LECTURES ON SPECIAL KÄHLER GEOMETRY AND
ELECTRIC–MAGNETIC DUALITY ROTATIONS

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In these lectures I review the general structure of electric–magnetic duality rotations in every even space–time dimension. In four dimensions, which is my main concern, I discuss the general issue of symplectic covariance and how it relates to the typical geometric structures involved by N=2 supersymmetry, namely Special Kähler geometry for the vector multiplets and either HyperKähler or Quaternionic geometry for the hypermultiplets. I discuss classical continuous dualities versus non–perturbative discrete dualities. How the moduli space geometry of an auxiliary dynamical Riemann surface (or Calabi–Yau threefold) relates to exact space–time dualities is exemplified in detail for the Seiberg Witten model of an SU(2) gauge theory.

1. Introduction and Historical Remarks

Electric–Magnetic Duality is an old idea. It was introduced by Dirac [1] in the context of an abelian gauge theory and lead to the derivation of the Dirac quantization rule for the magnetic charge \( q_m \) relative to the electric charge \( q_e \):

\[
\frac{q_e q_m}{4\pi \hbar} = \frac{n}{2} \quad n \in \mathbb{Z}
\]

(1)

In the seventies, after the discovery of the 't Hooft–Polyakov non–abelian monopole solution [2], the studies on electro–magnetic dualities were resurrected in a new context and lead to generalizations of eq. (1), where the interpretation of the magnetic charge as a topological quantum number plays a crucial role. The theory of monopoles made a substantial progress in the context of non–abelian gauge theories, spontaneously broken to an abelian phase by Higgs fields that transform in the adjoint representation of the gauge group. The Montonen–Olive conjecture [3] developed in this context establishes a well–defined correspondence between an electric spontaneously broken non abelian gauge theory and a dual magnetic gauge theory which is abelian. In the dual theory the unbroken electric subgroup \( \mathcal{H} \subset \mathcal{G} \) is replaced by a dual magnetic group \( \mathcal{H}^e \), such that the weight–lattice of the former is dual to the weight–lattice of the latter, (this is the suitable generalization of eq. (1)), the magnetic monopoles appear as elementary excitations rather than solitary waves while it is the massive non–abelian gauge bosons that are generated as soliton solutions of the non linear field equations. Finally the original coupling constant is replaced by its inverse:

\[
g^2 \longrightarrow \frac{1}{g^2}
\]

(2)

relating, in this way, the weak coupling regime of the magnetic theory to the strong coupling regime of the electric one and viceversa.

The very fact that scalars fields in the adjoint representation play a fundamental role shows the relation of electro–magnetic duality with supersymmetry, in particular with extended supersymmetry. Indeed it is a feature of \( N \geq 2 \) SUSY that the vector multiplets including the gauge bosons contain also scalar partners which, out of necessity, transform in the same representation as the vectors, i.e. the adjoint. This association was discovered early by Witten and Olive who identified the topological charges leading to the magnetic charges with the central charges of extended supersymmetry algebras [4].
The group of electric–magnetic duality rotations $\Gamma_D$ is, by definition, that group of transformations that, while implementing the exchange of strong–weak regimes, takes electric field strengths $F^A_{\mu\nu}$ into linear combinations of the same with their magnetic duals

$$F^\mu_\Lambda = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F^\Lambda_{\rho\sigma}$$

and vice versa. In such a process Bianchi identities are exchanged with field equations and electric currents are exchanged with magnetic currents. This has three immediate consequences:

a) The duality group $\Gamma_D$ is not, in general, a symmetry of the action since it involves a non-local transformation from electric gauge fields $A^\Lambda_{\mu}$ to magnetic ones $B^\Lambda_{\mu}$. Yet it is a symmetry of the complete quantum theory including all the topological sectors.

b) In view of the Dirac quantization condition \[1\], or of its non-abelian generalizations, $\Gamma_D$ should be a discrete rather than a continuous group. Indeed it should be an automorphism for the lattice $\Lambda_{em}$ containing the electric and the magnetic charges that are both quantized.

c) The determination of the exact duality group $\Gamma_D$ involves a complete control over the non-perturbative quantum corrections to the theory under consideration. Indeed by a duality rotation the perturbative expansion is mapped into a non-perturbative one.

Recently a great deal of work has been made in developing new ideas and new strategies apt to determine the exact duality group $\Gamma_D$, both in field theories and in string theories possessing an N=2 target supersymmetry [21], [40] and references therein) where the Montonen–Olive conjecture is particularly nice (differently from the N=2 case, here both the elementary gauge fields and the magnetic monopoles sit in the same type of supermultiplet that is in a vector multiplet) and where the vanishing of the beta–function guarantees the absence of true quantum corrections.

In these studies the focus of interest is on a non-abelian gauge group, characterized by a non-vanishing coupling constant $g^2 \neq 0$, and on the exact duality group $\Gamma_D$, which is discrete. Yet, the very reason why so much more of this programme can be done in extended supersymmetric theories, with respect to $N \leq 1$ theories, resides in the symbiotic relation existing between $N \geq 2$ supersymmetry and suitable groups $\Gamma_{cont}$ of continuous duality rotations that occurs for abelian (non-gauged) $g^2 = 0$ theories.

The line of thought is essentially the following. As we have seen, crucial for the existence of monopole–states and, hence, for the realization of non-perturbative dualities is the presence, in the elementary spectrum of the theory, of Higgs fields $\phi^I$ transforming in the adjoint representation. In extended ($N \geq 2$) SUSY, these latter do indeed exist and are the supersymmetric scalar partners of the gauge bosons $A^I_{\mu}$. Supersymmetry requires that the fields $\phi^I$ should be interpreted as coordinates of a suitable scalar manifold $M^v_{scalar}$ acting as target space in a 4D $\sigma$–model:

$$\phi : \ M_4 \rightarrow M^v_{scalar}$$

the kinetic term being

$$\mathcal{L}_{kin} = g_{IJ}(\phi) \partial_\mu \phi^I \partial^\nu \phi^J g^{\mu\nu}$$

where $g_{IJ}$ is a suitable metric on $M^v_{scalar}$. Suitable in the above sentence means appropriate to the fact that the scalars sit in the same multiplet as the gauge–bosons so that any transformations on the former should have an image on the latter. What are the transformations (symmetries or field redefinitions) one considers on the scalars of a $\sigma$–model ? Just the diffeomorphisms of the scalar manifold:

$$\forall t \in Diff[M^v_{scalar}] : \phi^I \rightarrow t^I(\phi)$$

$$t : M^v_{scalar} \rightarrow M^v_{scalar}$$

If the metric is invariant under some diffeomorphism $t$ of $M^v_{scalar}$, namely $t^* g = g$, than that particular diffeomorphism is an isometry and, as such, it is a global symmetry of the theory. Otherwise it is just a conceivable field redefinition. In any case, in order to commute with supersymmetry the transformation induced by $t$ on the scalar
fields \( \phi \) must have a counterpart on the vector fields. What are the conceivable transformation of vector fields? Just duality rotations. It follows that the constraints imposed by supersymmetry on the geometry of the scalar manifold is that it should be the base–manifold of a symplectic vector bundle \( SV \longrightarrow M_{\text{scalar}}^{v} \), whose structural group \( Sp(2n, \mathbb{R}) \) is the most general group of duality rotations (the sections of \( SV \) transform as the column vector of electric plus magnetic field strengths), and such that the tangent bundle \( T M_{\text{scalar}}^{v} \) is canonically embedded in \( SV \).

The isometries of \( M_{\text{scalar}}^{v} \) constitute therefore a subgroup \( I \subset Sp(2n, \mathbb{R}) \), to be identified with the previously mentioned continuous duality group \( \Gamma_{\text{cont}} \), which plays a fundamental role in the construction of the supersymmetric action and, eventually, also in the understanding of the quantum duality group \( \Gamma_{P} \).

The scheme is the following:

- Having fixed the number \( n_{V} \) of vector multiplets and \( m \) of matter multiplets (if permitted \( ^{2} \)), one sets the gauge coupling constant to zero \( (g^{2} = 0) \) so that the gauge group is abelian and all the matter fields are neutral: there are neither electric nor magnetic charges
- Next one tries to construct the globally or locally supersymmetric lagrangian with the above field content. As outlined above this involves the choice of a target manifold \( M_{\text{scalar}}^{v} \) for the 4D \( \sigma \)-model spanned by the scalar fields. It has a direct product structure:

\[
M_{\text{scalar}} = M_{\text{scalar}}^{v} \otimes M_{\text{scalar}}^{s}
\]  

(7)

the first factor containing the vector multiplet scalars, the second including those pertaining to the scalar multiplets (hypermultiplets in the \( N=2 \) case that is also the only relevant one, as already remarked).

- The scalar manifold \( M_{\text{scalar}}^{v} \) is restricted by supersymmetry to carry the symplectic vector bundle structure \( SV \longrightarrow M_{\text{scalar}}^{v} \) previously recalled and will admit a specific continuous group of isometries \( I \subset Sp(2\pi, \mathbb{R}) \) where \( \pi \) denotes the total number of vector fields in the theory. One has:

\[
\begin{align*}
N = 2 & \quad \pi = n_{V} + 1 \text{ graviphoton} \\
N = 3 & \quad \pi = n_{V} + 3 \text{ graviphotons} \\
N = 4 & \quad \pi = n_{V} + 6 \text{ graviphotons} \\
N = 5 & \quad \pi = 10 \text{ graviphotons} \\
N = 6 & \quad \pi = 16 \text{ graviphotons} \\
N = 7, 8 & \quad \pi = 56 \text{ graviphotons}
\end{align*}
\]

(8)

For \( N \geq 3 \) the solution of the constraints is unique. The scalar manifold is a uniquely chosen coset manifold \( I/S \). For \( N = 2 \) the duality constraints requested by supersymmetry allow more general solutions. They just select a class of complex manifolds with a peculiar geometry: The special Kähler manifolds that are precisely defined by the existence of a flat holomorphic vector bundle with structural group \( Sp(2\pi, \mathbb{R}) \), the norm of a typical section yielding the Kähler potential. There are in fact two kinds of special Kähler geometries, the rigid and the local one, depending on whether \( \pi \) equals \( n_{V} \) or \( n_{V} + 1 \). The rigid special manifolds accommodate the vector multiplet scalars in globally supersymmetric \( N=2 \) Yang–Mills theories, while the local special manifolds provide the appropriate description for the same scalar fields in locally supersymmetric theories, namely in matter coupled \( N = 2 \) supergravity. In the sequel when we say special manifolds without further specifications we always refer to the local ones. Differently from what happens in the \( N \geq 3 \) theories the class of special manifolds includes both homogeneous spaces as well as manifolds with no continuous isometries. It turns out that the structure of special geometry, as defined by supersymmetry, is also the geometric structure of the moduli spaces of Calabi–Yau
threefolds in the local case and of a special class of Riemann surfaces in the rigid case. This mathematical identity is the source of many important physical considerations developed both in the context of superstring compactifications and in the present context.

- When the choice of the scalar manifold has been performed on the basis of the continuous duality invariance and the corresponding supersymmetric lagrangian has been constructed at $g^2 = 0$, gauging of the model can be considered. Gauging, in this parlance, corresponds to two independent things: by switching on the gauge coupling constant $g^2 \neq 0$,
  
i) the gauge group can be made non–abelian
  
ii) electric charges can be given to the matter multiplets with respect to the gauge group, whether abelian or non abelian.

The possible gauge groups $G$ are subgroups of the symplectically embedded isometry group of the scalar manifold with dimension equal to the number of vector multiplets:

$$G \subset \mathcal{I}[\mathcal{M}_{\text{scalar}}] \subset Sp(2\pi, \mathbb{R})$$

$$\dim_{\mathbb{R}} G = n_V$$

and satisfying some extra constraints discussed further on.

In relation with this there is a crucial point that is not all the time fully appreciated by non supergravity experts. Non abelian gauge groups are introduced only in a second stage to turn global symmetries of the lagrangian into local ones. This is a familiar concept also in ordinary field theories. What is less familiar is that the gauge fields needed to gauge such global symmetries are already present in the lagrangian before gauging, while in ordinary field theories they are usually introduced at the same time with the gauging. This happens because vector and scalars are superpartners. Hence what one gauges is a subgroup of the duality symmetry group endowed with the further property of mixing electric charges only with electric charges.

- The gauging procedure introduces few more terms, both in the Lagrangian and in the supersymmetry transformation rules, that are completely controlled by the geometric structure of the scalar manifold and by the action of the gauge group on such a manifold.

At $g^2 \neq 0$ the continuous duality group $\Gamma_{\text{cont}} \subset Sp(2\pi, \mathbb{R})$ is broken. The question is: does any discrete subgroup

$$Sp(2\pi, \mathbb{Z}) \supset \Gamma_D \subset \Gamma_{\text{cont}}$$

survive in the quantum corrected theory?

- The recent exciting work on dualities, covered in many lectures at this Spring School, has been directed toward answering the above question in the affirmative sense. One considers the spontaneously broken phase of the globally or locally supersymmetric gauge theory and restricts one’s attention to the effective quantum action for the massless fields. All the perturbative and non perturbative corrections (space–time instanton contributions) are supposed to be taken into account. The scalar fields that are included in this action are the subset of neutral fields in the original lagrangian and are named moduli: they correspond to flat directions of the scalar potential generated by the gauging. Since, notwithstanding quantum corrections, supersymmetry is unbroken, the submanifold $\mathcal{M}_{\text{moduli}}$ of the scalar manifold must satisfy all the constraints of duality requested by supersymmetry. In the $N \geq 3$ case the fact that there is a unique choice provides the solution for the local geometry of $\mathcal{M}_{\text{moduli}}$. The covering space of the moduli–space is just the appropriate coset manifold $\mathcal{I}/\mathcal{S}$. There is only a question of global identifications that corresponds to modding by the exact duality group $\Gamma_D$. This latter is
nothing else but the discrete part of \( \mathcal{I} \). In a rather symbolic way we can write:

\[
\Gamma_D = \mathcal{I} \cap \mathbb{Z} \tag{11}
\]

The restriction to integers is due to the quantization of electric and magnetic charges, the latter being introduced by non-perturbative effects. In the \( N = 2 \) case things are more complicated by the fact that the duality constraints requested by supersymmetry allow wider choices. What we know a priori is that \( \mathcal{M}_{\text{moduli}} \) is a special Kähler manifold. Its global and local geometry, however, is quantum-corrected with respect to the geometry of the classical \( \mathcal{M}_{\text{scalar}} \) restricted to neutral fields. Since the exact duality group corresponds, essentially, to the group of global identifications for \( \mathcal{M}_{\text{moduli}} \), it follows that it cannot be just the restriction to integers of \( \mathcal{I} \), as it happens for the \( N \geq 3 \). In this case the solution for the geometry of \( \mathcal{M}_{\text{moduli}} \) and for \( \Gamma_D \) has been found relying on the previous observation that special geometry is the moduli-space geometry for a class of Riemann surfaces (in the rigid case) and for Calabi–Yau threefolds (in the local case). Solving the problem consists of guessing the right Riemann surface in the rigid case and the right Calabi–Yau manifold in the local case. What remains is a straightforward application of algebraic geometry techniques, in particular a derivation of the relevant Picard–Fuchs differential equations for the periods. The duality group \( \Gamma_D \) corresponds to the substitution group of integer homology bases and it has a semidirect product structure:

\[
\Gamma_D = \Gamma_W \times \mathcal{M}_{\text{mon}} \tag{12}
\]

where \( \mathcal{M}_{\text{mon}} \) is the monodromy group of the Picard–Fuchs differential system, while \( \Gamma_W \) is the symmetry group of the polynomial constraint \( W(X) = 0 \) defining the Riemann surface or Calabi–Yau threefold.

Having summarized the logical development that relates the issue of electric–magnetic duality to the geometry of special Kähler manifolds from the perspective of the most recent developments let me briefly summarize the history of the subject from the viewpoint of supergravity.

- **1978: Hidden symmetries** The prototype of duality symmetries of supergravity theories is given by the non-compact group \( E_7(-7) \) governing the structure of the \( N = 8 \) Lagrangian. It is discovered in 1978 by Cremmer and Julia \[23\]. At the time the duality symmetries are named hidden symmetries.

- **1981: General structure of the duality group** In this year Gaillard and Zumino \[24\] solve the general problem of constructing a lagrangian for a set of \( n \) abelian vector fields interacting with \( m \) scalar fields having \( \sigma \)-model interactions, in such a way that a group of electric–magnetic duality rotations is realized.

- **1981–1986: Duality and the construction of supergravities** In these years the role of duality rotations is fully exploited in the construction of all extended supergravity models. In particular just to quote few items of a very extensive literature: \( N = 4 \) \[25\], \( N = 5 \) \[26\], \( N = 3 \) \[27\], \( N = 8 \) \[28\].

- **1985-1986: Special Geometry in special coordinates** is discovered in the construction of the vector multiplet coupling to \( N=2 \) supergravity by using the so called superconformal tensor calculus \[29\]. In this approach the definition of special manifolds emerges in a preferred coordinate frame and it involves the existence of a holomorphic prepotential \( F(X) \). It will be only later on that the notion of special geometry will be freed from these limitations.

- **1986: Classification of Homogeneous Special Manifolds** The complete classification of those Special manifolds that are homogeneous spaces \( \mathcal{G}/\mathcal{H} \) is achieved in \[30\].
• **1989-1990: Relation of Special Geometry with Calabi–Yau manifolds** The special geometry structure of Calabi–Yau moduli spaces is first pointed out by Ferrara and Strominger in 1989 [31] with a supersymmetry argument. Subsequently it is rederived from (2,2) conformal theories by Kaplunovsky, Dixon and Luis [37] in 1990.

• **1990: Intrinsic definition of Special Geometry.** Relying on the relation of special geometry with Calabi Yau moduli spaces, Strominger arrives at its intrinsic definition [32].

• **1990-1991: Geometric Reformulation of N=2 Supergravity.** Using the intrinsic definitions of Special Geometry and of quaternionic geometry [35,36] N= 2 supergravity is reformulated geometrically by Castellani, D’Auria, Ferrara and Fré [33,34]. This allows the extension of the tensor calculus results to the most general case of couplings and gaugings.

• **1991-1994: Picard–Fuchs equations, Mirror Symmetry and Special Geometry** In these years there is an intense activity in studying the Special Geometry of Calabi–Yau moduli spaces and their physical and mathematical implications. Following the suggestion of mirror symmetry Candelas and collaborators [38] show that by means of algebraic geometry techniques one can derive the exact quantum moduli space of Kähler class deformations including all the world–sheet instanton corrections. At the same time one obtains the exact duality group. This result deals with what, in modern parlance, is named T–duality, yet the same mathematics is at work when a Calabi–Yau threefold is used to describe the S–duality invariance. The profound relation between Special Geometry and Picard Fuchs equations is investigated in 1993-1994 in [39].

• **1994-95: Trading T–duality for S–duality and second quantized mirror symmetry.** The paper by Seiberg and Witten [3] on the solution of rigid N=2 Yang–Mills theory by means of an auxiliary Riemann surface opens new perspectives on the use of a well established lore on Special Geometry and Duality Groups in a new context. T–duality is traded for S–duality and world–sheet instantons are traded for space–time instantons. The concept of second quantized mirror symmetry is introduced in [17].

2. **Plan of these Lectures**

In the present lectures my goal is to present the geometrical structures underlying matter coupled N = 2 supergravity and N = 2 Yang–Mills theories, emphasizing their role in the discussion of electric–magnetic duality rotations and strong–weak duality.

The very purpose is to show how all the recent exciting results that go under the collective name of S–duality or string–string duality are founded in these geometric structures and could not be properly understood without them. What is actually true is that the very formulation of N=2 supersymmetry, both in the local and in the global case, incorporates duality rotations as a necessary ingredient. This is most likely related with the other intriguing aspect of N=2 supersymmetry, namely its symbiotic relation with the topological twist to topological field–theories [45,46,47,48,51,49]. As these latter are the appropriate framework to single out instanton and monopole dominance in the path integral, duality symmetries that exchange elementary excitations with solitonic states could not fail to be deeply related with N = 2 theories and that from the very beginning.

The geometric structures underlying D = 4, N = 2 theories are:

1. **Special Kähler geometry** in its two versions, rigid and local. This is the geometry that pertains to the scalar fields belonging to the vector multiplets.

2. **Hypergeometry** in its two versions: HyperKähler and Quaternionic geometry.
This is the structure of the manifolds spanned by the scalar fields that sit in the hypermultiplets.

3. The momentum map. This is the geometrical construction that lifts the action of a Lie group on a symplectic manifold to a dual hamiltonian realization. Well known in classical mechanics, the concept of momentum map is the key-ingredient in the gauging of \( N = 2 \) supersymmetric theories.

The momentum map explains the nature of the auxiliary fields and it is used to write down the scalar potential of \( N = 2 \) theories. It also plays a major role in the topological twist. From the viewpoint taken in these lectures, gauging is that procedure that through the introduction of electric charges spoils a primeval continuous duality symmetry. Yet, at the same time, it introduces those non-linear interactions (typically the non-abelian couplings) that are responsible for the existence of classical non-perturbative solutions of the field equations with magnetic-like charges. Hence, after gauging, the primeval continuous duality symmetry is lost, but a new discrete one can emerge, due to the introduction of magnetic charges.

A major issue is to understand these quantum duality symmetries that emerge after gauging and their relations with the primeval continuous dualities. In some cases, like in the \( N = 4 \) theory the quantum dualities are just a discrete subgroup of the primeval classical duality group. In the \( N = 2 \) theories they are also a subgroup of the primeval classical ones, yet with a different symplectic embedding. In this difference resides non-perturbative physics.

As it might be expected the geometrical structures 1) 2) and 3), that so nicely marry with electric–magnetic dualities, are an yield of the supersymmetry algebra. So to see their origin one should deal with the fermions and work hard with Dirac algebra and gamma matrices. This has been done elsewhere \([13,12]\) and it is completely accomplished. I do not see any reason to repeat it here. Let us take it for given and let us rather illustrate the various geometrical items involved in the construction of the \( N = 2 \) theory. Fermions, although at the heart of the whole matter, obscure with their technicalities the profound meaning of the involved geometry and its relation with the physics of electric–magnetic duality rotations.

Hence in these lectures I shall never mention the fermions: of all supersymmetric lagrangians I shall write only the bosonic part. The structures of these bosonic lagrangians is determined by intercourse with the fermions, yet, once their form is established, it is their properties we want to investigate.

The final form of the bosonic action for \( N = 2 \) supergravity is as follows:

\[
\mathcal{L}_{\text{SUGRA}}^4 = \sqrt{-g} \left[ R[g] + g_{ij}(z,\overline{z}) \nabla\mu z^i \nabla\mu \overline{z}^j \right. \\
- 2 \lambda h_{uv}(q) \nabla\mu q^u \nabla\mu q^v + i \left( \langle \nabla_{\Lambda \Sigma} F_{\mu \nu} - \nabla_{\mu \nu} F^{\Lambda \Sigma} \rangle - g^2 \mathcal{V} \right) \right]
\]

(13)

where:

\[
g_{ij}(z,\overline{z}) = \text{special Kähler metric} \\
h_{uv}(q) = \text{quaternionic metric} \\
\langle \nabla_{\Lambda \Sigma} \rangle = \text{period matrix in spec.geom.} \\
\nabla\mu z^i = \partial_\mu z^i + g A_\mu^A k^A_i(z) \]

\[
k^A_i(z) = \text{holomorphic Killing vec.} \\
\nabla\mu q^u = \partial_\mu q^u + g A_\mu^A k^u_{A}(q) \]

\[
k^u_{A}(q) = \text{triholomorphic Killing vec.} \\
\mathcal{V} = \mathcal{L}^\Lambda \left( 4 k^i_A k^j_\Sigma h_{uv} + k^i_A k^j_\Sigma g_{ij} \right) L^\Sigma \\
+ \left( U^{A \Sigma} - 3 \mathcal{L}^\Lambda L^\Sigma \right) \mathcal{P}^\Lambda \mathcal{P}^\Sigma \\
\mathcal{P}^\Lambda = \text{triholomorphic mom. map} \\
\mathcal{L}^\Lambda = \text{cov. hol. sect.}
\]

(14)

Giving a precise geometrical meaning to the list of words appearing in eq. (13) is the purpose of the following lectures. They are organized as follows.

Lecture 3 is devoted to a study of the general structure, in any even space–time dimensions of those lagrangians, containing as elementary fields differential forms and scalars, that display duality covariance. The role of symplectic or pseudo–orthogonal transformations is explained.
Lecture 4 concentrates on the D=4 case where symplectic covariance is the relevant one and discusses the symplectic embedding of the scalar manifold diffeomorphism group. A very general formula for the *period matrix* is given in the case of homogeneous scalar manifolds: the Gaillard Zumino master formula.

Lecture 5 introduces the notion of Special Kähler geometry both in its local and rigid version.

Lecture 6 is devoted to a case study of some particular Special Kähler manifold of relevance in Kähler geometry both in its local and rigid version.

Lecture 7 explains hypergeometry, that is the space–time integration volume as it reduces a set of real scalar fields $\phi \in \mathbb{R}^{(p-1)\text{–indices}}$ to a particular Special Kähler manifold of relevance in Kähler geometry both in its local and rigid version.

Lecture 8 discusses the gauging after having drawn the distinction between classical, perturbative and non perturbative dualities.

Lecture 9 presents the last necessary geometrical ingredient of the N=2 construction, that is the symplectic covariance is the relevant one and distinguishes between the two instances of rigid and local hypergeometries and the corresponding rigid and local special geometries.

Lecture 10 presents a detailed derivation and discussion of the rigid non perturbative special geometries.

Finally, as an exemplification of the whole set of ideas discussed individually in previous lectures, lecture 11 presents a detailed derivation and discussion of the rigid non perturbative special geometries associated with the effective lagrangian of an SU(2) theory. This is the famous Seiberg Witten model already mentioned several times.

3. The Group of Continuous Duality Rotations in $D=2p$ Space–Time Dimensions

In this lecture I am going to review the general structure of an abelian theory of vectors and scalars displaying duality symmetries. The basic reference is the 1981 paper by Gaillard and Zumino, presentation, however, will be a little more general. Rather than restricting my attention to ordinary one–index gauge fields $A^\Lambda_{\mu}$ in $D=4$ I will consider a theory of antisymmetric gauge tensor with $(p-1)$–indices $A^\Lambda_{\mu_1... \mu_{p-1}}$, in a $D=2p$ dimensional space–time $M^{space–time}$.

$D \equiv \dim M^{space–time} = 2p$ (15)

The gauge tensors $A^\Lambda_{\mu_1... \mu_{p-1}}$ correspond to a set of differential $(p-1)$–forms:

$$A^\Lambda = A^\Lambda_{\mu_1... \mu_{p-1}} dx^{\mu_1} \wedge ... \wedge dx^{\mu_{p-1}}$$

$(\Lambda = 1, ..., \pi)$ (16)

The corresponding field strengths and their Hodge duals are defined as it follows:

$$F^\Lambda = \frac{1}{p!} F^\Lambda_{\mu_1... \mu_{p}} dx^{\mu_1} \wedge ... \wedge dx^{\mu_{p}}$$

$$F^\Lambda_{\mu_1... \mu_{p}} = \partial_{\mu_1} A^\Lambda_{\mu_2... \mu_{p}} + p-1 \text{ terms}$$

$$F^{\Lambda*} = \frac{1}{p!} \tilde{F}^\Lambda_{\mu_1... \mu_{p}} dx^{\mu_1} \wedge ... \wedge dx^{\mu_{p}}$$

$$\tilde{F}^\Lambda_{\mu_1... \mu_{p}} = \frac{1}{p!} \epsilon_{\mu_1... \mu_{p}\nu_1... \nu_{p}} F^{\Lambda|\nu_1... \nu_{p}}$$ (17)

Defining the space–time integration volume as it follows:

$$d^{D}x \equiv \frac{1}{D!} \epsilon_{\mu_1... \mu_D} dx^{\mu_1} \wedge ... \wedge dx^{\mu_{D}}$$ (18)

we obtain:

$$F^\Lambda \wedge F^{\Sigma} = \frac{1}{(p-1)!} \epsilon^{\mu_1... \mu_{\pi-1}\nu_1... \nu_{p}} F^\Lambda_{\mu_1... \mu_{p}} F^{\Sigma|\nu_1... \nu_{p}}$$

$$F^\Lambda \wedge F^{\Sigma*} = (-)^p \frac{1}{(p-1)!} \tilde{F}^\Lambda_{\mu_1... \mu_{p}} F^{\Sigma|\mu_1... \mu_{p}}$$ (19)

In addition to the $(p-1)$–forms let us also introduce a set of real scalar fields $\phi^{I}$ $(I = 1, ..., \pi)$ spanning an $\pi$–dimensional manifold $M^{\text{scalar}}$ endowed with a metric $g_{IJ}(\phi)$. Utilizing the above field content we can write the following action functional:

$$S = S_{\text{tens}} + S_{\text{scal}}$$

$$S_{\text{tens}} = \int \left[ \frac{1}{2} \gamma_{\Lambda \Sigma}(\phi) F^\Lambda \wedge F^{\Sigma*} + \frac{1}{2} \theta_{\Lambda \Sigma}(\phi) F^\Lambda \wedge F^{\Sigma} \right]$$

$$S_{\text{scal}} = \int \left[ \frac{1}{2} g_{IJ}(\phi) \partial_\mu \phi^I \partial^{\mu} \phi^J \right] d^{D}x$$ (20)

\(^3\text{whether the } \phi^J \text{ can be arranged into complex fields is not relevant at this level of the discussion.}
where the scalar field dependent $\overline{m} \times \overline{m}$ matrix $\gamma_\Lambda(\phi)$ generalizes the inverse of the squared coupling constant $\frac{1}{\theta}$ appearing in ordinary 4D–gauge theories. The field dependent matrix $\theta_\Lambda(\phi)$ is instead a generalization of the theta–angle of quantum chromodynamics. The matrix $\gamma$ is symmetric in every space–time dimension, while $\theta$ is symmetric or antisymmetric depending on whether $p = D/2$ is an even or odd number.

In view of this fact it is convenient to distinguish between the two cases by setting:

$$D = \begin{cases} 4\nu & \nu \in \mathbb{Z} \quad p = 2\nu \\ 4\nu + 2 & \nu \in \mathbb{Z} \quad p = 2\nu + 1 \end{cases} \quad (21)$$

Introducing a formal operator $j$ that maps a field strength into its Hodge dual:

$$(j\mathcal{F})_{\mu_1\ldots\mu_p} = \frac{1}{(p!)} \epsilon_{\mu_1\ldots\mu_p\nu_1\ldots\nu_p} \mathcal{F}^{\nu_1\ldots\nu_p} \quad (22)$$

and a formal scalar product:

$$(G, K) \equiv \mathcal{G}^T K = \frac{1}{(p!)} \sum_{\Lambda=1}^7 \mathcal{G}_\Lambda^{\mu_1\ldots\mu_p} \mathcal{K}^{\Lambda\mu_1\ldots\mu_p} \quad (23)$$

the total lagrangian of eq. (20) can be rewritten as

$$\mathcal{L}^{(\text{tot})} = \mathcal{F}^T \left( \gamma \otimes \mathbb{1} + \theta \otimes j \right) \mathcal{F}$$
$$+ \frac{1}{2} g J(\phi) \partial_\mu \phi \partial^\mu \phi \quad (24)$$

and the essential distinction between the two cases of eq. (21) is given, besides the symmetry of $\theta$, by the involutive property of $j$, namely we have:

$$D = 4\nu \quad \theta = \theta^T \quad j^2 = -\mathbb{1}$$
$$D = 4\nu + 2 \quad \theta = -\theta^T \quad j^2 = \mathbb{1} \quad (25)$$

Introducing dual and antiself–dual combinations:

$$D = 4\nu \left\{ \begin{array}{l} \mathcal{F}^\pm = \mathcal{F} \mp i j \mathcal{F} \\ j \mathcal{F}^\pm = \pm i \mathcal{F}^\pm \end{array} \right\}$$
$$D = 4\nu + 2 \left\{ \begin{array}{l} \mathcal{F}^\pm = \mathcal{F} \pm j \mathcal{F} \\ j \mathcal{F}^\pm = \pm \mathcal{F}^\pm \end{array} \right\} \quad (26)$$

and the field–dependent matrices:

$$D = 4\nu \left\{ \begin{array}{l} \mathcal{N} = \theta - i\gamma \\ \overline{\mathcal{N}} = \theta + i\gamma \end{array} \right\}$$
$$D = 4\nu + 2 \left\{ \begin{array}{l} \mathcal{N} = \theta + \gamma \\ -\overline{\mathcal{N}} = \theta - \gamma \end{array} \right\} \quad (27)$$

the vector part of the lagrangian can be rewritten in the following way in the two cases:

$$D = 4\nu \quad :$$
$$\mathcal{L}_{\text{vec}} = \frac{i}{8} \left[ \mathcal{F}^T \mathcal{N} \mathcal{F} - \mathcal{F}^{-T} \mathcal{N} \mathcal{F}^{-} \right]$$
$$D = 4\nu + 2 \quad :$$
$$\mathcal{L}_{\text{vec}} = \frac{1}{8} \left[ \mathcal{F}^T \mathcal{N} \mathcal{F}^+ + \mathcal{F}^{-T} \mathcal{N} \mathcal{F}^{-} \right] \quad (28)$$

Introducing the new tensor:

$$\tilde{G}_\mu^\Lambda \equiv -(p!)^{\frac{1}{2}} \frac{\partial \mathcal{L}}{\partial \mathcal{F}^\mu} \quad D = 4\nu$$
$$\tilde{G}_\mu^\Lambda \equiv (p!)^{\frac{1}{2}} \frac{\partial \mathcal{L}}{\partial \mathcal{F}^{-\mu}} \quad D = 4\nu + 2 \quad (29)$$

which, in matrix notation, corresponds to:

$$jG \equiv a \frac{\partial \mathcal{L}}{\partial \mathcal{F}} = \frac{2}{p} \left( \gamma \otimes \mathbb{1} + \theta \otimes j \right) \mathcal{F} \quad (30)$$

where $a = \mp$ depending on whether $D = 4\nu$ or $D = 4\nu + 2$, the Bianchi identities and field equations associated with the lagrangian can be written as follows:

$$\partial_\mu \tilde{G}_\mu^\Lambda = 0$$
$$\partial^\mu \tilde{G}_\mu^\Lambda = 0 \quad (31)$$

This suggests that we introduce the $2\overline{m}$ column vector:

$$\mathbf{V} \equiv \begin{pmatrix} j\mathcal{F} \\ j \mathcal{G} \end{pmatrix} \quad (32)$$

and that we consider general linear transformations on such a vector:

$$\begin{pmatrix} j\mathcal{F} \\ j \mathcal{G} \end{pmatrix}^T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} j\mathcal{F} \\ j \mathcal{G} \end{pmatrix} \quad (33)$$

For any matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2\overline{m}, \mathbb{R})$ the new vector $\mathbf{V}'$ of magnetic and electric field strengths satisfies the same equations as the old one. In a condensed notation we can write:

$$\partial \mathbf{V} = 0 \iff \partial \mathbf{V}' = 0 \quad (34)$$

Separating the self–dual and anti–self–dual parts

$$\mathcal{F} = \frac{1}{2} (\mathcal{F}^+ + \mathcal{F}^-)$$
$$\mathcal{G} = \frac{1}{2} (\mathcal{G}^+ + \mathcal{G}^-) \quad (35)$$
and taking into account that for $D = 4\nu$ we have:

\[ \mathcal{G}^+ = \mathcal{N}\mathcal{F}^+ \quad \mathcal{G}^- = \mathcal{N}\mathcal{F}^- \quad (36) \]

while for $D = 4\nu + 2$ the same equation reads:

\[ \mathcal{G}^+ = \mathcal{N}\mathcal{F}^+ \quad \mathcal{G}^- = -\mathcal{N}^T\mathcal{F}^- \quad (37) \]

the duality rotation of eq. 38 can be rewritten as:

\[ \begin{align*}
D &= 4\nu : \\
\begin{pmatrix} \mathcal{F}^+ \\ \mathcal{G}^+ \end{pmatrix}' &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^+ \\ \mathcal{N}\mathcal{F}^+ \end{pmatrix} \\
\begin{pmatrix} \mathcal{F}^- \\ \mathcal{G}^- \end{pmatrix}' &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^- \\ -\mathcal{N}^T\mathcal{F}^- \end{pmatrix}
\end{align*} \]

\[ \begin{align*}
D &= 4\nu + 2 : \\
\begin{pmatrix} \mathcal{F}^+ \\ \mathcal{G}^+ \end{pmatrix}' &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^+ \\ \mathcal{N}\mathcal{F}^+ \end{pmatrix} \\
\begin{pmatrix} \mathcal{F}^- \\ \mathcal{G}^- \end{pmatrix}' &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathcal{F}^- \\ -\mathcal{N}^T\mathcal{F}^- \end{pmatrix}
\end{align*} \quad (38) \]

In both cases the problem is that the transformation rule 38 of $\mathcal{G}^\pm$ must be consistent with the definition of the latter as variation of the Lagrangian with respect to $\mathcal{F}^\pm$ (see eq. 24). This request restricts the form of the matrix $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. As we are just going to show, in the $D = 4\nu$ case $\Lambda$ must belong to the symplectic subgroup $\text{Sp}(2\pi, \mathbb{R})$ of the special linear group, while in the $D = 4\nu + 2$ case it must be in the pseudo-orthogonal subgroup $\text{SO}(\pi, \pi)$:

\[ \begin{align*}
D &= 4\nu : \\
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &\in \text{Sp}(2\pi, \mathbb{R}) \subset \text{GL}(2\pi, \mathbb{R}) \\
D &= 4\nu + 2 : \\
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &\in \text{SO}(\pi, \pi) \subset \text{GL}(2\pi, \mathbb{R})
\end{align*} \quad (39) \]

the above subgroups being defined as the set of $2\pi \times 2\pi$ matrices satisfying, respectively, the following conditions:

$\Lambda \in \text{Sp}(2\pi, \mathbb{R}) \rightarrow \Lambda^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \Lambda = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ \quad (40)

$\Lambda \in \text{SO}(\pi, \pi) \rightarrow \Lambda^T \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \Lambda = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

To prove the statement we just made, we calculate the transformed lagrangian $\mathcal{L}'$ and then we compare its variation $\frac{d\mathcal{L}'}{d\mathcal{F}^\pm}$ with $\mathcal{G}^\pm$ as it follows from the postulated transformation rule 38. To perform such a calculation we rely on the following basic idea. While the duality rotation 38 is performed on the field strengths and on their duals, also the scalar fields are transformed by the action of some diffeomorphism $\xi \in \text{Diff}(\mathcal{M}_{\text{scalar}})$ of the scalar manifold and, as a consequence of that, also the matrix $\mathcal{N}$ changes. In other words given the scalar manifold $\mathcal{M}_{\text{scalar}}$ we assume that in the two cases of interest there exists a homomorphism of the following form:

$\iota_\delta : \text{Diff}(\mathcal{M}_{\text{scalar}}) \rightarrow \text{GL}(2\pi, \mathbb{R})$ \quad (41)

so that:

\[ \begin{align*}
\forall \quad \xi &\in \text{Diff}(\mathcal{M}_{\text{scalar}}) : \phi^I \stackrel{\xi}{\rightarrow} \phi'^I \\
\exists \quad \iota_\delta(\xi) &\equiv \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \in \text{GL}(2\pi, \mathbb{R})
\end{align*} \quad (42) \]

Using such a homomorphism we can define the simultaneous action of $\xi$ on all the fields of our theory by setting:

\[ \begin{align*}
\xi : \quad &\begin{cases}
\phi \rightarrow \xi(\phi) \\
\mathcal{V} \rightarrow \iota_\delta(\xi) \mathcal{V} \\
\mathcal{N}(\phi) \rightarrow \mathcal{N}'(\xi(\phi))
\end{cases}
\end{align*} \quad (43) \]

where the notation 32 has been utilized. In the tensor sector the transformed lagrangian, is

\[ \mathcal{L}'_{\text{tens}} = \iota_\delta^* \begin{pmatrix} \mathcal{F}^+ + (A + B\mathcal{N})^T \mathcal{N}'(A + B\mathcal{N}) \mathcal{F}^+ \\
- \mathcal{F}^- + (A + B\mathcal{N})^T \mathcal{N}'(A + B\mathcal{N}) \mathcal{F}^- \end{pmatrix} \quad (44) \]
for the $D = 4\nu$ case and
\[
\mathcal{L}_{\text{tens}} = \frac{i}{8} \left[ F^\top T (A + BN)^T N' (A + BN) F^+ - F^{-\top} (A - BN^T)^T N'^T (A - BN^T) F^- \right]
\] (45)
Consistency with the definition of $G^+$ requires, in both cases that
\[
N' \equiv N' (\xi (\phi)) = (C_\xi + D_\xi N' (\phi)) (A_\xi + B_\xi N (\phi))^{-1}
\] (46)
while consistency with the definition of $G^-$ imposes, in the $D = 4\nu$ case the transformation rule:
\[
\overline{N'} \equiv \overline{N'} (\xi (\phi)) = (C_\xi + D_\xi \overline{N} (\phi)) (A_\xi + B_\xi \overline{N} (\phi))^{-1}
\] (47)
and in the case $D = 4\nu + 2$ the other transformation rule:
\[
-N'^T \equiv -N'^T (\xi (\phi)) = (C_\xi - D_\xi N'^T (\phi)) (A_\xi - B_\xi N^T (\phi))^{-1}
\] (48)
It is from the transformation rules \[\] and \[\] that we derive a restriction on the form of the duality rotation matrix $\Lambda_\xi \equiv \iota_\delta (\xi)$. Indeed, in the $D = 4\nu$ case we have that by means of the fractional linear transformation $\Lambda_\xi$ must map an arbitrary complex symmetric matrix into another matrix of the same sort. It is straightforward to verify that this condition is the same as the first of conditions \[\] namely the definition of the symplectic group $Sp(2\pi, \mathbb{R})$. Similarly in the $D = 4\nu + 2$ case the matrix $\Lambda_\xi$ must obey the property that taking the negative of the transpose of an arbitrary real matrix $N$ before or after the fractional linear transformation induced by $\Lambda_\xi$ is immaterial. Once again, it is easy to verify that this condition is the same as the second property in eq. \[\] namely the definition of the pseudo–orthogonal group $SO(\pi, \pi)$. Consequently the homomorphism of eq. \[\] specializes as follows in the two relevant cases
\[
\iota_\delta : \begin{cases} 
\text{Diff (M}_{\text{scalar}} \rightarrow Sp(2\pi, \mathbb{R}) \\
\text{Diff (M}_{\text{scalar}} \rightarrow SO(\pi, \pi) \nend{cases}
\] (49)
Clearly, since both $Sp(2\pi, \mathbb{R})$ and $SO(\pi, \pi)$ are finite dimensional Lie groups, while Diff ($M_{\text{scalar}}$) is infinite–dimensional, the homomorphism $\iota_\delta$ can never be an isomorphism. Defining the Torelli group of the scalar manifold as:
\[
\text{Diff (M}_{\text{scalar}} \supset \text{Tor (M}_{\text{scalar}} \equiv \ker \iota_\delta \quad (50)
\]
we always have:
\[
\dim \text{Tor (M}_{\text{scalar}} = \infty \quad (51)
\]
The reason why we have given the name of Torelli to the group defined by eq. \[\] is because of its similarity with the Torelli group that occurs in algebraic geometry. There one deals with the moduli space $M_{\text{moduli}}$ of complex structures of a $(p+1)$–fold $M_{p+1}$ and considers the action of the diffeomorphism group $\text{Diff (M}_{\text{moduli}})$ on canonical homology bases of $(p+1)$–cycles. Since this action must be linear and must respect the intersection matrix, which is either symmetric or antisymmetric depending on the odd or even parity of $p$, it follows that one obtains a homomorphism similar to that in eq. \[\]
\[
\iota_h : \begin{cases} 
\text{Diff (M}_{\text{moduli}} \rightarrow Sp(2\pi, \mathbb{R}) \\
\text{Diff (M}_{\text{moduli}} \rightarrow SO(\pi, \pi) \nend{cases}
\] (52)
The Torelli group is usually defined as the kernel of such a homomorphism. When cohomology with real coefficients is replaced by cohomology with integer coefficients the homomorphism of eq. \[\] reduces to
\[
\iota_h : \begin{cases} 
\text{Diff (M}_{\text{moduli}} \rightarrow Sp(2\pi, \mathbb{Z}) \\
\text{Diff (M}_{\text{moduli}} \rightarrow SO(\pi, \pi, \mathbb{Z}) \nend{cases}
\] (53)
and the Torelli group becomes even larger.\[\]
This similarity between two problems that are, at first sight, totally disconnected is by no means accidental. In the theories recently considered in the literature and largely discussed at this Spring School, where electro–magnetic duality rotations are indeed realized as a quantum symmetry, the scalar manifold $M_{\text{scalar}}$ is identified with
\[\]
\[\]
\[\]
\[\]
\[\] To be precise one considers the mapping class group $\text{Diff}/\text{Diff}_0$ and defines the Torelli group as the kernel of the homomorphism $\iota_h : \text{Diff}/\text{Diff}_0 \rightarrow Sp(2\pi, \mathbb{Z})$ or $\rightarrow SO(\pi, \pi)$. Yet since $\text{Diff}_0$ is certainly in the kernel of $\iota_h$ we have slightly extended the notion. In other words, the Torelli group of algebraic geometry is $\text{Tor}/\text{Diff}_0$ with respect to the Torelli group defined $\text{Tor}$ here.
the moduli–space of complex structures for suitable complex \((p+1)\)–folds and the duality rotations are related with changes of integer homology basis. From the physical point of view what requires the restriction from the continuous duality groups \(Sp(2\pi,\mathbb{R}), SO(\pi,\pi,\mathbb{Z})\) to their discrete counterparts \(Sp(2\pi,\mathbb{Z}), SO(\pi,\pi,\mathbb{Z})\) is the Dirac quantization condition of electric and magnetic charges eq. \([1]\) which obviously occurs when electric and magnetic currents are introduced. Indeed the lattice spanned by electric and magnetic charges is eventually identified with the integer homology lattice of the corresponding \((p+1)\)–fold.

In view of this analogy, the natural question which arises is the following: what is the counterpart in algebraic geometry of the matrix \(\mathcal{N}\) that appears in the kinetic terms of the gauge fields? In view of its transformation property (see eq. \([42]\)) the answer is very simple: it is the period matrix. Consider for instance the situation, occurring in Calabi–Yau three–folds, where the middle cohomology group \(H_{DR}(\mathcal{M}_3)\) admits a Hodge–decomposition of the type:

\[
H_{DR}^{(3)}(\mathcal{M}_n) = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}
\]

and where the canonical bundle is trivial:

\[
c_1(T\mathcal{M}) = 0 \iff \dim H^{(3,0)} = 1
\]

Naming \(\Omega^{(3,0)}\) the unique (up to a multiplicative constant) holomorphic 3–form, and choosing a canonical homology basis of 3–cycles \((A^\alpha, B_\Sigma)\) satisfying:

\[
\begin{align*}
A^\alpha \cap A^\Sigma &= 0 & A^\alpha \cap B_\Delta &= \delta^\alpha_\Delta \\
B_\Gamma \cap A^\Sigma &= -\delta^\Sigma_\Gamma & B_\Gamma \cap B_\Delta &= 0
\end{align*}
\]

where \(\Lambda, \Sigma \ldots = 1, \ldots, \pi = 1 + h^{(2,1)}\)

we can define the periods:

\[
\begin{align*}
X^\Lambda(\phi) &= \int_{A^\Lambda} \Omega^{(3,0)}(\phi) \\
F_\Sigma(\phi) &= \int_{B_\Sigma} \Omega^{(3,0)}(\phi)
\end{align*}
\]

where \(\phi^i \ (i = 1, \ldots, h^{(2,1)})\) are the moduli of the complex structures and we can implicitly define the period matrix by the relation:

\[
\mathbf{F}_\Lambda = \mathbf{N}_{\Lambda\Sigma} \Omega^{\Sigma}, \quad \frac{\partial F_\Lambda}{\partial \phi^i} = \mathbf{N}_{\Lambda\Sigma} \frac{\partial \Omega^{\Sigma}}{\partial \phi^i}
\]

Under a diffeomorphism \(\xi\) of the manifold of complex structures the period vector

\[
\mathbf{V}(\phi) = \begin{pmatrix} X^\Lambda(\phi) \\
F_\Sigma(\phi) \end{pmatrix}
\]

will transform linearly through the \(Sp(2\pi,\mathbb{R})\) matrix \(\iota_h(\xi)\) defined by the homomorphism in eq. \([52]\) and the period matrix \(\mathcal{N}\) will obey the linear fractional transformation rule of eq. \([53]\). Indeed the intersection relations in eq. \([54]\) define the symplectic invariant metric \(\begin{pmatrix} 0 & \mathbb{I} \\
-\mathbb{I} & 0 \end{pmatrix}\).

Alternatively we can consider the more familiar example of a Riemann surface of genus \(g\). Introducing a basis of homology one–cycles \((A^\alpha, B_\beta)\) that satisfy the analogue of eq. \([53]\):

\[
A^\alpha \cap A^\beta = 0 \quad A^\alpha \cap B_\gamma = \delta^\alpha_\gamma \quad B_\mu \cap A^\beta = -\delta^\beta_\mu \quad B_\mu \cap B_\gamma = 0
\]

where \(\alpha, \beta \ldots = 1, \ldots, \pi = g\)

and a basis of holomorphic differentials \(\omega_i \ (i = 1, \ldots, g)\) we can set:

\[
\begin{align*}
f^\alpha_i &= \int_{A^\alpha} \omega_i & h^\alpha_{ij} &= \int_{B^\beta} \omega_j \\
&= \int_{A^\alpha} \bar{\omega}_i & h^\alpha_{ij} &= \int_{B^\beta} \bar{\omega}_j
\end{align*}
\]

which provides the standard definition of the period matrix \(\mathcal{N}\) in Riemann surface theory 5.

The two examples we have recalled here from algebraic geometry will be relevant for the issue of exact quantum duality symmetries in the context of local, respectively global \(N=2\) theories and we shall have more to say about them in the sequel. What should be clear from the above discussion is that a family of Lagrangians as in eq. \([24]\) will admit a group of duality–rotations/field–redefinitions that will map one into the other

5usually one normalizes the choice of holomorphic differentials in such a way that \(f^\alpha_i = \delta^\alpha_i\)
member of the family, as long as a kinetic matrix \( N_{\Lambda \Sigma} \) can be constructed that transforms as in eq. 66. A way to obtain such an object is to identify it with the period matrix occurring in problems of algebraic geometry. At the level of the present discussion, however, this identification is by no means essential: any construction of \( N_{\Lambda \Sigma} \) with the appropriate transformation properties is acceptable.

Note also that so far we have used the words *duality–rotations/field–redefinitions* and not the word duality symmetry. Indeed the diffeomorphisms of the scalar manifold we have considered were quite general and, as such had no pretension to be symmetries of the action, or of the theory. Indeed the question we have answered is the following: what are the appropriate transformation properties of the tensor gauge fields and of the generalized coupling constants under diffeomorphisms of the scalar manifold? The next question is obviously that of duality symmetries. Suppose that a certain diffeomorphism \( \xi \in \text{Diff}(M_\text{scalar}) \) is actually an isometry of the scalar metric \( g_{\mu \nu} \). Naming \( \xi^* : T M_\text{scalar} \rightarrow TM_\text{scalar} \) the push–forward of \( \xi \), this means that

\[
\forall X, Y \in TM_\text{scalar} \quad g(X, Y) = g(\xi^* X, \xi^* Y)
\] (64)

and \( \xi \) is an exact global symmetry of the scalar part of the Lagrangian in eq. 22. The obvious question is: "can this symmetry be extended to a symmetry of the complete action?" Clearly the answer is that, in general, this is not possible. The best we can do is to extend it to a symmetry of the field equations plus Bianchi identities letting it act as a duality rotation on the field-strengths plus their duals. This requires that the group of isometries of the scalar metric \( T(M_\text{scalar}) \) be suitably embedded into the duality group (either \( Sp(2\pi, \mathbb{R}) \) or \( SO(\mathbb{R}, \mathbb{R}) \) depending on the case) and that the kinetic matrix \( N_{\Lambda \Sigma} \) satisfies the covariance law:

\[
\mathcal{N} (\xi(\phi)) = (C_{\xi} + D_{\xi} N(\phi)) (A_{\xi} + B_{\xi} N(\phi))
\] (65)

A general class of solutions to this programme can be derived in the case where the scalar manifold is taken to be a homogeneous space \( G/H \). This is the subject of next section.

### 4. Symplectic embeddings of homogeneous spaces \( G/H \), the period matrix \( \mathcal{N} \) and the structure of extended supergravities

As remarked in the last section, the problem of constructing duality–symmetric lagrangians of type [22] admits general solutions when the scalar manifold is a homogeneous space \( G/H \). This is what happens in all extended supergravities for \( N \geq 3 \) and also in specific instances of \( N=2 \) theories. For this reason I devote the present section to a review of the construction of the *kinetic period matrix* \( \mathcal{N} \) in the case of homogeneous spaces. From now on I also concentrate on the case of \( D = 4 \) Minkowski space, so that the relevant homomorphism \( \iota_5 \) (see eq. 49) becomes:

\[
\iota_5 : \text{Diff} \left( \frac{G}{H} \right) \rightarrow Sp(2\pi, \mathbb{R})
\] (66)

In particular, focusing on the isometry group of the canonical metric defined on \( \frac{G}{H} \):

\[
\mathcal{I} \left( \frac{G}{H} \right) = G
\] (67)

we must consider the embedding:

\[
\iota_5 : G \rightarrow Sp(2\pi, \mathbb{R})
\] (68)

That in eq. 66 is a homomorphism of finite dimensional Lie groups and as such it constitutes a problem that can be solved in explicit form. What we just need to know is the dimension of the symplectic group, namely the number \( 2\pi \) of gauge fields appearing in the theory. Without supersymmetry the dimension \( m \) of the scalar manifold (namely the possible choices of \( \frac{G}{H} \)) and the number of vectors \( \pi \) are unrelated so that the possibilities covered by eq. 68 are infinitely many. In supersymmetric theories, instead, the two numbers \( m \) and \( \pi \) are related, so that there are finitely many cases to be studied corresponding to the possible embeddings of given groups \( G \) into a symplectic group \( Sp(2\pi, \mathbb{R}) \) of fixed dimension \( \pi \). Actually taking into account further conditions on the holonomy of the scalar manifold that are also

\[\text{Actually, in order to be true, eq. 66 requires that the normaliser of } H \text{ in } G \text{ be the identity group, a condition that is verified in all the relevant examples.}\]
imposed by supersymmetry, the solution for the symplectic embedding problem is unique for all extended supergravities with \( N \geq 3 \) as we have already remarked. This yields the unique scalar manifold choice displayed in Table 1.

Apart from the details of the specific case considered once a symplectic embedding is given there is a general formula one can write down for the period matrix \( \mathcal{N} \) that guarantees symmetry \( (\mathcal{N}^T = \mathcal{N}) \) and the required transformation property \[ \sim \]. This is the first result I want to present.

The real symplectic group \( Sp(2n, \mathbb{R}) \) is defined as the set of all real \( 2n \times 2n \) matrices

\[
\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

satisfying the first of equations [40], namely

\[
\Lambda^T \Phi \Lambda = \Phi
\]

where

\[
\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

If we relax the condition that the matrix should be real but we still impose eq. [70] we obtain the definition of the complex symplectic group \( Sp(2n, \mathbb{C}) \). It is a well known fact that the following isomorphism is true:

\[
Sp(2n, \mathbb{R}) \sim Usp(2n, \mathbb{C}) \equiv Sp(2n, \Phi) \cap Usp(2n, \mathbb{C}) \quad (72)
\]

By definition an element \( S \in Usp(\mathbb{C}, \mathbb{C}) \) is a complex matrix that satisfies simultaneously eq. [74] and a pseudo-unitarity condition, that is:

\[
S^T \Phi S = \Phi
\]

\[
S^\dagger H S = H
\]

\[
H \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

The general block form of the matrix \( S \) is:

\[
S = \begin{pmatrix} T & V^* \\ V & T^* \end{pmatrix}
\]

and eq.s [74] are equivalent to:

\[
T^\dagger T - V^\dagger V = 1
\]

\[
T^\dagger V^* - V^\dagger T^* = 0
\]

The isomorphism of eq. [72] is explicitly realized by the so called Cayley matrix:

\[
C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}
\]

via the relation:

\[
S = CAc^{-1}
\]

which yields:

\[
T = \frac{1}{2}(A - iB) + \frac{1}{2}(C + iD)
\]

\[
V = \frac{1}{2}(A - iB) - \frac{1}{2}(C + iD)
\]

When we set \( V = 0 \) we obtain the subgroup \( U(\mathbb{C}) \subset Usp(\mathbb{C}, \mathbb{C}) \), that in the real basis is given by the subset of symplectic matrices of the form

\[
\begin{pmatrix} A & B \\ -B & A \end{pmatrix}
\]

The basic idea, to obtain the general formula for the period matrix, is that the symplectic embedding of the isometry group \( \mathcal{G} \) will be such that the isotropy subgroup \( \mathcal{H} \subset \mathcal{G} \) gets embedded into the maximal compact subgroup \( U(\mathbb{C}) \), namely:

\[
\mathcal{G} \xrightarrow{\iota_1} Usp(n, \mathbb{C})
\]

\[
\mathcal{G} \subset \mathcal{H} \xrightarrow{\iota} U(\mathbb{C}) \subset Usp(n, \mathbb{C})
\]

If this condition is realized let \( L(\phi) \) be a parametrization of the coset \( \mathcal{G}/\mathcal{H} \) by means of coset representatives. By this we mean the following. Let \( \phi \) be local coordinates on the manifold \( \mathcal{G}/\mathcal{H} \): to each point \( \phi \in \mathcal{G}/\mathcal{H} \) we assign an element \( L(\phi) \in \mathcal{G} \) in such a way that if \( \phi' \neq \phi \), then no \( h \in \mathcal{H} \) can exist such that \( L(\phi') = L(\phi) \cdot h \). In other words for each equivalence class of the coset (labelled by the coordinate \( \phi \)) we choose one representative element \( L(\phi) \) of the class. Relying on the symplectic embedding of eq. [72] we obtain a map:

\[
L(\phi) \rightarrow O(\phi) = \begin{pmatrix} U_0(\phi) & U^*_1(\phi) \\ U_1(\phi) & U^*_0(\phi) \end{pmatrix} \in Usp(\mathbb{C}, \mathbb{C})
\]

that associates to \( L(\phi) \) a coset representative of \( Usp(\mathbb{C}, \mathbb{C})/U(\mathbb{C}) \). By construction if \( \phi' \neq \phi \) no unitary \( \mathbb{C} \times \mathbb{C} \) matrix \( W \) can exist such that:

\[
O(\phi') = O(\phi) \begin{pmatrix} W & 0 \\ 0 & W^* \end{pmatrix}
\]
Table 1
Scalar Manifolds of Extended Supergravities

| N  | # scal. in scal.m. | # scal. in vec. m. | # scal. in grav. m. | # vect. in vec. m. | # vect. in grav. m. | $\Gamma_{cont}$ | $\mathcal{M}_{scalar}$ |
|----|-------------------|-------------------|-------------------|-------------------|-------------------|----------------|------------------|
| 1  | 2 m               | n                 | n                 |                  |                  | $\mathcal{I}$ | $\subset Sp(2n, \mathbb{R})$ | Kähler           |
| 2  | 4 m               | 2 n               | n                 | 1                |                  | $\mathcal{I}$ | $\subset Sp(2n + 2, \mathbb{R})$ | Quaternionic $\otimes$ Special Kähler |
| 3  | 6 n               | n                 | n                 | 3                |                  | SU(3, n) | $\subset Sp(2n + 6, \mathbb{R})$ | SU(3, n) $\otimes$ SU(3) $\times$ U(n) |
| 4  | 6 n               | 2 n               | n                 | 6                |                  | SU(1, 1) $\otimes$ SO(6, n) | $\subset Sp(2n + 12, \mathbb{R})$ | SU(1, 1) $\otimes$ SO(6) $\times$ SO(n) |
| 5  |                  | 10                | 10                |                  |                  | SU(1, 5) | $\subset Sp(20, \mathbb{R})$ | SU(1, 5) $\otimes$ U(5) |
| 6  |                  | 30                | 16                |                  |                  | SO*(12) | $\subset Sp(32, \mathbb{R})$ | SO*(12) $\otimes$ U(1) $\times$ SU(6) |
| 7,8 |                  | 70                | 56                |                  |                  | $E_{7(-7)}$ | $\subset Sp(128, \mathbb{R})$ | $E_{7(-7)}$ $\otimes$ SU(8) |
On the other hand let $\xi \in \mathcal{G}$ be an element of the isometry group of $\mathcal{G}/\mathcal{H}$. Via the symplectic embedding of eq. (79) we obtain a $Usp(\pi, \pi)$ matrix

$$\mathcal{S}_{\xi} = \begin{pmatrix} T_\xi & V_\xi^* \\ V_\xi & T_\xi^* \end{pmatrix}$$

(82)

such that

$$\mathcal{S}_{\xi} \mathcal{O}(\phi) = \mathcal{O}(\xi(\phi)) \begin{pmatrix} W(\xi, \phi) & 0 \\ 0 & W^*(\xi, \phi) \end{pmatrix}$$

(83)

where $\xi(\phi)$ denotes the image of the point $\phi \in \mathcal{G}/\mathcal{H}$ through $\xi$ and $W(\xi, \phi)$ is a suitable $U(\pi)$ compensator depending both on $\xi$ and $\phi$. Combining eqs. (82) with eqs. (78) we immediately obtain:

$$U_0^\dagger (\xi(\phi)) + U_1^\dagger (\xi(\phi)) = W^+ \left[ U_0^\dagger (\phi) \left( A^T + iB^T \right) + U_1^\dagger (\phi) \left( A^T - iB^T \right) \right]$$

$$U_0^\dagger (\xi(\phi)) - U_1^\dagger (\xi(\phi)) = W^- \left[ U_0^\dagger (\phi) \left( D^T - iC^T \right) - U_1^\dagger (\phi) \left( D^T + iC^T \right) \right]$$

(84)

Setting:

$$\mathcal{N} \equiv \imath \left[ U_0^\dagger + U_1^\dagger \right]^{-1} \left[ U_0^\dagger - U_1^\dagger \right]$$

(85)

and using the result of eq. (84) one verifies that the transformation rule (83) is verified. It is also an immediate consequence of the analogue of eqs. (78) satisfied by $U_0$ and $U_1$ that the matrix in eq. (85) is symmetric

$$\mathcal{N}^T = \mathcal{N}$$

(86)

Eq. (85) is the masterformula derived in 1981 by Gaiûllard and Zumino [24]. It explains the structure of the gauge field kinetic terms in all $N \geq 3$ extended supergravity theories and also in those $N = 2$ theories where, using the parlance of my later lectures, the Special Kähler manifold $\mathcal{S}\mathcal{M}$ is a homogeneous manifold $\mathcal{G}/\mathcal{H}$. In particular, using eq. (85) we can easily retrieve the structure of $N = 4$ supergravity, extensively discussed in Sen’s lectures [10] as the basic example of theory where Strong–Weak duality is realized through Electric–Magnetic duality rotations. In Sen’s approach the structure of the $N = 4$ bosonic Lagrangian is derived by means of dimensional reduction from ten dimensions. Obviously the same structure can be directly derived in $D = 4$ by supersymmetry, since the $N = 4$ theory is unique. Actually, given the information (following from $N = 4$ supersymmetry) that the scalar manifold is the following coset manifold (see table 1):

$$\mathcal{M}^{N=4}_{scalar} = \mathcal{S}\mathcal{T} [6, n]$$

$$\mathcal{S}\mathcal{T} [m, n] = \frac{SU(1, 1)}{U(1)} \otimes \frac{SO(m, n)}{SO(m) \otimes SO(n)}$$

(87)

what we just need to study is the symplectic embedding of the coset manifolds $\mathcal{S}\mathcal{T} [6, n]$ where $n$ is the number vector multiplets in the theory. This is what I do in the next subsection where I actually consider the general case of $\mathcal{S}\mathcal{T} [m, n]$ manifolds.

4.1. Symplectic embedding of the $\mathcal{S}\mathcal{T} [m, n]$ homogeneous manifolds

The first thing I should do is to justify the name I have given to the particular class of coset manifolds I propose to study. The letters $\mathcal{S}\mathcal{T}$ stand for space–time and target space duality. Indeed, the isometry group of the $\mathcal{S}\mathcal{T} [m, n]$ manifolds defined in eq. (87) contains a factor ($SU(1, 1)$) whose transformations act as non–perturbative $S$–dualities and another factor $(SO(m, n))$ whose transformations act as $T$–dualities, holding true at each order in string perturbation theory. Furthermore $\mathcal{S}$ is the traditional name given, in superstring theory, to the complex field obtained by combining together the dilaton $D$ and axion $\mathcal{A}$:

$$S = A - i\exp[D]$$

$$\partial^\mu A = \epsilon^{\mu \nu \rho \sigma} \partial_\nu B_{\rho \sigma}$$

(88)

while $t^i$ is the name usually given to the moduli–fields of the compactified target space. Now in string and supergravity applications $S$ will be identified with the complex coordinate on the manifold $\frac{SU(1, 1)}{U(1)}$, while $t^i$ will be the coordinates of the coset space $\frac{SO(m, n)}{SO(m) \otimes SO(n)}$. Although as differentiable and metric manifolds the spaces $\mathcal{S}\mathcal{T} [m, n]$ are just direct products of two factors (corresponding to the above mentioned different physical interpretation of the coordinates $S$ and $t^i$), from the point of view of the symplectic embedding and duality rotations they have to be
regarded as a single entity. This is even more evident in the case \( m = 2, n = \) arbitrary, where the following theorem has been proven by Ferrara and Van Proeyen \([11]\): \( ST[2, n] \) are the only special Kähler manifolds with a direct product structure. For the definition of special Kähler manifolds I ask the audience to be patient and wait until my further lectures, yet the anticipation of this result should make clear that the special Kähler structure (encoding the duality rotations in the \( N = 2 \) case) is not a property of the individual factors but of the product as a whole. Neither factor is by itself a special manifold although the product is.

From this anticipation it also appears that the proposed study of symplectic embedding is relevant not only for the unique \( N = 4 \) theory, but also for a class of \( N = 2 \) theories. Are they physically relevant? Very much so, since the \( ST[2, r] \) special manifold is what emerges, at tree level, as the local structure. Indeed the only thing special about the global structure of the scalar manifold. What is the most exciting results is:

\[
\pi_1 \left( \hat{ST}[6, 22] \right) = SL(2, \mathbb{Z}) \otimes SO(6, 22, \mathbb{Z}) \]  

(91)
is just the restriction to the integers \( \mathbb{Z} \) of the original continuous duality group \( SL(2, \mathbb{R}) \otimes SO(6, 22, \mathbb{R}) \) associated with the manifold \( ST[6, 22] \). After modding by this discrete group the only duality–rotations that survive as exact duality symmetries of the quantum theory are those contained in \( \pi_1 \left( \hat{ST}[6, 22] \right) \) itself. This happens because of the Dirac quantization condition in eq. \([90]\) of electric and magnetic charges, the lattice spanned by these charges being invariant under the discrete group of eq. \([89]\). At this junction the relevance, in the quantum theory, of the symplectic embedding should appear. What does restriction to the integers exactly, mean? It means that the image in \( Sp(56, \mathbb{R}) \) of those matrices of \( SL(2, \mathbb{R}) \times SO(6, 22, \mathbb{R}) \) that are retained as elements of \( \pi_1 \left( \hat{ST}[6, 22] \right) \) should be integer valued. In other words we define:

\[
SL(2, \mathbb{Z}) \times SO(6, 22, \mathbb{Z}) \equiv \iota_3 \left( SL(2, \mathbb{R}) \times SO(6, 22, \mathbb{R}) \right) \cap Sp(56, \mathbb{Z}) \]  

(92)
we have determined an embedding $\iota_\delta$ that obeys the law in eq. 93, then:

$$\forall \mathcal{S} \in Sp(2\pi, \mathbb{R}) : \iota_\delta' \equiv \mathcal{S} \circ \iota_\delta \circ \mathcal{S}^{-1}$$

(94)

will obey the same law. That in eq. 94 is a symplectic transformation that corresponds to an allowed duality–rotation/field–redefinition in the abelian theory of type in eq. 20 discussed in the previous subsection. Therefore all abelian lagrangians related by such transformations are physically equivalent.

The matter changes in presence of gauging. When we switch on the gauge coupling constant and the electric charges, symplectic transformations cease to yield physically equivalent theories. This is the second issue in the above list. The choice of a symplectic gauge becomes physically significant. As I have emphasized in the introduction, the construction of supergravity theories proceeds in two steps. In the first step, which is the most extensive and complicated, one constructs the abelian theory: at that level the only relevant constraint is that encoded in eq. 93 and the choice of a symplectic gauge is immaterial. Actually one can write the entire theory in such a way that symplectic covariance is manifest. In the second step one gauges the theory. This breaks symplectic covariance and the choice of the correct symplectic gauge becomes a physical issue. This issue has been recently emphasized by the results in [42] where it has been shown that whether N=2 supersymmetry can be spontaneously broken to N=1 or not depends on the symplectic gauge.

These facts being cleared I proceed to discuss the symplectic embedding of the ST $[m, n]$ manifolds.

Let $\eta$ be the symmetric flat metric with signature $(m, n)$ that defines the $SO(m, n)$ group, via the relation

$$L \in SO(m, n) \iff L^T \eta L = \eta$$

(95)

Both in the $N = 4$ and in the $N = 2$ theory, the number of gauge fields in the theory is given by:

$\#\text{vector fields} = m \oplus n$

(96)

$m$ being the number of graviphotons and $n$ the number of vector multiplets. Hence we have to embed $SO(m, n)$ into $Sp(2m + 2n, \mathbb{R})$ and the explicit form of the decomposition in eq. 93 required by supersymmetry is:

$$2m + 2n \xrightarrow{SO(m, n)} m + n \oplus m + n$$

(97)

where $m + n$ denotes the fundamental representation of $SO(m, n)$. Eq. 97 is easily understood in physical terms. $SO(m, n)$ must be a T–duality group, namely a symmetry holding true order by order in perturbation theory. As such it must rotate electric field strengths into electric field strengths and magnetic field strengths into magnetic field strengths. The two irreducible representations into which the fundamental representation of the symplectic group decomposes when reduced to $SO(m, n)$ correspond precisely to electric and magnetic sectors, respectively. In the simplest gauge the symplectic embedding satisfying eq. 97 is block–diagonal and takes the form:

$$\forall L \in SO(m, n) \mapsto \left( \begin{array}{cc} L & 0 \\ 0 & (L^T)^{-1} \end{array} \right) \in Sp(2m + 2n, \mathbb{R})$$

(98)

Consider instead the group $SU(1, 1) \sim SL(2, \mathbb{R})$. This is the factor in the isometry group of $ST[m, n]$ that is going to act by means of S–duality non perturbative rotations. Typically it will rotate each electric field strength into its homologous magnetic one. Correspondingly supersymmetry implies that its embedding into the symplectic group must satisfy the following condition:

$$2m + 2n \xrightarrow{SL(2, \mathbb{R})} \bigoplus_{i=1}^{m+n} 2$$

(99)

where $2$ denotes the fundamental representation of $SL(2, \mathbb{R})$. In addition it must commute with the embedding of $SO(m, n)$ in eq. 98. Both conditions are fulfilled by setting:

$$\forall \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{R}) \mapsto \left( \begin{array}{cc} a & b \eta \\ c & d \end{array} \right) \in Sp(2m + 2n, \mathbb{R})$$

(100)

Utilizing eq.s 77 the corresponding embeddings into the group $Usp(m+n, m+n)$ are immediately
derived:
\[ \forall L \in SO(m, n) \quad \overset{\eta}{\mapsto} \]
\[ \left( \frac{1}{2} (L + \eta L\eta) \right) \]
\[ \left( \frac{1}{2} (L - \eta L\eta) \right) \]
\[ \in Usp(m + n, m + n) \]
\[ \forall \left( \begin{array}{cc} t & v \\ v^* & t^* \end{array} \right) \in SU(1, 1) \quad \overset{\eta}{\mapsto} \]
\[ \left( \begin{array}{cc} \text{Re} \mathbb{1} + i\text{Im}\eta & \text{Re} \mathbb{1} - i\text{Im}\eta \\ \text{Re} \mathbb{1} + i\text{Im}\eta & \text{Re} \mathbb{1} - i\text{Im}\eta \end{array} \right) \]
\[ \in Usp(m + n, m + n) \]  
(101)

where the relation between the entries of the $SU(1, 1)$ matrix and those of the corresponding $SL(2, \mathbb{R})$ matrix are provided by the relation in eq. 78.

Equipped with these relations we can proceed to derive the explicit form of the period matrix $\mathcal{N}$.

The homogeneous manifold $SU(1, 1)/U(1)$ can be conveniently parametrized in terms of a single complex coordinate $S$, whose physical interpretation will be that of an axion–dilaton, according to eq. 88. The custo metric parametrization appropriate for comparison with other constructions (Special Geometry (see later sections) or dimensional reduction (see for instance \[43\]) is given by the matrices:

\[ M(S) = \frac{1}{n(S)} \left( \begin{array}{cc} \mathbb{1} & iS \\ iS & \mathbb{1} \end{array} \right) \]  
(102)

\[ n(S) = \sqrt{\frac{4\text{Im} S}{1 + |S|^2 + 2\text{Im} S}} \]  
(103)

To parametrize the coset $SO(m, n)/SO(m) \times SO(n)$ we can instead take the usual coset representatives (see for instance \[13\]):

\[ L(X) = \left( \begin{array}{cc} \mathbb{1} + XX^T \end{array} \right)^{1/2} X \]
\[ \left( \begin{array}{cc} X^T \\ \mathbb{1} + X^TX \end{array} \right)^{1/2} \]  
(104)

where the $m \times n$ real matrix $X$ provides a set of independent coordinates. Inserting these matrices into the embedding formulae of eq.s 101 we obtain a matrix:

\[ Usp(n + m, n + m) \ni \tau(S, X) \in L(X) \]
\[ \left( \begin{array}{cc} U_0(S, X) & U_1(S, X) \\ U_1(S, X) & U_0(S, X) \end{array} \right) \]
(105)

that inserted into the master formula of eq. 86 yields the following result:

\[ \mathcal{N} = i\text{Im} S \eta L(X) L^T(X) \eta + \text{Re} S \eta \]  
(106)

Alternatively, remarking that if $L(X)$ is an $SO(m, n)$ matrix also $L(X)' = \eta L(X) \eta$ is such a matrix and represents the same equivalence class, we can rewrite eq. 105 in the simpler form:

\[ \mathcal{N} = i\text{Im} S L(X)' L^{T'}(X) + \text{Re} S \eta \]  
(107)

4.2. The bosonic lagrangian of $N = 4$ supergravity

As anticipated, eq.s 105 and 106 are the group–theoretical explanation of the form taken by $N = 2$ supergravity when the special manifold $ST[2, n]$ is used as scalar manifold in the vector multiplet sector. To this point I will come back after discussing the notion of special Kähler geometry. But eq. 105 is also the group theoretical explanation of the $N = 4$ lagrangian in the specific form discussed by Sen in his lectures at this school \[40\] and best suited to study $S$ and $T$ duality. Let me then show how such a lagrangian is retrieved from the previous discussion focusing on the case $ST[6, n]$.

Since we deal with the bosonic sector of $N = 4$ supergravity we have, in addition to the scalar and vector fields, also the graviton. Correspondingly we write:

\[ \mathcal{L}_{\text{bose}}^{N=4} = \sqrt{-g} \left[ R[g] \right. \]
\[ + \frac{1}{4(\text{Im} S)^2} \partial_{\mu} S \partial^{\rho} S \]
\[ - \frac{1}{4} \text{Im} S F^A_{\mu \nu} (\eta M \eta)_{\Lambda \Sigma} F^\Sigma_{\mu \nu} \]
\[ + \frac{1}{8\sqrt{-g}} \text{Re} S F^A_{\mu \nu} \eta_\Lambda \Sigma F_{\rho \sigma} \epsilon^{\mu \nu \rho \sigma} \]
\[ - \frac{1}{4} \text{Tr} (\partial_{\mu} M \partial^{\mu} M) \]  
(108)

where, by definition I have set:

\[ M = L(X) L^T(X) \]  
(109)

By construction the matrix $M(X)$ is symmetric. Comparison of eq.s 107 with eq.s 20 and
be the left invariant 1–form on the coset manifold \( \Theta \)
follows. Let the target manifold isumno\( m \) and choose for the pseudo–orthogonal metric \( \eta \) the following form:

\[
\eta = \begin{pmatrix}
O_{m \times m} & I_{m \times m} & O_{m \times r} \\
I_{m \times m} & 0_{m \times m} & O_{m \times r} \\
O_{r \times m} & O_{r \times m} & I_{r \times r}
\end{pmatrix}
\] (110)

In this basis the subalgebra

\[
SO(m) \otimes SO(m + r) \subset SO(m, m + r)
\] (111)
is made by those matrices \( \Lambda \) that in addition to satisfying the condition:

\[
\eta \Lambda^T + L \eta = 0
\] (112)
are also antisymmetric:

\[
\Lambda = -\Lambda^T
\] (113)

Relying on this fact the decomposition of the left invariant 1–form along the subalgebra and coset directions becomes very simple. The 1H–connection is just the antisymmetric part of \( \Theta \), while the vielbein on the coset manifold is given by its symmetric part:

\[
E = \frac{1}{2} (\Theta + \Theta^T)
\] (114)

It is then a matter of straightforward algebra to verify that the coset manifold metric

\[
ds^2 \equiv \text{Tr} (E \otimes E)
\] (115)
can be rewritten as

\[
ds^2 = -\frac{1}{4} \text{Tr} (dM \otimes dM)
\] (116)

the matrix \( M(X) \) being defined in eq. [108]. This proves why \(-\frac{1}{4} \text{Tr} (\partial_{\mu} M \partial^{\mu} M)\) is a permissible way of writing the sigma–model lagrangian when the target manifold is \( SO(m, n) / SO(m) \times SO(n) \).

Actually the entries of the symmetric matrix \( M \) can be taken as coordinates on the coset manifold.

In Sen’s lectures the lagrangian of eq. [107] is derived by dimensional reduction from \( D=10 \) supergravity and it is the starting point of all duality considerations. In my lectures I have derived it from duality–covariance and from the symplectic embedding of the sigma–model target manifold \( ST[m, n] \). For \( m = 6 \) it is indeed the bosonic lagrangian of \( N = 4 \) supergravity, but for \( m = 2 \) it is the bosonic action of a specific \( N = 2 \) supergravity: that with no hypermultiplets and with \( ST[2, n] \) as special manifold in the vector multiplet sector. It is then appropriate to turn to the general definition of special Kähler manifolds.

5. Special Kähler Geometry

Let me begin by reviewing the notions of Kähler and Hodge–Kähler manifolds that are the prerequisites to introduce the notion of Special Kähler manifolds.

5.1. Kähler manifolds

Let \( M \) be a 2n-dimensional manifold with a complex structure \( J : TM \rightarrow TM, J^2 = -I \).

A metric \( g \) on \( M \) is hermitian with respect to \( J \) if

\[
g(Ju, Jw) = g(u, w)
\] (117)

Given the metric \( g \) and the complex structure \( J \) we introduce the following differential 2-form \( K \):

\[
K(u, \bar{w}) = \frac{1}{2\pi} g(Ju, \bar{w})
\] (118)

The components \( K_{\alpha\beta} \) of \( K \) are given by

\[
K_{\alpha\beta} = g_{\gamma\beta} J^\gamma_\alpha
\] (119)

and by direct computation we can easily verify that \( g \) is hermitian if and only if \( K \) is anti-symmetric. By definition a hermitian complex manifold is a complex manifold endowed with a hermitian metric \( g \). In a well-adapted basis:

\[
J \frac{\partial}{\partial z^i} = i \frac{\partial}{\partial \bar{z}^i}
\] (120)

we can write \( g(u, w) = g_{ij} u^i w^j + g_{r, j} u^i w^j + g_{i, j} u^i w^j \). Reality of \( g(u, w) \) implies
\( g_{ij} = (g_{ij'})^*, \quad g_{i'j} = (g_{ij})^* \), symmetry yields
\( g_{ij} = g_{ji}, \quad g_{i'j'} = g_{ij'} \), while the hermiticity condition gives
\( g_{ij} = g_{i'j'} = 0 \). Hence in a well-adapted basis the 2-form \( K \) associated to the hermitian metric \( g \) can be written as follows:

\[
K = \frac{i}{2\pi} g_{ij} dz^i \wedge d\bar{z}^j \quad (121)
\]

A hermitian metric on a complex manifold \( \mathcal{M} \) is named a \( \Kahler \) metric if the associated 2-form \( K \) is closed:

\[
dK = 0 \quad (122)
\]

A hermitian complex manifold endowed with a \( \Kahler \) metric is called a \( \Kahler \) manifold and \( K \) is named its \( \Kahler \) 2–form. The Dolbeault cohomology class \([K]\) \( \in H^{(1,1)} \) of the \( \Kahler \) 2–form is named the \( \Kahler \) class of the metric. Equation (122) is a differential equation for \( g_{ij} \) whose general solution, in any local chart, is given by the following expression:

\[
g_{ij} = \partial_i \partial_j \mathcal{K} \quad (123)
\]

where \( \mathcal{K} = \mathcal{K}^* = \mathcal{K}(z, z^*) \) is a real function of \( z^i, z^{i'} \). The function \( \mathcal{K} \) is named the \( \Kahler \) potential and it is defined only up to the real part of a holomorphic function \( f(z) \). Indeed one sees that

\[
\mathcal{K}'(z, z^*) = \mathcal{K}(z, z^*) + \Re f(z) \quad (124)
\]

gives rise to the same metric \( g_{ij} \), as \( \mathcal{K} \). The transformation in eq. (122) is called a \( \Kahler \) transformation.

To fix our notations we write the formulae for the Levi–Civita connection 1–form and Riemann curvature 2–form on a \( \Kahler \) manifold:

\[
\begin{align*}
\Gamma^i_{jk} &= \Gamma^i_{kj} \quad \Gamma^i_{k'j'} = (\partial_j g_{k\ell})
\Gamma^i_{j'k'} = \Gamma^i_{k'j'} \quad \Gamma^i_{k'j'} = (\partial_j g_{k\ell})
\Gamma^i_{j'k'} = \Gamma^i_{k'j'}
\Gamma^i_{j'k'} &= (\partial_j g_{k\ell})
\Gamma^i_{j'k'} &= \partial_k \Gamma^i_{j\ell}
\Gamma^i_{j'k'} &= \partial_k \Gamma^i_{j\ell}
\Gamma^i_{j'k'} &= \partial_k \Gamma^i_{j\ell}
\end{align*}
\]

\[
\begin{align*}
R^i_{jk} &= R^{i}_{jk} = \partial_k \Gamma^i_{j\ell}
R^i_{jk} &= R^{i}_{jk} = \partial_k \Gamma^i_{j\ell}
R^i_{jk} &= R^{i}_{jk} = \partial_k \Gamma^i_{j\ell}
R^i_{jk} &= R^{i}_{jk} = \partial_k \Gamma^i_{j\ell}
R^i_{jk} &= \partial_k \Gamma^i_{j\ell}
\end{align*}
\]

(125)

The Ricci tensor has a remarkable simple expression:

\[
R^m_{ni} = R^m_{ni} = \partial_m \Gamma^i_{ni} = \partial_m \partial_n \log (\sqrt{g}) \quad (126)
\]

where \( g = \det |g_{\alpha\beta}| = (\det |g_{\alpha\beta}|)^2 \).

### 5.2. Hodge–Kähler manifolds

Consider next a line bundle \( \mathcal{L} \rightarrow \mathcal{M} \) over the \( \Kahler \) manifold. By definition this is a holomorphic vector bundle of rank \( r = 1 \). For such bundles the only available Chern class is the first:

\[
c_1(\mathcal{L}) = \frac{i}{2\pi} \bar{\partial} \partial \log h
\]

where the 1-component real function \( h(z, \bar{z}) \) is some hermitian fibre metric on \( \mathcal{L} \). Let \( \xi(z) \) be a holomorphic section of the line bundle \( \mathcal{L} \): noting that under the action of the operator \( \bar{\partial} \partial \) the term \( \log (\xi(z) \xi(z)) \) yields a vanishing contribution, we conclude that the formula in eq. (127) for the first Chern class can be re-expressed as follows:

\[
c_1(\mathcal{L}) = \frac{i}{2\pi} \bar{\partial} \partial \log || \xi(z) ||^2 \quad (128)
\]

where \( || \xi(z) ||^2 = h(z, \bar{z}) \xi(z) \) denotes the norm of the holomorphic section \( \xi(z) \).

Eq. (128) is the starting point for the definition of Hodge–Kähler manifolds, an essential notion in supergravity theory.

A \( \Kahler \) manifold \( \mathcal{M} \) is a Hodge manifold if and only if there exists a line bundle \( \mathcal{L} \rightarrow \mathcal{M} \) such that its first Chern class equals the cohomology class of the \( \Kahler \) 2–form \( K \):

\[
c_1(\mathcal{L}) = [K] \quad (129)
\]

In local terms this means that there is a holomorphic section \( W(z) \) such that we can write

\[
K = \frac{i}{2\pi} g_{ij} dz^i \wedge d\bar{z}^j
= \frac{i}{2\pi} \bar{\partial} \partial \log || W(z) ||^2 \quad (130)
\]

Recalling the local expression of the \( \Kahler \) metric in terms of the \( \Kahler \) potential \( g_{ij} = \partial_i \partial_j \mathcal{K}(z, \bar{z}) \), it follows from eq. (130) that
if the manifold $\mathcal{M}$ is a Hodge manifold, then the exponential of the Kähler potential can be interpreted as the metric $h(z, \overline{z}) = \exp(K(z, \overline{z}))$ on an appropriate line bundle $\mathcal{L}$.

This structure is precisely that advocated by the lagrangian of $N=1$ matter coupled supergravity: the holomorphic section $W(z)$ of the line bundle $\mathcal{L}$ is what, in N=1 supergravity theory is the superpotential and the logarithm of its norm $\log \| W(z) \|^2 = K(z, \overline{z}) + \log |W(z)|^2 = G(z, \overline{z})$ is precisely the invariant function in terms of which one writes the potential and Yukawa coupling terms of the supergravity action.

5.3. Special Kähler Manifolds: general discussion

As I have emphasized several times there are in fact two kinds of special Kähler geometry: the local and the rigid one. The former describes the scalar field sector of vector multiplets in $N=2$ supergravity while the latter describes the same sector in rigid $N=2$ Yang–Mills theories. Since $N=2$ includes $N=1$ supersymmetry, local and rigid special Kähler manifolds must be compatible with the geometric structures that are respectively enforced by local and rigid $N=1$ supersymmetry in the scalar sector. What is the distinction between the two cases in the $N=1$ theory? It deals with the first Chern–class of the line–bundle $\mathcal{L} \rightarrow \mathcal{M}$, whose sections are the possible superpotentials. In the local theory $c_1(\mathcal{L}) = [K]$ and this restricts $\mathcal{M}$ to be a Hodge–Kähler manifold. In the rigid theory, instead, we have $c_1(\mathcal{L}) = 0$. At the level of the lagrangian this reflects into a different behaviour of the fermion fields. These latter are sections of $\mathcal{L}^{1/2}$ and couple to the canonical hermitian connection defined on $\mathcal{L}$:

$$\theta \equiv h^{-1} \partial h = \frac{1}{h} \partial_i h \, dz^i$$
$$\overline{\theta} \equiv h^{-1} \overline{\partial} h = \frac{1}{h} \partial_i h \, d\overline{z}^i$$

In the local case where

$$[\overline{\partial} \theta] = c_1(\mathcal{L}) = [K]$$

the fibre metric $h$ can be identified with the exponential of the Kähler potential and we obtain:

$$\theta = \partial K = \partial_i K dz^i$$
$$\overline{\theta} = \partial \overline{K} = \partial_{\overline{i}} \overline{K} d\overline{z}^{\overline{i}}$$

In the rigid case, $\mathcal{L}$ is instead a flat bundle and its metric is unrelated to the Kähler potential. Actually one can choose a vanishing connection:

$$\theta = \overline{\theta} = 0$$

The distinction between rigid and local special manifolds is the $N=2$ generalization of this difference occurring at the $N=1$ level. In the $N=2$ case, in addition to the line–bundle $\mathcal{L}$ we need a flat holomorphic vector bundle $SV \rightarrow \mathcal{M}$ whose sections can be identified with the fermi–fermi components of electric and magnetic field–strengths. In this way, according to the discussion of previous sections the diffeomorphisms of the scalar manifolds will be lifted to produce an action on the gauge–field strengths as well. In a supersymmetric theory where scalars and gauge fields belong to the same multiplet this is a mandatory condition. However this symplectic bundle structure must be made compatible with the line–bundle structure already requested by $N=1$ supersymmetry. This leads to the existence of two kinds of special geometry. Another essential distinction between the two kind of geometries arises from the different number of vector fields in the theory. In the rigid case this number equals that of the vector multiplets so that

$$\# \text{vector fields} \equiv \pi = n$$
$$\# \text{vector multiplets} \equiv n = \dim_{\mathbb{C}} \mathcal{M}$$
$$\text{rank} SV \equiv 2\pi = 2n$$

On the other hand, in the local case, in addition to the vector fields arising from the vector multiplets we have also the graviphoton coming from the graviton multiplet. Hence we conclude:

$$\# \text{vector fields} \equiv \pi = n + 1$$
$$\# \text{vector multiplets} \equiv n = \dim_{\mathbb{C}} \mathcal{M}$$
$$\text{rank} SV \equiv 2\pi = 2n + 2$$
In the sequel we make extensive use of covariant derivatives with respect to the canonical connection of the line–bundle $\mathcal{L}$. Let us review its normalization. As it is well known there exists a correspondence between line–bundles and $U(1)$–bundles. If $\exp[f_{\alpha\beta}(z)]$ is the transition function between two local trivializations of the line–bundle $\mathcal{L} \rightarrow \mathcal{M}$, the transition function in the corresponding principal $U(1)$–bundle $\mathcal{U} \rightarrow \mathcal{M}$ is just $\exp[\text{Im}f_{\alpha\beta}(z)]$. At the level of connections this correspondence is formulated by setting:

$$\nabla_{(1)} \text{connection} \equiv Q = \text{Im}\theta = -\frac{i}{2}(\theta - \overline{\theta})$$ (137)

If we apply the above formula to the case of the $U(1)$–bundle $\mathcal{U} \rightarrow \mathcal{M}$ associated with the line–bundle $\mathcal{L}$ whose first Chern class equals the Kähler class, we get:

$$Q = -\frac{i}{2}(\partial_i K dz^i - \partial_i K dz^s)$$ (138)

Let now $\Phi(z, \overline{\tau})$ be a section of $\mathcal{U}^p$. By definition its covariant derivative is

$$\nabla \Phi = (d + ipQ)\Phi$$ (139)

or, in components,

$$\nabla_i \Phi = (\partial_i + \frac{1}{2}p\partial_i K)\Phi$$ (140)

$$\nabla_i^* \Phi = (\partial_i^* - \frac{1}{2}p\partial_i^* K)\Phi$$ (141)

A covariantly holomorphic section of $\mathcal{U}$ is defined by the equation:

$$\nabla_i^* \Phi = 0$$ (142)

We can easily map each section $\Phi(z, \overline{\tau})$ of $\mathcal{U}^p$ into a section of the line–bundle $\mathcal{L}$ by setting:

$$\tilde{\Phi} = e^{-pK/2}\Phi$$ (143)

With this position we obtain:

$$\nabla_i \tilde{\Phi} = (\partial_i + p\partial_i K)\tilde{\Phi}$$ (144)

$$\nabla_i^* \tilde{\Phi} = \partial_i^* \tilde{\Phi}$$ (145)

Under the map of eq. (143) covariantly holomorphic sections of $\mathcal{U}$ flow into holomorphic sections of $\mathcal{L}$ and vice versa.

### 5.4. Special Kähler manifolds: the local case

We are now ready to give the definition of local special Kähler manifolds and illustrate their properties. A first definition that does not make direct reference to the symplectic bundle is the following:

**Definition 5.1** A Hodge Kähler manifold is Special Kähler (of the local type) if there exists a completely symmetric holomorphic 3-index section $W_{ijk}$ of $(T^*\mathcal{M})^3 \otimes \mathcal{L}^2$ (and its antiholomorphic conjugate $W_{i\star j\star k\star}$) such that the following identity is satisfied by the Riemann tensor of the Levi–Civita connection:

$$\partial_{m^*} W_{ijk} = 0 \quad \partial_{m^*} W_{i\star j\star k\star} = 0$$

$$\nabla_{[m} W_{ijk]} = 0 \quad \nabla_{[m} W_{i\star j\star k\star]} = 0$$

$$\mathcal{R}^\ast_{i\star j\star k\star} = g_{i\star j} g_{k\star s} + g_{i\star k} g_{j\star s} + e^{2K} W_{i\star j\star s} W_{k\star j} g^{s\star t}$$ (146)

In the above equations $\nabla$ denotes the covariant derivative with respect to both the Levi–Civita and the $U(1)$ holomorphic connection of eq. (135). In the case of $W_{ijk}$, the $U(1)$ weight is $p = 2$.

The holomorphic sections $W_{ijk}$ have two different physical interpretations in the case that the special manifold is utilized as scalar manifold in an $N=1$ or $N=2$ theory. In the first case they correspond to the Yukawa couplings of Fermi families $E$. In the second case they provide the coefficients for the anomalous magnetic moments of the gauginos. Out of the $W_{ijk}$ we can construct covariantly holomorphic sections by setting:

$$C_{ijk} = W_{ijk} e^{K} ; \quad C_{i\star j\star k\star} = W_{i\star j\star k\star} e^{K}$$ (147)

Next we can give the second more intrinsic definition that relies on the notion of the flat symplectic bundle. Let $\mathcal{L} \rightarrow \mathcal{M}$ denote the complex line bundle whose first Chern class equals the Kähler form $K$ of an $n$-dimensional Hodge–Kähler manifold $\mathcal{M}$. Let $\mathcal{S}\mathcal{V} \rightarrow \mathcal{M}$ denote a holomorphic flat vector bundle of rank $2n + 2$ with structural group $Sp(2n + 2, \mathbb{R})$. Consider tensor bundles of the type $\mathcal{H} = \mathcal{S}\mathcal{V} \otimes \mathcal{L}$. A typical holomorphic section of such a bundle will be denoted by $\Omega$ and...
will have the following structure:

\[ \Omega = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \Lambda, \Sigma = 0,1, \ldots, n \] (148)

By definition the transition functions between two local trivializations \( U_i \subset M \) and \( U_j \subset M \) of the bundle \( \mathcal{H} \) have the following form:

\[
\left( \begin{array}{c} X \\ F \end{array} \right)_i = e^{f_{ij}} M_{ij} \left( \begin{array}{c} X \\ F \end{array} \right)_j
\] (149)

where \( f_{ij} \) are holomorphic maps \( U_i \cap U_j \rightarrow \mathfrak{g}^* \) while \( M_{ij} \) is a constant \( Sp(2n + 2, \mathbb{R}) \) matrix. For a consistent definition of the bundle the transition functions are obviously subject to the cocycle condition on a triple overlap:

\[
e^{f_{ij} + f_{jk} + f_{ki}} = 1
\]

\[M_{ij} M_{jk} M_{ki} = 1 \] (150)

Let \( i \langle \cdot \mid \cdot \rangle \) be the compatible hermitian metric on \( \mathcal{H} \)

\[
i\langle \Omega \mid \Omega \rangle = -i\Omega^\dagger \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \Omega
\] (151)

**Definition 5.2** We say that a Hodge–Kähler manifold \( M \) is special Kähler of the local type if there exists a bundle \( \mathcal{H} \) of the type described above such that for some section \( \Omega \in \Gamma(\mathcal{H}, \mathcal{M}) \) the Kähler two form is given by:

\[
K = \frac{i}{2\pi} \partial \bar{\partial} \log (i\langle \Omega \mid \Omega \rangle).
\] (152)

From the point of view of local properties, eq. 152 implies that we have an expression for the Kähler potential in terms of the holomorphic section \( \Omega \):

\[
\mathcal{K} = -\log (i\langle \Omega \mid \Omega \rangle) = -\log \left[ i \left( \frac{X^\Lambda}{F_\Sigma} F_\Lambda - F_\Sigma X^\Sigma \right) \right]
\] (153)

The relation between the two definitions of special manifolds is obtained by introducing a nonholomorphic section of the bundle \( \mathcal{H} \) according to:

\[
V = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv e^{\mathcal{K}/2} \Omega = e^{\mathcal{K}/2} \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix}
\] (154)

so that eq. 154 becomes:

\[
1 = i(V \mid V) = i \left( L^\Lambda M_\Lambda - M_\Sigma F^\Sigma \right)
\] (155)

Since \( V \) is related to a holomorphic section by eq. 154 it immediately follows that:

\[
\nabla_i V = \left( \partial_i - \frac{1}{2} \partial_i \mathcal{K} \right) V = 0
\] (156)

On the other hand, from eq. 155, defining:

\[
U_i = \nabla_i V = \left( \partial_i + \frac{1}{2} \partial_i \mathcal{K} \right) V \equiv \left( f_{i}^A \right)
\] (157)

it follows that:

\[
\nabla_i U_{j} = iC_{ijk} g^{kl} \gamma^l \nabla_k.
\] (158)

where \( \nabla_i \) denotes the covariant derivative containing both the Levi–Civita connection on the bundle \( \mathcal{T}M \) and the canonical connection \( \theta \) on the line bundle \( \mathcal{L} \). In eq. 158 the symbol \( C_{ijk} \) denotes a covariantly holomorphic ( \( \nabla_i C_{ijk} = 0 \) ) section of the bundle \( \mathcal{T}M^3 \otimes \mathcal{L}^2 \) that is totally symmetric in its indices. This tensor can be identified with the tensor of eq. 147 appearing in eq. 146. Indeed eq. 146 is just the integrability condition of eq. 153. The period matrix is now introduced via the relations:

\[
\bar{\mathcal{M}}_\Lambda = \bar{\mathcal{N}}_{\Lambda\Sigma} \mathcal{L}^\Sigma; \quad h_{\Sigma i} = \mathcal{N}_{\Lambda\Sigma} f_i^\Sigma
\] (159)

which can be solved introducing the two \((n+1) \times (n+1)\) vectors

\[
f_i^A = \begin{pmatrix} f_{i}^A \\ \mathcal{L}^A \end{pmatrix} ; \quad h_{\Lambda | I} = \begin{pmatrix} h_{\Lambda j} \\ \mathcal{M}_\Lambda \end{pmatrix}
\] (160)

and setting

\[
\mathcal{N}_{\Lambda \Sigma} = h_{\Lambda | I} \circ (f^{-1} \mid I)^{\dagger} \Sigma
\] (161)

As a consequence of its definition the matrix \( \mathcal{N} \) transforms, under diffeomorphisms of the base Kähler manifold exactly as it is requested by the rule in eq. 153. Indeed this is the very reason why the structure of special geometry has been introduced. The existence of the symplectic bundle \( \mathcal{H} \rightarrow \mathcal{M} \) is required in order to be able to pullback the action of diffeomorphisms on the field strengths and to construct the kinetic matrix \( \mathcal{N} \).
**Table 2**

*Homogeneous Symmetric Special Manifolds*

| n     | $G/H$                        | $Sp(2n + 2)$ | symp rep of $G$ |
|-------|------------------------------|--------------|-----------------|
| 1     | $SU(1,1)/U(1)$               | $Sp(4)$      | 4               |
| $n$   | $SU(1,n)/SU(n)\times U(1)$  | $Sp(2n + 2)$ | $n + 1 \oplus n + 1$ |
| $n + 1$ | $SU(1,1)/U(1) \otimes \frac{SO(2,n)}{SO(2) \times SO(n)}$ | $Sp(2n + 4)$ | $2 \otimes (n + 2 \oplus n + 2)$ |
| 6     | $Sp(6,\mathbb{R})/SU(3)\times U(1)$ | $Sp(14)$    | 14              |
| 9     | $SU(3,3)/SU(3)\times U(3)$  | $Sp(20)$     | 20              |
| 15    | $SO^*(12)/SU(6)\times U(1)$ | $Sp(32)$     | 32              |
| 27    | $E_{7(-6)}/E_6 \times SO(2)$ | $Sp(56)$     | 56              |
It is clear from our discussion that nowhere we have assumed the base Kähler manifold to be a homogeneous space. So, in general, special manifolds are not homogeneous spaces. Yet there is a subclass of homogenous special manifolds. The homogeneous symmetric ones were classified by Cremmer and Van Proeyen in [30] and are displayed in table 2. It goes without saying that for homogeneous special manifolds the two constructions of the period matrix, that provided by the master formula in eq. 85 and that given by eq. 161 must agree. We shall shortly verify it in the case of the manifolds $ST[2,n]$ that correspond to the second infinite family of homogeneous special manifolds displayed in table 2.

Anyhow, since special geometry guarantees the existence of a kinetic period matrix with the correct covariance property it is evident that to each existence of a kinetic period matrix with the cor-

7 the holomorphic sections of $\mathcal{L}$ would be the possible superpotentials if $\mathcal{M}$ were used as scalar manifold in an $N=1$ globally supersymmetric theory.
Just as in the local case eq. 168 yields an expression for the Kähler potential in terms of the holomorphic section \( \Omega \):

\[
\mathcal{K} = \left(i\left(\hat{\Omega} \tilde{\Omega}\right)\right) = \left[i\left(\nabla^\alpha F_\alpha - \mathcal{F}_\beta Y^\beta\right)\right] \tag{169}
\]

Similarly defining

\[
\hat{U}_i = \partial_i \hat{\Omega} \equiv \left(\frac{f_i}{h_{\beta ji}}\right) \tag{170}
\]

one finds:

\[
D_j \hat{U}_j = iC_{ijk} g^{k\ell} \hat{U}_{\ell} \tag{171}
\]

where \( D_j \) is the covariant derivative with respect to the Levi–Civita connection on \( T \mathcal{M} \) and where \( C_{ijk} \) is a totally symmetric holomorphic section of the bundle \( T \mathcal{M}^3 \otimes \mathcal{L}^2 \): \( \partial_\nu C_{ijk} = 0 \). Just as in the local case we obtain formulae that express the metric and the magnetic moments in terms of the symplectic sections:

\[
g_{ij^*} = -i(\hat{U}_i | \hat{U}_{j^*}) \quad C_{ijk} = \langle \partial_i \hat{U}_j | \hat{U}_k \rangle \tag{172}
\]

The integrability condition of eq. 171 is similar but different from eq. 146 due to the replacement of the covariant derivative on \( T \mathcal{M} \times \mathcal{L} \) by that on \( T \mathcal{M} \), due to the flatness of \( \mathcal{L} \). We get

\[
\partial_m C_{ijk} = 0 \quad \partial_m C_{ij^*k^*} = 0 \quad \nabla_{[m} C_{ijk]} = 0 \quad D_{[m} C_{ij^*k^*} = 0 \quad R_{ij^*k^*} = C_{ij^*k^*} g^{s^t} \tag{173}
\]

In a way similar to the local case, conditions 172 can be taken as an alternative definition of special geometry of the rigid type. The definition of the period matrix is obtained in full analogy to eq. 159:

\[
h_{\alpha i} = \mathcal{N}_{\alpha \beta} f_i^\beta \tag{174}
\]

that yields:

\[
\mathcal{N}_{\alpha \beta} = h_{\alpha i} \circ (f^{-1})^i_{\beta} \tag{175}
\]

### 5.6. Special Kähler manifolds: the issue of special coordinates

So far no privileged coordinate system has been chosen on the base Kähler manifold \( \mathcal{M} \) and no mention has been made of the holomorphic prepotential \( F(X) \) that is ubiquitous in the \( N = 2 \) literature and recently has been made famous, by Seiberg–Witten results, also to non supersymmetry experts. The simultaneous avoidance of privileged coordinates and of the prepotential is not accidental. Indeed, when the definition of special Kähler manifolds is given in intrinsic terms, as we did in the previous subsection, the holomorphic prepotential \( F(X) \) can be dispensed of. Although the structure of \( N = 2 \) supersymmetric theories is very often summarized by saying that the lagrangian is completely determined in terms of a single holomorphic function, the very existence of this function is not even guaranteed by special geometry which, alone, is the necessary and sufficient structure required to write down a supersymmetric lagrangian. Actually it appears that some of the physically most interesting cases are precisely instances where \( F(X) \) does not exist. Let us then see how the notion of \( F(X) \) emerges if we resort to special coordinate systems.

Note that under a Kähler transformation \( \mathcal{K} \rightarrow \mathcal{K} + \text{Re} f(z) \) the holomorphic section transforms, in the local case as \( \Omega \rightarrow \Omega e^{-f} \), so that we have \( X^\Lambda \rightarrow X^\Lambda e^{-f} \). This means that, at least locally, the upper half of \( \Omega \) associated with the electric field strengths can be regarded as a set \( X^\Lambda \) of homogeneous coordinates on \( \mathcal{M} \), provided that the jacobian matrix

\[
e^a_i(z) = \partial_i \left(\frac{X^a}{X^0}\right) \quad a = 1, \ldots, n \tag{176}
\]

is invertible. In this case, for the lower part of the symplectic section \( \Omega \) we obtain \( F_\Lambda = F_\Lambda (X) \). Recalling eqs. 155, 154 and 157 from which it follows

\[
0 = \langle V | U_i \rangle = X^\Lambda \partial_i F_\Lambda - \partial_i X^\Lambda F_\Lambda \tag{177}
\]

we obtain:

\[
X^\Sigma \partial_\Sigma F_\Lambda (x) = F_\Lambda (X) \tag{178}
\]
so that we can conclude:
\[ F_{\Lambda}(X) = \frac{\partial}{\partial X^\Lambda} F(X) \]  
where \( F(X) \) is a homogeneous function of degree 2 of the homogeneous coordinates \( X^\Lambda \). Therefore, when the condition in eq. (176) is verified we can use the special coordinates:
\[ t^a \equiv \frac{X^a}{X^0} \]  
and the whole geometric structure can be derived by a single holomorphic prepotential:
\[ F(t) \equiv (X^0)^{-2} F(X) \]  
In particular, eq. (154) for the Kähler potential becomes
\[ \mathcal{K}(t, \overline{t}) = -\log i \left[ 2 \left( F - \overline{F} \right) - \left( \partial_a F + \partial_{a'} \overline{F} \right) \left( t^a - \overline{t}^{a'} \right) \right] \]  
while eq. (162) for the magnetic moments simplifies into
\[ W_{abc} = \partial_a \partial_b \partial_c F(t) \]  

6. Examples of Special Manifolds of the local type

In this section I consider explicit examples of Special Kähler manifolds of the local type. In particular my goal is to let the audience appreciate the difference between the continuous duality groups that occur before the gauging of the theory and the discrete duality groups that occur in the effective quantum theory of the massless modes after symmetry breaking. On the concept of gauging I will make retour once I have explained also the other geometrical item needed to construct an \( N = 2 \) theory, namely hypergeometry. At this level of the discussion I just want to compare special manifolds with continuous or discrete isometry groups from a purely geometrical viewpoint.

The special manifolds with continuous dualities are, as it is obvious, homogeneous manifolds \( G/H \). Hence their special geometry is completely determined by the choice of the symplectic embedding. In particular the period matrix \( \mathcal{N} \) can be calculated by means of the master formula in eq. (85). As promised I want to show that such a calculation agrees with the definition of \( \mathcal{N} \) inside special geometry (see eq. (159)). I will do that in the particular case of the \( \mathbb{S}^7 [2, n] \) manifolds whose relevance for string theory effective actions has already been pointed out.

6.1. One–dimensional special manifolds

We begin with the simplest possible choice of special Kähler manifolds, those in complex dimension one. Let then
\[ \dim_{\mathbb{C}} SM = 1 \]  
A generic complex coordinate describing such a manifold will be denoted by \( z \) while a special coordinate will be denoted by \( t \). In an arbitrary symplectic basis the symplectic section \( \Omega \) has the following form:
\[ \Omega = \begin{pmatrix} X^0 \\ X^1 \\ F_0 \\ F_1 \end{pmatrix} \]  
In a symplectic basis where the holomorphic prepotential \( F(X) \) exists, setting, as in eq. (181):
\[ F(t) \equiv (X^0)^{-2} F(X) \]  
we obtain:
\[ \frac{1}{X^0} \Omega \equiv \tilde{\Omega}(t) = \begin{pmatrix} 1 \\ t \\ 2F(t) - t F'(t) \\ F'(t) \end{pmatrix} \]  
I will discuss three examples of one–dimensional special manifolds characterized by three different choices of the prepotential \( F(t) \). The first two cases are homogeneous manifolds and hence have a continuous duality group. The third case has only a discrete duality group. It is the moduli space of complex structures of the mirror quintic three–fold. Let us begin with the homogeneous manifolds. If we look at the classification of homogeneous symmetric special manifolds displayed in Table 3 we see that there are just two one–dimensional cases. They correspond to two
different symplectic embeddings:

\[ a) \quad \frac{SU(1,1)}{U(1)} \sim \frac{SL(2,\mathbb{R})}{SO(2)} \quad \mathcal{F}_2(t) = i \left( 1 - \frac{1}{2} t^2 \right) \]
\[ b) \quad \frac{SU(1,1)}{U(1)} \sim \frac{SL(2,\mathbb{R})}{SO(2)} \quad \mathcal{F}_3(t) = i \frac{1}{3!} d_0 t^3 \]

(188)

From the metric viewpoint the two manifolds are the same, but the Yukawa couplings/anomalous magnetic moments are different. In the first case one has \( W_{\mu} = 0 \), while in the second case one obtains \( W_{\mu} = d_0 = \text{const} \). Hence as special manifolds they are different. To prove what we have just stated I utilize the following strategy. Recalling eq. 102 that provides a coset parametrization for the manifold \( SU(1,1)/U(1) \), calculating the left–invariant one–form \( M^{-1}(s) dM(S) \) and extracting the vielbein component, we easily obtain the form of the invariant metric on such a coset:

\[ ds^2 = -p \frac{1}{(S - S')^2} dS \otimes dS \]

(189)

As far as the isometry group is concerned the real positive constant \( p > 0 \) appearing in eq. 189 is arbitrary since it just corresponds to an overall rescaling of the vielbein with a factor \( \sqrt{p} \). For any choice of this number \( p \in \mathbb{R}_+ \) the corresponding metric manifold is a copy of the homogeneous space \( SU(1,1)/U(1) \).

a) Upon the field identification

\[ t = \frac{S + i}{S - i} \]

(190)

the metric of eq. 189 can be obtained from the Kähler potential:

\[ K(t) = -\log \left[ 1 - |\mathcal{F}|^2 \right] \]

(191)

which follows from the choice a) of the holomorphic superpotential in eq. 188.

\[ \mathcal{F}(t) = \mathcal{F}_2(t) = (X^0)^2 F_2(X) \]
\[ F_2(X) = i (X_0^2 - X_1^2) \]

(192)

via the general formula in eq. 182

b) Alternatively, identifying

\[ S = t \]

(193)

the metric 189 can be retrieved from the following Kähler potential:

\[ K(t) = -p \log \left( t - \bar{t} \right) \]

(194)

For \( p = 3 \) eq. 194 is obtained from the general eq. 182 if the prepotential \( F(t) \) is chosen according to option b) of eq. 188:

\[ F(t) = F_3(t) = (X^0)^2 F_3(X) \]
\[ F_3(X) = \frac{1}{3!} d_0 \frac{X_1^3}{X_0} \]

(195)

From eqs. 193 and 192 utilizing the general formula in eq. 188 one easily verifies that in case b) the anomalous magnetic moment \( W_{\mu} \) is indeed equal to the non–zero constant \( d_0 \), while in the case a) it vanishes.

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\[ F_2(X) = i (X_0^2 - X_1^2) \]

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via the general formula in eq. 182

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the metric 189 can be retrieved from the following Kähler potential:

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For \( p = 3 \) eq. 194 is obtained from the general eq. 182 if the prepotential \( F(t) \) is chosen according to option b) of eq. 188:

\[ F(t) = F_3(t) = (X^0)^2 F_3(X) \]
\[ F_3(X) = \frac{1}{3!} d_0 \frac{X_1^3}{X_0} \]

(195)

From eqs. 193 and 192 utilizing the general formula in eq. 188 one easily verifies that in case b) the anomalous magnetic moment \( W_{\mu} \) is indeed equal to the non–zero constant \( d_0 \), while in the case a) it vanishes.

Group–theoretically case a) and case b) correspond to two different symplectic embeddings:

\[ M : SL(2, \mathbb{R}) \longrightarrow Sp(4, \mathbb{R}) \]

(196)

We have:

\[ a) \quad 4 \quad SL(2, \mathbb{R}) \longrightarrow 2 \oplus 2 \]
\[ b) \quad 4 \quad SL(2, \mathbb{R}) \sim \text{three-times symm.} \]

(197)

Explicitly the two embedding maps are displayed in Table 3. The proof that these two embeddings correspond to the two choices in eq. 192 and eq. 195 of the superpotential is done by checking that, for some overall rescaling \( \exp[\varphi_i(g, t)] \) the following transformation rule is true:

\[ M_i(g) \tilde{\Omega}_i(t) = \exp[\varphi_i(g, t)] \tilde{\Omega}_i \left( \frac{dt + c}{bt + a} \right) \]

(198)

\[ \forall g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{R}) \quad i = \left( \begin{array}{c} 2 \\ 3 \end{array} \right) \]

(199)

When one deals with string compactifications on Calabi–Yau threefolds \( M_3^{CY} \), the case b) of one–dimensional special manifolds emerges as the moduli space of Kähler class deformations \( \tilde{\Omega}_i \) whenever we have:

\[ h^{(1,1)} = \text{dim}_{\mathbb{C}} H^{(1,1)}(M_3^{CY}) = 1 \]

For an exhaustive review see, for instance the book by the present author.
Table 3
Symplectic embeddings of $SL(2, \mathbb{IR})$ in $Sp(4, \mathbb{IR})$

\[
\forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{IR}) \\
M_2(g) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \in Sp(4, \mathbb{IR})
\]

Yet the choice $F_3(t)$ is correct only in the large radius limit $m t \to \infty$. In this limit $d_0 \sim W_{ttt}$, is geometrically interpreted as the intersection number of three 2–cycles, (Poincaré duals to the $(1, 1)$–form $\omega^{(1,1)}$):

\[
d_0 = W_{ttt}^{(0)} = \int_{M_3^{CY}} \omega^{(1,1)} \wedge \omega^{(1,1)} \wedge \omega^{(1,1)}
\]

On the other hand at the quantum level we have infinitely many corrections due to world–sheet instantons \[38\]. Eq. 201 is replaced by:

\[
W_{ttt} = W_{ttt}^{(0)} + \sum_{k=1}^{\infty} n_k \frac{k^3 q^k}{1 - q} = \exp[2i\pi t]
\]

where $n_k = \text{number of rational curves of degree } k$ embedded in $M_3^{CY}$ \[38\]. This result is obtained from topological field–theories and for a review of the derivation I refer the audience to \[14\]. For the quintic threefold, the determination of the special manifold, with the values of $n_k$ appearing in the Yukawa coupling/anomalous magnetic moment explicitly evaluated, has been obtained by means of Picard–Fuchs equations on the mirror manifold \[38\]. A review of this procedure can also be found in \[14\]. Here we are not interested in these details. What we want to discuss is the form of the outcome special geometry. We have:

\[
\begin{align*}
F(t) &= F_\infty(t) + \Delta F(t) \\
F_\infty(t) &= \frac{1}{3!} 5 t^3 + \frac{11}{4} t^2 \\
&\quad - \frac{25}{12} t + i \frac{25 \zeta(3)}{\pi^3} \\
\Delta F(t) &= - i \sum_{N=1}^{\infty} \frac{d_N}{(2\pi N)^3} e^{2i\pi N t}
\end{align*}
\]

The quantum symplectic section

\[
\tilde{\Omega}_{\text{quantum}} (t) = \left( \begin{array}{c} 1 \\ 2 F(t) - t F'(t) \\ F'(t) \end{array} \right)
\]

has no longer a continuous group of duality rotations, but it rather admits a discrete, infinite, group of such rotations: $\Gamma_{\text{duality}} \subset Sp(4, \mathbb{Z})$ gen-
erated by the following two integer–valued symplectic matrices:

\[
A_q = \begin{pmatrix}
1 & 0 & -1 & 0 \\
-1 & 1 & 1 & 0 \\
5 & -8 & -4 & 1 \\
-3 & -5 & 3 & 1
\end{pmatrix},
\]

\[
T_q = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(206)

The interpretation of these generators of the duality group \( \Gamma \) is in terms of symmetries of the polynomial constraint \( W(X) = 0 \) defining the Calabi–Yau threefold as a vanishing locus in \( \mathbb{CP}^4 \)

\[
W(X, \psi) = \frac{1}{5} \sum_{i=1}^{5} X_i^5 - \psi \prod_{i=1}^{5} X_i
\]

(207)

and monodromies around the singular points of the Picard–Fuchs differential system. We have:

\[
A^5_q = \mathbb{I}
\]

\[\mathbb{Z}_5 \sim \Gamma_W \text{ symm of def. polyn.} \]

\[ T = T_0 \]

monodromy at \( \psi = \exp[2i\pi] = 1 \)

\[ T_k = A^{-k} T A^k \]

monodromy at \( \psi = \exp[2i\pi k/5] \)

\[ T_\infty = (AT)^5 \]

monodromy at \( \psi = \infty \)

\[ \mathbb{I} = T_0 T_1 T_2 T_3 T_4 T_\infty \]

\[ \Gamma_W = \frac{\Gamma_{duality}}{\Gamma_{monodromy}} \]

(208)

Monodromies, in particular refer to the singularities of the Picard–Fuchs equation in the non–special coordinate \( \psi \) that appears in eq. 207. The Picard–Fuchs equations precisely determine the relation between the special coordinate \( t \) and the non–special coordinate \( \psi \). Among the group elements of the duality group one determines the translation

\[ t \rightarrow t + 1 \]

(209)

of special coordinate. It is

\[
AT_q^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
-5 & 3 & 1 & -1 \\
8 & 5 & 0 & 1
\end{pmatrix}
\]

For comparison in the classical coset case the \( t \rightarrow t + 1 \) translation is effected by

\[
AT_{\text{class}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
-\frac{5}{2} & -\frac{3}{2} & 1 & -1 \\
\frac{5}{2} & \frac{3}{2} & 5 & 0 & 1
\end{pmatrix}
\]

(211)

which is symplectic but not integer! Similarly the matrix that performs a \( \mathbb{Z}_5 \) rotation in the classical coset manifold case is simply the \( Sp(4, \mathbb{R}) \) image of the \( O(2) \) rotation of an angle \( \theta = 2\pi/5 \) contained in \( SL(2, \mathbb{R}) \). This is displayed in Table 4 and it is far from being integer valued. What is the lesson learned from this example? The operations in the duality group have a geometrical origin in the cohomology of the target manifold \( M_{\text{3CY}} \) and are integer valued because they must map integer homology cycles into integer homology cycles. Yet if the Calabi–Yau threefold is utilized as a compactification space for a type II superstring all the operations of the duality group act as electric–magnetic duality in the sense discussed in previous lectures. Hence the integer–valuedness of the transformations can be alternatively traced back to the charge–lattice and to Dirac quantization condition in eq. 207. In the classical limit the theory has continuous duality symmetries that can be understood in terms of symplectic embeddings of Lie groups. The quantum theory has a discrete group of duality symmetries that, abstractly, is isomorphic to a discrete subgroup of the continuous one appearing in the classical case. Yet what changes from the classical to the quantum case is the symplectic embedding of this subgroup. This change of embedding is where all the non–perturbative effects of strong coupling physics reside.

I shall illustrate these ideas in my last lecture, presenting in minute details the case of Seiberg Witten solution of the N=2 SU(2) theory in terms of an auxiliary dynamical Riemann surface. Such a surface plays for rigid special geometry the role...
played by the Calabi–Yau threefolds in the case of local special geometry.

Let me now turn to another multidimensional example of special Kähler manifold of the local type.

6.2. The \(ST[2, n]\) special manifolds and the Calabi Visentini coordinates

When I studied the symplectic embeddings of the \(ST[m, n]\) manifolds, defined by eq. [5], a study that lead me to the general formula in eq. [106], I remarked that the subclass \(ST[2, n]\) constitutes a family of special Kähler manifolds: actually a quite relevant one. Here I survey the special geometry of this class.

Besides their applications in the large radius limit of superstring compactifications, the \(ST[2, n]\) manifolds are interesting under another respect. They provide an example where the holomorphic prepotential can be non–existing.

Consider a standard parametrization of the \(SO(2, n)/SO(2) \times SO(n)\) manifold, like for instance that in eq. [103]. In the \(m = 2\) case we can introduce a canonical complex structure on the manifold by setting:

\[
\Phi^\Lambda(X) = \frac{1}{\sqrt{2}} \left( L_0^\Lambda + i L_0^{\Sigma} \right)
\]

\(\Lambda = 0, 1, \alpha \quad \alpha = 2, \ldots, n + 1\) (212)

The relations satisfied by the upper two rows of the coset representative (consequence of \(L(X)\) being pseudo–orthogonal with respect to metric \(\eta_{\Lambda\Sigma} = \text{diag}(+, +, –, \ldots, –)\)):

\[
\begin{align*}
L_0^\Lambda L_0^{\Sigma} \eta_{\Lambda\Sigma} &= 1 \\
L_0^\Lambda L_1^{\Sigma} \eta_{\Lambda\Sigma} &= 0 \\
L_1^\Lambda L_1^{\Sigma} \eta_{\Lambda\Sigma} &= 1
\end{align*}
\]

(213)

can be summarized into the complex equations:

\[
\begin{align*}
\bar{\Phi}^\Lambda &\Phi^{\Sigma\eta_{\Lambda\Sigma}} = 1 \\
\Phi^\Lambda &\Phi^{\Sigma\eta_{\Lambda\Sigma}} = 0
\end{align*}
\]

(214)

Eq.s (214) are solved by posing:

\[
\Phi^\Lambda = \frac{X^\Lambda}{\sqrt{X^\Sigma X^\Lambda \eta_{\Lambda\Sigma}}}
\]

(215)

where \(X^\Lambda\) denotes any set of complex parameters, determined up to an overall multiplicative constant and satisfying the constraint:

\[
X^\Lambda X^\Sigma \eta_{\Lambda\Sigma} = 0
\]

(216)

In this way we have proved the identification, as differentiable manifolds, of the coset space \(SO(2, n)/SO(2) \times SO(n)\) with the vanishing locus of the quadric in eq. [214]. Taking any holomorphic solution of eq. [214], for instance:

\[
X^\Lambda(y) = \begin{pmatrix}
\frac{1}{2} (1 + y^2) \\
\frac{1}{2} (1 – y^2) \\
y^\alpha
\end{pmatrix}
\]

(217)

where \(y^\alpha\) is a set of \(n\) independent complex coordinates, inserting it into eq. [214] and comparing with eq. [213] we obtain the relation between whatever coordinates we had previously used to write the coset representative \(L(X)\) and the complex coordinates \(y^\alpha\). In other words we can regard the matrix \(L\) as a function of the \(y^\alpha\) that are named the Calabi Visentini coordinates.

Consider in addition the axion–dilaton field \(S\) that parametrizes the \(SU(1, 1)/U(1)\) coset according with eq. [103]. The special geometry of the manifold \(ST[2, n]\) is completely specified by

---

Table 4

The \(Z_5\) generator in the \(M_3\) embedding of \(SL(2, \mathbb{R})\)

| \(\begin{pmatrix} \cos[2\pi/5] & \sin[2\pi/5] \\ -\sin[2\pi/5] & \cos[2\pi/5] \end{pmatrix} \) | \(\begin{pmatrix} 0.0295085 & 0.272453 & -1.03229 & 0.33541 \\ -0.0908178 & -0.529508 & -0.33541 & -0.271441 \\ 0.716866 & -0.698771 & 0.0295085 & 0.0908178 \\ 0.698771 & 1.69651 & -0.272453 & -0.529508 \end{pmatrix} \) |
|---|---|---|---|

writing the holomorphic symplectic section $\Omega$ as follows (10):

$$\Omega(y, S) = \left( \frac{X^A}{F_A} \right) (218)$$

Notice that with the above choice, it is not possible to describe $F_A$ as derivatives of any prepotential. Yet everything else can be calculated utilizing the formulae I presented in previous lectures. The Kähler potential is:

$$\mathcal{K} = \mathcal{K}_1(S) + \mathcal{K}_2(y) = - \log(S - S) - \log X^T \eta X$$

The Kähler metric is block diagonal:

$$g_{ij} = \begin{pmatrix} g_{S \bar{S}} & 0 \\ 0 & g_{\alpha \bar{\alpha}} \end{pmatrix}$$

(220)

as expected. The anomalous magnetic moment-Yukawa couplings $C_{ijk}$ $(i = S, \alpha)$ have a very simple expression in the chosen coordinates:

$$C_{S\alpha} = - \exp[\mathcal{K}] \delta_{S\alpha},$$

all the other components being zero.

Using the definition of the period matrix given in eq. 164 we obtain

$$\mathcal{N}_{\Lambda \Sigma} = (S - S X S \bar{X} + X S X S \bar{X}) X^T \eta X$$

(223)

In order to see that eq. 223 just coincides with eq. 106 it suffices to note that as a consequence of its definition and of the pseudo–orthogonality of the coset representative $L(X)$, the vector $\Phi^A$ satisfies the following identity:

$$\Phi^A \bar{\Phi}^C + \Phi^C \bar{\Phi}^A = \frac{1}{2} L^A_{\Gamma} L^C_{\Delta} (\delta^{\Gamma \Delta} + \eta^{\Gamma \Delta})$$

(224)

Inserting eq. 224 into eq. 223 formula 106 is retrieved.

This completes the proof that the choice 218 of the special geometry holomorphic section corresponds to the symplectic embedding 28 and 106 of the coset manifold $\mathcal{S} T [2, n]$. In this symmetric gauge the symplectic transformations of the isometry group are the simplest possible ones and, anticipating the language of later lectures, the entire group $SO(2, n)$ is represented by means of classical transformations that do not mix electric fields with magnetic fields. The disadvantage of this basis, if any, is that there is no holomorphic prepotential. To find an $F(X)$ it suffices to make a symplectic rotation to a different basis.

If we set:

$$X^1 = \frac{1}{2}(1 + y^2) = - \frac{1}{2}(1 - \eta_{ij} t^i t^j)$$

$$X^2 = \frac{1}{2}(1 - y^2) = t^2$$

$$X^\alpha = \eta^\alpha = t^{2+\alpha} \quad \alpha = 1, \ldots, n - 1$$

(225)

where

$$\eta_{ij} = \text{diag}(+,-,\ldots, -) \quad i, j = 2, \ldots, n + 1$$

(226)

Then we can show that $\exists C \in Sp(2n + 2, \mathbb{R})$ such that:

$$C \left( \begin{array}{c} X^A \\ S_{\eta \Lambda \Sigma} X^A \end{array} \right) = \exp[\varphi(t)] \left( \begin{array}{c} 1 \\ S \\ t^i \\ \frac{S}{S \bar{S}} F \\ \frac{S \bar{S} F}{\bar{S} F} \end{array} \right)$$

(227)

with

$$F(S, t) = \frac{1}{2} S \eta_{ij} t^i t^j = \frac{1}{2} d_{IJK} t^I t^J t^K$$

$$t^i = S$$

$$d_{IJK} = \begin{cases} \eta_{ij} & \text{if } i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

(228)

and

$$W_{IJK} = d_{IJK} = \frac{\partial^3 F(S, t^i)}{\partial t^I \partial t^J \partial t^K}$$

(229)

This means that in the new basis the symplectic holomorphic section $C \Omega$ can be derived from the following cubic prepotential:

$$F(X) = \frac{1}{3!} d_{IJK} X^I X^J X^K$$

(230)
For instance in the case $n = 1$ the matrix which does such a job is:

$$
C = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

(231)

6.3. Comments on the $ST[2,2]$ case: S–duality and R–symmetry

To conclude let us focus on the case $ST[2,2]$. This manifold has two coordinates that we can either call $S$ and $t$, in the parametrization of eq. (228) or $S$ and $y$ in the Calabi Visentini basis. The relation between $t$ and $y$ simplifies enormously in this case:

$$
t = i \frac{y + 1}{y - 1}
$$

(232)

It is then a matter of choice to regard the holomorphic section in whatever basis as a function of $y$ or of $t$, in addition to $S$. Independently from this choice the manifold $ST[2,2]$ emerges as moduli space (at tree–level) in a locally N=2 supersymmetric gauge theory of a rank one gauge group, namely $SU(2)$. The two fields spanning the manifold have very different interpretations. The field $y$ is the scalar partner of the gauge field that remains massless after Higgs mechanism. Its vacuum expectation value is the modulus of the gauge theory. It is the same field that occurs also in a globally supersymmetric theory and which I shall amply discuss in my last lecture. On the other hand the field $S$ is the dilaton–axion. It plays the role of generalized coupling constant and generalized theta–angle. There are two $SL(2, \mathbb{R})$ groups embedded in $SP(6, \mathbb{R})$, they act as standard fractional linear transformations on the dilaton–axion $S$ and on the special coordinate $t$ for the gauge modulus. Using the Calabi–Visentini section of eq. (218) and the embedding eqs. (18) and (100) we have that

S–duality $S \rightarrow -1/S$ is generated by the symplectic matrix:

$$
S_{\text{duality}} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

(233)

while T–duality $t \rightarrow -1/t$ is generated by the symplectic matrix:

$$
R_{\text{symmetry}} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

(234)

If we think of the $t$–field as the modulus of some compact internal manifold then T–duality is just the transformation from small to large compactification radius. Looking at the same transformation in terms of the $y$ variable its meaning becomes more clear. It is $R$–symmetry $y \rightarrow -y$, an exact global symmetry of the microscopic lagrangian. The fact that the matrix generating T–duality or R–symmetry is block–diagonal agrees with the fact that this is a perturbative symmetry, holding at each order in perturbation theory and never exchanging electric with magnetic states. Very different is the nature of S–duality. Since it inverts the coupling constant it is by definition non–perturbative. It exchanges strong and weak coupling regimes and because of that it is supposed to exchange elementary states with soliton states. For this reason it must mix electric with magnetic field strengths and it is off–diagonal. These symmetries exist in the microscopic theory which is derived by gauging (see later lectures) the abelian theories possessing continuous duality symmetries (in this case the two $SL(2, \mathbb{R})$ groups). After gauging the continuous duality symmetries will be broken. The question is will the integer valued symplectic generators of S–duality and R–symmetry survive given that they respect the Dirac quantization condition? The answer is yes, but in the effective quantum theory they will be represented by new integer
valued elements of $Sp(6,\mathbb{Z})$ not derivable from the classical embedding. This is the same phenomenon already observed in the Calabi–Yau case with one modulus. The $\mathbb{Z}_5$ generator is not the image through the classical embedding map of the $\mathbb{Z}_5$ subgroup in the classical $O(2)$. Since the special geometry in the effective theory is corrected by the instanton contributions and has a new complicated transcendental structure, the duality generators must change basis to adapt themselves to the new situation and be integer valued in the new non–perturbative geometry. Alternatively one can turn matters around. If we know the new quantum symplectic embedding of the discrete duality group we have essentially determined the non perturbative geometry. It is this point of view that has proven very fruitful in the very recent literature.

7. Hypergeometry

Next we turn to the hypermultiplet sector of an $N = 2$ theory. Here there are 4 real scalar fields for each hypermultiplet and, at least locally, they can be regarded as the four components of a quaternion. The locality caveat is, in this case, very substantial because global quaternionic coordinates can be constructed only occasionally even on those manifolds that are denominated quaternionic in the mathematical literature. Anyhow, what is important is that, in the hypermultiplet sector, the scalar manifold $\mathcal{H}M$ has dimension multiple of four:

$$\dim_{\mathbb{R}} \mathcal{H}M = 4m$$

$m \equiv \# \text{ of hypermultiplets}$ (235)

and, in some appropriate sense, it has a quaternionic structure.

As Special Kähler is the collective name given to the vector multiplet geometry both in the rigid and in the local case, in the same way we name Hypergeometry that pertaining to the hypermultiplet sector, irrespectively whether we deal with global or local $N=2$ theories. Yet in the very same way as there are two kinds of special geometries, there are also two kind of hypergeometries and for a very similar reason. Supersymmetry requires the existence of a principal $SU(2)$–bundle

$$SU \rightarrow \mathcal{H}M$$

that plays for hypermultiplets the same role played by the the line–bundle $L \rightarrow SM$ in the case of vector multiplets. As it happens there the bundle $SU$ is flat in the rigid case while its curvature is proportional to the Kähler forms in the local case.

The difference with the case of vector multiplets is that rigid and local hypergeometries were already known in mathematics prior to their use in the context of $N = 2$ supersymmetry and had the following names:

rigid hypergeometry $\equiv$ HyperKähler geom.
local hypergeometry $=\text{ Quaternionic geom.}$ (237)

7.1. Quaternionic, versus HyperKähler manifolds

Both a quaternionic or a HyperKähler manifold $\mathcal{H}M$ is a $4m$-dimensional real manifold endowed with a metric $h$:

$$ds^2 = h_{uv}(q) dq^u \otimes dq^v \quad ; \quad u, v = 1, \ldots, 4m$$ (238)

and three complex structures

$$\left(J^x\right) : \ T(\mathcal{H}M) \rightarrow T(\mathcal{H}M)$$

$$(x = 1, 2, 3)$$ (239)

that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} \mathbb{I} + \epsilon^{xyz} J^z$$

(240)

and respect to which the metric is hermitian:

$$\forall X, Y \in T\mathcal{H}M : \ h(J^x X, J^y Y) = h(X, Y)$$

$$(x = 1, 2, 3)$$ (241)

From eq.$^{[21]}$ it follows that one can introduce a triplet of 2-forms

$$K^x = K^x_{uv} dq^u \wedge dq^v$$

$$K^x_{uv} = h_{uv}(J^x)^w_w$$

(242)

that provides the generalization of the concept of Kähler form occurring in the complex case. The triplet $K^x$ is named the HyperKähler form. It is an $SU(2)$ Lie–algebra valued 2–form in the same
way as the Kähler form is a $U(1)$ Lie-algebra valued 2–form. In the complex case the definition of Kähler manifold involves the statement that the Kähler 2–form is closed. At the same time in Hodge–Kähler manifolds (those appropriate to local supersymmetry) the Kähler 2–form can be identified with the curvature of a line–bundle which in the case of rigid supersymmetry is flat. Similar steps can be taken also here and lead to two possibilities: either HyperKähler or Quaternionic manifolds.

Let us introduce a principal $SU(2)$–bundle $SU$ as defined in eq. 236. Let $\omega^x$ denote a connection on such a bundle. To obtain either a HyperKähler or a quaternionic manifold we must impose the condition that the HyperKähler 2–form is covariantly closed with respect to the connection $\omega^z$:

$$\nabla K^x = dK^x + \epsilon^{xyz} \omega^y \wedge K^z = 0 \quad (243)$$

The only difference between the two kinds of geometries resides in the structure of the $SU$–bundle.

**Definition 7.1** A HyperKähler manifold is a 4$m$–dimensional manifold with the structure described above and such that the $SU$–bundle is flat.

Defining the $SU$–curvature by:

$$\Omega^x = d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z \quad (244)$$

in the HyperKähler case we have:

$$\Omega^x = 0 \quad (245)$$

Viceversa

**Definition 7.2** A quaternionic manifold is a 4$m$–dimensional manifold with the structure described above and such that the curvature of the $SU$–bundle is proportional to the HyperKähler 2–form.

Hence, in the quaternionic case we can write:

$$\Omega^x = \frac{1}{\lambda} K^x \quad (246)$$

where $\lambda$ is a non vanishing real number. Actually the limit of rigid supersymmetry can be identified with $\lambda \to \infty$.

As a consequence of the above structure the manifold $\mathcal{H}M$ has a holonomy group of the following type:

$$\text{Hol}(\mathcal{H}M) = SU(2) \otimes \mathcal{H} \quad \text{quater.}$$

$$\text{Hol}(\mathcal{H}M) = 1 \otimes \mathcal{H} \quad \text{HyperKähler}$$

$$\mathcal{H} \subset Sp(2m, \mathbb{R}) \quad (247)$$

In both cases, introducing flat indices $\{A, B, C = 1, 2\} \{\alpha, \beta, \gamma = 1, \ldots, 2m\}$ that run, respectively, in the fundamental representations of $SU(2)$ and $Sp(2m, \mathbb{R})$, we can find a vielbein 1-form

$$U^{A\alpha} = U^{A\alpha}_a (q) dq^a \quad (248)$$

such that

$$h_{uv} = U^{A\alpha}_a U^{B\beta}_B \epsilon_{AB} \quad (249)$$

where $C_{\alpha\beta} = -C_{\beta\alpha}$ and $\epsilon_{AB} = -\epsilon_{BA}$ are, respectively, the flat $Sp(2m)$ and $Sp(2) \sim SU(2)$ invariant metrics. The vielbein $\bar{U}^{A\alpha}$ is covariantly closed with respect to the $SU(2)$-connection $\omega^z$ and to some $Sp(2m, \mathbb{R})$-Lie Algebra valued connection $\Delta^{a\beta} = \Delta^{\beta a}$:

$$\nabla U^{A\alpha} \equiv dU^{A\alpha} + \frac{i}{2} \omega^{x} (\sigma^x \epsilon^{-1}) A_B \wedge U^{B\alpha} + \Delta^{a\beta} \wedge U^{A\gamma} C_{\beta\gamma} = 0 \quad (250)$$

where $(\sigma^x)_A^B$ are the standard Pauli matrices. Furthermore $\bar{U}^{A\alpha}$ satisfies the reality condition:

$$\bar{U}^{A\alpha} \equiv (U^{A\alpha})^* = \epsilon_{AB} C_{\alpha\beta} U^{B\beta} \quad (251)$$

Eq. (251) defines the rule to lower the symplectic indices by means of the flat symplectic metrics $\epsilon_{AB}$ and $C_{\alpha\beta}$. More specifically we can write a stronger version of eq. (249)

$$(U^{A\alpha}_u U^{B\beta}_v + U^{A\alpha}_v U^{B\beta}_u) C_{\alpha\beta} = h_{uv} \epsilon^{AB}$$

$$(U^{A\alpha}_u U^{B\beta}_v + U^{A\alpha}_v U^{B\beta}_u) \epsilon_{AB} = h_{uv} \frac{1}{m} \delta^{a\beta} \quad (252)$$

We have also the inverse vielbein $U^{A\alpha}_a$ defined by the equation

$$U^{A\alpha}_a U^{A\alpha}_a = \delta^a_v \quad (253)$$

Flattening a pair of indices of the Riemann tensor $R^{uv}_{ts}$ we obtain

$$R^{uv}_{ts} U^{A\alpha}_u U^{B\beta}_v = \Omega^x \frac{i}{2} (\epsilon^{-1} \sigma_x)^{AB} C^{\alpha\beta} + R^{ts}_{\beta a} \quad (254)$$
where $\mathbb{R}_{\alpha \beta}^{\alpha \beta}$ is the field strength of the $Sp(2m)$ connection:

$$d \mathcal{A} = \mathcal{A}^{\alpha \beta} \wedge \mathcal{A}^{\beta \delta} \mathcal{A}^{\gamma} \equiv \mathbb{R}_{\alpha \beta}^{\alpha \beta} dq^{\alpha} \wedge dq^{\beta} \quad (255)$$

Eq. (254) is the explicit statement that the Levi-Civita connection associated with the metric $h$ has a holonomy group contained in $SU(2) \otimes Sp(2m)$. Consider now eqs. 240, 242 and 246. We easily deduce the following relation:

$$h h_{uv}^{x} K_{x}^{y} = - \delta^{xy} h_{uv} + \epsilon^{xyz} K_{xyz} (256)$$

that holds true both in the Hyperkähler and in the Quaternionic case. In the latter case, where the $\lambda$ parameter is finite, eq. (256) can be rewritten as follows:

$$h h_{uv}^{x} \Omega_{x}^{y} = - \lambda^{x} h_{uv} + \lambda^{x} \Omega_{x}^{y} (257)$$

Eq. (257) implies that the intrinsic components of the curvature 2-form $\Omega^{x}$ yield a representation of the quaternion algebra. In the Hyperkähler case such a representation is provided only by the Hyperkähler form. In the quaternionic case we can write:

$$\Omega^{x}_{u, B} \equiv \Omega^{x}_{u, A} U^{u}_{A} U^{B} = -i M_{\alpha \beta}(\sigma^{x} \epsilon)_{AB} \quad (258)$$

Alternatively eq. (258) can be rewritten in an intrinsic form as

$$\Omega^{x} = i M_{\alpha \beta}(\sigma^{x} \epsilon)_{AB} U^{u}_{A} \wedge U^{B} \quad (259)$$

where from we also get:

$$\frac{i}{2} \Omega^{x}(\sigma^{x})_{A}^{B} = \lambda M_{\alpha \beta} \wedge U^{B} \quad (260)$$

Homogeneous symmetric quaternionic spaces are displayed in Table 3.

8. The Gauging

With the above discussion of Hyperkähler and Quaternionic manifolds I have completed my review of the geometric structures involved in the construction of abelian, ungauged $N = 2$ supergravity or of $N = 2$ rigid gauge theory. The relation of supersymmetry with duality–rotations has been repeatedly emphasized. Sticking to my promise of restricting my own attention to the bosonic sector of all lagrangians, the situation we have so far reached is the following. The bosonic Lagrangian of $N = 2$ supergravity coupled to $n$ abelian vector multiplets and $m$ hypermultiplets is the following:

$$\mathcal{L}_{\text{SUGRA}} = \sqrt{-g} \left[ R(g) + g_{ij}(z, \bar{z}) \partial^{i} z \partial^{j} \bar{z}^{i} + 2 \lambda h_{uv}(q) \partial^{\mu} q^{u} \partial^{\mu} q^{v} + i \left( \mathcal{N}_{\Lambda \Sigma} F_{\mu \nu}^{\Lambda} F^{-\Sigma} |_{\mu | \nu} - \mathcal{N}_{\Lambda \Sigma} F_{\mu \nu}^{\Lambda} F^{-\Sigma} |_{\mu | \nu} \right) \right] \quad (261)$$

where the $n$ complex fields $z^{i}$ span some special Kähler manifold of the local type $\mathcal{L}_{\text{SUGRA}}$ and the $4m$ real fields $q^{u}$ span a quaternionic manifold $\mathcal{L}_{\text{SUGRA}}$. By $g_{ij}$ and $h_{uv}$ we have denoted the metrics on these two manifolds. The proportionality constant between the $SU(2)$ curvature and the Hyperkähler form appearing in the lagrangian is fixed by to the value $\lambda = -1$ if we want canonical kinetic terms for the hypermultiplet scalars. The period matrix $\mathcal{N}_{\Lambda \Sigma}$ depends only on the special manifold coordinates $z^{i}$ and $z^{j}$ and it is expressed in terms of the symplectic sections of the flat symplectic bundle by eq. (63). On the other hand the bosonic Lagrangian of an $N = 2$ abelian gauge theory containing $n$ vector multiplets and coupled to $m$ hypermultiplets is the following one:

$$\mathcal{L}_{\text{YM}} = \sqrt{-g} \left[ g_{ij}(z, \bar{z}) \partial^{i} z \partial^{j} \bar{z}^{i} + 2 h_{uv}(q) \partial^{\mu} q^{u} \partial^{\mu} q^{v} + i \left( \mathcal{N}_{\alpha \beta} F_{\mu \nu}^{\alpha} F^{-\beta} |_{\mu | \nu} - \mathcal{N}_{\alpha \beta} F_{\mu \nu}^{\alpha} F^{-\beta} |_{\mu | \nu} \right) \right] \quad (262)$$

where the $n$ complex fields $z^{i}$ span some special Kähler manifold of the rigid type $\mathcal{L}_{\text{YM}}$ and the $4m$ real fields $q^{u}$ span a Quaternionic manifold $\mathcal{L}_{\text{YM}}$. By $g_{ij}$ and $h_{uv}$ we have denoted the metrics on these two manifolds. The period matrix $\mathcal{N}_{\alpha \beta}$ depends only on the special manifold coordinates $z^{i}$ and $z^{j}$ and it is expressed in terms of the symplectic sections of the flat symplectic bundle by eq. (73). In both theories there are no electric or magnetic currents and we have symplectic covariance. By means of the the first homomorphism in eq. (13) any diffeomorphism of the scalar manifold can be lifted to a symplectic transformation on
Table 5
Homogeneous symmetric quaternionic manifolds

| $m$ | $G/H$ |
|-----|-------|
| $m$ | $\frac{Sp(2m+2)}{Sp(2) \times Sp(2m)}$ |
| $m$ | $\frac{SU(m,2)}{SU(m) \times SU(2) \times U(1)}$ |
| $m$ | $\frac{SO(4,m)}{SO(4) \times SO(m)}$ |
| 2   | $\frac{G_2}{SO(3)}$ |
| 7   | $\frac{F_4}{Sp(6) \times Sp(2)}$ |
| 10  | $\frac{E_6}{SU(6) \times U(1)}$ |
| 16  | $\frac{E_7}{SO(12) \times SU(2)}$ |
| 28  | $\frac{E_8}{E_7 \times SU(2)}$ |
the electric–magnetic field strengths, the *period* matrix transforming, by construction covariantly as required by eq. [23]. Under this lifting any isometry of the scalar manifold becomes a symmetry not just of the lagrangian but of the differential system made by the equations of motions plus Bianchi identities. There are in fact three type of these isometries:

1. The *classical symmetries*, namely those isometries $\xi \in I(M_{\text{scalar}})$ whose image in the symplectic group is block–diagonal:

$$\iota_\delta(\xi) = \begin{pmatrix} A_\xi & 0 \\ 0 & (A_\xi^T)^{-1} \end{pmatrix}$$  \hspace{1cm} (263)

These transformations are exact ordinary symmetries of the lagrangian. They clearly form a subgroup

$$\text{Clas}(M_{\text{scalar}}) \subset I(M_{\text{scalar}})$$  \hspace{1cm} (264)

2. The *perturbative symmetries*, namely those isometries $\xi \in I(M_{\text{scalar}})$ whose image in the symplectic group is lower triangular:

$$\iota_\delta(\xi) = \begin{pmatrix} A_\xi & 0 \\ C_\xi & (A_\xi^T)^{-1} \end{pmatrix}$$  \hspace{1cm} (265)

These transformations map the electric field strengths into linear combinations of the gauge potentials. They are almost invariances of the action. Indeed the only non–invariance comes from the transformation of the period matrix

$$\mathcal{N} \longrightarrow (A_\xi^T)^{-1} \mathcal{N} (A_\xi)^{-1} + C_\xi (A_\xi^T)^{-1}$$  \hspace{1cm} (266)

Denoting collectively all the fields of the theory by $\Phi$ and utilizing eq.s [23, 24, 27] under a perturbative transformation the action changes as follows:

$$\int \mathcal{L}(\Phi) \, d^4x \rightarrow \int \mathcal{L}(\Phi') \, d^4x$$

$$+ 4 \Delta \theta_{\Lambda \Sigma} \int F^\Lambda \wedge F^\Sigma$$

$$\Delta \theta_{\Lambda \Sigma} = \left[ C_\xi (A_\xi^T)^{-1} \right]_{\Lambda \Sigma}$$  \hspace{1cm} (267)

The added term is a total derivative and does not affect the field equations. Quantum mechanically, however, it is relevant. It corresponds to a redefinition of the *theta–angle*. It yields a symmetry of the path–integral as long as the added term is an integer multiple of $2\pi \hbar$. This consideration will restrict the possible perturbative transformations to a discrete subgroup. In any case the group of perturbative isometries defined by eq. [265] contains the group of classical symmetries as a subgroup: $I(M_{\text{scalar}}) \supset \text{Pert}(M_{\text{scalar}}) \supset \text{Clas}(M_{\text{scalar}})$.

3. The *non–perturbative symmetries* namely those isometries $\xi \in I(M_{\text{scalar}})$ whose image in the symplectic group is of the form:

$$\iota_\delta(\xi) = \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix}$$  \hspace{1cm} (268)

with $B_\xi \neq 0$. These transformations are neither a symmetry of the classical action nor of the perturbative path integral. Yet they are a symmetry of the quantum theory. They exchange electric field strengths with magnetic ones, electric currents with magnetic ones and hence elementary excitations with soliton states.

The above discussion of duality symmetries may have left the audience intrigued about the following point. How can I talk about non–perturbative symmetries that exchange electric charges with magnetic charges if, so far, in the abelian theories described by eq.s [261] and [262] there are neither electric nor magnetic couplings? The answer is that the same general form of abelian theories encoded in these equations can be taken to represent two quite different things:

1. The fundamental theory prior to the gauging. It is neutral and abelian since the non–abelian interactions and the electric charges are introduced only by the gauging, but it contains all the fundamental fields.

2. The effective theory of the massless modes of the non–abelian theory. It is abelian and neutral because the only fields which remain
massless are, apart from the graviton, the multiplets in the Cartan subalgebra \( \mathcal{H} \subset \mathcal{G} \) of the gauge group and the neutral hypermultiplets corresponding to flat directions of the scalar potential.

What distinguishes the two cases is the type of scalar manifolds and their isometries.

In case 1) of the above list we have:

\[
\begin{align*}
\dim \mathcal{C} \mathcal{S} \mathcal{M} &= n \equiv \dim \mathcal{G} \\
\frac{1}{4} \dim \mathbb{R} \mathcal{H} \mathcal{M} &= \hat{m} \equiv \# \text{ of all hypermul.} \\
\mathcal{I} (\mathcal{S} \mathcal{M}) &= \text{cont. group } \supset \mathcal{G} \\
\mathcal{I} (\mathcal{H} \mathcal{M}) &= \text{cont. group } \supset \mathcal{G}
\end{align*}
\]

(269)

where \( \mathcal{G} \) denotes the gauge group.

In case 2) we have instead:

\[
\begin{align*}
\dim \mathcal{C} \mathcal{S} \mathcal{M} &= r \equiv \text{rank } \mathcal{G} \\
\frac{1}{4} \dim \mathbb{R} \mathcal{H} \mathcal{M} &= m \equiv \# \text{ of moduli hypermul.} \\
\mathcal{I} (\mathcal{S} \mathcal{M}) &= \text{discr. group} \\
\mathcal{I} (\mathcal{H} \mathcal{M}) &= \text{discr. group}
\end{align*}
\]

(270)

In the first instance (corresponding to eq. 269) the spaces \( \mathcal{S} \mathcal{M} \) and \( \mathcal{H} \mathcal{M} \) are chosen with a large continuous group of duality invariances that correspond to the concept of global symmetry of ordinary field theory. A subgroup of this global symmetry group can be made into a local symmetry of the theory by associating its generators with the vector fields already present in the theory. This is the above mentioned gauging procedure. Since \( \mathcal{G} \) is a subgroup of both isometry groups \( \mathcal{I} (\mathcal{S} \mathcal{M}) \) and \( \mathcal{I} (\mathcal{H} \mathcal{M}) \), it follows that \( \mathcal{G} \) acts non-trivially both on the vector multiplets and on the hypermultiplets. The former action introduces the non-abelian interactions, the latter action introduces the electric charges of the matter fields. For consistency of the gauging procedure the gauge group must be a subgroup of the group of classical symmetries \( \mathcal{G} \subset \text{Clas} (\mathcal{M}_{\text{scalar}}) \subset Sp(2n, \mathbb{R}) \) (271)

at the price of modifying the Lagrangian by the introduction of Chern–Simons like terms, de Wit and Van Proeyen have obtained the gauging also of perturbative transformations [53]. The gauge group in this case is non semisimple and the gauge algebra contains solvable subalgebras. In these lectures I confine my attention to the block diagonal gauging

As a consequence of gauging the lagrangians in eqs 261 and 262 get modified by the replacement of ordinary derivatives with covariant derivatives and by the introduction of new terms that are of two types:

1. fermion–fermion bilinears with scalar field dependent coefficients

2. A scalar potential \( \mathcal{V} \)

It is particularly nice and rewarding that all the modifications of the lagrangian and of the supersymmetry transformation rules can be described in terms of a very general geometric construction associated with the action of Lie–Groups on manifolds that admit a symplectic structure: the momentum map. In supersymmetry indeed, the geometric notion of momentum map has an exact correspondence with the notion of gauge multiplet auxiliary fields or D–fields. The next section is devoted to a review of the momentum map and to its applications in N=2 theories.

Prior to that, however, let me finish with my comparison of cases 1) and 2).

In the microscopic theory (eq. 269) the manifolds we start from to do the gauging are typically coset manifolds \( \mathcal{G}/\mathcal{H} \) for the supergravity case and flat manifolds for the rigid case. This comes from the need of an ample duality–symmetry group where the gauge group should be embedded. Indeed continuous isometries are allowed by non trivial Special Kähler manifolds of the local type and by Quaternionic manifolds. On the other hand Special Kähler manifolds of the rigid type or HyperKähler manifolds admit continuous isometries only if they are flat manifolds. Which particular choice of homogeneous Special Kähler manifold or quaternionic manifold is appropriate is predicted by tree–level string theory when we deal with the effective lagrangians of superstrings. In particular for the class of models dealt with in the recent literature on string–string duality the choice is:

\[
\mathcal{S} \mathcal{M} = ST[2, n]
\]

tions [53].
\( \mathcal{HM} = \frac{SO(4, \hat{m})}{SO(4) \otimes SO(\hat{m})} \)

(272)

After the gauging a scalar potential is introduced and most of the scalar fields acquire mass by Higgs effect. Yet the scalar potential admits flat directions. Those scalar fields that span the flat directions are the moduli and they fill the classical moduli space. For the vector multiplet sector this classical moduli space is easily deduced. It is:

\[ \mathcal{SM}_{\text{moduli}} = ST[2, r] \subset ST[2, n] \]

(273)

Indeed it suffices to restrict one's attention to the \( r \) fields in the Cartan's subalgebra of the gauge group. For the hypermultiplet moduli space we have:

\[ \mathcal{HM}_{\text{moduli}} = \left\{ \text{some hypersurface} \subset \frac{SO(4, \hat{m})}{SO(4) \otimes SO(\hat{m})} \right\} \]

(274)

When we deal with the effective lagrangian of the massless modes, which is the situation envisaged by eq.s 271, the two manifolds \( \mathcal{SM} \) and \( \mathcal{HM} \) are respectively identified with \( \mathcal{SM}_{\text{moduli}} \) and \( \mathcal{HM}_{\text{moduli}} \), the quantum moduli spaces of vector multiplets and hypermultiplets. There is no need of continuous isometries since there is no gauging to be done. This allows the manifolds to be non homogeneous, non flat manifolds depending on the case. Some discrete isometries however are usually present. They are of the utmost interest. Their symplectic lifting provides the non–perturbative duality symmetries discussed above that exchange elementary electric states with magnetic soliton states.

### 9. The Momentum Map

The momentum map is a construction that applies to all manifolds with a symplectic structure, in particular to Kähler, HyperKähler and Quaternionic manifolds.

Let us begin with the Kähler case, namely with the momentum map of holomorphic isometries. The HyperKähler and quaternionic case correspond, instead, to the momentum map of triholomorphic isometries.

#### 9.1. Holomorphic momentum map on Kähler manifolds

Let \( g_{ij} \) be the Kähler metric of a Kähler manifold \( \mathcal{M} \). As many time emphasized, it appears in the kinetic term of the scalar fields: the Wess–Zumino multiplet scalars in N=1 theories, the vector multiplet scalars in N=2 theories. If the metric \( g_{ij} \) has a non trivial group of continuous isometries \( G \) generated by Killing vectors \( k_A^i \) (\( A = 1, \ldots, \dim G \)), then the kinetic lagrangian admits \( G \) as a group of global space–time symmetries. Indeed under an infinitesimal variation

\[ z^i \rightarrow z^i + \epsilon^A k_A^i (z) \]

(275)

\( \mathcal{L}_{\text{kin}} \) remains invariant. Furthermore, if all the couplings of the scalar fields are performed in a diffeomorphic invariant way, then any isometry of \( g_{ij} \) extends from a symmetry of \( \mathcal{L}_{\text{kin}} \) to a symmetry of the whole lagrangian. Diffeomorphic invariance means that the scalar fields can appear only through the metric, the Christoffel symbol in the covariant derivative and through the curvature. Alternatively they can appear through sections of vector bundles constructed over \( \mathcal{M} \).

Typical case is the dependence on the scalar fields introduced by the period matrix \( \mathcal{N} \).

Let \( k_A^i (z) \) be a basis of holomorphic Killing vectors for the metric \( g_{ij} \). Holomorphicity means the following differential constraint:

\[ \partial_j k_A^i (z) = 0 \leftrightarrow \partial_j k_A^i (\mathbf{z}) = 0 \]

(276)

while the generic Killing equation (suppressing the gauge index \( \Lambda \)):

\[ \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0 \]

(277)

in holomorphic indices reads as follows:

\[ \nabla_i k_j + \nabla_j k_i = 0 \]

\[ \nabla_i k_j + \nabla_j k_i = 0 \]

(278)

where the covariant components are defined as \( k_j = g_{ji} k^i \) (and similarly for \( k_i \)).

The vectors \( k_A^i \) are generators of infinitesimal holomorphic coordinate transformations:

\[ \delta z^i = \epsilon^A k_A^i (z) \]

(279)

which leave the metric invariant. In the same way as the metric is the derivative of a more fundamental object, the Killing vectors in a Kähler
manifold are the derivatives of suitable prepotentials. Indeed the first of eq.s 278 is automatically satisfied by holomorphic vectors and the second equation reduces to the following one:

$$k^i_A = ig^{ij} \partial_j \mathcal{P}_A, \quad \mathcal{P}^*_A = \mathcal{P}_A$$  \hspace{1cm} (280)$$

In other words if we can find a real function $\mathcal{P}^*$ such that the expression $ig^{ij} \partial_j \mathcal{P}^*(\Lambda)$ is holomorphic, then eq. 280 defines a Killing vector.

The construction of the Killing prepotential can be stated in a more precise geometrical formulation which involves the notion of momentum map. Let me review this construction which reveals another deep connection between supersymmetry and geometry.

Consider a Kählerian manifold $\mathcal{M}$ of real dimension $2n$. Consider a compact Lie group $\mathcal{G}$ acting on $\mathcal{M}$ by means of Killing vector fields $\mathbf{X}$ which are holomorphic with respect to the complex structure $J$ of $\mathcal{M}$; then these vector fields preserve also the Kähler 2-form

$$\mathcal{L}_{\mathbf{X}} g = 0 \quad \Leftrightarrow \quad \nabla_{(\mu\nu)} X_{\nu} = 0 \bigg\} \Rightarrow \quad 0 = \mathcal{L}_{\mathbf{X}} K = i_{\mathbf{X}} dK + d(i_{\mathbf{X}} K) = d(i_{\mathbf{X}} K)$$  \hspace{1cm} (281)$$

Here $\mathcal{L}_{\mathbf{X}}$ and $i_{\mathbf{X}}$ denote respectively the Lie derivative along the vector field $\mathbf{X}$ and the contraction (of forms) with it.

If $\mathcal{M}$ is simply connected, $d(i_{\mathbf{X}} K) = 0$ implies the existence of a function $\mathcal{P}_{\mathbf{X}}$ such that

$$-\frac{1}{2\pi} d\mathcal{P}_{\mathbf{X}} = i_{\mathbf{X}} K$$  \hspace{1cm} (282)$$

The function $\mathcal{P}_{\mathbf{X}}$ is defined up to a constant, which can be arranged so as to make it equivariant:

$$\mathbf{X}\mathcal{P}_{\mathbf{Y}} = \mathcal{P}_{[\mathbf{X},\mathbf{Y}]}$$  \hspace{1cm} (283)$$

$\mathcal{P}_{\mathbf{X}}$ constitutes then a momentum map. This can be regarded as a map

$$\mathcal{P} : \mathcal{M} \longrightarrow \mathbb{R} \otimes \mathfrak{g}^*$$  \hspace{1cm} (284)$$

where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of the group $\mathcal{G}$. Indeed let $x \in \mathfrak{g}$ be the Lie algebra element corresponding to the Killing vector $\mathbf{X}$; then, for a given $m \in \mathcal{M}$

$$\mu(m) : x \longrightarrow \mathcal{P}_{\mathbf{X}}(m) \in \mathbb{R}$$  \hspace{1cm} (285)$$

is a linear functional on $\mathfrak{g}$. If we expand $\mathbf{X} = a^\Lambda k_\Lambda$ in a basis of Killing vectors $k_\Lambda$ such that

$$[k_\Lambda, k_\Gamma] = f^\Lambda_{\Lambda\Gamma} k_\Delta$$  \hspace{1cm} (286)$$

we have also

$$\mathcal{P}_{\mathbf{X}} = a^\Lambda \mathcal{P}_A$$  \hspace{1cm} (287)$$

In the following we use the shorthand notation $\mathcal{L}_\Lambda, i_\Lambda$ for the Lie derivative and the contraction along the chosen basis of Killing vectors $k_\Lambda$.

From a geometrical point of view the prepotential, or momentum map, $\mathcal{P}_\Lambda$ is the Hamiltonian function providing the Poissonian realization of the Lie algebra on the Kähler manifold. This is just another way of stating the already mentioned equivariance. Indeed the very existence of the closed 2-form $K$ guarantees that every Kähler space is a symplectic manifold and that we can define a Poisson bracket.

Consider Eqs. 280. To every generator of the abstract Lie algebra $\mathfrak{g}$ we have associated a function $\mathcal{P}_\Lambda$ on $\mathcal{M}$; the Poisson bracket of $\mathcal{P}_\Lambda$ with $\mathcal{P}_\Sigma$ is defined as follows:

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\} = 4\pi K(\Lambda, \Sigma)$$  \hspace{1cm} (288)$$

where $K(\Lambda, \Sigma) \equiv K(\vec{k}_\Lambda, \vec{k}_\Sigma)$ is the value of $K$ along the pair of Killing vectors.

We now prove the following lemma.

**Lemma 9.1** **The following identity is true:**

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\} = f^{\Gamma}_{\Lambda\Sigma} \mathcal{P}_\Gamma + C_{\Lambda\Sigma}$$  \hspace{1cm} (289)$$

where $C_{\Lambda\Sigma}$ is a constant fulfilling the cocycle condition

$$f^{\Gamma}_{\Lambda\Sigma} C_{\Gamma\Sigma} + f^{\Gamma}_{\Pi\Sigma} C_{\Gamma\Lambda} + f^{\Gamma}_{\Sigma\Lambda} C_{\Gamma\Pi} = 0$$  \hspace{1cm} (290)$$

**Proof.** Let us set $f_{\Lambda\Sigma} \equiv K(\Lambda, \Sigma)$. Using eq. 281 we get

$$4\pi f_{\Lambda\Sigma} = 4\pi f_{\Sigma\Lambda} = 2\pi i_{\Sigma} i_\Lambda K = -i_{\Sigma} d\mathcal{P}_\Lambda = i_\Lambda d\mathcal{P}_\Sigma = \frac{1}{2}(\mathcal{L}_\Lambda \mathcal{P}_\Sigma - \mathcal{L}_\Sigma \mathcal{P}_\Lambda)$$  \hspace{1cm} (291)$$
Let us now calculate $df_{\Lambda \Sigma}$. Since the exterior derivative commutes with the Lie derivative we find
\begin{align}
    df_{\Lambda \Sigma} &= \frac{1}{8\pi}(\mathcal{L}_\Lambda d\mathcal{P}_\Sigma - \mathcal{L}_\Sigma d\mathcal{P}_\Lambda) = \\
    &= \frac{1}{4}(-\mathcal{L}_\Lambda i_\Sigma K + \mathcal{L}_\Sigma i_\Lambda K) \tag{292}
\end{align}
Using now the identity
\begin{equation}
    [i_\Lambda, \mathcal{L}_\Sigma] = i_{[\Lambda, \Sigma]} \tag{294}
\end{equation}
and eq. 286, from eq. 291 we obtain
\begin{align}
    df_{\Lambda \Sigma} &= \frac{1}{2} i_{[\Sigma, \Lambda]} K = -\frac{1}{2} f_{\Lambda \Sigma} \Gamma^T K = \\
    &= \frac{1}{4\pi} f_{\Lambda \Sigma} \Gamma^T d\mathcal{P}_T \tag{295}
\end{align}
It follows that the difference
\begin{equation}
    C_{\Lambda \Sigma} = \{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\} - f_{\Lambda \Sigma} \Gamma^T \mathcal{P}_T \tag{297}
\end{equation}
is a constant since we have shown that its exterior derivative vanishes: $dC_{\Lambda \Sigma} = 0$. The cocycle condition in eq. 290 follows from the Jacobi identities fulfilled by the Poisson bracket of eq. 289. This concludes the proof of the lemma. If the Lie algebra $\mathfrak{g}$ has a trivial second cohomology group $H^2(\mathfrak{g}) = 0$, then the cocycle $C_{\Lambda \Sigma}$ is a coboundary; namely we have
\begin{equation}
    C_{\Lambda \Sigma} = f_{\Lambda \Sigma} \Gamma^T C_T \tag{298}
\end{equation}
where $C_T$ are suitable constants. Hence, assuming $H^2(\mathfrak{g}) = 0$ we can reabsorb $C_T$ in the definition of $\mathcal{P}_\Lambda$:
\begin{equation}
    \mathcal{P}_\Lambda \rightarrow \mathcal{P}_\Lambda + C_\Lambda \tag{299}
\end{equation}
and we obtain the stronger equation
\begin{equation}
    \{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\} = f_{\Lambda \Sigma} \Gamma^T \mathcal{P}_T \tag{300}
\end{equation}
Note that $H^2(\mathfrak{g}) = 0$ is true for all semi-simple Lie algebras. Using eq. 289, eq. 300 can be rewritten in components as follows:
\begin{equation}
    \frac{i}{2} g_{ij} \left( k_\Lambda^i k_\Sigma^j - k_\Sigma^i k_\Lambda^j \right) = \frac{1}{2} f_{\Lambda \Sigma} \Gamma^T \mathcal{P}_T \tag{301}
\end{equation}
Equation 301 is identical with the equivariance condition in eq. 283.

Comparing the definition of the Kähler potential in eq. 123 with the definition of the momentum function in eq. 280, we obtain an expression for the momentum map function in terms of derivatives of the Kähler potential:
\begin{align}
    i \mathcal{P}_\Lambda &= \frac{1}{2} \left( k_\Lambda^i \partial_i \mathcal{K} - k_\Lambda^i \partial_i \mathcal{K} \right) \\
    &= k_\Lambda^i \partial_i \mathcal{K} \\
    &= -k_\Lambda^i \partial_i \mathcal{K} \tag{302}
\end{align}
Eq. 302 is true if the Kähler potential is exactly invariant under the transformations of the isometry group $\mathcal{G}$ and not only up to a Kähler transformation as defined in eq. 124. In other words eq. 302 is true if
\begin{equation}
    0 = \mathcal{L}^A \mathcal{K} = k_\Lambda^i \partial_i \mathcal{K} + k_\Lambda^i \partial_i \mathcal{K} \tag{303}
\end{equation}
Not all the isometries of a general Kähler manifold have such a property, but those that in a suitable coordinate frame display a linear action on the coordinates certainly do. However, in Hodge–Kähler manifolds, eq. 303 can be replaced by the following one which is certainly true:
\begin{align}
    0 &= \mathcal{L}^A G = k_\Lambda^i \partial_i \mathcal{K} + k_\Lambda^i \partial_i \mathcal{K} \\
    G(z, \bar{z}) &= \log \| W(z) \|^2 \\
    &= \mathcal{K}(z, \bar{z}) + \text{Re} W(z) \tag{304}
\end{align}
where the superpotential $W(z)$ is any holomorphic section of the Hodge line–bundle. Indeed the transformation under the isometry of the Kähler potential is compensated by the transformation of the superpotential. Consequently, in Hodge–Kähler manifolds eq. 302 can be rewritten as
\begin{align}
    i \mathcal{P}_\Lambda &= \frac{1}{2} \left( k_\Lambda^i \partial_i G - k_\Lambda^i \partial_i \mathcal{G} \right) \\
    &= k_\Lambda^i \partial_i G \\
    &= -k_\Lambda^i \partial_i \mathcal{G} \tag{305}
\end{align}
and holds true for any isometry.

In $N = 1$ supersymmetry the Kählerian momentum maps $\mathcal{P}_T$ appear as auxiliary fields of the vector multiplets. For $N = 1$ supergravity the scalar manifold is of the Hodge type and eq. 303 can always be employed.

On the other hand, in $N = 2$ supersymmetry the auxiliary fields of the vector multiplets,
that form an $SU(2)$ triplet, are given by the
momentum map of triholomorphic isometries on
the hypermultiplet manifold (HyperKählerian or
quaternionic depending on the local or rigid na-
ture of supersymmetry). The triholomorphic mo-
momentum map is discussed in the subsection after
the next. Yet, although not identified with the
auxiliary fields, the holomorphic momentum map
plays a role also in $N = 2$ theories in the gaug-
ing of the $U(1)$ connection \[\mathbf{138}\] as I show shortly
from now.

9.2. Holomorphic momentum map on Special
Kähler manifolds

Here the Kähler manifold is not only Hodge
but it is special. Correspondingly we can write a
formula for $\mathcal{P}_\Lambda$ in terms of symplectic
invariants. In this context, to distinguish the holomorphic
momentum map from the triholomorphic one
that carries an $SU(2)$ index $x = 1, 2, 3$, we adopt
the notation $\mathcal{P}_\Lambda^0$. The request that the isometry
group should be embedded into the symplectic
vector bundle

$$\text{Sp}(2n + 2, \mathbb{R}) \ni T_\Lambda = \begin{pmatrix} a_\Lambda & b_\Lambda \\ c_\Lambda & d_\Lambda \end{pmatrix}$$

is some element of the real symplectic Lie algebra
and $f_\Lambda(z)$ corresponds to an infinitesimal Kähler
transformation.

The classical or perturbative isometries:

$$b_\Lambda = 0$$

that are relevant to the gauging procedure are
normally characterized by

$$f_\Lambda(z) = 0$$

Under condition \[\mathbf{309}\], recalling eqs \[\mathbf{153}\] and \[\mathbf{154}\],
from eq. \[\mathbf{306}\]
we obtain:

$$\mathcal{L}_\Lambda K = k^i_\Lambda \partial_i K + k^{i\ast}_\Lambda \partial_{i\ast} K = 0$$

(310)

that is identical with eq. \[\mathbf{303}\]. Hence we can use
eq \[\mathbf{302}\] that we rewrite as:

$$i \mathcal{P}^0_\Lambda = k^i_\Lambda \partial_i K = -k^{i\ast}_\Lambda \partial_{i\ast} K$$

(311)

Utilizing the definition in eq. \[\mathbf{157}\] we easily obtain:

$$k^i_\Lambda U^i = T_\Lambda V \exp[f_\Lambda(z)] + i \mathcal{P}^0_\Lambda V$$

(312)

Taking the symplectic scalar product of eq. \[\mathbf{312}\]
with $\overline{V}$ and recalling eq. \[\mathbf{153}\] we finally get \[\mathbf{10}\]

$$\mathcal{P}^0_\Lambda = \langle \overline{V} | T_\Lambda V \rangle = \langle V | T_\Lambda \overline{V} \rangle$$

$$= \exp[K] \langle \overline{\Omega} | T_\Lambda \Omega \rangle$$

(313)

In the gauging procedure we are interested in
groups the symplectic image of whose generators
is block-diagonal and coincides with the adjoint
representation in each block. Namely

$$T_\Lambda = \begin{pmatrix} f^{\Sigma} & 0 \\ 0 & -f^{\Sigma} \end{pmatrix}$$

Then eq. \[\mathbf{313}\] becomes

$$\mathcal{P}^0_\Lambda = e^K \left( F_\Delta f^{\Lambda \Sigma} \overline{X}^\Sigma + T_\Delta f^{\Lambda \Sigma} X^\Sigma \right)$$

(315)

9.3. The triholomorphic momentum map
on HyperKähler and Quaternionic
manifolds

Next I turn to a discussion of isometries of the
manifold $\mathcal{H}M$ associated with hypermultiplets.
As we know it can be either HyperKählerian or
quaternionic. For applications to $N = 2$ theo-
ories we must assume that on $\mathcal{H}M$ we have an ac-
tion by triholomorphic isometries of the same Lie
group $\mathcal{G}$ that acts on the Special Kähler manifold
$\mathcal{SM}$. This means that on $\mathcal{H}M$ we have Killing
vectors

$$\tilde{k}_\Lambda = k^u_\Lambda \frac{\partial}{\partial q^u}$$

(316)

satisfying the same Lie algebra as the corre-
sponding Killing vectors on $\mathcal{SM}$. In other words

$$\tilde{k}_\Lambda = \tilde{k}_\Lambda = k^i_\Lambda \partial_i + k^{i\ast}_\Lambda \partial_{i\ast} + k^u_\Lambda \partial_u$$

(317)

is a Killing vector of the block diagonal metric:

$$\tilde{g} = \begin{pmatrix} g_{ij} & 0 \\ 0 & h_{uv} \end{pmatrix}$$

(318)

defined on the product manifold $\mathcal{SM} \otimes \mathcal{H}M$.
Triholomorphicity means that the Killing vector

\[\text{The following and the next two formulae have been ob-
tained in private discussions of the author with A. Van
Proeyen and B. de Wit}\]
fields leave the HyperKähler structure invariant up to $SU(2)$ rotations in the $SU(2)$–bundle defined by eq. (238). Namely:

\[ \mathcal{L}_\Lambda K^x = \epsilon^{x y z} K^y W^z_\Lambda \]
\[ \mathcal{L}_\Lambda \omega^x = \nabla W^x_\Lambda \]

(319)

where $W^x_\Lambda$ is an $SU(2)$ compensator associated with the Killing vector $k^x_\Lambda$. The compensator $W^x_\Lambda$ necessarily fulfills the cocycle condition:

\[ \mathcal{L}_\Lambda W^z_\Sigma - \mathcal{L}_\Sigma W^z_\Lambda + \epsilon^{x y z} W^y_\Lambda W^z_\Sigma = f^z_{\Lambda \Sigma} W^x_\Gamma \]

(320)

In the HyperKähler case the $SU(2)$–bundle is flat and the compensator can be reabsorbed into the definition of the HyperKähler forms. In other words we can always find a map

\[ \mathcal{H} \mathcal{M} \rightarrow L^x_y(q) \in SO(3) \]

(321)

that trivializes the $SU$–bundle globally. Redefining:

\[ K^x = L^x_y(q) K^y \]

(322)

the new HyperKähler form obeys the stronger equation:

\[ \mathcal{L}_\Lambda K^{x y} = 0 \]

(323)

On the other hand, in the quaternionic case, the non–triviality of the $SU$–bundle forbids to eliminate the $W$–compensator completely. Due to the identification between HyperKähler forms and $SU(2)$ curvatures eq. (319) is rewritten as:

\[ \mathcal{L}_\Lambda \Omega^x = \epsilon^{x y z} \Omega^y W^z_\Lambda \]
\[ \mathcal{L}_\Lambda \omega^x = \nabla W^x_\Lambda \]

(324)

In both cases, anyhow, and in full analogy with the case of Kähler manifolds, to each Killing vector we can associate a triplet $\mathcal{P}_x(q)$ of 0-form prepotentials. Indeed we can set:

\[ i_\Lambda K^x = -\nabla \mathcal{P}_x^\Lambda \equiv -(d \mathcal{P}_x^\Lambda + \epsilon^{x y z} \omega^y \mathcal{P}_x^z) \]

(325)

where $\nabla$ denotes the $SU(2)$ covariant exterior derivative.

As in the Kähler case eq. (325) defines a momentum map:

\[ \mathcal{P} : \mathcal{M} \rightarrow \mathbb{R}^3 \otimes \mathfrak{g}^* \]

(326)

where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of the group $G$. Indeed let $x \in \mathfrak{g}$ be the Lie algebra element corresponding to the Killing vector $X$; then, for a given $m \in \mathcal{M}$

\[ \mu(m) : x \rightarrow \mathcal{P}_X(m) \in \mathbb{R}^3 \]

(327)

is a linear functional on $\mathfrak{g}$. If we expand $X = a^\Lambda k_\Lambda$ on a basis of Killing vectors $k_\Lambda$ such that

\[ [k_\Lambda, k_\Gamma] = f^z_{\Lambda \Sigma} k_\Delta \]

(328)

and we also a choose basis $i_x (x = 1, 2, 3)$ for $\mathbb{R}^3$ we get:

\[ \mathcal{P}_X = a^\Lambda \mathcal{P}_x^\Lambda i_x \]

(329)

Furthermore we need a generalization of the equivariance defined by eq. (330):

\[ X \circ \mathcal{P}_Y = \mathcal{P}_{[X, Y]} \]

(330)

In the HyperKähler case, the left–hand side of eq. (330) is defined as the usual action of a vector field on a 0–form:

\[ X \circ \mathcal{P}_Y = i_X d \mathcal{P}_Y = X^u \frac{\partial}{\partial u} \mathcal{P}_Y \]

(331)

The equivariance condition implies that we can introduce a triholomorphic Poisson bracket defined as follows:

\[ \{ \mathcal{P}_\Lambda, \mathcal{P}_\Sigma \}^z = 2 K^x (\Lambda, \Sigma) \]

(332)

leading to the triholomorphic Poissonian realization of the Lie algebra:

\[ \{ \mathcal{P}_\Lambda, \mathcal{P}_\Sigma \}^z = f^z_{\Lambda \Sigma} \mathcal{P}_\Delta \]

(333)

which in components reads:

\[ K^x_{uv}, k^z_\Lambda k^u_\Sigma = \frac{1}{2} f^z_{\Lambda \Sigma} \mathcal{P}_\Delta \]

(334)

In the quaternionic case, instead, the left–hand side of eq. (330) is interpreted as follows:

\[ X \circ \mathcal{P}_Y = i_X \nabla \mathcal{P}_Y = X^u \nabla_u \mathcal{P}_Y \]

(335)

where $\nabla$ is the $SU(2)$–covariant differential. Correspondingly, the triholomorphic Poisson bracket is defined as follows:

\[ \{ \mathcal{P}_\Lambda, \mathcal{P}_\Sigma \}^z = 2 K^x (\Lambda, \Sigma) - \frac{1}{2} \epsilon^{x y z} \mathcal{P}_Y^\Lambda \mathcal{P}_Y^x \]

(336)
and leads to the Poissonian realization of the Lie algebra
\[ \{ P_\Lambda, P_\Sigma \} = f^\Delta_{\Lambda\Sigma} P^\Delta \] (337)
which in components reads:
\[ K_{\alpha\beta}, k_\alpha^\lambda k_\beta^\Sigma - \frac{1}{2\Lambda} \varepsilon^{\alpha\beta\gamma} P^\gamma_{\Lambda\Sigma} P^\lambda_{\alpha\beta} = \frac{1}{2} f^\Delta_{\Lambda\Sigma} P^\Delta_{\alpha\beta} \] (338)
Eq. (338) which is the most convenient way of expressing equivariance in a coordinate basis plays a fundamental role in the construction of the supersymmetric action, supersymmetry transformation rules and of the superpotential for \( N = 2 \) supergravity on a general quaternionic manifold.

It is also very convenient to retrieve the rigid supersymmetry limit. Indeed, when \( \lambda \to \infty \) eq. (338) reduces to eq. (334). Eq. (338) was introduced in the physical literature in [34] where the general form of \( N = 2 \) supergravity beyond the limitations of tensor calculus was given.

### 9.4. The \( N = 2 \) supergravity lagrangian

Using the concepts and the geometric structures introduced so far the form of the bosonic lagrangian for \( N = 2 \) supergravity anticipated in eq.s (13) and (14) is now explained. In particular the scalar potential \( V \) is expressed, as one sees, in terms of the Killing vectors and of the momentum map functions. Indeed, differently from the \( N = 1 \) theory, in \( N = 2 \) supergravity the scalar potential is entirely due to the gauging of the theory, no superpotential being available without gauging.

Fermions are not discussed in these lectures, yet it is a legitimate curiosity to ask what happens to them in the gauging. Something very simple. Fermions behave as sections of the bundle \( \mathcal{L} \otimes \mathcal{TSM} \) in the gaugino case, as sections of the bundle \( \mathcal{THM} \otimes SU^{-1} \) in the hyperino case and as sections of the bundle \( \mathcal{L} \otimes SU \) in the gravitino case. Correspondingly the covariant derivatives of the fermions appearing in the action and in the transformation rules involves the composite connections \( \mathcal{Q}, \Gamma^i_j, \omega^x \) and \( \Delta^{\alpha\beta} \) defined on these bundles. Gauging just modifies these composite connections by means of Killing vectors and momentum map functions. Explicitly we have:

\[ \mathcal{TSM} : \text{tangent bundle} : \]

\[ \Gamma^i_j \to \Gamma^i_j + g A^\Lambda \partial_j k^i_\Lambda \]

\[ \mathcal{L} : \text{line bundle} : \]

\[ \mathcal{Q} \to \mathcal{Q} + g A^\Lambda P^\Lambda_\alpha \]

\[ SU : SU(2) \text{ bundle} : \]

\[ \omega^x \to \omega^x + g A^\Lambda \omega^{\Lambda^\alpha \beta} \]

\[ SU^{-1} \otimes \mathcal{THM} : Sp(2m) \text{ bundle} : \]

\[ \Delta^{\alpha\beta} \to \Delta^{\alpha\beta} + g A^\Lambda \partial \partial k^i_\Lambda \mathcal{U}^{\mu\nu\sigma A} \mathcal{U}^\beta_{\nu|A} \] (339)

### 10. An example of non–perturbative rigid special geometry: \( SU(r + 1) \) gauge theories

As an example of non–perturbative special geometry and of the associated electric–magnetic duality rotations, I choose the example of rigid \( N = 2 \) Yang–Mills theories for the gauge group \( SU(r + 1) \).

Here the essential idea that allows for the non–perturbative solution of the model is the introduction of an auxiliary dynamical Riemann surface. The moduli space of the gauge theory is identified with the moduli space of such a surface.

This is the realization of the correspondence already anticipated in eq.s (63). There we conceived the possibility that the components of the derivative \( \mathcal{U}^i_i = \partial_i \mathcal{U} \equiv \left( \frac{f^i_\alpha}{h_{ji}} \right) \) of the holomorphic symplectic section \( \mathcal{U} \) leading to rigid special geometry might be regarded as the periods of holomorphic differentials on a Riemann surface. The complete moduli–space of genus \( g \) Riemann surface is not a rigid special manifold. Yet a sublocus of this moduli space has such a property. The sublocus is spanned by the surfaces solving the dynamical problem under consideration.

In this lecture I summarize the results obtained for pure \( N = 2 \) gauge theories without hypermultiplets coupling in (4). My presentation follows (22). For the \( N = 2 \) microscopic gauge theory of a group \( G \), with Lie algebra \( \mathfrak{g} \); the rigid special geometry is encoded in a “minimal coupling” quadratic prepotential of the form:

\[ \mathcal{F}(\text{micro})(Y) = g_{ij}^{(K)} Y^I Y^J \]
\[ g^{(K)}_{ij} = \text{Killing metric on } \mathcal{G} \quad (340) \]

This choice is motivated by renormalizability, positivity of the energy and canonical normalization of the kinetic terms. It is also motivated by the need to have a continuous group of isometries where to embed the gauge group, as I have already explained. Consider next the effective lagrangian describing the dynamics of the massless modes. This is an abelian \( N = 2 \) gauge theory that admits the maximal torus \( H \subset G \) as new gauge group and is based on a new rigid special geometry:

\[
F^{(eff)}(Y) = g^{(K)}_{\alpha\beta} Y^\alpha Y^\beta + \Delta F^{(eff)} (Y^\alpha) \quad Y^\alpha \in \mathbb{H} \subset \mathcal{G} \quad (341)
\]

In general the effective prepotential \( F^{(eff)}(Y) \) has a transcendental dependence on the scalar fields \( Y^\alpha \) of the Cartan subalgebra multiplets, due to the correction \( \Delta F^{(eff)} (Y^\alpha) \). The main problem is the determination of this correction. Perturbatively one can get information on \( \Delta F^{(eff)} (Y^\alpha) \) and discover its logarithmic singularity for large values of the scalar fields \( Y^\alpha \). In particular one has a logarithmic correction to the gauge coupling period matrix

\[
\Delta \bar{N}_{\alpha\beta} = \frac{\partial^2}{\partial Y^\alpha \partial Y^\beta} \Delta F \quad \sim \quad \sum_\alpha \alpha^\alpha \omega^\alpha \omega^\alpha \mathcal{A}^2 \quad (342)
\]

where \( \alpha \) are the root vectors of the gauge Lie algebra and \( \mathcal{A}^2 \) is the dynamically generated scale. The perturbative monodromy following from

\[
\mathcal{N}_{\alpha\beta} \rightarrow [\mathcal{C} + DN](A + BN)^{-1}]_{\alpha\beta} \\
Sp(2r, \mathbb{R}) \ni \left( \begin{array}{ccc} A & B \\ C & D \end{array} \right) \sim \left( \begin{array}{ccc} 1 & \alpha^\alpha \alpha^\beta & 0 \\ \sum_\alpha \alpha^\alpha \omega^\alpha & 0 & 1 \end{array} \right) \quad (343)
\]

is assumed to be a part of the monodromy group of a genus \( r \) Riemann surface having a symplectic action on the periods of the surface. Guessing such a dynamical Riemann surface gives the non-perturbative structure \( \Delta F^{(eff)} \).

Denoting by \( r \) the rank of the original gauge group \( G \), one derives the structure of the effective gauge theory of the maximal torus \( \mathbb{H} \) from the geometry of an \( r \)-parameter family \( \mathcal{M}_1[r] \) of dynamical genus \( r \) Riemann surfaces. The essential steps of the procedure are as follows: naming \( u_i \) \((i = 1, \ldots, r)\) the \( r \) gauge invariant moduli of the family, (described as the vanishing locus of an appropriate polynomial) one makes the identifications:

\[
u_i \rightarrow (d_{a_1 \ldots a_{i+1}} Y^{a_1} \ldots Y^{a_{i+1}}) \quad (i = 1, \ldots, r) \quad (344)
\]

where \( Y^\alpha \) are the special coordinates of rigid special geometry and \( d_{a_1 \ldots a_{i+1}} \) is the restriction to the Cartan subalgebra of the rank \( i+1 \) symmetric tensor defining the \((i + 1)\)-th Casimir operator. The identification in eq. (344) is only an asymptotic equality for large values of \( u_i \) and \( Y^\alpha \); at finite values, the relation between the moduli \( u_i \) and the special coordinates (namely the elementary fields appearing in the lagrangian) is much more complicated. One considers the derivatives:

\[
\Omega_{u_i} \overset{\text{def}}{=} \partial_{u_i} \Omega = \partial_{u_i} \left( \frac{Y^\alpha}{\partial F} \right) \quad (345)
\]

where \( \Omega(u_i) \) is a section of the flat \( Sp(2r, \mathbb{R}) \) holomorphic vector bundle whose existence is encoded in the definition of rigid special Kähler geometry. This is eq. (172) written in a special coordinate basis. On one hand the Kähler metric is given by the general formula of eq. (170). On the other hand, one identifies the symplectic vectors \( \Omega_{u_i} \) with the period vectors:

\[
\Omega_{u_i} = \left( \int_{A^\alpha} \omega^i \right) \quad (i = 1, \ldots, r = \text{genus}) \quad (346)
\]

of the \( r \) holomorphic 1-forms \( \omega^i \) along a canonical homology basis, defined as in eq. (34) of a genus \( r \) dynamical Riemann surface \( \mathcal{M}_1[r] \). The generic moduli space \( \mathcal{M}_r \) of genus \( r \) surfaces is \( 3r - 3 \) dimensional. The dynamical Riemann surfaces \( \mathcal{M}_1[r] \) fill an \( r \)-dimensional sublocus \( L_R[r] \). The problem is that of characterizing intrinsically this locus. Let

\[
i : L_R[r] \rightarrow \mathcal{M}_r \quad (347)
\]

be the inclusion map of the wanted locus and let

\[
H \overset{\pi}{\rightarrow} \mathcal{M}_r \quad (348)
\]
be the Hodge bundle on \( M_r \), that is the rank \( r \) vector bundle whose sections are the holomorphic forms on the Riemann surface \( \Sigma_r \in M_r \). As fibre metric on this bundle one can take the imaginary part of the period matrix:

\[
\text{Im} \mathcal{N}_{\alpha \beta} = \int_{\Sigma_r} \omega^\alpha \wedge \overline{\omega}^\beta \tag{349}
\]

where \( \omega^\alpha \) is a basis holomorphic one-forms. The metric on this bundle one can take the imaginary part of the Hodge bundle and \( K \) is the Kähler class of \( M_r \).

Using very general techniques of algebraic geometry, the dependence of the periods (see eq.343) on the moduli parameters can be determined through the solutions of the Picard–Fuchs differential system, once \( \mathcal{M}_1[r] \) is explicitly described as the vanishing locus of a holomorphic superpotential \( \mathcal{W}(Z, X, Y; u_i) \). In particular one can study the monodromy group \( \Gamma_M \) of the differential system and the symmetry group of the potential \( \Gamma_W \), that are related to the full group of electric–magnetic duality rotations \( \Gamma_D \) as follows:

\[
\Gamma_W = \Gamma_D / \Gamma_M \tag{351}
\]

The elements of \( \Gamma_D \supset \Gamma_M \) are given by integer valued symplectic matrices \( \gamma \in Sp(2r, \mathbb{Z}) \) that act on the symplectic section \( \Omega \). Given the geometrical interpretation of these sections provided by eq.s 844, the elements \( \gamma \in \Gamma_D \subset Sp(2r, \mathbb{Z}) \) correspond to changes of the canonical homology basis respecting the intersection matrix in eq.31.

To be specific we mention the results obtained for the gauge groups \( G = SU(r+1) \). The rank \( r = 1 \) case, corresponding to \( G = SU(2) \), was studied by Seiberg and Witten in their original paper 11. The extension to the general case, with particular attention devoted to the \( SU(3) \) case, was obtained in 18, 18. In all these cases the dynamical Riemann surface \( \mathcal{M}_1[r] \) belongs to the hyperelliptic locus of genus \( r \) moduli space, the general form of a hyperelliptic surface being described (in inhomogeneous coordinates) by the following algebraic equation:

\[
w^2 = P_{(2+2r)}(z) = \prod_{i=1}^{2+2r} (z - \lambda_i) \tag{352}
\]

where \( \lambda_i \) are the \( 2 + 2r \) roots of a degree \( 2 + 2r \) polynomial. The hyperelliptic locus

\[
L_H[r] \subset M_r, \quad \dim L_H[r] = 2r - 1 \tag{353}
\]

is a closed submanifold of codimension \( r-2 \) in the \( 3r-3 \) dimensional moduli space of genus \( r \) Riemann surface \( \Sigma_r \). The \( 2r-1 \) hyperelliptic moduli are the \( 2r+2 \) roots of the polynomial appearing in eq.352, minus three of them that can be fixed at arbitrary points by means of fractional linear transformations on the variable \( z \). Because of their definition, however, the dynamical Riemann surfaces \( \mathcal{M}_1[r] \), must have \( r \) rather than \( 2r-1 \) moduli. We conclude that the \( r \)-parameter family \( \mathcal{M}_1[r] \) fills a locus \( L_R[r] \) of codimension \( r-1 \) in the hyperelliptic locus:

\[
L_R[r] \subset L_H[r], \quad \text{codim} \ L_R[r] = r - 1 \tag{354}
\]

This fact is expressed by additional conditions imposed on the form of the degree \( 2 + 2r \) polynomial of eq.352. In references 38, \( P_{(2+2r)}(z) \) was determined to be of the following form:

\[
P_{(2+2r)}(z) = P_{(r+1)}^2(z) - P_{(1)}^2(z) \tag{355}
\]

where \( P_{(r+1)}(z) \) and \( P_{(1)}(z) \) are two polynomials respectively of degree \( r+1 \) and \( 1 \). Altogether we have \( r + 3 \) parameters that we can identify with the \( r+1 \) roots of \( P_{(r+1)}(z) \) and with the two coefficients of \( P_{(1)}(z) \)

\[
P_{(r+1)}(z) = \prod_{i=1}^{r+1} (z - \lambda_i)
\]

\[
P_{(1)}(z) = \mu_1 z + \mu_0 \tag{356}
\]

Indeed, since the polynomial in eq.352 must be effectively of order \( 2 + 2r \), the highest order coefficient of \( P_{(r+1)}(z) \) can be fixed to 1 and the only independent parameters contained in \( P_{(r+1)}(z) \) are the roots. On the other hand, since \( P_{(1)}(z) \) contributes only subleading powers, both of its

\footnote{For genus 1, the moduli space is also 1-dimensional and the hyperelliptic locus is the full moduli space.}
coefficients $\mu_1$ and $\mu_0$ are effective parameters. Then, if we take into account fractional linear transformations, three gauge fixing conditions can be imposed on the $r+3$ parameters $\{\lambda_i\}, \{\mu_i\}$. In ref. (355) this freedom was used to set:
\[
\sum_{i=1}^{r+1} \lambda_i = 0
\]
\[
\mu_1 = 0
\]
\[
\mu_0 = \Lambda^{r+1}
\]  \hspace{1cm} (357)
where $\Lambda$ is the dynamically generated scale. With this choice the $r$–parameter family of dynamical Riemann surfaces is described by the equation:
\[
w^2 = \left( z^{r+1} - \sum_{i=1}^{r} u_i (\lambda_i) z^{r-i} \right)^2 - \Lambda^{2r+2} \]  \hspace{1cm} (358)
where the coefficients $u_i (\lambda_1, \ldots, \lambda_{r+1})$ \hspace{1cm} (i = 1, \ldots, r) \hspace{1cm} (359)
are symmetric functions of the $r + 1$ roots constrained by the first of eqs. 355 and can be identified with the moduli parameters introduced in eq. 344. In the particular case $r = 1$, the gauge–fixing of eq. 355 leads to the following quartic form for the elliptic curve studied in [3]:
\[
w^2 = (z^2 - u)^2 - \Lambda^4 = z^4 - 2u z^2 + u^2 - \Lambda^4 \]  \hspace{1cm} (360)
Of course other gauge fixings give equivalent descriptions of $\mathcal{M}_1[r]$; however, for our next purposes, it is particularly important to choose a gauge fixing of the $SL(2, \mathbb{C})$ symmetry such that the equation $\mathcal{M}_1[r]$ can be recast in the form of a Fermat polynomial in a weighted projective space deformed by the marginal operators of its chiral ring. In this way it is quite easy to study the symmetry group of the potential $\Gamma_w$ identifying the $R$-symmetry group and to derive the explicit form of the Picard-Fuchs equations satisfied by the periods. This is relevant for the embedding of the monodromy and $R$-symmetry groups in $Sp(2r, \mathbb{Z})$. The alternative gauge–fixing that we choose is the following:
\[
\sum_{i=1}^{r+1} \lambda_i = 0
\]
\[
\mu_1 \mu_0 + \left( \sum_{i=1}^{r+1} \frac{1}{\lambda_i} \right) \prod_{i=1}^{r+1} \lambda_i^2 = 0
\]
\[
-\mu_0^2 + \prod_{i=1}^{r+1} \lambda_i^2 = 1 \]  \hspace{1cm} (361)
To appreciate the convenience of this choice let us consider the general inhomogeneous form of the equation of the hyperelliptic surface eq. 355 and let us (quasi-)homogenize it by setting:
\[
w = \frac{z}{\Lambda} \quad \Rightarrow \quad z = \lambda \]  \hspace{1cm} (362)
With this procedure eq. 355 becomes a quasi–homogeneous polynomial constraint:
\[
0 = \mathcal{W}(Z,X,Y; \{\lambda\}, \{\mu\})
\]
\[
= -Z^2 + \left( \prod_{i=1}^{r+1} (X - \lambda_i Y) \right)^2
\]
\[
- (\mu_1 X Y^r + \mu_0 Y^{r+1})^2 \]  \hspace{1cm} (363)
of degree:
\[
\deg \mathcal{W} = 2r + 2
\]  \hspace{1cm} (364)
in a weighted projective space $WCP^{2; r+1, 1, 1}$, where the quasi–homogeneous coordinates $Z$, $X$, $Y$ have degrees $r+1$, 1 and 1, respectively. Adopting the notations of [5], namely denoting by $^{12}$
\[
WCP^n(d; q_1, q_2, \ldots, q_{n+1}) \]  \hspace{1cm} (365)
the zero locus (with Euler number $\chi$) of a quasi–homogeneous polynomial of degree $d$ in an $n$–dimensional weighted projective space, whose $n + 1$ quasi–homogeneous coordinates have weights $q_1, \ldots, q_{n+1}$:
\[
\mathcal{W}(\lambda^n X_1, \ldots, \lambda^{n+1} X_{n+1}) = \lambda^d \mathcal{W}(X_1, \ldots, X_{n+1}) \]  \hspace{1cm} (366)
we obtain the identification:
\[
\mathcal{M}_1[r] = WCP^{2(2r + 2; r + 1, 1, 1)} \]  \hspace{1cm} (367)
that yields, in particular:
\[
\mathcal{M}_1[1] = WCP^{2(4; 2, 1, 1)} \]
\[
\mathcal{M}_1[2] = WCP^{2(6; 3, 1, 1)} \]  \hspace{1cm} (368)
\text{Note the difference of notation: } WCP^{n; q_1, q_2, \ldots, q_{n+1}} \text{ is the full weighted projective space, in which eq. 365 is a hypersurface.
for the $SU(2)$ case studied in [1] and for the $SU(3)$ case studied in [8]. Using the alternative gauge fixing in eq. (363) the quasi-homogeneous Landau–Ginzburg superpotential of eq. (363), whose vanishing locus defines the dynamical Riemann surface, takes the standard form of a Fermat superpotential deformed by the marginal operators of its chiral ring:

$$ W(Z, X, Y; \{ \lambda \}, \{ \mu \}) = -Z^2 + X^{2r+2} + Y^{2r+2} + \sum_{i=1}^{2r-1} v_i(\lambda) X^{2r-1-i} Y^{r+1} \tag{369} $$

The coefficients $v_i$ $(i = 1, \ldots, 2r - 1)$ are the $2r - 1$ moduli of a hyperelliptic curve. In our case, however, they are expressed as functions of the $r$ independent roots $\lambda_i$ that remain free after the gauge-fixing of eq. (361) is imposed. The coefficients $v_i$ have a simple expression as symmetric functions of the $r + 1$ roots $\lambda_i$ subject to the constraint that their sum should vanish. For the general form of these expressions we refer the reader to [23]. In particular for the first case $r = 1$ we obtain

$$ \mathcal{M}_1[1] \rightarrow 0 = W(Z, X, Y; v = 2u) = -Z^2 + X^4 + Y^4 + v(\lambda) X^2 Y^2 \tag{370} $$

where

$$ \lambda_1 + \lambda_2 = 0 \quad \mu_1 = 0 \quad \mu_0 = \sqrt{\lambda_1^2 \lambda_2^2 - 1} \quad v = \lambda_1^2 + \lambda_2^2 + 4\lambda_1 \lambda_2 = -2 \lambda_1^2 \equiv 2u \tag{371} $$

Alternatively, using as independent parameters the coefficients $u_i(\lambda)$ appearing in eq. (358) we can characterize the locus $L_R[r]$ of dynamical Riemann surfaces by means of the following equations on the hyperelliptic moduli $v_i$:

\begin{align*}
  v_k &= -2u_k + \sum_{i+j=k-1} u_i u_j, \quad (k = 1, \ldots, r) \\
  v_{r+k} &= \sum_{i+j=r+k-1} u_i u_j - \delta_{r-1,k} \mu_1^2 \tag{372}
\end{align*}

Considering now the Hodge filtration of the middle cohomology group $H^{(1)}_{DR}(\mathcal{M}_1[r])$:

\begin{align*}
  F^0 &\subset F^1 \\
  F^0 &= H^{(1,0)} \\
  F^1 &\equiv H^{(1)}_{DR} = H^{(1,0)} + H^{(0,1)} \tag{373}
\end{align*}

the Griffiths residue map (351, 43) provides an association between elements of $F^k$ and polynomials $P^I_k(2r+2)(X)$ of the chiral ring $R(W) \equiv \{ [X]/\partial W \}$ of degree $k(2r + 2) = (k + 1)(2r + 2) - r - 3$ according to the following pattern:

| Cohom. deg | Polynom. |
|------------|----------|
| $F^0$ | $r - 1$ | $P^i_0(2r+2)$ | $i = 1, \ldots, r$ |
| $F^1$ | $3r + 1$ | $P^i_1(2r+2)$ | $i^* = 1, \ldots, r$ |

Explicitly, the periods of eq. (346) are represented by:

\begin{align*}
  \int_C \omega^i &= \int_C \frac{X^{r-i} Y^{i-1}}{W} \omega \\
  \int_C \omega^{i^*} &= \int_C \frac{X^{r+i} Y^{2r-i+1}}{W^2} \omega \tag{375}
\end{align*}

where $C$ denotes any of the homology cycles and $\omega = Z dX \wedge dY + \text{cycl}$. Using standard reduction techniques [44] one can obtain the first-order Picard-Fuchs differential system

$$ \left( \frac{\partial}{\partial v^I} \right) A_I(v) V = 0 \quad I = 1, \ldots, 2r - 1 \tag{376} $$

satisfied by the $2r$-component vector:

$$ V = \left( \begin{array}{c} \int_C \omega^i \\ \int_C \omega^{i^*} \end{array} \right) \tag{377} $$

in the $2r - 1$-dimensional moduli space of elliptic surfaces. Using the explicit embedding of the locus $L_R[r] \subset L_H[r]$ described by eqs (376), we obtain the Picard-Fuchs differential system of rigid special geometry by a trivial pull-back of eq. (376)

$$ \left( \frac{\partial}{\partial u^I} \right) A_I(v) \left( \frac{\partial}{\partial u^I} \right) V = 0. \tag{378} $$

The explicit solution of the Picard–Fuchs equations for $r = 1, 2$ has been given respectively in
The solution of the Picard–Fuchs equations for generic $r$ determines in principle the period of the surface and the monodromy group. I do not attempt to solve eq.s 378 for generic $r$, I will mostly concentrate on the $r = 1$ case as a pedagogical example.

Before plunging into the details of the $r = 1$ case I want to discuss the symmetry group of $M_1(r)$, which, together with the monodromy group $\Gamma_M$ defines the duality group $\Gamma_D$ according to eq. [351]. This symmetry group can be defined by considering those linear transformations $X \to M_A X$ of the quasi–homogeneous coordinate vector $X = (X, Y, Z)$ such that

$$\mathcal{W}(M_A X; v) = f_A(v) \mathcal{W}(X; \phi_A(v))$$

(379)

where $\phi_A(v)$ is a (generally non–linear) transformation of the moduli and $f_A(v)$ is a compensating overall rescaling of the superpotential that depends both on the moduli $v$ and on the chosen transformation $A$. Let us restrict our attention to the sublocus of the dynamical Riemann surface $\mathcal{W}(X; u) = 0$ defined by eq. [354], so that the moduli space geometry is a special Kähler geometry with Kähler potential

$$K = i(Y^\alpha \mathcal{F}_\alpha - \mathcal{F}^\alpha \mathcal{F}_\alpha).$$

(380)

In this case, only the subgroup $\Gamma^0_W \subset \Gamma_W$ given by the transformations that have a compensating rescaling factor $f(u) = e^{i \theta}$ acts as an isometry group for the moduli space, in contrast with curved special geometry, where the whole $\Gamma_W$ generates isometries. The hyperelliptic superpotential in eq. [365] admits a $\Gamma^0_W$ symmetry group which is isomorphic to the dihedral group $D_{2r+2}$, defined by the following relations on two generators $A, B$:

$$A^{2r+2} = 1; \quad B^2 = 1; \quad (AB)^2 = 1.$$  

(381)

The action of the generators on the moduli is the following. Let $\alpha^{2r+2} = 1$ be a $(2r+2)^{th}$ root of the unit and let the moduli $v_i$ be arranged into a column vector $v$. Then we have:

$$v' = Av, \quad A = \begin{pmatrix} \alpha^2 & 0 & \ldots & 0 \\ 0 & \alpha^3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha^{2r} \end{pmatrix}$$

(382)

For the transformations $A$ and $B$ the compensating transformations on the homogeneous coordinates $M_A$ and $M_B$ are

$$M_A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

(383)

Consequently the differential Picard–Fuchs system for the period (see eq. [377] of the generic hyperelliptic surface has a $\Gamma^0_W = D_{2r+2}$ symmetry as defined above and the generators $A$ and $B$ act by means of suitable $Sp(2r, \mathbb{Z})$ matrices on the period vector given by eq. [346]. However eq.s 378 are invariant only under the cyclic subgroup $\mathbb{Z}_{2r+2} \subset D_{2r+2}$ generated by $A$. Hence the potential $\mathcal{W}(u) = W(v(u))$ of the $r$-dimensional locus $L_R[r]$ of dynamical Riemann surfaces and the Picard-Fuchs first order system admits only the duality symmetry $\Gamma^0_W = \mathbb{Z}_{2r+2}$.

The physical interpretation of this group is $R$-symmetry. Indeed, recalling eq. [344] we see that when each of the elementary fields $Y^\alpha$ appearing in the lagrangian is rescaled as $Y^\beta \to \alpha Y^\beta$, then the first $u_i$ moduli are rescaled with the powers of $\alpha$ predicted by eq. [344]. According to the analysis of reference [19] this is precisely the requested R-symmetry for the topological twist. All the scalar components of the vector multiplets have the same R-charge ($q_R = 2$) under a $U_R(1)$ symmetry of the classical action, which is broken to a discrete subgroup in the quantum theory. Henceforth the integer symplectic matrix that realizes $A$ yields the R-symmetry matrix of rigid special geometry for $SU(r+1)$ gauge theories. An important problem is the derivation of the corresponding R-symmetry matrix in $Sp(2r + 4, \mathbb{Z})$, when the gauge theory is made locally supersymmetric by coupling it to supergravity including also the dilaton-axion vector multiplet suggested by string theory.
Another very important allied topic to electromagnetic dualities is indeed that of topological field theories: a topic that possibly captures their most profound meaning. Yet within the scope of these lectures I cannot open this Pandora’s vase and therefore I rather turn to discuss the details of Seiberg Witten example of rigid special geometry.

For convenience of later normalizations let us rewrite eq. (370) in the equivalent form:

$$0 = W(X,Y,Z; u) = -Z^2 + \frac{1}{4} (X^4 + Y^4) + \frac{3}{2} X^2 Y^2$$

(384)

One realizes that this potential has a $\Gamma_W = D_3$ symmetry group [56,18] defined by the following generators and relations

$$\hat{A}^2 = \mathbb{1}, \quad C^3 = \mathbb{1}, \quad (C\hat{A})^2 = \mathbb{1}$$

with the following action on the homogeneous coordinates and the modulus $u$:

$$\hat{A} : \begin{cases} M_\hat{A} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \\ \phi_\hat{A}(u) = -u \\ f_\hat{A}(u) = 1 \end{cases}$$

$$C : \begin{cases} M_C = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \\ \phi_C(u) = \frac{u^3}{u^3+1} \\ f_C(u) = \frac{i+u}{i+u} \end{cases}$$

(385)

(386)

Eq. (386) is given in reference [18], where the authors posed themselves the question why only the $\mathbb{Z}_2$ cyclic group generated by $\hat{A}$ is actually realized as an isometry group of the rigid special Kählerian metric. The answer is contained in the previous general discussion:

$$\mathbb{Z}_2 = \Gamma^{rig}_{\mathbb{W}} \subset \Gamma_W = D_3$$

(387)

Namely it is only $\mathbb{Z}_2$ that preserves the potential with a unit rescaling factor. The natural question at this point is what is the relation of this $\mathbb{Z}_2 \subset D_3$ with the dihedral $D_4$ symmetry expected for $r = 1$. The answer is simple: the $\mathbb{Z}_4$ action in $D_4$ becomes a $\mathbb{Z}_2$ action on the $u$ variable, $u \to \alpha^2 u$, ($\alpha^4 = 1$).

\footnote{We forget about the action on the $Z$ coordinate, which is immaterial, since it contributes only with a quadratic term to the polynomial $W$.}

As it has been shown in [4] the Picard–Fuchs equation associated, in the $SU(2)$ case, with the symplectic section:

$$\Omega_u = \partial_u \Omega = \partial_u \left( \frac{Y}{\partial Y} \right) = \left( \int_A^\omega F_i \right)$$

(388)

is

$$(\partial_u \mathbb{1} - A_u) V = 0,$$

(389)

where $V$ is defined in eq. (377) and the $2 \times 2$ matrix connection $A_u$ is given by:

$$A_u = \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(390)

with solutions

$$\partial_u Y \equiv f^{(1)}(u) = F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{-u}{2}, \frac{1}{2} \right)$$

$$\partial_u \frac{\partial Y}{\partial X} \equiv f^{(2)}(u) = iF\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{-u}{2}, \frac{1}{2} \right)$$

(391)

As it is obvious, $f^{(1)}(u)$ and $f^{(2)}(u)$ just provide a basis of two independent solutions of the linear second order differential equation derived from the linear system in eq. (389). Any other pair of linear combinations of the above functions would solve the same linear system. The reason why precisely $f^{(1)}(u)$ and $f^{(2)}(u)$ are respectively identified with $\partial_u Y$ and $\partial_u \frac{\partial Y}{\partial X}$ is given by the boundary conditions imposed at infinity. When $u \to \infty$, the special coordinate $Y(u)$ must approach the value it has in the original microscopic $SU(2)$ gauge theory. There the parameter $u$ was defined as the restriction to the Cartan subalgebra of the gauge invariant quadratic polynomial $tr(Y^2 \sigma_x^2)$ so that $u = \text{const}(Y)^2$, the special coordinate $Y(u)$ of the effective theory being $Y^2$ of the microscopic one. Correspondingly the boundary condition at infinity for $Y(u)$ is

$$Y(u) \approx 2 \sqrt{2u} + \ldots$$

(392)

At the same time when $u \to \infty$ the non perturbative rigid special geometry must approach its perturbative limit defined by the following pre-potential:

$$F_{\text{pert}}(Y) \equiv \frac{i}{2\pi} Y^2 \log Y^2$$

(393)
Combining eq. [392] and eq. [393] we obtain
\[ \frac{\partial F}{\partial Y} \approx \frac{1}{\pi} 2 \sqrt{2} u \log u + \ldots \quad \text{for} \quad u \to \infty \] (394)
so that for \( u \to \infty \) we have:
\[ \partial_u Y \approx \sqrt{\frac{2}{u}} + \ldots \quad \text{for} \]
\[ \partial_u \frac{\partial F}{\partial Y} \approx \frac{1}{\pi} 2 \left( \frac{1}{\sqrt{2} u} \log u + \sqrt{\frac{2}{u}} \right) + \ldots \] (395)
The boundary conditions in eqs. 395 are just realized by the choice of eq.s 391. Indeed using the
relation between hypergeometric functions and elliptic integrals:14
\[ \frac{\pi}{2} f \left( \frac{1}{2}, \frac{1}{2}, 1; x \right) = K(x) \]
\[ = \int_{0}^{\frac{\pi}{2}} (1 - x \sin^2 \theta)^{-\frac{1}{2}} d\theta \]
\[ = \frac{\pi}{2} f \left( \frac{1}{2}, \frac{1}{2}, 1; x \right) = E(x) \]
\[ = \int_{0}^{\frac{\pi}{2}} (1 - x \sin^2 \theta)^{-\frac{1}{2}} d\theta \]
\[ = \frac{\pi}{4} f \left( \frac{1}{2}, rac{1}{2}, 2; x \right) = B(x) \]
\[ = \left( \frac{E(x)}{x} + \frac{x - 1}{x} K(x) \right) \] (396)
and the relations
\[ \int_{0}^{x} K(t) dt = 2 x B(x) \]
\[ K \left( \frac{1}{x} \right) = \sqrt{x} \left( K(x) + i K \left( 1 - x \right) \right) \]
\[ \int_{0}^{x} \frac{A}{B} K \left( \frac{1}{x} \right) dt = 2 \sqrt{x} E \left( \frac{1}{x} \right) + 2 i \] (397)
the solutions of eq.s [391] can also be written as
\[ \partial_u Y \equiv f^{(1)}(u) \]
\[ = \frac{2}{\pi} [K \left( \frac{1+u}{2} \right) + i K \left( \frac{1-u}{2} \right)] \]
\[ = \frac{2}{\pi} \sqrt{\frac{2}{1+u}} K \left( \frac{2}{1+u} \right) \]
\[ \partial_u \frac{\partial F}{\partial Y} \equiv f^{(2)}(u) = \frac{2}{\pi} i K \left( \frac{1-u}{2} \right) \] (398)
By means of an integration one then obtains:
\[ Y(u) = \frac{s}{\pi} \int_{0}^{s} \frac{2}{1+t} K \left( \frac{2}{1+t} \right) dt = \]
\[ \frac{s}{\pi} \sqrt{\frac{1+u}{2}} E \left( \frac{2}{1+u} \right) + \text{const} \]
\[ \frac{\partial F}{\partial Y} = \frac{2}{\pi} i \int_{0}^{s} K \left( \frac{1-t}{2} \right) dt = \]
\[ -\frac{4}{\pi} (1 - u) B \left( \frac{1-u}{2} \right) + \text{const}. \] (399)
Choosing zero for the integration constants, the result in eq. [399] coincides with the integral representations originally given by Seiberg and Witten [3]:
\[ \begin{pmatrix} Y(u) = 2 a(u) = \frac{2 \sqrt{x}}{\pi} \int_{-1}^{1} \sqrt{\frac{u - x}{1 - x^2}} dx \\
\frac{\partial F}{\partial Y} = 2 a_D(u) = 2 i \frac{\sqrt{x}}{\pi} \int_{-1}^{1} \sqrt{\frac{u - x}{1 - x^2}} dx \end{pmatrix} \] (400)
Equipped with the above explicit solutions we can discuss duality, monodromy R symmetry and the explicit special metric on the rigid special manifold. The duality group of electric–magnetic rotations is, in this case [3]:
\[ \Gamma_D \equiv \overline{G}_\theta \subset PSL(2, \mathbb{Z}) \]
\[ \overline{G}_\theta \equiv \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} G_\theta \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \] (401)

14We use the notation \( K(x) \) for what is usually denoted as \( K(k) \) where \( x = k^2 \), and similar for other elliptic integrals.
isometry group should be symplectically embedded. Hence, naming \( u_i \rightarrow \phi_i(u) \) the transformations belonging to the discrete isometry group there must exist \( M_\phi \in Sp(2r, \mathbb{R}) \) such that

\[
\Omega(\phi(u)) = e^{i\theta_\phi} M_\phi \Omega(u) \quad (403)
\]

\[
\Omega_{u_i}(\phi(u)) \frac{\partial \phi_i}{\partial u_i} = e^{i\theta_\phi} M_\phi \Omega_{u_i}(u)
\]

The isometry \( u \rightarrow -u \), corresponding to \( R \)-symmetry (\( Y \rightarrow iY \) in the perturbative limit) induces, in this theory, the transformation

\[
-\Omega_u(-u) = i \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} f^{(1)}(u) \\ f^{(2)}(u) \end{pmatrix} = i R \Omega_u(u) \quad (404)
\]

while the monodromy transformation around \( u = 1 \) gives \((r \text{ small})\)

\[
\Omega_u(1 + re^{2\pi i}) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f^{(1)}(1 + r) \\ f^{(2)}(1 + r) \end{pmatrix} = T_1 \Omega_u(1 + r) \quad (405)
\]

Having recalled the explicit form of the isometry–duality group let us now study the structure of the rigid special metric. To this effect let us introduce the ratio of the two solutions to eq: \([389]\).

\[
\overline{\mathcal{N}}(u) = \frac{f^{(2)}(u)}{f^{(1)}(u)}. \quad (406)
\]

Recalling eq: \([63]\), we know that such a ratio is identified with the matrix \( \overline{\mathcal{N}} \) appearing in the vector field kinetic terms:

\[
\mathcal{L}_{\text{vector}} = \frac{1}{2} \overline{\mathcal{N}}(u) F_{\mu\nu}^- F_{\mu\nu}^- - \mathcal{N}(u) F_{\mu\nu}^+ F_{\mu\nu}^+ \quad (407)
\]

Recalling eq: \([172]\) we can now write the explicit form of the rigid special Kähler metric in the variable \( u \):

\[
ds^2 = g_{\overline{u}u} |du|^2
\]

\[
g_{\overline{u}u} = 2 \Im(\overline{\mathcal{N}}(u)) |f^{(1)}(u)|^2
\]

\[
(408)
\]

Calculating the Levi–Civita connection and Riemann tensor of this metric we obtain:

\[
\Gamma_{uu}^u = \frac{1}{2i} \frac{\partial \overline{\mathcal{N}}/\partial u}{\Im(\overline{\mathcal{N}}(u))} - \partial_u \log f^{(1)}(u)
\]

\[
R_{\overline{u}uu}^u = \frac{\partial_u \Gamma_{uu}^u}{\partial_u} = \frac{1}{4} \left( \frac{\partial \overline{\mathcal{N}}/\partial u}{\Im(\overline{\mathcal{N}}(u))} \right)^2 |\partial \overline{\mathcal{N}}/\partial u|^2
\]

\[
R_{\overline{u}u\overline{u}} = \frac{1}{2} \frac{\partial \overline{\mathcal{N}}/\partial u}{\Im(\overline{\mathcal{N}}(u))} |\partial \overline{\mathcal{N}}/\partial u| |f^{(1)}(u)|^2 \quad (409)
\]

so that we can verify that the above metric is indeed rigid special Kählerian, namely that it satisfies the constraint:

\[
R_{\overline{u}uu} - C_{u\overline{u}u} \overline{C}_{u\overline{u}u} g^{\overline{u}u} = 0 \quad (410)
\]

by calculating the Yukawa coupling or anomalous magnetic moment tensor:

\[
C_{u\overline{u}u} = \partial_u \overline{\mathcal{N}} \left( f^{(1)}(u) \right)^2 \quad (411)
\]

As one can notice from its explicit form (see eq: \([403]\)), the Kähler metric of the rigid \( N=2 \) gauge theory of rank \( r = 1 \) is not the Poincaré metric in the variable \( \overline{\mathcal{N}} \), as one might naively expect from the fact that \( \overline{\mathcal{N}} = \tau \) is the standard modulus of a torus and that \( G_\theta \subset PSL(2, \mathbb{Z}) \) linear fractional transformations are isometries. Indeed eq: \([408]\) is to be contrasted with the expression for the Poincaré metric:

\[
ds^2 = g^{\overline{u}u} \overline{\mathcal{N}} |d\overline{\mathcal{N}}|^2 \quad \left( \frac{1}{2i} \frac{1}{\Im(\overline{\mathcal{N}})} \right)^2 \quad (412)\]

From eq: \([104]\) however it is amusing to note that the Ricci form of the rigid metric is precisely the Poincaré metric.

\[
R_{\overline{\mathcal{N}} \overline{\mathcal{N}}} = g^{\overline{u}u} \overline{\mathcal{N}} \quad (413)
\]

This is a consequence of the general equation eq: \([354]\) in the case of one modulus where the period matrix \( \mathcal{N} \) can be used as a parameter.

10.1. The rigid special coordinates

In the special coordinate basis the anomalous magnetic moment tensor is given by:

\[
C_{YY} = C_{uu} \left( \frac{\partial u}{\partial Y} \right)^3 = -2i \frac{1}{\pi} \frac{1}{1-u^2} \left( \frac{\partial u}{\partial Y} \right)^3 \quad (414)
\]
The second of eqs. follows from the comparison between eq. and the Picard–Fuchs eq. satisfied by the periods that yields:

\[ C_{uuu} = -\frac{2i}{\pi} \frac{1}{1-u^2} \] (415)

In the large limit the asymptotic behaviour of the special coordinate is given by eq.:

\[ C_Y(u) = \frac{\partial^3 F}{\partial Y^3}(u) \approx \frac{i}{\sqrt{2\pi}} u^{-1/2} + \ldots \] (416)

and by triple integration one verifies consistency with the asymptotic behaviour of the prepotential \( F(Y) \) (see eq.):

\[ F(Y) \approx \text{const } Y^2 \log Y + \ldots \quad \text{for } Y \to \infty \] (417)

The formula in eq. contains the leading classical form of \( F(Y) \) plus the first perturbative correction calculated with standard techniques of quantum field–theory. Eq. was the starting point of the analysis of Seiberg and Witten who from the perturbative singularity structure inferred the monodromy group and then conjectured the dynamical Riemann surface. The same procedure has been followed to conjecture the dynamical Riemann surfaces of the higher rank gauge theories. The nonperturbative solution is given by

\[ F(Y) = \frac{i}{2\pi} Y^2 \log \frac{Y^2}{\Lambda^2} + \sum_{n=1}^{\infty} C_n \left( \frac{\Lambda^2}{Y^2} \right)^{2n} \] (418)

The infinite series in eq. corresponds to the sum over instanton corrections of all instanton numbers.

The important thing to note is that the special coordinates \( Y^\alpha(u) \) of rigid special geometry approach for large values of \( u \) the Calabi–Visentini coordinates of the manifold \( ST[2,n] \) discussed in previous lectures. As stressed there, the \( Y^\alpha \) are not special coordinates for local special geometry.

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