THE HOMFLY POLYNOMIAL FOR LINKS
IN RATIONAL HOMOLOGY 3-SPHERES

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ABSTRACT. We construct a polynomial invariant, for links in a large class of rational homology 3-spheres, which generalizes the 2-variable Jones polynomial (HOMFLY). As a consequence, we show that the dual of the HOMFLY skein module of a homotopy 3-sphere is isomorphic to that of the genuine 3-sphere.

§1. Introduction

The theory of quantum groups gives a systematic way of producing families of polynomial invariants, for knots and links in $\mathbb{R}^3$ or $S^3$ (see for example [17]). In particular, the Jones polynomial [9] and its generalizations ([5,10]), can be obtained that way. All these Jones-type invariants are defined as state models on knot diagrams or as traces of braid group representations, and the proofs of their topological invariance offer little insight into the underlying topology. The lack of a topological interpretation of these polynomial invariants always makes it hard, if not impossible, to generalize them for knots and links in other 3-manifolds.

In his study of the topology of the space of knots in $S^3$ [18], Vassiliev constructed a vast family of knot invariants which are now known as Vassiliev invariants or invariants of finite type. Some of the nice aspects of the theory of invariants of finite type are that they have a simple combinatorial description and that they provide a unifying way to view various knot polynomials ([2,3,13]). On the other hand, the fact that the theory of finite type invariants rests on topological foundations allows its generalization to knots in arbitrary 3-manifolds ([12, 15]). In this paper, using the machinery developed in [12,15], we will show the existence and uniqueness of link polynomials obeying the HOMFLY skein relation in a large class of rational homology 3-spheres.

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In order to state our main results we need to introduce some notation: Suppose that $M$ is an orientable 3-manifold. Let $\pi = \pi_1(M)$ and let $\hat{\pi}$ denote the set of non-trivial conjugancy classes of $\pi$. Notice that $\hat{\pi}$ can be identified with the set of non-trivial free homotopy classes of oriented loops in $M$. An $n$-component link is a collection of $n$ unordered oriented circles, tamely and disjointly embedded in $M$. Hence, a link is homotopically equivalent to an unordered $n$-tuple of elements in $\hat{\pi} \cup \{1\}$. In every homotopy class of links, we will fix, once and for all, a link $CL$ and call it a trivial link. If $CL$ has $k$ components which are homotopically trivial, our choice will be such that $CL = CL^* \coprod U^k$, where $U^k$ is the standard unlink with $k$ components in a small ball neighborhood disjoint from $CL^*$ and $U^1$ will be abbreviated to $U$ later on. We will denote by $CL^*$ the set of all trivial links with none of their components homotopically trivial. It is obvious that $CL^*$ is in 1-1 correspondence with unordered $n$-tuples of elements in $\hat{\pi}$, for all $n > 0$.

Every link $L$ is homotopic to a certain $CL^* \coprod U^k$ for some $CL^* \in CL^*$, possibly empty. But the aim of link theory in the 3-manifold $M$ is to understand how two links can differ up to a (tame) isotopy if they are homotopic. Let $\mathcal{L}$ be the set of isotopy classes of links in $M$ and let $R = \mathbb{C}[v^{\pm 1}, z^{\pm 1}]$ be the ring of Laurent polynomials in $v$ and $z$. A map $\mathcal{L} \to R$ will be called a link polynomial.

Now let $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ and let $I$ be the ideal of $R[t]$ generated by $v - v^{-1}$ and $t$. Let $\hat{R}$ be the pro-$I$ completion of $R[t]$, i.e. the inverse limit of

$$\cdots \to R[t]/I^n \to R[t]/I^{n-1} \to \cdots.$$ 

**Theorem A.** Let $M$ be a rational homology 3-sphere which is either atoroidal or a Seifert fibered space. Then, there is a unique map $J_M : \mathcal{L} \to \hat{R}$ satisfying the HOMFLY skein relation

$$v^{-1}J_M(L_+) - vJ_M(L_-) = zJ_M(L_o)$$

and with given values $J_M(U)$ and $J_M(CL^*)$ for every $CL^* \in CL^*$. Moreover, we may choose $J_M(U)$ and $J_M(CL^*)$ in $R$ appropriately such that $J_M$ is a link polynomial, i.e. $J_M(L) \in R$ for every $L \in \mathcal{L}$.

Here, as usual, the three links $L_+, L_-$ and $L_o$ appeared in the HOMFLY skein relation differ only in a small ball neighborhood in $M$ where, under a suitable projection, they intersect at a positive crossing, a negative crossing, and a smoothing of a crossing, respectively.
It seems to be worthwhile to comment on the relationship between Theorem A and the study of the HOMFLY skein modules of 3-manifolds (see [7] for a survey). With the notation as above, the HOMFLY skein module $S_3(M)$ is defined to be the $R$-module spanned by $L$, and subject to the HOMFLY skein relation

$$v^{-1}L_+ - vL_- = zL_o.$$ 

Let $S(R\hat{\pi})$ be the symmetric tensor algebra of the free $R$-module $R\hat{\pi}$ generated by $\hat{\pi}$. J. Przytycki proposed the following conjecture.

**Conjecture.** If $M$ is compact and contains no closed non-separating surfaces, then

$$S_3(M) \cong S(R\hat{\pi})$$

as $R$-modules.

In the special case of $M = X \times [0,1]$ where $X$ is a compact surface, it was proved by Przytycki (see [7] and references therein) that $S_3(M) \cong S(R\hat{\pi})$ as $R$-algebras. Many other partial verifications of the conjecture are known. In the presence of non-separating closed surfaces in $M$, one can construct examples in which $S_3(M)$ is not torsion free.

Notice that $\mathcal{CL}^* \cup \{U\}$ is in one-one correspondence with a basis of $S(R\hat{\pi})$. So Theorem A can be thought of providing a strong supporting evidence to (the dual version) of Przytycki’s conjecture. In the special case when $\pi_1(M) = 1$, Theorems A is of particular interest because of the Poincare conjecture. Also, in this special case, we may rephrase Theorem A in the language of skein modules since $\mathcal{CL}^* = \emptyset$ when $\pi_1(M) = 1$.

We denote by $S_3^*(M)$ the dual of $S_3(M)$, i.e. $S_3^* = \text{Hom}(S_3(M), R)$.

**Theorem B.** If $M$ is a homotopy 3-sphere, then

$$S_3^*(M) \cong S_3^*(S^3) \cong R$$

as $R$-modules.

Let us now briefly discuss the main ideas of the paper:

For an $L \in \mathcal{L}$ let $\mathcal{M}^L$ denote the space of maps, equipped with the compact-open topology, from a disjoint union of circles to $M$, which are homotopic to $L$. Any two links in some $\mathcal{M}^L$ are related by a sequence of “crossing changes”. When we make a crossing change from one link to another, we produce a singular link with
one transverse double point as an intermediate step. If we are given a link invariant $F : \mathcal{L} \rightarrow \mathcal{R}$, where $\mathcal{R}$ is a ring, we may define an invariant $f$ of singular links with one double point by

$$f(L_\times) = F(L_+) - F(L_-),$$

where $\times$ stands for a double point and $L_\pm$ are links obtained from resolving the double point into a positive or negative crossing, respectively.

A natural question is the following: Starting with a singular link invariant $f$, we want to find necessary and sufficient conditions that $f$ has to satisfy so that it is derived from a link invariant, via (2). This question and its general version was shown in [12] and [15] to play an important role in understanding invariants of finite type for links in 3-manifolds. In §3 we answer the question for links in rational homology spheres which are either atoroidal or Seifert fibered spaces. In §4 and §5, we use the results in §3 to prove Theorem A. Theorem B is a direct corollary of Theorem A.

The question of whether $f$ is induced by a link invariant turns out to be a question about the integrability of $f$ along paths in every $\mathcal{M}^L$. Let $\Phi$ be a path in $\mathcal{M}^L$ connecting two links. After perturbation, we may assume that there are only finitely many points on $\Phi$ where we see singular links with one double point. Moreover, we can assume that when the parameter of the path passes through a point where we see a singular link, the nearby links are changed by a crossing change. The sum of suitably signed values of $f$ on these singular links along the path $\Phi$, denoted here by $X_\Phi$, can be thought of as the integral of $f$ along $\Phi$.

In order for $f$ to be derived from a link invariant, it is necessary and sufficient that $X_\Phi$ is independent of $\Phi$ relative to the end points, or equivalently that $X_\Phi = 0$, for every loop $\Phi$ in $\mathcal{M}^L$. It turns out that the first thing that one has to do is to find a set of finitely many local integrability conditions which guarantee $X_\Phi = 0$ for every null-homotopic loop $\Phi$ in $\mathcal{M}^L$. Our technical assumptions about 3-manifolds will then imply that these local integrability conditions guarantee $X_\Phi = 0$ for every loop $\Phi$ in $\mathcal{M}^L$.

If $\Phi$ is null-homotopic in $\mathcal{M}^L$ then we achieve our goal by putting the null-homotopy into almost general position. This is done in [15] (see §3.2). The treatment for the general case is based on the machinery developed in [12]. Here we need to change our point of view and think of $\Phi$ as a map from a disjoint union of tori into $M$. This naturally leads to the study of tori in $M$ and to the use of the results in [8] and [16] in order to treat essential $\Phi$’s. More precisely, by employing the homotopy classification of essential tori in Seifert fibered spaces, we are able to homotope $\Phi$ into
a certain nice position so that the local integrable conditions imply $X_\Phi = 0$ (see §3.3 and §3.4).

We organize the paper as follows: In §2 we recall the generic picture of a family of maps from a compact 1-polyhedron into a 3-manifold, parameterized by a disc, and we give the preliminaries from the topology of 3-manifolds that we use in subsequent sections. In §3 we answer the integrability question addressed above. The main result of this section is Theorem 3.1.2. In §4 we use Theorem 3.1.2 to construct formal power series link invariants which satisfy the same crossing change formulae as the HOMFLY polynomial (Theorem 4.2.1). In §5 we use the surgery description of 3-manifolds to prove that, with an appropriate choice of the initial values, the power series obtained in §4 converge to Laurent polynomials. Our main result is Theorem 5.2.3. By combining Theorem 4.2.1 and Theorem 5.2.3, Theorem A, and subsequently Theorems B, will follow immediately.

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§2. Preliminaries

§2.1. Almost general position for a disjoint union of circles

In this section we summarize from [15] the results about the generic picture of a family of maps from a disjoint union of circles to a 3-manifold, parameterized by a disc.

Let $P$ be an 1-dimensional compact polyhedron. Let $M$ be a 3-manifold and let $D^2$ denote the 2-disc. A map $\Phi : P \times D^2 \to M$ gives rise to a family of maps $\{\phi_x : P \to M ; \ x \in D^2\}$ where $\phi_x(*) = \Phi(*,x)$ for $x \in D^2$. Suppose that every $\phi_x$ is a piecewise-linear map and let $S_\phi$ be the closure of the set $\{x \in D^2 ; \phi_x \text{ is not an embedding }\}$. One can see that $S_\phi$ is a sub-polyhedron of $F$.

Two maps $\phi_1, \phi_2 : P \to M$ are called ambient isotopic if there exists an isotopy $h_t : M \to M$, $t \in [0,1]$ with $h_0 = id$ and $h_1\phi_1 = \phi_2$.

Let us now introduce some terminology about 1-dimensional polyhedra in 3-manifolds.
Let $P$ be an 1–dimensional polyhedron. Every point $q \in P$ has a neighborhood homeomorphic to a bouquet of finitely many arcs such that $q$ is the common endpoint of these arcs. The number of arcs in the bouquet is called the valence of $q$. A point $q \in P$ with valence different than 2 is called a vertex of $P$. A component of the complement of vertices is called an edge of $P$.

A double point of a map $\phi : P \rightarrow M$ is a point $p \in M$ such that $\phi^{-1}(p)$ consists of two points. A double point of a piecewise linear map $\phi : P \rightarrow M$ is called transverse double point if there exist two 1–simplexes $\sigma_1, \sigma_2$ contained in the 1–skeleton of $P$ such that

1) $\phi$ is linear and non–degenerate on $\sigma_1$ and $\sigma_2$;
2) $\phi(\sigma_1) \cap \phi(\sigma_2)$ is the double in question;
3) $\phi(\sigma_1)$ and $\phi(\sigma_2)$ intersect transversally in their interiors and they do not lie on the same plane.

We call an 1–dimensional subpolyhedron $S \subset D^2$ neat if $S \cap \partial D^2$ consists of finitely many points and each of the them is a valence 1 vertex of $S$. We call these vertices boundary vertices of $S$ and we call the vertices of $S$ lying in the interior of $D^2$ interior vertices of $S$.

We suppose now that $P$ is a disjoint union of circles. Then we have

**Proposition 2.1.1.** ([15]) A map $\Phi : P \times D^2 \rightarrow M$ can be changed by an arbitrary small perturbation so that $S_{\phi}$ is a neat 1–dimensional subpolyhedron of $F$. Moreover, we have

1) If $x, x' \in D^2$ belong to the same component of $D^2 \setminus S_{\phi}$ or $S_{\phi} \setminus \{\text{interior vertices}\}$, then $\phi_x$ and $\phi_{x'}$ are ambient isotopic;
2) the interior vertices of $S_{\phi}$ are of valence either four or one;
3) If $x \in S_{\phi}$ lies on an edge of $S_{\phi}$ or is a boundary vertex, then $\phi_x$ has exactly one transverse double point;
4) If $x \in S_{\phi}$ is an interior vertex of valence four, then $\phi_x$ has exactly two transverse double points; and
5) If $x \in S_{\phi}$ is an interior vertex of valence one, then $\phi_x$ is an embedding ambient isotopic to the nearby embeddings.

We say that the resulting map in Proposition 2.1.1 is in almost general position. Figure 2.1 below illustrates $S_{\Phi} \subset D^2$ for a map $\Phi$ in almost general position.
Remark 2.1.2. a) The proposition above is true for maps $\Phi : P \times X \rightarrow M$, where $X$ is a compact surface with $\partial X \neq \emptyset$, as well as more general types of polyhedra $P$. It was used extensively, in [12] to define and study invariants of finite type for knots in 3-manifolds.
b) If $\Phi|_{P \times \partial D^2}$, is in almost general position already, then the perturbation in proposition 2.1.1 can be made relatively $\partial D^2$.

§2.2. Seifert fibered spaces

In this section we give some terminology from the topology of 3-manifolds and recall some results from [8] and [16] that are used in subsequent sections.

Definition 2.2.1. A surface $X \neq S^2$, properly embedded in a 3-manifold $M$ (or embedded in $\partial M$), is compressible if there exists a disc $D \subset M$ such that $D \cap X = \partial D$ and $\partial D$ is not homotopically trivial in $X$. Otherwise $X$ is called incompressible in $M$. A compact, orientable, irreducible 3-manifold is called a Haken (or sufficiently large) manifold, if it contains a two-sided incompressible surface.

Definition 2.2.2. Let $M$ be a closed 3-manifold and $X$ a surface. A map $\phi : X \rightarrow M$ is called essential if $\ker\{\phi_* : \pi_1(X) \rightarrow \pi_1(M)\} = 1$.

Let $(\mu, \nu)$ be a pair of relatively prime integers. Let

$$D^2 = \{(r, \theta); \ 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi\} \subset \mathbb{R}^2$$

A fibered solid torus of type $(\mu, \nu)$, is the quotient of the cylinder $D^2 \times I$ via the identification $((r, \theta), 1) = ((r, \theta + \frac{2\pi\nu}{\mu}), 0)$. If $\mu > 1$ the fibered solid torus is said to be exceptionally fibered and the core is the exceptional fiber. Otherwise the fibered solid torus is regularly fibered and each fiber is a regular fiber.
Definition 2.2.3. An orientable 3-manifold is said to be a Seifert fibered space, if it is a union of pairwise disjoint simple closed curves, called fibers, such that each one has a closed neighborhood, consisting of a union of fibers, which is homeomorphic to a fibered solid torus via a fiber preserving isomorphism.

In a Seifert fibered space $M$, a fiber is called exceptional if it has a neighborhood homeomorphic to an exceptionally fibered solid torus and the fiber in question corresponds to the exceptional fiber of the solid torus. The orbit space of $M$ is the quotient obtained by identifying every fiber of $M$ to a point.

Definition 2.2.4. Let $M$ be a Seifert fibered space, with a fixed fibration and let $p : M \to B$ be the fiber projection. Let $X$ be a surface. A map $\phi : X \to M$ is called vertical or saturated, with respect to $p$, if $p^{-1}(p\phi(X)) = \phi(X)$ and $\phi(X)$ contains no exceptional fibers.

Throughout this paper we are dealing with Seifert fibered spaces that are rational homology 3-spheres (i.e $H_1(M)$ is finite). It is known that if $\pi_1(M)$ is infinite then either $M$ is Haken or it fibers over the 2-sphere with three exceptional fibers. The following proposition gives a classification up to homotopy of singular essential tori in such manifolds.

Proposition 2.2.5. Suppose that $M$ is a Seifert fibered rational homology 3-sphere with infinite $\pi_1$. Let $\Phi : T = \bigsqcup T_i \to M$ be an essential map, where each $T_i$ is a torus. Then there exists a homotopy $\Phi_t : T \to S$, $t \in [0, 1]$, such that $\Phi_0 = \Phi$ and $\Phi_1$ is vertical with respect to the fibration of $M$.

The proof of the proposition is given in Proposition 5.13 of [8] for the case when $M$ is Haken. For the case that $M$ is not Haken, see Theorem 6.4 of [16].

§3. Integrating invariants of singular links

In this section we introduce singular links and study their invariants. Our purpose is to give conditions under which an invariant of singular links gives rise to a link invariant.

§3.1. Definitions and the statement of the main result

Let $M$ be an oriented 3-manifold and let $P$ be a disjoint union of oriented circles.

Definition 3.1.1. A singular link of order $n$ is a piecewise-linear map $L : P \to M$ that has exactly $n$ transverse double points. Two singular links $L$ and $L'$ are equivalent
if there is an isotopy $h_t : M \rightarrow M$, $t \in [0,1]$ such that $h_0 = id$, $L' = h_1(L)$ and the double points of $h_t(L)$ are transverse for every $t \in [0,1]$.

We will also use $L$ to denote $L(P)$. A singular link of order 0 is simply a link.

Let $p \in M$ be a transverse double point of a singular link $L$. Then $L^{-1}(p)$ consists of two points $p_1, p_2 \in L$. There are disjoint 1-simplices, $\sigma_1$ and $\sigma_2$, on $P$ with $p_i \in \text{int}(\sigma_i)$, $i = 1, 2$ such that for a small ball neighborhood $B$ of $p$ in $M$

$$L \cap B = L(\sigma_1) \cap L(\sigma_2)$$

Moreover, there is a proper 2-disc $D$ in $B$ such that $L(\sigma_1), L(\sigma_2) \subset D$ intersect transversally at $p$, and the isotopy $h_t$ of definition 3.1.1 carries the ball disc pair $(B,D)$ through for all the double points of $L$.

We can resolve a transverse double point of a singular knot of order $n$ in different ways. Notice that $L(\sigma_1) \cap L(\sigma_2)$ consists of four points on $\partial D$. Also, since $\sigma_i$ inherits an orientation from that of $P$ we can talk about the initial point and terminal point of $\sigma_i$ and $L(\sigma_i)$.

Now choose arcs $a_1, a_2, b_1, b_2$ with disjoint interiors such that

1. $a_1$ and $a_2$ go from the initial point of $L(\sigma_1)$ to the terminal point of $L(\sigma_1)$ and lie in distinct components of $\partial B \setminus \partial D$; and
2. $b_1$ and $b_2$ lie on $\partial D$ with $\beta_1$ going from the initial point of $L(\sigma_1)$ to the terminal point of $L(\sigma_2)$ and $b_2$ from the initial point of $L(\sigma_2)$ to the terminal point of $L(\sigma_1)$.

See Figure 3.1.

![Resolving a transverse double point.](image)

Figure 3.1

Define

$$L_+ = K \setminus K(\sigma_2) \cup a_1$$

$$L_- = K \setminus K(\sigma_2) \cup a_2$$
\[ L_0 = \overline{K \setminus K(\sigma_2)} \cup (b_1 \cup b_2) \]

Clearly \( L_+ \), \( L_- \) are well defined singular links of order \( n - 1 \). We call \( L_+ \) (respectively \( L_- \)) a positive (respectively a negative) resolution of \( L \).

We will denote by \( \mathcal{L}^n \) the set of equivalence classes of singular links of order \( n \) in \( M \). Let \( \mathcal{R} \) be a ring. A singular link invariant is a map \( \mathcal{L}^n \to \mathcal{R} \). Notice that for \( n = 0 \) we have a link invariant. From a link invariant \( F : \mathcal{L} \to \mathcal{R} \) we can always define a singular link invariant \( f : \mathcal{L}^1 \to \mathcal{R} \) as follows:

Let \( L_\times \in \mathcal{L}^1 \) where \( \times \) stands for the only double point. Then \( L_+, L_- \in \mathcal{L}^0 = \mathcal{L} \). We define \( f : \mathcal{L}^1 \to \mathcal{R} \) by

\[ f(L_\times) = F(L_+) - F(L_-) \quad (3.1.1) \]

In this section we will answer the following question: Suppose that we are given a singular link invariant \( f : \mathcal{L}^1 \to \mathcal{R} \). Under what conditions can we find a link invariant \( F : \mathcal{L} \to \mathcal{R} \) so that (3.1.1) holds for all \( L_\times \in \mathcal{L}^1 \).

In [2], D. Bar-Natan thinks of (3.1.1) when going from the link invariant \( F \) to the singular link invariant \( f \) as the “first derivative” of \( F \). In this spirit the question above concerns the “integrability” of a singular link invariant.

For the rest of the paper we will assume, unless otherwise stated, that \( M \) is a rational homology 3-sphere such that either i) it has trivial \( \pi_2 \) and there are no essential maps \( S^1 \times S^1 \to M \) or ii) it is a Seifert fibered space. If \( M \) is as in i) we will say that it is atoroidal. Notice that if \( M \) is as in ii) then it is irreducible and hence we have \( \pi_2(M) = 1 \) by the sphere theorem (see for example [7]).

We will also assume that \( \mathcal{R} \) is a ring which is torsion free as an abelian group. Our main result in this section is the following theorem, which answers the integrability question for a large class of rational homology 3-spheres.

**Theorem 3.1.2.** Suppose that \( M \) and \( \mathcal{R} \) are as above, and let \( f : \mathcal{L}^1 \to \mathcal{R} \) be a singular link invariant. There exists a link invariant \( F : \mathcal{L} \to \mathcal{R} \) so that (3.1.1) holds for all \( L_\times \in \mathcal{L}^1 \) if and only if \( f \) satisfies

\[ f(\infty) = 0 \quad (1) \]

\[ f(L_{\times+}) - f(L_{\times-}) = f(L_{+\times}) - f(L_{-\times}) \quad (2) \]

**Notation:** Before we proceed with the proof of Theorem 3.1.2, let us explain the notation above. In (1) the kink stands for a singular link \( L_\times \in \mathcal{L}^1 \) where there is a
2-disc $D \subset M$ such that $L_x \cap D = \partial D$, and the unique double point of $L_x$ lies on $\partial D$. In (2) we start with an arbitrary singular link $L_{\times \times} \in \mathcal{L}_2$. The four singular links in $\mathcal{L}_1$ are obtained by resolving one double point of $L_{\times \times}$ at a time. Following [15], we will call conditions (1) and (2) above the \textit{local integrability conditions}.

The proof of Theorem 3.1.2 will occupy the rest of §3.

**Proof of Theorem 3.1.2.** One direction of the theorem is clear. That is if a singular link invariant $f : \mathcal{L}_1 \to \mathcal{R}$ is derived from a link invariant $F : \mathcal{L} \to \mathcal{R}$ via (3.1.1), then it satisfies (1) and (2). To see that (1) is satisfied observe that the positive and the negative resolution of the double point in the kink are equivalent. For (2) observe that, using (3.1.1), both sides of (2) can be expressed as $F(L_{++}) - F(L_{-+}) - F(L_{+-}) + F(L_{- -})$.

We now turn into the proof of the other direction. Namely, assuming that a singular link invariant $f : \mathcal{L}_1 \to \mathcal{R}$ is given satisfying (1) and (2), we show that it can be derived from a link invariant $F : \mathcal{L} \to \mathcal{R}$ via (3.1.1), provided that $M$ is as in the statement of Theorem 3.1.2. The proof will be broken into several steps.

Let $L \in \mathcal{L}$ be a link in $M$. We also use $L$ to denote a representative $L : P \to M$, of $L$. Let $\mathcal{M}^L(P, M)$ denote the space of maps $P \to M$ homotopic to $L$, equipped with the compact-open topology. For every $L' \in \mathcal{M}^L(P, M)$, we choose a homotopy $\phi_t : P \times [0, 1] \to M$ such that $\phi_0 = L'$ and $\phi_1 = L$. After a small perturbation, we can assume that for only finitely many points $0 < t_1 < t_2 < \cdots < t_n < 1$, $\phi_t$ is not an embedding. Moreover, we can assume that $\phi_{t_i}$, for $i = 1, 2, \ldots, n$ are singular links of order 1 (i.e. $\phi_{t_i} \in \mathcal{L}(1)$). For different $t$’s in an interval of $[0, 1] \setminus \{t_1, t_2, \ldots, t_n\}$, the corresponding links are equivalent. When $t$ passes through $t_i$, $\phi_t$ changes from one resolution of $\phi_{t_i}$ to another.

We define

$$F(L') = F(L) + \sum_{i=1}^{n} \epsilon_i f(\phi_{t_i})$$

(3.1.2)

Here $\epsilon_i = \pm 1$ is determined as follows: If $\phi_{t_i + \delta}$, for $\delta > 0$ sufficiently small, is a positive resolution of $\phi_{t_i}$ then $\epsilon_i = 1$. Otherwise $\epsilon_i = -1$.

To prove that $F$ is well defined we have to show that modulo “the integration constant” $F(L)$, the definition of $F(L')$ by (3.1.2) is independent of the choice of the homotopy. For this we consider a closed homotopy $\Phi : P \times S^1 \to M$. After a small perturbation, we can assume that there are only finitely many points $x_1, x_2, \ldots, x_n \in S^1$, ordered cyclicly according to the orientation of $m$, so that $\phi_{x_i} \in \mathcal{L}_1$ and $\phi_y$ is equivalent to $\phi_x$ for all $x_i < x, y < x_{i+1}$. To prove that $F$ is well defined we need to
show that

\[ X_\Phi := \sum_{i=1}^{n} \epsilon_i f(\phi_{t_i}) = 0 \]  \hspace{1cm} (3.1.3)

where \( \epsilon_i = \pm 1 \) is determined by the same rule as above. We will call (3.1.3) the global integrability condition around \( \Phi \).

The proof of (3.1.3), which will be broken into many steps, occupies the rest of §3.

§3.2. The proof of the global integrability condition in some special cases

Assume that \( M \) is an oriented 3-manifold, with \( \pi_2(M) = 1 \), and that \( f : L^1 \to \mathcal{R} \) is a singular link invariant. Let \( L : P \to M \) be a link, and recall that \( \mathcal{M}^L(P,M) \) denotes the space of maps \( P \to M \) homotopic to \( L \), equipped with the compact-open topology. A closed homotopy \( \Phi : P \times S^1 \to M \) from \( L \) to itself, can be viewed as a loop in \( \mathcal{M}^L(P,M) \).

**Lemma 3.2.1.** Let \( M, P, \) and \( \Phi \) be as above. Moreover, suppose that \( \Phi \) can be extended to a map \( \hat{\Phi} : P \times D^2 \to M \), where \( D^2 \) is a 2-disc with \( \partial D^2 = \{\ast\} \times S^1 \). Then, \( \hat{\Phi} \) satisfies the global integrability condition, i.e. \( X_{\Phi} = 0 \).

**Proof.** We perturb \( \hat{\Phi} \) to an almost general position map as in Proposition 2.1.1. Then each edge of the set of singularities \( S_{\hat{\Phi}} \), corresponds to a singular link of order 1. So by using the invariant \( f \) we can assign an element of \( \mathcal{R} \) to every edge of \( S_{\hat{\Phi}} \). we can reduce the global integral condition around \( \hat{\Phi} \), to local integrable conditions around each interior vertex in \( S_{\hat{\Phi}} \).

![From global to local integrability conditions.](image)

**Figure 3.2**

More precisely, for every interior vertex of \( S_{\hat{\Phi}} \) draw a small circle \( C \) around it, so that the number of points in \( C \cap S_{\hat{\Phi}} \) is equal to the valence of the vertex. For a picture
see Figure 3.2. It suffices to show that
\[ \sum_{x \in C \cap S_{\hat{\Phi}}} \pm f(\hat{\phi}_x) = 0 \quad (3.2.1) \]
for every interior vertex of \( S_{\hat{\Phi}} \). Here \( \hat{\phi}_x(S^1) = \hat{\Phi}(P \times \{x\}) \).

**Case 1.** The valence of the interior vertex is one: In this case it is easy to see that for \( x \in S_{\hat{\Phi}} \), near that vertex, the unique double point of \( \hat{\phi}_x \) is at a kink. So (3.2.1) is implied by the local integrability condition (1)

**Case 2.** The valence of the interior vertex is four: In this case the four points in \( C \cap S_{\hat{\Phi}} \) correspond to the four singular knots appearing in the local integrability condition (2) and one can show that (3.2.1) is guaranteed by it. \( \square \)

**Lemma 3.2.2.** Let \( f : \mathcal{L}^1 \to \mathcal{R} \) be a singular link invariant and let \( \Phi : S^1 \to \mathcal{M}^L(P,M) \) a loop. Then \( X_{\Phi} \) only depends on the free homotopy class of \( \Phi \) in \( \mathcal{M}^L(P,M) \).

**Proof.** Let \( \Phi' \) be another closed homotopy in almost general position such that \( \Phi, \Phi' : S^1 \to \mathcal{M}^L(P,M) \) are freely homotopic loops in \( \mathcal{M}^L(P,M) \). Then there exists a homotopy \( \Phi_t : P \to \mathcal{M}^L(P,M) \) with \( t \in [0,1] \), such that \( \Phi_0 = \Phi \) and \( \Phi_1 = \Phi' \).

Let \( \gamma \) be the path in \( \mathcal{M}^L(P,M) \) defined by \( \gamma(t) = \Phi_t(L) \). After putting \( \gamma \) in almost general position we have
\[ X_{\gamma \Phi' \gamma^{-1}} = X_{\gamma} + X_{\Phi'} - X_{\gamma} = X_{\Phi'}. \]

Hence, we can assume that both \( X_{\Phi} \) and \( X_{\Phi'} \) are based at \( L \) and the homotopy \( \Phi_t \) is taken relatively \( L \). The homotopy \( \Phi_t \) gives rise to a map \( \mathcal{H} : P \times S^1 \times I \to M \). We cut the annulus \( S^1 \times I \) into a disc \( D \) along a proper arc \( \alpha \subset S^1 \times I \). Then, we have
\[ X_{\partial D} = \pm (X_{\Phi} - X_{\Phi'} - X_{\alpha} + X_{\alpha}). \]

By Lemma 3.2.1 we obtain \( X_{\partial D} = 0 \), and hence \( X_{\Phi} = X_{\Phi'} \). \( \square \)

To continue, we first need to introduce some notation. Suppose that \( P \) has \( m \) components; that is
\[ P = \coprod_{i=1}^{m} P_i \]
where each \( P_i \) is an oriented circle. Let \( L : P \to M \) be a link. Pick a base point \( p_i \in P_i \) and let \( a_i \) denote the homotopy class of \( L(P_i) \) in \( \pi_1(M,L(p_i)) \). Finally, we denote by \( Z(a_i) \) the centralizer of \( a_i \) in \( \pi_1(M,L(p_i)) \).

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Lemma 3.2.3. Suppose that $M$ is a rational homology 3-sphere, with $\pi_2(M) = 1$. Let $L : P \to M$ be a link and let $\Phi : P \times S^1 \to M$ be a closed homotopy from $L$ to itself. Moreover, assume that either $Z(a_i) = \pi_1(M, L(p_i))$ or $Z(a_i)$ is finite for some $i = 1, \ldots, m$. Then $X_\Phi = 0$.

**Proof.** Without loss of generality we may assume that either $Z(a_1) = \pi_1(M, L(p_1))$ or $Z(a_1)$ is finite. Let $L_1$ denote the restriction of $L$ on $P_1$ and let $\Phi_1$ denote the restriction of $\Phi$ on $P_1 \times S^1$. We denote by $\mathcal{M}_1 = \mathcal{M}^{L_1}(P_1, M)$ the space of maps $P_1 \to M$, which are homotopic to $L_1$, equipped with the compact-open topology. Let

$$\pi = \pi_1(\mathcal{M}^{L_1}(P_1, M), L_1).$$

One can see (see also Proposition 3.3 of [15]) that $\pi = Z(a_1)$. Clearly, $\Phi_1$ represents an element in $\pi$.

Let $\Psi : P_1 \times S^1 \to M$ be a loop in $\mathcal{M}_1$ based at $L_1$. We define $\tilde{\Psi} : P \times S^1 \to M$ by

$$\tilde{\Psi}|P_1 \times S^1 = \Psi$$

$$\tilde{\Psi}(P' \times S^1) = \Phi(P' \times S^1)$$

where $P' = P \setminus P_1$. Then $\tilde{\Psi}$ is the closed homotopy from $L$ to itself.

After a small perturbation, we can assume that there are only finitely many points $x_1, x_2, \ldots, x_n \in S^1$, ordered cyclicly according to the orientation of $S^1$, so that $\tilde{\psi}_{x_i} \in \mathcal{L}^1$ and $\tilde{\psi}_{x_i}$ is equivalent to $\tilde{\psi}_y$ for all $x_i < x, y < x_{i+1}$. (Here, $\tilde{\psi}_t$ denotes $\tilde{\Psi}(P \times \{t\})$).

Define

$$\chi(\Psi) := X_\Phi = \sum_{i=1}^n \epsilon_i f(\tilde{\psi}_{t_i}) \in \mathcal{R}$$

where $\epsilon = \pm 1$ is determined as in (3.1.3), and $\mathcal{R}$ is a ring which is torsion free as an abelian group. Notice that

$$\chi(\Phi_1) = X_\Phi$$

**Claim:** The assignment $\Psi \mapsto \chi(\Psi)$ is a group homomorphism $\chi : \pi \to \mathcal{R}$.

**Proof of the claim:** It is enough to show that $\chi(\Psi)$ is independent of the choice of the representative of $[\Psi] \in \pi$. Let $\Psi_1 : P_1 \times S^1 \to M$ be another loop in almost general position, which is homotopic to $\Psi$. Then clearly $\tilde{\Psi}$ and $\tilde{\Psi}_1$ are homotopic loops in $\mathcal{M}^L(P, M)$, and the claim follows from Lemma 3.2.2.
By the assumption, we have either \( \pi = \pi_1(M, L(p_1)) \) or \( \pi \) is finite. Since \( R \) is abelian, \( \chi \) must factor through a finite abelian group (as \( M \) is a rational homology 3-sphere). Thus, we must have \( \chi = 0 \) since \( R \) is torsion free. In particular,

\[
\chi(\Phi_1) = X_\Phi = 0
\]
as desired. \( \square \)

**Corollary 3.2.4.** Assume that \( M \) is as in Lemma 3.2.3 above. Let \( L : P \to M \) be a link and let \( \Phi : P \times S^1 \to M \) be a closed homotopy from \( L \) to itself.

1) If \( L \) has a component which is homotopically trivial in \( M \), then \( X_\Phi = 0 \).

2) If \( \pi_1(M) \) is finite, then \( X_\Phi = 0 \).

3) Assume that \( M \) is a Seifert fibered space, and that \( L \) has a component which is homotopic to a regular fiber of the fibration. Then \( X_\Phi = 0 \).

**Proof.** 1) and 2) follow immediately from Lemma 3.2.3.

3). By Lemma 32.8 of [8], we know that the centralizer of a regular fiber is \( \pi_1(M) \).

Hence the result follows from Lemma 3.2.3. \( \square \)

Thus, we may assume from now on that \( \pi_1(M) \) is infinite.

§3.3. Closed homotopy of links and essential tori

The purpose of this paragraph is study the topology of closed homotopy of links thought of singular tori in 3-manifolds. Since we are mainly interested in the global integrability condition, which in general will be reduced to these special cases discussed here, we may (and will) consider only 3-manifolds with infinite \( \pi_1(M) \) by Corollary 3.2.4.

Assume that \( M \) is a Seifert fibered space with orbit surface \( B \) and fiber projection \( p : M \to B \). Let \( \Phi : T = S^1 \times S^1 \to M \) be a closed homotopy of the knot \( \Phi|S^1 \times \{\ast\} \) which is vertical with respect to the given fibration. Let

\[
\tilde{M} \to M
\]

be the covering space corresponding to the cyclic normal subgroup generated by a regular fiber. We say that \( \Phi(S^1 \times \{\ast\}) \) doesn’t wrap around the fibers of \( M \) if \( \Phi|S^1 \times \{\ast\} \) lifts to \( \tilde{M} \).

**Lemma 3.3.1.** Let \( M \) and \( \Phi \) be as above. If \( \Phi(S^1 \times \{\ast\}) \) doesn’t wrap around the fibers of \( M \), then the closed homotopy \( \Phi \) is homotopic to another closed homotopy \( \Phi' \) with the following property: For every \( x_1, x_2 \in S^1 \), there is a homeomorphism \( h^{12} : M \to M \) such that

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1) \( h^{12} = id \) outside of a regular neighborhood of \( \Phi'(T) \) in \( M \);

2) \( h^{12}(\phi_x) = \phi_{x^2} \), where \( \phi_x = \Phi'|S^1 \times \{x\} \);

3) \( h^{12} \) is isotopic to the identity map \( id : M \rightarrow M \).

See Lemma 3.9 in [12].

**Lemma 3.3.2.** Let \( p : M \rightarrow B \) be as above and, in addition, let \( M \) be a rational homology 3-sphere. Let \( \Phi : T = S^1 \times S^1 \rightarrow M \) be an essential map. Then, there exists a map \( \Phi_1 : T \rightarrow M \) which is homotopic to \( \Phi \), and a finite covering \( \tau : S^1 \times S^1 \rightarrow S^1 \times S^1 \) such that the map \( \Phi_1 \circ \tau : S^1 \times S^1 \rightarrow M \) can be extended to a map \( \hat{\Phi} : S^1 \times X \rightarrow M \). Here \( X \) is a surface with \( \partial X = \{\ast\} \times S^1 \).

**Proof.** By Proposition 2.2.5, \( \Phi \) is homotopic to a map \( \Phi_1 : T \rightarrow M \) which is vertical with respect to the fibration of \( M \). Then, there exists a decomposition \( T = S^1 \times S^1 \) such that

a) \( \Phi_1(S^1 \times \{\ast\}) \) covers a regular fiber \( h \), of \( M \)

b) We have \( p(\Phi_1(\{\ast\} \times S^1)) = p(T) \).

Let \( H \) (respectively, \( Q \)) denote the curve \( S^1 \times \{\ast\} \) (respectively, \( \{\ast\} \times S^1 \) on \( T \), and let \( N \) be the cyclic normal subgroup \( \pi_1(M) \) generated by the regular fiber \( h \). Since \( M \) is a rational homology 3-sphere, the abelianization of the Fuchsian group \( \Delta = \pi_1(M)/N \) is finite. Let \( d \) be its order, and consider the \( d \)-fold covering \( \tau : \hat{T} \rightarrow T = H \times Q \), corresponding to the subgroup \( \mathbb{Z} \oplus d\mathbb{Z} \) of \( \pi_1(T) = \mathbb{Z} \oplus \mathbb{Z} \). Let \( \hat{H} \) and \( \hat{Q} \) denote the liftings, on \( \hat{T} \), of \( H \) and \( Q \) respectively. Then, the map \( \hat{\Phi}_1 \circ \tau \) extends to a map \( X \rightarrow B \) for some compact surface \( X \) with boundary \( \hat{Q} \). This gives us a map \( \pi_1(X) \rightarrow \Delta \), which in turn lifts to a map \( \hat{\Phi} : S^1 \times X \rightarrow M \) (recall that \( \pi_1(M) \) is an extension of \( \Delta \) by \( \mathbb{Z} \cong N \)), with \( \hat{\Phi}|S^1 \times \partial X = \Phi_1 \circ \tau \).

Let \( P \) be a disjoint union of oriented circles and let \( \Phi : P \times S^1 \rightarrow M \) be a closed homotopy from a link \( L : P \rightarrow M \) to itself. Let \( f : \mathcal{L} \rightarrow \mathcal{R} \) be a singular link invariant and let \( X_\Phi \) be the quantity defined in (3.1.3). Suppose that \( P \) has \( m \) components, that is

\[
P = \coprod_{i=1}^{m} P_i
\]

**Lemma 3.3.3.** Assume that \( M \) is a Seifert fibered rational homology 3-sphere and let \( P, \Phi \) be as above. Moreover, assume that \( \Phi|P_i \times S^1 \) is an essential map, for every \( i = 1, \ldots, m \). Then there exists a map \( \tilde{\Psi} : P \times D^2 \rightarrow M \) such that

\[
X_{\tilde{\Psi}} = aX_\Phi
\]  

(3.3.1)
for some $a \in \mathbb{Z}$. Here, $\partial \tilde{\Psi} = \tilde{\Psi}|P \times \partial D^2$ and $D^2$ is a 2-disc. In particular, we have $X_\Phi = 0$.

**Proof.** Let $T_i = P_i \times S^1$ and let $\Phi_i = \Phi|T_i$, for $i = 1, \ldots, m$. Denote by $l_i$ (respectively, $m_i$) the simple closed curve $P_i \times \{\ast\}$ (respectively, $\{\ast\} \times S^1$) on $T_i$.

By Lemma 3.3.2, and after a homotopy to a vertical position, there exist a finite covering $\tau_i : \hat{T}_i \rightarrow T_i$, such that $\Phi_i \circ \tau_i$ extends to a map $\hat{\Phi}_i : S^1 \times Y_i \rightarrow M$. Here $Y_i$ is a compact surface and $S^1 \times \partial Y_i = \hat{T}_i$. Moreover all the $\tau_i$’s can be taken to be of the same degree $d$.

**Case 1:** $d = 1$ so that $\hat{T}_i = T_i$. Notice, that this is always the case if $H_1(M) = 0$.

Recall, that the quantity $X_\Phi$ doesn’t change under homotopy (Lemma 3.2.2.), and let $H_i$ (respectively, $Q_i$) denote $S^1 \times \{\ast\}$ (respectively, $\{\ast\} \times \partial Y_i$). Suppose that $l_i = a_i H_i + b_i Q_i$, for some $a_i, b_i \in \mathbb{Z}$. We distinguish two subcases:

**Subcase 1:** Suppose that $a_i \neq 0$ for every $i = 1, \ldots, m$.

Let $q_i : \tilde{T}_i \rightarrow T_i$ be the covering of $T_i$ corresponding to the subgroup $a_i \mathbb{Z} \oplus \mathbb{Z}$ of $\pi_1(T_i) = \mathbb{Z} \oplus \mathbb{Z}$. Let $\tilde{l}_i, \tilde{Q}_i, \tilde{H}_i$ and $\tilde{m}_i$ denote the liftings of $l_i, Q_i, H_i$ and $m_i$, respectively. We have $\tilde{l}_i = \tilde{H}_i + b_i \tilde{Q}_i$.

Each map $q_i$ extends to an $|a_i|$-fold covering,

$$\tilde{q}_i : \tilde{l}_i \times \tilde{Y}_i \rightarrow S^1 \times Y_i$$

where $\tilde{Y}_i$ is a compact surface with $\partial \tilde{F}_i = \tilde{Q}_i$, and $\tilde{l}_i \times \tilde{Q}_i = \tilde{T}_i$.

Let $\bar{\Phi}_i = \hat{\Phi}_i \circ \tilde{q}_i$ and let

$$\bar{\Phi} = \prod_{i=1}^m \bar{\Phi}_i$$

**Claim:** We have that

$$X_{\partial \bar{\Phi}} = a X_\Phi$$

where $|a| = \max\{|a_1|, \ldots, |a_m|\}$.

**Proof of Claim:** Let $q_i*$ denote the map induced by $q_i$ on the fundamental groups. One can easily see that $q_i*(\tilde{m}_i) = a_i m_i$ and

$$\tilde{Q}_i = c_i \tilde{l}_i + \tilde{m}_i \quad (3.3.2)$$

for some $c_i \in \mathbb{Z}$. We identify the curves $\tilde{Q}_i$ by a common parameterization, and call the result $\tilde{Q}$. The parameterization should be such that corresponding points on the
\( \tilde{Q}_i \)'s map, under the \( q_i \)'s, to the same point on the parameter space of \( \Phi \). By (3.3.2) this induces a common parameterization of the curves \( \tilde{m}_i \). Identify them and call the result \( \tilde{m} \). Now, \( \tilde{\Phi} \) induces a map \( \tilde{l} \times \tilde{m} \rightarrow M \), where

\[
\tilde{l} = \prod_{i=1}^{m} \tilde{l}_i
\]

We continue to denote this map by \( \tilde{\Phi} \). Clearly, we have

\[
\tilde{\Phi}(\tilde{l} \times \{ x \}) = \prod_{i=1}^{m} \Phi_i(P_i \times \{ q_i(x) \})
\]

for every \( x \) on \( \tilde{m} \). Notice that each point, on the parameter space of \( \Phi \), for which \( \Phi(P) \) is not an embedding, corresponds to \( |a| \) points \( x \in \tilde{m} \) for which \( \tilde{\Phi}(\tilde{l} \times \{ x \}) \) is not an embedding. Now, the claim follows easily. Let us finally observe that, because of (3.3.2), the quantity \( X_{\partial \Phi} \) doesn't change if we replace the parameter space \( \tilde{m} \), by \( \tilde{Q} \).

To continue with the proof of the lemma, we choose a collection of proper arcs

\[
\{ \alpha^j_i \}_{j=1}^{m_i} \subset \tilde{Y}_i
\]

such that: a) each \( \tilde{Y}_i \) if cut along the \( \{ \alpha^j_i \}'s \) becomes a disc (as each \( \tilde{Y}_i \) can be chosen as connected), and b) the end points of the \( \{ \alpha^j_i \}'s \) avoid the points for which \( \tilde{\Phi}(\tilde{l} \times \{ \ast \}) \) is not an embedding. Let \( \Gamma_i \) denote the space obtained by cutting \( \tilde{l}_i \times \tilde{Y}_i \) along the collection of annuli

\[
\{ A^j_i \}_{j=1}^{m_i}
\]

where \( A^j_i = \tilde{l}_i \times \alpha^j_i \). Let us denote by \( \tilde{\Psi}_i \) the map induced on \( \Gamma_i \), by \( \tilde{\Phi}_i \circ \tilde{q}_i \). Finally, let

\[
\Gamma = \prod_{i=1}^{m} \Gamma_i
\]

and let

\[
\tilde{\Psi} = \prod_{i=1}^{m} \tilde{\Psi}_i
\]

The map induced on \( \Gamma \) by \( \tilde{\Psi} \), is the desired map.

**Subcase 2:** Suppose that \( a_i = 0 \) for some \( i = 1, \ldots, m \).

Suppose for example that \( a_1 = 0 \). Then \( \tilde{\Phi}(l_1) \) doesn't wrap around the fibers of \( M \). By Lemma 3.3.1, we may assume that \( \Phi_1(P_1 \times \{ x_1 \}) \) and \( \Phi_1(P_1 \times \{ x_2 \}) \) are isotopic for every \( x_1 \) and \( x_2 \in S^1 \). Let

\[
P' = \prod_{j \neq 1} P_j
\]
and let
\[ \Phi' = \prod_{j \neq 1} \Phi_j. \]

Observe that \( \Phi(P \times \{\ast\}) \) is not an embedding if either \( \Phi'(P' \times \{\ast\}) \) is not an embedding, or \( \Phi_1(P_1 \times \{\ast\}) \) intersects with some \( \Phi_j(P_j \times \{\ast\}) \). We may change \( \Phi' \) by composing it with the inverse of the isotopy of \( \Phi_1(P_1 \times \{\ast\}) \). Hence, \( X_\Phi \) doesn’t change if we assume that
\[ \Phi_1(P_1 \times \{x\}) = \Phi_1(P_1 \times \{x_0\}) \]
where \( x_0 \) is fixed and \( x \) runs on \( \{\ast\} \times S^1 \). Hence, \( \Phi_1 \) extends to a map \( \hat{\Phi}_1 : S^1 \times D_1 \to M \), where \( D_1 \) is a disc. Then we proceed as in Subcase 1 above.

**Case 2:** Now assume that \( d > 1 \), where \( d \) is the common degree of the coverings \( \tau_i : \hat{T}_i \to T_i \).

In this case we apply Lemma 3.3.2 to a suitable covering of \( T_i \) rather than \( T_i \) itself. More precisely suppose that \( T_i \) is in vertical position, and that \( l_i = a_i H_i + b_i Q_i \), for some \( a_i, b_i \in \mathbb{Z} \).

If \( a_i \neq 0 \) for every \( i = 1, \ldots, m \), let
\[ q_i : \bar{T}_i \to T_i \]
be the covering of \( T_i \) corresponding to the subgroup \( a_i \mathbb{Z} \oplus \mathbb{Z} \) of \( \pi_1(T_1) = \mathbb{Z} \oplus \mathbb{Z} \). Let \( \bar{l}_i, \bar{Q}_i, \bar{H}_i \) and \( \bar{m}_i \) denote the liftings of \( l_i, Q_i, H_i \) and \( m_i \), respectively. We have \( \bar{l}_i = \bar{H}_i + b_i \bar{Q}_i \) and we may choose \( \bar{L}_i \) and \( \bar{Q}_i \) as a system of generators of \( \pi_1(\bar{T}_i) \). Let
\[ \bar{\Phi} = \prod_{i=1}^{m} \Phi_i \circ q_i. \]

In view of the claim above, it is enough to prove the assertion in the statement of the lemma for \( \bar{\Phi} \). Now let \( \bar{\tau}_i : \bar{T}_i^* \to \bar{T}_i \) be coverings as in the proof of Lemma 3.3.2, and let \( d \) be their common degree. Let \( \Phi^*_i = \Phi_i \circ q_i \circ \bar{\tau}_i \) and let
\[ \Phi^* = \prod_{i=1}^{m} \Phi^*_i. \]

One can see that
\[ X_{\Phi^*} = dX_{\bar{\Phi}} \]
and proceed as in Case 1 above to prove the desired assertion for \( \Phi^* \).

If \( a_i = 0 \) for some \( i = 1, \ldots, m \), then we proceed as in Subcase 2 above. \( \square \)
Lemma 3.3.4. Assume that $M$ is an oriented 3-manifold with $\pi_2(M) = 1$. Let $\Phi$ be a closed homotopy such that

$$\Phi|P_i \times S^1$$

is an inessential map, for every $i = 1, \ldots, m$. Then $X_\Phi = 0$.

**Proof.** As before, we denote $T_i = P_i \times S^1$, $\Phi_i = \Phi|T_i$, $l_i = P_i \times \{\ast\}$, and $m_i = \{\ast\} \times S^1$ for $i = 1, \ldots, m$.

We have assumed that $\pi_1(M)$ is infinite. So it is torsion free (Theorem 9.8 of [6]). Hence, $\Phi_i$ extends to a map $\hat{\Phi}_i : S^1 \times D_i \to M$, where $D_i$ is a 2-disc and $S^1 \times \partial D_i = T_i$. Let $H_i$ (respectively, $Q_i$) denote $S^1 \times \{\ast\}$ (respectively, $\{\ast\} \times \partial D_i$). Suppose that $l_i = a_i H_i + b_i Q_i$, for some $a_i, b_i \in \mathbb{Z}$.

If $a_i = 0$ for some $i = 1, \ldots, m$, then one of the components of the link $\Phi(P \times \{\ast\})$ is homotopically trivial and the conclusion follows from Lemma 3.2.3. In the case that $a_i \neq 0$ for every $i = 1, \ldots, m$, one proceeds as in Case 1 of the proof of Lemma 3.3.3 to obtain a map $\tilde{\Psi} : P \times D^2 \to M$ such that

$$X_{\tilde{\Psi}} = a X_\Phi$$

(3.3.3)

for some $a \in \mathbb{Z}$. The desired conclusion then follows immediately. \qed

§3.4. Completing the proof of Theorem 3.1.2

Recall that $M$ is a rational homology 3-sphere which is either atoroidal or a Seifert fibered space. Also, $\pi_1(M)$ is infinite. As before, $P$ is a disjoint union of oriented circles and

$$\Phi : P \times S^1 \to M$$

is a closed homotopy from some link $L : P \to M$ to itself. Let $f : \mathcal{L}^1 \to \mathcal{R}$ be a singular link invariant as in the statement of theorem 3.1.2. We have to show that

$$X_\Phi = 0$$

(3.4.1)

where $X_\Phi$ is the signed sum of values of $f$ around $\Phi$ defined in (3.1.3).

If $M$ is atoroidal then by Lemma 3.3.4, (3.4.1) holds.

Suppose that $M$ is a Seifert fibered space. Let $E$ (resp. $I$) denote the set of components of $P \times S^1$ on which $\Phi$ is essential (resp. inessential). If $E$ or $I$ is empty the claim follows from Lemmas 3.3.3 or 3.3.4.

We first notice that as in the proof of Lemma 3.3.4, if there is a component in $I$ which is a closed homotopy of a homotopically trivial knot, then $X_\Phi = 0$ by Corollary 3.2.4. Otherwise, we have the following claim.
Calim: There exists a map $\tilde{\Psi} : P \times D^2 \rightarrow M$ such that

$$X_{\rho \tilde{\Psi}} = aX_\Phi$$

for some $a \in \mathbb{Z}$. In particular, we $X_\Phi = 0$.

Proof of the claim: The proof is very similar to the proofs of Lemmas 3.3.3 and 3.3.4. The only difference is that, for example, when cutting each surface $Y_i$ to discs (see the proof of Lemma 3.3.3), we have to make sure that the end points of cutting arcs also avoid singular links where the double points are other than intersections between components of $L$ in $E$. The details are left to the reader.

This finishes the proof of Theorem 3.1.2. □

Remark 3.4.1. A more general statement, than that of Theorem 3.1.2, was proved in [15] for links in manifolds with finite $\pi_1$ and in [12] for knots in irreducible 3-manifolds, as a first step in understanding finite type invariants for knots in these manifolds.

§4. An intrinsic definition of the HOMFLY power series

§4.1. Preliminaries

It is known ([11]) that the 2-variable Jones (or HOMFLY) polynomial ([5,10]) for links in $\mathbb{R}^3$ or $S^3$ is equivalent to sequence $\{J_n = J_n(t)\}_{n \in \mathbb{Z}}$ of 1-variable Laurent polynomials. They are completely determined by the following skein relations:

$$J_n(U) = 1$$

$$t^{\frac{n+1}{2}}J_n(L_+) - t^{-\frac{n+1}{2}}J_n(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_n(L_o)$$

where $L_+, L_-, L_o$ are the resolutions of a singular link $L_x \in \mathcal{L}(1)$ described in 3.1. In this context, the original Jones polynomial is $J_{-3}$.

Notice that the initial value $J_n(U) = 1$ is not essential. Any choice of the initial value together with (4.1.2) will determine a unique $J_n$.

Let

$$u_n(t) = \frac{t^{\frac{n+1}{2}} - t^{-\frac{n+1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}.$$  

By (4.1.2) one obtains

$$J_n(L \coprod U) = u_n(t) J_n(L)$$

where the link $L \coprod U$ is obtained from $L$ by adding an unknotted and unlinked component $U$.  

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The coefficients of the power series \( J_n(x) \), obtained from \( J_n(t) \) by substituting \( t = e^x \), are invariants of finite type ([2,3]). In the theorem below we reverse this procedure, and guided by (4.1.2) we will construct inductively power series invariants for links in 3-manifolds generalizing the \( J_n(x) \)'s.

**§4.2. The construction of the invariants**

Assume that \( M \) is an orientable, rational homology 3-sphere which is either atoroidal or a Seifert fibered space. For every \( n \in \mathbb{Z} \), we will construct a sequence of knot invariants

\[
v^0_n, v^1_n, \ldots, v^m_n, \ldots
\]

such that the formal power series

\[
J_{\{M,n\}}(L) = \sum_{m=0}^{\infty} v^m_n(L)x^m
\]

satisfy (4.1.2), under the change of variable \( t = e^x \), for every \( L \in \mathcal{L} \).

We will construct our invariants inductively (induction on \( m \)) by using Theorem 3.1.2. More precisely, each \( v^m_n \) is going to be obtained by integrating a suitable singular link invariant determined by the \( v^j_n \)'s with \( j < m \).

Recall that a link invariant obtained by integrating a singular link invariant is well defined up to a collection of “integral constants” (see the beginning of the proof of Theorem 3.1.2). This means that in order to define \( v^0_n, v^1_n, \ldots, v^m_n, \ldots \) uniquely, we need to make a choice of “initial links”.

Let \( L \) be an \( n \)-component link and recall from §3 that \( \mathcal{M}^L \) denotes the space of maps \( \coprod S^1 \to M \) which are homotopic \( L \). The spaces \( \mathcal{M}^k \) corresponding to links with \( n \) components are in one to one correspondence with the unordered \( n \)-tuples of conjugacy classes in \( \pi = \pi_1(M) \). In every such space we will fix, once and for all, a link \( CL \) and call it a trivial link. If \( CL \) has \( k \) components which are homotopically trivial, our choice will be such that \( CL = CL^* \coprod U^k \), where \( U^k \) is the standard unlink with \( k \) components in a small ball neighborhood disjoint from \( CL^* \). Let \( CL^* \) be the set of trivial links and \( CL^* \) be the set of trivial links with homotopically non-trivial components.

Notice that when \( M \) is simply connected there is a natural choice of trivial links. Namely, one chooses each \( CL \) to be the unlink \( U^k \). Thus \( CL^* = \emptyset \) in this case.

**Theorem 4.2.1.** Let \( M, \mathcal{L} \) and \( CL^* \) be as above. There exists a unique sequence of complex valued link invariants \( v^0_n, v^1_n, \ldots, v^m_n, \ldots \), with given values on the links in
CL* ∪ {U}, such that if we define a formal power series

\[ J_{\{M,n\}}(L) = \sum_{m=0}^{\infty} v^m_n(L)x^m \]

for \( L \in L \) then

\[ t^{\frac{n+1}{2}}J_{\{M,n\}}(L+) - t^{-\frac{n+1}{2}}J_{\{M,n\}}(L-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_{\{M,n\}}(L_o) \quad (4.2.1) \]

where \( t = e^x = 1 + x + \frac{x^2}{2} + \ldots \)

**Notation:** To simplify our notation, and throughout this proof, we will write \( J_n \) instead of \( J_{\{M,n\}} \).

**Proof.** By our assumption, the values \( v_0^0(CL^*), v_1^1(CL^*), \ldots, v_m^m(CL^*) \) are given for every \( CL^* \in CL^* \). Hence, we can form the power series \( J_n(CL^*) \). Also, we may form \( J_n(U) \) using the given values \( v_m^m(U) \)’s.

Guided by (4.1.3) we define

\[ J_n(CL \coprod U) = u_n(t) J_n(CL) \quad (4.2.2) \]

where

\[ u_n(t) = \frac{t^{\frac{n+1}{2}} - t^{-\frac{n+1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \quad (4.2.3) \]

Thus, \( J_n \) has been defined on all trivial links.

We define the link invariant \( v_n^0 \) by

\[ v_n^0(L) = v_n^0(CL), \]

where \( CL \) is the trivial link homotopic to \( L \).

Inductively, suppose that the invariants \( v_n^0, v_n^1, \ldots, v_n^{m-1} \) have been defined such that if we let

\[ J_{n}^{(m-1)}(L) = \sum_{i=1}^{m-1} v_i^m(L)x^i, \]

then

\[ J_n(L \coprod U) = u_n(t) J_n(L) \mod x^m, \quad (4.2.4) \]

\[ t^{\frac{n+1}{2}}J_n^{(m-1)}(L+) - t^{-\frac{n+1}{2}}J_n^{(m-1)}(L-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_n^{(m-1)}(L_o) \mod x^m. \quad (4.2.5) \]

¿From (4.2.5) we obtain

\[ J_n^{(m-1)}(L+) - J_n^{(m-1)}(L-) = \]

\[ (t^{-(n+1)} - 1)J_n^{(m-1)}(L_-) + t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_n^{(m-1)}(L_o) \mod x^m \]

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which leads us to define
\[ J_n^{(m)}(L_x) := (t^{-(n+1)} - 1)J_n^{(m-1)}(L_-) + t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_n^{(m-1)}(L_o) \mod x^{m+1}. \] (4.2.6)

This is a polynomial of degree \( m \), with trivial constant coefficient. The coefficients of \( x^i \), \( i = 1, 2, \ldots, m-1 \), in this polynomial are singular link invariants derived from \( v_n^i \), \( i = 1, 2, \ldots, m-1 \). However, the coefficient of \( x^m \) is a new singular link invariant. We are going to prove that it is derived from a knot invariant by using Theorem 3.1.2. For that we need to check that the local integrability conditions (1) and (2) are satisfied. It is enough to check them modulo \( x^{m+1} \). In what follows the symbol \( \equiv \) will denote calculation modulo \( x^{m+1} \).

To check (1), suppose we start with a singular link \( L^1 \in \mathcal{L}^1 \) that contains a kink \( \times \circ \). Let \( L \) be the link obtained by resolving the double point of \( L^1 \) and let \( U \) be unknot in \( M \). Then we have
\[
J_n^{(m)}(L^1) \equiv (t^{-(n+1)} - 1)J_n^{(m-1)}(L) + t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_n^{(m-1)}(L \bigcup U)
\]
\[
\equiv [(t^{-(n+1)} - 1) + t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) u_n(t)]J_n^{(m-1)}(L)
\]
\[
\equiv 0
\]

To check (2), we calculate
\[
J_n^{(m)}(L_{\times +}) - J_n^{(m)}(L_{\times -})
\]
\[
\equiv (t^{-(n+1)} - 1)J_n^{(m-1)}(L_{\times +}) + t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_n^{(m-1)}(L_{o+})
\]
\[
- (t^{-(n+1)} - 1)J_n^{(m-1)}(L_{\times -}) - t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_n^{(m-1)}(L_{o-})
\]
\[
\equiv (t^{-(n+1)} - 1) [(t^{-(n+1)} - 1)J_n^{(m-1)}(L_{\times +}) + t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_n^{(m-1)}(L_{o+}) + o(x^m)]
\]
\[
+ t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) [(t^{-(n+1)} - 1)J_n^{(m-1)}(L_{\times -}) + t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})J_n^{(m-1)}(L_{o-}) + o(x^m)]
\]
\[
\equiv (t^{-(n+1)} - 1)^2 J_n^{(m-1)}(L_{\times +}) + [t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})]^2 J_n^{(m-1)}(L_{o+})
\]
\[
+ (t^{-(n+1)} - 1) t^{-\frac{n+1}{2}}(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) [J_n^{(m-1)}(L_{\times -}) + J_n^{(m-1)}(L_{o-})].
\]

Since the result is symmetric with respect to the two double points we deduce that
\[
J_n^{(m)}(L_{\times +}) - J_n^{(m)}(L_{\times -}) \equiv J_n^{(m)}(L_{+\times}) - J_n^{(m)}(L_{-\times})
\]

Thus, the singular link invariant defined in (4.2.6) is induced by a link invariant. Using the given values \( \{v_n^m(CL) : CL \in \mathcal{C}L\} \), we can define a link invariant \( v_n^m \), such that if we let
\[
J_n^{(m)}(L) = \sum_{i=1}^{m} v_n^m(L) x^i
\]

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we have

\[ J_n^{(m)}(L_+) - J_n^{(m)}(L_-) = J_n^{(m)}(L_\times) \]

for \( L \in \mathcal{L} \) and \( L_\times \in \mathcal{L}^1 \). Therefore the invariant \( J_n^{(m)} \) defined in this way satisfies the inductive hypothesis (4.2.5).

Now, a straightforward calculation shows that

\[ J_n^{(m)}(L_\times \coprod U) \equiv u_n(t) J_n^{(m)}(L_\times) \]

which together with (4.2.2) shows that \( J_n^{(m)} \) satisfies the inductive hypothesis (4.2.4).

To finish our proof we need to show uniqueness. Inductively, we assume that \( v_n^0, v_n^1, \ldots, v_n^{m-1} \) are uniquely determined by (4.2.1) and their values on \( \mathcal{CL} \), for every \( n \in \mathbb{Z} \). Then, the conclusion for \( v_n^m \) follows from the fact that

\[ v_n^m(L) = v_n^m(CL) + \sum_{i=1}^s \pm v_n^m(L_i) \]

where \( L_1, \ldots, L_s \) are singular links in \( \mathcal{L}^1 \), and \( CL \) is the representative of \( L \) in \( \mathcal{CL} \).

To illustrate how the power series \( J_n = J_{\{M,n\}} \) depend on the choice of \( \mathcal{CL} \), let us restrict ourselves to the case of knots. Let \( \mathcal{C} \) denote the set of homotopy classes of loops in \( M \). Then \( \mathcal{CK} \) will contain exactly one knot, say \( K_C \), from every \( C \in \mathcal{C} \). Moreover, we suppose that the initial values of \( J_n \) are all equal to 1. To stress out the dependence on \( K_C \), let

\[ J_n^{K_C}(K) := J_n(K) \]

for every \( K \in \mathcal{M}^{K_C}(S^1, M) \).

Let \( \{\tilde{K}_C\}_{C \in \mathcal{C}} \) be a different choice of trivial knots. Then one can see that \( J_n(K) \) is well defined up to a multiplicative constant in the ring of formal power series with complex coefficients. More precisely,

**Proposition 4.2.2.** We have

\[ J_n^{\tilde{K}_C}(K) = J_n^{K_C}(K) J_n^{K_C}(K) \]

for every \( K \in \mathcal{M}^{K_C}(S^1, M) \) and \( c \in \mathcal{C} \).

**Proof.** Follows immediately from the definition. \( \square \)
Remark 4.2.4. Recall that an invariant \( f \), is called of finite type if there exists an integer \( m \), such that the singular link invariant derived from \( f \) is zero on all singular links with more than \( m \) double points. One can see that the invariants \( v_0^n, v_1^n, \ldots, v_m^n, \ldots \) constructed above are of finite type. It would be interesting to find a direct relation of the invariants constructed here with these coming from the \( SU(N) \)-perturbative Chern-Simons theory ([1,19]).

§5. The convergence of the power series

The purpose of this section is to show that, with an appropriate choice of the initial values, we have

\[
J_{\{M,n\}}(L) = J_{\{S^3,n\}}(L')
\]

for a certain link \( L' \) in \( S^3 \). See Theorem 5.2.3. Since \( J_{\{S^3,n\}} \) is a Laurent polynomial in

\[
v = t^{-\frac{n+1}{2}} \quad \text{and} \quad z = t^{\frac{1}{2}} - t^{-\frac{1}{2}},
\]

so does \( J_{\{M,n\}} \). Hence, we obtain an actual generalization of the 2-variable Jones-polynomial for links in atoroidal and Seifert fibered rational homology 3-spheres.

5.1. Some special cases

The reader might have noticed (see also [15]) that for \( M = S^3 \), Theorem 4.2.1 gives an intrinsically 3-dimensional proof of the existence of the HOMFLY “series”. To see that it converges to a Laurent polynomial in \( v \) and \( z \), though, we have to use some special features of \( S^3 \). Namely, every link in \( S^3 \) has a plane projection and when (4.2.1) can be applied to simplify that plane projection. So, if the initial value, \( J_{\{S^3,n\}}(U) \) in this case, is chosen to be a Laurent polynomial in \( v \) and \( z \), \( J_{\{S^3,n\}}(L) \) will be a Laurent polynomial for every \( L \). Thus, the main question we are facing now is the following.

**Question 5.1.1.** Do the power series \( J_{\{M,n\}}(t) \) converge to Laurent polynomials or rational functions in other manifolds \( M \), besides \( S^3 \)?

Theorem 5.2.3 gives a partial answer to this question. But we feel that it is still not clear how the convergence problem is related with the topology of the underlying manifold.

Let us see a special case first. Let us define

\[
\nabla_L = \nabla_L(t) := J_{\{M,-1\}}(L)(t^2)
\]
By (4.2.1) we see that $\nabla_L$ satisfies the following skein relation

$$\nabla_{L_+} - \nabla_{L_-} = (t - t^{-1})\nabla_{L_0}$$  \hspace{1cm} (5.1.1)

By (5.1.1) and the results in [4], we see that, possibly after some normalization, $\nabla_L$ is the Conway potential function, which is known to be a rational function in $t$. More precisely, it is known that

$$\nabla_L(t) = \frac{\Delta_L(t^2)}{t - t^{-1}}$$  \hspace{1cm} (5.1.2)

if $L$ is a knot, and

$$\nabla_L(t) = \Delta_L(t^2, \ldots, t^2)$$  \hspace{1cm} (5.1.3)

if $L$ is a link. Here $\Delta_L$ denotes the Alexander polynomial of $L$, and the number of variable entries in the right hand side of (5.1.3) is the same as the number of components of $L$. For more details see [4].

5.2. The existence of the 2-variable Jones polynomial

In this paragraph we will use the fact that $M$ is obtained by surgering $S^3$ along a link ([14]) together with the discussion in the beginning of 5.1 to show that the $J_{\{M,n\}}(t)$’s can be suitably normalized so that they converge to Laurent polynomials in $v$ and $z$, giving thereby a positive answer to Question 5.1.1. To simplify our notation we will write $J_n$ instead of $J_{\{S^3,n\}}$. We need the following

**Lemma 5.2.1.** There exists links $L^1 \subset M$ and $L^0 \subset S^3$, such that $M \setminus L^1$ is homeomorphic to $S^3 \setminus L^0$. We will write $M \setminus L^1 \cong S^3 \setminus L^0$.

**Proof.** There exists a framed link $L \subset S^3$ such that the result, say $\chi(L)$, of surgery of $S^3$ along $L$ is homeomorphic to $M$. Hence, there exists a homeomorphism $h_0 : \chi(L) \to M$. Then $L^0 = L$ and $L^1 = h_0(L)$ are as desired.  \hspace{1cm} $\blacksquare$

Let us fix $L^1$ and $L^0$ as in the lemma above and let $M_1 = M \setminus L^1$. Let $L \subset M$ be a link. After isotopy we may assume that $L \cap L^1 = \emptyset$ and hence $L \subset M_1$. Moreover, we have the following lemma whose proof is quite obvious.

**Lemma 5.2.2.** Let $L_1, L_2 \subset M^1$ be two links which are homotopic to each other in $M$. Then there exist a link $\tilde{L}$ in $M_1$ such that

1) $\tilde{L}$ is isotopic to $L_2$ in $M$; and
2) $\tilde{L}$ is homotopic to $L_1$ in $M_1$.

Now, let us fix a homeomorphism $h : M_1 \to S^3 \setminus L^0$. Suppose that we have chosen every $CL^*$ to be in $M^1$. By Theorem 4.2.1, there exists a unique sequence
of numerical link invariants $v_{\{M,n\}}^0, v_{\{M,n\}}^1, \ldots, v_{\{M,n\}}^m, \ldots$ such that the formal power series

$$J_{\{M,n\}}(L) = \sum_{m=0}^{\infty} v_{\{M,n\}}^m(L)x^m$$

has the following properties:

1) $J_{\{M,n\}}(CL^*) = J_n(h(CL^*))$ for every $CL^* \in CL^*$;
2) $J_{\{M,n\}}(U) = 1$; and
3) $J_{\{M,n\}}$ satisfies the skein relation (4.2.1).

Here $J_n = J_n(t) = \sum_{m=1}^{\infty} v_{n}^m x^m$ is the power series obtained from the polynomial $J_n(t)$ by the substituting $t = e^x$. We have,

**Theorem 5.2.3.** For every link $L \in \mathcal{L}(M)$ and $n \in \mathbb{Z}$, we have

$$J_{\{M,n\}}(L) = J_n(h(L))$$

Thus, since the power series $J_n$ converges to a Laurent polynomial in

$$v = t^{-\frac{n+i}{2}} \quad \text{and} \quad z = t^{\frac{1}{2}} - t^{-\frac{1}{2}},$$

so does $J_{\{M,n\}}$.

**Proof.** Let us assume inductively that the invariants $v_{\{M,n\}}^0, v_{\{M,n\}}^1, \ldots, v_{\{M,n\}}^{m-1}$ satisfy

$$v_{\{M,n\}}^i(L) = v_{n}^i(h(L)) \quad (5.2.1)$$

for every $L \in M^1 \subset M$ and $i = 1, \ldots, m - 1$. We will show that (5.2.1) is true for $i = m$.

By the assumption, we have

$$J_{\{M,n\}}(CL) = J_n(h(CL))$$

for every $CL \in CL$. Therefore,

$$J_{\{M,n\}}^m(CL) = v_{n}^m(h(CL)) \quad (5.2.2)$$

for every $CL \in CL$. Now, let $L \subset M^1 \subset M$ be a link. By our assumption, $L$ is homotopic to some $CL \in CL$. By Lemma 5.2.2 we may assume that $L$ is homotopic to $CL$ in $M^1$. Hence (see the proof of Theorem 4.2.1) there exist singular links $L^1, \ldots, L^s$ in $M^1$ such that

$$v_{\{M,n\}}^m(L) = v_{\{M,n\}}^m(CL) + \sum_{i=1}^{s} \pm v_{\{M,n\}}^m(L^i) \quad (5.2.3)$$
And we also have

\[ v^m_n(h(L)) = v^m_n(h(CL)) + \sum_{i=1}^{s} \pm v^m_n(h(L^i)) \quad (5.2.4) \]

Let \( L^i_- \) (respectively, \( L^i_o \)) denote the link obtained from \( L^i \) by resolving its double point negatively (respectively surgering the double point in a way consistent with the orientation). They are links in \( M^1 \). By (4.2.6) we see that \( v^m_{\{M,n\}}(L^i) \) is a linear combination of \( v^j_{\{M,n\}}(L^i_-) \)'s and \( v^j_{\{M,n\}}(L^i_o) \)'s with \( j = 0, \ldots, m - 1 \). Hence, by the induction hypothesis we have,

\[ v^j_{\{M,n\}}(L^i_-) = v^j_n(h(L^i_-)) \]

and

\[ v^j_{\{M,n\}}(L^i_o) = v^j_n(h(L^i_o)) \]

for every \( j = 0, \ldots, m - 1 \). One can see that \( h(L^i), h(L^i_-) \) and \( h(L^i_o) \) are related (in \( S^3 \)) in the same way that \( L^i, L^i_- \) and \( L^i_o \) are related in \( M^1 \). Hence, we obtain

\[ v^m_{\{M,n\}}(L^i) = v^m_n(h(L^i)) \quad (5.2.5) \]

Now, the desired conclusion follows immediately from (5.2.2) to (5.2.5). \( \blacksquare \)

**Proof of Theorems A and B.** Combine Theorem 4.2.1 and Theorem 5.2.3, to obtain Theorem A. Theorems B will follow immediately from Theorem A since \( CL^* = \emptyset \) when \( \pi_1(M) = 1 \). \( \blacksquare \)

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