EULER-CHOW SERIES FOR SCROLLS AND RULED SURFACES

E. JAVIER ELIZONDO AND ELADIO ESCOBEDO

Abstract. In this article we prove that the Euler-Chow series is rational for ruled surfaces and scrolls, an explicit computation is obtained.

CONTENTS

1. Introduction 1
2. Preliminaries 3
2.1. The Euler-Chow series of projective varieties 5
2.2. Projective bundle formulas 7
3. The Euler-Chow series for ruled surfaces 8
3.1. Decomposable ruled surfaces 9
3.2. Undecomposable ruled surfaces 10
4. Projective bundle formulas 11
5. The Euler-Chow series for normal rational scrolls 15
References 17

1. Introduction

Chow varieties has been of great interest to mathematicians for a long time. However, not much is known about them. The work of H. Blaine Lawson was a breakthrough in the subject and it had many implications in the area. But many other aspects of these varieties are still not known.

In this paper we are interested in computing what has been called the Euler-Chow series. Originally the series came up as a way to compute the Euler characteristic, but later on the series started to have interesting properties that show that it can be a much more important role to play. The

1991 Mathematics Subject Classification. Primary: 14C25 ; Secondary: 14C05.
Key words and phrases. Chow varieties, effective algebraic equivalence, monoid-graded algebras, generating functions, Chow quotients, invariant cycles, projective bundles, Grassmannians, flag varieties.
The first author was supported in part by grant UNAM-DGAPA IN102418.
The second author was partially supported by UNAM-DGAPA IN102418.
first example was done by Blaine Lawson and S.S.T. Yau (see [LY87]). In this case the series was a formal power series with coefficients the Euler characteristic of Chow varieties in the $n$-projective space.

In this same idea the series was formalized and computed for all dimensions for simplicial projective toric varieties in a paper in Compositio by Elizondo (see [Eli94]). The rationality of the series was proved and computed directly from the fan associated to the toric variety.

In the the paper of Elizondo and Lima-Filho (see [EL98]), it is observed that if the Picard group of a projective variety is isomorphic to the integer numbers the series is the Hilbert series. In some way it generalized it. There were other interested observation made there that started showing that the series was in a way generalizing other series of interest in algebraic geometry.

Elizondo an K. Kimura ([EK12]) introduce the motivic version: Instead of taking the Euler-characteristic as the coefficients of the series, they take the pure motive of the Chow varieties. This series generalizes, for dimension zero, the Weil Zeta series and other important series that were studied, like the one by Kapranov.

Finally, there is a strong relation between the Cox ring and the Euler-Chow series. The series is irrational for a family of varieties where the Cox ring is not noetherian. There is a conjecture and more results of the relation between Cox ring and the series in the paper of Xi Chen and E. Javier Elizondo [CE]

In this paper we compute the series for ruled surfaces and normal rational scrolls. Since we give an explicit generating function for the series, we are computing the Euler characteristic of the Chow varieties for this varieties. This is the first time that the series is computed in codimension one where the Picard group is not discrete. However the computation shows that the series behave quite well, the formulas that are obtained are quite natural if we compare with the previos cases, like Hirzebruch surfaces.

First, let us give a definition for the Euler-Chow series. Let $X$ be a projective variety, and let $M$ be the monoid of algebraic homological classes in $H_{2p}(X,\mathbb{Z})$. Let $C_\lambda(X)$ be the Chow variety of all effective cycles with homology class $\lambda$. We define the homological Euler-Chow series as

$$E_p(X) = \sum_{\lambda \in M} \chi(C_\lambda(X)) \cdot t^\lambda$$

We have that $E_p(X)$ is a function from $M$ to $\mathbb{Z}$, sending $\lambda$ to $\chi(C_\lambda)$, in other works it lives in the monoid ring $\mathbb{Z}[M] := \mathbb{Z}^M$. Compare with definition 2.10

Ruled surfaces and scrolls are defined as the projectivization of certain line bundles. We want to express the series of the total space in terms of Euler-Chow series of the bundles involved. In order to do that, we need to write the above definition of the series in a more general setting and define products of series living in different rings. This is done in section two following the paper of Elizondo and Lima-Filho [EL98]. We can say that it is the algebraic version of the homological
version of the series defined above in \([1]\). The algebraic version of the series takes the sum over the path-connected components of the monoid of \(p\)-effective cycles on \(X\). In this paper the Euler-Chow series will be the algebraic version that appears in definition \([2,10]\).

Section three is dedicate to compute and prove the following theorem.

**Theorem 1.1.** Let \(X\) be a ruled surface over curve \(C\), then its Euler-Chow series are given by

\[
E_0(X) = \left(\frac{1}{1-t} \right)^{\chi(X)} = \left(\frac{1}{1-t} \right)^{4-4g},
\]

\[
E_1(X) = \left(\frac{1}{1-t_0} \right)^{2-2g} \left(\frac{1}{1-t_1} \right) \left(\frac{1}{1-t_0-t_1} \right),
\]

\[
E_2(X) = \frac{1}{1-t}.
\]

Section four is dedicated to prove a formula that express the Euler-Chow series of the projectivization of a finite sum of vector bundles over a variety in terms of the Euler-Chow series of products of these vector bundles and the Euler-Chow series of each vector bundle. The terminology and technical details are given there. The main theorem is

**Theorem 1.2.** Let \(E_1, E_2 \text{ and } E_3\) be vector bundles over a complex projective variety \(W\), of ranks \(e_1, e_2 \text{ and } e_3\), respectively, with \(2 \leq p \leq e_1 + e_2 + e_3 - 1\). Then the \(p\)-th Euler-Chow series of \(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)\) is given by

\[
E_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) = \Psi_p^{\#} \left( E_{p-2}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \circ E_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \circ E_{p-1}(\mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \circ E_{p}(\mathbb{P}(E_1)) \circ E_{p}(\mathbb{P}(E_2)) \circ E_{p}(\mathbb{P}(E_3)) \right).
\]

Where the product \(\circ\) is defined in definition \([2,4]\). The generalization of this theorem for a finite sum follows directly and we write this down in theorem \([4,2]\) in page \([15]\).

Section five is dedicated to compute the Euler-Chow series for normal rational scrolls. Our main result is Corollary \([5,2]\) in page \([17]\). At the end we compute an example.

2. Preliminaries

In this section we recall constructions and definitions that were introduced in Elizondo and Lima-Filho \([2,3]\). We omit the proofs of the theorems and refer to the reader to the original paper for them.

Let \(M\) be an abelian monoid \(M\), we denote its multiplication by \(*_M : M \times M \to M\), and the additive operation by \(+ : M \times M \to M\). We say that \(M\) has finite multiplication if \(*_M\) has finite fibers.
Definition 2.1. Given a monoid with finite multiplication $M$, and a commutative ring $S$, denote by $S^M$ the set of all functions from $M$ to $S$. If $f$ and $F'$ are elements in $S^M$, let $f + f' \in S^M$ be defined by pointwise addition, i.e. $(f + f')(m) = f(m) + f'(m)$. Define the product $f * f' \in S^M$ as the “convolution”

$$(f * f')(m) = \sum_{a \ast_M b = m} f(a)f'(b).$$

It is easy to see that $S^M$ then becomes a commutative ring with unity, under these operations.

Definition 2.2. Given a monoid morphism $\Psi : M \to N$, $f \in S^M$ and $g \in S^N$, define $\Psi^\sharp g \in S^M$ and $\Psi^\flat f \in S^N$ by

$$\left(\Psi^\sharp g\right)(m) = g(\Psi(m))$$

and

$$(\Psi^\flat f)(n) = \sum_{m \in \Psi^{-1}(n)} f(m)$$

if $\Psi$ has finite fibers.

Proposition 2.3. Let $M$ and $N$ be monoids with finite multiplication, and let $\Psi : M \to N$ be a monoid morphism. Then

1. The pull-back map $\Psi^\sharp : S^N \to S^M$ is an $S$-module homomorphism.
2. If $\Psi$ has finite fibers then the push-forward map $\Psi^\flat : S^M \to S^N$ is a morphism of $S$-algebras.
3. Any ring homomorphism $\Psi : S \to S'$ induces a ring homomorphism $\Psi_* : S^M \to S'^N$.

The last operation we need to introduce is the following **exterior product**.

Definition 2.4. Given monoids $M$ and $N$, and a commutative ring $S$, one can define a map $\odot : S^M \otimes_S S^N \to S^{M \times N}$. This map sends $f \otimes g$ to the function $f \odot g \in S^{M \times N}$ which assigns to $(m, n)$ the element $f(m)g(n) \in S$.

Proposition 2.5. The operation $\odot$ is bilinear and associative. In other words, the following diagram commutes:

$$
\begin{align*}
(S^M \otimes_S S^N) \otimes_S S^P &\cong S^M \otimes_S (S^N \otimes_S S^P) \\
S^{M \times N} \otimes_S S^P &\cong S^M \otimes_S S^{N \times P}
\end{align*}
$$
2.1. **The Euler-Chow series of projective varieties.** Let $X$ be a projective algebraic variety over $\mathbb{C}$, and let $p$ be an integer such that $0 \leq p \leq \dim X$. The **Chow monoid** $\mathcal{C}_p(X)$ of effective $p$-cycles on $X$ is the free monoid generated by the irreducible $p$-dimensional subvarieties of $X$. It is known that $\mathcal{C}_p(X)$ can be written as a countable disjoint union of projective algebraic varieties $\mathcal{C}_{p,\alpha}(X)$, the so-called **Chow varieties**. We summarize, in the following statements, a few basic properties of the Chow monoids and varieties.

**Properties 2.6.** Let $X$ be a projective variety, and fix $0 \leq p \leq \dim X$.

1. The disjoint union topology on $\mathcal{C}_p(X)$, induced by the classical topology on the Chow varieties, is independent of the projective embedding of $X$; cf. [Hoy66].
2. The restriction of the monoid addition to products of Chow varieties is an algebraic map; [Fri91].
3. An algebraic map $f : X \to Y$ between projective varieties (hence a proper map), induces a natural monoid morphism $f^*: \mathcal{C}_p(X) \to \mathcal{C}_p(Y)$ which is an algebraic continuous map when restricted to a Chow variety; cf. [Fri91]. This is the **proper push-forward map**.
4. A flat map $f : X \to Y$ of relative dimension $k$, induces a natural monoid morphism $f^*: \mathcal{C}_p(Y) \to \mathcal{C}_{p+k}(X)$ which is an algebraic continuous map when restricted to a Chow variety; cf. [Fri91]. This is the **flat pull-back map**.

**Definition 2.7.**
1. We denote by $\Pi_p(X)$, the monoid $\pi_0(\mathcal{C}_p(X))$ of path-components of $\mathcal{C}_p(X)$. This is the monoid of “effective algebraic equivalence classes” of effective $p$-cycles on $X$. We use the notation $a \sim_{\text{alg}} b$ to express that two effective cycles $a, b$ are effectively algebraically equivalent.
2. The group of all algebraic $p$-cycles on $X$ modulo algebraic equivalence is denoted $A_p(X)$, and the submonoid of $A_p(X)$ generated by the classes of cycles with non-negative coefficients is denoted by $A^\geq_p(X)$; cf. Fulton [Ful84, §12]. We use the notation $a \sim_{\text{alg}} b$ to express that two cycles $a, b$ are algebraically equivalent.
3. Let $c : \mathcal{C}_p(X) \to H_{2p}(X,\mathbb{Z})$ be the cycle map into singular homology; cf. [Ful84, §19]. The image of $c$ is denoted by $M_p(X)$.

The following result explains the relation between the monoids above, the proof can be found in [EL98].

**Proposition 2.8.**

1. The Grothendieck group associated to the monoid $\Pi_p(X)$ is $A_p(X)$. In particular, there is a natural monoid morphism $\iota_p : \Pi_p(X) \to A_p(X)$ which satisfies the universal property that any monoid morphism $f : \Pi_p(X) \to G$, from $\Pi_p(X)$ into a group $G$, factors through $A_p(X)$.
2. The image of $\iota_p$ is $A^\geq_p(X)$, and the image of $A^\geq_p(X)$ under the cycle map is $M_p(X)$. 
The monoid surjection \( \tau_p : \Pi_p(X) \to A^\geq_p(X) \) induced by \( \iota_p \) is an isomorphism if and only if \( \Pi_p(X) \) has cancellation law.

Both \( \tau_p : \Pi_p(X) \to A^\geq_p(X) \) and \( c_p : A^\geq_p(X) \to M_p(X) \) are finite monoid morphisms.

The following result is found in Elizondo [Eli94].

**Proposition 2.9.** Given a complex projective algebraic variety \( X \) and \( 0 \leq p \leq \dim X \), the monoids \( C_p(X) \), \( \Pi_p(X) \), \( A^\geq_p(X) \) and \( M_p(X) \) all have finite multiplication.

**Definition 2.10.** The (algebraic) \( p \)-th Euler-Chow function of \( X \) is the function

\[
E_p(X) : \Pi_p(X) \to \mathbb{Z}
\]

\[ \alpha \mapsto \chi(C_{p,\alpha}(X)), \]

which sends \( \alpha \in \Pi_p(X) \) to the topological Euler characteristic of \( C_{p,\alpha}(X) \) (in the classical topology).

We associate a variable \( t^\alpha \) to each \( \alpha \in \Pi_p(X) \) and express the \( p \)-th Euler-Chow function as a formal power series

\[
E_p(X) = \sum_{\alpha \in \Pi_p(X)} \chi(C_{p,\alpha}(X)) \ t^\alpha.
\]

**Remark 2.11.**

One could in a similar fashion define the \( p \)-th Euler-Chow function mapping either \( A^\geq_p(X) \) or \( M_p(X) \) to \( \mathbb{Z} \). These would simply be the functions \( \iota_p(E_p(X)) \) and \( (c_p \circ \iota_p)(E_p(X)) \); cf. Definition 2.2.

**Example 2.12.**

1. If \( X \) is a connected variety, then \( C_0(X) = \bigsqcup_{d \in \mathbb{Z}_+} SP_d(X) \), where \( SP_d(X) \) is the \( d \)-fold symmetric product of \( X \). Therefore, the 0-th Euler-Chow function is given by

\[
E_0(X) = \sum_{d \geq 0} \chi(SP_d(X)) \ t^d = \left( \frac{1}{1-t} \right)^{\chi(X)},
\]

according to McDonald’s formula [Mac62].

2. For \( X = \mathbb{P}^n \), one has \( \Pi_p(\mathbb{P}^n) \cong \mathbb{Z}_+ \), with the isomorphism given by the degree of the cycles. In this case, the \( p \)-th Euler-Chow function was computed in [LY87]:

\[
E_p(\mathbb{P}^n) = \sum_{d \geq 0} \chi(C_{p,d}(\mathbb{P}^n)) \ t^d = \left( \frac{1}{1-t} \right)^{\binom{n+1}{p+1}}.
\]

3. Suppose that \( X \) is an \( n \)-dimensional variety such that \( Pic(X) \cong \mathbb{Z} \), generated by a very ample line bundle \( L \). Then, \( \Pi_{n-1}(X) \cong \mathbb{Z}_+ \) and \( E_{n-1}(X) \) is precisely the Hilbert function for the projective embedding of \( X \) induced by \( L \).
2.2. **Projective bundle formulas.** In this section we exhibit a formula for the Euler-Chow function of certain projective bundles over a variety \( W \). The basic setup is as follow. Consider two algebraic vector bundles \( E_1 \to W \) and \( E_2 \to W \) over a complex projective variety \( W \). The various maps involved in our discussion are displayed in the commutative diagram below:

\[
\begin{tikzcd}
\mathbb{P}(E_1) \arrow{r}{i_1} \arrow{d}[swap]{p_1} & \mathbb{P}(E_1 \oplus E_2) \arrow{r}{i_2} \arrow{d}[swap]{q} & \mathbb{P}(E_2) \arrow{d}{p_2} \\
W & & 
\end{tikzcd}
\]

where \( p_1, p_2 \) and \( q \) are projections from the indicated projective bundles, and \( i_1, i_2 \) are the canonical inclusions.

We introduce a monoid monomorphism \( t_p : \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \to \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2)) \) which is a closed inclusion.

Let \( L_1 \) and \( L_2 \) denote the tautological line bundles \( \mathcal{O}_{E_1}(-1) \) and \( \mathcal{O}_{E_2}(-1) \) over \( \mathbb{P}(E_1) \) and \( \mathbb{P}(E_2) \), respectively, and let \( \pi_1 \) and \( \pi_2 \) denote the respective projections from \( \mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \) onto \( \mathbb{P}(E_1) \) and \( \mathbb{P}(E_2) \). The \( \mathbb{P}^1 \)-bundle \( \pi : \mathbb{P}(\pi_1^*(L_1) \oplus \pi_2^*(L_2)) \to \mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \) is precisely the blow-up of \( \mathbb{P}(E_1 \oplus E_2) \) along \( \mathbb{P}(E_1) \Pi \mathbb{P}(E_2) \), which we denote by \( Q \), for short; see Lascou and Scott \[LS75\] for details. Let \( b : Q \to \mathbb{P}(E_1 \oplus E_2) \) denote the blow-up map.

Since \( \pi \) is a flat map of relative dimension 1, and \( b \) is a proper map, one has two algebraic continuous homomorphisms (cf. 2.6), given by the flat pull-back

\[
\pi^* : \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \to \mathcal{C}_p(Q)
\]

and the proper push-forward

\[
b_* : \mathcal{C}_p(Q) \to \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2)).
\]

**Definition 2.13.** The map \( t_p : \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \to \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2)) \) is defined as the composition \( t_p = b_* \circ \pi^* \).

In this way we become equipped with three morphisms of monoids with finite multiplication:

\[
i_1 p : \Pi_p(\mathbb{P}(E_1)) \to \Pi_p(\mathbb{P}(E_1 \oplus E_2))
\]

induced by \( i_1 \),

\[
i_2 p : \Pi_p(E_2) \to \Pi_p(\mathbb{P}(E_1 \oplus E_2))
\]

induced by \( i_2 \), and

\[
\phi_p : \Pi_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \to \Pi_p(\mathbb{P}(E_1 \oplus E_2))
\]

induced by \( t_p \).
These three maps induce a morphism (with finite fibers)

\[ \Psi_p : \Pi_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \times \Pi_p(\mathbb{P}(E_1)) \times \Pi_p(\mathbb{P}(E_2)) \longrightarrow \Pi_p(\mathbb{P}(E_1 \oplus E_2)) \]

by sending \((a, b, c)\) to \(\varphi_p(a) + i_1 p(b) + i_2 p(c)\).

The following result that appear in [EL98] will be needed and we will be going to be generalized.

**Theorem 2.14.** [EL98] Theorem 5.1 \(\) Let \(E_1\) and \(E_2\) be algebraic vector bundles over a connected projective variety \(W\), of ranks \(e_1\) and \(e_2\), respectively, and let \(0 \leq p \leq e_1 + e_2 - 1\). Then the \(p\)-th Euler-Chow function of \(\mathbb{P}(E_1 \oplus E_2)\) is given by

\[ E_p(\mathbb{P}(E_1 \oplus E_2)) = \Psi_{p\#}(E_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \circ E_p(\mathbb{P}(E_1)) \circ E_p(\mathbb{P}(E_2))). \]

**Corollary 2.15.** [EL98] Corollary 5.7 \(\) Let \(E\) be a line bundle over \(W\) which is generated by its global sections. Then the homomorphism

\[ \Psi_p : \Pi_{p-1}(W) \oplus \Pi_p(W) \oplus \Pi_p(W) \longrightarrow \Pi_p(\mathbb{P}(E \oplus 1)) \cong \Pi_{p-1}(W) \oplus \Pi_p(W) \]

sends \((\alpha, \beta, \gamma)\) to \((\alpha + \beta, \xi \cap \gamma + \alpha)\).

**Lemma 2.16.** [EL98] Lemma 5.4 \(\) Let \(X = \bigcup_{i \in \mathbb{N}} X_i\) and \(Y = \bigcup_{j \in \mathbb{N}} Y_j\) be spaces which are a countable disjoint union of connected projective varieties, and let \(f : X \to Y\) be a continuous map such that the restriction \(f|_{X_i}\) is an algebraic continuous map from \(X_i\) into some \(Y_j\). If \(f\) is a bijection, then it is a homeomorphism in the classical topology.

### 3. The Euler-Chow series for ruled surfaces

This section is dedicated to prove theorem 1.1 in page 3. First we prove it for the case of decomposable ruled surfaces, then we proceed to the case of undecomposable case, and we reduce the proof in this case to the decomposable situation. To help the reader we write down the theorem again.

**Theorem 1.1.** Let \(E_1, E_2\) and \(E_3\) be vector bundles over a complex projective variety \(W\), of ranks \(e_1, e_2\) and \(e_3\), respectively, with \(2 \leq p \leq e_1 + e_2 + e_3 - 1\). Then the \(p\)-th Euler-Chow series of \(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)\) is given by

\[ E_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) = \Psi_{p\#} \left( E_{p-2}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \circ E_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \right. \]

\[ \left. \circ E_p(\mathbb{P}(E_1)) \circ E_p(\mathbb{P}(E_2)) \right). \]

It is simple to compute \(E_0(X)\) and \(E_2(X)\). From [Mac62] we know that

\[ E_0(X) = \left( \frac{1}{1 - t} \right)^{\chi(X)} = \left( \frac{1}{1 - t} \right)^{4-4g} \]
and like a ruled surface is connected of dimension 2, we have

\[ E_2(X) = \frac{1}{1-t} \]

No we proceed to the proof of theorem \[ \text{I.1} \]

3.1. Decomposable ruled surfaces.

Proof. Let \( \pi : X \cong \mathbb{P}(\mathcal{E}) \to C \) be a ruled surface such that \( \mathcal{E} \) decompose as the direct sum of invertible sheaves over \( C \). For \[ \text{[Har77 V, 2.12 (a)]} \] we have that \( \mathcal{E} \cong \mathcal{O}_C \oplus \mathcal{L} \) for some \( \mathcal{L} \) with \( \deg \mathcal{L} \leq 0 \). Furthermore \( c = -\deg \mathcal{L} \geq 0 \). Since \( \mathcal{O}_C \) is the trivial line bundle over \( C \) we proceed to apply theorem \[ \text{2.14} \]. We have

\[ \mathbb{P}(\mathcal{O}_C) \cong \mathbb{P}(\mathcal{L}) \cong \mathbb{P}(\mathcal{O}_C) \times_C \mathbb{P}(\mathcal{L}) \cong C. \]

Using \[ \text{2.15 } \] (\[ \text{[EL98 Corollary 5.7]} \]) we obtain that

\[ \Psi_1 : \Pi_0(C) \times \Pi_1(C) \times \Pi_1(C) \to \Pi_1(\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C)) \cong \Pi_0(C) \times \Pi_1(C) \]

is given by \( \Psi(\alpha, \beta, \gamma) = (\xi \cap \gamma + \alpha, \beta + \gamma) \), where \( \xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C)}(1)) = c_1(\mathcal{L}(C_0)) = C_0. \)

Since \( \gamma \in \Pi_1(C) \) and \( C \) is a variety of dimension one, we have \( \gamma = c[C] = c[C_0] \). Therefore \( \xi \cap \gamma = C_0, nC_0 = -ne. \)

From here we obtain \( \Psi_1(a, b[C], c[C]) = (a - cc, (b - c)[C]) \), therefore we can identify \( \Psi_1 \) by \( (a, b, c) \mapsto (a - cc, b + c). \)

We have \( H_2(\mathbb{P}(\mathcal{E})) \cong \mathbb{Z}^2 \), and for each generator we associate variables \( t_0, t_1 \). We identify \( (\alpha_0, \alpha_1) \in \Pi_1(\mathbb{P}(\mathcal{E})) \) with \( t_0^{\alpha_0}t_1^{\alpha_1} \). We obtain

\[ E_1(\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C)) = \sum_{\alpha_0, \alpha_1} E_1(\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C))(\alpha_0, \alpha_1)t_0^{\alpha_0}t_1^{\alpha_1} \]

\[ = \sum_{\alpha_0, \alpha_1} \Psi_1(\mathcal{E}_0(\mathbb{P}(\mathcal{L})) \circ \mathcal{E}_1(\mathbb{P}(\mathcal{L})) \circ \mathcal{E}_1(C))(\alpha_0, \alpha_1)t_0^{\alpha_0}t_1^{\alpha_1} \]

\[ = \sum_{\alpha_0, \alpha_1} \left( \sum_{(a, b, c) \in \Psi_1^{-1}} (\mathcal{E}_0(C)(a) \cdot \mathcal{E}_1(C)(b) \cdot \mathcal{E}_1(C)(c)) \right) t_0^{\alpha_0}t_1^{\alpha_1} \]

\[ = \sum_{\alpha_0, \alpha_1} \left( \sum_{a - cc = \alpha_0 \atop b + c = \alpha_1} (\mathcal{E}_0(C)(a) \cdot \mathcal{E}_1(C)(b) \cdot \mathcal{E}_1(C)(c)) \right) t_0^{a-cc}t_1^{b+c} \]

\[ = \left( \sum_{a \geq 0} \mathcal{E}_0(C)(a) \cdot t_0^a \right) \left( \sum_{b \geq 0} \mathcal{E}_1(C)(b) \cdot t_1^b \right) \left( \sum_{c \geq 0} \mathcal{E}_1(C)(c) \cdot (t_0^{-c}t_1)^c \right). \]

From here we get
\[
E_1(\mathbb{P}(L \oplus \mathcal{O}_C)) = \left( \frac{1}{1-t_0} \right)^{2-2g} \left( \frac{1}{1-t_1} \right) \left( \frac{1}{1-t_0^{-1}t_1} \right)
\]

3.2. Undecomposable ruled surfaces.

Proof. As \( X \) is a ruled surface over \( C \), we have a map \( \pi: X \to C \), such that for all \( y \in C \) we have \( X_y = \pi^{-1}(y) \simeq \mathbb{P}^1 \). We denote this isomorphism by \( f_y \). Let \( C_0 \) be a section of minimum self-intersection, i.e. \( C_0^2 = -e \), and let \( C_1 \) be any other section. For each \( y \in C \) we can choose \( f_y \) such that \( f_y^{-1}(0 : 1) = C_1 \cap X_y \) and \( f_y^{-1}(1 : 0) = C_0 \cap X_y \) (unless that \( C_0 \cap X_y = C_1 \cap X_y \), in which case we only consider the point \((1 : 0))\).

We define an action of \( \mathbb{C}^* \) on \( X \) as follow: For \( x \in X \), \( \lambda \in \mathbb{C}^* \), we define \( \lambda \cdot x = f_y^{-1}(\lambda \cdot f_y(x)) \), were \( y = \pi(x) \) and we denote by \( \lambda \cdot f_y(x) \) the action of \( \mathbb{C}^* \) on \( \mathbb{P}^1 \) given by \( \lambda(a : b) = (a : \lambda b) \).

It is easy to see that the fixed point set of \( X \) under the action is

\[
X_{\mathbb{C}^*} = C_0 \cup C_1 \cup \{p_1, \ldots, p_r\} \text{ where } r = |C_0 \cap C_1|.
\]

This action induces an algebraic continuous action on \( \mathcal{E}_1(X) \), and we are interested in identifying the set of fixed points in this case. Consider the map

\[
\phi_1: \mathcal{E}_0(C) \times \mathcal{E}_1(C) \times \mathcal{E}_1(C) \to \mathcal{E}_1(X)
\]

defined by \( \phi_1(a, b, c) = \pi^*(a) + \sigma_0 * b + \sigma_1 * c \), where \( \sigma_0 \) and \( \sigma_1 \) are respectively the sections that correspond to \( C_0 \) and \( C_1 \).

We want to prove that \( \phi_1 \) is an homeomorphism on the set of fixed points \( \mathcal{E}_1(X)^{\mathbb{C}^*} \).

The invariant irreducible sub-varieties of dimension 1 are \( C_0, C_1 \), in fact, all their points are fixed under the action, and they are also the fibers of \( \pi \) (all their general points are in the same orbit under the action). From this, it is clear that \( \phi_1 \) is injective and surjective over \( \mathcal{E}_1(X) \).

Note that for each \( \alpha \in \Pi_1(X) \), the set of fixed points \( \mathcal{E}_{1,\alpha}(X)^{\mathbb{C}^*} \) is an algebraic subset if \( \mathcal{E}_{1,\alpha}(X) \) (not necessarily connected). Then it can be written as the disjoint union of projective varieties.

Therefore, for \([2.16] \) (\cite{EL98} Lemma 4.4)) we conclude that \( \phi_1 \) is an homeomorphism on its image.

We know by \([LY87] \) that the Euler characteristic of an algebraic variety is equal to the Euler characteristic of its fixed point set. Then we have

\[
\chi(\mathcal{E}_{1,\alpha}(X)) = \chi(\mathcal{E}_{1,\alpha}(X)^{\mathbb{C}^*}).
\]
On the other hand, let $\Psi_1$ be the morphism induced by $\phi_1$ between the monoids of the connected components, then we have

$$
\mathcal{E}_{1,\alpha}(X)^{\mathbb{C}^*} = \prod_{(a,b,c) \in \Psi_1^{-1}(\alpha)} \mathcal{E}_0.a(C) \times \mathcal{E}_1.b(C) \times \mathcal{E}_1.c(C),
$$

$$
\chi(\mathcal{E}_{1,\alpha}(X)) = \sum_{(a,b,c) \in \Psi_1^{-1}(\alpha)} \chi(\mathcal{E}_0.a(C)) \cdot \chi(\mathcal{E}_1.b(C)) \cdot \chi(\mathcal{E}_1.c(C)).
$$

Therefore

$$
E_1(X) = \Psi_1#(E_0(C) \odot E_1(C) \odot E_1(C)).
$$

Furthermore we have the following morphism

$$
A(X) \xrightarrow{\sim} A(\mathbb{P}(\mathcal{L}(e) \oplus \mathcal{O}_C)))
$$

$$
nC_0 + \pi^*b \mapsto nC'_0 + \pi'^*b.
$$

This morphism can be restricted to the effective part and to its pathwise connected components. Using the results obtained for decomposable ruled surfaces we obtain

$$
E_1(X) = \left(\frac{1}{1 - t_0}\right)^{2-2g} \left(\frac{1}{1 - t_1}\right) \left(\frac{1}{1 - t_0 - t_1}\right).
$$

□

4. Projective bundle formulas

In order to go to the case of scrolls we need to generalize the theorem 2.14 ([EL98 Theorem 5.1]).

We start by drawing the following diagram.
Here, $E_1, E_2$ and $E_3$ are vector bundles over a complex projective variety $W$. We have the following maps:

\[
\begin{align*}
  i_k &: \mathbb{P}(E_k) \to \mathbb{P}(E_1 \oplus E_2 \oplus E_3) & (k = 1, 2, 3) \\
i_{kl} &: \mathbb{P}(E_k \oplus E_l) \to \mathbb{P}(E_1 \oplus E_2 \oplus E_3) & (k < l, k, l = 1, 2, 3) \\
p_k &: \mathbb{P}(E_k) \to W & (k = 1, 2, 3) \\
p_{kl} &: \mathbb{P}(E_k \oplus E_l) \to W & (k < l, k, l = 1, 2, 3) \\
q &: \mathbb{P}(E_1 \oplus E_2 \oplus E_3) \to W \\
t_{klp} &: \mathbb{C}_{p-1}(\mathbb{P}(E_k) \times_W \mathbb{P}(E_l)) \to \mathbb{C}_{p}(\mathbb{P}(E_k \oplus E_l)) & (k < l, k, l = 1, 2, 3) \\
t_p &: \mathbb{C}_{p-2}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \to \mathbb{C}_{p}(\mathbb{P}(E_1 \oplus E_2 \oplus E_3))
\end{align*}
\]

where $p$'s and $q$'s are projections of the respective bundles, and the $i$'s are the canonical inclusions. Then we have the following morphisms of monoids with finite multiplication:

\[
\begin{align*}
i_{kp} &: \Pi_p(\mathbb{P}(E_k)) \to \Pi_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) & (k = 1, 2, 3) \\
i_{klp} &: \Pi_p(\mathbb{P}(E_k \oplus E_l)) \to \Pi_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) & (k < l, k, l = 1, 2, 3) \\
\varphi_{klp} &: \Pi_{p-1}(\mathbb{P}(E_k) \times_W \mathbb{P}(E_l)) \to \Pi_{p}(\mathbb{P}(E_k \oplus E_l)) & (k < l, k, l = 1, 2, 3) \\
\varphi_p &: \Pi_{p-2}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \to \Pi_{p}(\mathbb{P}(E_1 \oplus E_2 \oplus E_3))
\end{align*}
\]

Then we have the following induced morphism:

\[
\Psi_p : \left\{ \begin{array}{l}
\Pi_{p-2}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \times \Pi_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \\
\times \Pi_{p}(\mathbb{P}(E_1)) \times \Pi_{p}(\mathbb{P}(E_2)) \times \Pi_{p}(\mathbb{P}(E_3))
\end{array} \right\} \to \Pi_{p}(\mathbb{P}(E_1 \oplus E_2 \oplus E_3))
\]

Sending $(a, b, c, d, e, f, g)$ to $\Psi(a, b, c, d, e, f, g) = \varphi_p(a) + i_{12p} \varphi_{12p}(b) + i_{13p} \varphi_{13p}(c) + i_{23p} \varphi_{23p}(d) + i_1p(e) + i_{2p}(f) + i_{3p}(g)$.

Now we proceed to define the following maps $t_{klp}$ and $t_p$. Denote by $L_1$, $L_2$ and $L_3$ the lineal tautological bundles $\mathcal{O}_{E_1}(-1)$, $\mathcal{O}_{E_2}(-1)$ and $\mathcal{O}_{E_3}(-1)$ over $\mathbb{P}(E_1)$, $\mathbb{P}(E_2)$ and $\mathbb{P}(E_3)$, respectively and let $\pi_1$, $\pi_2$ and $\pi_3$ be the respectively projections $\mathbb{P}(E_1 \oplus E_2 \oplus E_3)$. The $\mathbb{P}^2$-bundle $\pi : \mathbb{P}(\pi_1^*(L_1) \oplus \pi_2^*(L_2) \oplus \pi_3^*(L_3)) \to \mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)$ is the blow-up of $\mathbb{P}(E_1 \oplus E_2 \oplus E_3)$ on $\mathbb{P}(E_1 \oplus E_2) \cup \mathbb{P}(E_1 \oplus E_3) \cup \mathbb{P}(E_2 \oplus E_3)$, which we denote by $Q$. Let $b : Q \to \mathbb{P}(E_1 \oplus E_2 \oplus E_3)$ be the blow-up map.

Since $\pi$ is a flat map of relative dimension 2, and $b$ is a proper map, we have two algebraic continuous homomorphisms given by the flat pull-back

\[
\pi^* : \mathbb{C}_{p-2}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \to \mathbb{C}_{p}(Q)
\]
and the proper push-forward

$$b_* : \mathcal{C}_p(Q) \to \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)).$$

In the same way, for \( k < l, \ k, l = 1, 2, 3 \), we denote by \( \pi_k^{kl} \) and \( \pi_l^{kl} \) the respectively projections of \( \mathbb{P}(E_k) \times_W \mathbb{P}(E_l) \) on \( \mathbb{P}(E_k) \) and \( \mathbb{P}(E_l) \). The \( \mathbb{P}^1 \)-bundle \( \pi_{kl} : \mathbb{P}(\pi_k^{kl}(L_k) \oplus \pi_l^{kl}(L_l)) \to \mathbb{P}(E_k \times_W \mathbb{P}(E_l)) \) is precisely the blow-up of \( \mathbb{P}(E_k \oplus E_l) \) on \( \mathbb{P}(E_k) \times_W \mathbb{P}(E_l) \), which we denote by \( Q_{kl} \). Let \( b_{kl} : Q_{kl} \to \mathbb{P}(E_k \oplus E_l) \) be the blow-up map.

Since \( \pi_{kl} \) is a flat map of relative dimension 1, and \( b_{kl} \) is a proper map, we have two continuous homomorphisms given by the flat pull-back

$$\pi_{kl}^* : \mathcal{C}_{p-1}(\mathbb{P}(E_k) \times_W \mathbb{P}(E_l)) \to \mathcal{C}_p(Q_{kl})$$

and the proper push-forward

$$b_{kl*} : \mathcal{C}_p(Q_{kl}) \to \mathcal{C}_p(\mathbb{P}(E_k \oplus E_l)).$$

**Definition 4.1.** The maps \( t_p \) and \( t_{klp} \) are defined as the following compositions \( t_p = b_* \circ \pi^* \) and \( t_{klp} = b_{kl*} \circ \pi_{kl}^* \).

Now we prove the main result in this section, Theorem 1.2 in page 3. We write down the theorem again to make easier the lecture.

**Theorem 1.2.** Let \( E_1, E_2 \) and \( E_3 \) be vector bundles over a complex projective variety \( W \), of ranks \( e_1, e_2 \) and \( e_3 \), respectively, with \( 2 \leq p \leq e_1 + e_2 + e_3 - 1 \). Then the \( p \)-th Euler-Chow series of \( \mathbb{P}(E_1 \oplus E_2 \oplus E_3) \) is given by

$$E_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) = \Psi_p \left( \begin{array}{c} E_{p-2}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \circ E_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \\ \circ E_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_3)) \circ E_{p-1}(\mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \\ \circ E_p(\mathbb{P}(E_1)) \circ E_p(\mathbb{P}(E_2)) \circ E_p(\mathbb{P}(E_3)) \end{array} \right).$$

**Proof.** Consider the action \( (\mathbb{C}^*)^2 \) on \( \mathbb{P}(E_1 \oplus E_2 \oplus E_3) \) given by taking the product of each scalar with two of the factors of \( E_1 \oplus E_2 \oplus E_3 \). The set of fixed points of \( \mathbb{P}(E_1 \oplus E_2 \oplus E_3)^{(\mathbb{C}^*)^2} \) is \( \mathbb{P}(E_1) \times \mathbb{P}(E_2) \times \mathbb{P}(E_3) \). This action induces a continuous algebraic action on \( \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) \); our next step is to identify its fixed points set. Consider the map

$$\psi_p : \left\{ \begin{array}{l} \mathcal{C}_{p-2}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \times \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \\ \times \mathcal{C}_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_3)) \times \mathcal{C}_{p-1}(\mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \\ \times \mathcal{C}_p(\mathbb{P}(E_1)) \times \mathcal{C}_p(\mathbb{P}(E_2)) \times \mathcal{C}_p(\mathbb{P}(E_3)) \end{array} \right\} \to \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3))$$

defined by \( \psi(a, b, c, d, e, f, g) = t_p(a) + i_{12p} \circ t_{12p}(b) + i_{13p} \circ t_{13p}(c) + i_{23p} \circ t_{23p}(d) + i_1(e) + i_2(f) + i_3(g) \).

It is proven that \( \psi_p \) is a homeomorphism in the set of fixed points. \( (\mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)))^{(\mathbb{C}^*)^2} \). If an element \( \sigma = \sum_i n_i V_i \in \mathcal{C}_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) \) is fixed by the action, then each of its irreducible
components must be invariant under the action. Furthermore, an irreducible invariant variety \( V \subset \mathbb{P}(E_1 \oplus E_2 \oplus E_3) \) can be of three types:

1. Those whose general points are fixed under the action and therefore all the points of \( V \) are fixed.
2. Those whose general points have non-trivial orbits of dimension 1.
3. Those whose general points have non-trivial orbits of dimension 2.

The Chow monoids \( C_p(X) \) of a variety \( X \) are free generated from the irreducible subvarieties of dimension \( p \) of \( X \). Given cycles \( \sigma_k \in C_p(\mathbb{P}(E_k)) \), \( k = 1, 2, 3 \), the support of \( i_{k*} \) is contained in \( \mathbb{P}(E_k) \), from this, the images of \( C_p(E_1), C_p(E_2) \) and \( C_p(E_3) \) under \( i_1, i_2 \) and \( i_3 \) are freely generated by the disjoint subset of the set of generators of \( C_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) \), which are varieties of the first type.

In the other hand, the support of \( i_{kkl*} \) is contained in \( \mathbb{P}(E_k \oplus E_l) \), \( k, l = 1, 2, 3 \) with \( k \leq l \), then the image of \( \mathbb{P}(E_k \oplus E_l) \) is freely generated by disjoint subset of the set of generators \( C_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) \), and since the restriction of the action to \( \mathbb{P}(E_k \oplus E_l) \) coincides with the action of \( \mathbb{C}^* \), we have that the elements of these set are of type 1 and 2, (EL98).

Since the irreducible varieties of type 1 that are contained in the image of \( i_{kkl*} \) coincide with some of the ones mentioned in the last paragraph, it only rest to see how are the ones of type 2. But these are of the form \( i_{kkl*} \circ t_{klp}(Z) \) for some \( (p - 1) \)-dimensional variety of \( \mathbb{P}(E_k) \times_W \mathbb{P}(E_l) \) by the proof of Theorem 2.14 in (EL98).

Now, given a \( (p - 2) \)-dimensional subvariety \( Z \) of \( \mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3) \) we have that the inverse image \( \pi^{-1}(Z) \) is a \( p \)-dimensional subvariety of \( Q \), whose points out of the exceptional divisor of the blow-up map \( b \) have irreducible orbits of dimension 2. Since \( b \) is a \( (\mathbb{C}^*)^2 \)-equivariant birational map, we have that the image \( b(\pi^{-1}(Z)) \) is an irreducible subvariety of \( \mathbb{P}(E_1 \oplus E_2 \oplus E_3) \) of type 3.

From this we obtain that the images of \( i_{1*}, i_{2*}, i_{3*}, i_{11*} \circ t_{12}, i_{13*} \circ t_{13p}, i_{23*} \circ t_{23p}, t_p \) are generated freely by the disjoint subsets of the generators of \( (C_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3))(\mathbb{C}^*)^2) \), therefor \( \psi_p \) is injective.

In order to prove the surjectivity we only need to show that all invariant irreducible subvarieties \( V \subset \mathbb{P}(E_1 \oplus E_2 \oplus E_3) \) of type 3 are of the form \( b(\pi^{-1}(Z)) \) for some subvariety \( (p - 2) \)-dimensional of \( \mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3) \); The rest of the cases are consequence of the proof of theorem 2.14 in (EL98) Theorem 5.1.

Let \( \tilde{V} \subset Q \) be the proper transform of \( V \) under \( b \), and define \( Z := \pi(\tilde{V}) \subset \mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3) \). The general points of \( V \) gave orbits of dimension 2, then the general points of \( \tilde{V} \) also have dimension 2, this shows that the general fibre of \( \pi_{\tilde{V}} : \tilde{V} \to Z \) has dimension 2. In particular we have that \( \tilde{V} = \pi^{-1}(Z) \) and since \( \pi |_{\tilde{V}} \) is a projective bundle, and \( b \) is a birational maps from \( \tilde{V} \) to \( V \), we conclude that \( t_p(Z) = b_* \circ \pi^*(Z) = V \).
The last reasoning shows that $\psi_p$ is a continuous algebraic bijection from its domain to the set of fixed points $(C_p(P(E_1 \oplus E_2 \oplus E_3))^{(C^*)^2}$. Observe that for each $\alpha \in \prod_p (P(E_1 \oplus E_2 \oplus E_3))$ the set of fixed points $(C_{p,\alpha}(P(E_1 \oplus E_2 \oplus E_3))^{(C^*)^2}$ is an algebraic subset of $(C_{p,\alpha}(P(E_1 \oplus E_2 \oplus E_3))$ (not necessarily connected), from here we can also write it as a countable disjoint union of projective varieties. Then we use Lemma 2.16 (Lemma 5.4 in [EL98]) to obtain that $\psi_p$ is a homeomorphism onto its image.

It is a general known result that the Euler characteristic of a variety with an algebraic torus action is equal to the Euler characteristic of the fixed points set (for example see Lawson and Yau [LY87]). Therefore, given $\alpha \in \prod_p ((P(E_1 \oplus E_2 \oplus E_3))$, it follows that

$$\chi(C_{p,\alpha}(P(E_1 \oplus E_2 \oplus E_3))) = \chi(C_{p,\alpha}(P(E_1 \oplus E_2 \oplus E_3))^{(C^*)^2}).$$

In the other hand, if $\Psi_p$ is the induced morphism by $\psi_p$ between the connected components of the monoids, then

$$C_{p,\alpha}(P(E_1 \oplus E_2 \oplus E_3))^{(C^*)^2} = \prod_{(a,b,c,d,e,f,g) \in \Psi_p^{-1}(\alpha)} \left( C_{p-2,\alpha}(P(E_1) \times_W P(E_2) \times_W P(E_3)) \times C_{p-1,b}(P(E_1) \times_W P(E_2)) \right)$$

since $\psi_p$ is a homeomorphism onto its image. We take the Euler characteristic in both sides and the theorem is proven.

The following theorem can be proved in similar way

**Theorem 4.2.** Let $E_i$, $i = 1, \ldots, n$, vector bundles over a complex projective variety $W$ of ranks $e_i$, respectively, and we assume that $n - 1 \leq p \leq e_1 + e_2 + e_3 - 1$. Then, the $p$-th Euler-Chow function of $P(\bigoplus_{i=1}^n E_i)$ is given by

$$E_p(P(\bigoplus_{i=1}^n E_i)) = \Psi_p^\# (E_{p-n+1}(\times_{i=1}^n P(E_i)) \circ (\circ_{i=1}^n E_{p-n+2}(\times_{j=1}^n P(E_j))) \circ \cdots \circ (\circ_{i=1}^n E_p(P(E_i))))$$

5. The Euler-Chow series for normal rational scrolls

Let $E_1$, $E_3$ be line bundles over a projective variety $W$, and let $E_2 = \mathbb{1}$ be the trivial line bundle. In this case we have

$$W = P(E_1) = P(E_2) = P(E_3) = P(E_1) \times_W P(E_2) = P(E_1) \times_W P(E_3) = P(E_2) \times_W P(E_3) = P(E_1) \times_W P(E_2) \times W P(E_3),$$
and the inclusion $i_k : \mathbb{P}(E_k) \to \mathbb{P}(E_1 \oplus E_2 \oplus E_3), \ k = 1, 2, 3,$ are sections of the projective bundle $\mathbb{P}(E_1 \oplus E_2 \oplus E_3)$ over $W$.

We write $\xi = c_1(\mathcal{O}_{\mathbb{P}(E_1 \oplus E_2 \oplus E_3)}(1))$ and $\xi' = c_1(\mathcal{O}_{\mathbb{P}(E_1 \oplus E_3)}(1))$. We have the following isomorphism

$$T : \mathbb{A}_{p-2}(W) \oplus \mathbb{A}_{p-1}(W) \oplus \mathbb{A}_p(W) \to \mathbb{A}_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3))$$

(12)

$$\alpha, b, \gamma \mapsto q^*\alpha + \xi \cap q^*b + \xi^2 \cap q^*\gamma = q^*\alpha + i_{12}(i^*_1p^*_1\gamma) = i_1p_1\gamma,$$

since

$$\xi^2 \cap q^*\gamma = \xi \cap i_{12}(i^*_1p^*_1\gamma) = i_1(i^*_1p^*_1\gamma) = i_1p_1\gamma$$

Therefore this isomorphism is equal to the composition

$$\mathbb{A}_{p-2}(W) \oplus \mathbb{A}_{p-1}(W) \oplus \mathbb{A}_p(W) \to \mathbb{A}_{p-2}(W) \oplus \mathbb{A}_p(\mathbb{P}(E_1 \oplus E_2)) \to \mathbb{A}_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)).$$

This isomorphism restrict to an injection

$$T^\geq : \mathbb{A}_{p-2}^\geq(W) \oplus \mathbb{A}_{p-1}^\geq(W) \oplus \mathbb{A}_p^\geq(W) \to \mathbb{A}_p^\geq(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)).$$

**Lemma 5.1.**

- The injection $T^\geq$ is an isomorphism.

- If $\Pi_*(W)$ are monoids with cancelation law for all $\ast$, then so are $\Pi_*(\mathbb{P}(E_1 \oplus E_2 \oplus E_3))$. Equivalently, if the natural surjective maps $\Pi_p(W) \to \mathbb{A}_p^\geq(\mathbb{P}(E_1 \oplus E_2 \oplus E_3))$ are isomorphisms for all $p$, then the following surjective maps are also isomorphisms $\Pi_p(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)) \to \mathbb{A}_p^\geq(\mathbb{P}(E_1 \oplus E_2 \oplus E_3)).$

**Proof.**

- All effective $p$-cycle $\alpha'$ is algebraic and effectively equivalent to a cycles of the form

$$q^*(a) + i_{12}(p^*_1(b) + i_{13}(p^*_3(c)) + i_{23}(p^*_2(d)) + i_{1}(e) + i_{2}(f) + i_{3}(g))$$

with $a \in \mathcal{C}_{p-2}(W), \ b, c, d \in \mathcal{C}_{p-1}(W), \ e, f, g \in \mathcal{C}_p(W)$. We have

$$i_{2}(f) = i_{12} \circ i_{2}(f) \equiv i_{12}(i_{1}(f) + p^*_2(Z' \cap f)) = i_{12}(i_{1}(f) + i_{12}(p^*_2(Z' \cap f)) = i_{1}(f) + i_{12}(p^*_2(Z' \cap f)).$$

In a similar form we obtain

$$i_{3}(g) \equiv i_{1}(g) + i_{12}(p^*_2(Z' \cap g)).$$

Since $E_1$ is generated by its sections, we can find a section $s : W \to E_1$ whose zero locus $Z \subset W$ intersects properly to $c$. Let $\hat{s} : W \to \mathbb{P}(E_1 \oplus \mathbb{I} \oplus E_3)$ be the composition $i \circ s,$
where \( \ell : E_1 \to \mathbb{P}(E_1 \oplus \mathbb{I} \oplus E_3) \) is the open inclusion. Then, the closure of the orbit of \( \tilde{s}_*c \) contains two fixed points: \( i_{12*} \circ p_{12}^*(c) + q^*(Z \cap c) \) and \( i_{13*} \circ p_{13}^*(c) \). Similarly for \( i_{23*} \circ p_{23}^*(d) \). Then

\[
a' \equiv_{a,b,c,d,e,f,g} q^*(a + Z \cap c + Z \cap d) + i_{12*} \circ p_{12}^*(b + c + d + Z' \cap f + Z' \cap g) + i_{1*}(e + f + g).
\]

This proves the surjectivity of \( T^\geq \).

- The injectivity is proven in the same way than the way was done in ([EL98] Lemma 5.6)].

\[ \square \]

**Corollary 5.2.** Under the same hypothesis, the homomorphism

\[
\Psi_p : \left\{
\begin{array}{c}
\Pi_{p-2}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2) \times_W \mathbb{P}(E_3)) \\
\Pi_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_2)) \times \Pi_{p-1}(\mathbb{P}(E_1) \times_W \mathbb{P}(E_3)) \\
\Pi_p(\mathbb{P}(E_1)) \times \Pi_p(\mathbb{P}(E_2)) \times \Pi_p(\mathbb{P}(E_3))
\end{array}
\right\} \to \Pi_p(\mathbb{P}(E_1 \oplus \mathbb{I} \oplus E_3)) \\
\simeq \Pi_{p-2}(W) \oplus \Pi_{p-1}(W) \oplus \Pi_p(W)
\]

sends \((a, b, c, d, e, f, g)\) in \((a + \xi \cap c + \xi \cap d, b + c + d + \xi' \cap f + \xi' \cap g, e + f + g)\).

**Example 5.3.**

\[ W = \mathbb{P}^1, \ E_1 = \mathcal{O}_{\mathbb{P}^1}(h) \text{ with } h \geq 0, \ E_2 = E_3 = \mathbb{I}. \]

We have

\[
\Psi_p(a[\mathbb{P}^{p-2}], b[\mathbb{P}^{p-1}], c[\mathbb{P}^{p-1}], d[\mathbb{P}^{p-1}], e[\mathbb{P}^p], f[\mathbb{P}^p], g[\mathbb{P}^p]) = ((a + h \cdot c + h \cdot d)[\mathbb{P}^{p-2}], (b + c + d + h \cdot f + h \cdot g)[\mathbb{P}^{p-1}], (e + f + g)[\mathbb{P}^p])
\]

Therefore

\[
E_p(\mathcal{O}_{\mathbb{P}^1}(h) \oplus \mathbb{I} \oplus \mathbb{I})) = \left(\frac{1}{1-t_0}\right)^{n+1}{p-1}\left(\frac{1}{1-t_1}\right)^{n+1}{p}\left(\frac{1}{1-t_2}\right)^{n+1}{p+1}\left(\frac{1}{1-t_0^n t_1}\right)^{2(n+1)}{p}\left(\frac{1}{1-t_1^n t_2}\right)^{2(n+1)}{p+1}
\]

**References**

[Bia73] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), 480–497.

[Bou89] N. Bourbaki, *General topology*, Springer-Verlag, New York, 1989.

[CE] Xi. Chen and E.J. Elizondo, *Rationality of Euler-Chow series and finite generation of Cox rings*, J. of Algebra **447** 2016, 206-239.

[EL94] E. J. Elizondo, *The Euler series of restricted Chow varieties*, Compositio Math. **94** (1994), no. 3, 297–310.

[EH96] E. J. Elizondo and R. Hain, *Chow varieties of Abelian varieties*, Bol. Soc. Mat. Mex. **2** (1996), no. 3, 95–99.

[EK12] E.-J. Elizondo and Shun-ichi Kimura, *Rationality of motivic Chow series modulo \( \mathbb{A}^1 \)-hotopy*, Adv. Math. **230** (2012), no. 3, 876893.

[EL98] E. J. Elizondo and P. Lima-Filho, *Euler-Chow series and projective bundles formulas*, J. Algebraic Geometry **7** (1998), 695–729.
[Fri91] E. Friedlander, *Algebraic cycles, Chow varieties and Lawson homology*, Compositio Math. 77 (1991), 55–93.
[FL92] E. Friedlander and H. B. Lawson, Jr., *A theory of algebraic cocycles*, Ann. of Math. (2) 136 (1992), 361–428.
[FM94a] E. Friedlander and B. Mazur, *Filtrations on the homology of algebraic varieties*, Mem. Amer. Math. Soc. 110 (1994), no. 529.
[Ful84] W. Fulton, *Intersection theory*, 1st ed., Springer-Verlag, Heidelberg, 1984.
[FH91] W. Fulton and J. Harris, *Representation theory, a first course*, 1st ed., Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.
[FM94b] W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) 139 (1994), 183–225.
[Ful97] W. Fulton, *Young tableaux*, Student Texts, vol. 35, London Mathematical Society, 1997.
[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer Verlag, 1977.
[Hoy66] W. Hoyt, *On the Chow bunches of different projective embeddings of a projective variety*, Amer. J. Math. 88 (1966), 273–278.
[Kap93] M. M. Kapranov, *Chow quotients of Grassmannians I*, Adv. Soviet Math. 16 (1993), no. 2, 29–110, I. M. Gelfand Seminar.
[KSZ91] M. Kapranov, *Quotients of toric varieties*, Mathematische Annalen 290 (1991), 643–655.
[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32, Sprin-Verlag, Berlin, 1996.
[LS75] A. T. Lascu and D. B. Scott, *An algebraic correspondence with applications to blowing up Chern classes*, Ann. di Mat. 102 (1975), 1–36.
[LY87] H. B. Lawson, Jr. and Steve S. T. Yau, *Holomorphic symmetries*, Ann. Sci. École Norm. Sup. (4) 20 (1987), 557–577.
[Law95] H. B. Lawson, Jr., *Spaces of Algebraic Cycles*, Surveys in Differential Geometry, vol. 2, 137–213, Surveys in Differential Geometry, International Press, 1995, pp. 137–213.
[Mac62] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Math. Proc. Cambridge Philos. Soc. 58 (1962), 563–568.
[Mac79] I. G. Macdonald, *Symmetric functions and hall polynomials*, Clarendon Press, Oxford, 1979.
[Sam71] P. Samuel, *Séminaire sur l’équivalence rationelle*, Paris-Orsay, 1971.

**Instituto de Matemáticas, UNAM, Mexico**

*E-mail address:* javier@im.unam.mx

**Instituto de Matemáticas, UNAM, Mexico**

*E-mail address:* eladio@matem.unam.mx