Scale of Homogeneity of the Universe from WMAP

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We review the physics of the Grishchuck-Zel’dovich effect which describes the impact of large amplitude, super-horizon gravitational field fluctuations on the Cosmic Microwave Background anisotropy power spectrum. Using the latest determination of the spectrum by WMAP, we infer a lower limit on the present length-scale of such fluctuations of $3.9 \times 10^3$ times the cosmological particle horizon (at the 95% confidence level).

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A definitive map of large scale fluctuations in the Cosmic Microwave Background (CMB) is currently being rendered by the WMAP satellite [1]. The preliminary results have been used to constrain a host of cosmological parameters within the context of a Big Bang universe with primordial adiabatic perturbations [2,3]. A candidate model, which consists of a cocktail of cold dark, matter, baryons, neutrinos, radiation and a cosmological constant (the ΛCDM model), seems to fit the data well, albeit with a larger goodness of fit than would be desired. The paradigm of an essentially homogeneous universe with mild deviations is systematically passing many of the observational tests with which it is confronted.

The homogeneity of space-time is one of the fundamental assumptions of our current working model of the universe. One possibility is that our observable universe is in a particularly quiet patch of an inhomogeneous space-time which has undergone a period of inflation [4–6]. Conceivably there may be indirect evidence of the primordial inhomogeneity within which our homogeneous patch resides. Grishchuck and Zel’dovich (GZ hereafter) have pointed out that such large scale inhomogeneity will have an impact on the amplitude of the quadrupole of the CMB [7]. In this report we combine their formalism with a goodness of fit analysis of the WMAP data to find a lower bound on the scale of inhomogeneity of the universe.

On large scales, the Sachs-Wolfe (SW) effect [8] relates metric perturbations with the CMB temperature anisotropies. It is given, along a direction $\hat{n}$ in the sky, in terms of the perturbations in the gravitational potential, $\Phi$, by

$$\frac{dT}{T}(\hat{n}) = \frac{1}{3} \Phi(\hat{n}(\eta_0 - \eta_{LS}), \eta_{LS}) + 2 \int_{\eta_{LS}}^{\eta_0} d\eta \frac{d\Phi}{d\eta}(\hat{n}(\eta - \eta_{LS}), \eta)$$

(1)

where $\eta$ is the conformal time defined today as $\eta_0$. The conformal time today is the comoving distance to the horizon. The second term is the Integrated Sachs-Wolfe (ISW) effect and accounts for the contributions to the fluctuations due to the possible variations of the gravitational potential along the line of sight whereas the first term is due to perturbations in the potential present at the last scattering (LS) surface.

By expanding the temperature fluctuations in spherical harmonics on the sky, it is straightforward to show that, in a flat Universe,

$$C_\ell = \frac{2}{\pi} \int_0^{\infty} k^2 dk P^{LS}_\Phi(k) \left[ \frac{1}{3} i \phi(k(\eta_0 - \eta_{LS})) \right] + 2 \int_{\eta_{LS}}^{\eta_0} d\eta \frac{d\phi}{d\eta}(k(\eta_0 - \eta_{LS}))$$

(2)

where we have used $\Phi(\mathbf{x}, \eta) = \int \Phi(\mathbf{k}, \eta) e^{-i k \cdot \mathbf{x}/(2\pi)^3}$ and $k$ is the comoving wave-number related to $x$ by $k = 2\pi/x$. For the suppression factor $g$, defined as $\Phi(\mathbf{k}, \eta) = g(\eta)\Phi(\mathbf{k}, \eta_{LS})$, we use the precise parametrization of [10]. Note that $g(\eta_{LS}) = 1$ and $g$ varies with time depending on the cosmological model considered. $P^{LS}_\Phi$ is the gravitational field power spectrum at the last scattering surface. With a flat spectrum $P^{LS}_\Phi(k) = A k^{-3}$, much larger than the horizon make little contribution to the $C_\ell$’s: If we consider only the SW term in Eq. (2), the wave-number contributions such that $k \ll 1/\eta_0$ scale with $k^2$. But a possible sharp and localized increase in the power spectrum at a certain super-horizon scale $k_{GCZ}$ of the type $P^{LS}_\Phi(k) = A(\delta(k - k_{GCZ})$ would leave a stronger imprint in the $C_\ell$’s at low multipoles as firstly shown by Grishchuck and Zel’dovich in [7]. This effect is more prominent for the monopole, dipole and quadrupole components. The monopole cannot be distinguished from a small shift in the measured temperature of the CMB because the density wave is super-horizon sized and we are averaging over our horizon. The dipole is contaminated by the Doppler effect caused by the motion of our Local Group with respect to the CMB rest-frame. The relevant multipole to study the GZ effect is therefore the quadrupole. For this multipole, contributions from super-horizon gravitational inhomogeneous modes of characteristic scale $k$ to the GZ effect get suppressed by a factor of $O(k^4)$, as will be shown later. By not seeing an increase of power in the quadrupole, one can therefore place constrains on the homogeneity properties of super-horizon perturbations namely on the minimum allowed wavelength. On larger scales, due to the suppression factor of $O(k^4)$, the universe can be anisotropic and inhomogeneous without being in disagreement with the available experimental constraints [9]. In fact, if we were to consider contributions from larger scales as well, (assuming, for example, a step function $P^{LS}_\Phi(k) = A\theta(k_{GCZ} - k)$),
the total contribution to the quadrupole value would decrease, doubling the value obtained for \( k_{\text{GZ}} \) from the data. This behaviour is due to the way the normalization constant \( A \) is derived. As will be explained below, in this case, when fixing \( A \), we would also be considering contributions from modes smaller than \( k_{\text{GZ}} \) which would reduce the relative amplitude of the potential fluctuations coming solely from the \( k_{\text{GZ}} \) mode, as compared to the standard Grishchuck and Zel’dovich calculation.

We consider that the homogeneity of the gravitational field fluctuations is satisfied as long as the characteristic amplitudes of the fluctuations are below one (in the linear regime boundary). In the limiting case of amplitudes of order 1, which we will investigate, the amplitude \( A \) of the power spectrum can be obtained by imposing that \( \langle |\Phi(L_{\text{GZ}})|^2 \rangle = 1 \), where \( \langle |\Phi(L_{\text{GZ}})|^2 \rangle \) is the average variance of the gravitational field fluctuations in spheres of comoving radius \( L_{\text{GZ}}/2 \) given by \( L_{\text{GZ}}/2 = 2\pi/k_{\text{GZ}} \). We point out that there is some confusion in the literature in this respect \([13,14]\): we are fixing the amplitude of the gravitational field perturbation associated with the density perturbation, not the amplitude of the density perturbation.

The normalized expression for the Grishchuck-Zel’dovich effect is the following

\[
C_\ell = \frac{2^6 \pi^5}{9} \left[ \frac{1}{3} j_2(k_{\text{GZ}}(\eta_0 - \eta_{\text{LS}})) + 2 \int_{\eta_{\text{LS}}}^{\eta_0} \frac{d\eta}{\eta} j_2(k_{\text{GZ}}(\eta_0 - \eta)) \right]^2
\]

At \( \ell = 2 \), for \( k_{\text{GZ}} \simeq 10^{-6} \text{Mpc}^{-1} \) and for the cosmologies considered here, the effect of the ISW corresponds to around 10% of the pure SW contributions. Looking at the purely SW term of this expression, which dominates over the ISW term, we can use the small-argument limit of the spherical Bessel function \( j_2(x) = x^2/15 \), to obtain the following relation for the quadrupole amplitude

\[
\frac{\Delta T}{T}(\ell = 2) \simeq (k_{\text{GZ}} \eta_0)^2 = \left( \frac{2\pi}{L_{\text{GZ}}} \right)^2
\]

where \( \Delta T/T = \sqrt{\ell(\ell+1)}C_\ell/(2\pi) \), \( L_0 = 2\eta_0 \) is the present size of the horizon diameter and \( L_{\text{GZ}} \) is the present length-scale of the super-horizon homogeneous patch of characteristic wave-number \( k_{\text{GZ}} \). This approximation was obtained previously by others \([9]\) without the \( 2\pi \) factor which was neglected. Having an upper limit on the value of the quadrupole, we can obtain a lower limit on the diameter size of the largest possible homogeneous scale of the universe. Similar expressions for the GZ effect were calculated for the open cosmology case \([13]\) for which the constraints on \( L_{\text{GZ}} \) can be even more stringent. Given the present experimental consensus around the flatness of our universe, we place ourselves in a flat geometry.

We proceed by using the recent WMAP temperature data to constrain the possible contribution of super-horizon gravitational field fluctuations (with an amplitude of order 1) to the quadrupole. For the analysis we consider the GZ contribution to the power spectrum in addition to the usual power law \( \Lambda-\text{CDM} \) spectrum. We compute the standard \( C_\ell \) spectrum by using CAMB code \([11]\), and add to it the GZ spectrum computed analytically as described in Eq. (3). We consider a grid of models corresponding to inflationary adiabatic perturbations in a flat cosmology by varying the following cosmological parameters around the best values found by the WMAP collaboration \([2]\): \( H_0, \Omega_\Lambda, \tau, \text{ and } n \) and \( L_{\text{GZ}}/L_0 \). The normalization of the “power law” \( C_\ell \) spectrum was left free and was marginalized over. No contributions from neutrinos or gravitational waves were considered. As the effect of the super-horizon perturbation is expected at very low \( \ell \) (mainly at the quadrupole), where the presence of the baryons is negligible, we fixed the baryon contribution at the best estimate from WMAP: \( \Omega_b h^2 = 0.023 \). We used the likelihood code provided by the WMAP team \([3]\) and modified by \([12]\) to compute the likelihood of each model of our grid. We then maximized over all the parameters, except \( L_{\text{GZ}}/L_0 \), in order to retrieve the likelihood as a function of the scale of the super-horizon perturbations. The result is plotted in Fig. 1.

We obtain an upper limit on the region of homogeneity (and isotropy) of \( L_{\text{GZ}} > 3927L_0 \) (at 95% CL). This correspond to \( L_{\text{GZ}} \sim 6.9 \times 10^7 \text{Mpc} \) for \( H_0 = 68 \text{Kms}^{-1}\text{Mpc}^{-1} \). As expected this value is comparable (except for the \( 2\pi \) factor, as detailed above) to previous constraints relying on the COBE data \([13–15]\), which also presented a low quadrupole value. The estimate from \([16]\) gives \( Q_{\text{COBE}} = 10 \pm 3 \pm 7 \text{ mK} \), where the quoted errors are statistical and systematic (from the foreground removal) respectively. The WMAP results \([17]\) give roughly the same upper limit of \( Q_{\text{WMAP}} = 12.3 \pm 3.1 \text{ mK} \) at 68% CL, where the systematics were assumed to be negligible. As stressed before, the contribution from the GZ effect to the quadrupole is proportional to \( k_{\text{GZ}}^4 \) such that observing a quadrupole amplitude higher by a factor of 2 decreases the length-scale \( L_{\text{GZ}} \) by only a factor of \( 2^{1/4} \). The small difference in the upper limit for the quadrupole between the WMAP and the COBE experiments induces an even smaller difference on the determination of the possible scale of the gravitational fluctuations considered. Nevertheless our result is more robust because we include the present uncertainty on most of the relevant cosmological parameters. It is worth mentioning that – as could have been predicted – nearly no degeneracies were found between the scale of the super-horizon fluctuations and all the other parameters explored in this analysis. This is due to the unique and strong localization of the GZ effect at the quadrupole component.

There has been some debate surrounding the statistical significance of the low quadrupole values obtained by the WMAP and the COBE DMR experiments as compared to the values predicted by the standard cosmological model \([2,17–21]\). In this context, some work has
been devoted to finding a physical mechanism able to explain the possible discrepancy [2, 19, 18, 22–29]. In this report, we take the quadrupole and octopole results from WMAP at face value and assume they are in accordance with the standard model within the cosmic variance errors, regardless of the present controversy. Of course, if any of these proposed mechanisms, modifying the standard scenario, are indeed lowering the quadrupole value our constraints on the homogeneity scale would be slightly affected.

To conclude we point out that one can estimate the number of e-foldings, $N_e$, which a patch of the universe has undergone during an inflationary period. Numerical studies of the onset of inflation in inhomogeneous universes indicate that the region of space time which inflates must start of with at most fluctuations of order one on the horizon scale [30, 31]. Mapping this scale onto $L_{GZ}$ today we find that $N_e \sim \ln(M_{\text{inf}}/(10^{-15} \text{ GeV}))$ where $M_{\text{inf}}$ is the energy scale of inflation. For inflation occurring at $M_{\text{inf}} > 10^{16} \text{ GeV}$ one has $N_e > 72$. It is interesting (but not surprising) that $N_e$ is similar to the number of e-folding required to get the correct amplitude and slope of fluctuations from the parametric amplification of quantum fluctuations during inflation [32, 33]. One can also reinterpret this result as a constraint on the duration of an uninterrupted period of inflation and in doing so limit the effects of primeval, non-thermal corrections to cosmological perturbations [34].

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