A complete $h$-vector for convex polytopes

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Abstract
This note defines a complete $h$-vector for convex polytopes, which extends the already known toric (or mph) $h$-vector and has many similar properties. Complete means that it encodes the whole of the flag vector.

First we define the concept of a generalised $h$-vector and state some properties that follow. The toric $h$-vector is given as an example. We then define a complete generalised $h$-vector, and again state properties.

Finally, we show that this complete $h$-vector and all with similar properties will sometimes have negative coefficients.

Most of the proofs, and further investigations, will appear elsewhere.

1 Generalised $h$-vectors

This note defines a complete $h$-vector for convex polytopes, and states some of its properties. Background, motivation and most of the proofs will be given elsewhere [3]. Prior knowledge of the toric $h$-vector, for example as in [6] or [2, §4.1], will help the reader. Throughout $\Delta$ will denote a convex polytope of dimension $d$. We study linear functions $h = h(\Delta)$ of the flag vector $f = f(\Delta)$ of $\Delta$. When conversely $f(\Delta)$ can be computed from $h(\Delta)$ we say that $h$ is complete.

Let $\delta$ be a face of $\Delta$, of dimension $i$. Associated with $\delta \subseteq \Delta$ there is the link $L_\delta$, a convex polytope of dimension $d - i - 1$, which encodes the local geometry of $\Delta$ around $\delta$. For example, if $\delta$ is a vertex then around that vertex $\Delta$ looks like $CL_\delta$, when $C$ is the cone or pyramid operator. Although $L_\delta$ is determined only up to projective equivalence, its flag vector is an invariant of $\delta \subseteq \Delta$. It is convenient to set $L_{\Delta} = \emptyset$ and $C\emptyset = pt.$

Throughout $C$ and $I$ denote the cone and cylinder (or pyramid and prism) operators, and we think of $D = IC - CC$ (1)

and $I$ and $C$ as operators on flag vectors. The total link vector $\ell = \ell(\Delta)$ has components $\ell_i = \sum_{\dim L_\delta = i} f(L_\delta)$. Many of the results rely on $\ell(I \Delta) = (1 + 2C)\ell(\Delta)$ and $\ell(C \Delta) = (1 + C)\ell(\Delta) + f(\Delta)$. In [3] the author shows $DI = ID$. This is a partial expression of the next result, upon which the definition of $h$ relies.

Theorem 1 (Bayer-Biller [1], generalised Dehn-Sommerville). The flag vectors produced by applying all words $W$ in $C$ and $D$ to a point are a basis for the vector space spanned by all convex polytope flag vectors.

Proposition 2. Suppose $g$ is a linear function of flag vectors. Then
$$h(\Delta) = \sum_{\delta \subseteq \Delta} (x - y)^{\dim \delta} g(L_\delta)$$

is also a linear function of flag vectors.

Definition 3. If $h(\Delta)$ is as in Proposition [2] and in addition both of

1. $g(\emptyset) = 1$ (or equivalently $h(pt) = 1$).
2. $g(CL) = yg(L)$.

hold then we will say that $h$ is a generalised $h$-vector.
Proposition 4. Suppose \( h \) is a generalised \( h \)-vector and \( \Delta \) is a convex polytope. Then

1. \( h(I\Delta) = (x + y)h(\Delta) \).
2. \( h(C\Delta) = g(\Delta) + x h(\Delta) \), and thus we can define \( g \) from \( h \).
3. \( h(D\Delta) = x y h(\Delta) \) (which follows from the two previous statements).
4. If \( \Delta \) is simple then \( h(\Delta) = \sum_{i=0}^{d} (x - y)^i y^i f_i(\Delta) \), the usual formula for simple polytopes.
5. If \( \Delta_1 \) is simple and \( \Delta_2 \) is any convex polytope then \( h(\Delta_1 \times \Delta_2) = h(\Delta_1) h(\Delta_2) \).

Corollary 5. Suppose there is a complete generalised \( h \)-vector. Then as operators on flag vectors \( DI = ID \).

Proof. We have \( h(DI\Delta) = h(ID\Delta) \). If \( h \) is complete \( f \) is a linear function of \( h \) and so \( f(DI\Delta) = f(ID\Delta) \).

Proposition 6. Suppose \( h \) is a linear function of flag vectors with \( h(pt) = 1 \), \( h(I\Delta) = (x + y)h(\Delta) \) and \( h(D\Delta) = x y h(\Delta) \) for any convex polytope \( \Delta \). Then \( h \) is a generalised \( h \)-vector (with \( g(\Delta) = h(C\Delta) - xh(\Delta) \)).

Proposition 7. Suppose \( g(DW pt) \) is known for all words \( W \) in \( C \) and \( D \). Then \( g \) determines a unique generalised \( h \)-vector. (This follows from Theorem \[4\] \( g(CL) = yg(L) \), \( g(\emptyset) = 1 \) and linearity.)

Proposition 8. The formula

\[
g(DL) = xy g(L)
\]

defines the toric (or middle perversity intersection homology or mpih) \( h \)-vector, as in \[6\] or \[2\] §4.1.

2 A complete \( h \)-vector

For simple polytopes \( h(x, y) = h(y, x) \) or in other words \( h \) is palindromic. This is a very important property. We will use the following notation. We denote, for example, \( ay^2 + bxy + c^2x \) by \( [a, b, c] \), and \([1, 1, 1] \) by \([2]\). We denote \((xy)^i[j] \) by \([i, j] \). Thus \([0, j] = [j] \) and \([1, 0] = [0, 1, 0] = xy \).

Definition 9 (Keyed and palindromic generalised \( h \)-vectors). Suppose a generalised \( h \)-vector has the form

\[
h(\Delta) = \sum h_k(\Delta) w_k
\]

where each \( h_k \) is a homogeneous polynomial, each \( k \) is a key as defined below and \( w_k \) its associated symbol. Suppose also that \( \dim \Delta = \deg h_k + \deg k \). If all this holds, we say that \( h \) is a keyed generalised \( h \)-vector. If each \( h_k \) is palindromic we say that \( h \) is palindromic.

For the rest of this note \( h \) denotes the palindromic keyed generalised \( h \)-vector we are about to define. Recall that by Proposition \[7\] it is enough to define \( g \) on polytopes of the form \( g(DW pt) \).

Definition 10 (Complete generalised \( h \)-vector). Suppose \( h(v) = [i, j] w_k \). Then

\[
g(Dv) = (xy)^{i+j+1} w_k + w_{k'}
\]

where \( k' \) is as below. This we extend linearly to \( h(v) = \sum \lambda_{ijk}[i, j] w_k \) and so to \( v = W pt \).

It is clear that \( \deg k' = \deg k + 2i + j + 3 \). The definition assumes that \( h(W pt) \) is palindromic. It turns out to be the same as \[3\] except for the addition of \( w_k \). This addition has, of course, recursive consequences. The author has developed software \[4\] for \( h \)-vector computations. In particular \[4\] contains a human and machine readable table of \( h \)-vectors for \( CD \) polytopes up to dimension 10.

Definition 11 (\( h \)-key). If \( k = ((d_1, \ldots, d_r), (c_1, \ldots, c_r)) \) then \( k' = ((i, d_1, \ldots, d_r), (j, c_1, \ldots, c_r)) \). As a shorthand we sometimes write, for example, \( k = ((1, 3, 2), (0, 2, 1)) \) as \( 132:021 \). We write \( e \) for the empty key \(((),())\), and set \( w_e = 1 \). Thus, \( h(pt) = [0, 0] w_e = 1 \). We use \( deg k = 2 \sum d_i + \sum c_i + 3r \) to define the degree of \( k \).
Example 12. Suppose \( h(v) = [1, 1]w_k \), which we can write as \([0, 1]w_k\). Then
\[
g(Dv) = (xy)^3y^2w_k + w_{k'} = [0, 1, 0, 0, 0]w_k + w_{k'} \tag{6}
\]
\[
h(CDv) = g(DV) + xh(Dv) = [0, 1, 0, 0, 0]w_k + w_{k'} + [0, 0, 1, 1, 0]w_k \\
= [0, 1, 1, 1, 0]w_k + w_{k'} = [1, 2]w_k + w_{k'} \tag{7}
\]

Proposition 13. If \( h(v) = [i, j]w_k \) then \( h(CDv) = [i + 1, j + 1]w_k + w_{k'} \).

Proposition 14. If \( h(Cv) = \lambda_{ijk} [i, j]w_k \) then \( h(CCv) = \lambda_{ijk} [i, j + 1]w_k \).

Example 15. If \( h(v) = w_{0, 0} \) then \( v = (CD - DC)pt \), because \( h(CDpt) = [1, 1] + w_{0, 0} \) and \( h(DCpt) = [1, 1] \). We write \( w_{0, 0} \) as \([0, 0]w_{0, 0}\). We now have
\[
h(CCDpt) = [1, 2] + [0, 1]w_{0, 0} \tag{9}
\]
\[
h(CDCpt) = [1, 2] + w_{0, 1} \tag{10}
\]
\[
h(Cv) = [0, 1]w_{0, 0} - w_{0, 1} \tag{11}
\]

This shows that sometimes \( h(v) = [i, j]w_k \) does not imply \( h(Cv) = [i, j + 1]w_k \).

Proposition 16 (Converse to Proposition 13). If \( h(Cv) = \sum \lambda_{ijk} [i, j + 1]w_k \) then \( v = Cv' \) for some \( v' \).

Theorem 17. Let \( g \) be as in Definition 10. Then
1. There is a unique extension of \( g \) to a generalised h-vector \( h \).
2. \( h(\Delta) \) is palindromic.
3. \( h_c \) is the toric h-vector.
4. The matrix for \( g \), for the \( CD \) and \([i, j]w_k \) bases, is upper triangular with ones along the diagonal.
5. \( h \) is complete.
6. Theorem 1 can be proved as part of the \( g \)-h recursion.

Unlike the toric/mpih h-vector, the complete h-vector \( h \) can have negative coefficients. This is unavoidable.

Example 18 (Bayer, personal communication). Let \( P \) be the bipyramid on the 3-simplex. Then
\[
h(P) = [1, 4, 10, 4, 1] + [6, 6]w_{0, 0} + [-4]w_{0, 1} \tag{12}
\]

Proposition 19. Suppose \( h' \) is a complete palindromic keyed generalised h-vector. Let \( P \) be as in Example 18 and let \( Q \) be \( CICCpt \). Write \( h'_{0,1}(P) = [a] \) and \( h'_{0,1}(Q) = [b] \). Then \( a > 0 \) and \( b < 0 \) or vice versa.

Sketch of proof. The general form of \( h'(\Delta) \) for \( \Delta = 3 \) is \([a_0, a_1, 1, a_0] + [b_0]w_{0, 0} \) and Proposition 1 determines \( h' \) on \( CCCpt \) and \( DCpt \). Both have \( b_0 = 0 \) and so, for \( h' \) to be complete, \( h'(CDpt) \) has \( b_0 = 0 \). Similarly the general form of \( h'(\Delta) \) for \( \Delta = 4 \) is \([a_0, a_1, a_2, a_1, a_0] + [b_0, b_1]w_{0, 0} + [c_0]w_{0, 1} \) and Proposition 1 determines \( h' \) on \( CCCCpt \) \( DCCpt \), \( DDpt \). They all have \( c_0 = 0 \). Proposition 14 shows the same for \( CDpt \).

For \( h' \) to be complete \( c_0 \) must be non-zero for some convex polytope. Think of \( c_0 \) as a linear function on 4-polytope flag vectors. It is non-zero and vanishes on the hyperplane \( H \) spanned by \( f(CCCCpt) \) \( f(DCCpt) \), \( f(DDpt) \) and \( f(CDpt) \). Finally, by a calculation we omit, \( f(P) \) and \( f(Q) \) are separated by \( H \). The result follows.

References

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