A class of Lie racks associated to symmetric Leibniz algebras

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Abstract

Given a symmetric Leibniz algebra \((L, .)\), the product is Lie-admissible and defines a Lie algebra bracket \([\ , \] \) on \(L\). Let \(G\) be the connected and simply-connected Lie group associated to \((L, [\ , \])\). We endow \(G\) with a Lie rack structure such that the right Leibniz algebra induced on \(T_eG\) is exactly \((L, .)\). The obtained Lie rack is said to be associated to the symmetric Leibniz algebra \((L, .)\). We classify symmetric Leibniz algebras in dimension 3 and 4 and we determine all the associated Lie racks. Some of such Lie racks give rise to non-trivial topological quandles. We study some algebraic properties of these quandles and we give a necessary and sufficient condition for them to be quasi-trivial.

1 Introduction

In the 1980’s, Joyce \cite{Joyce1982} and Matveev \cite{Matveev1995} introduced the notion of quandle. This notion has been derived from the knot theory, in the way that the axioms of a quandle are the algebraic interpretations of Reidemeister moves (I,II,III) for oriented knot diagrams \cite{Reidemeister1927}. The quandles provide many knot invariants. The fundamental quandle or knot quandle was introduced by Joyce who showed that it is a complete invariant of a knot (up to a weak equivalence). Racks which are a generalization of quandles were introduced by Brieskorn \cite{Brieskorn1987} and Fenn and Rourke \cite{Fenn1992}. Recently (see \cite{Joyce1999, Matveev2000}), there has been investigations on quandles and racks from an algebraic point of view and their relationship with other algebraic structures as Lie algebras, Leibniz algebras, Frobenius algebras, Yang Baxter equation, and Hopf algebras etc.

In 2007, Rubinsztein introduced the notion of topological quandles \cite{Rubinsztein2007}. Using a particular action of the braid group \(B_n\) on the Cartesian product of \(n\) copies of a topological quandle \((Q, \triangleright)\), he associated the space \(J_Q(L)\) of fixed points under the action of the braid \(\sigma \in B_n\) for the element \(\sigma\) whose closure is the oriented link \(L\). The main result of the paper was that the space \(J_Q(L)\) depends only on the isotopy class of the oriented link \(L\). One can extend the notion of topological quandles to topological racks in a trivial way. An important subclass of topological racks is the class of Lie racks consisting of rack structures on smooth manifolds such that the rack operation is smooth.

The main purpose of this paper is to take advantage of a known interaction between symmetric Leibniz algebras and Lie algebras to generate families of Lie rack structures on some Lie groups of dimensions 3 and 4. We derive a family of topological quandles and we study some of their algebraic structures. Furthermore, we give a necessary and sufficient condition for such topological quandles to be quasi-trivial and then constitute link-homotopy invariants.

Let us give a short overview of our method. We consider \((X, \triangleright, 1)\) a pointed Lie rack. That is a Lie rack with a fixed element \(1 \in X\), such that \(x \triangleright 1 = x\) and \(1 \triangleright x = 1\), for each \(x \in X\). It is known (see \cite{Joyce1999}) that
the tangent space $T_1 X$ has a structure of right Leibniz algebra. The problem of integrating Leibniz algebras to pointed Lie racks was formulated by J.-L. Loday in [28]. It consists in finding a generalization of the Lie’s third theorem for Leibniz algebras. There are only partial answers to this problem (see [12, 17]). However, S. Benayadi and M. Bordemann [6] gave a natural method for integrating symmetric Leibniz algebras, which are both right and left Leibniz algebras. This method is based on the characterization of symmetric Leibniz algebras given in [5]. More precisely, given a symmetric Leibniz algebra $(L, \cdot)$, the product is Lie-admissible and defines a Lie algebra bracket $[\cdot, \cdot]$ on $L$. Let $G$ be the connected and simply-connected Lie group associated to $(L, [\cdot, \cdot])$. Then, naturally one can build on $G$ a Lie rack structure such that the right Leibniz algebra on $T_e G$ is exactly $(L, \cdot)$. The obtained Lie rack is said to be associated to the symmetric Leibniz algebra $(L, \cdot)$. Having this method in mind, we determine all symmetric Leibniz algebras in dimension 3 and 4, up to an isomorphism, and for each of them we build the associated Lie rack. We get a family of Lie racks and some topological quandles. We study some algebraic properties of these quandles.

Our exposition is organized as follows. The Section 2 is devoted to preliminaries. We recall the notions of Lie racks, quandles and Leibniz algebras. In Section 3 we state our main result which introduces a Lie rack structure on the connected simply connected Lie group associated to the underlying Lie algebra of a given symmetric Leibniz algebra. We also investigate some algebraic properties of the associated topological quandle. In section 4, we first give all symmetric Leibniz algebras of dimension 3 and 4, and then we apply our method 1 to generate all the associated Lie racks. In Section 5, we study some algebraic properties of the derived topological quandles. In Section 6, we give an example of explicit calculations in dimension 4.

### 2 Preliminaries

#### 2.1 Lie racks and topological quandles

**Definition 1.**

1. A rack is a non-empty set $X$ together with a map $	riangleright : X \times X \longrightarrow X$, $(x, y) \mapsto x \triangleright y$ such that

   - for any fixed element $x \in X$, the map $R_x : X \longrightarrow X$, $y \mapsto y \triangleright x$ is a bijection,
   - for any $x, y, z \in X$, we have $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ (right self-distributivity).

2. A rack $X$ is called pointed, if there exists a distinguished element $1 \in X$ such that

   $$x \triangleright 1 = R_1(x) = id_X(x) = x \quad \text{and} \quad 1 \triangleright x = R_x(1) = 1, \quad \text{for each} \quad x \in X.$$

3. A rack $X$ is called a quandle if, for any $x \in X$, $x \triangleright x = x$.

4. A quandle $X$ is called a Kei if, for any $x, y \in X$, $(y \triangleright x) \triangleright x = y$, i.e., $R_x$ is an involution.

5. A topological quandle (rack) is a topological space $X$ with a quandle (rack) structure such that the product $\triangleright : X \times X \longrightarrow X$ is continuous and, for all $x \in X$, $R_x : X \longrightarrow X$, $y \mapsto y \triangleright x$ is a homeomorphism.

6. A Lie rack is a smooth manifold $X$ with a rack structure such that the product $\triangleright : X \times X \longrightarrow X$ is smooth and, for all $x \in X$, $R_x : X \longrightarrow X$, $y \mapsto y \triangleright x$ is a diffeomorphism.

When $X$ is a rack, sometimes we write $y \triangleright^{-1} x := R_x^{-1}(y)$.

**Remark 1.** The rack defined above is said to be a right distributive rack. There is also the notion of left distributive rack which is equivalent. In the following, we will consider the right version unless otherwise stated.
Examples 1. • Any non-empty set $X$ equipped with the operation $x \triangleright y := x$ for any $x, y \in X$ is a kei, which is called the trivial kei, the trivial quandle or the trivial rack.

• Let $G$ be a group. Then $G$ is a quandle under the operation of conjugation, i.e.

$$h \triangleright g = g^{-1}hg \quad \text{for all } g, h \in G.$$  

We denote this quandle by $\text{Conj}(G)$.

• Let $\mathbb{Z}_n$ be the ring of integers modulo $n \in \mathbb{N}^*$. For any $x, y \in \mathbb{Z}_n$, we define the operation $x \triangleright y = 2y - x$. The pair $(\mathbb{Z}_n, \triangleright)$ is a quandle which is called the dihedral quandle and is denoted by $R_n$.

• Let $\mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials in the variable $t$. Let $M$ be a $\mathbb{Z}[t, t^{-1}]$-module. The operation $x \triangleright y = tx + (1-t)y$ for any $x, y \in M$, makes $M$ into a quandle called the Alexander quandle.

A map $f$ between two racks $(X_1, \triangleright_1)$ and $(X_2, \triangleright_2)$ is a rack homomorphism if $f$ preserves the rack operations, i.e., $f(x \triangleright_1 y) = f(x) \triangleright_2 f(y)$ for all $x, y \in X_1$. If furthermore $f$ is a bijection it is called an isomorphism of racks. In particular, a bijective rack homomorphism $f : X \longrightarrow X$ is called a rack automorphism. One can define a quandle homomorphism in exactly the same way.

Remark 2. It is easy to see that for any rack $X$ and any $x \in X$, the right translation $R_x$ is a rack automorphism.

The last remark allows to show the following proposition.

Proposition 2.1. Let $(X, \triangleright)$ be a rack and $Q(X)$ be the set of its idempotents,

$$Q(X) = \{ x \in X \; , \; x \triangleright x = x \},$$  

then $(Q(X), \triangleright)$ is a quandle. In particular, if $(X, \triangleright)$ is a Lie rack then $(Q(X), \triangleright)$ is a topological quandle.

Proof. We note first that $Q(X)$ is closed by the binary operation $\triangleright$. Indeed, if $x, y \in Q(X)$, then

$$(x \triangleright y) \triangleright (x \triangleright y) = (x \triangleright x) \triangleright y = x \triangleright y.$$ 

The right distributivity is obviously satisfied.

For any $x, y \in Q(X)$, the restriction of the right translation $R_y$ to $Q(X)$ is injective by assumption. Actually $R_y$ is bijective when considered as a map defined on $X$. Let $y, z \in Q(X)$. There exists a unique $x \in X$ such that $R_y(x) = z$ and then, $x = R_y^{-1}(z) = z \triangleright y^{-1}$. Since $R_y^{-1}$ is a quandle morphism, we have

$$x \triangleright x = R_y^{-1}(z) \triangleright R_y^{-1}(z) = R_y^{-1}(z \triangleright z) = R_y^{-1}(z) = x.$$ 

Then the unique $x \in X$ such that $x = R_y^{-1}(z)$ belongs to $Q(X)$. This ends the proof. 

Furthermore, it is known that the set of all rack automorphisms of $X$ forms a group denoted by $\text{Aut}(X)$. The group of inner automorphisms $\text{Inn}(X)$ generated by all bijections $R_x$ is a normal subgroup of $\text{Aut}(X)$. Then, the map

$$\begin{array}{ccc}
R : & X & \longrightarrow & \text{Inn}(X) \\
& x & \longmapsto & R_x
\end{array}$$ 

induces a right action of the group $\text{Inn}(X)$ on $X$. The orbit $\Omega(x)$ of an element $x \in X$ is given by

$$\Omega(x) = \{ \varphi(x), \varphi \in \text{Inn}(X) \} = \{ R_y(x), y \in X \}.$$ 

Note that the notion of inner automorphisms of quandles is similarly defined.
Let \((Q, \triangleright)\) be a quandle. The subset \(Z(Q) = \{x \in Q : x \triangleright y = x \text{ } \forall y \in Q\}\) is called the **center** of the quandle \(Q\). In particular, if \(Q = \text{Conj}(G)\), then the center \(Z(Q)\) of the quandle \(Q\) coincides with the center \(Z(G)\) of the group \(G\).

We end this section by recalling the definition of three classes of quandles (see for instance [24, 25, 30]). We will show in the last section that some quandles we will obtain are in these classes.

**Definition 2.**
1. A quandle \((Q, \triangleright)\) is called **quasi-trivial** if \(x \triangleright \varphi(x) = x\) for any \(x \in Q\) and \(\varphi \in \text{Inn}(Q)\).
   This is equivalent to \(x \triangleright y = x\) for all \(x, y \in \Omega(x)\).

2. A quandle \(Q\) is called **medial**, if for any \(x, y, z, w \in Q\), we have
   \[(x \triangleright y) \triangleright (z \triangleright w) = (x \triangleright z) \triangleright (y \triangleright w)\].

### 2.2 Symmetric Leibniz algebras

In this subsection, we recall the definition of a Leibniz algebra with an emphasis on the structure of a symmetric Leibniz algebra for which we give a useful characterization and some immediate properties. For more details on Leibniz algebras one can see [17, 28].

Let \((L, \cdot)\) be an algebra. For any \(u \in L\), we denote by \(L_u\), respectively \(R_u\), the two endomorphisms of the vector space \(L\) defined by
\[L_u(v) = u.v\] and \[R_u(v) = v.u\], \(\forall v \in L\). The maps \(L_u\) and \(R_u\) are respectively called the left translation and the right translation by \(u\).

**Definition 3.**
1. An algebra \((L, \cdot)\) is said to be a **left Leibniz algebra**, if for each \(u \in L\), the left translation \(L_u\) is a derivation. That is, for any \(v, w \in L\) we have the following identity
   \[u.(v.w) = (u.v).w + v.(u.w)\] 

2. An algebra \((L, \cdot)\) is said to be a **right Leibniz algebra**, if for each \(u \in L\), the right translation \(R_u\) is a derivation. That is, for any \(v, w \in L\) we have the following identity
   \[(v.w).u = (v.u).w + v.(w.u)\].

3. If \((L, \cdot)\) is both a right and a left Leibniz algebra then it is called a **symmetric Leibniz algebra**.

Any Lie algebra is a symmetric Leibniz algebra. However, the class of symmetric Leibniz algebras is far more bigger than the class of Lie algebras as we will see later.

Let \(\mathfrak{L}\) be a real vector space equipped with a bilinear map \(\cdot : \mathfrak{L} \times \mathfrak{L} \rightarrow \mathfrak{L}\). Let \([,]\) and \(\circ\) be respectively its antisymmetric and symmetric parts. For all \(u, v \in \mathfrak{L}\), they are defined by:
\[[u, v] = \frac{1}{2}(u.v - v.u) \text{ and } u \circ v = \frac{1}{2}(u.v + v.u).\]

Thus
\[u.v = [u, v] + u \circ v.\]

The following proposition gives a useful characterization of symmetric Leibniz algebras (see [5], Proposition 2.11).

**Proposition 2.2.** Let \((\mathfrak{L}, \cdot)\) be an algebra. The following assertions are equivalent:

1. \((\mathfrak{L}, \cdot)\) is a symmetric Leibniz algebra.

2. The following conditions hold:
   (a) \((\mathfrak{L}, [\cdot, \cdot])\) is a Lie algebra.

   (b) For any \(u, v \in \mathfrak{L}\), \(u \circ v\) belongs to the center of \((\mathfrak{L}, [\cdot, \cdot])\).
(c) For any \( u, v, w \in \mathfrak{L} \), \( ([u, v]) \circ w = 0 \) and \( (u \circ v) \circ w = 0 \).

According to this proposition, any symmetric Leibniz algebra is given by a Lie algebra \((\mathfrak{L}, [\cdot, \cdot])\) and a bilinear symmetric form \( \omega : \mathfrak{L} \times \mathfrak{L} \rightarrow Z(\mathfrak{L}) \) where \( Z(\mathfrak{L}) \) is the center of the Lie algebra, such that, for any \( u, v, w \in \mathfrak{L} \),

\[
\omega([u, v], w) = \omega(u, v, w) = 0. \tag{5}
\]

Then the product of the symmetric Leibniz algebra is given by

\[
u.v = [u, v] + \omega(u, v), \quad u, v \in \mathfrak{L}. \tag{6}
\]

Note that if \( Z(\mathfrak{L}) = 0 \) or \([\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}\) then the solutions of (5) are trivial.

The following proposition is easy to prove.

**Proposition 2.3.** Let \((\mathfrak{g}, [\cdot, \cdot])\) a Lie algebra and \(\omega\) and \(\mu\) two solutions of (5). Then \((\mathfrak{g}, \cdot_\omega)\) is isomorphic to \((\mathfrak{g}, \cdot_\mu)\) (as symmetric Lie algebras) if and only if there exists an automorphism \(A\) of \((\mathfrak{g}, [\cdot, \cdot])\) such that

\[
\mu(u, v) = A^{-1}\omega(Au, Av). \]

### 3 Lie racks and topological quandles associated to symmetric Leibniz algebras

Let \((X, 1)\) be a pointed Lie rack with left distributivity. Kinyon showed in [27] that the tangent space \(T_1X\) carries a structure of left Leibniz algebra. In what follows, we show that, in the same way, one can get a structure of a right Leibniz algebra on the tangent space \(T_1X\) if the pointed Lie rack \((X, 1)\) is considered with right distributivity.

For each \(x \in X\), \(R_x(1) = 1\). We consider the linear map

\[
\text{Ad}_x = T_1R_x : T_1X \rightarrow T_1X.
\]

We have

\[
\text{Ad}_{x \triangleright y} = \text{Ad}_x \circ \text{Ad}_y \circ \text{Ad}^{-1}_x.
\]

Thus \(\text{Ad} : X \rightarrow \text{GL}(T_1X)\) is an homomorphism of Lie racks. If we put

\[
u.v := \text{ad}_u(v) = \frac{d}{dt}|_{t=0} \text{Ad}_{c(t)}(u), \quad \forall u, v \in T_1X,
\]

where \(c : \epsilon \rightarrow X\) is a smooth path in \(X\) such that \(c(0) = 1\) and \(c'(0) = v\). We have the following theorem.

**Theorem 3.1** ([27]). Let \((X, 1)\) be a pointed Lie rack. Then the tangent space \(T_1X\) endowed with the product

\[
u.v := \text{ad}_u(v) = \frac{d}{dt}|_{t=0} \text{Ad}_{c(t)}(u), \quad \forall u, v \in T_1X,
\]

is a right Leibniz algebra. Moreover, if \(X = \text{Conj}(G)\) where \(G\) is a Lie group then \((T_1X, \cdot)\) is the Lie algebra of \(G\).

The problem of integrating a Leibniz algebra into a pointed Lie rack was first formulated by J.-L. Loday in [28]. Though until now there is no natural answer to this problem, but there are many partial results. In [27], Kinyon gave a positive answer for split Leibniz algebras. In [17], S. Covez gave a local answer to the integration problem. He showed that every Leibniz algebra can be integrated into a local augmented Lie rack. In [12], Bordemann gave a global process of integrating Leibniz algebras but this process is, unfortunately, not functorial. For our purpose, there is a functorial process of integrating symmetric Leibniz algebras into pointed Lie racks.
algebras which was communicated to us privately by S. Benayadi and M. Bordemann and we will give it now in details.

Let \( (\mathfrak{L}, .) \) be a symmetric Leibniz algebra. According to Proposition 2.2, there exists a Lie bracket \([ , ]\) on \( \mathfrak{L} \) and a bilinear symmetric form \( \omega : \mathfrak{L} \times \mathfrak{L} \rightarrow Z(\mathfrak{L}) \), where \( Z(\mathfrak{L}) \) is the center of \( (\mathfrak{L}, [ , ]) \), such that,
\[
    u.v = [u, v] + \omega(u, v) \text{ for all } u, v \in \mathfrak{L}
\]
and \( \omega \) satisfies (5).

**Method 1.** Denote by \( \mathfrak{L}/\mathfrak{L} = \text{span}\{u, v, u, v \in \mathfrak{L}\}, \mathfrak{a} = \mathfrak{L}/\mathfrak{L}, q : \mathfrak{L} \rightarrow \mathfrak{a} \) and define \( \beta : \mathfrak{a} \times \mathfrak{a} \rightarrow \mathfrak{L} \) given by
\[
    \beta(q(u), q(v)) = \omega(u, v).
\]
By virtue of (5), \( \beta \) is well defined. Moreover, since \( [u, v] = \frac{1}{2}(u.v - v.u) \), \( q \) is a Lie algebra homomorphism when \( \mathfrak{a} \) is considered as an abelian Lie algebra. Consider \( G \) the connected and simply connected Lie group whose Lie algebra is \( (\mathfrak{L}, [ , ]) \) and \( \exp : \mathfrak{L} \rightarrow G \) its exponential. Hence there exists an homomorphism of Lie groups \( \kappa : G \rightarrow \mathfrak{a} \) such that \( d_{\kappa} = q \). Finally, consider \( \chi : G \times G \rightarrow G \) given by
\[
    \chi(h, g) = \exp(\beta(\kappa(h), \kappa(g))) \text{ for all } h, g \in G.
\]
Define now on \( G \) the binary product by putting
\[
    h \triangleright g := g^{-1}h\chi(h, g) \text{ for all } h, g \in G. \tag{7}
\]

We show the result obtained by S. Benayadi and M. Bordemann.

**Theorem 3.2.** Let \( (\mathfrak{L}, .) \) be a symmetric Leibniz algebra. Let \( G \) be the connected simply connected Lie group associated to the underlying Lie algebra endowed with the binary operation \( \triangleright \) defined by (7). Then \((G, \triangleright)\) is a pointed Lie rack whose associated right Leibniz algebra is exactly \((\mathfrak{L}, .)\).

**Proof.** At first, we note that the map \( \chi \) satisfies the following properties for all elements \( h, h_1, h_2, h_3 \in G \):
- The map \( \chi \) is symmetric, since \( \beta \) is symmetric.
- \( \chi(h_1, 1) = 1 = \chi(1, h) \) because \( \kappa(1) = 0 \) and \( \beta \) is bilinear.
- \[
    \chi(h_1h_2, h_3) = \exp(\beta(\kappa(h_1h_2), \kappa(h_3))) = \exp(\beta(\kappa(h_1) + \kappa(h_2), \kappa(h_3))) \\
    = \exp(\beta(\kappa(h_1), \kappa(h_3)))\exp(\beta(\kappa(h_2), \kappa(h_3))) \\
    = \chi(h_1, h_3)\chi(h_2, h_3)
    
    \]
- Since the map \( \beta \) takes its values in the centre of \( \mathfrak{L} \), it follows that,
\[
    1 = \chi(h_1^{-1}, h_2) = \chi(h_1^{-1}, h_2)\chi(h_1, h_2)
\]
and so \( \chi(h_1^{-1}, h_2) = \chi(h_1, h_2)^{-1} \).
- Since \( \kappa \) is a morphism between connected simply connected Lie groups, then for all \( x \in \mathfrak{L} \) we have \( \kappa(\exp(x)) = \exp_x(T_1\kappa(x)) \). The latter is equal to \( q(x) \) because the exponential map of the vector space Lie group \( \mathfrak{a} \) is the identity. Then
\[
    \kappa(\chi(h_1, h_2)) = \kappa(\exp(\beta(\kappa(h_1), \kappa(h_2)))) = q(\beta(\kappa(h_1), \kappa(h_2))) = 0,
\]
hence
\[
    \chi(h_1\chi(h_2, h_3), h_4) = \chi(h_1, h_4).
\]
Now we will use those properties to show the theorem.

First, it is easy to see that the binary operation \( \triangleright : G \times G \rightarrow G \) is a smooth map. Then for any \( g \in G \), the map \( R_g : G \rightarrow G \) which sends \( h \) to \( h \triangleright g \) is invertible with smooth inverse \( R_g^{-1} = R_{g^{-1}} \). This follows easily from the identity \( \chi(h \triangleright g, h_1) = \chi(h, h_1) \) for any \( h, g, h_1 \in G \).

Let us show that the self-distributivity condition is satisfied. For that we will use the following identity.

Finally, \( (z \triangleright h_2) \triangleright h_3 = (z h_2) \triangleright (h_3 h_2) = z h_2 \chi(h, h_2) \).

On the one hand we have

\[
(h_1 \triangleright h_2) \triangleright h_3 = (h_1^{-1} h_1 h_2 \chi(h_1, h_2)) \triangleright h_3
\]

This proves the self-distributivity condition. Moreover we have

\[ h_1 \triangleright 1 = h_1 \chi(1, h_1) = h_1 \quad \text{and} \quad 1 \triangleright h_1 = h_1^{-1} h_1 \chi(1, h_1) = 1 \]

Finally \( (G, \triangleright) \) is a pointed Lie rack.

For the last claim in the theorem we must show that the corresponding Leibniz product on \( \mathcal{L} \) is exactly the product we started with in \( \mathcal{L} \). Indeed, according to Theorem \( (3.1) \), we get for all \( u, v \in \mathcal{L} \) and \( x \in G \):

\[
\frac{\partial}{\partial t} \bigg |_{t=0} R_x(\exp(tu)) = \frac{\partial}{\partial t} \bigg |_{t=0} (\exp(tu) \triangleright x),
\]

\[
= \frac{\partial}{\partial t} \bigg |_{t=0} (x^{-1} \exp(tu)x \chi(x, \exp(tu))),
\]

\[
= \frac{\partial}{\partial t} \bigg |_{t=0} (\exp(tx^{-1}ux) \exp(\beta(x, \kappa(\exp(tu)))))
\]

\[
= \frac{\partial}{\partial t} \bigg |_{t=0} (\exp(tAd_{x^{-1}}(u)) \exp(\beta(x, tq(u))))
\]

\[
= Ad_{x^{-1}}(u) + \beta(x, q(u)).
\]
Replacing $x$ by the curve $t \mapsto \exp(tv)$, we obtain
\[
\frac{\partial}{\partial t} |_{t=0} T_1 R_{\exp(tv)}(u) = \frac{\partial}{\partial t} |_{t=0} (Ad_{\exp(-tv)}(u) + \beta(\exp(tv), q(u))),
\]
\[
= \frac{\partial}{\partial t} |_{t=0} (Ad_{\exp(-tv)}(u) + \beta(v, q(u))),
\]
\[
= [-v, u] + v \circ u,
\]
\[
= [u, v] + u \circ v = u . v.
\]
This establishes the formula (4) and completes the proof.

We say that the obtained Lie rack $(G, \rhd)$ is associated to the symmetric Leibniz algebra $(\mathfrak{L}, \cdot)$.

The Lie racks associated to symmetric Leibniz algebras give rise to a class of topological quandles in the following way. Let $(\mathfrak{L}, \cdot)$ be a symmetric Leibniz algebra and $(G, \rhd)$ be the associated Lie rack. According to Proposition 2.1,
\[
Q((G, \rhd)) = \{ g \in G ; \chi(g, g) = 1 \}
\]
is a topological quandle.

4 Lie racks associated to symmetric Leibniz algebras in dimensions 3 and 4

In this section, by using Proposition 2.2 we determine first all the symmetric Leibniz algebras of dimension 3 and 4, up to an isomorphism and, for each of them, we use Method 1 described in the last section to build the associated Lie racks.

4.1 Symmetric Leibniz algebras of dimension 3 and 4

We proceed in the following way:
1. We pick a Lie algebra $\mathfrak{g}$ with non trivial center in the list of [9].
2. By a direct computation, we determine the symmetric forms $\omega$ satisfying (5).
3. In the spirit of Proposition 2.3 we act by the group of automorphisms of $\mathfrak{g}$ on the obtained $\omega$ to reduce the parameters.

By doing so, we get for any Lie algebra $\mathfrak{g}$ of dimension 3 or 4 with non trivial center all non equivalent symmetric Leibniz structures for which $\mathfrak{g}$ is the underlying Lie algebra. In the last section, we give an example of detailed computations. The results are summarized in Table 1.

4.2 Lie racks

In this subsection, we determine by using Method 1 the Lie racks associated to the symmetric Leibniz algebras determined in the last subsection. Then we give the associated topological quandles defined in Proposition 2.1. In the last section, we explicit the computations for a particular example.

• $\mathfrak{g}_{3,1}$. The associated simply-connected Lie group is given by
\[
G_{3,1} = \left\{ \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} , x, y, z, \in \mathbb{R} \right\}.
\]
1. The Lie rack structure associated to $g_{1,1}$ and the associated topological quandle are respectively

$$M(x, y, z) \triangleright_1 M(a, b, c) = \begin{bmatrix} 1 & y & yb + 2cy + zc + x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{1,1})^1 = \{ M(a, b, c) \in G_{1,1} \mid b = -c \}.$$ 

2. The Lie rack structure associated to $g_{1,1}^2$ and the associated topological quandle are respectively

$$M(x, y, z) \triangleright_2 M(a, b, c) = \begin{bmatrix} 1 & y & -bz + cy + zc + x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{1,1})^2 = \{ M(a, b, c) \in G_{1,1} \mid c = 0 \}.$$ 

3. The Lie rack structure associated to $g_{1,1}^3$ and the associated topological quandle are respectively

$$M(x, y, z) \triangleright_3 M(a, b, c) = \begin{bmatrix} 1 & y & cy + yb - bz + cy + x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{1,1})^3 = \{ M(a, b, c) \in G_{1,1} \mid bc = 0 \}.$$ 

4. The Lie rack structure associated to $g_{1,1}^4$ and the associated topological quandle are respectively

$$M(x, y, z) \triangleright_4 M(a, b, c) = \begin{bmatrix} 1 & y & \gamma bz + \gamma cy - bz + cy + x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{1,1})^4 = \{ M(a, b, c) \in G_{1,1} \mid bc = 0 \}.$$ 

- $g_{2,1} \oplus 2g_1$. The associated simply-connected Lie group is given by

$$G_{2,1} \times \mathbb{R}^2 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ w & e^{-x} & 0 & 0 & 0 \\ 0 & 0 & e^y & 0 & 0 \\ 0 & 0 & 0 & e^z \\ w, x, y, z \in \mathbb{R} \right\}.$$ 

1. The Lie rack structure associated to $(g_{2,1} \oplus 2g_1)^1$ and the associated topological quandle are defined by the conjugation operation on $G_{2,1} \times \mathbb{R}^2$.

2. The Lie rack structure associated to $(g_{2,1} \oplus 2g_1)^2$ and the associated topological quandle are respectively

$$M(w, x, y, z) \triangleright_2 M(t, a, b, c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -te^a + e^a w + te^a - x & e^{-x} & 0 & 0 \\ 0 & 0 & e^{y+x} & 0 \\ 0 & 0 & 0 & e^{z+x} \end{bmatrix}$$

$$Q(G_{2,1} \times \mathbb{R}^2)^2 = \{ M(t, a, b, c) \in G_{2,1} \times \mathbb{R}^2 \mid a = 0 \}.$$
3. Lie rack structure associated to \((\mathfrak{g}_2 \oplus \mathfrak{g}_1)^3\) and the associated topological quandle

\[ M(w, x, y, z) \ni_3 M(t, a, b, c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -te^n + e^n w + te^n -x & e^{-x} & 0 & 0 \\ 0 & 0 & e^{ax+y} & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix} \]

\[ Q(G_{2,1} \times \mathbb{R}^2)^3 = \{ M(t, a, b, c) \in G_{2,1} \times \mathbb{R}^2 | a = 0 \} \].

4. The Lie rack structure associated to \((\mathfrak{g}_2 \oplus \mathfrak{g}_1)^4\) and the associated topological quandle are respectively

\[ M(w, x, y, z) \ni_4 M(t, a, b, c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -te^n + e^n w + te^n -x & e^{-x} & 0 & 0 \\ 0 & 0 & e^{ax+y} & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix} \]

\[ Q(G_{2,1} \times \mathbb{R}^2)^4 = \{ M(t, a, b, c) \in G_{2,1} \times \mathbb{R}^2 | ac = 0 \} \].

5. The Lie rack structure associated to \((\mathfrak{g}_2 \oplus \mathfrak{g}_1)^5\) and the associated topological quandle are respectively

\[ M(w, x, y, z) \ni_5 M(t, a, b, c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -te^n + e^n w + te^n -x & e^{-x} & 0 & 0 \\ 0 & 0 & e^{ax+y} & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix} \]

\[ Q(G_{2,1} \times \mathbb{R}^2)^5 = \{ M(t, a, b, c) \in G_{2,1} \times \mathbb{R}^2 | \gamma a = 0 \} \].

\[ \mathfrak{g}_{3,1} \oplus \mathfrak{g}_1 \]. The associated simply-connected Lie group is given by

\[ G_{3,1} \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & x & w & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix} \right\}, w, x, y, z, \in \mathbb{R} \}

1. The Lie rack structure associated to \((\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1)^1\) and the associated topological quandle are respectively

\[ M(w, x, y, z) \ni_1 M(t, a, b, c) = \begin{bmatrix} 1 & x & ax + 2 bx + yb + w & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix} \]

\[ Q(G_{3,1} \times \mathbb{R})^1 = \{ M(t, a, b, c) \in G_{3,1} \times \mathbb{R} | a = -b \} \].

2. The Lie rack structure associated to \((\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1)^2\) and the associated topological quandle are respectively

\[ M(w, x, y, z) \ni_2 M(t, a, b, c) = \begin{bmatrix} 1 & x & -ay + bx + yb + w & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix} \]

\[ Q(G_{3,1} \times \mathbb{R})^2 = \{ M(t, a, b, c) \in G_{3,1} \times \mathbb{R} | b = 0 \} \].
3. The Lie rack structure associated to \((\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1)^3\) and the associated topological quandle are respectively

\[
M(w, x, y, z) \triangleright_3 M(t, a, b, c) = \begin{bmatrix}
1 & x & xa - ya + xb + e^{g_{b,w}} + w & 0 \\
0 & 1 & y & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^z
\end{bmatrix}
\]

\[
Q(G_{3,1} \times \mathbb{R})^3 = \{M(t, a, b, c) \in G_{3,1} | b = 0\}.
\]

4. The Lie rack structure associated to \((\mathfrak{g}_{3,1} \oplus \mathfrak{g}_1)^4\) and the associated topological quandle are respectively

\[
M(w, x, y, z) \triangleright_4 M(t, a, b, c) = \begin{bmatrix}
1 & x & \gamma ay + \gamma bx - ya + xb + w & 0 \\
0 & 1 & y & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^z
\end{bmatrix}
\]

\[
Q(G_{3,1} \times \mathbb{R})^4 = \{M(t, a, b, c) \in G_{3,1} \times \mathbb{R} | ab = 0\}.
\]

- \(\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1\). The associated simply-connected Lie group is given by

\[
G_{3,2} \times \mathbb{R} = \left\{ \begin{bmatrix}
1 & 0 & 0 & 0 \\
x & e^y & 0 & 0 \\
w & -ye^y & e^y & 0 \\
0 & 0 & 0 & e^z
\end{bmatrix}, w, x, y, z \in \mathbb{R} \right\}.
\]

The Lie rack structure associated to \((\mathfrak{g}_{3,2} \oplus \mathfrak{g}_1)^1\) and the associated topological quandle are

\[
M(w, x, y, z) \triangleright M(t, a, b, c) = \begin{bmatrix}
1 & (x-a)e^{-b} + ae^{-b+y} & 0 & 0 \\
0 & 0 & 0 & 0 \\
(yb-ax)e^{-b} + e^{-b+y}(ab - ay) & -ye^y & e^y & 0 \\
0 & 0 & 0 & e^{y+b+z}
\end{bmatrix}
\]

\[
Q(G_{3,2} \times \mathbb{R}) = \{M(t, a, b, c) \in G_{3,2} \times \mathbb{R} | b = 0\}.
\]

- \(\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1\). The associated simply-connected Lie group is given by

\[
G_{3,3} \times \mathbb{R} = \left\{ \begin{bmatrix}
1 & 0 & 0 & 0 \\
x & e^y & 0 & 0 \\
w & 0 & e^y & 0 \\
0 & 0 & 0 & e^z
\end{bmatrix}, w, x, y, z \in \mathbb{R} \right\}.
\]

The Lie rack structure associated to \((\mathfrak{g}_{3,3} \oplus \mathfrak{g}_1)^1\) and the associated topological quandle are respectively

\[
M(w, x, y, z) \triangleright M(t, a, b, c) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
ae^{-b} + e^{-b}x + ae^{-b+y} & e^y & 0 & 0 \\
-tb^{-b} + e^{-b}t + te^{-b+y} & 0 & e^y & 0 \\
0 & 0 & 0 & e^{y+b+z}
\end{bmatrix}
\]

\[
Q(G_{3,3} \times \mathbb{R}) = \{M(t, a, b, c) \in G_{3,3} \times \mathbb{R} | b = 0\}.
\]
\( g_{3,4}^0 \oplus g_1 \). The associated simply-connected Lie group is given by

\[
G_{3,4}^0 \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ w & \cosh(y) & -\sinh(y) & 0 \\ x & -\sinh(y) & \cosh(y) & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix}, w, x, y, z \in \mathbb{R} \right\}.
\]

The Lie rack structure associated to \((g_{3,4}^0 \oplus g_1)^1\) and the associated topological quandle are respectively

\[
M(w, x, y, z) \triangleright M(t, a, b, c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \sinh(y-b)a + \cosh(y-b)t + \cosh(b)(w-t) + \cosh(b)(x-a) & \cosh(y) & -\sinh(y) & 0 \\ 0 & 0 & 0 & e^{yb+z} \end{bmatrix}.
\]

\( Q(G_{3,4}^0 \times \mathbb{R}) = \{ M(t, a, b, c) \in G_{3,4}^0 \times \mathbb{R} | b = 0 \} \).

\( g_{3,4}^\alpha \oplus g_1 \). The associated simply-connected Lie group is given by

\[
G_{3,4}^\alpha \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ w & e^{\alpha y} \cosh(y) & -e^{\alpha y} \sinh(y) & 0 \\ x & -e^{\alpha y} \sinh(y) & e^{\alpha y} \cosh(y) & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix}, w, x, y, z \in \mathbb{R} \right\}.
\]

The Lie rack structure associated to \(g_{3,4}^\alpha \oplus g_1\) and the associated topological quandle are respectively

\[
M(w, x, y, z) \triangleright M(t, a, b, c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \sinh(y-b)a + \cosh(y-b)te^{\alpha y}(y-b) & e^{\alpha y} \cosh(y) & -e^{\alpha y} \sinh(y) & 0 \\ 0 & 0 & 0 & e^{yb+z} \end{bmatrix}.
\]

\( Q(G_{3,4}^\alpha \times \mathbb{R}) = \{ M(t, a, b, c) \in G_{3,4}^\alpha \times \mathbb{R} | b = 0 \} \).

\( g_{3,5}^0 \oplus g_1 \). The associated simply-connected Lie group is given by

\[
G_{3,5}^0 \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ w & \cos(y) & -\sin(y) & 0 \\ x & \sin(y) & \cos(y) & 0 \\ 0 & 0 & 0 & e^y \end{bmatrix}, w, x, y, z \in \mathbb{R} \right\}.
\]

The Lie rack structure associated to \(g_{3,5}^0 \oplus g_1\) and the associated topological quandle are respectively

\[
M(w, x, y, z) \triangleright M(t, a, b, c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \sinh(y-b)a + \cosh(y-b)t + \cosh(y-b)te^{\alpha y}(y-b) & \cosh(y) & -\sinh(y) & 0 \\ 0 & 0 & 0 & e^{yb+z} \end{bmatrix}.
\]

\( Q(G_{3,5}^0 \times \mathbb{R}) = \{ M(t, a, b, c) \in G_{3,5}^0 \times \mathbb{R} | b = 0 \} \).
• $g_{3,5}^\alpha \oplus g_1$. The associated simply-connected Lie group is given by

$$G_{3,5}^\alpha \times \mathbb{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & e^{\alpha y} \cos(y) & -e^{\alpha y} \sin(y) & 0 \\
1 & 0 & e^{\alpha y} \sin(y) & e^{\alpha y} \cos(y) \\
0 & 0 & 0 & e^y \\
0 & 0 & 0 & e^z \end{bmatrix}, w, x, y, z \in \mathbb{R}.$$ 

The Lie rack structure associated to $g_{3,5}^\alpha \oplus g_1$ and the associated topological quandle are respectively

$$Q(G_{3,5}^\alpha \times \mathbb{R})^3 = \{ M(t, a, b, c) \in G_{3,5}^\alpha \times \mathbb{R} \mid b = 0 \}.$$

• $g_{4,1}$. The associated simply-connected Lie group is given by

$$G_{4,1} = \begin{bmatrix} 1 & z & w \\
0 & 1 & x \\
0 & 0 & y \\
0 & 0 & 1 \end{bmatrix}, w, x, y, z \in \mathbb{R}$$

1. The Lie rack structure associated to $g_{4,1}^1$ and the associated topological quandle are respectively

$$M(w, x, y, z) \rhd_1 M(t, a, b, c) = \begin{bmatrix} 1 & \frac{1}{2}z^2 & w + \frac{1}{2}y^2 + \frac{1}{2}bz^2 + (y + ez)(1 + z) \\
0 & 1 & b - cy + w - x + (y + ez) \\
0 & 0 & 1 \\
0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,1}^1) = \{ M(t, a, b, c) \in G_{4,1}^1 \mid b^2 + ec^2 = 0, e = 0, 1, -1 \}.$$

2. The Lie rack structure associated to $g_{4,1}^2$ and the associated topological quandle are respectively

$$M(w, x, y, z) \rhd_2 M(t, a, b, c) = \begin{bmatrix} 1 & \frac{1}{2}z^2 & w + \frac{1}{2}y^2 + \frac{1}{2}bz^2 + (y + ez)(1 + z) \\
0 & 1 & b - cy + w - x + (y + ez) \\
0 & 0 & 1 \\
0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,1}^2) = \{ M(t, a, b, c) \in G_{4,1}^1 \mid e = 0 \text{ or } c = 0 \}.$$

3. The Lie rack structure associated to $g_{4,1}^3$ and the associated topological quandle are respectively

$$M(w, x, y, z) \rhd_3 M(t, a, b, c) = \begin{bmatrix} 1 & \frac{1}{2}z^2 & w + \frac{1}{2}y^2 + \frac{1}{2}bz^2 + (y + ez)(1 + z) \\
0 & 1 & b - cy + w - x + (y + ez) \\
0 & 0 & 1 \\
0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,1}^3) = \{ M(t, a, b, c) \in G_{4,1}^1 \mid bc = 0 \}.$$
• $g_{4,3}$. The associated simply-connected Lie group is given by

$$G_{4,3} = \begin{bmatrix} e^{-z} & 0 & 0 & w \\ 0 & 1 & -z & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ w, x, y, z, \in \mathbb{R}$$

1. The Lie rack structure associated to $g_{4,3}^1$ and the associated topological quandle are respectively

$$M(w, x, y, z) \triangleright_1 M(t, a, b, c) = \begin{bmatrix} e^{-z} & 0 & 0 & te^{-z} + e^{c}(w - t) \\ 0 & 1 & -z & cy - bz + x + zc \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,3})^1 = \{ M(t, a, b, c) \in G_{4,3} | c = 0 \}.$$

2. The Lie rack structure associated to $g_{4,3}^2$ and the associated topological quandle are respectively

$$M(w, x, y, z) \triangleright_2 M(t, a, b, c) = \begin{bmatrix} e^{-z} & 0 & 0 & te^{-z} + e^{c}(w - t) \\ 0 & 1 & -z & cy - bz + x + (yb + ezc) \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,3})^2 = \{ M(t, a, b, c) \in G_{4,3} | b^2 + ec^2 = 0 \}.$$

3. The Lie rack structure associated to $g_{4,3}^3$ and the associated topological quandle are respectively

$$M(w, x, y, z) \triangleright_3 M(t, a, b, c) = \begin{bmatrix} e^{-z} & 0 & 0 & te^{-z} + e^{c}(w - t) \\ 0 & 1 & -z & cy - bz + x + (yc + zb) \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,3})^3 = \{ M(t, a, b, c) \in G_{4,3} | bc = 0 \}.$$

• $g_{4,8}^{-1}$. The associated simply-connected Lie group is given by

$$G_{4,8}^{-1} = \begin{bmatrix} 1 & x & w \\ 0 & e^z & y \\ 0 & 0 & 1 \end{bmatrix}, \ w, x, y, z, \in \mathbb{R}.$$ 

The Lie rack structure associated to $(g_{4,8}^{-1})^1$ and the associated topological quandle are respectively

$$M(w, x, y, z) \triangleright M(t, a, b, c) = \begin{bmatrix} 1 & a + xe^{-c} - ae^c & w + zc + ace^{-c}(b + y - be^c) \\ 0 & e^z & (y - b)e^{-c} + be^c - c \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q(G_{4,8}^{-1})^1 = \{ M(t, a, b, c) \in G_{4,8}^{-1} | c = 0 \}.$$ 

• $g_{4,9}^0$. The associated simply-connected Lie group is given by

$$G_{4,9}^0 = \begin{bmatrix} 1 & \cos(z) & yx - y \sin(z) - x \sin(z) & -2w \\ 0 & \nabla \cos(z) & \nabla \sin(z) & y \nabla 0 \\ 0 & -\sin(z) & \cos(z) & x \nabla 1 \nabla 0 \\ 0 & 0 & 0 & 0 & e^z \end{bmatrix}, \ w, x, y, z, \in \mathbb{R}.$$ 

The Lie rack structure associated to $(g_{4,9}^0)^1$ and the associated topological quandle are respectively

$$M(w, x, y, z) \triangleright M(t, a, b, c) =$$

$$\begin{bmatrix} 1 & \cos(z)(a - x) + \sin(z)(b - y) & \cos(z)(y - b) + \sin(z)(a - x) & -2w \\ 0 & \cos(z) & \nabla \sin(z) & y \nabla 0 \\ 0 & \nabla \sin(z) & \cos(z) & x \nabla 1 \nabla 0 \\ 0 & 0 & 0 & 0 & e^z \end{bmatrix}.$$ 

$$Q(G_{4,9}^0) = \{ M(t, a, b, c) \in G_{4,9}^0 | c = 0 \}.$$
5 Some algebraic properties of the obtained topological quandles

5.1 Quasi-triviality

We consider \((G, \triangleright)\) a Lie rack associated to a symmetric Leibniz algebra. The rack operation is given by
\[
h \triangleright g = g^{-1}hg\chi(h, g). \quad (8)
\]

We consider the associated topological quandle
\[Q(G) = \{g \in G, \chi(g, g) = 1\}.
\]

\(Q(G)\) is quasi-trivial if and only if, for any \(g, k \in Q(G)\)
\[g \triangleright (g \triangleright k) = g.
\]

Let \(g, k \in Q(G)\). We have
\[
g \triangleright (g \triangleright k) = g \triangleright (k^{-1}gk\chi(g, k)) = \chi(g, k)^{-1}k^{-1}g^{-1}gk^{-1}gk\chi(g, k)\chi(g, (k^{-1}gk\chi(g, k))
\]
\[= k^{-1}g^{-1}gk^{-1}gk\chi(g, (k^{-1}gk\chi(g, k))
\]
\[= k^{-1}g^{-1}gk^{-1}gk,
\]

Since
\[
\chi(g_1g_2, h) = \chi(h, g_1g_2) = \chi(g_1, h)\chi(g_2, h) \text{ and } \chi(g, \chi(g, k)) = \chi(g, g) = 1.
\]

Then \(Q(G)\) is quasi-trivial if and only if
\[[g, k^{-1}gk] = 1, \text{ for any } g, k \in Q(G).
\]

where \([a, b] = aba^{-1}b^{-1}\).

Note that if the quandle \(Conj(G)\) is quasi-trivial then \(Q(G)\) is quasi-trivial.

We use this characterization to obtain the following result by a direct computation (see [2]).

**Proposition 5.1.**

1. The topological quandles \(Q(G_3, 1)\), \(Q(G_3, 1)^2\), \(Q(G_3, 1)^3\), and \(Q(G_3, 1)^4\) are quasi-trivial.

2. The topological quandles \(Q(G_{2, 1} \times \mathbb{R}^2)^2\), \(Q(G_{2, 1} \times \mathbb{R}^2)^3\) are quasi-trivial.

3. The topological quandles \(Q(G_{3, 1} \times \mathbb{R}^1)\), \(Q(G_{3, 1} \times \mathbb{R}^1)^2\), \(Q(G_{3, 1} \times \mathbb{R}^1)^3\), and \(Q(G_{3, 1} \times \mathbb{R}^1)^4\) are quasi-trivial.

4. The topological quandles \(Q(G_{3, 2} \times \mathbb{R})\), \(Q(G_{3, 3} \times \mathbb{R})\), \(Q(G_{3, 4}^0 \times \mathbb{R})\), \(Q(G_{3, 4}^0 \times \mathbb{R})\), \(Q(G_{3, 5}^0 \times \mathbb{R})\) and \(Q(G_{3, 5}^0 \times \mathbb{R})\) are quasi-trivial.

5. * The topological quandle \(Q(G_{4, 1})^1\) is quasi-trivial if \(\epsilon = 0, 1\).
* The topological quandle \(Q(G_{4, 1})^2\) is quasi-trivial if \(\epsilon \neq 0\).
* The topological quandle \(Q(G_{4, 1})^3\) is quasi-trivial.

6. * The topological quandle \(Q(G_{4, 3})^1\) is quasi-trivial.
* The topological quandle \(Q(G_{4, 3})^2\) is quasi-trivial if \(\epsilon = 1\).

7. The topological quandles \(Q(G_{4, 5}^{-1})^1\) and \(Q(G_{4, 9}^0)\) are quasi-trivial.

It is important to point out that quasi-trivial quandles can be used to obtain link-homotopy invariants (see [2]).
5.2 Medial quandles

We prove that all classes of topological quandles \(Q(G_{3,1})\) and \(Q(G_{3,1} \times \mathbb{R})\) are medial. To do that we need the following lemma which can be easily shown.

**Lemma 5.2.** 1. For any \(M(a, b, c), M(x, y, z), M(t, u, w) \in G_{3,1}\), we have

\[
M(a, b, c) \triangleright [M(x, y, z) \triangleright M(t, u, w)] = M(a, b, c) \triangleright M(x, y, z),
\]

\[
M(a, b, c) \triangleright^{-1} [M(x, y, z) \triangleright^{-1} M(t, u, w)] = M(a, b, c) \triangleright^{-1} M(x, y, z),
\]

\[
M(a, b, c) \triangleright [M(x, y, z) \triangleright^{-1} M(t, u, w)] = M(a, b, c) \triangleright M(x, y, z).
\]

2. For any \(M(t, a, b, c), M(w, x, y, z), M(s, r, u, v) \in (G_{3,1} \times \mathbb{R})\), we have

\[
M(t, a, b, c) \triangleright [M(w, x, y, z) \triangleright M(s, r, u, v)] = M(t, a, b, c) \triangleright M(w, x, y, z),
\]

\[
M(t, a, b, c) \triangleright^{-1} [M(w, x, y, z) \triangleright^{-1} M(s, r, u, v)] = M(t, a, b, c) \triangleright^{-1} M(w, x, y, z),
\]

\[
M(t, a, b, c) \triangleright [M(w, x, y, z) \triangleright^{-1} M(s, r, u, v)] = M(t, a, b, c) \triangleright M(t, a, b, c).
\]

**Proof.** By straightforward computation.

Thus, we get

**Proposition 5.3.** Both classes of quandles \(Q(G_{3,1})\) and \(Q(G_{3,1} \times \mathbb{R})\) are medial.

**Proof.** Let \((M(a_1, b_1, c_1))_{1 \leq i \leq 4} \in (G_{3,1})^4\). Due to the above lemma, we have

\[
(M(a_1, b_1, c_1)) \triangleright (M(a_2, b_2, c_2)) \triangleright (M(a_3, b_3, c_3)) \triangleright M(a_4, b_4, c_4))
\]

\[
= (M(a_1, b_1, c_1) \triangleright M(a_2, b_2, c_2)) \triangleright M(a_3, b_3, c_3),
\]

\[
= (M(a_1, b_1, c_1) \triangleright M(a_3, b_3, c_3)) \triangleright (M(a_2, b_2, c_2) \triangleright M(a_3, b_3, c_3)), \text{ (self-distributivity)}
\]

\[
= (M(a_1, b_1, c_1) \triangleright M(a_3, b_3, c_3)) \triangleright (M(a_2, b_2, c_2)),
\]

\[
= (M(a_1, b_1, c_1) \triangleright M(a_3, b_3, c_3)) \triangleright (M(a_2, b_2, c_2) \triangleright M(a_4, b_4, c_4)).
\]

Hence, the quandles \(Q(G_{3,1})\) are medial. Similarly, we show that \(Q(G_{3,1} \times \mathbb{R})\) are medial.

6 Example of computation

In this section we apply Method\(^\text{1}\) to construct the Lie racks associated to symmetric Leibniz algebras with underlying Lie algebra \(g_{4,1}\). To start, we determine the symmetric Leibniz algebras associated to \(g_{4,1}\). For that, we proceed as described in \([4,1]\). Consider the Lie algebra \(g_{4,1}\) with non-vanishing Lie brackets given by

\[
[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2.
\]

The center is \(Z(g_{4,1}) = Re_1\). By applying Proposition\(^\text{2}\) and by doing a straightforward computations, we get the corresponding symmetric bilinear \(\omega\) satisfying the equation \(^\text{3}\).

\[
\omega(e_3, e_3) = \alpha e_1,
\]

\[
\omega(e_4, e_4) = \beta e_1,
\]

\[
\omega(e_3, e_4) = \gamma e_1
\]

where \((\alpha, \beta, \gamma) \neq 0\).
We use now Proposition 2.3 to complete the classification. We consider the group of the automorphisms of $g_{4,1}$ given by

$$
T = \begin{bmatrix}
  a_{3,3}a_{4,4}^2 & a_{2,3}a_{4,4} & a_{1,3} & a_{1,4} \\
  0 & a_{3,3}a_{4,4} & a_{2,3} & a_{2,4} \\
  0 & 0 & a_{3,3} & a_{3,4} \\
  0 & 0 & 0 & a_{4,4}
\end{bmatrix}
$$

with $\det(T) = a_{3,3}^2a_{4,4}^4$. We consider $\mu(u, v) = T^{-1}\omega(Tu, Tv)$. Then

$$
\mu(e_3, e_3) = \frac{a_{3,3}}{a_{4,4}^2}e_1,
$$

$$
\mu(e_4, e_4) = \frac{a_{3,3}a_{4,4}^2 + \beta a_{4,4}^2 + 2\gamma a_{4,4}a_{4,4}}{a_{3,3}a_{4,4}}e_1,
$$

$$
\mu(e_3, e_4) = \frac{a_{3,3}a_{4,4} + \gamma a_{4,4}}{a_{4,4}^2}e_1.
$$

Thus, we have three cases:

1. If $\alpha \neq 0$, we take $a_{3,3} = \frac{a_{3,3}}{\alpha}$ and $a_{3,4} = -\frac{\gamma a_{4,4}}{\alpha}$. Hence, we have

$$
\mu(e_3, e_3) = e_1,
$$

$$
\mu(e_4, e_4) = -\frac{\alpha\beta - \gamma^2}{a_{4,4}^2}e_1.
$$

2. If $\alpha = 0$ and $\gamma = 0$, we have $\mu(e_4, e_4) = \frac{2}{a_{3,4}}$.

3. If $\alpha = 0$ and $\gamma \neq 0$, we take $a_{4,4} = \gamma, a_{3,4} = -\frac{\beta}{2}$. Then we get $\mu(e_3, e_4) = e_1$.

Therefore, we obtain three classes of symmetric Leibniz algebras whose underlying Lie algebra is $g_{4,1}$ given in Table 1.

We finish this subsection by applying explicitly the method 1 to get the Lie rack structure on the Lie group $G_{4,1}$ associated to the symmetric Leibniz algebra $g_{4,1}$. We have $g_{4,1}/g_{4,1}^1 \simeq \langle e_1, e_2 \rangle$ and the quotient space $\alpha := g_{4,1}/g_{4,1}^1$ is identified to $\mathbb{R}^2$. So the projection $q_1 : g_{4,1} \rightarrow \alpha$ is given by

$$
q_1(w, x, y, z) = (y, z),
$$

it follows that the homomorphism $\kappa : G_{4,1} \rightarrow \mathbb{R}^2$ must be defined by $\kappa(w, x, y, z) = (y, z)$.

Now, $\beta_1 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow g_{4,1}$ is defined by

$$
\beta_1((y, z), (b, c)) = (yb + \varepsilon z)e_1, \quad \varepsilon = 0, 1, -1
$$

and the map $\chi_1 : G_{4,1} \times G_{4,1} \rightarrow G_{4,1}$ is given by

$$
\chi_1((w, x, y, z), (t, a, b, c)) = \begin{bmatrix}
  1 & 0 & 0 & (yb + \varepsilon z) \\
  0 & 1 & 0 & (yb + \varepsilon z) \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}.
$$

Finally, we get the Lie rack product on $G_{4,1}$

$$
M(w, x, y, z) \triangleright_1 M(t, a, b, c) = \begin{bmatrix}
  1 & z & \frac{1}{2}z^2 & w + \frac{1}{2}z^2y + \frac{1}{12}b(z^2 + (yb + \varepsilon z)(1 + z) \\
  0 & 1 & z & -bcz + c(x - w) + z(t - a) \\
  0 & 0 & 1 & bz - cy + w - x + (yb + \varepsilon z) \\
  0 & 0 & 0 & 1
\end{bmatrix}.
$$
| Lie algebra | Non-vanishing Lie brackets | Symmetric Leibniz Non-vanishing brackets | Name | Conditions |
|-------------|----------------------------|----------------------------------------|------|------------|
| $\mathfrak{g}_3.1$ | $[e_2, e_3] = e_1$ | $e_2. e_2 = e_1$, $e_3. e_3 = e_1$, $e_2. e_3 = 2 e_1$ | $\mathfrak{g}_3.1$ | $\epsilon = 0, 1$ |
| | | $e_3. e_3 = (\gamma + 1) e_1$, $e_1. e_2 = (\gamma - 1) e_1$ | | $\gamma \neq 0$ |
| $\mathfrak{g}_2.1 \oplus \mathfrak{g}_1$ | $[e_1, e_2] = e_1$ | $e_2. e_2 = e_3 + e_1$, $e_2. e_3 = e_3. e_2 = e_3 (e_3 + e_4)$, $e_1. e_2 = -2 e_1 = e_1$ | $(\mathfrak{g}_2.1 \oplus \mathfrak{g}_1)^1$ | $\epsilon \neq 0$ |
| | | | | $\gamma = 0, 1$, $\epsilon = 0, 1, -1$ |
| $\mathfrak{g}_3.3 \oplus \mathfrak{g}_1$ | $[e_2, e_3] = e_1$ | $e_2. e_2 = e_1$, $e_3. e_3 = e_1$, $e_2. e_3 = 3 e_1$, $e_3. e_2 = -e_1$ | $(\mathfrak{g}_3.3 \oplus \mathfrak{g}_1)^1$ | $\epsilon = 0, 1$ |
| | | | | $\gamma \neq 0$ |
| $\mathfrak{g}_2.1 \oplus \mathfrak{g}_1$ | $[e_2, e_3] = e_1$ | $e_2. e_2 = e_3 + e_1$, $e_2. e_3 = e_3. e_2 = e_3 (e_3 + e_4)$, $e_1. e_2 = -2 e_1 = e_1$ | $(\mathfrak{g}_2.1 \oplus \mathfrak{g}_1)^2$ | $\epsilon \neq 0$ |
| | | | | $\gamma = 0, 1$, $\epsilon = 0, 1, -1$ |
| $\mathfrak{g}_3.3 \oplus \mathfrak{g}_1$ | $[e_2, e_3] = e_1$ | $e_2. e_2 = e_3 + e_1$, $e_2. e_3 = e_3. e_2 = e_3 (e_3 + e_4)$, $e_1. e_2 = -2 e_1 = e_1$ | $(\mathfrak{g}_3.3 \oplus \mathfrak{g}_1)^2$ | $\epsilon \neq 0$ |
| | | | | $\gamma = 0, 1$, $\epsilon = 0, 1, -1$ |
| $\mathfrak{g}_3.3 \oplus \mathfrak{g}_1$ | $[e_2, e_3] = e_1$ | $e_2. e_2 = e_3 + e_1$, $e_2. e_3 = e_3. e_2 = e_3 (e_3 + e_4)$, $e_1. e_2 = -2 e_1 = e_1$ | $(\mathfrak{g}_3.3 \oplus \mathfrak{g}_1)^3$ | $\epsilon \neq 0$ |
| | | | | $\gamma = 0, 1$, $\epsilon = 0, 1, -1$ |
| $\mathfrak{g}_4.1$ | $[e_2, e_3] = e_1$ | $e_2. e_2 = e_3 + e_1$, $e_2. e_3 = e_3. e_2 = e_3 (e_3 + e_4)$, $e_1. e_2 = -2 e_1 = e_1$ | $(\mathfrak{g}_4.1)^1$ | $\epsilon = 0, 1, -1$ |
| | | | | $\gamma = 0, 1$, $\epsilon = 0, 1$ |
| $\mathfrak{g}_4.1$ | $[e_2, e_3] = e_1$ | $e_2. e_2 = e_3 + e_1$, $e_2. e_3 = e_3. e_2 = e_3 (e_3 + e_4)$, $e_1. e_2 = -2 e_1 = e_1$ | $(\mathfrak{g}_4.1)^2$ | $\epsilon \in \mathbb{R}$ |
| | | | | $\gamma = 0, 1$, $\epsilon \neq 0$ |
| $\mathfrak{g}_4.1$ | $[e_2, e_3] = e_1$ | $e_2. e_2 = e_3 + e_1$, $e_2. e_3 = e_3. e_2 = e_3 (e_3 + e_4)$, $e_1. e_2 = -2 e_1 = e_1$ | $(\mathfrak{g}_4.1)^3$ | $\epsilon \neq 0$ |
| | | | | $\gamma = 0, 1$, $\epsilon \neq 0$ |
| $\mathfrak{g}_4.1$ | $[e_2, e_3] = e_1$ | $e_2. e_2 = e_3 + e_1$, $e_2. e_3 = e_3. e_2 = e_3 (e_3 + e_4)$, $e_1. e_2 = -2 e_1 = e_1$ | $(\mathfrak{g}_4.1)^4$ | $\epsilon \neq 0$ |

Table 1: Symmetric Leibniz algebras of dimension 3 and 4.

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