Pseudo-Poincaré: A Unification Framework for Euclidean and Hyperbolic Graph Neural Networks

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Abstract

Hyperbolic neural networks have recently gained significant attention due to their promising results on several graph problems including node classification and link prediction. The primary reason for this success is the effectiveness of the hyperbolic space in capturing the inherent hierarchy of graph datasets. However, they are limited in terms of generalization, scalability, and have inferior performance when it comes to non-hierarchical datasets. In this paper, we take a completely orthogonal perspective for modeling hyperbolic networks. We use Poincaré disk to model the hyperbolic geometry and also treat it as if the disk itself is a tangent space at origin. This enables us to replace non-scalable Möbius gyrovector operations with an Euclidean approximation, and thus simplifying the entire hyperbolic model to a Euclidean model cascaded with a hyperbolic normalization function. Our approach does not adhere to Möbius math, yet it still works in the Riemannian manifold, hence we call it Pseudo-Poincaré framework. We applied our non-linear hyperbolic normalization to the current state-of-the-art homogeneous and multi-relational graph networks and demonstrate significant improvements in performance compared to both Euclidean and hyperbolic counterparts. The primary impact of this work lies in its ability to capture hierarchical features in the Euclidean space, and thus, can replace hyperbolic networks without loss in performance metrics while simultaneously leveraging the power of Euclidean networks such as interpretability and efficient execution of various model components.

1 Introduction

Graph data structures are ubiquitous. There had been several advancements in graph processing techniques across several domains such as citation networks [26], medicine [2], and e-commerce [8]. Recently, non-Euclidean geometries such as hyperbolic spaces have shown significant potential in representation learning and graph prediction [7]. Due to their exponential growth in volume with respect to radius [17], hyperbolic spaces are able to embed hierarchical structures with low distortion, i.e., increase in the number of nodes in a tree with increasing depth is analogous to moving outwards from the origin in a Poincaré ball (since both grow exponentially). This similarity enables the hyperbolic space to project the child nodes exponentially further away from the origin as compared to their parents (as shown in figure [1]a) which captures hierarchy in the embedding space. However, there are certain fundamental questions about hyperbolic models which are yet to be investigated. For example,
Figure 1: (a) Mapping hierarchical data onto a Poincaré ball. Starting with the root as origin (red), the children are continuously embedded further away with exponentially increasing distances. (b) Aggregation of vectors in Euclidean, Poincaré Ball and pseudo-Poincaré space (hyperbolic vectors in Euclidean space). Red points are the aggregation of blue points. In the pseudo-Poincaré paradigm, we reformulate hyperbolic operations to capture hierarchy in the Euclidean space.

- Hyperbolic models operate on completely new manifolds with their own gyrovector space \(^{31}\) (to be differentiated from the Euclidean vector space). Because they operate in different manifold, techniques such as dropout or L1/2 regularization does not guarantee to produce similar behavior.
- There are also some critical limitations in the extensibility of hyperbolic space. While the currently developed hyperbolic operations \(^{11}\) provide fair support for basic sequential neural networks, complex Euclidean architectures such as (multi-headed) attention networks \(^{32}\) require additional operations such as mean and concatenation. Accurate translation of the operations that are trivial in the Euclidean space becomes highly non-trivial when it comes to supporting them on the hyperbolic manifold due to the special properties of hyperbolic models \(^{27}\).
- Hyperbolic operators are computationally very expensive i.e., the number of complex mathematical operations for each primitive operator (see section 3) as well as limited GPU support for functions (such as \(\tanh^{-1}\)) limits their scalability as compared to the Euclidean graph neural networks.
- Lastly, current hyperbolic frameworks also perform well in the presence of inherent hierarchy in graphs (such as taxonomies or ontologies), but they lack their competitive edge when it comes to the non-hierarchical graphs (e.g., Hyperbolic networks perform worse than the Euclidean counterparts in citation networks dataset. The dataset is known to have low hyperbolicity \(^{21}\)).

Traditionally, Hyperbolic models use mixture of hyperbolic geodesics, Möbius math, and tangent space approximation at varying points. Instead of switching between multiple frameworks with potentially incongruent operations, in this work, we propose to use Poincaré disk model as our search space, and apply all approximations on the disk as if the disk is a tangent space derived from the origin. This enables us to replace non-scalable Möbius-math with an Euclidean approximation, and then simplify the whole hyperbolic model as an Euclidean model cascaded with a hyperbolic normalization function. In our approach we replace all Möbius math with Euclidean math, yet we still work in Riemannian manifold (use Riemannian optimizer to determine the direction of the gradient), thus, we call it Pseudo-Poincaré framework (shown in Figure 1(b)). Our approach enables the use of inexpensive operators also leads to scalability on existing hardware infrastructures well-optimized for machine-learning workloads.

Although this approximation can cause a possible distortion in the latent space, all the hyperbolic methods use tangent space approximation for their point aggregation, and we argue that fixing the tangent space at origin does not reduce the performance in general cases (when spatial locality for the aggregated points are not guaranteed). Given that our formulation lies in the Euclidean space, it is unbounded by the closure of hyperbolic space and can explore greater degrees of freedom in gradient descent. Additionally, the freedom also allows our model to explore Euclidean solution space, in congruence with the hyperbolic solution space, to get the best results in both the presence and absence of hierarchy in our datasets. For further evidence, we apply our formulation to construct a new pseudo-Poincaré graph convolution (NGCN) and attention network (NGAT), and a multi-relational graph embedding model (NMuR). We demonstrate that the reformulated models outperform several state-of-the-art Euclidean and non-Euclidean graph networks on standard graph tasks. To summarize, the main contributions of our paper are as follows:

1. We propose a generalized formulation of hyperbolic operations in a pseudo-Poincaré paradigm, that can effectively and efficiently capture the hierarchical features in a Euclidean space, on par with the hyperbolic space.
2. Our paradigm allows for greater degrees of freedom in gradient descent, satisfies the universal approximation theorem, and is also able to explore Euclidean solutions.
3. We demonstrate that Euclidean networks with the proposed hyperbolic normalization (NGCN, NGAT, and NMuR models) outperform the state-of-the-art Euclidean and non-Euclidean baselines across standard benchmarks on several graph problems including node classification, link prediction, and multi-relational embedding.

The rest of the paper is structured as follows: Section 2 discusses the related work in the area. Section 3 delves into the current hyperbolic formulation and section 4 reformulates the hyperbolic layers in the pseudo-Poincaré paradigm. Section 5 demonstrates our experimental results and insights into the performance of our paradigm. Section 6 concludes the paper.

2 Related Work

**Euclidean Networks.** Research into graph representation learning can be broadly classified into two categories based on the nature of the underlying graph dataset: (i) Homogeneous graphs - where the edges connect the nodes but do not have any underlying information and (ii) Heterogeneous or multi-relational graphs - where the edges contain information regarding the type of connection or a complex relation definition. Early research in this area relied on capturing information from the adjacency matrix through factorization and random walks. In matrix factorization based approaches [6], the adjacency matrix $A$ is factorized into low-dimensional dense matrices $L$ such that $\|L^T L - A\|$ is minimized. In random walk based approaches [12, 20, 21, 29, 34], the nodes’ information is aggregated through message passing over its neighbors through random or metapath [10] traversal. More recently, the focus has shifted towards neighborhood aggregation through neural networks, specifically, Graph Neural Networks (GNNs) [25]. In this line of approach, several architectures such as GraphSage [15], GCN [16], and GAT [33] have utilized popular deep learning frameworks to aggregate information from the nodes’ neighborhood. Alongside this, we also notice the concurrent development of architectures for multi-relational graphs. Generally focused on representation learning in knowledge graphs, approaches such as TransE [5], DistMult [35], RotatE [28], and MuRE [3] model the tail entities as head entities transformed by a relation embedding.

**Hyperbolic Networks.** While the given graph-specific approaches (in Euclidean space) show significant performance gains over their vanilla counterparts, they are limited to a certain depth of neighborhood and are also incapable of capturing the latent hierarchy in the graph datasets. They also suffer from high distortion [7]. The reason is that the representational space grows linearly while the number of nodes grows exponentially with increasing depth [11] in large graph datasets. This deficiency has led to the growth of hyperbolic networks which are shown to be an effective alternative. First proposed as basic Hyperbolic Neural Networks [11] for multinomial logistic regression, the use of hyperbolic space to capture hierarchy in graph datasets has led to the development of several architectures for representation learning in both homogeneous (HGCN [7], HAT [37]) and heterogeneous (MuRP [3], HypE [8]) graph datasets. These architectures, primarily, utilize the gyrovector space model to formulate the neural network operations in a Poincaré ball of curvature $c$ as Möbius addition $(\oplus_c)$, exponential map $(\exp_c^x)$, logarithmic map $(\log_c^x)$, Möbius scalar multiplication $(\otimes_c)$, and hyperbolic activation $(\sigma_c)$ [11].

While Hyperbolic Networks show promising results, switching back and forth between Möbius gyrovector operations and Euclidean tangent approximations (at various points) makes them computational very expensive, limits their extensibility, and makes them less predictable when applying techniques such as L1/2 regularization, drop-out. Furthermore, such back and forth between the manifolds (at variable points) comes with an implicit assumption that the embeddings have spatial locality in all dimensions for the models to be performant.

3 Hyperbolic Networks

3.1 Hyperbolic Transformation

**Hyperbolic space** is a Riemannian manifold with constant negative curvature. The coordinates in the hyperbolic space can be represented in several isometric models such as Minkowski space $\mathbb{R}^{n}_1$, Beltrami-Klein Model $\mathbb{K}^n$ and Poincaré Ball $\mathbb{B}^n$. In this paper, we consider the Poincaré Ball model because it is the most widely used one for hyperbolic networks. Since most of the raw data (and true outputs) are assumed to be in the Euclidean space, the first step to use hyperbolic models is to have
mapping between the Euclidean and hyperbolic space. For an embedding \( x \in \mathbb{R}^n \), its corresponding hyperbolic embedding in the Poincaré ball of curvature \( c \), \( p \in \mathbb{B}_{c}^{n} \) is calculated with the exponential map as \( p = \exp_{0}^{c}(x) = \frac{\tanh(\sqrt{c}\|x\|)}{\sqrt{c}\|x\|}x \). Conversely, logarithmic map is used to transform a hyperbolic embedding \( p \in \mathbb{B}^{n}_{c} \) to a Euclidean vector \( x \in \mathbb{R}^{n} \) is formulated as \( x = \log_{0}^{c}(p) = \frac{\tanh^{-1}(\sqrt{c}|p|)}{\sqrt{c}|p|}p \).

3.2 Hyperbolic Layers

Hyperbolic Feed-forward layer. For the Euclidean function \( f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m} \), its equivalent Möbius version \( f^{\circ c}: \mathbb{B}^{n}_{c} \rightarrow \mathbb{B}^{m}_{c} \) for the Poincaré ball is defined as \( f^{\circ c}(p) := \exp_{0}^{c}(f(\log_{0}^{c}(p))) \). Extending this, the hyperbolic equivalent \( h^{\mathbb{B}}(p) \) of the Euclidean linear layer \( h^{\mathbb{R}}(x) = Wx \) with weight matrix \( W \in \mathbb{R}^{n \times m} \) is formulated as:
\[
 h^{\mathbb{B}}(p) = W \otimes_{c} p = f^{\circ c}(Wp) = \frac{\tanh(\frac{|Wp|}{\|Wp\|})\tanh^{-1}(\sqrt{c}|p|)}{\sqrt{c}|p|}Wp \tag{1}
\]

Hyperbolic Convolution. For a hyperbolic graph node representation \( p_{0} \in \mathbb{B}^{n}_{c} \) with neighbors \( \{ {p_{i}} \}_{i=1}^{k} \). As given in [7], the hyperbolic convolution layer \( GCN^{\circ c}: \mathbb{B}^{n}_{c} \rightarrow \mathbb{B}^{m}_{c} \) constitutes of a linear transform, followed by neighborhood aggregation\(^1\) which is computed as:
\[
 h^{\mathbb{B}}(p_{i}) = W_{i} \otimes_{c} p_{i} \oplus_{c} b_{i}; \quad h^{\mathbb{B}}_{agg}(p_{0}) = \exp_{p_{0}}^{c}(\Sigma_{i=0}^{k}w_{ij}\log_{p_{0}}^{c}(h^{\mathbb{B}}(p_{i}))) \tag{2}
\]

\[
 GCN^{\circ c}(p_{0}) = \exp_{0}^{c}(\sigma(\log_{0}^{c}(h^{\mathbb{B}}_{agg}(p_{0})))) \tag{3}
\]

where \( w_{ij} = \text{softmax}_{i=1}^{k}(\text{MLP}(\log_{0}^{c}(p_{0}))||\log_{0}^{c}(p_{i}))) \) is an aggregation weight for neighbors.

Multi-relational Hyperbolic Representation. The multi-relational representations for knowledge graph \( KG \) are modeled using a scoring function. For a triplet \((h,r,t) \in KG\), where relation \( x_{r} \in \mathbb{R}^{n} \) connects the head entity \( x_{h} \in \mathbb{R}^{n} \) to the tail entity \( x_{t} \in \mathbb{R}^{n} \) and \( R \in \mathbb{R}^{n \times n} \) is the diagonal relation matrix, the hyperbolic equivalent \( \phi^{\mathbb{B}}(p_{h},p_{r},p_{t}) \) of the Euclidean scoring \( \phi(x_{h},x_{r},x_{t}) = \|Rx_{h} - (x_{t} + x_{r})\|^{2} \) function is computed using the hyperbolic distance \( d^{\mathbb{B}} \) as:
\[
 d^{\mathbb{B}}(p_{1},p_{2}) = \| \exp_{0}^{c}(-p_{1} \oplus_{c} p_{2}) \| = \frac{2}{\sqrt{c}} \tanh^{-1} \left( \sqrt{c}\| - p_{1} \oplus_{c} p_{2} \| \right) \tag{4}
\]

\[
 \phi^{\mathbb{B}}(p_{h},p_{r},p_{t}) = -d^{\mathbb{B}}(\exp_{0}^{c}(R\log_{0}^{c}(p_{h})), p_{t} \oplus_{c} p_{r})^{2} \tag{5}
\]

4 Pseudo-Poincaré Hyperbolic Networks

This section introduces our methodology for reformulating each of the aforementioned hyperbolic layers in Euclidean space. We will base our framework on the arguments unless the spatial locality is guaranteed, approximating on tangent space at a variable point does not necessarily increase the approximation accuracy. Therefore, fixating tangent space to be at the origin can bring both simplicity to the model and improve its performance in general cases. Moreover, only tangent space at origin has one-to-one mapping with the Poincaré (hyperbolic) half-plane model. This enables us to combine the tangent space approximation with Poincaré half-plane model and apply all approximations in the half-plane. This makes the pseudo-Poincaré networks much easier to understand, develop and optimize.

The notion of the curvature becomes more complicated in high-dimensional space since the manifold can curve in many different directions. Therefore, inline with the literature [18], we focus our arguments on a sub-space of the manifold (specifically, 2-D sub-space). Now, assume 2 points in a 2D Riemannian manifold with constant curvature (i.e., a ball or a hyperbolic-plan in the ambient space). There are infinite planes that cross-section the manifold and passes from these two points. The distance between these two points is the magnitude of the shortest curve (which lies in the optimal plane-sections). We further limit our argument to this plane-section (and curve) for the sake of simplicity. The arguments can be generalized to higher dimensions.

**Lemma 1.** For a tangent line from any point in the curve, the midpoint approximation error is relative to the angle between the a-x chord and the tangent line.

\(^{1}\)The aggregation occurs in the tangent space as concatenation is not well-defined for hyperbolic space.
Thus, if the approximation happens on a tangent line of point $p$, where $p$ is between the $x, y$, the approximation error is smaller than the one obtained using tangent lines at any of the points that are being aggregated. Now, on a half-circle, if we fixate $p$ to be the origin ($p_0$), there is a $50\%$ chance that $p$ lies between the aggregated points (assuming the given points’ coordinates have a uniform distribution).

**Pseudo-Poincaré Feed-forward Layer.** We state that cascading hyperbolic feed forward layers (including linear layers) is equivalent to cascading their Euclidean counterparts and then applying a hyperbolic normalization to the result.

**Lemma 2.** For a point in the tangent space at origin $x \in T_{0n} \mathbb{B}^n$, the exponential map to the hyperbolic space $\mathbb{B}^n$, $\exp_0^c : T_{0n} \mathbb{B}^n \rightarrow \mathbb{B}^n \exp_0(x)$, is equivalent to the input scaled by a non-linear scalar function $\omega : \mathbb{R}^n \rightarrow \mathbb{R}^n$, i.e.,

$$p = \exp_0^c(x) = \omega(x)x; \quad \omega(x) = \frac{\tanh(\sqrt{c|x|})}{\sqrt{c|x|}} \quad (6)$$

where $\omega(x)$ is the hyperbolic normalization for exponential maps. This follows from equation (1).

**Lemma 3.** Approximating on tangent space at origin for all stages causes the cascade of $n$ hyperbolic feed-forward layers $F_n^{\otimes c} \equiv \{f_i^{\otimes c}\}_{i=1}^{n}$ to be equivalent to cascading $n$ of their Euclidean counterparts $F_n = \{f_i\}_{i=1}^{n}$ encapsulated in $\log_0F_n$ and $\exp_0f$ functions, i.e.,

$$F_n(x) = f_n(\cdots f_1(x))$$

$$F_n^{\otimes c}(p) = \exp_0^c\left(\log_0\left(\cdots \log_0\left(f_n(\cdots f_1(p))\right)\right)\right) \quad (7)$$

**Proposition** Lemma 3 implies that the hyperbolic transformations $\log_0F_n$ and $\exp_0^cF_n$ are only needed at the initial and final stages to capture hierarchy in Euclidean networks.

**Lemma 4.** For given input features $x = \log_0(p) \in \mathbb{R}^n$, a given hyperbolic feed forward layer can be rewritten as $f^{\otimes c}(x) := \omega(f(x))f(x)$.

From lemmas 3 and 4 we arrive at the following theorem for cascaded hyperbolic layers.

**Theorem 1.** Fixing the tangent space to be always at origin, cascaded hyperbolic feed-forward layers $F_n^{\otimes c}$, for Euclidean input $x \in \mathbb{R}^n$ can be reformulated as $F_n^{\otimes c}(x) := \Omega(F_n(x))F_n(x)$, where $\Omega(F_n(x))$ is the hyperbolic normalization for a cascaded hyperbolic feed-forward network.

**Hyperbolic Normalization.** Theorem 1 allows us to state that cascaded hyperbolic layers can be reformulated in the Euclidean space using a hyperbolic normalization ($\Omega$) function. This approximation is more accurate at the first layers (where the adjacent inputs embeddings does not show strong spatial locality property, and hence allows us to approximate all operations on tangent space at origin accurately). The approximation resembles the technique proposed in [24] to normalize weights scaling function instead of $e^w$. However, both are similar in terms of using the feature vector magnitude as the denominator and hence remove the orthogonality of the dimensions.

The “unrolled deep hyperbolic network” (shown in Figure 2) presents the general idea of stacking multiple hyperbolic layers together. The input into each layer is passed through a logarithmic map to project into an Euclidean space and its output is projected back into the hyperbolic space after applying the computations. Equations (1)-(9) demonstrate how various hyperbolic layers are implemented. Algorithm 1 provides the generalized algorithm for our proposed hyperbolic model.

**Pseudo-Poincaré Graph Convolution.** The individual components of the Hyperbolic Graph Convolution, as defined in Eq. (3), can be reformulated as a cascaded hyperbolic feed forward layers. Thus, we approximate both the aggregation weights and the aggregation function in $T_{0n} \mathbb{B}^n$ as:

$$h_{\text{agg}}^R(p_i) = \exp_0^c\left(\sum_{i=0}^{n} w_{ij} \log_0^c (h^R(p_i))\right) \quad (7)$$

With this approximation, we apply theorem 1 to reformulate the $GCN^{\otimes c} : \mathbb{B}^n \rightarrow \mathbb{B}^m$ layers as Normalized GCN $NGCN^{\otimes c} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ over Euclidean inputs $x_0 \in \mathbb{R}^n$ with neighbors $\{x_j\}_{j=1}^{k}$. This is computed as $NGCN^{\otimes c} := \Omega(GCN(x_0))GCN(x_0)$, where $GCN(x_0)$ is the Euclidean Graph Convolution layer [16].
Algorithm 1: Normalized hyperbolic model. (Figure 2)

Input: Input Euclidean model $F_L$, sample node $x \in \mathbb{R}^n$

Output: Output of Normalized hyperbolic model $F_L^{\otimes c}(x)$

1 $f_0^{\otimes c}(x) = x$;
2 for layer $l : 1 \rightarrow L$ do
3 $x \leftarrow f_{l-1}^{\otimes c}(x)$;
4 $\# \text{Layer } l \text{ of the Euclidean model is } f_l(x)$
5 $\omega(f_l(x)) = \tanh(\sqrt{c}|f_l(x)|)$; using Eq. (6)
6 $f_l^{\otimes c}(x) = \omega(f_l(x)) f_l(x)$; using Lemma 4
7 end
8 $F^{\otimes c}(x) = f_L^{\otimes c}(x)$;
9 return $F_L^{\otimes c}(x)$

Figure 2: The top row labeled "Unrolled Deep Hyperbolic Network" presents the general idea of stacking deep hyperbolic network with $L$ layers. The input to each layer is passed through a logarithmic map to project onto an Euclidean space, and following the Euclidean layer computation, the output is projected back onto the hyperbolic space. Section 3.2 demonstrates how such an operator or function chaining is performed, and it is easy to see how stacking of such deep layers can become expensive. Our methodology shown in the lowest layer completely avoids the repeated application of these subspace transformations.

Note that the cascaded feed-forward functions of HGCN [7] operate on different tangent spaces, i.e., aggregation weights are calculated at the linear tangent space at origin $T_0B_n^c$ whereas the aggregation function as given in Eq. (7) is driven from a linear tangent space at the root node $T_{p_0}B_n^c$. An argument to use different tangent spaces for different function is based on the fact that Euclidean approximation is most accurate when the tangent space is close to the true resulting point. For aggregation, using tangent space at $p_0$ requires an implicit assumption that all aggregated hyperbolic points preserve a form of spatial locality, which implies similarity and not exactly the notion of hierarchy. However, we can still argue that hierarchical structures have a loose form of spatial locality, and hence using $log_{p_0}(p)$ will add some noise to the embedding compared to when we use $log_{p_0}(p)$.

Pseudo-Poincaré Graph Attention Layer. Here, our goal is to project the Euclidean GAT layer into the hyperbolic space by merely applying the hyperbolic normalization (e.g., Eq. (6)), then, study its approximation in relation to the existing hyperbolic graph attention layers as well as with a true hyperbolic graph attention mechanism.

Hyperbolic attention is formulated as a two-step process (as defined in [13]); (1) matching, which computes attention weights, and (2) the aggregation, which takes a weighted average of the values (attention coefficients as weights). This way, the attention layer can be seen as two cascaded layers, in which the first calculates the attention weights, while the second layer aggregates the weighted vectors. Existing hyperbolic graph attention method [32] $GAT^{\otimes c} : \mathbb{R}_n \rightarrow \mathbb{R}_m$ views the hyperbolic aggregation as a hyperbolic feed forward layer with a Euclidean weighted average as its function. Similarly $GAT : \mathbb{R}_n \rightarrow \mathbb{R}_m$ [33] also uses weighted-average for the aggregation.

Since we are using Euclidean weighted average over cascaded feed-forward layers, we can apply theorem [10] to hyperbolic GAT and approximate it in the hyperbolic space with a hyperbolic normalization function and Euclidean GAT as $NGAT^{\otimes c}(x) := \Omega(GAT(x))GAT(x)$.

Pseudo-Poincaré Multi-relational Representation. For Euclidean inputs $x_1, x_2 \in \mathbb{R}^n$ and L1-norm $d^\mathbb{E}(x_1, x_2)$, the hyperbolic formulation of the multi-relational hyperbolic distance $d^B(x, y)$,
Optimization. The scaling function in Eq. (6) uses the vector norm in the denominator and hence, each dimension depends on the value of other dimensions (i.e., non-orthogonal). Due to this, optimizers that are developed based on the orthogonality assumptions of the dimensions such as ADAM are not expected to produce the most optimal weights. Thus, we choose Riemannian optimizers such as Riemannian ADAM [4] for our model training. For traditional hyperbolic models, regularization techniques such as L2-regularization do not work as expected since the weights are in hyperbolic space. However, the trainable parameters in our formulation are in the Euclidean space. Hence, we apply regularization techniques that are applied to Euclidean networks.

5 Experimental Setup

In this section, we aim to answer following research questions (RQs) through our experiments:

- **RQ1:** How do the pseudo-Poincaré variants, NGAT and NGCN, compare to their Euclidean and hyperbolic counterparts in terms of performance on node classification and link prediction?
- **RQ2:** How does the pseudo-Poincaré variant, NMuR, compare to its Euclidean and hyperbolic counterpart in terms of performance on reasoning over knowledge graphs?
- **RQ3:** How do pseudo-Poincaré variants react to the choice of optimizers, and what are the effects on their execution time? How does the hyperbolic normalization compare to its counterparts?
- **RQ4:** What is the difference in the representations obtained through hyperbolic projection and hyperbolic normalization?

5.1 Problem Setting

For comparison of hyperbolic normalization with other methods, we conduct our experiments on the following tasks: (i) graph prediction (node classification and link prediction) and (ii) reasoning over knowledge graphs, which are described below:

**Graph Prediction.** Given a graph \( G = (V \times E) \), where \( v_i \in \mathbb{R}^n \), \( \{v_i\}_{i=1}^{|V|} \in V \) is the set of all nodes and \( E \in \{0, 1\}^{|V| \times |V|} \) is the boolean adjacency matrix, where \( |V| \) is the number of nodes in the graph and \( e_{ij} = 1 \), if link exists between node \( i \) and \( j \), and \( e_{ij} = 0 \), otherwise. Each node \( v_i \in V \) is also labeled with a class \( y_i \). In the task of **node classification**, the primary aim is to estimate a predictor model \( P_\theta \) with parameters \( \theta \) such that for an input node \( v_i \): \( P_\theta(v_i|E) = y_i \). In the task of **link prediction**, the primary aim is to estimate a predictor model \( P_\theta \) with parameters \( \theta \) such that for an input node-pair \( v_i, v_j \) and an incomplete adjacency matrix \( \hat{E} \): \( P_\theta(v_i, v_j|\hat{E}) = e_{ij} \), where \( e_{ij} \in E \) is an element in the complete adjacency matrix.

**Reasoning over Knowledge Graphs.** Let us say that a set of triplets constitutes a knowledge graph \( KG = \{(h_i, r_i, t_i)\}_{i=1}^{K \times G} \), where \( h_i \in \mathbb{R}^n \) and \( t_i \in \mathbb{R}^n \) are the head and tail entity, respectively connected by a relation \( r_i \in \mathbb{R}^n \). In the task of **reasoning** [3], our goal is to learn representations for the entities and relations using a scoring function that minimizes the distance between heads and tails using the relation as a transformation. The scoring function for Euclidean \( MURR \), hyperbolic \( MuRP \) and pseudo-Poincaré \( NMuR \) are presented in Eq. (8), Eq. (9) and Eq. (9), respectively.

5.2 Datasets and Baselines

**Datasets.** For comparing our model with the baselines, we chose the following standard benchmark datasets: (i) **Cora** [23] contains 2708 publications with paper and author information connected by citation links and classified into 7 classes based on their research areas, (ii) **Pubmed** [26] contains medical publications pertaining to diabetes labeled into 3 classes. The dataset is collected from the Pubmed database, (iii) **Citeseer** [26] contains 3312 annotated scientific publications connected by
citation links and classified into 6 classes of research areas, (iv) **WN18RR** [9] is a subset of the hierarchical WordNet relational graph that connects words with different types of semantic relations. WN18RR consists of 40,943 entities connected by 11 different semantic relations and (v) **FB15k-237** [5] [30] contains triple relations of the knowledge graph and textual mentions from the Freebase entity pairs, with all simple invertible relations are removed for better comparison. The dataset contains 14,541 entities connected by 237 relations. (vi) **Protein-Protein Interaction (PPI)** network dataset [14] contains 12M+ triples with 15 relations (preserving the original ratio of the relations in the dataset, we randomly sampled the dataset, and trained on the sampled subset).

We use Cora, Pubmed, and Citeseer for our experiments on homogeneous graph prediction, i.e., node classification and link prediction. For comparing the multi-relational networks on the task of reasoning, we utilize the FB15k-237 and WN18RR datasets. For a fair comparison of evaluation metrics, we use the same training, validation, and test splits as used in the previous methods, i.e., the splits for Cora, Pubmed, and Citeseer are as given in [7] and for FB15k-237 and WN18RR, the splits are in accordance with [8].

**Baselines.** For comparing the performance of our Pseudo-Poincaré models, we utilize the Euclidean and hyperbolic variants of the architecture as our baselines: (i) **Graph Convolution (GCN)** [16] aggregates the message of a node’s k-hop neighborhood using a convolution filter to compute the neighbor’s significance to the root, (ii) **Graph Attention (GAT)** [33] aggregates the messages from the neighborhood using learnable attention weights, (iii) **Hyperbolic Graph Convolution (HGCN)** [7] is a hyperbolic formulation of the GCN network that utilizes the hyperbolic space and consequently, Möbius operations to aggregate hierarchical features from the neighborhood, (iv) **Hyperbolic Graph Attention (HGAT)** [56] is a hyperbolic formulation of the GAT networks to capture hierarchical features in the attention weights, (v) **Multi-relational Embeddings (MuRE)** [3] transforms the head entity by a relation matrix and then learns representation by minimizing L1-norm between head and tail entities translated by the relation embedding and (vi) **Multi-relational Poincaré (MuRP)** [3] is a hyperbolic equivalent of MuRE that transforms the head entity by a relation matrix in the Poincaré ball and then learns representation by minimizing the hyperbolic distance between the head and tail entities translated by the relation embedding.

**Implementation Details.** We run our experiments on an Nvidia T4 GPU with 16 GB of VRAM. Our models are implemented on the Pytorch framework [22] and the baselines are adopted from [7] for GAT, GCN, and HGCN, and from [3] for MuRE and MuRP. Due to the unavailability of a public implementation, we use our own implementation of HGAT based on [56]. The implementations have been thoroughly tuned with different hyper-parameter settings for the best performance.

For GAT baseline, we used 64 dimensions with 4 and 8 attention heads (we found 4 heads perform better and hence we only present that result in our tables), with dropout rate of 0.6 and L2 regularization of $\lambda = 0.0005$ (for Pubmed and Cora), and $\lambda = 0.001$ (for Citeseer) to be inline with the hyper-parameters reported in [33]. We used same number of dimensions and hyper-parameters for our NGAT model. For GCN baseline, [33] used 64 dimensions, we used the same number of dimensions and hyperparameter setting as GAT baseline. It should be noted that, we observed poor performance when we applied regularization techniques (that GAT baseline used) to the hyperbolic models. For HGCN, the original paper used 16 hidden dimension, however, we tried 64, 32, and 16 for the hidden dimensions, we found 64 dimensions produces a better results. For the curvature choice, we used different curvatures for different tasks. For the task of node classification, we used $c = 0.3$ (for Cora), and $c = 1$ (for Pubmed and Citeseer) as they produced the best results, while for the link prediction task, $c = 1.5$ produced the best results. at the end, we empirically found that scaling the output of the hyperbolic normalization layer by the factor of 5 produces the best results when $c = 0.3 - 0.7$ (and for $c = 1.5$, we set the scaling factor to 3).

### 5.3 RQ1: Performance on Graph Prediction

To evaluate the performance of hyperbolic normalization with respect to Euclidean and hyperbolic alternatives, we compare the **NGCN** and **NGAT** models against the baselines on the tasks of node classification and link prediction. We utilize the standard evaluation metrics of accuracy and ROC-AUC for the tasks of node classification and link prediction, respectively. The results for our experiments are provided in Table[1]

[1] https://github.com/oom-debugger/pseudo-poincare
Table 1: Performance comparison results between hyperbolic normalization (ours) and the baseline methods on the graph prediction tasks of node classification and link prediction. The columns present the evaluation metrics, which are Accuracy (for node classification) and Area under ROC (ROC) (for link prediction), along with their corresponding 95% confidence intervals. The cells with the best performance are highlighted in bold and the second-best performance is marked with a box.

| Repr. Space | Datasets | Node Classification (Accuracy in %) | Link Prediction (ROC in %) |
|-------------|----------|------------------------------------|---------------------------|
|             |          | CORA | Pubmed | Citeseer | CORA | Pubmed | Citeseer |
| Euclidean   | GCN      | 80.1±0.3 | 78.5±0.3 | 71.4±0.3 | 90.2±0.2 | 92.6±0.2 | 91.3±0.2 |
|             | GAT      | 82.7±0.2 | 79.0±0.3 | 71.6±0.3 | 89.6±0.2 | 92.4±0.2 | 93.6±0.2 |
| Hyperbolic  | HGCN     | 77.9±0.3 | 78.9±0.3 | 69.6±0.3 | 91.4±0.2 | 95.0±0.1 | 92.8±0.3 |
|             | HGAT     | 79.6±0.3 | 79.2±0.3 | 68.1±0.3 | 90.8±0.2 | 93.9±0.2 | 92.2±0.2 |
| Pseudo-     | NGCN     | 82.4±0.2 | 78.8±0.3 | 71.9±0.3 | 91.3±0.2 | 94.7±0.1 | 92.8±0.2 |
| Poincaré    | NGAT     | 83.1±0.2 | 79.1±0.3 | 73.8±0.3 | 90.5±0.2 | 93.9±0.2 | 92.8±0.2 |

Table 2: Performance comparison results between Pseudo-Poincaré (ours) and the baseline methods on the reasoning task of multi-relational graph reasoning. The columns present the evaluation metrics of Hits@K (H@K) (%) and Mean Reciprocal Rank (MRR) (%) along with their corresponding 95% confidence intervals. The best results are highlighted in bold.

| Dim Models | Datasets | WN18RR | FB15K-237 | PPI |
|------------|----------|--------|-----------|-----|
|            |          | H@10   | H@3      | H@10 | H@3  |
| 40         | MuRE     | 40.9±0.3 | 49.7±0.3 | 44.4±0.3 | 29.0±0.3 | 46.2±0.3 | 31.9±0.3 | 2.8±0.3 | 9.2±0.3 | 4.1±0.3 |
|            | MuRP     | 42.5±0.3 | 52.2±0.3 | 45.5±0.3 | 29.8±0.3 | 47.4±0.3 | 32.8±0.3 | 5.5±0.3 | 10.2±0.3 | 4.4±0.3 |
|            | NMuR     | 43.6±0.3 | 57.5±0.3 | 47.4±0.3 | 31.5±0.3 | 49.7±0.3 | 34.7±0.3 | 8.2±0.3 | 14.3±0.3 | 11.7±0.3 |
| 200        | MuRE     | 44.2±0.3 | 51.0±0.3 | 48.5±0.3 | 31.4±0.3 | 49.5±0.3 | 34.8±0.3 | 10.6±0.3 | 14.5±0.3 | 11.9±0.3 |
|            | MuRP     | 44.6±0.3 | 52.4±0.3 | 46.2±0.3 | 31.5±0.3 | 49.8±0.3 | 34.8±0.3 | 8.2±0.3 | 15.0±0.3 | 9.9±0.3 |
|            | NMuR     | 44.7±0.3 | 57.9±0.3 | 48.1±0.3 | 32.2±0.3 | 50.6±0.3 | 35.5±0.3 | 12.6±0.3 | 15.0±0.3 | 13.8±0.3 |

From the results, in all cases (except link-prediction on Citeseer), pseudo-Poincaré variants outperform the Euclidean model. Pseudo-Poincaré variants perform better than their hyperbolic counterparts in node classification, but marginally lower in link-prediction. This is inline with our expectations as embeddings of adjacent points in link-prediction on homogeneous graphs tend to have spatial locality. However, spatial locality is less likely for node-classification (as far away nodes still can belong to the same class). It is also worth noting that Pseudo-Poincaré outperforms Euclidean in all node-classification datasets regardless of the hyperbolicity of the dataset (e.g. Cora has low hyperbolicity [7] which were previously suggested as a reason that inferior Hyperbolic networks over Euclidean counterparts). This observation suggests that Pseudo-Poincaré is more general purpose framework (i.e. it is performant in wider variety of tasks/datasets as compared to pure-Hyperbolic model). It should be noted that pseudo-Poincaré variants are consistently at least the second-best performing methods, closely following the top performer which makes it very appealing given the low-complexity, and speedup of the model (see Table 3b for execution time).

5.4 RQ2: Multi-Relational Reasoning

To compare the performance of hyperbolic normalization against the baselines for multi-relational graphs, we compare the NMuR model against MuRE and MuRP on the task of reasoning. We consider the standard evaluation metrics of Hits@K and Mean Reciprocal Rank (MRR) to compare our methods on the reasoning task. Let us say that the set of results for a head-relation query \((h, r)\) is denoted by \(R_{(h, r)}\), then the metrics are computed by \(MRR = \frac{1}{k} \sum_{i=1}^{n} \frac{1}{rank(i)}\) and \(Hits@K = \frac{1}{K} \sum_{k=1}^{K} e_k\), where \(e_k = 1\), if \(e_k \in R_{(h, r)}\) and 0, otherwise. Here \(rank(i)\) is the rank of the \(i^{\text{th}}\) retrieved sample in the ground truth. To study the effect of dimensionality, we evaluate the models with two values of the embedding dimensions: \(n = 40\) and \(n = 200\). The results from our comparison are given in Table 2.

From the results, note that NMuR outperforms the hyperbolic model MuRP, on an average, by \(\approx 5\%\) with 40 dimensions and \(\approx 3\%\) with 200 dimensions across various metrics and datasets. This shows that NMuR is able to effectively learn better representations at lower number of dimensions. Also, note that the difference in performance by increasing the number of dimensions is \(< 1\%\) for most cases, implying that NMuR is already able to capture the hierarchical space of the knowledge graphs at lower dimensions. This should limit the memory needed for reasoning without a significant loss in performance.
Table 3: (a) Comparison of normalization methods (Layer-Norm, Hyperbolic Norm, and Constant magnitude). The columns present results on Cora (CR), Pubmed (PM), and Citeseer (CS) datasets using GAT and GCN models. (b) The training epoch times (in seconds - lower is better) for Euclidean, hyperbolic, and Pseudo-Poincaré models. Note that the provided results are without any hyper-parameter tuning, thus the comparison is only valid column-wise. Table 1 provides the results for cross model comparison.

(a) Comparison of normalization methods.

| Base Model | CR    | PM    | CS    |
|------------|-------|-------|-------|
| GCN | Constant | 67.9 | 76.7 | 68.3 |
|       | Layer-Norm | 78.3 | 75.0 | 63.6 |
|       | Hyp-Norm | 80.9 | 78.0 | 72.1 |
| GAT | Constant | 75.3 | 73.0 | 61.2 |
|       | Layer-Norm | 75.6 | 74.9 | 65.3 |
|       | Hyp-Norm | 76.7 | 76.3 | 68.6 |

(b) Comparison of execution times.

| Model | CR    | PM    | CS    |
|-------|-------|-------|-------|
| Convolution | GCN | 0.07 | 0.13 | 0.04 |
|       | HGCN | 0.29 | 0.33 | 0.21 |
|       | NGCN | 0.08 | 0.14 | 0.06 |
| Attention | GAT | 0.16 | 0.38 | 0.09 |
|         | HGAT | 1.10 | 1.34 | 0.81 |
|         | NGAT | 0.19 | 0.37 | 0.12 |

Table 4: Performance of NGAT, GAT, and HGAT on node classification with RAdam and Adam optimizers.

| Models | Optimizers | Cora | Pubmed | Citeseer |
|--------|------------|------|--------|----------|
| GAT    | Adam       | 82.7±0.2 | 78.7±0.2 | 71.6±0.3 |
|        | RAdam      | 78.4±0.3 | 79.0±0.3 | 71.1±0.3 |
| HGAT   | Adam       | 76.8±0.3 | 76.0±0.3 | 68.1±0.3 |
|        | RAdam      | 79.6±0.3 | 78.9±0.3 | 68.1±0.3 |
| NGAT   | Adam       | 78.8±0.3 | 77.7±0.3 | 69.7±0.3 |
|        | RAdam      | 83.9±0.2 | 79.1±0.3 | 73.8±0.3 |

5.5 RQ3: Model Analysis

Hyperbolic Normalization vs. Layer Normalization. In this experiment, we study the difference between our hyperbolic normalization compared to other existing normalization approaches. For this, we select Layer-norm [19] as it is the closest in terms of input/output. Also, in order to study the effect of hyperbolic scaling in our normalization layer, we added constant-norm which is computed as follows: \( \text{norm}(x) = x/\|x\| \). In order to compare the effect of normalization layers, we did not use any extra hyper-parameter tuning and regularization. We set the embedding size to 64 for both GAT and GCN, and used 4 heads for GAT-models. Table 3a shows the comparison of different normalization methods on Cora, Pubmed, and Citeseer datasets, and confirms the outperformance of hyperbolic normalization. Since Layer-norm keeps the GAT model in the Euclidean space, we used ADAM optimizer. However, since both hyperbolic normalization and constant normalization have \( \|x\| \) as their denominator, we used RiemannianAdam (see Section 8).

Execution Time Comparison. In this analysis, we demonstrate the impact of hyperbolic normalization in improving the scalability of hyperbolic networks. We study the computational cost to train our model versus the hyperbolic and Euclidean counterparts. We ran all the experiments on Quadro RTX 8000 with 48GB memory, and reported the mean of \( \text{epoch/second} \) for each training experiment for link prediction task in Table 3b. Our model NGCN is 3 times faster than the HGCN on average over the entire training, while NGAT is 5 times faster than its own hyperbolic counterpart (HGAT). NGCN and NGAT are 25% and 12% slower compared to their Euclidean counterparts, respectively. We observed the smallest speedup gap when training Pubmed which is due to the very large dataset memory footprint. Even in this case, our NGCN model is still more than 2 times faster than HGCN and NGAT is more than 3.5 times faster than HGAT.

Optimizer Choice. In this study, we experimentally validate our observations regarding the choice of optimizers, as discussed in Section 8. Table 4 shows that NGAT works better when RiemannianAdam is used as an optimizer. This shows that, although the parameters are in the Euclidean space, their gradient updates occur in accordance with hyperbolic gradients. Therefore, NGAT leverages the architecture of GAT, but behaves more closely to HGAT when it comes to parameter optimization.

5.6 RQ4: Embedding Visualizations

The goal of this section is to visually confirm that our approach of learning in Euclidean space, indeed preserves the hierarchical characteristics of hyperbolic space. Towards this goal, we present the visualization of node embeddings from the Cora citation graph. We choose Cora, since it is a citation-based graph and does not contain explicit hierarchical (is-a) relations. We contrast the embeddings learned by GAT (a method based on Euclidean space), with HGAT (learning in hyperbolic space)
Figure 3: Embedding visualizations of the Cora citation graph for Euclidean (GAT), hyperbolic (HGAT), and our proposed Pseudo-Poincaré (NGAT) methods. NGAT achieves better separation between the node classes than the corresponding Euclidean and hyperbolic counterparts.

with the proposed NGAT approach. Figure 3 shows that NGAT ensures clearer separation between node classes in comparison to its Euclidean and hyperbolic counterparts.

6 Conclusion

In this paper, we showed that an Euclidean model can be converted to behave as a hyperbolic model by simply (1) connecting it to the hyperbolic normalization layer and (2) using a Riemannian optimizer. From the training perspective, this approach allows the trainable parameters to interact in the Euclidean vector space, consequently enabling the use of Euclidean regularization techniques for the model training. Furthermore, the training of our proposed approach is significantly faster compared to the existing hyperbolic methods. This allows the hyperbolic networks to be more flexible in application to a wider variety of graph problems. From task/application perspective, we removed the implicit assumption of having spatial locality of the adjacent nodes (in all dimensions), to be prerequisite of the hyperbolic model training. This allows the framework to be applied in a wider variety of tasks.

7 Broader Impact

Pseudo-Poincaré framework is the first of its kind approach that reformulates the hyperbolic network into a Euclidean network with an added normalization function for effectively capturing hierarchy of datasets. The method is generalizable to all hyperbolic networks and provides empirical evidence that any Euclidean deep learning model can be converted to its hyperbolic equivalent by using the normalization function. This extends its application to several domains including natural language processing, computer vision, and knowledge graphs. However, our framework is also sensitive to the nature of the dataset and it is necessary to understand the hyperbolicity of the underlying data and hierarchical nature of the problem before applying our framework as the solution. Additionally, the integrity of the training dataset needs to be ensured as attacks would propagate over the entire network and cause errors in the downstream applications.

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APPENDIX

A Proofs to Theorems and Lemmas

In this section, we provide the theoretical proofs to the lemmas and theorems used in our paper.

A.1 Lemma 1

For a tangent line from any point in the curve, the midpoint approximation error is relative to the angle between the x-y chord and the tangent line.

• We first state that the ideal tangent line is the line that is derived from the mid-point. **Explanation.** Note that curve with a constant curvature is a part of a circle, hence, the diameter from the midpoint (m) is the bisector of an xy arc. Based on the diameter-chord theorem, it is also the bisector of the isosceles triangle formed by xy chord and half-circle radius (xoy in 4.a). This means m-diameter is perpendicular to both ideal tangent line as well as the xy chords (xy chord is parallel to the ideal tangent line). This implies that one can calculate the mid-point of the x-y line (m in 4.a) and project it to the curve (perpendicular projection) and get the ideal midpoint. Now on, we use xy chord instead of the ideal tangent line.

• Then we try to find the mid-point approximation behavior with respect to other projections. Let’s start with the mid-point approximation of x,y on the curve on the tangent line derived from y (Figure 4 (b)). Let us refer to the projection of x on the Ty (tangent line) as xy. Also let us project the m on the Ty and call it mp. x,x,y,y and m,m,y,y are orthogonal triangles and m is mid-point of the hypotenuse xy. Based on the mid-segment theorem, we can infer that m,y,y is a mid-point of y,x,y, and hence its perpendicular projection on the curve gives us the approximate mid-point on the curve. Also, based on the properties of the orthogonal triangles, we can infer the ideal projection line and the approximated projected line form an angle \( \alpha \) equal to the angle between ideal tangent line and the tangent line from y.

• Note that, so far, the argument was scoped to the tangent line from one of the aggregated points (y in our case). However, we can easily generalize it for approximation on any tangent line, by drawing its parallel line that crosses the point y and apply the same trigonometry on the approximated point (Figure 4(c)).

Figure 4: (a) Calculating midpoint using the ideal tangent line. (b) Calculating the midpoint using tangent line that is passing from point y. \( \alpha \) is the angle between the tangent line (from point y) and the ideal tangent line. (c) Calculating the midpoint that uses a tangent line at arbitrary point p.
A.2 Lemma 2

Given \( x \in \mathbb{R}^n \) is a Euclidean equivalent of Poincaré ball’s point \( p \in \mathbb{B}_c^n \) in the tangent space at origin \( T_0 \mathbb{B}_c^n \). The relation between \( x \) and \( p \), as defined by Ganea et al. [11] as;

\[
p = \exp_0^c(x) = \tanh(\sqrt{c}||x||) \frac{x}{\sqrt{c}||x||}; \quad x = \log_0^c(p) = \tanh^{-1}(\sqrt{c}||p||) \frac{p}{\sqrt{c}||p||};
\]

(10)

Simplifying with, \( \omega(x) = \frac{\tanh(\sqrt{c}||x||)}{\sqrt{c}||x||} \) and \( \omega_\mathbb{B}^c(p) = \frac{\tanh^{-1}(\sqrt{c}||p||)}{\sqrt{c}||p||} \);

\[
p = \omega(x)x \text{ and } x = \omega_\mathbb{B}^c(p)p
\]

(11)

A.3 Proof of Lemma 3 and Theorem 1

Let us first look at the case of \( n = 1 \), i.e., a single hyperbolic feed-forward layer. This is defined by Ganea et al. [11] as;

\[
F_1^{\otimes c}(p) = f_1^{\otimes c}(p) = \exp_0^c(f_1(\log_0^c(p)))
\]

(12)

Reformulating this with Lemma 1,

\[
F_1^{\otimes c}(p) = \omega(f_1(\log_0^c(p)))f_1(\log_0^c(p));
\]

(13)

Substituting, \( x = \log_0^c(p) \);

\[
F_1^{\otimes c}(p) = \omega(f_1(x))f_1(x);
\]

(14)

Extending this to \( n = 2 \),

\[
F_2^{\otimes c}(p) = f_2^{\otimes c}(f_1^{\otimes c}(x)) = f_2^{\otimes c}(\omega(f_1(x)))f_1(x)
\]

Given that \( f_1(x) \) is a linear function; \( f(ax) = af(x) \);

\[
F_2^{\otimes c}(p) = \omega(f_1(x))f_2^{\otimes c}(f_1(x));
\]

\[
F_2^{\otimes c}(p) = \omega(f_1(x))\omega(f_2(x))f_2(f_1(x));
\]

\[
F_2^{\otimes c}(p) = \omega(f_2(x))\omega(f_1(x))F_2(x);
\]

Substituting; \( \Omega(F_2(x)) = \omega(f_1(x))\omega(f_2(x)) \);

\[
F_2^{\otimes c}(p) = \Omega(F_2(x))F_2(x);
\]

(15)

Extending the above formulation to arbitrary \( n \),

\[
F_n^{\otimes c}(p) = \omega(f_n(x))...\omega(f_2(x))\omega(f_1(x))F_n(x);
\]

Reformulating this with Lemma 1, we get the conclusion of Lemma 2;

\[
F_n^{\otimes c}(p) = \exp_0^c(F_n(\log_0^c(p)));
\]

(16)

Substituting; \( \Omega(F_n(x)) = \omega(f_n(x))...\omega(f_2(x))\omega(f_1(x)) \);

We get the conclusion of Theorem 1;

\[
F_n^{\otimes c}(p) = \Omega(F_n(x))F_n(x);
\]