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RESTRICTED RANGE SIMULTANEOUS APPROXIMATION AND
INTERPOLATION WITH PRESERVATION OF THE NORM

J.B. Prolla and S. Navarro

Abstract. Let $(F, | \cdot |)$ be a complete non-archimedean non-trivially valued division ring, with valuation ring $V$. Let $X$ be a compact 0-dimensional Hausdorff space, and let $D(X)$ be the ring of all continuous functions $f$ from $X$ into $V$ equipped with the supremum norm. Let $A \subseteq D(X)$. Assume that for every ordered pair $(s, t)$ of distinct elements of $X$, there is some multiplier of $A$, say $\varphi$, such that $\varphi(s) = 1$ and $\varphi(t) = 0$. Assume that $A$ contains the constants. We show that $A$ is uniformly dense in $D(X)$, and when $A$ is an interpolating family then simultaneous approximation and interpolation, with preservation of the norm, by elements of $A$ is always possible. We apply this to the case of von Neumann subsets and to the case of restricted range polynomial algebras.

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1. Introduction

Throughout this paper $X$ is a compact Hausdorff space which is 0-dimensional i.e., for any point $x$ and any open set $A$ containing $x$, there exists a closed and open set $N$ with $x \in N \subseteq A$, and $(F, | \cdot |)$ is a complete, non-Archimedean non-trivially valued division ring. We denote by $V$ the valuation ring of $F$, i.e., $V = \{t \in F : |t| \leq 1\}$, and by $D(X)$ the set of all continuous functions from the space $X$ into $V$, equipped with the topology of uniform convergence on $X$, determined by the metric $d$ defined by

$$d(f, g) = ||f - g|| = \sup\{|f(x) - g(x)| : x \in X\}$$

for every pair, $f$ and $g$, of elements of $D(X)$.

Our aim is to use the idea of T.J. Ransford (see [7]), to prove results in $D(X)$ that are analogous to those in $C(X; [0, 1])$ and $C(X; F)$, which were proved in [5] and [6], respectively. To avoid trivialities we assume that $X$ has at least two points.

Definition 1 A non-empty subset $A \subseteq D(X)$ is said to be a von Neumann subset if $\varphi \psi + (1 - \varphi) \eta$ belongs to $A$, whenever $\varphi, \psi$ and $\eta$ belong to $A$. 
Clearly, if \( A \subseteq D(X) \) is a von Neumann subset containing the constant functions 0 and 1, then the following properties are true:

(i) if \( \varphi \in A \), then \( 1 - \varphi \) belongs to \( A \);

(ii) if \( \varphi \) and \( \psi \) belong to \( A \), then \( \varphi \psi \) belongs to \( A \).

When \( A \subseteq D(X) \) has properties (i) and (ii), we say that \( A \) has property \( V \). This definition is motivated by the similar one introduced by R. I. Jewett, who in [1] proved the variation of the Weierstrass-Stone Theorem stated by von Neumann in [8].

**Definition 2** Let \( A \subseteq D(X) \) be a non-empty subset. We say that \( \varphi \in D(X) \) is a multiplier of \( A \) if \( \varphi \psi + (1 - \varphi)g \in A \).

Clearly, if \( M \) is the set of all multipliers of \( A \), then \( M \) satisfies property (i) above. The identity

\[
\varphi \psi f + (1 - \varphi \psi)g = \varphi[\psi f + (1 - \psi)g] + (1 - \varphi)g
\]

shows that \( M \) satisfies condition (ii) as well. Hence \( M \) has property \( V \).

**Definition 3** A subset \( A \subseteq D(X) \) is said to be strongly separating over \( X \), if given any ordered pair \((x, y) \in X \times X\), with \( x \neq y \), there exists a function \( \varphi \in A \) such that \( \varphi(x) = 1 \) and \( \varphi(y) = 0 \).

**Lemma 1** Let \( M \subseteq D(X) \) be a subset which has property \( V \) and is strongly separating over \( X \). Let \( N \) be a clopen proper subset of \( X \). For each \( \delta > 0 \), there is \( \varphi \in M \) such that

\[
|\varphi(t) - 1| < \delta, \text{ for all } t \in N, \tag{1}
\]

\[
|\varphi(t)| < \delta, \text{ for all } t \notin N. \tag{2}
\]

**Proof.** This result is essentially Lemma 1 of Prolla [6]. For the sake of completeness we include here its proof. Fix \( y \in X \), \( y \notin N \). Since \( M \) is strongly separating, for each \( t \in N \), there is \( \varphi_t \in M \) such that \( \varphi_t(y) = 1 \), \( \varphi_t(t) = 0 \). By continuity there is a neighborhood \( V(t) \) of \( t \) such that \( |\varphi_t(s)| < \delta \) for all \( s \in V(t) \). By compactness of \( N \) there are \( t_1, ..., t_n \in N \) such that \( N \subseteq V(t_1) \cup ... \cup V(t_n) \). Consider the function \( \psi_y = 1 - \varphi_{t_1} \varphi_{t_2} ... \varphi_{t_n} \). Clearly \( \psi_y \in M \) and \( \psi_y(y) = 0 \), while \( |\psi_y(t) - 1| < \delta \) for all \( t \in N \). Indeed, if \( t \in N \), then \( t \in V(t_i) \) for some index \( i \in \{1, 2, ..., n\} \). Hence

\[
|\psi_y(t) - 1| = |\varphi_{t_i}(t)| \cdot \prod_{j \neq i} |\varphi_{t_j}(t)| < \delta.
\]

By continuity, there is a neighborhood \( W(y) \) of \( y \) such that \( |\psi_y(s)| < \delta \) for all \( s \in W(y) \). By compactness of \( K = X \setminus N \), there are \( y_1, ..., y_m \in K \) such that \( K \subseteq W(y_1) \cup ... \cup W(y_m) \). Let \( \varphi = \psi_{y_1} \psi_{y_2} ... \psi_{y_m} \). Clearly \( \varphi \in M \). We claim that for each \( 1 \leq k \leq m \) we have

\[
|1 - \psi_{y_k}(t)| < \delta, \text{ for all } t \in N. \tag{3}
\]
We prove (3) by induction. For \( k = 1 \), (3) is clear, since \( |\psi_y(t) - 1| \leq \delta \) for all \( t \in N \) and all \( y \in K \). Assume (3) has been proved some \( k \). To simplify notation we write \( \psi_i = \psi_{y_i} \) for all \( 1 \leq i \leq m \). Then, for each \( t \in N \)
\[
|1 - \psi_1(t) \cdots \psi_{k+1}(t)| = \\
|1 - \psi_{k+1}(t) + \psi_{k+1}(t) - \psi_1(t) \cdots \psi_k(t) \cdot \psi_{k+1}(t)| \\
\leq \max \{ |1 - \psi_{k+1}(t)|, |\psi_{k+1}(t)|, |1 - \psi_1(t) \cdots \psi_k(t)| \} < \delta
\]
because \( |1 - \psi_{k+1}(t)| < \delta, |\psi_{k+1}(t)| \leq 1, \) and \( |1 - \psi_1(t) \cdots \psi_k(t)| < \delta \) by the induction hypothesis. Hence (3) is valid for \( k + 1 \).

Clearly (1) follows from (3) by taking \( k = m \). It remains to prove (2). Now if \( t \in K \) then \( t \in W(y_i) \) for some \( 1 \leq i \leq m \). Hence \( |\psi_i(t)| < \delta \), while \( |\psi_i(t)| \leq 1 \) for all \( j \neq i \). Therefore \( |\varphi(t)| < \delta \) and (2) is proved.

\[
\square
\]

**Remark.** If \( A \subset D(X) \) is a non-empty subset and \( f \in D(X) \), the distance of \( f \) from \( A \), denoted by \( \text{dist}(f,A) \), is defined as
\[
\text{dist} (f,A) = \inf \{ \| f - g \| : g \in A \}
\]

Clearly, \( f \) belongs to the uniform closure of \( A \) in \( D(X) \) if, and only if, \( \text{dist}(f;A) = 0 \).

If \( S \subset X \) is a non-empty closed subset of \( X \), we denote by \( f_S \in D(S) \). Similarly, \( A_S = \{ \varphi_S : \varphi \in A \} \), for each \( A \subset D(X) \). When \( S \) is a singleton set, say \( S = \{ x \} \), we identify \( f_S \) with its value \( f(x) \), and \( A_S \) with \( \{ \varphi(x) : \varphi \in A \} = A(x) \).

**Lemma 2** Let \( A \subset D(X) \) be a non-empty subset. For each \( f \in D(X) \), there exists a minimal closed and non-empty subset \( S \subset X \) such that
\[
\text{dist}(f_S;A_S) = \text{dist}(f;A)
\]

**Proof.** Since, for each \( x \in X \), we have
\[
\text{dist} (f(x); A(x)) \leq \text{dist} (f; A),
\]
we see that when \( \text{dist}(f;A) = 0 \), any singleton set \( S = \{ x \} \) satisfies
\[
\text{dist} (f_S; A_S) = \text{dist}(f;A)
\]

Assume now \( \text{dist}(f;A) > 0 \). Let us put \( d = \text{dist}(f;A) \). Define
\[
F(X) = \{ T \subset X ; \ T \text{ is closed and non-empty} \}
\]
and
\[ \mathcal{F} = \{ T \in \mathcal{F}(X) : \text{dist} (f_T; A_T) = d \}. \]

Clearly \( \mathcal{F} \neq \emptyset \) because \( X \in \mathcal{F} \). Let us order \( \mathcal{F} \) by set inclusion. Let \( \mathcal{C} \) be a totally ordered non-empty subset of \( \mathcal{F} \).

Let \( S = \cap \{ T : T \in \mathcal{C} \} \). Clearly, \( S \) is closed. If \( J \) is a finite subset of \( \mathcal{C} \), there is some \( T_0 \in J \) such that \( T_0 \subseteq T \) for all \( T \in J \). Hence
\[ T_0 = \cap \{ T : T \in J \}. \]

Now \( T_0 \neq \emptyset \) and by compactness \( S \neq \emptyset \). Hence \( S \in \mathcal{F}(X) \). We claim that \( S \in \mathcal{F} \). Clearly, \( \text{dist} (f_S; A_S) \leq d \). Suppose that \( \text{dist} (f_S; A_S) < d \) and choose a real number \( r \) such that \( \text{dist} (f_S; A_S) < r < d \). By definition of \( \text{dist} (f_S; A_S) \) there exists \( g \in A \) such that \( |f(x) - g(x)| < r \) for all \( x \in S \). Let
\[ U = \{ t \in X : |f(t) - g(t)| < r \}. \]

Then \( U \) is open and contains \( S \). By compactness, there is finite subset \( J \subseteq C \) such that \( \cap \{ T : T \in J \} \subseteq U \). Let \( T_0 \in J \) be such that \( T_0 \subseteq T \) for all \( T \in J \). Then \( \cap \{ T : T \in J \} = T_0 \) and so \( T_0 \subseteq U \). Hence \( |f(t) - g(t)| < r \) for all \( t \in T_0 \), and so \( \text{dist} (f_{T_0}; A_{T_0}) \leq r < d \), which contradicts the fact that \( T_0 \in \mathcal{F} \). This contradiction establishes our claim that \( \text{dist} (f_S; A_S) = d \). Therefore \( S \) is a lower bound for \( \mathcal{C} \) in \( \mathcal{F} \). By Zorn's Lemma there exists a minimal element in \( \mathcal{F} \) and this element satisfies all our requirements.

\[ \square \]

2. The Main Results

**Theorem 1** Let \( A \subseteq D(X) \) be a non-empty subset, whose set of multipliers is strongly separating over \( X \). For each \( f \in D(X) \), there is some \( x \in X \) such that

\[ (*) \quad \text{dist} (f(x); A(x)) = \text{dist} (f; A) \]

**Proof.** By Lemma 2 above, there is a minimal closed and non-empty subset \( S \subseteq X \) such that

\[ \text{dist} (f_S; A_S) = \text{dist} (f; A) \]

We claim that \( S = \{ x \} \) for some \( x \in X \). Since for any \( x \in X \), \( \text{dist} (f(x); A(x)) \leq \text{dist} (f; A) \) we see that when \( \text{dist} (f; A) = 0 \), then \((*)\) is true for all \( x \in X \). Hence we may assume \( d = \text{dist} (f; A) \) is strictly positive.

Assume that \( S \) contains at least two distinct points, say \( y \) and \( z \). Let \( N \) be a clopen subset of \( X \) such that \( y \in N \), while \( z \notin N \). Define
\[ Y = S \cap N, \]
$Z = S \cap K$, 

where $K = X \setminus N$. Notice that both $Y$ and $Z$ are closed. $Y \cap Z = \emptyset$ and $Y \cup Z = S$. Since $y \in Y$ and $z \in Z$, both $Y$ and $Z$ are non-empty. Furthermore, $z \notin Y$ and $y \notin Z$. Hence both $Y$ and $Z$ are proper subsets of $S$. By the minimality of $S$ we have 

$$d_Y := \text{dist}(f_Y : A_Y) < d;$$

$$d_Z := \text{dist}(f_Z : A_Z) < d.$$

Choose a real number $r$ such that 

$$\max\{d_Y, d_Z\} < r < d.$$

Since $d_Y < r$, there is some $g \in A$ such that $|f(t) - g(t)| < r$, for all $t \in Y$. Similarly, since $d_Z < r$, there is some $h \in A$ such that $|f(t) - h(t)| < r$, for all $t \in Z$. Choose $0 < \delta < r$. By Lemma 1, there is a multiplier of $A$, say $\varphi$, such that 

1. $|1 - \varphi(t)| < \delta$, for all $t \in N$.
2. $|\varphi(t)| < \delta$, for all $t \notin N$.

The function $k = \varphi g + (1 - \varphi)h$ belongs to $A$. We claim that $|f(t) - k(t)| < r$ for all $t \in S$. Let $t \in S$. There are two cases to consider, namely $t \in Y$ and $t \in Z$.

**Case I.** $t \in Y$

Let us write $g = \varphi g + (1 - \varphi)g$. Then 

$$|k(t) - g(t)| = |1 - \varphi(t)| \cdot |h(t) - g(t)| \leq |1 - \varphi(t)| < \delta$$

because $Y \subset N$ implies, by (1), that $|1 - \varphi(t)| < \delta$, and $|h(t) - g(t)| \leq \max\{|h(t)|, |g(t)|\} \leq 1$. Hence

$$|f(t) - k(t)| = |f(t) - g(t) + g(t) - k(t)| \leq \max\{|f(t) - g(t)|, |g(t) - k(t)|\} < r$$

**Case II.** $t \in Z$

Let us write $h = \varphi h + (1 - \varphi)h$. Then 

$$|k(t) - h(t)| = |\varphi(t)| \cdot |g(t) - h(t)| \leq |\varphi(t)| < \delta$$

because $Z \subset K = X \setminus N$ implies that $t \notin N$ and by (2), $|\varphi(t)| < \delta$. Hence

$$|f(t) - k(t)| = |f(t) - h(t) + h(t) - k(t)| \leq \max\{|f(t) - h(t)|, |h(t) - k(t)|\} < r.$$

Therefore $|f(t) - k(t)| < r$, for all $t \in S$ and dist $(f_S, A_S) \leq r < d$, a contradiction.
Remark. If $A \subset D(X)$ is as in Theorem 1 and $A(x) \supset \{0, 1\}$, for every $x \in X$, then it follows that the closure of $A$ contains the characteristic function of each clopen subset of $X$. Indeed, let $S \subset X$ be a clopen subset of $X$ and let $f$ be its characteristic function. Let $x \in X$ be given by Theorem 1. Now $f(x)$ is either 0 or 1 and therefore $A(x)$ contains $f(x)$ and so dist $(f, A) = 0$.

Corollary 1 Let $A \subset D(X)$ be a von Neumann subset which is strongly separating over $X$. For each $f \in D(X)$, there is some $x \in X$ such that

\[ \text{dist } (f(x); A(x)) = \text{dist } (f; A) \]

Proof. Let $M$ be the set of all multipliers of $A$. Since $A$ is a von Neumann subset, we see that $A \subset M$. Hence $M$ is strongly separating too, and the result follows from Theorem 1.

Theorem 2 Let $A \subset D(X)$ be a non-empty subset, whose set of multipliers is strongly separating over $X$. Let $f \in D(X)$ and $\varepsilon > 0$ be given. The following are equivalent:

1. there is some $g \in A$ such that $\|f - g\| < \varepsilon$.
2. for each $t \in X$, there is some $g_t \in A$ such that $|f(t) - g_t(t)| < \varepsilon$.

Proof. Clearly (1) $\Rightarrow$ (2). Conversely, assume that (2) holds. Let $x \in X$ be given by Theorem 1, i.e.,

\[ \text{dist } (f; A) = \text{dist } (f(x); A(x)). \]

By (2) applied to $t = x$, there is some $g_x \in A$ such that $|f(x) - g_x(x)| < \varepsilon$. Hence $\text{dist } (f(x); A(x)) < \varepsilon$. By (*) above, $\text{dist } (f; A) < \varepsilon$, and therefore some $g \in A$ such that $\|f - g\| < \varepsilon$ can be found. Hence (1) is valid.

Corollary 2 Let $A \subset D(X)$ be a von Neumann subset which is strongly separating over $X$. Let $f \in D(X)$ and $\varepsilon > 0$ be given. The following are equivalent:

1. there is some $g \in A$ such that $\|f - g\| < \varepsilon$.
2. for each $t \in X$, there is $g_t \in A$ such that $|f(t) - g_t| < \varepsilon$

Proof. Corollary 2 follows from Corollary 1 in the same way that Theorem 2 follows from Theorem 1. Or else, note that $A \subset M$ if $M$ denotes the set of all multipliers of $A$ and then apply Theorem 2 to $A$, since $M$ is strongly separating over $X$ because it contains $A$. 
Theorem 3 Let $A \subset D(X)$ be a non-empty subset such that the set $M$ of its multipliers is strongly separating, and for each $\lambda \in V$ and each $x \in X$, there is $\varphi \in A$ such that $\varphi(x) = \lambda$. Then $A$ is uniformly dense in $D(X)$.

Proof. Let $f \in D(X)$. By Theorem 1, there is some $x \in X$ such that

$$\text{dist } (f; A) = \text{dist } (f(x); A(x)).$$

Now, by hypothesis, $A(x) = V$. Hence $f(x) \in A(x)$ and so $\text{dist } (f(x); A(x)) = 0$. Hence $\text{dist } (f; A) = 0$ for all $f \in D(X)$, and $A$ is uniformly dense in $D(X)$.

Remark. If $A \subset D(X)$ is as in Theorem 1 and contains all the constant functions with values in $V$, then Theorem 3 applies trivially and $A$ is uniformly dense in $D(X)$.

Corollary 3 Let $A \subset D(X)$ be a von Neumann subset which is strongly separating over $X$, and for each $\lambda \in V$ and $x \in X$ there is $\varphi \in A$ such that $\varphi(x) = \lambda$. Then $A$ is uniformly dense in $D(X)$.

Corollary 4 Let $W$ be a subring of $D(X)$ which is strongly separating over $X$ and $W(x) = V$ for each $x \in V$. Then $W$ is uniformly dense in $D(X)$.

Proof. Clearly, every subring of $D(X)$ is a von Neumann subset.

Remark. The valuation ring $V$ is a topological ring with unit, and has a fundamental system of neighborhoods of 0 which are ideals in $V$. Hence Theorem 32 of Kaplansky [2] applies, giving an alternate proof for Corollary 4.

3. Examples

Let us give some examples of von Neumann subsets of $D(X)$ which are strongly separating over $X$. Let us first remark that a separating subring of $D(X)$ is not necessarily strongly separating over $X$. The set $W = \{ f \in D(X); |f(x)| < 1, \text{ for all } x \in X \}$ is an example of a separating subring of $D(X)$; in fact, it is a closed two-sided ideal of $D(X)$, which is not strongly separating. Indeed no function in $W$ can take the value 1 at any point in $X$. Further examples can be found. Indeed, for a fixed point $a \in X$ let us define $W_a = \{ f \in D(X); f(a) = 0 \}$. Clearly, $W_a$ is a subring of $D(X)$. Now $W_a$ is separating over $X$. Indeed, let $x \neq y$ be given in $X$. If $x = a$ or $y = a$, the function $\varphi \in D(X)$ which is zero at $a$ and one at the other point is such that $\varphi(x) \neq \varphi(y)$ and $\varphi \in W_a$. In case $x \neq a$ and $y \neq a$, let $\varphi \in D(X)$ be such that $\varphi(a) = 0$ and $\varphi(y) = 1$, and let $\psi \in D(X)$ be such that $\psi(x) = 0$ and $\psi(y) = 1.$
Then $\eta = \varphi \psi \in W_a$ and $\eta(x) = 0$ while $\eta(y) = 1$. On the other hand, $W_a$ is not strongly separating over $X$. For every ordered pair $(a, x)$, with $a \neq x$, there is no function $\varphi \in W_a$ such that $\varphi(a) = 1$ and $\varphi(x) = 0$. Indeed, $\varphi \in W_a$ implies $\varphi(a) = 0$, and so $W_a$ is not strongly separating over $X$.

**Example 1** The collection $A$ of the characteristic functions of all the clopen subsets of $X$ is a von Neumann subset of $D(X)$, containing 0 and 1, and moreover, since $X$ is a 0-dimensional compact Hausdorff space, $A$ is strongly separating over $X$.

**Example 2** Let $X = V = \{ t \in Q_p; |t|_p \leq 1 \}$, where $(Q_p, | \cdot |_p)$ is the p-adic field. Then the unitary subalgebra $W$ of all polynomials $q : Q_p \to Q_p$ is separating over $X$. By Proposition 1, Prolla [6], $A = \{ q \in W; q(X) \subset V \}$ is strongly separating over $X$. Clearly, $A$ is a von Neumann subset containing the constants in $D(X)$.

**Example 3** Let $n \geq 1$ be an integer and let $V = \{ t \in F; |t| \leq 1 \}$ and assume that $V$ is compact. Then the unitary subalgebra $W$ of all polynomials $q : F^n \to F$ in $n$-variables is separating over $X = V^n$, because $W$ contains all the $n$ projections. By Proposition 1, Prolla [6], $A = \{ q \in W; q(V^n) \subset V \}$ is a strongly separating von Neumann subset of $D(V^n)$ containing all constant functions with values in $V$.

**Example 4** Let $\{ S_i \}_{i \in I}$ be a finite partition of $X$ into clopen subsets, i.e., the set $I$ of indices is finite, each $S_i$ is a clopen set, $S_i \cap S_j = \emptyset$ for all $i \neq j$ and $X = \bigcup_{i \in I} S_i$. For each $i \in I$, let $\varphi_i$ be the characteristic function of $S_i$ and let $\lambda_i \in V$. Consider the function $\varphi \in D(X)$ defined by

$$\varphi(x) = \sum_{i \in I} \lambda_i \varphi_i(x)$$

for all $x \in X$. Let $A \subset D(X)$ be the collection of all functions $\varphi$ defined as above. Then $A$ satisfies all the hypothesis of Theorem 3 and therefore is uniformly dense in $D(X)$.

**Definition 4** A non-empty subset $A \subset D(X)$ is said to be a **restricted range polynomial algebra** if for every choice $\varphi_1, ..., \varphi_n \in A$ and $q : F^n \to F$ a polynomial in $n$-variables such that $|q(\varphi_1(x), \varphi_2(x), ..., \varphi_n(x))| \leq 1$ for all $x \in X$, the mapping $x \to q(\varphi_1(x), ..., \varphi_n(x))$ belongs to $A$.

Notice that the polynomials $(u_1, u_2) \to u_1 + u_2, (u_1, u_2) \to u_1 u_2$ and $(u_1, u_2) \to u_1 - u_2$ are such that $V \times V$ is mapped into $V$, and therefore any restricted range polynomial algebra is a subring of $D(X)$, and a fortiori a von Neumann subset. Notice that any restricted range polynomial algebra contains all the constant functions with values in $V$.

**Proposition 1** Let $A \subset D(X)$ be a restricted range polynomial algebra which is separating over $X$. Then $A$ is strongly separating over $X$.

**Proof.** Let $(s, t)$ be an ordered pair of distinct elements of $X$. By hypothesis, there exists $\varphi \in A$ such that $\varphi(s) \neq \varphi(t)$. 
Let \( q : F \to F \) be the linear function
\[
u \mapsto (\varphi(t) - \varphi(s))^{-1}(u - \varphi(s))
\]
Then \( q(\varphi(s)) = 0 \) and \( q((\varphi(t)) = 1. \) Since \( q \) is continuous, \( q(\varphi(X)) \) is a compact subset of \( F. \) By Kaplansky's Lemma (see Kaplansky [3] or Lemma 1.23, Prolla [4]) there is a polynomial \( p : F \to F \) such \( p(1) = 1 \) and \( p(0) = 0 \) and \( |p(t)| \leq 1 \) for all \( t \in q(\varphi(X)). \) Let \( r = p \circ q \) then \( r : F \to F \) is a polynomial such that \( r(\varphi(X)) \subset V. \) Hence \( r \circ \varphi = \psi \) belongs to \( A. \) Now \( \psi(s) = p(q(\varphi(s))) = p(0) = 0 \) and \( \varphi(t) = p(q(\varphi(t))) = p(1) = 1. \) Hence \( A \) is strongly separating.

**Corollary 5** Let \( A \subset D(X) \) be a restricted range polynomial algebra which is separating over \( X. \) Then \( A \) is uniformly dense in \( D(X). \)

**Proof.** By Proposition 1, \( A \) is strongly separating. On the other hand \( A \) contains all the constant functions with values in \( V. \) Hence \( A(x) = V, \) for every \( x \in X. \) Since \( A \) is a von Neumann set, the result follows from Corollary 3. Or else, notice that \( A \) is a subring and then apply Corollary 4.

\[ \square \]

4. Simultaneous Approximation and Interpolation

**Definition 5** A non-empty subset \( A \subset D(X) \) is called an interpolating family for \( D(X) \) if, for every \( f \in D(X) \) and every finite subset \( S \subset X, \) there exists \( g \in A \) such that \( g(x) = f(x) \) for all \( x \in S. \)

**Theorem 4** Let \( W \subset D(X) \) be an interpolating family for \( D(X), \) whose set of multipliers is strongly separating over \( X. \) Then, for every \( f \in D(X) \) every \( \varepsilon > 0 \) and every finite set \( S \subset X, \) there exists \( g \in A \) such that \( \|f - g\| < \varepsilon, \|g\| = \|f\| \) and \( g(t) = f(t) \) for all \( t \in S. \)

**Proof.** Let \( A = \{ g \in W; g(t) = f(t) \text{ for all } t \in S \}. \) Since \( W \) is an interpolating family for \( D(X), \) the set \( A \) is non-empty. It is easy to see that every multiplier of \( W \) is also a multiplier of \( A. \) Hence the set of multipliers of \( A \) is strongly separating over \( X. \) Consider the point \( x \in X \) given by Theorem 1, applied to \( A \) and \( f, \) i.e.,

\[
(*) \quad \text{dist} \, (f; A) = \text{dist} \, (f(x); A(x))
\]

Consider the finite set \( S \cup \{ x \}. \) Since \( W \) is an interpolating family for \( D(X), \) there is some \( g_x \in W \) such that \( g_x(t) = f(t) \) for all \( t \in S \cup \{ x \}. \) In particular, \( g_x(t) = f(t) \) for all \( t \in S \) and therefore \( g_x \in A. \) On the other hand \( g_x(x) = f(x) \) implies that \( f \{ x \} \in A(x). \) By (\( * \)), \( \text{dist} \, (f; A) = 0. \) Choose \( 0 < \delta \) such that \( \delta < \varepsilon \) and \( \delta < \|f\|. \)

There is some \( g \in A \) such that \( \|f - g\| < \delta. \) From the definition of \( A, \) it follows that \( g \in W \) and \( g(t) = f(t) \) for all \( t \in S. \) Moreover, \( \|f - g\| < \varepsilon \) and \( \|g\| = \|g - f + f\| = \|f\|, \) because \( \|g - f\| < \delta < \|f\|. \)
Corollary 6 Let $W \subset D(X)$ be an interpolating family for $D(X)$ which is a von Neumann subset and which is strongly separating over $X$. Then, for every $f \in D(X)$, every $\varepsilon > 0$ and every finite set $S \subset X$, there exists $g \in W$ such that $\|f - g\| < \varepsilon$, $\|g\| = \|f\|$, and $g(t) = f(t)$ for all $t \in S$.

**Proof.** The set $W$ is contained in the set $M$ of its multipliers and Corollary 6 follows from Theorem 4.

**Remark.** If $W \subset D(X)$ is an interpolating family for $D(X)$ which is strongly separating over $X$ and which is a subring of $D(X)$, then Corollary 6 applies to it.

Corollary 7 Let $W \subset D(X)$ be an interpolating family for $D(X)$ which is a restricted range polynomial algebra and which is separating over $X$. Then, for every $f \in D(X)$, every $\varepsilon > 0$, and every finite set $S \subset X$, there exists $g \in W$ such that $\|f - g\| < \varepsilon$, $\|g\| = \|f\|$ and $g(t) = f(t)$ for all $t \in S$.

**Proof.** We know that every restricted range polynomial algebra is a von Neumann subset. By Proposition 1, $W$ is strongly separating. The result now follows from the previous Corollary.

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