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On a Class of Almost Difference Sets Constructed by Using the Ding-Helleseth-Martinsens Constructions

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SUMMARY Pseudorandom binary sequences with optimal balance and autocorrelation have many applications in stream cipher, communication, coding theory, etc. It is known that binary sequences with three-level autocorrelation should have an almost difference set as their characteristic sets. How to construct new families of almost difference set is an important research topic in such fields as communication, coding theory and cryptograp- hy. In a work of Ding, Helleseth, and Martinsen in 2001, the authors developed a new method, known as the Ding-Helleseth-Martinsens Constructions in literature, of constructing an almost difference set from product sets of GF(2) and the union of two cyclotomic classes of order four. In the present paper, we have constructed two classes of almost difference set with product sets between GF(2) and union sets of the cyclotomic classes of order 12 using that method. In addition, we could find there do not exist the Ding-Helleseth-Martinsens Constructions for the cyclotomic classes of order six, eight and ten.

key words: Binary sequence, three-level autocorrelation, the Ding-Helleseth-Martinsens Constructions, almost difference set, cyclotomic classes of order twelve.

1. Introduction

Let \( (A, +) \) be an Abelian group with \( n \) elements and \( D \) be a \( k \)-subset of \( A \). Define the distance function \( d_D(e) = |(D + e) \cap D| \), where \( D + e = \{ x + e | x \in D \} \) and \( e \in D \setminus \{0\} \). \( D \) is referred to as an \((n, k, \lambda, t)\) almost difference set if \( d_D(e) \) takes on the value \( \lambda \) altogether \( t \) times and on the value \( \lambda + 1 \) altogether \( n - 1 - t \) times when \( e \) ranges over all the nonzero elements of \( A \). Binary sequences with three-level autocorrelation can be constructed from almost difference sets as their characteristic sets [1], [2]. Especially, almost difference sets with parameters \( (n, (n - 1)/2, (n - 5)/4, (n - 1)/4) \) give corresponding binary sequences with optimal three-level autocorrelation and optimal balance among 0’s and 1’s [1]. In that sense, constructing optimal binary sequences is equivalent to find corresponding almost difference sets. \( D \) is called a \((n, k, \lambda)\) difference set if the distance function \( d_D(e) \) takes on the value \( \lambda \) altogether \( n - 1 \) times when \( e \) ranges over all the nonzero elements of \( A \).

There are several ways of construction of almost difference set. We cite a few of those, for example, generalized cyclotomy can be used to construct almost difference set [3], the so-called Lempel-Cohn-Eastmans Construction uses set exponents and cyclotomy for the construction of almost difference set [4], [5]. Davis found almost difference sets with following parameters [6]:

\[
(1) \ (4 \cdot 3^2 n, \ 2(3^2 n - 3^m), \ 3^2 n - 2 \cdot 3^m, \ 3^2 n - 1) \ \text{in} \ H \times \mathbb{Z}_{3^m}^*,
\]

where \( H \) is a group of order four;

\[
(2) \ ((q + 1)q^2, \ q(q + 1), \ q, \ q^2 - 1) \ \text{in} \ H \times \text{EA}(q^2), \ \text{where} \ \text{EA}(q^2) \ \text{denotes the additive group} \ (GF(q^2), +) \ \text{and} \ H \ \text{is a group of order} \ q + 1.
\]

Perfect nonlinear functions (PN) have not only many applications in cryptography [8], [9], optimal constant-composition codes, and signal sets [10], [11], but also can be used to construct difference sets [7], and almost difference sets [2]. Almost difference sets can be constructed from difference sets as well, for instance see [2], [12].

Let \( q = df + 1 \) be a power of an odd prime, \( \alpha \) be a primitive element of extension field \( GF(q) \). Define the cosets \( D_{(d, \alpha)} = \{ \alpha^{kd+i} | 0 \leq k < f, 0 \leq i < d \} \), which are called the cyclotomic classes of order \( d \) with respect to \( GF(q) \). It is obvious that \( GF(q)^* = \bigcup_{i=0}^{d-1} D_{(d, \alpha)}^i \). The constants \( (m, n)_d = |(D_{(d, \alpha)}^m + 1) \cap D_{(d, \alpha)}^n| \) are known as the cyclotomic numbers of order \( d \) with respect to \( GF(q) \). It is possible to construct almost difference sets with cyclotomic classes of given order, for instance, see [1], [2], [13], [14].

In the present paper, we are concerned with the so-called Ding-Helleseth-Martinsens Constructions of almost difference set using the cyclotomic classes of order four [2], [14], and apply the method into the cyclotomic classes of order twelve. By computer investigation, we found that no almost difference sets could be constructed from the cyclotomic classes of order six, eight and ten using that method. However, two classes of almost difference sets can be obtained by using the Ding-Helleseth-Martinsens Constructions with the cyclotomic classes of order twelve. Let \( q \equiv 5 \pmod{8} \), then \( q \) can be expanded into \( q = s^2 + 4t^2 \) with \( s \equiv 1 \pmod{4} \) [16]. The Ding-Helleseth-Martinsens Constructions can be outlined by the following two theorems:

**Theorem 1.1** (Theorem 5 in [2], see also [14]). Let \( i, j, l \in \{0, 1, 2, 3\} \) be three pairwise distinct integers, and define

\[
C = \{ 0 \} \times \left( D_{i(4,q)}^j \cup D_{j(4,q)}^i \right) \cup \{ 1 \} \times \left( D_{i(4,q)}^j \cup D_{j(4,q)}^i \right).
\]

Then \( C \) is an \((n, (n - 2)/2, (n - 6)/4, (3n - 6)/4)\) almost difference set of \( A = GF(2) \times GF(q) \) if

\[
(1) \quad t = 1 \quad \text{and} \quad (i, j, l) = (0, 1, 3) \quad \text{or} \quad (0, 2, 1), \quad \text{or}
\]

\[
(2) \quad s = 1 \quad \text{and} \quad (i, j, l) = (1, 0, 3) \quad \text{or} \quad (0, 1, 2).
\]

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Then \( i, j, l \in \{0, 1, 2, 3\} \) be three pairwise distinct integers, and define

\[
C = \{0\} \times (D_i^{(4q)} \cup D_j^{(4q)}) \cup \{1\} \times (D_i^{(4q)} \cup D_j^{(4q)}) \cup \{0\}, 0).
\]

Then \( C \) is an \( (n, n/2, (n-2)/4, (3n-2)/4) \) almost difference set of \( A = GF(2) \times GF(q) \) if

1. \( t = 1 \) and
   \[
   (i, j, l) \in \{(0, 1, 3), (0, 2, 3), (1, 2, 0), (1, 3, 0)\}
   \]
   or
2. \( s = 1 \) and
   \[
   (i, j, l) \in \{(0, 1, 2), (0, 3, 2), (1, 0, 3), (1, 2, 3)\}
   \]

The next corollary extends Theorem 1.1 of which the proof method can be found in [14, eq.(2), eq.(3), and the proof of Lemma 4 and Theorem 1].

Corollary 1.1. Let \( i, j, l \in \{0, 1, 2, 3\} \) be three pairwise distinct integers, and define

\[
C = \{0\} \times (D_i^{(4q)} \cup D_j^{(4q)}) \cup \{1\} \times (D_i^{(4q)} \cup D_j^{(4q)}).
\]

Then \( C \) is an \( (n, (n-2)/2, (n-6)/4, (3n-6)/4) \) almost difference set of \( A = GF(2) \times GF(q) \) if

1. \( t = 1 \) and
   \[
   (i, j, l) \in \{(0, 1, 3), (0, 2, 1), (1, 2, 0), (1, 3, 2), (2, 0, 3), (2, 3, 1), (3, 1, 0), (3, 0, 2)\}
   \]
   or
2. \( t = -1 \) and
   \[
   (i, j, l) \in \{(0, 2, 3), (0, 3, 1), (1, 0, 2), (1, 3, 0), (2, 0, 1), (2, 1, 3), (3, 1, 2), (3, 2, 0)\}
   \]
   or
3. \( s = 1 \) and
   \[
   (i, j, l) \in \{(0, 1, 2), (0, 3, 2), (1, 0, 3), (1, 2, 3), (2, 1, 0), (2, 3, 0), (3, 0, 1), (3, 2, 1)\},
   \]

where \( n = |GF(2) \times GF(q)| = 2q \).

The rest of the present paper is structured as follows: in Section 2, basic concepts and data on the cyclotomic classes of order 12 will be introduced. In Section 3 we present the main results of the present paper: two new families of almost difference sets constructed by using the Ding-Helleseth-Marti nsens Constructions and the cyclotomic classes of order 12. In Section 4 a brief concluding remark will be given.

2. The cyclotomic classes of order twelve

Let \( p = 12f + 1 \) be an odd prime. Then, \( p \) can be expressed as \( p = x^2 + 4y^2 = A^2 + 3B^2 \), where \( x \equiv 1 \pmod{4} \) and \( A \equiv 1 \pmod{6} \). In an earlier work of Dickson than Whitman’s [16], Dickson pointed out that the 144 cyclotomic numbers of order twelve depend solely on the parameters \( p, A, B, x, \) and \( y \). For example, if \( 2 \) is a cubic residue of \( p, 3 \) is a biquadratic residue of \( p, \) and \( f \) is odd, then

\[
144(0, 2)_{12} = p + 1 - 2A + 24B - 12x.
\]

In order to define some more generally necessary concepts, let \( p = ef + 1 \) be an odd prime. Recall that we actually deal with the case \( e = 12 \). Let \( \beta = \exp(\frac{2\pi i}{12}) \) be a \( e^{th} \) root of unity where \( j = \sqrt{-1} \). Let \( a \in Z_p^\star \) where \( Z_p^\star \) denotes the multiplicative group modulo \( p \), and \( Ind(a) \) denote the index of \( a \) modulo \( p \) with respect to a given primitive root. In the theory of cyclotomy, the Jacobi sum plays a fundamental role that can be defined by the following equation for a pair of integers \( m \) and \( n \):

\[
\phi(p, e^n) = \sum_{a+b=1 \pmod{p}} \beta^{Ind(a)+Ind(b)}, \quad (1)
\]

where \( a, b \in Z_p^\star \). Another parameter is needed to classify the distinct cyclotomic numbers of order twelve according to \( f \) odd or even, and defined by
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Hereafter for the proof of lemmas and theorems of this section, we consider only the case where \( f \) is odd, \( M \equiv 0 \pmod{4} \), \( M \equiv 1 \pmod{6} \) and \( c = \beta^j \), since for other five cases with \( f \) odd (see Section 2), proof process is similar and it leads to the same results.

The distance functions \( d_c(w_1, w_2) \) and \( d_c^*(w_1, w_2) \) can be explicitly expanded out in \( d_1(w_2) \), \( d_2(w_2) \) and \( d_{1,2}(w_2) \), stated by the following two lemmas whose proofs can be found in [14, eq.(2) and eq.(4)].

**Lemma 3.1.**

\[
d_c(w_1, w_2) = \begin{cases} \left|D_I\right| + \left|D_j\right| & \text{if } w_1 = 0, w_2 = 0 \\ d_1(w_2) + d_2(w_2) & \text{if } w_1 = 0, w_2 \neq 0 \\ 2|D_I \cap D_J| & \text{if } w_1 = 1, w_2 = 0 \\ 0 & \text{if } w_1 = 1, w_2 \neq 0 \end{cases}
\]

**Lemma 3.2.**

\[
d_c^*(w_1, w_2) = d_c(w_1, w_2) + \begin{cases} \left|D_I \cap [w_2, -w_2]\right| & \text{if } w_1 = 0, w_2 \neq 0 \\ \left|D_I \cap [w_2, -w_2]\right| & \text{if } w_1 = 1, w_2 \neq 0 \\ 0 & \text{otherwise.} \end{cases}
\]

The following lemma gives the value of the distance function \( d_I(w) \) for a given index set \( I \):

**Lemma 3.3.** For a given index set \( I \), the distance function \( d_I(w) \) can be determined by

(1) Let \( I \in \{0, 1, 4, 5, 8, 9\}, \{2, 3, 6, 7, 10, 11\} \). Then,

\[
d_I(w) = \begin{cases} q - 2y - 3 & \text{if } w \in Q, \\ q + 4y - 3 & \text{if } w \in QN. \end{cases}
\]

(2) Let \( I \in \{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\} \). Then,

\[
d_I(w) = \begin{cases} q + 2y - 3 & \text{if } w \in Q, \\ q - 4y - 3 & \text{if } w \in QN. \end{cases}
\]

(3) Let \( I \in \{0, 2, 4, 6, 8, 10\} \). Then,

\[
d_I(w) = \begin{cases} q - 5 & \text{if } w \in Q, \\ q - 1 & \text{if } w \in QN. \end{cases}
\]

(4) Let \( I \in \{1, 3, 5, 7, 9, 11\} \). Then,

\[
d_I(w) = \begin{cases} q - 1 & \text{if } w \in Q, \\ q - 5 & \text{if } w \in QN. \end{cases}
\]

**Proof.** See the proof for Lemma 3.4. \( \square \)

Let \( I, J \in \{0, 1, 4, 5, 8, 9\}, \{2, 3, 6, 7, 10, 11\}, \{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\} \).

\( I \neq J \) and \( |I \cap J| = 3 \). Then, \( d_{I,J}(w) \) and \( d_{J,I}(w) \) can be expressed as functions only in \( q \) and \( x \), as the next lemma asserts:

**Lemma 3.4.** Let \( I = \{0, 1, 4, 5, 8, 9\} \) and \( J = \{0, 3, 4, 7, 8, 11\} \). Then,

\[
d_{I,J}(w) = \begin{cases} q_s - 2x - 3 & \text{if } w \in Q, \\ q - 2x - 3 & \text{if } w \in D_h^{(12,q)} \cup D_s^{(12,q)} \cup D_q^{(12,q)}, \\ q - 2x - 3 & \text{if } w \in D_3^{(12,q)} \cup D_7^{(12,q)} \cup D_{11}^{(12,q)}, \\ 0 & \text{if } w \in D_1^{(12,q)} \cup D_5^{(12,q)} \cup D_9^{(12,q)}. \end{cases}
\]

**Proof.** We only prove \( d_{I,J}(w) \). Let \( w \in GF(q) \setminus \{0\} \) and \( w^{-1} \in D_h^{(12,q)} \). Remark that \( (i+h, j+h)_{12} = ((i+h) \pmod{12}, (j+h) \pmod{12})_{12} \).

\[
d_{I,J}(w) = \left|\left(D_I + w\right) \cap D_J\right|
\]

\[
= \left|\bigcup_{i \in I} D_i^{(12,q)} + w\right| \cap \left(\bigcup_{j \in J} D_j^{(12,q)}\right)
\]

\[
= \left|\bigcup_{i \in I} (w^{-1}D_i^{(12,q)} + 1)\right| \cap \left(\bigcup_{j \in J} D_j^{(12,q)}\right)
\]

\[
= \sum_{i \in I} \sum_{j \in J} \left|D_i^{(12,q)} + 1\right| \cap D_j^{(12,q)}
\]

Let \( v_1, v_2, \ldots, v_{31} \) represent the 31 irreducible and distinct cyclotomic numbers of order 12 (see Table 2), then, from the last equation of eq.(3) and using Table 1, we can obtain

(1) \( D_h^{(12,q)} \subset Q \):

\[
d_{I,J}(w) = v_1 + v_{10} + v_{12} + v_{13} + v_{14} + 2v_{15} + 2v_{16} + v_{17} + v_{18} + v_9 + v_2 + 2v_{20} + 2v_{21} + 2v_{22} + 2v_3 + 2v_{24} + 2v_{25} + 2v_{26} + v_{27} + v_{28} + 2v_{29} + 2v_{30} + v_4 + v_5 + v_6 + v_8 + v_9.
\]

(2) \( D_h^{(12,q)} \subset D_1^{(12,q)} \cup D_5^{(12,q)} \cup D_9^{(12,q)} \):

\[
d_{I,J}(w) = v_1 + 3v_{13} + 3v_{14} + 3v_{17} + 3v_{18} + 3v_{19} + 2v_{22} + 3v_{23} + 2v_{24} + 3v_{25} + 2v_{26} + 3v_{27} + 3v_{28} + v_5 + v_9.
\]
(3) \( D^{(12,q)}_h \subset D^{(12,q)}_3 \cup D^{(12,q)}_7 \cup D^{(12,q)}_{11} \):

\[
\begin{align*}
d_{I,J}(w) &= v_{10} + 3v_{11} + v_{12} + 13 + v_{14} + \vspace{-1cm} \\
&2v_{15} + 2v_{16} + v_{17} + v_{18} + v_{19} + v_{2} + \\
&2v_{20} + 2v_{21} + v_{22} + v_{23} + v_{24} + v_{27} + v_{28} + \\
&2v_{29} + 3v_{3} + 2v_{30} + 2v_{31} + v_{4} + \vspace{-1cm} \\
&v_{6} + v_{7} + v_{8}.
\end{align*}
\]

(6)

From Table 2, we have

\[
\begin{align*}
v_1 &= (0, 0)_{12} = \frac{q - 6A - 16B - 23}{144}, \\
v_2 &= (0, 1)_{12} = \frac{q + 4A + 24B - 18x - 24y - 24}{144}, \\
v_3 &= (0, 2)_{12} = \frac{q - 2A - 24B - 12x - 24}{144}, \\
&\vdots \\
v_{30} &= (3, 2)_{12} = \frac{q - 2A + 12B + 12x - 2}{144}, \\
v_{31} &= (4, 2)_{12} = \frac{q + 4A + 24B - 18x - 24y - 24}{144}.
\end{align*}
\]

Substitute the formulae of \( v_1, v_2, \ldots, v_{31} \) in eq.(7) for \( v_1, v_2, \ldots, v_{31} \) in eq.(4)-eq.(6).

For two index sets \( I, J \) one of which is \( \{0, 2, 4, 6, 8, 10\} \) or \( \{1, 3, 5, 7, 9, 11\} \), the distance function \( d_{I,J}(w) \) is quite different. Let

\[
I, J \in (\{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\}, \{0, 2, 4, 6, 8, 10\})
\]

or

\[
I, J \in (\{0, 1, 4, 5, 8, 9\}, \{2, 3, 6, 7, 10, 11\}, \{1, 3, 5, 7, 9, 11\}),
\]

and \( |I \cap J| = 3 \). Then, the distance function \( d_{I,J}(w) \) is a function only in \( q, x \) and \( y \).

\[
\text{Lemma 3.5. Let } I = \{0, 3, 4, 7, 8, 11\} \text{ and } J = \{0, 2, 4, 6, 8, 10\}. \text{ Then,}
\]

\[
\begin{align*}
d_{I,J}(w) &= \begin{cases}
\frac{q-x-2y-2}{4} & \text{if } w \in D^{(12,q)}_0 \cup D^{(12,q)}_4 \cup D^{(12,q)}_8, \\
\frac{q+x-2y-4}{4} & \text{if } w \in D^{(12,q)}_2 \cup D^{(12,q)}_6 \cup D^{(12,q)}_{10}, \\
\frac{q-x-2y}{4} & \text{if } w \in D^{(12,q)}_3 \cup D^{(12,q)}_7 \cup D^{(12,q)}_{11}, \\
\frac{q-x-2y-2}{4} & \text{if } w \in D^{(12,q)}_1 \cup D^{(12,q)}_5 \cup D^{(12,q)}_9.
\end{cases}
\end{align*}
\]

Proof. Similar to Lemma 3.4 and omitted due to limited space.

We need another similar lemma to prove the main theorems. Let

\[
I, J \in (\{0, 1, 4, 5, 8, 9\}, \{2, 3, 6, 7, 10, 11\}, \{0, 2, 4, 6, 8, 10\})
\]

or

\[
I, J \in (\{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\}, \{1, 3, 5, 7, 9, 11\}),
\]

and \( |I \cap J| = 3 \). Then, the distance function \( d_{I,J}(w) \) is a function only in \( q, x \) and \( y \) as well.

\[
\text{Lemma 3.6. Let } I = \{0, 1, 4, 5, 8, 9\} \text{ and } J = \{0, 2, 4, 6, 8, 10\}. \text{ Then,}
\]

\[
\begin{align*}
d_{I,J}(w) &= \begin{cases}
\frac{q-x-2y-2}{4} & \text{if } w \in D^{(12,q)}_0 \cup D^{(12,q)}_4 \cup D^{(12,q)}_8, \\
\frac{q+x-2y-4}{4} & \text{if } w \in D^{(12,q)}_2 \cup D^{(12,q)}_6 \cup D^{(12,q)}_{10}, \\
\frac{q-x-2y}{4} & \text{if } w \in D^{(12,q)}_3 \cup D^{(12,q)}_7 \cup D^{(12,q)}_{11}, \\
\frac{q-x-2y-2}{4} & \text{if } w \in D^{(12,q)}_1 \cup D^{(12,q)}_5 \cup D^{(12,q)}_9.
\end{cases}
\end{align*}
\]

Proof. Similar to Lemma 3.4 and omitted due to limited space.

Now, we are ready to state and prove the first theorem.

\[
\text{Theorem 3.1. Let } C = \{0\} \times D_I \cup \{1\} \times D_J. \text{ Then, } C \text{ is an } (n, (n-2)/2, (n-6)/4, (3n-6)/4) \text{ almost difference set of } A = GF(2) \times GF(q) \text{ if}
\]

(1) \( x = 1 \) and

\[
I, J \in \{0, 1, 4, 5, 8, 9\}, \{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\}, \{2, 3, 6, 7, 10, 11\}, \{0, 2, 4, 6, 8, 10\}
\]

such that \( |I \cap J| = 3 \), or

(2) \( y = 1 \) and

\[
I, J \in \{0, 1, 4, 5, 8, 9\}, \{2, 3, 6, 7, 10, 11\}, \{1, 3, 5, 7, 9, 11\}
\]

such that \( |I \cap J| = 3 \), or

(3) \( y = 1 \) and

\[
I, J \in \{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\}, \{0, 2, 4, 6, 8, 10\}
\]

such that \( |I \cap J| = 3 \), or

(4) \( y = -1 \) and

\[
I, J \in \{0, 1, 4, 5, 8, 9\}, \{2, 3, 6, 7, 10, 11\}, \{0, 2, 4, 6, 8, 10\}
\]

such that \( |I \cap J| = 3 \), or

(5) \( y = -1 \) and

\[
I, J \in \{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\}, \{1, 3, 5, 7, 9, 11\}
\]

such that \( |I \cap J| = 3 \),
where \( n = |GF(2) \times GF(q)| = 2q \).

**Proof.** We only prove case (1) with \( I = \{0, 1, 4, 5, 8, 9\} \) and \( J = \{0, 3, 4, 7, 8, 11\} \) since proof for other cases is similar. By Lemma 3.1 for \( w_1 = 0 \) and \( w_2 \neq 0 \),

\[
d_c(w_1, w_2) = d_I(w_2) + d_J(w_2). \tag{8}
\]

From Lemma 3.3 (1) and (2), substitute the values of \( d_I(w_2), d_J(w_2) \) into eq.(8), we can obtain

\[
d_c(w_1, w_2) = \frac{q - 3}{2} \text{ for } w_2 \in Q \cup QN. \tag{9}
\]

By Lemma 3.1 for \( w_1 = 1 \) and \( w_2 \neq 0 \),

\[
d_c(w_1, w_2) = d_I(w_2) + d_J(w_2). \tag{10}
\]

Combining eq.(10) and Lemma 3.4, we get

\[
d_c(w_1, w_2) = \begin{cases} \frac{q - 2}{2} & \text{if } w_2 \in Q, \\ \frac{q - 3}{2} & \text{if } w_2 \in QN. \end{cases} \tag{11}
\]

Now consider the case \( w_1 = 1 \) and \( w_2 = 0 \) in Lemma 3.1.

\[
d_c(w_1, w_2) = 2|D_I \cap D_J| = 2|D_0^{(12,q)} \cup D_4^{(12,q)} \cup D_8^{(12,q)}| = 2 \cdot \frac{q - 1}{12} \cdot 3 \tag{13}
\]

Combining eq.(9), eq.(12) and eq.(13), we conclude that \( C \) is an \((2q, q - 1, \frac{q - 3}{2})\) almost difference set with respect to \( GF(2) \times GF(q) \). \( \square \)

In Lemma 3.2 two extra-terms \( \delta_I = |D_I \cap \{w_2, -w_2\}| \) and \( \delta_J = |D_J \cap \{w_2, -w_2\}| \) due to the presence of the element \((0, 0)\) in the product set \( C = \{0\} \times D_I \cup \{1\} \times D_J \cup \{(0, 0)\} \) can be explicitly computed for a given pair of distinct \( I \) and \( J \) by the following lemma:

**Lemma 3.7.** If \( D_0^{(12,q)} \cup D_6^{(12,q)} \subset D_I \) then \( \delta_I = |D_I \cap \{w_2, -w_2\}| = 2 \), else if \( D_0^{(12,q)} \subset D_I \) or \( D_6^{(12,q)} \subset D_I \) then \( \delta_I = |D_I \cap \{w_2, -w_2\}| = 1 \), else \( \delta_I = |D_I \cap \{w_2, -w_2\}| = 0 \).

**Proof.** Let \( w_2^{-1} \in D_6^{(12,q)} \). Recall that \( D_0^{(12,q)} = D_{8k+4}^{(12,q)} \pmod{12} \).

Since \( q = 12f + 1 \) with \( f \) being odd, \( -1 \in D_6^{(12,q)} \).

\[
\delta_I = |D_I \cap \{w_2, -w_2\}| = |\bigcup_{i \in I} D_i^{(12,q)} \cap \{w_2, -w_2\}|
\]

\[
= |\bigcup_{i \in I} w_2^{-1} D_i^{(12,q)} \cap \{1, -1\}|
\]

\[
= |\bigcup_{i \in I} D_i^{(12,q)} \cap \{1, -1\}|
\]

\[
= |D_{8k+4} \cap \{1, -1\}|
\]

where \( I + h = \{i + h \mid i \in I\} \).

\( \square \)

**Remark 3.1.** Let \( I = \{0, 1, 4, 5, 8, 9\} \) and \( J = \{0, 3, 4, 7, 8, 11\} \). Then, for all \( w_2 \in GF(q) \setminus \{0\} \)

\[
\begin{align*}
\delta_I &= |D_I \cap \{w_2, -w_2\}| = 1, \\
\delta_J &= |D_J \cap \{w_2, -w_2\}| = 1. \tag{14}
\end{align*}
\]

Now we are ready to state and prove the second theorem:

**Theorem 3.2.** Let \( C = \{0\} \times D_I \cup \{1\} \times D_J \cup \{(0, 0)\} \). Then, \( C \) is an \((n, n/2, (n - 2)/4, (3n - 2)/4)\) almost difference set of \( A = GF(2) \times GF(q) \) if

(1) \( x = 1 \) and

\[
I, J \in \{\{0, 1, 4, 5, 8, 9\}, \{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\}, \{2, 3, 6, 7, 10, 11\} \}
\]

such that \( |I \cap J| = 3 \), or

(2) \( y = 1 \) and

\[
I, J \in \{\{0, 1, 4, 5, 8, 9\}, \{2, 3, 6, 7, 10, 11\}, \{1, 3, 5, 7, 9, 11\} \}
\]

such that \( |I \cap J| = 3 \), or

(3) \( y = 1 \) and

\[
I, J \in \{\{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\}, \{0, 2, 4, 6, 8, 10\} \}
\]

such that \( |I \cap J| = 3 \), or

(4) \( y = -1 \) and

\[
I, J \in \{\{0, 1, 4, 5, 8, 9\}, \{2, 3, 6, 7, 10, 11\}, \{0, 2, 4, 6, 8, 10\} \}
\]

such that \( |I \cap J| = 3 \), or

(5) \( y = -1 \) and

\[
I, J \in \{\{0, 3, 4, 7, 8, 11\}, \{1, 2, 5, 6, 9, 10\}, \{1, 3, 5, 7, 9, 11\} \}
\]

such that \( |I \cap J| = 3 \),

where \( n = |GF(2) \times GF(q)| = 2q \).

**Proof.** We only prove case (1) with \( I = \{0, 1, 4, 5, 8, 9\} \) and \( J = \{0, 3, 4, 7, 8, 11\} \) since proof for other cases is similar.

(1) Case \( w_1 = 0 \) and \( w_2 \neq 0 \). By Lemma 3.2, eq.(9), and Remark 3.1 eq.(14), for all \( w_2 \in GF(q) \setminus \{0\} \)

\[
d_{C'}(w_1, w_2) = d_C(w_1, w_2) + \delta_I = \frac{q - 3}{2} + 1 \tag{15}
\]

(2) Case \( w_1 = 1 \) and \( w_2 \neq 0 \). By Lemma 3.2, eq.(11), and Remark 3.1 eq.(14),
From eq. (15), eq. (17) and eq. (18), it is clear that
\[
C = \{ \text{most diagonalizable classes} \}
\]
and
\[
\begin{cases}
q^{x+1} & \text{if } w_2 \in Q,
\\
q^{x-1} & \text{if } w_2 \in QN.
\end{cases}
\]
Set \( x = 1 \) into eq. (16), it leads to
\[
d_C(w_1, w_2) = d_C(w_1, w_2) + \delta_J
\]
\[
= \begin{cases}
q^x + 1 & \text{if } w_2 \in Q, \\
q^x + 1 & \text{if } w_2 \in QN.
\end{cases}
\] 
(16)

(3) Case \( w_1 = 1 \) and \( w_2 = 0 \). By Lemma 3.2, eq. (13),
\[
d_C(w_1, w_2) = 2|D_1 \cap D_J|
\]
\[
= 2[D_0^{(12,q)} \cup D_4^{(12,q)} \cup D_8^{(12,q)}]
\]
\[
= 2 \cdot \frac{q-1}{12} \cdot 3
\]
\[
= \frac{q-1}{2}.
\] 
(18)

From eq. (15), eq. (17) and eq. (18), it is clear that \( C' = \{0\} \times D_1 \cup \{1\} \times D_J \cup \{(0, 0)\} \) is an \((2q, q^{x+1}, \frac{q^x + 1}{3})\) almost difference set with respect to \( A = GF(2) \times GF(q) \) with \( n = |GF(2) \times GF(q)| = 2q \).

4. Conclusion

The Ding-Helleseth-Martinsen’s Constructions is an efficient method to find and construct new almost difference set. In the present paper, we have constructed two classes of almost difference set with product sets between \( GF(2) \) and union sets of the cyclotomic classes of order 12 using that method. Constructing binary sequences with optimal three-level autocorrelation and optimal balance is equivalent to finding almost difference sets as their support sets. It is possible to extend the Ding-Helleseth-Martinsen Constructions by using \( GF(4) \) instead of \( GF(2) \). In addition, by computer investigation we found that no the Ding-Helleseth-Martinsen Constructions exists for cyclotomic classes of order six, eight and ten.

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