Yang-Mills Theory for Noncommutative Flows

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Abstract

The moduli spaces of Yang-Mills connections on finitely generated projective modules associated with noncommutative flows are studied. It is actually shown that they are homeomorphic to those on dual modules associated with dual noncommutative flows. Moreover the method is also applicable to the case of noncommutative multi-flows.
§1. Introduction Among miscellaneous topics in super string theory or M-theory, one of their most important problems is concerned with the compactification of fields, which means that either 10 or 11 dimensional field theory would be reduced to 4 dimensional one by compactifying either 6 or 7 dimensional space time respectively. For instance, an 11 dimensional M-theory has a circle compactification to deduce a IIA-type super string theory, which describes a nonchiral field theory of closed strings due to BFSS ([4]). Moreover, this theory has also one more circle compactification to deduce a IIB-type superstring theory, which describes a chiral field theory of closed strings via the so-called T-transformations ([5]). Recently, Connes, Douglas and Schwarz have shown that the field theory to such a 2-torus compactification cited above has a complete solution by taking the moduli spaces of Yang-Mills connections of appropriate modules for the gauge action of the 2-torus on either commutative or noncommutative 2-torus ([2]). Actually, Connes and Rieffel have proved that the latter Yang-Mills moduli space is homeomorphic to the 2-torus ([3]). From this point of view, the problem of finding the Yang-Mills moduli space for a given smooth noncommutative dynamical system is a quite important one to determine the unified 4 dimensional field theory having the unique compactification. In this paper, we present a certain duality of Yang-Mills moduli spaces for noncommutative flows. More precisely, we show that the Yang-Mills moduli spaces for smooth noncommutative flows are homeomorphic to those for associated dual flows. This could be interpreted as no physical data is changed under dimension reduction of space time. The method itself is also applicable to noncommutative multi flows in principle. As a corollary, some basic examples are computed.

§2. Noncommutative Yang-Mills Theory In this section, we review the noncommutative Yang-Mills theory due to Connes-Rieffel[3]. Let $(A, G, \alpha)$ be a $C^\ast$-dynamical system, $A^\infty$ the set of all smooth elements of $A$ under $\alpha$ and $\alpha^\infty$ the restriction of $\alpha$ to $A^\infty$ where $G$ is a connected Lie group. Then the system $(A^\infty, G, \alpha^\infty)$ becomes a noncommutative smooth dynamical system. In what follows, we only treat such a dynamical system, so that we notationally write it by $(A, G, \alpha)$. Let $\delta$ be the differentiation map of $\alpha$. Then it is a Lie homomorphism from the Lie algebra $\mathcal{G}$ of $G$ to the Lie algebra $\text{Der}(A)$ of all $^\ast$-derivations of $A$. Let $\Xi$ be a finitely generated projective right $A$-module. Then it has a Hermitian structure...
$\langle \cdot \mid \cdot \rangle_A$ with the property that

$$<\xi \mid \eta >_A^* = <\eta \mid \xi >_A, \quad <\xi \mid \eta a >_A = <\xi \mid \eta >_A a$$

($\xi, \eta \in \Xi, a \in A$). Now we can define a noncommutative version of connections on vector bundles over manifolds in the following fashion: Let $\nabla$ be a linear map from $\Xi$ to $\Xi \otimes G^*$. Then it is called a connection of $\Xi$ if it satisfies

$$\nabla_X(\xi a) = \nabla_X(\xi) a + \xi \delta_X(a)$$

($\xi \in \Xi, a \in A, X \in G$). Moreover, a connection $\nabla$ is said to be compatible with respect to $\langle \cdot \mid \cdot \rangle_A$ (or compatible) if it satisfies

$$\delta_X(<\xi \mid \eta >) = <\nabla_X(\xi) \mid \eta > + <\xi \mid \nabla_X(\eta) >$$

($\xi \in \Xi, a \in A, X \in G$). We denote by $CC(\Xi)$ the set of all compatible connections of $\Xi$. Then it is nonempty because it contains the so-called Grassmann connection $\nabla^0$, which is defined as follows: By assumption, $\Xi = P(A^n)$ for some $n \geq 1$ and a projection $P \in M_n(A)$, so that $\nabla^0 = P[\delta^n]$ becomes a compatible connection of $\Xi$, where $\delta^n$ is the differentiation map of the action $\alpha^n = \alpha \otimes id_n$ on $M_n(A)$. Now for any $\nabla \in CC(\Xi)$, there exists an element $\Omega_X \in E = \text{End}_A(\Xi)$ such that

$$\nabla_X = \nabla^0_X + \Omega_X$$

($X \in G$), where $\text{End}_A(\Xi)$ is the set of all $A$-endomorphisms of $\Xi$. Since $\nabla$ and $\nabla^0$ are compatible, then $\Omega_X$ ($X \in G$) are all skew-adjoint. Given a $\nabla \in CC(\Xi)$, there exists a skew adjoint $E$-valued 2-form $\Theta_{\nabla}$ on $G$ such as

$$\Theta_{\nabla}(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

($X, Y \in G$). It is called the curvature of $\nabla$ associated with $(A, G, \alpha)$. Then $\nabla$ is said to be flat if there exists a 2-form $\omega$ on $G$ such that

$$\Theta_{\nabla}(X,Y) = \omega(X,Y) Id_E$$

($X, Y \in G$). We now assume an existence of a continuous $\alpha$-invariant faithful trace $\tau$ on $A$ as noncommutative version of integrability of manifolds. Then there also exists a continuous faithful trace $\tilde{\tau}$ on $E$ such that

$$\tilde{\tau}(<\xi \mid \eta >_E) = \tau(<\eta \mid \xi >_A)$$

($\xi, \eta \in \Xi$), where

$$<\xi \mid \eta >_E(\zeta) = \xi <\eta \mid \zeta >_A$$
(ξ, η, ζ ∈ Ξ). In fact, it is well defined because of the assumption of Ξ. Using ˜τ, we define a noncommutative version of the Yang-Mills functional on manifolds as follows:

$$YM(∇) = - ˜τ(\{Θ∇\}^2)$$

where

$$\{Θ∇\}^2 = \sum_{i<j} Θ∇(X_i ∧ X_j)^2 ∈ E$$

for an orthonormal basis \{X_i\}_i of G with respect to the Killing form. Since Θ∇(X_i ∧ X_j) are all skew adjoint, then \{Θ∇\}^2 has negative spectra only. Therefore YM(∇) ≥ 0 for all ∇ ∈ CC(Ξ). Moreover, it is independent of the choice of a hermitian structure < · | · >_A on Ξ. Now let U(E) be the set of all unitaries of E. It is called the gauge group of Ξ. For any u ∈ U(E), we define the gauge transformation γ_u on CC(Ξ) by

$$(γ_u(∇))_X(ξ) = u(∇_X)(u^∗ξ)$$

(u ∈ U(E), X ∈ G, ξ ∈ Ξ). Then γ calls the gauge action of U(E) on CC(Ξ). The Yang-Mills functional YM is γ-invariant, namely

$$YM(γ_u(∇)) = YM(∇)$$

(u ∈ U(E), ∇ ∈ CC(Ξ)). We then consider the first variational problem of YM, namely find a ∇ ∈ CC(Ξ) such that

$$\frac{d}{dt}(YM(∇_t)) \bigg|_{t=0} = 0$$

for any smooth path ∇_t ∈ CC(Ξ) (|t| < ε) with ∇_0 = ∇, which is called a Yang-Mills connection of Ξ with respect to the system (A, G, α, τ). Let MC(Ξ) be the set of all Yang-Mills connections of Ξ with respect to (A, G, α, τ). Then the orbit space \mathcal{M}^{(A,G,α,τ)}(Ξ) of MC(Ξ) by the gauge action γ of U(E) is called the moduli space of the Yang-Mills connections of Ξ with respect to the system (A, G, α, τ). We then state the following theorem due to Connes-Rieffel[3] which is quite powerful to construct a Yang-Mills connection:

**Theorem 2.1**[3] Let (A, G, α) be a C∞-dynamical system and τ be a faithful α-invariant continuous trace on A. Let Ξ be a finitely generated projective right A-module. If G is an abelian connected Lie group, then ∇ ∈ MC(Ξ) if and only if it is flat for any ∇ ∈ CC(Ξ).
§3. Dual Yang-Mills Moduli spaces

In this section, we only take Frechet flows (or multi-flows) as a special case of $C^\infty$-dynamical systems. According to Elliott-Natsume-Nest[7], let $(A, \mathbb{R}, \alpha)$ be a Frechet *-flow in the sense that

1. $(A, \{\| \cdot \|_n\}_{n \geq 1})$ is a Frechet *-algebra (which is dense in a $C^*$-algebra),
2. $t \mapsto \alpha_t(a)$ is $C^\infty$-class with respect to $\| \cdot \|_n$ $(n \geq 1)$,
3. For any $m, k \geq 1$, there exist $n, j \geq 1$ and $C > 0$ such that
   \[
   \left\| \frac{d^k}{dt^k} \alpha_t(a) \right\|_m \leq C(1 + t^2)^{j/2} \| a \|_n \quad (a \in A, \ t \in \mathbb{R})
   \]

In what follows, we state Frechet *-flows by $F^*$-flows. Typical are the following three examples as $F^*$-flows:

Example 3.1 Let $\mathcal{S}(\mathbb{R})$ be the abelian $F^*$-algebra of all complex valued rapidly decreasing smooth functions on $\mathbb{R}$ and $\lambda$ the shift action of $\mathbb{R}$ on $\mathcal{S}(\mathbb{R})$. Then the triplet $(\mathcal{S}(\mathbb{R}), \mathbb{R}, \lambda)$ is a $F^*$-flow.

Example 3.2 Let $\mathcal{K}^\infty(\mathbb{R})$ be the $F^*$-algebra consisting of all compact operators on $L^2(\mathbb{R})$ with their integral kernels in $\mathcal{S}(\mathbb{R}^2)$, and $Ad(\lambda)$ the adjoint action of $\mathbb{R}$ on $\mathcal{K}^\infty(\mathbb{R})$. Then the triplet $(\mathcal{K}^\infty(\mathbb{R}), \mathbb{R}, Ad(\lambda))$ is a $F^*$-flow.

Example 3.3 Let $\mathbb{R}^2_{\theta}$ be the $F^*$-algebra $\mathcal{S}(\mathbb{R}^2)$ with Moyal product $\star_{\theta}$ ($\theta \in \mathbb{R}$), and $\hat{\theta}$ the dual action of the canonical action $\theta$ on $\mathcal{S}(\mathbb{R})$. Then the triplet $(\mathbb{R}^2_{\theta}, \mathbb{R}, \hat{\theta})$ is a $F^*$-flow.

Now let $(A, \mathbb{R}, \alpha)$ be a $F^*$-flow with a continuous $\alpha$-invariant faithful trace $\tau$, and let $\mathcal{S}(\mathbb{R}, A)$ be the $F^*$-algebra consisting of all $A$-valued rapidly decreasing smooth functions on $\mathbb{R}$ with its seminorms $\| \cdot \|_{m,n}$ given by

\[
\| x \|_{m,n} = \sup_{t \in \mathbb{R}}(1 + t^2)^{m/2} \left\| \frac{d^n}{dt^n} x(t) \right\|_m
\]

($x \in \mathcal{S}(\mathbb{R}, A)$). Moreover it has the following product and involution:

1. $(x \star_{\alpha} y)(t) = \int_{\mathbb{R}} x(s)\alpha_s(y(t - s))ds$, (2) $x^*(t) = \alpha_t(x(-t))^*$

($x, y \in \mathcal{S}(\mathbb{R}, A)$). Then we call $\mathcal{S}(\mathbb{R}, A)$ the $F^*$-crossed product of $A$ by the action $\alpha$ of $\mathbb{R}$, which is written by $A \rtimes_{\alpha} \mathbb{R}$. In fact, the
three examples cited above are isomorphic to \( \mathbb{C} \rtimes \mathbb{R}, \mathcal{S}(\mathbb{R}) \rtimes_\lambda \mathbb{R} \) and \( \mathcal{S}(\mathbb{R}) \rtimes_\theta \mathbb{R} \) respectively. Then we define two actions \( \tilde{\alpha}, \tilde{\alpha} \) of \( \mathbb{R} \) on \( A \rtimes_\alpha \mathbb{R} \) given by

\[
\tilde{\alpha}_s(x)(t) = e^{2\pi ist}(x(t)), \quad \tilde{\alpha}_s(x)(t) = \alpha_s(x(t)) \quad (i = \sqrt{-1})
\]

\((x \in A \rtimes_\alpha \mathbb{R}, s, t \in \mathbb{R})\). The triplets \((A \rtimes_\alpha \mathbb{R}, \mathbb{R}, \tilde{\alpha})\) and \((A \rtimes_\alpha \mathbb{R}, \mathbb{R}, \tilde{\alpha})\) become \( F^* \)-flows. The former is called to be the dual \( F^* \)-flow of \((A, \mathbb{R}, \alpha)\). Then the same duality holds as in the case of \( C^* \)-crossed products in the following:

Theorem 3.4([7]) Given a \( F^* \)-flow \((A, \mathbb{R}, \alpha)\), its double dual \( F^* \)-flow \((A \rtimes_\alpha \mathbb{R} \rtimes_\hat{\alpha} \mathbb{R}, \mathbb{R}, \hat{\alpha})\) is isomorphic to the \( F^* \)-flow \((A \otimes \mathcal{K}^\infty(\mathbb{R}), \mathbb{R}, \alpha \otimes Ad(\lambda))\).

In fact, the equivariant isomorphism \( \Psi^0_\alpha : A \rtimes_\alpha \mathbb{R} \rtimes_\hat{\alpha} \mathbb{R} \leftrightarrow A \otimes \mathcal{K}^\infty(\mathbb{R}) \) is given by

\[
\Psi^0_\alpha(x)(t, s) = \int_{\mathbb{R}} e^{2\pi i rs} \alpha_{-s}(x(t - s, r)) \, dr
\]

\((x \in A \rtimes_\alpha \mathbb{R} \rtimes_\hat{\alpha} \mathbb{R}, t, s \in \mathbb{R})\). Then the inverse isomorphism \((\Psi^0_\alpha)^{-1}\) of \( \Psi^0_\alpha \) is given by

\[
(\Psi^0_\alpha)^{-1}(x)(t, s) = \int_{\mathbb{R}} e^{2\pi i (t-r)s} \alpha_r(x(r, r - t)) \, dr
\]

\((x \in A \otimes \mathcal{K}^\infty(\mathbb{R}), t, s \in \mathbb{R})\). Now let \((A \rtimes_\alpha \mathbb{R}, \mathbb{R}, \hat{\alpha})\) be the dual \( F^* \)-flow of \((A, \mathbb{R}, \alpha)\). If there exists a continuous faithful \( \alpha \)-invariant trace \( \tau \) on \( A \), then so does it for the \( F^* \)-flow \((\hat{A}, \mathbb{R}, \hat{\alpha})\) given by

\[
\hat{\tau}(x) = \tau(x(0)) \quad (x \in \hat{A})
\]

where \( \hat{A} = A \rtimes_\alpha \mathbb{R} \). Then \( \hat{\tau} \) is called the dual trace of \( \tau \). Then we consider the Yang-Mills moduli spaces for such dual systems. Namely, let \( \mathcal{E} \) be a finitely generated projective right \( A \)-module and \( \hat{\mathcal{E}} = \mathcal{S}(\mathbb{R}, \mathcal{E}) \) the set of all \( \mathcal{E} \)-valued rapidly decreasing smooth functions on \( \mathbb{R} \). Then it becomes a finitely generated projective right \( \hat{A} \)-module. Indeed, the action of \( \hat{A} \) on \( \hat{\mathcal{E}} \) is given by

\[
(\xi x)(t) = \int_{\mathbb{R}} \xi(s) \alpha_s(x(t - s)) \, ds
\]

\((\xi \in \hat{\mathcal{E}}, x \in \hat{A})\). On the other hand, the action \( \tilde{\alpha} \) is implemented by an unitary multiplier flow on \( \hat{A} \), namely there exists a strictly continuous unitary flow \( \tilde{u} \) of the multiplier algebra of \( \hat{A} \) such that
\[ \tilde{\alpha}_t = Ad(\tilde{u}_t) \text{ on } \tilde{A}. \] Then it follows that \( \tilde{\tau} \) is \( \tilde{\alpha} \)-invariant. Since the action \( \tilde{\alpha} \) commutes with \( \tilde{\alpha} \), we can define the action \( \tilde{\alpha} \) on \( \tilde{A} \) by \( \tilde{\alpha} \circ \tilde{\alpha} \) which makes \( \tilde{\tau} \) invariant. Then we propose another Yang-Mills moduli space \( \mathcal{M}^{(\tilde{A}, \mathbb{R}, \tilde{\tau})}(\tilde{\Xi}) \) of \( \tilde{\Xi} \) with respect to the \( F^* \)-flow \( (\tilde{A}, \mathbb{R}, \tilde{\tau}) \) and the dual trace \( \tilde{\tau} \), which is called the dual Yang-Mills moduli space of \( \mathcal{M}^{(A, \mathbb{R}, \alpha, \tau)}(\Xi) \).

§ 4. Main result

In this section, we prove the following theorem, which means physically that in quantum field theory, physical data are invariant under dimension reduction:

**Theorem 4.1** Let \((A, \mathbb{R}, \alpha)\) be a \( F^* \)-flow with a continuous \( \alpha \)-invariant faithful trace \( \tau \) and let \( \Xi \) be a finitely generated projective right \( A \)-module. Then there exist a \( F^* \)-flow \((\tilde{A}, \mathbb{R}, \tilde{\tau})\) with a dual trace \( \tilde{\tau} \) of \( \tau \) and a finitely generated projective right \( \tilde{A} \)-module \( \tilde{\Xi} \) whose Yang-Mills moduli space \( \mathcal{M}^{(\tilde{A}, \mathbb{R}, \tilde{\tau})}(\tilde{\Xi}) \) is homeomorphic to \( \mathcal{M}^{(A, \mathbb{R}, \alpha, \tau)}(\Xi) \).

Applying Theorems 3.4 and 4.1, we have the following corollary:

**Corollary 4.2** Let \((\tilde{A}, \mathbb{R}, \tilde{\tau})\) be the \( F^* \)-flow cited in Theorem 4.1 and \( \beta \) a smooth action commuting with \( \tilde{\tau} \). Suppose there exists a continuous faithful \( \beta \)-invariant trace \( \tau \), then given a finitely generated projective \( \beta \)-module \( \Xi \), there exists a \( F^* \)-flow \((A, \mathbb{R}, \beta_A)\) with a continuous faithful \( \beta_A \)-invariant trace \( \tau_A \) and a finitely generated projective \( A \)-module \( \Xi_A \) such that \( \mathcal{M}^{(\tilde{A}, \mathbb{R}, \tilde{\tau})}(\tilde{\Xi}) \) is homeomorphic to \( \mathcal{M}^{(A, \mathbb{R}, \beta_A, \tau_A)}(\Xi_A) \).

**Proof of Theorem 4.1:** By the assumption of \( \Xi \), there exist a natural number \( n \) and a projection \( P \in M_n(A) \) such that \( \Xi = P(A^n) \). Let us take a Hermitian structure \( < \cdot | \cdot >_A \) on \( \Xi \) by

\[
< \xi | \eta >_A = \sum_{j=1}^{n} \xi_j^* \eta_j
\]

\((\xi, \eta \in \Xi)\). Then if \( \nabla^0 \) is the Grassmann connection of \( \Xi \), then it belongs to \( CC(\Xi) \). Moreover it follows from Theorem 2.1 that \( \nabla^0 \in MC(\Xi) \). Now for any \( \nabla \in MC(\Xi) \) and \( X \in \text{Lie}(\mathbb{R}) \), there exists a skew adjoint element \( \Omega_X \in E \) such that

\[
\nabla_X = \nabla_X^0 + \Omega_X.
\]

As \( \Xi = P(A^n) \), it follows that \( \tilde{\Xi} = \hat{P}(\tilde{A}^n) \) where \( \hat{P} = P \otimes I_{S(\mathbb{R})} \).
Then we know that

\[ \text{End}_{\hat{\mathcal{A}}}(\hat{\Xi}) = \hat{\mathcal{P}} \mathcal{M}_n(\hat{\mathcal{A}}) \hat{\mathcal{P}}, \]

which is denoted by \( \hat{E} \). From now on, we want to define a mapping from \( \mathcal{M}^{(A, R, \alpha, \tau)}(\Xi) \) into \( \mathcal{M}^{(\hat{A}, \hat{R}, \hat{\alpha}, \hat{\tau})}(\hat{\Xi}) \) in the following way: Since \( E \) is no longer \( \alpha^n \)-invariant in general, it follows using the same idea in Connes[1] that there exists a \( F^* \)-flow \((\mathcal{M}_n(A), R, \beta)\) with the property that

1. \( \beta_t(P) = P \ (t \in \mathbb{R}), \)
2. \((\mathcal{M}_n(A), R, \beta)\) is outer equivalent to \((\mathcal{M}_n(A), R, \alpha^n)\).

By [1], let \( \iota_u \) be the equivariant isomorphism from \((\mathcal{M}_n(A) \rtimes_{\alpha^n} R, \alpha^n)\) onto \((\mathcal{M}_n(A) \rtimes_{\beta} R, \beta)\) such that

\[ \iota_u \circ \alpha^n \circ \iota_u^{-1} = \beta. \]

Then we have the following lemma which would be applied later:

Lemma 4.3([1]) The next two statements holds:

1. There is an equivariant isomorphism \( \iota_u \) from \((\mathcal{M}_n(A) \rtimes_{\alpha^n} R, \alpha^n)\) onto \((\mathcal{M}_n(A) \rtimes_{\beta} R, \beta)\).
2. There exists a unitary multiplier \( W \) of \( M_n(A) \otimes K^\infty(\mathbb{R}) \) such that

\[ \text{Ad}(W) \circ \Psi^\alpha_{0, \beta} \circ \iota_u = \Psi^\alpha_{0, \beta}, \]

where \( \Psi^\alpha_{0, \beta} \) are the equivariant isomorphisms as in Theorem 3.4 associated with \( \{ \cdot \} \), and \( \iota_u(a)(s) = \iota_u \{ a(s) \} \) for all \( a \in \mathcal{S}(\mathbb{R}, M_n(A) \rtimes_{\alpha^n} R). \)

Since \( M_n(\Xi) = (P \otimes I_n)([M_n(A)]^n) \), it is a finitely generated projective \( M_n(A) \)-module. Let \( d\beta \) be the infinitesimal generator of \( \beta \).

Now for any \( \nabla \in \text{MC}^{(M_n(A), R, \beta, \tau^n)}(M_n(\Xi)) \), there exists a skew adjoint element \( \Omega \in E_n = \text{End}_{M_n(A)}(M_n(\Xi)) \) such that

\[ \nabla = (P \otimes I_n)d\beta^n + \Omega. \]

Let \( \hat{E}_n = \text{End}_{M_n(A)}(M_n(\Xi)) \) be the set of all \( E_n \)-valued rapidly decreasing smooth functions on \( \mathbb{R} \). Then it becomes a \( F^* \)-algebra with respect to the \( \beta^n \)-twisted convolution product. By definition, we see that

\[ \hat{E}_n = \text{End}_{M_n(A)}(M_n(\Xi)), \]
where $\widehat{M_n(A)} = M_n(A) \rtimes_\beta \mathbb{R}$ and $\widehat{M_n(\Xi)} = S(\mathbb{R}, M_n(\Xi))$. Let us define the element $\widehat{\Omega} \in \widehat{E_n}$ by

$$\widehat{\Omega}(\xi)(t) = \Omega\{\xi(t)\}$$

($\xi \in \widehat{M_n(\Xi)}$, $t \in \mathbb{R}$) In fact, we check that

$$\widehat{\Omega}(\xi a)(t) = \Omega\{(\xi a)(t)\} = \int_\mathbb{R} \Omega\{\xi(s)\beta_s(a(t-s))\} \, ds$$

$$= \int_\mathbb{R} \Omega\{\xi(s)\} \beta_s(a(t-s)) \, ds$$

$$= \int_\mathbb{R} \widehat{\Omega}(\xi(s)) \beta_s(a(t-s)) \, ds$$

$$= (\widehat{\Omega}(\xi a))(t)$$

($\xi \in \widehat{M_n(\Xi)}$, $a \in \widehat{M_n(A)}$). As $\beta^n$ is used as the restriction of the natural extension of $\beta$ of $M_n^2(A)$ to $E_n$, then it follows from the definition that

$$\widehat{E_n} = E_n \rtimes_\beta^n \mathbb{R}.$$

Then we obtain that

$$\widehat{\Omega} \in E_n \rtimes_\beta^n \mathbb{R}.$$

We then have the following lemma:

**Lemma 4.4** $\widehat{\Omega} \in \widehat{E_n}$ is skew adjoint.

**Proof.** Since $\Omega \in E_n$ is skew adjoint, we compute that

$$< \widehat{\Omega}(\xi \otimes f) | \eta \otimes g >_{\widehat{M_n(A)}}$$

$$= \sum_{j=1}^n \widehat{\Omega}(\xi \otimes f)_j^* (\eta \otimes g)_j$$

where $\widehat{\Omega}(\xi \otimes f)_j, (\eta \otimes g)_j \in \widehat{M_n(A)}$. Then it is easy to check that

$$\widehat{\Omega}(\xi \otimes f)_j = \Omega(\xi)_j \otimes f$$

, using which we deduce that

$$< \widehat{\Omega}(\xi \otimes f) | \eta \otimes g >_{\widehat{M_n(A)}} (t)$$

$$= \sum_{j=1}^n \int_\mathbb{R} \beta_s(\Omega(\xi)_j^* \eta_j) f(-s)g(t - s) \, ds$$
\[
\int_{R} \beta_{s} \{ < \Omega(\xi) | \eta >_{M_{n}(A)} \} f(-s) g(t-s) \ ds
\]

As \( \Omega \) is skew adjoint, it follows that

\[
< \Omega(\xi) | \eta >_{M_{n}(A)} = - < \xi | \Omega(\eta) >_{M_{n}(A)}
\]

. Then we obtain that

\[
< \hat{\Omega}(\xi \otimes f) | \eta \otimes g >_{M_{n}(A)}(t) = - < \xi \otimes f | \hat{\Omega}(\eta \otimes g) >_{M_{n}(A)}(t)
\]

\((\xi, \eta \in M_{n}(\Xi), f, g \in S(\mathbb{R}))\). This implies the conclusion. Q.E.D.

Let \( d\hat{\beta}^{n} \) and \( d\tilde{\beta}^{n} \) be the infinitesimal generators of the dual action \( \hat{\beta}^{n} \) and the canonical extension \( \tilde{\beta}^{n} \) of \( \beta^{n} \) to \( \hat{E}_{n} \) respectively. Since \( \hat{\beta}^{n} \) commutes with \( \tilde{\beta}^{n} \), then \( d\hat{\beta}^{n} + d\tilde{\beta}^{n} \) is the infinitesimal generator of \( \beta^{n} \). Then we have the following lemma:

**Lemma 4.5**

\[
(\hat{P} \otimes I_{n})(d\beta^{n}) + \hat{\Omega} \in MC^{(\hat{M}_{n}(A), \mathbb{R}, \hat{\beta}, \tau_{n}^{\hat{M}_{n}(\Xi)}(\hat{M}_{n}(\Xi))},
\]

**Proof.** Since \( \hat{P} \otimes I_{n})(d\beta^{n}) \) is the Grassmann connection of \( \hat{M}_{n}(\Xi) \) with respect to the action \( \beta \), it belongs to \( MC^{(M_{n}(A), \mathbb{R}, \beta, \tau_{n}^{M_{n}(\Xi)}(M_{n}(\Xi))} \) by Theorem 2.1. As \( \hat{\Omega} \in \hat{E}_{n} \) is skew adjoint by Lemma 4.4, the conclusion follows from Theorem 2.1. Q.E.D.

By the Lemma 4.5, we then define a mapping

\[
\Phi_{\beta} : M^{(M_{n}(A), \mathbb{R}, \beta, \tau_{n})}(M_{n}(\Xi)) \mapsto M^{(\hat{M}_{n}(A), \mathbb{R}, \hat{\beta}, \tau_{n}^{\hat{M}_{n}(\Xi)}(\hat{M}_{n}(\Xi))}
\]

by the following fashion

\[
\Phi_{\beta}([\nabla]_{U(E_{n})}) = [(\hat{P} \otimes I_{n})(d\beta^{n}) + \hat{\Omega}]_{U(E_{n})},
\]

where \([\nabla]_{\{\ast\}}\) means the equivalence class of \( \nabla \) under the gauge action of \( \{\ast\} \).

We then check the following lemma:

**Lemma 4.6** \( \Phi_{\beta} \) is well defined.

**Proof.** Let \( \Omega, \Omega_{1} \) be two skew adjoint elements in \( E_{n} \), and suppose \( u(\nabla_{\beta}^{0} + \Omega)u^{\ast} = \nabla_{\beta}^{0} + \Omega_{1} \) for some unitary \( u \in E_{n} \), then

\[
\Omega_{1} = u\nabla_{\beta}^{0} u^{\ast} - \nabla_{\beta}^{0} + u\Omega u^{\ast}
\]
We have to show that \( \nabla^0_\beta + \Omega \) is equal to \( \nabla^0_\beta + \tilde{\Omega}_1 \) up to the gauge automorphisms of \( U(\hat{E}_n) \). Now we compute that

\[
(\nabla^0_\beta + \Omega)(\xi)(t) = (\hat{P} \otimes I_n)(\delta)(\xi)(t) + (u \nabla^0_\beta u^* - \nabla^0_\beta + u\Omega u^*)\{\xi(t)\} \\
= 2\pi i t \xi(t) + \tilde{\Omega} \tilde{u}(\xi)(t) + (u \nabla^0_\beta u^* - \nabla^0_\beta)\{\xi(t)\}
\]

(\( \xi \in \hat{M}_n(\hat{\Xi}) \)) where \( \tilde{u}(\xi)(t) = u\{\xi(t)\} \). Since we see that

\[
(u \nabla^0_\beta u^* - \nabla^0_\beta)\{\xi(t)\} = (\tilde{u} \nabla^0_\beta \tilde{u}^* - \nabla^0_\beta)(\xi)(t),
\]

and \( \tilde{u} \nabla^0_\beta \tilde{u}^*(\xi)(t) = 2\pi i t \xi(t) \), then we obtain that

\[
\nabla^0_\beta + \nabla^0_\beta + \tilde{\Omega}_1(\xi)(t) = \gamma \tilde{u}(\nabla^0_\beta + \nabla^0_\beta + \tilde{\Omega})(\xi)(t)
\]

(\( \xi \in \hat{M}_n(\hat{\Xi}) \)). As we know that

\[
\nabla^0_\beta + \nabla^0_\beta = \nabla^0_\beta
\]

, then the conclusion follows. Q.E.D.

By definition, \( \tilde{\beta} \) is a weakly inner action of \( \hat{M}_n(A) \) implemented by a unitary multiplier flow \( \mu \) of \( \hat{M}_n(A) \) faithfully acting on \( L^2(\mathbb{R}, H_{\tau^n}) \) for the Hilbert space \( L^2(\hat{M}_n(A), \tau^n) \). Actually, as \( \tilde{\beta}_t(a)(s) = \beta_t(a(s)) \) for all \( a \in \hat{M}_n(A), s, t \in \mathbb{R} \), then \( \mu_t(a)(s) = a(s - t) \) for all \( a \in \mathcal{S}(\mathbb{R}, \hat{M}_n(A)), s, t \in \mathbb{R} \). Then we have the following lemma:

**Lemma 4.7** The \( F^* \)-system \( (\hat{M}_n(A), \mathbb{R}, \tilde{\beta}) \) is inner conjugate to the \( F^* \)-system \( (\hat{M}_n(A), \mathbb{R}, \tilde{\beta}) \). Then it implies that the \( F^* \)-system \( (\hat{M}_n(A) \rtimes \beta \mathbb{R}, \mathbb{R}, \tilde{\beta}) \) is isomorphic to the system \( (\hat{M}_n(A), \mathbb{R}, \tilde{\beta}) \) via the map: \( \Lambda(x)(t) = \mu_{-t}x(t) \) for all \( x \in \hat{M}_n(A) \rtimes \beta \mathbb{R} \), where \( \hat{M}_n(A) = \hat{M}_n(A) \rtimes \hat{\beta} \mathbb{R} \).

By Lemmas 4.5 and 4.7, we deduce the following lemma:

**Lemma 4.8** Let \( \Lambda_\beta : \mathcal{M}^{(\hat{M}_n(A) \rtimes \beta \mathbb{R}, \mathbb{R}, \beta, \tau^n)}(\hat{M}_n(\hat{\Xi})) \rightarrow \mathcal{M}^{(\hat{M}_n(A), \mathbb{R}, \tilde{\beta} : \text{Ad}(\nu), \tau^n)}(\hat{M}_n(\hat{\Xi})) \) defined by

\[
\mathcal{M}^{(\hat{M}_n(A), \mathbb{R}, \tilde{\beta} : \text{Ad}(\nu), \tau^n)}(\hat{M}_n(\hat{\Xi}))
\]
\[ \Lambda_\beta([\nabla]_{U(M_n(A) \times \mathbb{R})}) = [\Lambda^n \circ \nabla \circ (\Lambda^n)^{-1}]_{U(M_n(A))} \]

where \( \nu \) is the unitary multiplier flow of \( \widehat{M}_n(A) \) implementing \( \widehat{\beta} \) on \( \widehat{M}_n(A) \), and \( \Lambda^n \) is the isomorphism from \( \operatorname{End}_{\widehat{M}_n(A) \times \mathbb{R}}(\widehat{M}_n(\Xi)) \) onto \( \operatorname{End}_{\widehat{M}_n(A)}(\widehat{M}_n(\Xi)) \) induced by \( \Lambda \). Then it implies that \( \Lambda_\beta \) is a homeomorphism.

**Proof.** By the definition of \( \Lambda \), we check that
\[
\Lambda \circ \widehat{\beta} \circ \Lambda^{-1} = \widehat{\beta}, \quad \Lambda \circ \widetilde{\beta} \circ \Lambda^{-1} = \widetilde{\beta}.
\]

By the same reason as for \( \widetilde{\beta} \), there exists a unitary multiplier flow \( \nu \) of \( \widehat{M}_n(A) \) such that \( \widetilde{\beta} = \text{Ad}(\nu) \) on \( \widehat{M}_n(A) \). The rest is easily seen. Q.E.D.

Let \( \Psi_0^\beta \) be the isomorphism from the \( \mathbb{F}^* \)-system \( (\widehat{M}_n(A), \mathbb{R}, \widehat{\beta}) \) onto the \( \mathbb{F}^* \)-system \( (M_n(A) \otimes \mathcal{K}^\infty(\mathbb{R}), \mathbb{R}, \beta \otimes \text{Ad}(\lambda)) \) defined by
\[
\Psi_0^\beta(x)(t, s) = \int_{\mathbb{R}} e^{2\pi i r s} \beta_{-1}(x(t - s, r)) \, dr,
\]
and
\[
(\Psi_0^\beta)^{-1}(x)(t, s) = \int_{\mathbb{R}} e^{2\pi i (t - r)s} \beta_r(x(r, r - t)) \, dr
\]
\((x \in M_n(A) \otimes \mathcal{K}^\infty(\mathbb{R}), t, s \in \mathbb{R})\). By definition, we compute that
\[
\Psi_0^\beta \circ \text{Ad}(\nu_p) \circ (\Psi_0^\beta)^{-1}(x)(t, s)
\]
\[= \int e^{2\pi i (rs + p(t-s))} \beta_{-1}((\Psi_0^\beta)^{-1}(x)(t - s, r)) \, dr,\]
which is equal to
\[\int e^{2\pi i (p(t-s) + (t-r')s)} \beta_{p-1}(x(r', r' - t + s)) \, dr' dr.\]
\((x \in M_n(A) \otimes \mathcal{K}^\infty(\mathbb{R}), t, s \in \mathbb{R})\). Therefore it follows that
\[
\Psi_0^\beta \circ \text{Ad}(\nu_p) \circ (\Psi_0^\beta)^{-1}(x)(t, s) = e^{2\pi i p(t-s)} x(t, s),
\]
\((x \in M_n(A) \otimes \mathcal{K}^\infty(\mathbb{R}), p, t, s \in \mathbb{R})\), which implies that there exists a unitary multiplier flow \( \nu_\beta \) of \( \mathcal{K}^\infty(\mathbb{R}) \) with the property that
\[
\Psi_0^\beta \circ \text{Ad}(\nu_p) \circ (\Psi_0^\beta)^{-1} = \text{Ad}(I \otimes (\nu_\beta)_p) \quad (p \in \mathbb{R})
\]
on $M_n(A) \otimes \mathcal{K}^\infty(\mathbb{R})$. Then it turns out that
\[
\Psi_0^\beta \circ (\hat{\beta} \circ Ad(\nu)) \circ (\Psi_0^\beta)^{-1} = \beta \otimes Ad(\lambda \circ \nu_\beta).
\]

Let $\Psi_\beta$ be the map: $\mathcal{M}(\hat{M}_n(A), \hat{\beta} \circ Ad(\nu), \hat{\tau^n})(\hat{M}_n(\Xi)) \mapsto M_n(A) \otimes K^\infty(\mathbb{R}),$ induced by the equivariant isomorphism $\Psi_0^\beta.$ Then we also show the following lemma by the same way as Lemma 4.8:

Lemma 4.9 $\Psi_\beta$ is a homeomorphism induced by the equivariant isomorphism $\Psi_0^\beta.$

Let us now consider the following map:
\[
\Pi_\beta : \mathcal{M}(\hat{M}_n(A) \otimes K^\infty(\mathbb{R}), \hat{\beta} \circ Ad(\lambda \circ \nu_\beta), \hat{\tau^n} \otimes Tr)(M_n(\Xi) \otimes K^\infty(\mathbb{R})) \mapsto \mathcal{M}(M_n(A), \mathbb{R}, \beta, \tau^n)(M_n(\Xi))
\]
defined by the natural one induced from the map $\Pi : M_n(A) \otimes K^\infty(\mathbb{R}) \mapsto M_n(A)$ given by $\Pi : x \mapsto (I \otimes e)x(I \otimes e),$ where $e$ is a rank one projection of $\mathcal{K}^\infty(\mathbb{R}),$ $Tr$ the canonical trace of $\mathcal{K}^\infty(\mathbb{R}).$ Now let $\nabla \in MC(M_n(\Xi) \otimes K^\infty(\mathbb{R}))$ and put
\[
\nabla_e(\xi) = (I_n \otimes e)\nabla(\xi \otimes e)
\]
($\xi \in M_n(\Xi)$), where $I_n$ is the identity of $E_n.$ Then $\nabla_e$ is well defined and independent of the choice of $e$ up to the gauge equivalence because of the existence of a unitary multiplier of $K^\infty(\mathbb{R})$ which sends $e$ to another rank one projection. Then we see that given any $u \in U(E_n \otimes K^\infty(\mathbb{R})),$
\[
(\gamma u(\nabla))_e = \gamma u_e(\nabla_e),
\]
where $u_e = (I_n \otimes e)u(I_n \otimes e) \in U(E_n).$ Here we define a map $\Pi_\beta$ by
\[
\Pi_\beta(\nabla)[U(E_n, \otimes K^\infty(\mathbb{R}))] = [\nabla_e][U(E_n)] = [\Pi^n \circ \nabla \circ (\Pi^n)^{-1}]_{U(E_n)}.
\]
Then it is well defined and independent of the choice of $e.$ Moreover, we have the following lemma:
Lemma 4.10

\[ \Pi_\beta : \mathcal{M}^{(M_n(A) \otimes K^\infty(\mathbb{R}), \mathbb{R}, \beta \otimes \text{Ad}(\lambda \circ \nu_\beta)), \tau^n \otimes \text{Tr}}(M_n(\Xi) \otimes K^\infty(\mathbb{R})) \]

\[ \longrightarrow \mathcal{M}^{(M_n(A), \mathbb{R}, \tau^n)}(M_n(\Xi)) \] is a homeomorphism.

Proof. Let us define the mapping \( \Pi_\beta^{-1} \) by

\[ \Pi_\beta^{-1}([\nabla]) = [\nabla \otimes I_{K^\infty(\mathbb{R})}] . \]

Then it is easily seen that both \( \Pi_\beta^{-1} \circ \Pi_\beta \) and \( \Pi_\beta \circ \Pi_\beta^{-1} \) are identities. Moreover if \( [\nabla^i] \longrightarrow [\nabla] \) with respect to \( \langle \cdot | \cdot >_{M_n(A)} \), then it follows from the definition that there exists a unitary net \( \{u_e\} \) (by choosing a subnet) of \( E_n \) such that \( \gamma_{u_e}(\nabla^i) \longrightarrow \nabla \), which implies that \( ([\nabla^i]_e) \longrightarrow [\nabla_e] \), so that \( \Pi_\beta \) is continuous and so is also \( \Pi_\beta^{-1} \) by the same way. Q.E.D.

Let \( \nabla^0 \) be the Grassmann connection of \( \cdot \). Then we easily check the following lemma:

Lemma 4.11

\[ \Psi_0^{\beta^m} \circ \nabla^{\beta^m}_{\beta^m \circ \text{Ad}(\nu)} \circ (\Psi_0^{\beta^m})^{-1} = \nabla^{\beta^m}_{\beta^m \circ \text{Ad}(\lambda \circ \nu_\beta)} \cdot \]

Proof. It follows from Lemma 4.9 that

\[ \Psi_0^{\beta^m} \circ (\hat{\beta} \circ \text{Ad}(\nu) \circ (\Psi_0^{\beta^m})^{-1} = \beta \otimes \text{Ad}(\lambda \circ \nu_\beta) \cdot \]

Since

\[ \widehat{M_n(\Xi)} = S(\mathbb{R}^2, M_n(\Xi)), \]

and

\[ \nabla^0_\beta = \hat{P} \ d \widehat{\beta^n}, \]

where

\[ \hat{P} = P \otimes I_n \otimes I_{S(\mathbb{R}^2)}, \]

and \( d \widehat{\beta^n} \) is the infinitesimal generator of \( \widehat{\beta^n} \), then this implies the conclusion. Q.E.D.

Moreover, we need the following lemma which is directly shown:

Lemma 4.12 There exists a

\[ U \in U(\text{End}_{M_n(A) \otimes K^\infty(\mathbb{R})}(M_n(\Xi) \otimes S(\mathbb{R}^2))) \]

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such that
\[
\Psi_0^\beta_n \circ \Lambda^n \circ \hat{\Omega} \circ (\Psi_0^\beta_n \circ \Lambda^n)^{-1} = \gamma_U(\Omega \otimes Id)
\]

Actually, \(U\) is defined as \(U(\xi)(s, t) = u^n_s \xi(s, t)\) for all \(\xi \in M_n(\Xi) \otimes S(\mathbb{R}^2)\), where \(u^n\) is the unitary flow to \(E_n\) induced by \(\beta^n\).

**Proof.** We know by definition that
\[
\Psi_0^\beta_n(\xi)(s, t) = \int_{\Xi} e^{2\pi i r t} u^n_{-s} \xi(s, t, r) \, dr
\]
(\(\xi \in \hat{M}_n(\Xi)\), and
\[
(\Psi_0^\beta_n)^{-1}(\xi)(s, t) = \int_{\Xi} e^{2\pi i (s-r)t} u^n_{r} \xi(r, r-s) \, dr
\]
(\(\xi \in M_n(\Xi) \otimes S(\mathbb{R}^2)\)). Then we compute that
\[
\Psi_0^\beta_n \circ \Lambda^n \circ \hat{\Omega}(\xi)(s, t) = \int_{\Xi} e^{2\pi i r t} u^n_{-s} (\Lambda^n \circ \hat{\Omega}(\xi)(s-t, r)) \, dr
\]
(\(\xi \in \hat{M}_n(\Xi)\)). Then as we know that
\[
\Lambda^n \circ \hat{\Omega}(\xi)(s-t, r) = \nu^n_{-s} \hat{\Omega}\{\xi(r)\}(s-t) = \Omega\{\xi(s-t+r, r)\}.
\]
Therefore we have that
\[
\Psi_0^\beta_n \circ \Lambda^n \circ \hat{\Omega}(\xi)(s, t)
\]
\[
= \int_{\Xi} e^{2\pi i r t} u^n_{-s} \Omega\{(\xi)(s-t+r, r)\} \, dr
\]
(\(\xi \in \hat{M}_n(\Xi)\)). Replacing \(\xi\) by \((\Psi_0^\beta_n \circ \Lambda^n)^{-1}(\xi)\), we obtain that
\[
(\Psi_0^\beta_n \circ \Lambda^n)^{-1}(\xi)(s-t+r, r)
\]
\[
= \int_{\Xi} e^{2\pi i (s-t-r')} u^n_{r'} \xi(r', r' - s + t) \, dr'
\]
(\(\xi \in M_n(\Xi) \otimes S(\mathbb{R}^2)\)). Combining the argument discussed above, we deduce that
\[
(\Psi_0^\beta_n \circ \Lambda^n) \circ \hat{\Omega} \circ (\Psi_0^\beta_n \circ \Lambda^n)^{-1}(\xi)(s, t)
\]
\[
= \int \int e^{2\pi i r' t} u^n_{-s} \Omega\{u^n_{r'} \xi(r', r' - s + t)\} \, dr' \, dr,
\]
which is equal to
\[ u^n_{s-s} \Omega \{ u^n_{s} \xi(s, t) \} = \gamma_U (\Omega \otimes \text{Id})(\xi)(s, t), \]
where \( U(\xi)(s, t) = u^n_{s-s} \xi(s, t) \). This implies the conclusion. Q.E.D.

We next show the following lemma which seems to be essential to prove our main theorem:

Lemma 4.13
\[ \Pi_\beta \circ \Psi_\beta \circ \Lambda_\beta \circ \Phi_\beta = \text{Id} \]
on \( \mathcal{M}^{(M_n(A), \mathbb{R}, \beta, \tau^n)}(M_n(\Xi)) \), where \( \Psi_\beta \) is the homeomorphism on the moduli space induced from the isomorphism \( \Psi_0^0 \beta \) given in Lemma 4.9.

Proof. Let \( \nabla = \nabla_0^0 + \Omega \in \mathcal{MC}^{(M_n(A), \mathbb{R}, \beta, \tau^n)}(M_n(\Xi)) \). Then we know by Lemmas 4.11 and 4.12 that there exist a \( U \in U(\text{End}_{M_n(A) \otimes K_\infty (\mathbb{R})} (M_n(\Xi) \otimes S(\mathbb{R}^2))) \) such that
\[ (\Psi_0^n_\beta \circ \Lambda^n_\beta) \circ \nabla_0^n_\beta \circ \nabla_0^n_\beta^{-1} = \nabla_0^n_\beta \circ \text{Ad}(\lambda_\nu_\beta), \]
and
\[ (\Psi_0^n_\beta \circ \Lambda^n_\beta) \circ \Omega \circ \nabla_0^n_\beta^{-1} = \gamma_U (\Omega \otimes \text{Id}). \]
By Lemmas 4.11 and 4.12, we obtain that
\[ \Psi_\beta \circ \Lambda_\beta \circ \Phi_\beta \circ \Phi_\beta ([\nabla]_{U(\mathbb{R}^2)}) = [(\nabla_0^n_\beta \circ \text{Ad}(\lambda_\nu_\beta))]_{U(\mathbb{R}^2)}. \]
then we see that
\[ (\nabla_0^n_\beta \circ \text{Ad}(\lambda_\nu_\beta)) e = \nabla_0^n_\beta \otimes e, \]
where \( e \) is a rank one projection of \( K_\infty (\mathbb{R}) \). In fact, we check that
\[ (P \otimes I_n \otimes e) d((\beta^n) \otimes \text{Ad}(\lambda \circ \nu_\beta))^n(\xi \otimes e) \]
\[ = (P \otimes I_n \otimes e) d((\beta^n) \otimes \text{Id})(\xi \otimes e)+(P \otimes I_n \otimes e) d(\text{Id} \otimes \text{Ad}(\lambda \circ \nu_\beta))^n(\xi \otimes e). \]
(\( \xi \in M_n(\Xi) \)). Then it follows that
\[ e \text{Ad}((\lambda_t \circ (\nu_\beta)_t)) (e) = e. \]
for all \( t \in \mathbb{R} \). Therefore, it follows that
\[ (P \otimes I_n \otimes e) d(\text{Id} \otimes \text{Ad}(\lambda \circ \nu_\beta))^n(\xi \otimes e) = 0. \]
\((\xi \in M_n(\Xi))\). On the other hand, we know that
\[
\gamma_U(\Omega \otimes Id)_e = \gamma_U(\Omega) \otimes e ,
\]
where \(U_e\) belongs to \(U(E_n)\) with \(U_e \otimes e = (I \otimes e)U(I \otimes e)\). By the definition of \(U\), \(U_e\) commutes with \(\beta^n\). Consequently, it follows that
\[
[\nabla^{0}_{\beta^n \otimes e} + \gamma_U(\Omega) \otimes e]U(e_\Omega) \otimes e = [\nabla^{0}_{\beta^n} + \Omega]U(E_n),
\]
which deduce the conclusion. Q.E.D.

Applying Lemma 4.13 to the system \((\widehat{M_n}(A), \mathbb{R}, \overline{\beta})\), we obtain the following corollary:

**Corollary 4.14**

\[
\Pi_{\overline{\beta}} \circ \Psi_{\overline{\beta}} \circ \Lambda_{\overline{\beta}} \circ \Phi_{\overline{\beta}} = Id
\]
on \(\mathcal{M}(\widehat{M_n}(A), \mathbb{R}, \overline{\beta}, \tau_n)\) on \(\widehat{M_n}(\Xi)\), where the map \(\Pi_{\overline{\beta}}\) is a homeomorphism from
\[
\mathcal{M}(\widehat{M_n}(A) \otimes \mathcal{K}^\infty(\mathbb{R}), \mathbb{R}, \overline{\beta} \otimes \Ad(\lambda \nu_{\overline{\beta}}), \tau_n \otimes \Tr) (\widehat{M_n}(\Xi) \otimes \mathcal{K}^\infty(\mathbb{R}))
\]
on to
\[
\mathcal{M}(\widehat{M_n}(A), \mathbb{R}, \overline{\beta}, \tau_n) (\widehat{M_n}(\Xi))\)

induced by the mapping:
\[
\widehat{M_n}(A) \otimes \mathcal{K}^\infty(\mathbb{R}) \mapsto \widehat{M_n}(A)
\]
given by \(x \mapsto (I \otimes e)x(I \otimes e)\).

As we have seen in Lemma 4.8,
\[
\Lambda \circ \overline{\beta} \circ \Lambda^{-1} = \overline{\beta} \circ \Ad(\nu)
\]
Then the relation between \(\Phi_{\overline{\beta} \circ \Ad(\nu)}\) and \(\Phi_{\overline{\beta}}\) is as follows:

**Lemma 4.15**

\[
\Phi_{\overline{\beta} \circ \Ad(\nu)} = \Lambda_{\beta} \circ (\Pi_{\overline{\beta}} \circ \Psi_{\overline{\beta}} \circ \Lambda_{\overline{\beta}})^{-1} \circ \Phi_{\overline{\beta}} \circ \Pi_{\overline{\beta}} \circ \Psi_{\overline{\beta}}
\]
on \(\mathcal{M}(\widehat{M_n}(A), \mathbb{R}, \overline{\beta} \circ \Ad(\nu), \tau_n) (\widehat{M_n}(\Xi))\).

**Proof.** By Lemmas 4.8 \sim 4.13 and Corollary 4.14, we have that
\[
(\Pi_{\overline{\beta}} \circ \Psi_{\overline{\beta}} \circ \Lambda_{\overline{\beta}})^{-1} = \Phi_{\overline{\beta}} \circ \Phi_{\overline{\beta}}.
\]
By the equality written just above this Lemma, we see that
\[ \Lambda_\beta \circ \Phi_\beta \circ (\Lambda_\beta)^{-1} = \Phi_\beta \circ \text{Ad}(\nu) . \]

By Lemma 4.13 and Corollary 4.14, \( \Phi_\beta \) is bijective. By Lemma 4.12, we have that
\[ \Phi_\beta^{-1} = \Phi_\beta \circ \Pi_\beta \circ \Psi_\beta , \]
which implies the conclusion. Q.E.D.

Summing up the above argument, we obtain the following lemma:

**Lemma 4.16** \( \Phi_\beta \) and \( \Phi_\beta \) are homeomorphisms.

**Proof.** By Lemmas 4.8 ~ 4.15, \( \Phi_\beta \) and \( \Phi_\beta \) are bijective and bicontinuous, which completes the proof. Q.E.D.

We then define a map \((\Phi_\alpha^n)^{-1}:
\[ M((M_n(A) \times K, R, \alpha^n, \tau^n)) \to M((M_n(\Xi))) \]
defined by
\[ \Pi_\alpha^n \circ \Psi_\alpha^n \circ (\tilde{\iota})^{-1} \circ \Lambda_\beta \circ \Phi_\beta \circ \iota , \]
where \( \iota \) is the extended map on \( M((M_n(A) \times K, R, \alpha^n, \tau^n)) \) induced by \( \iota_u \) in Lemma 4.3, and so is \( \tilde{\iota} \) on \( M((M_n(A) \times K, R, \alpha^n, \tau^n)) \) induced by \( \tilde{\iota}_u \), because the map \( \iota_u \) intertwines \( \overline{\alpha^n} \) and \( \overline{\beta} \), and the map \( \tilde{\iota}_u \) intertwines \( \overline{\alpha^n} \) and \( \overline{\beta} \).

**Lemma 4.17** \((\Phi_\alpha^n)^{-1} \) is a homeomorphism.

**Proof.** By Lemma 4.3 (2), there exists a unitary multiplier \( W \) of \( M_n(A) \otimes K^\infty(\mathbb{R}) \) such that
\[ \text{Ad}(W) \circ \Psi_\beta^0 \circ \tilde{\iota}_u = \Psi_\alpha^n , \]
which implies that \( \Psi_\beta \circ \tilde{\iota} = \Psi_\alpha^n \). Moreover, \( \iota \) and \( \tilde{\iota} \) are homeomorphisms. Then it follows from Lemma 4.16 that \((\Phi_\alpha^n)^{-1} \) is a homeomorphism. Q.E.D.

By Lemmas 4.17, we deduce the following corollary:

**Corollary 4.18** \( \Phi_\alpha^n \) is a homeomorphism:
\[ M((M_n(A) \times K, R, \alpha^n, \tau^n)) \to M((M_n(\Xi))) \to M((M_n(A) \times K, R, \alpha^n, \tau^n)) \to M((M_n(\Xi))) . \]
Finally, using $\Phi_\alpha^*$, we define the map $\Phi_\alpha$

$$\mathcal{M}^{(A,\mathbb{R},\alpha,\tau)}(\Xi) \longmapsto \mathcal{M}^{(\hat{A},\mathbb{R},\hat{\tau})}(\hat{\Xi})$$

by

$$\Phi_\alpha = \hat{\Pi}_n \circ \Phi_\alpha^* \circ (\Pi_n)^{-1},$$

where $\Pi_n$ is a homeomorphism:

$$\mathcal{M}^{(A \otimes M_n(\mathbb{C}),\mathbb{R},\alpha \otimes \text{Ad}(\lambda \circ \nu),\tau \otimes \text{Tr}n)}(\Xi \otimes M_n(\mathbb{C})) \longmapsto \mathcal{M}^{(A,\mathbb{R},\alpha,\tau)}(\Xi).$$

Finally, we show the following main lemma:

**Lemma 4.19** $\Phi_\alpha$ is a homeomorphism.

**Proof.** In Lemma 4.10, replacing $(M_n(A) \otimes K^\infty(\mathbb{R}), \beta \otimes \text{Ad}(\lambda \circ \nu_\alpha))$ and $M_n(\Xi) \otimes K^\infty(\mathbb{R})$ by $A \otimes M_n((C)), \alpha^n$ and $\Xi \otimes M_n(\mathbb{C})$ respectively, we deduce that both $\hat{\Pi}_n$ and $(\Pi_n)^{-1}$ are homeomorphisms. Then it implies the conclusion by Corollary 4.18. Q.E.D.

Summing up all the argument discussed above, we obtain the main result of Theorem 4.1

In what follows, we compute the moduli spaces of some concrete examples by means of Theorem 4.1:

**Example 4.20**

$$\mathcal{M}^{(K^\infty(\mathbb{R}),\mathbb{R},\text{Ad}(\lambda),\text{Tr})}(K^\infty(\mathbb{R})) \approx \mathcal{M}^{(\mathcal{S}(\mathbb{R}),\mathbb{R},\lambda,f)}(\mathcal{S}(\mathbb{R}))$$

$$\approx \mathcal{M}^{(\mathbb{C},\mathbb{R},\text{Id},1)}(\mathbb{C}) \approx \mathbb{R},$$

where $\approx$ means a symbol of homeomorphicity.

**Examples 4.21** Given a $\theta \in \mathbb{R}$, let us take the Moyal product $\star_\theta$ on $\mathcal{S}(\mathbb{R}^2)$. Then $(\mathcal{S}(\mathbb{R}^2), \star_\theta)$ becomes a $F^*$-algebra, which is denoted by $\mathbb{R}^2_\theta$. Since $\mathbb{R}^2_\theta$ is isomorphic to $\mathcal{S}(\mathbb{R}) \rtimes_\theta \mathbb{R}$, then it follows from Theorem 4.1 that

$$\mathcal{M}^{(\mathbb{R}^2_\theta,\mathbb{R},\tau_\theta)}(\mathbb{R}^2_\theta) \approx \mathbb{R},$$

where $\tau_\theta$ is the canonical trace of $\mathbb{R}^2_\theta$.

Even though changing $F^*$-flows into $F^*$-multiflows, the same result as Theorem 4.1 is obtained by using the ideas developed in Lemmas 4.3 $\sim$ 4.18 as follows:
Theorem 4.22  Let \((A, \mathbb{R}^n, \alpha)\) be a F*-multiflow with a faithful continuous \(\alpha\)-invariant trace \(\tau(n \geq 1)\), and \(\Xi\) a finitely generated projective \(A\)-module. Then there exist a F*-multiflow \((\hat{A}, \mathbb{R}^n, \hat{\alpha})\) with a dual trace \(\hat{\tau}\), and a dual \(\hat{A}\)-module \(\hat{\Xi}\) such that

\[ M^{(A, \mathbb{R}^n, \alpha, \tau)}(\Xi) \approx M^{(\hat{A}, \mathbb{R}^n, \hat{\alpha}, \hat{\tau})}(\hat{\Xi}). \]

Proof. Suppose \(n = 2\), we may choose a cocycle unitary multiplier with the property in Lemma 4.3 because \(\alpha(\mathbb{R}^2)\) is a commutative connected Lie group. We then choose an outer equivalent F*-multiflow \((M(A), \mathbb{R}^2, \beta)\) of \((A, \mathbb{R}^n, \alpha)\) such that \(\beta(t,s)(P) = P\) for all \((t, s) \in \mathbb{R}^2\). Then we see that the Grassmann connection \(\nabla_0\) satisfies the Yang-Mills condition. In fact, since \(\beta(t,s)(P) = P\) for all \((t, s) \in \mathbb{R}^2\), it follows that

\[ \Theta \nabla_0(X, Y) = \nabla_0^X \nabla_0^Y - \nabla_0^Y \nabla_0^X = P^n(d\beta^n)_X P^n(d\beta^n)_Y - P^n(d\beta^n)_Y P^n(d\beta^n)_X \]

\((X, Y \in \mathbb{R}^2)\), where \(P^n = P \otimes I_n\). As \(d\beta^n\) is a Lie homomorphism and \((d\beta^n)_X(P^n) = 0 \ (X \in \mathbb{R}^2)\), we have that

\[ \Theta \nabla_0(X, Y) = P^n \{ (d\beta^n)_X(d\beta^n)_Y - (d\beta^n)_Y(d\beta^n)_X \} = 0 \]

\((X, Y \in \mathbb{R}^2)\). Then it follows from Theorem 2.1 that \(\nabla_0\) is a Yang-Mills connection. By the same method as in the proof of Theorem 4.1, we deduce the conclusion. The way used above is also applicable to the case for \(n \geq 3\) by induction. Q.E.D.

The similar statement to Corollary 4.2 is in the following:

Corollary 4.23  Let \((A, \mathbb{R}^n, \alpha)\) be a F*-multiflow with a faithful continuous \(\alpha\)-invariant trace \(\tau\), and \((\hat{A}, \mathbb{R}^n, \hat{\alpha})\) its associated F*-flow with the dual trace \(\hat{\tau}\). Suppose \((\hat{A}, \mathbb{R}^n, \beta)\) is another F*-multiflow such that

\[ \hat{\tau} \cdot \beta = \hat{\tau}, \ \beta \circ \hat{\alpha} = \hat{\alpha} \circ \beta \]

then given a finitely generated projective \(\hat{A}\)-module \(\Xi\), there exist a F*-multiflow \((A, \mathbb{R}^n, \beta_A)\), a finitely generated projective \(A\)-module \(\Xi_A\) and a faithful \(\beta_A\)-invariant trace \(\tau_A\) of \(A\) such that

\[ M^{(\hat{A}, \mathbb{R}^n, \beta, \hat{\tau})}(\Xi) \approx M^{(A, \mathbb{R}^n, \beta_A, \tau_A)}(\Xi_A). \]
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