Warm inflation scenarios are studied with the dissipative coefficient computed in the equilibrium approximation. Use is made of the analytical expressions available in the low temperature regime with focus on the possibility of achieving strong dissipation within this approximation. Two different types of models are examined: monomial or equivalently chaotic type potentials, and hybrid like models where the energy density during inflation is dominated by the false vacuum. In both cases dissipation is shown to typically increase during inflation and bring the system into the strong dissipative regime. Observational consequences are explored for the amplitude of the primordial spectrum and the spectral index, which translate into constraints on the number of fields mediating the dissipative mechanism, and the number of light degrees of freedom produced during inflation.

This paper furthers the foundational development of warm inflation dynamics from first principles quantum field theory by calculating conservative lower bound estimates on dissipative effects during inflation using the well established thermal equilibrium approximation. This approximation does not completely represent the actual physical system and earlier work has shown relaxing both the equilibrium and low temperature constraints can substantially enlarge the warm inflation regime, but these improvements still need further theoretical development.

keywords: cosmology, inflation

PACS numbers: 98.80.Cq, 11.30.Pb, 12.60.Jv
Inevitably warm inflation dynamics is nonequilibrium and a complete treatment of the statistical state during warm inflation is probably not amenable to simple analytic approximations, with more numerical based methods such as [13, 18] needed. Nevertheless, progress must be made systematically, initially understanding what can be from analytic and near analytic treatments. There are two known analytic approximations that would be relevant to apply to this problem. First is a quasiparticle approximation following Morikawa and Sasaki [19]. The other is the equilibrium approximation, which is based on the assumption that the statistical state of all fields always remains in thermal equilibrium [14, 20]. The quasiparticle approximation has been developed for warm inflation in [13, 14, 15] and several interesting model building possibilities have been found in [5]. The equilibrium approximation has been developed for warm inflation in [16]. In this paper we will explore some of the interesting warm inflation models that such an approximation yields.

Testing the equilibrium approximation has an important general significance to the overall understanding of dissipative effects during inflation. In this approximation, the basic assumption is that the field system is minimally disturbed and so this approximation likely provides a lower bound on the degree of dissipative effects and particle production during inflation. Thus models that work under this approximation provide a minimal expectation on the overall robustness of warm inflation. Such solutions provide crucial existence proofs of the viability and consistency of warm inflation models with quantum field theory, which has been a basic question about these scenarios since they were first suggested [12, 21]. This of course is of general significance in the development of inflationary dynamics, since an alternative way to state these results is regimes which would have been regarded unquestionably to be governed by standard cold inflation dynamics in fact are not, since dissipative effects are shown to be significant. For the better part of the existence of the inflation idea, the blanket assumption has always been the dynamics is cold inflationary. The observation was made much later in the development of the inflation subject [1] that in fact this is not the unique situation and that dissipative effects can occur during inflation. Thus the results found in this paper provide a significant step in breaking these preset early notions about inflation dynamics and demonstrating that dissipation is a generic feature during inflationary expansion.

In the high temperature regime, in earlier work a class of warm inflation models were found for the equilibrium approximation in [12]; however one of the pronounced features of these models were the requirement of a very large number of fields $\sim 10^4$, and so not attractive for most model building purposes (although see [22] for string motivated models). In this paper, we will show that in the low-temperature regime, many types of reasonable warm inflation models can be found, in particular for a moderate number of fields and sensible values of the parameters.

The specific field theory models examined in this paper are the same as in [12, 14], in which the inflaton field is coupled to a heavy bosonic field which in turn is coupled to light fields. An important feature of such coupling configurations is that even for large perturbative couplings, in supersymmetric (susy) models, the quantum corrections to the effective potential from these terms can be controlled enough to maintain an adequately flat inflaton potential [14, 15], yet susy provides no cancellation of the time nonlocal terms, which are the dissipative terms, and so such terms can be quite significant.

The basic picture of dissipative dynamics for this class of models is as the background inflaton field moves down the potential, it excites the heavy bosonic fields which in turn decays into light degrees of freedom [13, 14, 15]. The latter quickly thermalize and become radiation. Consistency of the approximations then demands the thermalization timescale to be smaller than the evolution timescale of the inflation field and the expansion timescale of the Universe. In addition, the condition $T \gg H$ allows to neglect the expansion of the universe when computing $\Upsilon_\phi$. Under these conditions, the dissipative coefficient $\Upsilon_\phi$ has been recently computed for a generic supersymmetric inflationary model, in a thermal approximation [16]. It was shown that in the low $T$ limit, with $H \ll T < m_\chi$, the dissipative coefficient behaves as $T^3/m_\chi^2$, $m_\chi$ being the mass of the mediating field. Still, dissipation can be large enough to dominate over the expansion rate and bring the inflaton into the strong dissipative regime with $\Upsilon_\phi > H$.

In this paper we shall explore which kind of models of inflation can be brought into the strong dissipative regime within the low $T$ approximation. We want to explore the viability of the scenario by checking whether we can satisfy the conditions $\Upsilon_\phi > H$ and $H < T < m_\chi$ for at least 50-60 e-folds. Due to the $T$ dependence on the dissipative coefficient, the ratio $T/m_\chi$ tends to increase during inflation, and at some point the low $T$ approximation breaks down. The system would move then into the high $T$ regime, where the dissipative coefficient goes linearly with $T$, and soon after that $\Upsilon_\phi$ drops below $H$ and inflation ends. The transition from the low to the high $T$ regime may last some e-folds, but taking properly into account this period would require a more involved calculation of the dissipative coefficient beyond the analytical approximation. Therefore, as a first step in studying this kind of models, we only work within the low $T$ approximation, assuming that inflation ends soon after the condition $T < m_\chi$ is violated.

In section [11] we briefly review the basics of the warm inflationary dynamics in the strong dissipative regime, and applied this to some generic inflationary models, divided into two groups: (a) monomial potentials or chaotic inflation models, and (b) small field models where a constant term dominates the potential energy. In section [11] we summarize our findings and further comment on the consequences for model building.
II. DISSIPATION IN THE LOW T REGIME

We consider the interactions given in the superpotential:

\[ W = g \Phi X^2 + hXY^2, \]

(3)

where \( \Phi, X, \) and \( Y \) denote superfields, and \( \phi, \chi, \) and \( y \) will refer to their bosonic components. During inflation, the field \( y \) and its fermionic partner \( \bar{y} \) remain massless, while the mediating field \( \chi \) gets its mass from the interaction with the inflaton field \( \phi \), with \( m_\chi = \sqrt{2g\phi} \). Following Ref. [16], the dissipative coefficient in the low \( T \) regime is well approximated by:

\[ \Upsilon_\phi \simeq 0.64 \times g^2 h^4 \left( \frac{g\phi}{m_\chi} \right)^4 \frac{\tau^3}{m_\chi^2}. \]

(4)

Alternatively, we may consider a susy hybrid model of inflation, with the mass of the \( \chi \) field given by \( m_\chi^2 = 2g^2(\phi^2 - \phi_c^2) \), with \( \phi_c \) being the critical value. In either case, the dissipative coefficient given in Eq. (4) does not depend on the coupling \( g \) in this regime, but only on \( h \) which can be quite large, \( h \simeq O(1) \). The numerical coefficient in Eq. (4) was computed taking \( X, Y \) to be singlet complex fields. But in principle, they may belong to larger representations of a Grand Unification Theory (GUT) group, as it is typically assumed in susy hybrid models. This will give rise to an extra factor of \( N = N_\chi N_{\text{decay}}^2 \) decay in front of the dissipative coefficient, where \( N_\chi \) is the multiplicity of the \( X \) superfield, and \( N_{\text{decay}} \) counts the no. of decay channels available in \( X \)'s decays. Taking \( m_\chi^2 = 2g^2\phi^2 \), we have then:

\[ \Upsilon_\phi \simeq C_\phi \frac{T^3}{\phi^2}, \]

(5)

where \( C_\phi = 0.16 \times h^4 N_\chi \).

The conditions for slow-roll inflation are modified due to the extra friction term in Eq. (1). Demanding \( \ddot{\phi} < (3H + \Upsilon_\phi)\dot{\phi} \) and \( \dot{\phi}^2 < V \), they are given now by:

\[ \eta = \frac{\eta_H}{(1 + r)^2} < 1, \]

(6)

\[ \epsilon = \frac{\epsilon_H}{(1 + r)^2} < 1, \]

(7)

\[ \frac{V_\phi/\phi}{3H^2(1 + r)^3} < 1, \]

(8)

where \( \eta_H = m_p^2(V_{\phi\phi})/V \), \( \epsilon_H = m_p^2(V_\phi/V)^2/2 \) are the standard slow-roll parameters, and \( r = \Upsilon_\phi/(3H) \). In addition, once the source term for the radiation dominates in Eq. (2), this would reduce to:

\[ 4H\rho_R \simeq \Upsilon_\phi \phi^2. \]

(9)

However, in order to have \( \dot{\rho}_R < 4H\rho_R \) and then Eq. (9), we need to impose instead the slow-roll conditions:

\[ \frac{\eta_H}{(1 + r)} < 1, \]

(10)

\[ \frac{\epsilon_H}{(1 + r)} < 1, \]

(11)

\[ \frac{V_\phi/\phi}{3H^2(1 + r)} < 1. \]

(12)

For the kind of potentials considered in the following, we would have \( \epsilon_H \ll (V_\phi/\phi)/(3H^2) \simeq \eta_H \), and therefore the slow-roll conditions reduce mainly to:

\[ \frac{V_\phi/\phi}{3H^2(1 + r)} \simeq \frac{\eta_H}{(1 + r)} < 1. \]

(13)

Once this is fulfilled, also the conditions \( \eta < 1, \epsilon < 1 \) are satisfied. The above condition ensures that the radiation does not increase too fast during warm inflation, otherwise it will dominate too soon, not allowing inflation too proceed.
Therefore, in the strong dissipative regime with \( r > 1 \), the slow-roll evolution equations for the inflaton field and the radiation are given by:

\[
\dot{\phi} \simeq -\frac{V_\phi}{T_\phi}, \quad (14)
\]

\[
\rho_R \simeq \frac{V_\phi^2}{4HC_\phi C_R}. \quad (15)
\]

Alternatively, using \( \rho_R = C_R T^4 \), we have for the temperature of the thermal bath:

\[
T \simeq \left( \frac{(V_\phi \phi)^2}{4HC_\phi C_R} \right)^{1/7}, \quad (16)
\]

where \( C_R = \pi^2 g_*/30 \), and \( g_* \) is the effective number of light degrees of freedom. Once all the particles are in thermal equilibrium at a common \( T \) through rapid interactions, \( g_* \) counts the number of relativistic degrees of freedom in the model. However, thermalization is not an instantaneous process, and for a particle specie kinetic equilibrium with the thermal plasma is only reached once its interaction rate becomes larger than the Hubble rate. Taking into account the thermalization rate would translate into a lower effective \( T \) than expected \[23\], or equivalently a lower value \( g_*(T) \). Therefore, we may expect also \( g_*(T) \) varying while in the dissipative regime, from say \( O(10) \) to the value for the Minimal Supersymmetric Standard Model (MSSM) \( g_* = 228.75 \).

### A. Monomial potentials

We first study the inflationary trajectory for a general inflaton monomial potential:

\[
V(\phi) = V_0 \left( \frac{\phi}{m_P} \right)^n, \quad (17)
\]

with \( n > 0 \). Without enough dissipation, i.e., either for cold inflation with \( r = 0 \), or only weak dissipation with \( r < 1 \), these kind of models lead to inflation only for values of the inflaton field larger than the Planck mass \( m_P \). On the other hand, in the strong dissipative due to the larger friction term slow-roll conditions Eqs. (6) and (7) can be fulfilled for values of the field well below Planck. That is, we can regard now Eq. (17) from the effective field theory point of view, with the potential well define below the cut-off scale \( m_P \); higher order term contributions suppressed by \( m_P \) will be then negligible, without the need of fine-tuning the coefficients in front.

During slow-roll, once the system enters in the strong dissipative regime, we can integrate Eq. (14) using Eqs. (16) and (15). The potential decreases from its initial value \( V(0) \) as:

\[
\frac{V}{V(0)} \simeq \left( 1 - \frac{4\rho_R(0)}{7V(0)} N_e \right)^7, \quad (18)
\]

where \( N_e \) is the no. of e-folds from the beginning of inflation, an \( \rho_R(0) \) is the initial value for the radiation given by:

\[
\frac{\rho_R(0)}{V(0)} \simeq \left( \frac{9n^8 C_\phi^3}{4 C_R^4} \left( \frac{m_P^4}{V(0)} \right) \right)^{1/7}, \quad (19)
\]

Eq. (18) is only valid in the strong dissipative regime, \( r > 1 \), with:

\[
r \simeq \left( \frac{n^6 C_\phi^4}{576 C_R^5} \right)^{1/7} \left( \frac{m_P}{\phi(0)} \right)^2 \left( \frac{V(0)}{m_P^2} \right)^{1/7} \left( \frac{V(0)}{V} \right)^{2/n-1/7}, \quad (20)
\]

which increases during inflation for \( n < 14 \). On the other hand, the values of \( \eta_H/r, T/H \) and \( T/\phi \) are given respectively by:

\[
\frac{\eta_H}{r} \simeq 4 \left( \frac{n - 1}{n} \right) \left( \frac{\rho_R(0)}{V(0)} \right) \left( \frac{V(0)}{V} \right)^{1/7}, \quad (21)
\]

\[
\frac{T}{H} \simeq \left( \frac{81n^2}{4C_\phi C_R} \right)^{1/7} \left( \frac{m_P}{V(0)} \right)^{2/7} \left( \frac{V(0)}{V} \right)^{2/7}, \quad (22)
\]

\[
\frac{T}{\phi} \simeq \left( \frac{m_P}{\sqrt{3}\phi(0)} \right) \left( \frac{81n^2}{4C_\phi C_R} \right)^{1/7} \left( \frac{V(0)}{m_P^2} \right)^{3/14} \left( \frac{V}{V(0)} \right)^{3/14-1/n}. \quad (23)
\]
FIG. 1: Quartic potential: Allowed values for $g_*$ and $\mathcal{N} \equiv N_{\chi}^2 N_{\text{decay}}^2$ for having at least 50 e-folds of strong warm inflation: (a) enclosed region by the solid lines when $V(0)^{1/4}/m_P = 0.3$; (b) to the left of the dashed line when $V(0)^{1/4}/m_P = 0.1$. We have taken $\phi(0)/m_P = 1$.

Therefore, all the above ratios increase as inflation proceeds. This means that after some no. of e-folds, we will either have $T/\phi > 0.1$, and the low $T$ approximation will not longer hold, or $\eta_H/r > 1$, and inflation will end shortly after when $\rho_R > V$. From (21) and (22) we have the relation:

$$\frac{\eta_H}{r} \simeq 4 \left( \frac{n-1}{n} \right) \left( \frac{\rho_R}{V} \right),$$

and for any power $n > 1$ we have that the radiation would not dominate the total energy density in the slow-roll regime with $\eta_H/r < 1$. Once this ratio grows beyond one, so does the ratio $\rho_R/V$, and inflation ends when the radiation becomes larger than the inflaton potential.

Without a specific model at hand, we can always consider $\mathcal{N} = N_{\chi}^2 N_{\text{decay}}^2$ and $g_*$ as free parameters$^1$, and see for which values we can keep $\eta_H/r$ and $T/\phi$ small enough for at least 50 e-folds or so (and $T/H > 1$). In turn, these values will depend on the initial value for $V(0)$, and in the case of $\phi/T$ also on $\phi(0)$. For example, in Fig. 1 we have plotted the allowed region in the plane $g_* - \mathcal{N}$ for a quartic potential taking $V(0)^{1/4} \simeq 0.3m_P$ and $\phi(0) = m_P$; in this case, in order to satisfy all the constraints we require $g_* < 100$ but $\mathcal{N} > 2300$. Similar results are obtained for other powers of the potential. By lowering the value of the potential, it is easier to fulfill all conditions except that for the ratio $\eta_H/r$. Keeping the latter below one gives the lower bound:

$$\mathcal{N} > 8.4 \times 10^{-2} g_*^{3/4} \left( \frac{m_P}{V(0)^{1/4}} \right) \left( n^{1/7}(n-1) + \frac{n}{7} N_e \right)^{7/4},$$

and the lower $V(0)$ is, the larger $\mathcal{N}$ has to be. For example, for $n = 4$, $V(0)^{1/4}/m_P = 0.1$, and $g_* = 10$ we would need $\mathcal{N} > 2800$ (dashed line in Fig. 1), but getting to the no. of degrees of freedom for the MSSM, $g_* = 228.75$, would require $\mathcal{N} > 29000$, which looks rather large. In any case, curiously enough, due to the extra friction we can have inflation for values of the field below Planck, but the model prefers an initial value of the height of the potential only an order of magnitude or so below Planck.

On the other hand, the amplitude of the primordial spectrum is also affected by the strong dissipative friction term and the presence of a thermal bath. Approximately, when $T > H$ the fluctuations of the inflaton field are induced by

$^1$ We have also the value of the coupling $h$ among the heavy field $\chi$ and the light degrees of freedom, which hereon we fix to its maximum allowed value, $h = \sqrt{4\pi}$ when giving bound on the value of $\mathcal{N}$. 

the thermal fluctuations, instead of being vacuum fluctuations, with a spectrum proportional to the temperature of the thermal bath. In particular, when $r \gg 1$ we have for the spectrum of the inflaton fluctuations \[4\]:

$$P_{\phi^2}^{1/2} \simeq \left(\frac{\pi r}{4}\right)^{1/4} \sqrt{H},$$

(26)

with the amplitude of the primordial spectrum of the curvature perturbation given by:

$$P_{R}^{1/2} \simeq \left|\frac{H}{\phi}\right| P_{\phi^2}^{1/2} \simeq \frac{3H^3}{V_\phi} \left(\frac{\pi r}{4}\right)^{1/4} (1 + r) \sqrt{\frac{T}{H}},$$

(27)

which in the case of the monomial potential is given by:

$$P_{R}^{1/2} \simeq \left(\frac{\pi}{12}\right)^{1/4} \frac{n^3}{6} \frac{C_\phi^{9/14}}{C_R^{17/28}} \left(\frac{\phi}{m_P}\right)^{3/2} \frac{V}{m_P^4} \frac{15/28}{(1 + r) \sqrt{\frac{T}{H}}}.$$

(28)

In order to keep the amplitude of the primordial spectrum consistent with WMAP’s value \[24\], $P_{R}^{1/2} \simeq 5.5 \times 10^{-5}$, we would need a potential much smaller than $O(10^{-12} m_P^4)$. But for such a value of the potential, we would need roughly $\mathcal{N} \sim O(10^9)$ in order to get at least 50 e-folds in the strong dissipative regime, which seems rather large and unnatural\[2\]. Therefore, although in principle having strong dissipation during inflation is possible with a monomial potential, maintaining inflation for at least 50 e-folds requires rather large values of $\mathcal{N}$, and it tends to produce a too large amplitude for the primordial spectrum.

### B. Hybrid like models of inflation

We consider now small field models of inflation, by adding a constant term to the monomial potential:

$$V(\phi) = V_0 \left(1 + \left(\frac{\phi}{M}\right)^n\right), \quad n > 0$$

(29)

$$V(\phi) = V_0 \left(1 + \beta \ln\left(\frac{\phi}{M}\right)\right), \quad n = 0$$

(30)

Given that during inflation the potential is dominated by the constant term $V_0$, we can easily keep the value of the field below Planck in this class of models during slow-roll inflation. We regard this kind of potential as a generalization of a hybrid model \[6, 25\], where inflation ends once the inflaton field reaches the critical value, destabilizing the waterfall field coupled to it. Those interactions are not relevant to study the slow-roll dynamics, only to mark the end of inflation, and therefore we do not need to consider them in the inflationary potential Eqs. \[29, 30\]. However, the same interactions between the inflaton and the waterfall field required by the hybrid mechanism will give rise to dissipation, and leads to Eq. \[29\] in the low T regime. The case $n = 2$ would be the standard hybrid model \[6, 22\], with a mass term for the inflaton, whereas when $n = 0$ we have the susy model with the logarithmic correction coming from the 1-loop effective potential \[26, 27\]. In supersymmetric hybrid models, one needs to worry about the so called “eta” problem \[6, 7, 8, 9, 10\], i.e., the fact that generically su gra corrections give rise to scalar masses of the order of the Hubble parameter, including that of the inflaton, then forbidding slow-roll inflation. Different solution to this problem exist in the literature, for example by combining specific forms of the superpotential and the Kähler potential \[6, 26\]. Nevertheless, typically although we can avoid the quadratic correction, i.e. a mass contribution, su gra corrections manifest as higher powers in the inflaton field \[28\]. In the case of strong warm inflation, the presence of the extra friction term alleviates the problem: slow-roll conditions are fulfilled also for inflaton masses in the range $H < m_\phi^2 < \Upsilon_\phi$. In addition, the values of the field being smaller than in standard cold inflation, the effect of higher order su gra corrections is also suppressed.

The Hubble rate remains practically constant and given by $H \simeq V_0^{1/2}/(\sqrt{3} m_P)$ during inflation, so neglecting its variation we can integrate the slow-roll equations in the strong dissipative regime, Eq. \[13\] together with Eq. \[10\].

---

2 Strictly speaking, the numbers and order of magnitude estimations have been obtained for the quartic monomial, but similar values are obtained for the quadratic and other powers.
and we obtain:

\[
\left( \frac{\phi}{\phi(0)} \right) \simeq \left( 1 + \frac{n}{7} \left( \frac{64a^2 C_R^3}{C_\phi} \right)^{1/7} \left( \frac{\phi(0)}{H(0)} \right)^{n/7} N_e \right)^{-7/n},
\]

\[
\left( \frac{T}{H} \right)^7 \simeq \frac{a^4}{4C_\phi C_R} \left( \frac{\phi}{H} \right)^{2n},
\]

\[
\left( \frac{\phi}{T} \right) \simeq \left( \frac{\phi(0)}{T(0)} \right) \left( \frac{\phi}{\phi(0)} \right)^{1-2n/7},
\]

and therefore:

\[
\left( \frac{\eta}{r} \right) \simeq \frac{\eta_H}{r} \left( \frac{\phi}{\phi(0)} \right)^{n/7},
\]

\[
\left( \frac{\rho_R}{V} \right) \simeq \left( \frac{\rho_R}{V(0)} \right) \left( \frac{\phi}{\phi(0)} \right)^{2n/7},
\]

where the subscript “0” denotes the initial value and we have defined:

\[
a^2 = \left\{ \begin{array}{ll}
\left( \frac{\eta}{\eta_H} \right)^n, & n \neq 0, \\
\left( \frac{\eta}{\eta_H} \right)^n, & n = 0.
\end{array} \right.
\]

The amplitude of the primordial spectrum is given by Eq. (27), and the prediction for the spectral index \( n_S \) is given by:

\[
n_S - 1 \simeq 2 \frac{d \ln P_R^{1/2}}{d \ln k} \simeq 2 \phi \frac{d \ln P_R^{1/2}}{d \phi} \simeq - \frac{2V_H}{H^2(1+r)} \frac{d \ln P_R^{1/2}}{d \phi},
\]

evaluated at \( k_0 \), corresponding to horizon exit \((k = H a)\) at say 50-60 e-folds before the end of inflation. Taking into account the \( T \) field dependence, derived from \( 4H\rho_R \simeq \Upsilon_\phi \phi^2 \), we get:

\[
n_S - 1 \simeq \frac{1}{1 + r} \left( - \frac{3(3r-1)}{1+r} \epsilon_H - 3\eta_H + 6(1 + 3r \frac{m_P}{\phi} \sqrt{2\epsilon_H}) \right).
\]

where \( \eta_H \) and \( \epsilon_H \) are the standard slow-roll parameters without dissipation. By taking again the derivative of Eq. (39) with respect to \( \ln k \), i.e., with respect to the field, we obtain the running of the spectral index:

\[
n_S' \frac{d n_S}{d \ln k} \simeq \frac{1}{1 + r} \left( - \frac{3(3r-1)}{1+r} \epsilon_H' - 3\eta_H' + 6(1 + 3r \frac{m_P}{\phi} \sqrt{2\epsilon_H})' \right)
\]

\[
+ \frac{1}{(1+r)(1+7r)^2} \left( -30\epsilon_H + 18\eta_H - 24[m_P \phi \sqrt{2\epsilon_H}]' \right),
\]

with the derivatives of the slow-roll parameters and \( r \) given by:

\[
\eta_H' = -\frac{\epsilon_H}{1+r} (\xi_H - 2\eta_H),
\]

\[
\epsilon_H' = -\frac{2\epsilon_H}{1+r} (\eta_H - 2\epsilon_H),
\]

\[
(m_P \phi \sqrt{2\epsilon_H})' = \left( m_P \phi \right) \sqrt{2\epsilon_H} \frac{\sqrt{2\epsilon_H}}{1+r} \left( m_P \phi \sqrt{2\epsilon_H} + 2\epsilon_H - \eta_H \right),
\]

\[
r' = \frac{r}{1+7r} (10\epsilon_H + 8 \frac{m_P \phi \sqrt{2\epsilon_H}}{\phi \sqrt{2\epsilon_H} - 6\eta_H}),
\]

\[\text{Eq. (41)}\]

\[\text{Eq. (42)}\]

\[\text{Eq. (43)}\]

\[\text{Eq. (44)}\]
where \( \xi_H = 2m_p^2V'''/V' \). For the models considered in this section, we have that \( \epsilon_H = ((\phi/m_p)\eta_H/(n-1))^2/2 \), and therefore typically \( \epsilon_H \ll \eta_H \), so keeping only the leading terms in Eq. (39) and (40), we end with

\[
n_{S} - 1 \approx \frac{3\eta_H}{7r} \left( \frac{7-n}{n-1} + \left( \frac{\phi}{m_p} \right)^2 \frac{3\eta_H}{2(n-1)^2} \right),
\]

and

\[
n'_{S} \approx -3 \left( \frac{\eta_H}{7r} \right)^2 \left( \frac{n(7-n)}{(n-1)^2} + \left( \frac{\phi}{m_p} \right)^2 \frac{(14+10n-17n^2)}{(n-1)^4} \eta_H \right).
\]

Notice that the spectral index if of the order of \( O(\eta_H/r) \), whilst the running is of the order of \( O((\eta_H/r)^2) \). Therefore, the same condition needed to have slow-roll in the strong dissipative regime will avoid having a too large spectral index. We will have a blue-tilted spectrum when \( n \leq 7 \), including the case of \( n = 0 \), i.e., the logarithmic potential. This is disfavoured by the data \cite{24} when there is no running of the spectral index and no tensor contribution to the spectrum, otherwise the data is not conclusive. For example with non-negligible running, we have the allowed range \( 0.97 < n_S < 1.21 \) \cite{24}, with the running in the range \(-0.13 < dn_S/d\ln k < 0.007 \). The more negative running, the more blue-tilted the spectrum can be, which would be the case for \( 0 < n < 7 \) with \( n'_S \approx -(n_S-1)^2(n/3(7-n)) \).

Given that the field decreases during inflation, so does \( \eta_H/r \) and also \( \rho_R/V \) (or equivalently \( T/H \)) for any power \( n \neq 0 \), being constant for the logarithmic potential \( n = 0 \). Therefore, the energy density in radiation will never dominate in this regime. On the other hand, \( \phi/T \) diminishes for \( n < 4 \), but \( r \) only for \( n > 2 \). Therefore, for a logarithmic or quadratic potential once the system is brought into the strong dissipative regime stays there until the end. Indeed we may start in the weak dissipative regime, with \( T > H \) but \( r < 1 \), and it will evolve into \( r > 1 \). In the weak dissipative regime the inflaton evolution is given by the standard slow-roll equation, but the \( T \) behaves like:

\[
\frac{T}{H} \approx \frac{C_\phi a^4}{36C_R} \left( \frac{\phi}{H} \right)^{2(n-2)},
\]

and then

\[
r \approx \frac{C_\phi}{3} \left( \frac{C_\phi a^4}{36C_R} \right)^3 \left( \frac{\phi}{H} \right)^{2(3n-7)}.
\]

Therefore, for \( n = 0 \) both \( T \) and \( r \) grow until \( T \) reaches a constant value when \( r \geq 1 \); and we may have the interesting situation of a transition from “cold” \( \rightarrow \) “weak warm” \( \rightarrow \) “strong warm” inflation. For \( n = 2 \), \( T \) remains constant until it diminishes in the strong dissipative regime, so that the parameter space divides into either cold or warm inflation, but the system can evolve from weak to strong dissipation. In the weak dissipative regime, quantum fluctuations of the inflaton field will also have a thermal origin, with an amplitude \cite{2, 4}:

\[
\rho^{1/2}_R \approx \sqrt{3H^3 \frac{V_\phi}{V}} \left( \frac{T}{H} \right)^{3} \left( \frac{H}{\phi(0)} \right)^{2} \sqrt{\frac{C_\phi}{36C_R}},
\]

where in the second equality we have used Eq. (48); the spectral index in given by:

\[
n_S \simeq 1 - 2\eta_H + 2\epsilon_H,
\]

Notice that even in the weak dissipative regime a logarithmic potential gives rise to blue-tilted spectrum, while for \( n > 0 \) we would have a red-tilted spectrum, just the reverse than the standard cold predictions.

For larger powers \( n > 2 \), we have the opposite behavior, and even if inflation starts in the strong dissipative regime it will evolve towards the weak and the cold regime. When this happens before the last 50-e-folds of inflation, then dissipation becomes irrelevant.

In the following we will briefly consider the cases \( n = 0, 2, 4 \) separately. The same than for the monomial potentials, we are interested in getting the constraints on the parameter values \( N \) and \( g_* \), which will depend on the power \( n \), by demanding first having 50 e-folds of inflation with \( r \geq 1 \), and then getting the right primordial spectrum.

**Case \( n = 0 \): Hybrid logarithmic potential.** For the potential Eq. (39) we have that the field value decreases exponentially as:

\[
\phi \simeq \phi(0)\exp[(\eta_H/r)N_e],
\]

\( n \neq 0 \)
where the ratio $\eta_H/r$ is approximately constant, and given by:

$$\frac{\eta_H}{r} \simeq -\left( \frac{64a^2C^3_R}{C^4_\phi} \right)^{1/7},$$

with $a^2 = \beta V_0/H^4$. The temperature as mentioned before is approximately constant and given by Eq. (52) with $n = 0$, while the ratios $\phi/T$ and $r$ decreases/increases respectively like:

$$\frac{\phi}{T} \simeq \left( \frac{\phi(0)}{H} \right) \left( \frac{4C_\phi C_R}{a^4} \right)^{1/7} \exp[(\eta_H/r)N_e],$$

$$r \simeq \frac{1}{3} \left( \frac{H}{\phi(0)} \right)^2 \left( \frac{a^4C^{4/3}}{4C_R} \right)^{3/7} \exp[-(\eta_H/r)N_e]$$

where we have used Eqs. (53), (55) together with (22). The lower bound on $\mathcal{N}$ is obtained in the limiting case for slow-roll warm inflation, when $T/H \simeq 1$, $r(0) \simeq 1$ and $(\phi/T)_N \simeq 10$, where the subscript “N” means $N_e$ e-folds after, which gives:

$$\mathcal{N} \simeq 11.87 \exp[-0.23N_e g^1/2_e / N^{1/2}] .$$

For example, with $g_* \simeq 10$, $N_e = 50$ we have the lower bound: $\mathcal{N} \simeq 180$, $(\phi(0)/H) \simeq 150$, $a^2 \simeq 244$, $\eta_H/r \simeq -0.054$; with $g_* \simeq 228.75$, $N_e = 50$ we have: $\mathcal{N} \simeq 1350$, $(\phi(0)/H) \simeq 1.2 \times 10^4$, $a^2 \simeq 5.2 \times 10^4$, and $\eta_H/r \simeq -0.1$. Those values of $\mathcal{N} = N\chi^2_{\text{decay}}$ are quite in the range of a realistic model; for example with $\chi$ in the $126$ or $351$ of SO(10) ($E_6$), and $\chi^2_{\text{decay}} \approx O(10)$, one can expect having $\mathcal{N}$ in the range of a few thousands. However, for such values we always get a too large amplitude of the primordial spectrum, Eq. (24), which using Eqs. (53) and (55) can be written as:

$$P_R^{1/2} \simeq 0.4 \left( \frac{H}{\phi(0)} \right)^{3/2} \left( \frac{C^{18.4_\phi}}{C^2_R} \right)^{1/28}.$$

Therefore, in order to match the amplitude of the primordial spectrum with WMAP’s value we need a larger initial value of the field $\phi(0)$, but then the value of $C_\phi(N)$ has to be larger in order to stay within the low $T$ approximation, with $\phi/T \geq 10$; at the same time we need to increase $a^2$ in order to keep $r(0) \geq 1$. Satisfying WMAP’s constraints requires then $\mathcal{N} \gtrsim 5 \times 10^5$, rather large, with $\phi(0)/H \gtrsim 2 \times 10^6$ and $a^2 \gtrsim 8 \times 10^{11}$.

Having inflation in the weak warm regime, and a transition from weak to strong dissipation at the end, might in principle help in fixing the amplitude of the spectrum to lower values. We still need to impose that (a) $T/H > 1$, Eq. (13), (b) we get enough inflation, i.e., $N_e \approx 50$. These translates into $\mathcal{N} \gtrsim 0.05g_*/\eta_H$, with $\eta_H/1/(2N_e)$, and therefore we have the lower bound $\mathcal{N} \gtrsim 0.2g_*/N_e^2$. For example for $g_* = 288.75$ and $N_e \approx 50$ we have $\mathcal{N} \gtrsim 1.2 \times 10^5$, which again is rather large. Going from weak to strong dissipation earlier and having only 10 e-folds in the weak regime in principle is a possibility for smaller values of $\mathcal{N}$, but then again we cannot keep the low $T$ approximation and $\phi/T \geq 10$ for the remaining e-folds of inflation. However one has to bear in mind that we are not properly taking into account how inflation ends, which requires going beyond the analytical approximations. The transition from the low to the high $T$ is not necessarily instantaneous as we have implicitly assumed in this paper, and may last another few e-folds. Counting all together can bring again the values of the parameters into realistic values.

**Case $n = 2$: Hybrid Quadratic potential.** The main restriction for this model comes from keeping the ratios $T/H$ and $\phi/T$ within the range of the low $T$ approximations. During slow-roll the dissipative ratio $r$ increases, so its initial value $r_0$ becomes the lower bound on this parameter. On the other hand, we still require $(\eta_H/r_0) = a^2/(3r_0) < 1$ in order to have slow-roll warm inflation. By using Eqs. (51), and (54), we can rewrite Eqs. (52), (53), in terms of $r_0$ and $\eta_H/r_0$ like:

$$\frac{T}{H} \simeq \left( \frac{C_\phi}{C_R} \right) \left( \frac{\eta_H}{2r_0} (1 + 2\eta_H/7r_0 N_e)^{-1} \right)^2,$$

$$\frac{\phi}{T} \simeq \left( \frac{C_\phi}{C^{1/2}_R} \right) \left( \frac{3\eta_H}{2r_0} \right) \left( 3r_0 (1 + 2\eta_H/7r_0 N_e)^3 \right)^{-1/2}.$$
From the approximated expression for the spectral index, Eq. (45), if we want to keep $n_S$ within the observable range, we require $\eta_H/r_0 \leq 0.093$, which for $N_e \simeq 50$ gives the lower bound $N \gtrsim 32.5g_*$. Again, having slow-roll warm inflation for example with $g_* \simeq 228.75$ needs $N_e > \sim 750$, but for $g_* \simeq 10$ it only requires $N \gtrsim 325$. As an example, in fig. (2) we have plotted the predicted spectral index depending on the no. of e-folds left to the end of inflation, for $N \equiv N_{\chi N_{\text{decay}}}^2 = 10000, 20000, 30000$ and $g_* = 228.75$. The corresponding spectral index of the primordial spectrum would be that at around 50-55 e-folds, which is always $n_S < 1.2$. The value of the running can be obtained from Eq. (46), and it is given respectively by $n_S' \simeq -2.5 \times 10^{-3}$, $-8.5 \times 10^{-4}$, $-4.7 \times 10^{-4}$.

We can estimate the amplitude of the primordial spectrum, Eq. (27) in terms of the parameters $r_0$ and $\eta_H/r_0$:

$$
\mathcal{P}_R^{1/2} \simeq 4\pi^3 \left( \frac{\pi r_0^2}{3} \right)^{1/4} \left( \frac{r_0}{\eta_H} \right)^3 \frac{C_R^{1/2}}{C_0^{3/2}}.
$$

(61)

Having a not too large amplitude for the primordial spectrum prefers values of $\eta_H/r_0$ as large as possible, but not too large values of $r_0$. For values of $N$, $g_*$ within the range of Eq. (40), the amplitude remains below say $10^{-4}$ for values of $r_0$ of the order of $O(10)$. Therefore in this kind of models parameter values can be found giving rise to the right order of magnitude for the primordial spectrum in the strong dissipative regime, but the stronger constraint comes from getting a not too blue-tilded spectrum.

**Case $n \geq 4$: Hybrid quartic and higher powers.** In this case we have that $T/H$ and $r$, Eqs. (52) and (54), both decrease during inflation, while on the contrary $\phi/T$ increases. Therefore, it is not difficult to remain within the range of the low $T$ approximation, but on the other hand strong dissipation with $r \geq 1$ will last only a few e-folds. With $n = 4$, having 50 e-folds in the strong regime requires for example $N = N_{\chi N_{\text{decay}}}^2 \gtrsim 10^3$ for $g_* \simeq 10$, and $N \gtrsim 10^4$ for $g_* \simeq 228.75$. In addition if we want to get the right amplitude for the spectrum Eq. (27), and spectral index, this value increases by one order of magnitude, as we need to adjust the values of the quartic coupling $a^2 = \lambda$ to rather small values, and then the initial value of the field to larger values to have $\phi(0)/T \geq 10$. Numbers do not change much if we demand 10 or 50 e-folds of inflation in the strong dissipative regime.

**Comments: Ending inflation**

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3 Given the uncertainties in the analytical estimations, and in Eq. (27), it would not make sense to impose the specific WMAP value as a constraint, but an order of magnitude estimation would be sufficient. Furthermore, we have checked numerically that when the primordial spectrum is originated with $r_0$ not much larger than unity, the value of the amplitude tends to be lower than that given in Eq. (27).
We have seen that depending on the values of the field ($\phi(0)$) and couplings ($V_0$, $\beta$) in the inflationary potential with $n = 0, 2$, we may have either weak or strong dissipative regime during $N_e$ e-folds, but for rather large values of the parameter $N = N_e N_{\text{decay}}^2$. However, those large values are not required for having inflation in the weak/strong dissipative regime, but in order to match the predicted values of the spectrum with the cosmological observations, including having enough inflation in the low $T$ regime. With lower values of the multiplicity parameter, in the range of a few thousands or so, dissipative effects are relevant for the inflationary dynamics, and indeed the tendency is to bring the system into the strong dissipative regime, first in the low $T$ regime, and later into the high $T$ regime, where the analytical approximations break down. From this point of view, cosmological observations rule out large regions of the parameter space in this kind of models. Nevertheless, a quantitative statement on the values of the parameters needs in turn to properly take into account the dynamics at the end of inflation. There are some important corrections that have to be included before ruling out any of the models presented here because they do not fit cosmological observations. First of all, as already mentioned, how the system interpolates between the low and the high $T$ regime, and for how long (how many e-folds) this period lasts. In addition, in hybrid models as the inflaton evolves towards the critical value $\phi_c$, the mass of the waterfall field decreases accordingly, approaching the tachyonic instability, with $m^2 = 2g^2(\phi^2 - \phi_c^2)$. The dissipative coefficient Eq. (4) depends on the ratio of $T^3/m^2$, and thus close to the critical value it will get a rather large enhancement factor, with

$$\Upsilon_\phi \simeq C_\phi \left( \frac{\phi^2}{\phi^2 - \phi_c^2} \right)^3 \frac{T^3}{\phi^2}. \quad (62)$$

Although it may look at a first glance that this enhancement of $\Upsilon_\phi$, and therefore $r$, will bring the system faster into the high $T$ regime, the increase of the extra friction in the inflaton evolution tends to slow down the field, and as a consequence also the evolution of the ratios $T/H$ and $\phi/T$. This again provides some extra e-folds of inflation in the warm dissipative regime. Finally, we will have to take into account thermal corrections in the scalar masses, mainly $m_\chi$, which can be non negligible towards the end of inflation in the high $T$ regime. Nevertheless, in order to take into account these effects one has to resort to numerical calculations, which are beyond the scope of this paper. The results presented here can be seen as a kind of worst case scenario, and the bounds on the parameters will be relaxed once we give up the requirement of having the full 50-60 last e-folds in the low $T$ range.

### III. SUMMARY

This paper has made an important step in understanding dissipative effects and particle production during the inflationary expansion period. These results are of general significance in understanding inflationary dynamics, since they break preconceived ideas set early in the development of inflation that the dynamics during inflation has negligible dissipative and particle production effects. The results in this paper show that picture does not express the general case and that dissipative effects become important in very generic models. Moreover as discussed in this paper and elsewhere, the consequences of dissipation during inflation are nontrivial with respect to observable signatures and model building prospects.

The technical problem of determining dissipative effects during inflation should not be underestimated. To appreciate this point, it is useful to compare this warm inflation problem to the standard cold inflation problem. In the latter, the particle production phase is pictured as entirely separated from the inflationary expansion phase; this problem has been intensely studied for well over two decades and is still not fully understood. The warm inflation problem is technically much more difficult, since particle production and inflationary expansion are meant to be occurring concurrently and this problem has been examined for a much shorter amount of time.

The basic problem of determining dissipative effects during inflation can be posed as follows. A background inflaton field is evolving slowly in time as the Universe inflates. This field is coupled to other fields, thus in general dissipative and particle production effects can be expected. This is a nonequilibrium situation, with the central problem being to determine what is the statistical state of the Universe. To address this problem from first principles quantum field theory has required building up the necessary knowledge through an interplay of investigating nonequilibrium approximations and then testing their relevance to the actual inflation problem. Ideally one could imagine trying to solve this problem through numerical calculations and computer simulations, but the problem is too complex to immediately do a very general treatment this way. Ultimately the goal is to achieve some sort of reliable treatment of the problem by these methods. However before that can be achieved, a clearer idea of the approximate state of the system is needed. This is the direction that has been developing in the past few years. The one baseline one has to go on is the assumption the statistical state remains close to thermal equilibrium throughout evolution. There are obvious deviations from equilibrium due to the evolution of the background inflaton field and due to production of light particles and their rescattering with the background field. Nevertheless, thermal equilibrium is a good limiting
case to examine since reliable and unambiguous calculations can be made of dissipation and particle production. However the assumption of equilibrium is a conservative one, from which one anticipates dissipative effects will be minimal. As such this approximation is vitally important in determining a minimal level of warm inflation.

With these considerations in mind, the results in this paper are all the more encouraging. We have shown that all the simple inflation models, those with monomial and hybrid potentials, have warm inflation regimes. Moreover the two distinguishing model building features of warm inflation, inflaton mass bigger than the Hubble parameter so complete avoidance to the “eta problem” and field amplitudes below the Planck scale even in monomial potentials, have both been verified explicitly in models. Although improvements on the equilibrium approximation to obtain the correct nonequilibrium state will quantitatively change the results from those in this paper, these qualitative features will persist and so have been firmly established as realizable from realistic quantum field theory models.

The quantitative changes from improving on the equilibrium approximation should lead to a decrease in the total number of fields and so to even more simpler models that yield warm inflation. In this paper, within the equilibrium approximation, we found models realizing warm inflation requiring a minimum of \( N_{\chi} \approx O(10^3/N_{\text{decay}}^2) \) fields. This is within the realm of realistic model building with particle physics GUT models such as \( SO(10) \) or \( E_6 \), with the \( \chi \) field in the \( 210 \) or \( 351 \), and a decay factor \( N_{\text{decay}} \sim O(3 - 4) \). To gain an idea of how much these models can be improved from the nonequilibrium treatment can be gained by examining the results already studied for warm inflation using the quasiparticle approximation \( \frac{3}{4} \frac{12}{16} \frac{16}{12} \). This approximation has been studied in the context of quantum field theory and cosmology applications since the work of Morikawa and Sasaki in the mid 1980s [19]. It is motivated by generic features known of many-body systems in condensed matter physics. No derivation of this approximation has been made in quantum field theory, but there is some suggestive numerical evidence in support of it [33]. One of the next steps in the development of warm inflation dynamics is to understand by how much the equilibrium approximation deviates and in particular how closely it tends to the quasiparticle approximation.

In summary, this paper has accomplished two things. On the side of principle, due to the thermal equilibrium approximation underlying all the results, this paper has calculated minimal expectations of radiation production and dissipative effects during inflation. On the side of application, this paper has identified and developed a reliable methodology for performing warm inflation calculations. Although the results found here are only on the periphery of usefulness for model building, the methodology developed in this paper gives a foothold from which further improvements can be made. Aside from the major modifications already mentioned of expanding the equilibrium approximation to a more accurate nonequilibrium treatment, there are also several improves within the equilibrium approximation that can be made, and would lower the number of fields necessary, thus increase the scope of model building prospects. One modification is to move away from the strict low temperature regime into the intermediate temperature regime. This is technically much more difficult since now all finite temperature effective potential corrections will have to be accounted for. However the results of this paper already anticipate that the relaxation of this condition will increase dissipative effects, lower the total number of fields needed and help resolve issues of exiting the inflation epoch. A second improvement is treating higher order calculations of the dissipation coefficient. This dissipative coefficient is closely related to the shear viscosity, as observed in [12], for which resummation calculations [34] have shown provide as much as a factor four enhancement. One anticipates similar enhancements could occur also for the dissipative coefficients.

Acknowledgments

The authors thank Ian Moss for valuable discussions, and Lisa Hall for comments. AB was funded by the United Kingdom Particle Physics and Astronomy Research Council (PPARC).

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