Distillability via protocols respecting the positivity of partial transpose

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(March 31, 2022)

We show that all quantum states that do not have a positive partial transpose are distillable via channels, which preserve the positivity of the partial transpose. The question whether NPT bound entanglement exist is therefore closely related to the connection between the set of separable superoperators and PPT-preserving maps.

I. INTRODUCTION

One of the main tasks of quantum information theory is the systematic investigation of quantum entanglement, which is one of the key ingredients in quantum computation and quantum information processing. In spite of considerable research efforts, however, there are still many aspects of entanglement which are not fully understood. This is not only true for the quantitative theory (the explicit computation or at least the estimation of entanglement measures) but even for qualitative features.

These qualitative features are best explained by looking at the history of the problem. In 1989 [1] it was a new realization that there is a proper gap between obviously entangled states (those violating some Bell inequality) and the obviously non-entangled states, which are now called separable. The next step, made by Sandu Popescu in 1995 [2] was the striking result that this gap could be narrowed by distillation: By local filtering and classical communication one could sometimes get highly entangled states even from states not violating any Bell inequality. For a while it was everybody’s favourite conjecture that there should be no more gap, i.e., that all non-separable states should be distillable. This folk conjecture was shattered in 1998 by counterexamples [3], which are now called bound entangled states. The way these examples were established was by showing that the property of a density operator of having a positive partial transpose, i.e., of being a ppt-state, does not change under distillation. Therefore any non-separable state with positive partial transpose has to be “bound entangled”. The obvious white spot on the entanglement map is then:

Are these all bound entangled states, or are there undistillable states, whose partial transpose is not positive?

There were two recent papers [4–6] presenting some evidence for the existence of non-ppt bound entangled states. However, the matter is not decided [7], and in view of the rapidly growing dimensions of the Hilbert spaces involved, numerical evidence can be treacherous in this field. The latest development was an attempt by Pawel Horodecki [8] at showing the existence of a gap between ppt-states and distillable states, using a stronger protocol [10] of distillation. The attempt failed due to an error in another paper, but it remained unclear whether the idea could be made to work. What we show in the present note is that it cannot work: using the same distillation protocol [10], every non-ppt-state becomes distillable.

The rather subtle dependence of distillability on “protocols” requires some explanation. Typically a protocol fixes the amount of classical communication allowed to Alice and Bob in the process. Thus we may distinguish distillation with no communication allowed or with one-way or two-way communication. Even stronger protocols than two-way communication protocols exist: these are defined by requiring only a subset of the properties which are true for all two-way distillation procedures. One example is the requirement that the overall operation can be written as a sum of tensor products of local operations (“separable superoperator”). Another such property, which is the one we consider in this paper following Rains [11], is that operators with positive partial transpose are again taken to such operators. An example of such a ppt-preserving protocol is the case where Alice and Bob share a ppt bound entangled state and use a protocol consisting of local operations and classical communication (LOCC) [12].

Obviously, the weaker the requirements on the admissible transformations, the larger the set of distillable states. However, when we do not care about rates of distillation, the dependence on the protocol is not as strong as one might think. For example, distillability with two-way communication and with separable superoperators are known to be equivalent [3]. Moreover the stronger protocols have the virtue of being much more manageable and more easily parametrized than two-way communication processes, which may involve an arbitrarily large number of exchanges of classical information.

Therefore it seemed quite reasonable to study the problem of a proper gap between ppt states and distillable states under this “ppt preserving” protocol. Moreover,
since a proper gap is the currently favored conjecture, it was reasonable to expect a gap even with such a protocol. The main result of this note is, however, that the gap disappears, if we allow such a strong protocol. Unfortunately, this does not provide conclusive evidence about the gap for weaker protocols.

II. DISTILLATION VIA PPT-PRESERVING CHANNELS

For the sake of completeness we begin by recapitulating the result of Rains [10] for the fidelity of distillation via PPT-preserving channels. Let $\rho$ be a density operator corresponding to a quantum state on $\mathcal{C}^d \otimes \mathcal{C}^d$ and $\rho \mapsto T(\rho)$ a trace-preserving positive map, such that $\sigma \geq 0$ implies $T(\sigma_T)_{T_2} \geq 0$, where the superscript $T_2$ denotes the partial transposition with respect to a given basis. Let us further write $P_m = |\psi_m\rangle\langle\psi_m|$ for the projector onto the maximally entangled state in $m \times m$ dimensions, i.e., $|\psi_m\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^m |i\rangle \otimes |i\rangle$. Theorem 3.1 of [10] then reads:

**Lemma 1** The maximal fidelity of distillation via PPT and trace preserving positive maps with respect to the $m$-dimensional maximally entangled state is given by

$$F_m(\rho) := \max_{T} \text{tr}[P_m T(\rho)] = \max_{A} \text{tr}[\rho A],$$

where the maximum on the right side is taken over all hermitian operators $A$ satisfying

$$0 \leq A \leq 1 \quad \text{and} \quad -1 \leq m A_{T_2} \leq 1. \quad (2)$$

**Proof:** First note that since every unitary of the form $U \otimes \overline{U}$ commutes with $P_m$ it suffices to consider trace preserving positive maps mapping into the set of isotropic states, i.e., states, which are obtained by averaging over all these unitaries $U$:

$$T(\rho) = \text{tr}[\rho B](1 - P_m) + \text{tr}[\rho A]P_m. \quad (3)$$

The coefficients in Eq. (3) have to be linear functionals of $\rho$ so that we are free to write them as traces, and $T(\rho)$ being again a proper state requires that $0 \leq A, B \leq 1$, and $(m^2 - 1)B + A = 1$. In order to obtain a PPT-preserving channel we additionally have to demand that $\sigma \geq 0$ implies that

$$T(\sigma_{T_2})_{T_2} = \text{tr}[\sigma B_{T_2}](1 - \frac{1}{m} F) + \text{tr}[\sigma A_{T_2}] \frac{1}{m} F \geq 0, \quad (4)$$

where $F$ denotes the flip operator, i.e., $F |\phi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\phi\rangle$. Inequality (4) is satisfied if the absolute value of the coefficient of the flip operator is less or equal than the weight of the identity operator:

$$\pm \frac{1}{m} \text{tr}\left[\sigma \left(\frac{A_{T_2} - 1}{m^2 - 1} + A_{T_2}\right)\right] \leq \text{tr}\left[\sigma \frac{1 - A_{T_2}}{m^2 - 1}\right]. \quad (5)$$

Since this inequality has to hold for all positive operators $\sigma$ we can reformulate it as an operator inequality which is in turn equivalent to $-1 \leq m A_{T_2} \leq 1$.

Hence, there is a one-to-one correspondence between PPT-preserving maps $T$ of the form (3) and the respective hermitian operators $A$ satisfying the constraints specified in Lemma 1 given by $\text{tr}[P_m T(\rho)] = \text{tr}[\rho A]$. \[ \square \]

In fact positive maps of the form (3) are even completely positive, i.e. Lemma 1 holds also for ppt-preserving channels as can easily be seen by writing down a Kraus decomposition:

$$T(\rho) = \text{tr}[\rho B](1 - P_m) + \text{tr}[\rho A]P_m = (1 - P_m) \text{tr}\left[\sqrt{B \rho \sqrt{B}}\right] (1 - P_m) + P_m \text{tr}\left[\sqrt{A \rho \sqrt{A}}\right] P_m.$$ \[ \square \]

This however is a special property of positive maps of the form (3). Indeed positivity and ppt-preservation do not imply complete positivity in general, a counterexample being the transposition.

Now we can utilize Lemma 1 in order to prove the following

**Theorem 1** Any NPT state, i.e., state with not positive partial transpose, is distillable via PPT-preserving channels.

**Proof:** We recall that a state is known to be distillable via standard LOCC distillation protocols if $\text{tr}[\rho P_m] > \frac{1}{m}$ [13]. The task is therefore to find an appropriate operator $A$ such that $\text{tr}[\rho A] > \frac{1}{m}$.

Let $P_{\text{neg}}$ be the projector onto the negative eigenspace of $\rho_{T_2}$. We choose $A$ to be of the form

$$A = \frac{1}{m} (1 - \epsilon P_{\text{neg}}), \quad 0 < \epsilon \leq \min \left\{ 2, \frac{\|P_{\text{neg}}\|_{\infty}}{1} \right\}, \quad (6)$$

where $\| \cdot \|_{\infty}$ denotes the operator norm. Now we have to check, whether $A$ satisfies the constraints in Lemma 1.

Positivity of the parameter $\epsilon$ implies $m A_{T_2} \leq 1$. To ensure $A \leq 1$ it is sufficient that $\epsilon \leq (m - 1)\|P_{\text{neg}}\|_{\infty}$ but $0 \leq A$ requires the even stronger condition $\epsilon \leq \|P_{\text{neg}}\|_{\infty}$. Moreover, $m A_{T_2} \geq 1$ is equivalent to $\epsilon \leq 2$, which shows that Eq. (6) indeed defines an admissible operator $A$. With the above $A$ we obtain

$$\text{tr}[\rho A] = \frac{1 + \epsilon N(\rho)}{m}, \quad (7)$$

where $N(\rho)$ is the negativity [14], which is just the sum over the absolute values of the negative eigenvalues of $\rho_{T_2}$. Since the state has at least one such negative eigenvalue by assumption, we end up with a fidelity larger than $\frac{1}{m}$, which completes our proof. \[ \square \]

Of course one may further evaluate Eq. (6) for more specific states. Let us for instance consider states commuting with all unitaries of the form $U \otimes \overline{U}$, which can be written as
\[ \rho(p) = (1-p) \frac{P_+}{r_+} + p \frac{P_-}{r_-}, \quad 0 \leq p \leq 1, \]

where \( P_+ (P_-) \) is the projector onto the symmetric (antisymmetric) subspace of \( \mathbf{C}^d \otimes \mathbf{C}^d \) and \( r_\pm = \text{tr}(P_\pm) = \frac{d^2 \pm d}{2} \) are the respective dimensions. Evaluating Eq. (8) for these states \( (\epsilon = 2) \) then leads to

\[ \text{tr}[A \rho(p)] = \frac{||\rho(p)T_2||_1}{m} = \frac{d - 2 + 4p}{md}. \]

In fact this turns out to be already the maximal value for \( F_m(\rho(p)) \). This can easily be seen by decomposing the partial transpose of the state into its positive and negative part, i.e., \( \rho^{T_2} = \rho_+ - \rho_- \). Then

\[ \text{tr}(\rho A) = \text{tr}(A^{T_2} \rho^{T_2}) = \text{tr}(A^{T_2}(\rho_+ - \rho_-)) \leq \frac{1}{m} \text{tr}(\rho_+ - \rho_-) = \frac{1}{m} ||\rho^{T_2}||_1, \]

where the estimate is due to the constraint \( m||A||_1 \leq 1 \).

In fact this bound for the maximal fidelity can always be reached for states with \( ||P_{neg}^2||_\infty \leq \frac{1}{2} \).

### III. CONCLUSION

We have argued that enlarging the set of distillation protocols to PPT-preserving channels immediately implies that any NPT state can be distilled. Since we know that a state can be distilled via proper LOCC operations iff \( \text{tr}[P_m S(\rho)] > \frac{1}{m} \) for some separable superoperator \( S \) [13], this raises the question about the connection between the sets of separable superoperators and PPT-preserving channels. It is obvious that any separable superoperator is PPT-preserving but we do not know yet any efficient method for deciding whether a given operator \( A \) from Lemma 1 corresponds to a separable superoperator.

There is a standard argument telling us that NPT bound entangled states exist iff there exist undistillable entangled states of the form special \( U \otimes U \)-invariant form [8, 13]. So the question about the existence of NPT bound entangled states becomes to decide whether PPT-preserving channels that distill \( U \otimes U \)-invariant states near the separable boundary can be realized as separable superoperators or not.

Moreover, the above discussion raises the question whether it suffices to use LOCC operations and PPT bound entangled states as an additional resource in order to distill all NPT states.

Another interesting feature of the distillation we discussed is that we only needed a single copy of the given bipartite state, and not a tensor product of many identically prepared ones. This raises the question whether distillability under LOCC protocols can also be decided at the single copy level. All examples known to us would be consistent with this.

### ACKNOWLEDGEMENT

The authors would like to thank Barbara Terhal for interesting discussions and for pointing out reference [1]. Funding by the European Union project EQUIP (contract IST-1999-11053) and financial support from the DFG (Bonn) is gratefully acknowledged.

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