Lattice $\varphi^4$ theory of finite-size effects above the upper critical dimension

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Abstract

We present a perturbative calculation of finite-size effects near $T_c$ of the $\varphi^4$ lattice model in a $d$-dimensional cubic geometry of size $L$ with periodic boundary conditions for $d > 4$. The structural differences between the $\varphi^4$ lattice theory and the $\varphi^4$ field theory found previously in the spherical limit are shown to exist also for a finite number of components of the order parameter. The two-variable finite-size scaling functions of the field theory are nonuniversal whereas those of the lattice theory are independent of the nonuniversal model parameters. One-loop results for finite-size scaling functions are derived. Their structure disagrees with the single-variable scaling form of the lowest-mode approximation for any finite $\xi/L$ where $\xi$ is the bulk correlation length. At $T_c$, the large-$L$ behavior becomes lowest-mode like for the lattice model but not for the field-theoretic model. Characteristic temperatures close to $T_c$ of the lattice model, such as $T_{\text{max}}(L)$ of the maximum of the susceptibility $\chi$, are found to scale asymptotically as $T_c - T_{\text{max}}(L) \sim L^{-d/2}$, in agreement with previous Monte Carlo (MC) data for the five-dimensional Ising model. We also predict $\chi_{\text{max}} \sim L^{d/2}$ asymptotically. On a quantitative level, the asymptotic amplitudes of this large-$L$ behavior close to $T_c$ have not been observed in previous MC simulations at $d = 5$ because of nonnegligible finite-size terms $\sim L^{(4-d)/2}$ caused by the inhomogeneous modes. These terms identify the possible origin of a significant discrepancy between the lowest-mode approximation and previous MC data. MC data of larger systems would be desirable for testing the magnitude of the $L^{(4-d)/2}$ and $L^{4-d}$ terms predicted by our theory.
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1. Introduction

A detailed understanding of the range of applicability of the $\varphi^4$ field theory in $d$ dimensions is of fundamental interest to statistical and elementary particle physics [1]. The perturbative treatment of the critical behavior of the $\varphi^4$ field theory in $d \leq 4$ dimensions is known to be nontrivial because of the problem of infrared divergences. This problem is solved by the renormalization-group theory [1, 2]. Above four dimensions where the critical behavior is mean-field like, no infrared problems of perturbation theory arise and no necessity exists for invoking the renormalization group. Thus the $\varphi^4$ theory above four dimensions appears to be free of essential problems.

This is true, however, only for infinite systems. For the $\varphi^4$ field theory of confined systems above four dimensions [1, 3] there are several aspects of general interest — such as the nature of the fundamental reference lengths, the range of validity of universal finite-size scaling, the relevance of inhomogeneous fluctuations, and the significance of lattice effects — that have remained unresolved until recently. These issues have turned out [4-6] to be closely related to the longstanding problem regarding the verification of earlier phenomenological [7, 8] and analytical [3] predictions for $d > 4$ and regarding the various attempts to test these predictions by means of Monte Carlo (MC) simulations for the five-dimensional Ising model [7-12]. Clarifying these issues is of substantial interest for a better understanding of finite-size effects and of the concept of finite-size scaling [13-18], not only for $d > 4$ but also for the limit $d \to 4$.

Recently [4-6] we have shown, on the basis of exact results in the limit $n \to \infty$ of the $O(n)$ symmetric $\varphi^4$ theory, that finite-size effects for $d > 4$, for cubic geometry and periodic boundary conditions, are more complicated and less universal than predicted previously and that therefore the previous analyses of MC data were not conclusive. In particular we have found that lattice effects and inhomogeneous fluctuations of the order parameter play an unexpectedly important role and that two reference lengths must be employed in a finite-size scaling description. So far, however, no direct justification was given for our conclusions to be valid also for the more relevant case of lattice systems with a finite number $n$ of components of the order parameter.

It is the purpose of the present paper to provide this justification. We shall present a perturbative treatment of a $\varphi^4$ lattice model in one-loop order that leads to quantitative predictions of asymptotic finite-size effects for
$d > 4$ and $n = 1$. We shall show that the previous arguments [4] demonstrating the necessity of using two scaling variables (rather than a single scaling variable) remain valid also for finite $n$ for both the field-theoretic and the lattice model. We also confirm that the finite-size effects of the $\varphi^4$ lattice model differ fundamentally from those of the $\varphi^4$ field theory for general $n$. This implies that the Landau-Ginzburg-Wilson continuum Hamiltonian for an $n$ component order parameter does not correctly describe the finite-size effects of spin models on lattices with periodic boundary conditions above the upper critical dimension.

More specifically, we study the case of cubic geometry (volume $L^d$) with periodic boundary conditions and calculate the asymptotic finite-size scaling form of the order-parameter distribution function $P(\Phi)$ where $\Phi$ is the spatial average of the fluctuating local order parameter $\varphi$. From $P(\Phi)$ we derive the asymptotic finite-size scaling functions of the susceptibility, of the order parameter and of the Binder cumulant [19]. Two scaling variables

$$x = t(L/\xi_0)^2 \quad , \quad t = (T - T_c)/T_c$$

and

$$y = (L/l_0)^{4-d}$$

are needed where $\xi_0$ and $l_0$ are reference lengths related to the bulk correlation length $\xi$, similar to the case $n \to \infty$ [4-6]. $\xi_0$ is the amplitude of $\xi$ for $T > T_c$ at vanishing external field $h$ whereas $l_0$ is (proportional to) the amplitude of $\xi$ at $T = T_c$ for small $h$ [4]. The second length $l_0$ is connected with the four-point coupling $u_0$ according to $l_0 \sim u_0^{1/(d-4)}$. As an alternative choice of scaling variables we also employ $w$ and $y$ where

$$w = xy^{-1/2} = t(L/\bar{\ell})^{d/2}$$

with the reference length

$$\bar{\ell} = l_0(\xi_0/l_0)^{4/d}.$$ 

In addition to the lengths $\xi_0$, $l_0$ and $L$ there is the microscopic length $\bar{\alpha}$ or $\Lambda^{-1}$, i.e., the lattice spacing of the lattice model or the inverse cutoff of the field-theoretic model. For short-range interactions, $\xi_0$ and $l_0$ are expected to be of $O(\bar{\alpha})$ or $O(\Lambda^{-1})$. Our scaling functions presented in Section 4 are valid in the asymptotic range $L \gg \bar{\alpha}$, $\xi \gg \bar{\alpha}$ or $L\Lambda \gg 1$, $\xi\Lambda \gg 1$. 

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The role of the length \( l_0 \) is twofold. Since the dangerous irrelevant character of \( u_0 \) already exists at the mean-field level, the length \( l_0 \) appears via \( \tilde{\ell} \) in the variable \( w \sim u_0^{-1/2} t L^{d/2} \) already at the level of the lowest-mode approximation which takes only homogeneous fluctuations into account \[4\]. The second important role of \( l_0 \) originates from \( u_0 \) being the coupling of the inhomogeneous higher modes. Here \( u_0 \) does not have a dangerous irrelevant character. In fact, these higher modes are relevant for \( d > 4 \) as has been demonstrated for \( n \to \infty \) \[4\] and will be shown in the present paper to be valid for general \( n \), contrary to opposite statements in the literature \[1,3\]. The length \( l_0 \) sets the length scale of the finite-size effects arising from these modes.

Both scaling variables \( x \) and \( y \) or \( w \) and \( y \) must be employed in general, i.e., at any finite value of \( \xi/L < \infty \) in the entire asymptotic \( L^{-1} - \xi^{-1} \) plane (Fig.1) to provide a complete description of asymptotic finite-size effects of the \( \varphi^4 \) theory. Our description is consistent with the general scaling structure in terms of \( t L^2 \) and \( u_0 L^{4-d} \) proposed by Privman and Fisher \[21\] but inconsistent with the reduced structure proposed by Binder et al. \[7\] and with the lowest-mode approximation of Brézin and Zinn-Justin \[3\] in terms of a single scaling variable \( t L^{d/2} \) equivalent to \( w \). We find that it is only the region between the curved dotted lines in Fig.1 where the single-variable scaling forms of Refs. \[3\] and \[7\] are justifiable for the lattice model, but not for the field-theoretic model. The region between the curved lines corresponds to the special case \( \xi/L \to \infty \) in the limit \( L \to \infty \) and \( |t| \to 0 \) at finite \( w \) where the large \(-L\) behavior becomes lowest-mode like for the lattice model. For \( t = 0 \) \((w = 0)\) this was found previously \[4-6\] for the case of the susceptibility and of the Binder cumulant, as conjectured in Ref. \[22\]. As a consequence we shall show that characteristic temperatures, in the sense of "pseudocritical" temperatures \[13\] such as \( T_{\text{max}}(L) \) of the maxima of the susceptibility or the "effective critical temperature" \( T_c(L) \) where the magnetization has its maximum slope, scale asymptotically as \( T_c - T_{\text{max}}(L) \sim L^{-d/2} \) or \( T_c - T_c(L) \sim L^{-d/2} \) for the \( \varphi^4 \) lattice model, in agreement with previous Monte Carlo (MC) data \[4\]. Correspondingly we predict \( \chi_{\text{max}} \sim L^{d/2} \) asymptotically for the lattice model.

On a quantitative level, our theory predicts that the large \(-L\) behavior close to \( T_c \) is strongly affected by nonnegligible finite-size terms \( \sim L^{(4-d)/2} \) caused by the higher (inhomogeneous) modes, as demonstrated recently for
the Binder cumulant at $T_c$. In the analysis of Ref. [9] the observed “slow approach to the scaling limit” was considered to be the most significant discrepancy between the lowest-mode prediction [3] and the MC data. Our theory identifies the terms $\sim L^{(4-d)/2}$ as the possible origin of this discrepancy. We also show that, for the same reason, the successful method of determining bulk $T_c$ via the intersection point of the Binder cumulant [19] is not accurately applicable to finite spin models of small size in $d = 5$ dimensions, as demonstrated in Ref. [6]. New MC simulations over a larger range of $L$ would be desirable for testing the predictions of our theory regarding the magnitude of the $L^{(4-d)/2}$ and $L^{4-d}$ terms.

In Section II we derive some of the bulk properties of the lattice model for $n = 1$ above four dimensions in one-loop order. In particular we identify the amplitudes of the correlation length at $h = 0$ for $T > T_c$ as well as at $T = T_c$ for $h \neq 0$. In Section III we derive the effective Hamiltonian and the order-parameter distribution function in one-loop order. Applications to the asymptotic (large $L \gg \tilde{a}$, small $|t| \ll 1$) finite-size scaling functions and predictions of the lattice model for $d = 5$ are presented and discussed in Section IV. Results for the field-theoretic model are briefly presented in Section V. We summarize the results of our paper in Section VI.

2. Lattice model: Bulk properties for $d > 4$

We consider a $\varphi^4$ lattice Hamiltonian $H$ for the one-component variables $\varphi_i$ with $-\infty \leq \varphi_i \leq \infty$ on the lattice points $x_i$ of a simple-cubic lattice in a cube with volume $V = L^d$ and with periodic boundary conditions. We assume

$$H = \tilde{a}^d \left\{ \frac{1}{2} \sum_i \varphi_i^2 + u_0 (\varphi_i^2)^2 - h \varphi_i + \sum_{i,j} \frac{1}{2\tilde{a}^2} J_{ij} (\varphi_i - \varphi_j)^2 \right\}$$

where $\tilde{a}$ is the lattice spacing. The couplings $J_{ij}$ are dimensionless quantities. The variables $\varphi_j$ have the Fourier representation

$$\varphi_j = \frac{1}{L^d} \sum_\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}_j} \hat{\varphi}_\mathbf{k} .$$

In terms of the Fourier components

$$\hat{\varphi}_\mathbf{k} = \tilde{a}^d \sum_j e^{-i\mathbf{k} \cdot \mathbf{x}_j} \varphi_j$$

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the Hamiltonian $H$ reads

$$H = L^{-d} \sum_{k} \frac{1}{2} \left[ r_0 + \hat{J}_k \hat{\phi}_k \hat{\phi}_{-k} - h \hat{\phi}_0 \right]$$

$$+ u_0 L^{-3d} \sum_{k k' k''} \left( \hat{\phi}_k \hat{\phi}_{k'} \right) \left( \hat{\phi}_{k''} \hat{\phi}_{-k-k'-k''} \right) \tag{8}$$

where

$$\hat{J}_k = \frac{2}{\tilde{a}^2} [J(0) - J(k)] \tag{9}$$

with

$$J(k) = (\tilde{a}/L)^d \sum_{i,j} J_{ij} e^{-ik(x_i-x_j)} \tag{10}$$

The summation $\sum_k$ runs over discrete $k$ vectors with components $k_j = 2\pi m_j/L$, $m_j = 0, \pm 1, \pm 2, \cdots$, $j = 1, 2, \cdots, d$ with a finite cutoff $-\Lambda \equiv -\pi/\tilde{a} \leq k_j < \pi/\tilde{a} \equiv \Lambda$. We assume a finite-range pair interaction $J_{ij}$ such that its Fourier transform has the small $k$ behavior

$$\hat{J}_k = J_0 k^2 + O(k^4) \tag{11}$$

with

$$J_0 = \frac{1}{d} \frac{(\tilde{a}/L)^d}{\tilde{a}^2} \sum_{i,j} \left( J_{ij}/\tilde{a}^2 \right) (x_i - x_j)^2 \tag{12}$$

The complete information on thermodynamic properties is contained in the Gibbs free energy per unit volume (in units of $k_B T$)

$$f = -\frac{1}{L^d} \ln \int \mathcal{D}\varphi \exp(-H) \tag{13}$$

where the symbol $\int \mathcal{D}\varphi$ is the usual abbreviation for the multiple integral over the real and imaginary parts of (the finite number of) the Fourier components $\varphi_k$.

Recently we have found [5] in the large-$n$ limit that the two reference lengths of the finite-size scaling functions for $d > 4$ are determined by the two bulk correlation-length amplitudes $\xi_0$ at $h = 0$ for $T > T_c$ and $l_0$ at $T = T_c$ for small $h$. We shall see that this property remains valid, as far as $\xi_0$ is concerned, also for $n = 1$ in one-loop order. With regard to $l_0$, an additional $n$-dependent prefactor arises that is 1 in the large-$n$ limit and
$3^{-1/2}2^{-1/3}$ for $n = 1$ in one-loop order (see Eq. (28) below). For this purpose we need to identify the amplitudes of the bulk correlation length for $d > 4$. We also calculate the bulk amplitudes of the order parameter $M_b$ and of the susceptibilities $\chi^+_b$ and $\chi^-_b$ above and below $T_c$ as reference quantities for the corresponding finite-size effects.

All of our calculations are carried out at finite cutoff $\Lambda$ (finite lattice spacing $\tilde{a}$). First we derive the asymptotic form of the bulk susceptibility $\chi_b$ at $h = 0$ above and below $T_c$ as well as at $T = T_c$ for small $h$. The bulk Gibbs free energy density is denoted by $f_b = \lim_{L \to \infty} f$. In terms of the bulk order parameter

$$M_b = - \lim_{L \to \infty} \partial f / \partial h$$

the bulk Helmholtz free energy density $\Gamma_b = f_b + M_b h$ reads in one-loop order

$$\Gamma_b(r_0, M_b) = \frac{1}{2} r_0 M_b^2 + u_0 M_b^4 + \frac{1}{2} \int \ln(r_0 + 12 u_0 M_b^2 + \hat{J}_k)$$

where $J_k$ stands for $(2\pi)^{-d} \int d^d k$ with $|k_i| \leq \Lambda$. Above $T_c$ for $h = 0$, the inverse bulk susceptibility $(\chi^+_b)^{-1}$ is

$$(\chi^+_b)^{-1} = (\partial^2 \Gamma_b / \partial M_b^2)_{M_b=0} = r_0 + 12 u_0 \int \frac{\hat{J}_k}{r_0 + \hat{J}_k} + O(u_0^2)$$

which determines the critical value $r_{0c}$ of $r_0$ as

$$r_{0c} = - 12 u_0 \int \frac{\hat{J}_k^{-1}}{r_0 + \hat{J}_k} + O(u_0^2).$$

Thus we rewrite $(\chi^+_b)^{-1}$ above $T_c$ in terms of $r_0 - r_{0c}$ as

$$(\chi^+_b)^{-1} = (r_0 - r_{0c}) \left[ 1 - 12 u_0 \int \frac{\hat{J}_k^{-2}}{r_0 + \hat{J}_k} \right] + O(u_0^2)$$

where

$$r_0 - r_{0c} = a_0 t, \quad t = (T - T_c)/T_c.$$

Note that the integral in Eq. (18) exists only for $d > 4$ and only for finite cutoff. The spontaneous value $M_s$ of the bulk order parameter for $h \to 0$
below $T_c$ is determined by $\partial \Gamma_b / \partial M_b = 0$. This yields for $d > 4$

$$M_s^2 = (4u_0)^{-1}(r_0c - r_0) \left[ 1 + 24u_0 \int \frac{\tilde{J}_k^{-2}}{k} \right] + O(u_0^2).$$

(20)

The inverse susceptibility $(\chi_b^-)^{-1}$ for $h \to 0$ below $T_c$ is in one-loop order

$$(\chi_b^-)^{-1} = \left( \frac{\partial^2 \Gamma}{\partial M^2} \right)_{M_b = M_s} = 2(r_0c - r_0) \left[ 1 - 12u_0 \int \frac{\tilde{J}_k^{-2}}{k} \right] + O(u_0^2).$$

(21)

From the equation of state at $T_c$

$$\frac{\partial \Gamma_b}{\partial M_b} = h = 4u_0M_b^3 \left[ 1 - 36u_0 \int \frac{\tilde{J}_k^{-2}}{k} \right]$$

(22)

we obtain the $h$ dependence of the inverse bulk susceptibility $\chi_c^{-1}$ at $T_c$ as

$$\chi_c^{-1} = \frac{\partial^2 \Gamma_b}{\partial M^2} = 3h^{2/3} \left\{ 4u_0 \left[ 1 - 36u_0 \int \frac{\tilde{J}_k^{-2}}{k} \right] \right\}^{1/3} + O(u_0^3).$$

(23)

For $T \geq T_c$, the bulk susceptibility at finite wave vector $\mathbf{q}$

$$\chi_b(\mathbf{q}) = \lim_{L \to \infty} \frac{a^{2d}}{L^d} \sum_{i,j} \langle \varphi_i \varphi_j \rangle \ e^{-i\mathbf{q}(\mathbf{x}_i - \mathbf{x}_j)}$$

(24)

has the one-loop form (for both $h = 0$ and $h \neq 0$)

$$\chi_b(\mathbf{q})^{-1} = \chi_b(0)^{-1} + \tilde{J}_q \left[ 1 + O(u_0^2) \right].$$

(25)

Thus the square of the bulk correlation length for $T \geq T_c$ is given by

$$\xi^2 = \chi_b(0) \left[ \partial \chi_b(k)^{-1} / \partial k^2 \right]_{k=0} = J_0 \chi_b(0) + O(u_0^2).$$

(26)

Substituting Eqs. (18) and (23) into Eqs. (25) and (26) yields the asymptotic form for $d > 4$

$$\xi = \xi_0 t^{-1/2}, \quad t > 0, \quad h = 0,$$

(27)
and
\[ \xi = 3^{-1/2}2^{-1/3}l_0(h^2d^2J_0^{-1})^{-1/6} , \ t = 0 , \ h \neq 0 \]
with the lengths
\[ \xi_0 = a_0^{-1/2}J_0^{1/2} \left[ 1 + 12u_0 \int_k \left( \hat{J}_k^{-2} \right) \right]^{1/2} + O(u_0^2) \]
and
\[ l_0 = \left\{ u_0J_0^{-2} \left[ 1 + 36u_0 \int_k \left( \hat{J}_k^{-2} \right) \right]^{-1} \right\}^{1/(d-4)}. \]

These lengths will appear also in the finite-size scaling functions in Sect. IV. We see that for \( d > 4 \) the fluctuations (that enter via the one-loop integrals) only modify the amplitudes but do not change the mean-field \( t \) and \( h \) dependence. The ”dangerous” \( u_0 \) dependence \cite{2} of \( \xi \) at \( T_c \), Eqs. (28) and (30), is clearly seen in \( l_0 \sim u_0^{1/(d-4)} \). We note that both \( \xi_0 \) and \( l_0 \) are cutoff dependent via \( \int_k \hat{J}_k^{-2} \). In rewriting \( [1 - 12u_0 \int_k \hat{J}_k^{-2}]^{-1/2} \) as \( [1 + 12u_0 \int_k \hat{J}_k^{-2}]^{1/2} \) in \( \xi_0 \) (and similarly in \( l_0 \)) we have been guided by the resummed forms of \( \xi_0 \) and \( l_0 \) in the limit \( n \to \infty \) (Eqs. (141) and (142) in Ref. \cite{4}).

Corresponding results can be derived for the continuum \( \varphi^4 \) Hamiltonian (see Eq. (62) below) with periodic boundary conditions, similar to the case \( n \to \infty \) studied previously \cite{4}. This amounts essentially to replacing \( \hat{J}_k \) by \( k^2 \) in the equations given above. As far as bulk properties of \( \chi_+^0, \chi^- \), \( \xi \) and \( M_b \) are concerned, only the nonuniversal amplitudes are modified but the \( t \) and \( h \) dependence remains identical for the field-theoretic and the lattice \( \varphi^4 \) model. For the finite system, however, lattice effects become significant as we shall see in the subsequent Sections. For the specific heat even the (finite) bulk value at \( T_c \) turns out to be different for the field-theoretic and the lattice model, similar to the case \( n \to \infty \) \cite{4}.

3. Order-parameter distribution function for \( d > 4 \)

3.1. General form in one-loop order

A perturbation approach to finite-size effects of the \( \varphi^4 \) lattice model for \( d > 4 \) can be set up in a way similar to the previous finite-size perturbation theory.
for $d < 4$ above and below $T_c$ \[23\]. We decompose
\[\varphi_j = \Phi + \sigma_j\] (31)
and shall derive an effective Hamiltonian $H^{\text{eff}}$ \[3, 23\] for the lowest (homogeneous) mode
\[\Phi = \frac{\tilde{a}}{L^d} \sum_j \varphi_j\] (32)
by a perturbative treatment of the fluctuations of the higher modes
\[\sigma_j = L^{-d} \sum_{k \neq 0} \hat{\phi}_k e^{i k \cdot x_j} .\] (33)
Correspondingly we write the lattice Hamiltonian, Eq. (5), in the form
\[H = H_0(\Phi) + H_1 + H_2 ,\] (34)
with the lowest-mode Hamiltonian
\[H_0(\Phi) = L^d \left( \frac{1}{2} r_0 \Phi^2 + u_0 \Phi^4 - h \Phi \right) ,\] (35)
the Gaussian part
\[H_1 = L^{-d} \sum_{k \neq 0} \frac{1}{2} (\bar{r}_{0L} + \hat{J}_k) \hat{\sigma}_k \hat{\sigma}_{-k} ,\] (36)
with
\[\bar{r}_{0L} = r_0 + 12 u_0 M_0^2 ,\] (37)
and the perturbation part
\[H_2 = \tilde{a}^d \sum_j \left[ 6 u_0 (\Phi^2 - M_0^2) \sigma_j^2 + 4 u_0 \sigma_j^3 + u_0 \sigma_j^4 \right] .\] (38)
Unlike the case $d < 4$, we must work, for $d > 4$, at finite cutoff. Thus we shall incorporate the finite shift $r_{0c} \sim O(u_0)$, Eq. (17), of the parameter $r_0$ whereas in the previous \[23\] dimensional regularization at infinite cutoff there was no $O(u_0)$ contribution to $r_{0c}$. Thus we define
\[M_0^2 = \frac{1}{Z_0^{\text{eff}}} \int_{-\infty}^{\infty} d\Phi \, \Phi^2 \exp(-H_0^{\text{eff}})\] (39)
where now
\[ H_0^{\text{eff}} = L^d \left[ \frac{1}{2} (r_0 - r_{0c}) \Phi^2 + u_0 \Phi^4 - h \Phi \right] \] (40)
and
\[ Z_0^{\text{eff}} = \int_{-\infty}^{\infty} d\Phi \exp(-H_0^{\text{eff}}). \] (41)
contain the shifted variable \( r_0 - r_{0c} \).

The partition function is decomposed as
\[ Z = \int \mathcal{D}\phi \, e^{-H} = \int_{-\infty}^{\infty} d\Phi \exp[-(H_0 + \Gamma)], \] (42)
where
\[ \Gamma(\Phi) = -\ln \int \mathcal{D}\sigma \exp[-(H_1 + H_2)] \] (43)
is determined by the higher modes within perturbation theory. We rewrite
\[ H_0(\Phi) + \Gamma(\Phi) = \Gamma(0) + H^{\text{eff}}(\Phi) \] (44)
and define the order-parameter distribution function
\[ P(\Phi) = \frac{1}{Z^{\text{eff}}} \exp[-H^{\text{eff}}(\Phi)], \] (45)
\[ Z^{\text{eff}} = \int_{-\infty}^{\infty} d\Phi \exp[-H^{\text{eff}}(\Phi)]. \] (46)
In a perturbation calculation with respect to \( H_2 \) we obtain the effective Hamiltonian in one-loop order
\[ H^{\text{eff}}(\Phi) = L^d \left[ \frac{1}{2} r_0^{\text{eff}} \Phi^2 + u_0^{\text{eff}} \Phi^4 + O(\Phi^6) - h \Phi \right] \] (47)
where
\[ r_0^{\text{eff}} = r_0 - r_{0c} + 12u_0 \left[ L^{-d} \sum_{k \neq 0} (r_0 L + \hat{J}_k)^{-1} - \int_k \hat{J}_k^{-2} \right] + 144u_0^2 M_0^2 L^{-d} \sum_{k \neq 0} (r_0 L + \hat{J}_k)^{-2}, \] (48)
\[ u_0^{\text{eff}} = u_0 - 36u_0^2 L^{-d} \sum_{k \neq 0} (r_{0L} + \hat{j}_k)^{-2} . \] (49)

In Eq. (48) we have added and subtracted \( r_{0c} \) and have replaced \( r_{0L} \), in the \( O(u_0) \) terms, by
\[ r_{0L} = r_0 - r_{0c} + 12u_0 M_0^2 . \] (50)
This quantity is a positive function of \( r_0 - r_{0c} \) for arbitrary \( -\infty \leq r_0 - r_{0c} \leq \infty \) at any finite \( L \).

Moments of the distribution function can now be calculated as
\[ < \Phi^m > = \int_{-\infty}^{\infty} d\Phi \Phi^m P(\Phi) \] (51)
and
\[ < |\Phi|^m > = \int_{-\infty}^{\infty} d\Phi |\Phi|^m P(\Phi) . \] (52)

Because of the (one-loop) \( \Phi^4 \) structure of \( H^{\text{eff}} \), these averages can be expressed in terms of the well-known functions
\[ \vartheta_m(Y) = \frac{\int_0^\infty ds s^m \exp(-\frac{1}{2} Y s^2 - s^4)}{\int_0^\infty ds \exp(-\frac{1}{2} Y s^2 - s^4)} \] (53)
that appear also in the finite-size theory below four dimensions \([23, 24] \). The moments determine several thermodynamic quantities such as \([23] \) the susceptibilities
\[ \chi^+ = L^d < \Phi^2 > , \] (54)
\[ \chi^- = L^d (< \Phi^2 > - < |\Phi|^2 >) , \] (55)
the ”magnetization” \( M = < |\Phi| > \), and the Binder cumulant
\[ U = 1 - \frac{1}{3} < \Phi^4 > / < \Phi^2 >^2 . \] (56)
In terms of the effective parameters $r_0^{\text{eff}}$ and $u_0^{\text{eff}}$ they can be expressed in one-loop order as

\begin{align*}
\chi^+ &= (L^d/u_0^{\text{eff}})^{1/2} \partial_2 (Y^{\text{eff}}), \\
\chi^- &= (L^d/u_0^{\text{eff}})^{1/2} \left[ \partial_2 (Y^{\text{eff}}) - \partial_1 (Y^{\text{eff}})^2 \right], \\
M &= (L^d u_0^{\text{eff}})^{-1/4} \partial_1 (Y^{\text{eff}}), \\
U &= 1 - \frac{1}{3} \partial_4 (Y^{\text{eff}})/\partial_2 (Y^{\text{eff}})^2,
\end{align*}

with the dimensionless quantity

\[Y^{\text{eff}} = L^{d/2} r_0^{\text{eff}} (u_0^{\text{eff}})^{-1/2}.
\]

We note that at this stage of perturbation theory these expressions do not yet represent a systematic expansion with respect to the coupling $u_0$, compare Eqs. (5.19)-(5.22) of Ref. [23].

Corresponding formulas are obtained for the case of the Landau-Ginzburg-Wilson continuum Hamiltonian

\[H = \int d^d x \left[ \frac{1}{2} r_{0} \varphi^2 + \frac{1}{2} (\nabla \varphi)^2 + u_0 \varphi^4 - h \varphi \right]
\]

with the field $\varphi(x)$ by the replacement $\hat{J}_k \to k^2$ in the sums of the one-loop terms of the effective parameters $r_0^{\text{eff}}$ and $u_0^{\text{eff}}$.

A justification of the above perturbation theory can be given in terms of the order-parameter distribution function where all higher modes are treated in a nonperturbative way [24].

### 3.2 Asymptotic form of the effective parameters

In order to study the asymptotic finite-size critical behavior we shall consider the limit of $L \gg \tilde{a}$, $\xi \gg \tilde{a}$ or $L \Lambda \gg 1$, $\xi \Lambda \gg 1$. For this purpose we
decompose the perturbation part of $r_{0}^{\text{eff}}$ and $u_{0}^{\text{eff}}$ into bulk integrals and finite-size contributions in the following way,

$$
r_{0}^{\text{eff}} = (r_{0} - r_{0c}) \left\{ 1 - 12u_{0} \int_{k} [\hat{J}_{k}(r_{0L} + \hat{J}_{k})]^{-1} \right\}
+ 144u_{0}^{2}M_{0}^{2} \left\{ \int_{k} (r_{0L} + \hat{J}_{k})^{-2} - \int_{k} [\hat{J}_{k}(r_{0L} + \hat{J}_{k})]^{-1} \right\}
- 12u_{0}\Delta_{1}(r_{0L}) - 144u_{0}^{2}M_{0}^{2}\Delta_{2}(r_{0L}),
$$

$$
u_{0}^{\text{eff}} = u_{0} - 36u_{0}^{2} \int_{k} (r_{0L} + \hat{J}_{k})^{-2} + 36u_{0}^{2}\Delta_{2}(r_{0L}),
$$

with

$$
\Delta_{m}(r_{0L}) = \int_{k} (r_{0L} + \hat{J}_{k})^{-m} - L^{-d} \sum_{k \neq 0} (r_{0L} + \hat{J}_{k})^{-m}.
$$

In the lowest-mode approximation we would have simply $r_{0}^{\text{eff}} = r_{0}$, $u_{0}^{\text{eff}} = u_{0}$. Up to this point, the determination of the effective Hamiltonian for the field-theoretic model, Eq. (62), is still parallel to that of the lattice model. The corresponding formulas are simply obtained by the replacement $\hat{J}_{k} \rightarrow k^{2}$.

The crucial difference, however, comes from the large-$L$ behavior of $\Delta_{m}$, as shown recently for the special case $m = 1$ and $r_{0L} = 0$ [4]. For general $r_{0L}$ we find, for the lattice model, the cutoff-independent large-$L$ behavior

$$
\Delta_{m}(r_{0L}) = J_{0}^{-m}I_{m}(r_{0L}J_{0}^{-1}L^{2})L^{2m-d} + O(e^{-L/\tilde{a}})
$$

with

$$
I_{m}(x) = (2\pi)^{-2m} \int_{0}^{\infty} dy \, y^{m-1} e^{-(xy/4\pi^{2})} \left[ (\pi/y)^{d/2} - K(y)^{d} + 1 \right]
$$

where $K(y) = \sum_{j=-\infty}^{\infty} \exp(-yj^{2})$. For the field-theoretic model, however, the corresponding large-$L$ behavior differs significantly according to the cutoff dependent result

$$
\int_{k} (r_{0L} + k^{2})^{-m} - L^{-d} \sum_{k \neq 0} (r_{0L} + k^{2})^{-m} = I_{m}(r_{0L}L^{2})L^{2m-d}
+ \Lambda^{d-2m} \left\{ a_{m}(d,r_{0L}\Lambda^{-2})(\Lambda L)^{-2} + O\left[ (\Lambda L)^{-4} \right] \right\}
$$
where

\[ a_m(d, r_0 \Lambda^{-2}) = \frac{d}{3(2\pi)^{d-2}} \int_0^\infty dx x^m \left[ \int_{-1}^1 dy e^{-y^2 x} \right]^{d-1} \exp \left[ -(1 + r_0 \Lambda^{-2}) x \right]. \]

This leads to

\[
\begin{align*}
\begin{array}{c}
\text{(69)} \\

r_0^{eff} &= (r_0 - r_{0c}) \left\{ 1 - 12u_0 \int_k \left[ k^2 (r_0 L^2 + k^2)^{-2} \right]^{-1} \right\} \\
&+ \quad 144u_0^2 M_0^2 \left\{ \int_k (r_0 L^2)^{d-2} + \Lambda^{d-2} a_1(d, r_0 \Lambda^{-2}) (\Lambda L)^{-2} \right\} \\
&- \quad 12u_0 \left[ I_1(r_0 L^2)L^{2-d} + \Lambda^{d-2} a_1(d, r_0 \Lambda^{-2}) (\Lambda L)^{-2} \right] \\
&- \quad 144u_0^2 M_0^2 \left[ I_2(r_0 L^2)L^{4-d} + \Lambda^{d-4} a_2(d, r_0 \Lambda^{-2}) (\Lambda L)^{-2} \right], \tag{70}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{(71)} \\

u_0^{eff} = u_0 - 36u_0^2 \int_k (r_0 L^2 + k^2)^{-2} \\
&+ \quad 36u_0^2 \left[ I_2(r_0 L^2)L^{4-d} + \Lambda^{d-4} a_2(d, r_0 \Lambda^{-2}) (\Lambda L)^{-2} \right] \\
\end{array}
\end{align*}
\]

for the field-theoretic model. We note that for \( d < 4 \) the cutoff dependent terms in Eqs. (70) and (71) vanish in the limit \( \Lambda \to \infty \). For \( d > 4 \), however, part of these terms are divergent for \( \Lambda \to \infty \) and cannot be dropped. In particular these terms carry the important size dependence \( \sim L^{-2} \) of \( r_0^{eff} \) which is not present in Eq. (63) for the lattice model and which has been overlooked previously [1, 3]. Employing the method of dimensional regularization [1] in Eq. (68) would mean that the finite results for \( d < 4 \) at \( \Lambda = \infty \) are continued analytically to \( d > 4 \). This would yield the same (cutoff-independent) result for \( r_0^{eff} \) and \( u_0^{eff} \) of the field-theoretic case as of the lattice model. Thus dimensional regularization would omit the important analytic \( L^{-2} \) dependence in Eqs. (68) and (70). This omission, however, cannot be justified — unlike the omission of an analytic \( t \) dependence in bulk critical phenomena. Thus the method of dimensional regularization may yield misleading results in the finite-size field theory above the upper critical dimension.

The remaining bulk integrals in \( r_0^{eff} \) and \( u_0^{eff} \) have finite limits for \( r_{0L} \to 0 \) (large \( L \), small \( |t| \)) for both the lattice and field-theoretic model. Taking the
limit \( r_{0L} \to 0 \) in these integrals is justified only if \( |t| \ll 1 \) and \((L/\bar{a})^{-d/2} \ll 1 \) or \((\Lambda L)^{-d/2} \ll 1 \). This restriction should be kept in mind when applying our asymptotic scaling functions to MC data of spin models of small size. The asymptotic expressions for the lattice model for \( d > 4 \) read

\[
 r_{0}^{\text{eff}} = (r_{0} - r_{0c}) \left[ 1 - 12u_{0} \int_{k} \hat{J}_{k}^{-2} \right] - 12u_{0} J_{0}^{-1} L^{-d/2} I_{1}(r_{0L} J_{0}^{-1} L^{2}) \\
- 144u_{0}^{2} M_{0}^{2} J_{0}^{-2} L^{-d/2} I_{2}(r_{0L} J_{0}^{-1} L^{2}) \tag{72}
\]

\[
 u_{0}^{\text{eff}} = u_{0} \left[ 1 - 36u_{0} \int_{k} \hat{J}_{k}^{-2} + 36u_{0} J_{0}^{-2} I_{2}(r_{0L} J_{0}^{-1} L^{2}) L^{-d/2} \right] \tag{73}
\]

The corresponding results of the field-theoretic model for \( d > 4 \) are obtained from Eqs. (70) and (71) as

\[
 r_{0}^{\text{eff}} = (r_{0} - r_{0c}) \left[ 1 - 12u_{0} \int_{k} k^{-4} \right] - 12u_{0} L^{-d/2} I_{1}(r_{0L} L^{2}) \\
- 144u_{0}^{2} M_{0}^{2} L^{-d/2} I_{2}(r_{0L} L^{2}) - 12u_{0} A^{-d/4} a_{1}(d, r_{0L} \Lambda^{-2}) L^{-2} \\
- 144u_{0}^{2} A^{-d/4} M_{0}^{2} a_{2}(d, r_{0L} \Lambda^{-2})(\Lambda L)^{-2} \tag{74}
\]

\[
 u_{0}^{\text{eff}} = u_{0} \left[ 1 - 36u_{0} \int_{k} k^{-4} + 36u_{0} L^{-d/2} I_{2}(r_{0L} L^{2}) \\
+ 36u_{0} A^{-d/4} a_{2}(d, r_{0L} \Lambda^{-2})(\Lambda L)^{-2} \right] \tag{75}
\]

Substituting Eqs. (72) - (75) into Eqs. (47) - (49) completes our calculation of the asymptotic form of \( H_{\text{eff}} \) and of the order-parameter distribution function \( P(\Phi) \), Eq. (45), in one-loop order for \( d > 4 \) and \( n = 1 \). The restriction "asymptotic" means that these results, Eqs. (72)-(75), are applicable to arbitrary \( r_{0L} L^{2} \) only in the sense that \( L/\bar{a} \) must be large and \( |t| \) must be small.

As found already in the large-\( n \) limit [4], the leading "shift of \( T_{c} \)" [3] in \( r_{0}^{\text{eff}} \), Eq. (72), is not just a temperature independent constant \( \sim L^{2-d} \) for the lattice model but a more complicated function of \( r_{0L} L^{2} \). For the field-theoretic model the leading shift is proportional to \( L^{-2} \) according to Eq. (74) and is nonuniversal, i.e., explicitly cutoff-dependent. This result differs from the (temperature independent) shift \( \sim L^{2-d} \) predicted for the field-theoretic model [3] and from the shift \( \sim L^{-d/2} \) considered in previous work [7–9]. Our shifts are caused by the (inhomogeneous) higher modes of
the order-parameter fluctuations. They cannot be neglected even for large $L/\tilde{a}$ (except for the extreme case of the bulk limit) and cannot be regarded only as "corrections" to the lowest-mode approximation since they lead to a two-variable finite-size scaling structure for both the field-theoretic and the lattice model, in contrast to the single-variable scaling structure of the lowest-mode approximation, as will be further discussed in Section 4.2.

Our results can be generalized to $n > 1$ by means of a nonperturbative treatment of the order-parameter distribution function of the $O(n)$ symmetric $\varphi^4$ theory [24]. It is obvious that this does not change the conclusions regarding the structural differences between the finite-size effects of the field-theoretic and lattice versions of the $\varphi^4$ model.

4. Finite-size scaling functions of the lattice model

4.1 Analytic results

In the following we consider only the case $h = 0$. Inspection of the asymptotic expressions of $r_0^{\text{eff}}$ and $u_0^{\text{eff}}$, Eqs. (72) and (73), shows that they depend on three different lengths $\xi_0$, Eq.(29), $l_0$, Eq.(30), and $L$. Therefore there exist different ways of writing $H^{\text{eff}}$ in a finite-size scaling form.

Considering the ratio $\xi/L$ as a fundamental dimensionless variable [13-15] leads to the following scaling form

$$H^{\text{eff}} = F(z, x, y) = \frac{1}{2}r^{\text{eff}}(x, y)z^2 + u^{\text{eff}}(x, y)z^4,$$  \hspace{1cm} (76)

where $x$ and $y$ are the scaling variables given in Eqs. (1) and (2) and $z$ is the scaled order-parameter variable

$$z = J_0^{-1/2} L^{(d-2)/2} \Phi .$$  \hspace{1cm} (77)

The scaled effective parameters are in one-loop order

$$r^{\text{eff}}(x, y) = r_0^{\text{eff}} L^2 J_0^{-1} = x - 12I_1(\tilde{r})y - 144\partial_2(y_0)I_2(\tilde{r})y^{3/2}$$  \hspace{1cm} (78)

and

$$u^{\text{eff}}(x, y) = u_0^{\text{eff}} L^{4-d} J_0^{-2} = y + 36I_2(\tilde{r})y^2$$  \hspace{1cm} (79)

with

$$\tilde{r} = x + 12\partial_2(y_0)y^{1/2}$$  \hspace{1cm} (80)
and
\[ y_0 = xy^{-1/2} \quad . \]  
(81)

This leads to the finite-size scaling form
\[ \chi^\pm = L^2 P^\pm(x, y) \]  
(82)

and
\[ M = L^{(2-d)/2} P_M(x, y) \]  
(83)

with the two-variable scaling functions
\[
\begin{align*}
P^+_\chi(x, y) & = J_0^{-1} \left[ u^{\text{eff}}(x, y) \right]^{-1/2} \varphi_2(Y(x, y)), \\
P^-_\chi(x, y) & = J_0^{-1} \left[ u^{\text{eff}}(x, y) \right]^{-1/2} \left\{ \varphi_2(Y(x, y)) - [\varphi_1(Y(x, y))]^2 \right\}, \\
P_M(x, y) & = J_0^{-1/2} \left[ u^{\text{eff}}(x, y) \right]^{-1/4} \varphi_1(Y(x, y)), \\
U(x, y) & = 1 - \frac{1}{3} \frac{\varphi_4(Y(x, y))}{[\varphi_2(Y(x, y))]^2},
\end{align*}
\]
(84) \(\text{to} \) (87)

where
\[ Y(x, y) = r^{\text{eff}}(x, y) \left[ u^{\text{eff}}(x, y) \right]^{-1/2} \quad . \]  
(88)

The traditional finite-size scaling theories for \( d < 4 \) \([13-18]\) have no asymptotic dependence on a second scaling variable \( y \). We see that our scaling functions do not depend on the nonuniversal model parameters \( \tilde{a}, J_{ij}, a_0, u_0 \), except via the length scales \( \xi_0 \) and \( l_0 \) contained in \( x \) and \( y \), and apart from the metric prefactors \( J_0^{-1} \) and \( J_0^{-1/2} \) in Eqs. (84)-(86). Thus we may consider these functions to be universal in a restricted sense, i.e., for a certain class of lattice models (rather than continuum models, see below).

From the previous one-loop finite-size theory \([23]\) and the successful comparison with high-precision MC data in three dimensions \([23]\) it has become clear that careful consideration must be devoted to the appropriate form of evaluating these one-loop results. The previous analysis indicated that the prefactor \( (u^{\text{eff}})^{-1/2} \) in Eqs. (84)-(88) should be further expanded with respect to the coupling \( u_0 \), in the spirit of a systematic perturbation approach, see Eqs. (5.45), (5.46) and footnote 1 of Sec.7 of Ref. \([23]\), as well as Eqs. (6.15), (6.16), (6.31) and (6.32) of Ref. \([23]\). Thus the expanded forms\( (u^{\text{eff}})^{-1/2} = y^{-1/2}[1 + 18I_2(\bar{r})y]^{-1} \) or\( (u^{\text{eff}})^{-1/2} = y^{-1/2}[1 - 18I_2(\bar{r})y] \) should
be substituted into Eqs. (84), (85) and (88), respectively [and similarly for \((u^{\text{eff}})^{-1/4}\) in Eq. (86)]. These expanded forms should be taken into account in a future quantitative comparison of Eqs. (84)-(88) with MC data. In the present paper we confine ourselves, for simplicity, to the unexpanded form of Eqs. (84)-(88), as has been done in the result for \(U(x,y)\) presented in Ref. [6]. The same comment applies to Eqs. (99)-(103) below.

At \(T_c (x = 0)\) we obtain from Eqs. (78) - (88) the large \(-L\) behavior in one-loop order for \(d > 4\)

\[
\chi_c^+ \sim L^2 J_0^{-1} y^{-1/2} \varphi_2(0) \sim L^{d/2} ,
\]

\[
M_c \sim L^{(2-d)/2} J_0^{-1/2} y^{-1/4} \varphi_1(0) \sim L^{-d/4} ,
\]

\[
\lim_{L \to \infty} U(0, y) = 1 - \frac{1}{3} \varphi_4(0)/\varphi_2(0)^2 = 0.2705 .
\]  

The exponents in Eqs. (89) and (90) and the asymptotic value in Eq. (91) are identical with those obtained in the lowest-mode approximation at \(T_c\). The dangerous irrelevant character of \(u_0 \sim y\) is clearly exhibited in Eqs. (89) and (90) in the form of \(y^{-1/2}\) and \(y^{-1/4}\).

Alternatively we may employ, instead of \(x\) and \(y\), the variables \(w\) and \(y\) where \(w\) is given by Eq. (3). This implies the following scaling form

\[
H^{\text{eff}} = \tilde{F}(s, w, y) = \frac{1}{2} \tilde{r}^{\text{eff}}(w, y)s^2 + \tilde{u}^{\text{eff}}(w, y)s^4
\]

with the scaled order-parameter variable

\[
s = J_0^{-1/2} L^{d/4} l_0^{(d-4)/4} \Phi
\]  

The scaled effective parameters are

\[
\tilde{r}^{\text{eff}}(w, y) = r^{\text{eff}} y^{-1/2}
\]

\[
= w - 12 I_1(\tilde{r}) y^{1/2} - 144 \varphi_2(w) I_2(\tilde{r}) y
\]

and

\[
\tilde{u}^{\text{eff}}(w, y) = u^{\text{eff}} y^{-1} = 1 + 36 I_2(\tilde{r}) y
\]
with
\[ \tilde{r} = [w + 12\vartheta_2(w)]y^{1/2}. \]  
(96)

This leads to the finite-size scaling forms
\[ \chi^\pm = L^{d/2} \tilde{P}_\chi^\pm (w, y) \]  
(97)
and
\[ M = L^{-d/4} \tilde{P}_M (w, y) \]  
(98)
with the two-variable scaling functions
\[ \tilde{P}_\chi^+ (w, y) = A \left[ \tilde{u}_{\text{eff}}^e (w, y) \right]^{-1/2} \vartheta_2 (\tilde{Y} (w, y)), \]  
(99)
\[ \tilde{P}_\chi^- (w, y) = A \left[ \tilde{u}_{\text{eff}}^e (w, y) \right]^{-1/2} \left\{ \vartheta_2 (\tilde{Y} (w, y)) - [\vartheta_1 (\tilde{Y} (w, y))]^2 \right\}, \]  
(100)
\[ \tilde{P}_M (w, y) = \sqrt{A} \left[ \tilde{u}_{\text{eff}}^e (w, y) \right]^{-1/4} \vartheta_1 (\tilde{Y} (w, y)) \]  
(101)
\[ \tilde{U} (w, y) = 1 - \frac{1}{3} \vartheta_4 (\tilde{Y} (w, y)) / \left[ \vartheta_2 (\tilde{Y} (w, y)) \right]^2, \]  
(102)
where \( A = J_0^{-1} l_0^{(4-d)/2} \) and
\[ \tilde{Y} (w, y) = \tilde{r}_{\text{eff}} (w, y) \left[ \tilde{u}_{\text{eff}}^e (w, y) \right]^{-1/2}. \]  
(103)

In the lowest-mode approximation the \( y \) dependence in Eqs. (92) - (103) is dropped and Eqs. (94) and (103) are replaced by \( \tilde{Y} = \tilde{r}_{\text{eff}} = u_0^{-1/2} a_0 t L^{d/2} \).

These results will be discussed and applied to \( d = 5 \) in the following Subsections.

### 4.2 Discussion

Both sets of scaling variables \( (x, y) \) and \( (w, y) \) are useful in the analysis of the finite-size scaling structure. First we consider the \( (x, y) \) representation. In order to elucidate the effect of the fluctuations of the (inhomogeneous) higher modes above and below \( T_c \) we assume \( y \) to be small (large \( L/l_0 \)) and expand \( \chi^+ \) and \( \chi^- \) with respect to \( y \) at finite \( |x| > 0 \), i.e., \( T \neq T_c \). This
yields

\[
\chi^+ = \chi_b^+ \left\{ 1 - 12 \left[ x^{-1} - I_1(x) \right] \frac{y}{x} + O(y^2/x^2) \right\}, \quad x > 0 \quad (104)
\]

\[
\chi^- = \chi_b^- \left\{ 1 + \left[ 15x^{-1} + 12I_1(-2x) - 36xI_2(-2x) \right] \frac{y}{x} + O(y^2/x^2) \right\}, \quad x < 0 \quad (105)
\]

where \( \chi_b^+ \) and \( \chi_b^- \) are the bulk quantities given in Eqs. (18) and (21). Similar expressions can be derived for \( M \) and \( U \).

The terms \( \sim x^{-1} \) in the square brackets can be traced back to the lowest-mode contributions whereas the terms \( \sim I_1(x) \) and \( \sim I_m(-2x) \) arise from the higher (inhomogeneous) modes. If the latter terms were ignored one could rewrite \( \chi^\pm \) in a lowest-mode form with the single variable \( x/y^{1/2} \), as noted previously in the case \( n \to \infty \) [4–6]. For any finite \( |x| = L^2/\xi^2 \), however, i.e., along the straight dashed lines in the \( L^{-1} - \xi^{-1} \) plane (Fig. 1), there exists no argument that would allow one to ignore the \( I_m \) terms arising from the higher modes. This proves the necessity of including two separate scaling variables in general. In particular it is misleading to consider the finite-size effects of the higher modes as a ”correction” to the lowest-mode approximation — in the same sense as changes of mean-field exponents caused by critical fluctuations for \( d < 4 \) should not be considered as ”corrections”. The crucial point is that the higher modes cause a new structure of the finite-size scaling functions that cannot be written in terms of a single variable \( x/y^{1/2} \) (except for the special case \( x \to 0, y \to 0 \) at finite \( x/y^{1/2} \), see below). This structural aspect is a matter of principle, regardless of how large or small the effect of the higher modes might be.

In this context we take up a nontrivial aspect in the discussion in the previous literature about role played by the ”shift of \( T_c \)”. It was asserted that a term of the type \( \sim L^{2-d} \) in the parameter \( r_0^{\text{eff}} \) of \( H^{\text{eff}} \) represents a ”correction to scaling” [9] or a ”subdominant term” [12] that can be neglected in the large \(-L\) limit compared to the lowest mode part \( r_0 - r_{0c} = a_0t \). This assertion is incorrect, however, for the reasons just given in the preceding paragraph. The term \( \sim L^{2-d} \) in \( r_0^{\text{eff}} \), Eq. (72), is, in fact, the origin of the terms \( \sim I_1(x) \) of \( \chi^\pm \) in Eqs. (104) and (105) which we have shown to represent nonnegligible contributions rather than ”corrections”. Similarly, the term \( \sim M_0^2L^{4-d} \) in \( r_0^{\text{eff}} \), Eq. (72), is the origin of the nonnegligible higher-mode contribution \( I_2(-2x) \) to \( \chi^- \) in Eq. (105).
The expansion in Eqs. (104) and (105) breaks down in the limit of small $|x|$, i.e., large $\xi/L$. This includes the large -$L$ limit at $T = T_c$ where the exponent of the susceptibility $\chi_c \sim L^{d/2}$ and the Binder cumulant $U_c$ have been found [4–6] to agree with the lowest-mode approximation for the lattice model (but not for the field-theoretic model). Here this result is seen from the representation in terms of $w$ and $y$ as given in Eqs. (92)-(103). In this representation the single-variable lowest-mode like structure appears as the leading $w$ dependence whereas the higher-mode contributions $\sim I_1$ and $\sim I_2$ are multiplied by $y^{1/2}$ and $y$. As noted previously [4] these higher-mode contributions are not of a dangerous irrelevant character even though the dangerous irrelevant four-point coupling $u_0$ determines the length scale $l_0 \sim u_0^{1/(d-4)}$.

We shall see below that the $y^{1/2}$ terms are quantitatively important. Nevertheless, at first sight it seems justified to consider the latter contributions as asymptotically negligible in the limit $y \to 0$ corresponding to $L \to \infty$. In the terminology of the renormalization group, this limit corresponds to approaching the “Gaussian fixed point” of the dimensionless four-point coupling $u_0 L^{4-d}$. Neglecting the $y$-dependence in this limit is justified, however, only if $w$ is kept finite, i.e., if $|t|$ vanishes sufficiently strongly. Keeping $w$ finite for $L \to \infty$ is a special case corresponding to paths in the $L^{-1} - \xi^{-1}$ plane where the ratio $\xi/L$ diverges as $L^{(d-4)/4}$. Such paths become asymptotically parallel to the vertical axis (Fig. 1). This includes the special case $T = T_c$, $L \to \infty$. The description of the entire $L^{-1} - \xi^{-1}$ plane, on the other hand, requires both $w$ and $y$ as generic scaling variables in order to correctly include the $w \to \infty$ limit which corresponds to a finite ratio $\xi/L$ for $L \to \infty$. The latter limit $w \to \infty$ cannot be taken correctly within the single-variable scaling structure such as $\tilde{P}_{\chi}(w,0)$ and within the reduced scaling form $\tilde{F}(s,w,0)$. This structure does not capture the complete finite-size effects presented in Eq. (104) and (105) above. For this reason the inhomogeneous fluctuations must be considered as relevant, for finite $L$, in the sense of the renormalization group. For the same reason the reduced scaling form [equivalent to $\tilde{F}(s,w,0)$] that was proposed by Binder et al. [7] for general $\xi/L$ (not only for $\xi/L = \infty$), is not valid.

Our order-parameter distribution function $P(\Phi) \sim \exp(-H^{eff})$ can be compared with the zero-field probability distribution function of Binder [8]...
below $T_c$

$$P_L(s) = \text{const} \left\{ \exp \left[ -(s-m_b)^2L^d/2\chi_b \right] + \exp \left[ -(s+m_b)^2L^d/2\chi_b \right] \right\}$$ \hspace{1cm} (106)

where $m_b \sim |t|^{1/2}$ and $\chi_b \sim |t|^{-1}$ are $L$ independent bulk quantities. In $P_L(s)$ the temperature dependence enters in the form $[L/l(t)]^d$ with the ”thermodynamic length” \cite{8,9,19} $l(t) \sim |t|^{-2/d}$. Our theory identifies the relevant length scale $\tilde{\ell}$ of the corresponding variable $w$, Eq. (3), in terms of a combination of $l_0$ and $\xi_0$ according to Eq. (4). The distribution function $P_L(s)$ has been invoked as an argument in support of the single-variable scaling structure of the free energy (at $h = 0$) proposed by Binder et al. \cite{7}. For finite $\xi/L$, however, our results do not agree with the structure of $P_L(s)$ which does not contain the important $y$ dependence reflected in the shift $\sim I_1L^{2-d}$ in Eqs. (72), (78), (94) that is caused by the inhomogeneous modes. Thus the double Gaussian form of $P_L(s)$, Eq. (106), as well as the underlying theory of Gaussian thermodynamic fluctuations \cite{26}, are not applicable to finite systems with periodic boundary conditions in the critical region above the upper critical dimension. This is remarkable in view of the fact that mean-field theory becomes exact in the bulk limit for $d > 4$.

4.3 Predictions for $d = 5$

We illustrate and further discuss our results for the lattice model for the case $d = 5$ which can be compared with MC data of the five-dimensional Ising model. In Fig. 2 we plot the order-parameter distribution function in terms of $F$, Eqs. (76)-(81),

$$P(\Phi, t, L, u_0) d\Phi = \frac{\exp \left\{ -F(z, x, y) \right\}}{\int_{-\infty}^{\infty} dz \exp \left\{ -F'(z, x, y) \right\}} d\Phi,$$ \hspace{1cm} (107)

for typical values of $x$ and $y$ above, at and below $T_c$. The shape of these functions resembles that of the MC data shown in Fig. 1 of Ref. \cite{9}. In order to demonstrate the effect of the higher modes on these functions we show the order parameter distribution function in Fig. 3 in terms of $\tilde{F}$, Eqs.
\[ P(\Phi, t, L, u_0) \, d\Phi = \frac{\exp[-\tilde{F}(s, w, y)]}{\int_{-\infty}^{\infty} ds \exp[-\tilde{F}(\tilde{s}, w, y)]} \, ds, \quad (108) \]

with given \( w \) but for several values of \( L/l_0 \) including the limiting function for \( L/l_0 \to \infty \) at fixed \( w \). The latter function has the structure \( P_0(\Phi) \) of the lowest-mode approximation (with one-loop expressions for the reference lengths \( \xi_0 \) and \( l_0 \)). This does not mean, however, that \( P_0(\Phi) \) is the exact representation of \( P \) in the large -\( L \) limit in general. The constraint \( w = \text{const} < \infty \) restricts the validity of \( P_0(\Phi) \) only to the special region \(|t|L^{d/2} < \infty \) corresponding to \( \xi/L \to \infty \) (region between the curved dotted lines in Fig. 1). The width of this region in the \( L^{-1} - \xi^{-1} \) plane vanishes asymptotically for \( L \to \infty \). This special region is of interest because it contains characteristic ("pseudocritical" [13]) temperatures close to \( T_c \) such as \( T_{\text{max}}(L) \) and \( T_c(L) \) to be defined below.

The scaling functions \( \tilde{P}_\chi^\pm \), Eqs. (99) and (100), of \( \chi^+ \) and \( \chi^- \) corresponding to the order-parameter distribution function of Fig. 3 are shown in Fig. 4. Similar plots can be made for \( M \) and \( U \). For comparison with MC data we refer to Figs. 11-14 of Ref. [9]. At first sight, the changes due to the variation of \( L/l_0 \) appear to be small. At the level of accuracy of previous MC data [3], however, these changes and their disagreements with the lowest-mode predictions [3] have been clearly detected and have been considered as a major discrepancy, regardless of their smallness, because of their unexplained weak \( L \)-dependence.

Here we further elucidate the deviations from the lowest-mode predictions by plotting in Fig. 5 the scaling functions of \( U, \chi^\pm \) and \( M \) at and below \( T_c \) as functions of the reduced length \( L/l_0 \). As originally found in Ref. [3] for the example of the Binder cumulant at \( T_c \), the slow approach to the asymptotic \((L \to \infty)\) values arises from the \( y^{1/2} \sim L^{(4-d)/2} \) terms. This slow approach was observed in the previous MC data [3] and, at that time, gave sufficient reason to doubt the correctness of the lowest-mode predictions [3] which do not have a weak subleading \( L \)-dependence at \( T_c \). Our theory now shows that subsequent attempts [10-12] to explain the discrepancies did not resolve the problems. In particular, the bulk form of the renormalization-group flow equations employed by Blöte and Luijten [10] led to an apparent confirmation of the (incorrect) shift \( \sim L^{2-d} \) predicted in Ref. [3] for the field-theoretic
model. These bulk flow equations \[10\] do not correctly describe the finite-size effects of the $\varphi^4$ lattice model either. We believe that our theory identifies the origin of the previous discrepancy, apart from possible quantitative aspects which we shall address elsewhere, after a quantitative identification of the lengths $\xi_0$ and $l_0$.

An interesting consequence of the existence of the limiting function $P_0(\Phi)$ mentioned above is the existence of limiting scaling functions such as $\tilde{P}^{-\chi}(w,0)$ of $\chi^{-}$ for $L/l_0 \to \infty$ at fixed $w$ [Fig. 4 (b)]. For finite $L$, $\chi^{-}$ exhibits a maximum $\chi_{\text{max}}$ below $T_c$ at a temperature $T_{\text{max}}(L)$. The asymptotic $L$ dependence of $T_c - T_{\text{max}}(L)$ can be inferred from the fact that $\tilde{P}^{-\chi}(w,0)$ has a temperature dependence only of the form of $w \sim tL^{d/2}$. This implies the large -$L$ behavior $T_c - T_{\text{max}}(L) \sim L^{-d/2}$ and correspondingly $\chi_{\text{max}} \sim L^{d/2}$. Similar arguments lead to our prediction $T_c - T_{\text{c}}(L) \sim L^{-d/2}$ where $T_{\text{c}}(L)$ is the “effective critical temperature” \[3\] at which the magnetization has its maximum slope. The same power law is valid for the temperature at which the specific heat has its maximum. These power laws $\sim L^{-d/2}$ agree with the MC data \[9\]. The true asymptotic amplitudes of these power laws, however, have not been observed in previous MC simulations in $d = 5$ dimensions because of the slow approach of the subleading terms $\sim L^{(4-d)/2}$ towards $L \to \infty$ mentioned above. MC simulations of larger systems would be desirable for testing the magnitude of such subleading terms predicted by our theory.

As indicated already in Figs. 3 and 4 of Ref. \[6\], we also point to an additional interesting effect of practical importance. In Fig. 6 we have plotted the Binder cumulant as a function of $x$ for several values of $L/l_0$. Without the effect of the slowly decaying contribution $\sim y^{1/2} \sim L^{(4-d)/2}$ one would have expected \[4\] a well identifiable intersection point of these curves if $L$ is, say, larger than 10 $\tilde{a}$. For $d < 4$, this features has been a standard and successful empirical method of determining the value of bulk $T_c$ from MC data of finite systems. Our previous \[3\] and present figures demonstrate that this method is not accurately applicable, without additional information, to systems with $d = 5$.

5. Finite-size scaling functions of the $\varphi^4$ field theory

For the field-theoretic model the effective parameters $r_0^{\text{eff}}$ and $u_0^{\text{eff}}$, even in their asymptotic form in Eqs. (74) and (75), depend explicitly on the length $\Lambda^{-1}$, in addition to the lengths $\xi_0, l_0$ and $L$, as found already in the
limit \( n \to \infty \). This means that none of the original nonuniversal model parameters \( a_0, u_0 \) and \( \Lambda \) becomes unimportant even close to \( T_c \) and for large \( L \). In a scaled form the effective parameters read

\[
 r_{\text{eff}}^2 = r_{0 \text{eff}}^2 L^2 = x - 12 I_1(\bar{r}) y - 144 \vartheta_2(y_0) I_2(\bar{r}) y^{3/2} - 12 u_0 \Lambda^{d-4} a_1(d, \bar{r} \Lambda^{-2}) - 144(u_0 \Lambda^{d-4})^{3/2} \vartheta_2(y_0) a_2(d, \bar{r} \Lambda^{-2}) (\Lambda L)^{-d/2} \tag{109}
\]

and

\[
 u_{\text{eff}} = u_{0 \text{eff}} L^{4-d} = y + 36 I_2(\bar{r}) y^2 + 36(u_0 \Lambda^{d-4})^2 a_2(d, \bar{r} \Lambda^{-2}) (\Lambda L)^{2-d} \tag{110}
\]

where \( \bar{r} \) and \( y_0 \) are given in Eqs. (80) and (81). The last term \( \sim L^{2-d} \) in Eq. (110) can be neglected asymptotically.

Substituting these expressions into Eqs. (76), (84) – (88) and (107) (with \( J_0 = 1 \)) yields the finite-size scaling functions of the order-parameter distribution function and of the quantities \( \chi^\pm, M \) and \( U \). The two-variable finite-size scaling functions depend on \( x \) and \( y \) and, in addition, explicitly on the nonuniversal parameter \( u_0 \Lambda^{d-4} \). Thus the scaling functions are nonuniversal for \( n = 1 \), and obviously also for general \( n \). At \( T_c \), the asymptotic power laws of \( \chi \) and \( M \) are found to be

\[
 \chi_c^+(L) = L^2 P_{\chi}^+(0, y) \sim L^{d-2}, \tag{111}
\]

\[
 M_c(L) = L^{(2-d)/2} P_M(0, y) \sim L^{-1} \tag{112}
\]

for the field-theoretic model which differ from those of the lattice model in Eqs. (89) and (90). The asymptotic value of \( U \) at \( T_c \) for the field-theoretic model is in one-loop order

\[
 \lim_{L \to \infty} U(0, y) = 2/3 \tag{113}
\]

which is far from that of the lattice model in Eq. (91). The significant differences between Eqs. (89)-(91) and Eqs. (111)-(113) are due to the \( L \)-independent but cutoff-dependent term \( \sim u_0 \Lambda^{d-4} \) in Eq. (109), similar to
the constant additive term in Eq. (122) of Ref. [4]. Existing MC data for Ising models on \( d = 5 \) lattices (such as the MC result \( U_{MC} = 0.319 \pm 0.017 \) in Ref. [2]) clearly disagree with these field-theoretic results and rule out the possibility that the \( \varphi^4 \) field theory provides a correct description of finite lattice systems above the upper critical dimension. In particular, the prediction of a breakdown of universality for finite systems above the upper critical dimension constitutes a serious failure of the continuum approximation for lattice systems. The present results for finite \( n \) confirm our earlier [4,6,22] assertion regarding the applicability of the \( \varphi^4 \) field theory for \( d > 4 \).

6. Summary and conclusions

We summarize and further comment on the results of this paper as follows.

On the basis of a one-loop calculation for the \( \varphi^4 \) model on a lattice and for the \( \varphi^4 \) continuum model in a cubic geometry with periodic boundary conditions above four dimensions we have shown that our general conclusions regarding universality and finite-size scaling inferred from the large-\( n \) limit [4-6] remain valid for finite \( n \). In particular, \( \varphi^4 \) field theory based on the Landau-Ginzburg-Wilson continuum Hamiltonian [1] does not correctly describe the leading finite-size effects of spin systems on a lattice with \( d > 4 \).

Although the critical exponents of mean-field theory are exact for bulk systems above four dimensions, the thermodynamic theory of Gaussian fluctuations [26] is not applicable to finite systems with periodic boundary conditions in the critical region for \( d > 4 \).

Finite-size scaling in terms of a single scaling variable, as predicted by the phenomenological theory of Binder et al. [7] and by the lowest-mode approximation of Brézin and Zinn-Justin [3], is not valid for the \( \varphi^4 \) field theory. For the \( \varphi^4 \) lattice model it is not valid for any finite \( \xi/L \) where \( \xi \) is the bulk correlation length. As originally conjectured in Ref. [22], lowest-mode like large -\( L \) behavior is asymptotically correct for the lattice model at \( T_c \) as shown previously for the susceptibility [4–6] and the Binder cumulant [1]; furthermore, it is valid in the small region of finite \( |w| \sim |t|L^{d/2} \) in the large -\( L \) and small \( |t| \) limit (Fig. 1). This region, corresponding to a divergent ratio \( \xi/L \sim L^{(d-4)/4} \rightarrow \infty \), represents only a small part (between the two curved dotted lines of Fig.1) of the general finite-size scaling regime (of arbitrary finite \( \xi/L \)) for which earlier theories [3, 6] were originally thought to be valid. Our two-variable finite-size scaling structure is consistent with that
proposed by Privman and Fisher [21] but is significantly less universal than anticipated previously [15].

The inhomogeneous higher modes have been shown to be relevant above the upper critical dimension, contrary to different statements in the previous literature [1,3,10,12,15,27-43]. We have identified the characteristic length scale \( l_0 \), Eq. (30), of the finite-size effects of the higher modes in terms of the amplitude of the bulk correlation length \( \xi \) at \( T = T_c \) for small external field \( h \). The one-loop finite-size effects arising from the relevant higher modes do not represent "corrections" to the lowest-mode approximation but constitute a generic part of the correct finite-size scaling structure. By contrast, two-loop contributions are expected to represent only quantitative corrections that will not change the scaling structure.

The "shift of \( T_c \)" in the temperature variable \( r^{\text{eff}}_0 \) of the exponential order-parameter distribution function is proportional to \( L^{-2} \) for the field-theoretical model and proportional to \( I(t, L^{-1})L^{2-d} \) for the lattice model where the function \( I(t, L^{-1}) \) has a finite limit \( I(0, 0) \). The effects caused by these shifts remain nonnegligible at any finite ratio \( \xi/L \) even in the large \(-L \) limit as demonstrated in Eqs. (104) and (105) for the example of the susceptibility above and below \( T_c \).

The "shift of \( T_c \)" \( \sim L^{2-d} \) mentioned in the preceding paragraph must be distinguished from shifts of characteristic temperatures, in the sense of pseudocritical temperatures [13], such as the temperature \( T_{\text{max}}(L) \) at which the susceptibility \( \chi^-(t, L) \) has its maximum, or the "effective critical temperature" \( T_c(L) \) where the order parameter has its maximum slope. We find that these "shifts" have the asymptotic (large \( L \)) behavior \( T_c - T_{\text{max}}(L) \sim L^{-d/2} \) and \( T_c - T_{\text{c}}(L) \sim L^{-d/2} \) for the \( \varphi^4 \) lattice model. Similarly our theory implies \( \chi_{\text{max}} \sim L^{d/2} \) asymptotically. This is a simple consequence of the fact that the order-parameter distribution function shown in Fig.3 has a finite limit for \( L \to \infty \) at finite \( w \) and that the position of \( T_{\text{max}}(L) \) and \( T_c(L) \) remain located in the temperature region of finite \( w \) in the limit \( L \to \infty \).

Our theory identifies the possible origin of a significant discrepancy between MC data at \( d = 5 \) [9] and the lowest-mode prediction for the Binder cumulant at \( T_c \) in terms of slowly decaying finite-size terms \( \sim L^{(4-d)/2} \) (Fig. 5). These terms also mask the true asymptotic amplitudes of the power laws \( \chi_c \sim L^{d/2} \) and \( M_c \sim L^{-d/4} \) at \( T_c \). For the same reason the method of determining bulk \( T_c \) (from MC data via the intersection point of the Binder cumulant) is demonstrated in Fig. 6 to become quantitatively
inaccurate at $d = 5$, as found originally in Ref. [6].

Quantitative predictions for various asymptotic finite-size scaling functions have been made for $d = 5$ and $n = 1$ (Figs. 2−6). These predictions are expected to be valid for sufficiently large $L/a$ and small $|t|$. The true range of applicability remains to be explored by quantitative comparisons with MC data, after an appropriate identification of the nonuniversal lengths $\xi_0$ and $l_0$. As noted previously [4], it is not yet established whether the $\varphi^4$ model on a finite lattice is fully equivalent to finite spin models regarding the leading and subleading finite-size effects.

Our results indicate that in the limit $d - 4 \to 0^+$, for systems with periodic boundary conditions, different amplitudes of finite-size effects at $d = 4$ are obtained depending on whether a lattice model or a continuum model is considered. In view of this possible ambiguity at $d = 4$ the limiting behaviour for $4 - d \to 0^+$ (i.e., $\epsilon \to 0^+$ in the standard $\epsilon = 4 - d$ expansion) should also be reexamined for lattice models at finite lattice spacing and for continuum models at finite cutoff.

**Note added**

After completion of the present work we received a preprint "Finite-size scaling above the upper critical dimension revisited: The case of the five-dimensional Ising model" by E. Luijten, K. Binder, and H.W.J. Blöte where the authors compare the asymptotic result for $U(x, y)$ in the (unexpanded) form of Eqs. (87) and (88) with their Monte Carlo data of the five-dimensional Ising model. The authors confirm the "occurrence of spurious cumulant intersections" predicted in Figs. 3 and 4 of Ref. [6] and agree with the slow convergence of finite-size effects for $L \to \infty$ found in Ref. [6] which essentially resolves the longstanding discrepancies noted in the MC studies in Refs. [7-9] regarding the Binder cumulant. On a much more quantitative level than considered previously [3-12], however, the authors estimate the length $l_0$ as $l_0 = 0.603$ (13) and claim to find new "significant discrepancies" between their MC data for small $L$ and our asymptotic (large $L$) one-loop result for $\chi^+$ in the (unexpanded) form of Eq. (84).

We doubt the significance of the quantitative deviations for small system sizes $L = 4$ and 8 shown in their Figs. 7(a) and (b), except for the *sign and curvature* of the deviations from the large-$L$ behavior of the susceptibility shown in their Figs. 8 and 9. We propose that the latter issue can be resolved essentially on the basis of our complete non-asymptotic one-loop
expression for $H_{\text{eff}}$ presented in Eqs. (47)-(49) of this paper or on the basis of the underlying order-parameter distribution function [24], rather than by an asymptotic two-loop calculation suggested by Luijten et al. We note that similar non-asymptotic effects are well known for small spin systems at $T_c$ in three dimensions as discussed in the context of Fig. 14 of Ref. [18].

We doubt the reliability of the estimate of $l_0 = 0.603 (13)$ by Luijten et al. since it was found by applying the asymptotic ($L \to \infty$) expression for $\chi^c$ in their Eq. (31) to non-asymptotic MC data. Our Fig. 5b indicates that the apparent (6 percent) mismatch between theory and MC data for $L \leq 22$ in Fig. 9 of Luijten et al. is due to this inadequate estimate of $l_0$.

Part of the remarks by Luijten et al. regarding the limiting case $t \to 0$, $L \to \infty$ at fixed $tL^{d/2}$ agree with our earlier and present independent findings. We disagree, however, with their claim that ”there is no contradiction at all” between our finite-size theory and the ideas of Ref. [7]. First, we note that the ideas of Ref. [7] fail for the continuum $\varphi^4$ model. Second, we maintain that the single-variable scaling structure (for $h = 0$) proposed in Ref. [7] does not capture the correct structure of finite-size effects of the lattice model at any finite value of $\xi/L$ (see Fig.1). In particular the ideas of Ref. [7] do not lead to a correct description of the finite-size departures from bulk critical behavior at small but fixed $|t|$ in the asymptotic range $0 < |t| \ll 1$ [compare Eqs. (104) and (105)] to which an acceptable finite-size scaling structure should be applicable (such as the general structure proposed in Ref. [21]).

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Figure Captions

**Fig. 1.** Asymptotic $L^{-1} - \xi^{-1}$ plane (schematic plot) above ($t > 0$) and below ($t < 0$) $T_c$ for the lattice model above four dimensions where $L$ is the system size and $\xi$ is the bulk correlation length (in units of the lattice spacing). The straight dashed lines correspond to paths at constant finite ratio $\xi/L$. The curved dotted lines represent paths at constant finite $tL^{d/2}$ for $L \to \infty$ and $t \to 0$ corresponding to a divergent ratio $\xi/L \sim L^{(d-4)/4} \to \infty$. The single-variable scaling form in terms of $w = t(L/\ell)^{d/2}$, Eqs. (3) and (4), is valid only in the region between the curved dotted lines. The two-variable scaling form in terms of $x = tL^2/\xi^2$ and $y = (L/l_0)^{4-d}w$ or $w$ and $y$ is necessary in the entire $L^{-1} - \xi^{-1}$ plane where $\xi/L$ is finite.

**Fig. 2.** Theoretical prediction of the scaled order-parameter distribution function $P(\Phi, t, L, u_0)L^{-(d-2)/2}J_0^{1/2}$ of the $\varphi^4$ lattice model in $d = 5$ dimensions vs $z$, Eq. (77), in the form of Eq. (107) where $F(z, x, y)$ is given by Eqs. (76)–(81). The parameter values are $x = 0.25$ above $T_c$ (a), $x = 0$ at $T_c$ (b), and $x = -0.75$ below $T_c$ (c), where $|x| = L^2/\xi^2$. The reduced length is $y^{-1} = L/l_0 = 16$. Compare Fig. 1 of Ref. [9].

**Fig. 3.** Theoretical prediction of the scaled order-parameter distribution function $P(\Phi, t, L, u_0)L^{-d/4}J_0^{(4-d)/4}$ of the $\varphi^4$ lattice model in $d = 5$ dimensions vs $s$, Eq. (93), in the form of Eq. (108) where $\tilde{F}(s, w, y)$ is given by Eqs. (92) – (96). The parameter values are $w = 1$ above $T_c$ (a), $w = 0$ at $T_c$ (b), and $w = -3$ below $T_c$ (c) where $w = t(L/\ell)^{d/2}$. The curves are shown for two reduced lengths $L/l_0 = 8$ (dotted lines) and $L/l_0 = 32$ (dashed lines) as well as for the limiting case $L/l_0 = \infty$ (solid lines) at fixed $w$.

**Fig. 4.** Theoretical prediction of the finite-size scaling functions $\tilde{P}^+(w, y)J_0^{(d-4)/2}$ and $\tilde{P}^-(w, y)J_0^{(d-4)/2}$ of the susceptibilities $\chi^+$, Eq. (99), (a) and $\chi^-$, Eq. (100), (b) of the lattice model in $d = 5$ dimensions vs $w = t(L/\ell)^{d/2}$ for several values of the reduced length $L/l_0$, corresponding to Fig. 3, including the limiting case $L/l_0 = \infty$ at fixed $w$. In this representation the position of the maximum of $\tilde{P}^-\chi$ attains a finite value for $L/l_0 \to \infty$ which determines $T_c - T_{\max} \sim L^{-d/2}$. Compare Fig. 13 of Ref. [9].

**Fig. 5.** Theoretical prediction of various finite-size scaling functions of the lattice model in $d = 5$ dimensions, $U(0, y)$ (a), $\tilde{P}^+(0, y)J_0^{(d-4)/2}$ (b), $\tilde{P}^-_M(0, y)J_0^{(d-4)/4}$ (c), $\tilde{P}^-(-3, y)J_0^{(d-4)/2}$ (d), $\tilde{P}^-(-6, y)J_0^{(d-4)/2}$ (e) according to Eqs. (99)–(103), as a function of the scaled length $y^{-1} = L/l_0$. The arrows indicate the asymptotic one-loop values for $L \to \infty$. Compare
Fig. 6. Theoretical prediction of the Binder cumulant $U(x, y)$, Eq. (87), as a function of the scaled reduced temperature $x = t(L/\xi_0)^2$, Eq. (1), in the range $-4 \leq x \leq 4$ (a) and in the range $-0.4 \leq x \leq 0.4$ (b), for several values of the scaled length $y^{-1} = (L/l_0)^{d-4}$, at $d = 5$: $L/l_0 = 8$ (dotted line), $L/l_0 = 16$ (dot-dashed line), $L/l_0 = 32$ (dashed line). The bulk value [thin solid line in (a)] is $2/3$ below $T_c$ and 0 above $T_c$. The cross indicates the asymptotic one-loop value $U(0, 0) = 0.2705$ at $T_c, L \to \infty$, Eq. (91). Compare Figs. 3 and 4 of Ref. [6].
\[ P \frac{L^{(2-d)/2}}{J_0^{1/2}} \]

(a)

\[ x = 0.25 \]

\[ \frac{L}{l_0} = 16 \]
\[ \frac{P L}{L_{0}^{1/2}} = J_{0} \]

\[ x = -0.75 \]

\[ \frac{L}{L_{0}} = 16 \]
\[ P \frac{L}{4} J_0 \left( \frac{(4-d)}{4} \right) = w \]

(a)

- \( \cdots \) \( L / l_0 = 8 \)
- \( \cdots \cdots \) \( 32 \)
- \( \cdots \cdots \cdots \) \( \text{infinite} \)
\[ P L^{-d/4} J_0^{1/2} \left( \frac{(4d)^{1/4}}{l_0} \right) \]

(b)

\[ w = 0 \]

\[ L / l_0 = 8, \quad 32, \quad \text{infinite} \]
The diagram shows the results of a calculation involving the variable $w = -3$.

The equation $L / l_0 = 8$, $32$, and infinite are represented by dotted, dashed, and solid lines, respectively. The variable $s$ is plotted on the x-axis, while the function $P L^{1/4} J_0^{1/2} l_0^{(4d)/4}$ is plotted on the y-axis, with values ranging from 0 to 0.5.
\( \chi^L_{J_0} \text{ for } (d-4)/2 \)
(b)
\[ T = T_c \]
\[ T = T_C \]
$w = -3.0$

(d)
\( \chi L^{-d/2} J_0 (d/4)^2 \)

\( w = -6.0 \)
