The Permutation Groups and the Equivalence of Cyclic and Quasi-Cyclic Codes

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Abstract—We give the class of finite groups which arise as the permutation groups of cyclic codes over finite fields. Furthermore, we extend the results of Brand and Huffman et al. and we find the properties of the set of permutations by which two cyclic codes of length \( p^r \) can be equivalent. We also find the set of permutations by which two quasi-cyclic codes can be equivalent.

Index Terms—Permutation group, equivalency of codes, cyclic code, quasi-cyclic code, doubly transitive groups.

I. INTRODUCTION

A class of cyclic objects on \( n \) elements is a class of combinatorial objects on \( n \) elements, where automorphisms of objects in the class and isomorphisms between objects in the same class are permutations of \( S_n \), and where permutation group contains a complete cycle. Such classes include circulant graphs, circulant digraphs, cyclic designs and cyclic codes. N. Brand [9] characterised the set \( H(P) \) of permutations by which two combinatorial cyclic objects on \( p^r \) elements are equivalent. By using these results Huffman et al. [20] gave explicitly this set in the case \( n = p^2 \) and construct algorithms to find the equivalency between cyclic objects and extended cyclic objects. In this paper we also give explicitly the set \( H(P) \) for codes of length \( p^r \). We simplify the algorithms of Huffman et al. by proving some results on the order of some subgroups of the permutation group \( \text{Per}(C) \).

It is well known that we can construct from the cyclic codes many optimal codes with permutation groups sharing many properties [15] or [23]. With this motivation and also since the set \( H(P) \) depends essentially on the structure of the group \( \text{Per}(C) \), we give the class of finite groups which arise as the permutation groups of cyclic codes. Note that the permutation groups of cyclic codes are known only for few families, such as the Reed–Solomon codes, the Reed–Muller codes and some BCH codes [3], [6], [7]. Recently, R. Bienert and B. Klopsch [8] studied the permutation group of cyclic codes in the binary case. We generalize a result of [8] concerning the doubly transitive permutation groups with socle \( \text{PSL}(d, q) \) to the non-binary cases. Furthermore, we prove that if the length is a composed or a prime power number, then \( \text{Per}(C) \) is imprimitive or doubly transitive. Hence, we use the classification of the doubly primitive groups which contains a complete cycle, given by J. P. McSorley [24], [25] and our previous results to give the permutation groups in the doubly transitive cases. Further, we consider the permutation groups of cyclic MDS codes, and that by building on the results of [4]. Finally, we consider the quasi-cyclic codes. These codes are interesting; they are used in many powerful cryptosystems [26]. We characterize the set \( H'(P) \) of permutations by which two quasi-cyclic codes can be equivalent. We find some of its properties. But, we did not prove that \( H'(P) \) is a group. Even though, by using the software GAP, we find on several examples that \( H'(P) \) is an imprimitive group. Hence we conjecture that \( H'(P) \) is a group. Under this hypothesis, we prove that \( H'(P) \) is an imprimitive group, or the alternating group \( Alt(n) \) or the symmetric group \( S_n \). The last situation implies that the code is trivial.

This paper is organised as follow. In second section we deal with the permutation groups of cyclic codes. In the third section we consider the equivalency problem for the cyclic codes. We simplify and generalize some results of Huffman et al. [20] from the length \( p^2 \) to the length \( p^r \). We characterize the structure of \( H(P) \) and in some cases we give exactly the set \( H'(P) \) or prove that it is a group. Finally in the last section we consider the equivalency problem for the quasi-cyclic codes.

II. PERMUTATION GROUPS OF CYCLIC CODES

Let \( C \) be a linear code of length \( n \) over a finite field \( \mathbb{F}_q \), and \( \sigma \) a permutation of \( S_n \). To the code \( C \) we associate a linear code \( \sigma(C) \) defined by:

\[
\sigma(C) = \{(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \mid (x_1, \ldots, x_n) \in C\}.
\]

We say that the codes \( C \) and \( C' \) are equivalent by permutation if there exists a permutation \( \sigma \in S_n \) such that \( C' = \sigma(C) \). The permutation group of \( C \) is the subgroup of \( S_n \) given by:

\[
\text{Per}(C) = \{\sigma \in S_n \mid \sigma(C) = C\}.
\]

We recall that a linear code \( C \) over \( \mathbb{F}_q \) is cyclic if it verifies \( T \in \text{Per}(C) \), where \( T \) is a complete cycle of length \( n \).

The following elementary Lemma is a folklore, well-known in the area of group theory.

Lemma 1: Let \( C \) be a cyclic code of length \( n \). Then its permutation group \( \text{Per}(C) \) is a transitive group.

Proof: The group \( \text{Per}(C) \) operates on the set \( \{1, \ldots, n\} \) and contains the permutation shift \( T \). For \( i, j \in \{1, \ldots, n\} \), we have \( T^{j-i}(i) = j \); hence \( \text{Per}(C) \) is transitive. \( \blacksquare \)

A transitive group is either primitive or imprimitive. An interesting class of primitive group is the class of the doubly transitive groups. A doubly transitive permutation group \( G \) has a unique minimal normal subgroup \( N \), which is either regular and elementary abelian or simple and primitive and has \( |C_G(N)| = 1 \) [12, p. 202]. All simple groups which can occur as minimal normal subgroup of a doubly transitive group are known [13]. This result is due to the classification of finite
simple groups. By using this classification J. P. McSorley [24] gave the following result.

**Lemma 2**: A group $G$ of degree $n$ which is doubly transitive and contains a complete cycle, has as socle $N$, verifies $N \leq G \leq Aut(N)$ and is equal to one of the cases in the following Table.

| $G$ | $n$ | $N$ |
|-----|-----|-----|
| $AGL(1,p)$ | $p$ | $C_p$ |
| $S_4$ | $4$ | $C_2 \times C_2$ |
| $S_n, n \geq 5$ | $n$ | $Aut(n)$ |
| $Alt(n), n$ odd and $\geq 5$ | $n$ | $Alt(n)$ |
| $PGL(d,t) \leq G \leq PTL(d,t)$ | $(t^d - 1)/t - 1$ | $PSL(d,t)$ |
| $(d,t) \neq (2,2),(2,3),(2,4)$ | $11$ | $PSL(2,11)$ |
| $M_{11}$ | $11$ | $M_{11}$ |
| $M_{23}$ | $23$ | $M_{23}$ |

**Remark 1**: The projective semi-linear group $PTL(d,t)$ is the semi-direct product of the projective linear group $PGL(d,t)$ with the automorphism group $Z = Gal(F_t/F_{p^a})$ of finite field $F_t$, where $t = p^a, p'$ a prime, i.e.

$$PTL(d,t) = PGL(d,t) \rtimes Z.$$  

The order of these groups are

$$|PGL(d,t)| = (d,t-1)|PSL(d,t)|, |Z| = s$$

and $PTL(d,t) = sPGL(d,t)$. Hence if $t$ is a prime we have $PTL(d,t) = PGL(d,t)$. The zero code, the entire space, the repetition code and its dual are called elementary codes. The permutation group of these codes is $S_n$ [21, p. 1410]. Furthermore, there is no cyclic codes with permutation group equals to $Alt(n)$.

The following Lemma is a generalization of a part of [8, Theorem E] to the non binary cases.

**Lemma 3**: Let $C$ be a non elementary cyclic code of length $n = t^{d}\frac{p^a}{(p^a - 1)}$ over a finite field of characteristic $p'$, $d \geq 2$, and $t = p^a$ be the power of a prime, if the group $Per(C)$ verifies:

$$PGL(d,t) \leq Per(C) \leq PTL(d,t),$$

then we have,

$$d \geq 3, t = p^a \text{ and } Per(C) = PTL(d,t).$$

**Proof**: For $d = 2$, as the group $PGL(2,t)$ acts 3-transitively on 1-dimensional projective space $\mathbb{P}^1(F_t)$, we can deduce from [25, Table 1 and Lemma 2], that the underlying code is elementary, hence $Per(C) = S_n$, which is a contradiction. Hence $d \geq 3$, and similarly we deduce from [25, Table 1 and Lemma 2] that $t = p^a$. Let $V$ denotes the permutation module of $F_{p'}$, associated to the natural action of $PGL(d,t)$ on $(d-1)$ dimensional projective space $\mathbb{P}^{d-1}(F_t)$. Let $U_1$ be a $PGL(d,t)$-submodule of $V$. Hence $U_1$ is $PTL(d,t)$-invariant. That is because, for a generator of the cyclic group $PTL(d,t)/PGL(d,t) = Gal(F_t/F_{p'})$, then $U_2 = V_{\sigma}^*$ regarded as $PGL(d,t)$-module, is simply a twist of $U_1$. Let $F_{p'}$ be the algebraic closure of $F_{p'}$, we conclude that the composition factors of $F_{p'}PGL(d,t)$-modules $\overline{U_1} = F_{p'} \otimes U_1$ and $\overline{U_2} = F_{p'} \otimes U_2$ are the same. The submodules of the $F_{p'}PGL(d,t)$-module $V = F_{p'} \otimes V$ are uniquely determined by their composition factors [2]. Hence we have $\overline{U_1} = \overline{U_2}$; this implies $U_1 = U_2$. This gives $Per(C) = PTL(d,t)$.  

We recall that the group $AG(n) = \{ \tau_{a,b} : a \neq 0, (a,n) = 1, b \in \mathbb{Z}/n\mathbb{Z} \}$ is the group of the affine transformations defined as follow

$$\tau_{a,b} : z_n \mapsto z_n \quad x \mapsto (ax + b) \mod n.$$  

(1)

When $n = p$ a prime number, we have $AGL(1,p) = AG(p)$, and is called the affine group.

**Proposition 4**: Let $C$ be a non elementary cyclic code of length $p$ over $\mathbb{F}_q$, with $q = p^a$. Hence $Per(C)$ is a primitive group, and one of the following holds:

1) $Per(C) = PSL(2,11)$ of degree 11, $q = 3$ ( $C$ is the [11,6,5] ternary Golay code or its dual),
2) $Per(C) = M_{23}$ of degree 23, $q = 2$ (C is the binary Golay code [23,12,7], or its dual),
3) $Per(C) = PTL(d,t)$ of degree $p = (t^d - 1)/(t - 1)$, where $t$ is a power of the prime $p'$.
4) $C_p \leq Per(C) \leq AGL(1,p)$, with a normal Sylow subgroup $N$ of order $p$; and such that $p \geq 5$.

**Proof**: A transitive permutation group of prime degree is a primitive group [28, p. 195]. From a consequence of a result of Burnside [12], we have that a transitive group of prime degree is either a subgroup of the affine group $AGL(1,p)$ or a doubly transitive group.

In the doubly-transitive cases, from Lemma 2, Remark 1 and the Lemma 3 we have that these groups are $S_p, M_{11}, PSL(2,11)$ of degree 11, $M_{23}$ of degree 23, or $PTL(d,t)$ of degree $p = (t^d - 1)/(t - 1)$ over $\mathbb{F}_q$, with $q = p^a$ and $t$ is a prime power of $p'$. From Remark 1, the group $S_p$ corresponds to an elementary code. From [25, Table 1, Lemma 2 and (J)], $M_{11}$ rules out, $PSL(2,11)$ is the permutation group of a ternary code and $M_{23}$ is the permutation group of a binary code. From [21] p. 1410, these codes are unique, namely $PSL(2,11)$ corresponds to the [11,6,5] ternary Golay code or its dual. $M_{23}$ is the permutation group of the [23,12,7] binary Golay code and its dual.

Since $|AGL(1,p)| = (p-1)p$, then $Per(C) \leq AGL(1,p)$ admits as order $pm$ with $m(p-1)$, hence it contains a Sylow subgroup $N$ of order $p$. From Sylow’s Theorem, $N$ is unique, hence it is normal. For $p$ equals to 2 or 3, we have $AGL(1,p) = S_p$, hence the codes are elementary. Then we assume $p \geq 5$.

If $(q,p) = 1$, the number of cyclic codes of length $p$ over $\mathbb{F}_q$, is equal to $2 \frac{q-1}{\text{ord}_{p}(q)+1}$, where $\text{ord}_{p}(q)$ is the multiplicative order of $q$ modulo $p$ [18]. When $\text{ord}_{p}(q) = p-1$, there are only four codes, namely the elementary codes. This is the cases for $q = 13, p = 5, 11, 31, 37, 41$ or $q = 11$ and $p = 13, 17, 23, 31$.

In the following table we give examples of permutation groups of cyclic codes of length $p$ over $\mathbb{F}_q$ in the case $Per(C) \leq AGL(1,p)$. The integer $m$ is such that $Per(C) = C_m \times C_p$. The codes are given in pairs, corresponding to the code and its dual.
| $q$ | $p$ | $m$ | Codes |
|-----|-----|-----|--------|
| 11  | 5   | 2   | $[5,3,3],[5,2,4]$ |
| 5   | 1   | 2   | $[5,2,4],[5,3,3]$ |
| 7   | 3   | 2   | $[7,4,4],[7,3,5]$ |
| 19  | 3   | 2   | $[19,16,3],[19,3,6]$ |
| 37  | 6   | 2   | $[37,31,5],[37,6,27]$ |
| 13  | 7   | 2   | $[7,5,3],[7,2,6]$ |
|     |     |     | $[7,3,5],[7,4,4]$ |
| 17  | 4   | 2   | $[17,13,4],[17,4,12]$ |
|     |     |     | $[17,12,4],[17,5,11]$ |
| 17  | 8   | 2   | $[17,9,8],[17,9,7]$ |
| 23  | 11  | 2   | $[23,12,9],[23,11,10]$ |
| 29  | 14  | 2   | $[29,15,11],[29,14,12]$ |

**Theorem 5:** Let $C$ be a non elementary cyclic code over $\mathbb{F}_q$ of length $n = p^r$, where $r \geq 2$ and $q = p^s$. Then $\text{Per}(C)$ is either:
1) an imprimitive group, and admits a system of $p^i$ blocks of each of length $p^s, (s = i)$ formed by the orbits of $< T^{p^i} >$, a such system of block is complete, or
2) a doubly transitive group equals to $\text{PGL}(d,t)$, with $p^r = \frac{t^d - 1}{t - 1}, d \geq 3, t$ a power of $p^r$.

**Proof:** A simply transitive subgroup of $S_n$, of degree $p^r$ which contains a permutation of order $p^r$ (in our case $T$) must be imprimitive [16, p. 229]. Hence it admits a system of $p^i$ blocks of each of length $p^s, (s = i)$ formed by the orbits of $T^{p^i}$ [17] p. 67, a such system of block is complete.

From the previous we have that if $\text{Per}(C)$ is primitive it must be doubly transitive. From Lemma 2, Lemma 3 and Remark 1 we get the results.

**Corollary 6:** If $p \geq 5$ be a prime number, $C$ a cyclic codes of length $p^r, r \geq 2$ over $\mathbb{F}_p$. Then if $C$ is not an elementary code, the group $\text{Per}(C)$ is imprimitive.

**Proof:** From Theorem 5 if $\text{Per}(C)$ is primitive, hence it is doubly transitive. From Lemma 2 the only cases when the socle can be abelian are $N = C_p$ and $C_2 \times C_2$. In which cases $\text{Per}(C)$ must be equal to $\text{AGL}(1,p)$ or $\Sigma_2$; which is impossible. From [17, Lemma 22], if $\text{Per}(C)$ is doubly transitive with non abelian socle, then $\text{Soc}(\text{Per}(C)) = \text{Alt}(p^m)$, hence from Remark 1 the code is elementary.

**Theorem 7:** A non elementary cyclic code $C$ of composed length $n$ over $\mathbb{F}_q$, where $q = p^s$, admits a permutation group $\text{Per}(C)$ which is either
1) imprimitive, or
2) or doubly transitive; equals to $\text{Per}(C) = \text{PGL}(d,t)$, with $n = \frac{t^d - 1}{t - 1}, d \geq 3, t$ a power of $p^r$.

**Proof:** The group $\text{Per}(C)$ contains a full cycle and is with composed degree. Hence from Theorem of Burnside and Shur [30, p. 65], $\text{Per}(C)$ is either imprimitive or doubly transitive. In the doubly transitive from Lemma 2 and Lemma 3 this happens only if $\text{Per}(C)$ is $\text{Alt}(n)$, $S_n$ or $\text{PGL}(d,t)$.

From [8] the permutation group of the Hamming code $[15,12,3]_2$ is $\text{PGL}(4,2)$.

Now we will deal with the permutation group of $MDS$ codes.

T. P. Berger [4] proved that the only linear $MDS$ codes with minimum distance $d_C$ equals to 2 or $n$ are the repetitions codes and their duals. Hence in the following result we assume that $n > d_C > 2$.

**Theorem 8:** Let $C$ be a non elementary $[n,n-d_C + 1,d_C]$ $MDS$ cyclic code over $\mathbb{F}_q, q = p^s$, $n > 5, n > d_C > 2$, and $\text{gcd}(n-2,d_C-2) = 1$. If the group $\text{Per}(C)$ is doubly transitive, then
1) $\text{Per}(C) = \text{AGL}(1,p)$ with $n = p$, or
2) $\text{Per}(C) = \text{PSL}(d,p')$ with $n = \frac{p'^d - 1}{p' - 1}, d \geq 3, \text{gcd}(p' - 1, d) = 1$.

**Proof:** If $n = p$, hence from Lemma 2 $\text{Per}(C)$ is equal to $\text{AGL}(1,p)$, $\text{PSL}(2,11)$, $M_{11}$, $M_{23}$ or a subgroup of $\text{PGL}(d,p')$. From Proposition 4, we eliminate $\text{PSL}(2,11)$, $M_{11}$, $M_{23}$ since the $[11,6,5]_3$ and $[23,12,7]_2$ Golay codes are not $MDS$ and $M_{11}$ does not correspond to any cyclic code. Now we need the following Lemma.

**Lemma 9:** ([4]) If $C$ is an $MDS$ cyclic code with length $n$ and minimum distance $d_C > 2$ such that $\text{Per}(C)$ contains a doubly transitive subgroup $B$ and $\gcd(n-2,d_C-2) = 1$, then $B = \text{Per}(C)$.

If $\text{Per}(C)$ is a doubly transitive not equal to $\text{AGL}(1,p)$, then Theorem 5, Theorem 7, Lemma 9 and Remark 1 implies that the only solution is $\text{Per}(C) = \text{PGL}(d,p') = \text{PGL}(d,p') = \text{PGL}(d,p')$ for $\gcd(p' - 1, d) = 1$.

**Remark 2:** When $p$ a prime number any cyclic code of length $p$ over $\mathbb{F}_p$ is $MDS$ and is equivalent to an extended Reed–Solomon code [29]. The permutation group of the last codes is the affine group $\text{AGL}(1,p)$ [3].

### III. The Equivalence of Cyclic Codes

Let $C$ and $C'$ two cyclic codes of length $p^r$, with $p$ an odd prime number. The aim of this section is to find the permutations by which $C$ and $C'$ can be equivalent. Even if our results are also true for the cyclic combinatorial objects of length $p^r$, we consider only cyclic codes.

The multiplier $M_a$ is the affine transformation $M_a = \tau_a, 0$.

It is obvious that the image of a cyclic code by a multiplier is an equivalent cyclic code. If $\gcd(n, \phi(n)) = 1$ or $n = 4$, or $n = p^r, p > r$ are primes, and the Sylow $p$-subgroup of $\text{Per}(C)$ has order $p$. Then two cyclic codes $C$ and $C'$ of length $n$ are equivalent if and only if they are equivalent by a multiplier [1], [20], [27].

When $C$ is a cyclic code of length $p^r$, $P$ a Sylow $p$-subgroup of $\text{Per}(C)$, the following subset of $S_{p^r}$ was introduced by N. Brand [9]

$$H(P) = \{ \sigma \in S_{p^r}, \sigma^{-1}T \sigma \in P \}.$$  

The set $H(P)$ is well defined, since $< T >$ is a subgroup of $\text{Per}(C)$ of order $p^r$, hence it is a $p$-group of $\text{Per}(C)$.

From Sylow’s Theorem, there exists a Sylow $p$-subgroup $P$
of \( \text{Per}(C) \) such that \( < T > \leq P \). Furthermore in some cases the set \( H(P) \) is a group.

**Lemma 10:** (\cite[Lemma 3.1]{Huffman}) Let \( C \) and \( C' \) two cyclic codes of length \( p^r \). Let \( P \) be a Sylow \( p \)-subgroup of \( \text{Per}(C) \) which contains \( T \). Then \( C \) and \( C' \) are equivalent if and only if \( C \) and \( C' \) are equivalent by an element of \( H(P) \).

In the case of length \( p^2 \), \( H(P) \) has been given explicitly by Huffman et al. \cite{Huffman}. They proved that if \( P \) is a Sylow \( p \)-subgroup of \( \text{Per}(C) \) and \( |P| = p^2 \), then the cyclic codes \( C \) and \( C' \) are equivalent if only if they are equivalent by a multiplier. If \( p^2 < |P| \leq p^{r+1} \) the codes \( C \) and \( C' \) are equivalent by a power of a multiplier times a power of a generalized multiplier. A generalized multiplier is a permutation \( \mu(a,b)^t \in S_{p^r} \) defined as follows:

\[
i \rightarrow (ai) \mod p^k + bp^k.
\]

Now we are interested on the existence of the multiplier \( M_{p+1} \) in the group \( \text{Per}(C) \). Because it has as effect to simplify Algorithm 3.1 for the cyclic codes and Algorithm 6.11 for extended cyclic objects \cite{Huffman}.

**Lemma 11:** Let \( z \) be the largest integer such that \( p^t | (q^t - 1) \), with \( t \) order of \( q \) modulo \( p \). Hence if \( z = 1 \) we have:

1) \( \text{ord}_{p^r} q = p^{r-1}t \).

2) The multiplier \( M_{p+1} \in \text{Per}(C) \).

**Proof:** The proof of \( z = 1 \Rightarrow \text{ord}_{p^r} q = p^{r-1}t \), follows from \cite[Lemma 3.5.4]{Huffman}.

Since \( z = 1 \Rightarrow \text{ord}_{p^r} q = p^{r-1}t \), we choose integers \( a_k \), for \( k = 0, \ldots, p^{r-1} - 1 \), such that \( q^{a_k} = 1 \mod p^r \). Since \( \text{ord}_{p^r} q = p^{r-1}t \), the integers \( q^{a_k} = 1 \mod p^r \) are different modulo \( p^r \). This gives the integers \( a_k \), \( k = 0, \ldots, p^{r-1} - 1 \) are different modulo \( p^r \). Hence there exists \( a_k \) such that, \( a_{k_0} = 1 \mod p^r \), then \( a_{k_0}p = p \mod p^r \iff q^{a_{k_0}t} = 1 \mod p^r \). So \( q^{a_{k_0}t} = (1 + p) \mod p^r \).

Now by multiplying by \( i \) we obtain:

\[
(1 + p)q^i = iq^{i+k_{a_{k_0}}} \mod p^r,
\]

i.e., the cyclotomic class of \( i \) is invariant by the multiplier \( M_{p+1} \).

The following proposition gives some properties of the set \( H(P) \).

**Proposition 12:** Let \( P \) be a Sylow subgroup of \( \text{Per}(C) \) which contains \( T \). Hence the group \( H(P) \) verifies:

1. \( AG(p^r) = N_{S_{p^r}}(< T >) \subset H(P) \).

2. \( N_{S_{p^r}}(P) \subset H(P) \).

**Proof:** The fact \( AG(p^r) = N_{S_{p^r}}(< T >) \) follows from \cite[p. 710]{Huffman}.

Now we consider \( \sigma \in N_{S_{p^r}}(< T >) \), then

\[
\sigma^{-1} < T > \sigma = < T >.
\]

Since we assumed \( < T > \leq P \), hence \( \sigma \in H(P) \).

Let \( \sigma \in N_{S_{p^r}}(P) \), hence \( \sigma \in N_{S_{p^r}}(P) \) verifies \( \sigma^{-1}P \sigma = P \). Since we assumed \( < T > \leq P \), then \( \sigma^{-1}T \sigma \in P \), i.e.

\[
N_{S_{p^r}}(P) \subset H(P).
\]

**Lemma 13:** (\cite[Theorem 5.6]{Huffman}) Let \( p \) be an odd prime and let \( q \) be a prime power with \( p \mid q \). Let \( C \) be a cyclic code of length \( p^r \) over \( \mathbb{F}_q \) and \( t_k \) be the order of \( q \) modulo \( p^k \) and suppose that \( z = 1 \). For \( 1 \leq k < r \), \( \text{Per}(C) \) contains the group

\[
G_k = \{(\mu_{q^i,c}) \mid 0 \leq c < k < p^k \}
\]

which is of order \( t_kp^k \).

Furthermore, each element of \( H_k = \{(\mu_{q^i,c}) \mid 0 \leq c < k \} \) fixes the idempotent of the code \( C \).

**Theorem 14:** Let \( C \) be a cyclic code of length \( p^r, r > 1 \).

Hence a Sylow \( p \)-sous group of \( \text{Per}(C) \) is of order \( p^n \), such that

\[
r \leq s \leq p^{r-1} + p^{r-2} + \ldots + 1.
\]

Furthermore if \( z = 1 \), \( s \) verifies:

\[
2r - 1 \leq s \leq p^{r-1} + p^{r-2} + \ldots + 1.
\]

**Proof:** Let \( P \) be a Sylow \( p \)-subgroup of \( \text{Per}(C) \), hence \( P \) is a \( p \)-group of \( S_{p^r} \). From Sylow’s Theorem \( P \) is contained in a Sylow \( p \)-subgroup of \( S_{p^r} \). It is well known that a Sylow \( p \)-subgroup of \( S_{p^r} \) is of order \( p^{r-1}t^{p^{r-2}+\ldots+1} \).

It is isomorphic with the wreath product \( Z_{p^r} \wr \mathbb{Z}_p \) \cite[Kaluzhin’s Theorem]{Huffman}, hence \( r \leq s \leq p^{r-1} + p^{r-2} + \ldots + 1 \).

If \( z = 1 \), from Lemma 13 we have that \( \text{Per}(C) \) contains the group \( G_r \), of order \( t_r p^r \). From Lemma 11 we have that \( t_r = t_1p^{r-1} \), then we have \( |G_r| = t_1p^{2r-1} \). Hence \( p^{2r-1} \), divides \( \text{Per}(C) \), which means that \( \text{Per}(C) \) contains a \( p \)-subgroup of order at least \( p^{2r-1} \).

For \( n < p \), we define the following subsets of \( S_{p^r} \):

\[
Q^n = \{f : Z_{p^r} \rightarrow Z_{p^r} | f(x) = \sum_{i=0}^{n} a_ix^i, a_i \in Z_{p^r} \text{ for each } i, p \text{ is relatively prime to } a_1, \text{ and } p^{r-1} \text{ divides } a_i \text{ for } i = 2, 3, \ldots, n \}.
\]

The set \( Q^n \) and \( Q^n_1 \) are subgroups of \( S_{p^r} \) \cite[Lemma 2.1]{Huffman}.

**Theorem 15:** Let \( \text{Per}(C) \) be the permutation group of a cyclic code of length \( p^r \) and \( P \) a Sylow \( p \)-subgroup of \( \text{Per}(C) \) of order \( p^n \) such that \( T \leq P \). If \( r \leq s \leq p + 1 \), then \( H(P) \) is a group and we have:

(a) if \( s = r, P = < T > \leq H(P) \), and \( H(P) = AG(p^r) \),

(b) if \( s > r, P = Q^n_{s-2} \leq H(P) \), and \( H(Q^n_{s-2}) = Q^n_{s-1} \).

**Proof:** From Theorem 14 we have that \( r \leq s \leq p^{r-1} + p^{r-2} + \ldots + 1 \). In the case \( r = s, \) it is obvious to have \( P = < T > \), and \( H(T) < T > = N_{S_{p^r}}(T >) \).

Proposition 12 gives \( H(< T >) = AG(p^r) \). Furthermore, we remark that the part \( (b) \) of \cite[Theorem 2.1]{Huffman} can be generalized for the case \( p^r \).

Since it is based essentially on the \cite[Lemma 2.4]{Huffman} and the \cite[Lemma 3.1]{Huffman}.
Lemma. 3.2](the last Lemma was given for the length $p^r$).
The hypothesis $s \leq p + 1$ is to assure that $Q_{i}^{s-1} \leq Per(C)$.
Hence the result follows from [9, Theorem 2.2].

As noticed Brillhart et al. [10], it is quite unusual to have $z > 1$. Hence the importance of the following result.

**Theorem 16:** Let $C$ be a cyclic code of length $n = p^r$, over $\mathbb{F}_q$, such that $gcd(p, q) = 1, z = 1$ and $P$ be a Sylow $p$-subgroup of $Per(C)$ of order $p^{2r-1}$, then we have:

1) $P$ is normal in $G_T$,
2) $N_{S_n}(P) = G_T$,
3) $H(P) = \{ \tau \in S_n \mid \tau \mapsto q^{ij}a + q^{(i-1)}jc + q^{(i-2)}jc + \ldots + c, 0 \leq j < tp^{r-1}, a, c \in Z_n \}$

**Proof:** In this case we can assume that $P \leq G_T$, because we have $T = \mu_{1,1}^{(p^r)} \in G_T$. Let $N$ be the number of the Sylow $p$-subgroup in $G_T$. From the Sylow’s Theorem, $N \equiv 1 \mod p$ and $N$ divide $|G_T| = t_1p^{2r-1}$. Assume that $N = 1 + kp$, with $k > 0$ and $N\alpha = t_1p^{2r-1}, \alpha \geq 1$. Hence $(1+ kp)\alpha = t_1p^{2r-1}$, but $((1 + kp)p^{2r-1} = 1$. Then $\tau = \alpha$, absurd.

Because $t_1(p-1)$ and $N\alpha \geq 1$, hence $N = 1$. Since $N = 1$ and the Sylow subgroups are conjugate this gives that $P$ is normal in $G_T$.

Now, we will prove that $N_{S_n}(P) = G_T$.

Let $\sigma \in P \subseteq G_T$, then we can write $\sigma = \mu_{q,1}^{(p^r)} = T^{c}M_{q'},$ with $T$ the shift and $M_{q'}$ is a multiplier. Let $\tau \in N_{S_n}(P)$, hence there exists $\sigma' = T^{c}M_{q'} \in P$ such that $\tau \sigma = \sigma' \tau$, this is equivalent to $\tau T^{c}M_{q'} = T^{c'}M_{q'} \tau$.

For $0$ we have:

$$\tau T^{c}M_{q'}(0) = \tau T^{c}(0) = \tau(c),$$

hence we have:

$$\tau(c) = T^{c'}M_{q'} \tau(0) = q^{i'} \tau(0) + c'.

Then, $\tau(c) = q^{i' + d}$, where $d = c' + q^{i'}(\tau(0) - c)$. This implies $N_{S_n}(P) \leq G_T$.

Furthermore $P$ is normal in $G_T$, which gives that $N_{G_T}(P) = G_T$. But $N_{G_T}(P) \leq N_{Per(C)}(P) \leq N_{S_n}(P) \leq G_T$, then $N_{S_n}(P) = G_T$.

From the Lemma 12 we have that $N_{S_n}(P) \subset H(P)$. Hence $N_{S_n}(P) = G_T \subset H(P).

Now, let $\tau \in H(P)$, hence there exist $c$ and $j$ such that:

$$\tau T^{c-1} = T^{c}M_{q'}, \iff \tau T = T^{c}M_{q'} \tau$$

That is because $P \leq G_T$. For $0$ we obtain:

$$\tau T(0) = \tau(1) = q^{i'} \tau(0) + c.$$

$$\tau T(1) = \tau(2) = T^{c}M_{q'}(\tau(1)) = q^{i'} \tau(1) + c$$

$$\tau T(2) = \tau(3) = T^{c}M_{q'}(\tau(2)) = q^{i'} \tau(2) + c$$

Then the elements of $H(P)$ are $\tau \in S_n$ such that $\tau(i) = q^{ij}a + q^{(i-1)}jc + q^{(i-2)}jc + \ldots + c$.

IV. THE EQUIVALENCE FOR THE QUASI-CYCLIC CODES

A code $C$ of length $n = lm$ is said to be quasi-cyclic of order $l$ over $\mathbb{F}_q$, if it is invariant by the permutation

$$T^l : Z_n \rightarrow Z_n$$

$$i \mapsto i + l \mod n.$$ (6)

We consider the cycles $\sigma_i = (i, i + l, i + 2l, \ldots, i + (m-1)l)$ for $0 \leq i \leq l - 1$. The cycles $\sigma_i$ have order $m$. Furthermore we have

$$T^i = \sigma_0 \ldots \sigma_{l-1}.$$ (7)

This gives that $|\{T^l\}| = lcm(|\sigma_0|, \ldots, |\sigma_{l-1}|) = m$.

**Proposition 17:** Let $n = lm$, with $(m, l) = 1$ and $T^l > 0$ the subgroup of $S_n$ generated by the permutation $T^l$. Hence the normalizer of $T^l > 0$ in $S_n$ contains the following groups.

1) $Q = \langle \sigma_0, \ldots, \sigma_{l-1}, T >$.

2) $AG(n)$.

**Proof:** This is obvious that $T \in N_{S_n}(T^l > 0)$. Now we consider $\sigma_i \in Q$, from the relation (7) we have $\sigma_0 \ldots \sigma_{l-1} = T^l$. Furthermore the cycles $\sigma_i$ are disjoints, then commute. Hence $\sigma_i^{-1}T^l \sigma_i = T^l$.

We consider the affine transformation $\tau_{a,b} \in AG(n)$ proving that $\tau_{a,b} \in N_{S_n}(T^l > 0)$ is equivalent to prove the existence of an $a \in N^+$ such that,

$$\tau_{a,b} T^l \tau_{a,b}^{-1} = T^l a.$$ The permutation $\tau_{a,b}$ can be decomposed as follows, $\tau_{a,b} = \tau_{1,b} \tau_{a,0}$. Which gives with (7) the following equality:

$$\tau_{1,b} T^l \tau_{1,b}^{-1} = \tau_{1,b} \tau_{a,0} \sigma_0 \ldots \sigma_{l-1} \tau_{a,0}^{-1} \sigma_{1,b}^{-1}.$$ (8)

This is well known [19, Lemma. 5.1] if $\sigma = \sigma_0 \ldots \sigma_{l-1}$ is a product of the disjoint cycles and $S$ is a permutation of $S_n$. Hence $S \sigma S^{-1} = S(\sigma_0)(\sigma_1) \ldots S(\sigma_{l-1})$. For $r_a = a \mod l$ we obtain that $\tau_{a,0}(\sigma_i) = \sigma_{1,a}^{\sigma_{1,a}}$. This gives:

$$\tau_{1,b} \tau_{a,0}(\sigma_0) \tau_{a,0}(\sigma_1) \ldots \tau_{a,0}(\sigma_{l-1}) = \sigma_{1,a}^{\sigma_0 \sigma_{1,a} \ldots \sigma_{1,a}^{a \sigma_{r(a-1)}}} = T^{d_a}.$$ For $r_b = b \mod l$, we obtain

$$\tau_{1,b} \tau_{a,0} T^l \tau_{a,0}^{-1} = \tau_{1,b}$$

hence

$$\tau_{1,b} T^l = \prod_{i=0}^{l-1} \sigma_i \tau_{a,b} = T^l.$$ Finally we obtain:

$$\tau_{a,b} T^l \tau_{a,b}^{-1} = \tau_{1,b} \tau_{a,0} T^l \tau_{a,0}^{-1} \tau_{1,b}^{-1} = \tau_{1,b} T^l \tau_{1,b}^{-1} = T^{d_a}.$$ This gives $a = a$, hence $\tau_{a,b} \in N_{S_n}(T^l > 0)$.

In [14] C. Chabot gave explicitly the group $N_{S_n}(T^l > 0)$.  


A. Quasi-Cyclic Codes of length $p'l$

In the following, we consider the quasi-cyclic codes of length $n = p'l$, with $p$ a prime number which verifies $(p,l) = (p,q) = 1$. In this case $T^l \leqslant Per(C)$ is a subgroup of order $p'$. Hence it is contained in a Sylow $p$-subgroup $P$.

Lemma 18: Let $C$ and $C'$ two quasi-cyclic codes of length $n = p'l$ and $P$ a Sylow $p$-subgroup of $Per(C)$ such that $T^l \in P$. Hence $C$ and $C'$ are equivalent only if they are equivalent by the elements of the set

$$H'(P) = \{ \sigma \in S_n | \sigma^{-1} T^l \sigma \in P \}.$$ 

Proof: Since $C$ and $C'$ are equivalent, hence there exist a permutation $\sigma \in S_n$ such that $C' = \sigma(C)$. This gives the relation between the permutation groups $Per(C)$ and $Per(C')$.

$$Per(C') = \sigma Per(C) \sigma^{-1} \quad (9)$$

Let $P$ be a Sylow subgroup of $Per(C)$, hence from the relation $9$ we have $\sigma \sigma^{-1} = P'$ is a Sylow $p$-subgroup of $Per(C')$. From the Sylow Theorem there exists $\tau \in Per(C')$ such that $\tau P' \tau^{-1} = P'$. We can assume that $< T^l > \supseteq P'$, since $< T^l >$ is a $p$-group. Let $\gamma = \tau^{-1} \sigma$, then $\gamma$ is an isomorphism between $C$ and $C'$, because $\gamma(C) = \tau^{-1} \sigma(C) = \tau^{-1} C' = C'$ and $\gamma^{-1} T^l \gamma = \sigma^{-1} \tau T^l \tau^{-1} \sigma = \sigma^{-1} P' \sigma = P$ (because $\tau T^l \tau^{-1} \in \tau P' \tau^{-1}$), hence $\gamma \in H'(P)$.

It is obvious that if $P < T^l >$, then we have

$$N_{S_n}(< T^l >) = H'( < T^l >).$$

The following proposition gives other properties of $H'(P)$.

Proposition 19: Let $P$ be a Sylow $p$-subgroup of $Per(C)$, hence the group $H'(P)$ verifies the following properties:

- $N_{S_n}(< T^l >) \subset H'(P)$,
- $N_{S_n}(P) \subset H'(P)$.

Proof: We consider $N_{S_n}(< T^l >)$, the normalizer of $< T^l >$ in $S_n$. Then the permutation $\sigma \in N_{S_n}(< T^l >)$ verifies $\sigma^{-1} < T^l > \sigma = < T^l > \subset P$. Hence,

$$N_{S_n}(< T^l >) \subset H'(P). \quad (10)$$

We consider $N_{S_n}(P)$, the normalizer of $P$ in $S_n$. Then the permutation $\sigma \in N_{S_n}(P)$ verifies $\sigma^{-1} P \sigma = P$, hence for $T^l \in P$ we have $\sigma^{-1} T^l \sigma \in P$. Hence

$$N_{S_n}(P) \subset H'(P). \quad (11)$$

Corollary 20: The set $H'(P)$ verifies $AG(n) \subset H'(P)$.

Proof: From Proposition 17 we have $AG(n) \leqslant N_{S_n}(< T^l >)$. Furthermore, from Proposition 19 we have $N_{S_n}(< T^l >) \subset H'(P)$, hence the result.

By using the software GAP, we find on several examples that the set $H'(P)$ is an imprimitive group. By using this conjecture we prove the following result.

Proposition 21: If $n$ is even then, $H'(P)$ is either

1) imprimitive, 
2) or $S_n$ if the code is trivial.

If $n$ is odd then, $H'(P)$ is either

1) imprimitive, 
2) or $H'(P) = Alt(n)$, 
3) or $S_n$ if the code is trivial.

Proof: From Proposition 17 $H'(P)$ contains the group $Q = < \sigma_0, \ldots, \sigma_{l-1}, T >$,

by a similar proof to the Theorem 7 we obtains that the group $H'(P)$ is either imprimitive or doubly transitive. In the case doubly transitive we need the following Lemma.

Lemma 22: ([31]) Let $G$ be a primitive group of degree $n$ which contains a cycle of length $m > 1$. Hence if $m$ verifies $m < (n - m)!$, we have $G = Alt(n)$ or $S_n$.

If the group $H'(P)$ is doubly transitive, then it is primitive. From Proposition 19 it contains the cycles $\sigma_i$ of length $p'$. Furthermore we have $p' < (n - p')!$. Hence from Lemma 22 $H'(P)$ is either $Alt(n)$ or $S_n$. If $n$ is even, since $T \in H'(P)$ is a cycle of length $n$ is odd. Then $T \notin Alt(n)$, hence $H'(P) = S_n$. If $n$ is odd, $H'(P)$ is the group $Alt(n)$ or $S_n$.

V. Conclusion

The aim of this work is to find solutions of the three following problems:

- **Problem I** The classification of the permutation groups of cyclic codes.
- **Problem II** The determination of the permutations by which two cyclic codes of length $p'$ can be equivalent.
- **Problem III** The determination of the permutations by which two quasi-cyclic codes can be equivalent.

Our contribution to the solution of Problem allows to solve all the primitive cases and some imprimitive cases. Even though, there is still work and extension to do for the imprimitive cases.

As solution to Problem II we found explicitly the set $H(P)$ of permutations by which two cyclic codes of length $p'$ can be equivalent. We also proved that is sometimes a group.

We think that our contribution to solve Problem III can be refined by proving that the set $H'(P)$ of permutations by which two quasi-cyclic codes are equivalent is a group. If it is the case, since we proved that it will be an imprimitive group or $Alt(n)$ or $S_n$, hence it will be interesting to find some of other properties of $H'(P)$ in the imprimitive cases. Also, the results can be generalised to other situations of length.

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