A THEOREM ON AMPHICHEIRAL ALTERNATING LINKS

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Abstract
After reconsidering the Dasbach-Hougardy counterexample to the Kauffman Conjecture on alternating knots, we reformulate the conjecture and prove this reformulated conjecture for alternating links as the Mirror Theorem.

1. Introduction
We begin with a short description of the general background to the paper. Whitney [9] characterized spherical projections of planar graphs. He showed that two embeddings of a planar graph into a sphere are related by 2-isomorphisms. A result similar in spirit in knot theory is the Flyping Theorem that was conjectured by Tait and proven by Menasco and Thistlethwaite: Any two reduced alternating projections of a prime alternating link are related by flypes. A flype is a replacement of a subtangle in the knot diagram as shown in Figure 1. The replacement corresponds to turning the tangle around by 180 degrees and pulling its connecting arcs as well so that a crossing on one side of the tangle ends up on the other side, as shown in the Figure. Flyping is an ambient isotopy, and if a link is alternating and one performs a flype, then the resulting link is also alternating.

In Figure 3 we illustrate the formation of the checkerboard graph $G(K)$ of a link diagram $K$. The checkerboard graph of a link diagram $K$ in the plane is formed by first coloring the regions of the link diagram (shaded and unshaded) with two colors so that adjacent regions have distinct colors, and so that the outer unbounded region is unshaded. Then a node is assigned to each shaded region, and edges occur in the graph whenever two regions share a crossing in the link diagram. In the Figure 3 we have also illustrated how to decorate the graph $G(K)$ with signs so that the diagram $K$ can be reconstructed from it. In this paper we shall not need...
to keep track of these signs and so we intend by $G(K)$ the undecorated graph that is associated with the checkerboard coloring. The reader will find it easy to verify that the planar dual $G^*(K)$ of the graph $G(K)$ is obtained by forming a graph by the same prescription but using the unshaded regions. We let $K^*$ denote the mirror image of the diagram $K$ that is obtained by switching all the crossings of $K$. It is not hard to see that $G(K^*)$ is graph isomorphic on the surface of the two dimensional sphere with the dual graph $G^*(K)$. This leads to natural questions about the relationship of graphs and dual graphs for links that are amphichiral.

A link is said to be amphichiral if it is ambient isotopic to its mirror image. It should be remarked that the natural domain for graph isomorphisms in this paper is on the surface of a two dimensional sphere. Knots and links are standardized by planar diagram embeddings so that the checkerboard graphs are well-defined.

![Figure 1: Flyping.](image1)

![Figure 2: The Checkerboard Graph $G(K)$.](image2)
For denoting knots and links we use Conway notation [1, 4]. All tangles are finite compositions of elementary tangles. The elementary tangles are 0, 1 and \(-1\). Algebraic knots and links can be obtained by using three operations: sum, product, and ramification, leading from tangles \(a, b\) to the new tangles \(a + b, -a, ab,\) and \((a, b)\), respectively, where \(-a\) is the image of \(a\) in NW-SE mirror line, \(ab = -a + b, \quad -a = a 0, \quad \text{and} \quad (a, b) = -a - b\) (Fig. 3). Polyhedral knots and links can be obtained by substituting vertices of basic polyhedra by algebraic tangles.

![Figure 3](image)

**Figure 3:** (a) The elementary tangles; (b) sum of tangles \(a + b\); (c) tangle \(-a = a 0\); (d) product of tangles \(ab = -a + b\); (e) ramification of tangles \((a, b) = -a - b\); (f) the basic polyhedron 6".

Louis Kauffman conjectured in [6] (revised in [7]) that every amphicheiral alternating knot can be drawn so that the checkerboard graph of the knot diagram is self dual:

**Conjecture 1.1.** (Kauffman Conjecture) Let \(K\) be an alternating amphicheiral knot. Then there exists a reduced alternating knot diagram \(D\) of \(K\), such that \(G(D)\) is isomorphic to \(G^*(D)\), where \(G(D)\) is a checkerboard-graph of \(D\) and \(G^*(D)\) is its dual [7].

In the paper [2] Oliver Dasbach and Stephan Hougardy found a counterexample to this conjecture, an amphicheiral alternating knot 14\(a_{10435}\), given in Conway notation [1, 4] as \((2, 1, 3) 11 (2, 1, 3)\), which has four different minimal diagrams, but none of their checkerboard graphs is isomorphic to its dual.

The knot \((2, 1, 3) 11 (2, 1, 3)\) has four minimal diagrams (Fig. 4 a,b,c,d):

1. \(((((2, 1, 3), 1), 1)((2, 1), 3));
2. (((((1, 2), 3), 1), 1) ((2, 1), 3));
3. (((1, 3, (2, 1))), 1) ((2, 1), 3);
4. (1, (1, ((2, 1), 3))) ((2, 1), 3).

Figure 4: Four minimal diagrams of the knot \((2, 1) \mathbf{1} (2, 1, 3)\).

None of their corresponding checker-board graphs \(G_1, G_2, G_3, G_4\) is isomorphic to its dual \(G^*_1, G^*_2, G^*_3, G^*_4\). However, \(G_3 \simeq G^*_1\) (Fig. 5a) and \(G_4 \simeq G^*_2\) (Fig. 5b).

Figure 5: Isomorphic graphs (a) \(G_3 \simeq G^*_1\); (b) \(G_4 \simeq G^*_2\).

This motivates us to reformulate the original Kauffman conjecture for alternating knots and prove a *Mirror Theorem* for alternating links that we shall state below. This Theorem is based on using the Tait Flyping Conjecture, proved by Menasaco and Thistlethwaite [8]. This result tells us that two reduced alternating link diagrams are ambient isotopic if and only if they are related by a series of flypes (to be described below). Given a knot or link \(K\), let \(G(K)\) denote the checkerboard graph for \(K\). We will say that that two graphs \(G(K)\) and \(G(K')\) are *flype equivalent* if \(K\) and \(K'\) can be transformed into one another by a sequence of flypes.
Now suppose that $K$ is a reduced alternating diagram that is ambient isotopic to its mirror image diagram $K^*$. Let $G(K)$ denote the graph of $K$ and $G(K^*)$ denote the graph of the mirror image knot $K^*$. Then we know from construction that $G(K^*) = G^*(K)$ where $G^*(K)$ denotes the dual graph of $G(K)$ in the plane or on the surface of the two-dimensional sphere. Since we know that $K^*$ is related to $K$ by flypes, it follows that the graphs $G(K)$ and $G^*(K)$ are flype equivalent. This is the correct statement that can replace the original Kauffman conjecture, and becomes our Theorem 2.1. The interest in this simple reformulation, in the light of the Flyping Theorem, is that if we are given a minimal diagram for an alternating link, then this diagram is known [5] to be alternating and it is not hard to enumerate all the possible flype-equivalent diagrams.

2. Main Results

Here is the statement and proof of the Mirror Theorem described in the Introduction. The reader should note that in speaking of the checkerboard graphs of knots and links we assume that diagrams are taken in the plane, and the the unbounded planar region is not shaded in the checkerboard shading.

**Theorem 1.** (Mirror Theorem for Alternating Links) Let $K$ be an reduced alternating, prime, knot or link. Then the checkerboard graph $G(K)$ is flype equivalent to its dual graph $G^*(K)$ if and only if $K$ is amphicheiral. Note that this result implies that if $K$ is amphicheiral, then some minimal diagram of $K^*$ has checkerboard graph that is equivalent to the dual of the checkerboard graph of $K$.

**Proof:** In [8] Menasco and Thistlethwaite prove the Tait flyping conjecture. They prove that if $D$ and $D'$ are minimal alternating diagrams for prime links $K$ and $K'$, then these links are ambient isotopic if and only if there is a sequence of flypes connecting the two alternating diagrams. We know that a reduced alternating diagram of a link is minimal by the results in [5]. Furthermore, all minimal alternating diagrams in the link type of $D$ are obtained by applying sequences of flypes to $D$. Thus, if $D$ is ambient isotopic to its mirror image (amphicheiral), then there is a diagram $D'$, obtained by a sequence of flypes from $D$ such that $D'$ is exactly the result of reversing all the crossings in the diagram $D$. This means that if $G$ is the checkerboard graph of $D$, then $G^*$ is the checkerboard graph of $D'$. This proves the Theorem. □

**Remark.** Note that one can take the chiral 3-component link with $n = 16$ crossings $6^*(2,1,2)1.(2,2,1)1$, with 16 different minimal diagrams. Its minimal diagrams $D_1 = 6^*(2,1,2)1.(2,2,1)1$ (Fig. 6a) and $D_2 = 6^*(1,(2,(1,2)))((2,1),2),1$ (Fig. 6b) satisfy the relationship $G_1 \simeq G_2^*$ (Fig. 7a,b). This shows that some mirror image graphs may be equivalent to some graphs of the original link but that this does not, in itself, imply chirality.

For links with a single minimal diagram, the Mirror Theorem can be used as the criterion for recognition of amphicheiral alternating links: a link with a single minimal diagram $D$ is amphicheiral if $G(D) \simeq G^*(D)$. For example, among mutant alternating knots $K_1 = .(2,3).(3,2)$ and $K_2 = .(2,3).(2,3)$ with a single minimal
Figure 6: Two minimal diagrams (a) $D_1 = 6^*(21,2)1.(2,21)1$ and (b) $D_2 = 6^*(1,(2,1,2)).(((2,1),2),1)$ of the chiral link $6^*(21,2)1.(2,21)1$ with $G_1 \simeq G_2^*$. 

Figure 7: Graphs $G_1$ and $G_2^*$ with $G_1 \simeq G_2^*$.

diagram (Fig. 8) the first is amphicheiral, and the other is not, because the graph of the first knot is self-dual, and the other is not (Fig. 9).

**Definition 1.** An amphicheiral alternating knot or link $L$ is called Dasbach-Hougardy link if it has no minimal diagram for which $G(D)$ is isomorphic to $G^*(D)$.

According to a computer search, the smallest Dasbach-Hougardy link is 12-crossing alternating link $(21,2)11(21,2)$ which belongs to the same family $(p1,2)11(p1,2)$ $(p \geq 2)$ as Dasbach-Hougardy counterexample, knot $(21,3)11(21,3)$. In this way it is possible to obtain an infinite number of Dasbach-Hougardy links belonging to the same family.

Moreover, we propose the more general construction of Dasbach-Hougardy links:

**Definition 2.** Alternating pretzel (Montesinos) tangle $p_1,p_2,\ldots,p_n$, where $p_1$, $p_2$, $\ldots$, $p_n$ $(n \geq 2)$ are rational tangles not beginning by 1 is called oriented if it is not equal to its reverse. If all $p_i$ are integers, it is called integer tangle, and if at least one $p_i$ is not an integer it is called non-integer tangle.

The symbol $1^{4k-2}$ denotes $11\ldots1$, where 1 occurs $4k - 2$ times $(k \geq 1)$.

**Conjecture 1.2.** Every link given by Conway symbol of the form

$$(p_1,p_2,\ldots,p_n)1^{4k-2}(p_1,p_2,\ldots,p_n)$$

$(k \geq 1, n \geq 2)$, where $p_1,p_2,\ldots,p_n$ is oriented non-integer tangle is Dasbach-Hougardy link which satisfies Mirror Theorem. If $p_1,p_2,\ldots,p_n$ is oriented integer
Figure 8: Knots $K_1 = (2, 3), (3, 2)$ and $K_2 = (2, 3), (2, 3)$.

Figure 9: (a) Self-dual graph $G((2, 3), (3, 2))$; (b) graph $G((2, 3), (2, 3))$ which is not self-dual.

tangle, the obtained link is chiral. All knots or links of the form

$$(p_1, p_2, \ldots, p_n) t (p_1, p_2, \ldots, p_n)$$

where $p_1, p_2, \ldots, p_n$ is oriented integer or non-integer tangle and $t$ is palindromic rational tangle are amphicheiral and satisfy the original Kauffman conjecture.

The proposed general construction gives an infinite class of Dasbach-Hougardy links. For example, $(2, 1, 2, 2) \ 11 \ (21, 2, 2), \ (31, 2, 2) \ 11 \ (31, 2, 21)$ (Figs. 10, 11), or $(21, 2, 2) \ 11 \ 11 \ 11 \ (21, 2, 2)$ are examples of Dasbach-Hougardy links. Each of them can be used as the counterexample to the original Kauffman conjecture, but satisfies the Mirror Theorem.

All knots or links of the form

$$(p_1, p_2, \ldots, p_n) \ t (p_n, \ldots, p_2, p_1)$$

where $p_n, \ldots, p_2, p_1$ is reverse of $p_1, p_2, \ldots, p_n$, and $t$ is a palindromic rational tangle are amphicheiral and satisfy original Kauffman conjecture.

In the paper [3] the authors proved that the original Kauffman Conjecture is true for achiral alternating knots. They also announced the counterexample to

*A rational tangle is called palindromic if it is equal to its reverse.*
Figure 10: Minimal diagrams (a) \( D_1 = ((1, (2, 1), 2, (3, 1)), 1) \) \( ((3, 1), 2, (2, 1)) \) and (b) \( D_2 = (((3, 1), 2, (1, 2)), 1), 1)((3, 1), 2, (2, 1)) \) of the amphicheiral link \( (3 \overline{1}, \overline{2})_{11} (3 \overline{1}, 2, \overline{2}) \).

Figure 11: Graphs \( G_1 = G(((1, (2, 1), 2, (3, 1)), 1) \) \( ((3, 1), 2, (2, 1)) \) and \( G_2^* = G^*(((3, 1), 2, (1, 2)), 1), 1)((3, 1), 2, (2, 1)) \) with \( G_1 \cong G_2^* \).

the conjecture that every Dasbach-Hougardy knot is algebraic, which we proposed in the preceding version of this paper (arXiv:1005.3612v1 [math.GT]).
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