MIXED MOTIVES AND MOTIVIC BIRATIONAL COVERS

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Dedicated to Professor Charles Weibel on the occasion of his 65th birthday.

ABSTRACT. We introduce a tower of localizing subcategories in Voevodsky’s big (closed under infinite coproducts) triangulated category of motives. We show that the tower induces a finite filtration on the motivic cohomology groups of smooth schemes over a perfect field. With rational coefficients, this finite filtration satisfies several of the properties of the still conjectural Bloch-Beilinson-Murre filtration.

1. Introduction

The main goal of this paper is to present an alternative approach to the conjectural Bloch-Beilinson-Murre filtration [4], [6], [19] in the context of Voevodsky’s triangulated category of motives $DM$. Traditionally, the Bloch-Beilinson-Murre filtration is understood as an outcome of the conjectural motivic $t$-structure [4, p. 20-22], [11, Conj. 4.8], [2]. Due to the lack of progress, it seems to the author that it is worth it to relax the conditions and look instead for a tower where the truncation functors are triangulated. The great advantage of this approach is that the finiteness of the proposed filtration follows in a straightforward way from the construction, whereas in the traditional approach this property seems to be the most inaccessible one [24], [25]. This method was introduced by Voevodsky in his successful approach to the spectral sequence relating motivic cohomology and algebraic $K$-theory [29], [30].

Our approach can be sketched quickly as follows. For a smooth scheme $X$ of finite type over a perfect field $k$, the Chow groups can be computed in Voevodsky’s triangulated category of motives $DM$ [28]:

$$CH^q(X)_R \cong \text{Hom}_{DM}(M(X)(-q)[-2q], 1_R);$$

where $CH^q(X)_R$ is the Chow group with $R$-coefficients, and $1_R$ is the motive of a point with $R$-coefficients. Since $DM$ is a triangulated category, it is possible to construct the filtration by considering a tower in $DM$, (see [31]):

$$\cdots \rightarrow b\mathbb{C}_{\leq -3}(1_R) \rightarrow b\mathbb{C}_{\leq -2}(1_R) \rightarrow b\mathbb{C}_{\leq -1}(1_R) \rightarrow 1_R$$

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where $bc_{\leq -n}$ is a triangulated functor $DM \to DM$ and defining the $p$-component of the filtration $F^pCH^q(X)$ to be the image of the induced map:

$$\text{Hom}_{DM}(M(X)(-q)[-2q], bc_{\leq -p}1_R) \to \text{Hom}_{DM}(M(X)(-q)[-2q], 1_R) \cong CH^q(X)_R$$

The finiteness of the filtration $F^*CH^q(X)$ is proved in \[52.4\]. Another advantage of this approach is that the filtration is defined (and is finite) for any coefficient ring $R$ (not just the rationals) and for any smooth $k$-scheme of finite type (not necessarily projective), whereas it is known that with integral coefficients it is not possible to construct a motivic $t$-structure in $DM$ [27, Prop. 4.3.8].

Now, we describe the contents of the paper. In \[1.1\] we fix the notation and introduce the basic definitions that will be used in the rest of the paper. In \[2\] we review some general facts from triangulated categories that will be used to construct the birational towers. In \[3\] we introduce the birational covers and the birational tower which will be used to construct the filtrations we are interested in, see \[3.2.1\].

In \[4\] we compare the zero birational cover constructed in this paper with the Kahn-Sujatha unramified cohomology functor. In \[5\] we study the birational tower for motivic cohomology and describe the main properties of the induced filtration on the motivic cohomology groups of smooth (projective) $k$-schemes, see \[5.2.4\], \[5.3.2\], \[5.3.6\]. Finally, in \[6\] we include some comments on the Bloch-Beilinson-Murre filtration.

1.1. Definitions and Notation. In this paper $k$ will denote a perfect base field, $Sch_k$ the category of $k$-schemes of finite type and $Sm_k$ the full subcategory of $Sch_k$ consisting of smooth $k$-schemes regarded as a site with the Nisnevich topology.

Let $Cor_k$ denote the Suslin-Voevodsky category of finite correspondences over $k$, having the same objects as $Sm_k$, morphisms $c(U, V)$ given by the group of finite relative cycles on $U \times_k V$ over $U$ and composition as in \[31\], p. 673 diagram (2.1)]. By taking the graph of a morphism in $Sm_k$, we obtain a functor $\Gamma : Sm_k \to Cor_k$. A Nisnevich sheaf with transfers is an additive contravariant functor $F$ from $Cor_k$ to the category of abelian groups such that the restriction $F \circ \Gamma$ is a Nisnevich sheaf. We will write $Shv^{tr}$ for the category of Nisnevich sheaves with transfers which is an abelian category [18, 13.1]. For $X \in Sm_k$, let $Z_{tr}(X)$ be the Nisnevich sheaf with transfers represented by $X$ [18, 2.8 and 6.2].

Let $K(Shv^{tr})$ be the category of chain complexes (unbounded) on $Shv^{tr}$ equipped with the injective model structure [5, Prop. 3.13], and let $D(Shv^{tr})$ be its homotopy category. We will write $K^b(Shv^{tr})$ for the left Bousfield localization \[8, 3.3\] of $K(Shv^{tr})$ with respect to the set of maps $\{Z_{tr}(X \times_k A^1)[n] \to Z_{tr}(X)[n] : X \in Sm_k, n \in \mathbb{Z}\}$ induced by the projections $p : X \times_k A^1 \to X$. Voevodsky’s big triangulated category of effective motives $DM^{eff}$ is the homotopy category of $K^b(Shv^{tr})$ [27].

Let $T \in K^b(Shv^{tr})$ be the chain complex of the from $Z_{tr}(\mathbb{G}_m)[1]$ [18, 2.12], where $\mathbb{G}_m$ is the $k$-scheme $A^1\setminus\{0\}$ pointed by 1. We will write $Spt_T(Shv^{tr})$ for the category of symmetric $T$-spectra on $K^b(Shv^{tr})$ equipped with the model structure defined in [9, 8.7 and 8.11], [1, Def. 4.3.29]. Voevodsky’s big triangulated category of motives $DM$ is the homotopy category of $Spt_T(Shv^{tr})$ [27].

We will write $M(X)$ for the image of $Z_{tr}(X) \in D(Shv^{tr})$, $X \in Sm_k$ under the $A^1$-localization map $D(Shv^{tr}) \to DM^{eff}$. Let $\Sigma^\infty : DM^{eff} \to DM$ be the suspension functor [9, 7.3] (which is denoted by $F_0$ in loc. cit.), we will abuse notation.
and simply write $E$ for $\Sigma^\infty E$, $E \in DM_{\text{eff}}$. Given a map $f : X \to Y$ in $Sm_k$, we will write $f : M(X) \to M(Y)$ for the map induced by $f$ in $DM$.

By construction, $DM_{\text{eff}}$ and $DM$ are tensor triangulated categories [1, Thm. 4.3.76 and Prop. 4.3.77] with unit $1 = M(\text{Spec}(k))$. We will write $E(1)$ for $E \otimes \mathbb{Z}_{tr}(G_m)[-1]$, $E \in DM$ and inductively $E(n) = (E(n-1))(1)$, $n \geq 0$. We observe that the functor $DM \to DM$, $E \mapsto E(1)$ is an equivalence of categories [9, 8.10], [1, Thm. 4.3.38]; we will write $E \mapsto E(-1)$ for its inverse, and inductively $E(-n) = (E(-n+1))(-1)$, $n > 0$. By convention $E(0) = E$ for $E \in DM$.

Let $R$ be a commutative ring with $1$. We will write $E_R$ for $E \otimes 1_R$ where $E \in DM$, and $1_R$ is the motive of a point with $R$-coefficients $M(\text{Spec}(k)) \otimes R$.

We shall use freely the language of triangulated categories. Our main reference will be [21]. Given a triangulated category, we will write $[1]$ (resp. $[n]$) to denote its suspension (resp. desuspension) functor; and for $n > 0$, $[n]$ (resp. $[-n]$) will be the composition of $[1]$ (resp. $[-1]$) iterated $n$-times. If $n = 0$, $[0]$ will be the identity functor.

We will use the following notation in all the categories under consideration: $0$ will denote the zero object, and $\cong$ will denote that a map (resp. a functor) is an isomorphism (resp. an equivalence of categories).

2. Localizing and orthogonal subcategories

In this section we collect several results of Neeman which are the main technical tools for the construction of the filtration on motivic cohomology. All the constructions can be carried out as well at the level of model categories, see [23, §2].

2.1. Let $\mathcal{T}$ be a compactly generated triangulated category in the sense of Neeman [20, Def. 1.7] with set of compact generators $\mathcal{G}$. For $\mathcal{G}' \subseteq \mathcal{G}$, let $\text{Loc}(\mathcal{G}')$ denote the smallest full triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{G}'$ and is closed under arbitrary (infinite) coproducts.

**Definition 2.1.1.** Let $\mathcal{T}' \subseteq \mathcal{T}$ be a triangulated subcategory. We will write $\mathcal{T}'^\perp$ for the full subcategory of $\mathcal{T}$ consisting of the objects $E \in \mathcal{T}$ such that for every $K \in \mathcal{T}'$: $\text{Hom}_\mathcal{T}(K, E) = 0$.

If $\mathcal{T}' = \text{Loc}(\mathcal{G}')$ and $E \in \mathcal{T}'^\perp$, we will say that $E$ is $\mathcal{G}'$-orthogonal.

**Remark 2.1.2.** In order to check that $E \in \text{Loc}(\mathcal{G}')^\perp$ it suffices to see that for every $G \in \mathcal{G}'$ and for every $p \in \mathbb{Z}$: $\text{Hom}_\mathcal{T}(G[p], E) = 0$. In effect, let $\perp E$ be the full subcategory of $\mathcal{T}$ consisting of objects $F$ such that $\text{Hom}_\mathcal{T}(F[p], E) = 0$ for every $p \in \mathbb{Z}$. It is clear that $\perp E$ is closed under infinite coproducts and that it is a full triangulated subcategory of $\mathcal{T}$. Now, our hypothesis implies that $\mathcal{G}' \subseteq \perp E$. Hence, $\text{Loc}(\mathcal{G}') \subseteq \perp E$ by definition of $\text{Loc}(\mathcal{G}')$ (2.1), and this is equivalent to $E \in \text{Loc}(\mathcal{G}')^\perp$ (2.1.1).

**Theorem 2.1.3** (Neeman). Under the same hypothesis as in [21] Let $\mathcal{T}'$ be a triangulated subcategory of $\mathcal{T}$. Assume that $\mathcal{T}'$ is also compactly generated. Then the inclusion $i : \mathcal{T}' \to \mathcal{T}$ admits a right adjoint $r$, which is a triangulated functor.

Let $f = i \circ r$. Then there exists a triangulated functor $s : \mathcal{T} \to \mathcal{T}$ together with natural transformations:

\[
\begin{align*}
\pi : & \text{id} \longrightarrow s \\
\sigma : & s \longrightarrow [1] \circ f
\end{align*}
\]
such that for any $E$ in $T$ the following conditions hold:

(1) There is a natural distinguished triangle in $T$:

$$f(E) \rightarrow E \rightarrow s(E) \rightarrow f(E)[1]$$

(2) $s(E)$ is in $T'^\perp$ (see 2.1.4).

**Proof.** The triangulated categories $T$ and $T'$ are compactly generated. Hence, the existence of the right adjoint $r$ follows from theorem 4.1 in [20]. The remaining results follow from propositions 9.1.19 and 9.1.8 in [21]. □

2.1.5. With the notation of 2.1 and 2.1.3. If $T$ results follow from propositions 9.1.19 and 9.1.8 in [21]. □

**Lemma 2.1.7.** Under the same hypothesis as in 2.1 and with the notation of 2.1.6.

Consider the inclusion $j : Loc(G')^\perp \rightarrow T$.

(1) $Loc(G')^\perp$ is closed under infinite coproducts and it is a full triangulated subcategory of $T$.

(2) The inclusion $j$ is a triangulated functor which commutes with infinite coproducts. In addition, $Loc(G')^\perp$ is a compactly generated triangulated category in the sense of Neeman [20, Def. 1.7] with set of compact generators $G'^\perp = \{s_G : G \in G\setminus G'\}$.

(3) The inclusion $j$ admits a right adjoint $p : T \rightarrow Loc(G')^\perp$, which is also a triangulated functor.

**Proof.** (1): It is clear from the definition (2.1.1) that $Loc(G')^\perp$ is a full triangulated subcategory of $T$.

Now we proceed to show that $Loc(G')^\perp$ is closed under infinite coproducts. Consider an indexing set $\Lambda$ and let $E = \bigoplus_{\lambda \in \Lambda} E_\lambda$ where $E_\lambda \in Loc(G')^\perp$ for all $\lambda$. By 2.1.2 it suffices to see that for all $G \in G'$: $\text{Hom}_T(G[p], E) = 0$ for every $p \in \mathbb{Z}$. However, $G$ is a compact object in $T$: thus we conclude that $\text{Hom}_T(G[p], E) \cong \bigoplus_{\lambda \in \Lambda} \text{Hom}_T(G[p], E_\lambda) = 0$ since $G[p] \in Loc(G')$ and $E_\lambda \in Loc(G')^\perp$.

(2) By 2.1.7(1), $j$ is a triangulated functor that commutes with infinite coproducts. Now, let $G \in G'^\perp \subseteq G$. Since $G$ is compact [20, Def. 1.6] in $T$, it follows from (2.1.6) that $s_G : G$ is also compact in $Loc(G')^\perp$.

Thus, it only remains to see that $G'^\perp$ is a set of generators for $Loc(G')^\perp$ [20,Defs. 1.7, 1.8]. To check this, let $E \in Loc(G')^\perp$ be such that $\text{Hom}_{Loc(G')^\perp}(s_G, E) \cong \text{Hom}_T(G, E) = 0$ (2.1.6) for all $G \in G \setminus G'$. We observe that if $G \in G'$ then $\text{Hom}_T(G, E) = 0$ since $E \in Loc(G')^\perp$ (see 2.1.1). Therefore, $\text{Hom}_T(G, E) = 0$ for all $G \in G = G' \cup (G\setminus G')$, and we conclude that $G \cong 0$ since $G$ is a set of generators for $T$.

(3) This follows by combining 2.1.7(2) and 2.1.3. □

2.1.8. Since the inclusion $j : Loc(G')^\perp \rightarrow T$ is a full embedding, we deduce that the unit of the adjunction $id \rightarrow pj$ is a natural isomorphism.

2.2. **Slice towers and their duals.** Recall that $T$ is a compactly generated triangulated category with set of compact generators $G$. 
2.2.1. Consider a family of subsets of $G$: $S = \{G_n\}_{n \in \mathbb{Z}}$ such that $G_{n+1} \subseteq G_n \subseteq G$ for every $n \in \mathbb{Z}$.

Thus, we obtain a tower of full triangulated subcategories of $T$:

$$(2.2.2) \quad \cdots \subseteq \text{Loc}(G_{n+1}) \subseteq \text{Loc}(G_n) \subseteq \text{Loc}(G_{n-1}) \subseteq \cdots$$

We will call (2.2.2) the slice tower determined by $S$. The reason for this terminology is [29], [10], [32, p. 18]. If we consider the orthogonal categories $\text{Loc}(G_n)^\perp$ (2.1.1), we obtain a tower of full triangulated subcategories of $T$:

$$(2.2.3) \quad \cdots \subseteq \text{Loc}(G_{n-1})^\perp \subseteq \text{Loc}(G_n)^\perp \subseteq \text{Loc}(G_{n+1})^\perp \subseteq \cdots$$

3. Birational Coverings and the Birational Tower

In this section we apply the formalism of §2 to Voevodsky’s triangulated category of motives $DM$, in order to construct a tower of triangulated subcategories of $DM$ which will induce the filtration on the Chow groups that we are interested in. In [23] the construction was done in the Morel-Voevodsky motivic stable homotopy category $SH$. However, as pointed out by the anonymous referee, it is more natural to carry out the construction in $DM$ since various issues of functoriality with respect to Chow correspondences become straightforward in $DM$. Nevertheless the analogue tower in $SH$ is also interesting, since it is possible to show that with finite coefficients the algebraic cycles induced by the Steenrod operations of Voevodsky are elements of high order in the filtration, and it allows to study other theories which are not representable in $DM$ (integrally), e.g. homotopy invariant $K$-theory $KH$ and Voevodsky algebraic cobordism $MGL$. We will study this applications for $SH$ in a future work.

3.1. Generators. It is well known that $DM$ is a compactly generated triangulated category [21] with compact generators [1, Thm. 4.5.67]:

$$(3.1.1) \quad G_{DM} = \{M(X)(p) : X \in \text{Sm}_k; p \in \mathbb{Z}\}.$$  

Let $G_{eff} \subseteq G_{DM}$ be the set consisting of compact objects of the form:

$$(3.1.2) \quad G_{eff} = \{M(X)(p) : X \in \text{Sm}_k; p \geq 0\}.$$  

If $n \in \mathbb{Z}$, we will write $G_{eff}(n) \subseteq G_{DM}$ for the set consisting of compact objects of the form:

$$(3.1.3) \quad G_{eff}(n) = \{M(X)(p) : X \in \text{Sm}_k; p \geq n\}.$$  

3.1.4. By Voevodsky’s cancellation theorem [31], the suspension functor $\Sigma^\infty : DM_{eff} \to DM$ induces an equivalence of categories between $DM_{eff}$ and the full triangulated subcategory $\text{Loc}(G_{eff})$ of $DM$ (2.1). We will abuse notation and write $DM_{eff}$ for $\text{Loc}(G_{eff})$.

3.1.5. We will write $DM_{eff}(n)$ for the full triangulated subcategory $\text{Loc}(G_{eff}(n))$ of $DM_{eff}$ (2.1), and $DM_{eff}(n)^\perp$ for the orthogonal category $\text{Loc}(G_{eff}(n))^\perp$ (2.1.1). Notice that $DM_{eff}(n)$ is compactly generated with set of generators $G_{eff}(n)$ [20, Thm. 2.1(2.1)].
3.2. The birational tower. Consider the family $S = \{G^\text{eff}(n)\}_{n \in \mathbb{Z}}$ of subsets of $G_{DM}$. By construction $S$ satisfies the conditions of 2.2.1 hence we obtain a tower of full triangulated subcategories of $DM$ (2.2.3):

$$\cdots \subseteq DM^{+}(q-1) \subseteq DM^{+}(q) \subseteq DM^{+}(q+1) \subseteq \cdots$$

We will call the tower (3.2.1) the birational tower. The reason for this terminology is [13], [22] Thms. 3.6 and 1.4.(2)] where it is shown that the orthogonal categories have a geometric description in terms of birational conditions. This also explains the shift in the index in 3.2.3.

**Proposition 3.2.2.** The inclusion, $j_q : DM^{+}(q) \to DM$ admits a right adjoint:

$$p_q : DM \to DM^{+}(q),$$

which is also a triangulated functor.

**Proof.** Since $DM$ is compactly generated, we can apply 2.1.7(3). □

3.2.3. Birational covers. We define $bc_{\leq q} = j_{q+1} \circ p_{q+1}$.

The following proposition is well-known.

**Proposition 3.2.4.** The counit $bc_{\leq q} = j_q p_q \theta_q^q \to id$ of the adjunction constructed in 3.2.2 satisfies the following universal property:

For any $E$ in $DM$ and for any $F \in DM^{+}(q+1)$, the map $\theta_q^E : bc_{\leq q}E \to E$ in $DM$ induces an isomorphism of abelian groups:

$$\text{Hom}_{DM}(F, bc_{\leq q}E) \xrightarrow{\theta_q^E} \text{Hom}_{DM}(F, E)$$

**Proof.** If $F \in DM^{+}(q+1)$, then $\text{Hom}_{DM}(F, E) = \text{Hom}_{DM}(j_{q+1}F, E)$. By adjointness:

$$\text{Hom}_{DM}(j_{q+1}F, E) \cong \text{Hom}_{DM^{+}(q+1)}(F, p_{q+1}E).$$

Since $DM^{+}(q+1)$ is a full subcategory of $DM$, we deduce that:

$$\text{Hom}_{DM^{+}(q+1)}(F, p_{q+1}E) = \text{Hom}_{DM}(j_{q+1}F, j_q p_q p_{q+1}E) = \text{Hom}_{DM}(F, bc_{\leq q}E).$$

This finishes the proof. □

**Remark 3.2.5.** By construction $bc_{\leq q}E$ is in $DM^{+}(q+1)$ (3.2.2)-(3.2.3), thus we conclude that the universal property of 3.2.4 characterizes $bc_{\leq q}E$ up to a unique isomorphism.

3.2.6. Since $DM^{+}(q+1) \subseteq DM^{+}(q+2)$ (3.2.1), it follows from (3.2.4)-(3.2.5) that $bc_{\leq q} \circ bc_{\leq q+1} \cong bc_{\leq q}$ and that there exists a canonical natural transformation $bc_{\leq q} \to bc_{\leq q+1}$.

**Corollary 3.2.7.** Let $E$ be in $DM$. Then, the natural map $\theta_q^E : bc_{\leq q}E \to E$ is an isomorphism in $DM$ if and only if $E$ belongs to $DM^{+}(q+1)$.

**Proof.** First, we assume that $\theta_q^E$ is an isomorphism in $DM$. We observe that $bc_{\leq q}E$ is in $DM^{+}(q+1)$ (see 3.2.2 and 3.2.3). Hence, we deduce that $E$ is also in $DM^{+}(q+1)$ since it is a full triangulated subcategory of $DM$.

Finally, we assume that $E$ is in $DM^{+}(q+1)$. Thus, $\theta_q^E$ is a map in $DM^{+}(q+1)$ since $bc_{\leq q}E$ is in $DM^{+}(q+1)$ by construction. Therefore, by 3.2.4 we deduce that $\theta_q^E$ is an isomorphism in $DM^{+}(q+1)$, and hence an isomorphism in $DM$. □
Theorem 3.2.8. There exist triangulated functors:

\[ bc_{q+1/q} : DM \to DM \]

together with natural transformations:

\[ \pi_{q+1} : bc_{\leq q+1} \to bc_{q+1/q} \]
\[ \sigma_{q+1} : bc_{q+1/q} \to [1] \circ bc_{\leq q} \]

such that for any \( E \) in \( DM \) the following conditions hold:

1. There is a natural distinguished triangle in \( DM \):

\[
\begin{array}{c}
bc_{\leq q} E \\
\overset{bc_{\leq q+1} E}{\longrightarrow} \overset{\pi_{q+1}}{\longrightarrow} \overset{bc_{q+1/q} E}{\longrightarrow} \overset{\sigma_{q+1}}{\longrightarrow} (bc_{\leq q} E)[1]
\end{array}
\]

2. \( bc_{q+1/q} E \) is in \( DM^{\perp}(q + 2) \).
3. \( bc_{q+1/q} E \) is in \( (DM^{\perp}(q + 1))^\perp \). Namely, for any \( F \) in \( DM^{\perp}(q + 1) \):

\[ \text{Hom}_{DM}(F, bc_{q+1/q} E) = 0. \]

Proof. Since \( DM \) is compactly generated, we can apply 2.1.7(2) to conclude that the triangulated categories \( DM^{\perp}(q + 1) \subseteq DM^{\perp}(q + 2) \) (see 3.2.1) are compactly generated. Thus, the result follows by combining 2.1.3 (see 3.2.2-3.2.3 and 3.2.6).

\[ \square \]

Theorem 3.2.10. There exist triangulated functors:

\[ bc_{> q} : DM \to DM \]

together with natural transformations:

\[ \pi_{> q} : id \to bc_{> q} \]
\[ \sigma_{> q} : bc_{> q} \to [1] \circ bc_{\leq q} \]

such that for any \( E \) in \( DM \) the following conditions hold:

1. There is a natural distinguished triangle in \( DM \):

\[
\begin{array}{c}
bc_{\leq q} E \\
\overset{bc_{\leq q+1} E}{\longrightarrow} \overset{\pi_{> q}}{\longrightarrow} \overset{bc_{> q} E}{\longrightarrow} \overset{\sigma_{> q}}{\longrightarrow} (bc_{\leq q} E)[1]
\end{array}
\]

2. \( bc_{> q} E \) is in \( (DM^{\perp}(q + 1))^\perp \). Namely, for any \( F \) in \( DM^{\perp}(q + 1) \):

\[ \text{Hom}_{DM}(F, bc_{> q} E) = 0. \]

Proof. Combining 3.1 and 2.1.7(2), we deduce that the triangulated categories \( DM^{\perp}(q + 1) \subseteq DM \) are compactly generated. Thus, the result follows from 2.1.6 (see 3.2.2-3.2.3).

\[ \square \]
Theorem 3.2.12. For any $E$ in $\text{DM}$, there exists the following commutative diagram in $\text{DM}$:

\[
\begin{array}{ccccccccc}
bc_{\leq q}E & \rightarrow & bc_{\leq q+1}E & \rightarrow & bc_{q+1/q}E & \rightarrow & bc_{q+1/q+1}E & \rightarrow & (bc_{\leq q}E)[1] \\
| & & | & & | & & | & & | \\
bc_{\leq q}E & \rightarrow & E & \rightarrow & bc_{q+1}E & \rightarrow & bc_{q+1/q+1}E & \rightarrow & (bc_{\leq q}E)[1] \\
| & & | & & | & & | & & | \\
0 & \rightarrow & bc_{q+1}E & \rightarrow & bc_{q+1/q+1}E & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(bc_{\leq q}E)[1] & \rightarrow & (bc_{\leq q+1}E)[1] & \rightarrow & (bc_{q+1/q+1}E)[1] & \rightarrow & (bc_{\leq q}E)[2] \\
\end{array}
\]

where all the rows and columns are distinguished triangles in $\text{DM}$.

Proof. The result follows from 3.2.8 and the octahedral axiom applied to the following commutative diagram (see 3.2.6):

\[
\begin{array}{cccc}
bc_{\leq q}E & \rightarrow & bc_{\leq q+1}E \\
\downarrow & & \downarrow \\
E & \rightarrow & E \\
\end{array}
\]

\[
\begin{array}{cccc}
bc_{q+1/q}E & \rightarrow & bc_{q+1/q+1}E \\
| & & | \\
bc_{\leq q+1}E & \rightarrow & 0 \\
\downarrow & & \downarrow \\
(bc_{\leq q+1}E)[1] & \rightarrow & (bc_{q+1/q+1}E)[1] \\
\end{array}
\]

\[
\begin{array}{cccc}
E & \rightarrow & E \\
\downarrow & & \downarrow \\
(bc_{\leq q}E)[1] & \rightarrow & (bc_{\leq q}E)[2] \\
\end{array}
\]

3.2.13. The spectral sequence. By 3.2.12 for every $E$ in $\text{DM}$ there is a tower in $\text{DM}$:

(3.2.14) \[
\cdots \rightarrow bc_{\leq -1}(E) \rightarrow bc_{\leq 0}(E) \rightarrow bc_{\leq 1}(E) \rightarrow \cdots \\
\]

We will call (3.2.14) the birational tower of $E$.

Remark 3.2.15. By 3.2.2 and 3.2.3 the birational tower (3.2.14) is functorial with respect to morphisms in $\text{DM}$.

Theorem 3.2.16. Let $G, K$ be in $\text{DM}$. Then there is a spectral sequence of homological type with term $E_{p,q}^1 = \text{Hom}_{\text{DM}}(G, (bc_{p/p-1}K)[q-p])$ and where the abutment is given by the associated graded group for the filtration $F_\bullet$ of $\text{Hom}_{\text{DM}}(G, K)$ defined by the image of $\theta^{bc}_{p,p-1} : \text{Hom}_{\text{DM}}(G, bc_{p/p-1}K) \rightarrow \text{Hom}_{\text{DM}}(G, K)$, or equivalently the kernel of $\pi_{>q} : \text{Hom}_{\text{DM}}(G, bc_{\leq p}K) \rightarrow \text{Hom}_{\text{DM}}(G, K)$.  

Proof. Since $\text{DM}$ is a triangulated category, the result follows from 3.2.12 and 3.2.14. \qed
3.3. Effective covers. Consider again the family $S = \{G^{\text{eff}}(n)\}_{n \in \mathbb{Z}}$ of subsets of $\mathcal{G}_{DM}$ (3.1.1-3.1.3). Since $DM$ and $DM^{\text{eff}}(n)$ are compactly generated (see 3.1 and 3.1.5), it follows from 2.1.5 that the inclusion $i_n : DM^{\text{eff}}(n) \rightarrow DM$ admits a right adjoint $r_n : DM \rightarrow DM^{\text{eff}}(n)$ which is a triangulated functor. We will write $f_n$ for the triangulated functor $i_n \circ r_n : DM \rightarrow DM$.

The functor $f_n$ is the $(n-1)$-effective cover in the slice filtration studied in [29, 10] in the context of the Morel-Voevodsky motivic stable homotopy category $SH$ and Voevodsky’s effective triangulated category of motives $DM^{\text{eff}}$, respectively.

Remark 3.3.1. Since $f_qE \in DM^{\text{eff}}(q)$, an argument parallel to (3.2.4)-(3.2.5) shows that $f_qE$ is characterized up to a unique isomorphism by the following universal property:

For any $F \in DM^{\text{eff}}(q)$, the counit of the adjunction $(i_n, r_n)$, $\epsilon_q^E : f_qE \rightarrow E$ in $DM$ induces an isomorphism of abelian groups:

$$\text{Hom}_{DM}(F, f_qE) \xrightarrow{\epsilon_q^E} \text{Hom}_{DM}(F, E)$$

Lemma 3.3.2. Let $p, q \in \mathbb{Z}$ with $q \geq p$. Then:

1. $f_p(bc_{\leq q}E)$ is in $DM^{\text{eff}}(p) \cap DM^{\bot}(q+1)$.
2. $f_p(bc_{\leq q}E)$ is characterized up to a unique isomorphism by the following universal property:
   
   For any $F \in DM^{\text{eff}}(p) \cap DM^{\bot}(q+1)$, the composition of counits (see 3.2.4 and 3.3.1) $\theta_q^E \circ \epsilon_p^E : f_p(bc_{\leq q}E) \rightarrow E$ in $DM$ induces an isomorphism of abelian groups:

$$\text{Hom}_{DM}(F, f_p(bc_{\leq q}E)) \xrightarrow{\epsilon_p^E} \text{Hom}_{DM}(F, E)$$

Proof. [1]: We observe that $f_p(bc_{\leq q}E)$ is in $DM^{\text{eff}}(p)$ by construction. Now, let $F \in DM^{\text{eff}}(q+1)$. To conclude it suffices to check that $\text{Hom}_{DM}(F, f_p(bc_{\leq q}E)) = 0$.

We observe that $DM^{\text{eff}}(q+1) \subseteq DM^{\text{eff}}(p)$ since $q+1 > p$ (see 3.1.3 and 3.1.5), so $F$ is also in $DM^{\text{eff}}(p)$. Then, by the universal property 3.3.1

$$\text{Hom}_{DM}(F, f_p(bc_{\leq q}E)) \cong \text{Hom}_{DM}(F, bc_{\leq q}E).$$

Finally, $\text{Hom}_{DM}(F, bc_{\leq q}E) = 0$ since $F \in DM^{\text{eff}}(q+1)$ and $bc_{\leq q}E \in DM^{\bot}(q+1)$ (see 3.2.5).

2. Follows directly by combining 3.2.5, 3.3.1 and [1] above.

Proposition 3.3.3. Let $E \in DM$ and $n, q \in \mathbb{Z}$. Then:

1. There is a natural isomorphism $t_{bc}^E(n) : (bc_{\leq q}E)(n) \rightarrow bc_{\leq q+n}(E(n))$ such that $\theta_{q+n}^E \circ t_{bc}^E(n) = \theta_q^E(n)$ (see 3.2.4).
2. There is a natural isomorphism $t_{bc}^E(n) : (f_qE)(n) \rightarrow f_{q+n}(E(n))$ such that $\epsilon_{q+n}^E \circ t_{bc}^E(n) = \epsilon_q^E(n)$ (see 3.3.1).

Proof. We observe that $DM \rightarrow DM$, $E \mapsto E(n)$ is a triangulated equivalence of categories which maps $DM^{\text{eff}}(q)$ surjectively onto $DM^{\text{eff}}(q+n)$ (see 3.1.5 and 3.1.3), and hence also $DM^{\bot}(q+1)$ surjectively onto $DM^{\bot}(q+n+1)$. Hence, the result follows from 3.2.4 3.2.5 (resp. 3.3.1).
4. Kahn-Sujatha unramified cohomology

In this section, we will show that the right adjoint \( p_1 : DM \to DM^\perp(1) \) constructed in \([3.2.2]\) is a non-effective version of the Kahn-Sujatha unramified cohomology functor \( R_{nr} \) defined in \([13, \S 5]\). The author would like to thank the anonymous referee for bringing this crucial fact to our attention.

4.1. Kahn-Sujatha birational motives. Let \( DM^\circ \) be the Kahn-Sujatha triangulated category of birational motives defined in \([13, \text{Def. 3.2.1}]\), \( \nu_{\leq 0} : DM^{\text{eff}} \to DM^\circ \) the corresponding localization functor (see the diagram in \([13, \text{p. 20}]\)), and \( i^\circ : DM^\circ \to DM^{\text{eff}} \) its right adjoint \([13, \text{Thm. 3.3.5}]\).

Remark 4.1.1. Since the right adjoint \( i^\circ \) is fully faithful \([13, \text{Thm. 3.3.5}]\), we conclude that the counit \( \nu_{\leq 0} \circ i^\circ \to \text{id} \) of the adjunction \( (\nu_{\leq 0}, i^\circ) \) is a natural isomorphism.

4.1.2. We will write \( DM^{\text{eff}} \cap DM^\perp(1) \) for the full triangulated subcategory of \( DM^{\text{eff}} \) consisting of objects \( E \) which belong to \( DM^\circ \) and to \( DM^\perp(1) \) \([3.1.5]\). Let \( \iota \) denote the inclusion \( DM^{\text{eff}} \cap DM^\perp(1) \to DM^{\text{eff}} \).

Lemma 4.1.3. With the notation of \([4.1] \) and \([4.1.2] \)

1. The composition \( \nu_{\leq 0} \circ \iota : DM^{\text{eff}} \cap DM^\perp(1) \to DM^\circ \) is an equivalence of categories.

2. For every \( E \in DM^\circ \), \( i^\circ E \in DM^{\text{eff}} \cap DM^\perp(1) \). Moreover, the fully faithful functor \( i^\circ : DM^\circ \to DM^{\text{eff}} \) induces an equivalence \( DM^\circ \cong i^\circ(\nu_{\leq 0} \circ i^\circ) \cong DM^{\text{eff}} \cap DM^\perp(1) \) which is a quasi-inverse to \( \nu_{\leq 0} \circ \iota \).

3. The natural transformation \( \iota \to i^\circ \circ (\nu_{\leq 0} \circ \iota) \) deduced from the unit of the adjunction \( (\nu_{\leq 0}, i^\circ) \) is an isomorphism.

Proof. \([1]\) This follows from the definition of \( DM^\circ \) \([13, \text{Def. 3.2.1}]\) and \([21, \text{Thm. 9.1.16}]\).

\([2]\) Since \( i^\circ : DM^\circ \to DM^{\text{eff}} \) is fully faithful and a right adjoint of \( \nu_{\leq 0} \), it suffices to show that \( i^\circ E \) is in \( DM^\perp(1) \). Now, we observe that by construction \( DM^\circ \) is the localization for the pair \( DM^{\text{eff}}(1) \subseteq DM^{\text{eff}} \) \([21, \text{Def. 9.1.1}]\). The result then follows from \([21, \text{Lem. 9.1.2}]\).

\([3]\) Since \( i^\circ(\nu_{\leq 0} \circ i^\circ) = DM^{\text{eff}} \cap DM^\perp(1) \), it suffices to show that \( i^\circ \to i^\circ \circ (\nu_{\leq 0} \circ i^\circ) \) is an isomorphism which is clear since \( i^\circ \) is fully faithful \([4.1.1] \). \( \square \)

Definition 4.1.4. The Kahn-Sujatha unramified cohomology functor \( R_{nr} : DM^{\text{eff}} \to DM^\circ \) is the right adjoint of \( i^\circ : DM^\circ \to DM^{\text{eff}} \) \([13, \S 5]\).

Recall that \( i_0 \) is the inclusion \( DM^{\text{eff}} \to DM \), which admits a right adjoint \( r_0 \) and that \( f_0 = i_0 \circ r_0 \) \([3.3]\).

Proposition 4.1.5. The functor \( i^\circ \circ R_{nr} : DM^{\text{eff}} \to DM^{\text{eff}} \) is isomorphic to \( (r_0 \circ bc_{\leq 0}) \circ i_0 : DM^{\text{eff}} \to DM^{\text{eff}} \), where \( bc_{\leq 0} \) is the birational cover defined in \([3.2.3]\).

Proof. By \([4.1.3(2)]\), \( (i^\circ \circ R_{nr})E \in DM^{\text{eff}} \cap DM^\perp(1) \) for every \( E \in DM^{\text{eff}} \). Hence, it suffices to show that the natural transformation \( i_0 \circ i^\circ \circ R_{nr} \to i_0 \) deduced from the counit of the adjunction \( (i^\circ, R_{nr}) \) satisfies the universal property of \([3.3.2(2)]\).

Let \( E' \in DM^{\text{eff}} \cap DM^\perp(1) \). Then:

\[ \text{Hom}_{DM}(i_0 E', i_0 \circ i^\circ \circ R_{nr} E) \cong \text{Hom}_{DM^{\text{eff}}}(E', (i^\circ \circ R_{nr}) E) \]


and by adjointness:

\[ \text{Hom}_{DM^{\text{eff}}}(E', (i^o \circ R_{nr}) E) \cong \text{Hom}_{DM^{\text{eff}}}(\nu_{\leq 0} E', R_{nr} E) \cong \text{Hom}_{DM^{\text{eff}}}(i^o \nu_{\leq 0} E', E) \]

On the other hand \( \text{Hom}_{DM^{\text{eff}}}(i^0 \nu_{\leq 0} E', E) \cong \text{Hom}_{DM^{\text{eff}}}(E', E) \) by Exercise 3.8. Hence, we conclude that \( \text{Hom}_DM(i_0 E', i_0 \circ i^o \circ R_{nr} E) \cong \text{Hom}_DM(i_0 E', i_0 E) \). This finishes the proof.

5. The Birational Tower for Motivic Cohomology

We will study the birational tower \( \text{(5.2.14)} \) for the motive of a point \( 1_R \) with coefficients in a commutative ring \( R \). Our goal is to show that the tower induces a finite filtration on the Chow groups, and that it satisfies several of the properties of the still conjectural Bloch-Beilinson-Murre filtration.

5.1. Basic properties.

**Lemma 5.1.1.** \( 1_R \) belongs to \( DM^{\bot}(1) \) and to \( DM^{\text{eff}} \) (3.1.5).

**Proof.** It follows directly from the definition of \( DM^{\text{eff}} \) (3.1.5) that \( 1_R \) belongs to \( DM^{\text{eff}} \). It only remains to show that \( 1_R \in DM^{\bot}(1) \). Combining 3.1.5 and 2.1.2, we conclude that it suffices to see that for every \( X \in Sm_k \) and every \( p, q \in \mathbb{Z} \) with \( p \geq 1 \): \( \text{Hom}_{DM^{\text{eff}}}(M(X)(p)[q], 1_R) = 0 \).

Let \( U = \mathbb{A}^p \setminus \{0\} \) and consider the Gysin distinguished triangle in \( DM^{\text{eff}} \) [15, 15.15]: \( M(X \times U) \to M(X \times \mathbb{A}^p) \to M(X)(p)[2p] \). Hence, it suffices to show that the map induced by the open immersion \( X \times U \to X \times \mathbb{A}^p \): \( \text{Hom}_{DM^{\text{eff}}}(M(X \times \mathbb{A}^p)(q), 1_R) \to \text{Hom}_{DM^{\text{eff}}}(M(X \times U)(q), 1_R) \) is an isomorphism for every \( q \in \mathbb{Z} \). But this follows directly from the computation of motivic cohomology in weight zero [18, 4.2]. □

**Proposition 5.1.2.** Let \( n \geq 0 \) be an arbitrary integer. Then the natural map (see \( 3.2.14 \)), \( \theta_n^R : b_{c \leq n}(1_R) \to 1_R \) is an isomorphism in \( DM \).

**Proof.** Combining 3.2.7 and 3.2.1 it suffices to show that \( 1_R \) is in \( DM^{\bot}(1) \). This follows from 5.1.1. □

Hence, we conclude that the birational tower \( \text{(5.2.14)} \) for the motive of a point with \( R \) coefficients is as follows:

\[
\cdots \xrightarrow{b_{c \leq -3}(1_R)} b_{c \leq -2}(1_R) \xrightarrow{b_{c \leq -1}(1_R)} 1_R
\]

\[ \xrightarrow{\theta_1^R} \]

5.2. The filtration on the motivic cohomology of a scheme. Let \( X \in Sm_k \), and \( p, q \in \mathbb{Z} \). We will write \( H^{p,q}(X,R) \) for \( \text{Hom}_{DM}(M(X), 1_R(q)[p]) \), i.e. for the motivic cohomology of \( X \) with coefficients in \( R \) of degree \( p \) and weight \( q \). By adjointness, \( H^{p,q}(X,R) \cong \text{Hom}_{DM}(M(X)(-q)[-p], 1_R) \). Therefore, the abutment of the spectral sequence \( \text{(5.2.10)} \) for the tower \( \text{(5.1.3)} \) evaluated in \( M(X)(-q)[-p] \) induces a filtration on \( H^{p,q}(X,R) \). Namely:
Definition 5.2.1. Let $p, q$ be arbitrary integers, and let $X$ be in $Sm_k$. Consider
the decreasing filtration $F^\bullet$ on $H^{p,q}(X, R)$ where $F^n H^{p,q}(X, R)$ is given by the
image of $\theta^1_{-n} R$ (see 5.1.3), $n \geq 0$:

$$\begin{align*}
\text{Hom}_{DM}(M(X)(-q)[-p], bc_{\leq -n} 1_R) \\
\downarrow \theta^1_{-n} R \\
\text{Hom}_{DM}(M(X)(-q)[-p], 1_R) = H^{p,q}(X, R).
\end{align*}$$

By construction, the filtration $F^\bullet$ is functorial in $X$ with respect to
morphisms in $DM$.

Remark 5.2.2. The existence of the functor $\text{Chow}^{\text{eff}}(k) \to DM$ [27, Prop. 2.1.4, Thm. 3.2.6], where $\text{Chow}^{\text{eff}}(k)$ is the category of effective Chow motives over $k$;
implies that the filtration $F^\bullet$ constructed in 5.2.1 is functorial with respect to Chow
correspondences.

5.2.3. Finiteness of the filtration. Our goal is to show that for $X$ in $Sm_k$, the
filtration $F^\bullet$ on $H^{p,q}(X, R)$ defined in 5.2.1 is concentrated in the range $0 \leq n \leq q$.

Theorem 5.2.4. Let $p, q$ be arbitrary integers, and let $X$ be in $Sm_k$. Then the
decreasing filtration $F^\bullet$ on $H^{p,q}(X, R)$ constructed in 5.2.1 satisfies the following
properties:

1. $F^0 H^{p,q}(X, R) = H^{p,q}(X, R)$,
2. $F^{q+1} H^{p,q}(X, R) = 0$.

Proof. By 5.1.2 we deduce that $\theta^1_{0} R$ is an isomorphism in $DM$. This proves the first
claim. For the second claim, we observe that $M(X)(-q)[-p]$ is in $DM^{\text{eff}}(-q)$ (see
3.1.3) and 3.1.5). Thus, it suffices to show that $bc_{\leq -q-1} 1_R$ belongs to $DM^{\perp}(-q)$
(see 3.1.5). This follows from 3.2.2 and 3.2.3.

5.3. The components of the filtration. We will describe some properties that
an element in $H^{p,q}(X, R)$ needs to satisfy in order to be in the $n$-component of the
filtration $F^\bullet$ (5.2.1). First, we fix some notation.

5.3.1. Let $X \in Sm_k$ and $\alpha \in H^{p,q}(X, R)$ with $q \geq 0$. Set $A = M(X)(-q)[-p] \in
DM$. Then $\alpha$ induces a map $\alpha(-q)[-p] : A \to 1_R$. Let $n > 0$ and set $n' = -n + 1$.
Recall the $n'$-effective cover $\epsilon_{n'}^A : f_{n'}(A) \to A$ (5.2.1).

Proposition 5.3.2. With the notation of 5.3.1, $\alpha \in F^n H^{p,q}(X, R)$ (see 5.2.1) if
and only if $\alpha(-q)[-p] \circ \epsilon_{n'}^A = 0$.

Proof. ($\Rightarrow$): By construction 5.2.1, it suffices to show that:

$$\text{Hom}_{DM}(f_{-n+1} A, bc_{\leq -n} 1_R) = 0.$$ 

This follows directly from 3.2.5 and 3.3.1 (see 3.1.5). 

($\Leftarrow$): Let $s_{<n'}(A)$ be the cone of $\epsilon_{n'}^A$, and consider the following diagram in $DM$
where the top row is a distinguished triangle:

$$\begin{array}{ccc}
f_{n'}(A)[1] & \rightarrow & s_{<n'}(A) \rightarrow A \rightarrow f_{n'}(A) \\
\alpha' \downarrow & \alpha' \downarrow & \alpha' \downarrow \\
bc_{\leq -n} 1_R \rightarrow 1_R
\end{array}$$
Hence, $\alpha(-q)[-p] \circ \epsilon'_{n+1} = 0$ if and only if there exists $\alpha'$ which makes the right triangle commute. On the other hand, the universal property of $\epsilon'_{n+1}$ (3.3.1) implies that $s_{<n}A \in DM^+(n+1)$ (see 3.1.3). Thus, by the universal property of $\theta_{-n}^{1,q}$ (3.2.4) we conclude that the existence of $\alpha'$ such that the right triangle commutes is equivalent to the existence of $\alpha''$ such that the square commutes.

Hence, by construction (5.2.4) we conclude that $\alpha \in F^nH^{p,q}(X,R)$. \hfill $\square$

**Corollary 5.3.3.** With the notation of (5.3.1) Then $F^nH^{p,q}(X,R)$ (see 5.2.1) is isomorphic to:

$$\text{Ker}(\text{Hom}_{DM}(M(X)(-q)[-p], 1_R)) \text{Ann}_{DM}(f_{-n+1}(M(X)(-q)[-p]), 1_R)$$

which is also isomorphic to:

$$\text{Ker}(\text{Hom}_{DM}(M(X), 1_R[q][p])) \text{Ann}_{DM}(f_{q-n+1}(M(X)), 1_R(q)[p])$$

**Proof.** The first isomorphism follows directly from 5.3.2, the second one follows by adjointness and 5.3.3. \hfill $\square$

**Remark 5.3.4.** By 5.3.3 it is possible to construct the filtration $F^\bullet$ (5.2.1) using only the effective covers of the slice filtration.

5.3.5. The filtration for smooth projective varieties. We will show that an algebraic cycle $\alpha \in H^{2q,q}(X,R) \cong CH^q(X)$ is in $F^1H^{2q,q}(X,R)$ (5.2.1) if and only if $\alpha$ is numerically equivalent to zero. When the coefficient ring $R \neq \mathbb{Q}$, we will say that $\alpha$ is numerically equivalent to zero if for every field extension $K/k$ of finite transcendence degree and every cycle $\beta \in CH^q(X_K)_R$, the intersection multiplicity $\text{deg}(\alpha_K \cdot \beta) \in R$ is zero.

Let $\text{hom}^\text{eff}$ denote the internal Hom-functor in $DM^\text{eff}$. Notice that it does not coincide in general with the internal Hom-functor in $DM$ (see the remark after Cor. 4.3.6 in [27]).

The following result is based on the work of Kahn-Sujatha [14, Lem. 5.1 and its proof]. The fact that this approach does in fact compute the first component of the filtration was brought to my attention by the anonymous referee.

**Theorem 5.3.6.** Let $X$ be a smooth projective $k$-scheme and $q \geq 0$. Then via the isomorphism $H^{2q,q}(X,R) \cong CH^q(X)_R$, $F^1H^{2q,q}(X,R)$ (5.2.1) gets identified with the $R$-submodule of cycles numerically equivalent to zero.

**Proof.** Let $\alpha \in CH^q(X)_R$. By 5.3.3 it suffices to show that $\alpha$ is numerically equivalent to zero if and only if the composition:

$$f_qM(X) \xrightarrow{M(X)} M(X) \xrightarrow{\alpha} 1_R(q)[2q]$$

is zero in $DM$, or equivalently in $DM^\text{eff}$ since $f_qM(X)$, $M(X)$, $1_R(q)[2q]$ are in $DM^\text{eff}$. By [10, Prop. 1.1], $f_qM(X) \cong \text{hom}^\text{eff}(1(q)[2q], M(X)(q)[2q])$, so we are reduced to show that $\alpha$ is numerically equivalent to zero if and only if the composition:

$$\text{hom}^\text{eff}(1(q)[2q], M(X)(q)[2q]) \xrightarrow{\epsilon'_q} M(X) \xrightarrow{\alpha} 1_R(q)[2q]$$
Towards the Bloch-Beilinson-Murre filtration

In this section we include some remarks on the still conjectural Bloch-Beilinson-Murre filtration [11, 24, 25, 12].

6.1. Basic Setup. Recall that $k$ is a perfect base field. Let $SmProj_k$ be the full subcategory of $Sm_k$ where the objects are smooth projective varieties over $k$. We will write $CH^*(X)_Q$ for the Chow ring of $X$ with rational coefficients: $\bigoplus_{n=0}^{\dim X} CH^n(X)_Q$. We fix a Weil cohomology theory $H$ on $SmProj_k$ [15, 1.2], with cycle class map $\text{cl}_X : CH^p(X)_Q \rightarrow H^{2p}(X)$. Given a correspondence $\Lambda \in CH^q(X \times Y)_Q$, we will write respectively $CH^*_\Lambda : CH^{t+d_X-q}(X)_Q \rightarrow CH^t(Y)_Q$, $H^*_\Lambda : H^{t+2(d_X-q)}(X) \rightarrow H^t(Y)$ for the induced maps on the Chow groups and on cohomology [15, 1.3], where $d_X$ is the dimension of $X$. Namely, $CH^*_\Lambda (\alpha) = \pi_Y^*(\Lambda \cdot \pi_X^* \alpha)$, where $\pi_X$, $\pi_Y$ are the relative projections to $X$ and $Y$; and similarly for $H^*_\Lambda$. 

is zero in $DM^{\text{eff}}$, which is equivalent (by adjointness and Voevodsky’s cancellation theorem [31]) to show that the map induced by $\alpha$:

$$\text{hom}^{\text{eff}}(1(q)[2q], M(X)) \xrightarrow{\alpha_*} \text{hom}^{\text{eff}}(1_R(q)[2q], 1_R(q)[2q]) \cong 1_R$$

is zero. It follows from [14] Prop. 2.3] that $\text{hom}^{\text{eff}}(1(q)[2q], M(X))$ is in $(DM^{\text{eff}})_{\geq 0}$ for Voevodsky’s homotopy $t$-structure [27, p. 11], and by [18, 4.2] $1_R$ is in the heart of the homotopy $t$-structure. Hence, $\alpha_*$ descends to a map:

$$h_0(\text{hom}^{\text{eff}}(1(q)[2q], M(X))) \xrightarrow{\tilde{\alpha}_*} 1_R$$

which is zero if and only if $\alpha_*$ is zero. Now, $\tilde{\alpha}_*$ is a morphism in the heart of the homotopy $t$-structure, which is given by homotopy invariant Nisnevich sheaves with transfers. Thus, $\tilde{\alpha}_*$ is zero if and only if the induced map on sections [18, 11.2]:

$$\Gamma(K, h_0(\text{hom}^{\text{eff}}(1(q)[2q], M(X)))) \xrightarrow{\tilde{\alpha}_*} \Gamma(K, 1_R) \cong R$$

is zero for every field extension $K/k$ of finite transcendence degree. By [14] Prop. 2.3], $\Gamma(K, h_0(\text{hom}^{\text{eff}}(1(q)[2q], M(X)))) \cong CH^q(X_K)$; so we are reduced to show that $\tilde{\alpha}_K$, computes the intersection multiplicity with $\alpha_K$.

Let $Y \in Sm_k$ with function field $K$. Then, by adjointness $\tilde{\alpha}_K$ is the colimit indexed by the non-empty open subsets $U \subseteq Y$:

$$\text{hom}_{DM^{\text{eff}}}(M(U)1(q)[2q], M(X)) \xrightarrow{\alpha_{U,*}} \text{hom}_{DM^{\text{eff}}}(M(U)(q)[2q], 1_R(q)[2q]) \xrightarrow{\cong} CH^0(U)_R \cong R$$

Since $X$ is smooth projective, $\text{hom}_{DM^{\text{eff}}}(M(U)1(q)[2q], M(X)) \cong CH_{d_U+q}(U \times X)_R$ where $d_U$ is the dimension of $U$. Hence, it suffices to show that $\alpha_{U,*}$ gets identified with the map $\beta \mapsto p_{U,*}(\beta \cdot (U \times \alpha))$, $\beta \in CH_{d_U+q}(U \times X)_R$ where $p_U : U \times X \rightarrow U$.

Finally, this follows by combining the existence of the functor $\text{Chow}^{\text{eff}}(k) \rightarrow DM$ [27, Prop. 2.1.4, Thm. 3.2.6] (where $\text{Chow}^{\text{eff}}(k)$ is the category of effective Chow motives over $k$) with the projective bundle formula in $DM$ [18, 15.12] (which shows in particular that $M(\mathbb{P}^1)/M(\mathbb{P}^{n-1})$ is a model for $1(q)[2q]$ [18, 15.2]).


Definition 6.1.1. Let $X \in \text{SmProj}_k$. We will say that a descending filtration $F^\bullet$ on $CH^\bullet(X)_\mathbb{Q}$ is a filtration of Bloch-Beilinson-Murre type if the following conditions are satisfied:

**BBM1:** $F^0CH^q(X)_\mathbb{Q} = CH^q(X)_\mathbb{Q}$ for every $0 \leq q \leq \dim X$.
**BBM2:** $F^1CH^q(X)_\mathbb{Q} = CH^q_{\text{num}}(X)_\mathbb{Q}$, the group of cycles numerically equivalent to zero modulo rational equivalence.
**BBM3:** $F^\bullet$ is functorial with respect to Chow correspondences between smooth projective varieties, i.e. given $\Lambda \in CH^q(X \times Y)_\mathbb{Q}$:

$$CH^q_{\Lambda}(F^nCH^{q+d}X \times Y)_\mathbb{Q} \subseteq F^nCH^q(Y)_\mathbb{Q}.$$ 

We will write $Gr^n_{\Lambda}(CH^q(X)_\mathbb{Q})$ for the map induced by $CH^q_{\Lambda}$ on the graded groups:

$$Gr^n_{\Lambda}(CH^{q+d}X \times Y)_\mathbb{Q} \rightarrow Gr^n_{\Lambda}(CH^q(Y)_\mathbb{Q}).$$

**BBM4:** With the notation of BBM3 and 6.1. If $H^{2q-n}_\Lambda = 0$ then $Gr^n_{\Lambda}(CH^q(X)_\mathbb{Q}) = 0$.
**BBM5:** $F^{q+1}CH^q(X)_\mathbb{Q} = 0$.

6.1.2. Instead of BBM4 it is possible to consider the weaker condition BBM4\*:

If the class of $\Lambda \in CH^q(X \times Y)_\mathbb{Q}$ is homologous to zero, i.e. $c_l(X \times Y)(\Lambda) = 0 \in H^{2d}(X \times Y)$; then $Gr^n_{\Lambda}(CH^q(X)_\mathbb{Q}) = 0$ for every $n; 0 \leq n \leq q$ (see the graded condition in [33, p. 421]).

Remark 6.1.3. Assuming that homological and numerical equivalence coincide on $X$, i.e. the standard conjecture $D(X)$ [15, Prop. 3.6]; we observe that BBM2 simply says that $F^iCH^q(X)_\mathbb{Q}$ is given by the group of cycles homologically equivalent to zero modulo rational equivalence $CH^q_{\text{num}}(X)_\mathbb{Q}$. This is the condition that appears in [25, 12]. Notice that when the base field $k = \mathbb{C}$ is given by the complex numbers, then $D(X)$ is known to be true in the following cases:

1. $CH^q_{\text{hom}}(X)_\mathbb{Q} = CH^q_{\text{num}}(X)_\mathbb{Q}$; for $q = 0, 1, 2, \dim X - 1, \dim X$ [10, p. 369 Cor. 1]. In fact, when $q = 1$ (resp. $q = \dim X$) and $k$ is algebraically closed, homological and numerical equivalence coincide [17] (resp. non-triviality of the cycle map [15, p.363 1.2C(iii)]).
2. $X$ is an abelian variety [15, p. 372 Thm. 4],
3. $\dim X \leq 4$ [16, p. 369 Cor. 1].

Since $CH^q(X)_\mathbb{Q} \cong \text{Hom}_{DM}(M(X)(-q)[-2q], 1_\mathbb{Q})$, we can restrict the filtration $F^\bullet$ considered in (6.1.1) to the Chow groups with rational coefficients in order to obtain a filtration that satisfies several of the properties considered in (6.1.1):

**Theorem 6.1.4.** With the notation of (6.1) Let $X \in \text{SmProj}_k$. Then the filtration $F^\bullet$ on $CH^\bullet(X)_\mathbb{Q}$ defined in (5.2.1) satisfies the conditions BBM1 BBM2 BBM3 and BBM5 of (6.1.1).

**Proof.** In order to obtain the filtration we just take $R = \mathbb{Q}$ in (5.2.1), and observe that $CH^q(X)_\mathbb{Q} \cong \text{Hom}_{DM}(M(X)(-q)[-2q], 1_\mathbb{Q})$.

Now, property BBM1 follows from 5.2.1\*, property BBM2 follows from 5.3.1, property BBM3 follows from 5.2.1, 5.2.2, and property BBM5 follows from 5.2.1\*.

**Remark 6.1.5.** Notice that the filtration defined in 5.2.1 satisfies the conditions required in 6.1.4 for any coefficient ring $R$ and not just the rational numbers.
6.1.6. In general there is not much that can be said at the moment with respect to condition $\text{BBM}_4$. However, assuming that the tensor structure in $DM$ is compatible with the birational covers $bc_{\leq -m}1_R \to 1_R$ in the sense of 6.1.7 then it is possible to show that $\text{BBM}_4$ holds 6.1.8.

**Conjecture 6.1.7.** Consider the tower (5.1.3) in $DM$ for $R = \mathbb{Q}$, and let $n, m \geq 0$. Then there exists a map $\mu_{n,m} : (bc_{\leq -m}1_Q) \otimes (bc_{\leq -m}1_Q) \to bc_{\leq -m}1_Q$, such that the following diagram commutes:

$$(bc_{\leq -m}1_Q) \otimes (bc_{\leq -m}1_Q) \xrightarrow{\mu_{n,m}} (1_Q \otimes 1_Q) \cong 1_Q$$

$bc_{\leq -m}1_Q \xrightarrow{\theta_{n,m}} (1_Q \otimes 1_Q) \cong 1_Q$$

**Proposition 6.1.8.** Let $X \in \text{SmProj}_k$. Assume that the maps $\mu_{n,1}$ in 6.1.7 exist for $n \geq 0$. Then the filtration $F^*$ on $CH^*(X)_Q$ considered in 5.2.1 and (6.1.4) satisfies the remaining condition $\text{BBM}_4$ (6.1.7).

**Proof.** Let $Y \in \text{SmProj}_k$, and $\Lambda \in CH^q(X \times Y)_Q$ a correspondence homologous to zero, in particular numerically equivalent to zero by [15 Prop. 1.2.3]. Consider $\Lambda$ as a map $M(X \times Y) \to 1_Q(q)[2q]$. By 6.1.7 we conclude that there exists a map $\Lambda' : M(X \times Y)(-q')[-2q'] \to 1_{\leq -1}1_Q$ in $DM$ such that $\Lambda(-q')[-2q'] = \theta_{n,1} \circ \Lambda'$ (5.1.3).

Now let $\alpha \in F^nCH^q(X)_Q$, which we will consider as a map $\alpha : M(X) \to 1_Q(q)[2q]$. Hence, by construction (5.2.1) there exists a map $\alpha' : M(X)(-q)[-2q] \to bc_{\leq -1}1_Q$, such that $\alpha(-q)[-2q] = \theta_{n,1} \circ \alpha'$ (5.1.3).

Thus, it suffices to show that the following composition factors through $\theta_{n,1} : bc_{\leq -n-1}1_Q \to 1_Q$ (see 5.2.1):

$$(M(X)(-q)[-2q]) \otimes (M(X) \otimes M(Y))(-q')[-2q']$$

$$(bc_{\leq -n}1_Q) \otimes (bc_{\leq -1}1_Q) \xrightarrow{\theta_{n,1} \otimes \theta_{n,1}} (1_Q \otimes 1_Q) \cong 1_Q$$

But this follows directly from the existence of the map $\mu_{n,1}$ (6.1.7). \qed

6.1.9. The anonymous referee suggests the following heuristic argument to explain why the filtration considered in (6.1.4) is a reasonable candidate for a Bloch-Beilinson-Murre filtration: Let $X \in \text{SmProj}_k$ be of pure dimension $d$. Combining 3.3.3 and 3.3.2 we conclude that $\alpha \in CH^q(X)_Q$ is in $F^nCH^q(X)_Q$ if and only if the composition:

$$f_{q-n+1}(M(X)) \xrightarrow{M(X) \otimes \alpha} M(X) \otimes 1_R(q)[2q]$$

is zero in $DM_{\text{eff}}$. Let $r = q - n + 1$, by [10 Prop. 1.1], [32 Lem. 5.9] we conclude that $f_r(M(X)) \cong \text{hom}_{\text{eff}}(1_Q(r)[2r], M(X)) \otimes 1_Q(r)[2r]$ in $DM_{\text{eff}}$. Thus, it follows from Voevodsky’s cancellation theorem [31] that $\alpha \in F^nCH^q(X)_Q$ if and only if the map $\text{hom}_{\text{eff}}(1_Q(r)[2r], M(X)) \to 1_Q(n-1)[2n-2]$ in $DM_{\text{eff}}$ induced by $\alpha$ is zero.
Assuming a conjecture of Ayoub [3, 4.22], the functor $DM^{\text{eff}} \rightarrow DM^{\text{eff}}; A \mapsto \text{hom}^{\text{eff}}(1_Q(1), A)$ maps $DM^{\text{eff}}_{\leq n}$ to $DM^{\text{eff}}_{\leq n-1}$, where $DM^{\text{eff}}_{\leq n}$ is the smallest full triangulated subcategory of $DM^{\text{eff}}$ which is closed under arbitrary (infinite) coproducts and contains $M(Y)$ for $Y \in Sm_k$, dim $Y \leq n$. Thus, we conclude that $\text{hom}^{\text{eff}}(1_Q(r)[2r], M(X))$ is in $DM^{\text{eff}}_{\leq d-q+n-1}$. Hence, using the argument in 5.3.6 we conclude that $F^n CH^q(X)_0$ is contained in the kernel of the maps $CH^{n-1}_Y \rightarrow CH^{n-1}(Y)_0; \alpha \mapsto \pi_Y^*(\Lambda \cdot \pi_Y^* \alpha)$ (see [0]), where $\Lambda \in CH^{d-q+n-1}(Y \times X)_0$ and dim $Y \leq d-q+n-1$. This implies that $F^n CH^q(X)_0$ is contained in the filtration defined in [33, §1.1] which for zero cycles is a natural candidate for a Bloch-Beilinson-Murre filtration [33, Prop. 6].

For more details on Ayoub’s conjecture [3, 4.22], we refer the reader to the recent preprint [7].

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