Two approaches for Helmholtz equation: generalized Darboux Transformation and the method of $\bar{\partial}$-problem

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Abstract

Two approaches to solution of the two-dimensional Helmholtz equation with a "wave number" are proposed. The results can be applied both in numerical areas of physics and in the theory of nonlinear equations. The first approach is based on the requirement of the covariance of equation under the generalized Darboux transformation (Moutard transformation). It allows to construct a new solution of equation, using a given initial solution of the equation. Simultaneously we obtain the "dressing" relation for the "wave number". The simplest examples of the approach are considered in detail. In the second approach the Green-Cauchy formula (the $\bar{\partial}$-method) is applied to reduce the solution of the equation to the solution of a system of singular integral equations.

1. INTRODUCTION

An intensive development of the methods in the theory of nonlinear integrable equations stimulated as well the development of the so-called dressing methods for the solution of a series of linear equations having the important physical applications. One of the most popular and effective methods is the method of Darboux Transformation (DT) [1], which gives the exact solutions of different one-dimensional equations. The situation for the multidimensional equations is more complicate. In [2] the multidimensional variant of DT, which depends on spatial derivatives, was considered, but such approach is rather complicate and cumbersome.

In paper [3] the method of integration of the non-stationary Schrodinger and Fokker-Plank equations was proposed with using of the Darboux-like anzats and introducing of some functional-differential equation. It appeared, that in some sense this approach is similar to the so-called Moutard transformation (see, for example, [4]) for the variable real "wave number" (potential). In this case the functional-differential equation of the second order is replaced by the system of two differential equations of the first order for functional, included in Darboux’s anzats. For complex "wave number" the situation is more complicate - the solution can be found in some simple cases only.

In Section 2 of this paper the idea of a Darboux-type anzats - Moutard transformation is introduced in the example of the two-dimensional Helmholtz equation. The main advantage of such approach is independence of the procedure on the spatial derivatives, and hence it can straightforwardly extended onto higher dimensionality of space.

In Section 3 the simplest classical problem of diffraction of the electromagnetic waves on a half-plane is considered. We reformulate this problem in terms of a so-called method $\bar{\partial}$-problem. The idea of such approach (in another formulation of the problem) belongs to V.D.Lipovskii [5]. The powerful and effective possibilities of this method for integration of multidimensional systems such as Kadomtsev-Petviashvili-2 equation, Devey-Stewartson-2 system and so on were demonstrated (see, for example, [6]-[7]). Apparentely, up to now this approach was not applied in the theory of diffraction. Similar to the Wiener-Hopf method, it allows to formulate the initial boundary problem in terms of the integral equations, which admit the explicit exact solutions.
2. The generalized Darboux Transformation

a). The two-dimensional Helmholtz’s equation belongs to a wide class of the equations of the following form (see, for example, [8]):

$$\Psi_{pq} = V(p,q)\Psi,$$

where \(V = V(p,q)\) is the given, in general case, complex-valued function. Here and below we use the notations: \(\Psi_q = \partial \Psi / \partial q, \ldots\)

In the simplest particular case: \(p = x, q = y, V(x,y) \equiv 1\), this equation has the general solution

$$\Psi(x,y) = \int_0^x f_1(s)J_0(2i\sqrt{y(x-s)})ds + \int_0^y f_2(s)J_0(2i\sqrt{x(y-s)})ds + [f_1(0) + f_2(0)]J_0(2i\sqrt{xy}),$$

where \(J_0(.)\) is the Bessel’s function, and \(f_1, f_2\) are some differentiable functions.

The class of such equations is important both for the differential geometry and for a wide variety of the physical problems. Indeed, for example, for \(p = x + t, q = x - t\) we have the heterogeneous wave equation:

$$\Psi_{tt} - \Psi_{xx} = V(x,t)\Psi,$$

and for \(p = z = x + iy, q = \bar{p}\) we obtain the Helmholtz equation itself:

$$\triangle \Psi = V(x,y)\Psi.$$

We shall use it also in the complex form \((\partial_z = (1/2)(\partial_x - i\partial_y), \partial_{\bar{z}} = (1/2)(\partial_x + i\partial_y))\)

$$\Psi_{z\bar{z}} = \frac{1}{4}V(z, \bar{z})\Psi.$$

The equation (5) arises in the Quantum Mechanics (two-dimensional stationary Schrodinger equation), theory of diffraction of electromagnetic waves, an acoustic diffraction and other problems. From the nonlinear equations point of view the interest to (4) is caused by his role as the associated linear system for the well-known nonlinear completely integrable Novikov-Veselov equation [9].

It is not difficult to check by direct calculation that if \(\Phi = \Phi(z, \bar{z}), \chi = \chi(z, \bar{z})\) are two solutions of equation (4), then two important relations are true:

$$(\Phi_z\chi)_{\bar{z}} = (\Phi_{\bar{z}}\chi)_z, \quad (\Phi_z\chi - \Phi_{\bar{z}}\chi)_z = (\Phi_{\chi_{\bar{z}}} - \Phi_{\chi_z})_z.$$ 

It follows from the second of them, that the integral

$$\omega(\Phi, \chi) = \int_{(z_0, \bar{z}_0)}^{(z, \bar{z})} (\Phi_z\chi - \chi_z\Phi)dz + (\chi_z\Phi - \chi\Phi_z)d\bar{z},$$

does not depend on the integration path.

b). We shall proceed to the analysis of the equation (5) from the generalized Darboux Transformation point of view. Let \(\Psi, \Psi_1\) are two linearly independent solutions of (5). Following [1, 3], we assume that

$$\tilde{\Psi}(z, \bar{z}) = \Omega(\Psi, \Psi_1)$$

...
where $\Omega = \Omega(\Psi, \Psi_1) = \Omega(z, \bar{z})$ is the functional, given on direct product of two copies of spaces of "wave functions". After substitution of the anzats (8) in equation (5), and requiring covariance of this equation under transformation $\Psi \rightarrow \Psi$, $U \rightarrow \bar{U}$, we obtain the equation, which do not contain already the potential:

$$\Omega_{zz} - (\ln \Psi_1)_z \Omega_z - (\ln \Psi_1)_z \Omega_{\bar{z}} + q_1 \Omega = 0.$$  

Here $q_1 = q_1(z, \bar{z})$ is a complex-valued function ("parameter" of separation of variables), introduced in order to obtain the real potential $\tilde{V}(z, \bar{z})$ for the case of $V(z, \bar{z}) = \nu(z, \bar{z})$ (conservative medium). If $V(z, \bar{z}) \neq \tilde{V}(z, \bar{z})$ and $\tilde{V}(z, \bar{z}) \neq \nu(z, \bar{z})$ (nonconservative medium), one may set $q_1 = 0$ in (9).

Equation (9) allows the physical interpretation: it looks like the two-dimensional stationary Schrodinger equation for charged particle in a non-homogenous stationary electromagnetic field.

Actually, this equation can be written as (see, for example, [10]):

$$\Delta \psi - 2i(A \nabla \psi) - (A^2 + \phi - E)\psi = 0,$$

where $A(x, y) = (A_1, A_2)$ is the vector potential, with the gauge condition $\text{div} A = 0$, $\phi = \phi(z, \bar{z})$ is the scalar potential, $E$ is the energy value, and the system of units $\hbar = c = e = 1$, $m = 1$ was used ($m$ is the mass of the particle). In variables $z, \bar{z}$:

$$\psi_{zz} - \frac{1}{4}(iA_1 - A_2)\psi_z - \frac{1}{4}(iA_1 + A_2)\psi_{\bar{z}} - \frac{1}{4}[A^2 + \phi - E]\psi = 0.$$  

Comparing (11) and (9), we see, that analogues of coefficients at $\Omega_z, \Omega_{\bar{z}}$ are expressed as linear combinations of the components of the vector-potentiel, and the analogue of function $q_1$ in terms of the square of its module and of the scalar potential.

Besides, the separation of variables leads to the following dressing relation for potential $\tilde{V}(z, \bar{z})$ (number):

$$\tilde{V}(z, \bar{z}) = V(z, \bar{z}) - 8(\ln \Psi_1)_{z\bar{z}} - 4q_1(z, \bar{z}).$$

In the conservative case the requirement of reality reads:

$$(\arg \Psi_1)_{z\bar{z}} = -4 \Im q_1.$$  

For the double dressing procedure, with

$$\tilde{\Psi}(z, \bar{z}) = \frac{\Omega(\Psi_2[1], \Psi[1])}{\Psi_2[1]},$$

and taking into account (12), we obtain

$$\tilde{V}(z, \bar{z}) = V - 8(\ln \Omega(\Psi_1, \Psi_2))_{z\bar{z}} - 4\{q_1 + q_2[1]\},$$

where $\Psi_2[1]$ is the fixed solution of equation (5) with $\Psi_2[1] = \nu(1)$. In particular, for the $N$-fold iteration of the dressing procedure, one will obtain the relations, expressed in terms of the functions $\Omega(\Psi_i, \Psi_j)$ only [1].

It is clear, that similar to the case of standard DT, the dressing procedure proposed can be multiply iterated. Setting $\Psi = \Psi[0], \tilde{\Psi} = \Psi[1], \ldots, V = V[0], \bar{V} = V[1], \ldots$, we have chains: $\Psi[0] \rightarrow \Psi[1] \rightarrow \ldots \rightarrow \Psi[N] \rightarrow \ldots$, and $V[0] \rightarrow V[1] \rightarrow \ldots \rightarrow V[N] \rightarrow \ldots$. In particular, for the $N$-fold iteration of the dressing procedure, one will obtain the relations, expressed in terms of the functions $\Omega(\Psi_{i}, \Psi_j)$ only [1].
c). Now some simplest examples of realization of the proposed approach will be considered.

1. Let us take \( V(z, \bar{z}) = 0 \) in (5). Then \( \Psi_j(z, \bar{z}) = \psi_j^{(1)}(z) + \psi_j^{(2)}(\bar{z}) \), \( j = 1, 2, \ldots, N \). The relation (12) takes a form:

\[
V[1] = -8[\ln (\psi_1^{(1)}(z) + \psi_1^{(2)}(\bar{z}))]_{z \bar{z}} - 4q_1.
\]

In the case \( V[1] = \tilde{V}[1] \) the following condition

\[
\text{Im} \ q_1 = i \left[ \frac{\psi_2^{(1)}(\bar{z})}{(\psi^{(1)} + \psi^{(2)})^2} - \frac{\psi_1^{(1)}(\bar{z})}{(\psi^{(1)} + \psi^{(2)})^2} \right]
\]

must be satisfied. On the next step we have:

\[
V[2] = V - 8(\ln \Omega(\Psi_1, \Psi_2))_{z \bar{z}} - 4[q_1 + q_2[1]],
\]

and the functional \( \Omega(\Psi_1, \Psi_2) \) satisfies the equation (9), if to replace \( \Psi \to \Psi_1, \Psi_1 \to \Psi_2, q_1 \to q_1, q_2 \to q_2[1] \) in it. It is not difficult to check, that for \( q_1 = q_2[1] = 0 \) this equation is solved by the function \( \Omega(\Psi_1, \Psi_2) = \omega(\Psi_1, \Psi_2) \), where the expression for \( \omega(\Psi_1, \Psi_2) \) is defined in (7). After the simple calculations we obtain:

\[
\Omega(\Psi_1, \Psi_2) = \gamma_0 + 2(\psi_1^{(1)}\psi_2^{(2)} - \psi_2^{(1)}\psi_1^{(2)}) + \int_{(z_0, \bar{z}_0)}(\psi_1^{(1)}\psi_2^{(1)} - \psi_2^{(1)}\psi_1^{(1)})dz + (\psi_2^{(2)}\psi_1^{(2)} - \psi_1^{(2)}\psi_2^{(2)})d\bar{z},
\]

where \( \gamma_0 \) is a complex constant. This expression, without using of the equation (9), was obtained for the first time in [1].

2. Let’s assume in (11) that \( V = V(x, y) = 1 \) at \( -\infty < x < +\infty, y > 0 \). Applying the Fourier transformation in the variable \( x \), we shall construct the solution \( \Psi_j (j = 1, 2, \ldots, N) \), which is bounded in \( x \) and goes to zero at \( y \to \infty \):

\[
\Psi_j(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ikx - \sqrt{k^2 + 1}y} A_j(k).
\]

Here \( A_j(k) \) are some functions (functional parameters), such that \( A_j(k) \in \mathbb{L}_2(-\infty, +\infty) \) in the real case one obtains from (19) \( A_j(k) = \tilde{A}_j(-k) \) (in case of complex \( k \) we assume \( \text{Re} \sqrt{k^2 + 1} > 0 \)). Then

\[
V[1] = 1 - 2\Delta \ln \left\{ \int_{-\infty}^{+\infty} dk e^{ikx - \sqrt{k^2 + 1}y} A_1(k) \right\}.
\]

The analogue of the second relation in (5) yields:

\[
(\psi_x\psi_1 - \psi_{1x}\psi_x)_x = (\psi_{1y}\psi - \psi_1\psi_y)_y.
\]

Therefore the solution \( \Psi[1] \) is determined by a relation (8) at \( \Omega(\Psi, \Psi_1) = \omega(\Psi, \Psi_1) \), where

\[
\omega(\Psi, \Psi_1) = \int_{(x_0, y_0)}^{(x, y)} (\psi_{1y}\psi - \psi_1\psi_y)dx + (\psi_x\psi_1 - \psi_{1x}\psi)dy.
\]

On the next step we obtain:
In the particular case of 

\[ A_j(k) = c_j \delta(k - p_j) + d_j \delta(k - r_j), \]

with \( \delta(.) \) - the Dirac delta- function, \( c_j, d_j, p_j, r_j \in \mathbb{R} \) - constants, \( j = 1, 2, \ldots, N, \ldots \), taking (due to linearity of the equation) the real part of the solution \( \Psi_j \) we shall have:

\[ \Psi_j(x, y) = c_j e^{-\sqrt{p_j^2 + 1}y} \cos p_j x + d_j e^{-\sqrt{r_j^2 + 1}y} \cos r_j x. \]

Thus, the dressing formula (15) gives:

\[ V[1] = 1 - 2\Delta \ln \left( c_1 e^{-\sqrt{p_1^2 + 1}y} \cos p_1 x + d_1 e^{-\sqrt{r_1^2 + 1}y} \cos r_1 x \right). \]

Completely similarly, it is possible to obtain from (15), (24) the explicit representation for \( V[2] \), though rather cumbersome.

3. Let us consider a case, when the initial solution is determined by the Coulomb potential, i.e. in (4) we assume that \( V(x, y) = (\sqrt{x^2 + y^2})^{-1}, x \in (-\infty, \infty), y \in (-\infty, \infty). \) Solving this equation in the polar coordinates and coming back to the cartesian ones, (with requirement that the solution is finite at \( x = y = 0 \)), we obtain (11):

\[ \Psi_m(x, y) \sim e^{im \arctan \frac{y}{x}} J_\left( \frac{2i(x^2 + y^2)^{1/4}}{m} \right), \quad m = 1, 2, \ldots, N, \ldots, \]

where \( J_\nu(.) \) is the Bessel function. Then for real potential we find:

\[ V[1] = \frac{1}{\sqrt{x^2 + y^2}} - 2\Delta \ln \left\{ \Re \left[ e^{im \arctan \frac{y}{x}} J_\left( \frac{2i(x^2 + y^2)^{1/4}}{m} \right) \right] \right\}. \]

It is clear now, that already on a first step we obtain the potential which is not amenable for separation of variables. Hence, all subsequent functions of the chain \( \Psi[k], V[k] \) obey this property as well.

Potentials of the form (26) and (28), and of the corresponding chains, are integrable, i.e. the equations, produced by them, have the exact solutions.

In two following examples we consider a case \( q_1 \neq 0 \) when the equation (9) allows the simplest solutions.

4. Let \( \Omega \in \mathbb{D} \in \mathbb{C} \). From (6) have:

\[ (\ln \Omega)_{\bar{z}} = \frac{q_1}{(\ln \Psi_1)_{\bar{z}}} \quad \text{at the condition, that} \quad \left( \frac{q_1}{(\ln \Psi_1)_{\bar{z}}} \right) = 0. \]

To solve this equation we use the formula of \( \bar{\partial} \)-problem (see, for example, [6]). Assume that

\[ \Omega_{\bar{\partial}D} = C_0, \]

where \( C_0 \) is some complex constant, we find:

\[ \ln \Omega(\bar{z}) = C_0 + \frac{1}{2\pi i} \int \int_{\bar{D}} \frac{q_1}{(\ln \Psi_1)(\zeta - z)} d\zeta \wedge d\bar{\zeta}. \]

Here \( \zeta = \zeta_R + i\zeta_I, d\zeta \wedge d\bar{\zeta} = -2i d\zeta_R d\zeta_I. \) The correctness of this expression follows from the well-known relation of the theory of generalized functions (\( \bar{\partial} \equiv \bar{\partial}_z \)):

\[ \bar{\partial}(\frac{1}{\pi(z - z_0)}) = \delta(z - z_0). \]
5. Analogously, let $\Omega = 0$, i.e. $\Omega$ is an holomorphic function in $D \subset \mathbb{C}$. From (9) we obtain:

$$(\ln \Omega)_z = \frac{q_1}{(\ln \Psi_1)_z}$$

at the condition, that $$(\frac{q_1}{(\ln \Psi_1)_z})_\bar{z} = 0.$$ Assuming, that

$$\Omega|_{\partial D} = C_1,$$

where $C_1$ is complex constant, from (33) we have

$$\ln \Omega(z) = C_1 + \frac{1}{2\pi i} \int \int_D (\ln \Psi_1)_{\bar{z}}(\zeta - \bar{z}) d\zeta \wedge d\bar{\zeta},$$

(d). Now we shall find the quantum-mechanical sense of the transformations (8). For this purpose we shall consider more general, than (5), equation:

$$\Psi(z, \bar{z}) = 1 + 4(V(z, \bar{z}) - \lambda)\Psi,$$

(36)

where $\lambda \in \mathbb{C}$ is a complex parameter. It is possible also to use here the transformation (8) and dressing relation (15). Let us assume that $V(z, \bar{z}) = z\bar{z}$, corresponding to the two-dimensional isotropic harmonic oscillator. Then the equation (36) can be rewritten as

$$-4\Psi(z, \bar{z}, \lambda) + z\bar{z}\Psi(z, \bar{z}, \lambda) = \lambda \Psi(z, \bar{z}, \lambda).$$

For $\lambda = \lambda_1 = -2$ this equation has the solution $\Psi_1(z, \bar{z}) = \exp(\frac{1}{2}z\bar{z})$, and agrees with $\tilde{V}(z, \bar{z}) = z\bar{z} - 4 - 4q_1$. Thus we obtain the equation

$$-4\Psi[1](z, \bar{z}, \lambda) + (z\bar{z} - 4q_1)\Psi[1](z, \bar{z}, \lambda) = (\lambda + 4)\Psi[1](z, \bar{z}, \lambda).$$

Comparing (37) and (38) for $q_1 = 0$, we find

$$\Psi[1](z, \bar{z}, \lambda) = \Psi(z, \bar{z}, \lambda + 4),$$

i.e. the transformation (8), together with the requirement of covariance, acts as the quantum-mechanical creation operator of particles $^2$, and the quantity $-4q_1$ can be interpreted as an energy level shift (in a complex case). In this sense we have two-dimensional generalization of the standard DT [1].

Here the equation for functional $\Omega$ follows from (9):

$$\Omega_{xx} + \Omega_{yy} - 2x \Omega_x - 2y \Omega_y + q_1 \Omega = 0.$$  

This equation, which can be called the generalized Helmholtz equation, is known in the mathematical literature [12]. It arises under consideration of orthogonal polynomials in two independent variables. For $q_1 = 2(n + m)$ it has the solution

$$\Omega(x, y) = F_{n+m,m}(x, y) = H_n(x)H_m(y),$$

where $H_l(.)$ are polynomials of Chebychev-Ermit, $m, n = 0, 1, \ldots$. Here the generalized condition of orthogonality is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y)F_{nm}(x, y)F_{ks}(x, y) dx dy = \delta_{n-m,k-s} \delta_{ms},$$

where $h(x, y)$ is a weight function. This condition is satisfied for $h(x, y) = \exp(-|x|^2 - |y|^2)$.
where \( h(x, y) = \exp \left\{ -(x^2 + y^2) \right\} \) is the weight function, and \( \delta_{ij} \) is the Kronecker symbol.

For \( q_1 = 0 \) the solution of equation (40) also can be factorized. Setting \( \Omega(x, y) = \Omega_1(x) \Omega_2(y) \), we have:

\[
\Omega_{1xx} - 2x\Omega_{1x} - \beta\Omega_1 = 0, \quad \Omega_{2yy} - 2y\Omega_{2y} + \beta\Omega_2 = 0,
\]

where \( \beta \) is the "separation of variables parameter". For \( \beta = 2n \), where \( n = 1, 2, \ldots \), the solution of these equations has the solution in terms of the Chebychev-Hermit polynomials: \( \Omega_2(x, y) = H_n(y) \). Setting \( \Omega_1(x) \equiv \mathcal{P}(x) = \sum_{k=0}^{\infty} a_k x^k \), from the first of the equations (43) we find the recurrence relation:

\[
a_{k+2} = \frac{2(k + n)}{(k + 2)(k + 1)} a_k,
\]

where the numbers \( a_0, a_1 \) remain arbitrary, and the power series for function \( \mathcal{P}(x) \) converges uniformly on the entire axis.

Thus, in the dressing procedure the solution of equation (36) is

\[
\tilde{\Psi}(x, y) = \Psi[1](x, y) = c_n e^{x^2 + y^2} \mathcal{P}(x) H_n(y),
\]

where \( c_n \) is a constant. Besides that, similarly to the one-dimensional case (one-dimensional stationary Schrödinger equation), the states described by the relation (45) are not normalizable.

The detailed investigation of the creation and annihilation operators in two and more spatial dimensions was performed in the monograph [10].

### 3. A method of \( \bar{\partial} \) - problem.

In this Section we consider the application of the \( \bar{\partial} \)-problem method for the simplest case of diffraction on half-planes (see, for example, [14]):

\[
\Delta u + k^2 u = 0,
\]

and \(^3\)

\[
u = -u^{(i)}(x, y) = -e^{-ikx\cos \theta} \quad \text{at} \quad x < 0, \quad y \to \pm 0,
\]

\[
u = O\left(\frac{1}{\sqrt{r}}\right), \quad \frac{\partial \nu}{\partial r} -iku = o\left(\frac{1}{\sqrt{r}}\right) \quad \text{at} \quad r = \sqrt{x^2 + y^2} \to \infty.
\]

Here \( \nu = u(x, y) = u^{(tot)}(x, y) - u^{(i)}(x, y) \), \( u^{(tot)} \) is the total potential, \( u^{(i)} \) is the potential of a falling wave, \( u^{(i)} = \exp \left\{ -ikx\cos \theta - iky\sin \theta \right\} \), \( \Delta \) is the two-dimensional Laplace operator, \( k = k_1 + ik_2, \ k_1, k_2 > 0, \ 0 \leq \theta \leq \pi \).

Let us introduce the variables \( z \) and \( \bar{z} \), and let us assume \( u(z, \bar{z}) = v(z, \bar{z}) \exp \left\{ (k_+ z + k_- \bar{z}) \right\} \), where \( k_+ = -(i/2)k \exp(i\theta), \ k_- = -(i/2)k \exp(-i\theta) \). Then in the \( z \) - representation the problem (40)- (48) can be written in terms of function \( v = v(z, \bar{z}) \equiv v(z, \bar{z}, k_+, k_-) \) as

\[
v_{zz} + k_+ v_z + k_- v_{\bar{z}} = 0,
\]

\[
v(z, \bar{z}) = -1 \quad \text{at} \quad z + \bar{z} < 0, \quad z - \bar{z} \to \pm 0.
\]
The radiation condition (48) gives

\[ v(z, \bar{z}) = o(1) \quad \text{at} \quad |z| \to \infty. \]

It is not difficult to obtain the symmetry property from the equation (49):

\[ v(z, \bar{z}; k_+, k_-) = v(\bar{z}, z; k_-, k_+). \]

In order to derive the integral equations we note, that for any complex-valued function \( g(z, \bar{z}) \in C(\overline{\mathbb{D}}) \cap C^1(\overline{\mathbb{D}}), \mathbb{D} \subset \mathbb{C} \) the Green formulas [13] can be used:

\[ \oint_{\partial \mathbb{D}} g d\zeta = -\int_{\mathbb{D}} \int g \frac{d\zeta \wedge \bar{\zeta}}{2\pi i} \frac{\partial_w (\zeta, \bar{\zeta})}{\zeta - z}, \]

\[ \int_{\partial \mathbb{D}} g d\zeta = -\int_{\mathbb{D}} \int g \frac{d\zeta \wedge \bar{\zeta}}{2\pi i} \frac{\partial_w (\zeta, \bar{\zeta})}{\zeta - \bar{z}}. \]

Setting \( g(\zeta, \bar{\zeta}, l_1, l_2) = g_1(\zeta, \bar{\zeta}, l_1, l_2) = \frac{1}{(\zeta - z)}, g(\zeta, \bar{\zeta}, l_1, l_2) = g_2(\zeta, \bar{\zeta}, l_1, l_2) = \frac{1}{(\zeta - \bar{z})}, \) \( w(\zeta, \bar{\zeta}) \equiv w(\zeta, \bar{\zeta}, l_1, l_2) \in C(\overline{\mathbb{D}}) \cap C^1(\overline{\mathbb{D}}), \) where \( l_1, l_2 \in \mathbb{C} \) are parameters, applying to these functions the Cauchy’s formula with paths \( \partial \mathbb{D} \cup |-\zeta - z| = \varepsilon_1, \) and \( \partial \mathbb{D} \cup |\zeta - \bar{z}| = \varepsilon_2 \) accordingly, where \( \varepsilon_1, \varepsilon_2 > 0, \) and passing to limits \( \lim_{\varepsilon_1, \varepsilon_2 \to 0} \), we obtain the integral representations:

\[ w(z, \bar{z}) = \int_{\partial \mathbb{D}} \frac{d\zeta}{2\pi i} \frac{w(\zeta, \bar{\zeta})}{\zeta - z} + \int_{\mathbb{D}} \int \frac{d\zeta \wedge \bar{\zeta}}{2\pi i} \frac{\partial_w (\zeta, \bar{\zeta})}{\zeta - z}, \]

\[ w(z, \bar{z}) = -\int_{\partial \mathbb{D}} \frac{d\zeta}{2\pi i} \frac{w(\zeta, \bar{\zeta})}{\zeta - \bar{z}} + \int_{\mathbb{D}} \int \frac{d\zeta \wedge \bar{\zeta}}{2\pi i} \frac{\partial_w (\zeta, \bar{\zeta})}{\zeta - \bar{z}}. \]

To apply these representations, equation (49) can be rewritten as

\[ \partial_z (\partial_z + k_-) v(z, \bar{z}) = -k_+ \partial_z v(z, \bar{z}), \quad \partial_z (\partial_z + k_+) v(z, \bar{z}) = -k_- \partial_z v(z, \bar{z}). \]

We assume, that

\[ v_z(z, \bar{z}) = o(1), \quad v_{\bar{z}}(z, \bar{z}) = o(1) \quad \text{at} \quad |z| \to \infty. \]

Choosing the path as \( \partial \mathbb{D} = \Gamma \cup C_R, \) where \( C_R \) is the circle of radius \( R, \) and contour \( \Gamma \) consists of the straight line \( y = \pm i0, \) at \( x < 0, \) and the half-circle of small radius at \( x \geq 0, \) we obtain from (54)-(56):

\[ v_z(z, \bar{z}) + k_- v(z, \bar{z}) = \int_{\Gamma \cup C_R} \frac{d\zeta}{2\pi i} \frac{v_z(\zeta, \bar{\zeta}) + k_- v(\zeta, \bar{\zeta})}{\zeta - z} - \int_{\mathbb{D}} \int \frac{d\zeta \wedge \bar{\zeta}}{2\pi i} \frac{k_+ v(\zeta, \bar{\zeta})}{\zeta - z}, \]

\[ v_{\bar{z}}(z, \bar{z}) + k_+ v(z, \bar{z}) = -\int_{\Gamma \cup C_R} \frac{d\zeta}{2\pi i} \frac{v_{\bar{z}}(\zeta, \bar{\zeta}) + k_+ v(\zeta, \bar{\zeta})}{\zeta - \bar{z}} - \int_{\mathbb{D}} \int \frac{d\zeta \wedge \bar{\zeta}}{2\pi i} \frac{k_- v(\zeta, \bar{\zeta})}{\zeta - \bar{z}}. \]

These relations can be also simplified. Indeed, taking into account (51) and (57) we have:

\[ \lim_{R \to \infty} \int_{C_R} \frac{d\zeta}{2\pi i} \frac{v_z(\zeta, \bar{\zeta}) + k_- v(\zeta, \bar{\zeta})}{\zeta - z} = \lim_{R \to \infty} \int_{C_R} \frac{d\zeta}{2\pi i} \frac{v_{\bar{z}}(\zeta, \bar{\zeta}) + k_+ v(\zeta, \bar{\zeta})}{\zeta - \bar{z}} = 0. \]

Besides that, condition (50) gives:
\[
\int_\Gamma \frac{d\zeta}{2\pi i} \frac{v_\zeta(\zeta, \tilde{\zeta}) + k_+(\zeta, \tilde{\zeta})}{\zeta - \tilde{z}} = \int_{-\infty}^0 \frac{d\xi}{2\pi i} \frac{v_\zeta^+(\xi) - v_\zeta^-(\xi)}{\xi - \tilde{z}}.
\]

Then equations (58)-(59) yield:

\[
v_z(z, \tilde{z}) + k_- v(z, \tilde{z}) = \int_{-\infty}^0 \frac{d\xi}{2\pi i} \frac{v_\zeta^+(\xi) - v_\zeta^-(\xi)}{\xi - z} - k_+ \int_D \int \frac{d\zeta \wedge d\zeta}{2\pi i} \frac{v_\zeta(\zeta, \tilde{\zeta})}{\zeta - z},
\]

\[
v_\bar{z}(z, \tilde{z}) + k_+ v(z, \tilde{z}) = -\int_{-\infty}^0 \frac{d\xi}{2\pi i} \frac{v_\zeta^+(\xi) - v_\zeta^-(\xi)}{\xi - \bar{z}} - k_- \int_D \int \frac{d\zeta \wedge d\zeta}{2\pi i} \frac{v_\zeta(\zeta, \tilde{\zeta})}{\zeta - \bar{z}},
\]

where we used notations:

\[
v_\zeta^+(\xi) = \frac{1}{2}(\partial_\xi - i\partial_\eta) (\zeta + i\eta, \xi - i\eta)_{|\eta| \to 0}, \quad v_\zeta^-(\xi) = \frac{1}{2}(\partial_\xi + i\partial_\eta) (\zeta + i\eta, \xi - i\eta)_{|\eta| \to 0}.
\]

The equations (63)-(64) allow, in particular, to obtain the system of equations which connects a derivative of a field on a negative half-axis with derivative in the domain \(D\). Really, \(z = x < 0\) and taking into account the boundary condition (50), equations (63) can be written as

\[
v_z^+(x) - k_- = \int_{-\infty}^0 \frac{d\xi_1}{2\pi i} \frac{v_{\zeta_1}^+(\xi_1) - v_{\zeta_1}^-(\xi_1)}{\xi_1 - x - i0} - k_+ \int_D \int \frac{d\zeta_1 \wedge d\zeta_1}{2\pi i} \frac{v_{\zeta_1}(\zeta_1, \tilde{\zeta}_1)}{\zeta_1 - x},
\]

\[
v_z^-(x) - k_- = \int_{-\infty}^0 \frac{d\xi_1}{2\pi i} \frac{v_{\zeta_1}^+(\xi_1) - v_{\zeta_1}^-(\xi_1)}{\xi_1 - x + i0} - k_+ \int_D \int \frac{d\zeta_1 \wedge d\zeta_1}{2\pi i} \frac{v_{\zeta_1}(\zeta_1, \tilde{\zeta}_1)}{\zeta_1 - x},
\]

and equation (64) as

\[
v_{\bar{z}}^+(x) - k_+ = -\int_{-\infty}^0 \frac{d\xi_2}{2\pi i} \frac{v_{\bar{\zeta}_2}^+(\xi_2) - v_{\bar{\zeta}_2}^-(\xi_2)}{\xi_2 - x + i0} - k_- \int_D \int \frac{d\bar{\zeta}_2 \wedge d\bar{\zeta}_2}{2\pi i} \frac{v_{\bar{\zeta}_2}(\bar{\zeta}_2, \bar{\tilde{\zeta}}_2)}{\bar{\zeta}_2 - x},
\]

\[
v_{\bar{z}}^-(x) - k_+ = -\int_{-\infty}^0 \frac{d\xi_2}{2\pi i} \frac{v_{\bar{\zeta}_2}^+(\xi_2) - v_{\bar{\zeta}_2}^-(\xi_2)}{\xi_2 - x - i0} - k_- \int_D \int \frac{d\bar{\zeta}_2 \wedge d\bar{\zeta}_2}{2\pi i} \frac{v_{\bar{\zeta}_2}(\bar{\zeta}_2, \bar{\tilde{\zeta}}_2)}{\bar{\zeta}_2 - x}.
\]

The system of equations (60)-(69) is closed system of singular integral equations on the variable \(x\), to find the functions \(v_z^+, v_z^-\) and \(v_{\bar{z}}^+, v_{\bar{z}}^-\), respectively, from the values of \(v_z\) and \(v_{\bar{z}}\) at an arbitrary point of domain \(D\).

These functions can be also calculated using another approach. Indeed, equation (48) can be rewritten as

\[
(\frac{1}{2} \partial_{\bar{z}} + k_+) v_z(z, \bar{z}) = -(\frac{1}{2} \partial_{\bar{z}} + k_-) v_{\bar{z}}(z, \bar{z}).
\]

Then we construct the generalized function - fundamental solution of the operator \(((1/4) i\partial_y) + k_+\), which satisfies the equation

\[
[(\partial_x + i\partial_y) + 4k_+] G_1(x, y, k_+) = 4\delta(x)\delta(y).
\]

Thus from the equation (70) it follows:
Similarly we can find the generalized function - fundamental solution of the operator \(((1/4)(\partial_x - i\partial_y) + 4k_-):\)

\[
[(\partial_x - i\partial_y) + 4k_-]G_2(x, y, k_-) = 4\delta(x)\delta(y).
\]

From (70) and (73) we obtain:

\[
v_z(x, y) = -\int_D \int d\xi d\eta G_2(x - \xi, y - \eta, k_-)[\partial_{\xi} + i\partial_{\eta} + 4k_+]v_{\xi}(\xi, \eta).
\]

The solutions \(G_1\) and \(G_2\) are connected among themselves by the symmetry relation:

\[
G_1(z, \bar{z}, k_+) = -\bar{G}_2(-z, \bar{z}, -\bar{k}_+), \quad G_2(z, \bar{z}, -k_-) = -\bar{G}_1(-z, -\bar{z}, -\bar{k}_-).
\]

The system of equations (72), (74) is a system of integral equations of the convolution type, and it can be solved by the Fourier transformation (for generalized functions). Such calculations as well as the explicit expressions for functions \(G_1, G_2\) and the comparison with the results [14] will be given later.

The approach developed above in this Section seems to be an alternative to the traditional Wiener-Hopf approach. It would be interesting to generalize this method for the case of the three-dimensional problem of diffraction.

Acknowledgements

The author is grateful to M.V.Ioffe for attention and A.V.Tukhtin for useful discussions of questions of the theory of diffraction.

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