ON POINCARÉ SERIES OF HALF-INTEGRAL WEIGHT

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Abstract. We use Poincaré series of $K$-finite matrix coefficients of genuine integrable representations of the metaplectic cover of $SL_2(\mathbb{R})$ to construct a spanning set for the space of cusp forms $S_m(\Gamma, \chi)$, where $\Gamma$ is a discrete subgroup of finite covolume in the metaplectic cover of $SL_2(\mathbb{R})$, $\chi$ is a character of $\Gamma$ of finite order, and $m \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$. We give a result on the non-vanishing of the constructed cusp forms and compute their Petersson inner product with any $f \in S_m(\Gamma, \chi)$. Using this last result, we construct a Poincaré series $\Delta_{\Gamma, k, m, \xi, \chi} \in S_m(\Gamma, \chi)$ that corresponds, in the sense of the Riesz representation theorem, to the linear functional $f \mapsto f^{(k)}(\xi)$ on $S_m(\Gamma, \chi)$, where $\xi \in \mathbb{C}_{\Im(z) > 0}$ and $k \in \mathbb{Z}_{\geq 0}$. Under some additional conditions on $\Gamma$ and $\chi$, we provide the Fourier expansion of cusp forms $\Delta_{\Gamma, k, m, \xi, \chi}$ and their expansion in a series of classical Poincaré series.

1. Introduction

In this paper, we adapt representation-theoretic techniques developed for the group $SL_2(\mathbb{R})$ in [5] and [9] to the case of the metaplectic cover of $SL_2(\mathbb{R})$. Using this, we prove a few results on cusp forms of half-integral weight.

To give an overview of our results, we introduce the basic notation. The metaplectic cover of $SL_2(\mathbb{R})$ can be realized as the group

$$SL_2(\mathbb{R})^\sim := \left\{ \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \eta \right\} \in SL_2(\mathbb{R}) \times \text{Hol}(\mathcal{H}) : \eta^2(z) = cz + d \text{ for all } z \in \mathcal{H},$$

where Hol$(\mathcal{H})$ is the space of all holomorphic functions defined on the upper half-plane $\mathcal{H}$. The multiplication law and smooth structure of $SL_2(\mathbb{R})^\sim$ are defined in Section 2. $SL_2(\mathbb{R})^\sim$ acts on $\mathcal{H}$ by $\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \eta \right) : \eta := \frac{az + b}{cz + d}$. Moreover, for every $m \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$ we have the following right action of $SL_2(\mathbb{R})^\sim$ on $\mathbb{C}^\mathcal{H}$: $(f \mid_{m} \sigma)(z) := f(\sigma z)\eta^2_{\sigma}(z)^{-2m}$. Let $P : SL_2(\mathbb{R})^\sim \rightarrow SL_2(\mathbb{R})$ be the projection onto the first coordinate.

Next, let $\Gamma$ be a discrete subgroup of finite covolume in $SL_2(\mathbb{R})^\sim$, $\chi : \Gamma \rightarrow \mathbb{C} \times$ a character of finite order, and $m \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$. The space $S_m(\Gamma, \chi)$ of cusp forms of weight $m$ for $\Gamma$ with character $\chi$ by definition consists of all $f \in \text{Hol}(\mathcal{H})$ that satisfy $f\mid_{m} \gamma = \chi(\gamma) f$ for all $\gamma \in \Gamma$ and vanish at all cusps of $P(\Gamma)$. $S_m(\Gamma, \chi)$ is a finite-dimensional Hilbert space under the

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Petersson inner product \( \langle f_1, f_2 \rangle_{\Gamma} := |\Gamma \cap Z(\text{SL}_2(\mathbb{R}))|^{-1} \int_{\Gamma \backslash \mathcal{H}} f_1(z) \overline{f_2(z)}  \Im(z)^m \, dv(z) \), where \( dv(x + iy) := \frac{dx \, dy}{y^2} \). We write \( S_{m}(\Gamma) := S_m(\Gamma, 1) \). Let us denote by \( \Phi \) the classical lift \( S_{m}(\Gamma) \to \mathbb{C}^{\text{SL}_2(\mathbb{R})^\sim} \) that maps \( f \in S_{m}(\Gamma) \) to \( F_f : \text{SL}_2(\mathbb{R})^\sim \to \mathbb{C}, \) \( F_f(\sigma) := (f|_m \sigma)(i) \), where \( i \) is the imaginary unit. \( \Phi \) is a unitary isomorphism \( S_{m}(\Gamma) \to \Phi(S_{m}(\Gamma)) =: A(\Gamma \backslash \text{SL}_2(\mathbb{R})^\sim)_m \subseteq L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^\sim) \) (Theorem 4-5).

The starting point of this paper are results of [11], where we applied the techniques of [5] to compute certain \( K \)-finite matrix coefficients of genuine integrable representations of \( \text{SL}_2(\mathbb{R})^\sim \) and study their Poincaré series with respect to \( \Gamma \). In Lemma 5-1,(5), we show that the Poincaré series \( P_{1} F_{k,m} \) \((k \in \mathbb{Z}_{\geq 0}, m \in \frac{3}{2} + \mathbb{Z}_{\geq 0}) \) discussed in [11, Section 6] belong to \( A(\Gamma \backslash \text{SL}_2(\mathbb{R})^\sim)_m \). The main result of this paper is Theorem 5-6, in which we compute the Petersson inner product of \( P_{1} F_{k,m} \) with any \( \varphi \in A(\Gamma \backslash \text{SL}_2(\mathbb{R})^\sim)_m \) using the representation theory of \( \text{SL}_2(\mathbb{R})^\sim \). It is the \( \text{SL}_2(\mathbb{R})^\sim \)-variant of [9, Theorem 2-11].

In the rest of the paper, we use the facts of the previous paragraph to prove a few results about \( S_{m}(\Gamma, \chi) \) for \( m \in \frac{3}{2} + \mathbb{Z}_{\geq 0} \). Most of these results are half-integral weight variants of results of [5], [6], [7], and [9].

First, by considering the preimages of functions \( P_{1} F_{k,m} \) under \( \Phi \), in Theorem 6-1 we construct the following spanning set for \( S_{m}(\Gamma, \chi) \) (cf. [5, Lemma 4-2]):

\[
(1-1) \quad \langle P_{1,\chi} f_{k,m} \rangle(z) := (2i)^{m} \sum_{\gamma \in \Gamma} \chi(\gamma) \frac{(\gamma \cdot z - i)^k}{(\gamma \cdot z + i)^{m+k}} \eta_{\gamma}(z)^{-2m}, \quad z \in \mathcal{H}, \quad k \in \mathbb{Z}_{\geq 0}, \]

(see (2-1)). Moreover, we obtain results (Theorem 6-2 and Corollary 6-5) on the non-vanishing of cusp forms \( P_{1,\chi} f_{k,m} \) in the case when \( P(\Gamma) \subseteq \text{SL}_2(\mathbb{Z}) \) by adapting our study of the non-vanishing of functions \( P_{1} F_{k,m} \) conducted in [11, Section 6].

Next, Theorem 5-6 translates via the unitary isomorphism \( \Phi^{-1} \) to Theorem 6-1.(3), which states that, for every \( k \in \mathbb{Z}_{\geq 0}, \)

\[
\langle f, P_{1,\chi} f_{k,m} \rangle_{\Gamma} = \sum_{l=0}^{k} \binom{k}{l} (2i)^l \frac{4\pi}{\prod_{r=0}^{l}(m-1+r)} f^{(l)}(i), \quad f \in S_{m}(\Gamma, \chi). \]

It is a short way from this relation to the proof of the following fact in the case when \( \xi = i \): for every \( k \in \mathbb{Z}_{\geq 0}, \) the Poincaré series

\[
(1-2) \quad \Delta_{\Gamma,k,m,\xi,\chi}(z) := \frac{(2i)^{m}}{4\pi} \left( \prod_{r=0}^{k} (m-1+r) \right) \sum_{\gamma \in \Gamma} \frac{\chi(\gamma)}{\gamma \cdot z - \xi^{k+m}} \eta_{\gamma}(z)^{-2m}, \quad z \in \mathcal{H},
\]

belongs to \( S_{m}(\Gamma, \chi) \) and satisfies

\[
\langle f, \Delta_{\Gamma,k,m,\xi,\chi} \rangle_{\Gamma} = f^{(k)}(\xi), \quad f \in S_{m}(\Gamma, \chi).
\]

We prove that this holds for all \( \xi \in \mathcal{H} \) in Proposition 7-2 and Theorem 7-12 (cf. [9, Corollary 1-2]).
Incidentally, our proof of Theorem 7-12 proves the following integral formula (Corollary 7-16):

\[
(1-3) \quad f^{(k)}(\xi) = \frac{(-2i)^m}{4\pi} \left( \prod_{r=0}^{k} (m - 1 + r) \right) \int_{\mathcal{H}} \frac{f(z)}{(z - \xi)^{m+k}} \mathfrak{Z}(z)^m \, dv(z)
\]

for all \( f \in S_m(\Gamma, \chi) \), \( k \in \mathbb{Z}_{\geq 0} \), and \( \xi \in \mathcal{H} \). We use this formula in Corollary 7-16 to give a short proof that

\[
(1-4) \quad \sup_{\xi \in \mathcal{H}} |f^{(k)}(\xi)\mathfrak{Z}(\xi)^{\frac{m+k}{2}}| < \infty, \quad f \in S_m(\Gamma, \chi), \; k \in \mathbb{Z}_{\geq 0},
\]

which enables us to prove, in Proposition 7-17, that

\[
\sup_{z, \xi \in \mathcal{H}} \mathfrak{Z}(\xi)^{\frac{m+k}{2}} \mathfrak{Z}(z)^{\frac{m}{2}} |\Delta_{\Gamma, k, m, \xi, \chi}(z)| < \infty
\]

for every \( k \in \mathbb{Z}_{\geq 0} \) (cf. [7, (1-5)]).

Next, assume that \( \infty \) is a cusp of \( P(\Gamma) \) and that \( \eta_{\gamma}^{-2m} = \chi(\gamma) \) for all \( \gamma \in \Gamma_{\infty} \), so that we have the classical Poincaré series \( \psi_{\Gamma, n, m, \chi} \in S_m(\Gamma, \chi) \), \( n \in \mathbb{Z}_{>0} \), defined by

\[
\psi_{\Gamma, n, m, \chi}(z) := \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \overline{\chi(\gamma)} e^{2\pi i n \frac{z}{2}} \eta_{\gamma}(z)^{-2m}, \quad z \in \mathcal{H},
\]

where \( h \in \mathbb{R}_{>0} \) is such that the group \( \{ \pm 1 \} \; P(\Gamma_{\infty}) \) is generated by \( \left\{ \pm \left( \begin{array}{cc} 1 & h \\ 0 & 1 \end{array} \right) \right\} \). Theorem 8-4 gives the Fourier expansion of cusp forms \( \Delta_{\Gamma, k, m, \xi, \chi} \) and their expansion in a series of classical Poincaré series (cf. [6, Theorem 3-5]). In Corollary 8-7, this Fourier expansion combined with (1-4) provides a quick proof of some bounds on the derivatives of classical Poincaré series (cf. [7, Theorem 1-2]).

Finally, in Section 9 we apply our results to the standardly defined spaces \( S_m(N, \chi) \), where \( N \in 4\mathbb{Z}_{>0} \) and \( \chi \) is an even Dirichlet character modulo \( N \) (e.g., see [10]). We show that \( S_m(N, \chi) \) coincides with \( S_m(\Gamma_0(N), \chi) \), where \( \Gamma_0(N) \) is an appropriate discrete subgroup of \( \text{SL}_2(\mathbb{R})^\sim \), and \( \chi \) is identified with a suitable character of \( \Gamma_0(N) \). Corollary 9-3 gives a formula for the action of Hecke operators \( T_{p^2, m, \chi} \), for prime numbers \( p \nmid N \), on cusp forms \( \Delta_{\Gamma_0(N), k, m, \xi, \chi} \) in terms of their expansion in a series of classical Poincaré series (cf. [6, Lemma 5-8]).

Let us mention that a non-representation-theoretic proof of Proposition 7-2 and Theorem 7-12 in the case when \( k = 0 \) can be obtained by adapting the proof of [4, Theorem 6.3.3] to half-integral weights. The case when \( k \in \mathbb{Z}_{>0} \) can be derived from it essentially by taking the \( k \)th derivative (the details can be gleaned from the first sentence of the proof of Proposition 7-2 and from Lemma 7-9). Similarly, the integral formula (1-3) can be deduced from the half-integral weight variant of [4, Theorem 6.2.2]; the integral-weight variant of (1-3) for \( k = 0 \) is actually used in the proof of [4, Theorem 6.3.3] (see the last equality on [4, pg. 230]). On the other hand, our results on the non-vanishing of cusp forms \( P_{\Gamma, \chi} f_{k, m} \) are based on applying the integral criterion [8, Lemma 2-1] to the corresponding Poincaré series on \( \text{SL}_2(\mathbb{R})^\sim \). To do that, we used the Cartan decomposition of \( \text{SL}_2(\mathbb{R})^\sim \), which is not easily accessible when working directly in \( S_m(\Gamma, \chi) \).
2. Preliminaries on the metaplectic group

Let $\sqrt{\cdot} : \mathbb{C} \to \mathbb{C}$ be the branch of the complex square root with values in $\{ z \in \mathbb{C} : \Re(z) > 0 \} \cup \{ z \in \mathbb{C} : \Re(z) = 0, \Im(z) \geq 0 \}$. We write $i := \sqrt{-1}$ and

$$z^m := (\sqrt{z})^{2m}, \quad z \in \mathbb{C}^*, \ m \in \frac{1}{2} + \mathbb{Z}. \quad (2-1)$$

Next, we define $H := \{ z \in \mathbb{C} : \Im(z) > 0 \}$ and denote by $\text{Hol}(H)$ the space of all holomorphic functions $H \to \mathbb{C}$.

The group $\text{SL}_2(\mathbb{R})$ acts on $\mathbb{C} \cup \{ \infty \}$ by

$$g.z := \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \ z \in \mathbb{C} \cup \{ \infty \}. \quad (2-2)$$

We have

$$\Im(g.z) = \frac{\Im(z)}{|cz + d|^2}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \ z \in H. \quad (2-3)$$

For every $N \in \mathbb{Z}_{>0}$, we denote

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \ (\text{mod } N) \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0, \ a \equiv d \equiv 1 \ (\text{mod } N) \right\},$$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : b \equiv c \equiv 0, \ a \equiv d \equiv 1 \ (\text{mod } N) \right\}.$$

The group $\text{SL}_2(\mathbb{R})^\sim := \left\{ \sigma = \left( g_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \eta_\sigma \right) \in \text{SL}_2(\mathbb{R}) \times \text{Hol}(H) : \eta_\sigma^2(z) = cz + d \text{ for all } z \in H \right\}$ with multiplication law

$$\sigma_1 \sigma_2 := (g_{\sigma_1}g_{\sigma_2}, \eta_{\sigma_1}(g_{\sigma_2} \cdot z)\eta_{\sigma_2}(z)), \quad \sigma_1, \sigma_2 \in \text{SL}_2(\mathbb{R})^\sim, \quad (2-4)$$

acts on $\mathbb{C} \cup \{ \infty \}$ by

$$\sigma.z := g_\sigma \cdot z, \quad \sigma \in \text{SL}_2(\mathbb{R})^\sim, \ z \in \mathbb{C} \cup \{ \infty \},$$

and, for every $m \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, on the right on $\mathbb{C}^H$ by

$$\left( f \mid_m \sigma \right)(z) := f(\sigma.z) \eta_\sigma(z)^{-2m}, \quad z \in H, \ f \in \mathbb{C}^H, \ \sigma \in \text{SL}_2(\mathbb{R})^\sim. \quad (2-5)$$

In the following, we use shorthand notation $(g_\sigma, \eta_\sigma(i))$ for elements $\sigma = (g_\sigma, \eta_\sigma)$ of $\text{SL}_2(\mathbb{R})^\sim$. $\text{SL}_2(\mathbb{R})^\sim$ is a connected Lie group with a smooth (Iwasawa) parametrization $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}$.
\( \text{SL}_2(\mathbb{R})^\sim \),

\[
(x, y, t) \mapsto \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right), 1 \left( \begin{array}{cc} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{array} \right), \left( \begin{array}{cc} \cos t & -\sin t \\ \sin t & \cos t \end{array} \right), e^{\frac{it}{2}} \right).
\]

The projection \( P : \text{SL}_2(\mathbb{R})^\sim \to \text{SL}_2(\mathbb{R}) \) onto the first coordinate is a smooth covering homomorphism of degree 2. The center of \( \text{SL}_2(\mathbb{R})^\sim \) is \( Z(\text{SL}_2(\mathbb{R})^\sim) := P^{-1}(\{\pm 1\}) \cong (\mathbb{Z}/4\mathbb{Z}, +) \).

We will denote the three factors on the right-hand side of (2.4), from left to right, by \( n_x \), \( a_y \), and \( \kappa_t \). We also define

\[
h_t := \left( \begin{array}{cc} e^t & 0 \\ 0 & e^{-t} \end{array} \right), \quad t \in \mathbb{R}_{\geq 0}.
\]

Next, we recall the \( \text{SL}_2(\mathbb{R})^\sim \)-invariant Radon measure \( \nu \) on \( \mathcal{H} \) defined by \( d\nu(x + iy) := \frac{dx \, dy}{y^2}, \) \( x, y \in \mathbb{R}, y \in \mathbb{R}_{>0} \), and fix the following Haar measure on \( \text{SL}_2(\mathbb{R})^\sim \): for \( \varphi \in C_c(\text{SL}_2(\mathbb{R})^\sim), \)

\[
\int_{\text{SL}_2(\mathbb{R})^\sim} \varphi \, d\mu_{\text{SL}_2(\mathbb{R})^\sim} := \frac{1}{4\pi} \int_0^{2\pi} \int_{\mathcal{H}} \varphi(n_x a_y \kappa_t) \, dv(x + iy) \, dt
\]

(2.5)

\[
= \frac{1}{4\pi} \int_0^{2\pi} \int_{\mathcal{H}} \varphi(\kappa_t h_t \kappa_{t_2}) \sinh(2t) \, d\theta_1 \, dt \, d\theta_2.
\]

(2.6)

Furthermore, for a discrete subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{R})^\sim \), let \( \mu_{\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim} \) be the unique Radon measure on \( \Gamma \setminus \text{SL}_2(\mathbb{R})^\sim \) such that, for all \( \varphi \in C_c(\text{SL}_2(\mathbb{R})^\sim), \)

\[
\int_{\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim} \sum_{\gamma \in \Gamma} \varphi(\gamma \sigma) \, d\mu_{\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim}(\sigma) = \int_{\text{SL}_2(\mathbb{R})^\sim} \varphi \, d\mu_{\text{SL}_2(\mathbb{R})^\sim}.
\]

Equivalently, for all \( \varphi \in C_c(\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim), \)

\[
\int_{\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim} \varphi \, d\mu_{\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim} = \frac{1}{4\pi \varepsilon_{\Gamma}} \int_0^{2\pi} \int_{\Gamma \setminus \mathcal{H}} \varphi(n_x a_y \kappa_t) \, dv(x + iy) \, dt,
\]

where \( \varepsilon_{\Gamma} := |\{\Gamma \cap Z(\text{SL}_2(\mathbb{R})^\sim)\}|. \) For every \( p \in \mathbb{R}_{\geq 1} \), we define \( L^p(\text{SL}_2(\mathbb{R})^\sim) \) and \( L^p(\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim) \) using \( \mu_{\text{SL}_2(\mathbb{R})^\sim} \) and \( \mu_{\Gamma \setminus \text{SL}_2(\mathbb{R})^\sim} \), respectively.

We identify the Lie algebra \( \mathfrak{g} := \text{Lie}(\text{SL}_2(\mathbb{R})^\sim) \) with \( \text{Lie}(\text{SL}_2(\mathbb{R})) \equiv \mathfrak{sl}_2(\mathbb{R}) \) via the differential of \( P \) at 1 and extend this identification to that of the universal enveloping algebras of their complexifications: \( U(\mathfrak{g}_C) \equiv U(\mathfrak{sl}_2(\mathbb{C})) \). Now,

\[
k^o := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad n^+ := \frac{1}{2} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad n^- := \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}
\]

form a standard basis of \( \mathfrak{g}_C \) (we have \( [k^o, n^+] = 2n^+, \ [k^o, n^-] = -2n^-, \) and \( [n^+, n^-] = k^o \), and

\[
\mathcal{C} := \frac{1}{2} (k^o)^2 + n^+ n^- + n^- n^+.
\]
generates the center of $U(g_C)$. We will need the formulae \cite[(2-13)-(2-14)]{Gol1} giving the action of $\mathcal{C}$ and $n^+$ as left-invariant differential operators on $C^\infty(\text{SL}_2(\mathbb{R})^\sim)$ in Iwasawa coordinates:

\begin{equation}
\mathcal{C} = 2y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2y \frac{\partial^2}{\partial x \partial t},
\end{equation}

\begin{equation}
n^+ = iye^{-2it} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) + \frac{i}{2} e^{-2it} \frac{\partial}{\partial t}.
\end{equation}

$n^-$ acts as the complex conjugate of $n^+$:

\begin{equation}
n^- = -i ye^{2it} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) - \frac{i}{2} e^{2it} \frac{\partial}{\partial t}.
\end{equation}

$K := \{ \kappa_t : t \in \mathbb{R} \}$ is a maximal compact subgroup of $\text{SL}_2(\mathbb{R})^\sim$. It is isomorphic to $(\mathbb{R}/4\pi\mathbb{Z}, +)$ via $\kappa_t \mapsto t + 4\pi\mathbb{Z}$. Its unitary dual consists of the characters $\chi_n, n \in \frac{1}{2}\mathbb{Z}$, defined by

$$
\chi_n(\kappa_t) := e^{-int}, \quad t \in \mathbb{R}.
$$

We say that a function $F : \text{SL}_2(\mathbb{R})^\sim \to \mathbb{C}$ transforms on the left (resp., on the right) as $\chi_n$ if $F(\kappa \sigma) = \chi_n(\kappa) F(\sigma)$ (resp., $F(\sigma \kappa) = F(\sigma) \chi_n(\kappa)$) for all $\kappa \in K$ and $\sigma \in \text{SL}_2(\mathbb{R})^\sim$.

### 3. Preliminaries on cusp forms of half-integral weight

Let $m \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$. Let $\Gamma$ be a discrete subgroup of finite covolume in $\text{SL}_2(\mathbb{R})^\sim$. We denote by $S_m(\Gamma)$ the space of cusp forms for $\Gamma$ of weight $m$, i.e., the space of all $f \in \text{Hol}(\mathcal{H})$ that satisfy

\begin{equation}
(f|m|_\gamma = f, \quad \gamma \in \Gamma,
\end{equation}

and vanish at every cusp of $P(\Gamma)$. Let us explain the last condition. For a cusp $x$ of $P(\Gamma)$, let $\sigma \in \text{SL}_2(\mathbb{R})^\sim$ such that $\sigma \infty = x$. Then, it follows from \cite[Theorem 1.5.4.(2)]{Gol1} that

$$
Z(\text{SL}_2(\mathbb{R})^\sim) \sigma^{-1} \Gamma_x \sigma = Z(\text{SL}_2(\mathbb{R})^\sim) \langle n_h \rangle
$$

for some $h \in \mathbb{R}_{>0}$, hence $f|m|_\sigma$ has a Fourier expansion of the form

$$
(f|m|_\sigma)(z) = \sum_{n \in \mathbb{Z}} a_n e^{\pi in \frac{z}{m}}, \quad z \in \mathcal{H}.
$$

We say that $f$ vanishes at $x$ if $a_n = 0$ for all $n \in \mathbb{Z}_{\leq 0}$.

Next, we recall the half-integral weight variant of \cite[Theorems 2.1.5 and 6.3.1]{Gol1}:

**Lemma 3-2.** Let $f \in \text{Hol}(\mathcal{H})$ such that (3-1) holds. Then, the following claims are equivalent:

1. $f \in S_m(\Gamma)$.
2. $\sup_{z \in \mathcal{H}} |f(z) \Im z \frac{m}{\pi}| < \infty$.
3. $\int_{\Gamma \backslash \mathcal{H}} |f(z) \Im z \frac{m}{\pi}|^2 \, dv(z) < \infty$.

More generally, let $\chi : \Gamma \to \mathbb{C}^\times$ be a character of finite order. $S_m(\Gamma, \chi)$ is defined as the space of all $f \in S_m(\ker \chi)$ that satisfy

\begin{equation}
(f|m|_\gamma = \chi(\gamma) f, \quad \gamma \in \Gamma.
\end{equation}
Clearly, $S_m(\Gamma) = S_m(\Gamma, 1)$. $S_m(\Gamma, \chi)$ is a finite-dimensional Hilbert space under the Petersson inner product

\[ \langle f_1, f_2 \rangle_{\Gamma} := \varepsilon_{\Gamma}^{-1} \int_{\Gamma \backslash H} f_1(z) \overline{f_2(z)} \Theta(z)^m \, dv(z), \quad f_1, f_2 \in S_m(\Gamma, \chi). \]

The orthogonal projection $S_m(\ker \chi) \to S_m(\Gamma, \chi)$ is given by

\[ f \mapsto \sum_{\gamma \in \ker \chi \backslash \Gamma} \chi(\gamma) f|_m \gamma, \quad f \in S_m(\ker \chi). \]

We record another basic fact in the following lemma.

**Lemma 3-6.** Let $\sigma \in SL_2(\mathbb{R})^\sim$. Then, $f \mapsto f|_m \sigma$ defines a unitary isomorphism $S_m(\Gamma, \chi) \to S_m(\sigma^{-1} \Gamma \sigma, \chi^\sigma)$, where $\chi^\sigma := \chi(\sigma \cdot \sigma^{-1})$.

The main results of this paper concern elements of $S_m(\Gamma, \chi)$ constructed in the form of a Poincaré series

\[ P_{\Lambda \backslash \Gamma, \chi} f := \sum_{\gamma \in \Lambda \backslash \Gamma} \chi(\gamma) f|_m \gamma, \]

where $\Lambda$ is a subgroup of $\Gamma$, and $f : \mathcal{H} \to \mathbb{C}$ satisfies $f|_m \lambda = \chi(\lambda) f$ for all $\lambda \in \Lambda$. We write $P_{\Gamma, \chi} f := P_{\{1\} \backslash \Gamma, \chi} f$ and $P_{\gamma} f := P_{\Gamma, \chi} f$.

4. SOME REPRESENTATION-THEORETIC RESULTS

Throughout this section, let $m \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$.

Let $r$ be the right regular representation of $SL_2(\mathbb{R})^\sim$. For a discrete subgroup $\Gamma$ of $SL_2(\mathbb{R})^\sim$, let $r_{\Gamma}$ be the unitary representation of $SL_2(\mathbb{R})^\sim$ by right translations in $L^2(\Gamma \backslash SL_2(\mathbb{R})^\sim)$.

**Lemma 4-1.**

1. There exists a unique (up to unitary equivalence) irreducible unitary representation $\pi_m$ of $SL_2(\mathbb{R})^\sim$ that decomposes, as a representation of $K$, into the orthogonal sum $\bigoplus_{k \in \mathbb{Z}_{\geq 0}} \chi_{m+2k}$.

2. Let $v$ be a non-zero element of the $\chi_m$-isotypic component of $\pi_m$. Then, $\pi_m(n^-) v = 0$, and for every $k \in \mathbb{Z}_{\geq 0}$, $\pi_m(n^k) v$ spans the $\chi_{m+2k}$-isotypic component of $\pi_m$.

**Proof.** (1) is [11, Lemma 3-5.(1)], and (2) is clear from the proof of [11, Lemma 3-5].

The following lemma is central to our proof of Theorem 5-6.

**Lemma 4-2.** Let $\Gamma$ be a discrete subgroup of $SL_2(\mathbb{R})^\sim$. Suppose that $\varphi \in C^\infty(\Gamma \backslash SL_2(\mathbb{R})^\sim) \cap L^2(\Gamma \backslash SL_2(\mathbb{R})^\sim)$, $\varphi \neq 0$, has the following properties:

1. $\varphi$ transforms on the right as $\chi_m$.
2. $C \varphi = m \left( \frac{m}{2} - 1 \right) \varphi$.

Then, the minimal closed subrepresentation $H$ of $r_{\Gamma}$ containing $\varphi$ is unitarily equivalent to $\pi_m$, and $\varphi$ spans its $\chi_m$-isotypic component.

**Proof.** [1, Lemma 77] remains valid when the right regular representation of $G$ is replaced by the representation of $G$ by right translations in $L^2(\Lambda \backslash G)$, where $\Lambda$ is a discrete subgroup of $G$. By this result, $H$ is an orthogonal sum of finitely many closed irreducible $SL_2(\mathbb{R})^\sim$-invariant subspaces. Hence, its $(g, K)$-module of $K$-finite vectors, $H_K$, is a direct sum of
finitely many irreducible \((g, K)\)-modules, and it is generated by \(\varphi\) (see [2, Theorem 0.4]). From this it follows by an elementary computation in \(H_K\), using (1)–(2), that \(H_K\) is in fact an irreducible \((g, K)\)-module and that it is isomorphic, as a \(K\)-module, to \(\bigoplus_{k \in \mathbb{Z}_{\geq 0}} \chi_{m+2k}\). Thus, \(H\) is unitarily equivalent to \(\varpi_m\) by Lemma 4-1.(1). Since \(\varphi \neq 0\) belongs to its (one-dimensional) \(\chi_m\)-isotypic component by (1), the second claim is clear.

Next, we recall the classical lift of \(f : \mathcal{H} \to \mathbb{C}\) to \(F_f : SL_2(\mathbb{R})^\sim \to \mathbb{C}\) defined by
\[
F_f(\varphi) := (f \mid_m \varphi)(i), \quad \varphi \in SL_2(\mathbb{R})^\sim,
\]
i.e., in Iwasawa coordinates,
\[
F_f(n_x a_y \kappa_t) = f(x + iy) y^m e^{-imt}, \quad x, t \in \mathbb{R}, \quad y \in \mathbb{R}_{>0}.
\]

The following result is well-known, but we could not find a convenient reference, so we provide a short proof.

**Theorem 4-5.** Let \(\Gamma\) be a discrete subgroup of finite covolume in \(SL_2(\mathbb{R})^\sim\). Then, the lift \(f \mapsto F_f\) defines a unitary isomorphism \(S_m(\Gamma) \to \mathcal{A}(\Gamma \backslash SL_2(\mathbb{R})^\sim)_m\), where \(\mathcal{A}(\Gamma \backslash SL_2(\mathbb{R})^\sim)_m\) is the subspace of \(L^2(\Gamma \backslash SL_2(\mathbb{R})^\sim)\) consisting of all \(\varphi \in L^2(\Gamma \backslash SL_2(\mathbb{R})^\sim) \cap C^\infty(\Gamma \backslash SL_2(\mathbb{R})^\sim)\) with the following properties:

1. \(\varphi\) transforms on the right as \(\chi_m\).
2. \(C\varphi = m \left(\frac{\pi}{2} - 1\right) \varphi\).

Every \(\varphi \in \mathcal{A}(\Gamma \backslash SL_2(\mathbb{R})^\sim)_m\) is bounded.

**Proof.** An elementary computation using (4-3), (4-4), (2-7), (2-8), and Lemma 3-2 shows that \(f \mapsto F_f\) is a well-defined isometry \(S_m(\Gamma) \to \mathcal{A}(\Gamma \backslash SL_2(\mathbb{R})^\sim)_m\). To prove its surjectivity, let \(\varphi \in \mathcal{A}(\Gamma \backslash SL_2(\mathbb{R})^\sim)_m\), \(\varphi \neq 0\), and define \(f : \mathcal{H} \to \mathbb{C}\),
\[
f(x + iy) := \varphi(n_x a_y) y^{-\frac{3}{2}}.
\]
Obviously, \(f \in C^\infty(\mathcal{H})\) and \(F_f = \varphi\). Next, by Lemma 4-2 \(\varphi\) spans the \(\chi_m\)-isotypic component of a closed subrepresentation of \(\Gamma_f\) that is unitarily equivalent to \(\varpi_m\). Thus, \(n^{-\varphi} = 0\) by Lemma 4-1.(2), so \((\partial_x + i\partial_y) f = 0\) by (2-10), hence \(f\) is holomorphic. Furthermore, the fact that \(\varphi \in L^2(\Gamma \backslash SL_2(\mathbb{R})^\sim)\) implies that \(f\) satisfies (3-1) and, by (2-7), that
\[
\int_{\Gamma \backslash \mathcal{H}} \left| f(z) \Im(z)^\frac{m}{2} \right|^2 dv(z) < \infty,
\]
so \(f \in S_m(\Gamma)\) by Lemma 3-2. The same lemma implies that \(\sup_{z \in \mathcal{H}} \left| f(z) \Im(z)^{\frac{m}{2}} \right| < \infty\), so \(\varphi\) is bounded by (4-4).

Next, let \(\Gamma\) be a discrete subgroup of finite covolume in \(SL_2(\mathbb{R})^\sim\), and let \(\chi\) be a character of \(\Gamma\) of finite order. For \(\varphi : SL_2(\mathbb{R})^\sim \to \mathbb{C}\), we define the Poincaré series
\[
(P_{\Gamma, \chi} \varphi)(\sigma) := \sum_{\gamma \in \Gamma} \overline{\chi(\gamma)} \varphi(\gamma \sigma), \quad \sigma \in SL_2(\mathbb{R})^\sim.
\]
We write \(P_{\Gamma} \varphi := P_{\Gamma, 1} \varphi\). The following lemma is elementary. Its proof is left to the reader.

**Lemma 4-6.** Let \(f : \mathcal{H} \to \mathbb{C}\). Then, the series \(P_{\Gamma, \chi} f\) converges absolutely (resp., absolutely and uniformly on compact sets) on \(\mathcal{H}\) if and only if \(P_{\Gamma, \chi} F_f\) converges in the same way on \(SL_2(\mathbb{R})^\sim\), and in that case
\[
F_{P_{\Gamma, \chi} f} = P_{\Gamma, \chi} F_f.
\]
5. Proof of Theorem 5-6

In the following lemma we recall some results of [11].

**Lemma 5-1.** Let $m \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$.

1. We define $f_{k,m} : \mathcal{H} \to \mathbb{C}$,
   \[
   f_{k,m}(z) := (2i)^m \frac{(z-i)^k}{(z+i)^{m+k}}.
   \]
   $F_{k,m} := F_{f_{k,m}}$ is a (unique up to a multiplicative constant) matrix coefficient of $\overline{\pi_m}$ that transforms on the right as $\chi_m$ and on the left as $\chi_{m+2k}$.
2. $CF_{k,m} = m \left( \frac{m}{2} - 1 \right) F_{k,m}$.
3. We have
   \[
   F_{k,m}(\kappa_{\theta_1}h_{\theta_2}) = \chi_{m+2k}(\kappa_{\theta_1}) \frac{\tan^k(t)}{\cosh^m(t)} \chi_m(\kappa_{\theta_2}), \quad \theta_1, \theta_2 \in \mathbb{R}, \ t \in \mathbb{R}_{\geq 0}.
   \]

4. If $m \in \frac{5}{2} + \mathbb{Z}_{\geq 0}$, then $F_{k,m} \in L^1(\text{SL}_2(\mathbb{R})^\sim)$.

5. Suppose $m \in \frac{5}{2} + \mathbb{Z}_{\geq 0}$. Let $\Gamma$ be a discrete subgroup of finite covolume in $\text{SL}_2(\mathbb{R})^\sim$. Then, the series $\sum_{\gamma \in \Gamma} |F_{k,m}(\gamma \cdot)|$ converges uniformly on compact sets in $\text{SL}_2(\mathbb{R})^\sim$, and $P_{\Gamma}F_{k,m} \in A(\Gamma \backslash \text{SL}_2(\mathbb{R})^\sim)_m$.

**Proof.** (1) is [11, Proposition 4-7], (2) follows from [11, Lemma 4-4.(3)], (3) is [11, Lemma 4-9], (4) is [11, Lemma 4-10], and (5) is clear from the proof of [11, Lemma 6-2].

Next, we prepare a few technical results for the proof of Theorem 5-6.

**Lemma 5-2.** Let $m \in \frac{3}{2} + \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. Then, we have the following:

1. $\|F_{k,m}\|_{L^2(\text{SL}_2(\mathbb{R})^\sim)}^2 = \frac{4\pi k!}{\prod_{\ell=0}^{m-1} (m-1+\ell)}$.
2. Let $f \in \text{Hol}(\mathcal{H})$. Then, for all $x, y, t \in \mathbb{R}$ with $y > 0$,
   \[
   (n^+)^k F_f \left( n_x a_y \kappa_i \right) = \chi_{m+2k}(\kappa_i) y^\frac{m}{2} \sum_{l=0}^{k} \binom{k}{l} (2iy)^l \left( \prod_{r=l+1}^{k} (m-1+r) \right) f^{(l)}(x + iy).
   \]
3. $(n^+)^k F_{k,m}(1) = k!$.

**Proof.** (1) By Lemma 5-1.(3) and (2-6), we have
   \[
   \|F_{k,m}\|_{L^2(\text{SL}_2(\mathbb{R})^\sim)}^2 = \frac{1}{4\pi} \int_0^{4\pi} \int_0^{4\pi} \int_0^{4\pi} \frac{\tanh^k(t)}{\cosh^{2m}(t)} \sinh(2t) d\theta_1 dt d\theta_2,
   \]
   which, substituting $x = \tanh^2(t)$ and using the identities $\sinh(2t) = 2 \sinh(t) \cosh(t)$ and $\frac{1}{\cosh^2(t)} = 1 - \tanh^2(t)$, equals
   \[
   4\pi \int_0^1 x^k (1-x)^{m-2} dx = \frac{4\pi k!}{\prod_{r=0}^{m-1} (m-1+r)}.
   \]
   The last equality is obtained by $k$-fold partial integration.
(2) This is proved by induction on \( k \in \mathbb{Z}_{\geq 0} \) using (2-9) and noting that in the case when \( k = 0 \) (5-3) is (4-4).

(3) Since \( F_{k,m} = F_{k,m} \), (3) is just (2) applied to \( f = f_{k,m} \) with \( x = t = 0 \) and \( y = 1 \). □

Let \( \Gamma \) be a discrete subgroup of \( \text{SL}_2(\mathbb{R})^- \). For \( F \in L^1(\text{SL}_2(\mathbb{R})^-) \) and \( \varphi \in L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-) \), \( r_\Gamma(F) \varphi \in L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-) \) is standardly defined by the following condition:

\[
\langle r_\Gamma(F) \varphi, \phi \rangle_{L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-)} = \int_{\text{SL}_2(\mathbb{R})^-} F(y) \langle r_\Gamma(y) \varphi, \phi \rangle_{L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-)} \, d\mu_{\text{SL}_2(\mathbb{R})^-}(y)
\]

for all \( \phi \in L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-) \). It is well-known that

\[
(5-4) \quad (r_\Gamma(F) \varphi)(x) = \int_{\text{SL}_2(\mathbb{R})^-} F(y) \varphi(xy) \, d\mu_{\text{SL}_2(\mathbb{R})^-}(y)
\]

for almost all \( x \in \text{SL}_2(\mathbb{R})^- \). The following lemma is immediate.

**Lemma 5-5.** Let \( \Gamma \) be a discrete subgroup of \( \text{SL}_2(\mathbb{R})^- \). Let \( F \in L^1(\text{SL}_2(\mathbb{R})^-) \) and \( \varphi \in L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-) \). If \( \varphi \) is continuous and bounded, then the integral in (5-4) converges for every \( x \in \text{SL}_2(\mathbb{R})^- \), and (5-4) expresses the continuous representative of \( r_\Gamma(F) \varphi \).

Now we are ready to prove the main result of this paper:

**Theorem 5-6.** Let \( \Gamma \) be a discrete subgroup of finite covolume in \( \text{SL}_2(\mathbb{R})^- \). Let \( m \in \frac{3}{2} + \mathbb{Z}_{\geq 0} \), \( k \in \mathbb{Z}_{\geq 0} \), and \( \varphi \in \mathcal{A}(\Gamma \backslash \text{SL}_2(\mathbb{R})^-)_m \). Then,

\[
(5-7) \quad \langle \varphi, P_\Gamma F_{k,m} \rangle_{L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-)} = \frac{4\pi}{\prod_{r=0}^k (m - 1 + r)} \left( (n^+)^k \varphi \right)(1).
\]

**Proof.** The case when \( \varphi \equiv 0 \) is trivial, so suppose that \( \varphi \not\equiv 0 \). We have

\[
\langle \varphi, P_\Gamma F_{k,m} \rangle_{L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-)} = \int_{\Gamma \backslash \text{SL}_2(\mathbb{R})^-} \varphi(\sigma) \sum_{\gamma \in \Gamma} F_{k,m}(\gamma \sigma) \, d\mu_{\Gamma \backslash \text{SL}_2(\mathbb{R})^-}(\sigma)
\]

\[
= \int_{\Gamma \backslash \text{SL}_2(\mathbb{R})^-} \sum_{\gamma \in \Gamma} \varphi(\gamma \sigma) \overline{F_{k,m}(\gamma \sigma)} \, d\mu_{\Gamma \backslash \text{SL}_2(\mathbb{R})^-}(\sigma)
\]

\[
= \int_{\text{SL}_2(\mathbb{R})^-} \varphi F_{k,m} \, d\mu_{\text{SL}_2(\mathbb{R})^-}.
\]

Now, \( F_{k,m} \in L^1(\text{SL}_2(\mathbb{R})^-) \) by Lemma 5-1.(4), and \( \varphi \) is continuous and bounded by Theorem 4-5. Thus, by Lemma 5-5, \( r_\Gamma(F_{k,m}) \varphi \in L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-) \cap C(\Gamma \backslash \text{SL}_2(\mathbb{R})^-) \) is given by

\[
(5-9) \quad (r_\Gamma(F_{k,m}) \varphi)(x) = \int_{\text{SL}_2(\mathbb{R})^-} \overline{F_{k,m}(y)} \varphi(xy) \, d\mu_{\text{SL}_2(\mathbb{R})^-}(y), \quad x \in \text{SL}_2(\mathbb{R})^-.
\]

In particular,

\[
(5-10) \quad \langle \varphi, P_\Gamma F_{k,m} \rangle_{L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^-)} = (r_\Gamma(F_{k,m}) \varphi)(1).
\]

so (5-8) implies that
To compute \( r_\Gamma (\overline{F_{k,m}} \varphi) \) (1), we note that by Lemma 4-2 \( \varphi \) generates the \( \chi_m \)-isotypic component of a closed subrepresentation \( H_\varphi \) of \( r_\Gamma \) that is unitarily equivalent to \( \overline{\pi}_m \). Clearly, \( r_\Gamma (\overline{F_{k,m}} \varphi) \in H_\varphi \). In fact, \( r_\Gamma (\overline{F_{k,m}} \varphi) \) belongs to the \( \chi_{m+2k} \)-isotypic component of \( H_\varphi \); since \( \overline{F_{k,m}} \) transforms on the left as \( \chi_{m+2k} \) by Lemma 5-1.(1), we have

\[
(r_\Gamma (\overline{F_{k,m}} \varphi) (x\kappa) = \int_{\text{SL}_2(\mathbb{R})^-} \overline{F_{k,m}}(y) \varphi(xy) d\mu_{\text{SL}_2(\mathbb{R})^-}(y)
\]

\[
= \lambda \chi_{m+2k}(\kappa) \int_{\text{SL}_2(\mathbb{R})^-} \overline{F_{k,m}}(y) \varphi(xy) d\mu_{\text{SL}_2(\mathbb{R})^-}(y)
\]

for all \( x \in \text{SL}_2(\mathbb{R})^- \) and \( \kappa \in K \). Hence, by Lemma 4-1.(2),

\[
r_\Gamma (\overline{F_{k,m}} \varphi) = \lambda \left(n^+\right)^k \varphi \quad \text{for some } \lambda \in \mathbb{C}.
\]

To calculate \( \lambda \), we apply Lemma 4-2, with \( \Gamma = \{1\} \), to \( F_{k,m} \). (\( F_{k,m} \) satisfies all conditions of Lemma 4-2 by Lemmas 5-1.(1)-(2) and 5-2.(1).) We obtain that \( F_{k,m} \) spans the \( \chi_m \)-isotypic component of a closed subrepresentation \( H_{F_{k,m}} \) of \( r \) that is unitarily equivalent to \( \overline{\pi}_m \). Let \( \Phi : H_\varphi \rightarrow H_{F_{k,m}} \) be a unitary equivalence. Since \( \varphi \) and \( F_{k,m} \) span the \( \chi_m \)-isotypic components of, respectively, \( H_\varphi \) and \( H_{F_{k,m}} \), we have

\[
\Phi \varphi = s F_{k,m} \quad \text{for some } s \in \mathbb{C}^\times.
\]

By applying \( \Phi \) to both sides of (5-11), we obtain

\[
r(\overline{F_{k,m}}) \Phi \varphi = \lambda \left(n^+\right)^k \Phi \varphi,
\]

hence by (5-12)

\[
r(\overline{F_{k,m}}) F_{k,m} = \lambda \left(n^+\right)^k F_{k,m}.
\]

By evaluating (the continuous representatives of) both sides at \( 1 \in \text{SL}_2(\mathbb{R})^- \) and using that \( (r(\overline{F_{k,m}}) F_{k,m}) (1) = \|F_{k,m}\|_{L^2(\text{SL}_2(\mathbb{R})^-)}^2 \) by Lemma 5-5, we obtain

\[
\lambda = \frac{4\pi}{(n^+)^k \|F_{k,m}\|_{L^2(\text{SL}_2(\mathbb{R})^-)}^2 (1)} = \frac{4\pi}{\prod_{r=0}^k (m - 1 + r)}
\]

by Lemma 5-2.(1) and (3).

Thus,

\[
(\varphi, P_\Gamma F_{k,m})_{L^2(\text{SL}_2(\mathbb{R})^-)} = (r_\Gamma (\overline{F_{k,m}} \varphi) (1) \quad r_\Gamma (\overline{F_{k,m}} \varphi) (1) \quad \lambda \left(n^+\right)^k \varphi (1) \quad \lambda \left(n^+\right)^k \varphi (1). \]

\( \square \)
6. CUSP FORMS (1-1) AND THEIR NON-VANISHING

Throughout this section, let \( m \in \frac{5}{2} + \mathbb{Z}_{\geq 0} \).

**Theorem 6-1.** Let \( \Gamma \) be a discrete subgroup of finite covolume in \( \text{SL}_2(\mathbb{R})^\sim \), \( \chi : \Gamma \to \mathbb{C}^\times \) a character of finite order, and \( k \in \mathbb{Z}_{\geq 0} \). Then:

1. The series \( P_{\Gamma, \chi} f_{k,m} \) converges absolutely and uniformly on compact sets in \( \mathcal{H} \).
2. \( P_{\Gamma, \chi} f_{k,m} \in S_m(\Gamma, \chi) \).
3. For every \( f \in S_m(\Gamma, \chi) \),
   \[
   \langle f, P_{\Gamma, \chi} f_{k,m} \rangle_\Gamma = \sum_{l=0}^{k} \binom{k}{l} (2i)^l \frac{4\pi}{\prod_{r=0}^{l}(m-1+r)} f^{(l)}(i).
   \]
4. \( \{ P_{\Gamma, \chi} f_{n,m} : n \in \mathbb{Z}_{\geq 0} \} \) spans \( S_m(\Gamma, \chi) \).

**Proof.** (1) By Lemma 4-6 it suffices to prove that the series \( P_{\Gamma, \chi} F_{k,m} = P_{\Gamma, \chi} F_{k,m} \) converges absolutely and uniformly on compact sets in \( \text{SL}_2(\mathbb{R})^\sim \), which is clear from Lemma 5-1.(5).

Next, we prove (2)–(4) in the case when \( \chi = 1 \):

2. Since \( F_{\Gamma, f_{k,m}} = P_{\Gamma, f_{k,m}} \) belongs to \( \mathcal{A}(\Gamma \backslash \text{SL}_2(\mathbb{R})^\sim)_m \) by Lemma 5-1.(5), by Theorem 4-5 \( P_{\Gamma, f_{k,m}} \) belongs to \( S_m(\Gamma) \).

3. Let \( f \in S_m(\Gamma) \). We have, by Theorem 4-5,
   \[
   \langle f, P_{\Gamma, f_{k,m}} \rangle_\Gamma = \langle F_f, P_{\Gamma, f_{k,m}} \rangle_{L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^\sim)} \]
   \[
   = \langle F_f, P_{\Gamma, F_{k,m}} \rangle_{L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^\sim)} \]
   \[
   \overset{(4-7)}{=} \langle F_f, P_{\Gamma} F_{k,m} \rangle_{L^2(\Gamma \backslash \text{SL}_2(\mathbb{R})^\sim)} \]
   \[
   = \frac{4\pi}{\prod_{r=0}^{k}(m-1+r)} \binom{k}{l} \int_{\mathbb{R}} f^{(l)}(i) \prod_{r=0}^{l}(m-1+r) f^{(l)}(i). \]

4. It suffices to show that every \( f \in S_m(\Gamma) \) satisfying \( \langle f, P_{\Gamma, f_{n,m}} \rangle_\Gamma = 0 \) for all \( n \in \mathbb{Z}_{\geq 0} \) is identically zero. Indeed, from (3) it follows by induction on \( n \in \mathbb{Z}_{\geq 0} \) that such an \( f \) satisfies \( f^{(n)}(i) = 0 \) for all \( n \in \mathbb{Z}_{\geq 0} \), so \( f \) is identically zero since \( f \in \text{Hol}(\mathcal{H}) \).

Now, since the orthogonal projection \( S_m(\ker \chi) \to S_m(\Gamma, \chi) \) given by (3-5) maps \( P_{\ker \chi} f_{k,m} \) to \( P_{\Gamma, \chi} f_{k,m} \), the claims (2)–(4) in the case when \( \chi \neq 1 \) follow from the proven ones about \( P_{\ker \chi} f_{k,m} \).

Next, we give a result on the non-vanishing of cusp forms \( P_{\Gamma, \chi} f_{k,m} \) in the case when \( P(\Gamma) \subseteq \text{SL}_2(\mathbb{Z}) \). Let us denote by \( M(a, b) \) the median of the beta distribution with parameters \( a, b \in \mathbb{R}_{>0} \), i.e., the unique \( M(a, b) \in [0, 1] \) such that

\[
\int_0^{M(a, b)} x^{a-1}(1-x)^{b-1} dx = \int_0^1 x^{a-1}(1-x)^{b-1} dx.
\]
Theorem 6.2. Let $N \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}_{>0}$. Let $\Gamma$ be a subgroup of finite index in $P^{-1}(\Gamma(N))$, and let $\chi : \Gamma \to \mathbb{C}^\times$ be a character of finite order. Suppose that

\begin{equation}
\chi|_{\Gamma \cap Z(SL_2(\mathbb{R}))} = \chi^m|_{\Gamma \cap Z(SL_2(\mathbb{R}))},
\end{equation}

(otherwise $S_m(\Gamma, \chi) = 0$ by (3-3)). If

\begin{equation}
N > \frac{4M\left(\frac{k}{2} + 1, \frac{m}{2} - 1\right)^{\frac{1}{2}}}{1 - M\left(\frac{k}{2} + 1, \frac{m}{2} - 1\right)},
\end{equation}

then $P_{\Gamma, \chi} f_{k,m}$ is not identically zero.

Proof. It suffices to prove the non-vanishing of $F_{P_{\Gamma, \chi} f_{k,m}} = P_{\Gamma, \chi} F_{k,m}$. We do this by applying to $P_{\Gamma, \chi} F_{k,m}$ the non-vanishing criterion [8, Lemma 2-1] with $\Gamma_1 = \{1\}$ and $\Gamma_2 = \Gamma \cap Z(SL_2(\mathbb{R}))$: $F_{k,m}$ satisfies the condition (1) of [8, Lemma 2-1] since it transforms on the right as $\chi_m$ and (6-3) holds. A compact set $C$ satisfying the conditions (2)–(3) of [8, Lemma 2-1] can be found using (6-4) exactly as in the proof of [11, Proposition 6-7].

To illustrate the strength of Theorem 6-2, we can use some well-known properties of $M(a, b)$ [11, Lemma 6-12] to obtain the following variant of [11, Corollary 6-18].

Corollary 6-5. Let $N \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}_{>0}$. Let $\Gamma$ be a subgroup of finite index in $P^{-1}(\Gamma(N))$. Let $\chi : \Gamma \to \mathbb{C}^\times$ be a character of finite order such that (6-3) holds. Then, $P_{\Gamma, \chi} f_{k,m}$ is not identically zero if one of the following holds:

1. $k = 0$ and $N > 4 \cdot 2^{-\frac{1}{m-2}} \sqrt{4\frac{1}{m-2}} - 1$
2. $m = 4$ and $N > \frac{4}{2^{\frac{1}{m-2}} - 2^{-\frac{1}{m-2}}}$
3. $0 < k \leq m - 4$ and $N \geq 4 \sqrt{\frac{k+2}{m-2}\left(1 + \frac{k+2}{m-2}\right)}$
4. $0 < m - 4 \leq k$ and $N \geq 4 \sqrt{\frac{k}{m-4}\left(1 + \frac{k}{m-4}\right)}$.

7. Cusp forms (1-2)

Throughout this section, let $\Gamma$ be a discrete subgroup of finite covolume in $\text{SL}_2(\mathbb{R})$, $\chi : \Gamma \to \mathbb{C}^\times$ a character of finite order, and $m \in \frac{5}{2} + \mathbb{Z}_{>0}$.

For every $k \in \mathbb{Z}_{\geq 0}$ and $\xi \in \mathcal{H}$, we define $\delta_{k,m,\xi} : \mathcal{H} \to \mathbb{C}$,

\begin{equation}
\delta_{k,m,\xi}(z) := \frac{(2i)^m}{4\pi} \prod_{r=0}^{k} (m-1+r) \frac{1}{(z-\xi)^{m+k}}.
\end{equation}

Note that $\delta_{k,m,\xi}(z) = \left(\frac{z}{\xi}\right)^k \delta_{0,m,\xi}(z)$.

Proposition 7-2. Let $k \in \mathbb{Z}_{\geq 0}$ and $\xi \in \mathcal{H}$. Then, the Poincaré series

\begin{equation}
\Delta_{\Gamma, k,m,\xi,\chi}(z) := (P_{\Gamma, \chi} \delta_{k,m,\xi})(z) = \frac{(2i)^m}{4\pi} \left(\prod_{r=0}^{k} (m-1+r)\right) \sum_{\gamma \in \Gamma} \frac{\chi(\gamma)}{(\gamma.z - \xi)^{m+k}} \eta_{\gamma}(z)^{-2m}
\end{equation}

converges absolutely and uniformly on compact sets in $\mathcal{H}$ and belongs to $S_m(\Gamma, \chi)$. 

Proof. This can be proved by applying the obvious half-integral weight variant of [4, Theorems 2.6.6.(1) and 2.6.7]. We give an alternative proof. Note that

\begin{equation}
\sum_{l=0}^{k} \binom{k}{l} (-2i)^l \frac{4\pi}{\prod_{r=0}^{l}(m-1+r)} \delta_{l,m,i}, \quad k \in \mathbb{Z}_0,
\end{equation}

hence by the binomial inversion formula

\begin{equation}
\delta_{k,m,i} = \prod_{r=0}^{k}(m-1+r) \sum_{l=0}^{k} \binom{k}{l} (-1)^l f_{l,m}, \quad k \in \mathbb{Z}_0,
\end{equation}

so the claim in the case when \( \xi = i \) follows from Theorem 6-1.(1) and (2). Now the claim for general \( \xi = x + iy \in \mathcal{H} \) (with \( x, y \in \mathbb{R} \)) is clear, using Lemma 3-6, from the identity

\begin{equation}
\Delta_{\Gamma,k,m,\xi,\chi} = y^{-\frac{m}{2} - k} \Delta_{(n_x a_y)^{-1},\Gamma,n_x a_y,k,m,i,\chi} \big|_{m (n_x a_y)^{-1}}
\end{equation}

which is easily checked by following definitions. \( \square \)

The following technical lemmas will be used in our analytic proof of Theorem 7-12.

**Lemma 7-6.** Let \((X, dx)\) be a measure space. Let \(D \) be a domain in \( \mathbb{C} \). Suppose that \( f : D \times X \to \mathbb{C} \) is a measurable function with the following properties:

1. For every \( x \in X \), \( f(\cdot, x) \) is holomorphic on \( D \).
2. For every circle \( C \subseteq D \), \( \int_C |f(z, x)| \, dz < \infty \).

Then, \( F : D \to \mathbb{C} \),

\begin{equation}
F(z) := \int_X f(z, x) \, dx,
\end{equation}

is well-defined and holomorphic on \( D \), and we have

\begin{equation}
F^{(k)}(z) = \int_X \left( \frac{d}{d\zeta} \right)^k f(\zeta, x) \bigg|_{\zeta = z} \, dx, \quad z \in D, \quad k \in \mathbb{Z}_0.
\end{equation}

**Proof.** Without (7-8), this is [4, Lemma 6.1.5]. To prove (7-8), let \( z \in D \) and fix \( \delta \in \mathbb{R}_0 \) such that \( \{ \zeta \in \mathbb{C} : |\zeta - z| \leq \delta \} \subseteq D \). Let \( k \in \mathbb{Z}_0 \). We have

\begin{align*}
F^{(k)}(z) &= \frac{k!}{2\pi i} \int_{|\zeta - z| = \delta} \frac{F(\zeta)}{(\zeta - z)^{k+1}} \, d\zeta \\
&= \int_X \left( \frac{k!}{2\pi i} \int_{|\zeta - z| = \delta} \frac{f(\zeta, x)}{(\zeta - z)^{k+1}} \, d\zeta \right) \, dx \\
&= \int_X \left( \frac{d}{d\zeta} \right)^k f(\zeta, x) \bigg|_{\zeta = z} \, dx
\end{align*}

by applying the Cauchy integral formula for derivatives in the first and the last, and Fubini’s theorem in the second equality. \( \square \)

**Lemma 7-9.** Let \( f \in S_m(\Gamma, \chi) \). Then, the function \( I_f : \mathcal{H} \to \mathbb{C} \),

\[ I_f(\xi) := \langle f, \Delta_{\Gamma,0,m,\xi,\chi} \rangle_{\Gamma}, \]

is holomorphic, and \( I_f^{(k)}(\xi) = \langle f, \Delta_{\Gamma,k,m,\xi,\chi} \rangle_{\Gamma} \) for all \( \xi \in \mathcal{H} \) and \( k \in \mathbb{Z}_0 \).
Proof. For every $k \in \mathbb{Z}_{\geq 0}$ and $\xi \in \mathcal{H}$,
\[
\langle f, \Delta_{\Gamma,k,m,\xi,\chi} \rangle_G \overset{(3-4)}{=} \varepsilon^{-1}_G \int_{\Gamma \setminus \mathcal{H}} f(z) \sum_{\gamma \in \Gamma} \chi(\gamma) \overline{\delta_{k,m,\xi} \mid_{m} \gamma} (z) \Im(z)^m \, dv(z)
\]
\[
\overset{(3-3)}{=} \varepsilon^{-1}_G \int_{\Gamma \setminus \mathcal{H}} \sum_{\gamma \in \Gamma} (f \mid_{m} \gamma) (z) \overline{\delta_{k,m,\xi} \mid_{m} \gamma} (z) \Im(z)^m \, dv(z)
\]
\[
\overset{(2-2)}{=} \varepsilon^{-1}_G \int_{\Gamma \setminus \mathcal{H}} \sum_{\gamma \in \Gamma} f(\gamma,z) \overline{\delta_{k,m,\xi} \mid_{\gamma} \gamma} \Im(\gamma,z)^m \, dv(z)
\]
\[
= \int_{\mathcal{H}} f(z) \overline{\delta_{k,m,\xi} \mid z} \Im(z)^m \, dv(z)
\]
(7-10)
\[
= \frac{(-2i)^m}{\pi} \left( \prod_{r=0}^{k} (m - 1 + r) \right) \int_{\mathcal{H}} \frac{f(z)}{(z - \xi)^{m+k}} \Im(z)^m \, dv(z).
\]

The claim of the lemma follows from (7-10) by Lemma 7-6. The condition (2) of Lemma 7-6 is satisfied since
\[
(7-11) \quad \int_{\mathcal{H}} \frac{|f(z)|}{|z - \xi|^{m+k}} \Im(z)^m \, dv(z) \leq \left( \sup_{z \in \mathcal{H}} |f(z)\Im(z)^{\frac{m}{2}}| \right) \int_{\mathcal{H}} \frac{\Im(z)^{\frac{m}{2}}}{|z - \xi|^{m+k}} \, dv(z)
\]
\[
= \left( \sup_{z \in \mathcal{H}} |f(z)\Im(z)^{\frac{m}{2}}| \right) \int_{\mathcal{H}} \frac{\Im(z)^{\frac{m}{2}}}{|z + n\Re(z)|^{m+k}} \, dv(z) \frac{1}{\Im(z)^{\frac{m}{2} + k}}
\]
(applying the substitution $z \mapsto n\Re(z)a\Im(z), z$ for the last equality), and the right-hand side is obviously bounded for $\xi$ in any circle $C \subseteq \mathcal{H}$. \qed

**Theorem 7-12.** We have
\[
(7-13) \quad \langle f, \Delta_{\Gamma,k,m,\xi,\chi} \rangle_G = f^{(k)}(\xi), \quad f \in S_m(\Gamma, \chi), \quad k \in \mathbb{Z}_{\geq 0}, \quad \xi \in \mathcal{H}.
\]
For every $\xi \in \mathcal{H}$, $\{\Delta_{\Gamma,k,m,\xi,\chi} : k \in \mathbb{Z}_{\geq 0}\}$ spans $S_m(\Gamma, \chi)$.

Proof. Using (7-4), Theorem 6-1.(3) can be written in the following way: for all $f \in S_m(\Gamma, \chi)$ and $k \in \mathbb{Z}_{\geq 0},$
\[
\sum_{l=0}^{k} \binom{k}{l} (2i)^l \frac{4\pi}{\prod_{r=0}^{l} (m - 1 + r)} \langle f, \Delta_{\Gamma,l,m,\xi,\chi} \rangle_G = \sum_{l=0}^{k} \binom{k}{l} (2i)^l \frac{4\pi}{\prod_{r=0}^{l} (m - 1 + r)} f^{(l)}(i).
\]

This implies, by induction on $k \in \mathbb{Z}_{\geq 0}$, that, for all $f \in S_m(\Gamma, \chi)$ and $k \in \mathbb{Z}_{\geq 0},$
\[
(7-14) \quad \langle f, \Delta_{\Gamma,k,m,\xi,\chi} \rangle_G = f^{(k)}(i).
\]

From here, one can obtain (7-13) for general $\xi = x + iy \in \mathcal{H}$ (with $x, y \in \mathbb{R}$) in two ways. The first is algebraic (cf. the proof of [9, Lemma 3-8]): Let $f \in S_m(\Gamma, \chi)$. By taking the $k$th derivative at $z = i$ of the both sides of the equality $f(x + iy) = y^\frac{m}{2} (f \mid_{m} n_x a_y) (z)$, we
obtain, using Lemma 3-6,
\[ f^{(k)}(\xi) = y^{-\frac{m}{2} - k} (f|_{m} n_{x} a_{y})^{(k)}(i) \]
\[ \overset{(7-14)}{=} y^{-\frac{m}{2} - k} \left< f|_{m} n_{x} a_{y}, \Delta_{(n_{x} a_{y})}^{-1} \Gamma_{n_{x} a_{y}, k, m, i, \chi_{n_{x} a_{y}}} \right> (n_{x} a_{y})^{-1} \Gamma_{n_{x} a_{y}} \]
\[ \overset{(7-5)}{=} \left< f, \Delta_{\Gamma, k, m, \xi, \chi} \right> \Gamma. \]

A second way to obtain (7-13) from (7-14) is analytic: Let \( f \in S_{m}(\Gamma, \chi) \). By Lemma 7-9, (7-14) shows that
\[ I_{f}^{(k)}(i) = f^{(k)}(i), \quad k \in \mathbb{Z}_{\geq 0}, \]
i.e., \( f \) and \( I_{f} \) have the same Taylor expansion at \( i \). Since both are holomorphic on \( \mathcal{H} \), it follows by the uniqueness of analytic continuation that
\[ I_{f}^{(k)}(\xi) = f^{(k)}(\xi), \quad \xi \in \mathcal{H}, \quad k \in \mathbb{Z}_{\geq 0}, \]
and this is (7-13) by Lemma 7-9.

The second claim of the theorem follows from (7-13) as in the proof of Theorem 6.1.4. \( \square \)

(7-13) and (7-10) prove the following integral formula:

**Corollary 7-15.** Let \( f \in S_{m}(\Gamma, \chi) \). Then, for all \( k \in \mathbb{Z}_{\geq 0} \) and \( \xi \in \mathcal{H} \),
\[ f^{(k)}(\xi) = \frac{(-2i)^{m}}{4\pi} \left( \prod_{r=0}^{k} (m - 1 + r) \right) \int_{\mathcal{H}} \frac{f(z)}{(\overline{\xi} - \xi)^{m+k}} \Im(z)^{m} dv(z). \]

More generally, Corollary 7-15 holds for every \( f \in \text{Hol}(\mathcal{H}) \) such that \( \sup_{z \in \mathcal{H}} |f(z) \Im(z)^{\frac{m}{2}}| < \infty \). This follows from the half-integral weight version of [4, Theorem 6.2.2]. As a simple application of Corollary 7-15, we prove:

**Corollary 7-16.** Let \( f \in S_{m}(\Gamma, \chi) \). Then, for every \( k \in \mathbb{Z}_{\geq 0} \),
\[ \sup_{\xi \in \mathcal{H}} |f^{(k)}(\xi) \Im(\xi)^{\frac{m}{2} + k}| < \infty. \]

**Proof.** By Corollary 7-15 and (7-11),
\[ \sup_{\xi \in \mathcal{H}} |f^{(k)}(\xi) \Im(\xi)^{\frac{m}{2} + k}| \leq \frac{2m}{4\pi} \left( \prod_{r=0}^{k} (m - 1 + r) \right) \left( \sup_{z \in \mathcal{H}} |f(z) \Im(z)^{\frac{m}{2}}| \right) \int_{\mathcal{H}} \frac{\Im(z)^{\frac{m}{2}}}{|z + i|^{m+k}} dv(z), \]
and the right-hand side is finite by Lemma 3-2. \( \square \)

Now we can easily prove the following result (cf. [7, (1-5)]):

**Proposition 7-17.** Let \( k \in \mathbb{Z}_{\geq 0} \). Then,
\[ \sup_{z, \xi \in \mathcal{H}} \Im(\xi)^{\frac{m}{2} + k} \Im(z)^{\frac{m}{2}} |\Delta_{\Gamma, k, m, \xi, \chi}(z)| < \infty. \]
Proof. Let us fix an orthonormal basis \( \{ f_1, \ldots, f_d \} \) of \( S_m(\Gamma, \chi) \). We have
\[
\Delta_{\Gamma, k, m, \xi, \chi}(z) = \sum_{l=1}^{d} \langle \Delta_{\Gamma, k, m, \xi, \chi}, f_l \rangle_{\Gamma} f_l(z) = \sum_{l=1}^{d} f_l^{(k)}(\xi) f_l(z),
\]
hence
\[
\sup_{z, \xi \in \mathcal{H}} \Im(\xi)^{\frac{k}{2}} \Im(z)^{\frac{k}{2}} |\Delta_{\Gamma, k, m, \xi, \chi}(z)| \leq \sum_{l=1}^{d} \left( \sup_{\xi \in \mathcal{H}} |f_l^{(k)}(\xi)| \Im(\xi)^{\frac{k}{2}} \right) \left( \sup_{z \in \mathcal{H}} |f_l(z)| \Im(z)^{\frac{k}{2}} \right),
\]
and the right-hand side is finite by Corollary 7-16.
\[\square\]

8. Two expansions of cusp forms (1-2)

Throughout this section, let \( \Gamma \) be a discrete subgroup of finite covolume in \( \text{SL}_2(\mathbb{R}) \), \( \chi : \Gamma \to \mathbb{C}^\times \) a character of finite order, and \( m \in \frac{5}{2} + \mathbb{Z}_{\geq 0} \). Moreover, suppose that \( \infty \) is a cusp of \( P(\Gamma) \) and that
\[
\eta_\gamma(z)^{-2m} = \chi(\gamma), \quad \gamma \in \Gamma_\infty, \: z \in \mathcal{H}.
\]
Let \( h \in \mathbb{R}_{>0} \) such that
\[
Z(\text{SL}_2(\mathbb{R}) \sim) \Gamma_\infty = Z(\text{SL}_2(\mathbb{R}) \sim) \langle n_h \rangle.
\]

By the half-integral weight version of \([4, \text{Theorem 2.6.9}]\), for every \( n \in \mathbb{Z}_{>0} \) the classical Poincaré series
\[
\psi_{\Gamma, n, m, \chi} := P_{\Gamma_\infty \setminus \Gamma, \chi} e^{2\pi inz},
\]
converges absolutely and uniformly on compact sets in \( \mathcal{H} \), and \( \psi_{\Gamma, n, m, \chi} \in S_m(\Gamma, \chi) \). Moreover, by the half-integral weight version of \([4, \text{Theorem 2.6.10}]\), every \( f \in S_m(\Gamma, \chi) \) has the following Fourier expansion:
\[
f(z) = \frac{e^{\Gamma_1(4\pi)m^{-1}}}{\Gamma(m-1)h^m} \sum_{n=1}^{\infty} n^{m-1} \langle f, \psi_{\Gamma, n, m, \chi} \rangle_{\Gamma} e^{2\pi inz}, \quad z \in \mathcal{H}.
\]
(8-1)

Here we use the standard notation for the gamma function: \( \Gamma(x) := \int_{0}^{\infty} t^{x-1}e^{-t} \, dt, \: x \in \mathbb{R}_{>0} \).

Theorem 8-4 provides the Fourier expansion of cusp forms \( \Delta_{\Gamma, k, m, \xi, \chi} \) and their expansion in a series of classical Poincaré series. It is a half-integral weight variant of \([6, \text{Theorem 3-5}]\). Lemma 8-2 resolves the convergence issues of its proof.

We define a norm \( \| \cdot \|_{\Gamma, 1} \) on \( S_m(\Gamma, \chi) \) by
\[
\|f\|_{\Gamma, 1} := \int_{\Gamma \mathcal{H}} |f(z) \Im(z)^{\frac{m}{2}}| \, dv(z), \quad f \in S_m(\Gamma, \chi).
\]

Lemma 8-2. Let \( k \in \mathbb{Z}_{\geq 0} \) and \( \xi \in \mathcal{H} \). Then, the series
\[
\sum_{n=1}^{\infty} n^{m+k-1} e^{-2\pi inz} \psi_{\Gamma, n, m, \chi}
\]
converges:

(1) absolutely in the norm \( \| \cdot \|_{\Gamma, 1} \)
(2) absolutely and uniformly on compact sets in \( \mathcal{H} \)
(3) in the topology of \( S_m(\Gamma, \chi) \).
Proof. (1) implies the absolute convergence of (8.3) at every \( z \in \mathcal{H} \) by [4, Corollary 2.6.2]. (1) also implies the rest of the claims (2) and (3) since \( S_m(\Gamma, \chi) \) is finite-dimensional.

To prove (1), observe that

\[
\|\psi_{\Gamma,n,m,\chi}\|_{\Gamma,1} \leq \int_{\Gamma z \mathcal{H}} \sum_{\gamma \in \Gamma_{\infty} \mathcal{H}} \left| \left( e^{2\pi i n \frac{z}{m}} \right) \gamma (z) \mathfrak{S}(z) \right|^\frac{m}{2} dv(z)
\]

(2-2)

\[
\int_{\Gamma z \mathcal{H}} \sum_{\gamma \in \Gamma_{\infty} \mathcal{H}} \left| e^{2\pi i n \frac{\gamma}{m}} \mathfrak{S}(\gamma, z) \right|^\frac{m}{2} dv(z) = \int_{\Gamma_{\infty} \mathcal{H}} \left| e^{2\pi i n \frac{z}{m}} \mathfrak{S}(z) \right|^\frac{m}{2} dv(z)
\]

so

\[
\sum_{n=1}^{\infty} \left| n^{m+k-1} e^{-2\pi i n \frac{z}{m}} \psi_{\Gamma,n,m,\chi}(z) \right| \leq \frac{h^m}{(2\pi)^{m/2} - 1} \Gamma \left( \frac{m}{2} - 1 \right) \sum_{n=1}^{\infty} n^{m+k} e^{-2\pi n \frac{\mathfrak{S}(\xi)}{m}},
\]

and the right-hand side is finite by d’Alembert’s ratio test.

\[\Box\]

**Theorem 8-4.** Let \( k \in \mathbb{Z}_{\geq 0} \) and \( \xi \in \mathcal{H} \). Then:

1. \( \Delta_{\Gamma,k,m,\xi,\chi} \) has the following Fourier expansion:

\[
\Delta_{\Gamma,k,m,\xi,\chi}(z) = \frac{\varepsilon}{\Gamma(m-1)} \sum_{n=1}^{\infty} n^{m-1} \psi_{\Gamma,n,m,\chi}(\xi) e^{2\pi in \frac{z}{m}}, \quad z \in \mathcal{H}.
\]

2. We have

\[
\Delta_{\Gamma,k,m,\xi,\chi}(z) = \frac{\varepsilon}{\Gamma(m-1)} \sum_{n=1}^{\infty} n^{m+k-1} e^{-2\pi in \frac{z}{m}} \psi_{\Gamma,n,m,\chi}(z), \quad z \in \mathcal{H}.
\]

The right-hand side converges in \( S_m(\Gamma, \chi) \) and absolutely and uniformly on compact sets in \( \mathcal{H} \).

**Proof.** This can be proved analogously to the proof of [6, Theorem 3-5], all convergence issues being settled by Lemma 8-2. We provide a shorter proof:

1. (8-5) follows from (8-1) since

\[
\langle \Delta_{\Gamma,k,m,\xi,\chi}, \psi_{\Gamma,n,m,\chi} \rangle_{\Gamma} = \frac{\varepsilon}{\Gamma(m-1)} \sum_{n=1}^{\infty} n^{m-1} \psi_{\Gamma,n,m,\chi}(\xi) e^{2\pi in \frac{z}{m}}, \quad z \in \mathcal{H}.
\]

2. We have

\[
\Delta_{\Gamma,k,m,\xi,\chi}(z) = \frac{\varepsilon}{\Gamma(m-1)} \sum_{n=1}^{\infty} n^{m+k-1} e^{-2\pi in \frac{z}{m}} \psi_{\Gamma,n,m,\chi}(z), \quad z \in \mathcal{H}.
\]

The convergence claim follows from Lemma 8-2. \[\Box\]

Now we can easily prove some bounds on the derivatives of classical Poincaré series (cf. [7, Theorem 1-2]):
Corollary 8-7. Let $k \in \mathbb{Z}_{\geq 0}$. Then,

$$\sup_{n \in \mathbb{Z}_{>0}} n^{\frac{m}{2} - 1} \Im(\xi) \frac{m}{2} + k \left| \psi_{\Gamma, n, m, \chi}(\xi) \right| < \infty.$$ 

Proof. Let us fix an orthonormal basis $\{f_1, f_2, \ldots, f_d\}$ of $S_m(\Gamma, \chi)$, and for each $l \in \{1, 2, \ldots, d\}$ let $f_l(z) = \sum_{n=1}^{\infty} a_n(f_l) e^{2\pi i n \xi}$ be the Fourier expansion of $f_l$. We have

$$\Delta_{\Gamma, k, m, \xi, \chi}(z) = \sum_{d} \langle \Delta_{\Gamma, k, m, \xi, \chi}, f_l \rangle \Gamma, \chi \Gamma(z) = \sum_{n=1}^{\infty} \left( \sum_{l=1}^{d} f_l^{(k)}(\xi) a_n(f_l) \right) e^{2\pi i n \xi}, \quad z, \xi \in \mathcal{H}.$$ 

hence by Theorem 8-4,(1)

$$\frac{\xi(4\pi)^{m-1}}{\Gamma(m-1)h^m} n^{m-1} \psi_{\Gamma, n, m, \chi}(\xi) = \sum_{l=1}^{d} f_l^{(k)}(\xi) a_n(f_l), \quad n \in \mathbb{Z}_{>0}, \xi \in \mathcal{H}.$$ 

Thus,

$$\sup_{n \in \mathbb{Z}_{>0}} n^{\frac{m}{2} - 1} \Im(\xi) \frac{m}{2} + k \left| \psi_{\Gamma, n, m, \chi}(\xi) \right| \leq \frac{\Gamma(m-1)h^m}{\xi(4\pi)^{m-1}} \sum_{l=1}^{d} \left( \sup_{\xi \in \mathcal{H}} \left| f_l^{(k)}(\xi) \Im(\xi) \frac{m}{2} + k \right| \right) \left( \sup_{n \in \mathbb{Z}_{>0}} \left| a_n(f_l) \right| \right),$$

and the right-hand side is finite by Corollary 7-16 and by the half-integral weight version of [4, Corollary 2.1.6].

9. Application to Cusp Forms for $\Gamma_0(N)$

We define the automorphic factor $J : \Gamma_0(4) \times \mathcal{H} \to \mathbb{C}$,

$$J(\gamma, z) := \frac{\Theta(\gamma, z)}{\Theta(z)},$$

where $\Theta \in \text{Hol}(\mathcal{H})$ is given by $\Theta(z) := \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}$. An explicit formula for $J$ is given by [3, III.(4.2)]. It easily implies that for every $N \in 4\mathbb{Z}_{>0}$

$$\Gamma_0(N) := \{ (\gamma, J(\gamma, \cdot)) : \gamma \in \Gamma_0(N) \cap \Gamma_1(4) \}$$

is a discrete subgroup of finite covolume in $\text{SL}_2(\mathbb{R})^\ast$.

Let $m \in \frac{5}{2} + \mathbb{Z}_{\geq 0}$ and $N \in 4\mathbb{Z}_{>0}$. Let $\chi$ be an even Dirichlet character modulo $N$. We identify $\chi$ with the character of $\Gamma_0(N)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi(d)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and with the character of $\Gamma_0(N)$ given by $\langle \gamma, J(\gamma, \cdot) \rangle \mapsto \chi(\gamma)$ for all $\gamma \in \Gamma_0(N) \cap \Gamma_1(4)$.

Finally, we define

$$S_m(N, \chi) := S_m(\Gamma_0(N), \chi).$$

This definition of $S_m(N, \chi)$ is equivalent to the one given in [10]. (In [10], $S_m(N, \chi)$ is defined regardless of the parity of $\chi$, but turns out to be 0 if $\chi$ is odd.) The Petersson inner product on $S_m(N, \chi)$ is

$$\langle f, g \rangle_{\Gamma_0(N)} = \int_{\Gamma_0(N) \backslash \mathcal{H}} f(z) g(z) \Im(z)^m dv(z), \quad f, g \in S_m(N, \chi),$$
and we have, for all $k \in \mathbb{Z}_{\geq 0}$ and $\xi, z \in \mathcal{H}$,
\[
\Delta_{\Gamma_0(N), k, m, \xi, \chi}(z) = \frac{(2i)^m}{8\pi} \left( \prod_{r=0}^{k} (m - 1 + r) \right) \sum_{\gamma \in \Gamma_0(N)} \frac{\chi(\gamma)}{(\gamma, z - \xi)^m} J(\gamma, z)^{-2m}.
\]

$\Gamma_0(N)$ and $\chi$ satisfy the assumptions of the first paragraph of Section 8, hence we have the classical Poincaré series
\[
\psi_{\Gamma_0(N), n, m, \chi}(z) = \sum_{\gamma \in \Gamma(N) \cap \Gamma_0(N)} (\chi, e^{2\pi i n \gamma z}) J(\gamma, z)^{-2m}, \quad z \in \mathcal{H}, \ n \in \mathbb{Z}_{> 0},
\]
and the cusp forms $\Delta_{\Gamma_0(N), k, m, \xi, \chi}$ have the expansion (8-6) in a series of classical Poincaré series. As a final application of our results, in Corollary 9-3 we express the action of Hecke operators of half-integral weight on $\Delta_{\Gamma_0(N), k, m, \xi, \chi}$ in terms of (8-6) (cf. [6, Lemma 5-8]).

For every prime number $p$, the Hecke operator $T_{p^2, m, \chi} : S_m(N, \chi) \rightarrow S_m(N, \chi)$ is given by
\[
\sum_{n=1}^{\infty} a(n) e^{2\pi i n z} T_{p^2, m, \chi} := \sum_{n=1}^{\infty} b(n) e^{2\pi i n z},
\]
where
\begin{equation}
(9-1) \quad b(n) := a(p^2 n) + \left( \frac{-1}{p} \right)^{m-1} \chi(p) \left( \frac{n}{p} \right)^{m-\frac{1}{2}} a(n) + \chi(p^2) p^{2m-2} a\left(\frac{n}{p^2}\right)
\end{equation}

[10, Theorem 1.7]. Here we understand that $a(n/p^2) = 0$ if $p^2 \nmid n$, while $\left( \frac{1}{p} \right)$ is the usual Legendre symbol if $p$ is odd and is identically zero if $p = 2$.

If $p \nmid N$, then
\begin{equation}
(9-2) \quad \langle f | T_{p^2, m, \chi} | g \rangle_{\Gamma_0(N)} = \chi(p^2) \langle f, g \rangle_{T_{p^2, m, \chi}}_{\Gamma_0(N)}, \quad f, g \in S_m(N, \chi).
\end{equation}

This enables us to prove:

**Corollary 9-3.** Let $N \in 4\mathbb{Z}_{>0}$, $m \in \frac{5}{2} + \mathbb{Z}_{>0}$, $k \in \mathbb{Z}_{>0}$, and $\xi \in \mathcal{H}$. Let $\chi$ be an even Dirichlet character modulo $N$. Then, for every prime number $p$ such that $p \nmid N$, $T_{p^2, m, \chi}$ maps the cusp form
\begin{equation}
(9-4) \quad \Delta_{\Gamma_0(N), k, m, \xi, \chi}(z) = \left( \frac{4\pi}{\Gamma(m-1)} \right)^{m-1} \left( -2\pi i \right)^k \sum_{n=1}^{\infty} n^{m+k-1} e^{-2\pi i n z} \psi_{\Gamma_0(N), n, m, \chi}(z)
\end{equation}
to
\begin{equation}
(9-5) \quad \left( \Delta_{\Gamma_0(N), k, m, \xi, \chi} | T_{p^2, m, \chi} \right)(z) = \left( \frac{4\pi}{\Gamma(m-1)} \right)^{m-1} \left( -2\pi i \right)^k \sum_{n=1}^{\infty} n^{m+k-1} E_{p, k, n, m, \chi}(\xi) \psi_{\Gamma_0(N), n, m, \chi}(z),
\end{equation}
where
\[
E_{p, k, n, m, \chi}(\xi) := \mathbb{1}_{p^2 \mathbb{Z}}(n) \frac{\chi(p^2)}{p^{2k}} e^{-2\pi i \frac{n}{p^2} \xi} + \left( \frac{-1}{p} \right)^{m-\frac{1}{2}} \chi(p) \left( \frac{n}{p} \right)^{m-\frac{1}{2}} p^{m-\frac{1}{2}} e^{-2\pi i \frac{n}{p^2} \xi} + p^2 e^{-2\pi i \frac{p^2 n}{p^2} \xi}.
\]
Here $\mathbb{1}_{p^2 \mathbb{Z}}$ is the characteristic function of $p^2 \mathbb{Z} \subseteq \mathbb{Z}$, and $\left( \frac{1}{p} \right)$ is the usual Legendre symbol.
Proof. (9-4) is a special case of (8-6). The proof of (9-5) is analogous to that of [6, Lemma 5-8]. For every $z \in \mathcal{H}$, we have

$$
\left(\Delta \Gamma_0(N),k,m,\xi,\chi\right|T_{p^2,m,\chi}\right)(z) = \left\langle \Delta \Gamma_0(N),k,m,\xi,\chi\right|T_{p^2,m,\chi}\right)\Gamma_0(N)
$$

which equals

$$
\chi(p^2)\left\langle \Delta \Gamma_0(N),k,m,\xi,\chi\right|T_{p^2,m,\chi}\right)\Gamma_0(N)
$$

By (8-5) and (9-1), the right-hand side equals

$$
\frac{(4\pi)^{m-1}}{\Gamma(m-1)} \sum_{n=1}^{\infty} (-2\pi in)^{k} \chi\left(p^2\right) \left(p^2n\right)^{m-1} \psi_{\Gamma_0(N),p^2n,m,\chi}(z)
$$

$$
\quad + \left(\frac{-1}{p}\right)^{m-\frac{1}{2}} \chi(p) \left(\frac{n}{p}\right) p^{m-\frac{1}{2}}n^{m-1} \psi_{\Gamma_0(N),n,m,\chi}(z)
\quad + p^{2m-2} \left(\frac{n}{p^2}\right)^{m-1} \psi_{\Gamma_0(N),n/p^2,m,\chi}(z)e^{-2\pi in\xi}.
$$

By rearranging this sum to be over the index $n$ in $\psi_{\Gamma_0(N),n,m,\chi}(z)$, we obtain (9-5). The rearrangement is valid by Lemma 8-2.

\[\blacksquare\]

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