On irreducible components of Rapoport-Zink spaces
Yoichi Mieda

Abstract. Under a mild condition, we prove that the action of the group of self-quasi-isogenies on the set of irreducible components of a Rapoport-Zink space has finite orbits. Our method allows both ramified and non-basic cases. As a consequence, we obtain a finiteness result on the representation obtained from the ℓ-adic cohomology of a Rapoport-Zink tower.

1 Introduction

A Rapoport-Zink space, introduced in [RZ96], is a moduli space of deformations by quasi-isogenies of a fixed $p$-divisible group $X$ over $\mathbb{F}_p$ with various additional structures. It is a formal scheme $\mathcal{M}$ locally formally of finite type over the ring of integers $\mathcal{O}_{\bar{E}}$ of $\bar{E}$, which is the maximal unramified extension of a finite extension $E$ (called the local reflex field) of $\mathbb{Q}_p$. Rapoport-Zink spaces are local analogues of Shimura varieties of PEL type (cf. [Kot92]), and are also related to Shimura varieties themselves by the theory of $p$-adic uniformization (cf. [RZ96, Chapter 6]).

One of the main reasons we are interested in Rapoport-Zink spaces is the conjectural relation between the $\ell$-adic cohomology of the Rapoport-Zink tower and the local Langlands correspondence. Let us recall it briefly. We write $M$ for the rigid generic fiber of $\mathcal{M}$. By using level structures on the universal $p$-divisible group over $M$, we can construct a projective system of étale coverings $\{M_{K'}\}$ over $M$ called the Rapoport-Zink tower. Here $K'$ runs through compact open subgroups of $G' = G'(\mathbb{Q}_p)$, where $G'$ is a certain inner form of the reductive algebraic group $G$ over $\mathbb{Q}_p$ that is naturally attached to the linear-algebraic data appearing in the definition of $\mathcal{M}$. As in the case of Shimura varieties, $G'$ acts on the tower by Hecke correspondences. On the other hand, the group $J$ of self-quasi-isogenies of $X$ preserving additional structures naturally acts on $\mathcal{M}$, and this action lifts canonically to $M_{K'}$, for each $K' \subset G'$. Let $H^i_c(M_\infty)$ be the compactly supported $\ell$-adic étale cohomology of the tower $\{M_{K'}\}$. Then, it is naturally equipped with an action of $G' \times J \times W_E$, where $W_E$ is the Weil group of $E$. Roughly speaking, it is expected...
that the representation $H^i_c(M_\infty)$ of $G' \times J \times W_E$ is described by means of the local Langlands correspondence for $G'$ and $J$. There are many results in the classical setting, namely when $\mathcal{M}$ is either the Lubin-Tate space or the Drinfeld upper half space (cf. [Car90], [Har97], [HT01], [Boy09], [Dat07]). However, very little is known in other cases.

In this paper, we will study the underlying reduced scheme $\overline{\mathcal{M}} = \mathcal{M}^{\text{red}}$. There already have been a lot of results on $\overline{\mathcal{M}}$ in various particular cases; see [Vie08a], [Vie08b], [VW11], [RTW13] and [HP13] for example. In contrast to them, here we try to keep $\mathcal{M}$ as general as possible. The main theorem of this paper is as follows.

**Theorem 1.1 (Theorem 2.6)** Assume that the isogeny class of the $p$-divisible group $X$ with additional structures comes from an abelian variety (for the precise definition, see Definition 2.4). Then, the action of $J$ on the set of irreducible components of $\overline{\mathcal{M}}$ has finite orbits.

As a consequence of this theorem, we can obtain the following finiteness result on the cohomology $H^i_c(M_\infty)$.

**Corollary 1.2 (Theorem 4.4)** Assume the same condition as in Theorem 1.1. Then, for every compact open subgroup $K'$ of $G'$, the $K'$-invariant part $H^i_c(M_\infty)^{K'}$ of $H^i_c(M_\infty)$ is a finitely generated $J$-representation.

This finiteness result is very important when we apply the representation theory (such as results in [Ber84]) to $H^i_c(M_\infty)$; see [IM10], [Mie11] and [Mie12]. The author also expects that Theorem 1.1 has many other applications than Corollary 1.2. For example, it plays an important role in the study of the relation between the Zelevinsky involution and the cohomology $H^i_c(M_\infty)$ (cf. [Mie14]).

Note that the same results as Theorem 1.1 and Corollary 1.2 have been obtained by Fargues ([Far04, Théorème 2.4.13, Proposition 4.4.13]) under the condition that the data defining $\mathcal{M}$ is unramified. However, there are many interesting ramified settings, and usually the geometry of $\mathcal{M}$ attached to a ramified data is much more complicated than that in the unramified case. For example, in the unramified case $\mathcal{M}$ is formally smooth over $\mathcal{O}_E$, while in the ramified case $\mathcal{M}$ is sometimes not even flat over $\mathcal{O}_E$.

We would like to say a few words on the condition on $X$ in Theorem 1.1. First, the problem to find an abelian variety with additional structures which has the prescribed $p$-divisible group up to isogeny can be seen as a generalization of Manin’s problem, and is studied by many people (see Remark 2.5). By the results of them, it is natural for the author to expect that this condition is always the case. Second, this condition is obviously satisfied if $\mathcal{M}$ is related to a Shimura variety of PEL type by the theory of $p$-adic uniformization. At present, the most successful way to study $H^i_c(M_\infty)$ is relating it with the cohomology of a suitable Shimura variety and applying a global automorphic method. Rapoport-Zink spaces to which this technique is applicable automatically satisfy our condition. The author hopes that these two points guarantee the condition in Theorem 1.1 to be harmless in practice.
To explain the strategy of our proof of Theorem 1.1 first let us assume that the isocrystal with additional structures $b$ associated with $X$ is basic (cf. [Kot85, 5.1]). In this case, by the $p$-adic uniformization theorem of Rapoport-Zink [RZ96, Theorem 6.30], $\mathcal{M}$ uniformizes some moduli space of PEL type $X$ along an open and closed subscheme $Z$ of the basic stratum $\overline{X}^{(b)}$ of the special fiber $\overline{X} = X \otimes \overline{\mathbb{F}}_p$. More precisely, there exist finitely many torsion-free discrete cocompact subgroups $\Gamma_1, \ldots, \Gamma_m$ of $J$ such that the formal completion $X/Z$ of $X$ along $Z$ is isomorphic to $\bigoplus_{i=1}^m \Gamma_i \backslash \mathcal{M}$. Since $\overline{X}$ is a scheme of finite type over $\mathbb{F}_p$, $Z$ has only finitely many irreducible components. Hence the number of irreducible components of $\Gamma_1 \backslash \mathcal{M}$ is also finite, and thus in particular the number of $J$-orbits in the set of irreducible components of $\overline{\mathcal{M}}$ is finite. However, if $b$ is not basic, the $p$-adic uniformization theorem [RZ96, Theorem 6.23] becomes more complicated; it involves the formal completion of $X$ along a possibly infinite set of closed subschemes of $\overline{X}$. By this reason, we cannot extend the method above to the non-basic case. To overcome this problem, we will invoke results by Oort [Oor04] and Mantovan [Man04], [Man05]. Roughly speaking, they say that the Newton polygon stratum $\overline{X}^{(b)}$ is almost equal to the product of $\mathcal{M}$ with the Igusa variety. We will generalize a part of them to the ramified case. Combining this generalization with the fact that $\overline{X}^{(b)}$ is of finite type, we can conclude Theorem 1.1. The author thinks that this method itself has some importance, because it is a first step to a generalization of Mantovan’s formula (the main theorem of [Man05]) to the ramified case. Recall that Mantovan’s formula is a natural extension of the method in [HT01], and is one of the most powerful tools to investigate the $\ell$-adic cohomology of Shimura varieties. It is also useful to study the cohomology $H^r_c(M_{\infty})$ of Rapoport-Zink towers (cf. [Shi12]). We plan to pursue this problem in our future work.

The outline of this paper is as follows. In Section 2, we recall the definition of Rapoport-Zink spaces and explain the precise statement of the main theorem. Section 3 is devoted to the proof of it. In Section 4, we will give some applications of the main theorem, including Corollary 1.2.

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Notation For a field $k$, we denote its algebraic closure by $\overline{k}$. For a scheme $X$ and a point $x$ in $X$, we write $\kappa(x)$ for the residue field at $x$.

2 Rapoport-Zink spaces

Here we briefly recall the definition of Rapoport-Zink spaces. The main reference is [RZ96, Chapter 3]. Fix a prime number $p > 2$. A Rapoport-Zink space of PEL type is associated with a tuple $(B, \mathcal{O}_B, *, V, \langle , \rangle, \mathcal{L}, b, \mu)$ consisting of the following objects (cf. [RZ96, Definition 3.18]):

- $B$ is a finite-dimensional semisimple algebra over $\mathbb{Q}_p$.
- $\mathcal{O}_B$ is a maximal order of $B$. 

- $\ast$ is an involution on $B$ under which $\mathcal{O}_B$ is stable.
- $V$ is a finite faithful $B$-module.
- $\langle \ , \ \rangle : V \times V \to \mathbb{Q}_p$ is a non-degenerate alternating bilinear pairing satisfying $\langle av, w \rangle = \langle v, a^*w \rangle$ for every $a \in B$ and $v, w \in V$.
- $\mathcal{L}$ is a self-dual multi-chain of $\mathcal{O}_B$-lattices in $V$ (cf. [RZ96, Definition 3.1, Definition 3.13]).

To explain the remaining objects, we denote by $G$ the algebraic group over $\mathbb{Q}_p$ consisting of $B$-linear automorphisms of $V$ which preserve $\langle \ , \ \rangle$ up to a scalar multiple (cf. [RZ96, 1.38]).

- $b$ is an element of $G(K_0)$, where $K_0$ denotes the fraction field of $W(\overline{\mathbb{F}}_p)$.
- $\mu$ is a cocharacter $\mathbb{G}_m \to G$ defined over a finite extension $K$ of $K_0$.

We impose the following conditions:

(a) The isocrystal $(N_b, \Phi_b) = (V \otimes_{\mathbb{Q}_p} K_0, b\sigma)$ has slopes in the interval $[0, 1]$, where $\sigma$ is the Frobenius isomorphism on $K_0$.
(b) We have $\text{sim}(b) = p$, where $\text{sim} : G \to \mathbb{G}_m$ denotes the similitude character.
(c) The weight decomposition of $V \otimes_{\mathbb{Q}_p} K$ with respect to $\mu$ has only the weight 0 and 1 parts: $V \otimes_{\mathbb{Q}_p} K = V_0 \oplus V_1$.

Before proceeding, it is convenient to introduce notion of polarized $B$-isocrystals.

**Definition 2.1** A polarized $B$-isocrystal is an isocrystal $(N, \Phi)$ endowed with a $B$-action and a non-degenerate alternating bilinear pairing $\langle \ , \ \rangle : N \times N \to K_0$ satisfying $\langle ax, y \rangle = \langle x, a^*y \rangle$ and $\langle \Phi x, \Phi y \rangle = p\sigma(\langle x, y \rangle)$ for each $a \in B$ and $x, y \in N$. An isomorphism between two polarized $B$-isocrystals $(N, \Phi, \langle \ , \ \rangle)$ and $(N', \Phi', \langle \ , \ \rangle')$ is a $K_0$-linear isomorphism $f : N \to N'$ which is compatible with $B$-actions and maps $\langle \ , \ \rangle$ to a $\mathbb{Q}_p^\times$-multiple of $\langle \ , \ \rangle'$.

Consider a triple $(X, \lambda, \iota)$ consisting of a $p$-divisible group $X$ over $\overline{\mathbb{F}}_p$, a quasi-polarization $\lambda : X \to X^\vee$ (namely, a quasi-isogeny satisfying $\lambda^\vee = -\lambda$) and a homomorphism $\iota : B \to \text{End}(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ satisfying $\lambda \circ \iota(a^*) = \iota(a)^\vee \circ \lambda$ for every $a \in B$. Then the rational Dieudonné module $\mathbb{D}(X)_\mathbb{Q}$ is naturally endowed with a structure of a polarized $B$-isocrystal. We denote it by $\mathbb{D}(X, \lambda, \iota)_\mathbb{Q}$, or simply by $\mathbb{D}(X)_\mathbb{Q}$.

The condition (b) above implies that the induced $B$-action on $N_b$ and the pairing induced by $\langle \ , \ \rangle$ give a structure of a polarized $B$-isocrystal on $(N_b, \Phi_b)$. We call it the polarized $B$-isocrystal attached to $b$ and simply write $N_b$ for it. Let $\mathbf{J}$ be the algebraic group over $\mathbb{Q}_p$ of automorphisms of $N_b$; for a $\mathbb{Q}_p$-algebra $R$, $\mathbf{J}(R)$ consists of $h \in \text{Aut}_R(N_b \otimes_{\mathbb{Q}_p} R)$ such that

- $h \circ (\Phi_b \otimes \text{id}) = (\Phi_f \otimes \text{id}) \circ h$,
- and $h$ preserves the pairing $\langle \ , \ \rangle$ on $N_b \otimes_{\mathbb{Q}_p} R$ up to $R^\times$-multiplication.

For the representability of $\mathbf{J}$, see [RZ96, Proposition 1.12]. Moreover, if we denote by $\nu : \mathbb{D} \to G \otimes_{\mathbb{Q}_p} K_0$ the slope homomorphism attached to $b$ (cf. [Kot85, §4.2]) and
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by \( G_p \), the centralizer of the image of \( \nu \) in \( G \otimes_{\mathbb{Q}_p} K_0 \), then we have an isomorphism \( J \otimes_{\mathbb{Q}_p} K_0 \cong G_p \otimes_{K_0} K_0 \) (see [Kot97, §3.3, §A.2]). In particular, \( J \) is reductive. On the other hand, \( J \) is not connected in general.

The condition (a) ensures the existence of a \( p \)-divisible group \( X \) over \( \mathbb{F}_p \) whose rational Dieudonné module \( \mathbb{D}(X)_{\mathbb{Q}} \) is isomorphic to \( (N_b, \Phi_b) \). Therefore, we can find a triple \( (X, \lambda_0, \iota_0) \) as in Definition 2.1 such that \( \mathbb{D}(X, \lambda_0, \iota_0)_{\mathbb{Q}} \) is isomorphic to \( N_b \) as a polarized \( B \)-isocrystal. We fix such a triple \( (X, \lambda_0, \iota_0) \). By the Dieudonné theory, the group \( J = J(\mathbb{Q}_p) \) can be identified with the group of self-quasi-isogenies of \( X \) compatible with \( \iota_0 \) and preserving \( \lambda_0 \) up to \( \mathbb{Q}_p^* \)-multiplication.

Let \( E \) be the field of definition of the cocharacter \( \mu \). It is the subfield of \( K \) generated by \( \{ \text{Tr}(a; V_0) \mid a \in B \} \) over \( \mathbb{Q}_p \). In particular it is a finite extension of \( \mathbb{Q}_p \). We denote by \( \tilde{E} \) the composite field of \( E \) and \( K_0 \) inside \( K \). We write \( \text{Nilp}_{\mathcal{O}_{\tilde{E}}} \)

the category of \( \mathcal{O}_{\tilde{E}} \)-schemes on which \( a \) acts \( B \)-equivariantly. We denote it by \( \text{Nilp}_{\mathcal{O}_{\tilde{E}}} \).

Now we can give the definition of the Rapoport-Zink space (see also [RZ96, Definition 3.21, 3.23 c), d)]):

**Definition 2.2** Consider the functor \( M: \text{Nilp}_{\mathcal{O}_{\tilde{E}}} \to \text{Set} \) that associates \( S \) with the set of isomorphism classes of \( \{(X_L, \iota_L, \rho_L)\}_{L \in \mathcal{L}} \) where

(a) \( X_L \) is a \( p \)-divisible group over \( S \),

(b) \( \iota_L: \mathcal{O}_B \to \text{End}(X_L) \) is a homomorphism,

(c) \( \rho_L: X \otimes_{\mathbb{F}_p} \mathcal{S} \to X_L \times_S \mathcal{S} \) is a quasi-isogeny compatible with \( \mathcal{O}_B \)-actions, such that the following conditions are satisfied.

(a) For \( L, L' \in \mathcal{L} \) with \( L \subset L' \), the quasi-isogeny \( \rho_{L'} \circ \rho_{L'}^{-1}: X_L \times_S \mathcal{S} \to X_{L'} \times_S \mathcal{S} \) lifts to an isogeny \( X_L \to X_{L'} \) with height \( \log_{(L'/L)} \). Such a lift is automatically unique and \( \mathcal{O}_B \)-equivariant. We denote it by \( \tilde{\rho}_{L', L} \).

(b) For \( a \in B^\times \) which normalizes \( \mathcal{O}_B \) and \( L \in \mathcal{L} \), the quasi-isogeny \( \rho_{aL} \circ \iota_0(a) \circ \rho_{L}^{-1}: X_L \times_S \mathcal{S} \to X_{aL} \times_S \mathcal{S} \) lifts to an isomorphism \( X_L^a \to X_{aL} \). Here \( X_L^a \)

(d) For each \( L \in \mathcal{L} \), we have the following equality of polynomial functions on \( a \in \mathcal{O}_B \):

\[
\det_{\mathcal{O}_S}(a; \text{Lie } X_L) = \det_K(a; V_0).
\]
This equation is called the determinant condition. For a precise formulation, see [RZ96, 3.23 a)].

The functor $\mathcal{M}$ is represented by a formal scheme (denoted by the same symbol $\mathcal{M}$) which is locally formally of finite type over $\text{Spf } \mathcal{O}_E$ (see [RZ96, Theorem 3.25]). We write $\overline{\mathcal{M}}$ for the underlying reduced scheme $\mathcal{M}^{\text{red}}$ of $\mathcal{M}$. It is a scheme locally of finite type over $\mathcal{O}$, which is often not quasi-compact. Each irreducible component of $\overline{\mathcal{M}}$ is known to be projective over $\mathcal{O}$ (see [RZ96, Proposition 2.32]).

The group $J = J(\mathbb{Q}_p)$ acts naturally on the functor $\mathcal{M}$ on the left; the element $h \in J$, regarded as a self-quasi-isogeny on $X$, carries $\{(X_L, \iota_L, \rho_L)\}_{L \in \mathcal{L}}$ to $\{(X_L, \iota_L, \rho_L \circ h^{-1})\}_{L \in \mathcal{L}}$. Hence $J$ also acts on the formal scheme $\mathcal{M}$ and the scheme $\overline{\mathcal{M}}$.

**Remark 2.3** In the definition of $\mathcal{M}$, we fixed a triple $(X, \lambda_0, \iota_0)$. However, we can check that the formal scheme $\mathcal{M}$ with the action of $J$ is essentially independent of this choice. See the remark after [RZ96, Definition 3.21].

For a scheme $X$ locally of finite type over $\mathbb{F}_p$, we write $\text{Irr}(X)$ for the set of irreducible components. We investigate the action of $J$ on $\text{Irr}(\overline{\mathcal{M}})$ under the following condition:

**Definition 2.4** We say that $b$ comes from an abelian variety if there exist

- a $\mathbb{Q}$-subalgebra $\widetilde{B}$ which is stable under $*$ and satisfies $\widetilde{B} \otimes_{\mathbb{Q}} \mathbb{Q}_p = B$,
- an abelian variety $A_0$ over $\mathbb{F}_p$,
- a polarization $\lambda_0 : A_0 \rightarrow A_0^\vee$,
- and a homomorphism $\iota_0 : \widetilde{B} \rightarrow \text{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $\lambda_0 \circ \iota_0(a^*) = \iota_0(a)^\vee \circ \lambda_0$

for every $a \in B$

such that the polarized $B$-isocrystal $\mathcal{D}(A_0[p^\infty])_\mathbb{Q}$ is isomorphic to $N_b$.

Note that in this case $\widetilde{B}$ is a finite-dimensional semisimple algebra over $\mathbb{Q}$, $*$ is a positive involution on $\widetilde{B}$, and $\iota_0$ is an injective homomorphism (recall that we assume $V$ to be a faithful $B$-module).

**Remark 2.5** Assume that the data $(B, *, V, \langle , \rangle)$ comes from global objects. Then the problem of finding $(A_0, \lambda_0, \iota_0)$ as in Definition 2.4 is a generalization of Manin’s problem (cf. [Man63, Chapter IV, §5, Conjecture 1, Conjecture 2]). There are a lot of works in this direction; see [Tat71], [Oor00], [Vas11], [VW13], [SS13], [Kre12]. For example, if the data $(B, \mathcal{O}_B, *, V, \langle , \rangle)$ is unramified (namely, $B$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$ and there exists a $\mathbb{Z}_p$-lattice of $V$ which is self-dual for $\langle , \rangle$ and preserved by $\mathcal{O}_B$), then we can always find such $(A_0, \lambda_0, \iota_0)$ (see [VW13, Theorem 10.1]).

The main theorem of this paper is the following:

**Theorem 2.6** Assume that $b$ comes from an abelian variety. Then, $\text{Irr}(\overline{\mathcal{M}})$ has finitely many $J$-orbits.
Remark 2.7  i) If the data $(B, \mathcal{O}_B, *, V, \langle \cdot, \cdot \rangle)$ is unramified, the corresponding result is obtained in [Far04, Théorème 2.4.13]. Together with Remark 2.5, the theorem above gives an alternative proof of Fargues’ result.

ii) In some concrete cases, there are more precise results on irreducible components of $\mathcal{M}$. See [Vie08a], [Vie08b], [VW11], [RTW13] and [HP13] for instance.

Remark 2.8 In this paper, we only work on Rapoport-Zink spaces of PEL type. However, it will be possible to apply the same technique to Rapoport-Zink spaces of EL type.

3 Proof of the main theorem

3.1 First reduction

In the first step of our proof of Theorem 2.6, we will replace $\mathcal{M}$ with a simpler moduli space $\mathcal{N}$. We fix a lattice $L \in \mathcal{L}$ such that $L \subset L^\vee$.

Definition 3.1 Let $\mathcal{N} : \text{Nilp}_{W(\overline{\mathbb{F}}_p)} \rightarrow \text{Set}$ be the functor that associates $S$ with the set of isomorphism classes of $(X, \iota, \rho)$ where

- $X$ is a $p$-divisible group over $S$,
- $\iota : \mathcal{O}_B \rightarrow \text{End}(X)$ is a homomorphism,
- and $\rho : X \otimes_{\overline{\mathbb{F}}_p} S \rightarrow X \times_S S$ is a quasi-isogeny compatible with $\mathcal{O}_B$-actions,

such that the following condition is satisfied.

- Locally on $S$, there exist a constant $c \in \mathbb{Q}^\times$ and an isogeny $\lambda : X \rightarrow X^\vee$ with height $\lambda = \log_p \#(L^\vee/L)$ such that the following diagram is commutative:

$$
\begin{array}{c}
X \otimes_{\overline{\mathbb{F}}_p} S \\
\downarrow c \circ \lambda \otimes \text{id} \\
X^\vee \otimes_{\overline{\mathbb{F}}_p} S
\end{array}
\rightarrow
\begin{array}{c}
X \times_S S \\
\downarrow \lambda \times \text{id} \\
X^\vee \times_S S
\end{array}
\overset{\rho}{\longrightarrow}
\begin{array}{c}
X \times_S S \\
\downarrow \lambda \times \text{id} \\
X^\vee \times_S S
\end{array}
\overset{\rho^\vee}{\longrightarrow}
\begin{array}{c}
X \times_S S \\
\downarrow \lambda \times \text{id} \\
X^\vee \times_S S
\end{array}
$$

By a similar method as in the proof of [RZ96 Theorem 3.25], we can prove that $\mathcal{N}$ is represented by a formal scheme (denoted by the same symbol $\mathcal{N}$) which is locally formally of finite type over Spf $W(\overline{\mathbb{F}}_p)$. We write $\mathcal{N}^\text{red}$ for the underlying reduced scheme $\mathcal{N}^\text{red}$ of $\mathcal{N}$.

The group $J$ acts naturally on $\mathcal{N}$ on the left; an element $h \in J$ carries $(X, \iota, \rho)$ to $(X, \iota, \rho \circ h^{-1})$.

Lemma 3.2 There exists a proper morphism of formal schemes $\mathcal{M} \rightarrow \mathcal{N}$ compatible with $J$-actions.
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Proof. It suffices to construct a proper morphism $\mathcal{M} \to \mathcal{N} \otimes_{W(\overline{\mathbb{F}}_p)} \mathcal{O}_{\mathcal{E}}$. Let $S$ be an object of $\text{Nilp}_{\mathcal{O}_{\mathcal{E}}}$ and $\{(X_{L_i}, \iota_{L_i}, \rho_{L_i})\}_{L_i \in \mathcal{L}}$ an element of $\mathcal{M}(S)$. We will show that $(X_L, \iota_L, \rho_L)$ gives an element of $\mathcal{N}(S)$. We need to verify the condition on quasi-polarizations. By the condition (c) in Definition 2.2 locally on $S$ there exist $c \in \mathbb{Q}_p^\times$ and an isomorphism $p_L : X_L \to (X_{L'})^\vee$. Let $\lambda$ be the composite $X_L \xrightarrow{p_L} (X_{L'})^\vee \xrightarrow{(\rho_{L'}, \iota_{L'})\vee} X_L^\vee$. Then, we have

$$
\text{height } \lambda = \text{height } \tilde{\rho}_{L',L} = \log_p \#(L^\vee/L),
\quad c\lambda_0 \otimes \text{id} = \rho_{L'}^\vee \circ (p_L \times \text{id}) \circ \rho_L = \rho_L^\vee \circ (\lambda \times \text{id}) \circ \rho_L,
$$

as desired. Hence we obtain a morphism $\mathcal{M} \to \mathcal{N} \otimes_{W(\overline{\mathbb{F}}_p)} \mathcal{O}_{\mathcal{E}}$ of formal schemes, which we denote by $\psi$. Clearly $\psi$ is $J$-equivariant.

Next we prove that $\psi$ is proper. We denote the universal object over $\mathcal{N} \otimes_{W(\overline{\mathbb{F}}_p)} \mathcal{O}_{\mathcal{E}}$ by $(\tilde{X}, \tilde{\iota}, \tilde{\rho})$. We want to classify $\{(X_{L'_i}, \iota_{L'_i}, \rho_{L'_i})\}_{L'_i \in \mathcal{L}}$ as in Definition 2.2 such that $(X_L, \iota_L, \rho_L) = (\tilde{X}, \tilde{\iota}, \tilde{\rho})$. By the condition (b) in Definition 2.2 such an object is determined by $\{(X_{L'_i}, \iota_{L'_i}, \rho_{L'_i})\}_{L'_i \in \mathcal{L}}$ up to isomorphism. Furthermore, for each $L' \in \mathcal{L}$ with $L \subset L' \subseteq p^{-1}L$, $(X_{L'}, \iota_{L'}, \rho_{L'})$ is determined by $\text{Ker } \tilde{\rho}_{L',L}$, which is a finite flat subgroup scheme of $X_L[p]$ with degree $\#(L'/L)$. Clearly such a finite flat subgroup scheme is classified by a formal scheme $\mathcal{M}_{L'}$ which is proper over $\mathcal{N} \otimes_{W(\overline{\mathbb{F}}_p)} \mathcal{O}_{\mathcal{E}}$. Let $\mathcal{M}'$ be the fiber product over $\mathcal{N} \otimes_{W(\overline{\mathbb{F}}_p)} \mathcal{O}_{\mathcal{E}}$ of $\mathcal{M}_{L'}$ for $L' \in \mathcal{L}$ with $L \subset L' \subseteq p^{-1}L$. By [RZ96, Proposition 2.9], it is easy to observe that the natural morphism $\mathcal{M} \to \mathcal{M}'$ is a closed immersion (strictly speaking, we use the fact that [RZ96, Proposition 2.9] is valid for $\alpha \in \text{Hom}(X, Y) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ which is not necessary a quasi-isogeny). This concludes the proof.

Corollary 3.3 If the action of $J$ on $\text{Irr}(\overline{\mathcal{N}})$ has finite orbits, then so does the action of $J$ on $\text{Irr}(\overline{\mathcal{M}})$.

Proof. Take finitely many $\alpha_1, \ldots, \alpha_m \in \text{Irr}(\overline{\mathcal{N}})$ such that $\text{Irr}(\overline{\mathcal{N}}) = \bigcup_{i=1}^m J\alpha_i$. By Lemma 3.2, for each $i$, only finitely many components $\beta_1, \ldots, \beta_k_i \in \text{Irr}(\overline{\mathcal{M}})$ are mapped into $\alpha_i$ by the morphism $\overline{\mathcal{M}} \to \overline{\mathcal{N}}$. It is easy to observe that $\text{Irr}(\overline{\mathcal{M}}) = \bigcup_{i=1}^m \bigcup_{j=1}^{k_i} J\beta_j$.

By this corollary, we may consider $\overline{\mathcal{N}}$ in place of $\overline{\mathcal{M}}$.

3.2 A moduli space of PEL type

So far, we considered an arbitrary triple $(\mathcal{X}, \lambda_0, \iota_0)$ as in Section 2. Now we take $(A_0, \lambda_0, \iota_0)$ as in Definition 2.2 and consider the triple $(\overline{\mathcal{X}}, \lambda_0, \iota_0)$ attached to it; namely, we put $\overline{\mathcal{X}} = A_0[p^\infty]$ and denote the induced quasi-polarization $\overline{\mathcal{X}} \to \overline{\mathcal{X}}^\vee$ and homomorphism $B \to \text{End}(\overline{\mathcal{X}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by the same symbols $\lambda_0$ and $\iota_0$.

Lemma 3.4 Assume that $\mathcal{N}(\mathbb{F}_p) \neq \emptyset$. Then we can replace $(A_0, \lambda_0, \iota_0)$ so that the following conditions are satisfied:

$$
\text{B and homomorphism of Lemma 3.2, for each } i, \alpha_i \text{ mapped into So far, we considered an arbitrary triple (X, \lambda_0, \iota_0) as in Section 2. Now we take (A_0, \lambda_0, \iota_0)}$$

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- \( X = A_0[p^\infty] \) is completely slope divisible (cf. [Zin01, Definition 10]).
- \( \text{ord}_p \deg \lambda_0 = \log_p \#(L^\vee/L) \), where \( \text{ord}_p \) denotes the \( p \)-adic order.
- there exists an order \( \bar{\mathcal{O}} \) of \( \bar{B} \) which is contained in \( \iota_0^{-1}(\text{End}(A_0)) \), stable under * and satisfies \( \bar{\mathcal{O}} \otimes \mathbb{Z}_p = \mathcal{O}_B \).

Proof. First we prove that \( \mathcal{N}(\overline{\mathbb{F}_p}) \) contains an element \( (X, \iota, \rho) \) such that \( X \) is completely slope divisible. Take an arbitrary element \( (X, \iota, \rho) \) in \( \mathcal{N}(\overline{\mathbb{F}_p}) \). By [Zin01, Corollary 13], there exists a unique slope filtration \( 0 = X_0 \subset X_1 \subset \cdots \subset X_\infty = X \). Put \( X' = \bigoplus_{i=1}^\infty X_i/X_{i-1} \). Let us observe that \( X' \) is completely slope divisible. Since \( A_0 \) and \( X \) are defined over a finite field, so is \( X \). Therefore, by [Zin01, Corollary 13], \( X_i \) is also defined over a finite field. Hence [OZ02, Corollary 1.5] tells us that \( X' \) is completely slope divisible. As the category of isocrystals over \( \overline{\mathbb{F}_p} \) is semisimple, the rational Dieudonné modules \( \mathbb{D}(X)_\mathbb{Q} \) and \( \mathbb{D}(X')_\mathbb{Q} \) are canonically isomorphic. Thus a quasi-isogeny \( f : X \rightarrow X' \) is naturally induced. It is characterized by the property that the composite \( X_i \rightarrow X \xrightarrow{f} X' \xrightarrow{p^m} X_i/X_{i-1} \) is the canonical surjection for each \( i \).

The \( \mathcal{O}_B \)-action \( \iota \) on \( X \) induces an \( \mathcal{O}_B \)-action \( \iota' \) on \( X' \), and the quasi-isogeny \( f \) is \( \mathcal{O}_B \)-equivariant. Put \( \rho' = f \circ \rho \). We prove that \( (X', \iota', \rho') \) belongs to \( \mathcal{N}(\overline{\mathbb{F}_p}) \). It suffices to see the last condition in Definition 3.1. Since \( (X, \iota, \rho) \) lies in \( \mathcal{N}(\overline{\mathbb{F}_p}) \), we can find an element \( e \in \mathbb{Q}_p^\times \) and an isogeny \( \lambda : X \rightarrow X^\vee \) with height \( \lambda = \log_p \#(L^\vee/L) \) such that \( c \lambda_0 = \rho' \circ \lambda \circ \rho \). The isogeny \( \lambda : X \rightarrow X^\vee \) should be compatible with the slope filtrations on \( X \) and \( X^\vee \), and thus it induces the isogeny \( \lambda' : X' \rightarrow X'^\vee \). By the Dieudonné theory, we can easily see that \( \lambda = f^\vee \circ \lambda' \circ f \) and height \( \lambda' = \log_p \#(L^\vee/L) \). Hence we have \( c \lambda_0 = \rho^\vee \circ \lambda \circ \rho = \rho^\vee \circ \lambda' \circ \rho \), and thus conclude that \( (X', \iota', \rho') \in \mathcal{N}(\overline{\mathbb{F}_p}) \).

Now, fix \( (X, \iota, \rho) \in \mathcal{N}(\overline{\mathbb{F}_p}) \) where \( X \) is completely slope divisible. Then, there exist an abelian variety \( A'_0 \) over \( \overline{\mathbb{F}_p} \) and a \( p \)-quasi-isogeny \( \phi : A_0 \rightarrow A'_0 \) such that \( \phi[p^\infty] : A_0[p^\infty] \rightarrow A'_0[p^\infty] \) can be identified with \( \rho : X \rightarrow X \). By the last condition in Definition 3.1 there exists an integer \( m \) such that \( (\rho^\vee)^{-1} \circ p^m \lambda_0 \circ \rho^{-1} \) gives an isogeny \( X \rightarrow X^\vee \) of height \( \log_p \#(L^\vee/L) \). Consider the quasi-isogeny \( \lambda'_0 = (\phi^\vee)^{-1} \circ p^m \lambda_0 \circ \phi^{-1} : A'_0 \rightarrow A'_0 \). Passing to \( p \)-divisible groups, we can easily observe that it is a polarization on \( A'_0 \). By construction we have \( \text{ord}_p \deg \lambda_0 = \log_p \#(L^\vee/L) \).

As \( \phi \) induces \( \text{End}(A_0) \otimes \mathbb{Q} \xrightarrow{\iota} \text{End}(A'_0) \otimes \mathbb{Q} \), \( \iota_0 \) induces a homomorphism \( \iota'_0 : \bar{B} \rightarrow \text{End}(A'_0) \otimes \mathbb{Q} \). Let \( \bar{\mathcal{O}}' \) be the inverse image of \( \text{End}(A'_0) \) under \( \iota'_0 \). It is an order of \( \bar{B} \), for \( \iota'_0 \) is injective. We will show that \( \bar{\mathcal{O}}' \otimes \mathbb{Z}_p = \mathcal{O}_B \). For \( a \in \bar{B} \cap \mathcal{O}_B \), consider \( \iota'_0(a) \in \text{End}(A'_0) \otimes \mathbb{Q} \). Since the induced element \( \iota'_0(a)[p^\infty] \in \text{End}(A'_0[p^\infty]) \otimes \mathbb{Z}_p \) can be identified with \( \iota(a) \in \text{End}(X) \), it belongs to \( \text{End}(A'_0[p^\infty]) \). Therefore we conclude that \( \iota'_0(a) \in \text{End}(A'_0) \otimes \mathbb{Z}_p \), and thus \( a \in \bar{\mathcal{O}}' \otimes \mathbb{Z}_p \). Hence we have \( (\bar{B} \cap \mathcal{O}_B) \otimes \mathbb{Z}_p \subset \bar{\mathcal{O}}' \otimes \mathbb{Z}_p \). On the other hand, [Rei03, Theorem 5.2] for \( R = \mathbb{Z}_p \) tells us that \( (\bar{B} \cap \mathcal{O}_B) \otimes \mathbb{Z}_p \mathbb{Z}_p = \mathcal{O}_B \). As \( \mathcal{O}_B \) is a maximal order of \( B \), we conclude that \( \bar{\mathcal{O}}' \otimes \mathbb{Z}_p = \mathcal{O}_B \).
Take a \( \mathbb{Z} \)-basis \( e_1, \ldots, e_r \) of \( \mathcal{O}' \). Since \( \mathcal{O}' \otimes \mathbb{Z}_p = \mathcal{O}_B \), we can find an integer \( N > 0 \) which is prime to \( p \) such that \( Ne_i^* \in \mathcal{O}' \) for every \( 1 \leq i \leq r \) (recall that \( \mathcal{O}_B \) is stable under \( \ast \)). Let \( \mathcal{O} \) be the \( \mathbb{Z} \)-subalgebra of \( \mathcal{O}' \) generated by \( Ne_i \) and \( Ne_i^* \) for \( 1 \leq i \leq r \). Then \( \mathcal{O} \) is an order of \( \tilde{B} \) which is contained in \( \mathcal{O}' = \iota_0^{-1}(\text{End}(A'_0)) \), stable under \( \ast \) and satisfies \( \mathcal{O} \otimes \mathbb{Z}_p = \mathcal{O}_B \).

By construction the polarized \( B \)-isocrystal associated to \( (A'_0, \lambda'_0, \iota'_0) \) is isomorphic to \( N \); indeed the homomorphism \( \mathbb{D}(A_0[p^\infty])_{\mathbb{Q}} \rightarrow \mathbb{D}(A'_0[p^\infty])_{\mathbb{Q}} \) induced by \( \phi \) is an isomorphism of polarized \( B \)-isocrystals. Hence we may replace \( (A_0, \lambda_0, \iota_0) \) by \( (A'_0, \lambda'_0, \iota'_0) \).

If \( \mathcal{N}(\mathbb{F}_p) = \emptyset \), then \( \mathcal{N} = \emptyset \) and Theorem 2.4 is clear. In the sequel, we will assume that \( \mathcal{N}(\mathbb{F}_p) \neq \emptyset \), and take \( (A_0, \lambda_0, \iota_0) \) and \( \mathcal{O} \) as in Lemma 3.4. Put \( g = \dim A_0 \) and \( d = \deg \lambda_0 \).

**Definition 3.5** For an integer \( \delta \), let \( \mathcal{N}^{(\delta)} \) be the open and closed formal subscheme of \( \mathcal{N} \) consisting of \( (X, \iota, \rho) \) with \( g^{-1} \cdot \text{height} \rho = \delta \). Note that the left hand side is always an integer. Indeed, by the definition of \( \mathcal{N} \), at least locally, there exist \( c \in \mathbb{Q}_p^\times \) and an isogeny \( \lambda : X \rightarrow X^\vee \) with height \( \lambda = \log_p \#(L^\vee/L) \) such that \( \text{height}(c\lambda_0) = \text{height} \rho + \text{height} \lambda = \text{height} \rho^\vee \). Since \( \text{height} \lambda_0 = \log_p \#(L^\vee/L) = \text{height} \lambda \) and \( \text{height} \rho^\vee = \text{height} \rho \), we have \( \text{height} \rho = g \cdot \text{ord}_p(c) \). Hence \( g^{-1} \cdot \text{height} \rho = \text{ord}_p(c) \) is an integer.

The formal scheme \( \mathcal{N} \) is decomposed into the disjoint union \( \bigsqcup_{\delta \in \mathbb{Z}} \mathcal{N}^{(\delta)} \). Put \( \mathcal{N}^{(\delta)} = (\mathcal{N}^{(\delta)})^{\text{red}} \).

By the argument in the definition above, we can also prove the following:

**Lemma 3.6** Let \( \delta \) be an integer. For \( S \in \text{Nilp}_{W(\mathbb{F}_p)} \), and \( (X, \iota, \rho) \in \mathcal{N}^{(\delta)}(S) \), there exists a unique isogeny \( \lambda : X \rightarrow X^\vee \) with height \( \lambda = \log_p \#(L^\vee/L) \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
X \otimes_{W(\mathbb{F}_p)} S & \xrightarrow{\rho} & X \times_S S \\
\downarrow p^\delta \lambda_0 \otimes \text{id} & & \downarrow \lambda \times \text{id} \\
X^\vee \otimes_{W(\mathbb{F}_p)} S & \xrightarrow{\rho^\vee} & X^\vee \times_S S. \\
\end{array}
\]

**Proof.** We have only to check that the quasi-isogeny \( \lambda : X \rightarrow X^\vee \) lifting \( (\rho^\vee)^{-1} \circ (p^\delta \lambda_0 \otimes \text{id}) \circ \rho^{-1} \) is an isogeny. This is a local problem on \( S \), and thus we may assume that there exist \( c \in \mathbb{Q}_p^\times \) and an isogeny \( \lambda' : X \rightarrow X^\vee \) with height \( \lambda' = \log_p \#(L^\vee/L) \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
X \otimes_{W(\mathbb{F}_p)} S & \xrightarrow{\rho} & X \times_S S \\
\downarrow c \lambda_0 \otimes \text{id} & & \downarrow \lambda' \times \text{id} \\
X^\vee \otimes_{W(\mathbb{F}_p)} S & \xrightarrow{\rho^\vee} & X^\vee \times_S S. \\
\end{array}
\]
By the argument in Definition 3.5, we have \( \text{ord}_p(c) = \delta \). Therefore \( c' = p^\delta c^{-1} \) lies in \( \mathbb{Z}_p^\times \). On the other hand, we have \( \lambda = c' \lambda' \), as they coincide over \( \mathfrak{S} \). Hence \( \lambda \) is an isogeny, as desired.

Next we will construct a moduli space of abelian varieties with some additional structures.

**Definition 3.7** Fix an integer \( n \geq 3 \) which is prime to \( p \). Let \( X \) be the functor from the category of \( \mathbb{Z}_p \)-schemes to \( \text{Set} \) that associates \( S \) to the set of isomorphism classes of \((A, \lambda, \alpha, \iota)\) where

- \( A \) is a \( g \)-dimensional abelian scheme over \( S \),
- \( \lambda : A \to A^\vee \) is a polarization of \( A \) with \( \deg \lambda = d \),
- \( \alpha : (\mathbb{Z}/n\mathbb{Z})^{2g} \cong A[n] \) is an isomorphism of group schemes over \( S \) (an \( n \)-level structure on \( A \)),
- and \( \iota : \tilde{O} \to \text{End}(A) \) is a homomorphism such that \( \lambda \circ \iota(a^\ast) = \iota(a)^\vee \circ \lambda \) for every \( a \in \tilde{O} \).

An isomorphism between two quadruples \((A, \lambda, \alpha, \iota), (A', \lambda', \alpha', \iota')\) is an isomorphism \( f : A \to A' \) such that \( \lambda = f^\vee \circ \lambda' \circ f \), \( \alpha' = f \circ \alpha \), and \( \iota'(a) \circ f = f \circ \iota(a) \) for every \( a \in \tilde{O} \).

Note that, unlike in the definition of Shimura varieties of PEL type, we impose no compatibility condition on \( \alpha \) and \( \iota \). This is to avoid an auxiliary choice of linear-algebraic data outside \( p \).

**Proposition 3.8** The functor \( X \) is represented by a quasi-projective scheme over \( \mathbb{Z}_p \).

**Proof.** By [MFK94, Theorem 7.9], the moduli space of \((A, \lambda, \alpha)\) is represented by a quasi-projective scheme \( \mathcal{A}_{g,d,n} \) over \( \mathbb{Z}_p \). Furthermore, [Lan13, Proposition 1.3.3.7] tells us that \( X \) is represented by a scheme which is finite over \( \mathcal{A}_{g,d,n} \). This concludes the proof.

In the proof of Theorem 2.6, we focus on the geometric special fiber \( \overline{X} = X \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}_p} \) of \( X \). For \( x \in \overline{X} \) and a geometric point \( \overline{\pi} \) lying over \( x \), the corresponding quadruple \((A_{\overline{\pi}}, \lambda_{\overline{\pi}}, \alpha_{\overline{\pi}}, \iota_{\overline{\pi}})\) gives a polarized \( B \)-isocrystal \( \mathbb{D}(A_{\overline{\pi}}[p^\infty])_Q \) over \( \kappa(\overline{\pi}) \). The condition that \( \mathbb{D}(A_{\overline{\pi}}[p^\infty])_Q \) is isomorphic to \( N_0 \otimes_{\kappa_0} \text{Frac} W(\kappa(\overline{\pi})) \) is independent of the choice of \( \overline{\pi} \). Indeed, the following lemma holds:

**Lemma 3.9** Let \( k \) be an algebraically closed field of characteristic \( p \) and \( k' \) an algebraically closed extension field of \( k \). For polarized \( B \)-isocrystals \( N, N' \) over \( k, N \cong N' \) if and only if \( N \otimes_{\text{Frac} W(k)} \text{Frac} W(k') \cong N' \otimes_{\text{Frac} W(k)} \text{Frac} W(k') \).

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Proof. Assume that there exists an isomorphism

\[ f : N \otimes_{\text{Frac} \kappa} \text{Frac} \kappa(k') \xrightarrow{\sim} N' \otimes_{\text{Frac} \kappa} \text{Frac} \kappa(k') \]

of polarized \( B \)-isocrystals. By [RR96, Lemma 3.9], there exists an isomorphism \( f' : N \xrightarrow{\sim} N' \) of isocrystals (without any additional structure) such that \( f = f' \otimes \text{id} \). Further, by [RR96, Lemma 3.9] we can observe that \( f' \) is compatible with \( B \)-actions and polarizations. Hence \( f' \) gives an isomorphism of polarized \( B \)-isocrystals.

We write \( \overline{X}^{(b)} \) for the set of \( x \in \overline{X} \) satisfying \( \mathbb{D}(A_{\mathcal{P}^\infty})_q \cong N_b \otimes_{K_0} \text{Frac} \kappa(\mathcal{F}) \). The next lemma ensures that \( \overline{X}^{(b)} \) is a locally closed subset of \( \overline{X} \).

**Lemma 3.10** Let \( S \) be a locally noetherian scheme of characteristic \( p \) and \( M \) an \( F \)-isocrystal over \( S \) (cf. [Kat79, §2.1], [RR96, §3.1]) endowed with a \( B \)-action and a non-degenerate alternating bilinear pairing \( M \otimes_M \to \mathbf{1}(-1) \) satisfying the same conditions in Definition 2.4 (iii), Theorem 3.6 (ii), Theorem 3.8. We will adapt their argument to our case.

First, by Grothendieck’s specialization theorem [Kat79, Theorem 2.3.1], the set consisting of \( s \in S \) such that \( M_s \cong N_b \otimes_{K_0} \text{Frac} \kappa(\mathcal{F}) \) as isocrystals (without any additional structure) is locally closed in \( S \). Therefore, by replacing \( S \) with this subset endowed with the induced reduced scheme structure, we may assume that the Newton polygons of the isocrystals \( M_s \) for \( s \in S \) are constant. Further, we may also assume that \( S \) is connected. In this case, we shall prove that \( S^{(b)} \) is either \( S \) or empty. As in the proof of [RR96, Theorem 3.8], it suffices to show the following:

Assume that \( S = \text{Spec} \ k[[t]] \), where \( k \) is an algebraically closed field of characteristic \( p \). We denote by \( s_1 \) (resp. \( s_0 \)) the generic (resp. closed) point of \( S \). Then we have an isomorphism \( M_{s_1} \cong M_{s_0} \otimes_{K_0} \text{Frac} \kappa(\mathcal{F}) \) of polarized \( B \)-isocrystals.

Following [RR96], we write \( R \) for the perfect closure of \( k[[t]] \) and \( a \) for the composite \( \text{Spec} \ R \to \text{Spec} \ k \xrightarrow{s_0} S \). By [Kat79, Theorem 2.7.4] and [RR96, Lemma 3.9], there exists a unique isomorphism \( f : M_R \xrightarrow{\sim} a^*(M) \) of \( F \)-isocrystals over \( \text{Spec} \ R \) which induces the identity over \( s_0 \). We shall observe that \( f \) is compatible with the \( B \)-actions and the polarizations on \( M_R \) and \( a^*(M) \). For \( b \in B \), \( f \circ \iota_{M_R}(b) \) and \( \iota_{a^*(M)}(b) \circ f \) are elements of \( \text{Hom}(M_R, a^*(M)) \), whose images in \( \text{Hom}(M_{s_0}, M_{s_0}) \) are both equal to \( \iota_{M_{s_0}}(b) \). On the other hand, [RR96, Lemma 3.9] tells us that the pull-back map \( \text{Hom}(M_R, a^*(M)) \to \text{Hom}(M_{s_0}, M_{s_0}) \) is bijective (note that \( M_R \) is constant, for it is isomorphic to \( a^*(M) \)). Thus \( f \) is compatible with \( B \)-actions. The same method can be used to compare polarizations. Hence, by taking the fiber of
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At $s_1$, we obtain an isomorphism $f_{s_1} : M_{s_1} \cong M_{s_0} \otimes_{K_0} \text{Frac} W(\kappa(s_1))$ compatible with $B$-actions and polarizations. This concludes the proof.

We endow $\overline{X}(b)$ with the reduced scheme structure induced from $\overline{X}$. By definition, the fixed $(A_0, \lambda_0, \iota_0)$ and an arbitrary $n$-level structure $\alpha_0$ on $A_0$ give an $\mathbb{F}_p$-valued point of $\overline{X}(b)$. In the following we fix $\alpha_0$.

### 3.3 Oort’s leaf

Here we follow the construction in [Oor04], [Man04] and [Man05].

**Definition 3.11** Let $C$ be the subset of $\overline{X}(b)$ consisting of $x = (A_x, \lambda_x, \alpha_x, \iota_x)$ satisfying the following condition:

- for some algebraically closed field extension $k$ of $\kappa(x)$, there exists an isomorphism $X \otimes_{\mathbb{F}_p} k \cong A_x[p^\infty] \otimes_{\kappa(x)} k$ which carries $\lambda_0$ to a $\mathbb{Z}_p$-multiple of $\lambda_x$ and $\iota_0$ to $\iota_x$.

By definition, $(A_0, \lambda_0, \alpha_0, \iota_0)$ lies in $C$. In particular $C$ is non-empty. The goal of this subsection is to prove the following theorem, which is a generalization of [Oor04, Theorem 2.2].

**Theorem 3.12** The subset $C$ is closed in $\overline{X}(b)$.

The idea of our proof of Theorem 3.12 is to use the theory of Igusa towers developed in [Man04].

**Definition 3.13** Let $C_{\text{naive}}$ be the subset of $\overline{X}(b)$ consisting of $x = (A_x, \lambda_x, \alpha_x, \iota_x)$ satisfying the following condition:

- for some algebraically closed field extension $k$ of $\kappa(x)$, there exists an isomorphism $X \otimes_{\mathbb{F}_p} k \cong A_x[p^\infty] \otimes_{\kappa(x)} k$ as $p$-divisible groups (we impose no compatibility on quasi-polarizations and $O_B$-actions).

By [Oor04, Theorem 2.2], $C_{\text{naive}}$ is a closed subset of $\overline{X}(b)$. We endow it with the induced reduced scheme structure. Let $C_{\sim \text{naive}}$ be the normalization of $C_{\text{naive}}$.

Let $(\mathcal{A}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\iota})$ be the universal object over $\overline{X}$ and put $G = \mathcal{A}[p^\infty]$. We sometimes denote the pull-back of $\mathcal{A}$ and $G$ to various schemes by the same symbols $\mathcal{A}$ and $G$.

**Lemma 3.14** The $p$-divisible group $G$ over $C_{\sim \text{naive}}$ is completely slope divisible in the sense of [OZ02, Definition 1.2].

**Proof.** Recall that $X$ is completely slope divisible (cf. Lemma 3.4). Consider a minimal point $\eta$ of $C_{\sim \text{naive}}$. By the definition of $C_{\text{naive}}$, there exists an algebraically closed extension field $k$ of $\kappa(\eta)$ such that $X \otimes_{\mathbb{F}_p} k \cong G_\eta \otimes_{\kappa(\eta)} k$. Therefore [OZ02, Remark in p. 186] tells us that $G_\eta$ is completely slope divisible. Hence, by [OZ02, Proposition 2.3], we conclude that $G$ over $C_{\sim \text{naive}}$ is completely slope divisible. \qed
By the lemma above, we have a (unique) slope filtration $0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_s = \mathcal{G}$. Put $\mathcal{G}^i = \mathcal{G}_i/\mathcal{G}_{i-1}$. The action of $\mathcal{O}_B$ on $\mathcal{G}$ induces that on $\mathcal{G}^i$. The quasi-polarization $\lambda: \mathcal{G} \to \mathcal{G}^\vee$ induced from the universal polarization gives a morphism $\lambda_i: \mathcal{G}^i \to (\mathcal{G}^i)^\vee$, where $j$ is the integer such that slope $\mathcal{G}^j + \text{slope } \mathcal{G}^j = 1$.

We can also consider $0 = X_0 \subset X_1 \subset \cdots \subset X_s = X$, $X^i = X_i/X_{i-1}$, the $\mathcal{O}_B$-action on $X^i$ and $\lambda_{0;i}: X^i \to (X^i)^\vee$. They appear as the fibers of the previous objects at a point of $C_{\text{naive}}$ lying over $(A_0, \lambda_0, \alpha_0, \iota_0) \in C_{\text{naive}}$.

**Definition 3.15** For an integer $m \geq 0$, let $C_{\text{naive}, m}$ be the functor from the category of $C_{\text{naive}}$-schemes to $\text{Set}$ that associates $S$ to the set of $\{j_{m,i}\}_{1 \leq i \leq s}$ where $j_{m,i}: X^i[p^m] \otimes_{\mathbb{Z}_p} S \xrightarrow{\sim} \mathcal{G}_i[p^m] \times C_{\text{naive}}$ is an isomorphism of group schemes over $S$.

This functor is represented by a scheme of finite type over $C_{\text{naive}}$. We denote the universal isomorphisms over $C_{\text{naive}, m}$ by $j_{m,i}^{\text{univ}}$.

Let $C_{\text{naive}, m}$ be the intersection of the scheme-theoretic images of $C_{\text{naive}, m'} \to C_{\text{naive}, m}$ for $m' \geq m$ and put $I_{\text{naive}, m} = (C_{\text{naive}, m})^{\text{red}}$. We also denote the restriction of $j_{m,i}$ to $I_{\text{naive}, m}$ by the same symbol $j_{m,i}^{\text{univ}}$.

The following proposition is due to Harris-Taylor and Mantovan:

**Proposition 3.16** For each $m \geq 0$, the morphism $I_{\text{naive}, m+1} \to I_{\text{naive}, m}$ is finite étale and surjective.

**Proof.** This is essentially proved in [Man04, Proposition 3.3], but we should be careful since [Man04, Proposition 3.3] is stated in the case of unitary Shimura varieties. Recall that its proof is a combination of [HT01, Proposition II.1.7] and [Man04, Lemma 3.4]. The former is valid for any reduced excellent scheme over $\mathbb{F}_p$. The latter requires the fact that the completed local ring $\mathcal{O}_{C_{\text{naive}}}^{\sim}$ is a normal integral domain for each closed point $x \in C_{\text{naive}}$. This is true because $C_{\text{naive}}$ is normal and excellent (cf. [EGA] IV, 7.8.3 (v))).

**Definition 3.17** For $m \geq 0$, let $C_{\text{naive}} \subset I_{\text{naive}, m}$ be the locus on which $\{j_{m,i}^{\text{univ}}\}$ preserves the $(\mathbb{Z}/p^m\mathbb{Z})^\times$-homothety classes of the quasi-polarizations (i.e., the homomorphisms $\lambda_{0;i}$ and $\tilde{\lambda}_i$) and the $\mathcal{O}_B$-actions explained before Definition 3.15. Clearly it is a closed subscheme of $I_{\text{naive}, m}$.

Moreover, let $C_{m'}$ be the intersection of the scheme-theoretic images of $C_{m'} \to C_{m}$ for $m' \geq m$. Put $I_{m}^{\sim} = (C_{m})^{\text{red}}$ and $C_{m} = I_{g}^{\sim}$.

**Corollary 3.18** For each $m \geq 0$, the morphism $I_{g} \to I_{m}^{\sim}$ is finite and surjective.

**Proof.** Since $I_{g}^{\sim}$ (resp. $I_{m+1}^{\sim}$) is a closed subscheme of $I_{\text{naive}, m}$ (resp. $I_{\text{naive}, m+1}$), the finiteness follows from Proposition 3.16. To show the surjectivity, note that there exists an integer $m' \geq m + 1$ such that $C_{m+1}^{\sim}$ coincides with the scheme-theoretic image of $C_{m'} \to C_{m+1}^{\sim}$, for $C_{m+1}^{\sim}$ is a noetherian scheme. Take $x \in C_{m'}$.}

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As $C^{\sim}_{m'} \rightarrow C^{\sim}_m$ is finite by Proposition \ref{prop:finite-morphism}, $x$ lies in the set-theoretic image of this map. Hence we can find $x'$ in $C^\sim_m$ which is mapped to $x$. The image of $x'$ in $C^\sim_{m+1}$ lies in $C^\sim_{m+1}$, and is mapped to $x$. Therefore $\text{Im}_{s+m} = C^\sim_{m+1} \rightarrow C^\sim_m = \text{Im}_{s+m}$ is surjective, as desired.

The scheme $C^\sim$ is a closed subscheme of $C^\sim_{\text{naive}}$. Therefore, to prove Theorem \ref{thm:coincidence} it suffices to show the following proposition.

**Proposition 3.19** The image of $C^\sim$ under the finite morphism $C^\sim_{\text{naive}} \rightarrow C_{\text{naive}}$ coincides with $C$.

**Proof.** Take $x \in C^\sim_{\text{naive}}$. First assume that the image of $x$ in $C_{\text{naive}}$ lies in $C$. Then there exist an algebraically closed extension field $k$ of $\kappa(x)$ and an isomorphism $j : X \otimes_{\mathbb{F}_p} k \xrightarrow{\cong} G_x \otimes_{\kappa(x)} k$ preserving the $\mathbb{Z}_p^\times$-homotopy classes of the quasi-polarizations and the $O_B$-actions. It induces an isomorphism $j_{m,i} : X'[p^m] \otimes_{\mathbb{F}_p} k \xrightarrow{\cong} G_x'[p^m] \otimes_{\kappa(x)} k$ for each $m \geq 0$ and $1 \leq i \leq s$. Therefore we obtain a system of maps $\{\text{Spec } k \longrightarrow C^\sim_{\text{naive},m}\}_{m \geq 0}$ compatible with projections. It is easy to check that the morphism $\text{Spec } k \longrightarrow C^\sim_{\text{naive},m}$ factors through $\text{Im}_{s+m}$. In particular, the image of $\text{Spec } k \longrightarrow C^\sim_{\text{naive}}$, which is nothing but $x$, lies in $C^\sim$.

Conversely assume that $x$ lies in $C^\sim$. By Corollary \ref{cor:surjectivity} we can take a system of points $\{x_m \in \text{Im}_{s+m}\}_{m \geq 0}$ compatible with projections such that $x_0 = x$. Let $k$ be an algebraic closure of $\text{lim}_{\rightarrow} \kappa(x_m)$. Then, for each $m$ we have a collection of isomorphisms $\{j_{m,i} : X'[p^m] \otimes_{\mathbb{F}_p} k \xrightarrow{\cong} G_x'[p^m] \otimes_{\kappa(x)} k\}_{1 \leq i \leq s}$ compatible with the change of $m$.

By [Zin01] Corollary 11, the slope filtrations on $X$ and $G_x \otimes_{\kappa(x)} k$ split canonically. Namely, we have isomorphisms $X \cong \bigoplus_{i=1}^s X_i$ and $G_x \otimes_{\kappa(x)} k \cong \bigoplus_{i=1}^s G_{x_i} \otimes_{\kappa(x)} k$ which are compatible with the quasi-polarizations and the $O_B$-actions. Hence, $\{j_{m,i}\}_{1 \leq i \leq s}$ induces an isomorphism $j_m : X'[p^m] \otimes_{\mathbb{F}_p} k \xrightarrow{\cong} G_x'[p^m] \otimes_{\kappa(x)} k$ compatible with the $(\mathbb{Z}/p^m\mathbb{Z})^\times$-homotopy classes of the quasi-polarizations, the $O_B$-actions and the change of $m$. By taking inductive limit with respect to $m$, we obtain an isomorphism $j : X \otimes_{\mathbb{F}_p} k \xrightarrow{\cong} G_x \otimes_{\kappa(x)} k$ compatible with the $O_B$-actions. Since the projective limit of a projective system consisting of finite sets is non-empty, $j$ preserves the $\mathbb{Z}_p^\times$-homotopy classes of the quasi-polarizations. This means that the image of $x$ in $C^\sim_{\text{naive}}$ belongs to $C$.

Noting that $C^\sim_{\text{naive}} \rightarrow C_{\text{naive}}$ is surjective, we conclude the proof.

Now the proof of Theorem \ref{thm:coincidence} is complete. We endow the closed subset $C \subset \overline{X}(b)$ with the induced reduced scheme structure.

**Proposition 3.20** The scheme $C$ is smooth over $\overline{\mathbb{F}}_p$.

**Proof.** It can be proved in the same way as [Oor04] Theorem 3.13 (i).
By this proposition, we can apply the same construction as in Definition 3.15 and Definition 3.17 to \( G/C \). Hence we first obtain the naive Igusa tower \( \{ \text{Ig}_{naive,m} \} \), and after modifying it, the Igusa tower \( \{ \text{Ig}_m \} \). The following proposition can be proved in the same way as Corollary 3.18 and Proposition 3.19.

**Proposition 3.21**

i) The transition maps of \( \{ \text{Ig}_m \} \) is finite and surjective.

ii) We have \( \text{Ig}_0 = C \).

### 3.4 Almost product structure

In this subsection, an element \((X, \iota, \rho)\) of \( \mathcal{N}(S) \) for \( S \in \text{Nilp}_{W(F_p)} \) will be denoted by \((X, \rho)\). For integers \( r, m \) with \( m \geq 0 \), let \( \mathcal{N}^{r,m} \) be the closed formal subscheme of \( \mathcal{N} \) consisting of \((X, \rho)\) such that \( p^r \rho \) is an isogeny and \( \text{Ker}(p^r \rho) \) is killed by \( p^m \). We put \( \mathcal{N}^{r,m} = (\mathcal{N}^{r,m})^{\text{red}} \). It is a scheme of finite type over \( \overline{F}_p \) (cf. [RZ96, Corollary 2.31]).

As in [Oor04], [Man04] and [Man05], we will construct a morphism

\[
\text{Ig}_m \times_{\overline{F}_p} \mathcal{N}^{r,m} \rightarrow \mathcal{X}^{(b)}.
\]

Recall that we denote by \((A, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})\) the universal object over \( \text{Ig}_m \). For an integer \( N \geq 0 \), let \( A^{(p^N)} \) be the pull-back of \( A \) by the \( N \)th power of the absolute Frobenius morphism \( \text{Fr}: \text{Ig}_m \rightarrow \text{Ig}_m \). By [Man04, Lemma 4.1], there exists an integer \( \delta_m \geq 0 \) depending on \( m \) such that for \( N \geq \delta_m \) we have a canonical isomorphism

\[
A^{(p^N)}[p^m] = G^{(p^N)}[p^m] \cong \bigoplus_{i=1}^{s} (G^{(i)}(p^N))[p^m].
\]

Then, the universal Igusa structure \( \{ j^{(i,m)}_{univ} \} \) over \( \text{Ig}_m \) gives an isomorphism

\[
A^{(p^N)}[p^m] \cong \bigoplus_{i=1}^{s} (G^{(i)}(p^N))[p^m] \left\langle \bigoplus_{i=1}^{s} (X^{(i)}(p^N))[p^m] \otimes_{\overline{F}_p} \text{Ig}_m \cong X^{(p^N)}[p^m] \otimes_{\overline{F}_p} \text{Ig}_m,
\]

where \((X^{(i)}(p^N))\) and \(X^{(p^N)}\) are the pull-back of \( X^i \) and \( X \) by the \( N \)th power of the absolute Frobenius morphism on \( \text{Spec} \overline{F}_p \). It preserves the \((\mathbb{Z}/p^\infty)\)\(^{\times}\)-homothety classes of the quasi-polarizations and the \( O_B \)-actions.

On the other hand, over \( \mathcal{N}^{r,m} \) we have the universal \( p \)-divisible group \( \tilde{X} \) with an \( O_{B^p} \)-action and the universal \( O_{B^p} \)-quasi-isogeny \( \tilde{\rho}: X \otimes_{\overline{F}_p} \mathcal{N}^{r,m} \rightarrow \tilde{X} \). By the definition of \( \mathcal{N}^{r,m} \), \( p^r \rho \) is an isogeny and its kernel \( \text{Ker}(p^r \rho) \) is contained in \( X[p^m] \otimes_{\overline{F}_p} \mathcal{N}^{r,m} \). Hence \( \text{Ker}(p^r \rho)^{(p^N)} \) is a finite flat subgroup scheme of \( X^{(p^N)}[p^m] \otimes_{\overline{F}_p} \mathcal{N}^{r,m} \).

Now consider the abelian scheme \( \text{pr}_1^* A^{(p^N)} \) on \( \text{Ig}_m \times_{\overline{F}_p} \mathcal{N}^{r,m} \). Under the isomorphism \( \text{pr}_1^* A^{(p^N)}[p^m] \cong X^{(p^N)}[p^m] \otimes_{\overline{F}_p} (\text{Ig}_m \times_{\overline{F}_p} \mathcal{N}^{r,m}) \), \( \text{pr}_2^* \text{Ker}(p^r \rho)^{(p^N)} \) corresponds to a finite flat group scheme \( \mathcal{H} \) of \( \text{pr}_1^* A^{(p^N)}[p^m] \). We put \( A' = (\text{pr}_1^* A^{(p^N)}/\mathcal{H}) \). It is an abelian scheme over \( \text{Ig}_m \times_{\overline{F}_p} \mathcal{N}^{r,m} \) endowed with a \( p \)-isogeny \( \phi: \text{pr}_1^* A^{(p^N)} \rightarrow A' \). We
will find additional structures on \( \mathcal{A}' \) so that they give an element of \( \mathfrak{X}(\text{Ig}_m \times \mathbb{F}_p \mathbb{N}^{r,m}) \).

The \( \tilde{O} \)-action on \( \text{pr}_1^* \mathcal{A}^{(p^n)} \) induced from \( \tilde{t} \) gives an \( \tilde{O} \)-action \( \iota' \) on \( \mathcal{A}' \) by the isogeny \( \phi \).

Indeed, it suffices to observe that \( \mathcal{H} \) is stable under the \( \tilde{O} \)-action, which follows from the fact that \( \text{Ker}(p^n \tilde{\rho}) \) is stable under the \( \mathcal{O}_B \)-action. As \( n \) is assumed to be prime to \( p \), the quasi-isogeny \( p^{-r} \phi \) induces an isomorphism \( \text{pr}_1^* \mathcal{A}^{(p^n)}[n] \rightarrow \mathcal{A}'[n] \) on \( n \)-torsion points.

Let \( \alpha' \) be the level structure on \( \mathcal{A}' \) induced from \( \alpha \) by this isomorphism. We shall construct a polarization \( \lambda' \) on \( \mathcal{A}' \). Recall that we have a decomposition \( \mathcal{N} = \bigsqcup_{\delta \in \mathbb{Z}} \mathcal{N}(\delta) \) into open and closed subschemes. Put \( \mathcal{N}(\delta,r,m) = \mathcal{N}(\delta) \cap \mathcal{N}^{r,m} \).

It suffices to construct \( \lambda' \) over \( \text{Ig}_m \times \mathbb{F}_p \mathbb{N}^{(\delta),r,m} \) for each \( \delta \in \mathbb{Z} \). By Lemma 3.6 on \( \mathcal{N}(\delta,r,m) \) there exists an isogeny \( \lambda_{\mathcal{N}} : \mathcal{N} \rightarrow \mathcal{N}^\vee \) with height \( \log_p \#(L^\vee/L) \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{X} \otimes \mathbb{F}_p \mathcal{N}^{r,m} & \xrightarrow{\overline{\rho}} & \mathbb{X}^\vee \\
| \quad | \quad | \\
\rho \lambda_{\mathcal{N}} \otimes \text{id} & \lambda_{\mathcal{N}} & \\
\end{array}
\]

Put \( \lambda' = (\phi^\vee)^{-1} \circ p^{\delta+2r} \text{pr}_1^* \lambda_{\mathcal{N}} \circ \phi^{-1} \). Let us observe that the quasi-isogeny \( \lambda'[p^\infty] : \mathcal{A}'[p^\infty] \rightarrow \mathcal{A}'^{\vee}[p^\infty] \) induced by \( \lambda' \) is an isogeny with height \( \log_p \#(L^\vee/L) \).

As \( \text{Ig}_m \times \mathbb{F}_p \mathcal{N}^{r,m} \) is reduced, by [RZ96 Proposition 2.9], it suffices to show that, for each point \( z = (x,y) \) in \( (\text{Ig}_m \times \mathbb{F}_p \mathcal{N}^{r,m})(\mathbb{F}_p) \), the quasi-isogeny \( \lambda_z'[p^\infty] \) is an isogeny with height \( \log_p \#(L^\vee/L) \). By [Zin01 Corollary 11], we have an isomorphism \( \mathcal{A}_x^{(p^n)}[p^\infty] = \mathcal{G}_x^{(p^n)} \cong \bigoplus_{i=1}^s (\mathcal{G}_x^{(p^n)})_{i} \). Hence, Proposition 3.21 i) ensures that the isomorphism \( \mathcal{A}_x^{(p^n)}[p^m] \cong \mathbb{X}^{(p^n)}[p^m] \), the specialization at \( x \) of the isomorphism used above, can be extended to an isomorphism \( \mathcal{A}_x^{(p^n)}[p^\infty] \cong \mathbb{X}^{(p^n)}[p^\infty] \) between \( p \)-divisible groups which preserves the \( \mathbb{Z}_p^\times \)-homothety classes of the quasi-polarizations and the \( \mathcal{O}_B \)-actions (cf. the proof of Proposition 3.19). Under this isomorphism, \( \mathcal{A}_x'[p^{\infty}] \) is identified with \( \mathcal{X}_y^{(p^n)} \), and \( \lambda_z'[p^\infty] \) fits into the following diagram, which is commutative up to \( \mathbb{Z}_p^\times \)-multiplication:

\[
\begin{array}{ccc}
\mathbb{X}^{(p^n)} & \xrightarrow{\phi_z[p^\infty]=p^r \rho_y^{(p^n)}} & \mathcal{X}_y^{(p^n)} \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathbb{X}^{(p^n)} & \xrightarrow{\phi_y'[p^\infty]=p^r \rho_y'^{(p^n)}} & (\mathcal{X}_y^{(p^n)})^{(p^n)} \\
\end{array}
\]

Hence \( \lambda_z'[p^{\infty}] \) is identified with a \( \mathbb{Z}_p^\times \)-multiple of \( \lambda_{\mathcal{X}_{igy}}^{(p^n)} \), which is an isogeny with height \( \log_p \#(L^\vee/L) \). As deg \( \phi \) is a power of \( p \), we conclude that \( \lambda' \) is an isogeny with degree \( d \). Thus it gives a desired polarization on \( \mathcal{A}' \). Obviously the quadruple \( (\mathcal{A}', \lambda', \alpha', \iota') \) belongs to \( \mathfrak{X}(\text{Ig}_m \times \mathbb{F}_p \mathcal{N}^{r,m}) \).
Definition 3.22 Let \( \pi_N : \text{Ig}_{m} \times_{\mathbb{F}_p} \mathbb{N}^{r,m} \rightarrow \mathbb{F}_p \) be the morphism determined by the quadruple \((A', \lambda', \alpha', \iota')\). This morphism factors through \( \mathbb{F}_p(b) \) (note that the \( N \)th power of the relative Frobenius morphism gives an isogeny between \( X \) and \( \mathbb{F}_p(b) \)).

The following lemma, which is an analogue of [Man04, Proposition 4.3], is easily verified.

Lemma 3.23 i) For an integer \( N \geq \delta_m \), we have \( \pi_{N+1} = \text{Frob}_p \circ \pi_N \), where \( \text{Frob}_p : X \rightarrow X \) is the \( p \)th power Frobenius morphism over \( \mathbb{F}_p \) (namely, the base change to \( \mathbb{F}_p \) of the absolute Frobenius morphism on \( \mathbb{X} \)).

ii) For an integer \( N \geq \max\{\delta_m, \delta_{m+1}\} \), the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Ig}_{m+1} \times_{\mathbb{F}_p} \mathbb{N}^{r,m} & \xrightarrow{\pi_N} & \mathbb{F}_p(b) \\
\downarrow & & \\
\text{Ig}_m \times_{\mathbb{F}_p} \mathbb{N}^{r,m} & \overset{\pi_N}{\rightarrow} & \mathbb{F}_p(b) \\
\end{array}
\]

Let \( k \) be an algebraically closed field containing \( \mathbb{F}_p \). At the level of \( k \)-valued points, we can define a variant \( \Pi : \text{Ig}_m(k) \times \mathbb{N}^{r,m}(k) \rightarrow \mathbb{F}_p(b)(k) \) of \( \pi_N \). Let \( x = (A, \lambda, \alpha, \iota, \{j_{m,i}\}_{1 \leq i \leq s}) \) be an element of \( \text{Ig}_m(k) \) and \( y = (X, \rho) \) be an element of \( \mathbb{N}^{r,m}(k) \). In this case, by [Zin01, Corollary 11], we have a canonical isomorphism \( A[p^\infty] = G_x \cong \bigoplus_{i=1}^{s} G_i \). Hence \( \{j_{m,i}\}_{1 \leq i \leq s} \) induces an isomorphism \( A[p^m] \cong X[p^m] \otimes_{\mathbb{F}_p} k \) preserving the \((\mathbb{Z}/p^m\mathbb{Z})^\times\)-homothety classes of the quasi-polarizations and the \( \mathcal{O}_B \)-actions (in this argument, we need no restriction on \( m \geq 0 \)). By this isomorphism, the finite subgroup scheme \( \text{Ker}(p^r \rho) \) of \( X[p^m] \otimes_{\mathbb{F}_p} k \) corresponds to a subgroup scheme \( H \) of \( A[p^m] \). Put \( A' = A/H \). In the same way as in the definition of \( \pi_N \), we can find a polarization \( \lambda' \), a level structure \( \alpha' \) and an \( \mathcal{O} \)-action \( \iota' \) on \( A' \) induced from \( \lambda, \alpha \) and \( \iota \) respectively, so that \((A', \lambda', \alpha', \iota')\) belongs to the set \( \mathbb{F}_p(b)(k) \). We define the map \( \Pi : \text{Ig}_m(k) \times \mathbb{N}^{r,m}(k) \rightarrow \mathbb{F}_p(b)(k) \) by \( \Pi(x, y) = (A', \lambda', \alpha', \iota') \). The following lemma is clear.

Lemma 3.24 i) \( \Pi : \text{Ig}_m(k) \times \mathbb{N}^{r,m}(k) \rightarrow \mathbb{F}_p(b)(k) \) is compatible with the change of \( m \).

ii) \( \pi_N = \text{Frob}_p^N \circ \Pi \).

Lemma 3.25 Let \( k \) be an algebraically closed field containing \( \mathbb{F}_p \). For every point \( x = (A, \lambda, \alpha, \iota) \) of \( \mathbb{F}_p(b)(k) \), there exists an integer \( m \geq 0 \) such that \( x \) is contained in the image of \( \Pi : \text{Ig}_m(k) \times \mathbb{N}^{0,m}(k) \rightarrow \mathbb{F}_p(b)(k) \).

Proof. As \( x \) lies in \( \mathbb{F}_p(b) \), there is a quasi-isogeny \( \rho : X \otimes_{\mathbb{F}_p} k \rightarrow A[p^\infty] \) which is compatible with the \( \mathcal{O}_B \)-actions and preserves the quasi-polarizations up to \( \mathbb{Q}_p^\times \)-multiplication. Replacing \( \rho \) by \( p^r \rho \) if necessary, we may assume that \( \rho \) is an isogeny. Take an integer \( m \geq 0 \) such that \( \text{Ker} \rho \) is killed by \( p^m \). Then, there exists an isogeny \( \xi : A[p^\infty] \rightarrow X \otimes_{\mathbb{F}_p} k \) with \( \xi \circ \rho = p^m \). Put \( A' = A/\text{Ker} \xi \). Then \( A'[p^\infty] \) can be
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identified with \( \mathbb{X} \otimes_{\mathbb{F}_p} k \), and there exists a \( p \)-isogeny \( \phi: A' \rightarrow A \) corresponding to \( \rho \). As in the construction of \( \pi_N \), we can observe that the additional structures \( \lambda, \alpha, \iota \) on \( A \) induce additional structures \( \lambda', \alpha', \iota' \) on \( A' \) by the isogeny \( \phi \) so that \((A', \lambda', \alpha', \iota')\) gives an element of \( C(k) \). Let \( \{j_{m,i}\}_{1 \leq i \leq m} \) be the Igusa structure on \( A'[p^m] \) that comes from the identification \( A'[p^\infty] = \mathbb{X} \otimes_{\mathbb{F}_p} k \), and \( y \) be the point \((A', \lambda', \alpha', \iota', \{j_{m,i}\})\) in \( Ig_m(k) \). Now it is easy to check that \( z = (A[p^\infty], \rho) \) lies in \( \overline{N}^{0,m}(k) \) and \( \Pi(y, z) = x \).

**Proposition 3.26**

i) There exist integers \( m \geq 0 \) and \( N \geq \delta_m \) such that the morphism \( \pi_N: Ig_m \times_{\mathbb{F}_p} \overline{N}^{0,m} \rightarrow \overline{X}(b) \) is surjective.

ii) For \( m \) as in i) and an algebraically closed field \( k \) containing \( \overline{F}_p \), the map \( \Pi: Ig_m(k) \times \overline{N}^{0,m}(k) \rightarrow \overline{X}(b)(k) \) is surjective.

**Proof.** For each \( m \geq 0 \), take an integer \( N_m \geq \delta_m \), so that \( N_0 < N_1 < N_2 < \ldots \), and put \( T_m = (\text{Frob}_p^{N_m})^{-1} \pi_N(Ig_m \times_{\mathbb{F}_p} \overline{N}^{0,m}) \). By Lemma 3.23, we have

\[
T_m \subset \left( \text{Frob}_p^{N_{m+1}} \right)^{-1} \pi_N(Ig_m \times_{\mathbb{F}_p} \overline{N}^{0,m}) = \left( \text{Frob}_p^{N_{m+1}} \right)^{-1} \pi_N(Ig_{m+1} \times_{\mathbb{F}_p} \overline{N}^{0,m})
\]

\[
\subset \left( \text{Frob}_p^{N_{m+1}} \right)^{-1} \pi_N(Ig_{m+1} \times_{\mathbb{F}_p} \overline{N}^{0,m+1}) = T_{m+1}.
\]

Since \( \overline{X}(b) \) is a scheme of finite type over \( \overline{F}_p \), its underlying topological space is a spectral space in the sense of \( [\text{Hoc}69, \S 0] \). As in \( [\text{Hoc}69, \S 2] \), we consider the patch topology on \( \overline{X}(b) \), under which \( \overline{X}(b) \) becomes a compact space (\( [\text{Hoc}69, \text{Theorem 1}] \)). As \( \overline{N}^{0,m} \) is a scheme of finite type over \( \overline{F}_p \), \( T_m \) is a constructible subset of \( \overline{X}(b) \). In particular, it is an open set of \( \overline{X}(b) \) with respect to the patch topology. On the other hand, Lemma 3.24 ii) and Lemma 3.25 tell us that \( \bigcup_{m=0}^{\infty} T_m = \overline{X}(b) \). Hence there exists an integer \( m \geq 0 \) such that \( T_m = \overline{X}(b) \). As \( \text{Frob}_p \) is surjective, we have \( \pi_N(Ig_m \times_{\mathbb{F}_p} \overline{N}^{0,m}) = \overline{X}(b) \). This concludes the proof of i).

ii) follows from i), Lemma 3.24 ii) and the injectivity of \( \text{Frob}_p \) at the level of \( k \)-valued points.

**Proposition 3.27** Let \( m, m' \geq 0 \) be integers, and consider elements \((x, y) \in Ig_m(\overline{F}_p) \times \overline{N}^{0,m}(\overline{F}_p)\) and \((x', y') \in Ig_{m'}(\overline{F}_p) \times \overline{N}^{0,m'}(\overline{F}_p)\). If \( \Pi(x, y) = \Pi(x', y') \), then there exists \( h \in J \) such that \( y' = hy \).

**Proof.** By Proposition 3.21 i), we may assume that \( m = m' \). Let \((A, \lambda, \alpha, \iota, \{j_{m,i}\})\), \((A', \lambda', \alpha', \iota', \{j'_{m,i}\})\), \((X, \rho)\), \((X', \rho')\) be the objects corresponding to \( x, x', y, y' \), respectively. Recall that \( \{j_{m,i}\} \) induces an isomorphism \( j_m: X[p^m] \isom A[p^m] \). Again by Proposition 3.21 i), we can extend \( j_m \) to an isomorphism \( j: X \isom A[p^\infty] \) which preserves the \( Z_p^\times \)-homothety classes of the quasi-polarizations and the \( O_B \)-actions (cf. the proof of Proposition 3.19). Similarly we have isomorphisms \( j'_m: X[p^m] \isom A'[p^m] \) and \( j': X \isom A'[p^\infty] \). As \( \Pi(x, y) = \Pi(x', y') \), there exists an isomorphism
A/j(Ker ρ) ∼= A′/j′(Ker ρ′) compatible with the additional structures. On the other hand, j (resp. j′) induces an isomorphism \( X \cong X/\text{Ker } \rho \xrightarrow{\cong} (A/j(Ker \rho))[p^\infty] \) (resp. \( X' \cong X/\text{Ker } \rho' \xrightarrow{\cong} (A'/j'(Ker \rho'))[p^\infty] \)). Hence we obtain an isomorphism \( f : X \xrightarrow{\cong} X' \), which is compatible with the \( \mathcal{O}_B \)-actions. Consider the quasi-isogeny \( h = \rho'^{-1} \circ f \circ \rho : X \longrightarrow X \). It is straightforward to check that \( h \) in fact gives an element of \( J \) (use the fact that the isogenies \( A \longrightarrow A/j(Ker \rho) \) and \( A' \longrightarrow A'/j'(Ker \rho') \) preserve the polarizations up to multiplication by some powers of \( p \)). Now we conclude that \( y' = (X', \rho') = (X, f^{-1} \circ \rho') = (X, \rho \circ h^{-1}) = hy \), as desired.

Now we can give a proof of our main theorem.

**Proof of Theorem 2.6.** Take \( m \geq 0 \) as in Proposition 3.26 and let \( \mathcal{S} \subset \text{Irr}(N) \) be the subset consisting of irreducible components of \( N \) which intersect \( N^{0,m} \). Let us observe that \( \mathcal{S} \) is a finite set. Take a quasi-compact open subset \( U \) of \( N \) containing \( N^{0,m} \). If \( \alpha \in \mathcal{S} \), it intersects \( U \) and thus \( \alpha \cap U \) is an irreducible component of \( U \). Moreover the closure of \( \alpha \cap U \) in \( N \) coincides with \( \alpha \). Hence there exists an injection \( \mathcal{S} \hookrightarrow \text{Irr}(U) \). Since \( U \) is a scheme of finite type over \( \mathbb{F}_p \), \( \text{Irr}(U) \) is a finite set. Thus \( \mathcal{S} \) is also a finite set.

Therefore, it suffices to show that for every \( \alpha \in \text{Irr}(N) \) there exists \( h \in J \) such that \( h\alpha \in \mathcal{S} \). Fix \( y \in \alpha(\mathbb{F}_p) \). We can take integers \( r, m' \) with \( m' \geq 0 \) such that \( y \) lies in \( N^{r,m'}(\mathbb{F}_p) \). Then \( p^{-r}y \) lies in \( N^{0,m'}(\mathbb{F}_p) \). Let \( x \) be an arbitrary element of \( \text{Ig}_{m'}(\mathbb{F}_p) \); note that \( \text{Ig}_{m'} \neq \emptyset \), as \( C \) is non-empty and \( \text{Ig}_{m'} \longrightarrow C \) is surjective. By Proposition 3.26 ii), there exists \( (x', y') \in \text{Ig}_{m'}(\mathbb{F}_p) \times N^{0,m'}(\mathbb{F}_p) \) such that \( \Pi(x, p^{-r}y) = \Pi(x', y') \). By Proposition 3.27 there exists \( h' \in J \) such that \( y' = h'p^{-r}y \). Put \( h = h'p^{-r} \in J \). Then, \( h\alpha \in \text{Irr}(N) \) intersects \( N^{0,m} \) at \( y' \), and thus \( h\alpha \in \mathcal{S} \). This completes the proof.

## 4 Applications

In this section, we will give some applications of Theorem 2.6. Here we continue to use the notation introduced in Section 2. Let \( Z_J \) be the center of \( J \). First we will give a group-theoretic characterization of the quasi-compactness of \( Z_J/\mathcal{M} \).

**Theorem 4.1** Assume that \( b \) comes from an abelian variety and \( \mathcal{M} \) is non-empty. Then, the following are equivalent:

(a) The group \( J \) is compact-mod-center, namely, \( J/Z_J \) is compact.

(b) The quotient topological space \( Z_J/\mathcal{M} \) is quasi-compact.

**Proof of Theorem 4.1** (a) \( \implies \) (b). Take a compact open subgroup \( J^1 \) of \( J \). Since \( J/Z_J \) is compact, the image of \( J^1 \) in \( J/Z_J \) is a finite index subgroup of \( J/Z_J \). Therefore \( Z_JJ^1 \) is a finite index subgroup of \( J \). Take a system of representatives \( h_1, \ldots, h_k \in J \) of \( Z_JJ^1 \backslash J \).
By Theorem 2.6, we can choose \( \alpha_1, \ldots, \alpha_m \in \text{Irr}(\overline{\mathcal{M}}) \) such that \( \text{Irr}(\overline{\mathcal{M}}) = \bigcup_{i=1}^m J' \alpha_i = \bigcup_{i=1}^m \bigcup_{j=1}^k Z_j J' h_j \alpha_i \). By [Far04, Proposition 2.3.11] and the compactness of \( J', J' h_j \alpha_i \) consists of finitely many elements for each \( i \) and \( j \). Therefore, \( Z_j \setminus \overline{\mathcal{M}} \) is covered by the images of finitely many irreducible components of \( \overline{\mathcal{M}} \). In particular, \( Z_j \setminus \overline{\mathcal{M}} \) is quasi-compact.

To show the converse \((b) \implies (a)\), we use the subsequent lemma, which is similar to [Mie12, Lemma 5.1 iii)]).

**Lemma 4.2** Let \( T_1 \) and \( T_2 \) be quasi-compact subsets of \( \overline{\mathcal{M}} \). Then, the subset \( \{ h \in J \mid h T_1 \cap T_2 \neq \emptyset \} \) of \( J \) is contained in a compact subset of \( J \).

**Proof.** First note that \( T_i \) is contained in a finite union of irreducible components of \( \overline{\mathcal{M}} \). Indeed, take a quasi-compact open subscheme \( U \) of \( \overline{\mathcal{M}} \) containing \( T_i \); then, the closure \( \overline{\sigma} \) of each \( \alpha \in \text{Irr}(U) \) belongs to \( \text{Irr}(\overline{\mathcal{M}}) \), and \( T_i \) is contained in \( \bigcup_{\alpha \in \text{Irr}(U)} \overline{\sigma} \). Therefore, the closure \( \overline{T}_i \) of \( T_i \) in \( \overline{\mathcal{M}} \) is quasi-compact. Replacing \( T_1, T_2 \) by \( \overline{T}_1, \overline{T}_2 \), we may assume that \( T_1 \) and \( T_2 \) are closed. For \( T = T_1 \cup T_2 \), \( \{ h \in J \mid h T_1 \cap T_2 \neq \emptyset \} \) is contained in \( \{ h \in J \mid h T \cap T \neq \emptyset \} \). Therefore, it suffices to consider the case \( T = T_1 = T_2 \). By Lemma 3.2, we have a \( J \)-equivariant proper morphism \( f : \overline{\mathcal{M}} \to \overline{\mathcal{N}} \). As \( \{ h \in J \mid h T \cap T \neq \emptyset \} \) is contained in \( \{ h \in J \mid f(T) \cap f(T) \neq \emptyset \} \), we may replace \( \overline{\mathcal{M}} \) by \( \overline{\mathcal{N}} \) and \( \overline{\mathcal{N}}_{\text{GL}} \) by \( (\overline{\mathcal{N}}_{\text{GL}})_{\text{red}} \) and \( J \) by \( J_{\text{GL}} \). In this case, the claim is essentially proved in the proof of [RZ96, Proposition 2.34].

**Corollary 4.3** i) Let \( T \) be a non-empty quasi-compact subset of \( \overline{\mathcal{M}} \). Then, the subset \( \{ h \in J \mid h T = T \} \) of \( J \) is open and compact.

ii) Let \( T \) be a finite union of irreducible components of \( \overline{\mathcal{M}} \). Then, the subset \( \{ h \in J \mid h T \cap T \neq \emptyset \} \) of \( J \) is open and compact.

**Proof.** i) It can be proved in the same way as in [Mie12, Lemma 5.1 iv)].

We prove ii). The openness follows from i). Write \( T = \alpha_1 \cup \cdots \cup \alpha_m \), where \( \alpha_i \in \text{Irr}(\overline{\mathcal{M}}) \). Consider the subset \( S \) of \( \text{Irr}(\overline{\mathcal{M}}) \) consisting of \( \beta \) satisfying \( \beta \cap T \neq \emptyset \). It is a finite set (see the proof of Theorem 2.6 in p. 20). Write \( S = \{ \beta_1, \ldots, \beta_k \} \).

Then, we have \( \{ h \in J \mid h T \cap T \neq \emptyset \} = \bigcup_{i=1}^m \bigcup_{j=1}^k \{ h \in J \mid h \alpha_i = \beta_j \} \). Hence it suffices to show that \( \{ h \in J \mid h \alpha_i = \beta_j \} \) is compact for every \( i \) and \( j \). We may assume that there exists \( h_0 \in J \) such that \( h_0 \alpha_i = \beta_j \). In this case, we have \( \{ h \in J \mid h \alpha_i = \beta_j \} = h_0 J_{\alpha_i} \), where \( J_{\alpha_i} = \{ h \in J \mid h \alpha_i = \alpha_i \} \). By i), \( J_{\alpha_i} \) is compact, and thus \( h_0 J_{\alpha_i} \) is also compact. This concludes the proof.

**Proof of Theorem 4.1 (b) \( \implies \) (a).** Assume that \( Z_j \setminus \overline{\mathcal{M}} \) is quasi-compact. Then, there exist finitely many irreducible components \( \alpha_1, \ldots, \alpha_m \) whose images \( \overline{\alpha_1}, \ldots, \overline{\alpha_m} \) cover \( Z_j \setminus \overline{\mathcal{M}} \). Put \( T = \alpha_1 \cup \cdots \cup \alpha_m \) and \( K_T = \{ h \in J \mid h T \cap T \neq \emptyset \} \). Let us
observe that \( J = Z_f K_T \). Take any element \( h \in J \). Since \( Z_f \setminus \overline{M} \) is non-empty, we can find \( x \in T \). Then there exists \( z \in Z_f \) such that \( zh \in T \). Thus we have \( zh \in K_T \) and \( h \in Z_f K_T \), as desired.

By Corollary 4.3 ii), \( K_T \) is compact. Hence \( J/Z_f \) is also compact.

Next, we will prove a finiteness result on the \( \ell \)-adic cohomology of the Rapoport-Zink tower. Before stating the result, we recall some definitions. For a precise description, see [RZ96, Chapter 5].

We denote the rigid generic fiber \( t(\mathcal{M})_0 \) of \( \mathcal{M} \) by \( M \) (note that in this paper all rigid spaces are considered as adic spaces; cf. [Hub91], [Hub96]). The universal object on \( \mathcal{M} \) induces a system of étale \( p \)-divisible groups \( \{ \tilde{X}_L \}_{L \in \mathcal{L}} \) on \( M \). Assume \( M \) is non-empty, and fix a point \( x_0 \) of \( M \) and a geometric point \( \overline{x}_0 \) lying over \( x_0 \). Put \( V' = V_p \tilde{X}_{L,\overline{x}_0} \), which is independent of \( L \in \mathcal{L} \). Then, the universal quasi-polarization \( p_L : \tilde{X}_L \rightarrow (\tilde{X}_L)^{\vee} \) (cf. the condition (c) in Definition 2.2), well-defined up to \( \mathbb{Q}_{\ell}^{\times} \)-multiplication, induces an alternating bilinear pairing \( \langle \cdot, \cdot \rangle': V' \times V' \rightarrow \mathbb{Q}_{\ell} \), which is well-defined up to \( \mathbb{Q}_{\ell}^{\times} \)-multiplication (here we choose an isomorphism \( \mathbb{Q}_{\ell}(1) \cong \mathbb{Q}_{\ell} \), but the choice does not affect the \( \mathbb{Q}_{\ell}^{\times} \)-orbit of the pairing). Let \( G' \) be the algebraic group over \( \mathbb{Q}_p \) consisting of \( B \)-linear automorphisms of \( V' \) which preserve \( \langle \cdot, \cdot \rangle' \) up to a scalar multiple. By the comparison theorem for \( B \)-linear isomorphisms \( V \otimes_{\mathbb{Q}_p} B_{\text{crys}} \xrightarrow{\cong} V' \otimes_{\mathbb{Q}_p} B_{\text{crys}} \) which maps \( \langle \cdot, \cdot \rangle \) to a \( \mathbb{Q}_{\ell}^{\times} \)-multiple of \( \langle \cdot, \cdot \rangle' \). Thus, there exists a \( B \otimes_{\mathbb{Q}_p} \mathbb{Q}_{\ell}^{\times} \)-linear isomorphism \( V \otimes_{\mathbb{Q}_p} B_{\text{crys}} \xrightarrow{\cong} V' \otimes_{\mathbb{Q}_p} B_{\text{crys}} \) which maps \( \langle \cdot, \cdot \rangle \) to \( \mathbb{Q}_{\ell}^{\times} \)-multiple of \( \langle \cdot, \cdot \rangle' \).

Such a pair \( \langle V', \langle \cdot, \cdot \rangle' \rangle \) is classified by \( H^1(\mathbb{Q}_p, G) \). Let \( \xi \in H^1(\mathbb{Q}_p, G) \) be the element corresponding to the isomorphism class of \( \langle V', \langle \cdot, \cdot \rangle' \rangle \). Then, \( G' \) is the inner form of \( G \) corresponding to the image of \( \xi \) under the map \( H^1(\mathbb{Q}_p, G) \rightarrow H^1(\mathbb{Q}_p, G^{\text{ad}}) \). If \( G \) is connected, there is a formula that describes \( \xi \) by means of \( b \) and \( \mu \); see [RZ96, Proposition 1.20] and [Win97, §4.5.2]. In particular, \( \xi \) is independent of the choice of \( \overline{x}_0 \) in this case.

For \( L \in \mathcal{L} \), we denote the \( O_B \)-lattice \( T_p \tilde{X}_{L,\overline{x}_0} \subset V' \) by \( L' \). Then, \( \mathcal{L}' = \{ L' \mid L \in \mathcal{L} \} \) is a self-dual multi-chain of \( O_B \)-lattices of \( V' \). We denote by \( K'_{\mathcal{L}} \), the subgroup of \( G' = G'(\mathbb{Q}_p) \) consisting of \( g \) with \( gL = L' \) for every \( L \in \mathcal{L}' \). It is compact and open in \( G' \). For an open subgroup \( K' \) of \( K'_{\mathcal{L}} \), let \( M_{K'} \) be the rigid space over \( M \) classifying \( K' \)-level structures on \( \{ \tilde{X}_L \}_{L \in \mathcal{L}} \). Namely, for a connected rigid space \( S \) over \( M \), a morphism \( S \rightarrow M_{K'} \) over \( M \) is functorially in bijection with a \( \pi_1(S, \overline{\tau}) \)-invariant \( K' \)-orbit of systems of \( O_B \)-linear isomorphisms \( \{ \eta_L : L' \xrightarrow{\cong} T_p \tilde{X}_{L,\overline{\tau}} \}_{L \in \mathcal{L}} \) such that

- \( \eta_L \) is compatible with respect to \( L \),
- and \( \eta_L \) maps \( \langle \cdot, \cdot \rangle' \) to a \( \mathbb{Q}_{\ell}^{\times} \)-multiple of the alternating bilinear pairing on \( V_p \tilde{X}_{L,\overline{\tau}} \)

induced from the universal quasi-polarization.

Here \( \overline{\tau} \) is a geometric point in \( S \); the set of \( \pi_1(S, \overline{\tau}) \)-invariant \( K' \)-orbits of \( \{ \eta_L \}_{L \in \mathcal{L}} \) is essentially independent of the choice of \( \overline{\tau} \). We can easily observe that \( M_{K'} \) is finite.
étale over $M$. If $G$ is connected, $M_{K'} \rightarrow M$ is surjective. The action of $J$ on $M$ naturally lifts to an action on $M_{K'}$.

Varying $K'$, we obtain a projective system $\{M_{K'}\}_{K' \subset K''}$ of étale coverings over $M$. As usual, we can define a Hecke action of $G'$ on this tower; see [RZ96, 5.34]. By definition, the tower $\{M_{K'}\}_{K'}$ a priori depends on the point $\mathfrak{p}_0$. However, in fact it is known that $\{M_{K'}\}_{K'}$ depends only on the class $\xi \in H^1(\mathbb{Q}_p, G)$ (cf. [RZ96, 5.39]).

Fix a prime number $\ell \neq p$ and consider the compactly supported $\ell$-adic cohomology:

$$H^i_c(M_{K'}) = H^i_c(M_{K'} \otimes_E \mathbb{Q}_\ell), \quad H^i_c(M_{\xi,\infty}) = \lim_{\rightarrow} H^i_c(M_{K'}).$$

Then, $H^i_c(M_{\xi,\infty})$ becomes a $G' \times J$-representation. The action of $G'$ is clearly smooth. By [Far04, Corollaire 4.4.7], the action of $J$ is also smooth. In fact, it is also known that the Weil group $W_E$ naturally acts on $H^i_c(M_{\xi,\infty})$. These three actions are expected to be closely related to the local Langlands correspondence (see [Rap95]).

In the sequel, we will prove the following fundamental finiteness result on the representation $H^i_c(M_{\xi,\infty})$.

**Theorem 4.4** Assume that $b$ comes from an abelian variety. Then, for every integer $i \geq 0$ and every compact open subgroup $K'$ of $G'$, the $K'$-invariant part $H^i_c(M_{\xi,\infty})^{K'}$ is finitely generated as a $J$-representation.

In the unramified case, this theorem is proved in [Far04, Proposition 4.4.13]. The strategy of our proof is similar. First we will show that $\mathcal{M}$ is locally algebraizable in the sense of [Mic10, Definition 3.19]. Fix $\mathcal{B}$ and $(A_0, \lambda_0, \iota_0)$ as in Definition 2.4.

**Lemma 4.5** We can replace $(A_0, \lambda_0, \iota_0)$ so that the following conditions are satisfied:

- there exists an order $\mathfrak{O}$ of $\mathcal{B}$ which is contained in $\iota_0^{-1}(\text{End}(A_0))$, stable under $*$ and satisfies $\mathfrak{O} \otimes \mathbb{Z}_{\mathfrak{p}} = \mathcal{O}_B$.
- there exists a finite extension $F$ of $K_0 = \text{Frac}(\mathbb{F}_p)$ such that $(A_0, \lambda_0, \iota_0)$ lifts to an object $(\widetilde{A}_0, \widetilde{\lambda}_0, \widetilde{\iota}_0)$ over $\mathcal{O}_F$.

**Proof.** Take a finite extension $F$ of $\mathcal{E}$ such that $x_0 \in M(F)$. Then, the point $x_0$ corresponds to an object $\{(X_L, \tilde{i}_L, \tilde{\rho}_L)\}_{L \in \mathcal{L}}$ over $\mathcal{O}_F$ as in Definition 2.2. Let $\{(X_L, \iota_L, \rho_L)\}_{L \in \mathcal{L}}$ be the element in $\mathcal{M}(\mathbb{F}_p)$ that is obtained as the reduction of $\{(X_L, \tilde{i}_L, \tilde{\rho}_L)\}_{L \in \mathcal{L}}$.

Fix $L \in \mathcal{L}$ such that $L \subset L'$. There exist an abelian variety $A_0'$ over $\mathbb{F}_p$ and a $p$-quasi-isogeny $\phi: A_0 \rightarrow A_0'$ such that $\phi[p^\infty]: A_0[p^\infty] \rightarrow A_0'[p^\infty]$ can be identified with $\rho_L: X \rightarrow X_L$. By the conditions (a), (c) in Definition 2.2 there exists an integer $m$ such that $(\rho_L')^{-1} \circ p^m \lambda_0 \circ \rho_L$ gives an isogeny $X_L \rightarrow X^\vee_L$ of height $\log_p \#(L'/L)$. Consider the quasi-isogeny $\lambda^*_0 = (\phi^\vee)^{-1} \circ p^m \lambda_0 \circ \phi^{-1}: A_0' \rightarrow A_0$. Passing to $p$-divisible groups, we can easily observe that it is a polarization on $A_0'$.
On the other hand, let \( \iota_0 \) be the composite of \( \tilde{B} \xrightarrow{\iota} \End(A_0) \otimes \mathbb{Z} \mathbb{Q} \xrightarrow{(\iota)} \End(A'_0) \otimes \mathbb{Z} \mathbb{Q} \), where \( (\iota) \) is the isomorphism induced by \( \phi \). Then, by the same way as in the proof of Lemma \[3.4\], we can observe that \((A'_0, \lambda'_0, \iota'_0)\) satisfies the first condition in the lemma. By construction the polarized \( \mathcal{B} \)-isocrystal associated to \((A'_0, \lambda'_0, \iota'_0)\) is isomorphic to \( N_0 \). Hence we may replace \((A_0, \lambda_0, \iota_0)\) by \((A'_0, \lambda'_0, \iota'_0)\).

Finally, by the Serre-Tate theorem we obtain a formal lifting \((\tilde{A}_0, \tilde{\lambda}_0, \tilde{\iota}_0)\) to \( \mathcal{O}_F \) corresponding to \((X_L, \tilde{\iota}_L)\). The existence of the polarization \( \tilde{\lambda}_0 \) tells us that \( \tilde{A}_0 \) is algebraizable. This concludes the proof. \( \blacksquare \)

We take \((A_0, \lambda_0, \iota_0)\) and \( \tilde{\mathcal{O}} \) as in the lemma above, and fix a lift \((\tilde{A}_0, \tilde{\lambda}_0, \tilde{\iota}_0)\) over \( \mathcal{O}_F \). Let \( I \) be the algebraic group over \( \mathbb{Q} \) consisting of \( \tilde{\mathcal{O}} \)-linear self-quasi-isogenies of \( A_0 \) preserving \( \lambda_0 \) up to a scalar multiple. The functor of taking \( p \)-divisible groups induces an injection \( I(\mathbb{Q}) \hookrightarrow J(\mathbb{Q}_p) \).

We fix an embedding \( \mathcal{O}_F \hookrightarrow \mathbb{C} \) and take the base change \((\tilde{A}_0, \tilde{\lambda}_0, \tilde{\iota}_0)\) of \((A_0, \lambda_0, \iota_0)\) under this embedding. We put \( \Lambda = H_1(\tilde{A}_0, \mathbb{C}, \mathbb{Z}) \) and \( W = H_1(\tilde{A}_0, \mathbb{Q}, \mathbb{C}) \). The \( \tilde{\mathcal{O}} \)-action \( \tilde{\lambda}_0, \mathbb{C} \) on \( \tilde{A}_0, \mathbb{C} \) makes \( \Lambda \) (resp. \( W \)) an \( \tilde{\mathcal{O}} \)-module (resp. \( \tilde{\mathcal{B}} \)-module). The polarization \( \tilde{\lambda}_0, \mathbb{C} \) induces a \( * \)-Hermitian alternating bilinear pairing \( \langle \cdot, \cdot \rangle_{\tilde{\lambda}_0, \mathbb{C}} : \Lambda \times \Lambda \rightarrow \mathbb{Z} \). Clearly we have an isomorphism \( \Lambda \otimes \mathbb{Z} \tilde{\mathbb{Z}}^p \cong \lim_{\leftarrow (n, p) = 1} A_0[n]/(\mathbb{F}_p) \) compatible with additional structures, where we write \( \tilde{\mathbb{Z}}^p = \lim_{\leftarrow (n, p) = 1} \mathbb{Z}/n\mathbb{Z} \).

We denote by \( H \) the algebraic group over \( \mathbb{Q} \) consisting of \( \tilde{\mathcal{B}} \)-linear automorphisms of \( W \) which preserve \( \langle \cdot, \cdot \rangle_{\tilde{\lambda}_0, \mathbb{C}} \) up to a scalar multiple. We have a natural injection \( I(\mathbb{Q}) \hookrightarrow H(\mathbb{A}_f^p) \), where \( \mathbb{A}_f^p = \tilde{B}^p \otimes \mathbb{Z} \mathbb{Q} \).

Remark 4.6 If \( G \) is connected, we can also compare \( V \otimes_{\mathbb{Q}_p} K_0 = \mathbb{D}(A_0[p^\infty])_{\mathbb{Q}} \) with \( W \otimes_{\mathbb{Q}} K_0 \). By the comparison result between de Rham and crystalline theories (cf. [BOS83]), we have a canonical isomorphism \( V \otimes_{\mathbb{Q}_p} \mathbb{C} \cong H_1(\tilde{A}_0, \mathbb{C}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \). Therefore, by Steinberg’s theorem \( H^1(K_0, G) = 1 \), we have an isomorphism \( V \otimes_{\mathbb{Q}_p} K_0 \cong H_1(\tilde{A}_0, \mathbb{C}, \mathbb{Q}) \otimes_{\mathbb{Q}} K_0 \) compatible with various additional structures. However, we do not need this result.

We will use the following moduli space, which is a slight variant of [RZ96] Proposition 6.9.

Definition 4.7 For a compact open subgroup \( K^p \) of \( H(\mathbb{A}_f^p) \), let \( \mathcal{O}_K^p \) be the functor from the category of \( \mathcal{O}_E \)-schemes to \( \text{Set} \) that associates \( S \) to the set of isomorphism classes of \((A, \tilde{\lambda}, \alpha) \) where

- \( A = \{A_L\} \) is an \( \mathcal{L} \)-set of \( \tilde{\mathcal{O}} \)-abelian schemes over \( S \) (cf. [RZ96] Definition 6.5]; recall that we work on the category of \( \tilde{\mathcal{O}} \)-abelian schemes up to isogeny of order prime to \( p \)) such that for each \( L \in \mathcal{L} \) the determinant condition \( \det_{\mathcal{O}_S}(a; \text{Lie } A_L) = \det_{K}(a; V_0) \) in \( a \in \tilde{\mathcal{O}} \) holds,
- \( \tilde{\lambda} \) is a \( \mathbb{Q} \)-homogeneous principal polarization of \( A \) (cf. [RZ96] Definition 6.7)].
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– and $\alpha$ is a $K^p$-level structure

$$\alpha: H_1(A, A_f^p) \xrightarrow{\cong} W \otimes_{Q} A_f^p \pmod{K^p}$$

which is $\tilde{B} \otimes_{Q} A_f^p$-linear and preserves the pairings up to $(A_f^p)^\times$-multiplication (for the definition of $H_1(A, A_f^p)$, see [RZ96, 6.8]).

As in [RZ96, p. 279], we can easily show that the functor $\mathfrak{Z}_{K^p}$ is represented by a quasi-projective scheme over $\mathcal{O}_E$.

The difference between [RZ96, Definition 6.9] and this definition is that in our case $L$ is a self-dual multi-chain of $\mathcal{O}_B$-lattices in $V$, not in $W \otimes_{Q} Q_p$. However, the proof of [RZ96, Theorem 6.23] can be applied to our case without any change, and we obtain the following $p$-adic uniformization result.

**Proposition 4.8**  
1) Let $\widehat{\mathfrak{Z}}_{K^p}$ be the $p$-adic completion of $\mathfrak{Z}_{K^p}$. We have a morphism of pro-formal schemes over $\mathcal{O}_E$:

$$\Theta: \mathcal{M} \times H(A_f^p)/K^p \rightarrow \widehat{\mathfrak{Z}}_{K^p}.$$  

2) The group $I(Q)$ is discrete in $J(Q_p) \times H(A_f^p)$.

3) If $K^p$ is small enough, the quotient $I(Q)\backslash \mathcal{M} \times H(A_f^p)/K^p$ is a formal scheme. It is a countable disjoint union of spaces of the form $\Gamma \backslash \mathcal{M}$, where $\Gamma$ is a subgroup of $J(Q_p)$ of the form $(J(Q_p) \times hK^p h^{-1}) \cap I(Q)$ with $h \in H(A_f^p)$. Such a subgroup $\Gamma$ is torsion-free and discrete in $J(Q_p)$.

4) Let $\mathcal{T}$ be the set of closed subsets of $\mathfrak{Z}_{K^p} \otimes_{\mathcal{O}_E} \mathbb{F}_p$ which are the images of irreducible components of $\mathcal{M} \times H(A_f^p)/K^p$ under $\Theta$. Then, each $T \in \mathcal{T}$ intersects only finitely many members of $\mathcal{T}$.

5) The morphism $\Theta$ induces an isomorphism of formal schemes over $\mathcal{O}_E$:

$$\Theta: I(Q)\backslash \mathcal{M} \times H(A_f^p)/K^p \xrightarrow{\cong} (\mathfrak{Z}_{K^p})/\mathcal{T}.$$  

The target $(\mathfrak{Z}_{K^p})/\mathcal{T}$ is the formal completion of $\mathfrak{Z}_{K^p}$ along $\mathcal{T}$; see [RZ96, 6.22] for a precise definition.

The claim i) is proved in [RZ96, Theorem 6.21] (under the setting therein). The remaining statements are found in [RZ96, Theorem 6.23] and its proof.

By the same argument as in the proof of [Far04, Corollaire 3.1.4], we obtain the following algebraization result.

**Corollary 4.9** For every quasi-compact open formal subscheme $\mathcal{U}$ of $\mathcal{M}$, we can find a compact open subgroup $K^p$ of $H(A_f^p)$ and a locally closed subscheme $Z$ of $\mathfrak{Z}_{K^p} \otimes_{\mathcal{O}_E} \mathbb{F}_p$ such that $\mathcal{U}$ is isomorphic to the formal completion of $\mathfrak{Z}_{K^p}$ along $Z$. In particular, $\mathcal{M}$ is locally algebraizable in the sense of [Mic10, Definition 3.19].
Next we consider level structures at $p$. We write $Y_{K,p}^{\eta}$ for the generic fiber $Y_{K} \otimes_{\hat{O}_E} \mathcal{E}$, and $\tilde{A} = \{ \tilde{A}_L \}$ the universal $L$-set of $\hat{O}$-abelian schemes over $Y_{K,p}^{\eta}$. For an open subgroup $K' \subset K_{p'}$, consider a $K'$-level structure
\[ \{ \mathcal{L} \xrightarrow{\sim} T_p \tilde{A}_L \pmod{K'} \}_{L \in \mathcal{L}} \]
(the definition is similar as that in the definition of $M_{K'}$). Let $Y_{K,p}^{\eta}$ be the scheme classifying such level structures, which is finite étale over $Y_{K,p}^{\eta}$. The following corollary follows directly from the proof of [Far04, Corollaire 3.1.4]:

**Corollary 4.10** Let $\mathcal{U}$ be a quasi-compact open formal subscheme of $\mathcal{M}$, and $K'$ be an open subgroup of $K'_{p'}$. Put $U = t(\mathcal{U})_{\eta}$ and denote by $U_{K'}$ the inverse image of $U$ under $M_{K'} \rightarrow M$. The locally closed subscheme $Z$ of $Y_{K',p}^{\eta}$ in Corollary 4.9 can be taken so that we have the following cartesian diagram:

\[
\begin{array}{ccc}
U_{K'} & \rightarrow & (Y_{K',p}^{\eta})^{\text{ad}} \\
\downarrow & & \downarrow \\
U & \xrightarrow{\sim} & t((Y_{K'}^{\eta})/Z_{\eta})^{\text{copen}} (Y_{K,p}^{\eta})^{\text{ad}}.
\end{array}
\]

Here $(-)^{\text{ad}}$ denotes the associated adic space, and $(-)/Z$ the formal completion along $Z$.

**Corollary 4.11** Let the setting as in Corollary 4.10. Then, $H^i_c(U_{K'} \otimes_{\hat{O}_E} \mathcal{E}, \mathcal{F}_p)$ is a finite-dimensional $\mathbb{Q}_p$-vector space.

**Proof.** By Zariski’s main theorem, there exists a scheme $Y_{K,K'}^{\eta}$ finite over $Y_{K,p}^{\eta}$ that contains $Y_{K',p}^{\eta}$ as a dense open subscheme. Since $Y_{K,K'}^{\eta}$ is finite over $Y_{K,p}^{\eta}$, we conclude that the generic fiber of $Y_{K,K'}^{\eta}$ coincides with $Y_{K',p}^{\eta}$. We write $Z'$ for the inverse image of $Z$ under $Y_{K,K'}^{\eta} \rightarrow Y_{K,p}^{\eta}$. Then, by Corollary 4.10 we have an isomorphism $U_{K'} \cong t((Y_{K,K'}^{\eta})/Z_{\eta})$. Thus the finiteness follows from [Hub98b, Lemma 3.13 i], [Hub98a, Corollary 2.11] and [Hub98c, Lemma 3.3 (ii)].

Now we can give a proof of Theorem 4.4.

**Proof of Theorem 4.4.** The proof is similar as that of [Far04, Proposition 4.4.13] (see also [Mie12, §5]). We include it for the sake of completeness.

Recall that every $J$-submodule of a finitely generated smooth $J$-representation is again finitely generated. If $J$ is connected, this result is found in [Ber84, Remarque 3.12]; the general case is immediately reduced to the connected case. In particular we may assume that $K' \subset K_{p'}$. In this case, we have $H^i_c(M_{\infty}^{K'}) = H^i_c(M_{K'})$. We will show that it is finitely generated as a $J$-representation.

For simplicity, write $\mathcal{I} = \text{Irr}(\mathcal{M})$. For $\alpha \in \mathcal{I}$, put $\overline{U}_\alpha = \overline{\mathcal{M}} \setminus \bigcup_{\beta \in \mathcal{I}, \alpha \cap \beta \neq \emptyset} \beta$. It is a quasi-compact open subscheme of $\overline{\mathcal{M}}$. Note that there exists only finitely many
β ∈ I with \( \overline{U}_α \cap \overline{U}_β \neq \emptyset \) (cf. [Mic12, Corollary 5.2]). Let \( U_α \) be the open formal subscheme of \( M \) satisfying \( (U_α)^{\text{red}} = \overline{U}_α \), and \( U_α \) the rigid generic fiber of \( U_α \). The inverse image \( U_{α,K'} \) of \( U_α \) under \( M_K' \to M \) gives an open covering \( \{ U_{α,K'} \}_{α \in I} \) of \( M_{K'} \). For \( h \in J \), we have \( hU_{α,K'} = U_{ha,K'} \).

For a finite non-empty subset \( \underline{α} = \{ α_1, \ldots, α_m \} \) of \( I \), we put \( U_{\underline{α},K'} = \bigcap_{j=1}^m U_{α_j,K'} \). By Corollary 4.11 for \( \bigcap_{j=1}^m U_{α_j} \), \( H_c^i(U_{\underline{α},K'}) = H_c^i(U_{α_j,K'} \otimes E, \overline{Q}_ℓ) \) is a finite-dimensional \( \overline{Q}_ℓ \)-vector space. Put \( J_{\underline{α}} = \{ h \in J \mid h\overline{α} = \overline{α} \} \). By Corollary 4.3 i), \( J_{\underline{α}} \) is a compact open subgroup of \( J \). It acts smoothly on \( H_c^i(U_{\underline{α},K'}) \).

For an integer \( s \geq 1 \), let \( \mathcal{I}_s \) be the set of subsets \( \underline{α} \subseteq I \) such that \( \#\overline{α} = s \) and \( U_{\underline{α},K'} \neq \emptyset \). Note that there exists an integer \( N \) such that \( \mathcal{I}_s = \emptyset \) for \( s > N \). Indeed, by Theorem 2.6 we can take a finite system of representatives \( α_1, \ldots, α_k \) of \( J \setminus \text{Irr}(M) \); then we may take \( N \) as the maximum of \( \#\{ β \in \text{Irr}(M) \mid \overline{U}_α \cap \overline{U}_β \neq \emptyset \} \) for \( 1 \leq j \leq k \).

The group \( J \) naturally acts on \( \mathcal{I}_s \). We will observe that this action has finite orbits. Let \( \mathcal{I}_s^* \) be the subset of \( \mathcal{I}_s \) consisting of \( (α_1, \ldots, α_s) \) such that \( α_1, \ldots, α_s \) are mutually disjoint and \( U_{(α_1,\ldots,α_s),K'} \neq \emptyset \). As we have a \( J \)-equivariant surjection \( \mathcal{I}_s^* \to \mathcal{I}_s; (α_1, \ldots, α_s) \mapsto \{ α_1, \ldots, α_s \} \), it suffices to show that the action of \( J \) on \( \mathcal{I}_s^* \) has finite orbits. Consider the first projection \( \mathcal{I}_s^* \to \mathcal{I}_s \), which is obviously \( J \)-equivariant. The fiber of this map is finite, since for each \( α \in \mathcal{I}_s \) there exist only finitely many \( β \in \mathcal{I}_s \) such that \( U_{α,K'} \cap U_{β,K'} \neq \emptyset \). On the other hand, by Theorem 2.6 the action of \( J \) on \( \mathcal{I}_s \) has finite orbits. Hence the action of \( J \) on \( \mathcal{I}_s^* \) also has finite orbits, as desired. For each \( s \geq 1 \), we fix a system of representatives \( α_{s,1}, \ldots, α_{s,k_s} \) of \( J \setminus \mathcal{I}_s \).

Consider the Čech spectral sequence

\[
E_1^{-s,t} = \bigoplus_{\alpha \in \mathcal{I}_{s+1}} H_c^i(U_{\underline{α},K'}) \Rightarrow H_c^{-s+t}(M_{K'})
\]

with respect to the covering \( \{ U_{α,K'} \}_{α \in I} \). This spectral sequence is \( J \)-equivariant, and we can easily see that \( E_1^{-s,t} \cong \bigoplus_{j=1}^{k_s} \text{Ind}_J^J H_c^i(U_{α_j,K'}) \) as \( J \)-representations. The right hand side is obviously finitely generated. Thus so are the \( E_i \)-terms. Hence, by the property of finitely generated smooth \( J \)-representations recalled at the beginning of the proof, we may conclude that \( H_c^i(M_{K'}) \) is finitely generated as a \( J \)-representation for every \( i \).

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