ind}^Q_M$ for $\text{Ind}_{Q \cap M}^P M$ and consider the Steinberg $R$-representation $\text{St}^Q_M(R)$ of $M$, quotient of $\text{Ind}_{Q \cap M}^P M(R) (M \cap Q$ acts trivially on $R$) by the sum $\sum_{Q'} \text{Ind}^Q_{Q'}(R)$, $Q'$ running through the parabolic subgroups of $G$ with $Q \subset Q' \subset P$. The $R$-representation $\text{St}^Q_M(R)$ of $M$ is irreducible and admissible.

Writing $\text{St}^Q_M = \text{St}^Q_M(\mathbb{Z})$, $\text{St}^Q_M(R) \simeq R \otimes_\mathbb{Z} \text{St}^M_Q$. If $P_2 = M_2N_2$ is the parabolic subgroup corresponding to $\Delta_P \setminus \Delta_Q$, the inflation to $M^{is}_2$ of the restriction of $\text{St}^Q_M$ to $M'_2$ is $\text{St}^M_{Q \cap M_2}(R)$ [AHIV II.8 Proof of Proposition and Remark]. This is true for all $R$ so $\text{St}^Q_M(R)$ as an $R$-representation of $M'_2$ is absolutely irreducible.

To an $R$-representation $\sigma$ of $M$ are associated the following parabolic subgroups of $G$:

a) $P_\sigma = M_\sigma N_\sigma$ corresponding to the set $\Delta_\sigma$ of $\alpha \in \Delta \setminus \Delta_M$ such that $Z \cap M'_\alpha$ acts trivially on $\sigma$.

b) $P(\sigma) = M(\sigma)N(\sigma)$ corresponding to $\Delta(\sigma) = \Delta_P \cup \Delta_\sigma$. There exists an extension $\epsilon(\sigma)$ to $P(\sigma)$ of the inflation of $\sigma$ to $P$; it is trivial on $N(\sigma)$, write also $\epsilon(\sigma)$ for its restriction to $M(\sigma)$ [AHIV II.7 Proposition and Remark 2]. For $P \subset Q \subset P(\sigma)$, the
generalized Steinberg representation $\mathrm{St}_Q^{M(\sigma)}(\sigma)$ of $M(\sigma)$ \[(\text{II}3.2)\] is admissible, isomorphic to $e(\sigma) \otimes \mathbb{Z} \cdot \mathrm{St}_Q$.

c) $P_{min} = M_{min} \cdot N_{min}$ is the minimal parabolic subgroup of $G$ contained in $P$ such that $\sigma$ extends an $R$-representation $\sigma_{min}$ of $P_{min}$ trivial on $N_{min}$ \cite{AllenV1}, \cite{AllenV2} \S 2.2. We have $\Delta(\sigma_{min}) = \Delta(\sigma)$, $\epsilon_Q(\sigma) = \epsilon_Q(\sigma_{min})$, $\Delta_{\sigma_{min}}$ and $\Delta_{\sigma_{min}} \setminus \Delta_{P_{min}}$ are orthogonal. So $M(\sigma) = M_{min}M'_{\sigma_{min}}$, $M_{min}$ normalizes $M'_{\sigma_{min}}$ and $e(\sigma)$ is trivial on $M'_{\sigma_{min}}$.

When $P(\sigma) = G$, write $P_{min,2} = P_{\sigma_{min}}$ so $G = M_{min}M_{min,2}$. $M_{min}$ normalizes $M'_{min,2}$. For a parabolic subgroup $Q$ of $G$ containing $Q$, the action of $M_2'$ on $e(\sigma)$ is trivial and is absolutely irreducible on $\mathrm{St}_Q^G(R)$.

**Definition 1.** An $R$-triple $(P, \sigma, Q)$ of $G$ consists of a parabolic subgroup $P = MN$ of $G$, a smooth $R$-representation $\sigma$ of $M$, a parabolic subgroup $Q$ of $G$ with $P \subset Q \subset P(\sigma)$. To an $R$-triple $(P, \sigma, Q)$ of $G$ we attach the smooth $R$-representation of $G$

$$I_G(P, \sigma, Q) = \text{Ind}_{P(\sigma)}^G(\mathrm{St}_Q^{M(\sigma)}(\sigma)).$$

The representation $I_G(P, \sigma, Q) = I_G(P_{min}, \sigma_{min}, Q)$ is admissible when $\sigma$ is admissible \cite{AllenV1} Thm.4.21.

**Proposition 1.** Let $(P, \sigma, Q)$ be an $R$-triple of $G$ with $\sigma$ admissible of finite length such that for each irreducible subquotient $\tau$ of $\sigma$, $\tau(\sigma) = P(\tau)$ and $I_G(P, \tau, Q)$ is irreducible. Then $P(\sigma) = P(\sigma')$ for any non-zero subrepresentation $\sigma'$ of $\sigma$, and $I_G(P, \tau, Q)$ induces a lattice isomorphism $\mathcal{L}_\sigma \rightarrow \mathcal{L}_{I_G(P, \sigma, Q)}$.

**Proof.** Clearly $P(\sigma) \subset P(\sigma')$. As $\sigma'$ has finite length, it contains an irreducible subrepresentation $\tau$. From $P(\sigma) \subset P(\sigma') \subset P(\tau)$ and $\tau(\sigma) = P(\tau)$, we get $P(\sigma) = P(\sigma')$.

We are in the situation of Theorem \[\text{IV}\] with $A = R[M'_\sigma] \subset A = R[M(\sigma)]$ with the basis given by $M(\sigma)$, the $R$-representations $\mathrm{St}_Q^{M(\sigma)}(R)$ and $e(\sigma)$ of $M(\sigma)$. So the natural maps

$$e(\sigma) \rightarrow \text{Hom}_{R[M'_\sigma]}(\mathrm{St}_Q^{M(\sigma)}(R), \mathrm{St}_Q^{M(\sigma)}(\sigma))$$

$$\text{Hom}_{R[M]}(\mathrm{St}_Q^{M(\sigma)}(R), \mathrm{St}_Q^{M(\sigma)}(\sigma)) \otimes_R \mathrm{St}_Q^{M(\sigma)}(\sigma) \rightarrow \mathrm{St}_Q^{M(\sigma)}(\sigma)$$

are isomorphisms of $R$-representations of $M(\sigma)$, and we have the lattice isomorphism

$$\sigma' \rightarrow \mathrm{St}_Q^{M(\sigma)}(\sigma') : \mathcal{L}_\sigma \rightarrow \mathcal{L}_{\mathrm{St}_Q^{M(\sigma)}(\sigma')}.$$  

As \cite{Vigad} the functor $\text{Ind}_{P(\sigma)}^G \circ \text{Mod}_R(M(\sigma)) \rightarrow \text{Mod}_R(G)$ is exact fully faithful of right adjoint $R_{P(\sigma)}^G$, the unit of the adjunction is an isomorphism (Remark \[\text{III}\]). The length $\mathrm{St}_Q^{M(\sigma)}(\sigma)$ is equal to the (finite) length of $\sigma$, and its irreducible subquotients are $\mathrm{St}_Q^{M(\sigma)}(\tau)$ for the irreducible subquotients $\tau$ of $\sigma$; if $\text{Ind}_{P(\sigma)}^G(\mathrm{St}_Q^{M(\sigma)}(\tau)) = I_G(P, \tau, Q)$ is irreducible for all $\tau$, we are in the situation of Theorem \[\text{II}\] for $F = \text{Ind}_{P(\sigma)}^G$ and $W = \mathrm{St}_Q^{M(\sigma)}(\sigma)$, so the map $\sigma' \rightarrow I_G(P, \sigma', Q) : \mathcal{L}_\sigma \rightarrow \mathcal{L}_{I_G(P, \sigma, Q)}$ is a lattice isomorphism. \(\square\)

**Remark 8.** $I_G(P, \sigma, Q)$ determines the isomorphism class of $e(\sigma)$ because

$$e(\sigma) \simeq \text{Hom}_{R[M'_\sigma]}(\mathrm{St}_Q^{P(\sigma)}(R), R_{P(\sigma)}^G(I_G(P, \sigma, Q))).$$

(proof of Prop. \[\text{II}\] and $R_{P(\sigma)}^G(I_G(P, \sigma, Q)) \simeq \mathrm{St}_Q^{P(\sigma)}(\sigma)$).
We check now that the different steps of the construction of $I_G(P, \sigma, Q)$ commute with scalar extension.

Recalling that, for any commutative rings $R \subset R'$, the scalar extension from $R$ to $R'$ is the left adjoint of the scalar restriction from $R'$ to $R$ and that for a field extension $R'/R$ of characteristic $p$, the functor $\text{Ind}_F^G$ is fully faithful, we have:

**Proposition 2.** (i) For any commutative rings $R \subset R'$, the parabolic induction commutes with the scalar restriction from $R'$ to $R$ and with the scalar extension from $R$ to $R'$. Hence the left (resp. right) adjoint of the parabolic induction commutes with scalar extension (resp. restriction).

(ii) Let $R'/R$ be an extension of fields of characteristic $p$. Let $\sigma' \in \text{Mod}^F_R(M)$ and $\pi \in \text{Mod}^G_R(G)$ of scalar extension $\pi_{R'}$ from $R$ to $R'$ isomorphic to $\text{Ind}_F^G(\sigma')$. Then $\sigma'$ is isomorphic to the scalar extension $(L^G_{\pi})_{R'}$ of $L^G_{\pi}$ from $R$ to $R'$.

**Remark 9.** On admissible representations $R^G_P$ is the Emerton’s ordinary functor. We do not know if the ordinary functor admits a right adjoint or if it commute with scalar extension.

**Proposition 3.** [Strong compatibility of $I_G(P, - , Q)$ with scalar extension]

Let $(P, \sigma, Q)$ be an $R$-triple of $G$.

(i) Let $R'/R$ be an extension. Then $P(\sigma) = P(\sigma')$, $(P, \sigma, Q)$ is an $R'$-triple of $G$. If $\sigma$ is irreducible and $\sigma'$ is a non-zero subquotient of $\sigma_{R'}$, then $P(\sigma) = P(\sigma')$.

$(e(\sigma))_{R'} = e(\sigma_{R'})$, $\text{St}_Q^P(\sigma)_{R'} \simeq \text{St}_Q^P(\sigma_{R'})$ and $I_G(P, \sigma, Q)_{R'} \simeq I_G(P, \sigma, Q)$.

(ii) Let $R'$ a subfield of $R$. If $e(\sigma)$ or $\text{St}_Q^P(\sigma)$ or $I_G(P, \sigma, Q)$ descends to $R'$, then $\sigma$ descends to $R'$.

If $e(\sigma)$, resp. $\text{St}_Q^P(\sigma)$, resp. $I_G(P, \sigma, Q)$, is the scalar extension of an $R'$-representation $\tau'$, resp. $\rho'$, resp. $\pi'$, then $\sigma$ is the scalar extension of the natural $R'$-representation of $M$ on $\tau'$, resp. $\text{Hom}_{R'M'}(\text{St}_Q^P(\sigma)(R), \rho')$, resp. $\text{Hom}_{R'M'}(\text{St}_Q^P(\sigma)(R), L^G_{\pi'(\pi)})$.

**Proof.** (i) As an $R$-representation, $\sigma_{R'}$ is the direct sum of $R$-representations isomorphic to $\sigma$. If $\sigma$ is irreducible, any subquotient $\sigma'$ of $\sigma_{R'}$ is $\sigma$-isotypic. For $\alpha \in \Delta - \Delta_P$, $Z \cap M'$ acts trivially on an $R'$-representation $\tau$ if and only if it acts trivially on $\tau$ seen as an $R$-representation. So $P(\sigma) = P(\sigma_{R'})$ (hence $(P, \sigma, Q)$ is an $R'$-triple of $G$), and if $\sigma$ is irreducible $P(\sigma) = P(\sigma')$. It is clear from the definition that the extension commutes with scalar extension $(e(\sigma))_{R'} = e(\sigma_{R'})$.

(ii) If $I_G(P, \sigma, Q) = \pi_{R'}$, we have $\text{St}_Q^P(\sigma) \simeq (L^G_{\pi')(\pi})_{R'}$ (Proposition 2). If $\text{St}_Q^P(\sigma) \simeq \rho_{R'}$, we have $e(\sigma) \simeq (\text{Hom}_{R'M'}(\text{St}_Q^P(\sigma)(R), \rho')_{R'})$ because $e(\sigma) \simeq \text{Hom}_{R'M'}(\text{St}_Q^P(\sigma)(R', \rho')_{R'})$ (proof of Prop. 1). If $e(\sigma) \simeq \tau_R$ then $\sigma \simeq (\tau_{|M})_R$ because the restriction to $M$ commutes with scalar extension.

**III.4. Supersingular representations.**

In the setting of III.3 with an algebraically closed field $R$ of characteristic $p$, the definition of supersingularity for an irreducible admissible $R$-representation of $G$ [AHIV] uses the Hecke algebras of the irreducible smooth $R$-representations of the special parahoric subgroups of $G$. It is shown in [OV] that this definition is equivalent to a simpler one using
only the pro-$p$ Iwahori Hecke $R$-algebra of $G$. This latter definition has the advantage to extend easily to the non-algebraically closed case.

Let $R$ be a field of characteristic $p$. Let $I$ be a pro-$p$ Iwahori subgroup of $G$ compatible with $B$, so that $I \cap M$ is a pro-$p$ Iwahori subgroup of $M$ for a parabolic subgroup $P = MN$ (recall that $M$ contains $Z$, and that the $M$ are parametrized by the subsets $J = \Delta_M$ of $\Delta$). The pro-$p$ Iwahori Hecke ring $H(G, I) = H(G)$, the pro-$p$ Iwahori Hecke $R$-algebra $H(G)_R$, the categories $\text{Mod}_R(H(G))$ and $\text{Mod}_{\infty}^\infty(G)$ are defined as in $\S$ III.1.

The group $Z_1 = I \cap Z$ is the pro-$p$ Sylow subgroup of the unique parahoric subgroup $Z_0$ of $Z$ and $Z_k = Z_0/(I \cap Z)$ is finite of order prime to $p$. The elements in $H(G)$ with support in $G'$ form a subring $H(G')$ normalized by a subring of $H(G)$ isomorphic to $\mathbb{Z}[\Omega]$ for a commutative finitely generated subgroup $\Omega$, and

$$H(G) \simeq H(G')\mathbb{Z}[\Omega], \ H(G') \cap \mathbb{Z}[\Omega] \simeq \mathbb{Z}[Z'_k], \ Z'_k = (Z_0 \cap G')/(I \cap G').$$

To $M$ is associated a certain element $T_M$ in $H(G')$ which is central in $H(G)$ $\text{VigpIwss}$.  

**Definition 2.**  1. A (right) $H(G)_R$-module $\mathcal{X}$ is called supersingular if, for all $M \neq G$, the action of $T_M$ on $\mathcal{X}$ is locally nilpotent.

2. A smooth irreducible $R$-representation $\pi \in \text{Mod}^\infty_R(G)$ is called supersingular if the right $H(G)_R$-module $\pi^I \in \text{Mod}_R(H(G))$ is supersingular.

When $\pi$ is admissible, the definition is equivalent to the definition of $\text{AHHV}$ by $\text{OV}$.

**Remark 10.**  1. Let $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$ be an exact sequence of $H(G)_R$-modules. Then $\mathcal{V}$ is supersingular if and only if $\mathcal{V}'$ and $\mathcal{V}''$ are supersingular.

2. When $R$ contains a root of unity of order the exponent of $Z_k$, the simple supersingular $H(G)_R$-modules are classified $\text{VigpIwss}$ Thm. 6.18. As $H(G')_R$-modules, they are sums of supersingular characters.

3. The group $\text{Aut}(R)$ of automorphisms of $R$ acts on $\text{Mod}_R(G)$ and on $\text{Mod}_R(H(G))$. Clearly the action of $\text{Aut}(R)$ commutes with the $I$-invariant functor, and respects supersingularity, irreducibility, and admissibility.

We check easily that supersingularity commutes with scalar extension:

**Lemma 6.** Let $R'/R$ an extension, $\mathcal{X} \in \text{Mod}_R(H(G))$, $\pi \in \text{Mod}^\infty_R(G)$ irreducible and $\pi'$ an irreducible subquotient of $\pi_{R'}$. Then $\mathcal{X}$ is supersingular if and only if $\pi_{R'}$ is, and $\pi$ is supersingular if and only if $\pi'$ is.

**Proof** As an $H(G)_R$-module, $\mathcal{X}_{R'}$ is a direct sum of modules isomorphic to $\mathcal{X}$. As an $R$-representation of $G$, $\pi'$ is a direct sum of representations isomorphic to $\pi$. $\square$.

**Remark 11.** Let $\sigma$ be an irreducible supersingular $R$-representation of $M$. The scalar extension $\sigma_{R^{alg}}$ satisfies the conditions of Proposition $\text{[AHHV]}$ all irreducible subquotients $\tau$ of $\sigma_{R^{alg}}$ are supersingular (Lemma $\text{[AHHV]}$), $P(\tau) = P(\sigma) = P(\sigma_{R^{alg}})$ (Prop $\text{[AHHV]}$ (i)), and $I_G(P, \tau, Q)$ is irreducible (Classification theorem for $G$ over $R^{alg}$ $\text{AHHV}$).

**III.5. Classification of irreducible admissible $R$-representations of $G$.** Let $R$ be a field of characteristic $p$ of algebraic closure $R^{alg}$. We prove in this section the classification theorem for $G$ (Theorem $\text{[AHHV]}$). The arguments are formal and rely on the properties:

1. The decomposition theorem for $G$ (Thm. $\text{[AHHV]}$).
2. The classification theorem for $G$ (Thm. $\text{[AHHV]}$) over $R^{alg}$ $\text{AHHV}$. 

3 The compatibility of the scalar extension to $R^{\text{alg}}$ with supersingularity (Lem$[\text{I}_]$ and the strong compatibility with $I_G(P, -, Q)$ (Prop$[\text{III}_.])$.

4 The lattice isomorphism $L_{\sigma_{\text{R}^{\text{alg}}}} \rightarrow L_{I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q)}$ for the scalar extension $\sigma_{\text{R}^{\text{alg}}}$ to $R^{\text{alg}}$ of an irreducible admissible supersingular $R$-representation $\sigma$ (Prop$[\text{I}_]$ and Rem$[\text{III}_.]$).

We start the proof with an arbitrary irreducible admissible $R$-representation $\pi$ of $G$. By the decomposition theorem for $G$, the scalar extension $\pi_{\text{R}^{\text{alg}}}$ has finite length; we choose an irreducible subrepresentation $\pi_{\text{R}^{\text{alg}}}^{\text{alg}}$ of $\pi_{\text{R}^{\text{alg}}}$. By the decomposition theorem for $G$, $\pi_{\text{R}^{\text{alg}}}^{\text{alg}}$ is admissible, descends to a finite extension of $R$. By the classification theorem over $R^{\text{alg}}$,

$$\pi_{\text{alg}} \simeq I_G(P, \sigma_{\text{alg}}, Q)$$

for an $R^{\text{alg}}$-triple $(P = MN, \sigma_{\text{alg}}, Q)$ of $G$ with $\sigma_{\text{alg}}$ irreducible admissible supersingular. By the strong compatibility of $I_G(P, -, Q)$ with scalar extension, $\sigma_{\text{alg}}$ descends to a finite extension of $R$. By the decomposition theorem for $G$, $\sigma_{\text{alg}}$ is contained in the scalar extension $\sigma_{\text{R}^{\text{alg}}}$ of an irreducible admissible $R$-representation $\sigma$. By compatibility of the scalar extension with supersingularity and $I_G(P, -, Q)$, $(P, \sigma, Q)$ is an $R$-triple of $G$, $\sigma$ is supersingular and $I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q) \simeq I_G(P, \sigma, Q)_{\text{R}^{\text{alg}}}$. By the lattice isomorphism $L_{\sigma_{\text{R}^{\text{alg}}}} \rightarrow L_{I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q)}$, $I_G(P, \sigma_{\text{alg}}, Q)$ is contained in $I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q)$. As an irreducible subquotient $\pi_{\text{R}^{\text{alg}}}^{\text{alg}}$ of $\pi_{\text{R}^{\text{alg}}}$ is isomorphic to an irreducible subquotient $I_G(P, \sigma_{\text{alg}}, Q)$ of $I_G(P, \sigma, Q)_{\text{R}^{\text{alg}}} \simeq I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q)$, the decomposition theorem for $G$ implies that

$$\pi \simeq I_G(P, \sigma, Q).$$

Conversely, let $(P = MN, \sigma, Q)$ be an $R$-triple of $G$ with $\sigma$ irreducible admissible supersingular. We show that $I_G(P, \sigma, Q)$ is irreducible. By the decomposition theorem for $M$, $\sigma_{\text{R}^{\text{alg}}}$ has finite length, $I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q)$ also by the lattice isomorphism $L_{\sigma_{\text{R}^{\text{alg}}}} \rightarrow L_{I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q)}$, $I_G(P, \sigma, Q)_{\text{R}^{\text{alg}}} \simeq I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q)$ and the scalar extension is faithful and exact, hence $I_G(P, \sigma, Q)$ has also finite length. Let $\pi$ be an irreducible $R$-subrepresentation of $I_G(P, \sigma, Q)$. As $I_G(P, \sigma, Q)$ is admissible, $\pi$ is admissible. The scalar extension $\pi_{\text{R}^{\text{alg}}}$ is isomorphic to a subrepresentation of $I_G(P, \sigma, Q)_{\text{R}^{\text{alg}}} \simeq I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q)$. By the lattice isomorphism $L_{\sigma_{\text{R}^{\text{alg}}}} \rightarrow L_{I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q)}$, $\pi_{\text{R}^{\text{alg}}} \simeq I_G(P, \rho, Q)$ for a subrepresentation $\rho$ of $\sigma_{\text{R}^{\text{alg}}}$. The representation $\rho$ descends to $R$ because $I_G(P, \rho, Q)$ does, by the strong compatibility of $I_G(P, -, Q)$ with scalar extension. But $\sigma_{\text{R}^{\text{alg}}}$ has no proper subrepresentation descending to $R$ by the decomposition theorem for $G$, so $\rho = \sigma_{\text{R}^{\text{alg}}}$ and $\pi_{\text{R}^{\text{alg}}} = I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q) \simeq I_G(P, \sigma, Q)_{\text{R}^{\text{alg}}}$, or equivalently, $\pi \simeq I_G(P, \rho, Q)$.

Finally, let $(P, \sigma, Q)$ and $(P_1, \sigma_1, Q_1)$ be two $R$-triples of $G$ with $\sigma, \sigma_1$ irreducible admissible supersingular and $I_G(P, \sigma, Q) \simeq I_G(P_1, \sigma_1, Q_1)$. By scalar extension $I_G(P, \sigma_{\text{R}^{\text{alg}}}, Q) \simeq I_G(P_1, (\sigma_1)_{\text{R}^{\text{alg}}}, Q_1)$. The classification theorem over $R^{\text{alg}}$ implies $P = P_1, Q = Q_1$ and some irreducible subquotient $\sigma_{\text{alg}}$ of $\sigma_{\text{R}^{\text{alg}}}$ is isomorphic to some irreducible subquotient $\sigma_{\text{alg}}^1$ of $(\sigma_1)_{\text{R}^{\text{alg}}}$. As $R$-representations of $G$, $\sigma_{\text{alg}}$ and $\sigma_{\text{alg}}^1$ are $\text{isotypic}$ and $\sigma_{\text{alg}}^1$ is $\sigma_1$-isotypic, hence $\sigma, \sigma_1$ are isomorphic. This ends the proof of the classification theorem for $G$ (Theorem$[\text{III}_.]$).

IV. Classification theorem for $H(G)$

As in $\text{III}.$ $G$ is a $p$-adic reductive group and $R$ is a field of characteristic $p$. As in $\text{III}.$ $I$ is a pro-$p$ Iwahori subgroup of $G$ compatible with $B$, $H(G)$ is the pro-$p$ Iwahori Hecke ring, $H(G)_R$ the pro-$p$ Iwahori Hecke $R$-algebra, $Z_1$ the pro-$p$ Sylow of the unique
parahoric subgroup $Z_0$ of $Z$, $Z_k = Z_0/Z_1$. In this section we prove for the right $H(G)_R$-modules the results analogous to those of Section IV. Although $H(G)$ and $G$ are related by the $I$-invariant functor or its left adjoint, this relation in characteristic $p$ does not satisfy the same properties than in the complex case and does not permit to deduce the case of the pro-$p$ Iwahori Hecke algebra from the case of the group: the similar results for $H(G)$ and $G$ have to be proved separately.

IV.1. Pro-$p$ Iwahori Hecke ring.

The center $Z(H(G))$ of the pro-$p$ Iwahori Hecke ring $H(G)$ is a finitely generated ring and $H(G)$ is a finitely generated module over its center; the same is true for the $R$-algebra $H(G)_R$ and its center $Z(H(G)_R) = Z(H(G))_R$ [Vigplw]. This implies that the dimension over $R$ of a simple $H(G)_R$-module is finite [H1 2.8 Prop.].

Let $P = MN$ be a parabolic subgroup of $G$ (containing $B$ as in III.3). The pro-$p$ Iwahori Hecke ring $H(M)$ for the pro-$p$ Iwahori subgroup $I_M = I \cap M$ of $M$ (III.3) does not embed in the ring $H(G)$. However we are in the good situation where $H(M)$ is a localization of a subring $H(M^+)$ (of elements supported in the monoid $M^+ := \{m \in M \mid m(I \cap N)m^{-1} < I \cap N\}$) which embeds in $H(G)$. We explain this in more details after introducing some notations [Vigplw].

An index $M$ indicates an object defined for $M$; for $G$ we suppress the index. We write $N_M$ for the $F$-points of the normalizer of $T$ in $M$, $W_M = N_M/Z_1$, $W_{M'}$ for the image of $M' \cap N_M$ in $W_M$, $\Lambda = Z/Z_1$, $\ell_M$ for the length of $W_M$, $\Omega_M$ for the image in $W_M$ of the $m \in N_M$ normalizing $I_M$; $\Omega_M$ is the set of $u \in W_M$ of length $\ell_M(u) = 0$.

The natural map $W_M \to I_M \setminus M/I_M$ is bijective, $W_M = W_{M'}\Omega_M, W_{M'}\cap \Omega_M = W_{M'}\cap Z_k$, $W_{M'}$ is a normal subgroup $W_M$ and a quotient of $W_{M'}$ (via the quotient map $M/\supseteq M'$).

For $m \in M$ and $w = w(m) \in W_M$ image of $m \in N_M$ such that $I_M m I_M = I_M m_1 I_M$ (denoted also $I_M w I_M$), the characteristic function of $I_M m I_M$ seen as an element of $H(M)$ is written $T^M(m) = T^M(w)$. We have also the elements $T^{M,\ast}(m) = T^{M,\ast}(w)$ in $H(M)$ defined by $T^{M,\ast}(w) T^{M,\ast}(w^{-1}) = [I_M w I_M : I_M]$ [Vigplw Prop.4.13]. For $u \in \Omega_M$, $T^\ast(u) = T(u)$ is invertible of inverse $T(u^{-1})$. The $\mathbb{Z}$-module $H(M)$ is free of natural basis ($T^M(w))_{w \in W_M}$ and of $\ast$-basis ($T^{M,\ast}(w))_{w \in W_M}$. For the subring $H(M')$, the same is true with $W_{M'}$. The natural basis and the $\ast$-basis satisfy the same braid relations for $w_1, w_2 \in W_M, \ell_M(w) = \ell_M(w_1) + \ell_M(w_2)$ and the same quadratic relations with a change of sign for $s \in W_{M'}, \ell_M(s) = 1$:

$T^M(w_1) T^M(w_2) = T^M(w_1 w_2), T^{M,\ast}(w_1) T^{M,\ast}(w_2) = T^{M,\ast}(w_1 w_2)$,

$T^M(s)^2 = q_s + c_s T^M(s), T^{M,\ast}(s)^2 = q_s - c_s T^{M,\ast}(s)$,

where $q_s = [I_M s I_M : I_M] \equiv 0$ modulo $p$ and $c_s \in H(Z_0 \cap M')$ (the subring of elements supported on $Z_0$), $c_s \equiv -1$ modulo the ideal of $H(Z_0 \cap M')$ generated by $p$ and $T(u) - 1$ for $u \in Z_k \cap W_{M'}$ [Vigplw]. Both $q_s$ and $c_s$ do not depend on $M$ but $\ell_M$ depends on $M$. The quotient map $W_{M'} \to W_{M'}$ respects the length and the coefficients of the quadratic relations, the surjective natural linear map $H(M') \to H(M')$ is a ring homomorphism sending $T^{M,\ast}(w)$ to $T^M(w')$ and $T^{M,\ast,\ast}(w)$ to $T^{M,\ast,\ast}(w')$ if $w \in W_{M'}$ goes to $w' \in W_{M'}$ by the quotient map.

The injective linear maps associated to the bases

\[ T^M(m) \mapsto T(m) : H(M) \xrightarrow{\theta^G_{H(G)}} H(G), \quad T^{M,\ast}(m) \mapsto T^\ast(m) : H(M) \xrightarrow{\theta^G_{H(G)}} H(G), \]
generally do not respect the product but their restrictions to the subrings \(H(M^+)\) and \(H(M^-)\) (of elements supported on the inverse monoid \(M^-\) of \(M^+)\) do.

**Remark 12.** 1. For \(P \subset Q = M_QN_Q\), we have inclusions: if \(\epsilon \in \{+,-\}\), then

\[ M^\epsilon \subset M_Q^\epsilon \subset M_Q^\epsilon \cap M_Q^\epsilon \cap (H(M^\epsilon)) \subset \theta_{M^\epsilon}^G(H(M^\epsilon)) = \theta_{M^\epsilon}^{G*}(H(M^\epsilon)) \subset \theta_{M^\epsilon}^{G*}(H(M^\epsilon)). \]

2. When \(\Delta_M\) is orthogonal to \(\Delta \setminus \Delta_M\) the situation is simpler. Denoting by \(P_2 = M_2N_2\) the parabolic subgroup of \(G\) corresponding to \(\Delta \setminus \Delta_M\), we have: \(G'\) is the direct product of \(M'\) and of \(M'_2\), \(M' \subset M^+\) and \(M'_2 \subset M^+_2\), \(G = MM'_2\), \(W = W_{M'_2}W_{M^+}\Omega\) and \(W_{M'_2} \cap W_{M^+}\Omega = W_{M'_2} \cap Z_k\), the length \(\ell_G = \ell_M\) on \(W_{M'}\) and to \(\ell_{M'_2}\) on \(W_{M'_2}\). For \(w \in W_{M'}, w_2 \in W_{M'_2}, u \in \Omega, \ell(w) + \ell(w_2) = \ell(w_2u)\). The braided quadratic relations satisfied by \(T(w)\) for \(w \in W_M\) are the same than for \(T^M(w)\), also for the **basis**, and for \(M_2\). The maps \(\theta_{M_2}^G\) and \(\theta_{M'_2}^{G*}\) (resp. \(\theta_{M_2}^G\) and \(\theta_{M'_2}^{G*}\)) are equal, respect the product, and \(H(M') \times H(M'_2) \xrightarrow{\theta_{M_2}^G \times \theta_{M'_2}^{G*}} H(G')\) is a ring isomorphism.

**IV.2. Parabolic induction** \(\text{Ind}_P^{H(G)}\).

The parabolic induction functor:

\[
(2.4) \quad \text{Ind}_P^{H(G)} := - \otimes_{H(M^+), \theta_M^G} H(G) : \text{Mod}_R(H(M)) \rightarrow \text{Mod}_R(H(G))
\]
corresponds via the pro-\(p\) Iwahori invariant functor to \(\text{Ind}_P^{G} : \text{Mod}_R^G(M) \rightarrow \text{Mod}_R^G(G)\) [OV, Prop.4.4]:

\[
(-)^I \circ \text{Ind}_P^{G} \simeq \text{Ind}_P^{H(G)} \circ (-)^{\cap M} : \text{Mod}_R^G(M) \rightarrow \text{Mod}_R^G(H(G)).
\]

The parabolic induction functor \(\text{Ind}_P^{H(G)}\) has a right adjoint \(R_P^{H(G)}\) and a left adjoint \(L_P^{H(G)}\) [VigpIwst]. The right adjoint functors of \(\text{Ind}_P^{G}\) and of \(\text{Ind}_P^{H(G)}\) correspond via pro-\(p\) Iwahori invariant functor

\[
(-)^{\cap M} \circ R_P^{G} \simeq R_P^{H(G)} \circ (-)^I : \text{Mod}_R(G) \rightarrow \text{Mod}_R(H(M)),
\]
but the left adjoint functors do not [OV]. As \(- \otimes_{H(M^+), \theta} H(G) \simeq \text{Hom}_{H(M^+), \theta*}(H(G), -)\) (Proposition 8 in the appendix below), the left and right adjoints of \(\text{Ind}_P^{H(G)}\) are

\[
(2.5) \quad L_P^{H(G)} \simeq - \otimes_{H(M^+), \theta^{G*}} H(M), \quad R_P^{H(G)} = \text{Hom}_{H(M^+), \theta^{G*}}(H(M), -).
\]

**Remark 13.** For the pro-\(p\) Iwahori Hecke algebra, the left adjoint \(L_P^{H(G)}\) being a localization is exact but for the group, the left adjoint \(L_P^{G}\) is not.

**Proposition 4.** Let \(R\) be a field of characteristic \(p\). For two parabolic subgroups \(P = MN, P_1 = M_1N_1\) of \(G\) (containing \(B\)),

(i) \(R_P^{H(G)} \circ \text{Ind}_P^{H(G)} \simeq \text{Ind}_{P \cap P_1}^{H(M)} \circ R_{P \cap P_1}^{H(M)}\).

(ii) \(L_{P_1}^{H(G)} \circ \text{Ind}_P^{H(G)} \simeq \text{Ind}_{P \cap P_1}^{H(M)} \circ L_{P \cap P_1}^{H(M)}\).

(iii) The parabolic induction \(\text{Ind}_P^{H(G)}\) is fully faithful.

**Proof.** (i) is equivalent to the same relation for the parabolic coinduction and its right adjoint, which is proved in [Abeparind, Prop. 5.1]. What we call parabolic coinduction is denoted by \(I_P\) in [Abeparind, §4] (and called parabolic induction). The equivalence follows from the isomorphism [VigpIwst, Thm.1.8], [Abeparind Prop.2.21]:

\[
I_{P \cap P_1} \circ n_{w_{M\cap G}} \simeq \text{Ind}_P^{H(G)}
\]
where

\( w \mapsto n_w : W \to W \) is an injective homomorphism from the Weyl group \( W \) of \( \Delta \) to \( W \) satisfying the braid relations (there is no canonical choice), \( w_M \) is the longest element of the Weyl group of \( \Delta_M = \Delta_P \) for any parabolic subgroup \( P = MN \) of \( G \).

\( P^P = M^P N^P \) denotes the parabolic subgroup of \( G \) (containing \( B \)) with \( \Delta_{P^P} = \Delta_{P^P} = w_G w_P (\Delta_P) = w_G (\Delta_P) \) (image of \( \Delta_P \) by the opposition involution [1 1.5.1]).

The ring isomorphism [Abc §4.3]

\[
H(M) \to H(M^{op}) \quad T^M_w \mapsto T^{M^{op}}_{n_w G w_M w w_G = M} \quad \text{for } w \in W_M,
\]

induces by functoriality a functor \( \text{Mod}_R(H(M^{op})) \xrightarrow{n_{w G w_M}(-)} \text{Mod}_R(H(M)) \) of inverse \( \text{Mod}_R(H(M)) \xrightarrow{n_{w M w_G(-)}} \text{Mod}_R(H(M^{op})), \) as \( n_{w G w_M} = n_{w M w_G} = n_w^{-1} \).

(ii) This follows from (i) by left adjunction and exchanging \( P, P_i \).

(iii) This follows from (ii) when \( P_1 = P \). As the functorial morphism \( L_P^H(G) \circ \text{Ind}_P^H(G) \to \text{id} \) is an isomorphism, \( \text{Ind}_P^H(G) \) is fully faithful [Vigad"ojnt Prop.2.4].

**Remark 14.** The extension to \( H(M(V)) \) gives a lattice isomorphism \( \mathcal{L}_V \to \mathcal{L}_{\text{id}(V)} \).

**Lemma 7.** Assume that \( \Delta_M \) is orthogonal to \( \Delta \setminus \Delta_M \) and let \( P_2 = M_2 N_2 \) correspond to \( \Delta_2 := \Delta \setminus \Delta_M \).

For any right \( H(G)_R \)-module \( X \) extending an \( H(M)_R \)-module and any right \( H(G)_R \)-module \( Y \) extending an \( H(M)_R \)-module, there is a structure of right \( H(G)_R \)-module on

- \( X \otimes_R Y \) where the * basis of \( H(G) \) acts diagonally,
- \( \text{Hom}_{(H(M)_2)}(\mathcal{Y}, X \otimes_R Y) \), where \( T^*(w) \) acts by \( (T^*(w)X) \otimes (T^*(w)Y) = (T^*(w)X) \otimes (T^*(w)Y)^{-1} \)

for \( w \in W_{M_2} \) and trivially for \( w \in W_{M_2} \).

**Proof.** When \( \Delta_M \) is orthogonal to \( \Delta \setminus \Delta_M \), on \( H(M(V)) \) and \( H(M(V)) \), extending an \( H(M)_R \)-module \( X \) extending an \( H(M)_R \)-module, the action \( T^*(w)X \) of \( T^*(w)X \) for \( w \in W_{M_2} \) is trivial, hence \( T^*(wu)X = T^*(w)X \) is invertible for \( u \in \Omega \) ([IV, 1]).

For \( X \otimes_R Y \) see [Abc §4.15], [Abc §4.17].

For \( Z = \text{Hom}_{(H(M)_2)}(\mathcal{Y}, X \otimes_R Y) \), we check that the \( T^*(w)Z \) for \( w \in W_{M_2} \) respect the braid and quadratic relations ([IV, 1]). The braid relations follow from \( W = W_{M_2} W_{M'} \) and \( T^*(w_2 w u) = T^*(w_2 w u)T^*(w_2 u) \) if \( w \in W_{M'}, w_2 \in W_{M_2}, u \in \Omega \). For the quadratic relations, let \( s_2 \in W_{M_2} \) and \( s \in W_{M'} \) of length 1. Then \( T^*(s_2_2)X \), \( T^*(s_2)Z \) and \( T^*(s)Y \) are the identity. As \( T^*(s_2^2)Z = -T^*(s_2)Z \) and \( -c(s)Z \) is the identity, \( T^*(s_2)Z \) satisfies the
IV.4. The module $I_{H(G)}(P, V, Q)$.

As $T^*(s)_Z(f) = (T^*(s)_X \otimes \text{id}_Y)(f)$ for $f \in cZ$, $T^*(s)_Z$ satisfies the quadratic relation. □

The right $H(G)$-module $St_{P}^{H(G)} := (St_{P}^{G})^I$ is called a Steinberg $H(G)$-module and $St_{P}^{Q}(R) := R \otimes_{\mathbb{Z}} St_{P}^{G}$ a Steinberg $H(G)_{R}$-module. When $P \subset Q = M_{Q}N_{Q}$, we write $St_{P}^{H(Q)} := St_{P\cap M_{Q}}^{H(M_{Q})}$, recall that $(\text{Ind}_{Q}^{G}(R))^{I} \simeq \text{Ind}_{Q}^{H(G)}(R)$. It is known that [Ly]

- $St_{P}^{H(G)}(R)$ is absolutely simple and isomorphic to the cokernel of the natural map

$$\oplus_{P \subseteq Q \subseteq G} (\text{Ind}_{Q}^{G}(R))^{I} \rightarrow (\text{Ind}_{P}^{G}(R))^{I}.$$ (3.6)

- $T^*(z)$ acts trivially on $\text{Ind}_{P}^{H(G)}(Z), St_{P}^{H(G)}$ for $z \in Z \cap M'$ [AHenV2 Ex.3.14],

- When $P(V) = G, P_{2} = M_{2}N_{2}$ as in Lemma 7 and $P \subset Q$, we have $(St_{Q}^{G})^{I} = (St_{Q}^{G})^{I \cap M_{2}'}$ [AHenV2 §4.2, proof of theorem 4.7],

$$e(V \otimes_{R} \text{Ind}_{Q}^{H(G)}(R), e(V) \otimes_{R} St_{P}^{H(G)}(R), Hom_{\theta^{*}}(H(M_{2}'))(e(V), St_{Q}^{H(G)}(V))$$

are right $H(G)_{R}$-modules for the diagonal action of $(T^*(w))_{w \in W}$ on the first two ones, and for $T^*(w)$ acting on the other one by $T^*(w) \circ - \circ (T^*(w)e(V))^{-1}$ for $w \in W_{M} \Omega$ and by the identity for $w \in W_{M_{2}'}$ (Lemma 7). We have an $H(G)_{R}$-isomorphism ([AHenV2 Prop.4.5] where it is explicit):

$$\text{Ind}_{Q}^{H(G)}(e(V)) \simeq e(V) \otimes (\text{Ind}_{Q}^{G}(R))^{I}. (3.7)$$

These isomorphisms for $P \subset Q \subset Q_{1}$ and the inclusion $(\text{Ind}_{Q_{1}}^{G}(R))^{I} \subset (\text{Ind}_{Q}^{G}(R))^{I}$ give an injective $H(G)_{R}$-isomorphism $\text{Ind}_{Q_{1}}^{H(G)}(e(V_{1}))(V) \xrightarrow{\iota^{G}(Q_{1})} \text{Ind}_{Q}^{H(G)}(e(Q(V)))$.

The cokernel $St_{Q}^{H(G)}(V)$ of the $H(G)_{R}$-map

$$\oplus_{Q \subseteq Q_{1} \subseteq G} \text{Ind}_{Q_{1}}^{H(G)}(e(V)) \rightarrow \text{Ind}_{Q}^{H(G)}(e(V))$$ (3.8)

is isomorphic to $e(V) \otimes_{R} St_{Q}^{H(G)}(R)$ [AHenV2, Cor.3.17, Cor.3.6].

**Proposition 5.** Assume $P(V) = G$ and $P \subset Q$.

1. The natural maps $e(V) \rightarrow \text{Hom}_{H(M_{2})_{R}}(St_{Q}^{H(G)}(R), St_{Q}^{H(G)}(V))$ and

$$\text{Hom}_{H(M_{2})_{R}}(St_{Q}^{H(G)}(R), St_{Q}^{H(G)}(V)) \otimes_{\mathbb{Z}} St_{Q}^{H(G)} \rightarrow St_{Q}^{H(G)}(V)$$

are $H(G)_{R}$-isomorphisms.

2. The map $Y \rightarrow Y \otimes_{R} St_{Q}^{H(G)}(R) : L_{e(V)} \rightarrow L_{St_{Q}^{H(G)}(V)}$ is a lattice isomorphism of inverse $X \rightarrow \{y \in e(V), y \otimes St_{Q}^{H(G)}(X) \subset X\}$.

**Proof.** We are in the setting of Theorem 3 for $A = H(M_{2})_{R} \simeq \theta(H(M_{2})_{R}) \subset A' = H(G)_{R}$ with the $*$-basis, the right $H(G)_{R}$-module $St_{Q}^{H(G)}(R) = e(St_{Q}^{H(G)}(R))$, absolutely simple as an $\theta(H(M_{2})_{R})$-module where $T_{w}^{*}$ for $w \in W \setminus W_{M_{2}}$ (contained in $W_{M} \Omega$) acts invertibly, and the right $H(G)_{R}$-module $e(V)$ where $T_{w}^{*}$ for $w \in W_{M_{2}}$ acts by the identity. □

IV.4. The module $I_{H(G)}(P, V, Q)$. 

Definition 3. An $R$-triple $(P, V, Q)$ of $H(G)$ consists of a parabolic subgroup $P = MN$ of $G$, a right $H(M)_R$-module $V$, $Q$ a parabolic subgroup of $G$ with $P < Q < P(V)$. To an $R$-triple $(P, V, Q)$ of $H(G)$ corresponds a right $H(G)_R$-module

$$I_{H(G)}(P, V, Q) = \text{Ind}_{P(V)}^{H(G)}(\text{St}_Q^{H(M(V))}(V))$$

isomorphic to the cokernel of the $H(G)_R$-homomorphism

$$\oplus_{Q \subseteq Q_1 \subseteq P(V)} \text{Ind}_{Q_1}^{H(G)}(e_{Q_1}(V)) \twoheadrightarrow \oplus_{Q \subseteq Q_1 \subseteq P(V)} i_G^{H(G)}(Q_1, Q) \rightarrow \text{Ind}_Q^{H(G)}(e_Q(V))$$

where $i_G^{H(G)}(Q_1, Q) = \text{Ind}_{P(V)}^{H(G)}(e_{M(V)}(Q \cap M(V), Q_1 \cap M(V)))$.

We can recover $e(V)$ from $I_{H(G)}(P, V, Q)$ and $P(V)$:

$$(4.9) \quad e(V) \simeq \text{Hom}_{H(M'_P), \theta}^*(\text{St}_Q^{H(M(V'))}(R), L_{H(M(V))}^{H(G)}(I_{H(G)}(P, V, Q)))$$

by Proposition 5 and Proposition 4(iii):

$$(4.10) \quad I_{H(G)}^{H(G)}(I_{H(G)}(P, V, Q))) \simeq \text{St}_Q^{H(M(V'))}(V)$$

Proposition 6. Let $(P, V, Q)$ be an $R$-triple of $H(G)$ with $V$ of finite length and such that for each irreducible subquotient $\tau$ of $V$, $P(V) = P(\tau)$ and $I_{H(G)}(P, \tau, Q)$ is simple. Then $P(V) = P(V')$ for any non-zero $H(M)_R$-submodule $V'$ of $V$; the map $V' \mapsto I_{H(G)}(P, V', Q) : \mathcal{L}_V \rightarrow \mathcal{L}_{I_{H(G)}(P, V, Q)}$ is a lattice isomorphism.

Proof. $P(V) = P(V')$ is proved as Proposition 4(iii) when $A = H(M'_P)_R \simeq \theta H(M'_P)_R \subset A' = H(M(\sigma))_R$ with the $*$-basis, the $R$-representations $\text{St}_Q^{H(M(V))}(R)$ and $e(V)$ of $M(V)$. So $\text{St}_Q^{H(M(V))}(V)$ has finite length, and its irreducible subquotients are $\text{St}_Q^{H(M(V))}(\tau)$ for the irreducible subquotients $\tau$ of $V$. If $I_G(P, \tau, Q) = \text{Ind}_{P(V)}^G(\text{St}_Q^{M(V)}(\tau))$ is irreducible for all $\tau$, we are in the situation of Theorem 2 for $F = \text{Ind}_{P(V)}^G$ and $W = \text{St}_Q^{H(M(V))}(V)$ because $\text{Ind}_{P(V)}^G$ has a right adjoint and is exact fully faithful (Proposition 4(iii)) so the map $V' \mapsto I_G(P, V', Q) : \mathcal{L}_V \rightarrow \mathcal{L}_{I_{H(G)}(P, V, Q)}$ is a lattice isomorphism.

We check now that the compatibility of $I_{H(G)}(P, V, Q)$ with scalar extension, as for $I_G(P, \sigma, Q)$ (Proposition 3).

Proposition 7. (i) Let $R'/R$ be a field extension. The parabolic induction commutes with the scalar restriction from $R'$ to $R$ and with the scalar extension from $R$ to $R'$. Hence the left (resp. right) adjoint of the parabolic induction commutes with scalar extension (resp. restriction). This is valid for any commutative rings $R \subset R'$.

(ii) Let $R'/R$ be a field extension. For an $H(M)_R$-module $V$ and an $H(G)_R$-module $X$ such that $\text{Ind}_P^{H(G)}(V) \simeq X_{R'}$, we have $V' \simeq (L_P^{H(G)}X)_{R'}$.

(iii) Let $R'/R$ be an extension. For an $R$-triple $(P, V, Q)$ of $H(G)$ we have:

$P(V) = P(V'_R)$; if $V$ is simple and $V'$ a subquotient of $V_R'$, then $P(V) = P(V')$.

$(e(V))_{R'} = e(V'_R)$, $\text{St}_Q^{P(V)}(V')_{R'} \simeq \text{St}_Q^{P(V)}(V'_{R'})$, and $I_{H(G)}(P, V, Q)_{R'} \simeq I_{H(G)}(P, V'_R, Q)$.

(iii) Let $R'$ be a subfield of $R$ and $(P, V, Q)$ an $R$-triple of $H(G)$ such that $e(V)$, resp. $\text{St}_Q^{H(M(V))}(V)$, resp. $I_{H(G)}(P, V, Q)$, is the scalar extension of a $H(M(V))_{R'}$-module $\tau$, resp. $\rho$, resp. a $H(G)_{R'}$-module $\pi$. 

Then \( V \) is the scalar extension to \( R \) of the natural action of \( H(M)_{R'} \) on \( \tau \), resp. \( \text{St}^R_{\mathcal{V}}(\rho), \text{resp.} \text{Hom}_{H(M)_{R'}}(\text{St}^R_{\mathcal{V}}(R), L^{H(G)}_{\rho}) \).

**Proof.** (i) As for the group (Prop. 5.7), it is clear that the parabolic induction commutes with scalar extension (resp. restriction). Hence the left (resp. right) adjoint of the parabolic induction commutes with scalar extension (resp. restriction).

(ii) Let \( \mathcal{V} \) be an \( H(M)_{R'} \)-module. As an \( H(M)_{R'} \)-module, \( \mathcal{V}_{R'} \) is \( \mathcal{V} \)-isotypic and this holds true for any subquotient \( \mathcal{V}' \) of \( \mathcal{V}_{R'} \) if \( \mathcal{V} \) is simple. For \( \alpha \in \Delta \) orthogonal to \( \Delta_P \), \( T^{M,\ast}(z) \) for \( z \in Z \cap M' \) acts trivially on \( H(M)_{R'} \)-module if and only if it acts trivially on this module seen as an \( H(M)_{R'} \)-module. So \( P(\mathcal{V}) = P(\mathcal{V}_{R'}) \) and if \( \mathcal{V} \) is simple \( P(\mathcal{V}) = P(\mathcal{V}') \). It is clear that \( (e(\mathcal{V}))_{R'} \simeq e(\mathcal{V}_{R'}) \). As

\[
e(\mathcal{V}) \otimes_R \text{St}^R_{\mathcal{V}}(\mathcal{V}_{R'}) \simeq e(\mathcal{V}) \otimes_{\mathcal{Z}} \text{St}^R_{\mathcal{V}}(\mathcal{V}_{R'}) \simeq \text{St}^R_{\mathcal{V}}(\mathcal{V}) = \text{St}^R_{\mathcal{V}}(\mathcal{V}),
\]

we have \( \text{St}^R_{\mathcal{V}}(\mathcal{V})(\mathcal{V}) \simeq \text{St}^R_{\mathcal{V}}(\mathcal{V})(\mathcal{V}) \). As \( \text{Ind}^G_P \mathcal{V} \) commutes with scalar extension (i), we have \( I^G_H(\mathcal{P}, \mathcal{V}, Q)_{R'} = (\text{Ind}^G_P \mathcal{V})(\mathcal{V})(\mathcal{V})(\mathcal{V})(\mathcal{V}) \simeq I^G_H(\mathcal{P}, \mathcal{V}_{R'}, Q) \).

(ii) If \( I^G_H(\mathcal{P}, \mathcal{V}, Q) = \pi_R \) then \( \text{St}^R_{\mathcal{V}}(\mathcal{V}) = \rho_R \) where \( \rho \simeq L^G_{\mathcal{V}}(\pi) \) by (i) and (4.10).

If \( \text{St}^R_{\mathcal{V}}(\mathcal{V})(\mathcal{V}) = \rho_R \), then \( e(\mathcal{V}) = \tau_R \) where \( \tau \simeq \text{Hom}_{H(M)_{R'}}(\text{St}^R_{\mathcal{V}}(\mathcal{V})(\mathcal{V}), \rho) \) as \( e(\mathcal{V}) \simeq \text{Hom}(\mathcal{V}, \mathcal{V})(\mathcal{V})(\mathcal{V})(\mathcal{V})(\mathcal{V})(\mathcal{V}) \) (Prop. 5) and \( \text{St}^R_{\mathcal{V}}(\mathcal{V}) = \rho_R \).

If \( e(\mathcal{V}) = \tau_R \) then \( T^{\mathcal{V}}(m) \) acts trivially on \( \tau_R \) for \( m \in M' \) hence also on \( \tau \) and \( \mathcal{V} = \mathcal{Z} \) is the restriction of \( \tau \) to \( H(M)_{R'} \).

**Remark 15.** Proposition 6 applies to the scalar extension \( \mathcal{V}_{R_{alg}} \) to \( R_{alg} \) of a simple supersingular \( H(M)_{R_{alg}} \)-module \( \mathcal{V} \); the proof is the same as for the group (Remark 11). By the decomposition theorem of \( \mathcal{V} \) and Lemma 6 all irreducible subquotients \( \tau \) of \( \mathcal{V}_{R_{alg}} \) are supersingular, \( P(\tau) = P(\mathcal{V}) = P(\mathcal{V}_{R_{alg}}) \) (Prop 7(ii)), and \( I^G_H(\mathcal{P}, \tau, Q) \) is irreducible by the classification theorem for \( H(G) \) over \( R_{alg} \) (Thm 5 [AHenV2]).

**IV.5. Classification of simple modules over the pro-p Iwahori Hecke algebra.** As in [III.5] for \( G \) the classification theorem for \( H(G) \) over \( R_{alg} \) (Thm 5) descends to \( R \) by a formal proof relying on the properties:

1. The decomposition theorem for \( H(G) \) (Thm 1).
2. The classification theorem for \( H(G) \) over \( R_{alg} \) (Thm 5 [AHenV2]).
3. The compatibility of scalar extension with \( I^G_H(\mathcal{P}, -, Q) \) (Prop. 7) and supersingularity (Lemma 9).
4. The lattice isomorphism \( \mathcal{L}_{\mathcal{V}_{R_{alg}}} \rightarrow \mathcal{L}_{I^G_H(\mathcal{P}, \mathcal{V}_{R_{alg}}, Q)} \) for the scalar extension \( \mathcal{V}_{R_{alg}} \) to \( R_{alg} \) of a simple supersingular \( H(M)_{R_{alg}} \)-module \( \mathcal{V} \) (Prop 5 and Remark 13).

We start the proof with an arbitrary simple \( H(G)_{R} \)-module \( \mathcal{X} \). By the decomposition theorem, the \( H(G)_{R_{alg}} \)-module \( \mathcal{X}_{R_{alg}} \) has finite length; we choose a simple submodule \( \mathcal{X}_{R_{alg}} \) of \( \mathcal{X}_{R_{alg}} \). By the classification theorem over \( R_{alg} \),

\[
(5.11) \quad \mathcal{X}_{R_{alg}} \simeq I^G_H(\mathcal{P}, \mathcal{V}_{R_{alg}}, Q)
\]

for an \( R_{alg} \)-triple \( (\mathcal{P} = MN, \mathcal{V}_{R_{alg}}, Q) \) of \( G \) where \( \mathcal{V}_{R_{alg}} \) is a simple supersingular \( H(M)_{R_{alg}} \)-module. By the decomposition theorem, \( \mathcal{X}_{R_{alg}} \) descends to a finite extension of \( R \), and
also $\mathcal{V}^{alg}$ by compatibility of scalar extension with $I_{H(G)}(P, -, Q)$. By the decomposition theorem, $\mathcal{V}^{alg}$ is contained in the scalar extension $\mathcal{V}^{alg}_{Ralg}$ to $R^{alg}$ of a simple $H(M)_{R}$-module $\mathcal{V}$. By compatibility of scalar extension with $I_{H(G)}(P, -, Q)$ and supersingularity, $\mathcal{V}$ is supersingular, $(P, \mathcal{V}, Q)$ is an $R$-triple of $G$ and

$$(5.12) \quad I_{H(G)}(P, \mathcal{V}^{alg}, Q) \simeq I_{H(G)}(P, \mathcal{V}, Q)_{Ralg}.$$ 

We have $I_{H(G)}(P, \mathcal{V}^{alg}, Q) \subset I_{H(G)}(P, \mathcal{V}^{alg}, Q)$ by the lattice isomorphism $\mathcal{L}_{\mathcal{V}^{alg}} \to \mathcal{L}_{I_{H(G)}(P, \mathcal{V}^{alg}, Q)}$. The decomposition theorem implies

$$\mathcal{X} \simeq I_{H(G)}(P, \mathcal{V}, Q).$$ 

Conversely, we start with an $R$-triple $(P, \mathcal{V}, Q)$ of $H(G)$ with $\mathcal{V}$ simple supersingular and we prove that $I_{H(G)}(P, \mathcal{V}, Q)$ is simple. By the decomposition theorem, the $H(G)_{Ralg}$-module $\mathcal{V}^{alg}_{Ralg}$ has finite length, and $I_{H(G)}(P, \mathcal{V}^{alg}, Q)$ also by the lattice isomorphism $\mathcal{L}_{\mathcal{V}^{alg}} \to \mathcal{L}_{I_{H(G)}(P, \mathcal{V}^{alg}, Q)}$. The scalar extension is faithful and exact and $I_{H(G)}(P, \mathcal{V}, Q)_{Ralg} \simeq I_{H(G)}(P, \mathcal{V}, Q)$ so $I_{H(G)}(P, \mathcal{V}, Q)$ has also finite length. We choose a simple $H(G)_{R}$-submodule $\mathcal{X}^{alg}_{Ralg}$ of $I_{H(G)}(P, \mathcal{V}, Q)$. The $H(G)_{Ralg}$-module $\mathcal{X}^{alg}_{Ralg}$ is contained in $I_{H(G)}(P, \mathcal{V}, Q)_{Ralg}$ hence $\mathcal{X}^{alg}_{Ralg} \simeq I_{H(G)}(P, \mathcal{V}^{alg}, Q)$ for an $H(M)_{Ralg}$-submodule $\mathcal{V}^{alg}$ of $\mathcal{V}^{alg}_{Ralg}$ by $(5.12)$ and the lattice isomorphism $\mathcal{L}_{\mathcal{V}^{alg}} \to \mathcal{L}_{I_{H(G)}(P, \mathcal{V}^{alg}, Q)}$. As $I_{H(G)}(P, \mathcal{V}^{alg}, Q)$ descends to $R$, $\mathcal{V}^{alg}$ is also. But no proper $H(M)_{Ralg}$-submodule of $\mathcal{V}^{alg}_{Ralg}$ descends to $R$ by the decomposition theorem for $H(G)$, so $\mathcal{V}^{alg}_{Ralg} \simeq I_{H(G)}(P, \mathcal{V}^{alg}, Q)$ and $\mathcal{X}^{alg}_{Ralg} \simeq I_{H(G)}(P, \mathcal{V}, Q)_{Ralg}$ by compatibility of scalar extension with $I_{H(G)}(P, -, Q)$. So $\mathcal{X} \simeq I_{H(G)}(P, \mathcal{V}, Q)$ and $I_{H(G)}(P, \mathcal{V}, Q)$ is simple.

Finally, let $(P, \mathcal{V}, Q)$ and $(P, \mathcal{V}, Q)$ be two $R$-triples of $H(G)$ with $\mathcal{V}, \mathcal{V}$ simple supersingular and $I_{H(G)}(P, \mathcal{V}, Q) \simeq I_{H(G)}(P, \mathcal{V}, Q)$. The scalar extensions to $R^{alg}$ are isomorphic $(I_{H(G)}(P, \mathcal{V}, Q))^{alg}_{Ralg} \simeq (I_{H(G)}(P, \mathcal{V}, Q))^{alg}_{Ralg}$. The classification theorem for $H(G)$ over $R^{alg}$ and $(5.12)$ imply $P = P, Q = Q$ and some simple $H(M)_{Ralg}$-subquotient $\mathcal{V}^{alg}_{Ralg}$ of $\mathcal{V}^{alg}_{Ralg}$ is isomorphic to some simple $H(M)_{Ralg}$-subquotient $\mathcal{V}^{alg}_{Ralg}$ of $\mathcal{V}^{alg}_{Ralg}$. As $\mathcal{V}^{alg}$ is $\mathcal{V}$-isotypic and $\mathcal{V}^{alg}_{Ralg}$ is $\mathcal{V}^{alg}_{Ralg}$-isotypic as $H(M)_{Ralg}$-module, $\mathcal{V}$ and $\mathcal{V}$ are isomorphic.

This ends the proof of the classification theorem for $H(G)$ (Thm.5). □

V. Applications

Let $R$ be a field of characteristic $p$ and $G$ a reductive $p$-adic group as in $\{\text{III.3}\}$

V.1. Vanishing of the smooth dual. The dual of $\pi \in \text{Mod}_{R}(G)$ is $\text{Hom}_{R}(\pi, R)$ with the contragredient action of $G$, that is, $(gf)(gx) = f(x)$ for $g \in G, f \in \text{Hom}_{R}(\pi, R), x \in \pi$. The smooth dual of $\pi$ is $\pi^{\vee} := \cup_{K} \text{Hom}_{R}(\pi, R)^{K}$ where $K$ runs through the open compact subgroups of $G$.

A finite dimensional smooth $R$-representation of $G$ is fixed by an open compact subgroup, and its smooth dual is equal to its dual.

We prove Thm. 4 Let $R^{alg}/R$ be an algebraic closure and let $\pi$ be a non-zero irreducible admissible $R$-representation $\pi$ of $G$. By Remark 2, $(\pi^{\vee})^{alg}_{Ralg} \subset (\pi^{alg}_{Ralg})^{\vee}$. Assume that $\pi^{\vee} \neq 0$. Then, $(\pi^{\vee})^{alg}_{Ralg} \neq 0$, hence $(\pi^{alg}_{Ralg})^{\vee} \neq 0$ implying $\rho^{\vee} \neq 0$ for some irreducible subquotient $\rho$ of $\pi^{alg}_{Ralg}$. By the theorem over $R^{alg}$ [Aden1, V2 Thm.6.4], the $R^{alg}$-dimension of $\rho$ is finite. The $R^{alg}$-dimension is constant on the $\text{Aut}_{R}(R^{alg})$-orbit of $\rho$. By the decomposition theorem (Thm. 10), the $R^{alg}$-dimension of $\pi^{alg}_{Ralg}$ is finite. It is equal to the
or isomorphic when \( V \).

5.12, 5.13].

\( \text{Mod} \)

\( \sigma \)

\( \text{G} \)

Denote by \( \sigma \) a smooth \( R \)-representation of \( M \) and by \( V \) a \( H(M) \)-module.

1. The parabolic induction commutes with \((-)^R \) [OV Prop.4.4] and with \(- \otimes_{H(G)} R[I \backslash G] \) [OV Cor.4.7]:

\[ (\text{Ind}_P^G \sigma)^R \simeq \text{Ind}_P^H(\sigma^{I \cap M}), \text{Ind}_P^H(V \otimes_{H(G)} R[I \backslash G] \simeq \text{Ind}_P^G(V \otimes_{H(M)} R[(I \cap M) \backslash M]). \]

2. The last isomorphism and the faithfulness \( \text{Ind}_P^H \) (Prop.4.2) show that the natural map

\[ \eta_{\text{Ind}_P^H(V)} : \text{Ind}_P^H(V) \to (\text{Ind}_P^H(V) \otimes_{H(G)} Z[I \backslash G])^I \]

is bijective if and only if the natural map \( \eta_V : V \to (V \otimes_{H(M)} Z[(I \cap M) \backslash M])^{I \cap M} \) is.

3. The trivial \( R \)-representation of \( G \) is naturally isomorphic to \( R \otimes_{H(G)} Z[I \backslash G]^I \) [OV Lemma 2.25].

4. The \( I \)-invariants of \( I_G(P, \sigma, Q) \) is isomorphic to \( I_G(P, \sigma^{I \cap M}, Q) \) when \( \sigma = \sigma_{\min} \) (III.3) and \( P(\sigma) = P(\sigma^{I \cap M}) \).

**Lemma 8.** Let \( \sigma \) be an irreducible admissible supersingular \( R \)-representation of \( M \). Then \( \sigma = \sigma_{\min} \), \( P(\sigma) = P(\sigma^{I \cap M}) \), so \( I_G(P, \sigma, Q)^I \simeq I_H(G)(P, \sigma^{I \cap M}, Q) \).

**Proof.** The equality \( \sigma = \sigma_{\min} \) follows from the classification (Thm.4) because \( \sigma \) is supersingular (III.3). When \( \sigma = \sigma_{\min} \), then \( \Delta_\sigma \) is orthogonal to \( \Delta_M \) (III.3). As \( \sigma \) being irreducible is generated by \( \sigma^{I \cap M} \), \( P(\sigma) = P(\sigma^{I \cap M}) \) [AHenV2 Thm.3.13].

5. The representations \( I_H(G)(P, V, Q) \otimes_{H(G)} R[I \backslash G] \) and \( I_G(P, V \otimes_{H(M)} R[(I \cap M) \backslash M], Q) \) are isomorphic when \( V \) is simple and supersingular (when \( V \otimes_{H(M)} R[(I \cap M) \backslash M] = 0 \) or \( P(V) = P(V \otimes_{H(M)} R[(I \cap M) \backslash M]) \) when it is not 0, more generally) [AHenV2 Cor. 5.12, 5.13].

**V.2.2.** \( \text{Ind}_P^G(R) \) and \( \text{Ind}_P^H(G)(R) \). It is known that \( \text{Ind}_P^G(R) \) is multiplicity free of irreducible subquotients \( \text{St}_G^Q(R) \) and \( \text{Ind}_Q^G(R) \) is the subrepresentation of \( \text{Ind}_P^G(R) \) with cosocle \( \text{St}_Q^G(R) \), for \( P \subset Q \subset G \) [LV §9].

Therefore, sending \( \text{St}_Q^G(R) \) for \( P \subset Q \) to \( \Delta_Q \setminus \Delta_P \) induces a lattice isomorphism from \( \mathcal{L}_{\text{Ind}_P^G(R)} \) onto the set of upper sets in \( P(\Delta \setminus \Delta_P) \); to an upper set in \( P(\Delta \setminus \Delta_P) \) is associated the subrepresentation \( \sum J \text{Ind}_P^G(J \cup \Delta_P)(R) \) for \( J \) in the upper set [AHenV2 Prop.3.6].

The \( H(G) \)-module \( \text{Ind}_P^H(G)(R) \) has a filtration with quotients \( \text{St}_Q^H(G)(R) \) for \( P \subset Q \subset G \). By the classification theorem, the \( \text{St}_Q^H(G)(R) \) are simple not isomorphic. So \( \text{Ind}_P^H(G)(R) \) is multiplicity free of simple subquotients \( \text{St}_Q^H(G)(R) \) for \( P \subset Q \subset G \).

Applying 1, 2 and 3 in (V.2) the natural map

\[ \text{Ind}_P^H(G)(R) \otimes_{H(G)} Z[I \backslash G] \to \text{Ind}_P^G(R) \]
is an isomorphism and \( n_{\text{Ind}^H_G(R)} \) is bijective.

The properties a), b'), c') of Theorem 2 are satisfied for the functor \(- \otimes_{H(G)} \mathbb{Z}[I \setminus G] : \text{Mod}_R(H(G)) \rightarrow \text{Mod}_R(G)\) of right adjoint \((-)^t\), and the \( H(G)_R\)-module \( \text{Ind}^G_H(R) \). So \((- \otimes_{H(G)} \mathbb{Z}[I \setminus G], (-)^t\) give lattice isomorphisms between \( \mathcal{L}_{\text{Ind}^H_G(R)} \) and \( \mathcal{L}_{\text{Ind}^G_H(R)} \).

**V.2.3.** \( \text{Ind}^G_P(\text{St}^M_Q(R)) \) and \( \text{Ind}^H_G(\text{St}^M_Q(V)) \) for \( Q \subseteq P \). This case is a direct consequence of [V.2.2] because

\[
\text{Ind}^G_P(\text{St}^M_Q(R)) = \text{Ind}^G_Q(R)/ \sum_{Q \subseteq Q_1 \subseteq P} \text{Ind}^G_{Q_1}(R)
\]

is a quotient of \( \text{Ind}^G_Q(R) \). We deduce from [V.2.2] that \( \text{Ind}^G_P(\text{St}^M_Q(R)) \) is multiplicity free of irreducible subquotients \( \text{St}^G_{Q_1}(R) \) for \( Q \subseteq Q' \) but \( Q' \) does not contain any \( Q_1 \) such that \( Q \subseteq Q_1 \subseteq P \), that is, \( Q = Q' \cap P \). The subrepresentation \( \text{Ind}^G_P(\text{St}^M_Q(R)) \) of \( \text{Ind}^G_P(\text{St}^M_Q(R)) \) has cosocle \( \text{St}^G_{Q_1} \). Sending \( \text{St}^G_{Q_1} \) to \( \Delta_{Q_1} \cap (\Delta \setminus \Delta_P) \) gives a lattice isomorphism from \( \mathcal{L}_{\text{Ind}^G_P(\text{St}^M_Q(R))} \) onto the lattice of upper sets in \( \mathcal{P}(\Delta \setminus \Delta_P) \) (which does not depend on \( Q \)). We deduce also from [V.2.2] and Remark 3 that \(- \otimes_{H(G)} \mathbb{Z}[I \setminus G] \) and \((-)^t\) give lattice isomorphisms between \( \mathcal{L}_{\text{Ind}^H_G(\text{St}^M_Q(V))} \) and \( \mathcal{L}_{\text{Ind}^G_P(\text{St}^M_Q(R))} \).

**V.2.4.** \( \text{Ind}^G_P\sigma \) for \( \sigma \) irreducible admissible supersingular and \( \text{Ind}^H_G(V) \) for \( V \) simple supersingular. \( \text{Ind}^G_P\sigma \) admits a filtration with quotients \( I_G(P, \sigma, Q) = \text{Ind}^G_P(\text{St}^M_Q(\sigma)) \) for \( P \subset Q \subset P(\sigma) \) and by the classification theorem, the \( I_G(P, \sigma, Q) \) are irreducible and not isomorphic; so \( \text{Ind}^G_P(\sigma) \) is multiplicity free of irreducible subquotients \( I_G(P, \sigma, Q) \) for the \( R\)-triples \( (P, \sigma, Q) \) of \( G \). The maps

\[
X \mapsto e(\sigma) \otimes_R X \mapsto \text{Ind}^G_P(\sigma)(e(\sigma) \otimes_R X) : \mathcal{L}_{\text{Ind}^M_P(\sigma)(R)} \rightarrow \mathcal{L}_{e(\sigma) \otimes_R \text{Ind}^M_P(\sigma)(R)} \rightarrow \mathcal{L}_{\text{Ind}^G_P(\sigma)}
\]

are lattice isomorphisms: this follows from the lattice theorems and the classification theorem (Thm 2, Thm 3, Thm 4), as in Proposition 1 (When \( R \) is algebraically closed [Allen-V], Prop.3.8)).

For a simple supersingular \( H(M)_R\)-module \( V \), the same arguments show that \( \text{Ind}^H_G(V) \) is multiplicity free of simple subquotients \( I_{H(G)}(P, V, Q) \) for \( P \subset Q \subset P(V) \). The maps

\[
Y \mapsto e(V) \otimes_R Y \mapsto \text{Ind}^H_G(V)(e(V) \otimes_R Y) : \mathcal{L}_{\text{Ind}^{H(M)(V)}_P(\sigma)(R)} \rightarrow \mathcal{L}_{e(V) \otimes_R \text{Ind}^{H(M)(V)}_P(\sigma)(R)} \rightarrow \mathcal{L}_{\text{Ind}^H_G(V)}
\]

are lattice isomorphisms, by applying Thm 2, Thm 3, Thm 4, as in Proposition 3.

**V.2.5.** \( \text{Ind}^G_P(\text{St}^M_Q(\sigma_1)) \) and \( \text{Ind}^H_G(\text{St}^M_Q(\sigma_1)) \) for an \( R\)-triple \( (P_1, \sigma_1, P) \) of \( G \), \( P_1 \subset Q \subset P \), \( \sigma_1 \) irreducible admissible supersingular and similarly for \( V \). This is a direct consequence of [V.2.4] because

\[
\text{Ind}^G_P(\text{St}^M_Q(\sigma_1)) = (\text{Ind}^G_Q(e_Q(\sigma_1)))/(\sum_{Q \subseteq Q_1 \subset P} \text{Ind}^G_{Q_1}(e_Q(\sigma_1))
\]
is a subquotient of $\Ind_P^G(\sigma_1)$ as $e_Q(\sigma_1) \subset \Ind_{P_1}^M(\sigma_1)$ and similarly for $\mathcal{V}$. We have $\Ind_Q^G e_Q(\sigma_1) \simeq \Ind_{P_1}^G(\sigma_1 \otimes_R \Ind_{P_1}^M(\sigma_1)(R))$, and a lattice isomorphism (§V.2.4):

$$X \mapsto \Ind_{P_1}^G(\sigma_1) \otimes_R X : \mathcal{L}_{\Ind_{P_1}^M(\sigma_1)(R)} \to \mathcal{L}_{\Ind_{P_1}^G(\sigma_1)}$$

inducing a lattice isomorphism (Remark 5):

$$\mathcal{L}_{\Ind_{P_1}^M(\sigma_1)(\St_Q^M(R))} \to \mathcal{L}_{\Ind_{P_1}^G(\St_Q^M(\sigma_1))}.$$ 

The $R$-representation $\Ind_{P_1}^G(\St_Q^M(\sigma_1))$ is multiplicty free of irreducible subquotients $I_G(P_1, \sigma_1, Q')$ for the $R$-triples $(P_1, \sigma_1, Q')$ of $G$ with $Q' \cap P = Q$ (§V.2.3). And similarly for $\mathcal{V}$ with the same arguments and references.

**V.2.6. Ind$_P^G$ $\sigma$ for $\sigma$ irreducible admissible and Ind$_P^H(G)\mathcal{V}$ for $\mathcal{V}$ simple.** By the classification theorem, there exists an $R$-triple $(P_1, \sigma_1, Q)$ of $G$ with $Q \subset P$, $\sigma_1$ irreducible admissible supersingular such that

$$\sigma \simeq I_M(P_1 \cap M, \sigma_1, Q \cap M) = \Ind_{P_1 \cap M}^M(\St_Q^M(\sigma_1 \cap M)(\sigma_1)).$$

The transitivity of the induction implies $\Ind_{P_1 \cap M}^G \sigma \simeq \Ind_{P_1 \cap M}(\St_Q^M(\sigma_1 \cap M)(\sigma_1))$. This is the case (§V.2.5) with $P(\sigma_1) \cap P$. The $R$-representation $\Ind_{P_1 \cap M}^G \sigma$ of $G$ is multiplicity free of irreducible subquotients $I_G(P_1, \sigma_1, Q')$ for the $R$-triples $(P_1, \sigma_1, Q')$ of $G$ with $Q' \cap P = Q$ (note that $Q' \subset P(\sigma_1), Q \subset P$). The map

$$X \mapsto \Ind_{P_1 \cap M}^G(\sigma_1) \otimes_R X : \mathcal{L}_{\Ind_{P_1 \cap M}(\St_Q^M(\sigma_1 \cap M)(R))} \to \mathcal{L}_{\Ind_{P_1 \cap M}^G(\sigma_1)}$$

is a lattice isomorphism. And similarly for $\mathcal{V}$ with the same arguments and references.

**V.2.7. Invariants by the pro-$p$ Iwahori.** We keep the notations of (§V.2.6). The classification theorem shows that

$$\sigma^{I \cap M} \text{ is simple } \iff \sigma_1^{I \cap M_1} \text{ is simple}$$

because $\sigma^{I \cap M} \simeq I_{H(M)}(P_1 \cap M, \sigma_1^{I \cap M_1}, Q \cap M)$ (§V.2.1) and $\sigma_1^{I \cap M_1}$ is supersingular of finite length.

Assume first that $P(\sigma_1) = P(\mathcal{V}_1)$ in §V.2.6. In §V.2.3 we saw that the maps

$$X \mapsto X^{I \cap M(\sigma_1)}, \quad Y \mapsto Y \otimes_{H(M(\sigma_1))} \mathbb{Z}[I \cap M(\sigma_1) \backslash M(\sigma_1)]$$

between $\mathcal{L}_{\Ind_{P(\sigma_1) \cap P}(\St_Q^{M(\sigma_1) \cap M}(R))}$ and $\mathcal{L}_{\Ind_{P(\sigma_1) \cap P}(\St_Q^{H(M(\sigma_1) \cap M)(R)})}$ are lattice isomorphisms, inverse from each other. They induce lattice isomorphisms, inverse from each other, between $\mathcal{L}_{\Ind_{P(\sigma_1)}(\sigma)}$ and $\mathcal{L}_{\Ind_{P(\sigma_1)}^H(\mathcal{V})}$:

$$\mathcal{L}_{\Ind_{P(\sigma_1)}^G(\sigma), \Ind_{P(\mathcal{V}_1)}^H(\mathcal{V})} : \mathcal{L}_{\Ind_{P(\sigma_1)}^G(\sigma_1)} \otimes_R \mathcal{L}_{\Ind_{P(\mathcal{V}_1)}^H(\mathcal{V})} \to \mathcal{L}_{\Ind_{P(\sigma_1)}^H(\mathcal{V})}.$$

(2.14) $\Ind_{P(\sigma_1)}^G(e(\sigma_1) \otimes_R Y) \mapsto \Ind_{P(\mathcal{V}_1)}^H(\mathcal{V})$, Y \mapsto Y \otimes_{H(M(\sigma_1))} \mathbb{Z}[I \cap M(\sigma_1) \backslash M(\sigma_1)](2.15)$

by the lattice isomorphisms of (§V.2.6) with $\mathcal{L}_{\Ind_{P(\sigma_1)}^G(\sigma)}$ and $\mathcal{L}_{\Ind_{P(\sigma_1)}^H(\mathcal{V})}$.

Assume now (until the end of §V.2.7) that $\sigma^{I \cap M}$ and $\sigma_1^{I \cap M_1}$ are simple, and that the natural map $\sigma^{I \cap M} \otimes_{H(M)} \mathbb{Z}[I \cap M] \to \sigma$ is injective.
We write $P(\sigma_1) = P(\sigma_1^{I \cap M})$ (Lemma 8, 9). When $V_1 = \sigma_1^{I \cap M}$, the lattice isomorphisms (14, 15) between $\mathcal{L}^{G}_{\text{Ind}^G_{\ell}(\sigma)}$ and $\mathcal{L}^{H(G)}_{\text{Ind}^H_{\ell}(\sigma^{I \cap M})}$ are simply given by $(-)^I$ and $-\otimes_{H(M(G)} Z[I \cap G]$:

$$\text{(2.16)} \quad \text{Ind}^G_{\text{Ind}^G_{\ell}(\sigma)}(e(\sigma_1) \otimes_R X) \mapsto (\text{Ind}^G_{\text{Ind}^G_{\ell}(\sigma)}(e(\sigma_1) \otimes_R X))^I,$$

$$\text{(2.17)} \quad \text{Ind}^{H(G)}_{\text{Ind}^H_{\ell}(\sigma^{I \cap M})}(e(\sigma_1^{I \cap M}) \otimes_R Y) \mapsto (\text{Ind}^{H(G)}_{\text{Ind}^H_{\ell}(\sigma^{I \cap M})}(e(\sigma_1^{I \cap M}) \otimes_R Y)) \otimes_{H(M(G)} Z[I \cap G].$$

This follows from the lattice theorem (Thm 2) applied to the functor $-\otimes_{H(G)} Z[I \cap G]$ : $\text{Mod}_R(H(G)) \to \text{Mod}_R(G)$ of right adjoint $(-)^I$ and to $\text{Ind}^{H(G)}_{\text{Ind}^H_{\ell}(\sigma^{I \cap M})}$ after having checked that they satisfy the hypotheses a), b'), c') of this theorem.

As the map $\sigma^{I \cap M} \otimes_{H(M)} Z[I \cap M] \to \sigma$ is injective, $\sigma$ irreducible, and as the parabolic induction commutes with $-\otimes_{H(G)} Z[I \cap G]$, the natural map

$$\text{Ind}^{H(G)}_{\text{Ind}^H_{\ell}(\sigma^{I \cap M})} \otimes_{H(G)} Z[I \cap G] \to \text{Ind}^G_{\ell} \sigma$$

is an isomorphism. The commutativity of the parabolic induction with $(-)^I$ (IV, 2.1) implies that $\text{Ind}^{H(G)}(\sigma^{I \cap M}) \to (\text{Ind}^G_{\text{Ind}^G_{\ell}(\sigma^{I \cap M})})^I$ is an isomorphism, i.e. a).

The $H(G)$-module $\text{Ind}^{H(G)}_{\text{Ind}^H_{\ell}(\sigma^{I \cap M})}$ and $\text{Ind}^G_{\ell} \sigma$ have finite length (IV, 2.6), i.e. c'). For b'), i.e. $\pi^I$ is simple for any irreducible subquotient $\pi$ of $\text{Ind}^{H}_{\ell}(\sigma)$, we write $\pi \simeq H(P_M, \sigma, Q')$ for an $R$-triple $(P_M, \sigma, Q')$ of $G$ with $Q' \cap P = Q$. By lemma 3, $I_G(P_M, \sigma, Q') \simeq I_{H(G)}(P_M, \sigma^{I \cap M}, Q')$ and $I_{H(G)}(P_M, \sigma^{I \cap M}, Q')$ is simple by the classification theorem.

This ends the proof of Theorem 4.

VI. APPENDIX: EIGHT INDUCTIONS $\text{Mod}_R(H(M)) \to \text{Mod}_R(H(G))$

For a commutative ring $R$ and a parabolic subgroup $P = MN$ of $G$, there are eight different inductions $\text{Mod}_R(H(M)) \to \text{Mod}_R(H(G))$ mentioned in the text.

 associated to the eight elements of $\{\otimes, \text{Hom} \} \times \{+, -, \} \times \{\theta, \theta^\star\}$ where $\theta := \theta_M^{G}$ IV, 1. We We write $\{\theta^\eta, \theta^{\eta \star}\} = \{\theta, \theta^\star\}$ (as sets). The triple $(\otimes, +, \theta)$ corresponds to the parabolic induction $\text{Ind}^{H(G)}_{\ell}(\sigma^{I \cap M})$ and the triple $(\otimes, -, \theta^\star)$ corresponds to $CT^{H(G)}_{\ell}(\sigma^{I \cap M}) \to \text{Ind}^{H}_{\ell}(\sigma^{I \cap M}) \to \text{Ind}^{H}_{\ell}(\sigma^{I \cap M})$ that we call parabolic induction. The propositions (Prop 8, 9) comparing these eight inductions, are extracted from [Abeinv] and [Abeinv]. To formulate them we need first to define the “twist by $n_{w_G m_M}^\star$” and the involution $e_{\ell}^M$ of $H(M)$.

Twist by $n_{w_G M_M}^\star$. Let $w = w_P$ be the longest element of the Weyl group of $\Delta = \Delta_P$, and $w^{-1} : \mathbb{W} \to W$ is an injective homomorphism from the Weyl group $\mathbb{W}$ of $\Delta$ to $W$ satisfying the braid relations (there is no canonical choice).

Let $P_{\text{op}} = M_{\text{op}} N_{\text{op}}$ denote the parabolic subgroup of $G$ (containing $B$) with $\Delta_{\text{op}} = \Delta_{\text{op}} = w_G W_P (\Delta_P) = w_G (-\Delta_P)$ (image of $\Delta_P$ by the opposition involution $\alpha \mapsto w_G (-\alpha)$ [I, 1.5.1]). The twist by $n_{w_G M_M}^\star$ is the ring isomorphism [Abe] §4.3]

$$H(M) \to H(M_{\text{op}}) (T_w^M, T^{M_{\text{op}}, \star} w_{w_G M_M}^\star) \mapsto (T_{n_{w_G M_M}^\star w_{w_G M_M}^\star}^M, T^{M_{\text{op}}, \star} w_{w_G M_M}^\star)$$

It restricts to an isomorphism $H(M^\star) \to H(M_{\text{op}}, -\epsilon)$ [VigP20, Prop.2.20]. The inverse of the twist by $n_{w_G M_M}^\star$ is the twist by $n_{w_G W_P}^\star$, as $n_{w_G W_P}^\star = n_{w_G W_P}^\star$. 


By functoriality the twist by by $n_{wGw_M}$ gives a functor

$$\text{Mod}_R(H(M)) \xrightarrow{n_{wGw_M}(\cdot)} \text{Mod}_R(H(M^{op})).$$

**Involution** $\ell_{\ell - \ell_M}^M$ [Abeparind §4.1]. The two commuting involutions $\ell^M$ and $\ell_{\ell - \ell_M}$ of the ring $H(M)$:

$$(T_w^M, T_{w^*}^M) \xrightarrow{\ell^M} (-1)^{\ell^M(w)}(T_{w^*}^M, T_w^M),$$

$$(T_w^M, T_{w^*}^M) \xrightarrow{\ell_{\ell - \ell_M}} (-1)^{\ell_{\ell - \ell_M}(w)}(T_{w^*}^M, T_w^M),$$

[Vigplw, Prop. 4.23], [Abeparind Lemmas 4.2, 4.3, 4.4, 4.5].

give by composition an involution $\ell_{\ell - \ell_M}^M$ of $H(M)$

$$(T_w^M, T_{w^*}^M) \xrightarrow{\ell_{\ell - \ell_M}} (-1)^{\ell(w)}(T_{w^*}^M, T_w^M).$$

The twist by $n_{wGw_M}$ and the involution $\ell_{\ell - \ell_M}^M$ commute:

$$n_{wGw_M}(\cdot) \circ \ell_{\ell - \ell_M}^M = \ell_{\ell - \ell_M}^M \circ n_{wGw_M}(\cdot) : H(M) \rightarrow H(M^{op}),$$

send $T_w^M$ for $w \in W_M$ to

$$(-1)^{\ell(n_{wGw_M}^{-1}w)}n_{wGw_M}^{-1}w = (-1)^{\ell(w)}n_{wGw_M}^{-1}w$$

(for the equality, recall that the length $\ell_M$ of $W_M$ is invariant by conjugation by $w_M$, and $\ell(n_{wGw_M}^{-1}w) = \ell(n_{wGw_M}^{-1}w) = \ell(w)).$ By functoriality we get a functor

$$\text{Mod}_R(H(M)) \xrightarrow{(-\ell_{\ell - \ell_M}^M)} \text{Mod}_R(H(M)).$$

When $M = G$, we write simply $\ell^G$.

We are now ready for the comparison of the eight inductions, which follows from different propositions in [Abeparind]. In the following propositions, $\mathcal{V}$ is any right $H(M)_R$-module.

**Proposition 8.** Exchanging $+, -$ corresponds to the twist by $n_{wGw_M}$,

$$\mathcal{V} \otimes_{H(M^{op}),\theta^n} H(G) \simeq n_{wGw_M}(\mathcal{V}) \otimes_{H(M^{op},-),\theta^n} H(G),$$

$$\text{Hom}_{H(M^{op}),\theta^n}(H(G), \mathcal{V}) \simeq \text{Hom}_{H(M^{op},-),\theta^n}(H(G), n_{wGw_M}(\mathcal{V})).$$

Exchanging $\theta, \theta^*$ corresponds to the involutions $\ell_{\ell - \ell_M}^M$ and $\ell^G$,

$$\mathcal{V} \otimes_{H(M^{op}),\theta^n} H(G) \xrightarrow{\ell_{\ell - \ell_M}^M \otimes_{H(M^{op}),\theta^n}} H(G),$$

$$\text{Hom}_{H(M^{op}),\theta^n}(H(G), \mathcal{V}) \xrightarrow{\ell_{\ell - \ell_M}^M} \text{Hom}_{H(M^{op}),\theta^n}(H(G), \mathcal{V}^M_{\ell - \ell_M}).$$

Exchanging $\otimes$, $\text{Hom}$ corresponds to the involutions $\ell_{\ell - \ell_M}^M$ and $\ell^G$,

$$\mathcal{V} \otimes_{H(M^{op}),\theta^n} H(G) \xrightarrow{\ell^G} \text{Hom}_{H(M^{op}),\theta^n}(H(G), \mathcal{V}^M_{\ell - \ell_M}).$$

**Remark 16.** One can exchange $(\otimes, \theta^n)$ and $(\text{Hom}, \theta^n)$ without changing the isomorphism class:

$$\mathcal{V} \otimes_{H(M^{op}),\theta^n} H(G) \simeq \text{Hom}_{H(M^{op}),\theta^n}(H(G), \mathcal{V}).$$
**Proposition 9.** The dual exchanges \((\otimes, +)\) and \((\text{Hom}, -)\):

\[
\begin{align*}
0.24 & \quad (\mathcal{V} \otimes_{H(M^\mathfrak{a}), \theta^\mathfrak{a}} H(G))^* \simeq \text{Hom}_{H(M^\mathfrak{a}), \theta^\mathfrak{a}}(H(G), \mathcal{V}^*), \\
0.25 & \quad \mathcal{V}^* \otimes_{H(M^\mathfrak{a}), \theta^\mathfrak{a}} H(G) \simeq (\text{Hom}_{H(M^\mathfrak{a}), \theta^\mathfrak{a}}(H(G), \mathcal{V}))^*.
\end{align*}
\]

**Proof.** Applying (0.23), the upper isomorphism (0.24) for an arbitrary \((\epsilon, \theta^\mathfrak{a})\) is equivalent to the lower isomorphism (0.25) for an arbitrary \((\epsilon, \theta^\mathfrak{a})\).

We prove the upper isomorphism for an arbitrary \((\epsilon, \theta^\mathfrak{a})\). For \((+, \theta)\), it is implicit in [Abeinv] [§4.1]. Applying it to the twist by \(n_{wGwp}\) of \((M, \mathcal{V})\) and using (0.18) (0.19), we get (0.24) for \((- , \theta)\). The image by \(\iota^G\) of the upper isomorphism (0.24) for \((\epsilon, \theta)\) and \(\mathcal{V}^{\mathfrak{a}} \otimes_{\mathfrak{a}} \iota_M\) is (0.24) for \((\epsilon, \theta^\mathfrak{a})\) and \(\mathcal{V}\), because the anti-involution \(\zeta_M\) of \(H(M)\) commutes with the involution \(\iota_M^W\), their composite in any order sends \((T^M_w, T^{M^*, w})\) to \((-1)^{\ell(w)}(T^{M, w-1}, T^{M^*, w})\) for \(w \in W_M\), as \(\ell(w) = \ell(w^{-1})\).

\[
\square
\]

### References

[Abe] N. Abe. *Modulo p parabolic induction of pro-p-Iwahori Hecke algebra*, J. Reine Angew. Math., DOI:10.1515/crelle-2016-0043.

[Abeinv] N. Abe. *Parabolic induction for pro-p-Iwahori Hecke algebras*, [arXiv:1612.01312](http://arxiv.org/abs/1612.01312).

[Abeiparind] N. Abe. *Involutions on pro-p-Iwahori Hecke algebras*, [arXiv:1704.00408v1](http://arxiv.org/abs/1704.00408v1).

[AHHV] N. Abe, G. Henniart, F. Herzig, M.-F. Vigneras *A classification of irreducible admissible mod p representations of p-adic reductive groups*, J. Amer. Math. Soc. 30 (2017), 495-559.

[AHenV1] N. Abe, G. Henniart, M.-F. Vigneras *Modulo p representations of reductive p-adic groups: Functorial properties. Transactions of the AMS (to appear).*

[AHenV2] N. Abe, G. Henniart, M.-F. Vigneras *On pro-p-Iwahori invariants of R-representations of reductive p-adic groups.*

[BkiA2] N. Bourbaki. *Algèbre Chapitres 1 à 3* Hermann 1970.

[BkiA8] N. Bourbaki. *Algèbre Chapitre 8* Springer, 2012.

[CR] Curtis Reiner *Methods of Representation Theory, vol. I*, Wiley Interscience 1981.

[Eis] M. D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry* Springer, coll. GTM (no 150), 2004.

[Hn] G. Henniart. *Sur les représentations modulo p de groupes réductifs p-adiques. Automorphic forms and L functions II. Local aspects*, 41–55, Contemp. Math., 489, Amer. Math. Soc., Providence, RI, 2009.

[He] F. Herzig. *The classification of irreducible admissible mod p representations of a p-adic GL_n*, Invent. Math. 186 (2011), no. 2, 373–434.

[KS] M. Kashwara, P. Shapira. *Categories and Sheaves*, Grundlehren des mathematischen Wissenschaften, Vol. 332, Springer-Verlag 2006. Errata in [http://webusers.imj-prg.fr/~pierre.schapira/books](http://webusers.imj-prg.fr/~pierre.schapira/books)

[K] K. Kozlowski. *Pro-p-Iwahori invariants for SL_2 and L-packets of Hecke modules*. [arXiv:1308.0239v3](http://arxiv.org/abs/1308.0239).

[Kohl] J. Kohlhaase *Smooth duality in natural characteristic/ Advances in Math. 317 (2017) 1-49.*

[Lang] S. Lang. *Algebra*, Addison-Wesley 1984.

[Ly] T. Ly. *Représentations de Steinberg modulo p pour un groupe réductif sur un corps local*, Pacific J. Math. 277 (2015), no. 2, 425-462.

[Oss] R. Ollivier. *Compatibility between Satake and Bernstein isomorphisms in characteristic p* Algebra Number Theory 8 (2014), no.1071–1111.

[O] R. Ollivier. *Le foncteur des invariants sous l’action du pro-p-Iwahori de GL_2(F)* J. reine angew. Math. 635 (2009), 149–185.

[OS] R. Ollivier, P. Schneider. *A canonical torsion for pro-p Iwahori-Hecke modules*, [arXiv:1602.00383](http://arxiv.org/abs/1602.00383 [math.RT]).

[OV] R. Ollivier, M.-F. Vignéras. *Parabolic induction in characteristic p* 2017, ArXiv 1703.04921v1

[Vigivre] M.-F. Vignéras. *Représentations l-modulaires d’un groupe réductif p-adique avec l ≠ p*, Progress in Math. 137, Birkhauser, Boston (1996).
[T] J. Tits. *Classification of Algebraic Semisimple groups*. Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., 1966, 3362.

[Vigadjoint] M.-F. Vignéras. *The right adjoint of the parabolic induction*, in *Arbeitstagung Bonn 2013: In Memory of Friedrich Hirzebruch*. Progress in Math. Birkhauser, 2016.

[VigpIw] M.-F. Vignéras. *The pro-p-Iwahori Hecke algebra of a reductive p-adic group I*. Compositio mathematicae 152, vol.7, No1, 693–753 (2016).

[VigpIwc] M.-F. Vignéras. *The pro-p-Iwahori Hecke algebra of a reductive p-adic group II*. A volume in the honour of Peter Schneider. Muenster J. of Math. Vol. 7, No 1, 2014 (364-379).

[VigpIwss] M.-F. Vignéras. *The pro-p-Iwahori Hecke algebra of a reductive p-adic group III*. Journal of the Institute of Mathematics of Jussieu (2015) 1-38.

[VigpIwst] M.-F. Vignéras. *The pro-p-Iwahori Hecke algebra of a reductive p-adic group V*. In memoriam: Robert Steinberg. Pacific J. of Math. Vol. 279, No,1-2, (2015) 499-529.

(G. Henniart) Université de Paris-Sud, Laboratoire de Mathématiques d’Orsay, Orsay cedex F-91405 France; CNRS, Orsay cedex F-91405 France

E-mail address: Guy.Henniart@math.u-psud.fr

(M.-F. Vignéras) Institut de Mathématiques de Jussieu-Paris Rive Gauche, 4 Place Jussieu, Paris 75005 France

E-mail address: marie-france.vigneras@imj-prg.fr