String theory and classical integrable systems

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Abstract

We discuss different formulations and approaches to string theory and 2d quantum gravity. The generic idea to get a unique description of many different string vacua altogether is demonstrated on the examples in 2d conformal, topological and matrix formulations. The last one naturally brings us to the appearance of classical integrable systems in string theory. Physical meaning of the appearing structures is discussed and some attempts to find directions of generalizations to “higher-dimensional” models are made. We also speculate on the possible appearance of quantum integrable structures in string theory.
1 Introduction

String theory or theory of 2d gravity continues to be one of the main directions of investigation in mathematical physics. Recent years have brought us to some progress in understanding its relation to a much older field of interest for many mathematical physicists - integrable systems. At the moment we can already advocate that partially “integrable science” is directly related to string theory – that part connected with classical integrable equations and their hierarchies. The situation with quantum integrable systems is not yet as clear\(^1\) so I will almost skip this question below (except for some minor speculations at the end). In contrast, the appearance of hierarchies of classical integrable equations in description of non-perturbative string amplitudes is already a well-known fact at least for low-dimensional string models \([1, 2, 3, 4, 5]\).

Below I will try to stress the most essential points of this formalism when the hierarchies of integrable equations appear in string theory and discuss the parallels with other languages. I will also try to clarify the target-space picture for these models and demonstrating possible “generalizations” to the case of “higher-dimensional” string theories.

Let me remind you now some questions that can appear in the modern string theory from a physical point of view\(^2\). If we believe that string theory could have something to do with our reality then the idea is to find a convenient language suitable for computation of physical quantities - physical amplitudes. It would be marvelous if they could be computed exactly and in any background. Unfortunately nobody knows how to do this for most of string theories with the exception of those where amazing structure of integrable equations has appeared. However, there is no complete effective target space theory even for these models - if exists such a theory will be a good candidate for the role of string field theory \([6, 7, 8]\).

The simplest example of string theory (mostly well-known) is topological pure gravity (which has lots of different equivalent formulations and is mostly well-studied)\([3, 10, 11, 12]\). Naively such theory should not have target space at all. Then the natural question is what is the origin of the nice structure appearing in the form of the Virasoro constraints

\(^1\)though there are already many arguments making us believe that quantum integrable systems should play an essential role in formulation of string theory.
acting to a partition function \([13, 14, 15]\), the set of which is actually equivalent to the concept of integrability in these models?

Below we are going to pay attention to several points concerning integrability arising in description of low-dimensional string theory and try to show how the structure of hierarchies of classical integrable equations can be generalized to higher dimensional theories. The main idea is that the phase space of appearing classical integrable models may be considered as a phase space for effective string field theory, and its quantization can lead to the formulation of the second-quantized string theory in terms of (quantum?) integrable systems.

First, we review a little the 2\text{d} conformal field theory language – the basic Polyakov definition of string theory. We will concentrate on Liouville (physical) gravity coupled to most known non-critical string – so called \((p, q)\) models, and try to discuss its target-space structure when taking the quasiclassical limit. As in any covariant description this one requires a lot of “extra” information of the theory, this leads to the situation when the original formulation of string theory is not effective for answering questions about its target-space structure etc (like it occurs in the theory of particles). We are going to consider an example of motion in the space of \((p, q)\) theories - a step towards their effective description and then turn to the other ways to formulate the same theories.

At least part of these models \(((p, 1)\text{ and/or } (1, p))\) allow to consider them by topological theory language \([16, 17, 18]\). Coupling to reparameterization ghosts one gets total \(c = 0\) central charge which allows an interpretation in terms of the twisted \(N = 2\) theory \([19]\). Physical gravity is now included in “topological matter” – while topological gravity appears roughly speaking in the integration over module space - thus, generalization of the notion of \textit{critical} string. In principle even 26-dimensional bosonic string can be considered as topological theory, but low-dimensional examples allow one to demonstrate better the explicit \(N = 2\) cancellation of bosonic and fermionic two-dimensional determinants and target-space co-ordinates appear as corresponding zero modes. The result is very close to \textit{localization} formulas appeared to be a useful tool when studying topological, quantum-mechanical and integrable models \([20]\).

It appears that at least for topological models there exists a very effective exact for-
mulation on the language of matrix models. Indeed, the matrix integral is a sort of counting of possible two-dimensional diagrams thus being a natural object in the theory of pure gravity - or string theory with an empty target space. These integrals naturally come from an intersection theory on module space [12], what would be more interesting is if one can successfully realize the same idea for the moduli of target-spaces (see recent papers [21, 22]).

Matrix models brought us to the understanding of possible role of integrable systems in string theory. Till the moment only for these “empty models” but there exists a way to compute the exact amplitudes in these string theories by proving that their generation function is a $\tau$-function of KP (or in general Toda-lattice) hierarchy satisfying some natural additional constraint. This constraint is an analog of unitarity condition and can be also interpreted in terms of some flow in the space of different low-dimensional models.

However, there are some puzzles, arising along this way. They are directly connected with the question of interpretation of $p-q$ duality, which is in order repeated to the problem of string field theory background independence [23]. We are going to discuss them below (see also [24]).

2 2d conformal theories

Let us start with reminding that by canonical string theory one usually has in mind the induced two dimensional gravity, having the following form in the Polyakov’s path integral approach:

$$\int Dg D\phi e^{-S[\phi]} \sim \int D\gamma e^{-\int R_\gamma + R + \mu^2 \sqrt{\gamma}}$$

(1)

where $\phi$ stands for the integration over some 2d conformal field theory in the background world-sheet metric $g$.

It seems to be true that at the moment there is not still an existing consistent method of quantization the appearing in (1) Liouville theory (see however [25, 26] etc). The common belief is that the following ideology is right [27, 28, 29]. Consider the integral in
rameterization ghosts $b$ and $c$; (iii) conformal Liouville theory. The latter one should be defined as a conformal theory with the central charge $26 - c_{\text{matter}}$ in order to cancel the anomaly.

For the simplest example of so-called $(p, q)$ models coupled to 2d gravity $\phi$ can be simply considered as a “deformed” scalar field. Choosing the gauge for metric $g_{ab} = e^{\hat{g}_{ab}}$ we reduce the problem to a conformal theory of two fields $\phi$ and $\varphi$ with the stress-tensors

$$T_m = -\frac{1}{2} (\partial \phi)^2 + i\alpha_0 \partial^2 \phi$$
$$T_L = -\frac{1}{2} (\partial \varphi)^2 + \beta_0 \partial^2 \varphi$$

(2)

where

$$\partial \phi(z) \partial \phi(0) = -\frac{1}{z^2} + ...$$
$$\partial \varphi(z) \partial \varphi(0) = -\frac{1}{z^2} + ...$$

(3)

thus giving the Virasoro central charges

$$c_m = 1 - 12\alpha_0^2$$
$$c_L = 1 + 12\beta_0^2$$

(4)

satisfying $c_m + c_L - 26 = 0$ with $-26$ coming from the reparameterization ghosts contribution. For the $(p, q)$ theories

$$\alpha_0 = \sqrt{\frac{p}{2q}} - \sqrt{\frac{q}{2p}}$$
$$\beta_0 = \sqrt{\frac{p}{2q}} + \sqrt{\frac{q}{2p}}$$

(5)

or

$$\beta_0 = \sqrt{2} \cosh \theta$$
$$\alpha_0 = \sqrt{2} \sinh \theta$$

(6)

with
For such system one has the following matter Kac spectrum

\[
\begin{align*}
\alpha_+ &= \sqrt{\frac{2p}{q}} \\
\alpha_- &= -\sqrt{\frac{2q}{p}} \\
\alpha_{n,m} &= \frac{1-n}{2}\alpha_+ + \frac{1-m}{2}\alpha_- = \frac{(1-n)p - (1-m)q}{\sqrt{2pq}} \\
\Delta_{n,m} &= \frac{(np - mq)^2 - (p-q)^2}{4pq} \\
\Delta_{\text{min}} &= \frac{1-(p-q)^2}{4pq}
\end{align*}
\]

(8)

where in matter sector there exists a “periodicity” allowing one to restrict to

\[
\begin{align*}
n &= 1, \ldots, q-1 \\
m &= 1, \ldots, p-1
\end{align*}
\]

(9)

while the gravity sector is given by

\[
\begin{align*}
\beta_{n,m} &= \frac{p+q \pm (np-mq)}{\sqrt{2pq}} \rightarrow \frac{(1-n)p + (1+m)q}{\sqrt{2pq}} \\
\beta_{\text{min}} &= \frac{p+q \pm 1}{\sqrt{2pq}}
\end{align*}
\]

(10)

where we have chosen a sign in order to make correspondence to the proper quasiclassical limit.

Indeed, we see that the conformal \((p,q)\) model is totally symmetric under exchange of \(p\) and \(q\). However, the difference between \(p\) and \(q\) becomes crucial when coupling to 2d gravity, or better to say when considering string theory. In fact, the asymmetry appears

\footnote{and actually this “minimality” is broken by interacting with gravity (see for example [30])}
if one takes the quasiclassical limit \(^5\) then only half of the screening operators have well-defined limit (\(\alpha_+\) and \(\beta_+\) for \(q \to \infty\) and \(\alpha_-\) and \(\beta_-\) for \(p \to \infty\)). The simplest way to see it is to consider \((p, q)\) model as a Hamiltonian reduction of the WZNW theory \(^3\) and for the WZNW theory it is known \(^4\) that only one screening operator (having smooth limit for \(k \to \infty\)) appears naturally from classical action as a constraint on free fields. Physically it means that one has to choose the operator coupling to unity (or lowest dimensional one) - i.e. what is called the puncture operator in a theory.

2.1 Rotations in the space of free fields: the way to move in the space of theories

Now, let us turn to the question how one can describe the whole set of different \((p, q)\) models. In fact this is rather hard to do using the technique of present section - more or less complete description exists only based on the methods presented below. However, here we will try to use as much as possible of conformal methods in order to get understanding of possible flows in the space of theories.

One can consider the following rotation in the space of \((p, q)\) theories

\[
\tilde{\beta}_0 = \alpha_0 \sinh \vartheta + \beta_0 \cosh \vartheta \\
\tilde{\alpha}_0 = \alpha_0 \cosh \vartheta + \beta_0 \sinh \vartheta
\]  

(11)

The same rule one has for primary operators \(e^{i(\alpha_0 \phi + \beta \varphi)}\), labeled by (8), (10)

\[
\tilde{\beta} = \alpha \sinh \vartheta + \beta \cosh \vartheta \\
\tilde{\alpha} = \alpha \cosh \vartheta + \beta \sinh \vartheta
\]  

(12)

with parameter of the transformation

\[
\vartheta = \frac{1}{2} \log \frac{\tilde{p} \tilde{q}}{q \bar{p}}
\]  

(13)

Now it is easy to rewrite it in the space of fields \(^6\)

---

\(^5\)this quasiclassical limit is important if we want to discuss target-space properties of the theory

\(^6\)Note that in particular such rotation makes from real Liouville field for \(c = 1\) complex-valued for \(c < 1\).
\[ \Phi_{n,m} = \exp \left( i \alpha_{n,m} \phi + \beta_{n,m} \varphi \right) \] (14)

One finds that for \( \Phi_{n,m}^{(p,q)} \rightarrow \Phi_{n,m}^{(\tilde{p},\tilde{m})} \)

\[ \tilde{n} = \frac{p}{q} \]
\[ \tilde{m} = m \] (15)

Now, let us consider an explicit example how the transformation (15) works. First, let us take (10) and make there a substitution

\[ kq + r = np - mq \] (16)

for \((q,p)\) theory and

\[ kp + r = np - mq \] (17)

for \((p,q)\) theory. Then (we restrict ourselves to the second choice (17))

\[ k = n - \left\lfloor \frac{qm}{p} \right\rfloor \]
\[ r = mq - p \left\lfloor \frac{qm}{p} \right\rfloor \] (18)

where \([x]\) means integer part of \(x\), and this give the correspondence [39]

\[ \sigma_k(O_r) \sim \int \exp \left( i \alpha_{n,m} \phi + \beta_{n,m} \varphi \right) \] (19)

i.e. \(r\) enumerates “topological primary fields” and we have defined (16) and (17) in order to have exactly \(q - 1\) or \(p - 1\) of them. In such terminology \(k\) counts their “gravitational descendants”.

Now the transformation (15) works as follows: take for example \((p,1)\) theory and consider \(\sigma_1(1)\). 1 is given by zero-dimensional matter operator with

\[ r = m = p - 1 \]
\[ \Delta_1 = \Delta_{0,p-1} = 0 \]  

while for \( \sigma_1(1) \) itself one has

\[ r = m = p - 1 \]
\[ k = n = 1 \]  

(21)

Then, making transformation (15) we get

\[ \tilde{m} = m = p - 1 \]
\[ \tilde{n} = p \]  

(22)

It means that the rotated field becomes primary one in the \((\tilde{p}, \tilde{q})\) model with \( \tilde{p} = p \) and

\[ \tilde{k} = \tilde{n} - \left[ \frac{\tilde{q}m}{p} \right] = 0 \]  

(23)

which gives

\[ \tilde{q} = p + \left[ \frac{\tilde{q}}{p} \right] \]  

(24)

or just \( \tilde{q} = p + 1 \). For \( p = 2 \) such an operator drops us from pure topological gravity \((p, q) = (2, 1)\) to the pure physical gravity point \((\tilde{p}, \tilde{q}) = (2, 3)\).

The example considered above as just an illustration of the flow in the space of simplest string theories. We have seen that in the original conformal formulation they strongly depend on the basis one has to choose in the space of fields and/or observables. That is one of the reasons why more convenient target space description for string theory is necessary.

On the language of matrix models these relations can be rewritten in the form of the Virasoro-\( W \) constraints and generalized KdV flows. We will see below, that the effective target-space description based on integrable systems gives much stronger possibilities to investigate this phenomenon.

### 3 Topological language

Now, let us make a sort of an intermediate step – to reformulate the above picture in
gravity since metric contribution – Liouville and ghost sectors are also represented by certain conformal theories. Then one can generalize the above consideration restricting to the only requirement that the total central charge of a theory is equal to zero. The presence of gravity remains in the only fact that the result after all should be integrated over module space. Such object is usually meant by what is called topological gravity \cite{9,18}. From such point of view critical string is a good example of a topological theory interacting with topological gravity except for the only case that integral over module space might diverge.

Now consider this (conformal matter plus Liouville gravity plus reparameterization ghosts)

\[
T_{gh} = -2b\partial c + c\partial b
\]  \hspace{1cm} (25)

\[
(T = -j b\partial c - (1 - j)c\partial b \text{ for } j = 2)
\]

as a twisted $N = 2$ superconformal theory \cite{19}. In such case for $(q,p)$ and $(p,q)$ “untwisted” models two values of the central charge are

\[
c_{(q,p)} = 3(1 - \frac{2p}{q})
\]

\[
c_{(p,q)} = 3(1 - \frac{2q}{p})
\]  \hspace{1cm} (26)

This can be demonstrated for example as follows. First let us consider the $SU(2)_k$ WZNW model. Such theory possesses conformal symmetry with the Virasoro central charge

\[
c_{SU(2)_k} = \frac{3k}{k + 2}
\]  \hspace{1cm} (27)

One of the possible ways to get matter $(p,q)$ model is via the Drinfeld-Sokolov reduction. Then, one easily finds that

\[
k_{(q,p)} \equiv \tilde{k} = \frac{q}{p} - 2
\]

\[
k_{(p,q)} \equiv k = \frac{p}{q} - 2
\]  \hspace{1cm} (28)

where asymmetry between $p$ and $q$ appeared exactly as we mentioned above when one has to distinguish the classical screening operator. The relation
\[ k + 2 = \frac{1}{k + 2} \]  

in particular demonstrates the duality between two "classical" limits when \( k \to \infty \) corresponds to \( \tilde{k} \to -2 \) and vice versa. It can also clarify what is the meaning of the Wess-Zumino model with a rational central charge – considering it as a dual to that one with integer \( k \) in the above sense.

The most easy way to check the relations (26), (28) is using bosonization technique when performing the reduction [37]. Indeed, “twisting”

\[ T_{WZNW} \to \tilde{T}_{WZNW} = T_{WZNW} - \partial H \]  

where (see [38] for more detailed description of free field technique)

\[ T_{WZNW} = w\partial\chi - \frac{1}{2}(\partial\phi)^2 - \frac{i}{\sqrt{2(k+2)}}\partial^2\phi \]

\[ H = w\chi - i\sqrt{\frac{1}{2(k+2)}}\partial \phi \]  

one gets

\[ \tilde{T}_{WZNW} = -\partial w\chi - \frac{1}{2}(\partial\phi)^2 - \frac{i}{\sqrt{2}} \left( \frac{1}{\sqrt{k+2}} - \sqrt{k+2} \right) \partial^2 \phi \]  

(32)

and the field \( \phi \) stands now for minimal model, i.e. the corresponding screening operators become the screening charges of the \((p,q)\) model.

Another valuable relation exists between the \( SU(2)_k \) WZNW model and \( N = 2 \) minimal model \( A_k \). Namely the \( SU(2)_k \) Kac-Moody currents can be represented as

\[ J_{\pm} = e^{\pm i\sqrt{\frac{k+1}{2}}\phi}G_{\pm} \]  

(33)

with

\[ H = i\sqrt{\frac{k}{2}}\partial h \]  

(34)

– the Cartan current of the \( SU(2)_k \) while
being the $U(1)$ current of the $N = 2$ minimal model and $G_{\pm}$ denote the corresponding superconformal symmetry generators. The twisting of $N = 2$ gives

$$T_{N=2} \to T_{N=2}^{tw} = T_{N=2} - \frac{i}{2} \sqrt{\frac{k}{k+2}} \partial^2 \Phi$$

where

$$T_{N=2} = T_{WZNW} - T_h + T_\Phi$$

Eqs. (32), (36) and (37) altogether give

$$T_{N=2}^{tw} = T_{WZNW}^{tw} + \tilde{T}_\Phi - \tilde{T}_h$$

where from the first term in the r.h.s. one can single out the $(p, q)$ matter model, while the rest can be transformed by similar technique into the Liouville and ghost contributions.

### 3.1 Landau-Ginzburg models

The particular class of topological theories which includes $N = 2$ superconformal minimal models is given by the Landau-Ginzburg models. The action can be written in the form:

$$\int \partial X \bar{\partial} X^* + \psi \bar{\partial} \psi^* + \bar{\psi} \partial \bar{\psi}^* + FF^* + W'(X)F + \psi \bar{\psi} W''(X) + W'(X^*)F^* + \psi^* \bar{\psi}^* W''(X^*)$$

which is invariant under the $N = 2$ supersymmetry transformations, generated by

$$G = \psi \frac{\delta}{\delta X} + F \frac{\delta}{\delta \psi} - \partial X^* \frac{\delta}{\delta \bar{\psi}^*} - \bar{\partial} \psi^* \frac{\delta}{\delta F^*}$$

$$G = \bar{\psi} \frac{\delta}{\delta X^*} - F^* \frac{\delta}{\delta \bar{\psi}^*} - \bar{\partial} X^* \frac{\delta}{\delta \psi^*} + \partial \bar{\psi}^* \frac{\delta}{\delta F^*}$$

$$G^* = \psi^* \frac{\delta}{\delta X^*} + F^* \frac{\delta}{\delta \bar{\psi}^*} - \partial X \frac{\delta}{\delta \bar{\psi}} - \bar{\partial} \bar{\psi} \frac{\delta}{\delta F}$$

$$\bar{G}^* = \bar{\psi}^* \frac{\delta}{\delta X^*} - F^* \frac{\delta}{\delta \psi^*} - \bar{\partial} X \frac{\delta}{\delta \psi} + \partial \bar{\psi} \frac{\delta}{\delta F}$$

$$\{G, G^*\} = -2\partial$$
\{ \bar{G}, G^* \} = -2 \bar{\partial}

After twisting, the lagrangian takes the form

\[\int \bar{\partial}X \bar{\partial}X^* + \psi \bar{\partial}\psi^* + \bar{\psi} \partial \bar{\psi}^* + FF^* + \psi \bar{\psi}W''(X) + FW'(X) + \sqrt{g} \left[ F^*W'(X^*) + \psi^* \bar{\psi}^*W''(X^*) \right] = \]

\[= \int \frac{1}{2} \psi \bar{\partial}(\psi^* - \bar{\psi}^*) + \frac{1}{2} \bar{\psi} \partial (\bar{\psi}^* - \psi^*) + \psi \bar{\psi}W''(X) + FW'(X) + \{Q, V\} \]

with

\[Q = G^* + \bar{G}^* = \theta \frac{\delta}{\delta X^*} - F^* \frac{\delta}{\delta \eta} - \partial X \frac{\delta}{\delta \psi} - \bar{\partial}X \frac{\delta}{\delta \bar{\psi}} - (\partial \bar{\psi} + \bar{\partial} \psi) \frac{\delta}{\delta F} \]

\[\psi dz, \bar{\psi} \bar{d} \bar{z} \text{ and } Fdzd\bar{\bar{z}} \text{ are forms and } \psi^*, \bar{\psi}^* \text{ and } F^* \text{ are scalar functions and where} \]

\[V = -\int \frac{1}{2} (\psi \bar{\partial}X^* + \bar{\psi} \partial X^*) + \sqrt{g} \eta W'(X^*) \]

where we have introduced

\[\theta = \psi^* + \bar{\psi}^* \]

\[\eta = \frac{1}{2}(\psi^* - \bar{\psi}^*) \]

The integral with the action (41) can be computed by localization technique [20]. It localizes on \(Q = 0\), i.e.

\[\theta = 0 \quad F^* = 0 \left( = \frac{\partial W}{\partial X} \right) \quad \partial X = 0 \quad \bar{\partial}X = 0 \quad \partial \bar{\psi} + \bar{\partial} \psi = 0 \]

Computation of the path integral for (41) gives zero for the trivial potential \(W(X) = X\). This is the statement we will use below for \((1, p)\) models – strictly speaking the case \((1, p)\) should correspond to (41) with a trivial potential and non-trivial kinetic term,
bosonic-fermionic cancellation. Eqs. (43) demonstrate that actually the path integral is not still the most effective description for those models – it can be reduced to a more simple object. Such objects are directly related to integrable systems and we will pass to their description below.

4 Matrix models

To understand better the effective description of 2d gravity and string models let us for a moment trivialize the situation and return back from strings to particles, i.e. from surfaces to lines. A natural question is what is the analog of topological string models in the one-dimensional case and the answer should be very simple. Indeed, for the topological one-dimensional theory the only thing which can appear is something related to the points at the end of paths and their permutations, so these should be combinatorial numbers attached to the ends of Feynman diagrams.

The module space for one-dimensional theories consists of the lengths of world-lines, so inclusion of topological one-dimensional gravity should somehow take this into account. For the simplest case of the propagator one should get

$$G_{\alpha\beta} = \int_0^\infty dT f_{\alpha\beta}(T)$$

with $T$ being the length of the world-line while $\alpha$ and $\beta$ are indices running over the space attached to each point - end of the line, i.e. over the Hilbert space of the corresponding theory. The objects $G_{\alpha\beta}$ can be considered as building blocks for the theory.

In the case of absence of the target space the only choice for $f_{\alpha\beta}(T)$ is $\delta_{\alpha\beta}f(T)$, so instead of nontrivial propagators one gets just a number $G$ and the ”theory” reduces to a ”generation function” via the one-dimensional integral

$$\int d\phi \exp \left( -\frac{\phi^2}{2G^2} + t\phi + \sum g_n \phi^n \right)$$

where one should fix by hands what sort of one-dimensional ”branches” - i.e. geometries is allowed. This is a typical ”counting diagram” integral and it should be considered as a one-dimensional analog of generating function below.
From this point of view, two-dimensional topological gravity should naturally bring to “fat graphs” where generation function has a nice prescription to be computed via matrix models. A simple analog of (47) would look like

\[ Z_N = \int DM_{N \times N} \exp (-TrV(M)) \]  

which was proven (in the limit \( N \to \infty \)) to be an effective way to compute the integral over two-dimensional metrics, including the sum over topologies. The continuum integration (1) is approximated by triangulations of world-sheet in (18).

Below, we will concentrate mostly to a slightly different version of the integral (18) which rather has an interpretation of the target space theory. The exact expression is [5]

\[ Z(N)[V|M] \equiv C(N)[V|M] e^{TrV(M) - TrMV'(M)} \int DX e^{-TrV(X) + TrV'(M)X} \]  

where the integral is taken over \( N \times N \) “Hermitean” matrices, with the normalizing factor given by Gaussian integral

\[ C(N)[V|M]^{-1} \equiv \int DY e^{-TrU_2[M,Y]}, \]

\[ U_2 \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} Tr[V(M + \epsilon Y) - V(M) - \epsilon YV'(M)] \]  

Including an external matrix field, which can be considered as a source (\( \equiv \) coupling constants) it allows us do assign more or less concrete potential to a theory.

The formula (19) has in fact a lot of similarities with the Landau-Ginzburg model we discussed in the previous section. Both theories are determined by a potential and as we will see below there exists a simple relation between the potential in (19) and the superpotential of the Landau-Ginzburg model \( W(X) \), namely:

\[ W(X) = V'(X) \]

4.1 From matrix models to integrable systems

Now we are going to demonstrate that matrix models being an adequate formulation for certain very simple string theories naturally lead to appearance of the classical integrable
systems describing the exact solutions for such strings. Namely, we will show that introduced in the previous section model (49) is a particular solution to the KP (Toda lattice) hierarchy. That is:

(A) The partition function $Z^V_N[M]$ (49), if considered as a function of time-variables

$$T_k = \frac{1}{k} \text{Tr} M^{-k}, \quad k \geq 1$$

is a KP $\tau$-function for any value of $N$ and any potential $V[X]$.

(B) As soon as $V[X]$ is homogeneous polynomial of degree $p + 1$, $Z^{(v)}_N[M] = Z^{(p)}_N[M]$ is in fact a $\tau$-function of $p$-reduced KP hierarchy. \footnote{Moreover, $\partial Z^{(p)}_N[M] / \partial T_n = 0$.}

In order to prove these statements, first, we rewrite (49) in terms of determinant formula

$$Z^{(v)}_N[M] = \det(i_{ij}) \phi_i(\mu_j) \Delta(\mu)$$

$$i,j = 1, \ldots, N. \quad (53)$$

Then, we show that any KP $\tau$-function in the Miwa parameterization does have the same determinant form.

The main thing which distinguishes matrix models from the point of view of solutions to the KP-hierarchy is that the set of functions $\{\phi_i(\mu)\}$ in (53) is not arbitrary. This is the origin of $L_{-1}$ and other $W$-constraints (which in the context of KP-hierarchy may be considered as implications of $L_{-1}$).

The fact that the classical integrable system appear in string theory, of course has more deep reason that this simple illustration for low-dimensional models.

### 4.2 Integrability from the determinant formula

We begin with an evaluation of the integral $[5]$

$$\mathcal{F}^{(v)}_N[\Lambda] \equiv \int DX \ e^{-\text{Tr}[V(X) - \text{Tr}\Lambda X]}$$

The integral over the "angle" $U(N)$-matrices can be easily taken with the help of $[40]$ and if eigenvalues of $X$ and $\Lambda$ are denoted by $\{x_i\}$ and $\{\lambda_i\}$ respectively, the result is
\[ \frac{1}{\Delta(\Lambda)} \left[ \prod_{i=1}^{N} \int dx_i e^{-V(x_i) + \lambda x_i} \right] \Delta(X) \]  

(55)

\[ \Delta(X) \text{ and } \Delta(\Lambda) \text{ are Van-der-Monde determinants, e.g. } \Delta(X) = \prod_{i>j} (x_i - x_j). \]

The r.h.s. of (55) can be rewritten as

\[ \Delta^{-1}(\Lambda) \Delta(\frac{\partial}{\partial \Lambda}) \prod_{i} \int dx_i e^{-V(x_i) + \lambda x_i} = \]

\[ = \Delta^{-1}(\Lambda) \det_{(ij)} F_i(\lambda_j) \]  

(56)

with

\[ F_{i+1}(\lambda) \equiv \int dx x^i e^{-V(x) + \lambda x} = (\frac{\partial}{\partial \lambda})^i F_1(\lambda). \]  

(57)

Note that

\[ F_1(\lambda) = \mathcal{F}^{(V)}_{N=1}[\lambda]. \]  

(58)

If we recall that

\[ \Lambda = V'(M) = W(M) \]  

(59)

and denote the eigenvalues of \( M \) through \( \{\mu_i\} \), then:

\[ \mathcal{F}^{(V)}_{N}[W(M)] = \frac{\det \tilde{\Phi}_1(\mu_j)}{\prod_{i>j} (W(\mu_i) - W(\mu_j))}, \]  

(60)

with

\[ \tilde{\Phi}_i(\mu) = F_i(W(\mu)). \]  

(61)

Proceed now to the normalization (50). Indeed, it is given by the Gaussian integral:

\[ C^{(N)}[V|M]^{-1} \equiv \int DX \ e^{-U_2(M,X)}. \]  

(62)

Then for evaluation of (62) it remains to use the obvious rule of Gaussian integration,

\[ \int DX \ e^{-\sum_{i,j} U_{ij} x_i x_j} \sim \prod_i N^{-1/2} \]  

(63)
and substitute the explicit expression for \( U_{ij}(M) \). If potential is represented as a formal series,

\[
V(X) = \sum \frac{v_n}{n} X^n \\
W(X) = \sum v_n X^n
\]

we have

\[
U_2(M, X) = \sum_{n=0}^{\infty} v_{n+1} \left\{ \sum_{a+b=n-1} \text{Tr} M^a X M^b X \right\},
\]

and

\[
U_{ij} = \sum_{n=0}^{\infty} v_{n+1} \left\{ \sum_{a+b=n-1} \mu_i^a \mu_j^b \right\} = \sum_{n=0}^{\infty} v_{n+1} \frac{\mu_i^n - \mu_j^n}{\mu_i - \mu_j} = \frac{W(\mu_i) - W(\mu_j)}{\mu_i - \mu_j}.
\]

Coming back to (49), we conclude that

\[
Z_N^{\{V\}}[M] = e^{\text{Tr}[V(M)-MW(M)]} C^{(N)}[V|M] F_N[W(M)] \sim \sim [\det \Phi_i(\mu_j)] \prod_{i>j} U_{ij} \frac{\prod s(\mu_i)}{\Delta(M)} \prod_{i=1}^{N} s(\mu_i). \tag{65}
\]

The product of s-factors at the r.h.s. of (65) can be absorbed into \( \Phi \)-functions:

\[
Z_N^{\{V\}}[M] = \frac{\det \Phi_i(\mu_j)}{\Delta(M)}, \tag{67}
\]

where

\[
\Phi_i(\mu) = s(\mu) \Phi_i(\mu) \xrightarrow{\mu \to \infty} \mu^{i-1} (1 + O(1)). \tag{68}
\]

where the asymptotic is crucial for the determinant (67) to be a solution to the KP hierarchy in the sense of [42].

The Kac-Schwarz operator [35], [36]. From eqs. (61), (66) and (68) one can deduce that \( \Phi_i(\mu) \) can be derived from the basic function \( \Phi_1(\mu) \) by the relation
\[ \Phi_i(\mu) = [W'(\mu)]^{1/2} \int x^{i-1} e^{-V(x)+xV'(\mu)} dx = A_{\{V\}}^{i-1}(\mu) \Phi_1(\mu), \tag{69} \]

where \( A_{\{V\}}(\mu) \) is the first-order differential operator

\[
A_{\{V\}}(\mu) = s \frac{\partial}{\partial \lambda} s^{-1} = \frac{e^{V(\mu)-\mu W(m)}}{[W'(\mu)]^{1/2}} \frac{\partial}{\partial \mu} \frac{e^{-V(\mu)+\mu W(\mu)}}{[W'(\mu)]^{1/2}} = \frac{1}{W'(\mu)} \frac{\partial}{\partial \mu} + \mu - \frac{W''(\mu)}{2[W'(\mu)]^2}. \tag{70} \]

In the particular case of \( V(x) = \frac{x^{p+1}}{p+1} \)

\[ A_{\{p\}}(\mu) = \frac{1}{p\mu^{p-1}} \frac{\partial}{\partial \mu} + \mu - \frac{p-1}{2p\mu^p} \tag{71} \]

coincides (up to the scale transformation of \( \mu \) and \( A_{\{p\}}(\mu) \)) with the operator which determines the finite dimensional subspace of the Grassmannian in ref.\[35\]. We emphasize that the property

\[ \Phi_{i+1}(\mu) = A_{\{V\}}(\mu) \Phi_i(\mu) \quad (F_{i+1}(\lambda) = \frac{\partial}{\partial \lambda} F_i(\lambda)) \tag{72} \]

is exactly the thing which distinguishes partition functions of GKM from the expression for generic \( \tau \)-function in Miwa’s coordinates,

\[ \tau^{\{\phi_i\}}_N[M] = \frac{[\det \phi_i(\mu_j)]}{\Delta(M)}, \tag{73} \]

with arbitrary sets of functions \( \phi_i(\mu) \). In the next section we demonstrate that the quantity \( (73) \) is exactly a KP \( \tau \)-function in Miwa coordinates, and we return to the Kac-Schwarz operator in sect.5.

### 4.3 KP \( \tau \)-function in Miwa parameterization

A generic KP \( \tau \)-function is a correlator of a special form \[41\]:

\[ \tau^G\{T_n\} = \langle 0 | : e \sum T_n J_n : G | 0 \rangle \tag{74} \]

with
\[ J(z) = \bar{\psi}(z)\psi(z); \quad G = : \exp G_{mn} \bar{\psi}_m \psi_n : \]  

in the theory of free 2-dimensional fermionic fields \( \psi(z) , \bar{\psi}(z) \) with the action \( \int \bar{\psi} \partial \psi \). The vacuum states are defined by conditions

\[ \psi_n |0 \rangle = 0 \quad n < 0 , \quad \bar{\psi}_n |0 \rangle = 0 \quad n \geq 0 \]  

where \( \psi(z) = \sum \psi_n z^n \, dz^{1/2} , \quad \bar{\psi}(z) = \sum \bar{\psi}_n z^{-n-1} \, dz^{1/2} . \)

The crucial restriction on the form of the correlator, implied by (75) is that the operator \( : e^{\sum T_n J_n} : G \) is Gaussian exponential, so that the insertion of this operator may be considered just as a modification of \( \langle \bar{\psi} \psi \rangle \) propagator, and the Wick theorem is applicable. Namely, the correlators

\[ \langle 0 | \prod_i \bar{\psi}(\mu_i) \psi(\lambda_i) G |0 \rangle \]  

for any relevant \( G \) are expressed through the pair correlators of the same form:

\[ (77) = \det_{(ij)} \langle 0 | \bar{\psi}(\mu_i) \psi(\lambda_j) G |0 \rangle \]  

The simplest way to understand what happens to the operator \( e^{\sum T_n J_n} \) after the substitution of (52) is to use the free-boson representation of the current \( J(z) = \partial \varphi(z) \). Then

\[ \sum T_n J_n = \sum_i \left\{ \sum_n \frac{1}{n} \mu_n \varphi_n \right\} = \sum_i \varphi(\mu_i) , \text{ and} \]

\[ : e^{\sum_i \varphi(\mu_i)} := \frac{1}{\prod_{i<j} (\mu_i - \mu_j)} \prod_i : e^{\varphi(\mu_i)} : . \]  

In fermionic representation it is better to start from

\[ T_n = \frac{1}{n} \sum_i \left( \frac{1}{\mu_i^n} - \frac{1}{\bar{\mu}_i^n} \right) \]  

instead of (52). Then

\[ : e^{\sum T_n J_n} := \frac{\prod^N_{i<j} (\bar{\mu}_i - \mu_j)}{\prod_{i>j} (\mu_i - \mu_j) \prod_{i>j} (\bar{\mu}_i - \bar{\mu}_j)} \prod_i \bar{\psi}(\bar{\mu}_i) \psi(\mu_i) . \]  

In order to come back to (52) it is necessary to shift all \( \bar{\mu}_i^{\prime}s \) to infinity. This may be
\( \langle N \rangle \sim \langle 0 | \tilde{\psi}(\infty) \tilde{\psi}'(\infty) \ldots \tilde{\psi}(N-1)(\infty) \rangle \).

The \( \tau \)-function now can be represented in the form:

\[
\tau_N^G[M] = \langle 0 | e^{\sum T_n J_n} : G | 0 \rangle = \Delta(M)^{-1} \langle N | \prod_i e^{\varphi(\mu_i)} : G | 0 \rangle = \lim_{\tilde{\mu}_j \to \infty} \frac{\prod_{i,j} (\tilde{\mu}_i - \mu_j)}{\prod_{i>j} (\mu_i - \mu_j)} \langle 0 | \tilde{\psi}(\mu_i) \psi(\mu_i) G | 0 \rangle
\]

(82)

applying the Wick's theorem (77), (78) and taking the limit \( \tilde{\mu}_i \to \infty \) we obtain:

\[
\tau_N^G[M] = \det \frac{\phi_i(\mu_j)}{\Delta(M)}
\]

(83)

with functions

\[
\phi_i(\mu) \sim \langle 0 | \tilde{\psi}^{(i-1)}(\infty) \psi(\mu) G | 0 \rangle \to \mu \to \infty \mu^{-1}(1 + O(\frac{1}{\mu})).
\]

(84)

Thus, we proved that KP \( \tau \)-function in Miwa coordinates (52) has exactly the determinant form (53), or is a \( \tau \)-function of KP hierarchy.

### 4.4 Universal \( L_{-1} \)-constraint and string equation

Let us return to the question of specifying particular ”stringy” solutions to the KP hierarchy which we already demonstrated considering basis vectors (70). We will show that the matrix version of the Kac-Schwarz operator which is almost

\[
Tr \frac{\partial}{\partial \Lambda_{tr}} = Tr \frac{1}{W'(M)} \frac{\partial}{\partial M_{tr}}
\]

(85)

acting on \( \tau \)-function gives the string equation. Therefore it is natural to examine, how this operator acts on

\[
Z^{(V)}[M] = \frac{\det \tilde{\Phi}_i(\mu_j)}{\Delta(M)} \prod_i s(\mu_i),
\]

(86)

\[
s(\mu) = (W'(\mu))^{1/2} e^{V(\mu) - \mu W(\mu)},
\]

(87)
First of all, if \( Z^{(v)} \) is considered as a function of \( T \)-variables,

\[
\frac{1}{Z^{(v)}} Tr \frac{\partial}{\partial \Lambda} Z^{(v)} = - \sum_{n \geq 1} \operatorname{Tr} \left[ \frac{1}{W'(M)M^{n+1}} \frac{\partial \log Z^{(v)}}{\partial T_n} \right].
\]

(88)

On the other hand, if we apply (85) to explicit formula (86), we obtain:

\[
\frac{1}{Z^{(v)}} Tr \frac{\partial}{\partial \Lambda} Z^{(v)} = -Tr M + \frac{1}{2} \sum_{i,j} \frac{W'(\mu_i) - W'(\mu_j)}{\mu_i - \mu_j} + Tr \frac{\partial}{\partial \Lambda} \log \det F_i(\lambda_j),
\]

(89)

We can prove that

\[
\frac{1}{Z^{(v)}} \mathcal{L}_{-1} Z^{(v)} = - \frac{\partial}{\partial T_1} \log Z^{(v)} + Tr M - Tr \frac{\partial}{\partial \Lambda} \log \det F_i(\lambda_j).
\]

(90)

can be used in order to suggest the formula for the universal operator \( \mathcal{L}_{-1} \).

Here

\[
\mathcal{L}_{-1} = \sum_{n \geq 1} \operatorname{Tr} \left[ \frac{1}{W'(M)M^{n+1}} \frac{\partial}{\partial T_n} + \frac{1}{2} \sum_{i,j} \frac{1}{W'(\mu_i)W'(\mu_j)} \frac{W'(\mu_i) - W'(\mu_j)}{\mu_i - \mu_j} - \frac{\partial}{\partial T_1} \right],
\]

(91)

So, in order to prove the \( \mathcal{L}_{-1} \)-constraint, one should prove that the r.h.s. of (90) vanishes, i.e.

\[
\frac{\partial}{\partial T_1} \log Z^{(v)}_N = Tr M - Tr \frac{\partial}{\partial \Lambda} \log \det F_i(\lambda_j),
\]

(92)

This is possible to prove only if we remember that \( Z^{(v)}_N = \tau^{(v)}_N \). In this case the l.h.s. may be represented as residue of the ratio

\[
\text{res}_\mu \frac{\tau^{(v)}_N(T_n + \mu^{-n}/n)}{\tau^{(v)}_N(T_n)} = \frac{\partial}{\partial T_1} \log \tau^{(v)}_N(T_n).
\]

(93)

However, if expressed through Miwa coordinates, the \( \tau \)-function in the numerator is given by the same formula with one extra parameter \( \mu \), i.e. is in fact equal to \( \tau^{(v)}_{N+1} \). This idea is almost enough to deduce (92). For example, if \( N = 1 \)

\[
\frac{\partial}{\partial T_1} \log \tau^{(v)}_N(T_n) = \frac{\partial}{\partial T_1} \log \tau^{(v)}_{N+1}(T_n).
\]
\[ \tau_1^{(V)}(T_n + \mu^{-n}/n) = \tau_2^{(V)}[\mu_1, \mu] = \]
\[ = e^{V(\mu_1) - \mu W(\mu)} e^{V(\mu) - \mu W(\mu)} [W'(\mu_1)W'(\mu)]^{1/2} \frac{F(\lambda_1)\partial F(\lambda)/\partial \lambda - F(\lambda)\partial F(\lambda_1)/\partial \lambda_1}{\mu - \mu_1} = \]
\[ = \frac{e^{V(\mu) - \mu W(\mu)}[W'(\mu)]^{1/2} F(\lambda)}{\mu - \mu_1} \tau_1^{(V)}[\mu_1] \cdot [-\partial \log F(\lambda_1)/\partial \lambda_1 + \partial \log F(\lambda)/\partial \lambda]. \]

(94)

The function

\[ F(\lambda) = \int dx \ e^{-V(x) + \lambda x} \sim e^{V(\mu) - \mu W(\mu)}[W'(\mu)]^{-1/2} \{1 + O\left(\frac{W'''}{W'bW'}\right)\}. \]

(95)

If \( W(\mu) \) grows as \( \mu^p \) when \( \mu \to \infty \), then \( W''/(W')^2 \sim \mu^{-p-1} \), and for our purposes it is enough to have \( p > 0 \), so that in the braces at the r.h.s. stands \( \{1 + o(1/\mu)\}(\mu \cdot o(\mu) \to 0 \) as \( \mu \to \infty \)). Then numerator at the r.h.s. of (94) is \( \sim 1 + o(1/\mu) \), while the second item in square brackets behaves as \( \partial \log F(\lambda)/\partial \lambda \sim \mu(1 + o(1/\mu)) \). Combining all this, we obtain:

\[ \frac{\partial}{\partial T_1} \log \tau_1^{(V)} = \text{res}_\mu \left\{ \frac{1 + o(1/\mu)}{\mu - \mu_1} \left[-\partial \log F(\lambda_1)/\partial \lambda_1 + \mu(1 + o(1/\mu))\right] \right\} = \mu_1 - \partial \log F(\lambda_1)/\partial \lambda_1. \]

(96)

i.e. (94) is proved for the particular case of \( N = 1 \).

In the particular case of monomial potential \( V = \frac{\lambda^{p+1}}{p+1} \) (91) turns into more common form [2, 3]:

\[ L^{(p)}_{-1} = \frac{1}{p} \sum_{n \geq 1} (n + p)T_{n+p} \frac{\partial}{\partial T_n} + \frac{1}{2p} \sum_{a+b=p, a,b \geq 0} aT_aT_b - \frac{\partial}{\partial T_1}, \]

(97)

5 Canonical quantization and p-q duality

5.1 General ideology

Now, let us turn to somewhat more general question of how a generic string theory (first-quantized or second quantized) should look like. In the simplest case of topological string we can reduce ourselves to the question of basic module space. In the frames of this
background for the first-quantized theory while the second-quantized theory should be related to the quantization of module space.

In the well-known case of pure topological gravity we should expect nothing since that theory does not have any target space at all. This is somehow consistent with the observation that the partition function can be made trivial just by a choice of gauge (polarization).

We are going to demonstrate that the matrix model solution can be obtained within the frames of second quantization on a kind of “module space” for these theories (see [43] for more detailed information on this point).

Finally we will make some comments on the considered problem in the framework of mirror symmetry (see for example [47]). The important remark is that mirror manifolds should be distinguished classically and this effect is very closely related to that one we have in the case of \((p,q)\) models.

### 5.2 String equation and Heisenberg algebra

Now we are going directly to a problem of description of a particular representation of the Heisenberg algebra. One should start from [43] where the “phase space” for \((p,q)\) models is considered as a certain “generalized” module space for the Riemann surfaces with punctures. In the simplest case of sphere with the only puncture one might take the phase space with a symplectic structure

\[ \{W, Q\} = 1 \]  

which is actually generated by

\[ \{z, t_1\} = 1 \]

\[ \{\tilde{z}, \tilde{t}_1\} = 1 \]  

(\(z^p = W(\mu)\) and \(\tilde{z}^q = Q(\mu)\)). For trivial \((1, p)\) topological theories \(\tilde{z} = \mu\).

From this point of view what we consider is a quantization of a symplectic manifold
and we can perform it by standard methods.

The corresponding action is

\[ S = \int WdQ + S_0 \]
\[ dS = \delta W \wedge \delta Q \tag{101} \]

and \( S_0 \) parameterizes an “initial point”. Now, it is obvious that in the proposed quantization scheme the set of coupling constants depends on the way of quantization, so does the solutions (potentials) of the hierarchy, \( \tau \)- or the BA function etc.

Now the quantization gives the representation of the Heisenberg operators, satisfying the string equation

\[ [\hat{P}, \hat{Q}] = 1 \tag{102} \]

in the “momentum” (spectral) space

\[ \hat{P} = \lambda \]
\[ \hat{Q} = \frac{\partial}{\partial \lambda} + Q(\lambda) \tag{103} \]

From the point of view of the KP hierarchy, we will also add some additional requirements on the “spectral parameter” implying that

\[ \lambda = W(\mu) = \mu^p \tag{104} \]

then \((p, q)\) models correspond to the case where \(Q(\lambda)\) should be a polynomial of \(\mu\) of degree \(q\) \(\mathbb{Z}\): (while the corresponding wave functions should have specific asymptotics when \(\mu \to \infty\)).

Wave functions of this problem appear to be the Baker-Akhiezer functions of the corresponding integrable system and when acting on wave functions conditions \((103)\) get the form of the Kac-Schwarz equations \([35, 36]\):

\[ \lambda \varphi_i(\mu) = \sum_j W_{ij} \varphi_j(\mu) \]
\[ \hat{A}\varphi_i(\mu) = \sum_j A_{ij} \varphi_j(\mu) \]  (105)

where

\[ \lambda = W(\mu) \sim \mu^p \]

\[ A^{(W,Q)} \equiv s^{(W,Q)}(\mu) \frac{1}{W'(\mu)} \frac{\partial}{\partial \mu} \left[ s^{(W,Q)}(\mu) \right]^{-1} = \]

\[ = \frac{1}{W'(\mu)} \frac{\partial}{\partial \mu} - \frac{1}{2} W''(\mu) + Q(\mu) \]  (106)

The standard way to construct wave functions of the theory is to define the Fock vacuum by

\[ \hat{A} \Psi_0 = 0 \]  (107)

with an obvious solution

\[ \Psi_0 = \sqrt{W'(\mu)} \exp \int QdW \]  (108)

and the corresponding \( \tau \)-function is a determinant projection of higher states

\[ \Psi_n \sim W^n \Psi_0 \]  (109)

to the states with a canonical asymptotics

\[ \varphi_i(\mu) \rightarrow \mu^{i-1} \]  (110)

forming the conventional basis in the space of wave functions – the point of infinite-dimensional Grassmannian.

The only simple case arises when the Kac-Schwarz equations (105) have trivial solution, i.e. when \( p = 1 \). Starting from normalization \( \varphi_1(\mu) = 1 \) (corresponding to \( \Psi_0 = \exp \int Qd\mu \)), and using first of eqns. (105) one can always get \( \Psi_n = \mu^n \exp \int Qd\mu \rightarrow \varphi_i(\mu) = \mu^{i-1} \) exactly. Then the second condition of (105) is fulfilled automatically for any \( Q(\mu) \).

However, one can see that the corresponding solutions are related to topological models by a kind of Fourier transformation. Indeed, it has been observed [33, 34] that the system

\[ exp \int Qd\mu \]
of equations [105] possesses a duality symmetry which relates \((p,q)\) to \((q,p)\) solution. The duality transformation for the Baker-Akhiezer functions looks like

\[
\psi^{(P,Q)}(z) = [P'(z)]^{1/2} \int dQ \ e^{P(z)Q(x)} \psi^{(Q,P)}(x)[Q'(x)]^{-1/2}
\]  

(111)

and it can be also written for the basis vectors in the Grassmannian

\[
\phi_i(\mu) = [W'(\mu)]^{1/2} \exp(-S_{W,Q}|_{x=\mu}) \int dM_Q(x)f_i(x) \exp S_{W,Q}(x,\mu)
\]  

(112)

with

\[
dM_Q(x) = dx \sqrt{Q'(x)}
\]

\[
S_{W,Q}(x,\mu) = - \int^x WdQ + W(\mu)Q(x)
\]  

(113)

and for the partition functions

\[
\tau^{(W,Q)}[M] = \int DX \tau^{(Q,W)}[X] \exp \left\{ Tr[1/2 \log Q'(X) + \int_M X W(z)dQ(z) + W(M)Q(X)] \right\}
\]  

(114)

(here, better to consider normalized partition function \(\tau^{(W,Q)} \rightarrow Z^{(W,Q)} \rightarrow \Psi_{BA}^{(W,Q)}(t_k - \frac{1}{k}TrM^{-k})\). It makes possible to obtain solutions for nontrivial models – topological \((p,1)\) models [3] and their Landau-Ginzburg deformations [32].

\[
\varphi_i(\mu) = \sqrt{p\mu^{p-1}} \exp \left( - \sum t_k \mu^k \right) \int dx \ x^{i-1} \exp(-V(x) + x\mu^p)
\]  

(115)

which are dual to \((1,p)\) model in the above sense.

Here, we immediately run into a puzzle: how to interpret this from the point of view of quantization theory. Indeed, the duality transformation [112] is nothing but a transformation from \(\hat{p}\) to \(\hat{q}\) quantization procedure or from one to another representation of quantum algebra and as it is well-known the quantization should be independent of this. It means that \((p,1)\) and \((1,p)\) or trivial theory are in fact equivalent as string theories, i.e. the nontrivial partition functions for \((p,1)\) theories corresponding to some well-known topological theories (twisted \(N=2\) Landau-Ginzburg theories) give nothing new.
from a physical point of view\textsuperscript{9}. Thus, the first puzzle is that $\tau^{(1,p)} \equiv 1$ seems to contain all the “topological” information as a “dual” partition function does. Second, the topological numbers perhaps should not be considered as "real observables" of the theory – they rather correspond to a sort of combinatorial factors for Feynman diagrams in particle theories.

This is actually a new feature of string theory if we compare it to quantum field theory – i.e. even trivial target-space model can possess rich and nontrivial structure. The Virasoro action in these theories naturally follows from (103), (105).

Let us finally add few comments about holomorphic anomaly. The “quasiclassical” $\tau$-function obeys a homogeneous relation

$$ \sum t_j \frac{\partial}{\partial t_j} \log \tau_0 = 2 \log \tau_0 $$

spoilt by the contribution of the one-loop correction, having the form, for example, for the (2, 1) theory

$$ \sum t_j \frac{\partial}{\partial t_j} \log \tau - 2 \log \tau = -\frac{1}{24} $$

The similiar expressions appear when one considers the logarithm of the partition function for the higher-dimensional theories \textsuperscript{31} and this should mean that the expression (117) should have a similiar nature.

\section{Conclusion}

Now let us briefly summarize the main ideas presented above. We have tried to demonstrate that appearing in the context of matrix models effective target-space description of string theory can be a useful tool for constructing a nonperturbative string field theory. Indeed, the space of coupling constants $\{T_k\}$ may be considered for simplest string models as a space of background fields and one might hope to get a second-quantized theory by quantization of appearing there structures. It has been shown by Krichever

\textsuperscript{9}(2,1) model corresponds to pure topological gravity and generates intersection indices on module spaces of Riemann surfaces with punctures - it appears that the intersection indices in topological gravity carry a physical meaning.
that the "small phase space" in fact can be considered as a certain module space for a spectral surface with marked points if one restricts the order of singularities in these marked points. Then it is natural to consider the quantization of (98) as a quantization of this module space. In fact we have shown above that the particular example of (p, 1) models rather leads to a trivial theory – topological gravity (W-gravity) which is not too much interesting as a target space theory. However, the natural question that appears is a generalization of this approach to more interesting module spaces.

For example, there exists a quite interesting scheme of quantization of module spaces of flat connections and projective structures on Riemann surfaces with punctures [44]. This is not far from what we need in the case of string models: in fact module spaces of flat connections already appeared in the context of two-dimensional Yang-Mills theory and its relation to string theory [15, 46]. It is natural to think that the related string models should have partition (generating) functions more simple than the discussed above theories, being related thus from the point of view of integrable hierarchies with the, say, rational \( \tau \)-functions. The appearance of such \( \tau \)-functions can be interpreted in the way that a restricted amount of world-sheet topologies give contribution to the partition function. In fact [48] there exists another, so-called "character" phase of GKM considered above which is closely related to the Yang-Mills theory and rational \( \tau \)-functions.

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