Weak containment by restrictions of induced representations

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Let $\pi$ and $\sigma$ be representations of a locally compact group $G$.

Three notions for what it means for $\pi$ to contain $\sigma$:

- $\sigma$ is unitarily equivalent to a subrepresentation of $\pi$
- $\sigma$ is quasi-contained in $\pi$
- $\sigma$ is weakly contained in $\pi$
Quasi-containment

\[ \pi, \sigma - \text{representations of } G \]
\[ \text{VN}_\pi := \pi(G)'' \subset B(\mathcal{H}_\pi) \]

**Definition**

\( \sigma \) is *quasi-contained* in \( \pi \) if \( \sigma \) is unitarily equivalent to subrepresentation of some amplification of \( \pi \).

**Theorem**

\( \pi \) quasi-contains \( \sigma \) iff the identity map on \( G \) extends to a normal \( * \)-homomorphism \( \text{VN}_\pi \to \text{VN}_\sigma \).
Weak containment

\( \pi, \sigma \) – representations of \( G \)
\[ \pi_{\xi,\eta}: G \to \mathbb{C} \text{ defined by } \pi_{\xi,\eta}(s) = \langle \pi(s)\xi, \eta \rangle \text{ for } \xi, \eta \in \mathcal{H}_\pi \]
\( C^*_\pi := \pi(L^1(G)) \parallel \cdot \parallel \)

**Definition**

\( \sigma \) is *weakly contained* in \( \pi \) (write \( \sigma \prec \pi \)) if for every \( \xi \in \mathcal{H}_\sigma \), \( \sigma_{\xi,\xi} \) is the limit of positive definite functions of the form \( \sum_{i=1}^{N} \pi_{\eta_i,\eta_i} \) in the topology of uniform convergence on compact subsets of \( G \).

**Theorem**

\( \pi \) weakly contains \( \sigma \) if and only if the identity map on \( L^1(G) \) extends to *-homomorphism \( C^*_\pi \to C^*_\sigma \).
Main problem

Let $H$ be a closed subgroup of a locally compact group $G$ and $\pi$ a representation of $H$.

When is $\pi$ “contained” in $(\text{Ind}_{H}^{G}\pi)|_{H}$?
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Let $H$ be a closed subgroup of a locally compact group $G$ and $\pi$ a representation of $H$.

When is $\pi$ “contained” in $(\text{Ind}_H^G \pi)|_H$?

Easy exercise

If $G$ be a discrete group, then $\pi$ is unitarily equivalent to a subrepresentation of $(\text{Ind}_H^G \pi)|_H$. 
Aside: Classes of Locally Compact Groups

\[ \tau : G \rightarrow B(L^1(G)) \] defined by \( \tau(s)f(t) = f(s^{-1}ts)\Delta(s) \)

**Definition**

A locally compact group \( G \) is a \textit{SIN} group if the identity of \( G \) admits a neighbourhood base consisting of conjugation invariant compact sets \( K \), i.e., sets \( K \) such that \( s^{-1}Ks = K \) for all \( s \in G \).

**Example**

- Abelian groups,
- Discrete groups,
- Compact groups

**Theorem (Mosak)**

A locally compact group \( G \) is SIN if and only if \( L^1(G) \) has a \textit{central BAI}, i.e., a BAI \( \{ e_{\alpha} \} \subset L^1(G) \) such that \( \tau(s)e_{\alpha} = e_{\alpha} \) for all \( s \in G \).
Aside: Classes of Locally Compact Groups

**Definition**

A locally compact group $G$ is QSIN if $L^1(G)$ has a *quasi-central* BAI, i.e., a BAI $\{e_\alpha\} \subset L^1(G)$ such that $\|\tau(s)e_\alpha - e_\alpha\| \to 0$ uniformly on compact subsets of $G$.

**Theorem (Losert-Rindler)**

Every amenable group is QSIN.
Question

When does \((\text{Ind}^G_H \pi)|_H\) contain \(\pi\)?

Theorem (Cowling-Rodway)

Let \(G\) be a SIN group. Then \((\text{Ind}^G_H \pi)|_H\) quasi-contains \(\pi\) for every closed subgroup \(H\) of \(G\) and representation \(\pi\) of \(H\).

Example (Khalil)

The above result fails for \(G = \mathbb{R} \ltimes \mathbb{R}^+\) be the \(ax + b\) group and \(H\) be the subgroup \(\mathbb{R}\).
Question
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Main result

Theorem (W.)

Let $G$ be a QSIN group. Then $\pi \preceq (\text{Ind}^G_H \pi)|_H$ for every closed subgroup $H \leq G$ and representation $\pi$ of $H$. 

Example (Bekka)

The above result fails for $G = \text{SL}(2, \mathbb{R})$ and $H = \text{SL}(2, \mathbb{Z})$. 
Main result

**Theorem (W.)**

Let $G$ be a QSIN group. Then $\pi \prec (\text{Ind}^G_H \pi)|_H$ for every closed subgroup $H \leq G$ and representation $\pi$ of $H$.

**Example (Bekka)**

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Completely Positive Maps

**Definition**

Let $A$ and $B$ be C*-algebras. A linear map $\phi : A \to B$ is *completely positive* if

\[
[a_{ij}] \in M_n(A) \text{ is positive} \Rightarrow [\phi(a_{ij})] \in M_n(B) \text{ is positive.}
\]
Nuclear C*-algebras

Definition

A C*-algebra $A$ is _nuclear_ if $A \otimes_{\text{min}} B = A \otimes_{\text{max}} B$ for every C*-algebra $B$.

Definition

A C*-algebra $A$ has the _completely positive approximation property (CPAP)_ if there exist ccp maps $\varphi_i : A \to M_{n_i}(\mathbb{C})$ and $\psi_i : M_{n_i}(\mathbb{C}) \to A$ such that

$$\|\psi_i \circ \varphi_i(a) - a\| \to 0$$

for every $a \in A$. 

Theorem (Kirchberg)

A C*-algebra $A$ is nuclear iff it has the CPAP.
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**Theorem (Kirchberg)**

A C*-algebra $A$ is nuclear iff it has the CPAP.
Nuclearity of Group C*-algebras

\[ C^*_r(G) := C^*_\lambda = \overline{\lambda(L^1(G))} \| \cdot \| \]
\[ C^*(G) := C^*_{\pi_u}, \text{ where } \pi_u \text{ is universal representation of } G \]

**Theorem (Lance)**

Let \( G \) be a discrete group. Then \( G \) is amenable if and only if \( C^*_r(G) \) is nuclear.

**Theorem (Connes)**

Let \( G \) be a separable and connected. Then \( C^*(G) \) is nuclear.
Exact C*-algebras

**Definition**

A C*-algebra $A$ is exact if for every short exact sequence $0 \rightarrow J \rightarrow B \rightarrow C \rightarrow 0$ of C*-algebras, the sequence

$$0 \rightarrow A \otimes_{\text{min}} J \rightarrow A \otimes_{\text{min}} B \rightarrow A \otimes_{\text{min}} C \rightarrow 0$$

is exact.

**Theorem (Kirchberg)**

Let $A$ be a C*-algebra and suppose that $A \hookrightarrow B(\mathcal{H})$ is a faithful embedding. The C*-algebra $A$ has is exact if and only if there exists ccp maps $\varphi_i : A \rightarrow M_{n_i}(\mathbb{C})$ and $\psi_i : M_{n_i}(\mathbb{C}) \rightarrow B(\mathcal{H})$ such that $\|\psi_i \circ \varphi_i(a) - a\| \rightarrow 0$ for all $a \in A$. 

Nuclear $\Rightarrow$ Exact
Exact C*-algebras

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Let $A$ be a C*-algebra and suppose that $A \hookrightarrow B(\mathcal{H})$ is a faithful embedding. The C*-algebra $A$ has is exact if and only if there exists ccp maps $\varphi_i : A \to M_{n_i}(\mathbb{C})$ and $\psi_i : M_{n_i}(\mathbb{C}) \to B(\mathcal{H})$ such that $\|\psi_i \circ \varphi_i(a) - a\| \to 0$ for all $a \in A$.

**Nuclear $\Rightarrow$ Exact**
$\mathcal{C}^*(\mathbb{F}_2)$ is not exact

**Theorem (Wasserman)**

The sequence

$$0 \to \mathcal{C}^*(\mathbb{F}_2) \otimes_{\min} J \to \mathcal{C}^*(\mathbb{F}_2) \otimes_{\min} \mathcal{C}^*(\mathbb{F}_2) \to \mathcal{C}^*(\mathbb{F}_2) \otimes_{\min} \mathcal{C}^*_r(\mathbb{F}_2) \to 0$$

is not exact, where $J$ is the kernel of $\mathcal{C}^*(\mathbb{F}_2) \to \mathcal{C}^*_r(\mathbb{F}_2)$. 
Local properties of C*-algebras

*Local reflexivity* and the *local lifting property* (LLP) are C*-algebraic properties which are weaker than nuclearity.
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\[ \text{Exact} \Rightarrow \text{Locally Reflexive} \]
Local properties of C*-algebras

*Local reflexivity* and the *local lifting property* (LLP) are C*-algebraic properties which are weaker than nuclearity.

**Exact \Rightarrow Locally Reflexive**

| Definition |
|------------|
| A unital C*-algebra $A$ has the LLP if any ucp map $\varphi: A \rightarrow B/J$ is locally liftable, i.e., for any finite dimensional operator system $E \subset A$, there exists a ucp map $\psi: E \rightarrow B$ such that $\varphi = q \circ \psi$ (where $q: B \rightarrow B/J$ is the quotient map). A nonunital C*-algebra $A$ is said to have the LLP if its unitization does. |
Local properties of C*-algebras

*Local reflexivity* and the *local lifting property* (LLP) are C*-algebraic properties which are weaker than nuclearity.

\[ \text{Exact} \Rightarrow \text{Locally Reflexive} \]

**Definition**

A unital C*-algebra \( A \) has the LLP if any ucp map \( \varphi : A \to B/J \) is locally liftable, i.e., for any finite dimensional operator system \( E \subset A \), there exists a ucp map \( \psi : E \to B \) such that \( \varphi = q \circ \psi \) (where \( q : B \to B/J \) is the quotient map).

A nonunital C*-algebra \( A \) is said to have the LLP if its unitization does.

**Theorem (Kirchberg)**

A C*-algebra \( A \) has the LLP if and only if \( A \otimes_{\text{min}} \mathcal{B}(\mathcal{H}) = A \otimes_{\text{max}} \mathcal{B}(\mathcal{H}) \) canonically.
Local properties of C*-algebras

**Theorem (Effros-Haagerup)**

If $A$ is a locally reflexive C*-algebra, then the sequence

$$0 \to J \otimes_{\min} C \to A \otimes_{\min} C \to A/J \otimes_{\min} C \to 0$$

is exact for every closed two-sided ideal $J$ of $A$ and every C*-algebra $C$.

**Theorem (Effros-Haagerup)**

Let $B$ be a C*-algebra and $J$ a closed two sided ideal of $B$. If $A := B/J$ has the local lifting property, then the sequence

$$0 \to J \otimes_{\min} C \to B \otimes_{\min} C \to A \otimes_{\min} C \to 0$$

is exact for every C*-algebra $C$. 
Theorem (Wasserman)

The sequence

\[
0 \to \mathbb{C}^* (\mathbb{F}_2) \otimes_{\text{min}} J \to \mathbb{C}^* (\mathbb{F}_2) \otimes_{\text{min}} \mathbb{C}^* (\mathbb{F}_2) \to \mathbb{C}^* (\mathbb{F}_2) \otimes_{\text{min}} \mathbb{C}^r_1 (\mathbb{F}_2) \to 0
\]

is not exact, where \( J \) is the kernel of \( \mathbb{C}^* (\mathbb{F}_2) \to \mathbb{C}^r_1 (\mathbb{F}_2) \).
Local properties of group C*-algebras

Theorem (Wasserman)

The sequence

\[ 0 \to C^*(\mathbb{F}_2) \otimes_{\min} J \to C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2) \to C^*(\mathbb{F}_2) \otimes_{\min} C^*_l(\mathbb{F}_2) \to 0 \]

is not exact, where \( J \) is the kernel of \( C^*(\mathbb{F}_2) \to C^*_l(\mathbb{F}_2) \).

Corollary

\( C^*(\mathbb{F}_2) \) is not locally reflexive and \( C^*_l(\mathbb{F}_2) \) does not have the LLP.
Local properties of group C*-algebras

**Theorem (W.)**

Let $G$ be a QSIN group which contains $\mathbb{F}_2$ as a closed subgroup. Then

$$0 \to C^*(G) \otimes_{\text{min}} K \to C^*(G) \otimes_{\text{min}} C^*(G) \to C^*(G) \otimes_{\text{min}} C^*_r(G) \to 0$$

is not exact, where $K$ is the kernel of $C^*(G) \to C^*_r(G)$. 

**Key Fact:** ($\text{Ind}_G \times G \mathbb{F}_2 \times \mathbb{F}_2 \pi \big|_{\mathbb{F}_2 \times \mathbb{F}_2}$ weakly contains $\pi$ for every representation $\pi$ of $\mathbb{F}_2 \times \mathbb{F}_2$.)

**Corollary**

If $G$ is QSIN and contains $\mathbb{F}_2$ as a closed subgroup, then $C^*(G)$ is not locally reflexive and $C^*_r(G)$ does not have the LLP.
Local properties of group $C^*$-algebras

**Theorem (W.)**

Let $G$ be a QSIN group which contains $\mathbb{F}_2$ as a closed subgroup. Then

$$0 \to C^*(G) \otimes_{\text{min}} K \to C^*(G) \otimes_{\text{min}} C^*(G) \to C^*(G) \otimes_{\text{min}} C^*_r(G) \to 0$$

is not exact, where $K$ is the kernel of $C^*(G) \to C^*_r(G)$.

**Key Fact:** $(\text{Ind}_{\mathbb{F}_2 \times \mathbb{F}_2}^G \pi)_{|\mathbb{F}_2 \times \mathbb{F}_2}$ weakly contains $\pi$ for every representation $\pi$ of $\mathbb{F}_2 \times \mathbb{F}_2$. 
Local properties of group C*-algebras

**Theorem (W.)**

Let $G$ be a QSIN group which contains $\mathbb{F}_2$ as a closed subgroup. Then

$$0 \to C^*(G) \otimes_{\min} K \to C^*(G) \otimes_{\min} C^*(G) \to C^*(G) \otimes_{\min} C^r(G) \to 0$$

is not exact, where $K$ is the kernel of $C^*(G) \to C^r(G)$.

**Key Fact:** $(\text{Ind}_{F_2 \times F_2}^{G \times G} \pi)|_{F_2 \times F_2}$ weakly contains $\pi$ for every representation $\pi$ of $F_2 \times F_2$.

**Corollary**

If $G$ is QSIN and contains $\mathbb{F}_2$ as a closed subgroup, then $C^*(G)$ is not locally reflexive and $C^r(G)$ does not have the LLP.
Thank you!