GENERALIZED SOLUTIONS TO MODELS OF COMPRESSIBLE VISCOUS FLUIDS

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Abstract. We propose a new approach to models of general compressible viscous fluids based on the concept of dissipative solutions. These are weak solutions satisfying the underlying equations modulo a defect measure. A dissipative solution coincides with the strong solution as long as the latter exists (weak–strong uniqueness) and they solve the problem in the classical sense as soon as they are smooth (compatibility). We consider general models of compressible viscous fluids with non–linear viscosity tensor and non–homogeneous boundary conditions, for which the problem of existence of global–in–time weak/strong solutions is largely open.

1. Introduction. Fluids in continuum mechanics are characterized by Stokes’ law
\[ T = S - pI \]
relating the Cauchy stress \( T \) to the viscous stress \( S \) and the scalar pressure \( p \). For the sake of simplicity, we ignore the thermal effects and consider the basic system of field equations for the fluid density \( \varrho = \varrho(t,x) \) and the macroscopic velocity \( \mathbf{u} = \mathbf{u}(t,x) \):

- conservation of mass
  \[ \partial_t \varrho + \text{div}_x(\varrho \mathbf{u}) = 0; \] (1.1)
- balance of linear momentum
  \[ \partial_t (\varrho \mathbf{u}) + \text{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \text{div}_x S; \] (1.2)
• balance of energy

\[
\partial_t \left( \frac{1}{2} \rho |\mathbf{u}|^2 + P(\rho) \right) + \text{div}_x \left[ \left( \frac{1}{2} \rho |\mathbf{u}|^2 + P(\rho) + p(\rho) \right) \mathbf{u} \right] = \text{div}_x (S \cdot \mathbf{u}) - S : \nabla_x \mathbf{u},
\]

where \( P(\rho) \) is the so-called pressure potential related to the pressure \( p, \)

\[
P'(\rho) \rho - P(\rho) = p(\rho).
\]

We suppose the fluid is contained in a bounded domain \( \Omega \subset \mathbb{R}^d, \ d = 2, 3, \) with general inflow–outflow boundary conditions

\[
\mathbf{u}|_{\partial \Omega} = \mathbf{u}_B, \ \rho|_{\Gamma_{in}} = \rho_B, \ \Gamma_{in} = \left\{ x \in \partial \Omega \ \big| \ \mathbf{u}_B \cdot \mathbf{n} < 0 \right\},
\]

where \( \mathbf{n} \) is the outer normal vector to \( \partial \Omega. \) The viscous stress tensor \( S \) is related to the symmetric velocity gradient

\[
\mathbb{D}_x \mathbf{u} = \frac{\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t}{2}
\]

through a general implicit rheological law

\[
\mathbb{D}_x \mathbf{u} : S = F(\mathbb{D}_x \mathbf{u}) + F^*(S)
\]

for a suitable convex potential \( F \) and its conjugate \( F^*. \) Finally, we prescribe the initial conditions

\[
\rho(0, \cdot) = \rho_0, \ \rho \mathbf{u}(0, \cdot) = \mathbf{m}_0.
\]

The total energy of the fluid is not conserved due to the presence of the dissipative term \( S : \nabla_x \mathbf{u} \) on the right-hand side of (1.3). As a matter of fact, the equation (1.3) can be deduced from (1.1), (1.2) via a straightforward manipulation as long as all quantities involved are smooth enough. In the weak formulation used in the present paper, the equation (1.3) is replaced by a suitable form of energy inequality discussed below.

If \( F \) is a proper convex l.s.c. function, the rheological relation (1.6) can be interpreted in view of Fenchel–Young inequality as

\[
S \in \partial F(\mathbb{D}_x \mathbf{u}) \iff \mathbb{D}_x \mathbf{u} \in \partial F^*(S).
\]

Note that the standard linear Newton's rheological law

\[
S = \mu \mathbb{D}_x \mathbf{u} + \lambda \text{div}_x \mathbf{u} \mathbb{I}
\]

corresponds to

\[
F(\mathbb{D}_x \mathbf{u}) = \frac{\mu}{2} |\mathbb{D}_x \mathbf{u}|^2 + \frac{\lambda}{2} |\text{div}_x \mathbf{u}|^2, \ \mu > 0, \ \lambda + \frac{2}{d} \mu \geq 0.
\]

The resulting problem is the standard Navier–Stokes system governing the motion of a linearly viscous compressible barotropic fluid.

The iconic example of the pressure–density relation is the isentropic state equation,

\[
p(\rho) = a \rho^\gamma, \ a > 0, \ \gamma > 1.
\]

In this case, the Navier–Stokes system (1.1), (1.2), (1.8), (1.9) admits global in time weak solutions for \( \gamma > \frac{d}{d+1}, \) see Lions [14] and [10], for the homogeneous boundary conditions, and [12], [4], [13] for general inflow–outflow boundary conditions. The paper [13] contains also the weak strong uniqueness result. The existence of global weak solutions in the case \( d = 2, \gamma = 1 \) was proved by Plotnikov and Vaigant [18].
Much less is known for a general non–linear dependence of viscosity on the velocity gradient. To the best of our knowledge, there are only two large–time existence results in the class of weak solutions in the multidimensional case: (i) Mamontov [15], [16] considered the case of exponentially growing viscosity coefficients and linear pressure \( p(\rho) = a\rho \); (ii) [11], where the bulk viscosity \( \lambda = \lambda(|\nabla x u|) \) becomes singular for a finite value of \( |\nabla x u| \); (iii) [17], where the authors consider linear pressure and introduce a notion of measure–valued solutions.

Motivated by [1], [3], we introduce the concept of dissipative solution for problem (1.1)–(1.7). The dissipative solution satisfies the system of equations

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho u) & = 0, \\
\partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x p(\rho) & = \text{div}_x [S_{\text{eff}}].
\end{align*}
\]

(1.10)

The effective stress acting on the fluid can be decomposed as

\[
S_{\text{eff}} = S - \mathcal{R},
\]

(1.11)

where the “turbulent” component \( \mathcal{R} \) is called Reynolds stress and it is a positively semi–definite tensor. The problem is supplemented with the energy inequality

\[
\begin{align*}
\frac{d}{dt} \int_\Omega \left[ \frac{1}{2} |\rho u - u_B|^2 + P(\rho) + \mathcal{E} \right] dx + \int_\Omega \left[ F(D_x u) + F^*(S_{\text{eff}} + \mathcal{R}) \right] dx dt \\
+ \int_{\partial\Omega} P(\rho) u_B \cdot n dS_x dt & \leq - \int_\Omega \left[ \rho(\rho) I + \rho u \otimes u : \nabla_x u_B \right] dx dt \\
+ \frac{1}{2} \int_\Omega |\rho u \cdot \nabla_x u|^2 dx dt + \int_{\partial\Omega} S_{\text{eff}} : \nabla_x u_B dx dt.
\end{align*}
\]

(1.12)

Here the symbol \( \mathcal{E} \) denotes the energy dissipation defect directly related to the Reynolds stress \( \mathcal{R} \), more specifically, \( \mathcal{E} \approx \text{tr} \mathcal{R} \). The velocity \( u_B \) is a suitable extension of the boundary velocity to \( \overline{\Omega} \). The quantities \( \rho, u \) are called dissipative solution if they satisfy (1.10)–(1.12) in the sense of distributions for suitable \( S_{\text{eff}}, \mathcal{R} \), and \( \mathcal{E} \). It is easy to show that if

\[
\mathcal{R} = \mathcal{E} = 0, \quad S \in \partial F(D_x u),
\]

(1.13)

then \( [\rho, u] \) is the standard weak solution satisfying a variant of the energy inequality.

The exact definition of dissipative solution is given in Section 2. Then we plan to address the following issues:

- **Existence.** In Section 3, we show that problem (1.10)–(1.12) admits global–in–time solutions for any finite energy initial data. We consider a general bounded Lipschitz domain \( \Omega \) in view of possible applications in the analysis of numerical schemes.
- **Compatibility.** In Section 4, we show that dissipative solutions of (1.10)–(1.12) that are continuously differentiable are classical solutions of (1.1)–(1.7). In particular, (1.13) holds.
- **Weak–strong uniqueness.** A solution of (1.10)–(1.12) coincides with the strong solution of (1.1)–(1.7) emanating from the same initial data as long as the latter exists, see Sections, 5, 6.

2. **Weak formulation.** We consider a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d, d = 2, 3 \). We suppose that the boundary conditions for the velocity are given by \( u_B \in C^1(\mathbb{R}^d; \mathbb{R}^d) \). The boundary \( \partial\Omega \) is decomposed as

\[
\partial\Omega = \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \quad \Gamma_{\text{in}} = \left\{ x \in \partial\Omega \quad \text{the outer normal } n(x) \text{ exists, and } u_B(x) \cdot n(x) < 0 \right\}.
\]
On one hand, the required regularity of the field $u_B$ is not optimal and could be relaxed in the spirit of the work of Crippa et al. \cite{crippa20175} that is used in the existence proof in Section 3 below. On the other hand, the Lipschitz regularity of $\partial \Omega$ is relevant in possible applications to convergence of numerical schemes, where the underlying domain is a polygon.

2.1. **Structural restrictions on the constitutive equations.** We start by introducing the structural restrictions imposed on the pressure–density equation of state (EOS), and the specific form of the viscous stress.

2.1.1. **Pressure–density equation of state.** An iconic example of a barotropic EOS is the isentropic pressure–density relation

$$ p(\varrho) = a \varrho^\gamma, \quad a > 0, \quad \gamma > 1 $$

yielding the pressure potential $P(\varrho) = \frac{a}{\gamma - 1} \varrho^{\gamma}$. \hfill (2.1)

Here we allow for more general EOS retaining the essential features of (2.1):

$$ p \in C[0, \infty) \cap C^2(0, \infty), \quad p(0) = 0, \quad p'(\varrho) > 0 \text{ for } \varrho > 0; $$

the pressure potential $P$ determined by $P'(\varrho) \varrho - P(\varrho) = p(\varrho)$ satisfies $P(0) = 0$, and $P - \varrho p, \varpi p - P$ are convex functions for certain constants $a > 0, \varpi > 0$. \hfill (2.2)

As a matter of fact, certain hypotheses on the pressure can be relaxed. We shall discuss this issue in the concluding Section 7. It is easy to check that any $P$ satisfying (2.2) possesses certain coercivity similar to (2.1). More specifically,

$$ P(\varrho) \geq a \varrho^{\gamma} \text{ for certain } a > 0, \quad \gamma > 1 \text{ and all } \varrho \geq 1. \hfill (2.3) $$

Indeed as $\varpi p - P$ is a convex function and $P$ is strictly convex, we get

$$ \varpi p''(\varrho) \geq P''(\varrho) = \frac{p'(\varrho)}{\varrho}, \quad \varrho > 0. $$

This yields

$$ \log' \left( p'(\varrho) \right) \geq \log' \left( \varrho^{\frac{\gamma}{\gamma - 1}} \right) \Rightarrow p'(\varrho) \geq \varrho^{\frac{\gamma}{\gamma - 1}} \text{ for all } \varrho \text{ large enough;} $$

whence (2.3) holds for $\gamma = 1 + \frac{1}{\varpi}$.

2.1.2. **Viscous stress.** The relation between the symmetric velocity gradient $\mathbb{D}_x u$ and the viscous stress $\mathbb{S}$ is determined by the choice of the potential $F$. We suppose that

$$ F : P_{\text{sym}}^{d \times d} \to [0, \infty) \text{ is a (proper) convex function, } F(0) = 0, \hfill (2.4) $$

enjoying certain coercivity properties to render the symmetric gradient $\mathbb{D}_x u$, or at least its traceless part $\mathbb{D}_x u - \frac{1}{d} \text{div}_x u \|u\|$, integrable. To make the last statement more specific, we introduce the class of Young functions $A$:

- $A : [0, \infty) \to [0, \infty)$ convex,
- $A$ increasing,
- $A(0) = 0$.

Moreover, we shall say that $A$ satisfies $\Delta_2^2$–condition if there exist constants $a_1 > 2, a_2 < \infty$ such that

$$ a_1 A(z) \leq A(2z) \leq a_2 A(z) \text{ for any } z \in [0, \infty). \hfill (2.5) $$
In addition to (2.4), we suppose that for any $R > 0$, there exists a Young function $A_R$ satisfying the $\Delta_2^2$-condition (2.5) and such that

$$F(\mathbb{D} + \mathbb{Q}) - F(\mathbb{D}) - \mathbb{S} : \mathbb{Q} \geq A_R \left( \| \mathbb{Q} - \frac{1}{d} \text{tr}[\mathbb{Q}] \| \right)$$

(2.6)

for all $\mathbb{D}, \mathbb{S}, \mathbb{Q} \in \mathbb{R}^{d\times d}_{\text{sym}}$ such that $|\mathbb{D}| \leq R$, $\mathbb{S} \in \partial F(\mathbb{D})$.

Note that the standard Newtonian rheological law with the associated quadratic potential

$$F(\mathbb{D}) = \mu \| \mathbb{D} - \frac{1}{d} \text{tr}[\mathbb{D}] \|^2 + \eta \| \text{tr}[\mathbb{D}] \|^2, \quad \mu > 0, \quad \eta \geq 0,$$

satisfies (2.6) with

$$A_R(z) = A(z) = \frac{\mu}{2} z^2.$$

For a more general $p$-potential

$$F(\mathbb{D}) = \frac{\mu_0}{p} (\mu_1 + \| \mathbb{D}_0 \|^2)^{\frac{p}{2}}$$

with $\mu_0 > 0$, $\mu_1 \geq 0$, $p > 1$,

noticing that

$$F(\mathbb{A}) - F(\mathbb{B}) - \partial F(\mathbb{B}) : (\mathbb{A} - \mathbb{B}) = \int_0^1 (\partial F(\mathbb{B} + t(\mathbb{A} - \mathbb{B})) - \partial F(\mathbb{B})) : (\mathbb{A} - \mathbb{B}) \, dt$$

and making use of [7, Lemma 6.3] we get

$$F(\mathbb{A}) - F(\mathbb{B}) - \partial F(\mathbb{B}) : (\mathbb{A} - \mathbb{B}) \geq \begin{cases} \min \left\{ \frac{\mu_0}{p} (\mu_1 + 3R)^{p-2}\| \mathbb{A} - \mathbb{B} \|^2, \frac{\mu_0}{p} (\mu_1 + 3R)^{p-2}\| \mathbb{A} - \mathbb{B} \|^p \right\} & \text{if } p < 2, \\ \frac{\mu_0}{p} \| \mathbb{A} - \mathbb{B} \|^p & \text{if } p \geq 2, \end{cases}$$

for any $\mathbb{A}, \mathbb{B}$ such that $|\mathbb{B}| \leq R$. Thus relation (2.6) is satisfied with

$$A_R(z) = \begin{cases} \frac{\mu_0}{p} (\mu_1 + 3R)^{p-2}z^2 & \text{if } p < 2, \quad 0 \leq z \leq 1, \\ \frac{\mu_0}{p} (\mu_1 + 3R)^{p-2}(1-\alpha)z^2 + \alpha z - \beta & \text{if } p < 2, \quad z \geq 1, \\ \frac{\mu_0}{p} z^p & \text{if } p \geq 2, \end{cases}$$

where $0 < \alpha < 1$, $\beta > 0$ are constants fixed in order to make $A_R$ continuous and convex.

Finally, it follows from (2.6) that there exist $\mu > 0$ and $q > 1$ such that

$$F(\mathbb{D}) \geq \mu \left\| \mathbb{D} - \frac{1}{d} \text{tr}[\mathbb{D}] \right\|^q$$

for all $|\mathbb{D}| > 1$.

(2.7)

Indeed it is enough to consider $\mathbb{D} = \mathbb{S} = 0$ and $R = 1$ in (2.6). In view of (2.5), the function $A_1$ possesses the desired $q$-growth for large values of its arguments.

2.2. Weak formulation of the field equations. We introduce the weak formulation of the conservation laws (1.1), (1.2).
2.2.1. Equation of continuity - mass conservation. The density \( \varrho \geq 0 \) is non-negative and belongs to the class
\[
\varrho \in C_{\text{weak}}([0, T]; L^\infty(\Omega)) \cap L^\gamma(0, T; L^\gamma(\partial \Omega; |u_B \cdot n| dS_x)), \quad \gamma > 1,
\]
the momentum satisfies
\[
\varrho u \in C_{\text{weak}}([0, T]; L^{\frac{2\gamma}{\gamma + 1}}(\Omega; R^d)).
\]
The integral identity
\[
\left[ \int_{\Omega} \varrho \varphi \, dx \right]_{t=0}^{t=T} + \int_{0}^{T} \int_{\Gamma_{\text{out}}} \varrho u_B \cdot n \, dS_x \, dt + \int_{0}^{T} \int_{\Gamma_{\text{in}}} \varrho \varphi u_B \cdot n \, dS_x \, dt = \int_{0}^{T} \int_{\Omega} \left[ \varrho \partial_t \varphi + \varrho u \cdot \nabla \varphi \right] \, dx \, dt
\]
holds for any \( 0 \leq \tau \leq T \), and any test function \( \varphi \in C^1([0, T] \times \overline{\Omega}) \),
\[
\varrho(0, \cdot) = \varrho_0.
\]

2.2.2. Momentum balance. Let the symbol \( M^+(\overline{\Omega}; R_{\text{sym}}^{d \times d}) \) denote the set of all positively semi-definite tensor valued measures on \( \overline{\Omega} \). We suppose there exist
\[
S \in L^1((0, T) \times \Omega; R_{\text{sym}}^{d \times d}), \quad \mathfrak{R} \in L^\infty(0, T; M^+(\overline{\Omega}; R_{\text{sym}}^{d \times d})),
\]
such that the integral identity
\[
\left[ \int_{\Omega} \varrho u \cdot \varphi \, dx \right]_{t=0}^{t=T} = \int_{0}^{T} \int_{\Omega} \nabla \varphi : d\mathfrak{R}(t) \, dt + \int_{0}^{T} \int_{\Omega} \left[ \varrho u \cdot \partial_t \varphi + \varrho u \otimes u : \nabla \varphi + p(\varrho) \text{div} \varphi - S : \nabla \varphi \right] \, dx \, dt
\]
holds for any \( 0 \leq \tau \leq T \) and any test function \( \varphi \in C^1([0, T] \times \overline{\Omega}; R^d) \), \( \varphi|_{\partial \Omega} = 0 \),
\[
\varrho u(0, \cdot) = u_0
\]
Here we assume that all quantities appearing in (2.10) are at least integrable in \( (0, T) \times \Omega \). In accordance with (1.10), we may set
\[
S_{\text{eff}} = S - \mathfrak{R}.
\]

2.3. Energy inequality and defect compatibility condition. A proper form of the energy balance (1.3) is a cornerstone of the subsequent analysis. We first introduce the energy defect measure
\[
\mathcal{E} \in L^\infty(0, T; M^+(\overline{\Omega})).
\]
The energy inequality reads
\[
\left[ \frac{1}{2} \varrho \|u - u_B\|^2 + P(\varrho) \right]_{t=0}^{t=T} + \int_{0}^{T} \int_{\Omega} \left[ F(\nabla_x u) + F^*(S) \right] \, dx \, dt + \int_{0}^{T} \int_{\Gamma_{\text{out}}} P(\varrho_B) u_B \cdot n \, dS_x \, dt + \int_{0}^{T} \int_{\Gamma_{\text{in}}} P(\varrho_B) u_B \cdot n \, dS_x \, dt + \int_{\Omega} \mathfrak{R}(t) \, \text{d} \mathcal{E}(t)
\]
\[
\leq -\int_{0}^{T} \int_{\Omega} \left[ \varrho u \otimes u + p(\varrho) \text{div} \varphi \right] : \nabla_x u_B \, dx \, dt + \int_{0}^{T} \frac{1}{2} \int_{\Omega} \varrho u \cdot \nabla \varphi |u_B|^2 \, dx \, dt + \int_{0}^{T} \int_{\Omega} S : \nabla_x u_B \, dx \, dt - \int_{0}^{T} \int_{\Omega} \mathfrak{R}(t) \, \text{d} \mathcal{E}(t).
\]
for a.a. \( 0 \leq \tau \leq T \).
Finally, we suppose a compatibility conditions such that the energy defect $\mathcal{E}$ dominates the Reynolds stress $\mathcal{R}$, specifically,
\[
\underline{d}\mathcal{E} \leq \text{tr}[\mathcal{R}] \leq \overline{d}\mathcal{E},
\]
for certain constants $0 < \underline{d} \leq \overline{d}$.

(2.13)

This property is absolutely crucial for the weak–strong uniqueness principle stated in Section 6 below.

**Definition 2.1 (Dissipative solution).** Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded Lipschitz domain. The quantity $[\varrho, u]$ is called dissipative solution of the problem (1.1)–(1.7) if:

- $\varrho : [0, T] \times \Omega \to \mathbb{R}$, $\varrho \in C_{\text{weak}}([0, T]; L^\gamma(\Omega)) \cap L^\gamma(0, T; L^\gamma(\partial \Omega, |u_B \cdot n| dS_x))$, $\gamma > 1$, $u \in L^q(0, T; W^{1,q}(\Omega; \mathbb{R}^d))$, $(u - u_B) \in L^q(0, T; W^{1,q}_0(\Omega; \mathbb{R}^d))$, $q > 1$, $m \equiv \varrho u \in C_{\text{weak}}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d))$.

- There exist $S \in L^1((0, T) \times \Omega; R_{\text{sym}}^{d \times d})$, $E \in L^\infty(0, T; \mathcal{M}(\Omega))$, $\mathcal{R} \in L^\infty(0, T; \mathcal{M}(\Omega; R_{\text{sym}}^{d \times d}))$

such that the relations (2.8)–(2.13) are satisfied for any $0 \leq \tau \leq T$.

**Remark 2.2.** In view of (2.13), one can always consider
\[
\mathcal{E} \equiv \frac{1}{\underline{d}} \text{tr}[\mathcal{R}];
\]
whence, strictly speaking, the energy defect $\mathcal{E}$ can be completely omitted in the definition, see the discussion in Section 7 for details.

**Remark 2.3.** As we shall see in the existence proof below, the dissipative solutions can be constructed in such a way that the constant $\overline{d}$ depends solely on the dimension $d$ and the structural constants $\underline{d}, \overline{d}$ appearing in (2.2).

Although the concept of dissipative solution follows the same idea as in the incompressible case studied in [1], the compressible case is more challenging because the pressure $p = p(\varrho)$ depends explicitly on the density. Accordingly, the Reynolds stress involves a contribution from the pressure defect, while the energy defect is augmented by the contribution of the pressure potential. It is interesting to note that $\mathcal{R} \equiv 0$ in the case considered in [17], where the pressure is linear and $q > d$.

3. **Existence.** Our first goal is to show that the dissipative solutions exist globally in time for any finite energy initial data. The proof is based on a multilevel approximation scheme that shares certain features with the approximation of the compressible Navier–Stokes in [10]. First, we introduce a sequence of finite–dimensional spaces $X_n \subset L^2(\Omega; R^d)$,
\[
X_n = \text{span} \left\{ w_i \mid w_i \in C^\infty_c(\Omega; R^d), \; i = 1, \ldots, n \right\}.
\]
Without loss of generality, we may assume that $w_i$ are orthonormal with respect to the standard scalar product in $L^2$

Next, we regularize the convex potential $F$ to make it continuously differentiable. This may be done via convolution with a family of regularizing kernels
\[
F_\delta(\mathbb{D}) = \int_{R_{\text{sym}}^{d \times d}} \xi_\delta(|\mathbb{D} - Z|) F(Z) \; dZ - \int_{R_{\text{sym}}^{d \times d}} \xi_\delta(|Z|) F(Z) \; dZ.
\]
It is easy to check that $F_\delta$ are convex, non-negative, infinitely differentiable, $F_\delta(0) = 0$, and satisfy (2.7), specifically
\[ F_\delta(D) \geq \nu \left| D - \frac{1}{d} \text{tr}[D] \right|^q \quad \text{for all } |D| > 1, \tag{3.1} \]
with $q > 1$, $\nu > 0$ independent of $\delta \searrow 0$.

3.1. First level approximation. To begin, we suppose that the initial and boundary data are smooth. Specifically,
\begin{align*}
(A1) \quad & u_B \in C^1_t(R^d; R^d); \\
(A2) \quad & q_0 \in C^1_t(R^d), q_B \in C^1_t(\partial \Omega).
\end{align*}

3.1.1. Artificial viscosity approximation of the equation of continuity. Following Crippa, Donadello, Spinolo [6] we use a parabolic approximation of the equation of continuity,
\[ \partial_t \varrho + \text{div}_x (\varrho u) = \varepsilon \Delta_x \varrho \quad \text{in } (0, T) \times \Omega, \quad \varepsilon > 0, \tag{3.2} \]
supplemented with the boundary conditions
\[ \varepsilon \nabla_x \varrho \cdot n + (q_B - q)[u_B \cdot n^-] = 0 \quad \text{in } (0, T) \times \partial \Omega, \tag{3.3} \]
and the initial condition
\[ \varrho(0, \cdot) = q_0. \tag{3.4} \]
Here, $u = v + u_B$, with $v \in C([0, T]; X_n)$, in particular, $u_{|\partial \Omega} = u_B$. Note that for given $u, q_B, u_B$ this is a linear problem for the unknown $\varrho$.

As $\Omega$ is merely Lipschitz, the usual parabolic estimates fail at the level of the spatial derivatives and we are forced to use the weak formulation:
\[ \left[ \int_{\Omega} \varrho \phi \, dx \right]_{t=0}^{t=T} = \int_0^T \int_{\Omega} [\varrho \partial_t \varphi + \varrho u \cdot \nabla_x \varphi - \varepsilon \nabla_x \varrho \cdot \nabla_x \varphi] \, dx \, dt \\
- \int_0^T \int_{\partial \Omega} \varrho u_B \cdot n \, dS_x \, dt + \int_0^T \int_{\partial \Omega} \varphi(q - q_B)[u_B \cdot n^-] \, dS_x \, dt, \tag{3.5} \]
for any test function
\[ \varphi \in L^2(0, T; W^{1,2}(\Omega)), \quad \partial_t \varphi \in L^1(0, T; L^2(\Omega)). \]

**Lemma 3.1 (Crippa et al [6, Lemma 3.2]).** Let $\Omega \subset R^d$ be a bounded Lipschitz domain.

Given $u = v + u_B, v \in C([0, T]; X_n)$, the initial–boundary value problem (3.2)–(3.4) admits a weak solution $\varrho$ specified in (3.5), unique in the class
\[ \varrho \in L^2(0, T; W^{1,2}(\Omega)) \cap C([0, T]; L^2(\Omega)). \]
The norm in the aforementioned spaces is bounded only in terms of the data $q_B$, $q_0$, $u_B$ and
\[ \sup_{t \in (0, T)} \| v(t, \cdot) \|_{X_n}. \]

Next, we report another result of Crippa et al. [6, Lemma 3.4].

**Lemma 3.2 (Maximum principle).** Under the hypotheses of Lemma (3.1), the solution $\varrho$ satisfies
\[ \| \varrho \|_{L^\infty((0, T) \times \Omega)} \leq \max \{ \| q_0 \|_{L^\infty(\Omega)}; \| q_B \|_{L^\infty((0, T) \times \Gamma_n)}; \| u_B \|_{L^\infty((0, T) \times \Omega)} \} \cdot \exp \left( \tau \| \text{div}_x u \|_{L^\infty((0, T) \times \Omega)} \right). \]
Next, we perform renormalization of equation (3.2). In view of future applications, in particular the strong minimum principle, it is convenient to rewrite the integral identity (3.5) in terms of a new variable \( r = r(t, x) = \varrho(t, x) - \chi(t) \), where \( \chi \in W^{1,\infty}(0, T) \). After a straightforward manipulation, we obtain:

\[
\left[ \int_{\Omega} r \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \left[ r \partial_t \varphi - (\partial_t \chi + \chi \text{div}_x u) \varphi + r u \cdot \nabla_x \varphi - \varepsilon \nabla_x r \cdot \nabla_x \varphi \right] \, dx \, dt + \int_0^\tau \int_{\partial\Omega} \varphi ((r + \chi) - \varrho_B) [u_B \cdot n]^- \, dS_x \, dt,
\]

for any test function \( \varphi \in L^2(0, T; W^{1,2}(\Omega)) \), \( \partial_t \varphi \in L^1(0, T; L^2(\Omega)) \).

**Lemma 3.3 (Renormalization).** Under the conditions of Lemma 3.1, let \( B \in C^2(R) \), \( \chi \in W^{1,\infty}(0, T) \), and \( r = \varrho - \chi \).

Then, the (integrated) renormalized equation

\[
\left[ \int_{\Omega} B(r) \, dx \right]_{t=0}^{t=\tau} = - \int_0^\tau \int_{\Omega} \text{div}_x (ru) B'(r) \, dx \, dt - \varepsilon \int_0^\tau \int_{\Omega} |\nabla_x r|^2 B''(r) \, dx \, dt - \int_{\Omega} \left( \partial_t \chi + \chi \text{div}_x u \right) B'(r) \, dx + \int_0^\tau \int_{\partial\Omega} B'(r) ((r + \chi) - \varrho_B) [u_B \cdot n]^- \, dS_x \, dt
\]

holds for any \( 0 \leq \tau \leq T \).

**Proof.** To begin, in accordance the maximum principle stated in Lemma 3.2, we may assume \( B \in C^2_c(R) \). Moreover, as \( u = v + u_B \), we have

\[
\text{div}_x (gu) = \nabla_x g(v + u_B) + g \text{div}_x (v + u_B).
\]

Thus, in view of the bounds on \( \varrho \) obtained in Lemma 3.1, we have

\[
\|\text{div}_x (gu)\|_{L^2((0, T) \times \Omega)} \leq c
\]

in terms of the data only.

As

\[
\partial_t \varrho \in L^2(0, T; (W^{1,2})^*(\Omega)), \quad B'(r) \in L^2(0, T; W^{1,2}(\Omega))
\]

we deduce from the weak formulation (3.6) that

\[
\langle \partial_t r, B'(r) \rangle_{[W^{1,2}]^*, W^{1,2}} = - \int_{\Omega} \text{div}_x (ru) B'(r) \, dx - \varepsilon \int_{\Omega} |\nabla_x r|^2 B''(r) \, dx - \int_{\Omega} \left( \partial_t \chi + \chi \text{div}_x u \right) B'(r) \, dx + \int_{\partial\Omega} B'(r) ((r + \chi) - \varrho_B) [u_B \cdot n]^- \, dS_x
\]
Integrating over the interval \([\tau_1, \tau_2]\) we get
\[
\int_{\tau_1}^{\tau_2} (\partial_t r, B'(r))_{[W^{1,2}]; W^{1,2}} \, dt
= - \int_{\tau_1}^{\tau_2} \int_\Omega \text{div}_x (r u) B'(r) \, dx \, dt - \varepsilon \int_{\tau_1}^{\tau_2} \int_\Omega |\nabla_x r|^2 B''(r) \, dx \, dt
- \int_{\tau_1}^{\tau_2} \int_\Omega \left( \partial_t \chi + \chi \text{div}_x u \right) B'(r) \, dx \, dt
+ \int_{\tau_1}^{\tau_2} \int_{\partial \Omega} B'(r)((r + \chi) - \varrho_B)[u_B \cdot n]^- \, dS_x \, dt
\]

Finally, using the standard temporal regularization via a family of \(t\)--dependent convolution kernels, we find a sequence of functions
\[
\varrho_n \to \varrho \text{ in } L^2((\tau_1, \tau_2); W^{1,2}(\Omega)), \quad \varrho_n \in C^1((\tau_1, \tau_2]; W^{1,2}(\Omega)),
\partial_t \varrho_n \to \partial_t \varrho \text{ in } L^2(\tau_1, \tau_2; [W^{1,2}]^*(\Omega))
\]
for any \(0 < \tau_1 < \tau_2\), and, as \(\varrho \in C([0, T]; L^2(\Omega)) \cap L^\infty((0, T) \times \Omega),
B(\varrho_n - \chi)(\tau) \to B(\varrho(\tau) - \chi(\tau)) \text{ for any } \tau \in (0, T).

Thus we obtain
\[
\left[ \int_\Omega B(r) \, dx \right]_{t=\tau_1}^{t=\tau_2} = \lim_{n \to \infty} \left[ \int_\Omega B(\varrho_n - \chi) \, dx \right]_{t=\tau_1}^{t=\tau_2}
= \lim_{n \to \infty} \int_{\tau_1}^{\tau_2} \int_\Omega B'(\varrho_n - \chi) \partial_t (\varrho_n - \chi) \, dx \, dt = \int_{\tau_1}^{\tau_2} \langle \partial_t r, B'(r) \rangle \, dt
\]
for any \(0 < \tau_1 < \tau_2 < T\), which yields the desired conclusion. \(\square\)

Using Lemma 3.3 we obtain strict positivity of \(\varrho\) on condition that \(\varrho_B, \varrho_0\) enjoy the same property.

**Corollary 3.4.** Under the hypotheses of Lemma 3.1, we have
\[
\text{ess inf}_{t,x} \varrho(t, x) \geq \min \left\{ \min_{\Omega} \varrho_0; \min_{\Gamma_{\text{in}}} \varrho_B \right\} \exp \left( -T \|\text{div}_x u\|_{L^\infty((0, T) \times \Omega)} \right).
\]

Indeed it is enough to apply Lemma 3.3 to
\[
\chi(\tau) = \varrho \exp \left( -\int_0^\tau \|\text{div}_x u(t, \cdot)\|_{L^\infty(\Omega)} \, dt \right), \quad \varrho = \min \left\{ \min_{\Omega} \varrho_0; \min_{\Gamma_{\text{in}}} \varrho_B \right\}
\]
and
\[
B(r) = -r^+ = \begin{cases} -r & \text{if } r \leq 0, \\ 0 & \text{if } r > 0 \end{cases}
\]
to deduce
\[
\int_\Omega B(r)(\tau) \, dx = 0 \text{ for a.a. } \tau \in (0, T) \Rightarrow r = \varrho - \chi \geq 0.
\]

Finally, we observe that in the present setting, specifically for smooth and *time independent* vector fields \(\varrho_B, u_B\), the weak solution enjoys more regularity than in Crippa et al. [6]. In particular, we have the estimates
\[
\|\partial_t \varrho\|_{L^2((0, T) \times \Omega)} + \varepsilon \text{ess sup}_{\tau \in (0, T)} \|\nabla_x \varrho(\tau, \cdot)\|_{L^2(\Omega)}^2 \leq c, \quad (3.7)
\]
with the constant depending only on the data. The estimate (3.7) can be first obtained formally via multiplying the equation (3.2) on $\partial_t \rho$:

$$
\int_{\Omega} |\partial_t \rho|^2 \, dx + \int_{\Omega} \text{div} (\rho u) \partial_t \rho \, dx = \varepsilon \int_{\Omega} \Delta \rho \partial_t \rho \, dx
$$

$$
= -\frac{\varepsilon}{2} \frac{d}{dt} \int_{\Omega} |\nabla \rho|^2 \, dx + \varepsilon \int_{\partial \Omega} \nabla \rho \cdot n \partial_t \rho \, dS_x,
$$

where, in accordance with the boundary condition (3.3),

$$
\varepsilon \int_{\partial \Omega} \nabla \rho \cdot n \partial_t \rho \, dS_x = \int_{\partial \Omega} \partial_t \rho (\rho - \rho_B) |u_B \cdot n|^{-1} \, dS_x = \frac{1}{2} \frac{d}{dt} \int_{\partial \Omega} (\rho - \rho_B)^2 |u_B \cdot n|^{-1} \, dS_x.
$$

Consequently, the desired estimate (3.7) follows by integrating the above relation over time. As pointed out in Crippa et al. [6], the (unique) weak solution $\rho$ can be constructed by means of Faedo–Galerkin approximation. The latter being compatible with multiplication on $\partial_t \rho$, the above argument can be performed on the approximation and thus transferred to the limit via lower semi–continuity of the associated norms.

3.1.2. Galerkin approximation of the momentum balance. We look for approximate velocity field in the form

$$
u = \nu + u_B, \quad \nu \in C([0, T]; X_n).
$$

Accordingly, the approximate momentum balance reads

$$
\left[ \int_{\Omega} \rho u \cdot \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \rho u \cdot \partial_t \varphi \, dx \, dt - \varepsilon \int_0^\tau \int_{\Omega} \nabla \rho \cdot \nabla u \cdot \varphi \, dx \, dt
$$

$$
+ \int_0^\tau \int_{\partial \Omega} \left[ \rho u \otimes \nu : \nabla \varphi + p(\rho) \text{div} \varphi - \partial F_\delta(D_x u) : \nabla \varphi \right] \, dS_x \, dt
$$

(3.8)

for any $\varphi \in C^1([0, T]; X_n)$, with the initial condition

$$
\rho u(0, \cdot) = \rho_0 u_0, \quad u_0 = \nu_0 + u_B, \quad \nu_0 \in X_n.
$$

(3.9)

For fixed parameter $n$, $\delta > 0$, $\varepsilon > 0$, the first level approximation are solutions $[\rho, u]$ of the parabolic problem (3.2)–(3.4), and the Galerkin approximation (3.8), (3.9). The existence of the approximate solutions at this level can be proved in the same way as in [4]. Specifically, for $u = u_B + \nu$, $\nu \in C([0, T]; X_n)$, we identify the unique solution $\rho = \rho(u)$ of (3.2)–(3.4) and plug it as $\rho$ in (3.8). The unique solution $u = u(\rho)$ of (3.8) defines a mapping

$$
\mathcal{T} : \nu \in C([0, T]; X_n) \mapsto \mathcal{T}[\nu] = (u(\rho) - u_B) \in C([0, T]; X_n).
$$

The first level approximate solutions $\rho = \rho_{\delta, \varepsilon, n}$, $u = u_{\delta, \varepsilon, n}$ are obtained via a fixed point though the mapping $\mathcal{T}$. The exact procedure is detailed in [4] and in [13], from where we report the following result.\footnote{The energy inequality (3.10) in [13, Lemma 4.2] is derived under assumption $\Omega \in C^2$. This assumption is needed due to the treatment of the parabolic problem (3.2–3.4) via the classical maximal regularity methods. With Lemmas 3.1–3.3 and Corollary 3.4 at hand, the same proof can be carried out without modifications also in Lipschitz domains.}
Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain. Suppose that $p = p(\varrho)$ and $F = F(\varrho)$ satisfy (2.2), (2.4)–(2.6). Let the data belong to the class
\[ u_B \in C^1_c(R^d; \mathbb{R}^d), \quad \varrho_0 \in C^1(R^d), \quad \varrho_B \in C^1(\partial \Omega), \quad \varrho_0, \varrho_B \geq \varrho > 0, \]
\[ u_0 = v_0 + u_B, \quad \varrho_0 \in X_n. \]

Then for each fixed $\delta > 0$, $\varepsilon > 0$, $n > 0$, there exists a solution $\varrho$, $u$ of the approximate problem (3.2)–(3.4) and (3.8), (3.9). Moreover, the approximate energy inequality
\[
\left[ \int_{\Omega} \left[ \frac{1}{2} \varrho |u - u_B|^2 + P(\varrho) \right] \, dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{\Omega} \partial F_\delta(\nabla u) : \nabla u \, dx \, dt \\
+ \int_0^\tau \int_{\Gamma_{\text{out}}} P(\varrho) u_B \cdot n \, dS_x \, dt + \varepsilon \int_0^\tau \int_{\Omega} |P''(\varrho)| |\nabla \varrho|^2 \, dx \, dt \\
- \int_0^\tau \int_{\Gamma_{\text{in}}} [P(\varrho_B) - P'(\varrho)(\varrho_B - \varrho) - P(\varrho)] u_B \cdot n \, dS_x \, dt \\
\leq - \int_0^\tau \int_{\Omega} [g u \otimes u + p(\varrho)|u|^2] : \nabla u \, dx \, dt + \frac{1}{2} \int_0^\tau \int_{\Omega} g u \cdot \nabla |u|^2 \, dx \, dt \\
+ \int_0^\tau \int_{\Omega} \partial F_\delta(\nabla u) : \nabla u \, dx \, dt - \int_0^\tau \int_{\Gamma_{\text{in}}} P(\varrho_B) u_B \cdot n \, dS_x \, dt \\
+ \varepsilon \int_0^\tau \int_{\Omega} \nabla \varrho \cdot \nabla (u - u_B) \cdot u_B \, dx \, dt
\]
holds for any $0 \leq \tau \leq T$.

### 3.2. Second level approximation
The next step is to let $\delta \to 0$ in the regularization of the potential $F_\delta$. This is an easy task as all the necessary bounds from the previous step remain valid. Indeed, in view of the uniform bound (3.1) and the energy inequality (3.10), we deduce a uniform bound on the traceless part of the symmetric velocity gradient,
\[
\left\| \nabla u - \frac{1}{d} \text{div}_x u \right\|_{L^q((0,T) \times \Omega; \mathbb{R}^{d \times d})} \leq c, \quad q > 1
\]
uniformly for $\delta \to 0$. In view of the fact $u = v + u_B$, $u|_{\partial \Omega} = u_B$, we may use the $L^q$–version of Korn’s inequality to obtain
\[
\left\| \nabla u \right\|_{L^q((0,T) \times \Omega; \mathbb{R}^{d \times d})},
\]
which, combined with the standard Poincaré inequality, yields the final conclusion
\[
\left\| u \right\|_{L^q((0,T); W^{1,q}(\Omega; \mathbb{R}^d))} \leq c \text{ for some } q > 1.
\]

At this stage, $n$ is fixed and all norms are equivalent on the finite–dimensional space $X_n$. In particular,
\[
\left\| \nabla u \right\|_{L^\infty((0,T) \times \Omega; \mathbb{R}^{d \times d})}
\]
remains bounded uniformly for $\delta \searrow 0$. Therefore it is standard to perform the limit $\delta \to 0$. Accordingly, we have obtained the same conclusion as in Proposition 3.5.
with (3.8) replaced by
\[
\left[ \int_\Omega \varphi \cdot \varphi \, dx \right]_{t=0}^{t=T} = \int_0^T \int_\Omega \varphi \cdot \partial_t \varphi \, dx \, dt - \varepsilon \int_0^T \int_\Omega \nabla_x \varphi \cdot \nabla_x \varphi \, dx \, dt + \int_0^T \int_\Omega [\varphi \otimes \varphi + p(\varphi) \nabla \varphi - \mathbb{S} : \nabla \varphi] \, dx \, dt,
\]
(3.12)
for any \( \varphi \in C^1([0, T]; X_n) \), with \( S \in L^\infty((0, T) \times \Omega; R_\text{sym}^{d \times d}) \), and with the energy inequality in the form
\[
\left[ \int_\Omega \frac{1}{2} |u - u_B|^2 + P(\varphi) \right]_{t=0}^{t=T} + \int_0^T \int_\Omega [F(\nabla u) + F^*(S)] \, dx \, dt
+ \int_0^T \int_{\Gamma_{\text{out}}} P(\varphi)u_B \cdot n \, dS_x \, dt + \varepsilon \int_0^T \int_\Omega |\nabla \varphi|^2 \, dx \, dt
- \int_0^T \int_{\Gamma_{\text{in}}} [P(\varphi_B) - P'(\varphi_B)(\varphi_B - \varphi) - P(\varphi)] u_B \cdot n \, dS_x \, dt
\leq - \int_0^T \int_\Omega [\varphi \otimes |u + p(\varphi)|] : \nabla_x u_B \, dx \, dt + \frac{1}{2} \int_0^T \int_\Omega \varphi \cdot \nabla_x |u_B|^2 \, dx \, dt
+ \int_0^T \int_\Omega \mathbb{S} : \nabla_x u_B \, dx \, dt - \int_0^T \int_{\Gamma_{\text{in}}} P(\varphi_B)u_B \cdot n \, dS_x \, dt
+ \varepsilon \int_0^T \int_\Omega \nabla_x \varphi \cdot \nabla_x (u - u_B) \cdot u_B \, dx \, dt.
\]
(3.13)

### 3.3. The third level approximation.

Our next goal is to send \( \varepsilon \to 0 \) in the viscous approximation (3.2). This is a bit more delicate than the preceding step as we are loosing compactness of the approximate density in the spatial variable. We start by collecting the necessary estimates independent of \( \varepsilon \).

Similarly to the preceding section, we have (3.11), which, as \( n \) is still fixed, gives rise to
\[
\|u\|_{L^q(0, T; W^{1, \infty}(\Omega))} \leq c, \quad q > 1,
\]
(3.14)
yielding, in view of Lemma 3.2 and Corollary 3.4, the uniform bounds on the density
\[
0 < \varphi \leq \overline{\varphi}(t, x) \leq \underline{\varphi} \text{ for all } (t, x) \in [0, T] \times \overline{\Omega}.
\]
(3.15)

Note that at this stage \( \varphi \) possess a well defined trace on \( \partial \Omega \). This finally implies, by virtue of the energy inequality (3.13),
\[
\varepsilon \int_0^T \int_\Omega |\nabla \varphi|^2 \, dx \, dt \leq c,
\]
(3.16)
and
\[
\sup_{\tau \in [0, T]} \|u(\tau, \cdot)\|_{W^{1, \infty}(\Omega; R^d)} \leq c
\]
(3.17)

Now fix \( n > 0 \) and denote \([\varphi_\varepsilon, u_\varepsilon]\) the approximate solutions constructed in the previous step for each \( \varepsilon > 0 \). In view of the uniform bounds established above, we may assume
\[
\varphi_\varepsilon \to \varphi \text{ weakly-(*) in } L^\infty((0, T) \times \Omega) \text{ and weakly in } C_{\text{weak}}([0, T]; L^r(\Omega)) \forall r \in (1, \infty)
\]
passing to a suitable subsequence as the case may be. Note that the second convergence follows from the weak bound on the time derivative \( \partial_t \varphi_\varepsilon \) obtained from
equation (3.5). We also have
\[ \varrho_{\varepsilon} \to \varrho \text{ weakly-*(*) in } L^\infty((0, T) \times \partial \Omega; |u_B \cdot n|dS_x). \]
In addition, the limit density admits the same upper and lower bounds as in (3.15).
Similarly,
\[ u_{\varepsilon} \to u \text{ weakly-*(*) in } L^\infty((0, T; W^{1, \infty}(\Omega; R^d)), \]
and
\[ \varrho_{\varepsilon} u_{\varepsilon} \to m \text{ weakly-*(*) in } L^\infty((0, T) \times \Omega). \]
Moreover, an abstract version of Arzela–Ascoli theorem yields
\[ m = \varrho u \text{ a.a. in } (0, T) \times \Omega. \]

3.3.1. The limit in the approximate equation of continuity. Keeping the gradient estimate (3.17) in mind, it is easy to pass to the limit in the regularized continuity equation (3.5):
\[
\left[ \int_\Omega \varrho \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \varrho \partial_t \varphi + \varrho u \cdot \nabla_x \varphi \right] \, dx \, dt - \int_0^\tau \int_{\Gamma_{\text{out}}} \varphi \varrho u_B \cdot n \, dS_x \, dt - \int_0^\tau \int_{\Gamma_{\text{in}}} \varphi \varrho_B u_B \cdot n \, dS_x \, dt, \quad \varrho(0, \cdot) = \varrho_0
\]
for any \( \varphi \in C^1([0, T] \times \Omega) \), which is a weak formulation of the equation of continuity (1.1), with the boundary conditions (1.5), and the initial condition (1.7). Note that the quantity
\[ m = \begin{cases} \varrho u_B \cdot n & \text{on } \Gamma_{\text{out}}, \\ \varrho_B u_B \cdot n & \text{on } \Gamma_{\text{in}} \end{cases} \]
is the normal trace of the divergenceless vector field \( [\varrho, \varrho u] \) on the lateral boundary \( (0, T) \times \partial \Omega \) in the sense of Chen, Torres, and Ziemer [5].

3.3.2. The limit in the approximate momentum equation. The limit passage in the momentum equation (3.12) is more delicate. First observe that \( F^* \) is a superlinear function since \( F \) is proper convex, \( \text{Dom}[F] = R^{d \times d} \). In particular, we may assume
\[ S_{\varepsilon} \to S \text{ weakly in } L^1((0, T) \times \Omega; R^{d \times d}). \]
Next, we deduce from (3.12) that
\[ \partial_t \Pi_n [\varrho_{\varepsilon} u_{\varepsilon}] \text{ bounded in } L^2((0, T; X_n)), \]
where \( \Pi_n : L^2 \to X_n \) is the associated orthogonal projection. In particular, in view of (3.18), (3.19), we may infer that
\[ \varrho_{\varepsilon} u_{\varepsilon} \otimes u_{\varepsilon} \to \varrho u \otimes u \text{ weakly-*(*) in } L^\infty((0, T) \times \Omega). \]
Consequently, we may let \( \varepsilon \to 0 \) in (3.12) obtaining
\[
\left[ \int_\Omega \varrho u \cdot \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \varrho u \cdot \partial_t \varphi + \varrho u \otimes u : \nabla_x \varphi \right] \, dx \, dt + \int_0^\tau \int_\Omega \left[ \overline{p(\varrho)} \text{div}_x \varphi - S : \nabla_x \varphi \right] \, dx \, dt
\]
for any \( \varphi \in C^1([0, T]; X_n) \). Here \( \overline{p(\varrho)} \in L^\infty((0, T) \times \Omega) \) stands for the weak limit of the sequence \( \{ p(\varrho_{\varepsilon}) \}_{\varepsilon>0} \). As \( p = p(\varrho) \) is non-linear (convex), removing the bar is equivalent to showing pointwise convergence of the approximate densities.
This might be possible by manipulating the renormalized equation in Lemma 3.3. However, this is quite technical and we content ourselves with (3.21).

3.3.3. Conclusion. Finally, employing the weak lower semi-continuity of the potentials \( F \) and \( F^* \), we may perform the limit in the energy balance (3.13) obtaining

\[
\left[ \int_\Omega \left[ \frac{1}{2} \varrho |u - u_B|^2 + F(\varrho) \right] \, dx \right]_{t=0}^{t=T} + \int_0^T \int_\Omega \left[ F(\mathbb{D}_x u) + F^*(S) \right] \, dx \, dt \\
+ \int_0^T \int_{\Gamma_{out}} \overline{P(\varrho)} u_B \cdot n \, dS_x \, dt
\]

\[
\leq - \int_0^T \int_\Omega \left[ \varrho u \otimes u + \varrho(\varrho) \right] \cdot \nabla_x u_B \, dx \, dt + \frac{1}{2} \int_0^T \int_\Omega \varrho u \cdot \nabla x |u_B|^2 \, dx \, dt \\
+ \int_0^T \int_\Omega S : \nabla x u_B \, dx \, dt - \int_0^T \int_{\Gamma_{in}} P(\varrho_B) u_B \cdot n \, dS_x \, dt
\]

(3.22)

To conclude, we summarize the result obtained in the part.

**Proposition 3.6 (Approximate solutions, level III).** Let \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), be a bounded Lipschitz domain. Suppose that \( p = p(\varrho) \) and \( F = F(\mathbb{D}) \) satisfy (2.2), (2.4)–(2.6). Let the data belong to the class

\[ u_B \in C^1_c(\mathbb{R}^d; \mathbb{R}^d), \quad \varrho_0 \in C^1(\mathbb{R}^d), \quad \varrho_B \in C^1(\partial \Omega), \quad \varrho_0, \varrho_B \geq \varrho > 0, \quad u_0 = v_0 + u_B, \quad v_0 \in X_n. \]

Then for each fixed \( n > 0 \), there exists a solution \( \varrho, u, v \) of the approximate problem:

- \[
\left[ \int_\Omega \varrho \varphi \, dx \right]_{t=0}^{t=T} = \int_0^T \int_\Omega \left[ \varrho \partial_t \varphi + \varrho u \cdot \nabla x \varphi \right] \, dx \, dt \]

(3.23)

\[- \int_0^T \int_{\Gamma_{out}} \varrho \varphi u_B \cdot n \, dS_x \, dt - \int_0^T \int_{\partial \Gamma_{in}} \varrho \varphi_B u_B \cdot n \, dS_x \, dt, \quad \varrho(0, \cdot) = \varrho_0 \]

for any \( \varphi \in C^1([0, T] \times \overline{\Omega}); \)

- \[
\left[ \int_\Omega \varrho u \cdot \varphi \, dx \right]_{t=0}^{t=T} = \int_0^T \int_\Omega \left[ \varrho u \cdot \partial_t \varphi + \varrho u \otimes u : \nabla x \varphi \right] \, dx \, dt \\
+ \int_0^T \int_\Omega \left[ \varrho(\varrho) \text{div} x \varphi - S : \nabla x \varphi \right] \, dx \, dt
\]

(3.24)

for any \( \varphi \in C^1([0, T]; X_n); \)

- \[
\left[ \int_\Omega \left[ \frac{1}{2} \varrho |u - u_B|^2 + \overline{F(\varrho)} \right] \, dx \right]_{t=0}^{t=T} + \int_0^T \int_\Omega \left[ F(\mathbb{D}_x u) + F^*(S) \right] \, dx \, dt \\
+ \int_0^T \int_{\Gamma_{out}} \overline{P(\varrho)} u_B \cdot n \, dS_x \, dt
\]

\[
\leq - \int_0^T \int_\Omega \left[ \varrho u \otimes u + \varrho(\varrho) \right] \cdot \nabla_x u_B \, dx \, dt + \frac{1}{2} \int_0^T \int_\Omega \varrho u \cdot \nabla x |u_B|^2 \, dx \, dt \\
+ \int_0^T \int_\Omega S : \nabla x u_B \, dx \, dt - \int_0^T \int_{\Gamma_{in}} P(\varrho_B) u_B \cdot n \, dS_x \, dt.
\]

(3.25)
The symbol \( \bar{p}(\varrho) \) and \( \bar{P}(\varrho) \) stand for the weak limits of a bounded sequence \( \{ p(\varrho_n) \}_{\varepsilon > 0} \) and \( \{ P(\varrho_n) \}_{\varepsilon > 0} \), respectively.

3.4. **Final limit.** Our ultimate goal is to perform the limit \( n \to \infty \) in the family of approximate solutions obtained in Proposition 3.6. Our first observation uses the hypotheses (2.2), namely

\[
P''(\varrho) = \frac{P'(\varrho)}{\varrho} > 0 \quad \varrho > 0.
\]

Consequently, \( P \) is a strictly convex function and we have

\[
\bar{P}(\varrho) - P(\varrho) \geq 0.
\]

Next, again by virtue of hypothesis (2.2),

\[
\bar{P}(\varrho) - P(\varrho) \geq \overline{a}(\bar{p}(\varrho) - p(\varrho)), \quad \Pi(\bar{p}(\varrho) - p(\varrho)) \geq \bar{P}(\varrho) - P(\varrho).
\]

(3.26)

Thus we may rewrite (3.24) as

\[
\left[ \int_{\Omega} \varrho u \cdot \varphi \, dx \right]_{t=0}^{t=T} = \int_0^T \int_{\Omega} \left[ \varrho u \cdot \partial_t \varphi + \varrho u \otimes u : \nabla x \varphi + p(\varrho) \text{div}_x \varphi \right] \, dx \, dt
\]

\[
- \int_0^T \int_{\Omega} : \nabla x \varphi \, dx \, dt - \int_0^T \int_{\Omega} \left[ \bar{p}(\varrho) - p(\varrho) \right] \text{div}_x \varphi \, dx \, dt,
\]

while the energy inequality (3.25) reduces to

\[
\left[ \int_{\Omega} \left( \frac{1}{2} \frac{|m|^2}{\varrho} - m \cdot u_B + \frac{1}{2} \varrho |u_B|^2 + P(\varrho) \right) \, dx \right]_{t=0}^{t=T}
\]

\[
+ \int_0^T \int_{\Omega} [F(\varepsilon u) + F^*(\varepsilon)] \, dx \, dt + \int_0^T \int_{\Gamma_{in}} P(\varrho) u_B \cdot n \, dS_x \, dt
\]

\[
+ \int_{\Omega} \left[ \bar{P}(\varrho) - P(\varrho) \right] (\tau, \cdot) \, dx \leq - \int_0^T \int_{\Omega} \left[ \frac{m \otimes m}{\varrho} + p(\varrho) \Pi \right] : \nabla x u_B \, dx \, dt
\]

\[
+ \frac{1}{2} \int_0^T \int_{\Omega} \varrho u \cdot \nabla x |u_B|^2 \, dx \, dt + \int_0^T \int_{\Omega} : \nabla x u_B \, dx \, dt
\]

\[
- \int_0^T \int_{\Gamma_{in}} P(\varrho_B) u_B \cdot n \, dS_x \, dt - \int_0^T \int_{\Omega} \left( p(\varrho) - \bar{p}(\varrho) \right) \text{div}_x u_B \, dx \, dt,
\]

where the kinetic energy \( \frac{1}{2} \varrho |u - u_B|^2 \) in (3.25) is now more conveniently written as

\[
\frac{1}{2} \frac{|m|^2}{\varrho} - m \cdot u_B + \frac{1}{2} \varrho |u_B|^2
\]

and similarly \( \varrho u \otimes u \) is written as \( \frac{m \otimes m}{\varrho} \), where \( m = \varrho u \).

We set

\[
\mathcal{R} := \| \bar{p}(\varrho) - p(\varrho) \|, \quad \mathcal{E} := \bar{P}(\varrho) - P(\varrho).
\]

These terms may be interpreted as the Reynolds and energy defect measure, respectively. In accordance with (3.26), we have the defect compatibility condition

\[
\mathcal{E} = \bar{P}(\varrho) - P(\varrho) \leq \bar{a} \left( \bar{p}(\varrho) - p(\varrho) \right) = \frac{\pi}{d} \text{tr} \mathcal{R} \leq \frac{\pi}{a} \left( \bar{P}(\varrho) - P(\varrho) \right) = \frac{\pi}{a} \mathcal{E}.
\]

(3.29)

We are ready to perform the limit \( n \to \infty \). Let \( \{ g_n, m_n \} \) be a sequence of solutions obtained in Proposition 3.6, with the associated viscous stress tensors \( S_n \), the energy defects \( \mathcal{E}_n \), and the Reynolds tensors \( \mathcal{R}_n \). An easy application of Gronwall’s lemma shows that the total energy represented by the expression on the left–hand side of
the energy inequality (3.28) remains bounded uniformly for \( n \to \infty \). Consequently, extracting suitable subsequences if necessary, we may suppose
\[
\varrho_n \to \varrho \text{ in } C_{\text{weak}}([0, T]; L^1(\Omega)),
\]
\[
\varrho_n|_{T_{\text{out}}} \to \varrho \text{ weakly-(*) in } L^\infty(0, T; L^\gamma(\Gamma_{\text{out}}; |u_B \cdot n|dS_x)),
\]
\[
\mathbf{m}_n = \varrho_n \mathbf{u}_n \to \mathbf{m} \text{ weakly-(*) in } L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d)),
\]
where \( \gamma \) is given in (2.3). Next, repeating the arguments leading to (3.11), we get
\[
\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^q(0, T; W^{1,q}(\Omega; \mathbb{R}^d)),
\]
and also
\[
\mathbf{S}_n \to \mathbf{S} \text{ weakly in } L^1((0, T) \times \Omega; \mathbb{R}^d \times d\text{sym}).
\]
Our goal is to show that
\[
\mathbf{m} = \varrho \mathbf{u} \text{ a.a. in } (0, T) \times \Omega. \quad (3.30)
\]
To this end, we report the following result proved in Appendix.

**Lemma 3.7.** Let \( Q = (0, T) \times \Omega \), where \( \Omega \subset \mathbb{R}^d \) is a bounded domain. Suppose that
\[
r_n \to r \text{ weakly in } L^p(Q), \quad v_n \to v \text{ weakly in } L^q(Q), \quad p > 1, q > 1,
\]
and
\[
r_n v_n \to w \text{ weakly in } L^r(Q), \quad r > 1.
\]
In addition, let
\[
\partial_t r_n = \text{div}_x \mathbf{g}_n + h_n \text{ in } \mathcal{D}'(Q), \quad \|\mathbf{g}_n\|_{L^s(Q; \mathbb{R}^d)} \lesssim 1, \quad s > 1,
\]
\[
h_n \text{ precompact in } W^{-1, z}, \quad z > 1,
\]
and
\[
\|\nabla_x v_n\|_{M(Q; \mathbb{R}^d)} \lesssim 1 \text{ uniformly for } n \to \infty.
\]
Then
\[
w = rv \text{ a.a. in } Q.
\]
A direct application of Lemma 3.7 to
\[
r_n = \varrho_n, \quad v_n = u^i_n, \quad i = 1, \ldots, d, \quad \mathbf{g}_n = -\varrho_n \mathbf{u}_n, \quad h_n = 0, \quad p = \gamma, \quad r = s = \frac{2\gamma}{\gamma + 1}
\]
yields (3.30).

At this stage, we are able to perform the limit in the equation of continuity (3.23) to obtain (2.8). We can also approximate the initial data
\[
\varrho_{0,n} \to \varrho_0 \text{ in } L^\gamma(\Omega)
\]
to obtain (2.9) with the desired finite energy initial data.

The next step is to perform the same limit in the momentum equation (3.27). To this end, we first observe that
\[
\mathbf{R}_n = [p(\varrho) - p(\varrho_n)]n \to \mathbf{R}^1 \text{ weakly-(*) in } L^\infty(0, T; \mathcal{M}(\overline{\Omega})),
\]
\[
\mathbf{C}_n = [P(\varrho) - P(\varrho_n)]n \to \mathbf{C}^1 \text{ weakly-(*) in } L^\infty(0, T; \mathcal{M}(\overline{\Omega})),
\]
where the limit measures retain the compatibility condition (3.29)
\[
0 \leq \mathbf{C}^1 \leq \pi \mathbf{R}^1 \leq \frac{\pi}{2} \mathbf{C}^1.
\]
Similarly, we have
\[
\varrho\, u_n \otimes u_n + p(\varrho_n)I = 1_{\varrho_n > 0} \frac{m_n \otimes m_n}{\varrho_n} + p(\varrho_n)I \rightarrow \left[ 1_{\varrho > 0} \frac{m \otimes m}{\varrho} + p(\varrho)I \right]
\]
weakly- (*) in \( L^\infty(0, T; \mathcal{M}(\Omega; R^{d \times d}_{\text{sym}})) \),
and
\[
\frac{1}{2} \varrho_n |u_n|^2 + P(\varrho_n) = \frac{1}{2} \frac{|m_n|^2}{\varrho_n} + P(\varrho_n) \rightarrow \frac{1}{2} \frac{|m|^2}{\varrho} + P(\varrho)
\]
weakly- (*) in \( L^\infty(0, T; \mathcal{M}(\Omega)) \).

We set
\[
\mathcal{R}^2 = \left[ 1_{\varrho > 0} \frac{m \otimes m}{\varrho} + p(\varrho)I \right] - \left( 1_{\varrho > 0} \frac{m \otimes m}{\varrho} + p(\varrho)I \right)
\]
and
\[
\mathcal{E}^2 = \left[ \frac{1}{2} \frac{|m|^2}{\varrho} + P(\varrho) \right] - \left[ \frac{1}{2} \frac{|m|^2}{\varrho} + P(\varrho) \right] = \left[ \frac{1}{2} \frac{|m|^2}{\varrho} + P(\varrho) \right] - \left[ \frac{1}{2} \varrho |u|^2 + P(\varrho) \right]
\]
noting the relation
\[
d\mathcal{E}^2 \leq \text{tr}[\mathcal{R}^2] \leq \overline{d}\mathcal{E}^2, \quad \text{where } 0 < d \leq \overline{d}, \quad d = d(\varrho, \overline{\varrho}, d).
\]
Finally, we claim that
\[
\mathcal{R}^2 \in L^\infty(0, T; \mathcal{M}^+(\Omega; R^{d \times d}_{\text{sym}})).
\]
To see this, it is enough to observe that
\[
1_{\varrho > 0} \frac{m \otimes m}{\varrho} - 1_{\varrho > 0} \frac{m \otimes m}{\varrho} \geq 0.
\]
Indeed we compute
\[
\left[ 1_{\varrho > 0} \frac{m \otimes m}{\varrho} - 1_{\varrho > 0} \frac{m \otimes m}{\varrho} \right] : (\xi \otimes \xi) = \left[ \frac{|m \cdot \xi|^2}{\varrho} - \frac{|m \cdot \xi|^2}{\varrho} \right] \geq 0 \text{ for any } \xi \in \mathbb{R}^d,
\]
where the most right inequality follows from convexity of the l.s.c. function
\[
[\varrho, m] \mapsto \begin{cases} 
\frac{|m \cdot \xi|^2}{\varrho} & \text{if } \varrho > 0, \\
0 & \text{if } \varrho = 0, \ m = 0, \\
\infty & \text{otherwise}.
\end{cases}
\]

Having collected all necessary material, we are now ready to send \( n \rightarrow \infty \) in both the momentum balance (3.27) and the energy inequality (3.28). In particular, we obtain
\[
\left[ \int_\Omega \varrho u \cdot \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[ \varrho u \cdot \partial_t \varphi + \varrho u \otimes u : \nabla_x \varphi + p(\varrho) \text{div}_x \varphi \right] \, dx \, dt \\
- \int_0^\tau \int_\Omega S : \nabla_x \varphi \, dx \, dt + \int_0^\tau \left( \int_\Omega \nabla_x \varphi : d\mathcal{R}(t) \right) \, dt, \quad \mathcal{R} = \mathcal{R}^1 + \mathcal{R}^2,
\]
for any test function \( \varphi \in C^1([0,T];X_n) \), \( n \) arbitrary. It is a routine matter to
choose the spaces \( X_n \) in such a way that validity of (3.31) can be extended to
\( \varphi \in C^1([0,T];C^1_c(\Omega)) \) by density argument. Finally, for a function
\[
\varphi \in C^1([0,T] \times \overline{\Omega}; R^d), \quad \varphi|_{\partial \Omega} = 0,
\]
we construct a sequence \( \varphi_n \in C^1([0,T];C^1_c(\Omega)) \) such that
\[
\|\varphi_n\|_{W^{1,\infty}(0,T) \times \Omega; R^d} \leq c \quad \text{uniformly for } n \to \infty,
\]
\( \varphi_n(t,x) \to \varphi(t,x) \),
\( \partial_t \varphi_n(t,x) \to \partial_t \varphi(t,x) \), \( \nabla_x \varphi_n(t,x) \to \nabla_x \varphi(t,x) \) for any \( (t,x) \in (0,T) \times \Omega \).
In such a way we can extend validity of (3.31) to the class of test function (3.32).
We have shown the following existence result.

**Theorem 3.8 (Global existence of dissipative solutions).** Let \( \Omega \subset R^d \), \( d = 2,3 \) be a bounded Lipschitz domain. Suppose that \( p = p(\varrho) \) and \( F = F(D) \) satisfy
(2.2), (2.4)–(2.6). Let the data belong to the class
\[
\begin{align*}
\varrho_B & \in C^1(R^d; R^d), \quad \varrho_B \geq \varrho > 0, \\
\varrho_0 & \in L^\gamma(\Omega), \quad \varrho_0 \geq 0, \quad m_0 \in L^\frac{2}{n+2}(\Omega; R^d), \quad \int_\Omega \left[ \frac{1}{2} \frac{|m_0|^2}{\varrho_0} + P(\varrho_0) \right] \, dx < \infty,
\end{align*}
\]
where \( \gamma > 1 \) is given in (2.3).
Then the problem (1.1)–(1.7) admits at least one dissipative solution \( [\varrho, u] \) in
(0,T) \times \Omega in the sense specified in Definition 2.1.

4. **Compatibility.** We show that if a dissipative solution enjoys certain regularity,
specifically if
\[
\begin{align*}
\varrho & \in C^1([0,T] \times \overline{\Omega}; R^d), \quad \varrho \in C^1([0,T] \times \overline{\Omega}), \quad \inf_{(0,T) \times \Omega} \varrho > 0,
\end{align*}
\]
then \( [\varrho, u] \) is a classical solution, meaning \( \mathcal{E} = \mathcal{R} = 0 \).
To see this, we realize that \( (u - u_B) \) can be used as a test function in the
momentum equation (2.10), which, together with the equation of continuity (2.8),
yield the total energy equality:
\[
\begin{align*}
\left[ \int_\Omega \frac{1}{2} \varrho |u - u_B|^2 + P(\varrho) \, dx \right]_{t=0}^{t=T} & + \int_0^T \int_\Omega \mathbb{S} : \var{uxu} \, dt + \int_\Omega \mathbb{D} \mathbb{u} \, dt + \int_\Omega \mathbb{E} (\tau) \\
& + \int_0^T \int_{\partial \Omega} P(\varrho) u_B \cdot \var{n} S \, dt = -\int_0^T \int_\Omega [\var{uxu} + p(\varrho) \mathbb{I}] : \nabla_x u_B \, dx \, dt \\
& + \frac{1}{2} \int_0^T \int_\Omega \var{uxu}^2 \, dx \, dt + \int_0^T \int_\Omega \var{uxu} \, dx \, dt \\
& - \int_0^T \int_\Omega \nabla_x (u_B - u) : \var{n} \mathcal{R}(t) \, dt.
\end{align*}
\]
Relation (4.1) subtracted from the energy inequality (2.12) give rise to
\[
\int_\Omega \mathbb{D} \mathbb{u} \, dt + \int_0^T \left( F(\mathbb{D} u) + F^*(\mathbb{S}) - \mathbb{S} : \mathbb{D} u \right) \, dx \, dt \leq \int_0^T \int_\Omega \var{uxu} : \var{n} \mathcal{R}(t) \, dt,
\]
which, together with the compatibility hypothesis (2.13) and Gronwall lemma,
yields the desired conclusion \( \mathcal{E} = \mathcal{R} = 0 \) and \( \mathbb{S} \in \partial F(\mathbb{D} u) \).
Theorem 4.1 (Compatibility weak–strong). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Suppose that $[\varrho, \mathbf{u}]$ is a dissipative solution in the sense of Definition 2.1 belonging to the class

$$ u \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^d), \quad \varrho \in C^1([0, T] \times \overline{\Omega}), \quad \inf_{(0, T) \times \Omega} \varrho > 0. $$

Then $[\varrho, \mathbf{u}]$ is a classical solution, meaning $\mathcal{E} = \mathcal{R} = 0$ and the equations are satisfied in the classical sense.

5. Relative energy. The relative energy is a basic tool for showing the weak–strong uniqueness property. Let us introduce

$$ \mathcal{E}(\varrho, \mathbf{u} \mid \tilde{\varrho}, \tilde{\mathbf{u}}) = \frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho}) $$

that can be rewritten as

$$ \frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 + P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho}) $$

$$ = \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho}) $$

$$ = \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + P(\varrho) - \varrho \mathbf{u} \cdot (\tilde{\mathbf{u}} - \mathbf{u}_B) + \left[ \frac{1}{2} |\tilde{\mathbf{u}}|^2 - |\mathbf{u}_B|^2 \right] - P'(\tilde{\varrho}) \varrho + P(\tilde{\varrho}) $$

Our goal is to evaluate the time evolution of

$$ \int_{\Omega} \mathcal{E}(\varrho, \mathbf{u} \mid \tilde{\varrho}, \tilde{\mathbf{u}}) \, dx $$

where $[\varrho, \mathbf{u}]$ is a dissipative solutions and $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ are test functions in the class

$$ \tilde{\mathbf{u}} \in C^1([0, T] \times \overline{\Omega}; \mathbb{R}^d), \quad \tilde{\mathbf{u}}|_{\partial \Omega} = \mathbf{u}_B|_{\partial \Omega}, \quad \tilde{\varrho} \in C^1([0, T] \times \overline{\Omega}), \quad \inf_{(0, T) \times \Omega} \tilde{\varrho} > 0. $$

Step 1:

In accordance with the energy inequality (2.12), we get

$$ \left[ \int_{\Omega} \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + P(\varrho) \, dx \right]^{t=\tau}_{t=0} + \int_0^\tau \int_{\Omega} \left[ F(\nabla_x \mathbf{u}) + F^*(\mathbf{s}) \right] \, dx \, dt $$

$$ + \int_0^\tau \int_{\Gamma_{\text{out}}} P(\varrho) \mathbf{u}_B \cdot \mathbf{n} \, dS_x \, dt + \int_0^\tau \int_{\Gamma_{\text{in}}} P(\varrho_B) \mathbf{u}_B \cdot \mathbf{n} \, dS_x \, dt + \int_{\Pi} 1 d \mathcal{E}(\tau) $$

$$ \leq - \int_0^\tau \int_{\Omega} p(\varrho) \text{div}_x \mathbf{u}_B \, dx \, dt + \int_0^\tau \int_{\Omega} \varrho \mathbf{u} \cdot \nabla_x \mathbf{u}_B \cdot (\mathbf{u}_B - \mathbf{u}) \, dx \, dt $$

$$ + \int_0^\tau \int_{\Omega} \mathbf{S} : \nabla_x \mathbf{u}_B \, dx \, dt - \int_0^\tau \int_{\Pi} \nabla_x \mathbf{u}_B : d \mathcal{R}(t) \, dt $$

(5.1)

Step 2:

Plugging $\varphi = \tilde{\mathbf{u}} - \mathbf{u}_B$ in the momentum equation (2.10), we get

$$ \left[ \int_{\Omega} \varrho \mathbf{u} \cdot (\tilde{\mathbf{u}} - \mathbf{u}_B) \, dx \right]^{t=\tau}_{t=0} = \int_0^\tau \int_{\Omega} \left[ \varrho \mathbf{u} \cdot \partial_t \tilde{\mathbf{u}} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x (\tilde{\mathbf{u}} - \mathbf{u}_B) \right] \, dx \, dt $$

$$ + \int_0^\tau \int_{\Omega} \left[ p(\varrho) \text{div}_x (\tilde{\mathbf{u}} - \mathbf{u}_B) - \mathbf{S} : \nabla_x (\tilde{\mathbf{u}} - \mathbf{u}_B) \right] \, dx \, dt $$

$$ + \int_0^\tau \int_{\Pi} \nabla_x (\tilde{\mathbf{u}} - \mathbf{u}_B) : d \mathcal{R}(t) \, dt $$

(5.2)
Step 3:
Finally, we consider
\[\phi = \left[ \frac{1}{2} \left( |\tilde{u}|^2 - |u_B|^2 \right) - P'(\tilde{\theta}) \right]\]
in the equation of continuity (2.8) obtaining:
\[
\begin{align*}
\left[ \int_{\Omega} \phi \left[ \frac{1}{2} \left( |\tilde{u}|^2 - |u_B|^2 \right) - P'(\tilde{\theta}) \right] \, dx \right]_{t=0}^{t=T} \\
- \int_0^T \int_{\Gamma_{out}} P'(\tilde{\theta}) \varrho u_B \cdot n \, ds_x \, dt - \int_0^T \int_{\Gamma_{in}} P'(\tilde{\theta}) \varrho B \cdot n \, ds_x \, dt \\
= \int_0^T \int_{\Omega} \varrho \partial_t \left( \frac{1}{2} |\tilde{u}|^2 - P'(\tilde{\theta}) \right) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \varrho u \cdot \nabla_x \left( \frac{1}{2} |\tilde{u}|^2 - |u_B|^2 \right) - P'(\tilde{\theta}) \right) \, dx \, dt
\end{align*}
\]
Summing up (5.1) and (5.3) and subtracting (5.2) we get
\[
\begin{align*}
\left[ \int_{\Omega} \mathcal{E} \left( \varrho, u \mid \tilde{\varrho}, \tilde{u} \right) \, dx \right]_{t=0}^{t=T} + \int_0^T \int_{\Omega} \left[ F(D_x u) + F^*(S) \right] \, dx \, dt \\
- \int_0^T \int_{\Omega} S : \nabla_x \tilde{u} \, dx \, dt + \int_0^T \int_{\Gamma_{out}} \left[ P(\varrho) - P'(\tilde{\theta}) \varrho \right] u_B \cdot n \, ds_x \, dt \\
+ \int_0^T \int_{\Gamma_{in}} \left[ P(\varrho_B) - P'(\tilde{\theta}) \varrho_B \right] u_B \cdot n \, ds_x \, dt + \int_{\Gamma} \varrho(t) \, ds \, dt \\
\leq \int_0^T \int_{\Omega} \left[ \varrho u \cdot \partial_t \tilde{u} + \varrho \tilde{u} \cdot \nabla_x \tilde{u} + p(\varrho) \text{div}_x \tilde{u} \right] \, dx \, dt \\
- \int_0^T \int_{\Omega} \nabla_x \tilde{u} : \text{div}(t) \, dt + \int_0^T \int_{\Omega} \partial_t p(\tilde{\theta}) \, dx \, dt \\
+ \int_0^T \int_{\Omega} \left[ \varrho \partial_t \left( \frac{1}{2} |\tilde{u}|^2 - P'(\tilde{\theta}) \right) + \varrho u \cdot \nabla_x \left( \frac{1}{2} |\tilde{u}|^2 - P'(\tilde{\theta}) \right) \right] \, dx \, dt
\end{align*}
\]
Finally, regrouping several terms we conclude
\[
\begin{align*}
\left[ \int_{\Omega} \mathcal{E} \left( \varrho, u \mid \tilde{\varrho}, \tilde{u} \right) \, dx \right]_{t=0}^{t=T} + \int_0^T \int_{\Omega} \left[ F(D_x u) + F^*(S) \right] \, dx \, dt \\
- \int_0^T \int_{\Omega} S : \nabla_x \tilde{u} \, dx \, dt + \int_{\Gamma} \varrho(t) \, ds \, dt \\
+ \int_0^T \int_{\Gamma_{out}} \left[ P(\varrho) - P'(\tilde{\theta}) \varrho \right] u_B \cdot n \, ds_x \, dt \\
+ \int_0^T \int_{\Gamma_{in}} \left[ P(\varrho_B) - P'(\tilde{\theta}) \varrho_B \right] u_B \cdot n \, ds_x \, dt \\
\leq \int_0^T \int_{\Omega} \varrho(\tilde{u} - u) \cdot \nabla_x \tilde{u} \cdot (\tilde{u} - u) \, dx \, dt - \int_0^T \int_{\Omega} \nabla_x \tilde{u} : \text{div}(t) \, dt \\
- \int_0^T \int_{\Omega} \left[ p(\varrho) - P'(\tilde{\theta}) \varrho \right] \text{div}_x \tilde{u} \, dx \, dt \\
+ \int_0^T \int_{\Omega} \left[ \varrho \tilde{u} - u \right] \cdot \left[ \partial_t (\tilde{\theta} \tilde{u} ) + \text{div}_x (\tilde{\theta} \tilde{u} \otimes \tilde{u} ) + \nabla_x p(\tilde{\theta}) \right] \, dx \, dt \\
+ \int_0^T \int_{\Omega} \left( \frac{\varrho}{\tilde{\theta}} (u - \tilde{u}) \cdot \tilde{u} + p'(\tilde{\theta}) \left( 1 - \frac{\varrho}{\tilde{\theta}} \right) \right) \left[ \partial_t \tilde{\theta} + \text{div}_x (\tilde{\theta} \tilde{u} ) \right] \, dx \, dt.
\end{align*}
\]
We have shown the following result.

**Proposition 5.1 (Relative energy inequality).** Let \([\varrho, \mathbf{u}]\) be a dissipative solution in the sense of Definition 2.1. Suppose that
\[
\hat{\mathbf{u}} \in C^1([0, T] \times \Omega; R^d), \quad \hat{\mathbf{u}}_{|\partial \Omega} = \mathbf{u}_{|\partial \Omega}, \quad \hat{\varrho} \in C^1([0, T] \times \Omega), \quad \inf_{(0,T)\times\Omega} \hat{\varrho} > 0. \tag{5.6}
\]

Then the relative energy inequality \((5.5)\) holds for a.a. \(0 \leq \tau \leq T\).

6. **Weak–strong uniqueness.** Our goal is to show that a dissipative solutions coincide with the strong solution emanating from the same initial data and boundary conditions. Assuming the strong solution \([\check{\varrho}, \check{\mathbf{u}}]\) belongs to the class \((5.6)\), the obvious idea is to use the relative energy inequality \((5.5)\). Assuming regularity of the viscous stress \(\tilde{\mathbf{S}}\) related to the strong solution, specifically
\[
\tilde{\mathbf{S}} \in C([0, T] \times \Omega; P_{\text{sym}}^d), \quad \text{div}_x \tilde{\mathbf{S}} \in C([0, T] \times \Omega; R^d),
\]
we may rewrite \((5.5)\) as
\[
\begin{align*}
&\left[ \int_{\Omega} \mathcal{E}(\varrho, \mathbf{u} | \check{\varrho}, \check{\mathbf{u}}) \, dx \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{\Omega} \left[ F(\nabla_x \mathbf{u}) + F^*(\tilde{\mathbf{S}}) \right] \, dx \, dt \\
&\quad + \int_0^\tau \int_{\Omega} (\nabla \check{\varrho} \cdot \check{\mathbf{u}}) \, dx \, dt - \int_0^\tau \int_{\Omega} \nabla \check{\varrho} \cdot \mathbf{u} \, dx \, dt \\
&\quad + \int_0^\tau \int_{\Gamma_{\text{out}}} \left[ \mathbf{u}_B \cdot \mathbf{n} \right] \, dS_x \, dt + \int_{\Omega} 1 \, d \mathcal{E}(\tau) \tag{6.1}
\end{align*}
\]

where we have used the identity
\[
\int_0^\tau \int_{\Omega} (\check{\varrho} - \varrho) \, dx \, dt = - \int_0^\tau \int_{\Omega} \nabla_x \tilde{\mathbf{S}} \, dx \, dt.
\]

As \([\check{\varrho}, \check{\mathbf{u}}]\) is a strong solution of the problem with the same initial–boundary data, relation \((6.1)\) reduces to
\[
\begin{align*}
&\left[ \int_{\Omega} \mathcal{E}(\varrho, \mathbf{u} | \check{\varrho}, \check{\mathbf{u}}) \, dx \right]_{t=0}^{t=\tau} - \int_0^\tau \int_{\Omega} \nabla_x \tilde{\mathbf{S}} \, dx \, dt \\
&\quad + \int_0^\tau \int_{\Omega} \left[ F(\nabla_x \mathbf{u}) + F^*(\tilde{\mathbf{S}}) \right] \, dx \, dt + \int_0^\tau \int_{\Omega} \nabla \check{\varrho} \cdot \mathbf{u} \, dx \, dt \\
&\quad + \int_0^\tau \int_{\Gamma_{\text{out}}} \left[ \mathbf{u}_B \cdot \mathbf{n} \right] \, dS_x \, dt + \int_{\Omega} 1 \, d \mathcal{E}(\tau) \tag{6.2}
\end{align*}
\]

\[
\leq - \int_0^\tau \int_{\Omega} \varrho (\check{\varrho} - \varrho) \, dx \, dt
\]
Moreover, it follows from the structural hypothesis (2.2) and the compatibility condition (2.13) that

\[- \int_0^T \int_\Omega \left[ p(u) - p'(q)(q - \tilde{q}) - p(\tilde{q}) \right] \text{div}_x \tilde{u} \, dx \, dt \]

\[+ \int_0^T \int_\Omega \left( \frac{\alpha}{\beta} - 1 \right) (\tilde{u} - u) \cdot \text{div}_x \tilde{S} \, dx \, dt - \int_0^T \int_{\Gamma} \nabla_x \tilde{u} : d \mathcal{R}(t) \, dt. \]

Consequently, (6.2) gives rise to

\[\int_\Omega \mathcal{E} \left( \varrho, \mathbf{u} \mid \tilde{\varrho}, \tilde{\mathbf{u}} \right) (\tau, \cdot) \, dx - \int_0^T \int_\Omega \Sigma : \mathbb{D}_x \tilde{u} \, dx \, dt \]

\[+ \int_0^T \int_\Omega \left[ F(\mathbb{D}_x \mathbf{u}) + F^*(\Sigma) \right] \, dx \, dt + \int_0^T \int_\Omega \tilde{S} : (\mathbb{D}_x \tilde{u} - \mathbb{D}_x \mathbf{u}) \, dx \, dt \]

\[+ \int_0^T \int_{\Gamma_{\text{out}}} \left[ P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho}) \right] \mathbf{u}_B \cdot \mathbf{n} \, dS_x \, dt + \int_\Omega \int_\mathcal{C} \mathbf{u}(\tau) \, dx \, dt \]

\[\leq c(\|\nabla_x \tilde{u}\|_{L^\infty}) \left[ \int_0^T \int_\Omega \mathcal{E} \left( \varrho, \mathbf{u} \mid \tilde{\varrho}, \tilde{\mathbf{u}} \right) \, dx \, dt + \int_0^T \left( \int_\mathcal{C} \mathbf{u}(\tau) \, dx \, dt \right) \right] \]

\[+ \int_0^T \int_\Omega \left( \frac{\alpha}{\beta} - 1 \right) (\tilde{u} - u) \cdot \text{div}_x \tilde{S} \, dx \, dt. \]

To conclude, we regroup the dissipative integrals on the left hand side as

\[\int_0^T \int_\Omega \left[ F(\mathbb{D}_x \mathbf{u}) + F^*(\Sigma) \right] \, dx \, dt + \int_0^T \int_\Omega \tilde{S} : (\mathbb{D}_x \tilde{u} - \mathbb{D}_x \mathbf{u}) \, dx \, dt \]

\[\int_0^T \int_\Omega \Sigma : \mathbb{D}_x \tilde{u} \, dx \, dt = \int_0^T \int_\Omega \left[ F(\mathbb{D}_x \mathbf{u}) - \tilde{S} : (\mathbb{D}_x \mathbf{u} - \mathbb{D}_x \tilde{u}) - F(\mathbb{D}_x \tilde{u}) \right] \, dx \, dt \]

\[+ \int_0^T \int_\Omega \left[ F(\mathbb{D}_x \tilde{u}) + F^*(\Sigma) - \mathbb{D}_x \tilde{u} : \Sigma \right] \, dx \, dt \]

\[\geq \int_0^T \int_\Omega \left[ F(\mathbb{D}_x \mathbf{u}) - \tilde{S} : (\mathbb{D}_x \mathbf{u} - \mathbb{D}_x \tilde{u}) - F(\mathbb{D}_x \tilde{u}) \right] \, dx \, dt, \]

where we have used Fenchel–Young inequality. By virtue of the coercivity hypothesis (2.6), we have

\[\int_0^T \int_\Omega \left[ F(\mathbb{D}_x \mathbf{u}) - \tilde{S} : (\mathbb{D}_x \mathbf{u} - \mathbb{D}_x \tilde{u}) - F(\mathbb{D}_x \tilde{u}) \right] \, dx \, dt \]

\[\geq \int_0^T \int_\Omega A_R \left( \left| \mathbb{D}_x (\mathbf{u} - \tilde{u}) \right| - \frac{1}{d} \text{div}_x (\mathbf{u} - \tilde{u}) \right) \, dx \, dt, \]

where $R = R \left( \|\tilde{u}\|_{L^\infty}, \|\tilde{S}\|_{L^\infty} \right)$. Finally, we use the following two results concerning Korn and Poincaré inequalities:
Lemma 6.1 (Talenti [19, Lemma 3]). Let $A$ be a Young function, and let $\Omega \subset R^d$ be a bounded domain. Then
\[
\int_{\Omega} A \left( \frac{X_d}{d|\Omega|^\frac{1}{d}} |w| \right) \, dx \leq \frac{1}{d} \int_{\Omega} A (|\nabla_x w|) \, dx
\]
for any $w \in W^{1,1}_0(\Omega)$, where $X_d$ is a positive constant.

Lemma 6.2 (Breit, Cianchi, and Diening [2, Theorem 3.1]). Let $\Omega \subset R^d$, $d \geq 2$ be a bounded domain. Let $A$ be a Young function satisfying the $\Delta_2^p$–condition (2.5). Then there exists a constant $c > 0$ such that
\[
\int_{\Omega} A (|\nabla_x w|) \, dx \leq \int_{\Omega} A \left( c \left| \nabla_x w - \frac{1}{d} \text{div}_x w \right| \right) \, dx
\]
for any $w \in W^{1,1}_0(\Omega; R^d)$.

Combining Lemmas 6.1, 6.2 with (6.4) we may infer that
\[
\int_{\Omega} \mathcal{E} \left( \varrho, u \vline \begin{array}{c} \tilde{\varrho}, \tilde{u} \\ \tau, \cdot \end{array} \right) \, dx + \int_0^T \int_{\Omega} \tilde{A}_R (|\tilde{u} - u|) \, dx \, dt
\]
\[
+ \int_0^T \int_{\Gamma_{\text{out}}} \left[ P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho}) \right] u_B \cdot n \, ds + \int_\Gamma 1d \mathcal{E}(\tau) \leq c (\|
abla_x \tilde{u}\|_{L^\infty}) \left[ \int_0^T \int_{\Omega} \mathcal{E} \left( \varrho, u \vline \begin{array}{c} \varrho, \tilde{u} \\ \tau, \cdot \end{array} \right) \, dx \, dt + \int_0^T \left( \int_{\Gamma} 1d \mathcal{E}(t) \right) \, dt \right]
\]
\[
+ \int_0^T \int_{\Omega} \left( \frac{\varrho}{\tilde{\varrho}} - 1 \right) (\tilde{u} - u) \cdot \text{div}_x \tilde{S} \, dx \, dt,
\]
where $\tilde{A}_R$ is a Young function obtain by a simple rescaling of $A_R$ in (6.4).

Finally, we observe that
\[
\int_{\Omega} \left| \frac{\varrho}{\tilde{\varrho}} - 1 \right| |\tilde{u} - u| \, dx = \int_{\{\varrho < \delta\}} \left| \frac{\varrho}{\tilde{\varrho}} - 1 \right| |\tilde{u} - u| \, dx + \int_{\{\varrho \geq \delta\}} \left| \frac{\varrho}{\tilde{\varrho}} - 1 \right| |\tilde{u} - u| \, dx
\]
for any $\delta > 0$, where, on one hand,
\[
\int_{\{\varrho \geq \delta\}} \left| \frac{\varrho}{\tilde{\varrho}} - 1 \right| |\tilde{u} - u| \, dx \leq c(\delta) \int_{\Omega} \mathcal{E} \left( \varrho, m \vline \begin{array}{c} \varrho, \tilde{u} \\ \tau, \cdot \end{array} \right) \, dx.
\]
On the other hand,
\[
\int_{\{\varrho < \delta\}} \left| \frac{\varrho}{\tilde{\varrho}} - 1 \right| |\tilde{u} - u| \, dx \leq \frac{1}{2} \int_{\Omega} \tilde{A}_R (|\tilde{u} - u|) \, dx + c(\delta, \tilde{A}_R) \int_{\Omega} \mathcal{E} \left( \varrho, m \vline \begin{array}{c} \varrho, \tilde{u} \\ \tau, \cdot \end{array} \right) \, dx.
\]
Thus applying the standard Gronwall argument to (6.5) we obtain the desired conclusion:
\[
\varrho = \tilde{\varrho}, \quad u = \tilde{u}, \quad \mathcal{R} = \mathcal{E} = 0.
\]

We have proved the following result:

Theorem 6.3 (Weak–strong uniqueness). Let $\Omega \subset R^d$ be a bounded Lipschitz domain. Suppose that $p$ and $S$ satisfy the structural hypotheses (2.2), (2.4), and (2.6). Let $[\varrho, u]$ be a dissipative solution in the sense of Definition 2.1, and let $[\tilde{\varrho}, \tilde{u}]$ be a strong solution of the same problem belonging to the class
\[
\tilde{u} \in C^1([0, T] \times \Omega; R^d), \quad \tilde{\varrho} \in C^1([0, T] \times \Omega), \quad \inf_{(0, T) \times \Omega} \tilde{\varrho} > 0,
\]
with the stress tensor \( \tilde{\mathbb{S}} \in C([0, T] \times \overline{\Omega}; R^{d \times d}_{\text{sym}}) \), \( \text{div}_x \tilde{\mathbb{S}} \in C([0, T] \times \overline{\Omega}; R^d) \).

Then
\[
\rho = \tilde{\rho}, \quad u = \tilde{u} \text{ in } (0, T) \times \Omega, \quad \mathbb{E} = \mathcal{R} = 0.
\]

The existence of strong solution, even local in time, is a largely open problem solved only in particular cases. The case of a linearly viscous (Newtonian) fluid with rather general boundary conditions has been investigated by Valli and Zajaczkowski [20]. They show local in time existence of classical solution for large initial data satisfying the standard compatibility conditions. To the best of our knowledge, the existence of classical solutions in the non–Newtonian case is known only for problems in the space dimension \( d = 1 \), see Fang and Li [8], and Fang, Zhu, and Guo [9].

7. **Concluding remarks.** A short inspection of the proofs shows that both compatibility and weak–strong uniqueness property hold if the defect compatibility condition (2.13) is weakened to
\[
\text{tr}[\mathcal{R}] \leq \widetilde{d} \mathcal{E}, \quad \text{for a certain constant } \widetilde{d} > 0. \tag{7.1}
\]

The weak formulation can be then written in a concise way as:
\[
\left[ \int_{\Omega} \rho \phi \, dx \right]_{t=0}^{t=T} + \int_0^T \int_{\Gamma_{\text{out}}} \phi u_B \cdot n \, dS_x + \int_0^T \int_{\Gamma_{\text{in}}} \phi B_u \cdot n \, dS_x = \int_0^T \int_{\Omega} \left[ \phi \partial_t \phi + \phi \mathbf{u} \cdot \nabla \phi \right] \, dx \, dt, \quad \phi(0, \cdot) = \phi_0, \tag{7.2}
\]

for any \( \phi \in C^1([0, T] \times \overline{\Omega}) \);
\[
\left[ \int_{\Omega} \rho \mathbf{u} \cdot \phi \, dx \right]_{t=0}^{t=T} = \int_0^T \int_{\Omega} \left[ \rho \mathbf{u} \cdot \partial_t \phi + \phi \mathbf{u} \cdot \mathbf{u} \cdot \nabla \phi + \rho_{\phi} \div \phi \nabla \phi \right] \, dx \, dt - \int_0^T \int_{\Omega} \mathbb{S} : \nabla \phi \, dx \, dt + \int_0^T \int_{\Omega} \mathbb{S} \phi \, d \mathcal{R}(t) \, dt, \quad \phi(0, \cdot) = \phi_0, \tag{7.3}
\]

for any function \( \phi \in C^1([0, T] \times \overline{\Omega}; R^d) \), \( \phi|_{\partial \Omega} = 0 \);
\[
\left[ \int_{\Omega} \left[ \frac{1}{2} \rho |\mathbf{u} - \mathbf{u}_B|^2 + P(q) \right] \, dx \right]_{t=0}^{t=T} + \int_0^T \int_{\Omega} \left[ F(\mathbb{D}_x \mathbf{u}) + F^* (\mathbb{S}) \right] \, dx \, dt + \int_0^T \int_{\Gamma_{\text{out}}} P(q) \mathbf{u}_B \cdot n \, dS_x \, dt + \int_0^T \int_{\Gamma_{\text{in}}} P(q_B) \mathbf{u}_B \cdot n \, dS_x \, dt \tag{7.4}
\]
\[
+ \frac{1}{d} \int_0^T \int_{\partial \Omega} \text{tr}[\mathcal{R}] (\tau) \leq - \int_0^T \int_{\Omega} \left[ \rho \mathbf{u} \otimes \mathbf{u} + p(q) \mathbb{I} - \mathbb{S} \right] : \nabla_x \mathbf{u}_B \, dx \, dt - \int_0^T \int_{\Omega} \mathbf{u} \cdot \nabla_x \mathbf{u}_B \cdot \mathbf{u}_B \, dx \, dt, \quad \text{for some } \widetilde{d} > 0.
\]

Note that the energy defect measure \( \mathbb{E} \) is entirely eliminated and the only “free” quantity in (7.2)–(7.4) is the Reynolds stress \( \mathcal{R} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^d)) \). This new definition is in fact equivalent to Definition 2.1 as one can always define the “energy defect” as
\[
\mathbb{E} = \frac{1}{d} \text{trace}[\mathcal{R}].
\]

Convexity of the pressure \( p \) was necessary for the Reynolds stress \( \mathcal{R} \) to be a positively semi–definite tensor. From the point of view of physics, hypothesis (2.2)
may seem too restrictive. In particular, the physically relevant case of the isothermal pressure \( p(\varrho) = \theta \varrho, \theta > 0 \) is not included. A brief inspection on the proofs reveals that all principal results remain valid for any EOS of the form

\[
p(\varrho) + a \varrho, \ a \geq 0
\]

as long as \( p \) satisfies (2.2).

8. Appendix. Our goal is to show the following result.

Lemma 8.1. Let \( Q = (0,T) \times \Omega \), where \( \Omega \subset \mathbb{R}^d \) is a bounded domain. Suppose that

\[
r_n \to r \text{ weakly in } L^p(Q), \ v_n \to v \text{ weakly in } L^q(Q), \ p > 1, q > 1,
\]

and

\[
r_n v_n \to w \text{ weakly in } L^r(Q), \ r > 1.
\]

In addition, let

\[
\partial_t r_n = \text{div}_x g_n + h_n \text{ in } \mathcal{D}'(Q), \ \|g_n\|_{L^s(Q; \mathbb{R}^d)} \lesssim 1, \ s > 1,
\]

\[
h_n \text{ precompact in } W^{-1, z}, \ z > 1,
\]

and

\[
\| \nabla_x v_n \|_{M(Q; \mathbb{R}^d)} \lesssim 1 \text{ uniformly for } n \to \infty.
\]

Then

\[
w = rv \text{ a.a. in } Q.
\]

Proof. First, we introduce a cut-off function

\[
T_k(v) = kT \left( \frac{v}{k} \right), \ T \in C^\infty \cap C_B(\mathbb{R}),
\]

\[
T(Z) = T(-Z), \ T(Z) = Z \text{ if } |Z| \leq 1, \ 0 \leq T'(Z) \leq 1.
\]

Next, write

\[
v_n = T_k(v_n) + \left( v_n - T_k(v_n) \right),
\]

and

\[
r_n v_n = r_n T_k(v_n) + r_n \left( v_n - T_k(v_n) \right).
\]

Passing to a subsequence (not relabeled) we may assume

\[
T_k(v_n) \to T_k(v) \text{ weakly-* in } L^\infty(Q), \ r_n T_k(v_n) \to w_k \text{ weakly in } L^r(Q) \text{ as } n \to \infty.
\]

We claim that it is enough to show

\[
w_k = rT_k(v) \text{ a.a. in } Q \text{ for any } k \to \infty.
\]

Indeed we have

\[
\int_Q |v_n - T_k(v_n)| \, dx \, dt \leq \int_{|v_n| \geq k} |v_n|\]

\[
\leq |\{v_n \geq k\}| \frac{1}{k} \|v_n\|_{L^s(Q)} \to 0 \text{ as } k \to \infty \text{ uniformly in } n,
\]

and

\[
\|v - T_k(v)\|_{L^1(Q)} \leq \liminf_{n \to \infty} \|v_n - T_k(v_n)\|_{L^1(Q)} \to 0 \text{ as } k \to \infty.
\]
Similarly
\[
\int_Q |r_n(v_n - T_k(v_n))| \, dx \, dt \leq \int_{|v_n| \geq k} |r_n v_n| \\
\leq |\{v_n \geq k\}| \|r_n v_n\|_{L^\infty(Q)} \to 0 \text{ as } k \to \infty \text{ uniformly in } n.
\]

Since
\[
\|\nabla_x v_n\|_{\mathcal{M}(Q; R^d)} \lesssim 1 \Rightarrow \|\nabla_x T_k(v_n)\|_{\mathcal{M}(Q; R^d)} \lesssim 1 \text{ uniformly for } n \to \infty,
\]

it is enough to show the conclusion under the assumption
\[
v_n \to v \text{ weakly-(*) in } L^\infty(Q).
\]

To this end, we apply Div-Curl lemma to the vector fields
\[
U_n = [r_n, -g_n] : Q \to R^{d+1}, \quad \text{DIV}_{t,x} U_n = \partial_t r_n + \text{div}_x g_n = h_n,
\]
\[
U_n \to U = [r, g] \text{ weakly in } L^\min\{s,p\}(Q),
\]
and
\[
V_n = [v_n, 0] : Q \to R^{d+1}, \quad \text{CURL}_{t,x} V_n \approx \nabla_x v_n \text{ bounded in } \mathcal{M}(Q; R^{d \times d}).
\]

Applying Div–Curl Lemma we obtain the desired conclusion. \(\square\)

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