Analyticity for the (generalized) Navier-Stokes equations with rough initial data

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Abstract. We study the Cauchy problem for the (generalized) incompressible Navier-Stokes equations

\[ u_t + \nabla \cdot (\nabla u) + u \cdot \nabla u + \nabla p = 0, \quad \text{div} u = 0, \quad u(0, x) = u_0. \]

We show the analyticity of the local solutions of the Navier-Stokes equation (\( \alpha = 1 \)) with any initial data in critical Besov spaces \( \dot{B}^{n/p}_{p,q}(-1,p,q) \) with \( 1 < p \leq \infty \), \( 1 \leq q \leq \infty \) and the solution is global if \( u_0 \) is sufficiently small in \( \dot{B}^{n/p}_{p,q}(-1,p,q) \). In the case \( p = \infty \), the analyticity for the local solutions of the Navier-Stokes equation (\( \alpha = 1 \)) with any initial data in modulation space \( M_{\infty,1}^{-1} \) is obtained. We prove the global well-posedness for a fractional Navier-Stokes equation (\( \alpha = 1/2 \)) with small data in critical Besov spaces \( \dot{B}^0_{p,1} \) and show the analyticity of solutions with small initial data either in \( \dot{B}^0_{p,1} \) or in \( M_{\infty,1}^0 \). Similar results also hold for all \( \alpha \in (1/2, 1) \).

Key words and phrases. (Generalized) Navier-Stokes equations; Besov spaces, Modulation spaces, Gevrey analyticity.

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1 Introduction

In this paper, we study the Cauchy problem for the (generalized) incompressible Navier-Stokes equations (GNS):

\[ u_t + (-\Delta)^\alpha u + u \cdot \nabla u + \nabla p = 0, \quad \text{div} u = 0, \quad u(0, x) = u_0. \]  

where \( t \in \mathbb{R}^+ = [0, \infty), \ x \in \mathbb{R}^n \ (n \geq 2), \ \alpha > 0 \) is a real number. \( u = (u_1, ..., u_n) \) denotes the flow velocity vector and \( p(t, x) \) describes the scalar pressure. \( u_0(x) = \)
\((u_0^1, \ldots, u_0^n)\) is a given velocity with \(\text{div} u_0 = 0\). \(u_t = \partial u / \partial t\), \((-\Delta)^\alpha\) denotes the fractional Laplacian which is defined as \((-\Delta)^\alpha u(t, \xi) = |\xi|^{2\alpha} \hat{u}(t, \xi)\). It is easy to see that (1.1) can be rewritten as the following equivalent form:

\[
\begin{align*}
  u_t + (-\Delta)^\alpha u + \mathbb{P} \text{div}(u \otimes u) & = 0, \\
  u(0, x) & = u_0, 
\end{align*}
\]  

(1.2)

where \(\mathbb{P} = I - \text{div}\Delta^{-1}\nabla\) is the matrix operator projecting onto the divergence free vector fields, \(I\) is identity matrix.

It is known that (1.1) is essentially equivalent to the following integral equation:

\[
\begin{align*}
  u(t) & = U_{2\alpha}(t) u_0 + A_{2\alpha} \mathbb{P} \text{div}(u \otimes u), \\
  (A_{2\alpha} f)(t) & := \int_0^t U_{2\alpha}(t - \tau) f(\tau) d\tau. 
\end{align*}
\]  

(1.3)

Note that (1.1) is scaling invariant in the following sense: if \(u\) solves (1.1), so does \(u_\lambda(t, x) = \lambda^{2\alpha-1} u(\lambda^{2\alpha} t, \lambda x)\) with initial data \(\lambda^{2\alpha-1} u_0(\lambda x)\). A function space \(X\) defined in \(\mathbb{R}^n\) is said to be a critical space for (1.1) if the corresponding norm of \(u_\lambda(0, x)\) in \(X\) is invariant for all \(\lambda > 0\). It is easy to see that homogeneous type Besov space \(\dot{B}^{n/p-2\alpha+1}_{\infty,p,q}\) is a critical space for (1.1). In particular, \(\dot{B}^{n/p-1}_{p,q}\) is a critical space for GNS in the case \(\alpha = 1\) and \(\dot{B}^{n/p}_{p,q}\) is a critical space for GNS in the case \(\alpha = 1/2\).

If \(\alpha = 1\), GNS (1.1) is the well-known Navier-Stokes equation (NS) which have been extensively studied in the past twenty years; cf. [1, 6, 7, 10, 14, 15, 18, 20, 23, 25, 26, 27, 28, 34]. Kato [26] first used the semi-group estimates to obtain the strong \(L^p\) solutions of NS. Cannone [6] and Planchon [34] considered global solutions in 3D for small data in critical Besov spaces \(\dot{B}^{3/p-1}_{p,\infty}\) with \(3 < p \leq 6\). Chemin [10] obtained global solutions in 3D for small data in critical Besov spaces \(\dot{B}^{3/p-1}_{p,q}\) for all \(p < \infty\), \(q \leq \infty\). Koch and Tataru [27] studied local solutions for initial data in \(vmo^{-1}\) and global solutions for small initial data in \(BMO^{-1}\). Escauriaza, Serigin and Sverak were able to show the 3D global regularity if the solution has a uniform bound in the critical Lebesgue space \(L^3\). Iwabuchi [25] considered the well posedness of NS for initial data in modulation spaces \(\dot{M}^{3/p-1}_{p,q}\) especially in \(\dot{M}^{-1}_{p,1}\) with \(1 \leq p \leq \infty\). Foias and Temam [18] proved spatial analyticity for solutions in Sobolev spaces of periodical functions in an elementary way. The analyticity of solutions in \(L^p\) for NS was first shown by Grujić and Kukavica [23] and Lemarié-Rieusset [28] gave a different approach based on multilinear singular integrals. Using iterative derivative
estimates, the analyticity of NS for small initial data in $BMO^{-1}$ was obtained in Germain, Pavlovic and Staffilani [20] (see also Dong and Li [14], Miura and Sawada [33] on the iterative derivative techniques). Recently, Bae, Biswas and Tadmor [1] obtained the analyticity of the solutions of NS in 3D for the sufficiently small initial data in critical Besov spaces $\dot{B}^{3/p-1}_{p,q}$ with $1 < p, q < \infty$. Moreover, in the case $p = \infty$, they can show the analyticity of the solutions of NS in 3D for the sufficiently small initial data in $\dot{B}^{-1}_{\infty, q} \cap \dot{B}^{0}_{3, \infty}$ with $1 \leq q < \infty$. Since NS is ill-posed in $\dot{B}^{-1}_{\infty, \infty}$; cf. Bourgain and Pavlovic [5], Germain [19] and Yoneda [44], the largest Besov-type space on initial data for which NS is well-posed or analytically well-posed is still open.

For general $\alpha$, there are also some results on (1.1) in recent years. For $\alpha \geq 5/4, n = 3$, Lions [30] proved the global existence of classical solutions. Wu [43] proved (1.1) has a unique global solution with small data in $\dot{B}^{n/p-2\alpha+1}_{n/p, q}$ for $1 \leq q \leq \infty$ for $\alpha > 1/2$, $p = 2$, or $1/2 < \alpha < 1$, $2 < p < \infty$. Recently, Li, Zhai and Yang [31, 32] obtained the well-posedness of (1.1) in the case $1/2 < \alpha < 1$ in some $Q$-spaces. Yu and Zhai [45] showed the global existence and uniqueness with small initial data in $\dot{B}^{-2\alpha}_{1, \infty}$ for GNS in the case $1/2 < \alpha < 1$. As far as the authors can see, for $\alpha = 1/2$ and general $p$, the well-posedness and analyticity of solutions of (1.1) in critical Besov spaces $\dot{B}^{n/p}_{p,q}$ seems unsolved.

1.1 Notations

Throughout this paper, $C \geq 1$, $c \leq 1$ will denote constants which can be different at different places, we will use $A \lesssim B$ to denote $A \leq CB$. We denote $|x|_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p}$, $|x| = |x|_2$ for any $x \in \mathbb{R}^n$, by $L^p = L^p(\mathbb{R}^n)$ the Lebesgue space on which the norm is written as $\| \cdot \|_p$. Let $X$ be a quasi-Banach space. For any function space $X$ and the operator $T : X \to X$, we denote

$$\| u \|_{L^\gamma(I; X)} = \left( \int_I \| u(t, \cdot) \|_X^\gamma dt \right)^{1/\gamma}$$

for $1 \leq \gamma < \infty$ and with usual modifications for $\gamma = \infty$. In particular, if $X = L^p$, we will write $\| u \|_{L^\gamma(I; L^p)} = \| u \|_{L^\gamma(I; L^p)}$ and $\| u \|_{L^\gamma(L^\infty)} = \| u \|_{L^\gamma(0, \infty; L^p)}$. For any function space $X$ and the operator $T : X \to X$, we denote

$$TX = \{ Tf : f \in X \}, \quad \| f \|_{TX} := \| Tf \|_X.$$

(1.5)

Now let us recall the definition of dyadic decomposition in Littlewood-Paley theory [37]. Let $\psi : \mathbb{R}^n \to [0, 1]$ be a smooth cut-off function which equals 1 on the unit
ball and equals 0 outside the ball \( \{ \xi \in \mathbb{R}^n ; |\xi| \leq 2 \} \). Write \( \varphi(\xi) := \psi(\xi) - \psi(2\xi) \) and \( \varphi_k(\xi) = \varphi(2^{-k}\xi) \). \( \Delta_k := \mathcal{F}^{-1}\varphi_k\mathcal{F} \), \( k \in \mathbb{Z} \) are said to be the dyadic decomposition operators and satisfying the operator identity: \( I = \sum_{k=-\infty}^{+\infty} \Delta_k \). \( S_k := \sum_{j=-\infty}^{k} \Delta_j \) are called the low frequency projection operators. It is easy to see that \( S_ku \to u \) as \( k \to \infty \) in the sense of distributions. The norms in homogeneous Besov spaces are defined as follows:

\[
\|f\|_{B_{p,q}^s} = \left( \sum_{j=-\infty}^{+\infty} 2^{js} \|\Delta_j f\|_p^q \right)^{1/q}
\]

with usual modification if \( q = \infty \). We also need the following time-space norms:

\[
\|f\|_{\tilde{L}^q(I;B_{p,q}^s)} = \left( \sum_{j=-\infty}^{+\infty} 2^{js} \|\Delta_j f\|_{L^q(I;L^p)}^q \right)^{1/q} < \infty
\]

with the usual modification for \( q = \infty \) and by (1.5), for any \( T : \dot{B}_{p,q}^s \to \dot{B}_{p,q}^s \),

\[
\|f\|_{\tilde{L}^q(I;TB_{p,q}^s)} = \|Tf\|_{\tilde{L}^q(I;B_{p,q}^s)},
\]

Now we recall the definition of modulation spaces which was first introduced by Feichtinger [16] in 1983 (see also Gröchenig [22]). Let \( \sigma \) be a smooth cut-off function with \( \text{supp} \sigma \subset [-3/4, 3/4]^n \), \( \sigma_k = \sigma(\cdot - k) \) and \( \sum_{k \in \mathbb{Z}^n } \sigma_k = 1 \). We define the frequency-uniform decomposition operator as \( \Box_k = \mathcal{F}^{-1}\sigma_k\mathcal{F} \), \( \sigma \in \mathbb{Z}^n \). Let \( 0 < p, q \leq \infty \), \( s \in \mathbb{R} \) and

\[
\|f\|_{M_{p,q}^s} = \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\Box_k f\|_p^q \right)^{1/q},
\]

\( M_{p,q}^s \) is said to be a modulation space. There are exact inclusions between Besov spaces and modulation spaces (see [21, 36, 35, 40]), however, there are no straightforward inclusions between homogeneous Besov spaces and modulation spaces. Let \( 0 < p, q \leq \infty \), \( s \geq 0 \),

\[
\|f\|_{E_{p,q}^s} = \left( \sum_{k} 2^{qs|k|} \|\Box_k f\|_p^q \right)^{1/q},
\]

\( E_{p,q}^s \) was introduced in [12], which can be regarded as modulation spaces with analytic regularity. It is easy to see that \( M_{p,q}^0 = E_{p,q}^0 \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \alpha! = \alpha_1! \cdots \alpha_n! \) and \( \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \). Recall that the Gevrey class is defined as follows.

\[
G_{1,p} = \left\{ f \in C^\infty(\mathbb{R}^n) : \exists p, M > 0 \text{ s.t. } \|\partial^\alpha f\|_p \leq \frac{M \alpha!}{p^{\alpha!}}, \forall \alpha \in \mathbb{Z}^n_+ \right\}.
\]

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It is known that $G_{1,\infty}$ is the Gevrey 1-class. Moreover, we easily see that $G_{1,p_1} \subset G_{1,p_2}$ for any $p_1 \leq p_2$. There is a very beautiful relation between Gevrey class and exponential modulation spaces, we can show that $G_{1,p} = \bigcup_{s>0} E^s_{p,q}$, see Section 3. Similarly as in (1.7), we need the following norm

$$
\|f\|_{\overline{L}^{\gamma}(I;\overline{E}^n_{p,q}(t))} = \left( \sum_{k \in \mathbb{Z}^n} \|2^{\gamma(t)k}\hat{f}\|_{L^q(t\in I;L^p)}^q \right)^{1/q},
$$

(1.11)

with usual modification if $q = \infty$. Recently modulation spaces have been applied to the study of nonlinear evolutions; cf. [2, 3, 11, 12, 25, 41, 42].

As the end of this section, we recall the definition of multiplier space $M_p$ (see [4, 24]). Let $\rho \in \mathcal{S}'$. If there exists a $C > 0$ such that $\|\mathcal{F}^{-1}\rho \mathcal{F} f\|_{L^p} \leq C\|f\|_{L^p}$ holds for all $f \in \mathcal{S}$, then $\rho$ is called a Fourier multiplier on $L^p$. The linear space of all multipliers on $L^p$ is denoted by $M_p$ and the norm on which is defined as $\|\rho\|_{M_p} = \sup \{\|\mathcal{F}^{-1}\rho \mathcal{F} f\|_{L^p} : f \in \mathcal{S}, \|f\|_{L^p} = 1\}$. Using the multiplier estimates, we have the following Bernstein’s inequalities:

$$
\|\Delta_j(-\Delta)^{s/2}f\|_q \leq 2^{j(s+n(1/p-1/q))}\|f\|_p,
$$

$$
\|\Delta_j(-\Delta)^{s/2}f\|_p \sim 2^{js}\|\Delta_jf\|_p
$$

1.2 Main Results

The analyticity for the global solutions of NS in 3D with small initial data in critical Besov spaces $\dot{B}^{3/p-1}_{p,q} (1 \leq p, q < \infty)$ was obtained in [1]. In this paper, we show the analyticity of the local solutions of NS with large initial data in critical Besov spaces $\dot{B}^{n/p-1}_{p,q} (1 < p < \infty, 1 \leq q \leq \infty)$. Let us denote $\Lambda = (-\partial^2_{x_1})^{1/2} + \ldots + (-\partial^2_{x_n})^{1/2}$. Recall that $\|f\|_{\tilde{L}^q(I;e^{\sqrt{\Lambda}B^n_{p,q}})} = \|e^{\sqrt{\Lambda}f}\|_{\tilde{L}^q(I;\dot{B}^n_{p,q})}$. Our main results are the following theorems.

**Theorem 1.1** (Analyticity for NS (I): $p < \infty$) Let $n \geq 2, \alpha = 1, 1 < p < \infty, 1 \leq q \leq \infty$. Assume that $u_0 \in \dot{B}^{n/p-1}_{p,q}$. There exists a $T_{\text{max}} = T_{\text{max}}(u_0) > 0$ such that (1.11) has a unique solution $u \in \tilde{L}^3_{\text{loc}}(0, T_{\text{max}}; e^{\sqrt{\Lambda}B^n_{p,q}+1/3}) \cap \tilde{L}^{3/2}_{\text{loc}}(0, T_{\text{max}}; e^{\sqrt{\Lambda}B^n_{p,q}+1/3}) \cap \tilde{L}^{\infty}(0, T_{\text{max}}; e^{\sqrt{\Lambda}B^n_{p,q}})$. If $T_{\text{max}} < \infty$, then

$$
\|u\|_{\tilde{L}^3(0,T_{\text{max}};e^{\sqrt{\Lambda}B^n_{p,q}+1/3})} \cap \tilde{L}^{3/2}(0,T_{\text{max}};e^{\sqrt{\Lambda}B^n_{p,q}+1/3}) = \infty.
$$

Moreover, if $u_0 \in \dot{B}^{n/p-1}_{p,q}$ is sufficiently small, then $T_{\text{max}} = \infty$. 

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Theorem 1.1 has implied the analyticity of the solutions. In the case $p = \infty$, the analyticity of global solutions of NS with small initial data in $\dot{B}^{-1}_{\infty,q} \cap \dot{B}^0_{3,\infty}$ was shown in [1]. We use a different approach to get an analyticity result for local solutions with any initial data in the modulation space $M_{\infty,1}^{-1}$:

**Theorem 1.2** (Analyticity for NS (II): $p \leq \infty$) Let $n \geq 2$, $\alpha = 1$, $1 \leq p \leq \infty$. Assume that $u_0 \in M_{p,1}^{-1}$. There exists a $T_{\text{max}} = T_{\text{max}}(u_0) > 0$ such that (1.1) has a unique solution $u \in \tilde{L}_c^2(0, T_{\text{max}}; E_{p,1}^{ct})$ and $(I - \Delta)^{-1/2}u \in \tilde{L}^\infty([0, T_{\text{max}}); E_{p,1}^{ct})$, $c = 2^{-10}$. Moreover, if $T_{\text{max}} < \infty$, then $\|u\|_{L^2(0, T_{\text{max}}; E_{p,1}^{ct})} = \infty$.

Now we compare homogeneous Besov spaces with modulation spaces. Let $S_1$ be the low frequency projection operator defined in the above. Let $1 \leq p \leq \infty$. One easily sees that $S_1 \dot{B}^{-1}_{p,1} \subset S_1 M_{p,1}^{-1}$ and $(I - S_1) M_{p,1}^{-1} \subset (I - S_1) \dot{B}^{-1}_{p,1}$. Roughly speaking, the low frequency part of $M_{p,1}^{-1}$ is rougher than that of $\dot{B}^{-1}_{p,1}$, the high frequency part of $M_{p,1}^{-1}$ is smoother than that of $\dot{B}^{-1}_{p,1}$. In general there is no inclusion between $M_{p,1}^{-1}$ and $\dot{B}^{-1}_{p,1}$.

If $\alpha = 1/2$, we have

**Theorem 1.3** Let $n \geq 2$, $\alpha = 1/2$. Suppose $u_0 \in \dot{B}^{n/p}_{p,1}$ $(1 \leq p \leq \infty)$ and $\|u_0\|_{\dot{B}^{n/p}_{p,1}}$ is sufficiently small, then we have the following results:

i) (Well-posedness for $p < \infty$) If $1 \leq p < \infty$, then there exists a unique global solution $u$ to (1.1) satisfying $u \in \tilde{L}^\infty(\mathbb{R}^+, \dot{B}^{n/p}_{p,1})$.

ii) (Well-posedness for $p = \infty$) If $p = \infty$, then there exists a unique global solution $u$ to (1.1) satisfying $u \in \tilde{L}^\infty(\mathbb{R}^+, \dot{B}^{0}_{\infty,1}) \cap \tilde{L}^1(\mathbb{R}^+, \dot{B}^{1}_{\infty,1})$.

iii) (Analyticity for $p < \infty$) If $1 < p < \infty$, then the solution obtained in i) satisfying $u(t) \in \tilde{L}^\infty(\mathbb{R}^+, e^{\alpha t/2n} \dot{B}^{n/p}_{p,1})$.

iv) (Analyticity for $p = \infty$) Assume that $u_0 \in \dot{B}^{0}_{\infty,1} \cap M_{\infty,1}^0$ and $\|u_0\|_{\dot{B}^{0}_{\infty,1} \cap M_{\infty,1}^0}$ is sufficiently small. Then the solution obtained in ii) satisfies $u \in \tilde{L}^\infty(\mathbb{R}^+, E^{s(t)}_{\infty,1})$, $\nabla u \in \tilde{L}^1(\mathbb{R}^+, E^{s(t)}_{\infty,1})$ for $s(t) = 2^{-5}(1 \wedge t)$.

In Theorem 1.3 the uniqueness of the well-posedness of the case $p = \infty$ is worse than that of the case $p < \infty$. Moreover, the proof of Theorem 1.3 cannot be developed to the case for which the initial data belong to the other critical Besov spaces $\dot{B}^{n/p}_{p,q}$ with $q \neq 1$.

At the end of this section, we point out that the ideas used in this paper is also adapted to the case $1/2 < \alpha < 1$. In Section 7, we will generalize the analyticity results in Theorems 1.1 and 1.2 to the case $1/2 < \alpha < 1$, see Theorems 7.3 and 7.4.
2 Analyticity of NS: Proof of Theorem 1.1

First, we establish two linear estimates by following some ideas as in [10, 29, 1] and [39, 38]. If we remove $e^{\sqrt{t\Lambda}}$ in the following Lemmas 2.1 and 2.2, the results were essentially obtained in [39] (see also [41], Section 2.2.2). However, $e^{\sqrt{t\Lambda}}$ is of importance for the nonlinear estimates in the space $\tilde{L}^\gamma(\mathbb{R}^+; e^{\sqrt{t\Lambda}}\dot{B}^s_{p,q})$, see Lemarié-Rieusset [29] and [1].

Lemma 2.1 Let $1 < p < \infty$, $1 \leq q \leq \infty$, $1 \leq \gamma \leq \infty$. Then

$$\|U_2(t)u_0\|_{\tilde{L}^\gamma(\mathbb{R}^+; e^{\sqrt{t\Lambda}}\dot{B}^s_{p,q})} \lesssim \|u_0\|_{\dot{B}^{s-2/\gamma}_{p,q}}. \quad (2.1)$$

Proof. In view of $e^{-t|\xi|^2/2} + \sqrt{t}|\xi| \in M_p$ (see [1]), one easily sees that

$$\|\Delta_je^{\sqrt{t\Lambda}}U_2(t)u_0\|_p \lesssim \|\mathcal{F}^{-1}e^{-t|\xi|^2/2} + \sqrt{t}|\xi|\|_1\|\Delta_jU_2(t/2)u_0\|_p \lesssim \|\Delta_jU_2(t/2)u_0\|_p. \quad (2.2)$$

We have known that (see [10] and [41], Section 2.2.2)

$$\|\Delta_jU_2(t/2)f\|_{L^p} \lesssim e^{-ct2^j}\|\Delta_jf\|_{L^p}, \quad (2.3)$$

Hence, it follows from (2.2) and (2.3) that

$$\|\Delta_je^{\sqrt{t\Lambda}}U_2(t)u_0\|_p \lesssim e^{-ct2^j}\|\Delta_ju_0\|_p. \quad (2.4)$$

Taking $L^\gamma_t$ norm on inequality (2.4),

$$\|\Delta_je^{\sqrt{t\Lambda}}U_2(t)u_0\|_{L^\gamma_t L^p_x} \lesssim 2^{-2j/\gamma}\|\Delta_ju_0\|_p. \quad (2.5)$$

Taking sequence $l^q$ norms in (2.5), we have the result, as desired. \qed

Lemma 2.2 Let $1 < p < \infty$, $1 \leq q \leq \infty$, $1 \leq \gamma_1 \leq \gamma \leq \infty$, $I = [0, T)$, $T \leq \infty$. Then

$$\|\mathcal{A}_2f\|_{\tilde{L}^\gamma(I; e^{\sqrt{t\Lambda}}\dot{B}^s_{p,q})} \lesssim \|f\|_{\tilde{L}^{\gamma_1}(I; e^{\sqrt{t\Lambda}}\dot{B}^{s+2(1+1/\gamma^1-1/\gamma)}_{p,q})}. \quad (2.6)$$

In particular,

$$\|\mathcal{A}_2f\|_{\tilde{L}^\gamma(I; e^{\sqrt{t\Lambda}}\dot{B}^s_{p,q})} \lesssim \|f\|_{\tilde{L}^1(I; e^{\sqrt{t\Lambda}}\dot{B}^{s-2/\gamma}_{p,q})}. \quad (2.7)$$
Proof. In view of \( e^{(\sqrt{t}-\sqrt{\tau}-\sqrt{t-\tau})|\xi|} \in M_p \) (see \[1\]) and (2.4),

\[
\|\Delta_j e^{\sqrt{\Lambda} \cdot \rho^2 f} \|_p \lesssim \int_0^t \|\Delta_j e^{(\sqrt{t}-\sqrt{\tau}) \Lambda + (t-\tau) \Delta} e^{\sqrt{\Lambda}} f(\tau) \|_p d\tau
\]

\[
\lesssim \int_0^t \|\Delta_j e^{\sqrt{\Lambda} \cdot \rho^2 + (t-\tau) \Delta} e^{\sqrt{\Lambda}} f(\tau) \|_p d\tau
\]

\[
\lesssim \int_0^t e^{-c(t-\tau)^2j} \|\Delta_j e^{\sqrt{\Lambda}} f(\tau) \|_p d\tau. \tag{2.8}
\]

Using Young’s inequality, for \( \gamma_1 \leq \gamma \),

\[
\|\Delta_j e^{\sqrt{\Lambda} \cdot \rho^2 f} \|_{L_{t,\xi}^q L_{x}^p} \lesssim 2^{-2j(1+1/\gamma-1/\gamma_1)} \|\Delta_j e^{\sqrt{\Lambda}} f(\tau) \|_{L_{t,\xi}^{\gamma_1}, L_{x}^p}. \tag{2.9}
\]

Taking \( l^q \) norms in both sides of (2.9), we have the results, as desired. \( \square \)

For two tempered distributions \( f, g \), using the para-product decomposition \[9, 29\], one can decompose \( fg \) as the summation of the following two parts:

\[
fg = \sum_{i,j} \Delta_i f \Delta_j g = \sum_j S_{j-1} f \Delta_j g + \sum_i S_i g \Delta_i f,
\]

By the support property in frequency spaces, one can easily check that

\[
\Delta_i(S_j f \Delta_j g) = 0, \text{ for } i > j + 3.
\]

Therefore

\[
\Delta_i(fg) = \sum_{i \leq j+3} \Delta_i(S_{j-1} f \Delta_j g + S_j g \Delta_j f). \tag{2.10}
\]

**Lemma 2.3** Let \( 1 < p < \infty, 1 \leq q, \gamma, \gamma_1, \gamma_2 \leq \infty \) with \( 1/\gamma = 1/\gamma_1 + 1/\gamma_2 \), \( s, \varepsilon > 0 \).

Then

\[
\|fg\|_{L^{\gamma}(I, e^{\sqrt{\Lambda} \cdot \rho^2} B_{p,q}^s)} \lesssim \|f\|_{L^{\gamma_1}(I, e^{\sqrt{\Lambda} \cdot \rho^2/p-\varepsilon})} \|g\|_{L^{\gamma_2}(I, e^{\sqrt{\Lambda} \cdot \rho^2/p-\varepsilon})}
\]

\[
+ \|g\|_{L^{\gamma_1}(I, e^{\sqrt{\Lambda} \cdot \rho^2/p-\varepsilon})} \|f\|_{L^{\gamma_2}(I, e^{\sqrt{\Lambda} \cdot \rho^2/p-\varepsilon})}. \tag{2.11}
\]

Moreover, if \( \varepsilon = 0 \), then (2.11) also holds for \( q = 1 \).

**Proof.** Denote \( F = e^{\sqrt{\Lambda}} f, \ G = e^{\sqrt{\Lambda}} g \). In view of (2.10),

\[
e^{\sqrt{\Lambda}} \Delta_i(fg) = \sum_{i \leq j} e^{\sqrt{\Lambda}} \Delta_i(S_{j-1} f \Delta_j g + S_j g \Delta_j f)
\]

\[
= \sum_{i \leq j} \sum_{k \leq j-1} e^{\sqrt{\Lambda}} \Delta_i(\Delta_k e^{-\sqrt{\Lambda} F} \Delta_j e^{-\sqrt{\Lambda} G})
\]

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\[
+ \sum_{i=-3}^{3} \sum_{k \leq j} e^{\sqrt{\lambda}} \Delta_i (\Delta_k e^{-\sqrt{\lambda}} G \Delta_j e^{-\sqrt{\lambda}} F) \quad (2.12)
\]

Now we use an idea as in [29], Page 253 and [1], to consider the bilinear form

\[
B_t(u, v) = e^{\sqrt{\lambda}} (e^{-\sqrt{\lambda}} u e^{-\sqrt{\lambda}} v)
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix \xi} e^{\sqrt{\lambda}(|\xi| - |\eta| - |\eta_i|)} \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta d\xi. \quad (2.13)
\]

Denote for \( \lambda = (\lambda_1, ..., \lambda_n), \mu = (\mu_1, ..., \mu_n), \nu = (\nu_1, ..., \nu_n), \) and \( \lambda_i, \mu_i, \nu_i \in \{1, -1\}, \)

\[
D_\lambda = \{ \eta : \lambda_i \eta_i \geq 0, \ i = 1, ..., n \},
\]

\[
D_\mu = \{ \xi - \eta : \mu_i (\xi_i - \eta_i) \geq 0, \ i = 1, ..., n \},
\]

\[
D_\nu = \{ \xi : \nu_i \xi_i \geq 0, \ i = 1, ..., n \}.
\]

We denote by \( \chi_D \) the characteristic function on \( D \). Then we can rewrite \( B_t(u, v) \) as

\[
\sum_{\lambda, \mu, \nu \in \{1, -1\}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix \xi} \chi_{D_\nu}(\xi) e^{\sqrt{\lambda}(|\xi| - |\eta| - |\eta_i|)} \chi_{D_\mu}(\xi - \eta) \hat{u}(\xi - \eta) \hat{v}(\eta) d\eta d\xi. \quad (2.14)
\]

Noticing that for \( \eta \in D_\lambda, \xi - \eta \in D_\mu \) and \( \xi \in D_\mu, e^{\sqrt{\lambda}(|\xi| - |\eta| - |\eta_i|)} \) must belong to the following set

\[
\mathfrak{M} = \{ 1, e^{-2\sqrt{\lambda} |\xi|}, e^{-2\sqrt{\lambda} |\xi| - |\eta_i|}, e^{-2\sqrt{\lambda} |\eta|} \}, \ i = 1, ..., n. \quad (2.15)
\]

For \( 1 < p < \infty, \chi_{D_\lambda} \in M_p, m \in M_p \) for any \( m \in \mathfrak{M} \), it follows from the algebra property of \( M_p \) that

\[
\|B_t(u, v)\|_p \lesssim \|u\|_{p_1}\|v\|_{p_2}, \ 1/p_1 + 1/p_2 = 1. \quad (2.16)
\]

By (2.12), (2.13) and (2.16), then using Bernstein’s inequality, we have

\[
2^s \|e^{\sqrt{\lambda}} \Delta_i (fg)\|_{L_{t,1}^p L_x^p} \lesssim \sum_{i-3 \leq j} \sum_{k \leq j} 2^{s(i-j)} \|\Delta_k F\|_{L_{t,1}^p L_x^p} 2^{s(j)} \|\Delta_j G\|_{L_{t,1}^p L_x^p}
+ \sum_{i-3 \leq j} \sum_{k \leq j} 2^{s(i-j)} \|\Delta_k G\|_{L_{t,1}^p L_x^p} 2^{s(j)} \|\Delta_j F\|_{L_{t,1}^p L_x^p}
\lesssim \|F\|_{\tilde{L}_1(I;B_{p/q}^{p'/(p'-1)})} \sum_{i-3 \leq j} 2^{s(i-j)} 2^{(s+\epsilon)j} \|\Delta_j G\|_{L_{t,1}^p L_x^p}
+ \|G\|_{\tilde{L}_1(I;B_{p/q}^{p'/(p'-1)})} \sum_{i-3 \leq j} 2^{s(i-j)} 2^{(s+\epsilon)j} \|\Delta_j F\|_{L_{t,1}^p L_x^p}. \quad (2.17)
\]

Taking sequence \( L^q \) norm over all \( i \in \mathbb{Z} \) in (2.17) and using Young’s inequality, we immediately have the result, as desired. \( \square \)
Corollary 2.4 Let $1 < p < \infty$, $1 \leq q \leq \infty$. Then

\begin{align*}
\|fg\|_{\mathcal{L}^1(I;\mathbb{R}^n)} & \lesssim \|f\|_{\mathcal{L}^q(I;\mathbb{R}^n)} \|g\|_{\mathcal{L}^{q'}(I;\mathbb{R}^n)} \\
& + \|g\|_{\mathcal{L}^q(I;\mathbb{R}^n)} \|f\|_{\mathcal{L}^{q'}(I;\mathbb{R}^n)}
\end{align*}

(2.18)

Proof of Theorem 1.1. Now let us consider the map

$$
\mathcal{T} : u(t) \rightarrow U_2(t)u_0 + \mathcal{A}_2\mathcal{P}\text{div}(u \otimes u)
$$

(2.19)

in the metric space $(I = [0,T])$

$$
\mathcal{D} = \{u : \|u\|_{\mathcal{L}^3(I;\mathbb{R}^n,\mathcal{B}_{p,q}^n(\mathbb{R}^n))} \leq \delta\},
$$

$$
d(u,v) = \|u - v\|_{\mathcal{L}^3(I;\mathbb{R}^n,\mathcal{B}_{p,q}^n(\mathbb{R}^n))}.
$$

By (2.7) and Corollary 2.4, for any $u,v \in \mathcal{D}$,

$$
\|\mathcal{T}u\|_{\mathcal{L}^3(I;\mathbb{R}^n,\mathcal{B}_{p,q}^n(\mathbb{R}^n))} \lesssim \|U_2(t)u_0\|_{\mathcal{L}^3(I;\mathbb{R}^n,\mathcal{B}_{p,q}^n(\mathbb{R}^n))} + \|u\|_{\mathcal{L}^3(I;\mathbb{R}^n,\mathcal{B}_{p,q}^n(\mathbb{R}^n))} + \delta^2,
$$

(2.20)

$$
d(\mathcal{T}u, \mathcal{T}v) \leq \delta d(u,v).
$$

(2.21)

There exists a $T > 0$ such that

$$
\|U_2(t)u_0\|_{\mathcal{L}^3(I;\mathbb{R}^n,\mathcal{B}_{p,q}^n(\mathbb{R}^n))} \leq \delta/2.
$$

(2.22)

Indeed, we can choose a sufficiently large $J \in \mathbb{N}$ such that

$$
\left( \sum_{|j| \geq J} 2^{q(n/p-1)j} \|\triangle_j u_0\|^q_p \right)^{1/q} \leq \delta/4.
$$

It follows from (2.5) that

$$
\left( \sum_{|j| \geq J} 2^{q(n/p-1)j} \|\triangle_j e^{\sqrt{T}A} U_2(t) u_0\|^q_{L^3_{t}L^p_x} \right)^{1/q} \leq \left( \sum_{|j| \geq J} 2^{q(n/p-1)j} \|\triangle_j u_0\|^q_p \right)^{1/q} \leq \delta/4.
$$

On the other hand, one can choose $|I|$ small enough such that

$$
\left( \sum_{|j| \leq J} 2^{q(n/p-1)j} \|\triangle_j e^{\sqrt{T}A} U_2(t) u_0\|^q_{L^3_{t}L^p_x} \right)^{1/q} \leq \delta/4.
$$
So, we have (2.22). Applying the standard argument, we can show that \( \mathcal{F} \) is a contraction mapping from \( (\mathcal{D}, d) \) into itself. So, there is a \( u \in \mathcal{D} \) satisfying \( \mathcal{F}u = u \). Moreover, by Lemmas 2.1 and 2.2,

\[
\|u\|_{\tilde{L}^{\infty}(T,t;e^{\sqrt{\lambda t} \mathcal{H}^{n/p-1}})} \lesssim \|u_0\|_{\mathcal{B}^{n/p-1}} + \delta^2. \tag{2.23}
\]

The solution can be extended step by step and finally we have a maximal time \( T_{\text{max}} \) verifying

\[
u \in \tilde{L}_{\text{loc}}^3(0,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p-1}}) \cap \tilde{L}^{3/2}_\text{loc}(0,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p+1/3}}) \cap \tilde{L}^\infty([0,T_{\text{max}});e^{\sqrt{\lambda t} \mathcal{H}^{n/p-1}}).
\]

If \( T_{\text{max}} < \infty \) and \( \|u\|_{\tilde{L}^3(0,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p-1}}) \cap \tilde{L}^{3/2}(0,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p+1/3}})} < \infty \), we claim that the solution can be extended beyond \( T_{\text{max}} \). Indeed, let us consider the integral equation

\[
u(t) = U_2(t-T)\nu(T) + \int_T^t U_2(t-\tau)\mathbb{P}\text{div}(u \otimes u)(\tau)d\tau, \tag{2.24}
\]

we see that\n
\[
\|U_2(t-T)\nu(T)\|_{\tilde{L}^3(T,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p-1/3}}) \cap \tilde{L}^{3/2}(T,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p+1/3}})} \leq \|\nu\|_{\tilde{L}^3(T,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p-1/3}}) \cap \tilde{L}^{3/2}(T,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p+1/3}})} + C\|\nu\|^2_{\tilde{L}^3(T,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p-1/3}}) \cap \tilde{L}^{3/2}(T,T_{\text{max}};e^{\sqrt{\lambda t} \mathcal{H}^{n/p+1/3}})} \lesssim \delta/2. \tag{2.25}
\]

if \( T \) is sufficiently close to \( T_{\text{max}} \), (2.25) is analogous to (2.22), which implies that the solution exists on \( [T,T_{\text{max}}] \). A contradiction with \( T_{\text{max}} \) is maximal. Moreover, if \( \|\nu_0\|_{\mathcal{B}^{n/p-1}} \ll 1 \), we can directly choose \( T = \infty \) in (2.20) and (2.21).

\[
\square
\]

### 3 Gevrey Class and \( \mathcal{E}^{s}_{p,q} \)

In [38], it was shown that \( G_{r,2} = \bigcup_{s>0} E^{s}_{2,q} \). In this paper we generalize this equality to all \( p > 0 \).

**Proposition 3.1 ([37], Nikol’skij’s inequality)** Let \( \Omega \subset \mathbb{R}^n \) be a compact set, \( 0 < r \leq \infty \). Let us denote \( \sigma_r = n(1/(r \wedge 1) - 1/2) \) and assume that \( s > \sigma_r \). Then there exists a constant \( C > 0 \) such that

\[
\|\mathcal{F}^{-1} \varphi \mathcal{F} f\|_{r} \leq C \|\varphi\|_{H^s} \|f\|_{r}
\]

holds for all \( f \in L^{r}_{\Omega} := \{ f \in L^{r} : \text{supp}\hat{f} \subset \Omega \} \) and \( \varphi \in H^{s} \). Moreover, if \( r \geq 1 \), then the above inequality holds for all \( f \in L^{r} \).
Proposition 3.2 Let $0 < p, q \leq \infty$. Then

$$G_{1,p} = \bigcup_{s > 0} E^{s}_{p,q}. \quad (3.1)$$

Proof. We have

$$\| \partial^\alpha f \|_p \lesssim \sum_{k \in \mathbb{Z}^n} \| \Box_k \partial^\alpha f \|_p. \quad (3.2)$$

We easily see that

$$\| \Box_k \partial^\alpha f \|_p \lesssim \sum_{|l|_{\infty} \leq 1} \| \sigma_{k+l} \partial^\alpha \|_{M_p} \| \Box_k f \|_p. \quad (3.3)$$

Since $M_p$ is translation invariant, in view of Nikol’skij’s inequality we have

$$\| \sigma_k \xi \|_{M_p} = \| \sigma \cdot (\xi + k) \|_{M_p} \lesssim \| \sigma \cdot (\xi + k) \|_{H^L}, \quad L > (n/(1 \wedge p) - 1/2). \quad (3.4)$$

It is easy to calculate that

$$\| (\xi + k)^\alpha \sigma \|_{H^L} \lesssim \| (\xi + k)^\alpha \|_2 + \sum_{i=1}^n \| \partial^\alpha ((\xi + k)^\alpha \sigma) \|_2 \lesssim \prod_{i=1}^n (k_i)^{\alpha_i} + \sum_{i=1}^n \sum_{l_i=1}^L \prod_{m=0}^{l_i-1} (\alpha_i - m) \prod_{j=1,j \neq i}^n (k_j)^{\alpha_j} (k_i)^{\alpha_i-l_i}. \quad (3.5)$$

It follows that

$$\| \partial^\alpha f \|_p \lesssim \sum_{k \in \mathbb{Z}^n} \prod_{i=1}^n (k_i)^{\alpha_i} \| \Box_k f \|_p \lesssim \sum_{i=1}^n \sum_{l_i=1}^L \sum_{k \in \mathbb{Z}^n} \prod_{m=0}^{l_i-1} (\alpha_i - m) \prod_{j=1,j \neq i}^n (k_j)^{\alpha_j} (k_i)^{\alpha_i-l_i} \| \Box_k f \|_p. \quad (3.6)$$

If we can show that

$$\frac{s^{[\alpha]} \prod_{m=0}^{l_i-1} (\alpha_i - m) \prod_{j=1,j \neq i}^n (k_j)^{\alpha_j} (k_i)^{\alpha_i-l_i}}{\alpha!} e^{2s(|k_1| + \ldots + |k_n|)} \lesssim 1, \quad (3.6)$$

then together with $E^{s+\varepsilon}_{p,q} \subset E^{s}_{p,q}$ for all $s, \varepsilon > 0$ (cf. [42]), we immediately have

$$\| \partial^\alpha f \|_p \lesssim \frac{\alpha!}{s^{[\alpha]} \prod_{m=0}^{l_i-1} (\alpha_i - m) \prod_{j=1,j \neq i}^n (k_j)^{\alpha_j} (k_i)^{\alpha_i-l_i}} e^{2s(|k_1| + \ldots + |k_n|)} \| \Box_k f(x) \|_p \lesssim \frac{\alpha!}{s^{[\alpha]} \| f \|_{E^{s}_{p,1}}} \leq \frac{\alpha!}{s^{[\alpha]} \| f \|_{E^{s}_{p,q}}}. \quad (3.7)$$
for any $\hat{c} > c = 2n\log e$. Noticing that $s > 0$ is arbitrary, it follows from (3.7) that

$${\bigcup}_{s > 0} E^s_{p,q} \subset G_{1,p}. \quad (3.8)$$

Now we show (3.6). Using Taylor’s expansion, we see that

$$e^{2s|k_i|} \geq (2s|k_i|)^{\alpha_i - l_i}/(\alpha_i - l_i)!; \quad e^{2s|k_j|} \geq (2s|k_j|)^{\alpha_j}/\alpha_j!,$$  

(3.9)

from which we see that (3.6) holds.

Next, we show that $G_{1,p} \subset \bigcup_{s > 0} E^s_{p,\infty}$. We have

$$\|f\|_{E^s_{p,\infty}} = \sup_k e^{2s|k|}\|\Box_k f\|_p \lesssim \sup_k \sum_{m=0}^{\infty} (ns/m!)(|k|^m + \ldots + |k|^m)\|\Box_k f\|_p \quad (3.10)$$

If $|k_i| \leq 10$, we see that

$$\sum_{m=0}^{\infty} (ns/m!)|k|^m\|\Box_k f\|_p \lesssim \sum_{m=0}^{\infty} (10ns/m!)(1 + (m + L)^L)\|\Box_k f\|_p \quad (3.11)$$

If $|k_i| > 10$, in view of Nikol’skij’s inequality, for some $L > (n/(1 \wedge p) - 1/2),$

$$|k|^m\|\Box_k f\|_p \lesssim |k|^m\sum_{|\alpha| \leq 1} \|\sigma_{k+1}\xi^{m-\alpha}\|_{M_p}\|\Box_k \partial^{\alpha} f\|_p \lesssim 2^m(1 + (m + L)^L)\|\partial^{\alpha} f\|_p \quad (3.12)$$

By (3.10) and (3.12), if $f \in G_{1,p}$ and $|k_i| > 10$, then for any $s < \rho/2n,$

$$\sum_{m=0}^{\infty} (ns/m!)|k|^m\|\Box_k f\|_p \lesssim \sum_{m=0}^{\infty} (2ns/m!)(1 + (m + L)^L) \lesssim 1. \quad (3.13)$$

By (3.11) and (3.13), we have $f \in E^s_{p,\infty}$. It follows that $G_{1,p} \subset \bigcup_{s > 0} E^s_{p,\infty}$. Noticing that $E^s_{p,\infty} \subset E^{s/2}_{p,q}$, we immediately have $G_{1,p} \subset \bigcup_{s > 0} E^s_{p,q}$. \hfill $\Box$

4 Analyticity of NS: Proof of Theorem 1.2

In Section 2, the nonlinear mapping estimate is invalid for the case $p = \infty$, we cannot use the same way as in Section 2 to obtain the analyticity of the solutions.
if \( p = \infty \). Our idea is to use the frequency-uniform decomposition techniques to handle the case \( p = \infty \), where the corresponding space for the initial data should be modulation spaces. The local well posedness of NS in \( M^{-1}_{\infty,1} \) was established in [25].

Let \( 1 \leq p \leq \infty \). We show that there exists \( c > 0 \) (say \( 0 < c \leq 2^{-10} \)) such that

\[
\| \square_k U_2(t)f \|_p \lesssim e^{-2c|t|^2} \| \square_k f \|_p \tag{4.1}
\]

holds for all \( f \in L^p \) and \( k \in \mathbb{Z}^n \). Let \( \tilde{\sigma} : \mathbb{R}^n \to [0,1] \) be a smooth cut-off function satisfying \( \tilde{\sigma}(\xi) = 1 \) for \( |\xi| \leq 3/4 \) and \( \tilde{\sigma}(\xi) = 0 \) for \( |\xi| > 7/8 \). Let us observe that

\[
\| \square_k U_2(t)f \|_p = \| \mathcal{F}^{-1}\tilde{\sigma}(\xi - k)e^{-t|\xi|^2} \hat{k}f \|_p 
\lesssim \| \tilde{\sigma}(\xi - k)e^{-t|\xi|^2} \|_{M_p} \| \square_k f \|_p. \tag{4.2}
\]

In view of Nikol’skij’s inequality,

\[
\| \tilde{\sigma}(\xi - k)e^{-t|\xi|^2} \|_{M_p} \lesssim \| \tilde{\sigma}e^{-|\xi|+k|^2} \|_{H^L}, \quad L > n(1/(1 \wedge p) - 1/2). \tag{4.3}
\]

Since for \( |k| \geq 1 \),

\[
\partial_{\xi}^L (\tilde{\sigma}e^{P(\xi)}) = \sum_{\beta + \gamma = L} C_{L,\beta,\gamma} \partial_{\xi}^\beta \tilde{\sigma} \partial_{\xi}^\gamma e^{P(\xi)}, \tag{4.4}
\]

\[
|\partial_{\xi}^\gamma e^{P(\xi)}| \lesssim e^{P(\xi)} \sum_{\alpha_1 + \ldots + \alpha_q = \gamma} |\partial_{\xi}^{\alpha_1} P(\xi)\ldots\partial_{\xi}^{\alpha_q} P(\xi)|, \tag{4.5}
\]

we immediately have for \( |k| \geq 1 \),

\[
|\partial_{\xi}^\gamma e^{-t|\xi+k|^2}| \lesssim e^{-t|\xi+k|^2}/2, \quad \text{so,}
\]

\[
\| \tilde{\sigma}(\xi - k)e^{-t|\xi|^2} \|_{M_p} \lesssim e^{-t|k|^2}/2^L. \tag{4.6}
\]

We easily see that (4.6) also holds for \( k = 0 \). Hence, we have shown (4.1).

Now let \( 0 < t_0 \leq \infty \). We consider the estimate of \( \mathcal{A}_2 f \). By (4.1), we have

\[
\| \square_k \mathcal{A}_2 f \|_{L_t^p} \lesssim \int_0^t 2^{-2c(t-\tau)|k|^2} \| \square_k f(\tau) \|_{L_t^p} d\tau. \tag{4.7}
\]

It follows from (4.7) that

\[
2^{c|k|} \| \square_k \mathcal{A}_2 f \|_{L_t^p} \lesssim \int_0^t 2^{-c(t-\tau)|k|^2} 2^{c|k|} \| \square_k f(\tau) \|_{L_t^p} d\tau. \tag{4.8}
\]

By Young’s inequality, from (4.8) we have

\[
\| 2^{c|k|} \square_k \mathcal{A}_2 f \|_{L_{t_0}^p} \lesssim (k)^{-1} \| 2^{c|k|} \square_k f(\tau) \|_{L_{t_0}^p} \tag{4.9}
\]
Applying $\|\nabla k f\|_{L^p} \lesssim \langle k \rangle \|\nabla f\|_{L^p}$, we have
\begin{equation}
\|2^{s|k|} \nabla k \alpha_k f\|_{L^2([t_0,t]) L^p} \lesssim \|2^{s|k|} \nabla f\|_{L^1([t_0,t]) L^p}.
\end{equation}
(4.10)

For convenience, we denote for any $A \subset \mathbb{Z}^n$,
\begin{equation}
\|f\|_{\widetilde{L}^q(I; A; E_{p,q}^q)} = \left(\sum_{k \in A} 2^{s|k|} \|\nabla f\|_{L^q_{l=1} L^p_k}^q\right)^{1/q}.
\end{equation}
(4.11)

So, taking the sequence $l^q$ norm in both sides of (4.10), we have

**Proposition 4.1** Let $1 \leq p \leq \infty$. There exists a constant $c > 0$ ($0 < c \leq 2^{-10}$) such that for $0 < t_0 < \infty$ and $I = [0, t_0]$,
\begin{equation}
\|U_2(t) u_0\|_{L^2(I; A; E_{p,q}^q)} \lesssim \sum_{k \in \mathbb{Z}^n} \|\nabla k u_0\|_{L^p} , \quad A \subset \mathbb{Z}^n \setminus \{0\},
\end{equation}
(4.12)
\begin{equation}
\|\Box_0 U_2(t) u_0\|_{L^2_{l=1} L^p_k} \lesssim \|f\|_{L^1[I; A; E_{p,q}^q]} , \quad A \subset \mathbb{Z}^n \setminus \{0\},
\end{equation}
(4.13)
\begin{equation}
\|\nabla \alpha_k f\|_{L^2(I; A; E_{p,q}^q)} \lesssim \|f\|_{\widetilde{L}^q(I; A; E_{p,q}^q)} , \quad A \subset \mathbb{Z}^n \setminus \{0\},
\end{equation}
(4.14)
\begin{equation}
\|\Box_0 \alpha_k f\|_{L^2_{l=1} L^p_k} \lesssim \|f\|_{L^1[I; A; E_{p,q}^q]}
\end{equation}
(4.15)
holds for all $u_0 \in M_{p,1}^{-1}$ and $f \in \widetilde{L}^1(I; E_{p,1}^{ct})$.

Now we consider a nonlinear mapping estimate in $\widetilde{L}^q(I; E_{p,q}^{p(t)})$.

**Proposition 4.2** Let $1 \leq p, p_1, p_2, q, q_1, q_2, \tilde{q}, \tilde{q}_1, \tilde{q}_2 \leq \infty$, $1/p = 1/p_1 + 1/p_2$, $1/\tilde{q} = 1/\tilde{q}_1 + 1/\tilde{q}_2$ and $1/q = 1/q_1 + 1/q_2 - 1$, $s(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ and $I \subset \mathbb{R}_+$ with $\sup_{t \in I} s(t) < \infty$. Then we have
\begin{equation}
\|fg\|_{\widetilde{L}^q(I; E_{p,q}^{p(t)})} \lesssim \sup_{t \in I} 2^{s(t)} \|f\|_{\widetilde{L}^q_{l=1} I; E_{p_1,q_1}^{q_1}} \|g\|_{\widetilde{L}^q_{l=2} I; E_{p_2,q_2}^{q_2}}.
\end{equation}
(4.16)

**Proof.** By definition, we have
\begin{equation}
\|fg\|_{\widetilde{L}^q(I; E_{p,q}^{p(t)})} = \left(\sum_{k \in \mathbb{Z}^n} 2^{s(t)|k|} \|\nabla f\|_{L^q_{l=1} L^p_k}^q\right)^{1/q}.
\end{equation}
(4.17)

Using the fact
\begin{equation}
\Box_k (fg) = \sum_{i,j \in \mathbb{Z}^n} \Box_k (\Box_i f \Box_j g) , \quad \Box_k (\Box_i f \Box_j g) = 0 \text{ if } |k - i - j| > 4,
\end{equation}
(4.18)
by Hölder’s inequality, we have
\[
\|2^{s(t)}k|\Box_k(fg)\|_{L^q_{t\in I}L^2_x} \lesssim \sum_{i,j} \|2^{s(t)}k|\Box_k(\Box_i f \Box_j g)\|_{L^q_{t\in I}L^p_x} \chi_{|k-i-j| \leq 4}
\]
\[
\lesssim \sum_{|i| \leq 4} \sum_{j} \|2^{\tilde s(t)}|\Box_i f \Box_j g\|_{L^q_{t\in I}L^p_x} \|2^{s(t)|\Box_k-f|}g\|_{L^q_{t\in I}L^2_x}
\]
\[
\lesssim \sum_{|i| \leq 4} \sum_{j} \|2^{\tilde s(t)}|\Box_i f\|_{L^q_{t\in I}L^p_x} \|2^{s(t)|\Box_k-f|}g\|_{L^q_{t\in I}L^2_x}
\]
\[
\lesssim \sup_{t} 2^{4s(t)} \sum_{|i| \leq 4} \sum_{j} \|2^{s(t)|\Box_i f\|_{L^q_{t\in I}L^p_x} \times \|2^{s(t)|\Box_k-f|}g\|_{L^q_{t\in I}L^2_x}.
\]
(4.19)

Combining (4.17) and (4.19) and applying Young’s inequality, we immediately have the result, as desired. \hfill \Box

Now let us consider the map
\[
\mathcal{T} : u(t) \to U_2(t)u_0 + \mathcal{A}_2 \text{div}(u \otimes u).
\]
(4.20)

Let \( c = 2^{-10} \), \( I = [0,t_0] \). For any \( \delta > 0 \), one can choose \( J := J(u_0) > 0 \) satisfying \( \sum_{|k| > J} \|\Box_k u_0\|_p \leq \delta/4C \). By Proposition 4.1
\[
\|U_2(t)u_0\|_{L^2(I,\{k:|k| \geq J\};E^{\alpha}_{p,1})} \leq C \sum_{|k| \geq J} \|\Box_k u_0\|_p \leq \delta/4.
\]
(4.21)

On the other hand, one can choose \( I \) satisfying \( |I|^{1/2} \langle J \rangle \leq \delta/4C \|u_0\|_{M_{p,1}^{-1}} \), so,
\[
\|U_2(t)u_0\|_{L^2(I,\{k:|k| < J\};E^{\alpha}_{p,1})} \leq C \sum_{|k| < J} \|2^{-ct|k|^2} \Box_k u_0\|_p \|L^2_{t\in I}
\]
\[
\leq C |I|^{1/2} \langle J \rangle \sum_{|k| < J} \langle k \rangle^{-1} \|\Box_k u_0\|_p \leq \delta/4.
\]
(4.22)

Hence, in view of (4.21) and (4.22),
\[
\|U_2(t)u_0\|_{L^2(I;E^{\alpha}_{p,1})} \leq \delta/2.
\]
(4.23)

Denote \( \widetilde \sigma_0 = \mathcal{T}^{-1} \sigma \mathcal{F} \). Noticing that \( \mathcal{A}_0 \text{div} : L^p \to L^p \) for all \( p \in [1, \infty] \), we have from Propositions 4.1 and 4.2 that
\[
\|\mathcal{A}_2 \text{div}(u \otimes u)\|_{L^2(I;E^{\alpha}_{p,1})} \lesssim (1 + |I|^{1/2}) \|u \otimes u\|_{L^2(I;E^{\alpha}_{p,1})}
\]
\[
\lesssim (1 + |I|^{1/2}) 2^{4ct_0} \|u\|_{L^2(I;E^{\alpha}_{p,1})} \|u\|_{L^2(I;E^{\alpha}_{p,1})}
\]

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\[
\lesssim (1 + |I|^{1/2}) 2^{4c_{10}} \|u\|_{L^2(I; E_{p,1}^{ct})}^2. \tag{4.24}
\]

We can assume that \(|I| \leq 1\). Hence, collecting (4.23) and (4.24), we have
\[
\|\mathcal{T}u\|_{L^2(I; E_{p,1}^{ct})} \lesssim \delta/2 + C \|u\|_{L^2(I; E_{p,1}^{ct})}^2. \tag{4.25}
\]

Now we can fix \(\delta\) verifying \(C\delta \leq 1/4\). Put
\[
\mathcal{D} = \{u \in \tilde{L}^2(I; E_{p,1}^{ct}) : \|u\|_{L^2(I; E_{p,1}^{ct})} \leq \delta\} \tag{4.26}
\]

We have \(\mathcal{T}u \in \mathcal{D}\) if \(u \in \mathcal{D}\) and
\[
\|\mathcal{T}u - \mathcal{T}v\|_{L^2(I; E_{p,1}^{ct})} \lesssim \frac{1}{2}\|u - v\|_{L^2(I; E_{p,1}^{ct})}, \quad u, v \in \mathcal{D}. \tag{4.27}
\]

Next, we show that
\[
\sum_k \langle k \rangle^{-1}\|2^{ct|k|} \Box_k u\|_{L^\infty_{t \in (0, t_0)} L^p_x} \lesssim \|u_0\|_{M_{p,1}^{-1}} + \delta^2. \tag{4.28}
\]

In order to show (4.28), we need the following

**Proposition 4.3** Let \(1 \leq p \leq \infty\). There exists a constant \(c > 0\) (0 < \(c \leq 2^{-10}\)) such that for \(0 < t_0 < \infty\) and \(I = [0, t_0]\),
\[
\sum_k \langle k \rangle^{-1}\|2^{ct|k|} \Box_k U_2(t) u_0\|_{L^\infty_{t \in (0, t_0)} L^p_x} \lesssim \|u_0\|_{M_{p,1}^{-1}}, \tag{4.29}
\]
\[
\sum_{k \neq 0} \langle k \rangle^{-1}\|2^{ct|k|} \Box_k \varphi^2 \nabla f\|_{L^\infty_{t \in (0, t_0)} L^p_x} \lesssim \|f\|_{L^1(I; E_{p,1}^{ct})} \tag{4.30}
\]

holds for all \(u_0 \in M_{p,1}^{-1}\) and \(f \in \tilde{L}^1(I; E_{p,1}^{ct})\).

**Proof.** In view of (4.1), we immediately have (4.29) and (4.30). \(\square\)

Applying Proposition 4.3 and noticing that \(\Box_0 \text{div} : L^p \rightarrow L^p\) for all \(p \in [1, \infty]\), we have
\[
\sum_k \langle k \rangle^{-1}\|2^{ct|k|} \Box_k u\|_{L^\infty_{t \in (0, t_0)} L^p_x} \lesssim \|u_0\|_{M_{p,1}^{-1}} + \|u \otimes u\|_{L^1(I; E_{p,1}^{ct})} \lesssim \|u_0\|_{M_{p,1}^{-1}} + 2^{4c_{10}} \|u\|_{L^2(I; E_{p,1}^{ct})}^2, \tag{4.31}
\]

which implies (4.28).

We now extend the solution from \(I = [0, t_0]\) to \(I_1 = [t_0, t_1]\) for some \(t_1 > t_0\) and consider the mapping
\[
\mathcal{R}_1 : u(t) \rightarrow U_2(t - t_0) u(t_0) + \int_{t_0}^t U_2(t - \tau) \text{div}(u \otimes u)(\tau) d\tau. \tag{4.32}
\]
\mathcal{D}_1 = \{ u \in \tilde{L}^2(I_1; E_{p,1}^t) : \| u \|_{\tilde{L}^2(I_1; E_{p,1}^t)} \leq \delta_1 \}, \quad (4.33)

where \( \delta_1 \) will be chosen below. Taking \( t = t_0 \) in (4.28), one has that
\[
\sum_k \langle k \rangle^{-1} \| 2^{ct_0k} \delta_k u \|_p \lesssim \| u_0 \|_{M_{p,1}} + \delta^2.
\] (4.34)

For any \( \delta_1 > 0 \), in view of (4.34), we can choose a sufficiently large \( J \) such that
\[
C \sum_{|k| > J} \langle k \rangle^{-1} \| 2^{ct_0k} \delta_k u \|_p \leq \delta_1/4.
\] (4.35)

Hence, in view of Proposition 4.1,
\[
\| U_2(t - t_0)u(t_0) \|_{\tilde{L}^2(I_1; (k:|k| > J); E_{p,1}^t)} \leq C \sum_{|k| > J} \langle k \rangle^{-1} \| 2^{ct_0k} \delta_k u \|_p \leq \delta_1/4,
\] (4.36)

and one can choose \( t_1 > t_0 \) verifying \( C|I_1|^{1/2}\langle J \rangle (\| u_0 \|_{M_{p,1}} + \delta^2) \leq \delta_1/4 \), so,
\[
\| U_2(t - t_0)u(t_0) \|_{\tilde{L}^2(I_1; (k:|k| \leq J); E_{p,1}^t)} \leq C|I_1|^{1/2} \sum_{|k| \leq J} \| 2^{ct_0k} \delta_k u \|_p
\leq C|I_1|^{1/2}\langle J \rangle \sum_{|k| \leq J} \langle k \rangle^{-1} \| 2^{ct_0k} \delta_k u \|_p
\leq C|I_1|^{1/2}\langle J \rangle (\| u_0 \|_{M_{p,1}} + \delta^2) \leq \delta_1/4. \quad (4.37)
\]

In view of (4.36) and (4.37),
\[
\| U_2(t - t_0)u(t_0) \|_{\tilde{L}^2(I_1; E_{p,1}^t)} \leq \delta_1/2, \quad (4.38)
\]

It follows from Propositions 4.1 and 4.2
\[
\left\| \int_{t_0}^t U_2(t - \tau)\mathbb{P}\text{div}(u \otimes u)(\tau) d\tau \right\|_{\tilde{L}^2(I_1; E_{p,1}^t)} \leq C(1 + |I_1|^{1/2}) \| u \otimes u \|_{\tilde{L}^1(I_1; E_{p,1}^t)}
\leq C(1 + |I_1|^{1/2}) 2^{4ct_1} \| u \|_{\tilde{L}^2(I_1; E_{p,1}^t)}^2. \quad (4.39)
\]

We can assume that \( t_1 \leq 2 \). Hence, collecting (4.38) and (4.39), we have
\[
\| \mathcal{R}_1 u \|_{\tilde{L}^2(I_1; E_{p,1}^t)} \leq \delta_1/2 + C2^{8c} \| u \|_{\tilde{L}^2(I_1; E_{p,1}^t)}^2. \quad (4.40)
\]

Now we can choose \( \delta_1 > 0 \) satisfying
\[
C2^{8c}\delta_1 \leq 1/4. \quad (4.41)
\]

It follows that \( \mathcal{R}_1 u \in \mathcal{D}_1 \) for any \( u \in \mathcal{D}_1 \). So, we have extended the solution from \([0, t_0] \) to \([t_0, t_1] \). Noticing that
\[
\sum_k \langle k \rangle^{-1} \| 2^{ct_0k} \delta_k U_2(t)u_0 \|_{L^p_{t \in [0, t_1]} L^p} \lesssim \sum_k \langle k \rangle^{-1} \| 2^{ct_0k} \delta_k u_0 \|_p, \quad (4.42)
\]
we easily see that the solution can be extended to \([t_1, t_2], [t_2, t_3], \ldots \) and finally find a \( T_{\text{max}} > 0 \) verifying the conclusions of Theorem 1.2.
5 Well-posedness of GNS: $\alpha = 1/2$

In this section we give some time-space estimates for the solutions of the linear evolution equation

$$u_t + (-\Delta)^{1/2} u = f, \quad u(0, x) = u_0.$$  \hfill (5.1)

In the following, we will set up some estimates for $U_1(t)$ using exponential decay property of $e^{-t|\xi|}$ together with the dyadic decomposition.

**Lemma 5.1** Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $1 \leq \gamma \leq \infty$. Then

$$\|U_1(t)f\|_{L_\gamma^\gamma(R^n; \dot{B}^{s-1/\gamma}_{p,q})} \lesssim \|f\|_{\dot{B}^{s-1/\gamma}_{p,q}}.$$  \hfill (5.2)

Proof. By Nikol’skij’s inequality, we easily see that for some $L > n/2$

$$\|\Delta_j U_1(t)f\|_{L^p} \lesssim \|\varphi(\xi) e^{-c2^j|\xi|}\|_{M_p} \|f\|_{L^p} \lesssim \|\varphi e^{-c2^j|\xi|}\|_{H^s} \|f\|_{L^p} \lesssim e^{-c2^j} \|f\|_{L^p},$$

Note that $\Delta_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$, hence we have

$$\|\Delta_j U_1(t)f\|_{L^p} \lesssim e^{-c2^j} \|\Delta_j f\|_{L^p}. \quad (5.3)$$

Taking $L^\gamma_t$ norm on inequality \(5.3\), we get

$$\|\Delta_j U_1(t)f\|_{L^\gamma_t L^p} \lesssim e^{-c2^j} \|\Delta_j f\|_{L^\gamma_t L^p} \lesssim 2^{-j/\gamma} \|\Delta_j f\|_{L^p}.$$  \hfill (5.4)

Multiplying \(5.3\) by $2^{js}$ and taking $l^s$ norm, we obtain the results as desired. \qed

**Lemma 5.2** Let $1 \leq p, q \leq \infty$, $1 \leq \gamma_1 \leq \gamma \leq \infty$. Then

$$\|\mathcal{A}_1 f\|_{L_\gamma^\gamma(R^n; \dot{B}^{s-1/\gamma}_{p,q})} \lesssim \|f\|_{L_\gamma^\gamma(R^n; \dot{B}^{s-1/\gamma_1}_{p,q})}.$$  \hfill (5.5)

Proof. By Lemma 5.1, we have

$$\|\Delta_j \mathcal{A}_1 f\|_{L^p} \lesssim \int_0^t \|\Delta_j U_1(t-\tau)f(\tau, x)\|_{L^p} d\tau \lesssim \int_0^t e^{-c(t-\tau)^2} \|\Delta_j f(\tau)\|_{L^p} d\tau.$$  \hfill (5.6)

Taking $L^\gamma_t$ norm on time variable and using Young inequality,

$$\|\Delta_j \mathcal{A}_1 f\|_{L^\gamma_t L^p} \lesssim 2^{-j(1+1/\gamma-1/\gamma_1)} \|\Delta_j f\|_{L_\gamma^\gamma_t L^p}. \quad (5.7)$$

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Multiplying (5.4) by \(2^{js}\) and taking \(l'\) norm, we obtain the results.

Below, we prove the existence part of Theorem 1.3 and the analyticity part is left into the next section. We divide the proof into two cases.

Case I: \(1 \leq p < \infty\). We construct the resolution space as follows:

\[
\mathcal{D} = \{ u \in \mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^n) : \| u \|_{L^\infty(\mathbb{R}^+; \dot{B}_{p,1}^{n/p})} \leq 2C \| u_0 \|_{\dot{B}_{p,1}^{n/p}} \}
\]

with metric

\[
d(u, v) = \| u - v \|_{L^\infty(\mathbb{R}^+; \dot{B}_{p,1}^{n/p})}.
\]

Assume that \(u_0 \in \dot{B}_{p,1}^{n/p}\) is sufficiently small such that \(4C \| u_0 \|_{\dot{B}_{p,1}^{n/p}} = 1/2\), where \(C\) is the largest constant appeared in the following inequalities. Considering the mapping

\[
\mathcal{F} : u(t) \rightarrow U_1(t)u_0 + \mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u),
\]

we will verify that \(\mathcal{F} : (\mathcal{D}, d) \rightarrow (\mathcal{D}, d)\) is a contraction mapping. In fact, by Lemma 5.1 and Lemma 5.2, we have

\[
\| \mathcal{F} u \|_{L^\infty(\mathbb{R}^+; \dot{B}_{p,1}^{n/p})} \lesssim \| u_0 \|_{\dot{B}_{p,1}^{n/p}} + \| u \otimes u \|_{L^\infty(\mathbb{R}^+; \dot{B}_{p,1}^{n/p})}.
\]

(5.5)

We will use the decomposition (2.10) to deal with the estimates of \(\| u \otimes u \|_{L^\infty(\mathbb{R}^+; \dot{B}_{p,1}^{n/p})}\). We have

\[
\| \Delta_j (u \otimes u) \|_{L^p_x} = \left\| \sum_{j \leq k+3} \Delta_j (S_{k-1} u \Delta_k u + S_k u \Delta_k u) \right\|_{L^p_x}
\]

\[
\lesssim \sum_{j \leq k+3} \| S_k u \|_{L^\infty_x} \| \Delta_k u \|_{L^p_x}
\]

\[
\lesssim \sum_{j \leq k+3} \sum_{l \leq k} \| \Delta_l u \|_{L^\infty_x} \| \Delta_k u \|_{L^p_x}
\]

\[
\lesssim \sum_{j \leq k+3} \sum_{l \leq k} 2^{ln/p} \| \Delta_l u \|_{L^\infty_x} \| \Delta_k u \|_{L^p_x},
\]

(5.6)

where we used Bernstein’s inequality in the last estimate. Hence taking \(L^\infty_t\) norm to (5.6), we obtain

\[
\| \Delta_j (u \otimes u) \|_{L^\infty_t L^p_x} \lesssim \sum_{k \geq j-3} \sum_{l \leq k} 2^{ln/p} \| \Delta_l u \|_{L^\infty_t L^p_x} \| \Delta_k u \|_{L^\infty_t L^p_x}
\]

\[
\lesssim \| u \|_{L^\infty(\mathbb{R}^+; \dot{B}_{p,1}^{n/p})} \sum_{k \geq j-3} \| \Delta_k u \|_{L^\infty_t L^p_x}.
\]

(5.7)
Multiplying (5.7) by $2^{jn/p}$,

$$2^{jn/p} \| \Delta_j(u \otimes u) \|_{L^2(L^p_x)} \lesssim \| u \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)} \sum_{k \geq j-3} 2^{-(k-j)n/p} 2^{kn/p} \| \Delta_k u \|_{L^2(L^p_x)}.$$  \hspace{1cm} (5.8)

Taking $l^1$ norm in (5.8) and then using Young’s inequality,

$$\| u \otimes u \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)} \lesssim \| u \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)}^2.$$  \hspace{1cm} (5.9)

Inserting (5.9) into (5.5), we get

$$\| u \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)} \lesssim \| u \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)}^2.$$  \hspace{1cm} (5.10)

which means that $\mathcal{T} : \mathcal{D} \rightarrow \mathcal{D}$. Similarly,

$$\| \mathcal{T} u - \mathcal{T} v \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)} \leq \frac{1}{2} \| u - v \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)}.$$  

Hence there exists a $u \in \tilde{L}^\infty(\mathbb{R}^+;\dot{B}_p^1)$ satisfying

$$u(t) = U_1(t)u_0 + \mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u)$$

with

$$\| u \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)} \lesssim \| u_0 \|_{B_{p,1}^n/p}.$$  

In particular, we have

$$\| u \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)} \lesssim \| u \|_{L^\infty(\mathbb{R}^+;B_{p,1}^n/p)} \lesssim \| u_0 \|_{B_{p,1}^n/p}.$$  

The uniqueness is similar.

Case II: $p = \infty$. We now consider the following resolution space:

$$\mathcal{D}_1 = \{ u \in \mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^n) : \| u \|_{L^\infty(\mathbb{R}^+;B_{\infty,1}^0)} \cap \tilde{L}^1(\mathbb{R}^+;B_{\infty,1}^1) \leq 2C \| u_0 \|_{B_{\infty,1}^0} \}$$

with metric

$$d_1(u, v) = \| u - v \|_{L^\infty(\mathbb{R}^+;B_{\infty,1}^0) \cap \tilde{L}^1(\mathbb{R}^+;B_{\infty,1}^1)}.$$  

Suppose that $u_0 \in \dot{B}_{\infty,1}^0$ is sufficiently small such that $4C \| u_0 \|_{B_{\infty,1}^0} = 1/2$. We will verify that $\mathcal{T} : (\mathcal{D}_1, d_1) \rightarrow (\mathcal{D}_1, d_1)$ is a contractive map. Using Lemma 5.1 and Lemma 5.2

$$\| \mathcal{T} u \|_{L^\infty(\mathbb{R}^+;\dot{B}_{\infty,1}^0) \cap \tilde{L}^1(\mathbb{R}^+;B_{\infty,1}^1)} \lesssim \| u_0 \|_{B_{\infty,1}^0} + \| u \otimes u \|_{\tilde{L}^1(\mathbb{R}^+;B_{\infty,1}^1)},$$  \hspace{1cm} (5.10)
From (5.10), we see that
\[ \| \Delta_j(u \otimes u) \|_{L^\infty} \lesssim \sum_{k \geq j-3} \sum_{l \leq k} \| \Delta_l u \|_{L^\infty} \| \Delta_k u \|_{L^\infty}, \]  
(5.11)

Taking $L^1$ norm on (5.11), we obtain
\[ \| \Delta_j(u \otimes u) \|_{L^1L^\infty} \lesssim \| u \|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1})} \sum_{k \geq j-3} \| \Delta_k u \|_{L^1_t L^\infty_x}. \]  
(5.12)

Multiplying (5.12) by $2^j$ and then taking $L^1$ norm on (5.12),
\[ \| u \otimes u \|_{L^1(\mathbb{R}^+; \dot{B}^{1}_{\infty,1})} \lesssim \| u \|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1})} \left( \sum_{k \geq j-3} 2^{-(k-j)} 2^k \| \Delta_k u \|_{L^1_t L^\infty_x} \right)_{L^1_j}. \]

Therefore, in view of Young’s inequality, we have
\[ \| u \otimes u \|_{L^1(\mathbb{R}^+; \dot{B}^{1}_{\infty,1})} \lesssim \| u \|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1})} \| u \|_{L^1(\mathbb{R}^+; \dot{B}^{1}_{\infty,1})}. \]  
(5.13)

Combining (5.13) with (5.10), we get for any $u \in \mathcal{D}_1$,
\[
\| \mathcal{F} u \|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1}) \cap \tilde{L}^1(\mathbb{R}^+; \dot{B}^{1}_{\infty,1})} \lesssim \| u_0 \|_{\dot{B}^{0}_{\infty,1}} + \| u \|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1})} \| u \|_{\tilde{L}^1(\mathbb{R}^+; \dot{B}^{1}_{\infty,1})} \\
\leq C \| u_0 \|_{\dot{B}^{0}_{\infty,1}} + 4C^2 \| u_0 \|_{\dot{B}^{0}_{\infty,1}} \| u_0 \|_{\dot{B}^{0}_{\infty,1}} < 2C \| u_0 \|_{\dot{B}^{0}_{\infty,1}},
\]
which means that $\mathcal{F} : \mathcal{D}_1 \to \mathcal{D}_1$ and there exists a solution $u \in \tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1}) \cap \tilde{L}^1(\mathbb{R}^+; \dot{B}^{1}_{\infty,1})$ to (1.1) satisfying
\[ \| u \|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1}) \cap \tilde{L}^1(\mathbb{R}^+; \dot{B}^{1}_{\infty,1})} \lesssim \| u_0 \|_{\dot{B}^{0}_{\infty,1}}. \]

In particular, we have
\[ \| u \|_{L^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1})} \lesssim \| u \|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1})} \lesssim \| u_0 \|_{\dot{B}^{0}_{\infty,1}}. \]

The uniqueness is similar, we omit the details here. \(\square\)

**Remark 5.3** Since $\tilde{L}^\infty(\mathbb{R}^+; \dot{B}^{0}_{\infty,1})$ is not a Banach algebra, the method for the nonlinear estimates in case $1 \leq p < \infty$ doesn’t work for the case $p = \infty$.

## 6 Analyticity of GNS: $\alpha = 1/2$

First, we show that there exists $c > 0$ (say $0 < c \leq 2^{-5}$) such that
\[ \| \Box_k U_1(t)f \|_p \lesssim e^{-2\epsilon t|k|} \| \Box_k f \|_p \]  
(6.1)
holds for all \( f \in L^p \) and \( k \in \mathbb{Z}^n \). Let \( \hat{\sigma} : \mathbb{R}^n \to [0, 1] \) be the same cut-off function as in Section 3. In view of Nikol’skij’s inequality, we have

\[
||\square_k U_1(t)f||_p \lesssim ||\hat{\sigma}(\xi - k)e^{-t|\xi|}||_{M_p}||\square_k f||_p \\
\lesssim ||\hat{\sigma}e^{-t|\xi+k|}||_{H^L}||\square_k f||_p
\]  

(6.2)

for any \( L > n(1/(1 \wedge p) - 1/2) \). Similar to (6.1) and (6.5), we have

\[
|\partial^L_\xi (\hat{\sigma}e^{-t|\xi+k|})| \lesssim e^{-t|\xi+k|/2} \chi_{[-7/8, 7/8]^n}(\xi),
\]

we immediately have for \( |k| \geq 1 \),

\[
||\hat{\sigma}(\xi - k)e^{-t|\xi|}||_{M_p} \lesssim e^{-t|k|/16}.
\]  

(6.4)

We easily see that (6.4) also holds for \( k = 0 \). Hence, we have shown that

**Proposition 6.1** Let \( 0 < p \leq \infty \). There exists a constant \( c > 0 \) (say \( 0 < c \leq 2^{-5} \)) such that

\[
||\square_k U_1(t)f||_p \lesssim 2^{-2c|k|}||\square_k f||_p
\]  

(6.5)

uniformly holds for all \( k \in \mathbb{Z}^n \) and \( f \in L^p \).

Now let \( 0 < t_0 \leq \infty \). We consider the estimate of \( \mathcal{A}_1 f \). Applying Proposition 6.1, we have

\[
||\square_k \mathcal{A}_1 f||_{L^p_t} \lesssim \int_0^t 2^{-2c(t-\tau)|k|}||\square_k f(\tau)||_{L^p_t} d\tau.
\]  

(6.6)

It follows from (6.6) that

\[
2^{ct|k|}||\square_k \mathcal{A}_1 f||_{L^p_t} \lesssim \int_0^t 2^{-c(t-\tau)|k|/2}||\square_k f(\tau)||_{L^p_t} d\tau.
\]  

(6.7)

For \( |k| \geq 1 \) and \( q \geq \tilde{q}_1 \), using Young’s inequality, we have from (6.7) that

\[
||2^{ct|k|}||\square_k \mathcal{A}_1 f||_{L^p_t} \lesssim \langle k \rangle^{1/\tilde{q}_1 - 1/\tilde{q} - 1}||2^{ct|k|}||\square_k f||_{L^p_t} \lesssim \langle k \rangle^{1/\tilde{q}_1 - 1/\tilde{q} - 1}||2^{ct|k|}||\square_k f||_{L^p_t}.
\]  

(6.8)

So, taking the sequence \( L^q \) norm in both sides of (6.8), we have

**Proposition 6.2** Let \( 1 \leq p \leq \infty \), \( 1 \leq \tilde{q}_1 \leq \tilde{q} \leq \infty \). There exists a constant \( c > 0 \) (say, \( c \leq 2^{-5} \)) such that for \( 0 < t_0 \leq \infty \), \( f = [0, t_0) \) and \( \Lambda \subset \mathbb{Z}^n \setminus \{0\} \),

\[
||\mathcal{A}_1 f||_{L^q(\Lambda, L^p_{t_0})} \lesssim \sum_{k \in \Lambda} \langle k \rangle^{1/\tilde{q}_1 - 1/\tilde{q} - 1}||2^{ct|k|}||\square_k f||_{L^p_t} \lesssim \sum_{k \in \Lambda} \langle k \rangle^{1/\tilde{q}_1 - 1/\tilde{q} - 1}||2^{ct|k|}||\square_k f||_{L^p_t}.
\]  

(6.9)
holds for all $f \in \tilde{L}^2(I; E_{p,q}^{ct})$. In particular, we have

$$\left\| \nabla \mathcal{A}_f \right\|_{\tilde{L}^1(I; E_{p,q}^{ct})} \lesssim \left\| f \right\|_{\tilde{L}^1(I; E_{p,q}^{ct})},$$

(6.10)

$$\left\| \mathcal{A}_f \right\|_{\tilde{L}^\infty(I; E_{p,q}^{ct})} \lesssim \left\| f \right\|_{\tilde{L}^1(I; E_{p,q}^{ct})},$$

(6.11)

**Proof of iv) of Theorem 6.3** Denote $I = [0, 1]$. Put

$$\|u\|_X = \|u\|_{\tilde{L}^1(I; \mathbb{B}_{p,q}^{1}) \cap \tilde{L}^\infty(I; \mathbb{B}_{p,q}^{0})}, \quad \|u\|_Y = \|\nabla u\|_{\tilde{L}^1(I; E_{p,q}^{ct})} + \|u\|_{\tilde{L}^\infty(I; E_{p,q}^{ct})},$$

(6.12)

and

$$\mathcal{D} = \{u : \|u\|_X + \|u\|_Y \leq \delta\}, \quad d(u, v) = \|u - v\|_{X \cap Y}.$$  

(6.13)

We will show that

$$\mathcal{T} : u(t) \rightarrow U_1(t)u_0 + \mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u)$$

is a contraction mapping from $(\mathcal{D}, d)$ into itself. In Section 5 we have shown that

$$\|\mathcal{T} u\|_X \leq C \|u_0\|_{\tilde{B}_{p,q}^{0}} + C \|u\|^2_X.$$  

(6.15)

So, it suffices to estimate $\|\mathcal{T} u\|_Y$. Put $c = 2^{-5}$. By Proposition 6.1,

$$\|\nabla U_1(t)u_0\|_{\tilde{L}^1(I; E_{p,q}^{ct})} + \|U_1(t)u_0\|_{\tilde{L}^\infty(I; E_{p,q}^{ct})} \lesssim C \|u_0\|_{\mathcal{M}_{p,q}^{0}}.$$  

(6.16)

Since $\|\Box_0 f\|_\infty \lesssim \|f\|_\infty \lesssim \|f\|_{\tilde{B}_{p,q}^{0}}$, we have from Section 5 that

$$\|\Box_0 \nabla \mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u)\|_{L^1_{t,x} \cap L^\infty_{t,x}} + \|\Box_0 \mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u)\|_{L^1_{t,x} \cap L^\infty_{t,x}} \lesssim \|\mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u)\|_{L^1(I; \mathbb{B}_{p,q}^{1}) \cap \tilde{L}^\infty(I; \mathbb{B}_{p,q}^{0})} \lesssim \|u\|^2_X.$$  

(6.17)

Therefore, it suffices to consider the estimate of $\nabla \mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u)$ in the space $\tilde{L}^1(I, \mathbb{Z}_+; E_{p,q}^{ct})$ and $\mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u)$ in the space $\tilde{L}^\infty(I, \mathbb{Z}_+; E_{p,q}^{ct})$ for $\mathbb{Z}_+ = \mathbb{Z} \setminus \{0\}$. It follows from Propositions 6.2 and 4.2 that

$$\|\nabla \mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u)\|_{L^1(I; \mathbb{Z}_+; E_{p,q}^{ct})} + \|\mathcal{A}_1 \mathcal{P} \text{div}(u \otimes u)\|_{L^\infty(I; \mathbb{Z}_+; E_{p,q}^{ct})} \lesssim \|\nabla (u \otimes u)\|_{\tilde{L}^1(I; E_{p,q}^{ct})} \lesssim \|\nabla u\|_{\tilde{L}^1(I; E_{p,q}^{ct})} \lesssim \|u\|_{\tilde{L}^\infty(I; E_{p,q}^{ct})}.$$  

(6.18)

Hence, collecting (6.16) and (6.18), we have

$$\|\mathcal{T} u\|_Y \leq C \|u_0\|_{\mathcal{M}_{p,q}^{0}} + C \|\nabla u\|_{\tilde{L}^1(I; E_{p,q}^{ct})} \|u\|_{\tilde{L}^\infty(I; E_{p,q}^{ct})},$$

(6.19)

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By \((6.15)\) and \((6.19)\)

\[
\|\mathcal{T}u\|_X + \|\mathcal{T}u\|_Y \leq C\|u_0\|_{\tilde{B}^{0}_{\infty,1} \cap M^{0}_{\infty,1}} + C\|u\|_X^2 + C\|u\|_Y^2. 
\]  

\((6.20)\)

Assume that

\[
C\delta \leq 1/4, \quad C\|u_0\|_{\tilde{B}^{0}_{\infty,1} \cap M^{0}_{\infty,1}} \leq \delta/2.
\]  

\((6.21)\)

Using the above contraction mapping argument, we obtain that there exists a solution of the integral equation \(\mathcal{T}u = u\) in the space \(X \cap Y\). In particular, we have

\[
\|u\|_{\tilde{L}^{\infty}(0,1;\tilde{E}^t_{\infty,1})} \leq \delta.
\]  

\((6.22)\)

In particular, we have for \(u(1) = u(1, x)\),

\[
\|u(1)\|_{\tilde{E}^1_{\infty,1}} \leq \delta, \quad c = 2^{-5}.
\]  

\((6.23)\)

We now extend the regularity of solutions. Let us consider the mapping

\[
\mathcal{T}_1 : u(t) \rightarrow U_1(t - 1)u(1) + \int_1^t U_1(t - \tau)\mathbb{P}\text{div}(u \otimes u)(\tau)d\tau.
\]  

\((6.24)\)

Let \(I_1 = [1, \infty)\) and

\[
\|u\|_{X_1} = \|u\|_{\tilde{L}^1(I_1;\tilde{B}^{0}_{\infty,1}) \cap \tilde{L}^{\infty}(I_1;\tilde{B}^{0}_{\infty,1})}, \quad \|u\|_{Y_1} = \|\nabla u\|_{\tilde{L}^1(I_1;\tilde{E}^1_{\infty,1})} + \|u\|_{\tilde{L}^{\infty}(I_1;\tilde{E}^1_{\infty,1})}.
\]  

\((6.25)\)

\[
\mathcal{D}_1 = \{u : \|u\|_{X_1} + \|u\|_{Y_1} \leq \delta\}, \quad d_1(u, v) = \|u - v\|_{X_1 \cap Y_1}.
\]  

\((6.26)\)

We use the same notation \(\mathbb{Z}_n^* = \mathbb{Z}^n \setminus \{0\}\) as above. Using the same way as in \((6.17)\), we see that

\[
\|u\|_{Y_1} \lesssim \|u\|_{X_1} + \|\nabla u\|_{\tilde{L}^1(I_1;\mathbb{Z}_n^*;\tilde{E}^1_{\infty,1})} + \|u\|_{\tilde{L}^{\infty}(I_1;\mathbb{Z}_n^*;\tilde{E}^1_{\infty,1})}.
\]  

\((6.27)\)

In section \([5]\), we have shown that

\[
\|\mathcal{T}_1 u\|_{X_1} \leq C\|u_0\|_{\tilde{B}^{0}_{\infty,1}} + C\|u\|_{X_1}^2.
\]  

\((6.28)\)

So, it suffices to estimate \(\nabla \mathcal{T}_1 u\) in the space \(\tilde{L}^1(I_1;\mathbb{Z}_n^*;\tilde{E}^1_{\infty,1})\) and \(\mathcal{T}_1 u\) in the space \(\tilde{L}^{\infty}(I_1;\mathbb{Z}_n^*;\tilde{E}^1_{\infty,1})\). Put \(c = 2^{-5}\). By Proposition \([6.1]\)

\[
\|\nabla U_1(t - 1)u(1)\|_{\tilde{L}^1(I_1;\mathbb{Z}_n^*;\tilde{E}^1_{\infty,1})} + \|U_1(t - 1)u(1)\|_{\tilde{L}^{\infty}(I_1;\mathbb{Z}_n^*;\tilde{E}^1_{\infty,1})} \leq C\|u(1)\|_{\tilde{E}^1_{\infty,1}}.
\]  

\((6.29)\)
By (6.31) and Young’s inequality,
\[
\left\| \nabla \int_1^t U_1(t - \tau) f(\tau) d\tau \right\|_{\overline{L}^1(I_1, Z^c_{E_{1,1}})} \lesssim \left\| f \right\|_{\overline{L}^1(I_1, Z^c_{E_{1,1}})} \tag{6.30}
\]
\[
\left\| \int_1^t U_1(t - \tau) f(\tau) d\tau \right\|_{\overline{L}^\infty(I_1, Z^c_{E_{1,1}})} \lesssim \left\| f \right\|_{\overline{L}^1(I_1, Z^c_{E_{1,1}})}. \tag{6.31}
\]
Therefore, by (6.30) and Proposition 4.2 we have
\[
\left\| \nabla \int_1^t U_1(t - \tau) \mathbb{P} \text{div}(u \otimes u)(\tau) d\tau \right\|_{\overline{L}^1(I_1, Z^c_{E_{1,1}})} \lesssim \left\| \mathbb{P} \text{div}(u \otimes u) \right\|_{\overline{L}^1(I_1, Z^c_{E_{1,1}})} \lesssim \left\| \text{div}(u \otimes u) \right\|_{\overline{L}^1(I_1, Z^c_{E_{1,1}})} \lesssim \left\| \nabla u \right\|_{\overline{L}^1(I_1, E_{1,1})} \left\| u \right\|_{\overline{L}^\infty(I_1, E_{1,1})} \lesssim \left\| u \right\|_{Y_1}^2. \tag{6.32}
\]
By (6.31) we see that \( \left\| \int_1^t U_1(t - \tau) \mathbb{P} \text{div}(u \otimes u)(\tau) d\tau \right\|_{\overline{L}^\infty(I_1, Z^c_{E_{1,1}})} \) also has the upper-bound as in (6.32). Hence, collecting (6.23), (6.28), (6.29) and (6.32), we have
\[
\left\| \mathcal{I}_1 u \right\|_{Y_1} \lesssim C \left\| u_0 \right\|_{\dot{B}^0_{\infty,1}} + C \left\| u(1) \right\|_{E^c_{1,1}} + C \left\| u \right\|_{X_1} + C \left\| u\right\|_{Y_1}^2 \lesssim C \delta + C \left\| u \right\|_{X_1 \cap Y_1}. \tag{6.33}
\]
Using the contraction mapping argument, we can show that the integral equation \( \mathcal{I}_1 u = u \) has a solution in \( \mathcal{P}_1 \). The left part of the proof is standard and we omit the details.

\[\square\]

**Proof of iii) of Theorem 1.3** In view of \( e^{-\frac{t}{4} |\xi| + \frac{1}{2} t |\xi|^2} \in M_p \), one easily sees that
\[
\left\| \Delta_j e^{t \Lambda/2n} U_1(t) u_0 \right\|_p \lesssim \left\| \Delta_j U_1(t/4) u_0 \right\|_p \lesssim e^{-ct^2} \left\| \Delta_j u_0 \right\|_p. \tag{6.34}
\]
By (6.34),
\[
\left\| \Delta_j e^{t \Lambda/2n} \mathcal{A}_1 f \right\|_p \lesssim \int_0^t e^{-c(t-\tau)^2} \left\| \Delta_j e^{\tau \Lambda/2n} f(\tau) \right\|_p d\tau. \tag{6.35}
\]
Using Young’s inequality,
\[
\left\| \Delta_j e^{t \Lambda/2n} \mathcal{A}_1 f \right\|_{L^\infty_{t,X_1} L^p_x} \lesssim 2^{-j} \left\| \Delta_j e^{t \Lambda/2n} f \right\|_{L^\infty_{t,X_1} L^p_x}. \tag{6.36}
\]
By (6.36), we have
\[
\left\| \nabla \mathcal{A}_1 f \right\|_{\overline{L}^\infty(I_c e^{t \Lambda/2n} \dot{B}^p_{\infty,q})} \lesssim \left\| f \right\|_{\overline{L}^\infty(I_c e^{t \Lambda/2n} \dot{B}^p_{\infty,q})}. \tag{6.37}
\]
We consider the mapping
\[ \mathcal{T} : u(t) \rightarrow U_1(t)u_0 + \mathcal{A}_t \mathbb{P} \text{div}(u \otimes u) \quad (6.38) \]
in the metric space
\[ \mathcal{D} = \{ u : \| u \|_{L_\infty \left( \mathbb{R}^+; \mathbb{L}_L^{p,n} \right)} \leq \delta \}, \quad d(u, v) = \| u - v \|_{L_\infty \left( \mathbb{R}^+; \mathbb{L}_L^{p,n} \right)} \quad (6.39) \]
By (6.34), (6.37) and Lemma 2.3 (it is easy to see that one can replace \( \sqrt{t} \) by \( t/2 \) in Lemma 2.3), for any \( u, v \in \mathcal{D} \),
\[ \| \mathcal{T} u \|_{L_\infty \left( \mathbb{R}^+; \mathbb{L}_L^{p,n} \right)} \lesssim \| U_1(t)u_0 \|_{L_\infty \left( \mathbb{R}^+; \mathbb{L}_L^{p,n} \right)} + \| u \otimes u \|_{L_\infty \left( \mathbb{R}^+; \mathbb{L}_L^{p,n} \right)} \lesssim \| u_0 \|_{\mathbb{L}_L^{p,n}} + \| u \|_{\mathbb{L}_L^{p,n}} \]
\[ \| \mathcal{T} u - \mathcal{T} v \|_{L_\infty \left( \mathbb{R}^+; \mathbb{L}_L^{p,n} \right)} \lesssim \delta \| u - v \|_{L_\infty \left( \mathbb{R}^+; \mathbb{L}_L^{p,n} \right)} \quad (6.41) \]
Applying the standard argument, we can show that \( \mathcal{T} \) is a contraction mapping from \( (\mathcal{D}, d) \rightarrow (\mathcal{D}, d) \). So, there is a \( u \in \mathcal{D} \) satisfying \( \mathcal{T} u = u \). The left part of the proof is standard and we omit it. \( \square \)

7 Analyticity of GNS: \( 1/2 < \alpha < 1 \)

Lemma 7.1 Let \( 1/2 < \alpha < 1 \), \( 1 < p < \infty \), \( 1 \leq q \leq \infty \), \( 1 \leq \gamma \leq \infty \). Then
\[ \| U_{2\alpha}(t)u_0 \|_{L_\gamma \left( \mathbb{R}^+; c^{1/2\alpha L_{p,q}} \right)} \lesssim \| u_0 \|_{c^{1/2\alpha L_{p,q}}} \quad (7.1) \]
*Sketch of Proof.* We have
\[ \| \Delta_j U_{2\alpha}(t) f \|_{L_p^\gamma} \lesssim e^{-ct2^{2\alpha j}} \| \Delta_j f \|_{L_p^\gamma} \quad (7.2) \]
In view of (7.2), one easily sees that
\[ \| \Delta_j c^{1/2\alpha L_{p,q}} U_{2\alpha}(t) u_0 \|_p \lesssim e^{-ct2^\alpha j} \| \Delta_j u_0 \|_p \quad (7.3) \]
Taking \( L_p^\gamma \) norm on inequality (7.2),
\[ \| \Delta_j c^{1/2\alpha L} U_{2\alpha}(t) u_0 \|_{L_p^\gamma} \lesssim 2^{-2\alpha j/\gamma} \| \Delta_j u_0 \|_p \quad (7.4) \]
Taking sequence \( L^q \) norms in (7.4), we have the result, as desired. \( \square \)
Lemma 7.2 Let \( 1/2 < \alpha < 1, 1 < p < \infty, 1 \leq q \leq \infty, 1 \leq \gamma_1 \leq \gamma \leq \infty, I = [0, T), \ T \leq \infty. \) Then

\[
\| \partial_{2a} f \|_{L^\gamma(I; x^{1/2\alpha} B_{p,q}^r)} \lesssim \| f \|_{L^\gamma(I; x^{1/2\alpha} B_{p,q}^{-2\alpha(1+1/\gamma-1/\gamma_1)})}, \tag{7.5}
\]

Proof. In view of \( e^{(t^{1/2\alpha} - r^{1/2\alpha} - (t-r)^{1/2\alpha})/\gamma} \), we can use the same way as that of the proof of Lemma 7.2, we have for

\[
\| \Delta_j e^{1/2\alpha} \partial_{2a} f \|_p \lesssim \int_0^t e^{-c(t-\tau)^{2\alpha}} \| \Delta_j e^{1/2\alpha} f(\tau) \|_p \, d\tau. \tag{7.6}
\]

Using Young’s inequality, for \( \gamma_1 \leq \gamma \), we have

\[
\| \Delta_j e^{1/2\alpha} \partial_{2a} f \|_{L^{p,q}([0, T])} \lesssim 2^{-2\alpha j(1+1/\gamma_1-1/\gamma)} \| \Delta_j e^{1/2\alpha} f(\tau) \|_{L^{p,q}([0, T])}. \tag{7.7}
\]

Taking \( l^q \) norms in both sides of \((7.7)\), we have the results, as desired. \( \square \)

Theorem 7.3 (Analyticity for GNS: \( p < \infty \)) Let \( n \geq 2, 1/2 < \alpha < 1, 1 < p < \infty, 1 \leq q \leq \infty. \) Assume that \( u_0 \in B^{n/p+1-2\alpha}_{p,q}, \gamma_\pm = 2\alpha/(2\alpha - 1 \pm \varepsilon), 0 < \varepsilon \ll 1. \) There exists a \( T_{\text{max}} = T_{\text{max}}(u_0) > 0 \) such that \((1.1)\) has a unique solution \( u \in \tilde{L}^\gamma_{\text{loc}}(0, T_{\text{max}}; e^{(1/2\alpha \Lambda \| \partial_{2a} f \|^2_{p,q})}) \cap \tilde{L}^\infty([0, T_{\text{max}}); e^{(1/2\alpha \Lambda \| \partial_{2a} f \|^2_{p,q})}). \) If \( T_{\text{max}} < \infty \), then

\[
\| u \|_{L^\gamma+(0, T_{\text{max}}; e^{(1/2\alpha \Lambda \| \partial_{2a} f \|^2_{p,q})}) \cap L^\infty(0, T_{\text{max}}; e^{(1/2\alpha \Lambda \| \partial_{2a} f \|^2_{p,q})})} = \infty.
\]

Moreover, if \( u_0 \in B^{n/p+1-2\alpha}_{p,q} \) is sufficiently small, then \( T_{\text{max}} = \infty. \)

Sketch of Proof. Taking \( \gamma = 2\alpha/(2\alpha - 1), \gamma_1 = \gamma/2 \) in Lemmas 7.1 and 7.2 we have for \( I = [0, T), \)

\[
\| U_{2a}(t) u_0 \|_{L^\gamma(0, T; x^{1/2\alpha} B_{p,q}^{-2\alpha})} \lesssim \| u_0 \|_{B^{n/p+1-2\alpha}_{p,q}}, \tag{7.8}
\]

\[
\| \nabla \partial_{2a} f \|_{L^\gamma(0, T; x^{1/2\alpha} B_{p,q}^{n/p+\varepsilon})} \lesssim \| f \|_{L^\gamma/[2\alpha(2\alpha - 1)]^{1/2\alpha} B_{p,q}^{n/p}}, \tag{7.9}
\]

By \((7.9)\) and Lemma 2.3,

\[
\| \partial_{2a} \ \text{div}(u \otimes u) \|_{L^\gamma(0, T; x^{1/2\alpha} B_{p,q}^{n/p+\varepsilon})} \lesssim \| u \otimes u \|_{L^\gamma(0, T; x^{1/2\alpha} B_{p,q}^{n/p})} \lesssim \| u \|_{L^\gamma(0, T; x^{1/2\alpha} B_{p,q}^{n/p})}^2 \tag{7.10}
\]

Now, in view of \((7.8)\) and \((7.10)\), we can use the same way as that of the proof of Theorem 1.1 to get the results, as desired. \( \square \)
Theorem 7.4 (Analyticity for GNS: $p = \infty$) \( \text{Let } n \geq 2, \frac{1}{2} < \alpha < 1, 1 \leq p \leq \infty. \) Assume that \( u_0 \in M_{p,1}^{1-2\alpha}. \) There exists a \( T_{\text{max}} = T_{\text{max}}(u_0) > 0 \) such that (1.1) has a unique solution \( u \in \tilde{L}_{loc}^{2\alpha/(2\alpha-1)}(0,T_{\text{max}};E_{p,1}^{\text{ct}}) \) and \( (I-\Delta)^{(1-2\alpha)/2}u \in \tilde{L}^{\infty}(0,T_{\text{max}};E_{p,1}^{\text{ct}}). \) Moreover, if \( T_{\text{max}} < \infty, \) then
\[
\|u\|_{\tilde{L}^{2\alpha/(2\alpha-1)}(0,T_{\text{max}};E_{p,1}^{\text{ct}})} = \infty.
\]

The proof of Theorem 7.4 follows the ideas as in Theorems 1.1 and 7.3 and we omit the details of the proof.

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