A Note on Sectional Curvatures of Hermitian Manifolds

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Abstract First, we derive expression of the Chern sectional curvature of a Hermitian manifold in
local complex coordinates. As an application, we find that a Hermitian metric is Kähler if the Riemann
sectional curvature and the Chern sectional curvature coincide. Second, we prove that the sectional
curvature restricted to orthogonal 2-planes of a G-Kähler-like manifold with non-negative (resp. non-
positive) sectional curvature can take its maximum (resp. minimum) at a holomorphic plane section.
And we also prove that the holomorphic bisectional curvature of a Kähler-like manifold with non-
negative (resp. non-positive) Chern sectional curvature can take its maximum (resp. minimum) at
the holomorphic sectional curvature.

Key words Chern sectional curvature, G-Kähler-like manifold, Kähler-like manifold.

2020 MR Subject Classification: 53C55, 53B35

1 Introduction

Suppose \((M, h)\) is an \(n\)-dimensional Hermitian manifold, and \(g = \text{Re } h\) is the background Rieman-
nian metric associated to \(h\). Let us denote by \(D\) and \(\nabla\) the Chern (or Hermitian) connection and the
Levi-Civita (or Riemannian) connection, respectively. The curvature tensors of the Chern connection
and the Levi-Civita connection are denoted by \(R\) and \(\mathcal{R}\). It is well known that \(h\) is Kähler if and only
if the Chern connection \(D\) coincide with the Levi-Civita \(\nabla\) (refer to \([2, 7, 10]\), etc.). Hence \(\mathcal{R}\) is the
linear extension of \(R\) over \(\mathbb{C}\) under Kähler hypothesis.

Let \(z = (z^1, \cdots, z^n)\) and \(x = (x^1, \cdots, x^{2n})\) be local complex and real coordinates of a point \(p \in M\),
where \(z^\alpha = x^\alpha + \sqrt{-1}x^{n+\alpha}, 1 \leq \alpha \leq n\). In local coordinates, we denote by \(h = h_{\alpha\bar{\beta}}(z)dz^\alpha d\bar{z}^\beta\) and
\(g = \text{Re } h = g_{ij}(x)dx^i dx^j\). In this paper, we assume that lowercase Greek indices run from 1 to \(n\)
and lowercase Latin indices run from 1 to 2\(n\). We define a bundle isomorphism \(\varphi : TM \rightarrow T^{1,0}M\) by
\[
\varphi = \frac{1}{2}(u - \sqrt{-1}Ju), \quad \forall u \in TM,
\]
where \(J\) is the complex structure on \(M\). Let \(u, v \in T_pM\), we set \(\xi = u_o, \eta = v_o \in T_p^{1,0}M\). If \(h\) is
Kähler, then
\[
R(Ju, u, v, Jv) = 2\mathcal{R}\left(\xi, \eta, \eta\right), \quad (1.1)
\]
The first formula can be referred to [10], and the second formula can be referred to [6, 8]. Especially, if we take $v = u$ in (1.1) or take $v = Ju$ in (1.2), then (1.1) and (1.2) become

$$R(Ju, u, u, Ju) = 2R(\xi, \xi, \xi, \xi)$$  \hspace{1cm} (1.3)

under Kähler hypothesis.

In this paper, we will give a geometric interpretation of the right side in (1.2) when $h$ is non-Kähler. In local coordinates, the right side in (1.2) can be written as

$$\frac{1}{2} \mathcal{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}(\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta)(\eta^\gamma \bar{\xi}^\delta - \xi^\gamma \bar{\eta}^\delta),$$  \hspace{1cm} (1.4)

where $\mathcal{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \mathcal{R}\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta}, \frac{\partial}{\partial z^\gamma}, \frac{\partial}{\partial \bar{z}^\delta}\right)$. We find that

$$\frac{1}{2} \mathcal{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}(\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta)(\eta^\gamma \bar{\xi}^\delta - \xi^\gamma \bar{\eta}^\delta)
\left(\left\langle h_{\alpha\beta}\xi^\alpha \bar{\xi}^\beta, \left(\left\langle h_{\gamma\delta}\eta^\gamma \bar{\eta}^\delta, \eta^\gamma \bar{\xi}^\delta - \xi^\gamma \bar{\eta}^\delta\right\rangle\right\rangle - \frac{1}{4} \left\langle h_{\alpha\beta}\left(\xi^\alpha \bar{\xi}^\beta + \eta^\alpha \bar{\eta}^\beta\right)\right\rangle\right)^2 \hspace{1cm} (1.5)$$

is just the sectional curvature of the metric connection induced by the Chern connection, we call it the Chern sectional curvature of $(M, h)$.

**Theorem 1.1.** Let $(M, h)$ be a Hermitian manifold with the background Riemannian metric $g = \text{Re} h$. Suppose $D$ is the metric connection on the Riemannian manifold $(M, g)$ induced by the Chern connection. For arbitrary $u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i} \in T_p M$, we have

$$g((D^2 u)(v, u), v) = \frac{1}{2} \mathcal{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}(\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta)(\eta^\gamma \bar{\xi}^\delta - \xi^\gamma \bar{\eta}^\delta),$$  \hspace{1cm} (1.6)

where $\xi = u_o$ and $\eta = v_o$. Especially,

$$g((D^2 u)(Ju, u), Ju) = 2\mathcal{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^\alpha \bar{\xi}^\beta \xi^\gamma \bar{\xi}^\delta.$$  \hspace{1cm} (1.7)

From (1.6) in Theorem 1.1, we can see

$$g((D^2 u)(v, u), v) = g((D^2 v)(u, v), u).$$

Recently, the first author and Qiu [3] proved that a Hermitian manifold such that (1.3) is Kähler. Hence (1.7) in Theorem 1.1 can imply the following corollary.

**Corollary 1.2.** Let $(M, h)$ be a Hermitian manifold such that the Riemann sectional curvature and the Chern sectional curvature coincide. Then $h$ is a Kähler metric.

Does the sectional curvature restricted to orthogonal 2-planes of a Kähler manifold always take its maximum or minimum at some holomorphic plane section? According to Lu’s result in [5], we have a positive answer if the Kähler manifold has non-negative or non-positive sectional curvature. Next, we will generalize Lu’s result on G-Kähler-like manifolds.
Theorem 1.3. Let \((M, h)\) be a G-Kähler-like manifold with non-negative (resp. non-positive) sectional curvature. Then the sectional curvature restricted to orthogonal 2-planes of \((M, h)\) can take its maximum (resp. minimum) at some holomorphic plane section.

Moreover, we have

Theorem 1.4. Let \((M, h)\) be a Kähler-like manifold with non-negative (resp. non-positive) Chern sectional curvature. Then the holomorphic bisectional curvature of \((M, h)\) can take its maximum (resp. minimum) at the holomorphic sectional curvature.

G-Kähler-like manifolds and Kähler-like manifolds are introduced by Yang and Zheng \cite{[9]}, which are classes of non-Kähler manifolds. We extend the Riemannian curvature tensor \(R = (R_{ijkl})\) linearly over \(\mathbb{C}\). For a general Hermitian metric, Gray \cite{[1]} discovered \(R_{\alpha\beta\gamma\delta} = R_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} = 0\), but the types \(R_{\alpha\beta\gamma\bar{\delta}}\) and \(R_{\alpha\beta\bar{\gamma}\delta}\) may not vanish.

Definition 1.5. \cite{[9]} A Hermitian metric \(h\) is called Kähler-like, if its holomorphic sectional curvature tensor satisfies \(R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}}\) for all \(\alpha, \beta, \gamma, \delta = 1, 2, \cdots, n\). Similarly, \(h\) is called Gray-Kähler-like, or G-Kähler-like for short, if \(R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\bar{\gamma}\bar{\delta}} = 0\) for all \(\alpha, \beta, \gamma, \delta = 1, 2, \cdots, n\).

The rest of this paper is arranged as follows. In Section 2, we introduce the Chern sectional curvature and give proof of Theorem 1.1. In Section 3, we provide a brief overview of complexified Riemannian curvature and give proofs of Theorems 1.3 and 1.4.

2 The Chern sectional curvature

The Chern connection \(D\) is the unique connection on the holomorphic tangent bundle \(T^{1,0}M\) which is compatible with the Hermitian metric \(h\) and the complex structure \(J\). We denote by

\[
\theta = (\theta^\alpha_\beta) = (\Gamma^\alpha_{\beta\gamma}dz^\gamma)
\]

the connection 1-form matrix of \(D\), where \(\Gamma^\alpha_{\beta\gamma} = h^{\lambda\bar{\alpha}}\frac{\partial h_{\beta\lambda\bar{\delta}}}{\partial z^\gamma}\) are connection coefficients of \(D\). The curvature tensor of \(D\) is denoted by \(\mathfrak{R}\) with components

\[
\mathfrak{R}_{\alpha\beta\gamma\delta} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^\gamma\partial z^\delta} + \frac{\partial h_{\alpha\lambda\bar{\delta}}}{\partial z^\gamma}h^{\lambda\bar{\gamma}}\frac{\partial h_{\delta\bar{\lambda}}}{\partial z^\delta}. \tag{2.1}
\]

We define the holomorphic bisectional curvature by

\[
B(\xi, \eta) = \frac{\mathfrak{R}(\xi, \bar{\xi}, \eta, \bar{\eta})}{h(\xi, \xi) h(\eta, \eta)}, \quad \forall \xi, \eta \in T^{1,0}M. \tag{2.2}
\]

Ulteriorly, the holomorphic sectional curvature is defined by

\[
H(\xi) = B(\xi, \xi) = \frac{\mathfrak{R}(\xi, \bar{\xi}, \xi, \bar{\xi})}{h(\xi, \xi)^2}, \quad \forall \xi \in T^{1,0}M. \tag{2.3}
\]

Lemma 2.1. \cite{[2]} Under the bundle isomorphism \(\phi\) any complex connection on \(T^{1,0}M\) induces a metric connection on the Riemannian manifold \((M, g)\).
By a straightforward computation, we have
\[
D \frac{\partial}{\partial x^\alpha} = \frac{1}{2} \left( \tilde{\theta}_\alpha^\beta + \tilde{\theta}_\alpha^\beta \right) \frac{\partial}{\partial x^\beta} - \frac{\sqrt{-1}}{2} \left( \theta_\alpha^\beta - \tilde{\theta}_\alpha^\beta \right) \frac{\partial}{\partial x^{\beta+n}},
\]
\[
D \frac{\partial}{\partial x^{\alpha+n}} = \frac{\sqrt{-1}}{2} \left( \theta_\alpha^\beta - \tilde{\theta}_\alpha^\beta \right) \frac{\partial}{\partial x^\beta} + \frac{1}{2} \left( \theta_\alpha^\beta + \tilde{\theta}_\alpha^\beta \right) \frac{\partial}{\partial x^{\beta+n}}.
\]
For a general Hermitian manifold \((M, h)\), the induced metric connection \(D\) is compatible with the Riemannian metric \(g\), but not torsion free.

**Definition 2.2.** Let \((M, h)\) be a Hermitian manifold. We call
\[
K_D(u, v) = \frac{g((D^2 u)(v, u), v)}{g(u, u)g(v, v) - g(u, y)^2}
\]
the Chern sectional curvature of the 2-plane \(\Pi(u, v)\) spanned by two linearly independent tangent vectors \(u = u^i \frac{\partial}{\partial x^i}, \ v = v^i \frac{\partial}{\partial x^i} \in TM\).

Next we give expression of the Chern sectional curvature in local complex coordinates.

**Proof of Theorem 1.1** We denote by \(D \frac{\partial}{\partial x^i} = \tilde{\theta}_i^j \frac{\partial}{\partial x^j}\) and \(\tilde{\theta} = (\tilde{\theta}_i^j)\). It follows (2.1) that
\[
\tilde{\theta} = F \text{diag} \left\{ \theta, \tilde{\theta} \right\} F^{-1},
\]
where \(F = \left( \begin{array}{cc} I & I \\ I & -I \end{array} \right)\) and \(I\) is the \(n \times n\) unit matrix. In order to simplify the calculation process, we introduce the following notations. Denote by
\[
\frac{\partial}{\partial x} = \left( \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{2n}} \right), \quad \frac{\partial}{\partial z} = \left( \frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n} \right), \quad \frac{\partial}{\partial \bar{z}} = \left( \frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^n} \right),
\]
\[
U = (u^1, \cdots, u^{2n}), \quad \Xi = (\xi^1, \cdots, \xi^n).
\]
By a straightforward computation, we have
\[
D^2 u = u^j \left( \tilde{\theta}_j^i - \tilde{\theta}_k^i \right) \frac{\partial}{\partial x^i}
= U \left( \tilde{\theta} - \tilde{\theta} \otimes \tilde{\theta} \right) \left( \frac{\partial}{\partial x} \right)^t
= (\Xi, \Xi) \text{diag} \left\{ d\theta - \theta \otimes \theta, d\tilde{\theta} - \tilde{\theta} \otimes \tilde{\theta} \right\} \otimes \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)^t
= \Xi \alpha \left( d\theta^\mu \otimes \frac{\partial}{\partial z^\mu} + \tilde{\theta}^\alpha \left( d\tilde{\theta}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu} \right) \right)
= \Xi \alpha \left( \Theta^\mu_{\alpha \gamma} \delta^{\gamma} \delta^{\delta} \otimes \frac{\partial}{\partial z^\mu} - \xi^\beta \left( \Theta^\mu_{\beta \delta} \delta^{\gamma} \delta^{\delta} \otimes \frac{\partial}{\partial \bar{z}^\mu} \right) \right).
\]
Hence
\[
(D^2 u)(v, u) = \xi^\alpha \Theta^\mu_{\alpha \gamma} \delta^{\gamma} \delta^{\delta} \frac{\partial}{\partial z^\mu} - \xi^\beta \Theta^\mu_{\beta \delta} \delta^{\gamma} \delta^{\delta} \frac{\partial}{\partial \bar{z}^\mu},
\]
and
\[
g \left( (D^2 u)(v, u), y \right) = \frac{1}{2} \xi^\alpha \Theta^\mu_{\alpha \gamma} \delta^{\gamma} \delta^{\delta} \frac{\partial}{\partial z^\mu} - \frac{1}{2} \xi^\beta \Theta^\mu_{\beta \delta} \delta^{\gamma} \delta^{\delta} \frac{\partial}{\partial \bar{z}^\mu},
\]
\[
g \left( (D^2 u)(v, u), y \right) = \frac{1}{2} \xi^\alpha \Theta^\mu_{\alpha \gamma} \delta^{\gamma} \delta^{\delta} \frac{\partial}{\partial z^\mu} - \frac{1}{2} \xi^\beta \Theta^\mu_{\beta \delta} \delta^{\gamma} \delta^{\delta} \frac{\partial}{\partial \bar{z}^\mu}.
\]
Note that
\[ (Ju)_\alpha = \frac{1}{2} (Ju - \sqrt{-1}J^2u) = \frac{\sqrt{-1}}{2} (u - \sqrt{-1}Ju) = J(u_\alpha). \]

In order to prove (1.7), we only replace \( v \) and \( \eta \) in (1.6) with \( Ju \) and \( \sqrt{-1}v \), respectively.

According to above theorem, we have
\[ K_D(u, v) = \frac{1}{2} \frac{\partial^2 g_{ik}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{jl}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jl}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^j \partial x^l} + g^{st} ([jl, s][ik, t] - [jk, s][il, t]), \]

where
\[ [jk, s] = \frac{1}{2} \left( \frac{\partial g_{js}}{\partial x^j} + \frac{\partial g_{ks}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x_s} \right). \]

The sectional curvature of the 2-plane \( \Pi(u, v) \) is defined by
\[ K(u, v) = \frac{R(u, v, v, u)}{g(u, u)g(v, v) - g(v, u)^2}. \]

We call \( \Pi(u, Ju) \) a holomorphic plane section \([10]\).

If \((M, h)\) is a Kähler manifold, then the induced metric connection \( D \) and the Levi-Civita connection \( \nabla \) coincide, thus \( K_D(u, v) = K(u, v) \).

Now we extend the background Riemannian metric \( g = \Re h \) linearly over \( \mathbb{C} \) to the complexified tangent bundle \( T_{\mathbb{C}}M = TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M \), then
\[ g \left( \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta} \right) = g \left( \frac{\partial}{\partial \bar{z}^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right) = 0, \]
\[ g \left( \frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right) = g \left( \frac{\partial}{\partial \bar{z}^\alpha}, \frac{\partial}{\partial z^\beta} \right) = \frac{1}{2} h_{\alpha\bar{\beta}}. \]

Hence
\[ h(\xi, \eta) = 2g(\xi, \bar{\eta}), \quad \forall \xi, \eta \in T^{1,0}M. \]

For simplicity, we denote by \( h_{\alpha\beta} = h_{\overline{\alpha}\overline{\beta}} = 0 \).
where $A, B, C, D \in \{1, \cdots, n, \bar{1}, \cdots, \bar{n}\}$ and $z^A = z^\alpha$ if $A = \alpha$, $z^A = \bar{z}^\alpha$ if $A = \bar{\alpha}$. That is,

$$
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} h^{\lambda\alpha} \left( \frac{\partial h_{\beta\lambda}}{\partial z^\gamma} + \frac{\partial h_{\gamma\lambda}}{\partial z^\beta} \right), \quad \Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} = \frac{1}{2} h^{\lambda\alpha} \left( \frac{\partial h_{\bar{\beta}\lambda}}{\partial \bar{z}^\bar{\gamma}} - \frac{\partial h_{\bar{\gamma}\lambda}}{\partial \bar{z}^\bar{\beta}} \right).
$$

Then the complexified Riemannian curvature tensor components are defined by [4]

$$
R_{ABCD} = 2g \left( \left[ \nabla_{\frac{\partial}{\partial z^A}}, \nabla_{\frac{\partial}{\partial z^B}} \right] \frac{\partial}{\partial z^C}, \frac{\partial}{\partial z^D} \right) = h \left( \left[ \nabla_{\frac{\partial}{\partial z^A}}, \nabla_{\frac{\partial}{\partial z^B}} \right] \frac{\partial}{\partial z^C}, \frac{\partial}{\partial z^D} \right).
$$

According to [4],

$$
R_{\alpha\beta\mu\nu}^\gamma = R_{\alpha\beta\mu\nu} h^{\rho\gamma} = - \left( \frac{\partial \Gamma_{\alpha\mu}^\gamma}{\partial z^\beta} - \frac{\partial \Gamma_{\beta\mu}^\gamma}{\partial z^\alpha} + \frac{\partial \Gamma_{\alpha\nu}^\gamma}{\partial z^\beta} - \frac{\partial \Gamma_{\beta\nu}^\gamma}{\partial z^\alpha} \right) h^\lambda_{\kappa\rho} \left( \frac{\partial h_{\alpha\lambda}}{\partial z^\mu} + \frac{\partial h_{\beta\lambda}}{\partial z^\nu} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} \right).
$$

A direct computation shows

$$
\left( \frac{\partial \Gamma_{\alpha\mu}^\gamma}{\partial z^\beta} + \frac{\partial \Gamma_{\beta\mu}^\gamma}{\partial z^\alpha} \right) h^\gamma_{\rho\tau} = - \left( \frac{\partial^2 h_{\alpha\bar{\bar{\bar{\nu}}}}}{\partial z^\mu \partial \bar{z}^\beta} + \frac{\partial^2 h_{\beta\bar{\bar{\bar{\nu}}}}}{\partial z^\alpha \partial \bar{z}^\beta} \right) + \frac{1}{4} \left( \frac{\partial h_{\alpha\lambda}}{\partial z^\mu} + \frac{\partial h_{\beta\lambda}}{\partial z^\nu} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} \right) h^\lambda_{\kappa\rho} \left( \frac{\partial h_{\alpha\lambda}}{\partial z^\mu} + \frac{\partial h_{\beta\lambda}}{\partial z^\nu} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} \right).
$$

Hence

$$
R_{\alpha\beta\mu\nu} = - \frac{1}{2} \left( \frac{\partial^2 h_{\alpha\bar{\bar{\bar{\nu}}}}}{\partial z^\mu \partial \bar{z}^\beta} + \frac{\partial^2 h_{\beta\bar{\bar{\bar{\nu}}}}}{\partial z^\alpha \partial \bar{z}^\beta} \right) + \frac{1}{4} \left( \frac{\partial h_{\alpha\lambda}}{\partial z^\mu} + \frac{\partial h_{\beta\lambda}}{\partial z^\nu} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} \right) h^\lambda_{\kappa\rho} \left( \frac{\partial h_{\alpha\lambda}}{\partial z^\mu} + \frac{\partial h_{\beta\lambda}}{\partial z^\nu} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} - \frac{\partial h_{\nu\lambda}}{\partial \bar{z}^\kappa} \right). \tag{3.10}
$$

If $h$ is Kähler, then $R = \Re$. Recently, Yang and Zheng [4] proved $h$ is is Kähler if $R = \Re$. 

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6
The complexified Riemannian curvature tensor $R_{ABCD}$ is also skew-symmetric in the first two and last two entries, symmetric between the first two and last two entries, and satisfies the Bianchi identity. Hence

$$ R(u, v, v, u) = 2\text{Re} \left[ R(\xi, \eta, \eta, \bar{\xi}) + R(\xi, \eta, \bar{\eta}, \bar{\xi}) \right] + R(\xi, \bar{\eta}, \bar{\eta}, \bar{\xi}) 
$$

(3.11)

Moreover, the sectional curvature of the G-Kähler-like metric and thus

$$ R(\xi, \eta, \xi, \eta) - \frac{1}{2} \left[ R(\xi, \bar{\eta}, \bar{\eta}, \bar{\xi}) + R(\eta, \xi, \bar{\xi}, \bar{\eta}) \right] . $$

Especially,

$$ R(u, Jv, Jv, u) = 2R(\xi, \bar{\xi}, \xi). $$

(3.12)

For a G-Kähler-like metric, Yang and Zheng [9] pointed out the vanishing of $R_{\alpha \beta \gamma \delta}$ plus the first Bianchi identity imply $R_{\alpha \beta \gamma \delta} = R_{\gamma \beta \alpha \delta}$. In order to prove Theorems 1.3 and 1.4, we need to introduce the following theorem of Lu [5].

**Theorem 3.1.** [5] Let $A_{\alpha \beta \mu \nu}$ be a set complex numbers ($\alpha$, $\beta$, $\mu$, $\nu = 1, \ldots, n$) satisfying

1. $A_{\alpha \beta \mu \nu} = A_{\mu \beta \alpha \nu} = A_{\alpha \beta \nu \mu}$ and $A_{\alpha \beta \mu \nu} = \overline{A_{\beta \alpha \nu \mu}},$

2. $A_{\alpha \beta \mu \nu} (\xi^\alpha \eta^\beta - \eta^\alpha \bar{\xi}^\beta) (\xi^\nu \bar{\eta}^\mu - \eta^\nu \bar{\xi}^\mu) \geq 0$ (or $\leq 0$) for all complex numbers $\xi^\alpha$, $\eta^\alpha$ ($\alpha = 1, \ldots, n$).

Then we always have

$$ |A_{\alpha \beta \mu \nu} \xi^\alpha \bar{\xi}^\beta \eta^\mu \bar{\eta}^\nu|^2 \leq A_{\alpha \beta \mu \nu} \xi^\alpha \bar{\xi}^\beta \xi^\nu \bar{\eta}^\mu A_{\alpha \beta \mu \nu} \eta^\alpha \bar{\eta}^\beta \eta^\mu \bar{\eta}^\nu . $$

(3.13)

**Proof of Theorem 1.3** Yang and Zheng [9] pointed out $R_{\mu \beta \bar{\alpha} \bar{\nu}} = 0$ is equivalent to

$$ R_{\alpha \beta \mu \nu} = R_{\mu \beta \alpha \nu} \text{ or } R_{\alpha \beta \mu \nu} = R_{\alpha \beta \nu \mu} . $$

(3.14)

Hence the G-Kähler-like condition implies

$$ R_{\alpha \beta \mu \nu} = R_{\mu \beta \alpha \nu} = R_{\alpha \beta \nu \mu} = \overline{R_{\beta \alpha \nu \mu}}, $$

(3.15)

$$ R(u, v, v, u) = R(\xi, \eta, \bar{\eta}, \bar{\xi}) - \frac{1}{2} \left[ R(\xi, \bar{\eta}, \bar{\xi}, \eta) + R(\eta, \bar{\xi}, \bar{\eta}, \xi) \right] $$

(3.16)

$$ = \frac{1}{2} R_{\alpha \beta \mu \nu} (\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta) (\xi^\nu \bar{\eta}^\mu - \eta^\nu \bar{\xi}^\mu) . $$

If we replace $v$ and $\eta$ in (3.16) with $Jv$ and $\sqrt{-1} \eta$, then

$$ R(u, Jv, Jv, u) = R(\xi, \bar{\xi}, \xi, \bar{\eta}) + \frac{1}{2} \left[ R(\xi, \bar{\eta}, \bar{\xi}, \bar{\eta}) + R(\eta, \bar{\xi}, \bar{\eta}, \xi) \right] , $$

(3.17)

and thus

$$ R(u, v, v, u) + R(u, Jv, Jv, u) = 2R(\xi, \bar{\xi}, \xi, \bar{\eta}) . $$

(3.18)

Moreover, the sectional curvature of the G-Kähler-like metric $h$ is non-positive or non-negative, so is $R(\xi, \bar{\xi}, \xi, \bar{\eta})$. It follows (3.15), (3.16) and Theorem 3.1 that

$$ |R(\xi, \bar{\xi}, \xi, \bar{\eta})|^2 \leq R(\xi, \bar{\xi}, \xi, \bar{\xi}) R(\eta, \bar{\eta}, \eta, \bar{\eta}) . $$

(3.19)
And the non-positive or non-negative sectional curvature condition plus (3.16) and (3.17) imply
\[\left| R(\xi, \bar{\eta}, \xi, \bar{\eta}) + R(\eta, \bar{\xi}, \eta, \bar{\xi}) \right| \leq 2R(\xi, \bar{\xi}, \eta, \bar{\eta}),\]
which with (3.16) yields
\[R(u, v, v, u)^2 \leq 4R(\xi, \bar{\xi}, \eta, \bar{\eta}) \leq 4R(\eta, \bar{\xi}, \xi, \bar{\eta}) = R(Ju, u, u, Ju) R(Jv, v, v, Jv).\]
by using (3.12) and (3.19). We have
\[\left| R(u, v, v, u) \right| \leq \max \{ |R(Ju, u, u, Ju)|, |R(Jv, v, v, Jv)| \} . \quad (3.20)\]
Note that \(g(y, Jy) = 0\) for all \(y \in TM\). Hence when we assume \(g(u, v) = 0\), if \(h\) has non-negative sectional curvature then
\[K(u, v) \leq \max_{y \in TM} K(y, Jy),\]
and if \(h\) has non-positive sectional curvature then
\[K(u, v) \geq \min_{y \in TM} K(y, Jy).\]
where \(u, v, y\) are unit vectors. We complete the proof.

**Proof of Theorem 1.4** The Kähler-like condition implies
\[\mathcal{R}_{\alpha\beta\mu\nu} = \mathcal{R}_{\mu\beta\alpha\nu} = \mathcal{R}_{\alpha\nu\beta\mu} = \mathcal{R}_{\beta\alpha\nu\mu}. \quad (3.21)\]
By (1.6) in Theorem 1.1, the non-negative or non-positive Chern sectional curvature means
\[\mathcal{R}_{\alpha\beta\mu\nu}(\xi^\alpha \bar{\eta}^\beta - \eta^\alpha \bar{\xi}^\beta)(\xi^\nu \bar{\mu}^\mu - \eta^\nu \bar{\xi}^\mu) \geq 0 \quad \text{or} \quad 0. \quad (3.22)\]
By Theorem 3.1, we have
\[\left| R(\xi, \bar{\xi}, \eta, \bar{\eta}) \right|^2 \leq R(\xi, \bar{\xi}, \eta, \bar{\eta}) R(\eta, \bar{\xi}, \xi, \bar{\xi}) R(\eta, \bar{\eta}, \eta, \bar{\eta}) . \quad (3.23)\]
Therefore, if \(h\) has non-negative Chern sectional curvature then
\[R(\xi, \bar{\xi}, \eta, \bar{\eta}) \leq \max_{\zeta \in T^{1,0}M} R(\zeta, \bar{\zeta}, \zeta, \bar{\zeta}) ,\]
and if \(h\) has non-positive Chern sectional curvature then
\[R(\xi, \bar{\xi}, \eta, \bar{\eta}) \geq \min_{\zeta \in T^{1,0}M} R(\zeta, \bar{\zeta}, \zeta, \bar{\zeta}) ,\]
where \(\xi, \eta, \zeta\) are unit vectors. We complete the proof.

**Acknowledge:** This work is supported by the National Natural Science Foundation of China (Grant No. 12001165).
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