Asymptotic solutions of a fourth–order analogue for the Painlevé equations

I Yu Gaiur, N A Kudryashov
National Research Nuclear University MEPhI, Kashirskoe sh. 31, Moscow, 115409 Russia
E-mail: iygayur@gmail.com

Abstract. Asymptotic solutions of a fourth–order analogue for the Painlevé equations that is self–similar reduction of the modified Sawada–Kotera and Kaup–Kupershmidt equation is considered. The Boutroux variables of two types have been found which allows us to find asymptotic solutions of the equation in the neighbourhood of the infinity. It was shown that asymptotic of self–similar solution for the modified Sawada–Kotera and Kaup–Kupershmidt equations can be determined as solutions of autonomous differential equations. Asymptotic solutions expressed by elementary functions have been found too. Besides asymptotic solutions expressed by logarithmic derivative of two elliptic Weierstrass functions have been found. Connection between obtained asymptotic solutions and asymptotic solutions of the Sawada–Kotera and Kaup–Kupershmidt equations has been discussed.

1. Introduction
In this paper we consider a fourth–order analogue of Painlevé equations in the form [1]:

$$w_{zzzz} + 5w_z w_{zz} - 5w^2 z_w + 5w w_z^2 - 5w^5 - zw - \beta = 0.$$ (1)

This equation can be obtained taking into account the self–similar variables

$$u = \frac{1}{(5t)^{1/2}} w(z), \quad z = \frac{x}{(5t)^{1/2}},$$ (2)

from the following nonlinear partial differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( u_{xxxx} - 5 u_x u_{xx} + 5 u w_x^2 + 5 w^2 u_{xx} - w^5 \right) = 0.$$ (3)

Equation (3) is the modified Kaup–Kupershmidt and Sawada–Kotera equations. It is known that solutions of the Kaup–Kupershmidt and Sawada–Kotera equations can be found via the solutions of equation (3) by means of the Miura transformations. These equations plays significant role in the description of non–linear waves in a liquid with gas bubbles and in other application [2, 3].

Equation (1) is one of the fourth–order analogues of the Painlevé equations [4, 5]. It is known that solutions of equation (1) can determine new special functions [1, 7, 8]. It was found that solutions of equation (1) can be obtained using the Bäcklund transformations at some values of parameters [9]. The Cauchy problem for equation (1) can be solved using the Lax pair [10].
There are some rational solutions of equation (1) expressed via special polynomials at some values of parameters too [11, 12]. The Lagrangian description of equation (1) was obtained in work [13].

Asymptotic solutions of the first and the second Painlevé equations [14, 15] in the complex domain were obtained by Pierre Boutroux in his classical work [16]. At the present time there are a lot of works where asymptotic properties of the solutions for Painlevé equations were studied. One of approaches for finding the Boutroux variables of the Painleve equations was proposed in work by A. Bruno [17], that is also known as the three dimensional power geometry method. Using this method the author has found asymptotic solutions determined in terms of elliptic and periodic functions in neighborhood of infinity. Investigation of the asymptotic properties of solutions for the Painlevé equations in the complex domain using the method of isomonodromy deformation is presented in works s [18, 19, 20, 21].

Rational solutions of equation (1) have been found in paper [22]. Asymptotic solutions of equation (1) determined as series by the powers of $z$ have been obtained in paper [23]. However, asymptotic solutions determined in the Boutroux variables have not been found yet.

The aim of this paper is to find the Boutroux variables of equation (1) using power geometry methods and to determine Boutroux asymptotic solutions in the neighborhood of infinity.

2. Boutroux variables for asymptotic solutions of equation (1)

To determine asymptotic solutions of equation (1) we are going to look for solutions in the following form

$$w(z) = z^\alpha f(t), \quad t = z^{\frac{\gamma}{\gamma}}. \quad (4)$$

The main idea of the following approach to choose real parameters $\alpha$ and $\gamma$ thus substitution (4) transform equation (1) to the next form

$$F(f, f_t, f_{tt}, f_{ttt}, f_{tttt}) + \sum_{i \in \mathbb{Z}_+} \frac{G_i(f, f_t, f_{tt}, f_{ttt}, f_{tttt})}{t^i} = 0. \quad (5)$$

Thus we can propose that damping asymptotic solutions can be determined as solution of

$$F(f, f_t, f_{tt}, f_{ttt}, f_{tttt}) = 0, \quad (6)$$

when $|t| \to \infty$. It has been found that there are two types of the Boutroux variables for equation (1). The first one is

$$w(z) = \frac{f(t)}{z}, \quad t = \frac{4}{5} z^{\frac{5}{2}} \quad (7)$$

and the second one is

$$w(z) = z^{\frac{1}{4}} v(t), \quad t = \frac{4}{5} z^{\frac{5}{2}}. \quad (8)$$

Substitutions (7) and (8) transforms equation (1) to the form (5).

3. Trigonometric asymptotic solutions of equation (1)

Substituting (7) into equation (1) we have the following equality

$$\begin{align*}
w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5ww_z^2 + w^5 - zw - \beta &= \\
= f_{ttt} - f - \beta - 2 \frac{f_{tt} - f_t f_{tt}}{t} - \frac{16 f f_t - 27 f_{tt} + 28 f_t^2 + 16 f^2 f_t + 16 f f_t^2}{5 t^2}. \quad (9)
\end{align*}$$
At $|t| \to \infty$, we obtain that the asymptotic solution of equation (1) can be determined from the following equation

$$f_{tttt} - f - \beta = 0.$$  \hfill (10)

General solution of equation (10) can be written in form

$$f(t) = -\beta + C_1 e^{it} + C_2 e^{-it} + C_3 e^{-t} + C_4 e^t.$$ \hfill (11)

To obtain asymptotic solution of equation (1) we should choose constants in such a way that we obtain solution with growth behaviour less than the first power of $t$.

For example, to obtain asymptotic solutions of equation (1) on the real axes we should choose constants in the following way

$$C_4 = 0, \quad t \to \infty.$$  \hfill (12)

Thus, we obtain asymptotic solution on real axis of equation (1) in the following form

$$w_1(z) = -\beta + C_1 e^{i\frac{\sqrt{5}}{4}z^{5/4}} + C_2 e^{-i\frac{\sqrt{5}}{4}z^{5/4}} + C_3 e^{-\frac{\sqrt{5}}{4}z^{5/4}}, \quad z \to +\infty,$$  \hfill (13)

The asymptotic solution (13) is illustrated in Fig. 1.

**Figure 1.** Trigonometric asymptotic solution (13) of equation (1) at $\beta = 1$, $C_1 = C_2 = C_3 = 1, z \to \infty$.

4. **Elliptic asymptotic solutions of equation (1)**

Taking into account transformation (8) we obtain the following equality

$$w_{zzzz} + 5w_z w_{zz} - 5w^2 w_{zz} - 5ww_z^2 + w^5 - zw - \beta =$$

$$= v_{tttt} + 5v_t v_{tt} - 5v_t^2 - 5v^2 v_t + v^5 - v - \frac{5v^2 v_t - vv_{tt} + \frac{4}{t} \beta - 3v_t^2 - 2v_{tt}}{t} -$$

$$- \frac{3v_t - 2v^3}{5t^2} - \frac{3v^2 - 15v_t}{25t^3} - \frac{231}{625} \frac{v}{t^4} = 0.$$ \hfill (14)
At $|t| \to \infty$, we obtain asymptotic solution of (1) determined by the following equation
\[ v_{ttt} + 5v_tv_t - 5v_t^2v_t - 5v^2 + v^5 - v = 0. \] (15)

Equation (15) has solution in form of elliptic function. This asymptotic solution of (14) can be written as [24]
\[ v(t) = \frac{d}{dt} \ln [\Theta(t)], \quad \Theta(t) = \eta(t) - \mu(t) \] (16)
where $\eta$ and $\mu$ are any two solutions of the equation
\[ \gamma_{tt} = 6\gamma^2 - \frac{1}{24}. \] (17)

The solution of equation (17) is the Weierstrass elliptic function $\wp(t, \frac{1}{12}, g_3)$, where $g_3$ is an arbitrary constant. The asymptotic solution of equation (1) takes the form
\[ w_3(z) = \frac{\wp'\left(\frac{1}{2}z + \frac{i}{2}, \frac{1}{12}, g_3\right) - \wp'\left(\frac{1}{2}z - \frac{i}{2}, \frac{1}{12}, \tilde{g}_3\right)}{\wp\left(\frac{1}{2}z + \frac{i}{2}, \frac{1}{12}, g_3\right) - \wp\left(\frac{1}{2}z - \frac{i}{2}, \frac{1}{12}, \tilde{g}_3\right)} \] (18)
where $z_1, z_2, g_3, \tilde{g}_3$ - are arbitrary constants. The asymptotic solution (18) is illustrated in Fig. 2.

![Figure 2](image)

Figure 2. Real and imaginary part of solution (18) at $g_3 = 4, \tilde{g}_3 = 0, z_1 = 2i, z_2 = 2i + 4$.

5. The relationship with the Savada–Kotera equation and the Kaup–Kupershmit equation.
It is also known that solutions of equation (3) are connected with the Savada–Kotera and the Kaup–Kupershmit equations by the Miura transformation. Let us demonstrate the connection between equation (1) and self—similar solutions of the Savada–Kotera equation and the Kaup–Kupershmit equation.
The Savada–Kotera equation and the Kaup–Kupershmidt equation can be written in form [25]
\[ u_t + u_{xxxxx} + 10uu_{xxx} + 5\nu u_xu_{xx} + 20u^2u_x = 0, \]  
(19)
where we take \( \nu = 2 \) for the Kaup–Kupershmidt equation [26], and \( \nu = 5 \) for the Savada–Kotera equation [27, 28]. Using self–similar variables
\[ u = \frac{1}{(5t)^{\frac{3}{4}}} g(z), \quad z = \frac{x}{(5t)^{\frac{1}{4}}}, \]  
(20)
we obtain the following differential equation
\[ g_{zzzzz} + 10gg_{zzz} + 5\nu g_zg_{zz} + 20g^2g_z - 2g - zg_z = 0. \]  
(21)
Using the Miura transformation in the case \( \nu = 5 \) for the Sawada–Kotera equation in the form
\[ g = w_z - \frac{w^2}{2}, \]  
(22)
we obtain
\[ \left( \frac{d}{dz} - w \right) \frac{d}{dz} f(z, w) = 0. \]  
(23)
where \( f(z, w) \) is differential sum of equation (1).
In the case of the Kaup–Kupershmidt equation the Miura transformation takes form
\[ g = - w_z + \frac{w^2}{2}. \]  
(24)
Using transformation (24) we obtain that self–similar reduction of the Kaup–Kupershmidt equation can be written in the following form
\[ \frac{1}{2} \left( \frac{d}{dz} + 2w \right) \frac{d}{dz} f(z, w) = 0. \]  
(25)
Thus, we can obtain the asymptotic solutions of the Savada–Kotera and the Kaup–Kupershmidt equations (19) using formulae (22) and (24). This relationship gives us opportunity to obtain asymptotic solutions of the SK and KK equations by using Miura on the obtained asymptotic solutions (13) and (18).

6. Conclusion
In this work we have considered a fourth–order analogue for the Painlevé equations. This equation can be also obtained as a self–similar reduction of the modified Sawada–Kotera and Kaup–Kupershmidt equations. We have found two types of the Boutroux variables for equation (1). We have constructed asymptotic solutions of equation (1) via trigonometric functions (13) and Weierstrass elliptic functions (18). We have also demonstrated that asymptotic self–similar solutions of the Sawada–Kotera and Kaup–Kupershmidt equations can be obtained via asymptotic solutions (13) and (18) by means of the Miura transformation.

7. Acknowledgement
This research was supported by Russian Science Foundation grant No. 14–11–00258.
References
[1] Kudryashov N A 2002 J. Phys. A: Math. Gen. 35 4617
[2] Kudryashov N A and Sinelshchikov D I 2014 Regular and Chaotic Dynamics 19 576
[3] Kudryashov N A and Sinelshchikov D I 2014 International Journal of Non-Linear Mechanics 63 31
[4] Kudryashov N A 1999 Phys. Lett. A 252 173
[5] Kudryashov N A 2003 J. Math. Phys. 44 6160
[6] Gordon P R and Pickering A 2011 Applied Mathematics and Computation 218 3942
[7] Kudryashov N A and Soukharev M B 1998 Physics Letters A 237 206
[8] Kudryashov N A 1997 Physics Letters A 224 353
[9] Kudryashov N A 2000 Physics Letters A 273 194
[10] Kaup D J 1980 Studies in Applied Mathematics 62 189
[11] Kudryashov N A and Pickering A 1998 J. Phys. A: Math. Gen. 31 9505
[12] Kudryashov N A 2008 Physics Letters A 372 1945
[13] Choudhury A G, Guha P and Kudryashov N A 2012 Applied Mathematics and Computation 218 6612
[14] Painlevé P 1902 Acta Mathematica 25 1
[15] Polyanin A D and Zaitsev V F 2012 Handbook of Nonlinear Partial Differential Equations, Second ed. (Chapman and Hall/CRC, Boca Raton)
[16] Boutroux P 1913 Annales scientifiques de l’Ecole Normale Supérieure 30 255
[17] Bruno A D 2015 International Journal of Differential Equations Volume 2015 13
[18] Kitaev A V 1994 Russian Mathematical Surveys 49 81
[19] Novokshenov V Y 1984 Functional Analysis and Its Applications 18 260
[20] Its A R and Kapaev A A 1988 Mathematics of the USSR-Izvestiya 31 193
[21] Its A R, Fokas A S and Kapaev A A 1994 Nonlinearity 7 1291
[22] Kudryashov N A and Demina M V 2007 Phys. Lett. A 363 346
[23] Efimova O Y and Kudryashov N A 2007 Regular and Chaotic Dynamics 12 198
[24] Cosgrove C 2000 Studies in Applied Mathematics 104 1
[25] Foursov M V and Moreno Maza M 2001 Technical report, LIFL, Université de Lille-I, France
[26] Kupershmidt B A 1984 Phys. Lett. A 102 25
[27] Sawada K and Kotera T 1974 Prog. Theor. Phys. 51 1355
[28] Caudrey P J, Dodd R K and Gibbon J D 1976. R. Soc. London, Ser. A 351 407