Abstract: We propose a simple mechanism which enforces $|U_{\mu j}| = |U_{\tau j}| \forall j = 1, 2, 3$ in the lepton mixing matrix $U$. This implies maximal atmospheric neutrino mixing and a maximal CP-violating phase but does not constrain the reactor mixing angle $\theta_{13}$. We implement the proposed mechanism in two renormalizable seesaw models which have features strongly resembling those of models based on a flavour symmetry group $\Delta(27)$. Among the predictions of the models, there is a determination, although ambiguous, of the absolute neutrino mass scale, and a stringent correlation between the absolute neutrino mass scale and the effective Majorana mass in neutrinoless double-beta decay.

Keywords: Discrete and Finite Symmetries, Beyond Standard Model, Neutrino Physics
1 Introduction

With the recent results of the Double Chooz, Daya Bay, and RENO Collaborations [1–3] the earlier hints [4, 5] of a non-zero reactor mixing angle $\theta_{13}$ have been confirmed. The unexpectedly large value of $\theta_{13}$ [1–3, 6, 7] renders a $\mu-\tau$ interchange symmetry in the neutrino mass matrix [8–14], and therefore also tri-bimaximal mixing [15], highly unlikely, since that symmetry was tailored to achieve $\theta_{13} = 0$ at some energy scale. However, there is a different version of the $\mu-\tau$ interchange symmetry, which is based on a generalized CP transformation that includes the $\mu-\tau$ interchange [16, 17]. In this version, the maximal atmospheric mixing angle $\theta_{23}$ is not coupled with a vanishing $\theta_{13}$ but rather with a maximal CP-violating phase $\delta$ in the lepton mixing matrix. Phenomenologically, this scenario is fully viable.

In this letter we introduce a new mechanism for generating this type of lepton mixing. In order to establish our notation we firstly define the lepton mass Lagrangian as

$$L_{\text{mass}} = -\bar{\ell}_L M_\ell \ell_R + \frac{1}{2} \nu_L^T C^{-1} M_\nu \nu_L + \text{H.c.},$$

(1.1)

with $M_\ell$ and $M_\nu$ being the mass matrices of the charged leptons and of the neutrinos, respectively; the latter mass matrix is of the Majorana type. Those mass matrices are diagonalized by $3 \times 3$ unitary matrices $U_\ell$ and $U_\nu$ according to

$$U_\ell^T M_\ell U_\ell = \text{diag}(m^2_e, m^2_\mu, m^2_\tau),$$

(1.2a)

$$U_\nu^T M_\nu U_\nu = \text{diag}(m_1, m_2, m_3),$$

(1.2b)

respectively. Then the lepton mixing matrix $U$ is given by

$$U = U_\ell^T U_\nu.$$

(1.3)
Our idea is the following. Suppose that we have a model which gives a real matrix \( U_\nu \) and
\[
U_\ell = U_\omega \equiv \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix},
\]
(1.4)
where \( \omega = (-1 + i\sqrt{3})/2 \). Then, it is trivial to see that the mixing matrix \( U \) has the property
\[
|U_{\mu j}| = |U_{\tau j}|, \quad \forall j = 1, 2, 3.
\]
(1.5)
When using the standard parameterization of the mixing matrix [18], the relations (1.5) require [16, 17]
\[
\cos \theta_{23} = \sin \theta_{23} = \frac{1}{\sqrt{2}},
\]
\[
\sin \theta_{13} \cos \delta = 0,
\]
whence it follows
\[
\theta_{23} = \frac{\pi}{4},
\]
\[
\delta = \pm \frac{\pi}{2},
\]
(1.7a)
(1.7b)
since we know that \( \theta_{13} \neq 0 \).

This paper is organized as follows. In section 2 we develop two seesaw models based on the idea laid out above. These models make predictions beyond those in (1.7); the extra predictions are presented in section 3. The conclusions of the paper are summarized in section 4.

2 The models

The fermion sectors of our models contain the usual leptonic Standard Model multiplets, namely three left-handed gauge-SU(2) doublets, subsumed under the symbol \( D_L \), and three right-handed charged-lepton gauge singlets, subsumed under the symbol \( \ell_R \). The scalar sectors contain three Higgs doublets with weak hypercharge 1/2, which we subsume under the symbol \( \phi \). For the symmetry transformations of the models we make use of the matrices
\[
E = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad A = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad C = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}.
\]
(2.1)
We define the bilinears
\[
[\phi_\ell R]_0 = \phi_1 \ell_{1R} + \phi_2 \ell_{2R} + \phi_3 \ell_{3R},
\]
(2.2a)
\[
[\phi_\ell R]_1 = \phi_1 \ell_{1R} + \omega \phi_2 \ell_{2R} + \omega^2 \phi_3 \ell_{3R},
\]
(2.2b)
\[
[\phi_\ell R]_2 = \phi_1 \ell_{1R} + \omega^2 \phi_2 \ell_{2R} + \omega \phi_3 \ell_{3R},
\]
(2.2c)
which are analogous to the ones used in models based on the symmetry $A_4$ [19]. Indeed, under the transformations
\begin{align}
S : \quad \ell_R &\to A\ell_R, \quad \phi \to A\phi, \quad (2.3a) \\
T : \quad \ell_R &\to E\ell_R, \quad \phi \to E\phi, \quad (2.3b)
\end{align}
the bilinears transform as
\begin{align}
[\phi\ell_R]_j &\xrightarrow{S} [\phi\ell_R]_j, \quad (2.4a) \\
[\phi\ell_R]_j &\xrightarrow{T} \omega^{2j}[\phi\ell_R]_j. \quad (2.4b)
\end{align}
These transformation properties allow us to write down the charged-lepton Yukawa Lagrangian
\[ L_Y^{(\ell)} = -\sum_{j=1}^{3} h_j \bar{D}_j L [\phi\ell_R]_j + \text{H.c.} \quad (2.5) \]
if we supplement (2.3b) by
\[ T : \quad D_L \to C^2 D_L. \quad (2.6) \]
The charged-lepton Yukawa Lagrangian (2.5) looks very similar to the one of some $A_4$-based models [19–21], but that look is misleading — in our models the roles of $D_L$ and $\ell_R$ are reversed relative to the $A_4$-based models. This can also be seen by computing the mass matrix of the charged leptons, which in our models is
\[ M_\ell = \text{diag}(h_1, h_2, h_3)(\sqrt{3}U_\omega)\text{diag}(v_1, v_2, v_3), \quad (2.7) \]
where $v_j$ denotes the vacuum expectation value (VEV) of the neutral component of the Higgs doublet $\phi_j$. Thus, $U_\omega^\dagger M_\ell$ is diagonal if $h_1 = h_2 = h_3$, i.e. if the Yukawa coupling constants are all equal, whereas in the $A_4$ models one needs equality of VEVs. The equality of the $h_j$ is achieved by assuming invariance of $L_Y^{(\ell)}$ under
\[ T' : \quad D_L \to E D_L, \quad \ell_R \to C\ell_R. \quad (2.8) \]
Therefore, in the following we shall use
\begin{align}
&h_1 = h_2 = h_3 \equiv h, \quad (2.9a) \\
&U_\omega^\dagger M_\ell = \sqrt{3}h \text{diag}(v_1, v_2, v_3). \quad (2.9b)
\end{align}
We emphasize that in our models we do not make any assumption of alignment of the VEVs of the Higgs doublets, viz. we require neither any equality among the $v_j$ nor that any of them vanishes. On the other hand, we do require a strong hierarchy of the VEVs; indeed, it follows from equation (2.9b) that $m_e = \sqrt{3}|hv_1|$, etc., and, therefore,
\[ |v_1| : |v_2| : |v_3| = m_e : m_\mu : m_\tau. \quad (2.10) \]
2.1 Model I

In this model we use the type I seesaw mechanism [22–26]. The lepton sector contains three right-handed neutral gauge singlets $\nu_R$ and the scalar sector contains a fourth Higgs doublet, $\phi$. Additionally, there are three complex scalar singlets, which we subsume under the symbol $\eta$. The Higgs doublet $\phi$ is invariant under $S$, $T$, and $T'$, while $\nu_R$ and $\eta$ transform in the same way as $D_L$. A summary of the multiplets and their transformation properties is presented in table 1. Notice that the symmetries $T$ and $T'$ together generate a group $\Delta(27)$ under which $D_L$, $\ell_R$, $\nu_R$, and $\eta$ are identical triplets while $\phi$ decomposes into non-equivalent singlets.\(^1\)

In this way we obtain the neutrino Yukawa couplings

$$\mathcal{L}_Y^{(\nu)} = -y_\nu \bar{D}_L \tilde{\phi} \nu_R$$  \hspace{1cm} (2.11a)

$$+ \frac{y}{2} \sum_{j=1}^{3} \eta_j \nu_{jR}^T C^{-1} \nu_{jR}$$  \hspace{1cm} (2.11b)

$$+ y' (\nu_{2R}^T C^{-1} \nu_{3R} \eta_1 + \nu_{3R}^T C^{-1} \nu_{1R} \eta_2 + \nu_{1R}^T C^{-1} \nu_{2R} \eta_3) + \text{H.c.}$$  \hspace{1cm} (2.11c)

The neutrino Dirac mass matrix $M_D$, which originates in (2.11a), is proportional to the unit matrix. We assume the VEVs $\langle \eta_j \rangle_0 = s_j$ to be at a high (seesaw) scale. Therefore, the inverse neutrino mass matrix has the form

$$M^{-1}_\nu = \begin{pmatrix} \zeta a & c & b \\ c & \zeta b & a \\ b & a & \zeta c \end{pmatrix},$$  \hspace{1cm} (2.12)

with $\zeta^* = y/y'$. The inverse mass matrix (2.12) has the typical form of mass matrices in renormalizable models based on the group $\Delta(27)$ [28, 29]. This is understandable since, as we have pointed out, the fields $D_L$, $\nu_R$, and $\eta$ behave under $T$ and $T'$ as irreducible three-dimensional representations of $\Delta(27)$.

As explained in the introduction, we need the neutrino mass matrix $M_\nu$ to be real. The first step in this direction is to assume in the Lagrangian a CP symmetry which renders

\(^1\)We may speculate about the full symmetry group of the model. Taking into account also $S$, the column with caption $\ell_R$ of table 1 suggests $AC$ to be a generator of the symmetry group; it has the sixth root of unity, $-\omega^2$, in its diagonal. So we might naively guess $\Delta(108) \equiv \Delta(3 \times 6^2)$ to be the symmetry group. However, beyond the symmetries listed in table 1 the model possesses an accidental $2 \leftrightarrow 3$ interchange symmetry:

$$D_{2L} \leftrightarrow D_{3L}, \quad \ell_{2R} \leftrightarrow \ell_{3R}, \quad \phi_2 \leftrightarrow \phi_3, \quad \nu_{2R} \leftrightarrow \nu_{3R}, \quad \eta_2 \leftrightarrow \eta_3.$$  

Therefore, the full symmetry group is $\Delta(216) = \Delta(6 \times 6^2)$. For a discussion of the groups $\Delta(6n^2)$ see [27].
h, \ y_\nu, \ y, \text{ and } y' \text{ real. That symmetry is given by}

\begin{align}
\text{CP} : \quad & \begin{cases}
D_L \to iCD_L^*, & \ell_R \to iSC\ell_R^*, & \nu_R \to iC\nu_R^*, \\
\phi \to S\phi^*, & \phi_\nu \to \phi_\nu^*, & \eta \to \eta^*,
\end{cases} \\
& (2.13)
\end{align}

where C is the charge-conjugation matrix in Dirac space while

\begin{align}
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\end{align}

acts in flavour space. This matrix S is needed in order to interchange the terms \(\phi_2\ell_2R\) and \(\phi_3\ell_3R\) in each of equations (2.2).

Next we discuss the scalar potential of the three \(\eta_j\). It has six terms compatible with \(T, T', \text{ and the CP symmetry:}

\begin{equation}
V_\eta = \sum_{j=1}^{3} \left( \mu |\eta_j|^2 \right) + \lambda_1 \left( |\eta_1\eta_2|^2 + |\eta_1\eta_3|^2 + |\eta_2\eta_3|^2 \right) \\
+ M_1 (\eta_1\eta_2\eta_3 + \text{H.c.}) + M_2 (\eta_1^3 + \eta_2^3 + \eta_3^3 + \text{H.c.}) \\
+ \lambda_3 \left( \eta_1^2\eta_2\eta_3 + \eta_2^2\eta_1\eta_3 + \eta_3^2\eta_1\eta_2 + \text{H.c.} \right). \tag{2.15}
\end{equation}

All six constants in \(V_\eta\) are real: \(\mu, \lambda_1, \text{ and } \lambda_2\) are real because the potential is Hermitian and \(M_1, M_2, \text{ and } \lambda_3\) are real because of the CP symmetry (2.13). The latter three terms in the potential are responsible for the relative phases of the VEVs \(s_j\). If we choose negative \(M_1, M_2, \text{ and } \lambda_3\), then at the minimum of the potential those VEVs will have phases given by [30]

\begin{equation}
\arg s_j = \omega^{p_j}, \tag{2.16}
\end{equation}

where the \(p_j\) are integers such that \(p_1 + p_2 + p_3 = 0 \mod 3\). This is precisely what is needed in order for the matrix \(M_\nu^{-1}\) to be real apart from unphysical phases.

It can be shown that the potential (2.15) is rich enough to allow for the \(|s_j|\) to be all different, as needed in our model.

Since the model has four Higgs doublets, one might be tempted to argue that, after switching to a basis in the space of the Higgs doublets where only one of them has a non-vanishing VEV, precisely that doublet with VEV corresponds to the Higgs doublet of the Standard Model and all other doublets can be made heavy [31]. However, this argument is only applicable in the general case where the Higgs potential is not restricted by family symmetries. But, if one accepts the possibility of soft breaking of the CP and family symmetries through terms of dimension two in the scalar potential, then one can apply the above argument in a modified way. Ignoring, for the time being, the gauge singlets \(\eta_j\), we may assume that the term \(\phi_\nu^\dagger\phi_\nu\) has negative sign while the \(3 \times 3\) matrix of mass-squared
terms for the φ_j is positive definite. Then, the VEVs v_j are induced by the VEV of φ_ν through the soft-breaking terms φ_j^\dagger φ_j [32]. In this setting, the role of the Higgs doublet of the Standard Model is played by φ_ν, while the φ_j are additional doublets that can be made heavy. When one includes in the scalar potential terms containing products of both singlets η and doublets φ, the potential becomes even more versatile with respect to our goal.

The models in this paper are designed for the lepton sector. It is straightforward, however, to accommodate the quarks by using the doublet φ_ν to give them masses, thus playing the role of the Standard Model’s sole Higgs doublet. In this setting, the doublets φ_j do not play any role in the quark sector. A problem arises, however, since we have in the model a CP symmetry and we know CP to be violated in the hadron sector. This problem may be solved by adding to the model one or more extra scalar doublets transforming under the various symmetries in exactly the same way as φ_ν; all those doublets will have Yukawa couplings to the quarks and, if their VEVs acquire relative phases through the mechanism of spontaneous CP violation, then the quark mixing matrix will be complex, yet the predictions for the lepton sector will stay unchanged, because the Dirac mass matrix M_D will just acquire an overall phase which may be rotated away.

2.2 Model II

In this model we use the type II seesaw mechanism [33–35]. The lepton sector is identical with the one of the Standard Model, i.e. no right-handed neutrino singlets are present. The scalar sector contains, besides the three Higgs doublets in φ, three gauge-SU(2) triplets with weak hypercharge 1, which we subsume under the symbol ∆:

$$\Delta = \begin{pmatrix} \Delta^+ / \sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+ / \sqrt{2} \end{pmatrix}.$$  

(2.17)

We assume ∆ to transform under S, T, and T' in exactly the same way as D_L. This allows us to write down the Yukawa couplings [36–38]

$$\mathcal{L}_Y^{(\Delta)} = \tilde{y} \sum_{j=1}^{3} D_{jL}^T C^{-1} \varepsilon \Delta_j D_{jL} \quad (2.18a)$$

$$+ \tilde{y}' (D_{2L}^T C^{-1} \varepsilon \Delta_1 D_{3L} + D_{3L}^T C^{-1} \varepsilon \Delta_2 D_{1L} + D_{1L}^T C^{-1} \varepsilon \Delta_3 D_{2L}) + \text{H.c.},$$  

(2.18b)

where

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  

(2.19)

acts in gauge-SU(2) space. When the neutral components of ∆ acquire VEVs 〈Δ_j^0〉_0 = δ_j we obtain

$$\mathcal{M}_\nu = \begin{pmatrix} \tilde{y} \delta_1 & \tilde{y}' \delta_3 & \tilde{y}' \delta_2 \\ \tilde{y}' \delta_3 & \tilde{y} \delta_1 & \tilde{y}' \delta_1 \\ \tilde{y}' \delta_2 & \tilde{y}' \delta_1 & \tilde{y} \delta_3 \end{pmatrix}.$$  

(2.20)

This is of the same form as the M^{-1}_\nu in equation (2.12).
From the transformation properties of the scalar doublets $\phi$ and triplets $\Delta$, it is obvious that the scalar potential cannot have a trilinear term of the form $\phi^{\dagger} \Delta \phi^*$ invariant under $S$, $T$ and $T'$. Absence of such a term leads to a Goldstone boson since the scalar potential becomes invariant under separate phase transformations of the $\phi$ and $\Delta$. In order to avoid the Goldstone boson one must resort to soft breaking of the flavour symmetries. We shall not pursue this issue further and in the following we shall simply assume that there is a satisfying solution which leads, moreover, to real VEVs at the minimum of the potential.

3 Predictions for neutrino masses and lepton mixing

We first discuss the predictions of model I. It is convenient to use the weak basis where the charged-lepton mass matrix is diagonal. In that basis the neutrino mass matrix is

$$M_{\nu}^{(w)} = U^T \ell M_\nu U_\ell. \quad (3.1)$$

Since in our case $U_\ell = U_\omega$, we find

$$M_{\nu}^{(w)}^{-1} = U_\omega^\dagger M_\nu^{-1} U_\omega^* = \begin{pmatrix} \bar{\zeta} & \bar{\alpha} & \bar{b} \\ \bar{c} & \bar{\zeta} \bar{b} & \bar{a} \\ \bar{b} & \bar{a} & \bar{\zeta} \bar{c} \end{pmatrix}, \quad (3.2)$$

where

$$\bar{a} = \frac{\zeta - 1}{3}(a + b + c), \quad (3.3a)$$
$$\bar{b} = \frac{\zeta - 1}{3}(a + \omega^2 b + \omega c), \quad (3.3b)$$
$$\bar{c} = \frac{\zeta - 1}{3}(a + \omega b + \omega^2 c), \quad (3.3c)$$
$$\bar{\zeta} = \frac{\zeta + 2}{\zeta - 1}. \quad (3.3d)$$

We know that $a$, $b$, $c$, and $\zeta$ are real. Therefore, $\bar{a}$ and $\bar{\zeta}$ are real too, and $\bar{c} = \bar{b}^*$. Therefore, the inverse mass matrix (3.2) has the form

$$M_{\nu}^{(w)}^{-1} = \begin{pmatrix} x & y & y^* \\ y & z & w \\ y^* & w & z^* \end{pmatrix}, \text{ with } x \text{ and } w \text{ real.} \quad (3.4)$$

This type of matrices was discussed in [17], where it was shown that its diagonalizing unitary matrix, which in the present case is the complex conjugate of the lepton mixing matrix, i.e. $U^*$, is of the form [16, 17]

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ w'_1 & w'_2 & w'_3 \end{pmatrix}, \quad (3.5)$$

where the $u_j$ are real. Note that the phase convention inherent in equation (3.5) is that $U$ diagonalizes $M_{\nu}^{(w)}$ up to arbitrary signs of the masses. Thus, we have

$$U^\dagger M_{\nu}^{(w)}^{-1} U^* = \text{diag} \left( \frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{1}{\mu_3} \right), \text{ with } \mu_j = \epsilon_j m_j, \; \epsilon_j = \pm 1. \quad (3.6)$$
The phase factors appearing in the effective neutrino mass $m_{\beta\beta}$ of neutrinoless double-beta decay are identical with the $\epsilon_j$, i.e.

$$m_{\beta\beta} = \left| \sum_{j=1}^{3} u_j^2 \mu_j \right| = \left| \sum_{j=1}^{3} u_j^2 \epsilon_j m_j \right|. \tag{3.7}$$

There are nine physical quantities in the neutrino sector: three neutrino masses, three mixing angles, one Dirac-type CP-violating phase, and two Majorana phases. The mass matrix (3.2) has four real parameters, consequently the model must make five predictions: the atmospheric mixing angle $\theta_{23}$ is $\pi/4$, the Dirac-type phase $\delta$ is $\pm\pi/2$, each of the two Majorana phases is either 0 or $\pi$, and the remaining prediction can be gathered from inspection of equation (3.2), whence we deduce that

$$(M_{\nu}^{-1})_{11}(M_{\nu}^{-1})_{13} = \tilde{\zeta} \alpha \beta = (M_{\nu}^{-1})_{22}(M_{\nu}^{-1})_{23}. \tag{3.8}$$

This identity is not meaningful in its phase, because $M_{\nu}^{-1}$ may be rephased at will,

$$(M_{\nu}^{-1})_{\alpha\beta} \rightarrow (M_{\nu}^{-1})_{\alpha\beta} e^{i(\psi_\alpha + \psi_\beta)}. \tag{3.9}$$

However, the equality of the moduli of both sides of equation (3.8) is physically meaningful. The physical content of this relation will be worked out in the following.

Due to equations (3.5) and (3.6) one has

$$(M_{\nu}^{-1})_{11} = \sum_{j=1}^{3} \frac{u_j^2}{\mu_j}, \tag{3.10a}$$

$$(M_{\nu}^{-1})_{13} = \sum_{j=1}^{3} \frac{u_j w_j^*}{\mu_j}, \tag{3.10b}$$

$$(M_{\nu}^{-1})_{22} = \sum_{j=1}^{3} \frac{w_j^2}{\mu_j}, \tag{3.10c}$$

$$(M_{\nu}^{-1})_{23} = \sum_{j=1}^{3} \frac{|w_j|^2}{\mu_j}. \tag{3.10d}$$

Notice that $(M_{\nu}^{-1})_{11}$ and $(M_{\nu}^{-1})_{23}$ are real, cf. equation (3.4). The equality of the moduli of both sides of equation (3.8) then reads

$$\left( \sum_{j=1}^{3} \frac{u_j^2}{\mu_j} \right) \left( \sum_{j'=1}^{3} \frac{u_{j'} w_{j'}^*}{\mu_{j'}} \right)^2 = \left( \sum_{j=1}^{3} \frac{w_j^2}{\mu_j} \right) \left( \sum_{j'=1}^{3} \frac{|w_{j'}|^2}{\mu_{j'}} \right)^2. \tag{3.11}$$

At this point we have to exploit the unitarity condition of $U$

$$u_j u_{j'} + 2 \text{Re}(w_j w_{j'}^*) = \delta_{jj'}. \tag{3.12}$$
We can transform equation (3.11) into
\[
\left( \sum_{j} \frac{U_{j}^{2}}{m_{j}^{2}} + \sum_{j<j'} \frac{2U_{j}U_{j'}}{\mu_{j}\mu_{j'}} \right) \left[ \sum_{j} \frac{U_{j}(1 - U_{j})}{2m_{j}^{2}} - \sum_{j<j'} \frac{U_{j}U_{j'}}{\mu_{j}\mu_{j'}} \right] = \left[ \sum_{j} \frac{(1 - U_{j})^{2}}{4m_{j}^{2}} + \sum_{j<j'} \frac{1 + U_{j} + U_{j'} + U_{j}U_{j'}}{2\mu_{j}\mu_{j'}} \right] \times \left[ \sum_{j} \frac{(1 - U_{j})^{2}}{4m_{j}^{2}} + \sum_{j<j'} \frac{1 - U_{j} - U_{j'} + U_{j}U_{j'}}{2\mu_{j}\mu_{j'}} \right],
\]
where we have defined \( U_{j} \equiv u_{j}^{2} \). Taking into account that the first line of \( U \) is related to the mixing angles through
\[
U_{3} = \sin^{2} \theta_{13}, \tag{3.14a}
U_{2} = \cos^{2} \theta_{13} \sin^{2} \theta_{12}, \tag{3.14b}
U_{1} = \cos^{2} \theta_{13} \cos^{2} \theta_{12}, \tag{3.14c}
\]
the fifth prediction of our model, embodied in equation (3.13), is amenable to numerical analysis.

In the case of model II, the predictions for lepton mixing and for the Majorana phases, listed in the paragraph after equation (3.7), hold true as well. As for relation (3.13), we must make the replacement \( \mu_{j} \to 1/\mu_{j} \) in order to obtain the corresponding relation for model II.

In order to evaluate equation (3.13) numerically, we use as input the 2\( \sigma \) ranges of \( \Delta m_{21}^{2} \equiv m_{2}^{2} - m_{1}^{2}, \Delta m_{31}^{2} \equiv m_{3}^{2} - m_{1}^{2}, \) \( \sin^{2} \theta_{12}, \) and \( \sin^{2} \theta_{13} \) taken from table 1 of [6]. We allow the lightest neutrino mass, \( m_{0} \), to lie in between zero and 0.3 eV; this is inspired by the extant cosmological bounds on the sum of the light neutrino masses [18]. These five parameters, in the specified ranges, form our parameter space. In order to find points in this parameter space which are compatible with equation (3.13), we define a figure-of-merit function
\[ F = \frac{|R - L|}{|R + L|} + \Phi, \]
where \( R \) and \( L \) are the expressions in the right-hand and left-hand sides, respectively, of equation (3.13), and \( \Phi \) is a function which has the value zero if all the parameters lie in the ranges specified above and \( 10^{6} \) if at least one parameter is outside its range. We minimize \( F \) and declare a point to be allowed whenever \( F < 10^{-9} \). For the minimization we employ the Nelder-Mead algorithm, i.e. the downhill simplex method [39].\(^4\) In order to produce the scatter plots in figures 1–4, for every possible sign choice\(^5\) of the masses \( \mu_{j} \) — see equation (3.6) — we have run the Nelder-Mead algorithm with \( 10^{5} \) randomly chosen simplices in the parameter space.

\(^3\)Note that \( m_{0} \equiv m_{1} \) for a normal neutrino mass spectrum and \( m_{0} \equiv m_{3} \) for an inverted neutrino mass spectrum.

\(^4\)This is the method that we have used for producing the scatter plots displayed in this paper. However, it is also possible to treat equation (3.13) \textit{exactly}, since that equation produces, when using as input the neutrino masses and \( U_{3} \), a quartic equation for \( U_{2} \) (one must use \( U_{1} = 1 - U_{2} - U_{3} \)), which is solvable through an exact algorithm. We have used this exact method to confirm the numerical results presented in this paper.

\(^5\)Note that the sign choices \((\epsilon_{1}, \epsilon_{2}, \epsilon_{3})\) and \((-\epsilon_{1}, -\epsilon_{2}, -\epsilon_{3})\) are equivalent, as can be read off from equation (3.13).
Figure 1. Lightest neutrino mass as a function of $\sin^2 \theta_{12}$ in the case of model I. Here and in the following figures, dark (blue) and light (red) colour indicate the allowed range for the normal and inverted ordering of the neutrino mass spectrum, respectively.

Figure 2. Effective Majorana mass $m_{\beta\beta}$ versus lightest neutrino mass in the case of model I.
Figure 3. Lightest neutrino mass as a function of $\sin^2 \theta_{12}$ in the case of model II.

Figure 4. Effective Majorana mass $m_{\beta\beta}$ versus lightest neutrino mass in the case of model II.
Figures 1 and 2 are for model I. In figure 1 we have plotted the lightest neutrino mass $m_0$ as a function of $\sin^2 \theta_{12}$. Here and in all other figures, the blue (dark) colour corresponds to a normal and the red (light) colour to an inverted neutrino mass spectrum. The different bands in figure 1 are associated with different sign choices $\epsilon_j$. These bands correspond to the allowed spots and lines in figure 2, which can be deduced from the corresponding ranges of the smallest neutrino mass. Let us consider figure 2 for a detailed explanation of the sign choices associated with the allowed ranges of $m_0$. Of the two spots in the left of that figure, the upper one corresponds to $(+ + +)$ and the lower one to $(+ + -)$. In the middle of the scatter plot, both the upper and the lower stroke correspond to signs $(+ + -)$. On the right part of the figure, the upper line corresponds to both $(+ - +)$ and $(+ - -)$ but the second sign choice holds only in its upper three quarters; the lower line, too, is generated by two sign choices: $(+ - -)$ holds along the whole line and $(+ + +)$ in its upper half.

Figures 3 and 4 refer to model II. The major difference between models I and II is that the two lowest bands for $m_0$ disappear in model II: there is no band below $m_0 = 10^{-2}$ eV in figure 3 and there are no spots in the left side of figure 4. This also means that the sign choice $(+ + +)$ is not allowed in model II. Otherwise, the interpretation with respect to the signs $\epsilon_j$ is the same in both models.

In both models, the dependence of the allowed range of $m_0$ on $\sin^2 \theta_{13}$ is very faint and does not show up significantly in a plot. For this reason we refrain from showing those plots here.

4 Conclusions

The recent experimental results on neutrino oscillations have shown that the reactor mixing angle $\theta_{13}$ is not as small as previously thought. This disagrees with the standard version of $\mu-\tau$ interchange symmetry, but not with an alternative version which predicts a maximal atmospheric mixing angle and maximal CP violation in neutrino mixing while leaving $\theta_{13}$ arbitrary. In this paper we have introduced a novel mechanism which realizes this scenario. Our mechanism needs a left-handed diagonalization matrix $U_\ell = U_\omega$ in the charged-lepton sector — see equation (1.4) — where $U_\omega$ is the well-known maximal-mixing unitary matrix which also appears in ordinary $A_4$-based models; our mechanism moreover needs a real neutrino mass matrix. We have constructed two models, one based on the type I and the other one based on the type II seesaw mechanism. Since with regard to the neutrino sector the symmetry structure of our models bears resemblance with models based on $\Delta(27)$, we have obtained an additional constraint on the neutrino mass and mixing parameters, which can be approximately interpreted as a determination of the absolute mass scale of the neutrinos in terms of the mass-squared differences and of the solar mixing angle — see figures 1 and 3. Because several sign choices are possible for the neutrino masses — see equation (3.6) — this determination is, however, ambiguous. Since in this alternative version of $\mu-\tau$ interchange symmetry the Majorana phases are either zero or $\pi$, there is a rather stringent correlation between the neutrino mass scale and $m_{\beta\beta}$, the effective Majorana mass in neutrinoless double-beta decay — see figures 2 and 4. It should be
emphasized that \( m_{\beta\beta} \) may in our model assume quite large values, of order \( 0.1 \text{eV} \), which might render it observable in upcoming experiments — see \([40, 41]\) and references therein.

Besides the predictions for the lepton sector, the models have interesting features. They do not require any VEV alignment, only reality of the VEVs of the scalar gauge singlets in model I / gauge triplets in model II is essential. This reality is easily obtained at least in the case of model I, as we have shown here and also earlier in \([42, 43]\). Possible phases of the VEVs of the scalar doublets are irrelevant for CP violation in our models. Nevertheless, the models do feature CP violation in lepton mixing, which is indeed maximal, i.e. \( \delta = \pm \pi/2 \). The reason is that here the CP symmetry (2.13) is not broken by complex VEVs but rather by \(|v_2| \neq |v_3|\), which is responsible for the muon and tau masses being different, \((m_\mu \neq m_\tau)\) \([17, 44]\). Indeed, in our models the different masses of the charged leptons are not obtained from different Yukawa coupling constants but rather from different VEVs. This idea has been used before in several other models \([45–49]\); here we have followed \([46, 47]\) in this respect.

Acknowledgments

The work of P.M.F. is supported by the Portuguese Foundation for Science and Technology (FCT) through the contract PTDC/FIS/117951/2010, the FP7 Reintegration Grant PERG08-GA-2010-227025, and PEst.OE/FIS/UI0618/2011. The work of L.L. is funded by FCT through its unit 777 and through the project PTDC/FIS/098188/2008. W.G. and P.O.L. acknowledge support from the Austrian Science Fund (FWF), Project No. P 24161-N16.

References

[1] DOUBLE-CHOOZ collaboration, Y. Abe et al., Indication for the disappearance of reactor electron antineutrinos in the Double CHOOZ experiment, *Phys. Rev. Lett.* 108 (2012) 131801 [arXiv:1112.6353] [INSPIRE].

[2] DAYA-BAY collaboration, F.P. An et al., Observation of electron-antineutrino disappearance at Daya Bay, *Phys. Rev. Lett.* 108 (2012) 171803 [arXiv:1203.1669] [INSPIRE].

[3] RENO collaboration, J.K. Ahn et al., Observation of reactor electron antineutrino disappearance in the RENO experiment, *Phys. Rev. Lett.* 108 (2012) 191802 [arXiv:1204.0626] [INSPIRE].

[4] G.L. Fogli, E. Lisi, A. Marrone, A. Palazzo and A.M. Rotunno, Evidence of \( \theta_{13} > 0 \) from global neutrino data analysis, *Phys. Rev. D* 84 (2011) 053007 [arXiv:1106.6028] [INSPIRE].

[5] T. Schwetz, M. Tórtola and J.W.F. Valle, Where we are on \( \theta_{13} \): addendum to ‘Global neutrino data and recent reactor fluxes: status of three-flavour oscillation parameters’, *New J. Phys.* 13 (2011) 109401 [arXiv:1108.1376] [INSPIRE].

[6] D.V. Forero, M. Tórtola and J.W.F. Valle, Global status of neutrino oscillation parameters after Neutrino-2012, *arXiv:1205.4018* [INSPIRE].
[7] G.L. Fogli et al., Global analysis of neutrino masses, mixings and phases: entering the era of leptonic CP-violation searches, Phys. Rev. D 86 (2012) 013012 [arXiv:1205.5254] [inSPIRE].

[8] T. Fukuyama and H. Nishiura, Mass matrix of Majorana neutrinos, hep-ph/9702253 [inSPIRE].

[9] R.N. Mohapatra and S. Nussinov, Bimaximal neutrino mixing and neutrino mass matrix, Phys. Rev. D 60 (1999) 013002 [hep-ph/9809415] [inSPIRE].

[10] E. Ma and M. Raidal, Neutrino mass, muon anomalous magnetic moment and lepton flavor nonconservation, Phys. Rev. Lett. 87 (2001) 011802 [Erratum ibid. 87 (2001) 159901] [hep-ph/0102255] [inSPIRE].

[11] C.S. Lam, A 2–3 symmetry in neutrino oscillations, Phys. Lett. B 507 (2001) 214 [hep-ph/0104116] [inSPIRE].

[12] K.R.S. Balaji, W. Grimus and T. Schwetz, The solar LMA neutrino oscillation solution in the Zee model, Phys. Lett. B 508 (2001) 301 [hep-ph/0104035] [inSPIRE].

[13] E. Ma, The all-purpose neutrino mass matrix, Phys. Rev. D 66 (2002) 117301 [hep-ph/0207352] [inSPIRE].

[14] A. Ghosal, A neutrino mass model with reflection symmetry, Mod. Phys. Lett. A 19 (2004) 2579 [inSPIRE].

[15] P.F. Harrison, D.H. Perkins and W.G. Scott, Tri-bimaximal mixing and the neutrino oscillation data, Phys. Lett. B 530 (2002) 167 [hep-ph/0202074] [inSPIRE].

[16] P.F. Harrison and W.G. Scott, $\mu$–$\tau$ reflection symmetry in lepton mixing and neutrino oscillations, Phys. Lett. B 547 (2002) 219 [hep-ph/0210197] [inSPIRE].

[17] W. Grimus and L. Lavoura, A non-standard CP transformation leading to maximal atmospheric neutrino mixing, Phys. Lett. B 579 (2004) 113 [hep-ph/0305309] [inSPIRE].

[18] PARTICLE DATA GROUP collaboration, J. Beringer et al., Review of particle physics, Phys. Rev. D 86 (2012) 010001 [inSPIRE].

[19] E. Ma and G. Rajasekaran, Softly broken $A_4$ symmetry for nearly degenerate neutrino masses, Phys. Rev. D 64 (2001) 113012 [hep-ph/0106291] [inSPIRE].

[20] G. Altarelli and F. Feruglio, Discrete flavor symmetries and models of neutrino mixing, Rev. Mod. Phys. 82 (2010) 2701 [arXiv:1002.0211] [inSPIRE].

[21] A.Y. Smirnov, Discrete symmetries and models of flavor mixing, J. Phys. Conf. Ser. 335 (2011) 012006 [arXiv:1103.3461] [inSPIRE].

[22] P. Minkowski, $\mu \to e\gamma$ at a rate of one out of $10^9$ muon decays?, Phys. Lett. B 67 (1977) 421 [inSPIRE].

[23] T. Yanagida, Horizontal symmetry and masses of neutrinos, in Proceedings of the Workshop on the Baryon Number of the Universe and Unified Theories, Tsukuba Japan (1979), O. Sawada and A. Sugamoto eds., KEK Report 79-18-95 [inSPIRE].

[24] S.L. Glashow, The future of elementary particle physics, in Quarks and leptons. Proceedings of the Advanced Study Institute, Cargèse France (1979), J.-L. Basdevant et al. eds., Plenum Press, New York U.S.A. (1981).
[25] M. Gell-Mann, P. Ramond and R. Slansky, Complex spinors and unified theories, in Supergravity, D.Z. Freedman and F. van Nieuwenhuizen eds., North Holland, Amsterdam Netherlands (1979) [SPIRE].

[26] R.N. Mohapatra and G. Senjanović, Neutrino mass and spontaneous parity violation, Phys. Rev. Lett. 44 (1980) 912 [SPIRE].

[27] J.A. Escobar and C. Luhn, The flavor group $\Delta(6n^2)$, J. Math. Phys. 50 (2009) 013524 [arXiv:0809.0639] [SPIRE].

[28] E. Ma, Neutrino mass matrix from $\Delta(27)$ symmetry, Mod. Phys. Lett. A 21 (2006) 1917 [hep-ph/0607056] [SPIRE].

[29] E. Ma, Near tribimaximal neutrino mixing with $\Delta(27)$ symmetry, Phys. Lett. B 660 (2008) 505 [arXiv:0709.0507] [SPIRE].

[30] G.C. Branco, J.M. Gérard and W. Grimus, Geometrical T-violation, Phys. Lett. B 136 (1984) 383 [SPIRE].

[31] H. Georgi and D.V. Nanopoulos, Suppression of flavor changing effects from neutral spinless meson exchange in gauge theories, Phys. Lett. B 82 (1979) 95 [SPIRE].

[32] W. Grimus, L. Lavoura and B. Radovčič, Type II seesaw mechanism for Higgs doublets and the scale of new physics, Phys. Lett. B 674 (2009) 117 [arXiv:0902.2325] [SPIRE].

[33] J. Schechter and J.W.F. Valle, Neutrino masses in SU(2) × U(1) theories, Phys. Rev. D 22 (1980) 2227 [SPIRE].

[34] G. Lazarides, Q. Shafi and C. Wetterich, Proton lifetime and fermion masses in an SO(10) model, Nucl. Phys. B 181 (1981) 287 [SPIRE].

[35] R.N. Mohapatra and G. Senjanović, Neutrino masses and mixings in gauge models with spontaneous parity violation, Phys. Rev. D 23 (1981) 165 [SPIRE].

[36] W. Grimus and L. Lavoura, Nonconservation of total lepton number with scalar bosons, Phys. Lett. B 70 (1977) 433 [SPIRE].

[37] T.P. Cheng and L.-F. Li, Neutrino masses, mixings and oscillations in SU(2) × U(1) models of electroweak interactions, Phys. Rev. D 22 (1980) 2860 [SPIRE].

[38] G.B. Gelmini and M. Roncadelli, Left-handed neutrino mass scale and spontaneously broken lepton number, Phys. Lett. B 99 (1981) 411 [SPIRE].

[39] J.A. Nelder and R. Mead, A simplex method for function minimization, Comput. J. 7 (1965) 308.

[40] F.T. Avignone, Strategies for next generation neutrinoless double-beta decay experiments, Nucl. Phys. Proc. Suppl. 143 (2005) 233 [SPIRE].

[41] A. Faessler, A. Meroni, S.T. Petcov, F. Šimkovic and J. Vergados, Uncovering multiple CP-nonconserving mechanisms of $(\beta\beta)^{0\nu}_{0}$ decay, Phys. Rev. D 83 (2011) 113003 [arXiv:1103.2434] [SPIRE].

[42] W. Grimus and L. Lavoura, A three-parameter neutrino mass matrix with maximal CP violation, Phys. Lett. B 671 (2009) 456 [arXiv:0810.4516] [SPIRE].

[43] W. Grimus and L. Lavoura, Tri-bimaximal lepton mixing from symmetry only, JHEP 04 (2009) 013 [arXiv:0811.4766] [SPIRE].
[44] W. Grimus and H. Neufeld, *On spontaneous CP-violation in the lepton sector*, Phys. Lett. B 237 (1990) 521 [inSPIRE].

[45] H.E. Haber, G.L. Kane and T. Sterling, *The fermion mass scale and possible effects of Higgs bosons on experimental observables*, Nucl. Phys. B 161 (1979) 493 [inSPIRE].

[46] W. Grimus and L. Lavoura, *A model realizing the Harrison-Perkins-Scott lepton mixing matrix*, JHEP 01 (2006) 018 [hep-ph/0509239] [inSPIRE].

[47] W. Grimus and L. Lavoura, *A three-parameter model for the neutrino mass matrix*, J. Phys. G 34 (2007) 1757 [hep-ph/0611149] [inSPIRE].

[48] R.A. Porto and A. Zee, *The private Higgs*, Phys. Lett. B 666 (2008) 491 [arXiv:0712.0448] [inSPIRE].

[49] Y. BenTov and A. Zee, *Lepton private Higgs and the discrete group \( \Sigma(81) \)*, arXiv:1202.4234 [inSPIRE].