Information geometry and asymptotic geodesics on the space of normal distributions

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Abstract
The family \( \mathcal{N} \) of \( n \)-variate normal distributions is parameterized by the cone of positive definite symmetric \( n \times n \)-matrices and the \( n \)-dimensional real vector space. Equipped with the Fisher information metric, \( \mathcal{N} \) becomes a Riemannian manifold. As such, it is diffeomorphic, but not isometric, to the Riemannian symmetric space \( \text{Pos}_1(n+1, \mathbb{R}) \) of unimodular positive definite symmetric \( (n+1) \times (n+1) \)-matrices. As the computation of distances in the Fisher metric for \( n > 1 \) presents some difficulties, Lovrič et al. (J Multivar Anal 74:36–48, 2000) proposed to use the Killing metric on \( \text{Pos}_1(n+1, \mathbb{R}) \) as an alternative metric in which distances are easier to compute. In this work, we survey the geometric properties of the space \( \mathcal{N} \) and provide a quantitative analysis of the defect of certain geodesics for the Killing metric to be geodesics for the Fisher metric. We find that for these geodesics the use of the Killing metric as an approximation for the Fisher metric is indeed justified for long distances.

Keywords Gaussian distributions · Fisher metric · Cone of positive definite matrices · Symmetric spaces · Geodesics.

Mathematics Subject Classification Primary 53C35; Secondary 53C30 · 62H05 · 62B10

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1 Introduction and overview

A multivariate normal distribution is determined by its covariance matrix and its mean vector. So for a fixed \( n \geq 1 \), the family \( \mathcal{N} \) of \( n \)-variate normal distributions is a differentiable manifold which can be identified with the product of the space of positive definite symmetric \( n \times n \)-matrices by the vector space \( \mathbb{R}^n \). For various statistical purposes, it is desirable to have a measure of distance between the elements of \( \mathcal{N} \). Such a distance measure is provided by the Fisher metric on \( \mathcal{N} \), which is a Riemannian metric that appears naturally in a certain statistical framework. We briefly review some properties of Fisher metric on the normal distributions in Sect. 2.

Computing the distances on \( \mathcal{N} \), however, turns out to be a non-trivial task. Even though explicit forms for the geodesics of the Fisher metric on \( \mathcal{N} \) are known (due to Calvo and Oller [4]), these only yield explicit formulas for the distance in particular cases. So Lovrič, Min-Oo and Ruh [10] proposed the use of a different metric in which distances are easier to compute. They map \( \mathcal{N} \) diffeomorphically onto the Riemannian symmetric space \( \text{SL}(n+1, \mathbb{R})/\text{SO}(n+1) \). This map is not an isometry between the Fisher metric and the metric of the symmetric space, which we call the Killing metric, but nevertheless, the two metrics are quite similar in appearance. So it is reasonable to ask how different they really are.

In Sect. 3 we describe the geometry of \( \mathcal{N} \) as a Riemannian homogeneous but non-symmetric space with the Fisher metric. In Theorem A we show that \( \mathcal{N} \) is a bundle whose base is the cone \( \text{Pos}(n, \mathbb{R}) \) of symmetric positive definite \( n \times n \)-matrices and whose fiber is \( \mathbb{R}^n \). This also gives rise to two pointwise mutually orthogonal foliations, one with leaves isometric to \( \text{Pos}(n, \mathbb{R}) \), the other with leaves isometric to \( \mathbb{R}^n \). The bundle structure obtained in Theorem A can be compared with the Main Theorem from [11] where a general homogeneous Hessian manifold is considered. In this sense, our Theorem A can be seen as an explicit description for \( \mathcal{N} \) of the fiber bundle structure observed in the abstract setup of more general manifolds.

To make a case for using the Killing metric as a sensible approximation for the Fisher metric, we compare the geometry of the Fisher metric and the geometry of the Killing metric in Sect. 4. We find that the Levi-Civita connection for the Fisher metric on the leaves \( \text{Pos}(n, \mathbb{R}) \) is affinely equivalent to the Levi-Civita connection of the Killing metric. So unparameterized geodesics in these leaves are the same for the two metrics. In Theorem B we show that Killing geodesics orthogonal to a leaf \( \text{Pos}(n, \mathbb{R}) \) at some point are asymptotically geodesic in the Fisher metric, that is, their defect from being a Fisher geodesic tends to zero as their curve parameter tends to infinity. We achieve this by introducing the notion of geodesic defect, which is given as the limit of the average norm of the acceleration of a curve over increasingly larger intervals (see Sect. 4.4). In particular, it can be seen as a sort of limit of the average of the geodesic curvature of the curve (see Remark 3 details). So we find that for two important classes of unparameterized geodesics, the Killing geodesics approximate or are identical to the corresponding Fisher geodesics. Though this is not an exhaustive comparison, it provides some justification to consider the easier-to-compute Killing metric as a good approximation for the Fisher metric.
Notations and conventions

Throughout, we will assume matrices to be real-valued. For a matrix $X \in \mathbb{R}^{n \times n}$, we let $X^\top$ denote its transpose. We also write $X^{-\top} = (X^\top)^{-1}$. The identity matrix is denoted by $I$ or $I_n$. By $E_{ij}$ we denote the elementary matrix whose entry in row $i$, column $j$ is 1, and all other entries are 0. Its symmetrization is $S_{ij} = \frac{1}{2}(E_{ij} + E_{ji})$. The canonical basis vectors of $\mathbb{R}^n$ are denoted by $e_1, \ldots, e_n$.

As usual, $\text{GL}(n, \mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid \text{det}(A) \neq 0 \}$, $\text{SL}(n, \mathbb{R}) = \{ A \in \text{GL}(n, \mathbb{R}) \mid \text{det}(A) = 1 \}$, $\text{O}(n) = \{ A \in \text{GL}(n, \mathbb{R}) \mid A^\top = A^{-1} \}$, $\text{SO}(n) = \{ A \in \text{O}(n) \mid \text{det}(A) = 1 \}$ denote the general linear, special linear, and (special) orthogonal groups, respectively. The subgroup of $\text{GL}(n, \mathbb{R})$ of matrices with positive determinant is denoted by $\text{GL}^+(n, \mathbb{R})$. The affine group is the semidirect product

$$\text{Aff}(n, \mathbb{R}) = \text{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n,$$

where the semidirect product is given by $(A_1, b_1)(A_2, b_2) = (A_1A_2, b_1 + A_1b_2)$ for $(A_i, b_i) \in \text{Aff}(n, \mathbb{R})$. We also write $\text{Aff}^+(n, \mathbb{R}) = \text{GL}^+(n, \mathbb{R}) \rtimes \mathbb{R}^n$.

By $\text{Sym}(n, \mathbb{R})$ we denote the set of symmetric $n \times n$-matrices, $\text{Sym}(n, \mathbb{R}) = \{ S \in \mathbb{R}^{n \times n} \mid S = S^\top \}$. We write $\text{Sym}_0(n, \mathbb{R})$ for the corresponding subspaces of elements with trace 0. The subset of diagonal matrices in $\text{Sym}(n, \mathbb{R})$ is denoted by $\text{Diag}(n, \mathbb{R})$.

The set of positive definite symmetric matrices in $\text{Sym}(n, \mathbb{R})$ is denoted by $\text{Pos}(n, \mathbb{R})$, $\text{Pos}(n, \mathbb{R}) = \{ S \in \text{Sym}(n, \mathbb{R}) \mid x^\top Sx > 0 \text{ for all non-zero } x \in \mathbb{R}^n \}$. Its subset of unimodular elements is $\text{Pos}^1(n, \mathbb{R}) = \{ S \in \text{Pos}(n, \mathbb{R}) \mid \text{det}(S) = 1 \}$. Recall that $\text{Pos}(n, \mathbb{R}) = \text{GL}(n, \mathbb{R})/\text{O}(n)$ and $\text{Pos}^1(n, \mathbb{R}) = \text{SL}(n, \mathbb{R})/\text{SO}(n)$.

2 Some background on information geometry

In this section we briefly review the concepts from information geometry that we use in the following. We mainly follow Amari and Nagaoka’s [1] presentation.
### 2.1 The Fisher metric and dual connections

Information geometry provides a framework to study a class of probability distributions \( p(x; \theta) \) defined on a sample space \( \Omega \) and determined by finitely many parameters \( \theta = (\theta_1, \ldots, \theta_n) \), where we assume for simplicity that \( p \) depends smoothly on \( x \) and \( \theta \). For example, the set of univariate normal distributions is parametrized by the mean \( \theta_1 = \mu \) and the variance \( \theta_2 = \sigma^2 \).

In general, the set \( M \) of admissible values for \( \theta \) can be viewed as an \( n \)-dimensional differentiable manifold, and we can define a positive semidefinite bilinear tensor \( g = (g_{ij}) \) on \( M \) via

\[
 g_{ij}(\theta) = -\int_{\Omega} \frac{\partial^2 \log(p(x; \theta))}{\partial \theta_i \partial \theta_j} p(x; \theta) \, dx.
\]

In the following we assume that \( g \) is positive definite everywhere, so that \((M, g)\) is a Riemannian manifold. Then \( g \) is called the **Fisher metric** on \( M \), and \((M, g)\) is called a **statistical manifold**.

In addition to the Fisher metric, there are two particular torsion-free affine connections defined on \( M \), denoted by \( \nabla^{(e)} \) and \( \nabla^{(m)} \). We refer to [1] and [2] for the definitions and corresponding properties. These connections are **dual** to each other with respect to \( g \), which means that for all vector fields \( X, Y, Z \) on \( M \),

\[
 Zg(X, Y) = g(\nabla^{(e)}_Z X, Y) + g(X, \nabla^{(m)}_Z Y).
\]

Moreover, the affine combination

\[
 \nabla = \frac{1}{2} \nabla^{(e)} + \frac{1}{2} \nabla^{(m)}
\]

yields the Levi-Civita connection \( \nabla \) of the Fisher metric \( g \).

The letters “e” and “m” stand for “exponential” and “mixture”, respectively, referring to two families of probability distributions in which these connections appear naturally. More generally, there is a whole family of affine connections \( \nabla^{(\alpha)} \) with \( \alpha \in [-1, 1] \) associated to \( g \), and \( \nabla^{(e)} = \nabla^{(1)} \), \( \nabla^{(m)} = \nabla^{(-1)} \). However, we are not concerned with values \( \alpha \neq \pm 1 \) here.

### 2.2 Exponential families

An **exponential family** is a statistical manifold \( M \) that consists of probability distributions of the form

\[
 p(x; \theta) = \exp(c(x) + \theta_1 f_1(x) + \cdots + \theta_n f_n(x) - \psi(\theta))
\]
for given functions $c, f_1, \ldots, f_n : \Omega \to \mathbb{R}$ and $\psi : M \to \mathbb{R}$. The normalization of $p(x; \theta)$ implies

$$
\psi(\theta) = \log \left( \int_{\Omega} \exp(c(x) + \theta_1 f_1(x) + \cdots + \theta_n f_n(x)) \, dx \right).
$$

(3)

The connections $\nabla^{(e)}$ and $\nabla^{(m)}$ are distinguished on an exponential family (see Amari and Nagaoka [1, Sections 2.3 and 3.3]).

**Theorem 1** Let $M$ be an exponential family. Then $\nabla^{(e)}$ and $\nabla^{(m)}$ are flat torsion-free affine connections on $M$.

In fact, the $\theta_1, \ldots, \theta_n$ form a flat coordinate system in the sense that $\nabla^{(e)}_{\partial_i} \partial_j = 0$, $i, j = 1, \ldots, n$, for the coordinate vector fields $\partial_i = \frac{\partial}{\partial \theta_i}$. The flat coordinate system $\eta_1, \ldots, \eta_n$ for $\nabla^{(m)}$ is obtained via a Legendre transform of $\theta_1, \ldots, \theta_n$,

$$
\frac{\partial \psi}{\partial \theta_i} = \eta_i, \quad i = 1, \ldots, n.
$$

In the flat $\theta$-coordinates, the Fisher metric for an exponential family is given as a Hessian metric $g = \nabla^{(e)} d\psi$, or equivalently

$$
g_{ij}(\theta) = \frac{\partial^2 \psi(\theta)}{\partial \theta_i \partial \theta_j}.
$$

(4)

We call $\psi$ the potential of the Fisher metric. The dual potential $\psi^*$ is given by $\psi^* = \theta^\top \eta - \psi$, and in the flat $\eta$-coordinates, the inverse $g^{ij}$ is given as a Hessian metric

$$
g^{ij}(\eta) = \frac{\partial^2 \psi^*(\eta)}{\partial \eta_i \partial \eta_j}.
$$

(5)

Another important property of exponential families is the following (see Amari and Nagaoka [1, Theorem 2.5]).

**Theorem 2** A submanifold $N$ of an exponential family $M$ is totally geodesic in $M$ with respect to $\nabla^{(e)}$ if and only if $N$ is an exponential family itself.

### 2.3 Normal distributions

The most important exponential family is formed by the normal distributions. An $n$-variate normal distribution is determined by its covariance matrix $\Sigma \in \text{Pos}(n, \mathbb{R})$ and its mean $\mu \in \mathbb{R}^n$ by the following formula

$$
p(x; \Sigma, \mu) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp \left( -\frac{1}{2} (x - \mu)^\top \Sigma^{-1} (x - \mu) \right).
$$
so the manifold we are considering is the space \( \mathcal{N} = \text{Pos}(n, \mathbb{R}) \times \mathbb{R}^n \). The flat coordinates for the connection \( \nabla^{(m)} \) are \((\mathcal{E}, \xi)\), where

\[
\xi = \mu \in \mathbb{R}^n, \quad \mathcal{E} = \Sigma + \mu \mu^\top \in \text{Pos}(n, \mathbb{R}),
\]

and the flat coordinates for the connection \( \nabla^{(e)} \) are \((\Theta, \theta)\), where

\[
\theta = \Sigma^{-1} \mu \in \mathbb{R}^n, \quad \Theta = -\frac{1}{2} \Sigma^{-1} \in (-\text{Pos}(n, \mathbb{R})).
\]

The potential \( \psi \) in these coordinate systems is (compare (3))

\[
\begin{align*}
\psi(\Sigma, \mu) &= \frac{1}{2} \mu^\top \Sigma^{-1} \mu + \frac{1}{2} \log(\det(2\pi \Sigma)), \\
\psi(\mathcal{E}, \xi) &= \frac{1}{2} \xi^\top (\mathcal{E} - \xi \xi^\top)^{-1} \xi + \frac{1}{2} \log(\det(2\pi (\mathcal{E} - \xi \xi^\top))), \\
\psi(\Theta, \theta) &= -\frac{1}{4} \theta^\top \Theta \theta - \frac{1}{2} \log(\det(-\pi^{-1} \Theta)).
\end{align*}
\]

### 3 Geometry of the family of normal distributions

In this section we take a closer look at the information geometry of the manifold \( \mathcal{N} = \text{Pos}(n, \mathbb{R}) \times \mathbb{R}^n \). Note that \( \text{Pos}(n, \mathbb{R}) = \mathbb{R} \times \text{Pos}_1(n, \mathbb{R}) \) as a product of manifolds.

#### 3.1 Basic geometric properties of \( \mathcal{N} \)

Here, we state the explicit form of the Fisher metric, its Levi-Civita connection and its curvature tensor in the \((\Sigma, \mu)\)-coordinates. These were originally computed by Skovgaard [12,13].

If \( g \) is the Fisher metric on \( \mathcal{N} \), \( X, Y \) are two coordinate vector fields in the \( \Sigma \)-directions, and \( v, w \) are two coordinate vector fields in the \( \mu \)-directions, then the metric tensor is

\[
g_{(\Sigma, \mu)}((X, v), (Y, w)) = v^\top \Sigma^{-1} w + \frac{1}{2} \text{tr}(\Sigma^{-1} X \Sigma^{-1} Y),
\]

(6)

and the Levi-Civita connection is determined by

\[
\begin{align*}
\nabla_X Y &= \nabla_Y X = -\frac{1}{2} (X \Sigma^{-1} Y + Y \Sigma^{-1} X), \\
\nabla_v w &= \nabla_w v = \frac{1}{2} (v w^\top + w v^\top), \\
\nabla_X v &= \nabla_v X = -\frac{1}{2} X \Sigma^{-1} v.
\end{align*}
\]

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Note that the symmetry in these equations is due to the fact that we are looking at coordinate vector fields.

If $X_1, X_2, X_3, X_4$ and $v_1, v_2, v_3, v_4$ are coordinate vector fields in the $\Sigma$- and $\mu$-directions, respectively, then the curvature of the Fisher metric is determined by

$$R(v_1, v_2, v_3, v_4) = \frac{1}{4}(v_2^\top \Sigma^{-1} v_3)(v_1^\top \Sigma^{-1} v_4) - (v_1^\top \Sigma^{-1} v_3)(v_2^\top \Sigma^{-1} v_4),$$

$$R(X_1, X_2, X_3, X_4) = \frac{1}{4}(\text{tr}(X_2 \Sigma^{-1} X_1 \Sigma^{-1} X_3 \Sigma^{-1} X_4 \Sigma^{-1})$$
$$- \text{tr}(X_1 \Sigma^{-1} X_2 \Sigma^{-1} X_3 \Sigma^{-1} X_4 \Sigma^{-1})), (8)$$

$$R(v_1, v_2, X_1, X_2) = \frac{1}{4}(v_1^\top \Sigma^{-1} X_1 \Sigma^{-1} X_2 \Sigma^{-1} v_2 - v_1^\top \Sigma^{-1} X_2 \Sigma^{-1} X_1 \Sigma^{-1} v_2),$$

$$R(v_1, X_1, v_2, X_2) = \frac{1}{4} v_1^\top \Sigma^{-1} X_1 \Sigma^{-1} X_2 \Sigma^{-1} v_2.$$

We now consider the two foliations of $\mathcal{N}$ into submanifolds of fixed $\Sigma_0$ or $\mu_0$, respectively. For fixed $\Sigma_0 \in \text{Pos}(n, \mathbb{R}), \mu_0 \in \mathbb{R}^n$ we will write

$$\mathcal{N}(\cdot, \mu_0) = \{(\Sigma, \mu_0) \mid \Sigma \in \text{Pos}(n, \mathbb{R})\},$$

$$\mathcal{N}(\Sigma_0, \cdot) = \{(\Sigma_0, \mu) \mid \mu \in \mathbb{R}^n\}.$$

It follows from (6) that the two foliations determined by these submanifolds are orthogonal.

Recall that the second fundamental form $B$ of a submanifold $N$ of $M$ is the normal component of $\nabla_X Y$ in $TM$ for two vector fields $X, Y$ tangent to $N$. We let $\partial_{(ij)}$ denote the coordinate vector field in direction $S_{ij}$, and we let $\partial_m$ denote the coordinate vector field in direction $e_m$. We denote by $J_\Sigma = \{(i, j) \mid 1 \leq i \leq j \leq n\}$ the set enumerating the coordinates of $\text{Sym}(n, \mathbb{R})$ and by $J_\mu = \{i = 1, \ldots, n\}$ the set enumerating the coordinates of $\mathbb{R}^n$, and set $J = J_\Sigma \cup J_\mu$. When we refer to an index $p \in J$, it may mean either a single index from $J_\mu$ or an index pair from $J_\Sigma$. Then the Christoffel symbols for the Levi-Civita connection $\nabla$ are denoted by $\Gamma_{pq}^r$, with $p, q, r \in J$.

**Proposition 1** For any $\mu_0 \in \mathbb{R}^n$ and with respect to the Fisher metric of $\mathcal{N}$, the submanifold $\mathcal{N}(\cdot, \mu_0)$ is totally geodesic.

**Proof** By (7), $\nabla_{\partial_{(ij)}} \partial_{(kl)}$ is tangent to $\mathcal{N}(\cdot, \mu_0)$ for all $i, j, k, l$. An arbitrary tangent vector field $X$ to $\mathcal{N}(\cdot, \mu_0)$ can be written as $X = \sum_{(i,j) \in J_\Sigma} w_{ij} \partial_{(ij)}$, with $w_{ij} \in C^\infty(\mathcal{N})$. Then

$$\nabla_{\partial_{(ij)}} X = \sum_{p \in J} (\partial_{(ij)} w_p + \sum_{q \in J} \Gamma_{(ij)q}^p w_q) \partial_p$$

$$= \sum_{p \in J} (\partial_{(ij)} w_p + \sum_{(k,l) \in J_\Sigma} \Gamma_{(ij)(kl)}^p w_{kl}) \partial_p \quad (w_m = 0 \text{ for } m \in J_\mu)$$

$$= \sum_{(r,s) \in J_\Sigma} (\partial_{(ij)} w_{rs} + \sum_{(k,l) \in J_\Sigma} \Gamma_{(ij)(kl)}^{(rs)} w_{kl}) \partial_{(rs)},$$
where the last identity follows from the fact that $\Gamma^m_{(ij)(kl)} = 0 = w_m$ for $m \in J_{\mu}$. The last expression is the induced covariant derivative on the submanifold $N(\cdot, \mu_0)$, since the $\mu$- and $\Sigma$-directions are orthogonal everywhere. Hence the second fundamental form of $N(\cdot, \mu_0)$ vanishes, which means $N(\cdot, \mu_0)$ is totally geodesic. \qed

Proposition 2 For any $\Sigma_0 \in \text{Pos}(n, \mathbb{R})$ and with respect to the Fisher metric of $N$, the submanifold $N(\Sigma_0, \cdot)$ is parallel. Also, the second fundamental form $B$ of $N(\Sigma_0, \cdot)$ satisfies

$$B(e_i, e_j) = \frac{1}{2}(E_{ij} + E_{ji})$$

for all $i, j = 1, \ldots, n$.

Proof The second fundamental form of $N(\Sigma_0, \cdot)$ is given by

$$B(\partial_i, \partial_j) = \sum_{(k,l) \in J_{\Sigma}} \Gamma_{ij}^{(kl)} \partial_{(kl)},$$

where $i, j \in J_{\mu}$.

Denote by $\nabla^\perp$ and $\nabla$ the normal and induced connection for $N(\Sigma_0, \cdot)$, respectively. By (7), $\nabla$ is a flat connection on $N(\Sigma_0, \cdot)$. Then the covariant derivative of $B$ is given by (i, j, m \in J_{\mu})

$$(\nabla_{\partial_m} B)(\partial_i, \partial_j) = \nabla^\perp_{\partial_m} (B(\partial_i, \partial_j)) - B(\nabla_{\partial_m} \partial_i, \partial_j) - B(\partial_i, \overline{\nabla}_{\partial_m} \partial_j)$$

$$= \nabla^\perp_{\partial_m} (B(\partial_i, \partial_j)),$$

where the last identity holds since $\nabla$ is flat and $\partial_i$ come from affine coordinates. Hence, we have for all $i, j, m \in J_{\mu}$

$$(\nabla_{\partial_m} B)(\partial_i, \partial_j) = \nabla^\perp_{\partial_m} (B(\partial_i, \partial_j))$$

$$= \nabla^\perp_{\partial_m} \left( \sum_{(k,l) \in J_{\Sigma}} \Gamma_{ij}^{(kl)} \partial_{(kl)} \right)$$

$$= \sum_{(k,l) \in J_{\Sigma}} (\partial_m \Gamma_{ij}^{(kl)}) \partial_{(kl)} + \sum_{(k,l), (r,s) \in J_{\Sigma}} \Gamma_{ij}^{(kl)} \Gamma_{(kl)m}^{(rs)} \partial_{(rs)}.$$ 

In this expression, $\partial_m \Gamma_{ij}^{(kl)} = 0$ and $\Gamma_{(kl)m}^{(rs)} = 0$ due to equation in (7). These computations imply that $\nabla^\perp B = 0$, in other words that $N(\Sigma_0, \cdot)$ is parallel.

On the other hand, to compute $B$ we use (7),

$$B(e_i, e_j) = \frac{1}{2}(e_i e_j^\top + e_j e_i^\top) = \frac{1}{2}(E_{ij} + E_{ji}),$$

where we have used the identification of basis vector with their corresponding partial differential operators. \qed
From the previous result the submanifold $\mathcal{N}(\Sigma_0, \cdot)$ is not totally geodesic. Hence, $\mathcal{N}$ is not the Riemannian product of $\mathcal{N}(\cdot, \mu_0)$ and $\mathcal{N}(\Sigma_0, \cdot)$ even though they are mutually orthogonal.

### 3.2 $\mathcal{N}$ as a homogeneous space

It is well-known that the affine group $\text{Aff}(n, \mathbb{R})$ acts transitively on $\mathcal{N}$ by

$$(A, b) \cdot (\Sigma, \mu) = (A \Sigma A^\top, A \mu + b),$$  \hspace{1cm} (9)

where $A \in \text{GL}(n, \mathbb{R})$, $b \in \mathbb{R}^n$, $(\Sigma, \mu) \in \mathcal{N}$. Furthermore, the action remains transitive when restricted to $\text{Aff}^+(n, \mathbb{R})$. The tangent space $T_{(I, 0)} \mathcal{N}$ can be identified with the vector space $\text{Sym}(n, \mathbb{R}) \times \mathbb{R}^n$. Given $(\Sigma, \mu) \in \mathcal{N}$ and $(X, v) \in T_{(\Sigma, \mu)} \mathcal{N}$, the tangent action of $(A, b) \in \text{Aff}(n, \mathbb{R})$ is

$$(A, b) \cdot (X, v) = (AXA^\top, Av).$$  \hspace{1cm} (10)

Thus we can identify

$$T_{(\Sigma, \mu)} \mathcal{N} \cong (A \cdot \text{Sym}(n, \mathbb{R}) \cdot A^\top) \oplus A\mathbb{R}^n = (A \cdot \text{Sym}(n, \mathbb{R}) \cdot A^\top) \oplus \mathbb{R}^n,$$

where $AA^\top = \Sigma$.

**Lemma 1** The affine group $\text{Aff}(n, \mathbb{R})$ acts transitively and isometrically on $\mathcal{N}$ by (9). Moreover, if $R \subset \text{GL}(n, \mathbb{R})$ denotes the subgroup of lower triangular matrices with positive diagonal entries, then the subgroup $R \ltimes \mathbb{R}^n$ acts simply transitively on $\mathcal{N}$.

**Proof** The transitivity is a well-known fact. It remains to check that (9) is isometric. The tangent action of $(A, b) \in \text{Aff}(n, \mathbb{R})$ is (10), hence

$$g_{(A \Sigma A^\top, A \mu + b)}((AXA^\top, Av), (AXA^\top, Av)) = (Av)^\top (A \Sigma A^\top)^{-1} (Av) + \frac{1}{2} \text{tr}((A \Sigma A^\top)^{-1} AXA^\top (A \Sigma A^\top)^{-1} AXA^\top)$$

$$= v^\top \Sigma^{-1} v + \frac{1}{2} \text{tr}(A^{-1} \Sigma^{-1} X \Sigma^{-1} XA^\top) = v^\top \Sigma^{-1} v + \frac{1}{2} \text{tr}(\Sigma^{-1} X \Sigma^{-1} X) = g_{(\Sigma, \mu)}((X, v), (X, v)).$$

This shows that the action is isometric.

Note that $(A, b) \cdot (I, 0) = (I, 0)$ is equivalent to $A \in O(n), b = 0$. So the stabilizer of $\text{Aff}(n, \mathbb{R})$ at $(I, 0)$ is $O(n)$. From the Iwasawa decomposition $\text{GL}(n, \mathbb{R}) = O(n)R$ it follows that $R \ltimes \mathbb{R}^n$ acts simply transitively.

### 3.3 Geometry of $\text{Pos}(n, \mathbb{R})$

As a consequence of Proposition 1 and Theorem 2, the Fisher metric of the family $\mathcal{N}(\cdot, \mu_0)$ of normal distributions with mean $\mu_0$ coincides with the restriction of the
Fisher metric of $\mathscr{N}$ to $\mathscr{N}(\cdot, \mu_0)$. Since all of these submanifolds are isometric, we may take $\mu_0 = 0$ for convenience. In the following, we will make explicit how $\mathscr{N}(\cdot, 0)$ with its Fisher metric is isometric to a symmetric space $\text{Pos}(n, \mathbb{R}) = \text{GL}(n, \mathbb{R})/\text{O}(n)$ with a suitably scaled Killing metric.

Consider the product of irreducible Riemannian symmetric spaces

$$M = \mathbb{R} \times \text{Pos}_1(n, \mathbb{R}),$$

where its Riemannian metric $g_M = g_1 \times g_2$ is the product of the metric $g_1$, which is $\frac{1}{2n}$ times the multiplication on $\mathbb{R}$, and the metric $g_2$ on $\text{Pos}_1(n, \mathbb{R})$ given by $g_2, \Sigma (X, Y) = \frac{1}{2} \text{tr}(\Sigma^{-1}X \Sigma^{-1}Y)$. Let $\text{GL}(n, \mathbb{R})$ act on $M$ via

$$A \cdot (\alpha, \Sigma) = (\alpha + 2 \log(\det(A))), \det(A)^{-\frac{2}{n}} A \Sigma A^\top).$$

**Lemma 2** The $\text{GL}(n, \mathbb{R})$-action on $M$ given above is by isometries.

**Proof** The tangent action of $A \in \text{GL}(n, \mathbb{R})$ at $(\alpha, \Sigma)$ on $(t, X) \in T_{(\alpha, \Sigma)}M$ is

$$dA(\alpha, \Sigma)(t, X) = (t, \det(A)^{-\frac{2}{n}} AXA^\top).$$

Hence

$$g_{M, A \cdot (\alpha, \Sigma)}(dA(\alpha, \Sigma)(t_1, X_1), dA(\alpha, \Sigma)(t_2, X_2))
= g_{M, A \cdot (\alpha, \Sigma)}((t_1, \det(A)^{-\frac{2}{n}} AX_1A^\top), (t_2, \det(A)^{-\frac{2}{n}} AX_2A^\top))
= g_{1, \alpha+2 \log(\det(A))}(t_1, t_2)
+ g_{2, \det(A)^{-\frac{2}{n}} A \Sigma A^\top}(\det(A)^{-\frac{2}{n}} AX_1A^\top, \det(A)^{-\frac{2}{n}} AX_2A^\top)
= \frac{1}{2n}t_1t_2 + \frac{1}{2} \text{tr}((\det(A)^{-\frac{2}{n}} A \Sigma A^\top)^{-1} \det(A)^{-\frac{2}{n}} AX_1A^\top
\times (\det(A)^{-\frac{2}{n}} A \Sigma A^\top)^{-1} \det(A)^{-\frac{2}{n}} AX_2A^\top)
= \frac{1}{2n}t_1t_2 + \frac{1}{2} \text{tr}(\Sigma^{-1}X_1 \Sigma^{-1}X_2)
= g_{M, (\alpha, \Sigma)}((t_1, X_1), (t_2, X_2))$$

Hence the action of $A \in \text{GL}(n, \mathbb{R})$ is isometric. \qed

Now define a map

$$\Psi : \text{Pos}(n, \mathbb{R}) \to \mathbb{R} \times \text{Pos}_1(n, \mathbb{R}), \quad \Sigma \mapsto (\log(\det(\Sigma)), \det(\Sigma)^{-\frac{1}{n}} \Sigma). \quad (11)$$

Note that for $A \in \text{GL}(n, \mathbb{R})$,

$$\Psi(A \cdot \Sigma) = (\log(\det(A \Sigma A^\top)), \det(A \Sigma A^\top)^{-\frac{1}{n}} A \Sigma A^\top)$$
\[= \log(\det(\Sigma)) + 2 \log(|\det(A)|) = \det(A)^{-2} \cdot \frac{1}{n} \det(\Sigma) \cdot \frac{1}{n} A \Sigma A^\top \]

\[= A \cdot \log(\det(\Sigma)), \quad \det(\Sigma)^{-\frac{1}{n}} \Sigma \]

\[= A \cdot \Psi(\Sigma).\]

So the map \(\Psi\) is \(\text{GL}(n, \mathbb{R})\)-equivariant.

We equip the manifold \(\text{Pos}(n, \mathbb{R})\) with the restriction of the Fisher metric (6) of \(\mathcal{N}\) to \(\mathcal{N}(\cdot, 0)\), which is the Fisher metric \(g\) of \(\mathcal{N}(\cdot, 0)\) by Proposition 1. Then \(\text{GL}(n, \mathbb{R})\) acts isometrically on \(\text{Pos}(n, \mathbb{R})\) by Lemma 1.

**Proposition 3** The Riemannian manifold \((\text{Pos}(n, \mathbb{R}), g)\) is isometric to the product \((\mathbb{R} \times \text{Pos}_1(n, \mathbb{R}), g_1 \times g_2)\) of the irreducible Riemannian symmetric spaces \((\mathbb{R}, g_1)\) and \((\text{Pos}_1(n, \mathbb{R}), g_2)\). In particular, \((\text{Pos}(n, \mathbb{R}), g)\) is a Riemannian symmetric space.

**Proof** The map \(\Psi\) defined in (11) is the desired isometry. In fact, \(\Psi\) is \(\text{GL}(n, \mathbb{R})\)-equivariant with respect to the isometric \(\text{GL}(n, \mathbb{R})\)-actions on \(\text{Pos}(n, \mathbb{R})\) and \(M\), and since \(\Psi'(\Sigma) = \Psi(A \cdot I) = A \cdot \Psi(I)\) (where \(\Sigma = AA^\top\)), it is enough to show that \(\Psi\) is an isometry at \(I \in \text{Pos}(n, \mathbb{R})\). So let \(X, Y \in T_I \text{Pos}(n, \mathbb{R}) \cong \text{Sym}(n, \mathbb{R})\). The differential of \(\Psi\) at \(I\) is

\[d\Psi_I X = \left. \frac{d}{dt} (\log(\det(I + tX)) \cdot \det(I + tX)^{-\frac{1}{n}} (I + tX)) \right|_{t=0}

= \left. \left( \det(I + tX)^{-1} \frac{d}{dt} \det(I + tX), \quad -\frac{1}{n} \det(I + tX)^{-\frac{1}{n} - 1} \left( \frac{d}{dt} \det(I + tX) \right) (I + tX) \right. \right|_{t=0}

+ \left. \det(I + tX)^{-\frac{1}{n}} \frac{d}{dt} (I + tX) \right|_{t=0}

= \left( \text{tr}(X), \quad X - \frac{1}{n} \text{tr}(X)I \right) \cdot \left( \text{tr}(Y), \quad Y - \frac{1}{n} \text{tr}(Y)I \right) \]

Then

\[g_{M,\Psi(I)}(d\Psi_I X, d\Psi_I Y)\]

\[= g_{M,(0,I)} \left( \left( \text{tr}(X), X - \frac{1}{n} \text{tr}(X)I \right), \left( \text{tr}(Y), Y - \frac{1}{n} \text{tr}(Y)I \right) \right) \]

\[= \frac{1}{2n} \text{tr}(X)\text{tr}(Y) + \frac{1}{2} \text{tr} \left( \left( X - \frac{1}{n} \text{tr}(X)I \right) \left( Y - \frac{1}{n} \text{tr}(Y)I \right) \right) \]

\[= \frac{1}{2n} \text{tr}(X)\text{tr}(Y) + \frac{1}{2} \text{tr} \left( XY - \frac{1}{n} \text{tr}(X)Y - \frac{1}{n} \text{tr}(Y)X \right) + \frac{1}{2n} \text{tr}(X)\text{tr}(Y) \]

\[= \frac{1}{2n} \text{tr}(X)\text{tr}(Y) + \frac{1}{2} \text{tr}(XY) - \frac{1}{n} \text{tr}(X)\text{tr}(Y) + \frac{1}{2n} \text{tr}(X)\text{tr}(Y) \]

\[= \frac{1}{2} \text{tr}(XY) = g_I(X, Y).\]
This shows that $\Psi$ is an isometry and concludes the proof of the proposition. □

**Corollary 1** \( \text{Iso}(\text{Pos}(n, \mathbb{R}), g) = \text{GL}^+(n, \mathbb{R}) \).

**Proof** Let \( G = \text{Iso}(\text{Pos}(n, \mathbb{R}), g) \) and let \( K \) be a subgroup of \( G \) such that \( G/K = \text{Pos}(n, \mathbb{R}) \). Let \( \mathfrak{g}, \mathfrak{k} \) denote the respective Lie algebras of \( G \), \( K \), and \( \sigma \) the Cartan involution. Since \( G/K \) is a symmetric product by Proposition 3, \( \mathfrak{g} \) and \( \mathfrak{k} \) split as a products \( \mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \) and \( \mathfrak{k} = \mathfrak{k}_1 \times \mathfrak{k}_2 \), \( \mathfrak{k}_i \subset \mathfrak{g}_i \), such that \( (\mathfrak{g}_1, \sigma) \) and \( (\mathfrak{g}_2, \sigma) \) are the symmetric Lie algebras associated to \( \text{Pos}(n, \mathbb{R}) \) and \( \text{Pos}^1(n, \mathbb{R}) \), respectively (cf. Kobayashi and Nomizu [8, Section XI.5], KN1). Since \( \text{Pos}^1(n, \mathbb{R}) = \text{SL}(n, \mathbb{R})/\text{O}(n) \) and \( \text{SL}(n, \mathbb{R}) \) is simple, \( \mathfrak{g}_2 = \mathfrak{sl}(n, \mathbb{R}) \) by Helgason [7, Theorem V.4.1]. Hence

\[
\dim G = \dim \text{SL}(n, \mathbb{R}) + \dim \text{Iso}(\mathbb{R}, g_\mathbb{R}) = (n^2 - 1) + 1 = \dim \text{GL}(n, \mathbb{R})
\]

and clearly \( \text{GL}(n, \mathbb{R}) \subseteq G \), so that \( G^\circ = \text{GL}(n, \mathbb{R})^\circ = \text{GL}^+(n, \mathbb{R}) \). □

### 3.4 Bundle geometry and foliations on \( \mathcal{N} \)

Let \( g \) denote the Fisher metric on \( \mathcal{N} \). We can now describe the geometry of \( (\mathcal{N}, g) \) in terms of Riemannian symmetric spaces.

**Theorem A** Consider the family of \( n \)-variate normal distributions \( \mathcal{N} \) equipped with the Fisher metric \( g \), given by (6). The following hold:

1. \( (\mathcal{N}, g) \) is a vector bundle

\[
\mathbb{R}^n \longrightarrow \mathcal{N} \longrightarrow \text{Pos}(n, \mathbb{R}),
\]

where the base \( \text{Pos}(n, \mathbb{R}) \) is equipped with the Fisher metric and the fiber over \( \Sigma \) is \( \mathbb{R}^n \) with scalar product determined by \( \Sigma^{-1} \).

2. The base \( \text{Pos}(n, \mathbb{R}) \) can be identified with the totally geodesic submanifold \( \mathcal{N}(\cdot, \mu_0) \) for any \( \mu_0 \in \mathbb{R}^n \), and it is isometric to a product of irreducible Riemannian symmetric spaces

\[
\text{Pos}(n, \mathbb{R}) = \mathbb{R} \times \text{Pos}_1(n, \mathbb{R})
\]

with the metrics on the factors given in Proposition 3.

3. The fiber \( \mathbb{R}^n \) over \( \Sigma_0 \) can be embedded as a parallel submanifold \( \mathcal{N}(\Sigma_0, \cdot) \) for any fixed \( \Sigma_0 \in \text{Pos}(n, \mathbb{R}) \), and as such it is orthogonal at \( (\Sigma_0, \mu_0) \in \mathcal{N} \) to the embedding of the base as \( \mathcal{N}(\cdot, \mu_0) \).

4. The submanifolds \( \mathcal{N}(\cdot, \mu) \) for all \( \mu \in \mathbb{R}^n \) and the submanifolds \( \mathcal{N}(\Sigma, \cdot) \) for all \( \Sigma \in \text{Pos}(n, \mathbb{R}) \) form two foliations of \( \mathcal{N} \), the leaves of which are pointwise orthogonal to one another.

**Proof** \( \mathcal{N} = \mathbb{R}^n \times \text{Pos}(n, \mathbb{R}) \) is a product of differentiable manifolds, though not of Riemannian manifolds. As such, \( \mathcal{N} \) is trivally a vector bundle with base \( \text{Pos}(n, \mathbb{R}) \) and fiber \( \mathbb{R}^n \). By Propositions 1 and 2, the submanifold \( \mathcal{N}(\cdot, \mu_0) \) is totally geodesic.
and the submanifold $\mathcal{N}(\Sigma_0, \cdot)$ is parallel, and they are orthogonal to each other at $(\Sigma_0, \mu_0)$. Also, the base $\mathcal{N}(\cdot, \mu_0)$ is isometric to $\text{Pos}(n, \mathbb{R})$ for every $\mu_0 \in \mathbb{R}^n$. The metrics on base and fiber are clear from (6). This proves parts (1) and (3). Part (2) is Proposition 3. For part (4), it is clear that $\mathcal{N}$ is a union of either of these families of submanifolds, and their pointwise orthogonality is clear from (6). □

4 The symmetric space of normal distributions

Due to the difficulty of explicitly computing distances in the Fisher metric on $\mathcal{N}$, Lovrič, Min-Oo and Ruh [10] suggested to replace the Fisher metric of $\mathcal{N}$ by the Killing metric of the symmetric space $\text{Pos}_1(n+1, \mathbb{R})$. For any homogeneous manifold that admits a Riemannian metric $\kappa$ turning it into a symmetric space, we will call $\kappa$ the Killing metric. Although the Fisher and the Killing metric are not isometric for $n > 1$, they are still quite similar, and distances in the Killing metric can be computed rather easily by exploiting the geometry of the symmetric space, as explained in [10].

In this section, we briefly recall the approach by Lovrič et al. [10] and compare the Killing metric on $\text{Pos}_1(n+1, \mathbb{R})$ to the Fisher metric on $\mathcal{N}$. We will find that for a lot of geodesics in the Fisher metric on $\mathcal{N}$, the geodesics in the Killing metric are good approximations at long distances.

4.1 On the symmetric space $\text{Pos}_1(n+1, \mathbb{R})$

A diffeomorphism from $\mathcal{N}$ to $\text{Pos}_1(n+1, \mathbb{R})$ is given by

$$\Phi : \mathcal{N} \to \text{Pos}_1(n+1, \mathbb{R}), \quad (\Sigma, \mu) \mapsto \frac{1}{\sqrt{n+1} \det(\Sigma)} \left( \begin{array}{c} \Sigma + \mu \mu^\top \\ \mu \\ 1 \end{array} \right), \quad (12)$$

in particular, $\dim \mathcal{N} = \dim \text{Pos}_1(n+1, \mathbb{R})$. However, $\Phi$ is not an isometry.

The tangent space of $\mathcal{N}$ at $(\Sigma, \mu) = (I_n, 0)$ can be identified with $\text{Sym}(n, \mathbb{R}) \oplus \mathbb{R}^n$. The differential of $\Phi$ at $(I_n, 0)$ is given by

$$(X, v) \mapsto d\Phi_{(I_n, 0)}(X, v) = \left( \begin{array}{c} X - \frac{\text{tr}(X)}{n+1} I_n \\ \frac{v}{\text{tr}(X)} \\ -\frac{\text{tr}(X)}{n+1} \end{array} \right), \quad (13)$$

where $X \in \text{Sym}(n, \mathbb{R})$ and $v \in \mathbb{R}^n$.

Remark 1 $\Phi$ is not an isometry for every $n \geq 1$. To see this compare (6) with (17) below. However, it is well known that the spaces $\mathcal{N}$ and $\text{Pos}_1(n+1, \mathbb{R})$ are isometric precisely for $n = 1$, but not through $\Phi$.

The map $\Phi$ allows us to identify the spaces $\mathcal{N}$ and $\text{Pos}_1(n+1, \mathbb{R})$, and use the global coordinate system $(\Sigma, \mu)$ to describe the elements of $\text{Pos}_1(n+1, \mathbb{R})$ as well. In the following we will do so, while suppressing the dependence on $\Phi$ in the notation.
Remark 2 Note that we use a different coordinate system to the one in [10]. There, instead of \((\Sigma, \mu)\), the authors use coordinates \((A, \mu)\), where \(A\) is the symmetric square root of \(\Sigma\), that is \(\Sigma = AA^\top\). This explains the absence of certain scalar factors in our formulas (12) and (13). It also affects the appearance of the metric (17) below, where in addition we use the different scaling factor \(\frac{1}{2}\) rather than \(\frac{1}{4}\) for the trace to obtain a symmetric metric from the Killing form of \(\mathfrak{sl}(n + 1, \mathbb{R})\).

For \(n \geq 1\), the isometry group of \(\text{Pos}_1(n + 1, \mathbb{R})\) is \(\text{SL}(n + 1, \mathbb{R})\), which acts on \(P \in \text{Pos}_1(n + 1, \mathbb{R})\) by

\[
P \mapsto \text{SPS}^\top.
\] (14)

The affine group \(\text{Aff}^+(n, \mathbb{R})\) also acts isometrically on \(\text{Pos}_1(n + 1, \mathbb{R})\) via the homomorphic embedding

\[
\text{Aff}^+(n, \mathbb{R}) \hookrightarrow \text{SL}(n + 1, \mathbb{R}), \quad (A, b) \mapsto \frac{1}{\sqrt{n+1\det(A)}} \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}.
\] (15)

Lemma 3 The diffeomorphism \(\Phi\) is equivariant for the \(\text{Aff}^+(n, \mathbb{R})\)-actions on \(\mathcal{N}\) and \(\text{Pos}_1(n + 1, \mathbb{R})\). In particular, \(\text{Aff}^+(n, \mathbb{R})\) acts transitively on \(\text{Pos}_1(n + 1, \mathbb{R})\).

Proof For any \((A, b) \in \text{Aff}^+(n, \mathbb{R})\) and \((\Sigma, \mu) \in \mathcal{N}\), with (9),

\[
\Phi((A, b) \cdot (\Sigma, \mu)) = \Phi(\Sigma \Sigma A^\top, A \mu + b)
\]

\[
= \frac{1}{\sqrt{n+1\det(A)^2\det(\Sigma)}} \begin{pmatrix} A \Sigma \Sigma A^\top + (A \mu + b)(A \mu + b)^\top & A \mu + b \\ \mu^\top A^\top + b^\top A^\top & 1 \end{pmatrix}
\]

and with (14) and (15),

\[
(A, b) \cdot \Phi(\Sigma, \mu) = \frac{1}{\sqrt{n+1\det(A)^2\det(\Sigma)}} \begin{pmatrix} A^\top & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma + A \mu \mu^\top & \mu \\ \mu^\top & 1 \end{pmatrix} \begin{pmatrix} A \Sigma \Sigma A^\top + (A \mu + b)(A \mu + b)^\top & A \mu + b \\ \mu^\top A^\top + b^\top A^\top & 1 \end{pmatrix}
\]

Hence \(\Phi\) is \(\text{Aff}^+(n, \mathbb{R})\)-equivariant. Since \(\text{Aff}^+(n, \mathbb{R})\) acts transitively on \(\mathcal{N}\), it acts transitively on \(\text{Pos}_1(n + 1, \mathbb{R})\) as well. \(\square\)

The symmetric space \(\text{Pos}_1(n + 1, \mathbb{R})\) is irreducible, which means that its Killing metric is, up to a positive multiple, determined by the Killing form of the Lie algebra \(\mathfrak{sl}(n + 1, \mathbb{R})\). The diffeomorphism \(\Phi\) allows us to identify the Killing metric \(\kappa\) with its pullback to \(\mathcal{N}\), and thus express it in the \((\Sigma, \mu)\)-coordinates of \(\mathcal{N}\). We can choose \(\kappa\)
suitably scaled such that in the \((\Sigma, \mu)\)-coordinates on \(\mathcal{N}\), it is given at \((\Sigma, \mu) = (I_n, 0)\) by

\[
\kappa((I_n, 0), (X, v), (Y, w)) = \frac{1}{2} \text{tr}(d\Phi(I_n, 0)(X, v)d\Phi(I_n, 0)(Y, w)) = v^\top w + \frac{1}{2} \text{tr}(XY) - \frac{1}{2(n+1)} \text{tr}(Xtr(Y)). \tag{16}
\]

Here we used (13) for the differentials. Then at any point \((\Sigma, \mu) \in \mathcal{N}\), the Killing metric is given by transporting (16) by the action of the affine group. We obtain

\[
\kappa((\Sigma, \mu), (X, v), (Y, w)) = v^\top \Sigma^{-1} w + \frac{1}{2} \text{tr}(\Sigma^{-1} X \Sigma^{-1} Y) - \frac{1}{2(n+1)} \text{tr}(\Sigma^{-1} X tr(\Sigma^{-1} Y)). \tag{17}
\]

Note that we use a scaling of the Killing metric \(\kappa\) different from the one in [10], to make it resemble the Fisher metric on \(\mathcal{N}\) more closely. Namely, up to the term \(-\frac{1}{2(n+1)} \text{tr}(\Sigma^{-1} X tr(\Sigma^{-1} Y))\), (17) resembles the Fisher metric (6) on \(\mathcal{N}\). The similarity becomes more apparent in the following paragraph.

### 4.2 Killing geodesics and Fisher geodesics in \(\text{Pos}(n, \mathbb{R})\)

We will simply speak of Fisher geodesics and Killing geodesics when referring to geodesics of the Fisher metric \(g\) and the Killing metric \(\kappa\), respectively. Even though \((\mathcal{N}, g)\) and \((\text{Pos}_1(n+1, \mathbb{R}), \kappa)\) are not isometric, we will see that the corresponding embeddings of the symmetric cone \(\text{Pos}(n, \mathbb{R})\) in both spaces are affinely equivalent.

For any fixed \(\mu_0 \in \mathbb{R}^n\), define the submanifold

\[
P_n(\mu_0) = \left\{ \frac{1}{\sqrt{\det(\Sigma)}} \begin{pmatrix} \Sigma + \mu_0 \mu_0^\top \\ \mu_0 \end{pmatrix} \begin{pmatrix} \mu_0^\top \\ 1 \end{pmatrix} \bigg| \Sigma \in \text{Pos}(n, \mathbb{R}) \right\}
\]

of the symmetric space \(\text{Pos}_1(n+1, \mathbb{R})\) with Killing metric (17). Clearly, \(P_n(\mu_0)\), just like \(\mathcal{N}(\cdot, \mu_0)\), is diffeomorphic to \(\text{Pos}(n, \mathbb{R})\).

**Proposition 4** Consider \(P_n(\mu_0)\) for any fixed \(\mu_0 \in \mathbb{R}^n\).

1. The affine transformation \((I, \mu_0)\) maps \(P_n(0)\) isometrically to \(P_n(\mu_0)\). In particular, the \(P_n(\mu_0)\) are isometric to each other for all \(\mu_0\).
2. \(\Phi(\mathcal{N}(\cdot, \mu_0)) = P_n(\mu_0)\).
3. \(P_n(\mu_0)\) is a totally geodesic submanifold of \(\text{Pos}_1(n+1, \mathbb{R})\).
4. Let \(X, Y\) be coordinate vector fields in the \(\Sigma\)-coordinates on \(\text{Pos}_1(n+1, \mathbb{R})\). Then their covariant derivative with respect to the Killing metric at the point \((\Sigma, \mu_0)\) is

\[
\nabla_X^\kappa Y = -\frac{1}{2}(X \Sigma^{-1} Y + Y \Sigma^{-1} X). \tag{18}
\]
5. $\Phi|_{\mathcal{N}(\cdot, \mu_0)} : \mathcal{N}(\cdot, \mu_0) \to P_n(\mu_0)$ is an affine equivalence.

In the proof of this proposition, we use the following formulas by Skovgaard [12, Lemma 2.3 and its proof]. Let $X, Y, Z \in \text{Sym}(n, \mathbb{R})$ and $\Sigma \in \text{Pos}(n, \mathbb{R})$, and let $\partial_X$ denote the directional derivative in the direction of $X$. Then

$\partial_X \text{tr}(Y \Sigma^{-1}) = -\text{tr}(Y \Sigma^{-1} X \Sigma^{-1})$,
$\partial_X \text{tr}(Y \Sigma^{-1} Z \Sigma^{-1}) = -\left(\text{tr}(Y \Sigma^{-1} X \Sigma^{-1} Z \Sigma^{-1}) + \text{tr}(Y \Sigma^{-1} Z \Sigma^{-1} X \Sigma^{-1})\right)$. (19)

**Proof** (Proof of Proposition 4) Part (1) is straightforward to verify using (14), (15) and the definition of $P_n(\mu_0)$. Part (2) is straightforward from (12). If we use the relations for the Levi-Civita connection of $\kappa$ given in [10, (3.8)], the computation for part (3) is identical to the proof of Proposition 1.

For part (4), Let $X, Y, Z$ be coordinate vector fields in the $\Sigma$-coordinates. We interpret them as tangent vector fields of the totally geodesic submanifold $P_n(\mu_0)$. Define a covariant derivative $\tilde{\nabla}_X Y$ on $P_n(\mu_0)$ by (18). Use (19) together with (17) to find

$X \kappa(Y, Z) = \frac{1}{2} \partial_X \text{tr}(\Sigma^{-1} Y \Sigma^{-1} Z)$
$= -\frac{1}{2(n+1)} \left( (\partial_X \text{tr}(\Sigma^{-1} Y)) \text{tr}(\Sigma^{-1} Z) + \text{tr}(\Sigma^{-1} Y) (\partial_X \text{tr}(\Sigma^{-1} Z)) \right)$
$= -\frac{1}{2} \left( \text{tr}(\Sigma^{-1} X \Sigma^{-1} Y \Sigma^{-1} Z) + \text{tr}(\Sigma^{-1} Y \Sigma^{-1} X \Sigma^{-1} Z) \right)$
$+ \frac{1}{2(n+1)} \left( \text{tr}(\Sigma^{-1} Y \Sigma^{-1} X) \text{tr}(\Sigma^{-1} Z) + \text{tr}(\Sigma^{-1} Y) \text{tr}(\Sigma^{-1} Z) \right)$

and

$\kappa(\tilde{\nabla}_X Y, Z) = -\frac{1}{2} \kappa(X \Sigma^{-1} Y + Y \Sigma^{-1} X, Z)$
$= -\frac{1}{4} \text{tr}(\Sigma^{-1} X \Sigma^{-1} Y \Sigma^{-1} Z + \Sigma^{-1} Y \Sigma^{-1} X \Sigma^{-1} Z)$
$+ \frac{1}{4(n+1)} \left( \text{tr}(\Sigma^{-1} X \Sigma^{-1} Y) \text{tr}(\Sigma^{-1} Z) \right)$
$+ \text{tr}(\Sigma^{-1} Y \Sigma^{-1} X) \text{tr}(\Sigma^{-1} Z)$,
$= -\frac{1}{4} \left( \text{tr}(\Sigma^{-1} X \Sigma^{-1} Y \Sigma^{-1} Z) + \text{tr}(\Sigma^{-1} Y \Sigma^{-1} X \Sigma^{-1} Z) \right)$
$+ \frac{1}{2(n+1)} \text{tr}(\Sigma^{-1} X \Sigma^{-1} Y) \text{tr}(\Sigma^{-1} Z),$
\[
\kappa(Y, \tilde{\nabla}_X Z) = -\frac{1}{4} (\text{tr}(\Sigma^{-1} Y \Sigma^{-1} X \Sigma^{-1} Z) + \text{tr}(\Sigma^{-1} Y \Sigma^{-1} Z \Sigma^{-1} X)) + \frac{1}{2(n+1)} \text{tr}(\Sigma^{-1} Y) \text{tr}(\Sigma^{-1} X) \text{tr}(\Sigma^{-1} Z).
\]

After applying some identities for the trace and collecting terms, we find that indeed
\[
X\kappa(Y, Z) = \kappa(\tilde{\nabla}_X Y, Z) + \kappa(Y, \tilde{\nabla}_X Z).
\]

By evaluating \(\tilde{\nabla}_X Y\) on coordinate vector fields, we readily find that the torsion vanishes. Hence \(\tilde{\nabla}\) is the Levi-Civita connection of the restriction of \(\kappa\) to \(P_n(\mu_0)\), and since \(P_n(\mu_0)\) is a totally geodesic submanifold, \(\tilde{\nabla}\) is the restriction of the Levi-Civita connection \(\nabla^\kappa\) of \((\text{Pos}_1(n+1, \mathbb{R}), \kappa)\) to \(P_n(\mu_0)\).

For part (5), it is evident from comparing (7) and (18) that \(\Phi|_{\mathcal{N}(\cdot, \mu_0)}\) is indeed an affine equivalence from \(\mathcal{N}(\cdot, \mu_0)\) to \(P_n(\mu_0)\).

### 4.3 Distances in \(\text{Pos}(n, \mathbb{R})\)

Distances between points contained in \(\mathcal{N}(\cdot, \mu_0)\) for fixed \(\mu_0 \in \mathbb{R}\) are readily computed using the fact that \(\mathcal{N}(\cdot, \mu_0)\) is a totally geodesic submanifold of \(\mathcal{N}\), and also a Riemannian symmetric space isometric to \(\text{Pos}(n, \mathbb{R})\). The distances in this symmetric space are easy to compute, since they can be reduced to computations in a flat totally geodesic submanifold.

**Lemma 4** Let \(\Delta = \text{diag}(\delta_1, \ldots, \delta_n) \in \text{Diag}(n, \mathbb{R}) \cap \text{Pos}(n, \mathbb{R})\). The Fisher distance from the identity matrix \(I_n\) to \(\Delta\) is

\[
\text{dist}_g(I_n, \Delta) = \sqrt{\frac{1}{2} \sum_{i=1}^{n} \log(\delta_i)^2}.
\]

**Proof** It is well-known that \(\text{Diag}(n, \mathbb{R}) \cap \text{Pos}(n, \mathbb{R})\) is a maximal flat totally geodesic subspace in \(\text{Pos}(n, \mathbb{R})\). Thus the geodesic \(\gamma\) from \(I_n\) to \(\Delta\) is

\[
\gamma(t) = \exp(t\Lambda),
\]

where \(\Lambda = \log(\Delta)\) (this is well-defined since all eigenvalues of \(\Delta\) are positive). Then by (6) for all \(t\),

\[
\|\gamma'(t)\|_{\gamma(t)}^g = \|\Lambda \gamma(t)\|_{\gamma(t)}^g = \sqrt{\frac{\text{tr}(\Lambda^2)}{2}}.
\]

The distance from \(I_n\) to \(\Delta\) is then

\[
\text{dist}_g(I_n, \Delta) = \int_0^1 \|\gamma'(t)\|_{\gamma(t)}^g \, dt = \sqrt{\frac{\text{tr}(\Lambda^2)}{2}} = \sqrt{\frac{1}{2} \sum_{i=1}^{n} \log(\delta_i)^2}.
\]
with \( \Lambda = \text{diag}(\log(\delta_1), \ldots, \log(\delta_n)) \).

Similarly to the procedure described by Lovrič et al. [10, pp. 42-43] for \( \text{Pos}_1(n+1, \mathbb{R}) \) with the Killing metric, we describe the procedure to derive the Fisher distance formula for elements \( S_1 = (\Sigma_1, \mu_0), S_2 = (\Sigma_2, \mu_0) \) in \( \mathcal{N}(\cdot, \mu_0) \).

1. By applying the isometry \( (I_n, -\mu_0) \in \text{Aff}^+(n, \mathbb{R}) \), we may assume that \( S_1, S_2 \in \mathcal{N}(\cdot, 0) \). Under the identification of this submanifold with \( \text{Pos}(n, \mathbb{R}) \), we identify \( S_i \) with \( \Sigma_i \).

2. We can write \( \Sigma_1 = A_1 A_1^\top \) for some \( A_1 \in \text{SL}(n, \mathbb{R}) \).

3. When applying the isometry \( (A_1^{-1}, 0) \in \text{Aff}^+(n, \mathbb{R}) \), we have

\[
\text{dist}_g(\Sigma_1, \Sigma_2) = \text{dist}_g(I_n, A_1^{-1} \Sigma_2 A_1^{-\top}).
\]

We may thus assume that \( \Sigma_1 = I_n \). Note that this may change the eigenvalues of \( \Sigma_2 \).

4. By applying an isometry \( (T, 0) \) for some \( T \in \text{O}(n) \), we may assume that \( A_1^{-1} \Sigma_2 A_1^{-\top} = \Lambda \) is a diagonal matrix in \( \text{Diag}(n, \mathbb{R}) \cap \text{Pos}(n, \mathbb{R}) \). Now Lemma 4 applies, and we obtain

\[
\text{dist}_g(S_1, S_2) = \text{dist}_g(\Sigma_1, \Sigma_2) = \sqrt{\frac{1}{2} \sum_{i=1}^n \log(\lambda_i)^2}
\]

for the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of the matrix \( A_1^{-1} \Sigma_2 A_1^{-\top} \).

Up to a factor \( \frac{1}{\sqrt{2}} \), this coincides with the distance formula for elements in \( P_n(\mu_0) \) as computed in [10].

### 4.4 Asymptotic geodesics orthogonal to \( \text{Pos}(n, \mathbb{R}) \)

As we just saw, the lengths of geodesics in \( \mathcal{N} \) tangent to the symmetric submanifolds \( \mathcal{N}(\cdot, \mu_0) \) are relatively easy to compute. Unfortunately, the same cannot be said for geodesics transversal to \( \mathcal{N}(\cdot, \mu_0) \). Although explicit solutions for the Fisher metric’s geodesic equation have been found by Calvo and Oller [4, Section 3], they only yield explicit formulas for the distance between two points in some special cases. In this paragraph, we want to argue that Killing geodesics provide reasonable approximations whose lengths are easy to compute.

We introduce some terminology. Let \( c : \mathbb{R} \to \mathcal{N} \) be a differentiable curve and define the geodesic defect of \( c \) to be

\[
\delta(c) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \| \nabla_c c'(s) c'(s) \|_c^9 \, ds.
\]

If \( \delta(c) = 0 \), then we call \( c \) an asymptotic geodesic in the Fisher metric on \( \mathcal{N} \). Note that by this definition, \( \delta(c) \) is invariant under isometries of \( \mathcal{N} \). We restrict ourselves...
to curves with domain of definition \( \mathbb{R} \) here, since below we will only study Killing geodesics \( c \), which are complete.

**Remark 3** We recall that a curve \( c : \mathbb{R} \to N \) is a geodesic if and only if \( \nabla_{c'}(t)c'(t) = 0 \) for all \( t \in \mathbb{R} \). Hence, \( \| \nabla_{c'}(s)c'(s) \|^2_{c(s)} \) gives a numerical value that measures the failure or obstruction for the curve \( c \) to be a geodesic at \( s \). Hence, the integral

\[
\frac{1}{t} \int_0^t \| \nabla_{c'}(s)c'(s) \|^2_{c(s)} \, ds
\]

yields the average of the obstruction for the curve \( c \) to be a geodesic on the interval \([0, t]\). It follows that \( \delta(c) \) measures the asymptotic behaviour of such obstruction for \( c \) to be a geodesic. It is in this sense that we say that \( c \) is asymptotically geodesic when \( \delta(c) = 0 \). We also note that \( \| \nabla_{c'}(s)c'(s) \|^2_{c(s)} \) is sometimes called the geodesic curvature of the curve which also motivates our terminology.

Our goal in this paragraph is to compare the behaviour of such Killing geodesics with that of Fisher geodesics, and eventually we will show:

**Theorem B** Consider the family of \( n \)-variate normal distributions \( N \) equipped with the Fisher metric \( g \), given by (6). Let \( c : \mathbb{R} \to N \) be a geodesic for the Killing metric \( \kappa \) on \( N \), given by (17). Assume that \( c(0) = (\Sigma_0, \mu_0) \) and \( c'(0) \perp N(\cdot, \mu_0) \). Then \( c \) is an asymptotic geodesic for the Fisher metric.

The proof requires some preparations. For simplicity, we will assume that

\[
c(0) = (I_n, 0), \quad c'(0) = (0, e_1).
\]

In the proof of Theorem B below we see that it is sufficient to treat this case.

At the point \((I_n, 0)\), the tangent subspace orthogonal to \( T_{(I_n, 0)}N(\cdot, 0) \) is mapped by \( d\Phi \) to

\[
V = \left\{ \begin{pmatrix} 0 & v \\ v^T & 0 \end{pmatrix} \mid v \in \mathbb{R}^n \right\}.
\]

Incidentally, \( V \) is also the orthogonal space to \( T_{I_{n+1}}P_n(0) \) for the Killing metric on \( \text{Pos}_1(n+1, \mathbb{R}) \). Moreover, \( V \) lies in \( \text{Sym}_0(n+1, \mathbb{R}) \), the complement of the maximal subalgebra of compact type in the Cartan decomposition of \( \mathfrak{sl}(n+1, \mathbb{R}) \).

**Remark 4** Recall that in any Riemannian symmetric space \( M = G/K \), the geodesics through a point \( p \in M \) are given as the orbits of one-parameter subgroups

\[
\gamma(t) = \exp(tX)p \quad \text{for some } X \in \mathfrak{m},
\]

where \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) is a Cartan decomposition of the Lie algebra of \( G \) (cf. Kobayashi and Nomizu [8, Corollary X.2.5], KN1). In particular, for \( M = \text{Pos}(n, \mathbb{R}) \) and \( G = \text{GL}(n, \mathbb{R}) \), the subspace \( \mathfrak{m} \) is \( \text{Sym}(n, \mathbb{R}) \), and for \( M = \text{Pos}_1(n, \mathbb{R}) \) and \( G = \text{SL}(n, \mathbb{R}) \), the subspace \( \mathfrak{m} \) is \( \text{Sym}_0(n, \mathbb{R}) \).
By this remark, the Killing geodesics tangent to $V$ at $I_{n+1}$ in $\text{Pos}_1(n+1, \mathbb{R})$ is given by the action (14) of the one-parameter subgroups

$$\exp \begin{pmatrix} 0 & tv \\ tv^T & 0 \end{pmatrix}.$$ 

**Lemma 5** The Killing geodesic $\tilde{c}$ with

$$\tilde{c}(0) = I_{n+1}, \quad \tilde{c}'(0) = \begin{pmatrix} 0 & e_1 \\ e_1^T & 0 \end{pmatrix}$$

is given by

$$\tilde{c}(t) = \begin{pmatrix} \cosh(2t) & 0 & \sinh(2t) \\ 0 & I_{n-1} & 0 \\ \sinh(2t) & 0 & \cosh(2t) \end{pmatrix}. \quad (22)$$

Its preimage in $\mathcal{N}$ under the diffeomorphism $\Phi$ is

$$c(t) = (\Phi^{-1} \circ \tilde{c})(t) = \begin{pmatrix} \cosh(2t)^{-2} & 0 \\ 0 & \cosh(2t)^{-1}I_{n-1} \end{pmatrix}, \tanh(2t)e_1.$$ \quad (23)

**Proof** Write $X = \tilde{c}'(0)$. By induction, we find that the even and odd powers of $X$ are

$$X^{2k} = \begin{pmatrix} e_1e_1^T & 0 \\ 0 & 1 \end{pmatrix}, \quad k \geq 1, \quad X^{2k+1} = \begin{pmatrix} 0 & e_1 \\ e_1^T & 0 \end{pmatrix}, \quad k \geq 0.$$ 

Since $e_1e_1^T = E_{11}$ we have

$$\exp(tX) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} X^{2k+1} + \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} X^{2k}$$

$$= \begin{pmatrix} 0 & 0 & \sinh(t) \\ 0 & 0 & 0 \\ \sinh(t) & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cosh(t) & 0 & 0 \\ 0 & I_{n-1} & 0 \\ 0 & 0 & \cosh(t) \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix}.$$ 

This one-parameter subgroup acts on $I_{n+1}$ by

$$\exp(tX)I_{n+1}\exp(tX)^T = \exp(tX)^2 = \exp(2tX) = \begin{pmatrix} \cosh(2t) & 0 & \sinh(2t) \\ 0 & I_{n-1} & 0 \\ \sinh(2t) & 0 & \cosh(2t) \end{pmatrix}.$$
which is the desired expression (22) for the geodesic \( \tilde{c} \). To obtain the expression for \( c \), we need the \((\Sigma, \mu)\)-coordinates of \( \tilde{c} \). By (12),
\[
\frac{1}{n+1/\sqrt{\det(\Sigma)}} = \cosh(2t), \quad \mu = \tanh(2t)e_1,
\]
and thus
\[
\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & \cosh(2t)I_{n-1}^{-1} \end{pmatrix} = -\tanh(2t)^2e_1e_1^\top = \begin{pmatrix} \cosh(2t)^{-2} & 0 \\ 0 & \cosh(2t)^{-1}I_{n-1} \end{pmatrix}.
\]
This yields the expression (23) for \( c(t) \).

After applying some identities for the hyperbolic functions, we find:

**Lemma 6** The first and second derivatives of the Killing geodesic \( c \) are
\[
c'(t) = \begin{pmatrix} \frac{-4 \sinh(2t)}{\cosh(2t)^3} & 0 \\ 0 & -\frac{2 \sinh(2t)}{\cosh(2t)^2}I_{n-1}^{-1} \end{pmatrix}, \quad c''(t) = \begin{pmatrix} \frac{-8+16 \sinh(2t)^2}{\cosh(2t)^4} & 0 \\ 0 & -\frac{-4+4 \sinh(2t)^2}{\cosh(2t)^3}I_{n-1}^{-1} \end{pmatrix},
\]
(24)
\[
c''(t) = \begin{pmatrix} \frac{-8+16 \sinh(2t)^2}{\cosh(2t)^4} & 0 \\ 0 & -\frac{-4+4 \sinh(2t)^2}{\cosh(2t)^3}I_{n-1}^{-1} \end{pmatrix}, \quad \mu'(t) = \begin{pmatrix} \frac{4 \cosh(2t)^2}{\cosh(2t)^3} \\ 0 \end{pmatrix},
\]
(25)

Using Lemma 6 and (7), we can now compute the second covariant derivative of \( c(t) = (\Sigma(t), \mu(t)) \),
\[
\nabla_{c'(t)}c'(t) = c''(t) - \left( \Sigma'(t)\Sigma^{-1}(t)\Sigma'(t) - \mu'(t)\mu'(t)^\top + \Sigma'(t)\Sigma^{-1}(t)\mu'(t) \right),
\]
(26)

with
\[
\Sigma'(t)\Sigma^{-1}(t)\Sigma'(t) = \begin{pmatrix} \frac{-4 \sinh(2t)}{\cosh(2t)^3} & 0 \\ 0 & -\frac{2 \sinh(2t)}{\cosh(2t)^2}I_{n-1}^{-1} \end{pmatrix}, \quad \Sigma'(t)\Sigma^{-1}(t)\mu'(t) = \begin{pmatrix} \frac{4 \cosh(2t)^2}{\cosh(2t)^3} \\ 0 \end{pmatrix},
\]

\[
\mu'(t)\mu'(t)^\top = \begin{pmatrix} \frac{4 \cosh(2t)^2}{\cosh(2t)^3} \\ 0 \end{pmatrix}.
\]
Lemma 7  

The second covariant derivative \( \nabla_{c'(t)} c'(t) \) for the Fisher metric in \( \mathcal{N} \) of the Killing geodesic \( c \) is

\[
\nabla_{c'(t)} c'(t) = \left( \begin{array}{c} -\frac{4}{\cosh(2t)^3} \Sigma_0 - \frac{4}{\cosh(2t)^3} I_n \end{array} \right),
\]

for some \( v \in \mathbb{R}^n \).

In particular, \( c \) is not a Fisher geodesic.

With this lemma, we can prove Theorem B.

**Proof** (of Theorem B) Let \( c \) be a Killing geodesic beginning at a point \( c(0) = (\Sigma_0, \mu_0) \) whose initial direction \( c'(0) \) is orthogonal to \( \mathcal{N}(\cdot, \mu_0) \). That is, \( c'(0) = (0, v) \) for some \( v \in \mathbb{R}^n \).

1. We may reparameterize \( c \) by rescaling the parameter \( t \) such that \( \|v\|_{(\Sigma_0, \mu_0)} = 1 \).
   This affects the second covariant derivative of \( c \) only by a constant factor.
2. By applying an isometry \( (A, b) \in \text{Aff}^+(n, \mathbb{R}) \) with \( A^\top A = \Sigma_0^{-1} \) and \( b = -A\mu_0 \), we may assume that \( c(0) = (I_n, 0) \) and \( c'(0) \) is orthogonal to \( \mathcal{N}(\cdot, 0) \).
3. Then we may apply another isometry \( (T, 0) \in \text{Aff}(n, \mathbb{R}) \) with \( T \in O(n) \), so that we may assume \( c'(0) = (0, e_1) \), while \( c(0) = (I_n, 0) \) still holds.

Since the affine group acts isometrically for both the Fisher metric and the Killing metric, the resulting curve \( c \) is still a Killing geodesic.

By Lemma 7 and (6),

\[
\|\nabla_{c'(t)} c'(t)\|_{c(t)}^g = \frac{1}{2} \text{tr} \left( c(t)^{-1}(\nabla_{c'(t)} c'(t)) c(t)^{-1}(\nabla_{c'(t)} c'(t)) \right)
\]

\[
= \frac{1}{2} \text{tr} \left( \begin{array}{cc} 16 & 0 \\ 0 & 16 \cosh(2t)^3 I_n \end{array} \right)
\]

\[
= \frac{2\sqrt{2n}}{\cosh(2t)^2}.
\]

Now

\[
\int_0^t \|\nabla_{c'(t)} c'(s)\|_{c(s)}^g \, ds = 2\sqrt{2n} \int_0^t \frac{1}{\cosh(2s)^2} \, ds = \sqrt{2n} \tanh(2t).
\]

It follows that

\[
\delta(c) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \|\nabla_{c'(t)} c'(s)\|_{c(s)}^g \, ds
\]

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\[
\lim_{t \to \infty} \frac{\sqrt{2n} \tanh(2t)}{t} = 0.
\]

Hence the Killing geodesic \( c \) is an asymptotic Fisher geodesic. As the geodesic defect is invariant under isometries of the Fisher metric, this is true for any geodesic with \( c'(0) \) orthogonal to \( \mathcal{X}(\cdot, \mu_0) \).

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**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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