NOTES ON MITSUI’S PRIME NUMBER THEOREM WITH SIEGEL ZEROS

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Abstract. In these notes, we refine Mitsui’s Prime Number Theorem from 1957, which for a number field $K$ predicts how many prime elements there are in bounded convex sets in $K \otimes \mathbb{Q} \mathbb{R}$, by incorporating potential Siegel zeros of Hecke $L$-functions. This allows the norm of the modulus $N(q)$ to grow at a pseudopolynomial rate $e^{\sqrt{\log X}/O(1)}$ with respect to the size $X$ of the convex set as opposed to powers of $\log X$. The extra flexibility and precision will be essential in our future application to the study of linear patterns of prime elements. We also hope that our updated exposition will make Mitsui’s work accessible to a wider mathematical audience.

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1. Introduction

1.1. The Siegel–Walfisz prime number theorem. Prime Number Theorem states in one of the several equivalent forms:

$$\sum_{0 < p < X} \log(p) = X + o(X) \quad \text{as} \quad X \to +\infty.$$

Here the sum $\sum_{0 < p < X}$ is the sum over prime numbers $p$ satisfying $0 < p < X$.

A little more generally, the Siegel–Walfisz prime number theorem says for any modulus $1 < q < \log^A X$ ($A \geq 1$ is a prescribed constant) and $a \in (\mathbb{Z}/q\mathbb{Z})^\times$, we

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have the estimate

\[
\sum_{\substack{0 < p < X \\ p \equiv a \mod q}} \log(p) = \frac{1}{\varphi(q)} X + O_A(Xe^{-\sqrt{\log X/O_A(1)}}). \tag{1.1}
\]

Formulated this way, the restriction \( q < \log^A X \) cannot be improved by today’s human technology because the contemporary analytic number theory has not been able to exclude the existence of notorious Siegel zeros of Dirichlet \( L \)-functions. Siegel zeros are potential real zeros of Dirichlet \( L \)-functions slightly smaller than 1, which contradicts the Grand Riemann Hypothesis. This concept will be reviewed in \( \S 2 \).

Indeed, a Siegel zero \( 0 < \beta < 1 \) modulo \( q \) would give rise to a second main term \( X^{\beta}/\beta \varphi(q) \) to (1.1). While this can be absorbed in the error term if we stay in the realm \( q < \log^A X \), we have to take it seriously if we want to go beyond this bound. Indeed, by incorporating the term \( X^{\beta}/\beta \varphi(q) \) into the picture, the modulus \( q \) is allowed to be at least as large as \( e^{\sqrt{\log X/O(1)}} \). Namely, for all \( 1 \leq q < e^{\sqrt{\log X/O(1)}} \) we have

\[
\sum_{\substack{0 < p < X \\ p \equiv a \mod q}} \log(p) = \frac{1}{\varphi(q)} \left( X - \psi_{\text{Siegel}}(a) \frac{X^{\beta}}{\beta} \right) + O(Xe^{-\sqrt{\log X/O(1)}})
\]

where \( \psi_{\text{Siegel}} : (\mathbb{Z}/q\mathbb{Z})^\times \to \{\pm 1\} \) is the potential Siegel character mod \( q \). See e.g. \cite[Theorem 5.27 on p. 122]{6}.

1.2. Mitsui’s theorem and our main result. The number field analogue of the Siegel–Walfisz theorem has already been documented. It is known as Mitsui’s prime number theorem, which can be formulated as follows:

**Theorem 1.2.1** (Mitsui \cite[Corollary and Theorem on p. 35]{11}, 1956). Let \( K \) be a number field of degree \( n = [K : \mathbb{Q}] = \dim_{\mathbb{Q}} K \). Let \( A > 1 \) be a fixed positive number. Let \( X > 1 \) and consider a convex open set \( C \subset K \otimes_{\mathbb{Q}} \mathbb{R} \) contained in the region \( (K \otimes_{\mathbb{Q}} \mathbb{R})_X < X \) of size \( X \) (\( \S 2.4 \)). Let \( q \subset \mathcal{O}_K \) be a non-zero ideal such that

\[
N(q) < \log^A X
\]

and choose any \( \alpha \in (\mathcal{O}_K/q)^\times \). Then we have

\[
\sum_{\substack{\pi \in C, \\ \text{prime element,} \\ \pi \equiv \alpha \mod \mathcal{O}_K/q}} \log|N(\pi)| = \frac{w_K}{\varphi_K(q)2^{\tau_1} \tau_2 h_K R_K} \text{Vol}(C) + O_{K,A}(X^n e^{-\sqrt{\log X/O_{K,A}(1)}}). \tag{1.3}
\]

Here, there is a canonical \( \mathbb{R} \)-linear isomorphism \( K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^n \) and \( \text{Vol}(C) \) refers to the volume in the Lebesgue measure \( \mu \) on \( \mathbb{R}^n \).

The main purpose of this text is to improve Mitsui’s theorem by incorporating the secondary main term coming from potential Siegel zeros \( \beta \) of \( L \)-functions \( L(\psi, s) \) associated with mod \( q \) characters \( \psi \). To be specific, let \( \psi_{\text{Siegel}} \) be the potential Siegel character mod \( q \) and \( \beta \) the Siegel zero (see \( \S 2.2 \) for these notions). In our main
Theorem 3.1.1, the right hand side of (1.3) will be refined to the expression
\[
\varphi_K(q)2^{r_1} \pi^{r_2} h_K R_K \int_C 1 - \psi_{\text{Siegel}}(-; \alpha) N(-)^{\beta - 1} d\mu \\
+ O_K(N^{n} e^{-\sqrt{\log X}/O_K(1)}).
\]
This modification allows us to relax the restriction (1.2) to a pseudopolynomial growth
\[
N(q) < e^{\sqrt{\log X}/O_K(1)}.
\]
This quantitative improvement will be essential in our application in the forthcoming work, where based on the author’s former joint work [7] we extend results of Green–Tao–Ziegler [3, 4] and Tao–Teräväinen [14] on linear patterns of prime numbers to general number fields. To illustrate what it is about, consider polynomials of degree 1:
\[
ax + by + c \in \mathcal{O}_K[x, y],
\]
where the coefficients \((a, b, c)\) vary in a given finite subset \(T \subset \mathcal{O}_K^3\). Under some conditions on \(T\) including the hypothesis that the vectors \((a, b)\) \(\in \mathcal{O}_K^2\) are pairwise linearly independent, we will be able to conclude that for any large enough convex body \(C \subset (K \otimes \mathbb{Q} \mathbb{R})^{r_1 + r_2}\), there are points \((x, y)\) \(\in C \cap \mathcal{O}_K^2\) such that the values of (1.4) are prime elements of \(\mathcal{O}_K\) simultaneously for all \((a, b, c)\) \(\in T\).

This will furthermore be applied to a Hasse principle type problem on \(K\)-rational points on certain specific type of varieties as was done in [5] by Harpaz–Skorobogatov–Wittenberg over \(K = \mathbb{Q}\).

### 1.3. Remarks on the formulation.
In Mitsui’s original account [11, Theorem on p. 35], the theorem is formulated for sets \(C\) of the following form
\[
C = (K \otimes \mathbb{Q} \mathbb{R})_{<X_1, \ldots, X_{r_1 + r_2}} := \{x \in K \otimes \mathbb{Q} \mathbb{R} \mid \|\sigma_i(x)\| < X_i, 1 \leq i \leq r_1 + r_2\},
\]
where \(X_i\) are positive numbers satisfying \(X_i \leq X_j^n\) for all \(i, j\) for a prescribed constant \(a \geq 1\). Also, the error term in his formulation is
\[
O_{K, A, a}(Ne^{-\sqrt{\log X}/O_{K, A, a}(1)}),
\]
where \(N\) is the supremum of norms in the region \((K \otimes \mathbb{Q} \mathbb{R})_{<X_1, \ldots, X_{r_1 + r_2}}\). His arguments can easily be adapted to give Theorem 1.2.1. It is not clear if the theorem stays true for general convex bodies \(C \subset (K \otimes \mathbb{Q} \mathbb{R})_{<X_1, \ldots, X_{r_1 + r_2}}\).

We formulated our Theorem 3.1.1 for convex bodies \(C\) in \((K \otimes \mathbb{Q} \mathbb{R})_{<X_1, \ldots, X_{r_1 + r_2}}\) in parallel with Theorem 1.2.1 because our intended application requires this case. It can also be shown for \(C = (K \otimes \mathbb{Q} \mathbb{R})_{<X_1, \ldots, X_{r_1 + r_2}}\) with essentially the same proof but we do not present the argument specifically needed for this case.

In addition to prime elements, his theorem in [11] contains statements about how many prime ideal numbers there are—a concept which has something to do with the fact that the number ring \(\mathcal{O}_K\) is not necessarily a unique factorization domain.
(UFD)—in the convex body \( C \). Our result will also be shown in this generality, replacing prime ideal numbers by prime elements in ideals (§1.5.2). This generality will be useful for example when one wants to find linear patterns of points in \( \mathbb{Z}^2 \) at which a given quadratic form \( ax^2 + bxy + cy^2 \) takes primes values; see [7, §10].

1.4. Outline of the method. The method is largely the same as Mitsui’s in [11]. On top of the consideration of potential Siegel zeros, there are two expositional improvements which we hope are making this text more accessible to the modern reader. One is that we got rid of the Byzantine notion of ideal numbers (see e.g. [13, p. 486] for this notion) and used exclusively the usual notion of ideals to formulate and prove the results. The other is that Mitsui’s highly unpenetrable seemingly ad hoc computations and estimates has mostly been replaced by general discussions of idèles and Fourier analysis.

Now we outline Mitsui’s [11] method. The case of quadratic fields is also sketched in references like [6, p. 130] [9, p. 318].

Let us assume that there is no Siegel zeros because we want to focus on the main ideas here. First, from a known zero-free region of Hecke \( L \)-functions, we have the following estimate of the weighted sum of prime ideals (Theorem 2.3.2). Below, \( \psi \) is an arbitrary mod \( q \) Hecke character.

\[
\sum_{\psi(p)} \log N(p) = \begin{cases} 
N + O_K(Ne^{-\sqrt{\log N/O_K(1)}}) & \text{if } \psi \equiv 1, \\
O_K(Ne^{-\sqrt{\log N/O_K(1)}}) & \text{otherwise}
\end{cases}
\] (1.5)

For simplicity let us further assume \( \mathcal{O}_K \) is a UFD. If we choose an \( \mathcal{O}_K \)-fundamental domain \( D \subset \mathcal{O}_K \setminus \{0\} \), prime ideals (now all principal) correspond to unique prime elements in \( D \). Thus the left hand side of (1.5) becomes a sum over prime elements \( \pi \in D \) with \( N(\pi) < N \).

Though not strictly needed, there is a standard, geometric way of choosing \( D \). For example when \( K = \mathbb{Q}(\sqrt{2}) \), an instance of \( D \) is shown in Figure 1.

Mod \( q \) Hecke characters are characters of the mod \( q \) idèle class group \( C(q) \), which is a commutative Lie group. By modding out a subgroup \( \sim \mathbb{R}^0 \times \mathbb{R} \), we obtain a compact commutative Lie group \( C(q)/\mathbb{R}^0 \). It is a finite cover of another group

\[
C(q)/\mathbb{R}^0 \rightarrow (K \otimes \mathbb{Q} \mathbb{R})/(\mathcal{O}_K^\times \cdot \mathbb{R}^0),
\]

where the kernel is \( (\mathcal{O}_K/q)^\times \). Fourier analysis (§11) can be applied to this torus. This allows us to count prime elements in a specified mod \( q \) class and inside thin cones as in Figure 2. Such cones are written as \( \mathbb{R}^0_P \) in the main body of the text.

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\(^3\)Actually these two aspects have been preventing the author from being able to read through his paper to this day.

\(^4\)By the way it’s a pity that they don’t mention Mitsui’s work even though he pretty much predates these books.

\(^5\)Let us not forget that we are assuming there is no Siegel zero. Also, in the main body of the text \( \psi(p) \) is written \( \psi([p]) \) for good reason.
Figure 1. A fundamental domain for $K = \mathbb{Q}(\sqrt{2})$ truncated at some norm. The scale is adjusted so that the point $x + y\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ looks like $(x, y\sqrt{2}) \in \mathbb{R}^2$. The norm is $N(x + y\sqrt{2}) = |x^2 - (y\sqrt{2})^2|$. The hyperbolas pass through points of the same norm. In this example we know $(K \otimes \mathbb{Q} \mathbb{R})/O_K \cong \mathbb{R}_{>0} \times (\mathbb{R}/\mathbb{Z})$.

Thus we will obtain

\[
\sum_{\pi \equiv a \mod q \atop \pi \in \mathbb{R}_{>0}^\times P \atop |N(\pi)| < N} \log |N(\pi)| = \frac{\text{const.}}{\varphi_K(q)} N + O_K(N e^{-\sqrt{\log N}/O_K(1)}),
\]

where the constant depends only on $K$ and $\mathbb{R}_{>0}^\times P$, and is proportional to the volume of the set $\mathbb{R}_{>0}^\times P \cap \{x \in K \otimes \mathbb{Q} \mathbb{R} \mid |N(x)| < 1\}$. See Corollary 4.2.4.

In \([5]\) we will use (1.6) to count prime elements in a given convex set $C$ by approximating it with segments of thin cones (Figure 3). This approximation can be done within the claimed error.

Management of the error, notably its independence from $q$, requires somewhat careful and boring analysis, which is done in Appendices A–D.

1.5. General notation and terminology.

1.5.1. Multiplicative groups of complex numbers. Here are some subgroups of $\mathbb{C}^\times$ that we frequently use in this paper.

- We write $\mathbb{R}_{>0}^\times$ for the multiplicative group of positive real numbers. We know $\mathbb{R}_{>0}^\times \cong \mathbb{R}$ via the logarithm.
- We write $S^1 \subset \mathbb{C}^\times$ for the subgroup of complex numbers of absolute value 1. It is canonically isomorphic to $\mathbb{R}/\mathbb{Z}$ by the map $\theta \mod \mathbb{Z} \mapsto e^{2\pi i \theta}$.

\footnote{Let us repeat that we are assuming that there is no Siegel zero.}
Fourier analysis allows us to focus on prime elements inside thin cones like this (colored dark gray).

Approximation of a convex body $C$ by segments of thin cones

- We know
  
  $\mathbb{R}^\times \cong \mathbb{R}_{>0}^\times \times \{\pm 1\} \cong \mathbb{R} \times \{\pm 1\}$ and
  
  $C^\times \cong \mathbb{R}_{>0}^\times \times S^1 \cong \mathbb{R} \times (\mathbb{R}/\mathbb{Z})$, 

  where we choose the projection $C^\times \to \mathbb{R}_{>0}^\times$ to be $z \mapsto |z|^2$. 
1.5.2. Number rings. We write \( \text{Ideals}_K \) for the multiplicative monoid of non-zero ideals of \( \mathcal{O}_K \). We write \( h_K := \# \text{Cl}(K) \) for the class number. For each class \( \lambda \in \text{Cl}(K) \), we fix a representing fractional ideal \( a_\lambda \subset K \). For the trivial class \( 0 \in \text{Cl}(K) \), let us choose \( a_0 := \mathcal{O}_K \). Let \( a_\lambda^{-1} = \{ \alpha \in K \mid \alpha a_\lambda \subset \mathcal{O}_K \} \) be the inverse fractional ideal. We have a bijection

\[
(1.7) \quad \left\{ (0) \neq a \in \mathcal{O}_K \text{ ideals } \mid [a] = \lambda \in \text{Cl}(K) \right\} \cong \frac{(a_\lambda^{-1} \setminus \{0\})/\mathcal{O}_K^\times}{a = a a_\lambda \leftrightarrow a}.
\]

It follows that if we are given an \( \mathcal{O}_K^\times \)-fundamental domain \( \mathcal{D} \) for \( (K \otimes \mathbb{Q} \mathbb{R})^\times \), there is a bijection

\[
(1.8) \quad \text{Ideals}_K \cong \bigcup_{\lambda \in \text{Cl}(K)} a_\lambda^{-1} \cap \mathcal{D}.
\]

Let us call an element \( \pi \in a \) of a non-zero fractional ideal \( a \) a prime element of \( a \) if \( \pi a^{-1} \) is a non-zero prime ideal (where \( a^{-1} \) is the inverse fractional ideal of \( a \)).

Denote by \( \mathcal{P}(a) \) the set of prime elements of \( a \). In this paper, sums and products \( \bigoplus_p \sum_p \prod_p \) indexed by the letter \( p \) will always mean those indexed by non-zero prime ideals, often subject to some additional conditions. Likewise, sums and products \( \bigoplus_{\pi} \sum_{\pi} \prod_{\pi} \) indexed by \( \pi \) will mean those indexed by prime elements.

Euler’s totient function \( \varphi_K \) for \( K \) is defined as the function \( a \mapsto \#(\mathcal{O}_K/a)^\times \), \( \text{Ideals}_K \to \mathbb{N} \).

1.5.3. Landau’s \( O(-) \) symbol. The symbol \( O_{a,b,c}(1) \) denotes a complex value whose absolute value is bounded from above by a constant \( C_{a,b,c} > 0 \) depending only on the parameters \( a, b, c \). The specific value can be different from occasion to occasion. In particular, an \( O(1) \) means a complex value whose absolute value is bounded by an absolute constant independent of any parameter even if some parameters are at play.

When \( f(t) \) is a positively valued function of \( t \), the expression \( O_{a,b,c}(f(t)) \) means \( O_{a,b,c}(1) \cdot f(t) \). Often \( t \) will be a positive real parameter, and the bound may be valid only when \( t \) is large enough. We allow this threshold to depend on the parameters \( a, b, c \). We shall indicate it if the threshold depends on more (or less) parameters.

We make an exception of allowing ourselves to use \( O_K(-) \) even when the bounds depend on additional choices related to \( K \), as long as those choices are independent of the problem at hand and can be made once the number field \( K \) is given. Such choices include an isomorphism \( \mathcal{O}_K^\times \cong (\mathbb{Z}/a\mathbb{Z}) \times \mathbb{Z}' \) and representatives \( a_\lambda (\lambda \in \text{Cl}(K)) \) for the ideal class group.

The expression \( f(t) \ll_{a,b,c} g(t) \) means \( f(t) = O_{a,b,c}(g(t)) \). The expression \( f(t) \gg_{a,b,c} g(t) \) means that \( f(t) \ll_{a,b,c} g(t) \) and \( g(t) \ll_{a,b,c} f(t) \) both hold.

It goes without saying that a statement of the form

- If \( A = B/O_{a,b,c}(1) \), then \( X = O_{d,e,f}(Y) \), or
- If \( A \ll_{a,b,c} B \), then \( X \ll_{d,e,f} Y \)

\footnote{An algebro-geometric justification for this piece of terminology is that for any integral scheme \( X \) and a line bundle \( \mathcal{L} \) on it, we could call a non-zero section \( s : \mathcal{O}_X \to \mathcal{L} \) prime if its zero locus \( (s) \subset X \) is an integral scheme.}
should be read as the statement that there are large enough positive constants $C_{a,b,c} > 1$ and $C_{d,e,f} > 1$ depending only on the parameters indicated in subscript such that the following holds:

\[
\text{If } |A| < B/C_{a,b,c}, \text{ then } |X| < C_{d,e,f} Y.
\]

Observe that the larger the constants $C_{a,b,c}, C_{d,e,f}$ are, the more restrictive the hypothesis is and the weaker the conclusion. So the statement gets more likely to hold.

1.5.4. Usage of some letters. The letter $X$ is often used to indicate the length which characterizes the region in $K \otimes_{\mathbf{Q}} \mathbf{R}$ in which we search for prime elements.

The letter $N$ is used to bound the norm of elements. It is often set to be $N := X^n$, where $n = [K : \mathbf{Q}]$.

The letters $M, Y > 1$ will be used to denote generic positive parameters which grow at a pseudopolynomial rate $e^{\sqrt{\log N}/O_K(1)}$. The specific value of $O_K(1)$ may vary from occasion to occasion.

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2. Recollection of Prime Ideal Theorem

Let us recall the notion of Hecke characters, the associated $L$-functions, their known zero-free regions and the Prime Ideal Theorem 2.3.2 that results.

2.1. Hecke characters. We denote the group of idèles by $I_K$. A Hecke character is a continuous homomorphism

\[
\psi : I_K/K^\times \to S^1.
\]

In the literature it is also called a Gro\ss encharakter. Every such $\psi$ factors through a quotient of the following form for some non-zero ideal $q = \prod_p$ finitely many $p^{n_p}$:

\[
\psi : I_K/K^\times \to \left( (K \otimes_{\mathbf{Q}} \mathbf{R})^\times \times \bigoplus_{p | q} K^\times / (1 + p^{n_p} \mathcal{O}_{K,p}) \times \bigoplus_{p \nmid q} \mathbf{Z} \right) / K^\times
\]

\[
\to S^1.
\]

In this case we say $\psi$ is a mod $q$ Hecke character. It is called a primitive mod $q$ Hecke character if it is not a mod $q'$ Hecke character for any other divisor $q' | q$, $q' \neq q$. In this case $q$ is called the conductor of $\psi$. See [1, XV-§3], [9] VII-§3 and XVI-§7] or [13] VI-§1 and VII-§6] for more background.

Let us write this quotient (the so-called idèle class group mod $q$) in the middle as

\[
C(q) = C(q)/K^\times := \left( (K \otimes_{\mathbf{Q}} \mathbf{R})^\times \times \bigoplus_{p | q} K^\times / (1 + p^{n_p} \mathcal{O}_{K,p}) \times \bigoplus_{p \nmid q} \mathbf{Z} \right) / K^\times.
\]
2.2. **The L-function.** Let $\psi$ be a Hecke character with conductor $q$. For prime ideals $p \nmid q$, let $[p] \in \widehat{C}(q)$ be the element whose $p$-component is 1 and the other components are trivial. Sometimes the symbol $[p]$ is used to mean its image to $\bigoplus_{p \nmid q} \mathbb{Z}$ as well. Multiplicatively we can define $[a] \in \widehat{C}(q)$ or $\bigoplus_{p \nmid q} \mathbb{Z}$ for every ideal $a$ prime to $q$.

The Hecke $L$-function associated to $\psi$ is defined by (recall $q$ is the conductor)

$$L(s, \psi) := \sum_{\text{ideal prime to } q} \frac{\psi([a])}{N(a)^s} = \prod_{p \nmid q} \frac{1}{1 - \frac{\psi([p])}{N(p)^s}}$$

for $\Re(s) > 1$. $L(s, \psi)$ is known to be a meromorphic function on $\mathbb{C}$ (see e.g. [1, Main Theorem 4.4.1 and p. 346] [9, Theorem 12 on p. 295 and p. 299] or [13, VII-§8 (8.6)]).

2.3. **The zero-free region and prime ideal theorem.** A classical zero-free region is known:

**Theorem 2.3.1 (zero-free region).** There is a positive constant $0 < c_K < 1$ such that the following holds for every non-zero ideal $q \subset \mathcal{O}_K$.

1. For all but possibly one mod $q$ Hecke character $\psi$ (not necessarily primitive), the $L$-function $L(s, \psi)$ does not have zeros or poles in the region (where $s = \sigma + it$ as usual)

$$\sigma > 1 - \frac{c_K}{\log(N(q)(|t| + 4))}.$$ 

2. The potential exceptional $\psi$, if it exists, has to be real (i.e., takes values in $\{-1, +1\}$). This $\psi$ is called the Siegel (or exceptional) character mod $q$ and denoted by $\psi_{\text{Siegel}}$.

3. The $L$-function $L(\psi_{\text{Siegel}}, s)$ can have at most one zero $\beta$ and the zero has to be real:

$$1 - \frac{c_K}{\log(4N(q))} < \beta < 1.$$ 

The zero $\beta$ is called the Siegel (or exceptional) zero mod $q$.

**Proof.** See e.g. [6, Theorem 5.35] or [8, Lemma 2.3], and [12, Theorem 11.7] or [15, Theorem 1.9].

It is known that an $L$-function with a zero-free region as in Theorem 2.3.1 is accompanied by a Prime Ideal Theorem where the prime ideals are counted with weights. To us, the pseudopolynomial control of the error term in the following theorem is important.

**Theorem 2.3.2 (Prime Ideal Theorem).** Let $N > 1$ be a positive number and $q \in \text{Ideals}_K$ be an ideal with norm

$$N(q) < e^{\sqrt{\log N/O_K(1)}}.$$
Let \( \psi \) be a mod \( q \) Hecke character. Then we have
\[
\sum_{p | q, N(p) < N} \psi([p]) \log N(p) = N \cdot 1_{\psi=1} - \frac{N^\beta}{\beta} 1_{\psi=\text{Siegel}} + O_K(N e^{-\sqrt{\log N/O_K(1)}}),
\]
where \( \beta \) is the potential Siegel zero mod \( q \).

The terms \( 1_{[-]} \) are present only when the condition in the subscript is true and equal the constant function \( 1 \) in such cases.

Proof. See e.g. [6, Theorem 5.13].

\[\Box\]

3. The statement and first steps of the proof

Before stating our main result, we have to introduce a few pieces of notation. Let \( a, q \subset O_K \) be non-zero fractional ideals. Write \( (a/qa)^\times \subset a/qa \) for the set of residue classes which can be a single generator of \( a/qa \) as an \( O_K \)-module. By Proposition C.2.4, we have a canonical set map \( (K \otimes Q R)^\times \times (a/qa)^\times \to C(q) \). Thus for any function \( \psi \) on \( C(q) \), we may consider its pullback to \( (K \otimes Q R)^\times \times (a/qa)^\times \). We write this restriction as \( (x,\alpha) \mapsto \psi(x;\alpha) \). In particular if we fix a residue class \( \alpha \in (a/qa)^\times \), we obtain a function \( \psi(-;\alpha) : (K \otimes Q R)^\times \to C \).

3.1. The statement.

**Theorem 3.1.1.** Let \( K \) be a number field of degree \( n = [K : Q] = \dim_Q K \). Fix a non-zero fractional ideal \( a \subset O_K \).

Let \( X > 1 \) be a real number. Set \( N := X^n \).

Let \( C \subset (K \otimes Q R)^\times_{<XN(a)^{1/n}} \) be a convex open set and \( q \) be an ideal whose norm obeys a bound
\[
N(q) < e^{\sqrt{\log N/O_K(1)}} \tag{3.1}
\]
with a large enough constant \( O_K(1) > 1 \). Let \( \alpha \in (a/qa)^\times \). Then we have
\[
\sum_{\pi \in C \cap a, \pi=\alpha \mod a/qa} \log N(\pi a^{-1}) = \\
\frac{1}{N(a)^{\phi_K(q)}} 2^{r_1 r_2} h_K R_K \int_C 1 - \psi_{\text{Siegel}}(-;\alpha) N((-) a^{-1})^{\beta-1} d\mu_{\text{add}} \\
+ O_K \left( N e^{-\sqrt{\log N/O_K(1)}} \right), \tag{3.2}
\]
where by convention we ignore the term \( \psi_{\text{Siegel}}(-;\alpha) N((-) a^{-1})^{\beta-1} \) if there is no Siegel zero mod \( q \); in this case the integral is \( \text{Vol}_{\text{add}}(C) \).

Our convention for the measure \( \mu_{\text{add}}, \text{Vol}_{\text{add}} \) is in §A.5. Though this is not needed in the sequel, the coefficient \( w_K/(2^{r_1 r_2} h_K R_K) \) is equal to \( 2^{r_2}/(\text{res}_{s=1}(\zeta_K(s))/\sqrt{|D_K|}) \) by Class Number Formula [9, Theorem 5 on p. 161], where \( D_K \) is the discriminant of \( K \).

Requirements for the \( O_K(1) \) constant in (3.1) will be specified in §4.1.2, 5.2.5.
3.2. First reduction. Here we show that in Theorem 3.1.1 we may assume \( \mathcal{a} \) equals the pre-chosen representative \( \mathcal{a}_\lambda \) of its ideal class \( \lambda \) without loss of generality. This will also allow us to simplify the range of the convex set \( C \subset (K \otimes \mathbb{Q} \mathbb{R})_{<X} \), omitting \( N(\mathcal{a}_\lambda)^{1/n} \), because \( N(\mathcal{a}_\lambda) \) is an \( O_K(1) \) when representatives \( \mathcal{a}_\lambda \) are fixed beforehand. The reader who is not interested in Theorem 3.1.1 for a general fractional ideal \( \mathcal{a} \) may of course skip over this step to 3.3.

By the bijection \( \mathfrak{I} \)

\[
\mathfrak{I}(K) \cong \bigcup_{\lambda \in \mathcal{C}(K)} \left( \mathfrak{a}_\lambda^{-1} \setminus \{0\} \right) / \mathcal{O}_K^x
\]

there are an index \( \lambda \) and an element \( a_0 \in \mathfrak{a}_\lambda^{-1} \) such that \( \mathcal{a} = a_0 \mathcal{a}_\lambda \). We have

\[
3.3. \quad N(\mathcal{a}) = N(a_0)N(\mathcal{a}_\lambda).
\]

We argue that Theorem 3.1.1 for \( \mathcal{a} \) follows from the case of \( \mathcal{a}_\lambda \) if we multiply some parameters by \( a_0^{-1} \in K^\times \subset (K \otimes \mathbb{Q} \mathbb{R})^\times \).

3.2.1. The left hand side. Via the correspondence of prime elements \( \mathcal{P}(\mathcal{a}) \xrightarrow{\cong} \mathcal{P}(\mathcal{a}_\lambda) \), \( \pi \mapsto a_0^{-1}\pi =: \pi' \), the left hand sides of (3.2) for \( \mathcal{a} \) and \( \mathcal{a}_\lambda \) are equal:

\[
3.4. \quad \sum_{\pi \in \mathcal{C} \cap \mathfrak{a}, \pi \equiv \mathfrak{a} \mod \mathfrak{a} / \mathfrak{q}_\mathfrak{a}} \log N(\pi \mathfrak{a}^{-1}) = \sum_{\pi' \equiv \mathfrak{a}_\lambda^{-1} \cap \mathfrak{C} \cap \mathfrak{a}_\lambda, \pi' \in \mathfrak{a}_\lambda^{-1} \cap \mathfrak{C} \cap \mathfrak{a}_\lambda} \log N(\pi' \mathfrak{a}_\lambda^{-1}).
\]

There is a constant \( A = O_K(1) \) independent\(^8\) of \( \mathcal{a} \) such that \( a_0 \) can be taken from \((K \otimes \mathbb{Q} \mathbb{R})_{<AN(\mathcal{a}_\lambda^{-1})^{1/n}}\), see e.g. [7 Lemma 4.11] or [10 Lemma 4.2], the latter of which is stated for principal ideals but whose proof is valid for the general case if appropriately read. It follows that \( a_0^{-1} \mathcal{C} \subset (K \otimes \mathbb{Q} \mathbb{R})_{<AXN(a_0)^{-1}} \) for another \( A' = O_K(1) \). Write \( X' := XA' \) and

\[
3.5. \quad N' := (X')^n \asymp_K N.
\]

3.2.2. The right hand side. On the right hand sides, since the Jacobian of the multiplication map \([a_0]: a_0^{-1} \mathcal{C} \rightarrow \mathcal{C}\) is \( N(a_0) \) we have the compatibility of measures

\[
[a_0]^* \mu_{\text{add}} = N(a_0) d\mu_{\text{add}}.
\]

We can compute the pullback of the norm function as \([a_0]^* \mathcal{N}(-a^{-1}) = \mathcal{N}((-a)^{-1})\).

We have \( a_0^{-1} \alpha \in (\mathfrak{a}_\lambda / \mathfrak{q}_\mathfrak{a})^\times \). By the commutative diagram

\[
\begin{array}{ccc}
(K \otimes \mathbb{Q} \mathbb{R})^\times \times (\mathfrak{a}/\mathfrak{q})^\times & \cong & [a_0] \\
& & \downarrow \mathbb{C}(\mathfrak{q}) \\
(K \otimes \mathbb{Q} \mathbb{R})^\times \times (\mathfrak{a}_\lambda/\mathfrak{q}_\mathfrak{a})^\times & \rightarrow & \mathcal{C}(\mathfrak{q})
\end{array}
\]

we have \([a_0]^* \psi(-; \alpha) = \psi(-; a_0^{-1} \alpha)\).

\(^8\)The independence of \( A \) from \( \mathcal{a} \) is not needed if one is fine with the dependence on \( \mathcal{a} \) of the constants in the error term in the theorem. But this independence plays a role in our future work.
It follows that

\begin{equation}
(3.6) \quad \int_C 1 - \psi_{\text{Siegel}}(-; \alpha) N((-)a^{-1})^\beta d\mu_{\text{add}} = N(a_0) \int_{a_0^{-1}C} 1 - \psi_{\text{Siegel}}(-; a_0^{-1}\alpha) N((-)a_0^{-1})^\beta d\mu_{\text{add}}.
\end{equation}

3.2.3. **End of reduction to** \(a_\lambda\). Now if we assume the validity of Theorem 3.1.1 for \(a_\lambda\) and the convex body \(a_0^{-1}C \subset (K \otimes \mathbb{Q} R)_{< \lambda^{-n}}^{\times}\), we have

\[
\sum_{\pi' \in a_0^{-1}C \cap a_\lambda, \pi' \equiv a_0^{-1}\alpha \text{ in } a_\lambda/qa_\lambda} \log N(\pi' a_\lambda^{-1}) = \sum_{a_\lambda} \int_{a_\lambda^{-1}C} 1 - \psi_{\text{Siegel}}(-; a_0^{-1}\alpha) N((-)a_0^{-1})^\beta d\mu_{\text{add}}
\]

\[\approx \frac{w_K}{N(a_\lambda)\varphi_K(q)} 2^{r_1} \pi r_2 h_K R_K \int_{a_0^{-1}C} 1 - \psi_{\text{Siegel}}(-; a_0^{-1}\alpha) N((-)a_0^{-1})^\beta d\mu_{\text{add}} + O_K(N^{1/2})\sqrt{N}/O_K(1).
\]

By (3.3) (3.4) (3.5) (3.6), this is equivalent to Theorem 3.1.1 for \(a\) and \(C \subset (K \otimes \mathbb{Q} R)_{< \lambda^{-n}}^{\times}\). This reduces the problem to the case of \(a = a_\lambda\) for some \(\lambda \in \text{Cl}(K)\).

3.3. **From prime ideals to prime elements.** We already know how many prime ideals there are within bounded norms (Theorem 2.3.2). Here is the first step to deduce information on the number of prime elements in bounded regions.

Let the situation be as in Theorem 3.1.1 and let \(\psi\) be a mod \(\mathfrak{q}\) Hecke character.

3.3.1. **Prime ideals and prime elements.** Let \(D \subset (K \otimes \mathbb{Q} R)_{\mathbf{c}}^{\times}\) be an arbitrary \(\mathcal{O}_K^{\times}\)-fundamental domain.

We temporarily use the bijection (1.8) in the form

\[
\text{Ideals}_K \cong \bigsqcup_{\lambda \in \text{Cl}(K)} a_\lambda \cap D.
\]

Namely we use the complete set of representatives \(\{a_\lambda\}_\lambda\) as opposed to \(\{a_\lambda\}_\lambda\). It follows that for every \(p\), there are a unique \(\lambda \in \text{Cl}(K)\) and a \(\pi \in a_\lambda \cap D\) such that \(p = \pi a_\lambda^{-1}\). In this situation \(\pi\) is a prime element of \(a_\lambda\) by the very definition (1.5.2) of this notion. This establishes the bijection

\[
\{p \subset \mathcal{O}_K \mid \text{non-zero prime ideals} \} \cong \bigsqcup_{\lambda \in \text{Cl}(K)} \mathcal{P}(a_\lambda) \cap D.
\]

3.3.2. **To subtract a diagonal image.** Now let \(p\) be a non-zero prime ideal not dividing \(\mathfrak{q}\) and \(\pi \in a_\lambda \cap D\) the corresponding prime element. Let us write \(\text{diag}(\pi) \in \mathcal{C}(\mathfrak{q})\) for the canonical diagonal image

\[
\text{diag}(\pi) = (K \otimes \mathbb{Q} R)^{\times} \times \bigoplus_{p_1 | \mathfrak{q}} K^{\times}/(1 + p_1^{n_{p_1}} \mathcal{O}_{K,p_1}) \times \bigoplus_{p \neq \mathfrak{q}} \mathbb{Z}.
\]

We know \(r_{p_2}(\pi) = 0\) for \(p_2 \nmid \mathfrak{q}\) different from \(p\). This implies that \(\text{diag}(\pi) = (\pi; \text{diag}(\pi); [p])\), where \(\text{diag}(\pi) \in \bigoplus_{p_1 | \mathfrak{q}} K^{\times}/(1 + p_1^{n_{p_1}} \mathcal{O}_{K,p_1})\) is also the diagonal
image, which we will simply write as $\pi$ in the sequel. Since the Hecke character $\psi$ kills the image of $K^\times$, we know

$$
\psi([p]) = \psi([p] \text{ diag}(\pi)^{-1}) = \psi(\pi^{-1}; \pi^{-1}; 0) = \overline{\psi}(\pi; \pi; 0).
$$

By changing the notation $\psi \leftrightarrow \overline{\psi}$, from Theorem 2.3.2 we deduce:

**Corollary 3.3.3.** Under the notation of Theorem 2.3.2 and this §3.3, we have

$$
\sum_{\lambda \in \text{Cl}(K)} \sum_{\pi \in a_\lambda \cap D, N(\pi a_\lambda^{-1}) < N} \psi(\pi; \pi; 0) \log N(\pi a_\lambda^{-1}) < N \psi(\pi; \pi; 0) \log N(\pi^{-1} a_\lambda^{-1}) (3.7)
$$

4. **Main theorem for thin cones—Fourier analysis**

Here we will combine Fourier analysis recalled in §§B–C and Corollary 3.3.3 to prove Theorem 3.1.1 for a special type of subsets (Corollary 4.2.3).

Let the situation be as in Theorem 3.1.1. By Proposition C.2.4 we have a chart

$$
C(q)/\mathbb{R}_>^x \cong \bigcup_{\lambda \in \text{Cl}(K)} \frac{(K \otimes \mathbb{Q} \mathbb{R})^x / \mathbb{R}_>^x}{\mathcal{O}_K^x}.
$$

Let $\lambda \in \text{Cl}(K)$ be an ideal class and let $\alpha \in (a_\lambda / qa_\lambda)^x$.

4.1. **Thin cones.** Let $P \subset (K \otimes \mathbb{Q} \mathbb{R})^x / \mathbb{R}_>^x$ be a connected open set such that the map $P \to (K \otimes \mathbb{Q} \mathbb{R})^x / (\mathcal{O}_K^x \mathbb{R}_>^x)$ is injective. There is an $\mathcal{O}_K^x$-fundamental domain $D \subset (K \otimes \mathbb{Q} \mathbb{R})^x$ containing $\mathbb{R}_>^x P$. Fix any such $D$.

It follows that $P \times \{\alpha\} \to C(q)/\mathbb{R}_>^x$ is also injective. Write $P \times \{\alpha\} \subset C(q)/\mathbb{R}_>^x$ for the image as well. Write $(\mathbb{R}_>^x P) \times \{\alpha\}$ for the corresponding subset of either $(K \otimes \mathbb{Q} \mathbb{R})^x \times (a_\lambda / qa_\lambda)^x$ or $C(q)$.

4.1.1. **Applying Fourier analysis.** Suppose further $P$ is convex and small enough, depending on $\varphi_K(q)$, to satisfy the hypothesis of Proposition C.2.2. The term *thin cones* refers to subsets of the form $\mathbb{R}_>^x P$ for such a $P$. We then have a decomposition of the indicator function $1_{P \times \{\alpha\}}$ on $C(q)/\mathbb{R}_>^x$ as in Proposition C.2.2

$$
1_{P \times \{\alpha\}} = \sum_{\psi \in \Psi \subset C(q)/\mathbb{R}_>^x} c_\psi \psi + G + H. (4.1)
$$
Now we take the sum \( \sum_{\psi \in \Psi} c_{\psi} \) of (3.7). Combined with (4.1) we obtain the following

\[
(4.2) \quad \sum_{\pi \in \mathfrak{a}_\lambda \cap \mathcal{D}, \ \mathbf{N}(\pi^{a_{\lambda}}) < N, \ \pi \in \mathfrak{a}/q\mathfrak{a}} \log \mathbf{N}(\pi^{a_{\lambda}^{-1}}) + G((\pi;\pi;0)) + H((\pi;\pi;0))
\]

\[
= \frac{\text{Vol}_{\text{id} - \text{ele}}(P \times \{\alpha\})}{\text{Vol}_{\text{id} - \text{ele}}(C(q)/\mathbb{R}_{>0}^\times)} \left( N(1 + O_n(1/M)) 
- \frac{N^\beta}{\beta} (\psi_{\text{Siegel}}(P \times \{\alpha\}) + O_n(1/M)) 
+ O_K(\varphi_K(q)Y^{n-1})O_K(Ne^{-\sqrt{\log N}/O_K(1)}) \right).
\]

Here, note that as \( \psi_{\text{Siegel}} \) takes values in the discrete set \( \{\pm 1\} \) and \( P \times \{\alpha\} \) is connected, the value \( \psi_{\text{Siegel}}(P \times \{\alpha\}) \in \{\pm 1\} \) is well defined.

The parameters \( Y, M > 1 \) are from Proposition C.2.2. Let us specify their values.

4.1.2. Estimate of errors. First, we want that the term

\[
O_K(\varphi_K(q)Y^{n-1})O_K(Ne^{-\sqrt{\log N}/O_K(1)})
\]

in (4.2) remain in the form \( O_K(Ne^{-\sqrt{\log N}/O_K(1)}) \). For this, we declare that the \( O_K(1) \) constant in (3.1) in the statement of Theorem 3.1.1 be large enough (note that \( \varphi_K(q) < N(q) \)) and that the parameter \( Y \) be an \( e^{\sqrt{\log N}/O_K(1)} \) with \( O_K(1) \) large enough. (The thresholds for these choices of course depends on the \( O_K(1) \) constant in the exponent in the displayed formula.)

Having fixed \( Y \), we take \( M \) to be an \( e^{\sqrt{\log N}/O_K(1)} \) with \( O_K(1) \) even larger so that \( H((\pi;\pi;0)) = O_n(M\log Y) \) will be an \( O_K(e^{-\sqrt{\log N}/O_K(1)}) \); it follows then that in (4.2)

\[
\sum_{\pi} H((\pi;\pi;0)) = O_K(Ne^{-\sqrt{\log N}/O_K(1)})
\]

because the number of \( \pi \)'s is at most \( \#\{x \in \mathfrak{a}_\lambda \cap \mathcal{D} \mid \mathbf{N}(x^{a_{\lambda}}) < N\} = O_K(N) \).

The above choices allow us to conclude also

\[
\sum_{\pi} G((\pi;\pi;0)) = O_K(Ne^{-\sqrt{\log N}/O_K(1)}).
\]

As a result, for every \( \alpha, P \) as in the beginning of §4.1.1 we get:

\[
(4.3) \quad \sum_{\pi \in \mathfrak{a}_\lambda \cap \mathcal{D}, \ \mathbf{N}(\pi^{a_{\lambda}^{-1}}) < N, \ \pi \in \mathfrak{a}/q\mathfrak{a}} \log \mathbf{N}(\pi^{a_{\lambda}^{-1}}) = \frac{\text{Vol}_{\text{id} - \text{ele}}(P \times \{\alpha\})}{\text{Vol}_{\text{id} - \text{ele}}(C(q)/\mathbb{R}_{>0}^\times)} \left( N - \frac{N^\beta}{\beta} \psi_{\text{Siegel}}(P \times \{\alpha\}) \right) + O_K(Ne^{-\sqrt{\log N}/O_K(1)}).
\]

4.2. Slight change of notation. Let us rewrite the condition \( \mathbf{N}(\pi^{a_{\lambda}^{-1}}) < N \) in the left hand side of (4.3) as \( \mathbf{N}(\pi) < N_1(a_1)N =: N_1 \) and write this \( N_1 \) as \( N \) afresh. Also, we may consider this formula for two values \( N' < N \) and take the difference. This gives:
Corollary 4.2.1. Let \( N' < N \). Let \( q \) be a non-zero ideal whose norm satisfies
\[
N(q) < e^{\sqrt{\log N'/O_K(1)}}.
\]

Let \( \lambda \in \text{Cl}(K) \), \( \alpha \in (a_\lambda/q a_\lambda)^\times \) be arbitrary, and \( P \subset (K \otimes Q R)^\times /R^\times_{>0} \) be a convex connected open set small enough (depending on \( \varphi_K(q) \)) to satisfy the hypothesis of Proposition 4.2.2.

Then we have
\[
\sum_{\pi \in a_\lambda \cap R^\times_{>0} P, \ N' < N(\pi) < N, \ \pi \equiv \alpha \mod{a_\lambda/q a_\lambda}} \frac{1}{N(a_\lambda)} \text{Vol}_\text{dèôle}(P \times \{\alpha\}) \left( (N - N') - \psi_{\text{Siegel}}(P \times \{\alpha\}) \frac{N(\beta) - N(\beta')}{\beta N(a_\lambda)^{\beta-1}} \right)
+ O_K(N e^{-\sqrt{\log N'/O_K(1)}}).
\]

\[\text{(4.4)}\]

4.2.2. Description as an integral. It will be convenient to note that we can describe the previous quantity as an integral
\[
(N - N') - \psi_{\text{Siegel}}(P \times \{\alpha\}) \frac{N(\beta) - N(\beta')}{\beta N(a_\lambda)^{\beta-1}} = \int_{N'}^N 1 - \frac{\psi_{\text{Siegel}}(P \times \{\alpha\})}{N(a_\lambda)^{\beta-1}} x^{\beta-1} dx.
\]

Recall the basis \( u_1, \ldots, u_{r_1+r_2} \) and the associated coordinates \( \xi_1, \ldots, \xi_{r_1+r_2} \) of \( R^{r_1+r_2} \) from \( \text{(A.5)} \). The norm map \((K \otimes Q R)^\times \to R\) looks like the composite
\[
(K \otimes Q R)^\times \cong R^{r_1+r_2} \times (S^1)^{r_2} \times \{\pm 1\}^{r_1} \xrightarrow{\text{projection}} R^{r_1+r_2} \to R \quad (\xi_1, \ldots, \xi_{r_1+r_2}) \mapsto e^{\xi_{r_1+r_2}}.
\]

For the reader’s eyeballs’ comfort, write \( \xi := \xi_{r_1+r_2} \). By the substitution \( x = e^\xi \) we can compute the previous integral (multiplied by the constant factor \( \text{Vol}_{\text{mult}}(P) \)) as
\[
\text{Vol}_{\text{mult}}(P) \int_{N'}^N 1 - \frac{\psi_{\text{Siegel}}(P \times \{\alpha\})}{N(a_\lambda)^{\beta-1}} x^{\beta-1} dx = \int_{\log N', \log N} e^\xi - \frac{\psi_{\text{Siegel}}(-; \alpha)}{N(a_\lambda)^{\beta-1}} e^{\beta \xi} |d\xi d\xi_1 \ldots d\xi_{r_1+r_2-1}|.
\]

See \( \text{(A.2)} \) for the definition of the subset \([\log N', \log N] \times P \) of \( R \times H \times (S^1)^{r_2} \times \{\pm 1\}^{r_1} \).

By \( \text{(A.3)} \), noting \( N(-) = e^\xi \), this can be written as an integral in \( d\mu_{\text{add}} \) (let us also switch to the additive notation \([N', N] \cdot P \subset K \otimes Q R\) from \( \text{(A.3)} \):
\[
= 2^{r_2} \sqrt{r_1 + r_2} \int_{[N', N] \cdot P} 1 - \frac{\psi_{\text{Siegel}}(-; \alpha)}{N(a_\lambda)^{\beta-1}} N(-)^{\beta-1} d\mu_{\text{add}}.
\]

Now we can rewrite the main terms in \( \text{(4.4)} \). By the choice of \( \mu_{\text{dèôle}} \) we have \( \text{Vol}_{\text{dèôle}}(P \times \{\alpha\}) = \text{Vol}_{\text{mult}}(P) \). We also know the value \( \text{(C.3)} \) of \( \text{Vol}_{\text{dèôle}}(C(q)/R^\times_{>0}) \).
We conclude that the main terms of (4.4) are rewritten as
\begin{equation}
\frac{1}{N(a_\lambda)} \frac{\text{Vol}_{\text{Idèle}}(P)}{\text{Vol}_{\text{Idèle}}(C(q)/\mathbb{R}^2_{>0})} \left( (N - N') - \frac{\psi_{\text{Siegel}}(P \times \{\alpha\})}{\beta N(a_\lambda)} (N^\beta - N'^\beta) \right) \end{equation}
\begin{equation}
\frac{w_K}{N(a_\lambda) \phi_K(q)} 2^n \pi^2 h_K R_K \int_{[N', N], P} 1 - \psi_{\text{Siegel}}(-, \alpha) \left( \frac{N(-)}{N(a_\lambda)} \right)^{\beta - 1} d\mu_{\text{add}}.
\end{equation}

Namely:

**Corollary 4.2.3.** Let 1 < N' < N be positive numbers and q be an ideal whose norm obeys the bound:
\[
N(q) < e^{\sqrt{\log N}/O_K(1)}.
\]

Then Theorem 3.1.1 is true for subsets of the form \(C = [N', N] \cdot P\) where \(P \subset (K \otimes \mathbb{Q} \mathbb{R})^/\mathbb{R}_{>0}^\times\) is an open set which maps injectively into \((K \otimes \mathbb{Q} \mathbb{R})^/\mathbb{R}_{>0}^\times\mathcal{O}_K^\times\) and such that the image is a small convex set of size \(\ll_n \varphi(q)^{(n-2)}\) in the sense of Definition B.1.1.

Let us repeat that the definition of the set \([N', N] \cdot P\) can be found at 4.1.2.

5. END OF PROOF

Lastly, we will “integrate” the estimate (4.5) in various small convex subsets \(P \subset (K \otimes \mathbb{Q} \mathbb{R})^/\mathbb{R}_{>0}^\times\) to prove Theorem 3.1.1. Some care must be taken to make sure that the bound of the error will remain uniform in the ideal parameter q.

Let \(Y > 1\) be again an auxiliary parameter which will be set to be an \(e^{\sqrt{\log N}/O_K(1)}\).

5.1. Reduction to the case of annulus sectors. Let the notation be as in Theorem 3.1.1 in particular let \(X > 1\) be a positive number. Write \(N := X^n\). Let \(C \subset (K \otimes \mathbb{Q} \mathbb{R})_{<X}\) be a convex open set.

Let \(q \subset \mathcal{O}_K\) be an ideal whose norm obeys the upper bound 3.1 and let \(\alpha \in (a_\lambda/q a_\lambda)^\times\) for some chosen ideal class \(\lambda \in \text{Cl}(K)\).

5.1.1. Elements close to the axes. The number of elements of \(a_\lambda\) in the region
\[
\{ x \in (K \otimes \mathbb{Q} \mathbb{R})_{<X} \mid \text{some of the coordinates}
\]
\[
x = (x_1, \ldots, x_{r_1 + r_2}) \in K \otimes \mathbb{Q} \mathbb{R} \cong \mathbb{R}^r_1 \times \mathbb{C}^r_2
\]
\[
\text{has size } |x_i| < 1\}
\]
is \(O_K(X^{n-1}) = O_K(N/X)\). Hence the contribution of these points to the sum \(\sum \pi \log N(\pi a_\lambda^{-1}) \leq \sum \pi \log N\) is \(O_K(N\log X)\). Also, the contribution of this region to the integral \(\int_C (1 - \psi_{\text{Siegel}}(-, \alpha) N((-)/a_\lambda^{-1}) d\mu\) is \(O_n(N/X)\).

Therefore the contribution from points in 5.1.1 can be ignored.

**Definition 5.1.2.** An annulus sector of size \(X/Y\) is a connected component of a subset of \(K \otimes \mathbb{Q} \mathbb{R}\) of the following Baumkuchen-like shape, determined by the choice of parameters \(a_1, \ldots, a_{r_1 + r_2}, \theta_1, \ldots, \theta_{r_2} \in \mathbb{R}\):
\[
B = B(a_1, \ldots, a_{r_1 + r_2}; \theta_1, \ldots, \theta_{r_2})
\]
\[
= \{ x \in K \otimes \mathbb{Q} \mathbb{R} \mid 0 < |x|_{a_\lambda} - a_i < X/Y \text{ for } \forall i,
\]
\[
0 < \arg(|x|_{a_\lambda}) - \theta_i < 1/Y \text{ (mod } 2\pi \mathbb{Z} \text{) for } r_1 + 1 \leq \forall i \leq r_1 + r_2\}.\]
We want to approximate the convex body $C$ by the union of disjoint annulus sectors of size $\leq X/Y$. First we cover $(K \otimes \mathbb{Q} \mathbb{R})_{< X}$ with $O_n(Y^n)$ many disjoint annulus sectors $B_j$ of size $\leq X/Y$. Then let

$$B_j \ (j \in J)$$

be those annulus sectors contained in $C$ and not intersecting with the region $J$.\footnote{Here we could have used the letter $Y$ afresh but we prefer to use a different symbol because in the logical order, the $Y$ in (5.1) cannot be specified until we specify $Y_2$.}

**Lemma 5.1.3.** We have $\text{Vol}(C \setminus \bigcup_{j \in J} B_j) \ll N/Y$. Consequently, there are $\ll_K N/Y$ points in $C \setminus \bigcup_{j \in J} B_j$.

**Proof.** Let $B_j \ (j \in J')$ be those annulus sectors intersecting with $\partial C$ or the region $[\partial \lambda]$.

Since an annulus sector of size $\leq X/Y$ has diameter $O_n(X/Y)$, the volume covered by $B_j$’s ($j \in J'$) is $O_n(J/Y \cdot (\text{Area}(\partial C) + O_n(X^{n-1})))$. Because $\text{Area}(\partial C) = O_n(X^{n-1})$ (see e.g. [3, Lemma A.1]), this quantity is an $O_n(X^n/Y) = O_n(N/Y)$.

5.1.4. **Annulus sectors inside $C$.** By Lemma 5.1.3 we have

$$\sum_{\pi \in a_\lambda \cap C, \ \pi \equiv \alpha \text{ in } a_\lambda/q_\lambda} \log N(\pi a_\lambda^{-1}) = \left( \sum_{j \in J} \sum_{\pi \in a_\lambda \cap B_j, \ \pi \equiv \alpha \text{ in } a_\lambda/q_\lambda} \log N(\pi a_\lambda^{-1}) \right) + O_n((N \log N)/Y).$$

Assuming that the conclusion of Theorem 5.1.1 holds for $B_j, j \in J$ (with uniform $O_K(1)$ constants of course), we can write

$$\sum_{j \in J} \sum_{\pi \in a_\lambda \cap B_j, \ \pi \equiv \alpha \text{ in } a_\lambda/q_\lambda} \log N(\pi a_\lambda^{-1}) =$$

$$\frac{\nu_K}{N(a_\lambda)^{2}} \pi_2 \pi_2 h_K R_K \int_{\bigcup_{j \in J} B_j} 1 - \psi_{\text{Siegel}}(-; \alpha) N((-a_\lambda^{-1})^{\beta-1}) \, d\mu + \#J \cdot O_K(N e^{-\log N/O_K(1)}).$$

We know $\#J = O_n(Y^n)$ because the total number of annulus sectors obeys this bound. It follows that the last error term in (5.2) is an $O_K(N e^{-\log N/O_K(1)})$ if the constant in $Y = e^{\log N/O_K(1)}$ is taken large enough.

The difference between $\int_{\bigcup_{j \in J} B_j}$ and $\int_C$ is, again by Lemma 5.1.3 bounded by $O(1) \cdot \#(C \setminus \bigcup_{j \in J} B_j) = O_n(N/Y)$.

Thus we are reduced to proving Theorem 5.1.1 for $C = B_j$ an annulus sector of size $\leq X/Y$ disjoint from the region $J$.\footnote{Here we could have used the letter $Y$ afresh but we prefer to use a different symbol because in the logical order, the $Y$ in (5.1) cannot be specified until we specify $Y_2$.}

5.2. **Proof for annulus sectors.** Now we claim and prove that Theorem 5.1.1 is true for annulus sectors $B \subset K \otimes \mathbb{Q} \mathbb{R}_{< X}$ of size $< X/10$ say, disjoint from $J$.

Let $Y_2 > 1$ be a new auxiliary integer parameter\footnote{Here we could have used the letter $Y$ afresh but we prefer to use a different symbol because in the logical order, the $Y$ in (5.1) cannot be specified until we specify $Y_2$.} of the form $e^{-\log N/O_K(1)}$.
5.2.1. Annulus sectors in the multiplicative coordinates. Via the bijection $(K \otimes \mathbb{Q} \mathbb{R})^\times \simeq \mathbb{R}^{t_1+t_2} \times (\mathbb{R}/\mathbb{Z})^{r_2} \times \{\pm 1\}^{r_1}$, our annulus sectors look like cubes contained in $[0, \log X]^{r_1+r_2} \times (\mathbb{R}/\mathbb{Z})^{r_2} \times \{\pm 1\}^{r_1}$ whose facets are parallel to coordinate hyperplanes. The left-hand bound “0” in $[0, \log X]^{r_1+r_2}$ is due to the assumption that our set is disjoint from \( [0,1) \).

5.2.2. Cubes in $H \times (S^1)^{r_2} \times \{\pm 1\}^{r_1}$. We consider the image of $[0, \log X]^{r_1+r_2}$ by the projection $\mathbb{R}^{r_1+r_2} = \mathbb{R} \times H \to H$. We cover it with cubes of size $1/2$ with respect to a fixed $\mathbb{R}$-basis of $H$. This requires $O_n((Y_2 \log X)^{r_1+r_2-1})$ cubes.

We also cover the torus $(\mathbb{R}/\mathbb{Z})^{r_2}$ by $Y_2^{r_2}$ cubes of size $1/2$.

**Definition 5.2.3.** By a cube in $(K \otimes \mathbb{Q} \mathbb{R})^\times / \mathbb{R}_{>0}^\times \times \mathbb{Z}^\times$ let us mean a subset $P$ obtained as the product of a cube in $H$ and a cube in $(\mathbb{R}/\mathbb{Z})^{r_2}$ of sizes $1/2$ as in 5.2.2 placed in one of the connected components $\{\pm 1\}^{r_1}$. There are $O_n(Y_2^{r_2-1} \log^{r_1+r_2-1} X)$ of them.

5.2.4. Sandwiching by special type of subsets. Let $P$ be a cube in $(K \otimes \mathbb{Q} \mathbb{R})^\times / \mathbb{R}_{>0}^\times \times \mathbb{Z}^\times$ in the sense of Definition 5.2.3 and consider the intersection $B \cap \mathbb{R}_{>0}^\times P$ in $(K \otimes \mathbb{Q} \mathbb{R})^\times$.

For $1 < a < b$, recall from 3.A.3 that we write $[a, b] \cdot P \subset (K \otimes \mathbb{Q} \mathbb{R})^\times$ for the set of points where the projection to $H \times (S^1)^{r_2} \times \{\pm 1\}^{r_1}$ belong to $P$ and the value of $N(\cdot)$ is in $[a, b]$.

By 3.A.3, up to negligible error there are two real numbers $\log(N/X) < a < b < \log N$ satisfying

$$[a, b] \times P \subset B \cap (\mathbb{R} \times P) \subset [a - \varepsilon, b + \varepsilon] \times P \quad \text{in } \mathbb{R} \times H \times (S^1)^{r_2} \times \{\pm 1\}^{r_1},$$

where $\varepsilon = O_n(1/Y_2)$ by 5.2.1. In the additive notation, this means

$$[e^a, e^b] \cdot P \subset B \cap \mathbb{R}_{>0}^\times P \subset [e^a(1-\varepsilon), e^{b}(1+\varepsilon)] \cdot P \quad \text{in } K \otimes \mathbb{Q} \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$ 

It follows that

$$\sum_{\pi \in \mathfrak{a}_\lambda \cap \{e^a, e^b\} \cdot P} \log N(\pi a_\lambda^{-1}) \leq \sum_{\pi \in \mathfrak{a}_\lambda \cap (B \cap \mathbb{R}_{>0}^\times P)} \log N(\pi a_\lambda^{-1}) \leq \sum_{\pi \in \mathfrak{a}_\lambda \cap \{e^a(1-\varepsilon), e^{b}(1+\varepsilon)\} \cdot P} \log N(\pi a_\lambda^{-1}).$$

5.2.5. Applying the main theorem for the special case. We want to apply Corollary 4.2.3 to the sets $[e^a, e^b] \cdot P$ and $[e^a(1-\varepsilon), e^{b}(1+\varepsilon)] \cdot P$ to rewrite 5.3. We have to verify the hypotheses on $N(\mathfrak{q})$ and the size of $P$:

- We know $e^{\sqrt{\log(e^{a(1-\varepsilon)})}} > e^{\sqrt{\log(N/X)}} > e^{\sqrt{\log(N/2)}} > N(\mathfrak{q})$ by 5.1;
- By the construction of $P$, the image of $P \times \{\alpha\}$ to $(K \otimes \mathbb{Q} \mathbb{R})^\times / \mathbb{R}_{>0}^\times$ is a cube of size $\ll_K 1/Y_2$. By declaring that we shall take the $O_K(1)$ in the bound 5.1 of $N(\mathfrak{q})$ large enough depending on that in $Y_2$, we can ensure that the image of $P \times \{\alpha\}$ is small enough (and automatically convex) to satisfy $1/Y_2 \ll_K N(\mathfrak{q})^{-(n-2)} < \varphi_K(\mathfrak{q})^{-(n-2)}$ as required in Proposition 4.2.2.
Under these circumstances we can apply Corollary 4.2.3 and obtain an estimate of our sum from below:

\[
\sum_{\pi \in \mathfrak{a} \cap (B \cap \mathbb{R}^n_{>0}) \atop \pi \equiv \alpha \mod \mathfrak{a} / q \mathfrak{a}} \log N(\pi a_\lambda^{-1}) >
\]

\[
\frac{w_K}{N(\mathfrak{a}_\lambda) \varphi_K(\mathfrak{q}) 2^r \pi^{r^2} h_K R_K} \int_{[e^a, e^b] \cdot P} 1 - \psi_{\text{Siegel}}(-; \alpha) N((-) a_\lambda^{-1})^{\beta-1} \, d\mu_{\text{add}}
\]

\[
+ O_K(N^{-\sqrt{\log N}/O_K(1)})
\]

and from above by the same expression except that the domain of integration \([e^a, e^b] \cdot P\) is replaced by \([e^a(1 - \varepsilon), e^b(1 + \varepsilon)] \cdot P\).

5.2.6. Difference of upper and lower bounds. The difference of the main terms of the upper and lower bounds in (5.4) is bounded by

\[
\frac{w_K}{N(\mathfrak{a}_\lambda) \varphi_K(\mathfrak{q}) 2^r \pi^{r^2} h_K R_K} \int_{([e^a(1 - \varepsilon), e^b(1 + \varepsilon)] \cdot P) \backslash ([e^a, e^b] \cdot P)} 1 - \psi_{\text{Siegel}}(-; \alpha) N((-) a_\lambda^{-1})^{\beta-1} \, d\mu_{\text{add}}
\]

and is an \(O_K(N/(Y_2)^n)\) because: the factor in front of the integral is an \(O_K(1)\), the integrand always has absolute value \(\leq 2\) and the domain of integration has volume \(< 2N\varepsilon \text{Area}(P) \ll N/(Y_2)^n\). By (5.4) and the triangle inequality we conclude

\[
\sum_{\pi \in \mathfrak{a}_\lambda \cap (B \cap \mathbb{R}^n_{>0}) \atop \pi \equiv \alpha \mod \mathfrak{a} / q \mathfrak{a}} \log N(\pi a_\lambda^{-1}) =
\]

\[
\frac{w_K}{N(\mathfrak{a}_\lambda) \varphi_K(\mathfrak{q}) 2^r \pi^{r^2} h_K R_K} \int_B 1 - \psi_{\text{Siegel}}(-; \alpha) N((-) a_\lambda^{-1})^{\beta-1} \, d\mu_{\text{add}}
\]

\[
+ O_K(N^{-\sqrt{\log N}/O_K(1)})
\]

5.2.7. To take the sum over the cubes. Take the sum \(\sum_P\) of (5.5) over the cubes \(P\) in the sense of Definition 5.2.3. Since there are \(O_K((Y_2)^n \log^{r_1 + r_2 - 1} X)\) of them, we obtain

\[
\sum_{\pi \in \mathfrak{a}_\lambda \cap B \atop \pi \equiv \alpha \mod \mathfrak{a}_\lambda / q \mathfrak{a}} \log N(\pi a_\lambda^{-1}) =
\]

\[
\frac{w_K}{N(\mathfrak{a}_\lambda) \varphi_K(\mathfrak{q}) 2^r \pi^{r^2} h_K R_K} \int_B 1 - \psi_{\text{Siegel}}(-; \alpha) N((-) a_\lambda^{-1})^{\beta-1} \, d\mu_{\text{add}}
\]

\[
+ O_K\left(\frac{N^{\log^{r_1 + r_2 - 1} X}}{Y_2}\right)
\]

\[
+ O_K((Y_2)^n \log^{r_1 + r_2 - 1} X \cdot N^{-\sqrt{\log N}/O_K(1)})
\]
The first error term in (5.6) is an $O_K(N e^{-\log N/O_K(1)})$ with the $O_K(1)$ in the exponent being more or less the same as that in $Y_2 = e^{\log N/O_K(1)}$.

By taking the $O_K(1)$ constant in $Y_2$ large enough compared to the constant in $e^{-\log N/O_K(1)}$ in the second error term, we can make the second error term also an $O_K(N e^{-\log N/O_K(1)})$.

If follows that if we set the $O_K(1)$ constant in $Y_2$ large enough, the two error terms in (5.6) become a new $O_K(N e^{-\log N/O_K(1)})$. Having fixed $Y_2$, we can now finalize the $O_K(1)$ constant in the restriction (5.1) on $N(q)$ following the requirements from §2.4.1-2.5.2. This proves Theorem 3.1.1 for annulus sectors $B$ and accomplishes the goal of §5.2.

Then we run the process of §5.1 to fix the $O_K(1)$ constant in $Y = e^{\log N/O_K(1)}$ and deduce Theorem 3.1.1 in general. This completes the proof of Theorem 3.1.1.

**Appendix A. Conventions on $K \otimes Q R$**

We have to fix some conventions in order to be able to do analysis on $(K \otimes Q R)^{\times}$ and its quotients $(K \otimes Q R)^{\times}/O_K^{\times}$ and $(K \otimes Q R)^{\times}/O_K^{\times}R_{\sigma 0}$. Following the usual convention, we number the embeddings as

$$
\sigma_1, \ldots, \sigma_{r 1} \quad \sigma_{r 1 + 1}, \ldots, \sigma_{r 1 + r 2} \quad \sigma_{r 1 + r 2 + 1}, \ldots, \sigma_{r 1 + 2r 2}
$$

where $\sigma_1, \ldots, \sigma_{r 1}$ are the different real embeddings and $\sigma_{r 1 + 1}, \ldots, \sigma_{r 1 + r 2}$ represent the different conjugate pairs of imaginary embeddings.

**A.1. Norm functions.** For a non-zero ideal $a \subset O_K$, we write $N(a) := \#(O_K/a)$ for its norm. For a non-zero element $\alpha \in O_K$, by slight abuse of notation we write $N(\alpha) := N(\alpha O_K) = |N_{K/Q}(\alpha)|$. The norm functions $N(-)$ and $N_{K/Q}$ extend to $K \otimes Q R$ naturally,

$$
\begin{array}{c}
K \\
\downarrow \\
K \otimes Q R \\
\downarrow \\
Q
\end{array}
\quad \quad \quad
\begin{array}{c}
N or N_{K/Q} \\
\uparrow \quad \downarrow \\
K \otimes Q R \\
\downarrow \\
R
\end{array}
$$

where we have denoted the extensions by the same symbols.

Let us emphasize that $N(-)$ always takes positive values; not to be confused with the ring-theoretic norm $N_{K/Q}(-)$, which does take negative values unless $K$ is totally imaginary.

**A.2. The group structure of $(K \otimes Q R)^{\times}$.** Let $\sigma_i \otimes Q R : K \otimes Q R \to R \quad (1 \leq i \leq r_1)$ and $\sigma_i \otimes Q R : K \otimes Q R \to C \quad (r_1 + 1 \leq i \leq r_1 + r_2)$ be the unique $R$-linear extensions. The map they jointly define

$$
K \otimes Q R \xrightarrow{\sigma_i} R^{r_1} \times C^{r_2}
$$

is known to be an isomorphism of $R$-algebras.

It follows that

$$
(A.1) \quad (K \otimes Q R)^{\times} \cong (R^{\times})^{r_1} \times (C^{\times})^{r_2} \cong R^{r_1 + r_2} \times (S^{1})^{r_2} \times \{\pm 1\}^{r_1},
$$

where the second isomorphism follows the conventions in §1.5.1 in particular the map $C^{\times} \to R$ is defined by $z \mapsto \log(|z|^2)$.
A.3. Decomposition of $\mathbb{R}^{r_1+r_2}$. Write

$$u_{r_1+r_2} := \frac{1}{n} (1,\ldots,1,2,\ldots,2) \in \mathbb{R}^{r_1+r_2}.$$  

In [A.3], the diagonal image $\mathbb{R}^\times_{>0} \subset (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$ corresponds to the subspace

$$\mathbb{R} \cdot u_{r_1+r_2} \times \{1\}^{r_2} \times \{+1\}^{r_1} \subset \mathbb{R}^{r_1+r_2} \times (S^1)^{r_2} \times \{\pm1\}^{r_1}.$$  

Let $H \subset \mathbb{R}^{r_1+r_2}$ be the following $\mathbb{R}$-linear subspace of codimension 1:

$$H := \left\{ (x_1,\ldots,x_{r_1+r_2}) \left| \sum_{i=1}^{r_1+r_2} x_i = 0 \right. \right\}.$$  

This corresponds to elements $x = (x_1,\ldots,x_{r_1};z_1,\ldots,z_{r_2}) \in (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$ of norm $\pm 1$.

We have the direct sum decomposition

$$\mathbb{R}^{r_1+r_2} = \mathbb{R} \cdot u_{r_1+r_2} \oplus H$$  

and hence the isomorphism

$$(K \otimes_{\mathbb{Q}} \mathbb{R})^\times / \mathbb{R}^\times_{>0} \cong H \times (S^1)^{r_2} \times \{\pm1\}^{r_1}.$$  

This also gives a decomposition

$$(K \otimes_{\mathbb{Q}} \mathbb{R})^\times = \mathbb{R} \times H \times (S^1)^{r_2} \times \{\pm1\}^{r_1}$$

$$= \mathbb{R}^\times_{>0} \times (K \otimes_{\mathbb{Q}} \mathbb{R})^\times / \mathbb{R}^\times_{>0}.$$  

Given a subset $P \subset (K \otimes_{\mathbb{Q}} \mathbb{R})^\times / \mathbb{R}^\times_{>0}$, we write $\mathbb{R}^\times_{>0} \cdot P$ for its inverse image in $(K \otimes_{\mathbb{Q}} \mathbb{R})^\times$. When we see it as a subset of $\mathbb{R} \times H \times (S^1)^{r_2} \times \{\pm1\}^{r_1}$, we prefer to use the symbol $\mathbb{R} \times P$.

If furthermore $0 < a < b$ are given, we write

(A.2) $$[a,b] \cdot P := \{ x \in \mathbb{R}^\times_{>0} \cdot P \mid a \leq N(x) \leq b \}.$$  

When we see it as a subset of $\mathbb{R} \times H \times (S^1)^{r_2} \times \{\pm1\}^{r_1}$, we write it as $[\log a, \log b] \times P$.

A.4. The domain $(K \otimes_{\mathbb{Q}} \mathbb{R})_{<X}$. For an embedding $\sigma: K \hookrightarrow \mathbb{C}$ and $x \in K \otimes_{\mathbb{Q}} \mathbb{R}$, we set

$$|x|_{\sigma} := |(\sigma \otimes_{\mathbb{Q}} \mathbb{R})(x)|,$$  

where $| - |$ in the right hand side is the usual absolute value in $\mathbb{C}$. For a positive number $X > 1$, we define a subset $(K \otimes_{\mathbb{Q}} \mathbb{R})_{<X} \subset K \otimes_{\mathbb{Q}} \mathbb{R}$ by

$$(K \otimes_{\mathbb{Q}} \mathbb{R})_{<X} := \{ x \in K \otimes_{\mathbb{Q}} \mathbb{R} \mid |x|_{\sigma_i} < X \text{ for all } 1 \leq i \leq r_1 + r_2 \}.$$  

A.5. Measures. We will consider two measures on $(K \otimes_{\mathbb{Q}} \mathbb{R})^\times$. One is the restriction of the Lebesgue measure by the inclusion $(K \otimes_{\mathbb{Q}} \mathbb{R})^\times \subset K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^n$, where we took the isomorphism $\mathbb{C} \cong \mathbb{R}^2; x + iy \leftrightarrow (x,y)$. We denote it by $\mu_{\text{add}}$.

The other is obtained through $(K \otimes_{\mathbb{Q}} \mathbb{R})^\times \cong \mathbb{R}^{r_1+r_2} \times (S^1)^{r_2} \times \{\pm1\}^{r_1}$. We give $\mathbb{R}^{r_1+r_2}$ the Lebesgue measure and $(S^1)^{r_2}$ the Haar measure such that the volume of $(S^1)^{r_2}$ is $(2\pi)^{r_2}$. (In other words, as far as volume is concerned, we regard $S^1$ as $\mathbb{R}/2\pi\mathbb{Z}$, not $\mathbb{R}/\mathbb{Z}$.) Each point of $\{\pm1\}^{r_1}$ is given weight 1. We denote the resulting product measure on $(K \otimes_{\mathbb{Q}} \mathbb{R})^\times$ by $\mu_{\text{mult}}$. 
By some calculus one finds
\[ d\mu_{\text{mult}} = \frac{2^{r_2}}{N(-)} d\mu_{\text{add}} \quad \text{on} \quad (K \otimes \mathbb{Q} \mathbb{R})^\times, \]
where \( N(-) \) is the norm function in §A.1.

We endow \( H \subset \mathbb{R}^{r_1+r_2} \) with the induced metric and hence the measure \( \mu_H \). This endows \((K \otimes \mathbb{Q} \mathbb{R})^\times / \mathbb{R}_+^\times = H \times (S^1)^{r_2} \times \{\pm 1\}^{r_1} \) with the product measure. We denote it by \( \mu_{\text{mult}} \) as well.

Let \( u_1, \ldots, u_{r_1+r_2-1} \in H \) be any orthonormal basis and \((\xi_1, \ldots, \xi_{r_1+r_2}) \) be the coordinate system corresponding to the basis \( u_1, \ldots, u_{r_1+r_2} \) of \( \mathbb{R}^{r_1+r_2} \). (See §A.3 for the vector \( u_{r_1+r_2} \).) Note that the measure \( \mu_H \) on \( H \) is exactly the one associated with the volume form \( d\xi_1 \ldots d\xi_{r_1+r_2-1} \) on \( H \). The inner product of \( u_{r_1+r_2} \) and the unit normal vector \((1, \ldots, 1)/\sqrt{r_1+r_2} \) of \( H \) is \( 1/\sqrt{r_1+r_2} \). This implies the following relation of measures:
\[ |d\xi_1 \ldots d\xi_{r_1+r_2}| = \sqrt{r_1+r_2} d\mu_{\text{mult}} = \frac{2^{r_2} \sqrt{r_1+r_2}}{N(-)} d\mu_{\text{add}} \quad \text{(A.3)} \]

A.6. The regulator. Dirichlet’s Unit Theorem says that we have an isomorphism as follows, which we fix:
\[ \mathcal{O}_K^\times \cong (\mathbb{Z}/w_K \mathbb{Z}) \times \mathbb{Z}^{r_1+r_2-1}. \]

By its proof, we know that the inclusion map followed by the projection
\[ \mathcal{O}_K^\times \hookrightarrow (K \otimes \mathbb{Q} \mathbb{R})^\times \cong \mathbb{R} \times H \times (S^1)^{r_2} \times \{\pm 1\}^{r_1} \rightarrow H \]
has image which is a full-rank lattice. The regulator \( R_K \) of \( K \) is the real number satisfying \( \text{Vol}_{\text{mult}}(H/\mathbb{Z}^{r_1+r_2-1}) = R_K \sqrt{r_1+r_2} \).

A.7. The structure of \((K \otimes \mathbb{Q} \mathbb{R})^\times / \mathcal{O}_K^\times\). We take this opportunity to compute the abelian group \((K \otimes \mathbb{Q} \mathbb{R})^\times / \mathcal{O}_K^\times\) further. By §A.6 we know it decomposes as
\[ \mathbb{R} \times \frac{(K \otimes \mathbb{Q} \mathbb{R})^\times}{\mathcal{O}_K^\times \cdot \mathbb{R}_+^\times} = \mathbb{R} \times \frac{H \times \{\pm 1\}^{r_1} \times (S^1)^{r_2}}{\mathcal{O}_K^\times}. \]

We have the next commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
1 & \to & \mathbb{Z}/w_K \mathbb{Z} & \to & \mathcal{O}_K^\times & \to & \mathbb{Z}^{r_1+r_2-1} & \to & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \{\pm 1\}^{r_1} \times (S^1)^{r_2} & \to & H \times \{\pm 1\}^{r_1} \times (S^1)^{r_2} & \to & H & \to & 0
\end{array}
\]
This gives an exact sequence
\[ \text{(A.4)} \quad 1 \to \frac{\{\pm 1\}^{r_1} \times (S^1)^{r_2}}{(\mathcal{O}_K^\times)_{\text{tors}}} \to \frac{(K \otimes \mathbb{Q} \mathbb{R})^\times}{\mathcal{O}_K^\times \cdot \mathbb{R}_+^\times} \to (\mathbb{R}/\mathbb{Z})^{r_1+r_2-1} \to 0. \]

The left-hand term is isomorphic to \( \{\pm 1\}^{r_1-1} \times (S^1)^{r_2} \) if \( r_1 \neq 0 \) and \( (S^1)^{r_2} \) if \( r_1 = 0 \).
A.8. The neutral component of \((K \otimes \mathbb{Q} \mathbb{R})^\times / \mathcal{O}_K^\times \mathbb{R}_{>0}^\times\). From \([A.4]\) it is now clear that \((K \otimes \mathbb{Q} \mathbb{R})^\times / (\mathcal{O}_K^\times \mathbb{R}_{>0})\) is a torus (= commutative compact Lie group) of dimension \(n - 1\). As such, its neutral component \(((K \otimes \mathbb{Q} \mathbb{R})^\times / (\mathcal{O}_K^\times \mathbb{R}_{>0}))^0\) is isomorphic to \((S^1)^{n-1}\). Choose and fix any such isomorphism
\[
(A.5) \quad \left(\frac{(K \otimes \mathbb{Q} \mathbb{R})^\times}{\mathcal{O}_K^\times \mathbb{R}_{>0}}\right)^0 \cong (S^1)^{n-1}.
\]
Of course there are natural ways of specifying such an isomorphism in the spirit of \([A.4]\) relying only on mild choices. But this is not needed in this paper.

A.9. The volume of \((K \otimes \mathbb{Q} \mathbb{R})^\times / \mathcal{O}_K^\times \mathbb{R}_{>0}^\times\). Since \(\mathcal{O}_K^\times\) is a discrete subgroup of \((K \otimes \mathbb{Q} \mathbb{R})^\times / \mathbb{R}_{>0}^\times\), and translation by an element of \(\mathcal{O}_K^\times\) preserves the measure \(\mu\) (see \([A.5]\) for this measure), the quotient inherits the measure. Let us write the resulting one also by \(\mu\).

It will be convenient to know the volume of \((K \otimes \mathbb{Q} \mathbb{R})^\times / \mathcal{O}_K^\times \mathbb{R}_{>0}^\times\). The left-hand term of \([A.4]\) has volume \(2^{r_1} (2\pi)^{r_2} / w_K\). The right-hand term has volume \(R_K \sqrt{r_1 + r_2} \) (\([A.6]\)). By the definition of \(\mu_{\text{mult}}\) as the product measure, it follows that
\[
(A.6) \quad \text{Vol}_{\text{mult}}((K \otimes \mathbb{Q} \mathbb{R})^\times / \mathcal{O}_K^\times \mathbb{R}_{>0}^\times) = \frac{2^{r_1} (2\pi)^{r_2} R_K \sqrt{r_1 + r_2}}{w_K}.
\]

**Appendix B. Fourier analysis on a torus**

By a torus we mean a compact commutative Lie group. By so-called Lie’s three theorems, such a group \(T\) is known to be isomorphic to a product
\[
(B.1) \quad T \cong G \times (S^1)^d
\]
of a finite commutative group \(G\) and a connected torus \((S^1)^d\). Of course the isomorphism is usually non-canonical.

**B.1. Approximation of indicator functions of small convex subsets by characters.** Let \(T\) be a torus and
\[
(B.2) \quad \hat{T} := \text{Hom}_{\text{cont}}(T, S^1) \cong \hat{G} \times \mathbb{Z}^d
\]
be its Pontryagin dual. Every \(\xi \in \hat{T}\) will be seen as a \(C\)-valued function on \(T\) by \(T \xrightarrow{\xi} S^1 \subset C\). Fourier analysis on \(T\) roughly says that functions on \(T\) can be written as \(C\)-linear combinations of elements of \(\hat{T}\). The precise statement we use in this paper is Proposition \([B.1.2]\) below.

**Definition B.1.1.** Let \(T\) be a torus whose neutral component \(T^0\) is equipped with an isomorphism \(T^0 \cong (S^1)^d\). A small convex subset of \(T\) (resp. of size \(\leq \delta\), \(0 < \delta < 1/4\)) is a connected subset \(P \subset T\) such that for some \(p \in P\) the translate
\[
P - p \subset T^0 \cong (S^1)^d \cong (\mathbb{R}/\mathbb{Z})^d
\]
is the projection of an open convex subset \(\tilde{P} \subset [-\frac{1}{4}, \frac{1}{4}]^d \subset \mathbb{R}^d\) (resp. \(\tilde{P} \subset [-\delta, \delta]^d\)).
Proposition B.1.2. Let $M, Y > 1$ be positive parameters. Let $P$ be a small convex subset of a torus of size $< 1/5$ with respect to some identification $T^0 \cong (S^1)^d = \mathbb{R}^d/\mathbb{Z}^d$. Then the indicator function $1_P$ can be approximated as:

$$1_P = \sum_{\xi \in \Xi} c_{\xi} \cdot \xi + G + H,$$

where

- $\Xi \subseteq \hat{T}$ is a subset of size $O_d(\#\pi_0(T) \cdot Y^d)$,
- the coefficients $c_{\xi} \in \mathbb{C}$ obey bounds $|c_{\xi}| \leq 1$,
- the values of $G$ have absolute values $\leq 1$ and there is another small convex set $P'$ satisfying $\text{Supp}(G) \subseteq P' \setminus P$ and $\text{Vol}(P' \setminus P) = O_n(1/M)$,
- $H$ is a function satisfying $\|H\|_{\infty} \leq O_d(M \cdot \log Y)$.

Moreover, for $\xi \in (\pi_0(T))^- \subseteq \hat{T}$ the coefficient $c_{\xi}$ can be explicitly described: letting $\mu$ be any Haar measure on $T$, 

$$c_{\xi} = \frac{\mu(P)}{\mu(T)}\xi(P) + O_d(1/M),$$

where the value $\xi(P) \in \mathbb{C}$ is well defined because $\xi \in (\pi_0(T))^-$ is constant on every connected set.

Proof. This follows from [3] Corollary A.3 and [2] Lemma A.9.

Note that although [2] Lemma A.9] is stated for the connected torus $(R/Z)^d$, the corresponding statement for a non-connected commutative torus easily follows from the decompositions (B.1), (B.2). The specific choice of these decompositions affect the resulting objects $\Xi, G, H, \ldots$ but do not affect the bounds of the error terms. \hfill $\square$

In our application, the number $\#\pi_0(T)$ of connected components of $T$ will be allowed to grow at a pseudopolynomial rate $e^{\sqrt{log N}/O_K(1)}$ at the fastest.

Appendix C. Idèle class group as a torus

Let $q = \prod_p p^{a_p}$ be a non-zero ideal and let $C(q)$ be the mod $q$ idèle class group from §2.1. We perform Fourier analysis from §3 on the torus $C(q)/\mathbb{R}_{>0}^\times$. Its volume (C.3) explains the coefficient in Mitsui’s Prime Number Theorem 1.2.1, 3.1.1.

C.1. The structure of $C(q)/\mathbb{R}_{>0}^\times$. In this subsection we review how the quotient $C(q)/\mathbb{R}_{>0}^\times$ is a real torus and fix some notation on the way.

C.1.1. To factor out $\text{Cl}(K)$. Recall our notation

$$C(q) = \bar{C}(q)/K^\times = \left((K \otimes \mathbb{Q} \mathbb{R})^\times \times \bigoplus_{p | q} K^\times / (1 + p^{a_p} \mathcal{O}_{K,p}) \times \bigoplus_{p | q} \mathbb{Z}ight)/K^\times.$$

Consider the $p$-adic valuation map $v_p: K^\times / (1 + p^{a_p} \mathcal{O}_{K,p}) \to \mathbb{Z}$ for each $p \mid q$. Its kernel is identified with $\mathcal{O}_{K,p}^\times / (1 + p^{a_p} \mathcal{O}_{K,p}) = (\mathcal{O}_{K,p}/p^{a_p} \mathcal{O}_{K,p})^\times = (\mathcal{O}_K/p^{a_p})^\times$. If we take the direct sum over $p \mid q$, by Chinese Remainder Theorem we get $(\mathcal{O}_K/q)^\times$. 
It follows that we have the following commutative diagram with exact rows and injective vertical maps, where $P_K$ denotes the group of principal fractional ideals

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathcal{O}_K^\times & \longrightarrow & K^\times & \longrightarrow & P_K & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \frac{(K \otimes \mathbb{Q} \mathbb{R})^\times}{\mathbb{R}_{>0}} \times (\mathcal{O}_K/q)^\times & \longrightarrow & \tilde{C}(q)/\mathbb{R}_{>0}^\times & \longrightarrow & \bigoplus_{\text{p non-zero}} \mathbb{Z} & \longrightarrow & 0.
\end{array}
\]

By the snake lemma we obtain an exact sequence

\[(C.1) \quad 1 \rightarrow \frac{(K \otimes \mathbb{Q} \mathbb{R})^\times}{\mathbb{R}_{>0}} \times (\mathcal{O}_K/q)^\times \rightarrow C(q) \rightarrow \Cl(K) \rightarrow 0.\]

**C.1.2. To factor out $(\mathcal{O}_K/q)^\times$.** We want to compute the first term of (C.1) further. Consider the commutative diagram with exact rows:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathcal{O}_K^\times & \longrightarrow & \mathcal{O}_K^\times & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & (\mathcal{O}_K/q)^\times & \longrightarrow & \frac{(K \otimes \mathbb{Q} \mathbb{R})^\times}{\mathbb{R}_{>0}} \times (\mathcal{O}_K/q)^\times & \longrightarrow & 0.
\end{array}
\]

As the vertical maps are injective, we conclude that $\frac{(K \otimes \mathbb{Q} \mathbb{R})^\times}{\mathbb{R}_{>0}} \times (\mathcal{O}_K/q)^\times / \mathcal{O}_K^\times$ is an extension of $(\mathcal{O}_K/q)^\times$ by $(\mathcal{O}_K/q)^\times$, the first of which has been computed in §A.7.

**C.1.3. The structure of $C(q)/\mathbb{R}_{>0}^\times$—conclusion.** Our computation so far expresses $C(q)/\mathbb{R}_{>0}^\times$ as a twofold extension

\[(C.2) \quad 1 \rightarrow (\mathcal{O}_K/q)^\times \rightarrow \frac{(K \otimes \mathbb{Q} \mathbb{R})^\times}{\mathbb{R}_{>0}} \times (\mathcal{O}_K/q)^\times / \mathcal{O}_K^\times \rightarrow C(q)/\mathbb{R}_{>0}^\times \rightarrow \Cl(K) \rightarrow 0.
\]

In §A.7 we saw that $\frac{(K \otimes \mathbb{Q} \mathbb{R})^\times}{\mathcal{O}_K \mathbb{R}_{>0}^\times}$ is a torus. It follows that $C(q)/\mathbb{R}_{>0}^\times$ is also a torus.
Its neutral component \( (C[q]/\mathbb{R}_>^\times)^0 \) surjects to the neutral component of \( (K\otimes q\mathbb{R})^\times_{\mathcal{O}_K^*} \), and the size of the kernel of this map is bounded from above by \( \#(\mathcal{O}_K/q)^\times = \varphi_K(q) < N(q) \).

C.1.4. The volume of \( C[q]/\mathbb{R}_>^\times \). Via the diagram \( (\ref{C.2}) \), the volume form of \( (K\otimes q\mathbb{R})^\times_{\mathcal{O}_K^*} \) induces a volume form on \( C[q]/\mathbb{R}_>^\times \). Write the resulting measure as \( \mu_{\text{idèle}} \) and the image is a small convex subset of size \( [\text{Vol}_\text{idèle}(-) \text{ the volume of subsets}]^{10} \). By the diagram and \( (\ref{A.0}) \) we have

\[
\text{Vol}_\text{idèle}(C[q]/\mathbb{R}_>^\times) = h_K \varphi_K(q) \text{Vol}_{\text{mult}}\left( \frac{(K\otimes q\mathbb{R})^\times}{\mathcal{O}_K^*} \right)_{\mathbb{R}_>^\times} = \varphi_K(q)2^{n_1}(2\pi)^{n_2}R_Kh_K\sqrt{r_1 + r_2}/w_K.
\]

C.2. Indicator function of convex sets. Now we combine Fourier analysis from \( \S 3 \) and the description of \( C[q]/\mathbb{R}_>^\times \). The consequence, Proposition C.2.2, is used when we compute weighted sum of prime elements in thin cones as in Figure 2 in Introduction.

C.2.1. A trivialization of the neutral component. Consider the covering map induced on the neutral components \( (C[q]/\mathbb{R}_>^\times)^0 \to \left( \frac{(K\otimes q\mathbb{R})^\times}{\mathcal{O}_K^*} \right)_{\mathbb{R}_>^\times}^0 \). In \( (\ref{A.5}) \) we fixed an isomorphism between the target and \( (S^1)^n = (\mathbb{R}/\mathbb{Z})^n \). This identifies the covering map with a map of the form

\[
\mathbb{R}^{n-1}/\Lambda(q) \to (\mathbb{R}^{n-1}/\mathbb{Z}^{n-1})^n
\]

for some sublattice \( \Lambda(q) \subset \mathbb{Z}^{n-1} \) of index dividing \( \varphi_K(q) \). By \( \mathbb{Z} \)-linear algebra (Euclid’s algorithm, see Proposition D.11), one can find a basis

\[
f_1, \ldots, f_{n-1} \in \Lambda(q)
\]

whose coefficients as elements of \( \mathbb{Z}^{n-1} \) have absolute values \( \leq \varphi_K(q) \).

Now we are ready to state the following consequence of Fourier analysis.

Proposition C.2.2 (indicator function of a small convex set). Let \( M, Y > 1 \) be real numbers.

Let \( P \subset C[q]/\mathbb{R}_>^\times \) be a connected open subset such that its translate to \( (C[q]/\mathbb{R}_>^\times)^0 \) projects injectively by the map

\[
(C[q]/\mathbb{R}_>^\times)^0 \to \left( \frac{(K\otimes q\mathbb{R})^\times}{\mathcal{O}_K^*} \right)_{\mathbb{R}_>^\times}^0 \cong (S^1)^n
\]

and the image is a small convex subset of size \( \ll_n \varphi_K(q)^{(n-2)} \) in the sense of Definition B.1.1

Then one has a decomposition of the indicator function \( 1_P: C[q]/\mathbb{R}_>^\times \to \mathbb{C} \) of the form:

\[
1_P = \sum_{\psi \in \Xi \subset (C[q]/\mathbb{R}_>^\times)^0} c_{\psi} \psi + G + H,
\]

where

\[\text{Note that this may be different from the measure adopted in the reader’s favorite reference because every reference has its own convention on the choice of measures. Don’t worry, any two differ only by a constant factor because all are Haar measures.}\]
\( \# \Xi = O_K(\varphi_K(q)Y^{n-1}) \),
\( |\varepsilon| \leq 1 \),
the values of \( G \) have absolute value \( \leq 1 \), and there is another small convex set \( P' \) such that \( \text{Supp}(G) \subset P' \setminus P \) and \( \text{Vol}(P' \setminus P) = O_n(1/M) \),
\( \|H\|_\infty \leq O_n(M^{\log Y^c}) \).

Moreover, for \( \psi \in (\pi_0(C(q)/R_{>0}^\times))^{-} \subset (C(q)/R_\div^\times)^{-} \) we have
\[
c_\psi = \frac{\text{Vol}(P)}{\text{Vol}(C(q)/R_{>0}^\times)} \phi(P) + O_n(1/M).
\]

Proof. We continue to identify \((C(q)/R_{>0}^\times)^0\) with \( R_\div^{-1}/\Lambda(q) \) as in (C.4). Take a basis of \( \Lambda(q) \) as in (C.3). For this basis the matrix representing the linear map \( \phi: Z_1^{-1} \cong \Lambda(q) \to Z_1^{-1} \) has entries of absolute values \( \leq \varphi_K(q) \). It follows that \( \phi^{-1} \): \( R_\div^{-1} \to \Lambda(q) \otimes \mathbf{Z} R \cong R_\div^{-1} \) has coefficients \( \ll_n \varphi_K(q)^{-n} \). It follows that if the constant \( 0 < c_n < 1 \) is chosen small enough to match the implied constant, the inverse map \( \phi^{-1} \) sends the cube
\[
[-c_n\varphi_K(q)^{-n-2}, c_n\varphi_K(q)^{-n-2}]^{n-1}
\]
into \((-1/5,1/5)^n\).

Under such a choice of \( \Lambda(q) \cong Z_1^{-1} \) and \( 0 < c_n < 1 \) we see that \( P \) is a small convex set of size \( < 1/5 \) in \( C(q)/R_{>0}^\times \) with respect to the corresponding trivialization \((C(q)/R_{>0}^\times)^0 \cong (S^1)^n\).

Therefore Proposition \([3.1.2]\) applies and gives the desired result. Note that by (C.2)
\[
\# \pi_0(C(q)/R_{>0}^\times) \leq h_K \cdot \varphi_K(q) \cdot \# \pi_0 \left( \frac{(K \otimes_\mathbf{Q} R^\times)}{\Omega_K \cdot R_{>0}^\times} \right)
\]
and \( \# \pi_0 \left( \frac{(K \otimes_\mathbf{Q} R^\times)}{\Omega_K \cdot R_{>0}^\times} \right) \leq 2^{11} = O_n(1). \)

C.2.3. A chart. To produce connected open sets \( P \) as in Proposition \([3.2.2]\) it is useful to take the following kind of chart.

To state it, for non-zero ideals \( a, q \subset \mathcal{O}_K \) let us denote by \((a/qa)^\times\) the set of elements which is a generator of \( a/qa \) as an \( \mathcal{O}_K/q \)-module. For \( \mathcal{O}_K = a \), this recovers the set of invertible elements \( (\mathcal{O}_K/q)^\times \). Note that by Chinese Remainder Theorem \( a/qa = \bigoplus_{p \mid q} a/p^n a \), the condition that \( \alpha \in a/qa \) is in \((a/qa)^\times\) is equivalent to the condition that its image in \( a/p^n a \) is in \((a/p^n a)^\times\) for all \( p \mid q \).

Let \( v_p(a) \in \mathbf{Z} \) be the \( p \)-adic valuation of a non-zero (fractional) ideal \( a \). Let us quickly recall its definition: the scalar extension \( a\mathcal{O}_{K,p} \) is a free \( \mathcal{O}_{K,p} \)-module of rank 1. Take any generator \( \alpha \) of \( a\mathcal{O}_{K,p} \) and we write \( v_p(\alpha) := v_p(\alpha) \), which is independent of the choice of \( \alpha \).

Then \((a/qa)^\times\) is equivalently defined as the set of residue classes \( \overline{\alpha} \in a/qa \) such that some (or any) lift \( \alpha \in a \) satisfies
\[
(C.6) \quad v_p(\alpha) = v_p(a) \text{ for all } p \mid q.
\]

For each \( p \), this condition does not depend on the specific lift \( \alpha \).

Consider the map of sets
\[
(C.7) \quad (a/qa)^\times \to \left( \bigoplus_{p \mid q} K^\times/(1 + p^n \mathcal{O}_{K,p}) \right) \oplus \bigoplus_{p \mid q} \mathbf{Z}
\]
induced by \( a \setminus \{0\} \mapsto K^\times \) composed with \( K^\times \to K^\times/(1+p^n \mathcal{O}_{K,p}) \) for each \( p \mid q \) and

the valuation \( v_p: K^\times \to \mathbb{Z} \) for each \( p \mid q \). The map (C.7) is well defined on \((a/qa)^\times\) because if \( \alpha_1, \alpha_2 \in a \) are two elements satisfying (C.6) such that \( \varepsilon := \alpha_1 - \alpha_2 \in qa \), one can verify \( \alpha_1/\alpha_2 \in 1 + p^n \mathcal{O}_{K,p} \) for \( p \mid q \). Indeed, we have \((\alpha_1/\alpha_2) - 1 = \varepsilon/\alpha_2 \) and

\[
v_p(\varepsilon/\alpha_2) \geq (v_p(q) + v_p(a)) - v_p(a) = v_p(q) = n_p.
\]

**Proposition C.2.4.** Let \( D \subset (K \otimes \mathbb{Q} R)^\times \) be a fundamental domain for the \( \mathcal{O}_K^\times \)-action and \( \{a_\lambda\}_{\lambda \in \text{Cl}(K)} \) be a complete set of representatives.

The inclusions \( a_\lambda \setminus \{0\} \mapsto K^\times \) induce the following bijections of sets

\[
C(q) \cong \bigsqcup_{\lambda \in \text{Cl}(K)} (K \otimes \mathbb{Q} R)^\times \times (a_\lambda/qa_\lambda)^\times \cong \bigsqcup_{\lambda \in \text{Cl}(K)} D \times (a_\lambda/qa_\lambda)^\times.
\]

**Proof.** Recall \( C(q) = \tilde{C}(q)/K^\times \). Consider the following maps

\[
\tilde{C}(q) \to \bigoplus_{p \text{ all}} \mathbb{Z} \to \text{Cl}(K)
\]

and for each \( \lambda \in \text{Cl}(K) \) let \( \tilde{C}(q)_\lambda \) be the fiber of this map over \( \lambda \):

\[
\tilde{C}(q)_\lambda := \tilde{C}(q) \times_{\text{Cl}(K)} \{\lambda\}.
\]

Then we get a decomposition \( \tilde{C}(q) = \bigsqcup_{\lambda \in \text{Cl}(K)} \tilde{C}(q)_\lambda \). Since the action of \( K^\times \) preserves each \( \tilde{C}(q)_\lambda \), we get

(C.8) \[
C(q) = \bigsqcup_{\lambda \in \text{Cl}(K)} \tilde{C}(q)_\lambda/K^\times.
\]

For a non-zero (fractional) ideal \( a \) we consider the following constructions. We define subsets

\[
K_{q,a} := \left\{ (x_p)_{p|q}, (a_p)_{p|q} \in \left( \bigoplus_{p|q} K^\times/(1+p^n \mathcal{O}_{K,p}) \times \bigoplus_{p|q} \mathbb{Z} \right) \right\} \cap \left\{ \forall p \mid q : v_p(x_p) = v_p(a), \forall p \notmid q : a_p = v_p(a) \right\}.
\]

\[
\tilde{C}(q)_{a,\lambda} := (K \otimes \mathbb{Q} R)^\times \times K_{q,a} \subset \tilde{C}(q)_{[a]}, \text{ in particular } \tilde{C}(q)_{a,\lambda} \subset \tilde{C}(q)_\lambda.
\]

One sees that \( \tilde{C}(q)_{a,\lambda} \) is stable under the \( \mathcal{O}_K^\times \)-action. The induced map

(C.9) \[
\tilde{C}(q)_{a,\lambda}/\mathcal{O}_K^\times \to \tilde{C}(q)_\lambda/K^\times
\]

is a bijection. The inverse is described as follows: let \( (x; (x_p)_{p|q}; (a_p)_{p|q}) \in \tilde{C}(q)_\lambda \). By the definitions of the sets \( \tilde{C}(q)_{a,\lambda} \subset \tilde{C}(q)_\lambda \) and \( \text{Cl}(K) = \text{cok}(K^\times \to \bigoplus_{p \text{ all}} \mathbb{Z}) \) there is an \( \alpha \in K^\times \) such that if we let \( \text{diag}(\alpha) \in \tilde{C}(q) \) be its diagonal image

\[
\text{diag}(\alpha) \cdot (x; (x_p)_{p|q}; (a_p)_{p|q}) \in \tilde{C}(q)_{a,\lambda}.
\]

This gives an element of \( \tilde{C}(q)_{a,\lambda} \) whose class in \( \tilde{C}(q)_{a,\lambda}/\mathcal{O}_K^\times \) is independent of the choice of \( \alpha \in K^\times \).

Since \( D \subset (K \otimes \mathbb{Q} R)^\times \) is a fundamental domain for the \( \mathcal{O}_K^\times \)-action we see that

the composite

(C.10) \[
D \times K_{q,a} \hookrightarrow \tilde{C}(q)_{a,\lambda} \cong \tilde{C}(q)_{a,\lambda}/\mathcal{O}_K^\times
\]
is a bijection.

In view of the bijections (C.8) (C.9) (C.10), it remains to verify that the map (C.7) induces a bijection

\[(a_\lambda/q a_\lambda)^\times \xrightarrow{\cong} K_{q,a_\lambda} \].

Both sides naturally decompose into products \(\prod_{p | q} \). It suffices to show the following bijection

\[(a_\lambda/p^n a_\lambda)^\times \xrightarrow{\cong} \{x \in K^\times/(1 + p^n \mathcal{O}_{K,p}) \mid v_p(x) = v_p(a_\lambda)\}\]

for each \(p | q\).

The inverse map can be constructed as follows. Let \(\alpha \in K^\times\) be an element satisfying \(v_p(\alpha) = v_p(a_\lambda)\). This implies \(\alpha \in a_\lambda \mathcal{O}_{K,p}\) and that it generates \(a_\lambda \mathcal{O}_{K,p}\) as an \(\mathcal{O}_{K,p}\)-module. So it determines a class in \((a_\lambda \mathcal{O}_{K,p}/p^n a_\lambda \mathcal{O}_{K,p})^\times = (a_\lambda/p^n a_\lambda)^\times\).

Moreover, it is obvious that if the ratio of two such \(\alpha, \alpha'\) is in \(1 + p^n \mathcal{O}_{K,p}\), their difference is in \(p^n a_\lambda \mathcal{O}_{K,p}\).

It is routine to check that this is actually the inverse. \(\square\)

### Appendix D. Euclid’s algorithm

**Proposition D.1.** Let \(\Lambda \subset \mathbb{Z}^n\) be a subgroup of finite index \(\#(\mathbb{Z}^n/\Lambda) = D\). Then there is a basis \(v_1, \ldots, v_n\) of \(\Lambda\) whose entries have absolute values \(\leq D\).

**Proof.** Take an arbitrary basis of \(\Lambda\) and write it as a set of column vectors.

The assertion is translated to the following assertion on matrices: let \(A \in M_{n \times n}(\mathbb{Z})\) be an \(n \times n\) matrix with positive determinant \(D\). Then there is an invertible matrix \(U \in \text{GL}_n(\mathbb{Z})\) such that \(AU\) has entries whose absolute values are \(\leq D\).

When \(n = 1\) (or \(n = 0\)), the assertion is trivial. We proceed by induction on \(n\).

Let \(d_1 > 0\) be the greatest common divisor of the entries in the first row of \(A\). By Euclid’s algorithm, which can be realized as the right multiplication by invertible matrices, the matrix \(A\) can be transformed into the form

\[(d_1 \cdot 0 \ldots 0 \cdot A')^\prime\]

with \(A' \in M_{(n-1) \times (n-1)}(\mathbb{Z})\).

This also shows \(d_1 | D\). Set \(\det(A') = D' = D/d_1\).

From the induction hypothesis, by the right multiplication of invertible matrices which leaves the first column and row unchanged, one can transform \(A'\) so that it has entries whose absolute values are \(\leq D'\).

The entries * in (D.1) might a priori have absolute values \(> D\), but one can use the entries of the modified \(A'\) to make them smaller than \(D'\). This completes the proof. \(\square\)

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