More on half-wormholes and ensemble averages

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Abstract

We continue our study Half-Wormholes and Ensemble Averages about the half-wormhole proposal. By generalizing the original proposal of the half-wormhole, we propose a new way to detect half-wormholes. The crucial idea is to decompose the observables into self-averaged sectors and non-self-averaged sectors. We find the contributions from different sectors have interesting statistics in the semi-classical limit. In particular, dominant sectors tend to condense and the condensation explains the emergence of half-wormholes and we expect that the appearance of condensation is a signal of possible bulk description. We also initiate the study of multi-linked half-wormholes using our approach.

Keywords: half-wormholes, factorization problem, quantum gravity, ensemble averages, SYK

(Some figures may appear in colour only in the online journal)

1. Introduction

Recent progress in quantum gravity and black hole physics impresses on the fact that wormholes play important roles. Many evidences suggest an appealing conjectural duality between a bulk gravitation theory and an ensemble theory on the boundary [2–54]. For example, the seminal work [2] shows that Jackiw-Teitelboim (JT) gravity is equivalent to a random matrix theory. On the other hand, this new conjectural duality is not compatible with our general belief about the AdS/CFT correspondence. A sharp tension is the puzzle of factorization [55, 56]. In [57], this puzzle is studied within a toy model introduced in [37], where they find that (approximate) factorization can be restored if other saddles which are called half-wormholes are included. Motivated by this idea, in [58] a half-wormhole saddle is proposed in a 0-dimensional (0d) Sachdev–Ye–Kitaev (SYK) model, followed by further analyses in different models [59–66].

Another scenario where half-wormholes are crucial is in the example of the spectral form factor. Chaotic systems exhibit a linear ramp behavior in their spectral form factor at late times [67]. In gravity theory, the smooth portion of the ramp can be explained by the emergence of a ‘wormhole’ saddle [68]. The oscillation portion should also be captured by the gravity theory, and the half-wormhole saddle is a potential candidate. This phenomenon has been observed in other SYK-like models [66], even though their half-wormhole concept differs from ours [69]. In our previous works [44, 69], we pointed out the connection between the gravity computation in [57] and the field theory computation in [58] and tested the half-wormhole proposal in various models. The main difficulty of this proposal is the construction of the half-wormhole saddles. Furthermore, the ansatz proposed in [58, 62] seems to rely on the fact the ensemble is Gaussian with zero mean value. As a result, the 0d SYK model only has non-trivial cylinder wormhole amplitude. However for a generic gravity theory for example the JT gravity, disk and all kinds of wormhole amplitudes should exist. In our previous work [69], we find even turning on disk amplitude in 0d SYK model will change the half-wormhole ansatz dramatically.

In this work, we generalize the idea of [57] and propose a method of searching for half-wormhole saddles. In our proposal, the connection between [57] and [58] will manifest. One notable benefit of our approach is that it does not depend on the trick of introducing a resolution identity used in [58], the collective variables emerge automatically. More importantly, our proposal can be straightforwardly generalized to non-Gaussian ensemble theories.
2. Gaussian distribution or the CGS model

In [57], the main model is the Coleman and Giddings-Strominger (CGS) model. The CGS model is a toy model of describing spacetime wormholes and it is more suggestive to obtain it from the Marolf-Maxfield (MM) model [37] by restricting the sum over topologies to only include the disk and the cylinder [57].

Let the amplitudes of the disk and cylinder be $\mu$ and $r^2$, i.e.
\[
\langle \hat{Z} \rangle = \mu, \quad \langle \hat{Z}^2 \rangle = r^2, \tag{1}
\]
where $|\rangle = |\text{HH}\rangle$ denotes the no-boundary (Hartle-Hawking) state and $\hat{Z}$ denotes the boundary creation operator thus $\langle \hat{Z}^n \rangle$ computes the Euclidean path integral over all manifolds with $n$ boundaries. For the CGS model the gravity amplitude or the ‘correlation function of the partition function’ $\langle \hat{Z}^n \rangle$ is a polynomial of $\mu$ and $r^2$ and in particular its generating function is simply
\[
\langle e^{\mu \hat{Z}} \rangle = \exp \left( \mu \mu + \frac{\mu^2 r^2}{2} \right). \tag{2}
\]
Thus we can identify $\hat{Z}$ as a Gaussian random variable $Z$ such that the gravity amplitude $\langle \langle Z \rangle \rangle$ can be computed as the ensemble average $E(f(Z)) \equiv \langle f(Z) \rangle$. This equivalence is a baby version of gravity ensemble duality.

The crucial idea of [57] is that the correlation functions of partition function do not factorize in general but they factorize between $\alpha$-states which are the eigenstates of $\hat{Z}$
\[
\langle \alpha | \hat{Z}^2 | \alpha \rangle = \langle \alpha | \hat{Z} | \alpha \rangle^2 = Z^2. \tag{3}
\]
The $\alpha$-state is also created by a generation operator acting on $|\text{HH}\rangle$
\[
|\alpha \rangle = \psi_0|\text{HH}\rangle. \tag{4}
\]
Note that $\psi$ can be expressed in terms of $\hat{Z}$ in a very complicated way so $\psi$ commutes with $\hat{Z}$. Then (3) can be rewritten in a very suggestive way
\[
Z^2 = \langle \psi^2 \hat{Z}^2 \rangle = \langle \hat{Z}^2 \rangle + \langle \psi^2 \hat{Z}^2 \rangle, \tag{5}
\]
where we have assumed that $\alpha$-state is normalized $\langle \psi_\alpha^2 \rangle = 1$. This rewriting is interesting because it separates out the self-averaged part $\langle \hat{Z}^2 \rangle$ and non-self-averaged part $\langle \psi_\alpha^2 \hat{Z}^2 \rangle$. In the CGS model, since the eigenvalue of $\hat{Z}$ is continuous and supported on $\mathbb{R}$ so that we can express $\psi_\alpha$ in terms of $\hat{Z}$ schematically as
\[
\psi_\alpha = \delta(\hat{Z} - Z_\alpha) = \int \frac{dk}{2\pi} e^{ik(\hat{Z} - Z_\alpha)}, \tag{6}
\]
\[
\text{thus}
\langle Z^2 \psi_\alpha \rangle = \int \frac{dk}{2\pi} e^{-ikZ_\alpha} \langle Z^2 e^{ikZ_\alpha} \rangle \tag{7}
\]
\[
\Rightarrow Z^2_\alpha = \int \frac{dk}{2\pi} e^{-ikZ_\alpha} \langle Z^2 e^{ikZ_\alpha} \rangle = \frac{2}{\langle \psi_\alpha^2 \rangle} \tag{8}
\]
Noting that $\langle \psi_\alpha \rangle = P(Z_\alpha)$, where $P(Z)$ is the PDF of $Z$, we find that (8) coincides with the trick used in [69] and [62] of rewriting $Z_\alpha^2$ as a formal average
\[
Z^2_\alpha = \int dZ \delta(Z - Z_\alpha) \frac{Z^2 P(Z)}{P(Z_\alpha)} = \int \frac{dk}{2\pi} e^{-i\frac{k}{2\pi}(\hat{Z}^2 e^{ikZ_\alpha})}. \tag{9}
\]
From which we can derive some useful approximation formula $Z^2_\alpha \approx \langle Z^2 \rangle + \Phi$, where $\langle Z^2 \rangle$ and $\Phi$ are respectively recognized as the wormhole and half-wormhole contributions as shown in [62, 69]. So we can think of this trick as a refinement of the factorization proposal of [57]. We will elaborate on this below.

2.1. Half-Wormhole in CGS-like model

In the CGS model, because $Z$ satisfies the Gaussian distribution there is a more concrete expression for the half-wormhole saddle as shown in [57]. The key point is the fact that when $Z$ is Gaussian, it can be thought of as the position operator of a simple harmonic oscillator so there exists a natural orthogonal basis, the number basis $|n \rangle$ which is called the $n$-baby Universe basis in the context of the gravity model. If we insert the complete basis $\sum_i |i \rangle \langle i |$ into (9) we can get
\[
Z^2_\alpha = \int dZ \delta(Z - Z_\alpha) \frac{Z^2 P(Z)}{P(Z_\alpha)} = \int \frac{dk}{2\pi} e^{-i\frac{k}{2\pi}(\hat{Z}^2)} \langle Z^2 e^{ikZ_\alpha} \rangle \tag{10}
\]
\[
= \int \frac{dk}{2\pi} e^{-i\frac{k}{2\pi}Z^2} \sum_{i=0}^{n} \langle Z^2 i \rangle \langle i | e^{ikZ_\alpha} \rangle \tag{11}
\]
\[
= \int \frac{dk}{2\pi} e^{-i\frac{k}{2\pi}Z^2} \sum_{i=0}^{n} \langle Z^2 i \rangle \sqrt{i!} t^{i} \langle i | e^{ikZ_\alpha} \rangle \tag{12}
\]
\[
= \int \frac{dk}{2\pi} e^{-i\frac{k}{2\pi}Z^2} \sum_{i=0}^{n} \langle Z^2 i \rangle \frac{n}{i!} \langle Z^2 \rangle^{n-i} \frac{1}{i!} t^{i} \langle \delta^{n-i} \rangle \equiv \sum_{i=0}^{n} \langle Z^2 i \rangle \frac{n!}{i!} t^{i} \langle \delta^{n-i} \rangle \tag{13}
\]
\[
\text{where}
\theta^{(n-i)} = \int \frac{dk}{2\pi} e^{-i\frac{k}{2\pi}Z^2} \langle Z^2 i \rangle \langle \delta^{n-i} \rangle = \langle \delta^{n-i} \rangle \tag{14}
\]
\[
\text{Note that}
\langle \delta^{n-i} \rangle = \frac{\langle \delta^{n-i} \rangle}{\langle e^{ikZ_\alpha} \rangle} \equiv \phi_{n-i}^{(i)} = \langle ikZ_\alpha \rangle^{n-i} \tag{15}
\]
\[
\text{then (14) coincides with results in [69]. So we confirm the result that within the Gaussian approximation (only keep the first two cumulants), $Z^2_\alpha$ can be decomposed as (14) and it suggests that $\theta$'s are the convenient building blocks of possible half-wormhole saddles. Some examples of the}
\]
\[
\text{Note that our convention is $Z = \mu + r(a + d')$.}
\]
decomposition (14) are\(^3\)

\[
Z_n^3 = \theta^{(1)} + \langle Z \rangle, \\
Z_n^2 = \theta^{(2)} + 2\langle Z \rangle \theta^{(1)} + \langle Z^2 \rangle, \\
Z_n^3 = \theta^{(3)} + 3\langle Z \rangle \theta^{(2)} + 3\langle Z^2 \rangle \theta^{(1)} + \langle Z^3 \rangle, \\
Z_n^4 = \theta^{(4)} + 4\langle Z \rangle \theta^{(3)} + 6\langle Z^2 \rangle \theta^{(2)} + 4\langle Z^3 \rangle \theta^{(1)} + \langle Z^4 \rangle,
\]

with

\[
Y_n = \frac{1}{(2\pi)^n} \int \prod_i \left( \frac{dk_i}{\sqrt{2\pi / t^2}} e^{-\frac{i k_i^2}{2t^2}} \right) \langle e^{i \sum_i k_i x_i} \rangle \sum_{n_1, \ldots, n_N} \langle Y^a_{n_1, \ldots, n_N} \rangle \frac{m_{11} \ldots m_{NN}}{\langle e^{i \sum_i k_i x_i} \rangle}.
\]

\[\theta^{(1)} = -\mu + Z_n, \quad \theta^{(2)} = (\mu - Z_n)^2 - t^2 = \theta^{(1)^2} - t^2, \]

\[\theta^{(3)} = (\mu - Z_n)^3 + 3(\mu - Z_n)^2 t^2 = \theta^{(1)^3} - 3t^2 \theta^{(1)}, \]

\[\theta^{(4)} = 3t^4 + 6t^2(\mu - Z_n)^2 + (\mu - Z_n)^4 = \theta^{(1)^4} - 6t^2 \theta^{(1)^2} + 3t^4.
\]

In general we have

\[
\theta^{(i)} = \int \frac{dk}{2\pi} e^{\frac{ik}{\mu - Z_n}} \langle e^{iZ} \rangle (ik)^i = \int \frac{dk}{\sqrt{2\pi / t^2}} e^{-\frac{i k^2}{2t^2}} (ik)^i,
\]

so \(\theta^{(i)} / (i!)^i\) is the \(i\)th moment of ‘Gaussian distribution’ \(\mathcal{N}(\mu - Z_n, t^2, 1/t^2)\) and the generating function is

\[
\langle e^{ik} \rangle = e^{\frac{ik}{\mu - Z_n}} \frac{\mu}{r^2}.
\]

Considering the following ensemble average

\[
\langle \langle e^{ik_1} \rangle \langle e^{ik_2} \rangle \rangle_{Z_n} = e^{-\frac{ik_1 k_2}{\mu - Z_n}},
\]

and expanding both sides into Taylor series of \(u_1\) and \(u_2\) one can find

\[
\langle \theta^{(i)} \theta^{(j)} \rangle_{Z_n} = i! t^j 2^i \delta_{ij}.
\]

Due to this orthogonal condition we can directly tell which sector in the decomposition of \(Z_n^n\) is dominant by computing \(Z_n^n Z_n^n\)

\[
Z_n^n = \sum_i c_i \theta^{(i)}, \quad \langle Z_n^n Z_n^n \rangle = \sum_i c_i^2 i! t^j.
\]

In CGS model, since there is only a single random variable \(Z\) so it does not admit any approximation related to large \(N\) or small \(G_N\). Therefore the wormhole or half-wormhole are not true saddles in the usual sense. To breathe life into them we should consider a model with a large number \(N\) of random variables such as random matrix theory or SYK model which can be described by certain semi-classical collective variables like the \(G_N\) in SYK, which potentially have a dual gravity description. However, we find that it is illustrative to first apply the factorization proposal to some simple statistical models as we did in [69].

### 2.2. Statistical model

Let us consider a function \(Y(X)\) of a large number \(N\) independent random variables \(X_i\). Assuming that \(X_i\)s are drawn from the Gaussian distribution then we have the decomposition

\[
Y = \sum_{i=1}^N X_i.
\]

Appropriately for \(n = 1\) there are only two sectors

\[
y = \sum_{i=1}^N \mu + \sum_{i=1}^N \theta^{(1)} = \Theta_0 + \Theta_1, \quad \langle \Theta_0 \rangle = N/2, \quad \langle \Theta_1 \rangle = N/2,
\]

and for \(n = 2\) there are three sectors

\[
y^2 = \Phi_0 + \Phi_1 + \Phi_2, \quad \Phi_0 = \langle Y^2 \rangle, \quad \Phi_1 = 2N \Phi_{11}, \quad \Phi_2 = \sum_{y \neq y} \langle \theta_1 \theta_1 \rangle^2 - \delta_y t^2.
\]

In general the parameters \(\mu\) and \(t^2\) are \(N\) independent therefore \(Y^n\) is self-averaged \(Y^n \approx \langle Y^n \rangle\) in the large \(N\) limit. This is also true even \(X_i\) are not Gaussian because of the central limit theorem. But we also know in the literature that in order to have well-defined semi-classical approximation, the parameters \(\mu\) and \(t^2\) should depend on \(N\) in a certain way like in SYK model. Interestingly in this case if \(t^2 \sim \mu^2 N\), the self-averaged part and non-self-averaged part are comparable and we should keep them both. This is exactly what we have encountered in the 0-SYK model. But a crucial difference is that for this simple choice of observables, all the non-self-averaged sectors are also comparable so it is not fair to call any of them the half-wormhole saddle and to restore

\[\theta^{(i)}\] is simply the (unnormalized) Hermite polynomial.
factorization we have to include all the non-self-averaged sectors. The extremal case is \( r^2 \gg \mu^2 N \). In this limit we find that the sector with highest level dominates. For example,\( Y \approx \Theta_1, \quad Y^2 \approx \exp{Y} + \Phi_2, \quad \left( \Theta_1^2 \right) \approx \exp{Y}, \quad \left( \Theta_2^2 \right) \approx 2 \exp{Y}, \quad \) then it is reasonable to identify \( \Theta_1 \) with half-wormhole and identify \( \Phi_2 \) with the 2-linked half-wormhole. Similarly we can introduce \( n \)-linked half-wormholes. For example, in this extremal case, we can approximate \( Y^2 \) with
\[
Y^2 \approx 3 \exp{Y} + \Lambda_3, \tag{38}
\]
\[
\Lambda_3 = \sum_{i \neq j \neq k} \left( \theta_i^{(1)} \theta_j^{(1)} \theta_k^{(1)} \right) + 3 \sum_{i \neq j} \left( \theta_i^{(2)} \theta_j^{(1)} \right) + \sum_i \left( \theta_i^{(3)} \right), \tag{39}
\]
\[
= \sum_{i,j,k} \left( \theta_i^{(1)} \theta_j^{(1)} \theta_k^{(1)} \right) - 3 \beta r^2 N \sum_i \left( \theta_i^{(1)} \right), \tag{40}
\]
where the sector \( \Lambda_3 \) should describe the 3-linked half-wormhole. We will consider a similar construction in the 0-SYK model.

### 2.2.2. Exponential observables

In the Random Matrix Theory or quantum mechanics, the most relevant observable is the exponential operator \( \text{Tr}(e^{i\beta H}) \) since it relates to the partition function. So it may be interesting to consider a similar exponential operator
\[
Y = \sum_k e^{\beta X_k}, \tag{41}
\]
in the toy statistical model. By a Taylor expansion of the exponential operator we find the following decomposition
\[
e^{\beta X} = \sum_k \frac{\beta^k X^k}{k!}, \quad \theta^{(0)} = 1, \tag{42}
\]
thus
\[
Y = \sum_k \Theta_k, \quad \Theta_k = e^{\beta r^2} \cdot \sum_i \frac{\beta^k \Omega_i^k}{k!}, \tag{43}
\]
\[
\left( \Theta_k^2 \right) = N e^{2 \beta r^2 + \frac{k^2}{2}} e^{\beta r^2} / k!, \quad \left( \Theta_k^4 \right) = e^{-2 \beta r^2} \frac{e^{2 \beta r^4}}{k^4} \cdot \left( \left( \Theta_1^2 \right) \right)^2 \tag{44}
\]
Interestingly the ratio \( r_k \) follows the Poisson distribution \( \text{Pois} \), some examples are in figure 1. When \( \beta < 1 \) the dominant sector is \( \Theta_2 \) while for \( \beta \gg 1 \) the Poisson distribution approaches Gaussian distribution \( \mathcal{N}(\beta r^2, \beta r^4) \) so we have to include all the sectors in the peak \( k \in (\beta r^2 - \beta r^4 + \beta r^2) \) to have a good approximation. We can decompose \( Y^2 \) in a similar way
\[
Y^2 = \sum_k \Phi_k, \tag{45}
\]
\[
\left( \Theta_2^2 \right) = e^{-2 \beta r^2} \frac{e^{2 \beta r^4}}{k^4} \cdot \left( \left( \Theta_1^2 \right) \right)^2 \tag{46}
\]
\[
\left( \Theta_3^2 \right) = e^{-2 \beta r^2} \frac{e^{2 \beta r^4}}{k^4} \cdot \left( \left( \Theta_1^2 \right) \right)^2 \tag{47}
\]

The behavior is similar. When \( \beta < 1 \), the dominant sector is the self-averaged sector \( \Theta_0 \). When \( 2 \beta r^2 > \log N \) approaches the Gaussian \( \mathcal{N}(4 \beta r^2, 4 \beta^2 r^4) \). On the other hand, when \( 1 < 2 \beta r^2 \leq \log N \) approaches the Gaussian \( \mathcal{N}(2 \beta r^2, 2 \beta^2 r^4) \). In the end when \( 2 \beta r^2 \sim \log N \), (46) will have two comparable peaks, see figure 2 as an example. However the half-wormhole ansatz proposed in [62, 69] which can be written as
\[
\Phi = \sum_{k=0}^{\infty} \Phi_k, \tag{48}
\]
\[ \phi_k = \Phi_k + (e^{-\beta h} - 1) e^{2\omega \beta/2 + \beta h} (2\beta)^k \sum_i \frac{\theta^{(1)}_{ai}}{k!}, \]  

(49)

only works for small value of \( \beta h \).

To summarize our proposal, by introducing the basis \( \{ \theta \} \) which is the generalization of \( n \)-baby Universe basis \[57\] we can decompose the observables or partition functions into a single self-averaged sector and many non-self-averaged sectors. These sectors are independent in the sense of \( \{28\} \). The contributions from each sector have interesting statistics: in the large \( N \) limit leading contributing sectors may condense to peaks. This condensation is a signal that the observable potentially has a bulk description (or semi-classical description) in the large \( N \) limit. If the self-averaged sector survives then it means the observable is approximately self-averaging. The surviving non-self-averaged sectors in the large \( N \) limit are naturally interpreted as the \( (n\text{-linked}) \) half-wormholes which are the results of sector condensation. In the extremal case, only one non-self-averaging survives reminiscing the famous Bose–Einstein condensation.

2.3. 0-SYK model

In this section we apply our proposal to the 0-SYK model which has the ‘action’

\[ z = \int d^N \psi \exp \left( i/2 \sum J_{i_1...i_q} \psi_{i_1...i_q} \right), \]  

(50)

where \( \psi_{i_1...i_q} = \psi_{i_1} \psi_{i_2} ... \psi_{i_q} \) and \( \psi_i \) are Grassmann numbers. The random couplings \( J_{i_1...i_q} \) is drawn from a Gaussian distribution

\[ (J_{i_1...i_q}) = u, \quad (J_{i_1...i_q} J_{i_1'...i_q'}) = r^2 \delta_{i_1...i_q} \delta_{i_1'...i_q'}, \quad r^2 = \tau_q^2 (q-1)!/N^{q-1}, \]  

(51)

where we found in \[69\] in order to have a semi-classical description \( u \) should also have a proper dependence

\[ u = ( -i )^{q}/r (q/2 - 1)!/2 N^{q/2 - 1}. \]  

(52)

We sometimes use the collective indices \( A, B \) to simplify the

\[ z = \int d^N \psi \int d\mathbb{R} \int d\Sigma \frac{d\Sigma}{2\pi i/N} e^{i/2 \sum \frac{N \Sigma^2}{2 N^{2/3}} G^{2/3}} e^{-N e^{G} + \sum \frac{\theta^{(1)}_{i_1...i_q} \psi_{i_1...i_q}}{k!} + \sum_{c_1 c_2} \psi_{c_1 c_2}} \]  

(61)

Integrating out the Grassmann numbers directly gives \[4\]:

\[ z = \int d^N \psi \exp \left( i/2 \sum J_{i_1...i_q} \psi_{i_1...i_q} \right) = \sum_{A_1 < ... < A_p} \text{sgn}(A) J_{A_1...A_p}, \quad p = N/q, \]  

(54)

where the expression (54) is nothing but the hyperpfaffian \( \text{ Pf}(J) \). According to (30), we can similarly decompose it as

\[ z = \sum_{i} \Theta_i, \quad \Theta_0 = \{ \}, \]  

(55)

\[ \Theta_k = u^{N-k} \sum_{k < n < \ldots < n_{p-k}} \text{ Pf}(\theta^{(1)}_{A_1...A_p}) \]  

(56)

where the tensor \( \theta^{(1)}_{A_1...A_p} \) means that the index \( A \) is not in the set \( \{I_1, ..., I_{p-k}\} \). The expression (56) can be derived by a combinatorial method used in \[69\] or by using the \( G, \Sigma \) trick as follows. First we expand \( z \) into series of \( \theta^{(1)} \)

\[ z = \int d^N \psi \exp \left( i/2 \sum \frac{A \psi_{i_1...i_q}}{k!} \right) = \int d^N \psi e^{i/2 \sum_{A} \psi_{i_1...i_q}} \]  

(57)

\[ = \int d^N \psi \left( i/2 \sum_{A} \theta^{(1)}_{i_1...i_q} \right) e^{i/2 \sum_{A} \psi_{i_1...i_q}}, \]  

(58)

thus by matching the power of \( \theta^{(1)} \) we get an integral expression of \( \Theta_k \)

\[ \Theta_k = \int d^N \psi \left( i/2 \sum_{A} \theta^{(1)}_{i_1...i_q} \right) e^{i/2 \sum_{A} \psi_{i_1...i_q}} \]  

(59)

Next following \[69\] we can introduce \( G, \Sigma \) variables directly as

\[ G = \frac{1}{N} \sum_{i<j} \psi_i \psi_j, \]  

(60)

\[ \Sigma = \frac{1}{N} \sum_{i<j} \psi_i \psi_j. \]  

(61)

\[ \text{notation} \]

\[ A = \{ a_1 < ... < a_q \}, \quad J_A \psi_A \equiv J_{a_1...a_q} \psi_{a_1...a_q}, \]  

(53)
which implies that

\[ z_{2}^{(1)} \sim \frac{2^{N/2}}{N^{2}} z_{2}^{(0)}, \quad z_{2}^{(2)} \sim \frac{2^{N}}{N^{2} \sqrt{N}} z_{2}^{(0)} \sim \frac{2^{N/2}}{N^{2} \sqrt{N}} z_{2}^{(1)}, \]

so that we have the approximation

\[ z \sim \Theta_{2}. \]

Similarly when \( p = 3 \), we can find

\[ z = \langle z \rangle + \Theta_{1} + \Theta_{2} + \Theta_{3}, \]

and

\[ z_{2}^{(3)} \sim \frac{t^{2}}{u^{2}} \frac{1}{3} z_{2}^{(2)}, \quad z_{2}^{(2)} \sim \frac{t^{2}}{u^{2}} \sqrt{N} z_{2}^{(1)}, \quad z_{2}^{(1)} \sim \frac{t^{2}}{u^{2}} \sqrt{N} z_{2}^{(0)}, \]

thus

\[ z \sim \Theta_{3}. \]

This turns out be general: when \( p \ll N \) the dominant term is \( \Theta_{p} \). Therefore, the self-averaged \( \langle z \rangle \) will not survive. This behavior is the same as we found in the simple statistical model in the regime when the cylinder amplitude is much larger than the disk amplitude.

On the other hand, if \( q \ll N \) then

\[ \frac{t^{2}}{u^{2}} \sim \frac{1}{N}, \]

the situation is very different. As a simple demonstration let us consider the case of \( q = 2 \)

\[ z_{2}^{(p)} \sim \frac{N^{q/2} t^{N}}{u^{q} z_{2}^{(0)}}, \]

\[ z_{2}^{(2)} \sim \frac{N^{2} t^{2} z_{2}^{(0)}}{u^{2} z_{2}^{(0)}} = N^{2} z_{2}^{(0)}, \]

and

\[ z_{2}^{(3)} \sim \frac{N^{6} t^{6}}{u^{6} z_{2}^{(0)}} = N^{3} z_{2}^{(0)} \ldots, \]

which implies that

\[ z_{2}^{(k)} \sim s_{k} z_{2}^{(0)}, \quad s_{k} = \frac{1}{2^{k} N^{k/2} k!(N-2k)!}. \]
The dominant term is neither $\langle z \rangle$ nor $\Theta_p$ but some intermediate term $\Theta_k$ as argued in [69]. With this detailed analysis we find that we should also include some 'sub-leading' sectors. The distribution of the surviving sectors in the large $N$ limit has a peak centered at the 'dominant' sector with a width roughly $\sqrt{N}$. One possible interpretation of this result is the surviving sectors are only approximate saddles or constrained saddles with some free parameters. Even though each approximate saddle contribution is as tiny as $1/\sqrt{N}$ but after integrating over the free parameters the total contribution is significant. Note that similar approximate saddles are also found for the spectral form factor in the SYK model [68]. We plot the ratio $z_k^{(1)}/z_2^{(0)}$ as function of $k$ in figure 3. With increasing $q$ or equivalently decreasing $p$, the peak moves to the left (small $k$) and becomes sharper and sharper. This is consistent with our analysis of limit of small $p$ where there is only one dominant saddle, $\Theta_p$. So our result shows that the wormhole (actually disk in this case) does not persist but the half-wormhole appears. As we found in [69] $\langle z^2 \rangle$ can be computed by a trick of introducing the collective variables

$$G_{LR} = \frac{1}{N} \sum_i \psi_i^L \psi_i^R, \quad G_L = \frac{1}{N} \sum_{i<j} \psi_i^L \psi_j^L, \quad G_R = \frac{1}{N} \sum_{i<j} \psi_i^R \psi_j^R,$$

and doing the path integral. The final expression is

$$\langle z^2 \rangle = \int_R d^3 G_i \int_R d^3 \sum_i \sum_{m=0}^{N/2} \left( \gamma^m \Sigma_{LR} + i \sqrt{\Sigma_{LR}} \right)^N + \left( \gamma^m \Sigma_{LR} - i \sqrt{\Sigma_{LR}} \right)^N \left( \gamma^m \Sigma_{LR} + i \sqrt{\Sigma_{LR}} \right)^N$$

$$= \int_R d^3 G_i \int_R d^3 \sum_{m=0}^{N/2} \left( \gamma^m \Sigma_{LR} \right)^2 m \left( i \sum_{LR} \right)^2 \left( \gamma^m \Sigma_{LR} \right)^N \left( \gamma^m \Sigma_{LR} \right)^N \left( \gamma^m \Sigma_{LR} \right)^N.$$

In [69] we indeed find a new non-trivial saddle point whose saddle contribution is larger than the saddle contribution of the trivial disk saddle and wormhole saddle. The new non-trivial saddle should correspond to $\sum_k (\Theta_k^2)$ with $k$ in the peak. The
expression (87) of \( z^2 \) leads to a \( G, \sum \) expression of each \( z^2 \):

\[
\langle \Theta_k^2 \rangle = z^{2(k)}_2 = \left( \frac{N}{kq} \right) \int d^k G_i \int d^N \Sigma \left( \Sigma_{\mathcal{L},\mathcal{R}} + kq \sum_{\Sigma_k} \right)^{N-kq} e^{Nz^2 G_{\mathcal{L}}^k + \mu G_{\mathcal{L}}^{k^2} + \mu G_{\mathcal{R}}^{k^2}} e^{-N\langle \Sigma, G_i \rangle}. \quad (87)
\]

Actually we can derive a different \( G, \sum \) expression from \( \Theta_k \) directly in a more enlightening way. Because \( \psi_i \) are Grassmann numbers and \( q \) is even then the exponential in (57) factorizes

\[
e^{z^2/2 \sum_k \psi_k \psi_k} = \prod_A e^{z^2/2 \psi_k \psi_k}. \quad (88)
\]

Using Tyler expansion the definition of \( \theta^{i(1)} \) one can derive a useful identity

\[
e^{X} = \left( e^{X} \right)^{\sum_{i=0}^{\infty} \frac{\theta^{i(1)}}{n!}}, \quad (89)
\]

where \( X \) is the random variable. With the help of this identity and \( \psi_A^2 = 0 \), (88) can be decomposed into

\[
\prod_A e^{z^2/2 \psi_k \psi_k} = \left\{ e^{z^2/2 \sum_k \psi_k \psi_k} \right\} \left( 1 + \sum_A \theta^{(1)}_A (\psi^3/2 \psi_k) + \frac{1}{2!} \sum_{A,B} \theta^{(1)}_A (\psi^3/2 \psi_k) \theta^{(1)}_B (\psi^3/2 \psi_k) + \ldots \right)
\]

\[
= \left\{ e^{z^2/2 \sum_k \psi_k \psi_k} \right\} e^{\sum_A \psi^3/2 \psi_k}. \quad (90)
\]

Thus the we can express \( \langle \Theta_k^2 \rangle \) as

\[
\langle \Theta_k^2 \rangle = \int d^k G_i \int d^N \Sigma \left( \Sigma_{\mathcal{L},\mathcal{R}} + kq \sum_{\Sigma_k} \right)^{N-kq} e^{Nz^2 G_{\mathcal{L}}^k + \mu G_{\mathcal{L}}^{k^2} + \mu G_{\mathcal{R}}^{k^2}} e^{-N\langle \Sigma, G_i \rangle}. \quad (91)
\]

\[
= \int d^k G_i \int d^N \Sigma \left( \Sigma_{\mathcal{L},\mathcal{R}} + kq \sum_{\Sigma_k} \right)^{N-kq} e^{-Nz^2 G_{\mathcal{L}}^k + \mu G_{\mathcal{L}}^{k^2} + \mu G_{\mathcal{R}}^{k^2}} \frac{1}{k!} \left( \frac{Nz^2}{q} \right)^k.
\]

The integral (92) is not convergent but we can introduce the generating function

\[
F(v) = \int d^k G_i \int d^N \Sigma \left( \Sigma_{\mathcal{L},\mathcal{R}} + kq \sum_{\Sigma_k} \right)^{N-kq} e^{-Nz^2 G_{\mathcal{L}}^k + \mu G_{\mathcal{L}}^{k^2} + \mu G_{\mathcal{R}}^{k^2}} \frac{1}{k!} \left( \frac{Nz^2}{q} \right)^k.
\]

which can be computed with a saddle point approximation and the \( \langle \Theta_k^2 \rangle \) is given by

\[
\langle \Theta_k^2 \rangle = \frac{1}{k!} \frac{d^k F(v)_{\text{saddle}}}{dv_k} \bigg|_{v=0}. \quad (93)
\]

As a simple test, we know that the exact result of \( F(v) \) is just

\[
F(v) = \langle z^2 \rangle_{t \rightarrow t^2} = \sum_k c_k m^{2-k} t^2 u^{2p-2k} v^k,
\]

which indeed leads to

\[
\langle \Theta_k^2 \rangle = \frac{1}{k!} \frac{d^k F(v)_{\text{saddle}}}{dv_k} \bigg|_{v=0} = c_k m^{2-k} t^{2k} v^{2p-2k}. \quad (94)
\]

### 2.3.1. Half-wormhole in \( z^2 \)

To make the half-wormhole saddle manifest below we will set \( u = 0 \). In this case ‘Bose–Einstein’ condensation happens. As found in [58] for the square of partition function \( z^2 \) the wormhole persists and there is only one
dominant non-self-averaged sector. Applying (30) directly leads to the decomposition

$$z^2 = \sum_i \Phi_{2i},$$

(97)

with

$$\Phi_0 = \langle z^2 \rangle = \sum_{A(B_i)< \ldots < A(B_p)} \sgn(A) \sgn(B) i^2 \delta_{A,B_i} \ldots i^2 \delta_{A,B_p},$$

(98)

$$\Phi_2 = \sum_k \sum_{A(B_i)< \ldots < A(B_p)} \sgn(A) \sgn(B) i^2 \delta_{A,B_i} \ldots (\theta_{A_i}^{(1)} \theta_{B_i} - \theta_{A_i}^{(1)} \theta_{B_i}^{(2)}) \ldots (\theta_{A_p}^{(1)} \theta_{B_p} - \theta_{A_p}^{(1)} \theta_{B_p}^{(2)}),$$

(99)

$$\Phi_{2p} = \sum_{A(B_i)< \ldots < A(B_p)} \sgn(A) \sgn(B) \left( \theta_{A_i}^{(1)} \theta_{B_i}^{(1)} + \delta_{A,B_i} \left( \theta_{A_i}^{(2)} - \theta_{A_i}^{(1)} \theta_{B_i}^{(2)} \right) \right) \ldots \left( \theta_{A_p}^{(1)} \theta_{B_p}^{(1)} + \delta_{A,B_p} \left( \theta_{A_p}^{(2)} - \theta_{A_p}^{(1)} \theta_{B_p}^{(2)} \right) \right)$$

(100)

where $\Phi_{2p}$ is the half-wormhole saddle which is found in [58, 62] by noticing $\theta_{A_i}^{(1)} = J_A$ and $\theta_{A_i}^{(2)} = - \theta_{A_i}^{(1)} - t_i^2$. Actually the connection between the half-wormhole proposed in [58] and factorization proposal introduced in [57] has been pointed out in [44]. A useful way to derive the expression of $\Phi_i$ is to use (89) first

$$e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle = \prod_A e^{\imath \theta_{A}^{(1)} / 2} (\psi_A^{(1)} + \psi_A^{(2)} + \psi_A^{(3)}),$$

(101)

$$= \left( e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle \right) \prod_A (1 + \imath t_i^2 \theta_{A}^{(1)} / 2 + \psi_A^{(1)} - \psi_A^{(2)} - \psi_A^{(3)}),$$

(102)

$$= \left( e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle \right) e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle,$$

(103)

and then to substitute it into the integral form of $z^2$

$$z^2 = \int d^N \psi_0 \langle \psi_0 | e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle \rangle > e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle,$$

(104)

$$= \prod_{k=0}^P \int d^N \psi_0 \langle \psi_0 | e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle \rangle \left( \sum_A \langle \theta_{A}^{(1)} \theta_{B_i}^{(1)} \rangle + t_i^2 \right) e^{\imath \theta_{A_i}^{(2)} / 2} \left( \sum_A \langle \theta_{A_i}^{(1)} \theta_{A_i}^{(2)} \rangle \right) \frac{1}{(p-k)!} \frac{1}{(p-k)!},$$

(105)

$$= \sum_{A(B_i)< \ldots < A(B_p)} \sgn(A) \sgn(B) \prod_{i} \left( \theta_{A_i}^{(1)} \theta_{B_i}^{(1)} + \delta_{A,B_i} \left( \theta_{A_i}^{(2)} - \theta_{A_i}^{(1)} \theta_{B_i}^{(2)} \right) \right).$$

(106)

By matching the power of $t_i^2$ we can extract the expression of $\Phi_i$. Note that the expressions of $\Phi_i$ have been derived in [62] based on the proposal of [58]. In [62] the non-dominant sectors are derived as fluctuations of the dominant saddle $\Phi_{2p}$ with the help of introducing $G, \sum$ variables. Because our derivation here does not rely on $G, \sum$ trick so it can be used to derive possible $n$-linked half-wormholes in $z^n$. First we notice that $\langle z^2 \rangle = \langle \Phi_0^2 \rangle$ is in the same order of $\langle z^2 \rangle$ as proved in [58] so the wormhole saddle persists. To confirm that $\Phi_{2p}$ is the only dominant non-self-averaged saddle we only need to show

$$\langle z^4 \rangle \approx \langle \Phi_0^2 \rangle + \langle \Phi_{2p}^2 \rangle,$$

(107)

which also has been proved in [58, 62]. Another benefit of the rewriting (104) is that we can introduce $G, \sum$ variable directly if needed because the appearance of $\langle e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle \rangle$ instead of introducing them “by hand” by inserting an identity as proposed in [58]. As we argued in [69] when $a = 0$, $\Phi_{2p}$ will not be the dominant sector anymore. Instead, there will be a package of surviving non-self-averaged sectors.

2.3.2. Half-wormhole in $z^3$. As we argued in the statistical toy model, there should exist $n$-linked half-wormholes. For simplicity let us focus on 3-linked half-wormholes and $z^3$. Similar to (104), $z^3$ can be rewritten as

$$z^3 = \int d^N \psi e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle \langle \psi_A^3 \rangle$$

(108)

$$= \int d^N \psi \left( e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle \right) e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle \times e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle \times e^{\imath t_i^2} \sum_{A} \langle \psi_A^3 \rangle,$$

(109)
which is expected to be one of the dominant non-self-averaged sectors in the large limit.

As a consistency check, substituting the explicit expressions

and

which is simple to check for small

In general, we expect

is the analogue of

Therefore the approximation

is the analogue of (38). We believe that this analogy persists for all other higher moments \( z^n \). Recall that \( \theta^{(i)} \) can be thought of as moments thus it is reasonable to introduce the connected moments or the cumulants \( \bar{\theta}^{(i)} \) with those \( z^n \) can be cast into

In general, we expect

which is simple to check for small \( n \) by a direct calculation. Since \( \theta^{(i)} \) is Gaussian so the only non-vanishing cumulants are \( \bar{\theta}^{(1)} \) and \( \bar{\theta}^{(2)} \) thus

As a consistency check, substituting the explicit expressions \( \theta^{(1)}_A = J_A \) and \( \theta^{(2)}_A = -r^2 \) into (122) leads to \( z^n \) directly as it should be since (122) is nothing but a rewriting of \( z^n \) in a convenient way of extracting contributions from different sectors and it is a direct generalization of the trick introduced in [38]. In particular the highest level sector of \( z^n \) can be expressed as

which is expected to be one of the dominant non-self-averaged sector in the large \( N \) limit.
2.4. 0+1 SYK model

Now let us apply our proposal to the 1-SYK model. The partition function is defined as

\[ z(\beta) = \int \mathcal{D}\psi \exp\left\{ - \int_0^\beta d\tau (\psi_i \partial_\tau \psi_i + i\beta^2 J_A \psi_i) \right\}, \]

(124)

with \( J_A \)'s satisfy (51). We will assume that (122) is approximately valid at least semi-classically. In other words, the saddle point can be derived from (122). The possible problem of (122) in the one-dimensional SYK model is that the fermions are not Grassmann numbers but Majorana fermions. As a result, \( \psi_A \) does not commute with \( \psi_B \) if there are odd number of common indexes in the collective indexes \( A \) and \( B \). Therefore (89) is not exact anymore. The reason why we expect such subtlety is negligibly in the large \( N \) limit is that when we introduce standard \( G_{\Sigma \sum} \) variables in the SYK model we already ignore this fact and it is shown in [68] this approximation is correct in the large \( N \) limit.

2.4.1. Half-wormhole in \( z \) and complex coupling

First let us consider \( z(\beta + iT) \)

\[ z(\beta + iT) = \int \mathcal{D}\psi e^{- \int_0^{\beta+iT} d\tau (\psi_i \partial_\tau \psi_i + i\sum_{\alpha \neq i} J_{\alpha} \psi_i \psi_\alpha)}, \]

(125)

where we have defined the operator

\[ C_A(\beta + iT) \equiv i\beta^2 \int_0^{\beta+iT} d\tau_A \psi_i. \]

(126)

The reason we consider \( z(\beta + iT) \) is that its square \( (z(\beta + iT)z(\beta - iT)) \) is the spectral form factor (SFF) which has universal behaviors for chaotic systems like SYK model and random matrix theories. When \( T \) is small, SFF is self-averaged so it is dominated by disconnected piece \( \langle z(\beta + iT)z(\beta - iT) \rangle \approx \langle z(\beta + iT)z(\beta - iT) \rangle_0 \). Because the one point function decays with respect to time and so is SFF. This decay region of SFF is called the slope. Because of the chaotic behavior SFF should not vanish in the late time. It will be the non-self-averaged sector dominates which are responsible for the ramp of the SFF. Therefore, in the ramp region we expect the approximation

\[ z \approx \Theta(\beta + iT) \equiv \int \mathcal{D}\psi e^{- \int_0^{\beta+iT} d\tau (\psi_i \partial_\tau \psi_i + i\sum_{\alpha \neq i} J_{\alpha} \psi_i \psi_\alpha)} \]

(127)

\[ = \int \mathcal{D}\psi e^{- \int_0^{\beta+iT} d\tau (\psi_i \partial_\tau \psi_i + i\sum_{\alpha \neq i} J_{\alpha} \psi_i \psi_\alpha)} \approx \left( e^{\sum_{\alpha \neq i} J_{\alpha} \psi_i \psi_\alpha} \right)^{\beta+iT} \]

(128)

which is the analog of the highest level sector (123) in the 0d SYK model. It can also be written as \( \langle e^{\sum_{\alpha \neq i} J_{\alpha} \psi_i \psi_\alpha} \rangle_{\text{SYK}} \), where SYK can be thought of as the anti-SYK model which is an SYK model but with an opposite bi-linear coupling or it can be thought of as an SYK model with purely imaginary random coupling \( iJ_\alpha \). The relation between factorization and complex couplings in the SYK model was also proposed in [62]. To confirm this approximation, let us compute

\[ \langle \Theta(\beta + iT)\Theta(\beta - iT) \rangle = \int \mathcal{D}\psi_l \mathcal{D}\psi_r \approx \int_0^{\beta+iT} d\tau (\psi_l \partial_\tau \psi_l + \psi_r \partial_\tau \psi_r) \]

(129)

\[ = \int \mathcal{D}\psi_l \mathcal{D}\psi_r \approx \int_0^{\beta+iT} d\tau (\psi_l \partial_\tau \psi_l + \psi_r \partial_\tau \psi_r) \]

(130)

which describes the wormhole saddle considering that we can introduce the \( G_{LR} \) as

\[ t^2 \sum_A c_A^l c_A^r \approx \frac{Q^2}{q} \int d\tau_l \int d\tau_r \left( \sum_i \psi_i^l \psi_i^r \right)^q \]

(131)

so the saddle point solution of (130) is the same saddle point solution of \( \langle z^2 \rangle \) with \( G_{LL} = G_{RR} = 0 \). Such solutions are found in [68]. To be more precise, these solutions found in [68] are time-dependent and only in the ramp region we have \( G_{LL}, G_{RR} \rightarrow 0 \). This is why we stress that only in the ramp region our approximation is good. Away from this region, we have to include other
sectors which can be obtained by the expansion (125) as
\[
\Theta_k = \int \mathcal{D}\psi e^{-\int d^4x \, L(\psi, \bar{\psi})} e^{\sum_\alpha g_{\alpha}^2 \alpha \sum_\beta \bar{\alpha} \beta \alpha} \left( \frac{\left( \sum_\alpha \Theta_\alpha \Theta_\alpha \right)^k}{k!} \right),
\]
(132)

\[
\approx \frac{1}{k!} \left\langle e^{\sum_\alpha g_{\alpha}^2 \alpha \sum_\beta \bar{\alpha} \beta \alpha} \right\rangle_{\text{SYK}}.
\]
(133)

2.4.2. Half-wormhole in \( z^2 \) and factorization. Let us consider \( z(iT)z(-iT) \) and apply our decomposition proposal (122)
\[
z(iT)z(-iT) = \int \mathcal{D}\psi^{L(R)} e^{\sum_\alpha g_{\alpha}^2 \alpha \sum_\beta \bar{\alpha} \beta \alpha} \left( e^{\frac{i}{2} \sum_\alpha \bar{\alpha} \alpha \beta} + e^{\frac{i}{2} \sum_\alpha \alpha \beta \bar{\alpha}} \right).
\]
(134)

Motivated by the result of 0-SYK model, we expect that there is also a ramp region where the dominant non-self-averaged sector is given by the 2-linked half-wormhole\(^5\)
\[
\Phi = \int \mathcal{D}\psi^{L(R)} e^{\sum_\alpha g_{\alpha}^2 \alpha \sum_\beta \bar{\alpha} \beta \alpha} \left( e^{\frac{i}{2} \sum_\alpha \bar{\alpha} \alpha \beta} + e^{\frac{i}{2} \sum_\alpha \alpha \beta \bar{\alpha}} \right)
\]
(135)
\[
= \int \mathcal{D}\psi^{L(R)} e^{\sum_\alpha g_{\alpha}^2 \alpha \sum_\beta \bar{\alpha} \beta \alpha} e^{-\frac{i}{2} \sum_\alpha \bar{\alpha} \alpha \beta}
\]
(136)
\[
= \int \mathcal{D}\psi^{L(R)} e^{\sum_\alpha g_{\alpha}^2 \alpha \sum_\beta \bar{\alpha} \beta \alpha} e^{-\frac{i}{2} \sum_\alpha \bar{\alpha} \alpha \beta} \sum_\alpha \beta \alpha \bar{\alpha} \beta \alpha
\]

Our proposal (137) of the 2-linked half-wormhole is very close to the one proposed in [62] which has two more bi-linear terms \( e^{\frac{i}{2} \sum_\alpha \bar{\alpha} \alpha \beta} + e^{\frac{i}{2} \sum_\alpha \alpha \beta \bar{\alpha}} \) in the second exponent. It seems that our proposal is more proper considering that \( \langle \Phi^2 \rangle \) there are only bi-linear correlations between \( L(R) \) and \( L'(R') \)
\[
\langle \Phi^2 \rangle = \int \mathcal{D}\psi^{L} \mathcal{D}\psi^{R} e^{-\int d^4x \, L(\psi, \bar{\psi})} \int \mathcal{D}\psi^{L} \mathcal{D}\psi^{R} e^{-\int d^4x \, L'(\psi, \bar{\psi})} \sum_\alpha \beta \alpha \bar{\alpha} \beta \alpha
\]
(138)

as shown in figure 4. Thus it implies the approximate factorization
\[
\text{Error} = z^2 - \langle z^2 \rangle - \Phi,
\]
(139)
\[
\langle \text{Error}^2 \rangle \approx \langle z^2 \rangle - \langle z^2 \rangle^2 - 2 \langle \Phi z^2 \rangle + \langle \Phi^2 \rangle \approx (3 - 4 + 2) \langle z^2 \rangle^2 \approx 0,
\]
(140)

where we have assumed in the regime where the wormhole dominates the partition function \( z \) approximates a Gaussian random variable. The bulk point of view of the factorization is also interesting. The insertion of \( e^{\sum_\alpha g_{\alpha}^2 \alpha \sum_\beta \bar{\alpha} \beta \alpha} \) can be thought of as inserting spacetime branes in the gravity path integral and the opposite bi-linear coupling means the wormhole amplitudes connecting the branes are opposite to the usual spacetime wormhole amplitudes such that including all the effects of wormholes and branes factorization is achieved. In [65], it is proposed that JT gravity can be factorized by inserting such spacetime branes.

2.5. Random matrix theory

In this section, let us apply our proposal to the Random Matrix Theory: the GUE ensemble which can also be thought of the CGS model with end-of-world (EOW) branes. The random matrix element \( H_q \) is identified with an EOW brane \( \langle \psi, \bar{\psi} \rangle \) in the notation of [37] or the topological complex matter field \( Z_{\psi, \bar{\psi}} \) in the notation of [44] with the restriction that the disk amplitude of \( \langle \psi, \bar{\psi} \rangle \) vanishes i.e., \( \langle \langle \psi, \bar{\psi} \rangle \rangle = 0 \). The equivalence between these two models can be understood as the following. The correlation functions of \( H_q \) are computed by the Wick contractions which exactly describe how to connect different EOW branes \( \langle \hat{\psi}, \hat{\bar{\psi}} \rangle \) with spacetime wormholes in the Disk-Cylinder approximation. Therefore the correlation functions of the random matrix theory are equal to the gravity path integral as we have seen in the CGS model. In this theory, we are interested in the observable
\[
z(\beta) = \text{Tr}(e^{-\beta H}),
\]
(141)

\(^5\) Note that we have normalized the fermionic integral such that \( \int d\psi = 0 \) thus \( \langle \Phi \rangle = 0 \).
whose ensemble average is given by
\[ \langle z \rangle = \int dH e^{-\frac{1}{2}TrH^2} z, \]  
where \( r^2 \) is usually taken to be \( 1/N. \)

2.5.1. Half-wormhole. First let us consider the non-self-averaged sector in \( z \). It is useful to study a simpler observable \( TrH^n \) to get some intuitions about the non-self-averaged sector of matrix functions. For the random variable \( H \), we can not use the decomposition (30) directly. One possible way of adapting to (30) is to rewrite \( H \) as a linear combination of the Gaussian random variables. However this rewriting is not very convenient. Alternatively, we can transfer the matrix integral into the integral over eigenvalues
\[ Z = \int dH e^{-\frac{1}{2}TrH^2} \]  
where \( \Delta(\lambda) \) is the Vandermonde determinant
\[ \Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j). \]  
Then the simple single-trace observable translates to
\[ TrH^n = \sum_i \lambda_i^n. \]  
However those eigenvalues are not Gaussian random variables. As a result, even though we can still do the sector decomposition but the resulting different sectors are not orthogonal anymore. Although when the level is finite, we can obtain a new orthogonal basis by a direct diagonalization but it is still very cumbersome. We will make some preliminary analysis beyond Gaussian distribution in next section. Here we will take a similar approach as before. Considering the non-vanishing correlator \( \langle H_{ij}H_{ji} \rangle = r^2 \), we should define
\[ \Theta^{(1)}_{ij} = H_{ij}, \quad \Theta^{(2)}_{ij} = \Theta^{(1)}_{ij} - r^2, \]  
thus we have
\[ H_{ij}H_{km} = \langle H_{ij}H_{km} \rangle + \delta_{ik}\delta_{jm}(\Theta^{(1)}_{ij} - \Theta^{(1)}_{ji}) \]  
\[ \equiv \langle H_{ij}H_{km} \rangle + [H_{ij}H_{km}] \]  
and
\[ H_{ij}H_{jm} = \sum_{i=0}^{3} \Theta_i, \quad \Theta_0 = \langle H_{ij}H_{jm} \rangle, \quad \Theta_2 = 0, \]  
\[ \Theta_1 = \theta^{(1)}_{ij} \langle H_{ij}H_{jm} \rangle + \theta^{(1)}_{ij} \langle H_{ij}H_{jm} \rangle + \theta^{(1)}_{ij} \langle H_{ij}H_{jm} \rangle. \]  
\[ (146) \]
\[ (147) \]
\[ (148) \]
\[ (149) \]
\[ (150) \]
\[ (151) \]
\[ (152) \]
\[ (153) \]
\[ (154) \]
\[ (155) \]
\[ (156) \]
\[ (157) \]
\[ (158) \]
\[ (159) \]
\[ (160) \]
\[ (161) \]
\[ (162) \]
correlation functions factorize so the dominant sector is always the self-averaged sector. The more interesting observable is \( z(iT) \) whose expectation value is

\[
\lim_{N \to \infty} \langle z(iT) \rangle = \sum_{k=0}^{l} \frac{(iT)^{2k}}{(2k)!} \langle \text{Tr} H^{2k} \rangle = N \sum_{k=0}^{l} \frac{(iT \sqrt{N})^{2k}}{(2k)!} C_k
\]

(163)

\[
=N \frac{J_l(2\alpha T)}{\alpha T} \sim 0, \quad \text{when } T \gg 1,
\]

(164)

where \( C_k \) is a famous Catalan number and \( \alpha = \sqrt{N}t \). So in the late time, the non-self-averaged sector becomes important. The lowest sector can be simply obtained by expanding \( z \) and picking the term with \( \theta^{(1)} \):

\[
\Theta_1 = \text{Tr} \theta^{(1)} \frac{1}{N} \left( N i T + \frac{(iT)^3}{3!} \times \langle \text{Tr} H^2 \rangle + \frac{(iT)^5}{5!} \times 5 \times \langle \text{Tr} H^4 \rangle + \ldots \right)
\]

(165)

\[
= i \text{Tr} \theta^{(1)} \frac{J_l(2\alpha T)}{\alpha T},
\]

(166)

Similarly we find that the next sector is \( \theta^{(2)} \):

\[
\Theta_2 = \text{Tr} \theta^{(2)} \frac{1}{\alpha^2} \left( \frac{(iT\alpha)^2}{2} + \frac{(iT\alpha)^4}{4!} \times 4 + \frac{(iT\alpha)^6}{6!} \times (6 \times 2 + 3) + \ldots \right),
\]

(167)

\[
= -i \text{Tr} \theta^{(2)} \frac{J_l((2T\alpha))}{\alpha^2},
\]

(168)

where we have dropped the terms \( \text{Tr} \theta^{(1)} \text{Tr} \theta^{(1)} \) because they are suppressed by \( 1/N \). Comparing with the known results \( \Theta = \sum_{i=1}^{10} \Theta_i \) of the wormhole contribution to \( \langle z(iT)z(iT) \rangle \),

\[
\langle z(iT)z(iT) \rangle = \sum_{l=0}^{\infty} (l + 1)(-1)^{l+1} J_{l+1}(2\alpha T)J_{l+1}(2\alpha T),
\]

(169)

we will show in the Appendix that

\[
\Theta_k = (i)^k \frac{J_l(2\alpha T)}{\alpha^k} \text{Tr} \theta^{(k)}, \quad k > 0.
\]

(170)

We plot \( \Theta_3^2 + \Theta_4^2 \) in figure 6(a) and \( \Theta_5^2 \) in figure 6(b). The result is very interesting. We see that every curve has the typical slop, ramp and plateau regimes. Another interesting fact is that only the first few sectors contribute to the slop and ramp regions. For example, adding the first 20 sectors we find that the ramp region is roughly located at \([2.5/\alpha, 4/\alpha]\) and we plot the contribution of each sector in figure 5. Actually

6 There is a \( 1/N \) in front because one of the summation of indexes gives the trace of \( \theta^{(1)} \) instead of a factor of \( N \). For example \( \sum_{n,m,a,b} \theta^{(1)}_{nmab} \langle H_n H_m H_a H_b \rangle = \text{Tr} \theta^{(1)} \).

7 The factor \( 6 \times 2 \) comes from the adjacent terms like \( \theta^{(2)}_{nmab} \langle H_n H_m H_a H_b \rangle \) and factor \( 3 \) comes from the pairs like \( \theta^{(1)}_{nmab} \langle H_n H_m H_a H_b \rangle \).

8 For example see [65].

3. Beyond Gaussian distribution or the generalized CGS model

One of simplest way to go beyond CGS model is again starting from the MM model but including connected space-times with other topologies in the Euclidean path integral. So the next simplest case beyond CGS model is the Disk-Cylinder-Pants model. Let the amplitudes of the disk, cylinder and pants to be

\[
\langle \hat{Z} \rangle = \kappa_1, \quad \langle \hat{Z}^2 \rangle = \kappa_2, \quad \langle \hat{Z} \rangle - 3 \langle \hat{Z} \rangle \langle \hat{Z}^2 \rangle + 2 \langle \hat{Z}^2 \rangle = \kappa_3.
\]

(172)
\[ \alpha = 1, \ T = 3. \]

**Figure 5.** Contributions of different sectors (170), \( \alpha = 1, \ T = 3. \) The horizontal axis represents different sectors, and the vertical axis is the proportion of each sector.

The generating function is
\[ \langle e^{iaZ} \rangle = \exp \left( \sum_{n} \frac{a^{2n+1}}{2n+1} \right), \quad (173) \]
so we can also identify \( Z \) as a random variable albeit with a very complicated PDF. We can simply think of the distribution defined by the same generating function. In [69] we introduce the connected correlators to decompose \( \langle Z^n e^{iaZ} \rangle \) for example
\[ \langle Z e^{iaZ} \rangle = \langle Z \rangle e^{iaZ} + \langle Z e^{iaZ} \rangle_c, \quad (174) \]
\[ \langle Z^2 e^{iaZ} \rangle = \langle Z^2 \rangle e^{iaZ} + 2 \langle Z \rangle \langle Z e^{iaZ} \rangle_c + \langle Z^2 e^{iaZ} \rangle_c, \quad (175) \]
\[ \langle Z^3 e^{iaZ} \rangle = \langle Z^3 \rangle e^{iaZ} + 3 \langle Z^2 \rangle \langle Z e^{iaZ} \rangle_c + 3 \langle Z \rangle \langle Z^2 e^{iaZ} \rangle_c + \langle Z^3 e^{iaZ} \rangle_c, \ldots \quad (176) \]

such that using the trick (9) we can decompose \( Z^n \) into different sectors which are exactly like (18)-(21). In other words, the number basis or \( \{ \theta \} \) is still the basis for decomposition. But \( \{ \theta \} \) should be determined from the recursion relations (18)-(21). For example, in the Disk-Cylinder-Pants model the first few \( \theta \) are
\[ \theta^{(1)} = Z - \kappa_1, \quad \theta^{(2)} = \theta^{(1)} - \kappa_2, \quad \theta^{(3)} = \theta^{(1)} - 3 \kappa_2 \theta^{(1)} - \kappa_3, \quad (177) \]

Because of the inclusion of new wormholes, the pants, the basis is not orthogonal anymore in the sense
\[ \langle \theta^{(i)} \theta^{(j)} \rangle = \delta_{ij}. \quad (178) \]
It is easy to find that
\[ \langle \theta^{(1)} \theta^{(2)} \rangle = \kappa_3, \quad \langle \theta^{(2)} \theta^{(3)} \rangle = 6 \kappa_2 \kappa_3. \quad (179) \]
Moreover the matrix \( M^{(3)}_{ij} = \langle \theta^{(i)} \theta^{(j)} \rangle, \ i, j = 1, 2, 3 \)

\[ M^{(3)} = \begin{pmatrix} \kappa_2 & \kappa_3 & 0 \\ \kappa_3 & 2 \kappa_2 & 6 \kappa_2 \kappa_3 \\ 0 & 6 \kappa_2 \kappa_3 & 6 \kappa_2^2 + 9 \kappa_3^2 \end{pmatrix}. \quad (180) \]

Naturally \( \theta \) can be understood as the \( i \)-linked half wormhole as shown in figures 7 and 8.

**3.1. Toy statistical model**

We start from the simplest operator
\[ Y = \sum_i (X_i - \langle X_i \rangle). \quad (181) \]
The modification starts to show up in
\[ Y^3 = \Delta_0 + \Delta_1 + \Delta_3, \quad (182) \]
where
\[ \Delta_0 = \langle Y^3 \rangle, \quad (183) \]
\[ \Delta_1 = 3 N \kappa_2 \sum \theta^{(1)}, \quad (184) \]
\[ \Delta_3 = \sum_i \theta^{(1)} + 3 \sum_{i \neq j} \theta^{(1)} \theta^{(1)} + \sum_{i \neq j \neq k} \theta^{(1)} \theta^{(1)} \theta^{(1)}. \quad (185) \]
In this special case since \( \langle \theta^{(1)} \theta^{(3)} \rangle = 0 \), there is no cross terms in \( \langle Y^6 \rangle \)
\[ \langle Y^6 \rangle = \sum_i \langle \Delta_i \rangle, \quad (186) \]
\[ \langle \Delta_0 \rangle \sim N^2 \kappa_2^3, \quad \langle \Delta_1 \rangle \sim N^3 \kappa_2^3, \quad \langle \Delta_3 \rangle \sim N^3 \kappa_2^3 + N^2 \kappa_3^2, \quad (187) \]
where we only keep the possible leading terms. When \( \kappa_2, \kappa_3 \sim O(1) \), the operator \( Y^3 \) is not self-averaged and the effect of \( \kappa_3 \) is negligible. The interesting case is when \( N \kappa_2^2 > N \kappa_3^2 \) so that we have the approximation
\[ Y^3 \approx \langle Y^3 \rangle + \Delta_1, \quad (188) \]
which is the analog of (36).

**3.2. 0-SYK model**

Let us reconsider the 0-SYK model but assume the random couplings satisfying
\[ \langle J_{i_1 \ldots i_q} \rangle = 0, \quad \langle J_{i_1 \ldots i_q} J_{j_1 \ldots j_q} \rangle = \kappa_2 \delta_{i_1 j_1} \ldots \delta_{i_q j_q}, \quad \kappa_2 = \tau^2 (q-1)! / N^{q-1}, \quad (189) \]
\[ \langle J_A J_B J_C \rangle = \kappa_3 \delta_{ABC}, \quad (190) \]
we will determine the scaling of \( \kappa_3 \) in a moment. Then the averaged quantity is
\[ \langle \epsilon^3 \rangle = \int d^3k \frac{1}{\epsilon^2} \sum_{s} \langle \psi_s \psi_s \rangle \langle \psi_s \psi_s \rangle \langle \psi_s \psi_s \rangle \langle \psi_s \psi_s \rangle \langle \psi_s \psi_s \rangle \epsilon^3 \kappa_3 \sum \langle \psi_s \psi_s \rangle \langle \psi_s \psi_s \rangle \quad (191) \]
which can be computed by introducing the collective variables

\[ G_{ab} = \frac{1}{N} \sum_i \psi_i^a \psi_i^b, \quad (a, b) = (1, 2), (1, 3), (2, 3), \]  

(192)

\[ G_3 = \frac{1}{N} \sum_{i<j} \psi_i^a \psi_j^b \psi_j^c \psi_i^d, \]  

(193)

\[ \sum_A \psi_A^a \psi_A^b = \frac{N^q}{q!} G_{ab}^q, \]  

(194)

to rewrite \((z^3)\) as

\[ (z^3) = \int \frac{dG_{ab} d\Sigma_3}{2\pi i/N} \int \frac{dG_3 d\Sigma_3}{2\pi i/N} e^{(q+2)G_{ab}^q + 2qG_3^q} e^{-N(\sum_a G_{ab} + \Sigma_3 G_3)} \int d^2N e^{\sum_a \psi_i^a \psi_i^b + \Sigma_3 \sum_{i<j} \psi_i^a \psi_j^b \psi_i^c \psi_j^d} \]  

(195)

\[ = \int \frac{dG_{ab} d\Sigma_3}{2\pi i/N} \int \frac{dG_3 d\Sigma_3}{2\pi i/N} N^{q/2} e^{-N\Sigma G_3 + N^3/2} = \gamma^3 m_p, \]  

(196)

where \(m_p\) is defined in (69) and

\[ \gamma^3 \equiv \frac{N^{q/2-1}}{(q/2 - 1)!}, \quad \gamma \sim \mathcal{O}(1), \]  

(197)

thus

\[ \kappa^3 \sim \frac{(q/2 - 1)!}{N^{q/2} - 1}. \]  

(198)

Recall that

\[ z^3 = \sum_{A,B,C} \text{sgn}(A) \text{sgn}(B) \text{sgn}(C) J_A J_B J_C \]  

(199)

In general decomposing \(z^3\) is still very complicated. Let us consider some simple examples. If \(p = 2\), then we have

\[ z^3 = \sum_A \text{sgn}(A) J_A^2 + 3 \sum_{A,B,A = B} \text{sgn}(A) \text{sgn}(B) J_A^2 J_B J_C + \sum_{A,B,C} \text{sgn}(A) \text{sgn}(B) \text{sgn}(C) J_A J_B J_C, \]  

(200)

and there are seven different sectors. A simple way to derive the explicit expression of each sector is to first decompose each \(J_A^2\) as (18)–(21):

\[ J_A = \theta_A^{(1)}, \quad J_A^2 = \theta_A^{(2)} + (\kappa_2)_A, \quad J_A^3 = \theta_A^{(3)} + (\kappa_3)_A + 3(\kappa_2)_A \theta_A^{(1)}, \]  

(201)

then collect the terms in the same sector:

\[ \Delta_0 = \kappa_2^3 \sum_A \text{sgn}(A) = m_p \kappa_2^3 = (z^3), \]  

(202)

\[ \Delta_1 = 3(\kappa_2)_A \sum_A \theta_A^{(2)}_A (\kappa_2)_A + (\kappa_2)_A \theta_A^{(1)}_A, \]  

(203)

\[ \Delta_2 = (6 + 3M_p) \kappa_2^2 \sum_A \text{sgn}(A) \theta_A^{(1)} \theta_A^{(1)}_A, \]  

(204)

\[ \Delta_3 = \sum_A \text{sgn}(A) (\kappa_3)_A (\kappa_3)_A + (\kappa_3)_A \theta_A^{(3)}_A, \]  

(205)

\[ \Delta_4 = 3(\kappa_2)_A \sum_A \text{sgn}(A) \left( \theta_A^{(1)} \theta_A^{(1)}_A + \theta_A^{(1)} \theta_A^{(1)}_A + \sum_{B,B=A} \theta_B^{(1)} \theta_A^{(1)}_A \right), \]  

(206)
where $M_p = \frac{(pq)!}{p!(q!)^p}$. Now we are ready to compute $\langle \Delta_0 \Delta_y \rangle$ using the relation (180). It turns out that different sectors are still orthogonal for this case:

$$\langle \Delta_0^2 \rangle = M_0^2 \kappa_2^2, \quad \langle \Delta_y^2 \rangle = \langle \Delta_0^2 \rangle + \langle \Delta_y^2 \rangle = 18M_p \kappa_3^2 \kappa_2^2,$$

$$\langle \Delta_y^2 \rangle = (6 + 3M_p)^2 M_p \kappa_2^2, \quad \langle \Delta_y^2 \rangle = 2M_p \kappa_3^2 (6 \kappa_2^2 + 9 \kappa_3^2),$$

$$\langle \Delta_y^2 \rangle = 18M_p \kappa_3^2 (6 \kappa_2^2 + 9 \kappa_3^2) + 18M_p (2M_p - 2) \kappa_2^2,$$

$$\langle \Delta_y^2 \rangle = M_p (6 \kappa_2^2 + 9 \kappa_3^2)^2 + 9M_p (M_p - 1) 4 \kappa_2^2 + 9(m_p^2 - M_p) \kappa_3^4 + 6M_p (M_p - 1) (M_p - 2) \kappa_2^2.$$

In large $N$ limit the relevant parameters have the following asymptotic behaviors

$$\kappa_2 \sim \frac{(N/2 - 1)!}{N^{N/2 - 1}}, \quad \kappa_3 \sim \frac{(N/4 - 1)!}{N^{N/4 - 1}}, \quad \kappa_3 \sim \kappa_2^3 m_p e^N, \quad M_p \sim m_p^2 \sqrt{N},$$

then the approximation can be given as

$$z^3 \sim \Delta_3 + \Delta_6.$$  

In general we find that when $p \ll N(\text{or} \ q \gg 1)$, $z$ is not self-averaged, i.e. the wormhole does not persists, but the (three-linked) half-wormhole emerges. This fact can be intuitively understood as the following. In this limit because of the scaling (213), the three-mouth-wormhole amplitude is favored thus the possible dominate sectors are $\Delta_0$, $\Delta_{3p-3}$ and $\Delta_{3p}$:

$$\langle \Delta_{3p-3}^2 \rangle = \langle \Delta_0^2 \rangle + \langle \Delta_{3p-3}^2 \rangle \sim M_p \kappa_3^2,$$

$$\langle \Delta_{3p}^2 \rangle \sim M_p \kappa_3^2,$$

and since $M_p \gg m_p^2$ we conclude that $z^3 \approx \Delta_{3p-3} + \Delta_{3p}$. This is similar to the result obtained in section 2.3. In the same limit, $z$ is not self-averaged neither while the half-wormhole emerges.

### 4. Discussion

In this paper, we have generalized the factorization proposal introduced in [57]. The main idea is to decompose the observables into the self-averaging sector and non-self-averaging sectors. We find that the contributions from different sectors have
interesting statistics in the semi-classical limit. When the self-averaging sector survives in this limit, the observable is self-averaging. An interesting phenomenon is the sector condensation, meaning the surviving non-self-averaging trend to condense, and in the extreme case, only one non-self-averaging sector is left-over, resembling the Bose–Einstein condensation. Then the half-wormhole saddle is naturally understood as the condensed sectors. We apply this proposal to a simple statistical model, a 0-SYK model and a random matrix model. Half-wormhole saddles are identified and they are in agreement with the known results. With our proposal, we also show the equivalence between the results in [57] and [58]. We also studied multi-linked-half-wormholes and their relations. There are some future directions.

4.1. Sector condensation

It is interesting to understand the sector condensation better. We expect that it is some criterion for an ensemble theory or a statistical observable to potentially have a bulk description, and so it deserves to be studied in other gravity/ensemble theories. Definitively, the extreme case mimicking the Bose–Einstein condensation is the most interesting one. We have not understood when it will happen and could it be used as some order parameter. We expect by studying the ‘phase diagram’ in the sector space we can obtain more information about the observables and systems.

4.2. Complex coupling and half-wormholes

In [62], it shows that factorization is related to the complex couplings. In our approach, the complex coupling emerges as an auxiliary parameter to obtain the half-wormhole saddle. The trick here is similar to the one used by Coleman, Giddings and Strominger [70–72], where the non-local effect of spacetime wormhole is ‘localized’ with the price of introducing random couplings. But the current analysis shows that this is only possible when ‘Bose–Einstein’ happens such that the dominant sector can be obtained from this trick. So it would be interesting to explore the relation between complex coupling and half-wormhole further using our approach.

4.3. Relations to other factorization proposal

Besides the half-wormhole proposal, there exists other proposals of factorization. For example, in [65] it shows two-dimensional gravity can be factorized by including other branes in the gravitational path integral. These new branes correspond to specific operators in the dual matrix model. From the point of view of our approach, inserting operators may be related to adding back the contributions from non-self-averaging sectors. In [73], it is argued that factorization can be restored by adding other kinds of asymptotic boundaries corresponding to the degenerate vacua. It is clear that from our approach, this is equivalent to introducing new random variables. It would be interesting to see how this changes the statistic of contribution form different sectors.

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First let us rederive the non-self-averaged sectors of $z$ in a more systematic way. For simplicity let us set $t^2 = 1/N$. Therefore similar to the computation of (27) we have

$$ [G(u)] = \int \mathcal{D}H e^{\frac{i}{2} H u H} e^{-\frac{1}{t^2} Tr(H^2)} e^{\frac{i}{2} Tr(u H)} e^{\frac{i}{2} Tr u H^2}. $$ \hspace{1cm} (225)

Defining

$$ J_{ba} = \left\{ \frac{\delta z}{\delta H_{ab}} \right\} = \int dH e^{\frac{N}{2} Tr i \lambda} \frac{\delta z}{\delta H_{ab}}, \quad (177) $$

$$ = N \int dH e^{\frac{N}{2} Tr^2} H_{ba} Tr(e^{TH}) = N \langle H_{ba} z \rangle \quad (182) $$

then we can rewrite $\Theta_1$ as

$$ \Theta_1 = \sum_{ij} \theta_{ij}^1 J_{ji} \equiv \langle Tr(\theta^1) \partial z \rangle. $$ \hspace{1cm} (191)

By considering that in (182) $H_{ba}$ has to contract with other $H_{ab}$ or by a argument of symmetry it is obvious

$$ J_{ba} = \delta_{ab} J_{ba}, $$ \hspace{1cm} (220)

thus

$$ J_{ba} = \delta_{ab} N \langle H_{ba} z \rangle = \delta_{ab} \langle Tr H z \rangle, \quad \Theta_1 = \sum_{ij} \theta_{ij}^1 J_{ji} = Tr \theta^1 \langle Tr H z \rangle, $$ \hspace{1cm} (221)

where we have used the fact there is a permutation symmetry in the diagonal elements $\{ H_{ii} \}$. Similarly the second non-self-averaged sector $\Theta_2$ can be written as

$$ \Theta_2 = \frac{1}{2} Tr \theta^2 \langle (Tr H^2 - N) z \rangle. $$ \hspace{1cm} (222)

In general, it is

$$ \Theta_k = \frac{1}{k} Tr \theta^k \langle [Tr H^k] z \rangle, $$ \hspace{1cm} (223)

which simply means that $\{ [Tr H^k] \}$ is an orthogonal basis in the sense

$$ \langle [Tr H^k] [Tr H^l] \rangle = k \delta_{kl}. $$ \hspace{1cm} (224)

Recall (160) the generating function of the normal-ordered operator is

$$ G(u) = \langle Tr(e^{TH}) z \rangle = \langle z(u) \langle z(iT) \rangle + \sum_{l=0}^{\infty} (l + 1)(-1)^{l+1} J_{l+1}(-2iu) J_{l+1}(2T), $$ \hspace{1cm} (229)

Expanding the generating function gives

$$ \langle Tr(H) z \rangle = i J_1(2T), \quad \langle Tr(H^2) z \rangle = N \langle z \rangle - 2 J_2(2T), $$ \hspace{1cm} (231)

$$ \langle Tr(H^3) z \rangle = -3i J_3(2T) + 3i J_1(2T), $$ \hspace{1cm} (232)

$$ \langle Tr(H^4) z \rangle = 2N \langle z \rangle - 8 J_2(2T) + 4 J_4(2T), $$ \hspace{1cm} (233)

which indeed lead to (170).

It would be desired to derive a generating function of the normal ordered operators [Tr $H^k$] which has the integral form

$$ G(u) = \int \mathcal{D}H e^{\frac{i}{2} H u H} e^{\frac{1}{t^2} Tr(HiH)} e^{\frac{i}{2} Tr(u H)} e^{\frac{i}{2} Tr u H^2}. $$ \hspace{1cm} (234)

Note that (234) describes a GUE model coupled with an external source. As shown in [65] it can be rewritten as

$$ G(u) = \prod_{i} d\lambda_i e^{\frac{1}{2} \sum \lambda_i - \frac{1}{4} \sum_{ij} \lambda_i \lambda_j + \frac{1}{2} \sum_{i} \frac{\Delta(\lambda_i)}{\Delta(\lambda)} \sum k \delta_{kl} e^{u \lambda_i} $$ \hspace{1cm} (235)

$$ = \sum_{j} e^{-u^2} \sum \lambda_j - \frac{1}{2} \sum_{i} (\lambda_i-n_i) \frac{N}{\lambda_i} \left( 1 + \frac{u^2 t^2}{\lambda_i - \lambda_j} \right) $$ \hspace{1cm} (236)

$$ = e^{-u^2} \frac{1}{2} \int_{\frac{1}{2} \pi}^{\frac{1}{2} \pi} \sin^{N-1} \left( 1 + \frac{u^2 t^2}{\lambda_i - \lambda_j} \right) e^{u \lambda_i}. $$ \hspace{1cm} (237)
Notice that in the large $N$ limit $[\text{Tr} H^N]$ is a linear combination of single trace operator so we should expand each $1/(w - \lambda)$ into Taylor series and only keep terms with $\sum_{k} \lambda_k^k$:

$$ [G(u)] = e^{-\frac{u^2}{2N}} \left[ \frac{N}{u} \right] \frac{dw}{2\pi i} \left( 1 + \frac{-u/N}{w} \right)^N\sum_{k=0}^{N-1} \lambda_k^k e^{iuw}, $$

where we have substituted $r^2 = 1/N$. Sending $N$ to infinity gives

$$ e^{-\frac{u^2}{2N}} \sim 1, \quad \left( 1 + \frac{-u/N}{w} \right)^N \sim \left( 1 + \frac{u}{Nw} \right)^{N-1} \sim e^{-u/w}, $$

thus we arrive at the final result

$$ [G(u)] = \left( \frac{-N}{u} \right) \frac{dw}{2\pi i} e^{u(w-\frac{1}{4})} + \sum_{k=1}^{\infty} \frac{\text{Tr} H^k}{k} \frac{dw}{2\pi i} e^{u(w-\frac{1}{4})} \frac{1}{w^{k+1}}. $$

These contour integral can be evaluated exactly by using the expansion

$$ e^{u(w-\frac{1}{4})} = \sum_{i=-\infty}^{\infty} w^i J_i(2u), $$

which leads to

$$ [G(u)] = \frac{N}{u} J_1(2u) + \sum_{k} \text{Tr} H^k J_k(2u). $$

By expanding with respect to $u$, indeed we get the correct normal-ordered operators

$$ [G(u)] = N + u \text{Tr} H + \frac{u^2}{2!} (\text{Tr} H^2 - N) + \frac{u^3}{3!} (\text{Tr} H^3 - 3\text{Tr} H) + \frac{u^4}{4!} (\text{Tr} H^4 - 4\text{Tr} H^2 + 2N) + \frac{u^5}{5!} (\text{Tr} H^5 - 5\text{Tr} H^3 + 10\text{Tr} H) + \frac{u^6}{6!} (\text{Tr} H^6 - 6\text{Tr} H^4 + 15\text{Tr} H^2 - 5N) + \ldots $$

We can also obtain a generating function of $\Theta_k$

$$ \langle [G(u)] z \rangle = \int dH \int dH' \text{e}^{-N\text{Tr} H^2} \text{e}^{N\text{Tr}(H||H')} \text{Tr} e^{iH} \text{Tr} e^{iH'} $$

which, unfortunately, does not have a simple closed form but the ensemble average $\langle \text{Tr} H^k \text{Tr} e^{iH} \rangle$ can be computed with the generating function (230).

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