NEW LATTICE SPHERE PACKINGS DENSER THAN MORDELL-WEIL LATTICES

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Abstract. 1) We present new lattice sphere packings in Euclid spaces of many dimensions in the range 3332 – 4096, which are denser than known densest Mordell-Weil lattice sphere packings in these dimensions. Moreover it is proved that if there were some nice linear binary codes we could construct lattices denser than Mordell-Weil lattices of the many dimensions in the range 128 – 3272.

2) Lattice sphere packings of many dimensions in the range 4098 – 8232 better than present records are presented. Some new dense lattice sphere packings of moderate dimensions 84, 85, 86, 181 – 189 denser than any previously known sphere packings of these dimensions are also given.

3) New lattices with densities at least 8 times of the densities of Craig lattices of the dimensions $p-1$, where $p$ is a prime satisfying $p-1 \geq 1222$, are constructed. Some of these lattices provide new record sphere packings. The construction is based on the analogues of Craig lattices.

1. INTRODUCTION

The problem to find the dense packing of infinite equal non-overlapping spheres in Euclid space $\mathbb{R}^n$ is a classical mathematical problem ([27, 21, 11, 44]). Low dimension sphere packing problems seem to be understood better than the problems in higher dimensions. The root lattices in Euclid spaces of dimensions 1, 2, 3, 4, 5, 6, 7 and 8 had been proved to be the unique densest lattice sphere packings in these dimensions(see [11]). Kepler conjecture about the densest 3 dimensional sphere packing problem was proved in [26]. Many known densest sphere packings are lattice packings or packings from finitely many translates of lattices(see [11, 12, 31, 47, 48]). Constructing lattice sphere packings from error-correcting codes, algebraic number fields and algebraic geometry have been proposed by many authors and stimulated many further works([30, 31, 11, 15, 16, 2, 36, 37, 38, 33, 34, 23, 24, 17, 18, 19, 7, 20, 42, 43, 35, 47, 48, 10, 11]). Recently Leech lattice, which was found in 1965 in [30], has been proved to be the unique densest lattice packing.
in dimension 24 (see [9, 10]). For Rogers upper bound and Kabatiansky-Levenshtein upper bound on the densities of sphere packings we refer to [11] pages 19-20. The recent work [9] gave better upper bounds on densities of sphere packings. Table 3 in page 711 of [9] is the latest upper bound for center densities of sphere packings in dimensions 1-36. From Voronoi theory ([39, 32]), there are algorithms to determine the densest lattice sphere packings in each dimension. However the computational task for dimensions $n \geq 9$ is generally infeasible.

Laminated lattices were known from 1877 ([28]) and we refer to [11, 14] for a historic survey. In 1982 J. H. Conway and N. J. A. Sloane published [12] in which all densities of laminated lattices up to dimension 48 were determined. These laminated lattices are known densest lattices in dimensions 1-29 except dimensions 10, 11, 12, 13 at that time. For dimensions bigger than or equal to 25, there are many laminated lattices with the same density. The known densest sphere packing in dimension 12 is the Coxeter-Todd lattice packing and the known densest sphere packings in dimensions 10,11,13 were found by Leech and Sloane in 1970 by using non-linear binary codes(see [31]). This situation was changed in 1995 and 1997, when A. Vardy published [22] and R. Bacher published [1]. New non-lattice sphere packings better than laminated lattice packings were constructed in Euclid spaces of dimensions 20(see [17]), 22(see [13]), 27,28,29,30(see [18]) and 18(see [4]). Lattice sphere packings in Euclid spaces of dimensions 27, 28 and 29 which are better than laminated lattice packings were constructed in [1]. We refer to Nebe-Sloane list [45] for known densest sphere packings in low dimensions.

The knowledge about high dimension sphere packing problem is quite different. For the high dimensions $n$, in the range $80 \leq n \leq 4096$, $n = 2p - 2$ where $p$ is a prime number satisfying $p \equiv 5 \ mod \ 6$, or $n = 2^t$, where $7 \leq t \leq 12$, the known densest sphere packings are lattices from algebraic curves over function fields. That are Mordell-Weil lattices which were discovered by N. Elkies and T. Shioda in 1990’s (see [17, 18, 42, 43, 35]). For Mordell-Weil lattice in the dimension $n = 2p - 2$, where $p$ is a prime number satisfying $p \equiv 5 \ mod \ 6$, the center density is $\left(\frac{p+1}{12}\right)^{p-1}p^{(p-5)/6}$ (see [42, 43] for other cases). For the center densities of Mordell-Weil lattices in the dimensions $n = 2^t$, where $7 \leq t \leq 12$, we refer to [17, 11] ([11] Preface to Third Edition, page xviii). For example, the known densest sphere packing in dimension 4096 is the Mordell-Weil lattice with center density $2^{11527}$. After the discovery of Mordell-Weil lattices by Elkies and Shioda in 1990’s, there are a lot of effort for understanding its structure and construction(for example, [23, 33, 25]). For dimensions in the range $149 \leq n = p - 1 \leq 3001$ (where $p$ is a prime number, $149 \leq n \leq 3001$ except $p = 509, 513$ and 521), many of the known densest sphere packings in high $n = p - 1$ dimension Euclid spaces are
Craig lattice packings and their recent improvements (see [15, 16, 20, 11]). For the sphere packing problem in Euclidean spaces of many dimensions in the range $4100 \leq n \leq 12754608$ or $n \leq 8 \cdot 10^8$, the best known lattices were given by Bos-Conway-Sloane construction ([5], [11], page 17, Table 1.3 and Chapter 8 section 10). The higher dimensional sphere packing problem was remarked in N. J. A. Sloane’s talk in ICM 1998 as follows:” ...But we know very little about this range (of dimensions $80 \leq n \leq 4096$)....” ([44]).

In this paper we propose the analogous Craig lattice, which is a far-reaching extension of Craig lattice ([15, 16, 11], section 6, Chapter 8). These lattices are polynomial lattices and can be constructed for all dimensions. Our construction gives lattice sphere packings of dimensions $n$ in the range $n = p - 1 \geq 1222$ better than Craig lattices and their refinements in [20]. Thus some of these lattices provides new record sphere packings. Our construction also establishes a close relation between nice lattices and linear error-correcting codes. If there were some “good enough” linear codes over $\mathbb{F}_2, \mathbb{F}_4, \mathbb{F}_8$ (see [22]), then our construction would lead to new lattices denser than Mordell-Weil lattices in very many dimensions in the range $128 - 3272$.

In many high dimensions in the range $3332 - 4096$, we prove that some wanted codes in the above description exist. Thus we present some lattice sphere packings in these dimensions, which are denser than the Mordell-Weil lattices. New lattices denser than Shimada lattices in dimensions 84, 85 and 86 ([45, 40, 41]) are constructed in section 4. Dense lattice sphere packings in many high dimensions in the range $4098 - 8323$ which are denser than the known best lattices from Bos-Conway-Sloane construction are also presented in section 7. Other new dense lattice sphere packings of some moderate dimensions in the range $148 - 200$ are constructed in section 8.

All lattices constructed in this paper are integral lattices.

**Definition 1.1.** For a packing of infinite equal non-overlapping spheres in $\mathbb{R}^n$ with centers $x_1, x_2, \ldots, x_m, \ldots$, the packing radius $\rho$ is defined as $\frac{1}{2} \min_{i \neq j} ||x_i - x_j||$. The density $\Delta$ is $\lim_{t \to 0} \frac{\text{Vol}\{x \in \mathbb{R}^n : ||x|| < t, \exists x_i, ||x - x_i|| < \rho\}}{\text{Vol}\{x \in \mathbb{R}^n : ||x|| < t\}}$. Center density $\delta$ is defined as $\frac{\Delta}{V_n}$ where $V_n$ is the volume of the ball of radius 1 in $\mathbb{R}^n$.

Let $b_1, \ldots, b_m$ be $m$ linearly independent vectors in the Euclidean space $\mathbb{R}^n$ of dimension $n$. The discrete point sets $L = \{x_1b_1 + \cdots + x_nb_m : x_1, \ldots, x_m \in \mathbb{Z}\}$ is rank $m$ lattice in $\mathbb{R}^n$. The determinant of the lattice is defined as $\det(L) = \det(\langle b_i, b_j \rangle)$. The volume of the lattice is $\text{Vol}(L) = (\det(L))^\frac{n}{2}$. Let $\lambda(L)$ be the Euclidean norm of the shortest non-zero vectors in the lattice and the minimum norm of the lattice is just $\mu(L) = (\lambda(L))^2$. When the centers of the spheres are these lattice vectors in $L$ we have $\rho = \frac{\mu(L)}{2}$ and the center density $\delta(L) = \frac{n}{\text{Vol}(L)}$. The lattice $L^* = \{y \in \mathbb{R}^m : \langle y, x \rangle \in \mathbb{Z}\}$
is called the dual lattice of the lattice $L$. A lattice is called integral if the inner products between lattice vectors are integers. An unimodular lattice is the integral lattice $L$ satisfying $L^* = L$. The unimodular lattices whose minimum norm attain the largest possible bound are called extremal unimodular lattices. The problem to find extremal unimodular lattices seems very difficult and has attracted many works (see [2, 34, 11], Ch.7 or Nebe-Sloane database of lattices [45]).

Let $r$ be a prime power and $F_r$ be the finite field with $r$ elements. A linear (non-linear) error-correcting code $C \subset F_r^n$ is a $k$-dimensional subspace (or a subset of $M$ vectors). For a codeword $x \in C$, $wt(x)$ is the number of nonzero coordinates of $x$. The minimum Hamming weight (or distance) of the linear (or non-linear) code $C$ is defined as $d(C) = \min_{x \neq y, x, y \in C} \{wt(x - y)\}$. We refer to $[n, k, d]$ (or $(n, M, d)$) code as linear (or non-linear) code with length $n$, distance $d$, and dimension $k$ (or $M$ codewords). Given a binary code $C \subset F_2^n$, the construction A ([11, 31]) leads to a lattice in $R^n$. The lattice $L(C)$ is defined as the set of integral vectors $x = (x_1, ..., x_n) \in Z^n$ satisfying $x_i \equiv c_i \mod 2$ for some codeword $c = (c_1, ..., c_n) \in C$. It is easy to check $\rho = \frac{1}{2} \min\{\sqrt{d(C)}, 2\}$ and $Vol(L(C)) = 2^{n-k(C)}$ where $k(C)$ is the dimension of the code $C$. This gives a lattice sphere packing with the center density $\delta = \frac{\min\{\sqrt{d(C)}, 2\}^n}{2^{2n-k(C)}}$. This construction A leads to some best known densest lattice packings in low dimensions (see [11]). For a non-linear binary code, the same construction gives the non-lattice packing with center density $M \cdot \frac{\min\{\sqrt{d(C)}, 2\}^n}{2^{2n-k(C)}}$, where $M$ is number of codewords in the nonlinear binary code $C$. For example, the non-linear length 11, minimum distance 4 binary codes with 72 codewords implies a non-lattice packing with center density $\delta = \frac{9}{256} = 0.03516$, which is the known densest sphere packing in dimension 11 ([11]). The known densest sphere packings in Euclid spaces of dimensions 10 and 13 were constructed similarly in [31]. The packing in dimension 10 is from non-linear binary $(10, 40, 4)$ code ([11]). From the non-linear code $(12, 144, 4)$ a non-lattice sphere packing with center density $\frac{9}{256}$ (which is smaller than the center density $\frac{1}{27}$ of the Coxeter-Todd lattice) can be constructed ([31, 11]). The packing in dimension 13 is the union of infinitely many copies of this non-lattice dimension 12 packing from non-linear $(12, 144, 4)$ binary codes (see [31]). We refer to [45] for records of dense sphere packings in Euclid spaces of various dimensions.

2. Analogous Craig lattices

Let $\zeta$ be a primitive $p$-th root of unity, where $p$ is an odd prime. Then the ring of the algebraic integers in $Q[\zeta]$ is $Z[\zeta]$ ([15, 16]). The Craig lattice $A_{p-1}^{(i)}$ of rank $p-1$ introduced in [15] is the ideal in the ring $Z[\zeta]$ (free $Z$ module) generated by $(1 - \zeta)^{i}$, where $i$ is a positive integer. A cyclotomic
construction of Leech lattice was given by Craig in [16]. In [20] refinements of Craig lattices were constructed by adding some fractional numbers. The refinements of Craig lattices have their center density at least the three times of the center density of the original Craig lattices. These provided some new record lattice sphere packings in some dimensions (see [20]).

Another form of Craig lattices was given in [11] section 6 of Chapter 8. It is a cyclic lattice \( \mathbb{A}_n^{(m)} \) of rank \( n \) in the ring \( \mathbb{Z}[x]/(x^{n+1} - 1) \), the ideal generated by \( (x - 1)^m \) in the ring \( \mathbb{Z}[x]/(x^{n+1} - 1) \). The volume of the Craig lattice \( \mathbb{A}_n^{(m)} \) is \( (n + 1)^{m - \frac{1}{2}} \). When \( n + 1 = p \) is an odd prime and \( m < \frac{p}{2} \), the minimum norm \( \mu(\mathbb{A}_n^{(m)}) \geq 2m \) (see Theorem 7 at page 223 of [11]). This is just the original Craig lattices introduced in [15].

We propose the following analogue of the Craig lattice as a \( \mathbb{Z} \) sub-module of the \( \mathbb{Z} \) module \( R = \mathbb{Z} < 1, x, ..., x^n > \) spanned by \( 1, x, x^2, ..., x^n \). Given positive integers \( n, m, l \) satisfying \( m < \frac{n}{2} \) and \( l \geq n + 1 \), consider the sub-module \( \mathbb{A}_n^{(m,l)} = \mathbb{Z}[x]((x - 1)^n + \mathbb{Z}(x - 1)^{n+1} + \cdots + \mathbb{Z}(x - 1)^{m-1} + \mathbb{Z}(x - 1)^{m} + l\mathbb{Z}(x - 1)^{m+1} + \cdots + l\mathbb{Z}(x - 1)^{n-1} + l\mathbb{Z}(x - 1)^{n} \) in \( \mathbb{Z} < 1, x, ..., x^n > \). Then any element \( v \) in \( \mathbb{A}_n^{(m,l)} \) can be represented as \( v_0 + v_1 x + \cdots + v_n x^n \). The set of all coordinates \( (v_0, v_1, ..., v_n) \) of these vectors \( v \) in \( \mathbb{A}_n^{(m,l)} \) is a sub-lattice in \( \mathbb{Z}^{n+1} \).

Equivalently we take the base \( 1, x, ..., x^n \) as the orthogonal base of the \( \mathbb{Z} \) module \( R \) and the \( \mathbb{Z} \) sub-module described as above is just the lattice. For any polynomial \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R \) with integral coefficients satisfying \( f(1) = 0 \), \( f(x) \) is a linear combination of \( (x - 1), ..., (x - 1)^n \) with integral coefficients. If \( l \) has no prime factor smaller than \( m \), a polynomial \( f(x) = a_n x^n + \cdots + a_1 x + a_0 \) with integral coefficients is in the lattice \( \mathbb{A}_n^{(m,l)} \) if and only if \( f(1) = 0 \) and \( f^{(i)}(1) \equiv 0 \mod l \) for \( i = 1, ..., m - 1 \).

Theorem 2.1.

1. When \( n + 1 \) is a prime or \( m = 1 \), \( \mathbb{A}_n^{(m,n+1)} \) is just the Craig lattice;
2. Given positive integers \( n, m, l \) satisfying \( m < \frac{n}{2} \) and \( l \geq n + 1 \), the \( \mathbb{A}_n^{(m,l)} \) is a lattice of rank \( n \) and its volume is \( l^{m-1} (n + 1)^{\frac{1}{2}} \). When \( l \) is a prime number, the minimum norm of the lattice \( \mathbb{A}_n^{(m,l)} \) satisfies \( \mu(\mathbb{A}_n^{(m,l)}) \geq 2m \).

Proof. 1) When \( m = 1 \), \( \mathbb{A}_n^{(1,l)} \) has an integral base \( \{(x - 1), (x - 1)x, ..., (x - 1)x^{n-1}\} \). Thus \( \mathbb{A}_n^{(1,l)} = \mathbb{A}_n^{(1)} \).

We prove the \( \mathbb{A}_n^{(m,l)} \) is a cyclic lattice when \( l = n + 1 \) is a prime number. It is clear that \( (x - 1)^{j+1}x = (x - 1)^{j+1} + (x - 1)^j \) is in \( \mathbb{A}_n^{(m,l)} \) for \( m \leq j \leq n - 1 \) and \( l(x - 1)^jx = l(x - 1)^{j+1} + l(x - 1)^j \) is in \( \mathbb{A}_n^{(m,l)} \) for \( 1 \leq j \leq m - 1 \). We only need to check the element \( (x - 1)^n x - (x^{n+1} - 1) \), which is a shift of \( (x - 1)^n \),
are denser, since in their construction the prime number with the construction of Craig-like lattices in [7]. It is obvious our lattices dimensions with “not bad” densities. The following result can be compared the smallest prime \( q \) functions of \( \mathbb{Z} \) sub-sets in \( \mathbb{F} \) and \( \mathbb{A} \) as in [11] page 224. We take the analogue Craig lattice injective \( \mathbb{Z} \) and 2

Proof. From the Bertrand postulate there exists a prime number \( x \) Thus (3) analogous Craig lattice with density \( \mathbb{A}^{(1)} \) contains the Craig lattice \( \mathbb{A}^{(m)} \) as a sub-lattice. It has the same index as a sub-lattice of the lattice \( \mathbb{A}^{(1)} \). The conclusion is proved.

2) It is obvious that \( \mathbb{A}^{(m,l)} \) is a sub-lattice of the rank \( n \) Craig lattice \( \mathbb{A}^{(1)} = \{(v_0, v_1, ..., v_n) \in \mathbb{Z}^{n+1} : v_0 + v_1 + \cdots + v_n = 0 \} \). On the other hand \( \mathbb{A}^{(m,l)} \) has index \( l^{m-1} \) in the Craig lattice \( \mathbb{A}^{(1)} \). Thus it is a rank \( n \) lattice and has volume \( l^{m-1} \text{Vol}(\mathbb{A}^{(1)}) = l^{m-1}(n+1)^{\frac{1}{2}} \). The proof of the second conclusion is the same as the proof of Theorem 7 of page 223 of [11]. If \( \mu(\mathbb{A}^{(m,l)}) < 2m \) we have an element \( f(x) = \sum x^i - \sum x^j \in \mathbb{A}^{(m,l)} \), where \( S \) and \( T \) are two sub-sets in \( \{0, 1, ..., n\} \) satisfying \( h = |S| = |T| < m \). Here \( S = \{s_1, ..., s_h\} \) and \( T = \{t_1, ..., t_h\} \) may contain repeated elements. Then from the condition \( f(x) \in \mathbb{A}^{(m,l)} \) we have \( f_i(1) = 0 \mod l \), for \( i = 0, 1, ..., m-1 \). Then we have \( \sum_{j=1}^{h} s_j i^j \equiv \sum_{j=1}^{h} t_j i^j \mod l \), for \( i = 0, 1, ..., m-1 \). Since \( l \) is a prime number, from the Newton’s identities over the finite field \( \mathbb{Z}/l\mathbb{Z} \), the elementary symmetric functions of \( S \) and \( T \) of degree \( < m \) have to be the same. Thus \( S = T \) and \( f(x) = 0 \) since we have \( l \geq n + 1 \).

From Theorem 2.1 we have the following analogous Craig lattices for all dimensions with “not bad” densities. The following result can be compared with the construction of Craig-like lattices in [7]. It is obvious our lattices are denser, since in their construction the prime number \( q \) is required to be the smallest prime \( q \) satisfying \( q \equiv 1 \mod n \).

**Theorem 2.2.** For each dimension \( n \) and each \( m < \frac{n}{2} \) we have a analogous Craig lattice with density \( \Delta_n \geq \frac{m^2}{2^{n-1} \frac{m}{2} n^{m-1} (n+1)^{\frac{1}{2}}} \). When suitable \( m \) is taken we have \( \frac{1}{n} \log_2 \Delta_n \geq -\frac{1}{2} \log_2 \log_2 n + o(1) \).

**Proof.** From the Bertrand postulate there exists a prime number \( l \) between \( n \) and \( 2n \) for any positive integer \( n \). Set \( m \) to be the integer nearest to \( \frac{n}{2 \log_2 n} \) as in [11] page 224. We take the analogue Craig lattice \( \mathbb{A}^{(m,l)} \). A direct calculation gives us the result. \( \square \)

Let \( l \) be an odd number. We define a mapping \( \pi : \mathbb{A}^{(m,l)} / 2(\mathbb{A}^{(m,l)}) \to \mathbb{Z}^{n+1} / 2(\mathbb{Z}^{n+1}) \) by \( \pi(a_0 + a_1 x + \cdots + a_n x^n) = (a_0, ..., a_n) \mod 2 \). This is an injective \( \mathbb{Z} \) linear mapping. It is clear that \( 2\left(\frac{f(1)}{n!}\right) \) can be divided by \( l \) implies that \( \frac{f(1)}{n!} \) can be divided by \( l \), since \( l \) is an odd number. We can check that the image of \( \pi \) is the linear binary \([n+1, n, 2] \) code.
Theorem 2.3. Suppose the positive integers $n, m, l$ satisfy $m < \frac{n}{2}$, $l \geq n + 1$ and $l$ is a odd prime number. If there exists a linear binary sub-code of the $[n + 1, n, 2]$ code with parameters $[n + 1, k, \geq 8m]$, then we have a lattice with center density at least $\frac{2^{k - \frac{m}{2} - m}}{l^{m-1}(n+1)^2}$.

Proof. The $[n+1,k,\geq 8m]$ binary linear sub-code $V$ is in the image $\pi(A_n^{(m,l)})$ as a binary linear $[n + 1, n, 2]$ code. From the $\mathbb{Z}$-linearity of $\pi$, the inverse image $\pi^{-1}(V)$ is a lattice with volume $2^{n-k}Vol(A_n^{(m,l)})$. Let $v$ be a vector in $\phi^{-1}(V)$. If $\phi(v) = 0$, $v \in 2A_n^{(m,l)}$, then the Euclid norm of $v$ is at least $8m$. If $\pi(v) \neq 0$, then at least $8m$ coordinates of the vector $v$ are odd numbers and the Euclid norm of $v$ is at least $8m$. The conclusion is proved.

Theorem 2.4. Suppose the positive integers $n, m, l$ satisfy $m \leq \frac{n+1}{2}$, $l \geq n + 1$ and $l$ is a odd prime. If there exists a linear binary $[n,k,8m]$ code then we have a lattice with center density at least $\frac{2^{k - \frac{m}{2} - m}}{l^{m-1}(n+1)^2}$.

Proof. The extended code of the binary $[n,k,8m]$ code by adding a parity check column we get the linear sub-code in the Theorem 2.3. □

From Theorem 2.4 some lattices of rank $n = p^2 - 1$, where $p$ is a prime, can be constructed. We have a lattice $A_{120}^{(11,127)}$ of rank 120 with center density at least $2^{74.9640}$ (less than the center density $2^{76}$ of the Bos-Conway-Sloane construction $\eta(E_8)$, see [11] p.242), a lattice $A_{168}^{(13,173)}$ of rank 168 with center density at least $2^{133.9011}$ (larger than the center density $2^{120}$ of the Bos-Conway-Sloane construction $\eta(L_24)$, see [11] p.242), a lattice $A_{288}^{(17,293)}$ of rank 288 with center density at least $2^{309.3031}$ (larger than $2^{300}$ of the Bos-Conway construction $\eta(L_24)$, see [11] p.242), a lattice $A_{360}^{(19,367)}$ of rank 360 with center density at least $2^{427}$ (larger than $2^{408}$ of the $\eta(L_24)$ in $\mathbb{R}^{360}$, see [11] p.242).

We now apply Theorem 2.4 to construct lattices in dimensions 60, 96, 136 and 144. In the dimension 60, the two known good lattices are Kashichang-Pasupathy lattice $K_{60}$ with the center density $2^{17.4346}$ (see [11], page xivi) and the extremal 60 dimensional lattice with center density $(\frac{3}{2})^{30} \approx 2^{17.55}$ (see Nebe-Sloane database of lattices [45]). Applying Theorem 2.4 to $A_{60}^{(7,61)}$ and the binary linear $[60,1,60]$ code (see [45]) we get a lattice with center density $2^{16.672}$ (also from [20]). From [20] if there was a binary linear $[60,27,16]$ code, then a new denser lattice with center density $2^{18.1039}$ could be constructed. The Elkies lattice in dimension 60 has center density $2^{19.04}$, which is the section of Mordell-Weil lattice (see [45]). If there was a binary linear $[60,28,16]$ code (see [22]), we could construct a lattice sphere packing in dimension 60 with center density $2^{19.1039}$. The known densest sphere packing in dimension 96 is is the lattice $\eta(P_{96})$ with center density $2^{52.078}$ (see [11]).
From the table in [22], there exists a linear binary [96, 23, 32] code, we get a lattice with center density $2^{47.9003}$. If there was a linear binary [96, 30, 32] code [22]), we could construct a lattice sphere packing with center density $2^{47.9003}$. If there was a linear binary [136, 57, 32] code [22], it is possible that there was a [136, 37, 32] code, which would lead to a possible new lattice with center density $2^{100.1570}$. In dimension 144 there is a dense lattice $\eta(\Lambda_{24})$ with center density $2^{96}(11)$ p.242) from Bos-Conway-Sloane construction [5]. From the $A^{(14,149)}_{144}$ and the binary linear [144, 1, 144] code(see [22]) we get a new lattice with center density $2^{105.6736}$. In dimension $n = 160$, since $n + 1 = 161 = 23 \cdot 7$ is not a prime, we have no Craig lattice in this dimension. The analogous Craig lattice $A^{(16,163)}_{160}$ has its center density $\delta_{160} = \frac{8^{90}}{163^{1.77}} \approx 2^{126.4051}$. By using the trivial linear binary [160, 1, 160] code and Theorem 2.4 we get a dense lattice in dimension 160 with center density at least $2^{127.4051}$. The analogous Craig lattice $A^{(8,163)}_{160}$ has center density $2^{104.8847}$. Applying Theorem 2.4 to the linear binary code [160, 19, 64]([22]) then a lattice with center density $2^{123.8847}$ can be constructed. If there was a linear binary [160, 27, 64] code([22]), a lattice with center density $2^{131.8847}$ could be constructed. On the other hand, there is no Mordell-Weil lattice in dimension 160. The nearest Mordell-Weil lattice in dimension smaller than 160 (from Theorem 1.1 of [13]) is the Mordell-Weil lattice of dimension 140 with center density $2^{113.31}$. There is no child lattice $\eta(E_8)$ in dimension 160(11), page 241). The lattice in dimension 160 from Minkowski-Hlawka Theorem has center density approximately 111.2378. Thus our construction from analogous Craig lattice gives a new dense lattice in dimension 160.

Remark 2.1. When $l(>n)$ is a prime number, the analogous Craig lattice $A_{n}^{(m,l)}$ is just the section of the Craig lattice $A_{l-1}^{(m,l)}$ by imposing the condition that the last $l - n - 1$ coordinates are zero.

3. Improving Craig lattices and their refinements

The main result of this section is the following theorem.

Theorem 3.1. Let $p$ be a prime larger than or equal to 1223. Suppose $A_{p-1}^{(m)}$ is the densest Craig lattice of dimension $p - 1$. We can construct a new lattice with center density at least $8\delta(A_{p-1}^{(m)})$ from Theorem 2.4.
Proof. It is known the Craig lattice $A_n^{(m)}$, where $m$ is nearest integer of $\frac{n}{2\log_e(n+1)}$, is the densest Craig lattice in the dimension $n = p - 1$, where $p$ is a prime number. Since $n \geq 1222$, then $\frac{n}{2\log_e(n+1)} \leq \frac{4}{7}$. Concatenating linear $[\lfloor \frac{n}{7} \rfloor, 1, \lfloor \frac{n}{7} \rfloor]$ code over $\mathbb{F}_8$ with binary linear $[7, 3, 4]$ code and a suitable trivial extension we get a binary linear $[n, 3, 8(\frac{n}{2\log_e(n+1)} + 1)]$ code. From Theorem 2.4 we get the conclusion. □

In [20], the Craig lattices are refined to new lattices with center density at most $6\delta(A_n^{(m)})$ in the range $1298 \leq n \leq 3482$. Thus our constructed lattices are better than the lattices in [20]. Some of these new dense lattice sphere packings are better than any previously known ones.

| $\dim = n$ | $\text{new} - \log_2\delta$ | $\text{known} - \text{densest}$ | $\text{nearest} - \text{MW}(\dim < n)$ |
|-----------|-----------------|-----------------|------------------|
| 1398      | 2908.8254       | 2905.8254(Craig)| 2919.8743(1364)  |
| 1432      | 2980.6910       | 2977.6910(Craig)| 3012.6846(1400)  |
| 2178      | 5131.4554       | 5128.4554(Craig)| 5086.8746(2120)  |
| 2296      | 5592.5709       | 5589.5709(Craig)| 5377.7840(2216)  |

4. Lattices in dimensions 52, 68, 84, 85, 86, 120, 168, 242, 246, 248, 288 and 360 from analogous Craig lattices

The known densest lattices in dimension 48 and 56 are extremal unimodular lattices([11] Preface to Third Edition) and the known extremal unimodular lattices in dimension 80 has center density $2^{40}$ which is slightly less than the center density $2^{40.14}$ of the known densest lattice(Mordell-Weil lattice) in this dimension(see [2]). The recently constructed extremal unimodular lattice (see [34]) in dimension 72 is the known densest lattice with center density $2^{36}$. Therefore we compare the new lattices from Theorem 2.4 with known (extremal) unimodular lattices. The Gaborit extremal unimodular lattice in dimension 52 has its minimum norm 5 and center density $\left(\frac{5}{2}\right)^{26} \approx 2^{8.4552}$ ([15]). Applying Theorem 2.4 to the analogous Craig lattice $A_{52}^{(6,53)}$ and the $[52, 1, 52]$ linear binary code we can construct a integral lattice with center density $2^{10.7045}$. This is a new dense sphere packing in dimension 52. The dimension 68 extremal unimodular lattices with minimum norm 6 have their center densities $\left(\frac{5}{2}\right)^{34} \approx 2^{19.89}$. Applying Theorem 2.4 to analogous Craig lattice $A_{68}^{(4,71)}$ and binary linear $[68, 8, 32]$ code(see [22]) we get a new lattice with center density at least $2^{20.4757}$. The volume of this new lattice is $2^{60} \cdot 71^{3} \cdot 69^{\frac{1}{2}}$. The Thompson-Smith unimodular lattice


in dimension 248 has minimum norm 10 or 12. Thus its center density is at most $3^{124} \approx 2^{196.54}$ (see Nebe-Sloane database of lattices, [34]). Applying Theorem 2.4 to $A_{248}^{(4,251)}$ and binary linear [248,131,32] code (see [22]) we get a new lattice with center density at least $2^{227.0997}$ and the volume of this new lattice is $2^{137} \cdot 251^3 \cdot 249^1$. In dimension 240 the known densest sphere packing is the Craig lattice packing with center density $2^{245.0006}$.

Applying Theorem 2.4 to the analogous Craig lattice $A_{242}^{(12,251)}$ with center density at least $2^{221.1127}$ and the [242,23,96] linear binary code ([22]) we get a dense lattice sphere packing in dimension 242 with center density $2^{243.1127}$.

Applying Theorem 2.4 to the analogous Craig lattice $A_{246}^{(12,251)}$ with center density $2^{226.2827}$ and linear binary [246,23,96] code ([22]), we get a dense lattice sphere packing in dimension 246 with center density $2^{249.2827}$.

In dimension 104, the presently known densest sphere packing is the Mordell-Weil lattice with center density $\frac{9}{2} \approx 2^{67.0168}$ ([43], Theorem 1.3). Before the invention of Mordell-Weil lattices by Elkies and Shioda, the previously known densest sphere packing is the child lattice $\eta(E_8)$ with center density $2^{60}(\text{[11], page 17, Table 1.3})$. The analogous Craig lattice $A_{104}^{(5,107)}$ has center density 2.38.3949. Thus if there was a linear binary [104,22,40] code ([22]), a lattice denser than $\eta(E_8)$ with center density $2^{60.3949}$ could be constructed. However even the codes attaining the upper bound in the Table [22] exist, we cannot construct lattices denser than Mordell-Weil lattice in the dimension 104.

In [40, 41] long computation of algebraic geometry over finite fields was used to construct dense lattice sphere packings in dimensions 84, 85 and 86 with center densities $\delta_{84}^{Shimada} \approx 2^{30.795}$, $\delta_{85}^{Shimada} \approx 2^{32.5}$ and $\delta_{86}^{Shimada} \approx 2^{34.2075}$ (see Nebe-Sloane list [45], and the comment about Shimada’s 86 dimensional lattice there). We take $n = 84$, 85, 86 and $l = 89$, which is a prime, and $m = 4$. The corresponding dimension $n$ analogous Craig lattice has center density $\delta_n \approx 2^n / 2 \approx 2^{22.67}$. Since [84,16,32], [85,16,32] and [87,17,32] linear binary codes exist ([22]), we have better lattices in dimension 84 with center density at least $2^{35.4}$, better lattices in dimension 85 with center density at least $2^{35.83}$, and better lattice in dimension 86 with center density at least $2^{37.33}$, from Theorem 2.4. The analogous Craig lattice $A_{86}^{(10,89)}$ has center density $2^{38.3225}$, the analogous Craig lattice $A_{85}^{(10,89)}$ has center density $2^{37.1616}$ and the analogous Craig lattice $A_{84}^{(10,89)}$ has center density $2^{36.006}$. Applying Theorem 2.4 to the trivial linear binary [84,1,84], [85,1,85] and [86,1,86] codes we get new lattices in dimensions 84, 85 and 86 with center densities $2^{37.006}$, $2^{38.1616}$ and $2^{39.3225}$.

Applying Theorem 2.4 to the analogous Craig lattice $A_{120}^{(11,127)}$ and linear binary [120,1,120] code we get a lattice sphere packing in dimension 120
with center density $2^{75.0640}$. Applying Theorem 2.4 to the analogous Craig lattice $\mathbf{A}_{168}^{(13,173)}$ and linear binary [168,2,114] code ([22]) we get a lattice sphere packing in dimension 168 with center density a lattice of rank 168 $2^{135.9011}$. From [22], there is a linear binary [144,9,68] code. Thus we have a linear binary [288,9,136] code. Applying Theorem 2.4 to the analogous Craig lattice $\mathbf{A}_{288}^{(17,293)}$ and linear binary [288,9,136] code we get a lattice sphere packing in dimension 288 with center density $2^{318.3031}$. From [22], there is a linear binary [180,16,78] code. Thus we have a linear binary [360,16,156] code. Applying Theorem 2.4 to the analogous Craig lattice $\mathbf{A}_{360}^{(19,367)}$ and linear binary [360,16,156] code we get a lattice sphere packing in dimension 360 with center density $2^{443}$.

In the following table 3 we list some possible better lattices under the condition that some nice codes exist. The Elkies lattices of dimensions 57 – 60 are cross-sections of Mordell-Weil lattices, we refer to [48], page 278.

**Table 2.**

| dimension | new $- \log_2 \delta$ | known                                      |
|-----------|------------------------|--------------------------------------------|
| 52        | 10.7045                | 8.4552(Gaborit), 10.4578(MW)               |
| 60        | 16.672                 | 19.04(Elkies, section – MW), 17.55(extremal) |
| 68        | 20.6757                | 19.89(Gaborit + Harada – Kitazume)         |
| 84        | 37.006                 | 30.795(Shimada)                            |
| 85        | 38.1616                | 32.5(Shimada)                              |
| 86        | 39.3225                | 34.2075(Shimada)                           |
| 96        | 47.9003                | 52.078($\eta(P_{48q})$)                    |
| 120       | 75.0640                | 76($\eta(E_8)$)                            |
| 144       | 105.6736               | 96($\eta(\Lambda_{24})$)                  |
| 160       | 127.4051               | 111(Minkowski – Hlawka)                    |
| 168       | 135.9011               | 120($\eta(\Lambda_{24})$)                 |
| 246       | 249.2827               | 234.33039(Minkowski – Hlawka)              |
| 248       | 227.0997               | 196.54(Thompson – Smith)                   |
| 288       | 318.3031               | 300($\eta(\Lambda_{24})$)                 |
| 360       | 443                    | 408($\eta(\Lambda_{24})$)                 |

In the following we list some possible better lattices under the condition that some nice codes exist. The Elkies lattices of dimensions 57 – 60 are cross-sections of Mordell-Weil lattices, we refer to [48], page 278.
5. Possible new lattices denser than Mordell-Weil lattices

We prove the following result.

**Proposition 5.1.** Let \( p \) be a prime satisfying \( p \equiv 5 \mod 6 \). If there was a binary linear \([2^p - 2, 7^p - 5 - \lceil \frac{p-11}{12} \cdot \log_2 p \rceil, \geq \frac{2(p+1)}{3}]\) code, then we could construct a new lattice of dimension \( 2p - 2 \) with center density larger than the center density \( \frac{(p+1)/12}{p(p-5)/6} \) of Mordell-Weil lattice in this dimension.

**Proof.** Set \( m = \lceil \frac{p+1}{12} \rceil \). From Bertrand ’s postulate there exists a prime \( l \) between \( 2p - 1 \) and \( 4p - 2 \). Applying Theorem 2.4 to analogous Craig lattice \( A^{(m,l)}_{2p-2} \) and the code in the condition, we get the lattice. \( \square \)

The center density of the dimension 52 Mordell-Weil lattice is \( \frac{5/2}{253\cdot 253} \approx 2^{10.4578} \) from Theorem 1.2 [43] (p.933 of [43]). Craig lattice \( A^{(6)}_{52} \) has its center density \( \frac{3^{26}}{53^{27}} \approx 2^{9.7045} \). Applying Theorem 2.4 to this Craig lattice and the binary linear \([52,1,52]\) code we get a lattice with center density \( 2^{10.7045} \). This is a lattice with center density slightly larger than the center density of Mordell-Weil lattice from Theorem 1.2 of [43]. It should be indicated that in the refinement of [20] the least dimension is 57. The dimension 140 Mordell-Weil lattice has center density \( \frac{6^{70}}{4711} \approx 2^{113.31} \). Applying Theorem 2.4 to the binary \([140,50,32]\) code and the analogous Craig lattice \( A^{(4,151)}_{140} \) we get a lattice with center density \( 2^{94.6656} \), which is slightly less than the center density \( 2^{97} \) of the non-lattice packing constructed in [3]. If there was a binary linear \([140,69,32]\) code (see [22]), there would be a possible new lattice of rank 140 with center density at least \( 2^{114.6656} \), which is denser than Mordell-Weil lattice in this dimension.
**Proposition 5.2.** If there was binary linear $[128, 59, 32]$ code, then a new lattice of rank 128 with center density larger than the center density $2^{97.40}$ of Mordell-Weil lattice could be constructed from Theorem 2.4.

*Proof.* Applying Theorem 2.4 to $A_{128}^{(4,131)}$ and the possible binary linear $[128, 59, 32]$ code or to $A_{128}^{(6,131)}$, we could get the new lattice. □

From table in [22] there exists a binary linear $[128, 43, 32]$ code, we have a dimension 128 lattice with center density $2^{83.1784}$. The present upper bound for the minimum distance of linear binary $[128, 59]$ code is 32, but people do not know whether this code exists or not.

**Proposition 5.3.** If there was one of the binary linear $[256, 99, 64]$ code, $[256, 74, 80]$ code, $[256, 136, 48]$ code and $[256, 56, 96]$ code, then a new 256 dimension lattice with center density larger than the center density $2^{294.8}$ of Mordell-Weil lattice could be constructed from Theorem 2.4.

*Proof.* Applying Theorem 2.4 to analogous Craig lattice $A_{256}^{(m,257)}$, where $m = 8, 10, 6, 12$, and the possible binary linear code we could get the lattice. □

From table in [22] the present upper bound for the minimum Hamming weight of a binary linear $[256, 99]$ code is 74 and a binary linear $[256, 99, 48]$ code exists, this lead to a lattice with center density $2^{257.8492}$. We do not know whether a linear binary $[256, 99, 64]$ code exists or not. For the remaining cases, we can find binary linear $[256, 74, d_{74} = 56]$ code exists and the present upper bound for $d_{74}$ is 85, people do not know whether a binary linear $[256, 74, 80]$ code exists or not. Similarly $32 \leq d_{136} \leq 54$, people do not know whether a binary linear $[256, 136, 48]$ code exists or not. $68 \leq d_{56} \leq 96$, people do not know whether a binary linear $[256, 56, 96]$ code exists or not.

For the dimension 508 case we have the following result.

**Proposition 5.4.** If there was one of the linear $[254, 26, 112]$ code over $F_2$, $[254, 36, 104]$ code over $F_2$, $[254, 47, 96]$ code over $F_2$, $[254, 78, 80]$ code over $F_2$, $[169, 18, 104]$ code over $F_4$, $[169, 24, 96]$ code over $F_4$, $[169, 39, 80]$ code over $F_4$, $[127, 16, 96]$ code over $F_8$, $[127, 26, 80]$ code over $F_8$, $[127, 42, 64]$ code over $F_8$, then we could construct a new lattice in dimension 508 with center density larger than the center density $2^{745.62}$ of Mordell-Weil lattice in this dimension (see [43] p. 934).

*Proof.* If there was a linear $[169, 24, 96]$ code over $F_4$, a linear binary $[507, 48, 192]$ code could be constructed as the concatenated code with the binary linear $[3, 2, 2]$ code. From Theorem 2.4 we could have a new lattice with center density $2^{48}\delta(A_{508}^{(24,509)}) \approx 2^{48+699.2897} = 2^{747.2897}$. The other cases can be checked similarly. □
From the table \[22\] a linear \([169, 24, 85]\) code over \(\mathbb{F}_4\) has been constructed and the present upper bound for the linear \([169, 24, d_{24}]\) code over \(\mathbb{F}_4\) is 104. People do not know whether a \([169, 24, 96]\) code over \(\mathbb{F}_4\) exists or not.

If there was one of the binary linear \([512, 353, 64]\) code, \([512, 289, 80]\) code, \([512, 441, 48]\) code, \([512, 239, 96]\) code and \([512, 170, 128]\) code, then a new lattice of rank 512 with center density larger than \(2^{797.12}\) could be constructed from Theorem 2.4. We can apply Theorem 2.4 to the analogous Craig lattice \(A_{1024}^{(m,1031)}\), where \(m = 10, 12, 16, 24, 30, 32\), and the following possible codes we could get the possible new lattice. If there was one of the binary linear \([1024, 925, 80]\) code, \([1024, 810, 96]\) code, \([1024, 598, 128]\) code, \([1024, 419, 192]\) code, \([1024, 314, 240]\) code and \([1024, 286, 256]\) code, then a new dense lattice of rank 1024 with center density larger than \(2^{2018.2}\) could be constructed from Theorem 2.4. In the case of of dimension 2048, if there was one of the binary linear \([2048, 1479, 192]\) code, \([2048, 1216, 240]\) code, \([2048, 1142, 256]\) code, \([2048, 719, 384]\) code, \([2048, 522, 480]\) and \([2048, 471, 512]\) code, then a new dense lattice of rank 2048 with center density larger than \(2^{4891}\) could be constructed from Theorem 2.4. In the case of dimension 4096, applying Theorem 2.4 to the analogous Craig lattice \(A_{4096}^{(m,4099)}\), if there was one of the binary linear \([4096, 2708, 384]\) code, \([4096, 2192, 480]\) code, \([4096, 2050, 512]\) code, \([4096, 865, 960]\) and \([4096, 770, 1024]\) code, then a new lattice of rank 4096 with center density larger than \(2^{11527}\) could be constructed from Theorem 2.4.

For many lengths in the range 128 – 256 our knowledge about the linear codes over \(\mathbb{F}_2, \mathbb{F}_4, \mathbb{F}_8\) is not sufficient to determine whether these nice codes in the table in \[22\] can be constructed or not. The motivation for the past works on long binary codes are mainly from the construction of efficient McEliece public key cryptosystem(for example, see \([6]\)). This is the first time establishing an intimate relation between long linear binary codes and the known densest Mordell-Weil lattices in high dimensions. In the following table 4 some possible new lattices denser than Mordell-Weil lattices are listed.
Table 4.

| dimension | possible – $\log_2 \delta$ | known – densest | condition |
|-----------|-----------------------------|-----------------|-----------|
| 128       | 98.3831                     | 97.40(MW)       | [128, 59, 32] |
| 140       | 114.6656                    | 113.31(MW)      | [140, 69, 32] |
| 164       | 148.1570                    | 147.3318(MW)    | [164, 92, 32] |
| 176       | 147.3596                    | 147.3318(MW)    | [164, 58, 48] |
| 176       | 165.8067                    | 165.1474(MW)    | [176, 104, 32] |
| 176       | 166.3191                    | 165.1474(MW)    | [176, 68, 48] |
| 200       | 194.9761                    | 194.2188(MW)    | [200, 122, 32] |
| 200       | 195.0917                    | 194.2188(MW)    | [200, 53, 64] |
| 212       | 221.4932                    | 221.4145(MW)    | [212, 68, 64] |
| 224       | 241.3005                    | 241.0012(MW)    | [224, 76, 64] |
| 256       | 294.958                     | 294.8(MW)       | [256, 99, 64] |
| 256       | 295.15                      | 294.8(MW)       | [256, 74, 80] |
| 256       | 294.8492                    | 294.8(MW)       | [256, 136, 48] |
| 256       | 294.8156                    | 294.8(MW)       | [256, 56, 96] |
| 272       | 323.1472                    | 323.0536(MW)    | [272, 112, 64] |
| 380       | 525.4662                    | 525.1006(MW)    | [380, 133, 96] |
| 452       | 671.0404                    | 670.4412(MW)    | [452, 130, 128] |
| 508       | 747.2897                    | 745.62(MW)      | [508, 48, 192] |
| 512       | 797.3117                    | 797.12(MW)      | [512, 353, 64] |
| 692       | 1200.4738                   | 1199.8554(MW)   | [692, 309, 128] |
| 716       | 1260.9065                   | 1260.7960(MW)   | [716, 331, 128] |
| 1024      | 2018.2944                   | 2018.2(MW)      | [1024, 286, 256] |
| 1436      | 3112.5083                   | 3111.8561       | [1436, 571, 256] |
| 2048      | 4891.9666                   | 4891(MW)        | [2048, 471, 512] |
| 4096      | 11527.8215                  | 11527(MW)       | [4096, 770, 1024] |
6. Lattice sphere packings denser than Mordell-Weil lattices

**Theorem 6.1.** There exists a binary linear \([4096, 772, 1024]\) code. We can construct a lattice sphere packing in dimension 4096 with center density at least \(2^{11529}\), which is denser than the Mordell-Weil lattice in this dimension.

**Proof.** From Gilbert-Varshamov bound if \(V(4096, 1023) = \sum_{i=0}^{1023} \binom{4096}{i} < 2^{4097-k}\), the linear binary \([4096, k, 1024]\) code exists\([16]\). From the inequality \(\sum_{i=0}^{r} \binom{n}{i} < 2^{nH(r/n)} < 2^{H(\frac{1}{4})n}\), where \(H(x)\) is the binary entropy function\([16]\ p.21), then \(V(4096, 1023) < 3324\). We get the conclusion. \(\square\)

**Lemma 6.2.** The linear binary code of length \(8n\), dimension \([6\log_2 3 - 8)\ n]\) and minimum distance \(2n\) exists.

**Proof.** Let \(V(n, r) = \sum_{i=0}^{r} \binom{n}{i}\). From the Gilbert-Varshamov bound\([16]\), if \(V(n, d-1) < 2^{n-k+1}\), then a linear binary code with parameter \([n, k, d]\) exists. From Theorem 1.4.5 of \([16]\ page 21\) \(V(8n, 2n-1) < 2^{8H(\frac{1}{4})n}\). The conclusion follows directly. \(\square\)

Actually we have the following result.

**Theorem 6.3.** Let \(p\) be a prime number satisfying \(p \equiv 5 \mod 6\) and \(1667 \leq p \leq 2039\). We can construct new lattice sphere packing in dimension \(n = 2p-2\) with center density better than the \(\frac{(p+1)/12)^{p-1}}{p^{p-5/6}}\) of the corresponding Mordell-Weil lattice in dimension \(n\).

**Proof.** The center density of Mordell-Weil lattice in dimension \(2p-2\) is \(\frac{(p+1)/12)^{p-1}}{p^{p-5/6}}\). The analogous Craig lattice \(A_{2p-2}(\frac{p-1}{p})^{2p+t}\), where \(t\) is a positive integer such that \(2p+t\) is a prime number, has its center density \(\frac{(p-1)^{p-1}}{(2p+t)^{p-1}}\). It can be checked that \((2p+t) < 2^{1.001}p\). On the other hand there exists a \([2p-2, [0.3776(p-1)], \frac{p-1}{2}]\) linear binary code, the conclusion follows directly. \(\square\)

**Remark 6.1.** The construction of the wanted linear binary codes in Theorem 6.3 is proved from the Gilbert-Varshamov bound. From the point view of coding theory, it seems quite possible that better linear binary codes exist. If that was true, the method in this paper would give lattice sphere packings in more dimensions better than Mordell-Weil lattices. It would be desirable that the codes in Theorem 6.1 and Theorem 6.3 could be constructed explicitly.
In the following table 5 we list some new lattices denser than Mordell-Weil lattices.

### Table 5.

| dimension | new − log₂δ | known − densest |
|-----------|-------------|-----------------|
| 3332      | 8913        | 8897.0184(Moedell − Weil) |
| 3956      | 11035       | 10969.9654(Mordell − Weil) |
| 3992      | 11159       | 11099.6432(Mordell − Weil) |
| 4004      | 11208       | 11130.5560(Mordell − Weil) |
| 4052      | 11370       | 11294.2234(Mordell − Weil) |
| 4076      | 11455       | 11375.6625(Mordell − Weil) |
| 4096      | 11529       | 11527(Mordell − Weil) |
7. NEW DENSER LATTICES OF HIGH DIMENSIONS IN THE RANGE 4098-8640

The known densest sphere packing in dimension 4098 is the Craig lattice $A_{4098}^{(246,4099)}$ with center density $2^{11279}$ ([11], page 17, Table 1.3). By using the Craig lattice $A_{4098}^{(128,4099)}$ (4099 is a prime, [19]) and the linear binary $[4098, 773, 1024]$ code from Lemma 6.2, we can construct a new dense lattice of rank 4098 with center density at least $2^{11536}$. In dimension 4104 the lattice $\eta(A_{24})$ has the center density $2^{11400}$ ([11], page 242, Table 8.7). By using the analogous Craig lattice $A_{4104}^{(128,4111)}$ (4111 is a prime number) with center density at least $2^{10780}$ and the linear binary $[4104, 774, 1026]$ code, we can construct a new dense lattice of rank 4104 with center density at least $2^{11554}$. In dimension $4124 = 2^{2} \cdot 2063$ where $p = 2063$ is a prime satisfying $2063 \equiv 5 \mod 6$, the Mordell-Weil lattice sphere packing has the center density $\frac{172^{2062}}{2063^{2063}} \approx 2^{11537.183}$. The analogous Craig lattice $A_{4124}^{(128,4127)}$ (4127 is a prime number) has center density at least $2^{10840}$. From Lemma 6.2 there exists a linear binary $[4124, 778, 1031]$ code, we can construct a new denser lattice sphere packing with center density $2^{11618}$.

We have the following result.

**Theorem 7.1.** For each dimension $N = 24n$ in the range 4104 – 8640, we can construct a new lattice sphere packing which is denser than the rank $N$ child lattice of the Leech lattice $\eta(A_{24})$.

**Proof.** We take the analogous Craig lattice $A_{24n}^{([\frac{24n}{l}],l)}$, where $l$ is smallest prime number bigger than or equal to $24n + 1$. From Lemma 6.2, there exists a linear binary $[24n, [4.5312n], 6n]$ code. Then we can construct a lattice with center density $2^{[4.5312n]} \cdot \frac{(\frac{12n}{l})^{12n}}{\frac{l^{2n/4}}{2}}$. The conclusion follows from the direct computation \(\Box\)

**Remark 7.1.** It would be desirable if the codes (the existence is proved by Lemma 6.2) used in Theorem 7.1 could be constructed explicitly.

In dimension 16392, the known densest sphere packing is the child of the Leech lattice by Bos-Conway-Sloane construction with center density $2^{615168}$ ([11], page 17). We can have a lattice sphere packing in dimension 16392 with center density $2^{61497}$ from Theorem 2.4 and Lemma 6.2, which is less than the center density of the above known lattice sphere packing in dimension 16392.

Here is the table 6 of new lattices of dimensions in the range 4098 – 16380, which are better than the records in [11], page 17, Table 1.3.
8. New dense lattice sphere packings of odd dimensions

In many previous constructions of dense lattice sphere packing in the Euclid space of dimension \( n \), \( n \) is always assumed to be an even number. Craig lattice is of dimension \( p - 1 \), where \( p \) is a prime number. The Mordell-Weil lattices and the children of the Leech lattice in the Bos-Conway-Sloane construction are of even dimensions. Our construction Theorem 2.2 and 2.4 can be used for each dimension and the center density of the constructed lattice sphere packing depends "continuously" on the dimension. Thus we can have quite good lattice sphere packings in the "missing" or odd dimensions. In the following table 7, new dense lattice sphere packings in moderate dimensions 149 −159 and 183 −193 are listed. It should be noted that in these dimensions, no previous constructions like Craig, Bos-Conway-Sloane and Mordell-Weil can be applied, except in dimension 150, 156, 190 and 192, there are known Craig lattices and their refinements(20, 11, page 17). In the dimensions 150, 156, 190 and 192, the refinements of Craig lattices in 20 are slightly denser.

Remark 8.1. We observe that the above lattice sphere packings are much better than the lattice sphere packings from Minkowski-Hlawka Theorem. New dense lattice sphere packings can also be constructed from Theorem 2.4 for dimensions \( n = 241, \ldots, 251 \).

In table 8 we list some new lattice sphere packings in the dimension \( p, p - 2 \) and \( p + 2 \) where \( p \) is a prime number. The center density is quite close to the center density of the Craig lattice in the dimension \( p - 1 \) (11, page 17, Table 1.3). In table 9 some new lattice sphere packings in the dimensions \( 2p - 3 \), where \( p \) is a prime number satisfying \( p \equiv 5 \mod 6 \), are listed. We also list some new lattice sphere packings in dimensions \( 24n' - 1 \) in table 10.
### Table 7.

| dimension | new \(-\log_2 \delta\) | dimension | new \(-\log_2 \delta\) |
|-----------|---------------------|-----------|---------------------|
| 149       | 112.3048            | 183       | 158.4505            |
| 150       | 114.06 (<18>)       | 184       | 160.0355            |
| 151       | 113.7424            | 185       | 161.6205            |
| 152       | 115.2811            | 186       | 163.2055            |
| 153       | 116.8248            | 187       | 164.7905            |
| 154       | 118.3685            | 188       | 166.3755            |
| 155       | 119.9122            | 189       | 167.9605            |
| 156       | 121.4559 (<18>)     | 190       | 169.5455 (<18>)     |
| 157       | 122.1067            | 191       | 171.1305            |
| 158       | 123.6504            | 192       | 172.44 (<18>)       |
| 159       | 125.1941            | 193       | 173.5188            |

### Table 8.

| dimension | \(\log_2 \delta\) | dimension | \(\log_2 \delta\) | dimension | \(\log_2 \delta\) |
|-----------|------------------|-----------|------------------|-----------|------------------|
| 87        | 40.4835          | 89        | 42.5005          | 91        | 43.9503          |
| 149       | 112.3048         | 151       | 115.0103         | 153       | 117.4377         |
| 179       | 153.5829         | 181       | 155.3909         | 183       | 158.4792         |
| 189       | 167.7417         | 191       | 170.5800         | 193       | 173.1791         |
| 507       | 741.1263         | 509       | 744.4672         | 511       | 748.8247         |

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New lattices sphere packings denser than Mordell-Weil lattices

Table 9.

| dimension | new - $\log_2 \delta$ | known - densest (dim = n + 1) |
|-----------|----------------------|--------------------------------|
| 3331      | 8910.1498            | 8897.0184 (Mordell - Weil)    |
| 3955      | 11032.7450           | 10969.9654 (Mordell - Weil)   |
| 3991      | 11156.0229           | 11099.6432 (Mordell - Weil)   |
| 4003      | 11212.0171           | 11130.5560 (Mordell - Weil)   |
| 4051      | 11365                | 11294.2234 (Mordell - Weil)   |
| 4075      | 11452                | 11375.6625 (Mordell - Weil)   |
| 4095      | 11526                | 11527 (Mordell - Weil)        |

Table 10.

| dimension | new - $\log_2 \delta$ | old - record (dim = n + 1) |
|-----------|------------------------|-----------------------------|
| 4097      | 11533                  | 11279 (Craig)               |
| 4103      | 11551                  | 11400 (\eta(\mathbf{A}_{24})) |
| 4123      | 11615                  | 11537.1837 (Mordell - Weil) |
| 8183      | 26819                  | 26712 (\eta(\mathbf{A}_{24})) |
| 8189      | 26911                  | 26154 (Craig)               |
| 8207      | 26949                  | 26808 (\eta(\mathbf{A}_{24})) |
| 16379     | 61415                  | 59617 (Craig)               |

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