CHARACTERIZATION OF SOME CLASSES OF GRAPHS AND THEIR PRODUCTS AS FRAME GRAPHS

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Abstract. To each finite frame $\varphi$ in an inner product space $H$ we associate a simple graph $G(\varphi)$, called frame graph, with the vectors of the frame as vertices and there is an edge between vertices $f$ and $g$ provided that $\langle f, g \rangle \neq 0$. In this paper the relation between the order of $G(\varphi)$ and the dimension of $H$ is investigated for some well-known classes of graphs and their products.

1. Introduction

The study of frames, using the properties of graphs, is an exciting research topic and hopefully will become mutually useful for both frame and graph theory as well as in computer science. For example, in [26, 17, 4] the relation between equiangular tight frames and graphs was observed. A one-to-one correspondence between a subclass of equiangular tight frames and regular two-graphs was offered in [17] and another one between real equiangular frames of $n$ vectors and graphs of order $n$ was given in [29]. The authors of [27] found some restrictions on the existence of real equiangular tight frames by an equivalence between equiangular tight frames and strongly regular graphs with certain parameters.

In [1] we defined a natural connection between frames and graphs. This connection is made by the zero-nonzero pattern of the correlation between different elements of the frame. More precisely, for a finite frame $\varphi$ in an inner product space $H$ we associate a simple graph $G(\varphi)$, termed the frame graph, with the elements of a frame as vertices, two distinct vertices are adjacent if and only if the respective vectors are non-orthogonal. A simple graph $G$ is said to be frame graph in space $H$ if there exists a frame $\varphi$ for $H$ such that $G(\varphi) = G$. In [1] we studied some basic properties of frame graphs and identified some classes of tight frame graphs (a frame graph is a tight frame graph if the associated frame is tight). Among other things it is shown that complete graphs and the join of each graph with itself are tight frame graphs, whereas non-trivial trees and cycles with at least six vertices are not.

This interesting definition and the corresponding observations have received an extensive interest by many researchers, see [28, 8, 7, 12, 19] for example. Investigating the relation between the dimension of $H$ and the graph-theoretic properties of $G$ is the main purpose of this paper. This problem has a deep connection with a well known problem in graph theory, namely minimum positive semidefinite rank.

The outline of the paper is as follows. In Section 2, after fixing some notation and definitions, we discuss the relation between the dimension of $H$ and the minimum positive semidefinite rank problem which considers the minimum rank over all positive semidefinite...
Hermitian matrices whose $ij$th entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in $G$ and is zero otherwise. By this relation some well known classes of graphs such as trees and their complements, cycles and their complements, complete and complete bipartite graphs will be characterized as frame graphs. In Section 3, the relation between $\dim(H)$ and the order of a graph will be studied for the join of two graphs and also for the corona, Cartesian and strong product of some well-known classes of graphs. As a result we find the minimum positive semidefinite rank of some classes of graphs which, as far as we know, are new results. The results are summarized in Table 1.

| Result # | $G$ | Order | $mr^C_{\perp}(G)$ | $mr^R_{\perp}(G)$ |
|----------|-----|-------|------------------|------------------|
| [2], 3.3 | $P_n \boxtimes P_m$ | $nm$ | $(n-1)(m-1)$ | $(n-1)(m-1)$ |
| 3.4     | $T \square K_n$ | $nm$ | $mn - n$ | $mn - n$ |
| 3.5     | $C_3 \triangleleft P_n$ | $3n$ | $3n - 3$ | $3n - 3$ |
| 3.10    | $K_n \square K_m$ | $nm$ | $\leq n + m - 1\ n + m - 2$ or $n + m - 1$ |
| 3.13    | $T \circ T'$ | $mm' + m$ | $mm' - 1$ | $mm' - 1$ |
| 3.13    | $T \circ K_n, n \geq 2$ | $mn + m$ | $2m - 1$ | $2m - 1$ |
| 3.13    | $K_n \circ T$ | $mn + n$ | $nm - n + 1$ | $nm - n + 1$ |
| 3.13    | $K_n \circ K_m$ | $mn + n$ | $n + 1$ | $n + 1$ |
| 3.13    | $C_n \circ T$ | $mn + n$ | $nm - 2$ | $nm - 2$ |
| 3.13    | $T \circ C_n$ | $mn + m$ | $m(n - 1) - 1$ | $m(n - 1) - 1$ |
| 3.13    | $C_n \circ K_m$ | $mn + m$ | $2n - 2$ | $2n - 2$ |
| 3.13    | $K_m \circ C_n$ | $mn + m$ | $m(n - 2) + 1$ | $m(n - 2) + 1$ |
| 3.13    | $C_n \circ C_m$ | $mn + n$ | $n(m - 1) - 2$ | $n(m - 1) - 2$ |

Table 1. Summary of minimum positive semidefinite rank results established in this paper ($T$ and $T'$ are trees with $|T| = m$ and $|T'| = m'$).

2. Definitions, Preliminaries and basic facts

The following definitions and facts are standard and can be found in any text book about graph theory, see for example [10, 5, 30].

Recall that a graph $G = (V, E)$ consists of a set of vertices $V$ and a set of edges $E$, where the elements of $E$ are two-element sets of vertices. The order of a graph $G$, denoted by $|G|$, is the number of vertices. Distinct vertices $f_i$ and $f_j$ is said to be adjacent if $\{f_i, f_j\}$ is an edge. The degree of each vertex is the number of vertices adjacent to it and the minimum degree among all vertices is denoted by $\delta(G)$. Note that our graphs are simple, i.e., multiple edges and loops do not appear in them and all the edges are undirected. A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle of order $n$ is denoted by $C_n$.

A graph $G$ is connected if each pair of vertices in $G$ belongs to a path in $G$. A connected component of $G$ is a maximal connected subgraph of $G$. A connected graph without a cycle
is called a tree.

An independent set in a graph is a set of pairwise nonadjacent vertices. The independence number of a graph $G$, $\alpha(G)$, is the maximal size of an independent set of vertices. A graph $G$ is complete or a clique if $\alpha(G) = 1$. The complete graph of order $n$ is denoted by $K_n$. The graph $G = (V, E)$ is called bipartite if $V$ is the union of two disjoint (possibly empty) independent sets called partite sets of $G$. If every vertex of the first set with $n$ elements is adjacent to every vertex of the second set with $m$ elements, then it is called a complete bipartite graph and denoted by $K_{n,m}$.

By the complement $\overline{G}$ of a graph $G$, we mean the graph on the same vertex set where two vertices are adjacent if and only if they are not adjacent in $G$.

Given a graph $G = (V, E)$, the graph $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$ is called a subgraph of $G$. An induced subgraph $H$ is a subgraph in which two vertices are adjacent if and only if they are adjacent in $G$.

Suppose that $G$ is labeled and that $G_1, \ldots, G_k$ are (labeled) subgraphs of $G$, that is, each $G_i, i = 1, \ldots, k$ is the result of deleting some edges and/or vertices from $G$. We say that $G_1, \ldots, G_k$ cover $G$ if each edge (vertex) of $G$ is an edge (vertex) of at least one $G_i, 1 \leq i \leq k$ or $G = \bigcup_{i=1}^k G_i$. The cover $G_1, \ldots, G_k$ of $G$ is called a clique cover of $G$ if each of $G_1, \ldots, G_k$ is a clique of $G$. The clique cover number of $G$, $\text{cc}(G)$, is the minimum value of $k$ for which there is a clique cover $G_1, \ldots, G_k$ of $G$.

A graph is said to be chordal if it has no induced cycles $C_n$ with $n \geq 4$. As examples trees and complete graphs are chordal.

For an $n \times n$ Hermitian matrix $A$, the graph of $A$, denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, 2, \ldots, n\}$ such that two distinct vertices $i$ and $j$ are adjacent if and only if the $ij$-th entry of $A$ is non-zero. The set of all $n \times n$ complex Hermitian positive semidefinite matrices will be denoted by $\mathcal{H}^+_n$ and its subset consisting of all real matrices denoted by $S^+_n$.

The real minimum positive semidefinite rank of $G$, $\text{mr}^R_+(G)$, and complex Hermitian minimum positive semidefinite rank, $\text{mr}^C_+(G)$ are defined by

$$\text{mr}^R_+(G) = \min \{ \text{rank}(B) : B \in S^+_n \text{ and } \mathcal{G}(B) = G \},$$

and

$$\text{mr}^C_+(G) = \min \{ \text{rank}(B) : B \in \mathcal{H}^+_n \text{ and } \mathcal{G}(B) = G \}.$$  

If $\text{mr}^R_+(G) = \text{mr}^C_+(G)$, then we denote the common value $\text{mr}^R_+(G) = \text{mr}^C_+(G)$ by $\text{mr}_+(G)$.

Clearly $\text{mr}^C_+(G) \leq \text{mr}^R_+(G)$. A graph on 16 vertices for which these parameters are not identical was presented in [3]. For a graph $G$ of order $n$, it is well known and straightforward that $\text{mr}^R_+(G) \leq n - 1$. Other fundamental facts, that will be of interest to us, are that $\text{mr}_+(G) = n - 1$ if and only if $G$ is a tree and $\text{mr}_+(C_n) = n - 2$ [IS]. For more details on minimum positive semidefinite rank we refer the reader to [IS] [6] [11] [2] [14] [3].

A finite frame for a finite dimensional Hilbert space $\mathcal{H}$ (or inner product space) is a finite sequence $\{f_i\}_{i=1}^n$ in $\mathcal{H}$ such that there exist constants $0 < A \leq B < \infty$ with the property that

$$A \|f\|^2 \leq \sum_{i=1}^n |\langle f, f_i \rangle|^2 \leq B \|f\|^2$$

holds for all $f \in \mathcal{H}$ [9].
Given a frame $\varphi = \{f_i\}_{i=1}^m$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $B$ be the $n \times m$ matrix whose $j$th column is $f_j$. Then $B^TB (B^*B)$ is a positive semidefinite matrix called the Gramian matrix of $\varphi$ and denoted by $\text{Gram}(\varphi)(B^*)$ is the conjugate transpose of B). The rank of $B$, $n$, is equal to the rank of $\text{Gram}(\varphi)$ \cite{10} Theorem 7.2.10.

Let $\varphi$ be a finite frame for the inner product space $\mathcal{H}$. We associate a simple graph $G(\varphi)$, called a frame graph, whose vertices are the elements of $\varphi$ and two distinct vertices $a$ and $b$ are adjacent if and only if $\langle a, b \rangle \neq 0$. A simple graph $G$ is called a frame graph in the space $\mathcal{H}$ if there exists a frame $\varphi$ for $\mathcal{H}$ such that $G(\varphi) = G$. In this paper all graphs are non-trivial and connected, and so the associated frames do not include zero vectors.

For a given graph $G$, we interested to find the inner product $\mathcal{H}$ so that there exists a frame $\varphi$ for $\mathcal{H}$ such that $G(\varphi) = G$. At first, we show that this problem is deeply connected with the well known minimum positive semidefinite rank problem. We will find the minimum positive semidefinite rank for join of graphs and Cartesian, corona and strong products of some well known classes of graphs and use them to characterize these products as frame graphs.

**Lemma 2.1.** For a graph $G$, $\text{mr}_+^\mathbb{R}(G) = k$ (respectively, $\text{mr}_+^\mathbb{C}(G) = k$) if and only if $k$ is the minimum number such that $G$ is a frame graph in a real (respectively, complex) inner product space of dimension $k$.

**Proof.** Let $A$ be a positive semidefinite real (complex) $n \times n$ matrix of rank $k$ such that $G(A) = G$. The matrix $A$ has a decomposition of the form $A = B^TB$. The columns of $B$ constitute a frame $\varphi$ in a real (complex)$k$-dimensional space such that $G(\varphi) = G$. On the other hand, the Gramian matrix of each frame $\varphi$ in a real (complex) space $\mathcal{H}$ describes $G(\varphi)$, or $G(\text{Gram}(\varphi)) = G(\varphi)$, and its rank is $\dim(\mathcal{H})$. Hence the lowest dimension of a real (complex) space in which a frame associated to the graph $G$ is exactly the real (complex Hermitian) minimum positive semidefinite rank.

**Lemma 2.2.** Let $G$ be a frame graph in $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) with $m > n$ vertices. Then $G$ is a frame graph in $\mathbb{R}^{n+1}$ (respectively, $\mathbb{C}^{n+1}$).

**Proof.** Let $\varphi = \{f_1, f_2, \ldots, f_m\}$ be a frame for $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) such that $G(\varphi) = G$. Since $\varphi$ is a frame, a subset of $\varphi$ with $n$ elements spans $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$). Then, without loss of generality, we may assume that $\{f_1, f_2, \ldots, f_n\}$ spans $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$). Clearly

$$\varphi' := \{(f_1, 0), (f_2, 0), \ldots, (f_n, 0), (f_{n+1}, 1), (f_{n+2}, 0), \ldots, (f_m, 0)\},$$

spans $\mathbb{R}^{n+1}$ (respectively, $\mathbb{C}^{n+1}$) and $G(\varphi) = G(\varphi')$. Hence $G$ is a frame graph in $\mathbb{R}^{n+1}$ (respectively, $\mathbb{C}^{n+1}$).

For a given graph $G$ on $n$ vertices, the above lemmas conclude that $\text{mr}_+^\mathbb{R}(G) = k$ (respectively, $\text{mr}_+^\mathbb{C}(G) = k$) if and only if $G$ is just a frame graph in the real (respectively, complex) inner product spaces of dimension $k, k + 1, \ldots, n$. For example, $G$ is just a frame graph in spaces of dimension $n - 1$ and $n$ if and only if $G$ is a tree. Another example is a cycle of order $n$, $C_n$, which is just a frame graph in the spaces of dimension $n - 2, n - 1$ and $n$. Trees and cycles, as two well known classes of graphs, are not frame graph in the spaces of lower dimensions, whereas complete graphs are frame graphs in all possible spaces because $\text{mr}_+(K_n) = 1$. Clearly complete graphs are the only connected graphs which are frame graphs in all possible spaces. Let $G$ be a non-complete connected graph and $\varphi$ be a
frame for an inner product space $\mathcal{H}$ such that $G(\varphi) = G$. Then the dimension of $\mathcal{H}$ is at least two. The following example introduces two classes of connected graphs which achieve this bound.

**Example 2.1.** Let $G$ be a simple graph obtained from $K_n$ ($n \geq 3$) by deleting an edge and let $\{e_1, e_2\}$ be the standard orthogonal basis of $\mathbb{C}^2 (\mathbb{R}^2)$. Then the frame $\varphi = \{e_1 + e_2, e_1 - e_2\} \cup \{e_1\}_{i=1}^{n-2}$ makes $G$ a frame graph in $\mathbb{C}^2 (\mathbb{R}^2)$, i.e., $mr_+(G) = 2$. Another graph is $H_n$, the $(n - 2)$-regular graph on $n$ vertices. It is known that $mr_+(H_n) = 2$ (see Proposition 5.2 of [6]). Both $G$ and $H_n$ are frame graphs in the inner product spaces of dimension $2, 3, ..., n - 1$ and $n$.

Despite trees and complete graphs which are frame graphs in a minimum and maximum number of spaces, respectively, complete bipartite graphs with the parts of the same size are frame graphs in a medium number of possible spaces. Indeed, since the independence number of each complete bipartite graph $K_{m,n}$ is $m$ and it has no isolated vertex, it cannot be a frame graph in an inner product space of dimension less than $m$. Let $\{e_i\}_{i=1}^m$ be the standard orthogonal basis of $\mathbb{R}^m$. The set $\{e_i\}_{i=1}^m \cup \{e_1 + e_2 + \ldots + e_i - 1 + \frac{2\pi}{2m}e_i + e_{i+1} + \ldots + e_m\}_{i=1}^n (n \neq 2)$ is a frame for $\mathbb{R}^m$ and its frame graph is $K_{m,n}$. Now Lemma 2.2 guaranties that $K_{m,n}$ is just a frame graph in the spaces of dimension $m, m + 1, ..., n + m - 1, n + m$. Proposition 2.2 of [15] shows that this result holds for a larger class of bipartite graphs. We include it here:

**Proposition 2.3.** Let $G$ be a bipartite graph which contains $K_{n,n}$ as an induced subgraph and its partite sets $U$ and $V$ are of size $m$ and $n$ where $m \geq n$. Then $G$ is just a frame graph in the inner product spaces of dimension $m, m + 1, ..., m + n - 1, m + n$.

3. Product of graphs and frame graph

As we saw in the previous section, to characterize a graph as frame graph we just need to find its minimum positive semidefinite rank. Some well known classes of graphs such as trees and their complements, cycles and their complements, complete and complete bipartite graphs have known minimum positive semidefinite rank (see [20]), so the dimension of all the spaces which they are frame graphs in them can be obtained by Lemma 2.2. In this section, we try to find the minimum positive semidefinite rank of the product of some prominent classes of graphs and use them to characterize these graphs as frame graphs. The following definition and proposition, which can be found in [15], provides a lower bound for $mr_+(G)$ and will be useful later in this work.

Let $G = (V, E)$ be a connected graph and $S = \{v_1, ..., v_m\}$ be an ordered set of vertices of $G$. Denote by $G_k$ the subgraph induced by $v_1, v_2, ..., v_k$ for all $k = 1, 2, ..., m$. Let $H_k = (V_k, E_k)$ be the connected component of $G_k$ such that $v_k \in V_k$. If for each $k$, there exists $w_k \in V$ such that $w_k \neq v_l$ for $l \leq k$, $\{v_k, w_k\} \in E$ and $\{w_k, v_l\} \notin E$ for all $v_l \in V_k$ with $l \neq k$, then $S$ is called a vertex set of ordered subgraphs (or OS-vertex set). The **OS-number** of a graph $G$, denoted by $OS(G)$, is the maximum cardinality among all OS-vertex sets of $G$. For example $OS(T) = m - 1$ where $T$ is a tree on $m$ vertices (see [15] Proposition 3.9) and, clearly, $OS(K_n) = 1$. Note that the OS-number is not defined for a single vertex, $K_1$, and so, in this section, we assume that all graphs have more than one vertex.
Proposition 3.1. \cite{15} Proposition 3.3 and \cite{11} Observation 3.14. Let $G$ be a connected graph. Then $\text{OS}(G) \leq \text{mr}_C^O(G) \leq \text{mr}_R^O(G) \leq \text{cc}(G)$.

The vertex connectivity $\kappa(G)$ of a connected graph $G = (V,E)$ is the minimum size of $S \subseteq V$ such that $G - S$ is disconnected or a single vertex. The following result provides an upper bound for the real minimum positive semidefinite rank.

Proposition 3.2. \cite{21} \cite{22}. For each graph $G$ on $n$ vertices, $\text{mr}_R^O(G) \leq n - \kappa(G)$.

3.1. Strong product. The strong product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \boxtimes G_2$, is the graph with the vertex set $V_1 \times V_2$ such that $(u, v)$ is adjacent to $(u', v')$ if and only if (1) $u = u'$ and $\{v, v'\} \in E_2$ or (2) $v = v'$ and $\{u, u'\} \in E_1$ or (3) $\{u, u'\} \in E_1$ and $\{v, v'\} \in E_2$.

Theorem 3.3. Let $G$ be the strong product of $P_n$ and $P_m$, i.e., $G = P_n \boxtimes P_m$. Then $G$ is just a frame graph in the inner product spaces of dimension $(n-1)(m-1), (n-1)(m-1) + 1, ..., mn - 1, mn$.

Proof. If we prove $\text{mr}_R^O(G) = (n-1)(m-1)$, then the "lifting" argument of Lemma 2.2 completes the proof. $G$ has a clique cover of $(n-1)(m-1)$ copies of $K_4$, so, by Proposition 3.1, $\text{mr}_R^O(G) \leq (n-1)(m-1)$. Let $\{s_1, s_2, ..., s_n\}$ and $\{t_1, t_2, ..., t_m\}$ be the vertex sets of $P_n$ and $P_m$, respectively. The ordered set $\bigcup_{i=1}^{n-1} \bigcup_{j=1}^{m-1} \{v_{i,j}\}$, where $v_{i,j} = (s_i, t_j)$, with $v_{i,j} = (s_{i+1}, t_{j+1})$ is an OS-vertex set of length $(n-1)(m-1)$. Therefore $(n-1)(m-1) \leq \text{OS}(G)$. Now Proposition 3.1 implies $(n-1)(m-1) \leq \text{mr}_C^O(G)$. \hfill $\square$

3.2. Joins of graphs. The join $G_1 \vee G_2$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex and edge sets is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$ together with all the edges joining $V_1$ and $V_2$.

Suppose $G$ is decomposable into two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, sharing only one vertex $v$ such that if $u \in V_1$ and $w \in V_2$, then $\{u, w\} \in E$ only if $u = v$ or $w = v$. Then $G_1$ and $G_2$ are joined at vertex $v$ and $G$ is denoted by $G = G_1 \vee G_2$.

Let $G$ and $H$ be two connected graphs on two or more vertices. Proposition 2.4 of \cite{15} states that $\text{mr}_C^O(G \vee H) = \max\{\text{mr}_C^O(G), \text{mr}_C^O(H)\}$ and by Theorem 3.4 of \cite{6} we have $\text{mr}_R^O(G,H) = \text{mr}_C^O(G) + \text{mr}_C^O(H)$. These, combined with Lemma 2.2, give the following theorem.

Theorem 3.4. Let the graph $G$ be just a frame graph in spaces of dimension $n', n' + 1, ..., n - 1, n$ and $H$ be just a frame graph in the spaces of dimension $m', m' + 1, ..., m - 1, m$ where $n' \geq m'$. The followings are hold:

(i) $G \vee H$ is just a frame graph in the spaces of dimension $n', n' + 1, ..., m + n - 1$ and $m + n$.

(ii) $G.H$ is just a frame graph in the spaces of dimension $n' + m', n' + m' + 1, ..., m + n - 2$ and $m + n - 1$.

3.3. Cartesian product. The Cartesian product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \Box G_2$, is the graph with vertex set $V_1 \times V_2$ such that $(u, v)$ is adjacent to $(u', v')$ if and only if (1) $u = u'$ and $\{v, v'\} \in E_2$ or (2) $v = v'$ and $\{u, u'\} \in E_1$. The set of vertices associated with (the same) OS-vertex set in each copy of $G$ or $H$ is an OS-vertex set for $G \Box H$. Then we have the following lemma.
Lemma 3.5. Let $G$ and $H$ be the graphs of order $n$ and $m$, respectively. Then
\[
\max\{nOS(H), mOS(G)\} \leq OS(G \square H).
\]

The connectivity of the graph $G \square H$ can be determined by knowing the minimum degree and the connectivity of the components $G$ and $H$.

Proposition 3.6. [25] Let $G$ and $H$ be the graphs on $n$ and $m$ vertices, respectively. Then $\kappa(G \square H) = \min\{\kappa(G), \kappa(H), \delta(G) + \delta(H)\}$.

Theorem 3.7. The graph $G = T \square K_m$, where $T$ is a tree on $n$ vertices, is just a frame graph in the inner product spaces of dimension $mn - m, mn - m + 1, ..., mn - 1, mn$.

Proof. It is sufficient to show that $mr_+(G) = mn - m$ and use the “lifting argument” of Lemma 2.2

By Lemma 3.3 $mn - m \leq OS(G)$, so Proposition 3.1 implies $mn - m \leq mr_+^R(G)$. On the other hand, using Propositions 3.2 and 3.6 we have $mr_+^R(G) \leq mn - m$. Using $mr_+(G) \leq mr_+^R(G)$ implies that $mr_+^R(G) = mr_+^R(G) = mn - m$ which completes the proof. □

It is a conjecture that $mr_+(C_m \square P_n) = mn - \min\{m, 2n\}$ (See [23]). This conjecture is valid for the case $n = 2$ [23]. The above theorem shows that it also holds for the case $m = 3$.

Corollary 3.8. $mr_+(C_3 \square P_n) = 3n - 3$.

At the end of this subsection we consider the graph $K_n \square K_m$. Unfortunately the minimum positive semidefinite rank of this graph is not known until now, and hence we cannot characterize it as frame graph. Another conjecture in [23] states that $mr_+(K_n \square K_m) = n + m - 2$. In what follows we show that $mr_+^R(K_n \square K_m)$ is one of the numbers $n + m - 2$ or $n + m - 1$, therefore $K_n \square K_m$ is a frame graph in the real spaces of dimension $n + m - 1, n + m, ..., nm$ and the dimension $n + m - 2$ remains unsolved.

If $A$ is an $n \times n$ matrix and $B$ is an $m \times m$ matrix, then $A \otimes B$, the Kronecker product, is the $n \times n$ block matrix whose $(i, j)$th block is the $m \times m$ matrix $a_{ij}B$. Let $G$ and $H$ be the graphs on $n$ and $m$ vertices, respectively, and let $A$ and $B$ be two matrices such that $G(A) = G$ and $G(B) = H$. Then $G(A \otimes I_m + I_n \otimes B) = G \square H$ (see [14]). The matrix $A \otimes I_m + I_n \otimes B$ known as Kronecker sum and denoted by $A \oplus B$. If $\lambda$ is an eigenvalue of $A$ and $\mu$ is an eigenvalue of $B$, then $\lambda + \mu$ is an eigenvalue of $A \oplus B$ and any eigenvalue of $A \oplus B$ arises as such a sum of eigenvalues of $A$ and $B$ [14]. We use these facts to proof the next lemma.

Lemma 3.9. For the graphs $G$ and $H$ on $n$ and $m$ vertices, respectively,
\[
mr_+^R(G \square H) \leq mmr_+^R(G) + nmr_+^R(H) - mr_+^R(G)mr_+^R(H).
\]

Proof. Let $A$ and $B$ be two matrices in $S^+_n$ and $S^+_m$, respectively, such that $G(A) = G$, $G(B) = H$, rank$(A) = mr_+^R(G)$, and rank$(B) = mr_+^R(H)$. Then $A$ has $n - mr_+^R(G)$ zero eigenvalues, $B$ has $m - mr_+^R(H)$ zero eigenvalues and all the other eigenvalues of $A$ and $B$ are positive and non zero. Since the eigenvalues of $A \oplus B$ are the sum of the eigenvalues of $A$ and $B$, $A \oplus B$ has exactly $mmr_+^R(G) + nmr_+^R(H) - mr_+^R(G)mr_+^R(H)$ positive non zero eigenvalues, and so rank$(A \oplus B) = mmr_+^R(G) + nmr_+^R(H) - mr_+^R(G)mr_+^R(H)$. The matrix $A \oplus B$ belongs to $S^+_{nm}$ and describes $G \square H$, i.e., $G(A \oplus B) = G \square H$, so $mr_+^R(G \square H) \leq$ rank$(A \oplus B) = mmr_+^R(G) + nmr_+^R(H) - mr_+^R(G)mr_+^R(H)$. □
The minimum rank of a graph $G$ (over $\mathbb{R}$) is defined to be
\[ \text{mr}(G) = \min \{ \text{rank}(A) : A \in S_n \text{ and } G(A) = G \}, \]
where $S_n$ is the set of symmetric matrices over $\mathbb{R}$. Clearly
\[ \text{mr}(G) \leq \text{mr}_+^R(G). \]

**Theorem 3.10.** $\text{mr}_+^R(K_n \square K_m)$ is either $n + m - 2$ or $n + m - 1$.

*Proof.* It is known that $\text{mr}(K_n \square K_m) = m + n - 2$ (see [2, Corollary 3.11]). Since $\text{mr}(K_n \square K_m) \leq \text{mr}_+^R(K_n \square K_m)$, $n + m - 2 \leq \text{mr}_+^R(G)$. On the other hand, by Lemma 3.9 $\text{mr}_+^R(K_n \square K_m)$ cannot be more than $n + m - 1$. \qed

**3.4. Corona product.** The corona product of $G_1 = (V_1, E_1)$ with $G_2 = (V_2, E_2)$, denoted by $G_1 \circ G_2$, is the graph of order $|V_1||V_2| + |V_1|$ obtained by taking one copy of $G_1$ and $|V_1|$ copies of $G_2$, and joining all the vertices in the $i$th copy of $G_2$ to the $i$th vertex of $G_1$. The following proposition helps us to provide a short proof for the next theorem.

**Proposition 3.11.** ([11, Observation 3.14]) Let $G$ be a graph and $G_1, G_2, \ldots, G_n$ be a cover of (labeled) subgraphs of it, i.e., $G = \bigcup_{i=1}^n G_i$. Then $\text{mr}_+^R(G) \leq \sum_{i=1}^n \text{mr}_+^R(G_i)$.

**Theorem 3.12.** Let $G$ be a connected graph of order $n$ with $\text{OS}(G) = \text{mr}_+^R(G)$ and $H$ be a connected graph with $\text{OS}(H) = \text{mr}_+^R(H)$. Then $\text{mr}_+^R(G \circ H) = n(\text{mr}_+^R(H)) + \text{mr}_+^R(G)$.

*Proof.* Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. $G \circ H = (\bigcup_{i=1}^n H \vee v_i) \cup G$. Therefore, by Proposition 3.11
\[ \text{mr}_+^R(G \circ H) \leq \left(\sum_{i=1}^n \text{mr}_+^R(H \vee v_i)\right) + \text{mr}_+^R(G) = n(\text{mr}_+^R(H)) + \text{mr}_+^R(G). \]

On the other hand, the union of $n$ distinct $\text{OS}$-vertex sets of the copies of $H$ along with the $\text{OS}$-vertex set of a copy of $G$ in $G \circ H$ makes an $\text{OS}$-vertex set of length $n\text{OS}(H) + \text{OS}(G)$ for $G \circ H$, and so $n\text{OS}(H) + \text{OS}(G) \leq \text{OS}(G \circ H)$. Again by using Proposition 3.11 we have $n(\text{mr}_+^R(H)) + \text{mr}_+^R(G) \leq \text{mr}_+^C^R(G \circ H)$. \qed

It is easy to check that $\text{OS}(C_t) = \text{mr}_+^R(C_t) = t - 2$ and $\text{OS}(K_n) = \text{mr}_+^R(K_n) = 1$ and it is known that if $G$ is a connected chordal graph, then $\text{OS}(G) = \text{mr}_+^C^R(G) = \text{cc}(G)$ (see [15, Corollary 3.8]). If $T$ is a tree of order $m$, then it is chordal and $\text{OS}(T) = \text{mr}_+^R(T) = \text{cc}(T) = m - 1$. Now the following corollary is a direct consequence of Theorem 3.12.

**Corollary 3.13.** Let $T$ and $T'$ be trees of order $m \geq 2$ and $m' \geq 2$, respectively. The followings hold.

1. $\text{mr}_+^R(T \circ T') = mm' - 1$.
2. $\text{mr}_+^R(T \circ K_n) = 2m - 1 (n \geq 2)$.
3. $\text{mr}_+^R(K_n \circ T) = nm - n + 1$.
4. $\text{mr}_+^R(K_n \circ K_m) = n + 1$.
5. $\text{mr}_+^R(C_n \circ T) = nm - 2$.
6. $\text{mr}_+^R(T \circ C_n) = m(n - 1) - 1$.
7. $\text{mr}_+^R(C_n \circ K_m) = 2n - 2$.
8. $\text{mr}_+^R(K_m \circ C_n) = m(n - 2) + 1$. 

\[ mr_+(C_n \circ C_m) = n(m - 1) - 2. \]

As far as we know all of the results of Corollary 3.13 are new with the exception of 4 and 7 which was established earlier in [2] and [24].

Note that \( mr_+(H_n) = OS(H_n) = 2 \). Therefore the minimum positive semidefinite rank of the corona product of \( H_n \) with trees, cycles and complete graphs can be calculated easily.

The following theorem is an immediate consequence of Theorem 3.12, Lemma 2.1 and Lemma 2.2.

\textbf{Theorem 3.14.} Let \( G \) and \( H \) be connected graphs of order \( n \) and \( m \), respectively, which are either chordal or cycle graph. If \( G \) is just a frame graph in the spaces of dimension \( n', n' + 1, \ldots, n \) and \( H \) is just a frame graph in the spaces of dimension \( m', m' + 1, \ldots, m \), then \( G \circ H \) is just a frame graph in the spaces of dimension \( nm' + n', nm' + n' + 1, \ldots, nm + n - 1, nm + n \).

\textbf{References}

[1] F. Abdollahi and H. Najafi, Frame Graph, Linear and Multilinear Algebra, 66 (2018), 1229–1243.

[2] AIM Minimum Rank-Special Graphs Work Group, Zero forcing sets and the minimum rank of graphs, Linear Algebra and its Applications, 428 (2008), 1628–1648.

[3] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche and H. van der Holst, Zero forcing parameters and minimum rank problems, Linear Algebra and its Applications, 433 (2010), 401–411.

[4] B. G. Bodman and V. I. Paulsen, Frames, graphs and erasures, Linear Algebra and its Applications, 404 (2005), 118–146.

[5] B. Bollobas, Graph Theory, An Introductory Course, Springer-Verlag, New York, 1979.

[6] M. Booth, P. Hackney, B. Harris, C. R. Johnson, M. Lay, T. D. Lenker, L. H. Mitchell, S. K. Narayan, Amanda Pascoe and B. D. Sutton, On the minimum semidefinite rank of simple graphs, Linear and Multilinear Algebra, 59 (2011), 483–506.

[7] T. Y. Chien and S. Waldron, A characterisation of projective unitary equivalence of finite frames and applications, SIAM J. Discrete Mathematics, 30, (2016), 976–994.

[8] T. Y. Chien, Equiangular lines, projective symmetries and nice error frames, PhD Thesis, University of Auckland, 2015.

[9] O. Christensen, An introduction to frame and Riesz bases, Birkhauser, Boston, 2003.

[10] R. Diestel, Graph Theory, Springer-Verlag, New York, 1997.

[11] S. M. Fallat and L. Hogben, The minimum rank of symmetric matrices described by a graph: A survey, Linear Algebra and Its Applications, 426, (2007), 558–582.

[12] V. Furst and H. Grotts, Tight Frame Graphs Arising as Line Graphs, The pump journal of undergraduate research, 4 (2021), 1–19.

[13] R. Frucht and F. Harary, On the coronas of two graphs, Aequationes Math., 4 (1970), 322–325.

[14] C. Godsil and G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.

[15] P. Hackney, B. Harris, M. Lay, S. K. Narayan, L. H. Mitchell and A. Pascoe, Linerly independent vertices and minimum semidefinite rank, Linear Algebra and its Applications, 431 (2009), 1105–1115.

[16] R. A. Horn and C. R. Johnson Matrix Analysis, Cambridge University Press, Cambridge, 1990, corrected reprint of the 1985 original.

[17] R. B. Holmes and V. I Paulsen, Optimal frames for erasures, Linear Algebra and its Applications, 377 (2004), 31–51.

[18] H. van der Holst, Graphs whose positive semi-definite matrices have nullity at most two, Linear Algebra and its Applications, 375 (2003), 1–11.

[19] J. W. Iverson and D. G. Mixon, Doubly transitive lines II: Almost simple symmetries, (2019), arXiv:1905.06859v1.
[20] X. Li, M. Nathanson and R. Phillips, Minimum vector Rank and Complement Critical Graphs, Electronic Journal of Linear Algebra, 27 (2014), 100–123.

[21] L. Lovász, M. Saks and A. Schrijver, Orthogonal representations and connectivity of graphs, Linear Algebra and its Applications, 114/115 (1989), 439–454.

[22] L. Lovász, M. Saks and A. Schrijver, A correction: “Orthogonal representations and connectivity of graphs”, Linear Algebra and its Applications, 313 (2000), 101–105.

[23] T. Peters, Positive semidefinite maximum nullity and zero forcing number, Electronic Journal of Linear Algebra, 23 (2012), 815–830.

[24] T. Peters, Positive semidefinite maximum nullity and zero forcing number, PhD Thesis, Iowa State University, 2012.

[25] S. Spacapan, Connectivity of Cartesian products of graphs, Applied Mathematics Letters, 21 (2008), 682–685.

[26] T. Strohmer, R.W. Heath, Grassmannian frames with applications to coding and communication, Appl. Comput. Harmonic Anal., 14 (2003), 257–275.

[27] M.A. Sustik, J.A. Tropp, I.S. Dhillon and R.W. Heath Jr, On the existence of equiangular tight frames, Linear Algebra and its Applications, 426 (2007), 619–635.

[28] S. F. D. Waldron, An introduction to finite tight frames, Springer, 2018.

[29] S. F. D. Waldron, On the construction of equiangular frames from graphs, Linear Algebra and its applications, 431 (2009), 2228–2242.

[30] D. West, Introduction to graph theory, Second edition, University of Illinois, Urbana, 2001.