Vehicle Lateral Stability Regions for Control Applications

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ABSTRACT This article presents an improved method of obtaining lateral stability regions for road vehicles, considering the influence of steering angle, center of gravity, longitudinal speed, and tire-road friction coefficient on the vehicle dynamics. Comprehensive stability regions are obtained for a wide range of such parameters. Moreover, conservative stability regions are proposed, in the case of control applications that demand robust or safety-critical control actions. Next, a steering-angle-dependent region is used to implement a safety-critical electronic stability control system with active front steering as actuation. The resulting control system extends safety-critical control based on control-dependent barrier functions, introducing a control-dependent Lyapunov function to improve its steady-state behavior. Finally, we identify and propose workarounds for problems that arise from the number of inputs being less than the dimension of the desired safe set. The conservative stability regions and the extended safety-critical control system are validated by means of simulation results based on a nonlinear lateral stability model.

INDEX TERMS Active front steering, control barrier function, electronic stability control, safety-critical control, vehicle lateral stability, vehicle stability control.

I. INTRODUCTION

Vehicle stability has been studied throughout the development of the field of vehicle dynamics, in the twentieth century [1], [2], [3]. In particular, the phenomenon of lateral instability related to maximum lateral force of any wheeled vehicle is intuitive: as the tire loses contact with the ground, dynamic friction increasingly becomes a substitute for static friction, centripetal forces decrease, and the curve radius increases as a direct result. The tire force generation process is more elaborate, which justifies the study of tire dynamics [2].

Common approaches to vehicle stability deal with system stability and controllability, as they are inherent to one another, in a practical sense [4], [5]. A driver wants the vehicle to remain stable, so he is able to control it. And he needs the vehicle to be controllable, so as to remain stable on the desired trajectory [6], [7], [8]. The majority of the works on the subject considers either a phase portrait of a simplified system [4], [8], [9], [10], [11], [12] or the linearized system characteristic equation [4], [5], [13], [14] to evaluate system stability. This does not imply, naturally, that other approaches have not been presented, for instance, Lyapunov techniques [15], [16]. Finally, we mention that this work is concerned with lateral stability, and not roll stability, e.g., in [17]; or unified chassis control [18], [19].

Most works on vehicle stability present results for a specific vehicle operating at a given condition or, at best, a restricted set of conditions. In this study, alternatively, we present stability regions for varying center of gravity location along the longitudinal axis, road friction, vehicle speed and steering angle. Moreover, for control purposes, it is often necessary to obtain stability regions that are valid for a
set of conditions. Thus, we present conservative sets which remain true for a range of values. Here, we use the term “conservative” because the regions or sets simultaneously respect several parameters and conditions.

Furthermore, the existing literature uses several simplifications to obtain the linearized system that gives rise to the characteristic equation. These are related to slip angle, steering angle, and track width. They are valid under typical conditions, such as low steering angles at high speeds, but may not hold for more drastic maneuvers which may arise due to emergencies or driver misconduct.

We apply the lateral stability region results to the safety-critical control problem presented by [6], [7], where it is important that the safe set is conservative with respect to system stability and controllability. They present an electronic stability control (ESC) system which is restricted to lateral stability, with active steering (AS) as the control input method. Active steering consists of altering the steering angle that is expected from the driver’s steering wheel input to achieve vehicle stability. Differential braking is the more commonly used input method, with the yaw rate as the controlled variable. This latter technique is known as yaw stability control (YSC) [20], [21], [22]. Alternatively, steerable front wheels, or active front steering (AFS), are used in [6] and here.

This work aims to present a flexible, straightforward method for obtaining lateral stability regions for wheeled vehicles. Vehicle dimensions, center of gravity location, steering system type, tire parameters, friction coefficient and longitudinal speed can all be adjusted. A comprehensive set of conditions. Thus, we present conservative sets which remain true for a range of values. Here, we use the term “conservative” because the regions or sets simultaneously respect several parameters and conditions.

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Furthermore, we have identified limitations in the control method used in [6]. The problems are related to having a single control input (hence a single input direction) to steer the system state vector away from barriers over a 2D space. As a consequence, the state may either reach regions where the control input cannot satisfy conflicting safety restrictions, or where the restriction is invariant with respect to the control signal. (See Section IV.) One solution is given in [7], i.e., to use four-wheel steering (4WS), whereas here we present other control barrier functions to overcome this limitation.

The vehicle model and its linearization are presented in Section II. The procedure to obtain stability regions is presented in Section III, together with the regions themselves. The safety-critical control formulation and its extension follow in Section IV. Next, Section IV-A presents a numeric implementation, and discusses the results. Finally, Section V presents the conclusions and future works.

II. VEHICLE MODEL

We consider a four-wheel vehicle moving over a plane, as shown in Fig. 1. The wheelbase is given by \( l \), which is the sum of the distances between the center of gravity of the vehicle and front and rear wheels, \( l_1 \) and \( l_2 \), in the respective order. The track width is \( l_{w} = 2l_{w} \). The velocity of the vehicle is \( v_{x} \), which is composed of \( v_{x} \) and \( v_{y} \), whereas its side-slip angle is \( \beta \) and yaw rate \( \dot{\psi} \). The steering angles of the left and right wheels are, in the order, \( \delta_{l} \) and \( \delta_{r} \); and the velocity and slip angles of the wheels are noted as \( v_{i} \) and \( \alpha_{i} \), in the respective order, with \( i = \{fl, fr, rl, rr\} \) for the front left, front right, rear left and rear right wheels, respectively.

The lateral model can be obtained by considering a constant longitudinal speed \( v_{gx} \) and negligible longitudinal tire forces, or

\[
\dot{v}_{gy} = -v_{gx}\dot{\psi} + \frac{F_{yf}y_{f}}{m_{p}} + \frac{F_{yr}y_{r}}{m_{p}} + \frac{\cos(\delta_{r})F_{yf}}{m_{p}} + \frac{\cos(\delta_{l})F_{yr}}{m_{p}}, \tag{1a}
\]

\[
\dot{\psi} = -\frac{l_{z}}{I_{z}}(F_{yr} + F_{yf}) + \left(\frac{l_{z}}{I_{z}}\cos(\delta_{l}) + \frac{l_{w}}{I_{z}}\sin(\delta_{l})\right)F_{yf} + \left(\frac{l_{z}}{I_{z}}\cos(\delta_{r}) - \frac{l_{w}}{I_{z}}\sin(\delta_{r})\right)F_{yr}, \tag{1b}
\]

where \( m_{p} \) is the vehicle mass, \( I_{z} \) is the yaw moment of inertia, and \( F_{yf}, i = \{fl, fr, rl, rr\}, \) are the lateral forces on the wheels.

The lateral forces have been modeled by the magic formula model, as presented in [2], and summarized in Appendix A. The slip angles are the main arguments of the force equations, being given by

\[
\alpha_{fl} = \delta_{l} - \arctan\left(\frac{v_{gy} + l_{1}\dot{\psi}}{v_{gx} - l_{w}\dot{\psi}}\right), \tag{2}
\]

\[
\alpha_{fr} = \delta_{r} - \arctan\left(\frac{v_{gy} + l_{1}\dot{\psi}}{v_{gx} + l_{w}\dot{\psi}}\right), \tag{3}
\]

\[
\alpha_{rl} = -\arctan\left(\frac{v_{gy} - l_{2}\dot{\psi}}{v_{gx} - l_{w}\dot{\psi}}\right), \tag{4}
\]
\[ \alpha_{rr} = -\arctan \left( \frac{v_{gy} - l_{2} \dot{\psi}}{v_{gx} + I_{w} \dot{\psi}} \right). \] (5)

In [4], [5], and [14], simplifications were made so as to obtain a linearized model, which is the standard approach in the literature, e.g., [2], known as the bicycle model. They are, however, different models. The latter is a linearization around null steering angles, thus, the lateral stiffness is constant and must be obtained at null slip angles. The first one, in turn, has been linearized on the current state vector, where a local lateral stiffness should be considered. Alternatively, in order to assess the impact of such simplifications, we have chosen not to employ any of them, except the ones already used to obtain (1). In other words, sine and tangent trigonometric functions have not been approximated by fractions, the influence of steering angle on the dynamic matrix has not been disregarded, the lateral distance between wheels (track width) has been considered, and \( v_{gx} \) has not been considered to be much greater than \( v_{gy} \).

As mentioned, in addition to the aforementioned considerations, the set of magic formula equations for lateral force, from [2], has been used (as presented in Appendix A), in lieu of the 2D LuGre tire model, as in [4] and [5] or the Fiala model, considered by [14].

As such, (1) can be linearized around the current state \( x_{o} = [v_{gy}(t) \ \dot{\psi}(t)]^{T} \) by choosing the change in \( x_{o} \), i.e., \( x = [\Delta v_{gy} \ \Delta \dot{\psi}]^{T} \) as the state vector (as done in [4] and [14]) and the steering angle as input signal, i.e., \( u = \delta = \delta_{l} = \delta_{r} \). In addition to it, we can write (1) as

\[ \dot{x} = f(x, u) = \left[ \begin{array}{c} f_{1}(x_{o}, u) \\ f_{2}(x_{o}, u) \end{array} \right]. \] (6)

Hence,

\[ \dot{x} \approx Ax + Bu, \] (7)

where

\[ A \triangleq \frac{\partial f}{\partial x_{o}} = \left[ \begin{array}{cc} \frac{\partial f_{1}}{\partial v_{gy}} & \frac{\partial f_{1}}{\partial \dot{\psi}} \\ \frac{\partial f_{2}}{\partial v_{gy}} & \frac{\partial f_{2}}{\partial \dot{\psi}} \end{array} \right], \quad B \triangleq \frac{\partial f}{\partial u}. \] (8)

Most terms of (6) are not explicit functions of either \( v_{gy} \) or \( \dot{\psi} \) (see (1)). Let \( f(x_{o}, u) \) be rewritten as

\[ f(x_{o}, u) = f_{1}(x_{o}, u) + f_{ne}(x_{o}, u) = \left[ \begin{array}{c} -v_{gy} \dot{\psi} \\ 0 \end{array} \right] + \left[ \begin{array}{c} f_{ne1}(x_{o}, u) \\ f_{ne2}(x_{o}, u) \end{array} \right]. \] (9)

Therefore, \( A \) must be obtained by chain rule as

\[ A = \left[ \begin{array}{cc} 0 & -v_{gy} \\ 0 & 0 \end{array} \right] + \frac{\partial f_{ne}}{\partial F_{y}} \frac{\partial F_{y}}{\partial x_{o}}. \] (10)

The second term from the right-hand side of (10) can be obtained from

\[ \frac{\partial f_{ne}}{\partial F_{y}} \triangleq \left[ \begin{array}{cccc} \frac{\partial f_{ne1}}{\partial F_{y}} & \frac{\partial f_{ne1}}{\partial F_{y}} & \frac{\partial f_{ne1}}{\partial F_{y}} & \frac{\partial f_{ne2}}{\partial F_{y}} \\ \frac{\partial f_{ne1}}{\partial F_{y}} & \frac{\partial f_{ne1}}{\partial F_{y}} & \frac{\partial f_{ne1}}{\partial F_{y}} & \frac{\partial f_{ne2}}{\partial F_{y}} \end{array} \right] \] (11)

The derivatives of the tire forces are given by

\[ \frac{\partial F_{y}}{\partial x_{o}} = \left[ \begin{array}{cc} \frac{\partial F_{y1}}{\partial v_{gy}} & \frac{\partial F_{y1}}{\partial \dot{\psi}} \\ \frac{\partial F_{y2}}{\partial v_{gy}} & \frac{\partial F_{y2}}{\partial \dot{\psi}} \end{array} \right]. \] (12)

with \( C_{d1} \equiv (\partial F_{yi}/\partial \alpha) \) being the local lateral stiffness. The aforementioned derivatives are listed in Appendix C.

The characteristic equation of (7) is given by

\[ |\lambda I - A| = \lambda^{2} + a_{1}\lambda + a_{2}, \] (15)

which is Hurwitz if \( a_{1}, a_{2} > 0 \). However, as \( a_{1} > 0, v_{gy} > 0 \), the stability condition can be reduced to \( a_{2} > 0 \). We shall also consider a controllability condition related to avoiding negative local lateral stiffness values, regardless of whether such wheels are steerable or not. The results are presented in Section III.

### III. STABILITY REGIONS

The stability region of moving vehicles depends on both the operating conditions and the vehicle parameters. We present regions for different steering angle values, center of gravity locations alongside the longitudinal axis of the vehicle; friction coefficients between the tires and the pavement; and longitudinal speeds.

The vehicle parameters employed in this study are the default ones from the MATLAB Vehicle Dynamics Blockset (VDB) [23]. A subset of the parameters, yet enough to describe (1), is listed in Table 1. The tire parameters were also the default ones from VDB (See Appendix B).

The effective stability regions have been obtained in three steps. First, the stability condition \( (a_{2} > 0) \) is verified for every pair \((v_{gy}, \dot{\psi})\) on a lattice. The region where this condition holds is the stability region. After, the rear and front wheels’ controllability condition is checked, when we define the controllability of a wheel as it operating on a condition of positive local lateral stiffness, i.e., \( C_{d1} > 0 \). Points where this condition holds for every wheel constitute the controllability region. Finally, the intersection of the aforementioned

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**TABLE 1. Vehicle parameters.**

| Parameter | Value |
|-----------|-------|
| \( m_{b} \) | 1181 kg |
| \( I_{x} \) | 2066 kg·m² |
| \( l_{1} \) | 1.515 m |
| \( l_{2} \) | 1.504 m |
| \( L_{w} \) | 1.922 m |

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regions yields the effective stability region, which considers system stability and controllability. In other words, the region where $A$ is Hurwitz is first obtained. After that, the regions where the lateral force of each wheel is beyond the maximum absolute value are deducted. The remainder is the effective stability region, where the vehicle is still stable and can yet be controlled by the driver, with no oversteering.

It is now appropriate to justify the controllability definition used here. Strictly speaking, only the front local lateral stiffness is necessary to control the vehicle, according to (7) and (8), where $B$ does not depend on the rear local lateral stiffness. However, steering a vehicle under a condition of positive local front lateral stiffness and negative rear lateral stiffness means the vehicle is performing a drifting maneuver. This is unacceptable for a passenger vehicle transiting on a public road. Hence, such a condition will be regarded as unstable, even if it would be more appropriately classified as (very) unsafe otherwise, albeit still controllable by an expert driver.

The stability condition depends on the vehicle speed, dimensions, mass distribution, wheels, and friction coefficient with the ground. This is so because these are either parameters of $A$ or said parameters (namely, the local lateral stiffnesses of the wheels) depend on them. It also depends on road grade, road bank, and throttle or breaking, aspects that have not been covered by the vehicle model employed in this study, i.e., (1).

On the other hand, the controllability condition depends on any aspect that influences the tire force generation. Consequently, it depends on the mass distribution of the vehicle, since it affects the normal force on each wheel. Similarly, on the angle between the plane of the wheels and the ground normal, i.e., on the camber angle. Moreover, it depends on each wheel’s inflating pressure—but we will not present stability regions for varying inflating pressures on any tire. In addition to it, since the roll angle is not being considered in the model, the camber angle will be regarded as null. As previously mentioned, we have disregarded longitudinal accelerations on the wheels. Hence, lateral force equations have been used instead of either a combination of longitudinal and lateral wheel models, or combined slip models.

Previous analysis and control studies have considered lateral stability regions that maintain their shape for different steering angles [4], [5], [6], [7], [14]. This aspect was verified at small steering angles. Nonetheless, emergency maneuvers often call for near-limit steering angles, when the hourglass-like shapes of [4], [11] do not hold. Fig. 2 shows different stability regions for vehicles at 60 km h$^{-1}$ ($v_{gx} = 16.67$ m s$^{-1}$) on a dry asphalt road, with a typical friction coefficient $\mu = 0.8$, where stable regions are colored black.

The controllability condition related to the same case ($v_{gx} = 16.67$ m s$^{-1}$ or 60 km h$^{-1}$) is depicted in Fig. 3, also for steering angle increments of 20$^\circ$. These are the intersection of both front and rear wheels under positive local lateral stiffness. It can be seen that the diamond-like shape of low steering angles is deformed for large steering angles.

The effective stability region, which can be used for control applications and should be maintained when driving the vehicle, is shown in Fig. 4. For lower steering angles, the shape of the region is similar to an hourglass. As the steering angle increases in magnitude, it becomes deformed.

Moreover, the controllability condition was verified for each wheel in this work. We mention that in case the controllability is considered for each axle, some shapes become closer to those presented in [11] for nonzero road grades. We omit such results, since the consideration of an alternative condition would effectively duplicate the number of images shown hereby. Furthermore, the controllability condition can be changed to consider drifting. This is as simple as ignoring the rear wheels in the controllability analysis—which can be done either individually or by axle.

The stability regions in Figs. 2, 3 and 4 consider a constant longitudinal speed of 60 km h$^{-1}$. The longitudinal speed
that the lower the longitudinal speed, the smaller the neighborhood around the origin becomes. Particularly, one can observe changes for nonzero values of $v_{gy}$ around $\dot{\psi} \approx 0$: lower values of $l_1$ are related to valid points for $\dot{\psi} < 0$, whereas larger ones mean the controllability condition only holds for $\dot{\psi} > \epsilon, \epsilon > 0$.

greatly influences vehicle stability. Hence, we studied stability regions from low to high speeds. Particularly, from $v_{gx} \in \{2.778 \text{ m s}^{-1}, 27.78 \text{ m s}^{-1}\} (10 \text{ km h}^{-1}, 100 \text{ km h}^{-1})$. Control applications may deal with varying longitudinal speeds, either due to accelerating, braking, or longitudinal components of the lateral forces from the steering wheels.

The stability condition region for null steering angle and longitudinal speeds from $2.44 \text{ m s}^{-1}$ to $11.11 \text{ m s}^{-1}$ (10 km h$^{-1}$ to 40 km h$^{-1}$) is shown in Fig. 5. It can be seen that the lower the longitudinal speed, the smaller the neighborhood around the origin becomes.

Fig. 6 depicts the stability condition region, also for null steering angle, for longitudinal speeds between $19.44 \text{ m s}^{-1}$ to $27.78 \text{ m s}^{-1}$ (70 km h$^{-1}$ to 100 km h$^{-1}$). The neighborhood around the origin increases with $v_{gx}$, as in Fig. 5.

The controllability regions with respect to longitudinal speed can be seen in Fig. 7. The low-speed regions are shown in Fig. 7a, whereas the high-speed ones are in Fig. 7b. They increase with speed since slip angles are not altered if $v_{gy}$ and $\psi$ are proportional to $v_{gx}$.

The effective stability region pertaining Figs. 5, 6 and 7 is shown in Fig. 8. It is close to the controllability one for low speed values. As the longitudinal speed increases, it becomes closer to an increasingly thinner hourglass/like shape, as it results from the narrower stability region restricted by the larger controllability region.

Other variable which has a noticeable impact on the stability regions is the road/tire friction coefficient. All previous figures have been related to a friction coefficient $\mu = 0.8$, which is typical of dry asphalt. We show the stability regions for lower friction coefficient values in Figs. 9, 10 and 11. The respective regions appear to be proportional to $\mu$, even if we have used nonlinear equations for almost every aspect of the dynamics of the vehicle.

Finally, the effect of longitudinal displacement of the center of gravity on the lateral stability has been verified. This can be parameterized by $l_1$ and $l_2$. We have considered $l_1 = \{0.2l, 0.4l, 0.6l, 0.8l\}$, for constant $l = l_1 + l_2$. The stability condition is depicted in Fig. 12. It can be observed that the stability region around the origin is slightly reduced as the center of gravity approaches the rear axle.

The controllability condition is depicted in Fig. 13. As in Fig. 12, the deformation in the respective region is not pronounced. Particularly, one can observe changes for nonzero values of $v_{gy}$ around $\dot{\psi} \approx 0$: lower values of $l_1$ are related to valid points for $\dot{\psi} < 0$, whereas larger ones mean the controllability condition only holds for $\dot{\psi} > \epsilon, \epsilon > 0$. 

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**FIGURE 4.** Effective stability regions for $v_{gx} = 60 \text{ km h}^{-1}, \mu = 0.8$, and varying $\delta$. Dashed line: $\delta = -\pi/6 \text{ rad} (-30^\circ)$. Dotted line: $\delta = -\pi/18 \text{ rad} (-10^\circ)$. Dash/dot line: $\delta = \pi/18 \text{ rad} (10^\circ)$. Continuous line: $\delta = \pi/6 \text{ rad} (30^\circ)$.

**FIGURE 5.** Stability case for $\delta = 0 \text{ rad}, \mu = 0.8$, and varying $v_{gx}$.
(a) $v_{gx} = 2.778 \text{ m s}^{-1} (10 \text{ km h}^{-1})$ (b) $v_{gx} = 5.556 \text{ m s}^{-1} (20 \text{ km h}^{-1})$
(c) $v_{gx} = 8.333 \text{ m s}^{-1} (30 \text{ km h}^{-1})$ (d) $v_{gx} = 11.11 \text{ m s}^{-1} (40 \text{ km h}^{-1})$.

**FIGURE 6.** Stability case for $\delta = 0 \text{ rad}, \mu = 0.8$, and varying $v_{gx}$.
(a) $v_{gx} = 19.44 \text{ m s}^{-1} (70 \text{ km h}^{-1})$. (b) $v_{gx} = 22.22 \text{ m s}^{-1} (80 \text{ km h}^{-1})$.
(c) $v_{gx} = 25.00 \text{ m s}^{-1} (90 \text{ km h}^{-1})$. (d) $v_{gx} = 27.78 \text{ m s}^{-1} (100 \text{ km h}^{-1})$. 

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For a varying center of gravity, the effective stability region can be seen in Fig. 14. The respective deformations are a result of those mentioned for Fig. 12 and Fig. 13. They are less pronounced than the ones observed in Figs. 4, 8 or 11.

### A. CONSERVATIVE REGIONS

Some control techniques make use of operation regions, which are to be respected for either performance, robustness, or safety reasons. For instance, model predictive control (MPC) considers state and control regions which must be respected during the system operation [24], [25], [26], [27]. Here, we consider a safety-critical control application. Safety-critical control, in terms of CBFs, has been tackled in [28], [29]; we consider the extension presented and implemented in [6], [7].

Conservative stability regions for steering angle amplitudes of $\pi/18$ rad, $\pi/9$ rad, and $\pi/6$ rad ($10^\circ$, $20^\circ$, and $30^\circ$) in [6] and [7], in the form of safe sets. However, if either large steering angles are necessary, if the road-tire friction coefficient changes, or if the vehicle experiences center of gravity changes, the time-varying aspect of the safe set must be taken into account. On the other hand, there are no closed-form solutions for the boundary of such sets, which are used to obtain CBFs in safety-critical control. Hence, the alternative is to consider smaller sets, which are valid over the domain in which the parameters of the system vary. We designate them conservative stability regions.

We present conservative regions for relatively small and large steering angle changes, longitudinal speed changes, and road friction changes. More than one variable parameter can be taken into account, naturally. We omit such cases for the sake of brevity. Conservative stability regions for steering angle amplitudes of $\pi/18$ rad, $\pi/9$ rad, and $\pi/6$ rad ($10^\circ$, $20^\circ$, and $30^\circ$)
FIGURE 9. Stability case for $v_{gx} = 60 \text{ km h}^{-1}$, $\delta = 0 \text{ rad}$, and varying $\mu$. (a) $\mu = 0.2$. (b) $\mu = 0.4$. (c) $\mu = 0.6$. (d) $\mu = 0.8$.

FIGURE 10. Controllability case for $v_{gx} = 60 \text{ km h}^{-1}$, $\delta = 0 \text{ rad}$, and varying $\mu$. Dashed line: $\mu = 0.2$. Dotted line: $\mu = 0.4$. Dash/dot line: $\mu = 0.6$. Continuous line: $\mu = 0.8$.

are shown in Fig. 15, where we consider the shifting vector presented in [5] in order to account for the stability region translations over the $v_{gy} \times \dot{\psi}$ plane with respect to $\delta$, given by

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} v_{gy} \delta \\ v_{gx} \delta \end{bmatrix},$$

and defined as the state vector change $s$ such that the wheel slip angles for $x$ and $x - s$ are the same for $\delta = 0$ and nonzero, in the respective order. The effective stability region reduces as the maximum steering angle increases. The higher the steering angle, the more drastic the reduction becomes in the first and third quadrants of the $(v_{gy} - s_1) \times (\dot{\psi} - s_2)$ plane.

For varying $v_{gx}$ and $\mu$, the conservative regions will be those of the minimum value of either variable—reason why we shall refrain from showing them again. This is clear from Fig. 8 and 11, in the respective order: the stability regions increase as the longitudinal speed and road-tire friction coefficient increase. It is important, however, that the idea of larger stability areas being related to an easier safe
or robust control system is not necessarily true. Whereas this does hold for higher friction coefficients, it does not so for higher longitudinal speeds. In particular, the larger area in the direction of the $0v_{gy}$ axis is related to the same side-slip angle, in case $v_{gy}$ is proportional to $v_{gx}$. Similarly, points where $\psi$ is proportional to $v_{gx}$ are related to the same Ackermann radius. However, the necessary lateral force increases quadratically with $v_{gx}$. Hence, as is intuitive, it is harder to keep a vehicle stable at higher longitudinal speeds. From the perspective of the pertaining dynamic system, we recall that both $v_{gx}$ and $\mu$ are parameters of $A$ and $B$. However, $\mu$ is a factor of the wheels’ stiffnesses, which appears in the numerator of several terms in those matrices. On the other hand, $v_{gx}$ appears in denominators, so it is natural that the system stability is affected in a different way.

The influence of longitudinal center of gravity shifts is displayed in Fig. 16, for $l_f$ ranging from 2% to 98% of the wheelbase $l$. The lateral stability condition, in this case, is dictated by situations where the center of gravity approach the front axle (see Fig. 12). The controllability condition, however, is most restrictive when it is closer to the rear axle, as seen in Fig. 13. As a result, Fig. 16 combines the effects of both previously mentioned extreme locations of the center of gravity along $l$.

IV. CONTROL APPLICATION

The conservative regions presented in Section III-A can be used to design feedback systems that prevent the vehicle from losing lateral stability. In particular, they can be used to restrict the system states with MPC or safety-critical control, by choosing stable sets as constraints. However, the stability regions depend not only on the longitudinal speed and vehicle parameters, but on the driver’s inputs.

The safety-critical control strategy presented in [6] introduces the control-dependent barrier functions (CDBFs), which, as the name implies, are CBFs which depend not only on the state, but also on the control input.

Considering the system [6]

$$\dot{x} = f(x, u),$$  \hspace{1cm} (17)

where $x \in \mathcal{D} \subseteq \mathbb{R}^n$ is the state, $u \in \mathcal{U} \subseteq \mathbb{R}^m$ is the differentiable input, with $\dot{u} = \omega$ and $\omega \in \mathcal{O}$. Let the extended system be defined as

$$\dot{x} = \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} f(x, u) \\ \omega \end{bmatrix} = \hat{f}(\dot{x}, \omega),$$  \hspace{1cm} (18)

where $\dot{x} = [x \ u]$. We consider a set $\mathcal{C}(x, u)$ defined by a continuous differentiable function $h(x, u)$ as [6]

$$h(x, u) \geq 0, \ \forall x \in \mathcal{C}(x, u),$$  \hspace{1cm} (19)

$$h(x, u) = 0, \ \forall x \in \partial \mathcal{C}(x, u),$$  \hspace{1cm} (20)

$$h(x, u) > 0, \ \forall x \in \text{Int}(\mathcal{C}(x, u)).$$  \hspace{1cm} (21)

If there exist a control $u \in \mathcal{U}$, $\omega \in \mathcal{O}$ and an extended class $\kappa$ function $\zeta(h(x, u))$, such that $h(x, u)$ satisfies [6]

$$L_j h(x, u) + \zeta(x, u) \geq 0,$$  \hspace{1cm} (22)
with
\[ L_j h(x, u) = \frac{\partial h(x, u)}{\partial x} f(x, u) + \frac{\partial h(x, u)}{\partial u} \dot{u}, \]
then \( C(x, u) \) is a control/dependent invariant set and \( h(x, u) \) is a CDBF.

**Remark 1:** A continuous function \( \alpha_K : [0, a) \to [0, \infty) \)
for some \( a > 0 \) is said to belong to class \( K \) if it is strictly increasing and \( \alpha_K(0) = 0 \) [28].

As described in [6], the control input can be obtained by a quadratic programming (QP) problem expressed as
\[ u^* = \arg \min_{u \in \mathbb{R}^m} \frac{1}{2} u^T \underline{H} u + \underline{F}^T \dot{u}, \]
subject to
\[ \frac{\partial h(x, u)}{\partial x} f(x, u) + \frac{\partial h(x, u)}{\partial u} \dot{u} + \zeta(h(x, u)) \geq 0, \]
where \( H \in \mathbb{R}^{m \times m} \) and \( \underline{F} \in \mathbb{R}^m \).

Control/dependent barrier functions, as defined here, do not explicitly deal with a stability or tracking objective. However, it is desirable that when the driver completes a lane-change maneuver, he or she can return the steering wheel to a near-zero angle, and have the vehicle moving in a straight line afterward. The works [28], [29] propose a control structure that integrates stability/tracking objectives, expressed as a CLF, and safety constraints, expressed as a CBF, through QP. Here, we extend this methodology, however, expressing the stability/tracking objectives as a control/dependent Lyapunov function (CDLF) \( V(x, u) \), and the safety constraints as a CDBF \( h(x, u) \). Thus, the QP (24) is adapted and we can enunciate the safety-critical control problem as the CDLF-CDBF-QP problem
\[ u^* = \arg \min_{u_j \in \mathbb{R}^{m+1}} \frac{1}{2} u_j^T \underline{H} u_j + \underline{F}^T \dot{u}_j, \]
subject to
\[ \frac{\partial V(x, u)}{\partial x} f(x, u) + \frac{\partial V(x, u)}{\partial u} \dot{u} + \gamma(V(x, u)) - \delta_R \leq 0, \]
\[ \frac{\partial h(x, u)}{\partial x} f(x, u) + \frac{\partial h(x, u)}{\partial u} \dot{u} + \zeta(h(x, u)) \geq 0, \]
where \( u_j = [u \ \delta_R] \) and \( \delta_R \) is the relaxation parameter used to make the stability/tracking objectives as a soft constraint and the safety as a hard constraint.

The aforementioned tracking objective of AFS is to not interfere with the driver’s commands whenever the safety constraint is being satisfied. This is achieved with the quadratic CDLF
\[ V(x, u) = (u - u_r)^2, \]
where \( u_r \) is the reference value of \( u \).

For the active front steering lateral stability problem, the input signal is \( u = \delta_u \), added to the steering angle from the driver \( \delta_d \), so that
\[ \delta = \delta_d + \delta_u. \]
not having a safe set boundary parallel to \( s \). Fig. 20 depicts a possible solution.

The CDBF, which defines the safe set from Fig. 20, can be systematically obtained from any effective stability region. First, a parallelogram is obtained, such as Fig. 19a. An ellipse-shaped set is then inscribed in it. This can be done by means of linear transformations of the original parallelogram from Fig. 19a until it becomes a square (see Fig. 21a). Then an inscribed circle is transformed into the angled ellipse accordingly, as shown in Fig. 21b. The detailed procedure is presented in Appendix D.

The CDBF can be obtained as parabolidals that establish the safe set from Figs. 20 and 21b. The safe set is given by

\[
b(x) = -p_1 v_{gy}^2 - p_2 v_{gy} \dot{\psi} - p_3 \dot{\psi}^2 + 1,
\]

with \( b(x) = 0 \), where \( p_1, p_2 \), and \( p_3 \) are the coefficients of the quadratic polynomial. Hence, the CDBF is obtained by using the shifting vector \( s \) as

\[
h(x, u) = -p_1 (v_{gy} - s_1)^2 - p_2 (v_{gy} - s_1)(\dot{\psi} - s_2) - p_3 (\dot{\psi} - s_2)^2 + 1.
\]

Therefore, the CDLF-CDBF-QP problem for the AFS problem can be expressed, in standard form, as

\[
\mu^* = \arg \min_{\mu \in \mathbb{R}^{m+1}} \frac{1}{2} \mu^T H \mu + F^T \mu_f,
\]

subject to \( A_{qp} \mu_f \leq b_{qp} \),

where

\[
A_{qp} \triangleq \begin{bmatrix} \frac{\partial V(x, u)}{\partial x} & -1 \\ \frac{\partial V(x, u)}{\partial u} & 0 \end{bmatrix},
\]

\[
b_{qp} \triangleq \begin{bmatrix} \frac{\partial V(x, u)}{\partial x} f(x, u) - \gamma(V(x, u)) \end{bmatrix} + \zeta(h(x, u)).
\]
The use of conservative stability regions in safety-critical control is hereby presented, as formulated in Section IV. Table 2 presents the controller parameters.

The vehicle described by (1) and Table 1 was simulated to respond to a J-turn maneuver, with driver input depicted in Fig. 22. The standard/form CDLF-CDBF-QP problem was solved with the algorithm presented in [30], adapted from the implementation from [31]. The controller considers the conservative stability region for a maximum steering angle of $\pi/6$ rad ($30^\circ$), shown in Fig. 15.

The control-dependent barrier function describes the safety during the system operation. More specifically, safety is verified if the CDBF does not assume negative values over time, as (19), (20), and (21) describe. The CDBF is shown in Fig. 23, where we can observe that the barrier function reaches zero as the maneuver reaches the maximum steering angle. The CDBF assumes positive values otherwise.

The restrictions are more clearly observed in the $(v_{gy} - s_1) \times (\dot{\psi} - s_2)$ plane. The system state and the safe set are both functions of time. As a result, they move over the $v_{gy} \times \dot{\psi}$ plane. By using the shifting vector, the restrictions are fixed and centered at the origin of the plane, while only the shifted state moves over it. Fig. 24 shows that the shifted state vector remains inside the safe set for the entirety of the J-turn maneuver.

The quadratic problem has been extended to the CDLF-CDBF-QP problem in order to have the added steering angle vanish in steady-state. The CDLF restrictions, however, are only considered in safety-critical control when the CDBF

| Parameter | Value |
|-----------|-------|
| $p_a$     | 0.1   |
| $p_b$     | 200   |
| $c_V$     | 50    |
| $c_h$     | 100   |

### Table 2. Controller parameters.
restrictions are satisfied. The control input, obtained by integrating $\dot{u}$, is depicted in Fig. 25. The convergence to zero is slow enough to allow the human driver to react and compensate for it.

Finally, Fig. 26 presents the trajectory of the vehicle. The closed-loop trajectory is compared to the one that results from the same longitudinal speed $v_{gx}$ and the desired yaw rate $\dot{\psi}_d$, given by

$$\dot{\psi}_d = \min \left\{ \left| \frac{v_{gx} \delta}{l + k_{sl} v_{gx}^2} \right|, \left| \frac{\mu g}{v_{gx}} \right| \right\} \text{sign}(\delta), \quad (38)$$

where $g$ is the acceleration of gravity and

$$k_{sl} \triangleq \frac{m_g (l_C \alpha_f 0 - l_C \alpha_r 0)}{l C_{\alpha_f} C_{\alpha_r}} \quad (39)$$

is the understeer coefficient or gradient, with

$$C_{\alpha_f} 0 = \left. \frac{\partial F_{yf}}{\partial \alpha_{f0}} \right|_{\alpha_{f0}=0} + \left. \frac{\partial F_{yr}}{\partial \alpha_{r0}} \right|_{\alpha_{r0}=0},$$

$$C_{\alpha_r} 0 = \left. \frac{\partial F_{yr}}{\partial \alpha_{r0}} \right|_{\alpha_{r0}=0} + \left. \frac{\partial F_{yf}}{\partial \alpha_{f0}} \right|_{\alpha_{f0}=0}.$$

The result is surprising since nowhere in the text have we explicitly considered $\dot{\psi}_d$. Hence, one should expect drift error in steady-state, whereas YSC techniques often use $\dot{\psi}_d$ as a set point for $\dot{\psi}$.

The safety-critical control strategy differs from the commonly used discrete activation conditions based on $\beta$ and $\dot{\psi}$ in terms of safety in nominal conditions, activation frequency, stability recovery time, and robustness. Safety-critical control is guaranteed to satisfy the safety restrictions, viz., maintain stability, for the nominal system. Hence no stability recovery time will be necessary under these circumstances, whereas methods that demand activation have to take action to restore vehicle control to the driver. On the other hand, these methods are more robust to unexpected parameter variations, such as sudden friction coefficient reductions, as they inherently depend on a degree of loss of vehicle control to act. Methods based on optimization, e.g., safety-critical control and MPC, demand recovery measures in case the optimization restrictions are momentarily not satisfied due to such events.
V. CONCLUSION
A method of obtaining lateral stability regions for road vehicles that accounts for nonlinear aspects has been presented. The stability regions are obtained based on both dynamical stability and non-slip conditions, which are evaluated throughout a $y_{DS} \times \psi$ lattice which contains the lateral model equilibrium point. The influence of the steering angle has been assessed for both small and large values, when commonly used simplifications are not valid. With this method, stability regions for a variety of lateral model parameters have been presented.

Conservative stability regions over a range of lateral model parameter values have been presented. Their application in automatic control has been validated through a safety-critical control system for AFS. A minor extension to the CDBF formulation was introduced, for further driver convenience. In addition to it, a method of obtaining viable control barrier functions for arbitrary stability regions was presented, avoiding limitations from having steering wheels on a single axle. Simulation results validated the proposal.

Active front steering operates by avoiding loss of vehicle control, which it accomplishes by compensating for aggressive maneuvers. However, while such compensations keep wheels in safe local lateral stiffness regions, they are unable to, at the same time, track an arbitrary yaw rate. Moreover, AFS is unable to deal with harsh maneuvers at higher speeds or severe understeering. Indeed, it is well known that differential braking outperforms AFS in ESC applications. Hence, we wish to apply conservative stability regions and safety-critical control to differential braking ESC systems. Since turning and braking are complex vehicle aspects to model, higher-order models will be more adequate to validate the resulting system. Thus, either more sophisticated simulation software or experimental results shall be pursued.

We have assumed a human driver during the development of this study. Hence the stability region proposal and the control formulation and application can be either extended or adjusted to account for professional drivers, automated vehicles, or other control objectives. In particular, the controllability criterion depends on the actuation and degree of conservativeness of the closed-loop system. The control application, in turn, was made as simple as possible to showcase the application of the stability region proposal and the improvement of the available control literature. Several aspects can be accounted for, such as road grade and bank, lateral shifts of center gravity, optimality or robustness extensions, and stability recovery systems.

**APPENDIX A**

**MAGIC FORMULA MODEL**
The magic formula equation set which was used in this study is presented in [2], Chapter 4, for a $z$-down SAE J670 coordinate system [32]. For a ISO 8855 coordinate system [33], it is given by:

$$F_{xy} = D_y \sin \left[ C_y \arctan(\phi_1) \right] + S_{Vy}$$  \hspace{1cm} (40)

**APPENDIX B**

**TIRE PARAMETERS**
The pure side slip magic formula parameters are listed in Table 3. The scaling factors $\lambda$ have not been used, except $\lambda'_{\mu_y}$. Thus, they have a default value of 1. Similarly, the $\zeta$ parameters have all been set to 1, i.e., we have disregarded camber and turn slip.

| Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|
| $p_{Cy1}$ | 1.343000 | $p_{Ky6}$ | -0.88405 |
| $p_{Dy1}$ | 0.878268 | $p_{Ky7}$ | -0.237260 |
| $p_{Dy2}$ | -0.064460 | $p_{Hy1}$ | -0.001834 |
| $p_{Dy3}$ | 0 | $p_{Hy2}$ | 0.003464 |
| $p_{Ey1}$ | -0.809776 | $p_{Vy1}$ | -0.006754 |
| $p_{Ey2}$ | -0.600181 | $p_{Vy2}$ | 0.036379 |
| $p_{Ey3}$ | 0.099173 | $p_{Vy3}$ | -0.163544 |
| $p_{Ey4}$ | -6.575797 | $p_{Vy4}$ | -0.491003 |
| $p_{Ey5}$ | 0 | $p_{pp1}$ | -0.620596 |
| $p_{K_y1}$ | -15.57174 | $p_{pp2}$ | -0.064782 |
| $p_{K_y2}$ | 1.731265 | $p_{pp3}$ | -0.164649 |
| $p_{K_y3}$ | 0.365350 | $p_{pp4}$ | 0.283194 |
| $p_{K_y4}$ | 1.981768 | $p_{pp5}$ | 0 |
| $p_{K_y5}$ | 0 | $W_0$ | 4000 |

$$\phi_1 = B_y \alpha_y - E_y (B_y \alpha_y - \arctan(B_y \alpha_y))$$  \hspace{1cm} (41)

$$\alpha_y = \alpha^* + S_{Hy}$$  \hspace{1cm} (42)

$$\alpha^* = -\tan(\alpha) \text{sign}(v_x)$$  \hspace{1cm} (43)

$$C_y = p_{Cy1} \lambda C_y$$  \hspace{1cm} (44)

$$D_y = -\mu_y W_2$$  \hspace{1cm} (45)

$$\mu_y = (p_{Dy1} + p_{Dy2} \gamma^*) (1 + p_{Dy3} \delta_i + p_{Dy4} \gamma^*) \times (1 - p_{Dy3} \gamma^*) \lambda_{\mu y}$$  \hspace{1cm} (46)

$$E_y = (p_{Ey1} + p_{Ey2} \gamma^*) \times [1 + p_{Ey3} \gamma^*]$$  \hspace{1cm} (47)

$$B_y = \frac{K_y \alpha}{C_y D_y + \epsilon_y}$$  \hspace{1cm} (48)

$$K_y \alpha = p_{K_y1} W_0 (1 + p_{Hy1} (1 - p_{Kh3} \gamma^*)) \sin \left[ p_{K_y4} \arctan(\phi_2) \right]$$  \hspace{1cm} (49)

$$\lambda_2 = \frac{W/W_0}{(p_{K_y2} + p_{K_y4} \gamma^*) (1 + p_{Hy2} \delta_i)}$$  \hspace{1cm} (50)

$$S_{hy} = (p_{Hy1} + p_{Hy2} \gamma^*) (1 - p_{Hy3} \gamma^*) \lambda_{hy} + K_{hy0} \gamma^* - S_{VVY} \zeta_0 + \epsilon_K + \zeta_4 - 1$$  \hspace{1cm} (51)

$$S_{VVY} = W (p_{VV1} + p_{VV2} \gamma^*) \lambda_{VV} \lambda_{ee} \zeta_5 + S_{VVY}$$  \hspace{1cm} (52)

$$K_{yy0} = W (p_{Ky6} + p_{Ky7} \gamma^*) (1 + p_{Ky3} \delta_i)$$  \hspace{1cm} (53)

$$K_{yy} = C_{Fy}$$  \hspace{1cm} (54)
APPENDIX C
LINEARIZED SYSTEM
The tire slip angle derivatives with respect to \( v_{gy} \) and \( \psi \) are given by

\[
\frac{\partial \alpha_i}{\partial v_{gy}} = -\frac{1}{(v_{gy} - l_{w1})^2} \left[ \frac{v_{gy} + l_1 \dot{\psi}}{v_{gy} - l_{w1} \dot{\psi}} \right]^2 + 1,
\]

(55)

\[
\frac{\partial \alpha_i}{\partial \psi} = \frac{1}{(v_{gy} - l_{w1})^2} \left[ \frac{v_{gy} + l_1 \dot{\psi}}{v_{gy} - l_{w1} \dot{\psi}} \right]^2 + 1,
\]

(56)

\[
\frac{\partial \alpha_i}{\partial v_{gy}} = -\frac{1}{(v_{gy} - l_{w2})^2} \left[ \frac{v_{gy} - l_2 \dot{\psi}}{v_{gy} - l_{w2} \dot{\psi}} \right]^2 + 1.
\]

(57)

\[
\frac{\partial \alpha_i}{\partial \psi} = \frac{1}{(v_{gy} - l_{w2})^2} \left[ \frac{v_{gy} - l_2 \dot{\psi}}{v_{gy} - l_{w2} \dot{\psi}} \right]^2 + 1.
\]

(58)

where \( L_{w1} = 2l_{w1} \) is the front track width and \( L_{w2} = 2l_{w2} \) is the rear track width.

The tire force derivatives with respect to slip angles have been obtained by derivating (40) with respect to \( \alpha \), under nominal tire inflating pressure, null camber angle, and steady-state normal force for each tire. As a consequence, the local lateral stiffnesses are expressed by

\[
C_{\alpha i} = C_y D_y \cos \left[ C_y \arctan(\tilde{\phi}) \right] \tilde{\psi},
\]

(63)

with

\[
\tilde{\phi} = B_i \alpha + E_i (\arctan(B_i \alpha) - B_i \alpha),
\]

(64)

\[
\tilde{\psi} = B_y - E_y \left( B_y - \frac{B_y}{B_i^2 \alpha^2 + 1} \right),
\]

(65)

where \( B_i, C_i, D_y, \) and \( E_y \) are obtained for each wheel, according to Appendix A.

APPENDIX D
CONTROL BARRIER FUNCTION PROCEDURE
The elliptic paraboloid \( b \) from Section IV is obtained from chained linear transformations of a circular paraboloid opening to the bottom (i.e., it has a maximum point). The relation between the paraboloid and the safe set is that the latter is defined by \( b = 0 \), if \( u = 0 \) (see Section IV).

Let \( b_i \), \( i \in \{1, 2, 3, 4\} \) define the sides of a parallelogram centered at the origin of a 2D space as in Fig. 27a. Let the objective be to transform this parallelogram into an upright square by means of linear transformations. This can be achieved in three steps, using base changes.

First, let the vertices of the parallelogram be \( P_1, P_2, P_3, P_4 \), the intersections of \( b_1 \) and \( b_2, b_2 \) and \( b_3, b_3 \) and \( b_4 \), and \( b_1 \) and \( b_4 \), in the respective order. Then the change-of-base matrix

\[
P_1 \triangleq [v_2 \ v_1]^{-1}, \quad v_1 = P_1, \ v_2 = P_2
\]

(66)

turns the parallelogram into a rhombus, with \( v_1 \) on the ordinate axis and \( v_2 \) on the abscissa. The result of the change of base from (66) is shown in Fig. 27b. It is then straightforward that

\[
P_2 \triangleq \text{diag} \left\{ \frac{v_1}{v_2}, 1 \right\}
\]

(67)

transforms the rhombus into a 45°-inclined square. Finally, a 45° rotation matrix

\[
P_3 \triangleq \begin{bmatrix}
\cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\
-\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4})
\end{bmatrix}
\]

(68)
completes the procedure. The three transformations are depicted in Fig. 21a.

We now inscribe a circle in the upright square, described by
\[ x^2 + y^2 - a^2 = 0, \]  
(69)
where \( a \) is the radius of the circle, i.e., half the length of the side of the square. We can express (69) in matrix form as
\[
\begin{bmatrix}
P_3
\end{bmatrix}
\begin{bmatrix}
[1]
\frac{1}{a^2}
0
0
\end{bmatrix}
\begin{bmatrix}
P_3
\end{bmatrix}
- 1 = 0,  
(70)
\]
Then the circle is transformed into circles or ellipses inscribed in the inclined square, rhombus, and parallelogram, respectively, by
\[
\begin{bmatrix}
P_3
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
P_3
\end{bmatrix}
\begin{bmatrix}
[1]
\frac{1}{a^2}
0
0
\end{bmatrix}
\begin{bmatrix}
P_3
\end{bmatrix}
- 1 = 0,  
(71)
\]
\[
\begin{bmatrix}
P_3
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
P_3
\end{bmatrix}
\begin{bmatrix}
[1]
\frac{1}{a^2}
0
0
\end{bmatrix}
\begin{bmatrix}
P_3
\end{bmatrix}
- 1 = 0,  
(72)
\]
and
\[
\begin{bmatrix}
P_3
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
P_3
\end{bmatrix}
\begin{bmatrix}
[1]
\frac{1}{a^2}
0
0
\end{bmatrix}
\begin{bmatrix}
P_3
\end{bmatrix}
- 1 = 0,  
(73)
\]
as seen in Fig. 21b. Finally, we obtain \( h(x, u) \) as
\[
h(x, u) = \begin{bmatrix}
P_3
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
P_3
\end{bmatrix}
\begin{bmatrix}
[1]
\frac{1}{a^2}
0
0
\end{bmatrix}
\begin{bmatrix}
P_3
\end{bmatrix}
- 1 = 0.  
(74)
\]

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