Lattice Boltzmann scheme for relativistic fluids

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(Dated: December 15, 2009)

A Lattice Boltzmann formulation for relativistic fluids is presented and numerically verified through quantitative comparison with recent hydrodynamic simulations of relativistic shock-wave propagation in viscous quark-gluon plasmas. This formulation opens up the possibility of exporting the main advantages of Lattice Boltzmann methods to the relativistic context, which seems particularly useful for the simulation of relativistic fluids in complicated geometries.

PACS numbers: 47.11.-j, 12.38.Mh, 47.75.+f
Keywords: Lattice Boltzmann, quark-gluon plasma, relativistic fluid dynamics

In the last decade, the Lattice Boltzmann (LB) method has attracted considerable interest as an alternative computational fluid dynamics method, based on the solution of a minimal Boltzmann kinetic equation, rather than on the discretization of the equations of continuum fluid mechanics. To date, the overwhelming majority of LB applications are directed towards classical, i.e. non-quantum and non-relativistic, fluids. However, while quantum versions of the LB equation have existed for more than a decade, to the best of our knowledge, an LB equation capable of handling relativistic fluids has not yet been proposed. In this Letter, we fill this gap and present an LB formulation for relativistic fluids. Our procedure is based on two simple and yet apparently unpursued observations, i) the kinetic formalism is naturally covariant/hyperbolic/conservative, ii) being based by construction on a finite-velocity scheme, existing lattice Boltzmann methods naturally feature relativistic-like equations of state, in the sense that the sound speed, $c_s$, is a sizeable fraction of the speed of light $c$, i.e. the maximum velocity of mass transport ($c_s/c = K$, with $0.1 < K < 1$). Based on the above, one is led to propose that, upon choosing the lattice speed $c_l = \delta x / \delta t \sim c$, the current LB mathematical framework should allow for relativistic extensions, which is indeed the case as shown in this Letter.

Our relativistic LB scheme (RLB) relies upon a moment-matching procedure similar to the one originally used to derive the earliest LB models for classical hydrodynamics. That is, the local kinetic equilibria are expressed as parametric polynomials of the relativistic fluid velocity $\vec{\beta} = \vec{u} / c$, with the parameters fixed by the condition of matching the analytic expression of the relevant relativistic moments, namely the number density, energy density and energy-momentum. As anticipated, the possibility of a successful matching stems directly from the fact that, even in standard (non-relativistic) LB fluids, the sound speed $c_s$ is of the same order of the speed of light, typically $c_s = c / \sqrt{3}$, which is exactly the equation of state of relativistic fluids. As a result, $|\vec{\beta}| = Ma / \sqrt{3}$, so that $|\vec{\beta}|$ is of the same order as the Mach number $Ma = |\vec{u}| / c_s$. Owing to this simple, and yet crucial property, it is therefore possible to tackle weakly relativistic problems much the same way as traditional LB handles classical low-Mach fluids. This spawns the exciting opportunity of carrying the assets of LB over to the context of weakly relativistic fluids, such as the important case of quark-gluon plasmas generated by recent experiments on heavy-ions and hadron jets. The RLB scheme is verified through quantitative comparison with recent one dimensional hydrodynamic simulations of relativistic shock-wave propagation in viscous quark-gluon plasmas. We can also apply our scheme to three dimensional geometries as shown in Fig. 1.

Being based on a second-order moment-matching procedure, rather than on a high-order systematic expansion in $|\vec{\beta}|$ of the local relativistic equilibrium (Jüttner)
distribution\textsuperscript{12}, the present approach is in principle limited to weakly relativistic problems, with $|\tilde{\beta}| \sim 0.1$. However, by introducing artificial faster-than-light particles (numerical “tachyons”), the present RLB scheme is shown to produce quantitatively correct results up to $|\tilde{\beta}| \sim 0.6$, corresponding to Lorentz’s factors $\gamma = \sqrt{1 - |\tilde{\beta}|^2} \sim 1.4$. Although still far from state-of-the-art numerical methods for relativistic hydrodynamics\textsuperscript{13}, the RLB might nevertheless offer a fairly inexpensive alternative to more sophisticated methods at moderate values of $|\tilde{\beta}|$. In addition, since LB is recognized as an excellent solver for flows in complex geometries, like porous media, it is plausible to expect that the present RLB scheme may play a useful role for the simulation of relativistic fluids in non-idealized geometries.

To begin our model description, we focus on the relativistic fluid equations associated with the conservation of number of particles and momentum-energy. The energy-momentum tensor reads as follows\textsuperscript{14,15}: $T^\mu\nu = P\eta^{\mu\nu} + (\epsilon + P)u^\mu u^\nu$, with $\epsilon$ the energy density and $P$ the hydrostatic pressure. The velocity 4-vector is defined by $u^\mu = (\gamma, \gamma\tilde{\beta})^\mu$, where $\tilde{\beta} = \vec{u}/c$ is the velocity of the fluid in units of the light speed and $\gamma = 1/\sqrt{1 - |\tilde{\beta}|^2}$. The tensor $\eta^{\mu\nu}$ denotes the Minkowski metric. Additionally, we define the particle 4-flow, $N^\mu = (\gamma n, n\gamma\tilde{\beta})^\mu$, with $n$ the number of particles per volume. Applying the conservation rule to energy and momentum, $\partial_\mu T^\mu\nu = 0$, and to the 4-flow, $\partial_\mu N^\mu = 0$, we obtain the hydrodynamic equations.

Note that as opposed to a non-relativistic fluid, we have two scalar equations. To complete the set of equations, we need to define a state equation that relates, at least, two of the three quantities: $n, P$ and $\epsilon$.

The above hydrodynamic equations can be derived as a macroscopic limit of the following relativistic Boltzmann-BGK equation \textsuperscript{14,15}

$$\partial_\mu (p^\mu f) = \frac{f^{eq} - f}{\tau}$$ (1)

where $p^\mu = (E(p), \vec{p}c)$ is the particle 4-momentum with $E(p)$ the relativistic energy as function of the momentum modulus $p=|\vec{p}|$, $f^{eq}$ a local relativistic equilibrium and $\tau$ the relaxation time. Lattice Boltzmann theory for classical fluids shows that it may prove more convenient to solve fluid problems by numerically integrating the underlying kinetic equation rather than the macroscopic fluid equation themselves. The main condition for this to happen is that a sufficiently economic representation of the velocity space degrees of freedom be available. Following upon consolidated experience with non-relativistic fluids, such a representation is indeed provided by discrete lattices, whereby the particle velocity (momentum) is constrained to a handful of constant discrete velocities, with sufficient symmetry to secure the fundamental conservations of fluid flows, namely mass-momentum-energy conservation and rotational invariance. The main advantages of the kinetic representation for classical fluids have been discussed at length\textsuperscript{2}, and they amount basically to the fact that the information is transported along straight-streamlines (the discrete velocities are constant in space and time) rather than along space-time dependent trajectories generated by the flow itself, as it is case for hydrodynamic equations. Moreover, diffusive transport is not described by second-order spatial derivatives, but rather emerges as a collective property from adiabatic relaxation to local equilibria. This is crucial in securing a balance between first-order derivatives in both space and time, which is essential for relativistic equations.

In order to reproduce the relativistic hydrodynamic equations, we propose an LB model with the D3Q19 cell configuration, as shown in Fig. 2. We define two distribution functions $f_i$ and $g_i$ for each velocity vector $\vec{c}_i$, where the index $i$ labels the discrete momenta within each cell. The hydrodynamic variables are calculated by imposing the following macroscopic constraints, $n\gamma = \sum_{i=0}^{18} f_i$, $(\epsilon + P)\gamma^2 - P = \sum_{i=0}^{18} g_i$, and $(\epsilon + P)\gamma^2 \vec{u} = \sum_{i=0}^{18} g_i \vec{c}_i$. From these equations, we have to extract the physical quantities $n, \vec{u}, \epsilon$ and $P$, where we have only five equations for six unknowns. The problem is closed by choosing an equation of state for ultra-relativistic fluids, $\epsilon=3P$.

The distribution functions evolve according to the BGK Boltzmann evolution equation \textsuperscript{16} (full details in a future extended publication),

$$f_i(\vec{x} + \vec{c}_i \delta t, t + \delta t) - f_i(\vec{x}, t) = -\frac{\delta t}{\tau} (f_i - f_{i}^{eq}) ,$$ (2)

and,

$$g_i(\vec{x} + \vec{c}_i \delta t, t + \delta t) - g_i(\vec{x}, t) = -\frac{\delta t}{\tau} (g_i - g_{i}^{eq}) ,$$ (3)

where $f_{i}^{eq}$ and $g_{i}^{eq}$ are the equilibrium distribution functions.

The equilibrium distribution functions that recover the relativistic fluid equations in the continuum limit, read as follows:

$$f_{i}^{eq} = w_i n\gamma \left[1 + 3 (\vec{c}_i \cdot \vec{u}) \frac{c_t}{c_i} \right] ,$$ (4)
for $i \geq 0$ and,
\[
g_{i}^{eq} = w_{i}(\epsilon + P)\gamma^{2} \left[ \frac{3P}{(P + \epsilon)\gamma^{2}c_i^2} + \frac{3(c_i \cdot \vec{u})}{c_i^2} + \frac{9}{2} \frac{(c_i \cdot \vec{u})^2}{c_i^2} - \frac{3}{2} \frac{u_i}{c_i^2} \right] ,
\]

(5)

for $i > 0$ and,
\[
g_{0}^{eq} = w_{0}(\epsilon + P)\gamma^{2} \left[ \frac{3P(2 + c_0^2)}{(P + \epsilon)\gamma^{2}c_0^2} - \frac{3}{2} \frac{u_0}{c_0^2} \right] ,
\]

(6)

for the rest particles. Here, $c_i$ is the limiting velocity of the lattice which relates the cell size and the time step $c_i = \frac{\delta x}{\delta t}$, and we have rescaled the velocity units such that the speed of light $c=1$. The weights for this set of discrete speeds are defined by $w_0 = 1/3$ for the rest particles, $w_i = 1/18$ for the velocities $|c_i| = c_i$, and $w_1 = 1/36$ for $|c_i| = \sqrt{2}c_i$.

The choice of the state equation, $\epsilon = 3P$, simplifies the equilibrium functions as follows,
\[
f_i^{eq} = w_i n\gamma \left[ 1 + 3 \frac{(c_i \cdot \vec{u})}{c_i^2} \right] ,
\]

(7)

for $i \geq 0$ and,
\[
g_i^{eq} = w_i \epsilon\gamma^{2} \left[ \frac{1}{\gamma^2 c_i^2} + \frac{4(c_i \cdot \vec{u})}{c_i^2} + 6 \frac{(c_i \cdot \vec{u})^2}{c_i^2} - 2 \frac{|\vec{u}|^2}{c_i^2} \right] \]

(8)

for $i > 0$ and,
\[
g_0^{eq} = w_0 \epsilon\gamma^{2} \left[ \frac{4 - 2 + c_0^2}{\gamma^2 c_0^2} - 2 \frac{|\vec{u}|^2}{c_0^2} \right] ,
\]

(9)

for $i = 0$. Then, the equations for the macroscopic variables take the form: $n\gamma = \sum_{i=0}^{18} f_i$, $\frac{4}{3}\epsilon (\gamma^2 - \frac{1}{4}) = \sum_{i=0}^{18} g_i^2$ and $\frac{4}{3}\epsilon \gamma^2 \vec{u} = \sum_{i=0}^{18} g_i \vec{c}_i$. In our model, the shear viscosity is computed according to standard LB procedures as: $\eta = \frac{4}{5} \gamma^2 \epsilon (\tau - \delta t/2) c_i^2$.

To test the model we solve the Riemann problem in viscous gluon matter [11]. We use the state equation for ultra-relativistic fluids $\epsilon = 3P$, as before, and the relation between energy density and particle number density, $\epsilon = 3nT$, with $T$ the temperature [14]. The initial configuration consists of two regions divided by a membrane located at $z=0$. Both regions have thermodynamically equilibrated matter with different constant pressure, $P_0$ for $z<0$ and $P_1$ for $z>0$. At $t=0$ the membrane is removed.

We implement a one-dimensional simulation with an array of size $1 \times 1 \times 800$. In this case, the 4-velocity is given by $u^\mu = (\gamma, 0, 0, \gamma \beta)^\mu$. The velocity of the lattice is chosen $c_i = 1.0$, therefore the cell size $\delta x$ and time step $\delta t$ are fixed to unity. This corresponds in IS units to $\delta x = 0.008\text{fm}$ and $\delta t = 0.008\text{fm/c}$. The viscosity is calculated through $\eta = \frac{4}{5} \gamma^2 \epsilon (\tau - 1/2)$, and the entropy density by the approximation $s = 4n - n \ln \lambda$, with $\lambda = \frac{n}{n_\text{eq}}$ the
gluon fugacity and the equilibrium particle density \( n^\text{eq} \) are given by, \( n^\text{eq}=\frac{2c_2}{\pi^2} \), with \( c_2=16 \) for gluons. Now, we can calculate the ratio between the viscosity and entropy density, \( \eta/s \), that is used as a parameter to characterize the shock-wave. The pressures were chosen \( P_0=5.43\text{GeVfm}^{-3} \) and \( P_1=2.22\text{GeVfm}^{-3} \), corresponding to \( 7.9433\times10^{-6} \) and \( 3.2567\times10^{-6} \) lattice units, respectively. The initial temperature is \( T_0=350\text{MeV} \), corresponding to \( T_0=0.0287 \) lattice units. With these parameters, the conversion between physical and numerical units for the energy, is \( 1\text{MeV}=8.2\times10^{-5} \).

Fig. 4 shows the results for different values of \( \eta/s \) and the comparison with the BAMPS formulation (Boltzmann Approach of Multiparton Scattering) microscopic transport model simulations at time \( 3.2\text{fm/c} \). On the other hand, in Fig. 5 we can see the evolution of the system for \( \eta/s=0.1 \) comparing the two numerical models. In both cases, we find an excellent agreement with BAMPS. To simulate fluid moving at higher speed, \( \beta\sim0.6 \), we use numerical “tachyons” with \( c_l=10 \). Indeed, from Eqs. 4 and 5 it is seen that the positivity condition \( f_i^l>0 \) implies \( \vec{c}_l\cdot\vec{u}<\frac{c_l^2}{2} \). Now, the pressure \( P_1 \) is taken as \( 0.9532\text{GeVfm}^{-3} \) and we define two temperatures, \( T_0=0.0328 \) and \( T_1=0.0164 \), the first one for \( z<0 \) and the second one for \( z>0 \). Fig. 4 shows the shockwave for \( \eta/s=0.001 \) and the comparison with the BAMPS simulation, is again excellent. Our LB scheme easily extends to three dimensions, as is illustrated in Fig. 1 where we simulate the collision of a relativistic shock wave with a fixed spherical obstacle. A typical \( 200\times100\times100 \) lattice-site simulations spanning 1350 timesteps, takes about 1900 CPU seconds on a standard PC.

Summarizing, we have developed a Lattice Boltzmann formulation for (mildly) relativistic fluids. One of the major areas of application of non-relativistic LB schemes is flow through geometrically complex domains with internal obstacles, such as porous media. It is therefore expected that the present RLB scheme may become useful for the simulation of relativistic fluids in complicated geometries.

Acknowledgements

SS would like to acknowledge kind hospitality and financial support from ETH Zürich.