DYNAMIC DISCRETE-TIME PORTFOLIO SELECTION FOR DEFINED CONTRIBUTION PENSION FUNDS WITH INFLATION RISK

HAIXIANG YAO
School of Finance
Guangdong University of Foreign Studies, Guangzhou 510006, China

PING CHEN AND MIAO ZHANG
Actuarial Studies, Department of Economics
University of Melbourne, Australia

XUN LI*
Department of Applied Mathematics
The Hong Kong Polytechnic University, Hong Kong, China

(Communicated by Hailiang Yang)

ABSTRACT. This paper investigates a multi-period asset allocation problem for a defined contribution (DC) pension fund facing stochastic inflation under the Markowitz mean-variance criterion. The stochastic inflation rate is described by a discrete-time version of the Ornstein-Uhlenbeck process. To the best of our knowledge, the literature along the line of dynamic portfolio selection under inflation is dominated by continuous-time models. This paper is the first work to investigate the problem in a discrete-time setting. Using the techniques of state variable transformation, matrix theory, and dynamic programming, we derive the analytical expressions for the efficient investment strategy and the efficient frontier. Moreover, our model’s exceptional cases are discussed, indicating that our theoretical results are consistent with the existing literature. Finally, the results established are tested through empirical studies based on Australia’s data, where there is a typical DC pension system. The impacts of inflation, investment horizon, estimation error, and superannuation guarantee rate on the efficient frontier are illustrated.

1. Introduction. Generally speaking, there are two types of plans that employers adopt to fund retirement benefits. A defined benefit (DB) plan is what most people think of as a traditional pension plan. The benefit is based on a formula, taking into account the retiree’s years of service, final salary before retirement, and age. In a defined contribution (DC) plan, the employer contributes a certain percentage of the

2020 Mathematics Subject Classification. Primary: 90C26; Secondary: 91B28, 49N15.

Key words and phrases. Stochastic inflation rate, multi-period mean-variance formulation, portfolio selection, defined contribution pension fund, efficient frontier.

This research is partially supported by the National Natural Science Foundation of China (Nos. 71871071, 72071051, 71471045), the Innovative Research Group Project of National Natural Science Foundation of China (No. 71721001), the Natural Science Foundation of Guangdong Province of China (Nos. 2018B030311004, 2017A030313399), and Research Grants Council of Hong Kong under grants 15213218 and 15215319.

* Corresponding author: Xun Li.
employee’s salary to an account each year. The account grows with contributions and investment earnings. The fundamental difference between a DC plan and a DB plan is who takes the investment risk. The DB plan guarantees the pension for life so that the employer bears the investment risk. However, in the DC plan, the accumulated money in the pension fund account could be inadequate to support the employee’s retirement, that is, the employee takes the investment risk. In the past few decades, faced with inflation and sustained decreasing mortality rate, the DB system carries tremendous pressure to pay sufficient pensions to retirees. This work explains the worldwide trend of conversion from DB to DC plans. Nowadays, most newly established pension plans in the United States, Canada, and Australia are based on DC.

In the DC pension scheme, members’ retirement benefit is determined by the accumulation in the pension fund’s account, driven by two key factors: contribution rate and investment rate of return. The contribution rate is set by the employer or employee, usually subject to a minimum requirement by the legislation and hence is a relatively stable rate. The main source of risk born by the DC fund member comes from the uncertainty of investment return. Accordingly, an investment strategy in the DC pension plan is of essential importance. In this paper, the investment strategy for the DC pension fund is studied under the mean-variance framework\(^1\)

It is well known that inflation is an essential factor for individual investors, families, and fund managers to consider, especially for long-term investment durations. By adopting the Ornstein-Uhlenbeck process as the expected rate of inflation in a continuous-time setting, [5] and [26] developed a class of optimal dynamic asset allocation problems. In recent years, more factors such as consumption, investment, and life insurance policies are included\(^2\). The risk of inflation is also a crucial factor for DC pension funds for two reasons. First, the purpose of the pension scheme is to provide sufficient funds for post-retirement consumption. If inflation is unexpectedly high, the member’s accumulation will lose its purchasing power and fail to achieve adequacy. Second, the pension plan is usually managed on a long-term horizon (say from 20 to 40 years), which implies more inflation fluctuations. Therefore, it is crucial to take into account the inflation risk for the DC pension fund.

Portfolio selection problems involving the inflation risk in the DC pension fund have been widely considered. [34] studied the expected power utility maximization investment problem for a DC pension fund with inflation. [15] solved a similar problem, which also includes the interest rate risk. Assuming that the price follows a geometric Brownian motion, [24] treated an asset allocation problem for the DC pension. In a continuous-time setting, [33] modeled the consumer price index (CPI) by the Ornstein-Uhlenbeck process, then formulated an optimal investment problem of the DC pension scheme under the mean-variance criterion. [7] investigated an optimal investment strategy under the loss aversion and a minimum performance

---

\(^1\)The mean-variance portfolio selection model firstly proposed by [25] has become one of the foundations of modern finance theory and inspired many extensions and applications (e.g., [20], [22], [8], [32], and [9] ect.).

\(^2\)Based on continuous-time CRRA utility maximization model and by using martingale approach, [19] investigated an optimal consumption, investment, and life insurance decision problem of a family under inflation risk. [21] considered the time-consistent reinsurance-investment problem with stochastic interest rate and inflation risk for an insurer under the mean-variance criterion. [16] studied the optimal consumption, portfolio, and life insurance policies under interest rate and inflation risks in which the wage earner’s preferences are represented by the stochastic differential utility.
constraint for the DC pension plan with the inflation risk. By incorporating the stochastic interest rate and a minimum guarantee of inflation protection on the annuities, [29] obtained the closed-form solutions of an asset allocation problem for the DC pension fund.

All of the literature mentioned above on the DC pension fund management with inflation is limited to continuous-time settings. In practice, however, dynamic asset allocation strategies can only be implemented in a manner of discrete-time. To enhance the model’s practicability, this paper attempts to fill up the gap using a discrete-time formulation. Another feature of our model is that we incorporate stochastic inflation risk, where the inflation rate is described by a discrete-time version of Vasicek model (Ornstein-Uhlenbeck process). In addition to inflation, the salary level is also assumed to be stochastic. This adds two more state variables to the fund wealth process. These extensions significantly increase the computational complexity in obtaining the closed form solutions.

Our model points out that there are multiple numbers of risky assets, which results in a series of matrix computations, instead of scalar calculations, when we apply dynamic programming. The existence of closed form solutions depends on the invertibility, positivity, and comparability of the evolving matrices. Hence, it cannot be trivially proved. Using the mathematical property of the Moore-Penrose pseudoinverse of a matrix, symmetrical square matrix, and the technique of reduction to absurdity, we can obtain the desired property and then derive the explicit expression of the optimal investment strategy.

Besides, our model is a general one which includes many special cases. For example, by setting the contribution rate $g_k$ of the pension fund to 0 for time periods $k = 0, 1, \ldots, T - 1$, our model degenerates to an ordinary multi-period portfolio selection model under stochastic inflation, which is also yet to be studied in the existing literature. We refer to Section 6 for more details. As a direct application of our result, we establish a three-fund theorem in our general model. In another special case, when the stochastic inflation rate, the risky assets, and the wage income growth rate are uncorrelated for each other, we find that the conventional two-fund theorem for a multi-period mean-variance portfolio selection problem (Li and Ng, 2000) also holds in our model.

In the empirical studies, we use data from Australia to illustrate the impacts of the efficient frontier’s main parameters. In 2017, Australia had the highest proportion in the world of DC to DB assets at 87% to 13%. The 20-year growth in pension assets in Australia has been 12.1% per annum. The critical features in this success have been government-mandated pension contributions, a competitive institutional model, and the dominance of DC\textsuperscript{3}. By adopting the contribution rate as the current “superannuation guarantee (SG)” rate in Australia, we study the impacts of inflation, investment horizon, estimation error, and varying contribution rate on efficient frontier within the framework of mean-variance. Our main findings are: a longer time horizon is more likely to be capable of managing its investment risk than of a shorter one; given the same level of risk, the non-inflation return rate is higher than the inflationary case; our results are robust to the estimation bias; a higher contribution rate or initial salary not only results in a higher investment return but also brings more risk.

\textsuperscript{3}From the Global Pension Asset Study 2018 by Towers Watson, at www.towerswatson.com. Also available through: Australian Bureaus of Statistics, Managed Funds, Australia (cat 5655.0), March 2018.
We believe that this work will enrich the literature on dynamic investment problems for DC pension funds. Recently, this topic has attracted more and more attention in the academic world. By applying the discrete-time dynamic programming technique, [10] considered the DC pension with an annuity guarantee in the continuous-time model. Based on CARA utility maximization, [13] derived an optimal investment and contribution strategy of the DC pension fund when the scheme member’s income is stochastic. Under the mean-variance framework, [30] investigated a portfolio selection problem in the accumulation phase of a defined contribution (DC) pension scheme. [31] considered an asset allocation problem for the DC pension fund with both stochastic income and mortality risk under the multi-period mean-variance framework. [17] studied an optimal asset allocation and benefit outgo policies of the DC pension plan with compulsory conversion claims, and derive the closed form solutions for the optimal policy using the technique of HJB (Hamilton-Jacobi-Bellman) equation and variational inequality. [18] analyzed an optimal saving problem for individuals with DC pension plans and solved the optimization problem by combining the stochastic optimal control and multi-stage stochastic programming. Using the mean-variance criterion, [36] investigated an optimal multi-period investment management problem for a defined contribution pension fund with imperfect information. [12] considered an optimal investment problem for a defined contribution (DC) pension fund manager under loss aversion (S-shaped utility) and with trading and Value-at-Risk (VaR) constraints. For more literature on the asset allocation of the DC pension, the reader is referred to [14], [3], [4], [6], [28], and [11].

The remainder of this paper is organized as follows. Section 2 describes the market setting for our problem. Section 3 sets up the multi-period portfolio selection problem for a DC pension fund with inflation. In Section 4, we transform the original problem into a standard stochastic optimal control problem, and derive the corresponding analytical solution using dynamic programming approach. Closed form expressions for the efficient strategy and the efficient frontier are derived in Section 5 via Lagrange theory. Some special cases and their simplified results are presented in Section 6. In Section 7, the results obtained in this paper are illustrated via some empirical examples. Section 8 concludes this paper. To make our presentation concise, we place all the proofs in the appendices.

2. Market setting. We consider a financial market composed of $n + 1$ assets (indexed as asset 0, 1, $\cdots$, $n$) that may include a risk free asset (could be all risky). Let $r^i_k$ be the random gross returns of the $i$th stock over period $k$ (i.e., the period from time $k$ to $e k + 1$), $i = 0, 1, \cdots, n$ and $k = 0, 1, \cdots, T$. For convenience, let $r_k = (r^0_k, r^1_k, \cdots, r^n_k)'$ denote the random gross returns of these $n + 1$ assets over period $k$.

This paper considers a multi-period asset allocation problem for a DC pension fund. Suppose that a representative pension fund member enters the pension fund at time 0, and plans to retire at time $T$. Before the retirement, members need to pay contribution into the fund at the beginning of each period in a predefined way. Upon the retirement, the accumulated wealth in the retiree’s pension fund account is converted to an annuity, which will be paid to the retiree regularly after the retirement. If a member dies before the retirement, the accumulated money in the member’s pension account can be withdrawn. Denote the initial wealth of a member’s account by $x_0$, and the member’s initial income by $y_0 (> 0)$. Let $y_k$ be the wage income received at time $k$. Assume that the wage income is stochastic,
and satisfies the following dynamics:

\[ y_{k+1} = q_k y_k, \quad k = 0, 1, \cdots, T - 1, \]

where \( q_k \) is an exogenous random variable representing the stochastic growth rate of the wage income over period \( k \). Obviously, the wage income can not be negative, so we suppose that \( q_k > 0 \) almost surely for all \( k = 0, 1, \cdots, T - 1 \). Suppose that \( g_k y_k \) is the amount that the member contributes at time \( k \), where \( g_k \) is a deterministic variable depending only on \( k \). Let \( x_k \) and \( z_k \) be the wealth of the member's account just before and after the contribution at time \( k \), respectively. Then, we have \( z_k = x_k + g_k y_k \).

**Remark 1.** In order to let our model be more general, we do not assume that \( g_k \geq 0 \). For example, when \( g_k < 0 \), \( g_k y_k \) can be interpreted as the member’s consumption or the benefit received by the member from the pension fund over period \( k \), for \( k = 1, 2, \cdots, T \). Therefore, our model can also be used to study the asset allocation problem for the DC pension in the de-cumulation phase.

Suppose that the pension fund can be invested in the \( n + 1 \) assets in the market. Incorporating the contribution \( g_k y_k \) at the beginning of period \( k \), the total available amount for investment is \( (x_k + g_k y_k) \). Denote the proportion of member’s wealth in the pension fund’s account invested in the \( i \)th risky asset at time \( k \) by \( u^i_k \). Then, the proportion invested in the 0th asset at time \( k \) is \( 1 - \sum_{i=1}^{n} u^i_k \). Therefore, \( x_k \) follows the dynamics:

\[
\begin{align*}
x_{k+1} &= (x_k + g_k y_k) \left[ (1 - \sum_{i=1}^{n} u^i_k) r^0_k + \sum_{i=1}^{n} u^i_k r^i_k \right] \\
&= (x_k + g_k y_k) \left( r^0_k + \sum_{i=1}^{n} u^i_k p^i_k \right) \\
&= (x_k + g_k y_k) \left( r^0_k + P_k' u_k \right),
\end{align*}
\]

where \( P_k = (p^1_k, p^2_k, \cdots, p^n_k)' \) is \( (r^1_k - r^0_k, r^2_k - r^0_k, \cdots, r^n_k - r^0_k)' \). Here, \( A' \) denote the transpose of a matrix or vector \( A \).

In the long-term management of a pension fund, the effect of price inflation plays an important role in one’s living standard. The periodic consumer price index (CPI) is an official publication in many countries. Denote by \( P^0_k \) the consumer price index from at time \( k \) for \( k = 0, 1, ..., T - 1, T \). Then the corresponding inflation rate for the period from \( k \) to \( k + 1 \) is \( \Pi_k = \frac{P^0_{k+1} - P^0_k}{P^0_k} \cdot \). Let \( 1 + \Pi_k = e^{\psi_k} \), that is, \( I_k \) is the continuous inflation rate (or force of inflation rate) from \( k \) to \( k + 1 \). Following the spirit of [5] and [26], we assume that \( I_k \) follows a discrete-time Ornstein-Uhlenbeck (i.e., mean-reversion) process. Namely,

\[
I_{k+1} - I_k = \psi_k (I - I_k) + \sigma_k \varepsilon_k, \quad k = 0, 1, \cdots, T - 1,
\]

where \( I \) is the long-run mean of the inflation rate, \( \psi_k \) is the degree of mean reversion, \( \varepsilon_k \) is a random satisfying \( E[\varepsilon_k] = 0 \) and \( \text{Var}(\varepsilon_k) = 1 \), and \( \sigma_k \) is the volatility of the inflation rate. Then, the real wealth \( \bar{x}_k \) follows

\[
\bar{x}_k = x_k \frac{P^0_k}{P^0_0} = x_0 \frac{P^0_0}{P^0_1} \frac{P^0_1}{P^0_2} \cdots \frac{P^0_{k-1}}{P^0_k} \frac{P^0_k}{P^0_{k-1}} \cdots (1 + \Pi_0)^{-1} \cdots (1 + \Pi_{k-1})^{-1}
\]
where $\Lambda_k = e^{-t_k}$ for $k = 0, 1, \ldots, T - 1$. Let $\phi_k = 1 - \psi_k$, according to (3), we have
\[
\Lambda_{k+1} = e^{-t_{k+1}} = \Lambda_k \phi_k e^{-(1-\phi_k)\bar{f}_k - \sigma_k \varepsilon_k} = c_k \Lambda_k^{\phi_k},
\]
where $c_k = e^{-(1-\phi_k)\bar{f}_k - \sigma_k \varepsilon_k} > 0$.

Similarly, we have the real average wage from active members $\bar{y}_k$ follows,
\[
\bar{y}_k = y_k \Lambda_0 \Lambda_1 \cdots \Lambda_{k-1}.
\]

According to (1), (2), (4) and (6), it follows that
\[
\bar{x}_{k+1} = \Lambda_k (\bar{x}_k + g_k \bar{y}_k) (r_k^0 + P_k u_k), \quad (7)
\]
and
\[
\bar{y}_{k+1} = \Lambda_k \bar{y}_k y_k = q_k \Lambda_k \bar{y}_k. \quad (8)
\]

Note that now there are basically three stochastic processes involved in our optimization problem: $\bar{x}_k$, $\bar{y}_k$ and $\Lambda_k$, which will be incorporated in the objective function in the next section.

3. Model formulation. Let $\varphi_k$ be the information set up to time $k$. An investment strategy $u = \{u_k; k = 0, 1, \ldots, T - 1\}$ is called an admissible strategy, if $u_k$ is finite and progressive measurable with respect to $\varphi_k$ for $k = 0, 1, \ldots, T - 1$. Denote $\Theta(\bar{x}_k, \bar{y}_k, \Lambda_k)$ the set of all such admissible investment strategies starting from time $k$ to time $T$ with initial state $(\bar{x}_k, \bar{y}_k, \Lambda_k)$.

The multi-period mean-variance portfolio selection problem refers to the problem of finding an optimal admissible investment strategy such that variance of the terminal wealth is minimized for a given expected terminal wealth level $d$. With stochastic inflation rate, the multi-period mean-variance asset allocation for DC pension fund can be formulated as
\[
\begin{aligned}
\min_{u \in \Theta(\bar{x}_0, \bar{y}_0, \Lambda_0)} & \quad \text{Var}(\bar{x}_T) = E[\bar{x}_T - d]^2, \\
\text{s.t.} & \quad E[\bar{x}_T] = d, \text{ subject to } (5), (7) \text{ and } (8).
\end{aligned} \quad (9)
\]

For $d \geq d_{\sigma_{\text{min}}}$, the corresponding optimal solution $u^* = \{u^*_k; k = 0, 1, \ldots, T - 1\}$ of Problem (9) as called an efficient investment strategy, where $d_{\sigma_{\text{min}}}$ is the expected terminal wealth corresponding to the global minimum variance of the terminal wealth over all feasible strategies. Let $\text{Var}^*(\bar{x}_T)$ be the corresponding minimum value of Problem (9), then the point $(\text{Var}(\bar{x}_T), d)$ is called an efficient point. The set of all the efficient points is called the efficient frontier.

The constraint equality $E[\bar{x}_T] = d$ can be tackled by the Lagrange method. To derive the problem conveniently, we choose a Lagrange multiplier $2\gamma$ and transform the original optimization problem into
\[
\begin{aligned}
\min_{u \in \Theta(\bar{x}_0, \bar{y}_0, \Lambda_0)} & \quad E[\bar{x}_T - d]^2 + 2\gamma (E[\bar{x}_T] - d), \\
\text{s.t.} & \quad (5), (7) \text{ and } (8).
\end{aligned} \quad (10)
\]

In Problem (10), we have
\[
E[\bar{x}_T - d]^2 + 2\gamma (E[\bar{x}_T] - d) = E[\bar{x}_T^2 + 2a\bar{x}_T] - d^2 - 2ad, \quad (11)
\]
where \( a = \gamma - d \). Since \(-d^2 - 2ad \) is a fixed value, then Problem (10) is equivalent to the following optimization problem in the sense that they share the same optimal solution:

\[
\begin{aligned}
\min_{u \in \Theta(x, y_0, \Lambda)} & \quad \mathbb{E}[\bar{x}_T^2 + 2a\bar{x}_T], \\
\text{subject to} & \quad (5), (7) \text{ and } (8).
\end{aligned}
\]

In the next section, we solve Problem (12) using the technique of dynamic programming.

4. Solution of the Lagrange problem. In order to apply dynamic programming, we introduce the Bellman’s equation for Problem (12). We consider a truncated form of Problem (12) starting from time \( k \) with real wealth \( \bar{x}_k = \bar{x} \), real wage income \( \bar{y}_k = \bar{y} \), and gross deflation rate \( \Lambda_k = \Lambda > 0 \). Define the corresponding value function as follows

\[
f_k(\bar{x}, \bar{y}, \Lambda) = \min_{u \in \Theta(\bar{x}, \bar{y}, \Lambda)} \mathbb{E} \left[ \bar{x}_T^2 + 2a \bar{x}_T \mid \bar{x}_k = \bar{x}, \bar{y}_k = \bar{y}, \Lambda_k = \Lambda \right],
\]

subject to (5), (7) and (8). Then, according to the dynamic programming principle, we obtain Bellman’s equation for Problem (12) as follows:

\[
\begin{aligned}
f_k(\bar{x}, \bar{y}, \Lambda) &= \min_{u_k} \mathbb{E} \left[ f_{k+1}(\Lambda (\bar{x} + g_k \bar{y}) \left( r_k^0 + P_k' u_k \right), q_k \Lambda \bar{y}, c_k \lambda^k) \right], \\
f_T(\bar{x}, \bar{y}, \Lambda) &= \bar{x}^2 + 2a \bar{x}.
\end{aligned}
\]

Consequently, setting \( k = 0 \), we have \( f_0(\bar{x}_0, \bar{y}_0, \Lambda_0) \) and \( f_0(\bar{x}_0, \bar{y}_0, \Lambda_0) - d^2 - 2ad \) are the optimal value functions for Problem (12) and Problem (10), respectively.

For notational simplicity, we write \( \mathbb{E}^{-1}[\cdot] = (\mathbb{E}[\cdot])^{-1} \). To derive the expression for \( f_k(\bar{x}, \bar{y}, \Lambda) \), we construct series of \( \delta_k, w_k, \lambda_k, \alpha_k, \varpi_k, \eta_k \) and \( \theta_k \) \( (k = 0, 1, \ldots, T) \) satisfying the following recursions:

\[
\begin{aligned}
\delta_k &= 1 + \phi_k \delta_{k+1}, \quad w_k = w_{k+1} D_k, \quad \lambda_k = \lambda_{k+1} C_k, \\
\alpha_k &= \alpha_{k+1} - \frac{\lambda_k^2}{4w_k} G_k, \quad \varpi_k = \varpi_{k+1} F_k + 2g_k w_{k+1} D_k, \\
\eta_k &= \eta_{k+1} \mathbb{E}[c_k^2 + q_k^2] + g_k w_{k+1} D_k + \varpi_{k+1} g_k F_k - \frac{\varpi_k}{\lambda_k} J_k, \\
\theta_k &= \theta_{k+1} \mathbb{E}[c_k^2 + q_k^2] + g_k w_{k+1} D_k - \frac{\lambda_k \varpi_k}{\lambda_{k+1} w_{k+1}} H_k,
\end{aligned}
\]

where

\[
\begin{aligned}
D_k &= \mathbb{E}[r_k^0 r_{k+1}^2] - \mathbb{E}[r_k^0 r_{k+1}^2 P_k' \mathbb{E}^{-1}[c_k^2 + q_k^2 P_k' P_k]], \\
C_k &= \mathbb{E}[r_k^0 r_{k+1}^2 r_{k+1}^2 P_k' \mathbb{E}^{-1}[c_k^2 + q_k^2 P_k' P_k]], \\
G_k &= \mathbb{E}[c_{k+1}^2 + P_k' \mathbb{E}^{-1}[c_{k+1}^2 + q_k^2 P_k' P_k]], \\
F_k &= \mathbb{E}[r_k^0 c_{k+1}^2 + q_k^2 P_k' \mathbb{E}^{-1}[c_{k+1}^2 + q_k^2 P_k' P_k]], \\
J_k &= \mathbb{E}[c_{k+1}^2 + q_k^2 P_k' \mathbb{E}^{-1}[c_{k+1}^2 + q_k^2 P_k' P_k]], \\
H_k &= \mathbb{E}[c_{k+1}^2 + q_k^2 P_k' \mathbb{E}^{-1}[c_{k+1}^2 + q_k^2 P_k' P_k]],
\end{aligned}
\]

with boundary conditions \( \delta_T = 0, w_T = 1, \lambda_T = 2, \alpha_T = 0, \varpi_T = 0, \eta_T = 0, \theta_T = 0 \).

Suppose that \( M \) and \( N \) are symmetric matrices with the same order. Throughout this paper, we denote \( M > N \) \( (M \geq N) \) if only if \( (\text{iff}) M - N \) is positive definite (semidefinite). Especially, \( M > 0 \) \( (M \geq 0) \) iff \( M \) is positive definite (semidefinite).

Let \( |M| \) denote the determinant for square matrix \( M \). Following most existing literatures, throughout this paper, we make the following assumptions.

**Assumption 1.** \( \mathbb{E}[r_k r_k'] > 0 \), i.e., \( \mathbb{E}[r_k r_k'] \) is positive definite for \( k = 0, 1, \ldots, T - 1 \), where \( r_k = (r_k^0, r_k^1, r_k^2, \ldots, r_k^T)' \).
Assumption 2. Random series $Y_k = (r'_k, q_k, \varepsilon_k)$ are statistically independent for $k = 0, 1, \cdots, T - 1$, i.e., $Y_i$ and $Y_j$ are independent for $i \neq j$.

Assumption 3. $E[c_k^{\delta_k+1} P'_k] \neq 0$ for all $k = 0, 1, \cdots, T - 1$, where $0$ is zero vector of order $n$.

For self-completeness, we give the following lemmas before we proceed to the main results in this paper.

Lemma 1. [32] Let $\xi = (\xi_1, \xi_2, \cdots, \xi_n)'$ be a random vector, then $|E[\xi\xi']| = 0$ iff there is a nonzero vector $b = (b_1, b_2, \cdots, b_n)'$ such that $b'\xi = b_1\xi_1 + b_2\xi_2 + \cdots + b_n\xi_n = 0$ with probability 1.

Lemma 2. [32] Let $\xi = (\xi_1, \xi_2, \cdots, \xi_n)'$ be a random vector such that $E[\xi\xi']$ positive definite, and $\eta$ is another positive random variable. Then we have $E[\eta\xi\xi']$ is also a positive definite matrix.

For any matrix, denote by $M^+$ the Moore-Penrose pseudo inverse of $M$ satisfying:

$$MM^+M = M, \quad M^+MM^+ = M^+, \quad (MM^+)' = MM^+, \quad (M^+M)' = M^+M.$$ 

In particular, $M^+$ is unique for any matrix $M$, and if the inverse $M^{-1}$ of $M$ exists, then $M^+ = M^{-1}$.

Let $M$ be a symmetrical square matrix, and be partitioned as $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} \end{pmatrix}$, where $M_{11}$ and $M_{22}$ are also symmetrical square matrices. Then, the following lemmas hold.

Lemma 3 (35). If $M_{22} \succ 0$, then we have $|M| = |M_{22}||M_{11} - M_{12}M_{22}^{-1}M_{12}'|$. 

Lemma 4 (11). $M \succeq 0$ is equivalent to $M_{22} \succ 0$, $M_{22}M_{22}^+M_{12}' = M_{12}'$ and $M_{11} - M_{12}M_{22}^+M_{12}' \succeq 0$.

Lemma 5 (35). If $M \succeq N \succeq 0$, then we have $|M| \geq |N|$.

For convenience, Let 

$$B_k = \begin{pmatrix} r_k^0 \\ P_k \end{pmatrix}, \quad L_k = \begin{pmatrix} 1 \\ c_k^{\delta_k+1} P_k \end{pmatrix}, \quad Q_k = c_k^{\delta_k+1} B_k.$$ 

Before we derive the expressions for the optimal investment strategy and the value function, we need to present the following propositions.

Proposition 1. $E[c_k^{2\delta_k+1} P'_k], E[B_kB'_k]$ and $E[Q_kQ'_k]$ are all positive definite for $k = 0, 1, \cdots, T - 1$.

Proof 4.1. Refer to Appendix A.1.

The next proposition includes some useful properties regarding $D_k, C_k$ and $G_k$.

Proposition 2. For $k = 0, 1, \cdots, T - 1$, we have 

$$D_k = |E[Q'_k]||E[c_k^{2\delta_k+1} P'_k]|^{-1},$$

$$C_k = |E[Q_kL'_k]||E[c_k^{2\delta_k+1} P'_k]|^{-1},$$

$$1 - G_k = |E[L_kL'_k]||E[c_k^{2\delta_k+1} P'_k]|^{-1}.$$ 

Proof 4.2. Refer to Appendix A.2.
Proposition 3. For any $k = 0, 1, \ldots, T - 1$, we have $D_k > 0$, $0 < G_k \leq 1$ and $C_k^2 \leq D_k(1 - G_k) < D_k$.

Proof 4.3. Refer to Appendix A.3.

For convenience, define $\prod_{i=k}^{k-1} (\cdot) = 1$, $\sum_{i=k}^{k-1} (\cdot) = 0$. According to the recursive relations and boundary conditions about $\delta_k$, $w_k$, $\lambda_k$, $\alpha_k$, $\varpi_k$, $\eta_k$ and $\theta_k$ in (14), we obtain their expressions by repeated backward iterations.

Proposition 4. For all $k = 0, 1, \ldots, T$, we have

$$
\begin{align*}
\delta_k &= \sum_{i=k}^{T-1} \prod_{j=k}^{i-1} \phi_j, \quad w_k = \prod_{i=k}^{T-1} D_i > 0, \\
\lambda_k &= 2 \prod_{i=k}^{T-1} C_i, \quad \alpha_k = -\sum_{i=k}^{T-1} G_i \prod_{j=i+1}^{T-1} C_j^2, \\
\varpi_k &= 2 \sum_{i=k}^{T-1} g_i w_{i+1} D_i \prod_{j=k}^{i-1} F_j, \\
\eta_k &= \sum_{i=k}^{T-1} \left( g_i^2 w_{i+1} D_i + \varpi_{i+1} g_i F_i - \frac{\varpi_i}{w_{i+1}} F_i \right) \prod_{j=k}^{i-1} \mathbb{E} \left[ \mathcal{C}_{j+1}^2 q_j^2 \right], \\
\theta_k &= \sum_{i=k}^{T-1} \left( \lambda_{i+1} g_i C_i - \frac{\lambda_{i+1} \varpi_{i+1}}{w_{i+1}} H_i \right) \prod_{j=k}^{i-1} \mathbb{E} \left[ \mathcal{C}_{j+1}^2 q_j^2 \right].
\end{align*}
$$

(18)

Based on the above preliminary results, we present the following theorem which is the main result of this paper.

Theorem 1. For $k = 0, 1, \ldots, T - 1$, the optimal value function of Problem (12) is given by

$$
f_k(\bar{x}, \bar{y}, \Lambda) = w_k \bar{x}^2 \Lambda^{2k} + \lambda_k a \bar{x} \Lambda^{\delta_k} + \alpha_k a^2 + \varpi_k \Lambda^{2k} \bar{y}^2 + \eta_k a \Lambda^{2k} \bar{y}^2 + \theta_k a \Lambda^{2k} \bar{y},
$$

(19)

and the corresponding optimal investment strategy is given by

$$
u_k^*(\bar{x}, \bar{y}, \Lambda) = -E^{-1} \left[ \mathcal{C}_{k+1}^2 P_k P_k' \right] \times \left( \mathbb{E} \left[ v_k^0 \mathcal{C}_k^2 P_k \right] + \frac{\lambda_{k+1} a \Lambda^{-\delta_k} \mathbb{E} \left[ \mathcal{C}_{k+1}^2 P_k \right]}{2(\bar{x} + g_k \bar{y}) w_{k+1}} + \frac{\varpi_{k+1} \bar{y} \mathbb{E} \left[ \mathcal{C}_{k+1}^2 q_k P_k \right]}{(2(\bar{x} + g_k \bar{y}) w_{k+1})} \right),
$$

(20)

where $\delta_k$, $w_k$, $\lambda_k$, $\alpha_k$, $\varpi_k$, $\eta_k$ and $\theta_k$ are defined by (14) or (18).

Proof 4.4. Refer to Appendix A.4.

Remark 2. Note that $\Lambda_k = e^{-I_k}$, so (19) and (20) can be rewritten as

$$
f_k(\bar{x}, \bar{y}, I_k) = w_k \bar{x}^2 e^{-2k I_k} + \lambda_k a \bar{x} e^{-\delta_k I_k} + \alpha_k a^2 + \varpi_k e^{-2k I_k} \bar{y}^2 + \eta_k e^{-2k I_k} \bar{y}^2 + \theta_k a e^{-\delta_k I_k} \bar{y},
$$

(21)

$$
u_k^*(\bar{x}, \bar{y}, I_k) = -E^{-1} \left[ \mathcal{C}_{k+1}^2 P_k P_k' \right] \times \left( \mathbb{E} \left[ v_k^0 \mathcal{C}_k^2 P_k \right] + \frac{\lambda_{k+1} a \Lambda^{-\delta_k} \mathbb{E} \left[ \mathcal{C}_{k+1}^2 P_k \right]}{2(\bar{x} + g_k \bar{y}) w_{k+1}} + \frac{\varpi_{k+1} \bar{y} \mathbb{E} \left[ \mathcal{C}_{k+1}^2 q_k P_k \right]}{(2(\bar{x} + g_k \bar{y}) w_{k+1})} \right).
$$

(22)

In a continuous-time setting, Yao et al. (2013) also studied a portfolio selection problem for DC pension funds under inflation. They obtained the expression for the optimal value function and the corresponding optimal investment strategy as follows

$$
V(t, I, \bar{x}) = p(t, I) \bar{x}^2 + g(t, I)a \bar{x} + h(t) a^2 + \varpi(t, I) \bar{x} + \varphi(t, I) a + z(t, I),
$$

(23)
where, \( p(t, I), q(t, I), h(t, I), \omega(t, I), \phi(t, I) \) and \( z(t, I) \) are some functions defined in [33], and \( \delta(t), \tilde{B}(t), \sigma_0(t), \sigma_1(t), B_p(t) \) and \( B_g(t) \) are some market parameters or defined by the market parameter (see Yao et al. (2013) for more details). Comparing \( (23) \) and \( (24) \) with \( (21) \) and \( (22) \) in this paper, we find that both in continuous-time and discrete-time settings, the optimal value function is a quadratic function of \( \bar{x} \) and \( \alpha \), and the optimal investment strategy is an inverse proportional function of \( \bar{x} \). There are some differences between these two cases, which can be stated as follows.

To make the problem easier to be handled and solved, in the continuous-time model of [33], it is assumed that the pension contribution rate increases with the inflation rate, so the dynamic process of the wage income needs not to be set. Namely, the wage income is not a state variable in the model of [33]. As a result, the real wage level \( \bar{y} \) does not appear in the optimal value function and optimal strategy of [33], while the optimal value function and optimal strategy in our model are quadratic function and inverse proportional function of \( \bar{y} \), respectively. In addition, in the expression of the optimal value function and optimal strategy of our discrete-time model, the inflation rate \( I \) appears in an exponential form; while in the continuous-time case of [33], \( I \) could be in exponential form in some functions (e.g., both \( p(t, I) \) and \( q(t, I) \) are exponential functions of \( I \)), or in more complex form in some other functions (e.g., both \( \phi(t, I) \) and \( z(t, I) \) are complex nonlinear functions of \( I \)), please see [33] for more details.

5. Efficient investment strategy and efficient frontier. It is known from Theorem 1 that

\[
f_0(\bar{x}_0, \bar{y}_0, \Lambda_0) = w_0 \bar{x}_0^2 \Lambda_0^{2d_0} + \lambda_0 a \bar{x}_0 \Lambda_0^{d_0} + \alpha_0 a^2 + \omega_0 \Lambda_0^{2d_0} \bar{x}_0 \bar{y}_0 + \eta_0 \Lambda_0^{d_0} \bar{y}_0^2 + \theta_0 a \Lambda_0^{d_0} \bar{y}_0.
\]

Hence the optimal value of Problem (10) is given by

\[
L(\bar{x}_0, \bar{y}_0, \Lambda_0, a) = f_0(\bar{x}_0, \bar{y}_0, \Lambda_0) - d^2 - 2ad \\
= w_0 \bar{x}_0^2 \Lambda_0^{2d_0} + \lambda_0 a \bar{x}_0 \Lambda_0^{d_0} + \alpha_0 a^2 + \omega_0 \Lambda_0^{2d_0} \bar{x}_0 \bar{y}_0 + \eta_0 \Lambda_0^{d_0} \bar{y}_0^2 + \theta_0 a \Lambda_0^{d_0} \bar{y}_0 - d^2 - 2ad \\
= \alpha_0 a^2 + (\lambda_0 \bar{x}_0 \Lambda_0^{d_0} + \theta_0 \Lambda_0^{d_0} \bar{y}_0 - 2d) a + w_0 \bar{x}_0^2 \Lambda_0^{2d_0} + \omega_0 \Lambda_0^{2d_0} \bar{x}_0 \bar{y}_0 + \eta_0 \Lambda_0^{d_0} \bar{y}_0^2 - d^2,
\]

where \( a = \gamma - d \).

According to the Lagrange dual theory (see [23]), the optimal value for the original mean-variance Problem (9), namely the minimum variance \( \text{Var}^*(x_T) \), can be obtained by maximizing \( L(\bar{x}_0, \bar{y}_0, \Lambda_0, a) \) over \( \gamma \). Recall that \( a = \gamma - d \), this is equivalent to maximizing \( L(\bar{x}_0, \bar{y}_0, \Lambda_0, a) \) over \( a \), i.e.,

\[
\text{Var}^*(x_T) = \max_a L(\bar{x}_0, \bar{y}_0, \Lambda_0, a),
\]

where \( L(\bar{x}_0, \bar{y}_0, \Lambda_0, a) \) is given by (25).

In order to show that the existence and uniqueness of solutions for Problem (26), we first give the following proposition.

**Proposition 5.** For all \( k = 0, 1, \ldots, T - 1 \), we have
(i) $0 > \alpha_k \geq -1 + \frac{T-1}{2} \sum_{j=k}^{T} \frac{C_j^2}{D_j} = -1 + \frac{\lambda_j^2}{4\omega_k} \geq -1$;

(ii) $\alpha_k = -1 + \frac{\lambda_j^2}{4\omega_k} \iff C_j^2 = 1 - G_j$ for $j = k, k+1, \cdots, T-1$.

The proof of Proposition 5 is similar to that of Proposition 5 in [32]. So we omit it here.

Proposition 5 shows that $\alpha_0 < 0$, therefore, there exists a unique optimal solution to Problem (A.58). By the first-order condition we obtain the unique optimal solution as follows:

$$a^* = \frac{\lambda_0 \bar{x}_0 \Lambda_0^{\delta_0} + \theta_0 \Lambda_0^{\delta_0} \bar{y}_0 - 2d}{2\alpha_0}. \quad (27)$$

Substituting (27) into (20) with $\bar{x}_k = \bar{x}, \bar{y}_k = \bar{y}$ and $\Lambda_k = \Lambda$, the optimal investment strategy to the mean-variance model (9) can be presented by

$$u_k(\bar{x}_k, \bar{y}_k, \Lambda_k) = -\mathbb{E}^{-1}[c_k^{2\delta_k+1} P_k P_k'] \left( \mathbb{E} [c_k^{2\delta_k+1} P_k] + \frac{\mathbb{E} [c_k^{2\delta_k+1} q_k P_k]}{2(\bar{x}_k + \bar{y}_k) \bar{w}_{k+1}} \right). \quad (28)$$

Again, substituting (27) into (26), we obtain the minimum variance of the mean-variance model (9) as follows

$$\text{Var}^*(\bar{x}_T) = \begin{cases} 
-\frac{1+\alpha_0}{\alpha_0} \left( d - \frac{\lambda_0 \bar{x}_0 \Lambda_0^{\delta_0} + \theta_0 \Lambda_0^{\delta_0} \bar{y}_0}{2(1+\alpha_0)} \right)^2 - \frac{\left( \lambda_0 \bar{x}_0 \Lambda_0^{\delta_0} + \theta_0 \Lambda_0^{\delta_0} \bar{y}_0 \right)^2}{4(1+\alpha_0)} \\
+ \frac{u_0 \bar{x}_0^2 \Lambda_0^{\delta_0} + \bar{w}_0 \Lambda_0^{\delta_0} \bar{x}_0 \bar{y}_0 + \eta_0 \Lambda_0^{\delta_0} \bar{y}_0^2}{4} \quad \alpha_0 \neq -1, \\
- \left( \lambda_0 \bar{x}_0 \Lambda_0^{\delta_0} + \theta_0 \Lambda_0^{\delta_0} \bar{y}_0 \right) d + \frac{\left( \lambda_0 \bar{x}_0 \Lambda_0^{\delta_0} + \theta_0 \Lambda_0^{\delta_0} \bar{y}_0 \right)^2}{4} \\
+ \frac{u_0 \bar{x}_0^2 \Lambda_0^{\delta_0} + \bar{w}_0 \Lambda_0^{\delta_0} \bar{x}_0 \bar{y}_0 + \eta_0 \Lambda_0^{\delta_0} \bar{y}_0^2}{4} \quad \alpha_0 = -1.
\end{cases} \quad (29)$$

Since for any real number $d$, we should have $\text{Var}^*(\bar{x}_T) \geq 0$, then we exclude the case of $\alpha_0 = -1$ here. Thus, the minimum variance is given by

$$\text{Var}^*(\bar{x}_T) = -\frac{1+\alpha_0}{\alpha_0} \left( d - \frac{\lambda_0 \bar{x}_0 \Lambda_0^{\delta_0} + \theta_0 \Lambda_0^{\delta_0} \bar{y}_0}{2(1+\alpha_0)} \right)^2 \quad \alpha_0 \neq -1. \quad (30)$$

Setting $d = d_{\sigma_{\text{min}}} := \frac{\lambda_0 \bar{x}_0 \Lambda_0^{\delta_0} + \theta_0 \Lambda_0^{\delta_0} \bar{y}_0}{2(1+\alpha_0)}$, we obtain the global minimum variance

$$\text{Var}_{\text{min}}(\bar{x}_T) := \frac{u_0 \bar{x}_0^2 \Lambda_0^{\delta_0} + \bar{w}_0 \Lambda_0^{\delta_0} \bar{x}_0 \bar{y}_0 + \eta_0 \Lambda_0^{\delta_0} \bar{y}_0^2}{4(1+\alpha_0)} \quad \alpha_0 = -1. \quad (31)$$

Obviously, rational investors would not select an expected terminal wealth which is less than $d_{\sigma_{\text{min}}}$. We summarize the above results in the following theorem.

**Theorem 2.** For a given expected terminal wealth $\mathbb{E}[\bar{x}_T] = d (d \geq d_{\sigma_{\text{min}}})$, the efficient investment strategy and the efficient frontier of the multi-period mean-variance asset allocation problem (9) for a DC pension fund under stochastic inflation rate are given by (28) and (30), respectively.
Remark 3. From (28) and (30), we find that the efficient frontier is a parabolic function of the expected final wealth \( u \), and the efficient strategy is a linear function of \( u \). This conclusion is consistent with the continuous-time case considered by [33].

Let
\[
\begin{align*}
\zeta_k &= -E^{-1}[c_k^{2\delta_k+1}P_kP_k'\mathbb{E}[\delta_k^{2\delta_k+1}P_k]], \\
\xi_k &= -E^{-1}[c_k^{2\delta_k+1}P_kP_k'\mathbb{E}[\delta_k^{2\delta_k+1}q_kP_k]], \\
\tilde{\pi}_k &= \frac{\zeta_k}{\xi_k}, \\
\tilde{\xi}_k &= \frac{\zeta_k}{\xi_k}, \\
\tilde{\pi}_k &= \frac{\zeta_k}{\xi_k}, \\
\tilde{\xi}_k &= \frac{\zeta_k}{\xi_k}.
\end{align*}
\]

Then, the optimal investment strategy (28) can be rewritten as
\[
u_k^2(\tilde{x}_k, \tilde{y}_k, \Lambda_k) = (1'\zeta_k)\tilde{\xi}_k + \frac{\pi_{k+1}\tilde{y}_k(1'\xi_k)\tilde{\xi}_k + \frac{1}{2}\tilde{\lambda}_k\Lambda_k\tilde{\xi}_k}{2\tilde{\alpha}(\tilde{x}_k+\tilde{y}_k)\tilde{y}_k}\tilde{\xi}_k + \frac{\delta_k\Lambda_k\tilde{\lambda}_k\tilde{\xi}_k}{2\tilde{\alpha}(\tilde{x}_k+\tilde{y}_k)\tilde{y}_k}\tilde{\xi}_k + \frac{\delta_k\Lambda_k\tilde{\lambda}_k\tilde{\xi}_k}{2\tilde{\alpha}(\tilde{x}_k+\tilde{y}_k)\tilde{y}_k}\tilde{\xi}_k.
\]

It follows from (33) that the optimal strategy linearly depends on \( \tilde{\xi}_k, \tilde{\pi}_k \) and \( \tilde{\xi}_k \), which can be taken as three mutual funds. This leads to a “Three Fund” Theorem. This result is concluded in the following theorem.

Theorem 3. Under the stochastic inflation (3), the following hold:

The optimal portfolio involves an allocation linearly constructed by the three “artificial” mutual funds \( \tilde{\xi}_k, \tilde{\pi}_k \) and \( \tilde{\xi}_k \). Moreover, the optimal proportional allocations \( \mu_1^k, \mu_2^k \) and \( \mu_3^k \) of wealth in light of three mutual funds \( \zeta_k, \pi_k \) and \( \xi_k \) are presented by

\[
\begin{align*}
\mu_1^k &= (1'\zeta_k), \\
\mu_2^k &= \frac{\pi_{k+1}\tilde{y}_k(1'\xi_k)}{2\tilde{\alpha}(\tilde{x}_k+\tilde{y}_k)\tilde{y}_k}, \\
\mu_3^k &= \frac{\delta_k\Lambda_k\tilde{\xi}_k}{2\tilde{\alpha}(\tilde{x}_k+\tilde{y}_k)\tilde{y}_k}.
\end{align*}
\]

Theorem 3 shows that all the optimal portfolios can be generated from three mutual funds \( \tilde{\xi}_k, \tilde{\pi}_k \) and \( \tilde{\xi}_k \). From (34), we have the following interesting findings:

(i) The investment proportion \( \mu_1^k \) in the mutual fund \( \tilde{\xi}_k \) is determined by some market parameters only, and is independent of the initial real wealth \( \tilde{x}_0 \), the initial real wage income \( \tilde{y}_0 \), the current real wealth \( \tilde{x}_k \), the current real wage income \( \tilde{y}_k \), and the investor’s risk attitude dictated by the expected terminal wealth level \( d \);

(ii) The investment proportion \( \mu_2^k \) in the mutual fund \( \tilde{\xi}_k \) is affected by both the current real wealth \( \tilde{x}_k \) and the current real wage income \( \tilde{y}_k \), but is independent of the initial real wealth \( \tilde{x}_0 \), the initial real wage income \( \tilde{y}_0 \) and the expected terminal wealth level \( d \);

(iii) The proportion \( \mu_3^k \) invested in \( \tilde{\pi}_k \) is affected by all parameters including the initial real wealth \( \tilde{x}_0 \), the initial real wage income \( \tilde{y}_0 \), the current real wealth \( \tilde{x}_k \), the current real wage income \( \tilde{y}_k \), and the expected terminal wealth level \( d \).

Remark 4. Under the stochastic inflation, a three-fund theorem exists within the framework of continuous-time portfolio selection, such as [5]. Their paper pointed out that one fund arises from the uncertainty of the real interest rate, another one is the classical optimal growth fund, and the last one replicates the inflation uncertainty. We point out that [33] did not discuss the fund separation theorem in their continuous-time model, but based on their expression of the optimal investment strategy (see (24)), a three-fund theorem can be constructed by \( (\delta(t)\delta'(t))^{-1}B(t), (\delta(t)\delta'(t))^{-1}\delta(t)\sigma'\alpha(t) \) and \( (\delta(t)\delta'(t))^{-1}\delta(t)\sigma'_\alpha(t) \).
Let
\[
\chi^1_k = -E^{-1}[P_k P_k']E[P_k], \quad \chi^2_k = -E^{-1}[P_k P_k']E[r_k^0 P_k], \quad \bar{\chi}^1_k = \frac{\chi^1_k}{1'\chi^1_k}, \quad \bar{\chi}^2_k = \frac{\chi^2_k}{1'\chi^2_k}.
\]

(35)

We find that, if the stochastic inflation rate, the risky assets, and the stochastic growth rate of the wage income are uncorrelated with each other (i.e., \(c_k, q_k\) and \(P_k\) are uncorrelated), then fund \(\tilde{\chi}_k\) reduces to \(\bar{\chi}_k^2\), and both \(\tilde{\pi}_k\) and \(\tilde{\chi}_k\) reduce to \(\bar{\chi}_k^1\). As a result, the optimal portfolio \(u^*_k(x_k, y_k, A_k)\) is an affine function of \(\bar{\chi}_k^1\) and \(\bar{\chi}_k^2\), that is, the classical “Two Fund” Theorem (see [20]) also holds (see Section 6 for more details). This implies that, in the optimal investment strategy, the underlying reason on the transition from the two-fund property to the three-fund property comes from the correlation among the stochastic inflation rate, the risky assets and the stochastic wage income.

6. Special cases. To study the impacts of the inflation rate and pension contribution on both the optimal strategy and the mean-variance efficient frontier, this section discusses some special cases of our model. The corresponding simplified results are presented. For convenience, in this section, we denote \(\mathbb{E}^2[\cdot] = (\mathbb{E}[\cdot])^2\).

Special case 1: When the inflation rate, excess returns of the risky assets, and the growth rate of the wage income are independent with each other. Mathematically, \(\varepsilon_k, P_k\) and \(q_k\) are statistically independent with each other, \(k = 0, 1, 2, \ldots, T-1\). As a result, \(c_k = e^{-(1-\theta_k)T-\sigma_k}\varepsilon_k\), \(q_k\) and \(P_k\) are uncorrelated. According to (15), in this case, \(D_k, C_k, G_k, F_k, J_k\) and \(H_k\) can be simplified as follows

\[
\begin{align*}
D_k & = \mathbb{E}[c_k^{2k+1}]X_k, & C_k & = \mathbb{E}[c_k^{\delta k+1}]Y_k, & G_k & = \frac{g^2[c_k^{\delta k+1}]}{\mathbb{E}[c_k^{\delta k+1}]}A_k, \\
F_k & = \mathbb{E}[c_k^{2k+1}]\mathbb{E}[q_k]X_k, & J_k & = \mathbb{E}[c_k^{2k+1}]\mathbb{E}^2[q_k]A_k, & H_k & = \mathbb{E}[c_k^{\delta k+1}]\mathbb{E}[q_k]A_k, \\
\end{align*}
\]

(36)

where

\[
\begin{align*}
A_k & = \mathbb{E}[P_k'E^{-1}[P_k'P_k']E[P_k]], \\
X_k & = \mathbb{E}[(r_k^0)^2] - \mathbb{E}[r_k^0 P_k']E^{-1}[P_k P_k']E[r_k^0 P_k], \\
Y_k & = \mathbb{E}[r_k^0] - \mathbb{E}[r_k^0 P_k']E^{-1}[P_k P_k']E[P_k].
\end{align*}
\]

(37)

Hence, by (18), we have \(u_k, \lambda_k, \alpha_k, \omega_k, \eta_k\) and \(\theta_k\) can be expressed in terms of three symbols \(A_k, X_k\) and \(Y_k\), instead of six symbols \(D_k, C_k, G_k, F_k, J_k\) and \(H_k\),

\[
\begin{align*}
w_k & = \prod_{i=k}^{T-1} \mathbb{E}[c_i^{2i+1}]X_i, & \lambda_k & = 2 \prod_{i=k}^{T-1} \mathbb{E}[c_i^{\delta i+1}]Y_i, \\
\alpha_k & = 2 \prod_{i=k}^{T-1} \mathbb{E}[c_i^{\delta i+1}]A_i \prod_{j=i+1}^{T-1} \frac{\mathbb{E}[c_j^{\delta j+1}]Y_j^2}{\mathbb{E}[c_j^{\delta j+1}]X_j}, \\
\omega_k & = 2 \sum_{i=k}^{T-1} g_i w_i [\mathbb{E}[c_i^{2i+1}]X_i \prod_{j=k}^{T-1} \mathbb{E}[c_j^{2j+1}]E[q_j]Y_j], \\
\eta_k & = \frac{1}{T-k} \sum_{i=k}^{T-1} \mathbb{E}[c_i^{2i+1}] \left( g_i^2 w_i X_i + \omega_i E[q_i]Y_i - \frac{g_i^2 E^2[q_i]A_i}{2w_i+1} \right) \prod_{j=k}^{T-1} \mathbb{E}[c_j^{2j+1}]E[q_j^2], \\
\theta_k & = \frac{1}{T-k} \sum_{i=k}^{T-1} \mathbb{E}[c_i^{\delta i+1}] \left( \lambda_i g_i Y_i - \frac{\lambda_i \omega_i E^2[q_i]A_i}{w_i+1} \right) \prod_{j=k}^{T-1} \mathbb{E}[c_j^{2j+1}]E[q_j^2].
\end{align*}
\]

(38)

In this case, by (28), the efficient investment strategy can be simplified by
Hence, by (18),
\begin{equation}
\chi_k^2 = \left( \frac{\sigma_k + \bar{y}_k E[q_k]}{\sqrt{\Sigma_{k} + \bar{y}_k y_k}} \right)^{w_k} + \frac{\Lambda_{k+1}(\Lambda_0 \bar{y}_k \Lambda_k + \theta_0 \Lambda_0 \bar{y}_k - 2d) \Lambda_{k+1}^{-1}}{4d \sigma_k + \lambda k + \bar{y}_k y_k} \right)^{w_k} \chi_{k+1}^2 \chi_k^2,
\end{equation}
where \( \chi_1^2 \) and \( \chi_2^2 \) are defined by (35), and the efficient frontier is also given by (30).

By (39), we know that the efficient investment strategy is proportional to \( \chi_1^2 \) and \( \chi_2^2 \), so the “Two Fund” Theorem holds. That is, instead of allocating the wealth among \( n+1 \) assets, an investor only needs to allocate his wealth between the cash account and the two funds \( \chi_1^2 \) and \( \chi_2^2 \) to achieve the same investment performance.

**Special case 2:** There exists a risk-free asset in the market, and \( \epsilon_k, P_k \) and \( q_k \) are statistically independent with each other. Suppose that the 0th asset is the risk-free asset. Then \( r_0^k \) is a deterministic constant for \( k = 0, 1, 2, \ldots, T-1 \). According to (37), in this case, we further have
\begin{equation}
X_k = (r_0^k)^2(1 - A_k), \quad Y_k = r_0^k(1 - A_k),
\end{equation}
where \( A_k = E[P_k]E^{-1}[P_k P_k]E[P_k] \). Then, by (36), \( D_k, C_k, G_k, F_k, J_k \) and \( H_k \) can be simplified further by
\begin{equation}
\begin{aligned}
D_k &= E[(c_k^{2k+1})(r_0^k)^2(1 - A_k)], \quad C_k = E[(c_k^k)^2 r_0^k (1 - A_k)], \\
G_k &= \frac{r_0^k E[c_k^{2k+1}]}{E[c_k^{2k+1}]} A_k, \quad F_k = E[(c_k^{2k+1})] E[q_k] r_0^k (1 - A_k), \\
J_k &= E[c_k^{2k+1}] E^2[q_k] A_k, \quad H_k = E[c_k^{2k+1}] E[q_k] A_k.
\end{aligned}
\end{equation}

Hence, by (18), \( w_k, \lambda_k, \alpha_k, \varpi_k, \eta_k, \theta_k \) and \( \theta_k \) can be further expressed in terms of \( A_k \) only, rather than \( A_k, X_k \) and \( Y_k \):
\begin{equation}
\begin{aligned}
w_k &= \prod_{i=k}^{T-1} E[c_i^{2i}] (r_i^0)^2 (1 - A_i), \quad \lambda_k = 2 \prod_{i=k}^{T-1} E[c_i^{2i+1}] r_0^i (1 - A_i), \\
\alpha_k &= -\sum_{i=k}^{T-1} \frac{E[c_i^{2i+1}] A_i}{E[c_i^{2i+1}]}, \\
\varpi_k &= 2 \sum_{i=k}^{T-1} g_i w_i + E[c_i^{2i+1}] (r_0^i)^2 (1 - A_i) - E[c_i^{2i+1}] E[q_i] r_0^i (1 - A_j), \\
\eta_k &= \sum_{i=k}^{T-1} \left( g_i r_0^i (1 - A_i) - \frac{E[c_i^{2i}] E[q_i] A_i}{4w_i^2} \right) \prod_{j=k}^{T-1} E[c_j^{2j+1}] E[q_j^2], \\
\theta_k &= \sum_{i=k}^{T-1} \frac{E[c_i^{2i+1}] (r_i^0)^2 (1 - A_i)}{w_i^2} \prod_{j=k}^{T-1} E[c_j^{2j+1}] E[q_j^2].
\end{aligned}
\end{equation}

In this case, by (35), we have \( \chi_k^2 = -E^{-1}[P_k P_k]E[r_0^k P_k] = r_0^k \chi_1^2 \). Therefore, according to (39), the efficient investment strategy can be further simplified as
\begin{equation}
\begin{aligned}
u_k^*(\bar{x}_k, \bar{y}_k, A_k) &= \left( r_0^k + \frac{\sigma_k + \bar{y}_k E[q_k]}{\sqrt{\Sigma_{k} + \bar{y}_k y_k}} \right)^{w_k} + \frac{\Lambda_{k+1}(\Lambda_0 \bar{y}_k \Lambda_0 + \theta_0 \Lambda_0 \bar{y}_k - 2d) \Lambda_{k+1}^{-1}}{4d \sigma_k + \lambda k + \bar{y}_k y_k} \right)^{w_k} \chi_{k+1}^2 \chi_k^2,
\end{aligned}
\end{equation}
while the efficient frontier is still given by (30). We conclude from (43) that the efficient investment strategy is proportional to \( \chi_1^2 \). This leads to the well-known one-fund theorem in the conventional dynamic portfolio selection problem with a risk-free asset (see [20] for more details). This finding implies that the lack of a risk-free asset is the reason transforming the one-fund theorem to the two-fund theorem in the optimal investment strategy.
Special case 3: The case with no inflation. We only need to set \( \bar{I} = 0, \phi_k = \sigma_k = 0 \) for \( k = 0, 1, \ldots, T - 1 \). Then by (4), we have \( I_k = 0 \) for \( k = 0, 1, \cdots, T \). This means there is no inflation. By (5) and notice that \( c_k = e^{-(1-\phi_k)I \cdots \sigma_k \epsilon_k} \) and \( \Lambda_k = \prod_{i=1}^{K} = e^{-I_k} > 0 \), in this case, we have \( c_k = \Lambda_k = \prod_{i=1}^{K} = 1 \) for \( k = 0, 1, \cdots, T \), which leads to \( \bar{x}_k = x_k \) and \( \bar{y}_k = y_k \) for \( k = 0, 1, \cdots, T \). Then, according to (15), \( D_k, C_k, G_k, F_k, J_k \) and \( H_k \) can be simplified by

\[
\begin{align*}
D_k &= \mathbb{E}[(\bar{r}^2_k)] - \mathbb{E}[\bar{r}_k \bar{P}_k] \mathbb{E}^{-1}[P_k P_k'] \mathbb{E}[(\bar{r}_k^2) P_k], \\
C_k &= \mathbb{E}[\bar{r}_k^0] - \mathbb{E}[\bar{r}_k \bar{P}_k] \mathbb{E}^{-1}[P_k P_k'] \mathbb{E}[(\bar{r}_k^0) P_k], \\
G_k &= \mathbb{E}[P_k P_k'] \mathbb{E}^{-1}[P_k P_k'] \mathbb{E}[P_k], \\
F_k &= \mathbb{E}[\bar{r}_k \bar{q}_k] - \mathbb{E}[\bar{r}_k \bar{P}_k] \mathbb{E}^{-1}[P_k P_k'] \mathbb{E}[q_k P_k], \\
J_k &= \mathbb{E}[q_k P_k'] \mathbb{E}^{-1}[P_k P_k'] \mathbb{E}[q_k P_k], \\
H_k &= \mathbb{E}[P_k] \mathbb{E}^{-1}[P_k P_k'] \mathbb{E}[q_k P_k].
\end{align*}
\]

In addition, \( \delta_k, \eta_k \) and \( \theta_k \) can be simplified by

\[
\begin{align*}
\delta_T &= 0, \delta_k = 1, \ k = 0, 1, 2, \cdots, T - 1, \\
\eta_k &= \sum_{i=k}^{T-1} \left( g^i_k w_{i+1} D_i + \bar{w}_{i+1} F_i g_i - \frac{\bar{w}_{i+1}}{4w_{i+1}} J_i \right) \prod_{j=k}^{i-1} \mathbb{E}[q^2_j], \\
\theta_k &= \sum_{i=k}^{T-1} \left( \lambda_{i+1} c_i g_i - \frac{\lambda_{i+1} \bar{w}_{i+1}}{w_{i+1}} H_i \right) \prod_{j=k}^{i-1} \mathbb{E}[q^2_j],
\end{align*}
\]

and \( w_k, \lambda_k, \alpha_k \) and \( \bar{w}_k, \eta_k \) are also given by (18). In this case, the efficient investment strategy can be simplified by

\[
\begin{align*}
u_k^*(x_k, y_k) &= -\mathbb{E}^{-1}[P_k P_k'] \left( \mathbb{E}[\bar{r}_k^0 P_k] + \frac{\bar{w}_{k+1} y_k}{2(x_k + y_k y_k) w_{k+1}} \mathbb{E}[q_k P_k] \
+ \frac{\lambda_{k+1}(\lambda_0 x_0 + \theta_0 y_0) 2d}{4(x_k + y_k y_k) w_{k+1}} \mathbb{E}[P_k] \right),
\end{align*}
\]

and the efficient frontier is simplified by

\[
\begin{align*}
\text{Var}^*(x_T) &= - \frac{1 + \alpha_0}{\alpha_0} \left( d - \frac{\lambda_0 x_0 + \theta_0 y_0}{2(1 + \alpha_0)} \right)^2 \
&\quad + w_0 x_0^2 + \bar{w}_0 x_0 y_0 + \eta_0 y_0^2 - \frac{(\lambda_0 x_0 + \theta_0 y_0)^2}{4(1 + \alpha_0)}.
\end{align*}
\]

Special case 4: The case of no pension contribution. In this case, we only need to set the contribution rate \( y_k = 0 \) for \( k = 0, 1, \cdots, T - 1 \). Then, our model degenerates to an ordinary multi-period mean–variance portfolio selection model under inflation. By (18), for \( k = 0, 1, \cdots, T \), we have

\[
\bar{w}_k = 0, \quad \eta_k = 0, \quad \theta_k = 0.
\]

In this case, the symbols \( F_k, J_k \) and \( H_k \) appear as 0 in our expressions. According to (28) and (30), in this case, the efficient investment strategy and the efficient frontier are independent of the state variable \( \bar{y}_k \), and are simplified by

\[
\begin{align*}
u_k^*(\bar{x}_k, \Lambda_k) &= -\mathbb{E}^{-1}[c_k^{2\delta_k+1} P_k P_k'] \left( \mathbb{E}[\bar{r}_k^0 c_k^{2\delta_k+1} P_k] + \frac{\lambda_{k+1}(\lambda_0 \bar{x}_0 A_0^{\delta_k} - 2d \Lambda_0^{\delta_k} \Lambda_k^{\delta_k} \bar{x}_0 + 2d A_0^{\delta_k} \Lambda_k^{\delta_k})}{4(1 + \alpha_0)} \mathbb{E}[c_k^{\delta_k+1} P_k] \right),
\end{align*}
\]

and

\[
\text{Var}^*(\bar{x}_T) = - \frac{1 + \alpha_0}{\alpha_0} \left( d - \frac{\lambda_0 \bar{x}_0 A_0^{\delta_k}}{2(1 + \alpha_0)} \right)^2 + \left( w_0 - \frac{\lambda_0^2}{4(1 + \alpha_0)} \right) \bar{x}_0^2 A_0^{2\delta_k},
\]

respectively. For \( k = 0, 1, \cdots, T - 1 \), the expressions of \( \delta_k, w_k, \lambda_k \) and \( \alpha_k \) are also given by (18).
7. **Empirical examples.** In this section, we use data in Australia to illustrate how the pension fund can keep a tradeoff between its return and risk in terms of efficient frontiers.

To make the illustration more understandable, we present a brief outline of the Australian retirement funding system before analysis. The pension scheme in Australia is a typically defined contribution system, or called “superannuation” in its context. In this section, we interchangeably use superannuation and pension, referring to Australia’s retirement income system. The Australian government introduced the compulsory superannuation scheme in 1992. According to the Superannuation Guarantee (Administration) Act, the employer is required to pay a proportion of its employees’ salary to a superannuation fund as the “superannuation guarantee (SG)**, that is, $g_k$ in our model. This contribution rate has been rising from 3% in 1992 to 9.5% in 2017 and is supposed to continue the growth to 12% by 2025. Each member of the superannuation fund has an individual account managed by the fund trustee. Each year, the investment return and contribution from employers are credited to the individual account with administration fees deducted. Upon retirement, fund members will have access to the accumulated amount within their account for retirement spending.

**Data and estimation.** To implement our model, the following inputs are required: time unit, time horizon $T$, SG contribution rate $g_k$, salary growth rate $q_k$, inflation rate of price index $I_k$, the riskless rate $r_f$, the mean and covariance matrix of risk premiums $P_k$ for a basket of stocks.

We take quarters as the time unit and consider a time horizon of 5 quarters. In the first example, we explore how the efficient frontier is varied within different time horizons. The contribution rate is set at 9.5% as a benchmark. We adjust this ratio in the fourth example to reflect its future growth for a better forecast. The salary growth rate $q_k$ is assumed to be time-invariant based on the index of “average weekly earnings” (AWE) published by the ABS (Australian Bureau of Statistics) every half year. The sample covers the period from May 2012 to Nov 2016. To reflect the whole population characteristics, we choose the index of total earnings including full time and part time income for both male and female. The growth rate $q_k$ is directly estimated as the mean of the sample and adjusted to be a quarterly rate. Both samples and estimates are listed in Table 1.

As for the inflation rate of price index, we also use the data provided by the ABS which reports price index every quarter as an indicator of the price level across Australia. The series of index number starts from September 1948 until December 2016 including 274 observations. Our model assumes a mean-reverting log-inflation rate $I_k$. Three parameters are involved: the long-term mean $\bar{I}$, the autoregressive coefficient $\phi$ and the variance of random error $\sigma^2$. The process $I_k - \bar{I}$ belongs to AR(1) model as follows,

$$I_k - \bar{I} = \phi(I_{k-1} - \bar{I}) + \epsilon_k, \quad k = 1, 2, \ldots$$

To apply the ordinary least square (OLS) estimation, we assume that the random errors $\{\epsilon_k\}$ are independently and identically distributed (IID). We first estimate $\bar{I}$ with the mean of the 274 observations. Then, the centered data $I_k - \bar{I}$ is used to perform an OLS estimation of $\phi$ and $\sigma^2$. The IID assumption for $\{\epsilon_k\}$ guarantees three estimators’ consistency. The results of the regression analysis are summarized in Table 2.
We use the treasury note (TN) yield of duration between one and six months as a proxy for the risk free interest rate $r_f$. The historical data of TN yield is publicly available from the official website "Australian Office of Financial Management". Using the “Weighted Average Issue Yield” from 2009 to 2017, we estimate the riskless rate in each year, and then risk premiums can be computed from these estimated rates. We list the results in Table 3.

For the risky assets, we select three shares listed on the ASX (Australian Securities Exchange): ANZ (Australia and New Zealand Banking Group), BHP Billiton, and Telstra. According to their market share and capitalization, the three companies represent three important industry sectors in Australia. ANZ is one of Australia’s four largest banks. As a mining multinational, BHP Billiton is one of the largest companies by its market value. Telstra is a leader in the telecommunication and media industry. We collect the dividend-adjusted monthly stock price during the period of January 2009 to April 2017 from “Yahoo finance”. Then, combined with the risk-less rates $r_f$ calculated from TN yield, the time unvarying expectation vector $E[P_k]$ and covariance matrix $E[P_k P_k']$ are estimated by their sample counterparts as follows,

\[
\begin{align*}
E[P_k] &= [5.51\%, 3.78\%, 3.23\%]' , \\
E[P_k P_k'] &= \begin{bmatrix}
3.50\% & 2.38\% & 0.63\% \\
2.38\% & 8.14\% & -0.07\% \\
0.63\% & -0.07\% & 1.72\%
\end{bmatrix}.
\end{align*}
\]

Finally, for technical reasons, we assume that $\{\epsilon_t\}$ and $\{P_t\}$ are independent, that is, the equity premium is uncorrelated to the inflation rate.

Now we are ready to depict the shape of a pension fund’s efficient frontier. The reward and risk balance is captured in the mean-variance relation as follows,

\[
\text{Var}^*(\bar{x}_T) = -\frac{1 + \alpha_0}{\alpha_0}(d - d_{\sigma_{\min}})^2 + \text{Var}_{\min}(\bar{x}_T).
\]

The above relation is between the anticipatory accumulated asset value and its variance at time $T$. We can rewrite this equation by dividing both sides with $\bar{x}_0^2$, and it becomes

\[
\text{Var}^*\left(\frac{\bar{x}_T}{\bar{x}_0}\right) = -\frac{1 + \alpha_0}{\alpha_0} \left(\frac{d}{\bar{x}_0} - \frac{d_{\sigma_{\min}}}{\bar{x}_0}\right)^2 + \frac{\text{Var}_{\min}(\bar{x}_T)}{\bar{x}_0^2}.
\]

Note that at time 0, the inflation-adjusted money value is its nominal value, that is, $\bar{x}_0 = x_0, \bar{y}_0 = y_0$. Hence, we have

\[
\frac{d_{\sigma_{\min}}}{\bar{x}_0} = \frac{\Lambda \delta_0 (\lambda_0 + \theta_0 a)}{2(1 + \alpha_0)},
\]

\[
\frac{\text{Var}_{\min}(\bar{x}_T)}{\bar{x}_0^2} = \frac{\Lambda^2 \omega_0(a + \eta_0 a^2) - \Lambda \delta_0 (\lambda_0 + \theta_0 a)^2}{4(1 + \alpha_0)},
\]

where $a = \frac{\bar{y}_0}{\bar{x}_0}$. Apparently, the equation (51) describes the mean and variance of the pension fund’s accumulation rate during a period of time rather than the magnitude of its accumulated asset value. Let $R_T = \frac{\bar{x}_T}{\bar{x}_0} - 1$ denote the real accumulation rate of fund asset during the whole period from 0 to $T$. Note that this is different from the return rate in one time unit. Thus, we have $E[R_T] = \frac{d}{\bar{x}_0} - 1$, and the equation
Table 1. Salary growth rate

| Date      | Total earnings ($) | Growth rate |
|-----------|--------------------|-------------|
| May-2012  | 1053.20            | N/A         |
| Nov-2012  | 1081.30            | 1.0267      |
| May-2013  | 1105.00            | 1.0219      |
| Nov-2013  | 1114.20            | 1.0083      |
| May-2014  | 1123.00            | 1.0079      |
| Nov-2014  | 1128.70            | 1.0051      |
| May-2015  | 1136.90            | 1.0073      |
| Nov-2015  | 1145.70            | 1.0077      |
| May-2016  | 1160.90            | 1.0133      |
| Nov-2016  | 1163.50            | 1.0022      |
| $q_k$(half yearly) | N/A | 1.0112 |
| $q_k$(quarterly)    | N/A    | 1.0056    |

Table 2. Log-inflation rate

| $\hat{I}$ | $\hat{\sigma}$ | $\hat{\phi}$ | Confidence interval for $\hat{\phi}$ (95%) |
|------------|----------------|--------------|----------------------------------|
| 0.54%      | 0.45%          | 0.5709       | (0.4731,0.6687)                  |

Table 3. TN yield

| Year | Weighted Average Issue Yield (%) |
|------|----------------------------------|
| 2009 | 3.1537                           |
| 2010 | 4.4971                           |
| 2011 | 4.5861                           |
| 2012 | 3.4670                           |
| 2013 | 2.6450                           |
| 2014 | 2.5127                           |
| 2015 | 2.0541                           |
| 2016 | 1.8134                           |
| 2017 | 1.5807                           |

$r_k^0$(annually) = 2.9233

$r_k^0$(quarterly) = 0.7229

(51) is equivalent to

$$\text{Var}^*(R_T) = -\frac{1 + \alpha_0}{\alpha_0} \left[ \mathbb{E}[R_T] - \left( \frac{d_{\sigma_{\text{min}}}}{\bar{x}_0} - 1 \right) \right]^2 + \frac{\text{Var}_{\text{min}}(\bar{x}_T)}{\bar{x}_0^2}. \tag{52}$$

In the remainder of this section, we analyze the efficient frontier using equation (52). In addition to the parameters, we discussed previously, the impacts of ratio $a$ also deserves more investigation. The value of $a$, defined as the proportion between the initial scale of salary and fund asset, can be an indicator of how members’ future contributions would affect the pension fund’s accumulation for a given salary inflation rate. If $a$ is relatively small, members’ future salary may play little role in the accumulation phase, which could last for a few years before retirement. In the final example, we demonstrate how $a$ impacts the shape of the efficient frontier.
Example 7.1. Time horizon plays a crucial part when it comes to asset allocation, especially for a pension fund. Broadly speaking, stocks, compared to government bonds, have a relatively large long-term return as well as frequent short-term fluctuations. Considering both investment return and risk, a professional fund manager would adopt an aggressive strategy within a long time horizon by increasing equity investment weight. By contrast, more conservative portfolios should be constructed for a superannuation fund of a short time horizon. In this example, we intend to explore how the tradeoff between a pension fund’s return and risk varies under a set of time horizons. As shown in Figure 1, there is no clear-cut tendency for the movement of the efficient frontier. None of the curves is uniformly better or worse than another one. However, we can still learn two things by comparing these five curves. First, a rising $T$ leads to the increase of both $d_{\sigma_{\min}}$ and $\text{Var}_{\min}(\bar{x}_T)$. Whereas the former experiences a minor growth, the minimum standard deviation is more than doubled, rising from 6% to 13%. In addition, the slope of the efficient frontier is becoming higher as the time horizon increases. We can interpret the slope of an efficient frontier as follows. When the pension fund improves its investment target at a higher return rate, a larger slope will result in a proportionally smaller increase of risk it has to bear. Therefore, under the mean-variance analysis, a pension fund of a longer time horizon is more likely to be capable of managing its investment risk than that of a shorter one.
Example 7.2. Our second example aims at illustrating how price inflation can affect the shape of the efficient frontier. The introduction of inflation enables us to analyze a pension fund’s real return, which is more important than merely a nominal return since a pension fund aims to support members’ consumption after their retirements. A real return is a more accurate measure of the purchasing power of the fund asset. Figure 2 makes a comparison between inflation and non-inflation cases. Unsurprisingly, the efficient frontier performs better when there is no inflation. Given the same amount of risk, the non-inflation return rate would be roughly two percent higher than the inflationary case. The two percentage of excess return is consistent with our model assumes that the long-term inflation rate $\bar{I}$ is estimated to be 0.54% quarterly or around 2% annually. The independence of $\{\epsilon_k\}$ and $\{P_k\}$ implies such a degree of increase. Otherwise, if stock returns can inflate as the price level goes up, the real investment return will be expected to be more close to the nominal return. Another feature of the figure is the same level of minimum standard deviation at 10%. Usually, the minimum variance is achieved when the fund is fully invested in the risk-free asset or government bonds in this situation. The same minimum variance indicates that inflation does not incur any extra uncertainty except reducing the real investment return of risk-free assets.

Example 7.3. This example considers the impact of estimation error on the efficient frontier. We take five values from the estimator $\hat{\phi}$’s confidence interval at the level of 95%. The corresponding five efficient frontiers are drawn in Figure 3.
However, the efficient frontier is not sensitive to the variation of the parameter $\phi$. We should also emphasize that the unit on both horizontal and vertical axes has been scaled up to distinguish the little difference between the efficient frontiers. Thus, our result is very robust to the estimation bias. Figure 3 also demonstrates how the parameter $\phi$ affects the efficient frontier. The parameter $\psi$, defined as $1-\phi$, indicates how fast the inflation rate $I_k$ converges to its long-term mean. The smaller $\psi$ (greater $\phi$) is, the greater variance of $I_k$, resulting in a lower efficient frontier. This relation is reflected in the figure.

**Example 7.4.** The superannuation guarantee rate $g_k$ is another factor influencing a pension fund’s efficient frontier. A higher SG rate means more significant contributions to the pension fund and improves its accumulated value. Hence, as seen in Figure 4, the level of $d_{\sigma_{\text{min}}}$ has a slightly increasing tendency as the SG rate rises. Nevertheless, the movement of the whole curve may complicate our analysis. An increase in SG rate tends to shift the efficient frontier downward to a small extent, which implies that more contributions to the pension fund render its efficient frontier worse by bringing more future uncertainty. One possible explanation may be made from a technical perspective. As the size of a pension fund grows over time, its variance may also experience an increase generated by stochastic stock returns and the inflation rate’s randomness. Besides, when the expected return rate exceeds 15%, the five curves almost converge into one. Thus, if the fund targets at a
Figure 4. Impact of SG rate on efficient frontier

high level of investment return, the future SG rate can exert little influence on its efficient frontier based on our analysis.

Example 7.5. The last example provides to demonstrate how the proportion, \( a = \frac{y_0}{x_0} \), between initial salary and initial fund value, may affect the efficient frontier. By making a simple comparison, we can find that Figure 5 shares a similar pattern with Figure 4. Both figures can be interpreted in the same way, that is, how the incoming cash flow from SG contributions can alter a pension fund’s investment return and risk. Whereas a higher SG rate produces more contributions directly, a higher ratio, \( a \), would have a similar impact indirectly by increasing the amount of the initial salary. Thus, a similar argument can be made here, as in the previous example.

8. Conclusion. Unlike the continuous-time setting, which many papers adopt on the dynamic portfolio selection, we investigate the multi-period mean-variance portfolio selection problem for the DC pension fund under inflation in this paper. Using the techniques of state variable transformation, matrix theory, and dynamic programming, we derive the closed-form expressions for the efficient strategy and the efficient frontier. The simplified solutions for some special cases are also studied. Finally, we present some empirical examples based on the data of Australia to illustrate our results.

Due to the lack of iterated-expectation property of the mean-variance criterion, our model’s optimal investment strategies are dynamically time-inconsistent ([2]).
This means that the optimal strategy obtained at time $t$ may not be optimal for a later time $s > t$. Therefore, our strategy is called "precommitment" in the sense that the objective is pre-determined at time zero. It is an interesting topic to further study the time-consistent mean-variance investment strategy for the DC pension funds under stochastic inflation. Our work can also be extended in other aspects. For example, we can generalize our results to maximize the expected utility formulation under inflation, and incorporate a stochastic market environment, such as a Markov regime-switching market. It is also possible to apply our model to the optimal investment-reinsurance problems faced by insurance companies. These are our future research topics.

Appendix A.

Appendix A.1: Proof of proposition 1.

Proof. By [20], we know that we have $\mathbb{E}[P_k P'_k] > 0$ under Assumption 1. Notice that $c_k = e^{-\phi_k(l - \sigma_k \varepsilon_k)}$, then $c_k^{2^{\delta_k + 1}} = e^{-2\delta_{k+1}(l - \phi_k)l - 2\delta_{k+1}\sigma_k \varepsilon_k} > 0$. Therefore, it follows from Lemma 2 that $\mathbb{E}[c_k^{2^{\delta_k + 1}} P_k P'_k] > 0$.

Now we prove the positive definiteness of $\mathbb{E}[B_k B'_k]$ using the technique of reduction to absurdity. Assume that $\mathbb{E}[B_k B'_k] \succeq 0$. Since $\mathbb{E}[B_k B'_k]$ is positive semidefinite, then we have $|\mathbb{E}[B_k B'_k]| = 0$. According to Lemma 1, there exists a nonzero vector
\( \bar{b} = (b_0, b_1, b_2, \ldots, b_n)' = (b_0, b'_1)' \), where \( b = (b_1, b_2, \ldots, b_n)' \), such that
\[
\bar{b}'B_k = b_0 r_0^k + b'_1 P_k = b_0 r_0^k + \sum_{i=1}^{n} b_i (r_i^k - r_0^k)
\]
\[
= \left( b_0 - \sum_{i=1}^{n} b_i \right) r_0^k + b_1 r_1^k + b_2 r_2^k + \cdots + b_n r_n^k = 0 \quad (A.53)
\]

with probability 1. If \( b_0, b_1, b_2, \ldots, b_n \) are not all equal to zero, then \( \left( b_0 - \sum_{i=1}^{n} b_i \right) \), \( b_1, b_2, \ldots, b_n \) are also not all equal to zero. Hence it follows from Lemma 1 that \( |\mathbb{E}[r_k r_k']| = 0 \), which contradicts to the fact that \( \mathbb{E}[r_k r_k'] > 0 \) in Assumption 1. Therefore, \( \mathbb{E}[B_k B_k'] > 0 \).

Finally, we prove that \( \mathbb{E}[Q_k Q_k'] > 0 \). Note that \( \mathbb{E}[Q_k Q_k'] = \mathbb{E}[c_k^{2n+1} B_k B_k'] \) and \( c_k^{2n+1} > 0 \). As \( \mathbb{E}[B_k B_k'] > 0 \), it follows from Lemma 2 that \( \mathbb{E}[Q_k Q_k'] \) is positive definite. This completes the proof. \( \square \)

Appendix A.2: Proof of proposition 2.

Proof .2. According to (10), we obtain
\[
\mathbb{E}[Q_k Q_k'] = \mathbb{E}\left[ c_k^{\delta + 1} ( r_0^k / P_k ) c_k^{\delta + 1} ( r_0^k / P_k' ) \right] = \left( \mathbb{E}[r_0^k c_k^{\delta + 1}] \mathbb{E}[r_0^k c_k^{\delta + 1} P_k'] \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \right),
\]
\[
\mathbb{E}[Q_k L_k'] = \mathbb{E}\left[ c_k^{\delta + 1} \left( r_0^k / P_k \right) \left( 1 / c_k^{\delta + 1} P_k' \right) \right] = \left( \mathbb{E}[r_0^k c_k^{\delta + 1}] \mathbb{E}[r_0^k c_k^{\delta + 1} P_k'] \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \right),
\]
\[
\mathbb{E}[L_k L_k'] = \mathbb{E}\left[ \left( 1 / c_k^{\delta + 1} P_k \right) \left( 1 / c_k^{\delta + 1} P_k' \right) \right] = \left( \mathbb{E}[r_0^k c_k^{\delta + 1}] \mathbb{E}[r_0^k c_k^{\delta + 1} P_k'] \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \right).
\]

By Proposition 1, we have \( |\mathbb{E}[c_k^{2\delta + 1} P_k P_k']| > 0 \). Then, it follows Lemma 3 and (15) that
\[
|\mathbb{E}[Q_k Q_k']| = D_k \left| \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \right|, \quad |\mathbb{E}[Q_k L_k']| = C_k \left| \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \right|, \quad |\mathbb{E}[L_k L_k']| = (1 - G_k) \left| \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \right|.
\]

Therefore, we obtain
\[
D_k = \left| \mathbb{E}[Q_k Q_k'] \right| \left| \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \right|^{-1}, \quad C_k = \left| \mathbb{E}[Q_k L_k'] \right| \left| \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \right|^{-1}, \quad (1 - G_k) = \left| \mathbb{E}[L_k L_k'] \right| \left| \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \right|^{-1}.
\]

This implies the desired result. \( \square \)

Appendix A.3: Proof of proposition 3.

Proof .3. For any \( k = 0, 1, \ldots, T - 1 \), it follows from Proposition 1 that \( |\mathbb{E}[c_k^{2\delta + 1} P_k P_k']| > 0 \) and \( |\mathbb{E}[Q_k Q_k']| > 0 \). And obviously, \( |\mathbb{E}[L_k L_k']| \geq 0 \). By Proposition 2, we have \( D_k > 0 \), \( G_k \leq 1 \). Since \( \mathbb{E}[c_k^{2\delta + 1} P_k P_k'] \) is positive definite, we know
that $E^{-1}[c_k^{2\delta_k+1} P_k P_k]$ is positive definite. By Assumption 3, $E[c_k^{2\delta_k+1} P_k'] \neq 0$. Then we have

$$G_k = E[c_k^{2\delta_k+1} P_k'E^{-1}[c_k^{2\delta_k+1} P_k P_k']E[c_k^{2\delta_k+1} P_k] > 0.$$  

In the following, we prove that $D_k (1 - G_k) \geq C_k^2$. By Proposition 1, $E[Q_k Q_k'] > 0$, so the inverse $E^{-1}[Q_k Q_k']$ of $E[Q_k Q_k']$ exists, then $E^+[Q_k Q_k'] = E^{-1}[Q_k Q_k']$. Since

$$E \left[ \begin{pmatrix} L_k \\ Q_k \end{pmatrix} \right] = \begin{pmatrix} E[L_k L_k'] \\ E[Q_k L_k'] \\ E[Q_k Q_k'] \end{pmatrix} \geq 0,$$

Then, by Lemma 4, we have

$$E[L_k L_k'] - E[L_k Q_k']E^{-1}[Q_k Q_k']E[Q_k L_k'] \geq 0.$$  

Obviously, we have

$$E[L_k Q_k']E^{-1}[Q_k Q_k']E[Q_k L_k'] = E[L_k Q_k']E^{-1}[Q_k Q_k'] (E[L_k Q_k'])' \geq 0.$$  

Consequently, we obtain

$$E[L_k L_k'] \geq E[L_k Q_k']E^{-1}[Q_k Q_k']E[Q_k L_k'] \geq 0.$$ (A.54)

Then, it follows from (A.54) and Lemma 5 that

$$|E[L_k L_k']| \geq |E[L_k Q_k']|E^{-1}[Q_k Q_k']|E[Q_k L_k']||E[L_k Q_k']| |E[Q_k Q_k']| |E[Q_k L_k']|. \tag{A.55}$$

Notice that $|E[Q_k L_k']| = |E[L_k Q_k']|$ and $|E^{-1}[Q_k Q_k']| = |E[Q_k Q_k']|^{-1}$. Then, (A.55) implies

$$|E[Q_k Q_k']|^2 \leq |E[Q_k Q_k']| |E[L_k L_k']|,$$

which along with Proposition 2 gives $C_k^2 \leq D_k (1 - G_k)$. As $G_k > 0$ and $D_k > 0$, we have $C_k^2 \leq D_k G_k < D_k$. This completes the proof. 

\begin{appendix}
\section*{Appendix A.4: Proof of Theorem 1}

\textbf{Proof.4.} We prove this theorem by mathematical induction on $k$.

For $k = T - 1$, by Bellman equation (13), we have

$$f_{T-1}(\bar{x}, \bar{y}, \Lambda) = \min_{u_{T-1}} E[f_T(\Lambda (\bar{x} + g_{T-1} \bar{y}) (r_0^T_{T-1} + P_{T-1} T^{-1}) , q_k \Lambda \bar{y}, c_{T-1} \Lambda \phi_{T-1})]$$

$$= \min_{u_{T-1}} E$$

$$\left[ (\bar{x} + g_{T-1} \bar{y})^2 \Lambda^2 (r_0^T_{T-1} + P_{T-1} T^{-1})^2 + 2a (\bar{x} + g_{T-1} \bar{y}) \Lambda (r_0^T_{T-1} + P_{T-1} T^{-1}) \right]$$

$$= \left[ (\bar{x} + g_{T-1} \bar{y})^2 \Lambda^2 \Lambda (r_0^T_{T-1})^2 + 2a (\bar{x} + g_{T-1} \bar{y}) \Lambda \Lambda (r_0^T_{T-1}) \right]$$

$$+ \min_{u_{T-1}} \left[ (\bar{x} + g_{T-1} \bar{y})^2 \Lambda^2 u_{T-1} E[P_{T-1} P_{T-1} T^{-1}] u_{T-1} \right]$$

$$+ \min_{u_{T-1}} \left[ (\bar{x} + g_{T-1} \bar{y})^2 \Lambda^2 \Lambda \Lambda (r_0^T_{T-1}) \Lambda (r_0^T_{T-1}) \right] u_{T-1} \right]. \tag{A.56}$$

By Assumption 1, we know that $E[P_{T-1} P_{T-1} T^{-1}] > 0$. Hence, taking the first order with respect to $u_{T-1}$ leads to the optimal strategy:

$$u_{T-1} = -E^{-1}[P_{T-1} P_{T-1} T^{-1}] \left( E[r_0^T_{T-1} P_{T-1}] + \frac{a}{\Lambda (\bar{x} + g_{T-1} \bar{y})} E[P_{T-1}] \right). \tag{A.57}$$

\end{appendix}
Substituting (A.57) into (A.56) yields
\[
\begin{align*}
f_{T-1}(\bar{x}, \bar{y}, \Lambda) &= (\bar{x} + g_{T-1} \bar{y})^2 \Lambda^2 E[(r_{T-1}^0)^2] + 2a(\bar{x} + g_{T-1} \bar{y}) \Lambda E[r_{T-1}^0] \\
&\quad - ((\bar{x} + g_{T-1} \bar{y}) \Lambda^2 E[r_{T-1}^0 P_{T-1}' + a\Lambda E[P_{T-1}^0])] \\
&\quad \times E^{-1}[P_{T-1}' P_{T-1}^0]\left((\bar{x} + g_{T-1} \bar{y}) E[r_{T-1}^0 P_{T-1}'] + a\Lambda^{-1} E[P_{T-1}]ight) \\
&= (\bar{x} + g_{T-1} \bar{y})^2 \Lambda^2 E[(r_{T-1}^0)^2] - (\bar{x} + g_{T-1} \bar{y})^2 \Lambda^2 E[r_{T-1}^0 P_{T-1}'] E^{-1}[P_{T-1}' P_{T-1}^0] \\
&\quad E[r_{T-1}^0 P_{T-1}'] + 2a(\bar{x} + g_{T-1} \bar{y}) \Lambda E[r_{T-1}^0] - 2(\bar{x} + g_{T-1} \bar{y}) \Lambda a E[r_{T-1}^0 P_{T-1}'] \\
&\quad - a^2 E[P_{T-1}^0] E^{-1}[P_{T-1}' P_{T-1}^0] E[P_{T-1}] \\
&= D_{T-1}(\bar{x} + g_{T-1} \bar{y})^2 \Lambda^2 - 2aC_{T-1}(\bar{x} + g_{T-1} \bar{y}) \Lambda - G_{T-1} a^2. \\
\end{align*}
\] (A.58)

On the other hand, by (14) and its terminal conditions, we have
\[
\begin{align*}
\delta_{T-1} &= 1, \quad w_{T-1} = D_{T-1}, \quad \lambda_{T-1} = 2C_{T-1}, \\
\alpha_{T-1} &= -G_{T-1}, \quad \omega_{T-1} = 2g_{T-1}D_{T-1} = 2g_{T-1}w_{T-1}, \\
\eta_{T-1} &= g_{T-1}^2, \quad \tau_{T-1} = 2C_{T-1}g_{T-1} = \lambda_{T-1}g_{T-1}. \\
\end{align*}
\] (A.59)

Hence, we obtain
\[
\begin{align*}
f_{T-1}(\bar{x}, \bar{y}, \Lambda) &= D_{T-1}(\bar{x} + g_{T-1} \bar{y})^2 \Lambda^2 + 2aC_{T-1}(\bar{x} + g_{T-1} \bar{y}) \Lambda - G_{T-1} a^2 \\
&= w_{T-1}(\bar{x}^2 + 2g_{T-1}\bar{x}\bar{y} + g_{T-1}^2 \bar{y}^2) \Lambda^{2\delta_{T-1}} + \lambda_{T-1}a(\bar{x} + g_{T-1} \bar{y}) \Lambda^{\delta_{T-1}} + \alpha_{T-1} a^2 \\
&= w_{T-1}\Lambda^{2\delta_{T-1}} \bar{x}^2 + 2w_{T-1}g_{T-1}\Lambda^{2\delta_{T-1}} \bar{x}\bar{y} + g_{T-1}^2 w_{T-1}\Lambda^{2\delta_{T-1}} \bar{y}^2 \\
&\quad + \lambda_{T-1}a\Lambda^{\delta_{T-1}} \bar{x} + \lambda_{T-1}g_{T-1}a\Lambda^{\delta_{T-1}} \bar{y} + \alpha_{T-1} a^2 \\
&= w_{T-1}\Lambda^{2\delta_{T-1}} \bar{x}^2 + \omega_{T-1}\Lambda^{2\delta_{T-1}} \bar{x}\bar{y} + \eta_{T-1}\Lambda^{2\delta_{T-1}} \bar{y}^2 \\
&\quad + \lambda_{T-1}a\Lambda^{\delta_{T-1}} \bar{x} + \theta_{T-1} a\Lambda^{\delta_{T-1}} \bar{y} + \alpha_{T-1} a^2. \\
\end{align*}
\] (A.60)

According to (A.59) and (A.57), and noting that \(\delta_T = 0, w_T = 1, \lambda_T = 2, \alpha_T = 0, \omega_T = 0, \eta_T = 0, \theta_T = 0, a^2 = 0,\) we have
\[
\begin{align*}
&- E^{-1}[\Lambda^{2\delta_{T-1}} P_{T-1} P_{T-1}^0] \left(E[\Lambda^{2\delta_{T-1}} P_{T-1}'] + \frac{1}{2} \Lambda^{-\delta_{T-1}}ight) \\
&\quad \times \lambda_{T-1} a E[c_{T-1}' P_{T-1}'] + w_{T-1} \Lambda^{\delta_{T-1}} E[c_{T-1}' P_{T-1}'] \\
&= E^{-1}[P_{T-1} P_{T-1}^0] E[\Lambda^{2\delta_{T-1}} P_{T-1}'] + \frac{a}{\Lambda^{2\delta_{T-1}} P_{T-1}} E[P_{T-1}'] \\
&= u_{T-1}^*.
\end{align*}
\] (A.61)

Combining (A.60) and (A.61), we know that (19) and (20) hold for \(k = T - 1.\)

Now suppose that (19) and (20) hold for \(k + 1,\) i.e.,
\[
\begin{align*}
f_{k+1}(\bar{x}, \bar{y}, \Lambda) &= w_{k+1}\bar{x}^2 \Lambda^{2\delta_{k+1}} + \lambda_{k+1} a\bar{x} \Lambda^{\delta_{k+1}} + \alpha_{k+1} a^2 \\
&\quad + \omega_{k+1} \Lambda^{2\delta_{k+1}} \bar{x} \bar{y} + \eta_{k+1} \Lambda^{2\delta_{k+1}} \bar{y}^2 + \theta_{k+1} a\Lambda^{\delta_{k+1}} \bar{y}.
\end{align*}
\] (A.62)

Note that \(1 + \phi_k \delta_{k+1} = \delta_k,\) by Bellman equation (13), we have
\[
\begin{align*}
f_k(\bar{x}, \bar{y}, \Lambda) &= \min_{u_k} E[f_{k+1}(\Lambda (\bar{x} + g_k \bar{y}) (r_k^0 + P_k u_k), g_k \Lambda \bar{y}, c_k \Lambda^{\delta_k})] \\
&= \min_{u_k} E \left[w_{k+1}(\bar{x} + g_k \bar{y})^2 \Lambda^2 (r_k^0 + P_k u_k)^2 (c_k \Lambda^{\delta_k})^{2\delta_{k+1}}
\end{align*}
\]
Collecting the similar items, we obtain the expression of
\[
\begin{align*}
+ \lambda_{k+1} a (\bar{x} + y_k \bar{y}) \Lambda (r^k_d + P_k' u_k) (c_k \Lambda^\delta_k)^{u_k+1} + \alpha_{k+1} a^2 \\
+ \varpi_{k+1} (c_k \Lambda^\delta_k)^{u_k+1} (\bar{x} + g_k \bar{y}) \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y} \\
+ \gamma_{k+1} c_k \Lambda^\delta_k (q_k \Lambda \bar{y})^2 + \theta_{k+1} a (c_k \Lambda^\delta_k)^{u_k+1} q_k \Lambda \bar{y}
\end{align*}
\]
\[
= w_{k+1} E[r_k^d c_k^{2u_k+1}] E[(\bar{x} + y_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}] \\
+ \varpi_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y}) \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y} \\
+ \theta_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y} \\
\leq w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y}) \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}
\]
\[
= \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y}) \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}} \\
\leq w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y}) \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}
\]
Since \( E[r_k^d c_k^{2u_k+1}] P_k' P_k'^* u_k \geq 0 \), \( w_{k+1} > 0 \) and \( \Lambda = \Lambda_k = e^{-t_k} > 0 \), by Propositions 1 and 4, taking the first order with respect to \( u_k \) gives
\[
\begin{align*}
&u_k (\bar{x}, \bar{y}, \Lambda) = -E^{-1}[c_k^{2u_k+1} P_k' P_k'^* u_k] \\
&\times \left( E[r_k^d c_k^{2u_k+1} P_k'] + \lambda_{k+1} a E[c_k^{2u_k+1} P_k'] + \varpi_{k+1} \bar{y} E[c_k^{2u_k+1} q_k P_k'] \right).
\end{align*}
\]
(64)

Substituting (64) into (63) and simplifying the equation yields
\[
\begin{align*}
f_k(\bar{x}, \bar{y}, \Lambda) &= \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}} \\
&\leq \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}} \\
&\leq \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}} \\
&\leq \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}}
\end{align*}
\]
\[
\begin{align*}
&= \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}} \\
&\leq \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}} \\
&\leq \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}} \\
&\leq \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}}
\end{align*}
\]
Collecting the similar items, we obtain the expression of \( f_k(\bar{x}, \bar{y}, \Lambda) \) as follows:
\[
\begin{align*}
f_k(\bar{x}, \bar{y}, \Lambda) &= \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}} \\
&\leq \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}} \\
&\leq \underbrace{w_{k+1} E[r_k^d c_k^{2u_k+1}] (\bar{x} + g_k \bar{y})^2 \Lambda^\delta_k + \lambda_{k+1} a \Lambda (r^k_d + P_k' u_k) q_k \Lambda \bar{y}}_{\text{A.63}}
\end{align*}
\]
Therefore, by (9) and (8), we have

\[ f_k(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{\Lambda}}) = \left( \bar{\bar{x}} + g_k \bar{\bar{y}} \right)^2 \Lambda^{2k} w_{k+1} D_k + a \left( \bar{\bar{x}} + g_k \bar{\bar{y}} \right) \Lambda^{\delta_k} \lambda_{k+1} C_k + \left( \alpha_{k+1} - \frac{\lambda_{k+1}}{4w_{k+1}} G_k \right) a^2 \]

\[ + \omega_k + 1 F_k(\bar{\bar{x}} + g_k \bar{\bar{y}}) \bar{\bar{y}} \Lambda^{2k} + \left( \eta_{k+1} E[c_k^{2k+1} q_k^2] - \frac{\lambda_{k+1}}{4w_{k+1}} J_k \right) \Lambda^{2k} \bar{\bar{y}}^2 \]

\[ + \left( \theta_{k+1} E[C_k^{\delta_k+1} q_k] - \frac{\lambda_{k+1} w_{k+1}}{4w_{k+1}} H_k \right) a \bar{\bar{y}} \Lambda^{\delta_k} \]

\[ = \bar{\bar{x}} \bar{\bar{y}} \Lambda^{2k} w_{k+1} D_k + 2g_k \bar{\bar{x}} \bar{\bar{y}} \Lambda^{2k} w_{k+1} D_k + g_k^2 \bar{\bar{y}}^2 \Lambda^{2k} w_{k+1} D_k \]

\[ + a \Lambda^{\delta_k} \lambda_{k+1} C_k \bar{\bar{x}} + a \Lambda^{\delta_k} \lambda_{k+1} C_k g_k \bar{\bar{y}} + \left( \alpha_{k+1} - \frac{\lambda_{k+1}}{4w_{k+1}} G_k \right) a^2 \]

\[ + \omega_k + 1 F_k(\bar{\bar{x}} + g_k \bar{\bar{y}}) \bar{\bar{y}} + \omega_k + 1 F_k \Lambda^{2k} g_k \bar{\bar{y}}^2 \]

\[ + \left( \eta_{k+1} E[c_k^{2k+1} q_k^2] - \frac{\lambda_{k+1}}{4w_{k+1}} J_k \right) \Lambda^{2k} \bar{\bar{y}}^2 \]

\[ + \left( \theta_{k+1} E[C_k^{\delta_k+1} q_k] - \frac{\lambda_{k+1} w_{k+1}}{4w_{k+1}} H_k \right) a \bar{\bar{y}} \Lambda^{\delta_k} \]

\[ = w_k \bar{\bar{x}} \bar{\bar{y}} \Lambda^{2k} + \lambda_k a \bar{\bar{x}} \Lambda^{\delta_k} + \alpha_k a^2 + \omega_k \Lambda^{2k} \bar{\bar{y}} + \eta_k \Lambda^{2k} \bar{\bar{y}}^2 + \theta_k a \Lambda^{\delta_k} \bar{\bar{y}}. \]

(A.65)

Here, equations (A.65) and (A.64) indicate that (19) and (20) hold for \( k \). By the principle of mathematical induction, (19) and (20) hold true for all \( k = 0, 1, \cdots, T-1 \). Therefore, the desired result is proved. \( \square \)

REFERENCES

[1] A. Albert, Conditions for positive and nonnegative definiteness in terms of pseudoinverses, SIAM Journal on Applied Mathematics, 17 (1969), 434–440.

[2] S. Basak and G. Chabakauri, Dynamic mean-variance asset allocation, The Review of Financial Studies, 23 (2010), 2970–3016.

[3] D. Blake, D. Wright and Y. Zhang, Target-driven investing: Optimal investment strategies in defined contribution pension plans under loss aversion, Journal of Economic Dynamics and Control, 37 (2013), 195–209.

[4] D. Blake, D. Wright and Y. M. Zhang, Age-dependent investing: optimal funding and investment strategies in defined contribution pension plans when members are rational life cycle financial planners, Journal of Economic Dynamics and Control, 38 (2014), 105–124.

[5] M. J. Brennan and Y. Xia, Dynamic asset allocation under inflation, The Journal of Finance, 57 (2002), 1201–1238.

[6] A. Chen and L. Delong, Optimal investment for a defined-contribution pension scheme under a regime switching model, Astin Bulletin, 45 (2015), 397–419.

[7] Z. Chen, Z. F. Li, Y. Zeng and J. Y. Sun, Asset allocation under loss aversion and minimum performance constraint in a DC pension plan with inflation risk, Insurance: Mathematics and Economics, 75 (2017), 137–150.

[8] X. Y. Cui, J. J. Gao, X. Li and D. Li, Optimal multi-period mean-variance policy under no-shorting constraint, European Journal of Operational Research, 234 (2014), 459–468.

[9] X. Y. Cui, X. Li and D. Li, Mean-variance policy for discrete-time cone constrained markets: The consistency in efficiency and minimum-variance signed supermartingale measure, Mathematical Finance, 27 (2017), 471–504.

[10] P. Devolder, M. Bosch Princep and I. Dominguez Fabian, Stochastic optimal control of annuity contracts, Insurance: Mathematics and Economics, 33 (2003), 227–238.
[11] Y. Dong and H. Zheng, Optimal investment of DC pension plan under short-selling constraints and portfolio insurance, *Insurance: Mathematics and Economics*, 85 (2019), 47–59.

[12] Y. Dong and H. Zheng, Optimal investment with S-shaped utility and trading and Value at Risk constraints: An application to defined contribution pension plan, *European Journal of Operational Research*, 281 (2020), 341–356.

[13] P. Emms, Lifetime investment and consumption using a defined-contribution pension scheme, *Journal of Economic Dynamics and Control*, 36 (2012), 1303–1321.

[14] R. Gerrard, B. Hogaard and E. Vigna, Choosing the optimal annuitization time post retirement, *Quantitative Finance*, 12 (2012), 1143–1159.

[15] N. W. Han and M. W. Hung, Optimal asset allocation for DC pension plans under inflation, *Insurance: Mathematics and Economics*, 51 (2012), 172–181.

[16] N.-W. Han and M.-W. Hung, Optimal consumption, portfolio, and life insurance policies under interest rate and inflation risks, *Insurance: Mathematics and Economics*, 73 (2017), 54–67.

[17] L. He and Z. X. Liang, Optimal assets allocation and benefit outgo policies of DC pension plan with compulsory conversion claims, *Insurance: Mathematics and Economics*, 61 (2015), 227–234.

[18] A. K. Konicz and J. M. Mulvey, Optimal savings management for individuals with defined contribution pension plans, *European Journal of Operational Research*, 243 (2015), 233–247.

[19] M. Kwak and B. H. Lim, Optimal portfolio selection with life insurance under inflation risk, *Journal of Banking and Finance*, 46 (2014), 59–71.

[20] D. Li and W. L. Ng, Optimal dynamic portfolio selection: Multi-period mean-variance formulation, *Mathematical Finance*, 10 (2000), 387–406.

[21] D. P. Li, X. M. Rong and H. Zhao, Time-consistent reinsurance-investment strategy for a mean-variance insurer under stochastic interest rate model and inflation risk, *Insurance: Mathematics and Economics*, 64 (2015), 28–44.

[22] X. Li, X. Y. Zhou and A. E. B. Lim, Dynamic mean-variance portfolio selection with no-shorting constraints, *SIAM Journal on Control and Optimization*, 40 (2002), 1540–1555.

[23] D. G. Luenberger, *Optimization by Vector Space Methods*, John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[24] Q.-P. Ma, On optimal pension management in a stochastic framework with exponential utility, *Insurance: Mathematics and Economics*, 49 (2011), 61–69.

[25] H. Markowitz, Portfolio selection, *Journal of Finance*, 7 (1952), 77–91.

[26] C. Munk, C. Sørensen and T. N. Vinther, Dynamic asset allocation under mean-reverting returns, stochastic interest rates, and inflation uncertainty: Are popular recommendations consistent with rational behavior?, *International Review of Economics and Finance*, 13 (2004), 141–166.

[27] M. Simutin, Cash holding and mutual fund performance, *Review of Finance*, 18 (2014), 1425–1464.

[28] J. Y. Sun, Z. F. Li and Y. Zeng, Precommitment and equilibrium investment strategies for defined contribution pension plans under a jump-diffusion model, *Insurance: Mathematics and Economics*, 67 (2016), 158–172.

[29] M.-L. Tang, S.-N. Chen, G. C. Lai and T. P. Wu, Asset allocation for a DC pension fund under stochastic interest rates and inflation-protected guarantee, *Insurance: Mathematics and Economics*, 78 (2018), 87–104.

[30] E. Vigna, On efficiency of mean-variance based portfolio selection in defined contribution pension schemes, *Quantitative Finance*, 14 (2014), 237–258.

[31] H. X. Yao, Y. Z. Lai, Q. H. Ma and M. J. Jian, Asset allocation for a DC pension fund with stochastic income and mortality risk: A multi-period mean-variance framework, *Insurance: Mathematics and Economics*, 54 (2014), 84–92.

[32] H. X. Yao, Z. F. Li and D. Li, Multi-period mean-variance portfolio selection with stochastic interest rate and uncontrollable liability, *European Journal of Operational Research*, 252 (2016), 837–851.

[33] H. X. Yao, Z. Yang and P. Chen, Markowitz's mean-variance defined contribution pension fund management under inflation: A continuous-time model, *Insurance: Mathematics and Economics*, 53 (2013), 851–863.

[34] A. Zhang and C.-O. Ewald, Optimal investment for a pension fund under inflation risk, *Mathematical Methods of Operations Research*, 71 (2010), 353–369.
[35] F. Z. Zhang, Matrix Theory: Basic Results and Techniques, Second edition, Universitext. Springer, New York, 2011.

[36] L. Zhang, H. Zhang and H. X. Yao, Optimal investment management for a defined contribution pension fund under imperfect information, Insurance: Mathematics and Economics, 79 (2018), 210–224.

Received April 2020; revised September 2020.

E-mail address: yaohaixiang@gdufs.edu.cn
E-mail address: pche@unimelb.edu.au
E-mail address: zhangmiaounimelb@gmail.com
E-mail address: malixun@polyu.edu.hk