**Genuinely Multipartite Entanglement vias Quantum Communication**

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Multiparticle entanglement is of important resources for quantum communication and quantum computation. Our goal in this paper is to characterize general multipartite entanglement according to quantum biseparable channels. We firstly prove a no-go result for fully genuinely multipartite entanglement on finite-dimensional spaces with the biseparable channel model. We further propose a semi-device-independent entanglement model depending on the connection ability in quantum circuit. This implies a complete hierarchy of genuinely multipartite entanglements. It also provides a completely different multipartite nonlocality from quantum network entanglements. These results show new insights for multipartite entanglement, quantum network, and measurement-based quantum computation.

**Introduction.** Entanglement is an important quantum property of two or more systems in quantum mechanics associated with Schrödinger evolution equations [1]. There is only one bipartite entanglement model on finite-dimensional spaces for characterizing nonlocal correlations, that is, it cannot be decomposed into an ensemble of separable states. This allows to witness any bipartite entanglement beyond all separable states using entanglement witness [2]. Another device-independent method is inspired by Einstein-Podolsky-Rosen (EPR) steering [3] or Bell inequality [4, 5] which can witness stronger quantum nonlocality of entanglement [7].

Although the fully separable model is easily extended for multi-particle systems, it cannot justify the genuinely multipartite nonlocality. Instead, different models are constructed for special multipartite goals. One is the so-called biseparable model [8] which can verify the genuinely multipartite entanglement beyond biseparable states. This allows to witness stronger nonlocality than its from a fully separable model. If the local tensor decomposition is considered, a high-dimensional model [9–11] or quantum network entanglement rules out any network separable state consisting of small entanglements which are shared by partial parties [12–14]. Different from the biseparable model [8] the new model provides a device-independent verification of unknown entanglement devices. Another is from the particle-losing model for featuring the entanglement robustness of local systems [15–17]. It can follow a different hierarchy of well-known entanglements including GHZ state and Dicke states beyond other models [8, 12–14]. All of these entanglement models can only justify special multipartite systems. A natural problem is to explore new model for general many-body systems.

In general, a bipartite entanglement can be generated by quantum communication model, as shown in Fig.1(a). Each separable state can be obtained by passing a product state through quantum separable channel [18]. This suggests a novel model for multipartite entanglement by ruling out any outputs of fully separable quantum channels. A natural extension is to explore biseparable quantum channel [8] in multipartite scenarios, as shown in Fig.1(b). It can be further regarded as an adversarial model in cryptographic applications of quantum secret sharing [19], as shown in Fig.1(c). This intrigues a new way for exploring many-body systems.

Our goal in this work is to characterize multipartite entanglement in terms of biseparable quantum channels. We first propose a bipartite-generation model to characterize all the states generated by passing proper biseparable state through biseparable channel. This implies the first no-go result for fully genuinely multipartite entanglement. It further implies a complete hierarchy for characterizing any multipartite entanglement according
to the connection ability in generation circuits. The present model provides a simple standard to verify general multipartite entanglement using Schmidt numbers of reduced density matrices. This shows a complete different multipartite nonlocality from previous models [12–14, 17].

**MAIN RESULTS**

**Fully genuinely multipartite entanglement.** A general isolated d-dimensional quantum system is represented by a normalized vector $|\phi\rangle$ in Hilbert space $\mathcal{H}_d$. Instead, an open system is described by probabilistically mixing an ensemble of pure states $\{|\phi_i\rangle\}$, that is, $\rho = \sum p_i |\phi_i\rangle \langle \phi_i|$, where $\{p_i\}$ is a probability distribution. An n-particle pure state $|\Phi\rangle$ on Hilbert space $\otimes_{i=1}^n \mathcal{H}_{A_i}$ is biseparable $[8]$ if $|\Phi\rangle = |\phi\rangle \otimes |\psi\rangle$ with pure states $|\phi\rangle$ and $|\psi\rangle$, where $I$ and $\bar{I}$ are bipartitions of $\{A_1, \ldots, A_n\}$. Here, $|\phi\rangle$ can be generated from a fully separable state $|0\rangle^\otimes n$ passing through a biseparable quantum channel $\mathcal{E}(\cdot) := U_I \otimes V_{\bar{I}}(\cdot) U_I^\dagger \otimes V_{\bar{I}}^\dagger$, that is, $|\Phi\rangle = (U_I \otimes V_{\bar{I}})|0\rangle^\otimes n$, where $U_I(\otimes_{A_i \in I} |0\rangle_{A_i}) = |\phi\rangle_I$ and $V_{\bar{I}}(\otimes_{A_i \in \bar{I}} |0\rangle_{A_i}) = |\psi\rangle_{\bar{I}}$. This intrigues a multipartite entanglement model. Define a biseparable complete positive trace-preserving (BCPTP) channel [8, 18] on Hilbert space $\mathcal{H}_I \otimes \mathcal{H}_{\bar{I}}$ as

$$\mathcal{E}_I(\rho) = \sum_i (K_i \otimes S_i) \rho (K_i^\dagger \otimes S_i^\dagger)$$

(1)

where $K_i, S_i$ are respective Kraus operators on Hilbert space $\mathcal{H}_I$ and $\mathcal{H}_{\bar{I}}$ and satisfy $\sum_i K_i^\dagger K_i \otimes S_i S_i^\dagger = \mathbb{I}$ with the identity operator $\mathbb{I}$, $\mathcal{H}_I = \otimes_{A_i \in I} \mathcal{H}_{A_i}$, and $\mathcal{H}_{\bar{I}} = \otimes_{A_i \in \bar{I}} \mathcal{H}_{A_i}$. It is forward to show any biseparable state $\rho_{bs} = \sum p_i |\phi_i\rangle \otimes |\psi_i\rangle$ over a given bipartition $I$ and $\bar{I}$ can be represented by using BCPTP channel as $\rho_{bs} = \mathcal{E}_I(|0\rangle \langle 0|^\otimes n)$, where $K_i$ is defined by $|0\rangle \mapsto |\phi_i\rangle_I$, and $S_i$ is defined by $|0\rangle \mapsto |\psi_i\rangle_{\bar{I}}$. This means that any biseparable state $\rho_{bs}$ can be generated by a mixture of BCPTP channels, that is,

$$\rho_{bs} = \sum_I q_I \mathcal{E}_I(|0\rangle \langle 0|^\otimes n)$$

(2)

where the summation is over any proper subset of $\{A_1, \ldots, A_n\}$. Thus BCPTP channel provides an equivalent representation of the biseparable model [8]. Instead of fully separable states, in what follows we consider any biseparable state as the input of BCPTP channel.

**Definition 1.** An n-partite state is fully genuinely-generation entanglement if it is not bipartite-generation state given by

$$\rho_{bg} = \sum_I p_I \mathcal{E}_I(\rho_{bs})$$

(3)

where $I \subset \{A_1, \ldots, A_n\}$, $\mathcal{E}_I$ are BCPTP channels in terms of the bipartition $I$ and $\bar{I}$, $\{p_I\}$ is a probability distribution, and $\rho_{bs}$ is any biseparable state.

In Definition 1, $\rho_{bs}$ may be entanglement for different bipartitions $I$ and $\bar{I}$. This is different from the standard biseparable model input a fully separable state in Eq.(2). For any genuinely n-partite entanglement [8], assume that the Schmidt decomposition respect to the bipartition $I = \{A_1\}$ and $\bar{I} = \{A_2, \ldots, A_n\}$ is given by

$$|\Phi\rangle = \sum_{i=1}^d \sqrt{\lambda_i} |\phi_i\rangle_I |\psi_i\rangle_{\bar{I}}$$

(4)

where $\lambda_i$ are Schmidt coefficients satisfying $\sum_{i=1}^d \lambda_i = 1$, $\{|\phi_i\rangle_I\}$ are orthogonal states of the particle $A_1$, and $\{|\psi_i\rangle_{\bar{I}}\}$ are orthogonal states of the particles in $\bar{I}$. There is a unitary transformation $U$ on Hilbert space $\mathcal{H}_I$ satisfying (Appendix A):

$$U : |\psi_i\rangle_{\bar{I}} \mapsto |0\rangle_{A_2} |\psi_i\rangle_{J, i = 1, \ldots, d}$$

(5)

where $\{|\psi_i\rangle_{J}\}$ are orthogonal states of the particles in $J := \{A_3, \ldots, A_n\}$. Thus $|\Phi\rangle$ is a bipartite-generation state from Definition 1. This implies a general no-go result for any finite-dimensional state.

**Result 1.** There is no fully genuinely-generation multipartite entanglement on any finite-dimensional Hilbert space.

![FIG. 2. Schematic quantum circuit for generating a general state.](attachment:image1)

(a) Standard quantum circuit model with different depths. Here, $U_i$ are two-particle universal logical gates [20, 21]. (b) The present circuit model with two-depth. There are two BCPTP channels $\sum_I p_I \mathcal{E}_I(\cdot)$ and $\sum_J q_J \mathcal{E}_J(\cdot)$ over different bipartitions $I, \bar{I}$ and $J, \bar{J}$.

Result 1 holds for any pure or mixed state on finite dimensional Hilbert space. It means any state can be generated by passing a biseparable state through biseparable channels. Combing with Eqs.(1) and (2), Result 1 implies any state $\rho$ on Hilbert space $\otimes_{i=1}^n \mathcal{H}_{A_i}$ has the following decomposition

$$\rho = \sum_{I,J} p_{IJ} \mathcal{E}_I(\rho_{bs}) \circ \mathcal{E}_J(|0\rangle \langle 0|^\otimes n)$$

(6)

where $\{\mathcal{E}_I, \forall I\}$ and $\{\mathcal{E}_J, \forall J\}$ are BCPTP channels, and $\{p_{IJ}\}$ and $\{q_{IJ}\}$ are probability distributions. Eq.(6) implies a universal circuit with two-depth for building any state from a fully product state, as shown in Fig.2. The present model is stronger than the standard quantum
circuit model with unfixed depth [20, 21]. The present biseparable quantum channels may activate entanglement swapping is the core of quantum networks [22, 23].

Partial-connection multipartite entanglement. The fully genuinely-generation multipartite entanglement requires ruling out all biseparable states and biseparable quantum channels. This may be too strong both in theory and applications. Instead of partial joint operations in Definition 1, we consider a one-side biseparable channel \( E_I \), where joint operations may be performed on \( I \) while fully local operations are on \( T \). This can be regarded as a semi-device-independent scenario as shown in Fig.1(c), where partial adversaries in \( I \) may perform joint operations while others are not allowed. Denote \( \ell_I \) as the number of particles in \( I \). Define a \( k \)-connection BCPTP channel \( \mathcal{E}_I^{(k)} \) on Hilbert space \( \mathcal{H}_I \otimes \mathcal{H}_T \) as

\[
\mathcal{E}_I^{(k)}(\rho) = \sum_i (K_i \otimes (\otimes_j S_{ij})) \rho (K_i^\dagger \otimes (\otimes_j S_{ij}^\dagger))
\]

where \( K_i \) and \( S_{ij} \) are respective Kraus operators on Hilbert space \( \mathcal{H}_I \) and \( \mathcal{H}_{A_j} \), and satisfy \( \sum_i (K_i^\dagger \otimes (\otimes_j S_{ij}^\dagger)) = \mathbb{1} \). The present \( k \)-connection BCPTP channel is of state-dependent. Our goal here is for featuring genuinely multipartite entanglement with the present quantum channel (7).

**Definition 2.** An \( n \)-partite state is \( k \)-connection genuinely entanglement (\( k \)-CGE) if it is not a \( k \)-connection biseparable state given by

\[
\rho_{bg}^{(k)} = \sum_{\ell_I \leq k} p_{\ell_I} \mathcal{E}_I^{(\ell_I)}(\rho_{bs})
\]

where \( \mathcal{E}_I^{(\ell_I)} \) are \( \ell_I \)-connection BCPTP channels in terms of bipartition \( I \) and \( T \), \( \{p_I\} \) is a probability distribution, and \( \rho_{bs} \) is any biseparable state.

**FIG. 3.** Schematic hierarchy of genuinely \( n \)-partite entanglement. The set consisting of \( k \)-CGE is included in the set consisting of \( k-1 \)-CGE. The largest set is consisting of \( 1 \)-CGE while the smallest set contains \( k_{max} \)-CGE with \( k_{max} = \lfloor \frac{n-1}{2} \rfloor \), where \( \lfloor x \rfloor \) denotes the maximal integer no more than \( x \).

Similar to the proof of Theorem 1, the Schmidt decomposition of a given \( n \)-particle pure state \( |\Phi \rangle \) on \( d^n \)-dimensional Hilbert space \( \otimes_{i=1}^n \mathcal{H}_{A_i} \) is given by

\[
|\Phi \rangle = \sum_{i=1}^N \sqrt{\lambda_i} |\phi_i \rangle_I |\psi_i \rangle_T
\]

where \( \lambda_i \) are Schmidt coefficients satisfying \( \sum_{i=1}^N \lambda_i = 1 \), \( \{|\phi_i \rangle \} \) are orthogonal states of the particles in \( I \), and \( \{|\psi_i \rangle \} \) are orthogonal states of the particles in \( T \). For any integer \( \ell_I \) with \( \ell_I \geq n/2 + 1 \), it follows \( N \leq d^{n-\ell_I} < d^{n/2-1} \). There is a unitary transformation \( U \) on Hilbert space \( \mathcal{H}_I \) satisfying

\[
U : |\psi_i \rangle_I \mapsto |0 \rangle_{A_j} |\psi_i \rangle_I, i = 1, \cdots, N
\]

where \( A_j \in T \) and \( \{|\psi_i \rangle \} \) are orthogonal states of the particles in \( J := T - \{A_j\} \). So, the state \( |\Phi \rangle \) is an \( \ell_I \)-connection biseparable. This implies another no-go result as follows.

**Result 2.** There is no \( k \)-CGE on finite-dimensional space for any integer \( k \geq n/2 + 1 \).

Result 2 rules out the possibility of \( k \)-CGE for large \( k \). The situation is different for small \( k \). For special case of \( n = 2 \), Definition 2 is reduced to the standard separable model of bipartite systems [1, 2]. For each \( n \geq 3 \), from Definition 2 any biseparable state [8] is 1-connection biseparable state. A further fact is that any \( k \)-connection biseparable state is \( s \)-connection biseparable state for any \( s \geq k \). This implies a complete hierarchy for all the multipartite entanglements, as shown in Fig.3. The largest set contains 1-CGEs, that is, the genuinely multipartite entanglement in the biseparable model [8]. Instead, the smallest set consists of the strongest multipartite entanglement, that is, \( k_{max} \)-CGE with \( k_{max} = \lfloor \frac{n}{2} \rfloor \).

For any \( n \)-particle state \( |\Phi \rangle \) on Hilbert space \( \otimes_{i=1}^n \mathcal{H}_{A_i} \), from Eq.(9) the orthogonality of \( \{|\phi_i \rangle \} \) allows for constructing a unitary transformation (10) if and only if the Schmidt number satisfies \( N \geq d^{k-1} \) with \( \ell_I = k \). This implies a directive way to verify a given pure \( k \)-CGE by using the Schmidt number of reduced density matrices (Appendix B).

**Result 3.** An \( n \)-partite pure state on \( d^n \)-dimensional Hilbert space \( \otimes_{i=1}^n \mathcal{H}_{A_i} \) is \( k \)-CGE if and only if the Schmidt number of the reduced density matrix of any \( k \) particles is larger than \( d^{k-1} \).

From Definition 2 any genuinely multipartite entanglement in the biseparable model [8] is 1-CGE. Moreover, the present \( k \)-CGE is stronger than robust entanglement with the robustness-depth \( k \) since the particle-losing channel [17] is local CPTP channel. From Result 3 it generally requires to evaluate Schmidt numbers of almost all the reduced density matrices. This yields to a NP hard problem for general entanglement because of exponential number of reduced states. Instead, it is easy for some special states.

One example is \( n \)-partite Greenberger-Horne-Zeilinger (GHZ) states [24] given by

\[
|GHZ \rangle = \sum_{i=1}^d a_i |i \cdots i \rangle_{A_1 \cdots A_n}
\]

where \( a_i \) satisfy \( \sum_{i=1}^d a_i^2 = 1 \). It easy to verify the state (11) is 1-CGE from its symmetry. Moreover, for \( n \)-partite
W-type states [29]:

$$|W\rangle_{A_1\ldots A_n} = \sum_{i=1}^{d} a_i |1_i\rangle + a_{n+1} |1\ldots 1\rangle$$  \hspace{1cm} (12)$$

from Result 3 it is 2-CGE for \( a_i \neq 0, i = 1, \ldots, n + 1 \), where \( |1_i\rangle \) denotes the \( i \)-excitation defined by \(|1_i\rangle = |0\rangle^{i-1}\langle 1|0\rangle^{n-i}\). This can be extended for \( n \)-qudit Dicke state with \( s \) excitations [30] given by

$$|D_{s,n}\rangle = \frac{1}{\sqrt{L_s}} \sum_{i_1, \ldots, i_n = s} |i_1 \cdots i_n\rangle_{A_1\ldots A_n}$$  \hspace{1cm} (13)$$

where \( L_s \) is the normalization constant. It is \( k \)-CGE with \( k = \lfloor \log_d(s + 1) \rfloor + 1 \) (Appendix C). Moreover, \( |D_{s,n}\rangle \) is equivalent to \( |D_{((d-1)^{n-s,n})}\rangle \) under local unitary operations. This implies the strongest nonlocality of Dicke state with \( s = \lfloor (d-1)^n/2 \rfloor \).

Another entangled quantum networks which show different nonlocalities beyond single entanglement [27, 28]. One is resource states for quantum computation [25]. The so-called cluster states [25] consist of any generalized Einstein-Podolsky-Rosen (EPR) states [3] as 1-CGE (Appendix D). Similar result holds for graph states [26] consisting of generalized EPR [3] and GHZ states [24]. While some quantum networks show different generation abilities. One example is \( n \)-partite complete-connected network \( \mathcal{N}_c \) where each pair shares one bipartite entanglement \(|\psi_{ij}\rangle\). Recent result shows the joint state of any \( k \)-partite subnetwork is entangled for \( k \geq 2 \) [8, 17]. Hence, from Results 2 and 3 the joint state of \( \mathcal{N}_c \) is \( k_{\text{max}} \)-CGE with \( k_{\text{max}} = \lfloor \frac{n}{2} \rfloor \). This means \( \mathcal{N}_c \) with any bipartite entanglement shows stronger nonlocality than GHZ state (11) and W state (12) for any \( n > 4 \) in the present model. This is different from the robustness-entanglement model [17] where both the W state and \( \mathcal{N}_c \) has the same robust-depth. Remarkably, it is converse to the recent result [12–14] which shows both GHZ and W states are stronger. This shows a surprising feature of the genuinely multipartite nonlocality beyond bipartite scenarios, that is, it is model-dependent. For general quantum network we provide a polynomial-time algorithm for estimating the upper bound of \( k \) (Appendix E).

Robustness of \( k \)-CGE. From Eq.(8) all the \( k \)-connection biseparable states consist of a convex set \( S_k \). This allows for verifying a general entanglement near to \( k \)-CGE \(|\Phi\rangle\) by using linear entanglement witness [2, 31] defined as

$$\mathcal{W}_{\Phi} = r \mathbb{I} - |\Phi\rangle\langle \Phi|$$  \hspace{1cm} (14)$$

where \( r = \max_{\rho_{cb} \in S_k} D(\rho_{cb}, |\Phi\rangle\langle \Phi|) \) for any distance function \( D(\cdot, \cdot) \) of states [2], and \( \mathbb{I} \) is the identity operator. From Result 3 the entanglement witness for genuinely multipartite entanglement [2, 8] is useful for verifying any 1-CGE. One example is the GHZ state (11) with \( r = \max\{a_1^2, \ldots, a_4^2\} \).

For the W-type state (12) with \( n = 4 \), it is easy to get \( r = \max\{1 - a_i^2, 1 - a_j^2, 1 \leq i < j \leq 4\} \). This implies a different entanglement witness for verifying 2-CGE beyond the genuinely multipartite entanglement (or 1-CGE) [2, 8] with \( r = \max\{a_i^2, a_j^2 + a_j^2, 1 \leq i < j \leq 4\} \). Both visibilities are \( v > \frac{\sqrt{3}}{2} \) for Werner state [18] \( \rho_v = v|W\rangle\langle W| + \frac{1-v}{4^n} \mathbb{I} \), as shown in Fig. 4.

Conclusions. Result 3 provides an efficient way for verifying special multipartite entanglements. This intrigues a natural problem to explore new way for general multipartite entanglements. While the entanglement is the weakest nonlocality of quantum states, one may explore new hierarchies in terms of the so-called multipartite steering [32, 33] or Bell nonlocality [4, 24]. This is of special importance for recovering novel multipartite nonlocality beyond bipartite scenarios. Additionally, the present model shows the first example which shows converse multipartite nonlocality to recent quantum network model [12–14]. This intrigues a basic problem to explore the intrinsic nonlocality or the most reasonable model for multipartite systems.

In conclusion, we have investigated genuinely-generation multipartite entanglement with the help of biseparable quantum channel. We proposed a bipartite-generation model to characterize all the multipartite state generated by using proper biseparable state. This has implied a no-go result for the strongest multipartite entanglement in the present model. We further defined a biseparable model with local-connection. The one-side local joint operation has provided a general standard for characterizing the connecting ability of multipartite entanglement. We obtained a simple hierarchy of multipartite entanglement in terms of the connection ability. The new entanglement witness is used to verify the entanglement robustness. These results should be interesting in
multipartite entanglement theory, quantum communication, and quantum computation.

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APPENDIX A. PROOF OF EQ.(5)

The Schmidt decomposition of any n-partite pure state $|\Phi\rangle$ on Hilbert space $\otimes_{i=1}^{n} \mathcal{H}_A_i$ is given by

$$|\Phi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |\phi_i\rangle_{A_1} |\psi_i\rangle_{A_2}...|\phi_i\rangle_{A_n}$$  \hspace{1cm} (15)
where $\lambda_i$ are Schmidt coefficients satisfying $\sum_{i=1}^{d} \lambda_i = 1$, $\{|\phi_i\rangle\}$ are orthogonal states of the particle $A_1$, and $\{|\psi_i\rangle\}$ are orthogonal states of the particles $A_2, \ldots, A_n$. For a given set of orthogonal states $\{|\psi_i\rangle\}$, $i=1, \ldots, d$, there exists an orthogonal basis $\{|\phi_j\rangle\}$ of Hilbert space $\otimes_{i=1}^{n} \mathcal{H}_{A_i}$. Similarly, for a given set of orthogonal states $\{|\tilde{\psi}_i\rangle\}$ of the particles $A_3, \ldots, A_n$, there exists an orthogonal basis $\{|\tilde{\phi}_j\rangle\}$ of Hilbert space $\otimes_{i=1}^{n} \mathcal{H}_{A_i}$, such that $|\tilde{\phi}_i\rangle = |0\rangle_{A_2} |\tilde{\psi}_1\rangle_{A_3 \ldots A_n}$ for $i=1, \ldots, d$. This can be obtained by extending the set $\{|0\rangle_{A_2} |\tilde{\psi}_1\rangle_{A_3 \ldots A_n}\}$ into an orthogonal basis on Hilbert space $\otimes_{i=1}^{n} \mathcal{H}_{A_i}$. With this basis, the following mapping on Hilbert space $\otimes_{i=1}^{n} \mathcal{H}_{A_i}$ as

$$ U : |\psi_j\rangle_{A_2 \ldots A_n} \mapsto |0\rangle_{A_2} |\tilde{\psi}_1\rangle_{A_3 \ldots A_n}, \quad i = 1, \ldots, d $$

$$ |\psi_j\rangle_{A_2 \ldots A_n} \mapsto |\tilde{\phi}_j\rangle_{A_2 \ldots A_n}, \quad i = d+1, \ldots, d^{n-1} \quad (16) $$

Note that $\{|\psi_i\rangle_{A_2 \ldots A_n}\}$ and $\{|\tilde{\phi}_j\rangle_{A_2 \ldots A_n}\}$ are orthogonal bases of Hilbert space $\otimes_{i=1}^{n} \mathcal{H}_{A_i}$. Hence, $U$ is a unitary transformation on Hilbert space $\otimes_{i=1}^{n} \mathcal{H}_{A_i}$. This has completed the proof of Eq. (5).

### APPENDIX B. PROOF OF RESULT 3

For a given $n$-partite pure state $|\Phi\rangle$ on Hilbert space $\otimes_{i=1}^{n} \mathcal{H}_{A_i}$, assume that the Schmidt decomposition is given by

$$ |\Phi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} |\phi_i\rangle_I |\psi_i\rangle_T \quad (17) $$

for each bipartition $I$ and $T$ of $\{A_1, \ldots, A_n\}$, where $\lambda_i$ are Schmidt coefficients satisfying $\sum_i \lambda_i = 1$. $\{|\phi_i\rangle_I\}$ are orthogonal states of particles in $I$, and $\{|\psi_i\rangle_T\}$ are orthogonal states of particles in $T$. Here, $N$ is the Schmidt number of the reduced density matrix $\rho_I = \sum_i \lambda_i^2 |\phi_i\rangle_I \langle \phi_i|$. If $N \leq d^{k-1}$ with $\ell_I = k$, from Result 2, there exists a unitary transformation $U$ on $k$ particles $A_1, \ldots, A_k$ such that $U|\Phi\rangle = |\hat{\Phi}\rangle |0\rangle_{A_{k+1}} \ldots |0\rangle_{A_n}$ for some $n-1$-particle state $|\hat{\Phi}\rangle$. This means that $|\Phi\rangle$ is $k$-connection biseparable state, that is, it is not $k$-CGE.

Moreover, $N > d^{k-1}$ with $\ell_I = k$, we show that $|\Phi\rangle$ is $k$-CGE. The proof is completed by contradiction. Assume that $|\Phi\rangle$ is not $k$-CGE, that is, $|\Phi\rangle$ is $k$-connection biseparable state. Hence, there exists a unitary operation $U$ on $k$ particles $A_1, \ldots, A_k$ such that $U|\Phi\rangle = |\Phi\rangle |0\rangle_{A_{k+1}}$. Let $I = \{A_1, \ldots, A_k\}$. From Eq. (17), we have

$$ |\hat{\Phi}\rangle |0\rangle_{A_{k+1}} = \sum_{i=1}^{N} \sqrt{\lambda_i} U|\phi_i\rangle_I |\psi_i\rangle_T $$

$$ = \sum_{i=1}^{N} \sqrt{\lambda_i} |\hat{\phi}_i\rangle_{I - \{A_{k+1}\}} |0\rangle_{A_{k+1}} |\psi_i\rangle_T \quad (18) $$

for some orthogonal states $\{|\hat{\phi}_i\rangle_{I - \{A_{k+1}\}}\}$ on the particles in $I - \{A_{k+1}\}$. However, the dimension of Hilbert space $\otimes_{x \in I - \{A_{k+1}\}} \mathcal{H}_{A_x}$ is $d^{k-1}$. So, there are at most $d^{k-1}$ orthogonal states. This is contradicted to the orthogonal state set $\{|\phi_i\rangle_{I - \{A_{k+1}\}}, j=1, \ldots, N\}$ with $N > d^{k-1}$. It means that $|\Phi\rangle$ is $k$-connection biseparable state. This has completed the proof.

### APPENDIX C. CLUSTER STATES

In this section, we firstly prove any cluster state generated by any generalized EPR states [3] is 1-CGE. And then, it will be extended for any graph state generated by any generalized EPR states [3] and generalized GHZ states [24].

Consider an $n$-partite cluster state $|C\rangle$ generated by generalized EPR states [3] $|\phi_1\rangle, \ldots, |\phi_n\rangle$, where $|\phi_i\rangle = \cos \theta_i |00\rangle + \sin \theta_i |11\rangle$ with $\theta_i \in (0, \frac{\pi}{2})$, $i = 1, \ldots, m$. Assume $|C\rangle$ is shared by $n$ parties $A_1, \ldots, A_n$. Each party $A_i$ can perform local controlled-phase operation $C_f(\theta_i)$ on the shared two qubits, where $C_f(\theta_i)$ is defined by

$$ C_f(\theta_i) = |00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| + e^{i\theta_i} |11\rangle \langle 11| \quad (19) $$

One party may perform different $C_f(\theta_i)$’s on each pair of shared qubits. Assume that each party $A_i$ shares $k_i$ EPR states with others. Here, all the $k_i$ qubits shared by the party $A_i$ is jointly combined into a particle on $2^{k_i}$-dimensional Hilbert space $\mathcal{H}_{A_i}$. Denote $\rho_i$ as the reduced density matrix of the party $A_i$. It is easy to obtain the Schmidt number of $\rho_i$ is $2^{k_i}$. Moreover, any local controlled-phase operation (19) does not change its Schmidt number. This has proved that $|C\rangle$ is 1-connection state. Moreover, it is genuinely multipartite entanglement [8, 14]. So, $|C\rangle$ is 1-CGE.

Now, consider an $n$-partite graph state $|G\rangle$ generated by generalized EPR states [3] $|\phi_1\rangle, \ldots, |\phi_m\rangle$ and generalized GHZ states [24] $|\psi_1\rangle, \ldots, |\psi_n\rangle$, where $|\psi_i\rangle$ are $k_i$-qubit GHZ states defined by $|\psi_i\rangle = \cos \vartheta_i |1\rangle \otimes |k_i\rangle + \sin \vartheta_i |0\rangle \otimes |k_i\rangle$ with $\vartheta_i \in (0, \frac{\pi}{2})$. Assume each party $A_i$ shares $s_i$ EPR states and $t_i$ GHZ states with others. Each party $A_i$ can perform local joint controlled-phase operation $CC_f(\theta_i)$ on the shared $u$ qubits, where $CC_f(\theta_i)$ is defined by

$$ CC_f(\theta_i) = \sum_{i_1, \ldots, i_u \neq i} |i_1 \ldots i_u\rangle \langle i_1 \ldots i_u| $$

$$ + e^{i\theta_i} |1\rangle \langle 1| \otimes u \quad (20) $$

All the $s_i + t_i$ qubits shared by the party $A_i$ is jointly combined into a particle in $2^{s_i + t_i}$-dimensional Hilbert space $\mathcal{H}_{A_i}$. Denote $\varrho_i$ as the reduced density matrix of the party $A_i$. It is easy to obtain the Schmidt number of $\varrho_i$ is $2^{s_i + t_i}$. Moreover, any local operation (20) does not change the Schmidt number. This has proved that $|G\rangle$ is 1-connection state. Moreover, it is genuinely multipartite entanglement [8, 14]. Hence, $|G\rangle$ is 1-CGE.
A further result for connection ability of general networks will be proved in Appendix E.

**APPENDIX D. CONNECTION ABILITY OF DICKE STATE**

In this section, we prove any $n$-qudit Dicke state (14) is $k$-CGE with $k = \lceil \log_2(n+1) \rceil + 1$. Specially, for any given bipartition $I = \{A_{j_1}, \cdots, A_{j_l} \}$ and $T$ with $l_T = k+1$, the Schmidt decomposition of $|D_{s,n} \rangle$ is given by

\[
|D_{s,n} \rangle = \sum_{i=0}^{L_I} \sqrt{\gamma_i} |\phi_i \rangle_I |\psi_i \rangle_T
\]

where $\{|\phi_i \rangle_I \}$ are orthogonal states of particles in $I$, $\{|\psi_i \rangle_T \}$ are orthogonal states of particles in $T$, $\gamma_i$ are Schmidt coefficients satisfying $\sum_i \gamma_i = 1$, and $L_I$ denotes the Schmidt number. In fact, $|\phi_i \rangle_I$ are generalized $\ell_T$-qudit Dicke states defined by

\[
|\phi_i \rangle_I = \sum \alpha_{j_1, \cdots, j_{l_T}}^{(i)} |j_1, \cdots, j_{l_T} \rangle_{A_{j_1} \cdots A_{j_{l_T}}}
\]

where $\alpha_{j_1, \cdots, j_{l_T}}^{(i)}$ depend on proper coefficients of $\gamma_i$'s and satisfy $\sum_{j_1, \cdots, j_{l_T}} = (\alpha_{j_1, \cdots, j_{l_T}}^{(i)})^2 = 1$. Similarly, $|\psi_i \rangle_T$ are generalized $n-\ell_T$-qudit Dicke states defined by

\[
|\psi_i \rangle_T = \sum \beta_{r_2, \cdots, r_{n-\ell_T}}^{(i)} |r_1, \cdots, r_{n-\ell_T} \rangle_{A_{r_2} \cdots A_{r_{n-\ell_T}}}
\]

where $\beta_{r_2, \cdots, r_{n-\ell_T}}^{(i)}$ depend on proper coefficients of $\gamma_i$'s and satisfy $\sum_{A_{r_2} \cdots A_{r_{n-\ell_T}}} = (\beta_{r_2, \cdots, r_{n-\ell_T}}^{(i)})^2 = 1$.

Note that for each pair of $i, j$ with $i \neq j$, the states $|\phi_i \rangle_I$ and $|\phi_j \rangle_I$ are orthogonal because $\{j_1, \cdots, j_{l_T}\}$ with $j_1 + \cdots + j_{l_T} = i_1$ and $\{j'_1, \cdots, j'_{l_T}\}$ with $j'_1 + \cdots + j'_{l_T} = i_2$ are orthogonal. This implies $\{|\phi_i \rangle_I \}$ defined in Eq. (22) are orthogonal. Similarly, we can prove $\{|\psi_i \rangle_T \}$ defined in Eq. (23) are orthogonal. This has proved the Schmidt decomposition (21) with the orthogonal states in Eqs. (22) and (23). Moreover, it implies the Schmidt number $L_I = s + 1$. From Result 3, it has proved the result.

**APPENDIX E. ESTIMATING UPPER BOUND OF $k$ FOR ENTANGLED QUANTUM NETWORKS**

In this section, we estimate the upper bound of $k$ for an $n$-partite entangled quantum network using a polynomial-time algorithm. Here, for simplicity we assume each pair shares at least one bipartite entanglement on $d$-dimensional Hilbert spaces $\mathcal{H}_d \otimes \mathcal{H}_d$.

We firstly present a sufficient and necessary condition for verifying an entangled quantum network. Consider an $n$-partite quantum network $\mathcal{N}_q$ consisting of any bipartite entanglement on Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_d$. Denote $\mathcal{N}_q^{(A)}$ as the subnetwork consisting of all parties in $A$. For a given subnetwork $\mathcal{N}_q^{(A)}$, denote $s_{i;in}$ denotes the inner connectedness degree of the party $A_i$, that is, the number of bipartite entangled states shared with other parties in $A$. While $s_{i;out}$ denotes the outer connectedness degree of the party $A_i$, that is, the number of bipartite entangled states shared out of other parties in $A$. Denote $n_T$ as all the other inner connectedness degrees in $\mathcal{N}_q^{(A)}$, that is, the number of bipartite entangled states shared by two parties in $A - \{A_i\}$. We firstly prove the following lemma.

**Lemma 1.** An $n$-partite quantum network $\mathcal{N}_q$ is $k$-connection biseparable if $s_{i;in} + 2n_T \geq s_{i;out}$ for some $i$ and $k$-partite subnetwork.

**Proof of Lemma 1.** The proof is completed by two steps. One step is to transform all the entangled states shared by inner parties in a given subnetwork $\mathcal{N}_q^{(A)}$. The other is to change all the entangled states shared by one party into others in $A$. For simplicity, consider a $k$-partite subnetwork $\mathcal{N}_q^{(A)}$ with $A = \{A_1, \cdots, A_n\}$. In what follow, we only prove the result for the party $A_1$. Similar proof holds for other subnetworks and parties.

Denote all the entangled states shared by the party $A_1$ and parties in $A$ as (see red lines in Fig. 5(a))

\[
\varrho_{1;in} = \otimes_i \rho_{A_1 \rightarrow A_i \rightarrow 1}
\]

where $\rho_{A_1 \rightarrow A_i \rightarrow 1}$ are bipartite entangled states on Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_d$. Denote all the entangled states shared by the party $A_1$ and parties out of $A$ as (see orange lines in Fig. 5(a))

\[
\varrho_{1;out} = \otimes_j \rho_{A_1 \rightarrow A_j \rightarrow 1}
\]

where $\rho_{A_1 \rightarrow A_i \rightarrow 1}$ are bipartite entangled states on Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_d$. Denote all the entangled states shared by any two parties in $A_i := A - \{A_1\}$ as (see green lines in Fig. 5(a))

\[
\varrho_{A_i} = \otimes_i \rho_{A_i \rightarrow A_j \rightarrow 1}
\]

where $\rho_{A_i \rightarrow A_j \rightarrow 1}$ are bipartite entangled states on Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_d$. There exists a local CPTP mapping $\mathcal{E}_{A_i}$ such that

\[
\mathcal{E}_{A_i} : \varrho_{A_i} \mapsto \otimes_i \rho_{A_i \rightarrow A_j \rightarrow 1} |00 \rangle \langle 00|
\]

Denote $A_1$ consists of all particles $A_{1 \rightarrow j}$ in Eq. (25). Denote $B$ consists of all particles $A_{1 \rightarrow j}$ in Eq. (24) and all particles $A_{i \rightarrow j}$ and $A_{j \rightarrow i}$ in Eq. (25). Let $SWAP(A_1, B_i)$ be swapping operation of two parties $A_1$ and $B_i$, defined by

\[
SWAP(A_1, B_i) = \sum_i |i i \rangle \langle ii| + \sum_{i \neq j} |i j \rangle \langle j i|
\]

where $A_i \in A_1$ and $B_i \in B$, as shown in Fig. 5(b). After these swapping operations, all the bipartite entangled states shared with $A_1$ are disentangled with all the
FIG. 5. Schematic quantum network. (a) A general quantum network with subnetwork consisting of all parties in $\mathcal{A}$. Here, the orange lines denote entanglements shared by $A_1$ and others out of $\mathcal{A}$. The red lines denote entanglements shared by $A_1$ and others in $\mathcal{A}$. The green lines denote entanglements shared by parties (except for $A_1$) in $\mathcal{A}$. (b) Swapping two particles (red dotted lines) in $\mathcal{A}$.

FIG. 6. Schematic quantum network. A general quantum network with subnetwork consisting of all parties in $\mathcal{A}$. Here, there is a chain subnetwork shown in red lines.

**Lemma 2.** An $n$-partite $c$-connected quantum network $\mathcal{N}_q$ is $k$-connection biseparable state with $2k \geq c+1$, where the $c$-connectedness means that there are $c$ numbers of different chain subnetworks connecting each pair $(A_i, A_j)$ of $\mathcal{N}_q$.

In Lemma 2, a chain subnetwork consists of $A_{i_1}, \ldots, A_{i_c}$ such that each adjacent pair share one entanglement.

**Proof of Lemma 2.** For any $n$-partite $c$-connected quantum network $\mathcal{N}_q$, there is one party ($A_1$ for example) who shares $c$ bipartite entangled states with others. Moreover, for any $c$ and $k$ there exists a $k$-partite subnetwork $\mathcal{N}_q^{(A)}$ with $A_1 \in \mathcal{A}$ such that all the parties (except for $A_1$) in $\mathcal{A}$ have at least one chain subnetwork, as shown red lines in Fig. 6. This implies that

$$s_{1;\text{in}} + 2t_1 \geq 2k - 2$$

where an $m$-partite chain network has $2m - 2$ particles. From Lemma 1 and $s_{1;\text{out}} = c - 1$ we have completed the proof. □

**Algorithm 1:** Verifying any $n$-partite quantum network $\mathcal{N}_q$ consisting of bipartite entanglement

**Input:** Finite-size network $\mathcal{N}_q$

**Output:** $k$, satisfying that $\mathcal{N}_q$ is at most $k$-connection entanglement

(i) Find the connectedness degree $\ell_i$ for any each party $A_i$, with $i = 1, \ldots, n$.

(ii) Rearrange all parties with decreasing order into $A_1, \ldots, A_n$ (for simplicity).

(iii) Find $J$ such that $\ell_j = \min\{\ell_i\}$ with $j \in J$. Let $m = |J|$.

(iv) For $s = 1 : m$

(v) For $t = 1 : \lfloor \frac{s+1}{s-1} \rfloor$

(a) Let $A_t = \{A_j\}$ and $\mathcal{A}_t = \{A_1, \ldots, A_{s-1}\} - A_t$.

(b) Let $A_{t+1} = A_u \cup \{A_s\}$, where $A_u$ has shared the most bipartite entangled states with parties in $\mathcal{A}_t$ compared with other parties in $\mathcal{A}_t$ and $A_s \in \mathcal{A}_t$.

(c) Evaluate $s_{j;\text{in}}$ and $t_j$.

(d) If $s_{j;\text{in}} + 2t_j < s_{j;\text{out}}$

\[ t \rightarrow t + 1 \]

Otherwise

\[ v_s = \min\{v_1, \ldots, v_m\} \]

\[ s \rightarrow s + 1 \]

(vi) Output $k \leq \min\{v_1, \ldots, v_m\}$

It is general difficult to find the largest $k$ for any entangled quantum network $\mathcal{N}_q$ such that the total state of $\mathcal{N}_q$ is $k$-CGE. Here, from Lemmas 1 and 2, we present an efficient method for obtaining the upper bound of $k$, as shown in Algorithm 1.

For each party $A_j$ with the minimal connectedness degree $\ell_j$, from Lemma 1 if $s_{j;\text{in}} + 2t_j \geq s_{j;\text{out}}$ for some $t$ there is a CPTP mapping to disentangle all the particles shared by $A_j$. Hence, the total state of $\mathcal{N}_q$ is $t$-connection biseparable. The time complexity of the step (i) is at most $O(n^2)$. The time complexity of the step (iii) is at most $O(n)$. For a given party $A_j$ with $j \in J$, the time complexity of the step (b) is at most $O(n^2)$. Hence, the total time complexity is at most $O(n^3)$. 
Some examples are shown in Fig. 7. For the chain network in Fig. 7(a), the party $A_1$ shares one bipartite entanglement (red line) with other parties out of $\mathcal{A}$, and shares one bipartite entanglement (green line) with $A_2$. From Lemma 1 the chain network in Fig. 7(a) is 2-connection biseparable, where the party $A_1$ can be disentangled with other parties out of $\mathcal{A}$ by using joint operation of $A_1$ and $A_2$. Moreover, it is genuinely multipartite entanglement [8, 14], that is, any local operation cannot disentangle one party. Thus the chain network in Fig. 7(a) is 1-CGE. Similar result holds for the star network in Fig. 7(b) and cyclic network in Fig. 7(c). For a complete connected network in Fig. 7(d), the party $A_1$ shares two bipartite entanglements (red lines) with other parties out of $\mathcal{A}$. There are three bipartite entanglements (green lines) shared by parties in $\mathcal{A}$. From Lemma 1, it is 3-connection biseparable while any two parties cannot jointly disentangle one party. Hence, this network is 2-CGE, where the party $A_1$ can be disentangled with other parties out of $\mathcal{A}$ by using joint operation of the parties $A_1, A_2$ and $A_3$. In general, we can prove an $n$-partite complete connected network is $k_{\max}$-CGE with $k_{\max} = \lfloor \frac{n}{2} \rfloor$. Similarly, from Lemma 2 the planar network in Fig. 7(e) is 1-CGE while the cubic network is 2-CGE.