V-JACOBIAN AND V-CO-JACOBIAN FOR LIPSCHITZIAN MAPS

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Dedicated to the 60th birthday of Francis Clarke and Richard Vinter

Abstract. The notions of V-Jacobian and V-co-Jacobian are introduced for locally Lipschitzian functions acting between arbitrary normed spaces X and Y, where V is a subspace of the dual space Y*. The main results of this paper provide a characterization, calculus rules and also the computation of these Jacobians of piecewise smooth functions.

1. Introduction. Consider a Lipschitzian map f acting between two normed spaces X and Y. Motivated by the case of smooth maps, the task of finding a “good” generalized Jacobian notion at a point p focuses on searching for a derivative-like object, namely, a nonempty set of linear operators that serves as a reasonable approximation of the function change near p, and possesses “applicable” calculi.

The notion of Clarke’s generalized gradient ([4]) is known to be a satisfactory approximation for a real-valued, Lipschitzian function f at p by a nonempty subset of \( L(X, \mathbb{R}) \), which is the dual space \( X^* \). Similarly, when X and Y are of finite dimension, Clarke’s generalized Jacobian was defined for Lipschitzian maps in [3], [4] as a subset of the space \( \mathcal{L}(X, Y) \). Rademacher’s differentiability theorem is instrumental in obtaining the nonemptiness of this generalized Jacobian.

For general normed spaces X and Y, several notions of sets playing the role of a derivative have been constructed. For instance, the notion of derive containers in [31], [32]; the concepts of screens and fans in [6], [5]; the concept of shields [28]; the fan derivative [7], and the notion of coderivatives in [17]. Many of these notions are not given in terms of sets of linear operators, but rather in the form of set-valued maps.

On the other hand, in a series of papers [20], [19], [21], [22] and [23], notions of a generalized Jacobian and a co-Jacobian for Lipschitzian functions have recently

2000 Mathematics Subject Classification. 49J52, 49A52, 58C20.

Key words and phrases. Generalized Jacobian, Co-Jacobian, Characterization theorem, Chain rule, Sum rule, Continuous selection.

Research of the first author is supported by the Hungarian Scientific Research Fund (OTKA) Grant NK81402 and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project implemented through the New Hungary Development Plan co-financed by the European Social Fund, and the European Regional Development Fund. Research of the second author is supported by the National Science Foundation under grant DMS-0707789.
been developed as nonempty sets of linear operators when $X$ is any normed space. The notions are recalled in Section 3.

When $Y$ is finite dimensional, the generalized Jacobian $\partial f(p) \subseteq \mathcal{L}(X,Y)$ was constructed in [20] and [21] so that it extended both Clarke’s generalized Jacobian and gradient. Moreover, the following basic identity was established:

$$y^* \circ \partial f(p) = \partial f^*(y^* \circ f)(p) \quad (y^* \in Y^*).$$

(1.1)

This identity relates Clarke’s gradient to our generalized Jacobian and could be viewed as a special case of the chain rule.

Subsequently, this notion of generalized Jacobian was expanded in [19] to the case when, for some normed space $V$, we have $Y = V^*$ and $Y$ has the Radon–Nikodým property. In this more general setting, $\partial f(p)$ is kept in the space $\mathcal{L}(X,Y)$ (which equals $\mathcal{L}(X,V^*)$) and satisfies the linear chain rule (1.1). The Radon–Nikodým property of $Y$ guarantees the applicability of the generalized Rademacher differentiability theorem, while the range being a dual space ensures that the norm-closed unit ball of the space $\mathcal{L}(X,Y)$ is compact in a certain topology. A consequence of these two properties is that $\partial f(p)$ is a nonempty subset of $\mathcal{L}(X,Y)$. Therefore, when neither the range is a dual space nor the Radon–Nikodým property is present, one can naturally ask whether a concept of a generalized Jacobian $\partial f(p) \subseteq \mathcal{L}(X,Y)$ can be defined in such a way that it is nonempty, and enjoys a sufficiently rich and applicable calculus, in particular, identity (1.1) is satisfied. In fact, we will show in an example below that such a goal cannot be reached without imposing extra assumptions on the range space $Y$.

In order to avoid assuming any condition on the range space, we introduced in [23] the notion of a co-Jacobian, $\partial^* f(p)$, which is a nonempty set of operators in the space $\mathcal{L}(Y^*,X^*)$, as opposed to $\mathcal{L}(X,Y)$. This set $\partial^* f(p)$ satisfies the following linear chain rule:

$$\partial^* f(p) (y^*) = \partial f^*(y^* \circ f)(p) \quad (y^* \in Y^*),$$

(1.2)

which is a counterpart of (1.1). Furthermore, when $Y = V^*$ and $Y$ has the Radon–Nikodým property, it is shown that

$$\partial f(p) = (\partial^* f(p))|_V \quad \text{and} \quad \partial^* f(p) = (\partial f(p))^*.$$

(1.3)

When $Y = V^*$, but $Y$ does not have the Radon–Nikodým property, then the first equation in (1.3) motivates a definition for the generalized Jacobian that is still contained in $\mathcal{L}(X,Y)$ and naturally generalizes the concept known for the Radon–Nikodým setting. It follows that when $Y = V^*$ then, $V \subseteq Y^*$.

Consider the case when $Y$ is not the dual of a normed space $V$ but only $V \subseteq Y^*$ holds. Then, the first equation in (1.3) is still capable of defining a generalized Jacobian. However, in this case, the set of operators defining $\partial f(p)$ will no longer stay in $\mathcal{L}(X,Y)$, but rather in $\mathcal{L}(X,V^*)$. Observe that from Lemma 2.2 we know that $V \subseteq Y^*$ yields that $V^* = Y^*|_V \supseteq Y|_V$. If $V$ is also separating the elements of $Y$, then the same lemma implies that $Y|_V$ isomorphic to $Y$ and hence, in this case, we could consider the space $V^*$ as an enlargement of $Y$. The main advantage of this enlargement lies in the fact that $Y$ is embedded into a dual space which brings us back to the previous friendly setting. It appears that a natural candidate for the space $V$ is the largest possible choice, which is $V = Y^*$. However, in this case, $V^* = Y^{**}$, which can be significantly larger than $Y$ when $Y$ is non-reflexive. Thus, in the non-reflexive case, it is more rewarding to choose $V$ as an appropriate proper subspace of $Y^*$ so that the gap between $Y^{**}|_V$ and $Y|_V$ is not too big.
From the first glance, it may appear that finding a generalized Jacobian for \( f : X \rightarrow Y \) in the space \( \mathcal{L}(X,Y) \) instead of the space \( \mathcal{L}(X,Y) \) is somewhat unnatural and unreasonable. However, as we shall see in the simple example below, even if a minimal requirement is postulated for a concept of a generalized Jacobian, then this goal is not in general reachable.

Given any Lipschitz map \( f : X \rightarrow Y \), we require that a generalized Jacobian \( \Delta f : X \rightarrow 2^{\mathcal{L}(X,Y)} \) must satisfy the following natural assumptions:

(i) \( \Delta f(x) \) is a nonempty subset of \( \mathcal{L}(X,Y) \) for all \( x \in X \);

(ii) For every linear functional \( y^* \in Y^* \), the linear (inclusion) chain rule holds:

\[
y^* \circ \Delta f(x) \subseteq \partial^c(y^* \circ f)(x) \quad (x \in X).
\]

In the context when \( X \) and \( Y \) are the spaces of continuous functions, we furnish an example which shows that the existence of a generalized Jacobian satisfying conditions (i) and (ii) is not possible. The reason for the failure of such a construction is that the gap between \( Y \) and \( Y^{**} \) is significant (see [9]). It is worth mentioning that the conclusion of this example remains valid if we replace in the above condition (ii) Clarke’s gradient by any other smaller subdifferential.

**Example.** Denote by \( \mathcal{C}(I) \) the space of continuous real-valued functions equipped with the supremum norm, where \( I := [-1,1] \). Define the function \( f : \mathcal{C}(I) \rightarrow \mathcal{C}(I) \) by \( f(x)(t) := |x(t)| \), and let \( x_0(t) := t \). We show that \( \Delta f(x_0) \subseteq \mathcal{L}(\mathcal{C}(I),\mathcal{C}(I)) \) cannot be defined so that properties (i), (ii) be satisfied.

Clearly \( f \) is a Lipschitzian map with Lipschitz modulus 1. For a fixed \( \tau \in I \), consider the linear functional \( y^* \in (\mathcal{C}(I))^* \) (which is the evaluation at the point \( \tau \)) defined by

\[
(y^*_*, y) := y(\tau) \quad (y \in \mathcal{C}(I)).
\]

Then \( y^*_* \circ f \) is a Lipschitzian function and its Clarke’s generalized directional derivative at \( x \in \mathcal{C}(I) \) is computed in the following way:

\[
(y^*_* \circ f)^\circ(x,h) = \lim_{\eta \rightarrow y^*} \limsup_{\lambda \rightarrow 0^+} \frac{|y(\tau) + \lambda h(\tau)| - |y(\tau)|}{\lambda} = \limsup_{\eta \rightarrow y^*} \lim_{\lambda \rightarrow 0^+} \frac{|\eta + \lambda h(\tau)| - |\eta|}{\lambda}
\]

\[
= (|\cdot|)^\circ(x(\tau),h(\tau)) = \begin{cases} h(\tau) & \text{if } x(\tau) > 0, \\
|h(\tau)| & \text{if } x(\tau) = 0, \\
-h(\tau) & \text{if } x(\tau) < 0. \end{cases}
\]

Therefore, a linear functional of \( \mathcal{C}(I) \) represented by a bounded regular Borel measure \( \mu \) belongs to Clarke’s subgradient \( \partial^c(y^*_* \circ f)(x) \) for \( x \in \mathcal{C}(I) \) if and only if, for all \( h \in \mathcal{C}(I) \),

\[
\int_I h(t)d\mu(t) \leq \begin{cases} h(\tau) & \text{if } x(\tau) > 0, \\
|h(\tau)| & \text{if } x(\tau) = 0, \\
-h(\tau) & \text{if } x(\tau) < 0. \end{cases}
\]

This is equivalent to the property that the support of the measure \( \mu \) is the singleton \( \{\tau\} \) and

\[
\mu(\{\tau\}) = \begin{cases} 1 & \text{if } x(\tau) > 0, \\
[-1,1] & \text{if } x(\tau) = 0, \\
\{-1\} & \text{if } x(\tau) < 0. \end{cases}
\]

Hence \( \mu(\{\tau\}) = \text{sign}(x(\tau)) \) whenever \( x(\tau) \neq 0 \).
By property (i), the generalized Jacobian $\Delta f(x)$ is not empty for all $x \in \mathcal{C}(I)$. Let $\Phi \in \Delta f(x)$. Then $\Phi \in \mathcal{L}(\mathcal{C}(I), \mathcal{C}(I))$ and, by (ii), for all $\tau \in I$, there exists $x^* \in \partial' (y^* \circ f)(x)$ such that $y^* \circ \Phi = x^*$. If $x^*$ is represented by a regular Borel measure then, as we have seen above, $\mu$ is supported at $\{\tau\}$ and $\mu(\{\tau\}) = \text{sign}(x(\tau))$ whenever $x(\tau) \neq 0$. Hence, the equality $y^* \circ \Phi = x^*$ yields that

$$\Phi(h)(\tau) = \mu(\{\tau\})h(\tau) = \text{sign}(x(\tau))h(\tau) \quad (h \in \mathcal{C}(I))$$

whenever $x(\tau) \neq 0$. In particular, if $h(t) \equiv 1$ and $x = x_0$, then, for $\Phi \in \Delta f(x_0)$, we obtain that

$$\Phi(1)(\tau) = \text{sign}(\tau) \quad (\tau \neq 0),$$

which contradicts the continuity of $\Phi(1)$ at $t = 0$.

In other words, there is no reasonable form of a generalized Jacobian in the space $\mathcal{L}(\mathcal{C}(I), \mathcal{C}(I))$ for all Lipschitzian maps $f : \mathcal{C}(I) \rightarrow \mathcal{C}(I)$. Therefore, the desired generalized Jacobian should be constructed in this case in another space, which is more appropriate. In fact, we shall show in the last section of this paper that such a Jacobian could be successfully constructed as a subset of $\mathcal{L}(\mathcal{C}(I), \mathcal{B}(I))$, where $\mathcal{B}(I)$ denotes the space of bounded real-valued functions equipped with the sup-norm. For this setting, the space $V$ will be chosen as a proper subspace of $Y^*$, namely, the linear subspace spanned by all the Dirac measures on the interval $I$.

The aim of this paper is to introduce the concepts of the $V$-co-Jacobian, $\partial^c_n f(p) \subseteq \mathcal{L}(V, X^*)$, and the $V$-Jacobian, $\partial^V f(p) \subseteq \mathcal{L}(X, V^*)$ for a Lipschitzian map $f$ acting between two normed spaces $X$ and $Y$, where $V$ is any subspace of $Y^*$. More specifically, we define the set $\partial^V f(p)$ as the largest subset of $\mathcal{L}(X, V^*)$ such that the following linear chain rule holds

$$y^* \circ \partial^V f(p) = \partial(\mathcal{C}(I), \mathcal{B}(I))$$

for all $n \in \mathbb{N}$ and for all $y^* \in V^*$. When $n = 1$ and $V = Y^*$, this identity becomes the chain rule in (1.1). If (1.5) is required only for $n = 1$ then the set $\partial^V f(p)$ would be much larger and hence less informative.

In Sections 4 and 5 we establish the nonemptiness and a characterization of the $V$-co-Jacobian and the $V$-Jacobian, respectively. Furthermore, we derive an identity relating both notions. The results on the co-Jacobian obtained in [23] are employed. Differentiability properties and mean value theorems are obtained in Section 6. In Section 7 we derive the nonsmooth-smooth and the smooth-nonsmooth chain rules and hence, the sum rule. In Section 8 we establish the $V$-Jacobian, and hence $V$-co-Jacobian, for a continuous selection map. In the subsequent section we generalize Thibault’s limit set and Ioffe’s fan derivative so that they now take values in $V^*$ as opposed to $Y$. Furthermore, we establish the connections between our new notions and these modified concepts, as well as with Mordukhovich’s normal and mixed co-derivatives.

2. Auxiliary results. Throughout this paper, whenever $Z$ is a normed space, the symbols $Z^*$ and $Z^{**}$ denote the first and second dual spaces of $Z$, respectively. The space $Z$ is considered as a subset of $Z^{**}$ via the canonical embedding. The open and closed unit balls of $Z$ are denoted by $B_Z$ and $\overline{B}_Z$, respectively. The set $\Lambda(Z)$ will denote the family of finite dimensional subspaces of $Z$.

Let $X$ and $Y$ be normed spaces and denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from $X$ to $Y$ equipped with the standard operator norm. Given
a continuous linear operator $\Phi : X \to Y$, its adjoint operator $\Phi^* : Y^* \to X^*$ is defined via the equality
$$\langle \Phi^*(y^*), x \rangle = \langle y^*, \Phi(x) \rangle = \langle y^* \circ \Phi, x \rangle \quad (x \in X, y^* \in Y^*),$$
in other words, $\Phi^*$ is given by the formula
$$\Phi^*(y^*) = y^* \circ \Phi \quad (y^* \in Y^*). \quad (2.1)$$

It is well-known that the adjoint operator $\Phi^*$ is continuous linear and the map $\Phi \mapsto \Phi^*$ is a norm-preserving endomorphisms of $\mathcal{L}(X, Y)$ into $\mathcal{L}(Y^*, X^*)$. As we shall see below, these spaces are isometrically isomorphic whenever $Y$ is reflexive. To understand how the spaces $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y^*, X^*)$ are related to each other when $Y$ is a dual space of another normed space $V$, we need the following result.

**Lemma 2.1.** Let $X$ and $V$ be normed spaces. Then, for every $\Phi \in \mathcal{L}(X,V^*)$,
$$\Phi = (\Phi^*|_V)^*|_X. \quad (2.2)$$
The mapping $\Phi \mapsto \Phi^*|_V$ is an isometrical isomorphism from the space $\mathcal{L}(X,V^*)$ onto $\mathcal{L}(V,X^*)$ and its inverse is given by the mapping $\Psi \mapsto \Psi^*|_X$. Furthermore,
$$\mathcal{L}(X,V^*) = \mathcal{L}(X^{**}, V^*)|_X \quad \text{and} \quad \mathcal{L}(V,X^*) = \mathcal{L}(V^{**}, X^*)|_V \quad (2.3)$$
and the identity mapping between these spaces is a linear isometry.

**Proof of Lemma 2.1.** Equation (2.2) is equivalent to showing, for all $x \in X$ and $v \in V$, that
$$\langle v, \Phi(x) \rangle = \langle v, (\Phi^*|_V)^*|_X(x) \rangle. \quad (2.4)$$
Indeed, using the definition of the adjoint operator twice, we get
$$\langle v, (\Phi^*|_V)^*|_X(x) \rangle = \langle v, (\Phi^*|_V)^*(x) \rangle = \langle \Phi^*|_V(v), x \rangle = \langle \Phi^*(v), x \rangle = \langle v, \Phi(x) \rangle,$$
which proves (2.4).

The linearity of the mapping $\Phi \mapsto \Phi^*|_V$ is obvious. On the other hand, for every $\Phi \in \mathcal{L}(X,V^*)$, we have
$$\|\Phi^*|_V\| = \sup_{\|v\|=1} \|\Phi^*|_V(v)\| = \sup_{\|v\|=1} \|\Phi^*(v)\| = \sup_{\|v\|=1, \|x\|=1} |\langle \Phi^*(v), x \rangle|$$
$$= \sup_{\|x\|=1} \sup_{\|v\|=1} |\langle v, \Phi(x) \rangle| = \sup_{\|v\|=1} \|\Phi(x)\| = \|\Phi\|,$$
which proves that $\Phi \mapsto \Phi^*|_V$ is also an isometry. By the first part, we can see that the inverse of this mapping is given by $\Psi \mapsto \Psi^*|_X$ (which is also a linear isometry). Hence both mappings are isometric isomorphisms.

To prove the first identity in (2.3), observe that if $\Psi \in \mathcal{L}(X^{**}, V^*)$, then $\Psi|_X \in \mathcal{L}(X,V^*)$ which proves the inclusion “$\subset$”. On the other hand, if $\Phi \in \mathcal{L}(X,V^*)$ then $\Phi = (\Phi^*|_V)^*|_X \in \mathcal{L}(X^{**}, V^*)|_X$, which proves the reversed inclusion. \hfill $\square$

**Lemma 2.2.** If $Y = V^*$ holds for some normed space $V$, then $\mathcal{L}(X,Y)$ is isometrically isomorphic to the space $\mathcal{L}(Y^*, X^*)|_V$. In particular, if $Y$ is reflexive, then $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y^*, X^*)$ are isometrically isomorphic.

**Proof.** By Lemma 2.1, we have that $\mathcal{L}(X,V^*)$ is isometrically isomorphic to the space $\mathcal{L}(V,X^*) = \mathcal{L}(V^{**}, X^*)|_V$, which proves that $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y^*, X^*)|_V$ are isometrically isomorphic. If $Y$ is reflexive then $Y^* = V^{**} = V$, hence $\mathcal{L}(Y^*, X^*)|_V = \mathcal{L}(Y^*, X^*).$ \hfill $\square$
Remark 2.3. In the case $Y = V^*$, we have that $V \subseteq V^{**} = Y^*$, hence it is an even more general setting when only the condition $V \subseteq Y^*$ is satisfied. Then we have that $V^* = Y^{**}|_V$. Indeed, if $v^* \in V^*$, i.e., $v^*$ is a linear functional on $V$, then, by the Hahn–Banach extension theorem, $v^*$ can be extended to a linear functional on $Y^*$. Hence, $v^* \in Y^{**}|_V$ proving $V^* \subseteq Y^{**}|_V$. The reversed inclusion is trivial. The space $\mathcal{L}(Y^*, X^*)|_V$ is obviously a subspace of $\mathcal{L}(V, X^*)$ which is isometrically isomorphic to $\mathcal{L}(X, V^*) = \mathcal{L}(X, Y^{**}|_V)$. If we also assume that $Y^{**}|_V = Y|_V$, then we conclude that $\mathcal{L}(Y^*, X^*)|_V$ is isometrically isomorphic to a subspace of $\mathcal{L}(X, Y|_V)$.

Given two normed spaces $X$ and $V$, we equip the space $\mathcal{L}(X, V^*)$ with a topology in which the norm-closed unit ball of $\mathcal{L}(X, V^*)$ is compact. For all $x \in X$ and $v \in V$, define the map $x \otimes v : \mathcal{L}(X, V^*) \to \mathbb{R}$ by

$$(x \otimes v)(\Phi) := \langle \Phi(x), v \rangle \quad (\Phi \in \mathcal{L}(X, V^*)).$$  \hfill (2.5)

Then $x \otimes v$ is a continuous linear functional on $\mathcal{L}(X, V^*)$. Denote

$$X \otimes V := \{x \otimes v : x \in X, v \in V\}.$$  \hfill (2.6)

The weak topology induced by $X \otimes V$ on $\mathcal{L}(X, V^*)$, i.e., the topology $\sigma(\mathcal{L}(X, V^*), X \otimes V)$ will be called the weak-operator-topology and will be denoted by $\beta(X, V)$ throughout this paper. This notation indicates that the topology is described in terms of the elements of $X$ and $V$.

Obviously, the following sets form a neighborhood subbase for the origin in the $\beta(X, V)$-topology:

$$\{\Phi \in \mathcal{L}(X, Y) : |\langle \Phi(x), v \rangle| < \varepsilon\} \quad (x \in X, v \in V, \varepsilon > 0).$$

That is, the $\beta(X, V)$-topology is the topology of the pointwise convergence for the real valued bilinear functions defined on $X \times V$ by $(x, v) \mapsto \langle \Phi(x), v \rangle$. Trivially, the $\beta(X, V)$-topology is weaker than the norm-topology, hence $\beta(X, V)$-closed sets are automatically norm-closed, and norm-compact sets are automatically $\beta(X, V)$-compact.

The following theorem offers an analog of the Banach–Alaoglu theorem in the space $\mathcal{L}(X, V^*)$. Its proof can be found in [19, Thm. 2.1].

Theorem 2.4. ([19]) Let $X$ and $V$ be arbitrary normed spaces. Then the norm-closed unit ball of the Banach space $\mathcal{L}(X, V^*)$ is compact in the $\beta(X, V)$-topology.

Remark 2.5. As a consequence of the above theorem, the norm-closed unit balls of the Banach spaces $\mathcal{L}(Y^*, X^*)$ and $\mathcal{L}(X, Y^{**})$ are compact in the $\beta(Y^*, X)$- and $\beta(X, Y^{**})$-topology, respectively. Hence, whenever $Y$ is a dual of a normed space $V$, the norm-closed unit ball of $\mathcal{L}(X, Y)$ is compact in the $\beta(X, V)$-topology.

Given arbitrary subspaces $L \subseteq X$ and $H \subseteq V$, define the domain and image restriction maps $\text{dom}_L : \mathcal{L}(X, V^*) \to \mathcal{L}(L, V^*)$ and $\text{im}_H : \mathcal{L}(X, V^*) \to \mathcal{L}(X, H^*)$ by

$$\text{dom}_L(\Phi) := \Phi|_L \quad \text{and} \quad \text{im}_H(\Phi)(x) := \Phi(x)|_H \quad (x \in X),$$  \hfill (2.7)

respectively. The following result follows immediately by applying the definition of the $\beta$ topologies. It is also a consequence of [19, Proposition 2.2].

Lemma 2.6. ([19]) Let $X$ and $V$ be normed spaces. Then, for all subspaces $L \subseteq X$ and $H \subseteq V$, the maps $\text{dom}_L$ and $\text{im}_H$ are $(\beta(X, V), \beta(L, V))$-continuous and $(\beta(X, V), \beta(X, H))$-continuous linear operators, respectively. Furthermore, $\|\text{dom}_L\| \leq 1$ and $\|\text{im}_H\| \leq 1$. 
In order to prove inclusions between subsets of the space \( \mathcal{L}(V, X^*) \), the following result will be needed. It follows from [19, Theorem 2.4]. Given an operator \( \Phi \in \mathcal{L}(X, V^*) \) and \( v = (v_1, \ldots, v_n) \in V^n \), define \( \mathbf{v} \circ \Phi \in \mathcal{L}(X, \mathbb{R}^n) \) by

\[
\mathbf{v} \circ \Phi(x) := \langle \Phi(x), v_1 \rangle, \ldots, \langle \Phi(x), v_n \rangle
\]

In other words, \( \mathbf{v} \circ \Phi = \Phi^*(v_1, \ldots, \Phi^*(v_n)) \).

**Theorem 2.7.** ([19]) Let \( X \) and \( V \) be normed spaces and \( \mathcal{F}, \mathcal{G} \subseteq \mathcal{L}(X, V^*) \) such that \( \mathcal{G} \) is \( \beta(X, V) \)-compact. Then \( \mathcal{F} \subseteq \mathcal{G} \) holds if and only if, for all \( n \in \mathbb{N} \) and \( \mathbf{v} \in V^n \), for all \( L \in \Lambda(X) \),

\[
\mathbf{v} \circ \mathcal{F}|_L \subseteq \mathbf{v} \circ \mathcal{G}|_L.
\]  

(2.8)

3. **Known results.** Let \( X \) and \( Y \) be normed spaces, \( \mathcal{D} \) be a nonempty open subset of \( X \), \( p \) be an arbitrary point in \( \mathcal{D} \) and \( f : \mathcal{D} \to Y \). Define \( \ell_f(p) \), the Lipschitz modulus of \( f \) at \( p \), by

\[
\ell_f(p) := \inf_{\delta > 0} \sup \left\{ \frac{\|f(x) - f(y)\|}{\|x - y\|} \mid x, y \in p + \delta B_X, x \neq y \right\}.
\]  

(3.1)

It is obvious that \( \ell_f \) is an use function on \( \mathcal{D} \) and that \( f \) is Lipschitzian near \( p \) if and only if \( \ell_f(p) < +\infty \).

3.1. **The Generalized Jacobian.** For a real valued Lipschitzian function \( f : \mathcal{D} \to \mathbb{R} \), the generalized gradient of \( f \) at a point \( p \in \mathcal{D} \) is defined by

\[
\partial^c f(p) := \{ \zeta \in X^* \mid \langle \zeta, h \rangle \leq f^c(p, h), \forall h \in X \},
\]  

(3.2)

where

\[
f^c(p, h) := \lim_{\substack{x \to p \\text{in } X \\mathcal{D}}} \sup_{t \to 0^+} \frac{f(x + th) - f(x)}{t} \quad (h \in X)
\]  

(3.3)

is Clarke’s generalized directional derivative (see [4]).

When \( X \) and \( Y \) are both finite dimensional normed spaces and \( f : \mathcal{D} \to Y \) is a vector-valued Lipschitzian function, Clarke introduced in [3], [4] the notion of the generalized Jacobian of \( f \) at a point \( p \in \mathcal{D} \) by

\[
\partial f(p) := \text{co} \{ \Phi \in \mathcal{L}(X, Y) \mid \exists (x_i) \text{ in } \Omega(f) : \lim_{i \to \infty} (x_i, Df(x_i)) = (p, \Phi) \},
\]  

(3.4)

where \( \Omega(f) \) denotes the set of the points of \( \mathcal{D} \) where \( f \) is differentiable. Based on Rademacher’s celebrated differentiability theorem, we have that \( \Omega(f) \) is of full measure. In terms of the above generalized gradient and Jacobian, results have been derived pertaining optimality conditions, implicit functions theorems, metric regularity, and calculus rules including the sum rule and the chain rule. Thereby, it has already been shown that these objects are successful approximations of \( f \) by linear operators.

The notion of the generalized Jacobian introduced in [20] and [21] and which we recall here, will be the backbone for the co-Jacobian concept introduced in this paper. In those references, the difficulty caused by the infinite dimensionality of the domain was handled by introducing the following concepts of differentiability and Jacobian relative to finite dimensional subspaces of \( X \), so that Rademacher’s theorem remains applicable.
Let \( f : \mathcal{D} \to Y \) be a Lipschitzian function. Given a linear subspace \( L \subseteq X \), the function \( f \) is called \( L\)-Gâteaux differentiable at the point \( p \) if there exists a bounded linear map \( D_L f(p) : L \to Y \) such that, for all \( h \in L \),
\[
D_L f(p)(h) = f'(p, h) := \lim_{t \to 0} \frac{f(p + th) - f(p)}{t}.
\] (3.5)

We denote by \( \Omega_L(f) \) the set of those points \( p \in \mathcal{D} \) where \( f \) is \( L\)-differentiable.

Clearly, the \( X\)-Gâteaux differentiability is equivalent to the standard Gâteaux differentiability. In this case, the subscript \( X \) will be omitted from the notation, i.e., \( D_X f(p) \) will simply be denoted by \( Df(p) \). On the other hand, if \( L = (h) \), that is, the linear span of a nonzero vector \( h \in X \), then the \( L\)-Gâteaux differentiability of \( f \) means that the two-sided directional derivative, \( f'(p, h) \), of \( f \) at \( p \) in the direction \( h \) exists and \( D_{(h)} f(p)(h) = f'(p, h) \).

Let \( Y \) be a finite dimensional. If \( L \in \Lambda(X) \), then the function \( g : L \cap (\mathcal{D} - p) \to Y \) defined by \( g(\cdot) := f(p + \cdot) \) is Lipschitzian, therefore, by Rademacher’s theorem, \( g \) is almost everywhere differentiable on \( L \cap (\mathcal{D} - p) \). Thus, there exist a sequence \( u_i \in L \) such that \( u_i \to 0 \) and the sequence \( Dg(u_i) = Df(p + u_i) \) converges. Based on this observation, we introduce the the \( L\)-Jacobian of \( f \) at \( p \) via the following formula:
\[
\partial_L f(p) := \text{co} \left\{ \Phi \in L(L, Y) \mid \exists (x_i) \in \Omega_L(f) : \lim_{i \to \infty} (x_i, D_L f(x_i)) = (p, \Phi) \right\}.
\] (3.6)

Note that here the sequence \( (x_i) \) is not necessarily contained in the affine subspace \( p + L \), and hence, \( \partial_L f(p) \) can be significantly larger than Clarke’s generalized Jacobian of the restricted function \( f|_{p + L} \) at \( p \), which is \( \partial^c g(0) \), where \( g \) is the function defined above. We have that \( \partial_L f(p) \) is a nonempty compact convex set of the space \( L(L, Y) \). Finally, we are able to recall the definition of the generalized Jacobian:
\[
\partial f(p) := \left\{ \Phi \in L(X, Y) : \Phi|_L \in \partial_L f(p), \forall L \in \Lambda(X) \right\}.
\] (3.7)

If \( X \) is finite dimensional, then obviously \( \partial_X f(p) = \partial^c f(p) \). In this case, we have, for \( L \in \Lambda(X) \), that \( \partial_X f(p)|_L \subseteq \partial_L f(p) \). This yields that \( \partial^c f(p) \subseteq \partial f(p) \). Conversely, if \( \Phi \in \partial f(p) \), then \( \Phi = \Phi|_L \in \partial_X f(p) = \partial^c f(p) \), which implies the reversed inclusion \( \partial f(p) \subseteq \partial^c f(p) \). Therefore, the generalized Jacobian \( \partial f(\cdot) \) extends Clarke’s generalized Jacobian to the case when \( X \) is any normed space. If \( Y = \mathbb{R} \) and \( X \) is any normed space, then \( \partial f(\cdot) \) also coincides with Clarke’s generalized gradient (cf. [21]).

One of the main results established in the paper [21] is the theorem below that characterizes \( \partial f(\cdot) \) as a smallest operator set-valued mapping satisfying certain properties. If the dimension of \( Y \) is \( N \), then the space \( L(X, Y) \) is topologically isomorphic to the product space \( (X^*)^n \), hence the space \( L(X, Y) \) can be equipped with a the weak* topology inherited from \( (X^*)^n \).

Given a normed space \( Z \) equipped with a Hausdorff topology \( \tau \), a map \( \mathcal{F} : \mathcal{D} \to 2^Z \) is said to be sequentially \( \tau\)-usc at \( p \in \mathcal{D} \) if, whenever \( (x_i, z_i) \) is a sequence in \( \mathcal{D} \times Z \) such that \( z_i \in \mathcal{F}(x_i) \) for all \( i \), and \( (x_i) \) tends to \( p \), then \( \tau\)-clus \( z_i \subseteq \mathcal{F}(p) \), i.e., \( \mathcal{F}(p) \) contains all the \( \tau \)-cluster points of the sequence \( (z_i) \).

**Theorem 3.1.** ([21]) Let \( X \) be a normed space, \( Y \) be a finite dimensional normed space and \( f : \mathcal{D} \to Y \) be a Lipschitzian function. Then \( \mathcal{F} = \partial f \) is the smallest set-valued map \( \mathcal{F} : \mathcal{D} \to 2^{L(X, Y)} \) with the following three properties:

(i) \( \mathcal{F}(x) \) is a nonempty, convex and \( w^*\)-compact subset of the ball \( \ell_f(x)\overline{B}_{L(X, Y)} \) for all \( x \in \mathcal{D} \);
(ii) $\mathcal{F}$ is sequentially $w^*$-upper semicontinuous on $\mathcal{D}$;
(iii) For all $L \in \Lambda(X)$ and for all $x \in \Omega_L(f)$,
\[ D_L f(x) \in \mathcal{F}(x)|_L. \tag{3.8} \]

For further results, such as calculus rules, a mean value theorem, the computation rule of the generalized Jacobian of piecewise smooth functions, etc., we refer to the papers [20], [21].

In the case when $Y$ is an infinite dimensional dual space with the Radon–Nikodým property, an involved generalization of the above construction was elaborated in [19] which led to an extension of the generalized Jacobian $\partial f(\cdot)$ to this setting. The underlying technique employed in [19] is the generalization of Rademacher’s differentiability theorem proven in [2] and [1].

### 3.2. The Co-Jacobian

Based on the notion of the generalized Jacobian for functions with finite dimensional range, we have introduced in [23] the following notion of the co-Jacobian which will be useful in this paper.

Given $y^* = (y^*_1, \ldots, y^*_n) \in (Y^*)^n$, the finite dimensional-valued function
\[ g(x) := y^* \circ f(x) := (y^*_1 \circ f(x), \ldots, y^*_n \circ f(x)) \quad (x \in \mathcal{D}) \]
(called the vectorization of $f$) is Lipschitzian near $p$ provided that $f$ is Lipschitzian near $p$. Thus, the notion of generalized Jacobian recalled in (3.7) can be now applied to the function $g$.

As a result of this observation, the co-Jacobian of $f$ at the point $p$ was defined in [23] by
\[ \partial^* f(p) := \{ \Psi \in \mathcal{L}(Y^*, X^*) : \forall n \in \mathbb{N}, \forall y^* \in (Y^*)^n, \Psi(y^*) \in \partial(y^* \circ f)(p) \}. \tag{3.9} \]
For $H \in \Lambda(Y^*)$, define
\[ \partial^*_H f(p) := \{ \Psi \in \mathcal{L}(H, X^*) : \forall n \in \mathbb{N}, \forall y^* \in H^n, \Psi(y^*) \in \partial(y^* \circ f)(p) \}. \tag{3.10} \]
The next result was named as the restriction theorem of the co-Jacobian in [23].

**Theorem 3.2.** ([23]) Let $f : \mathcal{D} \to Y$ be a Lipschitzian function near $p$. Then, $\partial^* f(p)$ is a nonempty, convex and $\beta(Y^*, X^*)$-compact subset of $\mathcal{L}(Y^*, X^*)$ and, for all $H \in \Lambda(Y^*)$,
\[ \partial^* f(p)|_H = \partial^*_H f(p). \tag{3.11} \]
and hence, for all $n \in \mathbb{N}$ and $y^* \in (Y^*)^n$,
\[ \partial^* f(p)(y^*) = \partial(y^* \circ f)(p). \tag{3.12} \]

The following result from the paper [23] offers a complete characterization of the co-Jacobian as a set-valued map with certain properties. The result is analogous to Theorem 3.1.

**Theorem 3.3.** ([23]) Let $f : \mathcal{D} \to Y$ be a Lipschitzian function. Then $\mathcal{F} = \partial^* f$ is the smallest set-valued map $\mathcal{F} : \mathcal{D} \to 2^{\mathcal{L}(Y^*, X^*)}$ with the following properties:

(i) $\mathcal{F}(x)$ is a nonempty $\beta(Y^*, X^*)$-compact and convex subset of $\mathcal{L}(Y^*, X^*)$ for all $x \in \mathcal{D}$.
(ii) $\mathcal{F}$ is sequentially $\beta(Y^*, X^*)$-usc on $\mathcal{D}$.
(iii) For all $n \in \mathbb{N}$, $y^* \in (Y^*)^n$, $L \in \Lambda(X)$, and for all $x \in \Omega_L(y^* \circ f)$,
\[ D_L (y^* \circ f)(x) \in \mathcal{F}(x)|_L. \tag{3.13} \]

holds.
4. Main results: V-co-Jacobian. Let $X$ and $Y$ be normed spaces, $\mathcal{D}$ be a nonempty open subset of $X$, $p$ be an arbitrary point in $\mathcal{D}$ and $f : \mathcal{D} \to Y$ be a Lipschitzian function.

One of the main two notions of this paper is the V-co-Jacobian defined, for any subspace $V$ of $Y^*$, by

$$\partial_V^* f(p) := \{ \Psi \in \mathcal{L}(V, X^*) : \forall n \in \mathbb{N}, \forall y^* \in V^n, \Psi(y^*) \in \partial(y^* \circ f)(p) \}. \quad (4.1)$$

Observe that this definition extends the notion defined in (3.10) for finite dimensional subspaces to any subspace of $Y^*$. As it will be shown, it has an intimate relation to our new concept, the V-Jacobian $\partial^V f(p)$, introduced in the next section.

The following result is an extension of Theorem 3.2 to the setting of arbitrary subspaces of $V^*$. Its proof is based on the application of Theorem 3.2.

**Theorem 4.1.** Let $f : \mathcal{D} \to Y$ be a Lipschitzian function near $p$ and let $V$ be a subspace of $Y^*$. Then

$$\partial_V^* f(p) = \partial^* f(p) |_V. \quad (4.2)$$

**Proof.** The inclusion $\partial^* f(p)|_V \subseteq \partial_V^* f(p)$ is trivial. For the converse, let $\Phi_0 \in \partial^* f(p)$. Then, for any $H \in \Lambda(V)$ we have $\Psi_0|_H \in \partial^*_H f(p)$. Hence, by (3.11) of Theorem 3.2, it follows that there exists $\Phi_0 \in \partial^* f(p)$ such that $\Phi_0|_H = \Psi_0|_H$.

Define

$$S_H := \{ \Phi \in \partial^* f(p) : \Phi|_H = \Psi_0|_H \}.$$  

Since $\Phi_0 \in S_H$, then $S_H \neq \emptyset$. Furthermore, $S_H = \partial^* f(p) \cap \{ \Psi : \Psi|_H = \Psi_0|_H \}$, is $\beta(Y^*, X)$-compact for being the intersection of a $\beta(Y^*, X)$-compact set with a $\beta(Y^*, X)$-closed set. The family of sets $\{ S_H : H \in \Lambda(V) \}$ possesses the finite intersection property, due to the fact that

$$S_{H_1} \cap S_{H_2} \supseteq S_{H_1+H_2} \neq \emptyset, \quad (H_1, H_2 \in \Lambda(V)).$$

It results that $\bigcap_{H \in \Lambda(V)} S_H \neq \emptyset$. That is, there exists $\Psi \in \partial^* f(p)$ satisfying $\Psi|_V = \Psi_0$. Therefore, $\Psi_0 \in \partial^* f(p)|_V$. \hfill \square

As an obvious consequence of Theorem 4.1, we obtain a result that describes the connection between V-co-Jacobians belonging to different subspaces of $Y^*$.

**Corollary 4.2.** Let $f : \mathcal{D} \to Y$ be a Lipschitzian function near $p$ and let $V \subseteq W$ be subspaces of $Y^*$. Then

$$\partial_V^* f(p) = \partial_W^* f(p)|_V. \quad (4.3)$$

The result below describes the essential properties of the V-co-Jacobian as a set-valued map.

**Theorem 4.3.** Let $f : \mathcal{D} \to Y$ be a Lipschitzian function and let $V$ be a subspace of $Y^*$. Then $\mathcal{F}_V = \partial_V^* f$ is the smallest set-valued map $\mathcal{F}_V : \mathcal{D} \to 2^{\mathcal{L}(V, X^*)}$ with the following properties:

(i) $\mathcal{F}_V(x)$ is a nonempty $\beta(V, X)$-compact and convex subset of $\ell_j(x)\overline{\mathcal{B}_{L(V, X^*)}}$ for all $x \in \mathcal{D}$.

(ii) $\mathcal{F}_V$ is sequentially $\beta(V, X)$-usc on $\mathcal{D}$.

(iii) For all $n \in \mathbb{N}$, $y^* \in V^n$, $L \in \Lambda(X)$, and for all $x \in \Omega_L(y^* \circ f)$,

$$D_L(y^* \circ f)(x) \in \mathcal{F}_V(x)(y^*)|_L. \quad (4.4)$$

holds.
Proof. First we show that the set-valued map $\mathcal{F}_V := \partial^* f$ satisfies all the properties listed in (i)–(iii).

Consider the map $\Gamma : \mathcal{L}(Y^*, X^*) \to \mathcal{L}(V, X^*)$ given by
\[ \Gamma(\Phi) := \Phi|_V. \]
Then, by Lemma 2.6, $\Gamma$ is a $(\beta(Y^*, X), \beta(V, X))$-continuous linear operator with $\||\Gamma_1|| \leq 1$. Observe that, by Proposition 5.1, we have that
\[ \partial^*_V f(x) = \Gamma(\partial^* f(x)). \]
By Theorem 3.3, $\partial^* f(x)$ is a nonempty, convex and $\beta(Y^*, X)$-compact subset of $\ell_f(x) \mathcal{B}_{L(Y^*, X^*)}$. Hence, by the representation (4.5) and the continuity and linearity properties of the map $\Gamma$, property (i) follows.

To prove (ii), let $(x_i)$ be a sequence in $D$ converging to a point $x \in D$. Let $\Phi_i \in \partial^*_V f(x_i)$ be any sequence. Then, there exist $\Psi_i \in \partial^* f(x_i) \subseteq \ell_f(x_i) \mathcal{B}_{L(Y^*, X^*)}$ such that $\Phi_i = \Gamma(\Psi_i)$. The function $\ell_f$ being upper semicontinuous at $x$, it follows that $\ell_f$ is bounded by $\ell_f(x) + 1$ in a neighborhood of $x$. Thus, the sequence $(\Psi_i)$ is $\beta(Y^*, X)$-precompact. Applying now [19, Lemma 2.7], it follows that
\[ \beta(V, X)\text{-clus} \Phi_i = \Gamma\left( \beta(Y^*, X^*)\text{-clus} \Psi_i \right) \]
(4.6)
On the other hand, by the $\beta(Y^*, X^*)$-use property of the co-Jacobian map $\partial^*_V f$, we have that $\beta(Y^*, X^*)\text{-clus} \Psi_i \subseteq \partial^* f(x)$. Thus, (4.6) yields
\[ \beta(V, X)\text{-clus} \Phi_i \subseteq \Gamma(\partial^* f(x)), \]
which proves the $\beta(V, X)$-use property of the $V$-co-Jacobian map $\partial^*_V f$ at $x$.

For proving (iii), let $L \subseteq \Lambda(X)$, $n \in \mathbb{N}$, and $y^* \in V^n$ such that $y^* \circ f$ is $L$-differentiable at $x \in D$. Then, by property (ii) of Theorem 3.3 and Theorem 4.1, we obtain
\[ D_L[y^* \circ f](x) \in \partial^* f(x)(y^*) |_{L} \subseteq \partial^* f(x) |_{V}(y^*) |_{L} = \partial^*_V f(x)(y^*) |_{L}. \]
Thus the proof of property (iii) is complete.

Now assume that $\mathcal{F}_V$ is a set-valued map satisfying the conditions (i)–(iii) of the theorem. Let $p \in D$ be a fixed point where we want to show that the inclusion $\partial^*_V f(p) \subseteq \mathcal{F}_V(p)$ holds.

By property (i), $\text{co} \mathcal{F}_V(p)$ is bounded, therefore the right hand side of (4.4) is $\beta(V, X)$-compact. Thus, in view of Theorem 2.7, it suffices to show that, for all $n \in \mathbb{N}$, $y^* \in V^n$, and $L \subseteq \Lambda(X)$,
\[ \partial^*_V f(p)(y^*) |_{L} \subseteq \left( \text{co} \mathcal{F}_V(p) \right)(y^*) |_{L}. \]
(4.7)
By the linearity of the restriction and evaluation maps, we have
\[ \left( \text{co} \mathcal{F}_V(p) \right)(y^*) |_{L} = \left[ \text{co} (\mathcal{F}_V(p)(y^*)) \right] |_{L} = \text{co} (\mathcal{F}_V(p)(y^*)) |_{L}. \]
Applying also the continuity properties of these maps established in Lemma 2.6 and Lemma 2.1, we obtain
\[ \left( \text{co} \mathcal{F}_V(p) \right)(y^*) |_{L} = \left[ \text{co} (\mathcal{F}_V(p)(y^*)) \right] |_{L} = \text{co} (\mathcal{F}_V(p)(y^*)) |_{L}. \]
To simplify the left hand side of (4.7), let $\delta > 0$ be such that (4.4) holds for all $x \in (\Omega_L(y^* \circ f)) \cap (p + \delta B_X)$. By Theorem 3.2, we have
\[ \partial^*_V f(p)(y^*) |_{L} = \partial^* f(p)(y^*) |_{L} = \partial(y^* \circ f)(p) |_{L} = \partial_L(y^* \circ f)(p), \]
where $\partial_L(y^* \circ f)(p)$ is defined by
\[
\partial_L(y^* \circ f)(p) = \overline{\text{co}} \left\{ \Phi \mid \exists (x_i) \in \Omega_L(y^* \circ f) : \lim_{i \to \infty} (x_i, D_L(y^* \circ f)(x_i)) = (p, \Phi) \right\}.
\]

Hence (4.7) is equivalent to proving
\[
\partial_L(y^* \circ f)(p) \subseteq \overline{\text{co}} (F_V(p)(y^*))(L).
\] (4.8)

Thus, in order to verify (4.8), it suffices to show that
\[
\left\{ \Phi \mid \exists (x_i) \in \Omega_L(y^* \circ f) : \lim_{i \to \infty} (x_i, D_L(y^* \circ f)(x_i)) = (p, \Phi) \right\} \subseteq F_V(p)(y^*)|_L.
\] (4.9)

Indeed, if $\Phi$ is an element of the left hand side then there exists a sequence $(x_i)$ in $\Omega_L(y^* \circ f)$ such that $\lim_{i \to \infty} x_i = p$ and $\lim_{i \to \infty} D_L(y^* \circ f)(x_i) = \Phi$. By assumption (iii),
\[
D_L(y^* \circ f)(x_i) \in F_V(x_i)(y^*)|_L
\]
holds for large $i$. Thus, applying property (ii), it follows that
\[
\Phi = \lim_{i \to \infty} D_L(y^* \circ f)(x_i) \in F_V(p)(y^*)|_L,
\]
showing that (4.9) holds. □

5. **Main results: V-Jacobian.** Let $X$ and $Y$ be normed spaces, $\mathcal{D}$ be a nonempty open subset of $X$, $p$ be an arbitrary point in $\mathcal{D}$ and $f : \mathcal{D} \to Y$ be a Lipschitzian function.

Given a subspace $V$ of $Y^*$, we introduce the notion of the $V$-Jacobinan by
\[
\partial^V f(p) := \left\{ \Phi \in \mathcal{L}(X, V^*) : \forall n \in \mathbb{N}, \forall y^* \in V^n, \ y^* \circ \Phi \in \partial(y^* \circ f)(p) \right\}.
\] (5.1)

The following theorem enlightens the connection between the $V$-Jacobinan and the $V$-co-Jacobinan of $f$. As a consequence, we obtain that the generalized Jacobian introduced in the papers [20], [21], and [19], is completely determined by the co-Jacobinan.

**Proposition 5.1.** Let $f : \mathcal{D} \to Y$ be a Lipschitzian function $p \in \mathcal{D}$ and $V$ be a subspace of $Y^*$. Then,
\[
\partial^V f(p) = (\partial^* f(p)|_V)^*|_X \quad \text{and} \quad \partial^* f(p)|_V = (\partial^V f(p))^*|_V.
\] (5.2)

**Proof.** Using (2.1), it easily follows from the definitions of $\partial_V f(p)$ and $\partial^*_V f(p)$ that $\Phi \in \partial_V f(p)$ holds if and only if $\Psi = \Phi|_V \in \partial^*_V f(p)$, Therefore,
\[
\partial^*_V f(p) = (\partial^V f(p))^*|_V.
\]

Applying Theorem 4.1, the second equality in (5.2) follows immediately. The proof of first equation in (5.2) is analogous. □

**Corollary 5.2.** Let $f : \mathcal{D} \to Y$ be a Lipschitzian function, $p \in \mathcal{D}$ and $V$ be a subspace of $Y^*$. Then, for all $n \in \mathbb{N}$, $y^* \in V^n$,
\[
y^* \circ \partial^V f(p) = \partial^* f(p)(y^*).
\]

**Proof.** Using the identity (2.1) and the second formula in (5.2), it follows that
\[
y^* \circ \partial^V f(p) = (\partial^V f(p))^*(y^*) = (\partial^V f(p))^*|_V(y^*) = \partial^* f(p)|_V(y^*) = \partial^* f(p)(y^*).
\] □
Remark 5.3. Using also equality (3.12), Corollary 5.2 implies that, for all \( n \in \mathbb{N} \), \( y^* \in V^n \),

\[ y^* \circ \partial^V f(p) = \partial(y^* \circ f)(p), \tag{5.3} \]

which is a particular case of the smooth-nonsmooth chain rule formulated in Theorem 7.3 below.

Corollary 5.4. Assume that \( Y \) is the dual of a space \( V \) and it has the Radon–Nikod"ym property. Let \( f : D \to Y \) be a Lipschitzian function and \( p \in D \). Then

\[ \partial^V f(p) = \partial f(p), \tag{5.4} \]

where \( \partial f(p) \) denotes the generalized Jacobian of \( f \) at \( p \) defined in [19].

Proof. By[19, Theorem 3.7], it follows that, for all \( n \in \mathbb{N} \) and \( y^* \in V^n \),

\[ y^* \circ \partial^V f(p) = \partial(y^* \circ f)(p), \tag{5.5} \]

Hence, equations (5.3) and (5.5) imply

\[ y^* \circ \partial^V f(p) = y^* \circ \partial f(p) \]

for all \( n \in \mathbb{N} \) and \( y^* \in V^n \). Applying Theorem 2.7, the identity (5.4) results. \( \square \)

Theorem 5.5. Let \( f : D \to Y \) be a Lipschitzian function and \( V \) be a subspace of \( Y^* \). Then the \( V \)-Jacobian mapping \( \mathcal{F}^V = \partial^V f \) is the smallest set-valued mapping \( \mathcal{F}^V : D \to 2^{\mathcal{L}(X,V^*)} \) such that

(i) For all \( x \in D \), \( \mathcal{F}^V(x) \) is a nonempty, \( \beta(X,V) \)-compact and convex subset of \( \ell_f(x)B_{L(X,V^*)} \).

(ii) The set-valued map \( \mathcal{F}^V \) is sequentially \( \beta(X,V) \)-usc on \( D \).

(iii) For \( L \in \Lambda(X), n \in \mathbb{N} \), and \( y^* \in V^n \) such that \( y^* \circ f \) is \( L \)-differentiable at \( x \in D \), we have

\[ D_L(y^* \circ f)(x) \in y^* \circ \mathcal{F}^V(x)|_L, \tag{5.6} \]

Proof. Consider the map \( \Gamma : \mathcal{L}(V,X^*) \to \mathcal{L}(X,V^*) \) defined by

\[ \Gamma(\Psi) = \Psi|_X. \]

Observe that, by Proposition 5.1, we have that, for all \( x \in D \),

\[ \partial^V f(x) := \Gamma(\partial_x^V f(x)). \tag{5.7} \]

On the other hand, by Lemma 2.1, \( \Gamma \) is a \((\beta(V,X),\beta(X,V^*))\)-continuous isometric isomorphism between the spaces \( \mathcal{L}(V,X^*) \) and \( \mathcal{L}(X,V^*) \). Hence, the properties of the \( V \)-co-Jacobian map \( \partial_x^V f =: \mathcal{F}_V \) presented in Theorem 4.3 directly yield that the \( V \)-Jacobian map \( \partial^V f =: \mathcal{F}^V \) enjoys the properties listed in (i)–(iii) of Theorem 5.5.

To prove the reversed statement, assume that \( \mathcal{F}^V : D \to 2^{\mathcal{L}(X,V^*)} \) is a set-valued map satisfying conditions (i)–(iii). Then, it is easy to see that the set-valued map \( \mathcal{F}_V : D \to 2^{\mathcal{L}(V,X^*)} \) defined by

\[ \mathcal{F}_V(x) := (\mathcal{F}^V(x))^*|_V = \Gamma^{-1}(\mathcal{F}^V(x)) \]

satisfies all conditions (i)–(iii) of Theorem 4.3. Therefore, we get, for all \( x \in D \), that

\[ \partial_x^V f(x) \subseteq \mathcal{F}_V(x), \]

therefore,

\[ \partial^V f(x) = \Gamma(\partial_x^V f(x)) \subseteq \Gamma(\mathcal{F}_V(x)) = \mathcal{F}^V(x), \]

which completes the proof. \( \square \)
6. Differentiability properties and mean value Theorem. In this section we describe how the \( V \)-Jacobian is connected to the differentiability properties of a Lipschitzian function \( f : X \to Y \) at \( p \in D \).

For a given subspace \( V \subseteq Y^* \), we introduce the notions of strict \( V \)-Fréchet, \( V \)-Hadamard, and \( V \)-Gâteaux prederivatives. A set of operators \( \mathcal{F} \subseteq \mathcal{L}(X,Y^*) \) is called a strict \( V \)-Hadamard prederivative for the function \( f \) at \( p \) if, for all \( n \in \mathbb{N}, \) for all \( y^* \in V^n, \) for all \( \varepsilon > 0, \) and for all compact subsets \( C \) of the unit sphere of \( X, \) there exists \( \delta > 0 \) such that, for all \( x, y \in p + \delta B_X \) with \( y - x \in \|y - x\|C, \)

\[
\begin{align*}
y^* \circ f(y) - y^* \circ f(x) &\in y^* \circ \mathcal{F}(y - x) + \varepsilon\|y - x\|B_{R^m},
\end{align*}
\]

i.e., if \( y^* \circ \mathcal{F} \subseteq \mathcal{L}(X,R^n) \) is a strict Hadamard-prederivative for the function \( y^* \circ f \) at \( p \) for all \( y^* \in V^n. \) If the above requirements holds when \( C \) is the entire unit sphere, then \( \mathcal{F} \) is called a strict \( V \)-Fréchet prederivative. On the other hand, when the above definition holds only for finite \( C, \) then \( \mathcal{F} \) is called a strict \( V \)-Gâteaux prederivative. If \( \mathcal{F} \) is a singleton, i.e., \( \mathcal{F} = \{\Phi\} \) and \( \mathcal{F} \) is a strict \( V \)-Fréchet, \( V \)-Hadamard, or \( V \)-Gâteaux prederivative for \( f \) at \( p, \) then we say that \( f \) is strictly \( V \)-Fréchet, \( V \)-Hadamard, or \( V \)-Gâteaux differentiable at \( p \) with a \( V \)-derivative \( \Phi, \) respectively and the \( V \)-derivative will be denoted by \( D^V f(p). \)

**Theorem 6.1.** Let \( V \) be a subspace \( Y^* \), let \( p \in D \) and let \( f : D \to Y \) be Lipschitzian near \( p. \) Then \( \mathcal{F} := \partial^V f(p) \) is a strict \( V \)-Hadamard prederivative for \( f \) at \( p. \)

**Proof.** By [23], we know that \( \mathcal{F} := \partial^V f(p) \) is a strict \( w \)-Hadamard pre-coderivative for \( f \) at \( p. \) In other words, for all \( n \in \mathbb{N}, \) for all \( y^* \in (Y^*)^n, \) for all \( \varepsilon > 0, \) and for all compact subsets \( C \) of the unit sphere of \( X, \) there exists \( \delta > 0 \) such that, for all \( x, y \in p + \delta B_X \) with \( y - x \in \|y - x\|C, \)

\[
\begin{align*}
y^* \circ f(y) - y^* \circ f(x) &\in \partial^V f(p)(y^*)(y - x) + \varepsilon\|y - x\|B_{R^m},
\end{align*}
\]

In particular, taking \( y^* \in V^n \) and using (5.2), the statement follows. \(\square\)

As a corollary, we obtain a characterization of the case when \( \partial^V f(p) \) is a singleton. The result is analogous to what is known for Clarke's subgradient and for the generalized Jacobian introduced in [20], [20], [21].

**Corollary 6.2.** Let \( V \) be a subspace \( Y^* \), let \( p \in D \) and let \( f : D \to Y \) Lipschitzian near \( p. \) Then \( \partial^V f(p) \) (equivalently \( \partial^V f(p) \)) is a singleton if and only if \( f \) is strictly \( V \)-Hadamard differentiable at \( p. \)

**Proof.** Clearly, if \( \partial^V f(p) = \{\Phi\} \) is a singleton, then by Theorem 6.1, \( \Phi \) is the strict \( V \)-Hadamard derivative of \( f \) at \( p. \)

Conversely, if \( f \) is strictly \( V \)-Hadamard differentiable at \( p, \) then, for all and \( y^* \in V, \) the real-valued function \( y^* \circ f \) is Hadamard differentiable at \( p \) and hence, by the properties of generalized Jacobian, \( \partial(y^* \circ f)(p) \) is a singleton. Thus, in view of Theorem 3.2, \( y^* \circ \partial^V f(p) \) is also a singleton, for all \( y^* \in V. \) It easily follows that \( \partial^V f(p) \) must be a singleton, too. \(\square\)

The next result, which is phrased in terms of our co-Jacobian, is a counterpart of the mean value theorems in terms of the generalized gradient and Jacobian (see, e.g., [13], [19], [21]).

For an element \( y \in Y, \) we define \( y|_V \in V^* \) by the formula \( y|_V(y^*) := (y^*, y). \) In particular, if \( V = Y^*, \) then \( y|_V = y, \) where \( y \) is considered to be an element of \( Y^{**} \) via the canonical embedding.
Theorem 7.1. Let $V$ be an extension of $[21, \text{Thm. 4.2}]$, is called the nonsmooth-smooth and a smooth-nonsmooth chain rule. The next theorem, which 7. Chain rules and their consequences.

Chain rules and their consequences. In this section we shall establish a nonsmooth-smooth chain rule in terms of the co-Jacobian of $f$. Let $f$ be continuously differentiable at $p \in D$ and let $g : D \to Z$ be Lipschitzian near $f(p)$, where $D \subseteq Y$ is an open set containing $f(p)$. Then,

$$
\partial^W(g \circ f)(p) \subseteq \partial^W g(f(p)) \circ Df(p). \tag{7.1}
$$

If $g$ is strictly $W$-Hadamard differentiable at $f(p)$, then $g \circ f$ is $W$-Hadamard differentiable at $p$ and (7.1) holds with equality.

Proof. By the nonsmooth-smooth chain rule in terms of the co-Jacobian of $g$ at $f(p)$ derived in [23], we have that

$$
\partial^*(g \circ f)(p) \subseteq (Df(p))^* \circ \partial^* g(f(p)).
$$

Hence,

$$
\partial^*(g \circ f)(p)_W \subseteq (Df(p))^* \circ \partial^* g(f(p))_W. \tag{7.2}
$$
which, by Proposition 5.1, results that
\[
\partial^W (g \circ f)(p) = \left( \partial^* (g \circ f)(p)|_W^* \right)^* |_X \subseteq \left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right)^* |_X
\]
\[
= \left( \partial^* \left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right)^* |_X \right) \circ D f(p)
\]
\[
= \partial^W g\left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right) \circ D f(p)
\]
If \( g \) is strictly \( W \)-Hadamard differentiable at \( f(p) \) then \( \partial^W g\left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right) \) is a singleton and hence (7.1) yields that \( \partial^W (g \circ f)(p) \) is also a singleton and the equality in (7.1) automatically holds.

**Remark 7.2.** Using Theorem 4.1, the inclusion in (7.2) can be rewritten as
\[
\partial^W (g \circ f)(p) \subseteq \left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right) \circ D f(p)
\]
which states the nonsmooth-smooth chain rule in terms of the \( W \)-co-Jacobians of \( g \) and \( g \circ f \).

The following result is our smooth-nonsmooth chain rule.

**Theorem 7.3.** Let \( X, Y \) and \( Z \) be arbitrary normed spaces and let \( V \) and \( W \) be subspaces of \( Y^* \) and \( Z^* \), respectively. Let \( f : D \to Y \) be Lipschitzian near \( p \in D \) and let \( g : O \to Z \) be strictly Fréchet differentiable at \( f(p) \), where \( O \subseteq Y \) is an open set containing \( f(p) \). Assume that \( W \circ D g\left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right) \subseteq V \) and define the linear operator \( \Phi \in L(W, V) \) by
\[
\Phi(w) := w \circ D g\left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right) \quad (w \in W). \tag{7.3}
\]
Then,
\[
\partial^W (g \circ f)(p) = \Phi^* \circ \partial^V f(p). \tag{7.4}
\]
In particular, if \( V^* = Y \) and \( W^* = Z \) hold, then
\[
\partial^W (g \circ f)(p) = D g\left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right) \circ \partial^V f(p). \tag{7.5}
\]

**Proof.** Using the notation \( \Psi := D g\left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right) \), it follows from (7.3) that
\[
\Phi(w) = w \circ \Psi = \Psi^* (w) = (\Psi^* w)|_W \quad (w \in W). \tag{7.6}
\]
By the smooth-nonsmooth chain rule in terms of the co-Jacobian derived in [23], we have that
\[
\partial^* (g \circ f)(p) = \partial^* f(p) \circ (D g\left( (D f(p))^{**} \circ \partial^* g(f(p))|_W^* \right))^* = \partial^* f(p) \circ \Psi^*.
\]
Hence,
\[
\partial^* (g \circ f)(p)|_W = \partial^* f(p) \circ \Psi^*|_W = \partial^* f(p) \circ \Phi. \tag{7.7}
\]
Thus, applying Proposition 5.1, equation (7.7) implies that
\[
\partial^W (g \circ f)(p) = \left( \partial^* (g \circ f)(p)|_W^* \right)^* |_X = \left( \partial^* f(p)|_V \circ \Phi \right)^* |_X
\]
\[
= \Phi^* \circ \left( \partial^* f(p)|_V \right)^* |_X = \Phi^* \circ \partial^V f(p),
\]
which proves (7.4). If we also have \( V^* = Y \) and \( W^* = Z \), then
\[
\Phi^* = (\text{id}_V \circ (\Psi^*|_W)^*)^* = (\text{id}_V \circ (\Psi^*|_W))^*|_Y = \Psi,
\]
which yields (7.5). \( \square \)
Lipschitzian functions near $W$ if $→ D$. Let $\Phi$ be strictly Fréchet differentiable at $q := (f_1, \ldots, f_k)(p)$, where $O \subseteq Y_1 \times \cdots \times Y_k$ is an open set containing the point $q$. For all $j \in \{1, \ldots, k\}$, denote $D_j g(q)(y_j) := Dg(q)(0, \ldots, y_j, \ldots, 0)$ and assume that $W \circ D_j g(q) \subseteq V_j$. Then

$$\partial^W(g \circ (f_1, \ldots, f_k))(p) \subseteq \sum_{j=1}^k \Phi_j^* \circ \partial^V f_j(p),$$

(7.8)

where the linear operators $\Phi_j \in \mathcal{L}(W, V_j)$ are defined by

$$\Phi_j(w) = w \circ D_j g(q) \quad (w \in W).$$

(7.9)

If $W^* = Z$ and also $V_j^* = Y_j$ holds for all $j \in \{1, \ldots, k\}$, then

$$\partial^W(g \circ (f_1, \ldots, f_k))(p) \subseteq \sum_{j=1}^k D_j g(q) \circ \partial^V f_j(p).$$

(7.10)

**Proof.** The statement follows when Theorem 7.3 is applied to the functions $g$ and $f = (f_1, \ldots, f_k)$ and the easy-to-obtain inclusion

$$\partial^{V_1 \times \cdots \times V_k}(f_1, \ldots, f_k)(p) \subseteq \partial^V f_1(p) \times \cdots \times \partial^V f_k(p).$$

(7.11)

is used. \qed

As a particular case of Corollary 7.5 we obtain the so-called sum rule.

**Corollary 7.6.** Let $Y$ be a normed space and $V$ be a subspace of $Y^*$. Let $f_1, f_2 : D \to Y$ be Lipschitzian functions near $p \in D$. Then

$$\partial^V(f_1 + f_2)(p) \subseteq \partial^V f_1(p) + \partial^V f_2(p).$$

(7.12)

If either $f_1$ or $f_2$ are strictly $V$-Hadamard differentiable at $p$, then (7.12) holds with equality.

**Proof.** We apply Corollary 7.5 with the spaces $Y_1 := Y_2 := Z := Y$ and $V_1 := V_2 := W := V$ to the function $g : Y_1 \times Y_2 \to Z$ defined by $g(y_1, y_2) := y_1 + y_2$. Then, for the point $q = (f_1(p), f_2(p)) \in Y_1 \times Y_2$, we have that $D(g(q) = Dg(q) = id_Y$, where $id_S$ denotes the identity function of a set $S$ into itself. Therefore, for $j = 1, 2$, the equality $W \circ D_j g(q) = V_j$ trivially holds. On the other hand, for all $w \in W = V$, we get $\Phi_j(w) = w \circ D_j g(q) = w$. Hence, $\Phi_j = id_V$ and $\Phi_j^* = id_{V^*}$. Thus, formula (7.8) reduces to the sum rule (7.12).
For the equality, assume, say, that $f_2$ is $V$-Hadamard differentiable at $p$. Then, by Corollary 6.2, we have that $\partial^V f_2(p) = \{ D^V f_2(p) \}$. Thus applying (7.12), we get
\[
\partial^V f_1(p) = \partial^V (f_1 + f_2 - f_2)(p) \\
\subseteq \partial^V (f_1 + f_2)(p) + \partial^V (-f_2)(p) \\
= \partial^V (f_1 + f_2)(p) - D^V f_2(p),
\]
which implies $\partial^V (f_1 + f_2)(p) \supseteq \partial^V f_1(p) + D^V f_2(p)$ showing the equality in (7.12).

In the next result, we deduce a certain product rule, which can be useful in the applications.

**Lemma 7.7.** Let $X, Y_1, Y_2$, and $Z$ be normed spaces and assume that the operation $\circ : Y_1 \times Y_2 \to Z$ is a bounded bilinear function, i.e., there exists a constant $C$ such that, for all $(y_1, y_2) \in Y_1 \times Y_2$, the inequality $\|y_1 \circ y_2\| \leq C \|y_1\| \|y_2\|$ holds. Let $f = (f_1, f_2) : \mathcal{D} \to Y_1 \times Y_2$ be a Lipschitzian function. Then the function $F : \mathcal{D} \to Z$ defined by
\[
F(x) := f_1(x) \circ f_2(x) - f_1(x) \circ f_2(p) - f_1(p) \circ f_2(x) \quad (x \in \mathcal{D}) \tag{7.13}
\]
is strictly Fréchet differentiable at $p$ and $DF(p) = 0$.

**Proof.** Let $r > 0$ be chosen such that $f_1$ and $f_2$ are Lipschitzian on $p + rB_X$ with Lipschitz modulus $L_1$ and $L_2$, respectively. For $x, y \in p + rB_X$, we have the following estimate:
\[
\|F(x) - F(y)\| \\
= \|(f_1(x) - f_1(y)) \circ (f_2(x) - f_2(y)) + (f_1(y) - f_1(p)) \circ (f_2(x) - f_2(y))\| \\
\leq CL_1 L_2 \|x - y\| (\|x - p\| + \|y - p\|).
\]
Hence, with $\Phi = 0$, we get
\[
\lim_{(x,y) \to (p,p)} \frac{\|F(x) - F(y) - \Phi(x - y)\|}{\|x - y\|} = \lim_{(x,y) \to (p,p)} \frac{\|F(x) - F(y)\|}{\|x - y\|} = 0,
\]
which proves that $\Phi = 0$ is a strict Fréchet derivative for $F$ at $p$.

**Theorem 7.8.** Let $X, Y_1, Y_2$ and $Z$ be normed spaces, $W \subseteq Z^*$ be a subspace and let $f = (f_1, f_2) : \mathcal{D} \to Y_1 \times Y_2$ be a Lipschitzian function. Assume that $\circ : Y_1 \times Y_2 \to Z$ is a bounded bilinear function. Then
\[
\partial^W (f_1 \circ f_2)(p) = \partial^W (f_1 \circ f_2(p) + f_1(p) \circ f_2)(p). \tag{7.14}
\]

**Proof.** Define the function $F : \mathcal{D} \to Z$ by (7.13). Then, by Lemma 7.7, $F$ is strictly $W$-Hadamard differentiable and its $W$-Hadamard derivative is zero at $p$. On the other hand, we have that $f_1 \circ f_2 = F + (f_1 \circ f_2(p) + f_1(p) \circ f_2)$. Therefore, the sum rule (in the case when equality holds) applies, and (7.14) follows.

8. **V-Jacobian for continuous selections.** The notion of piecewise smooth function is a function whose domain can be partitioned into finitely many “pieces” relative on which smoothness holds and continuity holds across the joins of the pieces. In this section we consider functions that are piecewise locally Lipschitzian. Given a finite system of some Lipschitzian functions $g_1, \ldots, g_k : \mathcal{D} \to Y$, a continuous
function $f : D \to Y$ is called a continuous selection of $\{g_1, \ldots, g_k\}$ if, for all $x \in D$, there exists an index $j \in \{1, \ldots, k\}$ such that $f(x) = g_j(x)$, that is, for all $x \in D$, $f(x) \in \{g_1(x), \ldots, g_k(x)\}$.

When the functions $g_1, \ldots, g_k$ are differentiable (resp. $C^1$) on $D$, then we say that $f$ is piecewise differentiable (resp. piecewise smooth). The theory of continuous selections has been developed in some recent papers [8], [10], [11], [12], [18], [20], [21], [24], [25], [26], [27].

The main result of this section is Theorem 8.1 below which offers an inclusion for the generalized $V$-Jacobian of a Lipschitzian function which is decomposed in terms of finitely many Lipschitzian functions. The proof is based the analogous result obtained in terms of the co-Jacobian in our previous paper [23].

**Theorem 8.1.** Let $X$ and $Y$ be a normed spaces, $V$ be a subspace of $Y^*$ and let $f : D \to Y$ be a continuous selection of $\{g_1, \ldots, g_k\}$, where $g_1, \ldots, g_k : D \to Y$ are Lipschitzian functions near $p \in D$. Then $f$ is Lipschitzian near $p$ and

$$\partial^V f(p) \subseteq \text{co} \left( \bigcup_{j \in I(p)} \partial^V g_j(p) \right),$$

where

$$I(p) := \{ j \in \{1, \ldots, k\} \mid \exists U \subseteq D \text{ open : } p \in U, f|_U = g_j|_U \}.$$  (8.2)

Furthermore, if the functions $g_j$, for $j$ in $I(p)$, are strictly $V$-Hadamard differentiable at $p$, then (8.1) holds with equality.

**Proof.** By the result obtained in the setting of the co-Jacobian in [23], we have that

$$\partial^* f(p) \subseteq \text{co} \left( \bigcup_{j \in I(p)} \partial^* g_j(p) \right).$$

Hence, applying Proposition 5.1 twice and using obvious set theoretical and linear identities, we get

$$\partial^V f(p) = \left( \partial^* f(p)|_V \right)^*|_X \subseteq \left( \text{co} \left( \bigcup_{j \in I(p)} \partial^* g_j(p)|_V \right) \right)^*|_X$$

$$= \left( \text{co} \left( \bigcup_{j \in I(p)} \partial^* g_j(p)|_V \right) \right)^*|_X = \text{co} \left( \bigcup_{j \in I(p)} \left( \partial^* g_j(p)|_V \right)^* \right)^*|_X$$

$$= \text{co} \left( \bigcup_{j \in I(p)} \left( \partial^* g_j(p)|_V \right)^* \right)|_X = \left( \text{co} \left( \bigcup_{j \in I(p)} \partial^V g_j(p) \right) \right)|_X.$$

If the functions $g_j$, for $j$ in $I(p)$, are strictly $V$-Hadamard differentiable at the point $p$ then (8.3) holds with equality and, by the above argument, (8.1) is also satisfied with equality.

**Remark 8.2.** The inclusion in (8.1) can also be phrased in terms of the $V$-co-Jacobian as:

$$\partial^V f(p) \subseteq \text{co} \left( \bigcup_{j \in I(p)} \partial^V g_j(p) \right).$$
9. Connection of the \( V \)-Jacobian with other notions. The results in this section show how the \( V \)-Jacobiabian \( \partial V f(p) \) is connected to certain differentiability notions.

The limit points of directional difference quotients were introduced and investigated by Thibault [29], [30]. A notion more general than Thibault’s limit set, assuming that \( V \subseteq Y^* \), is defined as follows

\[
\delta V f(p, h) := \left\{ v^* \in V^* \mid \exists (x_i, t_i)_{i \in \mathbb{N}} \text{ in } \mathcal{D} \times \mathbb{R}_+ : \lim_{i \to \infty} (x_i, t_i) = (p, 0), \right.
\]

\[
\text{and } v^* \in \text{w}^*\text{-clus}_{i \to \infty} \frac{f(x_i + t_i h) - f(x_i)}{t_i} \bigg|_V \bigg. \right\}. \tag{9.1}
\]

We also introduce the following concept, which is related to Ioffe’s fan derivative (see [7]),

\[
D^*_V f(p)(h) := \{ v^* \in V^* : \langle v^*, y^* \rangle \leq (y^* \circ f)(p, h) \forall y^* \in V \}. \tag{9.2}
\]

Note that if \( Y \) is reflexive (i.e., when \( V = Y^* \)), then \( D^*_V f(p) \) coincides with Ioffe’s fan derivative.

The next result establishes a relationship between the co-Jacobian, and the notions defined in (9.1) and (9.2) without involving the generalized Jacobian which requires \( Y \) to be a Radon–Nikodým space.

**Theorem 9.1.** Let \( V \subseteq Y^* \) be a subspace and let \( f : \mathcal{D} \to Y \) be Lipschitzian near \( p \in \mathcal{D} \). Then, for all \( h \in X \),

\[
(\partial V f(p))^*(h) = \partial^V f(p)(h) = \overline{\text{co}}^{w^*} \delta V f(p, h) = D^*_V f(p)(h). \tag{9.3}
\]

**Proof.** The first equality in (9.3) follows by Proposition 5.1.

To prove the second equality in (9.3), by the \( w^* \)-compactness and convexity of the left and right hand sides of (9.3) it is enough to show that, for all \( y^* \in V \),

\[
\langle \partial V f(p)(h), y^* \rangle = (\overline{\text{co}}^{w^*} \delta V f(p, h), y^*). \]

In view of (5.3), the linearity and \( w^* \)-continuity of \( y^* \), this latter equation is equivalent to

\[
\partial(y^* \circ f)(p)(h) = \overline{\text{co}} (\delta V f(p, h), y^*). \tag{9.4}
\]

However, using again the linearity and \( w^* \)-continuity of \( y^* \) and [19, Lemma 2.7],

\[
\langle \delta V f(p, h), y^* \rangle = \left\{ \langle v^*, y^* \rangle \mid \exists (x_i, t_i)_{i \in \mathbb{N}} \text{ in } \mathcal{D} \times \mathbb{R}_+ : \right. \\
\text{lim}_{i \to \infty} (x_i, t_i) = (p, 0) \text{ and } v^* \in \text{w}^*\text{-clus}_{i \to \infty} \frac{f(x_i + t_i h) - f(x_i)}{t_i} \bigg|_V \bigg. \right\}
\]

\[
= \left\{ u \mid \exists (x_i, t_i)_{i \in \mathbb{N}} \text{ in } \mathcal{D} \times \mathbb{R}_+ : \right. \\
\text{lim}_{i \to \infty} (x_i, t_i) = (p, 0) \text{ and } u = \text{clus}_{i \to \infty} \left( y^*, \frac{f(x_i + t_i h) - f(x_i)}{t_i} \right) \bigg. \right\}
\]

\[
= \delta(y^* \circ f)(p, h).
\]

Thus, \( \overline{\text{co}} (\langle \delta V f(p, h), y^* \rangle) = \overline{\text{co}} (\delta(y^* \circ f)(p, h)). \) Hence, applying [21, Thm. 3.4] to the real-valued function \( y^* \circ f \), equation (9.4) follows and, consequently, the first equality in (9.3) holds.
To prove the third equality in (9.3), we show that $D^V_\circ f(p)(h) = \overline{co} \omega^V(\delta^V f(p, h))$. Indeed,

$$D^V_\circ f(p)(h)$$

and continuous functions over $T$ we get

An example. 10.

The first equalities are consequences of Proposition 5.1. Proof. If $p$ be a Lipschitzian function. In addition, let $f$ be a Lipschitzian function near $x$. Let $D^N_f(p)(y^*) = \{ x^* \in X^* \mid \langle x^*, -y^* \rangle \in N((p, f(p)); \text{graph f}) \}$, \hspace{1cm} (9.5)

$\frac{\partial f}{\partial f} (p)(y^*) = (\partial^V f(p))^\circ (y^*) = \overline{co} \omega^V D^M_f(p)(y^*)$.

If $X$ is also an Asplund space and $f$ is strictly Lipschitzian then, for all $y^* \in V$,

$$\frac{\partial f}{\partial f} (p)(y^*) = (\partial^V f(p))^\circ (y^*) = \overline{co} \omega^V D^N_f(p)(y^*)$$.

Proof. The first equalities are consequences of Proposition 5.1.

On the other hand, using Theorem 3.2, and also the result on the connection between the co-Jacobian and the mixed coderivative obtained in [23], for $y^* \in V$, we get

$$\frac{\partial f}{\partial f} (p)(y^*) = \frac{\partial f}{\partial f} (p)|_V (y^*) = \frac{\partial f}{\partial f} (p)(y^*) = \overline{co} \omega^V D^M_f(p)(y^*)$$

The proof for the normal coderivative is analogous.

10. An example. Let $X_0$ be a normed space, $D \subseteq X_0$ be a nonempty open set and let $\phi : D \rightarrow \mathbb{R}$ be a Lipschitzian function. In addition, let $T$ be a compact Hausdorff space and consider the normed space $X := C(T, X_0)$ of $X_0$-valued continuous functions over $T$. Set $D := \{ x \in C(T, X_0) \mid x(t) \in D, \forall t \in T \}$ and define,
in terms of the function ϕ, the (nonlinear) composition operator \( f_\varphi : \mathcal{D} \to \mathcal{C}(T, \mathbb{R}) \) by
\[
 f_\varphi(x) := \varphi \circ x \quad (x \in \mathcal{D}).
\] (10.1)

One can easily see that this operator is also Lipschitzian. As we have discussed it in the introduction, even in the particular case \( n = 1, \mathcal{D} = \mathbb{R}, \varphi = | \cdot | \), the Jacobian of the operator \( f_\varphi \) cannot be constructed as a subset of the space \( \mathcal{L}(\mathcal{C}(T, X_0), \mathcal{C}(T, \mathbb{R})) \).

In the rest of this section, we consider the subspace \( V \) of linear functionals over \( Y := \mathcal{C}(T, \mathbb{R}) \) expressed in terms of atomic measures, i.e., \( v \in V \) holds if and only if there exist an integer \( k \in \mathbb{N} \), real constants \( c_1, \ldots, c_k \), and pairwise distinct elements \( t_1, \ldots, t_k \in T \) such that
\[
v = c_1 y_{t_1}^* + \cdots + c_k y_{t_k}^*,
\]
where the linear functional \( y_{t}^* \in Y^* \) is defined by
\[
y_{t}^*(y) = y(t) \quad (y \in Y).
\]
In other words, \( V \) is the linear span of the set \( \{ y_{t}^* \mid t \in T \} \subseteq Y^* \). Obviously,
\[
\| c_1 y_{t_1}^* + \cdots + c_k y_{t_k}^* \| = |c_1| + \cdots + |c_k|.
\]
It is easy to see that, in terms of any bounded function \( \nu : T \to \mathbb{R} \), a linear functional \( v^* \in V^* \) can be defined via the following formula:
\[
v^*(c_1 y_{t_1}^* + \cdots + c_k y_{t_k}^*) := c_1 \nu(t_1) + \cdots + c_k \nu(t_k)
\quad (k \in \mathbb{N}, c_1, \ldots, c_k \in \mathbb{R}, t_1, \ldots, t_k \in T). \tag{10.2}
\]
Conversely, given an element \( v^* \in V^* \), define the function \( \nu : T \to \mathbb{R} \) by
\[
\nu(t) := v^*(y_{t}^*) \quad (t \in T).
\]
Then \( |\nu(t)| \leq \| v^* \| \| y_{t}^* \| = \| v^* \| \), hence \( \nu \) is a bounded real function on \( T \). We can also see that (10.2) holds, which proves that the elements of \( V^* \) can uniquely be represented via (10.2) in terms of the elements of \( \mathcal{B}(T, \mathbb{R}) \), the space of bounded real functions over \( T \). This justifies the choice of \( V \) because its dual can easily be computed.

Now assume that \( p \in \mathcal{D} \) and let \( \Phi \in \partial^V f_\varphi(p) \). Then, by the definition, for all \( n \in \mathbb{N}, \psi \in V^n \),
\[
\psi \circ \Phi \in \partial(\psi \circ f_\varphi)(p).
\]
In particular, for all \( t \in T \),
\[
y_{t}^* \circ \Phi \in \partial(y_{t}^* \circ f_\varphi)(p) = \partial^c(y_{t}^* \circ f_\varphi)(p). \tag{10.3}
\]
By definition, for \( x \in \mathcal{D} \)
\[
y_{t}^* \circ f_\varphi(x) = \varphi(x(t)) = \varphi \circ A_t(x),
\]
where the linear map \( A_t : X \to \mathbb{R}^m \) is defined by \( A_t(x) := x(t) \). Obviously, \( A_t \) is surjective, and hence the chain rule of the generalized gradient [4, Thm. 2.3.10] can be applied and we get
\[
\partial^c(y_{t}^* \circ f_\varphi)(p) = \partial^c(\varphi \circ A_t)(p) = \partial^c \varphi(A_t(p)) \circ A_t = \partial^c \varphi(p(t)) \circ A_t.
\]
By (10.3), for all \( t \in T \), there exists \( \psi_t \in \partial^c \varphi(p(t)) \subseteq X_0^* \) such that
\[
y_{t}^* \circ \Phi = \psi_t \circ A_t.
\]
Therefore, for all \( t \in T \) and for all \( x \in X \)
\[
(\Phi(x))(t) = y_{t}^* \circ \Phi(x) = \psi_t \circ A_t(x) = \psi_t(x(t)) = \langle \psi_t, x(t) \rangle.
\]
It easily follows from the known properties of Clarke’s subgradient that the set \( \bigcup_{t \in T} \partial c \varphi(p(t)) \) is bounded, hence the function \( t \mapsto \psi_t \) is bounded. Thus, we have shown the following result:

**Proposition 10.1.** Under the notations and assumptions above, for every point \( p \in D \) and for every linear map \( \Phi \in \partial V f \varphi(p) \), there exists a bounded function \( \psi : T \to X_0^* \) such that

\[
\psi(t) \in \partial c \varphi(p(t)) \quad (t \in T) \tag{10.4}
\]

and

\[
(\Phi(x))(t) = \langle \psi(t), x(t) \rangle \quad (x \in X, t \in T). \tag{10.5}
\]

It remains an interesting open question whether, for every bounded function \( \psi : T \to X_0^* \) satisfying (10.4), the linear map \( \Phi \in \mathcal{L}(X, V^*) \) defined by (10.5) belongs to \( \partial V f \varphi(p) \).

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Received September 2009; revised March 2010.

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