A TWO-PARAMETER DEFORMATION OF THE QUASI-SHUFFLE AND NEW BASES OF QUASI-SYMMETRIC FUNCTIONS

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Abstract. We define a two-parameter deformation of the quasi-shuffle by means of the formal group law associated with the exponential generating function of the homogeneous Eulerian polynomials, and construct bases of $QSym$ and $WQSym$ whose product rule is given by this operation.

1. Introduction

The Hopf algebra of quasi-symmetric functions [5] is historically the first example of a combinatorial Hopf algebra extending that of symmetric functions. While there is no general agreement on what should be the formal definition of a combinatorial Hopf algebra, its is clear that to be considered as combinatorial, such an algebra should, in addition to combinatorial product and coproduct rules, also have at least two bases related by a combinatorial rule.

Quasi-symmetric functions entered the scene with these requirements, having from the beginning two bases $M_f$ (monomial) and $F_f$ (fundamental). The product rule for the $M_f$ is the so-called quasi-shuffle (apparently first described in [28]), and its coproduct is given by deconcatenation.

At the time, the algebra of symmetric functions had already a lot of known bases (elementary, complete, Schur, power-sum, Hall-Littlewood, Jack, Macdonald etc., see [15]), and the subsequent realization of the dual of $QSym$ as noncommutative symmetric functions quickly led to the introduction of a lot of new bases on both sides (see e.g., [12, 7, 23, 14, 25, 27, 18, 24, 21] and the survey [16] for a sample).

Originally motivated by the study of posets, quasi-symmetric functions were next related to the combinatorics of descents in permutations and to the Solomon descent algebra, then to the 0-Hecke algebras and certain degeneracies of quantum groups [13]. More recently, they played a crucial role in the theory of Macdonald polynomials, via the quasi-symmetric expansions of certain LLT polynomials [6].

Another source of interest in quasi-symmetric functions stems in the theory of multiple zeta values (MZV), which are the specializations $x_n = \frac{1}{n}$ of the monomial functions $M_f$. Thus, the MZV satisfy a product formula given by the quasi-shuffle, but their representation as iterated integrals naturally labelled by binary words leads to another expression of the product, given by the ordinary shuffle of binary words.

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The equality between both expressions encodes algebraic relations satisfied by the MZV \([29]\).

A natural question is therefore: does there exist bases of \(QSym\) naturally labelled by binary words which multiply by the ordinary shuffle of binary words? This question has been answered affirmatively in \([18]\), where a general method for constructing such bases has been presented, and a special case investigated more in depth.

Recently, a variant of the quasi-shuffle called the block shuffle has been introduced by Hirose and Sato, also in relation with the MZV. It has been proved by Keilthy \([11]\) that the block shuffle algebra is isomorphic to the ordinary shuffle algebra, providing an isomorphism analogous to Hoffman’s exponential \([8]\), where the exponential is replaced by the hyperbolic tangent.

Actually, Hoffman’s classical proof of the isomorphism between shuffle and quasishuffle algebras is equivalent to the fact that \(QSym\) admits bases whose product rule is given by the ordinary shuffle of compositions\([4]\). The existence of such bases comes itself from the fact that \(Sym\) is the free graded associative algebra over a sequence of primitive elements.

Once more, a natural question is now to construct bases of quasi-symmetric functions whose product rule is given by the block shuffle. The existence of such bases is implied by the result of \([11]\). We shall reprove this result in a simpler way, starting on the dual side with a sequence of primitive generators, and investigate the bases obtained by a particular choice of these generators, corresponding to the Solomon idempotents of the descent algebra.

For a word \(w = a_1 \cdots a_n\) over the alphabet of positive integers, and an integer \(x\), define \(\zeta_x(w) = (a_1 + x)a_2 \cdots a_n\), where \(a_1 + x\) is the sum of integers. We shall see that the ordinary shuffle, the quasi-shuffle and the block shuffle are special cases of a two-parameter family of associative products defined by the recurrence

\[
au \star bv = a(u \star bv) + b(au \star v) + \alpha [a + b] \cdot (u \star v) + \beta \zeta_{a+b}(u \star v)
\]

associated with the formal group law

\[
F(x, y) = \frac{x + y + \alpha xy}{1 - \beta xy}.
\]

This is proved by introducing a basis of \(QSym\) satisfying this product rule. This basis is then lifted to \(WQSym\).

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We shall assume that the reader is familiar with the notations of \([4]\) and \([22]\).

\[\footnote{This proves the result for the quasi-shuffle algebra over the semigroup of nonnegative integers, but the argument is easily extended to quasi-symmetric functions with exponents in an appropriate additive semigroup \([17]\).} \]
2. **The block shuffle**

The block shuffle of words over the alphabet of positive integers is defined by

\[ w \hat{\cup} \epsilon = \epsilon \hat{\cup} w = w \]

where \( \epsilon \) is the empty word, and for letters \( a, b \)

\[ au \hat{\cup} bv = a(u \hat{\cup} bv) + b(au \hat{\cup} v) - \zeta_{a+b}(u \hat{\cup} v) \]

where \( \zeta_a(bv) = (a+b)v \) (\( a+b \) is the sum of the integers) and \( \zeta_a(\epsilon) = 0 \).

Suppose that \( X_I \) is a basis of \( QSym \) satisfying \( X_I X_J = X(I \hat{\cup} J) \) (where \( X \) is regarded as a linear map on the vector space spanned by compositions, and \( X(I) := X_I \)). Let \( Y_I \) be the dual basis of \( X_I \). From the product formula of the \( X_I \), we can calculate the coproduct of \( Y_I \). In particular, to get \( \Delta Y_n \), we see that \( X_n \) can appear in \( X_I X_J \) if and only if \( |\ell(I) - \ell(J)| = 1 \), and that its coefficient is a sign \( (-1)^r \), where \( r = \frac{1}{2}(|\ell(I) + \ell(J) - 1|) \).

\[ \Delta Y_n = \sum_{\ell(I) + \ell(J) = n} (-1)^{(\ell(I)-1)/2} Y_I \otimes Y_J. \]

Assuming that \( Y_I = Y^I \) is a multiplicative basis, and setting \( Y = \sum_{n \geq 1} Y_n \), this can be rewritten as

\[ \Delta Y = \frac{Y \otimes 1 + 1 \otimes Y}{1 \otimes 1 + Y \otimes Y}. \]

Starting with a sequence \( \Pi_n \) of primitive generators of \( \text{Sym} \), and setting

\[ Y = \sum_{n \geq 1} Y_n = \tanh \Pi \]

where \( \Pi = \sum_{n \geq 1} \Pi_n \)

we have from the addition formula

\[ \tanh(a + b) = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b} \]

\[ \Delta Y = Y(A + B) = \tanh(\Pi(A) + \Pi(B)) = \frac{Y \otimes 1 + 1 \otimes Y}{1 \otimes 1 + Y \otimes Y} \]

which is the generating series of \( (5) \). Thus, defining \( Y_n \) in this way does indeed yield a basis of \( \text{Sym} \) whose dual basis \( X_I \) mutiples by the block shuffle.

3. **A special choice**

Let us now choose \( \Pi = \varphi \), where

\[ \varphi = \log \sigma_1 = \sum_{n \geq 1} \varphi_n = \sum_{n \geq 1} \frac{\phi_n}{n} \]

so that

\[ Y = \frac{\sigma_1 - \lambda_1}{\sigma_1 + \lambda_1} = \frac{\sigma_1(2A) - 1}{2} \left( \frac{1 + \sigma_1(2A) - 1}{2} \right)^{-1} \]
whence
\begin{equation}
Y_n = \sum_{I^n} \frac{(-1)^{\ell(I)-1}}{2^\ell(I)} S^I(2A)
\end{equation}

\begin{equation}
= \sum_{I^n} \frac{(-1)^{\ell(I)-1}}{2^\ell(I)} \sum_{J\subseteq I} R_J(2A)
\end{equation}

\begin{equation}
= \sum_{J^n} \sum_{I^n \geq J} \frac{(-1)^{\ell(I)-1}}{2^\ell(I)} R_J(2A)
\end{equation}

\begin{equation}
= \frac{1}{2^n} \sum_{J^n} (-1)^{\ell(J)-1} R_J(2A).
\end{equation}

Thus,
\begin{equation}
Y_n = \frac{-1}{2^n} A_n(-1; 2A),
\end{equation}

where
\begin{equation}
A_n(t; A) = \sum_{I^n} t^{\ell(I)} R_I(A) = \sum_{k=1}^n t^k A(n, k)
\end{equation}

are the noncommutative Eulerian polynomials of [1, 20]. Therefore, with the notation of [1, 20]
\begin{equation}
Y_n = \frac{1}{2^n} \sum_{k=1}^n (-1)^{k-1} A(n, k) * S_n^{[2]},
\end{equation}

and using [22 (16)], we can compute
\begin{equation}
A(n, k) * S_n^{[2]} = \sum_{i=0}^k (-1)^i \binom{n+1}{i} S_n^{[k]} * S_n^{[2]}
\end{equation}

\begin{equation}
= \sum_{i=0}^k (-1)^i \binom{n+1}{i} \sum_{j=0}^{2k-2i} \binom{n+j}{j} A(n, 2k-2i-j)
\end{equation}

\begin{equation}
= \sum_{j=0}^k \sum_{i=0}^k (-1)^i \binom{n+j}{j} \binom{n+1}{i} A(n, 2k-2i-j)
\end{equation}

\begin{equation}
= \sum_r \left( \sum_{i=0}^k (-1)^i \binom{n+1}{i} \binom{n+2k-2i-r}{n} \right) A(n, r).
\end{equation}

\textbf{Lemma 3.1.} The coefficient of $A(n, r)$ in the above sum is
\begin{equation}
F(n, k, r) := \sum_{i=0}^k (-1)^i \binom{n+1}{i} \binom{n+2k-2i-r}{n} = \binom{n+1}{2k-r}.
\end{equation}

\textit{Proof} – One easily checks that
\begin{equation}
F(n, k, r) = F(n, k, r+1) + F(n-1, k, r) - F(n-1, k, r+2).
\end{equation}

This equation is also obviously satisfied by the right-hand side of (23). Moreover, the initial conditions are
\[ r = 2k; \text{ where Equation (23) reduces to } 1 = 1, \]
\[ r = 2k - 1; \text{ where Equation (23) reduces to } n + 1 = n + 1, \]
\[ n = 0; \text{ where Equation (23) reduces to } \binom{2k-r}{0} - \binom{2k-r-2}{0} = \binom{1}{2k-r} \text{ which is true for all values of } 2k - r. \]

Finally, we arrive at the following expression:

**Theorem 3.2.** On the ribbon basis,

\[ Y_n = \frac{1}{2^n} \sum_{r=1}^{n} \mathfrak{R} i^r (1 + i)^{n+1} \cdot A(n, r), \tag{25} \]

or, equivalently

\[ Y_n = \left( \frac{1}{\sqrt{2}} \right)^{n-1} \sum_{I=0}^{n} \cos(n + 1 + 2\ell(I)) \frac{\pi}{4} R_I. \tag{26} \]

**Proof –** The coefficient of \( A(n, r) \) in \( Y_n \) is \( 2^{-n} a(n, r) \), where

\[ a(n, r) = \sum_{k=1}^{n} (-1)^k \binom{n+1}{2k-r} = \mathfrak{R} \sum_{p \geq 0} i^{p+r} \binom{n+1}{p} = \mathfrak{R} i^r (1 + i)^{n+1}. \tag{27} \]

For example,

\[ Y_2 = -\frac{1}{2} R_{11} + \frac{1}{2} R_2 \tag{28} \]
\[ Y_3 = -\frac{1}{2} R_{12} - \frac{1}{2} R_{21} \tag{29} \]
\[ Y_4 = \frac{1}{4} R_{1111} + \frac{1}{4} R_{1112} + \frac{1}{4} R_{1211} + \frac{1}{4} R_{1311} + \frac{1}{4} R_{2111} - \frac{1}{4} R_{2211} - \frac{1}{4} R_{3111} - \frac{1}{4} R_{4} \tag{30} \]
\[ Y_5 = -\frac{1}{4} R_{11111} + \frac{1}{4} R_{11112} + \frac{1}{4} R_{11121} + \frac{1}{4} R_{11131} + \frac{1}{4} R_{12211} + \frac{1}{4} R_{13311} + \frac{1}{4} R_{22211} + \frac{1}{4} R_{33311} - \frac{1}{4} R_5 \tag{31} \]

Actually, the sums \( F(n, k, r) \) are the coefficients of the “amazing matrix” of [10]. It is proved in this reference that

\[ F_b(n, k, r) := \sum_{i=0}^{k} (-1)^i \binom{n + 1}{i} \binom{n + bk - bi - r}{n} = T_b(n + 1, bk - r) \tag{32} \]

where \( T_b(n, p) \) is a \( b \)-nomial coefficient, \( i.e., \) the coefficient of \( x^p \) in \((1 + x + \cdots + x^{b-1})^n\).

4. **Some transition matrices**

The construction of the basis \( X_I \) is a special case of the following one. Start with a multiplicative basis \( \varphi^I \) of \( \text{Sym} \), for example \( \varphi = \log \sigma_1 \). Choose a formal series \( f(z) = \sum_n c_n z^n \), and define \( Y := \sum_n Y_n = f(\varphi) \). We have then

\[ Y_n = \sum_{\ell=0}^{n} c_{\ell(I)} \varphi^I \tag{33} \]

so that
(34) \[ Y^J = \sum_{I_1=|I_1|, \ldots, I_r=|I_r|} c_{\ell(I_1)} \cdots c_{\ell(I_r)} \phi^{I_1 \cdots I_r}. \]

Let \( \phi_I \) be the dual basis of \( \varphi^I \) and \( X_I \) be the dual basis of \( Y^I \). By duality, (33) translates as

(35) \[ \phi_I = \sum_{I_1=\ldots=I_r} c_{\ell(I_1)} \cdots c_{\ell(I_r)} X_{|I_1|, \ldots, |I_r|}. \]

By construction, the coproduct of \( X_I \) is given by deconcatenation, since it is dual to a multiplicative basis, so that the map \( \Psi_f : \phi_I \mapsto X_I \) is a morphism of coalgebras. It is proved in [3] that all morphisms of coalgebras are actually of this type [3, Th. 2.2] and that more generally the group of coalgebra automorphisms of a shuffle algebra is isomorphic to the group of invertible formal power series in one variable under composition (in particular, \( \Psi_f \circ \Psi_g = \Psi_{fg} \)).

This construction has been rediscovered in [9], where the following convenient notation has been introduced. If we encode compositions by words over an auxiliary alphabet \( Z = \{ z_i | i \geq 1 \} \), endowed with the operation \( z_i \circ z_j := z_{i+j} \), and define a linear operator \( P \) by \( P(z^I) := \phi^I \) for any basis \( \phi^I \) of \( Qsym \), (35) can be recoded as in [9]

(36) \[ \phi \left( \frac{1}{1-\lambda z} \right) = X \left( \frac{1}{1-f_\circ(\lambda z)} \right) \]

where \( z = \sum_i z_i \) and \( f_\circ(z) \) means that powers of \( z \) are evaluated with the \( \circ \) operation.

If we choose \( f \) such that \( c_1 = 1 \), we have \( X_n = \phi_n \). We can then write

(37) \[ \sum_I \lambda^{\ell(I)} \phi_I = \exp \left\{ \lambda \sum_{k \geq 1} X_k \right\}. \]

Indeed, the l.h.s. is the Cauchy kernel for the pair of bases \( (\phi, \varphi) \) specialized at the virtual alphabet \( \Lambda \) defined by \( \varphi_n(\Lambda) = \lambda \) for all \( n \). Then,

(38) \[ \sigma_t(\Lambda) = \exp \left\{ \lambda \sum_{k \geq 1} t^k \right\} = \exp \left\{ \lambda \frac{t}{1-t} \right\}, \]

and

(39) \[ \sigma_1(X\Lambda) = \exp \left\{ \lambda \sum_{k \geq 1} \frac{x_i}{1-x_i} \right\} = \exp \left\{ \lambda \sum_{k \geq 1} X_k \right\}. \]

In the notation of [9, 11], this would be

(40) \[ X(\exp_*(\lambda z)) = \phi \left( \frac{1}{1-\lambda z} \right). \]

\(^2\)Actually, Hoffman and Ihara state this formula for an arbitrary linear combination of the \( z_i \), but this amounts only to a rescaling of the variables, or for us, of the \( \varphi_i \).
where $\ast$ is the product rule for the $X$-basis, for example the quasi-shuffle if $f(z) = e^z$ and the block shuffle of [11] if $f(z) = \tanh z$.

5. **Generalisation**

In order to obtain interesting bases, the series $f(z)$ should have an addition theorem of the type

$$f(x + y) = (f(x) + f(y))(1 + \sum_{i+j \geq 1} c_{ij} f(x)^i f(y)^j) := F(f(x), f(y));$$

(i.e., $F$ is a formal group law).

Tractable examples are rather scarce. To the list $e^z, \tanh z$ and $\tan z$, one can add $f(z) = \frac{1}{q}(e^{qz} - 1)$, which satisfies $f(x + y) = f(x) + f(y) + qf(x)f(y)$ and provides a deformation of the quasi-shuffle [11] satisfying the recurrence

$$au \ast bv = a(u \ast bv) + b(au \ast v) + q[a + b](u \ast v).$$

These examples can be interpolated by the (known) formal group law

$$F_{\alpha,\beta}(x, y) = \frac{x + y + \alpha xy}{1 - \beta xy}$$

corresponding to the function

$$f(x) = \frac{e^{sx} - e^{tx}}{se^{tx} - te^{sx}}$$

where $\alpha = s + t$ and $\beta = st$.

This is the exponential generating function of the homogeneous Eulerian polynomials

$$f(x) = \sum_{n \geq 1} E_n(s, t) \frac{x^n}{n!}$$

with

$$E_n(s, t) = \sum_{\sigma \in S_n} s^{r(\sigma)} t^{d(\sigma)}$$

where $r(\sigma)$ is the number of rises of $\sigma$ and $d(\sigma)$ its number of descents.

Define $Y_n$ by $Y = f(\varphi)$.

**Proposition 5.1.** The expansion of $Y_n$ on the $S$-basis is given by

$$Y_n = \sum_{r=1}^{n} \frac{1}{r!} \left( \sum_{k=1}^{r} s(r, k) E_k(s, t) \right) S_n^{[r]}$$

where $s(r, k)$ are the signed Stirling numbers of the first kind.

**Proof** – By definition,

$$Y = \sum_{k \geq 1} E_k(s, t) \frac{\varphi^k}{k!} = \sum_{k \geq 1} E_k(s, t) E^{[k]}$$
and since

\[
\sigma^r_I = \sum_k x^k E^k = \sum_{r \geq 0} \binom{x}{r} \sum_{l(I) = r} S^I = \sum_{r \geq 0} \frac{1}{r!} \sum_{k=0}^r s(r, k) x^k \sum_{l(I) = r} S^I
\]

we have

\[
E^k = \sum_r \frac{1}{r!} s(r, k) S^r
\]

whence the result.

The expansion of the ribbon basis follows from the identity

\[
\sum_{l(I) = r} S^I = \sum_{k=0}^{r-1} \binom{n-r+k}{k} A(n, r-k)
\]

Another expression can be obtained as above in terms of the amazing matrix. Imitating the calculation of Section 3, we have (treating \(s - t\) as a scalar)

\[
Y_n = \frac{1}{s-t} \sum_{i=0}^{\ell(I)-1} s^{\ell(I)-1} S^I((s-t)A) = \sum_{J=0}^{\ell(I)} \frac{s^{n-\ell(J)} s^{(\ell(J)-1)}}{(s-t)^n} R_J((s-t)A)
\]

so that

\[
Y_n = \frac{1}{(s-t)^n} \sum_j s^{n-j} t^{j-1} A(n, j) \ast S_n^{[s-t]}
\]

where

\[
A(n, j) \ast S_n^{[s-t]} = \sum_i P_{ij}(s-t)A(n, i).
\]

**Theorem 5.2.** Let \(X_I\) be the dual basis of \(Y^I\). Then,

\[
X_I X_J = X(I \ast J)
\]

where \(\ast\) is defined on words over the integers by the recursion

\[
a u \ast vb = a(u \ast bv) + b(au \ast v) + \alpha [a + b] \cdot u \ast v + \beta \zeta_{a+b}(u \ast v).
\]

In particular, the operation \(\ast\) is associative and commutative.

**Proof –** By (43), the coproduct of \(Y\) is

\[
\Delta Y = Y \otimes 1 + 1 \otimes Y + \alpha \sum_{\ell(I) = \ell(J)} \beta^{\ell(I)-1} Y^I \otimes Y^J + \sum_{|\ell(I) - \ell(J)| = 1} \beta^{(\ell(I)+\ell(J)-1)/2} Y^I \otimes Y^J
\]

Writing compositions as words to be in line with the notation of the theorem, we have

\[
\langle X_{au} X_{bv}, Y^{cw} \rangle = \langle X_{au} \otimes X_{bv}, \Delta Y_c \Delta Y^w \rangle.
\]
If \( c = a \), \( \Delta Y_c \) contains \( Y_a \otimes 1 \) with coefficient 1, so that

\[
\langle X_{au} X_{bv}, Y_c^{cw} \rangle = \left( X_{au} \otimes X_{bv}, (Y_a \otimes 1 + \ldots) \sum_{(w)} Y^{w(1)} \otimes Y^{w(2)} \right)
\]

(59)

\[
= \sum_{(w)} \langle X_{au}, Y_a Y^{w(1)} \rangle \langle X_{bv}, Y^{w(2)} \rangle
\]

(60)

\[
= \sum_{(w)} \langle X_u, Y^{w(1)} \rangle \langle X_{bv}, Y^{w(2)} \rangle
\]

(61)

\[
= \sum_{(w)} \langle X_u \otimes X_{bv}, Y^{w(1)} \otimes Y^{w(2)} \rangle
\]

(62)

\[
= \langle X_u X_{bv}, Y^w \rangle.
\]

(63)

Symmetrically, if \( c = b \), \( \langle X_{au} X_{bv}, Y_c^{cw} \rangle = \langle X_{au} X_v, Y_c^w \rangle \).

If \( c = a + b \), \( Y_a \otimes Y_b \) occurs in \( \Delta Y_c \), and a similar calculation shows that \( \langle X_{au} X_{bv}, Y_c^{cw} \rangle = \alpha \langle X_u X_v, Y^w \rangle \).

If \( c = a + b + d > a + b \), then the sum of the terms starting with \( Y_a \otimes Y_b \) in \( \Delta Y_c \) is

\[
\sum_{i+j=k, i,j \in I, |I|=1} Y^{as} \otimes Y^{bd} = \langle Y_a \otimes Y_b \rangle \Delta Y_d
\]

which occurs with coefficient \( \beta \). Thus,

\[
\langle X_{au} X_{bv}, Y_c^{cw} \rangle = \langle Y_a \otimes Y_b \rangle \Delta Y_d \Delta Y_c
\]

(64)

\[
\langle X_u \otimes X_v, \Delta \langle Y_d Y_c \rangle \rangle
\]

(65)

\[
\langle X_u X_v, Y_c^w \rangle = \langle \zeta_{a+b} \langle X_u X_v \rangle, Y_c^{cw} \rangle
\]

(66)

In all other cases, it is clear that \( \langle X_{au} X_{bv}, Y_c^{cw} \rangle = 0 \), whence the recurrence \[56].

\[
\sum_{(i) \subseteq I} \frac{n}{|I|} E_I(s,t) X_I
\]

(70)

where for \( I = (i_1, \ldots, i_r) \), \( \binom{n}{I} = \binom{n}{i_1, \ldots, i_r} \) is the multinomial coefficient and \( E_I = E_{i_1} \cdots E_{i_r} \) is a product of Eulerian polynomials.

For \( I = 1^n \), \( X_{1^n} \) is a symmetric function. The coefficients of its monomial expansion are

\[
\langle X_{1^n}, S^I \rangle = \langle \Delta X_{1^n}, S_{i_1} \otimes \cdots \otimes S_{i_r} \rangle = \prod_{k=1}^{r} \langle X_{1_{i_k}}, S_{i_k} \rangle = \prod_{k=1}^{r} a_{i_k}
\]

(71)

Clearly, \( X_1 = \phi_1 = F_1 = M_1 \) is the sum of the variables. It is always interesting to expand its powers on a new basis. Since \( \phi_{1^n} = \phi_1^n \), it follows from \[35\] that
where

\[
d_n = \langle X_{1^n}, S_n \rangle = [y^n] \left( \frac{1 + sy}{1 + ty} \right)^{\frac{1}{t}}.
\]

One can give a closed formula for the product \(X_{1^p}X_{1^q}\).

**Proposition 5.3.** For a composition \(I\), let \(\ell_0(I)\) denote its number of even parts, and \(\ell_1(I)\) its number of odd parts. Then

\[
X_{1^p}X_{1^q} = \sum_{I = p+q} \left( \frac{\ell_1(I)}{p - \frac{p+q-\ell_1(I)}{2}} \right) \alpha^{\ell_0(I)} \beta^{\frac{p+q-\ell_1(I)}{2}} X_I.
\]

**Proof –** By duality,

\[
\langle X_{1^p}X_{1^q}, Y^r \rangle = \langle \Delta^p(X_{1^p}X_{1^q}), Y_{i_1} \otimes \cdots \otimes Y_{i_r} \rangle
\]

\[
= \sum_{p_1 + \cdots + p_r = p, q_1 + \cdots + q_r = q} \prod_{k=1}^r \langle X_{1^{p_k}} X_{1^{q_k}}, Y_{i_k} \rangle.
\]

Now,

\[
\langle X_{1^p}X_{1^q}, Y_n \rangle = \langle X_{1^p} \otimes X_{1^q}, \Delta Y_n \rangle
\]

\[
= \left( X_{1^p} \otimes X_{1^q}, \frac{Y \otimes 1 + 1 \otimes Y + \alpha Y \otimes Y}{1 \otimes 1 - \beta Y \otimes Y} \right)
\]

\[
= \left( X_{1^p} \otimes X_{1^q}, \frac{Y_1 \otimes 1 + 1 \otimes Y_1 + \alpha Y_1 \otimes Y_1}{1 \otimes 1 - \beta Y_1 \otimes Y_1} \right)
\]

\[
= \begin{cases} 
\alpha \beta^{\frac{p+q}{2} - 1} & \text{if } p = q, \text{ so that } n = p + q \text{ is even,} \\
\beta^{\frac{p+q}{2}} & \text{if } |p - q| = 1, \text{ so that } n = p + q \text{ is odd.}
\end{cases}
\]

Thus, the coefficient of \(X_I\) is \(X_{1^p}X_{1^q}\) is equal to the number of \(2 \times r\) integer matrices

\[
\begin{bmatrix}
p_1 & p_2 & \cdots & p_r \\
q_1 & q_2 & \cdots & q_r
\end{bmatrix}
\]

with row sums \(p, q\), column sums \(i_1, \ldots, i_r\), and such that \(p_k = q_k\) if \(i_k\) is even, and \(|p_k - q_k| = 1\) of \(i_k\) is odd. On can form such a matrix by adding to the matrix \(p'_k = q'_k = \left\lfloor \frac{i_k}{2} \right\rfloor\) a matrix of 0 and 1, with 0 in the columns of the even \(i_k\), and exactly one 1 in the columns of the odd \(i_k\), such that the sum of the first row is \(p - \frac{p+q-\ell_1(I)}{2}\) and, equivalently, such that of the second row is \(q - \frac{p+q-\ell_1(I)}{2}\), whence the binomial coefficient.
6. Extension to other algebras

The algebra of quasi-symmetric functions has a noncommutative version \textbf{WQSym} (Word Quasi-Symmetric functions) consisting of the invariants of Hivert’s quasi-symmetrizing action of the symmetric group on the free associative algebra \([2]\). The product of the monomial basis of \textbf{WQSym} is described by convolution of packed words

\begin{equation}
M_u M_v = \sum_{w = u' \circ v'} M_w
\end{equation}

which can also be described in terms of the bigger algebra \textbf{MQSym} (Matrix Quasi-Symmetric functions) \([1]\), based on packed integer matrices. The product of two basis elements \(M_P M_Q\), where \(P\) is a \(p \times r\) matrix and \(Q\) a \(q \times s\) matrix is obtained by forming the block matrix

\begin{equation}
P \cdot Q = \begin{bmatrix}
P & 0 \\
0 & Q
\end{bmatrix}
\end{equation}

regarded as a word over the alphabet of rows, and taking the quasi-shuffle of the word \(P_1 \cdots P_r\), formed by its first \(p\) rows with the word formed by its last \(q\) rows, the contractions being given by vector addition of the rows.

Packed words can be encoded by packed \(0 - 1\)-matrices with exactly one 1 in each column. For example, \(u = 21321\) is encoded by

\begin{equation}
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\end{equation}

where \(m_{ij} = 1\) iff the \(j\)th letter is an \(i\). Then, packed convolution corresponds to the quasi-shuffle of these matrices, and so can be described by a recurrence

\begin{equation}
A P \uplus B Q = A'(P \uplus B Q) + B''(A U \uplus Q) + (A' + B'')(P \uplus Q)
\end{equation}

where \(A'\) and \(B''\) denote \(A\) and \(B\) completed by the appropriate number of zeros on the right or on the left.

The basis \(X_I\) can be lifted to \textbf{WQSym} by means of the (reverse) refinement order on packed words: if

\begin{equation}
X_I = \sum_{J \subseteq I} c_{IJ} M_J \quad \text{we define} \quad X_u = \sum_{v \subseteq u} c_{Ev(u)Ev(v)} M_v.
\end{equation}

To describe the product \(X_u X_v\), we shall embed \textbf{WQSym} and \textbf{MQSym} in the limit \(\ell \to \infty\) of the level \(\ell\) quasi-symmetric functions defined in \([19]\), where the noncommutative product of these algebras can be realized as a shifted version of the commutative product of a bigger algebra.

Let \(\Omega\) be the semigroup of nonnegative integer sequences \(i = (i_0, i_1, \ldots)\) with finite sum \(|i|\). We denote by \textbf{Sym} \((\mathbb{N})\) the Hopf algebra of \(\mathbb{N}\)-colored noncommutative symmetric functions, which is defined as the free associative algebra over indeterminates
$S_i$, with $S_0 = 1$, endowed with the coproduct

$$\Delta S_n = \sum_{i+j=n} S_i \otimes S_j.$$  

Departing from the notation of [19], we represent color sequences $i$ as row vectors, and regard the label $I$ of the basis element $S^I = S_{i_1} \cdots S_{i_r}$ as an $r \times \infty$ matrix. The multidegree $\|I\|$ of $I$ is defined as its column sum sequence. The Hopf algebra $QSym^{(N)}$ of $N$-colored quasi-symmetric functions is defined as its graded dual with respect to this multidegree.

Both algebras admit polynomial realizations, in terms of two colored alphabets

$$A = \bigsqcup_{c \geq 0} A^{(c)}, \quad X = \bigsqcup_{c \geq 0} X^{(c)},$$

In terms of $A$, the generating function of the complete functions can be written as

$$\sigma_X(A) = \prod_{i \geq 1} \left( 1 - \sum_{j \geq 0} x^{(j)} a_i^{(j)} \right)^{-1} = \sum_n S_n(A) x^n,$$

where $x^n = (x^{(0)})^{n_0} \cdots (x^{(k)})^{n_k} \cdots$

This realization gives rise to a Cauchy formula which in turn allows one to identify the dual of $\text{Sym}^{(N)}$ with the limit of an algebra introduced by S. Poirier in [26].

Let $X = X^{(0)} \sqcup \cdots \sqcup X^{(k)} \sqcup \cdots$, where $X^{(i)} = \{ x_j^{(i)}, j \geq 1 \}$, be an $N$-colored alphabet of commutative variables, also commuting with $A$. Imitating the level 1 case (see [1]),

we define the Cauchy kernel

$$K(X, A) = \prod_{j \geq 1} \sigma_{x_j^{(0)} \cdots x_j^{(k)} \cdots}(A).$$

Expanding on the basis $S^I$ of $\text{Sym}^{(N)}$, we get as coefficients what can be called the $N$-monomial quasi-symmetric functions $M_I(X)$

$$K(X, A) = \sum_I M_I(X) S^I(A),$$

defined by

$$M_I(X) = \sum_{j_1, \ldots, j_m} x_{j_1 \cdots j_m}^{i_1} \cdots x_{j_1 \cdots j_m}^{i_m},$$

with $I = (i_1, \ldots, i_m)$.

These functions form a basis of a subalgebra $QSym^{(N)}$ of $K[X]$, which we shall call the algebra of $N$-colored quasi-symmetric functions.

We can now define $\varphi_x = \log \sigma_x$, and get a family of primitive generators of $\text{Sym}^{(N)}$ by setting $\varphi_n = \text{coefficient of } x^n$ in $\varphi_x$.

With a formal series $f(x) = x + O(x^2)$ as above, we can now define a basis $Y^I$ by setting $Y^I_n = \text{coefficient of } x^n$ in $f(\varphi_x)$. The coproduct of $Y$ will then be given by the formal group law associated with $f$. 

Thus, if \( f \) is as above chosen as the exponential generating functions of the \( E_n(s, t) \), the product rule of basis \( X_1 \) will be the \((\alpha, \beta)\)-quasi-shuffle over the alphabet \( \Omega \).

The subspace of \( QSym^{(n)} \) spanned by the \( M_1 \) where \( I \) is a packed matrix (followed by zero columns) is stable for the product. This is not the case of the span of the matrices encoding packed words, but both are stable for the shifted product: if \( I \) is an \( r \times p \) packed matrix, and \( J \) an \( s \times q \) packed matrix, define

\[
M_1 \ast M_J = M_1 \cdot M_{0^p,J}
\]

that is, shift \( J \) by the appropriate number of zero columns so as to reproduce the product of \( MQSym \). Then, since \( X_1 \) is a linear combination of \( M_J \) involving only coarser \( J \) (obtained from \( I \) by adding consecutive rows), we have as well

\[
X_1 \ast X_J = X_1 \cdot X_{0^p,J}
\]

This is therefore an encoding of the product \( X_u X_v \) in \( WQSym \), which is thus given by the shifted \((\alpha, \beta)\)-quasi-shuffle of the corresponding matrices. Since in this case no two contractions can be equal, the product is multiplicity-free, and the coefficient of each \( X_u \) is just a single monomial \( \alpha^i \beta^j \), which can be explicitly computed.

Let \( f_{uv}^w \) be this coefficient. If there exists \( j \neq k \) such that \( u_j = u_k \) and \( w_j \neq w_k \), or \( v_j = v_k \) and \( w_{j+|u|} \neq w_{k+|u|} \), then \( f_{uv}^w = 0 \). Otherwise, let \( a_i \) be the number of different values \( u_j \) such that \( w_j = i \), and \( b_i \) be the number of different values \( v_j \) such that \( w = j + |u| = i \). Then,

\[
f_{uv}^w = \max(w) \prod_{i=1}^{\max(w)} f_{uv}^w(i), \quad \text{where} \quad f_{uv}^w(i) = \begin{cases} 0 & \text{if } |a_i - b_i| > 1, \\ \alpha \beta^{a_i-1} & \text{if } a_i = b_i, \\ \beta^{\min(a_i, b_i)} & \text{otherwise.} \end{cases}
\]

For example, the coefficient of \( X_{11211} \) in \( X_{123} X_{11} \) is \( \alpha \beta \), since \( a_1 = 2, b_1 = 2 \) and \( a_2 = 1, b_2 = 0 \).

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