COPRODUCTS OF FINITE GROUPS

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Abstract. We show that for any pair of non-trivial finite groups, their coproduct in the category of finite groups is not representable.

Given a category $\mathcal{C}$ and a pair of objects $X_1, X_2 \in \mathcal{C}$, the coproduct $X_1 \coprod X_2$ is representable in $\mathcal{C}$ iff

$$\exists X \in \mathcal{C}, \exists (\iota_{X_1}, \iota_{X_2}) \in \text{Hom}_\mathcal{C}(X_1, X) \times \text{Hom}_\mathcal{C}(X_2, X)$$

such that

$$\forall Y \in \mathcal{C}, \forall (f_1, f_2) \in \text{Hom}_\mathcal{C}(X_1, Y) \times \text{Hom}_\mathcal{C}(X_2, Y), \exists! f \in \text{Hom}_\mathcal{C}(X, Y)$$

making the diagram

```
    X_1
    /\  \f
X → f_1 \psi \iota_{X_1} \iota_{X_2} \iota_{X_2} \f_2 \f_2
```

commute.

Let $G, H$ be finite groups. If we regard them as members of the category of all groups, then $G \coprod H$ is representable: it is the free product $G \ast H$, and $G \to G \ast H$ and $H \to G \ast H$ are the canonical inclusions. The purpose of this note is to prove the following theorem:

**Theorem 1.** If $G, H$ are non-trivial groups, then the coproduct $G \coprod H$ in the category of finite groups is not representable.

The key will be the following proposition:

**Proposition 2.** Let $G, H$ be non-trivial groups, and let $g \in G \setminus \{1\}$ and $h \in H \setminus \{1\}$. For every $m \geq 1$, then there exist a finite group $T_m$ and a homomorphism $q_m : G \ast H \to T_m$ such that $|\langle q_m(gh) \rangle| > m$.

Before proving the proposition we show how it implies Theorem 1.

**Proof of Theorem 1.** Let $F$ be a finite group and

$$\iota_G : G \to F \text{ and } \iota_H : H \to F$$

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be homomorphisms. Let
\[ g \in G \setminus \{1\} \text{ and } h \in H \setminus \{1\}, \]
let \( m \) be the order of \( \iota_G(g) \iota_H(h) \), and let
\[ q_m : G \ast H \to T \]
be a homomorphism such that \( |\langle q_m(gh) \rangle| > m \) as in Proposition 2. Let
\[ f_G : G \to T_m \text{ and } f_H : H \to T_m \]
be the respective compositions of
\[ G \to G \ast H \text{ and } H \to G \ast H \]
with \( q_m \) so that
\[ \xymatrix{ F \ar[r]^{\iota_G} & G \ar[d]_{f_G} \ar[l]^{\iota_H} \\
T_m \ar[r]_{q_m} & G \ast H \ar[u]_{f_H} } \]
commutes. Observe that
\[ \{ f \in \text{Hom}(F, T_m) : f_G = \iota_G f \text{ and } f_H = \iota_H f \} \]
is empty since \( f_G(g)f_H(h) \) has order exceeding \( m \) while
\[ (f\iota_G(g))(f\iota_H(h)) = f(\iota_G(g)\iota_H(h)) \]
has order \( m \). Therefore one cannot find a morphism \( F \to T_m \) to complete the above diagram, and hence \( G \amalg H \) is not representable in the category of finite groups. \( \square \)

The proof of the proposition will occupy the remainder of this note.

Proof of Proposition 2
To start we recall a special case of a result of Marciniak:

**Theorem 3.** If there exists a faithful representation \( G \times H \to \text{GL}_n(Q) \), then there exists a faithful representation \( G \ast H \to \text{GL}_n(K) \) where \( K = Q(t) \).

**Proof.** See [1]. \( \square \)

Recall that a group is residually finite iff every non-identity element is contained in the complement of a finite-index normal subgroup.

**Corollary 4.** \( G \ast H \) is residually finite.

**Proof.** Let \( Q[G] \) and \( Q[H] \) be the respective group algebras of \( G \) and \( H \), and let \( n = |G| + |H| \). If \( V = Q[G] \oplus Q[H] \), then \( V \simeq Q^n \) and there exists a canonical faithful representation \( G \times H \to \text{GL}_n(Q) \). Let
\[ \rho : G \ast H \to \text{GL}_n(K) \]
be the corresponding representation given by Theorem 3. It is faithful and the image is finitely generated, so the corollary is a consequence of the fact that any finitely generated subgroup of $GL_n(\mathbb{C})$ is residually finite. Rather than appeal to this general fact though, we prove directly that $G \ast H$ is residually finite.

Let $w \in G \ast H$ be a non-identity element. We must show that there is a finite-index normal subgroup of $G \ast H$ whose complement contains $w$.

The set $\rho(G \cup H)$ is finite and contained in $GL(K)$, so the least common multiple of the denominators of all the entries of the matrices in $\rho(G \cup H)$ is a non-zero polynomial $D \in \mathbb{Q}[t]$. Moreover, the greatest common divisor of the numerators of all the entries of $\rho(w) - I_n$ is a non-zero polynomial $N \in \mathbb{Q}[t]$.

Let $Z \subset \mathbb{Q}$ be the finite set of zeros of $D \cdot N$ in $\mathbb{Q}$ and $U := \mathbb{Q} \setminus Z$. Let $S$ be the polynomial ring $\mathbb{Q}[t][1/DN]$ and

$$\text{GL}_n(S) \to \text{GL}_n(K)$$

be the homomorphism induced by the inclusion $S \to K$.

Observe that the elements of $\rho(G \cup H)$ all have finite order and generate $\rho(G \ast H)$. Therefore the elements of $\det(\rho(G \cup H))$ are roots of unity and generate $\det(\rho(G \ast H))$, so the latter is contained in $S^\times$. In particular, $\rho$ factors through (1).

For each $s \in U$, let $\mathbb{F}_s$ be the quotient field $S/(t-s)S$ and $\rho_s$ be the composition

$$G \ast H \to \text{GL}_n(S) \to \text{GL}_n(\mathbb{F}_s)$$

where the last homomorphism is induced by the quotient $S \to \mathbb{F}_s$. Observe that $\mathbb{F}_s$ is (canonically) isomorphic to $\mathbb{Q}$, so we can regard $\rho_s$ as a homomorphism

$$\rho_s: G \ast H \to \text{GL}_n(\mathbb{Q}).$$

Observe also that the least common multiple of the denominators of the entries of all matrices in $\rho_s(G \cup H)$ is a positive integer $D_s \in \mathbb{N}$, and that $\det(\rho_s(G \ast H)) \subseteq \mathbb{Z}[1/D_s]^\times$. Therefore the image of $\rho_s$ is contained in the image of the natural homomorphism

$$\text{GL}_n(\mathbb{Z}[1/D_s]) \to \text{GL}_n(\mathbb{Q}),$$

and hence there is a homomorphism

$$\rho_{s,p}: G \ast H \to \text{GL}_n(\mathbb{Z}/p)$$

for each $p \nmid D_s$. By construction, $\rho_{s,p}(w) \neq I_n$ and the kernel of $\rho_{s,p}$ has finite index in $G \ast H$. In particular, the complement of the latter is a finite-index subgroup containing $w$, so $G \ast H$ is residually finite as claimed.

Observe that $gh$ has infinite order in $G \ast H$. Therefore, for each $m \geq 1$, the element $w_m := (gh)^m$ is not the identity, hence Corollary 4 implies that there exists a finite-index normal subgroup $K_m \subseteq G \ast H$ whose complement contains $w_m$. 


To complete the proof of the proposition, we let $T_m := (G \ast H)/K_m$ and 

$$q_m : G \ast H \to T_m$$

be the canonical quotient. By definition, 

$$q_m(w^m!) = q_m(w_m) \neq 1,$$

hence $q_m(w)$ has order exceeding $m$ as claimed. \qed

References

1. Zbigniew S. Marciniak, *A note on free products of linear groups*, Proc. Amer. Math. Soc. 94 (1985), no. 1, 46–48. MR 781053