Relation identities in 3-distributive varieties

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Abstract. Let $\alpha$, $\beta$, $\gamma$, $\Theta$, $\Psi$, $R$, $S$, $T$, $\ldots$ be variables for, respectively, congruences, tolerances and reflexive admissible relations. Let juxtaposition denote intersection. We show that if the identity

$$\alpha(\beta \circ \Theta) \subseteq \alpha \beta \circ \alpha \Theta \circ \alpha \beta$$

holds in a variety $V$, then $V$ has a majority term, equivalently, $V$ satisfies $\alpha(\beta \circ \gamma) \subseteq \alpha \beta \circ \alpha \gamma$. The result is unexpected, since in the displayed identity we have one more factor on the right and, moreover, if we let $\Theta$ be a congruence, we get a condition equivalent to 3-distributivity, which is well-known to be strictly weaker than the existence of a majority term.

The above result is optimal in many senses; for example, we show that slight variations on the above identity, such as $R(S \circ T) \subseteq RS \circ RT \circ RT \circ RS$ hold in every 3-distributive variety. Similar identities are valid even in varieties with 2 non trivial Gumm terms, with no distributivity assumption. We also discuss relation identities in $n$-permutable varieties and present a few remarks about implication algebras, a classical example of a 3-distributive variety without a majority term.

1. Introduction

The famous and well-known characterizations by Jónsson [4] and Day [11] of, respectively, congruence distributive and congruence modular varieties are classical results in the general theory of algebraic systems. Though the results deal with congruences, the proofs make an essential use of reflexive and admissible relations. See, e. g., Jónsson [5], Tschantz [15] and [7, 8, 9, 10, 11] for further examples, details, comments and references about the use of reflexive and admissible relations.

In [11] we used relation identities in order to approach the problem of the relationships between the numbers of Gumm and of Day terms in a congruence modular variety. While the problem is still largely unsolved, the partial results confirm the usefulness of the approach. Though our main aim has been the study of relation identities satisfied in congruence modular varieties, we encountered delicate issues already in the relatively well-behaved case of 4-distributive varieties [10]. Here we show that the problem of the satisfaction of relation identities is not trivial even for identities related to 3-distributivity. A variety $V$ is $n$-distributive if $V$ satisfies the congruence

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identity $\alpha(\beta \circ \gamma) \subseteq \alpha \beta \circ \alpha \beta \circ \alpha \gamma \ldots$ ($n$ factors), where “$n$ factors” means $n - 1$ occurrences of $\circ$ on the right-hand side. In general, we say that some identity holds in a variety $\mathcal{V}$ if the identity holds in the set of reflexive and admissible relations on every algebra in $\mathcal{V}$. We shall adopt the conventions introduced in the first paragraph of the abstract, in particular, juxtaposition denotes intersection and $R, S \ldots$ are interpreted as reflexive and admissible relations. All the binary relations considered in this note are assumed to be reflexive, hence we shall sometimes simply say admissible in place of reflexive and admissible.

The definition of $n$-distributivity as given above is equivalent to the classical notion introduced in Jónsson [4]; there he showed that a variety is congruence distributive if and only if it is $n$-distributive for some $n$. As remarked in [8], a recent result by Kazda, Kozik, McKenzie, Moore [6] can be used to strengthen Jónsson’s Theorem to the effect that a variety $\mathcal{V}$ is congruence distributive if and only if $\mathcal{V}$ satisfies the relation identity

$$\alpha(S \circ T) \subseteq \alpha S \circ \alpha T \circ \alpha S \ldots \quad (k \text{ factors}), \quad (1.1)$$

for some $k$. Hence it is interesting to study relation identities satisfied by congruence distributive varieties; in particular, to evaluate the best possible value of $k$ for which $(1.1)$ holds in any given variety.

It is easy to see that if $\mathcal{V}$ has a majority term (and is not a trivial variety) then the best possible value of $k$ for $\mathcal{V}$ in $(1.1)$ is 2. In [10] we provided examples of 4-distributive varieties for which the best possible value of $k$ is 4. The main results of the present note assert that there is no variety for which the best possible value of $k$ is 3, even if we allow $T$ to be a tolerance; moreover, the best value of $k$ is 4 for every 3-distributive not 2-distributive variety. Combining the present results with [10], we get that the best value of $k$ in $(1.1)$ does not determine the distributivity level of a variety.

We do not know whether there is a variety for which $k = 5$ is the best value. More generally, we know little about the best possible values of $k$ for arbitrary congruence distributive varieties, though some bounds are provided in [8] [10] for, say, varieties with $h$ directed Jónsson terms and for varieties with an $n$-ary near-unanimity term. Bounds are provided in Section 4 below for congruence distributive $n$-permutable varieties.

The identities we consider in this note are extremely sensible to minimal variations. For example, if $k = 3$ and we let $T$ be a congruence in $(1.1)$, we get a condition equivalent to 3-distributivity. See identity (3.3) in Theorem 3.1. It is well-known that there are 3-distributive not 2-distributive varieties; in particular, any such variety fails to satisfy the identity displayed in the abstract, but does satisfy $\alpha(S \circ \gamma) \subseteq \alpha S \circ \alpha \gamma \circ \alpha S$. With respect to the displayed formula in the abstract, the symmetry is only apparent. Here the relation assumed to be a congruence is placed in the middle of the right-hand side; in the displayed formula in the abstract it appears two times on the edges.

Further relation identities satisfied by 3-distributive varieties are presented in Section 3. There we show that, in many cases, we do not even need the
equation \( j_2(z, y, z) = z \) from the set of Jónsson’s equations characterizing 3-distributivity; namely, the results apply to congruence modular varieties with 2 non trivial Gumm terms. Congruence distributive and congruence modular varieties which are further assumed to be \( n \)-permutable are discussed in Section \( 4 \). Finally, in Section \( 5 \) we show directly by an example that the displayed identity in the abstract fails in the variety of implication algebras. We also show that most properties of implication algebras are preserved if we add a 4-ary near-unanimity term.

2. A tolerance identity implying 2-distributivity

Theorem 2.1. If \( V \) is a variety and the identity

\[
\alpha(\beta \circ \Theta) \subseteq \alpha\beta \circ \alpha\Theta \circ \alpha\beta
\]

(2.1)

holds in every algebra \( A \) in \( V \), for all congruences \( \alpha \), \( \beta \) and tolerance \( \Theta \) on \( A \), then \( V \) has a majority term. In fact, it is enough to assume that (2.1) holds in \( F_V(3) \), the free algebra in \( V \) generated by 3 elements.

Theorem 2.1 shall be proved after a lemma, whose proof is standard, but not completely usual, since it involves a tolerance, not only congruences. Before stating the lemma we recall the Jónsson terms for 3-distributivity. These terms will appear only marginally in the proof of the lemma and will assume a prominent role in Section 3. According to [4], a variety \( V \) is \( \Delta_3 \) if and only if \( V \) has terms \( j_1 \) and \( j_2 \) satisfying the following set of equations:

\[
\begin{align*}
(J_L) \ x &= j_1(x, x, z), & (J_C) \ j_1(x, z, z) &= j_2(x, z, z), & (J_R) \ j_2(x, x, z) &= z, \\
(J_1) \ x &= j_1(x, y, x), & (J_2) \ j_2(z, y, z) &= z.
\end{align*}
\]

It is implicit in [4], and only a minor part of it, that a variety \( V \) is \( \Delta_3 \) if and only if \( V \) obeys the definition of 3-distributivity that we have given in the introduction. In the equations (J) we could have done with just two variables, but we shall not need this observation, which is important and relevant in different contexts. We believe that here maintaining three variables looks intuitively clearer; the same shall apply to some other sets of equations below.

Notice the asymmetry between \( j_1 \) and \( j_2 \) in the equations (J). We get different identities if we reverse both the order of the terms and of the variables. A probably better way to appreciate the asymmetry goes as follows. Recall that a majority term is a term \( j_1 \) satisfying the equations \( (J_L), (J_1) \), as well as \( j_1(x, z, z) = z \). In (J), \( j_1 \) is required to satisfy two thirds of the majority condition. On the other hand, a Maltsev term for permutability is a term \( j_2 \) satisfying \( (J_R) \), as well as \( x = j_2(x, z, z) \). In (J), \( j_2 \) is required to satisfy one third of the majority condition and one half of the permutability condition. The terms \( j_1 \) and \( j_2 \) are tied by equation \( (J_C) \), but \( (J_C) \) is not symmetric, either. Had we required \( j_1(x, x, z) = j_2(x, x, z) \) in place of \( (J_C) \), we would have obtained \( x = z \), by \( (J_L) \) and \( (J_R) \). In spite of the apparent similarity, \( j_1 \) and \( j_2 \) behave in a quite different way! The above observation is only part
of much more general considerations, see [11, Remark 4.2]. Now we state and prove the lemma we need. In some of the equations below we shall use a semicolon in place of a comma in order to improve readability.

**Lemma 2.2.** If $F_V(3)$ satisfies the identity (2.1), then $V$ has a 5-ary term $w$ such that the following equations hold in $V$:

\begin{equation}
\begin{align*}
(A) & \quad x = w(x, x, z; x, z) \\
(B) & \quad w(x, x, z; z, x) = z \\
(C) & \quad x = w(x, y, x; y, x) \\
(D) & \quad w(x, y, x; y, y) = x.
\end{align*}
\end{equation}

**Proof.** Let $x$, $y$ and $z$ be the generators of $F_V(3)$ and let $\alpha$ and $\beta$ be the congruences generated by, respectively, the pairs $(x, z)$ and $(x, y)$. Let $\Theta$ be the smallest tolerance containing $(y, z)$, thus $(x, z) \in \alpha(\beta \circ \Theta)$, as witnessed by the element $y$. By assumption, $(x, z) \in \alpha \beta \circ \alpha \Theta \circ \alpha \beta$, hence $x \alpha \beta j_1 \alpha \Theta j_2 \alpha \beta z$, for certain elements $j_1, j_2 \in F_V(3)$. Notice that the condition $j_1 \alpha j_2$ is redundant, since it follows from the assumption that $\alpha$ is a congruence and $j_1 \alpha x \alpha z \alpha j_2$. In fact, we shall not use $j_1 \alpha j_2$ explicitly.

Since we are working in the free algebra generated by $x$, $y$ and $z$, we can think of $j_1$ and $j_2$ as ternary terms. Classically, the $\alpha$-relations imply that $j_1$ and $j_2$ satisfy the equations $(J_1)$ and $(J_2)$ from (2.1), while the $\beta$-relations entail $(J_L)$ and $(J_R)$. Were $\Theta$ assumed to be a congruence, we had also $(J_C)$, thus getting the Jónsson’s condition for 3-distributivity. Since $\Theta$ is only assumed to be a tolerance, we proceed in a different fashion. It is easy to check that

$$\Theta = \{(w(x, y, z; y, z), w(x, y, z; z, y)) \mid w \text{ a 5-ary term of } V\}.$$  

(2.3)

Indeed, the relation defined by the condition on the right in (2.3) is obviously reflexive, admissible, symmetrical and contains $(y, z)$. On the other hand any reflexive, admissible and symmetrical relation containing $(y, z)$ has to contain all the pairs appearing on the right in (2.3). Hence equality follows.

Since $j_1(x, y, z) \Theta j_2(x, y, z)$ by construction, then by (2.3) there exists some 5-ary term $w$ such that

$$j_1(x, y, z) = w(x, y, z; y, z) \quad \text{and} \quad w(x, y, z; z, y) = j_2(x, y, z).$$  

(2.4)

Since the equations in (2.4) involve only three variables and hold in $F_V(3)$, these equations hold throughout $V$. Notice that in (2.3) we actually need three variables; two variables are not enough. Substituting the equations (2.4) in $(J_L)$, $(J_R)$, $(J_1)$ and $(J_2)$, we get respectively the equations (A), (B), (C) and (D) in (2.2).

**Remark 2.3.** Of course, the proof of Lemma 2.2 provides an “if and only if” condition, since we can retrieve $j_1$ and $j_2$ from $w$, using the equations (2.4); then the classical homomorphism argument shows that these three terms witness that (2.1) holds throughout $V$. However, we do not need this argument, since the conclusion of Theorem (2.1) is much stronger; indeed, the existence of a majority term implies the tolerance identity $\alpha(\beta \circ \Theta) \subseteq \alpha \beta \circ \alpha \Theta$, actually, the relation identity $R(S \circ T) \subseteq RS \circ RT$. In other words, we get the quite
surprising result that, globally, that is, within a variety, the locally weaker tolerance identity $\alpha(\beta \circ \Theta) \subseteq \alpha \beta \circ \alpha \Theta \circ \alpha \beta$ implies (and hence is equivalent to) the much stronger relation identity $R(S \circ T) \subseteq RS \circ RT$. Notice that the latter identity is stronger in two senses: first, we have 2 factors instead of 3 factors on the right; moreover, it is more general, since it deals with reflexive and admissible relations, rather than with tolerances or congruences.

But we are getting too far ahead! A proof of Theorem 2.1 is necessary in order to justify the above comment.

**Proof of Theorem 2.1.** Suppose that $V$ satisfies (2.1). By Lemma 2.2, we have a 5-ary term $w$ satisfying the equations (2.2). Define

$$m(x, y, z) = w(w(x, y, z; y, z), y; z, w(x, z, y; y, z)).$$

(2.5)

Using (2.5) and (2.2) repeatedly, we get

$$m(x, x, z) = w(w(x, x, z; x, z), x; z, w(x, z, x; x, z)) = w(x, z, x; x, z) = z,$$

where the subscripts indicate the equations of (2.2) that we have used, in the respective order. Hence $m$ is a majority term and the theorem is proved. □

3. Relation identities satisfied in 3-distributive varieties

Since there are 3-distributive varieties without a majority term (see, e. g., Section 5 below), we have that 3-distributive varieties do not necessarily satisfy identity (2.1). On the other hand in the present section we show that 3-distributive varieties satisfy identities which are slightly weaker than (2.1), but very similar to it. For example, 3-distributive varieties do satisfy $R(S \circ T) \subseteq RS \circ RT \circ RT \circ RS$. Notice that if we take $R$, $S$ and $T$ to be congruences in the above identity, we get back 3-distributivity, hence the result is optimal.

As mentioned in the introduction, the results from Kazda, Kozik, McKenzie and Moore [6] and the observations in [8] show that, for every $n$, there is some (possibly quite large) $k$ such that $\alpha(S \circ T) \subseteq \alpha S \circ \alpha T \circ \alpha S \circ \ldots \circ \alpha S$ (k factors) holds in every $n$-distributive variety. We show that in the case $n = 3$ the best possible value of $k$ is 4, hence not particularly large. Moreover, we can equivalently consider a reflexive and admissible relation $R$ in place of the congruence $\alpha$.

**Theorem 3.1.** If $V$ is a 3-distributive variety, then $V$ satisfies

$$R(S \circ T) \subseteq RS \circ RT \circ RT \circ RS,$$

(3.1)

$$R(S \circ T) \subseteq RS \circ RT \circ RS \circ RT,$$

(3.2)

and

$$\alpha(S \circ \gamma) \subseteq \alpha S \circ \alpha \gamma \circ \alpha S.$$  

(3.3)
Proof. Suppose that \((a, c) \in R(S \circ T)\), hence \(a R c\) and \(a S b T c\), for some \(b\). Because of 3-distributivity, we have Jónsson’s terms \(j_1\) and \(j_2\) from [4], as recalled in [4] at the beginning of the previous section. Consider the elements 
\[ e = j_1(j_1(a, b, c), b, c), \quad f = j_1(j_1(a, c, c), c, c) = (j_1c)\quad j_2(j_2(a, c, c), c, c) \quad \text{and} \quad g = j_2(j_2(a, c, c), c, c). \]
In order to prove (3.1) we shall show that \(a RS e RT f RT g RS c\). First, we perform the following computations in which, for clarity, we underline the elements which are moved by the appropriate relations and, as usual, we indicate the identities we are using by superscripts.

\[ a = (j_1c) j_1(j_1(a, a, c), a, c) S j_1(j_1(a, b, c), b, c) = e, \]
\[ a = (j_1) j_1(j_1(a, b, c), b, c) R j_1(j_1(a, b, c), b, c) = e, \]
\[ e = j_1(j_1(a, b, c), b, c) T j_1(j_1(a, c, c), c, c) = f. \]

In order to show \(e R f\) we perform a preliminary computation. We have
\[ e = j_1(j_1(a, b, c), b, c) R j_1(j_1(a, b, c), b, c) = (j_1) c, \]

hence \(e R c\) and
\[ e = (j_1c) j_2(j_2(a, a, c), a, c) R j_2(j_2(a, c, c), a, c) = f. \]

So far, we have showed \(a RS e RT f\). Notice that we have not used the identity \((j_2)\), yet. Next, we have
\[ f = (j_1c) j_2(j_2(a, c, c), j_2(b, c, c), c) T j_2(j_2(a, c, c), j_2(b, c, c), c) = g, \]
\[ f = (j_1c) j_2(j_2(a, c, c), c, j_2(a, a, c)) R j_2(j_2(a, c, c), c, j_2(a, a, c)) = (j_2) j_2(a, c, c), \]

hence \(f R j_2(a, c, c)\) and \(f R c\), so
\[ f = (j_2) j_2(f, j_2(b, c, c), f) R j_2(j_2(a, c, c), j_2(b, c, c), c) = g. \]
Finally,
\[ g = j_2(j_2(a, c, c), j_2(b, c, c), c) S j_2(j_2(b, c, c), j_2(b, c, c), c) = (j_1c) c, \]
\[ g = j_2(j_2(a, c, c), j_2(b, c, c), c) R j_2(j_2(c, c, c), j_2(b, c, c), c) = (j_2) c. \]

Thus we have proved identity (3.1). Identity \((3.2)\) is proved by a small variation; this time we consider the same elements \(e\) and \(f\), but we take \(g^* = j_2(j_2(a, c, c), j_2(a, b, c), c)\) in place of \(g\). We already proved \(a RS e RT f\). In order to complete the proof of (3.2) we compute
\[ f = (j_1c) j_2(j_2(a, c, c), j_2(a, a, c), c) S j_2(j_2(a, c, c), j_2(a, b, c), c) = (j_1c) j_2(j_2(a, c, c), j_2(a, b, c), c) = g^*, \]
\[ f = (j_2) j_2(f, j_2(a, b, c), f) R j_2(j_2(a, a, c), j_2(a, b, c), c) = g^*, \]
\[ g^* = j_2(j_2(a, c, c), j_2(a, b, c), c) T j_2(j_2(a, a, c), j_2(a, c, c), c) = (j_1c) c, \]
\[ g^* = j_2(j_2(a, c, c), j_2(a, b, c), c) R j_2(j_2(c, c, c), j_2(b, c, c), c) = (j_2) c, \]

where we have used the previously proved identities \(f R j_2(a, c, c)\) and \(f R c\).

Thus \(f RS g^* RT c\) and \((3.2)\) follows.

Identity \((3.3)\) is immediate from \((3.1)\), since \(\alpha\) and \(\gamma\) are assumed to be congruences, hence \(\alpha \gamma\) is a transitive relation. There is a direct simpler proof.
of \((3.3)\): just notice that if \((a, c) \in \alpha(S \circ \gamma)\), as witnessed by \(b\), then \(a \in \alpha S j_1(a, b, c) \circ \gamma j_2(a, b, c)\) \(\alpha S c\). For example, \(j_1(a, b, c) \gamma j_1(a, c, c) = j_2(a, c, c) \gamma j_2(a, b, c)\). The rest is more direct and easier. Cf. also [7] Remark 17.

Let us say that a variety \(\mathcal{V}\) has two non trivial Gumm terms if \(\mathcal{V}\) has terms \(j_1\) and \(j_2\) satisfying the identities \((J_L)\), \((J_C)\), \((J_R)\) and \((J_1)\) from [11] in Section 2 thus we are leaving out identity \((J_2)\). Sometimes a variety as above is said to have 3 Gumm terms, since a trivial projection \(j_0\) onto the first coordinate is counted. Notice that some authors give the definition of Gumm terms with the ordering of variables and of terms reversed. This applies to some papers of ours, too, but here it is more convenient to maintain the analogy with the common condition for 3-distributivity. A variety \(\mathcal{V}\) has two non trivial Gumm terms if and only if \(\mathcal{V}\) satisfies the congruence identity \(\alpha(\beta \circ \gamma) \subseteq \alpha \beta \circ \alpha(\gamma \circ \beta)\). Notice that if we take converses and exchange \(\beta\) and \(\gamma\), we get \(\alpha(\beta \circ \gamma) \subseteq \alpha(\gamma \circ \beta) \circ \alpha \gamma\). See, e. g., [8] [11] for further details and information.

If \(S\) is a reflexive binary relation on some algebra, we denote by \(\overline{S}\) the smallest admissible relation containing \(S\).

**Theorem 3.2.** If \(\mathcal{V}\) has two non trivial Gumm terms, then \(\mathcal{V}\) satisfies

\[
R(S \circ T) \circ RW \subseteq RS \circ RT \circ R(S \cup T \cup W). \tag{3.4}
\]

**Proof.** Suppose that \((a, c^*) \in R(S \circ T) \circ RW\) with \(a R c^*\) and \(a S b T c RW c^*\).

Consider the elements \(e\) and \(f\) introduced in the proof of Theorem 3.1. We have \(a RS e RT f\) under the present assumptions, too, since the identity \((J_2)\) has not been used in the proof of the above relations. In order to prove \((3.4)\) it is then enough to compute

\[
f = j_2(j_2(a, c, c^*), j_2(b, b, c^*), c) \overline{S \cup T \cup W} j_2(j_2(a, c, c^*), j_2(b, b, c^*), c) = (\overline{J_L}) c^*,
\]

\[
f = j_2(j_2(a, c, c^*), c, c) R j_2(j_2(a, c, c^*), c, c) = (\overline{J_R}) c^*.
\]

If \(R\) is a binary relation, we let \(R^*\) denote the transitive closure of \(R\).

**Corollary 3.3.** (1) A 3-distributive variety satisfies

\[
R^*(S \circ T)^* = (RS \circ RT)^* \text{ and, more generally,} \tag{3.5}
\]

\[
(R \circ V)^*(S \circ T)^* = (RS \circ RT \circ VS \circ VT)^* . \tag{3.6}
\]

(2) If \(\mathcal{V}\) has two non trivial Gumm terms, then \(\mathcal{V}\) satisfies

\[
R^*(S \circ T)^* = (RS \circ RT)^* \circ R(S \cup T), \text{ and} \tag{3.7}
\]

\[
R^*S^* = (RS)^*. \tag{3.8}
\]

**Proof.** We first prove \((3.3)\). Let \(S^n\) denote the iterated relational composition of \(S\) with itself with a total of \(n\) factors. By taking \(W = 0\) and \(S^n\) in place of both \(S\) and \(T\) in \((3.4)\), we get \(RS^{2n} = R(S^n \circ S^n) \subseteq RS^n \circ RS^n \circ RS^n\). Since \(RS^* = \bigcup_{n \in \omega} RS^n\), we get by induction that \(RS^* \subseteq (RS)^*\). By taking \(R^*\) in place of \(R\) in the above identity and then exchanging the role of \(R\) and \(S\) we get \(R^*S^* \subseteq (R^*S)^* \subseteq ((RS)^*)^* = (RS)^*\). The reverse inclusion is trivial.
Having proved (3.8), then by taking $S \circ T$ in place of $S$ in (3.8), we get $R^*(S \circ T)^* = (R(S \circ T))^*$. By taking $W = S \circ T$ in (3.4), we get $R(S \circ T) \circ R(S \circ T) \subseteq RS \circ RT \circ R(S \circ T)$. Then by induction, always factorizing out the first compound factor on the left, we get $(R(S \circ T))^* \subseteq (RS \circ RT)^* \circ R(S \circ T)$. Applying again (3.4) with $W = 0$, we get $R(S \circ T) \subseteq RS \circ RT \circ R(S \cup T)$. Summing everything up, we get the $\subseteq$ inclusion in (3.7). Again, the reverse inclusion is trivial, since $S \circ T \supseteq S \cup T$. Now (3.5) is immediate from (3.7) and (3.2). Then (3.2) follows by applying (3.5) twice. □

Remark 3.4. Theorems 3.1 and 3.2 can be improved in many ways. First, notice that the “middle” element $j_1(j_1(a, c, c), c, c)$ in the above computations does not depend on $b$. Hence the proofs provide a bound for $R(S_1 \circ T_1)(S_2 \circ T_2)(S_3 \circ T_3)\ldots$. Moreover, the element $j_1(j_1(a, c, c), c, c)$ is the same in the proofs of 3.1 and 3.2. In addition, the arguments in the proof of Theorem 3.2 clearly apply to 3-distributive varieties, as well. Summing everything up, we get that a 3-distributive variety satisfies

$$R(S_1 \circ T_1)(S_2 \circ T_2)(S_3 \circ T_3)\ldots \subseteq R(RS_1 \circ RT_1)(RS_2 \circ RT_2)(RS_3 \circ RT_3)\ldots \circ R(RS_1 \circ RT_1)(RT_1 \circ RS_1)(RS_2 \circ RT_2)(RT_2 \circ RS_2)\ldots (S_1 \cup T_1)(S_2 \cup T_2)\ldots$$

and that a variety with 2 Gumm terms satisfies

$$R(S_1 \circ T_1)(S_2 \circ T_2)(S_3 \circ T_3)\ldots \subseteq R(RS_1 \circ RT_1)(RS_2 \circ RT_2)(RS_3 \circ RT_3)\ldots \circ R(S_1 \cup T_1)(S_2 \cup T_2)(S_3 \cup T_3)\ldots$$

Corollary 3.3 solves affirmatively [2] Problem 2.12] in the special cases of 3-distributive varieties and of varieties with 2 non-trivial Gumm terms. Namely, in such varieties, lattices of admissible preorders (i.e., reflexive and transitive relations) are, respectively, congruence distributive and congruence modular. In passing, let us mention that the assumption that $V$ is locally finite can be weakened to $F_V(3)$ finite in [2] Theorem 2.6, by essentially the same proof.

We have not yet thoroughly checked how much the methods in the present section overlap with [0].

4. Assuming $n$-permutability

The length of a chain of iterated relational compositions is bounded in an $n$-permutable variety. Hence, assuming also congruence distributivity, we get bounds for the value of $k$ in the identity (1.1) from the introduction. This observation is dealt with in the following Proposition 4.1. Since we are dealing with admissible relations, not congruences, the bound is not exactly $n$, in general. Recall that a variety $V$ is $n$-permutable if the congruence identity $\beta \circ \gamma \circ \beta \cdots = \gamma \circ \beta \circ \gamma \cdots$ (n factors on both sides) holds in $V$.

Recall that if $R$ is a reflexive binary relation on some algebra, $\overline{R}$ denotes the smallest admissible relation containing $R$ and $R^*$ denotes the transitive closure
of $R$. Parts of the following proposition are known. See, e. g., Hagemann, Mitschke \footnote{[3]} and a reference there.

**Proposition 4.1.** Suppose that $n \geq 2$ and $\mathcal{V}$ is a variety.

1. Each of the following identities is equivalent to $n$-permutability of $\mathcal{V}$.

\[ R_1 \circ R_2 \circ \cdots \circ R_n \subseteq R_1 \cup R_2 \cup R_3 \circ \cdots \circ R_{n-1} \cup R_n, \quad (4.1) \]
\[ R^n = R \circ R \circ \ldots \quad (n - 1 \text{ factors}), \quad (4.2) \]
\[ (S \circ T)^n = S \circ T \circ S \circ T \circ \ldots \quad (2n - 2 \text{ factors}), \quad (4.3) \]
\[ (S \circ T)^* = S \circ T \circ S \circ S \circ T \circ \ldots \quad (2n - 2 \text{ factors}). \quad (4.4) \]

2. Each of the following identities is equivalent to the conjunction of congruence distributivity and of $n$-permutability of $\mathcal{V}$.

\[ (\Theta(S \circ T))^* = \Theta S \circ \Theta T \circ \Theta S \circ \Theta T \circ \ldots \quad (2n - 2 \text{ factors}), \quad (4.5) \]
\[ (\Theta(S \circ T))^* = \Theta S \circ \Theta T \circ \Theta S \circ \Theta S \circ \Theta T \circ \ldots \quad (2n - 2 \text{ factors}). \quad (4.6) \]

3. Each of the following identities is equivalent to the conjunction of congruence modularity and of $n$-permutability of $\mathcal{V}$.

\[ (\Theta R^*)^* = \Theta R \circ \Theta R \circ \ldots \quad (n - 1 \text{ factors}), \quad (4.7) \]
\[ \Theta R^n \subseteq \Theta R \circ \Theta R \circ \ldots \quad (n - 1 \text{ factors}). \quad (4.8) \]

In items (2) and (3) we can equivalently consider a congruence $\alpha$ in place of the tolerance $\Theta$. In 3-distributive varieties we can consider a reflexive and admissible relation $R$ in place of $\Theta$.

**Proof.** If, say, $\mathcal{V}$ is a 3-permutable variety, then, by \footnote{[3]}, there are terms $t_1$ and $t_2$ such that $x = t_1(x, y, y)$, $t_1(x, x, y) = t_2(x, y, y)$ and $t_2(x, x, y) = y$ are equations valid in $\mathcal{V}$. If $a R_1 b R_2 c R_3 d$, then $a = t_1(b, c, c) R_1 \cup R_2 \cup R_3 t_2(b, c, d) = d$, hence (4.1) follows in the case $n = 3$. In general, for $n \geq 2$, the same argument shows that every $n$-permutable variety satisfies (4.1).

We now show that (4.1) $\Rightarrow$ (4.2) $\Rightarrow$ (4.3) $\Rightarrow$ (4.4) $\Rightarrow$ (4.1) hold in every algebra. Notice that in (4.2), the left-hand side is always larger than the right-hand side, hence it is enough to prove the reverse inclusion.

Taking $R_1 = R_2 = \cdots = R$ in (4.1), we get $R \circ R \circ \ldots \ (n \text{ factors}) \subseteq R \circ R \circ \ldots \ (n - 1 \text{ factors})$; then (4.2) follows from a trivial induction.

If $S$ and $T$ are reflexive and admissible, then $S \circ T$ is reflexive and admissible, too, hence (4.3) is the special case of (4.2) obtained by considering $S \circ T$ in place of $R$. In turn, if $0$ denotes the minimal congruence, i.e., the identity relation, then taking $T = 0$ in (4.3) we get back (4.2).

In order to prove that (4.2) implies (4.4), observe that $S \circ T$ is a reflexive and admissible relation which contains both $S$ and $T$, thus $S \circ T \supseteq S \cup T$. Since $S, T \subseteq S \cup T$, hence $S \circ T \subseteq S \cup T \circ S \cup T$, we get $(S \circ T)^* = (S \cup T)^*$. By taking the admissible relation $S \cup T$ in place of $R$ in (4.2) and using alternatively $S \cup T \subseteq S \cup T \supseteq T \circ S$, we get (4.4).
Finally, by taking $S$ and $T$ congruences in (4.4), we get 

$$\alpha + \beta = \alpha \circ \beta \circ \alpha \circ \ldots$$

($n$ factors), an identity obviously equivalent to $n$-permutability. Notice that, since congruences are transitive, $n-2$ factors annihilate in (4.4), hence we end up with $n$ actual factors. This completes the proof of (1).

If $\mathcal{V}$ is congruence distributive, then, by Kazda, Kozik, McKenzie and Moore [6] and by [8], $\Theta(S \circ T \circ S \circ T \circ \ldots) \subseteq \Theta S \circ \Theta T \circ \Theta S \circ \ldots$, where, for every number of factors on the left, we might have a much larger number of factors on the right. If in addition $\mathcal{V}$ is $n$-permutable, we can apply the identities (4.3) - (4.4) with $\Theta S$ and $\Theta T$ in place of, respectively, $S$ and $T$, obtaining a bounded number of factors on the right. Hence (4.5) - (4.6) follow from the assumptions.

Conversely, by taking $\Theta, S$ and $T$ congruences in either (4.5) or (4.6), we get identities implying congruence distributivity. If we take $\Theta = 1$, that is, the largest congruence, in (4.5) or (4.6), we get, respectively, (4.3) and (4.4), which imply $n$-permutability by (1).

In order to prove (3), we have from [9, Theorem 1] that a congruence modular variety satisfies $\Theta R^* \subseteq (\Theta R)^*$, thus by $n$-permutability and (1.2) we get (4.7). Moreover, (4.7) implies (4.8) and the latter identity implies congruence modularity by [9] (or just take $\Theta = \alpha$ and $R = \beta \circ \alpha \gamma$). As above, (4.7) and (4.8) imply $n$-permutability, taking $\Theta = 1$ and using (1.2).  

For each item in Proposition 4.1, there are obviously many other intermediate equivalent conditions. In order to keep the proposition within a reasonable length, we have not explicitly stated such equivalent conditions. For example, as follows from the proof, the identity $R \circ R \circ \ldots \ (m \text{ factors}) \subseteq R \circ R \circ \ldots \ (n-1 \text{ factors})$ is equivalent to $n$-permutability, for every $m \geq n$. Similarly, we can replace transitive closure with a sufficiently long iteration of compositions in (4.3) - (4.7). Moreover, in (4.5) - (4.6) we can replace $(\Theta(S \circ T))^* = \Theta S \circ \Theta T \circ \ldots$ with either $(\Theta(S \circ T))^* = \Theta S \circ \Theta T \circ \ldots$ or $\Theta(S \circ T)^* \subseteq \Theta S \circ \Theta T \circ \ldots$

5. Implication algebras and a 4-ary near-unanimity term

We denote by $+$, $\cdot$ and $'$ the operations of a Boolean algebra. The variety $\mathcal{I}$ of implication algebras is the variety generated by polynomial reducts of Boolean algebras in which $i(x, y) = xy'$ is the only basic operation. Equivalently, $\mathcal{I}$ is the variety of algebras with a binary operation $i$ which satisfies all the equations satisfied by the term $xy'$ in Boolean algebras. A more frequent description of implication algebras uses the term $x + y'$, instead but Boolean duality implies that we get the same variety. Mitschke [13] showed that the variety of implication algebras is 3-distributive, not 2-distributive, 3-permutable and not permutable.

In the above notations, $i(x, i(y, z))$ represents the Boolean term $f(x, y, z) = x(yz)' = x(y' + z)$. Sometimes it is simpler to deal with the corresponding reduct $\mathcal{I}^-$ of Boolean algebras. Namely, $\mathcal{I}^-$ is the variety generated by reducts
of Boolean algebras having $f$ as the only basic operation. The varieties $\mathcal{I}$ and $\mathcal{I}^-$ have many properties in common, for example, $\mathcal{I}^-$ is still 3-distributive, 3-permutable and, obviously, not 2-distributive and not permutable. The terms $j_1 = f(x, f(x, y, z), z) = x(x' + y' + z) = x(y + z)$ and $j_2 = f(z, y, x) = z(y' + x)$ are Jónsson terms witnessing 3-distributivity. The terms $f$ and $j_2$ are Hagemann-Mitschke terms \[4\] for 3-permutability.

On the other hand, $\mathcal{I}^-$ is much simpler to deal with. For example, free algebras in $\mathcal{I}^-$ are much smaller and more easily describable than free algebras in $\mathcal{I}$. Further details about $\mathcal{I}^-$ can be found in the former version \[12\] of this note. Now \[12\] is largely subsumed by the present version. By Proposition \[4.1\] both $\mathcal{I}$ and $\mathcal{I}^-$ satisfy the identities \[4.5\] and \[4.6\] with four factors on the right and moreover we can take an admissible relation $R$ in place of $\Theta$ in these identities. A direct proof of related identities appeared in \[12\]. By Theorem \[2.1\] we get that $\mathcal{I}$ fails to satisfy the identity \[2.1\]. Again, a direct proof appeared in \[12\]. Some features of the counterexample presented in \[12\] are reported below.

Example 5.1. Let $2$ be the 2-elements Boolean algebra and consider the following elements of $2^5$: $x = (1, 1, 1, 0, 0); y = (1, 0, 0, 1, 0)$ and $z = (0, 1, 0, 1, 1)$. Let $A$ be the subset of $2^5$ consisting of those elements which are $\leq$ than at least one among $x$, $y$ or $z$. Then $A = (A, i)$ is an implication algebra. Let $\alpha$ be the kernel of the second projection, $\beta$ be the intersection of the kernels of the first and of the fifth projections, $\gamma$ be the intersection of the kernels of the third and of the fourth projections.

Let $\Psi$ be the binary relation on $A$ defined as follows: two elements $a, b \in A$ are $\Psi$-related if and only if at least one of the following conditions holds:
(a) both $a \leq x$ and $b \leq x$, or
(b) (either $a \leq y$ or $a \leq z$, possibly both) and (either $b \leq y$ or $b \leq z$, possibly both).

The relation $\Psi$ is trivially symmetric; $\Psi$ is also reflexive, since, by construction, every element of $A$ is $\leq$ than either $x$, $y$ or $z$. We claim that $\Psi$ is admissible in $A$, thus a tolerance. Indeed, if $a \Psi c$ is witnessed by (a), then $ab' \leq a \leq x$ and $cd' \leq c \leq x$, for all $b, d \in A$, hence $ab' \Psi cd'$ (we do not even need the assumption that $c$ and $d$ are $\Psi$-related). Similarly, if $a \Psi c$ is given by (b), then $ab'$ is $\leq$ than either $y$ or $z$ and the same holds for $cd'$, hence $ab' \Psi cd'$. We have proved that $\Psi$ is a tolerance on $A$.

Let $\Theta = \gamma \Psi$. Then $(x, z) \in \alpha(\beta \circ \Theta)$, as witnessed by $y$. The only (other) element of $A$ which is $\alpha \beta$-related to $x$ is $(1, 1, 0, 0, 0) = x(y + z)$ and the only element $\alpha \beta$-related to $z$ is $(0, 1, 0, 0, 1) = z(y' + x)$. No non trivial $\Theta$ relation holds among the above elements, hence $(x, z) \notin \alpha \beta \circ \alpha \Theta \circ \alpha \beta$ and this shows that the identity \[2.1\] fails in $A$. See \[12\] for further details, comments and variations related to the present example.

Mitschke \[14\] showed that $\mathcal{I}$ has no near-unanimity term. It is thus natural to ask whether there exists a 3-permutable, 3-distributive not 2-distributive
variety with a near-unanimity term. We show that some expansion of $I$ has these properties.

Let $u$ be the lattice term defined by $u(x_1, x_2, x_3, x_4) = \prod_{j \neq j}(x_i + x_j)$, where the indices on the product vary on the set $\{1, 2, 3, 4\}$. The term $u$ is clearly a near-unanimity term in every lattice, in particular, in Boolean algebras. Let $I_{nu}$ denote the variety generated by polynomial reducts of Boolean algebras in which both $i$ and $u$ are taken as basic operations. We denote by $I_{nu^-}$ the variety in which only $f$ and $u$ are considered.

**Proposition 5.2.** Both $I_{nu}$ and $I_{nu^-}$ are 3-distributive and 3-permutable varieties with a 4-ary near-unanimity term. The varieties $I_{nu}$ and $I_{nu^-}$ are neither permutative, nor 2-distributive.

**Proof.** Since any permutative congruence distributive variety is arithmetical, hence 2-distributive, it is enough to show that $I_{nu}$ is not 2-distributive. All the rest follows from previous remarks and the mentioned results from [13].

Essentially, we are going to show that the counterexample to 2-distributivity presented in Mitschke [13], and credited in that form to the referee, is closed under $u$ (but notice that we are working with the dual). The subset $B = 2^3 \setminus \{(1, 1, 1)\}$ of the 8-elements Boolean algebra $2^3$ is clearly closed under $i$. It is also closed under $u$, since if $b_1, b_2, b_3, b_4 \in B$, then at least two $b_j$’s have a 0 in the same position, hence $u(b_1, b_2, b_3, b_4)$ has 0 in the same position. If $\alpha$, $\beta$ and $\gamma$ are, respectively, the kernels of the 2nd, 1st and 3rd projections, then $((1, 1, 0), (0, 1, 1)) \in \alpha(\beta \circ \gamma)$, as witnessed by the element $(1, 0, 1)$. But $((1, 1, 0), (0, 1, 1)) \notin \alpha \circ \alpha \circ \alpha$, since the only element which could do the job is $(1, 1, 1)$ and $(1, 1, 1) \notin B$. □

**Remark 5.3.** In [7] we showed that, under a fairly general hypothesis, a variety $V$ satisfies an identity for congruences if and only if $V$ satisfies the same identity for tolerances, provided that only tolerances representable as $R \circ R^-$ are taken into account.

By [13], $I$ is 3-distributive that is, $I$ satisfies the congruence identity $\alpha(\beta \circ \gamma) \subseteq \alpha \circ \alpha \circ \alpha$. On the other hand, $I$ fails to satisfy this identity when $\gamma$ is interpreted as a tolerance, by Example 5.1. Alternatively, use Theorem 2.1 and again [13], where implication algebras are shown not to be 2-distributive. Hence the assumption of representability is necessary in [7], even in the case of 3-distributive 3-permutable varieties. A similar counterexample in the case of 4-distributive varieties has been presented in [10].

The author considers that it is highly inappropriate, and strongly discourages, the use of indicators extracted from the list below (even in aggregate forms in combination with similar lists) in decisions about individuals (job opportunities, career progressions etc.), attributions of funds and selections or evaluations of research projects.

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